Oriented chromatic number of Halin graphs

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Abstract
Oriented chromatic number of an oriented graph $G$ is the minimum order of an oriented graph $H$ such that $G$ admits a homomorphism to $H$. The oriented chromatic number of an unoriented graph $G$ is the maximal chromatic number over all possible orientations of $G$. In this paper, we prove that every Halin graph has oriented chromatic number at most 8, improving a previous bound by Hosseini Dolama and Sopena, and confirming the conjecture given by Vignal.

Keywords: Graph coloring, oriented graph coloring, Halin graph, oriented chromatic number

1. Introduction

Oriented coloring is a coloring of the vertices of an oriented graph $G$ such that: (1) no two neighbors have the same color, (2) for every two arcs $(t, u)$ and $(v, w)$, either $\beta(t) \neq \beta(w)$ or $\beta(u) \neq \beta(v)$. In other words, if there is an arc leading from the color $\beta_1$ to $\beta_2$, then no arc leads from $\beta_2$ to $\beta_1$.

It is easy to see that an oriented graph $G$ can be colored by $k$ colors if and only if there exists a homomorphism from $G$ to an oriented graph $H$ with $k$ vertices. In this case we shall say that $G$ is colored by $H$.

The oriented chromatic number $\chi'(G)$ of an oriented graph $G$ is the smallest number $k$ of colors needed to color $G$, and the oriented chromatic number $\chi'(G)$ of an unoriented graph $G$ is the maximal chromatic number over all possible orientations of $G$. The oriented chromatic number of a family of graphs is the maximal chromatic number over all possible graphs of the family.

Oriented coloring has been studied in recent years [2, 3, 5, 7, 8, 9, 10, 12, 13, 14, 15], see [11] for a short survey of the main results. Several authors established or gave bounds on the oriented chromatic number for some families of graphs, such as: oriented planar graphs [9], outerplanar graphs [12, 13], graphs with bounded degree three [8, 12, 14], $k$-trees [12], graphs with given excess [3], grids [4, 5, 15] or hexagonal grids [1].

In this paper we focus on the oriented chromatic number of Halin graphs. A Halin graph $H$ is an unoriented planar graph which admits a planar embedding such that deleting the edges of its external face ($F_0$) gives a tree with at least three leaves. The vertices on $F_0$ are called exterior vertices of $H$, and the remaining vertices are called interior vertices of $H$.

In [16] Vignal proved that every oriented Halin graph has oriented chromatic number at most 11 and conjectured that the oriented chromatic number of every oriented Halin graph is at most 8. Hosseini Dolama and Sopena proved in [7] that every oriented Halin graph has oriented chromatic number at most 9 and they presented an oriented Halin graph with oriented chromatic number equal to 8. Figure 1 presents another example of Halin graph with oriented chromatic number equal to 8. Determining the exact value of oriented chromatic number of Halin graph is an open problem presented by Sopena in [11].

2. Preliminaries

We recall that $T_7$ is the tournament build from the non-zero quadratic residues of 7, see [2, 3, 5, 12, 14]. More precisely, $T_7$ is the graph with vertex set $\{0, 1, \ldots, 6\}$ and such that $(i, j)$ is an arc if and only if $j - i = 1, 2, 4 \pmod{7}$.

Figure 1: Halin graph with oriented chromatic number equal to 8.

In this paper we shall prove that every oriented Halin graph can be colored with at most 8 colors. Hence, the oriented chromatic number of the family of Halin graphs is equal to 8.

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Lemma 1 (see [15]). For every \( a \in \{1, 2, 4\} \) and \( b \in \{0, 1, \ldots, 6\} \), the function \( \phi(x) = ax + b \pmod{7} \) is an automorphism in \( T_7 \).

Lemma 2 (see [15]). Let \( G \) be an oriented graph, \( h \) a homomorphism from \( G \) into \( T_7 \), and let \( G' \) be the oriented graph obtained from \( G \) by reversing all arcs. More precisely, \((u, v)\) is an arc in \( G'\) if and only if \((v, u)\) is an arc in \( G \).

Then the function \( f(x) = -h(x) \pmod{7} \) is a homomorphism from \( G' \) into \( T_7 \).

In the sequel we shall simply write \( ax + b \) instead of \( ax + b \pmod{7} \) when writing about homomorphisms of \( T_7 \).

A fan \( F \) is an oriented planar graph which consists of a rooted oriented tree with a root \( r \) and leaves \( x_1, \ldots, x_m \); and for every \( 1 \leq i \leq m - 1 \), the leaves \( x_i \) and \( x_{i+1} \) are connected by an arc: \((x_i, x_{i+1})\) or \((x_{i+1}, x_i)\). We shall denote the root of \( F \) by \( r(F) \), the first leaf \( x_1 \) by \( fl(F) \), and the last leaf \( x_m \) by \( ll(F) \).

Note that if we remove one vertex or arc from the exterior cycle of a Halin graph, then we obtain a fan. But we shall also consider other fans, e.g. with one leaf, \( m = 1 \), or with the root having only one son.

Suppose we have two fans \( F_1 \) and \( F_2 \). We can compose them in one fan \( F \), denoted by \( F_1 + F_2 \), in the following way, see Fig. 2.

- root of \( F_1 \) becomes the root of \( F \), i.e. \( r(F) := r(F_1) \),
- \( r(F_1) \) is joined with \( r(F_2) \) by an arc \( s_1 \), i.e. \( s_1 = (r(F_1), r(F_2)) \) or \( s_1 = (r(F_2), r(F_1)) \),
- \( ll(F_1) \) is joined with \( fl(F_2) \) by an arc \( s_2 \),
- the first leaf of \( F_1 \) becomes the first leaf of \( F \), i.e. \( fl(F) := fl(F_1) \),
- the last leaf of \( F_2 \) becomes the last leaf of \( F \), i.e. \( ll(F) := ll(F_2) \), and \( r(F_2) \) becomes the last son of \( r(F_1) \).

Proof. The main idea of the proof is to find two automorphisms \( \phi_1 \) and \( \phi_2 \) of \( T_7 \) which change the coloring \( c_2 \) on \( F_2 \) in such a way that the new colorings \( \phi_1 \circ c_2 \) and \( \phi_2 \circ c_2 \) will fit to coloring \( c_1 \) on \( F_1 \), and be different from each other on \( ll(F_2) \). More precisely, we shall show that for any colors \( c_1(ll(F_1)) \), \( c_2(ll(F_2)) \), \( c_2(ll(F_2)) \) and any direction of \( s_1 \) and \( s_2 \), there exist two automorphisms

\[ \phi_1, \phi_2 : T_7 \to T_7 \]

such that the colorings \( d_1 \) and \( d_2 \) defined by:

\[
d_1(x) = \begin{cases} 
  c_1(x) & \text{if } x \in F_1 \\
  \phi_1(c_2(x)) & \text{if } x \in F_2 
\end{cases}
\]

and

\[
d_2(x) = \begin{cases} 
  c_1(x) & \text{if } x \in F_1 \\
  \phi_2(c_2(x)) & \text{if } x \in F_2 
\end{cases}
\]

color the composition \( F = F_1 + F_2 \) in a proper way and satisfy (c2).

We can assume that \( s_1 \) goes from \( r(F_1) \) to \( r(F_2) \). Otherwise we can reverse all arcs, negate all colors, color \( F_1 + F_2 \), and reverse arcs and negate colors back. We can also assume that \( c_1(ll(F_1)) \), and \( c_2(ll(F_2)) \), are both in \( \{1, 3\} \). Otherwise we can multiply all colors in \( F_1 \), or \( F_2 \), by 2 or 4.

Consider first the case when \( c_1(ll(F_1)) = 3 \), \( c_2(ll(F_2)) = 1 \), and \( s_2 \) goes from \( ll(F_1) \) to \( ll(F_2) \). In this case we first consider automorphisms \( \phi_1(x) = x + 4 \) and \( \phi_2(x) = 2x + 2 \).

The arc \( s_1 = (r(F_1), r(F_2)) \) has colors \( (0, \phi_1(0)) = (0, 4) \) or \( (0, \phi_2(0)) = (0, 2) \), which are proper. The arc \( s_2 = (ll(F_1), ll(F_2)) \) has colors \( (c_1(ll(F_1)), c_1(ll(F_2))) = (3, \phi_1(1)) = (3, 5) \) or \( (3, \phi_2(1)) = (3, 4) \) which are proper.

Moreover, for every color \( x \neq 2, \phi_1(x) \neq \phi_2(x) \). Hence, if \( c_2(ll(F_2)) \neq 2 \), then \( \phi_1(c_2(ll(F_2))) \neq \phi_2(c_2(ll(F_2))) \), so \( d_1(ll(F_2)) \neq d_2(ll(F_2)) \).

If \( c_2(ll(F_2)) = 2 \), then we make change and set \( \phi_2(x) = 4x + 4 \). The new \( \phi_2 \) also gives a proper coloring \( d_2 \), and \( d_1(ll(F_2)) \neq d_2(ll(F_2)) \), if \( c_2(ll(F_2)) = 2 \). For all other cases the definitions of \( \phi_1 \) and \( \phi_2 \) are given in Table 4.
every $\phi_i$ from the table, the arc $s_1 = (r(F_1), r(F_2))$ has colors $(0, \phi_i(0)) = (0, 1), (0, 2)$ or $(0, 4)$ which are proper. It is easy to see that in every case, the coloring of the arc $s_2$, with color $c_1(ll(F_1))$ on one end and $\phi_i(c_2(ll(F_2)))$ on the other, is proper. In every line in the table, $\phi_i(x) \neq \phi_2(x)$, for every $x \neq 0$. Hence, $d_1(ll(F_2)) = \phi_i(c_2(ll(F_2))) \neq \phi_2(c_2(ll(F_2))) = d_2(ll(F_2))$, for every $c_2(ll(F_2))$. From the lemma assumptions, $c_2(ll(F_2)) \neq 0$. \hfill $\Box$

**Lemma 4.** For every fan $F$, there is a coloring $c : F \to T_7$ such that $c(r(F)) = 0$, $c(ll(F)) \neq 0$, and $c(ll(F)) = 0$.

**Proof.** Proof by induction on $n$ — the number of vertices of $F$. If $n = 2$, then $F$ consists of the root and one leaf and the lemma is obvious. If $n \geq 3$, take $r$ — the root of $F$ and let $x_1, \ldots, x_k$ be its sons. We have three cases.

1. $k = 1$,
2. $k \geq 2$ and the last son $x_k$ belongs to the exterior path,
3. $k \geq 2$ and the last son $x_k$ belongs to the interior tree.

Case 1. $r$ has only one son $x_1$. This son $x_1$ belongs to the interior of $F$, because $F$ has more than two vertices. Let $F_1$ be the fan rooted in $x_1$, and suppose that arc is going from $r$ to $x_1$. By induction, there is a coloring $c_1 : F_1 \to T_7$ such that $c_1(x_1) = 0$, and $c_1(ll(F_1)) = 0$. Now define the coloring $c : F \to T_7$ as follows:

$$c(x) = \begin{cases} 0 & \text{if } x = r \\ c_1(x) + b & \text{if } x \in F_1 \\ \end{cases}$$

where $b \in \{1, 2, 4\}$ is a constant satisfying conditions $c_1(ll(F_1)) + b \neq 0$ and $c_1(ll(F_1)) + b \neq 0$, such $b$ exists.

Case 2. Let $F_1$ be the fan obtained by removing $x_k$. By induction, there is a proper coloring $c_1 : F_1 \to T_7$. The vertex $x_k$ is connected by arcs only with $r$ (having color $c_1(r) = 0$) and $ll(F_1)$ (having color $c_1(ll(F_1)) = 0$). It is easy to see that $x_k$ can be colored in a proper way.

Case 3. Let $F_2$ be the fan rooted in $x_k$ and $F_1$ the fan obtained from $F$ by removing $x_k$ and its descendants. By induction, $F_1$ and $F_2$ can be properly colored and, by Lemma 5, also their composition $F$ can be colored in a proper way. \hfill $\Box$

### 3. Main result

**Theorem 5.** Every oriented Halin $H$ graph can be colored with 8 colors.

**Proof.** If $H$ has only 3, 4, or 5 vertices on the exterior cycle, then we can color them with at most five colors, each vertex with different color, and the interior tree with additional three colors. Hence, in the sequel we shall consider Halin graphs with at least six vertices on the exterior cycle.

If not all arcs on the exterior cycle are going in the same direction, then we have three vertices $v_1, v_2, v_3$ on the exterior cycle and arcs $(v_1, v_2), (v_3, v_2)$; and let $r$ be the father of $v_2$ in the interior tree ($r$ does not have to be the father of $v_1$ or $v_3$), see Fig. 3.

Remove $v_2$, and consider the fan $F$ with the root in $r$ and leaves going from $v_1$ to $v_3$ around the whole graph $H$. By Lemma 4 there is a coloring $c : F \to T_7$ such that $c(r) = 0, c(v_1) \neq 0$, and $c(v_3) \neq 0$. Now put back $v_2$ with color 7.

In the sequel we shall consider Halin graphs with all arcs on the exterior cycle going in the same direction. Suppose first that there are at least two vertices in the interior. Let $r$ be one on the lowest level of the interior tree, $p$ be its father in the interior, and $x_1, \ldots, x_k$ the sons of $r$ on the cycle, see Fig. 3. Let $x_0$ be the predecessor of $x_1$ on the cycle, and $x_{k+1}$ the successor of $x_k$ (it is possible that $x_0 = x_{k+1}$). The arcs on the cycle are going from $x_0$ to $x_1$ and so on. We have four cases:

1. $k = 1$,
2. $k \geq 2$ and there is the arc $(r, x_1)$,
3. $k \geq 2$ and there is the arc $(x_k, r)$,
4. $k \geq 2$ and there are arcs $(x_i, r)$ and $(r, x_{i+1})$, for some $1 \leq i \leq k - 1$.

Case 1. If $k = 1$, see Fig. 3 we remove $x_1$ and $r$, and obtain the fan with the root in $p$. By Lemma 4 we can color it with $c(p) = 0, c(x_0) \neq 0$, and $c(x_2) \neq 0$. If $c(x_0) \neq c(x_2)$, then we set $x_1$ to 7, and for $r$ we choose the color that fits to the color in $p$ and is different from $c(x_0)$ and $c(x_2)$ (there are three colors for $r$ that fit to the color in $p$).

If $c(x_0) = c(x_2)$, consider first colors for $r$ and $x_1$ which are in accordance only with two arcs: one joining $p$ with $r$ and the other, joining $r$ with $x_1$. We can obtain at least 6 different colors for $x_1$, and either three of them are proper for the arc $(x_0, x_1)$ or three are proper for $(x_1, x_2)$. In the former case we set $x_2$ to 7, put back $x_1$ and $r$, and color $x_1$ and $r$ in such a way that color in $x_1$ is different from the colors of the neighbors of $x_2$. In the later case we set
By Lemma 3, there are two colorings such that: 

**Case 3.**

If \( d_3(x_0) = 0 \) (or \( d_2(x_0) = 0 \)), then we simply put the arc \((x_0, x_1)\) back. If \( d_1(x_0) \neq 0 \) and \( d_2(x_0) \neq 0 \), then we choose coloring, say \( d_1 \), which gives \( d_1(x_0) \neq d_1(x_2) \), put the arc \((x_0, x_1)\) back, and set \( x_1 \) to 7.

**Case 4.** \( k \geq 2 \) and the arc is going from \( r \) to \( x_1 \), see Fig. 7. We remove the arc \((r, x_1)\) from \( r \) and its sons on exterior cycle, and \( F_2 \) formed by \( p \) and the other vertices in \( H \). By Lemma 3 they can be colored by \( T_7 \). Now compose the fans by adding the arc \((x_1, r)\) back, and set \( x_1 \) to 7.

|       | \( F(F_1) \) | \( F(F_2) \) | \( s_2 \) | \( \phi_1(x) \) | \( \phi_2(x) \) | \( s_2 \) colored by \( d_1 \) | \( s_2 \) colored by \( d_2 \) |
|-------|------------|------------|----------|----------------|----------------|----------------|----------------|
| 1     | 1          | \( F(F_1) \lor F(F_2) \) | \( x + 1 \) | \( x + 2 \) | \( 1 \lor 2 \) | \( 1 \lor 3 \) | \( \) |
| 1     | 1          | \( F(F_1) \lor F(F_2) \) | \( 2x + 2 \) | \( 2x + 4 \) | \( 1 \lor 4 \) | \( 1 \lor 6 \) | \( \) |
| 1     | 3          | \( F(F_1) \lor F(F_2) \) | \( 2x + 4 \) | \( 4x + 4 \) | \( 1 \lor 3 \) | \( 1 \lor 2 \) | \( \) |
| 1     | 3          | \( F(F_1) \lor F(F_2) \) | \( x + 1 \) | \( x + 4 \) | \( 1 \lor 4 \) | \( 1 \lor 0 \) | \( \) |
| 3     | 1          | \( F(F_1) \lor F(F_2) \) | \( 4x + 2 \) | \( 4x + 4 \) | \( 3 \lor 6 \) | \( 3 \lor 1 \) | \( \) |
| 3     | 3          | \( F(F_1) \lor F(F_2) \) | \( x + 1 \) | \( x + 2 \) | \( 3 \lor 4 \) | \( 3 \lor 5 \) | \( \) |
| 3     | 3          | \( F(F_1) \lor F(F_2) \) | \( 4x + 1 \) | \( 4x + 4 \) | \( 3 \lor 6 \) | \( 3 \lor 2 \) | \( \) |

Table 1: Definition of \( \phi_1(x) \) and \( \phi_2(x) \).
What is left is the case where there is only one vertex \( r \) in the interior and all arcs on the exterior cycle are going in the same direction. There are two cases:

- all arcs incident with \( r \) are going in one direction. Then color exterior with 5 colors and add sixth color to \( r \),

- not all arcs incident with \( r \) are going in the same direction. Then we have the situation described in Case 4 above.

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