PERIODIC SOLUTIONS FOR IMPULSIVE NEUTRAL DYNAMIC EQUATIONS WITH INFINITE DELAY ON TIME SCALES

A. ARDJOUNI¹ AND A. DJOUDI²

Abstract. Let \( T \) be a periodic time scale. We use the Krasnoselskii’s fixed point theorem to show that the impulsive neutral dynamic equations with infinite delay

\[
x^\Delta(t) = -A(t)x^\sigma(t) + g^\Delta(t, x(t-h(t))) + \int_{-\infty}^{t} D(t, u) f(x(u)) \Delta u, \quad t \neq t_j, \quad t \in T,
\]

\[
x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \quad j \in \mathbb{Z}^+
\]

have a periodic solution. Under a slightly more stringent conditions we show that the periodic solution is unique using the contraction mapping principle.

1. Introduction

In 1988, Stephan Hilger [9] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger’s initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems; see [7, 8, 17] and references therein. The study of impulsive initial and boundary value problems is extensive. For the theory and classical results, we direct the reader to the monographs [6, 16, 18].

Recently Althubiti, Makhzoum and Raffoul [2] investigated the existence and uniqueness of periodic solutions for the neutral differential equation with infinite delay

\[
x'(t) = -a(t)x(t) + \frac{d}{dt} g(t, x(t-h(t))) + \int_{-\infty}^{t} D(t, u) f(x(u)) du.
\]

By employing the Krasnoselskii’s fixed point theorem and the contraction mapping principle, the authors obtained existence and uniqueness results for periodic solutions.

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The nonlinear impulsive dynamic equation
\[
    x^\Delta(t) = -a(t)x^\sigma(t) + f(t, x(t)), \quad t \neq t_j, \, t \in \mathbb{T},
\]
\[
    x(t_j^+) = x(t_j^-) + I_j(t_j, x(t_j)), \quad j = 1, 2, \ldots, n,
\]
has been investigated in [10]. By using Schaeffer’s theorem, the existence of periodic solutions has been established.

In this article, we are interested in the analysis of qualitative theory of periodic solutions of impulsive neutral dynamic equations. Inspired and motivated by the works mentioned above and the papers [1–5, 10–15, 20–22] and the references therein, we are concerned with the system
\[
    x^\Delta(t) = -A(t)x^\sigma(t) + g^\Delta(t, x(t - h(t))) + \int_{\infty}^{t} D(t, u) f(x(u))\Delta u, \quad t \neq t_j, \, t \in \mathbb{T},
\]
\[
    x(t_j^+) = x(t_j^-) + I_j(x(t_j)), \quad j \in \mathbb{Z}^+,
\]
where \( \mathbb{T} \) is an \( \omega \)-periodic time scale, \( 0 \in \mathbb{T} \) and \( x^\sigma = x \circ \sigma \). For each interval \( U \) of \( \mathbb{R} \), we denote by \( U_\mathbb{T} = U \cap \mathbb{T} \), \( x(t_j^+) \) and \( x(t_j^-) \) represent the right and the left limit of \( x(t_j) \) in the sense of time scales, in addition, if \( t_j \) is left-scattered, then \( x(t_j^-) = x(t_j) \), \( A(t) = \text{diag}(a_i(t))_{n \times n} \) and \( D(t, u) = \text{diag}(D_i(t, u))_{n \times n} \) are diagonal matrices with continuous real-valued functions as its elements, \( \mathbb{R}^+ = \{ a \in C(\mathbb{T}, \mathbb{R}) : 1 + \mu(t)a(t) > 0 \} \) where \( \mu(t) = \sigma(t) - t, \, h \in C(\mathbb{T}, \mathbb{T}) \), \( g = (g_1, g_2, \ldots, g_n) \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n) \), \( f = (f_1, f_2, \ldots, f_n) \in C(\mathbb{R}^n, \mathbb{R}^n) \), \( I_j = (I_j^{(1)}, I_j^{(2)}, \ldots, I_j^{(n)}) \in C(\mathbb{R}^n, \mathbb{R}^n) \) and \( A(t), h(t), g(t, x(t - h(t))) \) are all \( \omega \)-periodic functions with respect to \( t \), \( D(t + \omega, u + \omega) = D(t, u) \), \( \omega > 0 \) is a constant. There exists a positive integer \( p \) such that \( t_{j+p} = t_j + \omega \), \( I_{j+p} = I_j \), \( j \in \mathbb{Z}^+ \), without loss of generality, we also assume that \( [0, \omega)_\mathbb{T} \cap \{ t_j, \, j \in \mathbb{Z}^+ \} = \{ t_1, t_2, \ldots, t_p \} \).

To reach our desired end we have to transform the system (1.1) into an integral system and then use Krasnoselskii’s fixed point theorem to show the existence of periodic solutions. The obtained integral system is the sum of two mappings, one is a contraction and the other is a compact. Also, transforming system (1.1) to an integral system enables us to show the uniqueness of the periodic solution by appealing to the contraction mapping principle.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary results needed in later sections, then we give the Green’s function of (1.1), which plays an important role in this paper. In Section 3, we establish our main results for periodic solutions by applying the Krasnoselskii’s fixed point theorem and the contraction mapping principle.

2. Preliminaries

In this section, we shall recall some basic definitions and lemmas which are used in what follows.
Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$ 

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous (rd-continuous) provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If $f$ is continuous at each right-dense points and each left-dense point, then $f$ is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T})$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $x(t)$, $x^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood $U_T$ of $t$ such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U_T$.

If $x$ is continuous, then $x$ is right-dense continuous, and if $x$ is delta differentiable at $t$, then $x$ is continuous at $t$.

**Remark 2.1.** $x : \mathbb{T} \to \mathbb{R}^n$ is delta derivable or right-dense continuous or continuous if each entry of $x$ is delta derivable or right-dense continuous or continuous.

Let $x$ be right-dense continuous. If $X^\Delta(t) = x(t)$, then we define the delta integral by

$$\int_a^t x(s)\Delta s = X(t) - X(a).$$

**Definition 2.1** ([12]). We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $p$ is called the period of the time scale.

Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $p$. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period $\omega$ if there exists a natural number $n$ such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$ and $\omega$ is the smallest positive number such that $f(t + \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that $f$ is periodic with period $\omega > 0$ if $\omega$ is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

**Remark 2.2.** According to [12], if $\mathbb{T}$ is a periodic time scale with period $p$, then $\sigma(t + np) = \sigma(t) + np$ and the graininess function $\mu$ is a periodic function with period $p$. 
Then the initial value problem
\[ y(t) = A(t)y(t) + f(t), \quad y(t_0) = y_0, \]
is invertible for all \( t \in T^k \).

Let \( A, B : T \to R^{n \times n} \) be two \( n \times n \)-matrix-valued regressive functions on \( T \), we define
\[
(A \oplus B)(t) := A(t) + B(t) + \mu(t)A(t)B(t),
\]
\[
(\ominus A)(t) := -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1},
\]
\[
(A(t)) \ominus (B(t)) := (A(t)) \ominus (\ominus (B(t))),
\]
for all \( t \in T^k \).

**Theorem 2.1** ([8]). Let \( A \) be an regressive and rd-continuous \( n \times n \)-matrix-valued function on \( T \) and suppose that \( f : T \to R^n \) is rd-continuous. Let \( t_0 \in T \) and \( y_0 \in R^n \). Then the initial value problem
\[
y(t) = A(t)y(t), \quad y(t_0) = y_0,
\]
has a unique solution \( y : T \to R^n \).

**Definition 2.3** ([8]). Let \( t_0 \in T \) and assume that \( A \) is an regressive and rd-continuous \( n \times n \)-matrix-valued function. The unique matrix-valued solution of the initial value problem
\[
x(t) = A(t)x(t), \quad x(t_0) = I,
\]
where \( I \) denotes as usual the \( n \times n \)-identity matrix, is called the matrix exponential function (at \( t_0 \)), and it is denoted by \( e_A(\cdot, t_0) \).

**Remark 2.3.** Assume that \( A \) is a constant \( n \times n \)-matrix. If \( T = R \), then
\[
e_A(t, t_0) = e^{A(t-t_0)},
\]
while if \( T = Z \) and \( I + A \) is invertible, then
\[
e_A(t, t_0) = (I + A)^{t-t_0}.
\]

In the following lemma, we give some properties of the matrix exponential function.

**Lemma 2.1** ([8]). Assume that \( A, B : T \to R^{n \times n} \) are regressive and rd-continuous matrix-valued functions on \( T \). Then

(i) \( e_0(t, s) \equiv I \) and \( e_A(t, t) \equiv I \);
(ii) \( e_A(\sigma(t, s)) = (I + \mu(t)A(t))e_A(t, s) \);
(iii) \( e_A^{-1}(t, s) = e_A^{\ominus A^*}(t, s) \);
(iv) \( e_A(t, s) = e_A^{\ominus A^*}(s, t) = e_A^*(s, t) \);
(v) \( e_A(t, s)e_A(s, r) = e_A(t, r) \);
(vi) \( e_A(t, s)e_B(t, s) = e_{A \oplus B}(t, s) \), if \( A(t) \) and \( B(t) \) commute,

where \( A^* \) denotes the conjugate transpose of \( A \).
Lemma 2.2 ([8]). Suppose $A$ and $B$ are regressive matrix-valued functions, then

(i) $A^*$ is regressive;
(ii) $\ominus A^* = (\ominus A)^*$;
(iii) $(A^\Delta)^* = (A^\Delta)^*$ holds for any differential matrix-valued function $A$.

Next, we state Krasnosel’skiĭ’s fixed point theorem which enables us to prove the existence of a periodic solution of (1.1). For its proof we refer the reader to [19].

Theorem 2.2 (Krasnosel’skiĭ). Let $M$ be a closed convex nonempty subset of Banach space $(B, \| \cdot \|)$. Suppose that $\Phi$ and $\Psi$ map $M$ into $B$ such that

(i) $x, y \in M$ imply $\Phi x + \Psi y \in M$;
(ii) $\Psi$ is compact and continuous;
(iii) $\Phi$ is a contraction mapping.

Then there exists $z \in M$ with $z = \Phi z + \Psi z$.

Lemma 2.3. A function $x$ is an $\omega$-periodic solution of (1.1) if and only if $x$ is an $\omega$-periodic solution of the equation

\[ x(t) = g(t, x(t - h(t))) + \int_{t}^{t+\omega} \left[ \int_{-\infty}^{s} D(s, u)f(x(u))\Delta u - A(s)g^\sigma(s, x(s - h(s))) \right] \Delta s + \sum_{j:t_j \in [t, t+\omega)} G(t, t_j)I_j(x(t_j)), \]

where

\[ G(t, s) = \text{diag}(G_i(t, s))_{n \times n}, \quad G_i(t, s) = (1 - e_{\ominus a_i}(\omega, 0))^{-1} e_{\ominus a_i}(t + \omega, s), \]
\[ A(t) = \text{diag}(a_i(t))_{n \times n}, \quad e_{\ominus a_i}(t, s) = \frac{1}{e_{a_i}(t, s)}, \]
\[ \ominus a_i(t) = -\frac{a_i(t)}{1 + \mu(t)a_i(t)}, \quad g^\sigma(t, x(t - h(t))) = g(\sigma(t), x^\sigma(t - h(t))). \]

**Proof.** If $x$ is an $\omega$-periodic solution of (1.1). For any $t \in \mathbb{T}$, there exists $j \in \mathbb{Z}$ such that $t_j$ is the first impulsive point after $t$. Then for $i = 1, 2, \ldots, n$, $x_i$ is an $\omega$-periodic solution of the equation

\[ x_i^\Delta(t) + a_i(t)x_i^\sigma(t) = g_i^\Delta(t, x_i(t - h(t))) + \int_{-\infty}^{t} D_i(t, u)f_i(x(u))\Delta u. \]

Multiply both sides of (2.1) by $e_{a_i}(t, 0)$ and then integrate from $t$ to $s \in [t, t_j]_\mathbb{T}$, we obtain

\[ \int_{t}^{s} [e_{a_i}(\tau, 0)x_i(\tau)]^\Delta \Delta \tau = \int_{t}^{s} e_{a_i}(\tau, 0) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) + \int_{-\infty}^{\tau} D_i(\tau, u)f_i(x_i(u))\Delta u \right] \Delta \tau, \]
or

\[ e_{a_i}(s,0)x_i(s) = e_{a_i}(t,0)x_i(t) + \int_t^s e_{a_i}(\tau,0) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) ight] \Delta \tau, \]

\[ + \int_{-\infty}^\tau D_i(\tau,u) f_i(x_i(u)) \Delta u \Delta \tau, \]

then

\[ x_i(s) = e_{\Xi a_i}(s,t)x_i(t) + \int_t^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau, \quad i = 1, 2, \ldots, n, \]

hence

\[ (2.2) \quad x_i(t_j) = e_{\Xi a_i}(t_j,t)\ x_i(t) + \int_{t_j}^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau, \quad i = 1, 2, \ldots, n. \]

Similarly, for \( s \in (t_j, t_{j+1}] \), we have

\[ x_i(s) = e_{\Xi a_i}(s,t_j)x_i(t_j) + \int_{t_j}^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau \]

\[ + \int_{-\infty}^\tau D_i(\tau,u) f_i(x_i(u)) \Delta u \Delta \tau, \]

\[ = e_{\Xi a_i}(s,t_j)x_i(t_j) + \int_{t_j}^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau + e_{\Xi a_i}(s,t_j) I_j^{(i)}(x_i(t_j)) \]

\[ + \int_{-\infty}^\tau D_i(\tau,u) f_i(x_i(u)) \Delta u \Delta \tau + e_{\Xi a_i}(s,t_j) I_j^{(i)}(x_i(t_j)), \]

for \( i = 1, 2, \ldots, n \). Substituting (2.2) in the above equality, we obtain

\[ x_i(s) = e_{\Xi a_i}(s,t)x_i(t) + \int_t^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau \]

\[ + \int_{-\infty}^\tau D_i(\tau,u) f_i(x_i(u)) \Delta u \Delta \tau + e_{\Xi a_i}(s,t_j) I_j^{(i)}(x_i(t_j)). \]

Repeating the above process for \( s \in [t, t + \omega]_T \), we have

\[ x_i(s) = e_{\Xi a_i}(s,t)x_i(t) + \int_t^s e_{\Xi a_i}(s,\tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right] \Delta \tau \]

\[ + \int_{-\infty}^\tau D_i(\tau,u) f_i(x_i(u)) \Delta u \Delta \tau + e_{\Xi a_i}(s,t_j) I_j^{(i)}(x_i(t_j)). \]
It follows from (2.3) and (2.4) that obtain for $i = 1, 2, \ldots, n$. Let $s = t + \omega$ in the above equality, we have

$$x_i(t + \omega) = e_{\oplus a_i}(t + \omega, t)x_i(t) + \int_t^{t+\omega} e_{\oplus a_i}(t + \omega, \tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right. + \int_{-\infty}^{t} \left. D_i(\tau, u) f_i(x_i(u)) \Delta u \right] \Delta \tau + \sum_{j: t_j \in [t, t+\omega)} e_{\oplus a_i}(t + \omega, t_j) I_j^{(i)}(x_i(t_j)),
$$

for $i = 1, 2, \ldots, n$. Noticing that $x_i(t + \omega) = x_i(t)$ and $e_{\oplus a_i}(t + \omega, t) = e_{\oplus a_i}(\omega, 0)$, we obtain

$$(2.3) \quad (1 - e_{\oplus a_i}(\omega, 0))x_i(t) = \int_t^{t+\omega} e_{\oplus a_i}(t + \omega, \tau) \left[ g_i^\Delta(\tau, x_i(\tau - h(\tau))) \right. + \int_{-\infty}^{\tau} \left. D_i(\tau, u) f_i(x_i(u)) \Delta u \right] \Delta \tau + \sum_{j: t_j \in [t, t+\omega)} e_{\oplus a_i}(t + \omega, t_j) I_j^{(i)}(x_i(t_j)),$$

for $i = 1, 2, \ldots, n$. Notice that

$$(2.4) \quad \int_t^{t+\omega} e_{\oplus a_i}(t + \omega, \tau) g_i^\Delta(\tau, x_i(\tau - h(\tau))) \Delta \tau$$

$$= e_{\oplus a_i}(t + \omega, \omega) g_i(t + \omega, x_i(t + \omega - h(t + \omega)))$$

$$- e_{\oplus a_i}(t + \omega, t) g_i(t, x_i(t - h(t)))$$

$$- \int_t^{t+\omega} e_{\oplus a_i}(t + \omega, \tau) a_i(\tau) g_i^\sigma(\tau, x_i(\tau - h(\tau))) \Delta \tau$$

$$= [1 - e_{\oplus a_i}(\omega, 0)] g_i(t, x_i(t - h(t)))$$

$$- \int_t^{t+\omega} e_{\oplus a_i}(t + \omega, \tau) a_i(\tau) g_i^\sigma(\tau, x_i(\tau - h(\tau))) \Delta \tau, \quad i = 1, 2, \ldots, n.$$

It follows from (2.3) and (2.4) that

$$x_i(t) = g_i(t, x_i(t - h(t))) + \int_t^{t+\omega} [1 - e_{\oplus a_i}(\omega, 0)]^{-1} e_{\oplus a_i}(t + \omega, \tau)$$

$$\times \left[ \int_{-\infty}^{\tau} D_i(\tau, u) f_i(x_i(u)) \Delta u - a_i(\tau) g_i^\sigma(\tau, x_i(\tau - h(\tau))) \right] + \sum_{j: t_j \in [t, t+\omega)} [1 - e_{\oplus a_i}(\omega, 0)]^{-1} e_{\oplus a_i}(t + \omega, t_j) I_j^{(i)}(x_i(t_j))$$

$$= g_i(t, x_i(t - h(t))) + \int_t^{t+\omega} G_i(t, \tau) \left[ \int_{-\infty}^{\tau} D_i(\tau, u) f_i(x_i(u)) \Delta u \right.$$
\[-a_i(\tau)g_i^\alpha(\tau, x_i(\tau - h(\tau)))] \Delta \tau + \sum_{j:t_j \in [t,t+\omega)} G_i(t, t_j)I_j^{(i)}(x_i(t_j)) \]

for \(i = 1, 2, \ldots, n\). Next, we prove the converse. Let

\[x_i(t) = g_i(t, x_i(t - h(t))) + \int_t^{t+\omega} G_i(t, s) \left[ \int_{-\infty}^{s} D_i(s, u) f_i(x_i(u)) \Delta u - a_i(s)g_i^\alpha(s, x_i(s - h(s))) \right] \Delta s + \sum_{j:t_j \in [t,t+\omega]} G_i(t, t_j)I_j^{(i)}(x_i(t_j)),\]

where

\[G_i(t, s) = (1 - e^{\Theta_{a_i}(\omega, 0)})^{-1} e^{\Theta_{a_i}(t + \omega, s)} , \quad i = 1, 2, \ldots, n.\]

Then if \(t \neq t_i, i \in \mathbb{Z}^+\), we have

\begin{align*}
x_i^\Delta(t) & = g_i^\Delta(t, x_i(t - h(t))) \\
& + \int_t^{t+\omega} \left\{ G_i(t, s) \left[ \int_{-\infty}^{s} D_i(s, u) f_i(x_i(u)) \Delta u - a_i(s)g_i^\alpha(s, x_i(s - h(s))) \right] \right\} \Delta s \\
& + G_i(t, t + \omega) \left[ \int_{-\infty}^{t+\omega} D_i(t + \omega, u) f_i(x_i(u)) \Delta u \\
& \quad - a_i(t + \omega)g_i^\alpha(t + \omega, x_i(t + \omega - h(t + \omega))) \right] \\
& - G_i(t, t) \left[ \int_{-\infty}^{t} D_i(t, u) f_i(x_i(u)) \Delta u - a_i(t)g_i^\alpha(t, x_i(t - h(t))) \right] \\
& = g_i^\Delta(t, x_i(t - h(t))) + \int_t^{t+\omega} \left\{ G_i(t, s) \left[ \int_{-\infty}^{s} D_i(s, u) f_i(x_i(u)) \Delta u - a_i(s)g_i^\alpha(s, x_i(s - h(s))) \right] \right\} \Delta s \\
& + \int_t^{t+\omega} \left\{ G_i(t, s) \left[ \int_{-\infty}^{s} D_i(s, u) f_i(x_i(u)) \Delta u - a_i(s)g_i^\alpha(s, x_i(s - h(s))) \right] \right\} \Delta s \\
& = g_i^\Delta(t, x_i(t - h(t))) + \int_t^{t+\omega} D_i(t, u) f_i(x_i(u)) \Delta u - a_i(t)x_i^\sigma(t) \\
& = - a_i(t)x_i^\sigma(t) + g_i^\Delta(t, x_i(t - h(t))) + \int_t^{t+\omega} D_i(t, u) f_i(x_i(u)) \Delta u, \quad i = 1, 2, \ldots, n.
\end{align*}

If \(t = t_i, i \in \mathbb{Z}^+\), we obtain

\begin{align*}
x_i(t_i^+) - x_i(t_i^-) & = \sum_{j:t_j \in [t_i^+, t_i^+ + \omega)} G_i(t_i, t_j)I_j^{(i)}(x_i(t_j)) - \sum_{j:t_j \in [t_i^-, t_i^- + \omega)} G_i(t_i, t_j)I_j^{(i)}(x_i(t_j)) \\
& = G_i(t_i, t_i + \omega)I_i^{(i)}(x_i(t_i + \omega)) - G_i(t_i, t_i)I_i^{(i)}(x_i(t_i)) \\
& = I_i^{(i)}(x_i(t_i)), \quad i = 1, 2, \ldots, n.
\end{align*}
So we know that, $x$ is also an $\omega$-periodic solution of (1.1). This completes the proof. □

Throughout this paper, we make the following assumptions.

(H1) The function $g = (g_1, g_2, \ldots, g_n)$ satisfies a Lipschitz condition in $x$. That is, for $i \in \{1, 2, \ldots, n\}$, there exists a positive constant $L_i$ such that
\[
|g_i(t, x) - g_i(t, y)| \leq L_i \|x - y\|, \quad \text{for all } t \in \mathbb{T}, \ x, y \in \mathbb{R}^n.
\]

(H2) The function $f = (f_1, f_2, \ldots, f_n)$ satisfies a Lipschitz condition in $x$. That is, for $i \in \{1, 2, \ldots, n\}$, there exists a positive constant $M_i$ such that
\[
|f_i(x) - f_i(y)| \leq M_i \|x - y\|, \quad \text{for all } t \in \mathbb{T}, \ x, y \in \mathbb{R}^n.
\]

(H3) For $j \in \mathbb{Z}$, $I_j = (I_j^{(1)}, I_j^{(2)}, \ldots, I_j^{(n)})$ satisfies Lipschitz condition. That is, for $i \in \{1, 2, \ldots, n\}$ there exists a positive constant $P_j^{(i)}$ such that
\[
\left| I_j^{(i)}(x) - I_j^{(i)}(y) \right| \leq P_j^{(i)} \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.
\]

(H4) There exists a positive constant $N_i$ such that
\[
\int_{-\infty}^{t} |D_i(t, u)| \Delta u \leq N_i.
\]

To apply Theorem 2.2 to (1.1), we define
\[
PC(\mathbb{T}) = \{x : \mathbb{T} \to \mathbb{R}^n : x|_{(t_j, t_{j+1})} \in C(t_j, t_{j+1})\mathbb{T}, \ \exists x(t_j) = x(t_j), x(t_j^+)\}, \ j \in \mathbb{Z}^+\}.
\]

Consider the Banach space
\[
X = \{x \in PC(\mathbb{T}) : x(t + \omega) = x(t)\},
\]
with the norm $\|x\| = \max_{t \in [0, \omega]} |x(t)|_0$, where $|x(t)|_0 = \max_{1 \leq i \leq n} |x_i(t)|$.

**Lemma 2.4** ([12]). Let $x \in X$. Then there exists $\|x^p\|$, and $\|x^p\| = \|x\|$.

Noticing that
\[
G_i(t, s) \leq (1 - e_{\mathbb{Z}^+}(\omega, 0))^{-1} = \eta_i,
\]
for convenience, we introduce the notation
\[
\bar{\eta} := \max_{1 \leq i \leq n} \eta_i, \quad \gamma := \max_{1 \leq i \leq n} \max_{t \in [0, \omega]} |a_i(t)|, \quad L := \max_{1 \leq i \leq n} L_i, \quad M := \max_{1 \leq i \leq n} M_i,
\]
\[
N := \max_{1 \leq i \leq n} N_i, \quad P_j := \max_{1 \leq i \leq n} P_j^{(i)}, \quad P := \max_{1 \leq j \leq p} P_j.
\]

Define the mapping $H : X \to X$ by
\[
(H \varphi)(t) = g(t, \varphi(t - h(t))) + \int_{t}^{t+\omega} G(t, s) \left[ \int_{-\infty}^{s} D(s, u) f(\varphi(u)) \Delta u - A(s)g^\sigma(s, \varphi(s - h(s))) \right] \Delta s + \sum_{j:t_j \in [t,t+\omega)} G(t, t_j)I_j(x(t_j)).
\]
To apply Theorem 2.2, we need to construct two mappings: one is a contraction and the other is continuous and compact. We express (2.5) as

\[(H\varphi(t) = (\Phi\varphi)(t) + (\Psi\varphi)(t),\]

where

\[(\Phi\varphi)(t) = g(t, \varphi(t-h(t))),\]

and

\[(\Psi\varphi)(t) = \int_t^{t+\omega} G(t,s) \left[ \int_{-\infty}^{s} D(s,u) f(\varphi(u)) \Delta u - A(s)g^{\sigma}(s, \varphi(s-h(s))) \right] \Delta s + \sum_{j:t_j \in [t,t+\omega)} G(t,t_j)I_j(\varphi(t_j)).\]

**Lemma 2.5.** Suppose (H1) holds and \(L < 1\), then \(\Phi : X \to X\), as defined by (2.6), is a contraction.

**Proof.** Trivially, \(\Phi : X \to X\). For \(\varphi, \psi \in X\), we have

\[\|\Phi(\varphi) - \Phi(\psi)\| = \max_{t \in [0,\omega]} \max_{1 \leq i \leq n} |g_i(t, \varphi_i(t-h(t))) - g_i(t, \psi_i(t-h(t)))| \leq L\|\varphi - \psi\|,\]

Hence \(\Phi\) defines a contraction mapping with contraction constant \(L\).

**Lemma 2.6.** Suppose (H1)–(H4) hold, then \(\Psi : X \to X\), as defined by (2.7), is continuous and compact.

**Proof.** Evaluating (2.7) at \(t+\omega\) gives

\[(\Psi\varphi)(t + \omega) = \int_t^{t+\omega} G(t+\omega,s) \left[ \int_{-\infty}^{s} D(s,u) f(\varphi(u)) \Delta u - A(s)g^{\sigma}(s, \varphi(s-h(s))) \right] \Delta s + \sum_{j:\omega \in [t+\omega,t+2\omega)} G(t+\omega,t_j)I_j(\varphi(t_j)).\]

\[= \int_t^{t+\omega} G(t+\omega,v+\omega) \left[ \int_{-\infty}^{v+\omega} D(v+\omega,u) f(\varphi(u)) \Delta u \right.\]

\[- A(v+\omega)g^{\sigma}(v+\omega, \varphi(v+\omega-h(v+\omega))) \left] \Delta v + \sum_{k:t_k \in [t,t+\omega)} G(t,t_k)I_j(\varphi(t_k))\right.\]

\[= \int_t^{t+\omega} G(t,v) \left[ \int_{-\infty}^{v} D(v,u) f(\varphi(u)) \Delta u \right.\]

\[- A(v)g^{\sigma}(v, \varphi(v-h(v))) \left] \Delta v + \sum_{k:t_k \in [t,t+\omega)} G(t,t_k)I_j(\varphi(t_k)).\]
This proves that $\Psi$ is continuous. Let $\varphi, \psi \in X$, given $\varepsilon > 0$, take

$$\delta = \frac{\varepsilon}{\eta[\omega(MN + L\gamma) + P]} \quad \text{such that for } ||\varphi - \psi|| \leq \delta.$$ 

By using the Lipschitz condition, we obtain

$$\|\Psi \varphi - \Psi \psi\| \leq \max_{t \in [0,\omega]_T} \left| \int_t^{t+\omega} G(t, s) \left[ \int_{-\infty}^s D(s, u) f(\varphi(u)) \Delta u - \int_{-\infty}^s D(s, u) f(\psi(u)) \Delta u \right] \Delta s \right|_0$$

$$+ \max_{t \in [0,\omega]_T} \left| \int_t^{t+\omega} G(t, s) A(s)[g^\sigma(s, \varphi(s - h(s))) - g^\sigma(s, \psi(s - h(s)))] \Delta s \right|_0$$

$$+ \max_{t \in [0,\omega]_T} \sum_{j, t_j \in [t, t+\omega]} |G(t, t_j)[I_j(\varphi(t_j)) - I_j(\psi(t_j))]|_0$$

$$\leq \eta \int_0^\omega \int_{-\infty}^s \left| D(s, u) [f(\varphi(u)) - f(\psi(u))] \right|_0 \Delta u \Delta s$$

$$+ \eta \gamma \int_0^\omega \left| g^\sigma(s, \varphi(s - h(s))) - g^\sigma(s, \psi(s - h(s))) \right|_0 \Delta s$$

$$+ \eta \max_{1 \leq j \leq p} |I_j(\varphi(t_j)) - I_j(\psi(t_j))|_0$$

$$\leq \eta[\omega(MN + L\gamma) + P] ||\varphi - \psi|| < \varepsilon.$$

This proves $\Psi$ is continuous. Next, we need to show that $\Psi$ is compact. Consider the sequence of periodic functions $\{\varphi_n\} \subset X$ and assume that the sequence is uniformly bounded. Let $\Theta > 0$ be such that $||\varphi_n|| \leq \Theta$, for all $n \in N$. In view of (H1)–(H3), we arrive at

$$\|g^\sigma(t, x)\| \leq \|g^\sigma(t, x) - g^\sigma(t, 0)\| + \|g^\sigma(t, 0)\|$$

$$= \max_{t \in [0,\omega]_T} \max_{1 \leq i \leq n} |g_i^\sigma(t, x) - g_i^\sigma(t, 0)| + \alpha_g$$

$$\leq L\|x\| + \alpha_g,$$

$$\|f(x)\| \leq \|f(x) - f(0)\| + \|f(0)\|$$

$$= \max_{1 \leq i \leq n} |f_i(x) - f_i(0)| + \alpha_f$$

$$\leq M\|x\| + \alpha_f,$$

$$\|I_j(x)\| \leq \|I_j(x) - I_j(0)\| + \|I_j(0)\|$$

$$= \max_{1 \leq i \leq n} n|I_{ij}^{(i)}(x) - I_{ij}^{(i)}(0)| + \alpha_{I_j}$$

$$\leq P_j\|x\| + \alpha_{I_j}, \text{ for } j \in \mathbb{Z}^+, \text{ hence,}$$

where $\alpha_g = \|g^\sigma(t, 0)\|$, $\alpha_f = \|f(0)\|$ and $\alpha_{I_j} = \|I_j(0)\|$. Hence,
the inequality

\[ (1.1) \]

holds. Then

\[ \text{Assume that} \]

Theorem 3.1.

\[ \text{has an} \]

\[ \text{1-periodic solution}. \]

Our main results reads as follows.

**Theorem 3.1.** Assume that (H1)–(H4) hold and \( L < 1 \). Suppose that there is a positive constant \( G \) such that all solutions \( x \) of (1.1), \( x \in X \), satisfy \( \|x\| \leq G \), and the inequality

\[ \frac{\gamma \omega \alpha_g + \omega N \alpha_f + \alpha}{1/\eta - \omega (\gamma L + MN) - L/\eta - P} \leq G, \]

holds. Then (1.1) has an \( \omega \)-periodic solution.
Proof. Define $M = \{ \varphi \in X : \| \varphi \| \leq G \}$. Then Lemma 2.6 implies $\Psi : X \to X$ and $\Phi : X \to X$. We need to show that if $\varphi, \psi \in M$, then $\| \Phi \varphi + \Psi \psi \| \leq G$. Let $\varphi, \psi \in M$ with $\| \varphi \|, \| \psi \| \leq G$, from (2.9)–(2.11), we have
\[
\| \Phi \varphi + \Psi \psi \| \leq \| \Phi \varphi \| + \| \Psi \psi \|
\leq LG + \overline{\eta} \omega (\gamma L + MN) + \overline{\eta} (\gamma \omega K + \omega N K f + GP + \alpha) \leq G.
\]
Thus $\Phi \varphi + \Psi \psi \in M$. We see that all the conditions of Krasnoselskii theorem are satisfied on the set $M$. Hence there exists a fixed point $z$ in $M$ such that $z = \Phi z + \Psi z$. By Lemma 2.3, this fixed point is a solution of (1.1). \hfill \square

**Theorem 3.2.** Suppose that (H1)–(H4) hold. If
\[
\Upsilon := \overline{\eta} [\omega (\gamma L + MN) + P] < 1,
\]
then (1.1) has an unique $\omega$-periodic solution.

Proof. For $\varphi, \psi \in X$, we have
\[
\| H \varphi - H \psi \| \leq \overline{\eta} \int_{0}^{\omega} \int_{-\infty}^{s} |D(s, u) f(\varphi(u)) - f(\psi(u))|_0 \triangle u \triangle s
\]
\[
+ \overline{\eta} \gamma \int_{0}^{\omega} |g^\sigma(s, \varphi(s - h(s))) - g^\sigma(s, \psi(s - h(s)))|_0 \triangle s
\]
\[
+ \overline{\eta} \sum_{j=1}^{p} |I_j(\varphi(t_j)) - I_j(\psi(t_j))|_0
\]
\[
\leq \overline{\eta} \omega MN \| \varphi - \psi \| + \overline{\eta} \gamma L \| \varphi - \psi \| + \overline{\eta} P \| \varphi - \psi \|
\]
\[
< \overline{\eta} [\omega (\gamma L + MN) + P] \| \varphi - \psi \|
\]
\[
= \Upsilon \| \varphi - \psi \|.
\]
This completes the proof. \hfill \square

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