Dynamical r-matrices and Separation of Variables: The Generalised Calogero-Moser Model

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ABSTRACT A generalisation of the classical Calogero-Moser model obtained by coupling it to the Gaudin model is considered. The recently found classical dynamical r-matrix [E. Billey, J. Avan and O. Babelon, PAR LPTHE 93-55] for the Euler-Calogero-Moser model is used to separate variables for this generalised Calogero-Moser model in the case in which there are two Calogero-Moser particles. The model is then canonically quantised and the same classical r-matrix is employed to separate variables in the Schrödinger equations.

1 Introduction

Separation of variables in the Hamilton-Jacobi equation is one of the methods of solving completely integrable models of classical mechanics (see e.g. [1]). If the model admits a Lax pair representation and hence is described by a classical Yang-Baxter algebra [2], the separation of variables can be achieved by the functional Bethe Ansatz [3] [4]. By a classical Yang-Baxter algebra we mean an infinite-dimensional Poisson bracket algebra given by an $N \times N$ matrix $L(\lambda)$, of which components are dynamical variables, and an $N^2 \times N^2$ matrix $r_{12}(\lambda, \mu)$, which may depend on dynamical variables [5] [6] [7] [8] [9] and is such that

$$\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\mu, \lambda), L_2(\mu)].$$

1 Benefactors’ Scholar of St. John’s College, Cambridge
Here $\lambda, \mu$ are free parameters, $L_1(\lambda) = L(\lambda) \otimes I$ and $L_2(\mu) = I \otimes L(\mu)$, and $r_{21}(\lambda, \mu) = Pr_{12}(\lambda, \mu)P$, where $P$ is an $N^2 \times N^2$ permutation matrix. $L(\lambda)$ is called a Lax matrix and $r_{12}(\lambda, \mu)$ is called an r-matrix. Matrix $r_{12}(\lambda, \mu)$ has to satisfy a relation necessary to ensure that the Poisson bracket (1) obey the Jacobi identity. This relation can take a form of either the classical Yang-Baxter equation (see e.g. [10]) or its dynamical generalisations [2] [8] [11] if $r_{12}(\lambda, \mu)$ depends on dynamical variables. We deal with this last case in the present paper.

Once the classical dynamical system is written in the form (1), we can define a generating function for integrals of motion,

$$t(\lambda) = \frac{1}{2} \text{tr} L^2(\lambda).$$

From (1) it follows that $\{t(\lambda), t(\mu)\} = 0$, hence the integrals generated by $t(\lambda)$ are in involution and the system is integrable. The main goal of the functional Bethe Ansatz is to separate the variables in the Hamilton-Jacobi equations for the integrals generated by $t(\lambda)$.

For models with $sl(2)$ symmetry the functional Bethe Ansatz can be described as follows. We begin with a Lax matrix

$$L(\lambda) = \begin{pmatrix} A(\lambda) & C(\lambda) \\ B(\lambda) & -A(\lambda) \end{pmatrix}$$

and, for simplicity, we assume the following form of a classical r-matrix

$$r_{12}(\lambda, \mu) = \begin{pmatrix} a(\lambda, \mu) & 0 & 0 & 0 \\ 0 & 0 & b(\lambda, \mu) & 0 \\ 0 & c(\lambda, \mu) & 0 & 0 \\ 0 & 0 & 0 & a(\lambda, \mu) \end{pmatrix},$$

for some functions $a(\lambda, \mu)$, $b(\lambda, \mu)$ and $c(\lambda, \mu)$. Functions $a(\lambda, \mu)$, $b(\lambda, \mu)$ and $c(\lambda, \mu)$ satisfy relations necessary to ensure that the Poisson bracket (1) obey the Jacobi identity. The r-matrices of this type appear in several integrable models such as homogeneous spin chains, heavy tops and systems on Riemannian manifolds of constant curvature. For any Lax matrix (3) and any r-matrix (4) the Poisson bracket algebra (1) takes the form

$$\{A(\lambda), B(\mu)\} = b(\lambda, \mu)B(\lambda) + a(\mu, \lambda)B(\mu),$$

$$\{A(\lambda), C(\mu)\} = -c(\lambda, \mu)C(\lambda) - a(\mu, \lambda)C(\mu),$$

$$\{B(\lambda), C(\mu)\} = 2c(\lambda, \mu)A(\lambda) + 2b(\mu, \lambda)A(\mu),$$

$$\{C(\lambda), C(\mu)\} = 0.$$
and zero for the remaining brackets. The separation variables are defined by
\[ B(X_i) = 0, \quad P_i = A(X_i). \] (6)

Using the algebra (5) we find
\[ \{X_i, X_j\} = \{P_i, P_j\} = 0, \quad \{X_i, P_j\} = \delta_{ij} \lim_{\lambda \to X_i} \frac{a(\lambda, X_i) B(\lambda)}{B'(X_i)}, \]
so that a sufficient condition for \((X_i, P_i)\) to be canonical variables reads
\[ \lim_{\lambda \to X_i} \frac{a(\lambda, X_i) B(\lambda)}{B'(X_i)} = 1, \]
or equivalently
\[ a(X_i + h, X_i) = \frac{1}{h} + O(h). \] (7)

We stress that condition (7) remains the same for systems described by non-dynamical and dynamical r-matrices. This in particular allows one to apply the functional Bethe Ansatz procedure to the latter, as discovered in [11]. This observation seems to be of great importance and interest, since there are integrable models, such as the Calogero-Moser model, for which the non-dynamical r-matrices do not exist. The models of this kind have recently attracted much attention and the theory of dynamical r-matrices requires closer investigation [8].

Having defined the separation variables we easily find the separated equations
\[ P_i^2 = t(X_i), \] (8)
which follow directly from (6).

In this paper we employ this procedure of separation of variables to solve the generalised Calogero-Moser model. In order to do so we use the recently introduced r-matrix [9] which depends on dynamical variables. In the next section we describe a generalisation of the Calogero-Moser model which is achieved by coupling the Gaudin model to it. Then we specialise to the case in which there are two Calogero-Moser particles. In this case the system has \(sl(2)\) symmetry and is governed by an r-matrix of the form [10] which in addition satisfies condition (7), so that the separation of variables procedure can be applied. We do it in Section 4. In Section 5 we quantise the model and use the quantum version of the functional Bethe Ansatz to separate variables in the Schrödinger equation [13]. The quantum functional Bethe Ansatz we use is obtained by the canonical quantisation of the classical procedure just described.
2 Description of the Model

In [9] Billey, Avan and Babelon proposed the parameter dependent classical dynamical r-matrix for the generalisation of the Calogero-Moser model constructed by Gibbons and Hermsen [14]. The model is governed by the Hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - \frac{1}{2} \sum_{i,j=1}^{N} \frac{f_{ij} f_{ji}}{(q_i - q_j)^2}, \]  

(9)

where the dynamical variables \((q_i, p_i)_{i=1,...,N}\), \((f_{ij})_{i,j=1,...,N}\) satisfy the Poisson bracket algebra

\[ \{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}. \]  

(10)

The model is integrable, when restricted to the surfaces \((f_{ii} = \text{const})_{i=1,...,N}\). In this section we propose a further generalisation of (9) which is achieved by introducing a non-trivial internal structure to the dynamical variables \((f_{ij})_{i,j=1,...,N}\). We consider a system governed by the Hamiltonian (9) but with the dynamical variables \((q_i, p_i)_{i=1,...,N}\), \((f_{ij}^\alpha)_{i,j=1,...,N}^{\alpha=1,...,M}\) which satisfy

\[ \{f_{ij}^\alpha, f_{kl}^\beta\} = \delta_{\alpha\beta}(\delta_{jk} f_{il}^\alpha - \delta_{il} f_{kj}^\alpha) \]  

(11)

and all but the last of relations (10). The dynamical variables \((f_{ij})_{i,j=1,...,N}\) are now defined by \(f_{ij} = \sum_{\alpha=1}^{M} f_{ij}^\alpha\). To see that the model (9)-(11) is integrable when \((f_{ii} = \text{const})_{i=1,...,N}\) we construct the corresponding Lax matrix

\[ L(\lambda) = \sum_{i=1}^{N} \left( p_i + \sum_{\alpha=1}^{M} f_{ii}^\alpha \frac{1}{\lambda - \epsilon_\alpha} \right) e_{ii} + \sum_{i,j=1}^{N} \left( \frac{f_{ij}}{q_i - q_j} + \sum_{\alpha=1}^{M} f_{ij}^\alpha \frac{1}{\lambda - \epsilon_\alpha} \right) e_{ij}, \]  

(12)

where \((e_{ij})_{kl} = \delta_{ik}\delta_{jl}\) and \(\epsilon_1 > \epsilon_2 > \cdots > \epsilon_M\) are arbitrary parameters. This form of \(L(\lambda)\) immediately reveals that the generalisation of the Calogero-Moser model we discuss here is achieved by the coupling of the \(M\)-particle Gaudin model [15] to the Calogero-Moser system. Using equations (10) and (11) we can write the Poisson brackets of the components of the Lax matrix (12) in the unified form [4]

\[ \{L_1(\lambda), L_2(\mu)\} = \left[ r_{12}(\lambda, \mu), L_1(\lambda) + L_2(\mu) \right] - \sum_{i,j=1}^{N} \frac{f_{ii} - f_{jj}}{(q_i - q_j)^2} e_{ij} \otimes e_{ji}. \]  

(13)

The r-matrix here is the dynamical one considered in [9]

\[ r_{12}(\lambda, \mu) = \sum_{i,j=1}^{N} \left( \frac{1}{\lambda - \mu} + \frac{1}{q_i - q_j} \right) e_{ij} \otimes e_{ji} + \frac{1}{\lambda - \mu} \sum_{i} e_{ii} \otimes e_{ii}. \]  

(14)
We note that \( r_{12}(\lambda, \mu) = -r_{21}(\mu, \lambda) \). From (13) we immediately learn that the model (9) is integrable provided \( f_{ii} = f_{jj}, \ i, j = 1, \ldots, N \), since in this case the Poisson bracket algebra (13) takes the form (1). In particular, the model is integrable on the surfaces \( (f_{ii} = \text{const})_{i=1,\ldots, N} \). This final reduction is possible, because each of \( f_{ii} \) Poisson commutes with the Hamiltonian \( H \). The Hamiltonian \( H \) is recovered as \( H = \frac{1}{2} \int_C \frac{\mathrm{d}X}{2\pi i} \text{tr} L^2(\lambda) \), where \( \lambda \) is considered as a complex variable and the contour \( C \) encloses the origin.

3 The N=2 Case

Now we focus on the \( N = 2 \) case for which we write a complete set of integrals of motion. In the next section we will use the functional Bethe Ansatz procedure to separate the variables in the Hamilton-Jacobi equations corresponding to these integrals of motion.

In the centre of mass frame, the Lax matrix in this case has the form (9), with

\[
A(\lambda) = p + \sum_{\alpha=1}^{M} \frac{S_3^\alpha}{\lambda - \epsilon_\alpha}, \quad B(\lambda) = -\frac{S_-}{q} + \sum_{\alpha=1}^{M} \frac{S_-^\alpha}{\lambda - \epsilon_\alpha}, \quad C(\lambda) = \frac{S_+}{q} + \sum_{\alpha=1}^{M} \frac{S_+^\alpha}{\lambda - \epsilon_\alpha}
\]

where \( S_3^\alpha, S_\pm^\alpha \) are generators of the so(2, 1) Poisson algebra,

\[
\{ S_3^\alpha, S_\pm^\beta \} = \pm \delta_{\alpha\beta} S_\pm^\beta, \quad \{ S_+^\alpha, S_-^\beta \} = 2 \delta_{\alpha\beta} S_3^\beta
\]

and \( (q, p) \) are the relative coordinates, \( \{ q, p \} = 1 \), \( q = q_1 - q_2 \), \( p = \frac{1}{2}(p_1 - p_2) \). The matrix (15) is obtained from (12) by subtracting the centre of mass motion. The non-zero Poisson brackets of the components of the Lax matrix (15) take the explicit form

\[
\begin{align*}
\{ A(\lambda), B(\mu) \} &= \frac{B(\lambda) - B(\mu)}{\lambda - \mu} + \frac{B(\lambda)}{q}, \\
\{ A(\lambda), C(\mu) \} &= -\frac{C(\lambda) - C(\mu)}{\lambda - \mu} + \frac{C(\lambda)}{q}, \\
\{ B(\lambda), C(\mu) \} &= 2 \left( \frac{A(\lambda) - A(\mu)}{\lambda - \mu} - \frac{A(\lambda) - A(\mu)}{q} + \frac{S_3}{q^2} \right).
\end{align*}
\]

The Poisson algebra (16) can be written in compact form (1) with \( r_{12}(\lambda, \mu) \) of the form (4), where

\[
a(\lambda, \mu) = \frac{1}{\lambda - \mu}, \quad b(\lambda, \mu) = \frac{1}{\lambda - \mu} + \frac{1}{q}, \quad c(\lambda, \mu) = \frac{1}{\lambda - \mu} - \frac{1}{q},
\]

provided that \( S_3 = 0 \). This follows immediately from (13). Writing explicitly the generating function (2),

\[
t(\lambda) = p^2 - \frac{S_- S_+}{q^2} + \sum_{\alpha, \beta=1}^{M} \frac{S_3^\alpha S_3^\beta + S_+^\alpha S_+^\beta}{(\lambda - \epsilon_\alpha)(\lambda - \epsilon_\beta)} + \sum_{\alpha=1}^{M} \frac{2pS_3^\alpha + \frac{1}{q}(S_+^\alpha S_- - S_-^\alpha S_+)}{\lambda - \epsilon_\alpha},
\]

\[
5
\]
we can compute that \( \{t(\lambda), S_3\} = 0 \), hence the reduction \( S_3 = 0 \) can be performed. A complete set of integrals of motion \( H, G_\alpha, H_\alpha, \alpha = 1, \ldots, M \) is found by rewriting the generating function \( t(\lambda) \) in the following form

\[
t(\lambda) = H + \sum_{\alpha=1}^{M} \frac{H_\alpha}{\lambda - \epsilon_\alpha} + \sum_{\alpha=1}^{M} \frac{G_\alpha}{(\lambda - \epsilon_\alpha)^2},
\]

where

\[
H = p^2 - \frac{S_- S_+}{q^2},
\]

\[
G_\alpha = (S_3^\alpha)^2 + S_+^{\alpha} S_-^{\alpha},
\]

\[
H_\alpha = \sum_{\beta=1 \atop \beta \neq \alpha}^{M} \frac{2 S_3^{\alpha} S_3^{\beta} + (S_+^{\alpha} S_+^{\beta} + S_-^{\alpha} S_-^{\beta})}{\epsilon_\alpha - \epsilon_\beta} + 2 p S_3^{\alpha} + \frac{1}{q} (S_-^{\alpha} S_+^{\alpha} - S_+^{\alpha} S_-^{\alpha}).
\]

Integrals of motion (18) are in involution provided that \( S_3 = 0 \). Notice that if this is the case, then \( \sum_{\alpha=1}^{M} H_\alpha = 0 \). The first of the integrals of motion (18) is the Hamiltonian of our system and for each \( \alpha = 1, \ldots, M \), \( G_\alpha \) is a Casimir function hence it remains constant on each symplectic leaf of the manifold on which the system is realised. Therefore we have \( M \) independent integrals of motion. We can represent the variables \( S^\alpha \) in terms of the canonical coordinates and momenta \( (x_\alpha, p_\alpha) \), \( \{x_\alpha, p_\beta\} = \delta_{\alpha\beta}, \alpha, \beta = 1, \ldots, M \) as follows

\[
S_3^\alpha = \frac{1}{2} x_\alpha p_\alpha, \quad S_+^\alpha = \frac{1}{2} p_\alpha^2, \quad S_-^\alpha = -\frac{1}{2} x_\alpha^2.
\]

In the representation (19) the first integrals (18) take the form

\[
H = p^2 + \frac{R^2}{4q^2} \sum_{\alpha=1}^{M} p_\alpha^2, \quad G_\alpha = 0, \quad \alpha = 1, \ldots, M,
\]

\[
H_\alpha = -\frac{1}{4} \sum_{\beta=1 \atop \beta \neq \alpha}^{M} \frac{M_{\alpha\beta}^2}{\epsilon_\alpha - \epsilon_\beta} + p x_\alpha p_\alpha + \frac{1}{4q} \sum_{\beta=1 \atop \beta \neq \alpha}^{M} \left(p_\alpha^2 x_\beta^2 - x_\alpha^2 p_\beta^2\right),
\]

where \( R^2 = \sum_{\alpha=1}^{M} x_\alpha^2 \) and \( M_{\alpha\beta} = p_\alpha x_\beta - x_\alpha p_\beta \). The constraint \( S_3 = 0 \) translates to

\[
\sum_{\alpha=1}^{M} x_\alpha p_\alpha = 0.
\]

If we choose \( H \) as the Hamiltonian of the system, as in fact we are doing here, then the constraint (21) implies that \( R^2 = \text{const} \) and we have a two particle Calogero-Moser model coupled to the free motion on the sphere \( S^{M-1} \).
4 Separation of Variables

The dynamical system (20) can be solved by separation of variables in the Hamilton-Jacobi equations. The separation is carried out in the framework of the functional Bethe Ansatz [3] as described in the introduction. First we consider the first of Eqs. (6), which in our case reads

\[ R^2 q - \sum_{\alpha=1}^{M} \frac{x^2_{\alpha}}{\lambda - \epsilon_{\alpha}} = 0. \]

It is a polynomial equation of degree \( M \) and it has \( M \) different solutions \( X_i, i = 1, \ldots, M \).

We use the Vieta theorem to derive the following expressions for \( q \) and \( x^2_{\alpha} \), \( \alpha = 1, \ldots, M \)

\[ q = \sum_{i=1}^{M} X_i - \sum_{\alpha=1}^{M} \epsilon_{\alpha}, \quad x^2_{\alpha} = \frac{R^2 \prod_{i=1}^{M} (X_i - \epsilon_{\alpha})}{q \prod_{\beta=1, \beta \neq \alpha}^{M} (\epsilon_{\beta} - \epsilon_{\alpha})}. \] (22)

We can choose the roots \( X_i, i = 1, \ldots, M \) in such a way that \( X_1 > \ldots > X_M \). From (22) we then learn that the separation coordinates \( X_i, i = 1, \ldots, M \) satisfy the inequalities

\[ X_M < \epsilon_M < X_{M-1} < \ldots < X_1 < \epsilon_1 \quad \text{or} \quad \epsilon_M < X_M < \ldots < \epsilon_1 < X_1. \] (23)

Since \( a(\lambda, \mu) \) obeys (7) it is clear that the canonical momenta \( P_i, i = 1, \ldots, M \) can be defined by the second of Eqs. (6). Therefore Eqs. (6) give a full set of canonical variables \((X_i, P_i)_{i=1,\ldots,M}, \{X_i, P_j\} = \delta_{ij}\), and we can proceed to separation of variables. We seek the common solution \( S(X_1, \ldots, X_M) \) of the Hamilton-Jacobi equations

\[ H \left( \frac{\partial S}{\partial X_1}, \ldots, \frac{\partial S}{\partial X_M}, X_1, \ldots, X_M \right) = E, \quad H_\alpha \left( \frac{\partial S}{\partial X_1}, \ldots, \frac{\partial S}{\partial X_M}, X_1, \ldots, X_M \right) = E_\alpha \]

in the separated form \( S(X_1, \ldots, X_M) = \sum_{i=1}^{M} S_i(X_i) \). Here \( E_\alpha \) are such that \( \sum_{\alpha=1}^{M} E_\alpha = 0 \). From (8) we find the separated equations

\[ \left( \frac{dS_i}{dX_i} \right)^2 - E - \sum_{\alpha=1}^{M} \frac{E_\alpha}{X_i - \epsilon_{\alpha}} = 0, \quad i = 1, \ldots, M. \] (24)

If we consider only a flow generated by the Hamiltonian \( H \), then \( E_\alpha, \alpha = 1, \ldots, M \) have the meaning of arbitrary separation constants.

The separated Hamilton-Jacobi equations (24) can be also used to express the Hamiltonian \( H \) in terms of the separation coordinates \((X_i, P_i)_{i=1,\ldots,M}\). Eliminating constants \( E_\alpha, \alpha = 1, \ldots, M \) from Eqs. (24), we find that

\[ H = \frac{\sum_{i=1}^{M} P_i^2 \prod_{\alpha=1}^{M} (X_i - \epsilon_{\alpha}) \prod_{j \neq i}^{M} (X_i - X_j)^{-1}}{\sum_{i=1}^{M} \prod_{\alpha=1}^{M} (X_i - \epsilon_{\alpha}) \prod_{j \neq i}^{M} (X_i - X_j)^{-1}}. \]
Finally we would like to stress that to derive separated equations such as Eqs. (24) we do not have to specify the representation (19) of $S^\alpha$, $\alpha = 1, \ldots, M$. In other words we can also separate the Hamilton-Jacobi equations for integrals (18). In this general case the separated equations read
\[
\left( \frac{dS_i}{dX_i} \right)^2 - E - \sum_{\alpha=1}^{M} \left( \frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{G_\alpha}{(X_i - \epsilon_\alpha)^2} \right) = 0.
\]

## 5 Quantisation

The classical system (9) can be quantised in the canonical way by replacing the Poisson brackets $\{ , \}$ with the commutators $-i[ , ]$. Also, the functional Bethe Ansatz method can be quantised in this way [13]. We can consider the Lax matrix (3) which generates the Gaudin algebra (see e.g. [16])

\[
\left[ L_1(\lambda), L_2(\mu) \right] = i [r_{12}(\lambda, \mu), L_1(\lambda)] - i [r_{21}(\mu, \lambda), L_2(\mu)].
\] (25)

A generating function $t(\lambda)$ for commuting integrals of motion is given by (4). We consider $r$-matrices of the form (4). Functions $a(\lambda, \mu)$, $b(\lambda, \mu)$, $c(\lambda, \mu)$ may depend on dynamical variables, hence they are operator valued functions in general. They satisfy similar conditions as before to ensure that the commutator (24) satisfies the Jacobi identity. In contrast to the classical case however, functions $a(\lambda, \mu)$, $b(\lambda, \mu)$ and $c(\lambda, \mu)$ must satisfy additional conditions to make the Gaudin algebra (25) consistent. The sufficient ones are

\[
[a(\lambda, \mu), A(\lambda)] = [a(\lambda, \mu), B(\lambda)] = [a(\lambda, \mu), C(\lambda)] = 0,
\]
\[
[b(\lambda, \mu), B(\lambda)] = [c(\lambda, \mu), C(\lambda)] = 0. \tag{26}
\]

If the conditions (26) are satisfied, then the algebra (25) takes the explicit form

\[
[A(\lambda), B(\mu)] = i (b(\lambda, \mu)B(\lambda) + a(\mu, \lambda)B(\mu)),
\]
\[
[A(\lambda), C(\mu)] = -i (c(\lambda, \mu)C(\lambda) + a(\mu, \lambda)C(\mu)), \tag{27}
\]
\[
[B(\lambda), C(\mu)] = i \left( [c(\lambda, \mu), A(\lambda)]_+ + [b(\mu, \lambda), A(\mu)]_+ \right),
\]

where $[ , ]_+$ denotes the anticommutator.

The separation of variables is then conducted in the way similar to the classical case. We define separation variables $(X_i, P_i)$ by (6). They are canonical if the condition (7) is
satisfied. In the quantum case however, Eqs. (3) are operator equations, hence we have to specify the order of operators appearing in $A(X_i)$ and $B(X_i)$. We always assume that the position operators precede the momenta. We also assume that all substitutions are done from the left. From this it follows that the new momenta $P_i$ are not true observables since $P_i \neq P_i^\dagger$ in general. Now we show how the hermitian separation variables can be defined and the separated Schrödinger equations obtained for the Calogero-Moser-Gaudin model discussed in this paper.

We consider the $N = 2$ case and we take the L-operator (15) as a Lax matrix. We observe that the Gaudin algebra for this model is obtained directly from (16) by replacing the Poisson brackets $\{ , \}$ with the commutators $-i[ , ]$. From definition (2) we find a quantum generating function for commuting integrals of motion

$$t(\lambda) = p^2 - \frac{[S_-,S_+]_+}{2q^2} + \sum_{\alpha,\beta=1}^{M} \frac{S_\alpha^3 S_\beta^3}{(\lambda - \epsilon_\alpha)(\lambda - \epsilon_\beta)} + \sum_{\alpha=1}^{M} \frac{2pS_\alpha^3 + \frac{1}{2q} \left( [S_\alpha^-,S_+]_+ - [S_\alpha^+,S_-]_+ \right)}{\lambda - \epsilon_\alpha}.$$

The full set of conserved quantities reads

$$H = p^2 - \frac{[S_-,S_+]_+}{2q^2},$$

$$G_\alpha = (S_3^\alpha)^2 + \frac{1}{2} [S_-^\alpha, S_+^\alpha]_+,$$

$$H_\alpha = \sum_{\beta=1}^{M} \frac{2S_\alpha^3 S_\beta^3 + (S_\alpha^3 S_\beta^3 + S_\beta^3 S_\alpha^3)}{\epsilon_\alpha - \epsilon_\beta} + 2pS_3^\alpha + \frac{1}{2q} \left( [S_\alpha^-,S_+]_+ - [S_\alpha^+,S_-]_+ \right),$$

and $t(\lambda)$ is given by (17). Similarly to the classical case, the first of the integrals of motion (28) plays the role of the Hamiltonian of the quantum system and $\sum_{\alpha=1}^{M} H_\alpha = 0$. For each $\alpha = 1, \ldots, M$, the integral $G_\alpha$ is a quadratic Casimir operator of the algebra $so_\alpha(2,1)$ generated by $S^\alpha$. Hence each of $G_\alpha$ is a number in any irreducible representation of the system. The particular representation is obtained by realising spin variables $S^\alpha$, $\alpha = 1, \ldots, M$ in terms of the canonical variables $(x_\alpha, p_\alpha)$, $[x_\alpha, p_\beta] = i\delta_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, M$ as follows:

$$S_3^\alpha = \frac{1}{4} (x_\alpha p_\alpha + p_\alpha x_\alpha), \quad S_\alpha^+ = \frac{1}{2} p_\alpha^2, \quad S_\alpha^- = -\frac{1}{2} x_\alpha^2.$$  

In the representation (29) the first integrals (28) take the form (compare Eqs.(20))

$$H = p^2 + \frac{R^2}{4q^2} \sum_{\alpha=1}^{M} p_\alpha^2,$$

$$G_\alpha = \frac{3}{16}, \quad \alpha = 1, \ldots, M,$$

and
\[ H_\alpha = -\frac{1}{4} \sum_{\beta=1}^{M} \frac{M^2_{\alpha\beta} + 1/2}{\epsilon_\alpha - \epsilon_\beta} + \frac{1}{2} p[x_\alpha, p_\alpha] + \frac{1}{4q} \sum_{\beta=1}^{M} (p^2_\alpha x^2_\beta - x^2_\alpha p^2_\beta), \]

Notice that \( R \) is an operator now. We can proceed to separation of variables in the Schrödinger equations, defining new canonical coordinates by (3). As we observed, the momenta \( P_i, i = 1, \ldots, M \) are not hermitian, therefore true separation momenta are still to be defined. We use Eqs.(22) to find that

\[ B(\lambda) = \frac{R^2}{2q} \frac{\prod_{i=1}^{M} (\lambda - X_i)}{\prod_{\alpha=1}^{M} (\lambda - \epsilon_\alpha)}, \quad A(\lambda) = \frac{2q}{R^2} B(\lambda) \left( p + q \sum_{i=1}^{M} \frac{1}{\lambda - X_i} D_i P_i \right), \]

where

\[ D_i = \frac{\prod_{\alpha=1}^{M} (X_i - \epsilon_\alpha)}{q \prod_{j=1}^{M} (X_i - X_j)}, \quad i = 1, \ldots, M. \]

The expression for \( A(\lambda) \) is derived by the analysis of the behaviour of \( A(\lambda) \) at \( \lambda = X_1, \ldots, X_M \) and \( \lambda = \infty \). From (23) we learn that each \( D_i \) is positive. Using the fact that \([X_i, P_j] = i \delta_{ij}\) and also that

\[ \sum_{i=1}^{M} \frac{1}{\lambda - X_i} \prod_{\alpha=1}^{M} (X_i - \epsilon_\alpha) = \frac{\prod_{\alpha=1}^{M} (\lambda - \epsilon_\alpha)}{\prod_{i=1}^{M} (\lambda - X_i)} - 1, \]

we find that

\[ A^\dagger(\lambda) = \frac{2q}{R^2} B(\lambda) \left( p + q \sum_{i=1}^{M} \frac{1}{\lambda - X_i} P^\dagger_i D_i \right). \]

Thus

\[ D_i P_i = P^\dagger_i D_i, \quad \text{(30)} \]

because \( A^\dagger(\lambda) = A(\lambda) \). Therefore we can define hermitian operators

\[ \Pi_i \equiv \sqrt{D_i P_i \frac{1}{\sqrt{D_i}}}, \quad \text{(31)} \]

The operators (31) are canonically conjugate to \( X_i \) and play the role of true separation momenta. Directly from the definition of the \( P_i \) we can derive equations of motion (8), which in terms of the \( \Pi_i \) read

\[ \frac{1}{\sqrt{D_i}} \Pi^2_i \sqrt{D_i} \Psi(X_1, \ldots, X_M) - \left( E + \sum_{\alpha=1}^{M} \left( \frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{3/16}{(X_i - \epsilon_\alpha)^2} \right) \right) \Psi(X_1, \ldots, X_M) = 0, \quad \text{(32)} \]

\( i = 1, \ldots, M \). Here \( E, E_\alpha \) are eigenvalues of the operators \( H, H_\alpha, \alpha = 1, \ldots, M \). To see that Eqs.(32) are really separation equations for the model we set

\[ \Psi(X_1, \ldots, X_M) = \sqrt{qV} \prod_{i=1}^{M} \Psi_i(X_i), \quad \text{(33)} \]
where $V$ denotes the Vandermonde determinant $V = \prod_{i<j}^{M}(X_i - X_j)$. Inserting the wave function (33) into equations (32) and representing $\Pi$ by $-i\frac{d}{dX_i}$, $i = 1, \ldots, M$, we obtain

$$
\frac{1}{\sqrt{C_i}} \frac{d^2}{dX_i^2} \left( \sqrt{C_i} \Psi_i \right) + E \Psi_i + \sum_{\alpha=1}^{M} \left( \frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{3/16}{(X_i - \epsilon_\alpha)^2} \right) \Psi_i = 0, \quad i = 1, \ldots, M,
$$

where

$$
C_i = \left| \prod_{\alpha=1}^{M} (X_i - \epsilon_\alpha) \right|.
$$

Equations (34) are then the separated Schrödinger equations for the generalised Calogero-Moser model. Finally we notice that the separated equations in the general representations of the algebras $so_\alpha(2, 1)$ can be obtained from (34) by replacing $3/16$ with the eigenvalues $g_\alpha$ of $G_\alpha$ which characterise these representations, i.e.

$$
\frac{1}{\sqrt{C_i}} \frac{d^2}{dX_i^2} \left( \sqrt{C_i} \Psi_i \right) + E \Psi_i + \sum_{\alpha=1}^{M} \left( \frac{E_\alpha}{X_i - \epsilon_\alpha} + \frac{g_\alpha}{(X_i - \epsilon_\alpha)^2} \right) \Psi_i = 0, \quad i = 1, \ldots, M.
$$

6 Conclusion

In this paper we have described an integrable generalisation of the Calogero-Moser model, which is achieved by coupling the $M$-particle Gaudin system to the Calogero-Moser model. The integrability of the model has been shown by using the recently introduced dynamical r-matrix [9]. We have also shown that the functional Bethe-Ansatz of [3] can be employed to separate the variables in models governed by this r-matrix despite its dependence on dynamical variables. The situation is similar to the one discussed in [11]. We have used this separation of variables procedure in the case in which there are two Calogero-Moser particles. We have also used the quantum counterpart of this procedure [13] to separate variables in the Schrödinger equations.

The model discussed in this paper is simply an example of a system which is governed by a dynamical r-matrix and can be solved by separation of variables in the framework of the functional Bethe Ansatz. By an analogy to the non-dynamical r-matrix case, one can expect that there are several other models which are governed by dynamical r-matrices of the type discussed here, and hence can be solved in a described way. This observation opens up new possibilities for constructing and solving integrable models as well as for getting a deeper insight into the nature of the models governed by dynamical r-matrices.
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