GENUS-CORRESPONDENCES RESPECTING SPINOR GENUS

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Abstract. For two positive definite integral ternary quadratic forms $f$ and $g$ and a positive integer $n$, if $n \cdot g$ is represented by $f$ and $n \cdot dg = df$, then the pair $(f, g)$ is called a representable pair by scaling $n$. The set of all representable pairs in $\text{gen}(f) \times \text{gen}(g)$ is called a genus-correspondence. In [7], Jagy conjectured that if $n$ is square free and the number of spinor genera in the genus of $f$ equals to the number of spinor genera in the genus of $g$, then such a genus-correspondence respects spinor genus in the sense that for any representable pairs $(f, g), (f', g')$ by scaling $n$, $f' \in \text{spn}(f)$ if and only if $g' \in \text{spn}(g)$. In this article, we show that by giving a counter example, Jagy’s conjecture does not hold. Furthermore, we provide a necessary and sufficient condition for a genus-correspondence to respect spinor genus.

1. Introduction

For a positive definite integral ternary quadratic form

$$f(x, y, z) = ax^2 + by^2 + cz^2 + pyz + qzx + rxy \quad (a, b, c, p, q, r \in \mathbb{Z}),$$

it is quite an old problem determining the set $Q(f)$ of all positive integers $k$ such that $f(x, y, z) = k$ has an integer solution. If the class number of $f$ is one, then one may easily compute the set $Q(f)$ by using, so called, the local-global principle. However, if the class number of $f$ is bigger than 1, then determining the set $Q(f)$ exactly seems to be quite difficult, except some very special ternary quadratic forms. If the integer $k$ is sufficiently large, then the theorem of Duke and Schulze-Pillot in [4] implies that if $k$ is primitively represented by the spinor genus of $f$, then $k$ is represented by $f$ itself.

Recently, W. Jagy proved in [7] that for any square free integer $k$ that is represented by a sum of two integral squares, it is represented by any ternary quadratic form in the spinor genus $x^2 + y^2 + 16kz^2$. To prove this, he introduced, so called a genus-correspondence, and proved some interesting properties on the genus-correspondence. To be more precise, let $\text{gen}(f)$ ($\text{spn}(f)$) be the set of genus (spinor genus, respectively) of $f$, for any ternary quadratic form $f$. Let $f$ and $g$ be positive definite integral ternary quadratic forms, and assume that there is a positive integer $n$ such that

$$n \cdot g \text{ is represented by } f \quad \text{and} \quad n \cdot dg = df.$$

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In this article, we denote such a pair \((f, g)\) by a representable pair by scaling \(n\). Note that \(n \cdot f\) is also represented by \(g\) for any representable pair \((f, g)\) by scaling \(n\). As stated in \([7]\), W. K. Chan proved that for any ternary quadratic form \(f' \in \gen(f)\), there is a ternary quadratic form \(g' \in \gen(g)\) such that \((f', g')\) is a representable pair by scaling \(n\), and conversely for any \(\tilde{g} \in \gen(g)\), there is an \(\tilde{f} \in \gen(f)\) such that \((\tilde{f}, \tilde{g})\) is also a representable pair by scaling \(n\). Jagy defined the set of representable pairs by scaling \(n\) by a genus-correspondence and proved some properties on a genus-correspondence. He also conjectured that if \(n\) is square free and the number of spinor genera in the genus of \(f\) equals to the number of spinor genera in the genus of \(g\), then such a genus-correspondence respects spinor genus in the sense that for any representable pairs \((f, g), (f', g')\) by scaling \(n\), where \(f' \in \gen(f)\) and \(g' \in \gen(g)\),

\[
f' \in \spn(f) \quad \text{if and only if} \quad g' \in \spn(g).
\]

In this article, we give an example such that Jagy’s conjecture does not hold. In fact, the concept of “genus-correspondence” in \([7]\) is a little bit ambiguous. We modify the notion of a genus-correspondence as follows: For a positive integer \(n\), let \(\mathcal{C}\) be a subset of \(\gen(f) \times \gen(g)\) such that each pair in \(\mathcal{C}\) is a representable pair by scaling \(n\). We say \(\mathcal{C}\) is a genus-correspondence if for any \(f' \in \gen(f)\), there is an \(g' \in \gen(g)\) such that \((f', g') \in \mathcal{C}\), and conversely, for any \(\tilde{g} \in \gen(g)\), there is an \(\tilde{f} \in \gen(f)\) such that \((\tilde{f}, \tilde{g}) \in \mathcal{C}\). Note that the set of all representable pairs by scaling \(n\) is a genus-correspondence. We show that without assumption that \(n\) is square free, there is a genus-correspondence respecting spinor genus if the number of spinor genera in \(\gen(f)\) is equal to the number of spinor genera in \(\gen(g)\).

In Section 5, we discuss when Jagy’s original conjecture is true. We provide a necessary and sufficient condition for the genus-correspondence consisting of all representable pairs by scaling \(n\) in \(\gen(f) \times \gen(g)\) to respect spinor genus under the assumption that \(n\) is square free and the number of spinor genera in \(\gen(f)\) is equal to the number of spinor genera in \(\gen(g)\).

The subsequent discussion will be conducted in the better adapted geometric language of quadratic spaces and lattices. The term lattice will always refer to a non-classic integral \(\mathbb{Z}\)-lattice on an \(n\)-dimensional positive definite quadratic space \(Q\). Here a \(\mathbb{Z}\)-lattice \(L\) is said to be non-classic integral if the norm ideal \(\mathfrak{n}(L)\) of \(L\) is contained in \(\mathbb{Z}\). The discriminant of a lattice \(L\) is denoted by \(dL\) and the number of (proper) spinor genera in \(\gen(L)\) is denoted by \(g(L)\). For any rational number \(a\), \(L^a\) is the lattice whose bilinear map \(B\) is scaled by \(a\).

Let \(L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_k\) be a \(\mathbb{Z}\)-lattice of rank \(k\). We write

\[
L = (B(x_i, x_j)).
\]

The right hand side matrix is called a matrix presentation of \(L\). If \(L\) admits an orthogonal basis \(\{x_1, x_2, \ldots, x_k\}\), then we simply write

\[
L = \langle Q(x_1), Q(x_2), \ldots, Q(x_k) \rangle.
\]

Throughout this paper, we say a \(\mathbb{Z}\)-lattice \(L\) is primitive if the norm ideal \(\mathfrak{n}(L)\) is exactly \(\mathbb{Z}\). For a prime \(p\), the group of units in \(\mathbb{Z}_p\) is denoted by \(\mathbb{Z}_p^\times\). Unless
confusion arises, we will simply use $\Delta_p$ to denote a non-square element in $\mathbb{Z}_p^\times$, when $p$ is odd.

We denote by $\langle a, b, c, s, t, u \rangle$ for the ternary $\mathbb{Z}$-lattice with a matrix presentation

$$\begin{pmatrix} a & u & t \\ u & b & s \\ t & s & c \end{pmatrix},$$

for convenience. For any $\mathbb{Z}$-lattice $L$, the equivalence class containing $L$ up to isometry is denoted by $[L]$. For any integer $a$, we say that $\frac{a}{q}$ is divisible by a prime $p$ if $p$ is odd and $a \equiv 0 \pmod{p}$, or $p = 2$ and $a \equiv 0 \pmod{4}$.

Any unexplained notations and terminologies can be found in [9] or [10].

2. Watson’s transformations on the set of spinor genera

Let $L$ be a non-classic integral $\mathbb{Z}$-lattice on a quadratic space $V$. For a prime $p$, we define

$$\Lambda_p(L) = \{ x \in L : Q(x + z) \equiv Q(z) \pmod{p} \text{ for all } z \in L \}.$$

Let $\lambda_p(L)$ be the primitive lattice obtained from $\Lambda_p(L)$ by scaling $V = L \otimes \mathbb{Q}$ by a suitable rational number. Recall that a $\mathbb{Z}$-lattice $L$ is called primitive if $n(L) = \mathbb{Z}$. For general properties of $\Lambda_p$-transformation, see [2] and [3].

For $L^1 \in \text{gen}(L)$ ($L^1 \in \text{spn}(L)$) and any prime $p$, one may easily show that $\lambda_p(L^1) \in \text{gen}(\lambda_p(L))$ ($\lambda_p(L^1) \in \text{spn}(\lambda_p(L))$, respectively). It is well known that as a map,

$$\lambda_p : \text{gen}(L) \longrightarrow \text{gen}(\lambda_p(L))$$

is surjective. Furthermore, $\lambda_p(\text{spn}(L)) = \text{spn}(\lambda_p(L))$. If we define $\text{gen}(L)_S$ the set of all spinor genera in $\text{gen}(L)$, then the map

$$\lambda_p : \text{gen}(L)_S \longrightarrow \text{gen}(\lambda_p(L))_S$$

given by $\text{spn}(L^1) \mapsto \text{spn}(\lambda_p(L^1))$ for any $\text{spn}(L^1) \in \text{gen}(L)_S$ is well-defined and surjective. In particular, $g(L) \geq g(\lambda_p(L))$ for any prime $p$.

Henceforth, $L$ is always a positive definite non-classic integral ternary $\mathbb{Z}$-lattice.

**Definition 2.1.** For a $\mathbb{Z}$-lattice $L$ and a prime $p$, if $g(L) = g(\lambda_p(L))$, then we say the lattice $L$ is of $H$-type at $p$.

From the definition, if $L$ is of $H$-type at $p$, then so is $L'$ for any $L' \in \text{gen}(L)$.

**Lemma 2.2.** Let $L$ be a primitive ternary $\mathbb{Z}$-lattice and let $p$ be an odd prime. Assume that after scaling by a unit in $\mathbb{Z}_p$ suitably,

$$L_p \simeq \langle 1, p^\alpha \epsilon_1, p^\beta \epsilon_2 \rangle,$$

where $\alpha, \beta (\alpha \leq \beta)$ are nonnegative integers and $\epsilon_1, \epsilon_2 \in \{1, \Delta_p\}$. If $L$ is not of $H$-type at $p$, then the pairs $(\alpha, \beta), (\epsilon_1, \epsilon_2)$ satisfy one of the conditions in Table 1.
Table 1 (odd case)

| $(\alpha, \beta)$ | $(\epsilon_1, \epsilon_2)$ |
|-------------------|---------------------------|
| (1, 2)            | (1, 1)                    |
| (1, 2)            | ($\Delta$, 1)            |
| (2, $k$), ($k \geq 3$) | (1, 1)                    |
| (2, 2$k$ + 1), ($k \geq 1$) | (1, $\Delta$) |

Proof. By 1027 of [10], we know that

\[ g(L) = [J_Q : P_D J_Q^2] \quad \text{and} \quad g(\lambda_p(L)) = [J_Q : P_D J_Q^\lambda_p(L)], \]

where $D$ is the set of positive rational numbers. Clearly, $\theta(O^+(\lambda_p(L)_q)) = \theta(O^+(L_q))$ for any prime $q \neq p$. Now one may easily check that if the pairs $(\alpha, \beta)$, $(\epsilon_1, \epsilon_2)$ do not satisfy one of the conditions in Table 1, then $\theta(O^+(\lambda_p(L)_q)) = \theta(O^+(L_p))$. This implies that $g(L) = g(\lambda_p(L))$. \qed

Lemma 2.3. Let $L$ be a primitive ternary $\mathbb{Z}$-lattice. If $L$ is not of $H$-type at 2, then there is an $\eta \in \mathbb{Z}_2^\times$ such that

\[ (L^\eta)_2 \cong \langle 1, 2^\alpha \epsilon_1, 2^\beta \epsilon_2 \rangle, \]

where $\alpha, \beta (\alpha \leq \beta)$ are nonnegative integers and $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2^\times$, and the pairs $(\alpha, \beta)$, $(\epsilon_1, \epsilon_2)$ satisfy one of the conditions in Table 2.

Proof. The proof is quite similar to the above lemma. For the computation of the spinor norm map, see [5]. \qed

Table 2 (Even case)

| $(\alpha, \beta)$ | $(\epsilon_1, \epsilon_2)$ | $(\alpha, \beta)$ | $(\epsilon_1, \epsilon_2)$ |
|-------------------|---------------------------|-------------------|---------------------------|
| (0, 4)            | $\epsilon_1 \equiv 2 \epsilon_2 \equiv 1 \ (4)$ | (5, 6)            | $2 \epsilon_1 + \epsilon_2 \in Q(\langle 1, 2 \epsilon_1 \rangle)$ |
| (1, 6)            | $\epsilon_2 \in Q(\langle 1, 2 \epsilon_1 \rangle)$ | (5, 7)            | $\epsilon_1 \epsilon_2 \equiv 1 \ (4)$ |
| (2, 2)            | $\epsilon_1 \equiv 1$, $\epsilon_2 \equiv 3 \ (4)$ | (5, 8)            | $\epsilon_2 \equiv 2 \epsilon_1 + 5 \ (8)$ |
| (2, 4)            | $\epsilon_1 \equiv 1 \ (4)$ | (5, 9)            | $\epsilon_1 \epsilon_2 \equiv 1 \ (4)$ |
| (2, 6)            | $\epsilon_1 \equiv 1 \ (4)$ | (5, 2$k$), ($k \geq 5$) | $1 + 2 \epsilon_1 \not\equiv \epsilon_2 \ (8)$ |
| (2, 2$k$ + 1), ($k \geq 4$) | $\epsilon_1 \equiv 2 \epsilon_2 + 3 \ (8)$ | (5, 2$k$ + 1), ($k \geq 5$) | $1 + 2 \epsilon_1 \not\equiv \epsilon_1 \epsilon_2 \ (8)$ |
| (2, 2$k$), ($k \geq 4$) | $\epsilon_1 \equiv 1 \ (4)$ | (6, 7)            | $5 \not\in Q(\langle \epsilon_1, 2 \epsilon_2 \rangle)$ |
| (3, 6)            | $\epsilon_2 \equiv 1 \ (8)$ | (6, 9)            | $5 \not\in Q(\langle \epsilon_1, 2 \epsilon_2 \rangle)$ |
| (4, 4)            | $\epsilon_1 \equiv 2 \epsilon_2 \equiv 1 \ (4)$ | (6, 2$k$ – 1), ($k \geq 6$) | $\epsilon_1 \not\equiv 5 \ (8)$ and $\epsilon_1 \not\equiv \epsilon_2 \Rightarrow \epsilon_1$ or $\epsilon_2 \equiv 1 \ (8)$ |
| (5, 5)            | $\epsilon_2 \equiv 3 \epsilon_1 + 6 \ (8)$ | (6, 2$k$), ($k \geq 6$) | $\epsilon_1, \epsilon_2 \not\equiv 5 \ (8)$ and $\epsilon_1 \not\equiv \epsilon_2 \Rightarrow \epsilon_1$ or $\epsilon_2 \equiv 1 \ (8)$ |

Lemma 2.4. Let $p$ be a prime and let $L$ be a primitive ternary $\mathbb{Z}$-lattice. As a function from $\text{gen}(L)_S$ to $\text{gen}(\lambda_p(L))_S$, $\lambda_p$ is a $2^a$ to one function for some $a = 0, 1$ or 2.
Proof. Note that if $L$ is not of $H$-type at $p$, then

$$|\theta(O^+(\lambda_p(L_p))))| = \begin{cases} 4 \cdot |\theta(O^+(L_p)))| & \text{if } p = 2, \ (\alpha, \beta) = (2, 4) \text{ and } \epsilon_1 \equiv \epsilon_2 \equiv 1 \ (4), \\ 2 \cdot |\theta(O^+(L_p))| & \text{otherwise.} \end{cases}$$

Suppose that $\lambda_p(\text{spn}(L)) = \text{spn}(M)$ with $\lambda_p(L) = M$. For any $\text{spn}(M') \in \text{gen}(M)s$, there is a split rotation $\Sigma \in J_p$ such that $M' = \Sigma M$. Since

$$\lambda_p(\Sigma L) = \lambda_p(L) = \Sigma M = M',$$

we have $\lambda_p(\text{spn}(\Sigma L)) = \text{spn}(M')$. Note that $L' \in \text{spn}(L'')$ if and only if $\Sigma L' \in \text{spn}(\Sigma L'')$, for any $L', L'' \in \text{gen}(L)$. Therefore $|\lambda_p^{-1}(\text{spn}(M))|$ is independent of the choices of $M \in \text{gen}(\lambda_p(L))$. The lemma follows from this and the fact that $\lambda_p$ is surjective and the number of spinor genera in any genus of a ternary quadratic form is a power of 2. \qed

3. $\Gamma_p$-transformations on the set of spinor genera

Let $V$ be a (positive definite) ternary quadratic space and let $L$ be a primitive ternary $\mathbb{Z}$-lattice on $V$. Let $p$ be a prime such that

$$L_p \simeq \left( \begin{array}{ccc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right) \perp \langle \epsilon \rangle, \text{ where } \epsilon \in \mathbb{Z}_p^\times.$$ 

For any nonnegative integer $m$, let $G_{L,p}(m)$ be a genus on $W$ such that each $\mathbb{Z}$-lattice $M \in G_{L,p}(m)$ satisfies

$$M_p \simeq \left( \begin{array}{ccc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right) \perp \langle p^m \rangle \text{ and } M_q \simeq (L^p)^m \text{ for any } q \neq p.$$ 

Here $W = V$ if $m$ is even, and $W = Vp$ otherwise.

For a nonnegative integer $m$, let $N \in G_{L,p}(m+1)$ be a primitive ternary $\mathbb{Z}$-lattice. By Weak Approximation Theorem, there exists a basis $\{x_1, x_2, x_3\}$ for $N$ such that

$$(B(x_i, x_j)) \equiv \left( \begin{array}{ccc} 0 & 1/2 \\ 1/2 & 0 \end{array} \right) \perp \langle p^{m+1} \delta \rangle (\text{mod } p^{m+2}),$$

where $\delta$ is an integer not divisible by $p$. We define two sublattices of $N$

$$\Gamma_{p,1}(N) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \Gamma_{p,2}(N) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3.$$ 

Note that $\Gamma_{p,i}(N)$ for $i = 1, 2$ depends on the choices of basis for $N$. However, the set $\{\Gamma_{p,1}(N), \Gamma_{p,2}(N)\}$ of sublattices of $N$ is independent of the choices of basis for $N$. In fact, these two sublattices are unique sublattices of $N$ with index $p$ whose norm is $p\mathbb{Z}$. We say that a $\mathbb{Z}$-lattice $M$ is a $\Gamma_p$-descendant of $N$ if $M \simeq \Gamma_{p,i}(N)$ for some $i = 1, 2$.

Lemma 3.1. Let $p, q$ be distinct primes and let $N \in G_{L,p}(m+1)$ for some nonnegative integer $m$.

1. If $M$ is a $\Gamma_p$-descendant of $N$, then $\lambda_q(M)$ is a $\Gamma_p$-descendant of $\lambda_q(N)$.

2. Assume that $N \in G_{L',q}(m'+1)$ for some nonnegative integer $m'$. Then any $\Gamma_q$-descendant of a $\Gamma_p$-descendant of $N$ is a $\Gamma_p$-descendant of some $\Gamma_q$-descendant of $N$. 

Proof. Note that if \( p, q \) are distinct primes, then \((\Gamma_{p, i}(N))_q = N_q\) and \((\Lambda_p(N))_q = N_q\). The lemma follows directly from this.

In [8], we defined a multi-graph \( G_{L,p}(m) \) and proved some properties of this graph. For those who are unfamiliar with the notations, we introduce the definition of this multi-graph briefly: the set of vertices in \( G_{L,p}(m) \) is the set of equivalence classes in \( G_{L,p}(m) \), say, \([M_1], [M_2], \ldots, [M_h]\). The set of edges is exactly the set of equivalence classes in \( G_{L,p}(m+1) \), say, \([N_1], [N_2], \ldots, [N_k]\). For each equivalence class \([N_i]\) in \( G_{L,p}(m+1) \), two vertices contained in the edge \([N_i]\) are defined by \([\Gamma_{p, 1}(N_i)]^\pm\) and \([\Gamma_{p, 2}(N_i)]^\pm\) that are defined above. Note that both lattices are contained in \( G_{L,p}(m') \). Hence this graph might have loops or multiple edges.

Two vertices \([T_i], [T_j]\) in \( G_{L,p}(0) \) are connected by an edge if and only if there are \( \mathbb{Z} \)-lattices \( T'_i \in [T_i] \) and \( T'_j \in [T_j] \) such that \( T'_i \) and \( T'_j \) are connected by an edge in the graph \( Z(T, p) \) which is defined in [11]. If two lattices \( T_i, T_j \in G_{L,p}(0) \) are spinor equivalent, then both \([T_i]\) and \([T_j]\) are contained in the same connected component. Moreover, the set of vertices in each connected component of \( G_{L,p}(0) \) consists of at most two spinor genera, and it consists of only one spinor genus if and only if \( j(p) \in P_{DJ}^\pm \), where \( D \) is the set of positive rational numbers and

\[
j(p) = (j_q) \in J_Q \quad \text{such that} \quad j_p = p \text{ and } j_q = 1 \text{ for any prime } q \neq p.
\]

We say that \( G_{L,p}(0) \) is of \( O \)-type if the set of vertices in the connected component of the graph \( G_{L,p}(0) \) consists of only one spinor genus, and it is of \( E \)-type otherwise.

Now, we consider the general case. For any positive integer \( m \), we say that a graph \( G_{L,p}(m) \) is of \( E \)-type if \( m \) is even and \( G_{L,p}(0) \) is of \( E \)-type, and it is of \( O \)-type otherwise.

Assume that \( G_{L,p}(m) \) is of \( E \)-type and \( M \in G_{L,p}(m) \). Since the map

\[
\lambda_p^\pm : \text{spn}(T) \to \text{spn}(\lambda_p^\pm(T))
\]

is surjective for any \( T \in G_{L,p}(m) \), there is a \( \mathbb{Z} \)-lattice \( M' \in G_{L,p}(m) \) such that \( M' \neq \text{spn}(M) \) and \([M']\) is connected to \([M]\) by a path by Lemma 3.5 in [8]. Note that \( g(G_{L,p}(m)) = g(G_{L,p}(m')) \) if and only if \( m \equiv m' \pmod{2} \), where \( g(G_{L,p}(m)) \) is the number of spinor genera contained in \( G_{L,p}(m) \). In particular, \( g(G_{L,p}(m)) = g(G_{L,p}(0)) \) for any even \( m \). So, every \( \mathbb{Z} \)-lattice \( M' \) satisfying the above condition is contained in a single spinor genus. From the existence of such a \( \mathbb{Z} \)-lattice \( M' \), we may define

\[
\text{Cspn}(M) = \begin{cases} 
\text{spn}(M) & \text{if } G_{L,p}(m) \text{ is of } O \text{-type}, \\
\text{spn}(M) \cup \text{spn}(M') & \text{otherwise}.
\end{cases}
\]

In Lemma 3.10 of [8], we proved that the set of all vertices in the connected component of \( G_{L,p}(m) \) containing \([M]\) is the set of equivalence classes in \( \text{Cspn}(M) \).

**Lemma 3.2.** For an integer \( m \geq 0 \), let \([N]\) in \( G_{L,p}(m+1) \) be an edge of the graph \( G_{L,p}(m) \). Then the set of all edges in the connected component of \( G_{L,p}(m) \) containing \([N]\) is the set of all classes in \( \text{Cspn}(N) \).
vice versa. We say a genus-correspondence \( C \) then any genus-correspondence respects spinor genus. However, the following ex-

Concerning this, Jagy conjectured in [7] that if \( N \) consisting of representable pairs by scaling

It suffices to show that the set of edges in the connected component of \( \mathcal{G}_{L,p}(m) \) containing \([N]\) is exactly the set of vertices in the connected component of \( \mathcal{G}_{L,p}(m+1) \) containing the vertex \([N]\) by Lemma 3.10 of [8]. Note that if \( N_1 \) and \( N_2 \) are different \( \Gamma_p \)-descendant of \( K \) for some \( K \in \mathcal{G}_{L,p}(m+2) \), then \( \lambda_p(K) \) is a \( \Gamma_p \)-descendant of both \( N_1 \) and \( N_2 \). This implies that every class in \( \text{Cspn}(N) \) is contained in the set of edges in the connected component of \( \mathcal{G}_{L,p}(m) \) containing \([N]\). Conversely, assume that \([N']\) is contained in the set of edges in the connected component of \( \mathcal{G}_{L,p}(m) \) containing \([N]\). Without loss of generality, we may assume that there is a \( \mathbb{Z} \)-lattice \( M \) that is a \( \Gamma_p \)-descendant of both \( N \) and \( N' \). If \( m = 0 \) or \( m \geq 1 \) and \( \lambda_p(N) \neq \lambda_p(N') \), then there is a \( \mathbb{Z} \)-lattice \( K \) whose \( \Gamma_p \)-descendants are both \( N \) and \( N' \) by Lemmas 3.2 and 3.3 of [8], that is, as vertices, \([N]\) and \([N']\) are contained in the edge \([K]\). Now suppose that \( m \geq 1 \) and \( \lambda_p(N) = \lambda_p(N') \). Then in this case, there is a \( \mathbb{Z} \)-lattice \( S \in \mathcal{G}_{L,p}(m+1) \) such that \( \lambda_p(S) \neq \lambda_p(N) \) and one of \( \Gamma_p \)-descendants of \( S \) is \( M \) (see Lemma 3.3 of [8]). Hence there are edges containing \([N], [S]\) and \([S], [N']\) in the graph \( \mathcal{G}_{L,p}(m+1) \). This completes the proof. \( \square \)

4. GENUS-CORRESPONDENCES

Let \( n \) be a positive integer. Let \( M \) be a ternary \( \mathbb{Z} \)-lattice on a quadratic space \( V \) and let \( N \) be a \( \mathbb{Z} \)-lattice on \( V^n \). Assume that there is a representation

\[
\phi : M^n \to N \quad \text{such that} \quad [N : \phi(M^n)] = n.
\]

Then clearly, \( N^n \) is also represented by \( M \). For any \( \mathbb{Z} \)-lattice \( M_1 \in \text{gen}(M) \), since \( (M^n)_p \sim (M^n)_p \to N_p \) for any prime \( p \), there is a \( \mathbb{Z} \)-lattice \( N_1 \in \text{gen}(N) \) that represents \((M_1)^n\). Conversely, for any \( \mathbb{Z} \)-lattice \( N' \in \text{gen}(N) \), there is a \( \mathbb{Z} \)-lattice \( M' \in \text{gen}(M) \) such that \((M')^n \to N' \) (see [7]). For \( M_1 \in \text{gen}(M) \) and \( N_1 \in \text{gen}(N) \) such that \((M_1)^n \to N_1 \), the pair \( ([N_1], [M_1]) \in \text{gen}(N) / \sim \times \text{gen}(M) / \sim \) is called a representable pair by scaling \( n \). A subset \( \mathcal{C} \subset \text{gen}(N) / \sim \times \text{gen}(M) / \sim \) consisting of representable pairs by scaling \( n \) is called a genus-correspondence if for any \( N' \in \text{gen}(N) \), there is an \( M' \in \text{gen}(M) \) such that \([N'], [M'] \in \mathcal{C} \), and vice versa. We say a genus-correspondence \( \mathcal{C} \) respects spinor genus if for any two \(([N_1], [M_1]), ([N_2], [M_2]) \) \( \in \mathcal{C} \),

\[
N_1 \in \text{spn}(N_2) \quad \text{if and only if} \quad M_1 \in \text{spn}(M_2).
\]

Concerning this, Jagy conjectured in [7] that if \( n \) is square free and \( g(N) = g(M) \), then any genus-correspondence respects spinor genus. However, the following example shows that the conjecture is not true.

**Example 4.1.** Let \( N_1 = \langle 12 \rangle \downarrow \left( \begin{array}{c} 15 \\ 5 \\ 135 \end{array} \right) \) and \( M_1 = \langle 1, 20, 80 \rangle \). Then one may easily check that \( g(M_1) = g(N_1) = 2 \), \( dN_1 = 15 \cdot dM_1 \) and \( M_1^{15} \) is represented by \( N_1 \). The genus of \( N_1 \) consists of the following 12 lattices up to isometry:

\[
\begin{align*}
N_1 & = \langle 12, 15, 135, 5, 0, 0 \rangle, & N_2 & = \langle 3, 7, 1200, 0, 0, 1 \rangle, & N_3 & = \langle 3, 60, 140, 20, 0, 0 \rangle, \\
N_4 & = \langle 3, 27, 300, 0, 0, 1 \rangle, & N_5 & = \langle 27, 27, 40, 10, 10, 3 \rangle, & N_6 & = \langle 12, 28, 83, 12, 4, -4 \rangle, \\
N_7 & = \langle 12, 28, 75, 0, 0, 4 \rangle, & N_8 & = \langle 15, 35, 48, 0, 0, 5 \rangle, & N_9 & = \langle 7, 12, 300, 0, 0, 2 \rangle, \\
N_{10} & = \langle 12, 43, 60, 20, 0, 6 \rangle, & N_{11} & = \langle 8, 12, 303, 4, -2, 4 \rangle, & N_{12} & = \langle 12, 35, 60, 10, 0, 0 \rangle.
\end{align*}
\]
Note that up to isometry,
\[ \text{spn}(N_1) = \{ N_i : 1 \leq i \leq 6 \} \text{ and } \text{spn}(N_7) = \{ N_i : 7 \leq i \leq 12 \}. \]

The genus of \( M_1 \) consists of the following 6 lattices up to isometry:
\[
M_1 = \langle 1, 20, 80, 0, 0, 0 \rangle, \quad M_2 = \langle 5, 16, 20, 0, 0, 0 \rangle, \quad M_3 = \langle 4, 20, 25, 10, 0, 0 \rangle, \\
M_4 = \langle 4.5, 80, 0, 0, 0 \rangle, \quad M_5 = \langle 9, 9, 20, 0, 0, 1 \rangle, \quad M_6 = \langle 4, 20, 21, 0, 2, 0 \rangle.
\]

Note that up to isometry,
\[ \text{spn}(M_1) = \{ M_i : 1 \leq i \leq 3 \} \text{ and } \text{spn}(M_4) = \{ M_i : 4 \leq i \leq 6 \}. \]

Define a genus-correspondence \( \mathcal{G} \) as follows:
\[
\mathcal{G} = \{(N_1], [M_1]), ([N_3], [M_2]), ([N_7], [M_2]), \\
([N_5], [M_3]), ([N_{11}], [M_3]), ([N_2], [M_4]), ([N_8], [M_4]), \\
([N_6], [M_5]), ([N_{10}], [M_5]), ([N_4], [M_6]), ([N_{12}], [M_6]).\}
\]

Then one may easily check that \( \mathcal{G} \) does not respect spinor genus.

In the remaining, we show that if we take a genus-correspondence suitably, then it respects spinor genus under the assumption that \( g(M) = g(N) \). We do not assume that \( n \) is square free for a while.

**Lemma 4.2.** For ternary \( \mathbb{Z} \)-lattices \( N \) and \( M \), assume that \( ([N], [M]) \) is a representable pair by scaling \( n \). Then for any \( N' \in \text{spn}(N) \), there is a \( \mathbb{Z} \)-lattice \( M' \in \text{spn}(M) \) such that \( ([N'], [M']) \) is a representable pair by scaling \( n \). Conversely, for any \( M'' \in \text{spn}(M) \) there is a \( \mathbb{Z} \)-lattice \( N'' \in \text{spn}(N) \) such that \( ([N''], [M'']) \) is a representable pair by scaling \( n \).

**Proof.** Since \( ([N], [M]) \) is a representable pair by scaling \( n \), \( \sigma(M'') \subseteq N \), for some isometry \( \sigma \in O(V) \). Let \( N' \in \text{spn}(N) \). Then there are \( \sigma' \in O(V) \) and \( \Sigma \in J \) such that \( N' = \sigma' \Sigma N \). If we define \( M' = \sigma' \Sigma \sigma(M) = \sigma' \sigma(\sigma^{-1} \Sigma \sigma) M \in \text{spn}(M) \), then
\[
(M')^n = \sigma' \Sigma \sigma(M'') \subseteq \sigma' \Sigma N = N'.
\]
The converse can be proved similarly. \( \square \)

A bipartite graph with partitions \( U \) and \( V \) of vertices and with \( E \) of edges is denoted by \( \mathcal{G}(U, V, E) \), or simply \( \mathcal{G}(U, V) \). For each vertex \( u \in U \) of the bipartite graph \( \mathcal{G}(U, V, E) \), we define \( N(u) = \{ v : uv \in E \} \). For a vertex \( v \in V \), \( N(v) \) is defined similarly. The graph \( \mathcal{G}(U, V, E) \) is called \((a, b)\)-regular if \( N(u) = a \) for any \( u \in U \), and \( N(v) = b \) for any \( v \in V \).

For two bipartite graphs \( \mathcal{G}(U, V, E) \) and \( \mathcal{G}(V, W, E') \), we define a *juxtaposition bipartite graph* of two bipartite graphs, denoted by \( \mathcal{G}(U, W, \tilde{E}) \), as follows: \( U \) and \( W \) are partitions of vertices and there is an edge \( uv \in \tilde{E} \) for \( u \in U \) and \( w \in W \) if and only if there is a vertex \( v \in V \) such that \( uv \in E \) and \( vw \in E' \).

For a representable pair \( ([N], [M]) \) by scaling \( n \), we define a bipartite graph
\[
\mathcal{G}(N, M) = \mathcal{G}(\text{gen}(N)_S, \text{gen}(M)_S)
\]
such that two vertices \( \text{spn}(N') \in \text{gen}(N)_S \) and \( \text{spn}(M') \in \text{gen}(M)_S \) are connected by an edge if and only if there are lattices \( N'' \in \text{spn}(N') \) and \( M'' \in \text{spn}(M') \) such that \( ([N''], [M'']) \) is a representable pair by scaling \( n \).

**Lemma 4.3.** Let \( N \) and \( M \) be two \( \mathbb{Z} \)-lattices such that \( ([N], [M]) \) is a representable pair by scaling \( n \). Then for some positive integers \( u, v \) such that \( u\gamma(N) = v\gamma(M) \), the graph \( \mathfrak{G}(N, M) \) is \( (u, v) \)-regular. In particular, if \( g(N) = g(M) \), then the graph \( \mathfrak{G}(N, M) \) is a regular bipartite graph.

**Proof.** Let \( \text{spn}(N') \in \text{gen}(N)_S \) and \( \text{spn}(M') \in \text{gen}(M)_S \) be two vertices the graph \( \mathfrak{G}(N, M) \) such that \( \text{spn}(M') \in \mathcal{N}(\text{spn}(N')) \). By Lemma 4.2, we may assume that \( (N', M') \) is a representable pair by scaling \( n \), that is, there is a representation \( \phi \in O(V) \) such that \( \phi((M')^n) \subset N' \). Let \( \text{spn}(N'') \) be another vertex in \( \text{gen}(N)_S \). Choose a split rotation \( \Sigma \in J_V \) such that \( N'' = \Sigma N' \). Since

\[
(\Sigma N', \phi^{-1}\Sigma\phi^{-1}(\Sigma(M')^n)) \text{ is a representable pair by scaling } n.
\]

Furthermore, since \( \phi^{-1}\Sigma\phi^{-1} \in J_V \), we have \( \text{spn}(\Sigma M') \in \mathcal{N}(\text{spn}(N'')) \). Note that for any two lattices \( M', M'' \in \text{gen}(M) \), \( M' \in \text{spn}(M'') \) if and only if \( \Sigma M' \in \text{spn}(\Sigma M'') \). Therefore

\[
|\mathcal{N}(\text{spn}(N'))| = |\mathcal{N}(\text{spn}(N''))|.
\]

Similarly, we also have \( |\mathcal{N}(\text{spn}(M'))| = |\mathcal{N}(\text{spn}(M''))| \) for any \( M', M'' \in \text{gen}(M) \). The lemma follows from this.

**Theorem 4.4.** Let \( N \) and \( M \) be two \( \mathbb{Z} \)-lattices such that \( ([N], [M]) \) is a representable pair by scaling \( n \). If \( g(N) = g(M) \), then there is a genus-correspondence respecting spinor genus.

**Proof.** We may assume that

\[
\text{gen}(N)_S = \{ \text{spn}(N_i) : i = 1, 2, \ldots, g \} \quad \text{and} \quad \text{gen}(M)_S = \{ \text{spn}(M_i) : i = 1, 2, \ldots, g \}.
\]

Since the graph \( \mathfrak{G}(N, M) \) defined above is a regular bipartite graph, there is a perfect matching by Hall's marriage theorem. Hence, without loss of generality, we may assume that each \( ([N_i], [M_i]) \) is a representable pair by scaling \( n \). We define a genus-correspondence \( \mathfrak{G} \) as follows: for \( ([N'], [M']) \in \text{gen}(N)/ \sim \times \text{gen}(M)/ \sim \), \( ([N'], [M']) \in \mathfrak{G} \) if and only if \( ([N'], [M']) \) is a representable pair by scaling \( n \) and there is an \( i \) \( (1 \leq i \leq g) \) such that \( N' \in \text{spn}(N_i) \) and \( M' \in \text{spn}(M_i) \). Then by Lemma 4.2 \( \mathfrak{G} \) is a genus-correspondence respecting spinor genus.

5. **Genus-correspondence respecting spinor genus**

From now on, we assume that \( n \) is a square free positive integer. Let \( N \) and \( M \) be ternary \( \mathbb{Z} \)-lattices such that the pair \( ([N], [M]) \) is a representable pair by scaling \( n \). In this section, We find a necessary and sufficient condition for the genus-correspondence \( \text{gen}(N)/ \sim \times \text{gen}(M)/ \sim \) to respect spinor genus under the assumption that \( g(M) = g(N) \).
Lemma 5.1. Let $p$ be a prime and let $N$ and $M$ be primitive ternary $\mathbb{Z}$-lattices. Then $([N],[M])$ is a representable pair by scaling $p$ if and only if $M^p \cong \Lambda_p(N)$ or $M$ is a $\Gamma_p$-descendent of $N$.

Proof. Without loss of generality, we may assume that $M^p$ is a sublattice of $N$ with index $p$. Then there is a basis $\{x_1, x_2, x_3\}$ for $N$ such that $M^p = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(px_3)$. Hence there are integers $a, b, c, s, t, u$ such that

$$\begin{pmatrix} pa \\ \frac{pu}{2} \\ \frac{pb}{2} \\ s \\ t \\ c \end{pmatrix}.$$ 

Hence if $s \equiv t \equiv 0 \pmod{p}$, then clearly, $M^p = \Lambda_p(N)$. If $s$ or $t$ is not divisible by $p$, then a Jordan decomposition of $N_p$ has an isotropic $\frac{1}{2}\mathbb{Z}_p$-modular component. Furthermore, $M^p$ is a sublattice of $N$ with index $p$ whose norm is $p\mathbb{Z}$. Therefore $M$ is a $\Gamma_p$-descendent of $N$. Note that the converse is almost trivial.

□

Lemma 5.2. For two ternary $\mathbb{Z}$-lattices $N$ and $M$, assume that $([N],[M])$ is a representable pair by scaling $p$. For any prime $p$ dividing $n$, there is a $\mathbb{Z}$-lattice $N(p)$ such that $([N],[N(p)])$ is a representable pair by scaling $p$, and $([N(p)],[M])$ is a representable pair by scaling $n_p$.

Proof. By assumption, we may assume that $M^n$ is a sublattice of $N$ with index $n$. Choose a basis $\{x_1, x_2, x_3\}$ for $N$ such that $M^n = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(nx_3)$. Define $N(p) = (\mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(px_3))^\perp$. Then one may easily show that $n(N(p)) = Z$ and $N(p)$ satisfies all conditions given above.

□

Corollary 5.3. Let $a, b$ be positive integers such that $ab$ is square free. Let $([N],[L])$ and $([L],[M])$ be representable pairs of ternary $\mathbb{Z}$-lattices by scaling $a$ and $b$, respectively. Then the graph $G(N,M)$ is exactly same to the juxtaposition bipartite graph $G\gen{\gen{L},S}(\gen{N},S;\gen{M},S)$.

Proof. It suffices to show that each set of edges for both graphs is same, which follows directly from the above lemma.

□

Let $([N],[M])$ be a representable pair of ternary $\mathbb{Z}$-lattices by scaling $p$, where $p$ is a prime. Then $\lambda_p(N) \cong M$ or $M$ is a $\Gamma_p$-descendent of $N$ by Lemma 5.1. If the former holds, then the bipartite graph $G(N,M)$ is $(1,1)$-regular or $(1,2)$-regular by Lemma 2.3. Furthermore, it is $(1,1)$-regular if and only if $N$ is of $H$-type at $p$. Note that $(1,4)$-regularity is impossible in our situation.

Now, assume that the latter holds. Then there is a $\mathbb{Z}$-lattice $L$ and a nonnegative integer $m$ such that $N \in G_{L,p}(m+1)$ and $M \in G_{L,p}(m)$. If the graph $G_{L,p}(m+1)$ is of $E$-type, then the bipartite graph $G(N,M)$ is $(1,2)$-regular by Lemma 3.2 and if the graph $G_{L,p}(m)$ is of $E$-type, then the bipartite graph $G(N,M)$ is $(2,1)$-regular. Finally, if both $G_{L,p}(m+1)$ and $G_{L,p}(m)$ are of $O$-type, then the bipartite graph $G(N,M)$ is $(1,1)$-regular. Note that both $G_{L,p}(m+1)$ and $G_{L,p}(m)$ cannot be of $E$-type simultaneously. We say $N$ is of $(E,O)$-type ($(O,E)$-type) at $p$ if the graph $G_{L,p}(m+1)$ ($G_{L,p}(m)$, respectively) is of $E$-type. Finally, we say $N$ is of $(O,O)$-type if both $G_{L,p}(m+1)$ and $G_{L,p}(m)$ are of $O$-type.
Let \( ([N], [M]) \) be a representable pair of ternary \( \mathbb{Z} \)-lattices by scaling \( n \), where \( n \) is a square free positive integer. Without loss of generality, we assume that \( M^n \) is a sublattice of \( N \) with index \( n \). Choose a basis \( \{x_1, x_2, x_3\} \) for \( N \) such that \( M^n = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(nx_3) \). Let \( n_2 \) be a product of primes \( p \) dividing \( n \) such that the rank of \( \frac{1}{x_3} \mathbb{Z}_p \)-modular component in a Jordan decomposition of \( N_p \) is two, and let \( n_1 \) be the integer satisfying \( n = n_1 n_2 \). Let \( n_2(e) \) be a product of primes \( q \) dividing \( n_2 \) such that \( \text{ord}_q(4 \cdot dM) \equiv 0 \pmod{2} \), and let \( n_2(o) \) be the integer satisfying \( n_2 = n_2(e)n_2(o) \). Define a ternary \( \mathbb{Z} \)-lattice

\[
L_{N,M} = (\mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(n_1n_2(e)x_3))^{\frac{1}{n_1n_2(o)}},
\]

Note that pair \( ([N], [L_{N,M}]) \) \( ([L_{N,M}], [M]) \) is a representable pair by scaling \( n_1n_2(e) \) \( n_2(o) \), respectively. Let \( g_{N,M} = g(L_{N,M}) \) be the number of (proper) spinor genera in the genus of \( L_{N,M} \).

**Theorem 5.4.** Let \( n \) be a square free positive integer and let \( ([N], [M]) \) be a representable pair by scaling \( n \). Then, any connected component of the bipartite graph \( \mathcal{G}(N,M) \) is a complete \( K_{\alpha, \beta} \)-graph, where

\[
\alpha = \frac{g(M)}{g_{N,M}} \quad \text{and} \quad \beta = \frac{g(N)}{g_{N,M}}.
\]

**Proof.** Without loss of generality, we assume that \( M^n \) is a sublattice of \( N \) with index \( n \). Let \( n_1n_2(e) = p_1p_2 \ldots p_s \) and \( n_2(o) = q_1q_2 \ldots q_t \), where each \( p_i \) and \( q_j \) is a prime. By Lemma 5.3, the graph \( \mathcal{G}(N,M) \) is a juxtaposition of the graphs \( \mathcal{G}(N, L_{N,M}) \) and \( \mathcal{G}(L_{N,M}, M) \), where \( L_{N,M} \) is a \( \mathbb{Z} \)-lattice defined above. Let \( \{x_1, x_2, x_3\} \) be a basis for \( N \) such that \( L_{N,M} = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(n_1n_2(e)x_3) \). Define

\[
N(i) = (\mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(p_1p_2 \ldots p_ix_3))^{\frac{1}{p_1p_2 \ldots p_i}}.
\]

Then the graph \( \mathcal{G}(N, L_{N,M}) \) is a juxtaposition of the graphs

\[
\mathcal{G}(N, N(1)), \mathcal{G}(N(1), N(2)), \ldots, \mathcal{G}(N(s - 1), L_{N,M}),
\]

all of which are either a \((1,1)\)-regular graph or a \((1,2)\)-regular graph. Therefore the graph \( \mathcal{G}(N, L_{N,M}) \) is a \( (1, \frac{g(M)}{g_{N,M}}) \)-regular graph. Similarly, one may easily check that the graph \( \mathcal{G}(L_{N,M}, M) \) is a \( (\frac{g(M)}{g_{N,M}}, 1) \)-regular graph. The theorem follows from these two observations. \( \square \)

**Corollary 5.5.** Let \( n \) be a square free positive integer and let \( ([N], [M]) \) be a representable pair by scaling \( n \). Assume that \( g(N) = g(M) \). Then \( g(N) = g_{N,M} \) if and only if the genus-correspondence \( \text{gen}(N)/ \sim \times \text{gen}(M)/ \sim \) respects spinor genus.

**Proof.** Note that the genus-correspondence \( \text{gen}(N)/ \sim \times \text{gen}(M)/ \sim \) respects spinor genus if and only if the graph \( \mathcal{G}(N, M) \) is \((1,1)\)-regular. Hence the corollary follows directly from the above theorem. \( \square \)

Recall that we are assuming that \( n \) is a square free positive integer and \( ([N], [M]) \) is a representable pair by scaling \( n \). Now we further assume that \( g(N) = g_{N,M} = \)
\( g(M) \), that is, the genus-correspondence \( \text{gen}(N)/ \sim \times \text{gen}(M)/ \sim \) respects spinor genus by Theorem 5.4. Assume that
\[
\text{gen}(N)_{S} = \{ \text{spn}(N_{i}) : i = 1, 2, \ldots, g \} \quad \text{and} \quad \text{gen}(M)_{S} = \{ \text{spn}(M_{i}) : i = 1, 2, \ldots, g \},
\]
and there is a unique edge containing \( \text{spn}(N_{i}) \) and \( \text{spn}(M_{i}) \) in the graph \( \mathcal{G}(N, M) \) for any \( i = 1, 2, \ldots, g \).

**Lemma 5.6.** Let \( k \) be a positive integer. Under the assumptions given above, \( \text{spn}(N_{i}) \) represents \( nk \) if and only if \( \text{spn}(M_{i}) \) represents \( k \), for any \( i = 1, 2, \ldots, g \).

**Proof.** First assume that \( n = p \) is a prime. Then \( \lambda_{p}(N) \simeq M \) or \( M \) is a \( \Gamma_{p} \)-descendant of \( N \) by Lemma 5.1. If the former holds, one may easily show that \( N \) represents \( pk \) if and only if \( M \) represents \( k \). If the latter holds, then also one may easily show that \( N \) represents \( pk \) if and only if at least one of \( \Gamma_{p} \)-descendants of \( N \) represents \( k \). Hence the lemma follows directly from this.

Let \( n = p_{1}p_{2}\cdots p_{r} \), where each \( p_{i} \) is a prime. We may assume that \( M^{n} \) is a sublattice of \( N \) with index \( n \). Let \( \{ x_{1}, x_{2}, x_{3} \} \) be a basis for \( N \) such that \( M^{n} = \mathbb{Z}x_{1} + \mathbb{Z}x_{2} + \mathbb{Z}(nx_{3}) \). Define
\[
N(i) = (\mathbb{Z}x_{1} + \mathbb{Z}x_{2} + \mathbb{Z}(p_{1}p_{2}\cdots p_{i}x_{3}))_{i}.\]

Then the graph \( \mathcal{G}(N, M) \) is a juxtaposition of the graphs
\[
\mathcal{G}(N, N(1)), \mathcal{G}(N(1), N(2)), \ldots, \mathcal{G}(N(r - 1), M).
\]
Since \( g(N) = g_{N, M} = g(M) \), we know that each graph \( \mathcal{G}(N(i), N(i + 1)) \) is a \((1, 1)\)-regular graph. Hence the lemma follows directly from the induction on \( r \).

A set \( S = \{ c_{1}, c_{2}, \ldots, c_{g} \} \) of integers is said to be a complete system of spinor exceptional integers for \( \text{gen}(L) \), for some ternary \( \mathbb{Z} \)-lattice \( L \), if for any subset \( U \subset S \), there is a unique \( \text{spn}(L') \in \text{gen}(L)_{S} \) such that every integer in \( U \) is represented by \( \text{spn}(L') \) and every integer in \( S - U \) is not represented by \( \text{spn}(L') \). For details, see [1].

**Corollary 5.7.** Under the same assumptions given above, suppose that there is a complete system \( \{ c_{1}, c_{2}, \ldots, c_{g} \} \) of spinor exceptional integers for \( \text{gen}(M) \). Then \( \{ nc_{1}, nc_{2}, \ldots, nc_{g} \} \) is a complete system of spinor exceptional integers for \( \text{gen}(N) \).

**Proof.** The corollary follows directly form the above lemma.

The following example was first introduced by Jagy in [7].

**Example 5.8.** Let \( n \) be a square free integer that is represented by a sum of two integral squares. Define ternary \( \mathbb{Z} \)-lattices
\[
N = \langle 1, 1, 16n \rangle \quad \text{and} \quad M = \langle 1, 1, 16 \rangle.
\]
Then one may easily check that \( \langle [N], [M] \rangle \) is a representable pair by scaling \( n \) and \( g(N) = g_{N, M} = g(M) = 2 \). Note that \( \{ 1 \} \) is a complete system of spinor exceptional integers for \( \text{gen}(M) \). By Corollary 5.7, we know that \( \{ n \} \) is a complete system of spinor exceptional integers for \( \text{gen}(N) \). In fact, Jagy proved in [7] that \( n \) is represented by any \( \mathbb{Z} \)-lattice in \( \text{spn}(N) \). However one may easily show that it holds even if \( n \) is not square free.
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