Can we see gravitational collapse in (quantum) gravity perturbation theory?

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Abstract

In this paper, by making use of the perturbative expansion around topological field theory we are trying to understand why the standard perturbation theory for General Relativity, which starts with linearized gravity does not see gravitational collapse. We start with investigating classical equations of motion. For zero Immirzi parameter the ambiguity of the standard perturbative expansion is reproduced. This ambiguity is related to the appearance of the linearized diffeomorphism symmetry, which becomes unlinked from the original diffeomorphism symmetry. Introducing Immirzi parameter makes it possible to restore the link between these two symmetries and thus removes the ambiguity, but at the cost of making classical perturbation theory rather intractable. Then we argue that the two main sources of complexity of perturbation theory, infinite number of degrees of freedom and non-trivial curvature of the phase space of General Relativity could be disentangled when studying quantum amplitudes. As an illustration we consider zero order approximation in quantum perturbation theory. We identify relevant observables, and sketch their quantization. We find some indications that this zero order approximation might be described by Doubly Special Relativity.

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1 Introduction

One of the simplest models describing a collapse of a spacetime is \(2 + 1\) dimensional gravity coupled to point particles [1]. A particle induces a conical singularity with the deficit angle proportional to its mass. If cosmological constant is zero spacetime collapses when deficit angle becomes larger than \(2\pi\). Physically it results in a bound on energy, or in the appearance of an invariant energy scale in the relativistic symmetries of particles propagation in such spacetime. The latter are known as Doubly Special Relativity symmetry [2] (see also [3].)

The picture of collapsing spacetime becomes more elaborate if we add cosmological constant. If \(\Lambda > 0\) spacetime has an outer cosmological horizon. When we add a massive particle the cosmological horizon shrinks around it, and if the mass of the particle exceeds Planckian the horizon shrinks to a point. If \(\Lambda < 0\), the energy is not bounded from above. However as soon as mass of a particle gets larger than \(m_{\text{Pl}}\), it becomes surrounded by a horizon known as the BTZ black hole.

In four spacetime dimensions the picture of gravitational collapse is even more intricate. Classically, any point particles creates a black hole around it with its size proportional to its mass. In the presence of positive cosmological constant one has also a cosmological horizon which shrinks as the mass grows. One can even imagine a situation when the horizon of a black hole collides with cosmological horizon. This could be a bound on energy in four dimensional GR analogous to the energy bound in \(2+1\) gravity mentioned above.

It is worth mentioning here that as it is understood at least in \(2+1\) gravity the phenomena described above are vital for consistent quantization of the theory. The energy bound is responsible for the appearance of fixed point in renormalization group.\(^1\)

The problem is that the standard perturbative approach which starts by linearizing gravity around a fixed background can see none of these phenomena. If in \(2+1\) dimensions one can hope to solve the theory exactly, this seems hardly possible in dimension four. There the complexity of the theory is multiplied by the presence of infinite number of degrees of freedom. The natural question then arises: is it possible to develop an approximate method to solving (quantum) Einstein’s equations which would be able to

\(^1\)Sometimes it is said that \(2+1\) gravity owes its renormalizibility to the absence of local degrees of freedom. This is not the case however. A simple counterexample is a quantization of \(2+1\) gravity by expanding around a fixed background. It knows about the absence of propagating modes of gravitational field, but it is still non-renormalizable
see gravitational collapse, maybe in some simplified form? The first step towards answering this question would be to understand the reason why the standard perturbative approach does not manage to do it.

We try to address this question in the present paper. For this we utilize the perturbative framework of [4] which uses a certain topological field theory (namely BF theory) as a starting point. It requires a non-zero cosmological constant and also admits inclusion of Immirzi parameter.

We start by studying such perturbative expansion of classical equations of motion with zero Immirzi parameter. It appears that it reproduces the results of standard perturbation theory which starts with linearization. The linearized Einstein’s equations emerge in it as certain integrability conditions for the first order equations.

The immediate problem that appears at this stage is the invariance of this equations with respect to linearized diffeomorphisms which become unlinked from the actual diffeomorphisms and therefore do not correspond to any physical symmetry. This leads to ambiguities in the physical results obtained from such perturbation theory.

This may seem strange for an approach starting from BF theory as it is invariant with respect to exact diffeomorphisms. However linearized diffeomorphism transformation now appear in a different guise: as a part of ”translational” symmetry group of BF-theory which is dual to the local gauge symmetry.

A small modification of BF theory, namely introducing Immirzi parameter provides a natural solution to this problem. It connects the ”translational” symmetry and the local gauge symmetry into a single one, and thus, to first approximation, reestablishes the link between linearized and actual diffeomorphisms. It has a physical effect that stringy degrees of freedom coming from breaking ”translational” symmetry now become coupled to particles degrees of freedom coming from breaking local gauge symmetry.

However the next problem emerges here. The first order integrability conditions turn into the form of exact Einstein’s equations and as such cannot be solved in a generic situations. This makes the classical perturbation theory not useful in practice.

Fortunately the situation is very different in quantum theory where we are interested not in calculating the values of the fields everywhere in space, but only in scattering amplitudes for finite number of particles. At a finite order of perturbation theory only finite number of modes of the field contribute to such amplitudes. This allows us to reduce the problem to finite-dimensional, while keeping track of some non-linearities of General Relativity at the same time.
One of the effects of introducing Immirzi parameter in this case is that quantum perturbation theory becomes “semi-holographical”. Its Feynman diagrams map on a boundary of space and they braid as in 3 dimensions. At the same time every particle in a diagram has 3 degrees of freedom as in 4 dimensions. In Euclidian theory for example the braiding effect makes energy and momenta bounded from above. The bound is controlled by a certain combination of Newton constant, cosmological constant, and Immirzi parameter.

In section 2 we review the setup of [4] and [5] for perturbation theory for gravity and its coupling to particles.

In section 3 we solve it to the first order for zero Immirzi parameter and identify the problem related to the appearance of an extra gauge symmetry.

In section 4 we discuss physical implication of this extra gauge symmetry, which is decoupling of some of the degrees of freedom linked in the full theory, and show how introducing Immirzi parameter restores this coupling. We also show that Immirzi parameter mixes mass with Taub-NUT charge, which results in the appearance of non-local singularities – gravitational Dirac strings.

In section 5 we give a brief perspective on how to calculate particle scattering amplitudes in this perturbation theory. We identify the observables which contribute to perturbative calculation of the amplitudes and show that at any finite order there is only a finite number of them, which makes quantum perturbation theory tractable. For non-zero Immirzi parameter this observables become non-local due to Dirac strings, and we argue that this leads to a certain kind of ”dimensional reduction” in quantum amplitudes. Finally, we take a closer look at the observables which are relevant at zero order of perturbation theory. We show that the variables canonically conjugate to particle positions form de Sitter space, which is an indication of the presence of Doubly Special Relativity effects in the theory.

In section 6 we discuss some mathematical problems which has to be solved to complete the definition of these perturbation theory.

2 Equations for gravity with particle source

To set the stage, let us recall, following [4] and [5] the formulation of gravity as a constrained topological field theory and it coupling to point particle(s). Let us start with the pure gravitational field, which is encoded in a $\text{SO}(4,1)$
(de Sitter) connection

$$A_\mu = A^{IJ} T_{IJ} = \left( \frac{1}{\ell} \epsilon_\mu^a T_{a4} + \frac{1}{4} \omega_\mu^{ab} T_{ab} \right)$$

(1)

where index \(a\) runs from 0 to 3. In this formula \(T_{IJ}\), \(I, J = 0, \ldots, 4\) are generators of Lie algebra \(so(4,1)\). \(\epsilon_\mu^a\) is the frame field from which the metric is constructed and \(\omega_\mu^{ab}\) is the spin connection. Connection has a natural mass dimension 1 whereas the frame field is dimensionless; this is the reason why a length scale \(\ell\) appears in the expression of the components of \(A\) representing the frame field. This length scale has to be the cosmological length \(\ell\) related to the cosmological constant by

$$\frac{1}{\ell^2} = \frac{\Lambda}{3}$$

To formulate the theory we also need a two form valued in the Lie algebra \(so(4,1)\) denoted by \(B = B_{\mu\nu}^{IJ} T_{IJ} dx^\mu \wedge dx^\nu\). In terms of this two form and connection \(A\) the action takes the form

$$S = \int \left( B_{IJ} \wedge F^{IJ} - \alpha \frac{\epsilon^{IJKL} B_{IJ} \wedge B_{KL} v_M}{4} - \beta \frac{B^{IJ} \wedge B_{IJ}}{2} \right)$$

with \(v_M\) being a constant vector in fundamental representation of the gauge group, forcing the gauge symmetry breaking from \(SO(4,1)\) down to \(SO(3,1)\) (see [6].) This action can be shown [4] to be equivalent to the action of General Relativity with nonzero cosmological constant and a nonzero, dimensionless Immirzi parameter \(\gamma\). The initial parameters dimensionless \(\alpha, \beta, \ell\) are related to the physical ones as follows

$$\frac{1}{\ell^2} = \frac{\Lambda}{3}, \quad \alpha = \frac{G\Lambda}{3} \frac{1}{(1 + \gamma^2)}, \quad \beta = \frac{G\Lambda}{3} \frac{\gamma}{(1 + \gamma^2)}$$

(3)

Let us now turn to the particle(s) coupling. As it was shown in [5] matter can arise in the most natural way by introducing the simplest possible term breaking the gauge symmetry of the theory in a localized way. The gauge degrees of freedom are then promoted to dynamical degree of freedom, which reproduce the dynamics of a relativistic particle coupled to gravity. This idea is realized is obtained by choosing a worldline \(P\) and a fixed element \(K\) of the \(so(4,1)\) Lie algebra with generators \(T^{IJ}\), depending on the particle rest mass and spin

$$K \equiv m\ell T^{04} + s T^{23}$$

(4)

so as to have

$$S_P(A) = - \int d\tau \text{Tr} \left( K A_{\tau}(\tau) \right)$$

(5)
where \( \tau \) parameterizes the world line \( z^\mu(\tau) \) and \( A_\tau(\tau) \equiv A_\mu(z(\tau)) \dot{z}^\mu \).

The action for the particle is obtained from (5) by adding gauge degrees of freedom that become dynamical at the particle worldline. To this end we take \( A^h = h^{-1}Ah + h^{-1}dh \) to be the gauge transformation of \( A \). Then the particle lagrangian takes the simple form

\[
L(z, h; A) = -\text{Tr} \left( K h(\tau) \right) \quad S = \int d\tau \, L(z, h; A) \tag{6}
\]

which can be rewritten as

\[
L(z, h; A) = -\text{Tr} (JA_\tau) + L_1(z, h) \tag{7}
\]

where in the first term \( J \) is given by

\[
J \equiv hK h^{-1} \tag{8}
\]

The first term in equation (7) describes the covariant coupling between the particle and the \( A \) connection of the (constrained) BF theory, while the second

\[
L_1(z, h) = -\text{Tr}(h^{-1}hK), \tag{9}
\]

describes the dynamics of the particle.

The field equations for gravity coupled to point particle, resulting from (2) and (7) in the gauge, in which the particle is at rest at the origin of the coordinate system take the form

\[
F^{IJ} = \alpha \epsilon^{IJKLM} B_{JK} v_M + \beta B^{IJ} \tag{10}
\]

\[
\mathcal{D}_A B^{IJ} = J^{IJ} \delta_P, \quad \delta_P = \delta^3(x) \varepsilon \tag{11}
\]

where \( \mathcal{D}_A \) is covariant derivative of connection \( A \) and \( \varepsilon \) is the volume three-form on constant time surface.

In this paper we will be primarily interested in solving these equations in the limit \( \alpha \to 0 \). It would seem that it suffices to put \( \alpha = 0 \) in (10) and solve the resulting equations; however there could be contributions from higher order equations affecting this limit. Therefore more care is needed and we consider expansion of the equations up to the first order.

Consider the expansion in \( \alpha \)

\[
A = A^{(0)} + \alpha A^{(1)} + \ldots, \quad B = B^{(0)} + \alpha B^{(1)} + \ldots
\]

In the zero order in \( \alpha \) we get

\[
F^{IJ}(A^{(0)}) = \beta B^{(0)IJ} \tag{12}
\]
\[ D_{A^{(0)}} B^{(0)IJ} = J^{IJ} \] (13)

In the first order in \( \alpha \) we find
\[ D_{A^{(0)}} A^{(1)} = \epsilon^{IJKLM} B^{(0)JK} v_M + \beta B^{(1)IJ} \] (14)
\[ D_{A^{(0)}} B^{(1)IJ} + [A^{(1)}, B^{(0)}]^{IJ} = 0 \] (15)

If we now take the covariant differential \( D_{A^{(0)}} \) of (14) and compare it with (15), we find the first integrability condition
\[ \epsilon^{IJKLM} D_{A^{(0)}} (B^{(0)JK} v_M) = 0. \] (16)

The second integrability condition comes from acting by \( D_{A^{(0)}} \) on eq.(15) and reads
\[ [B^{(0)}, B^{(0)*}]^{IJ} + [A^{(1)}, J]^{IJ} = 0, \] (17)
where
\[ B^{(0)*IJ} = \epsilon^{IJKLM} B^{(0)JK} v_M \] (18)

Equations (16), (17) affect the zero order contributions to the field \( B^{(0)} \) and cannot be ignored even in the limit \( \alpha \to 0 \).

Thus the topological limit of gravity, coupled to a point particle is governed by four equations (12), (13), (16), (17). In what follows we will drop the \(^{(0)}\) superscript to simplify the notation.

### 3 Zero Immirzi parameter and ambiguities in the solution

In this section we solve the above equations to the first order for \( \beta = 0 \) and identify the problem arising in such approximation.

Zero order is described by equations (12),(13). For \( \beta = 0 \) those equations possess the following gauge symmetry:
\[ A \to g^{-1} A g + g^{-1} dg, \quad B \to g^{-1} B g \] (19)
and
\[ B \to B + D_A \Phi \] (20)
where \( g \in SO(4,1) \), and \( \Phi \) is a \( so(4,1) \)-valued one form.

The solution in the zeroth order is thus
\[ A = g^{-1} dg \quad B = B_p + g^{-1} \Phi g, \] (21)
where $B_p$ is a particular solution depending on $J$. We consider spinless particle, so we choose

$$J = g^{-1}K g = m\ell g^{-1}T^{04} g.$$ 

If we define $B_p = gB_p g^{-1}$ the above equation simplifies to

$$d B_p = K$$

(22)

and a possible solution is

$$B_p = m\ell T^{04} \epsilon_{abc} \frac{x^a}{r^3} dx^b \wedge dx^c$$

(23)

So far $g$ and $\Phi$ were arbitrary, but from the first order we have extra equations (16), (17). The origin of this equations can also be explained the following way. We take the general solution of zeroth order equation (21) and plug it into the symmetry breaking term in (2) (the one which is proportional to $\alpha$). Then by varying the resulting expression w.r.t. $g$ we obtain equation (17) and by varying it w.r.t. $\Phi$ we obtain (16).

However because the whole action is diffeomorphism invariant and because the translational part of $g$ is closely related to diffeomorphism transformations we can start by making an arbitrary choice of $g$ which will simply result in a choice of coordinate system.

For spherical static coordinates in deSitter space we have to choose

$$g =
\begin{pmatrix}
\cosh \frac{t}{\ell} & \frac{r}{\ell} \sinh \frac{t}{\ell} & 0 & 0 & \sqrt{1 - \frac{r^2}{\ell^2}} \sinh \frac{t}{\ell} \\
0 & \sqrt{1 - \frac{r^2}{\ell^2}} \sin \theta \cos \phi & -\sin \phi & -\cos \theta \sin \phi & -\frac{r}{\ell} \sin \theta \cos \phi \\
0 & \sqrt{1 - \frac{r^2}{\ell^2}} \sin \theta \sin \phi & \cos \phi & -\cos \theta \sin \phi & -\frac{r}{\ell} \sin \theta \sin \phi \\
0 & \sqrt{1 - \frac{r^2}{\ell^2}} \cos \theta & 0 & \sin \theta & -\frac{r}{\ell} \cos \theta \\
\sinh \frac{t}{\ell} & \frac{r}{\ell} \cosh \frac{t}{\ell} & 0 & 0 & \sqrt{1 - \frac{r^2}{\ell^2}} \cosh \frac{t}{\ell}
\end{pmatrix}
$$

(24)

It remains now to plug the solution (23) to first integrability condition (16) and solve it for $\Phi$. For $\beta = 0$ $D_\lambda$ in (16) becomes ordinary differential $d$, and the equation reminds the linearized Einstein equation, but with unusual type of source. In this case the solution for $\Phi$ reads

$$\Phi = -\ell \left( 1 - \frac{r^2}{\ell^2} \right) \epsilon_{ijb} \frac{x^b}{r^3} dt T^{ij}$$

$$-\ell \sqrt{1 - \frac{r^2}{\ell^2}} \epsilon_{iab} \frac{x^b}{r^3} dx^a \left( \sinh \frac{t}{\ell} T^{4i} + \cosh \frac{t}{\ell} T^{0i} \right)$$

(25)
Notice that to this point the presence of non-zero cosmological constant was required. For $\Lambda \to 0$ one has $l \to \infty$ and the expression (25) for $\Phi$ diverges.

Having solved the condition (16) one can turn to finding $A^{(1)}$ from (14). The later equation has a regular limit as $l \to \infty$ and we can take it at this point. We will write down explicitly only the metrical part of it, i.e. the part proportional to $T^{4a}$

$$A^{(1)} = -\frac{1}{r}T^{40}dt + \frac{1}{r}T^{4i}dx^i + (...)T^{ab} + D_{A^{(0)}}\Psi,$$  \hspace{1cm} (26)

where $\Psi$ is an arbitrary 0-form field. This expression contains the standard Newtonian potential term, but there is also some ambiguity in it. Different choices of $\Psi$ lead to different metric that cannot be related by coordinate transformation. This is a source of ambiguity in physical results.

Similar problem can be observed in linearized gravity as well. Linearized Einstein’s equations

$$\nabla^m \nabla_m h_{ab} - 2\nabla^m \nabla_{(a} h_{b)m} + \nabla_a \nabla_b(g^{mn} h_{mn}) = 0$$  \hspace{1cm} (27)

are invariant with respect to exact diffeomorphisms when one simultaneously transforms the background field $g^{ab} \to g^{mn}_{0} \frac{\partial x^m}{\partial x_a} \frac{\partial x^n}{\partial x_b}$ and the fluctuation $h_{ab} \to h_{mn} \frac{\partial x^m}{\partial x_a} \frac{\partial x^n}{\partial x_b}$. At the same time they are invariant with respect to linearized diffeomorphisms when one transforms fluctuations only $h_{ab} \to h_{ab} + \nabla_{(a} \xi_{b)}$, where $\xi_a$ is an increment of a coordinate $x_a$, keeping the background field fixed. Linearized gravity doesn’t “remember” that the two above symmetries have common origin. Therefore the gauge-fixing for both symmetries can be done independently.

To illustrate one of the consequences of the above ambiguity let us recall one curious fact that the Schwarzschild solution of the Einstein’s equations in Kerr-Schild parameterization is also a solution of linearized Einstein’s equations (27)[7]. It is related to the usual Schwarzschild coordinates by transformation $T = t - 2m \ln(r/2m - 1)$, and the metric in this parameterization looks like

$$ds^2 = -(1 - \frac{2m}{r})dT^2 - \frac{4m}{r}dTdr + (1 + \frac{2m}{r})dr^2 + r^2d\Omega.$$  \hspace{1cm} (28)

It is a solution of equations (27), and it looks quite similar to the standard Newtonian metric

$$ds^2 = -(1 - \frac{2m}{r})dT^2 + (1 + \frac{2m}{r})dr^2 + r^2d\Omega.$$  \hspace{1cm} (29)
which is also a solution of (27). The two however cannot be related by a coordinate transformation, and they are physically different with different Penrose diagrams. They can be related by linearized diffeomorphism transformation with \( \xi_0 = -4m \ln(r/4m) \) and \( \xi_i = 0 \).

Clearly the arbitrariness in the choice of \( \Psi \) in (26) leads to exactly the same ambiguity as above. By adjusting \( \Psi \) we can have either standard newtonian potential or a black hole. This is a problem because perturbation theory cannot distinguish between the two.

However we also have to remember about the second integrability condition (17). By direct calculation one can check that away from particle sources it is automatically satisfied. This was to be expected as a result of diffeomorphism invariance. At the points of the particle sources this condition reduces to

\[
[A^{(1)} + \Phi^*, J]^{IJ} = 0,
\]

where

\[
\Phi^{*IJ} = \epsilon^{IJKLM} \Phi_{JK} v_M
\]

(31)

As it was shown in [5] equation (30) is directly related to geodesic equation for the test particles.

One can see that in equation (30) the problem of invariance of equation (14) with respect to linearized gauge transformation \( A^{(1)} \rightarrow A^{(1)} + D_A \Psi \) is combined with another ambiguity which is the invariance of equation (16) for \( \Psi \) with respect to similar transformation

\[
\Phi \rightarrow \Phi + D_A \chi,
\]

(32)

where \( \chi \) is an arbitrary scalar field. But the later symmetry is a subgroup of the symmetry of \( BF \)-theory, namely (20). Thus, the symmetry that causes the problem in the linearized gravity has its counterpart in \( BF \) theory. Therefore, the perturbative approach starting with \( BF \) theory considered in this section will suffer from the same problem as the standard perturbation theory that starts from linearized gravity. Similar conclusion has been reached also in [8] in the case of an analogous treatment of Yang-Mills theory.

However the considerations of this section provide a clear hint on how this problem is to be cured within perturbative framework. This is the subject of the next section.
4 Non-zero Immirzi parameter and restoration of unique gauge symmetry

Let us first discuss the physical meaning of the gauge symmetries that appear in BF theory. Initially the gauge parameters physically irrelevant as such, of course. However, after introducing matter by applying the symmetry breaking perturbation some of the gauge parameters turn into physical degrees of freedom. The easiest way to look at them is to introduce localized symmetry breaking terms, which results in the appearance of finite number of degrees of freedom, similar to those we used introducing particles in section 1. Then one can ask: what is the physical meaning of the degrees of freedom resulting from such symmetry breaking.

Consider BF theory coupled to a most general type of localized sources, which includes particles considered above as well as strings considered in [9].

\[
S = \int B_{IJ} \wedge F^{IJ} + S_P(A) + S_S(B),
\]

(33)

where \(S_P(A)\) is the same as in (5), and the string action reads

\[
S_S(B) = \int d\tau d\sigma \text{Tr}(LB_{\tau\sigma}),
\]

(34)

where \(\tau\) and \(\sigma\) are variables parameterizing the string worldsheet, \(B_{\tau\sigma} \equiv B_{\mu\nu} \frac{\partial x^\mu}{\partial \tau} \frac{\partial x^\nu}{\partial \sigma}\), and \(L\) is a charge analogous to (4) but so far with different mass and spin.

Equations of motion for particles were described in section 2. To obtain equations of motion for stringy source one has to perform a gauge transformation \(B \rightarrow h^{-1}(B + d\Phi)h\) and substitute it back to the string action (34). After integration by part the string action will look like

\[
S_S = \int_W d\tau d\sigma \text{Tr}(PB_{\tau\sigma}) + \int_{\partial W} d\tau \text{Tr}(\Phi P) - \int_W d\tau d\sigma \text{Tr}(\Phi d_A P),
\]

(35)

where \(P = h L h^{-1}\) is the string momentum density. The equations following from action (35) have to be supplied with boundary conditions at the string endpoints. To allow for non-vanishing string momentum density the natural condition is

\[
\Phi \bigg|_{\partial W} = D_A \chi.
\]

(36)

Thus, from the third term in (35) we get conservation of string momentum density along the whole worldsheet,

\[
d_A P = 0,
\]

(37)
and from the second term in (35) we get the same condition, but restricted to string endpoints. The variational principle thus become consistent for arbitrary string momentum $P$. From what the second term in (35) with boundary condition (36) turns into one can deduce that the gauge parameter $\chi$ is canonically conjugate to the string momentum $P$ at its endpoints. Notice that the gauge parameter $\chi$ can be turned into a physical degree of freedom only at the endpoint of the string. Thus, it becomes somewhat analogous to Chan-Patton degrees of freedom in string theory.

So far the particle positions $h$ and momenta $J$ are completely independent from string momenta $P$ and their canonical conjugates $\chi$. This is a consequence of a split between exact gauge transformations encoded in $h$ and linearized gauge transformations encoded in $\chi$. Below we consider how the link between the two can be restored already at zero order of perturbation theory.

Consider the constraints of the theory described by action (33). These are equations (12), (13) reduced on the spacial slice $\Sigma$ with sources on the r.h.s. (so far we are considering $\beta = 0$ case).

$$F^{IJ}(A) = P^{IJ} \delta S$$  
(38)

where $\delta S = \int_{W \cap \Sigma} d\sigma \delta^3(x - x(\sigma))$, and

$$\mathcal{D}_A B^{IJ} = J^{IJ} \delta P,$$  
(39)

where $J^{IJ}$ is a particle source described in section 2. Constraint (39) generates gauge transformations (19), and constraint (38) – transformations (20). The later contains a subgroup (32), which is directly linearized diffeomorphism transformations that were detected in the previous section as a source of the problem.

The obvious approach to solving this problem is to make the theory remember that the two symmetries have common origin, i.e. that the constraints (38) and (39) generating them are not actually independent. This is what naturally happens when we introduce a $\beta$ term in (38)

$$F^{IJ}(A) + \beta B^{IJ} = P^{IJ} \delta S$$  
(40)

(39) becomes a consequence of (40) due to Bianchi identity, provided that particle sources are located at the endpoints of the strings. The string becomes something like the Dirac string for "magnetic” charge $J$. The role of the Dirac strings in this context we will discuss below in detail. Positions of the particles and positions of the strings are thus no longer independent. At
the same time the gauge transformations (20) get deformed and also start affecting connection. The constraint (40) generates the following transformations

$$B \rightarrow B + \mathcal{D}_A \Phi + \beta [\Phi, \Phi], \quad A \rightarrow A + \beta \Phi,$$  

and the analog of the subgroup (32) is now

$$A + \beta \Phi \rightarrow g^{-1}dg + g^{-1}(A + \beta \Phi)g,$$  

which is the same as the gauge transformation (19). The link between the original gauge transformation and the linearized gauge transformation is thus restored. One of the resulting effects is that particles degrees of freedom and Chan-Patton degrees of freedom at the strings endpoints have merged into one.

Let us now take a closer look at equations (12), (13), and (14) for $\beta \neq 0$. The first, rather obvious thing to notice is that, as a result of Bianchi identity, eqs. (12) and (13) cannot be solved for non-trivial source if the connection is nonsingular. This problem can be solved by recalling the Dirac magnetic monopole solution (see, for example, [10]): we assume that our connection contains the part proportional to Dirac monopole connection

$$A = g^{-1}dg + \beta g^{-1}(A_D + \Phi)g, \quad A_D \sim a_D \equiv (1 - \cos \theta)d\phi$$  

where $g \in \text{SO}(4,1)$, and $\Phi$ is a $\text{so}(4,1)$-valued one form. We will use the ansatz (43) below. In fact the Dirac string and the string described in the beginning of this section are the same.

Consider now eq. (16), which is an integrability condition for the first order equations. Computing covariant differential, and noticing that since $v_M$ has only one non-zero component $v_4 = 1$ we find

$$\mathcal{D}_A v_M = A_M^4 = e_i, \quad i = 0, \ldots 4$$  

where $e_i$ is the tetrad one form. Now it follows from eqs. (16) and (12), (13) that

$$\frac{1}{\beta} \epsilon^{IJKLi} F_{JK} \wedge e_i + \epsilon^{IJKLM} v_M J_{KL} = 0$$  

It can be shown that for one particle the second term in the expression above vanishes, and thus we are left with

$$\epsilon^{IJKLi} F_{JK} \wedge e_i = 0$$  

(44)
Decomposing $F$ into torsion $F_{4i} \equiv T_i$ and curvature $F_{ij} \equiv R_{ij}$ two-forms one finds that any connection being a solution of our problem must be torsion-free $T_i = 0$ and satisfy the vacuum Einstein–de Sitter equations\(^2\)

$$\epsilon^{ijkl} R_{jk} \wedge e_l = 0$$

(45)

Having obtained the exact Einstein’s equations already at the first order of perturbation theory, one can be sure that non-perturbative effects, including gravitational collapse cannot be missed. But this is not a solution of any problem, because doing such perturbation theory is the same as solving General Relativity non-perturbatively. Thus, in classical theory, the present perturbative approach does not lead to any simplification. However it can bring about some new intuition in quantum theory which is the subject of the next section.

Before finishing this section let us point out some another consequence of introducing Immirzi parameter. Let us solve eq.(45) for the ansatz (43) to the leading order in $\beta$, which is the same as linear approximation. In this approximation the solution for $\Phi$ will be (25). One can check that $g^{-1}\Phi g$ does not have terms proportional to $T^4 c$ so that the contribution of it to the tetrad one-form is zero. The only contribution to the tetrad (in the leading order in $\beta$) is therefore (cf. (43)

$$A_{i\hat{4}} = e_i = \left( g^{-1} \frac{d}{\ell^2} + J a_D \right)_{i\hat{4}}$$

(46)

One can then easily find what the metric $g = \eta^{ij} e_i \otimes e_j$ is

$$g = -\left( 1 - \frac{r^2}{\ell^2} \right) (dt + N(1 - \cos \theta)d\phi)^2 + \left( 1 - \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(47)

This is de Sitter–Taub–NUT metric linearized with the Taub–NUT charge $N$ and with no mass. If one goes beyond linear approximation one obtains the exact Taub–NUT solution. This means that the Immirzi parameter mixes mass with Taub–NUT charge. The presence of a non-local (Dirac string) singularity has a profound consequences in quantum theory. This will be discussed in the next section.

5 Quantum amplitudes versus classical solutions

The two main sources of difficulties with four dimensional General Relativity are infinite number of degrees of freedom and non-linearity of its equations.

\(^2\)That is, Einstein equations with a positive cosmological constant.
One of the consequences of the latter is a non-trivial curvature of the phase space of the theory. One of the lessons from 2+1 gravity coupled to matter is that momentum part of its phase space is a group manifold rather than linear space and the information about its curvature is essential for consistent quantization of the theory.

In standard perturbation theory one starts by choosing a background - a certain point in the phase space of the theory. One then proceeds by linearizing the theory around this background, which in particular implies that one works in a tangent space of the phase space of the theory instead of the actual phase space. This space remains linear even after removing gauge degrees of freedom by taking its quotient by linearized diffeomorphism transformations.

Another property of quantum perturbation theory is that when we are interested in scattering of a finite number of particles in a finite order of perturbation theory we have to calculate Feynman diagrams. The expressions for Feynman diagrams are finite dimensional integrals instead of infinite dimensional path integrals. Thus quantum theory allows us to select a set of questions which could be answered by taking into account only finite number of modes of quantum field. This option is not available in classical perturbation theory where we have to solve infinite-dimensional field equations in every order. This is a big advantage of quantum perturbation theory over the classical one.

Now one can notice that the ability of quantum perturbation theory to reduce the problem of calculating scattering amplitudes to finite dimensional one, and its inability to see the curvature of the phase space of the theory are not logically linked which each other. Therefore one can ask if there is a quantum perturbation theory which allows to reduce the problem to a certain finite-dimensional subspace of the phase space and if that subspace is curved the perturbation theory still can see it.

Below we show that the quantum analog of the perturbation theory studied here does precisely this job, and it allows us to see some non-trivial effect already in zeroth order.

We start by explaining how the perturbative expressions for quantum amplitudes get reduced to finite-dimensional integrals.

The general perturbative expression for the partition function coupled to arbitrary finite number of particles (where we neglect all the interactions except gravitational) after integrating out $B$-field looks like

$$ Z(\{x_p\}, \{x_{pf}\}) = \int D\alpha \sum_n \frac{(i\alpha)^n}{n!} \left( \int v_{ABCDE} F_{BC}(x) \wedge F_{DE}(x) \right)^n $$
\times \exp\left[i \int_M F^{IJ} \wedge F_{IJ} + \sum S_p(x_{p_i}, x_{p_f})\right], \quad (48)

Where \( S_p \) are actions for particles of the type (5), and \( x_{p_i} \) and \( x_{p_f} \) are initial and final points of \( p \)-th particle. They are presumably located on initial and final slice of \( M \). The expression (48) can be rewritten in terms of n-point functions as

\[
Z(\{x_{p_i}\}, \{x_{p_f}\}) = \sum_n \frac{(i\alpha)^n}{n!} \int dx_1...dx_n W(x_1, ..., x_n, \{x_{p_i}\}, \{x_{p_f}\}), \quad (49)
\]

where

\[
W(x_1, ..., x_n, \{x_{p_i}\}, \{x_{p_f}\}) = \int \mathcal{D}A \prod_n (v_{AE}^{ABCD} F_{BC}(x_n) \wedge F_{DE}(x_n))
\]

\[
\exp\left[i \frac{1}{\beta} \int_M F^{IJ} \wedge F_{IJ} + \sum S_p(x_{p_i}, x_{p_f})\right]. \quad (50)
\]

Now one has to recall one of the key properties of n-point functions of diffeomorphism invariant theories, i.e. when the action and the measure are diffeomorphism invariant, (see e.g. [11] and references therein) that they do not actually depend on \( x_n \), as coordinates do not carry any physical information. The only dependence may come from possible coincidences of \( x_n \), but it yields a measure zero contribution. Of course n-point functions depend on relative positions of the points \( x_n \), but the latter are encoded not in the coordinates but in the fields.

This means that the integrands in (49) are constants and therefore the integrals can simply be dropped, i.e. absorbed into renormalization of \( \alpha \).

As a consequence, the \( n \)-order contribution to the partition function can be evaluated as a path integral of topological field theory with \( n \) localized symmetry breaking insertions. As it was discussed e.g. in [5]: breaking a symmetry of topological field theory in a localized way releases only a finite number of degrees of freedom. This is the central observation which allows us to reduce the procedure of perturbative calculation of quantum scattering amplitudes to a finite dimensional problem. Notice that this would not be possible in non-perturbative calculation as in such case the integral with a symmetry breaking term would stay in the exponent and as a result could not be replaced with a local symmetry breaking insertion.

Now let us try to compose a list of variables which become relevant for the calculation of the n-point function (50). To simplify the job let us first integrate out the particles degrees of freedom, \( g_p \), which is the gauge
parameter evaluated at the location of the particle. It can be done by replacing particle terms in the action with particle propagator.

\[ G_p(g(x_{p_i}), g(x_{p_f}), A_p) = \int \mathcal{D} g \exp \left( \int dt \text{Tr}(A_t J) + S_p(g) \right) \]  

(51)

We will not dwell on calculating this expression here. However, as it was stressed in [5], the propagator (51) is a matrix element of the open Wilson line operator whose representation is defined by the mass and the spin of the particle. The states, between which the matrix element has to be taken are \( SO(4,1) \) analogs of Wigner functions, with corresponding representation indices. Alternatively, one can insert Wilson lines directly. This would correspond to working in momentum representation, as the matrix indices of the Wilson line are momenta of the particles. Eq. (50) now reads:

\[ W(x_1, \ldots, x_n, \{x_{p_i}\}, \{x_{p_f}\}) = \int \mathcal{D} A \prod_p (G_p(g(x_{p_i}), g(x_{p_f}))) \prod_n (v_A e^{ABCDEF} F_{BC}(x_n) F_{DE}(x_n)) \exp[i \int_M F^{IJ} \wedge F_{IJ}], \]  

(52)

The dependence of this expression on \( \{x_n\}, \{x_{p_i}\}, \) and \( \{x_{p_f}\} \) should be understood as the dependence on the connection field evaluated at these points.

Now if instead of connection field \( A \) we use its holonomies connecting different point in space, it is clear that due to \( SO(4,1) \) gauge symmetry of the action in the exponent the integrand in (52) cannot depend on holonomies ending anywhere else but at the points \( x_{p_i}, x_{p_f}, \) and \( x_n \). For convenience however let us choose one common reference point \( x_0 \) putting it for example somewhere on the boundary of initial slice, \( x_0 \in \partial M \cap \Sigma_i \). For basis holonomies we choose \( g_n \) connecting \( x_n \) to \( x_0 \), \( g_{p_i} \) connecting \( x_{p_i} \) to \( x_0 \), and \( g_{p_f} \) connecting \( x_{p_f} \) to \( x_0 \). Any holonomy connecting any pair of points can be expressed as a combination of those from the above list. If we took \( \beta = 0 \) limit, the free action would force the connection to be flat, and the holonomies would become independent of paths along which they are taken. That means that these holonomies would comprise the full list of variables relevant for calculating the amplitude (52).

The situation is very different when we consider \( \beta \neq 0 \). First of all as it was discussed in [5] the \( BF \) theory with nonzero \( \beta \) cannot be consistently coupled to a local symmetry breaking source without introducing introducing an extended connection singularity – the Dirac string.
Generally, if in classical theory the equations of motion are inconsistent because of a charge non-conservation, in quantum theory in analogous situation one obtains a vanishing partition function. The situation is similar here. For example, if a particle located away from the boundary, and the connection is non-singular, in the partition function

\[ Z = \int \mathcal{D}A \exp \left( i \int F^{IJ} \wedge F_{IJ} + i \text{Tr}(AK) \delta^3(x - z(t)) \right) \]  \hspace{1cm} (53)

The first term in the exponent will be independent of \( A(z(t)) \), and the contribution from integrating over it will be of the form \( \delta(K) \). This means that if \( K \neq 0 \) the partition function is zero, and the only way to get non-zero contribution to the partition function is to either introduce the Dirac string, or put the particle on a spatial boundary. Below we will argue that the two are equivalent.

For similar reason the local symmetry breaking insertion, say at the point \( x \), also has to be connected by Dirac string to the boundary. The easiest way to see this is to extract an integral over \( dg(x) \) from the measure \( \mathcal{D}A \), where \( g(x) \) is some holonomy connecting \( x \) to the reference point along some path. If \( \mathcal{D}A \) is the Ashtekar-Lewandowski measure \([12]\), it has to contain \( dg(x) \). For simplicity consider 1-point function. The argument however will hold for n-point functions for non-coincident \( x_n \) as well. We will extract explicitly \( g(x) \) variable, after which the 1-point function can be written as

\[ W(x) = \int \mathcal{D}Adg(x) \text{Tr}(g(x)^{-1}vg(x)F(x) \wedge F(x)) \exp \left( i \int F^{IJ} \wedge F_{IJ} \right), \]  \hspace{1cm} (54)

where \( v \) is the matrix representation of the vector \( v^A \), in an appropriate representation. Now one can notice that \( g(x) \) is a functional of connection in the bulk, while the exponent term in the absence of Dirac string does not depend on the bulk connection. This means that the only dependence of the integrand in (54) on \( g \) is in pre-exponential term. The integral over \( dg(x) \) can be therefore explicitly calculated. It equals zero because it is an averaging of a vector over all possible directions. The only way to get a nonzero contribution is to introduce the Dirac string as in the case of particle.

The presence of the Dirac strings makes the holonomies path-dependent: the holonomy passing to the left from the string will be different from the one passing to the right. To keep track of this information we should to the list of all holonomies relevant for the \( \beta = 0 \) case described above add those around each string. A holonomy around the string ending at \( x_n, x_{p_i}, \) or \( x_{p_f} \) with the initial point connected to the tip of the string will be called \( g^*_n, g^*_p, \)
and \( g_{p_i} \), respectively. The natural choice of paths for holonomies \( g_n, g_{p_i} \), and \( g_{p_f} \) will be then the following. \( g_n \) for example will start at \( x_n \) which is the tip of one Dirac string, then follow this string towards the boundary, and then go along the boundary to the reference point \( x_0 \).

One can easily see that the above picture is in fact three dimensional instead of four. A Dirac string (more precisely a thin tube cut around it) can be considered as an extension of the boundary of space. The tip of each Dirac string can be considered as a puncture on the boundary. The holonomies along the strings can be then considered as holonomies taken along the boundary and connecting punctures. And the holonomies around strings just become holonomies around punctures. This dimensional reduction is, of course, just a reflection of the fact that the action in the exponent in (52) is the action of three dimensional theory, namely the Chern-Simons theory.

In the present paper we will be interested only in the limit \( \alpha \to 0 \). Quantum theory allows us to consistently take that limit just by dropping all the terms proportional to \( \alpha \) in the expansion (49), no care about integrability conditions as those encountered in classical theory is needed.

The relevant \( n \)-point (actually 0-point) function will simply read:

\[
W(\{x_{p_i}\}, \{x_{p_f}\}) = \int DA \prod_p (W_p(x_{p_i}, x_{p_f})) \exp\left[i \int_{\partial M} Y_{CS}(A)\right],
\]

Following the discussion above in this expression we have taken into account that all the points \( x_{p_i} \) and \( x_{p_f} \) and all the Wilson lines connecting them can be mapped on the three dimensional boundary of space.

Instead of doing path integral we will try to deduce physics by making canonical analysis of the theory encoded in (55). For this it is enough to consider the variable residing on the initial slice of \( \partial M \), ie those connecting \( x_{p_i} \). Let \( x_{p_0} \) be a reference point, and \( x_{p_1} \) is the particle of interest. The position of the particle with respect to the reference point will be described by an \( SO(4,1) \) group element – the holonomy \( g_{01} \) connecting \( x_{p_0} \) and \( x_{p_1} \).

Notice that although the expression for the amplitudes is described in terms of three dimensional theory the particle still has 3 translational degrees of freedom as does four dimensional particle.

Now one can ask what is the momentum of the particle, i.e. what is the variable canonically conjugate to translational part of \( g_{01} \). The Poisson structure of Chern-Simons theory

\[
\{A_i^{AB}, A_j^{CD}\} = \beta \epsilon_{ij} \delta^{[AC} \delta^{BD]},
\]

19
tells us that it has to be the translational part of the holonomy \( g_1^* \) which is taken around the particle \( x_{p_1} \) which starts and ends at \( x_{p_0} \). \( g_1^* \) is also an element of \( SO(4, 1) \) group, so the momentum space of the particle is de Sitter space. This strongly indicates that a theory describing relativistic symmetries of such a system of particles, in the flat space, \( \Lambda \to 0 \) limit will be of DSR type, like in the 2+1 dimensional case.

This expectation is also motivated by strong results on Hamiltonian quantization of Chern-Simon theories with arbitrary gauge group on Riemann surfaces [13]. On the basis of these results one expects that after quantization the symmetry group \( SO(4, 1) \) will be replaced by its quantum counter part \( U_q(SO(4, 1)) \) (Note however that Fock–Rosly theorem applies, strictly speaking, to compact gauge group and its extension to the non-compact case is highly nontrivial, see [14] for the analysis of \( SL(2, C) \) case.) Then the limit of cosmological constant going to zero will presumably lead to emergence of \( \kappa \)-Poincaré algebra [15], which is central for DSR program.

To finish this section let us say a few words about euclidian version of this model. For \( SO(5) \) gauge group the momentum space is a sphere, both energy and momenta are bounded, and the theory has a natural UV cutoff. Spacetime collapses at the energy higher than \( \sqrt{\Lambda/\beta} \). It is not yet clear what is the precise mechanism of this collapse. There are two possibilities in principle. Either all spacetime disappears at one instant, as it is the case for 2 + 1 gravity with zero cosmological constant. Or it starts collapsing in some region around a particle, and the region will be growing as the mass of the particle is growing. In the later case there may be some effects not suppressed by the smallness of the cosmological constant. This will be studied in the subsequent papers.

6 Discussion

To this point we do not have explicit expressions for higher order corrections to scattering amplitudes. In this section we will briefly mention the problems one has to face in calculating them.

Let us try to reexpress (52) in terms of finite dimensional integrals over holonomies instead of path integral over connections. First one can replace the action in the exponent in (52) by Chern-Simons action. Then we can take a locally flat connection connection \( A(\{g\}) \) whose moduli space is defined by holonomies \( g_n, g_{p_1}, g_{p_f} \) and \( g^*_n, g^*_{p_1}, g^*_{p_f} \) (we will denote the whole set by \( \{g\} \)), and plug it into the Chern-Simons action. We will have a functional depending on only finite number of parameters.
\[
\int Y_{CS}(A(\{g\})) \equiv \bar{S}_{CS}(\{g\})
\] (57)

A more complicated treatment is needed for preexponent terms. The local curvature has somehow to be read off from the set of holonomies. Clearly the two relevant holonomies at a point \(x_n\) are \(g_n\) and \(g_n^*\). They are dual to each other and therefore should give rise to two dual elements of curvature entering a local \(F_{BC}(x_n) \wedge F_{DE}(x_n)\) insertion in the action. The difficulty here is that holonomies are group elements and curvature is an algebra element. Therefore we need a certain map \(\mathcal{F}^{IJ}(g)\) from Lie group to Lie algebra. Having such a map we can substitute in our path integral

\[
F_{BC}(x_n) \wedge F_{DE}(x_n) \rightarrow \mathcal{F}^{BC}(g_n)\mathcal{F}^{DE}(g_n^*)
\] (58)
to obtain

\[
W(x_1, \ldots, x_n, \{x_{p_i}\}, \{x_{p_f}\}) = \int \prod \{g\} \prod g_p \left(G_p(g_{p_i}^{-1}g_{p_f})\right) \prod_{p} (v_A \epsilon^{ABCDE} F_{BC}(g_n) \mathcal{F}^{DE}(g_n^*)) \exp[i \bar{S}_{CS}(\{g\})],
\] (59)

This is a “Feynman diagram” expression for an \(n\)-point function in this perturbation theory. The only missing ingredient is the exact form of the map \(\mathcal{F}\). The only natural requirement on this map is gauge invariance

\[
\mathcal{F}^{IJ}(h^{-1} gh) = h_K^{I} F^{KL}(g) h_L^{J}
\] (60)

This however does not specify the map uniquely and it has to be derived from other requirements. Our proposal would be to derive the exact form of \(\mathcal{F}\) by comparing an analog of the expression (59) but without symmetry breaking with derivatives of Crane-Yetter invariants with respect to its coupling constant.

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