Some characterizations of multiple selfdecomposability, with extensions and an application to the Gamma function

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Abstract Inspirations for this paper can be traced to Urbanik [51, 52] where convolution semigroups of multiple decomposable distributions were introduced. In particular, the classical Gamma $G_t$ and $\log G_t$, $t > 0$, variables are selfdecomposable (i.e. have distributions in $L_0(R)$). In fact, we show that $\log G_t$ is twice selfdecomposable (i.e. have distributions in $L_1(R)$) if, and only if, $t \geq t_1$ where $t_1$ is an explicit critical value, and this an answer to a problem raised by Akita & Maejima [3]. Moreover, we provide several new factorizations of the Gamma function and the Gamma distributions that extend many known ones in the literature. To this end, we revisit the class of multiple selfdecomposable distributions, denoted $L_m(R)$, and propose handy tools for its characterization, mainly based on the Mellin-Euler operator and on the Hadamard fractional integral. Finally, we give a perspective for the generalization of the class $L_m(R)$, based on linear operators or on stochastic integral representations.

Key words: Bernstein functions; Difference-differential operators; Factorizations of the Gamma function; Gamma distributions; Hadamard fractional integral; Kanter’s factorization; Infinite divisibility; Integral stochastic representation; Laplace transform; Lévy-Laplace exponents; Lévy processes; Mellin-Euler differential operator; Multiplicative convolution; Multiple selfdecomposability; Stable distributions; Spectrally negative Lévy processes; Subordinators.

In memoriam of Kazimierz Urbanik (February 5, 1930 – May 29, 2005)

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1 Introduction

The distribution $\mu$ of a real-valued random variable $X$, is said to be \textit{ininitely divisible}, and we denote $\mu \in \text{ID}(\mathbb{R})$ or $X \sim \text{ID}(\mathbb{R})$, if

for each natural $n \geq 2$ there exits $\mu_n$ an $n$-th fold convolution: $\mu_n^n = \mu$.

In other terms, $\mu_n$ is the distribution of independent and identically distributed random variables $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ and $X \overset{d}{=} X_{1,n} + X_{2,n} + \ldots + X_{n,n}$. It is known that every r.v. $X \sim \text{ID}(\mathbb{R})$ is embedded into a \textit{Lévy process} $(Z_t)_{t \geq 0}$, i.e. a process with independent and stationary increments, such that $Z_0 = 0$ and $X \overset{d}{=} Z_1$, in order that the so-called Lévy-Khintchine formula holds:

$$X \sim \text{ID}(\mathbb{R}) \iff \mathbb{E}[e^{iuX}] = \mathbb{E}[e^{iuZ_t}] = e^{t \Phi(u)}, \quad t > 0, \quad u \in \mathbb{R},$$

where $\Phi$ is given by the following expression: for a fixed truncation function $h$ (i.e. a bounded function such that $\lim_{x \to 0} (h(x) - x)/x^2$ exists, for example $h(x) = x_{\{x\leq 1\}}$, or $x/(1+x^2)$), we have

$$\Phi(u) = iau - bu^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iu\lambda} - 1 - iu\lambda\right) \pi(dx),$$

where $a \in \mathbb{R}$, $b \geq 0$ are respectively called \textit{drift term} and \textit{Brownian coefficient}. Clearly, if one changes the truncation function to a new one $g$, the new drift term becomes $a_g := a + \int_{\mathbb{R}\setminus\{0\}} (g(x) - h(x)) \pi(dx)$. The measure $\Pi$ is called the \textit{Lévy measure} and satisfies the integrability condition

$$\int_{\mathbb{R}\setminus\{0\}} (x^2 \wedge 1) \pi(dx) < \infty.$$  

- \textit{Spectrally negative} Lévy processes on the real line have a particular interest. They correspond to Lévy processes $(Z_t)_{t \geq 0}$ with no positive jumps, viz. such that the associated Lévy measure $\Pi$ in (2) gives no mass to $(0, \infty)$. Defining

$$\Pi$$ be the image of $\pi$ by the reflection $x \mapsto -x$, and $\chi(x) := -h(-x)$, $h$ given by (2),

it becomes more handy to use the so-called \textit{Lévy-Laplace exponent} $\Psi$ instead of the Lévy-

Fourier exponent $\Phi$:

$$\Psi(\lambda) := \lim_{t \to 0} \frac{\log E[e^{\lambda X_t}]}{t} = \Phi(-i\lambda) = a\lambda + b\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\chi(x)\right) \Pi(dx), \quad \lambda \geq 0,$$

where $\Pi$ satisfies the integrability condition (3). The class $\mathcal{LE}$ of \textit{Laplace exponents} is defined as the set of functions $\Psi$ of the form (5) with $\chi(x) = x$, i.e.

$$\mathcal{LE} := \{\Psi(\lambda) = a\lambda + b\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right) \Pi(dx), \quad \lambda \geq 0\},$$

where $a \in \mathbb{R}$, $b \geq 0$, and the Lévy measure $\Pi$ satisfies integrability condition

$$\int_{(0,\infty)} (x^2 \wedge 1) \Pi(dx) < \infty.$$

We define the subclass $\mathcal{ID}_-(\mathbb{R})$ and $\text{ID}_-\mathbb{R}$ of $\text{ID}(\mathbb{R})$ as follows:

$$X \sim \mathcal{ID}_-(\mathbb{R}) \quad \text{(respectively } \text{ID}_-\mathbb{R}) \quad \text{if} \quad E[e^{\lambda X}] = E[e^{\lambda Z_t}] = e^{t\Psi(\lambda)}, \quad t > 0 \quad \text{and } \Psi \text{ is as in (5) \quad (respectively } \Psi \in \mathcal{LE}).$$

(8)
For Lévy processes, the references are numerous; we suggest the books of Bertoin [7], Kyprianou [29] or Sato [42]. The following proposition explains why $\overline{\text{ID}}_+(\mathbb{R})$ is the vague closure of $\text{ID}_-(\mathbb{R})$:

**Proposition 1.1** We have the equivalences:

1) $X \sim \overline{\text{ID}}_+(\mathbb{R}) \iff X \overset{d}{\Rightarrow} \lim_n X_n$, with $X_n \sim \text{ID}_-(\mathbb{R})$.
2) $X \sim \text{ID}(\mathbb{R}) \iff \text{there exist two sequences of independent r.v's} \ X'_n, X''_n \sim \text{ID}_-(\mathbb{R}) \text{ such that} \ X \overset{d}{\Rightarrow} \lim_n (X'_n - X''_n)$.

From the latter, we see that $X \sim \text{ID}(\mathbb{R})$ if, and only if $X = X' - X''$, where $X'$ and $X''$ are independent and $X', X'' \overset{d}{\Rightarrow} \overline{\text{ID}}_+(\mathbb{R})$.

- The class $\mathcal{BF}$ of Bernstein functions is defined by

$$\mathcal{BF} := \{ \phi(\lambda) = d\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi(dx), \ \ \lambda \geq 0 \}.$$  \hspace{1cm} (9)

where $d \geq 0$ and the Lévy measure $\Pi$ satisfies the integrability condition

$$\int_{(0,\infty)} (x \wedge 1) \Pi(dx) < \infty.$$  \hspace{1cm} (10)

We emphasize that, traditionally, a function of the form $\phi + q$ is called Bernstein function, when $\phi \in \mathcal{BF}$ and the addition of so-called killing term $q \geq 0$, transforms probability measures into sub-probability measures. In order to avoid discussions on sub-probability measures, we only consider the class $\mathcal{BF}$. Observe that a differentiable function $\Psi$ on $(0,\infty)$ belongs to $\mathcal{LE}$ if, and only if, $E[Z_1] = \Psi'(0)$ is finite and $\Psi' - \Psi'(0) \in \mathcal{BF}$. Also observe that and if that $\phi \in \mathcal{BF}$ is equivalent to the fact that $\phi$ is nonnegative and $\phi'$ belongs to the class of completely monotone functions and this yields $-\phi \in \mathcal{LE}$. The class $\mathcal{CM}$ of completely monotone functions corresponds to those infinitely differentiable functions $f : (0,\infty) \to (0,\infty)$, such that

$$(-1)^n f^{(n)}(x) \geq 0, \ \ \text{for all} \ n = 0, 1, 2\ldots.$$  

Bernstein characterized the class $\mathcal{CM}$ by

$$\mathcal{CM} := \{ f(\lambda) = \int_{[0,\infty)} e^{-\lambda x} v(dx), \ \ \lambda > 0, \ \ \text{where} \ v \ \text{is some Radon measure on} \ [0,\infty) \}.$$  \hspace{1cm} (11)

As for Lévy-Laplace exponents, the class $\mathcal{BF}$ is one-to-one with the set of infinitely divisible distributions supported by $\mathbb{R}_+$:

we denote $X \sim \text{ID}(\mathbb{R}_+)$, if $X \geq 0$ and $X \sim \text{ID}(\mathbb{R})$,

which is equivalent to $X \geq 0$ and $-X \sim \text{ID}_-(\mathbb{R})$. In this case, the embedding Lévy process $(Z_t)_{t \geq 0}$ in (1) is a subordinator (i.e. a Lévy process with increasing paths) and the Lévy-Khintchine formula takes the form

$$X \sim \text{ID}(\mathbb{R}_+) \iff E[e^{-\lambda X}] = E[e^{-\lambda Z_t}] = e^{-\lambda \phi(\lambda)}, \ \ t \geq 0, \ \ \lambda \geq 0, \ \ \text{and} \ \phi \in \mathcal{BF}.  \hspace{1cm} (12)$$

The class $\mathcal{L}_0(\mathbb{R})$ of self-decomposable distributions has the following description, among others:

we denote $\mu \in \mathcal{L}_0(\mathbb{R})$, if, for all $c \in (0,1)$, there exists a measure $\mu_c$ s.t. $\mu = T_c \mu * \nu_c$.  \hspace{1cm} (13)

where, for Borel sets $B$, $T_c(\mu(B)) = \mu(e^{-c}B)$. The latter has several equivalent formulations that can be found in [25] and in the book of Sato [42]:
\[ X \sim L_0(\mathbb{R}) \iff X = cX + V_c, \text{ where } V_c \text{ is independent of } X \text{ (necessarily } V_c \sim \text{ID}(\mathbb{R})), \]

\[ \iff X \sim \text{ID}(\mathbb{R}), \text{ and for each } 0 < c < 1, \text{ and each Borel set } B \subset \mathbb{R}, \text{ the Lévy measure satisfies, } \Pi(B) - \Pi(c^{-1}B) \geq 0, \]

\[ \iff \Pi(dx) = \frac{k(x)}{x} \text{, where the function } k \text{ is non-increasing on } (-\infty, 0) \text{ and on } (0, \infty). \]

Observe that

\[ X \sim L_0(\mathbb{R}) \text{ and } X \geq 0 \iff \text{the } k \text{-function in (15) is such that } k(x) = 0, \text{ for } x < 0. \]

In 1973, Urbanik [51, 52], introduced a decreasing sequence \( L_n(\mathbb{R}), n = 1, \ldots, \) of limiting distributions (for some specified triangular arrays) contained in the class \( L_0(\mathbb{R}), \) that is,

\[ L_{\infty}(\mathbb{R}) = \bigcap_{k \geq 0} L_k(\mathbb{R}) \subset \ldots \subset L_n(\mathbb{R}) \subset \ldots \subset L_1(\mathbb{R}) \subset L_0(\mathbb{R}). \] (16)

Probability measures in \( L_n(\mathbb{R}) \) are called \( n \)-times selfdecomposable measures. Urbanik characterized multiple selfdecomposable distributions by their characteristic functions in [48] and [52, Theorem 1 and 2]. Then Urbanik gave in [51] the results without detailed proofs. His proofs in [48, 49, 50] used the Choquet-Krein-Milman theorem on extreme points in compact convex sets; as a reference for this cf. Phelps [39]. A probabilistic proof, using random integral representations, is in [19]. Urbanik’s description of the classes \( L_n(\mathbb{R}), \) in terms of the convolution factorization is given in [52, Proposition 1]: with the convention \( L_{-1}(\mathbb{R}) := \text{ID}(\mathbb{R}), \)

\[ \mu \in L_n(\mathbb{R}), \text{ if, for all } c \in (0, 1), \text{ there exists } \mu_c \in L_{n-1}(\mathbb{R}) \text{ s.t. } \mu = T_c \mu \ast \nu_c, \] (17)

For more convenience, we denote

\[ L_n(\mathbb{R}_+) := \{ \mu \in \text{ID}(\mathbb{R}_+); \text{ such that } \mu \in L_n(\mathbb{R}) \}. \]

Note that \( \text{ID}(\mathbb{A}) \text{ and } L_n(\mathbb{A}), \mathbb{A} = \mathbb{R}_+, \mathbb{R}, \) are closed (in the weak convergence topology) convolutions semigroups. If \( \mu \in L_n(\mathbb{R}) \setminus L_{n+1}(\mathbb{R}), \) then we say that \( \mu \) is exactly \( n \)-times selfdecomposable.

For operator-selfdecomposability problems, we refer the book of Jurek & Mason [23] and Meerschaert & Scheffler [36], for distributional properties of selfdecomposable distributions we recommend the book of Sato [42]. For analytic properties related to infinitely divisible distributions on the half-real line, we recommend the book of Schilling, Song & Vondraček [45] and also Stein & van Harn [47]. Several other proofs for the characterization of \( L_n(\mathbb{R}_+) \) are \( L_n(\mathbb{R}) \) available and the references are multiple, cf. Berg & Forst [6], Jurek [19, 21, 23, 24], Van Thu [53] and also Stein & van Harn [47] and the references therein.

In Section 2, we provide several new decomposability properties for the Gamma function, the Gamma distributions, and the positive stable distribution. For this purpose, consider, for \( \alpha \in (0, 1) \cup (1, \infty) \) and \( t > 0, \) the following functions:

\[ G_\alpha(\lambda) = \frac{\Gamma(\frac{\lambda}{\alpha})}{\Gamma(\frac{\lambda}{\alpha})}, \lambda > 0 \text{ and } G_{\alpha,t}(\lambda) = \Gamma(t)^{1-\alpha} \frac{\lambda^{\alpha \lambda}}{\Gamma(t^{\alpha})}, \lambda \geq 0, \] (18)

and notice the following relation

\[ G_{\alpha,t}(\lambda) = \frac{1}{G_{\alpha,t}(\alpha \lambda)^{\alpha}}. \] (19)

1. Theorem 2.1 retrieves Berg, Çetinkaya & Karp’s result [4, Theorem 3.13] and gives a stochastic interpretation. The latter and Theorem 2.4 provide the exact range of the parameters \( \alpha \) and \( t \) for which
2. Corollary 2.5 proves that if \( \alpha_1, \alpha_2, \ldots, \alpha_n \in (0,1) \), \( \sum_{k=1}^n \alpha_k = 1 \), and if \( \mathcal{G}_1, \mathcal{G}_1, \ldots, \mathcal{G}_n \), are independent random variables with Gamma distribution with shape parameter \( t > 0 \), then

\[
\mathcal{G}_t := d(\mathcal{G}_1) \mathcal{G}_{1,t} \ldots \mathcal{G}_{n,t} e^{-Xt} \iff t \geq 1/2 \quad \text{and} \quad d(\mathcal{G}_1) \mathcal{G}_{1,t} \ldots \mathcal{G}_{n,t} = d(\mathcal{G}_1) e^{-Yt} \iff t < 1/2,
\]

where \( d(\mathcal{G}_1) := \prod_{k=1}^n \alpha_k^{\alpha_k} \) and all r.v.'s are assumed to be independent in each side of the identities and \( X_{\alpha,t}, Y_{\alpha,t} \sim \text{ID}(\mathbb{R}_+) \). Analytically, the latter is transcripted in terms of the Gamma function with the following remarkable fact:

\[
\lambda \mapsto \left( \frac{\Gamma(\lambda + t)}{d(\mathcal{G})^t \prod_{k=1}^n \Gamma(\alpha_k \lambda + t)} \right)^t \quad \text{is completely monotone,}
\]

if, and only if \( t \geq \frac{1}{2} \) and \( r > 0 \) or \( t < \frac{1}{2} \) and \( r < 0 \). This results generalizes several known results in the literature, c.f Alzer & Berg [7], Bertoin & Yor [8], Li & Chen [30], Mehrez [35], Pestana, Shanbhag & Sreehari [38], for instance. We also explore whenever \( X_{\alpha,t}, Y_{\alpha,t} \sim \text{ID}(\mathbb{R}_+) \) or \( \text{BO} \) (BO being the Bondesson class of distributions, cf. [45, Definition 9.1]).

3. We improve Akita & Maejima’s [3] results who proved that \( \log G_t \) is twice selfdecomposable for any \( t \geq 1/2 \) and that there exists a universal constant \( t_1 \in (0,1/2) \) such that the last property could be extended for \( t \in (t_1,+\infty) \). Using tools developed in Section 4, we were able in Proposition 2.7 to show that the value of \( t_1 \) is the maximum of an explicit elementary function, and is approximately 0.151649938034.

Section 3 prepares for the results of Section 4 and Lemma 3.4 gives the full characterization of the class

\[
\mathcal{M}_n := \{ k, \text{ s.t. } x \mapsto k(e^x) \text{ is } n\text{-monotone on } \mathbb{R} \},
\]

cf. Williamson [54] for \( n \)-monotone functions. Using the so-called Euler-Mellin differential operator \( \Theta = x d/dx \) and the so-called Hadamard fractional integral, we provide the converse of [45, Proposition 1.16] and, the analog of [45, Theorem 4.11] for the class \( \mathcal{M}_n \).

In Section 4, mainly in Corollary 4.4, we shall provide a simple proof for the characterization of the classes \( \text{L}_n(\mathbb{R}) \) based on the class \( \mathcal{M}_n \) given in (21). Jurek had provided the same characterization [18, Theorem 6.2 and Theorem 7.1] for infinitely divisible measures on Banach spaces. Our approach may have the merit to exhibit the tight link between the Mellin-Euler differential \( \Theta \) operator and the classes \( \text{L}_n(\mathbb{R}) \) and suggests that a mechanism producing new classes of infinitely divisible distribution via linear operators \( \Omega \) other than \( \Theta \) could be implemented as follows: assume \( \Omega \) is a linear operator on the space of functions \( f : [0,\infty) \rightarrow [0,\infty) \) which commute with the dilations, i.e. there exists

\[
\Omega(x \mapsto f(ax)) = \Omega(f)(ax), \quad \text{for all } a, u > 0.
\]

For instance, if \( \Omega = \Theta \), then \( \Omega(\mathcal{B}(-\mathcal{B}) \cap \mathcal{B}F \) is one-to-one with \( \text{L}_n(\mathbb{R}) \) and if \( \Omega = I - \Theta \), then \( \Omega(\mathcal{B}F) \cap \mathcal{B}F \) is not void. Now, let \( \phi \) be a Bernstein function of the form

\[
\phi(\lambda) = d \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \frac{k(x)}{x} dx = d \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \frac{\tilde{k}(\lambda/x)}{x} dx, \quad \lambda \geq 0.
\]

where \( d \geq 0 \) and assume that \( \tilde{k}(x) := \tilde{k}(1/x) \) is “good enough” so that we can swap \( \Omega \) and the integral. If \( \Omega^n, n = 1, 2, \ldots \), is the \( n \)-th iterate of \( \Omega \) and \( \omega := \Omega(\text{Identity})(1) \), then,

\[
\Omega^n(\phi)(\lambda) = \omega d \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) k_n(x) dx, \quad \text{where } k_n(x) := \Omega^n(\tilde{k})(1/x).
\]
Thus, it is simply seen that \( \Omega^n(\theta) \) remains a Bernstein function if \( k_n \) complies with the non-negativity and integrability conditions imposed on Lévy measures. With this mechanism, one is able to build new subclasses of \( \text{ID}(\mathbb{R}_+) \), eventually nested. When \( \Omega = \Theta \), we obtain in Corollary 4.4 the exact shape of the function \( k_n \) (for Bernstein or Lévy-Laplace exponents) of \( X \sim \text{ID}(\mathbb{R}_+) \) or \( \text{ID}_- (\mathbb{R}) \) or \( \bar{\text{ID}}_- (\mathbb{R}) \) for which it holds that

\[
X \sim L_n(\mathbb{R}_+) \text{ or } L_n(\mathbb{R}), \quad n = 1, 2, \ldots, \infty.
\]

Section 5 improves the last approach with a method based on integral stochastic representations like (113). All the proofs were postponed to Section 6 and Section 7 gives some background on the classes \( \text{ID}(\mathbb{R}) \) and \( L_0(\mathbb{R}) \).

## 2 New decomposability properties for the Gamma function, the Gamma distributions and the positive stable distribution

This section provides new multiple selfdecomposability properties involving the Gamma function and the Gamma and positive stable distributions.

Recall the Digamma function is defined by \( \psi(t) = \Gamma'(t)/\Gamma(t), \ t > 0 \) and is given by formula 5 p.903 [14]:

\[
\psi(t) = -\gamma + \int_0^\infty \frac{e^{-xt} - e^{-it}}{1 - e^{-x}} \, dx, \quad t > 0,
\]

where \( \gamma \) is the Euler-Mascheroni constant. Since \( \log \Gamma(\lambda) = \int_1^\lambda \psi(t) \, dt, \ \lambda > 0, \) we recover the following representations,

\[
\Gamma(\lambda) = \exp\left\{ -\gamma(\lambda - 1) + \int_0^\infty \left( e^{-\lambda u} - e^{-u} - (\lambda - 1) u e^{-u} \right) \frac{du}{u(1 - e^{-u})} \right\}, \quad \lambda > 0
\]

and then,

\[
\frac{\Gamma(\lambda + t)}{\Gamma(t)} = \exp\left\{ \psi(t)\lambda + \int_0^\infty \left( e^{-\lambda u} - 1 + \lambda u \right) \frac{e^{-tu}}{u(1 - e^{-u})} \, du \right\}, \quad \lambda \geq 0, \ t > 0. \tag{22}
\]

The \( q \)-Gamma function \( \Gamma_q(x), \ x > 0, \) defined by

\[
\Gamma_q(x) = \begin{cases} 
(1 - q)^{1 - x} \prod_{j=0}^{x-1} \frac{1 - q^{j+1}}{1 - q^j}, & \text{if } 0 < q < 1, \\
(q - 1)^{1 - x} q^{x(x-1)/2} \prod_{j=0}^{x-1} \frac{1 - q^{j+1}}{1 - q^j}, & \text{if } q > 1,
\end{cases} \tag{23}
\]

enjoys the basic property: \( \lim_{q \to 1^-} \Gamma_q(x) = \lim_{q \to 1^+} \Gamma_q(x) = \Gamma(x) \).

From now on, we denote by \( \mathcal{G}_t \) a random variable with the standard Gamma distribution with shape parameter \( t > 0 \), which has the density function, Laplace and Mellin transforms respectively given by

\[
f_{\mathcal{G}_t}(x) = \frac{x^{t-1}}{\Gamma(t)} e^{-x}, \quad \text{E}[e^{-\lambda \mathcal{G}_t}] = \frac{1}{(1 + \lambda)^t} \quad \text{and} \quad \text{E}[\mathcal{G}_t^2] = \frac{\Gamma(t + \lambda)}{\Gamma(t)}, \quad x > 0, \ \lambda > -t. \tag{24}
\]

The function \( \lambda \mapsto \lambda^a, \ 0 < a < 1, \) is a generic example of a Bernstein functions. It is not difficult to derive the representation
\[ \lambda^x = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty (1 - e^{-\lambda x}) \frac{dx}{x^{\alpha+1}}, \quad \lambda \geq 0, \]  

and this function is associated to the so-called standard positive stable r.v. \( S_\alpha \) with stability parameter \( \alpha \in (0, 1) \) via:

\[ E[e^{-\lambda S_\alpha}] = e^{-\lambda^\alpha}, \quad \lambda \geq 0, \quad \text{and} \quad E[(S_\alpha)^{-\lambda}] = \frac{\Gamma(1 + \frac{\lambda}{\alpha})}{\Gamma(1 + \lambda)}, \quad \lambda > -\alpha. \]  

The p.d.f. of \( S_\alpha \) is not explicit, except for \( \alpha = 1/2 \) where \( S_{1/2} \overset{d}{=} \frac{1}{4} G_{1/2} \), cf. the monograph of Zolotarev [55] for more account.

### 2.1 Elementary decomposability properties for Gamma and stable distributions

Using the representations (22) and (24), we deduce that the functions, given for \( \lambda \geq 0 \), by

\[ \phi_\alpha(\lambda) = -\log E[e^{-\lambda S_\alpha}] = \int_0^\infty (1 - e^{-\lambda x}) \frac{k_1(x)}{x} dx, \quad k_1(x) = e^{-x}, \]

\[ \Psi_\alpha(\lambda) = \log E[(S_\alpha)^{\lambda}] = \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \frac{k_2(x)}{x} dx, \quad k_2(x) = \frac{e^{-x}}{1 - e^{-x}}, \]  

are respectively in \( \mathcal{B} \mathcal{F} \) and \( \mathcal{L} \mathcal{E} \), and that \( k_1, k_2 \) are non-increasing (cf. also [20, p. 98] for \( k_1 \)). By (15), it is immediate that \( G_\alpha \sim L_0(\mathbb{R}_+) \) and \( \log G_\alpha \sim \tilde{T \mathcal{D}} \_\dashv (\mathbb{R}) \cap L_0(\mathbb{R}) \); cf. Jurek [20, p. 98] for the function \( k_2 \), see [22] for the Back driving Lévy process (BDLP) of \( \log G_\alpha \), cf. Section for BDLP’s.

Recall that the distribution of positive stable r.v. \( S_\alpha \) is given by (26). As for the r.v. \( G_\alpha \), one can deduce from identity (22), that the functions, given for \( \lambda \geq 0 \), by

\[ \phi_\alpha(\lambda) = -\log E[e^{-\lambda S_\alpha}] = \lambda^\alpha = \int_0^\infty (1 - e^{-\lambda x}) \frac{k_3(x)}{x} dx, \quad k_3(x) = \frac{\alpha}{\Gamma(1 - \alpha)x^\alpha}, \]

\[ \Psi_\alpha(\lambda) = \log E[(S_\alpha)^{-\lambda}] = \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \frac{k_4(x)}{x} dx, \quad k_4(x) = \frac{1}{1 - e^{-x}} - \frac{1}{1 - e^{-x}}, \]  

are respectively in \( \mathcal{B} \mathcal{F} \) and \( \mathcal{L} \mathcal{E} \), and that \( k_3, k_4 \) are non-increasing. By Corollary 4.4, it is immediate that \( S_\alpha \sim L_0(\mathbb{R}_+) \) and \( -\log S_\alpha \sim \tilde{I}_\mathcal{D} \_\dashv (\mathbb{R}) \cap L_0(\mathbb{R}) \).

### 2.2 Main results, new decomposability properties

After completing this work, we discovered the recent result of Berg, Çetinkayaş & Karp [4, Theorem 3.13] on the function \( G_{\alpha t} \), given in (18), which is equivalent to Theorem 2.1. Our Theorem provides a stochastic interpretation of [4, Theorem 3.13] and which is shown with a different proof. Further, our investigation of the function \( G_{\alpha t} \) goes beyond by seeking for selfdecomposability properties in Theorem 2.4, Corollary 2.5 and Proposition 2.7 below.

**Theorem 2.1** For the function \( G_{\alpha t} \), \( t > 0 \), given in (18), we have the equivalences between the assertions (i), (ii) and (iii) in each of the points 1) and 2).
1) (i) for all \( r > 0 \), the function \((G_{a,t})^r \in \mathcal{CM}\);
(ii) the function \(G_{a,t}\) is the Laplace transform of some positive infinitely divisible random variable \(X_{a,t}\):
\[
G_{a,t}(\lambda) = \mathbf{E}\left[e^{-\lambda X_{a,t}}\right], \quad \lambda \geq 0;
\]
(iii) \( a \in (0,1) \) and \( t \geq 1/2 \) or \( a \in (1,\infty) \) and \( t < 1/2 \).

2) (i) the function \((G_{a,t})^{-r} \in \mathcal{CM}\), for all \( r > 0 \);
(ii) the function \((G_{a,t})^{-1}\) is the Laplace transform of some positive infinitely divisible random variable \(Y_{a,t}\):
\[
\frac{1}{G_{a,t}(\lambda)} = \mathbf{E}\left[e^{-\lambda Y_{a,t}}\right], \quad \lambda \geq 0;
\]
(iii) \( a \in (0,1) \) and \( t < 1/2 \) or \( a \in (1,\infty) \) and \( t \geq 1/2 \).

Remark 2.2 Equality (19), is equivalent to
\[
\mathbf{E}\left[e^{-\lambda X_{a,t}}\right] = \mathbf{E}\left[e^{-\alpha \lambda Y_{1/a,t}}\right]^a, \quad \lambda \geq 0,
\]
and this strengthens the fact that \(X_{a,t}\) and \(Y_{1/a,t}\) are concomitantly infinitely divisible.

Theorem 2.1 provides the same result as the one of Ly et al. [32, Theorem 1.10] for \(G_{a,t}\). Using the fact that the pointwise limit of a sequence of completely monotone functions is also completely monotone, then using Theorem 2.1 and the fact that the limit
\[
G_a(\lambda) = \lim_{t \to 0} \Gamma(t)^{\alpha-1} \lambda^{-\alpha} G_{a,t}(\lambda),
\]
we immediately retrieve the results of Li & Chen [30, Theorem 9] and also of Alzer & Berg [2, Theorem 3.5] on the functions \(G_a\). Mehrez [35, Theorem 1] obtained the same result in the context of the \(q\)-analog of the function \(G_a\), i.e. when it is build with the \(q\)-Gamma function defined in (23).

Corollary 2.3 We have the following results for the function \(G_a\) given in (18).
1) The function \(\lambda \mapsto G_a(\lambda)^r\), is completely monotone for all \( r > 0 \), if, and only if, \( a \in (1,\infty) \);
2) The function \(\lambda \mapsto G_a(\lambda)^{-r}\), is completely monotone for all \( r > 0 \), if, and only if, \( a \in (0,1) \).

Theorem 2.4 Let \( t_0 = \frac{1}{2} + \frac{1}{2\sqrt{a}} \). It holds that
1) \(X_{a,t} \sim L_0(\mathbb{R}_+)\) if, and only if, \( a \in (0,1) \) and \( t \geq t_0 \);
2) \(Y_{a,t} \sim L_0(\mathbb{R}_+)\) if, and only if, \( a \in (1,\infty) \) and \( t \geq t_0 \).

In [45, Lemma 92, Theorem 9.7], it is shown that
\[
\mathbf{ME} \subset \mathbf{BO} \subset \mathbf{ID}(\mathbb{R}_+)
\]
where the Bondesson class \(BO\) is characterized as the smallest class of probability measures on \((0,\infty)\) which contains \(\mathbf{ME}\), the class of mixture of exponentials distribution, and which is closed under convolutions and vague limits. Actually, the class \(BO\) is one-to-one to the class of complete Bernstein functions, viz. of functions of the form
\[
\phi(\lambda) = d\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + x} \frac{v(dx)}{x} = d\lambda + \int_0^\infty (1 - e^{-\lambda x}) \mathcal{L}v(x) dx,
\]  
where \(\mathcal{L}v(x) = \int_{(0,\infty)} e^{-\lambda x} v(du)\). Furthermore, the representation
\[
v(dx) = x \eta(x), \quad \text{where} \quad \eta : (0,\infty) \to [0,1], \text{ is measurable},
\]
is equivalent to say that the Bernstein function \( \phi \) is associated to a distribution in ME, cf. [45, Theorem 9.5]. As a consequence of Theorems 2.1 and 2.4, we obtain the following factorizations for the Gamma distributions.

**Corollary 2.5** Let \( t > 0 \), \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1)^n \) such that \( \sum_{i=1}^n \alpha_i = 1 \), \( d(\alpha) := \prod_{i=1}^n \alpha_i^{\alpha_i} \) and recall the random variables \( X_{\alpha_j} \) and \( Y_{\alpha_j} \) given by Theorem 2.1. Let \( G_t \) be a Gamma distributed random variable with shape parameter \( t \) and \( G_{\alpha_1}, \ldots, G_{\alpha_n} \) denote independent copies of \( G_t \). Assuming that the random variables involved in the next factorizations in law are independent, we have the following results.

1) If \( t \geq 1/2 \), then we have the factorization in law
\[
G_t \overset{d}{=} d(\alpha) \prod_{j=1}^n G_{\alpha_j} \overset{d}{=} \prod_{j=1}^n e^{-u_j \lambda_j} G_{\alpha_j} \quad \text{where} \quad X_{\alpha_j} := \sum_{k=1}^n X_{\alpha_k, j},
\] (30)

2) If \( 0 < t < 1/2 \), then we have the factorization in law
\[
d(\alpha) \prod_{j=1}^n G_{\alpha_j} \overset{d}{=} \prod_{j=1}^n e^{-u_j \lambda_j} G_{\alpha_j} \quad \text{where} \quad Y_{\alpha_j} := \sum_{k=1}^n Y_{\alpha_k, j}.
\] (31)

3) If \( t \geq t_0 = \frac{1}{2} + \frac{1}{\sqrt{t}} \), then \( X_{\alpha_k, j}, Y_{\alpha_k, j} \sim \text{LO}(R_+) \).

4) We have \( X_{\alpha_k, j}, Y_{\alpha_k, j} \sim \text{BO} \), if, and only if, \( t \geq 1 \). In this case, their \( (1+t)^{-1} \)-fold convolutions, in the sense of (12), have distributions in ME.

**Remark 2.6** 1) A direct application of Corollary 2.5 is
\[
\log G_t \in \text{ID}_-(R) \cap L_0(R), \quad \text{if} \ t > 1/2 \quad \text{and} \quad \log G_t \in \text{ID}_-(R) \cap L_1(R), \quad \text{if} \ t > t_0.
\] (32)

2) The identities (30) and (31) have an interpretation in terms of the Gamma function by taking the Mellin-transforms in both sides: for all \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1)^n, \sum_{i=1}^n \alpha_i = 1 \) and all \( r > 0 \), we have
\[
\lambda \mapsto \mathbb{E}[e^{-\lambda X_{\alpha_j}}] = \left( \frac{\Gamma(\lambda + t)}{d(\alpha) \prod_{i=1}^n \Gamma(\alpha_i \lambda + t)} \right)^r \in \mathcal{CM}, \quad \text{if} \ t \geq \frac{1}{2}
\] (33)
\[
\lambda \mapsto \mathbb{E}[e^{-\lambda X_{\alpha_j}}] = \left( \frac{d(\alpha) \prod_{i=1}^n \Gamma(\alpha_i \lambda + t)}{\Gamma(\lambda + t)} \right)^r \in \mathcal{CM}, \quad \text{if} \ t < \frac{1}{2}.
\] (34)

The case \( t = 1 \) has been treated by Simon [46, subsection 3.4]. The latter has to be approached to Karp & Prilepkin’s recent result [27, Theorem 4], which states that the function
\[
\lambda \mapsto \frac{\prod_{i=1}^p \Gamma(A_i + \lambda + b_i)}{\prod_{j=1}^q \Gamma(B_j + \lambda + b_j)} A_i, B_j > 0, \quad a_i, b_j \geq 0,
\]
is completely monotone if, and only if,
\[
\sum_{k=1}^p A_k = \sum_{j=1}^q B_j, \quad \prod_{k=1}^p A_k^{a_k} = \sum_{j=1}^q B_j^{b_j} \quad \text{and} \quad P(u) := \sum_{j=1}^q e^{-u_j a_j / A_j} - \sum_{k=1}^p e^{-u / b_k} - \sum_{k=1}^p e^{-u / b_k} u / b_k \geq 0, \quad \text{for all} \ u > 0.
\]

As we see, our conditions in Corollary 2.5, for \( r = 1 \), are not expressed in terms of the function \( P \), since we assume \( p = 1, t = u_k = b_k \) and we require the stronger property of logarithmic complete monotonicity.

Using iterates of the shift operators \( \Delta f(x) = f(x + c) - f(x) \), Akita & Maejima [3, Theorem 1] have shown that \( \log G_t \sim L(R) \) if \( t > 1/2 \). In their Remark 2, they claimed:
"It is possible to extend to this property to \((t_1, \infty)\) for some \(t_1 \in (0, 1/2)\)."

that they evaluated, with numerical calculations, by \(t_1 \leq 0.152\). The next result and (92) in the proof of Proposition (6.1) below provides the value of \(t_1\) as the maximum of an explicit elementary function, a quantity that could not be computed by hand and that we evaluated by Maple.

**Proposition 2.7** For every \(t > 0\) and \(\alpha \in (0, 1)\), we have \(\log G_t \sim 1 \text{D}_{-}(R) \cap L_0(R)\) and the identity in law

\[
\log G_t \overset{d}{=} \alpha \log G_t + T_{t, \alpha}.
\]

(35)

Further, \(\log G_t \sim 1 \text{D}_{-}(R) \cap L_1(R)\), i.e. \(T_{t, \alpha} \sim 1 \text{D}_{-}(R) \cap L_0(R)\), if, and only if \(t > t_1 \approx 0.151649938034\).

The r.v. \(T_{t, \alpha}\) corresponds to \(\log J_{t, \alpha}\) in (36).

### 2.3 Comments on the factorizations

We start by observing the following:

**Proposition 2.8** If \((\alpha_k)_{k \geq 1}\) is a sequence of non increasing positive numbers such that \(\sum_{k=1}^{\infty} \alpha_k = 1\) then the sequences \(X_{\alpha_k}\) and \(Y_{\alpha_k}\) of Corollary 2.5 converge in distribution as \(n \to \infty\).

Let \(B_{s, t} > 0\), denotes a Beta-distributed random variable with probability density function

\[
\frac{1}{B(s, s+t)} t^{s-1}(1-t)^{s+t-1}, \quad 0 < t < 1.
\]

It is also worth noticing the following facts.

1. In a private communication, the first author provided to Bertoin & Yor their Lemma 1 in [8], which shows a factorization close to (30). This factorization states that if \(S_{\alpha, t} > 0 < \alpha < 1\), is a standard positive stable r.v. and if \(0 < \alpha t < s\), then we have the following factorization in law: with the convention \(B_{t, 0} = 1\), and

\[
\mathbb{P}(S_{\alpha, t} \in dx) = \frac{\mathbb{P}(S_{\alpha} > dx)}{E[S_{\alpha}^2]}, \quad x > 0,
\]

we have

\[
G_t \overset{d}{=} G_s \overset{d}{=} J_{\alpha, t}.
\]

where \(J_{\alpha, t} \overset{d}{=} \frac{B_{t, \alpha}}{E[S_{\alpha}^2] x^t}\), (on the r.h.s, the r.v.‘s are independent). (36)

2. Identity (30) has to be compared with Gordon’s one [13, Theorem 6]: if \(p \geq 2\) is an integer, then

\[
\frac{G_{mt}}{p} \overset{d}{=} \left( \frac{G_t G_{t+\frac{1}{p}} \ldots G_{t+\frac{p-1}{p}}}{p} \right)^\frac{1}{p}, \quad \text{(on the r.h.s, the r.v.‘s are assumed to be independent).}
\]

(37)

As an immediate consequence of Gordon’s factorization and the one in (30), we recover a new independent factorization in law for the Beta distributions. Using the r.v. \(X_{\alpha, t}\) in (30) and the Gamma-Beta algebra, we get

\[
p B_{t, (p-1)t} \overset{d}{=} e^{-X_{\alpha, t}} \left( \frac{B_{t, \alpha}}{p} + \frac{B_{t, \alpha}}{p^2} + \ldots + \frac{B_{t, \alpha}}{p^{p-1}} \right)^\frac{1}{p}, \quad \text{(on the r.h.s, the r.v.‘s are independent).}
\]

(38)
3. Let $0 < \alpha \leq 1$ and $\beta = 1 - \alpha$. Motivated by the selfdecomposability property of the r.v. $\log(S_\alpha)$, Pestana, Shanbhag & Sreehari [38], explored the structure of the r.v. $V_\alpha$ intervening in the so-called Kanter’s identity which involves the exponential and the positive stable distributions:

$$G_\alpha^{-\alpha} \overset{d}{=} G_\alpha^\beta e^{-V_\alpha}, \quad \text{(on the r.h.s, the r.v.’s are assumed to be independent).} \quad (39)$$

Using the well known independent factorisation in law $G_\alpha \overset{d}{=} (G_\alpha/S_\alpha)^\alpha$, which is easily justified by

$$P(G_\alpha^{1/\alpha} > \lambda) = P(G_\alpha > \lambda^\alpha) = e^{-\lambda^\alpha} = E[e^{-\lambda S_\alpha}] = P(G_\alpha > \lambda S_\alpha) = P\left(\frac{G_\alpha}{S_\alpha} > \lambda\right), \quad \lambda \geq 0,$$

then taking $n = 2$ and $(\alpha_1, \alpha_2) = (\alpha, \beta)$ in (30), we retrieve:

$$G_1 \overset{d}{=} \left(\frac{G_1}{S_\alpha}\right) = \alpha^{-\alpha} \beta^{-\beta} G_{1,1} \alpha^{-\alpha} G_{1,2} \beta^{-\beta} e^{-X_{\alpha,1}} \implies S_\alpha^{-\alpha} \overset{d}{=} \alpha^{-\alpha} \beta^{-\beta} G_{1} \beta^{-\beta} e^{-V_\alpha} \implies V_\alpha \overset{d}{=} X_{\alpha,1} + \alpha \log \alpha + \beta \log \beta,$$

where, on the r.h.s, the r.v.’s are assumed to be independent. Observe that

$$E[e^{-\lambda V_\alpha}] = \frac{\Gamma(1 + \lambda)}{\Gamma(1 + \lambda \alpha) \Gamma(1 + \lambda \beta)}, \quad \lambda \geq 0, \quad \text{and that} \quad X_{\alpha,1} \sim L_0(\mathbb{R}_+) \cap \text{ME},$$

constitutes an additional information to Kanter’s factorization (39).

3 The Multiplicative convolution, the Euler-Mellin differential operator $\Theta$ and $n$-monotone functions

To provide a simple characterization of multiple selfdecomposability, we give some account on Euler-Mellin differential operator $\Theta = x d/dx$ and its relationship with the multiplicative convolution and the concept of $n$-monotone functions.

We recall that the Mellin convolution (or multiplicative convolution) of two measures $\mu$ and $\nu$ on $(0, \infty)$ is defined by:

$$\mu \odot \nu(A) = \int_{(0,\infty)^2} 1_A(xy) \mu(dx) \nu(dy), \quad \text{if} A \text{ is a Borel set of} (0, \infty).$$

If $\mu$ is absolutely continues with density function $f$, then $\mu \odot \nu$ is the function given by

$$\mu \odot \nu(x) = f \odot \nu(x) = \int_{(0,\infty)} f\left(\frac{x}{y}\right) \frac{\nu(dy)}{y}, \quad x > 0.$$

Notice that the integrals above may be infinite when $\mu$ (and/or $\nu$) is not a finite measure.

The Euler-Mellin differential $\Theta$ and its discrete version $\theta_c$, defined by

$$\Theta(g)(x) = xg'(x) \quad \text{and} \quad \theta_c(g)(x) = g(x) - g(x/c), \quad x, c > 0. \quad (40)$$

will be needed in the sequel, for an account on $\Theta$ operators, we suggest [10]. The iterates of $\Theta$ are denoted by $\Theta^n, n \geq 2$ and $\theta_{t_1} \theta_{t_2} \ldots \theta_{t_n}$ denotes the composition of $\theta_{t_1}, \theta_{t_2}, \ldots, \theta_{t_n}$, $c_1, c_2, \ldots, c_n \in (0, 1)$. The following result will be also needed in the sequel.
Lemma 3.1 The following is true for every function \( g : (0, \infty) \rightarrow \mathbb{R} \) and \( c \in (0, 1) \cup (1, \infty) \):

1) If \( g \) is decreasing and \( g(0+) < \infty \), then the Frullani integral

\[
\int_0^\infty \frac{g(x) - g(x/c)}{x} \, dx = \int_0^\infty \frac{\Theta_c(g)(x)}{x} \, dx,
\]

equals to \((g(\infty) - g(0+))\log c\).

2) Assume further that \( g \) is differentiable. Then,

(i) the function \( x \mapsto \Theta_c(g)(x) \) is increasing (resp. decreasing), for every fixed \( c \in (0, 1) \), if, and only if, \( g \) is convex (resp. concave);

(ii) the function \( x \mapsto \Theta_c(g)(x) \) is increasing (resp. decreasing), for every fixed \( c \in (0, 1) \), if, and only if, \( x \mapsto g(x) \) is decreasing (resp. increasing).

Remark 3.2 Since the operators \( \Theta \) and \( \Theta_c \) commute, it is easily seen that statement 2)(ii) in Lemma 3.1 extends to \( n \)-times differentiable functions \( g \) via the iterates of \( \Theta \) and the compositions of the operators \( \Theta_c \). Then, the two following statements are equivalent:

(i) The functions \( x \mapsto \Theta_c \Theta_2 \cdots \Theta_n(g)(x) \) are increasing (resp. decreasing), for every \( c_1, c_2, \ldots, c_n \in (0, 1) \);

(ii) The function \( x \mapsto (-1)^n \Theta^n(g)(x) \) is increasing (resp. decreasing).

It what follows, our aim is to characterise the class \( \mathcal{M}_n \) defined in (21).

- Williamson [54] introduced the class of \( n \)-monotone functions on \((0, \infty)\) that one can extend to functions \( f : (a, \infty) \rightarrow \mathbb{R}, \ a \in [-\infty, \infty) \), by reproducing the same arguments of Schilling, Song & Vondraček, in [45, Theorem 1.11 and the discussion p.12 given in case \( a = -\infty \)]. For this, just observe that \( f \) is \( n \)-monotone on \((a, \infty)\) if, and only if, \( f(x + x_0) \) is \( n \)-monotone on \((0, \infty)\) for every \( x_0 > a \). Hence, we will say that \( f \) is \( 1 \)-monotone on \((0, \infty)\) for all \( x > a \) and if \( f \) is non-increasing and right-continuous. The function \( f \) is \( n \)-monotone on \((a, \infty)\), \( n = 2, 3, \ldots \), if it is \( n - 2 \) times differentiable,

\[
(-1)^j f^{(j)}(x) \geq 0, \quad \text{for all } x > a, \quad j = 0, 1, \ldots, n-2,
\]

and \((-1)^{n-1} f^{(n-2)}\) is non-negative, non-increasing and convex on \((a, \infty)\).

- Further, with the adaptation of [45, Theorem 1.11], we can affirm that \( f \) is \( n \)-monotone on \((a, \infty)\) if, and only if, \( f \) has the representation

\[
f(x) = c + \int_{(a, \infty)} (u - x)^{n-1} \nu(du), \quad x > a
\]

for some \( c \geq 0 \) and some measure \( \nu \) on \((a, \infty)\).

- Similarly, \( f \) is completely monotone on \(\mathbb{R}\) if and only if, it is \( n \)-monotone on \((a, \infty)\), for all \( n \geq 1 \), and all \( a < 0 \); and the latter ensures that \( f(x), \ x \in \mathbb{R} \), is also represented as in (11), with some measure \( \nu \) on \((0, \infty)\).

By (42), observe that a function \( f \) is \( n \)-monotone on \((0, \infty)\) if, and only if, it is represented by

\[
f(x) = c + \left((1-u)^{n-1} \ast \mu\right)(x), \quad x > 0.
\]

for some \( c \geq 0 \) and some measure \( \mu \) (take \( \mu(du) = u^n \nu(du) \) in (42)).

We will now illustrate to which extent the class of \( n \)-monotone functions is intimately related to Euler-Mellin’s operator. The iterates of the usual differential operator and of \( \Theta \) are linked by these relations: if \( g \) is \( n \) times differentiable on some interval \( I \), then
\[
x^n g^{(n)}(x) = \sum_{m=0}^{n} s(n,m) \Theta^m(g)(x) \quad \text{and} \quad \Theta^m(g)(x) = \sum_{m=0}^{n} S(n,m) x^n g^{(m)}(x), \quad x \in I, \quad (43)
\]

where \( s(n,m) \) and \( S(n,m) \), \( 0 \leq k \leq n \), denote the Stirling numbers of the first and second kind, respectively given by the positive numbers

\[
s(n,k) = \frac{1}{k!} \frac{d^n}{dx^n} (\log(x+1))^k \text{ at } x=0 \quad \text{and} \quad S(n,m) = \frac{1}{m!} \frac{d^n}{dx^n} (e^x - 1)^m \text{ at } x=0.
\]

cf. [10]. Notice that \( S(n,k) \) is also defined as the number of partitions of the set \( \{1, \ldots, n\} \) into exactly \( k \) nonempty subsets. It is also known that \( s(n,m) = (-1)^{n-m} [n]_m \), where \([n]_m\) is the number of permutations in the symmetric group of order \( n \) with exactly \( k \) cycles. Using (43), write

\[
x^n (-1)^n g^{(n)}(x) = \sum_{m=0}^{n} \binom{n}{m} (-1)^m \Theta^m(g)(x)
\]

and it is immediate that

\[
(-1)^m \Theta^m(g)(x) \geq 0, \quad \forall m = 0, 1, \ldots, n \quad \Rightarrow \quad (-1)^n g^{(n)}(x) \geq 0. \quad (44)
\]

Now, assuming that \( g(0) = 0 \) and \( g \) is \( n \) times differentiable on \((0, \infty)\), consider \( h_n := (-1)^{n-1} \Theta^{n-1}(h_{n-1}) = (-1)^{n-1} \Theta^n(g) \) are such that \( h_n(\infty) = 0 \) and \( h_n(u)/u \) is integrable at \( \infty \), for all \( m = 1, \ldots, n \), then

\[
h_{n-1}(x) = \int_{e}^{\infty} \frac{h_n(u)}{u} \, du = \int_{0}^{\infty} h_n\left( \frac{x}{u} \right) \frac{1}{u} \, du = (1_{[0,1]} \Theta h_n)(x), \quad x > 0,
\]

then, iterating, we get following inversion formulae

\[
g = (1_{[0,1]})^{\otimes n-1} h_n = (1_{[0,1]})^{\otimes n} \Theta h_n.
\]

Observing that

\[
(1_{[0,1]})^{\otimes n}(x) = \frac{\log^n(x)}{n!} 1_{[0,1]}(x), \quad (45)
\]

one retrieves that \( g \) is expressed by

\[
g(x) = \int_{x}^{\infty} \log^n \left( \frac{u}{x} \right) \frac{h_n(u)}{u} \, du, \quad x > 0. \quad (46)
\]

Of course, this discussion is informal, but it illustrates the fact that the transform \( h \mapsto g \) in (46) is the inverse of the operator \((-1)^n \Theta^n\). Hence, for “good” functions \( g \), for instance if \((-1)^n \Theta^n(g)(x) \geq 0, \quad m = 1, \ldots, n \), we will have

\[
g(x) = \int_{x}^{\infty} \log^n \left( \frac{u}{x} \right) (-1)^n \Theta^n(g)(u) \, du, \quad x > 0.
\]

The transform \( h \mapsto g \) in (46), is known as the Hadamard integral of order \( n+1 \), cf. [40], (18.43) p. 330] and [11]. Further, observing that the difference operators \( \Delta_n(f)(x) = f(x+c) - f(x), \quad c > 0 \) are linked to our \( \Theta^m \) operators by

\[
\Delta_n \left( (y \mapsto k(e^y)) \right)(x) = k(e^x) - k(e^c) = \theta_c \left( k \right)(e^{x+c}), \quad c > 0, \quad (47)
\]

and observing that if \( k \) is \( m \)-times differentiable on \((0, \infty)\), then

\[
(-1)^n u \frac{du}{dx} k(e^u) = (-1)^m \Theta^m(k)(e^u), \quad x \in \mathbb{R}, \quad (48)
\]

Title Suppressed Due to Excessive Length
and it appears natural to introduce the following class:

**Definition 3.3** Let \( n = 1, 2, \ldots \). A function \( k : (0, \infty) \to (0, \infty) \), is said to be \( \Theta_n \)-monotone functions, and we denote \( k \in \mathcal{M}_n \), if \( x \mapsto k(x^n) \) is \( n \)-monotone on \( \mathbb{R} \). The function \( k \) is \( \Theta_{\infty} \)-completely monotone if \( x \mapsto k(x^n) \) is completely monotone.

By (41) and (44), clearly,  
\[
\begin{aligned}
k \in \mathcal{M}_n & \iff \begin{cases} 
{k_0 := (-1)^m \Theta^m(k) \geq 0,} & \text{for all } m = 0, 1, \ldots, n - 2, \\
\text{and} & \\
k_{n-2} \text{ is non-negative, non-increasing and } x \mapsto k_{n-2}(x^2) \text{ is convex}
\end{cases} \quad (49)
\end{aligned}
\]

With these conditions, necessarily \( k_n \) is also convex and, by (47), it becomes clear that  
\[
k \in \mathcal{M}_n \text{ (resp. } \mathcal{M}_{\infty} \text{) } \implies k \text{ is } n\text{-monotone on } (0, \infty) \text{ (resp. completely monotone).} \quad (50)
\]

If \( k \) is \( n \)-times differentiable, then by Remark 3.2,  
\[
k \in \mathcal{M}_n \iff (-1)^n \Theta^n(k). \quad (51)
\]

The implication (50), left to right, was observed in [45, Proposition 1.16] and the proof there is based on induction on \( n \), without formalizing the class \( \mathcal{M}_n \).

Finally, after our discussion, adapting [45, Theorem 4.11] for \( n \)-monotone functions, using (47) and the \( n \)-th power multiplicative convolution of the function \( I_{(0,1)} \) given by (45), we easily deduce the full characterization of \( \mathcal{M}_n \), i.e., the converse to [45, Proposition 1.16] and the analog of [45, Theorem 4.11] on multiply monotone functions, for the class \( \mathcal{M}_n \):

**Lemma 3.4** Let \( k : (0, \infty) \to (0, \infty) \) and \( n \geq 1 \). Then, the following assertions are equivalent.

1) \( k \in \mathcal{M}_n \);
2) \( \Theta_1 \cdots \Theta_n(k) \geq 0 \), for all \( m = 1, 2, \ldots, n \) and \( c_1, \ldots, c_n \in (0, 1) \);
3) \( k \) is of the form  
\[
k(x) = c + \left( (I_{(0,1)})^{n-1} \otimes \mu \right)(x) = c + \frac{1}{(n-1)!} \int_{(0,\infty)} \log^{n-1} \left( \frac{y}{x} \right) \mu(dy), \quad x > 0,
\]

where \( c \geq 0 \) and the measure \( \mu \) is such that \( \int_1^\infty \log^{n-1}(y) \frac{\mu(dy)}{y} < \infty. \)

Furthermore, \( k \in \mathcal{M}_{\infty} \) if, and only if, it is represented by  
\[
k(x) = c + \int_{(0,\infty)} \frac{1}{x^n} \nu(dx), \quad x > 0,
\]

with some finite measure \( \nu \) on \((0, \infty)\).

**Remark 3.5** The integral transform of \( \mu \), in (52), is the measure version of the Hadamard integral of order \( n \) in (46).

### 4 Simple characterization of multiple selfdecomposable distributions

By the relations
\[ \Theta(\phi)(\lambda) = \frac{d}{dc} \left( e^{-\theta_{l/c}(\phi)(\lambda)} \right)_{c=1} \]

\[ \theta_{l/c}(\phi)(\lambda) = \phi(\lambda) - \phi(c\lambda) = \int_{c\lambda}^{\lambda} \phi'(t) dt = \int_{c}^{1} \lambda \phi'(s\lambda) ds = \int_{c}^{1} \Theta(\phi)(s\lambda) \frac{ds}{s} \]

and the fact that \( \mathcal{B}\mathcal{F} \) is a closed convex cone, a simple proof for the characterization of \( L_0(\mathbb{R}_+) \) could be provided. For instance, see Aguech and Jedidi [1], Behme [5] and Mai, Schenk & Scherer [33] for the following characterization.

**Theorem 4.1 ([1, 5, 33])** Let \( X \) be a nonnegative r.v. with cumulant function \( \phi(\lambda) = -\log \mathbf{E}[e^{-\lambda X}] \), \( \lambda \geq 0 \). Recall that \( \Theta(\phi) \) and \( \theta_d(\phi) \) are given by (40).

1) If \( \Theta(\phi) \in \mathcal{B}\mathcal{F} \) for some \( c > 1 \) or if \( \Theta(\phi) \in \mathcal{B}\mathcal{F} \) then \( \phi \in \mathcal{B}\mathcal{F} \).

2) The following assertions are equivalent.

   (i) \( X \sim L_0(\mathbb{R}_+) \);
   (ii) \( \Theta(\phi) \) is a cumulant function, for all \( d > 1 \);
   (iii) \( \theta_d(\phi) \in \mathcal{B}\mathcal{F} \) for all \( d > 1 \);
   (iv) \( \Theta(\phi) \in \mathcal{B}\mathcal{F} \);
   (v) \( \phi \in \mathcal{B}\mathcal{F} \) and is represented by

\[ \phi(\lambda) = d\lambda + \int_{0}^{\infty} (1 - e^{-\lambda x}) \frac{k(x)}{x} dx, \quad \lambda \geq 0 \]  

(54)

where \( d \geq 0 \) and \( k \) is a non-increasing function such that \( \int_{1}^{\infty} k(x) dx + \int_{1}^{\infty} \frac{k(x)}{x} dx < \infty \).

**Remark 4.2** Representation (54) explains identity (14). Indeed, we have

\[ \mathbf{E}[e^{-\lambda Y}] = \frac{\mathbf{E}[e^{-\lambda X}]}{\mathbf{E}[e^{-\lambda X}]} = e^{-\theta_{l/c}(\phi)(\lambda)}, \]

and then, after an elementary change of variable, we obtain the representation

\[ \theta_{l/c}(\phi)(\lambda) = (1 - c) d\lambda + \int_{0}^{\infty} (1 - e^{-\lambda x}) \frac{\theta_d(k)(x)}{x} dx. \]

(55)

Since \( Y \sim L_0(\mathbb{R}_+) \), then \( \theta_d(k)(x)/x \) is necessarily the density function of a Lévy measure.

Using Proposition 1.1 and mimicking the proof of Theorem 4.1, we can state this proposition without proof:

**Proposition 4.3** We have \( L_0(\mathbb{R}) \subset \text{ID}(\mathbb{R}) \) and the following holds true.

1) Let \( X \sim \text{ID}_-(\mathbb{R}) \). Then, the following assertions are equivalent.

   (i) \( X \sim L_0(\mathbb{R}) \);
   (ii) The Lévy-Laplace exponent \( \Psi \in \mathcal{L}\mathcal{E} \) associated to \( X \) satisfies \( \theta_{\Psi} \Psi \in \mathcal{L}\mathcal{E} \) for every \( c > 0 \);
   (iii) \( \Theta(\Psi) \) has the form

\[ \Psi(\lambda) = a\lambda + b\lambda^2 + \int_{0}^{\infty} \left( e^{-\lambda x} - 1 + \lambda \chi(x) \right) \frac{k(x)}{x} dx, \quad \lambda \geq 0 \]

(56)

for some \( a \in \mathbb{R} \), \( b \geq 0 \), some truncation function \( \chi \) as in (4) and some non-increasing function \( k \).

2) \( X \sim \text{ID}_+(\mathbb{R}) \cap L_0(\mathbb{R}) \) if, and only if, its associated Lévy-Laplace exponent \( \Psi \) has the form (56) with \( \chi(x) = x \), or equivalently \( \Theta(\Psi) \in \mathcal{L}\mathcal{E} \).
3) Vague closure: \( X \sim L_0(R) \) if, and only if, there exist two sequences of independent r.v.’s \( X'_n, X''_n \) \( \sim \text{IID}_-(R) \cap L_0(R) \), such that \( X \overset{d}{=} \lim_n (X'_n - X''_n) \).

To characterize the class \( L_n(R) \), it is enough to use Proposition 4.3 and to focus on the class \( \text{ID}_-(R) \cap L_0(R) \) of real-valued selfdecomposable r.v.’s \( X \), having a Lévy-Laplace exponent \( \Psi \) of the form

\[
\Psi(\lambda) = a \lambda + b \lambda^2 + \int_{(0,\infty)} \left( e^{-\lambda x} - 1 + \lambda x \right) \frac{k(x)}{x} \, dx, \quad \lambda \geq 0.
\]

for some \( a \in R, b \geq 0 \) and some non-increasing function \( k \) satisfying the condition (7), i.e.:

\[
\int_{(0,1)} x^k(x) \, dx + \int_{[1,\infty)} k(x) \, dx < \infty.
\]

Using the compositions of the Euler-Mellin operators \( \Theta, \theta \) given by (40) and right after, and also the class of \( \mathcal{M}_{n+1} \) given by Definition 3.3, multiple selfdecomposability property is simply characterized as follows:

**Corollary 4.4 (Characterization of multiple selfdecomposability by the Euler-Mellin operator)**

Let \( n \geq 1 \).

1) Assume one of the following

(a) \( X \sim L_0(R_+) \) and is associated with a Bernstein function \( \phi \) and with the \( k \)-function given by (54);

(b) \( X \sim \text{ID}_-(R) \cap L_0(R) \) and is associated to a Lévy-Laplace exponent \( \Psi \) to the \( k \)-function given by (57).

Then, the following assertions are equivalent.

(i) \( X \sim L_n(R_+) \) (respectively \( X \sim \text{ID}_-(R) \cap L_n(R) \));

(ii) \( k \in \mathcal{M}_{n+1}, \) i.e., is represented by

\[
k(x) = \left( \left( 1_{[0,1]} \right)^{\otimes n-1} \otimes \mu \right)(x) = \frac{1}{(n-1)!} \int_{(1,\infty)} \log^{n-1} \left( \frac{y}{x} \right) \frac{\mu(dx)}{y}, \quad x > 0 \quad (58)
\]

and \( \mu(dx)/x \) is a measure satisfying (10) and the additional integrability conditions at infinity

\[
\int_{(1,\infty)} \log^{n+1} (x) \frac{\mu(dx)}{x} < \infty, \quad (59)
\]

(respectively satisfying only (3) if \( X \sim \text{ID}_-(R) \cap L_n(R) \));

(iii) For all \( m = 1, \ldots, n+1 \) and \( d_1, \ldots, d_{n+1} > 1 \), the function

\[
(\Theta_{d_1} \ldots \Theta_{d_n})(\phi) \quad (\text{respectively, } (\theta_{d_1} \ldots \theta_{d_n})(\Psi))
\]

is the Bernstein function of some r.v. \( Y_m \sim \text{ID}(R_+) \) (respectively, is the Laplace-exponent of some r.v. \( Y_m \sim \text{ID}_-(R) \));

(iv) For all \( m = 1, \ldots, n+1, \Theta^m(\phi) \in \mathcal{E} \) (respectively, \( \Theta^m(\Psi) \in \mathcal{E} \));

(v) Let \( f = \phi \) (respectively, \( f = \Psi \)). There exists \( Y \sim \text{ID}(R_+) \) (respectively, \( Y \sim \text{ID}_-(R) \)) such that we have the representation

\[
f(\lambda) = \frac{1}{(n-1)!} \int_1^\infty \phi \left( \frac{\lambda}{x} \right) \log^e(x) \frac{dx}{x}, \quad \lambda \geq 0, \quad (60)
\]

where \( g \) is the Bernstein function (respectively Laplace exponent) of \( Y \).
(vi) \( X \overset{d}{=} \int_{(0,\infty)} e^{-x/|Z_t|} \, dZ_t \), where \( Z \) is some subordinator (respectively, some spectrally negative Lévy process) such that
\[
E[\log^{n+1}(1+|Z_1|)] < \infty \quad \text{(respectively } E[\log^{n+1}(1+|Z_1|)] < \infty \).
\]

Furthermore, in (v) and (vi), the r.v.’s \( Y \) and \( Z_1 \) have the same distribution.

2) \( X \sim \overline{\mathcal{D}}_c(\mathbb{R}) \cap \mathcal{L}_\mathcal{D}(\mathbb{R}) \) if, and only if, the associated \( k \)-function and \( \mu \)-measure satisfy (58) and (59).

3) We have the same equivalences as in 1) for \( n = \infty \), with the following additional conditions on support and integrability on the \( \nu \)-measure in the representation (53) of \( k \in \mathcal{M}_\infty \):

(i) \( \nu \) is supported by \((0,1)\), in case \( X \in \mathcal{L}_\mathcal{D}(\mathbb{R}) \) (respectively \((1,2)\), in case \( X \in \overline{\mathcal{D}}_c(\mathbb{R}) \cap \mathcal{L}_\mathcal{D}(\mathbb{R}) \), and satisfies
\[
\int_{(0,1)} \frac{1}{x(1-x)} \nu(dx) < \infty \quad \text{(respectively, } \int_{(1,2)} \frac{1}{(x-1)(2-x)} \nu(dx) < \infty \).
\]

The latter is equivalent to the representation
\[
\phi(\lambda) = d\lambda + \int_{(0,1)} \lambda x f(x) \nu(dx) \quad \text{(respectively, } \Psi(\lambda) = a\lambda + b\lambda^2 + \int_{(1,2)} \frac{\lambda x}{(x-1)f(x)} \nu(dx) \),
\]
for some \( d, b > 0, a \in \mathbb{R} \).

(ii) \( \nu \) is supported by \((0,2)\), in case \( X \in \overline{\mathcal{D}}_c(\mathbb{R}) \cap \mathcal{L}_\mathcal{D}(\mathbb{R}) \), and satisfies
\[
\int_{(0,2)} \frac{1}{x(2-x)} \nu(dx) < \infty.
\]

As a straightforward consequence, we recover the general case:

**Corollary 4.5** Assume \( 1 \leq n \leq \infty \) and \( X \sim \mathcal{L}_0(\mathbb{R}) \). Then, \( X \sim \mathcal{L}_0(\mathbb{R}) \) if, and only if \( X = X^+ - X^- \), where \( X^+ \) and \( X^- \) are independent and have distributions in \( \overline{\mathcal{D}}_c(\mathbb{R}) \cap \mathcal{L}_\mathcal{D}(\mathbb{R}) \).

**Remark 4.6** Formula like the above (58) appeared in Urbanik [51, 52]. The stochastic integral representation in I(vi) of Corollary 4.4 was proved in Jurek [19, Corollary 2.11].

5 Possible extensions of multiple selfdecomposability through integral stochastic representations

Observe that the integral stochastic representation (113) admits several extensions as noticed by Jurek & Vervaat [25]. As in the proof of [25, Proposition 1], we have the following: for \( 0 < a < b \), a function of bounded variation \( h : (a,b) \to \mathbb{R} \), a monotone function \( r : (0,\infty) \to (0,\infty) \), and a Lévy process \( Z(t) \), the random integral
\[
X = \int_{(a,b)} h(s) dZ(s),
\]
is well defined and
\[
\Phi_X(u) := \log E[e^{iuX}] = \int_{(a,b)} \log E[e^{iu h(s)Z_s}] \, dr(s), \quad u \in \mathbb{R}.
\]

By Jurek [21], the Fourier Lévy-exponents \( \Phi_{Z(1)} \) and \( \Phi_{Z} \) are linked by
\( \Phi_X(u) = \int_{(a,b)} \Phi_Z(ab(s)) dr(s). \)

The following may be viewed as a particular of the above scheme. For subordinators, Maejima [34] and Schilling, Song & Vondraček [45] had the same approach: let \( f : (0,A) \to (0,B) \) be a strictly decreasing function and \( Z = (Z_t)_{t \geq 0} \) be a subordinator with associated Bernstein function \( \phi_Z \). Then, the r.v.

\[ X = \int_{(0,A)} f(s) dZ_s \]  

is a well-defined r.v. on \([0, \infty)\]. Reasoning by Riemann approximation of the stochastic integral, [45, Lemma 10.1] provides the Laplace representation

\[ \mathbb{E}[e^{-\lambda X}] = e^{-\phi_X(\lambda)}, \quad \Phi_X(\lambda) = \int_0^A \phi_Z(\lambda, f(s)) ds, \quad \lambda \geq 0. \]

Since \( f \) admits a strictly decreasing inverse \( F = (0,B) \to (0,A) \), then, by the change of variable \( s = G(y) := F(1/y) \), we see that \( \phi_X \) takes the form of a Mellin convolution

\[ \Phi_X(\lambda) = \int_{(1/B, \infty)} \phi_Z \left( \frac{\lambda}{y} \right) dG(y). \]  

Since \( \mathcal{BF} \) is a closed convex cone, then \( \Phi_X \in \mathcal{BF} \) whenever it is a well defined function on \( \mathbb{R}_+ \), the latter shows that \( X \sim \text{ID}(\mathbb{R}_+) \). Two particular cases arise for the choice of \( f \) in (63):

- Taking a continuous nonnegative r.v. \( Y \) independent of \( Z \) and \( f(s) = \mathbb{P}(Y > s) \), we see that \( X \) corresponds to the conditional expectation \( X = \mathbb{E}[Z_t | Z] \), and the integral stochastic representation (113) corresponds to the case where \( Z \) has the standard exponential distribution. In particular, one can choose

\[ f(s) = \mathbb{E}[e^{-\lambda Y}] = e^{-\phi_X(\lambda)} = \mathbb{P}(G_1/Y > s), \quad s > 0, \]  

where \( G_1 \) is exponentially distributed and independent of \( Z \) and we are also in the previous situation: \( X = \mathbb{E}[Z_{G_1/Y} | Z] \).

- A possible extension of the latter is to take \( f \) of the form

\[ f = e^{-\phi} \quad \text{and} \quad \phi : (0,A) \to (-\log B, \infty) \]  

differentiable and strictly concave. (66)

Observe that \( \phi \) is necessarily increasing and that the inverse function \( \Psi' \), of \( \phi \), is a differentiable increasing and strictly convex function on \((0, \infty)\). Making the change of variable \( s = G(y) = \Psi'(\log y), \quad y > 1/B \), the Bernstein function of \( X \) in (64) takes the form

\[ \Phi_X(\lambda) = \int_{1/B}^\infty \phi_Z \left( \frac{\lambda}{y} \right) \Psi'(\log y) \frac{dy}{y}, \quad \lambda \geq 0. \]  

Let \( d\xi, \Pi_Z \) denote the drift term and the Lévy measure in the representation (9) of \( \phi_Z \). Since \( \Psi' \geq 0 \), then Fubini-Tonelli applies in (67) and by a change of variable, we obtain the following Mellin convolution representations:

\[ \phi_X(\lambda) = d\psi c\psi \lambda + \int_0^\infty (1 - e^{-\lambda x} ) l_x(x) dx, \quad c\psi := \int_{1/B}^\infty \Psi'(\log y) \frac{dy}{y^2}, \]

\[ l_x(x) := \int_{(1/B, \infty)} \Psi' \left( \log \frac{u}{x} \right) \Pi_Z(du), \quad x > 0. \]  

Assuming (66), the following is worth to be noticed:

(a) \( c\psi = 0 \) if \( B = \infty \).
(b) The definiteness of \( \phi_X \), i.e., the finiteness of the r.v. \( X \) given (63), is guaranteed by the following: if \( d \rho > 0 \), then the finiteness of the integral \( c \rho \) is required for the definiteness of \( \phi_X \) (i.e. \( X \) is a well-defined r.v. on \( \mathbb{R}_+ \)). Thus, \( \phi_X \) is a well-defined function on \( \mathbb{R}_+ \) if, and only if the Lévy measure \( \Pi_x \) satisfies the additional integrability condition

\[
\int_0^\infty (x \wedge 1) l_x(x) \frac{dx}{x} = \int_{(0, \infty)} a_\rho(u) \Pi_x(du) < \infty, \tag{70}
\]

where

\[
a_\rho(u) := \int_0^{Bu} (x \wedge 1) \Psi' \left( \log \frac{u}{x} \right) \frac{dx}{x}. \tag{71}
\]

Then, with some computation, we obtain that \( a_\rho \) is well defined if, and only if \( a_\rho(1) = c \rho < \infty \). In this case, we have

\[
a_\rho(u) = \begin{cases} c \rho u, & \text{if } 0 < Bu \leq 1 \\ \Psi'(\log u) - \Psi(-\log B) + u \int_0^\infty \Psi'(\log y) \frac{dy}{y^2}, & \text{if } Bu > 1. \end{cases} \tag{72}
\]

(c) One can release the differentiability assumption on \( \phi \) in case \( B = 1 \). Indeed, the assumption of strict concavity of \( \phi \) together with its positivity, insures that \( \phi \) is strictly increasing. Thus, almost everywhere, \( \phi \) is differentiable with a strictly decreasing derivative. Hence, almost everywhere, \( \Psi \) is differentiable and \( \Psi' > 0 \).

(d) One can link (72) with Lemma 6.4 below, through the observation

\[
x_0 = \frac{1}{B} \quad \text{and} \quad \rho_u(dy) = \frac{\Psi'(\log y)}{y^2} dy, \quad y > \frac{1}{B} \quad \implies \quad \chi_u(u) = a_\rho(u), \quad u > \frac{1}{B}, \tag{73}
\]

and notice that \( Z_s, s > 0 \), has the Lévy measure \( s \Pi_x \). Then, we immediately obtain the following consequence which constitutes an improvement and a simplification of the statement of Sato’s theorem [43, Theorem 2.6 and Theorem 3.5] in the case of subordinators.

**Corollary 5.1** Let \( X \) be a random variable represented by the stochastic integral in (63). Then the following assertions are equivalent.

1) \( X \) is a well defined r.v. on \([0, \infty)\);
2) \( \mathbb{E}[a_\rho(Z_1) 1_{Z > 1/B}] < \infty \);
3) \( \mathbb{E}[a_\rho(Z_s) 1_{Z > 1/B}] < \infty \) for all \( s > 0 \);
4) \( \int_{(1/B, \infty)} a_\rho(u) \Pi_x(du) < \infty \).

(e) The stochastic integral representation (63) for selfdecomposable distributions corresponds to the case where \( \varphi(x) = x \), hence \( \Psi' \equiv 1 \). In all cases, the function \( l_x \) is decreasing, because \( \Psi' \) is increasing. Thus,

\[
X \text{ is represented by } (63), \text{ with } f \text{ as in } (66) \implies X \sim \text{L}_0(\mathbb{R}_+). \tag{74}
\]

(f) By Corollary 4.4, we know that if \( X \in \text{L}_0(\mathbb{R}_+) \), then \( \phi_X \) takes the form

\[
\phi_X(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \frac{k_x(x)}{x} dx,
\]

such that \( k \in \mathcal{M}_{B+1} \) and is represented by (52):

\[
k_x(x) = \frac{1}{m!} \int_{(1, \infty)} \log^m \left( \frac{x}{y} \right) \frac{\mu_x(dy)}{y}, \quad x > 0.
\]

Identifying in (65), (68) and (69), we necessarily have there
\[ k_x = l_x, \quad \frac{\mu_x(dy)}{y} = \Pi_x(dy), \quad B = 1 \] and \( \Psi = \Psi_f \) s.t. \( \Psi'_f(s) = \frac{x^n}{n!}, x > 0. \)

Since \( \Psi_f(0) = 0, \) then

\[ \Psi_f(x) = \frac{x^{n+1}}{(n+1)!}, x > 0 \quad \text{and} \quad \phi_f(x) = \Psi_f^{-1}(x) = ((n+1)!/x) \frac{1}{\Gamma(n+1)}, s > 0. \]

Observe that \( \phi_f \in \mathcal{BF} \), and most of all, \( Y \overset{\mathcal{D}}{=} S_1/(n+1) \) the positive \( 1/(n+1) \)-stable r.v. given by (26). The change of variable, \( s \mapsto \Psi_f(s) = s^{n+1}/(n+1)! \), in (63) with \( f \) as in (65), amounts to change in the clock in the process \( Z : Z_t \sim Z_{\Psi_f(t)} \), and we retrieve the following result: there exist a subordinator \( Z \), such that

\[ X \sim \Lambda_0(\mathbb{R}_+) \iff X = \int_0^\infty e^{-((n+1)!/x)^{1/(n+1)}} dZ = \int_0^\infty e^{-x} dZ_{\Psi_f(t)}. \quad (75) \]

Further, since

\[ a_{\phi}(u) := \frac{1}{n!} \int_0^u (x \wedge 1) \left( \frac{\log u}{x} \right)^n \frac{dx}{x} = \frac{1}{(n+1)!} (u \log_{n=1} + \log^{n+1}(u) \log_{n=1}). \]

then Corollary 5.1 insures that \( X \) is well defined if, and only if, one of the equivalent conditions holds

\[ \int_{(1, \infty)} a_{\phi}(u) \Pi_2(du) < \infty \iff \mathbb{E}[\log^{n+1}(1+Z_1)] < \infty. \quad (76) \]

Also observe that the change of variable, \( s \mapsto s/(n+1)! \) amounts to change the scale of time in the process \( Z : Z_t \sim Y_t := Z_{\Psi_f(t)} \), we simply express \( \Lambda_0 \) property by:

\[ X \sim \Lambda_0(\mathbb{R}_+) \iff X = \int_0^\infty e^{-s^{1/(n+1)}} dY_s, \quad \text{for some subordinator } Y. \]

We emphasize that the last discussion could be rephrased identically for \( X \sim \text{TD}_\alpha(\mathbb{R}) \cap \Lambda_0(\mathbb{R}), \) and it suggests that a fractional selfdecomposability property of type \( \text{L}_{1/\alpha}(\mathbb{R}), \) with \( \alpha \in (0, 1) \) might be interesting to be formalized, from the stochastic point of view, since it would be associated to the fractional version of (75):

\[ X \sim \text{L}_{1/\alpha}(\mathbb{R}) \iff X = \int_0^\infty e^{-\theta^{1/\alpha}} dZ_\alpha = \int_0^\infty \mathbb{E}[e^{-\theta Y}] dZ_\alpha, \]

where, with our notations in (65), \( Y \overset{\mathcal{D}}{=} S_\alpha \) the positive \( \alpha \)-stable r.v. given by (26). Seeking more involved stochastic and analytical properties for the class of r.v.'s in (63) will be, hopefully, the scope of future work.

(g) Theorem 1.1 in [9] is closely connected to the class of Bernstein functions obtained in form (67) and, particularly, investigates whenever the function \( x \mapsto x^{\alpha} l_x(x) \), \( \alpha > -1 \) is completely monotone. With our approach, it is immediate that \( x \mapsto x^{\alpha} l_x(x) \in \mathcal{CM} \) for arbitrary Lévy measures \( \Pi_x \), if and only if, in (68),

\[ B = \infty \quad \text{and} \quad \Psi_f(u) = \Psi_f(u) := e^{au-ue^{-\alpha}}, \quad u \in \mathbb{R}. \]

Assuming the latter, we have the representation

\[ x^{\alpha} l_x(x) = \int_{(0, \infty)} e^{-\frac{x}{\alpha} u} \Pi_x(du), \quad \Pi_x(du) = \frac{l_x(x)}{x} dx, \quad x > 0. \quad (77) \]

For \( \alpha = 1 \) (respectively \( \alpha = 0 \)), the shape of the Lévy measure \( \Pi_x \) corresponds to the well known class \( \mathcal{CB} \) complete Bernstein functions (respectively \( \mathcal{TB} \) of Thorin Bernstein functions), cf.
Choosing \( \Psi_{a}(-\infty) = 0 \), observe that the inverse function of \( f_{a} \), given by (66), is provided by

\[
\Psi_{a}(x) = \int_{e^{-\infty}}^{e^{x}} \frac{e^{-u}}{u^{a+1}} du, \quad x \in \mathbb{R}, \quad f_{a}^{-1}(s) = \Psi_{a}(-\log s) = \int_{s}^{e^{\infty}} \frac{e^{-u}}{u^{a+1}} du, \quad s > 0.
\]

Thus,

\[
\Psi_{a}(+\infty) = \int_{0}^{\infty} \frac{e^{-u}}{u^{a+1}} du < \infty \iff a \in (-1, 0) \quad \text{and} \quad \Psi_{a}(+\infty) = \Gamma(1-a).
\]

Further, using the exponential integral \( Ei \) function and [14, 8.214 (1,2)], we have

\[
\Psi_{0}(x) = -Ei(-e^{-x}), \quad x > 0 \quad \lim_{x \to +\infty} x - \Psi_{0}(x) = \gamma \lim_{x \to +\infty} \frac{\Psi_{0}(x)}{x} = 1, \quad (78)
\]

where \( \gamma \) is the Euler-Mascheroni constant. If \( a > 0 \)

\[
\Psi_{a}(x) = \int_{e^{-x}}^{1} \frac{e^{-u}}{u^{a+1}} du + \Psi_{a}(0) \geq \frac{e^{ax} - 1}{ae}, \quad x > 0 \quad \lim_{x \to +\infty} \frac{\Psi_{a}(x)}{x} = +\infty. \quad (79)
\]

For \( a \geq 0 \), additional investigations on the inverse function \( \Phi_{a} \) of \( \Psi_{a} \) are feasible. Like \( f_{a} \), the function \( \Phi_{a} \) is also not explicit, but it certainly increases and satisfies

\[
\Phi_{a}(0+) = -\infty, \quad \Phi_{a}(\infty) = +\infty \quad \text{and} \quad \Phi_{a}(s_{a}) = 0, \quad \text{where} \quad s_{a} := \Psi_{a}(0) = \int_{e^{-1}}^{e^{\infty}} \frac{e^{-u}}{u^{a+1}} du, \quad a \geq 0.
\]

Further, since

\[
\Phi_{a}'(x) = \frac{1}{\Psi_{a}''(\Phi_{a}(x))} e^{-a\Phi_{a}'(x)} e^{-b_{a}(x)} = h_{a}(\Phi_{a}(x)), \quad x > 0, \quad h_{a}(u) := e^{-au} e^{-u} \sum_{n=0}^{\infty} e^{-(a+n)u},
\]

Thus, the function \( x \mapsto \hat{\Phi}_{a}(\lambda) := \Phi_{a}(s_{a} + \lambda), \lambda \geq 0 \), satisfies

\[
\hat{\Phi}_{a}(0) = 0, \quad \hat{\Phi}_{a}'(0) = \frac{1}{\Psi_{a}''(0)} = e \quad \text{and} \quad (\hat{\Phi}_{a})^{-1}(x) = \hat{\Psi}_{a}(x) = \Psi_{a}(x) - \Phi_{a}(0) = \int_{e^{-1}}^{1} \frac{e^{-u}}{u^{a+1}} du, \quad x \geq 0,
\]

Observing that \( h_{a}(u), \quad u > 0 \), is a completely monotone function, then, using Faa di Bruno’s formula and induction, we retrieve that \( \hat{\Phi}_{a}'(\infty, \infty) \in \mathcal{C}M \), hence we have

\[
\hat{\Phi}_{a}' = e^{-a} \hat{\Phi}_{a} e^{-\hat{b}_{a}} \quad \text{and} \quad \hat{\Phi}_{a} \in \mathcal{B}F.
\]

Further, by (78) and (79), we have

\[
d_{a} := \lim_{\lambda \to +\infty} \frac{\hat{\Phi}_{a}(\lambda)}{\lambda} = 1, \quad \text{if} \quad a = 0 \quad \text{and} \quad 0 \quad \text{otherwise}.
\]

We deduce that \( \hat{\Phi}_{a} \) has the drift term \( d_{a} \) and is associated to a Lévy measure \( \Pi_{a} \) and to a subordinator \( X_{a} = (Y_{a,t})_{t\geq0} \) through the representations:

\[
\hat{\Phi}_{a}(\lambda) = d_{a} \lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi_{a}(dx), \quad \lambda \geq 0,
\]

and
\[ \hat{\phi}_0'(\lambda) = d_0 + \int_{(0, \infty)} e^{-\lambda x} x \mathcal{P}_0(dx) = e^{-\lambda \hat{\phi}_0(\lambda)} = \frac{\sum_{n=0}^{\infty} e^{-(\mu+n)\hat{\phi}_0(\lambda)}}{n!} = \frac{\sum_{n=0}^{\infty} E\left[e^{-\lambda Y_{a,n+1}}\right]}{n!}. \]

Thus, letting \( \lambda \) to 0, we get

\[ \lim_{\lambda \to +\infty} \hat{\phi}_0'(\lambda) = d_0 = 1 + \frac{\sum_{n=0}^{\infty} \mathcal{P}(Y_{0,n} = 0)}{n!} = 1 + \frac{\sum_{n=1}^{\infty} \mathcal{P}(Y_{0,n} = 0)}{n!}, \]

which entails

\[ \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{P}(Y_{0,n} = 0) = 0 \quad \Rightarrow \quad \mathcal{P}(Y_{0,n} = 0) = 0, \quad \forall n = 0, 1, \ldots. \quad \Rightarrow \quad \mathcal{P}(Y_{a,t} = 0) = 0, \quad \forall t \geq 0. \]

Then, applying the Laplace inversion in (80), we get

\[ x \mathcal{P}_0(dx) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{P}(Y_{0,n} \in dx), \quad x > 0. \]

Similarly, if \( a > 0 \),

\[ d_a = \frac{\sum_{n=0}^{\infty} \mathcal{P}(Y_{a,a+n} = 0)}{n!} \quad \Rightarrow \quad \mathcal{P}(Y_{a,t} = 0) = 0, \quad \forall t \geq 0, \]

\[ x \mathcal{P}_a(dx) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{P}(Y_{a,a+n} \in dx), \quad x > 0. \]

If \( N \) is a r.v. with Poisson distribution, with rate parameter equal to 1, independent of the subordinator \( Y_a \), then, by (80), we have the closed forms obtained with the help of the subordinated r.v. \( Y_{a,a+N} \): for all \( \lambda \geq 0 \),

\[ \hat{\phi}_0'(\lambda) = 1 + e \sum_{n=1}^{\infty} \frac{1}{n!} E\left[e^{-\lambda Y_{0,n}}\right] = 1 + e \left\{ 1 - \frac{e^{-\lambda Y_{0,N}}}{Y_{0,N}} \right\} \quad \Rightarrow \quad \hat{\phi}_0(\lambda) = \lambda + e \left\{ 1 - \frac{e^{-\lambda Y_{0,N}}}{Y_{0,N}} \right\}, \]

and if \( a > 0 \),

\[ \hat{\phi}_a'(\lambda) = e \sum_{n=0}^{\infty} \frac{1}{n!} E\left[e^{-\lambda Y_{a,n+1}}\right] = e \left\{ 1 - \frac{e^{-\lambda Y_{a,N}}}{Y_{a,N}} \right\} \quad \Rightarrow \quad \hat{\phi}_a(\lambda) = e \left\{ 1 - \frac{e^{-\lambda Y_{a,N}}}{Y_{a,N}} \right\}. \]

Additionally, by (80) again, we have that for all \( a \geq 0, \lambda \geq 0 \) and \( t \geq 0 \),

\[ E\left[Y_{a,t} e^{-\lambda Y_{a,t}}\right] = \frac{\sum_{n=0}^{\infty} E\left[e^{-\lambda Y_{a,n+1}}\right]}{n!} = \frac{e \left\{ 1 - \frac{e^{-\lambda Y_{a,N}}}{Y_{a,N}} \right\}}{t} E\left[e^{-\lambda Y_{a,N}}\right]. \]

If \( Y_{a,t} \) is a version of the size biased distribution of \( Y_{a,t} \), defined for \( t > 0 \), by

\[ Y_{a,t} = \frac{x P(Y_{a,t} \in dx)}{E[Y_{a,t}]} = \frac{x P(Y_{a,t} \in dx)}{et}, \]

then, from (83) we get for \( a \geq 0 \):

\[ E\left[e^{-\lambda Y_{a,t}}\right] = E\left[e^{-\lambda Y_{a,a+N}}\right], \quad \forall \lambda \geq 0 \quad \Rightarrow \quad Y_{a,t} \overset{d}{=} Y_{a,a+t+N} \overset{d}{=} Y_{a,t} + Y_{a,a+N}, \quad \forall t > 0, \]

where in the last identity, the r.v.'s \( Y_{a,t} \) and \( Y_{a,a+N} \) are assumed to be independent.
Finally, by (63), we conclude that a nonnegative infinitely divisible r.v. \(X\) has a Bernstein function \(\phi\) with Lévy measure of the form (77) and \(a \geq 0\), if, and only if, for some subordinator \(Z\), it has the following stochastic representation

\[
X = \int_{(0, \infty)} f(s) dZ_s, \quad f(s) = e^{-\phi(s)}, \quad s \geq 0
\]

and recall that \(f\) coincides with

\[
f(s) = e^{-\hat{\phi}(s-a)} = E[e^{(\lambda-a)s} Y_a], \quad \text{if} \quad s \geq s_a.
\]

In [45, Chapter 10], several forms of the function \(f\) were investigated in the purpose of characterizing the induced classes of distribution. The cases \(a = 0\) and \(a = 1\) leading to the classes \(TB\) and \(CB\) were also investigated there but no closed-form was proposed for the corresponding function \(f\). For this reason, and because of the remarkable relations (81), (82), and (84), the distribution of \(Y_{a,1}\) would benefit from being investigated in more detail.

6 The proofs

Proof of Proposition 1.1. 1) Consider the Lévy-Laplace exponent \(\Psi\) of \(X \sim \text{ID}_{-}(\mathbb{R})\). The Lévy measure \(\pi\) gives no mass to \((0, \infty)\) and its image \(\Pi\) given by (4) satisfies (3). Observe that the truncated Lévy measures \(\Pi_n \coloneqq \Pi/\text{divides.alt0(0, n)}\), \(n \in \mathbb{N}\), satisfies (7), so that the quantities

\[
a_n \coloneqq a + \int_{(0, n]} (\chi(x) - x) \Pi(dx)
\]

are finite, and each of the functions,

\[
\Psi_n(\lambda) \coloneqq a_n \lambda + b \lambda^2 + \int_{(0, \infty)} (e^{-\lambda x} - 1 + \lambda x) \Pi_n(dx), \quad (85)
\]

belongs to \(LE\), hence, is associated with a r.v. \(X_n \sim \text{ID}_{-}(\mathbb{R})\). Since \(\Psi = \lim_n \Psi_n\), deduce that \(X \overset{d}{=} \lim_n X_n\). The converse is obvious because \(\text{ID}_{-}(\mathbb{R}) \subset \text{ID}_{-}(\mathbb{R})\). Statement 2) is an immediate consequence of 1). \(\square\)

6.1 Additional results, preliminaries for the proofs of the main results

The following technical result will be used in the proof of Lemma 6.3.

Proposition 6.1 Let \(t \geq 0\), \(\alpha \in (0, 1) \cup (1, \infty)\) and define for \(u > 0\),

\[
e\alpha \lambda(u) \coloneqq \frac{e^{-\alpha u}}{1 - e^{-\alpha u}} \quad (86)
\]

\[
g\alpha \lambda(u) \coloneqq \alpha e\alpha \lambda(u) - e\alpha \lambda(u/\alpha), \quad (87)
\]

\[
h\alpha \lambda(u) \coloneqq e\alpha \lambda(u) - e\alpha \lambda(u/\alpha). \quad (88)
\]

1) The following holds for \(g\alpha \lambda\):
The application of Lemma 3.1 and is positive, we recover two positive roots:

\[ \text{discriminant} \]

**Proof of Proposition 6.1.**

1) Since \( g_{α,ι}(u) = -αg_{ι,αι}(u/α) \), it is enough to consider the case \( α < 1 \). To understand the rest of the proof of 1), notice the following expression:

\[
g_{α,ι}(αu) = \frac{1}{u} \left[ \frac{αue^{-tαu}}{1-e^{-u}} - \frac{ue^{-tαu}}{1-e^{-u}} \right] = -\frac{θ_{1/α}(g_ι(u))}{u}, \quad g_ι(u) = \frac{ue^{-tαu}}{1-e^{-u}} \quad (89)
\]

1) (i) The necessary and sufficient condition \( t > 0 \) is trivial. The value of the integral is a direct application of 1) in Lemma 3.1.

1) (ii) The necessary part stems from \( \lim_{t \to 0^+} g_{α,ι}(u) = (1-α)(t-\frac{1}{2}) \). For the sufficient part, just notice that \( \lim_{t \to \infty} g_{α,ι}(u) = 0 \) and write

\[
g_ι(u) = \frac{ue^{-tαu}}{1-e^{-u}} = \frac{ue^{-tαu}/2}{1-e^{-u}} \cdot \frac{u/2}{\sinh(u/2)} \cdot e^{u(u+1/2)},
\]

in order to obtain that \( g_ι \) is decreasing and to conclude that \( g_{α,ι} \) is nonnegative.

1) (iii) According to Lemma 3.1 2) (i), we only need to check whenever \( g_ι \) is convex. Standard calculations lead to

\[
g_ι''(x) = \frac{g_ι(x)}{x(e^x-1)^2} P(x, t), \quad x, t > 0,
\]

where

\[
P(x, t) = x(e^x - 1)^2t^2 - 2x(e^x - 1)(e^x - 1 - x) + x(e^x + 1) - 2(e^x - 1),
\]

and it is clear that the convexity of \( g_ι \) is equivalent to the positivity of the function \( P \). Since the discriminant \( Δ_P(x) \) of the polynomial \( t \mapsto P(x, t) \) equals to

\[
Δ_P(x) = 4(e^x - 1)^2 \left[ (e^x - 1)^2 - x^2 e^x \right] = 16(e^x - 1)^2 \left[ \sinh(x/2)^2 - \left( \frac{x}{2} \right)^2 \right],
\]

and is positive, we recover two positive roots:

\[
t_κ(x) = \frac{1}{x(e^x-1)} \left[ (e^x - 1 - x) \pm \sqrt{(e^x - 1)^2 - x^2 e^x} \right] = \left[ \frac{1}{x} \pm \frac{1}{e^x - 1} \pm \sqrt{\frac{e^x}{x^2} - \frac{1}{(e^x-1)^2}} \right].
\]

Since

\[
\left( \frac{1}{x} - \frac{1}{e^x - 1} \right)^2 = \frac{(e^x - 1 - x)^2 - x^2 e^x}{x^2(e^x-1)^2}, \quad \left( \frac{1}{x^2} - \frac{e^x}{(e^x-1)^2} \right)^2 = \frac{e^x(e^x + 1)}{(e^x-1)^3} - \frac{2}{x^3}, \quad x > 0,
\]

we see that the functions

\[
x \mapsto \frac{1}{x} - \frac{1}{e^x - 1} \quad \text{and} \quad x \mapsto \frac{1}{x^2} - \frac{e^x}{(e^x-1)^2},
\]

are both decreasing, and then so is \( x \mapsto t_κ(x) \). By expansion near 0, we obtain

\[
\lim_{x \to 0^+} t_κ(x) = \lim_{x \to 0^+} \left[ \frac{1}{x} \pm \frac{1}{e^x - 1} \pm \sqrt{\frac{e^x}{x^2} - \frac{1}{(e^x-1)^2}} \right].
\]
\[
t_0 := \max_{x > 0} t_+(x) = t_+(0+) = \frac{1}{2} + \frac{1}{\sqrt{12}}.
\]

Since \( \min_{t > 0} t_-(x) = 0 \), we deduce that \( P(x, t) \) is positive for every \( x > 0 \) if, and only if, \( t \geq t_0 \).

2) Observe that
\[
\Theta(\epsilon_t) = \Theta_2(\epsilon_t)(u), \quad \epsilon_t(u) = e^{-tu} \epsilon_0(u), \quad u > 0,
\]
and recall that we are looking for the range of \( t \) for which we have \( \Theta_2(\epsilon_t) \) is decreasing. By Lemma 3.4, we have
\[
\theta_1, \theta_2(\epsilon_t) \geq 0, \quad \forall c_1, c_2 \in (0, 1) \iff \epsilon_t \in \mathcal{M}_2 \iff (-1)^n \Theta^n(\epsilon_t) \geq 0, \quad \text{for } n = 1, 2. \quad (91)
\]

Using the fact that \( \epsilon_t \) is decreasing, and using (43), it only remains to check whenever
\[
\Theta^n(\epsilon_t)(x) = x^n e^{\epsilon_t(x)} = x^n \left[ x \epsilon_0(x) t^2 - (\epsilon_0(x) + 2x \epsilon'_0(x)) t + \epsilon'_0(x) + x \epsilon''_0(x) \right] \geq 0.
\]

Since \( \epsilon'_0 = e_0(1-e_0) \) and \( \epsilon''_0 = e_0(1-e_0)(1-2e_0) \), we find that
\[
\Theta^n(\epsilon_t)(x) \geq 0 \iff Q(x,t) = x^n - (1 + 2x(1-e_0(x)) t + (1-e_0(x))(1+x(1-2e_0(x)) \geq 0.
\]

As in 1), the discriminant \( \Delta_Q(x) \) of the polynomial \( t \mapsto Q(x,t) \), is given by
\[
\Delta_Q(x) = 1 - 4x^2 \epsilon_0(x)(\epsilon_0(x) - 1) = 1 - \frac{4x^2 e^{\epsilon_t}}{(e^{\epsilon_t} - 1)^2} = \left( 1 - \frac{x}{\sinh(x/2)} \right) \left( 1 + \frac{x}{\sinh(x/2)} \right)
\]

Paradoxically, things are not as smooth as for \( g_{2t} \): \( \Delta_Q(x) \) is positive if, and only if, \( x \) is bigger than some value \( x_0 \) which can not be expressed by hand, but could be evaluated by Maple with the value \( x_0 \approx 3.5463796993 \). Hence, for \( x > x_0 \), we recover two positive roots
\[
t_+(x) = \frac{1}{2x} \left[ 1 - \frac{2x}{e^{\epsilon_t} - 1} + \sqrt{1 - \frac{4x^2 e^{\epsilon_t}}{(e^{\epsilon_t} - 1)^2}} \right]
\]
and finally, using Maple again, we get
\[
Q(x,t) \geq 0, \quad \forall x > 0 \iff t \geq t_1 = \max_{x>x_0} t_+(x) \approx 0.151463487259. \quad (92)
\]

\[\square\]

Let \( t > 0, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1)^n \). Recall \( e_t \) is given by (86) and let the functions
\[
h_{2t}(u) = \epsilon_t(u) - \sum_{k=1}^{n} \epsilon_t(u/\alpha_k), \quad u > 0. \quad (93)
\]

Alzer & Berg [2, Lemma 2.7] provided a Petrović-type inequality for the function \( \epsilon_1 \):
\[
\sum_{k=1}^{n} \epsilon_1(u/\alpha_k) - e_1(u) \geq 0, \quad \text{for all } u > 0.
\]

We shall provide additional information for the function \( h_{2t} \) in Lemma 6.3 below. For this purpose, we need some preliminary results. We denote by \( \delta_a \) the Dirac measure in \( a \) and by \( \lfloor \rfloor \) and \( \{ \} \) the integer and the fractional part functions, respectively.

**Lemma 6.2** Let \( t > 0, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (0,1)^n \) such that \( \sum_{k=1}^{n} \alpha_k = 1 \) and let \( \mu_{2t} \) and \( m_{2t} \) be the signed measure and the function defined by
\[
\mu_n(x) := \sum_{i=0}^{\infty} \left( \delta_{[t,x]}(dx) - \sum_{i=1}^{n} \delta_{\frac{\lfloor\left[\frac{1}{i}\right]\alpha - x\right]}(dx) \right) \quad \text{and} \quad m_n(x) := \mu_n([0,x)), \quad x \geq t.
\]
Then, \( m_n \) is positive for every fixed \( \alpha \) if, and only if, \( t \geq 1 \). In this case, we have the control
\[
0 \leq m_n(x) \leq 1 + t.
\] (94)

**Proof.** We treat only the case \( n = 2 \), the case \( n \geq 3 \) is treated similarly. By symmetry of the problem, we may assume that \( 0 < \alpha < \beta = 1 - \alpha < 1 \) and take \( \alpha = (\alpha, \beta) \). Since for every \( y > 0 \),
\[
\sum_{i=0}^{\infty} \delta_{\frac{\lfloor\left[\frac{1}{i}\right]\alpha - x\right]}([0,x)) = \lfloor y + 1 - t \rfloor I_{(\alpha, \beta)}(y),
\]
then
\[
m_n(x) = \mu_n([0,x)) = \begin{cases} 0 & \text{if } 0 < x < t \\ \lfloor x - t \rfloor + 1 & \text{if } t \leq x < t/\beta \\ \lfloor x - t \rfloor - \lfloor \beta x - t \rfloor & \text{if } t/\beta \leq x < t/\alpha \\ \lfloor x - t \rfloor - \lfloor \alpha x - t \rfloor & \text{if } t/\alpha \leq x. \end{cases}
\] (95)

1) Assume \( t \geq 1 \). Using the fact that \( y - 1 < \lfloor y \rfloor \leq y \), for all \( y \in \mathbb{R} \), the upper bound in (94) easily follows. Because the integer part function is increasing, we just need to study the positivity of the function \( m_n \) restricted to \((t/\alpha, \infty)\). Since \( [y + z] \geq [y] + [z] \), \( \forall y, z \in \mathbb{R} \), we deduce that, for every \( x > t/\alpha \),
\[
m_n(x) = \lfloor x + 1 - t \rfloor - \lfloor \alpha x + 1 - t + \beta x + 1 - t \rfloor = \lfloor x + 1 - t \rfloor - \lfloor x + 2(1 - t) \rfloor \geq 0.
\]

2) If \( t < 1 \), then the function \( m_n(x) \) could take negative values if \( x > t/\alpha \). Indeed, choose \( x \) such that for some positive integers \( k \) and \( l \), we have
\[
k \leq \lfloor \alpha x - t \rfloor < k + \frac{1 - t}{2} \quad \text{and} \quad l \leq \alpha x - t < l + \frac{1 - t}{2},
\]
so that
\[
\{\alpha x - t\} < \frac{1 - t}{2} \quad \text{and} \quad \{\alpha x - t\} < \frac{1 - t}{2}
\]
and then
\[
m_n(x) = \lfloor x - t \rfloor - \{\alpha x - t\} - \lfloor \beta x - t \rfloor - \lfloor \alpha x - t \rfloor + \{\alpha x - t\} + \{\beta x - t \} - 1 - \{x - t\} \\
\leq t + \{\alpha x - t\} + \{\beta x - t \} - 1 < 0.
\]
\( \square \)

This Lemma will be used in the proof of Corollary 2.5.

**Lemma 6.3** Let \( \theta_0 \) be the universal constant of Theorem 2.4. The function \( h_\alpha \) defined by (93) with \( \sum_{k=1}^{\infty} \alpha_k = 1 \) satisfies the following:
1) If \( t \geq 1/2 \), then \( h_\alpha(u) > 0 \) for all \( u > 0 \).
2) If \( t \geq 0 \), then \( h_\alpha \) is decreasing.
3) The function \( u \mapsto h_\alpha(u)/u \) is completely monotone if, and only if, \( t \geq 1 \). In this case, we have the representation
\[
h_\alpha(u) = (1 + t) \int_0^\infty e^{-ux} \eta(x)dx, \quad u > 0,
\]
Title Suppressed Due to Excessive Length

for some measurable function $\eta : (0, \infty) \to [0,1]$.

**Proof.** 1) and 2) Just notice the expression $h_{U,t}(u) = \sum_{k=0}^{n} g_{k,t}$, with $g_{k,t}$ given by (87), and apply Proposition 6.1.

3) Expanding the terms in $g_{U,t}$, and using the signed measure and the positive function $\mu_{U,t}$ and $m_{U,t}$ of Lemma 6.2, obtain the expression

$$
\frac{h_{U,t}(u)}{u} = \sum_{k=0}^{\infty} \left( e^{-(t+1)u} - \sum_{i=1}^{n} e^{-(u+i)\sigma_{i}} \right) = \frac{1}{u} \int_{[1,\infty)} e^{-ux} \mu_{U,t}(dx) = \int_{1}^{\infty} e^{-ux} m_{U,t}(x) dx, \quad u > 0,
$$

and conclude with the nonnegativity of $m_{U,t}$. $\square$

### 6.2 Linking the integrability of infinitely divisible distributions with their Lévy measure

Let $C$ be the class of functions $h : [0, \infty) \to [0, \infty)$, differentiable, such that

$$
\lim_{x \to \infty} h'(x) = 0 \quad \text{there exists } x_0 \geq 0 \text{ s.t. } h \text{ is concave on } [x_0, \infty). \quad (96)
$$

Note that for such functions, there exists a finite positive measure $\rho_0$ on $[x_0, \infty)$ such that the derivative of $h$ is represented by

$$
h'(x) = \rho_0([x, \infty)), \quad x \geq x_0. \quad (97)
$$

The following lemma gives an interpretation of the integrability condition (70) and constitutes a variant of [24, Theorem 2], which is stated with sub-multiplicative functions $h$.

**Lemma 6.4** Let $Z$ be nonnegative rv. with cumulant function $\phi(\lambda) = -\log E[e^{\lambda Z}]$ and let a function $h$ in the class $C$ defined by (96) and associated to the pair $(x_0, \rho_0)$ by (97).

1) We have the equivalence:

$$
E[h(Z)] < \infty \iff \int_{x_0}^{\infty} \min \{1, \phi(1/x)\} \rho_0(dx) < \infty. \quad (98)
$$

2) Assume $Z \sim $ ID($\mathbb{R}_+$) and has characteristics $(d, \Pi)$ in the representation (9) of its Bernstein function $\phi$, then the following assertions are equivalent.

1. $E[h(Z)] < \infty$;
2. $\int_{[x_0,\infty)} \chi_d(u) \Pi(du) < \infty$, where

$$
\chi_d(u) = \int_{[x_0,\infty)} (x \wedge u) \rho_0(dx) = \begin{cases} 
0, & \text{if } u \leq x_0, \\
x_0 h'(x_0) u, & \text{if } 0 < u < x_0, \\
h(u) - h(x_0) + x_0 h'(x_0), & \text{if } u \geq x_0.
\end{cases} \quad (99)
$$
3. $\int_{[x_0,\infty)} (h(u) - h(x_0)) \Pi(du) < \infty$.

**Proof of** 1) Since $h'(\infty) = 0$, obtain by Tonelli-Fubini’s theorem, that
\[ E[h(Z)] = h(0) + \int_0^\infty h'(u) \, P(Z > u) \, du \]
\[ = h(0) + \int_0^{x_0} h'(u) \, P(Z > u) \, du + \int_{x_0}^\infty \int_{x_0}^x P(Z > u) \, du \, \rho_h(dx), \]
\[ = h(0) + \int_0^{x_0} h'(u) \, P(Z > u) \, du - h'(x_0) \int_0^{x_0} P(Z > u) \, du + \int_{x_0}^\infty \int_0^x P(Z > u) \, du \, \rho_h(dx). \]

Since the first three terms in last equality are finite, deduce that
\[ E[h(Z)] < \infty \iff \int_0^\infty \int_0^x P(Z > u) \, du \, \rho_h(dx) < \infty. \quad (100) \]

Now, observe that for all \( \lambda \geq 0, \)
\[ (1 - e^{-1}) (\lambda \wedge 1) \leq (1 - e^{-\lambda}) \leq (\lambda \wedge 1). \quad (101) \]

Thus, write for all \( x > 0, \)
\[ (1 - e^{-1}) x (\phi(1/x) \wedge 1) \leq x (1 - e^{-\phi(1/x)}) = \int_0^\infty e^{-u/x} P(Z > u) \, du \leq x (\phi(1/x) \wedge 1), \quad (102) \]
\[ \int_0^\infty e^{-u/x} P(Z > u) \, du \geq \int_0^x e^{-u/x} P(Z > u) \, du \geq e^{-1} \int_0^x P(Z > u) \, du, \quad (103) \]
and deduce that
\[ \int_0^\infty e^{-u/x} P(Z > u) \, du = \int_0^x e^{-u/x} P(Z > u) \, du + \int_x^\infty e^{-u/x} P(Z > u) \, du \]
\[ \leq \int_0^x P(Z > u) \, du + P(Z > x) \int_x^\infty e^{-u/x} \, du = \int_0^x P(Z > u) \, du + e^{-1} x P(Z > x) \]
\[ \leq (1 + e^{-1}) \int_0^x P(Z > u) \, du. \quad (104) \]

From (102), (103) (104), deduce that
\[ \frac{e^{-1}}{e+1} (\phi(1/x) \wedge 1) \geq \int_0^x P(Z > u) \, du \leq e (\phi(1/x) \wedge 1), \]
and from equivalence (100), obtain the one in (98).

2) Let \( x_1 := \inf\{x > x_0; \phi(1/x) > 1\} > 0. \) By (98) and by the fact that
\[ x \mapsto x \phi(1/x) = d + \int_0^\infty e^{-u/x} \Pi(u, \infty) \, du \]
is non-decreasing and starts from \( \lim_{x \to x_0^+} x \phi(1/x) = d, \)
deduce the following steps:

a) Assume \( x_1 = \infty, \) then
\[ E[h(Z)] < \infty \iff \int_{(x_0, \infty)} \phi(1/x) x \, \rho_h(dx) < \infty, \]
and there is no problem of integrability of the last expression at 0 if \( x_0 = 0. \)

b) Assume \( x_1 < \infty, \) then,
\[ E[h(Z)] < \infty \iff \int_{(x_0, x_1)} x \, \rho_h(dx) + \int_{(x_1, \infty)} \phi(1/x) x \, \rho_h(dx) < \infty. \]
c) Deduce that in all cases,
\[ E[h(Z)] < \infty \iff I(x_0) := \int_{[x_0, \infty)} \phi(1/x) \rho_h(dx) < \infty. \] (105)

d) By representation (9) of \( \phi \) and Tonelli-Fubini’s theorem, deduce the representation,
\[ I(x_0) = d h'(x_0) + \int_{[0, \infty)} \int_{[0, \infty)} (1 - e^{-u/x}) \rho_h(dx) \Pi(du), \]
and from (101), deduce that
\[ I(x_0) < \infty \iff \int_{(0, \infty)} \chi_h(u) \Pi(du), \quad \text{where } \chi_h(u) \text{ is given by (99).} \]

If \( x_0 > 0 \), then as a Lévy measure, \( \Pi \) always integrates \( \chi_h \) on \( (0, x_0) \) and the constants on \( [x_0, \infty) \). The latter gives the equivalence \((i) \iff (ii)\). If \( x_0 = 0 \), \( \chi_h(u) = h(u) - h(0) \). Finally, from (105) deduce the equivalence \((ii) \iff (iii)\). \( \square \)

### 6.3 Proofs of the main results

**Proof of Theorem 2.1.** Using representations (22) for the Gamma function and (18) for Bernstein functions, then performing an obvious change of variable, using 1) in Lemma 6.1, and the fact that \( \int_0^\infty g_{\alpha,t}(x)dx = -d_{\alpha} \) provided by Proposition 6.1 1)(i), obtain the following representation for \( \alpha \in (0, 1) \) and \( t > 0 \):
\[ G_{\alpha,t}(\lambda) = \exp\left\{ d_{\alpha} \lambda + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u) \frac{g_{\alpha,t}(u)}{u} \, du \right\} = \exp\left\{ \int_0^\infty (1 - e^{-\lambda u}) \frac{g_{\alpha,t}(u)}{u} \, du \right\}, \] (106)
where \( d_{\alpha} \) is given by (18). Thus, since \( G_{\alpha,t}(0) = 1 \) and using (9), retrieve the equivalences in case \( \alpha \in (0, 1) \):
\[ \lambda \mapsto (G_{\alpha,t})' \in C_\mathcal{M}, \quad \forall r > 0 \iff \lambda \mapsto \int_0^\infty (1 - e^{-\lambda u}) \frac{g_{\alpha,t}(u)}{u} \, du \in B_\mathcal{F} \iff g_{\alpha,t} \geq 0 \]
\[ \iff G_{\alpha,t}(\lambda) = E[e^{-\lambda X_{\alpha,t}}], \quad \lambda \geq 0, \quad \text{and } X \sim \text{ID}(R_+). \]

By point 2) in Proposition 6.1, we deduce that \( g_{\alpha,t} \geq 0 \iff t \geq 1/2 \). The rest of the statements are obtained by the reflexive relation (19). \( \square \)

**Proof of Theorem 2.4.** By (54), the functions \( g_{\alpha,t}(x)/x \) is the density of the Lévy measure of \( X_{\alpha,t}, \alpha \in (0, 1) \). By point 1)(iii) in Proposition 6.1 is nondecreasing if, and only if \( t \geq t_0 \). The reasoning is analog for \( Y_{\alpha,t}, \alpha > 1 \). \( \square \)

**Proof of Corollary 2.5.** 1), 2) and 3) are a straightforward consequence of the Mellin transform representations (33) and of Theorems 2.1 and 2.4.

4) For the claim on \( X_{\alpha,t} \), we only need to use (106), to observe that
\[ -\log E[e^{-\lambda X_{\alpha,t}}] = \int_0^\infty (1 - e^{-\lambda u}) \frac{g_{\alpha,t}(u)}{u} \, du \] (107)
where \( h_{\mathcal{I}}(u) \) is given by (93) and to apply point 3) of Lemma 6.3 to finally obtain that
\[
-\log \mathbb{E}[e^{-\lambda X_{\mathcal{I}}}]^1/(1+\epsilon) = \int_0^\infty \frac{\lambda}{\lambda + x} \eta(x) dx, \quad \text{with} \quad \eta(x) := \frac{m_{\mathcal{I}}(x)}{(1+\epsilon)} \leq 1
\]
is a Bernstein function that meets the form (28). The assertion for \( Y_{\mathcal{I}} \) is shown identically. \( \square \)

Proof of Proposition 2.7. Using formula (27) and performing the change of variable \( u \to u/\alpha \), we write
\[
\Gamma(\lambda + t) = \exp \left\{ (1 - \alpha)\Psi(t)\lambda + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \frac{h_{\mathcal{I}}(u)}{u} du \right\} = \mathbb{E}[e^{\lambda T_{\mathcal{I}}}], \quad \lambda \geq 0 \quad (108)
\]
where \( h_{\mathcal{I}} \) is the nonnegative function given by (88) and \( T_{\mathcal{I}} \sim \text{ID}_.(\mathbb{R}) \) is such that the identity in law (35) holds. As a consequence of Proposition 6.1, we obtain that selfdecomposability of \( T_{\mathcal{I}} \) is equivalent to the decreaseness of \( h_{\mathcal{I}} \). The latter is also equivalent to \( t \geq t_1 \). \( \square \)

Proof of Proposition 2.8. Due to the representation (107) for the Bernstein function of \( X_{\mathcal{I}} \), it suffices to show that the function \( h_{\mathcal{I}} \) given by (93), converges as \( n \to \infty \). This not difficult to obtain, because for \( t > 0 \), the function \( u \to u e_t(u) = e^{\epsilon_t u}/(1 - e^{-\epsilon_t}) \) is bounded on \((0, \infty)\) by a constant, say \( C \). Since \( \sum_{k=1}^n \alpha_k = 1 \), we have
\[
e_t(u) - h_{\mathcal{I}}(u) = \frac{1}{n} \sum_{u=1}^n \frac{u}{\alpha_k} h_t \left( \frac{u}{\alpha_k} \right) \alpha_k \leq C, \quad u > 0.
\]
Thus, for fixed \( u, t > 0 \), the bounded and increasing sequence \( e_t(u) - h_{\mathcal{I}}(u) \) is convergent as \( n \to \infty \). \( \square \)

Proof of Lemma 3.1. 1) It is enough to consider the case \( c \in (0, 1) \). By Tonelli-Fubini’s theorem, we get
\[
\int_0^\infty \frac{g(x) - g(c|x|)}{x} dx = \int_0^1 \frac{1}{x} \int_{[x,c]} (-g(y)) dy dx = \int_{(0,\infty)} \int_{[x,c]} \frac{dx}{x} d(-g)(y) = (g(\infty) - g(0+)) \log c.
\]
2) Both assertions (i) and (ii) stem from
\[
\Theta_c(g)(x) = -\int_1^{1/c} xg'(sx) ds = -\int_1^{1/c} \Theta(g)(sx) ds \quad \text{and} \quad \lim_{c \to 1^-} \frac{\Theta_c(g)(x)}{1 - c} = -\Theta(g)(x).
\]
\( \square \)

Proof of Corollary 4.4. 1) Assuming that \( X \sim \text{L}_0(\mathbb{R}_+) \) (resp. \( X \sim \text{ID}_-.(\mathbb{R}) \cap \text{L}_0(\mathbb{R}) \)), then, as in (55), the associated Bernstein function \( \Phi \) represented by (54) (respectively, the Laplace exponent \( \Psi \) represented by (57)), satisfies the following: if \( c_1, c_2, \ldots, c_m \in (0, 1) \), and \( e_m, e_m = 1 - c_1 \ldots (1 - e_m) \), then the representations
\[
\theta_{1/e_1} \theta_{1/e_2} \ldots \theta_{1/e_m}(\phi)(\lambda) = e_m, 1 d\lambda \int_0^{\infty} \frac{(1 - e^{-\lambda}) \theta_{1/e_1} \ldots \theta_{1/e_m}(k)(x)}{x} dx
\]
\[
\theta_{1/e_1} \theta_{1/e_2} \ldots \theta_{1/e_m}(\Psi)(\lambda) = e_m, 1 d\lambda \int_0^{\infty} \frac{(e^{-\lambda x} - 1 + \lambda x) \theta_{1/e_1} \ldots \theta_{1/e_m}(k)(x)}{x} dx
\]
are straightforward and Lemma 3.4 clarifies the equivalences between (i), (ii), (iii) and (iv). Due to the integrability condition (10) (respectively (3)), we see that in the representation (32) of \( k \),
necessarily \( c = 0 \) and, after some calculus, we see that the \( \mu \)-measure should satisfy
\[
\int_{0}^{(\infty \wedge 1)} \frac{k(x)}{x} \, dx = \int_{0}^{(\infty \wedge 1)} \frac{a_{n,l}(y)}{y} \, \mu(dy) < \infty, \tag{109}
\]
\[
a_{n,l}(y) := \frac{1}{n!} \int_{0}^{y} (x \wedge 1) \log^{n} \left( \frac{y}{x} \right) \, dx = \begin{cases} \frac{1}{n+1}, & \text{if } x < 1 \\ \frac{\log^{n+1}(y)}{(n+1)!} + \frac{1}{n!} \int_{0}^{\infty} e^{-tz} \, dz, & \text{if } x \geq 1 \end{cases} \tag{110}
\]
with \( l = 1 \) if \( X \sim L_{n}(\mathbb{R}_{+}) \) and \( l = 2 \) if \( X \sim \text{ID}_{-}(\mathbb{R}) \cap L_{n}(\mathbb{R}) \). Observing that \( a_{n,l}(y) \sim \log^{n+1}(y)/(n+1)! \), as \( y \to \infty \), we recover the condition (59) which is satisfied if \( X \sim \text{ID}_{-}(\mathbb{R}) \cap L_{n}(\mathbb{R}) \) due to (7) and to the fact that \( \lim_{n \to \infty} a_{n,2}(y)/y = 0 \). The equivalence with (iv) and (v) are due to (75) in the case that \( X \sim L_{n}(\mathbb{R}_{+}) \). The proof is identical in the case that \( X \sim \text{ID}_{-}(\mathbb{R}) \cap L_{n}(\mathbb{R}) \). The equivalence with (vi) is provided by (75) and (76).

2) The assertion is immediate due to the decomposition and the approximation observed in point 3) of Proposition 4.3, taking into account (109).

3) The first assertion is evident from point 1). The conditions on the support of \( \mu \) are due to (3), (10) and to Lemma 3.4, which read as follows: taking \( l = 0 \) if \( X \sim L_{\infty}(\mathbb{R}_{+}) \) (respectively \( l = 1 \) if \( X \sim \text{ID}_{-}(\mathbb{R}) \cap L_{\infty}(\mathbb{R}) \)), then
\[
\int_{0}^{\infty} x(x \wedge 1) \frac{k(x)}{x} \, dx = \int_{0}^{(\infty \wedge 1)} \left( \int_{0}^{1} x^u \, dx + \int_{1}^{\infty} \frac{1}{x^{u+l+1}} \, dx \right) v(du) < \infty
\]
if, and only if, \( \text{support}(v) = (0, 1) \) if \( l = 0 \) (respectively \( \text{support}(v) = (1, 2) \)), and then necessarily
\[
\int_{(0,1)} \frac{1}{u(1-u)} v(du) < \infty, \quad \text{if } l = 0 \quad \text{(respectively } \int_{(1,2)} \frac{1}{(u-1)(2-u)} v(du) < \infty, \text{ if } l = 0 \)). \]

\[ \square \]

7 Some background on infinitely divisible and self-decomposable distributions

An equivalent definition for the infinite divisibility of a probability measure \( \mu \) is, that there exists a real row-wise independent random variables \( \xi_{n,k}, k = 1, 2, \ldots, k_n, n \geq 1, k_n \to \infty \), satisfying the infinitesimally condition
\[
\lim_{n \to \infty} \max_{1 \leq k_n} P(|\xi_{n,k_n}| > \varepsilon) \to 0, \quad \text{for each } \varepsilon > 0,
\]
and such that we have the limit in distribution
\[
\xi_{n,1} + \xi_{n,2} + \ldots + \xi_{n,k_n} \overset{d}{\to} \mu. \tag{111}
\]

Conversely, any infinitely divisible distribution can be obtained in the scheme (111); see Gnedenko & Kolmogorov [12] or Loève [31] or Parthasarathy [37], for multi-dimensional spaces. Similarly, the distribution \( \mu \) is self-decomposable, if there exists two sequences \( a_n > 0, b_n \in \mathbb{R} \) and an infinitesimal triangular array of independent real random variables of the form \( \xi_{n,k} := a_n Z_k \), such that
The terminology of selfdecomposability is due to the fact that the limiting distribution \( \mu \) corresponds to an infinitely divisible r.v. \( X \) that satisfies (13). The most general case of limits (112), where the \( Z_k \)'s are Banach space-valued random variables and the \( a_k \)'s are chosen from a group of bounded linear operators on a Banach space in question, was studied in Jurek [18]. In [18, Theorem 3.1 and Section 4], taking reals as a Banach space \( \mathbb{B} \), the multiplication by positive scalars as linear operators and all probability measures as a set \( \mathcal{Q} \), it is shown that

\[
\mu \in \text{L}_m(\mathbb{B}), \quad \text{if, and only if, (112) holds and } Z_k \sim \text{L}_{m-1}(\mathbb{B}), \quad k \geq 1, \ m \geq 1.
\]

Conversely, each selfdecomposable distribution can be obtained via the limiting scheme (112). It might be worth to remember that all selfdecomposable distributions are absolutely continuous with respect to Lebesgue measure, cf. [25, Section 3.8, p.162]. Additionally, if we assume that the r.v.'s \( Z_1, Z_2, \ldots, Z_n \) have the same distribution, then we get, at the limit, the class \( \text{S}(\mathbb{R}) \) of stable distributions on \( \mathbb{R} \), see the monograph of Zolotarev [55] or (26) for stable distributions. Furthermore, if \( a_0 := (\sigma \sqrt{n})^{-1} \) where \( \sigma^2 \) is the variance of \( Z_k \) and \( b_0 = -nE[Z_1] \), then in (112) we get Central Limit Theorem, i.e., \( \mu \) is the standard normal distribution \( \mathcal{N}(0,1) \).

From the above way of reasoning we have the inclusions:

\[
\text{(normal distributions)} \subseteq \text{S}(\mathbb{R}) \subseteq \text{L}_0(\mathbb{R}) \subseteq \text{ID}(\mathbb{R}).
\]

The class \( \text{L}_0(\mathbb{R}) \) is quite large and contains among others \( \chi^2 \), Fisher, gamma, log-gamma, etc; see Jurek [20].

In [19, Corollary 2.11], Jurek showed that when taking his operator equal to the identity, we have \( X \sim \text{L}_m(\mathbb{R}) \), \( m \geq 0 \), if, and only if, there exists a Lévy process \( \{Y_t\}_{t \geq 0} \), such that \( \mathbb{E}[\log^{m+1}(1 + |Y_t|)] < \infty \) and such that we have the integral stochastic representation

\[
X \overset{d}{=} \int_0^\infty e^{-t}dY_t(t), \quad r(t) = \frac{\log^{m+1}(1 + |Y_t|)}{(m+1)!}.
\]  

(113)

Note that these integral representations allow descriptions of classes \( \text{L}_m(\mathbb{R}) \) in terms of characteristic functions, see [19, Theorem 3.1], as Urbanik [50, 52] obtained by the extreme point method. For \( m = 0 \), the constructed Lévy process \( Y \) in (113) is such that

\[
Y(t + s) - Y(t) = e^{-s}V_{e^{-s}} + V_{e^{-s}}, \quad s, t > 0, \quad \text{where } V_{e^{-s}} \text{ is given by (14)},
\]

and the random integral characterization of selfdecomposable distributions is from Jurek & Ver-vaat [25, pp. 252-253]. The process \( Y \) is coined as the background driving Lévy process of \( X \), in short, BDLP. Other constructions of BDLP’s for selfdecomposable random variables are given in Jeanblanc, Pitman & Yor [15].

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