TWISTED LOGARITHMIC COMPLEXES OF POSITIVELY WEIGHTED HOMOGENEOUS DIVISORS

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Abstract. For a rank 1 local system on the complement of a divisor on a complex manifold $X$, its cohomology is calculated by the twisted meromorphic de Rham complex. Assuming the divisor is everywhere positively weighted homogeneous, we study conditions for a quasi-isomorphism from its twisted logarithmic subcomplex, called the logarithmic comparison theorem (LCT), by using a stronger version in terms of the associated complex of $D_X$-modules. If the connection is a pullback by a defining function $f$ of the divisor with residue $\alpha$, we show that LCT implies that the annihilator of $f^{\alpha-1}$ in $D_X$ is generated by first order differential operators and $\alpha-1-j$ is not a root of the Bernstein-Sato polynomial for any positive integer $j$. The converse holds in case the associated complex of $D_X$-modules is acyclic except for the top degree. We also give a simple proof of LCT in the hyperplane arrangement case under appropriate assumptions on residues as an immediate corollary of higher cohomology vanishing associated with Castelnuovo-Mumford regularity, where the zero-extension case is also treated.

Introduction

Let $D$ be a reduced divisor on a complex manifold $X$ of dimension $n$. Let $L$ be a rank 1 local system on the complement $U := X \setminus D$ with $\lambda_k$ eigenvalues of the local monodromies of $L$ around each global irreducible component $D_k$ of $D$. Choosing complex numbers $\alpha_k$ with $e^{-2\pi i \alpha_k} = \lambda_k$, we have a locally free $\mathcal{O}_X$-module $\mathcal{L}_X$ of rank 1 endowed with a meromorphic integrable connection $\nabla^{(\alpha)}$ which has a pole along $D$ and calculates the local system $L$ on $U$. Choosing local defining functions $f_k$ of $D_k \subset X$, it can be defined locally by using the twisted differential

$$\sum_k \alpha_k \omega_k \wedge : \mathcal{O}_X \rightarrow \Omega^1_X(\log D) \quad \text{with} \quad \omega_k := df_k/f_k,$$

and trivializing $\mathcal{L}_X$, where $\Omega^1_X(\log D)$ is the sheaf of logarithmic forms, see [SaK80]. We can apply the Hartogs extension theorem to see the independence of $\mathcal{L}_X$. (This is well known at the normal crossing locus of $D$, see [De70].)

Setting $\mathcal{M}_X(L) := \mathcal{L}_X(*D)$, we get the meromorphic de Rham complex

$$\mathcal{M}_X^\bullet(L), \nabla^{(\alpha)} := DR_X(\mathcal{M}_X(L), \nabla^{(\alpha)})[-n].$$

This is isomorphic to $R(j_U)_L$ in the derived category $D^b(X, \mathbb{C})$ with $j_U : U \hookrightarrow X$ the inclusion (by reducing to the normal crossing case, see [De70]). Note that $\mathcal{M}^0_X(L) = \mathcal{M}_X(L)$, since $DR_X$ is shifted by $n$.

Employing the logarithmic differential forms, we can get the logarithmic subcomplex

$$\mathcal{M}_{X, \log}^j(L), \nabla^{(\alpha)} \hookrightarrow (\mathcal{M}_X^\bullet(L), \nabla^{(\alpha)}),$$

with $\mathcal{M}_{X, \log}^j(L) := \Omega^j_X(\log D) \otimes_{\mathcal{O}_X} \mathcal{L}_X$ for $j \in \mathbb{Z}$. It is interesting whether $\mathcal{M}^j_{X, \log}(L)$ is a quasi-isomorphism. There is a lot of work about this, see for instance [CNM96], [WiYu97], [HoMo98], [To07], [CaNa09], [Ba22] (and also [Sa88], [Sa89]).

Since $\mathcal{M}^j_{X, \log}(L)$ is a morphism of differential complexes in the sense of [Sa88], [Sa89], we may consider a stronger version: Is $\mathcal{M}^j_{X, \log}(L)$ a $D$-quasi-isomorphism? Here a $D$-quasi-isomorphism

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means that we get a quasi-isomorphism applying the functor $\text{DR}^{-1}$ to the morphism (2), see [Sa89 (1.3.2)]. This stronger version is equivalent to the canonical quasi-isomorphism

$$3) \quad \text{DR}^{-1}_X(\mathcal{M}^*_X, \log(L), \nabla^{(\alpha)})[n] \xrightarrow{\sim} \mathcal{M}_X(L).$$

Here $\text{DR}^{-1}_X$ is the composition of $\text{DR}^{-1}_X$ with $\otimes O_X \omega_X^N$, and the latter is the inverse of the transformation from left $D_X$-modules to the corresponding right $D_X$-modules. (For the advantage of considering this stronger version, see for instance Theorem 2 below.) We can verify that the above two questions are equivalent to each other in the everywhere positively weighted homogeneous (or more generally, locally finite logarithmic stratification) case, see Lemma 1.7 and Proposition 1.7 below. We can solve these equivalent questions in the hyperplane arrangement case as an immediate corollary of higher cohomology vanishing for tame arrangements, see [CNM96, CaNa09, Ba22] (and also Corollary 2.1 below. Some more information is available in special cases of free divisors or hyperplane arrangements, see [Sa89, (1.3.2)]. This stronger version is equivalent to the canonical quasi-isomorphism $\text{DR}^{-1}_X(\mathcal{M}^*_X, \log(L), \nabla^{(\alpha)})[n] \xrightarrow{\sim} \mathcal{M}_X(L)$ for any $n \geq 0$.)

We say that a reduced divisor $D$ is positively weighted homogeneous around $p \in D$ if there are local coordinates $x_1, \ldots, x_n$ of $X$ with center $p$ such that $D \subset X$ is locally defined by a weighted homogeneous polynomial $f_p$ of strictly positive weights $w_{p,i}$, that is, $f_p$ is a linear combination of monomials $\prod_{i=1}^n x_i^{\nu_i}$ with $\sum_{i=1}^n w_{p,i} \nu_i = 1$. We say that $D$ is everywhere positively weighted homogeneous if the above condition is satisfied at any $p \in D$. (This is called locally (or strongly) quasi-homogeneous in [CNM96, CaNa09, Na15, etc.].)

There is an irreducible factorization $f_p = \prod_k f_{p,k}$, and we have $\alpha_{p,k} \in \mathbb{C}$ by choosing globally the $\alpha_k$ as above. Put

$$d_{p,k} := \deg_w f_{p,k}.$$

This is the weighted degree associated with the weights $w_{p,i}$ so that $\sum_k d_{p,k} = 1$. (In the hyperplane arrangement case, $d_{p,k} = (\deg f_p)^{-1}$.) Set

$$\tilde{\alpha}_p := \sum_k d_{p,k} \alpha_{p,k}, \quad e_p := \min\{ e \in \mathbb{Z}_{>0} \mid e w_{p,i} \in \mathbb{Z} \}.$$

These are locally constant on each stratum of the logarithmic stratification (which is locally finite in this case), see [1.6] below. In this paper we prove the following.

**Theorem 1.** Assume $D$ is positively weighted homogeneous around $p$. Then the stalks at $p$ of the source and target of (2) are both acyclic (hence (2) is a quasi-isomorphism at $p$) if

$$e_p \tilde{\alpha}_p \notin \mathbb{Z}.$$

This follows from a calculation of the twisted logarithmic complex in Proposition 1.5, see 2.7 below. Some more information is available in special cases of free divisors or hyperplane arrangements, see [CNM96, CaNa09, Ba22] (and also 3.1 below).

We then prove the following, which is very much inspired by Theorem A in Appendix, and gives a partial answer to a generalization of a question in [To07, 3.3].

**Theorem 2.** Assume $\alpha_k = \alpha$ (\forall k) for some $\alpha \in \mathbb{C}$. Consider the following conditions:

(a) The comparison morphism (2) is a $D$-quasi-isomorphism.

(b) The comparison morphism (2) is a quasi-isomorphism.

(c) The annihilator $\text{Ann}_{D_X}(f^{\alpha-1})$ is generated by $\tilde{\Theta}_{f,\alpha-1}$, see Remark 1.7a below.

(d) The annihilator $\text{Ann}_{D_X}(f^{\alpha-1})$ is generated by first order differential operators.

(e) We have $b_f(\alpha-1-j) \neq 0$ for any $j \in \mathbb{Z}_{>0}$.

(f) The $D_X$-module $O_X(*D)f^{\alpha-1}$ is generated by $f^{\alpha-1}$.

In general, condition (a) implies the other conditions, and we have the equivalences (c) $\Leftrightarrow$ (d) and (e) $\Leftrightarrow$ (f) unconditionally. If $D$ is everywhere positively weighted homogeneous and is tame (or more generally, $\text{DR}^{-1}_X(\mathcal{M}^*_X, \log(L), \nabla^{(\alpha)})[n] \xrightarrow{\sim} \mathcal{M}_X(L)$ is quasi-isomorphic to a $D_X$-module, see Corollary 1.7b below), then the above six conditions are equivalent to each other.
Here we assume the existence of Bernstein-Sato polynomial $b_f(s)$, which is called the BS polynomial for short in this paper, shrinking $X$ if necessary. For the definition of tame, see Corollary 1.4 below. The equivalence (e) $\Leftrightarrow$ (f) follows for instance from [Sa 21 Thm. 1], see Proposition 2.2 below. In case the two assumptions on $D$ for the last assertion are satisfied, we can show (c) $\Rightarrow$ (a) and (f) $\Rightarrow$ (a) as in 2.2 below. We also give a proof of (c) $\Leftrightarrow$ (e) using a quite different method in Appendix (where the tameness assumption cannot be weakened). It is not necessarily easy to construct an example where one can prove that $\text{dR}_X^m(\mathcal{M}_{X,\log}^\bullet(L), \nabla^{(\alpha)})[n]$ is not quasi-isomorphic to a $\mathcal{D}_X$-module (even in the non-free divisor case). This is partly related to [DiSa 12 (4.3.6)]. See [CaNa 05 Ex. 5.1] for such an example.

From now on, we assume that the local system $L$ is constant and the residues of connection are 0 in the introduction. In the isolated singularity case, Theorem 2 for $\alpha = 0$ combined with [To 04 Thm. 1.2], [HoMo 98] and Proposition 1.6 below implies the following.

**Corollary 1.** If the divisor $D$ has only isolated singularities and the logarithmic comparison theorem holds, that is, the comparison morphism (2) for $\alpha = 0$ is a quasi-isomorphism, then $D$ is locally positively weighted homogeneous, and it is a $\mathbb{Q}$-homology manifold for $n \geq 3$.

Returning to the general singularity case, consider the morphisms

$$H^j(\Omega_{X,p}^\bullet(\log D), d) \rightarrow H^j(\Omega_{X,p}^\bullet(D), d) \quad (j \in \mathbb{N}, p \in D),$$

where $(*)D$ denotes the localization along $D$. (In the hyperplane arrangement case, these are surjective as a consequence of [Br 73 Lem. 5].) We have the following.

**Theorem 3.** Assume $D \subset X$ is defined by a positively weighted homogeneous polynomial $f$ around $p \in D$.

(i) For $j \in \mathbb{Z}$, the morphism (6) is injective if and only if the weighted degree 1 part of the Brieskorn module $\mathcal{H}_{f,p}^j$ has no torsion, see 1.2 below for the notation.

(ii) Assume that $-1$ is the unique integral root of the local BS polynomial $b_{f,q}(s)$ at $q \neq p$ sufficiently near $p$. Then the morphism (6) for $j = n$ is surjective if and only if $-1$ is the unique integral root of the BS polynomial $b_f(s)$.

This follows from [BaSa 07 Thm. 1] (see 1.2 below) and [Sa 07 Thm. 2] (or Theorem 3 below) together with Proposition 1.4 below (showing a close relation between logarithmic forms and Brieskorn modules [Br 70], [BaSa 07]) by using the pole order filtration ([DiSa 12], see 2.3 below). Note that the roots of $b_f(s)$ and $b_f(s)$ are strictly negative, see [Ka 76].

By the same argument as in [DiSa 12 (4.3.6)], the Brieskorn module $\mathcal{H}_{f,p}^j$ is torsion-free for $1 < j \leq \text{codim}_X \text{Sing}_f$ and vanishes for $1 < j < \text{codim}_X \text{Sing}_f$. (Indeed, its corresponding Milnor fiber cohomology vanishes as a consequence of semi-perversity of the vanishing cycle complex, see [DiSa 04]. We can also use the microlocal Gauss-Manin system as in [DiSa 12] together with [JKSY 22 Prop. 2.]) For $j = 1$, $\mathcal{H}_{f,p}^1$ is a free $\mathbb{C}\{t\}$-module of rank 1, which is generated by $df$, and is isomorphic to $\mathbb{C}\{t\}$ endowed with the natural action of $\partial_t$, that is, $\partial_t(df) = 0$ (since $f$ is reduced), see for instance [BaSa 07 §2]. We have the following.

**Corollary 2.** Assume $X = \mathbb{C}^n$ with $n \geq 3$, $D$ is defined by a homogeneous polynomial $f$, the projective hypersurface $Z \subset \mathbb{P}^{n-1}$ defined by $f$ has only isolated singularities, and these are all weighted homogeneous. Then the morphism (6) is injective for any $j \in \mathbb{Z}$ and $p \in D$. It is surjective for any $j \in \mathbb{Z}$ and $p \in D$ if and only if $-1$ is the unique integral root of $b_f(s)$.

This is a corollary of Theorem 3 combined with [Sa 16b Thm. 2] showing that the pole order spectral sequence degenerates at $E_2$ (or equivalently, the Brieskorn module $\mathcal{H}_{f,0}^j$ is torsion-free, see [DiSa 12 Cor. 4.7]) under the hypothesis of Corollary 2. Since $b_f(s)$ is divisible by $b_{f,p}(s)$ for $p \neq 0$, the condition about roots of $b_f(s)$ in Corollary 2 implies that (2) induces an isomorphism on $X \setminus \{0\}$. For $\mathcal{H}_{f,0}^{n-1}$, we can use [DiSa 04 Thm. 0.1] (see Remark 1.3c below) together with Remark 1.4a below and also the inclusion $F \subset P$ (see [Sa 07 Prop. 4.4]) for the
case \( n = 3 \). Under the hypotheses of Corollary 2, we can show that \( \text{DR}_X^{-1}(\Omega_X^k/(\log D), d)[n] \) is quasi-isomorphic to a \textit{regular holonomic} \( \mathcal{D}_X \)-module (which is not necessarily a submodule of \( \mathcal{O}_X^k/(\log D) \)), see Remark 1.7c below. Note also that \(-n/d\) is not necessarily a roof of \( b_f(s) \) under the hypotheses of Corollary 2, for instance if \( f = x_1^a x_2^b + \sum_{i=3}^{n} x_i^d \) for \( a, b \geq 2 \) with \( a, b \) mutually prime and \( a + b = d \).

For \( n \geq 4 \), the morphism (8) is \textit{not} surjective under the hypothesis of Corollary 2 in almost all cases. There is, however, a \textit{quite exceptional case} for \( n = 4 \), where the morphism (2) is a quasi-isomorphism with assumptions of Corollary 2 satisfied; in particular, \(-1\) is the unique integral root of \( b_f(s) \). This happens if \( f = x^d + g(y, z) \) with \( g(y, z) \) a reduced homogeneous polynomial of degree \( d-1 \) in \( y, z \) and \( d \geq 3 \) (for instance, \( g = y^{d-1} + z^{d-1} \)), where [Sa 20b, Rem. 3.2] and the Thom-Sebastiani type theorem [Sa 94, Thm. 0.8] are used. It is unknown if there are other such examples for \( n \geq 4 \). If \( n = 3 \), there are many (see for instance [DiSt 17, Sa 16b]), and we have the following.

**Proposition 1.** Under the notation and assumptions of Corollary 2, assume further \( n = 3 \). Then the following conditions are equivalent:

(a) The morphism (2) is a quasi-isomorphism on \( X \).

(b) The morphism (6) for \( j = 3 \) is surjective at \( 0 \in X \).

(c) \(-1\) is the unique integral root of \( b_f(s) \).

(d) \( H^0_m(\mathcal{O}_{X,0}/(\partial f))_{d-3} = 0 \), or equivalently, \( M'_d = 0 \), see (2.4.3) below.

(e) All the roots of \( b_f(s) \) are contained in \((-2, 0)\).

Here \( m \) and \( (\partial f) \) are respectively the maximal and Jacobian ideals of \( \mathcal{O}_{X,0} \), and \( d := \deg f \).

This follows from Corollary 2 and Proposition 1.4 below using \textit{symmetries} of the \( \mu'_k \) and \( \delta_k := \mu'_k - \nu_{k+d} \) with center \( 3d/2 \) and \( d \) respectively (see Sa 16b (16)) together with [DiPo 16 Thm. 4.1] for (c) \( \iff \) (e), see 2.5 below.

Proposition 1 does not hold for \( n \geq 4 \), and there are many counterexamples (since the center of symmetry of the \( \mu'_k \) is at least \( 2d \)), see for instance [Sa 16b, Ex. 5.6], which shows the failure of (d) \( \Rightarrow \) (c) with \( n = 4, d = 6, M'_6 = M'_8 = 0, M'_{12} \neq 0 \). We can also set \( m = d-2 = 3, 4, 5 \ldots \) with \( n = 4 \) in the example written just after Corollary 3 below using [Sa 16b A.3], where \( \gamma_d \leq \mu_2 \). As for the failure of (c) \( \Rightarrow \) (e), consider \( f = x^d + y^{d-1}w + z^{d-1}w \) explained before Proposition 1 with \( d \geq 4 \), where the roots of \( b_h(s) \) with \( h := f|_{w=1} \) (and hence those of \( b_f(s) \)) are not contained in \((-2, 0)\), although \(-1\) is the unique integral root of \( b_f(s) \), see Remarks 1.4b and 2.5 below. (For \( n = 4 \), there is a counterexample to [DiPo 16 Thm. 4.1], but the assumption of Corollary 2 is not satisfied.) Note also that under the first two hypotheses of Corollary 2 with \( n = 3 \), we have \( H^0_m(\mathcal{O}_{X,0}/(\partial f)) = M' = 0 \) if and only if \( D \subset X \) is a free divisor, see [DiSt 17], etc. (This also fails for \( n \geq 4 \).)

By [DiSa 12, Thm. 5.2], the pole order spectral sequence \textit{never} degenerates at \( E_2 \) if the first two assumptions in Corollary 2 are satisfied, but not the last one (that is, some of the isolated singularities is not weighted homogeneous). However, this does not immediately imply the non-injectivity of (8), since we need the non-degeneration exactly at the degree \( d \) part as is shown by Theorem 3(i) (see also Remark 2.4 below).

Corollary 2 combined with Proposition 1.4 below, [Sa 16b Thm. 3], [DiSa 12 Cor. 1] gives the following.

**Corollary 3.** In the notation and assumption of Corollary 2 let \( \mu_Z \) be the sum of Milnor (or Tjurina) numbers of isolated singularities of \( Z \). Assume \( \left( \frac{d-1}{n-1} \right) > \mu_Z \) with \( d := \deg f \), or more generally, \( \gamma_{dn'} > \mu_Z \) with \( n' := \lfloor n/2 \rfloor \) in the notation of (2.4.3) below. Then the morphism (8) is injective for any \( j \in \mathbb{Z}, \) but not surjective for \( j = n-1 \) or \( n \).

The hypothesis is satisfied in the case \( Z \) has only one singular point which is a \textit{homogeneous} ordinary \( m \)-ple point (for instance, \( f = \sum_{i=1}^{n-1} x_i^m (x_i^{d-m} + x_n^{d-m}) \)) with \( \left( \frac{d-1}{n-1} \right) > (m-1)^{n-1} \) and
n ≥ 3. Note that \((d_{n-1}) = \gamma_d \leq \gamma_{dn'}\) in the notation of (2.4.3) below, see for instance [Sa16b (4.11.1)].

Combining Theorem 3(ii) with Lemma 1.6 and Remark 1.6a below, we can get the following.

**Corollary 4.** Assume that \(D\) is everywhere positively weighted homogeneous, and the BS polynomial \(b_f(s)\) of a local defining function \(f\) of \(D\) has an integral root which is strictly smaller than \(-1\). Then the morphism \((\mathcal{B}_f)\) is not an isomorphism for some \(j \in [1, n]\) (which does not necessarily coincide with \(n\)).

In the isolated singularity case, this is essentially known by [HoMo98], since the roots of \(b_f(s)\) are described by the Jacobian algebra in the weighted homogeneous isolated singularity case, see Remark 1.4b below. So Corollary 4 may be viewed as a partial generalization of [HoMo98]. Note that the last hypothesis on integral roots cannot be satisfied in the case of free divisors or hyperplane arrangements by [Na15], [Wa05] (or [Sa16a]), and a quasi-isomorphism holds in these cases, see [CNM96], [Ba22] (and 3 below).

In Section 1 we recall some basics of Brieskorn modules, Gauss-Manin systems, pole order filtration, and logarithmic complexes. In Section 2 we prove the main theorems applying the assertions in the previous section. In Section 3 we give a simple proof of a stronger version of the comparison theorem for hyperplane arrangements as an immediate corollary of [Sc03], [DeSi04]. In Appendix the annihilator of \(f^\alpha\) is studied.

### 1. Preliminaries

In this section, we recall some basics of Brieskorn modules, Gauss-Manin systems, pole order filtration, and logarithmic complexes

#### 1.1. Gauss-Manin systems.

Let \(f\) be a holomorphic function on a complex manifold \(X\) of dimension \(n ≥ 2\). Put \(D := f^{-1}(0) \subset X\). Let \(i_f : X \hookrightarrow X \times \mathbb{C}\) be the graph embedding by \(f\) with \(t\) the coordinate of \(\mathbb{C}\). Set

\[
\mathcal{B}_f := (i_f)_*^D \mathcal{O}_X = \mathcal{O}_X[\partial_t] \delta(t-f),
\]

where \((i_f)_*^D\) is the direct image as \(\mathcal{D}\)-module (and the sheaf-theoretic direct image by \(i_f\) is omitted to simplify the notation). The last term is a free module over \(\mathcal{O}_X[\partial_t]\) generated by \(\delta(t-f)\), and the actions of \(\partial_{x_i}\), \(t\) on \(\delta(t-f)\) are given by

\[
\partial_{x_i} \delta(t-f) = -(\partial_{x_i} f) \delta(t-f), \quad t \delta(t-f) = f \delta(t-f).
\]

The Gauss-Manin systems are defined by

\[
\mathcal{G}^j_{f,p} := \mathcal{H}^j \mathcal{K}^*_f \quad \text{with} \quad \mathcal{K}^*_f := \text{DR}_{X \times \mathbb{C}/\mathbb{C}}(\mathcal{B}_f) \quad (j \in \mathbb{Z},\ p \in D),
\]

where \(\text{DR}_{X \times \mathbb{C}/\mathbb{C}}\) is the unshifted de Rham complex so that \(\mathcal{K}^j_f = 0\) \((j \notin [0, n])\). By (1.1.1) the differential of \(\mathcal{K}^*_f\) is given by

\[
d(\eta \partial^k \delta(t-f)) = (d\eta) \partial^k t \delta(t-f) - (df \wedge \eta) \partial^{k+1} t \delta(t-f) \quad (k \in \mathbb{N}),
\]

and \(\mathcal{K}^*_f\) is essentially the double complex associated with \(d\) and \(df \wedge\). The \(\mathcal{G}^j_{f,p}\) are regular holonomic \(\mathcal{D}_{\mathbb{C},0}\)-modules (with \(\mathcal{D}_{\mathbb{C},0} = \mathcal{C}(t) \langle \partial_t \rangle\)), and correspond to the Milnor cohomology groups of \(f\) at \(p\) using the de Rham functor \(\text{DR}_{\mathbb{C}}\). They are finite free over \(\mathbb{C}\{\{t^{-1}\}\}[\partial_t]\) for \(j \neq 1\) (since the Milnor fibers are contractible). We have the isomorphisms

\[
\text{Gr}^\alpha_{j} \mathcal{G}^j_{f,p} = H^{j-1}(F_{f,p}, \mathbb{C})_{\lambda} \quad (j \in \mathbb{Z}, \alpha \in \mathbb{Q}, \lambda = e^{-2\pi i \alpha}).
\]
Here $F_{j,p}$ is the Milnor fiber of $f$ around $p$, $E_\lambda$ denotes the $\lambda$-eigenspace for a vector space $E$ endowed with a monodromy action, and $V$ is the filtration of Kashiwara and Malgrange, see [BaSa07], [DiSa12], [Sa20a], etc.

1.2. Brieskorn modules. In the above notation, set

$$\mathcal{A}_f^j := \text{Ker}(df \wedge : \Omega_X^j \to \Omega_X^{j+1}).$$

Then $(\mathcal{A}_f^\bullet, d)$ is a subcomplex of $\mathcal{K}_f^\bullet$, and we have the canonical morphisms

$$i^j_{f,p} : \mathcal{H}_f^j := H^j(\mathcal{A}_f^\bullet, d) \to G_{f,p}^j \quad (j \in \mathbb{Z}, p \in D).$$

The $\mathcal{H}_f^j$ are called the Brieskorn modules of $f$ at $p$, see [Br70], [BaSa07]. They are modules over $\mathbb{C}\{t\}$ and also over $\mathbb{C}\{[\partial_t^{-1}]\}$. By [BaSa07] Thm. 1], Ker $i^j_{f,p}$ coincides with the $t$-torsion and also with the $\partial_t^{-1}$-torsion, and $\text{Im} i^j_{f,p}$ is a finite free module over $\mathbb{C}\{t\}$ and also over $\mathbb{C}\{[\partial_t^{-1}]\}$. Its rank coincides with the dimension of the Milnor fiber cohomology, and it generates $G_{f,p}^j$ over $\mathbb{C}[\partial_t]$ (more precisely, $G_{f,p}^j = \bigcup_k \partial_t^k(\text{Im} i^j_{f,p})$, and is contained in $V^{>0} G_{f,p}^j$ with $V$ the filtration of Kashiwara and Malgrange (which is shifted by 1 in [BaSa07]).

The action of $\partial_t^{-1}$ on $\mathcal{H}_f^j$ is given by

$$\partial_t^{-1}[\eta] = [df \wedge \eta'] \quad \text{with} \quad d\eta' = \eta.$$  

This is compatible with (1.1.3) for $k = 0$.

Remark 1.2. Assume $f$ is positively weighted homogeneous with weights $w_1, \ldots, w_\alpha$ around a point $p \in D$ as in the introduction. Then $\mathcal{H}_f^j$, $G_{f,p}^j$ are completions of graded modules (with degrees in $\mathbb{Q}$) so that

$$\mathcal{H}_f^j = \bigoplus_{\alpha \in \mathbb{Q}} (\mathcal{H}_f^j)_{\alpha}, \quad G_{f,p}^j = \bigoplus_{\alpha \in \mathbb{Q}} (G_{f,p}^j)_{\alpha}.$$  

Here $(\mathcal{H}_f^j)_{\alpha}$, $(G_{f,p}^j)_{\alpha}$ denote the degree $\alpha$ part on which the Lie derivation $L_{t_\alpha}$ is given by multiplication by $\alpha$ (with $t_\alpha$ as in (1.4) below). Combining (1.2.2) with (1.2.3) below, we see that $\mathcal{H}_f^j$ is stable by the action of $\partial_t$, and the latter is given by multiplication by $\alpha$ on the degree $\alpha$ part (hence this holds also for $G_{f,p}^j$). We then get the canonical isomorphisms

$$\text{Gr}_V^\alpha \mathcal{H}_f^j = (\mathcal{H}_f^j)_\alpha, \quad \text{Gr}_V^\alpha G_{f,p}^j = (G_{f,p}^j)_\alpha \quad (\alpha \in \mathbb{Q}),$$

where $V$ is the filtration of Kashiwara and Malgrange such that the action of $\partial_t - \alpha$ is nilpotent on $\text{Gr}_V^\alpha$.

Via (1.1.4) we have the canonical isomorphisms

$$\text{Im} \left( \text{Gr}_V^\alpha \mathcal{H}_f^j \to \text{Gr}_V^\alpha G_{f,p}^j \right) = P^k H^{j-1}(F_{j,p}, \mathbb{C})_\lambda$$

(\alpha \in \mathbb{Q}, \ [j - \alpha] = k, \ \lambda = e^{-2\pi i \alpha}),

with $P$ the pole order filtration explained in (1.3) just below. (This isomorphism can be shown using acyclicity of the complex $(\Omega_X^\bullet, d)$.) In the isolated singularity case, this is quite well known (see for instance [SeSt85]), where the pole order filtration $P$ coincides with the Hodge filtration $F$.

1.3. Pole order spectral sequences. We have the pole order filtration $P$ on $\mathcal{K}_f^\bullet$ defined by

$$P_k \mathcal{K}_f^j := F_{k+j} \mathcal{B}_f \otimes_{\sigma_X} \Omega_X^j \quad (k \in \mathbb{Z}, \ j \in [0, n]),$$

where the filtration $F$ on $\mathcal{B}_f$ is by the order of $\partial_t$. The $\text{Gr}_k^P \mathcal{K}_f^j$ are truncated Koszul complexes for the action $df \wedge$ on $\Omega_X^\bullet$. (In the isolated singularity case, it gives the Hodge filtration $F$, but this does not hold in the non-isolated singularity case. We have only the inclusion $F \subset P$, see for instance [Sa07], Prop. 4.4.) We then get the pole order spectral sequence, which is essentially the spectral sequence for a double complex with differential given by $d$ and $df \wedge$, see [Sa20a], etc.
Assume \( f \) is positively weighted homogeneous with weights \( w_1, \ldots, w_n \) around a point \( p \in D \) as in Remark 1.2 above. The spectral sequence is compatible with the weighted grading. We get the induced pole order spectral sequence on each degree \( \alpha \) part. By (1.1.4) and (1.2.4), this spectral sequence defines the pole order filtration \( P \) on \( H^j(F_{f,p}, \mathbb{C})_\lambda \) via the isomorphism (1.1.4) with \( \alpha \in (-1,0] \), where \( \lambda = p_k = P_{-k} \). We have the following.

**Theorem 1.3** ([Sa07] Thm. 2). Assume \(-\alpha - k \) is not a root of \( b_{f,q}(s) \) for any \( k \in \mathbb{N} \) and \( q \neq p \) sufficiently near \( q \). Then \(-\alpha \) is a root of \( b_{f,p}(s) \) if and only if

\[
Gr^\alpha_P H^{n-1}(F_{f,p}, \mathbb{C})_\lambda \neq 0 \quad (j = [n-\alpha], \lambda = e^{-2\pi i \alpha}).
\]

Here \( b_{f,p}(s) \) is the local BS polynomial of \( f \) at \( p \), and the filtration \( \tilde{P} \) coincides with \( P \) in the weighted homogeneous, see a remark after [Sa07] (4.1.6)].

**Remark 1.3a.** Assume \( f \) is a homogeneous polynomial, and the projective hypersurface \( Z \) defined by \( f \) has only weighted homogeneous isolated singularities so that the pole order spectral sequence degenerates at \( E_2 \) (see [Sa16b]). Setting

\[
M := H^n_{dl} \Omega^*, \quad M^{(2)} := H^d_{dl}(H^*_{dl}(\Omega^*)),
\]

\[
N := H^{n-1}_{dl}(\Omega^*)(-d), \quad N^{(2)} := H^d_{dl}(H^*_{dl}(\Omega^*))(d),
\]

we have the isomorphisms for \( k \in [1, d], \ j \in \mathbb{N}, \) and \( \lambda := e^{-2\pi ik/d} : 

\[
M_{k+j}^{(2)} = Gr^\lambda_P H^{n-1}(F_{f,0}, \mathbb{C})_\lambda,
\]

\[
N_{k+j}^{(2)} = Gr^{n-1-j}_P H^{n-2}(F_{f,0}, \mathbb{C})_\lambda.
\]

Here \( \Omega^* \) denotes the complex of algebraic differential forms on \( X = \mathbb{C}^n \) (which is identified with the graded quotients of the \( m \)-adic filtration on \( \Omega^*_X \)), and \( H^*_{dl} \) means that we take the cohomology using the differential \( df \wedge \) (similarly for \( H^*_{dl} \) with \( df \wedge \) replaced by \( d \)), see for instance [Sa20a, (3)], etc. Under the hypothesis of Theorem 1.3, the first isomorphism in (1.3.2) means that \(-k/d \) is a root of \( b_{f,s}(s) \) if and only if \( M_k^{(2)} = 0 \).

**Remark 1.3b.** The above construction can be generalized to the case where the condition on the projective hypersurface \( Z \) is not satisfied so that the pole order spectral sequence does not necessarily degenerate at \( E_2 \). If the projective hypersurface \( Z \) has only isolated singularities, there are graded subquotients \( M^{(r+1)} \), \( N^{(r+1)} \) of \( M^r \), \( N^r \) (with \( M^{(1)} = M \), \( N^{(1)} = N \)) obtained by taking the cohomology of the \( E_r \)-differential \( df \) (shifting the degree by \(-rd \)) inductively for \( r \geq 1 \), see for instance the introduction of [Sa20a].

**Remark 1.3c.** Assume \( f \) is a homogeneous polynomial, and the projective hypersurface \( Z \subset \mathbb{P}^{n-1} \) has only isolated singularities. Then we have the inclusion

\[
H^{n-2}(F_{f,0}, \mathbb{C}) \hookrightarrow \bigoplus_i H^{n-2}(F_{f,p_i}, \mathbb{C})^T_i.
\]

Here \( f_i \) is the restriction of \( f \) to a transversal slice \( X_i \) to each irreducible component \( \Sigma_i \) of the singular locus \( \Sigma := \text{Sing} f \) with \( \{p_i\} = \Sigma_i \cap X_i \), and \( T_i \) is the horizontal monodromy around \( 0 \in \Sigma_i (\cong \mathbb{C}) \), see for instance [DiSa04, Thm. 0.1].

### 1.4. Logarithmic complexes in the weighted homogeneous case

Assume \( X = \mathbb{C}^n \), and \( D \subset X \) is a divisor defined by a reduced weighted homogeneous polynomial \( f \) with strictly positive weights \( w_i \) as in the introduction. Set

\[
\xi := \sum_{i=1}^n w_i x_i \partial_{x_i}.
\]

Then

\[
\iota_\xi (df) = \xi (f) = f,
\]

with \( \iota_\xi \) the *interior product*. There are well-known relations

\[
\iota_\xi \circ d + d \circ \iota_\xi = L_\xi,
\]

\[
\iota_\xi \circ (df/f \wedge) + (df/f \wedge) \circ \iota_\xi = \text{id},
\]
where \(L_\xi\) denotes the Lie derivation. (The last equality follows from the Leibniz rule.)

The logarithmic forms \(\Omega^*_X(\log D)\) are defined by the conditions: \(f\eta \in \Omega^*_X\) and \(fd\eta \in \Omega^{i+1}_X\) for \(\eta \in \Omega^1_X(*D)\), see \(\text{SaK80}\). The last condition is equivalent to that \(df/\eta \in \Omega^{i+1}_X\) assuming the first. Using (1.4.3), it is easy to see the following (see also \(\text{HoMo98}\)).

**Lemma 1.4.** Let \(A^j_f\) be as in (1.2). There are decompositions

\[
\Omega^j_X(\log D) = A^j_f f^{-1} \oplus \iota_\xi A^{j+1}_f f^{-1} \quad (j \in \mathbb{Z}),
\]

together with the isomorphisms

\[
\iota_\xi : A^{j+1}_f f^{-1} \xrightarrow{\sim} \iota_\xi A^{j+1}_f f^{-1} \quad (j \in \mathbb{Z}).
\]

**Proof.** By (1.4.3) the identity on the complex \((\Omega^*_X(\log D), df/f)\) is homotopic to 0. Hence the complex is acyclic, and a splitting of complex is given by \(\iota_\xi\) using (1.4.3). Lemma 1.4 thus follows.

**Proposition 1.4.** Let \((\Omega^*_X(\log D)f^{-r}, d)\) be the logarithmic complex multiplied by \(f^{-r}\) for \(r \in \mathbb{N}\). This complex is isomorphic to the mapping cone

\[
C(L_\xi : (\mathcal{A}^*_f f^{-r-1}, d) \to (\mathcal{A}^*_f f^{-r-1}, d)),
\]

and in the notation of Remark 1.2 we have the isomorphisms for \(j \in \mathbb{Z}\):

\[
H^j(\Omega^*_X,0(\log D)f^{-r}, d) = (\mathcal{H}^j_f,0)_{r+1} \oplus (\mathcal{H}^{j+1}_f,0)_{r+1}.
\]

**Proof.** It is enough to show the first assertion (using Remark 1.2). Indeed, the action of \(L_\xi\) on the degree \(\alpha\) part is given by multiplication by \(\alpha\). So we may restrict to the degree 0 part (since the other part is acyclic). Here the mapping cone is associated with the zero map, hence it is a direct sum of two complexes. Note also that the differential \(d\) on \(\mathcal{A}^*_f,0\) commutes with multiplication by \(f^r\) \((r \in \mathbb{N})\).

To show the first assertion, we first see that \((\Omega^*_X(\log D)f^{-r}, d)\) contains \((\mathcal{A}^*_f f^{-r-1}, d)\) as a subcomplex by Lemma 1.4 since \(d\) and \(df/f\) anti-commute.

On the other hand, the restriction of \(d\) to \(\iota_\xi A^{j+1}_f f^{-r-1}\) is the sum of

\[
d' : \iota_\xi A^{j+1}_f f^{-r-1} \to A^{j+1}_f f^{-r-1} \quad \text{and} \quad d'' : \iota_\xi A^{j+1}_f f^{-r-1} \to \iota_\xi A^{j+1}_f f^{-r-1}.
\]

Using (1.4.2) and the diagram below, we see that \(d', d''\) are identified respectively with \(L_\xi\) and the restriction of \(d\) to \(A^{j+1}_f f^{-r-1}\) up to sign via the isomorphism (1.4.3).

\[
\begin{array}{ccc}
\tau_\xi & d & d \\
\downarrow & \downarrow & \downarrow \\
\iota_\xi A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & d \\
\tau_\xi & d' & d' \\
\downarrow & \downarrow & \downarrow \\
\iota_\xi A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & A^{j+1}_f f^{-r-1} & d'' \\
\end{array}
\]

So Proposition 1.4 follows.

**Remark 1.4a.** If \(f\) is a weighted homogeneous polynomial with an isolated singularity at 0 and \(n \geq 3\), then Proposition 1.4 together with (1.2.3) and Remark 1.4b just below implies that the morphism (2) is a quasi-isomorphism if and only if the unipotent monodromy part of the vanishing cohomology vanishes, or equivalently, there is no integral spectral number (or no integral root of the reduced BS polynomial \(\tilde{b}_f(s) := b_f(s)/(s+1)\)). In the weighted homogeneous isolated singularity case with \(n = 2\), the morphism (2) is always a quasi-isomorphism using Proposition 1.4 etc. (where the link may be disconnected). These imply another proof of Theorem in \(\text{HoMo98}\).

**Remark 1.4b.** Assume \(f\) is a weighted homogeneous polynomial with an isolated singularity at 0 and with weights \(w_i\). The spectrum \(Sp_f(t) = \sum_k t^{\alpha_{f,k}}\) has a symmetry

\[
Sp_f(t) = Sp_f(t^{-1})t^n,
\]
which holds in the general hypersurface isolated singularity case, and it coincides with the
Poincaré series of the Jacobian ring $\mathbb{C}\{x\}/(\partial f)$ shifted by $\sum_i w_i$. More precisely, we have
\begin{equation}
(1.4.9) \quad \text{Sp}_f(t) = \prod_{i=1}^{n} \frac{t - t^{w_i}}{w_i - 1},
\end{equation}
see [St 77a, St 77b] (and also [JKSY 22a]).

The spectral numbers $\alpha_{f,k}$ coincide with the roots of the reduced BS polynomial $\tilde{b}_f(s)$
up to sign (forgetting multiplicities). This follows by combining [Va 82] (or [ScSt 85]) with
[Ma 75]. Note that there is no integral spectral number (that is, the unipotent
monodromy part of the vanishing cohomology vanishes) if and only if $D$ is a $\mathbb{Q}$-homology manifold.
This follows from the Wang sequence, see [Mi 68].

Remark 1.4c. The inductive limit of the isomorphism $(1.4.6)$ over $r \in \mathbb{N}$ is closely related
to the assertion that $X \setminus D$ is a $\mathbb{C}^*$-bundle over $\mathbb{P}^{n-1} \setminus Z$ in the $f$ homogeneous polynomial
case (using for instance [BuSa 10, §1.3]).

Lemma 1.4 has the following.

Corollary 1.4. Assume $X = \mathbb{C}^n$ with $D \subset X$ defined by a positively weighted homogeneous
reduced polynomial $f$ as in 1.4 and $\dim \text{Sing} D \leq 1$ (for instance, $n = 3$). Then $D$ is tame,
that is, the logarithmic differential forms $\Omega^j_X(\log D)$ have at most projective dimension $j$ for
any $j \in \mathbb{Z}$.

Proof. By Lemma 1.4 it is enough to show that
\begin{equation}
(1.4.10) \quad \text{pd}_{\mathcal{O}_X} \mathcal{A}^j_f < j \quad \text{if} \quad j \leq n-1,
\end{equation}
since this is clear for $j = n$ by definition (that is, $\mathcal{A}^n_f = \Omega^n_X(D)$). Here we use algebraic
coherent sheaves, and $\text{pd}_{\mathcal{O}_X}$ denotes the projective dimension over $\mathcal{O}_X$. It is well known
that we have acyclicity of the Koszul complex
\begin{equation}
(1.4.11) \quad \mathcal{H}^j(\Omega^*_X, df \wedge) = 0 \quad \text{if} \quad j < n-1,
\end{equation}
since $\dim \text{Sing} D \leq 1$, see for instance [JKSY 22b, Prop. 2]. This implies the assertion $(1.4.10)$
for $j < n-1$, since $\mathcal{A}^j_f = \ker df \wedge = \text{im} df \wedge (\subset \Omega^j_X)$ for such $j$.

For $j = n-1$, we have the exact sequence
\begin{equation}
(1.4.12) \quad 0 \to \mathcal{A}^{n-1}_f \to \Omega^{n-1}_X df \wedge \to \Omega^n_X \to M^\sim \to 0,
\end{equation}
where $M^\sim$ is the coherent sheaf associated to the graded $\mathbb{C}[x_1, \ldots, x_n]$-module $M$
studied in [DiSa 12], [Sa 16b], [Sa 20a], etc. (and is defined by the exact sequence). We then get that
\begin{equation}
(1.4.13) \quad \text{pd}_{\mathcal{O}_X} \mathcal{A}^{n-1}_f = \text{pd}_{\mathcal{O}_X} \mathcal{I}^{(n)} - 1
\end{equation}
\begin{equation}
= \text{pd}_{\mathcal{O}_X} M^\sim - 2 \leq n-2,
\end{equation}
since $X$ is smooth, where we denote by $\mathcal{I}^{(n)} \subset \Omega^*_X$ the image of $df \wedge$. This finishes the proof
of Corollary 1.4.

1.5. Twisted logarithmic complexes. Assume $D \subset X$ is everywhere positively weighted homogenous.
In the notation of the introduction, set
\begin{equation}
\omega_p := \sum_k \alpha_{p,k} \omega_{p,k} \quad \text{with} \quad \omega_{p,k} := df_{p,k}/f_{p,k}.
\end{equation}
We have the following.

Proposition 1.5. The complex $(\Omega^*_X, (log D) f^{-r}, d + \omega_p \wedge)$ for $p \in D$ and $r \in \mathbb{N}$
is isomorphic to the mapping cone
\begin{equation}
(1.5.1) \quad C(L_{\xi_p} + \tilde{\alpha}_p : (\mathcal{A}^*_f, f^{-r-1}, d + \omega_p \wedge) \to (\mathcal{A}^*_f, f^{-r-1}.d + \omega_p \wedge)).
\end{equation}
Here $\xi_p$ is the vector field associated with a positively weighted homogeneous polynomial $f_p$
as in 1.4 and $\tilde{\alpha}_p$ is defined in 14.
Proof. The argument is essentially the same as in the proof of Proposition 1.3. We can calculate the action of $\omega_{p,k}\wedge$ in a similar way to the case of the differential $d$ using (1.4.3) instead of (1.4.2). Here $f$ is replaced by $f_{p,k}$, and the right-hand side of (1.4.3) becomes $d_{p,k}$. Proposition 1.5 then follows.

Corollary 1.5. Assume $D$ is everywhere positively weighted homogenous. Let $\mathcal{C}^\bullet$ be the mapping cone of the comparison morphism (2). For $p \in D, j \in \mathbb{Z}$, we have

\[
\dim \mathcal{H}^i\mathcal{C}^\bullet_p \geq \dim \mathcal{H}^i\mathcal{C}^\bullet_p \quad \text{if} \quad \mathcal{H}^{i+1}\mathcal{C}^\bullet_p = 0.
\]

Proof. Proposition 1.3 implies that $\mathcal{C}^\bullet_p$ is isomorphic to the mapping cone of the action of $L_{\xi_p + \tilde{\alpha}_p}$ on the complex

\[
\mathcal{K}^\bullet_p := \left( \lim_{\tau \to 0} \mathcal{A}^\bullet_{f,p} f^{-r-1}/\mathcal{A}^\bullet_{f,p} f^{-1}, d + \omega_{p}\wedge \right).
\]

Here the action of $L_{\xi_p}$ on the modified degree $\beta$ part of $\mathcal{A}^\bullet_{f,p} f^{-r}$ is given by multiplication by $\beta \in \mathbb{C}$ by the definition of modified degree, see Remark 1.2 for the untwisted case. Taking the kernel or cokernel of the action of $L_{\xi_p + \tilde{\alpha}_p}$ is then the same as the restriction to the modified degree $-\tilde{\alpha}_p$ part. (Note that the differential $d + \omega_{p}\wedge$ preserves the modified degree.) Using the long exact sequence associated with the mapping cone of the action of $L_{\xi_p + \tilde{\alpha}_p}$ on $\mathcal{K}^\bullet_p$, we get the inequality (1.5.2). This finishes the proof of Corollary 1.5.

1.6. Logarithmic stratification. For a reduced divisor $D$ on a complex manifold $X$, we denote by $\Theta_X(-\log D)$ the sheaf of logarithmic vector fields. By definition a holomorphic vector field $v \in \Theta_X$ belongs to $\Theta_X(-\log D)$ if and only if $\xi(f) \subset (f)$ with $f$ a holomorphic function defining locally $D \subset X$, see [SaK 80].

For $p \in D$, let $E_p \subset T_p X$ be the image of $\Theta_{X,p}(\log D)$ in the tangent space $T_p X$ which is identified with $\Theta_{X,p} \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$, where $\mathfrak{m}_{X,p} \subset \mathcal{O}_{X,p}$ is the maximal ideal. Set

\[
D^{(k)} := \{ p \in D \mid \dim E_p = k \} \quad (k \geq 0).
\]

Taking local generators of $\Theta_X(-\log D)$, we get a matrix with coefficients in $\mathcal{O}_X$, and the $D^{(k)}$ are determined by the rank of this matrix at each $p$. This implies that the union $D^{(\leq j)} := \bigcup_{k \leq j} D^{(k)}$ is a closed analytic subset for any $j \in \mathbb{N}$.

Definition 1.6. We say that a divisor $D \subset X$ has the locally finite logarithmic stratification, if $\dim D^{(k)} \leq k$ for any $k \geq 0$. (This is equivalent to that the “logarithmic stratification” in the sense of [SaK 80] is everywhere locally finite.)

Remark 1.6a. Assume $D$ has the locally finite logarithmic stratification. Then, choosing a basis $(v_1, \ldots, v_k)$ of $E_p$ with $k := \dim E_p$, there are local analytic isomorphisms $(j \in [1, k])$:

\[
\begin{align*}
(X, p) &\cong (S_p^{(j)}, p) \times (\Delta^j, 0) \quad \text{inducing} \\
(D, p) &\cong (S_p^{(j)} \cap D, p) \times (\Delta^j, 0),
\end{align*}
\]

such that the subspace of $T_p X$ spanned by $v_1, \ldots, v_j$ is identified with the tangent space of $(p) \times \Delta^j$. Here $(S_p^{(j)}, p) \subset (X, p)$ is a submanifold of codimension $j$ such that $v_1, \ldots, v_j$ form a basis of $T_p X/T_p S_p^{(j)}$ (and $\Delta$ is an open disk). Indeed, we can show the following isomorphisms by decreasing induction on $j$ using integral curves of vector fields $\xi_j$ whose images in $T_p X$ are $v_j$ $(j \in [1, k])$:

\[
\begin{align*}
(S_p^{(j-1)}, p) &\cong (S_p^{(j)}, p) \times (\Delta, 0) \quad \text{inducing} \\
(S_p^{(j-1)} \cap D, p) &\cong (S_p^{(j)} \cap D, p) \times (\Delta, 0).
\end{align*}
\]

Here $S_p^{(j-1)}$ is the union of integral curves of $\xi_j$ passing through a point of $S_p^{(j)}$ (with $S_p^{(0)} = X$). This construction is compatible with $D$ by the uniqueness of integral curves (at the smooth points of $D$). Note that $S_p^{(j)}$ $(j \in [1, k-1])$ is determined by the $\xi_i$ $(i \in [j+1, k])$ and $S_p^{(k)}$. 

In the case $D$ has the locally finite logarithmic stratification, \(1.6.1\) imply that the $D^{(k)}$ are smooth and purely $k$-dimensional (unless $D^{(k)} = \emptyset$) for any $k \geq 0$, see also \[SaK 80\].

**Remark 1.6b.** If $D$ is positively weighted homogeneous at $p$, we may assume that so is $S_p^k \cap D \subset S_p^k$ in Remark \[1.6a\]. Indeed, we can define $S_p^k$ by $\bigcap_{i \in I} \{x_i = 0\}$ for some subset $I \subset \{1, \ldots, n\}$ with $x_1, \ldots, x_n$ weighted coordinates for $f_p$. (It does not seem clear whether this can be used for another proof of Lemma \[1.6\] below, since the situation at $q \in S_p^k \cap D^{(k)} \setminus \{p\}$ seems rather unclear when $k < \dim D^{(k)}$.)

The following seems to be known to specialists (see \[CNM 96\], etc.).

**Lemma 1.6.** Assume the divisor $D$ is everywhere positively weighted homogeneous as in the introduction. Then $D$ has the locally finite logarithmic stratification.

**Proof.** We have to show that $\dim D^{(k)} \leq k$ for any $k \geq 0$. Take a smooth point $p \in D^{(k)}$. We have the vector field $\xi = \sum_i w_i x_i \partial_{x_i}$ associated to $f_p$ so that $\xi(f_p) = f_p$. Using the trivialization of the tangent bundle by the coordinates $x_1, \ldots, x_n$, this vector field is identified with the map

$$\begin{array}{ccc}
(x_1, \ldots, x_n) & \mapsto & (w_1 x_1, \ldots, w_n x_n).
\end{array}$$

If $\dim D^{(k)} > k (= \dim E_p)$, we then see that its image cannot be contained in $E_q$ for some $q \in D^{(k)}$ sufficiently near $p$, since the $E_q$ depend on $q$ continuously in the Grassmannian. This is, however, a contradiction, since $\xi \in \Theta_X(-\log D)$. So Lemma \[1.6\] follows.

We have the following.

**Proposition 1.6.** Assume the divisor $D$ has the locally finite logarithmic stratification (for instance, $D$ has only isolated singularities). Then the morphism \(2\) is a quasi-isomorphism if and only if it is a $D$-quasi-isomorphism.

**Proof.** Apply the functor $^!\text{DR}_X^1$ to the morphism \(2\) (see a remark after \(3\) and \[Sa 89\]), and consider the mapping cone

$$\begin{array}{c}
C^* := C\big(^!\text{DR}_X^1(\mathcal{M}^\bullet_{X,\log}(L), \nabla^{(\alpha)}) \to ^!\text{DR}_X^1(\mathcal{M}^\bullet_X(L), \nabla^{(\alpha)})\big).
\end{array}$$

This is a bounded complex of coherent $\mathcal{D}_X$-modules. We have to show that this is acyclic if $\text{DR}_X(C^*)$ is acyclic. Set $Z := \bigcup_j \text{Supp} \, \mathcal{H}^j C^*$, and assume $Z \neq \emptyset$. By Remark \[1.6a\], $Z$ is a union of strata of $\mathcal{S}$. Let $V$ be a maximal-dimensional stratum contained in $Z$. By the local analytic triviality along $V$, we may restrict to an appropriate transversal slice to $V$ as in Remark \[1.6a\] so that the assertion is reduced to the case $Z$ is a point, denoted by 0. The cohomology sheaves $\mathcal{H}^j C^*$ are then finite direct sums of $\mathcal{B}_0 := \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]$, see for instance \[Ma 75\]. This is a special case of Kashiwara’s equivalence, see for instance \[Sa 89\].

We now get a contradiction, since $\text{DR}_X \mathcal{B}_0 = \mathbb{C}\{0\}$. This finishes the proof of Proposition \[1.6\].

1.7. Holonomicity and regular holonomicity. Using the assertions in \[1.6\] we can prove the following.

**Proposition 1.7a.** Assume the divisor $D$ has the locally finite logarithmic stratification. Then the complex $^!\text{DR}_X^1(\mathcal{M}^\bullet_{X,\log}(L), \nabla^{(\alpha)})$ is holonomic, that is, it belongs to $D^b_{\text{hol}}(\mathcal{D}_X)$; more precisely, the characteristic varieties of its cohomology $\mathcal{D}$-modules are contained in the union of the conormal bundles of $D^{(j)}$ ($j \in [0, n]$).

**Proof.** We argue by induction on strata. Assuming the assertion is proved on the complement of $D^{(j)}$, we have to show the assertion on the complement of $D^{(j-1)}$, that is, on a neighborhood of any point $p \in D^{(j)}$. By the analytic triviality along $D^{(j)}$, we may assume that $j = 0$ restricting to a transversal slice as in Remark \[1.6a\]. Then the assertion is clear since we have

$$T^* X \setminus T^*(X \setminus \{p\}) = T^*_p X,$$

and $^!\text{DR}_X^1(\mathcal{M}^\bullet_{X,\log}(L), \nabla^{(\alpha)})$ is a complex of coherent $\mathcal{D}_X$-modules by definition. Note that the locally finite logarithmic stratification satisfies the Whitney condition (b) by the local
analytic triviality along strata (see Remark \[1.6a\]), and the union of the conormal bundles of \(D^{(j)} \ (j \in [0, n])\) is a closed subset. This finishes the proof of Proposition \[1.7a\].

**Corollary 1.7a.** Assume \(D\) has the locally finite logarithmic stratification, and is tame, see Corollary \[1.4\]. Then the complex \(l^1 \text{DR}^{-1}(\mathcal{M}^\bullet_{X, \log}(L), \nabla^{(a)})[n]\) is quasi-isomorphic to a holonomic \(D_X\)-module.

**Proof.** Set \(\mathcal{K}^\bullet := l^1 \text{DR}^{-1}(\mathcal{M}^\bullet_{X, \log}(L), \nabla^{(a)})\). This is locally quasi-isomorphic to a complex of finite free \(D_X\)-modules \(\mathcal{F}^\bullet\) with \(\mathcal{F}^j = 0\) for \(j \notin [-n, 0]\) by the tameness assumption (using a double complex consisting of free resolutions locally). Applying the functor \(D\), this property is also satisfied for \(D\mathcal{K}^\bullet\), since \(\mathcal{H}^j D\mathcal{F} = 0\) for \(j \neq -n\) if \(\mathcal{F}\) is a finite free \(D_X\)-module. We thus get that \(D\mathcal{H}^j \mathcal{K}^\bullet = \mathcal{H}^{-j} D\mathcal{K}^\bullet = 0\) for \(j < 0\) (since \(\mathcal{H}^j D\mathcal{M} = 0\) for \(j \neq 0\) if \(\mathcal{M}\) is holonomic). Hence \(\mathcal{H}^j \mathcal{K}^\bullet = 0\) for \(j \neq 0\). This finishes the proof of Corollary \[1.7a\].

**Proposition 1.7b.** Assume \(D\) is everywhere defined by a homogeneous polynomial locally on \(X\) (as in the hyperplane arrangement case). Then the complex \(l^1 \text{DR}^{-1}(\mathcal{M}^\bullet_{X, \log}(L), \nabla^{(a)})\) is regular holonomic, that is, it belongs to \(D_{\text{rh}}^b(D_X)\).

**Proof.** We argue by induction on strata of the locally finite logarithmic stratification, see Lemma \[1.6\]. Assume the assertion is proved on the complement of \(D^{(j)}\). By the same argument as in the proof of Proposition \[1.7a\] we may assume \(j = 0\) cutting by a transversal slice to a stratum. (If there is a homogeneous polynomial \(f\) together with a vector field \(\xi\) not vanishing at 0 and such that \(\xi(f) \in (f)\), then we can show that \(f\) is a polynomial of fewer variables changing linearly the variables if necessary.)

Let \(\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)\) be the blow-up at the origin with \(\widetilde{D} = \rho^{-1}(D)\). Set \(E := \rho^{-1}(0)\). Note that \(\widetilde{X}\) is a line bundle over \(E\), where \(E\) is identified with the zero-section, and \(\widetilde{D}\) is the union of the pull-back of \(D\) with the zero-section. We can then apply the K"unneth formula for logarithmic forms locally by trivializing the line bundle.

For a sufficiently large integer \(m\), there are canonical morphisms

\[
\begin{align*}
\left(\mathcal{M}^\bullet_{X, \log}, \nabla^{(a)}\right) & \to \rho_* \left(\mathcal{M}^\bullet_{\widetilde{X}, \log}(mE), \nabla^{(a)}\right) \\
& \to \mathcal{R}\rho_* \left(\mathcal{M}^\bullet_{\widetilde{X}, \log}(mE), \nabla^{(a)}\right).
\end{align*}
\]

(1.7.1)

Here \(\mathcal{R}\rho_*\) is defined by taking the canonical flasque resolution (using discontinuous sections) by Godement, and \((mE)\) means \(\otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}(mE)\). The functor \(l^1 \text{DR}^{-1}\) (see \(3\) for definition) commutes with the direct image by \(\rho\) (see \(\text{[Sa 89, 3.3–5]}\)), and we have

\[
l^1 \text{DR}^{-1}(\mathcal{M}^\bullet_{\widetilde{X}, \log}(mE), \nabla^{(a)}) \in D_{\text{rh}}^b(D_{\widetilde{X}}),
\]

using the K"unneth formula for logarithmic forms together with the inductive hypothesis.

Let \(\mathcal{C}^\bullet\) be the mapping cone of the composition (1.7.1). This is cohomologically supported at the origin. By an argument similar to the proof of Proposition \[1.6\] (using a special case of Kashiwara’s equivalence), we can then verify that

\[l^1 \text{DR}^{-1}\mathcal{C}^\bullet \in D_{\text{rh}}^b(D_X)\]

We thus get that

\[
l^1 \text{DR}^{-1}(\mathcal{M}^\bullet_{X, \log}, \nabla^{(a)}) \in D_{\text{rh}}^b(D_X),
\]

since regular holonomic \(D\)-modules are stable by subquotients and extensions. This finishes the proof of Proposition \[1.7b\].
From Proposition 1.7b, Corollary 1.7a, and Lemma 1.6, we get the following.

**Corollary 1.7b.** Assume $D$ is tame (see Corollary 1.4, and is everywhere defined by a homogeneous polynomial locally on $X$ (as in the hyperplane arrangement case). Then the complex $\mathcal{D}^{\text{reg}}_X(\mathcal{M}^*_X, \nabla^{(\alpha)})[n]$ is quasi-isomorphic to a regular holonomic $\mathcal{D}_X$-module.

**Remark 1.7a.** The highest cohomology of $\mathcal{D}^{\text{reg}}_X(\mathcal{M}^*_X, \nabla^{(\alpha)})$ is the quotient of $\mathcal{D}_X$ divided by an ideal generated by logarithmic vector fields with certain $\mathcal{O}$-linear terms. In the case $\alpha_k = \alpha$ $(\forall k)$, this ideal is generated by $\Theta_{f, \alpha-1}$ with

$$\Theta_{f, \alpha} := \left\{ \xi - \alpha \xi(f)/f \mid \xi \in \Theta_X(-\log D) \right\} \ (\alpha \in \mathbb{C}),$$

using (3.1.8) below. The shift of $\alpha$ by $-1$ comes from the isomorphism $\Omega^n_X(\log D) = \Omega^n_X(D)$. We use the anti-involution $^*$ of $\mathcal{D}_X$ such that $x_i^* = x_i$, $\partial_{x_i}^* = -\partial_{x_i}$, and $(PQ)^* = Q^*P^*$ as is well known. It is interesting whether this quotient coincides with the $\mathcal{D}_X$-module generated by $f^\alpha$, see Theorem 2 and also Corollary 3.2a, Theorem A below.

**Remark 1.7b.** It does not seem trivial to show the regularity of the highest cohomology of $\mathcal{D}^{\text{reg}}_X(\mathcal{M}^*_X, \nabla^{(\alpha)})$ even under the assumption of Corollary 1.7a (for instance, if it has a $\mathcal{D}_X$-submodule whose support has dimension 1, where we cannot apply Hartogs theorem).

Note that the principal symbols of logarithmic vector fields do not necessarily generate the reduced ideal of the characteristic variety; for instance, in the case $f = x^d + y^d$ $(d \geq 3)$, the logarithmic vector fields are generated by the Euler field $x \partial_x + y \partial_y$ and $y^{d-1} \partial_y - x^{d-1} \partial_y$ (since $f_x, f_y$ form a regular sequence), but $\alpha x + \beta y$ and $\alpha y^{d-1} - \beta x^{d-1}$ (with $\alpha, \beta \in \mathbb{C}$) do not generate the maximal ideal of $\mathbb{C}[x, y]$.

**Remark 1.7c.** We can extend Corollary 1.7b to the case where the transversal slice may be defined by a positively weighted homogeneous polynomial if it has an isolated singularity. (In this case, the blowing-up at the origin can be replaced by an embedded resolution.) This includes the case where the hypothesis of Corollary 2 is satisfied, that is, $f$ is a homogeneous polynomial and $Z := \{ f = 0 \} \subset \mathbb{P}^{n-1}$ has only weighted homogenous isolated singularities (using Corollary 1.4). Assuming the local system is constant, we can prove the assertion in the latter case using the theory of $t$-structure [BBD82] and the strictness of the Hodge filtration $F$ together with [DiSt20] Prop. 2.2. (Here the highest cohomology sheaf may contain a $\mathcal{D}_X$-submodule whose support has dimension 1.)

## 2. Proof of the main theorems

In this section, we prove the main theorems applying the assertions in the previous section.

### 2.1. Proof of Theorem 1

The eigenvalues of the action of $L_{\xi_p}$ on the $A^j_{f,p} f^{-1}$ are contained in $e^{-1} \mathbb{Z}$ by definition, where $e_p$ as in [1]. By Proposition 1.5, the stalk of the twisted logarithmic complex at $p$ is then acyclic under the assumption [5]. So it is enough to show the vanishing

$$H^j(V \setminus D, L|_{V \setminus D}) = 0,$$

where $V$ is a sufficiently small open neighborhood of $p$ in $X$. Using the $\mathbb{C}^*$-action, we may assume $V = \mathbb{C}^n$ with $D$ defined by a weighted homogeneous polynomial $f_p = \prod_k f_{p,k} \in \mathbb{C}^n$.

Set

$$a_i := e_p a_{p,i} \ (i \in [1, n]).$$

Consider the finite map

$$\pi : V' := \mathbb{C}^n \ni (x_1, \ldots, x_n) \mapsto (x_1^{a_1}, \ldots, x_n^{a_n}) \in V = \mathbb{C}^n.$$

Put

$$g := \pi^* f_p, \quad g_k := \pi^* f_{p,k}.$$

Then $g, g_k$ are homogeneous polynomials of degree $d := e_p$ and $d_k := e_p d_{p,k}$ respectively.
Let $\rho : \tilde{V} \to V'$ be the blow-up along $0 \in V'$. Let $D', L'$ be the inverse image or pullback of $D, L$ by $\pi$, and similarly for $\tilde{D}, \tilde{L}$ replacing $\pi$ with $\tilde{\pi} := \pi \circ \rho$. Here $\tilde{L}$ is identified with $L'$. Since $L$ is a direct factor of the direct image of $L'$ by the finite morphism $\pi$, it is sufficient to show that

$$H^j(\tilde{V} \setminus \tilde{D}, \tilde{L})_{\tilde{V} \setminus \tilde{D}} = 0.$$  

Let $\lambda_0$ be the local monodromy (eigenvalue) of the local system $\tilde{L}$ around the exceptional divisor $E$ of $\rho$. Since $\tilde{V} \setminus \tilde{D}$ is a $\mathbb{C}^*$-bundle over $\mathbb{P}^{n-1} \setminus \{g = 0\}$, the vanishing (2.1.2) is reduced to the inequality

$$\lambda_0 \neq 1,$$

using the Leray-type spectral sequence.

In order to calculate $\lambda_0$, consider a generic line $\ell$ in $V'$ passing through the origin. Its pullback to $\tilde{V}$ intersects $E$ transversally. We perturb this $\ell$ slightly so that it intersects $D'$ transversally at smooth points. The above $\lambda_0$ is the product of the local monodromy eigenvalues for all the intersection points. (Note that $\ell$ is a complex line.) Let $\lambda_k$ be the contribution coming from the intersection points with $\{g_k = 0\}$ so that $\lambda_0 = \prod_k \lambda_k$. In view of the hypothesis $[\lambda]$, the inequality (2.1.3) is then reduced to the following.

$$\lambda_k = e^{-2\pi i d_k \alpha_{p,k}}.$$  

In the case $f_{p,k}$ does not coincide with any $x_i$ (up to non-zero constant multiple), the intersection $\{g_k = 0\} \cap (\mathbb{C}^*)^n$ is non-empty, and we get (2.1.4), since $\pi$ is unramified over $(\mathbb{C}^*)^n$ and the intersection number of $\{g_k = 0\}$ and $\ell$ coincides with the degree of $g_k$, that is, $d_k$.

In the other case, we have $f_{p,k} = x_i$ (up to non-zero constant multiple) for some $i$. The intersection number of $\{x_i = 0\}$ with $\ell$ is 1, and the local monodromy of $L'$ around $\{x_i = 0\}$ is given by $\lambda_i^{a_i}$ with $\lambda_i = e^{-2\pi i \alpha_{p,k}}$ and $d_{p,k} = w_{p,i}$ (since $f_{p,k} = x_i$). So (2.1.4) follows. This finishes the proof of Theorem [\lambda].

2.2. Proof of Theorem [\lambda]. We have the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^n((^t\text{DR}_X^1(\mathcal{M}_{X, \log}^*(L), \nabla^{(a)})) & \longrightarrow & \mathcal{H}^n(^t\text{DR}_X^1(\mathcal{M}_{X}^*(L), \nabla^{(a)})) \\
\begin{array}{c}
D_X / D_X \tilde{\Theta}_{\alpha_{f-1}} \\
\end{array} & \longrightarrow & \begin{array}{c}
D_X f^{-1} \\
\begin{array}{c}
\begin{array}{c}
\cup \\
\end{array}
\end{array}
\end{array}
\end{array}$$

(2.2.1)

see Remark [\lambda.7]. If condition (a) holds, then the upper horizontal morphism is bijective, hence the lower horizontal morphisms $\rho, \iota$ are. Thus conditions (c) and (d) hold. We can verify the equivalence (c) $\Leftrightarrow$ (d) using the definitions of $\tilde{\Theta}_{f, \alpha}$ and $\Theta_X(- \log D)$, see Remark [\lambda.7]. The equivalence (c) $\Leftrightarrow$ (d) follows for instance from [Sa2], Thm. 1], see Proposition [\lambda.2] below and also [Bu1].

We may then assume that the divisor $D$ is everywhere positively weighted homogeneous and the complex $^t\text{DR}_X^1(\mathcal{M}_{X, \log}^*(L), \nabla^{(a)})(n)$ is quasi-isomorphic to a $\mathcal{D}_X$-module. The first assumption implies the equivalence (c) $\Leftrightarrow$ (d) by Lemma [\lambda.6] and Proposition [\lambda.6]. From the second one, we can deduce that

$$\mathcal{H}^j \mathcal{C}^* = 0 \quad \text{if} \quad j \neq n-1, n,$$

with $\mathcal{C}^*$ the mapping cone of the comparison morphism (2) applied by $^t\text{DR}_X^1$. (This follows from the long exact sequence associated with the mapping cone.)

Assume condition (c) holds (that is, $\rho$ is injective), but (m) does not. We then get that

$$\mathcal{H}^n \mathcal{C}^* \neq 0, \quad \mathcal{H}^n \mathcal{C}^* = 0 \quad (\forall j \neq n).$$

We may assume that the cohomology $\mathcal{D}$-module $\mathcal{H}^n \mathcal{C}^*$ is supported at a point $p \in D$ using Lemma [\lambda.6] and cutting $D$ by a transversal slice to a maximal-dimensional stratum contained
in the support. Then this $\mathcal{D}$-module is a finite direct sum of $\mathcal{B}_p = \mathbb{C}[\partial_{x_1}, \ldots, \partial_{x_n}]$ supported at $p$ (using a special case of Kashiwara’s equivalence as in the proof of Proposition 1.6). This implies that the mapping cone of the comparison morphism (2) is acyclic except at the highest degree $n$, since $\text{DR}_X(\mathcal{B}_p) = \mathbb{C}(p)$. However, this contradicts Corollary 1.5. We thus get that (8) $\Rightarrow$ (a).

We can prove the implication (6) $\Rightarrow$ (a) by an argument similar to the above one, where $n$ is replaced by $n-1$ in (2.2.3). This finishes the proof of Theorem 2.

Proposition 2.2. For a holomorphic function $f$ on a complex manifold $X$, we have

$$\begin{align*}
\mathcal{D}_X f^\alpha &= \mathcal{D}_X f^{\alpha+1} & \text{if } b_f(\alpha) \neq 0, \\
\mathcal{D}_X f^\alpha &\neq \mathcal{D}_X f^{\alpha+1} & \text{if } b_f(\alpha) = 0, b_f(\alpha - k) \neq 0 (\forall k \in \mathbb{Z}_{>0}).
\end{align*}$$

(2.2.4)

Proof. By [Sa 21, Thm. 1] there are regular holonomic $\mathcal{D}_X$-modules $\mathcal{M}_f^\beta$ endowed with a finite increasing filtration $G_\bullet$ and a nilpotent endomorphism $N$ for $\beta \in (0, 1]$ such that $G_\bullet$ is stable by $N$ and

$$\begin{align*}
\text{Gr}_G^j \mathcal{M}_f^\beta &= 0 \iff b_f(-\beta - j) \neq 0, \\
\text{Gr}_G^j (\mathcal{M}_f^\beta/N \mathcal{M}_f^\beta) &= 0 \iff \mathcal{D}_X f^{-\beta-j} = \mathcal{D}_X f^{-\beta-j+1}.
\end{align*}$$

(2.2.5)

(Here $N$ is not strictly compatible with $G$ in general.) The first assertion then follows. One can also get this by setting $s = \alpha$ in the functional equation associated with $b_f(s)$.

For the second assertion, we see that the second hypothesis implies the equalities

$$\begin{align*}
\mathcal{D}_X f^\alpha &= \mathcal{D}_X f^{\alpha-k} (\forall k \in \mathbb{Z}_{>0}), & \mathcal{M}_f^\beta &= G_j \mathcal{M}_f^\beta,
\end{align*}$$

(2.2.6)

where $\alpha = -\beta - j$. From the second equality of (2.2.6) we can deduce the isomorphism

$$\text{Gr}_G^j (\mathcal{M}_f^\beta/N \mathcal{M}_f^\beta) = \text{Gr}_G^j \mathcal{M}_f^\beta/N \text{Gr}_G^j \mathcal{M}_f^\beta.$$

(2.2.7)

So the assertion follows (since the cokernel of a nilpotent endomorphism of a nonzero object of an abelian category is nonzero). This finishes the proof of Proposition 2.2.

Remark 2.2. In the case the divisor $D$ is everywhere positively weighted homogeneous and the complex $\text{DR}_X^{-1}(\mathcal{M}_X, \nabla^{(\alpha)}_X)[n]$ is quasi-isomorphic to a $\mathcal{D}_X$-module, this can be shown using Theorem 1.3, Proposition 1.5, and Remark 1.2 by induction on strata. Indeed, the argument is quite similar to the proof of Theorem 3 below. Here we have to replace the unipotent monodromy part in (2.3.1) below by the $\lambda$-eigenspace with $\lambda = e^{2\pi i \alpha}$, and the degree 1 part in (2.3.2) below by the degree 1 $-\alpha$ part. Note that $\omega_D \Lambda$ vanishes, see also Remark 3.2 below.

2.3. Proof of Theorem 3. We first show the assertion (ii). Under the hypothesis on the integral roots of $b_{f,q}(s)$ for $q \neq p$, the BS polynomial $b_f(s)$ has an integral root which is strictly smaller than $-1$ if and only if

$$\begin{align*}
P^{n-1} H^{n-1}(F_f, \mathbb{C})_1 \neq H^{n-1}(F_f, \mathbb{C})_1,
\end{align*}$$

(2.3.1)

see Theorem 1.3 (that is, [Sa 07, Thm. 2]). By (1.2.4–5), the left-hand side of (2.3.1) is identified with the image of

$$\begin{align*}
(H_t f)_p^n \to (G^n f)_p^n.
\end{align*}$$

(2.3.2)

Here we can replace $G^n f_p$ with $G^n f_p[t^{-1}]$ (the localization by $t$), since we take $\text{Gr}_V^1$. (Indeed, the kernel and cokernel of the localization morphism are unions of subgroups annihilated by $t^k (k \gg 0)$, hence $\text{Gr}_V^1$ does not change under the localization by $t$.) The morphism (2.3.2) is then identified with

$$\begin{align*}
H^n(A'_{f,p} f^{-1}, d)_0 \to \lim_{r \to \infty} H^n(A_{f,p} f^{-r}, d)_0.
\end{align*}$$

(2.3.3)
where 0 denotes the degree 0 part. (Note that the differential d on \(A_{f,p}^r\) commutes with multiplication by \(f^k\) for \(k \in \mathbb{N}\).) Since the meromorphic de Rham complex is the inductive limit of \((\Omega^r_X (\log D) f^{-r}, d)\), the assertion (ii) now follows from Proposition 1.4.

The assertion (i) also follows from Proposition 1.4 using the coincidence of \(\text{Ker} \ i_{f,p}^r\) with the torsion (see [BaSa 07, Thm. 1] or 1.2), since the morphism \((2.3.2)\) is identified with \((2.3.3)\). This finishes the proof of Theorem 3.

### 2.4. Proofs of Corollaries 2 and 3

Corollary 2 follows from Theorem 3 using [Sa 16b, Thm. 2], [DiSa 12, Cor. 4.7] as is explained after Corollary 2, since the morphism \((2.3.2)\) is identified with \((2.3.3)\) in 2.3.

For the proof of Corollary 3, it is enough to show that \(M_{kd}^{(2)} \neq 0\) for some \(k \in \mathbb{Z}_{\geq 2}\) in the notation of Remark 1.3a using Proposition 1.4. This non-vanishing follows from injectivity of the morphism

\[
M' \hookrightarrow M^{(2)} \quad \text{(or equivalently, } M' \cap \text{Im} \ d_1 = 0),
\]

(see [Sa 16b, Thm. 3]) combined with a symmetry of the \(\mu_k\) with center \(nd/2\) (see [DiSa 12, Cor. 1]), where \(n \geq 3\). Indeed, the hypothesis implies that \(\mu_{kd'} > 0\), since

\[
\mu_k' + \mu_k'' = \nu_k + \gamma_k \quad (k \in \mathbb{Z}),
\]

(see [DiSa 12 (0.3)]) and \(\mu_k'' \leq \tau_z = \mu_z\). Here

\[
\sum_k \gamma_k t^k := \left(\sum_{k=1}^{d-1} t^k\right)^n \quad \text{(in particular, } \gamma_d = \left(\frac{d-1}{n-1}\right)\text{),}
\]

\[
M' := H^0_{mM}, \quad M'' := M/M', \quad \mu_k' := \dim M_k', \quad \mu_k'' := \dim M_k'',
\]

with \(m \subseteq \mathbb{C}[x_1, \ldots, x_n]\) the maximal ideal, \(M\) is as in Remark 1.3a and we have \(\gamma_d \leq \gamma_{kd'}\), see for instance [Sa 16b, (4.11.1)]. This finishes the proofs of Corollaries 2 and 3.

**Remark 2.4.** The morphism \((\mathbb{G})\) can be injective even if the pole order spectral sequence does not degenerate at \(E_2\), since we need the non-degeneration exactly at the degree \(d\) part for the non-injectivity as is shown by Theorem 3(i). Set, for instance,

\[
f = x^4 + y^3 z + z^3 w + xyzw.
\]

The associated projective hypersurface has a unique singular point which has type \(T_{3,4,8}\) (not \(T_{3,3,4}\)) with Milnor number 14 and Tjurina number 13 (according to calculations by a small computer program for non-degenerate functions and also by Singular [DGPS 20]). Using a small computer program based on the algorithm explained in [Sa 20a], the numerical data of the pole order spectral sequence are given as follows:

| \(k\) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|
| \(\gamma_k\) | 1 | 4 | 10 | 16 | 19 | 16 | 10 | 4 | 1 |
| \(\mu_k\) | 1 | 4 | 10 | 16 | 19 | 17 | 13 | 13 | 13 |
| \(\mu_k^{(2)}\) | 1 | 4 | 8 | 8 | 7 | 4 | 1 | 1 | 1 |
| \(\mu_k^{(3)}\) | 1 | 4 | 7 | 7 | 6 | 3 |
| \(\nu_k\) | 1 | 3 | 9 | 12 |
| \(\nu_k^{(2)}\) | 1 | 1 | 1 |
| \(\nu_k^{(3)}\) | 1 | 1 | 1 |

Here \(\mu_k^{(r)} = \dim M_k^{(r)}\) (and similarly for \(\nu_k^{(r)}\)), see Remark 1.3b. In this case we have \(\mu_4^{(2)} = \mu_4^{(3)}\). This implies that we may have the injectivity of \((\mathbb{G})\). (Some more calculation would be needed to see if the spectral sequence degenerates at \(E_3\).)

On the other hand, let

\[
f = x^5 + y^4 z + x^3 y^2 + w^5,
\]
so that \( f = g + w^5 \) with \( g \in \mathbb{C}[x, y, z] \) treated in [Sa20a, Ex. 5.1]. We have

\[
\begin{align*}
k & : 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \\
\gamma_k & : 1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 40 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \\
\mu_k & : 1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 40 \quad 44 \quad 45 \quad 46 \quad 44 \quad 44 \quad 44 \quad 44 \\
\mu_k^{(2)} & : 1 \quad 3 \quad 4 \quad 6 \quad 7 \quad 7 \quad 6 \quad 5 \quad 4 \quad 4 \quad 4 \\
\nu_k & : 1 \quad 4 \quad 11 \quad 21 \quad 32 \quad 42 \quad 48 \quad 51 \quad 52 \quad 52 \quad 52 \quad 52 \quad 52 \\
\nu_k^{(2)} & : 1 \quad 2 \quad 3 \quad 4 \quad 4 \quad 4 \quad 4 \\
\end{align*}
\]

Here \( \mu_k^{(2)} \neq \mu_k^{(3)} \). This implies the non-injectivity of \( \mathcal{F} \) for \( j = 4 \).

2.5. Proof of Proposition [1]. The equivalence \((\mathcal{F}) \iff (\mathcal{G})\) follows from Corollary [2] and Proposition [1.4] using [DiSa 04, Thm. 0.1] (see Remark [1.3c]) and the inclusion \( F \subset P \) for \( \mathcal{H} \).

Remark 2.5. Assume \( f = x^d + g(y, z)w \) with \( g(y, z) \) a reduced homogeneous polynomial of degree \( d-1 \) in \( y, z \) and \( d \geq 3 \). The unipotent monodromy part of the vanishing cycle complex \( \varphi_{f,1}Q_X \) vanishes (using for instance [Sa20b, Rem. 3.2] and the Thom-Sebastiani type theorem [Sa94, Thm. 0.8]). In the case \( f = x^5 + y^4 + z^4 \), the pole order spectral sequence can be calculated as below (using a small computer program explained in Remark [2.4]):

\[
\begin{align*}
k & : 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \\
\gamma_k & : 1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 40 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \quad 44 \\
\mu_k & : 1 \quad 4 \quad 10 \quad 20 \quad 31 \quad 40 \quad 44 \quad 45 \quad 46 \quad 44 \quad 44 \quad 44 \quad 44 \\
\mu_k^{(2)} & : 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \quad 3 \\
\nu_k & : 1 \quad 4 \quad 11 \quad 21 \quad 32 \quad 42 \quad 48 \quad 51 \quad 52 \quad 52 \quad 52 \quad 52 \quad 52 \\
\nu_k^{(2)} & : 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \\
\end{align*}
\]

3. Hyperplane arrangement case

In this section, we give a simple proof of a stronger version of the comparison theorem for hyperplane arrangements as an immediate corollary of [Sc 03], [DeSi 04].

3.1. Castelnuovo-Mumford regularity. For a finitely generated graded \( R \)-module \( M \) with \( R := \mathbb{C}[x, \ldots, x_n] \), the Castelnuovo-Mumford regularity can be defined as

\[
(3.1.1) \quad \text{reg} M := \max_{j,k} (c_{j,k} - j),
\]

by taking a minimal graded free resolution

\[
(3.1.2) \quad \cdots \to F_{j} \to \cdots \to F_{0} \to M \to 0,
\]

with \( F_{j} = \bigoplus_{k} R(-c_{j,k}) \), \( c_{j,k} \in \mathbb{Z} \), \( j \in \mathbb{N} \), see for instance [Et 03, §4A].
By H. Derksen and J. Sidman, we have the following.

**Proposition 3.1** ([DeSi 04, Cor. 3.7]). Let $F$ be a free graded $R$-module which is freely generated in degree $m$ (that is, $F$ is isomorphic to a direct sum of copies of $R(-m)$). Let $M, M_i \subset F$ be graded $R$-submodules ($i \in [1, n]$). Assume
\begin{equation}
(3.1.3) \quad x_i M_i \subset M \subset M_i \quad (i \in [1, n]),
\end{equation}
and $\text{reg} M \leq r - 1$ ($i \in [1, n]$) for some $r \geq m + 2$. Then $\text{reg} M \leq r$.

By the same argument as in the proof of [Sa 19, Prop. 1.3], which was inspired by [Sc 03] and a remark before [DeSi 04, Thm. 5.5] (where the degree is shifted by 1), we can immediately get the following (compare to [Ba 22], where the proof is formulated in a rather complicated manner).

**Corollary 3.1.** Let $D$ be an essential reduced central hyperplane arrangement in $X := \mathbb{C}^n$. Then we have the estimate
\begin{equation}
(3.1.4) \quad \text{reg} \Gamma(X, \Omega^1_X(\log D)) \leq 0 \quad (\forall j \in \mathbb{Z}),
\end{equation}
where algebraic logarithmic forms are used.

**Proof.** We argue by induction on $n$ and $d := \text{deg} D$ as in the proof of [Sa 19, Prop. 1.3]. Here we may assume $d > n$ (since the normal crossing case with $d = n$ is trivial) and $j \in [1, n - 1]$ (since $\Omega^1_X(\log D) = \mathcal{O}_X$ or $\mathcal{O}_X(d - n)$ if $j = 0$ or $n$). We may assume also that the coordinate hyperplanes $D_i := \{x_i = 0\}$ are contained in $D$ changing the coordinates if necessary (since $D$ is assumed to be essential). Let $D^{(i)}$ be the closure of $D \setminus D_i$ in $X$. There are inclusions
\begin{equation}
(3.1.5) \quad \Omega^i_X(\log D^{(i)}) \subset \Omega^i_X(\log D) \subset x_i^{-1} \Omega^i_X(\log D^{(i)}) \quad (j \in \mathbb{Z}, i \in [1, n]).
\end{equation}
(It is enough to verify these at smooth points of $D$ using the Hartogs-type theorem for logarithmic forms.) If the $D^{(i)}$ are all essential, then the assertion follows from the inductive hypothesis applying Proposition 3.1 to
\begin{equation}
(3.1.6) \quad M_i := x_i^{-1} \Gamma(X, \Omega^i_X(\log D^{(i)})) \quad (i \in [1, n]),
\end{equation}
where $F := \Gamma(X, \Omega^i_X(D))$ with $m = j - d - 2$ (since $j < n < d$).

Assume $D^{(n)}$ is not essential (replacing the order of coordinates if necessary). Then we have $f = x_n g$ with $g \in R^r := \mathbb{C}[x_1, \ldots, x_{n-1}]$ (changing the coordinates if necessary). Set
\[
D' := g^{-1}(0) \subset X' := \mathbb{C}^{n-1}, \quad D'' := \{0\} \subset X'' := \mathbb{C}.
\]
We have the Künneth formula (see for instance [CDFVIII])
\begin{equation}
(3.1.7) \quad \Omega^i_X(\log D) = \Omega^i_{X'}(\log D') \boxtimes \mathcal{O}_{X''} \oplus \Omega^{i-1}_{X'}(\log D') \boxtimes \Omega^1_{X''}(D'').
\end{equation}
(This also follows from the Hartogs-type theorem.) Here $\Omega^1_{X''}(D'')$ is identified with a free graded $\mathbb{C}[x_n]$-module generated freely by $dx_n/x_n$ (which has degree 0) taking the global sections. So the assertion follows from the inductive hypothesis. We can thus proceed by induction on $n, d$. This finishes the proof of Corollary 3.1.

**Remark 3.1.** The estimate of regularity (3.1.4) for $j = n-1$ is equivalent to the one for logarithmic vector fields [Sa 19, Prop. 1.3]. Indeed, there is an isomorphism induced by the interior product
\begin{equation}
(3.1.8) \quad \Theta_X(-\log D) \ni \xi \mapsto f^{-1} \iota_\xi \omega_0 \in \Omega^{n-1}_X(\log D),
\end{equation}
with $\omega_0 := dx_1 \wedge \cdots \wedge dx_n$. Here the degree is shifted by $n-d$.

### 3.2. Higher cohomology vanishing.

**Theorem 3.2.** Let $D$ be an essential reduced central hyperplane arrangement in $X := \mathbb{C}^n$. Let $\pi : \tilde{X} \to X$ be the blow-up at $0 \in X$ with $E := \pi^{-1}(0) (\cong \mathbb{P}^{n-1})$ the exceptional divisor. Setting $\tilde{D} := \pi^{-1}(D)$, we have the higher cohomology vanishing
\begin{equation}
(3.2.1) \quad H^i(E, \Omega^j_X(\log \tilde{D}) \otimes \mathcal{O}_X \mathcal{O}_E(k)) = 0 \quad (i > 0, k \geq -1, j \in \mathbb{Z}).
\end{equation}
Proof. Set

\[ M^j := \Gamma(X, \Omega_X^j(\log D)) \quad (j \in \mathbb{Z}). \]

The graded \( R \)-module \( M^j \) corresponds to the \( \mathcal{O}_X \)-module \( \Omega_X^j(\log D) \), and also to the \( \mathcal{O}_E \)-
module \( \mathcal{F}^j \) such that

\[ \Gamma(E, \mathcal{F}^j(k)) = M^j_k \quad (\forall k \in \mathbb{Z}). \]

Indeed, we have the isomorphisms at least for \( k \gg 0 \) (since \( E = \text{Proj} \, R \), see [Ha 77]).

The graded \( R \)-module \( \bigoplus_{k \in \mathbb{Z}} \Gamma(E, \mathcal{F}^j(k)) \) corresponds to the direct image by the inclusion \( X \setminus \{0\} \hookrightarrow X \), and we can apply the Hartogs-type extension theorem.

We have moreover the isomorphisms

\[ \mathcal{F}^j(k) = \Omega_X^j(\log \tilde{D}) \otimes_{\mathcal{O}_X} \mathcal{I}_E^k / \mathcal{I}_E^{k+1} \quad (k \geq 0), \]

where \( \mathcal{I}_E \subset \mathcal{O}_{\tilde{X}} \) is the ideal sheaf of \( E \subset \tilde{X} \). Indeed, the right-hand side of (3.2.3) can be identified with the subsheaf of \( \Omega_X^j(\log \tilde{D})|_E \) on which the action of the Lie derivation \( L_\xi \) is given by multiplication by \( k \) with \( \xi \) the pull-back of the vector field

\[ \xi := \sum_{i=1}^n x_i \partial_{x_i}, \]

under the birational morphism \( \pi \). Here we use locally the analytic Künneth formula for logarithmic forms as in (3.1.7) around each point of \( E \subset \tilde{X} \) together with GAGA. Note that the action of \( L_\xi \) on \( M^j_k \) is given by multiplication by \( k \).

Since the normal bundle of \( E \subset \tilde{X} \) is \( \mathcal{O}_E(-1) \), we get that

\[ \mathcal{I}_E^k / \mathcal{I}_E^{k+1} \cong \mathcal{O}_E(k) \quad (k \geq 0). \]

The vanishing (3.2.4) now follows from Corollary 3.1 taking a minimal graded free resolution as in (3.1.2), since we have for \( i > 0 \)

\[ H^i(E, \mathcal{O}_E(k)) = 0 \quad \text{unless} \quad i = n-1, \quad k \leq -n. \]

This finishes the proof of Theorem 3.2.

As a corollary, we get the following.

**Corollary 3.2a.** In the above notation, assume \( m \in \mathbb{Z}_{\geq -1} \). Then we have the isomorphism in \( D^b_c(X, \mathbb{C}) \):

\[ \mathcal{M}_{X, \log}^*(L)_{\geq m} = \begin{cases} \mathcal{R}(j_0)_* \mathcal{M}_{X, \log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\geq 1}, \\ (j_0)_! \mathcal{M}_{X, \log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\leq 0}, \end{cases} \]

where \( \nabla^{(\alpha)} \) is omitted to simplify the notation.

**Proof.** Let \( \pi : \tilde{X} \to X \), and \( \tilde{D}, E \) be as in Theorem 3.2. Set

\[ \mathcal{L}_{\tilde{X}} := \pi^* \mathcal{L}_X(-mE). \]

Here \( \mathcal{L}_X \) is trivialized by using the twisted differential as in the introduction, and \( (-mE) \) means the tensor product with \( \mathcal{O}_{\tilde{X}}(-mE) \). The residue \( \alpha_E \) along the exceptional divisor \( E \) is given by

\[ \alpha_E = \alpha_{(0)} + m \quad (\sum_{k=1}^d \alpha_k + m). \]

This can be shown by taking the pull-backs of \( df_k / f_k \) to \( \tilde{X} \), where the \( f_k \) are defining linear functions of \( D_k \) (Note that \( +m \) on the right-hand side of (3.2.7) corresponds to \( (-mE) \) in the definition of \( \mathcal{L}_{\tilde{X}} \)). Define \( \mathcal{M}_{X, \log}^*(L) \) using this \( \mathcal{L}_{\tilde{X}} \). Let \( \tilde{j} : \tilde{X} \setminus E \hookrightarrow \tilde{X} \) be the inclusion.

We have the isomorphisms in \( D^b_c(\tilde{X}, \mathbb{C}) \):

\[ \mathcal{M}_{X, \log}^*(L) = \begin{cases} \mathcal{R}_{\tilde{j}*} \mathcal{M}_{X, \log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\geq 1}, \\ \tilde{j}_! \mathcal{M}_{X, \log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\leq 0}, \end{cases} \]
using the Künneth formula, see (3.1.7). Indeed, there is a well-known isomorphism

\begin{equation}
C(t\partial_t + \beta : \mathcal{O}_\Delta \to \mathcal{O}_\Delta) = \begin{cases} 
Rj_!^*L'[1] & \text{if } \beta \notin \mathbb{Z}_{\geq 1}, \\
j_!^*L'[1] & \text{if } \beta \notin \mathbb{Z}_{\leq 0}.
\end{cases}
\end{equation}

Here \( \Delta \) is an open disk with coordinate \( t \), and \( L' \) is a local system on \( \Delta^* := \Delta \setminus \{0\} \) with \( j' : \Delta^* \hookrightarrow \Delta \) the inclusion.

By Theorem 3.2 together with [Ha 77, III, Thm. 11.1] (using (3.2.4) and GAGA), we get the vanishing

\begin{equation}
R^i\pi_*\mathcal{M}_{X,\log}^j(L) = 0 \quad (i > 0, j \in \mathbb{Z}),
\end{equation}

since \( \mathcal{O}_X(E) \otimes_{\mathcal{O}_X} \mathcal{O}_E \cong \mathcal{O}_E(-1) \) and the range of \( k \) in (3.2.1) is given by \( k \geq -1 \) (and we assume \( m \in \mathbb{Z}_{\geq -1} \)). By (3.2.8) this implies the isomorphisms in \( D_+(X, \mathbb{C}) \):

\begin{equation}
\pi_*\mathcal{M}_{X,\log}^*(L) = R\pi_*\mathcal{M}_{X,\log}^*(L)
= \begin{cases} 
R(j_0)_*\mathcal{M}_{X,\log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\geq 1}, \\
(j_0)_*\mathcal{M}_{X,\log}^*(L) & \text{if } \alpha_{(0)} + m \notin \mathbb{Z}_{\leq 0}.
\end{cases}
\end{equation}

On the other hand we have the isomorphism of complexes

\begin{equation}
\pi_*\mathcal{M}_{X,\log}^*(L) = \mathcal{M}_{X,\log}^*(L)_{\geq m},
\end{equation}

using the actions of Lie derivations \( L_{\ast \xi} \) and \( L_\xi \) (together with the Hartogs-type extension theorem for logarithmic forms), where \( \xi \) is as in the proof of Theorem 3.2. So the isomorphism (3.2.6) follows. This completes the proof of Corollary 3.2a.

Corollary 3.2a implies a stronger version of the comparison theorem as follows (compare to [Ba 22] where the argument is more complicated using local cohomology).

**Corollary 3.2b.** In the notation of Corollary 3.2a assume \( \alpha_k \notin \mathbb{Z}_{\geq 1} \) for any \( k \), and \( \alpha_Z \notin \mathbb{Z}_{\geq 2} \) for any dense edge \( Z \). Then the comparison morphism (2) is a \( D \)-quasi-isomorphism.

**Proof.** We argue by induction on the codimension of strata of the stratification associated to the hyperplane arrangement \( D \). The assertion is well known in the codimension 1 case, see [De 70]. Assume the assertion is proved outside a given stratum locally. Here we may assume that the closure of the stratum is a dense edge. In the other case, we can apply the compatibility of the canonical morphism (3) with external product \( \boxtimes \) using the Künneth formula and the inductive hypothesis. (The argument becomes more complicated if we consider only the morphism (2), since there is some difference between external products of \( \mathcal{O} \)-modules and \( \mathbb{C} \)-complexes.)

Cutting \( D \) by a transversal space passing through a general point of the stratum, we may assume that the stratum is 0-dimensional. So \( D \) is an essential indecomposable central hyperplane arrangement, and the assertion holds outside the origin.

Since \( \alpha_{(0)} \notin \mathbb{Z}_{\geq 2} \), we get the quasi-isomorphism

\begin{equation}
\mathcal{M}_{X,\log}^*(L)_{\geq -1} \sim \to \mathcal{M}_{X,\log}^*(L),
\end{equation}

using Proposition 1.5 (where the grading of \( A_{\ast 0} \) is indexed by \( \frac{1}{d} \mathbb{Z} \) with \( d := \deg f \) instead of \( \mathbb{Z} \) and \( \tilde{\alpha}_0 = \frac{1}{d} \alpha_{(0)} \)). The isomorphism (3) then follows from Corollary 3.2a for \( m = -1 \) using Lemma 1.6, Proposition 1.6 and the inductive hypothesis. This terminates the proof of Corollary 3.2b.

**Remark 3.2a.** It is easy to see that the condition \( \alpha_Z \notin \mathbb{Z}_{\geq 2} \) is optimal in the case codim \( Z = 2 \), see Example 3.2 below.
We can prove also the following (see [CaNa09] for the free divisor case).

**Corollary 3.2c.** Let $D \subset X$ be a reduced affine or projective hyperplane arrangement. For a dense edge $Z \subset D$, let $Z^0 \subset Z$ be the complement of the union of hyperplanes of $D$ not containing $Z$. Set $\delta_Z := \max_{j \in \mathbb{Z}} \delta^{(j)}_Z$ with

\[
(3.2.14) \quad \delta^{(j)}_Z := \mult_Z D - \min\{k \in \mathbb{Z} \mid (A^j_{f,p})_k \neq 0, p \in Z^0\} \quad (j \in \mathbb{Z}).
\]

In the notation of Corollary 3.2b assume $\alpha_Z \notin \mathbb{Z}_{\leq \delta_Z}$ for any dense edge $Z \subset D$ (where $\alpha_Z = \alpha_k$ with $\delta_Z = 0$ if $Z = D_k$). Then we have the canonical isomorphism in $D^b_{\text{hol}}(D_X)$:

\[
(3.2.15) \quad 1^{\text{DR}_{X}^{-1}}(\mathcal{M}^*_{X, \log}(L), \nabla^{(\alpha)})[n] \xrightarrow{\sim} D(\mathcal{M}_X(L^\vee)).
\]

Here $L^\vee$ is the dual local system of $L$ and $D$ is the dual functor. In particular, there is a natural quasi-isomorphism

\[
(3.2.16) \quad (j_U)_* L \xrightarrow{\sim} (\mathcal{M}^*_{X, \log}(L), \nabla^{(\alpha)}) \quad \text{in} \quad D^b_c(X, \mathbb{C}),
\]

or equivalently, $H^j(\mathcal{M}^*_{X, \log}(L), \nabla^{(\alpha)})_p = 0$ for any $p \in D$, $j \in \mathbb{Z}$.

**Proof.** In the definition (3.2.14), the multiplicity $\mult_Z D$ of $D$ along $Z$ coincides with the number of hyperplanes in $D$ containing $Z$, and the grading of $A^j_{f,p}$ is indexed by $\mathbb{Z}$ (counted by using the vector field $\xi$ as in the proof of Theorem 3.2). We have

\[
(3.2.17) \quad \delta_Z \in \mathbb{Z}_{\geq 1} \quad \text{ unless } \quad \text{codim}_X Z = 1.
\]

This is verified by reducing to the indecomposable essential central arrangement case, and considering the highest forms (that is, $j = n$).

The proof of Corollary 3.2c is quite similar to that of Corollary 3.2b. We argue by induction on the codimension of strata. In the codimension 1 case, we have the isomorphism noted after (3.2.8).

For the inductive argument, we may assume that $D$ is an essential central arrangement with $Z = \{0\}$. Since $\alpha_{(0)} \notin \mathbb{Z}_{\leq \delta_{(0)}}$, we can prove the quasi-isomorphism

\[
(3.2.18) \quad \mathcal{M}^*_{X, \log}(L)_{\geq 0} \xrightarrow{\sim} \mathcal{M}^*_{X, \log}(L),
\]

using Proposition 1.5. The assertion then follows from Corollary 3.2a for $m = 0$ using an argument similar to the proof of Proposition 1.6 and the inductive hypothesis. This completes the proof of Corollary 3.2c.

**Remark 3.2b.** Corollary 3.2c is optimal in the following sense: Assume there is a dense edge $Z \subset D$ with $\alpha_Z = \delta_Z$ and $\alpha_{Z'} \notin \mathbb{Z}_{\leq \delta_{Z'}}$ for any dense edge $Z' \subset D$ strictly containing $Z$. Then the morphism (3.2.18) is not a quasi-isomorphism at any $p \in Z^0$, since (3.2.15) is not. This argument may become quite complicated if the condition $\alpha_Z = \delta_Z$ is replaced by $\alpha_Z \in \mathbb{Z}_{\leq \delta_Z}$ and we have $\alpha_Z \leq \delta^{(j)}_Z$ for several $j \in \mathbb{Z}$, see Example 3.2 below for a simple case.

**Remark 3.2c.** It is easy to verify that

\[
(3.2.19) \quad \delta_Z \leq \mult_Z D - 3,
\]

by reducing to the case of an essential central arrangement with $Z = \{0\}$. Here $\delta^{(j)}_{(0)} \leq d - j$ ($j \in [2, \dim X]$) with $d := \deg D = \deg f$ by definition (3.2.14), and we have $\delta^{(1)}_{(0)} = 0$ since $D$ is reduced so that $A^1_f = O_X df$, see for instance [JKSY22b, Prop. 2]. If $\delta_{(0)} = \delta^{(2)}_{(0)} = d - 2$, we look at $(A^2_{f,0})_2$ after restricting $D$ to a sufficiently general 3-dimensional affine subspace.
Example 3.2. Let $X = \mathbb{C}^2$ with $f$ a reduced homogeneous polynomial in two variables of degree $d \geq 3$. Assume $\alpha_{(0)} = \sum_{i=k}^d \alpha_k \in \mathbb{Z} \cap [1, d-1]$. We have $\delta_{(0)} = d-2 = -\chi(\mathbb{P}^1 \setminus Z)$, and

$$\dim H^j(X \setminus D, L) = \begin{cases} d-2 & \text{if } j = 1 \text{ or } 2, \\ 0 & \text{otherwise,} \end{cases}$$

using the Leray-type spectral sequence associated with a $\mathbb{C}^*$-bundle (see Remark 1.4c or the Thom-Gysin sequence (see for instance [RSW21 Sect. 1.3])). Indeed, the local system $L$ is the pullback of a local system $L'$ of rank 1 on $\mathbb{P}^1 \setminus Z$, since $\alpha_{(0)} \in \mathbb{Z}$. The spectral sequence degenerates at $E_2$, since the Euler class of $L'$ vanishes by restricting it to $\mathbb{P}^1 \setminus \{p\}$ for $p \in Z$. Moreover the restriction of $(\Omega_X^* \log D, \nabla^{(a)})$ to the complement of $0 \in X$ is quasi-isomorphic to the pullback of $(\Omega_{\mathbb{P}^1}^* \log Z, \nabla^{(a)})$, which is quasi-isomorphic to $K' := Rj''_*j'_*L'$. Here $j = j'' \circ j' : \mathbb{P}^1 \setminus Z \hookrightarrow \mathbb{P}^1$ is a factorization depending on the $\alpha_k (k \in [1, d])$, and we may assume $j', j''$ are not the identity, since $\alpha_{(0)} = \sum_{k=1}^d \alpha_k \in [1, d-1]$. We then get that $H^i(\mathbb{P}^1, K') = 0 (i \neq 1)$, and $H^i(\mathbb{P}^1, K') = d-2$, since $\chi(K') = \chi(Rj_*L') = 2-d$.

On the other hand, Proposition 1.5 implies that

$$\dim H^j(\Omega_X^* \log D, \nabla^{(a)})_0 = \begin{cases} d-1-\alpha_{(0)} & \text{if } j = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $A_k = O_X df$ (since $D$ has an isolated singularity), and hence $(A_k^j)_{(0)} = 0$ for $k < d$.

So the morphisms (2) and (3.2.16) are both non-quasi-isomorphisms if $\alpha_{(0)} \in \mathbb{Z} \cap [2, d-2]$. Note that the higher vanishing is irrelevant to the twist of the differential operator (unless one tries to use the direct image of the canonical Deligne extension whose residues are contained either in $[0, 1)$ or in $(0, 1]$). If one applies Corollary 3.2e with $m = 0$ assuming $\alpha_{(0)} \in \mathbb{Z}_{\geq 0}$ with no condition on each $\alpha_k$, then one gets the zero-extension by the inclusion $X \setminus \{0\} \hookrightarrow X$, but (3.2.18) does not necessarily hold unless $\alpha_{(0)} \geq d-1$.

Note finally that Corollary 3.2d implies the following generalization of [Wa17 Thm. 5.3] (where $\alpha = 0$).

**Corollary 3.2d.** Let $\alpha \in \mathbb{C} \setminus (\mathbb{Z}_{\geq 1} \cup \bigcup_z (m_z)^{-1} \mathbb{Z}_{\geq 2})$. Here $Z$ runs over the dense edges of $D$, and $m_z$ is the number of hyperplanes of $D$ containing $Z$. Then the annihilator $\text{Ann}_{D_X}(f^{\alpha-1})$ of $f^{\alpha-1}$ in $D_X$ is generated by $\tilde{\Theta}_{f, \alpha-1}$ in the notation of Remark 1.7a.

**Proof.** Set $\alpha_k := \alpha (\forall k)$. The hypothesis of Corollary 3.2d is satisfied by the assumption on $\alpha$. So the assertion follows from this corollary together with Theorem 2 in the general case.

**Remark 3.2d.** The hypothesis of Corollary 3.2d implies that $b_f(\alpha-1-j) \neq 0$ for any $j \in \mathbb{Z}_{\geq 0}$ using [Sa16a Thm. 1]. Indeed, the latter theorem actually says that the roots of $b_f(s)$ supported at the origin are contained in $\mathbb{Z} \cap [n, 2d-2]$ after multiplied by $-d$ (with $d := \deg f$). Here $2d-2$ is closely related to $\mathbb{Z}_{\geq 2}$ in the assumption on $\alpha$. However, it does not seem easy to deduce Corollary 3.2d from Theorem 2 using [Sa16a Thm. 1] (without using Corollary 3.2d) unless $D$ is tame. Indeed, it is not necessarily easy to show that $\text{D}^{1}_{DR} \mathcal{M}_{X, \log}(L, \nabla^{(a)})[n]$ is quasi-isomorphic to a $D_X$-module.

**Remark 3.2e.** In the case $\alpha_k = \alpha (\forall k)$, Corollary 3.2a is compatible with Proposition 1.5 via the calculation of vanishing cycles using [BuSa10 1.3]. Here $\omega_p \wedge$ is given by $\alpha df \wedge$ (where $\alpha$ is not $\alpha-1$), and its action on $A_{p, \alpha}$ vanishes by definition.

**Remark 3.2f.** One can extend Corollary 3.2d to the case where the condition $\alpha_k = \alpha (\forall k)$ does not hold considering the action of $D_X$ on $\prod_k f_{k}^{\alpha_k}$ and modifying $\tilde{\Theta}_{f, \alpha-1}$ appropriately (where Hartogs theorem may be needed).
Appendix: Annihilator generated by vector fields

In this Appendix we give a proof of Theorem [A] below using an argument which is quite different from the proof of Theorem [2] and is closely related to the theory of BS polynomials in [Ka 76], [Bj 93]. In the case where the assumptions of Corollary [2] are satisfied, Theorem [A] together with this corollary implies a positive answer to a question in [To 07, 3.3] (using Corollary [1.4], where \( \alpha = -1 \).

**Theorem A.** Let \( D \) be a reduced divisor on a complex manifold \( X \) which is everywhere positively weighted homogeneous and tame. Then for \( \alpha \in \mathbb{C} \), the annihilator \( \text{Ann}_{D_X}(f^\alpha) \) of \( f^\alpha \) in \( D_X \) is generated by logarithmic vector fields with \( \mathcal{O} \)-linear terms if and only if \( b_f(\alpha-j) \neq 0 \) for any \( j \in \mathbb{Z}_{>0} \).

(Here we use analytic \( \mathcal{D} \)-modules. The corresponding assertion for algebraic \( \mathcal{D} \)-modules follows by using the full faithfulness of \( \mathcal{O}_{X,x} \) over \( \mathcal{O}_{X_{\text{alg}},x} \) for closed points \( x \in X_{\text{alg}} \).) We say that the annihilator \( \text{Ann}_{D_X}(f^\alpha) \) is generated by logarithmic vector fields with \( \mathcal{O} \)-linear terms if it is generated over \( D_X \) by

\[
\tilde{\Theta}_{f,\alpha} := \{ \xi - \alpha \xi(f)/f \mid \xi \in \Theta_X(-\log D) \} \quad (\alpha \in \mathbb{C}),
\]

where \( \Theta_X(-\log D) \) denotes the sheaf of logarithmic vector fields [SaK 80], see also [1.6]. Note that \( \tilde{\Theta}_{f,\alpha} \) is the specialization at \( s = \alpha \) of

\[
\tilde{\Theta}_f := \{ \xi - s \xi(f)/f \mid \xi \in \Theta_X(-\log D) \} \subset D_X[s],
\]

which is contained in \( \text{Ann}_{D_X[s]}(f^s) \). We have the isomorphisms as \( \mathcal{O}_X \)-modules

\[
(A.1) \quad \tilde{\Theta}_f = \tilde{\Theta}_{f,\alpha} = \Theta_X(-\log D).
\]

Theorem [A] is essentially a corollary of [Wa 17, Thm.3.26] where \( \tilde{\Theta}_f \) is used as is explained below.

For the proof of Theorem [A] we recall some basic of BS polynomials, see [Ka 76], [Bj 93]. For a holomorphic function \( f \) on a complex manifold \( X \), set \( D := f^{-1}(0) \subset X \), and

\[
N_f := D_X[s]f^s \subset \mathcal{O}_X(*D)[s]f^s),
\]

\[
N_{\alpha} := N_f/(s-\alpha)N_f \quad (\alpha \in \mathbb{C}).
\]

There are canonical \( \mathcal{D}_X \)-linear surjective morphisms

\[
(A.2) \quad r_{\alpha} : N_{\alpha} \twoheadrightarrow D_Xf^{\alpha} \subset \mathcal{O}_X(*D)f^{\alpha} \quad (\alpha \in \mathbb{C}).
\]

We have also a \( \mathcal{D}_X \)-linear injective endomorphism \( t : N_f \hookrightarrow N_f \) defined by

\[
(A.3) \quad t(P(s)f^s) = P(s+1)f^{s+1} \quad (P(s) \in D_X[s]),
\]

see [Ka 76]. By definition \( b_f(s) \) is the minimal polynomial of the action of \( s \) on the holonomic \( \mathcal{D}_X \)-module \( N_f/tN_f \) (which is induced by multiplication by \( s \) on \( D_X[s] \)). We assume \( b_f(s) \) exists shrinking \( X \) if necessary.

Since \( ts = (s+1)t \), the morphism \( t \) induces the \( \mathcal{D}_X \)-linear morphisms

\[
(A.4) \quad t_{\alpha} : N_{\alpha} \rightarrow N_{\alpha-1} \quad (\alpha \in \mathbb{C}),
\]

(see also (A.6) below) together with the commutative diagram

\[
(A.5) \quad \begin{array}{cccccc}
N_{\alpha} & \xrightarrow{t_{\alpha}} & N_{\alpha-1} & \xrightarrow{t_{\alpha-1}} & N_{\alpha-2} & \rightarrow & \cdots \\
\downarrow r_{\alpha} & & \downarrow r_{\alpha-1} & & \downarrow r_{\alpha-2} & & \\
D_Xf^{\alpha} & \xrightarrow{1_{\alpha}} & D_Xf^{\alpha-1} & \xrightarrow{1_{\alpha-1}} & D_Xf^{\alpha-2} & \rightarrow & \cdots
\end{array}
\]

Here the \( t_{\alpha} \) are natural inclusions \( (\alpha \in \mathbb{C}) \). It is easy to see that \( t_{\alpha} \) is surjective if \( b_f(\alpha-1) \neq 0 \), although the converse does not necessarily hold, see [Sa 21] Ex.4.2].
We have the following (which slightly improves [Bj93 Prop. 6.3.15]).

**Lemma A.** For \( \alpha \in \mathbb{C} \), the following conditions are equivalent:

(a) \( t_\alpha \) is injective,
(b) \( t_\alpha \) is surjective,
(c) \( b_f(\alpha-1) \neq 0 \).

**Proof.** This follows applying the snake lemma to the diagram below in view of the definition of \( b_f(s) \) noted after (A.3).

\[
\begin{array}{ccc}
0 & \to & \text{Ker } t_\alpha \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{N}_f \\
\downarrow & & \downarrow \\
\mathcal{N}_f & \xrightarrow{s-\alpha} & \mathcal{N}_f \\
\downarrow & & \downarrow t_\alpha \\
0 & \to & \mathcal{N}_\alpha \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{N}_f \\
\downarrow & & \downarrow \\
\mathcal{N}_f/t\mathcal{N}_f & \xrightarrow{s-\alpha+1} & \mathcal{N}_f/t\mathcal{N}_f \\
\downarrow & & \downarrow \\
\mathcal{N}_f/t\mathcal{N}_f & \to & \text{Coker } t_\alpha \\
\end{array}
\]

Indeed, \( \mathcal{N}_f/t\mathcal{N}_f \) is a holonomic \( \mathcal{D}_X \)-module (having locally finite length) so that the action of \( s-\alpha+1 \) is injective if and only if it is surjective. This finishes the proof of Lemma A.

We have the following (which shows that the converse of [Ka76 Prop. 6.2] holds, see also [Oa18] for the case \( f \) is a polynomial).

**Proposition A.** For \( \alpha \in \mathbb{C} \), the surjective morphism \( r_\alpha : \mathcal{N}_\alpha \to \mathcal{D}_X f^\alpha \) in (A.2) is bijective if and only if \( b_f(\alpha-j) \neq 0 \) for any \( j \in \mathbb{Z}_{>0} \).

**Proof.** It is enough to show that \( b_f(\alpha-j) \neq 0 \) for any \( j \in \mathbb{Z}_{>0} \) assuming the bijectivity of \( r_\alpha \), since the converse is proved in [Ka76 Prop. 6.2]. The desired assertion, however, follows easily from Lemma A using the diagram (A.5) inductively. This finishes the proof of Proposition A.

**Proof of Theorem A** The annihilator \( \text{Ann}_{\mathcal{D}_X[s]} f^s \) is generated by \( \tilde{\Theta}_f \) over \( \mathcal{D}_X[s] \), see [Wa17 Thm. 3.26]. (Here the tameness assumption cannot be weakened as in Theorem 2.) This implies that the annihilator of the image of \( f^s \) in \( \mathcal{N}_\alpha \) is generated by \( \tilde{\Theta}_{f,\alpha} \) over \( \mathcal{D}_X \). Indeed, we have the commutative diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{D}_X[s] \otimes \tilde{\Theta}_f \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{D}_X[s] \\
\downarrow & & \downarrow \phi' \\
0 & \to & \mathcal{N}_f \\
\downarrow & & \downarrow \\
0 & \to & \mathcal{D}_X[\star] \\
\end{array}
\]

where \( \phi', \phi, \phi'' \) are induced by multiplication by \( s-\alpha \) on \( \mathcal{D}_X[s] \). We see that its rows and columns are exact using the snake lemma together with the isomorphisms in (A.1), where \( \phi'' \) is injective using the inclusion \( \mathcal{N}_f \subset \mathcal{O}_X(\star D)[s] f^s \). From the diagram we thus get the canonical isomorphism

\[
\mathcal{N}_{f,\alpha} = \mathcal{D}_X/\mathcal{D}_X \tilde{\Theta}_{f,\alpha}.
\]

So the assertion follows from Proposition A. This completes the proof of Theorem A.

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