MODULATION APPROXIMATION FOR THE QUANTUM EULER-POISSON EQUATION

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ABSTRACT. The nonlinear Schrödinger (NLS) equation is used to describe the envelopes of slowly modulated spatially and temporally oscillating wave packet-like solutions, which can be derived as a formal approximation equation of the quantum Euler-Poisson equation. In this paper, we rigorously justify such an approximation by taking a modified energy functional and a space-time resonance method to overcome the difficulties induced by the quadratic terms, resonance and quasilinearity.

1. Introduction. In the current paper, we consider modulation approximation of the following quantum Euler-Poisson system (c.f. [11])

\[
\begin{align}
\partial_t n_i + \partial_x (n_i u_i) &= 0, \quad (1a) \\
\partial_t u_i + u_i \partial_x u_i &= -\partial_x \phi + \frac{\partial_x n_i}{n_i}, \quad (1b) \\
\partial_x^2 \phi &= n_e - n_i, \quad (1c) \\
\phi &= -\frac{1}{2} + \frac{1}{2} n_e^2 - \frac{H^2}{2 \sqrt{n_e}} \partial_x \sqrt{n_e}, \quad (1d)
\end{align}
\]

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where \( n_{e,i} \) are the electronic and ionic number densities, \( u_i \) is the ionic velocities, \( \phi \) is the scalar potential, \( H > 0 \) is the ratio between the electron plasmon energy and the electron Fermi energy and is used to describe the quantum diffraction effects. The model (1) is a reduced quantum model from a one-dimensional two species quantum plasma system made by one electronic and one ionic fluid, in the electrostatic approximation \([11]\). The equation (1d) is obtained by taking the electron fluid pressure \( P = P(n_e) \), modeled by the equation of state for a one dimensional zero-temperature Fermi gas, as

\[
P(n_e) = \frac{m_e v_{F_e}^2}{3 n_0^2} n_e^3,
\]

where \( n_0 \) is the equilibrium density for both electrons and ions, and \( v_{F_e} \) is the electron Fermi velocity, which is related to the Fermi temperature \( T_F \) by \( m_e v_{F_e}^2 = \kappa_B T_F \) with \( \kappa_B \) the Boltzmann constant. Here, the case \( \gamma = 3 \) for the electron fluid pressure is an (possibly the most) important case in dimension 1 \([13]\).

In the present paper, we aim to consider the modulation approximation for (1) and derive the nonlinear Schrödinger (NLS) equation with justification. The NLS equation plays an important role in describing approximately slow modulations in time and space of an underlying spatially and temporally oscillating wave packet in many dispersive systems. In order to derive the NLS approximation, we take \( \rho = n_i - 1 \), \( v = u_i \) and

\[
\begin{pmatrix} \rho \\ v \end{pmatrix} = e \Psi_{NLS} + O(\epsilon^2),
\]

with

\[
e \Psi_{NLS} = e A(\epsilon(x + c_g t), \epsilon^2 t) e^{i(k_0 x + \omega_0 t)} \varphi(k_0) + \text{c.c.},
\]

then the NLS equation can be derived for the complex amplitude \( A \),

\[
\partial_T A = i \nu_1 \partial_X A + i \nu_2 |A|^2,
\]

where \( T = \epsilon^2 t \in \mathbb{R} \) is the slow time scale and \( X = \epsilon(x + c_g t) \in \mathbb{R} \) is the slow spatial scale and coefficients \( \nu_1 = \nu_j(k_0) \in \mathbb{R} \) with \( j \in \{1, 2\} \). In the above modulation approximation, \( 0 < \epsilon \ll 1 \) is a small perturbation parameter, \( \omega_0 > 0 \) is the basic temporal wave number associated to the basic spatial wave number \( k_0 > 0 \) of the underlying temporally and spatially oscillating wave train \( e^{i(k_0 x + \omega_0 t)} \), \( c_g \) is the group velocity and ‘c.c.’ denotes the complex conjugate. The NLS equation is derived in order to describe the slow modulations in time and in space of the wave train \( e^{i(k_0 x + \omega_0 t)} \) and the time and space scales of the modulations are \( O(1/\epsilon^2) \) and \( O(1/\epsilon) \), respectively. Our ansatz leads to waves moving to the left. To obtain waves moving to the right, \( \omega_0 \) and \( c_g \) have to be replaced with \(-\omega_0\) and \(-c_g\).

For later use, we compute the dispersion relation here. Indeed, for the Euler-Poisson equation (1), it can be computed the following dispersion relation

\[
\omega(k) = k \sqrt{\frac{2 + (1 + \frac{H^2}{4}) k_0^2 + \frac{H^2}{4} k^4}{1 + k^2 + \frac{H^2}{4} k^4}} =: k \tilde{q}(k).
\]

From this dispersion relation, the group velocity \( c_g = \partial_k \omega \big|_{k=k_0} \) of the wave packet can be found. As a side remark, we also note that a NLS equation has been obtained formally for the amplitude of the ion oscillation mode based on the Euler-Poisson equation with cold ions and warm electrons in \([25]\).

The main result of this paper is the following
Theorem 1.1. For all $s_A \geq 6$, $k_0 \neq 0$ and all $C_1$, $T_0 > 0$, there exist $C_2 > 0$, $\epsilon_0 > 0$ such that for all solutions $A \in C([0,T_0], H^{s_A}(\mathbb{R}, \mathbb{C}))$ of the NLS equation (4) with
\[ \sup_{t \in [0,T_0]} \| A_0(t) \|_{H^{s_A}(\mathbb{R}, \mathbb{C})} \leq C_1, \]
the following holds. For all $\epsilon \in (0,\epsilon_0)$, there are solutions
\[ \left( \frac{n_i - 1}{u_i} \right) \in C([0,T_0/\epsilon^2], H^2(\mathbb{R}, \mathbb{R}))^2, \]
of the Euler-Poisson equation (1) with parameter $0 < H \leq 2$, which satisfy
\[ \sup_{t \in [0,T_0/\epsilon^2]} \left\| \left( \frac{n_i - 1}{u_i} \right) - e^{\Psi_{NLS}(\cdot, t)} \right\|_{H^2(\mathbb{R}, \mathbb{R})}^2 \leq C_2 \epsilon^{3/2}, \]
where $e^{\Psi_{NLS}}$ is given by (3) with $\varphi(k_0) = (1, -\hat{q}(k_0))^T$.

Remark 1. Note that we limit the parameter $0 < H \leq 2$ in the Theorem 1.1. When $H > 2$, there are more resonances appearing such that we can not control them by using the methods in this paper (Details refer to the Section 5). Furthermore, in the momentum equation (1b) of the ion-Euler-Poisson equation, we choose the ion pressure $p_i = n_i$. However, the result in this paper can be generalized to the general $\gamma$-law of the ion pressure $p_i$, i.e., when $p_i(n_i) = n_i^\gamma$ for $\gamma > 1$.

Note that the existence time scale for the quantum Euler-Poisson equation is $O(\epsilon^{-2})$, long enough such that the dynamics of the NLS equation can be seen in the quantum Euler-Poisson equation (1), and compared with the solution $(n_i - 1, u_i)$ and the approximation $e^{\Psi_{NLS}}$, which are both of order $O(1)$ in $L^\infty$, the error of order $O(\epsilon^{3/2})$ is small enough. Therefore, as $\epsilon$ is very small, the Euler-Poisson equation can be approximated by the NLS equation on a fixed $O(1)$ time interval of the ‘slow’ (i.e., NLS) time scale.

The Euler-Poisson equation is a good source of some dispersive equations, such as the Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), Zakharov-Kuznetsov (ZK) and the Nonlinear Schrödinger (NLS) equation. For the long wavelength approximation of the Euler-Poisson system, Guo and Pu [10] firstly established rigorously the KdV limit for the ion Euler-Poisson system in 1D, where the electron density satisfies the classical Maxwell-Boltzmann law and both the cold and hot plasma cases were considered. Then Pu generalized this result to the higher dimensional cases, i.e. the 2D KP-II equation and the 3D ZK equation are derived under different scalings [20]. Almost at the same time, Lannes, Linares and Saut [17] obtained rigorously the ZK equation in 3D for the Euler-Poisson system. Recently, Liu and Pu [18] established rigorously the QKdV limit in 1D and the QKP limit in 2D for the quantum Euler-Poisson system, where the electron equilibrium is given by a Fermi-Dirac distribution, and both the cold and hot plasma cases were considered.

Since the oscillatory solution of the KdV equation is a solution of the NLS equation in the small wave number region with frequency induced from the dispersive relation, the NLS approximation for the Euler-Poisson system (1) is a more interesting problem in some sense. However, this is a nontrivial problem for the justification of the NLS approximation for dispersive systems with quasilinear quadratic terms. Generally speaking, we will face at least the following problems when justifying the NLS approximation, say, quadratic nonlinear terms, quasilinearity, resonances and loss of regularities. A simple application of Gronwall’s inequality yields the desired result in the absence of quadratic terms [15]. If semilinear quadratic terms are
present in the original system and the eigenvalues of the linearized problem satisfy a non-resonance condition, then the approximation can be justified by the normal-form transformation, a near identity change of variables. A general approach to justify modulation equations like NLS was developed in [14], where quasilinear quadratic terms were excluded explicitly. However, since the linearized problem of many physics systems has continuous spectrum, so it is hard to avoid resonances, such as in the quantum ion-acoustic wave problem in this paper. Besides, the loss of regularities leads to another principal difficulty. If the quasilinear quadratic terms lose no more than half a derivative, the right-hand side of the normal-form transformed system will lose at most one derivative and the NLS approximation can be obtained with the help of the Cauchy-Kowalevskaya theorem [7, 22, 23]. If the quasilinear terms lose more than a half derivative, the justification will be much involved. Recently, the NLS approximation for a quasilinear Klein-Gordon equation that loses one derivative has been justified in [5] by using the normal form transformation to define a modified energy. Note that there is no resonance in such a quasilinear Klein-Gordon equation [5], i.e., the non-resonance condition is satisfied.

Very recently, Liu and Pu [19] obtained the NLS approximation for the ion Euler-Poisson equation without quantum effect, where resonance is present and the nonlinear terms lose one derivative. However, the equation in the present paper is different since the electron equilibrium is given by a Fermi-Dirac distribution, but not the Boltzmann distribution as in [19]. Besides, the higher derivative quantum terms also lead to more difficulties and complex computations. Different from [19], in the present paper we employ the modified energy method from Hunter et al [12] and Craig [2]. For the high frequency part belonging to the non-resonance region, we apply the space-time resonance method introduced in [9] but do not apply the normal-form transformation since it loses derivatives itself as in [5, 19]. For the low frequency part where the resonance appears, we construct modified energy functional by applying the normal-form transformation, but do not use it to eliminate the quasilinear quadratic terms directly as in [19]. It is worth noting that the first use of Hunter’s modified energy method to prove an approximation result of the modulational type is due to Düll [4], who studied a quasilinear wave equation without resonances. In addition, by working with the modified energy functional and some of the space-time resonance method, Cummings and Wayne [3] simplified and strengthened the results of [23]. However, the energy, especially $E_1$ (30), is differently defined in the present paper from that in [3], due to different structures and the fact that the quasilinear terms only lose half a derivative in [3], but the quasilinear terms from the quantum Euler-Poisson equation (1) lose one derivative and eventually causes the loss of two derivatives if we use the space-time resonance method. Thus it is much more difficult and interesting to consider the NLS approximation for the quantum Euler-Poisson equation (1).

We would like to mention some other modulational approximation results for various models which shares different as well as similar difficulties to the equation in the present paper, and for which various results and methods were developed in the past decades. But the list is far from being complete. Düll and Heß [6] justified the NLS approximation from a quasilinear dispersive scalar equation with non-trivial resonances. Totz and Wu [27] justified the NLS approximation for the 2D water wave problem in the special case of zero surface tension and infinite depth by finding a special transformation that allows to eliminate all quadratic terms. Totz [26] obtained the two dimensional hyperbolic NLS approximation rigorously
for the 3D water wave problem. In the context of the Korteweg-de Vries equation, the modulational approximation result can be obtained by applying a Miura transformation [21] to eliminate the dangerous quadratic terms. These results also lend some important idea to the present paper.

We end the introduction by giving the structure of this paper. In Section 2 we set up the problem and give estimates on the residual. In Section 3 we define the energy and prove Theorem 1.1. In Section 4 we look at the evolution of the energy. In particular, we separate the energy into three pieces, each of which will be dealt with separately. In Sections 5 and 6, we work with the three pieces and use some of the space-time resonance methods developed by Germain-Masmoudi-Shatah to obtain a closed energy estimation.

**Notation.** We denote the Fourier transform of a function \( u \in L^2(\mathbb{R}, K) \), with \( K = \mathbb{R} \) or \( K = \mathbb{C} \) by

\[
\hat{u}(k) = \frac{1}{2\pi} \int_{\mathbb{R}} u(x) e^{-ikx} dx, \quad k \in \mathbb{R}.
\]

Let \( H^s(\mathbb{R}, K) \) be the space of functions mapping from \( \mathbb{R} \) into \( K \) for which the norm

\[
\|u\|_{H^s(\mathbb{R}, K)} = \left( \int_{\mathbb{R}} |\hat{u}(k)|^2 (1 + |k|^2)^s dk \right)^{1/2}
\]

is finite. We also write \( L^2 \) and \( H^s \) instead of \( L^2(\mathbb{R}, \mathbb{R}) \) and \( L^2(\mathbb{R}, \mathbb{C}) \). Furthermore, define the space \( L^p(m)(\mathbb{R}, K) \) as

\[
L^p(m)(\mathbb{R}, K) = \{ u | \sigma^m u \in L^p(\mathbb{R}, K), \sigma(x) = (1 + x^2)^{1/2} \}.
\]

In addition, we write \( A \lesssim B \), if \( A \leq CB \) for a constant \( C > 0 \), and \( A = O(B) \), if \( |A| \lesssim B \).

**2. Setup and preliminaries.** An NLS type equation has been derived formally for the one dimensional motion of plasma composed of cold ions and isothermal electrons in [25]. Similarly, in the following we also obtain the NLS equation for the Euler-Poisson equation (1). In order to isolate (1) into linear, quadratic and higher order terms, we rewrite the parameter normalized Euler-Poisson system (1) with \( 0 < H \leq 0 \) as follows.

Taking \( \rho = n_i - 1, \quad \varrho = n_e - 1, \quad v = u_i \), we have

\[
\begin{aligned}
\partial_t \rho + \partial_x v + \partial_x (\rho v) &= 0, \quad (6a) \\
\partial_t v + \partial_x \rho + \partial_x \phi + v \partial_x v - \partial_x \rho^2 = -\partial_x \left[ \ln(1 + \rho) - \rho + \frac{\rho^2}{2} \right], \quad (6b) \\
\partial_x^2 \phi &= \varrho - \rho, \quad (6c) \\
\phi &= -\frac{1}{2} + \frac{1}{2} (\varrho + 1)^2 - \frac{H^2}{4(\varrho + 1)} \partial_x^2 \varrho + \frac{H^2}{8(\varrho + 1)^2} (\partial_x \varrho)^2. \quad (6d)
\end{aligned}
\]

For small \( \rho \), (6c) and (6d) define an inverse operator \( \rho \mapsto \varphi(\rho) \), which can be further expanded up to third order as

\[
\varphi(\rho) = \frac{1}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^2 \rho} + \frac{2 + \frac{H^2}{4} \partial_x^2 \rho}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4} \left( \frac{\partial_x}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^2 \rho} \right)^2 + D(\rho). \quad (7)
\]
Therefore for small $\rho$, (6d) and (7) define an inverse operator $\rho \mapsto \phi(\rho)$. Plugging $\varrho = \partial_x^2 \phi + \rho$ into (6d), we obtain

$$\phi(\rho) = \frac{1}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} - \frac{H^2}{4} \frac{\partial_x^2}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} + T(\rho, \rho) + \mathcal{M}(\rho),$$

(8)

with

$$T(\rho, \rho) = \frac{1}{2} \left( \frac{1}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \right)^2 + \frac{2 + H^2}{4} \frac{\partial_x^2}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \left( \frac{\partial_x}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \right)^2$$

$$+ \frac{H^2}{8} \left( \frac{\partial_x}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \right)^2 + \frac{2 + H^2}{4} \frac{\partial_x^2}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \left( \frac{\partial_x}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho} \right)^2,$$

(9)

where $\mathcal{D}$ and $\mathcal{M}$ are three or higher order terms for $\rho$, which do not lose derivatives and then satisfy some good properties.

We can rewrite the above system (6) as

$$\partial_t \begin{pmatrix} \rho \\ v \end{pmatrix} + \begin{pmatrix} \partial_x(1 + \frac{\partial_x^2}{1 - \partial_x^2 + \frac{H^2}{4} \partial_x^4 \rho}) \\ 0 \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\partial_x(\rho v) \end{pmatrix}$$

(10)

with $S = \begin{pmatrix} 1 & 0 \\ -q(\partial_x) & q(\partial_x) \end{pmatrix}$ and $\begin{pmatrix} \rho \\ v \end{pmatrix} = S \begin{pmatrix} U_1 \\ U_{-1} \end{pmatrix}$, where the operator $q$ satisfies $\overline{qu(x)} = \hat{q}(k)\hat{u}(k)$ for some function $u$ with $\hat{q}(k)$ defined in the equation (5). We define the operator $\Omega$ by $\Omega u(k) = i\omega(k)\hat{u}(k)$ and diagonalize the linear part of the equation (10)

$$\partial_t U_j = j\Omega U_j + Q_j(U,U) + N_j,$$

(11)

where $j \in \{1, -1\}$ and the quadratic term $Q_j$ and the high order term $N_j$ take the form

$$Q_j = -\frac{\partial_x(\rho v)}{2} - \frac{\partial_x}{4q(\partial_x)} (-|v|^2 + \rho^2 + 2T(\rho, \rho)),$$

$$N_j = j\frac{\partial_x}{2q(\partial_x)} (\ln(1 + \rho) - \rho + \frac{\rho^2}{2} + \mathcal{M}(\rho)).$$

(12)

Plugging $\rho = U_1 + U_{-1}$ and $v = -q(\partial_x)(U_1 - U_{-1})$ into $Q_j$ and $N_j$, we now compute the Fourier transform of $U_j$ as

$$\partial_t \hat{U}_j = i\omega(k)\hat{U}_j + \frac{i k}{2} \int \hat{q}(k) \ell (\hat{U}_1(k - \ell) + \hat{U}_{-1}(k - \ell))(\hat{U}_1(\ell) - \hat{U}_{-1}(\ell)) d\ell$$

$$+ \frac{i k}{4\hat{q}(k)} \int \hat{q}(k - \ell) \hat{q}(\ell) (\hat{U}_1(k - \ell) - \hat{U}_{-1}(k - \ell))(\hat{U}_1(\ell) - \hat{U}_{-1}(\ell)) d\ell$$

$$- \frac{i k}{4\hat{q}(k)} \int \hat{q}(k - \ell) \hat{q}(\ell) (\hat{U}_1(k - \ell) + \hat{U}_{-1}(k - \ell))(\hat{U}_1(\ell) + \hat{U}_{-1}(\ell)) d\ell$$

$$- \frac{i k}{2\hat{q}(k)} \int \hat{q}(k, k - \ell, \ell) (\hat{U}_1(k - \ell) + \hat{U}_{-1}(k - \ell))(\hat{U}_1(\ell) + \hat{U}_{-1}(\ell)) d\ell$$

$$+ \frac{i k}{2\hat{q}(k)} \sum_{l \geq 3} (-1)^{l+1} \frac{i}{l} (\hat{U}_1 + \hat{U}_{-1})^{*l} + \frac{i k}{2\hat{q}(k)} \mathcal{M}(\hat{U}_1 + \hat{U}_{-1})(k)$$

(13)
\[= ij\omega(k)\tilde{U}_j + \sum_{m,n,j \in \{\pm 1\}} \tilde{\alpha}_{m,n}^j(k, k - \ell, \ell)\tilde{U}_m \ast \tilde{U}_n \]
\[+ \frac{ijk}{2q(k)} \sum_{n \geq 3} (-1)^{n+1} \frac{1}{n}(\tilde{U}_1 + \tilde{U}_{-1})^n + \frac{ijk}{2q(k)} M(\tilde{U}_1 + \tilde{U}_{-1})(k), \tag{13} \]

with \( j \in \{\pm 1\}, \) where \( \tilde{\alpha}_{m,n}^j(k, k - \ell, \ell) \) is the multiplier of \( Q_j(U, U) \) for \( j, m, n \in \{\pm 1\}, \) and \( \tilde{\eta}(k, k - m, m) \) is the multiplier of the term \( T(\rho, \rho) \) from (9), thus we have

\[\tilde{\alpha}_{m,n}^j(k, k - \ell, \ell) = \frac{ik}{4} \left[ n\tilde{\eta}(\ell) + m\tilde{\eta}(k - \ell) + \frac{jmn\tilde{\eta}(k - \ell)\tilde{\eta}(\ell)}{q(k)} - \frac{j}{q(k)} - \frac{2j\tilde{\eta}(k - \ell)}{q(k)} \right], \tag{14} \]

\[\tilde{\eta}(k, k - m, m) = \frac{1}{2} \frac{1}{1 + (k - m)^2 + \frac{H^2}{4}(k - m)^4} \frac{1}{1 + m^2 + \frac{H^2}{4}m^4} \]
\[- \left( \frac{H^2}{8} + \frac{2 - H^2k^2}{1 + k^2 + \frac{H^2}{4}k^4} \right) \frac{k - m}{1 + (k - m)^2 + \frac{H^2}{4}(k - m)^4} \frac{m}{1 + m^2 + \frac{H^2}{4}m^4}, \tag{15} \]

From the form of \( \tilde{\eta}(k, k - m, m), \) we see that \( \tilde{\eta}(k, k - m, m) = O((k + m)^{-3}) \) for large enough \( k \) and \( m. \) In order to derive the NLS equation as an approximation equation for system (13), we make the ansatz

\[\begin{pmatrix} U_1 \\ U_{-1} \end{pmatrix} = \epsilon \tilde{\Psi} = \epsilon \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_{-1} \end{pmatrix} = \epsilon \tilde{\Phi}_1 + \epsilon \tilde{\Phi}_{-1} + \epsilon^2 \tilde{\Psi}_0 + \epsilon^2 \tilde{\Psi}_2 + \epsilon^2 \tilde{\Psi}_{-2}, \tag{16} \]

with

\[\epsilon \tilde{\Phi}_{\pm 1} = \epsilon \tilde{\psi}_{\pm 1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \epsilon \tilde{A}_{\pm 1}(\epsilon(x + cg_t, \epsilon^2t)E^{\pm 1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]

\[\epsilon^2 \tilde{\Psi}_0 = \begin{pmatrix} \epsilon^2 \tilde{\psi}_0 \\ \epsilon^2 \tilde{\psi}_0 \end{pmatrix} = \begin{pmatrix} \epsilon^2 \tilde{A}_{01}(\epsilon(x + cg_t, \epsilon^2t) \\ \epsilon^2 \tilde{A}_{02}(\epsilon(x + cg_t, \epsilon^2t)) \end{pmatrix}, \]

\[\epsilon^2 \tilde{\Psi}_{\pm 2} = \begin{pmatrix} \epsilon^2 \tilde{\psi}_{(\pm 2)1} \\ \epsilon^2 \tilde{\psi}_{(\pm 2)2} \end{pmatrix} = \begin{pmatrix} \epsilon^2 \tilde{A}_{(\pm 2)1}(\epsilon(x + cg_t, \epsilon^2t)E^{\pm 2} \\ \epsilon^2 \tilde{A}_{(\pm 2)2}(\epsilon(x + cg_t, \epsilon^2t)E^{\pm 2}) \end{pmatrix}, \]

where \( 0 < \epsilon \ll 1, E^{\pm j} = e^{\pm j(kax + \omega_0t)}, \omega_0 = \omega(k_0), \tilde{A}_{-j} = \tilde{A}_j \) and \( \tilde{A}_{-j}^* = \tilde{A}_j. \)

We insert our ansatz (16) into the system (13) and then replace the dispersion relation \( \omega = \omega(k) \) in all terms of the form \( \omega \tilde{A}_j E^j \) or \( \omega \tilde{A}_j E^j \) by their Taylor expansions around \( k = jk_0. \) By equating the coefficients of the \( \epsilon^m E^j \) to zero, we will obtain the NLS equation

\[\partial_T \tilde{A}_1 = i \frac{\partial^2 \omega(k_0)}{2} \partial_X^2 \tilde{A}_1 + i\nu_2(k_0)\tilde{A}_1|\tilde{A}_1|^2, \tag{17} \]

with some \( \nu_2(k_0) \in \mathbb{R}. \)

Now if we define the error \( R \) by

\[U = e\tilde{\Psi} + \epsilon^8 R, \]
and insert this approximation into (13), then we obtain the following equation for $R$

$$\partial_t R_j = j\Omega R_j + 2\epsilon Q_j(\tilde{\Psi}, R) + \epsilon^2 N_j(R) + \epsilon^{-\beta} \text{Res}_j(\epsilon^{\tilde{\Psi}}),$$  \hfill (18)

where the higher order terms and the residual term are defined respectively as follows

$$N_j(R) = N_j(U) - N_j(\tilde{\Psi}),$$

$$\text{Res}(\epsilon^{\tilde{\Psi}}) = -\epsilon\partial_t \tilde{\Psi} + \epsilon\tilde{\Omega} \tilde{\Psi} + \epsilon^2 Q(\tilde{\Psi}, \tilde{\Psi}) + \epsilon^3 N(\tilde{\Psi}).$$

Note that the linear operator $\Omega$ generates a uniformly bounded semigroup, thus the term $j\Omega R_j$ will not cause essential difficulties. And if we focus on the evolution of $||R||_{H^2}$, the terms $\epsilon^2 N_j(R)$ and $\epsilon^3 Q_j(R, R)$ of (18) are of at least $O(\epsilon^2)$ in terms of $\epsilon$ by choosing $3 \leq \beta \leq 9/2$. Besides, we can modify the approximation $\tilde{\Psi}$ to make the residual term $\epsilon^{-\beta} \text{Res}_j(\epsilon^{\tilde{\Psi}})$ to be arbitrarily small in the following.

To show that $R$ is small for times of $O(\epsilon^{-2})$ for the equation (18), however, the quasilinear term $2\epsilon Q_j(\tilde{\Psi}, R)$ is the biggest obstacle. On one hand, the quasilinearity loses one derivative, which will prevent the energy estimates from closing. On the other hand, the term $2\epsilon Q_j(\tilde{\Psi}, R)$ of (18) is only $O(\epsilon)$ and a direct application of Gronwall’s inequality will only control growth of energy for times of $O(\epsilon^{-1})$. In the present paper, we will overcome this difficulty by combining the method of space-time resonances with the method of modifying energy, both requiring us to avoid two resonances, one at $k = 0$ and one at $k = \pm k_0$. For the resonance at $k = 0$, we can deal with it by using the fact that the Fourier coefficients of the nonlinearity equal to zero at $k = 0$. Such property is a so-called transparency condition and this resonance is also referred to be trivial in some contexts. For the resonances at $k = \pm k_0$, we will use a weight function $\vartheta$ which was first introduced in [23], and the fact that the Fourier coefficients of the nonlinearity vanishes near $k = 0$.

Now we begin to modify the approximation $\epsilon^{\tilde{\Psi}}$ to make $\epsilon^{-\beta} \text{Res}_j(\epsilon^{\tilde{\Psi}})$ to be arbitrarily small. Similar to [7], we extend $\tilde{\Psi}$ to be higher order terms $\Psi$ and then make the support of the modified approximation $\Psi$ be restricted to small neighborhoods of a finite number of integer multiples of the basic wave number $k_0 > 0$ by some cut-off function.

More precisely, we define $A$ such that

$$A_{\pm 1} : \hat{A}_{\pm 1}(k) = \hat{A}_{\pm 1}(k) \text{ for } \{k \in \mathbb{R} \mid |k \pm k_0| \leq \delta\}; \hat{A}_{\pm 1}(k) = 0 \text{ otherwise},$$

$$A_{0} : \hat{A}_{0}(k) = \hat{A}_{0}(k) \text{ for } \{k \in \mathbb{R} \mid |k| \leq \delta\}; \hat{A}_{0}(k) = 0 \text{ otherwise},$$

$$A_{\pm 2l} : \hat{A}_{\pm 2l}(k) = \hat{A}_{\pm 2l}(k) \text{ for } \{k \in \mathbb{R} \mid |k \pm 2k_0| \leq \delta\}; \hat{A}_{\pm 2l}(k) = 0 \text{ otherwise},$$

$$A_{\pm j}^n : \hat{A}_{\pm j}^n(k) = \hat{A}_{\pm j}^n(k) \text{ for } \{k \in \mathbb{R} \mid |k \pm jk_0| \leq \delta\}; \hat{A}_{\pm j}^n(k) = 0 \text{ otherwise},$$

where $|j| + n \leq 4$, $l = 1, 2$ and $\delta > 0$ is a constant independent of $0 < \epsilon \ll 1$. Then we can proceed to replace $\epsilon^{\tilde{\Psi}}$ by a new approximation $\epsilon^{\Psi}$ of the form

$$\epsilon^{\Psi} = \epsilon^{\Phi_1} + \epsilon^{\Phi_{-1}} + \epsilon^2 \Psi_p,$$  \hfill (20)

where

$$\epsilon^{\Phi_{\pm 1}} = \epsilon^{\psi_{\pm 1}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \epsilon A_{\pm 1}(\epsilon (x + c_\gamma t), \epsilon^2 t) E_{\pm 1} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
\( e^2 \Psi_p = \left( \frac{\mathcal{E}^2 \psi_{p_1}}{\mathcal{E}^2 \psi_{p_{-1}}} \right) = e^2 \Psi_0 + e^2 \Psi_2 + e^2 \Psi_0 + e^2 \Psi_h, \)

\( e^2 \Psi_0 = \left( \frac{\mathcal{E}^2 \psi_{01}}{\mathcal{E}^2 \psi_{02}} \right) = e^2 \mathcal{A}_{01}(\epsilon(x + c_\gamma t), e^2 t), \)

\( e^2 \Psi = \left( \frac{\mathcal{E}^2 \psi_{(\pm)21}}{\mathcal{E}^2 \psi_{(\pm)22}} \right) = e^2 \mathcal{A}_{(\pm)21}(\epsilon(x + c_\gamma t), e^2 t) E^{\pm 2} \)

\( e^2 \Psi_h = \sum_{j=\pm 1, n=1, 2, 3} \left( \epsilon^{1+n} \mathcal{A}^n_{j1}(\epsilon(x + c_\gamma t), e^2 t) E^j \right) \)

\( + \sum_{j=\pm 2, n=1, 2} \left( \epsilon^{2+n} \mathcal{A}^n_{j2}(\epsilon(x + c_\gamma t), e^2 t) E^j \right) \)

\( + \sum_{n=1, 2} \left( \epsilon^{2+n} \mathcal{A}^n_{11}(\epsilon(x + c_\gamma t), e^2 t) \right) \)

\( + \sum_{j=\pm 3, n=0, 1} \left( \epsilon^{3+n} \mathcal{A}^n_{j2}(\epsilon(x + c_\gamma t), e^2 t) E^j \right) \)

\( + \sum_{j=\pm 4} \left( \epsilon^{4+n} \mathcal{A}^n_{j2}(\epsilon(x + c_\gamma t), e^2 t) E^j \right), \)

where \( A_j = \mathcal{A}_j \) and \( A_{j\ell} = \mathcal{A}_{j\ell} \) have compact support in Fourier space for all \( 0 < \epsilon \ll 1 \). Although we add higher order terms to our approximation, the lowest order terms remain those from \( \Psi_{NLS} \) such that we can write

\[ \hat{\Psi}(k) = \hat{\Psi}^c(k) + e\hat{\Psi}^s(k), \]

with

\[ \hat{\Psi}^c(k) = \left( \frac{\hat{\Psi}^c(k)}{\hat{\Psi}^c_{-1}(k)} \right) = \hat{\Phi}_1 + \hat{\Phi}_{-1} = \left( \begin{array}{cc} \hat{\psi}_1 & 0 \\ 0 & \hat{\psi}_{-1} \end{array} \right) = : \left( \begin{array}{cc} 0 \\ \phi_c \end{array} \right), \]

(21)

and

\[ \text{supp}(\hat{\Psi}^c) \subset \{ k \mid |k \pm k_0| < \delta \}, \]

due to the fact we cut the approximation off in Fourier space.

Then, by such a modification, exactly as in Sect. 2 of [7], the approximation will not change too much, but will lead to a simpler control of the error and make the approximation an analytic function, as shown in the following.

**Lemma 2.1.** For all \( s_A \geq 6 \), let \( \bar{A}_1 \in C([0, T_0], H^{s_A}(\mathbb{R}, \mathbb{C})) \) be a solution of the NLS equation (17) with

\[ \sup_{T \in [0, T_0]} \| \bar{A}_1 \|_{H^{s_A}} \leq C_A. \]

Then for all \( s > 0 \), there exist \( C_{Res}, C_{\Psi}, \epsilon_0 > 0 \) depending on \( C_A \) such that the following holds for all \( \epsilon \in (0, \epsilon_0) \). The approximation \( e^2 \Psi \) exists for all \( t \in [0, T_0/e^2] \)
such that

\[
\begin{align*}
\sup_{t \in [0, T_0/\epsilon^2]} \left\| \text{Res}_U(\epsilon \Psi) \right\|_{H^s} &\leq C_{\text{Res}} \epsilon^{9/2}, \quad (22a) \\
\sup_{t \in [0, T_0/\epsilon^2]} \left\| \epsilon \Psi - \epsilon (\hat{\Phi}_1 + \Phi_{-1}) \right\|_{H^{s-1}} &\leq C_{\Psi} \epsilon^{3/2}, \quad (22b) \\
\sup_{t \in [0, T_0/\epsilon^2]} \left\| \hat{\Psi}^{\epsilon,s} \right\|_{L^1(s+1)(\mathbb{R}, \mathbb{C})} &\leq C_{\Psi}. \quad (22c)
\end{align*}
\]

The proof of Lemma 2.1 is analogous to that of Lemma 2.6 in [7] (see also [5]) by noting the estimate

\[
\left\| (\chi_{[-\delta, \delta]} - 1) \epsilon^{-1} \hat{f}(\epsilon^{-1} \cdot) \right\|_{L^2(m)} \leq C \epsilon^{m+M-1/2} \| f \|_{H^{m+M}},
\]

for all \( m, M \geq 0 \), where \( \chi_{[-\delta, \delta]} \) is the characteristic function on \([ -\delta, \delta] \). The bound (22c) will be used to estimate

\[
\| \psi_j f \|_{H^s} \leq C \| \psi_j \|_{C^k} \| f \|_{H^s} \leq C \| \psi_j \|_{L^1(s)(\mathbb{R}, \mathbb{C})} \| f \|_{H^s},
\]

without loss of powers in \( \epsilon \) as it would be the case with \( \| \psi_j \|_{L^2(s)(\mathbb{R}, \mathbb{C})} \). Moreover, by an analogous argument as in the proof of Lemma 3.3 in [7], we have the following lemma.

**Lemma 2.2.** For all \( s > 0 \) there exists a constant \( C_{\Psi} > 0 \) such that

\[
\left\| \partial_t \hat{\psi}_{\pm 1} - i \omega \hat{\psi}_{\pm 1} \right\|_{L^1(s)} \leq C_{\Psi} \epsilon^2. \quad (23)
\]

Now we need to construct a new approximation for \( U \) by applying the modified ansatz \( \hat{\Psi} \). Let’s first define a weight function \( \vartheta \) to reflect the fact that the nonlinearity vanishes at \( k = 0 \),

\[
\hat{\vartheta}(k) = \begin{cases} 
1 & \text{for } |k| > \delta, \\
\frac{1}{\epsilon + (1 - \epsilon)|k|/\delta} & \text{for } |k| \leq \delta,
\end{cases} \quad (24)
\]

for some \( \delta > 0 \) above sufficiently small, but independent of \( 0 < \epsilon \ll 1 \). Then write the solution \( U \) of (13) as a sum of the approximation and error, i.e.,

\[
U = \epsilon \Psi + \epsilon^2 \vartheta R, \quad (25)
\]

where \( \vartheta R \) is defined by \( \hat{\vartheta} R = \hat{\vartheta} R \) to avoid writing the convolution \( \vartheta * R \), in a slight abuse of notation. Note that \( \hat{\vartheta}(k) R(k) \) is small at the wave numbers close to zero, since the nonlinearity vanishes at \( k = 0 \).

In essence, defining the weight function \( \vartheta \) is to deal with the nontrivial resonance \( \ell = 0 \), for details we refer to Section 6. But when we make such an approximation of (25), the weight function \( \vartheta \) will induce some troubles in the region of \( |k| < \delta \), which is the reason why we define the modified energy \( E_2 \) in Section 3, and further reflected in Section 6. Inserting this approximation (25) into (13) we obtain the following equation for \( \hat{R} \)

\[
\begin{align*}
\partial_t \hat{R} = &\Lambda R + 2 \epsilon \vartheta^{-1} Q(\Psi, \vartheta R) + 2 \epsilon^2 \vartheta^{-1} Q(\Psi^*, \vartheta R) + \epsilon^2 \vartheta^{-1} R(\vartheta R, \vartheta R) \\
&+ \epsilon^2 \vartheta^{-1} N(\vartheta R) + \epsilon^{-\beta} \vartheta^{-1} \text{Res}(\epsilon \Psi), \quad (26)
\end{align*}
\]

with

\[
\Lambda = \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega \end{pmatrix}, \quad (27)
\]

and \( \hat{\Omega} \hat{u}(k) = i \omega(k) \hat{u}(k) \).
By recalling (12), (14), (15) and (21) we have

$$Q_j(\hat{\Psi}^c, \hat{R})(k) = \sum_{m, n \in \{\pm 1\}} \int_{\mathbb{R}} \hat{\alpha}^j_{m,n}(k, k - \ell, \ell) \hat{\phi}(\ell) \hat{\Psi}^c_m(k - \ell, \ell) \hat{R}_n(\ell) d\ell$$

$$= \sum_{n \in \{\pm 1\}} \int_{\mathbb{R}} \hat{\alpha}^j_{-1,n}(k, k - \ell, \ell) \hat{\phi}(\ell) \hat{\phi}_n(k - \ell) \hat{R}_n(\ell) d\ell$$

$$= \frac{ik}{4} \int_{\mathbb{R}} \left[ n\hat{q}(\ell) - \hat{q}(k - \ell) - jn\hat{q}(k - \ell)\hat{q}(\ell) - j\frac{\hat{q}(k) - \hat{q}(k - \ell)}{\hat{q}(k)} \right] \times \hat{\phi}(\ell) \hat{\phi}_n(k - \ell) \hat{R}_n(\ell) d\ell. \quad (28)$$

3. **Definition of the energy and proof of theorem 1.1.** Define the energy $E$ as follows,

$$E = E_0 + E_1 + E_2,$$

with

$$E_0 = \frac{1}{2} \sum_{j \in \{\pm 1\}} \int_{\mathbb{R}} \left( R_j^2 + (\partial_x^2 R_j)^2 \right) dx,$$

$$E_1 = \frac{1}{2} \sum_{j \in \{\pm 1\}} \int_{\mathbb{R}} \left( -c_j q \phi_c(\partial_x^2 R_j)^2 - c \phi_c \partial_x^2 R_j \partial_x^2 R_{-j} \right) dx,$$

$$E_2 = \frac{1}{2} \sum_{j \in \{\pm 1\}} \int_{|k| < \delta} c \hat{R}_j B_j(\hat{\Psi}^c, \hat{R}) + c \hat{R}_j B_j(\hat{\Psi}^c, \hat{R}) + 2c^2 \hat{R}_j(\hat{\Psi}^c, \hat{R}) \hat{R}_j(\hat{\Psi}^c, \hat{R}) dk.$$

Here

$$B_j(\hat{\Psi}^c, \hat{R}) = \sum_{m, n \in \{\pm 1\}} \int_{\mathbb{R}} \frac{\hat{\alpha}^j_{m,n}(k, k - \ell, \ell) \hat{\phi}(\ell)}{\hat{\phi}_n(k - \ell)} \hat{\Psi}^c_m(k - \ell, \ell) \hat{R}_n(\ell) d\ell,$$

where

$$\Theta^j_{m,n}(k, \ell) = -j \omega(k) + m\omega(k - \ell) + n\omega(\ell), \quad (29)$$

with $j, m, n \in \{\pm 1\}$, recalling the equation (21) for $\Psi^c$ and $\phi_c$. Although the energy $E$ is similar to that from [3] formally, they are indeed different. The error variable $R$ in [3] in the energy $E_0$ and $E_1$ comes from the original first order system which is antisymmetric, and the nonlinear quadratic term does not involve crossing terms in the original error system. However, the quantum Euler-Poisson system (1) does not satisfy these features, thus we cannot take the same form of the energy $E$, especially the second term of $E_1$. The terms in $E_0$ are equivalent to the $H^2$ norm. The terms in $E_1$ are chosen to cancel the terms with the highest order derivatives from the $H^2$ norm. In fact, the bilinear transformation $B$ in $E_2$ comes from the normal-form transformation. The terms in $E_2$ are chosen to cancel quadratic terms in a very small region $|k| < \delta$. For other terms in that region as well as those outside, we will use the method of space-time resonances.

By applying Lemma 2.1, for any $p > 2$, we have $\|A\|_{L^p} < C$ for $t \in [0, T_0]$ and $\epsilon$ small enough, and there exists a constant $C_1$ such that

$$E_1 = -\frac{c}{2} \sum_{j \in \{\pm 1\}} \int_{\mathbb{R}} \left( j q \phi_c(\partial_x^2 R_j)^2 + \phi_c \partial_x^2 R_j \partial_x^2 R_{-j} \right) dx$$

$$\lesssim \epsilon \|\Psi\|_{L^\infty} \|R\|_{H^2}^2 \leq \epsilon C_1 \|R\|_{H^2}^2. \quad (30)$$
Lemma 3.1. For any $p > 2$ and $\delta > 0$ small enough, there exist constants $C_2 > 0$ and $\gamma > 0$ such that

$$E_2 = \frac{1}{2} \sum_{j \in \{\pm 1\}} \int_{|k| \leq \delta} \epsilon \hat{R}_j \hat{B}_j(\hat{\Psi}^e, \hat{R}) + \epsilon \hat{R}_j \hat{B}_j(\hat{\Psi}^e, \hat{R}) + 2\epsilon^2 \hat{B}_j(\hat{\Psi}^e, \hat{R}) \hat{B}_j(\hat{\Psi}^e, \hat{R}) dk$$

$$\lesssim \delta^\gamma (\|\hat{A}\|_{L^p} + \|\hat{A}\|_{L^p}^2) \|R\|_{L^2}^2 \leq C_2 \delta^\gamma \|R\|_{L^2}^2. \quad (31)$$

One can refer to [3, Prop. A.7] for the proof of Lemma 3.1. Combining the (30) with (31), one has

$$(1 - C_1 \epsilon - C_2 \delta^\gamma) E \leq \|R\|_{H^2}^2 \leq (1 + C_1 \epsilon + C_2 \delta^\gamma) E. \quad (32)$$

Therefore the energy $E$ is equivalent to $\|R\|_{H^2}^2$.

The goal of the rest of the paper is to show that

$$\partial_t E \lesssim \epsilon^2 (1 + E) + \epsilon^3 E^2 + \epsilon^4 E^3. \quad (33)$$

We then integrate to have

$$E(t) \leq C \left( E(0) + \int_0^t \epsilon^2 (1 + E(s)) + \epsilon^3 E(s)^2 + \epsilon^4 E(s)^3 ds \right),$$

and then for all $t$ such that $\epsilon E(t) + \epsilon^2 E(t)^2 \leq 1$, one has

$$E(t) \leq C E(0) + C \int_0^t (1 + 2E(s)) ds$$

$$= C(E(0) + \epsilon^2 t) + 2C \epsilon^2 \int_0^t E(s) ds.$$

An application of Gronwall’ inequality then gives us

$$E(t) \leq C (E(0) + \epsilon^2 t)e^{2C\epsilon^2 t}.$$

For $t = T_0/\epsilon^2$ we have

$$E(T_0/\epsilon^2) \leq C (E(0) + T_0)e^{2CT_0}.$$

Therefore, there exist constant $\tilde{C}$ such that $\sup_{t \in [0, T_0/\epsilon^2]} E(t) \leq \tilde{C}$. Then according to the inequality (32) for the equivalence between $E(t)$ with $\|R\|_{H^2}^2$, there exist constant $C$ such that

$$\sup_{t \in [0, T_0/\epsilon^2]} \|R\|_{H^2} \leq C,$$

independent of $\epsilon$ as desired. Theorem 1.1 is proved.

4. Evolution of the energy $E$. To prove (33), we calculate the evolution of the energy. For simplicity, we ignore those terms that involve the residual term $\text{Res}(\epsilon \Psi)$ and terms of order $\mathcal{O}(\epsilon^3)$ or higher in $\partial_t E_2$, which can be estimated directly. By computation,

$$\partial_t E_0 = \sum_{j \in \{\pm 1\}} \int R_j(j \Omega R_j) + \partial_x^2 R_j(j \Omega \partial_x^2 R_j) dx$$

$$+ 2\epsilon \sum_{j \in \{\pm 1\}} \int R_j \partial^{-1} Q_j(\Psi^e, \partial R) + \partial_x^2 R_j \partial^{-1} \partial_x^2 Q_j(\Psi^e, \partial R) dx.$$
\[ + \epsilon^2 \sum_{j \in \{\pm 1\}} \int \frac{1}{2} j(q\phi_c)(\partial^2_x R_j)^2 - \frac{1}{2} (\Omega\phi_c)\partial^2_x R_j \partial^2_x R_{-j} dx \]

\[ - \epsilon \sum_{j \in \{\pm 1\}} \int j(q\phi_c)\partial^2_x R_j j\Omega\partial^2_x R_j + \phi_c(j\Omega\partial^2_x R_j)\partial^2_x R_{-j} dx \]

\[ + \epsilon \sum_{j \in \{\pm 1\}} \int -\frac{1}{2} j q (\partial_t \phi_c - \Omega \phi_c)(\partial^2_x R_j)^2 - (\partial_t \phi_c - \Omega \phi_c)\partial^2_x R_j \partial^2_x R_{-j} dx \]

\[ + \epsilon^2 \sum_{j \in \{\pm 1\}} \int -j q \phi_c \partial^2_x R_j \left\{ 2\partial^{-1} \partial^2_x Q_j(\Psi, \partial R) + \epsilon^\beta - 1 \partial^3_x Q_j(\partial R, \partial R) \right\} dx, \]

and

\[ \partial_t E_2 = \frac{\epsilon}{2} \sum_{j \in \{\pm 1\}} \int \frac{ij\omega(k)}{|k| < \delta} \overline{B_j(\hat{\Psi}^c, \hat{R}) + \frac{1}{R_j} B_j(\Lambda \hat{\Psi}^c, \hat{R})} \]

\[ + \frac{1}{R_j} B_j(\partial_t \hat{\Psi}^c - \Lambda \hat{\Psi}^c, \hat{R}) \]

\[ + \epsilon^2 \sum_{j \in \{\pm 1\}} \int \frac{1}{|k| < \delta} \overline{\partial^{-1} k Q_j(\hat{\Psi}^c, \partial \hat{R}) B_j(\hat{\Psi}^c, \hat{R}) + B_j(\hat{\Psi}^c, \Lambda \hat{R}) B_j(\hat{\Psi}^c, \hat{R})} \]

\[ + \frac{1}{R_j} B_j(\hat{\Psi}^c, \partial^{-1} Q_j(\hat{\Psi}^c, \partial \hat{R})) \]

\[ + 2\epsilon^3 \sum_{j \in \{\pm 1\}} \int \frac{1}{|k| < \delta} B_j(\hat{\Psi}^c, (\partial^{-1} Q_j(\hat{\Psi}^c, \partial \hat{R}))) B_j(\hat{\Psi}^c, \hat{R}) \] + c.c.. \]

It seems that the terms with the coefficients of \( \epsilon^2 \) and \( \epsilon^3 \) in the expression for \( \partial_t E_2 \) can be bounded directly by \( O(\epsilon^2) \). However, these terms are only of \( O(\epsilon) \) because \( \partial^{-1}(k) = O(\epsilon^{-1}) \) for \( |k| < \delta \). Thus we must keep and analyze them carefully.

Note that the \( O(1) \) terms from \( \partial_t E_0 \) cancel due to the antisymmetry of \( \Omega \). By Lemma 2.2, the third lines from \( \partial_t E_1 \) and \( \partial_t E_0 \) all can be bounded by \( \epsilon^2(1 + E) + \epsilon^3 E^2 \). For the other terms of order \( O(\epsilon^2) \) or higher from \( \partial_t E_0 \) and \( \partial_t E_1 \), we have two derivatives falling on each power of \( R \). However, since both \( Q_j \) and \( N_j \) lose one regularity, there will be three derivatives on one of the factors of \( R \), which is different from [3] since there the quasilinear terms only lose 1/2 derivative for the quasilinear water wave system. To solve this difficulty, we make full use of the equation (26) and some other useful inequalities.

For convenience, we show the following lemmas
Lemma 4.1. Let \( j \in \{ \pm 1 \}, a_j \in H^2(\mathbb{R}, \mathbb{R}), \) and \( f_j \in H^1(\mathbb{R}, \mathbb{R}). \) Then we have
\[
\int_{\mathbb{R}} a_j f_j \partial_x f_j dx = -\frac{1}{2} \int_{\mathbb{R}} \partial_x a_j f_j^2 dx, \quad (34)
\]
\[
\sum_{j \in \{ \pm 1 \}} \int_{\mathbb{R}} a_j f_j \partial_x f_{-j} dx = \frac{1}{2} \int_{\mathbb{R}} (a_{-1} - a_1) (f_1 + f_{-1}) \partial_x (f_1 - f_{-1}) dx + \mathcal{O}(\| a_1 \|_{H^2(\mathbb{R}, \mathbb{R})} + \| a_{-1} \|_{H^2(\mathbb{R}, \mathbb{R})} (\| f_1 \|_{L^2(\mathbb{R}, \mathbb{R})}^2 + \| f_{-1} \|_{L^2(\mathbb{R}, \mathbb{R})}^2)), \quad (35)
\]
\[
\sum_{j \in \{ \pm 1 \}} j \int_{\mathbb{R}} a_j f_j \partial_x f_{-j} dx = \frac{1}{2} \int_{\mathbb{R}} (a_1 + a_{-1}) (f_1 + f_{-1}) \partial_x (f_1 - f_{-1}) dx + \mathcal{O}(\| a_1 \|_{H^2(\mathbb{R}, \mathbb{R})} + \| a_{-1} \|_{H^2(\mathbb{R}, \mathbb{R})} (\| f_1 \|_{L^2(\mathbb{R}, \mathbb{R})}^2 + \| f_{-1} \|_{L^2(\mathbb{R}, \mathbb{R})}^2)). \quad (36)
\]

One can refer to [5] for the proof of the above lemma.

Lemma 4.2 (Commutator Estimate). Let \( m \geq 1 \) be an integer, and then the commutator which is defined by the following
\[
[\nabla^m, f]g := \nabla^m (fg) - f \nabla^m g, \quad (37)
\]
can be bounded by
\[
\| [\nabla^m, f]g \|_{L^p} \leq \| \nabla f \|_{L^{p_1}} \| \nabla^{m-1} g \|_{L^{p_2}} + \| \nabla^m f \|_{L^{p_3}} \| g \|_{L^{p_4}}, \quad (38)
\]
where \( p, p_2, p_3 \in (1, \infty) \) and
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]

Proof. The proof can be found in [1], for example. \qed

The first step is to estimate \( \mathcal{O}(\epsilon^2) \) and higher terms from \( \partial_t E_0 \) and \( \partial_t E_1. \) When \( |k| < \delta \) or \( |\ell| < \delta, \) we can bound these \( \mathcal{O}(\epsilon^2) \) and higher terms from \( \partial_t E_0 \) and \( \partial_t E_1 \) directly by Young’s inequality and Cauchy-Schwartz inequality. At first glance it would seem we should have lost a power of \( \epsilon \) due to the fact that \( \tilde{\partial}^{-1}(k) \sim \mathcal{O}(\epsilon^{-1}) \) for \( k \approx 0; \) however, each term with a \( \tilde{\partial}^{-1} \) also has a term of the form \( ik \) in Fourier space coming from \( Q_j \) and \( N_j. \) This vanishes at \( k = 0 \) and we therefore do not lose a power of \( \epsilon. \) On the other hand, when \( |k| \geq \delta \) and \( |\ell| \geq \delta, \) we just need to estimate these terms by using \( \tilde{\partial}(k) = \tilde{\partial}(\ell) = 1 \) according to the definition (24) of \( \tilde{\partial}. \) We take the following term from \( \partial_t E_0 \) as an example,
\[
2 \epsilon^2 \sum_{j \in \{ \pm 1 \}} \int \partial_x^2 R_j \partial^{-1} \partial_x^2 Q_j (\Psi^+, \vartheta R) dx
\]
\[
= 2 \epsilon^2 \sum_{j, m, n \in \{ \pm 1 \}} \int k^4 \alpha^j_{mn}(k, k - \ell, \ell) \tilde{R}_j(k) \tilde{\Psi}^n_m(k - \ell) \tilde{R}_m(\ell) d\ell dk.
\]
Recall (14),
\[
\alpha^j_{mn}(k, k - \ell, \ell) = \frac{ik}{4} \left( n\tilde{q}(\ell) + m\tilde{q}(k - \ell) + \frac{jmn\tilde{q}(k - \ell)\tilde{q}(\ell)}{\tilde{q}(k)} \right) - \frac{j}{\tilde{q}(k)} + \frac{2j\tilde{q}(k, k - \ell, \ell)}{\tilde{q}(k)}.
\]
We only take one factor $ikn\hat{q}(\ell)/4$ of $\alpha'_{m}$ to finish the estimate. The others are similar.

\[
\frac{\epsilon^2}{2} \sum_{j,m,n \in \{\pm 1\}} \int \int ik^n \hat{q}(\ell) R_j(k) \overline{\hat{\Psi}_m^\alpha(k - \ell)} \tilde{R}_n(\ell) d\ell dk = \frac{\epsilon^2}{2} \sum_{j,m,n \in \{\pm 1\}} \int n \partial_x^2 R_j \partial_x^3 (\hat{\Psi}_m^\alpha q R_n) dx
\]

\[
= \frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j (\Psi_j^\alpha + \Psi_{-j}^\alpha) \partial_x^3 (q R_j - q R_{-j}) dx
\]

\[
+ \frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j [\partial_x^3 (\Psi_j^\alpha + \Psi_{-j}^\alpha)] (q R_j - q R_{-j}) dx
\]

\[
= \frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j (\Psi_j^\alpha + \Psi_{-j}^\alpha) \partial_x^3 (q - 1)(R_j - R_{-j}) dx
\]

\[
+ \frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j (\Psi_j^\alpha + \Psi_{-j}^\alpha) \partial_x^3 (R_j - R_{-j}) dx
\]

\[
+ \frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j [\partial_x^3 (\Psi_j^\alpha + \Psi_{-j}^\alpha)] (q R_j - q R_{-j}) dx
\]

By applying commutator estimate from Lemma 4.2, Cauchy inequality, Hölder’s inequality and Sobolev embedding, the last two terms can be bounded by $\epsilon^2 E$. Recalling the form of the dispersive relation (5), we have

\[
\hat{q}(k) = \sqrt{1 + 1 + \frac{\mu^2 k^2}{1 + k^2} + \frac{\mu^2 k^2}{4 k^4}} = \mathcal{O}(1), \quad |k| \to \infty, \quad (39)
\]

\[
\hat{q}(k) - 1 = \frac{\hat{q}(k)^2 - 1}{\hat{q}(k) + 1} = \frac{1 + \frac{\mu^2 k^2}{1 + k^2 + \frac{\mu^2 k^4}}}{\hat{q}(k) + 1} = \mathcal{O}(k^{-2}), \quad |k| \to \infty, \quad (40)
\]

Thus the first term can be controlled by $\epsilon^2 E$. Furthermore, by using Lemma 4.1, we have

\[
\frac{\epsilon^2}{2} \sum_{j \in \{\pm 1\}} \int j \partial_x^2 R_j (\Psi_j^\alpha + \Psi_{-j}^\alpha) \partial_x^3 R_{-j} dx
\]
Recall $U_j = \epsilon \Psi_j + \epsilon^\beta \partial R_j$ for $j \in \{\pm 1\}$, where we take $\gamma = 1$. Thus we have
\[
\partial_t(R_1 + R_{-1}) = \Omega(R_1 - R_{-1}) + \epsilon \partial_x \left[ (\Psi_1 + \Psi_{-1}) q(R_1 - R_{-1}) \right] \\
+ \epsilon \partial_x \left[ q(\Psi_1 - \Psi_{-1})(R_1 + R_{-1}) \right] \\
+ \epsilon^\beta \partial_x \left[ (R_1 + R_{-1}) q(R_1 - R_{-1}) \right] \\
+ \epsilon^{\beta - 1} \text{Res}_1(\epsilon \Psi) + \epsilon^{-\beta} \text{Res}_{-1}(\epsilon \Psi).
\]
From (41), there exist some constant $C$ such that the following estimate for $\partial_t(R_1 + R_{-1})$ hold,
\[
\|\partial_t(R_1 + R_{-1})\|^2 = \mathcal{O}(\|R\|_{H^1}^2) + \epsilon^2 \mathcal{O}(1).
\] 
Taking $\partial_x^2$ on the equation (41), we have
\[
\partial_x^2(R_1 - R_{-1}) = \frac{1}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}))} \left[ \partial_x^2 \partial_t(R_1 + R_{-1}) \\
- (eq(\Psi_1 - \Psi_{-1}) + \epsilon^\beta q(R_1 - R_{-1})) \partial_x^2(R_1 + R_{-1}) \\
- \epsilon(\partial_x^3, \Psi_1 + \Psi_{-1}) [q(R_1 - R_{-1}) - \epsilon(\partial_x^3, q(\Psi_1 - \Psi_{-1}))(R_1 + R_{-1})] \\
- \epsilon^{\beta - 2} \sum_{i=0}^2 C_i^2 \partial_x^2(R_1 + R_{-1}) \partial_x^2 \partial_t(R_1 - R_{-1}) \\
- \epsilon^{-\beta} \partial_x^2(\text{Res}_1(\epsilon \Psi) + \text{Res}_{-1}(\epsilon \Psi)) \right].
\] 
Note that
\[
1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}) = 1 + U_1 + U_{-1} = 1 + \rho = n_i,
\]
with $n_i$ bounded. Then by (43), we have
\[
- \frac{\epsilon^2}{2} \int (\Psi_1^s + \Psi_{-1}^s) \cdot \partial_x^2(R_1 + R_{-1}) \cdot \partial_x^3(R_1 - R_{-1}) dx \\
= - \frac{\epsilon^2}{2} \int \frac{\Psi_1^s + \Psi_{-1}^s}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}))} \times \partial_x^2(R_1 + R_{-1}) \partial_x^3(R_1 + R_{-1}) dx \\
+ \frac{\epsilon^2}{2} \int \frac{(\Psi_1^s + \Psi_{-1}^s)(eq(\Psi_1 - \Psi_{-1}) + \epsilon^\beta q(R_1 - R_{-1}))}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}))} \times \partial_x^2(R_1 + R_{-1}) \partial_x^3(R_1 + R_{-1}) dx \\
+ \frac{\epsilon^3}{2} \int \frac{\Psi_1^s + \Psi_{-1}^s}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}))} \partial_x^2(R_1 + R_{-1}) \times \partial_x^3(\Psi_1 + \Psi_{-1})(q(R_1 - R_{-1})) dx \\
+ \frac{\epsilon^3}{2} \int \frac{\Psi_1^s + \Psi_{-1}^s}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^\beta (R_1 + R_{-1}))} \partial_x^2(R_1 + R_{-1}) \times [\partial_x^3, q(\Psi_1 - \Psi_{-1})](R_1 + R_{-1}) dx
\]
\[
+ \frac{\epsilon^{\beta+2}}{2} \int \frac{\Psi_1^+ + \Psi_{-1}^-}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^3(R_1 + R_{-1})))} \partial_x^2(R_1 + R_{-1}) \] 
\times \sum_{i=0}^{2} C_i^2 \partial_x^2(R_1 + R_{-1}) \partial_x^{-i} q(R_1 - R_{-1}) dx 
\] 
\[+ \frac{\epsilon^{2-\beta}}{2} \int \frac{\Psi_1^+ + \Psi_{-1}^-}{q(1 + \epsilon(\Psi_1 + \Psi_{-1}) + \epsilon^3(R_1 + R_{-1})))} \partial_x^2(R_1 + R_{-1}) \] 
\times \partial_x^2(\text{Re}s_1(\epsilon \Psi) + \text{Re}s_{-1}(\epsilon \Psi)) dx 
\] 
\[= \frac{\epsilon^2}{2} \frac{d}{dt} \int \frac{\Psi_1^+ + \Psi_{-1}^-}{q(1 + \rho)}(\partial_x^2 R_1 + \partial_x^2 R_{-1})^2 dx + \epsilon^2 O(1 + E + \epsilon E^2 + \epsilon^2 E^3), \]

where we have used commutator estimate from Lemma 4.2, Cauchy inequality, H"{o}lder’s inequality, Sobolev embedding, the estimate (42) and Lemma 2.2. Focus on the first term on the right-hand side of (44), note that the term \( \frac{\epsilon^2}{4} \int \frac{\Psi_1^+ + \Psi_{-1}^-}{q(1 + \rho)}(\partial_x^2 R_1 + \partial_x^2 R_{-1})^2 dx \) = \( \epsilon^2 O(||R||^2_{l^2}) \) as long as \( ||R||^2_{l^2} = O(1) \). Thus we can add such terms to the energy \( E \) as \( O(\epsilon^2) \) corrections, which will not affect the equivalence between the corrected energy and \( ||R||^2_{l^2} \). For convenience, we still denote the corrected energy as \( E \). So far, we obtain the estimates for the terms of \( O(\epsilon^2) \) and higher from \( \partial_t E_0 \) and \( \partial_t E_1 \).

The second step is to consider the terms of the \( O(\epsilon) \) terms from \( \partial_t E_0 \) and \( \partial_t E_1 \),

\[
2\epsilon \sum_{j \in \{\pm 1\}} \int R_j \cdot \theta^{-1} Q_j(\Psi^c, \theta R) + \partial_x^2 R_j \cdot \theta^{-1} \partial_x^2 Q_j(\Psi^c, \theta R) dx 
\] 
\[+ \epsilon \sum_{j \in \{\pm 1\}} \int \frac{1}{2}(\theta^{-1} Q_j(\Psi^c, \theta R))^2 - \frac{1}{2}(\Omega \phi_c) \partial_x^2 R_j \cdot \partial_x^2 R_{-j} dx 
\] 
\[+ \epsilon \sum_{j \in \{\pm 1\}} \int -\frac{1}{2}(\theta^{-1} Q_j(\Psi^c, \theta R))^2 - \frac{1}{2}(\Omega \phi_c) \partial_x^2 R_j \cdot \partial_x^2 R_{-j} dx 
\] 
\[= 2\epsilon \sum_{j \in \{\pm 1\}} \int R_j \cdot \theta^{-1} Q_j(\Psi^c, \theta R) + \partial_x^2 R_j \cdot \theta^{-1} Q_j(\partial_x^2 \Psi^c, \theta R) dx 
\] 
\[+ \epsilon \sum_{j \in \{\pm 1\}} \int 4\partial_x^2 R_j \cdot \theta^{-1} Q_j(\partial_x \Psi^c, \theta \partial_x R) dx - \frac{1}{2}(\theta^{-1} Q_j(\Psi^c, \theta R))^2 
\] 
\[- \frac{1}{2}(\Omega \phi_c) \partial_x^2 R_j \partial_x^2 R_{-j} dx 
\] 
\[+ \epsilon \sum_{j \in \{\pm 1\}} \int 2\partial_x^2 R_j \cdot \theta^{-1} Q_j(\Psi^c, \theta \partial_x^2 R) dx + \Omega((\theta \phi_c) \partial_x^2 R_j) \partial_x^2 R_{-j} 
\] 
\[+ j(\Omega(\phi_c) \partial_x^2 R_{-j}) \partial_x^2 R_{-j} dx, \]

where we have applied the antisymmetry of \( \Omega \) to integrate by parts. For each term of the last line of (45), there are three derivatives on one of the factors of \( R \). However, a combination of them will reduce the derivatives on \( R \). In fact, recalling (28), we have for the last line of (45) that

\[
2\epsilon \sum_{j \in \{\pm 1\}} \int \left( \partial_x^2 R_j \cdot \theta^{-1} Q_j(\Psi^c, \theta \partial_x^2 R) + \frac{1}{2}(\theta^{-1} Q_j(\Psi^c, \theta R))^2 \right) dx 
\] 
\[+ \frac{1}{2} j(\Omega(\phi_c) \partial_x^2 R_{-j}) \partial_x^2 R_{-j} dx, \]
\[= \epsilon \sum_{j \in \{\pm 1\}} \int \frac{\partial \ell}{\partial k}(k \partial^{2} \alpha_{-1, j}(k, k - \ell, \ell) + i k \tilde{q}(k) \tilde{q}(k - \ell)) \tilde{R}_{j}(k) \tilde{\phi}_{c}(k - \ell) \tilde{R}_{j}(\ell) d\ell dkd\]

According to (28), (5) and (15), we know \(1 - \tilde{q}(k) = \mathcal{O}(k^{-2})\) for \(|k| > \delta\), and

\[\hat{\alpha}_{-1, n} = \begin{cases} 
\frac{1}{2} \left( i j k (\tilde{q}(\ell) - \frac{1}{q(k)}) - i k \tilde{q}(k - \ell)(1 + \frac{\tilde{q}(\ell)}{q(k)}) + \frac{2 i j k \tilde{q}(k - \ell)}{q(k)} \right) & \text{for } n = j, \\
\frac{1}{2} \left( -i j k (\tilde{q}(\ell) + \frac{1}{q(k)}) - i k \tilde{q}(k - \ell)(1 - \frac{\tilde{q}(\ell)}{q(k)}) + \frac{2 i j k \tilde{q}(k - \ell)}{q(k)} \right) & \text{for } n = -j.
\end{cases}
\]

Note that

\[|2 \hat{\alpha}_{-1, j}(k, k - \ell, \ell) + i k \tilde{q}(k) \tilde{q}(k - \ell)| = \mathcal{O}(k^{-2}) + \mathcal{O}(\ell^{-2}), \quad \text{for } |k|, |\ell| > \delta,
\]

\[|2 \hat{\alpha}_{-1, -j}(k, k - \ell, \ell) + i j k \tilde{q}(k)| = \mathcal{O}(k^{-2}) + \mathcal{O}(\ell^{-2}), \quad \text{for } |k|, |\ell| > \delta.
\]

Therefore, all terms of (45) can be bounded by \(\epsilon \mathcal{O}(E)\). However, To show that \(R\) is small for times of \(\mathcal{O}(\epsilon^{-2})\), we still need to cancel all the \(\mathcal{O}(\epsilon)\) terms from (45).

The last step is to use the space-time approach to transfer all terms of (45) and \(\partial_{t} E_{2}\) to the terms of \(\mathcal{O}(\epsilon^{2})\). Before we begin the space-time approach it will be convenient to analyze the remaining terms in Fourier space. Since each term is real, we can split \(\mathcal{O}(\epsilon)\) from (45) into two parts and use Plancherel’s theorem to get

\[2 \epsilon \sum_{j \in \{\pm 1\}} \int \left( R_{j} \vartheta^{-1} Q_{j}(\Psi^{c}, \vartheta R) + \partial_{k}^{2} R_{j} \vartheta^{-1} Q_{j}(\partial_{k}^{2} \Psi^{c}, \vartheta R) \right) dx
\]

\[+ \epsilon \sum_{j \in \{\pm 1\}} \int \left( 4 \partial_{k}^{2} R_{j} \vartheta^{-1} Q_{j}(\partial_{k} \Psi^{c}, \vartheta \partial_{k} R) - \frac{1}{2} j(q \Omega \varphi_{c})(\partial_{k}^{2} R_{j})^{2}
\]

\[- \frac{1}{2} (\Omega \varphi_{c}) \partial_{k}^{2} R_{j} \cdot \partial_{k}^{2} R_{-j} \right) dx
\]

\[+ \epsilon \sum_{j \in \{\pm 1\}} \int \left( 2 \partial_{k}^{2} R_{j} \cdot \partial \vartheta^{-1} Q_{j}(\Psi^{c}, \vartheta \partial_{k}^{2} R) + \Omega((q \varphi_{c}) \partial_{k}^{2} R_{j}) \partial_{k}^{2} R_{j}
\]

\[+ j \Omega(\varphi_{c} \partial_{k}^{2} R_{-j}) \partial_{k}^{2} R_{j} \right) dx
\]

\[= 2 \epsilon \sum_{j,m,n \in \{\pm 1\}} \int \left( \frac{\partial \ell}{\partial k} \frac{\partial \alpha_{mn}}{\partial k}(k, k - \ell, \ell) + \frac{\partial \ell}{\partial k} k^{2}(k - \ell)^{2} \alpha_{mn}(k, k - \ell, \ell)
\]

\[+ 2 \frac{\partial \ell}{\partial k} k^{2}(k - \ell) \alpha_{mn}(k, k - \ell, \ell) \tilde{R}_{j}(k) \tilde{\Psi}_{m}(k - \ell) \tilde{R}_{n}(k) d\ell dkd\]

\[+ 2 \epsilon \sum_{j,m,n \in \{\pm 1\}} \int \left( - \frac{i j k^{2} \ell^{2}}{4} \tilde{q}(k - \ell) \omega(k - \ell) + \frac{\partial \ell}{\partial k} k^{2} \ell^{2} \alpha_{mnj}(k, k - \ell, \ell)
\]

\[+ i k^{2} \omega(k) \ell^{2} \tilde{q}(k - \ell) \tilde{R}_{j}(k) \tilde{\Psi}_{m}(k - \ell) \tilde{R}_{j}(k) d\ell dkd\]

\[+ 2 \epsilon \sum_{j,m \in \{\pm 1\}} \int \left( - \frac{i j k^{2} \ell^{2}}{4} \omega(k - \ell) + \frac{\partial \ell}{\partial k} k^{2} \ell^{2} \alpha_{mn-1}(k, k - \ell, \ell)
\]

\[+ i j k^{2} \omega(k) \ell^{2} \tilde{R}_{j}(k) \tilde{\Psi}_{m}(k - \ell) \tilde{R}_{j}(k) d\ell dkd\]

5. **Applying the space-time resonance.** The space-time resonance method has a wide range of applications in the well-posedness of solutions. It was first introduced by Germain, Masmoudi and Shatah, in order to understand global existence.
for nonlinear dispersive equations in the whole space and with small data. The space-time resonance approach extends Shatah’s normal form method and is also related to Klainerman’s vector-field method \([16, 24]\). In this section, we will only sketch the time resonance part of the space-time resonance method. Ignoring higher order terms \(N(U)\) of the diagonalized equation (11), we consider the following simplified example

\[
\partial_t U = \Lambda U + \epsilon Q(U, U),
\]

\(\left. U(t) \right|_{t=0} = U_0.\)

Recall that \(\alpha(k, k - \ell, \ell)\) is the multiplier of nonlinear quadratic term \(Q(U, U)\) in Fourier space in (13). We will write \(U\) in a rotating coordinate frame. Set \(f = e^{-\Lambda t} U\), then we have

\[
\partial_t f = \epsilon e^{-\Lambda t} Q(e^{\Lambda t} f, e^{\Lambda t} f),
\]

\(f(t)|_{t=0} = U_0.\)

Note that the \(O(1)\) term no longer appears in this coordinate frame. Using Duhamel’s formula we can write the Fourier transform of the solution \(f\) as

\[
\hat{f}_j(t, k) = \hat{U}_j(k) + \epsilon \int_0^t \int \left. e^{i\Theta_{mn}^j(k, \ell, s) k - \ell, \ell} \hat{f}_m(k - \ell, \ell) \hat{f}_n(\ell, s) \right|_{\ell=0} d\ell ds \tag{48}
\]

with

\[
\Theta_{mn}^j(k, \ell) = -j\omega(k) + m\omega(k - \ell) + n\omega(\ell).
\]

The main idea of the space-time resonance approach is to analyze the above using the method of stationary phase for both \(s\) and \(\ell\). We will call the set the time resonances

\[
\mathcal{T} = \{(k, \ell) \mid \Theta_{mn}^j(k, \ell) = 0\}.
\]

If these resonances do not exist, or if we are integrating in some region bounded away from \(\mathcal{T}\), then we can integrate (48) by parts with respect to \(s\) and get

\[
\hat{f}_j(t, k) = \hat{U}_j(k) + \epsilon \int_0^t \int e^{i\Theta_{mn}^j(k, \ell, s) k - \ell, \ell} \frac{\alpha_{mn}^j(k, k - \ell, \ell)}{i\Theta_{mn}^j(k, \ell)} \hat{f}_m(k - \ell, \ell) \hat{f}_n(\ell, s) d\ell ds \tag{49}
\]

For the equation (49), the first integrand on the right-hand side is small and no longer grows with respect to \(t\), the second is cubic in \(f\) and \(O(\epsilon^2)\) when the time derivative is applied to either factor of \(f\). Besides, if there exists some resonances, and the multiplier \(\alpha_{mn}^j\) consists of some factor which can cancel these resonances, then we can also obtain the equation (49). In essence, the presence of the cutoff function \(\vartheta\) from (24) is to cancel the resonance \(\ell = 0\) such that the equation (49) is well-defined.

To apply the space-time resonance methods to deal with the \(O(\epsilon)\) term such that \(R\) is small for times of \(O(\epsilon^{-2})\), it will be slightly easier to move into a rotating coordinate frame. Therefore we define

\[
\begin{pmatrix} f_1 \\ f_{-1} \end{pmatrix} = e^{-\Lambda t} \begin{pmatrix} R_1 \\ R_{-1} \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ g_{-1} \end{pmatrix} = e^{-\Lambda t} \begin{pmatrix} \Psi_1 \\ \Psi_{-1} \end{pmatrix}, \tag{50}
\]
certainly, we have
\[
\left(\frac{g_1^c}{g_{-1}^c}\right) = e^{-\Lambda t}\left(\Psi_1^c\right).
\]
For these variables we have
\[
\partial_t f = 2\epsilon e^{-\Lambda t - 1}Q(e^{\Lambda t} g, e^{\Lambda t} f) + \epsilon^\beta e^{-\Lambda t} \vartheta - 1\mathcal{Q}(e^{\Lambda t} g) + \epsilon e^{-\Lambda t} \vartheta - 1\mathcal{Q}(e^{\Lambda t} f) + \epsilon e^{-\Lambda t} \vartheta - 1\mathcal{R}(e^{\Lambda t} g).
\]
Note that the \(O(1)\) term no longer appears in this coordinate frame. According to the form of \(\omega(k)\) from (5) and the condition \(0 < H \leq 2\), we obtain
\[
\mathcal{T} = \{(k, \ell) \mid k = 0, \ell = 0 \text{ or } k = \ell\}
\] (52)
is the only resonance. Indeed, putting \(a = H^2/4\), we have
\[
\hat{q}(k)^2 = 1 + \frac{1 + ak^2}{1 + k^2 + ak^4} = 1 + \frac{1}{r(k^2)},
\]
where
\[
 r(k^2) = r(s) = \frac{1}{1 + as} + s, \quad s \geq 0,
\]
with
\[
r'(s) = -\frac{a}{(1 + as)^2} + 1.
\]
Then, when \(a \leq 1\), it can be shown that \(r\) is strictly increasing for \(s \geq 0\), so \(\hat{q}\) is strictly decreasing, and then \(\mathcal{T} = \{(k, \ell) \mid k = 0, \ell = 0 \text{ or } k = \ell\}\) is the only resonance. When \(a > 1\), however, \(r\) first decreases then increases, so \(\hat{q}\) first increases then decreases. Then it is easy to see the equation \(\hat{q}(2k) = \hat{q}(k)\) has (at least) one positive solution \(k\), and for such a solution it holds that \(\omega(2k) = 2\omega(k)\). Therefore, we limit the parameter \(0 < H \leq 2\) in the Theorem 1.1 in order to avoid dealing with more unknown resonances.

In the following we will use the transparency condition to deal with the resonances from (52). In detail, for the region \(k = 0\), we can use a transparency condition directly by applying the form of the nonlinear term. For the region \(\ell = 0\), we create a further transparency condition by using the rescaling \(\vartheta\). For the region \(k = \ell\), recall that
\[
\text{supp}(\hat{\Psi}_c^c(k - \ell)) = \{(k, \ell) \mid |k - \ell| \leq 2\delta\}.
\]
Therefore, in the process of using the space-time resonance approach, we only need to consider \(k, \ell\) in the region \(\text{supp}(\hat{\Psi}_c^c(k - \ell))\) which means we can avoid the region where \(k = \ell\) completely. Since each of the aforementioned regions will be dealt with differently, we define the regions
\[
\mathcal{W} = \{(k, \ell) \in \text{supp}(\hat{\Psi}_c^c(k - \ell)) \mid |\ell| < \delta\},
\]
\[
\mathcal{Z} = \{(k, \ell) \in \text{supp}(\hat{\Psi}_c^c(k - \ell)) \mid |k| < \delta\},
\]
and
\[
\mathcal{V} = \text{supp}(\hat{\Psi}_c^c(k - \ell)) \setminus (\mathcal{W} \cup \mathcal{Z}).
\]
We integrate the terms from (47) with respect to $t$ and adopt the notation used in (50), leading to

$$\epsilon \sum_{j,m,n \in \{\pm 1\}} \int_0^t \int_{\mathcal{V}} \hat{\zeta}(k, k - \ell, \ell) e^{i \Theta_{mn}(k, k - \ell) s} \hat{f}_j(k) \hat{g}_m(k - \ell) \hat{f}_n(\ell) d\ell dks,$$

(53)

with the following kernel functions $\hat{\zeta}(k, k - \ell, \ell)$ coming from (47)

$$\frac{\partial (\ell)}{\partial (k)} \hat{\alpha}_{mn}^j(k, k - \ell, \ell), \quad \frac{\partial (\ell)}{\partial (k)} k^2 (k - \ell) \hat{\alpha}_{mn}^j(k, k - \ell, \ell),$$

$$2 \frac{\partial (\ell)}{\partial (k)} k^2 (k - \ell) \ell \hat{\alpha}_{mn}^j(k, k - \ell, \ell), -\frac{ijk^2 \ell^2}{4} \bar{q}(k - \ell) \omega(k - \ell), -\frac{i k^2 \ell^2}{4} \omega(k - \ell),$$

$$\frac{\partial (\ell)}{\partial (k)} k^2 \ell^2 \hat{\alpha}_{mj}^j(k, k - \ell, \ell) + i k^2 \omega(k) \ell^2 \bar{q}(k - \ell),$$

$$\frac{\partial (\ell)}{\partial (k)} k^2 \ell^2 \hat{\alpha}_{m,-j}^j(k, k - \ell, \ell) + i k^2 \omega(k) \ell^2.$$

In the region $\mathcal{V}$ and $\mathcal{W}$, we only need to estimate (53). In the region $\mathcal{Z}$, we need to estimate (53) and the terms from $\partial_t E_2$.

6. The estimates in the regions $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{Z}$. First of all, let us estimate (53) in the region $\mathcal{V}$. Note that $\hat{\alpha}^{-1}(k) = \hat{\alpha}(\ell) = 1$ and $\Theta_{mn} \neq 0$ in the region $\mathcal{V}$. Then we integrate by parts with respect to time $s$ for the equation (53), leading us to the following equations of the form

$$\epsilon \sum_{j,m,n \in \{\pm 1\}} \int_0^t \int_{\mathcal{V}} e^{i \Theta_{mn}(k, k - \ell) s} \hat{\zeta}(k, k - \ell, \ell) \hat{f}_j(k) \hat{g}_m(k - \ell) \hat{f}_n(\ell) d\ell dks,$$

(54)

$$- \epsilon \int_0^t \int_{\mathcal{V}} e^{i \Theta_{mn}(k, k - \ell) s} \hat{\zeta}(k, k - \ell, \ell) \frac{\partial (\ell)}{\partial (k)} \hat{f}_j(k) \hat{g}_m(k - \ell) \hat{f}_n(\ell) d\ell dks,$$

(55)

with the multiplier $\hat{\zeta}(k, k - \ell, \ell)$ taking the following forms

$$\hat{\alpha}_{mn}^j(k, k - \ell, \ell), k^2 (k - \ell) \hat{\alpha}_{mn}^j(k, k - \ell, \ell), 2k^2 (k - \ell) \ell \hat{\alpha}_{mn}^j(k, k - \ell, \ell),$$

$$-\frac{ijk^2 \ell^2}{4} \bar{q}(k - \ell) \omega(k - \ell), k^2 \ell^2 \hat{\alpha}_{mj}^j(k, k - \ell, \ell) + i k^2 \omega(k) \ell^2 \bar{q}(k - \ell),$$

$$-\frac{i k^2 \ell^2}{4} \omega(k - \ell), k^2 \ell^2 \hat{\alpha}_{m,-j}^j(k, k - \ell, \ell) + i k^2 \omega(k) \ell^2.$$

(56)

We note that there are similar complex conjugate terms that can be dealt with in the same manner.

Note that (54) and (55) are well-defined because $T \cap \mathcal{V} = \emptyset$. Since the boundary term (54) is bounded by $C \epsilon E$, then we can subtract it from the left-hand side of our energy $E$, this does not change the energy in a significant way. We still take the notation $E$. For (55), we will focus on one term and the rest terms are similar. Ignoring the summation for the moment and using the equation (51), we have

$$\epsilon \int_0^t \int_{\mathcal{V}} e^{i \Theta_{mn}(k, k - \ell) s} \hat{\zeta}(k, k - \ell, \ell) \frac{\partial (\ell)}{\partial (k)} \hat{f}_j(k) \hat{g}_m(k - \ell) \hat{f}_n(\ell) d\ell dks.$$
\[ 2 \varepsilon \int_0^t \left[ \int_V e^{i\omega_1(k-\ell) + \omega_2(\ell)} \tilde{\Psi}(k, k-\ell, \ell) \frac{iQ_j(e^{\Lambda^s} \tilde{g}, e^{\Lambda^s} \tilde{f})(k)}{\Theta_{mn}(k, \ell)} \times \tilde{g}_m(k-\ell) \partial_{\ell}(\ell) d\ell dk ds \right] \]

\[ + \varepsilon^3 \int_0^t \left[ \int_V e^{i\omega_1(k-\ell) + \omega_2(\ell)} \tilde{\Psi}(k, k-\ell) \frac{N_j(e^{\Lambda^s} \tilde{g} \tilde{f})(k) \tilde{g}_m(k-\ell) \partial_{\ell}(\ell) d\ell dk ds}{i\Theta_{mn}(k, \ell)} \right] \]

\[ + \varepsilon^{\beta+1} \int_0^t \left[ \int_V e^{i\omega_1(k-\ell) + \omega_2(\ell)} \tilde{\Psi}(k, k-\ell) \frac{Q_j(e^{\Lambda^s} \tilde{g} \tilde{f})(k)}{i\Theta_{mn}(k, \ell)} \times \tilde{g}_m(k-\ell) \partial_{\ell}(\ell) d\ell dk ds \right] \]

\[ + \varepsilon^{-\beta} \int_0^t \left[ \int_V e^{i\omega_1(k-\ell) + \omega_2(\ell)} \tilde{\Psi}(k, k-\ell) \frac{\text{Res}(e^{\Lambda^s} \tilde{g} \tilde{f})(k)}{i\Theta_{mn}(k, \ell)} \times \tilde{g}_m(k-\ell) \partial_{\ell}(\ell) d\ell dk ds \right]. \]  

The last term can be bounded directly by \( C \varepsilon^2 E^{1/2} \) by using Young’s inequality and Cauchy-Schwarz inequality.

Furthermore, for the first three terms on the right-hand side of (57), recall (56), (14) and (46), then all of the multipliers are bounded with the exception of the following (the algebraic power sum of \( k \) and \( \ell \) equals to 4)

\[ 2 \frac{\partial(\ell)}{\partial(k)} k^2(k-\ell)q_{mn}(k, k-\ell, \ell), \quad \frac{-ik^2q_{mn}(k, k-\ell, \ell)}{4q(k)}, \quad -ik^3q_{mn}(k, k-\ell, \ell), \]

i.e.

\[ 2ik^3(k-\ell)q(k-\ell), \quad 2ik^3(k-\ell)q(k-\ell)q(k), \quad 2ik^3(k-\ell)q(k-\ell), \]

\[ 2ik^3(k-\ell)q(k-\ell)q(k), \quad -ik^2\omega(k-\ell), \]

i.e.

\[ \eta(k-\ell), \quad i\eta(k-\ell), \quad \eta(k-\ell). \]  

In essence, except for the kernel functions from (58), the sum of exponents of \( k \) and \( \ell \) at most reach to 3 for the other kernel functions. We only take one multiplier \( k^3(k-\ell)^2q(\ell) \) and one term of the first three terms from (57) as an example to obtain

\[ e^{\beta+1} \int_0^t \left[ \int_V 2\frac{\partial(\ell)}{\partial(k)} k^2(k-\ell)q_{mn}(k, k-\ell, \ell) \frac{Q_j(e^{\Lambda^s} \tilde{g} \tilde{f})(k)}{\Theta_{mn}(k, \ell)} \times \tilde{g}_m(k-\ell) \partial_{\ell}(\ell) d\ell dk ds \right]. \]
Recall (29), \( \Theta \) and (5), we have
\[
\alpha = \frac{\partial^2}{\partial t^2} q(t) \frac{\partial^2 q(t)}{\partial t \partial x} - \frac{\partial^2}{\partial t \partial x} q(t) \frac{\partial^2 q(t)}{\partial t^2} + \int_0^t \int_{\mathbb{R}^2} \frac{n j k^4 (k - \ell)^2}{2i \Theta_{\text{int}}(k, \ell)} \left( \frac{\partial^2}{\partial t^2} q(t) \frac{\partial^2 q(t)}{\partial t \partial x} - \frac{\partial^2}{\partial t \partial x} q(t) \frac{\partial^2 q(t)}{\partial t^2} \right) dk \, dt
\]
\[
\times \frac{\partial^2}{\partial t^2} R_n(\ell) \frac{\partial^2 R_n(\ell)}{\partial t \partial x} \, dp \, dk
\]
\[
\lesssim \epsilon^{\beta + 1} \int \left( \int |k|^2 \frac{\partial^2}{\partial t^2} q(t) \frac{\partial^2 q(t)}{\partial t \partial x} - \frac{\partial^2}{\partial t \partial x} q(t) \frac{\partial^2 q(t)}{\partial t^2} |dp| \right) \left( \int |k|^2 \frac{\partial^2}{\partial t^2} R_n(\ell) \frac{\partial^2 R_n(\ell)}{\partial t \partial x} |\,d\ell\right) dk
\]
\[
\lesssim \epsilon^{\beta + 1} \|R\|_L^2 \|R\|_H^2 \lesssim \epsilon^{\beta + 1} E^{3/2},
\]
where we have used integration by parts and \( k = (k - \ell) + \ell \), as well as Cauchy-Schwartz and Young’s inequality. In the second to the last step, we have used Cauchy-Schwartz, Young’s equality, and the fact \( \|\hat{\Psi}\|_L^2 \leq C \) independent of \( \epsilon \). The terms with the multiplier from (58) cannot be bounded this way due to the high powers of \( k \) and \( \ell \). Since each power of \( k \) and \( \ell \) is essentially one derivative in real-space, what we have is two derivatives falling on each power of \( R \). When we consider the nonlinear term acting on \( R \) as well this puts 3-derivatives on one of the factors of \( R \). Therefore, for this reason we must take advantage of Lemma 4.1 and (43). Similar to Section 4 for the estimation of the terms of \( \mathcal{O}(\epsilon^2) \) and even higher order terms, we firstly only take one term of (57) and one multiplier such as \( 2 \, \imath k^3 (k - \ell) \hat{q}(\ell) \) from (58) to simply the estimation, the others are similar,
\[
2 \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \sum_{m, n \in \{\pm 1\}} \int_\mathbb{V} \frac{2 \imath j k^2 (k - \ell) \ell \hat{q}(\ell)}{i \Theta_{\text{int}}(k, \ell)} Q_j(k) \hat{R}_m(k) \hat{R}_n(\ell) \, d\ell \, dk \, ds
\]
\[
= 2 \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \sum_{m, n \in \{\pm 1\}} \int_\mathbb{V} \frac{2 \imath j k^2 (k - \ell) \ell \hat{q}(\ell)}{i \Theta_{\text{int}}(k, \ell)} \hat{R}_m(k) \hat{R}_n(\ell) \, d\ell \, dk \, ds
\]
\[
\times \sum_{d, r \in \{\pm 1\}} \hat{\Theta}_{\text{dr}}(k - p, p) \hat{\Psi}_d(k - p) \hat{\partial}(p) \hat{R}_r(p) \, dp \, d\ell \, ds.
\]
Recall the form of \( \alpha_{\text{mn}}^{j} \) of (14), now we also only take one factor such as \( \imath k^2 \psi(p) \) of \( \alpha_{\text{dr}}^{j} \), then we have
\[
\epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \sum_{m, n \in \{\pm 1\}} \int_\mathbb{V} \frac{\imath j k^2 (k - \ell) \ell}{i \Theta_{\text{int}}(k, \ell)} \hat{R}_m(k) \hat{R}_n(\ell) \, d\ell \, dk \, ds
\]
\[
\times \sum_{d, r \in \{\pm 1\}} (\imath k^2 \hat{\Psi}_d(k - p) \hat{\partial}(p) \hat{R}_r(p) \, dp \, d\ell \, ds.
\]
Recall (5), we have
\[
\omega(k) = k \hat{q}(k) = k + \mathcal{O}(|k|^{-1}), \text{ for } |k| \to \infty,
\]
and
\[
\omega'(k) = 1 + \mathcal{O}(|k|^{-2}), \text{ for } |k| \to \infty.
\]
Recall (29), \( \Theta_{\text{mn}}^{j}(k, \ell)(k, \ell) = -j \omega(k) + m \omega(k - \ell) + n \omega(\ell) \), then for \( |k| \to \infty \), we have
\[
\Theta_{\text{mn}}^{j} = -j \omega(k) - \omega(\ell) + m \omega(k - \ell) - n \omega(\ell) - j \omega(k)(k - \ell) + m \omega(k - \ell),
\]
\[
\Theta_{\text{mn}}^{\ell} = -j \omega(k) + \omega(\ell) - n \omega(k - \ell) = -j \omega(k)(1 + \mathcal{O}(|k|^{-1})),
\]
for some \( \theta \in [0, 1] \). Then when \( n = -j \), we can approximate the multiplier using the fact that \( k - \ell \approx \pm k_0 \). This gives us \( |\Theta_{\text{mn}}| \sim |k| \), for all \( (k, \ell) \in \mathbb{V} \). Using this
estimate on the multiplier, the integrand can then be estimated directly using the method shown for the other kernel functions. When \( n = j \), we approximate the kernel again with

\[
\left| \frac{j(k - \ell)}{\Theta_{mn}(k, \ell)} \right| \leq C,
\]

for all \((k, \ell) \in \mathbb{V}\). In this case there is still an extra one-derivative and so this cannot be estimated directly. We can drop \( \hat{\phi}(p) \) since it only affects this term with \( p \) small, in which case the error can be estimated directly. We can also integrate over all of \( \mathbb{R}^2 \) instead of \( \mathbb{V} \) for the same reason. Then we only need to estimate the following by using Plancharel’s Theorem and the fact that \( R \) and \( \Psi \) are real,

\[
\epsilon^2 \int_0^t \sum_{m,j,d,r \in \{\pm 1\}} \int k^2 \hat{\Psi}_m(k - \ell) i t \hat{R}_j(\ell) d\ell \int (rk^2 \hat{\Psi}_d(k - p) \hat{q}(p) \hat{R}_r(p) dp) dkd\ell
\]

\[
= \epsilon^2 \int_0^t \left( \sum_{m,j \in \{\pm 1\}} \partial_x^2 (\Psi_m \partial_x R_j) \right) \left( \partial_x^2 \left( \sum_{d,r \in \{\pm 1\}} (r \Psi_d q R_r) \right) \right) dxds
\]

\[
= \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \partial_x^2 (\phi_c \partial_x R_j) \partial_x^2 (\phi_c q (R_j + R_{-j})) dxds
\]

\[
+ \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \left( [\partial_x^2, \phi_c] \partial_x R_j \right) [\phi_c q \partial_x^2 (R_j + R_{-j})] dxds
\]

By applying Lemma 4.2, the last two commutators can be estimated directly. Then by (43) and the Lemma 4.1, we have

\[
\epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \partial_x^2 (\phi_c \partial_x R_j) \partial_x^2 (R_j + R_{-j}) dxds
\]

\[
= \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \partial_x^2 (\phi_c \partial_x R_j) (q - 1) \partial_x^2 (R_j + R_{-j}) dxds
\]

\[
+ \epsilon^2 \sum_{j \in \{\pm 1\}} \int_0^t \partial_x^2 (\phi_c \partial_x R_j) \partial_x^2 (R_j + R_{-j}) dxds
\]

\[
= 2\epsilon^2 \int_0^t \partial_x^2 (\phi_c \partial_x R_j) (R_j + R_{-j}) dxds + \epsilon^2 O(1)
\]

\[
= 2\epsilon^2 \int_0^t \frac{\phi_c^2}{q(1 + \rho)} (\partial_x^2 (R_j + R_{-j}))^2 dx \bigg|_0^t + \epsilon^2 O(1 + E + \epsilon E^2 + \epsilon^2 E^3).
\]
Now the first term can be treated as boundary term, we subtract it from the left-hand side of our estimate using the fact that it is both small and bounded independent of \( t \). Thus, over the region \( \mathcal{V} \) we have

\[
\int_0^t \partial_s E(s) ds \lesssim \int_0^t e^2(1 + E) + e^3E^2 + e^4E^3 ds.
\]

Secondly, let us estimate (53) in the region \( \mathcal{W} \). Note that \( \hat{\vartheta}(k) = 1 \) in \( \mathcal{W} \). Then we have

\[
\epsilon \sum_{j,m,n \in \{\pm 1\}} \int_0^t \int_\mathcal{W} \tilde{\zeta}(k, k - \ell, \ell)e^{i\Theta_{\ell,m,n}(k,k-\ell)t} \frac{f_j(k)}{\Theta_{\ell,m,n}(k,k-\ell)} \tilde{\vartheta}_m(k-\ell) \tilde{f}_n(\ell) d\ell dk ds,
\]

with the following kernel functions \( \tilde{\zeta}(k, k - \ell, \ell) \)

\[
\hat{\vartheta}(\ell) \hat{\alpha}_{mn}^j(k, k - \ell, \ell), \quad \hat{\vartheta}(\ell) k^2(k - \ell)^2 \hat{\alpha}_{mn}^j(k, k - \ell, \ell),
\]

\[
2\hat{\vartheta}(\ell) k^2(k - \ell) \hat{\alpha}_{mn}^j(k, k - \ell, \ell), \quad -\frac{i j k^2 \omega}{4} \vartheta(k - \ell) \omega(k - \ell), \quad -\frac{i k^2 \omega}{4} \omega(k - \ell),
\]

\[
\hat{\vartheta}(\ell) k^2 \omega^2 \hat{\alpha}_{mn}^j(k, k - \ell, \ell), \quad ik^2 \omega(k - \ell) \hat{\vartheta}(\ell) k^2 \omega^2 \hat{\alpha}_{mn, -j}^j(k, k - \ell, \ell), \quad i j k^2 \omega(k - \ell)^2.
\]

There exist two groups of kernel functions, some with \( \hat{\vartheta}(\ell) \) and some without.

First, for the terms with \( \hat{\vartheta}(\ell) \), define \( \vartheta = \vartheta_0 + \epsilon \). This splits the integral up into two terms. For the term with \( \vartheta_0(\ell) \), let \( \tilde{\zeta}(k, k - \ell, \ell) = \vartheta_0(\ell) \tilde{\vartheta}(k, k - \ell, \ell) \). Since \( \vartheta_0(\ell) = 0 \) when \( \ell = 0 \), we have

\[
\left| \frac{\tilde{\zeta}(k, k - \ell, \ell)}{i \Theta_{mn}^j(k, k)} \right| = \left| \frac{\vartheta_0(\ell) \tilde{\vartheta}(k, k - \ell, \ell)}{i \Theta_{mn}^j(k, k)} \right| \leq C,
\]

for some constant \( C \) and all \( (k, \ell) \in \mathcal{W} \). Thus integration by parts with respect to \( s \) for (61) leads to

\[
\epsilon \sum_{j,m,n \in \{\pm 1\}} \int_0^t \int_\mathcal{W} e^{i\Theta_{\ell,m,n}(k,k-\ell)t} \frac{\tilde{\vartheta}_m(k-\ell)}{i \Theta_{\ell,m,n}(k,k-\ell)} \tilde{f}_n(\ell) d\ell dk ds \int_0^t \vartheta_0(\ell) \tilde{\vartheta}_m(k-\ell) \tilde{f}_n(\ell) d\ell dk ds.
\]

So the terms (62) and (63) are well-defined because \( \mathcal{T} \cap \mathcal{W} = \{ \ell = 0 \} \) and \( \tilde{\vartheta}_0 = 0 \) for \( (k, \ell) \in \mathcal{W} \). Besides, the boundary term (62) is bounded by \( \epsilon E \). We can subtract it to the left-hand side of our energy \( E \) as we did with the boundary term in \( \mathcal{V} \). For the non-boundary term (63) we can use the fact that \( \mathcal{W} \) is compact to bound all the terms directly using Young’s inequality and Cauchy-Schwartz. In particular, since all the kernels can be bounded by a constant, we do not need any of the extra integration by parts techniques that we have used in \( \mathcal{V} \). The other term is of order \( \mathcal{O}(\epsilon^2) \) and so can be estimated directly without any other integration by parts.

Second, for the terms without \( \hat{\vartheta}(\ell) \) we note that these terms do not need the transparency condition because of the presence of a factor \( \ell^2 \) in the kernel, which leads to

\[
\left| \frac{\tilde{\zeta}(k, k - \ell, \ell)}{i \Theta_{mn}^j(k, k)} \right| < C |\ell|.
\]

We can thus bound these terms directly after integrating by parts as well.
Finally, let us estimate (53) and the terms from $\partial_t E_2$ in the region $Z$. The problem we have to deal with in essence is the lack of $\epsilon$ due to $\hat{\theta}(\ell) = O(\epsilon^{-1})$ near $k = 0$ over the region $Z$. However, we have a advantage of $\hat{\theta}(\ell) = 1$. The method in this section is similar to [3]. In fact, as to [19] (also refer to [23] and [7]), we also can apply the normal-form transformation twice without defining $E_2$ to the original quasilinear $O(\epsilon)$ terms such that the terms from (47) become $O(\epsilon^2)$ terms. These two treatments are actually figured out. We need to consider both the terms from (47) as well as those from $\partial_t E_2$

\[
2\epsilon \sum_{j,m,n \in \{\pm 1\}} \iint_Z \left( \frac{\partial(\ell)}{\partial(k)} \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) + \frac{\partial(\ell)}{\partial(k)} k^2(k - \ell)^2 \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) \right.
\]
\[
+ 2 \frac{\partial(\ell)}{\partial(k)} k^2(k - \ell) \ell \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) - \frac{i j k^2 \ell^2}{8} \hat{q}(k - \ell) \omega(k - \ell) - \frac{i k^2 \ell^2}{8} \omega(k - \ell)
\]
\[
+ \frac{\partial(\ell)}{\partial(k)} k^2 \ell^2 \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) + \frac{ik^2}{2} \omega k \ell^2 \hat{q}(k - \ell) + \frac{ijk^2}{2} \omega k \ell^2 \right)
\]
\[
\times \overline{R_j(k)} \hat{\Psi}_m^c(k - \ell) \overline{R_n(k)} d\ell dk
\]
\[
+ \epsilon \sum_{j \in \{\pm 1\}} \int_{|k|<\delta} \frac{\hat{\theta}(\ell)}{\theta(k)} \hat{\theta} B_j(\hat{\Psi}, \overline{R}) + \overline{R_j} B_j(\Lambda \hat{\Psi}, \overline{\Lambda R})
\]
\[
+ \hat{R}_j B_j(\hat{\Psi}, \Lambda \overline{R}) + \tilde{R}_j B_j(\hat{\Psi}, \Lambda \overline{R}) d\ell dk
\]
\[
+ \epsilon^2 \sum_{j \in \{\pm 1\}} \int_{|k|<\delta} \frac{\hat{\theta}(\ell)}{\theta(k)} Q_j(\hat{\Psi}, \hat{\theta} R) B_j(\hat{\Psi}, \overline{R}) + \overline{B_j(\Lambda \hat{\Psi}, \overline{R})} B_j(\hat{\Psi}, \overline{R})
\]
\[
+ \hat{R}_j B_j(\hat{\Psi}, \Lambda \overline{R}) + \tilde{R}_j B_j(\Lambda \hat{\Psi}, \overline{R}) d\ell dk
\]
\[
+ 2\epsilon^3 \sum_{j \in \{\pm 1\}} \int_{|k|<\delta} B_j \left( \hat{\Psi}, \left\{ \hat{\theta}^{-1} Q_j(\hat{\Psi}, \hat{\theta} R) \right\} \right) B_j(\hat{\Psi}, \overline{R}) d\ell dk
\]
\[
= \left\{ 2\epsilon \sum_{j,m,n \in \{\pm 1\}} \iint_Z \left( \frac{1}{\partial(k)} k^2(k - \ell)^2 \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) \right.
\]
\[
+ 2 \frac{1}{\partial(k)} k^2(k - \ell) \ell \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) - \frac{i j k^2 \ell^2}{8} \hat{q}(k - \ell) \omega(k - \ell) - \frac{i k^2 \ell^2}{8} \omega(k - \ell)
\]
\[
+ \frac{1}{\partial(k)} k^2 \ell^2 \tilde{\alpha}_{mn}^{j}(k, k - \ell, \ell) + \frac{ik^2}{2} \omega k \ell^2 \hat{q}(k - \ell) + \frac{ijk^2}{2} \omega k \ell^2 \right)
\]
\[
\times \overline{R_j(k)} \hat{\Psi}_m^c(k - \ell) \overline{R_n(k)} d\ell dk \right\}
\]
\[
+ \left\{ \frac{\epsilon}{2} \sum_{j \in \{\pm 1\}} \int_{|k|<\delta} \overline{R_j} \left( \frac{1}{\partial(k)} Q_j(\hat{\Psi}, \hat{\theta} R) - j \omega(k) B_j(\hat{\Psi}, \overline{R}) \right) \right\}
\]
\[
= \left\{ \frac{\epsilon}{2} \sum_{j \in \{\pm 1\}} \int_{|k|<\delta} \hat{R}_j \left( \frac{1}{\partial(k)} Q_j(\hat{\Psi}, \hat{\theta} R) - j \omega(k) B_j(\hat{\Psi}, \overline{R}) \right) \right\}
\]
\[ + B_j(\Lambda \hat{\Psi}^c, \hat{R}) + B_j(\hat{\Psi}^c, \Lambda \hat{R}) \right) dk \]  
\[ + \left\{ \frac{\epsilon}{2} \sum_{j \in \{\pm 1\}} \int \left| k \right| < \delta \, R_j B_j(\partial_j \hat{\Psi}^c - \Lambda \hat{\Psi}^c, \hat{R}) dk \right\} \]  
\[ + \left\{ \epsilon^2 \sum_{j \in \{\pm 1\}} \int \left| k \right| < \delta \, \partial^{-1}(k) Q_j(\hat{\Psi}^c, \partial \hat{R}) B_j(\hat{\Psi}^c, \hat{R}) + B_j(\hat{\Psi}^c, \Lambda \hat{R}) B_j(\hat{\Psi}^c, \hat{R}) \right\} \]  
\[ + \left\{ \epsilon^2 \sum_{j \in \{\pm 1\}} \int \left| k \right| < \delta \, \partial^{-1}(k) Q_j(\hat{\Psi}^c, \partial \hat{R}) B_j(\hat{\Psi}^c, \hat{R}) + B_j(\hat{\Psi}^c, \Lambda \hat{R}) B_j(\hat{\Psi}^c, \hat{R}) \right\} \]  
\[ + 2 \epsilon^3 \sum_{j \in \{\pm 1\}} \int \left| k \right| < \delta \, B_j(\hat{\Psi}^c, \partial^{-1} Q(\hat{\Psi}^c, \partial \hat{R})) B_j(\hat{\Psi}^c, \hat{R}) dk, \]  
with similar complex conjugate terms. We will use space-time resonance methods on (64); the form of \( E_2 \) was chosen to ensure that the terms in (65) all cancel; and as shown in [19], \( \| (\partial_j \hat{\Psi}^c - \Lambda \hat{\Psi}^c \|_{L^2} \leq C \epsilon^2 \), and so the term (66) can be bounded directly. Due to the fact that \( \partial^{-1}(k) \) is \( \mathcal{O}(\epsilon^{-1}) \) near \( k = 0 \) we need to show that terms of formal order as high as \( \mathcal{O}(\epsilon^3) \) are bounded by \( \epsilon^2 (1 + E) + \epsilon^3 E^2 \). As we will see, the terms in \( E_2 \) were chosen precisely because they give cancellations of terms that are not of \( \mathcal{O}(\epsilon^2 E) \) and to which we can’t apply the method of space-time resonance for (67).  

For (64), there exist two groups of kernel functions, some with \( \partial^{-1}(k) \) and some without. We first integrate with respect to \( t \) and move to a rotating coordinate frame for the terms with \( \partial^{-1}(k) \), denoting \( \hat{\zeta}(k, k - \ell, \ell) = \partial^{-1}(k) \hat{\Theta}(k, k - \ell, \ell) \), then we have  
\[ \epsilon \sum_{j, m, n \in \{\pm 1\}} \int_0^t \int Z \hat{\Theta}(k, k - \ell, \ell) e^{i \Theta_{mn}(k, \ell) s} \hat{\partial}^{-1}(k) \bar{f}_j(k) \bar{g}_m(k - \ell) \hat{f}_n(\ell) dl dk ds \]  
as well as four others without a factor of \( \hat{\partial}^{-1}(k) \). We note that \( T \cap Z = \{ k = 0 \} \). However, \( \hat{\Theta}(0, -\ell, \ell) = 0 \) and it approaches 0 cubically, that is  
\[ \left| \hat{\Theta}(k, k - \ell, \ell) \right| < C |k|^2, \]  
for some constant \( C \) and all \( (k, \ell) \in Z \). Thus we can integrate by parts with respect to \( s \) for (68),  
\[ \epsilon \sum_{j, m, n \in \{\pm 1\}} \int Z e^{i \Theta_{mn}(k, \ell) s} \hat{\Theta}(k, k - \ell, \ell) \partial_k \hat{\Theta}^{-1}(k) \bar{f}_j(k) \bar{g}_m(k - \ell) \hat{f}_n(\ell) dl dk  
\]  
\[ - \epsilon \int_0^t \int Z e^{i \Theta_{mn}(k, \ell) s} \hat{\Theta}(k, k - \ell, \ell) \partial_k \hat{\Theta}^{-1}(k) \bar{f}_j(k) \bar{g}_m(k - \ell) \hat{f}_n(\ell) dl dk ds. \]
Therefore, both (69) and (70) are well-defined. Moreover, each of the kernel functions approaches zero so rapidly that we have

\[
|\hat{\zeta}(k, k - \ell, \ell)| \leq |\hat{\theta}(k, k - \ell, \ell)\hat{\theta}^{-1}(k)| \leq C|k|.
\]

Therefore the boundary term (69) is bounded by \(\epsilon E\). We can subtract it to the left-hand side of our energy \(E\) as we did with the boundary term in \(W\). Besides, we use the fact that \(Z\) is compact to bound all the terms from (70) directly as we did in \(W\).

For those last terms that don’t contain a factor of \(\hat{\theta}^{-1}(k)\), we note that

\[
|\hat{\zeta}(k, k - \ell, \ell)| \leq C|k|,
\]

for \((k, \ell) \in Z\). Thus these terms can still be bounded directly once we integrate by parts. So far, we have obtained the estimation of (64).

The terms from (65) equal to zero because of bilinear map \(B\), the term (66) and the fourth term of (67) can be bounded directly by the lemma 2.2.

In the following we deal with the term of (67) in the region \(Z\). For the \(O(\epsilon^2)\) terms of (67), we have

\[
e^2 \sum_{j \in \{\pm 1\}} \int_{|k| < \delta} \left( \hat{\theta}^{-1}Q_j(\hat{\psi}^c, \hat{\theta}^R) + B_j(\hat{\psi}^c, \hat{\theta}^R) + B_j(\hat{\psi}^c, \Lambda \hat{\theta}^R) \right) B_j(\hat{\psi}^c, \Lambda \hat{\theta}^R) dk + c.c.
\]

(71)

After careful computation, we find the first term, the second term and the third term as well as their complex conjugates of (71) cancel. The last term of (71) takes the form

\[
e^2 \sum_{j \in \{\pm 1\}} \int_{|k| < \delta} \bar{R}_j B_j \left( \hat{\psi}^c, \hat{\psi}^c, \hat{\theta}^{-1}Q(\hat{\psi}^c, \hat{\theta}^R) \right) dk
\]

\[= e^2 \sum_{j,m,n,q,r \in \{\pm 1\}} \int_{|k| < \delta} \bar{R}_j(k) \frac{\hat{\alpha}^l_m(k, k - \ell, \ell)}{i\Theta^l_{mn}(k, \ell)} \frac{\hat{\theta}(\ell)}{\theta(k)} \hat{\psi}^c_m(k - \ell) \times \frac{\hat{\theta}(p)}{\theta(p)} \hat{\psi}^c_q(\ell - p)\bar{R}_r(p)dpd\ell dk
\]

(72)

\[= e^2 \sum_{j,m,n,q,r \in \{\pm 1\}} \int_{|k| < \delta} \bar{R}_j(k) \frac{\hat{\alpha}^l_m(k, k - \ell, \ell)}{i\Theta^l_{mn}(k, \ell)} \frac{\hat{\theta}(\ell)}{\theta(k)} \times \bar{R}_j(k) \hat{\psi}^c_m(k - \ell) \hat{\psi}^c_q(\ell - p)\bar{R}_r(p)dpd\ell dk.
\]

We can split this integral (72) up into two different regions depending on the value of \(p\) by using the support of \(\hat{\psi}^c\). Since \(|k| < \delta\), the factor of \(\hat{\psi}^c_m(k - \ell)\) above requires that \(\ell \approx \pm k_0\). Then similarly, the factor of \(\hat{\psi}^c_q(\ell - p)\) requires that \(p \approx 0\) or \(p \approx \pm 2k_0\). In details, we define

\[
\hat{\psi}^c = \sum_{(i=\pm)} \hat{\psi}^{c,i}
\]

(73)
Lemma 6.1. Assume that \( \Psi^{c,l} \) is defined from the approximation (73) and that \( R \in L^2 \). Then there exists \( C > 0 \) such that
\[
\left\| \int (\hat{\sigma}(\ell) - \hat{\sigma}(\ell - lk_0)) \hat{\Psi}^{c,l}(\ell - \ell) \hat{R}(\ell) d\ell \right\|_{L^2} \leq C\|R\|_{L^2}.
\]

One can refer to [19, Lemma 4.1] for the proof.

We can get the estimates of the first two terms of (74) directly from the Lemma (6.1). Since \( \hat{\sigma}(k - (l + n)k_0) = \hat{\sigma}(k) \), the third term of (74) can be bounded directly for \( l = n \). Besides, we can integrate by parts with respect to \( t \) as we have done for \( E_1 \) for \( l = n \). By moving to a rotating coordinate frame and integrating with respect to \( \ell \), we obtain
\[
\epsilon^2 \int_0^t \int_{|k|,|p| < 3\delta} \frac{\hat{\sigma}_{lm}^n(k, k - \ell, \ell) \hat{\sigma}_{qr}^n(\ell, \ell - p, p)}{i\Theta_{mn}(k, \ell)} \hat{\sigma}(k \pm 2\delta) e^{i\Theta_{ngr}(k, \ell, p)s} \frac{\hat{\sigma}(k \pm 2\delta)}{\hat{\sigma}(k)} d\ell d\ell dkdsd
\]
\[
\times \int f_j(k) \xi_m^\text{c} \xi_q^\text{c} (k - \ell) \xi_q^\text{c} (\ell - p) f_j(p) d\ell d\ell dk ds
\]
with
\[
\Theta_{ngr}(k, \ell, p) = -j\omega(k) + m\omega(k - \ell) + q\omega(\ell - p) + r\omega(p).
\]
Since \( \Theta_{ngr}(k, \ell, p) \neq 0 \) in the appropriate region, we can integrate by parts with respect to \( s \) to get
\[
\epsilon^2 \int_0^t \int_{|k|,|p| < 3\delta} \frac{\hat{\sigma}_{lm}^n(k, k - \ell, \ell) \hat{\sigma}_{qr}^n(\ell, \ell - p, p)}{i\Theta_{mn}(k, \ell)\Theta_{ngr}(k, \ell, p)} \hat{\sigma}(k \pm 2\delta) e^{i\Theta_{ngr}(k, \ell, p)s} \frac{\hat{\sigma}(k \pm 2\delta)}{\hat{\sigma}(k)} d\ell d\ell dkdsd
\]
\[ \partial_t \left( \mathcal{F}_j(k) \mathcal{G}_m^c(k-\ell) \mathcal{G}_q^c(\ell-p) \mathcal{F}_r(p) \right) dp dq dk ds \]
and a boundary term, which can be bounded directly. The \( O(\epsilon^3) \) terms of (67) can also be bounded exactly as (72).

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