Linear Response Theory and Fluctuation Dissipation Theorem for Systems with Absorbing States

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The Fluctuation Dissipation Theorem (FDT) is one of the fundamental results of statistical mechanics and is a powerful tool that connects the macroscopic properties to microscopic dynamics. The FDT and the linear response theory are mainly restricted to systems in the vicinity of stationary states. However, frequently, physical systems do not conserve the total probability. In systems with absorbing states, the net flux out of the system is positive, and the total probability decays with time. In this case the stationary distribution is trivially zero throughout and the tools provided by standard linear response theory fail. Here we present a new FDT for decaying systems which connects the response of observables conditioned on survival to conditional correlations without perturbations. The results have been verified through simulations and numerics in various important examples.

Statistical physics has been an invaluable bridge between microscopic dynamics and macroscopic observables. The Einstein-Smoluchowski equation connects diffusion to mobility, and the regression hypothesis by Onsager connects fluctuations in equilibrium dynamics to triggered deviations away from equilibrium [1–3]. These are closely connected to linear response theory initially proposed by Kubo for equilibrium dynamics [4, 5] and subsequent analogous fluctuation-dissipation relations in stochastic systems [6, 7]. These relations connect the response of an observable in the presence of a perturbation to the correlation of variables in unperturbed stationary dynamics. Although equilibrium dynamics is a useful description, studies into non-equilibrium systems have gained traction and rightly so, considering the vast number of systems that show non-equilibrium behaviour. This is not just limited to traditional systems, but include examples from biological systems, epidemiology, turbulence, and chemical systems amongst others [8–11].

A large body of literature exists on the violation of the fluctuation-dissipation theorem in glassy systems [12]. Even in non glassy systems, there have also been multiple attempts at extending the validity of fluctuation-dissipation relations to non equilibrium systems using the concepts of asymmetry [13, 14], frenesy [15], local currents [16–18]. In particular, successes have been obtained at predicting response to perturbation in an arbitrary non-stationary state [19]. However, the focus has been mainly on systems that have some form of long-time steady state. Instead, the focus of this Letter is on an entirely different class of non-equilibrium systems with absorbing states which, as a particular case, include systems with absorbing boundaries.

Systems with absorbing states are widespread in many fields. Some of the examples include chemical reactions, epidemics, and population dynamics [20–23]. In such systems, the net flux out of the system is positive, and hence the total probability distribution decays with time, rendering the steady-state trivial. However, such systems are also frequently found in a quasi-stationary state for a long time before reaching extinction. There exists extensive mathematical literature analyzing the properties of quasi-stationary systems [23–25]. The properties of time taken to extinction, the existence of quasi-stationary distribution [26, 27], simulation methods [28], etc have been studied with applications to cellular automata [29], birth-death processes [30], Brownian motion [31], contact process [32], and many others.

There exists a vast amount of literature on quasi-stationary systems, but there has been no foray into looking at the change in fluctuation-dissipation theorem in such systems. Although linear response theory in “quasi stationary” states has been considered [33, 34], the states referred to are metastable states in which systems spend a long time before reaching the equilibrium state, rather than systems with absorbing boundaries. The fields of quasi-stationary systems and non-equilibrium fluctuation-dissipation relations are crucial and significant, but they have not been connected.

In the following work, we build the bridge between these two fields by considering the modifications needed to linear response theory in the presence of absorbing states. We proceed by considering the long-time evolution of a finite system and the resulting decaying distribution in terms of the eigenfunctions and eigenvalues of the time evolution operator. We then consider all observable averages over trajectories that have not yet hit the absorbing state, calling this operation conditioning to survival. We find that a new fluctuation-dissipation theorem arises by introducing a new observable that accounts for the survival of trajectories. Furthermore, we generalize the FDT to infinite and continuous systems under some plausible and general hypothesis accounting for the change in the eigenvalue spectrum. We also show the validity of the new FDT when the perturbation is applied at arbitrary times, thereby giving a robust understanding of linear response theory in decaying systems.

Initial work by Kubo in response theory [4, 5] provides a way to calculate the response of an observable in
Hamiltonian dynamics, connecting the response to a perturbation to fluctuations in equilibrium dynamics. For an observable $A$, the response to a perturbation can be written as

$$\langle \Delta A(t) \rangle = \int_{-\infty}^{\infty} dt' R_{A,L}(t-t') F(t')$$

(1)

where $R_{A,L}(t-t')$ is the response function and $F(t)$ is the time component of the external perturbation.

If a system is described by a Hamiltonian $\mathcal{H}(p,q)$ ($\mathcal{L}_0$ representing the corresponding Liouville operator) and a perturbation to the Hamiltonian is given by $V(p,q)F(t)$, the response function for the observable $A(p,q)$ can be computed by

$$R_{A,L}(t,t') = \int dp dq \ e^{-i(t-t')\mathcal{L}_0} \times \left\{ V(p,q), p \right\} A(p,q)$$

(2)

where $\{A, B\}$ represents the Poisson bracket of $A$ and $B$ and $p \equiv \exp(-\beta \mathcal{H}(p,q))$.

The analogue of Eq. (2) in stochastic systems can be calculated using the Fokker-Planck equation with the Fokker-Planck operator taking the role of the Liouville operator [6]. For $\vec{x} = (x_1, ..., x_N)$, if $P_{st}(\vec{x})$ represents the stationary probability distribution of a stochastic system under the Fokker-Planck operator $L_{FP}(\vec{x}, t)$, then the response function of an observable $A$ to a perturbation $L_{ext}(\vec{x}) F(t)$ is given by

$$R_{A,L}(t) = \Theta(t) \int dN \ A(\vec{x}) \ e^{L_{FP}(\vec{x})t} \ L_{ext}(\vec{x}) \ P_{st}(\vec{x})$$

(3)

where $R_{A,L}$ is the response function and $\Theta$ is the Heaviside step function. Furthermore, if the operator $L_{FP}$ depends on some parameters $f = (f_1, f_2, ..., f)$, and the perturbation changes at least one them, $f \rightarrow f + \Delta f(t)$, then Eq. (3) leads to a Fluctuation-Dissipation relation which connects the response function to the time derivative of a two time correlation function [16–18, 35].

$$R_{A,f}(t-t') = \Theta(t-t') \frac{\partial}{\partial t} \left\langle A(t) \frac{\partial \phi}{\partial f}(t') \right\rangle$$

(4)

where $\phi(\vec{x})$ is the generalized potential characterising the stationary solution of the Fokker-Planck equation of the system, i.e., $P_{st}(\vec{x}) = e^{-\phi(\vec{x})}/N$, where $N$ is an appropriate normalization factor. As long as the system starts from the stationary state, the response function only depends on difference between the time of observation and the time of perturbation. In decaying systems, Eq. (4) does not hold because $P_{st}(\vec{x})$ is trivially zero everywhere. The average of observables share the same fate because the distribution itself decays with time.

Here, we first show a fluctuation dissipation theorem that holds for discrete and finite systems with an absorbing boundary/state. We begin by considering a finite space of states, $\Omega$, with an absorbing boundary, $\partial\Omega$, and an interior set, $\Omega^c = \Omega - \partial\Omega$. The absorbing boundary is characterised by outgoing transition rates that are zero in the master equation. Let $W_{x,y}$ be the transition rate from the state $y$ to the state $x$, and let $P_x(t)$ be the probability of finding the system in the state $x$ at time $t$. Thus, the equation satisfied by $P_x(t)$ is

$$\dot{P}_x(t) = \sum_{y \in \Omega^c} [W_{xy} P_y(t) - W_{yx} P_x(t)]$$

$$W_{x,y} = 0 \quad \forall x \in \Omega, \ y \in \partial\Omega$$

(5)

where $\partial\Omega$ may contain multiple states. We are interested in how the observables can be measured in the interior $\Omega^c$. So, to avoid degeneracy, we collapse all the absorbing states into a single representative state $o$. The probability of finding the system in the $o$ state is the sum of probabilities of finding the system on any one of the boundary state, $P_o(t) = \sum_{x \in \partial\Omega} P_x(t)$ and the transition rate into $o$ is the total transition rate into the boundary, $W_{xo} = \sum_{y \in \partial\Omega} W_{yx}$. Since the rates of transition out of the states of $\partial\Omega$ are zero, $W_{x,o} = 0$ for all $x \in \Omega^c$. Defining a new space of states, $\Omega' = \Omega^c \cup \{o\}$, the new master equation reads

$$\dot{P}_x(t) = H_{xy} P_y(t)$$

$$H_{xy} = W_{xy} - \delta_{xy} \sum_z W_{zy}$$

(6)

The right eigenvectors and eigenvalues of $H$ will be denoted as $\varphi_n$ and $\lambda_n$, respectively. The entries of the first column of $H$ are all zeros; hence the first eigenvalue is $\lambda_0 = 0$ and the corresponding first right eigenvector is $\varphi_{0,x} = \delta_{x,o}$. This corresponds to the long time limit of the probability distribution, which is zero everywhere on the interior. We will only deal with an irreducible matrix $W$, which means there always exists a non trivial path from any interior state to any other interior state. From the Perron Frobenius Theorem, the eigenvalue spectrum is

$$0 = \lambda_0 < \Re(\lambda_1) < \Re(\lambda_2) \leq \Re(\lambda_3)$$

(7)

where $\Re(\cdot)$ denotes the real part of the argument. Furthermore, the second eigenvalue $\lambda_1$ has to be real and the corresponding left and right eigenvectors are real with positive components in the interior [See Supplementary Information for more details [36]. See also [37]]. The orthogonality between left and right eigenvectors leads to

$$\sum_{x \in \Omega^c} \varphi_{1,x} = 1$$

We now turn our interest to the observables which are defined on the interior. To ensure that the system has not reached the boundary yet, we introduce an observable that characterizes the survival, namely, $\chi_x = 1 \ \forall x \in \Omega^c$ and $\chi_o = 0$. We stress that we use observables only on the interior and hence are zero on the boundary. In the event of an observable $B$ not satisfying the condition, it can be redefined as $B \chi$ to be zero on the boundary.
The conditional average of an observable $A$ is then represented by
\[
\langle A \| \chi \rangle_t \equiv \langle A(t) \| \chi(t) \rangle = \sum_{x \neq o} A_x P_x(t) \sum_{x} \chi_x P_x(t)
\]
(8)

It is clear from this definition that we are conditioning the observable to survive on the interior. As $t \to \infty$, the conditional average reaches a stationary value[36].

\[
\langle A \| \chi \rangle_{lt} = \sum_{x \neq o} A_x \varphi_{1x}
\]
(9)

The superscript $lt$ is used to denote the conditional observable at large time. Similarly, we write the conditional correlation between two observables $A$ and $B$ at times $t$ and $t'$, with $t' > t'$,
\[
\langle A(t) B(t') \| \chi(t) \rangle = \frac{\langle A(t) B(t') \| \chi(t) \rangle}{\langle \chi(t) \rangle} = \frac{\sum_{x,y} A_x P_x(t) y \sum_{x} \chi_x P_x(t)}{\sum_{x} \chi_x P_x(t)}
\]
(10)

where $P(x, t| y, t')$ is the probability of being in the state $x$ at time $t$ starting from $y$ at time $t'$ and $P(x, t| y, t') = [e^{H(t-t')}]|_{x,y}$ from Eq. (6). The generalization to multiple time correlations follows similarly as far as they are conditioned with the denominator $\langle \chi \rangle_t$, $t$ being the largest time appearing in the numerator. For sufficiently large starting times, namely, $t' \to \infty$ with $t-t' > 0$ finite, Eq. (10) reduces to
\[
\langle A(t) B(t') \| \chi(t) \rangle_{lt} = \sum_{x,y} A_x B_y P(x, t| y, t') \varphi_{1y} e^{\lambda_{1}(t-t')}
\]
(11)

The solution to Eq. (6) can be written in terms of the eigenvectors and eigenvalues. Let $c_{n,x,o}$ be the constants depending on the initial condition. Then,
\[
P(x, t|x_0, 0) = \sum_{n} c_{n,x,o} \varphi_{n,x} e^{-\lambda_n t}
\]
(12)

For times $t$ such that $t(\Re(\lambda_2) - \lambda_1) > 1$, the probability distribution is approximated by $P_{st}^{lt}(t) = \varphi_{ox} + c_1 \varphi_{1x} e^{-\lambda_1 t}$. Notice that the averages in Eq. (9) and Eq. (11) are equal to averaging the observables over $P_{st}^{lt}(t)$. This self consistency is expected, since both are considered at long times.

The conditional correlation and average give us the necessary tools to calculate the linear response. A perturbation in the parameters can be represented by a perturbation in the generator $H \to H + \delta H(t)$ and a change in the probability distribution due to the perturbation, namely, $P(t) = P^{lt}(t) + \delta P(t)$. We consider the system to be in the long time state before applying the perturbation, i.e., starting from $P^{lt}(t)$. Then, to the linear order,
\[
\delta P(t) = \int_0^t ds e^{H(t-s)} \delta H(s) P^{lt}(s)
\]
(13)

The change in an observable conditioned to survival is
\[
\frac{\delta \langle A \| \chi \rangle_t}{\langle \chi \rangle_t} = \frac{\delta \langle A \| \chi \rangle_t}{\langle \chi \rangle_t} - \langle A \| \chi \rangle_{lt} \delta \langle \chi \rangle_t + \text{higher order terms}
\]
(14)

We have used the superscript $lt$ to denote that we are starting from the observable approximately given by Eq. (9). Assuming that the perturbation can be factorized into the time and state components separately, i.e., $\delta H(s) = V F(s)$, and defining $B_x = \frac{1}{\tau} \sum_y V_{x,y} \varphi_{1y}$, the response function can be formally written as a correlation function,
\[
\delta \langle A \| \chi \rangle_t = \int_0^\infty ds \hat{R}_{A,V}(t-s) \delta F(s)
\]
(15)

We highlight that this result is different from the standard case, where the second term is absent and $H^{\text{standard}} = (\sum_y V_{x,y} P_y)/P_x$. The presence of the second term is due to the normalization over all interior states: it is important as it ensures that the average is performed over trajectories that survive at least till time $t$.

We are now in a position to obtain a fluctuation dissipation theorem. If the perturbation affects a particular parameter and $f \to f + \Delta f(t)$, then $V = \partial f H(f)$ which yields
\[
\hat{R}_{A,V}(t-s) = \hat{R}_{A,f}(t-s)
\]
\[
= -\Theta(t-s) \frac{\partial}{\partial f} \left( \left( A(t) - \chi(t) \langle A \| \chi \rangle_{lt} \right) \frac{\partial f}{\partial (s)} \| \chi \rangle_{lt} \right)
\]
(16)

where $\phi$ is now the potential associated to the second right eigenvector, rather than the stationary state, $\varphi_{1x} = e^{f x}$ $x \neq o$ and $\varphi_{1o} \equiv 0$ (we have used that $\varphi_{1x} > 0$, $\forall x \in \Omega$). While referring to $\langle A(t) \partial f \phi(s) \rangle$ and other such quantities, it does not necessarily mean that the observables depend on time but rather that we are referring to two time correlation defined in Eq. (10). In the previous equation one can substitute $\partial/\partial s$ with $-\partial/\partial s$ because the conditional correlation function depends only on $t-s$. The difference between the standard case is immediately apparent from the appearance of the second term which does not exist in Eq. (4).

The state space considered is finite. But in infinite discrete systems, the nature of eigenvalue spectrum can still satisfy the condition in Eq (7). In such cases, Eq. (16) still gives the correct response function formula.
A natural generalization leads us from infinite discrete states to a space of continuous states. The evolution of the system is thus described by a Langevin equation and the corresponding evolution of the probability density function is governed by a Fokker-Planck equation. In this latter case, we introduce $Q(x,t)$, which is the conditional probability distribution, conditioned to survival. It is defined as

$$Q(x,t|x_0,t_0) = \frac{P(x,t|x_0,t_0)}{\int dx \; P(x,t|x_0,t_0)} \quad (17)$$

At large times, i.e., $t - t_0 \to \infty$, we assume the conditional distribution converges to the quasi stationary distribution, $Q_{st}(x)$, which is analogous to the stationary distribution in the standard case [38, 39]. Under reasonable assumptions on the spectrum of eigenvalues, the quasi stationary distribution depends only on the first eigenfunction and eigenvalue. [see Supplementary Information for more details].

Then, the average of an observable and the correlation between two observables conditioned to survival modifies accordingly to

$$\langle A \rangle_s \equiv \int dx \; A(x) \; Q_{st}(x) \quad (18)$$

$$\langle A(t) B(t') \rangle_s = \int dx \; A(x) \; e^{(L(x)+\lambda_1)(t-t')} \; B(x) \; Q_{st}(x) \quad (19)$$

where $\lambda_1$ is the dominant eigenvalue of $L(x)$. Introducing the observable $I(x) = 1$, the response function is given by

$$\hat{R}_{A,f}(t) = -\Theta(t) \frac{\partial}{\partial t} \left[ \langle A(t) - \langle A \rangle_s I(t) \rangle \right] \left( t' = 0 \right)_s \quad (20)$$

which has similar form as Eq. (16). With the solution of the Fokker-Planck equation, the asymptotic long time limit also gives the leading eigenvalue and, if it exists, the quasi stationary distribution. This is another way to reach Eq. (20) without explicitly going through the eigenfunctions and eigenvalues.

We show the example of the Birth Death process to verify the FDT Eq. (16). The process is on a lattice of sites $n = 0, 1, 2, \ldots$ and absorbing boundary at 0. The transition rates are given by $W_{n+1,n} = b_n = bn$ and $W_{n-1,n} = d_n = dn$ with $d > b$. Though the system is on infinite lattice, the eigenvalues are given by $\lambda_k = (d-b)k$ for $k \geq 1$ and hence satisfies Eq. (7). With $b_{-1} = 0$, the Master equation is given by

$$\hat{P}_i(t) = b_{i-1}P_{i-1}(t)+d_{i+1}P_{i+1}(t)-(b_i+d_i)P_i(t) \quad i \geq 0 \quad (21)$$

The second eigenvector and the eigenvalue can be obtained by the generating function [21]

$$G(z,t) = \sum_{n \geq 0} P_n(t) z^n \quad (22)$$

with $P_n(t = 0) = \delta_{n,n_0}$, $\nu = 1 - b/d$, $A(z,t) = (1-z)(1-\left(1-\nu\right)z)^{-1}e^{-\nu t}$. At large times $A(z,t)$ is small enough to use the approximation $1/(1-x)^\nu \approx 1 - yx$. Since the coefficient of the slowest exponential in the long time state gives the second eigenvector, we obtain $\varphi_{1n} = \nu(1-\nu)^{n-1}$. The time dependant solution to the Master equation is calculated using Meixner Polynomials [40]. See SI for the full form of the time dependant solution.

Setting $s = 0$ and using $(e^{Ht})_{x,y} = P(x,t|y,0)$, we estimate Eq. (16). This is compared to simulations of the Birth Death process, where, after a large time of simulation, the perturbation is introduced in parameter $b$ as a delta perturbation. From Eq. (1), under a delta perturbation, the value of observable is equal to the response function. We start with $10^8$ trajectories and consider the averages only over surviving trajectories. The importance of the new FDT can be seen from the comparison, where, without the contribution from the second term, the response is not predicted correctly.

![FIG. 1. Simulated vs predicted response of $A_n = n^2$ in the Birth-Death Process. Parameters used are $b = 0.4$, $d = 0.5$, $\Delta b = 0.04$](image)

In the SI, we consider important examples in discrete and continuous systems and we find good agreement between the theory and the simulations in the examples.

We have been considering perturbations occurring at $t - t_0 \to \infty$ which is the quasi stationary limit (in the standard case, the stationary limit). But it is also possible for the perturbation to occur to any general non quasi-stationary distribution [19]. Following similar procedures as we have done earlier, this result can also be extended to perturbation applied to non quasi-stationary
states. The details of the calculations can be found in the Supplementary Information [36]. Figure 2 shows comparison between response observed and predicted for perturbation when a generic three state system is in non quasi-stationary state. At large times, the system is expected to be close to quasi-stationarity and the response of the observable to non quasi stationary perturbation is converging to that predicted by Eq. (16).

In summary, we have obtained a new fluctuation-dissipation theorem for systems with absorbing boundaries/states. Though the probability distribution decays with time, all is not lost. Starting from finite and discrete systems, we have shown the validity in infinite systems and continuous systems as well. Furthermore, we have generalized the new FDT to apply to cases when the perturbation occurs at an arbitrary time in the system’s evolution. We have also considered some important and often used examples to verify the results through numerical simulations. Our results provide tools for computing responses in systems with absorbing states. The theory opens new paths of study in the search for universal principles in non-equilibrium statistical mechanics. Modified fluctuation-dissipation theorems have already been verified in real-world systems [41, 42]. Applications such as microswimmers for resource capture can serve as a ground to use our theory to accurately predict and manipulate systems to desired results [43]. Considering the ubiquitous nature of systems with absorbing states, the scope is high for both experimental and theoretical work involving the new fluctuation-dissipation theorem.

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FIG. 2. Verification of modification of FDT to perturbation at non stationary times in a three state system with absorbing boundary. The legend shows the times at which perturbation was applied with the dashed lines being predictions and solid lines being simulations. Refer to Supplementary Information for more details.

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Supplementary Information : Linear Response Theory and Fluctuation Dissipation Theorem for Systems with Absorbing States

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1 Perturbation of parameters in stationary systems

Consider a system in steady state or equilibrium. Any external perturbation will shift the system away from steady state. Assuming that the perturbation is small, the response of the system can be analyzed within the linear approximation.

Let us consider the Fokker-Planck equation describing our system. The perturbations to the operator can then be written as
\[
L(x,t) = L_0(x) + \delta L(x,t)
\]

Here, \(L_0(x)\) is the unperturbed Fokker-Planck operator and the external perturbation is small and hence, we only consider upto first order. Similarly, we also write the solution to the perturbed operator in terms of stationary solution of the unperturbed operator and the perturbation:
\[
P(x,t) = P_{st}(x) + \delta P(x,t)
\]

Since we are looking at small perturbations, we also assume that the deviation \(\delta P(x,t)\) is small. With \(L_0(x)P_{st}(x) = 0\), we can write:
\[
\dot{P}(x,t) = \delta P(x,t) = L_0(x)\delta P(x,t) + \delta L(x,t)P_{st}(x) + \delta L(x,t)\delta P(x,t)
\]

of which the last term is negligible since we only consider linear approximation, i.e, the leading order. Solving this first order differential equation with the condition \(\delta P(x,-\infty) = 0\), we obtain
\[
\delta P(x,t) = \int_{-\infty}^{t} dt' e^{L_0(x)(t-t')} \delta L(x,t)(x,t') P_{st}(x)
\]

Having obtained this, we also assume that \(\delta \mathcal{L}(x,t)\) has same form as \(L_0(x)\) \cite{1} and can be split into spatial and temporal component.
\[
\delta \mathcal{L}(x,t) = \delta \mathcal{L}(x) F(t)
\]

Let \(A\) be an observable. Then, the deviation of \(A\) caused due to the perturbation around its mean value \(\langle A \rangle\) can be written as
\[
\Delta A(t) = \int_{-\infty}^{\infty} R_{A,L}(t-t')F(t') \, dt' \quad \text{where}
\]
\[
R_{A,L}(t) = \Theta(t) \int A(x) e^{L_0(x)t} \delta \mathcal{L}(x) P_{st}(x) \, dx
\]
This can also be expressed as a correlation function using generalized potential \[1\]. That is one form of fluctuation dissipation theorem. For our purposes, we can go further and express the response function in terms of time derivative of other correlations.

We now assume that the perturbation occurs in one of the parameters of the system, \( f \equiv (f_1, f_2, \ldots, f_N) \), i.e. \( f \rightarrow f + \Delta f(t) \). If \( \mathcal{L}_0(x) = \mathcal{L}(x, f) \) and,

\[
\mathcal{L}(x, f) P(x, f) = 0 \quad (7a)
\]

\[
P_{\text{st}}(x) \equiv P(x, f) = \frac{1}{Z(f)} e^{-\phi(x, f)} \quad (7b)
\]

where \( P_{\text{st}}(x) \) is the stationary solution of the Fokker-Planck operator with the dependance on the parameters \( f \). Then,

\[
\mathcal{L}(x, f + \Delta f(t)) = \mathcal{L}(x, f) + \Delta f(t) \delta \mathcal{L}(x) + \mathcal{O}(\Delta f^2) \quad (8)
\]

\[
\delta \mathcal{L}(x) = \frac{\partial \mathcal{L}(x, f)}{\partial f} \quad (9)
\]

From (7) and (9), we can obtain the following

\[
\delta \mathcal{L}(x) P(x, f) = -\mathcal{L}(x, f) \frac{\partial P(x, f)}{\partial f} \quad (10)
\]

Subsequently, the response function in (6) can now be written as

\[
R_{A,f}(t) = -\Theta(t) \int A(x) e^{\mathcal{L}(x, f)t} \mathcal{L}(x, f) \frac{\partial P(x, f)}{\partial f} \, dx \quad (11)
\]

\[
= -\Theta(t) \frac{d}{dt} \int A(x) e^{\mathcal{L}(x, f)t} \frac{\partial P(x, f)}{\partial f} \bigg|_{\Delta f=0} \, dx \quad (12)
\]

However, from (7b), we can write

\[
\frac{\partial P(x, f)}{\partial f} = P(x, f) \left[ -\frac{\partial \phi(x, f)}{\partial f} - \frac{\partial \log(Z(f))}{\partial f} \right] \quad (13)
\]

Since \( e^{\mathcal{L}(x, f)t} P(x, f) = P(x, f) \), and the second term in (13) is not time dependant when applied in (12), it is zero and does not contribute to the response function. We can now write (12) as

\[
R_{A,f}(t) = \Theta(t) \frac{d}{dt} \int A(x) e^{\mathcal{L}(x, f)t} P(x, f) \frac{\partial \phi(x, f)}{\partial f} \, dx
\]

\[
= \Theta(t) \frac{d}{dt} \left\langle A(t) \frac{\partial \phi(t')}{\partial f} \bigg|_{\phi(t')=0} \right\rangle \quad (14)
\]

Suppose we denote \( X_f = \frac{\partial \phi(t=0)}{\partial f} \), then, (14) can be written as

\[
R_{A,X_f}(t) = \Theta(t) \frac{d}{dt} K_{A,X_f}(t) \quad (15)
\]

where \( K_{A,B}(t) \) is the two time correlation function between observables \( A \) and \( B \). Equation (15) forms the main Fluctuation Dissipation Theorem we wish to pursue. We are considering the perturbation happening to a stationary distribution. For the case of perturbation at a general non-stationary state, similar form of the Fluctuation-Dissipation Theorem holds \[2\].
2 Discrete System with Finite States

In physical systems in the natural world, there are numerous examples of systems with absorbing states. One primary example is that of ecological systems where once a species goes extinct, there is no chance of revival. In such examples, due to the lack of a source in the system, the stationary distribution is trivially zero and hence, existing linear response theory cannot be applied. While the probability distribution of non boundary states goes to zero, the concept of quasi stationarity is still possible, where, if averaged over surviving trajectories only, we see non trivial results, both in the probability distribution and the corresponding quantities such as correlations and responses. Exploiting this fact, we can formulate a theory to calculate response of different observables even when the absorbing state is present.

2.1 Master Equation and Preliminaries

Let $\Omega$ represent the space of all the states. We can denote the boundary of this space by $\partial\Omega$ and the interior by $\Omega^\circ = \Omega - \partial\Omega$. The transition rates between different states shall be represented by $W_{xy}$. The absorbing boundary condition can be represented by setting the rate of transition out of the boundary states to be zero.

$$W_{xy} = 0 \forall x \in \Omega, y \in \partial\Omega$$

$$W_{xy} \geq 0$$

With these, we can now write the master equation for the time evolution of probability of being in state $x$ at time $t$.

$$\dot{P}_x(t) = \sum_{y \in \Omega} [W_{xy} P_y(t) - W_{yx} P_x(t)]$$

If $x \in \partial\Omega$

$$= \sum_{y \in \Omega^\circ} W_{xy} P_y(t)$$

Or

$$= \sum_{y \in \Omega^\circ} W_{xy} P_y(t) - \sum_{y \in \Omega} W_{yx} P_x(t)$$

if $x \in \Omega^\circ$ (16)

To make expressions simpler, we can introduce a single state 0 to represent the entire boundary. The corresponding entries for this state that would go into the ME would be

$$P_0(t) = \sum_{x \in \partial \Omega} P_x(t)$$

(17)

$$W_{0x} = \sum_{y \in \partial \Omega} W_{yx} \quad \text{and} \quad W_{x0} = \sum_{y \in \partial \Omega} W_{xy} = 0$$

(18)

These represent the total probability of being on the boundary and the total rates of transition into and out of the boundary respectively. Note that these also imply $W_{0x} = 0$ if $x \in \partial \Omega$.

Let us now redefine our state space to include this special 0 state. The corresponding rates are defined in Eq. (17) and (18) and

$$\Omega' = \Omega^\circ \cup \{0\}$$

(19)

In this new space, we can rewrite our ME as follows

$$\dot{P}_x(t) = \sum_{y \in \Omega'} [W_{xy} P_y(t) - W_{yx} P_x(t)]$$

(20)

which can similarly be broken down depending on whether $x = 0$ or otherwise. Note that since $P_0$ never enters on the r.h.s. of the previous equation due to our rates, we only need to solve the redefined ME for non-zero states, i.e., only the interior. It is always possible to get back the individual boundary states on $\partial\Omega$ by integrating

$$\dot{P}_y(t) = \sum_{x \in \Omega^\circ} W_{yx} P_x(t)$$

We continue representing the entire boundary by state 0. This also helps avoid degenerate ground states for each boundary state in the initial $\Omega$. 

Eigenvales and Eigenvectors

Note that $\sum_x P_x(t) = 0$, i.e., the total probability is conserved (and equivalently we can normalize it). We can define the operator $H$ with which we have $P(t) = H \, P(t)$

$$H_{xy} = W_{xy} - \delta_{xy} \sum_z W_{zy} \quad (21)$$

Normalizing the left and right eigenvectors of $H$ such that $(\psi^n|\varphi_m) = \delta_{m,n}$, the first left eigenvector is $\psi_0^0 = 1 \ \forall x \in \Omega$ for the corresponding eigenvalue $\lambda_0 = 0$. The corresponding right eigenvector is $\varphi_0^x = \delta_{x,0}$ as it can be easily verified.

In general, for the $n^{th}$ right eigenvector and eigenvalue, with $\Re(z)$ representing the real part of a complex number $z$,

$$H\varphi_n = -\lambda_n \varphi_n$$

and hence $0 = \lambda_0 < \Re(\lambda_1) < \Re(\lambda_2) \leq \Re(\lambda_3)$...

(22)

We will always work with an irreducible $W$, i.e. for any pair of interior nodes, there is always a path of non zero $W$’s from one to the other. This implies that the boundary state is not visited. In such a system, (22) can be proved using Perron-Frobenius Theorem.

Eigenvalue Spectrum

Consider an infinitesimal increment in time $\epsilon$. Then, the Master equation can be discretized to the Markov chain

$$P(t + \epsilon) = P(t) + \epsilon H \, P(t) \quad (23)$$

Consequently, define $T \equiv (\mathbb{I} + \epsilon H)$ where $\mathbb{I}$ is the identity matrix. We are considering an irreducible system. Hence, $T$ is a non negative, irreducible matrix of size $N \times N$ where $N$ is the number of states.

The eigenvalues of $T$ are simply $\alpha_i = 1 - \epsilon \lambda_i$. We consider the submatrix of $T$ corresponding to the interior. Let $T^*$ be a matrix of size $(N - 1) \times (N - 1)$ accounting for the interior states. Since we are in an irreducible system, there exists a path from one state to another, and hence, there always exists a $k_{xy} > 0$ for which the $(W_{xy})^{k_{xy}}$ is non zero. There are $(N - 1)^2$ such positive numbers. Choosing the largest of these numbers ensures that $T^*$ is primitive, i.e, $(T^*)^{k_{largest}} > 0$, if $\epsilon$ is sufficiently small (see below).

The characteristic function of the eigenvalues of $T^*$ is the characteristic function of $T$ without the $(\alpha - 1)$ factor coming from the first eigenvalue. Hence, finding the eigenvalues of $T^*$ is equivalent to finding the eigenvalues of $T$. Note that $T^*_{xy} \geq 0$ when $x \neq y$ and $T^*_{xx} = 1 - \epsilon \sum_z W_{zx}$ which is bounded allowing us to choose sufficiently small $\epsilon$ such that $T^*_{xy} \geq 0 \ \forall x, y$. Thus, we can apply the Perron–Frobenius theorem [3] on the matrix $T^*$. Then, we have a leading real eigenvalue $\alpha_1 > 0$. For any other eigenvalue $\alpha_2$, then, $\alpha_1 > |\alpha_2|$. Since $T = \mathbb{I} + \epsilon H$, and from the eigenvalue definition in (22), the eigenvalues are related as $\alpha = 1 - \epsilon \lambda$.

Then, with $\Re(z)$ and $\Im(z)$ representing the real and imaginary parts of $z$,

$$\alpha_1 > |\alpha_2| \Rightarrow (1 - \epsilon \lambda_1)^2 > |1 - \epsilon \Re(\lambda_2) - i \epsilon \Im(\lambda_2)|^2$$

$$\Rightarrow \epsilon^2 \lambda_1^2 - 2 \epsilon \lambda_1 > \epsilon^2 (\Re(\lambda_2)^2 + \Im(\lambda_2)^2) - 2 \epsilon \Re(\lambda_2) \quad (24)$$

In the limit $\epsilon \to 0$, which is the correct limit to get the Master equation back, we have $\Re(\lambda_2) > \lambda_1$.

The Perron–Frobenius Theorem also guarantees that the left and the right eigenvector of $T^*$ corresponding to the eigenvalue $\alpha_1$ (equivalently, $\lambda_1$) are non-degenerate and have components strictly positive.

If $\varphi_1$ and $\psi_1$ are the right and left eigenvectors of $H$ corresponding to $-\lambda_1$, then, $\varphi_1 x, \psi_1^T > 0 \ \forall x \neq 0$. Hence, we can choose the normalization such that $\sum_{x \neq 0} \varphi_1 x = 1$. Since $0 = \sum_{x} \psi_1^0 \varphi_1 x = \sum_{x} \varphi_1 x \Rightarrow \varphi_{10} = -1$. Then,

$$\lambda_1 = -\lambda_1 \varphi_{10} = \sum_y H_{0y} \varphi_{1y} = \sum_{y \neq 0} W_{0y} \varphi_{1y} > 0 \quad (25)$$
if at least one of the $W_{0y}$ is positive.

With the preliminaries we have listed above, we can start looking at how this affects observable quantities like correlation functions. We find results that differ from the more standard case where there is no absorbing boundary while looking at survived trajectories only.

**Correlation Functions**

Let us consider an initial condition to be $P^H(t) = \varphi_0 + \varphi_1$. This is a valid initial condition since it satisfies all requirements for a physical initial condition.

$$
\begin{align*}
P^H_x(0) &= \varphi_{1x} \geq 0 \quad x \neq 0 \\
P^H_x(0) &= 0 \quad x = 0 \\
\sum_x P^H_x(0) &= \langle \psi^0 | P^H(0) \rangle = 1
\end{align*}
$$

Then, $P^H(t) = \varphi_{0x} + e^{-\lambda_1 t} \varphi_{1x}$. We also define an observable $\chi_x = 1 - \delta_{x,0}$, whose average value at time $t$ gives the survival probability, i.e. the probability that the system has not yet been absorbed by the boundary. For any other observable $A$, we require that $A_x = 0$ on the boundary state, i.e. $x = 0$. If the observable does not have this criterion, we can always multiply it by $\chi$ and have $\chi_x A_x$ satisfying this criterion.

$$
\begin{align*}
\langle A \rangle^H_t &= \sum_x A_x P^H_x(t) = e^{-\lambda_1 t} \sum_{x \neq 0} A_x \varphi_{1x} \\
\langle \chi \rangle^H_t &= e^{-\lambda_1 t} \sum_{x \neq 0} \varphi_{1x} = e^{-\lambda_1 t} \quad \text{this represents the survival probability}
\end{align*}
$$

**Conditional Average**

$$
\frac{\langle A \rangle^H_t}{\langle \chi \rangle^H_t} = \sum_{x \neq 0} A_x \varphi_{1x} \equiv \langle A | \chi \rangle^H_t \tag{26}
$$

We use double delimiter in $\langle \bullet | \bullet \rangle$ to avoid confusion with the inner product earlier defined $\langle \bullet \rangle$. For an arbitrary initial condition (IC), at large times, it is equivalent to averaging over the defined $P^H$ as seen from below. For arbitrary $P(0)$,

$$
\begin{align*}
P(t) &= \varphi_0 + c_1 e^{-\lambda_1 t} \left[ \varphi_1 + O(e^{-t(R\lambda_2 - \lambda_1)}) \right] \\
\langle A \rangle_t &= \sum_x A_x P_x(t) = c_1 e^{-\lambda_1 t} \sum_{x \neq 0} A_x \varphi_{1x} \left[ 1 + O(e^{-t(R\lambda_2 - \lambda_1)}) \right] \\
\langle \chi \rangle_t &= c_1 e^{-\lambda_1 t} \sum_{x \neq 0} \varphi_{1x} \left[ 1 + O(e^{-t(R\lambda_2 - \lambda_1)}) \right] = c_1 e^{-\lambda_1 t} \left[ 1 + O(e^{-t(R\lambda_2 - \lambda_1)}) \right] \\
\frac{\langle A \rangle_t}{\langle \chi \rangle_t} &= e^{-\lambda_1 t} \sum_{x \neq 0} A_x \varphi_{1x} = \langle A | \chi \rangle^H_t
\end{align*}
$$

This gives us the conditional average. For the time correlation function, we need to consider the propagator.

**Time Evolution**

The ME can be written as matrix operation with the $H$ defined earlier.

$$
\dot{P}(t) = H \ P(t) \quad \Rightarrow \quad P(t) = e^{Ht} P(0)
$$

If $P_x(0) = \delta_{x,x_0}$, we obtain the propagator $P(x, t|x_0, 0) = [e^{Ht}]_{x,x_0}$. Hence, for a general IC,

$$
P_x(t) = \sum_{x_0} P(x, t|x_0, 0) P_{x_0}(0)
$$
Since \( H_{x,0} = 0 \) we have
\[
P(x, t|x_0 = 0, 0) = [e^{Ht}]_{x,0} = \delta_{x,0} \quad \forall t \tag{27}
\]
Correlation functions are defined as \( \langle A(t)B(t') \rangle = \sum_{x,y} A_x P(x,t|y,t') B_y P_y(t') \) and similarly for correlations involving more than two observables evaluated at different times. Since we have defined observables to be zero at node 0 (boundary) it follows from (27) and the Chapman–Kolmogorov equation that
\[
\langle A(t)B(t') \cdots C(t'') \rangle(x,t''') = \langle A(t)B(t') \cdots C(t'') \rangle \quad \text{if} \quad t''' \leq \max\{t', t''\} \tag{28}
\]
This has to be true logically too. Since we are calculating the correlation till \( \max\{t, t', \ldots t'''\} \), the only contributing trajectories have to last at least till time \( \max\{t, t', \ldots t'''\} \). Hence, at time \( t''' \) the trajectories have not yet hit the boundary, i.e. \( \chi(t''') = 1 \). But for the contrary case, this is not true
\[
\langle A(t)B(t') \cdots C(t'') \rangle(x,t''') \neq \langle A(t)B(t') \cdots C(t'') \rangle \quad \text{if} \quad t''' > \max\{t, t', \ldots t'''\} \tag{29}
\]
since on the l.h.s. we are requiring that trajectories have not hit the boundary at least till time \( t''' \) whereas in the r.h.s. we require that they only survive till a time \( t''' \). This is a consequence of having absorbing boundary. In the standard no absorbing boundary case, \( \chi = 1 \) trivially and \( \sum_x P(x,t|y,t') = 1 \). With the absorbing boundary, we are forced to neglect the 0 state and hence the sum over \( x \neq 0 \) is no longer unity. If the IC is \( P^{lt} \),
\[
\langle A(t)B(t') \rangle^{lt} = \sum_{x,y} A_x B_y P(x,t|y,t') P^{lt}_y(t')
\]

In this case, a conditional correlation function, which represents the correlation with only surviving trajectories is written as
\[
\frac{\langle A(t)B(t') \rangle^{lt}}{\chi^{lt}} = \sum_{x,y} A_x B_y P(x,t|y,t') \phi_{1y} e^{-\lambda_1 t'} \quad t' < t \tag{30}
\]
where \( I \) is the identity matrix. In this, we define \( A_{xy} = A_x \delta_{xy} \) and \( B_{xy} = B_y \delta_{xy} \). \( (\Phi_1)_{xy} = \phi_{1y} \delta_{xy} \). Similar to the conditional average we can start from arbitrary IC and at large times, it becomes equivalent to averaging over the long time state. Hence, in further calculations, we can just use \( P^{lt}(t) \) to determine correlations at large times.

At large time limit, \( t' \to \infty \) and denoting \( \Delta t = t - t' \), the propagator can be approximated as
\[
[e^{H\Delta t}]_{xy} = \phi_y \phi_{0,x} + \phi_y \phi_{1x} e^{-\lambda_1 \Delta t} + O(e^{-\Re(\lambda_2) \Delta t})
\]

Using this in (31),
\[
\langle A(t)B(t') \rangle^{lt} = \sum_x A_x \phi_{1x} \sum_y B_y \phi_{1y} \phi_y^\dagger (1 + O(e^{-\Re(\lambda_2) - \lambda_1} \Delta t)) \tag{33}
\]
By defining \( \langle B|\chi \rangle^{lt}_{\text{cond}} \equiv \sum_y B_y \phi_{1y} \psi_y^\dagger = \langle B \psi^\dagger | \chi \rangle^{lt} \). Therefore, at large time limit, \( t - t' \to \infty \),
\[
\langle A(t)B(t') \rangle^{lt} = \langle A|\chi \rangle^{lt} \langle B|\chi \rangle^{lt}_{\text{cond}} (1 + O(e^{-\Re(\lambda_2) - \lambda_1} \Delta t)) \tag{34}
\]
Since \( \psi_y^\dagger > 0 \), we also have \( \langle B|\chi \rangle^{lt}_{\text{cond}} > 0 \) if \( B \geq 0 \). If \( B = \chi \), \( \langle \chi|\chi \rangle^{lt}_{\text{cond}} = \sum_{y \neq 0} \phi_{1y} \psi_y^\dagger = \sum_y \phi_{1y} \psi_y^\dagger = 1 \) (since \( \psi_0^\dagger = 0 \) thanks to the normalization of left and right eigenvectors). It is important to note that

\(^1\)In practice in order to calculate conditional averages at large times, as defined above (see for example (31)), we can use the following formula
\[
\langle A(t)B(t') | \chi \rangle^{lt} = \lim_{t_0 \to -\infty} \frac{\sum_{x,y,z} A_x P(x,t|y,t') B_y P(y,t'|z,t_0) P_{\text{initial}}(z)}{\sum_{x,z} P(x,t|z,t_0) P_{\text{initial}}(z)} \tag{32}
\]
and similarly for multiple time correlation functions. The initial condition \( P_{\text{initial}} \) at time \( t_0 \) is arbitrary as far as \( P_{\text{initial}}(0) < 1 \). But for our purposes, it is easier to choose \( P_{\text{initial}}(0) = 0 \).
this is a significant difference from the standard treatment where \( \langle A(t)B(t') \rangle_{st} = \langle A \rangle_{st} \langle B \rangle_{st} \) at large time separation. This is because in the standard case with a non trivial stationary state, \( \varphi_0 \), one has \( \psi_x^0 = 1 \) for all \( x \). This can also be generalized to multiple correlations

\[
\langle A(t)B(t') \cdots C(t'') \rangle_{st} = \langle A \rangle_{st}^{t'} \langle B \rangle_{st}^{t''} \cdots \langle C \rangle_{st}^{t'''} \times (1 + O(e^{-\sum_j (\lambda_j - \lambda_1) \min(\Delta t, \Delta t', \cdots)})
\]

where \( t'' < t' < t \) in the \( t - t' = \Delta t \) and \( t' - t'' = \Delta t' \) large limit.

Using some examples illustrates this point and provides further insight. If we have \( A = \chi \), then, \( \langle \chi(t)B(t') \rangle_{st} \rightarrow \langle B \rangle_{st}^{t''} \). This is in contrast to the standard treatment. Correlation in standard treatment would be defined using the stationary state, i.e. the non trivial \( P_{st} = \varphi_0 \). Standard correlation, \( K_{st} = \sum_{x,y} A_x P(x,t|y,s) P_{st}(y) \). Therefore, if \( A_x = 1 \), \( K_{st} = \sum_{x,y} P(x,t|y,s) P_{st}(y) = \sum_y P_{st}(y) = \langle B \rangle_{st} \).

**Remark.** Looking at the previous two examples, we can see that using \( A = \chi \) the average is over trajectories that survive in the interior and don’t get absorbed by the boundary at least till time \( t \) where \( A = \chi \) is evaluated. So, we are essentially averaging over only the remaining/surviving particles/trajectories.

With these preliminaries in place, we can now look at how a perturbation in the system affects the observables. We now proceed to look at linear response to perturbation in such systems.

### 2.2 Linear Response

To capture the effect of perturbation, let us consider the perturbation to be represented by a change in the rates, i.e. consequently a change in \( H \). The effect of this perturbation will be a change in the long time state. The general case at all times will be treated in Section 5

\[
H \rightarrow H + \delta H(t)
\]

\[
P^\Omega(t) \rightarrow P^\Omega(t) + \delta P(t)
\]

where \( P^\Omega(t) = \varphi_0 + c_1 e^{\lambda t} \varphi_1 \) which is a more general version of the long time state starting from \( P^\Omega(0) \) defined in the earlier subsection. Therefore, we can write an equation for the time evolution of \( \delta P(t) \) and solve it as a first order ODE.

\[
\dot{\delta P(t)} + \delta P(t) = (H + \delta H(t)) P^\Omega(t) + \delta \Pi(t)
\]

\[
\delta P(t) = H \delta P(t) + \delta H(t) P^\Omega(t) + \text{higher order terms}
\]

The solution of this with initial condition, \( \delta P(0) = 0 \) is

\[
\delta P(t) = \int_0^t ds \ e^{H(t-s)} \delta H(s) \ P^\Omega(s)
\]

Then, the change in our conditional average of observable is going to be

\[
\frac{\delta \langle A \rangle_t}{\langle \chi \rangle_t^t} = \frac{\delta \langle A \rangle_t}{\langle \chi \rangle_t^t} - \langle A \rangle_{\chi(t)}^{tt} \frac{\delta \chi(t)}{\langle \chi \rangle_t^t} + \text{higher order terms}
\]

\[
\frac{\delta \langle A \rangle_t}{\langle \chi \rangle_t^t} = \sum_x A_x \delta P_x(t)/\langle \chi \rangle_t^t = \frac{1}{\langle \chi \rangle_t^t} \int_0^t ds \sum_{x',y} A_x P(x,t|x',s) \delta H_{x',y} \ P^\Omega_y(s)
\]

Let us assume that the perturbation can be split into time and state components, i.e. \( \delta H(s) = V \delta F(s) \), with \( V_{x,0} = 0 \). Then, we can write the perturbation of the operator average as a response function times the time component of the perturbation. That is,

\[
\frac{\delta \langle A \rangle_t}{\langle \chi \rangle_t^t} = \int_0^\infty ds \ R_{A,V}(t,s) \delta F(s)
\]
with \( R_{A,V}(t,s) = \Theta(t-s) \sum_{x,x',y} A_x P(x,t|x',s) V_{x',y} P_{ly}^t (s) \equiv \Theta(t-s) \langle A(t) B(s) || \chi \rangle^lt \)

and

\[
B_{x'} = \frac{1}{\varphi_1 x'} \sum_y V_{x',y} \varphi_1 y
\] (41)

Similarly we also write \( \delta \langle \chi \rangle^lt = \int_0^\infty ds R_{A,V}(t,s) \delta F(s) \). Ultimately, with the defined \( B \), we have

\[
\delta \langle \chi \rangle^lt = \int_0^\infty ds \hat{R}_{A,V}(t-s) \delta F(s)
\] (42)

\[
\hat{R}_{A,V}(t-s) = \Theta(t-s) [\langle A(t) B(s) || \chi \rangle^lt - \langle A || \chi \rangle^lt \langle \chi(t) B(s) || \chi \rangle^lt]
\] (43)

Notice that, thanks to Eq.(34) and \( \langle \chi || \chi \rangle^lt = 1 \), we have \( \lim_{t \to \infty} \hat{R}_{A,V}(t) = 0 \). Now that we have the form of the response function with conditional correlations, we can try and understand the fluctuation dissipation theorems associated with it.

**Fluctuation Dissipation Theorem**

Similar to the standard case, let us assume that \( H \) depends on some parameter \( f \) (can also be a set of parameters. For simplicity, we consider one, but the calculations remain the same for multiple parameters). We can now Taylor expand \( H \) around no perturbation in \( f \)

\[
H(f + \Delta f) = H(f) + \frac{\partial H(f)}{\partial f} \bigg|_{\Delta f = 0} \Delta f + O(\Delta f^2)
\]

We drop the subscript \( \Delta f = 0 \) for visual simplicity. We see that \( V = \partial_f H(f) \). Since \( (H + \lambda_1 \mathbb{1}) \varphi_1 = 0 \), differentiating,

\[
\left( V + \frac{\partial \lambda_1 \mathbb{1}}{\partial f} \right) \varphi_1(f) + (H(f) + \lambda_1 \mathbb{1}) \frac{\partial \varphi_1}{\partial f}(f) = 0
\] (44)

Using the above and Eq (31) and (41), we can write the correlation function

\[
\langle A(t) B(s) || \chi \rangle^lt = \sum_{xy} A_x \left( e^{(H(f) + \lambda_1 \mathbb{1})(t-s)} \right)_{xy} (V \varphi_1)_y
\]

\[
= \left[ - \frac{\partial}{\partial t} \langle A(t) \frac{\partial \phi(s)}{\partial f} || \chi \rangle^lt - \langle A || \chi \rangle^lt \frac{\partial \lambda_1(f)}{\partial f} \right]
\] (45)

In (45), if we use \( A = \chi \), we get a similar form for the second term in (43) and we finally obtain (we have dropped the \( f \) dependence in \( \phi \) only for simplicity of expression)

\[
\hat{R}_{A,V}(t-s) = -\Theta(t-s) \frac{\partial}{\partial t} \left( \langle A(t) - \chi(t) \langle A || \chi \rangle^lt \rangle \frac{\partial \phi(s)}{\partial f} || \chi \rangle^lt \right)
\] (46)

Equation (46) is the **Fluctuation Dissipation Theorem** for our case of absorbing state in finite and discrete systems! Notice that

\[
\left\langle \left( A(t) - \chi(t) \langle A || \chi \rangle^lt \right) || \chi \right\rangle^lt = 0.
\] (47)

In the standard treatment, the second term involving \( \chi(t) \langle A || \chi \rangle \) in eq.(46), is absent. Its presence is important since, from, the remark at the end of the previous section, we expect that, in general

\[
\frac{\partial}{\partial t} \left( \chi(t) \frac{\partial \phi(s)}{\partial f} || \chi \right)\rangle^lt \neq 0
\] (48)
3 Continuous systems

The Fokker-Planck Equation (FPE) describes the time evolution of probability density function under drift and diffusive forces [1]. Though the FPE can be derived from the Master equation assuming small jumps, the derivations we have for discrete state systems may no longer hold in the case of continuous space. One of the primary reason is the change in the eigenvalue spectrum. The formulation developed earlier assumes finite set of states and hence has a finite number of eigenvalues. The spectrum of eigenvalues is infinite or even continuous in certain cases as we shall see.

3.1 Response Function

For simplicity, we consider a one dimensional case. Let us consider the system to be defined on a space \( S \subseteq \mathbb{R} \). \( S \) contains absorbing boundaries at \( S_a \subseteq S \). Then, using the Fokker-Planck Equation, we can write the probability of finding a particle at \( x \) at time \( t \)

\[
\frac{\partial P(x,t)}{\partial t} = \mathcal{L}(x) P(x,t) \tag{49}
\]

where \( \mathcal{L}(x) \) is the Fokker Planck Operator given by

\[
\mathcal{L}(x) = -\frac{\partial}{\partial x} D_1(x) + \frac{\partial^2}{\partial x^2} D_2(x) \tag{50}
\]

In the absence of a sufficient source term, if an absorbing boundary is present, then, the probability \( P(x,t|x_0,t_0) \to 0 \) for \( t \to \infty \). In such cases, the probability is not normalized at all times, i.e, the total probability is not conserved. But, like with discrete case, certain considerations on the asymptotic nature of the probability distribution allows us to make progress.

Let us consider a perturbation occurring at time \( t = 0 \) and consider \( t_0 < 0 \). The solution to (49) can be written with \( P(x,t_0) \) being the initial condition,

\[
P(x,t) = e^{\mathcal{L}(x)(t-t_0)} P(x,t_0) \tag{51}
\]

Similar to the standard case, we represent the perturbation by a change in the Fokker Planck Operator and equivalently the resulting change in the solution to the FPE, i.e, the probability

\[
P_{perturb}(x,t) = P(x,t) + \delta P(x,t) \tag{52}
\]

\( P(x,t) \) and \( \mathcal{L}(x) \) are used to represent the unperturbed quantities. We assume that the F-P operator has the same form as without the perturbation, i.e.

\[
\delta \mathcal{L}(x,t) = -\frac{\partial}{\partial x} \delta D_1(x,t) + \frac{\partial^2}{\partial x^2} \delta D_2(x,t) \tag{53}
\]

At the leading order, we get,

\[
\delta P(x,t) = \delta \mathcal{L}(x,t) P(x,t) + \mathcal{L}(x) \delta P(x,t) \tag{54}
\]

Because of causality, the solution to (54) can be written down as follows

\[
\delta P(x,t) = \int_{-\infty}^{t} d\tau e^{\mathcal{L}(x)(t-\tau)} \left( \delta \mathcal{L}(x,\tau) P(x,\tau) \right), \tag{55}
\]

since \( \delta \mathcal{L} = 0 \) for all negative times. Until this stage, it has a common form as that of the standard case. The survival probability of finding a non-absorbed trajectory at time \( t \) starting from the initial condition at \( t_0 \) is

\[
\Pi(t) = \int_{\text{S}} dx \ P(x,t) \tag{56}
\]
We can also see the total probability conservation failing from (56) given absorbing boundary conditions. For an observable \(A(x)\), the average at time \(t\) is given by

\[
\langle A(t) \rangle = \int dx \ A(x) \ P(x,t)
\]  

(57)

This average decays to zero due to the decay of the probability. But, if we condition the observable average to be calculated on only the surviving trajectories, we can expect a non-trivial result. Then, we define

\[
\langle A(t) \rangle_s = \frac{\langle A(t) \rangle}{\text{prob of survival}} = \frac{\langle A(t) \rangle}{\Pi(t)}
\]  

(58)

It is now useful to define a conditional distribution \(Q(x,t)\). Eventually, at long times, this will give us the quasi stationary distribution \([4]\).

\[
Q(x,t) = \frac{P(x,t)}{\int dx' P(x',t)} \stackrel{t \to \infty}{\longrightarrow} Q_{st}(x)
\]  

(59)

At this point, we are assuming that a quasi stationary distribution (QSD) exists, for the unperturbed state. This is equivalent to saying \(\Pi(t)\) and \(P(x,t)\) have the same rate of decay. The implications of this can be seen in terms of averages and correlation of observables. In terms of (59), the average can be written as

\[
\langle A(t) \rangle_s = \int dx \ A(x) \ Q(x,t)
\]

Since we assume QSD exists, as \(t \to \infty\), \(\langle A(t) \rangle_s \to \langle A \rangle_s \equiv \int dx \ A(x) \ Q_{st}(x)\)

For correlation between two observables, we can also perform similar operation conditioned on survival. If \(A(x)\) and \(B(x')\) are two observables, assuming \(t' < t\) without loss of generality, the correlation is written as

\[
\langle A(t) \ B(t') \rangle = \int \int dx dx' \ A(x) \ P(x,t|x',t') \ B(x') \ P(x',t')
\]  

(60)

Since \(P(x,t|x',t') = \exp\{\mathcal{L}(x)(t-t')\} \delta(x-x')\) we get

\[
\langle A(t) \ B(t') \rangle = \int dx \ A(x) \ e^{\mathcal{L}(x)(t-t')} \ (B(x) \ P(x,t'))
\]  

(61)

Correlation conditioned on survived trajectories have to be divided by the survival probability at time \(t\). The caveat is that the survival probability should be considering when the system started, i.e., at \(t_0\) and not just from \(t'\). Then, we have

\[
\langle A(t) \ B(t') \rangle_s = \frac{\langle A(t) \ B(t') \rangle}{\int dx' P(x',t)} = \frac{1}{\Pi(t)} \int dx \ A(x) \ e^{\mathcal{L}(x)(t-t')} \ (B(x) \ P(x,t'))
\]

\[
= \int dx \ A(x) \ e^{\mathcal{L}(x)(t-t')} \ (B(x) \ Q(x,t')) \frac{\Pi(t')}{\Pi(t)}
\]

(62)

Since (51) depends on \(t-t_0\), what we need is the difference to go to infinity to reach QSD. Hence, we can also take the limit \(t_0 \to -\infty\). This is consistent with our formulation of the perturbation happening at \(t = 0\). With \(t_0 \to -\infty\), we can assume that \(\frac{\Pi(t')}{\Pi(t)}\) should approach the form \(\exp\{\lambda(t-t')\}\) for some \(\lambda\) (which will turn out to be the leading eigenvalue of \(\mathcal{L}(x)\)). Further in the calculations, we consider different cases where we make some deeper assumptions based on the eigenvalue spectrum from which the exponential behaviour immediately follows. Although the long time ratio does not depend on \(t_0\), this does not mean that the individual \(\Pi(t)\) do not depend on \(t_0\). This is a heuristic assumption based on the nature of eigenvalues and a broad class of examples which show this behaviour.

Then,

\[
\langle A(t) \ B(t') \rangle_s =_{t_0 \to -\infty} e^{\lambda(t-t')} \int dx \ A(x) \ e^{\mathcal{L}(x)(t-t')} \ (B(x) \ Q_{st}(x))
\]

(63)
Having seen how conditioning on survival changes the original definitions, we can use them to modify our response function. Change in the observable caused by perturbation in the operator can then be written as

\[ \delta \langle A(t) \rangle_s = \delta \left[ \frac{1}{\int dx' P(x',t)} \int dx \ A(x) \ P(x,t) \right] \]

\[ = \frac{1}{\int dx' P(x',t)} \int dx \ A(x) \ \delta P(x,t) - \frac{\int dx \ \delta P(x,t)}{\int dx \ P(x,t)}^2 \int dx \ A(x) \ P(x,t) \]  

(64)

Using (55),

\[ \delta \langle A(t) \rangle_s = \int dx \int_{-\infty}^{t} d\tau \ A(x) \ e^{\mathcal{L}(x)(t-\tau)} \left( \delta \mathcal{L}(x,\tau) \ Q(x,\tau) \right) \ \Pi(\tau) \frac{\Pi(t)}{\Pi(\tau)} - \int dx \ A(x) \ Q(x,t) \]

\[ \times \left( \int dx \int_{-\infty}^{t} d\tau \ e^{\mathcal{L}(x)(t-\tau)} \left( \delta \mathcal{L}(x,\tau) \ Q(x,\tau) \right) \ \Pi(\tau) \frac{\Pi(t)}{\Pi(\tau)} \right) \]  

(65)

We assume that the perturbation is separable, i.e., the spatial and temporal components are described separately. Assuming \( \delta \mathcal{L}(x,t) = \delta \mathcal{L}(x) \ F(t) \), we can then write (65) in terms of a response function. We also assume \( t_0 \to -\infty \). This is equivalent to allowing the system to evolve for a long period of time before introducing the perturbation. In the standard case, i.e., when a non-trivial stationary state exists, it is equivalent to starting at stationarity.

\[ \delta \langle A(t) \rangle_s = \int_{-\infty}^{\infty} d\tau R_{A,L}(t,\tau) \ F(\tau) \]

\[ R_{A,L}(t,\tau) \equiv_{t_0 \to -\infty} \Theta(t-\tau) \left[ \int dx \ A(x) \ e^{\mathcal{L}(x)(t-\tau)} \left( \delta \mathcal{L}(x) \ Q_{st}(x) \right) \ e^{\lambda(t-\tau)} \right. \]

\[ - \left. \langle A \rangle_s \int dx \ e^{\mathcal{L}(x)(t-\tau)} \left( \delta \mathcal{L}(x) \ Q_{st}(x) \right) \ e^{\lambda(t-\tau)} \right] \]  

(66)

In the standard case the last term of (66) would be

\[ I(t) \equiv \int dx \ e^{\mathcal{L}(x)t} \delta \mathcal{L}(x) P_{st}(x) \]  

(67)

and it is zero due to the normalization of \( P_{st}(x) \).

If we define the observable \( B(x) = Q_{st}^{-1}(x) \delta \mathcal{L}(x) \ Q_{st}(x) \), then, we can write the response function in terms of correlation like in the standard case. In doing so, we assume that \( Q_{st}(x) > 0 \ \forall x \). We define an additional variable that accounts for survival of the trajectories. Let \( \mathcal{I}(x) = 1 \ \forall x \in S^- \) (absorbing boundary). Then,

\[ R_{A,B}(t,t') = \langle A(t) \ B(t') \rangle_s - \langle A \rangle_s \langle \mathcal{I}(t) \ B(t') \rangle_s \]  

(68)

Here it becomes clear that \( \mathcal{I} \) accounts for survival of the trajectories till the time \( t \), according to the definition given in eq.(63).

### 3.2 Fluctuation Dissipation Theorem

While (68) connects the response of an observable to a correlation, it is quite uninformative since we do not know anything about the perturbation. Like in the standard case in Section 1, we can analyze the effect of change in parameters and how this affects the response function.

Before we proceed further, it is necessary to take into account the decaying nature of the distribution. Systems that reach a stationary state do not need such considerations. In the standard case, the leading eigenvalue of the Fokker-Planck operator, \( \lambda_0 = 0 \). The associated eigenfunction is the stationary state. But in decaying systems, while the solution to the FPE is still determined by the eigenvalues and the eigenfunctions, we no longer know anything about the leading eigenvalue except that it is non zero. Since that is the case, we also need the information about the spectrum of eigenvalues. Unlike in discrete space, the eigenvalue spectrum can be discrete or continuous in the continuous space case. This changes the problem significantly but also helps us in arriving at a FDT!
3.2.1 Discrete Eigenvalue Spectrum

Let us assume that the eigenvalue spectrum of $\mathcal{L}(x)$ is discrete. If $\lambda$ is an eigenvalue and $\psi$ is the associated right eigenfunction,

$$\mathcal{L}\psi = -\lambda\psi$$  \hspace{1cm} (69)

Then, we assume that the spectrum is of the form

$$0 < \lambda_1 < Re(\lambda_2) \leq Re(\lambda_3) \ldots$$  \hspace{1cm} (70)

**Note:** In discrete and finite systems, we proved the analogous of (70) starting from irreducible matrices. In continuous systems, in cases where there is a stationary solution and $x$-variables are even under time reversal, then, all the eigenvalues of the system are real. This is because we can write a measure such that the operator $\mathcal{L}$ is Hermitian (see V.7 in [5]). Consequently, if our system remains the same except for the boundary conditions changed to ensure decaying distribution, we can still use the same measure (Section 5.4 of [1]) to make the operator Hermitian and hence, all eigenvalues and eigenfunctions are real. But if we are more general, then, the eigenvalues can be complex. Asymptotically, the leading eigenvalue is what we expect to have dominant contribution to the probability. Since probability is real and non-negative, we can expect the leading eigenvalue to be real and corresponding eigenfunction to be real and non-negative. Hence, (70) is a general assumption encompassing different examples.

With (70), the solution to the FPE (51), with initial condition $\delta(x-x_0)$ can be written in term of the eigenfunctions and eigenvalues [1].

$$P(x,t) = \sum_{k=1}^{\infty} c_k(x_0) \psi_k(x) e^{-\lambda_k(t-t_0)}$$  \hspace{1cm} (71)

where $c_k(x_0)$ are determined by the initial conditions. They are also the left eigenfunctions of the FP operator, i.e, eigenfunctions of adjoint of $\mathcal{L}(x)$. Hence, at long times, $t-t_0 \to \infty$, (71) and the corresponding survival probability are

$$P(x,t) = c_1(x_0) \psi_1(x) e^{-\lambda_1(t-t_0)} [1 + O(e^{-Re(\lambda_2)-\lambda_1}(t-t_0)))]$$

$$\Pi(t) = c_1(x_0) e^{-\lambda_1(t-t_0)} \int dx' \psi_1(x') [1 + O(e^{-Re(\lambda_2)-\lambda_1}(t-t_0)))]$$  \hspace{1cm} (72)

From this, it can be seen that $\Pi(t)$ is equivalent to $\langle \chi \rangle_t$ in the finite system case. Hence, from Eq (59) and (56),

$$Q_{at}(x) = \lim_{t_0 \to -\infty} \frac{P(x,t)}{\Pi(t)} = \frac{\psi_1(x)}{\int dx' \psi_1(x')}$$  \hspace{1cm} (73)

and

$$\lim_{t_0 \to -\infty} \frac{\Pi(t)}{\Pi(t)} = e^{-\lambda_1(t-t)}$$  \hspace{1cm} (74)

and hence, (74) is no longer a primary assumption, but follows from a more basic assumption about the eigenvalue spectrum!

3.2.2 More General Eigenvalue Spectrum

Unlike discrete space with finite states, there is no reason for the eigenvalue spectrum to be discrete. The spectrum can be continuous and discrete together. To be general, we consider a spectral density $\rho(\lambda)$. Solely discrete spectrum is a particular case of the spectral density being a sum of Dirac $\delta$ functions. Then, in general, for $\lambda \in D \subset C$ with $Re(\lambda) > 0$ due to the absorbing boundary, Eq. (71) changes to

$$P(x,t) = \int_D \rho(\lambda) \psi_k(\lambda) c_{x_0}(\lambda) e^{-\lambda(t-t_0)} dRe(\lambda) \ dIm(\lambda)$$  \hspace{1cm} (75)
While naively, we might try to use asymptotic methods to calculate the dominant value, that would be based on the assumptions that $\rho(\lambda)$, $\psi(\lambda)$ and $e(\lambda)$ are well behaved and non trivial at $\lambda = \lambda_1 \equiv \inf \{ \text{Re}(\lambda) \}$. As we shall see in some physical examples, like the biased random walk on semi infinite line, the spectrum is real and near the edge of the spectrum, the density of states has power law behaviour $(\lambda - \lambda_1)^{-\beta}$ where $\beta > 0$ and the right and left eigenfunctions depend on the eigenvalues through $\sin(\sqrt{\lambda - \lambda_1}x)$ and $\sin(\sqrt{\lambda - \lambda_1}x_0)$.

For simplicity, we assume that $\lambda$ is real near the left edge of the spectrum and $\rho(\lambda)$ has the form

$$\rho(\lambda) \sim (\lambda - \lambda_1)^{-\beta} g(\lambda)$$

(76)

where $g(\lambda)$ is a well behaved function, finite at $\lambda_1$. The right edge of the spectrum is irrelevant since at asymptotically large times, the contribution is exponentially smaller. Assumption (76) has the advantage that in case there is no divergence, the exponent can be set to zero.

Similarly, for the right and left eigenfunctions (which are $\psi$ and $e$), we can make similar assumptions near the edge of the spectrum. But in this case, the functions don’t diverge and instead can go to zero.

For the case of eigenfunctions going to zero near the edge, let us assume

$$\psi_x(\lambda) = (\lambda - \lambda_1)^{\alpha} \hat{\psi}_x(\lambda)$$

$$c_{x_0}(\lambda) = (\lambda - \lambda_1)^{\beta} \hat{c}_{x_0}(\lambda)$$

(77)

In principle, we expect $\alpha = \gamma$, but to be more general, we can allow them to be different. Similar to (76), $\hat{\psi}_x(\lambda)$ and $\hat{c}_{x_0}(\lambda)$ are well behaved near the edge of the spectrum. Assumptions (76) and (77) can actually be presented in a more elegant manner.

Let $\Lambda = \lambda - \lambda_1$ and $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$ be the smallest numbers such that

$$\lim_{\Lambda \to 0} \Lambda^{-\alpha} \psi_x(\Lambda + \lambda_1) \neq 0$$

$$\lim_{\Lambda \to 0} \Lambda^{-\gamma} c_{x_0}(\Lambda + \lambda_1) \neq 0$$

$$\lim_{\Lambda \to 0} \Lambda^{\beta} \rho(\Lambda + \lambda_1) \in \mathbb{R}$$

(78)

The assumptions (76) and (77) are equivalent to saying $\alpha$, $\beta$ and $\gamma$ are finite. With these assumptions, we are in a position to tackle the beast that is (75).

First, we note that at asymptotically large times, the dominant contribution comes from the largest value of the exponential, i.e., the smallest value of $\Lambda(t - t_0)$, which would be $\lambda_1(t - t_0)$. But since our functions diverge/decay at that point, we first need to consider the integral in a small range $\lambda_1$ to $\lambda_1 + \epsilon$. Then, we are able to use our assumptions.

$$P(x, t) \sim \int_{\Lambda_1}^{\Lambda_1 + \epsilon} \rho(\Lambda) \psi_x(\Lambda) c_{x_0}(\Lambda) e^{-\Lambda(t - t_0)} d\Lambda$$

$$\sim \int_{\Lambda_1}^{\Lambda_1 + \epsilon} g(\Lambda) \hat{\psi}_x(\Lambda) \hat{c}_{x_0}(\Lambda) e^{-\Lambda(t - t_0)} \Lambda^{\alpha + \gamma - \beta} d\Lambda$$

$$\left( u = \Lambda(t - t_0) \right) \sim \frac{e^{-\Lambda_1(t - t_0)}}{(t - t_0)^{1 + \alpha + \gamma - \beta}} \int_0^{e(t - t_0)} du \frac{u}{(t - t_0) + \lambda_1}$$

$$\times \hat{\psi}_x \left( \frac{u}{(t - t_0) + \lambda_1} \right) \hat{c}_{x_0} \left( \frac{u}{(t - t_0) + \lambda_1} \right) e^{-u^{\alpha + \gamma - \beta}}$$

(79)

We now consider the limit $t_0 \to -\infty$. Let $h(y) = g(y) \hat{\psi}_x(y) \hat{c}_{x_0}(y)$. Then, Taylor expanding $h$ around $\lambda_1$. 

13
\[ P(x,t) \sim \frac{e^{-\lambda_1(t-t_0)}}{(t-t_0)^{1+\alpha+\gamma-\beta}} \int_0^\infty du \left[ h(\lambda_1) + \frac{u}{(t-t_0)} h'(\lambda_1) + \mathcal{O}\left( \frac{u^2}{(t-t_0)^2} \right) \right] e^{-u^\alpha+\gamma-\beta} \]

Similarly, we can also perform the asymptotic limit for the survival probability \( \Pi(t) = \int dx \ P(x,t) \) obtaining

\[ \Pi(t) \sim \frac{e^{-\lambda_1(t-t_0)}}{(t-t_0)^{1+\alpha+\gamma-\beta}} \Gamma(1+\alpha+\gamma-\beta) \ g(\lambda_1) \ \hat{\psi}_x(\lambda_1) \ \hat{\phi}_x(\lambda_1) + \text{h.o.t} \] (80)

Hence, the conditional probability density leads us to the quasi stationary distribution. We denote \( \hat{\psi}_x(\lambda_1) \) by \( \hat{\psi}_1(x) \).

\[ Q(x,t) = \frac{P(x,t)}{\Pi(t)} \xrightarrow{t_0 \to -\infty} \frac{\hat{\psi}_1(x)}{\int dx' \hat{\psi}_1(x')} = Q_{st}(x) \]

\[ \frac{\Pi(t')}{\Pi(t)} \xrightarrow{t_0 \to -\infty} e^{\lambda_1(t-t')} \] (82)

\[ \mathcal{L}(x)Q_{st}(x) = -\lambda_1 \ Q_{st}(x) \] (83)

It should be noted that the above relation also holds when the eigenvalue spectrum is discrete since it directly follows from (73).

### 3.2.3 Perturbation of Parameters

We now have all the necessary tools to arrive at a FDT. Like in the standard case, let us consider that \( \mathcal{L}(x) \) depends on a set of parameters \( f \equiv \{f_1, f_2, \ldots, f_n\} \). We show the calculations for a perturbation in one of these parameters because perturbation in multiple parameters follows in a similar manner. Assuming a perturbation in a parameter \( f \to f + \Delta f \), the Fokker-Planck operator can then be written as

\[ \mathcal{L}(x,t) = \mathcal{L}(x) + \frac{\partial \mathcal{L}(x)}{\partial f} \Delta f(t) + \mathcal{O}(\Delta f^2) \] (84)

Assuming perturbation till linear order, this is of the form

\[ \mathcal{L}(x,t) = \mathcal{L}(x) + \delta \mathcal{L}(x) \ F(t), \]

with \( \delta \mathcal{L}(x) = \frac{\partial \mathcal{L}(x)}{\partial f}, \quad F(t) = \Delta f(t) \) (85)

From (83), differentiating with respect to the parameter, we have

\[ \delta \mathcal{L}(x)Q_{st}(x) = -\mathcal{L}(x) + \lambda_1 \frac{\partial Q_{st}}{\partial f} - \frac{\partial \lambda_1}{\partial b} Q_{st} \] (86)

Because of (83) and \( \Pi(\tau)/\Pi(t) \xrightarrow{t_0 \to -\infty} e^{\lambda_1(t-\tau)} \) in (66) with \( \lambda = \lambda_1 \), the last term in (86) does not contribute to the response function. Substituting the rest into (66), we get
\[ R_{A,f}(t-\tau) = -\Theta(t-\tau) \left[ \int dx \ A(x) \ e^{(L(x)+\lambda_1)(t-\tau)} \ (L(x) + \lambda_1) \frac{\partial Q^{st}}{\partial f} + + \langle A \rangle_s \int dx \ e^{(L(x)+\lambda_1)(t-\tau)} \ (L(x) + \lambda_1) \frac{\partial Q^{st}}{\partial f} \right] = \]
\[
= -\Theta(t-\tau) \frac{\partial}{\partial t} \left[ \int dx \ A(x) \ e^{(L(x)+\lambda_1)(t-\tau)} \frac{\partial \phi(x)}{\partial f} Q^{st}(x) \right]
\]
\[
- \langle A \rangle_s \int dx \ e^{(L(x)+\lambda_1)(t-\tau)} \frac{\partial \phi(x)}{\partial f} Q^{st}(x) \right] = (87)
\]

Since \( x \) dependence of (80) is given by \( \dot{\psi}_1(x) \), we assume that it has to be positive for the probability to be positive. This is equivalent to our earlier statement about \( Q^{st}(x) > 0 \). Therefore, we are able to write \( Q^{st}(x) = e^{\phi(x)} \) where \( \phi \) is some potential. Setting the perturbation to happen at \( \tau = 0 \) and assuming \( t > 0 \) in (87), we can write it as

\[ R_{A,f}(t) = -\frac{\partial}{\partial t} \left[ \int dx \ A(x) \ e^{(L(x)+\lambda_1)(t-\tau)} \frac{\partial \phi(x)}{\partial f} Q^{st}(x) \right]
\]
\[ - \langle A \rangle_s \int dx \ e^{(L(x)+\lambda_1)(t-\tau)} \frac{\partial \phi(x)}{\partial f} Q^{st}(x) \right] = (88)
\]

where we have used (63).

Equation (88) is the Fluctuation Dissipation theorem we desire and we see that this is of the same form as Eq. (46). This connects the response of the perturbation of a parameter to a given observable that we can compute if we know the potential, i.e., the QSD. We have made a significant number of assumptions that we believe is based on physical systems. It might very well be possible to arrive at a very particular set of \( D_1 \) and \( D_2 \) in the Fokker-Planck operator that renders our assumptions invalid. But for the systems we believe the assumptions to hold, it now remains that we take those examples and verify whether the fluctuation dissipation theorem holds.

4 Examples and Verification

We want to take different examples to check whether the formalism developed earlier is correct. To do this, we analytically/numerically compute the response functions from the appropriate formula given the system. Then we consider a delta perturbation in the parameters, i.e,

\[ \Delta f(t) = \Delta f \times \delta(t) \quad \delta(t) \text{ is a Dirac Delta function} \]

This would reduce our response in the observable to perturbation in a parameter to just the response function, i.e,

\[ \delta \langle A(t) \rangle_{\text{survived}} = R_{A,f}(t) \Delta f \]

We perform simulation of the system to verify the results obtained analytically. For discrete systems, we perform the simulations using Gillespie algorithm. For continuous systems, we consider the Langevin equation and evolve it in a small time interval \( dt \) using the Euler-Maruyama algorithm. The system is allowed to run for a long time before the perturbation is applied, following which the responses are calculated. The response is calculated by measuring the conditional observable, i.e, mean value of the observable computed using only the particles that have survived till that time instant. We then compare it to the unperturbed observable and plot how the perturbation response changes with time.
4.1 Three State System

We consider a simple three state system from which all the eigenvectors and eigenvalues can be computed.

\[
\begin{array}{ccc}
0 & d & 1 \\
\downarrow & \downarrow & \downarrow \\
1 & b & 2 \\
\end{array}
\]

Corresponding to this, we can also write Master Equation as

\[\dot{P}(t) = HP(t)\]

where \(H\) is

\[
\begin{bmatrix}
0 & d & 0 \\
0 & -(b+d) & c \\
0 & b & -c
\end{bmatrix}
\]

It is possible to rescale time and all the other rates. This way, we will only have to work with two parameters instead of three. Writing \(t \equiv t' = ct\), \(b \equiv b' = b/c\), \(d \equiv d' = d/c\), we can set \(c = 1\) in our system.

The first eigenvalue is 0 as expected. The other two eigenvalues are both real and

\[
\lambda_{\pm} = \frac{b + d + 1 \pm \sqrt{(b + d + 1)^2 - 4d}}{2}
\]

\(\lambda_1\) is the smaller of the two eigenvalues. It is also seen that the condition on the eigenvalue spectrum is satisfied. Starting from this, for the eigenvector corresponding to this, we note that, like in the formalism, we set \(\phi_{10} = -1\). Computing the other two components, we obtain

\[
\varphi_1 = \begin{bmatrix}
-1 \\
\frac{\lambda_1}{d} \\
\frac{b\lambda_1}{d(1-\lambda_1)}
\end{bmatrix}
\]

It is also worth noting that the sum of the components is 0 and this is self consistent with our choice of \(-1\) for first component. By writing \(\varphi_1 = e^{\phi_1}\), for non zero states, and defining \(\phi_{10} = 0\), we have all the required elements to calculate response functions. To compute the propagator, i.e, the exponential of \(H\), we do diagonal decomposition of \((H + \lambda_1 I)(t - s)\) and exponentiate the diagonal matrix.

Observables and Responses

We consider the observable \(A_n = n\) and \(A_n = n^2\). These satisfy our requirement of \(A_0 = 0\). We have two parameters to perturb, the birth rate \(b\) and the death rate \(d\). We observe a timescale in the system which we denote by \(1/\tau = \sqrt{(b + d + 1)^2 - 4d}\). The response functions for perturbations in the parameters then are (for \(t > 0\))

\[
R_{n,b}(t) = \frac{\lambda_1}{d} \frac{e^{-t/\tau}}{d}
\]

\[
R_{n^2,d}(t) = \frac{3(d - 1 - b^2 - b(2 + d) + (b + 1)/\tau)}{2d^2} e^{-t/\tau}
\]

We verify these using simulations of the three state system. Initially we start from \(N = 10^7\) particles. We simulate the system for \(T_{st} = 10\tau\). The perturbations are in the \(b\) or \(d\) parameters by \(\Delta b/dt\) or \(\Delta d/dt\). The strength is chosen as such because it represents a discrete delta perturbation. We follow Gillespie Algorithm after the perturbation to simulate the system.
We see a good agreement of the simulations with the predictions, even considering the low value of the response function (due to which the statistics at long times are worse, especially since the number of particles in the system are continuously decaying).
4.2 Contact Process

The Contact Process is one of the important models in continuous time stochastic processes. A simplified model of the Directed Percolation class, the contact process shows many interesting non-equilibrium properties and is a useful tool in fields like epidemics. The model consists of $N$ sites, which can be infected {1} or not {0}. Infected sites have chance at infecting their healthy neighbours with rate $\lambda/2d$ ($\lambda$ is the standard notation used in the Contact Process model. It has no connection to the eigenvalues we have discussed) where $d$ is the dimension of the system. The infected sites heal into healthy sites at rate $1$ (any other rate $R$ can be considered to be equal to $1$ by appropriate rescaling of time). A representation of the model with the transition rates in 1D is given below. The filled circles represent infected sites and the empty sites are healthy sites.

\[ \lambda/2 \quad \lambda/2 \quad 1 \]  
\[ \ldots \quad \ldots \quad \ldots \quad \ldots \]  
\[ \ldots \quad \ldots \quad \ldots \quad \ldots \]  
\[ \ldots \quad \ldots \quad \ldots \quad \ldots \]  
\[ t \]

In the infinite size limit this model shows phase transition. For $\lambda < \lambda_c$, all sites eventually heal. Once all sites are healthy, there is no infected site to arise and hence, this is an absorbing state. For $\lambda > \lambda_c$, there exists a non-trivial value of the stationary value of average number of infected sites. We consider the absorbing regime to test the validity of our response theory. Notice, however, that as far as the system is finite we always end up in the absorbing state. The master equation for the contact process is

$$ \frac{\partial P_i}{\partial t} = \sum_j \left( W_{i,j} P_j - W_{j,i} P_i \right) $$  \hspace{1cm} (89)

where $i = (i_1, i_2, \ldots, i_N)$, $j = (j_1, j_2, \ldots, j_N)$ with $i_k, j_k \in \{0, 1\}$ being the possible states of node $k$, whereas $i$ and $j$ representing the state of the entire system. $i = 0$ is the absorbing state corresponding to $i_k = 0$, $\forall k$. We consider 1D model of the contact process with $N = 10$ lattice sites and $\lambda = 0.5$ and periodic boundary conditions. There are a total of $2^{10}$ possible states with absorbing state denoted by 0. The transition matrix is constructed based on the rules presented above. We consider only 1 event per transition. This means if there are two healthy neighbours of an infected site, only one of them can be infected per transition. From the transition matrix, we numerically calculate the second eigenvalue, the second right eigenvector and also the derivative of the eigenvector. Then, we can use this formula to calculate the response function for average number of infected sites, i.e. $A_i = x_i = \sum_{k=1}^{N} i_k$ being the number of infected sites in the state $i$:

$$ R_{x,\lambda}(t) = \frac{\partial}{\partial t} \left( -\sum_{i,j} x_i (e^{(H+\lambda I)t})_{ij} \frac{\partial \varphi_{ij}}{\partial \lambda} + \sum_j x_j \varphi_{1j} \sum_{i,j} \chi_i (e^{(H+\lambda I)t})_{ij} \frac{\partial \varphi_{ij}}{\partial \lambda} \right) $$

$$ = -\sum_{i,j} x_i (H + \lambda I) e^{(H+\lambda I)t} \chi_{ij} \frac{\partial \varphi_{ij}}{\partial \lambda} $$

$$ + \sum_j x_j \varphi_{1j} \sum_{i,j} \chi_i (H + \lambda I) e^{(H+\lambda I)t} \chi_{ij} \frac{\partial \varphi_{ij}}{\partial \lambda} $$

$\chi_i = 1$ for all non-zero states and zero for $i = 0$. $\varphi_1$ is the second right eigenvector and $\lambda_1$ is the corresponding eigenvalue of $H$ (see Eq. (46)).

For the simulation, we randomly start $10^7$ runs from one of the possible $2^{10} - 1$ states (ignoring the 0 state). We introduce a perturbation $\delta \lambda = 0.05 \lambda$ using discrete time evolution of $dt = 10^{-4}$ and the results are shown in Figure 3.
4.3 Birth Death Process

The formalism developed earlier assumes that $\Omega$ is finite. We have seen good agreement with the examples of three state system and the contact process, both of which are finite. In the case of infinite systems, the matrix $H$ becomes infinite dimensional. It is not necessarily guaranteed that one can find orthonormal eigenstates. Nevertheless, given a system for which unique eigenvalues and eigenstates exist, it would be interesting to see whether the theory remains valid even beyond the regime of finiteness of states, as long as the eigenvalue spectrum satisfies the condition we have for finite systems.

One such system is the Birth Death Process with absorbing boundary [6]. The lattice sites range from $n = 0, 1, 2, 3, ...$ with 0 being absorbing state. The transition rates at $n^{th}$ lattice site are given by

$$W_{n+1,n} = b_n = bn \quad W_{n-1,n} = d_n = dn$$

This automatically ensures that 0 is an absorbing state. The Master Equation for this system can be written as (setting $b_{-1} = 0$)

$$\dot{P}_i(t) = b_{i-1}P_{i-1}(t) + d_{i+1}P_{i+1}(t) - (b_i + d_i)P_i(t) \quad i \geq 0$$

To obtain the eigenvector, we can look at the generating function of the probability distribution [6]. The generating function is defined as

$$G(z,t) = \sum_{n \geq 0} P_n(t)z^n$$

$$= \left( \frac{1 - A(z,t)}{1 - (1 - \nu)A(z,t)} \right)^{n_n}$$

with $P_n(t = 0) = \delta_{n,0}$, $\nu = 1 - b/d$, $A(z,t) = \frac{1 - z}{1 - (1 - \nu)z}e^{-\nu t}$

At large times $A(z,t)$ can be considered to be small enough to approximate as $1/(1 - x)^n \approx 1 - nx$ in $G(z,t)$. With this, we find that the leading eigenvector is of the form $\varphi_{1n} = \nu(1 - \nu)^{n-1}$. Note that we can set $\varphi_{10} = -1$ since $\varphi_{1n}$ is normalized over the interior states.
The eigenvalues for the Birth Death Process are \( \lambda_i = (d-b)i \) [7] which can also be deduced from \( G(z,t) \). We also have the solution for the Master Equation [7]. A point of note is that we differ in the absorbing state used in [7]. While -1 is an absorbing state in [7], we choose 0 as the absorbing state. Therefore, the solution to the Master equation will be same as in [7] but with the index values shifted by 1 unit. We represent \( P_{n,n_0}(t) \) as probability of being at \( n^{th} \) state at time \( t \) starting from state \( n_0 \) at time \( t = 0 \). We define

\[
q_{n_0,n}(t) = \left( 1 - \frac{b}{d} \right)^2 \left( \frac{b}{d} \right)^{n_0-1} \frac{n_0}{n} (1 - e^{-\lambda_1 t})^{n_0-1} \left( 1 - \frac{b}{d} e^{-\lambda_1 t} \right)^{-n_0-1} \\
\times \sum_{k=0}^{n_0-1} \binom{n_0-1}{k} (-1)^k \left[ \frac{1 - \frac{d}{b} e^{-\lambda_1 t}}{1 - e^{-\lambda_1 t}} \right]^k \left[ \frac{1 - e^{-\lambda_1 t}}{1 - \frac{d}{b} e^{-\lambda_1 t}} \right]^{n_0-1-k} \frac{(n_0 + n + k)!}{(n_0 + n - k)! n_0!} 
\]

(91)

The summation over \( k \) can be written in a compact form as

\[
\frac{\Gamma(n_0 + n) (\frac{d(e^{-\lambda_1 t} - 1)}{be^{-\lambda_1 t} - d})^{n_0-1} 2F_1 (-n_0 - 1, -n_0 - 1; -n_0 - n - 3; \frac{(de^{\lambda_1 t} - be^{\lambda_1 t})(de^{\lambda_1 t} - de^{\lambda_1 t})}{be^{\lambda_1 t} - e^{\lambda_1 t})^2})}{\Gamma(n_0 + 1) \Gamma(n)}
\]

where \( 2F_1 \) is the hypergeometric function. In terms of the above definition we have:

\[
P_{n,n_0}(t) = \left( \frac{b}{d} \right)^{n-n_0} \frac{n_0}{n} q_{n_0,n}(t) \quad \text{if} \quad n_0 > n
\]
\[
= q_{n_0,n}(t) \quad \text{otherwise}
\]

(92)

**Response Function**

Since we are dealing with an infinite dimensional operator, we do not know how \( e^{Ht} \) acts on \( \partial f \varphi_1 \). Hence, we rewrite (46) slightly.

\[
R_{A,f}(t) = -\frac{\partial}{\partial t} \left( \langle A(t) \frac{\partial \phi(0)}{\partial f} || \chi \rangle - \langle A || \chi \rangle \langle \chi(t) \frac{\partial \phi(0)}{\partial f} || \chi \rangle \right) \]

(93)

\[
\langle A(t) \frac{\partial \phi(s)}{\partial f} || \chi \rangle = \sum_{x,y} A_x \langle e^{\lambda_1 (t-s)} \frac{\partial \phi_1 y(s)}{\partial f} \varphi_1 y \rangle
\]

(94)

Since \( P(t) = e^{Ht} P(0) \), with \( P_n(0) = \delta_{n,n_0} \), we have \( (e^{Ht})_{x,y} = P_{x,y}(t) \).

\[
\frac{\partial}{\partial t} \left( A(t) \frac{\partial \phi(s = 0)}{\partial f} || \chi \rangle \right) = \frac{\partial}{\partial t} \sum_{x,y} A_x P_{x,y}(t) e^{\lambda_1 t} \frac{\partial \phi_1 y}{\partial f} \varphi_1 y
\]

\[
= \sum_{x,y} A_x (\lambda_1 P_{x,y}(t) + \dot{P}_{x,y}(t)) e^{\lambda_1 t} \frac{\partial \phi_1 y}{\partial f} \varphi_1 y
\]

(95)

where we can use (90) for \( \dot{P}_{x,y}(t) \).

While closed form expression for \( R_{A,f} \) is difficult to calculate, we can compute it approximately using the explicit formula for \( P_{x,y}(t) \) from Eq. (92) and (91). An example is shown below. We consider \( A_n = n \) and a perturbation in \( b \). This response function represents the response of the mean population to change in birth rate. We see that the simulation follows our prediction.
It would be interesting now to compare the results of absorbing boundary to a similar non absorbing case. Instead of $b_n = b n$, if we consider a linear birth rate $b_n = (n + \beta) b$, the state 0 is no longer absorbing. Hence, if $b < d$, a stationary solution exists. The solution to this can be found in [7]. The stationary solution to non absorbing case with the given rates can easily be calculated and is

$$P_{st}(n) = (1 - b/d)^n (\beta)_n$$

where $(\beta)_n$ represents the Pochhammer notation.

While calculating averages and correlation, we do not use the formulation for absorbing case since $\phi_0 \neq \delta_{x,0}$ but is equal to $P_{st}$. Explicitly written,

$$\left\langle A(t) \frac{\partial \phi(s = 0)}{\partial f} \right\rangle = \sum_{x,y} A_x P_{x,y}(t) \frac{\partial \phi_{0,y}}{\partial f} P_{st}(y)$$

where $\phi_0$ is obtained from $P_{st} = e^{\phi_0}$.

If we choose $\beta = 1$, the results simplify considerably. Interestingly, in this case, the response is the same as absorbing case! This can be qualitatively understood looking at the similar form of $\phi_0$ in non reflecting case and $\phi_1$ in the absorbing case. Additionally, the solution to the master equation is also quite similar in form for $\beta = 1$ case.
But this turns out to be because of the choice of $\beta = 1$. If we choose $\beta = 2$, then, the responses are no longer the same. We show that with an example of different observable below.
The Ornstein-Uhlenbeck process is one of the most standard examples in stochastic processes, describing the evolution of velocity in the overdamped limit. While the process can be considered over the entire real line in 1D, for our purposes, we limit the domain of velocity to be \([0, \infty)\) with an absorbing boundary at zero. The Langevin equation is given by

\[
\dot{v}(t) = -\gamma v(t) + \sqrt{D} \eta(t)
\]  

with \(\eta(t)\) being a Gaussian white noise with zero mean and delta correlation, \(\langle \eta(t) \rangle = 0\) and \(\langle \eta(t) \eta(t') \rangle = 2\delta(t - t')\).

The Fokker Planck operator can be transformed into a Hermitian operator. This is achieved by the use of an appropriate function \(\phi\).

\[
\mathcal{L} = e^{\phi_s/2} \mathcal{L} e^{-\phi_s/2}
\]

The potential \(\phi_s\) is the same one that controls the stationary solution, i.e., \(P_{st} = N e^{-\phi_s}\), where \(N\) is the normalization constant. In the presence of an absorbing boundary, the stationary solution does not exist, but the function \(\phi_s\) can still be written in certain cases. Usually, this can occur from the stationary solution which exists if the boundary condition is changed from absorbing to reflecting. With the Hermitian form of the operator, it is possible to transform the Fokker-Planck equation to a Schrödinger like equation (see 5.4 in [1]). Therefore, problems in stochastic processes can be solved by identifying the appropriate potential that enters into the Schrödinger equation.

For the Ornstein-Uhlenbeck process on the real line with natural boundary conditions, the solution is given by the eigenfunctions which are Hermite polynomials, \(H_n\).
Ornstein-Uhlenbeck on \((-\infty, \infty)\)

\[ \mathcal{L}_0 \psi_n = -\lambda_n \psi_n \]

\[ \lambda_n = \gamma n \]

\[ \psi_0(v) = 4 \sqrt{\frac{\gamma}{2\pi D}} e^{-\gamma v^2/2} = \sqrt{P_{st}(v)} = \sqrt{Ne^{-\phi_s(v)/2}} \]

\[ \psi_n(v) = 4 \sqrt{\frac{\gamma}{2\pi D}} \frac{1}{\sqrt{2^n n!}} H_n \left( \sqrt{\frac{\gamma}{2D}} v \right) e^{-\gamma v^2/2} \]

\[ P(v, t|v_0, 0) = e^{\phi_s(v_0)/2-\phi_s(v)/2} \sum_n \psi_n(v) \psi_n(v_0) e^{-\lambda_n t} \tag{98} \]

For absorbing boundary at zero, only the eigenfunctions which are zero are zero should be retained. Furthermore, the flux at zero should be non zero. The odd Hermite polynomials satisfy these conditions. Hence, only the odd eigenfunctions are eigenfunctions for the problem with absorbing boundary, i.e, only \( n = 2m - 1 \) for \( n \geq 1 \) eigenvalues are eigenfunctions of (98) are solutions. For our problem, we only need the first non zero eigenvalue and the eigenfunction, which corresponds to \( n = m = 1 \).

While there is no summation formula for the odd Hermite polynomials, we are able to use the method of images to construct the solution to the problem. To find the probability distribution of finding the velocity to be \( v \) at a time \( t \), consider two initial conditions, \( v_0 \) and \(-v_0\). We know the solution to the OU process on the real line. Using the two initial conditions and consequently the two solutions, imposing the absorbing boundary condition implies that we need the combination of the two solutions to be zero at zero. This is achieved simply by taking the difference of the two. It can be verified that the solution satisfies the corresponding FPE to (96) with absorbing BC.

**Ornstein-Uhlenbeck on \([0, \infty)\) with absorbing boundary at 0**

\[
\begin{align*}
P(v, t|v_0, 0) &= \sqrt{\frac{\gamma}{2\pi D(1-e^{-2\gamma t})}} \left\{ e^{\gamma(v-v_0 e^{-\gamma t})^2/2D(1-e^{-2\gamma t})} - e^{\gamma(v+v_0 e^{-\gamma t})^2/2D(1-e^{-2\gamma t})} \right\} \tag{99}
\end{align*}
\]

The quasi stationary distribution can be evaluated, both from long time limit of (99) and the first eigenfunction. They both match and give the result

\[ Q_{st}(v) = \frac{\gamma v}{2D} e^{-\gamma v^2/2} \tag{100} \]

The simulations of the Langevin equation were performed using Euler-Maruyama method. After starting with sufficient number of iterations \((N = 10^8)\), all instances were allowed to run till time \( T = 10/\gamma \) when the perturbation was applied in a small \( dt \) window following which the responses were calculated using only the survived trajectories. The predicted response was calculated from numerically computed correlation function using (99). The comparison is showed in Fig 7.
Figure 7: Response of $\delta \langle v \rangle$ to perturbation in $\gamma$. Parameter values used are $\gamma = 0.5$, $D = 0.1$ and $\delta \gamma = 0.1$ with $dt = 10^{-4}$ for simulations. The matching can be further improved by using smaller $dt$ in simulations.

4.5 Biased Diffusion

Diffusion problems are central in the study of stochastic processes with applications to many fields. One of the common extension to diffusion is the presence of a constant drift term, thereby biasing the diffusion in a certain direction. The presence of a drift term modifies the Langevin equation to

$$\dot{x}(t) = -v + \sqrt{D} \eta(t)$$  \hspace{1cm} (101)

where $v > 0$ and $\eta(t)$ is a Gaussian white noise as described in the earlier example. There is no stationary solution for the diffusion problem on the entire real line since the distribution decays to zero with larger time. For a biased diffusion, we restrict the domain to the positive part of the real line with absorbing boundary to be at zero. The solution to this problem can be found through the method of images [8, 9].

$$P(x, t|x_0, 0) = \frac{1}{\sqrt{4\pi Dt}} \left( e^{-\frac{(x-x_0+v t)^2}{4Dt}} - e^{-\frac{(x+x_0+v t)^2}{4Dt}} \right)$$  \hspace{1cm} (102)

But the solution does not give us the eigenfunctions or the eigenvalues. We can start to find the solution by initially considering the domain to be closed between $[0, L]$. Then, for reflecting BC, the stationary solution gives us the potential with which we can transform the FP operator to a Hermitian operator.

**Biased Diffusion in closed interval with absorbing BC**

Let the system be on the domain $[0, L]$. Then, the corresponding FPE of (101) is given by

$$\dot{P} = v \partial_x P + D \partial_x^2 P$$  \hspace{1cm} (103)

In the case of domain being $[0, \infty)$ with reflecting boundary condition at $x = 0$, the stationary solution of (103) is $P_{st} = v/D e^{-vx/D}$. Since the Fokker-Planck operator has the same form for different boundary conditions and domains, we are able to get the potential $\phi(x)$ from the stationary solution in semi-infinite problem. Because of the existence of the potential, we are also able to transform the Fokker-Planck operator into a Hermitian operator in the case of closed domain $[0, L]$ with absorbing boundary conditions. Hence, the eigenvalues and the eigenfunctions of the operator are real. We stress that this might not be possible in all examples, but only in examples where the potential can be computed (See [10] for an example of non Hermitian form).
With a closed domain, we split the spatial and the temporal components of the solution to the FPE.

\[ P(x, t) = P_\lambda(x) e^{-\lambda t} \]  \hspace{1cm} (104)

\( P_\lambda(x) \) are the eigenfunctions of the FPE. Then, if \( e^{-\mu x} \) is a guess solution for the equation obeyed by \( P_\lambda(x) \),

\[ \mu = \frac{v^2}{2D} \pm \sqrt{\frac{v^4}{4D^2} - \frac{\lambda}{D}} \]  \hspace{1cm} (105)

Then, applying the boundary conditions, i.e, \( P(x, t) = 0 \) at \( x = 0 \) and \( x = L \), with the knowledge that the eigenfunctions and eigenvalues are real, we obtain

\[ P_\lambda(x) = \sqrt{\frac{2}{L}} e^{-\frac{vx}{2D}} \sin \left( \frac{\pi k}{L} x \right) \]

\[ \lambda_k = \frac{v^2}{4D} + \frac{D\pi^2 k^2}{L^2} \]  \hspace{1cm} (106)

The eigenvalues in this case are discrete. But note that in the limit \( L \to \infty \), the gap between successive eigenvalues continues to shrink. They then lead to a continuous spectrum. Such mixed spectrum problems can also be solved [11, 12]. Solving for the continuous spectrum, we obtain

\[ P(x, t|x_0, 0) = \int_{v^2/4D}^{\infty} \lambda d\lambda \left( \frac{\lambda}{D} - \frac{v^2}{4D^2} \right)^{-1/2} e^{-\frac{vx + \mu}{2D}} e^{-\lambda t} \]

\[ \sin \left( \sqrt{\frac{\lambda}{D} - \frac{v^2}{4D^2}} x \right) \sin \left( \sqrt{\frac{\lambda}{D} - \frac{v^2}{4D^2}} x_0 \right) \]  \hspace{1cm} (107)

Note that the integral when evaluated also gives the same solution as the one obtained from method of images. The problem can also be solved by making the change into QM perspective [1]. Considering a V shaped potential in the FPE, we get the biased diffusion problem. Imposing absorbing BC is equivalent to restricting the domain and considering eigenfunctions satisfying the BC. This method gives us the same eigenfunctions and the eigenvalues as earlier.

The eigenfunctions approach zero near the edge of the spectrum, i.e, near \( \lambda = v^2/4D \). Then, expanding the eigenfunctions and taking the limit as mentioned in the theory, we obtain the quasi stationary distribution. This agrees with the simulations of the systems when we take the conditional distribution at large times from simulation.

\[ Q_{st}(x) = \frac{v^2 x}{4D^2} e^{-\frac{vx}{2D}} \]  \hspace{1cm} (108)

The verification in this example is split into two parts. Since the spectrum of eigenvalues is continuous, the long time asymptotic behaviour of the probability distribution is not just exponential. Hence, obtaining an approximate quasi stationary distribution with sufficient number of particles is time consuming. Instead, we split the problem into verifying the approach to quasi stationary distribution and using a Monte Carlo generated QSD to verify the FDT.
Initially, we find that the conditional probability distribution approaches the quasi stationary distribution at large times (Fig 8). Then, we use Monte Carlo sampling to generate the required number of data points following the analytically predicted QSD which we use as the initial distribution onto which the perturbation is applied. The response is calculated as discussed in earlier examples and the results shown in Fig 9.

Figure 9: Prediction vs Simulation of response of average $x$ conditioned on survival to a perturbation in $v$. Parameter values are same as in Fig 8 with $\Delta v = 0.05$. The prediction starts from a time larger than the perturbation time because numerically computing the correlation at small times close to the perturbation is prone to floating point errors since the probability distribution approaches a Dirac Delta distribution.
4.6 Geometric Brownian Motion

The noise in Brownian Motion has a constant diffusion constant. But the diffusion coefficient can also depend on the variable under evolution. Geometric Brownian Motion is a model in which the logarithm of the position follows biased diffusion [13] and consequently, the position has a demographic noise component to it. The model has been widely used in finance to model stock prices [14]. The model is usually considered with positive drift and hence, solutions can be found to be a lognormal distribution.

\[
\dot{x}(t) = \mu x(t) + \sqrt{2} \sigma x(t) \eta(t)
\]  

(109)

For \( \mu > 0 \) and natural boundary conditions for \( \log(x) \), the time dependant distribution is

\[
P(x, t|x_0, 0) = \frac{1}{x \sqrt{4\pi \sigma^2 t}} e^{-\frac{(\log(x/x_0) - (\mu - \sigma^2) t)^2}{4\sigma^2 t}}
\]  

(110)

This is obtained by considering a change of variables \( y = \log(x/x_0) \) under Ito prescription. This leads to the SDE

\[
\dot{y}(t) = (\mu - \sigma^2) + \sqrt{2} \sigma \eta(t)
\]  

(111)

Note that this is just a biased diffusion Langevin equation, for which we know the solution. By using the solution and changing the variables back to \( x \), we obtain (110). But it is important to note that at \( x = 0, y = -\infty \). This means, applying natural boundary conditions on \( y \), i.e. the flux and the probability are zero at infinities, we also impose that on changing the variables back to \( x \), the flux and probability are zero at \( x = 0 \). This leads to complications, such as, if \( \mu < 0 \), then, the particle never reaches zero. The closer it gets to zero, the slower it moves. Hence, although it asymptotically reaches zero, there is no hitting the boundary.

Under this model, it is not possible to impose absorbing boundary condition at \( x = 0 \) since the particle is effectively never absorbed. We introduce a small \( \epsilon > 0 \) where the boundary is present. This changes the boundary conditions of the model, but allows us to impose an absorbing boundary at \( x = \epsilon \). We replace \( \mu \) with \( -\mu \) so that \( \mu > 0 \) and then follow similar procedure with the change of varibales. Then, under new boundary conditions, the time dependant distribution is

\[
\dot{x}(t) = -\mu x(t) + \sqrt{2} \sigma x(t) \eta(t)
\]  

(112)

\[
P(x, t|x_0, 0) = \frac{1}{x \sqrt{4\pi \sigma^2 t}} \left[ e^{-\frac{(\log(x/x_0) - \epsilon)^2}{4\sigma^2 t}} - e^{\frac{\mu \epsilon}{\sigma^2}} e^{-\frac{(\log(x/x_0) - \epsilon + \mu \epsilon t)^2}{4\sigma^2 t}} \right]
\]  

(113)

where \( \nu = -(\mu + \sigma^2) \) is the effective drift velocity. The flux at \( x = \epsilon \) is negative. Hence, the boundary is absorbing. Furthermore, the original model is recovered in the limit \( \epsilon \to 0 \) as expected.

Since we have the time dependant distribution, we can also compute the long time distribution and use it to calculate the quasi stationary distribution and the eigenvalue. This has the benefit of avoid the machinery that one needs to deal with to calculate the eigenvalues and the spectrum.

\[
\lambda_1 = \frac{\nu^2}{4\sigma^2}
\]  

(114)

\[
Q_{st}(x) = \frac{\nu^2}{4\sigma^4} \left( \frac{x}{\epsilon} \right)^{-\frac{\nu^2}{4\sigma^2}} \log \left( \frac{x}{\epsilon} \right)
\]  

(115)

We follow similar procedure as done with Biased Diffusion. The evolution and the convergence to the quasi stationary distribution was initially checked. It was followed with using Monte Carlo sampling to generate the required number of particles starting from quasi stationary state. In particular, it needs to be noted that the distribution has a slow decay. Due to this, the sampling can be done from \( y = \log(x) \) for a more representative distribution. Furthermore, all the calculations can be done with the change of variables to increase the numerical efficiency.

The verification is shown in Fig 10. The prediction starts at a later time owing to the same reason described in Ornstein-Uhlenbeck example. Nevertheless, after some time, when the time dependant distribution is no longer close to a delta function, the prediction matches very well with the simulations.
Figure 10: Prediction vs Simulation of response of survived $x$ conditioned to average on survival for Geometric Brownian Motion with absorbing boundary at $x = \epsilon$. The parameters used are $\epsilon = 0.1$, $x_0 = 1$, $\sigma = \sqrt{0.5}$, $\mu = 1$, $dt = 10^{-6}$ and $N = 10^8$.

5 MFDT with absorbing boundaries

In the standard case in Section 1 and in the extension to absorbing systems, the perturbation occurs either at stationarity or at long times when the solution to the FPE/ME is dominated by the leading eigenvalue and the eigenfunction. But the perturbation can also occur at a general non-stationary state. In the standard case with non-absorbing boundaries, the modified fluctuation dissipation theorem was derived by [2]. In a discrete system with an external field $h$, the response function is given by

$$R_A(t,t') = -\frac{d}{dt}(A(t) \partial_h \psi(t')|_{h=0})$$ (116)

Here, $\{c\}$ are the set of states, $h_t$ is the time dependent perturbation and the probability distribution, with the time dependent perturbation replaced by a constant perturbation $h$ is given by $\rho_t(c,h) = e^{-\psi_t(c,h)}$.

The deviation from the observable $A$ is then given by

$$\delta\langle A_t \rangle_h = \int_0^t dt' h(t') R_A(t,t')$$ (117)

where the subscript of $h$ is used to identify perturbed observable.

Similar to previous procedures, we condition the observable to survival. Then,

$$\delta\langle A_t \rangle_h = \frac{1}{\langle \chi_t \rangle_0} \delta\langle A_t \rangle_h - \frac{\langle A_t \rangle_0 \delta \langle \chi_t \rangle_h}{\langle \chi_t \rangle_0 \langle \chi_t \rangle_0}$$ (118)

The subscript 0 is used to denote unperturbed dynamics. Writing the conditioned observable with response functions,

$$\delta\langle A_t \rangle_h = \int_0^t dt' h(t') \left[ R_A(t,t') \frac{\langle A_t \rangle_0 \langle R_A(t,t') \rangle}{\langle \chi_t \rangle_0 \langle \chi_t \rangle_0} \right]$$ (119)
Since the distribution decays to zero in the presence of absorbing boundary, the solution to ME with constant perturbation also needs to be conditioned to survival. Defining a new potential as
\[ \phi_t(c) = \psi_t(c) + \log\langle \chi_t \rangle \]
we obtain the conditioned distribution \( \rho/\langle \chi \rangle \). Since \( \log\langle \chi_t \rangle \) is independent of states,
\[
\frac{1}{\langle \chi_t \rangle} \langle A_t \ [\partial_h \phi_t - \log(\langle \chi_t \rangle)] \rangle = 
\frac{1}{\langle \chi_t \rangle} \langle A_t \ \partial \phi_t \rangle - \frac{\langle A_t \rangle}{\langle \chi_t \rangle} \log(\langle \chi_t \rangle) 
\] (121)
Similarly, the term involving \( \chi \) results in
\[
\frac{\langle A_t \rangle}{\langle \chi_t \rangle^2} \langle \chi_t \ [\partial_h \phi_t - \log(\langle \chi_t \rangle)] \rangle = 
\frac{\langle A_t \rangle}{\langle \chi_t \rangle^2} \langle \chi_t \ \partial \phi_t \rangle - \frac{\langle A_t \rangle}{\langle \chi_t \rangle^2} \langle \chi_t \rangle \log(\langle \chi_t \rangle) 
\] (122)
We have dropped the subscript 0 and the evaluation at \( h = 0 \) of \( \partial_h \phi \) for visual simplicity in calculations. The last two terms in (121) and (122) cancel out. Then, the modified FDT has two contributions to the effective response function.
\[
\hat{R}_A(t, t') = -\frac{d}{dt'} \left[ \frac{1}{\langle \chi_t \rangle^2} \langle A(t) \ \partial \phi(t') \rangle - \frac{\langle A_t \rangle}{\langle \chi_t \rangle^2} \langle \chi(t) \ \partial \phi(t') \rangle \right] 
\] (123)
At the long time limit, i.e., \( t_0 \to -\infty \), the form of \( \chi_t \) depends only on the second right eigenvector and eigenvalue giving us result of previous calculations. It is to be noted that this is not entirely straightforward from the initial formula without introducing survival. The quantities in (116) end up becoming trivial under long time limit. Furthermore, the considerations of entropy production in [2] do not hold valid when there is absorbing boundary and therefore outgoing rate out of boundary state is zero.

We verify the modification of the FDT for a general non-stationary perturbation. The calculations of the quantities are more intensive, since there is no simplification arising out of asymptotic long time limit. Hence, we consider the Three-State System in which we know the eigensystem completely.
Figure 11: Verification of modification of FDT to non stationary perturbation with absorbing boundary in three state system. The perturbation is applied at time $t'$ with the system being prepared at $t_0 = 0$. The legend shows the different perturbation times with the system starting at $x_0 = 2$. The dashed line represents the predictions from (123). The gray lines represent the prediction and the simulation from the quasi stationary state, Figure 1. The parameter values used were the same ones in Figure 1.

Figure 11 shows remarkable agreement with the simulations at various perturbation times. Furthermore, the evolution of the response function with changing perturbation times is clearly seen, with the approach to the analytically predicted response from quasi stationary state.
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