FINITE NUMBERS OF INITIAL IDEALS IN NON-NOETHERIAN POLYNOMIAL RINGS

FELICITAS LINDNER

ABSTRACT. In this article, we generalize the well-known result that ideals of Noetherian polynomial rings have only finitely many initial ideals to the situation of ascending ideal chains in non-Noetherian polynomial rings. More precisely, we study ideal chains in the polynomial ring \( R = \mathbb{K}[x_{i,j} | 1 \leq i \leq c, j \in \mathbb{N}] \) that are invariant under the action of the monoid \( \text{Inc}(\mathbb{N}) \) of strictly increasing functions on \( \mathbb{N} \), which acts on \( R \) by shifting the second variable index. We show that for every such ideal chain, the number of initial ideal chains with respect to term orders on \( R \) that are compatible with the action of \( \text{Inc}(\mathbb{N}) \) is finite. As a consequence of this, we will see that \( \text{Inc}(\mathbb{N}) \)-invariant ideals of \( R \) have only finitely many initial ideals with respect to \( \text{Inc}(\mathbb{N}) \)-compatible term orders.

The article also addresses the question of how many such term orders exist. We give a complete list of the \( \text{Inc}(\mathbb{N}) \)-compatible term orders for the case \( c = 1 \) and show that there are infinitely many for \( c > 1 \). This answers a question by Hillar, Kroner, Leykin.

1. INTRODUCTION

It has long been known that for ideals of polynomial rings in finitely many variables, the number of initial ideals with respect to arbitrary term orders is finite (e.g. \cite{M}, Lemma 2.6). As this result relies on the Noetherianity of such polynomial rings, it cannot be transferred to ideals of polynomial rings in infinitely many variables in general. However, more recent results show that for certain non-Noetherian polynomial rings, there are classes of ideals satisfying a weaker kind of Noetherianity, namely Noetherianity up to the action of certain monoids. Thus, it seems worthwhile to try and generalize the result on finiteness of numbers of initial ideals in the Noetherian case to this class of ideals.

Let \( R := \mathbb{K}[x_{i,j} | i \in [c], j \in \mathbb{N}] \) be the polynomial ring over an arbitrary field \( \mathbb{K} \) in the variables indexed by \([c] \times \mathbb{N} \), where \( \mathbb{N} := \{1, 2, 3, \ldots \} \) denotes the set of natural numbers, \( c \in \mathbb{N} \) is any fixed number and \([c] := \{1, \ldots, c\} \). On \( R \), we can define an action of the monoid \( \text{Inc}(\mathbb{N}) := \{p : \mathbb{N} \rightarrow \mathbb{N} | p(n) < p(n + 1) \text{ for all } n \in \mathbb{N}\} \) of strictly increasing functions on \( \mathbb{N} \) by \( \mathbb{K} \)-linear extension of the map

\[
x_{i_1,j_1} \cdot \ldots \cdot x_{i_r,j_r} \mapsto p \cdot (x_{i_1,j_1} \cdot \ldots \cdot x_{i_r,j_r}) := x_{i_1,p(j_1)} \cdot \ldots \cdot x_{i_r,p(j_r)}
\]

for every \( p \in \text{Inc}(\mathbb{N}) \). Let \( R_n := \mathbb{K}[x_{i,j} | i \in [c], j \in [n]] \) and

\[
\text{Inc}(\mathbb{N})_{m,n} := \{p \in \text{Inc}(\mathbb{N}) | p(m) \leq n \}
\]

for each pair of natural numbers \( m \leq n \). We call a sequence of ideals \( J_0 = J_1 \subseteq J_2 \subseteq \ldots \), where each \( J_n \) is an ideal of \( R_n \), an \( \text{Inc}(\mathbb{N}) \)-invariant ideal chain in \( R \) if for every \( m \leq n \), we have

\[
\text{Inc}(\mathbb{N})_{m,n} \cdot J_m \subseteq J_n.
\]

In \cite{HS} it was shown that every \( \text{Inc}(\mathbb{N}) \)-invariant ideal chain \( J_0 \) in \( R \) stabilizes up to the action of \( \text{Inc}(\mathbb{N}) \), i.e. there is an index \( N \in \mathbb{N} \) satisfying

\[
\langle \text{Inc}(\mathbb{N})_{N,n} \cdot J_N \rangle_{R_n} = J_n
\]

for every \( n \geq N \).
for every \( n \geq N \). We call the minimal \( N \) satisfying \( \text{Ind}(J_{\circ}) = 1 \) the stability index of \( J_{\circ} \) and denote it by \( \text{Ind}(J_{\circ}) \).

Let \( \preceq \) be a term order on \( R \), i.e. a total order on the monomials of \( R \) respecting multiplication and satisfying \( 1 \preceq f \) for every monomial \( f \). If \( \preceq \) has the additional property that
\[
 f \preceq g \Rightarrow p \cdot f \preceq p \cdot g
\]
for all monomials \( f, g \in R \) and every \( p \in \operatorname{Inc}(\mathbb{N}) \), then we call \( \preceq \) an \( \operatorname{Inc}(\mathbb{N}) \)-compatible term order on \( R \). If \( \preceq \) is \( \operatorname{Inc}(\mathbb{N}) \)-compatible, then for every polynomial \( f \in R \), the leading monomial \( \text{in}_{\preceq}(f) \) of \( f \) with respect to \( \preceq \) satisfies
\[
\text{in}_{\preceq}(p \cdot f) = p \cdot \text{in}_{\preceq}(f).
\]
This implies that for every \( \operatorname{Inc}(\mathbb{N}) \)-invariant ideal chain \( J_{\circ} \) in \( R \), the chain of initial ideals
\[
\text{in}_{\preceq}(J_{\circ}) := \text{in}_{\preceq}(J_1) \subseteq \text{in}_{\preceq}(J_2) \subseteq \ldots
\]
is \( \operatorname{Inc}(\mathbb{N}) \)-invariant, too, and therefore stabilizes. Thus, we can define the set
\[
I(J_{\circ}) := \{ \text{Ind}(\text{in}_{\preceq}(J_{\circ})) \mid \preceq \text{ is an } \operatorname{Inc}(\mathbb{N}) \text{-compatible term order on } R \}
\]
of stability indices of initial ideal chains of \( J_{\circ} \) with respect to \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders. In this article, we will prove the following statement:

**Theorem 1.2.** For every \( \operatorname{Inc}(\mathbb{N}) \)-invariant ideal chain \( J_{\circ} \) in \( R \), the set \( I(J_{\circ}) \) is bounded above (and, thus, finite).

Note that as the global stability index \( \text{Ind}(J_{\circ}) \) of the ideal chain \( J_{\circ} \) can be smaller than \( \max(I(J_{\circ})) \) (see Remark 3.10), the perhaps obvious idea to prove Theorem 1.2 by showing that \( I(J_{\circ}) \) is bounded by \( \text{Ind}(J_{\circ}) \) must fail. Therefore, we have to use a different approach.

Theorem 1.2 has two interesting consequences in terms of statements on finiteness of numbers of initial ideals or, respectively, initial ideal chains: In Theorem 3.13, we will see that the number of initial ideal chains of \( \operatorname{Inc}(\mathbb{N}) \)-invariant ideal chains in \( R \) with respect to \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders is finite. As a consequence of this, the number of initial ideals of \( \operatorname{Inc}(\mathbb{N}) \)-invariant ideals of \( R \) with respect to \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders is finite, too, see Corollary 3.18.

Of course, Theorem 1.2 would be insubstantial if there were only finitely many \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders on \( R \). For \( c \geq 2 \), we can easily construct an infinite number of \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders: Choose any term order \( \preceq' \) on the polynomial ring \( K[x,1,\ldots,x_{c,1}] \). For every monomial \( f \in R \), there is a decomposition \( f = x_1^{a(1)} \cdot \ldots \cdot x_n^{a(n)} \), where \( a(i) = (a(i)_1,\ldots,a(i)_c) \in \mathbb{N}_0^c \) and \( x_1^{a(i)} := x_1^{a(i)_1} \cdot \ldots \cdot x_c^{a(i)_c} \cdot x_i^{(1)} \cdot \ldots \cdot x_i^{(n)} \). Set
\[
 x_1^{a(1)} \cdot \ldots \cdot x_n^{a(n)} \quad x^{b(1)}_1 \cdot \ldots \cdot x^{b(n)}_n \quad x_1^{a(i)} \cdot x_i^{(1)} \quad x_1^{a(i)} \quad x_1^{(1)}
\]
for \( i = \min\{ j \mid a(j) \neq b(j) \} \).

This obviously defines an \( \operatorname{Inc}(\mathbb{N}) \)-compatible term order on \( R \), and if we choose two distinct term orders \( \preceq_1', \preceq_2' \) of \( K[x,1,\ldots,x_{c,1}] \), then the corresponding term orders \( \preceq_1, \preceq_2 \) on \( R \) are distinct, too. As there are uncountably many distinct term orders on \( K[x,1,\ldots,x_{c,1}] \), our claim follows.

For \( c = 1 \), in contrast, there are only finitely many \( \operatorname{Inc}(\mathbb{N}) \)-compatible term orders on \( R \). Note that the above example of an \( \operatorname{Inc}(\mathbb{N}) \)-compatible term order can be applied to the case \( c = 1 \), too, using for \( \preceq' \) the only term order there is on the polynomial ring \( K[x] \) in one variable. This yields the term order
\[
x_1^{a_1} \cdot \ldots \cdot x_n^{a_n} \quad x_1^{b_1} \cdot \ldots \cdot x_n^{b_n} \quad a_i < b_i \quad \text{for } i = \min\{ j \mid a_j \neq b_j \},
\]
i.e. a term order of lexicographic type. We will see in Theorem 4.28 that, essentially, every \( \text{Inc}(\mathbb{N}) \)-compatible term order on \( R \) for \( c = 1 \) is of this type, resulting in a number of only six distinct \( \text{Inc}(\mathbb{N}) \)-compatible term orders. This answers Question 5.5 [HKL] by Hillar, Krone, Leykin.

The article is organized as follows: We begin with some technical preparations in Section 2 needed for the proof of Theorem 4.2, which we will give in Section 3. Section 3 also contains the proofs for the finiteness results mentioned above. In Section 4 we will then study the question of what the \( \text{Inc}(\mathbb{N}) \)-compatible term orders are in the case \( c = 1 \). Here, we will not only consider term orders but the larger class of monomial preorders, where we use the concept of a monomial preorder introduced in [KTV]. In Theorem 4.28 we will give a full classification of \( \text{Inc}(\mathbb{N}) \)-compatible monomial preorders on \( R \) for \( c = 1 \), comprising a complete list of the \( \text{Inc}(\mathbb{N}) \)-compatible term orders.

2. Technical preparations

Here and in the section that follows, the number \( c \) from the definitions of \( R \) and \( R_n \) is an arbitrary natural number. We start this section with some observations concerning the monoid \( \text{Inc}(\mathbb{N}) \) and its action on \( R \).

**Lemma 2.3** (cf. [NR], Proposition 4.6). Let \( l \leq m \leq n \) be natural numbers. Then
\[
\text{Inc}(\mathbb{N})_{l,m} = \text{Inc}(\mathbb{N})_{m,n} \circ \text{Inc}(\mathbb{N})_{l,m},
\]
meaning that for every \( p_1 \in \text{Inc}(\mathbb{N})_{l,m} \), \( p_2 \in \text{Inc}(\mathbb{N})_{m,n} \) we have \( p_2 \circ p_1 \in \text{Inc}(\mathbb{N})_{l,n} \), and every element \( p \in \text{Inc}(\mathbb{N})_{l,n} \) has such a decomposition.

**Lemma 2.4.** Let \( N, l \in \mathbb{N}, n \geq N \) and \( i_1 < \ldots < i_l \leq N, j_1 < \ldots < j_l \leq n \) be two ascending sequences of natural numbers. Then there is \( p \in \text{Inc}(\mathbb{N})_{N,n} \) such that \( p(i_r) = j_r \) for all \( r \in [l] \) if and only if \( j_r - j_{r-1} \geq i_r - i_{r-1} \) for all \( r \in [l+1] \), where we set \( i_0 = j_0 = 0 \) and \( i_{l+1} = N + 1, j_{l+1} = n + 1 \).

**Proof.** We use induction on \( n \geq N \). For \( n = N \), the restriction of each element from \( \text{Inc}(\mathbb{N})_{N,n} \) to \( [N] \) is the identity on \( [N] \). So the identities \( p(i_r) = j_r \) imply \( i_r = j_r \) for all \( r \in [l] \) and therefore \( j_r - j_{r-1} = i_r - i_{r-1} \) for all \( r \in [l+1] \). Conversely, assume that \( j_r - j_{r-1} \geq i_r - i_{r-1} \) holds for all \( r \in [l+1] \). If one of these inequalities was strict, we would obtain:
\[
N + 1 = j_{l+1} = \sum_{r=1}^{l+1} (j_r - j_{r-1}) > \sum_{r=1}^{l+1} (i_r - i_{r-1}) = i_{l+1} = N + 1,
\]
which is a contradiction. We conclude that \( j_r - j_{r-1} = i_r - i_{r-1} \) for all \( r \in [l+1] \), so \( j_r = i_r = \text{id}_N(i_r) \) for all \( r \in [l] \).

Now assume that our claim holds for \( n \). Let \( j_1 < \ldots < j_l \leq n + 1 \) and \( p \in \text{Inc}(\mathbb{N})_{N,n+1} \) with \( j_r = p(i_r) \) for all \( r \in [l] \). By Lemma 2.3 \( p \) has a decomposition \( p = p_2 \circ p_1 \) with \( p_1 \in \text{Inc}(\mathbb{N})_{N,n} \) and \( p_2 \in \text{Inc}(\mathbb{N})_{n,n+1} \). Let \( k_r := p_1(i_r) \) for all \( r \in [l] \). As \( \{k_1, \ldots, k_l\} \) is a subset of \([n]\), there is either \( s \in [l] \) with \( p_2(k_r) = k_r \) for all \( r < s \) and \( p_2(k_r) = k_r + 1 \) for all \( r \geq s \) or the restriction of \( p_2 \) to \([l]\) is the identity on \([k]\). Setting \( s := l + 1 \) in the second case and letting \( k_0 := 0, k_{l+1} := n + 1 \), we obtain for both cases, for every \( r \in [l+1] \):
\[
j_r - j_{r-1} = \begin{cases} 
  k_r - k_{r-1} & , r < s \\
  k_r - k_{r-1} + 1 & , r = s 
\end{cases},
\]
hence, by induction, \( j_r - j_{r-1} \geq i_r - i_{r-1} \).

Now let \( j_1 < \ldots < j_l \leq n + 1 \) be a sequence of natural numbers satisfying \( j_r - j_{r-1} \geq i_r - i_{r-1} \).
for all \( r \in [l + 1] \). We have

\[
\sum_{r=1}^{l+1} (j_r - j_{r-1}) - (i_r - i_{r-1}) = j_{l+1} - i_{l+1} = (n + 2) - (N + 1) \geq 1,
\]

so there is \( s \in [l + 1] \) with \( j_s - j_{s-1} > i_s - i_{s-1} \geq 1 \). Define \( p \in \text{Inc}(N) \) by

\[
p(k) := \begin{cases}
    k & k \leq j_s - 2 \\
    k + 1 & k \geq j_s - 1
\end{cases}.
\]

By the choice of \( s \), we have \( j_s - 1 \leq j_s - 2 \), so \( j_1, \ldots, j_{l+1} \) are contained in the image of \( p \) and we can define the sequence \( k_1 < \ldots < k_{l+1} \) by setting \( k_r := p^{-1}(j_r) \). For \( r \in [l + 1] \), we have

\[
k_r = \begin{cases}
    j_r & r \leq s - 1 \\
    j_r - 1 & r \geq s
\end{cases}.
\]

In particular, \( l + 1 \geq s \) implies \( k_{l+1} = j_{l+1} - 1 = n + 1 \). Setting \( k_0 := 0 \), we obtain

\[
(k_r - k_{r-1}) - (i_r - i_{r-1}) = \begin{cases}
    (j_r - j_{r-1}) - (i_r - i_{r-1}) & r \neq s \\
    (j_r - j_{r-1}) - (i_r - i_{r-1}) - 1 & r = s
\end{cases}
\]

for all \( r \in [l + 1] \) and thus \( k_r - k_{r-1} \geq i_r - i_{r-1} \) by the choice of \( s \). Hence by induction, there is \( q \in \text{Inc}(N)_{N,n} \) with \( q(i_r) = k_r \) for all \( r \in [l] \), yielding for every \( r \in [l] \) the identity \( j_r = (p \circ q)(i_r) \). By Lemma 2.3, \( p \circ q \) is contained in \( \text{Inc}(N)_{N,n+1} \), so the claim follows for \( n + 1 \).

**Lemma 2.5.** Let \( m \leq n \) be natural numbers, \( f \in R_m \) a polynomial of degree \( \deg(f) > 0 \) and \( p \in \text{Inc}(N) \) with \( p \cdot f \in R_n \). Let \( m', n' \in N \) be minimal with \( f \in R_{m'} \) and \( p \cdot f \in R_{n'} \). Then the following equivalence holds: There is \( p' \in \text{Inc}(N)_{m,n} \) such that \( p' \cdot f = p \cdot f \) if and only if \( m - m' \leq n - n' \).

**Proof.** Let \( i_1 < \ldots < i_l \leq m \) be the indices for which there is \( k_r \in [c] \) such that \( f \) contains the variable \( x_{k_r,i_r} \), and let \( j_1 < \ldots < j_l \leq n \) be the corresponding indices for \( p \cdot f \). By assumption, we have \( l \geq 1 \), \( i_l = m' \) and \( j_l = n' \). As \( p(i_r) = j_r \) for all \( r \in [l] \), we have \( p \in \text{Inc}(N)_{m',n'} \) and by Lemma 2.4 we obtain:

\[
j_r - j_{r-1} \geq i_r - i_{r-1} \quad \text{for all } r \in [l + 1]
\]

with \( i_0 = j_0 = 0 \) and \( i_{l+1} = m' + 1, \ j_{l+1} = n' + 1 \). So again by Lemma 2.4 there is \( p' \in \text{Inc}(N)_{m,n} \) with \( p'(i_r) = j_r \) for all \( r \in [l] \) if and only if \( n + 1 - n' = n + 1 - j_l \geq m + 1 - i_l = m + 1 - m' \).

**Lemma 2.6.** Let \( i_1 \leq i_2 \leq \ldots \) be an ascending sequence of natural numbers and \( g_{i_n} \in R_{i_n} \) be monomials. Then there are indices \( j < k \) such that \( g_{i_k} \) is contained in \( (\text{Inc}(N)_{i_j,i_k} \cdot g_{i_j})_{R_{i_k}} \).

**Proof.** There is nothing to show if \( g_{i_n} \in K \) for some \( n \in N \), so assume \( \deg(g_{i_n}) > 0 \) for all \( n \). By Theorem 3.1 in [HS], there is an infinite subsequence \( (g_{i_n})_{k \geq 1} \) of \( (g_{i_n})_{n \geq 1} \) such that for each \( k \in N \) we have \( g_{i_{n+k}} = f_k(p_k \cdot g_{i_n}) \) for a monomial \( f_k \in R_{i_{n+k}} \) and \( p_k \in \text{Inc}(N) \). We claim that one of the \( p_k \) can be substituted for an element from \( \text{Inc}(N)_{i_{n+k},i_{n+k+1}} \). By contradiction, assume that this is not the case. For each \( k \) let \( m_k \leq i_{n+k} \) be minimal with \( g_{i_{n+k}} \in R_{m_k} \). By Lemma 2.3 we have \( i_{n+k} - m_k > i_{n+k+1} - m_{k+1} \) for every \( k \geq 1 \). But this contradicts the fact that there are no infinite, strictly decreasing sequences of natural numbers.

We now return to our problem of stability indices of initial ideal chains with respect to \( \text{Inc}(N) \)-compatible term orders. We begin with the remark that if \( J_0 \) is an \( \text{Inc}(N) \)-invariant ideal chain, then every \( N \geq \text{Ind}(J_0) \) satisfies the stability condition (1.1).
Lemma 2.7. Let $J_o$ be an $\text{Inc}(\mathbb{N})$-invariant ideal chain in $R$ and let $N \geq \text{Ind}(J_o)$. Then 
\[
\langle \text{Inc}(\mathbb{N})N,n \cdot J_N \rangle_{R_n} = J_n
\]
for all $n \geq N$.

Proof. Let $N \geq \text{Ind}(J_o)$. Then by Lemma 2.3 and the $\text{Inc}(\mathbb{N})$-invariance of $J_o$, we have
\[
J_n = \langle \text{Inc}(\mathbb{N})\text{Ind}(J_o),n \cdot J_{\text{Ind}(J_o)} \rangle_{R_n}
= \langle \text{Inc}(\mathbb{N})N,n \cdot (\text{Inc}(\mathbb{N})\text{Ind}(J_o),N \cdot J_{\text{Ind}(J_o)}) \rangle_{R_n} \subseteq \langle \text{Inc}(\mathbb{N})N,n \cdot J_N \rangle_{R_n}.
\]

The key to our proof of Theorem 1.2 is the following proposition.

Proposition 2.8. Let $J_o = J_1 \subseteq J_2 \subseteq \ldots$ be an $\text{Inc}(\mathbb{N})$-invariant ideal chain in $R$ and $N \geq \text{Ind}(J_o)$. Then for every $\text{Inc}(\mathbb{N})$-compatible term order $\preceq$, the identity $\text{in}_\preceq(J_{2N}) = \langle \text{Inc}(\mathbb{N})N,2N \cdot \text{in}_\preceq(J_N) \rangle_{R_{2N}}$ implies that $\text{Ind}(\text{in}_\preceq(J_o)) \leq 2N$.

Proof. Let $N \geq \text{Ind}(J_o)$ and $\preceq$ be an $\text{Inc}(\mathbb{N})$-compatible term order. Suppose that $\text{in}_\preceq(J_{2N}) = \langle \text{Inc}(\mathbb{N})N,2N \cdot \text{in}_\preceq(J_N) \rangle_{R_{2N}}$. To prove the proposition, it is enough to show that the corresponding identity holds for every $n > 2N$, as this implies
\[
\text{in}_\preceq(J_n) = \langle \text{Inc}(\mathbb{N})N,n \cdot \text{in}_\preceq(J_N) \rangle_{R_n}
= \langle \text{Inc}(\mathbb{N})N,2N \cdot \text{in}_\preceq(J_N) \rangle_{R_n}
\subseteq \langle \text{Inc}(\mathbb{N})N,n \cdot \text{in}_\preceq(J_N) \rangle_{R_n},
\]
where we used Lemma 2.3 in the second and the $\text{Inc}(\mathbb{N})$-invariance of $\text{in}_\preceq(J_n)$ in the third line. To this end, it suffices to show that if $G$ is a Gröbner basis of $J_N$ with respect to $\preceq$, then $G' := \text{Inc}(\mathbb{N})N,n \cdot G$ is a Gröbner basis of $J_n$ with respect to $\preceq$, because this in turn implies
\[
\text{in}_\preceq(J_n) = \langle \text{in}_\preceq(g') \mid g' \in G' \rangle_{R_n}
= \langle \text{Inc}(\mathbb{N})N,n \cdot \text{in}_\preceq(g) \mid g \in G \rangle_{R_n}
\subseteq \langle \text{Inc}(\mathbb{N})N,n \cdot \text{in}_\preceq(J_N) \rangle_{R_n},
\]
where the $\text{Inc}(\mathbb{N})$-compatibility of $\preceq$ guarantees the validity of the second identity.

So let $n > 2N$. As $G$ generates $J_N$ and $N \geq \text{Ind}(J_o)$, $G'$ is a generating set for $J_n$ by Lemma 2.7. Thus, we only have to show that the $S$-polynomials of the elements of $G'$ reduce to zero with respect to $G'$. Choose $f', g' \in G'$ and write $f' = p_1 \cdot f$, $g' = p_2 \cdot g$ with $p_1, p_2 \in \text{Inc}(\mathbb{N})N,n$ and $f, g \in G$. Let $j_1 < \ldots < j_N \leq n$, $k_1 < \ldots < k_N \leq n$ be natural numbers satisfying $p_1([N]) = \{j_1, \ldots, j_N\}$ and $p_2([N]) = \{k_1, \ldots, k_N\}$ and let $i_1 < \ldots < i_{2N} \leq n$ be natural numbers such that $\{j_1, \ldots, j_N\} \cup \{k_1, \ldots, k_N\} \subseteq \{i_1, \ldots, i_{2N}\}$. Define the map $p$ by
\[
p(j) := \begin{cases} i_j & n + j \leq 2N \neq i_j \in [2N] \\ j & j > 2N \end{cases}.
\]
Then $p$ is an element of $\text{Inc}(\mathbb{N})N,2N$ satisfying $p_1([N]), p_2([N]) \subseteq p([N])$. We first want to show that $p^{-1} \cdot f'$ and $p^{-1} \cdot g'$ lie in $J_{2N}$. Due to the $\text{Inc}(\mathbb{N})$-invariance of $J_o$, this can be achieved by proving that the maps $(p^{-1} \circ p_1)|([N])$ and $(p^{-1} \circ p_2)|([N])$ can be extended to elements from $\text{Inc}(\mathbb{N})N,2N$. As $p_1, p_2$ and $(p_{p(0)})^{-1}$ are strictly increasing, the same is true for the restrictions of $p^{-1} \circ p_1$ and $p^{-1} \circ p_2$ to $[N]$. Furthermore, we have
\[
p^{-1}(p_1([N])) \leq p^{-1}(i_{2N}) = 2N
\]
Proof of Theorem 1.2. By contradiction, assume the existence of a sequence \((\preceq_n)_{n \geq 1}\) of \(
abla\)-compatible term orders on \(R\) with \(\lim_{n \to \infty} \text{Ind}(\preceq_n(J_n)) = \infty\). Set \(N_0 := \text{Ind}(J_0)\) and \(N_i := 2N_{i-1}\) for \(i \geq 1\). We claim that there is a collection \((\preceq_n^i)_{n \geq 1}\) of infinite subsequences of \((\preceq_n)_{n \geq 1}\), where \(i\) ranges over \(\mathbb{N} \cup \{0\}\), such that

1. \((\preceq_n^i)_{n \geq 1}\) is a subsequence of \((\preceq_n^{i-1})_{n \geq 1}\) for all \(i \geq 1\);
2. \(\in_{\preceq_n^i}(J_{N_i}) \subseteq (\text{Inc}(\mathbb{N})_{N_i-1} \cdot \text{in}_{\preceq_n^i}(J_{N_{i-1}}))_{R_{N_i}}\) for all \(i, n \geq 1\).
(3) $\text{in}_{\nabla_n}^j (J_{N_i}) = \text{in}_{\nabla_1}^j (J_{N_i})$ for all $i, n \geq 1$.

Indeed, we can construct these subsequences as follows: Set $(\leq_n^0)_{n \geq 1} := (\leq_n)_{n \geq 1}$. By induction, assume that the subsequence $(\leq_n^i)_{n \geq 1}$ has already been defined for some $i \geq 0$. Then $\lim_{n \to \infty} \text{Ind}(\text{in}_{\nabla_n}^i (J_{n})) = \infty$, so in particular, there are infinitely many indices $n$ such that $\text{Ind}(\text{in}_{\nabla_n}^i (J_{n})) > N_{i+1}$. By Proposition 2.3, these indices satisfy $\text{in}_{\nabla_n}^i (J_{N_{i+1}}) \subseteq \langle \text{Inc}(N)_{N_i, N_{i+1}} \cdot \text{in}_{\nabla_n}^i (J_{N_i}) \rangle_{R_{N_{i+1}}}$. Hence, we obtain an infinite subsequence of $(\leq_n^i)_{n \geq 1}$ satisfying (2). As the total number of initial ideals of $J_{N_{i+1}}$ is finite, this subsequence contains another infinite subsequence $(\leq_n^{i+1})_{n \geq 1}$ such that $\text{in}_{\nabla_n}^{i+1} (J_{N_{i+1}}) = \text{in}_{\nabla_1}^{i+1} (J_{N_{i+1}})$ for all $n$ and we are done.

For every $i \geq 1$, choose a monomial $g_i \in \text{in}_{\nabla_1}^i (J_{N_i})$ that is not contained in $\langle \text{Inc}(N)_{N_{i-1}, N_i} \cdot \text{in}_{\nabla_1}^i (J_{N_{i-1}}) \rangle_{R_{N_i}}$. Then for any pair $i < j$ of natural numbers, we have

$$g_j \notin \left\langle \text{Inc}(N)_{N_{j-1}, N_j} \cdot \text{in}_{\nabla_j}^j (J_{N_{j-1}}) \right\rangle_{R_{N_j}} \supseteq \left\langle \text{Inc}(N)_{N_{j-1}, N_j} \cdot \langle \text{Inc}(N)_{N_{j-1}, N_{j-1}} \cdot \text{in}_{\nabla_j}^j (J_{N_{j-1}}) \rangle \right\rangle_{R_{N_j}} = \left\langle \text{Inc}(N)_{N_{j-1}, N_j} \cdot \text{in}_{\nabla_j}^j (J_{N_{j-1}}) \right\rangle_{R_{N_j}} \supseteq \left\langle \text{Inc}(N)_{N_{j-1}, N_j} \cdot g_i \right\rangle_{R_{N_j}},$$

where we used properties (1) and (3) in the third and Lemma 2.3 in the fourth line. But by Lemma 2.0 such a sequence $(g_i)_{i \geq 1}$ cannot exist, and we have arrived at a contradiction.

\[ \square \]

**Remark 3.10.** The global stability index $\text{Ind}(J_o)$ of an $\text{Inc}(N)$-invariant ideal chain can be smaller than $\text{max}(\text{I}(J_o))$: Let $c = 1$, $J_1 = J_2 = J_3 = \{0\}$, $J_4 = \langle x_1 + x_3 \rangle_{R_4}$ and $J_n = \langle \text{Inc}(N)_{N_{i,n}} \cdot J_{i} \rangle_{R_n}$ for $n \geq 5$. Let $x_n \leq x_{n+1}$ for all $n \in N$. As $(x_2 + x_4) = (x_1 + x_4) = x_2 - x_1$ lies in $J_3$, we conclude that $x_2 \notin \text{in}_{\nabla_1}(J_3)$. On the other hand, we have $\text{in}_{\nabla_1}(J_4) = \langle x_3 \rangle_{R_4}$. Thus, $x_2$ is not an element of $\langle \text{Inc}(N)_{4,5} \cdot \text{in}_{\nabla_1}(J_4) \rangle_{R_5}$, so $\text{Ind}(\text{in}_{\nabla_1}(J_5)) > 4 = \text{Ind}(J_5)$.

We next want to study some of the consequences of Theorem 1.2, which include the statements on finiteness of numbers of initial ideals and initial ideal chains described in the introduction of this article. To this end, we need a few more preparations. Setting $S_n := \{ \sigma : N \to N \mid \sigma \text{ is bijective}, \sigma(i) = i \text{ for all } i \geq n + 1 \}$ and $S_\infty := \bigcup_{n \geq 1} S_n$, we can define an action of $S_\infty$ on $R$ by $\mathbb{K}$-linear extension of the maps

$$\sigma \cdot (x_{i_1,j_1} e_{r_1}, \ldots, x_{i_r,j_r} e_{r_r}) := x_{i_1,\sigma(j_1)} e_{\sigma(r_1)}, \ldots, x_{i_r,\sigma(j_r)} e_{\sigma(r_r)}$$

for every $\sigma \in S_\infty$. There is the following inclusion of orbits:

**Lemma 3.11** (cf. [NR], Lemma 7.5). For every pair of natural numbers $m \leq n$ and $f \in R_m$, we have $\text{Inc}(N)_{m,n} \cdot f \subseteq S_n \cdot f$.

Lemma 3.11 ensures that every $S_\infty$-invariant ideal chain $J_0 = J_1 \subseteq J_2 \subseteq \ldots$ in $R$, i.e. every ideal chain satisfying $S_n \cdot J_m \subseteq J_n$ for all $m \leq n$, is also $\text{Inc}(N)$-invariant. Note that the ideals $J_n$ of an $S_\infty$-invariant ideal chain are themselves $S_n$-invariant, i.e. they satisfy $S_n \cdot J_n \subseteq J_n$. 
For any subset $A \subseteq \mathbb{N}$, let $R_A$ be the polynomial ring over $\mathbb{K}$ in the variables indexed by $[\varepsilon] \times A$.

**Lemma 3.12.** Let $J \subseteq R_n$ be an ideal satisfying $S_n \cdot J \subseteq J$, $m \leq n$ and $p \in \text{Inc}(\mathbb{N})_{m,n}$. Then
\[
p \cdot (J \cap R_m) = J \cap R_{p([m])}.
\]
In particular, for every $f \in J \cap R_m$, the polynomial $p \cdot f$ is contained in $J$.

**Proof.** The inclusion $p \cdot (J \cap R_m) \subseteq J \cap R_{p([m])}$ follows from the $S_n$-invariance of $J$ and from Lemma 3.11. Conversely, let $f \in J \cap R_{p([m])}$ and $\sigma \in S_n$ satisfying $\sigma([m]) = p([m])$. Then $\sigma^{-1}_{p([m])} = p^{-1}_{[m]}$, and due to the $S_n$-invariance of $J$ we obtain
\[
f = p \cdot (p^{-1} \cdot f) = p \cdot (\sigma^{-1} \cdot f) \in p \cdot (J \cap R_m).
\]

**Lemma 3.13.** Let $n \in \mathbb{N}$ and $J \subseteq R_n$ be an ideal. Then for any Inc($\mathbb{N}$)-compatible term order $\preceq$, the following identity holds:
\[
(3.14) \quad p \cdot \text{in}_\preceq(J) = \text{in}_\preceq(p \cdot J).
\]
In this equation, $p \cdot J$ is regarded as an ideal of $R_{p([n])}$.

**Proof.** The left side of equation (3.14) is generated, as an ideal of $R_{p([n])}$, by the set $G_1 := \{p \cdot \text{in}_\preceq(f) \mid f \in J\}$, whereas the right side is generated by $G_2 := \{\text{in}_\preceq(p \cdot f) \mid f \in J\}$. The Inc($\mathbb{N}$)-compatibility of $\preceq$ yields $G_1 = G_2$, and the identity in (3.14) follows.

**Theorem 3.15.** For an Inc($\mathbb{N}$)-invariant ideal chain $J_o$, the following statements are equivalent:

1. $I(J_o)$ is bounded above.
2. The set of ideal chains $\{\text{in}_\preceq(J_o) \mid \preceq$ is an Inc($\mathbb{N}$)-compatible term order on $R\}$ is finite.

Furthermore, the two above statements imply:

3. There is $N \in \mathbb{N}$ such that
\[
(3.16) \quad \text{in}_\preceq(J_n) = \sum_{1 \leq i_1, \ldots, i_N \leq n} \text{in}_\preceq(J_n \cap R_{\{i_1,\ldots,i_N\}})_{R_n}
\]
for all $n \geq N$ and every Inc($\mathbb{N}$)-compatible term order $\preceq$ on $R$. Here, we regard the intersections $J_n \cap R_{\{i_1,\ldots,i_N\}}$ as ideals of $R_{\{i_1,\ldots,i_N\}}$.

If $J_o$ is not only Inc($\mathbb{N}$)- but also $S_\infty$-invariant then (3) is equivalent to (1) and (2).

**Proof.** We first show the equivalence of (1) and (2) for Inc($\mathbb{N}$)-invariant ideal chains $J_o$. The implication (2) $\Rightarrow$ (1) is clear. For the reverse implication, let $N := \max(I(J_o))$. Then by Lemma 2.7 for Inc($\mathbb{N}$)-compatible term orders $\preceq$, $\preceq'$ we have $\text{in}_\preceq(J_o) = \text{in}_{\preceq'}(J_o)$ if and only if $\text{in}_\preceq(J_n) = \text{in}_{\preceq'}(J_n)$ for all $n \in [N]$. As $J_1, \ldots, J_N$ each have only finitely many initial ideals, there are only finitely many sequences $L_1 \subseteq \ldots \subseteq L_N$ such that $L_n = \text{in}_\preceq(J_n)$ for all $n$ for any term order $\preceq'$ on $R$. Hence, assertion (2) follows.

Next, we show the implication (1) $\Rightarrow$ (3) for Inc($\mathbb{N}$)-invariant $J_o$. Let again $N := \max(I(J_o))$ and choose any Inc($\mathbb{N}$)-compatible term order $\preceq$ on $R$. Then by Remark 2.7 and the Inc($\mathbb{N}$)-compatibility of $\preceq$, $\text{in}_\preceq(J_n)$ is generated by
\[
\{\text{in}_\preceq(p \cdot f) \mid p \in \text{Inc}(\mathbb{N})_{n,n}, f \in J_N\}.
\]
As $J_0$ is Inc$(N)$-invariant, each of the polynomials $p \cdot f$ in the above set lies in one of the intersections $J_n \cap R_{(i_1, \ldots, i_N)}$, where $i_1 < \ldots < i_N$ ranges over all strictly ascending sequences of $[n]$. This proves the inclusion $\subseteq$ in (3.16). The reverse inclusion is obvious. Now assume $J_0$ to be $S_\infty$-invariant and that (3) holds. By the Noetherianity of $R_N$, there is an index $N' \geq N$ such that $J_n \cap R_N = J_{N'} \cap R_N =: J$ for all $n \geq N'$. Let $n \geq N'$ and $\preceq$ be any Inc$(N)$-compatible term order on $R$. For a sequence $1 \leq i_1 < \ldots < i_N \leq n$, let $p_{i_1, \ldots, i_N} \in$ Inc$(N)$ be any function satisfying $p_{i_1, \ldots, i_N}([N]) = \{i_1, \ldots, i_N\}$. Then by Lemmata 3.12 and 3.13 we have
\begin{align*}
in_{\preceq}(J_n) &= \sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle \text{in}_{\preceq}(J_n \cap R_{(i_1, \ldots, i_N)}) \rangle \rangle_{R_n} \\
&= \sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle \text{in}_{\preceq}(p_{i_1, \ldots, i_N} \cdot J) \rangle \rangle_{R_n} \\
&= \sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle p_{i_1, \ldots, i_N} \cdot \text{in}_{\preceq}(J) \rangle \rangle_{R_n}.
\end{align*}

By Lemma 2.3 each of the $p_{i_1, \ldots, i_N}$ has a decomposition $p_{i_1, \ldots, i_N} = p_{i_1, \ldots, i_N}^{(2)} \circ p_{i_1, \ldots, i_N}^{(1)}$ with $p_{i_1, \ldots, i_N}^{(1)} \in$ Inc$(N, N')$ and $p_{i_1, \ldots, i_N}^{(2)} \in$ Inc$(N')$. Thus, we obtain
\begin{align*}
&\sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle p_{i_1, \ldots, i_N} \cdot \text{in}_{\preceq}(J) \rangle \rangle_{R_n} \\
= &\sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle p_{i_1, \ldots, i_N}^{(2)} \cdot (p_{i_1, \ldots, i_N}^{(1)} \cdot \text{in}_{\preceq}(J)) \rangle \rangle_{R_n} \\
= &\sum_{1 \leq i_1 < \ldots < i_N \leq n} \langle \langle p_{i_1, \ldots, i_N}^{(2)} \cdot \text{in}_{\preceq}(p_{i_1, \ldots, i_N}^{(1)} \cdot J) \rangle \rangle_{R_n} \\
\subseteq &\langle \langle \text{Inc}(N) \rangle \rangle_{R_n} \cdot \langle \langle \text{in}_{\preceq}(J_{N'}) \rangle \rangle_{R_n},
\end{align*}
where we again used Lemma 3.13 for the second identity and Lemma 3.12 for the last inclusion. This shows that $\text{Ind}(\text{in}_{\preceq}(J_n)) \leq N'$ for any Inc$(N)$-compatible term order $\preceq$ and (1) follows.

**Remark 3.17.** By equation (3.16), for every Inc$(N)$-invariant ideal chain $J_n$ in $R$, there is a natural number $N$ such that for every $n \geq N$ and every Inc$(N)$-compatible term order $\preceq$ on $R$, there is a Gröbner basis of $J_n$ with respect to $\preceq$ whose elements each contain no more than $cN$ distinct variables. This is not the case for arbitrary ideal chains in $R$. For instance, set $c = 1$ and consider the ideal chain $J_n$ defined by
\begin{align*}
J_1 &:= \{0\}, \\
J_{2^n} &:= \langle J_{2^{n-1}}, x_{2^{n-1}+1} + \ldots + x_{2^n} \rangle_{R_{2^n}} \quad \text{for } n \geq 1, \\
J_m &:= J_{2^n} \quad \text{for } 2^n \leq m < 2^{n+1}.
\end{align*}

Then for any term order $\preceq$ on $R$ and $n \geq 1$, every polynomial $f \in J_{2^n}$ with $\text{in}_{\preceq}(f) \mid \text{in}_{\preceq}(x_{2^n+1} + \ldots + x_{2^{n+1}})$ must contain a non-trivial $\mathbb{K}$-multiple of $x_{2^n+1} + \ldots + x_{2^{n+1}}$, and, hence, at least $2^{n-1}$ distinct variables.

**Corollary 3.18.** Let $J$ be an ideal of $R$ satisfying Inc$(N)$ $\cdot J \subseteq J$. Then $J$ has only finitely many initial ideals with respect to Inc$(N)$-compatible term orders on $R$.

**Proof.** For every term order $\preceq$ on $R$, in$_{\preceq}(J)$ is generated by the union of all initial ideals in$_{\preceq}(J \cap R_n) \subseteq R_n$. Thus, if $\preceq, \preceq'$ are term orders on $R$ with in$_{\preceq}(J \cap R_n) = \text{in}_{\preceq'}(J \cap R_n)$ for all $n$, then in$_{\preceq}(J) = \text{in}_{\preceq'}(J)$. As the ideal chain $J_n := J \cap R_1 \subseteq J \cap R_2 \subseteq \ldots$ is Inc$(N)$-invariant, Theorem 3.15(2) tells us that there exists a finite number of Inc$(N)$-compatible
term orders $\preceq_1, \ldots, \preceq_N$ on $R$ such that for every Inc($\mathbb{N}$)-compatible term order $\preceq$ on $R$ there is $i \in [N]$ with $\text{in}_{\preceq}(J_o) = \text{in}_{\preceq_i}(J_o)$. This proves our claim. \hfill \Box

Remark 3.19. There is a more direct way to prove Corollary 3.18 which does not rely on Theorem 3.15. Namely, one can transfer the proof of finiteness of the number of initial ideals for ideals in polynomial rings in finitely many variables given in [M], Lemma 2.6, to the situation of Inc($\mathbb{N}$)-invariant ideals in $R$. Just substitute the ideals $m_i$ defined in [M] for $(\text{Inc}($$\mathbb{N}$$) \cdot m_i)_R$ and use the fact that Inc($\mathbb{N}$)-divisibility in $R$ is a well-partial-order ([HS], Theorem 3.1) as a substitute for Noetherianity. This raises the question whether Theorem 1.2 is just a simple consequence of Corollary 3.18.

Indeed, for any Inc($\mathbb{N}$)-invariant ideal chain $J_o$ in $R$, the ideal $J := \bigcup_{n \geq 1} J_n$ is an Inc($\mathbb{N}$)-invariant ideal of $R$, and for every term order $\preceq$ on $R$, we have $\text{in}_{\preceq}(J) = \bigcup_{n \geq 1} \text{in}_{\preceq_n}(J_n)$. Hence, Corollary 3.18 yields

$$\#\left\{ \bigcup_{n \geq 1} \text{in}_{\preceq_n}(J_n) \mid \preceq \text{ is Inc($\mathbb{N}$)}-\text{compatible} \right\} < \infty.$$

However, Theorem 1.2 provides more information than that: By Theorem 3.15(2), not only the number of unions of the initial ideals of the $J_n$ with respect to Inc($\mathbb{N}$)-compatible term orders is finite, but also the number of sequences $(\text{in}_{\preceq_n}(J_n))_{n \geq 1}$ giving rise to the same union.

Remark 3.20. Corollary 3.18 does not hold for the number of initial ideals with respect to arbitrary term orders on $R$: Let $c = 1$ and $J := (\text{Inc}($$\mathbb{N}$$) \cdot (x_1^2 x_2 + x_1 x_2^2))_R$ be the ideal that is generated by the Inc($\mathbb{N}$)-orbits of the polynomial $x_1^2 x_2 + x_1 x_2^2$. For every $n \in \mathbb{N}$, define the term order $\preceq_n$ by

$$x_{\sigma_n(1)}^{a_1} \cdots x_{\sigma_n(k)}^{a_k} \prec_n x_{\sigma_n(1)}^{b_1} \cdots x_{\sigma_n(k)}^{b_k} : \iff a_i < b_i \text{ for } i = \min\{ j \mid a_j \neq b_j \},$$

where the map $\sigma_n \in S_\infty$ is defined by

$$\sigma_n(j) = \begin{cases} n + 1 - j & , j \leq n \\ j & , j > n \end{cases}.$$

For example, if $n = 3$, then $(\sigma_3(1), \sigma_3(2), \sigma_3(3), \sigma_3(4), \sigma_3(5)) = (3, 2, 1, 4, 5)$. We claim that for every pair $n < n'$ of natural numbers, $x_1^2 x_{n'} \in \text{in}_{\preceq_n}(J) \setminus \text{in}_{\preceq_n}(J)$. We have $\text{in}_{\preceq_n}(x_1^2 x_{n'} + x_1 x_{n''}) = x_1^2 x_{n''}$ as $\sigma_n^{-1}(n') = n > n = \sigma_n^{-1}(1)$, so $x_1^2 x_{n''} \in \text{in}_{\preceq_n}(J)$. Let $f$ be a polynomial in $J$ that contains the monomial $x_1^2 x_{n''}$. We may assume $f$ to be homogeneous, so $f = \sum_{i=1}^k c_i p_i \cdot (x_1^2 x_2 + x_1 x_2^2)$ with $c_i \in K \setminus \{0\}$ and $p_i \in \text{Inc}($$\mathbb{N}$$)$, where

$$p_i \cdot (x_1^2 x_2 + x_1 x_2^2) \neq p_j \cdot (x_1^2 x_2 + x_1 x_2^2) \text{ for } i \neq j.$$  

As $f$ contains $x_1^2 x_{n''}$, there is exactly one $i$ with $p_i \cdot (x_1^2 x_2 + x_1 x_2^2) = x_1^2 x_{n''}$. Therefore, $f$ contains the monomial $x_1 x_{n''}^2$. But $x_1^2 x_{n''} \prec_{n''} x_1 x_{n''}^2$, so $x_1^2 x_{n''} \notin \text{in}_{\preceq_n}(J)$. We conclude that the initial ideals in $\preceq_n(J)$ are pairwise distinct. Thus, $J$ has infinitely many distinct initial ideals with respect to arbitrary term orders.

4. Classification of Inc($\mathbb{N}$)-Compatible Monomial Preorders for $c = 1$

In this section, we will always assume $c = 1$. Following the definition in [KTV], we call a strict partial order $\prec$ on $R$ or $R_n$ a monomial preorder if it satisfies the following conditions:

- Multiplicativity: For monomials $f, g, h \in R$ or $R_n$, $f \prec g$ implies $hf \prec hg$.
- Cancellativeness: For monomials $f, g, h \in R$ or $R_n$, $f \prec g$ implies $f \prec h$.
- Incomparability with respect to $\prec$ is transitive.
For every monomial $f \in R_n$ there is $a = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$ with $f = x^a := x_1^{a_1} \cdots x_n^{a_n}$. In [KTV] it was shown that for every monomial preorder $\prec$ on $R_n$, there is some $m \in \mathbb{N}$ and a matrix $M \in \mathbb{R}^{m \times n}$ such that for monomials $x^a, x^b \in R_n$ we have

$$x^a \prec x^b \Leftrightarrow M \cdot a \prec_{\text{lex}} M \cdot b,$$

where $\prec_{\text{lex}}$ denotes the lexicographic order on $\mathbb{R}^n$, i.e.

$$(\lambda_1, \ldots, \lambda_n) \prec_{\text{lex}} (\mu_1, \ldots, \mu_n) \Leftrightarrow \lambda_i < \mu_i \text{ for } i = \min \{j \mid \lambda_j \neq \mu_j\}.$$

Obviously, one can assume the rows of $M$ to be orthogonal and non-zero (and, consequently, $m \leq n$), and we will do so from now on.

Our goal for this section is to classify the $\text{Inc}(\mathbb{N})$-compatible monomial preorders on $R$, i.e. the monomial preorders $\prec$ which additionally satisfy the condition

$$(4.21) \quad f \prec g \Rightarrow p \cdot f \prec p \cdot g$$

for all monomials $f, g \in R$ and every $p \in \text{Inc}(\mathbb{N})$. Our strategy is to first classify the $\text{Inc}(\mathbb{N})$-compatible monomial preorders on $R_4$ (the question why we have to use $n = 4$ is addressed in Remark 4.27). By shifting variable indices and using Equation (4.21), we will then be able to deduce from this what the $\text{Inc}(\mathbb{N})$-compatible monomial preorders on $R$ are.

**Lemma 4.22.** Let $M \in \mathbb{R}^{m \times 4}$ be a matrix representing an $\text{Inc}(\mathbb{N})$-compatible monomial preorder $\prec$ on $R_4$. Then there is a real number $\lambda \neq 0$ such that the first row $r_1 \in \mathbb{R}^4$ of $M$ satisfies

$$(4.23) \quad r_1 \in \{(\lambda, \lambda, \lambda, \lambda), (\lambda, 0, 0, 0), (0, 0, 0, \lambda)\}.$$

If $r_1 = (\lambda, \lambda, \lambda, \lambda)$ and $m \geq 2$, then the second row $r_2$ of $M$ satisfies

$$(4.24) \quad r_2 \in \{(-\mu, -\mu, -\mu, 3\mu), (3\mu, -\mu, -\mu, -\mu)\}$$

for some $\mu \neq 0$.

**Proof.** Write $r_1 = (a_1, a_2, a_3, a_4)$. Due to the $\text{Inc}(\mathbb{N})$-compatibility of $\prec$, for any vector $(v_1, v_2, v_3) \in \mathbb{Z}^3$ the inequality $a_1v_1 + a_2v_2 + a_3v_3 > 0$ implies $a_2v_1 + a_3v_2 + a_4v_3 \geq 0$ and $a_1v_1 + a_3v_2 + a_4v_3 \geq 0$. Therefore, the matrices

\[
A_1 := \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}, \quad A_2 := \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_3 & a_4 \end{pmatrix}
\]

must have rank $\leq 1$. Assume $a_1 \neq 0$. Then from $\text{rank}(A_2) \leq 1$ we deduce $a_2 = a_3 = a_4$, which due to $\text{rank}(A_1) \leq 1$ implies either $a_2 = a_3 = a_4 = 0$ or $a_1 = a_2$. On the other hand, if $a_1 = 0$, $\text{rank}(A_1) \leq 1$ implies $a_2 = a_3 = a_4 = 0$. This proves (4.23).

Now assume that $r_1 = (\lambda, \lambda, \lambda, \lambda)$ and $m \geq 2$. Write $r_2 = (b_1, b_2, b_3, b_4)$. Again, the $\text{Inc}(\mathbb{N})$-compatibility of $\prec$ implies that if $(v_1, v_2, v_3) \in \mathbb{Z}^3$ satisfies $b_1v_1 + b_2v_2 + b_3(-v_1 - v_2) > 0$, then $b_2v_1 + b_3v_2 + b_4(-v_1 - v_2) \geq 0$ and $b_1v_1 + b_3v_2 + b_4(-v_1 - v_2) \geq 0$. Hence the matrices

\[
B_1 := \begin{pmatrix} b_1 - b_3 & b_2 - b_3 \\ b_2 - b_4 & b_3 - b_4 \end{pmatrix}, \quad B_2 := \begin{pmatrix} b_1 - b_3 & b_2 - b_3 \\ b_1 - b_4 & b_3 - b_4 \end{pmatrix}
\]

must have rank $\leq 1$. Assume $b_2 - b_3 \neq 0$. Then there are $x, y \in \mathbb{R}$ satisfying $(b_2 - b_4, b_3 - b_4) = x(b_1 - b_3, b_2 - b_3)$, $(b_1 - b_4, b_3 - b_4) = y(b_1 - b_3, b_2 - b_3)$. As the second columns of $B_1$ and $B_2$ agree, we have $x = y$ and, thus, $b_1 = b_2$. But then, $\text{rank}(B_1) \leq 1$ implies $b_2 = b_3$, which is a contradiction. We therefore may assume $b_2 - b_3 = 0$. If $b_1 - b_3 \neq 0$, $\text{rank}(B_1) \leq 1$ then yields $b_3 - b_4 = 0$, so $b_2 = b_3 = b_4$. On the other hand, if $b_1 - b_3 = 0$, we obtain $b_1 = b_2 = b_3$. As we assume $r_2$ to be orthogonal to $r_1$, this proves (4.24). \(\square\)

**Lemma 4.25.** Let $i \in \mathbb{N}$, $n \geq i$ and $\prec$ be a monomial preorder on $R_n$. 

(1) If $x_1$ and $x_j$ are incomparable for all $j \in [i]$, then any two monomials $f, g \in R_i$ with $\deg(f) = \deg(g)$ are incomparable.

(2) If 1 and $x_j$ are incomparable for all $j \in [i]$, then every pair of monomials $f, g \in R_i$ is incomparable.

Proof. Note that if $f, g$ and $f', g'$ are two pairs of incomparable monomials in $R_n$, then the pair $ff', gg'$ is incomparable, too. In case (1), this implies that $x_i^{\deg(f)}$ and $f$ are incomparable for every monomial $f \in R_i$; in case (2), we obtain that 1 and $f$ are incomparable for all monomials $f \in R_i$. Transitivity of incomparability now yields the desired statements.

For a monomial preorder $\prec$ on any polynomial ring, we denote by $\prec^{-1}$ the inverse of $\prec$, i.e. the monomial preorder which is defined by $f \prec^{-1} g \iff f \succ g$. We call a monomial preorder trivial if every pair of monomials $f, g$ is incomparable. A degree order is a monomial preorder $\prec$ satisfying $\deg(f) < \deg(g) \Rightarrow f \prec g$, and a reverse degree order is a monomial preorder which is the inverse of a degree order.

In what follows, we will write $f \preceq g$ instead of $f \not\succ g$, and we set $R_0 := \mathbb{K}$.

Proposition 4.26. For monomials $x^a \neq x^b \in R_i$ let $A := \{j | a_j \neq b_j\}$. The Inc($\mathbb{N}$)-compatible, non-trivial monomial preorders on $R_i$ are:

(1) $\prec^d$: $x^a \prec^d x^b :\iff \deg(x^a) < \deg(x^b)$;
(2) $\prec_{\min}^d$: $x^a \prec_{\min} x^b :\iff a_{\min(A)} < b_{\min(A)}$;
(3) $\prec_{\min}^d$: $x^a \prec_{\min} x^b :\iff \deg(x^a) < \deg(x^b)$ or $(\deg(x^a) = \deg(x^b) \text{ and } a_{\min(A)} < b_{\min(A)})$;
(4) $\prec_{\min}^d$: $x^a \prec_{\min} x^b :\iff \deg(x^a) > \deg(x^b)$ or $(\deg(x^a) = \deg(x^b) \text{ and } a_{\min(A)} < b_{\min(A)})$;
(5) $\prec_{\max,i}^d$: $i \in \{2, 3, 4\}$: $x^a \prec_{\max,i} x^b :\iff \max(A) \geq i \text{ and } a_{\max(A)} < b_{\max(A)}$;
(6) $\prec_{\max,i}^d$: $i \in \{2, 3, 4\}$: $x^a \prec_{\max,i} x^b :\iff \deg(x^a) < \deg(x^b)$ or $(\deg(x^a) = \deg(x^b), \text{max}(A) \geq i \text{ and } a_{\max(A)} < b_{\max(A)})$;
(7) $\prec_{\max,i}^d$: $i \in \{2, 3, 4\}$: $x^a \prec_{\max,i} x^b :\iff \deg(x^a) > \deg(x^b)$ or $(\deg(x^a) = \deg(x^b), \text{max}(A) \geq i \text{ and } a_{\max(A)} < b_{\max(A)})$;

and their inverses.

Proof. Let $\prec$ be an Inc($\mathbb{N}$)-compatible monomial preorder on $R_i$ and let $r_j$ be the $j$th row of a matrix representing it. We first consider the case $r_1 = (\lambda, \lambda, \lambda, \lambda), r_2 = (3\mu, -\mu, -\mu, -\mu)$ and the case $r_1 = (\lambda, 0, 0, 0)$, assuming that $\lambda, \mu > 0$. In the first case, $\prec$ is a degree order with the additional property that $(a_1 < b_1 \Rightarrow x^a \prec x^b)$ for monomials $x^a, x^b$ of the same degree. In the second case, this implication is valid for any pair of monomials $x^a, x^b$. Let $x^a, x^b \in R_i$ be monomials such that $a_{\min(A)} < b_{\min(A)}$ and, in the first case, $\deg(x^a) = \deg(x^b)$. We may assume that $a_i = b_i = 0$ for $1 \leq i < \min(A)$. Choose $p \in \text{Inc}(\mathbb{N})$ with $\{\min(A), ..., 4\} \subseteq p([4])$ and $p(1) = \min(A)$. Then we have $p^{i-1} \cdot x^a < p^{i-1} \cdot x^b$, and by Inc($\mathbb{N}$)-compatibility we conclude that this relation holds for $x^a$ and $x^b$, too. Thus, we obtain $\prec = \prec_{\min}^d$ in the first and $\prec = \prec_{\min}^{-1}$ in the second case. If $\lambda < 0$ or $\mu < 0$, an analogous argument shows that in the first case, $\prec$ is one of the monomial preorders $(\prec_{\min}^{-1})^d, -\prec_{\min}^{-1}$, and in the second case, we have $\prec = (\prec_{\min}^{-1})^d$.

Now assume that $r_1 = (\lambda, \lambda, \lambda, \lambda)$ and $r_2 = (-\mu, -\mu, -\mu, 3\mu)$ with $\lambda, \mu > 0$ (as above, the cases $\lambda < 0$ or $\mu < 0$ can be dealt with similarly). Then again, $\prec$ is a degree order, and for monomials $x^a, x^b$ with $\deg(x^a) = \deg(x^b)$ we have $(a_1 < b_1 \Rightarrow x^a \prec x^b)$. By Inc($\mathbb{N}$)-compatibility, the relation $x_1 \preceq x_i$ holds for all $i \in [4]$. Let $i \in \{2, 3, 4\}$ be minimal such
that $x_1 \prec x_i$, and let $f \in R_{i-1}$, $g \in R_i \setminus R_{i-1}$ be any monomials with $\deg(f) = \deg(g)$. Writing $g = g_1 x_i^e$ with $g_1 \in R_{i-1}$ and $f = f_1 f_2$ such that $\deg(f_1) = \deg(g_1)$, Lemma 4.25(1) tells us that $f_1$ and $g_1$ are incomparable and $x_i^e \succ f_2$, hence we obtain $f \prec g$.

Now let $i < j \leq 4$ and $f \in R_{j-1}$, $g \in R_j \setminus R_{j-1}$ be monomials of the same degree. Suppose that $f$ and $g$ are incomparable. As by Inc($\mathbb{N}$)-compatibility we have $x_1 \prec x_{j-1}$, this yields $x_1 g \prec x_{j-1} f$. Let $p \in \text{Inc}(\mathbb{N})$ be any function satisfying $p(j) = 4$. Then by Inc($\mathbb{N}$)-compatibility, we have $p \cdot x_1 g \prec p \cdot x_{j-1} f$, which is a contradiction. Thus, Lemma 4.25(1) and the Inc($\mathbb{N}$)-compatibility of $\prec$ yield $\prec \preccurlyeq \prec_{\text{max}, i}$.

Finally, let $r_1 = (0, 0, 0, \lambda)$ and assume that $\lambda > 0$. Then $\prec$ satisfies $(a_4 < b_4 \Rightarrow x^a \prec x^b)$, so in particular $1 \prec x_4$ and, thus, by Inc($\mathbb{N}$)-compatibility $1 \preceq x_i$ for all $i \in [4]$. Let $i \in [4]$ be minimal such that $1 \prec x_i$ and let $f \in R_{i-1}$, $g = g_1 x_i^e$ with $g_1 \in R_{i-1}$ and $e > 0$ be any monomials. By Lemma 4.25(2), $f$ and $g_1$ are incomparable, so we obtain $f \prec g$. Now let $i < j \leq 4$ and $f \in R_{j-1}$, $g \in R_j \setminus R_{j-1}$. Supposing that $f$ and $g$ are incomparable, we obtain $g \prec x_{j-1} f$. Arguing as in the paragraph above, this contradicts the Inc($\mathbb{N}$)-compatibility of $\prec$. Hence, Lemma 4.25(2) and the Inc($\mathbb{N}$)-compatibility of $\prec$ let us conclude that $\prec \preccurlyeq \prec_{\text{max}, i}$.

**Remark 4.27.** For $n = 2$ and $n = 3$ there is an infinite number of Inc($\mathbb{N}$)-compatible monomial preorders on $R_n$: For $n = 2$, choose any irrational number $\lambda > 0$. Then the matrix

$$
A(\lambda) := \begin{pmatrix} 1 & \lambda \\
1 + \lambda & -1 & -\lambda \end{pmatrix}
$$

defines an Inc($\mathbb{N}$)-compatible term order on $R_2$, and if $0 < \lambda' \neq \lambda$ is another irrational number, the term orders represented by $A(\lambda)$ and $A(\lambda')$ are distinct.

For $n = 3$, let $\lambda > 1$ be any irrational number and consider the matrix

$$
B(\lambda) := \begin{pmatrix} 1 & 1 & 1 \\
1 + \lambda & -1 & -\lambda \end{pmatrix}
$$

Then one can easily check that $B(\lambda)$ represents an Inc($\mathbb{N}$)-compatible term order on $R_3$, and for distinct irrational numbers $\lambda, \lambda' > 1$, the term orders represented by $B(\lambda)$ and $B(\lambda')$ are distinct, too.

**Theorem 4.28.** The Inc($\mathbb{N}$)-compatible monomial preorders on $R$ are the same as on $R_4$, with the exception that the number $i$ used in the definitions of preorders (5), (6) and (7) can take arbitrary values in $\mathbb{N}$. In particular, there are only six Inc($\mathbb{N}$)-compatible term orders on $R$, namely $\prec_{\text{min}}, \prec_{\text{dmin}}, (\prec_{\text{ndmin}})^{-1}, \prec_{\text{max}}, \prec_{\text{max}, 2}, (\prec_{\text{ndmax}, 2})^{-1}$.

**Proof.** Let $\prec$ be an Inc($\mathbb{N}$)-compatible monomial preorder on $R$. Note that by Inc($\mathbb{N}$)-compatibility, we either have $x_1 \preceq x_i$ for all $i \in \mathbb{N}$ or $x_1 \succeq x_i$ for all $i \in \mathbb{N}$. We will only consider the former case. By Lemma 4.25(1), if $x_1$ and $x_i$ are incomparable for all $i$, then so are $x_1$ and $x_j$ for every pair of natural numbers $i, j$. On the other hand, if $i \in \mathbb{N}$ is minimal such that $x_1 \prec x_i$, then Lemma 4.25(1) yields $x_{i-1} \prec x_i$, so by Inc($\mathbb{N}$)-compatibility we obtain $x_k \prec x_l$ for all $l \geq i$ and $k < l$. Thus, in any case we have $x_1 \preceq x_j$ for all $i \leq j$.

We will use the following notation: For a monomial $f \in R \setminus \{1\}$, let $m(f)$ and $M(f)$ denote the minimal or, respectively, maximal index of a variable occurring in $f$. By $e(f)$ and $E(f)$ we denote the exponents of these variables in $f$. For $f = 1$, we set $m(f) = \infty$, $M(f) = 0$ and define $x_0 := x_{\infty} := 1$. By the above observation, we have $x_{m(f)}^{\deg(f)} \preceq f$, $x_{M(f)}^{\deg(f)} \succeq f$, $x_{m(f)}^{\deg(f)} - E(f) \leq f$ and $x_{M(f)}^{\deg(f)} - E(f) \geq f$. Therefore, in order to show that for any monomials $f, g \in R$ the relation $f \prec g$ holds, it suffices to show one of the following relations:
\[ x \leq_{\deg} y \] where \( x \) and \( y \) are elements of some ordered set.

\[ x \cdot y \leq_{\deg} z \] if and only if \( x \) and \( y \) are incomparable.

Let \( \prec \) be the restriction of \( \leq \) to \( R_4 \). We will first show that if \( \prec \) is a degree or a reverse degree order, then the same holds for \( \preceq \). Assume that \( f' \prec g' \) whenever \( \deg(f') < \deg(g') \) and let \( f, g \in R \) be any monomials satisfying \( \deg(f) < \deg(g) \). Choose \( p \in \text{Inc}(\mathbb{N}) \) with \( M(f), m(g) \in p([2]_0) \). Then we have \( p^{-1} \cdot x_{M(f)}^{\deg(f)} \prec p^{-1} \cdot x_{m(g)}^{\deg(g)} \), which by \( \text{Inc}(\mathbb{N}) \)-compatibility of \( \prec \) implies that \( x_{M(f)}^{\deg(f)} \prec x_{m(g)}^{\deg(g)} \) and, thus, \( f \prec g \). If \( f' \prec g' \) whenever \( \deg(f') > \deg(g') \), an analogous argument shows that \( \preceq \), too, satisfies this condition, and we are done.

Assume that \( \prec \) is not a total order. By Proposition 4.26 this implies that \( \prec \) is either a degree or a reverse degree order or 1 and \( x_1 \) are incomparable.

Suppose that \( x_1 \) and \( x_i \) are incomparable for all \( i \in \mathbb{N} \). Then Lemma 4.25 implies that either \( \prec \in \{\preceq, (\preceq)^{-1}\} \) or, using the transitivity of incomparability, \( \prec \) is trivial.

Now let \( i \in \mathbb{N} \) be minimal such that \( x_1 \prec x_i \). If \( i = 2 \), Proposition 4.26 yields \( \prec = \preceq \).

Let \( j \geq 5 \) and \( f \in R_{j-1}, g \in R_j \setminus R_{j-1} \) and choose \( p \in \text{Inc}(\mathbb{N}) \) with \( p(3) = M(g) \) and \( M(f), m(g) \in p([3]_0) \). Then \( p^{-1} \cdot x_{M(f)}^{\deg(f)} \prec p^{-1} \cdot x_{m(g)}^{\deg(g)} \). So, by \( \text{Inc}(\mathbb{N}) \)-compatibility, \( x_{M(f)}^{\deg(f)} \prec x_{m(g)}^{\deg(g)} \) \( \leq_{\deg} E_1 \). We conclude that \( f \prec g \) and obtain \( \prec = \preceq \).

Now assume \( i > 2 \) and \( \prec \) to be a degree or a reverse degree order. Choose any \( j \geq i \) and monomials \( f \in R_{j-1}, g \in R_j \setminus R_{j-1} \) with \( \deg(f) = \deg(g) = : D \). Let \( p \in \text{Inc}(\mathbb{N}) \) be any function satisfying \( p(i) = M(g) \), \( M(f), m(g) \in p([i]_0) \). Then by Lemma 4.25(1), \( p^{-1} \cdot x_{M(f)}^{D-E(g)} \) and \( p^{-1} \cdot x_{m(g)}^{D-E(g)} \) are incomparable and \( p^{-1} \cdot x_{M(f)}^{E(g)} \prec p^{-1} \cdot x_{m(g)}^{E(g)} \) so \( p^{-1} \cdot x_{M(f)}^{D-E(g)} \prec p^{-1} \cdot x_{m(g)}^{D-E(g)} \). By \( \text{Inc}(\mathbb{N}) \)-compatibility, we conclude \( f \prec g \), which implies \( \prec = \preceq \).

Finally, suppose that \( \prec \) is a total order. If \( \prec \in \{\preceq, (\preceq)^{-1}\} \), let \( f, g \in R \) be any monomials satisfying \( M(g) > M(f) \) and, if \( f' \prec g' \) or \( f' \preceq g' \), \( \deg(f) = \deg(g) \). Let \( p \in \text{Inc}(\mathbb{N}) \) be such that \( M(f), m(g) \in p([3]_0) \). Then \( p^{-1} \cdot x_{M(f)}^{\deg(f)} \prec p^{-1} \cdot x_{m(g)}^{\deg(g)} \). Yielding \( f \prec g \) and, thus, \( \prec \in \{\preceq, (\preceq)^{-1}\} \).

On the other hand, if \( \prec \in \{(\preceq)^{-1}, (\preceq)^{-1}, (\preceq)^{-1}\} \), choose any monomials \( f, g \in R \) with \( M(f) < m(g) \) and, if \( f' \preceq (\preceq)^{-1} \) or \( f' \preceq (\preceq)^{-1} \), \( \deg(f) = \deg(g) \). Let \( p \in \text{Inc}(\mathbb{N}) \) be such that \( M(f), m(g) \in p([3]_\infty) \). Then \( p^{-1} \cdot x_{m(g)}^{\deg(g)} \prec p^{-1} \cdot x_{M(f)}^{\deg(f)} \). So \( f \prec g \) and, therefore, \( \prec \in \{(\preceq)^{-1}, (\preceq)^{-1}, (\preceq)^{-1}\} \). \( \square \)
Remark 4.29. Whereas by Corollary 3.18 the number of initial ideals of any \(\text{Inc}(\mathbb{N})\)-invariant ideal in \(R\) with respect to \(\text{Inc}(\mathbb{N})\)-compatible term orders is finite, this is not true for the number of initial ideals with respect to \(\text{Inc}(\mathbb{N})\)-compatible monomial preorders. Let \(J\) be as in Remark 3.20 and choose any numbers \(n < n'\). Then \(x_1 x_n^2 = \text{in}_{\text{max}, n}(x_1 x_n^2 + x_1 x_n^2) \in \text{in}_{\text{max}, n}(J)\). On the other hand, any polynomial \(f \in J\) either contains both \(x_1 x_n^2\) and \(x_1 x_n^2\) or neither of the two monomials. As \(x_1 x_n^2\) and \(x_1 x_n^2\) are incomparable with respect to \(\preceq_{\text{max}, n'}\), this implies that every element from \(\text{in}_{\text{max}, n'}(J)\) containing \(x_1 x_n^2\) must also contain \(x_1 x_n^2\) and, hence, \(x_1 x_n^2 \not\in \text{in}_{\text{max}, n'}(J)\). We conclude that the initial ideals \(\text{in}_{\text{max}, n}(J), n \in \mathbb{N}\), are pairwise distinct.

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Universität Marburg, Fachbereich Mathematik und Informatik, 35032 Marburg, Germany

E-mail address: lindner5@mathematik.uni-marburg.de