Feedback Gains for Gaussian Massive Multiple-Access Channels

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Abstract—Feedback is shown to increase the sum-rate capacity of K-user Gaussian multiple-access channels by at most a factor of approximately 1.54, improving Thomas’ doubling bound (1987). The new bound is the best possible in the sense that it can be approached as closely as desired for a massive number of users. Moreover, feedback provides unbounded power gain in K for a fixed transmit power per user.

I. INTRODUCTION

The K-user (K ≥ 2) complex-alphabet Gaussian multiple-access channel (MAC) has output

\[ Y = Z + \sum_{k=1}^{K} X_k \]

where the \( X_k \), \( k = 1, \ldots, K \), are complex channel inputs and \( Z \) is complex, circularly-symmetric, Gaussian noise with unit variance. Consider the average block power constraints

\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} |X_{k,i}|^2 \leq P_k, \quad k = 1, \ldots, K \]

where \( \mathbb{E}[.] \) denotes expectation, \( X_{k,i} \) represents the channel input of user \( k \) at time \( i \), \( i = 1, \ldots, n \), and \( P_k > 0 \) for all \( k \). We remark that real-alphabet channels can be treated as complex-alphabet channels by grouping the real symbols into pairs that are treated as complex numbers.

The sum-rate capacity of the MAC without feedback in nats per channel use is known to be (see [1])

\[ C(P_1, \ldots, P_K) = \ln(1 + KP) \]

where \( P = \frac{1}{K} \sum_{k=1}^{K} P_k \) is the average transmitter power. The sum-rate capacity with feedback is unknown in general, even for the most informative causal feedback, i.e., the past channel outputs \( Y_1, \ldots, Y_{i-1} \) at time \( i \). A classic paper by Thomas [1] shows that feedback can at most double the sum-rate capacity. The result follows by treating symmetric power constraints, \( P_k = P \) for all \( k \), and then generalizing via a concavity argument [1, Sec. V]. More precisely, the feedback sum-rate capacity for general power constraints satisfies

\[ C_{FB}(P_1, \ldots, P_K) \leq C_{FB}(P, \ldots, P) \leq 2C(P, \ldots, P) = 2C(P_1, \ldots, P_K). \]

The sum-rate capacity is known for some interesting special cases. Ozarow [2] determined the capacity region for \( K = 2 \) and Kramer [3] found the sum-rate capacity for \( K \geq 3 \) under symmetric power constraints, \( P_k = P \) for all \( k \), and if \( P \) is beyond some threshold. For both these cases, the capacity matches a cut-set bound. Recent work by Sula et al. [4] builds on [3, 5–8] and shows that

\[ C_{FB}(P, \ldots, P) = \ln(1 + K P \lambda^*) \]

where \( \lambda^* \) is the unique real number in the interval \([1, K]\) that satisfies the dependence-balance bound (see [5–7])

\[ \frac{1}{K} \ln(1 + K P \lambda) \leq \frac{1}{K-1} \ln(1 + (K-\lambda) P \lambda) \]

with equality. The coding theorem was established in [3] and the converse is proved in [4].

This paper studies the power gain factor \( \lambda^* \) in (5) and the capacity gain factor

\[ F(\pi) = \frac{\ln(1 + \pi \lambda^*)}{\ln(1 + \pi)} \]

where \( \pi = K P \) is the total transmit power. Sec. II develops bounds on these factors and gives numerical results. For example, we improve the second inequality in (4) to

\[ C_{FB}(P, \ldots, P) \leq 1.54 \cdot C(P, \ldots, P) \]

which also improves (4) for general power constraints. Sec. III concludes the paper.

II. FEEDBACK GAINS

Fig. 1 and Fig. 2 plot \( \lambda^* \) and \( F(\pi) \) against \( \pi \) in decibels for selected \( K \). We prove several properties of these curves. Consider the simple bounds

\[ \frac{x}{1 + x} \leq \ln(1 + x) \leq x \]

for \( x > -1 \) with equality on both sides if and only if \( x = 0 \). We follow the proof steps of [3, Thm. 3] and write (6) with equality as in [3, Eq. (70)]:

\[ \ln(1 + \pi \lambda) = K \ln \left( 1 + \frac{P \lambda^2}{1 + (K-\lambda) P \lambda} \right) \]

(10)

Now apply (9) to bound

\[ \frac{\pi \lambda^2}{1 + \pi \lambda} \leq \ln(1 + \pi \lambda) \leq \frac{\pi \lambda^2}{1 + (K-\lambda) P \lambda} \]

(11)
To prove Property 1 fix \( \pi \) and use the left-hand side (LHS) of (9) to show that

\[
f(\pi, 1) > 1, \quad 0 < \frac{\partial f(\pi, \lambda)}{\partial \lambda} < \frac{\pi}{1 + \pi \lambda} \leq 1anumber{14}
\]

for \( P > 0 \). The bounds (14) imply that the equation

\[
\lambda = f(\pi, \lambda) = \frac{\pi}{1 + \pi \lambda}
\]

has a unique and finite solution for \( \lambda \geq 1 \). Moreover, equality holds on both sides of (11), and on the RHS of (12), for \( P \to 0 \). Thus, the maximal \( \lambda \) is attained by \( K \to \infty \), i.e., the uppermost curves in Fig. 1 and Fig. 2 are defined by (15).

Properties 2 and 3 follow by the LHS of (12) and derive bounds (12) also imply that \( \log \lambda \) scales as \( \log \log K \) for fixed \( P \), see [4 Thm. 3].

**Properties of \( F(\pi) \)**

Fig. 2 shows that \( F(\pi) \) has the following properties:

4) \( F(\pi) \to 1 \) for \( \pi \to 0 \) or \( \pi \to \infty \);

5) the maximal capacity gain factor is \( F(\pi) \approx 1.537 \) at \( \pi \approx 5.38 \) (or \( \pi = 7.3 \) dB);

6) \( F(\pi) \) is not too sensitive to \( \pi \) for \( \pi \geq 5 \) dB.

To prove Property 4 for small \( \pi \), apply the RHS of (9) to the RHS of (12) and rearrange terms to obtain

\[
(1 - \pi) \lambda \leq 1
\]

with equality if and only if \( \pi = 0 \). Now apply (9) and (16) to (7) to bound

\[
F(\pi) \leq (1 + \pi) \lambda \leq \frac{1 + \pi}{1 - \pi}, \quad \text{if } 0 \leq \pi < 1.
\]

We thus have \( \lambda \to 1 \) and \( F(\pi) \to 1 \) as \( \pi \to 0 \). Next, for large \( \pi \) we have \( f(\pi, \lambda) \approx \ln(1 + \pi \lambda) \) and in the limit of large \( K \) and \( \pi \) we have \( \lambda = \ln(1 + \pi \lambda) \). We thus have

\[
\lim_{\pi \to \infty} \frac{\lambda}{\lambda - \ln (1 + (e^\lambda - 1) / \lambda)} = 1.
\]

Property 5 is trickier to prove because the uppermost curve in Fig. 2 is defined by the implicit equation (15). We thus numerically evaluate \( F(\pi) \) for a range of \( \pi \) and derive bounds for the other \( \pi \). Fig. 2 shows that the uppermost curve is maximally \( F(\pi) \approx 1.537 \) at \( \pi \approx 7.3 \) dB in the interval \(-10 dB \leq \pi \leq 30 dB\). The Appendix treats \( \pi \leq -10 \) dB and \( \pi \geq 30 \) dB and shows that \( F(\pi) \leq 1.321 \) for these \( \pi \). In fact, numerical calculations show that all curves in Fig. 2 are unimodal, i.e., all curves increase with \( \pi \) from \( F(0) = 1 \) up to a peak and then decrease with \( \pi \) back to \( F(\infty) = 1 \).

Property 5 strengthens Thomas’ factor-of-two bound. Moreover, the new bound is the best possible in the sense that it can be approached as closely as desired as \( K \to \infty \). For example, for \( K = 100 \) and \( P = 0 \) dB we have \( \lambda^* \approx 8 \) dB and \( F(\pi) \approx 1.4 \). Also, for finite \( K \) we have tighter bounds than for massive \( K \), e.g., for \( K = 10 \) the maximal capacity gain factor is \( F(\pi) \approx 1.446 \) at \( \pi \approx 7.2 \) dB.
III. CONCLUSIONS

Feedback was shown to increase the sum-rate capacity of $K$-user Gaussian MACs by at most a factor of approximately 1.54. The new bound can be approached for a massive number of users at a total transmit power of $\pi \approx 7.3$ dB. Also, feedback provides unbounded power gain in $K$ for a fixed transmit power per user. Note that we have studied memoryless channels and it is interesting to consider channels with memory, see [9] for references on single-user channels and [10], [11] for MACs.

Finally, we remark that channel-output feedback is usually considered unrealistic for classic communications applications because of its high precision that requires high rate. However, future radio standards such as 6G also target control applications for mechanical systems that react slowly as compared to wireless communication speeds. If we view the uplink as being mechanical and slow, then the wireless downlink could have high rate. A second interesting application is internet-of-things (IoT) systems with large $K$ and low uplink communication rates but where the downlink may have high capacity.

APPENDIX

Implicit differentiation of (15) gives

$$\frac{d\lambda}{d\pi} = \frac{\pi}{\lambda} \cdot \frac{(\pi - 1)\lambda + 1}{\pi\lambda(\lambda - 1) + 2\lambda - 1}$$

which is positive for both $\pi \geq 1$ and $0 < \pi < 1$ via (16). The power gain factor $\lambda$ thus increases with $\pi$, i.e., the uppermost curve in Fig. 1 is increasing. In fact, all curves in Fig. 1 are increasing.

Consider now the range $\pi \leq -10$ dB. The RHS of (17) increases with $\pi$ and we thus have $F(\pi) \leq 11/9 \approx 1.222$. We see that $F(\pi) \leq 1.321$, as claimed, and the bound is loose since $F(0.1) \approx 1.048$.

Consider next the range $\pi \geq 30$ dB. Equation (15) gives

$$\lambda \geq \ln(1 + \pi\lambda)$$

and therefore

$$F(\pi) \leq \frac{\lambda}{\ln(1 + \pi\lambda)}$$

which is increasing with $\pi$ for $\lambda > e$.

The term in square brackets in (22) increases with $\pi$ if $\lambda > e$ because:

- $\lambda$ increases with $\pi$;
- $\pi\lambda/(1 + \pi\lambda)$ increases with $\pi\lambda$;
- $(\ln\lambda)/\lambda$ decreases in $\lambda$ for $\lambda > e$.

But for $\pi = 30$ dB we have $\lambda \approx 9.119 > e$, see Fig. 1 and (22) gives $F(\pi) \leq 1.321$, as claimed. We compute $F(1000) \approx 1.320$ so the bound is almost tight.

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