Average Age of Information for a Multi-Source M/M/1 Queueing Model With Packet Management

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Background: Definition and Appliances

- Internet of Things (IoT) in 5G and beyond
- Cyber-physical control applications
- One key enabler for these services is the freshness of the sensor’s information at the destinations

A status update packet contains
- The measured value of the monitored process
- A time stamp representing the time when the sample was generated

Generated at random times
- Takes a random time to traverse the network
Background: Age of Information

- The traditional metrics (throughput and delay) can not fully characterize the information freshness
- AoI (at the destination) is the time elapsed since the last received status update was generated, i.e., the random process

\[ \Delta(t) = t - u(t) \]  

- \( u(t) \) is the time stamp of the most recently received update
- The most commonly used metrics for evaluating the AoI
  - Average AoI

![Graph showing age of information](image-url)
Background: Packet management in AoI Analysis

1. S. K. Kaul, R. D. Yates, and M. Gruteser, “Status updates through queues,” in Proc. Conf. Inform. Sciences Syst. (CISS), Princeton, NJ, USA, Mar. 2123, 2012, pp. 16.
2. M. Costa, M. Codreanu, and A. Ephremides, “On the age of information in status update systems with packet management,” IEEE Trans. Inform. Theory, vol. 62, no. 4, pp. 18971910, Apr. 2016.
3. R. D. Yates and S. K. Kaul, “The age of information: Real-time status updating by multiple sources,” IEEE Trans. Inform. Theory, vol. 65, no. 3, pp. 18071827, Mar. 2019.
4. A. Javani and Z. Wang, “Age of information in multiple sensing” [Online]. Available: http://arxiv.org/abs/1902.01975, 2019.
System Model

- Two independent sources, one server, and one sink
- The packets of source $i$ are generated according to the Poisson process with rate $\lambda_i$, $i \in \{1, 2\}$
- The packets are served according to an exponentially distributed service time with mean $1/\mu$
- The load of source $i$ is defined as $\rho_i = \lambda_i/\mu$, $i \in \{1, 2\}$
- The packet generation in the system follows the Poisson process with rate $\lambda = \lambda_1 + \lambda_2$
- The overall load in the system is $\rho = \rho_1 + \rho_2 = \lambda/\mu$
Packet Management Policy

- The queue can contain at most two packets at the same time, one packet of source 1 and one packet of source 2.
- When the system is empty, any arriving packet immediately enters the server.
- When the server is busy, a packet of a source $i \in \{1, 2\}$ waiting in the queue is replaced if a new packet of the \textbf{same source} arrives.
Aol analysis using the SHS technique (1/3)

- Models a queueing system through the states \((q(t), x(t))\)
  - \(q(t) \in Q = \{0, 1, \ldots, m\}\) is a continuous-time finite-state Markov chain that describes the occupancy
  - \(x(t) = [x_0(t) \ x_1(t) \cdots x_n(t)] \in \mathbb{R}^{1 \times (n+1)}\) is a continuous process that describes the evolution of age-related processes (for instance AoI of source one)

- \(q(t)\) can be presented as a graph \((Q, \mathcal{L})\)
  - A discrete state \(q(t) \in Q\) is a node of the chain
  - A (directed) link \(l \in \mathcal{L}\) from node \(q_l\) to node \(q'_l\) indicates a transition from state \(q_l \in Q\) to state \(q'_l \in Q\)

- A transition occurs when a packet arrives or departs in the system

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5 R. D. Yates and S. K. Kaul, “The age of information: Real-time status updating by multiple sources,” IEEE Trans. Inform. Theory, vol. 65, no. 3, pp. 1807-1827, Mar. 2019.
Aol analysis using the SHS technique (2/3)

- When a transition $l$ occurs
  - The discrete state $q_l$ changes to state $q'_l$
  - The continuous state $x$ is reset to $x'$; $x' = xA_l$, $A_l \in \mathbb{B}^{(n+1)\times(n+1)}$

- The continuous state $x$ evolves as a piece-wise linear function through the differential equation

$$\dot{x}(t) \triangleq \frac{\partial x(t)}{\partial t} = b_q$$

- $b_q = [b_{q,0} b_{q,1} \ldots b_{q,n}] \in \mathbb{B}^{1\times(n+1)}$, $b_{q,j} \in \{0, 1\}$, $\forall j \in \{0, \ldots, n\}$, $q \in Q$

- If the age process $x_j(t)$ increases at a unit rate, we have $b_{q,j} = 1$; otherwise, $b_{q,j} = 0$

- To calculate the average Aol using the SHS technique
  - The state probabilities of the Markov chain;
    $\pi_q(t) = \Pr(q(t) = q), \ \forall q \in Q$
  - The correlation vector between the discrete state $q(t)$ and the continuous state $x(t)$; $\mathbf{v}_q(t) = [v_{q0}(t) \ldots v_{qn}(t)]$, $\forall q \in Q$
**Aol analysis using the SHS technique (3/3)**

- $\mathcal{L}_q'$: set of incoming transitions, and $\mathcal{L}_q$: set of outgoing transitions
- Following the ergodicity assumption of the Markov chain $q(t)$,
  - $\boldsymbol{\pi}(t) = [\pi_0(t) \cdots \pi_m(t)]$ converges uniquely to the stationary vector $\bar{\boldsymbol{\pi}} = [\bar{\pi}_0 \cdots \bar{\pi}_m]$ satisfying
    \[
    \bar{\pi}_q \sum_{l \in \mathcal{L}_q} \lambda^{(l)} = \sum_{l \in \mathcal{L}_q'} \lambda^{(l)} \bar{\pi}_{ql}, \quad \forall q \in \mathcal{Q}, \quad \sum_{q \in \mathcal{Q}} \bar{\pi}_q = 1, \tag{2}
    \]
  - The correlation vector $\boldsymbol{v}_q(t)$ converges to a nonnegative limit $\bar{\boldsymbol{v}}_q = [\bar{v}_{q0} \cdots \bar{v}_{qn}], \forall q \in \mathcal{Q}$, as $t \to \infty$ such that
    \[
    \bar{v}_q \sum_{l \in \mathcal{L}_q} \lambda^{(l)} = b_q \bar{\pi}_q + \sum_{l \in \mathcal{L}_q'} \lambda^{(l)} \bar{v}_{ql} \mathbf{A}_l, \quad \forall q \in \mathcal{Q} \tag{3}
    \]
- The average Aol of source 1 is calculated by \(^6\)
  \[
  \Delta_1 = \sum_{q \in \mathcal{Q}} \bar{v}_{q0} \tag{4}
  \]

\(^6\) R. D. Yates and S. K. Kaul, “The age of information: Real-time status updating by multiple sources,” IEEE Trans. Inform. Theory, vol. 65, no. 3, pp. 1807–1827, Mar. 2019.
The state space of the Markov chain is $Q = \{0, 1, \cdots, 10\}$

| State | Index of the second packet | Index of the first packet | Index of the packet under service |
|-------|----------------------------|---------------------------|----------------------------------|
| 0     | -                          | -                         | -                                |
| 1     | -                          | -                         | 1                                |
| 2     | -                          | -                         | 2                                |
| 3     | -                          | 1                         | 1                                |
| 4     | -                          | 2                         | 1                                |
| 5     | 2                          | 1                         | 1                                |
| 6     | 1                          | 2                         | 1                                |
| 7     | -                          | 1                         | 2                                |
| 8     | -                          | 2                         | 2                                |
| 9     | 2                          | 1                         | 2                                |
| 10    | 1                          | 2                         | 2                                |
SHS Analysis for the Proposed System Model (2/6)

- The continuous process is $\mathbf{x}(t) = [x_0(t) \ x_1(t) \ x_2(t) \ x_3(t)]$
  - $x_0(t)$: the current AoI of source 1 at time instant $t$, $\Delta_1(t)$
  - $x_1(t)$ encodes what $\Delta_1(t)$ would become if the packet that is under service is delivered to the sink at time instant $t$
  - $x_2(t)$ encodes what $\Delta_1(t)$ would become if the first packet in the queue is delivered to the sink at time instant $t$
  - $x_3(t)$ encodes what $\Delta_1(t)$ would become if the second packet in the queue is delivered to the sink at time instant $t$

- Our goal is to find $\bar{v}_q, \forall q \in Q$ ($\Delta_1 = \sum_{q \in Q} \bar{v}_q$) by solving
  \[
  \bar{v}_q \sum_{l \in \mathcal{L}_q} \lambda^{(l)} = b_q \bar{\pi}_q + \sum_{l \in \mathcal{L}_q'} \lambda^{(l)} \bar{v}_{ql} \mathbf{A}_l, \ \forall q \in Q \tag{5}
  \]

- To form the system of linear equations, we need to determine
  - $b_q, \bar{\pi}_q, \forall q \in Q$
  - $\bar{v}_{ql} \mathbf{A}_l$ for each incoming transition $l \in \mathcal{L}_q', \forall q \in Q$
SHS Analysis for the Proposed System Model (3/6)

Calculating $\bar{\pi}_q$

$$(\bar{\pi}_q \sum_{l \in \mathcal{L}_q} \lambda^{(l)} = \sum_{l \in \mathcal{L}_q'} \lambda^{(l)} \bar{\pi}_q l, \ \forall q \in Q, \ \sum_{q \in Q} \bar{\pi}_q = 1,)$$

$\bar{\pi}_0 = \frac{1}{\beta}$,

$\bar{\pi}_1 = \frac{\rho_1}{\beta}$,

$\bar{\pi}_2 = \frac{\rho_2}{\beta}$,

$\bar{\pi}_3 = \frac{\rho_1^2}{\beta}$,

$\bar{\pi}_4 = \frac{\rho_1 \rho_2 (1 + \rho)}{\beta (1 + \rho_1)}$,

$\bar{\pi}_5 = \frac{\rho_2^2 \rho_2}{\beta (1 + \rho_2)}$,

$\bar{\pi}_6 = \frac{\rho_1^2 \rho_2 (1 + \rho)}{\beta (1 + \rho_1)}$,

$\bar{\pi}_7 = \frac{\rho_1 \rho_2 (1 + \rho)}{\beta (1 + \rho_2)}$,

$\bar{\pi}_8 = \frac{\rho_2^2}{\beta (1 + \rho_2)}$,

$\bar{\pi}_9 = \frac{\rho_1 \rho_2^2 (2 + \rho)}{\beta (1 + \rho_2)}$,

$\bar{\pi}_{10} = \frac{\rho_1 \rho_2^2}{\beta}$,

where $\beta = \rho^2 + \rho(2\rho_1 \rho_2 + 1) + 1$
Calculating $b_q$ ($\dot{x} = b_q$)

- $b_{q,1} = 1$, $\forall q \in Q$: the AoI of source 1, $\Delta_1(t) = x_0(t)$, increases at a unit rate with time in all discrete states
- $b_{q,i}$ is equal to 1 if the $i$th packet in the queue is a source one packet

\[
b_q = \begin{cases} 
[1 0 0 0], & q = 0, \\
[1 1 0 0], & q = 1, \\
[1 0 0 0], & q = 2, \\
[1 1 1 0], & q = 3, \\
[1 1 0 0], & q = 4, \\
[1 1 1 0], & q = 5, \\
[1 1 0 0], & q = 6, \\
[1 0 1 0], & q = 7, \\
[1 0 0 0], & q = 8, \\
[1 0 1 0], & q = 9, \\
[1 0 0 1], & q = 10 
\end{cases}
\]
SHS Analysis for the Proposed System Model (5/6)

- Calculating $\bar{v}_{ql} A_l$ for each incoming transition $l \in \mathcal{L}_q$, $\forall q \in Q$
  - There are 32 transitions
  - For instance transition $l : 3 \rightarrow 5$ in the chain

\[
x' = [x_0 \ x_1 \ x_2 \ 0]
\]

\[
x' = [x_0 \ x_1 \ x_2 \ x_3] A_l = [x_0 \ x_1 \ x_2 \ 0] \Rightarrow A_l = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\bar{v}_3 A_l = [v_{30} \ v_{31} \ v_{32} \ v_{33}] A_l = [v_{30} \ v_{31} \ v_{32} \ 0]
\]
After solving the system of linear equations we have

\[
\Delta_1 = \frac{\sum_{k=0}^{7} \rho_1^k \eta_k}{\mu \rho_1 (1 + \rho_1)^2 \sum_{j=0}^{4} \rho_1^j \xi_j},
\]

\[
\eta_0 = \rho_2^4 + 2\rho_2^3 + 3\rho_2^2 + 2\rho_2 + 1, \quad \eta_1 = 7\rho_2^4 + 15\rho_2^3 + 21\rho_2^2 + 14\rho_2 + 6,
\]

\[
\eta_2 = 17\rho_2^4 + 46\rho_2^3 + 64\rho_2^2 + 42\rho_2 + 16, \quad \eta_3 = 15\rho_2^4 + 73\rho_2^3 + 118\rho_2^2 + 78\rho_2 + 26,
\]

\[
\eta_4 = 5\rho_2^4 + 52\rho_2^3 + 124\rho_2^2 + 102\rho_2 + 30, \quad \eta_5 = 15\rho_2^3 + 66\rho_2^2 + 79\rho_2 + 24,
\]

\[
\eta_6 = 15\rho_2^2 + 31\rho_2 + 11, \quad \eta_7 = 5\rho_2 + 2,
\]

\[
\xi_0 = \rho_2^4 + 2\rho_2^3 + 3\rho_2^2 + 2\rho_2 + 1, \quad \xi_1 = 2\rho_2^4 + 6\rho_2^3 + 9\rho_2^2 + 7\rho_2 + 3,
\]

\[
\xi_2 = 6\rho_2^3 + 12\rho_2^2 + 10\rho_2 + 4, \quad \xi_3 = 6\rho_2^2 + 8\rho_2 + 3,
\]

\[
\xi_4 = 2\rho_2 + 1.
\]
Results

- Sum average AoI for different values of $\rho^7$ with $\mu = 1$

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7 R. D. Yates and S. K. Kaul, “The age of information: Real-time status updating by multiple sources,” IEEE Trans. Inform. Theory, vol. 65, no. 3, pp. 1807-1827, Mar. 2019.
Thank You For Your Attention!

Questions?
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