Potts Model Partition Functions for Self-Dual Families of Strip Graphs

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Abstract

We consider the $q$-state Potts model on families of self-dual strip graphs $G_D$ of the square lattice of width $L_y$ and arbitrarily great length $L_x$, with periodic longitudinal boundary conditions. The general partition function $Z$ and the $T = 0$ antiferromagnetic special case $P$ (chromatic polynomial) have the respective forms $\sum_{j=1}^{N_{F,L_y}} c_{F,L_y,j}^{L_x} \lambda_{F,L_y,j}^{L_x}$, with $F = Z, P$. For arbitrary $L_y$, we determine (i) the general coefficient $c_{F,L_y,j}$ in terms of Chebyshev polynomials, (ii) the number $n_F(L_y, d)$ of terms with each type of coefficient, and (iii) the total number of terms $N_{F,L_y,\lambda}$. We point out interesting connections between the $n_Z(L_y, d)$ and Temperley-Lieb algebras, and between the $N_{F,L_y,\lambda}$ and enumerations of directed lattice animals. Exact calculations of $P$ are presented for $2 \leq L_y \leq 4$. In the limit of infinite length, we calculate the ground state degeneracy per site (exponent of the ground state entropy), $W(q)$. Generalizing $q$ from $\mathbb{Z}_+$ to $\mathbb{C}$, we determine the continuous locus $B$ in the complex $q$ plane where $W(q)$ is singular. We find the interesting result that for all $L_y$ values considered, the maximal point at which $B$ crosses the real $q$ axis, denoted $q_c$, is the same, and is equal to the value for the infinite square lattice, $q_c = 3$. This is the first family of strip graphs of which we are aware that exhibits this type of universality of $q_c$.

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1 Introduction

The $q$-state Potts antiferromagnet (AF) \cite{1, 2} exhibits nonzero ground state entropy, $S_0 > 0$ (without frustration) for sufficiently large $q$ on a given lattice $\Lambda$ or, more generally, on a graph $G = (V, E)$ defined by its set of vertices $V$ and edges joining these vertices $E$. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. There is a close connection with graph theory here, since the zero-temperature partition function of the above-mentioned $q$-state Potts antiferromagnet on a graph $G$ satisfies

\[ Z(G, q, T = 0)_{PAF} = P(G, q) \quad (1.1) \]

where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices have the same color (for reviews, see \cite{3}-\cite{5}). The minimum number of colors necessary for such a coloring of $G$ is called the chromatic number, $\chi(G)$. Thus

\[ W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} \quad (1.2) \]

where $n = |V|$ is the number of vertices of $G$ and $\{G\} = \lim_{n \to \infty} G$. At certain special points $q_s$ (typically $q_s = 0, 1, ..., \chi(G)$), one has the noncommutativity of limits

\[ \lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n} \quad (1.3) \]

and hence it is necessary to specify the order of the limits in the definition of $W(\{G\}, q_s)$ \cite{6}. Denoting $W_{qn}$ and $W_{nq}$ as the functions defined by the different order of limits on the left and right-hand sides of (1.3), we take $W \equiv W_{qn}$ here; this has the advantage of removing certain isolated discontinuities that are present in $W_{nq}$.

Using the expression for $P(G, q)$, one can generalize $q$ from $\mathbb{Z}_+$ to $\mathbb{C}$. The zeros of $P(G, q)$ in the complex $q$ plane are called chromatic zeros; a subset of these may form an accumulation set in the $n \to \infty$ limit, denoted $\mathcal{B}$ \cite{9}, which is the continuous locus of points where $W(\{G\}, q)$ is nonanalytic. (For some families of graphs $\mathcal{B}$ may be null, and $W$ may also be nonanalytic at certain discrete points; this is not relevant for the present paper.) The maximal region in the complex $q$ plane to which one can analytically continue the function $W(\{G\}, q)$ from physical values where there is nonzero ground state entropy is denoted $R_1$.

The maximal value of $q$ where $\mathcal{B}$ intersects the (positive) real axis is labelled $q_c(\{G\})$. This point is important since it separates the interval of physical $q \geq q_c(\{G\})$ on the positive real $q$ axis where the Potts model exhibits nonzero ground state entropy (without frustration) from the interval $0 \leq q \leq q_c(\{G\})$ in which $W$ has different analytic form(s).
In this paper we shall present a number of exact results on the Potts model partition function and the special case comprised by the zero-temperature Potts antiferromagnet for families of self-dual strip graphs of the square lattice with (i) a fixed transverse width \( L_y \), (ii) arbitrarily great length \( L_x \), (iii) periodic longitudinal boundary conditions, and (iv) such that each vertex on one side of the strip, which we take to be the upper side (with the strip oriented so that the longitudinal, \( x \) direction is horizontal) are joined by edges to a single external vertex. A strip graph of this type will be denoted generically as \( G_D \) (where the subscript \( D \) refers to the self-duality) and, when its size is indicated, as \( G_D(L_y \times L_x) \). We shall present a number of structural formulas that hold for arbitrary \( L_y \). After recalling our earlier work for the case \( L_y = 1 \) [6, 7], we shall present exact results for the chromatic polynomial for \( L_y = 2 \) through \( L_y = 4 \) and shall study the zeros of these polynomials and determine the continuous accumulation set \( \mathcal{B} \) of these zeros in the limit \( L_x \to \infty \). In Fig. 1 we show an illustrative example of this family of strip graphs, for the case \( L_x = 4 \) and \( L_y = 3 \).

In general, the graph \( G_D(L_y \times L_x) \) has

\[
\begin{align*}
n &\equiv |V| = L_x L_y + 1 = f & (1.4) \\
e &\equiv |E| = 2L_x L_y & (1.5)
\end{align*}
\]

vertices, equal to the number of faces, \( f \), and edges. We recall that the dual \( G^* \) of a planar graph \( G = (V, E) \) is defined as the graph obtained by replacing each vertex (face) of \( G \) by a face (vertex) of \( G^* \) and connecting the vertices of the resultant \( G^* \) by edges (bonds). The graph \( G \) is self-dual if and only if \( G = G^* \).
It is easily checked that the family \( G_D(L_y \times L_x) \) is self-dual. We recall that the degree \( \Delta \) of a vertex is defined as the number of vertices to which it is adjacent (i.e., the number of nearest neighbors). In the graph \( G_D(L_y \times L_x) \) there are \( L_x(L_y - 1) \) vertices with degree \( \Delta = 4 \), \( L_x \) vertices with \( \Delta = 3 \), and finally the one external vertex with \( \Delta = L_x \). The chromatic number is
\[
\chi[G_D(L_y \times L_x)] = \chi[(Wh)_{L_x+1}] = \begin{cases} 
3 & \text{if } L_x \text{ is even} \\
4 & \text{if } L_x \text{ is odd} 
\end{cases} . \tag{1.6}
\]
The (infinite) square lattice (interpreted as the infinite limit of a graph) is self-dual, and this property has been useful in studies of the Ising, and general \( q \)-state Potts model on this lattice. An early example of this was the calculation by Kramers and Wannier of the critical temperature of the (zero-field) Ising model on the square lattice \(^8\) before Onsager derived a closed-form expression for the free energy \(^9\). The Potts model partition function and the equivalent Tutte polynomial \(^{10, 11}\) of a planar graph \( G \) satisfy certain symmetry relations when one changes \( G \) to \( G^* \) \(^2, 12\). Let us define \( v = e^K - 1 \), where \( K = J/(k_B T) \), with \( T \) being the temperature and \( J \) the spin-spin exchange constant for the Potts model. Then \(^2\)
\[
Z(G, q, v) = v^{\varepsilon(G)} q^{-c(G)} Z(G^*, q, q/v) . \tag{1.7}
\]
If \( G = G^* \), this reduces to an identity relating the \( q \)-state Potts model on \( G \) to the same model with \( v \to q/v \). In previous studies of complex-temperature (Fisher) zeros of the Potts model, it has proved useful to employ families of sections of the square lattice that are self-dual so that one can take advantage of the symmetry property \(^{1.7}\) \(^{13-18}\. The resulting patterns of zeros can be compared with those obtained with other types of boundary conditions \(^{13, 14, 20}\. It may be recalled that in finite-lattice studies one often chooses periodic boundary conditions to minimize boundary effects, but these do not preserve the self-duality property of the infinite square lattice. There are several types of self-dual boundary conditions for strips of the square lattice \(^{13, 18, 19}\). (The boundary conditions involved in the family \( G_D(L_y \times L_x) \) were denoted DBC2 in \(^{16}\. In this paper, instead of studying complex-temperature zeros, we shall focus on zeros of \( P(G, q) \) in the complex \( q \) plane. As we shall show, these exhibit some very interesting features. In particular, we shall show that for all of the strip widths considered, we find a universal value of \( q_c \), namely \( q_c = 3 \), the value for the infinite square lattice \(^{21}\. 

2 Some General Properties

For a recursive family of graphs, such as the strip graphs considered in this paper, comprised of \( L_x \) repetitions of a basic subgraph, the \( q \)-state Potts model partition function for arbitrary
\( q \) (not necessarily in \( \mathbb{Z}_+ \)) and \( v \) has the form \[12\]

\[
Z(G, q, v) = \sum_{j=1}^{N_{Z,G,\lambda}} c_{Z,G,j}(\lambda_{Z,G,j})^{L_x}
\]

(2.1)

where the coefficients \( c_{Z,G,j} \) and the terms \( \lambda_{Z,G,j} \) depend on the lattice type, the boundary conditions, and the width, but not the length. Since we are only dealing in this paper with a particular type of strip graph, \( G_D \), we shall henceforth usually suppress the \( G_D \) in the notation, but shall make explicit the dependence on \( L_y \), writing \( N_{Z,L_y,\lambda} \) for the total number of terms, \( \lambda_{Z,L_y,j} \) for a given term, \( c_{Z,L_y,j} \) for the corresponding coefficient, and similarly for the chromatic polynomial. Since the chromatic polynomial is a special case of the Potts model partition function, it also has this form \[26\]

\[
P(G_D(L_y \times L_x), q) = \sum_{j=1}^{N_{P,L_y,\lambda}} c_{P,L_y,j}(\lambda_{P,L_y,j})^{L_x}.
\]

(2.2)

Following our earlier nomenclature \[3\], we denote a \( \lambda \) as leading (= dominant) if it has a magnitude greater than or equal to the magnitude of other \( \lambda \)'s. In the limit \( n \to \infty \) the leading \( \lambda \) in \( Z \) determines the free energy per site \( f = \lim_{n \to \infty} Z^{1/n} \) and similarly the leading \( \lambda \) in \( P \) determines the function \( W \) defined in \[12\]. The continuous locus \( B \) where \( f \) or \( W \) is nonanalytic thus occurs where there is a switching of dominant \( \lambda \)'s in \( Z \) and \( P \), respectively, and is the solution of the equation of degeneracy in magnitude of these dominant \( \lambda \)'s.

In general one can regard the \( \lambda \)'s as the (nonzero) eigenvalues of a certain coloring matrix \[23, 24, 25\]. In our earlier calculations, we have found examples of zero eigenvalues \[25\], but for the present family we only obtain nonzero eigenvalues. We note that while the coefficients \( c_{P,L_y,j} \) and \( c_{Z,L_y,j} \) can be obtained as multiplicities of the distinct eigenvalues \( \lambda_{P,L_y,j} \) and \( \lambda_{Z,L_y,j} \) for sufficiently large integer \( q \), they may be zero or negative for some positive integer values of \( q \). More generally, while they play the role of eigenvalue multiplicities for sufficiently large integer \( q \), the domain of their definition may be generalized to \( q \in \mathbb{R}_+ \) or, indeed, to \( q \in \mathbb{C} \).

The dimension of the space of coloring configurations, \( \mathcal{N} \), is equal to the sum of the multiplicities of each distinct eigenvalue, i.e., the sum of the dimensions of the invariant subspaces corresponding to each of these distinct eigenvalues. For the chromatic polynomial, this is \( \mathcal{N} \), which is equal to the sum

\[
C_{P,L_y} = \sum_{j=1}^{N_{P,L_y,\lambda}} c_{P,L_y,j}
\]

(2.3)
while for the full Potts model partition function, we shall denote it as

$$C_{Z,L \gamma} = \sum_{j=1}^{N_{Z,L \gamma,\lambda}} c_{Z,L \gamma,j}.$$  \hspace{1cm} (2.4)

Some previous literature on chromatic polynomials and Potts model partition functions on lattice strips is in [27]-[58].

3 Structural Theorems

3.1 Coefficients

We find that the coefficients that enter into (2.2) for $P(G_D, q)$ and (2.1) for $Z(G_D, q, \nu)$ are polynomials in $q$ that consist of a special restricted set with the property that the coefficient of maximal degree $d \geq 1$ in $q$ is

$$\kappa^{(d)} = 2 \left[ U_{2d} \left( \frac{\sqrt{q}}{2} \right) - T_{2d} \left( \frac{\sqrt{q}}{2} \right) \right]$$

$$= \sqrt{q} U_{2d-1} \left( \frac{\sqrt{q}}{2} \right)$$

$$= \sum_{j=0}^{d-1} (-1)^j \binom{2d-1-j}{j} q^{d-j}$$  \hspace{1cm} (3.1.1)

where $T_n(x)$ and $U_n(x)$ are the Chebyshev polynomials of the first and second kinds, defined by

$$T_n(x) = \frac{1}{2} \sum_{j=0}^{\left[ \frac{n}{2} \right]} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (2x)^{n-2j}$$  \hspace{1cm} (3.1.2)

and

$$U_n(x) = \sum_{j=0}^{\left[ \frac{n}{2} \right]} (-1)^j \binom{n-j}{j} (2x)^{n-2j}$$  \hspace{1cm} (3.1.3)

where in eqs. (3.1.2) and (3.1.2) $\left[ \frac{n}{2} \right]$ in the upper limit on the summand means the integral part of $\frac{n}{2}$. The first few of these coefficients are

$$\kappa^{(1)} = q$$  \hspace{1cm} (3.1.4)

$$\kappa^{(2)} = q(q - 2)$$  \hspace{1cm} (3.1.5)

$$\kappa^{(3)} = q(q - 1)(q - 3)$$  \hspace{1cm} (3.1.6)
\[ \kappa^{(4)} = q(q-2)(q^2-4q+2) \]  
\[ \kappa^{(5)} = q(q^2-3q+1)(q^2-5q+5) \]  
\[ \kappa^{(6)} = q(q-1)(q-2)(q-3)(q^2-4q+1). \]  

We have established the following factorization:

\[ \kappa^{(d)} = \prod_{k=1}^{d}(q - s_{d,k}) \]  

where

\[ s_{d,k} = 2 + 2 \cos \left( \frac{\pi k}{d} \right) = 4 \cos^2 \left( \frac{\pi k}{2d} \right) \text{ for } k = 1, 2, \ldots d \]  

In [25], we showed that the coefficients \( c^{(d)} \) that entered into \( Z(G_s, q, v) \) and hence also \( P(G_s, q) \) for cyclic strips \( G_s \) of the square or triangular lattices (and Möbius strips of the square lattice) were of the form [25]

\[ c^{(d)} = U_{2d}\left( \sqrt{q} \right) = \sum_{j=0}^{d} (-1)^j \binom{2d - j}{j} q^{d-j} = \prod_{k=1}^{d}(q - q_{d,k}) \]  

where

\[ q_{d,k} \equiv 2 + 2 \cos \left( \frac{2\pi k}{2d+1} \right) = 4 \cos^2 \left( \frac{\pi k}{2d+1} \right) \text{ for } k = 1, 2, \ldots d. \]  

The \( \kappa^{(d)} \) can be expressed as differences of these \( c^{(d)} \) coefficients:

\[ \kappa^{(d)} = c^{(d)} - c^{(d-1)} \text{ for } d = 1, 2, \ldots \]  

The \( \kappa^{(d)} \) coefficients have the following general properties.

- Given the expression of \( \kappa^{(d)} \), (3.1.12) in terms of Chebyshev polynomials, and the recursion relation for these Chebyshev polynomials (for \( n \geq 1 \))

\[ Ch_{n+1}(x) = 2xCh_{n}(x) - Ch_{n-1}(x) \text{ for } Ch_{n}(x) = T_{n}(x) \text{ or } U_{n}(x) \]  

we obtain the recursion relation for the \( \kappa^{(d)} \) for \( d \geq 2 \),

\[ \kappa^{(d+1)} = (q-2)\kappa^{(d)} - \kappa^{(d-1)}. \]
• $\kappa^{(d)}$ is a polynomial of degree d in $q$ whose coefficients alternate in sign.

• $\kappa^{(d)}$ has the factor $q$.

• If and only if $d$ is even and $d \geq 2$, then $\kappa^{(d)}$ has the factor $(q - 2)$.

• If and only if $d = 0 \mod 3$ and $d \geq 3$, then $\kappa^{(d)}$ has the factor $(q - 3)$.

• The coefficient of the term $q^d$ in $\kappa^{(d)}$ is 1.

• The coefficient of the term $q$ in $\kappa^{(d)}$ is $(-1)^{d+1}d$.

• The $d$ zeros of $\kappa^{(d)}$ occur at the (real) values $s_{d,k}$, $k = 1, ..., d$, where $s_{d,k}$ was given in eq. (3.1.11). Clearly, these lie in the interval $0 \leq q < 4$. In passing, we note that the $k = 2$ special case $s_{d,2}$ is the Tutte-Beraha number $B_d$.

If one lets

$$q = 2 + 2 \cos \theta = 4 \cos^2 \left( \frac{\theta}{2} \right)$$

(3.1.17)

one sees that the argument of the Chebyshev polynomials in (3.1.1) is given by $\frac{\sqrt{q}}{2} = \cos(\theta/2)$. Using the identities (with $\omega = \theta/2$ here)

$$T_n(\cos \omega) = \cos(n\omega)$$

(3.1.18)

and

$$U_n(\cos \omega) = \frac{\sin[(n + 1)\omega]}{\sin \omega}$$

(3.1.19)

we have an alternate formula for $\kappa^{(d)}$,

$$\kappa^{(d)} = 2 \cot \omega \sin(2d\omega)$$

(3.1.20)

### 3.2 Determination of $n_P(L_y, d)$ and $N_{P,L_y,\lambda}$

Let us define $n_P(L_y, d)$ as the number of terms $\lambda_{P,L_y,j}$ in $P(G_D(L_y \times L_x), q)$ that have as their coefficients $c_{P,L_y,j} = \kappa^{(d)}$. For $G_D(L_y \times L_x)$ strips, these coefficients are nonzero for $1 \leq d \leq L_y + 1$. (In our earlier work [25], for different strip graphs, the coefficients were defined to start with $c^{(0)} = 1$, so that the range went from $0 \leq d \leq L_y$; in both cases, the resultant chromatic polynomial has terms in $q$ ranging from $q^n$ to $q$.) The total number, $N_{P,L_y,\lambda}$, of different terms $\lambda_{P,L_y,j}$ in (2.2) is then given by

$$N_{P,L_y,\lambda} = \sum_{d=1}^{L_y+1} n_P(L_y, d) .$$

(3.2.1)
For the sum, (2.3), of the coefficients in (2.2) we have

\[ C_{P,L_y} = \sum_{j=1}^{N_{P,L_y}} c_{P,L_y,j} = \sum_{d=1}^{L_y+1} n_P(L_y, d) \kappa^{(d)}. \]  

(3.2.2)

Using the same methods as in [25], we have determined the numbers \( n_P(L_y, d) \) of \( \lambda_{P,L_y,j} \)'s that have each type of coefficient, \( \kappa^{(d)} \). From coloring matrix arguments, using the fact that a transverse slice along the strip is a path (tree) graph \( T_{L_y+1} \), for which the chromatic polynomial is \( P(T_{L_y+1}, q) = q(q - 1)^{L_y} \), it follows that

\[ \sum_{d=1}^{L_y+1} n_P(L_y, d) \kappa^{(d)} = P(T_{L_y+1}, q) = q(q - 1)^{L_y}. \]  

(3.2.3)

Differentiating eq. (3.2.3) \( L_y \) times we obtain \( L_y + 1 \) linear equations in the \( L_y + 1 \) unknowns \( n_P(L_y, d) \) for \( 1 \leq d \leq L_y + 1 \). Solving these equations, we find that

\[ n_P(L_y, d) = 0 \quad \text{for} \quad d > L_y + 1 \]  

(3.2.4)

\[ n_P(L_y, L_y + 1) = 1 \]  

(3.2.5)

and

\[ n_P(1, 1) = 1 \]  

(3.2.6)

with all other numbers \( n_P(L_y, d) \) being determined by the recursion relation

\[ n_P(L_y + 1, d) = n_P(L_y, d - 1) + n_P(L_y, d) + n_P(L_y, d + 1) \quad \text{for} \quad 1 \leq d \leq L_y + 1. \]  

(3.2.7)

In particular, we find

\[ n_P(L_y, L_y) = L_y \]  

(3.2.8)

and

\[ n_P(L_y, 1) = M_{L_y} \]  

(3.2.9)

where \( M_n \) is the Motzkin number, given by

\[ M_n = \sum_{j=0}^{n} (-1)^j C_{n+1-j} \binom{n}{j} \]  

(3.2.10)

and

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]  

(3.2.11)

is the Catalan number. (The Catalan and Motzkin numbers occur in many combinatoric applications [59]-[61].)
Table 1: Table of numbers $n_P(L_y, d)$ and their sums, $N_{P,L_y,\lambda}$ for $G_D(L_y \times L_x)$. Blank entries are zero.

| $L_y$ ↓ | $d \rightarrow$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | $N_{P,L_y,\lambda}$ |
|--------|----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---------------------|
| 1      | 1              | 1   | 1   |     |     |     |     |     |     |     |     |     | 2                   |
| 2      | 2              | 2   | 2   | 1   |     |     |     |     |     |     |     |     | 5                   |
| 3      | 4              | 5   | 3   | 1   |     |     |     |     |     |     |     |     | 13                  |
| 4      | 9              | 12  | 9   | 4   | 1   |     |     |     |     |     |     |     | 35                  |
| 5      | 21             | 30  | 25  | 14  | 5   | 1   |     |     |     |     |     |     | 96                  |
| 6      | 51             | 76  | 69  | 44  | 20  | 6   | 1   |     |     |     |     |     | 267                 |
| 7      | 127            | 196 | 189 | 133 | 70  | 27  | 7   | 1   |     |     |     |     | 750                 |
| 8      | 323            | 512 | 518 | 392 | 230 | 104 | 35  | 8   | 1   |     |     |     | 2123                |
| 9      | 835            | 1353 | 1422 | 1140 | 726 | 369 | 147 | 44  | 9   | 1   |     | 6046                |
| 10     | 2188           | 3610 | 3915 | 3288 | 2235 | 1242 | 560 | 200 | 54  | 10  | 1   | 17303               |

We find that the $n_P(L_y, d)$ are closely related to certain numbers enumerating random walks. To explain this, consider a random walk on the nonnegative integers such that in each step the walker moves by $+1$, $-1$, or 0 units. Denote $m(n, k)$ as the number of walks of length $n$ steps starting at 0 and ending at $k$. The case $k = 0$ describes the number of walks defined above that return to the origin after $n$ steps. This is given by the Motzkin number; $m(n, 0) = M_n$. Define the sum

$$S_n = \sum_{k=0}^{n} m(n, k) . \quad (3.2.12)$$

We find that the numbers $n_P(L_y, d)$ are precisely equal to these enumerations of random walks:

$$n_P(L_y, d) = m(L_y, d - 1) \quad \text{for} \quad 1 \leq d \leq L_y + 1 \quad (3.2.13)$$

The values of the $n_P(L_y, d)$ for $L_y = 1$ to $L_y = 10$ are listed in Table 1. (Note that $S_9$ should read 6046 in the related Table 3 of Ref. [25].)

From the identity (3.2.13), it follows that the total number of terms $N_{P,L_y,\lambda}$ is

$$N_{P,L_y,\lambda} = S_{L_y} . \quad (3.2.14)$$

These numbers are listed in Table 1.

We have determined a generating function whose expansion yields the numbers $N_{P,L_y,\lambda}$:

$$\frac{1}{2} \left[ \left( \frac{1 + x}{1 - 3x} \right)^{1/2} - 1 \right] - x = \sum_{L_y=1}^{\infty} N_{P,L_y,\lambda} x^{L_y+1} . \quad (3.2.15)$$
Combining this with our results from [27], we observe the interesting fact that the total numbers of terms \(N_{P,L_y,\lambda}\) for the chromatic polynomial of a \(G_D(L_y \times L_x)\) strip graph (for any \(L_x\)) is equal to 1/2 the corresponding total number of terms for the cyclic or Möbius strips of the square or triangular lattice with the next larger width, \(L_y + 1\), i.e.,

\[
N_{P,L_y,\lambda} = \frac{1}{2} N_{P,A,cyc/Mb,L_y+1,\Lambda = sq,tri}
\]  

(3.2.16)

Further, we observe the intriguing relation

\[
N_{P,L_y,\lambda} = N_{DA,sq,L_y+1}
\]

(3.2.17)

where \(N_{DA,sq,n}\) denotes the number of directed \(n\)-site lattice animals on the square lattice. We prove this by noting that the generating function (3.2.13) that we have found for \(N_{P,L_y,\lambda}\) is the same as the generating function for directed \(n\)-site lattice animals on the square lattice given in [62].

From (3.2.13), it follows that for large width \(L_y\), \(N_{P,L_y,\lambda}\) grows exponentially fast, with the leading asymptotic behavior:

\[
N_{P,L_y,\lambda} \sim L_y^{-1/2} 3^{L_y} \text{ as } L_y \to \infty.
\]

(3.2.18)

This is the same asymptotic behavior that we found in [25] for the total number of terms entering in the chromatic polynomials for the cyclic or Möbius strips of the square or triangular lattice of width \(L_y\).

### 3.3 Determination of \(n_Z(L_y,d)\) and \(N_{Z,L_y,\lambda}\)

From the same type of coloring matrix arguments used in [25], we have

\[
\sum_{d=1}^{L_y+1} n_Z(L_y,d) \kappa(d) = q^{L_y+1}.
\]

(3.3.1)

Differentiating eq. (3.3.1) \(L_y\) times we obtain \(L_y + 1\) linear equations in the \(L_y + 1\) unknowns \(n_Z(L_y,d)\) for \(1 \leq d \leq L_y + 1\). Solving these equations, we obtain the general results

\[
n_Z(L_y,d) = 0 \text{ for } d > L_y + 1
\]

(3.3.2)

\[
n_Z(L_y,L_y + 1) = 1
\]

(3.3.3)

and

\[
n_P(1,1) = 2
\]

(3.3.4)
with all other numbers $n_Z(L_y, d)$ being determined by the recursion relation

$$n_Z(L_y, d + 1) = n_Z(L_y, d - 1) + 2n_Z(L_y, d) + n_Z(L_y, d + 1) \quad \text{for} \quad 1 \leq d \leq L_y + 1 . \quad (3.3.5)$$

We solve this recursion relation with the conditions (3.3.2)-(3.3.4) in closed form and obtain

$$n_Z(L_y, d) = \frac{2d}{L_y + d + 1} \left( \frac{2L_y + 1}{L_y - d + 1} \right) . \quad (3.3.6)$$

Summing these numbers, we find, for the total number of terms

$$N_{Z,L_y,\lambda} = \left( \frac{2L_y + 1}{L_y + 1} \right) . \quad (3.3.7)$$

We observe that the total numbers of terms $N_{Z,L_y,\lambda}$ for the Potts model partition function on the $G_D(L_y \times L_x)$ strip graph (for any $L_x$) is equal to 1/2 the corresponding total number of terms for the cyclic or Möbius strips of the square or triangular lattice with the next larger width, $L_y + 1$, i.e.,

$$N_{Z,L_y,\lambda} = \frac{1}{2} N_{Z,A,cyc/Mb,L_y+1} \quad \Lambda = sq, tri . \quad (3.3.8)$$

There is also an interesting connection with directed lattice animals:

$$N_{Z,L_y,\lambda} = N_{DA,tri,L_y+1} \quad (3.3.9)$$

where $N_{DA,tri,n}$ denotes the number of directed $n$-site lattice animals on the triangular lattice.

As $L_y \to \infty$, $N_{Z,L_y,\lambda}$ has the leading asymptotic behavior

$$N_{Z,L_y,\lambda} \sim \pi^{-1/2} L_y^{-1/2} 4^{L_y} \quad \text{as} \quad L_y \to \infty . \quad (3.3.10)$$

The values of $n_Z(L_y, d)$ and $N_{Z,L_y,d}$ are given for $1 \leq L_y \leq 10$ in Table 2.

In what follows, we shall need to refer to our calculation of the numbers $n_Z(L_y, d)$ of $\lambda$’s with associated coefficient $c^{(d)}$ for cyclic strips $G_s$ of the square or triangular lattice of width $L_y$ (and arbitrarily $L_x$ in [25]). We found

$$n_Z(G_s, L_y, d) = \frac{2d + 1}{L_y + d + 1} \left( \frac{2L_y}{L_y - d} \right) \quad (3.3.11)$$

for $0 \leq d \leq L_y$ and zero otherwise. These values will be needed below and are listed in Table 3.
Table 2: Table of numbers \( n_Z(L_y, d) \) and their sums, \( N_{Z,L_y,\lambda} \) for \( G_D(L_y \times L_x) \). Blank entries are zero.

| \( L_y \downarrow \quad d \rightarrow \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | \( N_{P,L_y,\lambda} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | | | | | | | | | | 3 |
| 2 | 5 | 4 | 1 | | | | | | | | | 10 |
| 3 | 14 | 14 | 6 | 1 | | | | | | | | 35 |
| 4 | 42 | 48 | 27 | 8 | 1 | | | | | | | 126 |
| 5 | 132 | 165 | 110 | 44 | 10 | 1 | | | | | | 462 |
| 6 | 429 | 572 | 429 | 208 | 65 | 12 | 1 | | | | | 1716 |
| 7 | 1430 | 2002 | 1638 | 910 | 350 | 90 | 14 | 1 | | | | 6435 |
| 8 | 4862 | 7072 | 6188 | 3808 | 1700 | 544 | 119 | 16 | 1 | | | 24310 |
| 9 | 16796 | 25194 | 23256 | 15504 | 7752 | 2907 | 798 | 152 | 18 | 1 | | 92378 |
| 10 | 58786 | 90440 | 87210 | 62016 | 33915 | 14364 | 4655 | 1120 | 189 | 20 | 1 | 352716 |

Table 3: Table of numbers \( n_Z(L_y, d) \) and their sums, \( N_{Z,G,\lambda} \) for cyclic strips \( G_s(L_y \times L_x) \) of the square or triangular lattice of width \( L_y \) and arbitrary length \( L_x \) [23].

| \( L_y \downarrow \quad d \rightarrow \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | \( N_{Z,L_y,\lambda} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | | | 2 |
| 2 | 2 | 3 | 1 | | | | | | | | | 6 |
| 3 | 5 | 9 | 5 | 1 | | | | | | | | 20 |
| 4 | 14 | 28 | 20 | 7 | 1 | | | | | | | 70 |
| 5 | 42 | 90 | 75 | 35 | 9 | 1 | | | | | | 252 |
| 6 | 132 | 297 | 275 | 154 | 54 | 11 | 1 | | | | | 924 |
| 7 | 429 | 1001 | 1001 | 637 | 273 | 77 | 13 | 1 | | | | 3432 |
| 8 | 1430 | 3432 | 3640 | 2548 | 1260 | 440 | 104 | 15 | 1 | | | 12870 |
| 9 | 4862 | 11934 | 13260 | 9996 | 5508 | 2244 | 663 | 135 | 17 | 1 | | 48620 |
| 10 | 16796 | 41990 | 48450 | 38760 | 23256 | 10659 | 3705 | 950 | 170 | 19 | 1 | 184756 |
Table 4: Generic Bratteli diagram relevant for $TL_n(q)$.

| $n \downarrow m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|------------------|---|---|---|---|---|---|---|---|---|----|----|
| 1                | 1 |   |   |   |   |   |   |   |   |    |    |
| 2                | 1 | 1 |   |   |   |   |   |   |   |    |    |
| 3                | 2 | 1 |   |   |   |   |   |   |   |    |    |
| 4                | 2 | 3 | 1 |   |   |   |   |   |   |    |    |
| 5                | 5 | 4 | 1 |   |   |   |   |   |   |    |    |
| 6                | 5 | 9 | 5 | 1 |   |   |   |   |   |    |    |
| 7                | 14| 14| 6 | 1 |   |   |   |   |   |    |    |
| 8                | 14| 28| 20| 7 | 1 |   |   |   |   |    |    |
| 9                | 42| 48| 27| 8 | 1 |   |   |   |   |    |    |
| 10               | 42| 90| 75| 35| 9 | 1 |   |   |   |    |    |

4 Connection with Bratteli Diagrams for $TL_n(q)$

There are a number of interesting algebraic properties of structural elements and properties of the chromatic polynomials and Potts model partition functions equivalently Tutte polynomials. The interpretation of the coefficients $c_{P,G,s,j}$ in chromatic polynomials as dimensions of invariant subspaces of coloring matrices was discussed in [23, 24] and its generalization to the full Potts model partition function was discussed in [25]. Thus, $C_{P,G,s,L_y}$ and $C_{Z,G,s,L_y}$ are sums of these dimensions, a fact that we have used in [25] and eqs. (3.2.3) and (3.3.1) here.

In [25] we pointed out another connection, namely the fact that the numbers $n_Z(G_s, L_y, d)$ for $0 \leq d \leq L_y$ for cyclic strips $G_s$ of the square or triangular lattice of width $L_y$ were related to Temperley-Lieb algebras. We have found a very interesting generalization of this for the $n_Z(G_D, L_y, d)$ calculated here. Let us denote $TL_n(q)$ as the Temperley-Lieb algebra of operators $U_i$, $n = 1, ..., n$ satisfying [33, 64]

$$U_i^2 = q^{1/2}U_i$$  \hspace{1cm} (4.1)

$$[U_i, U_j] = 0 \text{ if } |i - j| \neq 1$$  \hspace{1cm} (4.2)

where $[A, B] = AB - BA$, and

$$U_iU_{i \pm 1}U_i = U_i$$  \hspace{1cm} (4.3)

Associated with the structural decomposition of this algebra $TL_n(q)$ there is what is known as a Bratteli diagram [35, 64, 65], given in Table 4.

We observe the following interesting properties:
• For even \( n = 2L_y \), the entries in this row of the Bratteli diagram are equal to the \( n_Z(L_y, d) \) for the cyclic strip of the square lattice of width \( L_y \) and arbitrary length \( L_x \) for \( m = 2d + 1 \) with \( d \) in the range \( 0 \leq d \leq L_y \), i.e., \( 1 \leq m \leq 2L_y + 1 = n + 1 \).

• For odd \( n = 2L_y + 1 \), the entries in this row of the Bratteli diagram are equal to the \( n_Z(L_y, d) \) for the strip \( G_D \) of width \( L_y \) and arbitrary length \( L_x \) for \( m = 2d \) with \( d \) in the range \( 1 \leq d \leq L_y + 1 \), i.e., \( 2 \leq m \leq 2(L_y + 1) = n + 1 \).

As this makes clear, the numbers \( n_Z(L_y, d) \) for the cyclic strip and the \( G_D \) strip can be expressed via a single formula. Define

\[
C_{n,m} = \binom{n-1}{m} - \binom{n-1}{m-2},
\]

so that the \( C_{n,m} \) satisfy the recursion relation

\[
C_{n,m} = C_{n-1,m-1} + C_{n-1,m+1}.
\]

For \( G_{cyc}(L_y \times L_x) \), let \( n = 2L_y \) and \( m = L_y - d \) with \( 0 \leq d \leq L_y \). For \( G_D(L_y \times L_x) \), let \( n = 2L_y + 1 \) and \( m = L_y + 1 - d \) with \( 1 \leq d \leq L_y + 1 \). Then with these substitutions, \( C_{n,m} \) reduces to the respective expressions (3.3.11) and (3.3.6).

Indeed, the fact that our results in [25] related \( n_Z(L_y, d) \) for the cyclic strips of the square and triangular lattices to the even-\( n \) rows of the Bratteli diagram, i.e., the rows for \( TL_{2L_y}(q) \), raised a question: is there a family of graphs with the property that the coefficients in the generic formula (2.1) are restricted to a simple set such that the number of \( \lambda \)'s with coefficients of a particular type fill out the odd-\( n \) rows of the Bratteli diagram. We had searched for such a family of graphs and had finally found it in the form of the \( G_D(L_y \times L_x) \) graphs, which have provided an affirmative answer to this question: the entries in the odd \( n = 2L_y + 1 \) rows are given by the \( n_Z(L_y, d) \), as described above. Thus, our current results are of interest both for the connection with Bratteli diagrams and for the fact that, together with our previous findings in [25], they saturate this connection, i.e., the entries in all of the Bratteli diagram are now accounted for in terms of the 1-1 correspondence with \( n_Z(L_y, d) \) numbers for these cyclic and \( G_D \) strip graphs.

We observe that the Bratteli diagram also has a combinatoric interpretation. Consider random walks on the non-negative integers that begin at the origin and have the property that the walker moves by +1 or −1 spaces at each step. The \((n,m)\) entry in the Bratteli diagram is the number of random walks of this type, consisting of \( n \) steps, that end at \( m-1 \). Note that the number of random walks of this type that consist of \( 2n \) steps (a misprint in [25] had this as \( n \) steps) is the Catalan number \( C_n \). The recursion relation (4.5) follows immediately from this.
5 Width $L_y = 1$

It is of interest to calculate Potts model partition functions and chromatic polynomials for these families of self-dual graphs. In this section we shall discuss the chromatic polynomial for the lowest case, $L_y = 1$, and in the next sections we shall present calculations of the chromatic polynomials for $2 \leq L_y \leq 4$ and shall analyze the zeros of these polynomials in the complex $q$ plane and their accumulation sets as $L_x \to \infty$. In a separate publication we shall present more lengthy calculations of the full Potts model partition functions.

We recall that the $L_y = 1$ family is comprised of the wheel graphs $(Wh)$ with $L_x$ vertices on the rim and a central vertex, with edges forming “spokes” connecting this central graph to the vertices on the rim. The $n$-vertex wheel graph can be constructed as the “join” $(Wh)_{n+1} = K_1 + C_n$, where $K_n$ and $C_n$ are the complete graph and the circuit graph on $n$ vertices, respectively. (The join of two graphs $G$ and $H$ is defined as the graph obtained by connecting each vertex of $G$ to every vertex of $H$ by edges. The complete graph $K_n$ is defined as the graph with $n$ vertices such that each vertex is adjacent to every other vertex.)

The chromatic polynomial has the form of (2.2) with $N_{P,L_y=1,\lambda} = 2$ and the terms

\[ \lambda_{1,1} = -1 \] (5.1)
\[ \lambda_{1,2} = q - 2 \] (5.2)

and coefficients

\[ c_{1,1} = \kappa^{(2)} \] (5.3)
\[ c_{1,2} = \kappa^{(1)} . \] (5.4)

Here and below for brevity we write $\lambda_{L_y,j}$ for $\lambda_{P,L_y,j}$ and $c_{L_y,j}$ for $c_{P,L_y,j}$. Thus,

\[ P(1 \times L_x, q) = (Wh)_{L_x+1} = q \left[ (q - 2)(-1)^{L_x} + (q - 2)^{L_x} \right] . \] (5.5)

The continuous locus $\mathcal{B}$ formed as the continuous accumulation set of chromatic zeros in the limit $L_x \to \infty$ limit, is the unit circle centered at $q = 2$ [8, 7]

\[ \mathcal{B} : |q - 2| = 1 \text{ for } L_y = 1 . \] (5.6)

Evidently, this intersects the real axis at $q = 1$ and $q = 3$. The locus $\mathcal{B}$ separates the $q$ plane into two regions, $R_1$ and $R_2$, which are, respectively, the exterior and the interior of the circle (5.6). In region $R_1$

\[ W = q - 2 \text{ for } q \in R_1 . \] (5.7)
In any other region than $R_1$, only the magnitude $|W|$ can be determined unambiguously \[6\], and we have

$$|W| = 1 \quad \text{for } q \in R_2 . \quad (5.8)$$

Except for the single discrete zero at $q = 2$, the chromatic zeros lie exactly on the circle $|q - 2| = 1$ and occur with a density that is a constant as a function of angular position on this circle \[7\].

6 Width $L_y = 2$

Our general formula (3.2.15) yields the result that there are $N_{P,L_y=2,\lambda} = 5$ terms for this width $L_y = 2$. The chromatic polynomial has the form (2.2) with the following terms, listed in order of descending degree of their associated coefficients.

$$\lambda_{2,1} = 1 \quad (6.1)$$

$$\lambda_{2,(2,3)} = \frac{1}{2} (5 - 2q \pm \sqrt{5}) \quad (6.2)$$

$$\lambda_{2,(4,5)} = \frac{1}{2} \left[ q^2 - 5q + 7 \pm (q^4 - 6q^3 + 15q^2 - 22q + 17)^{1/2} \right] \quad (6.3)$$

and the corresponding coefficients

$$c_{2,1} = \kappa^{(3)} \quad (6.4)$$

$$c_{2,j} = \kappa^{(2)} \quad \text{for } j = 2, 3 \quad (6.5)$$

$$c_{2,j} = \kappa^{(1)} \quad \text{for } j = 4, 5 . \quad (6.6)$$

The locus $B$ is shown in Fig. 2, together with chromatic zeros for a long finite strip. Evidently, the chromatic zeros (except the discrete zero at $q = 2$), which eventually merge to form this locus in the $L_x \to \infty$ limit, generally lie close to the curves on $B$. This locus divides the $q$ plane into three regions: (i) region $R_1$ including the intervals $q > 3$ and $q < 1$ on the real axis and extending outward to infinitely large $|q|$; (ii) the region $R_2$ including the real interval $5/2 < q < 3$, and (iii) the innermost region $R_3$ including the real interval $1 < q < 5/2$. There are triple points at which all three regions are contiguous at $q = q_t, q_t^*$, where $q_t \approx 5/2 + 1.27i$. Note that the part of the boundary $B$ separating regions $R_2$ and $R_3$ is the line segment $Re(q) = 5/2, -Im(q_t) < Im(q) < Im(q_t)$. In region $R_1$, $\lambda_{2,4}$ is dominant, so

$$W = (\lambda_{2,4})^{1/2} \quad \text{for } q \in R_1 \quad (6.7)$$

$$|W| = |\lambda_{2,3}|^{1/2} \quad \text{for } q \in R_2 \quad (6.8)$$

$$|W| = |\lambda_{2,2}|^{1/2} \quad \text{for } q \in R_3 . \quad (6.9)$$

As is evident in Fig. 3, the locus $B$ crosses the real $q$ axis at $q = 1$, $q = 5/2$, and $q = 3$. 

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Figure 2: Singular locus $B$ for the $L_x \to \infty$ limit of $G_D(2 \times L_x)$. For comparison, chromatic zeros are shown for $L_x = 30$ (i.e., $n = 61$).
\textbf{7 Width } $L_y = 3$

Here our general formula \([3.2.15]\) yields $N_{P,L_y=3,\lambda} = 13$. We have calculated the chromatic polynomial and obtain the following results, where again the $\lambda$’s are listed in order of descending degree of their coefficients. The first term is

\[ \lambda_{3,1} = -1 . \]  

(7.1)

Next, the $\lambda_{3,j}$ for $2 \leq j \leq 4$ are the roots of the equation

\[ \xi^3 - (3q - 8)\xi^2 + (3q^2 - 16q + 19)\xi - (q^3 - 8q^2 + 19q - 13) = 0 . \]  

(7.2)

The $\lambda_{3,j}$ for $5 \leq j \leq 9$ are the roots of the equation

\[ \xi^5 + (3q^2 - 16q + 23)\xi^4 + (3q^4 - 34q^3 + 142q^2 - 261q + 177)\xi^3 + (q^6 - 20q^5 + 153q^4 - 591q^3 + 1234q^2 - 1332q + 581)\xi^2 - (2q^7 - 35q^6 + 257q^5 - 1028q^4 + 2423q^3 - 3371q^2 + 2566q - 823)\xi + (q - 2)^2(q^6 - 14q^5 + 79q^4 - 230q^3 + 366q^2 - 304q + 103) = 0 . \]  

(7.3)

Finally, the $\lambda_{3,j}$ for $10 \leq j \leq 13$ are the roots of the equation

\[ \xi^4 - (q^3 - 8q^2 + 24q - 26)\xi^3 - (2q^5 - 23q^4 + 111q^3 - 279q^2 + 362q - 191)\xi^2 - (q - 3)(q^6 - 14q^5 + 81q^4 - 251q^3 + 441q^2 - 415q + 161)\xi + (q - 2)^3(q^5 - 11q^4 + 48q^3 - 104q^2 + 112q - 47) = 0 . \]  

(7.4)

The corresponding coefficients are

\[ c_{3,1} = \kappa^{(4)} \]  

(7.5)

\[ c_{3,j} = \kappa^{(3)} \text{ for } 2 \leq j \leq 4 \]  

(7.6)

\[ c_{3,j} = \kappa^{(2)} \text{ for } 5 \leq j \leq 9 \]  

(7.7)

\[ c_{3,j} = \kappa^{(1)} \text{ for } 10 \leq j \leq 13 . \]  

(7.8)

The locus $\mathcal{B}$ defined in the $L_x \to \infty$ limit is shown in Fig. 3, together with chromatic zeros for a long finite strip. This locus divides the $q$ plane several regions, including (i) $R_1$, which contains the real intervals $q < 1$ and $q > 3$ and extends outwards to infinite $|q|$; (ii)
Figure 3: Singular locus \( \mathcal{B} \) for the \( L_x \to \infty \) limit of \( G_D(3 \times L_x) \). For comparison, chromatic zeros are shown for \( L_x = 30 \) (i.e., \( n = 91 \)).
$R_2$, which contains the real interval from $q \approx 2.637$ to $q = 3$; (iii) $R_3$, which contains the real interval $1 < q < 2.637$; and (iv) the complex-conjugate pair of small regions $R_4, R_4^*$, centered approximately at $q \approx 0.44 \pm 1.67i$. The density of chromatic zeros varies in different parts of $\mathcal{B}$; this density is low on the portion of $\mathcal{B}$ that passes through $q = 1$ and on the inner sides of the small regions $R_4, R_4^*$. In $R_1$, the dominant term is a root of the quartic equation (7.4), which we denote $\lambda_{3,R_1}$, so that

$$W = (\lambda_{3,R_1})^{1/3} \quad \text{for } q \in R_1 . \quad (7.9)$$

In regions $R_2$ and $R_3$, the dominant terms are two different roots of the fifth-degree equation (7.3), and a root of this equation is also dominant in the complex-conjugate pairs of regions $R_4, R_4^*$. There are three complex-conjugate pairs of triple points on $\mathcal{B}$. Our experience with the $L_x \to \infty$ limits of other families of strip graphs showed that there can often be additional extremely small regions [13]; we have not made an exhaustive search for these.

8 $L_y = 4$

Here our general formula (3.2.13) gives $N_{P,4,\lambda} = 35$. We have

$$\lambda_{4,1} = 1 \quad (8.1)$$

with coefficient

$$c_{4,1} = \kappa^{(5)} \quad (8.2)$$

and

$$\lambda_{4,2} = 2 - q . \quad (8.3)$$

The $\lambda_{4,j}$ for $3 \leq j \leq 5$ are roots of the equation

$$\xi^3 + 3(q - 3)\xi^2 - 3(q - 2)(q - 4)\xi + (q^3 - 9q^2 + 24q - 17) = 0 . \quad (8.4)$$

These $\lambda_{4,j}$, $2 \leq j \leq 5$ have the coefficient

$$c_{4,j} = \kappa^{(4)} \quad \text{for } 2 \leq j \leq 5 . \quad (8.5)$$

The other 30 terms consist of roots of two equations of degree 9 and an equation of degree 12. Since these are rather lengthy, they are relegated to the appendix of the copy of this paper in the cond-mat archive. As before for the other $G_D$ strips studied in this paper, these equations are in 1-1 correspondence with the coefficients listed in Table 1, so that all of the roots of a given equation have the same coefficient, as specified in this table.
Figure 4: Singular locus $\mathcal{B}$ for the $L_x \to \infty$ limit of $G_D(4 \times L_x)$. For comparison, chromatic zeros are shown for $L_x = 20$ (i.e., $n = 81$).
The locus $\mathcal{B}$ defined in the $L_x \to \infty$ limit is shown in Fig. 4, together with chromatic zeros for a long finite strip. This locus $\mathcal{B}$ divides the $q$ plane into a number of regions. These include (i) $R_1$, which contains the real intervals $q < 1$ and $q > 3$ and extends outwards to infinite $|q|$ as before; (ii) $R_2$, which contains the real interval from $2.746 \lesssim q < 3$; (iii) a narrow region $R_3$, which contains the real interval $(1/2)(3 + \sqrt{5}) \lesssim q \lesssim 2.746$, and (iv) $R_4$, which contains the real interval $1 < q \lesssim (1/2)(3 + \sqrt{5})$; (v) the complex-conjugate pair of small regions $R_5$, $R_5^*$, centered approximately at $q \simeq \pm 1.4i$; and (vi) a complex-conjugate pair of tiny sliver regions $R_6$, $R_6^*$ located around $q \simeq 5/2 \pm 1.55i$. We note in passing that $(1/2)(3 + \sqrt{5}) = B_5 = 2.61803...$, where $B_r = 4 \cos^2(\pi/r)$ is the Tutte-Beraha number. The same caveat given above again applies; we have not made an exhaustive search for other even tinier sliver regions. The small regions $R_5$ and $R_5^*$ extend into the $\text{Re}(q) < 0$ half-plane. In region $R_1$ the dominant term is a root of the ninth-order equation with coefficient $\kappa^{(1)}$ which we denote $\lambda_{4,R1}$, so that $W = (\lambda_{4,R1})^{1/4}$. The density of chromatic zeros is lowest on the portions of $\mathcal{B}$ that pass through $q = 1$ and $q = B_5$ and is intermediate on the portion that passes through $q = 3$.

9 General Features

Our exact results exhibit a number of features:

- For a given width $L_y$, the $\lambda_{L_y,j}$ with coefficient $c_{L_y,j} = \kappa^{(d)}$, where $1 \leq d \leq L_y + 1$, of which there are $n_P(L_y,d)$, are roots of an algebraic equation of degree at most $n_P(L_y,d)$. For example, for $L_y = 3$, the $\lambda_{3,j}$ with coefficients $\kappa^{(d)}$, $d = 4,3,2,$ and 1 are, respectively, roots of equations of degree 1,3,5, and 4. This case illustrates the possibility that the degree of the corresponding equation is equal to $n_P(L_y,d)$. A case illustrating the possibility that it is less is, e.g., that for $L_y = 4$, where the $\lambda_{4,j}$ for $2 \leq j \leq 5$ with coefficients $\kappa^{(4)}$ are roots of a cubic equation and a linear equation.

- Our general structural analysis shows that for a given $L_y$, there is a unique term $\lambda_{L_y,1}$ with coefficient of maximal degree, and this coefficient is $\kappa^{(L_y+1)}$. From our calculations we infer the general result that $\lambda_{L_y,1} = (-1)^{L_y}$.

- Another inference concerns the set of terms $\lambda_{L_y,j}$ with coefficients $\kappa^{(L_y)}$. There are $L_y$ of these terms as shown in (3.2.8). Let $\tilde{\lambda}_{L_y,j} = (-1)^{L_y+1}\lambda_{L_y,j}$. Then the $\tilde{\lambda}_{L_y,j}$'s and hence the $\lambda_{L_y,j}$'s with coefficients $\kappa^{(L_y)}$ can be calculated as follows. Denote the equation whose solution is $\tilde{\lambda}_{L_y,j}$ as $f(L_y,\xi)$. Thus, $f(1,\xi) = \xi - (q - 2)$ and

$$f(2,\xi) = f(1,\xi)(\xi - q + 3) - 1 \quad (9.1)$$
The $\lambda_{L_y,j}$’s for higher values of $L_y$ are then given by

$$f(L_y, \xi) = f(L_y - 1, \xi)(\xi - q + 3) - f(L_y - 2, \xi) \quad \text{for} \quad L_y \geq 3.$$  \hspace{1cm} (9.2)

We find that (i) if $L_y = 1 \mod 3$, one of the $\lambda_{L_y,j}$’s with coefficient $\kappa^{(L_y)}$ is $\lambda_{L_y,j} = (-1)^{L_y+1}(q-2)$; (ii) if $L_y = 2 \mod 5$, two others are $(-1)^{L_y+1}$ times the roots of $f(2, \xi)$; (iii) if $L_y = 3 \mod 7$, three others are $(-1)^{L_y+1}$ times the roots of $f(3, \xi)$; and so forth.

- The term that is dominant in region $R_1$ is the (real) root of an equation with degree $n_p(L_y, 1) = M_{L_y}$ of maximal magnitude and has coefficient $\kappa^{(1)} = q$.

- The locus $B$ for the $L_x \to \infty$ limit of the family $G_D(L_y \times L_x)$ crosses the real axis and divides the $\xi$ plane into various regions. This is in accordance with our earlier conjecture that a sufficient (not necessary) condition for families of graphs to yield a $B$ that divides the $\xi$ plane into different regions is that it contain global circuits, equivalent to periodic longitudinal boundary conditions for strip graphs. (Recall that for strip graphs with free longitudinal boundary conditions, $B$ does not, in general, cross the real axis or divide the $\xi$ plane into regions containing this axis [37, 38].)

- In each case where we have an exact solution, we find that $B$ crosses the real axis at $q = 1$ and $q = 3$.

The finding that the locus $B$ crosses the real axis on the left at $q = 1$ is different than, but related to, our earlier finding [6, 43], [45]-[49], [52, 53] that for the $L_x \to \infty$ limits of lattice strip graphs with periodic longitudinal boundary conditions and either free or transverse transverse boundary conditions, $B$ crossed the real axis on the left at $q = 0$. For brevity of notation, we shall denote periodic and free longitudinal boundary conditions as PBC$_x$ and FBC$_x$, respectively, and similarly, periodic and free transverse boundary conditions as PBC$_y$ and FBC$_y$. The present behavior can be understood as a consequence of the property that all of the vertices on the upper side of the strip are connected to a single external vertex. This is especially evident for the lowest case $L_y = 1$; here, as discussed in [3, 7], it simply shifts the $B$ for the $L_x \to \infty$ limit of the circuit graph, $|q - 1| = 1$, one unit to the right in the $\xi$ plane to form the locus (5.6). Hence, in particular, the crossing that would be present at $q = 0$ for the circuit graph is shifted to a crossing at $q = 1$. This suggests that the crossings that are present at $q = 1$ for all of the cases for which we have exact solutions are universal in the same sense that the crossing on the left at $q = 0$ was universal for the strip graphs with PBC$_x$ and either FBC$_y$ or PBC$_y$.

One of the most interesting features of the present work is the property that for all of the cases that we have studied, the point at which $B$ crosses the real axis on the right, $q_c$,
is also the same for different strip widths $L_y$. This is different from our earlier findings with lattice strip graphs with PBC$_x$ and either FBC$_y$ or PBC$_y$. There, we found that the generic behavior was, for a given set of these boundary conditions, and a given type of lattice, that $q_c(L_y)$ depended on $L_y$. For example, for strips of the square lattice with PBC$_x$ and FBC$_x$, $q_c = 2$ for $L_y = 1$ and for $L_y = 2$ \[\text{[6]}\], while $q_c \simeq 2.34$ for $L_y = 3$ \[\text{[43]}\] and $q_c \simeq 2.49$ for $L_y = 4$ \[\text{[52]}\]. From these results, we conjectured that $q_c$ is a monotonically non-decreasing function of $L_y$ for lattice strips with PBC$_x$ and FBC$_y$ \[\text{[49]}\]. In contrast, we showed that $q_c$ was not a monotonic function of $L_y$ for lattice strips with PBC$_x$ and PBC$_y$. For example, $q_c = 2$ for $L_y = 2$ (here the PBC$_y$ and FBC$_y$ results coincide), and $q_c$ then increases to the value $q_c = 3$ for $L_y = 3$, but decreases to $q_c \simeq 2.78$ for the next greater width, $L_y = 4$ \[\text{[53]}\]. We inferred from these results that the crucial property (aside from having PBC$_x$ so that $B$ is guaranteed to cross the real axis and define a $q_c$) is the type of transverse boundary conditions. The transverse boundary conditions for the $G_D$ family of strip graphs are of neither pure free nor periodic type; the lower side of the strip has free boundary conditions while the vertices on the upper side are all connected to a single external vertex. We can characterize the strips with PBC$_x$ and the various transverse boundary conditions by the number of sides with free boundary conditions; this number is 2 for FBC$_y$, 1 for our present $G_D$ family, and 0 for PBC$_y$. It is therefore plausible that the above-mentioned conjecture that $q_c$ is a non-decreasing function of $L_y$ for (PBC$_x$,FBC$_y$) strips also holds for $G_D$ strips. Accepting the validity of this conjecture, one can immediately infer the universality of the property $q_c = 3$, independent of $L_y$, as follows. This property $q_c = 3$ holds for at least one value of $L_y$, say, $L_y = 1$. One also knows that in the limit $L_y \to \infty$, $q_c = 3$ for the (infinite) square lattice \[\text{[21]}\]. Therefore, since the conjecture states that $q_c$ cannot decrease as $L_y$ increases and since exact results show that it has already achieved its $L_y = \infty$ value at $L_y = 1$, it must remain at this value for all larger widths $L_y$.

As $L_y$ increases toward infinity, one would expect that, in a certain sense, the effect of the single external vertex connected to each of the vertices on the upper side of the strip would become negligible. It would follow from this reasoning that in the limit $L_y \to \infty$, the left-hand complex-conjugate arms of $B$ would curve around, approaching the point $q = 0$, and finally meet at this point to form a closed component on $B$. Our results are consistent with this conjecture; we find that for the two largest values of $L_y$, where these left-hand arms are present (each with its own small region), they do approach the origin more closely for $L_y = 4$ than for $L_y = 3$. 

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10 Comparisons of $W(L_y, q)$ for different $L_y$

As we have before for various infinite-length, finite lattice strips [40, 52, 53], it is of interest to compare our exact solutions for $W$ at a given $q$, for various widths $L_y$, with the corresponding value of $W$ for the infinite lattice, equivalent here to the limit $L_y \to \infty$. The range of $q$ values for which this comparison is useful is determined by the values for which there is nonzero ground state entropy, $W > 1$ associated with the property that the number of ways of coloring the vertices of the strip such that no two adjacent vertices have the same color increases exponentially fast as $W^n$ as the number of vertices $n \to \infty$ (cf. eq. 1.2). Here the limit $n \to \infty$ is taken by sending $L_x \to \infty$. Although the chromatic number $\chi$ alternates between 3 and 4 depending on whether $L_x$ is even or odd, one can take the limit of infinite length with $L_x$ even, and hence this sequence of families of graphs has $\chi = 3$, i.e., the above-mentioned proper coloring can be carried out for $q \geq 3$. We thus make the comparison of the values of $W(q)$ for various widths with the values of $W(sq, q)$ for the infinite square lattice for this range $q \geq 3$. Except for $W(sq, q = 3) = (4/3)^{3/2}$ [21], the values $W(sq, q)$ are not known exactly; we use the Monte Carlo calculations that were performed as part of [33] for this purpose. For the comparison, as before, we define the ratios

$$R_W(q) = \frac{W(G_D(L_y \times \infty, q))}{W(sq, q)}.$$

(10.1)

The results are shown in Table 5.
Table 5: Comparison of values of $W(G_D(L_y \times \infty), q)$ with $W(sq, q)$ for $3 \leq q \leq 10$. For each value of $q$, the quantities in the upper line are identified at the top and the quantities in the lower line are the values of $Rw(L_y, q)$.

| $q$ | $W(1, q)$ | $W(2, q)$ | $W(3, q)$ | $W(4, q)$ | $W(sq, q)$ |
|-----|-----------|-----------|-----------|-----------|-----------|
| 3   |           |           |           |           | $1.53960..$ |
|     | 1         | 1.27202   | 1.36138   | 1.40596   |           |
|     | 0.6495    | 0.8262    | 0.8842    | 0.9132    | 1         |
| 4   |           |           |           |           |           |
|     | 2         | 2.16831   | 2.22345   | 2.25117   | 2.3370(7) |
|     | 0.8558    | 0.9278    | 0.9514    | 0.9632    | 1         |
| 5   |           |           |           |           |           |
|     | 3         | 3.12490   | 3.16628   | 3.18711   | 3.2510(10) |
|     | 0.9228    | 0.9612    | 0.9739    | 0.9803    | 1         |
| 6   |           |           |           |           |           |
|     | 4         | 4.09973   | 4.13294   | 4.14963   | 4.2003(12) |
|     | 0.9523    | 0.9761    | 0.9840    | 0.9879    | 1         |
| 7   |           |           |           |           |           |
|     | 5         | 5.08310   | 5.11082   | 5.12473   | 5.1669(15) |
|     | 0.9677    | 0.9838    | 0.9891    | 0.9918    | 1         |
| 8   |           |           |           |           |           |
|     | 6         | 6.07124   | 6.09503   | 6.10695   | 6.1431(20) |
|     | 0.9767    | 0.9883    | 0.9922    | 0.9941    | 1         |
| 9   |           |           |           |           |           |
|     | 7         | 7.06236   | 7.08318   | 7.09361   | 7.1254(22) |
|     | 0.9824    | 0.9911    | 0.9941    | 0.9955    | 1         |
| 10  |           |           |           |           |           |
|     | 8         | 8.05545   | 8.07396   | 8.08323   | 8.1122(25) |
|     | 0.9862    | 0.9930    | 0.9953    | 0.9964    | 1         |
The property that all of the vertices on the upper surface of the strip, as represented in Fig. [1] are connected to an external vertex has the effect of restricting the coloring of the strip and thereby reducing the value of $W(G_D(L_y \times \infty), q)$ relative to the value for a strip of the square lattice with the same periodic longitudinal boundary conditions, width, and $q$, but free transverse boundary conditions, $W(G_s(L_y \times \infty), FBC_y, PBC_x, q)$. In [40] a theorem was proved that for fixed $q$, $W(G_s(L_y \times \infty), FBC_y, PBC_x, q)$ is a monotonically decreasing function of $L_y$ which thus approaches the value of $W(sq, q)$ for the infinite square lattice from above. Here, because of the effect of the coloring restriction due to the connections of all of the vertices on the upper surface to the single external vertex, the situation is different. Indeed, we find numerically that (for $q > q_c$) over the range $1 \leq L_y \leq 4$ where we have obtained exact solutions, $W(G_D(L_y \times \infty), q)$ is an increasing function of $L_y$ for fixed $q$ in the range $q \geq q_c$ where . There are two effects depending on the strip width $L_y$ that one can identify here, and these act in opposite directions. The first is that as the width increases, the restriction, per site, due to the connections of the vertices on the upper side to the external vertex is ameliorated, since these vertices occupy a smaller fraction of the total number of vertices as $L_y$ increases. This is the dominant effect and tends to increase $W$. The second is that on the other (lower) side of the strip there is more freedom in coloring the vertices since they have no neighbors below them. These lower-side vertices occupy a smaller fraction of the total as $L_y$ increases, and this tends to decrease $W$. It may be recalled that the proof given in [40] for the monotonic decrease of $W(G_s(L_y \times \infty), q)$ for fixed $q \geq q_c$ as a function of $L_y$ for the case of free transverse boundary conditions relied upon this second effect. In that case, the vertices on both the upper and lower sides of the strip, which had greater freedom of coloring because of their reduced degree (number of neighboring vertices) occupied a commensurately smaller fraction of the total number of vertices as the strip width increased, and this yielded the result of the theorem. Here, however, one has a more complicated situation in which there are the two above-mentioned countervailing effects.

As in previous works, we also record the value of $W$ at certain other points in region $R_1$, in particular, $q = 0$ and $q = 1$; these are given in Table 6. Here we recall the order of limits used in our definition of $W$, (1.3). With the opposite order of limits, one has $W(q)_{nq} = 0$ for $q = 0, 1$, and 2, for any $L_y$. 

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Table 6: Some additional values of values of $|W(G_D(L_y \times \infty), q)|$.

| $q$ | $W(1, q)$ | $W(2, q)$ | $W(3, q)$ | $W(4, q)$ |
|-----|-----------|-----------|-----------|-----------|
| 0   | 2         | 2.35829   | 2.50361   | 2.58283   |
| 1   | 1         | 1.61803   | 1.85334   | 1.97672   |

11 Conclusions

Summarizing, in this paper we have presented a number of exact results for the partition function $Z(G_D, q, v)$ of the $q$-state Potts model and the $T = 0$ antiferromagnetic special case given by the chromatic polynomial $P(G_D, q)$, on families of self-dual strip graphs $G_D$ of the square lattice of width $L_y$ and arbitrarily great length $L_x$ with periodic longitudinal boundary conditions. We determined (i) the general coefficients $c_{Z,L_y,j}$ and $c_{P,L_y,d}$ in terms of Chebyshev polynomials, (ii) the numbers $n_Z(L_y, d)$ and $n_P(L_y, d)$ of terms in $Z$ and $P$ with each type of coefficient, and (iii) the total number of terms $N_{Z,L_y,\lambda}$ and $N_{P,L_y,\lambda}$. We have pointed out interesting connections between the $n_Z(L_y, d)$ and Temperley-Lieb algebras and between the $N_{F,L_y,\lambda}$, $F = Z, P$, and enumerations of directed lattice animals. We proceeded to present exact calculations of $P$ for $2 \leq L_y \leq 4$ and to study the analytic structure of the resultant $W$ functions in the complex $q$ plane. A particularly interesting finding is a universal value, $q_c = 3$, for all of the widths $L_y$ considered, which, furthermore, is equal to the value for the infinite square lattice. This is the first family of finite-width, infinite-length lattice strips for which we have found this type of universality of $q_c$.

Acknowledgments

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12 Appendix

In the text, we gave the $\lambda_{4,j}$ or the equations defining them for $1 \leq j \leq 5$ for the $L_y = 4$ strip. In this appendix we give the equations for the terms $\lambda_{4,j}$ for $6 \leq j \leq 35$. The $\lambda_{4,j}$, $6 \leq j \leq 14$, are roots of a 9-degree equation

$$
\xi^9 - (q^4 - 11q^3 + 51q^2 - 114q + 101)\xi^8
$$

$$
-(3q^7 - 51q^6 + 392q^5 - 1756q^4 + 4925q^3 - 8597q^2 + 8586q - 3750)\xi^7
$$
\[-(3q^{10} - 77q^9 + 899q^8 - 6307q^7 + 29495q^6 - 96135q^5 + 221090q^4 - 353821q^3
+376297q^2 - 239411q + 68889)\xi^6
\]
\[-(q^{13} - 40q^{12} + 701q^{11} - 7302q^{10} + 51090q^9 - 255451q^8 + 943367q^7 - 2612569q^6
+5434452q^5 - 8391440q^4 + 9348370q^3 - 7109828q^2 + 3304026q - 707059)\xi^5
\]
\[+(4q^{15} - 149q^{14} + 2590q^{13} - 27903q^{12} + 208559q^{11} - 1146435q^{10} + 4790250q^9 - 15497328q^8 + 39142459q^7 - 77176435q^6 + 117779005q^5 - 136551251q^4
+116336276q^3 - 68690109q^2 + 25102423q - 4273553)\xi^4
\]
\[-(q - 2)(6q^{16} - 226q^{15} + 4004q^{14} - 44291q^{13} + 342389q^{12} - 1961444q^{11}
+8613782q^{10} - 29579506q^9 + 80262537q^8 - 172631424q^7 + 293248190q^6
-389119576q^5 + 395172545q^4 - 296710120q^3 + 155196867q^2 - 50470243q + 7677301)\xi^3
\]
\[+(q - 2)(4q^{18} - 167q^{17} + 3293q^{16} - 40754q^{15} + 354781q^{14} - 2307336q^{13}
+11619216q^{12} - 46328775q^{11} + 148278798q^{10} - 383788784q^9 + 805173797q^8
-1365777711q^7 + 1858809772q^6 - 2001697759q^5 + 1667438591q^4
-1036284383q^3 + 452215858q^2 - 123566922q + 15898678)\xi^2
\]
\[-(q - 2)^2(q^{19} - 46q^{18} + 994q^{17} - 13427q^{16} + 127231q^{15} - 899290q^{14}
+4920180q^{13} - 21334829q^{12} + 74438607q^{11} - 210900517q^{10} + 487345238q^9
-918665623q^8 + 1407005787q^7 - 1735609937q^6 + 1699139075q^5
-1289913357q^4 + 732092287q^3 - 292275110q^2 + 73185494q - 8642781)\xi
\]
\[-(q - 2)^7 (q^{15} - 34q^{14} + 536q^{13} - 5198q^{12} + 34686q^{11} - 168742q^{10} \\
+ 618410q^9 - 1738952q^8 + 3783595q^7 - 6370774q^6 + 8234267q^5 - 8022902q^4 \\
+ 5703476q^3 - 2792146q^2 + 841318q - 117542) = 0. \tag{12.1}
\]

The \(\lambda_{4, j}, 15 \leq j \leq 26,\) are roots of a 12-degree equation

\[
\xi^{12} + (4q^3 - 33q^2 + 98q - 103)\xi^{11} \\
+ (6q^6 - 105q^5 + 771q^4 - 3057q^3 + 6914q^2 - 8454q + 4350)\xi^{10} \\
+ (4q^9 - 117q^8 + 1470q^7 - 10589q^6 + 48615q^5 - 148230q^4 + 300934q^3 \\
- 392705q^2 + 298868q - 100913)\xi^9 \\
+ (q^{12} - 51q^{11} + 1034q^{10} - 11831q^9 + 87645q^8 - 449620q^7 + 1652070q^6 \\
- 4404021q^5 + 8481267q^4 - 11530332q^3 + 10515134q^2 - 5776925q + 1445103)\xi^8 \\
- (6q^{14} - 252q^{13} + 4813q^{12} - 55790q^{11} + 440374q^{10} - 2511072q^9 + 10686844q^8 \\
- 34528528q^7 + 85175369q^6 - 159716148q^5 + 224144839q^4 - 228278980q^3 \\
+ 159441815q^2 - 68321288q + 13538233)\xi^7 \\
+ (15q^{16} - 637q^{15} + 12667q^{14} - 156625q^{13} + 1348190q^{12} - 8567954q^{11} \\
+ 41590251q^{10} - 157306151q^9 + 468514489q^8 - 1102386000q^7 + 2042068500q^6 \\
- 2946106890q^5 + 3244203521q^4 - 2634966875q^3 + 1487907358q^2 \\
- 521546134q + 85418708)\xi^6 \\
- (20q^{18} - 910q^{17} + 19575q^{16} - 264460q^{15} + 2514176q^{14} - 17858194q^{13} \\
+ 98204545q^{12} - 427439488q^{11} + 1492563991q^{10} - 4211869748q^9 + 9626229067q^8
\]
\[-17773020220q^7 + 26306043996q^6 - 30781790215q^5 + 27840711649q^4 \\
-1877330488q^3 + 8883458702q^2 - 2630912469q + 366761469)\xi^5 \\
+(15q^{20} - 749q^{19} + 17749q^{18} - 265398q^{17} + 2808406q^{16} - 22355292q^{15} \\
+138893776q^{14} - 689698353q^{13} + 2779911180q^{12} - 9184226013q^{11} \\
+25005667770q^{10} - 56201345479q^9 + 104079239038q^8 - 157931128836q^7 \\
+194413953469q^6 - 191128460116q^5 + 146506854315q^4 - 84366457483q^3 \\
+34322906908q^2 - 8792343237q + 1066038269)\xi^4 \\
-(q - 2)(6q^{21} - 324q^{20} + 8279q^{19} - 133226q^{18} + 1515528q^{17} - 12965452q^{16} \\
+86626388q^{15} - 463261467q^{14} + 2015823729q^{13} - 7215111460q^{12} \\
+2138413741q^{11} - 52649552494q^{10} + 107694098636q^9 - 182459513387q^8 \\
+254392095608q^7 - 288780678892q^6 + 262603966158q^5 - 186683744725q^4 \\
+99924573125q^3 - 37861221344q^2 + 9048442752q - 1025103882)\xi^3 \\
+(q - 2)(q^{23} - 65q^{22} + 1965q^{21} - 36996q^{20} + 488991q^{19} - 4842151q^{18} \\
+37397267q^{17} - 231374113q^{16} + 1168070271q^{15} - 4873964424q^{14} \\
+16957771126q^{13} - 49471796934q^{12} + 121369147461q^{11} - 250500702268q^{10} \\
+43409394035q^9 - 628647524990q^8 + 754930635327q^7 - 743026096643q^6 \\
+589226915539q^5 - 36716228065q^4 + 173056035863q^3 - 57980197360q^2 \\
+12299658707q - 1241302954)\xi^2 \]
\[(q - 2)^2(q - 3)(2q^{22} - 106q^{21} + 2662q^{20} - 42156q^{19} + 472587q^{18} - 3990653q^{17} + 26367347q^{16} - 139766257q^{15} + 604533952q^{14} - 2158390643q^{13} + 6409094853q^{12} - 15896051134q^{11} + 32980281410q^{10} - 57170276403q^9 + 82469188225q^8 - 98281609520q^7 + 95679352146q^6 - 74828758316q^5 + 45863717844q^4 - 21212341252q^3 + 6958799896q^2 - 1442606958q + 142022490)\xi
+(q - 2)^4(q^2 - 5q + 5)(q^{20} - 46q^{19} + 999q^{18} - 13621q^{17} + 130785q^{16} - 940159q^{15} + 5250952q^{14} - 23336758q^{13} + 83832989q^{12} - 245863743q^{11} + 591990845q^{10} - 1172460575q^9 + 1906935986q^8 - 253336002q^7 + 2722446125q^6 - 2330163861q^5 + 1551214058q^4 - 774015177q^3 + 27228617q^2 - 60200487q + 6289073) = 0 .
\] (12.2)

Finally, the \(\lambda_{4,j}, 27 \leq j \leq 35\), are roots of another 9-degree equation
\[
\xi^9 - (6q^2 - 33q + 48)\xi^8 + (15q^4 - 168q^3 + 707q^2 - 1327q + 933)\xi^7
- (20q^6 - 345q^5 + 2450q^4 - 9190q^3 + 19218q^2 - 21235q + 9659)\xi^6
+ (15q^8 - 360q^7 + 3693q^6 - 21232q^5 + 75012q^4 - 167016q^3 + 229025q^2
- 176806q + 58742)\xi^5
- (6q^{10} - 195q^9 + 2732q^8 - 21931q^7 + 112366q^6 - 385466q^5 + 899022q^4
- 1410181q^3 + 1425238q^2 - 838361q + 217832)\xi^4
\]
\[(q - 3)(q^{11} - 45q^{10} + 806q^9 - 7979q^8 + 49672q^7 - 206895q^6 + 593204q^5 \\
-1177217q^4 + 1590681q^3 - 1397396q^2 + 719432q - 164543)\xi^3 \]
\[+(q - 2)(3q^{12} - 108q^{11} + 1726q^{10} - 16260q^9 + 100858q^8 - 434873q^7 + 1338650q^6 \\
-2967964q^5 + 4708729q^4 - 5217472q^3 + 3834660q^2 - 1678782q + 330985)\xi^2 \]
\[+(q - 2)(q - 3)(3q^{12} - 93q^{11} + 1295q^{10} - 10716q^9 + 58721q^8 - 224620q^7 \\
+615435q^6 - 1217860q^5 + 1728820q^4 - 1718182q^3 + 1135486q^2 - 448175q + 79896)\xi \]
\[+(q - 2)^2(q^{13} - 32q^{12} + 465q^{11} - 4062q^{10} + 23798q^9 - 98757q^8 \\
+298728q^7 - 667187q^6 + 1100852q^5 - 1326203q^4 + 1134621q^3 \\
-653205q^2 + 226903q - 35923) = 0 \]  

(12.3)

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