IDA AND HANKEL OPERATORS ON FOCK SPACES

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Abstract. We introduce a new space IDA of locally integrable functions whose integral distance to holomorphic functions is finite, and use it to completely characterize boundedness and compactness of Hankel operators on weighted Fock spaces. As an application, for bounded symbols, we show that the Hankel operator $H_f$ is compact if and only if $\overline{H_f}$ is compact, which complements the classical compactness result of Berger and Coburn. Motivated by recent work of Bauer, Coburn, and Hagger, we also apply our results to the Berezin-Toeplitz quantization.

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1. Introduction

Denote by $L^2$ the Hilbert space of all Gaussian square-integrable functions $f$ on $\mathbb{C}^n$, that is,

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} dv(z) < \infty$$

where $v$ is the standard Lebesgue measure on $\mathbb{C}^n$. The Fock space $F^2$ (aka Segal-Bargmann space) consists of all holomorphic functions in $L^2$. The orthogonal projection of $L^2$ onto $F^2$ is denoted by $P$ and called the Bergman projection. For a suitable function $f : \mathbb{C}^n \to \mathbb{C}$, the Hankel operator $H_f$ and the Toeplitz operator $T_f$ are defined on $F^2$ by

$$H_f = (I - P)M_f \quad \text{and} \quad T_f = PM_f.$$

The function $f$ is referred to as the symbol of $H_f$ and $T_f$. Since $P$ is a bounded operator, it follows that both $H_f$ and $T_f$ are well defined and bounded on $F^2$ if $f$ is a bounded function. For unbounded symbols, despite considerable efforts, see, e.g. [4, 9, 15, 24], characterization of boundedness or compactness of these operators has remained an open problem for more than 20 years.

In this paper, as a natural evolution from BMO (see [28, 32]), we introduce a notion of integral distance to holomorphic (aka analytic) functions IDA and use it to completely characterize boundedness and compactness of Hankel operators on Fock spaces. Recently, in [23], which continues our present work, we use IDA in the Hilbert space setting to characterize the Schatten class properties of Hankel operators. Indeed, the space IDA is broad in
scope, and should have more applications, which we hope to demonstrate in future work in connection with Toeplitz operators.

All our results are proved for weighted Fock spaces $F^p(\varphi)$ consisting of holomorphic functions for which

$$
\int_{\mathbb{C}^n} |f(z)|^p e^{-p\varphi(z)} dv(z) < \infty,
$$

where $0 < p < \infty$ and $\varphi$ is a suitable weight function (see Section 2 for further details). Obviously, with $p = 2$ and $\varphi(z) = \frac{\alpha}{2}|z|^2$, we obtain the weighted Fock space $F^2_\alpha$. The study of $L^p$-type Fock spaces was initiated in [25] and has since grown considerably as seen in [42].

We also revisit and complement a surprising result due to Berger and Coburn [8] which states that for bounded symbols $Hf : F^2 \rightarrow L^2$ is compact if and only if $\overline{Hf}$ is compact.

In particular, we give a new proof and show that this phenomenon remains true for Hankel operators from $F^p(\varphi)$ to $L^q(\varphi)$ for general weights. What also makes this result striking is that it is not true for Hankel operators acting on other important function spaces, such as Hardy or Bergman spaces.

As an application, we will apply our results to the Berezin-Toeplitz quantization, which complements the results in [4].

1.1. Main results. We introduce the following new function spaces to characterize bounded and compact Hankel operators. Let $0 < s \leq \infty$ and $0 < q < \infty$. For $f \in L^q_{\text{loc}}$, set

$$(G_{q,r}(f)) = \inf_{h \in H(B(z,r))} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f - h|^q dv \quad (z \in \mathbb{C}^n)$$

where $H(B(z,r))$ stands for the set of holomorphic functions in the ball $B(z,r)$. We say that $f \in L^q_{\text{loc}}$ is in $\text{IDA}^{s,q}$ if

$$\|f\|_{\text{IDA}^{s,q}} = \|G_{q,1}(f)\|_{L^s} < \infty.$$

We further write $\text{BDA}^q$ for $\text{IDA}^{\infty,q}$ and say that $f \in \text{VDA}^q$ if

$$\lim_{z \to \infty} G_{q,1}(f)(z) = 0.$$

The properties of these spaces will be studied in Section 3.

We denote by $\mathcal{S}$ the set of all measurable functions $f$ that satisfy the condition in (2.7), which ensures that the Hankel operator $H_f$ is densely defined on $F^p(\varphi)$ provided that $0 < p < \infty$ and $\varphi$ is a suitable weight. Notice that the symbol class $\mathcal{S}$ contains all bounded functions. Further, we write $\text{Hess}_R \varphi$ for the Hessian of $\varphi$ and $E$ for the $2n \times 2n$ identity matrix—these concepts will be discussed in more detail in Section 2. It is important to notice that the condition $\text{Hess}_R \varphi \simeq E$ in the following theorems is satisfied by the classical Fock space $F^2$, the Fock spaces $F^2_\alpha$ generated by standard
weights \( \varphi(z) = \frac{\alpha}{2} |z|^2 \) \((\alpha > 0)\), Fock-Sobolev spaces, and a large class of non-radial weights.

**Theorem 1.1.** Let \( f \in S \) and suppose that \( \text{Hess}_R \varphi \simeq E \) as in (2.1).

(a) For \( 0 < p \leq q < \infty \) and \( q \geq 1 \), \( H_f : F^p(\varphi) \to L^q(\varphi) \) is bounded if and only if \( f \in \text{BDA}^q \), and \( H_f \) is compact if and only if \( f \in \text{VDA}^q \). For the operator norm of \( H_f \), we have the estimate

\[ \|H_f\| \simeq \|f\|_{\text{BDA}^q}. \]  

(b) For \( 1 \leq q < p < \infty \), \( H_f : F^p(\varphi) \to L^q(\varphi) \) is bounded if and only if it is compact, which is equivalent to \( f \in \text{IDA}^{s,q} \), where \( s = \frac{pq}{p-q} \), and

\[ \|H_f\| \simeq \|f\|_{\text{IDA}^{s,q}}. \]  

(c) For \( 0 < p \leq q \leq 1 \) and \( f \in L^\infty \), \( H_f : F^p(\varphi) \to L^q(\varphi) \) is bounded with

\[ \|H_f\| \leq C\|f\|_{L^\infty} \]  

and compact if and only if \( f \in \text{VDA}^q \).

We first note that Theorem 1.1 is new even for Hankel operators acting from \( F^2 \) to \( L^2 \). Previously only characterizations for \( H_f \) and \( H_f \) to be simultaneously bounded (or simultaneously compact) were known. These were given in terms of the bounded (or vanishing) mean oscillation of \( f \) by Bauer [4] for \( F^2 \) and by Hu and Wang [24] for Hankel operators from \( F^p \) to \( L^q \). In Theorem 7.1 of Section 7, we obtain these results as a simple consequence of Theorem 1.1. We also mention our recent work [23], which gives a complete characterization of Schatten class Hankel operators.

Theorem 1.1 should also be compared with the results for Hankel operators on Bergman spaces \( A^p \). Indeed, characterizations for boundedness and compactness can be found in [3] for anti-analytic symbols, in [18] for bounded symbols, and in [20, 28, 30, 32] for unbounded symbols. What makes the study of the two cases different are the properties such as \( F^p \subset F^q \) for \( p \leq q \) (as opposed to \( A^q \subset A^p \)) and certain nice geometry on the boundary of these bounded domains, which in turn helps with the treatment of the \( \bar{\partial} \)-problem.

What is very different about the results on Hankel operators acting on these two types of spaces is that our next result is only true in Fock spaces (see [18] for an interesting counterexample for the Bergman space).

**Theorem 1.2.** Let \( f \in L^\infty \) and suppose that \( \text{Hess}_R \varphi \simeq E \) as in (2.1). If \( 0 < p \leq q < \infty \) or \( 1 \leq q < p < \infty \), then \( H_f : F^p(\varphi) \to L^q(\varphi) \) is compact if and only if \( H_f \) is compact.

For Hankel operators on the Fock space \( F^2 \), Theorem 1.2 was proved by Berger and Coburn [8] using C*-algebra and Hilbert space techniques and by Stroethoff [38] using elementary methods. More recently in [18], limit operator techniques were used to treat the reflexive Fock spaces \( F^p_a \). However, our result is new even in the Hilbert space case because of the more general weights that we consider. As a natural continuation of our
present work, in [23], we prove that, for \( f \in L^\infty \), the Hankel operator \( H_f \) is in the Schatten class \( S_p \) if and only if \( \overline{H}_f \) is in the Schatten class \( S_p \) provided that \( 1 < p < \infty \).

As an application and further generalization of our results, in Section 6, we provide a complete characterization of those \( f \in L^\infty \) for which

\[
\lim_{t \to 0} \| T^{(t)}_f g - T^{(t)}_f g \|_t = 0
\]

for all \( g \in L^\infty \), where \( T^{(t)}_f = P^{(t)} M_f : F^2_t(\varphi) \to F^2_t(\varphi) \) and \( P^{(t)} \) is the orthogonal projection of \( L^2_t(\varphi) \) onto \( F^2_t(\varphi) \). Here \( L^2_t = L^2(\mathbb{C}^n, d\mu_t) \) and

\[
d\mu_t(z) = \frac{1}{t^n} \exp \left\{ -2\varphi \left( \frac{z}{\sqrt{t}} \right) \right\} dv(z).
\]

The importance of the semi-classical limit in (1.4) stems from the fact that it is one of the essential ingredients of the deformation quantization of Rieffel [34, 35] in mathematical physics. Our conclusion related to (1.4) extends and complements the main result in [6].

1.2. Approach. A careful inspection shows that the methods and techniques used in [7, 8, 18, 33, 38] depend heavily upon the following three aspects. First, the explicit representation of the Bergman kernel \( K(z, w) \) for standard weights \( \varphi(z) = 2\|z\|^2 \) has the property that

\[
K(z, w) e^{-\frac{\alpha}{2}\|z\|^2 - \frac{\alpha}{2}\|w\|^2} = e^{2\frac{\alpha}{\sqrt{t}}|z-w|^2}.
\]

However, for the class of weights we consider, this quadratic decay is known not to hold (even in dimension \( n = 1 \)), and is expected to be very rare [13]. The second aspect involves the Weyl unitary operator \( W_a \) defined as

\[
W_a f = f \circ \tau_a k_a,
\]

where \( \tau_a \) is the translation by \( a \) and \( k_a \) is the normalized reproducing kernel. As a unitary operator on \( F^p_\alpha \) (or on \( L^2_\alpha \)), \( W_a \) plays a very important role in the theory of the Fock spaces \( F^p_\alpha \) (see [42]). Unfortunately, no analogue of Weyl operators is currently available for \( F^p(\varphi) \) when \( \varphi \neq \frac{\alpha}{2}|w|^2 \). The third aspect we mention is Banach (or Hilbert) space techniques, such as the adjoint (for example, \( H^*_f \)) and the duality. However, when \( 0 < p < 1 \), \( F^p(\varphi) \) is only an \( F \)-space (in the sense of [36]) and the usual Banach space techniques can no longer be applied.

To overcome the three difficulties mentioned above, we introduce function spaces IDA, BDA and VDA, and develop their theory, which we use to characterize those symbols \( f \) such that \( H_f \) are bounded (or compact) from \( F^p(\varphi) \) to \( L^q(\varphi) \). Our characterization of the boundedness of \( H_f \) extends the main results of [4, 24, 33]. It is also worth noting that as a natural generalization of BMO, the space IDA will have its own interest and will likely be useful to study other (related) operators (such as Toeplitz operators).

In our analysis, we appeal to the \( \overline{\partial} \)-techniques several times. As the canonical solution to \( \overline{\partial} u = g\partial f, \overline{H}_f g \) is naturally connected with the \( \overline{\partial} \)-theory.
Hörmander’s theory provides us with the $L^2$-estimate, but less is known about $L^p$-estimates on $\mathbb{C}^n$ when $p \neq 2$. With the help of a certain auxiliary integral operator, we obtain $L^p$-estimates of the Berndtsson-Anderson’s solution \cite{10} to the $\overline{\partial}$-equation. Our approach to handling weights whose curvature is uniformly comparable to the Euclidean metric form is similar to the treatment in \cite{37}, which was initiated by Berndtsson and Ortega-Cerdà in \cite{11}, and a number of the techniques we use here were inspired by this approach. Although the work in \cite{11} is restricted to $n = 1$, some of the results were extended by Lindholm to higher dimensions in \cite{29}, and the others are easy to modify.

The outline of the paper is as follows. In Section 2 we study preliminary results on the Bergman kernel which are needed throughout the paper, and we also establish estimates for the $\overline{\partial}$-solution developed in \cite{10}. In Section 3, a notion of function spaces $IDA^{s,q}$ is introduced. We obtain a useful decomposition for functions in $IDA^{s,q}$ (compare with the decompositions of BMO and VMO). Using this decomposition, we obtain the completeness of $IDA^{s,q}/H(\mathbb{C}^n)$ in $\| \cdot \|_{IDA^{s,q}}$. In Sections 4 and 5 we prove Theorems 1.1 and 1.2 respectively. For the latter theorem, we also appeal to the Calderón-Zygmund theory of singular integrals, and in particular employ the Ahlfors-Beurling operator to obtain certain estimates on $\partial$ and $\overline{\partial}$ derivatives. In Section 6, we present an application of our results to quantization. In the last section, we give further remarks together with two conjectures.

Throughout the paper, $C$ stands for positive constants which may change from line to line, but does not depend on functions being considered. Two quantities $A$ and $B$ are called equivalent, denoted by $A \simeq B$, if there exists some $C$ such that $C^{-1}A \leq B \leq CA$.

2. Preliminaries

Let $\mathbb{C}^n = \mathbb{R}^{2n}$ be the $n$-dimensional complex Euclidean space and denote by $v$ the Lebesgue measure on $\mathbb{C}^n$. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we write $z \cdot \overline{w} = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and $|z| = \sqrt{z \cdot \overline{z}}$. Let $H(\mathbb{C}^n)$ be the family of all holomorphic functions on $\mathbb{C}^n$. Given a domain $\Omega$ in $\mathbb{C}^n$ and a positive Borel measure $\mu$ on $\Omega$, we denote by $L^p(\Omega, d\mu)$ the space of all Lebesgue measurable functions $f$ on $\Omega$ for which

$$\|f\|_{L^p(\Omega, d\mu)} = \left\{ \int_{\Omega} |f|^p d\mu \right\}^{\frac{1}{p}} < \infty \quad \text{for} \quad 0 < p < \infty$$

and $\|f\|_{L^\infty(\Omega, dv)} = \text{ess sup}_{z \in \Omega} \|f(z)\| < \infty$ for $p = \infty$. For ease of notation, we simply write $L^p$ for the space $L^p(\mathbb{C}^n, dv)$.

2.1. Weighted Fock spaces. For a real-valued weight $\varphi \in C^2(\mathbb{C}^n)$ and $0 < p < \infty$, denote by $L^p(\varphi)$ the space $L^p(\mathbb{C}^n, e^{-p\varphi} d\mu)$ with norm $\| \cdot \|_{p, \varphi} = \| \cdot \|_{L^p(\mathbb{C}^n, e^{-p\varphi} d\mu)}$. Then the Fock space $F^p(\varphi)$ is defined as

$$F^p(\varphi) = L^p(\varphi) \cap H(\mathbb{C}^n)$$
When (2.1) is satisfied, we write \( \text{Hess} \varphi \) and \( \varphi \) the example Notice that the weights which gives the so-called Fock-Sobolev spaces studied for example in \([14]\). Formally convex weights \( \varphi \) form the \( 2 \times 2 \) identity matrix; above, for symmetric matrices \( A \) and \( B \), we used the convention that \( A \leq B \) if \( B - A \) is positive semi-definite. When (2.1) is satisfied, we write \( \text{Hess}_{\mathbb{R}} \varphi \simeq E \). A typical model of such weights is given by \( \varphi(z) = \frac{1}{2} |z|^2 \) for \( z = (z_1, z_2, \ldots, z_n) \) with \( z_j = x_{2j-1} + i x_{2j} \), which induces the weighted Fock space \( F^p_\alpha \) studied by many authors (see, e.g., \([42]\)). Another popular example is \( \varphi(z) = |z|^2 - \frac{1}{2} \log(1 + |z|^2) \), which gives the so-called Fock-Sobolev spaces studied for example in \([14]\). Notice that the weights \( \varphi \) satisfying (2.1) are not only radial functions as the example \( \varphi(z) = |z|^2 + \sin[(z_1 + \overline{z}_1)/2] \) clearly shows.

For \( x = (x_1, x_2, \ldots, x_{2n}) \), \( t = (t_1, t_2, \ldots, t_{2n}) \) \( x \in \mathbb{R}^{2n} \), write \( z_j = x_{2j-1} + i x_{2j} \), \( \xi_j = t_{2j-1} + i t_{2j} \) and \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \). An elementary calculation similar to that on page 125 of \([27]\) shows

\[
\text{Re} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z) \xi_j \xi_k + \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z) \xi_j \overline{\xi}_k = \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) t_j t_k \\
\geq \frac{1}{2} m |\xi|^2.
\]

Replacing \( \xi \) with \( i \xi \) in the above inequality gives

\[-\text{Re} \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z) \xi_j \xi_k + \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z) \xi_j \overline{\xi}_k \geq \frac{1}{2} m |\xi|^2.\]

Thus,

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z) \xi_j \overline{\xi}_k \geq \frac{1}{2} m |\xi|^2.
\]

Similarly, we have an upper bound for the complex Hessian of \( \varphi \). Therefore, \( m \omega_0 \leq dd^c \varphi \leq M \omega_0 \), where \( \omega_0 = dd^c |z|^2 \) is the Euclidean Kähler form on
\[ C^n \text{ and } d^c = \frac{1}{4}(\overline{\partial} - \partial). \] This implies that the theory in [37] and [19] is applicable in the present setting.

For \( z \in C^n \) and \( r > 0 \), let \( B(z, r) = \{ w \in C^n : |w - z| < r \} \) be the ball with center at \( z \) with radius \( r \). For the proof of the following weighted Bergman inequality, we refer to Proposition 2.3 of [37].

**Lemma 2.1.** Suppose \( 0 < p \leq \infty \). For each \( r > 0 \) there is some \( C > 0 \) such that if \( f \in F^p(\varphi) \) then

\[
|f(z)e^{-\varphi(z)}|^p \leq C \int_{B(z, r)} |f(\xi)e^{-\varphi(\xi)}|^p \, dv(\xi).
\]

It follows from the preceding lemma that \( \|f\|_{q, \varphi} \leq C\|f\|_{p, \varphi} \) and

\[
(2.2) \quad F^p(\varphi) \subseteq F^q(\varphi) \quad \text{for } 0 < p \leq q < \infty.
\]

This inclusion is completely different from that of the Bergman spaces.

**Lemma 2.2.** There exist positive constants \( \theta \) and \( C_1 \), depending only on \( n \), \( m \) and \( M \) such that

\[
|K(z, w)| \leq C_1 e^{\varphi(z) + \varphi(w)} e^{-\theta|z - w|} \quad \text{for all } z, w \in C^n,
\]

and there exists positive constants \( C_2 \) and \( r_0 \) such that

\[
|K(z, w)| \geq C_2 e^{\varphi(z) + \varphi(w)}
\]

for \( z \in C^n \) and \( w \in B(z, r_0) \).

The estimate \( (2.3) \) appeared in [13] for \( n = 1 \) and in [16] for \( n \geq 2 \), while the inequality \( (2.4) \) can be found in [37].

For \( z \in C^n \), write \( k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z, z)}} \) for the normalized Bergman kernel. Then Lemma \( 2.2 \) implies that

\[
(2.5) \quad \frac{1}{C} e^{\varphi(z)} \leq \|K(\cdot, z)\|_{p, \varphi} \leq Ce^{\varphi(z)} \quad \text{and} \quad \frac{1}{C} \leq \|k_z\|_{p, \varphi} \leq C, \quad \text{for } z \in C^n,
\]

and \( \lim_{|z| \to \infty} k_z(\xi) = 0 \) uniformly in \( \xi \) on compact subsets of \( C^n \).

### 2.2. The Bergman projection.

For Fock spaces, we denote by \( P \) the orthogonal projection of \( L^2(\varphi) \) onto \( F^2(\varphi) \), and refer to it as the Bergman projection. It is well known that \( P \) can be represented as an integral operator

\[
(2.6) \quad Pf(z) = \int_{C^n} K(z, w)f(w)e^{-2\varphi(w)} \, dv(w)
\]

for \( z \in C^n \), where \( K(\cdot, \cdot) \) is the Bergman (reproducing) kernel of \( F^2(\varphi) \).

As a consequence of Lemma \( 2.2 \) it follows that the Bergman projection \( P \) is bounded on \( L^p(\varphi) \) for \( 1 \leq p \leq \infty \), and \( P|_{F^p(\varphi)} = I \) for \( 0 < p \leq \infty \); for further details, see Proposition 3.4 and Corollary 3.7 of [37].
2.3. **Hankel operators.** To define Hankel operators with unbounded symbols, consider

\[
\Gamma = \left\{ \sum_{j=1}^{N} a_j K(\cdot, z_j) : N \in \mathbb{N}, a_j \in \mathbb{C}, z_j \in \mathbb{C}^n, \text{ for } 1 \leq j \leq N \right\},
\]

and the symbol class

\[
S = \{ f \text{ measurable on } \mathbb{C}^n : fg \in L^1(\varphi) \text{ for } g \in \Gamma \}.
\]

Given \( f \in S \), the Hankel operator \( H_f = (I - P)M_f \) with symbol \( f \) is well defined on \( \Gamma \). According to Proposition 2.5 of [22], for \( 0 < p < \infty \), the set \( \Gamma \) is dense in \( F^p(\varphi) \), and hence the Hankel operator \( H_f \) is densely defined on \( F^p(\varphi) \).

2.4. **Lattices in \( \mathbb{C}^n \).** Given \( r > 0 \), a sequence \( \{a_k\}_{k=1}^{\infty} \) in \( \mathbb{C}^n \) is called an \( r \)-lattice if the balls \( \{B(a_k, r)\}_{k=1}^{\infty} \) cover \( \mathbb{C}^n \) and \( \left\{ B\left(a_k, \frac{2r}{\sqrt{n}}\right)\right\}_{k=1}^{\infty} \) are pairwise disjoint. A typical model of an \( r \)-lattice is the sequence

\[
\left\{ \frac{r}{\sqrt{n}}(m_1 + k_1i, m_2 + k_2i, \ldots, m_n + k_ni) \in \mathbb{C}^n : m_j, k_j \in \mathbb{Z}, j = 1, 2, \ldots, n \right\}.
\]

Notice that there exists an integer \( N \) depending only on the dimension of \( \mathbb{C}^n \) such that, for any \( r \)-lattice \( \{a_k\}_{k=1}^{\infty} \),

\[
1 \leq \sum_{k=1}^{\infty} \chi_{B(a_k, 2r)}(z) \leq N
\]

for \( z \in \mathbb{C}^n \), where \( \chi_E \) is the characteristic function of \( E \subset \mathbb{C}^n \). These well known facts are explained in [42] when \( n = 1 \) and they can be easily generalized to any \( n \in \mathbb{N} \).

2.5. **Fock Carleson measures.** In the theory of Bergman spaces, Carleson measures provide an essential tool for treating various problems, especially in connection with bounded operators, functions of bounded mean oscillation, and their applications; see, e.g. [40]. In Fock spaces, Carleson measures play a similar role—see [42] for the Fock spaces \( F^p_\alpha \). Carleson measures for Fock-Sobolev spaces were described in [14]. In [37], Carleson measures for generalized Fock spaces (which include the weights considered in the present work) were used to study bounded and compact Toeplitz operators. Finally, their generalization to \((p,q)\)-Fock Carleson measures was carried out in [19], which is indispensable to the study of operators between distinct Banach spaces and will be applied to analyze Hankel operators acting from \( F^p(\varphi) \) to \( L^q(\varphi) \) in our work.

We recall the basic theory of these measures. Let \( 0 < p, q < \infty \) and let \( \mu \geq 0 \) be a positive Borel measure on \( \mathbb{C}^n \). We call \( \mu \) a \((p,q)\)-Fock Carleson measure if the embedding \( I : F^p(\varphi) \rightarrow L^q(\mathbb{C}^n, e^{-q\varphi} d\mu) \) is bounded. Further,
the measure $\mu$ is referred to as a vanishing $(p,q)$-Fock Carleson measure if in addition
\[
\lim_{j \to \infty} \int_{\mathbb{C}^n} \left| f_j(z)e^{-\varphi(z)} \right|^q d\mu(z) = 0
\]
whenever $\{f_j\}_{j=1}^\infty$ is bounded in $F^p(\varphi)$ and converges to 0 uniformly on any compact subset of $\mathbb{C}^n$ as $j \to \infty$. Fock Carleson measures have been completely characterized in [19] and we only add the following simple result, which is trivial for Banach spaces and can be easily proved in the other cases.

**Proposition 2.3.** Let $0 < p, q < \infty$ and $\mu$ be a positive Borel measure on $\mathbb{C}^n$. Then $\mu$ is a vanishing $(p,q)$-Fock Carleson measure if and only if the inclusion map $I$ is compact from $F^p(\varphi) \to L^q(\mathbb{C}^n, d\mu)$.

**Proof.** It is not difficult to show that the image of the unit ball of $F^p(\varphi)$ under the inclusion is relatively compact in $L^q(\mathbb{C}^n, e^{\varphi(z)} d\mu)$. We leave out the details. \[\Box\]

2.6. **Differential forms and an auxiliary integral operator.** As in [27], given two nonnegative integers $s, t \leq n$, we write
\[
(2.10) \quad \omega = \sum_{|\alpha| = s, |\beta| = t} \omega_{\alpha,\beta} dz^\alpha \wedge d\bar{z}^\beta
\]
for a differential form of type $(s, t)$. We denote by $L_{s,t}$ the family of all $(s, t)$-forms $\omega$ as in (2.10) with coefficients $\omega_{\alpha,\beta}$ measurable on $\mathbb{C}^n$ and set
\[
(2.11) \quad |\omega| = \sum_{|\alpha| = s, |\beta| = t} |\omega_{\alpha,\beta}| \quad \text{and} \quad \|\omega\|_{p,\varphi} = \||\omega||_{p,\varphi}.
\]

Given a weight function $\varphi$ satisfying (2.1), we define an integral operator $A_\varphi$ as
\[
(2.12) \quad A_\varphi(\omega)(z) = \int_{\mathbb{C}^n} e^{(2\partial \varphi(z) - \xi)} \times \sum_{j<n} \omega(\xi) \wedge \frac{\partial|\xi - z|^2 \wedge (2\overline{\partial} \varphi(\xi))^j \wedge (\overline{\partial} \varphi(\xi) - z)^2)^{n-1-j}}{j!|\xi - z|^{2n-2j}}
\]
for $\omega \in L_{0,1}$, where $\langle \partial \varphi(\xi), z - \xi \rangle = \sum_{j=1}^n \frac{\partial \varphi}{\partial \xi_j}(\xi_j - z_j)$ as denoted on page 92 in [10].

For $(s_1, t_1)$-form $\omega_A$ and $(s_2, t_2)$-form $\omega_B$ with $s_1 + s_2 \leq n, t_1 + t_2 \leq n$, it is easy to verify that $|\omega_A \wedge \omega_B| \leq |\omega_A| |\omega_B|$. Therefore, for the $(n, n)$-form inside the integral of the right hand side of (2.12), we obtain
\[
\left| \omega(\xi) \wedge \frac{\partial|\xi - z|^2 \wedge (2\overline{\partial} \varphi(\xi))^j \wedge (\overline{\partial} \varphi(\xi) - z)^2)^{n-1-j}}{j!|\xi - z|^{2n-2j}} \right| \leq \frac{C}{|\xi - z|^{2n-2j-1}} |\omega(\xi)|
\]
because $i\overline{\partial} \varphi(\xi) \simeq i\overline{\partial} |\xi|^2$. 

Recall that
\[
\Gamma = \left\{ \sum_{j=1}^{N} a_j K_{z_j} : N \in \mathbb{N}, a_j \in \mathbb{C}, z_j \in \mathbb{C}^n \text{ for } 1 \leq j \leq N \right\}
\]
is dense in $F^p(\varphi)$ for all $0 < p < \infty$.

**Lemma 2.4.** Suppose $1 \leq p \leq \infty$.

(A) There is a constant $C$ such that $\|A_\varphi(\omega)\|_{p,\varphi} \leq C \|\omega\|_{p,\varphi}$ for $\omega \in L_{0,1}$.

(B) For $g \in \Gamma$ and $f \in C^2(\mathbb{C}^n)$ satisfying $|\partial f| \in L^p$, it holds that $\overline{\partial} A_\varphi(g \partial f) = g\partial f$.

**Proof.** Let $z \in \mathbb{C}^n$. By (2.1), using Taylor expansion of $\varphi$ at $\xi$, we get
\[
\varphi(z) - \varphi(\xi) \geq 2\Re \sum_{j=1}^{n} \frac{\partial \varphi(\xi)}{\partial \xi_j} (z_j - \xi_j) + m|z - \xi|^2.
\]
Then (2.12) gives
\[
(2.13) \quad \left| A_\varphi(\omega)(z)e^{-\varphi(z)} \right| \leq C \int_{\mathbb{C}^n} |\omega(\xi)|e^{-\varphi(\xi)} \left\{ \frac{1}{|\xi - z|} + \frac{1}{|\xi - z|^{2n-1}} \right\} e^{-m|\xi - z|^2} dv(\xi).
\]
For $l < 2n$ fixed, define another integral operator $\mathcal{A}_l$ as
\[
\mathcal{A}_l : h \mapsto \int_{\mathbb{C}^n} h(\xi) \frac{e^{-m|\xi - z|^2}}{|\xi - z|^l} dv(\xi).
\]
It is easy to verify, by interpolation, that $\mathcal{A}_l$ is bounded on $L^p$ for $1 \leq p \leq \infty$. Therefore,
\[
\|A_\varphi(\omega)\|_{p,\varphi} \leq C \|A_1 + A_{2n-1}\|_{L^p} (|\omega|e^{-\varphi})\|_{L^p} \leq C \left( \|A_1\|_{L^p \rightarrow L^p} + A_{2n-1}\|_{L^p \rightarrow L^p} \right) \|\omega\|_{p,\varphi},
\]
which completes the proof of part (A).

Notice that the convexity assumption in (2.1) yields $dd^c \varphi \simeq \omega_0$, which in turn means that $|\partial \overline{\partial} \varphi(\xi)| \simeq 1$. We use $p'$ to denote the conjugate of $p$, $\frac{1}{p} + \frac{1}{p'} = 1$. Now, for $f \in C^2(\mathbb{C}^n)$ satisfying $|\overline{\partial} f| \in L^p$, and $z, z_0 \in \mathbb{C}^n$, we
have
\[
\int_{C^n} |K(\xi, z_0) \overline{\partial f}(\xi)| \sum_{j=0}^{n-1} \frac{e^{-\varphi(\xi)} |\overline{\partial \varphi}(\xi)|^j}{|\xi - z|^{2n-2j-1}} dv(\xi)
\]
\[
\leq C \left\{ \sup_{\xi \in B(z_1,1)} |K(\xi, z_0) \overline{\partial f}(\xi)| e^{-\varphi(\xi)} \right\}
\]
\[
+ \int_{C^n \setminus B(z_1,1)} |K(\xi, z_0) \overline{\partial f}(\xi)| e^{-\varphi(\xi)} dv(\xi)
\]
\[
\leq C e^{\varphi(z_0)} \left\{ \sup_{\xi \in B(z_1,1)} |\overline{\partial f}(\xi)| + \|\overline{\partial f}\|_{L^s} \| K(\cdot, z_0) \|_{s', \varphi} \right\} < \infty.
\]
Hence, for \( g \in \Gamma \) and \( z \in C^n \), it holds that
\[
\int_{C^n} |g(\xi) \overline{\partial f}(\xi)| \sum_{j=0}^{n-1} \frac{e^{-\varphi(\xi)} |\overline{\partial \varphi}(\xi)|^j}{|\xi - z|^{2n-2j-1}} dv(\xi) < \infty.
\]
From Proposition 10 of [10], we get (B) (pay attention to the mistake in the last line of Proposition 10 in [10] where \( f \) is left out on the right hand side). The proof is completed. \( \square \)

**Corollary 2.5.** Suppose \( f \in \mathcal{S} \cap C^1(C^n) \) and \( |\overline{\partial f}| \in L^s \) with some \( 1 \leq s \leq \infty \). For \( g \in \Gamma \), it holds that
\[
(2.14) \quad H_f(g) = A_\varphi(g \overline{\partial f}) - P(A_\varphi(g \overline{\partial f})).
\]

**Proof.** Given \( f \in \mathcal{S} \cap C^1(C^n) \) with \( |\overline{\partial f}| \in L^s \) and \( g \in \Gamma \), \( \|g \overline{\partial f}\|_{1, \varphi} \leq \|g\|_{s', \varphi} \|\overline{\partial f}\|_{L^s} < \infty \), where \( s' \) is the conjugate of \( s \). Lemma 2.4 implies that \( u = A_\varphi(g \overline{\partial f}) \in L^1(\varphi) \) and \( \overline{\partial u} = g \overline{\partial f} \). Then \( fg - u \in L^1(\varphi) \). Notice that \( \overline{\partial} (fg - u) = g \overline{\partial f} - \overline{\partial u} = 0 \), and so \( fg - u \in F^1(\varphi) \). Since \( P|_{F^1_\varphi} = I \), we have
\[
fg - u = P(fg - u) = P(fg) - P(u).
\]
This shows that \( H_f = u - P(u) \). \( \square \)

### 3. The space IDA

In this section we introduce a new space to characterize boundedness and compactness of Hankel operators. The space IDA is related to the space of bounded mean oscillation BMO (see, e.g. [26, 42]), which has played an important role in many branches of analysis and their applications for decades. We find that IDA is also broad in scope and should have more applications in operator theory and related areas.
3.1. Definitions and preliminary lemmas. Let $0 < q < \infty$ and $r > 0$. For $f \in L^q_{\text{loc}}$ (the collection of $q$-th locally Lebesgue integrable functions on $\mathbb{C}^n$), following Luecking in [30], we define $G_{q,r}(f)$ as

\begin{equation}
G_{q,r}(f)(z) = \inf \left\{ \left( \frac{1}{|B(z,r)|} \int_{B(z,r)} |f - h|^q \, dv \right)^{\frac{1}{q}} : h \in H(B(z,r)) \right\}
\end{equation}

for $z \in \mathbb{C}^n$.

**Definition 3.1.** Suppose $0 < s \leq \infty$ and $0 < q < \infty$. The space IDA$^{s,q}$ (integral distance to holomorphic functions) consists of all $f \in L^q_{\text{loc}}$ such that

$$
\|f\|_{\text{IDA}^{s,q}} = \|G_{q,1}(f)\|_{L^s} < \infty.
$$

The space IDA$^{\infty,q}$ is also denoted by BDA$^q$. The space VDA$^q$ consists of all $f \in \text{BDA}^q$ such that

$$
\lim_{z \to \infty} G_{q,1}(f)(z) = 0.
$$

We will see in Section 6 that IDA$^{s,q}$ is an extension of the space IMO$^{s,q}$ introduced in [24].

Notice that the space BDA$^2$ was first introduced in the context of the Bergman spaces of the unit disk by Luecking [30], who called it the space of functions with bounded distance to analytic functions (BDA).

**Remark 3.2.** As is the case with the classical BMO$^q$ and VMO$^q$ spaces, we have

$$
\text{BDA}^{q_2} \subset \text{BDA}^{q_1} \quad \text{and} \quad \text{VDA}^{q_2} \subset \text{VDA}^{q_1}
$$

properly for $0 < q_1 < q_2 < \infty$.

Let $0 < q < \infty$. For $z \in \mathbb{C}^n$, $f \in L^q(B(z,r), dv)$ and $r > 0$, we define the $q$-th mean of $|f|$ over $B(z,r)$ by setting

$$
M_{q,r}(f)(z) = \left( \frac{1}{|B(z,r)|} \int_{B(z,r)} |f|^q \, dv \right)^{\frac{1}{q}}.
$$

For $\omega \in L_{0,1}$, we set $M_{q,r}(\omega)(z) = M_{q,r}(|\omega|)(z)$.

**Lemma 3.3.** Suppose $0 < q < \infty$. Then for $f \in L^q_{\text{loc}}$, $z \in \mathbb{C}^n$ and $r > 0$, there is some $h \in H(B(z,r))$ such that

\begin{equation}
M_{q,r}(f - h)(z) = G_{q,r}(f)(z)
\end{equation}

and

\begin{equation}
\sup_{w \in B(z,r/2)} |h(w)| \leq C \|f\|_{L^q(B(z,r), dv)}
\end{equation}

where the constant $C$ is independent of $f$ and $r$. 
Proof. Let $f \in L^q_{\text{loc}}, z \in \mathbb{C}^n$ and $r > 0$. Taking $h = 0$ in the integrand of (3.1), we get

$$G_{q,r}(f)(z) = M_{q,r}(f)(z) < \infty.$$ 

Then for $j = 1, 2, \ldots$, we can pick $h_j \in H(B(z,r))$ such that

$$M_{q,r}(f - h_j)(z) \to G_{q,r}(f)(z)$$

as $j \to \infty$. Hence, for $j$ sufficiently large,

$$M_{q,r}(h_j)(z) \leq C \{M_{q,r}(f - h_j)(z) + M_{q,r}(f)(z)\} \leq CM_{q,r}(f)(z).$$

This shows that $\{h_j\}_{j=1}^{\infty}$ is a normal family. Thus, we can find a subsequence $\{h_{j_k}\}_{k=1}^{\infty}$ and a function $h \in H(B(z,r))$ so that $\lim_{k \to \infty} h_{j_k}(w) \to h(w)$ for $w \in B(z,r)$. By (3.4), applying Fatou’s lemma, we have

$$G_{q,r}(f)(z) \leq M_{q,r}(f - h)(z) \leq \liminf_{k \to \infty} M_{q,r}(f - h_{j_k})(z) = G_{q,r}(f)(z),$$

which proves (3.2). It remains to note that, with the plurisubharmonicity of $|h|^q$, for $w \in B(z,r/2)$, we have

$$|h(w)| \leq M_{q,r}/2(h)(w) \leq CM_{q,r}(h)(z) \leq CM_{q,r}(f)(z),$$

which completes the proof.

Corollary 3.4. For $0 < s < r$, there is a constant $C > 0$ such that for $f \in L^q_{\text{loc}}$ and $w \in B(z,r-s)$, it holds that

$$G_{q,s}(f)(w) \leq M_{q,s}(f - h)(w) \leq CG_{q,r}(f)(z),$$

where $h$ is as in Lemma 3.3.

Proof. For $0 < s < r$ and $w \in B(z,r-s), B(w,s) \subset B(z,r)$. Then, the first estimate in (3.6) comes from the definition of $G_{q,s}(f)$, while (3.2) yields

$$M_{q,s}(f - h)(w) \leq CM_{q,r}(f - h)(z) = CG_{q,r}(f)(z)$$

which completes the proof.

For $z \in \mathbb{C}^n$ and $r > 0$, let

$$A^q(B(z,r), dv) = L^q(B(z,r), dv) \cap H(B(z,r))$$

be the $q$-th Bergman space over $B(z,r)$. Denote by $P_{z,r}$ the corresponding Bergman projection induced by the Bergman kernel for $A^2(B(z,r), dv)$. It is well known that $P_{z,r}(f)$ is well defined for $f \in L^1((B(z,r), dv)$.

Lemma 3.5. Suppose $1 \leq q < \infty$ and $0 < s < r$. There is a constant $C > 0$ such that, for $f \in L^q_{\text{loc}}$ and $w \in B(z, \frac{r-s}{2})$,

$$G_{q,s}(f)(w) \leq M_{q,s}(f - P_{z,r}(f))(w) \leq CG_{q,r}(f)(z)$$

for $z \in \mathbb{C}^n$. 


Proof. We only need to prove the second inequality. Suppose $1 < q < \infty$. Notice that $P_{0,1}$ is the standard Bergman projection on the unit ball of $\mathbb{C}^n$. Theorem 2.11 of [40] implies that

$$\|P_{0,1}\|_{L^q(B(0,1),dv) \rightarrow A^q(B(0,1),dv)} < \infty.$$ 

Now for $r > 0$ fixed and $f \in L^q((B(0,r), dv)$, set $f_r(w) = f(rw)$. Then

$$\|f_r\|_{L^q(B(0,1),dv)} = r^{-2n/q}\|f\|_{L^q(B(0,1),dv)}.$$ 

Furthermore, it is easy to verify that the operator $f \mapsto P_{0,1}(f_r)\left(\frac{z}{r}\right)$ is self-adjoint and idempotent, and it maps $L^2((B(0,r), dv)$ onto $A^2((B(0,r), dv)$. Therefore,

$$P_{0,r}(f)(z) = P_{0,1}(f_r)\left(\frac{z}{r}\right) \text{ for } f \in L^q(B(0,r), dv),$$

and hence

$$\|P_{0,r}\|_{L^q(B(0,r),dv) \rightarrow A^q(B(0,r),dv)} = \|P_{0,1}\|_{L^q(B(0,1),dv) \rightarrow A^q(B(0,1),dv)}.$$ 

Now for $z \in \mathbb{C}^n$ and $r > 0$, using a suitable dilation, it follows that

$$\|P_{z,r}\|_{L^q(B(z,r),dv) \rightarrow A^q(B(z,r),dv)} = \|P_{0,1}\|_{L^q(B(0,1),dv) \rightarrow A^q(B(0,1),dv)} < \infty,$$

Unfortunately, $P_{z,r}$ is not bounded on $L^1(B(z,r), dv)$, but with the same approach as above, by Theorem 1.12 of [40] and Fubini’s theorem, we have

$$\|P_{z,r}\|_{L^1(B(z,r),dv) \rightarrow A^1(B(z,r),(r^2-|z|^2)dv)} \leq C$$

for $z \in \mathbb{C}^n$ and $r > 0$.

Choose $h$ as in Lemma [33]. Then $h \in A^q(B(z,r), dv)$ because $f \in L^q_{\text{loc}}$. Thus, $P_{z,r}(h) = h$. Now for $w \in B(z,(r-s)/2)$ and $1 \leq q < \infty$,

$$\left\{ \int_{B(w,s)} |f - P_{z,r}(f)|^q dv \right\}^{\frac{1}{q}} \leq C \left\{ \int_{B(z,(r+s)/2)} |f - P_{z,r}(f)|^q dv \right\}^{\frac{1}{q}} \leq C \left\{ \int_{B(z,r)} |f(\xi) - P_{z,r}(f(\xi))|^q (r^2 - |\xi - z|^2)dv(\xi) \right\}^{\frac{1}{q}} \leq C \left\[ \left( \int_{B(z,r)} |f - h|^q dv \right)^{\frac{1}{q}} + \left( \int_{B(z,r)} |P_{z,r}(f - h)(\xi)|^q (r^2 - |\xi - z|^2)dv(\xi) \right)^{\frac{1}{q}} \right\} \leq C \left\{ \int_{B(z,r)} |f - h|^q dv \right\}^{\frac{1}{q}}.$$
From this and Lemma 3.3, (3.7) follows. □

Given $t > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a $\frac{1}{2}$-lattice, set $J_z = \{j : z \in B(a_j, t)\}$ and denote by $|J_z|$ the cardinal number of $J_z$. By (2.9), $|J_z| = \sum_{j=1}^{\infty} \chi_{B(a_j, t)}(z) \leq N$. Choose a partition of unity $\{\psi_j\}_{j=1}^{\infty}$ subordinate to $\{B(a_j, t/2)\}$ such that

$$\text{supp } \psi_j \subset B(a_j, t/2), \quad \psi_j(z) \geq 0, \quad \sum_{j=1}^{\infty} \psi_j(z) = 1,$$

(3.11)

$$|\partial \psi_j(z)| \leq Ct^{-1}, \quad \sum_{j=1}^{\infty} |\partial \psi_j(z)| = 0.$$

Given $f \in L^q_{\text{loc}}$, for $j = 1, 2, \ldots$, pick $h_j \in H(B(a_j, t))$ as in Lemma 3.3 so that

$$M_{q,t}(f - h_j)(a_j) = G_{q,t}(f)(a_j).$$

Define

$$f_1 = \sum_{j=1}^{\infty} h_j \psi_j \quad \text{and} \quad f_2 = f - f_1.$$

(3.12)

Notice that $f_1(z)$ is a finite sum for every $z \in \mathbb{C}^n$ and hence well defined because we have $\text{supp } \psi_j \subset B(a_j, t/2) \subset B(a_j, t)$.

Inspired by a similar treatment on pages 254–255 of [30], using the partition of unity, we can prove the following estimate.

**Lemma 3.6.** Suppose $0 < q < \infty$. For $f \in L^q_{\text{loc}}$ and $t > 0$, decomposing $f = f_1 + f_2$ as in (3.12), we have $f_1 \in C^2(\mathbb{C}^n)$ and

$$|\partial f_1(z)| + M_{q,t/2}(\partial f_1)(z) + M_{q,t/2}(f_2)(z) \leq CG_{q,2t}(f)(z)$$

(3.13)

for $z \in \mathbb{C}^n$, where the constant $C$ is independent of $f$.

**Proof.** Observe first that $f_1 \in C^2(\mathbb{C}^n)$ follows directly from the properties of the functions $h_j$ and $\psi_j$. For $z \in \mathbb{C}^n$, we may assume $z \in B(a_1, t/2)$ without loss of generality. Then for those $j$ that satisfy $|\partial \psi_j(z)| \neq 0$, $|h_j - h_1|^q$ is
plurisubharmonic on $B(z, t/2) \subset B(a_j, t)$. Hence, by Corollary 3.4,

$$|\overline{\partial}f_1(z)| = \left| \sum_{j=1}^{\infty} (h_j(z) - h_1(z)) \overline{\partial} \psi_j(z) \right|$$

$$\leq \sum_{j=1}^{\infty} |h_j(z) - h_1(w)| |\overline{\partial} \psi_j(z)|$$

$$\leq C \sum_{\{j:|a_j - z|<t/2\}} M_{q,t/4}(h_j - h_1)(z)$$

$$\leq C \sum_{\{j:|a_j - z|<t/2\}} \left[ M_{q,t/4}(f - h_j)(z) + M_{q,t/4}(f - h_1)(w) \right]$$

$$\leq C \sum_{\{j:|a_j - z|<t/2\}} G_{q,t}(f)(a_j).$$

Thus, using Corollary 3.4 again, we get

$$|\overline{\partial}f_1(z)| \leq CG_{q,3t/2}(f)(z) \text{ for } z \in \mathbb{C}^n,$$ 

and so,

$$M_{q,t/2}(\overline{\partial}f_1)(z)^q \leq C \frac{1}{|B(z, t/2)|} \int_{B(z, t/2)} G_{q,3t/2}(f)(w)^q \, dw$$

$$\leq CG_{q,2t}(f)(z)^q.$$

Similarly, we have $|f_2(\xi)|^q \leq C \sum_{j=1}^{\infty} |f(\xi) - h_j(\xi)|^q \psi_j(\xi)^q$, and so

$$M_{q,t/2}(f_2)(z)^q \leq C \sum_{j=1}^{\infty} \frac{1}{|B(z, t/2)|} \int_{B(z, t/2)} |f - h_j|^q \psi_j^q \, dv$$

$$\leq C \sum_{\{j:|a_j - z|<t/2\}} G_{q,t}(f)(a_j)^q.$$

Therefore,

$$M_{q,t/2}(f_2)(z) \leq CG_{q,3t/2}(f)(z).$$

Combining this and the other two estimates above gives (3.13). □

Given $\{\psi_j\}$ as in (3.11), we have another decomposition $f = \mathfrak{F}_1 + \mathfrak{F}_2$, where

$$\mathfrak{F}_1 = \sum_{j=1}^{\infty} P_{a_j,t}(f) \psi_j \text{ and } \mathfrak{F}_2 = f - \mathfrak{F}_1.$$

When $q = 2$, the two decompositions coincide.

**Corollary 3.7.** Suppose $1 \leq q < \infty$. For $f \in L^q_{\text{loc}}$ and $t > 0$, we have $\mathfrak{F}_1 \in C^2(\mathbb{C}^n)$ and

$$|\overline{\partial}\mathfrak{F}_1(z)| + M_{q,t/2}(\overline{\partial}\mathfrak{F}_1)(z) + M_{q,t/2}(\mathfrak{F}_2)(z) \leq CG_{q,2t}(f)(z)$$

for $z \in \mathbb{C}^n$, where the constant $C$ is independent of $f$. 
Finally for \( q > 1 \) and \( p < q \), we observe that Theorem 10.3.9 of \[27\] implies that, for \( 1 < q < \infty \),

\[
\|H_\Omega(S)\|_{L^q(\Omega, dv)} \leq C\|S\|_{L^p(\Omega, dv)},
\]

where the constant \( C \) is independent of \( S \) and of "small" perturbations of the boundary. (We note that the second item in Theorem 10.3.9 of \[27\] is stated incorrectly and should read \( \|u\|_{L^q} \leq C_p\|f\|_p \) instead.) Indeed, to deduce (3.16), we consider three cases. First, for \( 1 \leq q < \frac{2n+2}{2n+4} \),

\[
\|H_\Omega(S)\|_{L^q(\Omega, dv)} \leq C\|S\|_{L^{1}(\Omega, dv)} \leq C\|S\|_{L^q(\Omega, dv)}.
\]

For \( q = \frac{2n+2}{2n+4} \), take \( 1 < p = q < 2n+2 \) and \( \frac{q_1}{q} = \frac{2n+2}{2n+4} > q \). Then \( \frac{1}{q_1} = \frac{1}{p} - \frac{1}{2n+2} \), and by the second item in Theorem 10.3.9 of \[27\], we have

\[
\|H_\Omega(S)\|_{L^{q_1}(\Omega, dv)} \leq C\|H_\Omega(S)\|_{L^q(\Omega, dv)} \leq C\|S\|_{L^p(\Omega, dv)}.
\]

Finally, for \( q > \frac{2n+2}{2n+4} \), choose \( p \) so that \( \frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2} \). Then \( 1 < p < 2n+2 \) and \( p < q \). Now Theorem 10.3.9 of \[27\] implies

\[
\|H_\Omega(S)\|_{L^q(\Omega, dv)} \leq C\|S\|_{L^p(\Omega, dv)} \leq C\|S\|_{L^q(\Omega, dv)}.
\]

**Theorem 3.8.** Suppose \( 1 \leq q < \infty \), \( 0 < s < \infty \), and \( f \in L^q_{\text{loc}} \). Then \( f \in \text{IDA}_{s,q} \) if and only if \( f \) admits a decomposition \( f = f_1 + f_2 \) such that

\[
f_1 \in C^2(\mathbb{C}^n), \quad M_{q,r}(\overline{\partial} f_1) + M_{q,r}(f_2) \in L^s
\]

for some (or any) \( r > 0 \). Furthermore, for fixed \( \tau, r > 0 \), it holds that

\[
\|f\|_{\text{IDA}_{s,q}} \simeq \|G_{q,r}(f)\|_{L^s} \simeq \inf \left\{ \|M_{q,r}(\overline{\partial} f_1)\|_{L^s} + \|M_{q,r}(f_2)\|_{L^s} \right\}
\]

where the infimum is taken over all possible decompositions \( f = f_1 + f_2 \) that satisfy (3.17) with a fixed \( r \).

**Proof.** First, given \( 0 < r < R < \infty \), we have some \( a_1, a_2, \ldots, a_m \in \mathbb{C}^n \) so that \( B(0, R) \subset \bigcup_{j=1}^m B(a_j, r) \). Then, for \( g \in L^q_{\text{loc}} \),

\[
M_{q,R}(g)(z)^s \leq C \sum_{j=1}^m M_{q,r}(g)(z + a_j)^s, \quad z \in \mathbb{C}^n,
\]

and

\[
\int_{\mathbb{C}^n} M_{q,R}(g)(z)^s dv(z) \leq C \sum_{j=1}^m \int_{\mathbb{C}^n} M_{q,r}(g)(z + a_j)^s dv(z)
\]

\[
\leq C \int_{\mathbb{C}^n} M_{q,r}(g)(z)^s dv(z).
\]

**□**
This implies that \( (3.17) \) holds for some \( r \) if and only if it holds for any \( r \).
Suppose that \( f \in L^1_{\text{loc}} \) with \( \|G_{q,\tau}(f)\|_{L^s} < \infty \) for some \( \tau > 0 \) and decompose \( f = f_1 + f_2 \) as in Lemma 3.9 with \( t = \frac{1}{t} \). Then \( f_1 \in C^2(\mathbb{C}^n) \) and
\[
|\partial f_1(z)| + M_{q,\tau/4}(\partial f_1)(z) + M_{q,\tau/4}(f_2)(z) \leq CG_{q,\tau}(f)(z).
\]
Now for any \( r > 0 \), we have
\[
(3.20) \quad \|M_{q,\tau}(\partial f_1)\|_{L^s} + \|M_{q,\tau}(f_2)\|_{L^s} \leq C\|G_{q,\tau}(f)\|_{L^s}.
\]
This implies that, \( f = f_1 + f_2 \) satisfies \( (3.17) \).
Conversely, suppose \( f = f_1 + f_2 \) with \( f_1 \in C^2(\mathbb{C}^n) \) and \( M_{q,\tau}(\partial f_1) + M_{q,\tau}(f_2) \in L^s \) for some \( \tau > 0 \) as in Theorem \( 3.8 \). Then for any \( \tau > 0 \),
\[
(3.21) \quad \|G_{q,\tau}(f_2)\|_{L^s} \leq C\|M_{q,\tau}(f_2)\|_{L^s} \leq C\|M_{q,\tau}(f_2)\|_{L^s}.
\]
So \( f_2 \in \text{IDA}^{s,q} \). To consider \( f_1 \), we write \( u = H_{B(z,2\tau)}(\partial f_1) \) for the Henkin solution of the equation \( \partial u = \partial f_1 \) on \( B(z,2\tau) \). From \( (3.16) \) and \( (3.17) \), \( u \) satisfies
\[
(3.22) \quad M_{q,2\tau}(u)(z) \leq CM_{q,2\tau}(\partial f_1)(z) \quad \text{for} \quad z \in \mathbb{C}^n,
\]
which implies that \( u \in L^q(B(z,2\tau),dv) \). Similarly to \( (3.10) \),
\[
M_{q,\tau}(P_{z,2\tau}(u))(z) \leq CM_{q,2\tau}(u)(z).
\]
Thus,
\[
(3.23) \quad M_{q,\tau}(u - P_{z,2\tau}(u))(z) \leq CM_{q,\tau}(u)(z).
\]
Since
\[
f_1 - u \in L^q(B(z,2\tau),dv) \quad \text{and} \quad \partial (f_1 - u) = 0,
\]
we have
\[
f_1 - u \in A^q(B(z,2\tau),dv).
\]
Notice that \( P_{z,2\tau}|_{A^q(B(z,2\tau),dv)} = 1 \), and so
\[
(3.24) \quad f_1(\xi) - P_{z,2\tau}(f_1)(\xi) = u(\xi) - P_{z,2\tau}(u)(\xi) \quad \text{for} \quad \xi \in B(z,2\tau).
\]
Combining \( (3.22) \), \( (3.23) \) and \( (3.24) \), we get
\[
M_{q,\tau}(f_1 - P_{z,2\tau}(f_1))(z) = M_{q,\tau}(u - P_{z,2\tau}(u))(z)
\]
\[
\leq CM_{q,2\tau}(u)(z) \leq CM_{q,2\tau}(\partial f_1)(z).
\]
Therefore, by \( (3.19) \),
\[
\|G_{q,\tau}(f_1)\|_{L^s} \leq \|M_{q,\tau}(f_1 - P_{z,2\tau}(f_1))\|_{L^s}
\]
\[
\leq C\|M_{q,2\tau}(\partial f_1)\|_{L^s} \leq C\|M_{q,\tau}(\partial f_1)\|_{L^s}.
\]
This and \( (3.21) \) yield
\[
(3.25) \quad \|G_{q,\tau}(f)\|_{L^s} \leq C \left\{ \|M_{q,\tau}(\partial f_1)\|_{L^s} + \|M_{q,\tau}(f_2)\|_{L^s} \right\}.
\]
Thus, \( f = f_1 + f_2 \in \text{IDA}^{s,q} \).
It remains to note that the norm equivalence \((3.18)\) follows from \((3.20)\) and \((3.25)\).

With a similar proof we have the following corollary.

**Corollary 3.9.** Suppose \(1 \leq q < \infty\), and \(f \in L^q_{\text{loc}}\). Then \(f \in \text{BDA}^q\) (or \(\text{VDA}^q\)) if and only if \(f = f_1 + f_2\), where
\[
 f_1 \in C^2(\mathbb{C}^n), \quad \overline{\Delta}f_1 \in L^\infty_{0,1} \quad (\text{or } \lim_{z \to \infty} |\overline{\Delta}f_1| = 0)
\]
and
\[
 M_{q,r}(f_2) \in L^\infty \quad (\text{or } \lim_{z \to \infty} M_{q,r}(f_2) = 0)
\]
for some (or any) \(r > 0\). Furthermore,
\[
 \|f\|_{\text{BDA}^q} \simeq \inf \{ \|\overline{\Delta}f_1\|_{L^\infty_{0,1}} + \|M_{q,r}(f_2)\|_{L^\infty} \},
\]
where the infimum is taken over all possible decompositions \(f = f_1 + f_2\) with \(f_1\) and \(f_2\) satisfying the conditions in \((3.26)\) and \((3.27)\).

**Corollary 3.10.** Suppose \(1 \leq q < \infty\). Different values of \(r\) give equivalent seminorms \(\|G_{q,r}(\cdot)\|_{L^s}\) on \(\text{IDA}^{s,q}\) when \(0 < s < \infty\) and on both \(\text{BDA}^q\) and \(\text{VDA}^q\) when \(s = \infty\).

**Remark 3.11.** Recall that each \(f\) in \(\text{BMO}^q\) can be decomposed as \(f = f_1 + f_2\), where \(f_1\) is of bounded oscillation \(\text{BO}\) and \(f_2\) has a bounded average \(\text{BA}^q\) (see [42] for the one-dimensional case and [31] for the general case). Furthermore, we may choose \(f_1\) to be a Lipschitz function in \(C^2(\mathbb{C}^n)\) (see Corollary 3.37 of [42]), that is, \(f \in \text{BMO}^q\) if and only if \(f = f_1 + f_2\) with all \(\frac{\partial f_1}{\partial x_j} \in L^\infty\) for \(j = 1, 2, \ldots, 2n\) and \(f_2 \in \text{BA}^q\), or in the language of complex analysis both \(\overline{\Delta}f_1\) and \(\Delta f_1\) are bounded. Therefore, \(f \in \text{BMO}^q\) if and only if \(f, f' \in \text{BDA}^q\). For a similar relationship between \(\text{IMO}^q\) and the \(\text{IDA}\) spaces, see Lemma 6.1 of [23] and Theorem 7.1 below.

### 3.3. IDA as a Banach space.
We next prove that \(\text{IDA}^{s,q}/H(\mathbb{C}^n)\) with \(1 \leq s, q < \infty\) is a Banach space when equipped with the induced norm
\[
 (3.28) \quad \|f + H(\mathbb{C}^n)\| = \|f\|_{\text{IDA}^{s,q}}
\]
for \(f \in \text{IDA}^{s,q}\).

**Theorem 3.12.** For \(1 \leq s, q < \infty\), the quotient space \(\text{IDA}^{s,q}/H(\mathbb{C}^n)\) is a Banach space with the norm induced by \(\|\cdot\|_{\text{IDA}^{s,q}}\).

**Proof.** Obviously \(H(\mathbb{C}^n) \subset \text{IDA}^{s,q}\). Now given \(f \in \text{IDA}^{s,q}\) and \(h \in H(\mathbb{C}^n)\), \(G_{q,r}(f) = G_{q,r}(f + h)\). This means that the norm in \((3.28)\) is well defined on \(\text{IDA}^{s,q}/H(\mathbb{C}^n)\). If \(\|f\|_{\text{IDA}^{s,q}} = 0\), then \(G_{q,r}(f)(z) = 0\) in \(\mathbb{C}^n\). By Lemma 3.3 \(f \in H(B(z, r))\) and hence \(f \in H(\mathbb{C}^n)\).

Let \(f_1, f_2 \in \text{IDA}^{s,q}\) and \(z \in \mathbb{C}^n\). According to Lemma 3.3 there are functions \(h_j\) holomorphic in \(B(z, r)\) such that
\[
 M_{q,r}(f_j - h_j)(z) = G_{q,r}(f_j)(z) \quad \text{for } j = 1, 2.
\]
Then, since
\[ M_{q,r} ((f_1 + f_2) - (h_1 + h_2))(z) \leq M_{q,r} (f_1 - h_1)(z) + M_{q,r} (f_2 - h_2)(z), \]
we have
\[ G_{q,r}(f_1 + f_2)(z) \leq G_{q,r}(f_1)(z) + G_{q,r}(f_2)(z) \quad \text{for} \quad z \in \mathbb{C}^n. \]
Hence, \( \| f_1 + f_2 \|_{IDA^{s,q}} \leq \| f_1 \|_{IDA^{s,q}} + \| f_2 \|_{IDA^{s,q}} \). In addition, \( \| f \|_{IDA^{s,q}} \geq 0 \) and \( \| a f \|_{IDA^{s,q}} = |a| \| f \|_{IDA^{s,q}} \) for \( a \in \mathbb{C} \). Therefore, \( \cdot \|_{IDA^{s,q}} \) induces a norm on \( IDA^{s,q}/H(\mathbb{C}^n) \).

It remains to prove that the norm is complete. Suppose that \( \{ f_m \}_{m=1}^\infty \) is a Cauchy sequence in
\[ \| \cdot \|_{IDA^{s,q}} = \| G_{q,1}(\cdot) \|_{L^s}. \]
According to Corollary 3.10, we may assume that \( \{ f_m \}_{m=1}^\infty \) is a Cauchy sequence in \( \| G_{q,r}(\cdot) \|_{L^s} \) with \( r > 0 \) fixed. We now embark on proving that, for some \( f \in IDA^{s,q} \), \( \lim_{m \to \infty} \| G_{q,r/2}(f_m - f) \|_{L^s} = 0 \), which implies \( \{ f_m \}_{m=1}^\infty \) converges to some \( f \in IDA^{s,q} \) in \( \| \cdot \|_{IDA^{s,q}} \)-topology. For this purpose, let \( \{ a_j \}_{j=1}^\infty \) be some \( t = r/4 \)-lattice. We decompose each \( f_m \) similarly to (3.14) as follows
\[ f_{m,1} = \sum_{j=1}^\infty P_{a,j}(f_m) \psi_j \quad \text{and} \quad f_{m,2} = f_m - f_{m,1}, \]
where \( \{ \psi_j \}_{j=1}^\infty \) is the partition of unity subordinate to \( \{ B(a_j, r/4) \}_{j=1}^\infty \) as in (3.11). It follows from Corollary 3.7 that
\[ M_{q,r/8}(f_{m,2} - f_{k,2})(z)^s = M_{q,r/8} \left( (f_m - f_k) - \sum_{j=1}^\infty P_{a,j}(f_m - f_k) \psi_j \right)(z)^s \]
\[ \leq C G_{q,r/2}(f_m - f_k)(z)^s \]
\[ \leq C \int_{B(z,r/2)} G_{q,r}(f_m - f_k)(\xi)^s d\nu(\xi). \]
This implies that \( \{ f_{m,2} \}_{j=1}^\infty \) converges to some function \( f_2 \) in \( L^q_{\text{loc}} \)-topology. In addition, by Lemma 3.3, we have
\[ M_{q,r/2} (f_{m,2} - f_{k,2} - P_{z,r}(f_{m,2} - f_{k,2}))(z) \leq C G_{q,r}(f_{m,2} - f_{k,2})(z). \]
Letting \( k \to \infty \) and applying Fatou’s Lemma, we get
\[ G_{q,r/2}(f_{m,2} - f_2)(z)^s \leq M_{q,r/2} (f_{m,2} - f_2 - P_{z,r}(f_{m,2} - f_2))(z)^s \]
\[ \leq C \liminf_{k \to \infty} G_{q,r}(f_{m,2} - f_{k,2})(z)^s. \]
Integrate both sides over \( \mathbb{C}^n \) and apply Fatou’s lemma again to obtain the estimate
\[ \int_{\mathbb{C}^n} G_{q,r/2}(f_{m,2} - f_2)^s d\nu \leq C \liminf_{k \to \infty} \| f_{m,2} - f_{k,2} \|_{IDA^{s,q}}. \]
Therefore,
\begin{equation}
\lim_{m \to \infty} \|f_{m,2} - f_2\|_{\text{IDA}^s} = 0.
\end{equation}

Next we consider \(\{f_{m,1}\}_{m=1}^{\infty}\). Applying the estimate (3.15) to \(f_m - f_k\),
\begin{equation}
|\overline{\partial}(f_{m,1} - f_{k,1})(z)| \leq CG_{q,r/2} (f_m - f_k)(z).
\end{equation}
Hence, \(\{\overline{\partial}f_{m,1}\}_{m=1}^{\infty}\) is a Cauchy sequence in \(L^q_{0,1}\) (see (2.11)). We may assume \(\overline{\partial}f_{m,1} \to S = \sum_{j=1}^{n} S_j \overline{\partial}z_j\) under the \(L^q_{0,1}\)-norm. Since \(\overline{\partial}^2 = 0\), \(\overline{\partial}f_{m,1}\) is trivially \(\overline{\partial}\)-closed, and so, as the \(L^q_{0,1}\) limit of \(\{\overline{\partial}f_{m,1}\}_{m=1}^{\infty}\), \(S\) is also \(\overline{\partial}\)-closed weakly. Let \(\phi(z) = \frac{1}{2} |z|^2\) and \(g = 1 \in \Gamma\), and define
\[f_1(z) = A_\phi(S),\] and \(f^*_1 = A_\phi(\overline{\partial}f_{m,1})\).

Then, by Lemma 2.4,
\[f_1, f^*_1 \in L^q (\phi) \subset L^q_{\text{loc}}, \quad \overline{\partial}f^*_1 = \overline{\partial}f_{m,1},\]
and \(\{f^*_1\}_{m=1}^{\infty}\) converges to \(f_1\) in \(L^q (\phi)\). Therefore, for \(\psi \in C_c^\infty(\mathbb{C}^n)\) (the family of all \(C^\infty\) functions with compact support) and \(j = 1, 2, \cdots, n\), it holds that
\[-\left\langle f_1, \frac{\partial \psi}{\partial z_j} \right\rangle_{L^2} = \lim_{m \to \infty} \left\langle f^*_1, \frac{\partial \psi}{\partial z_j} \right\rangle_{L^2} = \lim_{m \to \infty} \left\langle \overline{\partial}f^*_1, \psi \right\rangle_{L^2} = \left\langle S_j, \psi \right\rangle_{L^2}.
\]
Hence, \(\overline{\partial}f_1 = S\) weakly. Then for \(H_{B(z,r)}(\overline{\partial}f_{m,1} - S)\), the Henkin solution to the equation \(\overline{\partial}u = \overline{\partial}f_{m,1} - S\) on \(B(z,r)\), (3.16) gives
\begin{equation}
\|H_{B(z,r)}(\overline{\partial}f_{m,1} - S)\|_{L^q(B(z,r), dv)} \leq C\|\overline{\partial}f_{m,1} - S\|_{L^q(B(z,r), dv)}.
\end{equation}
In addition, according to (3.24), it holds that
\[(f_{m,1} - f_1) - P_{z,r}(f_{m,1} - f_1) = H_{B(z,r)}(\overline{\partial}f_{m,1} - S) - P_{z,r}(H_{B(z,r)}(\overline{\partial}f_{m,1} - S))\]
on \(B(z,r)\). Therefore, by (3.24), (3.9), and (3.31) we have
\begin{equation}
\|(f_{m,1} - f_1) - P_{z,r}(f_{m,1} - f_1)\|_{L^q(B(z,r/2), dv)}^q = \|H_{B(z,r)}(\overline{\partial}f_{m,1} - S) - P_{z,r}(H_{B(z,r)}(\overline{\partial}f_{m,1} - S))\|_{L^q(B(z,r/2), dv)}^q \\
\leq C\|H_{B(z,r)}(\overline{\partial}f_{m,1} - S)\|_{L^q(B(z,r), dv)}^q \\
\leq C\|\overline{\partial}f_{m,1} - S\|_{L^q(B(z,r), dv)}^q.
\end{equation}
Since \(S = \lim_{k \to \infty} \overline{\partial}f_{k,1}\) in \(L^q_{0,1}\), by Fatou’s lemma,
\begin{equation}
\|\overline{\partial}f_{m,1} - S\|_{L^q(B(z,r), dv)}^q \leq C \lim_{k \to \infty} \|\overline{\partial}(f_{m,1} - f_{k,1})\|_{L^q(B(z,r), dv)}^q \\
\leq C \lim_{k \to \infty} G_{q,2r}(f_{m,1} - f_{k,1})(z)^0,
\end{equation}
A linear operator $T$ from $X$ to $Y$ is bounded (or compact) if $T(B)$ is bounded (or relatively compact) in $Y$. The collection of all bounded (and compact) operators from $X$ to $Y$ is denoted by $B(X,Y)$ (and by $K(X,Y)$ respectively). We use $\parallel \cdot \parallel_{BDA}$ to denote the corresponding operator norm.

Banach space in $\parallel \cdot \parallel$ is a closed subspace of $\text{BDA}^q$. The proof of Theorem 3.12 works for $BDA^q$. Therefore, $\lim_{m \to \infty} \parallel f_m \parallel_{\text{BDA}^q} = 0$. Set $f = f_1 + f_2 \in L^q_{\text{loc}}$. From (3.29) and (3.31) it follows that

$$\lim_{m \to \infty} \parallel f_m \parallel_{\text{IDAA}^q} \leq \lim_{m \to \infty} (\parallel f_m \parallel_{\text{IDAA}^q} + \parallel f_m - f \parallel_{\text{IDAA}^q}) = 0,$$

which completes the proof of the completeness and of the theorem. \hfill \Box

**Corollary 3.13.** Let $1 \leq q < \infty$. With the norm induced by $\parallel \cdot \parallel_{\text{BDA}^q}$, the quotient space $\text{BDA}^q / H(\mathbb{C}^n)$ is a Banach space and $\text{VDA}^q$ is a closed subspace of $\text{BDA}^q$.

**Proof.** The proof of Theorem 3.12 works for $s = \infty$, so $\text{BDA}^q / H(\mathbb{C}^n)$ is a Banach space in $\parallel \cdot \parallel_{\text{BDA}^q}$. That $\text{VDA}^q$ is a closed subspace of $\text{BDA}^q$ can be proved in a standard way. \hfill \Box

## 4. Proof of Theorem 1.1

Given two $F$-spaces $X$ and $Y$, we write $B(X)$ for the unit ball of $X$. A linear operator $T$ from $X$ to $Y$ is bounded (or compact) if $T(B(X))$ is bounded (or relatively compact) in $Y$. The collection of all bounded (and compact) operators from $X$ to $Y$ is denoted by $B(X,Y)$ (and by $K(X,Y)$ respectively). We use $\parallel T \parallel_{X \to Y}$ to denote the corresponding operator norm. In particular, we recall that when $0 < p < 1$, the Fock space $F^p(\varphi)$ with the metric given by $d(f,g) = \parallel f - g \parallel^p_{p,\varphi}$ is an $F$-space.

To deal with the boundedness and compactness of Hankel operators, we need an additional result involving positive measures and their averages. More precisely, given a positive Borel measure $\mu$ on $\mathbb{C}^n$ and $r > 0$, we write $\hat{\mu}_r(z) = \mu(B(z,r))$. Notice, in particular, $\hat{\mu}_r$ is a constant multiple of the averaging function induced by the measure $\mu$.

**Lemma 4.1.** Suppose $0 < p \leq 1$ and $r > 0$. There is a constant $C$ such that, for $\mu$ a positive Borel measure on $\mathbb{C}^n$, $\Omega$ a domain in $\mathbb{C}^n$, and $g \in H(\mathbb{C}^n)$,
it holds that
\[
\left( \int_{\Omega} |g(\xi)e^{-\varphi(\xi)}| \, d\mu(\xi) \right)^p \leq C \int_{\Omega^+} |g(\xi)e^{-\varphi(\xi)}|^p \, \tilde{\mu}_r(\xi) \, dv(\xi),
\]
where \( \Omega^+_r = \bigcup_{\{z \in \Omega\}} B(z, r) \).

Proof. Let \( \{a_j\}_{j=1}^{\infty} \) be an \( r/4 \)-lattice. Notice that
\[
\tilde{\mu}_{r/4}(a_j) \leq C \inf_{w \in B(a_j, r/2)} \tilde{\mu}_r(w)
\]
for all \( j \in \mathbb{N} \) and \( (a + b)^p \leq a^p + b^p \) for \( a, b \geq 0 \). Then
\[
\left( \int_{\Omega} |g(\xi)e^{-\varphi(\xi)}| \, d\mu(\xi) \right)^p \leq \sum_{j=1}^{\infty} \left( \int_{B(a_j, r/4) \cap \Omega} |g(\xi)e^{-\varphi(\xi)}| \, d\mu(\xi) \right)^p \leq C \sum_{\{j: B(a_j, r/4) \cap \Omega \neq \emptyset\}} \sup_{x \in B(a_j, r/4) \cap \Omega} \left| g(\xi)e^{-\varphi(\xi)} \right|^p \tilde{\mu}_{r/4}(a_j)^p \leq C \sum_{\{j: B(a_j, r/4) \cap \Omega \neq \emptyset\}} \int_{B(a_j, r/2)} \left| g(\xi)e^{-\varphi(\xi)} \right|^p \, dv(\xi) \leq C \int_{\Omega^+} \left| g(\xi)e^{-\varphi(\xi)} \right|^p \, \tilde{\mu}_r(\xi) \, dv(\xi),
\]
which completes the proof. \( \square \)

Remark 4.2. To prove compactness of Hankel operators on spaces that are not necessarily Banach spaces, we use the following result. For \( 0 < p, q < \infty \), \( H_f : F^p(\varphi) \rightarrow L^q(\varphi) \) is compact if and only if
\[
\lim_{m \to \infty} \| H_f(g_m) \|_{q, \varphi} = 0
\]
for any sequences \( \{g_m\}_{m=1}^{\infty} \) in \( B(F^p(\varphi)) \) satisfying
\[
\lim_{m \to \infty} \sup_{w \in E} |g_m(w)| = 0
\]
for compact subsets \( E \) in \( \mathbb{C}^n \).

Necessity is trivial. To prove sufficiency, we notice that \( B(F^p(\varphi)) \) is a normal family, so for any sequence \( \{g_m\}_{m=1}^{\infty} \subset B(F^p(\varphi)) \), there exist a holomorphic function \( g_0 \) on \( \mathbb{C}^n \) and a subsequence \( \{g_{m_j}\}_{j=1}^{\infty} \) such that
\[
\lim_{j \to \infty} \sup_{w \in E} |g_{m_j}(w) - g_0(w)| = 0.
\]
This and Fatou’s Lemma imply that \( g_0 \in B(F^p(\varphi)) \), and hence by the hypothesis, we get
\[
\lim_{j \to \infty} \| H_f(g_{m_j}) - H_f(g_0) \|_{q,\varphi} = \lim_{j \to \infty} \| H_f(g_{m_j} - g_0) \|_{q,\varphi} = 0.
\]
Therefore, \( H_f(B(F^p(\varphi))) \) is sequential compact in \( L^q(\varphi) \), that is, the Hankel operator \( H_f : F^p(\varphi) \to L^q(\varphi) \) is compact.

4.1. The case \( 0 < p \leq q < \infty \) and \( q \geq 1 \).

**Proof of Theorem 1.1 (a).** By (2.3) – (2.5),
\[
\| k_z \|_{p,\varphi} \leq C, \quad \sup_{\zeta \in B(z,r)} |k_z(\zeta)| e^{-\varphi(\zeta)} \geq C \quad \text{and} \quad \lim_{z \to \infty} \sup_{w \in E} |k_z(w)| = 0
\]
for any compact subset \( E \subset \mathbb{C}^n \). As in the proof of Theorem 4.2 of [20], there is an \( r_0 \) such that for all \( z \in \mathbb{C}^n \), we have
\[
\| H_f(z) \|_{q,\varphi}^q \geq \int_{B(z,r_0)} |f k_z - P(f k_z)|^q e^{-q\varphi} \, dv \\
\geq C \frac{1}{|B(z,r_0)|} \int_{B(z,r_0)} \left| f - \frac{1}{k_z} P(f k_z) \right|^q \, dv \geq C G_{q,r_0}(f)(z).
\]
If \( H_f \in B(F^p(\varphi), L^q(\varphi)) \),
\[
\| f \|_{BDA^q} \leq C \| H_f \|_{F^p(\varphi) \to L^q(\varphi)} < \infty;
\]
if \( H_f \in K(F^p(\varphi), L^q(\varphi)) \), then \( f \in VDA^q \) because
\[
\lim_{z \to \infty} G_{q,r_0}(f)(z) \leq C \lim_{z \to \infty} \| H_f(k_z) \|_{q,\varphi} = 0.
\]
Next we prove sufficiency. Suppose that \( f \in BDA^q \) and decompose \( f = f_1 + f_2 \) as in (3.12). Write \( d\mu = |f_2|^q \, dv \) and \( d\nu = |\partial f_1|^q \, dv \). According to Theorem 2.6 of [19] and Corollary 3.9, both \( d\mu \) and \( d\nu \) are \( (p,q) \)-Fock Carleson measures. We claim that both \( f_1, f_2 \in \mathcal{S} \). Indeed, since \( q \geq 1 \), we can use Lemma 4.1 with \( \Omega = \mathbb{C}^n \) and the measure \( |f_2| \, dv \) to get
\[
\int_{\mathbb{C}^n} |f_2(\xi)| K(\xi, z) e^{-\varphi(\xi)} \, dv(\xi) \leq C \int_{\mathbb{C}^n} M_{1,p}(f_2)(\xi) |K(\xi, z)| e^{-\varphi(\xi)} \, dv(\xi) \\
\leq C \int_{\mathbb{C}^n} M_{q,r}(f_2)(\xi) |K(\xi, z)| e^{-\varphi(\xi)} \, dv(\xi).
\]
Since \( f \in BDA^q \), Lemma 3.6 implies that
\[
\int_{\mathbb{C}^n} |f_2(\xi)| K(\xi, z) e^{-\varphi(\xi)} \, dv(\xi) \leq C \| f \|_{BDA^q} \int_{\mathbb{C}^n} |K(\xi, z)| e^{-\varphi(\xi)} \, dv(\xi) < \infty
\]
for \( z \in \mathbb{C}^n \). Hence, \( f_2 \in \mathcal{S} \), and so also \( f_1 = f - f_2 \in \mathcal{S} \) because \( f \in \mathcal{S} \) by the hypothesis. Since the Bergman projection \( P \) is bounded on \( L^q(\varphi) \) when
$q \geq 1$, we have for $g \in \Gamma$,
\[
\|H_{f_2}(g)\|_{q,\varphi} \leq (1 + \|P\|_{L^q(\varphi) \to F^q(\varphi)})\|f_2g\|_{q,\varphi} \\
\leq C\|M_{q,r}(f_2)\|_{L^\infty}\|g\|_{q,\varphi} \leq C\|M_{q,r}(f_2)\|_{L^\infty}\|g\|_{p,\varphi},
\]
where the second inequality follows from Lemma 4.1. For $H_{f_1}(g)$ with $g \in \Gamma$, Corollary 2.5 shows that $H_{f_1}(g) = A_\varphi(g\overline{f_1}) - P(A_\varphi(g\overline{f_1}))$. Lemma 2.4 implies
\[
(4.6) \quad \|H_{f_1}(g)\|_{q,\varphi} \leq C\|g\|_{q,\varphi} \leq C\|\overline{f_1}\|_{L^\infty}\|g\|_{q,\varphi} \leq C\|\overline{f_1}\|_{L^\infty}\|g\|_{p,\varphi}.
\]
From the above estimates and the fact that $\Gamma$ is dense in $F^p(\varphi)$, it follows that for $0 < p < q < \infty$, we have
\[
(4.7) \quad \|H_f\|_{F^p(\varphi) \to L^q(\varphi)} \leq C\{\|\overline{f_1}\|_{L^\infty} + \|M_{q,r}(f_2)\|_{L^\infty}\} \leq C\|f\|_{\text{VDA}}^q
\]
where the latter inequality follows from Lemma 3.6.

For compactness, suppose $f \in \text{VDA}^q$ so that $f = f_1 + f_2$ as in (3.12). Notice that both $d\mu = |f_2|^q d\nu$ and $d\nu = |\overline{f_1}|^q d\nu$ are vanishing $(p,q)$-Fock Carleson measures. Let $\{g_m\}$ be a bounded sequence in $F^p(\varphi)$ converging to zero uniformly on compact subsets of $\mathbb{C}^n$. Then
\[
\|H_{f_2}(g_m)\|_{L^q(\varphi)} \leq \|g_m f_2\|_{q,\varphi} + \|P(g_m f_2)\|_{q,\varphi} \\
\leq C\left(\int_{\mathbb{C}} |g_m e^{-\varphi}|^q d\mu\right)^{\frac{1}{q}} \to 0
\]
as $m \to \infty$. To prove $\lim_{m \to \infty} \|H_{f_1}(g_m)\|_{L^q(\varphi)} = 0$, for each $m$ we pick some $g^*_m \in \Gamma$ so that $\|g_m - g_m^*\|_{p,\varphi} < \frac{1}{m}$. Clearly, $\{g_m^*\}_{m=1}^\infty$ is bounded in $F^p(\varphi)$, and $\lim_{m \to \infty} \sup_{w \in E} \|g_m^*(w)\| = 0$ for any compact subset $E$. Again by Corollary 2.5,
\[
\|H_{f_1}(g_m)\|_{L^q(\varphi)} \leq C\|g_m^* f_1\|_{L^q(\varphi)} \leq C\|g_m^*\|_{L^q(\mathbb{C}^n,d\nu)} \to 0 \text{ as } m \to \infty.
\]
Therefore, since (3.6) guarantees that $H_{f_1} \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$, it follows that $\lim_{m \to \infty} \|H_{f_1}(g_m)\|_{L^q(\varphi)} = 0$, and so
\[
H_f = H_{f_1} + H_{f_2} \in \mathcal{K}(F^p(\varphi), L^q(\varphi)).
\]
Finally, it remains to notice that the norm equivalence (1.1) follows from (4.3) and (4.7). \hfill \Box

4.2. The case $1 \leq q < p < \infty$. We can now prove the case $q < p$ under the assumption that $q \geq 1$.

Proof of Theorem 1.1 (b). Suppose that $H_f \in \mathcal{B}(F^p(\varphi), L^q(\varphi))$. Because the proof of sufficiency is similar to the implication $(A) \Rightarrow (C)$ of Theorem 4.4 in [20], we only give the sketch here.

Indeed, take $r_0$ as in (4.1), and set $t = r_0/4$. Let $\{a_j\}_{j=1}^\infty$ be a $t/2$-lattice.

By Lemma 2.4 of [19], $\left\|\sum_{j=1}^\infty \lambda_j k_{a_j}\right\|_{p,\varphi} \leq C\|\lambda_j\|_{p}$ for all $\{\lambda_j\}_{j=1}^\infty \in \ell^p$, where the constant $C$ is independent of $\{\lambda_j\}_{j=1}^\infty$. Let $\{\phi_j\}_{j=1}^\infty$ be the sequence...
Rademacher functions on the interval \([0, 1]\). Using the boundedness of \(H_f\), we get
\[
(4.8) \quad \left\| H_f \left( \sum_{j=1}^{\infty} \lambda_j \phi_j(s)k_{a_j}(\cdot) \right) \right\|_{q,\varphi} \leq C \|H_f\|_{F_p(\varphi) \to L_q(\varphi)} \|\{\lambda_j\|^q\|_I^{\frac{1}{q}}}
\]
for \(s \in \left(0, 1\right]\). On the other hand,
\[
(4.9) \quad \int_{B(a_j, t)} H_f(k_z)(\xi) e^{-\varphi(\xi)} d\xi \geq C G_{q,t}(f)(a_j)^q.
\]
This and Khintchine’s inequality yield
\[
\int_0^1 H_f \left( \sum_{j=1}^{\infty} \lambda_j \phi_j(s)k_{a_j}(\cdot) \right) \right\|_{q,\varphi}^q dt \geq C \sum_{j=1}^{\infty} |\lambda_j|^q G_{q,t}(f)(a_j)^q.
\]
Combining this with (4.8) gives
\[
\sum_{j=1}^{\infty} |\lambda_j|^q G_{q,t}(f)(a_j)^q \leq C \|H_f\|_{F_p(\varphi) \to L_q(\varphi)} \|\{\lambda_j\|^q\|_I^{\frac{1}{q}}}
\]
for all \(\{\lambda_j\}^\infty_{j=1} \in l^2_t\). By duality with the exponentials \(\frac{p}{q}\) and its conjugate,
\[
\sum_{j=1}^{\infty} G_{q,t}(f)(a_j) \frac{p}{p-q} \leq C \|H_f\|_{F_p(\varphi) \to L_q(\varphi)} \|\frac{p}{p-q}\|
\]
Therefore, by (3.7),
\[
(4.10) \quad \int_{C^2} G_{q,t}(f)(z) \frac{p}{p-q} d\nu(z) \leq \sum_{j=1}^{\infty} \int_{B(a_j, t/2)} G_{q,t}(f)(z) \frac{p}{p-q} d\nu(z) \leq C \|H_f\|_{F_p(\varphi) \to L_q(\varphi)},
\]
which means that \(f \in \text{IDA}^{s,q}\) with the estimate \(\|f\|_{\text{IDA}^{s,q}} \leq C \|H_f\|\).

It should be pointed out that the right hand side of the estimate (4.24) (the analog of (4.10) above) in [20] should read \(C \|H_f\|_{A_p^{\alpha,q} \to L_q}\), and not \(C \|H_f\|_{A_p^{\alpha,q} \to L_q}\) as stated there.

Conversely, suppose \(f \in \text{IDA}^{s,q}\). As before, decompose \(f = f_1 + f_2\) as in (3.12). From Lemma 3.6 we know that \(\|M_{q,r}(f_1)\|_{p\alpha} \leq C \|f\|_{\text{IDA}^{s,q}}\).

Applying Hölder’s inequality to the right hand side integral in (4.5) with exponent \(\frac{pq}{p-q}\) and its conjugate exponent \(t\), since we have \(\|K(\cdot, z)\|_{t,\varphi} < \infty\), it follows that
\[
\int_{C^2} |f_2(\xi)K_z(\xi)| e^{-\varphi(\xi)} d\nu(\xi) \leq C \|M_{q,r}(f_2)\|_{p\alpha} \cdot |K_z|_{t,\varphi} < \infty.
\]
This implies \(f_2 \in S\), and so also \(f_1 \in S\).
Now for $d\nu = |\bar{\partial}f_1|^q dv$, applying Hölder’s inequality again with $\frac{p}{p-q}$ and its conjugate exponent $p/q$, we get

$$\|\tilde{\nu}\|_{L^{\frac{p}{p-q}}} = C \int_{\mathbb{C}^n} \left\{ \int_{B(\xi,r)} |\bar{\partial}f_1(\zeta)|^q dv(\zeta) \right\}^{\frac{p}{p-q}} dv(\xi)$$

(4.11)

$$\leq C \int_{\mathbb{C}^n} dv(\xi) \int_{B(\xi,r)} |\bar{\partial}f_1(\zeta)|^q dv(\zeta)$$

$$\simeq C \int_{\mathbb{C}^n} |\bar{\partial}f_1(\zeta)|^\frac{pq}{p-q} dv(\zeta) < \infty.$$ 

Theorem 2.8 of [19] shows that $\nu$ is a vanishing $(p,q)$-Fock Carleson measure, that is, the multiplier $M_{f_1} : g \mapsto g|\bar{\partial}f_1|$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$ (see Proposition 2.3). Therefore, by Lemma 3.6, $A_{\varphi}(\cdot|\bar{\partial}f_1)$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$. Moreover, $\Gamma$ is dense in $F^p(\varphi)$ and, by Corollary 2.3, $H_{f_1}(g) = A_{\varphi}(g|\bar{\partial}f_1) - P \circ A_{\varphi}(g|\bar{\partial}f_1)$ for $g \in \Gamma$. Hence, $H_{f_1} : F^p(\varphi) \to L^q(\varphi)$ is compact and we obtain the norm estimate

$$\|H_{f_1}\|_{F^p(\varphi) \to L^q(\varphi)} \leq C \sup_{\{g \in F^p(\varphi) : \|g\|_{p,\varphi} \leq 1\}} \|A_{\varphi}(g|\bar{\partial}f_1)\|_{q,\varphi} \leq C\|\bar{\partial}f_1\|_{\frac{pq}{p-q}}.$$ 

Similarly to (4.11), using Lemma 3.6 for $d\mu = |f_2|^q dv$, we get

$$\|\tilde{\mu}\|_{L^{\frac{p}{p-q}}} = C \int_{\mathbb{C}^n} \left\{ \int_{B(\xi,r)} |f_2(\zeta)|^q dv(\zeta) \right\}^{\frac{p}{p-q}} dv(\xi)$$

$$= C \|M_{\eta,\zeta}(f_2)\|_{\frac{pq}{p-q}} \leq C \|f\|^q_{IDA^{p,q}} < \infty.$$ 

Hence, $d\mu = |f_2|^q dv$ is a vanishing $(p,q)$-Fock Carleson measure. It follows from Proposition 2.3 that the identity operator

$$I : F^p(\varphi) \to L^q(\varbb{C}^n, e^{-q\varphi} d\mu)$$

is compact. Using the inequality

(4.13) $\|H_{f_2}(g)\|_{q,\varphi} \leq C\|f_2g\|_{q,\varphi} = C\|I(g)\|_{L^q(\varbb{C}^n, e^{-q\varphi} d\mu)},$

we see that $H_{f_2}$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$.

It remains to notice that the norm equivalence in (1.2) follows from combining the estimates in (4.10), (4.12), and (4.13).

**Remark 4.3.** In [38], it was proved that for bounded symbols $f$, the Hankel operator $H_f : F^2 \to L^2$ is compact if and only if

(4.14) $\|(I-P)(f \circ \phi_\lambda)\| \to 0$

as $|\lambda| \to \infty$, where $\phi_\lambda(z) = z + \lambda$. This characterization was recently generalized to $F^p_\alpha$ with $1 < p < \infty$ in [18]. Here we note that, using a generalization of Lemma 8.2 of [42] to the setting of $\varbb{C}^n$, one can prove that Stroethoff’s result remains true for Hankel operators acting from $F^p_\alpha$ to $L^q_\alpha$ whenever $1 \leq p, q < \infty$ even for unbounded symbols.
4.3. The case $0 < p \leq q \leq 1$ with bounded symbols. We start with the following preliminary lemma whose proof can be completed with a standard $\varepsilon$ argument.

**Lemma 4.4.** Suppose that $0 < p < \infty$, $h \in L^\infty$ and $\lim_{n \to \infty} h(z) = 0$. Then for any bounded sequence $\{g_j\}_{j=1}^\infty$ in $L^p_\varphi$ satisfying $\lim_{j \to \infty} g_j(z) = 0$ uniformly on compact subsets of $\mathbb{C}^n$, it holds that $\lim_{j \to \infty} \|g_j h\|_{p,\varphi} = 0$.

**Proof.** If $R$ is sufficiently large, there is a $C > 0$ such that

\[
\|g_j h\|_{p,\varphi}^p = \left( \int_{B(0,R)} + \int_{\mathbb{C}^n \setminus B(0,R)} \right) |g_j(\xi) h(\xi) e^{-\varphi(\xi)}|^p \, d\nu(\xi)
\]

\[
\leq \|h\|_{L^\infty}^p \sup_{|\xi| \leq R} |g_j(\xi) e^{-\varphi(\xi)}|^p + C\|g_j\|_{p,\varphi}^p \to 0
\]
as $j \to \infty$. \hfill \Box

**Proof of Theorem 1.1 (c).** Suppose that $f \in \mathcal{S}$. Then $f \in L^q_{\text{loc}}$ for $0 < q \leq 1$, and we may decompose $f = f_1 + f_2$ as in (3.12) with $t = r/2$. We claim that, for $g \in \Gamma$,

\[
\|H_{f_1}(g)\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} \left| g(\xi) e^{-\varphi(\xi)} \right|^q \|\overline{\partial} f_1\|_{L^\infty(B(\xi, r), d\nu)}^q \, d\nu(\xi)
\]

and

\[
\|H_{f_2}(g)\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} \left| g(\xi) e^{-\varphi(\xi)} \right|^q M_{1, r}(f_2)(\xi)^q \, d\nu(\xi).
\]

To estimate $\|H_{f_1}(g)\|_{q,\varphi}$, we use the representation

\[
H_{f_1}(g) = A_{\varphi}(g \overline{\partial} f_1) - P(A_{\varphi}(g \overline{\partial} f_1)),
\]

(see (2.11)), which suggests that we define a measure $d\mu_z$ as follows

\[
d\mu_z(\xi) = \left| \frac{1}{|\xi - z|^2} + \frac{1}{|\xi - z|^{2n-1}} \right| e^{-m|\xi - z|} \, d\nu(\xi).
\]

Then there is a constant $C$ such that, for $w \in \mathbb{C}^n$,

\[
\int_{B(w, r)} \left| \overline{\partial} f_1(\xi) \right|^q \left\{ \frac{1}{|\xi - z|} + \frac{1}{|\xi - z|^{2n-1}} \right\} e^{-m|\xi - z|^2} \, d\nu(\xi) \leq C \int_{B(w, r)} d\mu_z(\xi).
\]

Also, it is easy to verify that

\[
\left( \mu_z \right)_r(w) \leq C \sup_{\eta \in B(w, r)} \left| \overline{\partial} f_1(\eta) \right| e^{-m|w - z|},
\]

where the constant $C$ is independent of $z, w \in \mathbb{C}^n$. Recall that

\[
A_{\varphi}(g \overline{\partial} f_1)(z) = \int_{\mathbb{C}^n} e^{(2\varphi z - \xi)}
\]

\[
\times \sum_{j<n} g(\xi) \overline{\partial} f_1(\xi) \wedge \frac{\partial |\xi - z|^2 \wedge (2\overline{\partial} \varphi(\xi))^j \wedge (\overline{\partial} \varphi(\xi) - z^2)^{n-1-j}}{j! |\xi - z|^{2n-2j}}.
\]
Therefore, using (2.13) and Lemma 4.1 we get
\begin{equation}
|A_\varphi(g\overline{\partial}f_1)(z)e^{-\varphi(z)}|^q \leq C \left( \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}| \, d\mu_\xi(\xi) \right)^q
\end{equation}
\begin{equation*}
\leq C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r),dv)}^q e^{-qm|\xi-z|} \, dv(\xi).
\end{equation*}
Fubini’s theorem yields
\begin{equation}
\|A_\varphi(g\overline{\partial}f_1)\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} dv(z) \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r),dv)}^q e^{-qm|\xi-z|} \, dv(\xi)
\end{equation}
\begin{equation*}
\leq C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r),dv)}^q \, dv(\xi).
\end{equation*}
To deal with $P(A_\varphi(g\overline{\partial}f_1))$, we use Lemma 2.2 to obtain positive constants $\theta$ and $C$ so that, for $z \in \mathbb{C}^n$, we have
\begin{equation*}
\int_{\mathbb{C}^n} |K(w,z)|e^{-m|\xi-z|}e^{-\varphi(z)} \, dv(z)
\end{equation*}
\begin{equation*}
\leq Ce^{\varphi(w)} \int_{\mathbb{C}^n} e^{-m|\xi-z|}e^{-\theta|w-z|} \, dv(z)
\end{equation*}
\begin{equation*}
= Ce^{\varphi(w)} \left( \int_{\{z:|z-\xi| \geq |z-w|\}} + \int_{\{z:|z-\xi| < |z-w|\}} \right) e^{-m|w-z|}e^{-\theta|\xi-z|} \, dv(z)
\end{equation*}
\begin{equation*}
\leq Ce^{\varphi(w)} e^{-\tau|\xi-w|},
\end{equation*}
where $\tau = \min\{\theta, m\}$. Therefore, (4.17) and Fubini’s theorem yield
\begin{equation*}
|P(A_\varphi(g\overline{\partial}f_1))(w)|
\leq C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}| \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r/2),dv)} \, dv(\xi)
\end{equation*}
\begin{equation*}
\times \int_{\mathbb{C}^n} |K(w,z)|e^{-\theta|\xi-z|}e^{-\varphi(z)} \, dv(z)
\end{equation*}
\begin{equation*}
\leq Ce^{\varphi(w)} \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}| \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r/2),dv)} e^{-\tau|\xi-w|} \, dv(\xi).
\end{equation*}
Lemma 4.1 again gives
\begin{equation*}
\|P(A_\varphi(g\overline{\partial}f_1))(w)\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q \|\overline{\partial}f_1\|_{L^\infty(B(\xi,r),dv)}^q \, dv(\xi).
\end{equation*}
Combining this and (4.18), we get (4.15). For (4.16), notice first that
\begin{equation}
\|f_2g\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} |g(\xi)e^{-\varphi(\xi)}|^q M_{q,r}^q(f_2)(\xi) \, dv(\xi),
\end{equation}
and, by Lemma 4.1 with the measure $M_{1,r/2}(f_2)dv$, we have

$$|P(f_2g)(z)|^q \leq C \left( \int_{\mathbb{C}^n} \left| g(\xi)K(z,\xi)e^{-2\varphi(\xi)} \right| M_{1,r/2}(f_2)(\xi)dv(\xi) \right)^q,$$

(4.20)

\[ \leq C \int_{\mathbb{C}^n} \left| g(\xi)K(z,\xi)e^{-2\varphi(\xi)} \right|^q M_{1,r}(f_2)(\xi)^q dv(\xi). \]

Integrating both sides of (4.20) against $e^{-q\varphi}dv$ over $\mathbb{C}^n$ and using (4.19), we get

$$\|P(f_2g)\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} \left| g(\xi)e^{-\varphi(\xi)} \right|^q M_{1,r}(f_2)(\xi)^q dv(\xi).$$

(4.21)

This and (4.19) imply (4.16).

Now we suppose that $f \in L^\infty$ and $0 < p \leq q < 1$. For $g \in H(\mathbb{C}^n)$, similarly to the proof of (4.16), we have

$$\|H_f(g)\|_{q,\varphi} \leq C \left( \int_{\mathbb{C}^n} \left| g(\xi)e^{-\varphi(\xi)} \right|^q M_{1,r}(f)(\xi)^q dv(\xi) \right)^{1/q} \leq C \|f\|_{L^\infty} \|g\|_{p,\varphi}.$$

This implies boundedness of $H_f$ with the norm estimate (1.3).

For the second assertion, suppose first that $\lim_{|z| \to \infty} G_{q,r}(f)(z) = 0$ for some $r > 0$ and write $f = f_1 + f_2$ as above. Since the unit ball $B(F^p(\varphi))$ of $F^p(\varphi)$ is a normal family, to show that $H_f$ is compact from $F^p(\varphi)$ to $L^q(\varphi)$, it suffices to prove that for $k = 1, 2$,

$$\lim_{j \to \infty} \|H_{f_k}(g_j)\|_{q,\varphi} = \lim_{j \to \infty} \|f_k g_j - P(f_k g_j)\|_{q,\varphi} = 0$$

for any bounded sequence $\{g_j\}_{j=1}^\infty$ in $F^p(\varphi)$ with the property that

$$\lim_{j \to \infty} \sup_{w \in E} |g_j(w)| = 0$$

for $E$ compact in $\mathbb{C}^n$. From the assumption that $\lim_{|z| \to \infty} M_{q,r}(f_2)(z) = 0$, it follows that $d\mu = |f_2|^q dv$ is a vanishing $(p, q)$-Fock Carleson measure (see Theorem 2.7 of [19] and Proposition 2.3). Therefore, we get

$$\|f_2 g_j\|_{q,\varphi} = \|g_j\|_{L^q(\mathbb{C}^n, |f_2|^q dv)} \to 0 \text{ as } j \to \infty.$$  

Notice also that $\|g\|_{q,\varphi} \leq C\|g\|_{p,\varphi}$ for $g \in F^q(\varphi)$ and $p \leq q$. Further, by (4.16), we obtain

$$M_{1,r}(f_2)(\xi) \leq \|f_2\|_{L^\infty} M_{q,r}(f_2)(\xi)^q,$$

and applying Lemma 4.4 to $h = M_{q,r}(f_2)^q$, we get

$$\|H_{f_2 g_j}\|_{q,\varphi}^q \leq C \int_{\mathbb{C}^n} \left| g_j(\xi)e^{-\varphi(\xi)} \right|^q M_{1,r}(f_2)(\xi)^q dv(\xi) \leq C \|f_2\|_{L^\infty} \|g_j(\xi)e^{-\varphi(\xi)}\|_{q,\varphi}^q M_{q,r}(f_2)(\xi)^{q^2} dv(\xi) \to 0.$$
as \( j \to \infty \). So \( H_{f_2} \in \mathcal{K}(F^p(\varphi), L^q(\varphi)) \). As for \( H_{f_1} \), it follows from Lemma 3.6 that
\[
\|\overline{\partial} f_1\|_{L^\infty(B(\xi,r),dv)} \leq CG_{q,r}(f)(\xi) \to 0 \quad \text{when} \quad \xi \to \infty.
\]
Therefore, by (4.15),
\[
\|H_{f_1}(g_j)\|_{q,\varphi} \leq C \int_{\mathbb{C}^n} |g_j(\xi)e^{-\varphi(\xi)}|^{q} \|\overline{\partial} f_1\|_{L^\infty(B(\xi,r),dv)}^q dv(\xi) \to 0
\]
as \( j \to \infty \), and hence we have \( H_{f_1} \in \mathcal{K}(F^p(\varphi), L^q(\varphi)) \).

Conversely, suppose that \( H_f \) is compact from \( F^p(\varphi) \) to \( L^q(\varphi) \). Then, as in (4.4), we have
\[
(4.22) \quad \lim_{z \to \infty} G_{q,r}(f)(z) \leq C \lim_{z \to \infty} \|H_f(k_z)\|_{q,\varphi} = 0
\]
for \( r \in (0, r_0] \) fixed. We claim that (4.22) is valid for any \( r > 0 \). To see this, we consider the Hankel operator \( H_f \) on the Fock space \( F^p(\alpha) \). From (4.22), using the sufficiency part, it follows that \( H_f \) is compact from \( F^p(\alpha) \) to \( L^q(\mathbb{C}^n, e^{-\frac{2\alpha z^2}{R^2}} dv) \). Notice that the equality (1.5) yields
\[
\inf_{w \in B(z,r)} |K(w,z)| \geq C > 0
\]
for any \( r > 0 \) fixed, where the constant \( C \) is independent of \( z \in \mathbb{C}^n \). As in (4.2), we have
\[
\lim_{z \to \infty} G_{q,r}(f)(z) \leq C \lim_{z \to \infty} \|H_f(k_z)\|_{L^q(\mathbb{C}^n, e^{-\frac{2\alpha z^2}{R^2}} dv)} = 0.
\]
Thus, \( f \in \text{VDA}^q \). This completes the proof. \( \square \)

The following Corollary 4.5 is a direct consequence of the proof of Theorem 1.1 (c) which we use to complement and extend the classical result of Berger and Coburn in the next section.

**Corollary 4.5.** Suppose that \( 0 < q < 1 \) and \( f \in L^\infty \). Then the limit
\[
\lim_{z \to \infty} G_{q,r}(f)(z) = 0
\]
is independent of \( r > 0 \).

5. **Proof of Theorem 1.2**

*Proof of the case \( 0 < p \leq q < \infty \).* For \( R > 0 \), let \( \{a_k\}_{k=1}^\infty \) be the \( R/2 \)-lattice
\[
\left\{ \frac{R}{2\sqrt{n}}(m_1 + k_1i, m_2 + k_2i, \ldots, m_n + k_ni) \in \mathbb{C}^n : m_j, k_j \in \mathbb{Z}, j = 1, 2, \ldots, n \right\}.
\]
Choose \( \rho \in C^\infty(\mathbb{C}^n) \) such that
\[
0 \leq \rho \leq 1, \quad \rho|_{B(0,1/2)} \equiv 1, \quad \text{supp} \rho \subseteq B(0,3/4).
\]
Then \( \|\nabla \rho\|_{L^\infty} < \infty \) and
\[
0 < \sum_{k=1}^\infty \rho((z - a_k)/R) \leq C
\]
for \( z \in \mathbb{C}^n \). Define \( \psi_{j,R} \in C^\infty(\mathbb{C}^n) \) by
\[
\psi_{j,R}(z) = \sum_{k=1}^{\infty} \rho((z - a_j)/R).
\]
Then \( \{\psi_{j,R}\}_{j=1}^{\infty} \) is a partition of unity subordinate to \( \{B(a_j, R)\}_{j=1}^{\infty} \) and
\[
R \|\nabla \psi_{j,R}(\cdot)\|_{L^\infty} \leq C,
\]
where the constant \( C \) is independent of \( j \) and \( R \).

Now we suppose that \( f \in L^\infty \) and \( Hf \in K(F_p(\varphi), L_q(\varphi)) \). Theorem 1.1 and Corollary 4.5 imply that
\[
\lim_{z \to \infty} M_{q,R}(f)(z) = 0
\]
for \( R > 0 \) fixed. As in (3.2), pick \( h_{j,R} \in H(B(a_j, 2R)) \) so that
\[
\sup_{z \in B(a_j, R)} |h_{j,R}(z)| \leq C \|f\|_{L^\infty}.
\]
Set
\[
f_{1,R} = \sum_{j=1}^{\infty} \psi_{j,R} h_{j,R} \quad \text{and} \quad f_{2,R} = f - f_{1,R}.
\]
From estimates (2.9) and (3.3), it follows that there is a positive constant \( C \) such that
\[
\|f_{1,R}\|_{L^\infty} + \|f_{2,R}\|_{L^\infty} \leq C \|f\|_{L^\infty}
\]
for \( R > 0 \). Lemma 3.6 and (5.2) imply that
\[
\lim_{z \to \infty} M_{q,R}(f_{2,R})(z) = 0,
\]
and so
\[
H_{f_{2,R}} \in K(F_p(\varphi), L_q(\varphi)).
\]
Recall that \( P_{z,R} \) is the standard Bergman projection from \( L^2(B(z, R), dv) \) to \( A^2(B(z, R), dv) \). Since \( h_{j,R} \) is bounded on \( B(a_j, R) \), we have \( h_{j,R} = P_{a_j,R}(h_{j,R}) \), that is,
\[
\frac{1}{\pi} \int_{B(a_j, R)} \frac{R^2 h_{j,R}(\xi)dv(\xi)}{R^2 - (\xi - a_j) \cdot (z - a_j)^{n+1}}, \quad z \in B(a_j, R).
\]
Hence,
\[
\frac{\partial}{\partial \bar{z}} h_{j,R}(z) \leq C \frac{\|h_{j,R}\|_{L^\infty(B(z,R),dv)}}{R} \quad \text{for} \quad z \in B(a_j, 3R/4).
\]
Notice that \( \text{supp} \psi_{j,R} h_{j,R} \subseteq B(a_j, 3R/4) \), and the estimates (5.1) and (5.6) imply that

\[
|\partial f_{j,R}| \leq \sum_{j=1}^{\infty} |(\partial \psi_{j,R}) h_{j,R}| + \sum_{j=1}^{\infty} \psi_{j,R} |\partial (h_{j,R})| \leq C \frac{\|f\|_{L^\infty}}{R}.
\]

Therefore, using (4.6) (when \( q \geq 1 \)) and (4.15) (when \( q < 1 \)), we have

\[
\|H_{f_{j,R}}\|_{F^p(\varphi) \rightarrow L^q(\varphi)} \leq C \|\partial f_{j,R}\|_{L^\infty} \leq C \frac{\|f\|_{L^\infty}}{R}.
\]

The constants \( C \) above are all independent of \( f \) and \( R \). Therefore,

\[
\|H_T - H_{f_{j,R}}\|_{F^p(\varphi) \rightarrow L^q(\varphi)} = \|H_{f_{j,R}}\|_{F^p(\varphi) \rightarrow L^q(\varphi)} \leq C \frac{\|f\|_{L^\infty}}{R} \rightarrow 0
\]
as \( R \to \infty \). Finally, using (5.5) and the fact that \( \mathcal{K}(F^p(\varphi), L^q(\varphi)) \) is closed under the operator norm, we see that \( \mathcal{K}_T \in \mathcal{K}(F^p(\varphi), L^q(\varphi)) \), which completes the proof.

To deal with the case \( 1 \leq q < p < \infty \), we use the Ahlfors-Beurling operator, which is a very well-known Calderón-Zygmund operator on \( L^p(\mathbb{C}) \), \( 1 < p < \infty \), defined as follows

\[
\mathcal{T}(f)(z) = p.v. - \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(\xi)}{\xi - z} d\nu(\xi),
\]

where \( p.v. \) means the Cauchy principal value. The Ahlfors-Beurling operator connects harmonic analysis and complex analysis, and it is of fundamental importance in several areas of mathematics including PDE and quasiconformal mappings. See [1] and [2] for further details and examples.

**Lemma 5.1.** Suppose \( 1 < s < \infty \). Then there is some constant \( C \), depending only on \( s \), such that, for \( f \in C^2(\mathbb{C}^n) \cap L^\infty \) and \( j = 1, 2, \ldots, n \),

\[
(5.7) \quad \left\| \frac{\partial f}{\partial z_j} \right\|_{L^s} \leq C \left\| \frac{\partial f}{\partial z_j} \right\|_{L^s}.
\]

**Proof.** We consider the case \( n = 1 \) first. Let \( f \in C^2(\mathbb{C}) \cap L^\infty \). If \( \left\| \frac{\partial f}{\partial z} \right\|_{L^s} = 0 \), then \( f \in H(\mathbb{C}) \cap L^\infty \), which implies that the function \( f \) is constant and the estimate (5.7) follows. Next we suppose that \( \left\| \frac{\partial f}{\partial z} \right\|_{L^s} > 0 \). Take \( \psi(r) \in C^\infty(\mathbb{R}) \) to be decreasing such that \( \psi(x) = 1 \) for \( x \leq 0 \), \( \psi(x) = 0 \) for \( x \geq 1 \), and \( 0 \leq -\psi'(x) \leq 2 \) for \( x \in \mathbb{R} \). For \( R > 0 \) fixed, we set \( \psi_R(x) = \psi(x - R) \) for \( x \in \mathbb{R} \) and define \( f_R(z) = f(z) \psi_R(|z|) \) for \( z \in \mathbb{C} \). Since \( f \in C^2(\mathbb{C}) \cap L^\infty \), it is obvious that \( f_R(z) \in C^2(\mathbb{C}) \), the set of \( C^2 \) functions on \( \mathbb{R}^2 \) with compact support. From Theorem 2.1.1 of [12], it follows that

\[
f_R(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial f}{\partial z} \frac{d\xi}{\xi - z} \land d\xi.
\]
Notice that \( \frac{\partial R}{\partial \bar{z}} = \psi_R \frac{\partial R}{\partial z} + \frac{\partial \psi_R}{\partial \bar{z}} \). By Lemma 2 on page 52 of [1], we get
\[
(5.8) \quad \frac{\partial f}{\partial z}(z) = \mathcal{T} \left( \frac{\partial f}{\partial z} \right)(z) = \mathcal{T} \left( \psi_R \frac{\partial f}{\partial z} \right)(z) + \mathcal{T} \left( \frac{\partial \psi_R}{\partial \bar{z}} \right)(z).
\]
Now for \( r > 0 \) and \( |z| < r \), when \( R \) is sufficiently large, it holds that
\[
\left| \mathcal{T} \left( \frac{\partial \psi_R}{\partial \bar{z}} \right) \right|(z) \leq \frac{\|f\|_{L^\infty}}{\pi(R-r)^2} \int_{|\xi| \leq R+1} dv(\xi) \leq \frac{3R\|f\|_{L^\infty}}{(R-r)^2},
\]
and hence
\[
(5.9) \quad \left\| \mathcal{T} \left( \frac{\partial \psi_R}{\partial \bar{z}} \right) \right\|_{L^s(D(0,r),dv)} \leq \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s},
\]
where \( D(0,r) = \{ z \in \mathbb{C} : |z| < r \} \). In addition, by the boundedness of \( \mathcal{T} \) on \( L^s \) (see, for example, the estimate (11) on page 53 in [1]), we get
\[
(5.10) \quad \left\| \mathcal{T} \left( \psi_R \frac{\partial f}{\partial z} \right) \right\|_{L^s} \leq C \left\| \psi_R \frac{\partial f}{\partial z} \right\|_{L^s} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s}.
\]
For \( R \) sufficiently large, from (5.8), (5.9) and (5.10) it follows that
\[
\left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s(D(0,r),dv)} = \left\| \frac{\partial f_R}{\partial \bar{z}} \right\|_{L^s(D(0,r),dv)} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s}.
\]
Therefore,
\[
(5.11) \quad \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s} \leq C \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{L^s}.
\]
Now for \( n \geq 2 \) and \( f \in L^\infty \cap \text{C}(\mathbb{C}^n) \), by (5.11), we have
\[
\int_{\mathbb{C}^n} \left| \frac{\partial f}{\partial z_1} \right|^s dv(\xi) = \int_{\mathbb{C}^n} dv(\xi) \int_{\mathbb{C}} \left| \frac{\partial f}{\partial z_1} \right|^s dv(\xi_1)
\]
\[
\leq C \int_{\mathbb{C}^{n-1}} dv(\xi) \int_{\mathbb{C}} \left| \frac{\partial f}{\partial z_1} \right|^s dv(\xi_1).
\]
This implies (5.7) for \( j = 1 \). Similarly, (5.7) holds for \( j = 2, \ldots, n \), and the proof is complete. \( \square \)

Proof of the case \( 1 \leq q < p < \infty \). Notice first that if \( H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi)) \), then by Theorem 1.1 we have \( f \in \text{IDA}^{s,q} \) with \( s = \frac{p}{p-q} > 1 \). We use a decomposition \( f = f_1 + f_2 \) as in (3.17) with \( r = 1 \). Furthermore, by (5.4), we may assume that \( \|f_1\|_{L^\infty} \leq C\|f\|_{L^\infty} \). Then, from Lemma 5.1 it follows that
\[
\|\frac{\partial f_1}{\partial \bar{z}}\|_{L^s} \leq C \sum_{j=1}^n \left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{L^s} \leq C \sum_{j=1}^n \left\| \frac{\partial f}{\partial \bar{z}_j} \right\|_{L^s} \leq C \|\bar{\partial} f_1\|_{L^s}.
\]
We also observe that \( \|M_{q,r}(f_2)\|_{L^s} = \|M_{q,r}(f_2)\|_{L^s} < \infty \). Now Theorem 3.8 implies that \( \bar{f} = f_1 + f_2 \in \text{IDA}^{s,q} \), and hence, by Theorem 1.1 we get \( H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi)) \). \( \square \)
Remark 5.2. Notice that it follows from the preceding proof that
\[ \|H_f\|_{F^p(\varphi) \to L^q(\varphi)} \leq C\|H_f\|_{F^p(\varphi) \to L^q(\varphi)}. \]

6. Application to Berezin-Toeplitz quantization

As an application and further generalization of our results, we consider
deformation quantization in the sense of Rieffel \[34, 35\] and focus on one of
its essential ingredients in the non-compact setting of \( \mathbb{C}^n \) that involves the
limit condition
\[ \lim_{t \to 0} \left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\|_{F^2(\varphi) \to F^2(\varphi)} = 0. \]
Recently this and related questions were studied in \[5, 6, 17\], which also
provide further physical background and references for this type of quantization.

Recall that \( \varphi \in C^2(\mathbb{C}^n) \) is real valued and \( \text{Hess}_2 \varphi \simeq E \), where \( E \) is the
2n × 2n-unit matrix. For \( t > 0 \), we set
\[ d\mu_t(z) = \frac{1}{t^n} \exp \left\{ -2\varphi \left( \frac{z}{\sqrt{t}} \right) \right\} dv(z) \]
and denote by \( L^2_t(\varphi) \) the space of all Lebesgue measurable functions \( f \) in
\( \mathbb{C}^n \) such that
\[ \|f\|_t = \left\{ \int_{\mathbb{C}^n} |f|^2 d\mu_t(z) \right\}^{\frac{1}{2}}. \]
Further, we let \( F^2_t(\varphi) = L^2_t(\varphi) \cap H(\mathbb{C}^n) \). Then clearly \( F^2_t(\varphi) = F^2(\varphi) \) and
\( L^2_t(\varphi) = L^2(\varphi) \) in terms of the spaces that were considered in the previous
sections. Given \( f \in L^\infty \), we use the orthogonal projection \( P^{(t)} \) from \( L^2_t(\varphi) \)
onto \( F^2_t(\varphi) \) to define the Toeplitz operator \( T_f^{(t)} \) and the Hankel operator
\( H_f^{(t)} \), respectively, by
\[ T_f^{(t)} = P^{(t)} M_f \quad \text{and} \quad H_f^{(t)} = (I - P^{(t)}) M_f. \]
Let \( U_t \) be the dilation acting on measurable functions in \( \mathbb{C}^n \) as
\[ U_t : f \mapsto f(\cdot \sqrt{t}). \]
It is easy to verify that \( U_t \) is a unitary operator from \( L^2_t(\varphi) \) to \( L^2(\varphi) \) (as well
as a unitary operator from \( F^2_t(\varphi) \) to \( F^2(\varphi) \)). Further, we have \( U_t P^{(t)} U_t^{-1} = P^{(1)} \), which implies that
\[ (6.1) \quad U_t T_f^{(t)} U_t^{-1} = T_f(\sqrt{t}), \quad U_t H_f^{(t)} U_t^{-1} = H_f(\sqrt{t}). \]
Therefore,
\[ (6.2) \quad \|T_f^{(t)}\|_{F^2_t(\varphi) \to F^2_t(\varphi)} = \|T_f(\sqrt{t})\|_{F^2(\varphi) \to F^2(\varphi)} \]
and
\[ (6.3) \quad \|H_f^{(t)}\|_{F^2_t(\varphi) \to L^2_t(\varphi)} = \|H_f(\sqrt{t})\|_{F^2(\varphi) \to L^2(\varphi)}. \]
Given \( f \in L^2_{\text{loc}} \), for \( z \in \mathbb{C}^n \) and \( r > 0 \) set

\[
MO_{2,r}(f)(z) = \left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} \left| f - f_{B(z,r)} \right|^2 \, dv \right\}^{\frac{1}{2}}
\]

where \( f_{S} = \frac{1}{|S|} \int_{S} f \, dv \) for \( S \subset \mathbb{C}^n \) measurable.

The following definitions of BMO and VMO are analogous to the classical definition introduced by John and Nirenberg [26], but they differ from those widely used in the study of Bergman and Fock spaces.

**Definition 6.1.** We denote by BMO the set of all \( f \in L^2_{\text{loc}} \) such that

\[
\|f\|_{\text{BMO}} = \sup_{z \in \mathbb{C}^n, r > 0} MO_{2,r}(f)(z) < \infty
\]

and by VMO the set of all \( f \in \text{BMO} \) such that

\[
\lim_{r \to 0} \sup_{z \in \mathbb{C}^n} MO_{2,r}(f)(z) = 0.
\]

**Definition 6.2.** We define BDA* to be the family of all \( f \in L^2_{\text{loc}} \) such that

\[
\|f\|_{\text{BDA}^*} = \sup_{z \in \mathbb{C}^n, r > 0} G_{2,r}(f)(z) < \infty
\]

and VDA* to be the subspace of all \( f \in \text{BDA}^* \) such that

\[
\lim_{r \to 0} \sup_{z \in \mathbb{C}^n} G_{2,r}(f)(z) = 0.
\]

Given a family \( X \) of functions on \( \mathbb{C}^n \), we set \( \overline{X} = \{ \bar{f} : f \in X \} \).

**Proposition 6.3.** It holds that

\[
\text{BMO} = \text{BDA}^* \cap \overline{\text{BDA}^*} \quad \text{and} \quad \text{VMO} = \text{VDA}^* \cap \overline{\text{VDA}^*}.
\]

Furthermore, we have

\[
(6.4) \quad \|f\|_{\text{BMO}} \simeq \|f\|_{\text{BDA}^*} + \|\bar{f}\|_{\text{BDA}^*}
\]

for \( f \in L^2_{\text{loc}} \).

**Proof.** From a careful inspection of the proof of Proposition 2.5 in [24], it follows that there is a constant \( C > 0 \) such that, for \( f \in L^2_{\text{loc}} \) and \( z \in \mathbb{C}^n \), \( r > 0 \), there is a constant \( c(z) \) for which

\[
\left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} \left| f - c(z) \right|^2 \, dv \right\}^{\frac{1}{2}} \leq C \left\{ G_{2,r}(f)(z) + G_{2,r}(\bar{f})(z) \right\}.
\]

It is easy to verify that

\[
MO_{2,r}(f)(z) \leq \left\{ \frac{1}{|B(z,r)|} \int_{B(z,r)} \left| f - c(z) \right|^2 \, dv \right\}^{\frac{1}{2}},
\]

and hence

\[
MO_{2,r}(f)(z) \leq C \left\{ G_{2,r}(f)(z) + G_{2,r}(\bar{f})(z) \right\}.
\]
On the other hand, by definition, we have

\[ G_{2,r}(f)(z) \leq MO_{2,r}(f)(z) . \]

Thus, we have \( C_1 \) and \( C_2 \), independent of \( f, r \) and \( z \), such that

\[
C_1 \left\{ G_{2,r}(f)(z) + G_{2,r}(\overline{f})(z) \right\} \leq MO_{2,r}(f)(z)
\]

\[
\leq C_2 \left\{ G_{2,r}(f)(z) + G_{2,r}(\overline{f})(z) \right\} .
\]

Therefore, \( f \in \text{BMO} \) (or \( f \in \text{VMO} \)) if and only if \( f \in \text{BDA}_* \cap \overline{\text{BDX}_*} \) (or \( f \in \text{VDA}_* \cap \overline{\text{VDA}_*} \)). The estimate in (6.4) follows from (6.5).

**Theorem 6.4.** Suppose \( f \in L^\infty \). Then for all \( g \in L^\infty \), it holds that

\[
\lim_{t \to 0} \left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\|_{L^2(\varphi) \to L^2(\varphi)} = 0
\]

if and only if \( f \in \overline{\text{VDA}_*} \).

**Proof.** Given \( f \in L^\infty \), it follows from (6.3) that

\[
\left\| \left( H_f^{(t)} \right)^* \right\|_{L^2(\varphi) \to L^2(\varphi)} = \left\| H_f^{(t)} \right\|_{F^2_1(\varphi) \to L^2_1(\varphi)} = \left\| H_f(\cdot \sqrt{t}) \right\|_{F^2(\varphi) \to L^2(\varphi)} .
\]

This and Theorem 1.1 imply

\[
\frac{1}{C} \| G_{2,1}(f(\cdot \sqrt{t})) \|_{L^\infty} \leq \left\| \left( H_f^{(t)} \right)^* \right\|_{L^2(\varphi) \to F^2_1(\varphi)} \leq C \| G_{2,1}(f(\cdot \sqrt{t})) \|_{L^\infty} ;
\]

where the constant \( C \) is independent of \( f \) and \( t \).

Suppose \( f \in \overline{\text{VDA}_*} \). Then, by definition, we have

\[
\lim_{t \to 0} \sup_{r \in \mathbb{C}_n} G_{2,r}(\overline{f})(z) = 0 .
\]

It is easy to verify that

\[
G_{2,1} \left( f(\cdot \sqrt{t}) \right)(z) = G_{2,\sqrt{t}}(f) \left( z \sqrt{t} \right) .
\]

Now by (6.7), we get

\[
\lim_{t \to 0} \left\| \left( H_f^{(t)} \right)^* \right\|_{L^2(\varphi) \to F^2_1(\varphi)} \leq C \lim_{t \to 0} \| G_{2,\sqrt{t}}(\overline{f}) \|_{L^\infty} = 0.
\]

In addition, for \( f, g \in L^\infty \), it is easy to verify that

\[
T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} = - \left( H_f^{(t)} \right)^* H_g^{(t)} .
\]

Therefore, for all \( g \in L^\infty \),

\[
\lim_{t \to 0} \left\| T_f^{(t)} T_g^{(t)} - T_{fg}^{(t)} \right\|_{L^2(\varphi) \to L^2(\varphi)} \leq \| g \|_{L^\infty} \lim_{t \to 0} \left\| \left( H_f^{(t)} \right)^* \right\|_{L^2(\varphi) \to L^2(\varphi)} = 0,
\]

which gives (6.6).
Conversely, suppose that (6.6) holds for every \( g \in L^\infty \). Let \( g = \tilde{f} \in L^\infty \). Then it follows from (6.9) that

\[
\lim_{t \to 0} \left\| H_f^{(t)} \right\|_{F^2_t(\varphi) \to L^\infty_t(\varphi)}^2 = \lim_{t \to 0} \left( \left( H_f^{(t)} \right)^* \right)_t \left. H_f^{(t)} \right|_{F^2_t(\varphi) \to F^2_t(\varphi)} = \lim_{t \to 0} \left\| T_f^{(t)} T_g^{(t)} \right\|_{F^2_t(\varphi) \to F^2_t(\varphi)} = 0.
\]

This and (6.7) imply that \( f \in \text{VDA}_s \). \( \square \)

Combining Proposition 6.3 with Theorem 6.4 we obtain the following corollary, which is the main result of [6] when \( \varphi(z) = \frac{1}{|z|^2} \).

**Corollary 6.5.** Suppose \( f \in L^\infty \). Then for all \( g \in L^\infty \), it holds that

\[
(6.10) \quad \lim_{t \to 0} \left\| T_f^{(t)} T_g^{(t)} - T_f^{(t)} \right\| = 0 \quad \text{and} \quad \lim_{t \to 0} \left\| T_g^{(t)} T_f^{(t)} - T_f^{(t)} \right\| = 0
\]

if and only if \( g \in \text{VMO} \). Here \( \| \cdot \| = \| \cdot \|_{F^2_t(\varphi) \to F^2_t(\varphi)} \).

7. Further remarks

For \( 1 \leq p,q < \infty \), we have characterized those \( f \in S \) for which \( H_f : F^p(\varphi) \to L^q(\varphi) \) is bounded (or compact). For small exponents \( 0 < p < q < 1 \), we have proved that this characterization remains true for compactness when \( f \in L^\infty \). We also note that when \( p \leq q \) and \( q \geq 1 \), boundedness and compactness of Hankel operators \( H_f : F^p(\varphi) \to L^p(\varphi) \) depend on \( q \) (see Remark 3.2 and Theorem 1.1) while for \( p > q \) we cannot say the same—we note that we have no statement analogous to Remark 3.2 for \( \text{IDA}^{s,q} \).

Moreover, for harmonic symbols \( f \in S \) and \( 0 < p,q < \infty \), using the Hardy-Littlewood theorem on the sub-mean value (see Lemma 2.1 of [21], for example), we are able to characterize boundedness of \( H_f : F^p(\varphi) \to L^q(\varphi) \) with the space \( \text{IDA}^{s,q} \). We will return to this topic in a future publication.

We also note that the space \( F^\infty(\varphi) \) does not appear in our results because \( \Gamma \) is not dense in it. Instead, it may be possible to consider the space

\[
f^\infty(\varphi) = \{ f \in F^\infty(\varphi) : fe^{-\varphi} \in C_0(\mathbb{C}^n) \},
\]

which can be viewed as the closure of \( \Gamma \) in \( F^\infty(\varphi) \), and extend our results to this setting.

Regarding weights, the Fock spaces studied in this paper are defined with weights \( \varphi \in C(\mathbb{C}^n) \) satisfying \( \text{Hess}_{\varphi} \varphi \simeq E \). As stated in Section 2.1, these weights are contained in the class considered in [37]. Now, we note that for the weights \( \varphi \) in [37], \( i\partial \bar{\partial} \varphi \simeq \omega_0 \), and from Hörmander’s theorem on the canonical solution to \( \bar{\partial} \)-equation it follows that

\[
\| H_f g \|_{2,\varphi}^2 \leq \int_{\mathbb{C}^n} |g \bar{\partial} f |^2 \partial \bar{\partial} e^{-2\varphi} dv \leq C \| g \bar{\partial} f \|_{2,\varphi}^2,
\]

and hence we know that the conclusions of Theorem 1.1 remain true when \( q = 2 \) (see Theorem 4.3 of [23]). Upon these observations, we raise the following conjecture.
Conjecture 1. Suppose $\varphi \in C^2(\mathbb{C}^n)$ satisfying $i\partial\overline{\partial}\varphi \simeq \omega_0$. Then for $f \in \mathcal{S}$ and $0 < p, q < \infty$, $H_f \in B(F^p(\varphi), L^q(\varphi))$ if and only if $f \in \text{IDA}^{s,q}$ where $s = \frac{pq}{p-q}$ if $p > q$ and $s = \infty$ if $p \leq q$.

In the literature, there are a number of interesting results on the simultaneous boundedness (and compactness) of Hankel operators $H_f$ and $H_{\overline{f}}$. These types of characterizations often involve the function spaces $\text{BMO}^q$ and $\text{IMO}^{s,q}$ in their conditions; see, e.g., [24, 42] and the references therein. For $1 \leq q < \infty$ and $1 \leq s \leq \infty$, set $\text{IDA}^{s,q} = \{f : f \in \text{IDA}^{s,q}\}$. Then Proposition 2.5 of [24] shows that $\text{IDA}^{s,q} \cap \text{L}^\infty$ and the results of Section 4 provide a description of the simultaneous boundedness (or compactness) of $H_f$ and $H_{\overline{f}}$ as seen in the following theorem, where as before, we set $s = \frac{pq}{p-q}$ if $p > q$ and $s = \infty$ if $p \leq q$.

Theorem 7.1. Let $\varphi \in C^2(\mathbb{C}^n)$ be real valued, $\text{Hess}_R \varphi \simeq E$, and let $f \in \mathcal{S}$. For $1 \leq p, q < \infty$, Hankel operators $H_f$ and $H_{\overline{f}}$ are simultaneously bounded from $F^p(\varphi)$ to $L^q(\varphi)$ if and only if $f \in \text{IMO}^{s,q}$.

We state one more conjecture related to Theorem 1.2 in which we proved that for $f \in L^\infty$ and $0 < p < \infty$, $H_f$ is compact on $F^p(\varphi)$ if and only if $H_{\overline{f}}$ in compact on $F^p(\varphi)$. Recall that this phenomenon does not occur for Hankel operators on the Bergman space or on the Hardy space. As predicted by Zhu [42], and verified for Hankel operators on the weighted Fock spaces $F^p(\alpha)$ with $1 < p < \infty$ in [18], a partial explanation for this difference is the lack of bounded holomorphic or harmonic functions on the entire complex plane. From this point of view it is natural to suggest that a similar result should remain true for Hankel operators mapping from $F^p(\varphi)$ to $L^q(\varphi)$.

Conjecture 2. Suppose that $\varphi \in C^2(\mathbb{C}^n)$ satisfies $i\partial\overline{\partial}\varphi \simeq \omega_0$ and $0 < p, q < \infty$. Then for $f \in L^\infty$, $H_f \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$ if and only if $H_{\overline{f}} \in \mathcal{K}(F^p(\varphi), L^q(\varphi))$.

Notice that $\text{IDA}^{s,q} \cap L^\infty$ is a Banach algebra under the norm $\| \cdot \|_{\text{IDA}^{s,q}} + \| \cdot \|_\infty$. We can also express Conjecture 2 in algebraic terms, that is, we conjecture that $\text{IDA}^{s,q} \cap L^\infty$ on $\mathbb{C}^n$ is closed under the conjugate operation $f \mapsto \overline{f}$, where $1 < s \leq \infty$ and $0 < q < \infty$.

Related to our work on quantization and Theorem 6.4 in particular, we conclude this section with the following problem: Characterize those $f \in L^\infty$ for which it holds that

$$\lim_{t \to 0} \| T^{(t)}_f T^{(t)}_g - T^{(t)}_{fg} \|_{S_2} = 0$$

for all $g \in L^\infty$, where $\| \cdot \|_{S_2}$ stands for the Hilbert-Schmidt norm. It would also be important to consider this question for other Schatten classes $S_p$. 
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