Spectroscopic tests for short-range modifications of Newtonian and post-Newtonian potentials

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Abstract

There are theoretical frameworks, such as the large extra dimension models, which predict the strengthening of the gravitational field in short distances. Here we obtain new empiric constraints for deviations of standard gravity in the atomic length scale from analyses of recent and accurate data of hydrogen spectroscopy. The new bounds, extracted from $1S - 3S$ transition, are compared with previous limits given by antiprotonic Helium spectroscopy. Independent constraints are also determined by investigating the effects of gravitational spin-orbit coupling on the atomic spectrum. We show that the analysis of the influence of that interaction, which is responsible for the spin precession phenomena, on the fine structure of the states can be employed as a test of a post-Newtonian potential in the atomic domain. The constraints obtained here from $2P_{1/2} - 2P_{3/2}$ transition in hydrogen are tighter than previous bounds determined from measurements of the spin precession in an electron-nucleus scattering.
I. INTRODUCTION

The existence of extra dimensions has been speculated on a modern scientific basis since the advent of the Kaluza-Klein theory in the early twentieth century. This theory was an attempt to unify gravity and electromagnetism, the interactions known at that time, within a single formalism that postulated the presence of an additional spatial dimension. To avoid empirical contradictions, it is assumed that the fifth dimension has the topology of a circle with a radius of the order of the Planck length ($10^{-35}$ m). With such a tiny radius, however, there were no prospects for experimentally testing the existence of hidden dimensions neither at that time nor in a foreseeable future.

At the end of the last century, the subject of extra dimensions has emerged with renewed interest due to the braneworld models, which proposes a new scenario, inspired from developments in string theory, in which matter and all the standard model fields are confined in a space with three spatial dimensions (the 3-brane), while the gravitational field can propagate in all directions [1–4].

Initially, this scenario drew a lot of attention because the predicted ”spreading” of gravity to other dimensions could be an explanation for the hierarchy problem, i.e., for the question of why the gravitational interaction is so weak compared to the other forces, at long distances [1–4].

Another very interesting feature brought by some braneworld models, such as the ADD model [1, 2], is the possibility that empirical signals of extra dimensions could be the object of experimental search at present days. Indeed, as gravity is the only field that has access to the extra dimensions, then the hypothesis that the compactification radius $R$ could be much greater than the Planck length scale is phenomenological feasible, since gravity is being tested in a small length scale just recently.

It is well known that, if the supplementary space has compact topology, the gravitational field recovers its traditional four-dimensional behavior for distances $r$ much greater than the compactification radius $R$. On the other hand, the force gets strengthened at short distances ($r < R$). Direct laboratory tests of the inverse-square law, based on modern versions of torsion balances, put the upper limit $R < 44 \mu m$ [5, 6] if there is only one extra dimension. Regarding more codimensions, the most stringent constraints come from high-energy particle collision data and analysis of some astrophysics processes [7–13]. In all cases,
however, experimental upper bounds for $R$ are much weaker than the Planck length.

The amplification of gravity at short distances as predicted by higher-dimension theories has motivated many investigations about the behavior of the gravitational field in microscopic domains [14–21]. It is worthy of mention that, concerning the proton radius puzzle [23–27], there are conjectures on the possibility that higher-dimensional gravity could be an explanation for this issue [28].

A very common way of expressing modifications of gravity is through the so-called Yukawa parametrization, in which a Yukawa-like term is added to the Newtonian potential [6]. In this parametrization, the modified gravitational potential is written as $\varphi = -GM/r (1 + \alpha \exp(-r/\lambda))$, where the dimensionless parameter $\alpha$ measures the amplification of the interaction strength and $\lambda$ determines the length scale where the modifications are significant.

The Yukawa parametrization is very useful because it can account for deviations of Newtonian gravity which may have different physics origins. As we have mentioned above, hidden dimensions is a possible cause, however, there are extensions of the standard model of particle physics that predict the existence of additional bosons that indirectly could interfere in the inverse square law of gravity in certain domains [29, 30]. Some F(R)-theories make similar predictions too [31].

Facing these theoretical possibilities, here we are interested in using recent spectroscopic data of the hydrogen in order to searching for modifications of the gravitational field on the atomic scale. One of the tightest constraints for the parameter $\alpha$, derived from the atomic spectroscopy, is imposed by measurements of transition frequencies of the antiprotonic Helium [6, 32, 33], by exploring the gravitational interaction between the antiproton and the Helium nucleus. In section II, we shall see that new data of the $1S - 3S$ transition in the hydrogen establishes empiric bounds on deviations of the Newtonian potential that are slightly stronger than those obtained from that exotic atom.

Besides spectroscopic constraints, other independent bounds for hypothetical short-range interactions around the Angstrom length scale can be inferred from experiments that examine different physics phenomena such as neutron scattering [34], for instance. Another interesting example is the MTV-G experiment [35, 36] that intends to test the existence of a strong gravitational field produced by atomic nuclei by measuring, with a Mott polarimeter, the spin precession of an electron in a scattering process with a heavy nucleus. Preliminary
results were obtained by treating the geodetic precession of the spin from a classical point of view [6, 35, 36].

Here, inspired by the MTV-G experiment, we intend to inspect gravitational effects on the spin precession of an electron which is found in an atomic bound state, adopting the quantum perspective. As it is known, in the Hamiltonian formalism, the geodetic spin precession is dictated by the gravitational spin-orbit coupling [37, 38]. In section III, considering the Dirac equation in the curved spacetime, we discuss the influence of the gravitational spin-orbit coupling in the fine splitting between the states $2P_{1/2}$ and $2P_{3/2}$. As we shall see, this analysis does not put empirical limits on deviations of the Newtonian potential only, but actually it allows us to investigate the influence of a Post-Newtonian potential associated with spatial components of the metric in atomic length scale. This new potential is related to a specific parameter of the PPN-formalism (parametrized Post-Newtonian formalism) [39], whose geometrical meaning is connected to the curvature of the sections $t = \text{const}$. The constraints obtained here are stronger than those derived from the MTV-G scattering experiment [6, 35].

II. SPECTROSCOPIC CONSTRAINTS

In a certain range of length close to the Angstrom scale, the strongest empirical constraint on deviations of the gravitational field, imposed by atomic spectroscopy, as far as we know, comes from the analysis of transitions in the antiprotonic Helium ($\bar{p}He^+$) [6, 32, 33]. This exotic atom is formed in laboratory by replacing one of the two electrons of a natural Helium by an antiproton [40]. With this change, the Coulombian interaction between the particles and the nucleus is not directly altered, but the gravitational interaction between the antiproton, with mass $m_p$, and the Helium’s nucleus becomes almost two thousand times bigger than that between the nucleus and the electron, since $m_p \simeq 1836 \, m$, where $m$ is the electron mass. Hence, this exotic atom seems to be a system with adequate features to investigate the behavior of gravity in the atomic domain.

At first sight, another positive characteristic of this system would be the possibility of probing the gravitational interaction in a range of length that could reach thousands of the Angstrom, once the relative distance between the nucleus and the antiproton, which depends on the inverse of the antiproton mass, could be much smaller than traditional Bohr
radius \( (a_0 \simeq 0.5 \, \text{Å}) \). However, regarding this point, there is a downside aspect. In fact, in states in which the antiproton is found very close to the nucleus, the matter-antimatter annihilation process abbreviates the lifetime of the antiprotonic Helium to few picoseconds \[41\], preventing, therefore, any possibility of studying the spectroscopy of this atom with present technology.

It happens that, in a fraction of the antiprotonic helium atoms that are synthesized in laboratory, \( \bar{p} \) is found in Rydberg states with high principal quantum number \( n \) and high angular momentum \( l \sim n - 1 \ [41] \). In states with \( n \sim l \sim 40 \), the average distance of the antiproton to the nucleus is approximately around \( a_0 \ [32] \) and the overlap of the wave function with the nucleus is drastically suppressed. As a consequence, the lifetime of the antiprotonic helium increases to the order of microseconds \[41\]. These Rydberg states, with this longer lifetime, are amenable to be investigated by laser spectroscopy and, indeed, the transition frequencies between these metastable states were measured with a relative precision of some parts in \( 10^{-9} \ [41, 42] \). Comparing the experimental data \[41, 42\] with the theoretical calculations \[43, 44\], based on the theory of Quantum Electrodynamics (QED), one verifies a precise agreement between them \[6, 32, 33\]. This result put some bounds on the values that the parameters \( \alpha \) and \( \lambda \) could assume. For instance, at 1σ confidence-level, \( \alpha < 10^{28} \) for \( \lambda \sim 1 \, \text{Å} \ [33] \).

Here, we investigate possible deviations of the gravitational Newtonian potential in the atomic scale by using new data from hydrogen spectroscopy. The intention is to compare the constraints for the Yukawa parameters determined by the hydrogen spectroscopy with that established by the antiprotonic Helium. Although the gravitational interaction between proton and electron in the hydrogen atom is much weaker than the antiproton-nucleus gravitational interaction in \( \bar{p}He^+ \), the available spectroscopic data of the hydrogen are much more accurate. For instance, the experimental value of the \( 1S - 2S \) transition frequency, \( f_{1S-2S}^{\text{exp}} = 2466061413187035 \, \text{Hz} \), was recently measured with an error of \( \delta_{\text{exp}} = 10 \, \text{Hz} \), which corresponds to a relative precision of the order of \( 10^{-14} \ [45] \). If the theoretical value \( f_{1S-2S}^{\text{th}} \), predicted by QED, had an uncertainty \( \delta_{\text{th}} \) of the same magnitude order, then the agreement between \( f_{1S-2S}^{\text{th}} \) and \( f_{1S-2S}^{\text{exp}} \) would impose a much tighter constraint for the Yukawa parameters (see Figure 1). However the experimental value of the \( 1S - 2S \) transition frequency is the most precise value of the data set that is employed to determine the values of certain fundamental spectroscopic constants, such as the Rydberg constant (see Table XVIII of Ref.
As the theoretical predictions depend on these constants, the comparison between the calculated value with the measured frequency of this specific transition should be viewed with caution \[47\].

Because of this, let us consider the $1S - 3S$ transition. The relative precision achieved in the most recent measurement of the frequency of this transition, $f_{1S-3S}^{\text{exp}}$, is of the order of $10^{-12}$ \[48\]. The theoretical value calculated in Ref. \[47\] is of the same order. Although $f_{1S-3S}^{\text{exp}}$ is not so accurate as $f_{1S-2S}^{\text{exp}}$, the advantage of using that frequency to test QED is the fact that the isolated value of $1S - 3S$ transition frequency does not belong to the input data used in the least-squares adjustment of the values of the fundamental constants recommended by CODATA-2002 \[49\], which were employed by Ref. \[47\] to calculate the frequency of that transition.

The theoretical \[47\] and experimental \[48\] values are respectively:

\[
\begin{align*}
f_{1S-3S}^{\text{th}} &= 2922743278671.6(1.4) \text{ kHz}, \\
f_{1S-3S}^{\text{exp}} &= 2922743278671.5(2.6) \text{ kHz}.
\end{align*}
\]

The uncertainty is expressed in parenthesis. Admitting that the theoretical and experimental uncertainties are independent, then the combined error is $\delta f = \sqrt{\delta_{\text{th}}^2 + \delta_{\text{exp}}^2} \simeq 3.0$ kHz. It is clear that the theoretical prediction $f_{1S-3S}^{\text{th}}$ agrees very well with the measured frequency $f_{1S-3S}^{\text{exp}}$ within the combined error $\delta f$. Therefore, any new hypothetical interaction, such as the modified gravitational interaction, should not introduce corrections for the transition frequency in an amount $\Delta f$ greater than the error $\delta f$. The condition $\Delta f < \delta f$ imposes certain empirical limits for the parameters $\alpha$ and $\lambda$.

The supposed correction, $\Delta f$, that the modified gravity provides for the $1S - 3S$ transition can be calculated by using the perturbation method. The new gravitational interaction between the proton and electron is described, in the leading order, by the Hamiltonian $H_{G}^{(0)} = \frac{m_e \varphi}{r}$, where $\varphi = -Gm_p/r (1 + \alpha \exp(-r/\lambda))$ is the modified gravitational potential produced by the proton. This Hamiltonian $H_{G}^{(0)}$ should be considered as a small term of the atomic Hamiltonian. According to the perturbation formalism, in the first order, this new interaction will decrease the energy of each state $\psi$ by the amount $\langle H_{G}^{(0)} \rangle$, the average value of $H_{G}^{(0)}$ in the state $\psi$. The gravitational interaction will change the energy of the states $1S$ and $3S$ by different amounts, increasing the energy gap between these states. This implies
a correction of the transition frequency which, in the first approximation order, is given by:

$$\Delta f = \frac{\langle H_G^{(0)} \rangle_{3S} - \langle H_G^{(0)} \rangle_{1S}}{\hbar}.$$  \hspace{1cm} (3)

In Figure 1, we show the constraints on the Yukawa parameters imposed by the condition $\Delta f < \delta f$. As we can see, the new bounds are slightly stronger than those obtained from the spectroscopy of the atom $\bar{p}He^+$. For $\lambda = 1 \, \text{Å}$, for instance, the data demand that $\alpha < 1.7 \times 10^{27}$ at 1σ confidence-level.

Figure 1. Constraints for deviations of Newtonian potential determined here from the $1S - 3S$ transition in the hydrogen ($H_{1S-3S}$ line) are compared to other bounds imposed by spectroscopy of $dd\mu^+$, $\bar{p}He^+$ and $HD^+$ (data extracted from Ref. [32]). The $H_{1S-2S}$ curve is just a reference line that shows how stringent the spectroscopy of hydrogen could be, considering the current relative empirical precision ($10^{-14}$) of the $1S - 2S$ transition.

Actually, according to Figure 1, for $\lambda < 0.6 \, \text{Å}$, the $H_{1S-3S}$ constraints are tighter than several spectroscopic bounds, which also include empirical limits determined by the spectroscopy of $dd\mu^+$ (an exotic molecule formed by two deuterons and one muon) and also from the $HD^+$ (the ionized molecule constituted by a hydrogen and a deuterium).

The frequency of the $1S - 3S$ transition that we use here is one of the most accurate measurements in hydrogen spectroscopy, only surpassed by the $1S - 2S$ transition [50]. Based on these two transitions, it is possible to infer the proton charge radius. The value found in Ref. [50] is in agreement with the CODATA-2014 recommended value and differs by 2.8σ from the value extracted from the muonic hydrogen spectroscopy. Given this result, it seems interesting to resort to Rydberg states if we are aiming for a spectroscopic analysis that is less dependent on the proton size [51].
The effects of hidden dimensions in certain Rydberg states were studied in Ref. [52], by using a power-law parametrization for the modified gravitational potential. More recently, in Ref. [53] (we thank one of the referees to call our attention to this paper) Rydberg states were also considered with the purpose of constraining non-standard interactions by using Yukawa parametrization. As expected, the strongest restrictions are found in a length scale beyond the Bohr radius, since the studied states have a large principal quantum number. Admitting a relative precision of the order of $10^{-12}$ in the energy levels, it was found that the strength of the new interaction should be lesser than $10^{28}$ for $\lambda > 10^{-9}\text{m}$ (after converting data of figure 3 of Ref. [53] to units used here), which is very close to our result. They also considered the Rydberg states combined with data from other transitions. In this case, the constraint for $\alpha$ is almost of the order of $10^{27}$ in the range $10^{-10} - 10^{-7}\text{m}$ at a 95% confidence level, which is clearly compatible with our result.

So far, we have considered only spectroscopic constraints, since the main objective of this work is to use new data from hydrogen spectroscopy to put independent constraints on deviations of standard gravity. However, it is also interesting to compare these constraints with empirical limits imposed by sources of different natures. Figure 2, in addition to the spectroscopic constraints, includes other empirical limits set by data with distinct origins such as particle colliders, Casimir effect, torsion balance and Lunar Laser Ranging experiment, among others.

Figure 2. In this figure, the spectroscopic constraints for $\alpha$ determined here ($H_{1S-3S}$ line) are compared to empirical limits established by data from different origins (see Ref. [6] for collider data and Ref. [32] for all other data).

Almost all data in Figure 2 were extracted from Ref. [32], except for the collider data, which are obtained from Figure 1 of Ref. [6], and the line $H_{3S-1S}$ which was determined
here. Each distinct bound stands out on different length scale. As we can see, in this more
general context, the $H_{1S-3S}$ constraint is surpassed by the collisor bound for short $\lambda$ and
by HD$^+$ limit for $\lambda > 0.6$ Å. It is also important to remark that constraints from neutron
scattering, claimed in Ref. 34, are stronger than spectroscopic limit.

III. GRAVITATIONAL SPIN-ORBIT COUPLING

In a curved space, the final direction of a vector that is parallel transported along a closed
path may not coincide with its initial direction. Based on this fact, it is expected, in accord-
ance with the theory of General Relativity, that when a particle moves in a gravitational
field its spin will precess.

This effect in the classical regime was directly verified by some experiments such as
GRAVITY PROBE B, which measured the orientation changes of a gyroscope’s axis
moving along a geodesic around the Earth.

In the microscopic domain, the precession of an electron’s spin induced by the gravita-
tional field of an atomic nucleus was preliminarily investigated in the so-called MTV-G
experiment. This experiment tests the existence of a strong gravitational field in
the nuclear domain by studying the influence of a modified gravity on the spin precession
of an electron that is scattered by a nucleus. Following a classical description of the geodetic
spin precession of the electron, some preliminary results were obtained.

However, as the spin of an elementary particle is a quantum quantity, it is more appropri-
ate to treat the spin precession phenomena induced by a gravitational field in the quantum
mechanics formalism.

For this purpose, let us consider the Dirac equation in curved spacetime. The coupling
between fermions and the gravitational field is implemented through the tetrad fields $e^\mu_A$,
which consists of components of four orthonormal vector fields (each one is identified by the
index $A = 0, 1, 2$ or 3) written in some coordinate system $(x^\mu)$ (here, $\mu = 0, 1, 2$ and 3). The
tetrad fields satisfy the orthonormality condition $g_{\mu\nu}e^\mu_A e^\nu_B = \eta_{AB}$, where $g_{\mu\nu}$ denotes the
metric of the spacetime and $\eta_{AB}$ is the Minkowski metric.

With the help of the tetrad fields and the Dirac matrices defined on the Minkowski
spacetime, $\gamma^A$, we can construct the matrices $\gamma^\mu (x) = e^\mu_A (x) \gamma^A$, which satisfy the anti-
commutation algebra of the Dirac matrices adapted to the curved space: $\{ \gamma^\mu (x), \gamma^\nu (x) \} =$
The Dirac equation that describes the state $\psi$ of a particle of mass $m$ in a gravitational field is given by

$$[i\gamma^\mu (x) \nabla_\mu - mc/\hbar] \psi (x) = 0, \quad (4)$$

where $c$ is the speed of light, $\hbar$ is the reduced Planck constant and $\nabla_\mu$ is the covariant derivative of the spinor $\psi$, which depends on the spinorial connection $\Gamma_\mu (x)$ as follows:

$$\nabla_\mu \psi (x) = [\partial_\mu + \Gamma_\mu (x)] \psi (x). \quad (5)$$

Admitting the compatibility between the spinorial connection and the metric, it is possible to write $\Gamma_\mu (x)$ in terms of the Levi-Civita covariant derivative of the tetrad fields, $e^\nu_{A;\mu} (x)$, according to the expression:

$$\Gamma_\mu (x) = -i/4 \sigma^{AB} g_{\alpha\nu} e^\alpha_A (x) e^\nu_B;\mu (x), \quad (6)$$

where $\sigma^{AB} = i/2 [\gamma^A, \gamma^B]$ is a representation of the Lorentz Lie Algebra in the spinor space, written in terms of the commutating operator $[,]$.

From the Dirac equation, we can study the influence of the gravitational field produced by the proton on the behavior of an electron in the hydrogen atom. In the first approximation approach, it is reasonable to assume that the proton produces a static gravitational field with spherical symmetry. Under this condition, the spacetime metric can be put in the following form:

$$ds^2 = -c^2 w^2 dt^2 + v^2 (dx^2 + dy^2 + dz^2), \quad (7)$$

in the isotropic coordinates. The functions $w$ and $v$ depend only on the coordinate $r = (x^2 + y^2 + z^2)^{1/2}$. Associated to this metric, the non-null tetrad fields components are:

$$e^0_0 (x) = w^{-1}, \quad (8)$$

$$e^i_j (x) = \delta^i_j v^{-1}. \quad (9)$$

In the weak-field regime the function $v$ and $w$ can be expressed in terms of gravitational potentials as:

$$w = 1 + \varphi/c^2, \quad (10)$$

$$v = 1 - \tilde{\varphi}/c^2. \quad (11)$$
According to the General Relativity theory, $\phi = \tilde{\phi}$. However, as we are investigating modifications of the gravitational field in the atomic domain, let us admit that $\phi$ and $\tilde{\phi}$ can be different functions, or more precisely, that the Yukawa parameter ($\tilde{\alpha}$) associated to the potential $\tilde{\phi}$ is not necessarily equal to $\alpha$ (the parameter investigated in the previous section) and should be determined experimentally.

The potential $\tilde{\phi}$ is directly related to the curvature of the spatial section ($t = \text{const.}$) of the spacetime according to the geometric viewpoint. The possibility that $\tilde{\phi}$ is not necessarily equal to the Newtonian potential is also embodied in the parametrized post-Newtonian (PPN) formalism, which is a theoretical framework properly developed for the purpose of testing metric theories, such as the General Relativity and Brans-Dicke theory, in the weak-field limit [39].

The parameter $\tilde{\alpha}$ that we are investigating here can be put in correspondence with a certain PPN-parameter. Considering the Yukawa parametrization of the potential, we can see that $\tilde{\phi} = GM/r(1+\tilde{\alpha})$, in the limit $r \ll \lambda$. Thus, we can conclude that the combination $(\tilde{\alpha} + 1)$ plays the role of the parameter $\gamma$ of the PPN-formalism [39].

In the astrophysics domain, empirical bounds for the PPN-parameter $\gamma$ are extracted from time-delay and light deflection experiments, for example. Recent experiments were performed with the help of Cassini spacecraft and find that $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ [56], by studying the behavior of radio waves under the influence of the gravitational field of the Sun. This constraint is valid in the length scale of the solar radius and is compatible with the value predicted by the theory of General Relativity ($\gamma = 1$).

As we shall see later in this section, this Post-Newtonian parameter can be investigated in the atomic domain through the study of the influence of the gravitational spin-orbit coupling on the fine structure of the atom.

To show this, let us turn our attention to the Dirac equation again, now using the tetrad fields given above (Eqs. [8] and [9]). The Dirac equation can be rewritten in the form $i\hbar \frac{\partial \psi}{\partial t} = H_G \psi$, where $H_G$ is the operator that contains the gravitational sector of the atomic Hamiltonian. In a convenient representation, $H_G$ assumes the following form in the first order of the gravitational potentials [37, 38, 57]:

$$H_G = \beta mc^2 + \beta \varphi m + \frac{1}{2} \left\{ \tilde{\alpha} \cdot \vec{p}, [1 + (\phi + \tilde{\phi})/c^2] \right\},$$

(12)

where $\vec{p}$ is the usual three-dimensional momentum operator in flat spacetime, $\alpha^i = \gamma^0 \gamma^i$ and
\[ \beta = \gamma^0. \] The first term of \( H_G \) is related to the rest energy of the electron, the second term is associated with the usual potential energy of the proton-electron gravitational interaction \( (H_G^{(0)}) \), which was examined in the last section, and the third one gives rise to the kinetic term and also to relativistic and quantum corrections. In the form (12), it is important to stress that \( H_G \) is Hermitian in the Hilbert space of square-integrable functions endowed with the usual flat inner product \( \int d^3x \).

Admitting that the rest energy of the test particle is the leading term of the Hamiltonian, a semi-relativistic expansion of \( H_G \) (12) can be obtained by following the Foldy-Wouthuysen procedure, which consists in the elimination of odd operators of \( H_G \), in each order of \( 1/mc^2 \), by a convenient sequence of unitary transformations of the Hamiltonian [58]. Among many terms that arise in the expansion, here we want to focus our attention on the Hamiltonian term associated with the gravitational spin-orbit coupling, which can be expressed as [37, 38]:

\[
H_{Gso} = \frac{1}{mc^2} \frac{1}{r} \left( \frac{1}{2} \frac{d\varphi}{dr} + \frac{d\tilde{\varphi}}{dr} \right) (\vec{S} \cdot \vec{L}), \tag{13}
\]

where \( \vec{L} \) is the orbital angular momentum of the particle and \( \vec{S} \) is the spin operator, which can be written in terms of the Pauli matrices in the form \( \vec{S} = \left( \frac{\hbar}{2} \right) \vec{\sigma} \).

As we have already mentioned, this term (13), in the classical regime, is responsible for the geodetic precession of the axis of a gyroscope in curved spacetime [37, 38, 54]. Considering \( \vec{S} \) as a classical angular momentum measured by a co-moving geodesic observer, it can be shown [37, 54] that the spin dynamics is governed by the equation:

\[
\frac{d\vec{S}}{dt} = \left[ \left( \frac{1}{2} + \gamma \right) \frac{GM}{mc^2 r^3} \vec{L} \right] \times \vec{S}, \tag{14}
\]

assuming that \( \varphi = \tilde{\varphi}/\gamma = -GM/r \).

The analysis of the spin precession in the MTV-G experiment was based on the equation (14) taking \( \gamma = 1 \) [35, 36]. Therefore, without making any distinction between the Newtonian and the post-Newtonian potentials.

In the present work, we want to study the effect of the gravitational spin-orbit coupling on the energy levels of the hydrogen. In the atom, the electron is not a free-falling particle, but it is found in a bound state due to the electromagnetic interaction with the proton. This dominant interaction establishes a spin-orbit coupling too, described by a Hamiltonian that can be put in the form [58]:

12
\[ H_{Eso} = -\frac{q}{2m^2c^2} \frac{1}{r} \frac{d\phi_E}{dr} \left( \mathbf{S} \cdot \mathbf{L} \right), \] (15)

where \(-q\) is the electron charge and the function \(\phi_E\) is the electric potential produced by the proton in the curved space. In a spacetime with the metric (7), the electric field equations can be written in the same form of the Maxwell equations defined in a flat space endowed with a new electric permittivity given by \(\varepsilon = \varepsilon_0 v/w\), where \(\varepsilon_0\) is the permittivity of free space [38, 52]. Thus, if the electrostatic potential has spherical symmetry, it satisfies, in the first-order approximation, the following equation:

\[
\frac{d\phi_E}{dr} = -\frac{q}{4\pi\varepsilon_0 r^2} \left( 1 + \varphi/c^2 + \tilde{\varphi}/c^2 \right). \tag{16}
\]

As we can see, the spacetime curvature changes the electric potential. Through this correction of \(\phi_E\), the gravitational field can indirectly influence the atomic spin-orbit coupling from Hamiltonian \(H_{Eso}\). However, as we show in appendix, this contribution is five magnitude orders \((10^{-5})\) lesser than the direct contribution coming from (13). So, for our purposes, the indirect gravitational contribution can be ignored hereafter.

It is well known that the electromagnetic spin-orbit coupling is responsible for a fine splitting of energy levels with the same angular momentum \(l\), such as the \(2P_{1/2}\) and \(2P_{3/2}\) states, for example. The gravitational analogous (13), considered here as a weaker interaction in comparison to the Hamiltonian \(H_{Eso}\), will provide an additional shift in those states.

There are precise theoretical calculations of the energy of the hydrogen states, based on the QED theory. In Ref. [46], aiming to test the QED predictions, it was explicitly determined the transitions frequency between levels with \(n = 2\) in hydrogen. Specifically, the frequency transition between \(2P_{1/2}\) and \(2P_{3/2}\) is (pp. 1540 of Ref. [46]):

\[
f_{2P_{1/2}-2P_{3/2}}^{th} = 10969041.571(41) \text{ kHz}. \tag{17}
\]

This calculation is based on the recommended values for the fundamental constants which are extracted from the input data of CODATA-2010, but excluding the experimental values of \(2S_{1/2} - 2P_{1/2}\) and \(2P_{3/2} - 2S_{1/2}\) transition frequencies, which corresponds to the items A39, A40.1, and A40.2 in Table XVIII of CODATA-2010 [46].

On its turn, there are measurements of the centroid transition frequencies between \(2S_{1/2} - 2P_{1/2}\) [59] and \(2P_{3/2} - 2S_{1/2}\) [60]. Based on these experimental values, from which
the hyperfine structure of the states has already been excluded, we can determine the experimental frequency of the transition between $2P_{1/2}$ and $2P_{3/2}$:

$$f_{2P_{1/2}-2P_{3/2}}^{\text{exp}} = 10969045(15) \text{ kHz}.$$  

(18)

Of course, the theoretical and experimental values coincide within the combined error $\delta f = 15$ kHz. So the contribution provided by the gravitational spin-orbit interaction, $\Delta f_{so}$, cannot exceed this empirical error.

Now, in order to estimate $\Delta f_{so}$, let us remember that, due to the spin-orbit interaction, the operators $\vec{L}$ and $\vec{S}$ no longer commute with the atomic Hamiltonian. On the other hand, the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ is a conserved quantity. Therefore, atomic stationary states are labeled by the eigenvalues of the total angular momentum. When the orbital angular momentum and the spin are, let us say, aligned, the total momentum is $j = l + 1/2$ and, in this case, $H_{Gso}$ will provide a positive energy shift for the state. On the other hand, for states with $j = l - 1/2$, the spin-orbit interaction will give a negative contribution to the energy.

These effects increase the energy gap between those states. In the first approximation, the additional separation between states $(n, l, j = l + 1/2)$ and $(n, l, j = l - 1/2)$ due to gravitational spin-orbit interaction leads to the following change in the frequency transition:

$$\Delta f_{so} = \frac{\Delta E_{Gso}}{\hbar} = \frac{1}{hmc^2} \left< \frac{1}{r} \frac{d}{dr} (\phi/2 + \tilde{\phi}) \right>_{n,l} (l + 1/2) \hbar^2,$$  

(19)

where the average $\langle \rangle_{n,l}$ is calculated with respect to the radial solution of the Schrödinger equation, $R_{n,l}(r)$, for states with a principal quantum number $n$ and orbital angular momentum $l$.

Now let us discuss the implications of the condition $\Delta f_{so} < \delta f$ in the case of the $2P_{1/2} - 2P_{3/2}$ transition, for which $R_{21} = \left( 1/\sqrt{24a_o^2} \right) (r/a_o) e^{-r/2a_o}$. First, let us emphasize that in the previous section, we have tested modifications of the Newtonian potential, but not the potential $\tilde{\phi}$, since, in the $1S - 3S$ transition, the curvature of the spatial sections has no influence on the energy of the $S$-states in the leading order.

The expression (19) indicates that the effects of the gravitational spin-orbit coupling on the spectroscopy depend on a linear combination of the Newtonian potential and the potential $\tilde{\phi}$. Therefore, the spectroscopic analysis will put empirical bounds on the mixed parameter $\left( \alpha/2 + \tilde{\alpha} \right)$. However, taking into account the empirical bounds for the parameter
\( \alpha \) previously established, we can verify that the contribution of the potential \( \varphi \) in the transition \( 2P_{1/2} - 2P_{3/2} \) is smaller than the experimental error. Therefore, for practical purposes, we can neglect the potential \( \varphi \) in the expression (19). Thus, we can conclude that the analysis of the influence of the gravitational spin-orbit interaction on the fine structure of the states is actually a test of the Post-Newtonian potential \( \bar{\varphi} \) in the atomic domain.

In Figure 3, we show the constraints for the mixed parameter \((\alpha/2 + \bar{\alpha})\) in terms of \( \lambda \). As we can see, for \( \lambda > 1.5 \times 10^{-3} \) Å, the bounds determined by the \( 2P_{1/2} - 2P_{3/2} \) transition are more stringent than the empirical limits put by MTV-G experiment. In particular, for \( \lambda = 1 \) Å, we find \( \bar{\alpha} < 2.1 \times 10^{23} \), which is stronger than the MTV-G constraint by four magnitude orders.

Figure 3. The dashed line is extracted from the analysis of the influence of the gravitational spin-orbit coupling on the separation between the states \( 2P_{1/2} \) and \( 2P_{3/2} \) of hydrogen. It sets an empirical constraint for the mixed parameter \((\alpha/2 + \bar{\alpha})\), where \( \bar{\alpha} \) is related to the \( \gamma \)-parameter in the PPN-formalism. For \( \lambda > 1.5 \times 10^{-3} \) Å, the \( 2P_{1/2} - 2P_{3/2} \) constraint is stronger than the empirical bounds extracted from the MTV-G scattering experiment [36].

The analysis based on the influence of the gravitational spin-orbit coupling on the \( 2P_{1/2} - 2P_{3/2} \) transition provides a weaker limit compared to the bounds determined from the \( 1S - 3S \) transition. However, we should have in mind that these two tests probe different physical quantities. In fact, the \( 1S - 3S \) transition, as well as all tests described in the Figure 2, sets upper limits on deviation of the Newtonian potential, while the \( 2P_{1/2} - 2P_{3/2} \) actually allows us to constrain the behavior of a post-Newtonian potential, which has a proper geometric meaning and it is not necessarily equal to the Newtonian potential in some metric theories.
IV. FINAL REMARKS

Considering accurate data of hydrogen spectroscopy (more specifically, recent measurement of the $1S - 3S$ transition frequency \[48\]) we find constraints for short-range modifications of the Newtonian potential in the atomic domain. The bounds obtained here are tighter than several empirical limits imposed by the spectroscopy of some exotic atom such as the antiprotonic-Helium. Although the interaction between electron and proton is almost two thousand times weaker compared to antiproton-nucleus gravitational interaction, we have seen that the accuracy achieved in the hydrogen spectroscopy is high enough to determine an improved constraint for deviations of the Newtonian potential in range $\lambda < 0.6 \text{Å}$.

In the present discussion, we have adopted the comprehensive Yukawa parametrization to express modifications of gravity. The reason is that several models, such as large extra dimension models \[61\] and some F(R)-theories \[31\], predict an exponentially decreasing correction of the Newtonian potential in a domain beyond a certain length scale. Although the Yukawa parametrization is useful, it has some limitations as any approximation scheme \[22\]. Indeed, as we are exploring atomic spectroscopy data, the best results are found when the supposed deviations occur around the Bohr radius ($a_0$). However, if the modifications are significant in a length scale much lesser than $a_0$, then the Yukawa parametrization will not be capable to appropriately capture their effects. In this case, each model should be considered separately and studied in detail. In Ref. \[20\], for instance, the power-law parametrization is adopted to study the ADD model in thick branes scenarios.

In this paper, we have also investigated non-standard gravitational effects on the fine separation between the $2P_{3/2}$ and $2P_{1/2}$ states. The energy difference between these states, which have the same principal quantum number and the same orbital angular momentum, is mainly determined by the spin-orbit coupling. This interaction, which is responsible for the spin precession phenomena, was explored by MTV-G experiment in order to investigate a possible strong gravitational field produced by the nucleus, by measuring the spin precession of an electron in a scattering process with a Mott polarimeter.

Inspired by the MTV-G experiment, we have discussed the effects of gravitational spin-orbit coupling on the electron in a bound state. From the Dirac equation, we study the influence of that interaction on the fine structure of the $2P$-state of the hydrogen. We have shown that the empirical constraints determined by this analysis are numerically weaker.
than those put by the $3S - 1S$ transition, however, it is important to emphasize that, as the separation between $2P_{3/2}$ and $2P_{1/2}$ states has essentially a relativistic origin, then the analysis of the $2P_{3/2} - 2P_{1/2}$ transition yields not a test of Newtonian potential, but, actually, it is a test of a post-Newtonian potential, which is related to the $\gamma$-parameter in the PPN-formalism.

We have seen that, for $\lambda > 1.5 \times 10^{-3}$ Å, the empirical limits on the post-Newtonian potential determined by the spectroscopic data are stronger than those extracted from the MTV-G scattering experiment.

At this point, we would like to mention that in a recent measurement of $2S - 4P$ transition frequency, the splitting between the states $4P_{3/2}$ and $4P_{1/2}$ was determined with a precision of 4.3 kHz [62]. In principle, this more accurate measurement could be used to set a better constraint for the post-Newtonian potential.

Another perspective to test modifications of gravity in the atomic length scale is to consider the spectroscopy of heavier atoms or ions, especially that of the element He whose transitions are being measured with a relative precision of the order of $10^{-12}$ [63], and, for this reason, are being used to probe new physics in the microscopic domain [64].

Finally, we would like to highlight that a supposed modified gravitational interaction would affect the isotope shift of atomic transitions. Therefore, taking advantage of the well-studied theoretical framework of isotope shift spectroscopy as well as the precision measurements of the correspondent frequencies [65], we could, from this complementary method, set new constraints for the strength of non-standard gravitational interactions in the atomic domain [65, 66].

V. APPENDIX

Here we compare the magnitude order of the indirect contribution of the gravitational field through the Hamiltonian $H_{Es0}$ [15] and the direct contribution given by $H_{Gso}$ to the fine structure of $P$ -states, as discussed in section III.

Substituting (16) in (15), we may verify that the gravitational correction of $H_{Es0}$ is given by:

$$H_{Es0}^{(G)} = \frac{q^2}{8\pi\varepsilon_0 m^2 c^2 r^3} (\varphi/c^2 + \ddot{\varphi}/c^2) \left( \vec{S} \cdot \vec{L} \right). \quad (20)$$

Now let us write the potential $\varphi$ explicitly in terms of $r$ in both Hamiltonians $H_{Gso}$ and
$H_{Eso}^{(G)}$. Considering just the Yukawa-term, since the standard part is negligible, we obtain
the following expressions in magnitude order:

$$H_{Eso}^{(G)} \approx \frac{q^2}{8\pi\varepsilon_0 m^2 c^4} \frac{\alpha GM}{r^4} e^{-r/\lambda} \left( \vec{S} \cdot \vec{L} \right),$$

$$H_{Gso} \sim \frac{1}{mc^2} \left( 1 + \frac{r}{\lambda} \right) \frac{\alpha GM}{r^3} e^{-r/\lambda} \left( \vec{S} \cdot \vec{L} \right).$$

For the sake of simplicity, here we have assumed $\alpha \sim \tilde{\alpha}$. Now taking the average of the
Hamiltonians in the 2P-state, for instance, we obtain the following relation:

$$\langle H_{Eso}^{(G)} \rangle \sim \frac{q^2/4\pi\varepsilon_0 a_0}{mc^2} \left( \frac{\lambda}{\lambda + a_0} \right) \langle H_{Gso} \rangle.$$

Notice that the coefficient is proportional to the ratio between the energy of the hydrogen
ground state and the rest energy of the electron, which is of the order of $10^{-5}$. It also
depends on a certain relation between $\lambda$ and $a_0$, which is lesser than 1 for all value of $\lambda$.

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