A lending scheme for a system of interconnected banks
with probabilistic constraints of failure

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\textbf{Abstract}

We derive a closed form solution for an optimal control of interbank lending subject to terminal probability constraints on the failure of a bank. The solution can be applied to a network of banks providing a general solution when aforementioned probability constraints are assumed for all the banks in the system. We also show a direct method to compute the systemic relevance of any node within the financial network itself. Such a parameter being the fundamental one in deciding the accepted probability of failure, hence modifying the final optimal strategy adopted by a financial supervisor aiming at controlling the system.

\textbf{1 Introduction}

One of the most relevant changes within the financial world has been caused by the worldwide crisis that dates back to the biennium 2007/2008. In particular, starting from that breaking event, financial analysts, bank practitioners, applied mathematicians and economists, have been pushed to rethink the models they had used to work with in order to stop relying on a series of assumptions turned up to be too far from the rude concreteness of real markets, the mutated financial worldwide scenario and its changed functioning. In particular, such a \textit{big crunch} along with its consequences within worldwide markets, forced both investors and financial institutions to take consciousness that almost every financial quantity is exposed to a failure risk.

As an example, the standard Black and Scholes (BS) model, whose unrealistic restriction on coefficients have been the focus of several studies which have determined a plethora of alternative yet effective approaches, has shown its limit also from the time point of view. In fact, it is based on assuming a fixed terminal time, a particular that the above mentioned crisis has simply wipe out. Within such an upset scenario, the \textit{credit risk} analysis has seen an increasing interest within the theoretic financial community, implying the development of rigorous models able to suitably taking into account the \textit{risk exposure} factor and then appropriately treat \textit{default events}.

Along aforementioned lines, two main approaches have been developed: \textit{totally unexpected failure} and \textit{triggered failure}. Mathematically speaking, the first scenario is realized by considering a default event which is completely inaccessible for the probabilistic reference filtration. To deal latter situation, a typical method is based on \textit{filtration enlargement}, see, e.g., \cite{1,20}. While, allowing for a default triggered by an event which is accessible for the reference filtration, typically implies the assumption that it happens as soon as the trajectory of the main driving equation exits from a predetermined domain, see, e.g., \cite{1,16}.
The present work aims at analyse the latter approach, taking into consideration a network of interconnected financial entities, as banks or general economic agents, willing to lend money one to each other. The failure event for each agent, being represented by its capital hitting a barrier whose value is derived according to the general setting characterizing the whole system. As a main reference setting we refer to the one first introduced in [13], then generalized in [7, 15, 21]. In particular, along the lines defined in [7], we consider a financial supervisor, usually referred as lender of last resort (LOLR), aiming at guarantee the wellness of the financial network, hence lending money to those agents which are near to default, but also minimizing a given cost function. Therefore, the entire framework will be mathematically treated as a stochastic optimal control problem, defined over a set of communicating elements interacting with a central controller, provided specific rules and an aggregate cost function.

The main novelty of the present work is that, in addition to consider a LOLR that lend money aiming at minimizing a given cost, we also assume that there are fixed probability constraints the banks should satisfy at terminal time, or from a financial point of view the optimal strategy of the LOLR will be derived on the strict assumption that each bank has a certain probability of failure. In particular, as done in [16], we assume that a bank may fail only at a fixed terminal time, so that the financial entity will face bankruptcy if at terminal time its wealth is below a given threshold. We would like to underline that this work represents a first step within the analysed financial scenario, several generalizations being already in our research agenda. In particular, we shall consider a sequence of fixed times of possible failures, in a setting similar to [8], each with a given probability constraints, also considering constraints for any time $t \in [0, T]$. In what follows the the optimal strategy will be derived exploiting techniques related to stochastic target problems. Let us recall that the first results in this direction has been derived in [22], where an ad hoc dynamic programming principle has been provided. Later, several papers appeared generalizing such result to consider different cases of constraints, spanning from expectation constraints at fixed time, to almost sure constraints, see, e.g., [3, 4, 5, 6, 14], while in [18], an optimal solution is derived within a similar setting, but without using the stochastic target problem approach. Let us underling that, in the above mentioned papers, examples of concrete solutions often missing. This is why, at the end of this work, we consider a realistic example, comparing our result with the one obtained in [7], where, for the sake of clarity, we limit ourselves to a small set of interconnected banks, the case of larger network being of easy derivation. Moreover, since the construction of the model is strongly based on the mathematical theory of network, we will exploit its characteristics to derive a method based on the notion of page rank, first introduced in [19], to derive the relative importance of any bank in the network. This quantity will be then used to decide the admitted probability of each bank’s failure, requiring that important banks have the larger non-failure probability, hence adopting a too big to fail approach.

The present work is structured as follow: in Section 2 we introduce the main setting, giving the mathematical and financial definitions; in Section 3 we introduce the optimal control problem with probability constraints and we provide its solution; in Section 3.1 we apply derived results to a toy example.

## 2 The general setting

Before considering the financial setting, let us first introduce the mathematical notation needed to properly treat the general financial scenario. Let us consider a finite connected financial network identified with a graph $\mathcal{G}$ composed by $n \in \mathbb{N}$ vertices $v_1, \ldots, v_n$, corresponding to $n$ banks, and $m \in \mathbb{N}$ edges $e_1, \ldots, e_m$, assumed to be normalized on the interval $[0, 1]$, which represents interaction between the $n$ banks. In what follows we will use Greeks letters $\alpha, \beta, \gamma = 1, \ldots, m$ to denote edges, whereas Latin letters $i, j, k = 1, \ldots, n$, will denote vertexes. We refer to [10, 11, 17], for further details.

The structure of the graph is based on the incidence matrix $\Phi := \Phi^+ - \Phi^-$, where the sum is intended componentwise and $\Phi = (\phi_i,\alpha)_{n \times m}$, together with the incoming incidence matrix...
The general setting

Let us notice that the matrix \( I_n \) is row stochastic, in the sense that

\[
\Pi(v_i) = \{ \alpha \in \{1, \ldots, m\} : |\phi_{i,\alpha}| = 1 \},
\]

represents the set of incident edges to the vertex \( v_i \). We also introduce the adjacency matrix \( I = (i_{i,j})_{n \times n} \), defined as \( I := I^+ + I^- \), where \( I^+ = (i^+_{i,j})_{n \times n} \), resp. \( I^- = (i^-_{i,j})_{n \times n} \), is the incoming adjacency matrix, resp. outgoing adjacency matrix, defined as

\[
i^+_{i,j} = \begin{cases} 1 & \text{it exists } \alpha = 1, \ldots, m : v_j = e_\alpha(1), v_i = e_\alpha(0), \\ 0 & \text{otherwise} \end{cases},
\]

\[
i^-_{i,j} = \begin{cases} 1 & \text{it exists } \alpha = 1, \ldots, m : v_j = e_\alpha(0), v_i = e_\alpha(1). \\ 0 & \text{otherwise} \end{cases}
\]

Let us notice that \( I^+ = (I^-)^T \), then \( I \) is symmetric with null entries on the main diagonal.

Following the financial network setting proposed in [13, 21], we consider a network composed by \( n \) vertexes, each of them representing a different financial agent, and we denote by \( X_i(t) \) the value of the \( i \)-th agent at time \( t \in [0, T] \), being \( T < \infty \) a fixed positive terminal time. Each node may have nominal liabilities to other nodes directly connected with it. In this case, we denote by \( L_{i,j}(t) \) the payment that the bank \( i \) owes to the bank \( j \), at time \( t \in [0, T] \). Then, we introduce the time-dependent liabilities matrix \( L(t) = (L_{i,j}(t))_{n \times n} \), defined as

\[
L_{i,j}(t) = \begin{cases} L_{i,j}(t) & i^+_{i,j} \neq 0, \\ 0 & \text{otherwise} \end{cases},
\]

so that there cannot be any cash flow between two banks that are not edge-connected. At any time \( t \in [0, T] \), vertex \( v_i \) may also have an exogenous cash inflow \( X^i(t) \geq 0 \).

We will denote by \( u_i(t) \) the payment made at time \( t \in [0, T] \) by \( v_i \), whereas \( \bar{u}_i(t) = \sum_{j=1}^n L_{i,j}(t) \) is the total nominal obligation of the node \( i \) towards all other nodes. Clearly, we have that if \( \bar{u}_i(t) = u_i(t) \), then node \( i \) has satisfied all its liabilities. We also introduce the relative liabilities matrix \( \Pi(t) = (\pi_{i,j}(t)) \) defined as

\[
\begin{cases} \frac{L_{i,j}(t)}{u_i(t)} & \bar{u}_i(t) > 0, \\ 0 & \text{otherwise} \end{cases}.
\]

Let us notice that the matrix \( \Pi(t) \) is row stochastic, in the sense that \( \sum_{j=1}^n \pi_{i,j}(t) = 1 \), so that \( \pi_{i,j}(t) \) represents the proportion of the total debt at time \( t \) that the node \( i \) owes to the node \( j \). Similarly, we can define the cash inflow of the node \( i \) as the sum of the total payment that node \( i \) receives at time \( t \) by other nodes, that is \( \sum_{j=1}^n (\Pi_{i,j}(t))^T u_j(t) \), i.e. \( \sum_{j=1}^n \Pi_{i,j}(t)u_j(t) \), plus the exogenous cash inflow \( X^i(t) \).

We thus have that the value of \( v_i \) at time \( t \in [0, T] \) is given by

\[
V^i(t) = \sum_{j=1}^n (\Pi_{i,j}(t))^T u_j(t) + X^i(t) - u_i(t).
\]

Let us now introduce the notion of clearing vector as in [13, Definition 1], see also, e.g., [21, Definition 2.6]. In what follows, if not otherwise specified, we will use standard pointwise ordering for vectors in \( \mathbb{R}^n \), namely for every \( x, y \in \mathbb{R}^n \) it holds \( x \leq y \) if and only of \( x_i \leq y_i \), for any \( i = 1, \ldots, n \).
**Definition 2.0.1.** In a financial setting defined as above, a clearing vector is a vector \( u^*(t) \in [0, \bar{u}(t)] \) such that it satisfies the following:

**Limited liabilities**

\[
u_i^*(t) \leq \sum_{j=1}^{n} (\Pi_{i,j}(t))^T u_j^*(t) + X^i(t), \quad i = 1, \ldots, n;
\]

**Absolute priority** it happens that either obligations are paid in full, or all value is paid to creditors, that is \( u_i^*(t) = \bar{u}_i(t) \) if \( \bar{u}_i(t) \leq \sum_{j=1}^{n} (\Pi_{i,j}(t))^T u_j^*(t) + X^i(t) \) and \( u_i^*(t) = \sum_{j=1}^{n} (\Pi_{i,j}(t))^T u_j^*(t) + X^i(t) \) otherwise.

Existence and uniqueness of a clearing vector, in the sense of Def. 2.0.1, is treated in [13, 21]. In particular, in [13] it is shown that \( u^*(t) \) is a clearing vector if and only if

\[
u^*(t) = \bar{u}_i(t) \land \left( \sum_{j=1}^{n} (\Pi_{i,j}(t))^T u_j^*(t) + X^i(t) \right), \tag{2}
\]

Equation (2) can be interpreted as follows: the term \( \bar{u}_i(t) \) represents what the node \( i \) owes to other nodes at time \( t \in [0, T] \), whereas the second term \( \left( \sum_{j=1}^{n} (\Pi_{i,j}(t))^T u_j^*(t) + X^i(t) \right) \) represents the cash inflow of the node \( i \) at time \( t \in [0, T] \). In this setting a clearing vector represents the payment at time \( t \) of each node, hence each node pays the minimum between what it has and what it owes. Therefore, combining equation (1) and (2), we say that the bank \( i \) is in default if it is not able to meet all of its obligations. Consequently, the value of a bank equals

\[
V^i(t) = \left( \sum_{j=1}^{n} (\Pi_{i,j}(t))^T \bar{u}_j(t) + X^i(t) - \bar{u}_i(t) \right)^+, \tag{3}
\]

where \( (f(x))^+ \) denotes the positive part of the function \( f \). It follows that if \( V^i(t) \leq 0 \), then the bank \( i \) is in default, and we set its value to 0.

For the ease of notation, let us define the matrix \( \bar{L} = \left( \bar{L}_{i,j} \right)_{n \times n} := L - \mu u(t) \), where \( 1 \) stands for the \( n \times n \) identity matrix and \( u(t) = (u_1(t), \ldots, u_n(t)) \). The matrix \( \bar{L} \) has entry \( L_{i,j}(t) \) off diagonal, and \( -\sum_{j=1}^{n} L_{i,j}(t) \), representing the total payment that the bank \( i \) owes at time \( t \) to other nodes, on the main diagonal.

According to [15], we assume that liabilities between banks evolve according to the following deterministic equation

\[
\frac{d}{dt} L_{i,j}(t) = \mu_{i,j} L_{i,j}(t), \tag{4}
\]

for a fixed positive growth rate \( \mu > 0 \), non necessarily risk-neutral. Similarly, we assume the bank \( i \), at any time \( t \), invests the difference between cash inflow and cash outflow in an exogenous asset \( X^i(t) \) that evolves according to the following stochastic differential equation

\[
dX^i(t) = X^i(t) \left( \mu^i dt + \sigma^i dW^i(t) \right), \quad i = 1, \ldots, n.
\]

Moreover, again following [15], we introduce continuous (deterministic) default boundaries as follows

\[
X^i(t) \leq v^i(t),
\]

with

\[
v^i(t) := \begin{cases} R^i \left( \bar{u}_i(t) - \sum_{j=1}^{n} (\Pi_{i,j}(t))^T \bar{u}_j(t) \right) & t < T, \\ \bar{u}_i(t) - \sum_{j=1}^{n} (\Pi_{i,j}(t))^T \bar{u}_j(t) & t = T, \end{cases}
\]

being \( R^i \in (0,1), i = 1, \ldots, n \), suitable constants representing the recovery rate of the bank \( i \).
3 The stochastic optimal control with probability constraints

In what follows, we introduce the mathematical formulation of the optimal control problem we are interested in. Following [7], we consider a financial supervisor, called lender of last resort (LOLR), connected to any node belonging to the financial network. The LOLR aims at saving the network from default, being assumed to have full information about the network state. In particular, at any time $t$ the LOLR can lend money to the bank $i$, $i = 1, \ldots, n$, so that the controlled evolution of the bank $i$ satisfies

$$
dX^i_t = (\mu^i X^i_t + \alpha^i(t)) \, dt + \sigma^i X^i_t \, dW^i_t, \quad i = 1, \ldots, n,$$

being $\alpha^i(t)$ the loan from the LOLR to the bank $i$, at time $t \in [0, T]$ and such that $\alpha \in \mathcal{A}$, where

$$\mathcal{A} := \{ \alpha = (\alpha(t))_{t \in [0, T]} : \alpha(t) \geq 0 \text{ is } \mathcal{F}_t \text{-adapted}, \quad \int_0^T |\alpha(t)| \, dt < \infty \}.$$ 

In order to derive a closed form solution, we will consider the setting proposed originally by Merton in [10]. Therefore, we assume that default can happen only at some fixed time $t_i$, $i = 1, \ldots, l$, $l < \infty$, hence allowing to consider constraints defined only at terminal time, with no needs to introduce strong bonds at each time $t \in [0, T]$.

In particular, for a fixed terminal time $T$, we will assume that the bank $i$ can fail only at terminal time. Let us note that the above result can be generalized to the case where banks may only fail at some discrete time $t_1 < t_2 < \cdots < t_M = T$, if one is willing to separately consider any control problem between two fixed time $[t_i, t_{i+1}]$. This assumption allows to obtain a global control solution by gluing together solutions to a sequence of ordered optimal control problems, by exploiting results presented along subsequent sections. We shall study latter scenario in a future research, by mean of the results here provided to derive the global optimal solution via a backward induction methods, hence using what has been done in in [8, 20].

Assuming that the LOLR aims at minimizing lend resources implies that he try to minimize the functional

$$J(\alpha) = \frac{1}{2} \int_0^T \|\alpha^i(s)\|^2 \, ds, \quad i = 1, \ldots, n.$$  

Moreover, the LOLR minimizes [7] over the probabilistic constraint

$$\mathbb{P}(X^i(T) \geq v^i(T)) \geq q^i, \quad i = 1, \ldots, N,$$

for some suitable constant $q^i \in (0, 1), i = 1, \ldots, N$. For the ease of notation, in what follows we will drop the index $i$. Hence, with respect to the agent $i$, we will solve the general control problem, then we apply such result to all banks in the system.

Let us thus consider the value of a bank evolving according to

$$dX(t) = (\mu X(t) + \alpha(t)) \, dt + \sigma X(t) \, dW(t), \quad t \in [0, T],$$

$$X(0) = x,$$

and the corresponding default value $v := v(T)$ at terminal time. Let us assume that the external supervisor chooses the control $\alpha$ minimizing the following criterion

$$J(t, \alpha) = \frac{1}{2} \int_t^T \|\alpha(s)\|^2 \, ds, \quad \text{s.t. } \mathbb{P}(X(T) \geq v) \geq q,$$  

for a fixed probability $q \in (0, 1)$. 

The stochastic optimal control with probability constraints

5
3.1 Reduction to a stochastic target problem

In the current section we are going to formally introduce the Hamilton–Jacobi–Bellman (HJB) equation associated to the control problem defined by eq. 7, subjected to constraint given in eq. 8, reducing the related optimal control problem to a stochastic target one. The statement of an equivalent stochastic optimal control will allow us to derive a HJB equation.

Since the optimal control reads as follow

\[
J(t, \alpha) = \frac{1}{2} \int_t^T \|\alpha(s)\|^2 \, ds, \quad \text{s.t.} \quad \mathbb{P}(X(T) \geq v) \geq q, \quad (PC)
\]

and because we can rewrite the terminal probability in eq. (PC) as an expectation, namely

\[
\mathbb{P}(X(T) \geq v) = \mathbb{E} \left[ \mathbb{1}_{\{X(T) \geq v\}} \right],
\]

then we have the following

**Lemma 3.1.** Given the stochastic optimal control problem with terminal probability constraints (PC), then the terminal probability constraints holds if and only if there exists an adapted sub-martingale \( (P(s))_{s \in [t,T]} \) such that

\[
P(t) = q, \quad P(T) = \mathbb{1}_{\{X(T) \geq v\}}.
\]

*Proof.* Let us first prove \((\Rightarrow)\): since \( P(s) \) is a sub-martingale we have that

\[
\mathbb{E} \left[ \mathbb{1}_{\{X(T) \geq v\}} \right] \geq \mathbb{E} [P(T)] \geq P(t) = q.
\]

To prove the converse implication \((\Rightarrow)\), let us first denote

\[
q_0 := \mathbb{E} \left[ \mathbb{1}_{\{X^*(T) \geq v\}} \right],
\]

\[
P(s) := \mathbb{E} \left[ \mathbb{1}_{\{X^*(T) \geq v\}} | \mathcal{F}_s \right] - (q_0 - q),
\]

where \( X^* \) represents the solution with initial time \( s \in [t,T] \), then \( P \) is an adapted martingale and the claim follows.

Let us note that when the probability constraints is active, the submartingale \( P \) is given by

\[
P(s) = q + \int_t^T P(s) (1 - P(s)) \, dW(s), \quad (11)
\]

where \( \alpha_P \), taking values in \( \mathbb{R} \), is a new control which, a priori, cannot be assumed to be bounded, because it comes from the martingale representation theorem.

**Remark 3.2.** Since \( P \) represents the probability required to satisfy a terminal constraint, we could have defined \( P \) in equation (11) as

\[
P(s) = q + \int_t^T P(s) (1 - P(s)) \alpha_P(s) \, dW(s),
\]

so that \( P \) lies in \( [0,1] \).

We are now to explicitly derive the associated HJB equation that must be solved, see [4, 5, 22] for more details as well as for both existence and uniqueness proofs of associated viscosity solution.

The problem we are interested in can be further reduced introducing the set

\[
D = \{(t, x, q) \in [0,T] \times \mathbb{R} \times [0,1] : \mathbb{1}_{\{X(T) \geq v\}} - P(T) \geq 0 \quad \mathbb{P} \text{ a.s.}\},
\]
along with considering the new state variable $P$, see equation [11], in such a way that, via the geometric dynamic programming principle proved in [22], we can define the value function

$$ V(t, x, q) = \inf \left\{ \frac{1}{2} \int_t^T \alpha^2(s) \, ds \quad \text{s.t.} \quad \mathbb{I}_{[X(T) \geq 0]} - P(T) \geq 0 \quad \mathbb{P} \text{-a.s.} \right\}. $$

(12)

Since $V$ is non-decreasing in $q$, we have

$$ V(t, x, 0) \leq V(t, x, q) \leq V(t, x, 1), \quad q \in (0, 1), $$

therefore $V(t, x, 0)$ corresponds to the unconstrained problem, hence its value function is given by $V(t, x, 0) = 0$. As regards to the upper bound, we set $V(t, x, 1) = \infty$, and we prolong the value function outside $[0, 1]$, setting $V(t, x, q) = 0$, resp. $V(t, x, q) = \infty$, for $q < 0$, resp. for $q > 1$.

Let us then introduce the Hamiltonian that must be satisfied by the unconstrained optimal control

$$ H^X(x, \alpha, p, Q_x) = (\mu x + \alpha) p + \frac{1}{2} \sigma^2 x^2 Q_x + \alpha^2. $$

(13)

Intuitively, we are expecting that, when the terminal constraint is satisfied, one can solve the classical associated HJB equation whose Hamiltonian is given in equation (13), deriving that the optimal control coincides with the unconstrained case. Notice that the optimal solution to the present problem is $\alpha = 0$.

As regard the constrained case, taking into account the new martingale process $P$, we have to consider the couple

$$ dX(s) = (\mu X(s) + \alpha(s)) \, ds + \sigma X(s) \, dW(s), $$

$$ dP(s) = \alpha P(s) \, dW(s), $$

so that we can define the constrained Hamiltonian as

$$ H^{(X, P)}(x, \alpha, p, Q_x, \alpha P, Q_{xP}, Q_q) = (\mu x + \alpha)p + \frac{1}{2} \sigma^2 x^2 Q + \alpha^2 + \sigma x Q_{xP} \alpha P + \frac{1}{2} \alpha^2 P Q_q, $$

(14)

which should play the role of the Hamiltonian of the associated problem when the constraint is binding. Therefore, the HJB associated to the optimal control is

$$ - \partial_t V - \inf_{\alpha \in A} \inf_{\alpha P \in \mathbb{R}} H^{(X, P)}(x, \alpha, \partial_x V, \partial_x^2 V, \alpha P, \partial_{xP}^2 V, \partial^2_q V) = 0, $$

where above and in what follows, for the ease of notation, we avoided writing explicitly the dependencies of $V(t, x, q)$.

As mentioned above, $\alpha P$ could be unbounded, implying that the associated Hamiltonian may be infinite. Since the following holds

$$ H^{(X, P)}(x, \alpha, p, Q_x, \alpha P, Q_{xP}, Q_q) \geq H^X(x, \alpha, p, Q_x), $$

to evaluate the minimum of $H^{(X, P)}$ w.r.t. $\alpha P$, we can exploit a first order optimality condition that

$$ \alpha P = -\sigma x \frac{Q_{xP}}{Q_q}, $$

which, when plugged into equation (14), gives us the minimum of $H^{(X, P)}$ as follows

$$ \begin{cases} 
(\mu x + \alpha)p + \frac{1}{2} \sigma^2 x^2 Q + \alpha^2 - \frac{1}{2} \sigma^2 x^2 Q_{xP} & \quad Q_q > 0, \ 
(\mu x + \alpha)p + \frac{1}{2} \sigma^2 x^2 Q + \alpha^2 & \quad Q_q = 0 = Q_{xP}, \ 
-\infty & \quad \text{otherwise}.
\end{cases} $$

(15)
Therefore the associated value function has to solve the following HJB
\[-\partial_t V - \inf_{\alpha \in A} \bar{H}(x, \alpha, \partial_x V, \partial^2_x V, \partial^2_{xq} V) = 0,\] (16)
where the Hamiltonian \( \bar{H} \) is defined as in equation (15).

It follows that the value function introduced in equation (12) solves the following HJB equation
\[-\partial_t V - \inf_{\alpha \in A} \bar{H}(x, \alpha, \partial_x V, \partial^2_x V, \partial^2_{xq} V) = 0,\] (17)
subjected to the terminal condition
\[V(T, x, q) = \begin{cases} 0 & x \geq v, \\ \infty & \text{otherwise}. \end{cases}\]

Let us further assume that the LOLR can change the interest rate at which the banks assets can accrue, hence restricting the admissible control to be of the form
\[\alpha(t) = \psi X(t),\]
for a fixed constant \( \psi \in [0, \Psi] \), \( \Psi \in \mathbb{R}_+ \cup \{\infty\} \). In what follows we derive explicit solution to the optimal rate \( \psi \) that the LOLR shall give to the each bank to guarantee its terminal probability of survival.

The strategy is as follows: we compute the solution to the above problem in terms of contour line of a function \( \psi(t, x) \). In particular, defining
\[D(\bar{\psi}) := \{(t, x) : \psi(t, x) = \bar{\psi}\},\]
we have that, for any point \( (t, x) \in D(\bar{\psi}) \), the optimal control is given by \( \bar{\psi} \in [0, \Psi] \).

Then, we compute the reachability set with a fixed constant control \( \bar{\psi} \), that is
\[W^{\bar{\psi}}(t, x) = \mathbb{P}\left(X^{t, x; \bar{\psi}}(T) \geq v(T)\right) = \mathbb{E}\left[1_{\{v(T) \leq \bar{\psi}\}}(X^{t, x; \bar{\psi}}(T))\right],\]
where \( X^{t, x; \bar{\psi}}(T) \) denotes the value at time \( T \) with initial datum \( (t, x) \) and control \( \bar{\psi} \in [0, \Psi] \). By the Feynman–Kac theorem, we have that \( W^{\bar{\psi}}(t, x) \) solves the parabolic PDE
\[
\begin{cases}
W^{\bar{\psi}}(T, x) = 1_{\{\bar{\psi}(T) \leq \bar{\psi}\}}(x), \\
-\partial_t W^{\bar{\psi}}(t, x) = \partial_x W^{\bar{\psi}}(t, x) (\mu + \bar{\psi}) x + \frac{1}{2} \sigma^2 x^2 \partial^2_x W^{\bar{\psi}}(t, x),
\end{cases}
\]
whose solution can be explicitly computed to be
\[W^{\bar{\psi}}(t, x) = \mathbb{P}\left(\log X^{t, x; \bar{\psi}}(T) \geq \log v(T)\right) =
\begin{aligned}
&= \mathbb{P}\left(W(T - t) \geq \frac{1}{\sigma} \left(\log \frac{v(T)}{x} - \left(\mu + \bar{\psi} - \frac{\sigma^2}{2}\right)(T - t)\right)\right) = \\
&= \frac{1}{2} \left(1 - \text{Erf}\left(\frac{\log \frac{v(T)}{x} - \left(\mu + \bar{\psi} - \frac{\sigma^2}{2}\right)(T - t)}{\sqrt{2\sigma^2(T - t)}}\right)\right) = \\
&= \frac{1}{2} \left(1 - \text{Erf}\left(d(\mu, \bar{\psi}, \sigma, T - t)\right)\right),
\end{aligned}\] (18)
with
\[d(\mu, \bar{\psi}, \sigma, T - t) := \frac{\log \frac{v(T)}{x} - \left(\mu + \bar{\psi} - \frac{\sigma^2}{2}\right)(T - t)}{\sqrt{2\sigma^2(T - t)}}.\]
3.1 Reduction to a stochastic target problem

We can further derive two quantities of interest w.r.t. the explicit solution of the optimal control problem, namely the highest reachable probability and the switching region limiting no-action region. Concerning the first quantity, we first define

\[ W^H(t,x) := \sup \{ q : V(t,x,q) < \infty \} = \sup_{\psi \in [0,\Psi]} \mathbb{P} \left( X^{t,x;\psi}(T) \geq v(T) \right) ; \]

then the highest reachable probability is attained when considering the maximum admissible control \( \Psi \), and we have two possible cases: (1) if \( \Psi = \infty \), then

\[ W^H(t,x) = \mathbb{P} \left( X^{t,x;\Psi}(T) \geq v(T) \right) = 1 ; \]

otherwise, case (2), since \( \Psi < \infty \), analogously reasoning we have

\[ W^H(t,x) = \mathbb{P} \left( X^{t,x;\Psi}(T) \geq v(T) \right) = \frac{1}{2} \left( 1 - \text{Erf} \left( d(\mu,\Psi,\sigma,T-t) \right) \right) . \]

Assuming that \( W^H(t,x) = \tilde{q} \in (0,1) \), we have that

\[ \frac{1}{2} \left( 1 - \text{Erf} \left( d(\mu,\Psi,\sigma,T-t) \right) \right) = \tilde{q} , \]

hence, solving for \( \Psi \), we obtain the boundary region in implicit form

\[ \Psi(t,x;\tilde{q}) = \left( \frac{\sigma^2}{2} - \mu \right) + \frac{\log \left( \frac{v(T)}{x} \right)}{T-t} - \frac{\sigma \rho}{\sqrt{T-t}} , \quad (19) \]

with

\[ \rho := \sqrt{2} \text{Erf}^{-1} \left( 1 - 2\tilde{q} \right) , \]

implying that, for a required probability of success \( \tilde{q} \), the control problem is feasible for starting data \((t,x)\) that are to the right hand side of \( \Psi(t,x) \), as in equation (19). Clearly, if \( \Psi = \infty \), that is the LOLR is willing to give a possibly infinite return rate, any point is controllable, so that we can always find an admissible control such that the terminal probability constrain is attained.

About the second quantity mentioned before, namely the level set above which the null control \( \psi_0 \equiv 0 \) is optimal, we start computing

\[ W^0(t,x) = \mathbb{P} \left( X^{t,x;\psi_0}(T) \geq v(T) \right) = \frac{1}{2} \left( 1 - \text{Erf} \left( d(\mu,\psi_0,\sigma,T-t) \right) \right) , \]

then, solving explicitly such boundary, and by assuming that \( W^0(t,x) = \tilde{q} \in (0,1) \), we have

\[ \frac{1}{2} \left( 1 - \text{Erf} \left( d(\mu,\psi_0,\sigma,T-t) \right) \right) = \tilde{q} , \]

and, solving for \( \psi_0 \), we obtain the boundary region

\[ 0 = \psi_0(t,x;\tilde{q}) = \left( \frac{\sigma^2}{2} - \mu \right) + \frac{\log \left( \frac{v(T)}{x} \right)}{T-t} - \frac{\sigma \rho}{\sqrt{T-t}} , \quad (20) \]

where

\[ \rho := \sqrt{2} \text{Erf}^{-1} \left( 1 - 2\tilde{q} \right) . \]

Let us note that previous result has to be intended as follows: if the uncontrolled process \( X^{t,x;\psi_0}(T) \) already satisfies the terminal probability constraint, then it is optimal to solve the control problem with no terminal constraint, which is represented by the null control in the present case.
The above reasoning suggests that, for a fixed $q \in (0, 1)$, the optimal control $\psi$, exists if $(t, x)$ is to the right hand side of $\Psi(t, x; q)$. In particular, the optimal solution $\psi$ equals 0 if $(t, x)$ lies to the right hand side of $\psi_0(t, x; 0)$. Also, we have that, if $q \in (0, 1)$, then the optimal control is given by $\tilde{\psi}$ which reaches the required value $q$. In fact, equating $W^\tilde{\psi}(t, x) = q$, and solving for $\tilde{\psi}$, we obtain that for $(t, x)$, the optimal control $\tilde{\psi}$ is given by

$$
\tilde{\psi} = \psi(t, x; q) = \left( \frac{\sigma^2}{2} - \mu \right) + \frac{\log \frac{\psi(T)}{T - t}}{T - t} - \frac{\sigma \rho}{\sqrt{T - t}}.
$$

Moreover, the value function for the above control problem, associated with the control $\tilde{\psi} \in [0, \Psi]$, is given by

$$
V(t, x, W^\tilde{\psi}(t, x)) = -\psi^2 x^2 \left( \frac{2(\mu + \bar{\psi})(T - t) - 1}{4(\mu + \psi)} \right).
$$

Observing that the map

$$
q \mapsto V(t, x, q),
$$

is non-decreasing, together with the fact that $W^\tilde{\psi}(t, x) > W^\psi(t, x)$, for $\tilde{\psi} > \psi$, we have that

$$
V(t, x, W^\psi(t, x)) = -\infty, \quad \psi < \tilde{\psi},
$$

because the terminal constraint in equation (12) is not satisfied. Analogously, if $\psi > \tilde{\psi}$, then $W^\tilde{\psi}(t, x) < W^\psi(t, x)$, hence, once again, the non-decreasing property of $V$ w.r.t. the third argument $q$, implies

$$
V(t, x, W^\psi(t, x)) > V(t, x, W^\tilde{\psi}(t, x)),
$$

and the minimum is attained for the control $\tilde{\psi}$ implicitly given by

$$
\tilde{\psi} \equiv \tilde{\psi}(t, x; q) = \left( \frac{\sigma^2}{2} - \mu \right) + \frac{\log \frac{\psi(T)}{T - t}}{T - t} - \frac{\sigma \rho}{\sqrt{T - t}}.
$$

4 Application to a network of financial banks

In the present section, we use previously obtained results to study a real-world application characterized by an interconnected network of banks.

4.1 PageRank

Along previous sections, we have stated an optimal control problem, deriving its solution, by assuming that the accepted probability of failure $q^i$ is a fixed parameter to be chosen endogenously. In what follows we propose a general automatic criterion to deduce the global importance of each node in the system. Next computations exploit results on network analysis already used, e.g., to set the functioning logic of the Google research engine, see, e.g., [19]. According to the network formulation introduced in Section 2, and using results derived in [19], we show how to score the relative importance of any bank in the network, computing its so called Page Rank, as then to choose the best survival probability $q$.

Let us consider a system of $n$ banks and associate to it the usual bank enumeration, so that there is a one-to-one correspondence relation between the set of banks and the set of vertexes $V := \{v_1, v_2, \ldots, v_n\}$, being $I := \{1, 2, \ldots, n\}$, the associated set of indexes. Moreover, consider a LOLR’s strategy in which for each $v_i \in V$ the default probability constraint parameter $q^i$ depends on a predetermined rank $R^i$ associated to the $i$-th bank, hence representing its systemic importance in the network.

In what follows we are considering graphs as defined in Section 2. In particular: to each node $v_i \in V$ corresponds a bank, while, to the edges connecting nodes $(v_i, v_j) \in V \times V$, the following quantities are assigned

$$
\gamma^+(i, j) = \frac{c^+ L_{i,j}}{N_j - \min(N) + 1}, \quad \gamma^-(i, j) = \frac{c^- L_{j,i}}{N_i - \min(N) + 1}.
$$

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where, letting
\[ L_j^+ = \sum_{i \sim j} L_{ij}, \quad L_j^- = \sum_{i \sim j} L_{ji}, \]
we define \( N_j \) as the net amount of money held by bank \( j \) if it would pay its debts at the actual time, namely: \( N_j := X_j + L_j^- - L_j^+ \). Moreover \( c^+ \) and \( c^- \) are two non-negative constants, chosen as to confer more importance to the debts due or to credits owed. For the sake of simplicity, we set \( c^+ = c^- = 1 \). Notice that \( \gamma_{(i,i)} = \gamma_{(i,j)} = 0 \) and \( \gamma_{(j,i)} = \gamma_{(j,j)} \), for all \( i, j \in I \).

Let us then introduce the notion of outdegree \( \text{deg}^+ \), resp. indegree \( \text{deg}^- \), for any vertex \( v_i \in V \), i.e.
\[
\text{deg}^+(v_i) = \sum_{j \in \mathcal{I}} \gamma^{+}_{(i,j)}, \quad \text{deg}^-(v_i) = \sum_{j \in \mathcal{I}} \gamma^{-}_{(i,j)},
\]
and normalize the quantities defined in (22) associated to any couple \((i, j)\) of edges in the graph
\[
\bar{\gamma}^{+}_{(i,j)} = \frac{\gamma^{+}_{(i,j)}}{\text{deg}^+(v_j)}, \quad \bar{\gamma}^{-}_{(i,j)} = \frac{\gamma^{-}_{(i,j)}}{\text{deg}^-(v_j)}
\]
corresponding to the ratio of a linear combination on the liabilities between bank \( i \) and bank \( j \), and the asset value of bank \( j \). Further, we define the matrix \( \bar{G} \) as the matrix whose entries are \( \bar{\gamma}^{+}_{(i,j)} \) for \( i, j \in \mathcal{I} \), the quantities \( \bar{\gamma}^{-}_{(i,j)} \) being the weights assigned to each oriented edge. Then the rating value associated to any node/bank \( v_i \), is given by the following recursive formula
\[
R^+_{d} = d \sum_{j \sim i} \bar{\gamma}^{+}_{(i,j)} R^+_{d}, \tag{23}
\]
where \( d \in (0, 1) \) is a parameter to be chosen, typically \( d = 0.85 \), see, e.g., [17]. To compute equation (23), we introduce the so called Google-matrix, see, e.g., [17, ch 2].

We assume that our network is composed by banks not solely owing liabilities if \( c^+ = 1 \), resp. not solely owning liabilities if \( c^+ = 0 \), and at least connected for \( c^+ \in (0, 1) \). Of course, banks that are non connected to others belonging to the network, are not ranked, since their default cannot affect the system. On the other hand, even if the conditions for \( c^+ \in \{0, 1\} \) are not required, they guarantee the boundedness of all the elements of the matrix defined in the next Definition 4.0.1. We stress that, to avoid above restrictions, one can modify the values assigned to edges by equation (22), e.g., as follows: for \( c^+ = 1 \) and for every \( i \sim j \), define \( \tilde{\gamma}^{+}_{(i,j)} = \gamma_{i,j} / (N_j - \min(N) + 1) + \epsilon \) as the modified value assigned to the edges.

**Definition 4.0.1 (Google-matrix).** Let \( J \) be a \( n \times n \)-matrix whose entries are all ones. A Google-matrix is a \( n \times n \)-matrix given by
\[ G_d := \frac{1 - d}{n} J + d \bar{G}, \]
where \( d \in (0, 1) \) can be chosen to guarantee irreducibility of \( G_d \), while \( J \) is the \( n \times n \) matrix whose all entry are 1.

Since the matrix defined in (24) is positive we can apply the Perron–Frobenius Theorem which assures us that there exists a maximum real eigenvalue \( \lambda > 0 \) of \( G_d \), indeed \( \lambda \) is the so-called dominant Perron–Frobenius eigenvalue. Moreover, there exists one of the associated eigenvectors, denoted by \( R_d \) and usually called Perron-Frobenius dominant vector, which is both strictly positive and normalized and whose components represent the rating of each bank. Let us recall that \( d \) is usually chosen to be approximately equals to 0.85, see, e.g., [17]. It follows that proposed ranking procedure consists in computing the following series
\[ R_d = d \sum_{k=0}^{\infty} (1 - d)^k (G_d)^k 1, \]
1 being a \( n \)-dimensional vector whose entries are all equal to one.
4.2 A concrete case study

In what follows we consider a system of banks aiming at computing their ranking. First of all, we are considering a LOLR willing to save banks whose failure would cause insolvency and no ability to pay back their liabilities, i.e. $c^+ = 0$, therefore $c^- = 1$. According to what we have already seen within previous section, we fix $d = 0.85$ and we consider a system of banks whose liability matrix and cash vector are as follows, see Figure 1 for the associated graph,

\[
\mathbf{L} = \begin{bmatrix}
0 & 0 & 10 & 0 \\
5 & 0 & 5 & 5 \\
0 & 0 & 0 & 0 \\
10 & 4 & 0 & 0 \\
\end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix}
5.2 \\
6 \\
13 \\
3 \\
\end{bmatrix}, \quad \Rightarrow
\]

\[
\mathbf{G}_d = \begin{bmatrix}
0.0375 & 0.8344 & 0.0375 & 2.3042 \\
0.0375 & 0.0375 & 0.0375 & 0.9442 \\
2.9352 & 0.8344 & 0.0375 & 0.0375 \\
0.0375 & 0.8344 & 0.0375 & 0.0375 \\
\end{bmatrix},
\]

where the eigenvalues of the matrix $\mathbf{G}_d$ are $\lambda_1 = 1.2892$, $\lambda_2 = -0.8449$, $\lambda_3 = -0.1472 + 0.3982i$ and $\lambda_4 = \overline{\lambda_3}$. The absolute value of the eigenvector corresponding to the highest eigenvalue is

\[
\mathbf{R} = v_1 = \begin{bmatrix}
0.3516 \\
0.1342 \\
0.9177 \\
0.1275 \\
\end{bmatrix}^T.
\]

The third bank is the one with the highest ranking. Indeed, it is easy to note that its default would cause the default of the first bank and then an insolvency cascade. This is due to the fact that the third bank is systematically more important than the others. Notice that the amount of money due is the most important aspect to be taken into account for the safety of the system. We have reported in Table 1 some further considerations.

**Remark 4.1.** Looking at Figure 1 and Table 1 we can see that although the first and the third bank are owning the same amount of money to other banks, nonetheless their rankings $\mathbf{R}$ are extremely different. This is due to the fact that Bank 3 owns to Bank 1 and its insolvency would probably cause the default of Bank 1. In this example the cascade effect caused by the default of Bank 3 would stop with the default of two banks because of the small dimension of the system, while, on the contrary, such an effect can be amplified in big networks.

| Banks ($i$) | 1 | 2 | 3 | 4 |
|-------------|---|---|---|---|
| $X_i$       | 5.2 | 6 | 13 | 3 |
| $\sum_{j \sim i} L_{ji}$ | 15 | 4 | 15 | 5 |
| $R_i$       | 0.3516 | 0.1342 | 0.9177 | 0.1275 |

Table 1: Comparison among the banks rankings.
4.3 LOLR strategy under the PageRank approach

We will see in the present subsection, how to adapt the LOLR problem seen in section 3 to guarantee more flexibility to the banks that are more important for the network’s health. Such type of strategies are often referred to as Systemic importance driven (SID) strategies, see appendix A for more details.

We recall that the aim of the LOLR is to minimize the expenditure on banks bailout (7) constrained by (8), i.e. guaranteeing a probability $q^i$ that the bank $i$ will not default. Fixing, for all the banks, an identical probability constraint $q \in [0,1)$ would be an equality policy that can be reconducted to the max liquidity (ML) strategy introduced in [7], also analysed in appendix A. On one hand a ML strategy guarantee no privileges for any of the banks, on the other hand this would lead the LOLR to lend same amount of money for systematically important banks as for banks whose failure would not cause a snowball effect.

The main idea of the subsequent analysis consists in defining the probability constraints as an increasing function of the rank assigned to each bank. Namely, we have

$$q^i = f(R^i), \quad \text{for } f : \mathbb{R}^+ \to [0,1) \text{ increasing function},$$

where $R^i$ is the ranking of the bank $i$, as seen in Section 4.1. Notice that requiring $f' = 0$ the LOLR will again be restricted to the ML strategy.

In [7] we have shown that choosing $f$ to be an increasing function leads to a more convenient scenario for the health of networks which have a core-periphery structure, whereas, normally, banks networks have a dense cohesive core, with a periphery less connected.
Let us coming back to the network of banks already defined in subsection 4.2, see figure 1. We assume that the LOLR assigns the following probability constraints

\[
q_i = 0.9 + 0.05 \cdot \mathbb{1}_{\{R_i > 0.5\}} + 0.04 \cdot \mathbb{1}_{\{R_i > 0.75\}};
\]

and we perform a one period simulation of the network in figure 1 taking \(t_0 = 0\) and \(T = 1\). For the sake of simplicity, we also assume that all the liabilities expire at time \(T\), and that they exponentially increase in time with fixed growth rate \(r = 0.08\), i.e.

\[
L(t) = Le^{rt}, \quad \text{for } t \in [0,1].
\]

Furthermore, we assume that the cash vectors are described by geometric Brownian motions evolving according to the following equations

\[
\begin{align*}
\text{d}X_1 &= X_1 (0.2 \text{d}t + 0.1 \text{d}W_t), \\
\text{d}X_2 &= X_2 (0.15 \text{d}t + 0.25 \text{d}W_t), \\
\text{d}X_3 &= X_3 (0.3 \text{d}t + 0.2 \text{d}W_t), \\
\text{d}X_4 &= X_4 (0.05 \text{d}t + 0.4 \text{d}W_t).
\end{align*}
\]

Then we can compute the log-switching regions \(y_i\) for each banks accordingly to equation (21)

\[
\begin{align*}
y_1 &= 1.622593, \quad y_2 = 0, \quad y_3 = 2.97332, \quad y_4 = 0, \\
q_1 &= 0.9, \quad q_2 = 0.9, \quad q_3 = 0.99, \quad q_4 = 0.9.
\end{align*}
\]

recalling that they have to be less than the log initial wealth \(X_i(0)\) to guarantee the fulfillment of the probability constraint. Therefore, since

\[
\begin{align*}
\log(X_1(0)) &= 1.6487, \quad \log(X_2(0)) = 1.7918, \\
\log(X_3(0)) &= 2.5649, \quad \log(X_4(0)) = 1.0986,
\end{align*}
\]

we have that the LOLR has to intervene controlling Bank 3. Notice that the LOLR has not to intervene in banks 2 and 4, since they have more credits than debts, hence they cannot face bankruptcy, while the opposite is true for banks 1 and 3. For \(q_1 = 0.95\), we would have \(\tilde{y}_1 = 1.6589\) and there would need a LOLR intervention injecting money also in bank 1.

Figure 2 (top panel) represents 100 simulation for the evolution of Banks 1 and 3, with and without LOLR intervention. Since the probability of Bank 1 to survive is greater than \(q_1 = 90\%\), the LOLR is not going to intervene, whereas indeed its probability to default is approximately \(6.2\%\).

Clearly, requiring \(q_1 = 95\%\) would imply that the LOLR has to intervene lending money to Bank 1. In the left-lower Figure 2 there are represented 100 simulations of the process associated to Bank 3; since \(q_3 = 99\%\) and the default probability of Bank 3 is 38.8%, the LOLR is going to intervene injecting capital in its cash reserve. After the optimal injection of capital, Bank 3 has probability 1% to face the default event, see the right-lower Figure 2 for the representation of 100 simulations of Bank 3 in the case in which the LOLR is going to intervene. Let us underline that the simple case-study we analysed has been settled to provide an example as clear as possible, nonetheless, because all the analytical results we derived are in closed form, general complex networks can be theoretically treated as well. Clearly, increasing the graph’s connection grade, we have an exponential growth in computing the quantities of interest.

5 Conclusions

In the present work, we have derived a closed form solution for an optimal control of interbank lending subjected to specific terminal probability constraints on the failure of a bank. The obtained result can be applied to a system of interconnected banks, providing a network solution. We have also shown a simple and direct method to calculate the relative importance of any
node within the studied network. We would like to underline that such a ranking value is fundamental in deciding the accepted probability of failure which modifies the final optimal strategy of a financial supervisor aiming at controlling the system to prevent global crisis as generalized default.

The results here presented constitute a first of a wider research program. In particular, in future works we shall consider sequence of checking times each of which characterized by possibly different constraints to be considered by the supervisor. In this setting, a solution can be obtained by backward induction, see [8, 20], applied to retults here shown. Moreover, we will consider a framework where the failure can happen continuously in time, hence imposing strict constraints at any time before the terminal one T.

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As mentioned above, the financial setting has been mainly borrowed by [13] as concerns the lending system formulation, and from [7] for the optimal control problem with an external supervisor aiming at guaranteeing the overall sanity of the system.

This section is devoted to a comparison with [7]. We stress that our assumptions on the optimal control are in the spirit of [7], in the sense that we consider failure at discrete times; also we will not consider a global optimal control, deriving a control for the whole time interval but rather we derive a series optimal control and then gluing together the resulting optimal controls. As mentioned we leave the optimal global control to future research being this latter point mathematically more demanding.

This comparison is significant since their work is based on a similar framework: a multi-period controlled system of banks, represented by a network, in which an outside entity, named LOLR, provides liquidity assistance loans to financially unstable banks in order to reduce the level of systemic risk within the whole network of banks. To analyze the systemic risk in interbank networks their work follows a clearing system framework consistent with bankruptcy laws. In particular they generalize the single period clearing system in the paper by Eisenberg and Thomas, see [13], by a multi-period controlled clearing payment system assuming limited liability of equity, priority over equity, and proportional repayments of liabilities after the default event. This generalization leads to a better insight in the propagation and aftershocks of defaults. The main feature in [7] is the comparison between two possible LOLR strategies:

- the Systemic Importance Driven (SID) strategy, in which liquidity assistance is available only to banks considered systemically important, i.e. the banks whose default would cause significant losses to the financial system (because of their size, complexity and systemic interconnectedness);

- the Max-Liquidity (ML) strategy, in which the regulators aim to maximize the instantaneous total liquidity of the system.

By the analysis of these two different strategies they showed that the SID strategy is preferred when the network has a core-periphery structure, i.e. consisting of a dense cohesive core and
Comparison with \cite{7}

a sparse, loosely connected periphery. This is due by the fact that the ML strategy increases the default probability for systematically important banks. Although these two strategies are simplified and do not consider the amount of capital that the LOLR has to inject in the banks network, this comparison is useful because the numerical approach fits easily through simulations and systemic risk analysis.

Our work has some important similarities with the one by Capponi et al., in particular we also have considered a finite connected multiperiod financial network representing the banks system and the assumptions guaranteeing the consistency with the bankruptcy laws. But, despite this, instead of comparing the two strategies, SID and ML, we considered a LOLR wishing to minimize the square of the lend resources over the probabilistic constraint; i.e. we did not gave an initial budget at disposal to the LOLR as in \cite{7}, but took into consideration regulators aiming to find the loan control \( \{ \alpha^i(t) \}_{i=1,\ldots,N,t \in [t_k,t_{k+1}]} \) minimizing the functional given by equation (7) for each time interval, i.e. \( \forall k = 1,\ldots,M-1 \), ensuring that the probability for each exogenous asset value to be greater than the default boundary is greater than a given constants \( q^i \) for each bank \( i \in \{1,\ldots,N\} \).

Ultimately, while \cite{7} is meant to compare two strategies for the LOLR, our work follows a different path in searching the optimal budget consumption in order to guarantee a prescribed level of safety of the financial network, given by the parameters \( q^i \) with \( i = 1,\ldots,N \). Therefore we do not put strong constraint over the regulators budget, which depends on the default probability constraint parameters \( q^i \). To switch on a similar comparison as in \cite{7}, i.e. considering banks networks of the type: core-periphery and baseline random networks, and regulator policies of the type: SID and ML, it suffices to fix the probability constraint depending on the systematic importance of the banks; that is, banks whose failure would cause significant losses to the financial network, because of their size and systemic interconnectedness, should be endorsed with greater default probability parameters \( q^i \). Thence our study can be seen as an extension of the admissible policies, through considering an optimal control theory approach.