Triangular Gatzouras–Lalley-type planar carpets with overlaps

István Kolossváry1,2 and Károly Simon1

1 MTA-BME Stochastics Research Group, Budapest University of Technology and Economics, PO Box 91, 1521 Budapest, Hungary
2 MTA Alfréd Rényi Institute of Mathematics, Budapest, Hungary

E-mail: istvanko@math.bme.hu and simonk@math.bme.hu

Abstract
We construct a family of planar self-affine carpets with overlaps using lower triangular matrices in a way that generalizes the original Gatzouras–Lalley carpets (Gatzouras and Lalley 1992 Indiana Univ. Math. J. 41 533–68) defined by diagonal matrices. Assuming the rectangular open set condition, Barański (2008 Discrete Continuous Dyn. Syst. A 21 1015–23) proved for this construction that for typical parameters, which can be explicitly checked, the inequalities between the Hausdorff, box and affinity dimension of the attractor are strict. We generalize this result to overlapping constructions, where we allow complete columns to be shifted along the horizontal axis (as in Fraser and Shmerkin (2016 Ergod. Theor. Dyn. Syst. 36 2463–81) and Pardo-Simón (2019 Ergod. Theor. Dyn. Syst. pp 733–63) or allow parallelograms to overlap within a column in a transversal way. Our main result is to show sufficient conditions under which these overlaps do not cause a drop of the dimension of the attractor. Several examples are provided to illustrate the results, including a self-affine smiley, a family of self-affine continuous curves, examples with overlaps and an application of our results to some three-dimensional systems.

Keywords: self-affine set with overlaps, Gatzouras–Lalley planar carpet, Hausdorff dimension, box dimension, Ledrappier–Young formula
Mathematics Subject Classification numbers: Primary 28A80
Secondary 28A78

(Some figures may appear in colour only in the online journal)
Informal introduction

Gatzouras–Lalley carpets [21] are the attractors of self-affine iterated function systems (IFS) on the plane whose first level cylinders are aligned into columns using orientation preserving maps with linear parts given by diagonal matrices, see left-hand side of figure 1. In this paper, we consider a natural generalization of such carpets by replacing the diagonal matrices with lower triangular ones so that the column structure is preserved, see right-hand side of figure 1 and definition 1.1.

We call them Triangular Gatzouras–Lalley-type (TGL) planar carpets, indicating that the linear parts of the maps defining the IFS are triangular matrices and it is a natural generalization of the Gatzouras–Lalley construction. Such a particular TGL carpet (see the left-hand side of figure 2) appeared in the paper of Falconer and Miao [13] and later in the paper of Bárány [4]. For this particular example, the box and the Hausdorff dimension of the attractor are the same. However, Barański [3] showed that this is not true in general for TGL carpets under the assumption that the interiors of the first level cylinders are disjoint (like the image in the right-hand side of figure 1). In this paper, we further generalize this result by allowing different types of overlaps between the cylinders.

We distinguish three different kinds of TGL carpets with overlapping cylinders, see figure 3. Under some conditions, we compute the (typically different values of the) box- and Hausdorff dimension for carpets like the first two on figure 3. If the overlaps are as sophisticated as on the right-hand side of figure 3 then we can compute only the Hausdorff dimension. The Hausdorff dimension is not equal to the box dimension in the examples of figure 3, but they are equal for the example on the right-hand side of figure 2.

0.1. Methods used to handle overlaps

For each type of overlap and dimension we use different methods:

- The upper bounds on the Hausdorff and box dimensions (after some simple observations) follow from suitable adaptations of the results of Gatzouras–Lalley [21] and Fraser [19], respectively.
- To estimate the Hausdorff dimension from below we use the Ledrappier–Young formula of Bárány and Käenmäki [6] (cited in theorem 3.4) for self-affine measures. We show that this lower bound equals the upper bound
  - in case of overlapping like on the second figure of figure 3 by an argument inspired by the transversality method introduced in [8];
  - in case of overlapping like on the third figure of figure 3, we introduce a new separation condition for the self-similar IFS obtained as the projection of the TGL carpet under consideration to the horizontal line. This separation condition is a non-trivial consequence of Hochman’s exponential separation condition (HESC) [22]. We prove this in appendix since it can be of separate interest.
- To estimate the box dimension from below we could not simply use the Hausdorff dimension of the attractor, because, in our case, it is (typically) strictly smaller than the box dimension. Therefore,
  - in case of overlapping like on the first figure of figure 3, we use the method of Fraser and Shmerkin [20]: the main idea is to pass to a special subsystem of a higher iterate of the IFS which has non-overlapping columns;
Figure 1. The IFS defining a Gatzouras–Lalley carpet on the left and a triangular Gatzouras–Lalley-type carpet on the right.

Figure 2. The attractor $\Lambda_a$ from section 7.2 with parameter $a = 3/10$ (left) and section 7.3 with parameter $a = 3/20$ (right), shown together with the outlines of the images $f_i([0,1]^2)$.

Figure 3. Triangular Gatzouras–Lalley-type carpets with different overlaps. Left: shifted columns satisfying Hochman’s exponential separation condition (HESC). Center: non-overlapping columns, transversality condition. Right: mixture of both.
– in case of overlapping like on the second figure of figure 3, we introduce a new argument to count overlapping boxes. It uses transversality and a result of Lalley [26] based on renewal theory, which gives the precise asymptotics of the number of boxes needed to cover the projection of the attractor to the horizontal line.

**Remark 0.1.** An extended version of this paper can be found on the arXiv [25], which contains details of the proofs omitted in this manuscript.

1. Formal introduction

A self-affine IFS is a finite list of contracting affine mappings on \( \mathbb{R}^d \) of the form

\[
\mathcal{F} = \{ f_i(x) := A_i x + t_i \}_{i=1}^N
\]

where \( A_i \) are non-singular \( d \times d \) matrices and \( t_i \in \mathbb{R}^d \) are translation vectors. It is well-known that there exists a unique non-empty compact subset \( \Lambda_\mathcal{F} = \Lambda \) of \( \mathbb{R}^d \), called the self-affine set or the attractor associated to \( \mathcal{F} \), such that

\[
\Lambda = \bigcup_{i=1}^N f_i(\Lambda).
\]

For basic dimension theoretic definitions such as the Hausdorff, packing and (lower and upper) box dimension of a set and the Hausdorff and local dimension of measures we refer to Falconer [15]. Throughout, the Hausdorff, packing, lower and upper box dimension will be denoted by \( \dim_H, \dim_P, \dim_B, \overline{\dim}_B \), and \( \underline{\dim}_B \) respectively.

A general upper bound for all aforementioned dimensions is given by the affinity dimension \( \dim_{aff} \), introduced by Falconer [14, theorem 5.3], which comes from the ‘most natural’ cover of the set. All self-affine sets satisfy

\[
\dim_H \Lambda \leq \dim_P \Lambda \leq \overline{\dim}_B \Lambda \leq \min\{ \dim_{aff} \Lambda, d \}.
\]

In a generic sense, equality of dimensions is typical. Falconer proved in his seminal paper [14] that for fixed linear parts \( \{ A_1, \ldots, A_N \} \) if \( ||A_i|| < 1/3 \) and the translations are chosen randomly according to \( N \times d \) dimensional Lebesgue measure then all the aforementioned dimensions of the self-affine set are equal. The 1/3 bound was later relaxed by Solomyak [35] to 1/2, which is sharp due to an example of Przytycki and Urbański [33]. Very recently Bárány, Hochman and Rapaport [5] greatly improved these results in two dimensions by giving specific, but mild conditions on \( \{ A_1, \ldots, A_N \} \) under which the dimensions are equal. However, in some cases, which do not fall under these conditions, strict inequality is possible. Planar carpets form a large class of examples in \( \mathbb{R}^2 \) for which this exceptional behavior is typical. The highly regular column and/or row structure causes the drop of the Hausdorff dimension. We continue with the formal definition of TGL carpets and then present some pictures to informally explain our contribution.

1.1. Triangular Gatzouras–Lalley-type carpets

Denote the closed unit square by \( R = [0, 1] \times [0, 1] \). Let \( \mathcal{A} = \{ A_1, \ldots, A_N \} \) be a family of \( 2 \times 2 \) invertible, strictly contractive, real-valued lower triangular matrices. The corresponding self-affine IFS is the collection of affine maps

\[
\mathcal{F} = \{ f_i(x) := A_i x + t_i \}_{i=1}^N, \quad \text{where} \quad A_i = \begin{pmatrix} b_i & 0 \\ d_i & a_i \end{pmatrix} \quad \text{and} \quad t_i = \begin{pmatrix} t_{i,1} \\ t_{i,2} \end{pmatrix},
\]

wherein

\[ a_i = 0, \quad b_i = 1, \quad d_i = -1, \quad t_{i,1} = 0, \quad t_{i,2} = 0. \]
for translation vectors \( t_i \), with \( t_{i,1}, t_{i,2} \geq 0 \). We assume that \( a_i, b_i \in (0, 1) \).

The orthogonal projection of \( F \) to the horizontal \( x \)-axis, denoted \( \text{proj}_x \), generates an important self-similar IFS on the line

\[
\tilde{H} = \{ \tilde{h}_i(x) := b_i x + t_{i,1} \}_{i=1}^N.
\]

We denote the attractor of \( F \) and \( \tilde{H} \) by \( \Lambda = \Lambda_F \) and \( \Lambda_{\tilde{H}} \) respectively.

**Definition 1.1.** We say that an IFS of the form (1.1) is **triangular Gatzouras–Lalley-type (TGL)** and we call its attractor \( \Lambda \) a triangular Gatzouras–Lalley-type planar carpet (TGL carpet for short) if the following conditions hold:

(a) direction- \( x \) dominates, i.e.

\[
0 < a_i < b_i < 1 \quad \text{for all } i \in [N] := \{1, 2, \ldots, N\},
\]

(b) column structure: there exists a partition of \( [N] \) into \( M > 1 \) sets \( \mathcal{I}_1, \ldots, \mathcal{I}_M \) with cardinality \( |\mathcal{I}_i| = N_i > 0 \) so that

\[
\mathcal{I}_1 = \{1, \ldots, N_1\} \quad \text{and} \quad \mathcal{I}_i = \{N_1 + \ldots + N_{i-1} + 1, \ldots, N_1 + \ldots + N_i\}
\]

for \( i = 2, \ldots, M \). Assume that for any two distinct indices \( k, \ell \in \{1, \ldots, N\} \)

\[
\begin{cases}
  b_k = b_\ell :=: r_i, \\
  \ell_{k,1} = \ell_{\ell,1} :=: u_i.
\end{cases}
\]

(c) we assume that \( \sum_{j \in \mathcal{I}_i} a_j \leq 1 \) holds for every \( i \in \{1, \ldots, M\} \) and we also assume non-overlapping column structure. This means that

\[
u_i + r_i \leq u_i + 1 \quad \text{for} \quad i = 1, \ldots, M - 1 \quad \text{and} \quad u_M + r_M \leq 1.
\]

(d) Without loss of generality, we always assume in this paper that

- \((A1)\) \( f_i(R) \subset R \) for all \( i \in [N] \) and
- \((A2)\) The smallest and the largest fixed points of the functions of \( H \) are 0 and 1 respectively.

Observe that the definition allows overlaps within columns (like in the second part of figure 3), but the interiors of different columns do not overlap.

We say that \( \Lambda \) is a **shifted TGL carpet** if we drop the assumption (1.7), that is non-overlapping column structure is NOT assumed, we require only that \( \sum_{i=1}^M r_i \leq 1 \) (like in the first part of figure 3).

We often consider the following special cases:

**Definition 1.2.** We say that a shifted TGL carpet \( \Lambda \) has **uniform vertical fibres** if

\[
\sum_{j \in \mathcal{I}_i} a_j^{i-s} = 1 \quad \text{for every } i \in [M],
\]

where \( s = \dim B \Lambda \) and \( s_{\tilde{H}} = \dim B \Lambda_{\tilde{H}} \).
Furthermore, we call $\Lambda$ a diagonally homogeneous shifted TGL carpet if

$$b_i \equiv b \text{ and } a_i \equiv a \text{ for every } i \in [N].$$

In particular, a diagonally homogeneous carpet has uniform vertical fibres if $N/M \in \mathbb{N}$ and $N_i = N/M$ for every $i \in \{1, \ldots, M\}$.

The special case when $N_i = 1$ for all $i = 1, \ldots, M$ is treated in the paper of Bárányn et al [8, lemma 3.1].

1.1.1. Some notation. The map $f_i$ is indexed by $i \in [N]$. To indicate which column $i$ belongs to in the partition (1.4) we use the function

$$\phi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, M\}, \quad \phi(i) := i \text{ if } i \in I_i. \quad (1.9)$$

With this notation, we can formulate the column structure (1.5) as

$$\text{if } \phi(k) = \phi(\ell) = i, \text{ then } b_k = b_\ell = : r_i \text{ and } t_{i, 1} = t_{\ell, 1} = : u_i. \quad (1.10)$$

Throughout, $i$ is an index from $[N]$, while $\hat{i}$ with the hat is an index corresponding to a column from $\{1, \ldots, M\}$. We use analogous notation for infinite sequences $i = i_1i_2 \ldots$ and $\hat{i} = \hat{i}_1\hat{i}_2 \ldots$, see section 3.1 for details.

For compositions of maps, we use the standard notation $f_{i_1 \ldots i_k} := f_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ f_{i_1}$, where $i_j \in \{1, \ldots, N\}$. Similarly, for products of matrices we write

$$A_{i_1 \ldots i_k} := A_{i_1} \cdot \cdots \cdot A_{i_k} := \begin{pmatrix} b_{i_1 \ldots i_k} & 0 \\ d_{i_1 \ldots i_k} & a_{i_1 \ldots i_k} \end{pmatrix}.$$  

Immediate calculations give $b_{i_1 \ldots i_k} = b_{i_1} \cdot \cdots \cdot b_{i_k}$, $a_{i_1 \ldots i_k} = a_{i_1} \cdot \cdots \cdot a_{i_k}$ and

$$d_{i_1 \ldots i_k} = \sum_{\ell=1}^n d_i \cdot \prod_{k<\ell} a_k \cdot \prod_{r=\ell+1}^n b_r, \quad (1.11)$$

where by definition $\prod_{k<1} a_k := 1$ and $\prod_{r=n+1}^n b_r := 1$. The image $R_{i_1 \ldots i_k} := f_{i_1 \ldots i_k}(R)$ is a parallelogram with two vertical sides. We refer to $b_{i_1 \ldots i_k}$ as the width, $a_{i_1 \ldots i_k}$ as the height and $\gamma_{i_1 \ldots i_k}$ as the angle of the longer side of the parallelogram $R_{i_1 \ldots i_k}$, in other words,

$$\tan \gamma_{i_1 \ldots i_k} := \frac{d_{i_1 \ldots i_k}}{b_{i_1 \ldots i_k}}. \quad (1.12)$$

Both the width and height of $R_{i_1 \ldots i_k}$ are exponentially small, however, since direction-$x$ dominates, the width is exponentially larger than the height. A simple argument gives that $|\gamma_{i_1 \ldots i_k}|$ remains uniformly bounded away from $+\infty$.

**Lemma 1.3.** There exists a non-negative constant $K_0 < \infty$ such that for every $n$ and every finite length word $i_1 \ldots i_n$

$$\left| \frac{d_{i_1 \ldots i_n}}{b_{i_1 \ldots i_n}} \right| \leq K_0.$$

**Proof.** Since direction-$x$ dominates, $\max \{a_i/b_i\} < 1$, hence using (1.11)

$$\left| \frac{d_{i_1 \ldots i_n}}{b_{i_1 \ldots i_n}} \right| \leq \frac{|d_{i_1}|}{b_{i_1}} + \sum_{k=2}^n \frac{|d_{i_k}|}{b_{i_k}} \prod_{j=1}^{k-1} \frac{a_{i_j}}{b_{i_j}} \leq \max \{a_i/b_i\} \left| \frac{|d_i|/b_i}{1 - \max \{a_i/b_i\}} \right| < \infty. \quad \square$$

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1.2. Our contribution explained with pictures

A natural way to depict an IFS $\mathcal{F} = \{f_i\}$ is to provide the images $f_i(R)$, where $R$ is the smallest rectangle which contains $\Lambda$. Without loss of generality, we always assume in this paper that $R = [0,1]^2$. The correspondence between the shifted TGL IFS and a figure showing the collection of images of $R$ is unique.

The shaded rectangles and parallelograms in figure 1 show the images of $R$ under the orientation preserving affine maps defining a Gatzouras–Lalley (GL) carpet [21] on the left, see definition 1.4, and a triangular Gatzouras–Lalley-type (TGL) carpet on the right. These are typical examples which satisfy the rectangular open set condition (ROSC), see definition 1.6. Furthermore, there is a correspondence between the rectangles and parallelograms so that the height and width of corresponding ones coincide. We call the Gatzouras–Lalley carpet the GL-brother of the TGL carpet, see definition 1.5. Even though the ROSC holds, it is not immediate that the dimension of the two attractors should be the same. The parallelograms can be placed in a way that there is no bi-Lipschitz map between the two attractors. Nevertheless, Barański essentially shows in [3] that assuming the ROSC the Hausdorff and box dimension of a TGL carpet is equal to the respective dimension of its GL brother.

The IFS in figure 1 is an example for which $\dim_H \Lambda < \dim_B \Lambda < \dim_{\text{Aff}} \Lambda$. If the orthogonal projection of $\Lambda$ to the $x$-axis is the whole $[0,1]$ interval, then the box- and affinity dimensions are equal, see corollary 2.5. Figure 4 shows such an example, where the outlines of $f_j(R)$ are shown together with the attractor, which we call the ‘self-affine smiley’.

A special class of examples consists of the diagonally homogeneous carpets, recall definition 1.2. The well-known Bedford–McMullen carpets [10, 29] form a proper subclass of these TGL carpets. The attractor on the left-hand side of figure 2 first appeared in [13, figure 1a] and then again in [4, section 4.3]. It is exceptional in the class of TGL carpets, since $\dim_H \Lambda = \dim_B \Lambda = \dim_{\text{Aff}} \Lambda$. This is because it has uniform vertical fibres. In all these examples only the boundary of the cylinder sets $f_j(R)$ could intersect.

The main contribution of the present paper is that different types of overlaps are allowed in our construction, recall figure 3. On the left, the columns are shifted in a way that the IFS $\mathcal{H}$ on the $x$-axis generated by the columns satisfies Hochman’s exponential separation condition, see section 7.1.

Figure 4. The ‘self-affine smiley’, whose $\dim_H = 1.20665 \ldots < 1.21340 \ldots = \dim_B = \dim_{\text{Aff}}$, see section 7.1.
This type of shifted columns was considered by Fraser and Shmerkin [20] and Pardo-Simón [31] for different types of carpets. In the center, columns do not overlap, however, parallelograms within a column may do so if a certain transversality like condition holds, see definition 1.6. The system on the right-hand side of figure 3 has both types of overlaps.

By modifying the translation vectors in the example on the left-hand side of figure 2, we get a brother with overlaps seen on the right-hand side, for which we show in section 7.3 that transversality holds. Another concrete overlapping example satisfying transversality is $X \equiv X'$ in figure 5, for which there is a strict inequality between the Hausdorff, box and affinity dimensions. If there are just two first level cylinders in the middle column, resembling $X = X'$, then the Hausdorff and box dimensions are equal. Moreover, if there are no empty columns in this example, then the box and affinity dimensions coincide.

Section 2 contains the formal statements of all our main results. Roughly speaking, we show that for any TGL carpet $\Lambda$

$$\dim \Lambda \leq \dim \tilde{\Lambda},$$

where $\tilde{\Lambda}$ is the GL brother of $\Lambda$, see definition 1.5, and $\dim$ means either box or Hausdorff dimension, see theorems 2.1 and 2.4. When ROSC holds, and $\mathcal{H}$ satisfies Hochman’s condition, then the equality can be deduced from the recent works [6, 19]. Our main contribution is that in the presence of overlaps described above, we give sufficient conditions under which $\dim \Lambda$ does not drop below $\dim \tilde{\Lambda}$, see theorems 2.2 and 2.7. In particular, for the Hausdorff dimension we allow both types of overlaps simultaneously, however for the box dimension we can prove our results only if at most one of the two types of overlaps occurs.

For a discussion on a generalization to the case of orientation reversing maps, see section 7.5. In particular, we calculate the dimension of a family of self-affine continuous curves $\Lambda_\alpha$, generated by a family of IFS-s $\mathcal{F}_\alpha$ containing a map that reflects on the y-axis, see figure 6. The formal treatment of all these examples is done in section 7.

One motivation to study self-affine fractals of overlapping construction is that sometimes the dimension of a higher dimensional fractal of non-overlapping construction coincides with its lower dimensional orthogonal projection which can be a self-affine fractal of overlapping
construction. We obtain such a set in 3D by starting from a TGL carpet with overlaps on the \( xy \)-plane and then ‘lift’ it to 3D so that the interiors of the first level cylinders are disjoint. Figure 7 shows such an example with the first level cylinders (left), the attractor (center) and the projection of the cylinders and attractor to the \( xy \)-plane (right). Section 8 contains the formal treatment of this type of construction.

### 1.3. Brief overview of planar carpets

Independently of each other, Bedford [10] and McMullen [29] were the first to study non-self-similar planar carpets. They split the unit square \( R \) into \( m \) columns of equal width and \( n \) rows of equal height for some integers \( n > m \geq 2 \) and considered IFSs of the form

\[
f_{(i,j)}(x) := \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} x + \begin{pmatrix} i/m \\ j/n \end{pmatrix}
\]

for \( (i,j) \in A \subseteq \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \). They gave explicit formulas for the Hausdorff and box-counting dimension of the corresponding attractor \( \Lambda \). It turns out that
is atypical, namely, equality holds if and only if \( \Lambda \) has uniform vertical fibres.

Later Gatzouras and Lalley [21] generalized the results to the following class of IFSs.

**Definition 1.4.** A self-affine IFS \( \tilde{\mathcal{F}} = \{ \tilde{f}_i \}_{i=1}^{N} \) is a Gatzouras–Lalley (GL) IFS and its attractor \( \tilde{\Lambda} \) is a GL carpet if \( \tilde{\mathcal{F}} \) is a TGL IFS as in definition 1.1 with the additional assumptions that all off-diagonal elements \( d_i = 0 \) and the rectangular open set condition (ROSC) holds, i.e.

\[
\tilde{f}_i((0,1)^2) \cap \tilde{f}_j((0,1)^2) = \emptyset \quad \text{for all } i \not= j.
\] (1.13)

**Definition 1.5.** Let \( \Lambda \) be a shifted TGL carpet generated from the IFS \( \mathcal{F} \) of the form (1.1). We say that the Gatzouras–Lalley IFS \( \tilde{\mathcal{F}} = \{ \tilde{f}_i(x) : \tilde{A}_i + \tilde{t}_i \}_{i=1}^{N} \), where \( \tilde{A}_i = (\tilde{b}_i, \tilde{a}_i) \) and \( \tilde{t}_i = (\tilde{t}_{i1}, \tilde{t}_{i2}) \), and its attractor \( \tilde{\Lambda} \) is the GL brother of \( \mathcal{F} \) and \( \Lambda \), respectively, if \( \tilde{a}_i = a_i \) and \( \tilde{b}_i = b_i \) for every \( i \in [N] \), furthermore, \( \tilde{\mathcal{F}} \) has the same column structure (1.10) as \( \mathcal{F} \). If the shifted TGL carpet \( \Lambda \) is actually a TGL carpet (that is \( \Lambda \) has non-overlapping column structure), then we also require that \( \tilde{t}_{i1} = t_{i1} \) holds for all \( i \in [N] \).

There always exists such a brother since we assume the condition (c) of definition 1.1 and \( \sum_{i=1}^{M} r_i \leq 1 \). Throughout, the GL brother of \( \Lambda \) will always be denoted with the extra tilde \( \tilde{\Lambda} \).

A standard technique to give a lower bound for the Hausdorff dimension of the attractor \( \tilde{\Lambda} = \bigcup_{i \in [N]} \tilde{f}_i(\tilde{\Lambda}) \) is to study self-affine measures \( \nu_p \), i.e. compactly supported measures with support \( \tilde{\Lambda} \) satisfying

\[
\nu_p = \sum_{i=1}^{N} p_i \nu_p \circ \tilde{f}_i^{-1},
\]

for some probability vector \( p = (p_1, \ldots, p_N) \). Let \( \mathcal{P} \) be the set of all probability distributions on the set \([N]\) and \( \mathcal{P}_0 \subset \mathcal{P} \) consists of all distributions with \( p_i > 0 \). By definition

\[
\sup_{p \in \mathcal{P}} \dim_{H} \nu_p \leq \dim_{H} \tilde{\Lambda}.
\]

Gatzouras and Lalley proved that there always exists a \( p^* \) for which the supremum is attained, furthermore \( p^* \in \mathcal{P}_0 \). Let

\[
\alpha^* := \dim_{H} \nu_{p^*} = \sup_{p \in \mathcal{P}} \dim_{H} \nu_p.
\]

They explicitly calculated

\[
\dim_{H} \nu_p = \sum_{i=1}^{N} p_i \log p_i \sum_{i=1}^{N} p_i \log a_i + \left(1 - \frac{\sum_{j=1}^{N} p_j \log b_j}{\sum_{j=1}^{N} p_j \log a_j} \right) \sum_{i=1}^{M} q_i \log q_i \sum_{i=1}^{N} p_i \log b_i,
\] (1.14)

where \( q_i = \sum_{j \in I_i} p_j \). This formula is a special case of the Ledrappier–Young formula, see section 3.3 for details and references. For Bedford–McMullen carpets the optimal \( p^* \) can be given by routine use of the Lagrange multipliers method. The main result of [21] is that for a GL carpet the \( \alpha^* \) bound is sharp, i.e.

\[
\alpha^* = \dim_{H} \tilde{\Lambda}.
\]
In [21] Gatzouras and Lalley also gave an implicit formula to calculate the box dimension of their carpet. Let $s_i$ be the unique real such that $\sum_{i=1}^M r_i^{s_i} = 1$ ($r_i$ was defined in (1.5)). Then $\dim_B \Lambda = s$ is the unique real such that
$$\sum_{i=1}^N b_i^{s_i} q_i^{-s_i} = 1.$$ Again, equality of $\dim_H \tilde{\Lambda}$ and $\dim_B \tilde{\Lambda}$ is highly atypical. It holds if and only if the $\alpha^*$-dimensional Hausdorff measure of $\tilde{\Lambda}$, denoted $H^\alpha^*(\tilde{\Lambda})$, is positive and finite, which is equivalent to $\Lambda$ having uniform vertical fibres, recall (1.8). For Bedford–McMullen carpets Peres showed in [32] that $H^\alpha^*(\Lambda) = \infty$ when $\dim_H \Lambda < \dim_B \Lambda$.

More recently, Barański [2] kept the row and column structure but relaxed (1.3) by allowing an arbitrary subdivision of the horizontal and vertical axis. After appropriately choosing which direction is ‘dominant’, the results resemble that of [21]. Continuing this work, Barański showed in [3] how the result in [2] could be adapted to obtain the dimension result for TGL carpets assuming ROSC (1.13). Diagonal systems assuming only ROSC and no further restrictions on the translations were studied by Feng–Wang [18] and Fraser [19]. Former determined the $L^q$ spectrum of self-affine measures $\nu_p$ and in particular the box dimension of the attractor. In [19] linear isometries which map $[-1,1]^2$ to itself are allowed and the box dimension is determined. Fraser called these box-like sets. Observe that in all the mentioned papers the ROSC was assumed.

Carpets with overlaps were not studied until the last few years. Fraser and Shmerkin [20] shift the columns of Bedford–McMullen carpets to get overlaps, while Pardo-Simón [31] allows shifts in both directions of Barański carpets. Relying on a recent breakthrough by Hochman [22] on the dimension of self-similar measures on the line, both papers show that apart from a small exceptional set of parameters the results in [10, 29] and [2] remain valid in the overlapping case. This is the type of shifted columns that appear in figure 3.

We finish the section by formalizing the separation conditions between cylinders sets.

### 1.4. Separation conditions

In our main results, we assume different extents of separation for the parallelograms $f_i(R)$, recall figures 1 and 3. This will be considered in section 1.4.1. In section 1.4.2 we consider separation conditions for $\mathcal{H}$ which are actually conditions about the extent of separation of the column structure.

#### 1.4.1. Separation of the cylinder parallelograms.

**Definition 1.6 (Separation conditions for a shifted TGL IFS $F$).** We say that

- $\mathcal{F}$ satisfies the **rectangular open set condition (ROSC)** if the strong open set condition (SOSC) holds for $\mathcal{F}$ with $U = (0,1)^2$, i.e. $\Lambda \cap U \neq \emptyset$ with
  $$\bigcup_{i=1}^N f_i(U) \subseteq U \text{ and } f_i(U) \cap f_j(U) = \emptyset \text{ for every } i \neq j.$$ 

- each column independently satisfies the ROSC if for every $i \in [M]$ and $k, \ell \in \mathcal{I}_i$ we have $f_i(U) \cap f_{k}(U) = \emptyset$. In other words, if the interior of two first level cylinders intersect, then they are from different columns.
• $\mathcal{F}$ satisfies the **transversality condition** if there exists a $K_1 > 0$ such that for every $n$ and words $(i_1 \ldots i_n), (j_1 \ldots j_n) \in \{1, \ldots, N\}^n$ with $\phi(\omega) = \phi(\omega)$ for $k = 1, \ldots, n$ and $i_1 \neq j_1$ ($\phi$ was defined in (1.9)), we have

$$\left| \text{proj}_z(\text{int}(R_{i_1 \ldots i_n}) \cap \text{int}(R_{j_1 \ldots j_n})) \right| < K_1 \cdot \max\{a_{i_1} \cdot \ldots \cdot a_{i_n}, a_{j_1} \cdot \ldots \cdot a_{j_n}\}.$$  

(1.15)

Given two finite words $i_1 \ldots i_n$ and $j_1 \ldots j_n$, $i_1 \neq j_1$, the angle of the two corresponding parallelograms $R_{i_1 \ldots i_n}$ and $R_{j_1 \ldots j_n}$ can be defined as the angle between their non-vertical sides. The transversality condition ensures that any such pair of parallelograms in the same column have either disjoint interior or have an angle uniformly separated from zero.

Observe that this definition of transversality coincides in the diagonally homogeneous case with the one in [8]. In [8, section 1.5] a sufficient condition for the transversality condition was given. Namely, the authors introduced a self-affine IFS $\tilde{S}$ in $\mathbb{R}^2$ which is (in our setup)

$$\tilde{S} := \left\{ \tilde{S}_i(x, z) := (f_i(x), T_i(z)) \right\}_{j=1}^N, \quad (x, z) \in [0, 1]^2 \times \mathbb{R},$$

where $\{f_i\}_{j=1}^N$ was defined in (1.1) and $T_i : \mathbb{R} \to \mathbb{R}$ is given by

$$T_i(z) := \frac{a_i}{b_i} \cdot z + \frac{b_i}{d_i}.$$  

The relevance of the IFS $\mathcal{T}$ is that

$$\tan \gamma_{i_1 \ldots i_n} = T_{i_1 \ldots i_n}(0).$$  

(1.16)

Indeed, from the definition (1.12) of $\tan \gamma_{i_1 \ldots i_n}$ and formula (1.11) it immediately follows that

$$\tan \gamma_{i_1 \ldots i_n} = \frac{d_{i_1}}{b_{i_1}} + \sum_{\ell=2}^n \frac{d_{i_\ell}}{b_{i_\ell}} \cdot \prod_{k=1}^{\ell-1} \frac{a_{i_k}}{b_{i_k}} = T_{i_1 \ldots i_n}(0).$$

Using the same argument as in the proof of [8, lemma 1.2] we obtain that

**Lemma 1.7.** If $\tilde{S}$ satisfies the strong separation property (that is $\tilde{S}_i(\Lambda) \cap \tilde{S}_j(\Lambda) = \emptyset$ if $i \neq j$ and $\Lambda$ is the attractor of the IFS $\tilde{S}$) then the transversality condition holds.

The next lemma gives a different, easy-to-check sufficient condition for transversality.

**Lemma 1.8.** Let $\mathcal{P}_j := \{(k, \ell) : k, \ell \in I_j, \quad k \neq \ell, \quad R_k \cap R_\ell \neq \emptyset\}$, where $A^c$ denotes the interior of a set $A$. Moreover, we introduce

$$s_k := \frac{d_k}{b_k}, \quad r_k := \frac{a_k}{b_k}, \quad r^* := \max_{1 \leq k \leq N} r_k, \quad b_{\min} := \min_{1 \leq k \leq N} b_k$$  

and $s_* := \min_{1 \leq k \leq N} \min_{(k, \ell) \in \mathcal{P}_j} |s_k - s_\ell|.$

Assume that

$$s_* > \frac{1}{b_{\min}} \cdot \frac{r^*}{1 - r^*} \quad \text{or equivalently} \quad \frac{s_* b_{\min}}{2 + s_* b_{\min}} > r^*.$$  

Then the transversality condition holds.

In particular, in the diagonally homogeneous case transversality holds if

$$\frac{a}{b} < \frac{d_*}{2 + d_*}.$$  

(1.17)

where $d_* := \min_{1 \leq k \leq N} \min_{(k, \ell) \in \mathcal{P}_j} |d_k - d_\ell|$
Proof. The transversality condition holds if there exists $c > 0$ such that

for every $n$, for all $m \leq M$ with $P_m \neq \emptyset$ and for all $i, j \in \{1, \ldots, N\}^N$ with $(i_1, j_1) \in P_m$

we have: $|\gamma_{i_1} - \gamma_{j_1}| > c$. This can be deduced from (1.16). The details are left to the reader or can be found in [25]. □

14.2. Separation of the columns. We will also need some separation conditions for the column structure which are represented by separation properties of $\mathcal{H}$, recall (1.6).

The symbolic spaces for $\mathcal{F}$ and $\mathcal{H}$ are

$$\Sigma := \{1, \ldots, N\}^N \quad \text{and} \quad \Sigma_{\mathcal{H}} := \{1, \ldots, M\}^N.$$  

The natural projections form $\Sigma \to \Lambda$ and $\Sigma_{\mathcal{H}} \to \Lambda_{\mathcal{H}}$ are $\Pi$ and $\Pi_{\mathcal{H}}$ respectively, see section 3.1 for details. Whenever we are given a probability vector $q$ we associate to it another probability vector $\hat{q}$ on $\{1, \ldots, M\}$ such that

$$q_i := \sum_{j \in E_i} p_j, \quad (1.18)$$

Slightly abusing the notation, we write $P_0$ for both the set of the probability vectors of positive components on $\{1, \ldots, N\}$ and $\{1, \ldots, M\}$. The Bernoulli measure $p^\Sigma$ on $\Sigma$ is denoted $\mu_p$ and its push forward is $\nu_p = \Pi \circ \mu_p = \mu_p \circ \Pi^{-1}$. Analogously, $\mu_q = q^\Sigma$ on $\Sigma_{\mathcal{H}}$ and $\nu_q = (\Pi_{\mathcal{H}}) \circ \mu_q$.

Definition 1.9 (Separation conditions for $\mathcal{H}$). We say that $\mathcal{H}$ satisfies

- **Hochman’s exponential separation condition (HESC)** (see [22, p 775]) if there exist an $\varepsilon > 0$ and $n_\varepsilon \to \infty$ such that for

  $$\Delta_n := \min_{\tau \in (1, \mu_\tau)} \frac{1}{\log \tau} \left\{ \begin{array}{ll}
  |h_\tau(0) - h_\tau(0)|, & \text{if } h_\tau'(0) = h_\tau'(0); \\
  \infty, & \text{otherwise}
  \end{array} \right.$$  

  we have $\Delta_n > e^{-\varepsilon \cdot n_\varepsilon}$. Here $h'$ denotes the derivative of the function $h$.

- **Weak almost unique coding (WAUC)** if for all Bernoulli measures $\mu_q$ there exists $B_{\mathcal{H}} \subset \Sigma_{\mathcal{H}}$ (which may depend on $q$), such that

  $$\mu_q(B_{\mathcal{H}}) = 0$$

  and for every $\hat{i} \in \Sigma_{\mathcal{H}} \setminus B_{\mathcal{H}} : \#(\Pi_{\mathcal{H}}^{-1}(\hat{i}) \setminus B_{\mathcal{H}}) = 1$.

- **Almost unique coding (AUC)** holds if for every Bernoulli measure $\mu_q$ and for $\mu_q$-a.e. $\hat{i} \in \Sigma_{\mathcal{H}} : \#(\Pi_{\mathcal{H}}^{-1}(\hat{i}) \setminus B_{\mathcal{H}}) = 1$.

- **No dimension drop (NDD)** if for all push forward measures $\nu_q = (\Pi_{\mathcal{H}}) \circ \mu_q$

  $$\dim_{\mathcal{H}} \nu_q = \frac{-\sum_{i=1}^M q_i \log q_i}{\sum_{i=1}^M q_i \log r_i}.$$  

The following implications hold between these conditions

$$\text{HESC} \implies \text{NDD} \iff \text{WAUC}.$$  

(1.19)

HESC $\implies$ NDD follows from Hochman’s work [22, theorem 1.1]. AUC implies NDD from Feng–Hu [17, theorem 2.8 and corollary 4.16], but we do not know if the reverse direction NDD $\implies$ AUC holds or not. After the submission of our manuscript, Feng [16, corollary 4.7] proved the equivalence NDD $\iff$ WAUC for ergodic measures. Since we use
it only for Bernoulli measures, for completeness, we kept our own self-contained proof of NDD $\iff$ WAUC for Bernoulli measures in the appendix.

The set $\mathcal{U}$ of translations $(u_1, \ldots, u_M)$ defining $\mathcal{H}$ for which HESC does not hold is small. It is stated in [31, proposition 2.7] that it essentially follows from [23, theorem 1.10] that the Hausdorff and packing dimension of $\mathcal{U}$ is $M - 1$, in particular, $\mathcal{U}$ has $0M$-dimensional Lebesgue measure. Moreover, [22, theorem 1.5] states that if the parameters $(r_1, \ldots, r_M, u_1, \ldots, u_M)$ defining $\mathcal{H}$ are all algebraic, then HESC does not hold if and only if there is an exact overlap, i.e. $\Delta_n = 0$ for some $n$.

2. Main results

We now state our main results for the Hausdorff dimension of shifted TGL carpets in section 2.1, the box dimension in section 2.2 and discuss diagonally homogeneous carpets in section 2.3. For a discussion on generalizing towards negative entries in the main diagonal, see section 7.5.

2.1. Hausdorff dimension

For any vector $\mathbf{c} = (c_1, \ldots, c_K)$ with strictly positive entries and a probability vector $(p_1, \ldots, p_K)$ we write

$$\langle \mathbf{c} \rangle_p := \prod_{i=1}^{K} c_i^{p_i}.$$  

When no confusion is made, we suppress $p$ and write $\langle \mathbf{c} \rangle := \langle \mathbf{c} \rangle_p$. Throughout, we use this notation for the vectors $\mathbf{a} = (a_1, \ldots, a_N)$, $\mathbf{b} = (b_1, \ldots, b_N)$, $\mathbf{p} = (p_1, \ldots, p_N)$, $\mathbf{N} = (N_1, \ldots, N_M)$ and $\mathbf{q} = (q_1, \ldots, q_M)$, where $\mathbf{q}$ is derived from $\mathbf{p}$ via (1.18). Using this notation let us denote the function on the right-hand side of (1.14) by

$$D(\mathbf{p}) := \frac{\log \langle \mathbf{p} \rangle_p}{\log \langle \mathbf{a} \rangle_p} + \left( 1 - \frac{\log \langle \mathbf{b} \rangle_p}{\log \langle \mathbf{a} \rangle_p} \right) \frac{\log \langle \mathbf{q} \rangle_q}{\log \langle \mathbf{b} \rangle_p} = \frac{\log \langle \mathbf{q} \rangle_q}{\log \langle \mathbf{b} \rangle_p} + \frac{\log \langle \mathbf{p} \rangle - \log \langle \mathbf{q} \rangle}{\log \langle \mathbf{a} \rangle_p}. \quad (2.1)$$

**Theorem 2.1 (Upper bound).** Regardless of overlaps, for any shifted triangular Gatzouras–Lalley-type planar carpet $\Lambda$

$$\dim_H \Lambda \leq \sup_{\mathbf{p} \in \mathcal{P}} D(\mathbf{p}) := \alpha^*.$$  

Furthermore, there always exists a $\mathbf{p}^* \in \mathcal{P}_0$ for which $D(\mathbf{p}^*) = \alpha^*$.

The proof is given in section 4. Throughout, let $\mathbf{q}^*$ denote the vector $q_i^* = \sum_{j \in I_i} p_j^*$. The next theorem states sufficient conditions under which the Hausdorff dimension of a self-affine measure $\nu_\mathbf{p}$ on $\Lambda$ is equal to $D(\mathbf{p})$.

**Theorem 2.2.** Let $\mathbf{p} \in \mathcal{P}_0$, $\mu_\mathbf{p} := \mathbf{p}^\mathbf{p}$ and $\nu_\mathbf{p} := \prod \mu_\mathbf{p}$. For a shifted triangular Gatzouras–Lalley-type planar carpet $\Lambda$ we have

$$\dim_H \nu_\mathbf{p} = D(\mathbf{p})$$

if the horizontal IFS $\mathcal{H}$ satisfies HESC (in particular, always holds for non-overlapping columns) and
(i) either each column independently satisfies the ROSC or
(ii) \( \Lambda \) satisfies transversality (see definition 1.6), and the following inequality holds:

\[
\frac{\log(a)}{\log(b)} > 1 + \frac{\log(N)}{\log(q)}.
\]

We remark that proposition 2.10 provides a simple way to check condition (2.2) in the diagonally homogeneous case. Section 5 is devoted to the proof of this theorem. As an immediate corollary, we get the following.

**Corollary 2.3 (Sufficient conditions).** Whenever a shifted TGL carpet \( \Lambda \) satisfies the conditions of theorem 2.2 with \( p \) and \( q \) in (2.2) replaced by \( p^* \) and \( q^* \), then

\[
\dim H_\Lambda = \alpha^*.
\]

2.2. Box dimension

Recall the IFSs \( \tilde{H} \) (1.2) and \( H \) (1.6) obtained by projecting \( F \) to the \( x \)-axis. Recall \( s_x \) was defined so that \( \sum_{i=1}^{M} b_i^{s_x} = 1 \) and let \( \tilde{s}_x \) be the unique real such that \( \sum_{i=1}^{N} b_i^{\tilde{s}_x} = 1 \). Furthermore, introduce

\[
s_H := \dim_B \Lambda_{\tilde{H}} = \dim_B \Lambda_\tilde{H}.
\]

Since \( \Lambda_H \) is a self-similar set, \( s_H \) is well defined. If \( \Lambda \) is a TGL carpet then \( s_H = s_x \), otherwise \( s_H \leq s_x \). The value of the affinity dimension \( \dim_{Aff} \) of \( \Lambda \) can be deduced from the result of Falconer–Miao [13, corollary 2.6] together with the description in [8, section 1.3] and the fact that direction-\( x \)-dominates: \( \dim_{Aff} \Lambda = s_A \) is the unique real such that

\[
\sum_{i=1}^{N} b_i^{s_A} a_i^{-s_A} = 1.
\]

We extend its scope to triangular IFSs. In a different context, Hu [24] studied a related problem, where a version of Bowen’s formula determines the box dimension.

**Theorem 2.4 (Upper bound).** Regardless of overlaps, for any shifted triangular Gatzouras–Lalley-type planar carpet \( \Lambda \)

\[
\dim_B \Lambda = \dim_{Aff} \Lambda \leq s \leq s_A,
\]

where \( s \) is the unique solution of the equation

\[
\sum_{i=1}^{N} b_i^{s} a_i^{-s} = 1.
\]

In particular, if \( \Lambda \) satisfies the ROSC, then \( \dim_B \Lambda = \dim_{Aff} \Lambda = s \).
Corollary 2.5 (Equality of box- and affinity dimension). For any shifted TGL carpet \( \Lambda \)

\[ s = s_A \iff s_H = \min\{s, 1\}. \]

Proof. Follows immediately from comparing equations (2.3) and (2.4) defining \( s_A \) and \( s \), respectively, together with the fact that \( a_i < 1 \) and \( b_i/a_i > 1 \) for every \( i = 1, \ldots, N \). \( \square \)

Remark 2.6.

(a) The proof of Fraser [19] does not make use of any column structure (1.10). Hence, theorem 2.4 immediately extends to an IFS \( F \) of the form (1.1) as long as direction-x dominates \( 0 < a_i < b_j < 1 \) and the ROSC holds.

(b) Since \( \Lambda \) is compact and every open set intersecting \( \Lambda \) contains a bi-Lipschitz image of \( \Lambda \),

we get that \( \dim_B \Lambda = \dim_B \Lambda \), see [15, corollary 3.9].

Handling overlaps to calculate the box dimension is a greater challenge, since typically \( \dim_B \Lambda < \dim_H \Lambda \) and thus the usual technique of giving a lower bound by bounding the Hausdorff dimension from below does not suffice. Hence, a new counting argument was necessary.

Theorem 2.7 (Box dimension with overlaps). For a shifted TGL carpet \( \Lambda \) we have \( \dim_B \Lambda \geq s \), hence \( \dim_H \Lambda = s \), if either of the following hold:

(i) \( H \) satisfies HESC and each column independently satisfies the ROSC or

(ii) \( \Lambda \) is a TGL carpet, satisfies transversality and the following inequality:

\[ -\log(p) + \log(q) < s_H(\log(b) - \log(a)), \quad \text{(2.5)} \]

where \( \tilde{p} := (\tilde{p}_1, \ldots, \tilde{p}_N) \) and \( \tilde{q} := (\tilde{q}_1, \ldots, \tilde{q}_M) \) are defined by equation (2.4):

\[ \tilde{p}_i = b_i^{s_H} a_i^{1-s_H} \quad \text{and} \quad \tilde{q}_i = \sum_{j \in I_i} b_j^{s_H} d_j^{1-s_H}. \quad \text{(2.6)} \]

The analogue of the following sufficient and necessary condition for the equality of the box- and Hausdorff dimensions was proved in [21, theorem 4.6].

Theorem 2.8 (Equality of box- and Hausdorff dimension). Assume the shifted TGL carpet \( \Lambda \) satisfies ROSC and \( H \) satisfies No dimension drop (NDD). Then the following three conditions are equivalent,

\[ \dim_B \Lambda = \dim_H \Lambda \iff s_H = \dim_H v_q \iff \sum_{j \in I_i} d_j^{1-s_H} = 1 \text{ for every } i \in [M]. \quad \text{(2.7)} \]

All results for box dimension are proved in section 6.

2.3. Diagonally homogeneous carpets

We show how the conditions and formulas of our main results simplify in the diagonally homogeneous case. Recall the easy-to-check sufficient condition (1.17) for transversality in lemma 1.8. Moreover, observe that the vector \( \tilde{p} \) becomes the uniform vector \( \tilde{p}_i = 1/N \) and thus \( \tilde{q}_i = N_i/N \). A routine use of the Lagrange multipliers method gives the optimal \( \mathbf{p}^* \).
\[ p_k^* = N_j^{\log a_{\log b}^{-1} \cdot \left( \sum_{j=1}^M N_j^{\log a_{\log b}^{-1}} \right)^{-1}} \text{ if } k \in \mathcal{I}_i. \]  

(2.8)

Thus, conditions (2.2) and (2.5) become

\[
\frac{\log (p^*)_{p^*}}{\log (q^*)_{q^*}} < \frac{\log a}{\log b} \quad \text{and} \quad \frac{\log N}{\log M} + 1 + \frac{\log (\tilde{q})_{\tilde{q}}}{\log M} < \frac{\log a}{\log b},
\]

(2.9)

respectively. If in addition, the system has uniform vertical fibres, then \( \tilde{p}_i = p_i^* = 1/N \) also \( \tilde{q}_i = q_i^* = 1/M \). Hence, both conditions (2.2) and (2.5) become

\[
\frac{\log N}{\log M} < \frac{\log a}{\log b}.
\]

(2.10)

Next, we give an equivalent explicit formulation of condition (2.2). Let \( \varphi(y) := y \log y \) and for \( x \in (0, 1) \) define

\[
R(x) := x + (r(x) - 1)^{-1}, \quad \text{where} \quad r(x) = \frac{\varphi\left( \sum_{i=1}^M N_i \right)}{\sum_{j=1}^M \varphi(N_j)}.
\]

**Lemma 2.9.** \( R(x) \) is a continuous, strictly monotone increasing function.

**Proof.** Continuity is obvious. It is enough to show that \( r(x) \) is strictly monotone decreasing. This is rather tedious but straightforward. For details see [25]. \( \square \)

**Proposition 2.10.** The solution of the equation \( R(x) = 1 \) is unique. Let \( x_0 \) denote this solution. Then in the diagonally homogeneous case

(2.2) holds \( \iff \frac{\log b}{\log a} < x_0. \)

**Remark 2.11.** Observe that all the conditions for transversality, (2.2) and (2.5) are satisfied if the heights of the parallelograms \( R_i \) are ‘small enough’ compared to their width. See the examples with overlaps in section 7 for some explicit calculations.

**Proof of proposition 2.10.** Let \( x := \log b/\log a < 1 \). In the diagonally homogeneous case (2.2) simplifies to

\[
\frac{\log a}{\log b} = x > 1 + \frac{\sum_{i=1}^M q_i^* \log N_i}{\sum_{i=1}^M q_i^* \log q_i^*},
\]

where \( q_i^* = N_i^*/\sum_j N_j^* \). Multiplying each side by \( x \) we get

\[
1 > x + \frac{\sum_{i=1}^M q_i^* \log N_i^*}{\sum_{j=1}^M q_i^* \log q_i^*}.
\]

(2.11)

It is straightforward to check that for any \( y_1, \ldots, y_M \in \mathbb{R} \) and \( q_i := e^{y_i}/\sum_j e^{y_j} \)

\[
- \sum_{i=1}^M q_i \log q_i + \sum_{i=1}^M q_i \cdot y_i = \log \left( \sum_{i=1}^M e^{y_i} \right).
\]
Applying this with $\hat{y} = \log N^i_\hat{x}$ (then $q_i = q_i^*_\hat{x}$) in the denominator of (2.11) we get that (2.2) is equivalent to

$$1 > x + \frac{\sum_{i=1}^{M} q_i^* \log N^i_\hat{x}}{\log \sum_{i=1}^{M} N^i_\hat{x} - \sum_{i=1}^{M} q_i^* \log N^i_i} = R(x).$$

For $x$ small enough (2.2) holds, since $1/x$ tends to infinity while the right-hand side remains finite. On the other hand, for $x = 1$ it does not hold. Hence, $R(x) < 1$ for small enough $x$, while $R(1) \geq 1$. Thus, lemma 2.9 implies that there exists a unique $x_0 \in (0, 1)$ such that $R(x_0) = 1$. So any $x < x_0$ satisfies (2.2).

Finally, in the diagonally homogeneous case, the dimension formulas agree with the ones for Bedford–McMullen carpets.

**Corollary 2.12.** If a diagonally homogeneous shifted TGL carpet $\Lambda$ satisfies the conditions of theorems 2.2 and 2.7, then

$$\dim_H \Lambda = \frac{1}{\log b} \log \sum_{j=1}^{M} N^{\log_{\frac{b}{a}} x_j} \quad \text{and} \quad \dim_B \Lambda = \log N \left(1 - \frac{\log b}{\log a}\right) \frac{\log M}{\log b}.$$  

In particular, $\dim_H \Lambda = \dim_B \Lambda$ if and only if $\Lambda$ has uniform vertical fibres.

**Proof.** For diagonally homogeneous shifted TGL carpets the expression (2.1) for $D(p)$ simplifies to

$$D(p) = \frac{\log(\langle p \rangle_a)}{\log a} + \left(1 - \frac{\log b}{\log a}\right) \frac{\log(\langle q \rangle_a)}{\log b}.$$  

Applying this for $p^*$ from (2.8) gives the result $\dim_H \Lambda = D(p^*)$.

The equation for the box dimension $s = \dim_B \Lambda$, recall (2.4), simplifies to

$$N_i \cdot b^n \cdot a^{-s \cdot n} = 1. \quad (2.12)$$

Since $H$ satisfies No dimension drop property (recall definition 1.9), we have $s_H = \log M / (-\log b)$. Substituting this back into (2.12) and expressing $s$ from the equation gives the desired formula for $\dim_B \Lambda$.

Comparing the formula for $\dim_B \Lambda$ with the one for $D(p)$, we immediately get that equality holds if and only if $N_i = N/M$ for every $i \in \{1, \ldots, M\}$.  

### 3. Preliminaries

In this section, we collect important notation, definitions, preliminary lemmas and cite results used in the proofs of the subsequent sections.

#### 3.1. Symbolic notation

Throughout, we work simultaneously with the IFSs $F$, $\hat{H}$ and $H$, which are defined in (1.1), (1.2) and (1.6) respectively. Their attractors are $\Lambda$, $\Lambda_{\hat{H}} = \Lambda_{\hat{x}}$ respectively. We define the symbolic spaces
Finite words of length $n$ are either denoted with a ‘bar’ like $\overline{\imath} = i_1 \ldots i_n \in \Sigma_n$ or as a truncation $\imath[n] = i_1 \ldots i_n$ of an infinite word $\imath$, the length is denoted $|\imath|$. The set of all finite length words is denoted by $\Sigma^* = \bigcup_n \Sigma_n$ and analogously $\Sigma^*_n$. The left shift operator on $\Sigma$ and $\Sigma^*_n$ is $\sigma$, i.e. $\sigma(\imath) = i_2 i_3 \ldots$ and $\sigma(\bar{\imath}) = \bar{i_2 i_3} \ldots$.

The longest common prefix of $\imath$ and $\jmath$ is denoted $\imath \wedge \jmath$, i.e. its length is $|\imath \wedge \jmath| = \min\{k : i_k \neq j_k\} - 1$. This is also valid if one of them has or both have finite length. The $n$th level cylinder set of $\imath \in \Sigma$ is $[\imath][n] := \{j \in \Sigma : |\imath \wedge j| \geq n\}$. Similarly for $\tau \in \Sigma_n$ and $\bar{\imath} \in \Sigma^*_\mathcal{H}$. Recall that $R = [0,1]^2$. We use the standard notation $A_{i:n} = A_{i_1} \ldots A_{i_n}$ and $f_{i:n} = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}$ to write

$$A_{i:n} = f_{i:n}(\Lambda) \quad \text{and} \quad R_{i:n} := f_{i:n}(R)$$

for the $n$th level cylinder corresponding to $\imath$. The sets $\{R_{i:n}\}_{n=1}^\infty$ form a nested sequence of compact sets with diameter tending to zero, hence their intersection is a unique point $x \in \Lambda$. This defines the natural projection $\Pi : \Sigma \to \Lambda$

$$\Pi(\imath) := \lim_{n \to \infty} \bigcap_{n=1}^\infty R_{i:n} = \lim_{n \to \infty} f_{i:n}(\emptyset) = t_i + \sum_{n=2}^\infty A_{i,n-1} \cdot t_n. \quad (3.2)$$

The natural projections generated by $\mathcal{H}$ and $\mathcal{H}$ are

$$\Pi_{\mathcal{H}}(\imath) := \lim_{n \to \infty} h_{i:n}(0), \quad \imath \in \Sigma; \quad \text{and} \quad \Pi_{\mathcal{H}}(\bar{\imath}) := \lim_{n \to \infty} h_{\bar{i}:n}(0), \quad \bar{\imath} \in \Sigma^*_\mathcal{H}.$$

The following commutative diagram summarizes these notations:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Phi} & \Sigma^*_{\mathcal{H}} \\ \Pi \downarrow & \quad & \downarrow \Pi_{\mathcal{H}} \\ \Lambda & \xrightarrow{\text{proj}_{\mathcal{H}}} & \Lambda^*_\mathcal{H} \end{array} \quad (3.3)$$

We also introduce the measurable partitions $\alpha$ and $\beta$ of $\Sigma$ whose classes containing an $\imath \in \Sigma$ are defined

$$\alpha(\imath) := \Pi^{-1}(\Pi(\imath)) \quad \text{and} \quad \beta(\imath) := \Phi^{-1}(\Phi(\imath)). \quad (3.4)$$

The fact that these partitions are measurable are immediate consequences of the definition of measurability of a partition. Alternatively, this also follows from [34, theorem 2.2]. Thus, $\alpha(\imath)$ contains those $\jmath \in \Sigma$ for which $\Pi(\imath) = \Pi(\jmath) \in \Lambda$ and $\beta(\imath)$ corresponds to the ‘symbolic column’ of $\imath$, i.e. for $\jmath \in \beta(\imath)$ we have $\Pi_{\mathcal{H}}(\imath) = \Pi_{\mathcal{H}}(\jmath)$. These partitions play an important role when handling overlaps.

Bernoulli measures on $\Sigma$ are key in obtaining the lower bound for $\dim_{\mathcal{H}} \Lambda$. Recall the set

$$\mathcal{P} := \{\mathbf{p} = (p_1, \ldots, p_N) : p_i \geq 0, \quad \sum_{i=1}^N p_i = 1\}$$
of all probability distributions on the set \( \{1, 2, \ldots, N\} \) and let \( \mathcal{P}_0 \) denote the subset when all \( p_i > 0 \). The Bernoulli measure on \( \Sigma \) corresponding to \( \mathbf{p} \in \mathcal{P} \) is the product measure \( \mu_\mathbf{p} = \mathbf{p}^\Sigma \), i.e. the measure of a cylinder set is \( \mu_\mathbf{p}(\mathbf{1}[n]) = p_1 \cdot \ldots \cdot p_n \). All Bernoulli measures can be uniquely disintegrated according to the family of conditional measures \( \mu_{\mathbf{p},\alpha(i)} = \mu_{\alpha(i)} \) generated by the measurable partition \( \alpha \). That is for all Borel sets \( U \subset \Sigma \)

\[
\mu_{\mathbf{p}}(U) = \int \mu_{\alpha(i)}(U) d\mu_{\mathbf{p}}(i). \tag{3.5}
\]

The entropy of a Bernoulli measure \( \mu_\mathbf{p} \) is

\[
h_{\mu_\mathbf{p}} = -\sum_{i=1}^{N} p_i \log p_i = -\log(\mathbf{p})_\mathbf{p}. \tag{3.6}
\]

The push forward \( \nu_\mathbf{p} := \Pi_* \mu_\mathbf{p} \) is the self-affine measure on \( \Lambda \) defined by \( \nu_\mathbf{p} = \mu_\mathbf{p} \circ \Pi^{-1} \) or equivalently

\[
\nu_\mathbf{p} = \sum_{i=1}^{N} p_i \mu_\mathbf{p} \circ f_i^{-1}.
\]

Recall that a \( \mathbf{p} \in \mathcal{P} \) defines another distribution \( \mathbf{q} = (q_1, \ldots, q_M) \) via (1.18). Then \( \mu_\mathbf{q} = \mathbf{q}^\Sigma \) is a Bernoulli measure on \( \Sigma_\mathbf{q} \). Moreover, the self-similar measure on \( \Lambda_\mathbf{q} \) is \( \nu_\mathbf{q} = (\Pi_{\mathbf{q}})_* \mu_\mathbf{p} = (\Pi f_j)_* \nu_\mathbf{p} \). Our convention is that \( \mu \) always denotes a measure on (some) symbolic space, while \( \nu \) is supported on (a part of) \( R \).

### 3.2. Atypical parallelograms

The exponential rate of growth of the size of \( n \)th level parallelograms, the number of parallelograms in a column and the column’s measure can vary a lot for different \( \mathbf{i} \in \Sigma \). However, in a measure-theoretic sense those \( \mathbf{i} \) which behave atypically form a small set. Define the function

\[
X : \Sigma \rightarrow \mathbb{R}^+, \quad X(\mathbf{i}) := c_{\mathbf{i}},
\]

where \( \mathbf{c} = (c_1, \ldots, c_N) \) is an arbitrary vector with strictly positive elements. Let

\[
X_n(\mathbf{i}) := \prod_{j=0}^{n-1} X(\sigma^j\mathbf{i}) = \prod_{j=1}^{n} c_{\mathbf{i}_j}.
\]

In particular, if \( \mathbf{c} = \mathbf{a} := (a_1, \ldots, a_N) \) or \( \mathbf{b} := (b_1, \ldots, b_N) \), then \( X_n(\mathbf{i}) \) is the height and width of the parallelogram \( R_{\mathbf{a},\mathbf{b}} \). If \( \mathbf{c} = \mathbf{N} := (N_0(1), \ldots, N_0(N)) \) or \( \mathbf{q} := (q_{0(1)}, \ldots, q_{0(N)}) \), then \( X_n(\mathbf{i}) \) gives the number of parallelograms in and the measure of the column \( \Phi(\mathbf{i})n \).

Fix an arbitrary \( \mathbf{p} \in \mathcal{P} \). Recall the notation \( \langle \mathbf{c} \rangle_{\mathbf{p}} := \prod_{j=1}^{N} c_j^{p_j} \). When no confusion is made, we suppress \( \mathbf{p} \) and write \( \langle \mathbf{c} \rangle = \langle \mathbf{c} \rangle_{\mathbf{p}} \). In the rest of the subsection \( \delta > 0 \) is fixed. Define

\[
\text{Bad}_{\delta,n}(\mathbf{c}) := \begin{cases} \{ \mathbf{i} \in \Sigma : X_n(\mathbf{i}) < \langle \mathbf{c} \rangle^{(1-\delta)n} \text{ or } X_n(\mathbf{i}) > \langle \mathbf{c} \rangle^{(1+\delta)n}, \text{ if } \langle \mathbf{c} \rangle > 1, \\ \{ \mathbf{i} \in \Sigma : X_n(\mathbf{i}) < \langle \mathbf{c} \rangle^{(1+\delta)n} \text{ or } X_n(\mathbf{i}) > \langle \mathbf{c} \rangle^{(1-\delta)n}, \text{ if } \langle \mathbf{c} \rangle < 1. \end{cases} \tag{3.7}
\]

The definition can be extended to a positive real \( t \), by setting \( \text{Bad}_{t,n}(\mathbf{c}) := \text{Bad}_{t,n}(\mathbf{c})_{\mathbf{p}} \). Let \( \mu_\mathbf{p} \) be the Bernoulli measure on \( \Sigma \) defined by \( \mathbf{p} \in \mathcal{P} \).

**Lemma 3.1.** IF \( \langle \mathbf{c} \rangle_{\mathbf{p}} \neq 1 \) then there exists a constant \( C \) and an \( r \in (0, 1) \) such that

\[
\mu_\mathbf{p}(\text{Bad}_{\delta,n}(\mathbf{c})) < C \cdot r^n \quad \text{for every } n \geq 1.
\]
Hence, the Borel–Cantelli lemma immediately implies that 
\[ \mu_{\mathbf{p}}(i \in \text{Bad}_{\delta_{n}}(\mathbf{c})) \text{ for infinitely many } n) = 0. \]

**Proof.** Assume \( \langle \mathbf{c} \rangle > 1 \). Let \( S_{n}(X) := \frac{1}{n} \sum_{i=0}^{n-1} \log X(\sigma^{i}) \). Then 
\[ \mu_{\mathbf{p}}(X_{n}(i) < \langle \mathbf{c} \rangle^{(1-\delta)n}) = \mu_{\mathbf{p}}(S_{n}(X) < (1-\delta) \log \langle \mathbf{c} \rangle). \]

The \( \{\log X(\sigma^{i})\}_{i} \) are independent and identically distributed with expectation 
\[ E(\log X) = \sum_{j=1}^{N} p_{j} \log c_{j} = \log \langle \mathbf{c} \rangle. \]

Hence, Cramér’s large deviation theorem [12, theorem 2.1.24.] implies that 
\[ \mu_{\mathbf{p}}(X_{n}(i) < \langle \mathbf{c} \rangle^{(1-\delta)n}) \text{ decays exponentially fast in } n. \]

The argument for \( X_{n}(i) > \langle \mathbf{c} \rangle^{(1+\delta)n} \) is exactly the same. The proof is analogous when \( \langle \mathbf{c} \rangle < 1. \)

### 3.3. Ledrappier–Young formula

Let \( 0 < \alpha_{2}(A) \leq \alpha_{1}(A) < 1 \) denote the two singular values of a \( 2 \times 2 \) non-singular matrix \( A \). Namely, \( \alpha_{i}(A) \) is the positive square root of the \( i \)th largest eigenvalue of \( A^{T}A \), where \( A^{T} \) is the transpose of \( A \). The geometric interpretation of the singular values is that the linear map \( \mathbf{x} \mapsto A\mathbf{x} \) maps the unit disk to an ellipse with principal semi-axes of length \( \alpha_{2}(A) \) and \( \alpha_{1}(A) \). The singular values can also be expressed with the matrix norm: \( \alpha_{1}(A) = ||A|| \) and \( \alpha_{2}(A) = ||A^{-1}||^{-1} \). For a family of matrices \( \mathcal{A} = \{A_{1}, \ldots, A_{N}\} \), the asymptotic exponential growth rate of the semi-axes of the ellipses determined by the maps \( \mathbf{x} \mapsto A_{1}, \ldots, A_{N} \mathbf{x} \) is given by the Oseledec theorem.

**Theorem 3.2 (Oseledets [30]).** Let \( \mathcal{A} = \{A_{1}, \ldots, A_{N}\} \) be a set of non-singular \( 2 \times 2 \) matrices with \( ||A_{i}|| < 1 \) for \( i \in \{1, \ldots, N\} \). Then for any ergodic \( \sigma \)-invariant measure \( \mu \) on \( \Sigma \) there exist constants \( 0 < \chi_{1}^{\mu} \leq \chi_{2}^{\mu} \) such that for \( \mu \)-almost every \( i \)

\[ \lim_{n \to \infty} \frac{1}{n} \log \alpha_{1}(A_{i}^{n}) = \lim_{n \to \infty} \frac{1}{n} \log ||A_{i}^{n}|| = -\chi_{1}^{\mu}, \]

\[ \lim_{n \to \infty} \frac{1}{n} \log \alpha_{2}(A_{i}^{n}) = \lim_{n \to \infty} \frac{1}{n} \log ||(A_{i}^{n})^{-1}||^{-1} = -\chi_{2}^{\mu}. \]

The numbers \( \chi_{1}^{\mu} \) and \( \chi_{2}^{\mu} \) are called the Lyapunov-exponents of \( \nu = 1_{\Sigma} \mu \). If \( \chi_{1}^{\mu} \neq \chi_{2}^{\mu} \) then we say that \( \mu \) has simple Lyapunov spectrum.

It is an easy exercise to calculate the Lyapunov exponents of Bernoulli measures \( \mu_{\mathbf{p}} \) for a family of lower triangular matrices for which direction-\( x \)-dominates. For greater generality see Falconer–Miao [13].

**Lemma 3.3.** Fix any \( \mathbf{p} \in \mathcal{P} \) and a family of lower triangular matrices \( \mathcal{A} = \{A_{1}, \ldots, A_{N}\} \) for which direction-\( x \)-dominates. Then the Lyapunov spectrum of the Bernoulli measure \( \mu_{\mathbf{p}} \) is simple and the exponents equal 
\[ \chi_{1_{\mathbf{p}}}^{1} = -\sum_{i=1}^{N} p_{i} \log b_{i} = -\log \langle \mathbf{b} \rangle_{\mathbf{p}} \quad \text{and} \quad \chi_{1_{\mathbf{p}}}^{2} = -\sum_{i=1}^{N} p_{i} \log a_{i} = -\log \langle \mathbf{a} \rangle_{\mathbf{p}}. \]
Sketch of proof. Both the singular values or the norm of $A_{i_1,...,i_l}$ can be calculated directly. Since direction-$x$ dominates, the off-diagonal element does not play a role. An application of Oseledets theorem and the strong law of large numbers concludes the proof.

The Ledrappier–Young formula originates from the seminal work of Ledrappier and Young [27, 28] on determining the Hausdorff dimension of invariant measures of diffeomorphisms on compact manifolds. Through a succession of papers by Przytycki–Urbaniński [33], Feng–Hu [17], Bárány–Káemik [6] the formula was proved for the Hausdorff dimension of wider and wider classes of self-affine measures. In fact, Feng [16] recently announced that the Hausdorff dimension of the push-forward of a shift-invariant, ergodic measure $\mu$ satisfies a Ledrappier–Young type formula in full generality for any self-affine IFS on $\mathbb{R}^d$ which is contracting on average with respect to $\mu$. Also observe that the formulas proved in the earlier works of [2, 3, 10, 21, 29] are all special cases of the Ledrappier–Young formula. The main result of [6, theorem 2.4, corollary 2.8] can be stated in a simpler form in our context when direction-$x$ dominates.

Theorem 3.4 ([6], direction-$x$ dominates). Let $F$ be a shifted TGL-type IFS of the form (1.1). Furthermore, using the notation from section 3.1, let $\nu_p$ be any Bernoulli measure on $\Sigma$, $\nu_q = (\text{proj}_x)_* \nu_p$. Then, regardless of overlaps, $\nu_p$ is exact dimensional and satisfies the Ledrappier–Young formula

$$\dim H \nu_p = \frac{h_{\mu_p} - H}{\chi_{i_1}} + \left(1 - \frac{\chi_{i_2}}{\chi_{i_1}}\right) \dim H \nu_q,$$

where $H = -\int \log \mu_p(x) d\mu_p(x)$. Recall $\{\mu_{\alpha(i)}\}$ is the family of conditional measures of $\mu_p$ defined by the measurable partition $\alpha(i) = \Pi^{-1}(\Pi(i))$.

Moreover, if the IFS satisfies the ROSC and $p \in P_0$, then $H = 0$.

4. Upper bound for $\dim H \Lambda$

Consider a shifted triangular Gatzouras–Lalley-type planar carpet $\Lambda$ without any separation condition. To prove theorem 2.1 we essentially lift the original argument in [21], formulated on the attractor $\Lambda$, to the symbolic space $\Sigma$. This can be done because the method in [21] is completely symbolic in nature. Therefore, we only give a short sketch. More complete details can be found in the extended version [25].

The first step is to define a proper metric on $\Sigma$, which captures the distance between points on the attractor. Observe that for two points $i, j \in \Sigma$ the distance $|\Pi(i) - \Pi(j)|$ (recall (3.2)) can be small even if $|i \wedge j|$ is small. This occurs if $|\Phi(i) \wedge \Phi(j)|$ (recall (3.1)) is much larger than $|i \wedge j|$, i.e. the corresponding cylinders belong to the same column for a long time.

Lemma 4.1. $(\Sigma, d)$ is a metric space, where the distance between $i, j \in \Sigma$ is defined

$$d(i, j) := \prod_{k=1}^{l} b_{i_k} + \prod_{k=1}^{l} a_{i_k}.$$ 

Proof. It is straightforward to check that $d$ defines a metric on $\Sigma$.

The next step is to prove that the natural projection is Lipschitz with respect to this metric, which implies the following lemma.
Lemma 4.2. For any shifted triangular Gatzouras–Lalley-type planar carpet
\[ \dim_h \Lambda \leq \dim_H(\Sigma, d). \]

Proof. Using lemma 1.3 it is straightforward to check that there exists a \( C > 0 \) such that
\[ |\Pi(i) - \Pi(j)| \leq C \cdot d(i, j). \] Details can be found in [25]. \( \square \)

It remains to show that the value \( \alpha^* \) maximizing the expression for \( D(p) \) in (2.1) is an upper bound for the Hausdorff dimension of \( (\Sigma, d) \).

Proposition 4.3. For any choice of parameters defining a shifted triangular Gatzouras–Lalley-type triangular carpet
\[ \dim_h(\Sigma, d) \leq \alpha^*. \]

Proof of theorem 2.1. The upper bound is a corollary of lemma 4.2 and proposition 4.3. The compactness of \( \mathcal{P} \) and the continuity of \( D(p) \) implies that \( \sup_p D(p) \) is attained for some \( p^* \in \mathcal{P} \). Moreover, it is easy to check that \( p^* \in \mathcal{P}_0 \), see [21, proposition 3.4]. \( \square \)

Proposition 4.3 is essentially proved in [21, section 5] formulated on the attractor \( \Lambda \). We do not repeat the argument here, a sketch of the main steps is available in [25]. We merely comment that the balls in \( (\Sigma, d) \) are exactly the ‘approximate squares’ defined in [21, equation (1.2)]
\[ B_k(i) := \{ j \in \Sigma : |i \wedge j| \geq L_k(i) \text{ and } |i \wedge j| \geq k \}, \]
where
\[ L_k(i) := \max \left\{ n \geq 0 : \prod_{j=1}^{k} b_j^i \leq \prod_{j=1}^{n} a_j \right\}. \]
Furthermore, the Bernoulli measure \( \mu_{p^*} \) for which the maximum of \( D(p) \) is attained is the same family of Gatzouras–Lalley Bernoulli measures introduced in [21, equation (5.2)]. Namely, define the probability vector \( p = (p_1, \ldots, p_N) \) by
\[ p_i = p_i(\vartheta, \lambda, \rho) := C(\vartheta, \lambda, \rho) a_i^\lambda b_i^\vartheta \gamma_i(\vartheta)^{\rho-1}, \]
where \( \gamma_i(\vartheta) = \sum_{j \in A_i(\vartheta)} a_j^\vartheta \).

5. Proof of theorem 2.2

Our goal is to show that the Ledrappier–Young formula (3.8) of [6] for \( \dim_H \mu_\varrho \), cited in theorem 3.4, always equals the formula for \( D(p) \) in (2.1) under the conditions of theorem 2.2. For the rest of this proof, we fix a \( p \in \mathcal{P}_0 \) and assume \( \mathcal{H} \) satisfies HESC and either each column independently satisfies the ROSC or \( \Lambda \) satisfies transversality and (2.2).

The entropy of the system is \( h_{\nu_\varrho} = -\log \langle p \rangle_p \) (recall (3.6)), the Lyapunov-exponents from lemma 3.3 are \( \chi_{\nu_\varrho}^1 = -\log \langle a \rangle_p \) and \( \chi_{\nu_\varrho}^2 = -\log \langle b \rangle_p \). HESC for \( \mathcal{H} \) implies NDD for \( \nu_\varrho \), recall (1.19), hence \( \dim_H \nu_\varrho = \log \langle \rho \rangle_\varrho / \log \langle b \rangle_p \). As a result, to prove the theorem it is enough to show that the integral
\[ H = -\int \log \mu_{\alpha(i)}([i]) d\mu_\varrho(i) = 0, \]
where \( \{ \mu_{\alpha(i)} \} \) is the family of conditional measures of \( \mu_p \) defined by the measurable partition \( \{ \alpha(i) = \Pi^{-1}(\Pi(i)) \} \), recall (3.5). Since \( -\log \mu_{\alpha(i)}([i]) \geq 0 \), we have that \( H = 0 \) if and only if
\[ \mu_{\alpha(i)}([i]) = 1 \text{ for } \mu_p - \text{a.a. } i. \]  \hfill (5.1)

Thus, it suffices to show that \( \mu_{\alpha(i)} \) is concentrated on \( i \) for \( \mu_p \)-typical \( i \). Overlaps arising from the translations of columns or from intersections within a column can in theory cause problems. However, the next two results ensure that there is a full measure subset of \( \Sigma \) for which \( \mu_{\alpha(i)} \) is a point mass distribution.

Recall from (3.4) that \( \beta(i) = \Phi^{-1}(\Phi(i)) \) is the ‘symbolic column’ if \( i \). The first claim ensures that there is a full measure subset \( \Sigma_1 \subset \Sigma \) where the translations of the columns have no effect.

**Claim 5.1.** Assume Weak almost unique coding (WAUC) holds for \( \Sigma_H \), recall definition 1.9. Then there exists a full measure subset \( \Sigma_1 \subset \Sigma \) such that for all \( i \in \Sigma_1 \) and for all \( (j_1, \ldots, j_n) \neq (i_1, \ldots, i_n) \)

\[ \mu_{\alpha(i)}([j_1, \ldots, j_n]) = 0, \]  \hfill (5.2)

where \( \phi(i_k) = i_k \) and \( \phi(j_k) = j_k \) for \( k = 1, \ldots, n \), recall (1.9) for the definition of \( \phi \).

Consequently, for every \( i \in \Sigma_1 \) we have

\[ \mu_{\alpha(i)}(\beta(i)) = 0. \]  \hfill (5.3)

The second claim defines the full-measure set \( \Sigma_2 \subset \Sigma \) where intersections within columns have no effect.

**Proposition 5.2.** Assume that the conditions of theorem 2.2 hold. Then there exists a \( \Sigma_2 \subset \Sigma \), with \( \mu_p(\Sigma_2) = 1 \) such that for every \( i \in \Sigma_2 \) and \( k \in I_{\phi(i_1)} \setminus \{i_1\} \)

\[ \mu_{\alpha[i]}(\beta(i) \cap \alpha(i) \cap [k]) = 0. \]

Theorem 2.2 is a corollary of these two results. Sometimes we use the following notation:

**Definition 5.3.** Let \( F \subset \Sigma \) be a subset of full measure. Then we define

\[ \tilde{F} := \{ i \in F : \mu_{\alpha(i)}(\Sigma \setminus F) = 0 \}. \]

Since \( \mu_p(F) = 1 \), the disintegration formula (3.5) implies that \( \mu_p(\tilde{F}) = 1 \).

5.1. The proof of theorem 2.2 assuming claim 5.1 and proposition 5.2

**Proof of theorem 2.2 assuming claim 5.1 and proposition 5.2.** As we established above in (5.1) that to prove the theorem it is enough to check that

\[ \mu_{\alpha(i)}(\alpha(i) \cap [i]) = 0, \text{ for } \mu - \text{a.a. } i. \]  \hfill (5.4)

Clearly,

\[ \alpha(i) \cap [i] = \left( \bigcup_{k \in I_{\phi(i_1)}} (\alpha(i) \cap [k]) \right) \cup \left( \bigcup_{k \in I_{\phi(i_1)} \setminus \{i_1\}} (\alpha(i) \cap [k]) \right). \]

It follows from (5.2) that for every \( i \in \Sigma_1 \)

\[ \mu_{\alpha(i)}\left( \bigcup_{k \in I_{\phi(i_1)}} (\alpha(i) \cap [k]) \right) = 0, \]  \hfill (5.5)
where $\Sigma_1$ is defined in claim 5.1. Thus, to prove the theorem we only need to verify that
\[
\mu_{\alpha(i)} \left( \bigcup_{k \in \mathcal{I}(\alpha(i)) \setminus \{i\}} (\alpha(i) \cap [k]) \right) = 0 \quad \text{for } \mu \text{-a.a. } i.
\]  
(5.6)

We can write
\[
\bigcup_{k \in \mathcal{I}(\alpha(i)) \setminus \{i\}} (\alpha(i) \cap [k]) \subset \Sigma_1^c \cup \Sigma_2^c
\]
\[
\bigcup_{k \in \mathcal{I}(\alpha(i)) \setminus \{i\}} (\alpha(i) \cap \beta(i) \cap [k]) \cup \bigcup_{k \in \mathcal{I}(\alpha(i)) \setminus \{i\}} (\alpha(i) \cap \beta(i)^c \cap [k]).
\]

It follows from proposition 5.2 that $\mu_{\alpha(i)}(U) = 0$ for all $i \in \Sigma_2$, and it follows from claim 5.1 that $\mu_{\alpha(i)}(V) = 0$ for all $i \in \Sigma_1$. So, for all $i \in \Sigma_1 \cap \Sigma_2$ (5.6) holds, which together with (5.5) yields that (5.4) holds. This completes the proof of theorem 2.2 assuming claim 5.1 and proposition 5.2.

\[\square\]

5.2. The proof of claim 5.1

**Proof of claim 5.1.** In the definition of WAUC, recall definition 1.9, there is a set $\mathcal{B}_\Sigma \subset \Sigma_1$ defined in such a way that for $\Sigma' := \Sigma_1 \setminus \mathcal{B}_\Sigma$ we have $\mu_{\alpha}(\Sigma') = 1$ and
\[
i \in \Sigma' \iff \Sigma' \cap (\Pi_1^{-1}(B_{\Sigma} i)) = \{i\},
\]
where $\Pi_1$ is the natural projection from $\Sigma_1$ to $\Lambda_1$. Let
\[B := \Phi^{-1}(\mathcal{B}_\Sigma) \text{ and } \Sigma' := \Phi^{-1}(\Sigma').\]

Since $\mu_{\alpha}(\Sigma') = 1$ we can define $\Sigma_1 := \Sigma'$ (recall the notation $\sim$ from definition 5.3) so that $\mu_{\alpha}(\Sigma_1) = 1$ and
\[
\mu_{\alpha(i)}(B) = 0 \quad \text{for all } i \in \Sigma_1.
\]
(5.7)

Recall $\Pi_1$ is the natural projection from $\Sigma$ to $\Lambda_1$. Observe that by definition
\[
i \in \Sigma' \Rightarrow \Sigma' \cap (\Pi_1^{-1}(\Pi_1(i))) = \beta(i).
\]
(5.8)

Since $\alpha(i) \subset \Pi_1^{-1}(\Pi_1(i))$, we get from (5.8) that $i \in \Sigma' \Rightarrow \Sigma' \cap \alpha(i) \subset \beta(i)$. Equivalently,
\[
i \in \Sigma' \Rightarrow \alpha(i) \subset \beta(i) \cup B.
\]

By definition
\[\{j_1, \ldots, j_n\} \cap \beta(i) = 0 \iff (j_1, \ldots, j_n) \neq (i_1, \ldots, i_n).
\]

That is for $i \in \Sigma_1$ whenever $(j_1, \ldots, j_n) \neq (i_1, \ldots, i_n)$ then $[j_1, \ldots, j_n] \cap \alpha(i) \subset B$. So, (5.7) implies that (5.2) holds.

To obtain (5.3) from (5.2), we write $\beta(i)^c$ as a countable union
\[
\beta(i)^c = \bigcup_{\ell=0}^{\infty} \{j \in \Sigma : [i \wedge j] = \ell\} = \bigcup_{\ell=0}^{\infty} \bigcup_{j_{\ell+1} \neq i} [j_1, \ldots, j_{\ell+1}].
\]
By (5.2) the measure of each cylinder of the right-hand side is
\[ \mu_{a[i]}([j_1, \ldots, j_{\ell+1}]) = 0 \text{ if } [j_1, \ldots, j_{\ell+1}] \neq [i_1, \ldots, i_{\ell+1}], \quad i \in \Sigma_1.\]

5.3. Proof of proposition 5.2

If the columns independently satisfy ROSC, then the proof of [6, corollary 2.8] can be repeated in this setting, therefore we omit it. In the remainder we assume the shifted TGL carpet \( \Lambda \) satisfies transversality and (2.2):
\[ \frac{\log \langle a \rangle_p}{\log \langle b \rangle_p} > 1 + \frac{\log \langle N \rangle_q}{-\log \langle q \rangle_q}.\]

Throughout this proof, we fix \( \delta > 0 \) small enough such that
\[ 1 + \delta + \frac{(1 + \delta) \log \langle N \rangle_q}{\delta \log \langle b \rangle_p - \log \langle q \rangle_q} < (1 - \delta) \frac{\log \langle a \rangle_p}{\log \langle b \rangle_p}. \tag{5.9} \]

This can be achieved since the expression is continuous in \( \delta \) and we assume (2.2). The reason that we require this is that for such a \( \delta \) and
\[ u := (1 - \delta) \frac{\log \langle a \rangle_p}{\log \langle b \rangle_p} - (1 + \delta), \tag{5.10} \]
the inequality in (5.9) is equivalent to
\[ \langle N \rangle^{(1+\delta)} \cdot \langle q \rangle^{n} \cdot \langle b \rangle^{-\delta n} < 1. \tag{5.11} \]

At the very end of this proof we will need this. The importance of \( u \) defined above comes from the fact that for an arbitrary \( \ell \) and \( k = u \cdot \ell, \)
\[ \langle b \rangle_{\ell}^{k} = \frac{\langle a \rangle_{\ell}^{(1-\delta)\ell}}{\langle b \rangle_{\ell}^{(1+\delta)\ell}}. \tag{5.12} \]

Recall \( \alpha(i) = \Pi^{-1} \Pi(i), \beta(i) = \Phi^{-1} \Phi(i), \) that \( \Pi_K \) is the natural projection from \( \Sigma \) to \( \Lambda_K \) and that in (3.7) we define \( \text{Bad}_{\delta,n}(c) \) for a \( c = (c_1, \ldots, c_N) \) with \( \langle c \rangle_p \neq 1. \)

5.3.1 Further notation. Recall that Hochman’s exponential separation condition implies that for the self-similar measure \( \nu_q \) on \( \Lambda_K \) we have \( \dim_H \nu_q = \log \langle q \rangle_q / \log \langle b \rangle_p. \) Feng and Hu [17] proved that \( \nu_q \) is exact dimensional. That is for \( K_1 \) defined in (1.15) and
\[ S_{m_0} := \{ i \in \Sigma : \forall n \geq n_0, \nu_q \left( \left( \Pi_K(i) - 3K_1\langle b \rangle^{n}, \Pi_K(i) + 3K_1\langle b \rangle^{n} \right) \right) \in \left( \langle q \rangle^{n} \cdot \langle b \rangle^{-\delta n}, \langle q \rangle^{n} \cdot \langle b \rangle^{-\delta n} \right) \} \]
we have
\[ \mu_p \left( \bigcup_{n=1}^{\infty} S_{m_0} \right) = \lim_{m_0 \to \infty} \mu_p (S_{m_0}) = 1. \tag{5.13} \]

We define the set of symbols which are ‘good’ from level \( m \) on:
\[ \text{Good}_{m} := \bigcap_{n \geq m} \left( \text{Bad}_{\delta,n}(a) \cup \text{Bad}_{\delta,n}(b) \cup \text{Bad}_{\delta,n}(N) \cup \text{Bad}_{\delta,n}(q) \right)^c. \]
Note that it follows from lemma 3.1 that for
\[
\text{Good} := \bigcup_{m=1}^{\infty} \text{Good}_m, \quad \text{we have } \mu_p(\text{Good}) = 1. \tag{5.15}
\]

To measure vertical distance and neighborhood on $\Lambda$ we define
\[
\text{dist}_\alpha((x_0, y_0), (x, y)) := \begin{cases} |y - y_0|, & \text{if } x = x_0; \\ \infty, & \text{otherwise.} \end{cases}
\]

For every $m \geq 1$ the function $L_m : \text{Good} \to [0, 1]$ is defined as follows: if there exists no $j \in \text{Good}_m$ with $j_1 \neq i_1$ and $\Phi(i) = \Phi(j)$ then $L_m(i) := 1$. Otherwise we define
\[
L_m(i) := \inf \{ \text{dist}_\alpha(\Pi(i), \Pi(j)) : j \in \text{Good}_m \cap \beta(i) \text{ such that } j_1 \neq i_1 \}.
\]

Let
\[
V^m_\ell := \{ i \in \text{Good} : L_m(i) < \langle a \rangle^\ell \}
\]
\[
= \{ i \in \text{Good} : \exists j \in \beta(i) \cap [i_1]^\ell \cap \text{Good}_m, \ \text{dist}_\alpha(\Pi(i), \Pi(j)) < \langle a \rangle^\ell \}.
\]

Also define
\[
B^m_\ell := \{ i \in \text{Good} : L_m(i) = 0 \} = \{ i \in \text{Good} : \alpha(i) \cap \beta(i) \cap [i_1]^\ell \cap \text{Good}_m \neq \emptyset \}. \tag{5.16}
\]

Clearly, $B^m_\ell \subset B^{m+1}_\ell$ since $B^m_\ell = \cap_{\ell \geq m} V^m_\ell$. The key lemma states the following.

**Lemma 5.4.** For arbitrary $m \geq 1$ we have $\mu_p(B^m_\ell) = 0$.

**Proof of proposition 5.2 assuming lemma 5.4.** Let
\[
B_2 := \bigcup_{m=1}^{\infty} B^m_\ell = \{ i \in \text{Good} : \alpha(i) \cap \beta(i) \cap [i_1]^\ell \cap \text{Good} \neq \emptyset \}.
\]

By lemma 5.4, $\mu_p(B_2) = 0$. That is if $i \in \Sigma_2 := \widehat{\text{Good}} \cap B^c_2$ then, on the one hand, $\mu_{\alpha(i)}(\text{Good}^c) = 0$, on the other hand, $\alpha(i) \cap \beta(i) \cap [i_1]^\ell \cap \text{Good} = \emptyset$. This implies that $\mu_{\alpha(i)}(\alpha(i) \cap \beta(i) \cap [i_1]^\ell \cap \text{Good} = \emptyset)$, which completes the proof of proposition 5.2. \qed

It remains to show lemma 5.4. The method of the proof was inspired by [8, lemma 4.7], however, there are significant differences. On the one hand, in [8] the measure corresponding to $\nu_q$ is absolutely continuous with $L^q$ density and on the other hand, in [8] the diagonal part of all the linear parts of all the mappings are identical. These differences required a much more subtle argument in this paper.

**Proof of lemma 5.4.** Recall that we fixed an $m$. Let $\ell \geq m$. All sets and numbers from now on in this proof can be dependent of $m$ but $m$ is fixed so we omit it from notation.

We cover $V^m_\ell$ by the union of the $\Pi^{-1}$ pre-images of the parallelograms like the blue one ($R_{77}$) on the right-hand side of figure 8. These are parallelograms slightly bigger than the intersection of $R_7$ and the $\langle a \rangle^\ell$ neighborhood of $R_7$ for $r, j \in \Sigma_\ell$ with $\Phi(r) = \Phi(j)$.

To control the size of $\ell$th level parallelograms and the number of parallelograms in any given $\ell$th level column set
\[
\text{Bad}_{\delta, \ell} := \text{Bad}_{\delta, \ell}(a) \cup \text{Bad}_{\delta, \ell}(b) \cup \text{Bad}_{\delta, \ell}(N) \quad \text{and} \quad \text{Bad}_{\delta, \ell}^{\ell+1} := \{ i \ell : i \in \text{Bad}_{\delta, \ell}^1 \},
\]

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where $\text{Bad}_{\delta,n}(c)$ was defined in (3.7). Observe that $\text{Bad}_{\delta,1}^I$ is the union of complete $\ell$-cylinders. That is

$$\text{Bad}_{\delta,1}^I = \bigcup_{\omega \in \text{Bad}_{\delta,1}^I} \{ \omega \}.$$ 

The level $\ell$-cylinders of the symbolic spaces excluding these bad cylinders are:

$$\text{Good}_{\ell}^*: \{ 1, \ldots, N \}^\ell \setminus \text{Bad}_{\delta,1}^I \quad \text{and} \quad \text{Good}_{\ell,\tau}^*: \{ \tau \in \text{Good}_{\ell}^*: j_1 \neq i_1, \Phi(\gamma) = \Phi(\tau) \} \setminus \text{Bad}_{\delta,1}^I.$$ 

For $H \subset [0,1]^2$ let

$$U_\ell(H, r) := \bigcup_{(x_0,y_0) \in H} \{ (x,y) : x = x_0 \text{ and } |y - y_0| < r \}.$$ 

Choose $\tau \in \text{Good}_{\ell}^*$, $\gamma \in \text{Good}_{\ell,\tau}^*$ and define

$$I_{\tau,\gamma} := \text{proj}_\gamma(R_{\tau} \cap U_\ell(R_{\tau}, (\gamma)^{(1-\delta)}_\ell)), \quad R_{\tau,\gamma} := (I_{\tau,\gamma} \times [0,1]) \cap R_{\tau}, \quad \tilde{R}_{\tau,\gamma} := ([\tau] \cap \Pi^{-1}(R_{\tau,\gamma})).$$

$R_{\tau,\gamma}$ consists of those elements of $R_{\tau}$ which are physically ‘too close’ to $R_{\tau}$, see figure 8. As a result, we get a cover of $V_{\ell}^m$:

$$V_{\ell}^m \subset \text{Bad}_{\delta,1}^I \cup \bigcup_{\tau \in \text{Good}_{\ell}^*} \bigcup_{\gamma \in \text{Good}_{\ell,\tau}^*} \tilde{R}_{\tau,\gamma}. \quad (5.17)$$

Namely, if $i \in V_{\ell}^m$ then either $i \in \text{Bad}_{\delta,1}^I$ or $\tau := i|\ell \in \text{Good}_{\ell}^*$. In the second case, there is a $j \in \beta(i) \cap [i]^\ell \cap \text{Good}_{\delta,1}^* \text{ with dist}(\Pi(I), \Pi(J)) < (a)^\ell$. Hence, $\gamma := j|\ell \in \text{Good}_{\ell,\tau}^*$. As a result, with these notations, we have $i \in \tilde{R}_{\tau,\gamma}$.

If $I_{\tau,\gamma} \neq \emptyset$, then there exists a non-empty interval $J_{\tau,\gamma}$ such that

$$f_{\tau}^{-1}(R_{\tau,\gamma}) = J_{\tau,\gamma} \times [0,1].$$

With symbolic notation $H_{\tau,\gamma} := \Pi^{-1}(f_{\tau}^{-1}(R_{\tau,\gamma}))$ we can represent $\tilde{R}_{\tau,\gamma}$ as the concatenation

$$\tilde{R}_{\tau,\gamma} = u_\gamma H_{\tau,\gamma}. \quad (5.18)$$

On the other hand, $H_{\tau,\gamma} \subset \Pi_H^{-1}(J_{\tau,\gamma})$. Hence,
\[ \mu_p(H_{\tau,\gamma}) \leq \mu_p(\tilde{\Pi}_{H}^{-1}(J_{\tau,\gamma})) = \nu_q(J_{\tau,\gamma}). \] (5.19)

To continue, we give an upper bound for \( \nu_q(J_{\tau,\gamma}). \)

**Claim 5.5.** Let \( \tilde{\tau} \in \text{Good}_\tau^\ast \) and \( \tilde{\gamma} \in \text{Good}_\gamma^\ast \) and let \( k := u \cdot \ell, \) where \( u \) was defined in (5.10). If \( \tilde{\Pi}_{H}^{-1}(J_{\tau,\gamma}) \cap S_k \neq \emptyset \) (recall (5.13) for the definition of \( S_k \)), then
\[ \nu_q(J_{\tau,\gamma}) \leq \langle q \rangle^k \cdot \langle b \rangle^{-k\delta}. \]

**Proof of claim 5.5.** If \( J_{\tau,\gamma} \neq \emptyset \) then transversality (recall definition 1.6) implies that
\[ |J_{\tau,\gamma}| < 3K_1 \cdot \langle a \rangle^{(1-\delta)\ell}. \]

This is the very important point where we use that neither \( \tilde{\tau} \) nor \( \tilde{\gamma} \) is contained in \( \text{Bad}_{\tau,\gamma}^\ast \). Furthermore, \( \tilde{f}_{\tau}^{-1} \) expands along the \( x \)-axis by a factor between \( \langle b \rangle^{-(1-\delta)\ell} \) and \( \langle b \rangle^{-(1+\delta)\ell} \), hence
\[ |J_{\tau,\gamma}| < 3K_1 \cdot \frac{\langle a \rangle^{1-\delta}}{\langle b \rangle^{1+\delta}} \ell. \] (5.20)

If we set \( k \) as in claim 5.5 then as we mentioned in (5.12) the right-hand side of (5.20) is less than \( 3K_1 \cdot \langle b \rangle^k \):
\[ |J_{\tau,\gamma}| < 3K_1 \cdot \langle b \rangle^k. \]

Now assume that \( \tilde{\Pi}_{H}^{-1}(J_{\tau,\gamma}) \cap S_k \neq \emptyset \). Pick an arbitrary \( \omega \in \tilde{\Pi}_{H}^{-1}(J_{\tau,\gamma}) \cap S_k \). Then
\[ J_{\tau,\gamma} \subset \left( \tilde{\Pi}_{H}^{-1}(\omega) - 3K_1 \cdot \langle b \rangle^k, \tilde{\Pi}_{H}(\omega) + 3K_1 \cdot \langle b \rangle^k \right). \]

Using that \( \omega \in S_k \), we get that
\[ \nu_q(\tilde{\Pi}_{H}^{-1}(\omega) - 3K_1 \cdot \langle b \rangle^k, \tilde{\Pi}_{H}(\omega) + 3K_1 \cdot \langle b \rangle^k) \leq \langle q \rangle^k \cdot \langle b \rangle^{-k\delta}. \]

Now we conclude the proof of lemma 5.4. From the cover (5.17) of \( V_{\tau}^m \) together with (5.18) we obtain that for \( \ell \geq m \)
\[ \mu_p(V_{\tau}^m) \leq \mu_p(\text{Bad}_{\delta,\ell}) + \sum_{\tau \in \text{Good}_\tau^\ast} \mu_p([\tilde{\tau}]) \mu_p\left( \bigcup_{\tilde{\gamma} \in \text{Good}_\gamma^\ast} H_{\tau,\gamma} \right). \] (5.21)

To further bound (5.21), first observe that
\[ \# \text{Good}_\tau^\ast \leq \langle N \rangle^{(1+\delta)\ell}, \] whenever \( \tau \in \text{Good}_\tau^\ast \).

Moreover, using (5.19) and claim 5.5, for an arbitrary \( \tau \in \text{Good}_\tau^\ast \) we have
\[ \mu_p\left( \bigcup_{\tilde{\gamma} \in \text{Good}_\gamma^\ast} H_{\tau,\gamma} \right) \leq \mu_p(S_k) + \sum_{\tau \in \text{Good}_\tau^\ast} \mu_p(H_{\tau,\gamma}) \leq \mu_p(S_k) + \sum_{\tau \in \text{Good}_\tau^\ast} \nu_q(J_{\tau,\gamma}) \]
\[ \leq \mu_p(S_k) + \sum_{\tau \in \text{Good}_\tau^\ast} \langle q \rangle^k \cdot \langle b \rangle^{-k\delta} \]
\[ \leq \mu_p(S_k) + \left( \langle N \rangle^{(1+\delta)} \cdot \langle q \rangle^u \cdot \langle b \rangle^{-a\delta} \right)^\ell. \]
Plugging this back into (5.21), we deduce from lemma 3.1, (5.14) and (5.11) that for every \( m \),
\[
\lim_{\ell \to \infty} \mu_p(V^m_\ell) = 0.
\]
By (5.16), this implies that \( \mu_p(B^n_m) = 0 \).

6. Proof of results for box dimension

We begin the section by briefly commenting on how the upper bound for \( \dim B \Lambda \), recall theorem 2.4, follows directly from the work of Fraser [19] and then prove theorem 2.7 in section 6.3.

For \( \delta > 0 \) and a bounded set \( F \subset \mathbb{R}^2 \) let \( N_\delta(F) \) denote the minimal number of closed axes parallel rectangles for which the vertical sides are not shorter than the horizontal sides but the vertical sides are not longer than \((K_0 + 1)\)-times the horizontal sides, where \( K_0 \) was defined in lemma 1.3. Then
\[
\dim B F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \quad \text{and} \quad \overline{\dim B} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.
\]
In particular, it is enough to consider \( \delta \to 0 \) through the sequence \( \delta_k = c_k \) for some \( 0 < c < 1 \), see [15, section 3.1].

For \( t \geq 0 \) and any finite length word \( \bar{t} \in \Sigma^* \), Fraser defined the modified singular value function \( \psi^t \), which in our context is
\[
\psi^t(f_t) := b^{s_H}_t \cdot a^{-s_H}_t,
\]
where \( s_H = \dim B \Lambda_H \). He showed that the unique solution \( s \) of the equation
\[
\lim_{n \to \infty} \left( \sum_{\bar{t} \in \Sigma^n} \psi^t(f_t) \right)^{1/n} = 1
\]
is an upper bound for \( \dim B \Lambda \) and equals \( \dim B \Lambda \) if \( \Lambda \) satisfies the ROSC. In our context this equation simply becomes (2.4): \( \sum_{i=1}^{N} b^{s_H}_i \cdot a^{-s_H}_i = 1 \). The slight modification of the GL brother \( \tilde{\Lambda} \) ensures that the solution of (2.4) for \( \Lambda \) and \( \tilde{\Lambda} \) is the same.

For any TGL carpet \( \Lambda \), it follows from lemma 1.3 that the longer side of any parallelogram \( R_{\bar{t}} \) is at most \((K_0 + 1) \cdot b_{\bar{t}}\). This implies that there exists a constant \( C \) (independent of \( \delta \)) such that \( N_\delta(\Lambda) \leq C \cdot N_\delta(\tilde{\Lambda}) \). Hence, \( \overline{\dim B}(\Lambda) \leq \overline{\dim B}(\tilde{\Lambda}) \leq s \). Furthermore, when the ROSC is assumed, it is clear that the reversed inequalities also hold. This implies \( \overline{\dim B}(\Lambda) \geq \overline{\dim B}(\tilde{\Lambda}) = s \). This proves theorem 2.4.

In the presence of overlaps, one must be more careful when counting the intersections. The next subsection shows how to select a diagonally homogeneous subsystem from a higher iterate of \( F \).

6.1. Diagonally homogeneous subsystems

Recall for \( \bar{p} \in \mathcal{P}_0 \)
\[
h_{\bar{p}} = -\sum_{i=1}^{N} p_i \log p_i = -\log(p_{\bar{p}}),
\]
\[
\chi^1_{\bar{p}} = -\sum_{i=1}^{N} p_i \log b_i = -\log(b_{\bar{p}}) \quad \text{and} \quad \chi^2_{\bar{p}} = -\sum_{i=1}^{N} p_i \log a_i = -\log(a_{\bar{p}}).
\]
The following is a Ledrappier–Young like formula for the solution $s$ of (2.4). It generalizes the formula in corollary 2.12 for the diagonally homogeneous case. A similar result for Bedford–McMullen like systems in arbitrary dimension was proved in [17, theorem 2.15].

**Claim 6.1.** For $\tilde{\mathbf{p}} := (\tilde{p}_1, \ldots, \tilde{p}_N)$ defined by $\tilde{p}_i = b_i^{sn} a_i^{-sn}$, recall (2.6), we have

$$s = \frac{h_{\tilde{\mathbf{p}}}}{\lambda_{\tilde{\mathbf{p}}}} + \left(1 - \frac{\chi_{\mathbf{p}}^1}{\lambda_{\tilde{\mathbf{p}}}}\right)s_H,$$

where $s_H = \dim B \Lambda_H$.

**Proof.** Immediately follows from the observation that $h_{\tilde{\mathbf{p}}} = s_H \lambda_{\tilde{\mathbf{p}}}^1 + (s - s_H) \lambda_{\tilde{\mathbf{p}}}^2$. □

The following line of thought is an adaptation of [20, section 6] in order to extract from an arbitrary shifted TGL IFS $\mathcal{F} = \{f_i\}_{i=1}^N$ with $M$ columns a subsystem of a high enough iterate of $\mathcal{F}^k$, which has some nice properties required to prove the theorem.

Let $\mathcal{F}^k := \{f_{i_1 \ldots i_N} : i_1, \ldots, i_N \in \Sigma_k\}$. The first step is to pass from $\mathcal{F}$ to a diagonally homogeneous subsystem of $\mathcal{F}^k$. Analogous arguments appear for example in [9, lemma 5.2, 20, lemma 6.2] or [31, lemma 4.9].

**Definition 6.2.** A subsystem $G^{(k)} \subset \mathcal{F}^k$ is called a diagonally homogeneous subsystem if there exists $a^{(k)}$ and $b^{(k)}$ for which

$$g_i^{(k)}(\chi) = \begin{pmatrix} b^{(k)}_i & 0 \\ d_i^{(k)} & a^{(k)} \end{pmatrix} \chi + t_i, \quad \text{for every } g_i^{(k)} \in G^{(k)}.$$

Fix an arbitrary vector $\mathbf{v} = (v_1, \ldots, v_N)$, where $v_i \in \mathbb{N}$. Let $V_j := \sum_{i \in \Sigma_j} v_i$, $V := (V_1, \ldots, V_M)$, $V = V_1 + \ldots + V_M$ and define

$$\mathcal{M}_\mathbf{v} := \{(i_1, \ldots, i_N) \in \Sigma_V : \#\{\ell \in V : i_\ell = r\} = v_r \text{ for every } r = 1, \ldots, N\}. \quad (6.1)$$

**Claim 6.3.** The subsystem $G_\mathbf{v} = \{f_{i_1 \ldots i_N} : (i_1, \ldots, i_N) \in \mathcal{M}_\mathbf{v}\} \subset \mathcal{F}^V$ with $M_\mathbf{v}$ columns

(i) is a diagonally homogeneous subsystem with $a^{(v)} = \prod_{r=1}^M a_r^{v_r}$ and $b^{(v)} = \prod_{r=1}^M b_r^{v_r}$,

(ii) has uniform vertical fibres with $\prod_{r=1}^M \prod_{i \in \Sigma_r} a_i^{v_i}$ maps in each column and

(iii) for the probability vectors $\mathbf{v} = (v_1, \ldots, v_N)$ and $\overline{\mathbf{v}} = V/\mathbf{v}$

$$-N \log(V + 1) + V \cdot h_{\mathbf{v}} \leq \log \#\mathcal{M}_\mathbf{v} \leq V \cdot h_{\overline{\mathbf{v}}},$$

$$-N \log(V + 1) + V \cdot h_{\overline{\mathbf{v}}} \leq \log M_\mathbf{v} \leq V \cdot h_{\mathbf{v}}.$$

**Proof.** Parts (i) and (ii) are immediate. Part (iii) follows directly from [12, lemma 2.1.8]. □

**Lemma 6.4.** Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a shifted TGL IFS with $M$ columns. For every $k$ choose $v_k = (v_{1,k}, \ldots, v_{N,k})$ such that

$$v_{i,k} = [k\tilde{p}_i] \text{ for every } i = 1, \ldots, N, \quad (6.2)$$

where $\tilde{p}_i$ was defined in (2.6). Let $\tilde{p}_i := b_i^{sn} a_i^{-sn}$ and define the subsystem

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where $\mathcal{M}_{v_k}$ is defined by (6.1). Then $\mathcal{G}^{(k)}$ satisfies the assertions of claim 6.3 with $v_k$. For brevity we write $a^{(v_k)} = a^{(k)}$ and $b^{(v_k)} = b^{(k)}$. Let

$$N^{(k)} = \# \mathcal{M}_{v_k}$$

and denote the number of maps and columns in $\mathcal{G}^{(k)}$.

Moreover, $\lim_{k \to \infty} s^{(k)} = s$, where $s^{(k)}$ is the solution of $N^{(k)}(b^{(k)})^{s^{(k)}}(a^{(k)})^{-s^{(k)}} = 1$, i.e.

$$s^{(k)} = \frac{\log N^{(k)}}{\log b^{(k)}} + \left(1 - \frac{\log b^{(k)}}{\log a^{(k)}}\right)s_H.$$

**Remark 6.5.** The box dimension of the attractor of the IFS $\mathcal{G}^{(k)}$ is NOT equal to $s^{(k)}$, because $s_H$ is not the box dimension $s_H^{(k)}$ of the attractor of the IFS generated by the columns of $\mathcal{G}^{(k)}$. The problem is that $s_H^{(k)} \not\to s_H$ as $k \to \infty$ (except when $\dim H = \dim B$).

**Proof.** It follows from (6.2) that $k - N \leq V^{(k)} \leq k$ and

$$k \log(a) - \sum_{i=1}^{N} \log a_i \leq \log(a) \leq k \log(a).$$

Same holds for $\log b$. Furthermore, for the probability vector $v_k = v_k / V^{(k)}$

$$\widetilde{p}_i = \frac{1}{k} \leq \frac{\nu^{(k)}_i}{V^{(k)}} \leq \frac{\tilde{p}_i N}{k - N}.$$ 

Thus $\lim_{k \to \infty} h_{v_k} = h_p$. We can use claim 6.3 (iii) to bound $\log \# \mathcal{M}_{v_k}$. Hence, putting together all the above we get

$$\lim_{k \to \infty} s^{(k)} = \frac{h_p}{-\log(a)} + \left(1 - \frac{\log(b)}{\log(a)}\right)s_H,$$

which is equal to $s$ due to claim 6.1.\qed

If $\mathcal{G}^{(k)}$ already has non-overlapping columns, then the rest of the construction is not necessary. Otherwise, we can pass further to a subsystem $\mathcal{G}^{(k,\ell)} \subset (\mathcal{G}^{(k)})^{\ell}$ by throwing away ‘not too many’ columns of $(\mathcal{G}^{(k)})^{\ell}$ in order to ensure that $\mathcal{G}^{(k,\ell)}$ has non-overlapping columns.

Projecting $\mathcal{G}^{(k)}$ to the $x$-axis gives a subsystem of $\mathcal{H}^{(v)}$

$$\mathcal{G}^{(k)}_H := \{h_{i_1 \ldots i_{v(k)}} : \text{there exists } (i_1, \ldots, i_{v(k)}) \in \mathcal{M}_{v_k} \text{ s. t. } \Phi(i_1 \ldots i_{v(k)}) = i_1 \ldots i_{v(k)}\},$$

which has a total of $M^{(k)}$ maps, each with contracting ratio $b^{(k)}$. Observe that $\mathcal{G}^{(k)}_H$ also satisfies Hochman’s exponential separation condition, because this condition is assumed for $\mathcal{H}$ and this property passes on to any subsystem. Hence, the Hausdorff and box dimension of $\mathcal{G}^{(k)}_H$ are equal to

$$s^{(k)} = \frac{\log M^{(k)}}{-\log b^{(k)}}.$$ 

(6.3)

It follows from the definition of box dimension that for every $\varepsilon > 0$ there exists a subset of the columns of $(\mathcal{G}^{(k)})^{\ell}$, which are non-overlapping and have cardinality
\[
M^{(k,\ell)} \supseteq C_\varepsilon \left( (b^{(k)})^\varepsilon \right)^{-(\nu_\Omega-\varepsilon)} \quad (6.3) \quad \subseteq C_\varepsilon \cdot \left( M^{(k)} \right)^\varepsilon \left( b^{(k)} \right)^{\ell \varepsilon}.
\] (6.4)

This is the subsystem \( G^{(k,\ell)} \) which we will use in the proof of theorem 2.7 under condition (i). When condition (ii) of theorem 2.7 is assumed we use \( G^{(k,\ell)} = \left( G^{(k)} \right)^\ell \) since in this case non-overlapping columns are assumed for \( \Lambda \). Next, we present our argument to count the number of intersections within a column when \( \Lambda \) has non-overlapping columns.

6.2. Counting intersections

Let \( J \) be an arbitrary TGL IFS and \( G^{(k,\ell)} = \left( G^{(k)} \right)^\ell \) be the subsystem defined in the previous subsection. Then \( G^{(k,\ell)} \) is diagonally homogeneous with main diagonal \( \left( (b^{(k)})^\ell, (a^{(k)})^\ell \right) \), has uniform vertical fibres with \( (N^{(k)}/M^{(k)})^\ell \) maps in each column and the columns are non-overlapping. For every \( f_\tau \in G^{(k,\ell)} \), \( \tau \) can be written

\[
\tau = \tau_1 \tau_2 \ldots \tau_\ell, \text{ where } \tau_j \in \mathcal{M}_{\lambda} \text{ for } j = 1, \ldots, \ell.
\]

Let \( \Sigma^{(k,\ell)} := \{ \tau : f_\tau \in G^{(k,\ell)} \} \) and for the rest of the subsection fix such an \( \tau \in \Sigma^{(k,\ell)} \). Let

\[
\Sigma^{\tau}_- := \{ j = j_1 \ldots j_\ell \in \Sigma^{(k,\ell)} : \Phi(j) = \Phi(\tau) \text{ and } j \neq \tau \},
\]

i.e. \( \Sigma^{\tau}_- \) collects those \( j \) which belong to the symbolic column of \( \tau \). Recall \( \Lambda_\tau = f_\tau(\Lambda) \), \( R_\tau = f_\tau([0,1]^2) \). Let

\[
\tilde{R}_\tau := \left( \text{proj}_\ast(f_\tau(\Lambda)) \times [0,1] \right) \cap R_\tau \text{ and } \delta_\tau^{(k)} := (a^{(k)})^\ell.
\]

Our aim is to give a uniform upper bound for \( N_{\delta^{(k)}}(\tilde{R}_\tau \cap (\cup_{j \in \Sigma^{\tau}_-} \tilde{R}_j)) \). Observe that for every \( j \in \Sigma^{\tau}_- \)

\[
N_{\delta^{(k)}}(\tilde{R}_\tau \cap \tilde{R}_j) = N_{\delta^{(k)}}(\text{proj}_\ast(\Lambda_\tau \cap \Lambda_j)) = N_{\delta^{(k)}}(h^{-1}(\text{proj}_\ast(\Lambda_\tau \cap \Lambda_j)) \cup (\cup_{l \in \mathcal{M}_{\lambda}} R_\tau)) \quad (6.5)
\]

We state a result of Lalley [26, theorem 1], which gives the precise asymptotic of \( N_{\delta}(\Lambda_{\eta}) \). A set \( \{r_1, \ldots, r_M\} \) of positive reals is \( \tau \)-arithmetic if \( \tau > 0 \) is the greatest number such that each \( r_i \) is an integer multiple of \( \tau \), and non-arithmetic if no such \( \tau \) exists. We use the notation \( f(\delta) \sim g(\delta) \) to denote that \( \lim_{\delta \to 0} f(\delta)/g(\delta) = 1 \). Let \( F \) be a self-similar set on \([0,1]\) with contracting ratios \( \{r_1, \ldots, r_M\} \). Assume \( F \) satisfies the strong OSC and let \( \dim_{\text{H}} F = \dim_{\text{B}} F = t \), where \( t \) is the solution of \( \sum_{i=1}^M r_i^t = 1 \).

**Proposition 6.6 ([26, theorem 1]).** If \( \{\log r_1^{-1}, \ldots, \log r_M^{-1}\} \) is a non-arithmetic set, then for some \( K > 0 \)

\[
N_{\delta}(F) \sim K \delta^{-t} \text{ as } \delta \to 0.
\]

On the other hand, if \( \{\log r_1^{-1}, \ldots, \log r_M^{-1}\} \) is \( \tau \)-arithmetic, then for the subsequence \( \delta_n = e^{-nt} \) there exists a constant \( K' > 0 \) such that

\[
N_{\delta_n}(F) \sim K' \delta_n^{-t} \text{ as } n \to \infty.
\]

**Remark 6.7.** The reason why we can not handle both types of overlaps simultaneously for the box dimension is that we are unaware of an analogous result in the case that SOSC is not assumed. This question could be of independent interest.
We use the proposition for the self-similar set \( \Lambda_H \) with contracting ratios \( (r_1, \ldots, r_M) \). If \( \{\log r_1^{-1}, \ldots, \log r_M^{-1}\} \) is \( \tau \)-arithmetic, then we can choose \( \ell = \ell(n) \) so that

\[
\min\{e^{-\tau}, 1\} \cdot e^{-\tau n} < \delta^{(k)}_{\ell} \leq \max\{e^{-\tau}, 1\} \cdot e^{-\tau n},
\]

which implies that

\[
\lim_{n \to \infty} \frac{\ell(n)}{n} = \frac{\tau}{-\log a^{(k)}} \quad \text{and} \quad \lim_{n \to \infty} \frac{N_{e^{-\tau n}}(\Lambda_H)}{N_{a^{(k)}}(\Lambda_H)} = c
\]

for some universal constant \( c \). Thus the proposition implies that

\[
N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau) = N_{\delta^{(k)}_{\ell}/(b^{(k)})^\ell}(\Lambda_H) = (C + o(1))(b^{(k)}/a^{(k)})^{\ell z_n},
\]

where the constant \( C \) only depends on whether \( \{\log r_1^{-1}, \ldots, \log r_M^{-1}\} \) is \( \tau \)-arithmetic or not and the \( o(1) \to 0 \) as \( \ell \to \infty \). The next lemma ensures that a positive proportion of these boxes do not get covered by boxes coming from the cover of \( \tilde{R}_\tau \) for some \( j \in \Sigma_\tau^\infty \).

**Lemma 6.8.** If \( \mathcal{F} \) satisfies transversality and

\[
\frac{N^{(k)}}{M^{(k)}}(1 + K_1^{z_n}) < \left(\frac{b^{(k)}}{a^{(k)}}\right)^{\tau z_n},
\]

then there exists \( K_3 < 1 \) such that for \( \ell \) large enough and every \( \tau \in \Sigma^{(k,\ell)} \) we have

\[
N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau \cap \bigcup_{j \in \Sigma_\tau^\infty} \tilde{R}_j) \leq K_3 N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau).
\]

**Proof.** Fix \( j \in \Sigma^{(k,\ell)} \) such that \( |j \& \tau| = \zeta \), where we count \( \tau_{m,j} \in \mathcal{M}_{v_\zeta} \) as one symbol. Thus, \( \zeta \in \{0, 1, \ldots, \ell - 1\} \). Since \( \mathcal{F} \) satisfies transversality, then so do all of its subsystems, in particular \( \mathcal{G}^{(k,\ell)} \) as well. Hence,

\[
|\text{proj}_j(\tilde{R}_\tau \cap \tilde{R}_j)| \leq K_1 (b^{(k)})^\zeta (a^{(k)})^{\ell - \zeta},
\]

see figure 9. This together with (6.5) and proposition 6.6 yields that

\[
N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau \cap \tilde{R}_j) \leq (C + o(1))K_1^{z_n} \left(\frac{b^{(k)}}{a^{(k)}}\right)^{z_n}.
\]

Since \( \mathcal{G}^{(k,\ell)} \) has uniform vertical fibres, it follows that \( \#\{j \in \Sigma_\tau^\infty : |j \& \tau| = \zeta\} \leq (N^{(k)}/M^{(k)})^{\ell - \zeta} \).

Thus from a simple union bound we get

\[
N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau \cap \bigcup_{j \in \Sigma_\tau^\infty} \tilde{R}_j) \leq \sum_{\zeta = 0}^{\ell - 1} \left(\frac{N^{(k)}}{M^{(k)}}\right)^{\ell - \zeta} (C + o(1))K_1^{z_n} \left(\frac{b^{(k)}}{a^{(k)}}\right)^{z_n}
\]

\[
= \sum_{\zeta = 0}^{\ell - 1} \frac{K_1^{z_n}}{M^{(k)}/N^{(k)}} \left(\frac{b^{(k)}}{a^{(k)}}\right)^{z_n} \left(\frac{M^{(k)}/N^{(k)}}{a^{(k)}}\right)^\ell
\]

\[
= K_3 N_{\delta^{(k)}_{\ell}}(\tilde{R}_\tau),
\]

where the last inequality holds if \( N^{(k)}/M^{(k)} \leq (b^{(k)}/a^{(k)})^{z_n} \). This holds, because (6.7) is an even stronger assumption. Furthermore, simple arithmetic shows that \( K_3 < 1 \) if and only if (6.7) holds. \( \square \)
3.3. Proof of theorem 2.7

Throughout the proof, \( s \) is the target box dimension defined as the solution of (2.4):

\[
\sum_{i=1}^{N} b^{\ell_i} d_i^{-e_{x_i}} = 1.
\]

Fix \( \varepsilon > 0 \). We work with the subsystem \( \mathcal{G}^{(k, \ell)} \) defined in section 6.1. It will be enough to cover the subset \( \bigcup_{\mathcal{T} \in \mathcal{G}^{(k, \ell)}} f_{\mathcal{T}}(\Lambda) \subseteq \Lambda \), with boxes of size \( \delta_{k}^{(\ell)} := (a^{(k)})^{\ell} \). Recall \( \tilde{\mathcal{R}}_{\mathcal{T}} = (\text{proj}_x(f_{\mathcal{T}}(\Lambda)) \times [0, 1]) \cap (f_{\mathcal{T}}([0, 1]^2)) \).

Conclusion of proof assuming condition (i) of theorem 2.7. Assume \( \mathcal{F} \) generates a shifted TGL carpet \( \Lambda \) for which \( \mathcal{H} \) satisfies HESC and the columns independently satisfy ROSC. In this case it is enough to use the definition of box dimension to bound \( N_{d_1^{(k)}}(\tilde{\mathcal{R}}_{\mathcal{T}}) \geq C \varepsilon (b^{(k)}/a^{(k)})^{(s_{x} - \varepsilon)} \) for some constant \( C \varepsilon \) depending only on \( \varepsilon \). Recall from lemma 6.4 that \( s^{(k)} \to s \). We choose \( k \) so large that \( s^{(k)} \geq s - \varepsilon \) and we bound

\[
\lim_{\ell \to \infty} \frac{\log N_{d_1^{(k)}}(\Lambda)}{-\log a^{(k)}} \geq \lim_{\ell \to \infty} \frac{\log (M^{(k, \ell)}(N^{(k)}/M^{(k)})^{(b^{(k)}/a^{(k)})^{(s_{x} - \varepsilon)}})}{-\ell \log a^{(k)}}
\]

\[
\geq \frac{\log N^{(k)}}{-\log a^{(k)}} + \left(1 - \frac{\log b^{(k)}}{\log a^{(k)}} \right) s_{x} - \varepsilon \geq s - 2\varepsilon, \quad (6.8)
\]

where for the second inequality we substituted the lower bound for \( M^{(k, \ell)} \) from (6.4). Letting \( \varepsilon \searrow 0 \) yields \( \dim_B \Lambda \geq s \) as claimed.

Conclusion of proof assuming condition (ii) of theorem 2.7. For the remainder, we assume that \( \mathcal{F} \) has non-overlapping columns, satisfies transversality and (2.5):

\[
h_{\bar{p}} - h_{\bar{q}} < s_{x} (\log(b^{(\bar{p})}) - \log(a^{(\bar{p})})).
\]
where \( h_p = -\log(\bar{p}) \) and \( \bar{p}_i = h^\nu_i a_i^{1-s_H} \). We need to check that condition (6.7) of lemma 6.8 is satisfied, since it ensures that a positive proportion of the boxes needed to cover \( f_{\Xi}(\Lambda) \) are not intersected by any boxes covering \( f_{\Xi}(\Lambda) \) for \( \Xi \neq \Xi \).

**Claim 6.9.** Assumption (2.5) implies condition (6.7) of lemma 6.8 for all \( k \) large enough.

**Proof of claim 6.9.** We know from section 6.1 that

\[
\log a^{(k)} = k \log(\bar{a}) + O(1), \quad \log N^{(k)} = kh_p + o(k),
\]

\[
\log b^{(k)} = k \log(\bar{b}) + O(1), \quad \log M^{(k)} = kh_q + o(k).
\]

Taking the logarithm of each side of (6.7), substituting these values and dividing by \( k \) gives

\[
h_p - h_q + \frac{1}{k} \log (1 + K^{(k)}_i) < s_H(\log(\bar{b}) - \log(\bar{a})).
\]

with an error of \( o(1) \) as \( k \to \infty \) on either side. The second term on the left-hand side also tends to zero as \( k \to \infty \), thus (2.5) indeed implies (6.7) for large \( k \).

The conclusion of the proof of theorem 2.7 is now analogous to the calculation of (6.8) with the exception that we need the precise value of \( N^{(k)}_\delta^{(k)}(\tilde{R}_\ell) \) from (6.6) and we can use \( G^{(k)}_\ell = (G^{(k)}_\ell)^t \), so the number of columns \( M^{(k)}_\ell = (M^{(k)}_\ell)^t \). Choose \( k \) so large that \( s^{(k)} \geq s - \varepsilon \) and condition (6.7) hold simultaneously. Using lemma 6.8 we can basically repeat the calculation of (6.8)

\[
\liminf_{\ell \to \infty} \frac{\log N^{(k)}_\delta^{(k)}(\Lambda)}{-\log \delta^{(k)}_\ell} \geq \liminf_{\ell \to \infty} \frac{\log ((N^{(k)}_\ell)(1 + O(1))(b^{(k)}/a^{(k)}))^{(s_H)}}{-\ell \log a^{(k)}} = s^{(k)}.
\]

This concludes the proof of theorem 2.7. \( \square \)

**6.4. Proof of theorem 2.8**

The theorem claims that for a shifted TGL carpet \( \Lambda \)

\( (i) \dim_H \Lambda = \dim_B \Lambda \quad (ii) s_H = \dim_H \nu_q \quad (iii) \sum_{j \in \mathbb{Z}_i} a_j^{1-s_H} = 1 \text{ for every } i \in [M] \)

are equivalent, provided ROSC and no dimension drop (NDD, recall definition 1.9) hold. We show that \( (i) \iff (iii) \iff (ii) \iff (i) \).

Proof of \( (i) \iff (iii) \). Let \( \hat{\Lambda} \) be the GL brother of \( \Lambda \), recall definition 1.5. For a \( p \in \mathcal{P}_0 \) let \( \tilde{\nu}_p \) denote the push forward of the Bernoulli measure \( \mu_p \) on \( \hat{\Lambda} \). We have \( \dim_H \nu_p = \dim_H \tilde{\nu}_p \) for every \( p \in \mathcal{P}_0 \). Indeed, at the beginning of section 5 we proved \( \dim_H \nu_p = D(p) \) assuming ROSC and NDD, furthermore, Gatzouras–Lalley proved \( \dim_H \tilde{\nu}_p = D(p) \) [21, proposition 3.3]. Hence, \( \dim_H \Lambda = \dim_H \hat{\Lambda} \). Also, assuming NDD, \( s_H \) is the unique real which satisfies \( \sum_{i=1}^M r_i^{s_H} = 1 \). This implies \( \dim_M \Lambda = \dim_M \hat{\Lambda} \). The analogous claim of \( (i) \iff (iii) \) for \( \hat{\Lambda} \) was proved in [21, theorem 4.6]. Thus \( (i) \iff (iii) \) in our setting as well.

Proof of \( (iii) \implies (ii) \). Condition \( (iii) \) implies that the vector \( \tilde{q} \) is simply \( \tilde{q}_i = r_i^{s_H} \) for \( i \in [M] \), where \( r_j = b_j \) if \( j \in \mathbb{Z}_i \). NDD is assumed, thus \( \dim_H \nu_q = h_q/\chi_q^1 = s_H \chi_q^1/\chi_q^1 = s_H \).

Proof of \( (ii) \implies (i) \). We can use claim 6.1 and (3.8) to see that

\[
0 \leq \dim_H \Lambda - \dim_H \hat{\Lambda} \leq \dim_H \Lambda - \dim_H \nu_p = \left(1 - \frac{1}{\chi_p^1/\chi_p} \right) (s_H - \dim_H \nu_q).
\]
Clearly, (ii) implies $\dim_H \Lambda = \dim_B \Lambda$. This concludes the proof of theorem 2.8.

7. Examples

We now treat the examples presented in section 1.2 in detail.

We do not calculate numerically the exact value of the dimensions for the TGL carpet of figure 1, rather just comment why $\dim_H \Lambda < \dim_B \Lambda < \dim_{Aff} \Lambda$. It satisfies the ROSC, thus its dimensions are equal to its GL brother. Clearly, the IFSs on $[0, 1]$ generated from a vertical line in each of the columns do not have the same dimension. Hence, the third condition of (2.7) of theorem 2.8 does not hold. Furthermore, $\dim_B \Lambda_{Aff} < 1$ because there is an empty column. Thus, corollary 2.5 implies that $\dim_B \Lambda < \dim_{Aff} \Lambda$.

Except for the ‘$X \equiv X$’ example, all the other ones of section 1.2 satisfy $\Lambda_{Aff} = [0, 1]$, hence corollary 2.5 implies $\dim_B \Lambda = \dim_{Aff} \Lambda$.

7.1. The self-affine smiley: a non diagonally homogeneous example

The smiley is constructed from the TGL IFS

$$F = \left\{ f_i(x) = \left( \begin{array}{cc} b & 0 \\ d_i & a_i \end{array} \right) x + t_i \right\}_{i=1}^8,$$

where $b = 0.2$, $a_1 = \ldots = a_5 = 0.1$, $a_6 = a_7 = a_8 = 0.13$ and the off-diagonal elements $d_1 = -0.2$, $d_2 = -0.1$, $d_3 = d_7 = d_8 = 0$, $d_4 = 0.1$, $d_5 = d_6 = 0.2$. The translations were chosen so that the mouth is constructed from $f_1, \ldots, f_3$, the nose from $f_6$, and the eyes from $f_7$ and $f_8$. It is not diagonally homogeneous since the mouth is thinner than the nose and eyes. Clearly, $\Lambda$ does not have uniform vertical fibres, thus theorem 2.8 implies $\dim_H \Lambda < \dim_B \Lambda$. The numerical values of the dimensions given in figure 4 were obtained using Wolfram Mathematica 11.2. The box dimension was calculated from $\sum_{i=1}^{N} b_i^{n_i} a_i^{-n_i} = 1$, recall (2.4), while the maximization of $D(p)$ (2.1) gave the Hausdorff dimension.

7.2. Example for $\dim_H \Lambda = \dim_B \Lambda$

Define the matrices

$$A_1 := \left( \begin{array}{cc} 1/3 & 0 \\ 0 & a \end{array} \right), \quad A_2 := \left( \begin{array}{cc} 1/3 & 0 \\ 1/2 - a & a \end{array} \right), \quad A_3 := \left( \begin{array}{cc} 1/3 & 0 \\ a - 1/2 & a \end{array} \right).$$

For $a \in (0, 1/3)$ define the IFS $F_a$ consisting of

$$f_1(x) = A_1 x + \left( \begin{array}{c} 1/3 \\ 0 \end{array} \right), \quad f_2(x) = A_1 x + \left( \begin{array}{c} 1/3 \\ 1 - a \end{array} \right), \quad f_3(x) = A_2 x + \left( \begin{array}{c} 0 \\ 1/2 \end{array} \right),$$

$$f_4(x) = A_2 x + \left( \begin{array}{c} 2/3 \\ 0 \end{array} \right), \quad f_5(x) = A_3 x + \left( \begin{array}{c} 0 \\ 1/2 - a \end{array} \right), \quad f_6(x) = A_3 x + \left( \begin{array}{c} 2/3 \\ 1 - a \end{array} \right).$$

The attractor $\Lambda_a$ is shown in figure 2 for $a = 3/10$. Falconer and Miao showed in [13] how to calculate the box dimension and later Bárány in [4] showed that the same value is a lower bound for the Hausdorff dimension. Hence, $\dim_H \Lambda_a = \dim_B \Lambda_a$.

Alternatively, we can now argue that $\Lambda_a$ is a diagonally homogeneous TGL carpet for every $a \in (0, 1/3)$ satisfying ROSC with uniform vertical fibres. Hence, our results apply. After some basic arithmetic, the dimension formula simplifies to...
\[
\dim_H \Lambda_a = \dim_B \Lambda_a = 1 - \frac{\log 2}{\log a}, \tag{7.1}
\]

7.3. Overlapping example

With a modification of the translation vectors in the previous example, we construct a carpet with overlapping cylinders, see figure 2. Define

\[
f_1(\mathbf{x}) = A_{11} \mathbf{x} + \begin{pmatrix} 1/3 \\ 1/4 \end{pmatrix}, \quad f_2(\mathbf{x}) = A_{12} \mathbf{x} + \begin{pmatrix} 1/3 \\ 3/4 - a \end{pmatrix}, \quad f_3(\mathbf{x}) = A_{21} \mathbf{x} + \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}, \quad f_4(\mathbf{x}) = A_{22} \mathbf{x} + \begin{pmatrix} 2/3 \\ 1/4 \end{pmatrix}, \quad f_5(\mathbf{x}) = A_{31} \mathbf{x} + \begin{pmatrix} 0 \\ 3/4 - a \end{pmatrix}, \quad f_6(\mathbf{x}) = A_{32} \mathbf{x} + \begin{pmatrix} 2/3 \\ 3/4 - a \end{pmatrix},
\]

where the matrices \(A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, \) and \(A_{32}\) are from section 7.2. For \(a \in (0, 1/3)\) the attractor \(\Lambda_a\) is a diagonally homogeneous TGL carpet with uniform vertical fibres and non-overlapping columns. Transversality must be satisfied in order to apply our results. It would suffice to check (1.17) in lemma 1.8, but in fact, the constant \(K_1\) in definition 1.6 of transversality can be directly bounded in this example.

**Claim 7.1.** Transversality holds for every \(a < 1/6\) with

\[
K_1 < \frac{1/9 - a/3}{(1/2 - a)(1/3 - 2a)}.
\]

**Proof.** For brevity, we write \(d := 1/2 - a\) and \(b = 1/3\). Let \(\mathbf{t}\) and \(\mathbf{j}\) be two words of length \(n\) such that \(t_i \neq j_i\) and \(\phi(t_1) \ldots \phi(t_n) = \phi(j_1) \ldots \phi(j_n)\). Since \(R_\mathbf{t} \cap R_\mathbf{j} \neq \emptyset\) and due to the symmetry in the construction, we may assume \(i_1 = 3\) and \(j_1 = 5\), hence \(d_{11} = d\). A simple geometric exercise gives that \(K_1 \leq (\min \tan \gamma_\mathbf{t})^{-1}\), where \(\tan \gamma_\mathbf{t} = d/\ell_\mathbf{t}\). We need a lower bound for \(\tan \gamma_\mathbf{t}\). From (1.11) we get that

\[
\tan \gamma_{t_1 \ldots t_n} = \frac{d_{t_1 \ldots t_n}}{b_{t_1 \ldots t_n}} = \frac{1}{a} \sum_{\ell=1}^n d_{\ell} \left(\frac{a}{b}\right)^\ell = \frac{1}{a} \left(\frac{da}{b} + \sum_{\ell=2}^n d_{\ell} \left(\frac{a}{b}\right)^\ell\right).
\]

This is minimal if \(d_{\ell} = -d\) for every \(\ell \geq 2\). Thus, we obtain the lower bound

\[
\tan \gamma_{t_1 \ldots t_n} \geq d/b \left(1 - \sum_{\ell=2}^n \left(\frac{a}{b}\right)^{\ell-1}\right) \geq d \left(1 - \frac{a/b}{1 - a/b}\right) = \frac{d(b - 2a)}{b(b - a)}.
\]

This remains positive iff \(a < b/2 = 1/6\). Substituting \(d\) and \(b\) gives the bound for \(K_1\). \(\square\)

**Corollary 7.2.** For every \(a < 1/6\) : \(\dim_H \Lambda_a = \dim_B \Lambda_a = 1 - \log 2/\log a\).

**Proof.** For uniform vertical fibres both conditions (2.2) and (2.5) simplify to

\[
\frac{\log a}{\log b} > \frac{\log N}{\log M},
\]

which is satisfied here iff \(a \in (0, 1/6)\).

Thus, claim 7.1 and corollary 2.12 together imply that for every \(a \in (0, 1/6)\) we have \(\dim_H \Lambda_a = \dim_B \Lambda_a = 1 - \log 2/\log a\). \(\square\)
7.4. Example ‘X \equiv X’

This diagonally homogeneous carpet, recall figure 5, is a modification of the previous from section 7.3 in order to show an overlapping example for which all dimensions are different. Indeed, clearly it does not have uniform vertical fibres and there are empty columns.

The main diagonal of each matrix in the TGL IFS is \( b \equiv b = 0.28 \) and \( a \equiv a \). The off-diagonal elements are either \( d_i = \pm (1/2 - a) \) or 0. The translation vectors were chosen so that \( \Lambda_a \) is symmetric on both lines \( x = 1/2 \) and \( y = 1/2 \). In figure 5 \( a = 0.045 \).

Transversality for the system can be checked the same way as in claim 7.1, to obtain that transversality holds for every \( a < b/2 = 0.14 \) with

\[
K_1 < \frac{0.28(0.28 - a)}{(1/2 - a)(0.28 - 2a)}.
\]

**Corollary 7.3.** We have \( \dim_H \Lambda_a < \dim_B \Lambda_a < \dim_{\text{Aff}} \Lambda_a \), where

\[
\dim_H \Lambda_a = 0.78556 \cdot \log \left( 2 \cdot 2^{\frac{1 + 1}{1 + 2}} + 3^{\frac{1 + 1}{1 + 2}} \right), \quad \text{for every } a < 0.10405 \ldots,
\]

\[
\dim_B \Lambda_a = 0.84730 - \log a + 0.86303, \quad \text{for every } a < 0.10254 \ldots,
\]

\[
\dim_{\text{Aff}} \Lambda_a = 1 + 0.67294 - \log a, \quad \text{for every } a < 0.28.
\]

**Proof.** The formulas are applications of the ones in corollary 2.12 and (2.3). The affinity dimension is independent of overlaps. The bound for the parameter \( a \) in case of the Hausdorff dimension was obtained using proposition 2.10. The value \( x_0 = 0.56255 \ldots \) for which \( R(x_0) = 1 \) was calculated using Wolfram Mathematica 11.2. Then (2.2) holds for every \( a < 0.28 \).

\[\square\]

7.5. Negative entries in the main diagonal

Throughout we assumed that \( 0 < a_i < b_i < 1 \). We now comment on letting \( a_i \) or \( b_i \) < 0. For convenience, assume ROSC and non-overlapping columns.

**Proposition 7.4.** The dimension results of theorems 2.2 and 2.4 extend to TGL carpets satisfying the ROSC under the weaker condition that \( 0 < |a_i| < |b_i| < 1 \) and for every fixed \( \hat{i} \in \{1, \ldots, M\} \) and every \( k, \ell \in I_{\hat{i}} : b_k = b_\ell \).

**Sketch of proof.** All lower triangular matrices can be written as

\[
\begin{pmatrix}
   b_i & 0 \\
   d_i & a_i
\end{pmatrix} = \begin{pmatrix}
   |b_i| & 0 \\
   d_i & |a_i|
\end{pmatrix} \cdot L
\]

where \( d_i = d_i \) or \( -d_i \) and \( L \) is a reflection on one or both of the coordinate axis. Since \( L([-1,1]^2) = [-1,1]^2 \), such compositions fit into the framework of Fraser’s box-like sets [19]. Furthermore, the direction-x dominates property is preserved. Hence, the proof of the box dimension from section 6 immediately extends to this setting.

The lower bound for the Hausdorff dimension follows from Bárány–Käenmäki [6] cited in theorem 3.4. Since in any given column, all \( b_i \) have the same sign and we have ROSC, the column structure is preserved for every level. Thus, the dimension of the projected measure \( \nu_{q} \)
is not affected by the negative $a_i$, $b_i$. For the upper bound, we can modify the metric defined on $\Sigma$ in lemma 4.1 to be

$$d(i, j) := \prod_{k=1}^{[i \wedge j]} |b_k| + \prod_{k=1}^{[i \wedge j]} |a_k|.$$ 

One can easily check that $d(i, j)$ is indeed a metric and the natural projection $\Pi : \Sigma \to \Lambda$ is Lipschitz. Only the lengths of the sides of a parallelogram are important, its orientation is not.

The Bernoulli measure defined in (4.1) can be modified by again putting $a_i$ and $b_i$ in absolute value. The original proof of Gatzouras and Lalley [21] does not use that $a_i, b_i > 0$, only that $0 < |a_i| < |b_i| < 1$. □

In general, if a column has $b_i$ of different signs, then the initial column structure can easily be destroyed. This is true even if $|b_i| \equiv b$ and possibly empty columns also have width $b$, see figure 10. This motivates us to call a TGL carpet symmetric if

$$N_{ij} = N_{M-i+1} \text{ for } i = 1, \ldots, \lfloor M/2 \rfloor$$

(empty columns are allowed) and $|b_i| \equiv 1/M$. For a particular symmetric carpet, in the next subsection, we show that the dimension formulas hold.

7.6. A family of self-affine continuous curves

Let $a \in (0, 1/5]$ and $d = (1 - 5a)/4$. Define the matrices

$$A = \begin{pmatrix} 1/3 & a \\ d & 0 \end{pmatrix} \quad \text{and} \quad A^- = \begin{pmatrix} -1/3 & 0 \\ 0 & a \end{pmatrix}.$$

$A^-$ is orientation reversing. We introduce the parameterized family of IFSs $F_a$ given by the functions

$$f_1(x) = Ax, \quad f_2(x) = Ax + \begin{pmatrix} 1/3 \\ a + d \end{pmatrix}, \quad f_3(x) = A^-x + \begin{pmatrix} 2/3 \\ 2(a + d) \end{pmatrix},$$

$$f_4(x) = Ax + \begin{pmatrix} 1/3 \\ 3a + 2d \end{pmatrix}, \quad f_5(x) = A^-x + \begin{pmatrix} 2/3 \\ 4a + 3d \end{pmatrix}.$$

The translation vectors are chosen so that $f_1(0) = 0$, $f_5((1, 1)) = (1, 1)$ and $f_i((1, 1)) = f_{i+1}(0)$. This ensures that $\Lambda_a$ is a continuous curve in $\mathbb{R}^2$, see figure 6. Curves satisfying this property are also called affine zippers in the literature, see for example [1, 7]. Clearly, the attractor $\Lambda_a$ is a symmetric, diagonally homogeneous TGL carpet satisfying the ROSC for every value of $a$. For $a = 1/5$ all cylinders $R_{iln}$ are rectangles, however, it is not a classical Bedford–McMullen carpet, since $A^-$ contains a negative element.
Proposition 7.5. For every \( a \in (0, 1/5] \), the Hausdorff and box dimension of \( \Lambda_a \) are given by the continuous, strictly increasing functions

\[
\frac{1}{\log 3} \log \left( 2 + 3^{\frac{\log_3 a}{\log a}} \right) = \dim_H \Lambda_a < \dim_B \Lambda_a = 1 + \frac{\log(3/5)}{\log a}.
\]

Proof. \( A^- \) can be written as the composition of the reflection on the vertical axis with the diagonal matrix \( \text{Diag}(1/3, a) \). Hence, the proof of the box dimension carries over without difficulty.

The argument for the Hausdorff dimension follows that in proposition 7.4, with an extra argument why the dimension of \( \nu \) is not affected by \( A^- \).

The symbolic space \( \Sigma = \{1, \ldots, 5\}^\mathbb{N} \) codes the IFS \( \mathcal{F}_a \) on \([0, 1]^2\) and \( \hat{\mathcal{H}}_a \) on \([0, 1] \) (recall (1.2)). Fix a \( p = (p_1, \ldots, p_5) \in \mathcal{P} \). Due to the symmetry and diagonally homogeneous property we may assume that \( p_1 = p_5 \). Let \( \mu_p \) be the Bernoulli measure on \( \mathcal{P} \) and \( r_p = \Pi_p \mu_p \) its push forward. Define the IFS \( \mathcal{H}_a := \{ h_i(x) = x/3 + (i - 1)/3, i = 1, 2, 3 \} \), which is coded by \( \Sigma_\mathcal{H} = \{1', 2', 3'\}^\mathbb{N} \). The map \( \phi : \{1, \ldots, 5\} \to \{1', 2', 3'\} \) is defined

\[
\phi(1) = 1', \quad \phi(2) = \phi(3) = \phi(4) = 2', \quad \phi(5) = 3'.
\]

For \( i = i_1 \cdots i_n \in \{1', 2', 3'\}^n \) let us denote \( J^i \phi := \{ j : i_j = k, j \leq |i| \}, \quad \#^i \phi := |J^i \phi| \) and define \( \nu_q := (\text{proj}_i)_* \nu_p \). We claim that

\[
\nu_q(h_\Sigma([0, 1])) = p_1^{\#^i \phi(\tau)} \cdot (p_2 + p_3 + p_4)^{\#^i \phi^i(\tau)},
\]

i.e. \( \nu_q \) is the push forward (\( \Pi_\Sigma \)*\( \mu_q \)) of the Bernoulli measure \( \mu_q \) on \( \Sigma_\mathcal{H} \) defined by the vector \( q = (q_1, q_2, q_3) = (p_1, p_2 + p_3 + p_4, p_5) \). This implies that

\[
\dim_H \nu_q = \frac{\log(q)_q}{\log 3}.
\]

To see (7.2), choose an arbitrary \( \tau \in \{1', 2', 3'\}^n \). We determine those \( \hat{i} \in \{1, \ldots, 5\}^n \) for which \( \text{proj}_i h_\tau([0, 1]) = h_\tau([0, 1]) \). For indices \( j \in J^i \phi \) we can choose 2, 3 or 4 in \( \hat{i} \). Let \( J^i_2 \hat{i} = \{ j : i_j = 2 \} \). Orientation is reversed at each \( j \in J^i_2 \hat{i} \). \( |J^i_2 \hat{i}| \) uniquely determines \( \hat{i}_j \) if \( i_j = 1' \) or \( 3' \). Namely, whenever

\[
|J^i_2 \hat{i}| = \begin{cases} \text{odd}, & \text{if } i_j = 1' \text{ then necessarily } \hat{i}_j = 5 \text{ and if } i_j = 3' \text{ then } \hat{i}_j = 1; \\ \text{even}, & \text{if } i_j = 1' \text{ then necessarily } \hat{i}_j = 1 \text{ and if } i_j = 3' \text{ then } \hat{i}_j = 5. \end{cases}
\]

For indices \( j \in J^i \phi \setminus J^i_2 \hat{i} \) we can freely choose \( \hat{i}_j = 2 \) or 4. These are precisely the \( \hat{i} \) for which \( \text{proj}_i h_\tau([0, 1]^2) = h_\tau([0, 1]) \). Using that \( p_1 = p_5 \), the measure equals

\[
\nu_q(h_\Sigma([0, 1])) = p_1^{\#^i \phi(\tau)} \cdot \left( \left( \frac{\#^i \phi(\tau)}{\#^i \phi^i(\tau)} \right) p_2 + p_3 + p_4 \right)^{\#^i \phi^i(\tau)},
\]

which after two applications of the binomial theorem yields (7.2).

Finally, we conclude that \( \dim_H \Lambda_a < \dim_B \Lambda_a \) since \( \Lambda_a \) does not have uniform vertical fibres. \( \square \)
8. Three-dimensional applications

We can compute the Hausdorff dimension of some self-affine sponges in \( \mathbb{R}^3 \). We do not aim for full generality, rather just demonstrate how our results can be applied. Throughout this section we always use the following definitions:

**Definition 8.1.** Let \( \mathcal{F} \) be a TGL carpet on \([0,1]^2\) of the form (1.1), that is

\[
\mathcal{F} = \{ f_i(\mathbf{x}) := A_i \cdot \mathbf{x} + t_i \}_{i=1}^N, \quad \text{where } A_i = \begin{pmatrix} b_i & 0 \\ d_i & a_i \end{pmatrix} \quad \text{and} \quad t_i = \begin{pmatrix} t_{i,1} \\ t_{i,2} \end{pmatrix}, \quad \mathbf{x} \in [0,1]^2.
\]

Furthermore, let the vectors \( \mathbf{u} = (u_1, \ldots, u_N), \mathbf{v} = (v_1, \ldots, v_N), \mathbf{\lambda} = (\lambda_1, \ldots, \lambda_N) \) be such that for every \( 1 \leq i \leq N \)

\[
u_i, \lambda_i \in \mathbb{R} \quad \text{and} \quad \lambda_i \in (-1,1) \setminus \{0\}.
\]

We say that the three-dimensional self-affine IFS

\[
\widehat{\mathcal{F}} := \left\{ F_i(\hat{x}) := \hat{A}_i \cdot \hat{x} + \hat{t}_i \right\}_{i=1}^N, \quad \text{where } \hat{A}_i = \begin{pmatrix} b_i & 0 \\ d_i & a_i \end{pmatrix}, \quad \hat{t}_i := \begin{pmatrix} t_{i,1} \\ t_{i,2} \\ t_{i,3} \end{pmatrix}
\]

on \([0,1]^3\) is an uplift of \( \mathcal{F} \) corresponding to \((\mathbf{u}, \mathbf{v}, \mathbf{\lambda})\) if the following conditions hold:

(C1) For all \( 1 \leq i \leq N \) we have

\[
0 < |\lambda_i| < a_i < b_i < 1. \tag{8.1}
\]

(C2) \( \widehat{\mathcal{F}} \) satisfies the ROSC (see definition 1.4).

Let \( \Lambda \) and \( \hat{\Lambda} \) be the attractor of \( \mathcal{F} \) and \( \widehat{\mathcal{F}} \) respectively. We write \( \Pi \) and \( \hat{\Pi} \) for the natural projection from \( \Sigma := \{1,\ldots,N\}^N \) to \( \Lambda \) and \( \hat{\Lambda} \) respectively. For a probability vector \( \mathbf{p} = (p_1, \ldots, p_N) \) we set \( \nu_\mathbf{p} := \Pi_* (\mathbf{p}^N) \) and \( \hat{\nu}_\mathbf{p} := \hat{\Pi}_* (\hat{\mathbf{p}}^N) \).

We get the next corollary from [6, theorem 2.3, proposition 5.8 and proposition 5.9].

**Corollary 8.2 (Bárány, Käenmäki).** Assume that for an uplift \( \widehat{\mathcal{F}} \) of \( \mathcal{F} \) we have \( \mathbf{u} = \mathbf{v} = \mathbf{0} \) and \( \lambda_i = \lambda \) for all \( i \). Moreover, assume that for a probability vector \( \mathbf{p} = (p_1, \ldots, p_N) \) we have \( h_\mathbf{p} < \lambda^0 + \lambda_0^1 \) (i.e. the entropy is less than the sum of the Lyapunov exponents). Then

\[
\dim_H \nu_\mathbf{p} = \dim_H \hat{\nu}_\mathbf{p}.
\]

That is, the computation of the Hausdorff dimension of a Bernoulli measure for the three-dimensional non-overlapping system \( \widehat{\mathcal{F}} \) is traced back to the corresponding two-dimensional possibly overlapping system \( \mathcal{F} \). In this way, if \( \mathcal{F} \) satisfies the conditions of theorem 2.2, then we can determine \( \dim_H (\hat{\nu}_\mathbf{p}) \) for the three-dimensional system.

In general, we cannot approximate the Hausdorff dimension of a self-affine set in \( \mathbb{R}^3 \) by the Hausdorff dimension of self-affine (or even ergodic) measures (see [11, theorem 2.8]). However, this is possible in some special cases.

**Theorem 8.3.** Given a diagonally homogeneous TGL of the form

\[
\mathcal{F} = \{ f_i(\mathbf{x}) := A_i \cdot \mathbf{x} + t_i \}_{i=1}^N, \quad \text{where } A_i = \begin{pmatrix} b_i & 0 \\ d_i & a_i \end{pmatrix} \quad \text{and} \quad t_i = \begin{pmatrix} t_{i,1} \\ t_{i,2} \end{pmatrix}, \quad \mathbf{x} \in [0,1]^2.
\]
we assume that
(i): $\mathcal{F}$ has uniform vertical fibres (i.e. each column has the same number of maps).
(ii): The projection of $\Lambda$ to the x-axis is the whole interval $[0, 1]$ (this means that $1/b$ is equal to
the number of columns $M$). We assume this to guarantee that the box and affinity dimensions of $\Lambda$
coincide (see corollary 2.5).
(iii): Moreover, we assume that the parameter $a$ is sufficiently small so that both conditions
\((2.10), (1.17)\) and the transversality condition hold:
\[ a < \min \left\{ \frac{\log N}{b \log M}, \frac{bd^*}{2 + d^*} \right\}, \]
where $d^*$ was defined in lemma 1.8 as
\[ d^* := \min_{1 \leq \ell \leq M} \min_{(k, \ell) \in P} |d_k - d_\ell|, \]
where $(k, \ell) \in P$ if $f_k([0, 1]^2)$ and $f_\ell([0, 1]^2)$ belong to the same column and have disjoint
interiors.
We consider the self-affine IFS $\hat{\mathcal{F}}$ which is an uplift of $\mathcal{F}$ corresponding to $(u, v, \lambda)$
according to definition 8.1. That is (8.1) holds and $u$, $v$ and $\lambda$ are chosen such that
$\hat{\mathcal{F}} := \{ \hat{F}_i(x) := \hat{A}_i \cdot \hat{x} + \hat{t}_i \}_{i=1}^N$, where \( \hat{A}_i = \begin{pmatrix} b & 0 & 0 \\ d_i & a & 0 \\ u_i & v_i & \lambda_i \end{pmatrix} \), \( \hat{t}_i = \begin{pmatrix} t_{i,1} \\ t_{i,2} \\ t_{i,3} \end{pmatrix} \), $\hat{x} \in [0, 1]^3$
satisfies:
• $F_i([0, 1]^3) \subset [0, 1]^3$ holds for all $i \in \{1, \ldots, N\}$ and
• the set $F_i([0, 1]^3) \cap F_j([0, 1]^3)$ has an empty interior for all $i \neq j \in \{1, \ldots, N\}$.

Let $p := \left( \frac{1}{N}, \ldots, \frac{1}{N} \right)$. Then, using the notation of definition 8.1 we have
\[ \dim_H \hat{\nu}_p = \dim_H \hat{\Lambda} = \dim_B \hat{\Lambda} = \dim_{\text{Aff}} \hat{\Lambda} = 1 + \frac{\log(Nb)}{\log a}. \] (8.3)

To give the upper bound in the proof of this theorem, first we need to extend the scope of
lemma 1.3 to $\mathbb{R}^3$.

**Lemma 8.4.** There exist $K_x, K_y$ and $K_z$ such that for an arbitrary $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$
we have
\[ \hat{\Lambda}_{i_1 \ldots i_n} \leq \begin{pmatrix} b^n & 0 & 0 \\ K_x \cdot b^n & a^n & 0 \\ K_y \cdot b^n & K_z \cdot b^n & \lambda_{i_1 \ldots i_n} \end{pmatrix}, \]
that is each element of the matrix on the right-hand side is greater than or equal to the corre-
sponding element on the left-hand side.

**Proof.** For every $n$ and $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$ we introduce $x_{i_1 \ldots i_n}, y_{i_1 \ldots i_n}$ and $z_{i_1 \ldots i_n}$ such that
\[ \tilde{A}_{i_1 \ldots i_k} = \begin{pmatrix} b^{n_0} & 0 & 0 \\ a^n & b^{n_0} & 0 \\ z_{i_1 \ldots i_k} \cdot b^n & \lambda_{i_1 \ldots i_k} \end{pmatrix}. \]

Since the existence of \( K_x \) was proved in lemma 1.3, it suffices to prove that \( y_{i_1 \ldots i_k} \) and \( z_{i_1 \ldots i_k} \) are uniformly bounded in \( (i_1, \ldots, i_k) \in \Sigma^* \). To do so, observe that

\[ z_{i_1 \ldots i_k} = z_{i_1 \ldots i_k} + \lambda_{i_1 \ldots i_k} \cdot b^n, \]

\[ y_{i_1 \ldots i_k} = y_{i_1 \ldots i_k} + z_{i_1 \ldots i_k} \cdot b^n + \lambda_{i_1 \ldots i_k} \cdot u_{i_k+1}. \]

By (8.1) we obtain from (8.4) that there is a \( r \in (0,1) \) and \( c > 0 \) such that

\[ z_{i_1 \ldots i_k} < c \cdot r^n \text{ for all } n \text{ and } (i_1, \ldots, i_k) \in \{1, \ldots, N\}^n. \]

Namely, we can write down the formula for \( z_{i_1 \ldots i_k} \) inductively and thus we get that

\[ z_{i_1 \ldots i_k} \leq (a/b)^n \cdot \max \left\{ v_i + n \cdot \max \{v_i\} \right\}. \]

From here we get that (8.6) holds. This settles the existence of \( K_y \). Substituting (8.6) into (8.5) and using (8.1) again we obtain the existence of \( K_y \). Namely, the second and third summands in (8.5) are exponentially small. More precisely,

\[ K_y = \max \{u_i\} + \sum_{n=1}^{\infty} \left( c \cdot r^n \cdot \max \{d_i\} + \left( \max \{|\lambda_i|\} \right)^n \cdot \max \{u_i\} \right), \]

where all of the maximums are taken for \( i \in \{1, \ldots, N\} \).

\[ \square \]

**Proof of theorem 8.3.**

**Lower bound:** Observe that if condition (8.2) holds, then it follows from lemma 1.8 that the transversality condition holds. Moreover, as we noted in section 2.3, condition (8.2) also implies that conditions (2.2) and (2.5) hold when \( p \) is chosen as above to be the uniform vector. In this way the conditions of theorems 2.2 and 2.7 are satisfied. As an application of these theorems, we obtain that

\[ \dim \nu_p = \frac{-\log N}{-\log a} + \left( 1 - \frac{\log b}{\log a} \right) \frac{\log M}{-\log b} = 1 + \frac{\log(Nb)}{-\log a}. \]

Hence, we get the lower bound

\[ 1 + \frac{\log(Nb)}{-\log a} = \dim \nu_p \leq \dim \nu_{\tilde{A}} \leq \dim \tilde{A}. \]

**Upper bound:** It is enough to prove that

\[ \dim \text{Aff} \tilde{\Lambda} \leq 1 + \frac{\log(Nb)}{-\log a}. \]

This follows from lemma 8.4 since the cylinder \( F_{i_1 \ldots i_k} \left( [0,1]^3 \right) \) can be covered by \( N^n \cdot b^n/a^n \) axes parallel rectangular box of dimensions \( a^n \times K_x \cdot a^n \times (K_y + K_z) \cdot a^n \). This immediately implies that (8.7) holds.

\[ \square \]
Example 8.5. Recall the attractor in the center of figure 7. It is defined by an IFS

\[ \hat{F} = \{ F_i(\hat{x}) := \hat{A}_i \cdot \hat{x} + \hat{t}_i \}_{i=1}^6 \],

where \( 0 < \lambda < a < 1/3 \)

\[ \hat{A}_1 = \hat{A}_3 = \begin{pmatrix} 1/3 & 0 & 0 \\ 1 - a & a & 0 \\ 1 - \lambda & 0 & \lambda \end{pmatrix}, \quad \hat{A}_2 = \hat{A}_6 = \begin{pmatrix} 1/3 & 0 & 0 \\ a - 1 & a & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \hat{A}_3 = \hat{A}_4 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & a & 0 \\ \lambda - 1 & 0 & \lambda \end{pmatrix}. \]

The translations are chosen appropriately so that \( \hat{F} \) satisfies the ROSC and the projection to the \( xy \)-plane looks like the one on the right-hand side of figure 7. If \( \lambda < a < 1/6 \), then the conditions of theorem 8.3 hold and we have from (8.3) that for \( p = (1/6, \ldots, 1/6) \)

\[ \dim \hat{H} \hat{\nu}_p = \dim \hat{H} \hat{\lambda} = \dim A_{Aff} \hat{\lambda} = 1 = \frac{-\log 2}{\log a}. \]

Acknowledgment

We thank the anonymous referees for their useful comments which helped us to significantly improve the presentation of the paper. Both authors acknowledge support from the grant OTKA K123782 and MTA-BME Stochastic Research Group. Furthermore, IK was partially supported by the ÚNKP–17–3–IV. New National Excellence Program of the Ministry of Human Capacities.

Appendix. No dimension drop is equivalent to WAUC

In this appendix, we prove that for self-similar IFSs on the line and Bernoulli measures the separation conditions NDD and WAUC are equivalent. We recall notation and definitions.

A.1. Notation

Let \( \mathcal{H} = \{ h_i(x) := r_i x + a_i \}_{i=1}^M \) be a contractive self-similar IFS on the real line with attractor \( \Lambda_{\mathcal{H}} \). The symbolic space is \( \Sigma_{\mathcal{H}} = \{ 1, 2, \ldots, M \}^\mathbb{N} \) and the natural projection is \( \Pi_{\mathcal{H}}(\hat{i}) := \lim_{n \to \infty} h_{i(n)}(0) \) for \( \hat{i} \in \Sigma_{\mathcal{H}} \). Define a partition of \( \Sigma_{\mathcal{H}} \) by

\[ \xi(\hat{i}) := \Pi_{\mathcal{H}}^{-1} \Pi_{\mathcal{H}}(\hat{i}). \]

As we noted earlier in this paper, \( \xi \) is a measurable partition of \( \Sigma \). We write \( \hat{\xi} \) for the \( \sigma \)-algebra generated by the measurable partition \( \xi \). For a probability vector \( \mathbf{q} = (q_1, \ldots, q_M) \) we denote the Bernoulli measure on \( \Sigma_{\mathcal{H}} \) by \( \mu_{\mathbf{q}} \). Then there exists a \( \hat{\Sigma}_{\mathcal{H}} \subset \Sigma_{\mathcal{H}} \), with \( \mu_{\mathbf{q}}(\hat{\Sigma}_{\mathcal{H}}) = 1 \) such that for all \( \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \) there exists a probability measure \( \mu_{\xi(\hat{i})} \) defined on \( \xi(\hat{i}) \) such that

- For all \( A \subset \Sigma \) Borel set the mapping \( \hat{i} \mapsto \mu_{\xi(\hat{i})}(A) \) is \( \hat{\xi} \)-measurable and
- for all Borel sets \( U \subset \Sigma_{\mathcal{H}} \) we have

\[ \mu_{\mathbf{q}}(U) = \int \mu_{\xi(\hat{i})}(U) d\mu_{\mathbf{q}}(\hat{i}). \]
The push forward measure \( \nu_q = (\Pi_{\mathcal{H}})_* \mu_q \) is the self-similar measure with support \( \Lambda_{\mathcal{H}} \). The entropy and Lyapunov exponent of the system are

\[
h_{\mu_q} = -\log \langle q \rangle_q \quad \text{and} \quad \chi_{\nu_q} = -\sum_{i=1}^{M} q_i \log r_i = -\log \langle r \rangle_q,
\]

respectively, where \( \langle q \rangle_q = \prod_{i=1}^{M} q_j^\gamma_i \). Now we recall two separation conditions from definition 1.9.

### A.2. Definitions

We say that \( \mathcal{H} \) has **no dimension drop (NDD)** if for all probability vectors \( q \) with strictly positive entries we have

\[
\dim_{\text{H}} \nu_q = \frac{h_{\mu_q}}{\chi_{\nu_q}}.
\]

We say that \( \mathcal{H} \) has **weak almost unique coding (WAUC)** if for all probability vectors \( q \) with strictly positive entries there exists a set \( B_{\mathcal{H}} \subset \Sigma_{\mathcal{H}} \) (may depend on \( q \)) for which

\[
\mu_q(B_{\mathcal{H}}) = 0 \quad \text{and for every } \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \setminus B_{\mathcal{H}} : \#(\xi(\hat{i}) \setminus B_{\mathcal{H}}) = 1.
\]

**Proposition A.1.** For any self-similar IFS on the line the conditions NDD and WAUC are equivalent.

Let \( \delta_{\hat{i}} \) denote the Dirac-delta measure concentrated on the point \( \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \). We show the assertion in two steps. Namely, we prove that

\[
\text{NDD} \iff \mu_{\xi(\hat{i})} = \delta_{\hat{i}} \text{ for } \mu_q\text{-a.e. } \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \iff \text{WAUC}. \tag{A.2}
\]

**Proof of first equivalence in (A.2).** The result of Bárány–Käenmäki [6, theorem 2.3.] for dimension \( d = 1 \) states that for every Bernoulli measure \( \mu_q \) its push forward \( \nu_q \) is exact dimensional. Moreover,

\[
\dim_{\text{H}} \nu_q = \frac{h_{\mu_q} - H}{\chi_{\nu_q}}, \quad \text{where} \quad H = -\int \log \mu_{\xi(\hat{i})}([i]) \, d\mu_q(\hat{i}) \geq 0.
\]

From the definition of NDD, we get that

\[
\text{NDD} \iff H = 0 \iff \mu_{\xi(\hat{i})}([i]) = 1 \text{ for } \mu_q\text{-a.e. } \hat{i} \in \hat{\Sigma}_{\mathcal{H}}.
\]

Thus it suffices to show that

\[
\mu_{\xi(\hat{i})}([i]) = 1 \text{ for } \mu_q\text{-a.e. } \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \iff \mu_{\xi(\hat{i})} = \delta_{\hat{i}} \text{ for } \mu_q\text{-a.e. } \hat{i} \in \hat{\Sigma}_{\mathcal{H}}. \tag{A.3}
\]

The \( \iff \) direction in (A.3) is obvious. In the other direction, we show that for \( \mu_q\text{-a.e. } \hat{i} \in \hat{\Sigma}_{\mathcal{H}} \)

\[
\mu_{\xi(\hat{i})}([i]) = 1 \implies \mu_{\xi(\hat{i})}([i \ldots i]) = 1 \text{ for every } n \implies \mu_{\xi(\hat{i})} = \delta_{\hat{i}}.
\]

To see the first implication fix \( n \). Let \( \mathcal{H}^{(n)} := \{ (i_1, \ldots, i_n) : (i_1, \ldots, i_n) \in \{1, \ldots, M\}^n \} \) and \( \Sigma_{\mathcal{H}}^{(n)} \) be the symbolic space of infinite sequences of \( n \)-tuples \((i_1, \ldots, i_n)\). There is a natural one-to-one bijection between the elements of \( \Sigma_{\mathcal{H}} \) and \( \Sigma_{\mathcal{H}}^{(n)} \). A Bernoulli measure \( \mu_q \) on \( \Sigma_{\mathcal{H}} \) naturally defines a Bernoulli measure \( \mu_q^{(n)} \) on \( \Sigma_{\mathcal{H}}^{(n)} \) by \( \mu_q^{(n)}([i_1 \ldots i_n]) = \prod_{j=1}^{n} \mu_q([i_j]) \). Applying [6, theorem 2.3.] to this system yields the first implication. The second implication follows from the
Monotone convergence theorem
\[
1 = \lim_{n \to \infty} \mu_{\xi(i)}([i_1 \ldots i_n]) = \mu_{\xi(i)}(\hat{i}) \implies \mu_{\xi(i)} = \delta_{\hat{i}} \quad \text{for } \mu_q\text{-a.e. } \hat{i} \in \Sigma_H. \]

**Proof of second equivalence in (A.2).**

\[\implies \text{direction:}\]
We claim that the conditions in the definition of WAUC are satisfied with
\[
B_H := (\Sigma_H \setminus \hat{\Sigma}_H) \cup \{ \hat{i} \in \Sigma_H : \mu_{\xi(i)} \neq \delta_{\hat{i}} \}. \]

By assumption \(\mu_q(B_H) = 0\). Moreover, for any \(\hat{i} \in \Sigma_H \setminus B_H\) we have \(\hat{i} \in \hat{\Sigma}_H\), so the probability measure \(\mu_{\xi(i)}\) exists and \(\mu_{\xi(i)} = \delta_{\hat{i}}\). If \(\hat{j} \in \xi(\hat{i}) \setminus B_H\) then \(\xi(\hat{j}) = \xi(\hat{i})\), thus
\[
\delta_{\hat{i}} = \mu_{\xi(i)} = \mu_{\xi(j)} = \delta_{\hat{j}}. \]

That is \(\hat{i} = \hat{j}\). We showed that for every \(\hat{i} \in \Sigma_H \setminus B_H\) :
\[
\xi(\hat{i}) \setminus B_H = \{ \hat{i} \}. \tag{A.4}\]

\[\iff \text{direction:}\]
Clearly, WAUC is equivalent to the existence of \(B_H\) with \(\mu_q(B_H) = 0\) such that
\[
\text{if } \hat{i} \notin B_H \text{ then } \xi(\hat{i}) \setminus B_H = \{ \hat{i} \}. \]

Using (A.1) for \(B_H\) we obtain that the set
\[
\hat{\Sigma}_H := \left\{ \hat{i} \in \hat{\Sigma}_H : \mu_{\xi(i)}(B_H) = 0 \right\} \]

has full measure:
\[
\mu_q(\hat{\Sigma}_H) = 1, \tag{A.5}\]

where we remind the reader that \(\hat{\Sigma}_H\) is the set of those \(\hat{i} \in \Sigma_H\) for which the conditional probability measure \(\mu_{\xi(i)}\) exists. Assume that \(\hat{i} \in \hat{\Sigma}_H\). Then
\[
\mu_{\xi(i)}(\xi(\hat{i}) \setminus B_H) = 1 \quad \text{and by (A.4)}: \quad \xi(\hat{i}) \setminus B_H = \{ \hat{i} \}. \]

That is \(\mu_{\xi(i)}(\{ \hat{i} \}) = 1\) whenever \(\hat{i} \in \hat{\Sigma}_H\). Combining this with (A.5) we get that for a \(\mu_q\)-full measure set of \(\hat{i}\) we have \(\mu_{\xi(i)} = \delta_{\hat{i}}\). \qed

**ORCID iDs**

István Kolossváry https://orcid.org/0000-0002-2216-305X

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