Eccentric black hole mergers and zoom-whirl behavior from elliptic inspirals to hyperbolic encounters

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We perform a parameter study of non-spinning, equal and unequal mass black hole binaries on generic, eccentric orbits in numerical relativity. The linear momentum considered ranges from that of a circular orbit to ten times that value. We discuss the different manifestations of zoom-whirl behavior in the hyperbolic and the elliptic regime. The hyperbolic data set applies to dynamical capture scenarios (e.g. in globular clusters). Evolutions in the elliptic regime correspond to possible end states of supermassive black hole binaries. We spot zoom-whirl behavior for eccentricities as low as $e \sim 0.5$, i.e. within the expected range of eccentricities in massive black hole binaries from galaxy mergers and binaries near galactic centers. The resulting gravitational waveforms reveal a rich structure, which will effectively break degeneracies in parameter space improving parameter estimation.

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I. INTRODUCTION

Zoom-whirl orbits arise as a general relativistic phenomenon of the two-body problem. Such orbits do not exist in Newtonian gravity, where the orbits are Kepler’s conic sections, hence they represent an important facet of one of the fundamental problems in general relativity (GR). The term zoom-whirl was first used in [1][2]. It refers to the orbits of eccentric binaries where tight and fast revolutions (the whirls) are separated by phases in which the two objects move out to larger distances and back in (the zooms).

The physics behind this effect is precession. For bound orbits, one can define the precession per orbit as the angle between two consecutive apocenters. In general, precession accumulates continuously (with respect to some external reference frame) and amounts to an excess angle beyond the Newtonian motion. Precession is strongest for small separations and is therefore significant especially for eccentric orbits. In the solar system, the precession of the orbit of Mercury due to general relativity is 43 arcseconds per century, or $(2.9 \cdot 10^{-5})^\circ$ per orbit. For binary pulsars, precession is not necessarily much larger. For the Hulse-Taylor pulsar the precession is $4.2^\circ$ per year, but due to its short orbital period this amounts to $(3.7 \cdot 10^{-3})^\circ$ per orbit. Observed are on the order of $(2 \cdot 10^{-2})^\circ$ per orbit in some cases [3][5]. A SMBH binary model [6][7] fitted to the optical light curve of the quasar OJ-287 predicts (model dependent) orbital parameters with precession as large as $\sim 40^\circ$ per orbit.

In theory, general relativistic orbits with yet larger precession can be easily constructed by choosing appropriate orbital parameters. This is possible for test particles following geodesics around a black hole, but also for comparable mass compact objects in the post-Newtonian (PN) approximation [8]. As long as the particle or compact object orbits well outside the innermost stable circular orbit (ISCO), the classical picture of a slowly precessing ellipse applies. If the orbital parameters are chosen such that the object approaches distances close to or even inside the ISCO, it may follow an unstable circular orbit for some time. After this it either plunges or escapes to larger distances (infinity if the motion is unbound), which is the zoom-whirl behavior we are interested in. In the whirl regime orbits exhibit extreme precession with precession angles comparable to or larger than $2\pi$, wrapping the inner part of the orbit once or even several times around its center.

The main question about zoom-whirl orbits in full GR is how the classic, well known picture of zoom-whirl geodesics changes for binaries with comparable masses in configurations where radiation damping becomes significant. Naively, we do not expect the binary to radiate away more than its total mass, i.e. the number of orbits is finite since it is limited by the energy and angular momentum radiated away during each whirl. In fact, for comparable masses one might have questioned whether it is possible to obtain even a single (full) whirl. Since the whirls happen at high velocity and small separation (even inside the innermost stable circular orbit), the PN approximation is not directly applicable, e.g. [5]. However, recently some groups have performed numerical evolutions in full general relativity of eccentric black hole binaries (BHBs). Zoom-whirl orbits have indeed been found, although the number of whirls in these experi-
ments is less than three.

In [18], Pretorius and Klurana present the first example of a whirl orbit for an equal mass binary. In [19–23], several examples for the transition from inspiral to plunge, radiated energy, angular momentum and the resulting final spin are investigated. In [24] longer evolutions of unequal masses and non-vanishing spin with up to three elliptic orbits which transition through the zoom-whirl regime prior to merger are studied. The notion of marginally stable circular orbits in background spacetimes was shown to be in close resemblance to whirl orbits in numerical evolutions of finite mass ratio [15–25]. The consequences for kicks are addressed in [26]. Implications for data analysis are studied in e.g. [27–29]. In particular [29] point out the potentially deteriorating effects in signal processing when eccentricity is ignored in the waveform models.

Eccentric neutron star and mixed binaries in dynamical spacetime have been studied in [33–36], and in all cases zoom-whirl behavior has been identified. The focus in [27–31] is on high-energy collision. Among the key results so far is that the total energy radiated can easily exceed the 4% of the total mass radiated during the last stage of a quasi-circular inspiral. For high-energy collisions, up to 35 ± 5% have been found [39]. In [22], we found at low momentum multiple extrema in the radiated energy as a function of the initial data, and that only a modest amount of fine-tuning is required to spot these extrema. These extrema should be compared to the variations in the mass and spin of the merger remnant noted in [24].

Choosing different initial data and also different tuning strategies, these investigations have been performed in different regions in parameter space. In the present work we focus on an area that has received relatively little attention so far, namely intermediate momenta and comparable but not necessarily equal masses. We extend the discussion of [22], specifically we consider mass ratios 1:1, 1:2, and 1:3, and linear momenta that are 1 to 6, and in one case 10 times the value of a circular orbit, although not in all possible combinations.

Zoom-whirl is sometimes thought to occur beyond a certain, rather large eccentricity. Our results instead show that whirls can also be found for modest eccentricities. We give an analysis of the gravitational waves (GWs) and how specific features in the radiated energy are related to orbital characteristics.

A prerequisite for zoom-whirl orbits is eccentricity. Isolated black hole binaries formed at typical separations perform a sufficiently large number of orbits such that the orbits become circularized long before entering the strong field regime [42]. However, it cannot be expected that all binary GW sources are sufficiently isolated and hence other effects have to be taken into account. In fact, SMBH binaries are expected to be formed in gas/star rich environments [43] with potentially large eccentricities [44–45]. It is well understood that such binaries can gain eccentricity as a consequence of gravitational torques exchanged with the circumbinary disk [46–49]. Likewise, gravitational interaction with additional bodies (Kozai-oscillations, Hill-mechanism, mass segregation, gravitational focussing, etc.) generically induce eccentricity growth on a binary system. Numerous studies of such effects [27–30] suggest that the eccentricity of binaries, which emits significant gravitational radiation, cannot in general be ignored. Event rate estimates for eccentric compact object binaries [53–56] suffer from large uncertainties and vary considerably. Some studies predict that advanced LIGO should detect such sources, but given the large uncertainties this should be taken with care. For third generation detectors the detection range will be larger. Eccentric binary mergers will therefore become more interesting sources in the future. For supermassive BHBs, pulsar timing arrays will soon be able to resolve individual sources in a regime where many binaries are still expected to be eccentric [60].

Given a population of eccentric binaries, whether zoom-whirl orbits are of relevance to gravitational wave astronomy depends on several factors. Even if the signals are stronger, if excessive fine-tuning is required, then the population of strong sources might amount to a very small corner of parameter space. Conversely, if little tuning is involved, then zoom-whirl orbits can be potential GW sources even for ground-based detectors [27–29, 52]. With regard to GWs, for comparable mass binaries with astrophysical momenta we loose the unlimited number of whirl orbits due to gravitational radiation, but what is lost corresponds to a very small part of parameter space anyway. In any case, as a matter of principle we should be prepared to detect and recognize GWs from all corners of parameter space including zoom-whirl orbits.

The paper is organized as follows. In Sec. II, we describe the basics of our numerical methods, our choice of initial configurations, and give error estimates for typical runs. We discuss orbital properties in Sec. III A, resulting waveforms and radiated energy in III B, and phase space trajectories in III C. We conclude with Sec. IV.

II. NUMERICAL METHODS AND SUMMARY OF SIMULATIONS

A. Method

We performed a parameter study of the black hole binary problem using 3d numerical simulations obtained with the BAM code [67–69]. Initial data for black holes is computed by the puncture method [70] using a pseudo-spectral code [71], and evolved with the $\chi$-variant of the moving-puncture [72–73] version of the BSSN [74–75] formulation of the 3+1 Einstein evolution equations. We use a 4th order Runge-Kutta method with 6th order finite differencing. The wave extraction and the calculation of the radiated energy is done using a 4th order accurate implementation of the Newman-Penrose formalism. We extract $\Psi_4$ and thus also $E_{\text{rad}}$ at extraction radii.
of $r_{GW} = 60M, 80M, 100M$, where $M$ the total puncture mass $M$ (see below). Our grid is a box of typically $\gtrsim (640M)^3$ size, which is sufficient to keep the boundaries causally disconnected from the GWs for most of our runs. We employ bitant or quadrant symmetry when possible. Usually, the grid consists of 9 levels of mesh refinement starting at the coarsest level with resolution of $h = 5M$ and increases by factors of two, resulting in the resolution of $h \approx M/50$ at the finest level. The inner, finer levels are evolved according to Berger-Oliger timestepping while the outer levels do not follow the motion of the punctures and are evolved at the fixed timestep given by the innermost fixed level $\Delta t_{\text{fix}}$ [68]. The fixed boxes have twice as many grid points (for a more accurate wave propagation). Since some runs have exceptional settings the parameters of our simulations are summarized in Table I. For our analysis we measured the GW emission and radiated energy (normalized to the initial ADM-mass), studied the shape of the event horizons, the coordinate distance over time and how much time the binary spends at a separation $D$. Moreover we investigated a new way of analyzing binary evolutions, namely to look at the phase space with the coordinate velocity and the separation of the punctures $(v, D)$ serving as generalized coordinates. The velocity of the puncture is computed from the shift as $v = \sqrt{\beta_i(x_p)}\beta^i(x_p)$, where $x_p$ is the coordinate location of the puncture. $x_p$ is - as a diagnostic - tracked by integrating $\partial_t x_p^i = -\beta^i(x_p)$ using the ICN method as in [72].

B. Black hole parameters

The initial data for black hole binaries is characterized by a choice of the following parameters. In this work we set the spins to zero. Input for the computation of the initial data are parameters $m_i$ for the (bare) puncture masses, $\vec{P}$ for the momenta, and $\vec{x}$ for the positions. The total puncture mass $M$ is defined as $M = \sum_i m_i$. Since the global mass scale in vacuum is arbitrary, the masses can be characterized by one number, say the symmetric mass ratio denoted by $\nu = m_1 m_2 / (m_1 + m_2)^2$.

We choose coordinates in which the punctures are initially located on the $x$-axis, see Fig. 1. For equal masses we set $x_{1,2} = \pm D/2$ for a coordinate separation $D$. For unequal masses we leave $x_1$ unchanged but set $x_2 = x_1 m_1 / m_2$. For the momenta we choose $\vec{P}_{1,2} = \pm \vec{P}$. This implies together with the choice of $x_{1,2}$ that initially the center of mass is at rest and that mergers happen at the origin (except for a small merger kick due to unequal masses). Concretely, we consider momenta in the $x$-$y$ plane given by their magnitude $P$ and an angle $\Theta$ such that $\vec{P} = (- P \cos \Theta, P \sin \Theta, 0)$. Specifying the “shooting angle” $\Theta$ is equivalent to the choice of an impact parameter. The magnitude $P$ of the momenta is chosen as a multiple of $P_{qc}$, which denotes the magnitude of the momentum for a quasi-circular inspiral at separation $D$.

Given the configuration in Fig. 1 numerical simulations are parameterized by specific choices for $x_1$, $\nu$, $P$, and $\Theta$. We set $x_1 = 10M$ for all runs, which implies $D = 20M$ for equal masses. For unequal masses, we position the larger mass at $x_2$, i.e. $m_2 > m_1$ and $|x_2| < x_1$. Most of the simulations we discuss are for equal masses ($\nu = 1/4$), but we also consider a few examples for mass ratios 1:2 ($\nu = 2/9$) and 1:3 ($\nu = 3/16$). Following [71], for equal masses at $D = 20M$ the magnitude of the momentum for quasi-circular inspiral is $P_{qc} = 0.061747M$. We consider $P/P_{qc} = 1, 2, \ldots, 6$, and in one example as the most extreme case $P/P_{qc} = 10$. The direction of the momenta is given by $\Theta \in [0, 90^\circ]$. Here $\Theta = 0$ corresponds to a head-on collision, while for quasi-circular inspiraling orbits $\Theta$ is slightly smaller than 90° because the momentum has a small radial component. The case $\Theta > 90^\circ$ with initially radially outgoing motion can be ignored [78].

The ADM mass at the $i$-th puncture and at infinity is

$$M_{ADM}^i = (1 + u(x_i)) m_i + \frac{m_1 m_2}{2D},$$

$$M_{ADM}^\infty = M_{ADM}^1 + M_{ADM}^2 + E_{\text{bind}}$$

$$= m_1 + m_2 + \lim_{r \to \infty} (2ru),$$

respectively, where $u$ is the correction to the conformal factor in the puncture framework and $E_{\text{bind}}$ the binding energy. Values for $M_{ADM}^i$ range from 0.994 for 1$P_{qc}$ to 1.2 for 6$P_{qc}$. Since the momenta are non-zero, we obtain larger physical masses $M_{ADM}$ at the inner asymptotically flat ends of the punctures. The difference between the masses $m_i$ and $M_{ADM}^i$ ranges from 7·10$^{-3}M$ for 1$P_{qc}$ to 3.5·10$^{-2}M$ for 6$P_{qc}$, and is essentially independent of $\Theta$.

For the main part of this work, we first choose a specific mass ratio, in particular we choose between equal and unequal masses. Second, we choose one of several (low) momentum cases. Third, we vary the shooting angle systematically, in particular searching for maxima and minima in the total radiated energy, examining the number of whirls, etc. There are some obvious alternatives to set up such parameter scans, say fixing $\Theta$ [21], using some measure of eccentricity, the angular momentum [23], or the binding energy [19] as parameter. Apart from having a simple interpretation as scattering experiment with fixed momentum size, our setup also describes simulations at roughly constant total energy, if in analogy to classical point masses the total energy is defined as the sum of the kinetic and potential energy (since $P$
TABLE I. List of selected runs with corresponding initial data, order of convergence, estimated errors and merger times. A 3PN estimate \[19, 80\] of our lowest eccentric run \(1P_{qc}, \Theta = 60^\circ\) gives \(e \approx 0.5 - 0.6\) (depending on which definition of \(e\) is used).

| Model       | \(\Delta E_{rad} / E_{rad}\) | \(\Delta A / A\) | Ord. | \(t_m [M]\) |
|-------------|-------------------------------|------------------|------|-------------|
| \(1P_{qc}, \Theta = 10^\circ\) | 0.01                          | 0.008            | 4    | 40.5        |
| \(1P_{qc}, \Theta = 40^\circ\) | 0.013                         | 0.010            | 4    | 92.7        |
| \(1P_{qc}, \Theta = 52^\circ\) | 0.015                         | 0.012            | 4    | 766         |
| \(2P_{qc}, \Theta = 23.9^\circ\) | 0.017                         | 0.016            | 4    | 79.9        |
| \(2P_{qc}, \Theta = 26^\circ\) | 0.0013                        | 0.002            | 4    | \(\infty\) |
| \(3P_{qc}, \Theta = 18^\circ\) | 0.0095                        | 0.003            | 5    | 69.4        |
| \(4P_{qc}, \Theta = 15.6^\circ\) | 0.0095                        | 0.012            | 4    | 39.0        |
| \(10P_{qc}, \Theta_{\phi 50} = 5.4^\circ\) | 0.03                          | 0.012            | 4    | 40.5        |

and \(D\) are constant while varying \(\Theta\). Each run amounts to \(500 - 30000\) CPUh (the latter one for \(1P_{qc}, \Theta = 60^\circ\)), which is strongly dependent on how far and how many times the orbits zoom out. We implemented Brent’s method to bracket local extrema in the efficiency of converting energy into outgoing gravitational radiation, for which a small number of runs sufficed. This reduced the total number of runs to about a 130 while still sampling the parameter space in an adaptive and accurate way. In retrospect, we found that a golden section search is, for the finite accuracy we required, a better choice despite being only first order convergent. The parabolic interpolation inside Brent’s method chooses the new guesses systematically towards the flatter part of the asymmetric maxima.

C. Convergence and error estimates

We performed a convergence analysis for a representative subset of our runs and in general found 4th order convergence in the 22-mode of \(r \Psi_4\) and in the radiated energy \(E_{rad}\) demonstrating the overall consistency of the code with respect to the order of the Runge-Kutta integrator and the wave extraction routine. The errors due to the finite radius of our wave extraction sphere are quantified by the deviation from a \(1/r\) fall-off as measured from the data taken at three different extraction radii.

Error estimates based on this analysis are shown in Tab. 1 and selected convergence plots are shown in Figs. 2 and 3. In general, highly eccentric orbits are accurately treated by the BAM code, and also the presence of the rather high momenta considered here can be dealt with consistently. The relative errors in the 22-mode and the radiated energy both due to finite resolution and extraction radius is around 1%. For larger momenta the error from a finite extraction radius becomes the dominating error (\(\approx 2\%\)). Increasing the initial momentum leads to higher amounts of artificial (junk) radiation, enhances the ADM mass of the initial time slice and reduces the BH horizons. At some point these effects contaminate the solution in the sense that its physical relevance becomes questionable. However, we limited our data set to those regimes where the artificial radiation is either entirely negligible or at least small in comparison to the physical radiation. The \(P = 10P_{qc}\) sequence represents an exception, but we only used it in the context of the Hawking limit and as an approximate extrapolation of
our data set to the ones obtained by other groups at larger momenta.

The error due to the junk radiation arising from the conformally flat initial data can be reduced to some extent by choosing a sufficiently large initial separation. In the very high momentum case we needed both large separations and also much larger resolution until the radiated energy results converged, but once the appropriate resolution is used the accuracy compares favourably with the other results. For yet higher momentum runs we refer to [37, 39], who study momenta beyond the rest-mass dominated regime.

Convergence for unequal masses can be shown only at the higher resolutions. For the resolution we used the \( l = 2, m = 2 \) mode converges at second order – a common tendency when being at the edge of the convergent regime. The somewhat lower accuracy for larger mass ratios is a well-known effect of the gamma-driver condition we use (\( \eta = \text{const} \)). In [31, 33] it is shown that a generalization of this condition (\( \eta \) dependent on the local mass) leads to an improvement in accuracy.

Analyzing the dependence on resolution shows that the derived errors in the energy are not behaving according to a Gaussian distribution. There is a skewness in the actual (unknown) distribution of our measurements such that higher resolutions systematically produce higher energies. Hence, our errorbars should be slightly more extended towards larger values of \( E_{\text{rad}} \).

Summarizing, the simulations presented here do not pose new challenges to the numerical scheme, although there are specific requirements for accuracy in the presence of whirls together with long runtimes. In these cases there is a high sensitivity to the parameters and during the long evolutions numerical errors accumulate. Nevertheless, those evolutions have similar convergence behavior and error estimates, and only require a higher resolution to obtain convergence.

III. RESULTS

A. Orbital properties

To prime the discussion of the orbits, we first consider several examples of puncture tracks for equal mass binaries with \( P = P_{\text{qc}} \) and \( P = 5P_{\text{qc}} \), see Figs. 4 – 10. It is helpful to read the captions of these figures in sequence. Shown are the puncture tracks in the \( x-y \)-coordinate plane in the upper panels, and the 22-mode of the waveforms in the lower panels. The waveforms are further discussed in Sec. III B. The figures show two sequences of runs for two momenta that explore how the orbits change when the shooting angle is varied from small to large.

1. Classification of orbits

For any choice of mass ratio \( \nu \) and initial separation \( D \), we can in principle fill in a “phase-diagram” as shown in Fig. 11, which labels orbits in a \( P-\Theta \) plot. The main classification is whether initial parameters \( P \) and \( \Theta \) lead to orbits that are bound (implying capture and merger) or unbound (escape to infinity). In Newtonian gravity, we only have to check whether the kinetic energy exceeds the potential energy, or equivalently whether the binding energy is positive or negative. In general relativity, this
Fig. 6. $P = P_{qc}, \Theta = 50^\circ$. For a shooting angle two degrees larger than that leading to the strong whirl, there is a close encounter with a precession of about half an orbit, followed by a zoom out to about three times the radius at pericenter, followed by a short inspiral and merger that starts with significantly reduced eccentricity. Note the comparatively small and short wave pulse associated with the close encounter, again at about $200M$ of evolution.

Fig. 7. $P = P_{qc}, \Theta = 60^\circ$. Increasing the shooting angle beyond $50^\circ$, one can find an increasing number of elliptic orbits. Early on the orbit resembles the classical picture of a (strongly) precessing ellipse. The plot shows a transition through plunge through a full whirl phase at the onset of merger with a clear corresponding wave signal.

Fig. 8. $P = 5P_{qc}, \Theta = 14.15^\circ$. Example for initial momentum which is significantly larger than that of quasi-circular orbits, and which can easily produce unbound orbits. Zoom-whirl orbits are found for much smaller shooting angles than in Fig. 6. There is one whirl, and a short zoom followed by a merger. Due to the additional kinetic energy, the whirl signal increases in amplitude and exceeds the merger signal.

Fig. 9. $P = 5P_{qc}, \Theta = 14.20^\circ$. The larger the momentum, the more sensitive the orbit becomes to the choice of the shooting angle. A small change in angle compared to Fig. 8 leads to a much larger zoom out to an apocenter distance of $12M$ before it merges at the next encounter. The initial whirl, however, is almost unchanged, highlighting the analogy to unstable circular orbits.

Distinction is sometimes only possible a posteriori since the gravitational waves and the associated loss of energy and angular momentum are only known after the Einstein equations have been solved. Solutions to the evolution problem define the dividing line $\tilde{P} = \tilde{P}_{bu}(\nu, D, \Theta)$ in Fig. 11. Orbits with $P > \tilde{P}_{bu}$ are unbound, orbits with $P < \tilde{P}_{bu}$ are bound.

A simplified, a priori upper limit on the momentum $P$ that ensures boundedness is $\tilde{P}_h := P_{bu}(\nu, D, \Theta = 180^\circ)$, which is independent of $\Theta$, cmp. Fig. 11. If the momentum $P$ does not suffice to escape in the direction $\Theta = 180^\circ$ (for which radiation losses are minimized), then the orbits are bound for all $\Theta$. Here we use the assumption that the black holes are not spinning. Approximating the minimal radiation loss in the “head-off” direction by zero, we compute a simple estimate of $\tilde{P}_h$ based on the binding energy in [2]. Fixing $\Theta = 180^\circ$ and $D = 20M$ we iteratively compute initial data with varying $P$ to obtain the binding energy $E_{bind}$. $\tilde{P}$ is then defined as $\tilde{P} := P(E_{bind} = 0, \Theta = 180^\circ)$ resulting in $\tilde{P} = 0.085(4 \pm 3)M$. There is some error since we end
the iteration at some point and since radiation effects are ignored. For example, the momentum of the quasi-circular orbit leads to bound orbits for all angles since \( P_{qc} < \tilde{P} \approx 1.377P_{qc} \).

For brevity, we will refer to the set of configurations satisfying \( P < \tilde{P} \) as the elliptic regime. On the other hand, orbits with \( P > \tilde{P} \) form the hyperbolic class. Note that this terminology skips over the fact that orbits in the hyperbolic regime may still lead to a merger provided \( \Theta \) is small enough.

Inverting \( P_{bu}(\nu, D, \Theta) \), we define \( \Theta_{bu}(\nu, D, P) \), the shooting angle between bound and unbound orbits, as a function of \( P \). An a priori upper limit for bound orbits is given by \( \Theta_{geom} = \Theta_{ba}(\nu, D, P = \infty) \), see Fig. 11. In practice it is tricky to study large \( P \) due to limitations in the construction of initial data. Conceptually, however, we can think of this limit as a geometric constraint based on the finite size of the black holes, i.e. the idea is that the two black holes must merge when their event horizons touch. Using Euclidean geometry, \( \Theta_{geom} \) is given by \( \sin(\Theta_{geom}) = d_{merger}/D \), where \( d_{merger} \) is the separation of the punctures at the time of the merger. However, the size of the black holes depends on the gauge. The Schwarzschild radius for a mass \( m \) is \( 2m \) in Schwarzschild coordinates, \( m/2 \) for isotropic coordinates, and depending on the moving puncture gauge somewhere in between for the numerical evolutions. We therefore use the numerical result for \( d_{merger} \). For equal masses we find that a common event horizon appears at a coordinate distance of about \( d_{merger} \approx 1.76 \) to \( 1.95 \) \( M \) (with a slight drift towards smaller values with increasing momentum). For an initial separation of \( D = 20M \) this estimate leads to a geometric limit of \( \Theta_{geom} = 10.5^\circ \) using the Euclidean formula. This limit does not appear to be very restrictive for low momenta, but it is not in contradiction to the runs of this study, either. All our simulations with \( \Theta < \Theta_{geom} \) end in a merger.

The determination of the ultimate fate of a system outside the above ranges requires a full numerical evolution. Here a bound system can be defined by the (future) formation of a single event horizon, which is expensive to compute numerically. In our evolutions we use a criterion on the lapse at the center of our grid to determine a merger time. We justify this approach by a direct comparison with an event horizon finder \([84, 85]\). The merger time \( t_m \) is approximated by \( t_m^\alpha \), the time by which the lapse at the center of our grid has dropped below \( \alpha = 0.3 \). This is near the analytical value of a single Schwarzschild black hole in the same and similar gauges \([86–88]\). We have chosen a moderately long evolution among the elliptic category and get \( t_m = 484.175M \) and \( t_m^\alpha = 485.524 \), accurate to within \( \Delta t_m/t_m = 0.0028 \). We should mention, however, that the lapse criterion gives worse answers when the punctures move too fast, because the value \( \alpha = 0.3 \) is motivated by a Schwarzschild spacetime and hence is not well adapted to a boosted black hole. We used the lapse criterion to estimate the merger times and list them in Tab. 1. Those values are also used in Figs. 19 and 20.

Even if one performs a numerical evolution it can be difficult to determine whether an orbit is unbound. The absence of a common horizon is only a necessary but not a sufficient condition for unboundedness. If a merger does not occur after a given finite time, the question is for how long the simulation has to be continued to settle whether the binary is bound or unbound, and in principle this time can be infinite. A practical, approximate criterion can be given in terms of the initial binding energy \( E_{bind} \) and the energy radiated in GWs (see Sec. 11B) during the first encounter. Without gravitational radiation \( E_{bind} \) is a constant of motion and the orbits are

FIG. 10. \( P = 5P_{qc}, \Theta = 14.30^\circ \). Enlarging the shooting angle further compared to Fig. 9 results in a full whirl followed by a zoom to infinity (unbound orbit, no merger).

FIG. 11. This plot sketches the end state of eccentric black hole binaries in the plane spanned by our parameter choice for the initial data. The evolutions located in the grey shaded regions can be judged bound solely based on the initial data.
unbound for $E_{\text{bind}} > 0$ and bound if $E_{\text{bind}} < 0$. We find, unsurprisingly, that all orbits with $E_{\text{bind}} < 0$ also merge in our evolutions. We judge an orbit to be dynamically captured when the energy radiated during the first encounter exceeds the initial (positive) binding energy. This shortens the runtime to determine whether a run is unbound significantly because we do not have to track the black holes to larger and larger distances. Such a criterion is applicable close to the threshold between bound and unbound runs, although a few marginally bound runs may be incorrectly labeled unbound (but runs are labeled bound correctly).

We conclude with remarks on the relation to periodic orbits. Within the category of bound orbits there is a detailed classification scheme based on periodic orbits which is complete when neglecting radiation effects. In this classification one indexes all closed orbits with a triplet of integers $(z, w, v)$, where $z$ is the number of zooms within an approximate $2\pi$ period (i.e. the number of “leaves”), $v$ is the stride over the leaves $(1 \leq v \leq z - 1)$, and $w$ is the number of whirls. The total precession angle is $2\pi(w + \frac{v}{z})$. The question is whether this classification still works in an approximate sense for BHBs with radiation effects. Especially near the merger of comparable mass BHBs, the orbits shrink significantly and may not be well represented by a single periodic orbit, but rather by a sequence of them. Our findings imply that the longest whirls associated with the largest precession angles (largest $w$) occur for momenta with $P$ slightly larger than $P_c$ and are very close to a precession of $2\pi$. We also find that the dependence on $P$ is weak and beyond $P \gtrsim 2P_c$ compatible with the statement that it only depends on the mass ratio. Radiation damping seems to limit the length of the whirl phase for larger $P$, although there may be artifacts due to the initial data. In terms of periodic tables this means that we typically find preferred subsets of periodic orbits that approximate our evolutions best. The number of whirl-orbits $w$ is clearly limited by the efficiency of gravitational radiation. For equal masses $w = 1$ seems to be the largest $w$ one can obtain. For larger mass ratios $w = 2$ should also become possible somewhere beyond a mass ratio of 1:3. In the regime we are probing orbits with $z = 2, z = 3$ and $v = 1$ are favored. However, our data set contains too few data points on different mass ratios to make a strong statement.

2. Examples for orbital dynamics of BHBs

We describe the main aspects of the orbital dynamics that we find in our data set using the categorization introduced in the previous section. First we consider equal mass BHBs in the elliptic regime. All equal mass runs start at $D = 20M$ ($P = 10P_c$ has $D = 50M$) in such a way that $D$ shrinks. Obviously, the ensuing evolution depends on the values of $P$ and $\Theta$.

We discuss the orbital dynamics from low to high $\Theta$ for $P = P_c$. Puncture tracks for some values of $\Theta$ are shown in Fig. 11 while Fig. 12 shows the coordinate distance $D$. The insets of Fig. 14 show puncture tracks for some additional values of $\Theta$.

At low $\Theta$ (or equivalently for high eccentricities) $D$ monotonically shrinks leading to a rather prompt merger without completing a single orbit. The runs with larger $\Theta$ have correspondingly higher initial orbital angular momentum and manage to resist the strong gravitational pull for longer so that the merger time steadily grows. For $\Theta \approx 46^\circ$ the punctures complete one orbit before merger. At yet larger $\Theta \gtrsim 48^\circ$ the orbits begin to exhibit a circular phase (the whirl) which is maintained for longer as $\Theta$ is increased. However, at $\Theta \approx 48.5^\circ$ the orbit leaves the circle again towards larger radii (the zoom) delaying the merger significantly. In this range of $\Theta$ there is high sensitivity to the initial data (concerning merger time as a function of $\Theta$). A mild increase in $\Theta$ leads to a much larger $t_m$ because the BHs slow down as they move out before falling back. In the limit $\Theta \rightarrow 90^\circ$ the pericenter passages become shorter while the apocenters and pericenters become increasingly degenerate. The pericenter moves out with $\Theta$, hence the BHs do not cross their mutual gravitational potential as deeply and consequently not as much radiation occurs, enabling more and more orbits before merger.

Concerning the amount of precession, we see that although our evolutions start somewhere beyond the apocenter (e.g. Fig. 14), the orbits exhibit a huge precession of roughly $\pi$ and close to $2\pi$ for $\Theta = 48.5^\circ$ (followed by a tiny zoom). Even for the smallest eccentricity we studied $\Theta = 60^\circ$ ($e \sim 0.5$) one can see that the ellipses still have precessions as large as $2\pi/3$, meaning that over the
course of the whole evolution the accumulated precession
amounts to more than two entire orbits. Those values by
far exceed the amounts of precession known from mildly
relativistic systems like the famous Hulse-Taylor pulsar
c3 with a precession of $q = 0.0037^\circ$ per orbit or the bi-
nary pulsar h4 with $q = 0.0044^\circ$ per orbit. Furthermore
for these systems and their correspondingly milder grav-
itational radiation there is precession not only from one
apocenter to the next but also precession of the multi-leaf
clover as a whole very similar to the findings of studies of
periodic orbits [11,13] (or nearby aperiodic orbits). This
additional, peculiar precession effect is indeed small for
our evolutions as well, though not negligible.

Switching to the hyperbolic class the additional pos-
sibility arises that the BHs just fly past each other, de-
flecting their trajectories and escaping to infinity. This
gives rise to the merger / fly-by threshold (see insets in
Fig. 15) which we will discuss later.

We again describe the orbital phenomenology from
low to high values of $\Theta$. The qualitative features of
low $\Theta$ evolutions are the same as in the elliptic cate-
gory. The actual values of $\Theta$ that lead to analogous fea-
tures/characteristics (one complete orbit, a whirl, maxi-

We proceed by analyzing precession effects and discuss
resemblances to periodic orbits. For a given $P$ the pre-
cession angle shrinks with increasing $\Theta$ when approach-
ing the threshold as expected. The maximal amount
of precession we find is slightly larger than in the elliptic
category. We clearly recognize patterns known from
periodic orbits. For the $P = 2P_{qc}$ sequence we find
$z = 2, z = 3, z = 4$ orbits. The main difference to pe-
riodic orbits is that the orbits end in a merger after the
first leave has been traversed because of the severe radi-
ation losses. For instance $P = 2P_{qc}, \Theta = 25.1^\circ$ resembles
the $z = 3, v = 1, w = 0$ orbit with $q = 2\pi/3$. When
decreasing $\Theta$ by small amounts, the resulting orbits typ-
cally show the same amount of precession (only $D_{per}$
shrinks with $\Theta$). At some point there is a transition to
another multi-leaf clover and the precession amounts to
a value of $q = \pi$ and is now similar to the periodic orbit
labeled $z = 2, w = 0, v = 1$.

3. Unequal mass BHB and geodesic limit

Next we extend the discussion to unequal mass BHBs. By
doing so we move towards a region in parameter space
which can be increasingly well described by geodesics. In
fact, it has been in the latter regime where zoom-whirl
behavior was studied first [1]. This begs the following
question: Given a binary at a finite mass ratio, how far
away is it from the geodesic limit?

The fact that zoom whirls can be found not only for
geodesics, but also for equal masses suggests that zoom
whirls also occur for intermediate mass ratios and adds
to their expected astrophysical relevance. Indeed we
can confirm (see also [24, 30]) the presence of zoom-
whirl behavior for mass ratios 1:2 ($\nu \approx 0.2222$) and 1:3
($\nu = 0.1875$) (see Fig. 24).

As the mass ratio departs from unity, gravitational
radiation decreases, e.g. [89], which is consistent with
the trend to the geodesic limit. In the eccentric case
we find that qualitatively a similar statement still holds.
We point out, though, that there is a non-trivial depen-
dence on $\Theta$ (or inverse eccentricity). In particular, the
maximum in $E_{rad}/M_{ADM}$ (see Sec. III B 2 and Fig. 18)
is close to the equal mass values for the mass ratios we
have probed.

For lower symmetric mass ratio $\nu$ we do not find sig-
ificantly longer whirl phases in our data sets. It is to be
expected of course that for some mass ratio beyond 1:3
the whirl phases eventually will be longer and asymptote
to the geodesic limit. Highly eccentric binaries with mass
ratios up to 1:3 are in this sense still far away from the
geodesic limit.

We find evidence for the analogy of zoom-whirl dy-
namics and unstable circular orbits by investigating the
orbital radius during the whirl phase for various configu-
rations. Consistently, the whirl radius decreases with in-
creasing $P$, which we will refer to as the tightening of
the whirl. This is consistent with earlier studies [19, 21, 23].
in which it was found that the spin of the merger remnant increases with the initial angular momentum parameter, which implies a smaller radius for the unstable circular orbits.

Next, we investigate geodesics to derive lower limits on the shooting angle that separates merging from non-merging evolutions, \( \Theta_{\text{bu}} \). (The corresponding values from our evolutions can be seen as a vertical dividing line in Figs. 13, 17, and 18.) Since we find that \( \Theta_{\text{bu}} \) decreases monotonically with increasing \( P \), we expect this lower limit to be most restrictive for large \( P \). The idea is analogous to the capture/escape cavities for a photon in Schwarzschild spacetime in [90].

A null geodesic in the Schwarzschild spacetime on a circular orbit is located at a radius equal to the so-called Schwarzschild spacetime in [90]. analogous to the capture/escape cavities for a photon in Schwarzschild spacetime [90].

The proper computation of a null geodesic in Schwarzschild spacetime [90] leads to

\[
\Theta_{\text{bu}}^{\text{prod}} = 180\degree - \arcsin \left( \frac{3\sqrt{3}m}{D/2} \sqrt{1 - \frac{2m}{D/2}} \right) = 27.7\degree
\]

The same calculation for marginally bound circular orbits yields 21.8\degree. Fig. [7] indicates that neither of these limits apply to our evolutions, because there are unbound orbits with \( \Theta < \Theta_{\text{bu}}^{\text{prod}} \). Clearly, the assumption of a Schwarzschild spacetime is not a good one.

From [21, 23] we know that the merger remnant in our settings will settle down to a Kerr solution with spin parameters between 0.6 < \( a < 0.823 \) with only weak dependence on the initial conditions. Despite the fact that the Kerr metric does not describe the spacetime at merger, it may be a better approximation than Schwarzschild. The same estimate as above for Kerr spacetime yields \( \Theta_{\text{bu}}^{\text{prod}} = 14.9\degree \) for \( a = 0.6 \), \( \Theta_{\text{bu}}^{\text{prod}} = 11.4\degree \) for \( a = 0.0823 \) and \( \Theta_{\text{bu}}^{\text{prod}} = 5.7\degree \) for \( a = 1 \). The \( \Theta \)-values for 0.6 < \( a < 0.823 \) correspond rather well to the shooting angles separating bound from unbound runs in the higher momentum cases despite the fact that the limit from null-geodesics to (finite-size) equal mass binaries is by no means straightforward. We will use this analogy in interpreting our results on the radiated energy in Sec. III B 2 based on the tightening of the whirl orbits associated with a larger spin of the merger remnant.

\section{B. Radiation properties}

\subsection{1. Waveforms}

The methods used to compute quantities characterizing the GW content of the spacetime are described in e.g. [67]. Here we demonstrate how the orbital dynamics as described in Sec. III A are reflected in the GW signals.

The waveforms of quasi-circular binaries are rather well understood. To a certain extent merger waveforms as they arise from evolving quasi-circular binaries can be very similar to the ones seen in low eccentricity evolutions provided the binary circularizes before merger. For large eccentricities it is, however, natural to expect deviations from a quasi-circular BHB. We observe differences in the waveforms throughout the evolution including inspiral, onset of merger, coalescence and ring-down. Any imprints left from the eccentric inspiral have to be radiated away during this process, because the final spacetime can be described by the Kerr metric. In fact, the merger remnant reveals a different signal during ring-down [22]. In particular, quite generically high eccentricity is correlated with an amplified ring-down signal.

The inspiral features show some level of agreement with [91] and PN models for such waveforms are known analytically to 2PN order [92] (see the first comparison between numerical waveforms and Post-Newtonian ones in the eccentric regime [93]). However, features associated with zoom-whirl behavior (see Fig. 7 prior to merger) are exclusive to the strong field and thus have to be dealt with using the tools of numerical relativity. These inspiral signals will be observable by future GW interferometers such as LISA [94, 95] or eLISA/NGO [96], DECIGO [97] or the ET-telescope [98, 99].

We discuss typical waveforms of a representative subset of our evolutions. It is illustrative to go through Figs. 3, 10, 13 and their captions. Our main focus is on the richness in information stored in eccentric BHB waveforms in contrast to quasi-circular ones because of the promising implications for data analysis, see [32].

FIG. 13. Higher modes of \( r\Re(\Psi_4) \) summed over \( m \) for the equal mass case. Clearly one can see that \( l = 2 \) modes are - just as in the quasi-circular case - the dominant contribution. The plot also reveals that higher modes exhibit a much more significant contribution from higher \( l \)-modes. All odd \( l \) vanish within numerical error as expected from the quadrant symmetry of equal mass, non-spinning BHB.
Already the 22-mode shows obvious differences which become larger in other modes. For example the $l = 2$ $m = 0$-mode of a quasi-circular orbit looks just like a smaller amplitude version of the $l = 2$ $m = 2$-mode. In the eccentric case they contain completely different features. We plot the higher $l$ modes summed over $m$ in Fig. [13] for the equal mass case, for which (without BH spin) only even $l$-modes contribute by symmetry. We have computed the $l \leq 8$ modes and find that the $l = 2$ is still the largest contribution, but $l = 4$ has a significant contribution throughout the merger and $l = 6$ and $l = 8$ close to the maximum.

As an example of how different waveforms of binaries with eccentricity and mass ratio away from unity can be, we show in Fig. [24] the waveform and orbital trajectories for the mass ratio 1:3. Clearly the features induced by eccentric unequal mass BHB give rise to waveforms which effectively break degeneracies in parameter space [32, 100].

In this work we do not construct waveform templates. Longer runs will be needed in order to achieve a match to a PN waveform because of the small separations at pericenter. Performing wave extraction at larger radii is also clearly desirable in this context. With current codes this could be done at an acceptable computational cost.

### 2. Radiated energy

We compute the energy $E_{\text{rad}}$ radiated away in GWs and analyze these results together with the orbital dynamics. For the elliptic orbits we add an estimate of the radiated energy of the past evolution $E_{\text{rad}}^{\text{past}} \approx -E_{\text{bind}}(t = 0)$ to $E_{\text{rad}}$. Using this estimate we implicitly assume that the binary was isolated in its entire past. The actual value $E_{\text{bind}}(t = 0)$ for the $P = P_{\text{qc}}$ sequence turns out to be $E_{\text{rad}}^{\text{past}} \approx -E_{\text{bind}}(t = 0) \approx 0.0057 \pm 0.0001$. We normalize $E_{\text{rad}}$ by the ADM-mass of the initial time slice, $M_{\text{ADM}}(t = 0)$. The resulting quantity is what we call the “efficiency” of gravitational radiation.

The results of all our evolutions are presented in Figs. [14, 15, 16, 17, 24]. The different lines (colors, symbols) in these plots correspond to different initial momenta and each line shows the efficiency of gravitational radiation as a function of $\Theta$.

The first global feature to notice is that gravitational radiation becomes much more efficient for higher momenta. We give the maximal efficiency for 7 initial momenta $P$. So far the largest value $35 \pm 5\%$ was reported in [19, 37, 39, 102] (in which the punctures have coordinate velocities of $v = 0.94$). In our data set we come rather close to this limit (see Fig. [16]). The challenge in these studies arises from the growing significance of unphysical radiation content that is associated with the construction of initial data. Here we did not intend to push this limit further, but this shows that we have probed part of the parameter space close to the limits of former investigations. As we shall demonstrate (see Fig. [17]) the sampling is quite exhaustive and allows us to probe zoom-whirl behavior in a large class of orbits. In particular, one of our important findings is model $P = 1P_{\text{qc}}, \Theta = 60^\circ$ (elliptic class) with several close encounters before merger, see Fig. [7]. The initial eccentricity is as low as $e \sim 0.5$. This is a value within typical estimates of supermassive BHBs in galaxy merger scenarios following star- or disk-driven hardening [49] and also a value found for inspiraling binaries near galactic cores [64] which are driven to very similar eccentricities via the Kozai mechanism.

For low momenta and the mass ratios under consideration the shooting angles for the largest number of orbits in general neither coincide with the maxima in $E_{\text{rad}}$ nor do they coincide with the unstable, circular (whirl-like) orbits merging right thereafter. Generally, the maximum in $E_{\text{rad}}$ inside the hyperbolic regime lies close to the merger/fly-by threshold. However, in the limit $P \to \infty$ there appears to be a growing amount of degeneracy: the unstable-circular orbits actually seem to coincide with the most efficient radiators. In the next section we will give an interpretation for this behavior.

For low $\Theta$ we find, in agreement with previous studies [19], that the radiated energy quickly drops to the small amounts known from head-on collisions [39]. This drop can clearly be seen for every initial momentum considered in Fig. [17].

In addition, the shape of the transition from large to small $\Theta$ is by no means trivial. One of the key features in the radiated energy is that, especially in the $P = 1P_{\text{qc}}$ sequence but also for $P = 2P_{\text{qc}}$, there appear additional local extrema which match the number of encounters. The observed structure in $E_{\text{rad}}$ shows a remarkably clean periodicity as a function of $\Theta$ and should be compared with corresponding features in the final spin and mass in [23]. We find that these features are determined entirely by the dynamics during the last encounter. Zoom whirl effects in the $P = 1P_{\text{qc}}$ sequence minimize radiated energy. We find that the energy is less than that of a quasi-circular binary in direct contrast to [51]. We will interpret these observations in the next section.

Looking at our findings presented in Fig. [15] and [17] one may wonder why the additional peaks, i.e. additional encounters, are present in the lowest momentum sequence, but not in the higher momentum ones. The answer lies in the initial binding energy. For the large $P$ cases only those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures). As it turns out, the radius of capture for those evolutions which radiate a lot of energy during the first encounter will be bound orbits (dynamical captures).
Efficiency of gravitational radiation ($P=1P_{qc}$)

![Graph showing efficiency as a function of shooting angle Θ](image)

FIG. 14. Radiated energy as a function of the shooting angle Θ for the $P = 1P_{qc}$ runs. The small insets illustrate the corresponding orbital dynamics. One can see that the global maximum does not correspond to a zoom-whirl orbit. Strongest zoom-whirl behavior is rather associated with the local minimum in $E_{rad}$ near $\Theta = 48.5^\circ$. Compare with Fig. 12.

contributions during the second (and last) encounter. While we do observe similar orbital dynamics also for the higher $P$ cases, we however do not see a corresponding peak. The reason is obvious once one compares the GW amplitudes during the capturing first encounter with the amplitude during merger, see Fig. 8 and 9. For the large $P$ evolutions the mergers on the second encounter only have a negligible contribution to the radiated energy, but the whirly, capturing encounter dominates the energy loss.

Results for unequal mass runs are shown in Fig. 18. According to our findings the scaling of radiated energy with mass ratio is eccentricity-dependent. Comparing the maxima in $E_{rad}$ between equal mass and unequal mass runs we find that a mass ratio 1:2 still gives a maximal efficiency which is not too far away from the corresponding equal mass run with the same $P/M_{ADM}$. This result is in contradiction to our expectation from quasi-circular binaries where $E_{rad}$ decreases steeply with mass ratio. Also our results for mass ratios 1:3 show a similar trend suggesting that a such mass ratios still are (in the above sense) far away from the geodesic limit. Our results suggest further parameter studies to analyze the scaling in the eccentric regime along the mass ratio axis. Clearly, $E_{rad}$ is much more sensitive to $P$ rather than $\nu$.

C. New diagnostics

Many interesting questions about BHB cannot be tackled by just looking at gauge-invariant quantities. In this section we suggest new diagnostics that are helpful to interpret these spacetimes.

A first example is the observation in [22] that maxima in $E_{rad}$ coincide with a particular orbital configuration at the time of merger. Whenever the angle between the tangent vector of the puncture and the separation vector $\vec{D}$ at the time of merger is largest, the radiated energy is maximized. Here, we confirm this behavior also for $P = 2P_{qc}$ orbits in Fig. 19 demonstrating the robustness of our gauge-dependent conclusions in [22].

We interpret this empirical finding in the following way. Maximizing the above mentioned angle translates into maximizing $\vec{L} = \vec{D} \times \vec{P}$, the Newtonian expression for the angular momentum of two point masses. We therefore conjecture based on our data set that the strongest ring-down signals are caused by those evolutions which maximize the angular momentum at the moment of merger.

Another useful diagnostic is the histogram of $D(t)$ (see Fig. 20 and [22]). It measures the time the binary spends within an interval $D \pm \Delta D$ of coordinate separation. We focus on two important conclusions drawn from this plot. First, as already reported in [22] the whirls show up as a sharp and well-defined peak allowing us to measure the radii of unstable circular orbits in these highly non-
linear spacetimes. Second, the whirl radii are becoming systematically tighter as \( P \) increases (see Fig. 20), which is related to a higher Kerr-parameter of the merger remnant \([19, 24]\). We checked the coordinate separation as a function of time separately to exclude a possible issue with our merger time estimate which is not well suited for large \( P \).

For low momenta \([21, 23]\) showed that the final spin parameter lies within \( 0.6 < a < 0.0832 \) with the tendency that the spin parameter grows with the initial momentum. Thus the resulting background spacetime will have a tighter ISCO. As the binary spends considerably more time at the whirl radius than at a Newtonian pericenter at the same distance (see Fig. 20), the binary radiates much more efficiently if this whirl occurs at a smaller radius. Thus the geodesic analog together with our gauge-dependent diagnostics give a natural explanation for our earlier observations: In the high momentum case zoom-whirl orbits do coincide with the most efficient radiators where the whirls are tight, while this is not the case in the low momentum regime, where the whirl radii are significantly larger.

As a final diagnostic we present trajectories of the binaries through phase space, see Figs. 21, 22, 23 and 24. We choose \( D(t) \) and the coordinate velocity \( v(t) \) of the punctures as generalized coordinates. This construction is explicitly coordinate dependent, and switching to another gauge will lead to different trajectories. However, previous investigations led to the conclusion that the moving puncture gauge leads to puncture tracks that correspond rather well to what an observer sees from infinity, e.g. \([72, 80, 103]\). In particular, for orbiting motion one can argue based on the shift condition that this should be the case \([80]\), although for linear motion the situation is different. Hence, one has to keep the gauge issue in mind, but the moving puncture gauge leads to rather robust features in the phase space trajectories as we will discuss next.

To familiarize oneself with the trajectories in phase space consider a circular motion with constant velocity. This motion corresponds to a single point in a \( D-v \) phase space diagram. A Kepler ellipse corresponds to a line, which is curved according to Kepler’s third law.

In Fig. 21 we show runs for \( P = P_{qc} \) for four different \( \Theta \), while Fig. 22 gives a global impression of many different angles for \( P = 2 P_{qc} \) (each for equal masses). These simulations start at the lower right corner at \( D = 20 M, v = 0 \) and quickly rise due to the initial gauge adjustment, so that the coordinate velocity of each BH approaches the value given expected from the intial data. Then the trajectory moves towards the upper left as the orbits shrink and the punctures move faster. When the black holes merge the trajectory ends in the origin at \( D = 0, v = 0 \). Whirls or parts of tight circular orbits are indicated by approximately constant \( D \) but decreasing \( v \), with \( D \approx 2M-5M \). Zooms follow roughly the shape of elliptic orbits, with \( D \) varying between \( 5M-10M \) at pericenter out to an apocenter at \( 15M-27M \) in Fig. 21 or up to \( 40M \) in Fig. 22. Fig. 22 also shows one orbit for \( P = 2P_{qc} \) which escapes to infinity.
A head-on collision in phase space looks very much like an eccentric binary starting on the $v = 0$ line at the value of $D$ that corresponds to the same total energy. We note that a head-on collision always constitutes an upper envelope in the phase space, i.e. fixing $P$ and comparing $\Theta = 0$ runs with $\Theta \neq 0$ evolutions we always find $v(t, \Theta = 0) > v(t, \Theta \neq 0)$ (at least for $D(t) \leq D(t = 0)$).

We note that the motion of the punctures at the onset of merger is still rather mildly relativistic. This finding turns out to be surprisingly insensitive to the initial momentum. Fig. 23 shows runs for different $P$ with angles chosen for maximum radiation efficiency. Increasing the initial momentum of the punctures leads to a motion which is rather relativistic when entering the whirl phase, up to $v \sim 0.5$ and Lorentz factor $W \sim 1.155$ for $P = 6P_{qc}$ (see Fig. 23), but in the whirl they decelerate by large amounts (for $P = 6P_{qc}$ the decrease in velocity is as large as during the merger). At merger however all equal mass evolutions approach $v = 0.22 \pm 0.02$ or $W = 1.025 \pm 0.005$.

For a detailed look at the transition zone between bound and unbound evolutions, see Fig. 22 for the $P = 2P_{qc}$ sequence varying $\Theta$. First, we recognize the two extreme cases of merger and unbound motion. Inbetween we find a very complex transition which is governed by several additional (non-closed) loops corresponding to orbits that zoom out after a first whirl phase thereby slowing down (i.e. move to the lower right) before returning towards the upper left. Note, that during the next approach the binary must follow a path further to the lower left because the system is dissipative. The set of apocenters from all runs in this sequence forms a lower envelope (as the upper envelope mentioned before) that is never crossed by any of our evolutions at the same initial $D$. Note how far the zooms may extend when the binary approaches the merger / fly-by threshold resulting in an ever larger runtime. We face difficulties evolving orbits near $\Theta \rightarrow \Theta_{fly-by}$ because those orbits need very long evolutions (decrease of accuracy) and the black holes may reach distances close to or even beyond the wave extraction sphere, thereby producing artificial features in $E_{rad}$.

An investigation of the phase space trajectories of unequal mass BHBs reveals that the general shape of the trajectories is quite robust with respect to the mass ratio. An example is given in Fig. 24. There are two basic differences: (1) Unequal mass orbits move systematically slower compared to a corresponding equal mass binary with comparable initial momentum. (2) Unequal mass mergers start their final plunge from an increasingly larger coordinate separation than equal mass mergers. With re-
FIG. 18. Radiated energy $E_{\text{rad}}$ for mass ratios 1:1, 1:2 and 1:3 and some particular values of $P$. Qualitatively the results for unequal masses are similar to those for equal masses. For example, note the drop in $E_{\text{rad}}$ after a global maximum, the global maximum corresponds to an orbit that roughly completes 1 orbit, and zoom-whirl behavior for low $P$ is associated with inefficient radiation as in the equal mass regime. For $P = 2P_{qc}$, zoom-whirl behavior for mass ratio 1:2 occurs for larger $\Theta$ than in the 1:1 case.

FIG. 19. The inner region of puncture tracks from two different runs. The $P = 2P_{qc}$, $\Theta = 24.1^\circ$ run (blue solid line) corresponds to the global maximum in $E_{\text{rad}}$. The yellow straight lines through the origin represent the separation vectors $\vec{D}$ at the time when a common horizon forms. The result [22] that most efficient mergers occur when the tangent vectors of the orbits are closest to being orthogonal to $\vec{D}(t_m)$ carries over to larger $P$.

FIG. 20. Histogram of the coordinate separation $D$ for three different evolutions with $P = 2, 4, 6P_{qc}$. This plot shows how much time a binary has spent at a given separation $D$. All runs shown here correspond to the longest whirl phase found at each fixed momentum. Clearly visible is the tightening of the whirl radius for larger $P$ (compare with [22]). The overlap of the shaded region in $P = 6P_{qc}$ with the histogram happens because during the whirl the separation is indeed shorter than at the onset of merger.

FIG. 21. Phase space for $P = 1P_{qc}$ sequence. The $\Theta = 48.2^\circ$ evolution takes a detour in phase space thereby avoiding the region where radiation is most efficient. The $\Theta = 47^\circ$ evolution radiates more efficiently because it reaches further towards the upper left and at the same time spends considerable time at low $D$ (see Fig. 6 in [22]).

gard to gravitational radiation, unequal mass binaries do not tap as deeply into the gravitational potential as equal mass binaries, thus cannot extract as much energy from the spacetime. Furthermore, the deceleration is milder for higher mass ratios, again suggesting weaker signals from higher mass ratios.
FIG. 22. Phase space for $P = 2P_{qc}$ sequence. All runs except $\Theta = 25.5^\circ$ are bound. Near the threshold of immediate merger the zooms ($24.3^\circ \leq \Theta \lesssim 24.9^\circ$) may extend far out. The inset shows a close-up of the inner region where whirls occur (the point density is enlarged there).

FIG. 23. Phase space for different initial momenta $P$ with angles chosen for near maximal $E_{\text{rad}}$. Interestingly, close to the merger all binaries reach the same coordinate velocity independent of the large differences in initial conditions. This effect appears as a blurred "focal" point in phase space. Hence the whirl itself becomes more important for $E_{\text{rad}}$ than the actual merger. The deceleration in the $6P_{qc}$ whirl phase is larger than during the merger.

IV. CONCLUSIONS

Numerical relativity has confirmed the existence of zoom-whirl orbits beyond the geodesic and PN regime [18–22, 24, 26], thereby emphasizing their general relativistic origin. Previous studies explored the rich extension to the phenomenology of the GR two-body problem offered by zoom-whirl dynamics in various ways. In this work we performed numerical relativity simulations to investigate the parameter space of comparable mass, non-spinning, eccentric BHB for low and intermediate momenta more comprehensively than before. We explored zoom-whirl behavior in both the hyperbolic and elliptic regime carrying out more than 100 numerical evolutions in order to obtain a decent sampling of the underlying parameter space. We discussed various features of the orbits and characterized the corresponding GW emission, and we developed new diagnostics to analyze binary spacetimes by using phase space trajectories and a histogram of the coordinate separation.

For elliptic orbits, we discover zoom-whirls with imprints in the GWs that are comparable in amplitude to the merger waveform for eccentricities as low as $e \sim 0.5$. This is an important finding for the astrophysical relevance of zoom-whirl orbits. In particular, such values are within expected eccentricities of supermassive BHB that have resulted from galaxy mergers and subsequent star- or gas-driven hardening [49]. For low momenta, there occur several minima and maxima in the radiated energy when varying the shooting angle from head-on to quasi-circular orbits. We demonstrate that zoom-whirl dynamics may actually minimize the radiated energy in sharp contrast to [42].

In the elliptic regime whirls are found only during their last encounter. They emerge in disjunct intervals of the initial angular momentum (i.e. shooting angle). Apparently, as long as the binaries are not circularized just prior to plunge, zoom-whirls can always be found during the last encounter by a very modest amount of fine-tuning. In the hyperbolic regime we find that all evolutions that
lead to dynamical capture reveal whirl features during the capturing encounter and then simply plunge during the following encounter, potentially from a large separation.

High-momentum zoom-whirls maximize the radiated energy. The Kerr spacetime that the merger remnant will settle down to exhibits a larger spin parameter. This translates into a tighter unstable circular orbit (i.e. the whirl radius) resulting in more pronounced dynamics of the mass quadrupole. Especially the first, capturing close-encounter burst in high momentum evolutions can easily overwhelm the merger signal.

In the unequal mass case eccentric BHB are found to be more efficient radiators than expected from quasi-circular studies of unequal mass BHB. As a consequence the whirs are not significantly longer for the mass ratios under consideration. We note that numerical relativity is seeing improvements in dealing with large mass ratios [104, 105]. More detailed studies for unequal masses and also the inclusion of spin are promising directions for eccentric BHB simulations in the future.

The present work as well as other studies strongly suggest to include eccentricity in the waveform templates used in the data analysis of GW detectors, since eccentricity effectively breaks degeneracies in parameter space.

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