ON SOME QUOTIENTS MODULO NONREDUCTIVE GROUPS

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1. Introduction

Quotients modulo nonreductive groups arise in the study of moduli spaces of sheaves. Let \( k \) be an algebraically closed field of characteristic zero. We fix coherent algebraic sheaves \( E \) and \( F \) on the projective space \( \mathbb{P}^r = \mathbb{P}^r(k) \), which are direct sums of simple sheaves. The linear algebraic group \( G = \text{Aut}(E) \times \text{Aut}(F) \) acts by conjugation on the finite dimensional vector space \( W = \text{Hom}(E,F) \). To every character of \( G \) Drézet and Trautmann associate subsets \( W^{ss} \subset W \) of semistable points and \( W^s \subset W \) of stable points. One expects there to be categorical quotients \( W^{ss}/G \) and \( W^s/G \). In [4] the author has found many examples of locally closed subsets inside moduli spaces of semistable sheaves (in the sense of Gieseker) on \( \mathbb{P}^2 \), with linear Hilbert polynomial, that are isomorphic to quotients of the form \( W^{ss}/G \).

This paper is not concerned with the study of moduli spaces but only with the question of existence of quotients modulo \( G \). If \( E \) has more than one kind of simple sheaf in its decomposition, then \( \text{Aut}(E) \) is nonreductive, so this situation falls outside Mumford’s Geometric Invariant Theory. Drézet and Trautmann address this difficulty in [3] by embedding the action of \( G \) on \( W \) into the action of a reductive group \( \tilde{G} \) onto a larger space \( \tilde{W} \). Their main result states that if certain compatibility conditions relating the semistable points of \( W \) and \( \tilde{W} \) are satisfied, then \( W^{ss}/G \) and \( W^s/G \) exist and are projective, respectively quasiprojective varieties. Moreover, they find sufficient conditions, expressed in terms of linear algebra constants, under which the compatibility conditions are fulfilled. This allows Drézet and Trautmann to establish the existence of quotients for certain classes of morphisms, notably for morphisms of the form

\[ m_1 \mathcal{O}(-2) \oplus m_2 \mathcal{O}(-1) \rightarrow n \mathcal{O}, \]
cf. 6.4 in [2]. The purpose of this paper is to give more examples to the Drézet-
Trautmann theory. We use their embedding into the action of $G$ but we do not use
their linear algebra constants. We are concerned only with the geometric quotients
$W_s/G$ and we do not discuss properly semistable morphisms, i.e. morphisms which
are semistable but not stable. Applying 6.6.1 from [3] we establish the existence of
geometric quotients in the following situations:

- $mO(-d_1) \oplus 2O(-d_2) \rightarrow nO$ on $\mathbb{P}^r$ such that $0 < \lambda_1 < 1/(2a + m)$ and
either the conditions
  
  $$m < \left(\frac{r - 1 + d_1 - d_2}{r - 1}\right), \quad \lambda_1 \leq \frac{1}{a + m - 1} \left(1 - \frac{1}{n} \left(r + d_2 - 1\right)\right),$$
  
or the conditions
  
  $$m < \left(\frac{r + d_1 - d_2}{r}\right), \quad \lambda_1 \leq \frac{1}{3m} - \frac{2}{3mn} \left(r + d_2 - 1\right)$$
  
  are satisfied. Here $m$ and $n$ are not both even;

- $O(-d - 1) \oplus 3O(-d) \rightarrow nO$ on $\mathbb{P}^2$ such that
  
  $$0 < \lambda_1 < \frac{1}{10},$$
  
  $$-\frac{1}{2} + \frac{3}{4n} (d^2 + d) \leq \lambda_1 \leq \frac{2}{5} - \frac{3}{10n} (d^2 + d),$$
  
  $$-2 + \frac{3}{n} (d^2 + 2d) \leq \lambda_1 \leq 1 - \frac{3}{4n} (d^2 + 3d);$$

- $mO(-d - 1) \oplus 3O(-1) \rightarrow nO$ on $\mathbb{P}^2$ with $m < a$ and
  
  $$0 < \lambda_1 < \frac{1}{3a + m}, \quad \lambda_1 (4m - 3a + 3b) \leq \frac{n - 3}{n}, \quad \lambda_1 \leq \frac{n - 6}{mn}.$$ 

Here $a = (d + 1)(d + 2)/2$ and $b = d(d + 1)/2$ and $3 \nmid \gcd(m, n);$
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The precise meaning of $\lambda_1$ and $\lambda_2$ is revealed in section 2: they encode the character of $G$ relative to which the sets of stable points have been defined. Beside the above, we have at (3.3) a more general criterion giving sufficient conditions, expressed in terms of linear algebra constants, under which morphisms of type $(2,1)$

$$m_1\mathcal{O}(-d_1) \oplus m_2\mathcal{O}(-d_2) \longrightarrow n\mathcal{O}$$

satisfy the compatibility conditions from [3] that lead to existence of geometric quotients.

The paper is organized as follows: in section 2 we supply background material about quotients modulo nonreductive groups. As this is not part of mainstream Geometric Invariant Theory, we felt it necessary to reproduce the main results and definitions from [3]. Section 3 contains our criterion for morphisms of type $(2,1)$. We apply this criterion in section 7. The remaining sections are indirect applications of (3.3), by which we mean applications of the method of proof, rather than the statement of (3.3). All polarizations we discuss are assumed to be nonsingular in a sense that is made precise in our very brief discussion of the Geometric Invariant Theory fan, cf. the beginning of section 3.

2. QUOTIENTS MODULO NONREDUCTIVE GROUPS

In this section we reproduce some notations and definitions and we quote the main result from [3]. We also quote two particular cases of King’s Criterion of Semistability, as formulated in [3], which we will use in the subsequent sections.

Fix a vector space $V$ over $k$ of dimension $n + 1$. Drézet and Trautmann consider coherent algebraic sheaves $\mathcal{E}$ and $\mathcal{F}$ on the projective space $\mathbb{P}^n = \mathbb{P}(V)$ having decompositions

$$\mathcal{E} = \bigoplus_{1 \leq i \leq r} M_i \otimes \mathcal{E}_i, \quad \mathcal{F} = \bigoplus_{1 \leq l \leq s} N_l \otimes \mathcal{F}_l.$$ 

Here $M_i, N_l$ are vector spaces over $k$ of dimensions $m_i, n_l$. In [3] it is assumed that $\mathcal{E}_i$ and $\mathcal{F}_l$ are simple sheaves, but for the purposes of this paper we will assume that they are line bundles:

$$\mathcal{E}_i = \mathcal{O}(e_i), \quad e_1 < \ldots < e_r, \quad \mathcal{F}_l = \mathcal{O}(f_l), \quad f_1 < \ldots < f_s.$$ 

The linear algebraic group $\text{Aut}(\mathcal{E}) \times \text{Aut}(\mathcal{F})$ acts by conjugation on the finite dimensional vector space

$$W = \text{Hom}(\mathcal{E}, \mathcal{F}).$$

The subgroup of homotheties, which we identify with $k^*$, acts trivially so, without losing any information, we can instead consider the action of the quotient

$$G = \text{Aut}(\mathcal{E}) \times \text{Aut}(\mathcal{F})/k^*.$$ 

If $r > 1$, or if $s > 1$, the group $G$ is nonreductive, however, it contains the reductive subgroup

$$G_{\text{red}} = \text{GL}(M_1) \times \ldots \times \text{GL}(M_r) \times \text{GL}(N_1) \times \ldots \times \text{GL}(N_s)/k^*.$$ 

We represent elements of $G_{\text{red}}$ by pairs $(g, h)$, with

$$g = (g_1, \ldots, g_r), \quad h = (h_1, \ldots, h_s), \quad g_i \in \text{GL}(M_i), \quad h_j \in \text{GL}(N_j).$$

The characters of $G_{\text{red}}$ are of the form

$$\chi(g, h) = \prod_{1 \leq i \leq r} \det(g_i)^{-\lambda_i} \cdot \prod_{1 \leq j \leq s} \det(h_j)^{\mu_j}.$$
for integers $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$. As $\chi$ must be trivial on the subgroup of homotheties, we require the condition
\[ \sum_{1 \leq i \leq r} m_i \lambda_i = d = \sum_{1 \leq l \leq s} n_l \mu_l. \]
Clearly $\chi$ extends to a character of $G$, which we also denote by $\chi$. Drézet and Trautmann call a polarization the tuple
\[ \Lambda = (-\lambda_1/d, \ldots, -\lambda_r/d, \mu_1/d, \ldots, \mu_s/d). \]
They consider a semistability notion for $G_{\text{red}}$ depending on $\Lambda$. We quote below the definition. We point out that, relative to a certain linearization, this is the usual notion of semistability from Geometric Invariant Theory. This is made precise at lemma 3.4.1 in [3].

(2.1) Definition: Let $\Lambda$ be a fixed polarization. A point $\varphi \in W$ is called
(i) semistable with respect to $G_{\text{red}}$ and $\Lambda$ if there are an integer $m \geq 1$ and an algebraic function $f$ on $W$ satisfying $f(g.w) = \chi^m(g)f(w)$ for all $g \in G_{\text{red}}$ and $w \in W$, such that $f(\varphi) \neq 0$;
(ii) stable with respect to $G_{\text{red}}$ and $\Lambda$ if the isotropy group of $\varphi$ in $G_{\text{red}}$ is finite and there is $f$ as above, but with the additional property that the action of $G_{\text{red}}$ on the set $\{w \in W, f(w) \neq 0\}$ has closed orbits.

This definition is consistent because proportional tuples of integers give rise to the same sets of semistable (stable) points. Let $T$ be the maximal torus of $G_{\text{red}}$. Note that $T$ is also a maximal torus in $G$. A point $\varphi \in W$ is semistable (stable) with respect to $G_{\text{red}}$ and $\Lambda$ if and only if every point in its $G_{\text{red}}$-orbit is semistable (stable) with respect to $T$ and the restriction of $\chi$ to $T$. Taking this equivalence as definition for semistability (stability) with respect to $G$ we arrive at the following concept introduced by Drézet and Trautmann:

(2.2) Definition: A point $\varphi \in W$ is called semistable (stable) with respect to $G$ and $\Lambda$ if every point in its $G$-orbit is semistable (stable) with respect to $G_{\text{red}}$ and $\Lambda$. We denote by $W^{ss}(G, \Lambda)$ and $W^{s}(G, \Lambda)$ the sets of semistable, respectively stable points in $W$.

For checking semistability in concrete situations we need a criterion derived by A. King in [1] from Mumford’s Numerical Criterion. We use its formulation from [3]. Below we quote only a particular case. Let us represent a point $\varphi \in W$ by a matrix
\[ (\varphi_{li})_{1 \leq l \leq s, 1 \leq i \leq r} \quad \text{with} \quad \varphi_{li} \in \text{Hom}(M_i \otimes H^*_{li}, N_l), \quad H_{li} = \text{Hom}(E_i, F_l). \]
A family of subspaces $M'_i \subset M_i$, $N'_l \subset N_l$ will be called admissible if not all subspaces are zero and we do not have $M'_i = M_i$, $N'_l = N_l$ for all $i$, $l$.

(2.3) Proposition: A morphism $\varphi \in W$ is semistable (stable) with respect to $G$ and $\Lambda$ if and only if for each admissible family of subspaces $M'_i \subset M_i$, $N'_l \subset N_l$, which satisfies
\[ \varphi_{li}(M'_i \otimes H^*_{li}) \subset N'_l \quad \text{for all} \quad i, l, \]
we have
\[ \sum_{l=1}^{s} \mu_l \dim(N'_l) \geq (>) \sum_{i=1}^{r} \lambda_i \dim(M'_i). \]

Drézet and Trautmann embed the action of \( G \) on \( W \) into the action of a reductive group \( \tilde{G} \) on a finite dimensional vector space \( \tilde{W} \). They introduce associated polarizations
\[ \tilde{\Lambda} = (\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) \]
and define sets of semistable and stable points \( \tilde{W}^{ss}(\tilde{G}, \tilde{\Lambda}) \), respectively \( \tilde{W}^*(\tilde{G}, \tilde{\Lambda}) \), as at (2.1). In general they determine the following relationships:

\[ \zeta^{-1}(\tilde{W}^{ss}(\tilde{G}, \tilde{\Lambda})) \subset W^{ss}(G, \Lambda) \quad \text{and} \quad \zeta^{-1}(\tilde{W}^*(\tilde{G}, \tilde{\Lambda})) \subset W^*(G, \Lambda). \]

They also find sufficient conditions under which the reverse inclusions hold but, in this paper, we will not use them. The main result from [3] states that, if the sets of semistable (stable) points in \( W \) and \( \tilde{W} \) are compatible, then there are good or geometric quotients modulo \( G \). We quote below the part that will be used in the sequel:

\[ \text{(2.5) Proposition: If } \zeta^{-1}(\tilde{W}^*(\tilde{G}, \tilde{\Lambda})) = W^*(G, \Lambda), \text{ then there exists a geometric quotient } W^*(G, \Lambda)/G \text{ which is a smooth, quasi-projective variety.} \]

Combining (2.4) and (2.5) one arrives at the following:

\[ \text{(2.6) Proposition: If } \zeta(W^*(G, \Lambda)) \subset \tilde{W}^*(\tilde{G}, \tilde{\Lambda}), \text{ then there exists a geometric quotient } W^*(G, \Lambda)/G \text{ which is a smooth, quasi-projective variety.} \]

The goal of this paper is to give classes of examples of geometric quotients that result as applications of (2.6). We will avoid any discussion of properly semistable morphisms, i.e. morphisms which are semistable but not stable. Drézet and Trautmann’s theory works under certain a priori restrictions on the polarization \( \Lambda \). First, it is noticed that the set of semistable (stable) points in \( W \) is nonempty only if
\[ \lambda_i \geq (>) 0 \quad \text{and} \quad \mu_l \geq (>) 0 \quad \text{for all} \quad i, l. \]

Other conditions can be found at 5.4.1 in [3]: \( \tilde{W}^*(\tilde{G}, \tilde{\Lambda}) \) is not empty only if
\[ \begin{align*}
\alpha_2 &> 0 & \text{for } (r, s) = (2, 1), \\
\alpha_3 &> 0, \lambda_1 p_1 < 1 & \text{for } (r, s) = (3, 1).
\end{align*} \]

We will describe the embedding only in the case \( (r, s) = (2, 1) \). Let us write
\[ \mathcal{E}_1 = \mathcal{O}(-d_1), \quad \mathcal{E}_2 = \mathcal{O}(-d_2), \quad \mathcal{F}_1 = \mathcal{O}. \]

The polarization \( \Lambda \) is a triple
\[ \Lambda = (-\lambda_1, -\lambda_2, \mu_1), \quad \text{with} \quad \mu_1 = \frac{1}{n} \quad \text{and} \quad m_1 \lambda_1 + m_2 \lambda_2 = 1. \]
Recall that
\[ H_{11} = \text{Hom}(E_1, F_1) = S^{d_1}V^*, \quad H_{12} = \text{Hom}(E_2, F_1) = S^{d_2}V^*. \]
Consider also the spaces
\[ A_{21} = \text{Hom}(E_1, E_2) = S^{d_1-d_2}V^*, \]
\[ P_1 = \text{Hom}(E_1, E) = M_1 \oplus M_2 \oplus A_{21}, \]
\[ P_2 = \text{Hom}(E_2, E) = M_2, \]
\[ \tilde{W} = \text{Hom}(P_2 \otimes A_{21}, P_1) \oplus \text{Hom}(P_1, H_{11} \otimes N_1). \]
Small case letters \( a = a_{21}, p_1 = m_1 + m_2a_{21}, p_2 = m_2 \) denote the dimensions of the corresponding spaces. The group \( \tilde{G} \) acting by conjugation on \( \tilde{W} \) is
\[ \tilde{G} = \text{GL}(P_1) \times \text{GL}(P_2) \times \text{GL}(N_1)/k^*, \]
and the associated polarization is
\[ \tilde{\Lambda} = (-\alpha_1, -\alpha_2, \beta_1) \quad \text{with} \quad \alpha_1 = \lambda_1, \ \alpha_2 = \lambda_2 - a\lambda_1, \ \beta_1 = \mu_1 = \frac{1}{n_1}. \]
We represent morphisms in \( W \) as matrices \( \varphi = (\varphi', \varphi'') \),
\[ \varphi' = (\varphi_{ij})_{1 \leq i \leq n_1, 1 \leq j \leq m_1}, \quad \varphi_{ij} \in H_{11}, \]
\[ \varphi'' = (\varphi_{ij})_{1 \leq i \leq n_1, m_1 < j \leq m_1+m_2}, \quad \varphi_{ij} \in H_{12}. \]
We write \( \varphi_j \) for the \( j^{th} \) column of \( \varphi \). We consider the row vector
\[ X = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix} \]
with entries forming a basis of \( A_{21} \). The \( p_1 \times p_2 \)-matrix with entries in \( A_{21} \)
\[ \xi = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ X^T & 0 & \ldots & 0 \\ 0 & X^T & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X^T \end{bmatrix} \]
represents an element in \( \text{Hom}(P_2 \otimes A_{21}, P_1) \). The \( n_1 \times p_1 \)-matrix with entries in \( H_{11} \)
\[ \gamma(\varphi) = \begin{bmatrix} \varphi' & \varphi_{m_1+1}X & \cdots & \varphi_{m_1+m_2}X \end{bmatrix} \]
represents an element in \( \text{Hom}(P_1, H_{11} \otimes N_1) \). We put \( \zeta(\varphi) = (\xi, \gamma(\varphi)) \). It is clear that \( \gamma(h_1 \varphi) = h_1 \gamma(\varphi) \) for all \( h_1 \in \text{GL}(N_1) \). For this reason, when it comes to semistability considerations, we can, and we will assume that \( h_1 \) is the identity automorphism.

We finish this section with the particular case of King’s Criterion of Semistability that applies to \( \tilde{W} \):

**Proposition:** A point \((x, \gamma)\) of \( \tilde{W} \) is semistable (stable) with respect to \( \tilde{G} \) and \( \tilde{\Lambda} \) if and only if for each admissible family of subspaces \( P'_1 \subset P_1, P'_2 \subset P_2, N'_1 \subset N_1 \) satisfying
\[ x(P'_2 \otimes A_{21}) \subset P'_1, \quad \gamma(P'_1 \otimes H'_{11}) \subset N'_1, \]
we have
\[ \beta_1 \dim(N'_1) \geq (> \alpha_1 \dim(P'_1) + \alpha_2 \dim(P'_2). \]
3. A Criterion for Morphisms of Type (2,1)

We fix integers $d_1 > d_2 > 0$, and we fix a vector space $V$ of dimension $r + 1$ and we consider morphisms

$$\varphi = (\varphi', \varphi'') : m_1 \mathcal{O}(-d_1) \oplus m_2 \mathcal{O}(-d_2) \rightarrow n \mathcal{O} \quad \text{on} \quad \mathbb{P}^r = \mathbb{P}(V).$$

The polarization $\Lambda$ is uniquely determined by $\lambda_1 \in [0, 1/m_1]$. The theory of the GIT-fan, as developed in [5] and other works, informs us that there are finitely many values $0 = s_0 < s_1 < \ldots < s_q = 1/m_1$ such that when $\lambda_1$ varies in an interval $(s_k, s_{k+1})$ the set of semistable morphisms does not change and each open interval $(s_k, s_{k+1})$ is maximal with this property. The intervals $(s_k, s_{k+1})$ are called chambers. The points $s_k$ are called singular values for $\lambda_1$. Either $\lambda_1$ or $\lambda_2$ uniquely determine $\Lambda$, so we can talk of singular polarizations $\Lambda$, or singular values for $\lambda_2$.

According to King’s Criterion of Semistability (2.3), whenever the set of properly semistable morphisms with respect to $\Lambda$ is nonempty, there is an equality

$$\sum_{i=1}^s \mu_i \dim(N'_i) = \sum_{i=1}^r \lambda_i \dim(M'_i).$$

Those polarizations for which there is an equality as above, for some choice of subspaces $N'_i$ and $M'_i$, will be called irregular, or we may say that $\lambda_1$ or $\lambda_2$ is irregular. There are situations in which all polarizations are irregular, but those situations will not be addressed in this paper. The other possibility for morphisms of type (2,1), which we assume henceforth, is a finite set of irregular polarizations.

From King’s Criterion of Semistability we see that $W^s(G, \Lambda)$ depends only on the set of tuples of integers $(a_1, \ldots, a_r, b_1, \ldots, b_s)$, $0 \leq a_i \leq m_i$, $0 \leq b_i \leq n_i$, for which there is an inequality

$$\sum_{i=1}^s \mu_i b_i > \sum_{i=1}^r \lambda_i a_i.$$  

Let now $\Lambda^0$ be a fixed regular polarization. If $\Lambda$ is sufficiently close to $\Lambda^0$, meaning that $\lambda_1$ is sufficiently close to $\lambda_1^0$, then $\Lambda$ is also regular and the sets of tuples of integers with the above property for $\Lambda$ and $\Lambda^0$ are the same. Thus, the sets of stable morphisms, which for regular polarizations coincide with the sets of semistable morphisms, are the same. In other words, any open interval bounded by two consecutive irregular values for $\lambda_1$ is contained in a chamber. Thus, all singular polarizations are irregular. The author does not know if the converse statement is also true, but he will give below an example in which the irregular polarizations are singular. These considerations also show that for $\Lambda$ in a chamber we have $W^{ss}(G, \Lambda) = W^s(G, \Lambda)$ because, if $\Lambda$ happens to be irregular, we can perturb it slightly to a regular polarization in the same chamber.

Given integers $0 \leq \kappa_1 \leq m_1$ and $0 \leq \kappa_2 \leq m_2$ we denote by $l_{\kappa_1 \kappa_2}$ the smallest integer satisfying

$$\frac{l_{\kappa_1 \kappa_2}}{n} > \kappa_1 \lambda_1 + \kappa_2 \lambda_2,$$

and we consider morphisms of the form $\varphi_{\kappa_1 \kappa_2} = (\varphi'_{\kappa_1 \kappa_2}, \varphi''_{\kappa_1 \kappa_2})$, where

$$\varphi'_{\kappa_1 \kappa_2} = \begin{bmatrix} \ast & 0_{l_{\kappa_1 \kappa_2}, m_1 - \kappa_1} \\ \ast & \ast \end{bmatrix}, \quad \varphi''_{\kappa_1 \kappa_2} = \begin{bmatrix} \ast & 0_{l_{\kappa_1 \kappa_2}, m_2 - \kappa_2} \\ \ast & \ast \end{bmatrix}.$$
By $0_{lk}$ we denote the identically zero $l \times k$-matrix. According to King’s Criterion (2.3), the morphism $\varphi$ is semistable if and only if it is not equivalent to $\varphi_{\kappa_1 \kappa_2}$ for any choice of $\kappa_1$ and $\kappa_2$.

**Example (3.1)**: In the simplest case $m_1 = m_2 = 1$ the irregular polarizations are of the form $\Lambda = (\kappa/n, 1 - \kappa/n, 1/n), 0 \leq \kappa \leq n$. The set of semistable morphisms may be empty for some polarizations, for instance, if $n > \kappa + \dim(S^{d_2}V^*)$ and $\kappa/n < \lambda_1 < (\kappa + 1)/n$. We see this from the semistability conditions: $\varphi$ is in $W^{\ss}(G, \Lambda)$ if and only if $\varphi$ is not equivalent to a morphism of the form

\[
\begin{bmatrix}
* & 0_{\kappa+1,1} \\
* & * \\
* & *
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
0_{n-\kappa,1} & * \\
* & *
\end{bmatrix}.
\]

The following two conditions are sufficient to guarantee the existence of semistable morphisms corresponding to all polarizations:

\[
n \leq \left( \frac{r + d_2 - 1}{r} \right) \quad \text{and} \quad n \leq \left( \frac{r - 1 + d_1}{r - 1} \right).
\]

To see this we choose a nonzero linear form $\psi \in V^*$ and a linear complement $U \subset V^*$ of the subspace generated by $\psi$. We choose linearly independent elements $\varphi_{11}, \ldots, \varphi_{n1} \in S^{d_1}U$ and $\psi_{12}, \ldots, \psi_{n2} \in S^{d_2-1}V^*$.

We put $\varphi_{i2} = \psi_{i2} \psi$ for $1 \leq i \leq n$. We claim that $\varphi$ is not equivalent to a matrix having a zero entry. Indeed, the entries from the second column are linearly independent, and no linear combination of entries from the second column can divide a linear combination of entries from the first column, because the former is divisible by $\psi$, whereas the latter is not.

Under the above conditions on $n$ we see that the singular polarizations are precisely the irregular ones. Indeed, for $\lambda_1 = \kappa/n$ there exist properly semistable morphisms: just choose $\varphi_{11}, \ldots, \varphi_{\kappa 1}$ linearly independent in $S^{d_1}U$, put $\varphi_{i1} = 0$ for $\kappa + 1 \leq i \leq n$ and $\varphi_{i2} = \psi_{i2} \psi$ as above.

We now turn to the embedding into the action of the reductive group. In order to apply the theory from section 2 we need to assume a priori that $\alpha_2 > 0$, that is $\lambda_2 > a\lambda_1$. According to (2.8), a point $(\xi, \gamma) \in \tilde{W}$ is semistable if and only if

\[
(\xi, \gamma) \sim (\xi_{ki}, \gamma_{lk}) \quad \text{with} \quad \frac{l}{n} > k\lambda_1 + i\alpha_2.
\]

Here $\xi_{ki}, \gamma_{lk}$ are matrices of the form

\[
\xi_{ki} = \begin{bmatrix}
* & 0_{k,m_2-i} \\
* & * \\
* & *
\end{bmatrix}, \quad \gamma_{lk} = \begin{bmatrix}
* & 0_{l,p_1-k} \\
* & * \\
* & *
\end{bmatrix}.
\]

Because of the special form of $\xi$ we must have $k \leq m_1 + ai$. We conclude that, in order to ensure the semistability of $(\xi, \gamma)$, it is enough to require the condition

\[
\gamma \sim \gamma_{lk} \quad \text{with} \quad \frac{l}{n} > k\lambda_1 + i\alpha_2 \quad \text{and} \quad k \leq m_1 + ai.
\]

Actually, it is enough to require

\[
(3.2) \gamma \sim \gamma_{lk} \quad \text{with} \quad \frac{l}{n} > k\lambda_1 + i\alpha_2, \quad m_1 + (i - 1)a < k \leq m_1 + ia, \quad 0 \leq i \leq m_2.
\]

Before we state the main result of this section we introduce some linear algebra constants very similar in definition to the constants $c_i$ and $d_i$ from [3]. Given
integers $1 \leq i \leq m_2 - 1$ and $1 \leq j \leq m_2 a$ we denote by $k(i, j)$ the maximal dimension of a vector space $U \subset M_2 \otimes H_{i2}$ which is not contained in $M_2' \otimes H_{i2}$ for any subspace $M_2' \subset M_2$ of dimension $i$ and for which there is a subspace in $M_2' \otimes A_{21}$ of dimension at least $j$ orthogonal to $U$ under the canonical bilinear map

$$(M_2 \otimes H_{12}) \times (M_2' \otimes A_{21}) \rightarrow H_{11}. $$

Given an integer $2 \leq i \leq m_2$ and a vector space $M$ of dimension $i$ we let $k(i)$ be the maximal dimension of a subspace $U \subset M \otimes H_{i2}$ which is not contained in $M' \otimes H_{i2}$ for any proper subspace $M' \subset M$, and for which there is a nonzero subspace in $M' \otimes A_{21}$, orthogonal under the canonical bilinear map

$$(M \otimes H_{12}) \times (M' \otimes A_{21}) \rightarrow H_{11}. $$

(3.3) Claim: Let $W$ be the space of morphisms

$$\varphi : m_1 \mathcal{O}(-d_1) \oplus m_2 \mathcal{O}(-d_2) \rightarrow n \mathcal{O} \quad \text{on} \quad P^r = P(V).$$

Assume that $m_1 < a = \dim(S^{d_1 - d_2} V^*)$. Let $\Lambda$ be a nonsingular polarization satisfying the following conditions:

$$\lambda_2 > a \lambda_1,$$

$$i n \lambda_2 \geq n m_1 \lambda_1 + k(i, m_2 a - i a - m_1) \quad \text{for} \quad 1 \leq i \leq m_2 - 1,$$

$$i n \lambda_2 \geq n m_1 \lambda_1 + k(i) \quad \text{for} \quad 2 \leq i \leq m_2,$$

$$n \lambda_1 + i n \lambda_2 \geq k(i, m_2 a - i a - a + 1) \quad \text{for} \quad 1 \leq i \leq m_2 - 1.$$

Then $W^{ss}(G, \Lambda)$ admits a geometric quotient modulo $G$, which is a quasiprojective variety.

Proof: Let $\varphi$ be in $W^{ss}(G, \Lambda)$. According to (2.6), we need to show that $(\xi, \gamma(\varphi))$ is stable. Perturbing slightly $\Lambda$ we can ensure that $\Lambda$ is a nonsingular polarization and, at the same time, that $W^{ss}(G, \Lambda)$ has not changed. Thus $\tilde{W}^{ss}(\tilde{G}, \tilde{\Lambda}) = \tilde{W}^{ss}(\tilde{G}, \tilde{\Lambda})$ and it is enough to show that $\gamma(\varphi)$ satisfies condition (3.2).

We will argue by contradiction. Assume that $\gamma(\varphi)$ is equivalent to some $\gamma_{lk}$ with

$$\frac{l}{n} > k \lambda_1 + i \alpha_2, \quad m_1 + (i - 1)a < k \leq m_1 + ia, \quad 0 \leq i \leq m_2.$$

Let $\psi, \psi', \psi''$ denote the truncated matrices consisting of the first $l$ rows of $\varphi, \varphi', \varphi''$. In view of the comments before (2.8), we may assume that the vector subspace in $M_1 \otimes H_{11} \oplus M_2 \otimes H_{12}$ spanned by the rows of $\psi$ is orthogonal to a subspace, denoted $\text{Ker}(\psi)$, inside $M_1' \oplus M_2' \otimes A_{21}$, of dimension at least $p_1 - k$. Orthogonality here is understood under the canonical bilinear map

$$(M_1 \otimes H_{11} \oplus M_2 \otimes H_{12}) \times (M_1' \oplus M_2' \otimes A_{21}) \rightarrow H_{11}. $$

By analogy, considering the pairings

$$(M_1 \otimes H_{11}) \times M_1' \rightarrow H_{11} \quad \text{and} \quad (M_2 \otimes H_{12}) \times (M_2' \otimes A_{21}) \rightarrow H_{11},$$

we define the orthogonal subspaces $\text{Ker}(\psi')$ and $\text{Ker}(\psi'')$. The dimensions of the corresponding spaces are denoted by $\text{ker}(\psi), \text{ker}(\psi'), \text{ker}(\psi'')$.

Case $i = 0, \ 1 \leq k \leq m_1$. As $k \geq m_2 a - k > (m_2 - 1)a$ we see that $\text{Ker}(\psi'')$ intersects nontrivially every copy of $A_{21}$ inside $M_2' \otimes A_{21} \simeq k^{m_2} \otimes A_{21}$. Thus $\psi'' = 0$. Replacing possibly $\varphi$ with an equivalent morphism we may assume that
$\ker(\varphi') \geq m_1 - k$. We obtain that $\varphi$ is equivalent to $\varphi_{k0}$, which contradicts the semistability of $\varphi$.

Case $i = 1$, $m_1 < k < a$. Again $\psi'' = 0$. As $l/n > m_1 \lambda_1 + \alpha_2 > m_1 \lambda_1$ we arrive at the contradiction $\varphi \sim \varphi_{m_1,0}$.

Case $i = 1$, $a \leq k \leq m_1 + a$. If $\psi'' \sim \psi''_i$, then $\ker(\psi'') = (m_2 - 1)a$ hence, after possibly replacing $\varphi$ with an equivalent morphism, we may assume that $\ker(\psi') \geq m_1 + m_2a - k - (m_2 - 1)a = m_1 + a - k$.

Moreover, from $l/n > k \lambda_1 + \alpha_2 = (k-a)\lambda_1 + \lambda_2$ we get $l \geq l_{k-a,1}$. Thus $\varphi \sim \varphi_{k-a,1}$, contradiction.

If $\psi'' \sim \psi''_1$, then the rows of $\psi''$ span a space of dimension at most

$$k(1, m_2a - k) \leq k(1, m_2a - a - m_1).$$

From the hypotheses of the claim we have $l > nk\lambda_1 + n\alpha_2 = n\alpha_2 = n\lambda_2 \geq nm_1 \lambda_1 + k(1, m_2a - k)$ forcing $l \geq l_{m_1,0} + k(1, m_2a - k)$. We conclude that $\varphi \sim \varphi_{m_1,0}$, contradiction.

Case $i \geq 2$, $m_1 + (i-1)a < k < ia$. If $\psi'' \sim \psi''_{i-1}$, then, taking into account the inequalities

$$l/n > m_1 \lambda_1 + (i-1)a \lambda_1 + i\alpha_2 > m_1 \lambda_1 + (i-1)\lambda_2,$$

we obtain the contradiction $\varphi \sim \varphi_{m_1,i-1}$. Assume now that $\psi'' \sim \psi''_{i-1}$. The rows of $\psi''$ must span a vector space of dimension at most $k(i-1, m_2a - k)$. From the hypotheses of the claim we have the inequalities

$$l > nk\lambda_1 + nia_2 \geq n(m_1 + (i-1)a + 1)\lambda_1 + nia_2$$

$$> nm_1 \lambda_1 + n\lambda_1 + n(i-1)\lambda_2$$

$$\geq nm_1 \lambda_1 + k(i-1, m_2a - ia + 1)$$

$$\geq nm_1 \lambda_1 + k(i-1, m_2a - k).$$

From the above we get $l \geq l_{m_1,0} + k(i-1, m_2a - k)$, leading to the contradiction $\varphi \sim \varphi_{m_1,0}$.

Case $i \geq 2$, $ia \leq k \leq m_1 + ia$. As above, if $\psi'' \sim \psi''_{i-1}$ we get a contradiction. Assume now that $\psi'' \sim \psi''_{i-1}$ and $\psi'' \sim \psi''_{i-1}$. If $\ker(\psi'') = (m_2 - i)a$, then, replacing possibly $\varphi$ with an equivalent morphism, we may assume that $\ker(\psi') \geq m_1 + m_2a - k - (m_2 - i)a = m_1 + ia - k$.

Moreover, from the inequalities $l/n > k \lambda_1 + i\alpha_2 = (k-ia)\lambda_1 + i\lambda_2$ we get $l \geq l_{k-ia,i}$. These lead to the contradiction $\varphi \sim \varphi_{k-ia,i}$.

If $\ker(\psi'') > (m_2 - i)a$, then the rows of $\psi''$ span a space of dimension at most $k(i)$. From the hypotheses of the claim we have the inequalities

$$l > nk\lambda_1 + nia_2 \geq nia\lambda_1 + nia_2 = nia_2 \geq nm_1 \lambda_1 + k(i).$$

Thus $l \geq l_{m_1,0} + k(i)$ leading to the contradiction $\varphi \sim \varphi_{m_1,0}$. 

Finally, assume that \( \psi'' \sim \psi'' \). Then the rows of \( \psi'' \) span a space of dimension at most \( k(i, m_2a - k) \). From the hypotheses of the claim we have the inequalities

\[
\begin{align*}
l \geq nk\lambda_1 + nia \lambda_1 + nia \lambda_2 &= nm\lambda_1 + k(i, m_2a - ia - m_1) \\
\geq nm\lambda_1 + k(i, m_2a - k).
\end{align*}
\]

From the above we get \( l \geq l_{m, a} + k(i, m_2a - k) \), hence the contradiction \( \varphi \sim \varphi_{m, a} \).

The disadvantage of (3.3) is that the linear algebra constants \( k(i, j) \) and \( k(i) \) are difficult to compute in general. We carry out the computations only for a particular kind of morphisms in section 7. Sections 4, 5, 6 and 8 are not direct applications of (3.3): in the above proof a lot of information is lost in the course of estimating \( l \). We can get more precise statements by examining each case separately, yet the arguments will be just reworkings of the arguments from the proof of (3.3).

4. Morphisms of the Form \( mO(-d_1) \oplus 2O(-d_2) \rightarrow nO \)

We fix integers \( d_1 > d_2 > 0 \), we fix a vector space \( V \) of dimension \( r + 1 \) and we consider morphisms

\[
\varphi = (\varphi', \varphi'') : mO(-d_1) \oplus 2O(-d_2) \rightarrow nO \quad \text{on} \quad \mathbb{P}^r = \mathbb{P}(V).
\]

The singular values for \( \lambda_1 \) are among those values for which there are integers \( 0 \leq \kappa \leq n \) and \( 0 \leq p \leq m \) such that

\[
\frac{\kappa}{n} = p\lambda_1 \quad \text{or} \quad \frac{\kappa}{n} = p\lambda_1 + \lambda_2 \quad \text{or} \quad \frac{\kappa}{n} = p\lambda_1 + 2\lambda_2.
\]

If both \( m \) and \( n \) are even we can choose \( \kappa = n/2, p = m/2 \) and the second equation will be satisfied for all \( \lambda_1 \), in other words all polarizations will be irregular. To avoid this, we will assume in the sequel that either \( m \) or \( n \) is odd. Under this assumption the singular values for \( \lambda_1 \) are among the numbers \( \kappa/nm \), \( 0 \leq \kappa \leq n \), \( 1 \leq p \leq m \).

The semistability conditions in the special case \( m = 1 \) read as follows: for \( \kappa/n < \lambda_1 < (\kappa + 1)/n \) the morphism \( \varphi \) is semistable if and only if \( \varphi \) is not equivalent to a matrix having one of the following forms:

\[
\varphi_1 = \begin{bmatrix}
\ast & 0_{l_1, 2} \\
\ast & \ast
\end{bmatrix}, \quad \varphi_2 = \begin{bmatrix}
0_{l_2, 2} & \ast \\
\ast & \ast
\end{bmatrix}, \quad \varphi_3 = \begin{bmatrix}
\ast & 0_{l_3, 1} \\
\ast & \ast
\end{bmatrix}, \quad \varphi_4 = \begin{bmatrix}
0_{l_4, 1} & \ast \\
\ast & \ast
\end{bmatrix},
\]

where

\[
l_1 = \kappa + 1, \quad l_2 = \left\lfloor \frac{n - \kappa + 1}{2} \right\rfloor, \quad l_3 = \left\lfloor \frac{n + \kappa}{2} \right\rfloor + 1, \quad l_4 = n - \kappa.
\]

The set of semistable morphisms may be empty, for instance when \( n > \kappa + 2 \dim(S^{d_2}V^*) \). However, the following three conditions are enough to guarantee the existence of semistable morphisms corresponding to all chambers (we are still in the case \( m = 1 \)):

\[
(4.1) \quad n \leq \dim(S^{d_2-1}V^*) + l_3 - 1, \quad n \leq 2 \dim(S^{d_2-1}V^*) + l_1 - 1, \quad n \leq \dim(S^{d_1}U).
\]
We refer to (3.1) for the meaning of $U$. Explicitly, the above conditions take the form

$$
n \leq 2 \left( \frac{r + d_2 - 1}{r} \right) + \kappa - 1 \quad \text{when } n + \kappa \text{ is odd,}
$$

$$
n \leq 2 \left( \frac{r + d_2 - 1}{r} \right) + \kappa \quad \text{when } n + \kappa \text{ is even,}
$$

$$
n \leq \left( \frac{r + d_1 - 1}{r - 1} \right).
$$

We are not able to say precisely what are the singular values for $\lambda_1$ in general, however, in the special case $m = 1$, and under the assumption (4.1), one can see as at (3.1) that the singular values are $\kappa/n$, $0 \leq \kappa \leq n$.

We now turn to the embedding into the action of the reductive group. In order to apply the theory from section 2 we need to assume a priori that $\alpha_2 > 0$, that is $\lambda_1 < 1/(2a + m)$. According to (3.2), a point $(\xi, \gamma) \in W$ is semistable if $\gamma \sim \gamma_{lk}$

- with $\frac{l}{n} > k\lambda_1$ and $0 \leq k \leq m$,
- or with $\frac{l}{n} > k\lambda_1 + \alpha_2$ and $m + 1 \leq k \leq a + m$,
- or with $\frac{l}{n} > k\lambda_1 + 2\alpha_2$ and $a + m + 1 \leq k \leq 2a + m - 1$.

(4.2) Claim: Let $m$ and $n$ be positive integers at least one of which is odd. Let $W$ be the space of morphisms

$$
\varphi : m\mathcal{O}(-d_1) \oplus 2\mathcal{O}(-d_2) \longrightarrow n\mathcal{O} \quad \text{on} \quad \mathbb{P}^r = \mathbb{P}(V).
$$

Let $0 < \lambda_1 < 1/(2a + m)$ be a nonsingular value. Assume that either the conditions

(i) $m < \left( \frac{r - 1 + d_1 - d_2}{r - 1} \right)$ and $\lambda_1 \leq \frac{n - \dim(S^{d_2-1} V^{*})}{(a + m - 1)n}$

or the conditions

(ii) $m < \left( \frac{r + d_1 - d_2}{r} \right)$ and $\lambda_1 \leq \frac{n - 2 \dim(S^{d_2-1} V^{*})}{3mn}$

are satisfied. Then the set of semistable morphisms admits a geometric quotient $W^{ss}(G, \Lambda)/G$, which is a quasiprojective variety.

Proof: Let $\varphi$ be in $W^{ss}(G, \Lambda)$. According to (2.6), we need to show that $(\xi, \gamma(\varphi))$ is semistable. We argue by contradiction. Assume that $\gamma(\varphi) \sim \gamma_{lk}$ with $l/n > k\lambda_1$ and $0 \leq k \leq m$. Let $\psi = (\psi', \psi'')$ denote the truncated matrix consisting of the first $l$ rows of $\varphi$. By assumption $\psi$ has kernel inside $k^m \oplus S^{d_1-d_2} V^{*} \oplus S^{d_1-d_2} V^{*}$ of dimension at least $2a+m-k$ which is greater than $a+m$ by hypothesis, $m < a$. This shows that the kernel of $\psi$ intersects each copy of $S^{d_1-d_2} V^{*}$ nontrivially, forcing $\psi'' = 0$. Moreover, there are at least $m - k$ linearly independent elements in the kernel of $\psi'$ viewed as a subspace of $k^m$. We get $\varphi \sim \varphi_{k,0}$, which contradicts the semistability of $\varphi$.

Assume now that $\gamma(\varphi) \sim \gamma_{lk}$ with $l/n > k\lambda_1 + \alpha_2$ and $m + 1 \leq k \leq a + m$. Note that automatically $l \geq l_{m,0}$, thus excluding those $\gamma_{lk}$ with $k < a$ because, as we saw above, the condition $k < a$ forces $\psi'' = 0$, yielding the contradiction $\varphi \sim \varphi_{m,0}$. 

Assume that \( a \leq k \leq a + m \). We have \( \ker(\psi) \geq 2a + m - k > m \), which forces \( \ker(\psi'') \geq 1 \). Let \((f, g)\) be a nonzero vector of \( \text{Ker}(\psi'') \) regarded as a subspace of \( S_{d_1 - d_2} V^* \oplus S_{d_1 - d_2} V^* \). Assume that \( f, g \) are linearly dependent. Replacing possibly \( \varphi \) with an equivalent morphism we may assume that \( g = 0, f \neq 0 \) so the first column of \( \psi'' \) is zero. The second column of \( \psi'' \) is not zero because \( \varphi \sim \varphi_{m,0} \). Now we have \( \ker(\psi'') = a \) and, replacing possibly \( \varphi \) with an equivalent morphism, we may assume that \( \ker(\psi') \geq a + m - k \). As \( l/n > (k - a)\lambda_1 + \lambda_2 \) we arrive at \( \varphi \sim \varphi_{k-a,1} \), contradiction.

Assume now that \( f, g \) are linearly independent. We write \( f = hf_1, g = hg_1 \) with \( f_1, g_1 \) relatively prime and \( \text{max}\{0, d_1 - 2d_2\} \leq d = \text{deg}(h) < d_1 - d_2 \). The rows of \( \psi'' \) are of the form \((-g_1u, f_1u)\), hence they are vectors in a space of dimension equal to \( \dim(S_{2d_2+d-d_1} V^*) \). As \( \psi'' \neq 0 \) we see that the kernel of \( \psi'' \) consists of vectors of the form \((vf_1, vg_1)\), hence \( \ker(\psi'') = \dim(S_{d} V^*) \). Writing \( b = \dim(S_{d_1 - d_2 - 1} V^*) \) we have
\[
m \geq \ker(\psi') \geq 2a + m - k - \dim(S_{d} V^*) \geq 2a + m - k - b
\]
hence
\[
b + k \geq 2a \quad \text{so} \quad b + a + m \geq 2a \quad \text{giving} \quad m \geq a - b.
\]
This contradicts hypothesis (i) from the statement of the claim. Under hypothesis (ii), we would get a contradiction if we could show that the inequality \( l/n > k\lambda_1 + \alpha_2 \) implies the inequality
\[
l \geq l_{m,0} + \dim(S_{2d_2+d-d_1} V^*).
\]
Indeed, we would arrive at the contradiction \( \varphi \sim \varphi_{m,0} \). Thus, for all \( a \leq k \leq a + m \) and \( d \) we need the inequality
\[
nk\lambda_1 + n\alpha_2 \geq mn\lambda_1 + \dim(S_{2d_2+d-d_1} V^*).
\]
This would follow from the inequality
\[
n\lambda_1 + n\alpha_2 \geq mn\lambda_1 + \dim(S_{d_2} V^*).
\]
But the above is equivalent to the condition on \( \lambda_1 \) from hypothesis (ii).

We are left to examine the situation \( \gamma(\varphi) \sim \gamma_{l/k} \) with \( l/n > k\lambda_1 + 2\alpha_2 \) and \( a + m + 1 \leq k \leq 2a + m - 1 \). As \( k\lambda_1 + 2\alpha_2 \geq (m + 1)\lambda_1 + \lambda_2 + \alpha_2 \), we see that \( l \geq l_{m,1} \). If \( k < 2a \) then, as above, we get \( \ker(\psi'') \geq 1 \). Let \( f, g \) be as above. If \( f, g \) are linearly independent, we get the contradiction \( \varphi \sim \varphi_{m,1} \). If \( f, g \) are linearly independent, we get a contradiction as above under hypothesis (ii). Under hypothesis (i) we would also get a contradiction if we could show that the inequality \( l/n > k\lambda_1 + 2\alpha_2 \) implies the inequality
\[
l \geq l_{m,0} + \dim(S_{2d_2+d-d_1} V^*).
\]
Thus we need the estimate
\[
n(a + m + 1)\lambda_1 + 2n\alpha_2 \geq mn\lambda_1 + \dim(S_{d_2} V^*).
\]
Using the relations \( \alpha_2 = \lambda_2 - a\lambda_1 \) and \( 2\lambda_2 = 1 - m\lambda_1 \) we see that the above is equivalent to the estimate on \( \lambda_1 \) from hypothesis (i).

Finally, assume that \( 2a \leq k \leq 2a + m - 1 \). If \( \ker(\psi'') \geq 1 \) we get a contradiction as above. If \( \ker(\psi'') = 0 \), then \( \ker(\psi') \geq 2a + m - k \). As \( l/n > (k - 2a)\lambda_1 + 2\lambda_2 \) we arrive at the contradiction \( \varphi \sim \varphi_{k-2a,2} \). This finishes the proof of the claim.
According to corollary 7.2.2 in [3], if $\alpha_2 > 0$ and if $\lambda_2 \geq \alpha_2 c_1(2)/n$, then the conclusion of (4.2) is true. The constant $c_1(2)$ can be computed as in the proof of lemma 9.1.2 (loc. cit.) One has

$$c_1(2) = \frac{\dim(S^{d_2}V)}{\dim(S^{d_1-d_2}V)}.$$ 

Thus, according to Drézet and Trautmann, the conclusion of claim (4.2) holds if

$$\lambda_2 \geq \frac{1}{n} \left( r + d_2 \right), \quad \text{that is} \quad \lambda_1 \leq \frac{1}{m} \left( 1 - \frac{2}{n} \left( r + d_2 \right) \right).$$

Our result is not contained in Drézet and Trautmann’s result.

In the special case $m = 1$, $r \geq 2$ the estimates on $m$ from (4.2)(i) and (ii) are automatically fulfilled, so we have:

(4.3) Claim: Let $W$ be the space of morphisms

$$\varphi : \mathcal{O}(-d_1) \oplus 2\mathcal{O}(-d_2) \longrightarrow n\mathcal{O} \quad \text{on} \quad \mathbb{P}^r, \quad r \geq 2.$$ 

Let $0 < \lambda_1 < 1/(2a + 1)$ be a nonsingular value satisfying one of the following two conditions:

$$\lambda_1 \leq \frac{1}{a} - \frac{1}{3a} \left( r + d_2 - 1 \right) \quad \text{or} \quad \lambda_1 \leq \frac{1}{3} - \frac{2}{3n} \left( r + d_2 - 1 \right).$$

Then the set of semistable morphisms admits a geometric quotient modulo $G$, which is a quasiprojective variety.

At the end of this section we would like to spell out a simple case in which we can say for sure that the above claim is nonvacuous, that is, in which we know that $W^{ss}(G, \Lambda)$ is not empty. The case of the left-most chamber $0 < \lambda_1 < 1/n$ is completely understood, cf. lemma 9.3.1 in [3]. Let us take $1/n < \lambda_1 < 2/n$ and

(4.4) 

$$n = 2 \left( \frac{r + d_2 - 1}{r} \right) + 1, \quad d_1 \leq 2d_2 - 2.$$ 

Condition (4.1) is satisfied for $\kappa = 1$ because $n + \kappa$ is even. The conditions on $\lambda_1$ from (4.3) reduce to $0 < \lambda_1 < 1/(2a + 1)$. We have $1/n < 1/(2a + 1)$ because of the second inequality in (4.4). We conclude that claim (4.3) is nonvacuous for morphisms satisfying (4.4).

5. Morphisms of the Form $\mathcal{O}(-d - 1) \oplus 3\mathcal{O}(-d) \longrightarrow n\mathcal{O}$

We fix an integer $d > 0$, we fix a vector space $V$ over $k$ of dimension 3 and we consider morphisms

$$\varphi = (\varphi', \varphi'') : \mathcal{O}(-d - 1) \oplus 3\mathcal{O}(-d) \longrightarrow n\mathcal{O} \quad \text{on} \quad \mathbb{P}^2 = \mathbb{P}(V).$$

Keeping the notations from section 2 we have:

$$a = 3, \quad m_1 = 1, \quad m_2 = 3, \quad p_1 = 10, \quad \alpha_2 = \lambda_2 - 3\lambda_1 = \frac{1 - 10\lambda_1}{3}.$$ 

The singular values for $\lambda_1$ are among those values for which there are integers $0 \leq \kappa \leq n$ and $0 \leq p \leq 3$ such that

$$\frac{\kappa}{n} = p\lambda_2 \quad \text{or} \quad \frac{\kappa}{n} = \lambda_1 + p\lambda_2.$$
Using the relation \( \lambda_1 + 3\lambda_2 = 1 \) we see that the singular values for \( \lambda_1 \) are among the numbers \( \kappa/2n, 0 \leq \kappa \leq 2n \).

We write \( \varphi \) as a matrix \((\varphi_{ij})_{1 \leq i \leq n, 1 \leq j \leq 4}\) with \( \varphi_{11} \in S^{d+1}V^* \) while \( \varphi_{ij} \in S^dV^* \) for \( j = 2, 3, 4 \). The morphism \( \varphi \) is semistable if and only if it is not equivalent to a matrix having one of the following forms:

\[
\varphi_1 = \begin{bmatrix} * & 0_{t_1,3} \\ * & * \\ * & * \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0_{t_2,3} & * \\ * & * \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} * & 0_{t_3,2} \\ * & * \end{bmatrix},
\]

\[
\varphi_4 = \begin{bmatrix} 0_{t_4,2} & * \\ * & * \end{bmatrix}, \quad \varphi_5 = \begin{bmatrix} * & 0_{t_5,1} \\ * & * \end{bmatrix}, \quad \varphi_6 = \begin{bmatrix} 0_{t_6,1} & * \\ * & * \end{bmatrix}.
\]

Here each \( \varphi_i \) has a zero submatrix with \( l_i \) rows. The integers \( l \) are the smallest integers satisfying

\[
\frac{l_1}{n} > \lambda_1, \quad \frac{l_2}{n} > \lambda_2, \quad \frac{l_3}{n} > \lambda_1 + \lambda_2, \quad \frac{l_4}{n} > 2\lambda_2, \quad \frac{l_5}{n} > \lambda_1 + 2\lambda_2, \quad \frac{l_6}{n} > 3\lambda_2.
\]

In order to apply the theory from section 2 we need to have \( \alpha_2 > 0 \), that is we need to assume a priori that \( \lambda_1 < 1/10 \). According to (3.2), in order to show that \( (\xi, \gamma) \) is semistable, it is enough to show that \( \gamma \sim \gamma_{lk} \)

with \( \frac{l}{n} > \lambda_1 \) and \( k = 1 \),

or with \( \frac{l}{n} > k\lambda_1 + \alpha_2 \) and \( k = 2, 3, 4 \),

or with \( \frac{l}{n} > k\lambda_1 + 2\alpha_2 \) and \( k = 5, 6, 7 \),

or with \( \frac{l}{n} > k\lambda_1 + 3\alpha_2 \) and \( k = 8, 9 \).

(5.1) Claim: Let \( W \) be the space of morphisms of sheaves on \( \mathbb{P}^2 \) of the form

\[
\varphi : O(-d-1) \oplus 3O(-d) \rightarrow nO.
\]

Then for any nonsingular value for \( \lambda_1 \) satisfying the conditions

\[
0 < \lambda_1 < \frac{1}{10}, \quad \lambda_1 \geq -\frac{2}{3n} + \frac{3}{10n}(d^2 + d), \quad \lambda_1 \geq -\frac{1}{2} + \frac{3}{4n}(d^2 + d), \quad \lambda_1 \geq -2 + \frac{3}{n}(d^2 + 2d)
\]

the set of semistable morphisms admits a geometric quotient modulo \( G \), which is a quasiprojective variety.

Notice that for \( d \) and \( n \) large, and for \( n \) of order \( 3d^2/2 \) the last four inequalities follow from the first.

Proof: Let \( \varphi \) be in \( W^{ss}(G, \Lambda) \). According to (2.6), we need to show that \( (\xi, \gamma(\varphi)) \) is semistable. We argue by contradiction. Assume that \( \gamma(\varphi) \sim \gamma_{lk} \) with \( \frac{l}{n} > \lambda_1 \) and \( k = 1, 2 \). Note that \( l \geq l_1 \). Let \( \psi = (\psi', \psi'') \) denote the truncated matrix consisting
of the first $l$ rows of $\varphi$. By assumption $\psi$ has kernel inside $k \oplus V^* \oplus V^* \oplus V^*$ of dimension at least $10 - k$. For dimension reasons $\text{Ker}(\psi)$ intersects each copy of $V^*$ nontrivially, forcing $\psi'' = 0$. We get $\varphi \sim \varphi_1$, which contradicts the semistability of $\varphi$. The same argument also works in the case $k = 3$ and $\text{ker}(\psi'') = 7$.

Assume now that $k = 3$, $l/n > 3\lambda_1 + \alpha_2 = \lambda_2$ and $\text{ker}(\psi'') = 6$. Replacing $\varphi$ with an equivalent morphism we may assume that $\psi' = 0$. From remark (5.2) below we see that two columns of $\psi''$ vanish. As $l \geq l_2$ we get $\varphi \sim \varphi_2$, contradiction.

Assume that $k = 4$ and $l/n > 4\lambda_1 + \alpha_2 = \lambda_1 + \lambda_2$. As $\text{ker}(\psi'') \geq 5$ and $l \geq l_3$, we can apply (5.2) and we get the contradiction $\varphi \sim \varphi_3$. The same argument works if $k = 5$ and $\text{ker}(\psi'') \geq 5$.

Assume now that $k = 5$, $l/n > 5\lambda_1 + 2\alpha_2 = (2 - 5\lambda_1)/3$ and that $\text{ker}(\psi'') = 4$. The matrix $\psi''$ has the form given at remark (5.3) below, so its rows are elements in a vector space of dimension at most equal to the dimension of $S^{d-1}V^*$. If we could show that $l > \dim(S^{d-1}V^*)$, then we would conclude that $\varphi$ has a zero row, which would be a contradiction. Thus we need the inequality

$$\frac{n(2 - 5\lambda_1)}{3} \geq \binom{d + 1}{2}.$$ 

But this is equivalent to the second condition on $\lambda_1$ from the statement of the claim.

Assume that $k = 6$ and $l/n > 6\lambda_1 + 2\alpha_2 = 2\lambda_2$. Thus $l \geq l_4$ and, from the above inequality, $l \geq l_1 + \dim(S^{d-1}V^*)$. If $\text{ker}(\psi'') = 4$, then $\psi''$ has the form given at remark (5.3) below and we arrive at the contradiction $\varphi \sim \varphi_1$. If $\text{ker}(\psi'') = 3$, then we may assume that $\psi' = 0$. Let $\eta$ be a $3 \times 3$-matrix with entries in $V^*$ whose columns are linearly independent vectors in $\text{Ker}(\psi'')$. Each column of $\eta$ must contain at least two linearly independent elements, otherwise we get the contradiction $\varphi \sim \varphi_4$. Also, each row of $\eta$ must contain at least two linearly independent elements. Indeed, if

$$\eta = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ 0 & 0 & w_3 \end{bmatrix}$$

with $u_1$, $v_1$ linearly independent, $u_2$, $v_2$ linearly independent, then

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0$$

and we conclude that the first two columns of $\psi''$ are zero, again a contradiction.

Thus $\eta$ satisfies the hypotheses of remark (5.4) from below. We must have $\eta = \eta_1$ or $\eta = \eta_2$. In the first case the rows of $\psi''$ are elements in a vector space of dimension at most equal to the dimension of $S^{d-1}V^*$. As $l > \dim(S^{d-1}V^*)$, we conclude that $\varphi$ has a zero row, contradiction.

Assume now that $\eta = \eta_2$. From remark (5.5) we know that the rows of $\psi''$ are elements in a vector space of dimension at most $(d^2 + 3d)/2$. The third condition on $\lambda_1$ from the statement of the claim is equivalent to the inequality $2n\lambda_2 \geq (d^2 + 3d)/2$. This shows that $l > (d^2 + 3d)/2$, forcing $\varphi$ to have a zero row, again a contradiction.

The case $k = 7$ follows from the last two conditions on $\lambda_1$ from the claim and the cases $k = 8$ and $k = 9$ are analogous. This finishes the proof of the claim.

(5.2) Remark: Let $\psi$ be an $l \times 3$-matrix with entries in $S^dV^*$. We consider the
space of $3 \times 1$-matrices $\eta$ with entries in $V^*$ such that $\psi\eta = 0$ and we call it $\ker(\psi)$. Let $\ker(\psi)$ denote its dimension. If $\ker(\psi) \geq 5$, then two columns of $\psi$ must vanish.

(5.3) **Remark:** With the notations from (5.2), if $\ker(\psi) = 4$, then $\psi$ has the form

$$
\begin{bmatrix}
0 & uf_1 - vf_1 \\
\vdots & \vdots \\
0 & uf_l - vf_l
\end{bmatrix}
$$

with $u, v \in V^*$ linearly independent one-forms and $f_i \in S^{d-1}V^*$ for $1 \leq i \leq l$.

Next we quote 5.6 from [4]:

(5.4) **Remark:** Let $\eta$ be a $3 \times 3$-matrix with entries in $V^*$. Assume that $\det(\psi) = 0$ and that $\eta$ is not equivalent, modulo elementary operations on rows or on columns, to a matrix having a zero row, or having a zero column, or having a zero $2 \times 2$ submatrix. Then $\eta$ is equivalent to one of the following matrices:

$$
\eta_1 = \begin{bmatrix}
X & Y & 0 \\
Z & 0 & Y \\
0 & -Z & X
\end{bmatrix}
$$

or

$$
\eta_2 = \begin{bmatrix}
X & Y & Z \\
Y & a_1X + a_2Y & a_3X + a_4Y + a_5Z \\
Z & a_6X + a_7Y + a_8Z & a_9X + a_{10}Z
\end{bmatrix}
$$

with nonzero constants $a_1, \ldots, a_{10}$ in the ground field $k$. Here $\{X, Y, Z\}$ is a basis of $V^*$.

(5.5) **Remark:** The kernel of $\eta_2$ inside $S^dV^* \oplus S^dV^* \oplus S^dV^*$ is of dimension at most $(d^2 + 3d)/2$.

*Proof:* The elements of $\ker(\eta_2)$ are of the form

$$
f(-Y, X, 0) + g(-Z, 0, X) + h(0, -Z, Y)
$$

with $(f, g, h)$ determined modulo multiples of $(Z, -Y, X)$. Without loss of generality we may assume that $h$ depends only on $Y$ and $Z$. We have

$$
0 = f(-Y^2 + a_1X^2 + a_2XY) + g(-YZ + a_6X^2 + a_7XY + a_8XZ) + h(-a_1XZ - a_2YZ + a_6XY + a_7Y^2 + a_8YZ).
$$

This shows that $f$ is uniquely determined by $g$ and $h$. Hence $\ker(\eta_2)$ is of dimension at most

$$
\binom{d+1}{2} + d = \frac{d^2 + 3d}{2}.
$$

6. **Morphisms of the Form** $m\mathcal{O}(-d - 1) \oplus 3\mathcal{O}(-1) \rightarrow n\mathcal{O}$

We fix an integer $d > 0$, we fix a vector space $V$ of dimension 3, and we consider morphisms

$$
\varphi = (\varphi', \varphi'') : m\mathcal{O}(-d - 1) \oplus 3\mathcal{O}(-1) \rightarrow n\mathcal{O} \quad \text{on} \quad \mathbb{P}^2 = \text{P}(V).
$$

We write $b = \dim(S^{d-1}V^*)$ and, using the notations from section 2, we have

$$
m_1 = m, \quad m_2 = 3, \quad p_1 = 3a + m, \quad \alpha_2 = \lambda_2 - a\lambda_1 = \frac{1 - (3a + m)\lambda_1}{3}.
$$
The singular values for $\lambda_1$ are among those values for which there are integers $0 \leq \kappa \leq n$, $0 \leq p \leq m$ and $0 \leq q \leq 3$ such that $\kappa/n = p\lambda_1 + q\lambda_2$. If both $m$ and $n$ are divisible by 3 we can take $\kappa = n/3$, $p = m/3$, $q = 1$ and we see that all values for $\lambda_1$ are irregular. If either $m$ or $n$ is not divisible by 3, which will be our assumption in the sequel, then, using the relation $m\lambda_1 + 3\lambda_2 = 1$, we see that the singular values for $\lambda_1$ are among the numbers $\kappa/np$ with $0 \leq \kappa \leq 2n$, $1 \leq p \leq 2m$.

In order to apply the theory from section 2 we need to assume a priori that $a_2 > 0$, that is $\lambda_1 < 1/(3a + m)$. According to (3.2), in order to show that $(\xi, \gamma)$ is semistable, it is enough to show that $\gamma \sim \gamma_{lk}$

\begin{align*}
\text{with } \frac{1}{n} > k\lambda_1 & \quad \text{and } 0 \leq k \leq m, \\
\text{or with } \frac{1}{n} > k\lambda_1 + \alpha_2 & \quad \text{and } m + 1 \leq k \leq a + m, \\
\text{or with } \frac{1}{n} > k\lambda_1 + 2\alpha_2 & \quad \text{and } a + m + 1 \leq k \leq 2a + m, \\
\text{or with } \frac{1}{n} > k\lambda_1 + 3\alpha_2 & \quad \text{and } 2a + m + 1 \leq k \leq 3a + m - 1.
\end{align*}

(6.1) Claim: Let $m$ and $n$ be positive integers one of which is not divisible by 3. Let $W$ be the space of morphisms

$$\varphi : mO(-d - 1) \oplus 3O(-1) \to nO \quad \text{on} \quad \mathbb{P}^2 = \mathbb{P}(V).$$

Assume that $m < a$. Then for any nonsingular value $\lambda_1$ satisfying

$$0 < \lambda_1 \leq \frac{1}{3a + m}, \quad \lambda_1(4m - 3a + 3b) \leq \frac{n - 3}{n}, \quad \lambda_1 \leq \frac{n - 6}{mn}$$

the set of semistable morphisms admits a geometric quotient $W^{ss}(G, \Lambda)/G$ which is a quasiprojective variety.

Proof: Let $\varphi$ be in $W^{ss}(G, \Lambda)$. The first part of the proof proceeds as at (5.1). So let us assume that $\gamma(\varphi) \sim \gamma_{lk}$ with $1/n > k\lambda_1 + \alpha_2$ and $a \leq k \leq a + m$. We have $\ker(\psi''') \geq a + 1$. Let $r$ be the dimension of the vector space spanned by the rows of $\psi'''$. According to (5.2) we would have $\ker(\psi''') \leq a$ if $r \geq 5$, which is not the case. According to (5.3) we would have $\ker(\psi''') \leq \dim(S^{d-1}V^*)$ if $r = 4$, which is not the case either. Thus $r \leq 3$.

Assume that $r = 3$. Let $\eta$ be a $3 \times 3$-matrix with entries in $V^*$ formed from three linearly independent rows of $\psi'''$. We cannot have

$$\eta = \begin{bmatrix} 0 & 0 & u \\ v_1 & v_2 & * \\ w_1 & w_2 & *
\end{bmatrix} \quad \text{with } u \neq 0, \quad \begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \neq 0$$

because this would lead to $\ker(\psi''') \leq \ker(\eta) \leq a$. Since $\det(\eta) = 0$, we can apply (5.4) to deduce that $\eta = \eta_1$, $\eta = \eta_2$ or that $\eta$ is equivalent to a matrix having a zero column. The last case leads to the contradiction $\varphi \sim \varphi_{k-a,1}$. If $\eta = \eta_1$, we get $\ker(\eta) \leq \dim(S^{d-1}V^*)$. If $\eta = \eta_2$, we get, in view of (5.5), the inequality $\ker(\eta) \leq (d^2 + 3dl)/2 = a - 1$. Both of them are contrary to $\ker(\psi''') \geq a + 1$.

Assume now that $r = 2$. Let $\eta$ be a $2 \times 3$-matrix formed from two linearly independent rows of $\psi'''$. If $\eta$ had two zero columns we would arrive as before at
the contradiction $\varphi \sim \varphi_{k-a,1}$. Moreover, $\eta$ has two linearly independent elements in each row, otherwise we would get $\ker(\psi'') \leq a$. If

$$\eta = \begin{bmatrix} * & * & u \\ v & w & 0 \end{bmatrix}$$

with $u \neq 0$, then $\ker(\psi'')$ consists of matrices of the form

$$\begin{bmatrix} -w f_1 & \cdots & -w f_s \\ v f_1 & \cdots & v f_s \\ g_1 & \cdots & g_s \end{bmatrix}$$

with each $g_j$ uniquely determined by $f_j$. This shows that $\ker(\psi'') \leq \dim(S^{d-1}V^*)$, contradiction. We conclude that $\eta$ is not equivalent to a matrix having a zero entry, i.e. $\eta$ satisfies the hypothesis of remark (6.2) from below. According to remark (6.3) we again arrive at a contradiction: $\ker(\psi'') \leq a - 1$.

The case $r = 1$ and the remaining possibilities for $\gamma(\varphi)$ are dealt with in a similar manner. This finishes the proof of the claim.

(6.2) Remark: Let $\eta$ be a $2 \times 3$-matrix with entries in $V^*$. Assume that, under the canonical action of $\text{GL}(2) \times \text{GL}(3)$, $\eta$ is not equivalent to a matrix having a zero entry. Then $\eta$ is equivalent to one of the following two matrices:

$$\eta_3 = \begin{bmatrix} X & Y & Z \\ Y & b_1 X + b_2 Y + b_3 Z & c_2 Y + c_3 Z \end{bmatrix}$$

with $b_1, c_2, c_3 \neq 0$, or

$$\eta_4 = \begin{bmatrix} X & Y & Z \\ Y & b_2 Y + b_3 Z & c_1 X + c_2 Y + c_3 Z \end{bmatrix}$$

with $b_3, c_1 \neq 0$. Here $\{X, Y, Z\}$ is a basis of $V^*$.

Repeating the arguments from the proof of (5.5) we can estimate the dimensions of the kernels for the above matrices:

(6.3) Remark: The kernels of $\eta_3$ and $\eta_4$ inside $S^{dV^*} \oplus S^{dV^*} \oplus S^{dV^*}$ are of dimension at most $(d^2 + 3d)/2$. For $d = 2$ the kernel is of dimension at most 4.

Proof: We need to do only the case $d = 2$. Keeping the notations from (5.5) we have, say in the case $\eta = \eta_3$, the relation

$$f = -\frac{(-YZ + c_2 XY + c_3 XZ)g + (-b_1 XZ - b_2 YZ - b_3 Z^2 + c_2 Y^2 + c_3 YZ)h}{-Y^2 + b_1 X^2 + b_2 XY + b_3 XZ}.$$  

The kernel of $\eta_3$ can thus be parametrized by five parameters in the ground field $k$, namely the coefficients of $g$ and $h$, because we can assume that the coefficient of $X$ in $h$ is zero. The requirement that the above ratio be a polynomial gives algebraic conditions in the space of parameters. For $g = Y$, $h = Y$ the numerator is not divisible by the denominator because no monomial of the numerator is divisible by $X^2$. Thus the algebraic conditions are nontrivial, i.e. the kernel of $\eta_3$ is parametrized by a proper subvariety of $k^5$ and, as such, it has dimension at most 4.

The argument can be repeated for the kernel of $\eta_4$: choose $f = Y$, $h = Y$ and there will be no possibility for $g$. 


In the simplest nontrivial case $1/2mn < \lambda_1 < 1/(2m - 1)n$ the conditions from the claim take the form

$$\frac{3a}{2m} + \frac{1}{2} < n, \quad 5 \leq n + \frac{3(a - b)}{2m}, \quad 7 \leq n.$$ 

The following conditions are also necessary to ensure the nonemptiness of the set of semistable points:

$$n < l_{m,0} + 9, \quad n < l_{m,1} + 6, \quad n < l_{m,2} + 3.$$ 

Here $l_{m,0} = 1$ while $l_{m,1}$, $l_{m,2}$ are the smallest integers satisfying

$$l_{m,1} > \frac{n + 1}{3}, \quad l_{m,2} > \frac{4n + 1}{6}.$$ 

These three conditions on $n$ are satisfied for $n \leq 9$. For instance, if $m = [a/2]$, all the above conditions on $n$ reduce to $n \in \{7, 8, 9\}$. It is clear that for $d$ sufficiently large the set of semistable morphisms is not empty.

7. Morphisms of the Form $\mathcal{O}(-d - 2) \oplus 3\mathcal{O}(-d) \rightarrow n\mathcal{O}$

We fix an integer $d > 0$, we fix a vector space $V$ over $k$ of dimension 3 and we consider morphisms

$$\varphi = (\varphi', \varphi'') : \mathcal{O}(-d - 2) \oplus 3\mathcal{O}(-d) \rightarrow n\mathcal{O} \quad \text{on} \quad \mathbb{P}^2 = \mathbb{P}(V).$$

Here $m_1 = 1$, $m_2 = 3$, $a = 6$ and the singular values for $\lambda_1$ are as in section 5. The morphism $\varphi$ is semistable if and only if it is not equivalent to one of the morphisms of the form $\varphi_1, \ldots, \varphi_6$ from section 5. From this we see that $W^{ss}(G, \Lambda)$ is nonempty if and only if the following conditions are satisfied:

$$n < l_1 + 3\left(\frac{d + 2}{2}\right),$$

$$n < l_3 + 2\left(\frac{d + 2}{2}\right),$$

$$n < l_5 + \left(\frac{d + 2}{2}\right),$$

$$n < l_2 + \left(\frac{d + 4}{2}\right) + 2\left(\frac{d + 2}{2}\right),$$

$$n < l_4 + \left(\frac{d + 4}{2}\right) + \left(\frac{d + 2}{2}\right),$$

$$n < l_6 + \left(\frac{d + 4}{2}\right).$$
Taking into account the definitions of $l_1, \ldots, l_6$ from section 5, the above conditions can be rewritten as

\begin{align*}
n &\leq n\lambda_1 + 3 \left( \frac{d+2}{2} \right), \\
n &\leq n\lambda_1 + n\lambda_2 + 2 \left( \frac{d+2}{2} \right), \\
n &\leq n\lambda_1 + 2n\lambda_2 + \left( \frac{d+2}{2} \right), \\
(7.1) \\
n &\leq n\lambda_2 + \left( \frac{d+4}{2} \right) + 2 \left( \frac{d+2}{2} \right), \\
n &\leq 2n\lambda_2 + \left( \frac{d+4}{2} \right) + \left( \frac{d+2}{2} \right), \\
n &\leq 3n\lambda_2 + \left( \frac{d+4}{2} \right).
\end{align*}

Taking into account the relation $\lambda_1 + 3\lambda_2 = 1$, the first three conditions are equivalent to

\begin{align*}
\lambda_2 &\leq \frac{1}{n} \left( \frac{d+2}{2} \right).
\end{align*}

(7.2)

The conditions on $\Lambda$ from (3.3) for the existence of the quotient modulo $G$ read as follows:

\begin{align*}
\lambda_2 &> 6\lambda_1, \\
n\lambda_2 &\geq n\lambda_1 + k(1,11), \\
2n\lambda_2 &\geq n\lambda_1 + k(2,5), \\
(7.3) \\
2n\lambda_2 &\geq n\lambda_1 + k(2), \\
3n\lambda_2 &\geq n\lambda_1 + k(3), \\
n\lambda_1 + n\lambda_2 &\geq k(1,7), \\
n\lambda_1 + 2n\lambda_2 &\geq k(2,1).
\end{align*}

Next we will compute the linear algebra constants $k(i,j)$ and $k(i)$ using elementary operations with matrices of homogeneous polynomials. We do only the most laborious case:

\begin{itemize}
  \item [(7.4) Claim:] $k(2,5) = \dim(S^{d-1}V^*) = \left( \frac{d+1}{2} \right)$.
\end{itemize}

\textit{Proof:} In the sequel $\alpha$ will be a matrix with 3 columns and entries in $S^dV^*$ having linearly independent rows and linearly independent columns. Also, $\beta$ will be a $3 \times 5$-matrix with entries in $S^2V^*$ having linearly independent columns, such that $\alpha\beta = 0$. The constant $k(2,5)$ is the maximal number of rows that $\alpha$ could have for all choices of $\alpha$ and $\beta$. 

Let us fix a basis \( \{X, Y, Z\} \) of \( V^* \) and a basis \( \{u_1, \ldots, u_q\} \) of \( S^{d-1}V^* \). We consider the matrices
\[
\alpha_0 = \begin{bmatrix}
u_1 \\ \vdots \\ u_q \end{bmatrix} \begin{bmatrix}X & Y & Z \end{bmatrix} \quad \text{and} \quad \beta_0 = \begin{bmatrix}
-XY & -XZ & 0 & -Y^2 & -YZ \\
X^2 & 0 & -XZ & XY & 0 \\
0 & X^2 & XY & 0 & XY \\
\end{bmatrix}.
\]
The choice \( \alpha = \alpha_0, \beta = \beta_0 \) shows that \( k(2, 5) \geq q \). We will prove the converse inequality using induction on \( d \), but before that we need to make a few observations about \( \alpha \) and \( \beta \). Firstly, \( \beta \) cannot be equivalent to a matrix having two zeros on a column, otherwise the columns of \( \alpha \) would be linearly dependent. Secondly, \( \beta \) must have linearly independent rows. If, say, the third row of \( \beta \) is zero, then the matrix \( \beta_1 \) made of the first two rows of \( \beta \) has all maximal minors equal to 0. As \( \beta_1 \) has linearly independent columns, we see that \( \beta_1 \) is forced to have linearly dependent rows, contradicting observation one. Thirdly, \( \alpha \) cannot be equivalent to a matrix having a zero entry. If, say \( \alpha_{13} = 0 \), then all maximal minors of \( \beta_1 \) are zero, forcing \( \beta_1 \) to have linearly dependent columns or linearly dependent rows. This contradicts observation one, respectively observation two.

To begin the induction assume that \( d = 1 \). We have to show that \( \alpha \) cannot have more than one row. Assume that \( \alpha \) has two rows. The third observation from above says that \( \alpha \) satisfies the hypotheses of remark (6.2). Then (6.3) denies the existence of \( \beta \), contradiction.

Assume now that \( d \geq 2 \) and that \( \alpha \) has \( q + 1 = \dim(S^{d-1}V^*) + 1 \) rows. Let us put \( \alpha' = \alpha \mod Z, \beta' = \beta \mod Z \) and let us assume that the columns of \( \beta' \) are linearly independent. If \( \beta' \) has a zero row, then we may write
\[
\beta = \begin{bmatrix}
\beta_{11} & 0 \\
\beta_{21} & \beta_{22} \\
\end{bmatrix}
\]
with a \( 1 \times 3 \)-matrix \( \beta_{11} \) that, according to observation two from above, is not zero. As all \( 3 \times 3 \)-minors of \( \beta \) are zero, we get \( \det(\beta_{22}) = 0 \). Moreover, \( \beta_{22} \) cannot have linearly dependent rows or columns, otherwise observation one from above would be violated. Thus we may write
\[
\beta_{22} = \begin{bmatrix}X \\ Y \end{bmatrix} \begin{bmatrix}u & v \end{bmatrix}
\]
with linearly independent \( u, v \in V^* \). It follows that
\[
\alpha \sim \begin{bmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{21} & 0 \\
\end{bmatrix}
with \quad \alpha_{12} = \begin{bmatrix}v_1 \\ \vdots \\ v_q \end{bmatrix} \begin{bmatrix}
-Y \\ X \\
\end{bmatrix}
\]
for some \( v_1, \ldots, v_q \in S^{d-1}V^* \). This contradicts observation three from above. Thus far we have reached the conclusion that \( \beta' \) cannot have a zero row. As above, we may assume that
\[
\beta' = \begin{bmatrix}
\beta'_{11} & 0 \\
\beta'_{12} & \beta'_{22} \\
\end{bmatrix}
with \quad \beta'_{22} = \begin{bmatrix}X^2 & XY \\ XY & Y^2 \end{bmatrix}
\]
and we get

\[ \alpha' \sim \begin{bmatrix} \alpha'_{11} & \alpha'_{12} \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad \alpha'_{12} = \begin{bmatrix} v_1 \\ \vdots \\ v_d \end{bmatrix} \begin{bmatrix} -Y & X \end{bmatrix} \]

for some homogeneous polynomials \( v_1, \ldots, v_d \) of degree \( d - 1 \) in \( X \) and \( Y \). It follows that we can write

\[ \alpha = \begin{bmatrix} \alpha_1 \\ Z \alpha_2 \end{bmatrix} \quad \text{with} \quad \alpha_2 \text{ having } q + 1 - d = \binom{d+1}{2} + 1 = \binom{d}{2} + 1 \text{ rows.} \]

From the third observation at the beginning of this proof we see that \( \alpha_2 \) satisfies the induction hypothesis. However, \( \alpha_2 \) violates the conclusion of the induction hypothesis, so we have arrived at a contradiction.

It remains to examine the situation in which the columns of \( \beta' \) are linearly dependent. Let us write

\[ \beta = \begin{bmatrix} Z \beta_1 & \beta_2 \end{bmatrix} \quad \text{with} \quad \beta_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \]

By the first observation at the beginning of this proof we see that at least two among \( u, v, w \) must be linearly independent, say \( u = X, v = Y \) and either \( w = Z \) or \( w = 0 \). From the relation

\[ \alpha' \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} = 0 \]

we deduce, as before, the existence of \( \alpha_2 \), leading to a contradiction. This finishes the proof of the claim.

Using similar considerations, which we omit, we can compute the remaining linear algebra constants:

\[ k(1,11) = 0, \quad k(2) = k(1,7) = \binom{d+1}{2}, \quad k(3) = k(2,1) = \binom{d+2}{2} + \binom{d+1}{2}. \]

Conditions (7.3) now take the form

\[ \lambda_2 > 6 \lambda_1, \]

\[ 2n \lambda_2 \geq n \lambda_1 + \binom{d+1}{2}, \]

\[ 3n \lambda_2 \geq n \lambda_1 + \binom{d+2}{2} + \binom{d+1}{2}, \]

\[ n \lambda_1 + n \lambda_2 \geq \binom{d+1}{2}, \]

\[ n \lambda_1 + 2n \lambda_2 \geq \binom{d+2}{2} + \binom{d+1}{2}. \]

The second inequality follows from the first and the fourth. The third inequality
follows from the first and the fifth. Using the relation $\lambda_1 = 1 - 3\lambda_2$ we see that the first inequality is equivalent to $\lambda_2 > 6/19$. From this and (7.2) we obtain that

$$n < \frac{19}{6} \left( \frac{d + 2}{2} \right).$$

It becomes now clear that the last three inequalities in (7.1) are superfluous, namely they are satisfied if we set $\lambda_2 = 6/19$ and $n = 19(d^2 + 3d + 2)/12$. Eliminating all unnecessary conditions from (7.1) and (7.3), we finally arrive at:

(7.5) Claim: Let $W$ be the space of morphisms of sheaves on $\mathbb{P}^2$ of the form

$$\varphi : \mathcal{O}(-d - 2) \oplus 3\mathcal{O}(-d) \rightarrow n\mathcal{O}.$$

Then for any nonsingular polarization $\Lambda$ satisfying

$$\frac{6}{19} < \lambda_2 < \frac{1}{3},$$

$$\lambda_2 \leq 1 - \frac{1}{2n} \left( \frac{d + 1}{2} \right),$$

$$\lambda_2 \leq 1 - \frac{1}{n} \left( \frac{d + 2}{2} \right) - \frac{1}{n} \left( \frac{d + 1}{2} \right),$$

$$\lambda_2 \leq \frac{1}{n} \left( \frac{d + 2}{2} \right)$$

the set of semistable morphisms is nonempty and admits a geometric quotient $W^{ss}(G, \Lambda)/G$, which is a quasiprojective variety.

Note that necessarily $n$ must satisfy the conditions

$$\frac{19}{13} \left( \frac{d + 1}{2} \right) + \frac{19}{13} \left( \frac{d + 2}{2} \right) < n < \frac{19}{6} \left( \frac{d + 2}{2} \right).$$

8. Morphisms of the Form $m\mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \mathcal{O}(-d_3) \rightarrow n\mathcal{O}$

We fix integers $d_1 > d_2 > d_3 > 0$, we fix a vector space $V$ over $k$ of dimension $r + 1$ and we consider morphisms of sheaves on $\mathbb{P}^r = \mathbb{P}(V)$ of the form

$$\varphi = (\varphi^1, \varphi^2, \varphi^3) : m\mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \mathcal{O}(-d_3) \rightarrow n\mathcal{O}.$$

Employing the notations from [3] we have:

$$m_1 = m, \quad m_2 = 1, \quad m_3 = 1, \quad p_1 = m + a_{21} + a_{31}, \quad p_2 = 1 + a_{32}, \quad p_3 = 1,$$

$$m\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \alpha_1 = \lambda_1, \quad \alpha_2 = \lambda_2 - a_{21}\lambda_1, \quad \alpha_3 = \lambda_3 - a_{31}\lambda_1 - a_{32}\lambda_2 + a_{32}a_{21}\lambda_1.$$

The polarization $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \mu_1)$ is uniquely determined by the pair $(\lambda_1, \lambda_2)$ in $[0, 1] \times [0, 1]$. The singular polarizations are among those polarizations for which there are integers $0 \leq \kappa \leq n$ and $0 \leq p \leq m$ such that

$$\frac{\kappa}{n} = p\lambda_1$$

or

$$\frac{\kappa}{n} = p\lambda_1 + \lambda_2$$

or

$$\frac{\kappa}{n} = p\lambda_1 + \lambda_3.$$

Equivalently, for singular polarizations the pair $(\lambda_1, \lambda_2)$ lies on one of the lines $\lambda_1 = \kappa/np$ or $\lambda_2 = \kappa/n - p\lambda_1$. 
Given integers $0 \leq \kappa_1 \leq m$, $0 \leq \kappa_2 \leq 1$, $0 \leq \kappa_3 \leq 1$, we denote by $l_{\kappa_1\kappa_2\kappa_3}$ the smallest integer satisfying
\[
\frac{l_{\kappa_1\kappa_2\kappa_3}}{n} > \kappa_1 \lambda_1 + \kappa_2 \lambda_2 + \kappa_3 \lambda_3
\]
and we consider matrices of the form
\[
\varphi_{\kappa_1\kappa_2\kappa_3}^1 = \begin{bmatrix} * & 0 \kappa_1\kappa_2\kappa_3, m-\kappa_1 \\ * & \end{bmatrix}, \quad \varphi_{\kappa_1\kappa_2\kappa_3}^i = \begin{bmatrix} 0 \kappa_1\kappa_2\kappa_3, 1-i \\ * & \end{bmatrix}, \quad i = 2, 3.
\]
We put $\varphi_{\kappa_1\kappa_2\kappa_3} = (\varphi_{\kappa_1\kappa_2\kappa_3}^1, \varphi_{\kappa_1\kappa_2\kappa_3}^2, \varphi_{\kappa_1\kappa_2\kappa_3}^3)$. According to King’s Criterion of Semistability from [1], the morphism $\varphi$ is semistable if and only if it is not equivalent to a morphism of the form $\varphi_{\kappa_1\kappa_2\kappa_3}$ for any choice of $\kappa_1, \kappa_2, \kappa_3$.

We now turn to the embedding into the action of the reductive group. The map $\zeta : W \rightarrow \tilde{W}$ can be described explicitly: $\zeta(\varphi) = (\xi_2, \xi_3, \gamma(f))$ where
\[
\xi_2 \in M_{p_1,p_2}(S^{d_1-d_2}V^*), \quad \xi_3 \in M_{p_2,p_3}(S^{d_2-d_3}V^*), \quad \gamma(\varphi) \in M_{p_3,p_1}(S^{d_1}V^*).
\]
Concretely, let us choose a basis $\{X_0, \ldots, X_r\}$ of $V^*$ and let us choose bases
\[
\{U_i\}_{1 \leq i \leq a_{21}} \text{ of } S^{d_1-d_2}V^*, \quad \{V_j\}_{1 \leq j \leq a_{32}} \text{ of } S^{d_2-d_3}V^*, \quad \{W_k\}_{1 \leq k \leq a_{31}} \text{ of } S^{d_1}V^*
\]
made of monomials $X_0^{i_0} \cdots X_r^{i_r}$. Then
\[
\xi_2 = \begin{bmatrix} 0 & 0 \\ U & 0 \\ 0 & W \end{bmatrix} \quad \text{and} \quad \xi_3 = \begin{bmatrix} 0 & V_1 & \cdots & V_{a_{32}} \end{bmatrix}^T
\]
where
\[
U = \begin{bmatrix} U_1 & \cdots & U_{a_{21}} \end{bmatrix}^T
\]
while $W$ is an $a_{31} \times a_{32}$-matrix with entries $W_{kj} = W_k/V_j$ if $V_j$ divides $W_k$, otherwise $W_{kj} = 0$.

In order to apply the theory from section 2 we need to assume a priori that $\alpha_3 > 0$ and $p_1 \alpha_1 < 1$, see remark (2.7). The second condition follows from the condition $\alpha_2 > 0$, which we will assume in the sequel. According to King’s Criterion of Semistability, a point $(\xi_2, \xi_3, \gamma) \in \tilde{W}$ is semistable if and only if the following conditions are satisfied:

(i) $(\xi_2, \gamma) \sim ((\xi_2)_k, \gamma_{lk})$ with $l/n > k \lambda_1 + i \alpha_2 + \alpha_3$, $0 \leq k \leq m + a_{21} + a_{31} - 1, 1 \leq i \leq 1 + a_{32}$. Here $\gamma_{lk}$ and $(\xi_2)_k$ are matrices of the form
\[
\gamma_{lk} = \begin{bmatrix} 0, l+m+a_{21}+a_{31}-k & \star \\ \star & \end{bmatrix}, \quad (\xi_2)_k = \begin{bmatrix} \star \\ \star & 0, k+1+a_{32}-i \end{bmatrix}.
\]

(ii) $(\xi_2, \xi_3, \gamma) \sim ((\xi_2)_k, (\xi_3)_l, \gamma_{lk})$, with $l/n > k \lambda_1 + i \alpha_2$, $0 \leq k \leq m + a_{21} + a_{31} - 1, 1 \leq i \leq 1 + a_{32}$. Here $(\xi_3)_l$ denotes a matrix having zero entries on the last $i$ rows.

(iii) $(\xi_2, \gamma) \sim ((\xi_2)_k, \gamma_{lk})$ with $l/n > k \lambda_1$, $0 \leq k \leq m + a_{21} + a_{31} - 1$. Here $(\xi_2)_k$ denotes a matrix with zero entries on the last $k$ rows.

(8.1) Remark: Let $W'$ be a matrix obtained by performing elementary row and column operations on $W$. Assume that $W'$ has a zero submatrix with $a_{32} - 1$ columns. Then the zero submatrix has at most $a_{21}$ rows.
Proof: We notice first that the $a_{21}$ nonzero entries of $W$ on each column are linearly independent. The claim will follow if we show that the matrix $W_j$, obtained by deleting the $j$th column of $W$ and those $i$th rows for which $W_{ij} \neq 0$, has linearly independent rows. But, by construction, each row of $W_j$ is not zero and the nonzero entries of $W_j$ on each column are linearly independent. This finishes the argument.

As a direct consequence of the above remark we get the following:

(8.2) Remark: Assume that $\xi_2 \sim (\xi_2)_{k1}$. Then $k \leq m + a_{21}$.

At the other extreme, we would like to know what is the largest $k$ for which $\xi_2 \sim (\xi_2)_{k,a_{32}}$. For this we need the following analog of (8.1). Its proof will be included in the proof of (8.5):

(8.3) Remark: Let $W'$ be as at (8.1). Then each column of $W'$ has $a_{21}$ linearly independent elements, in other words it spans $S^{d_1-d_2}V^*$. 

(8.4) Remark: Assume that $\xi_2 \sim (\xi_2)_{k,a_{32}}$. Then $k \leq m + a_{31}$.

Given integers $1 \leq i < j \leq a_{32}$ we let $\omega_{ij}$ be the number of nonzero rows of the matrix made of the columns $i$ and $j$ of $W$. Let $\omega$ be the smallest among the numbers $\omega_{ij}$. Note that $2a_{21} < \omega < 2a_{21}$. In fact, we have the obvious formula

$$\omega = 2 \left( d_1 - d_2 + r \right) - \left( d_1 - d_2 + r - 1 \right).$$

(8.5) Remark: Let $W'$ be as at (8.1). Assume that $W'$ has a zero submatrix with two or more columns. Then the zero submatrix has at most $a_{31} - \omega$ rows.

Proof: We have to show that any matrix made of two columns of $W'$ has $\omega$ linearly independent rows. Let $W_j$ be the $j$th column of $W$. Let $W'_j$ be a linear combination of the columns of $W$ of the form

$$W'_j = W_j + \sum_{l<j} c_l W_l.$$ 

Given integers $1 \leq p < q \leq a_{32}$ we have to show that the matrix $W''$ made of the columns $W'_p$ and $W'_q$ has $\omega$ linearly independent rows.

We choose the lexicographic ordering on the monomials $X_0^r \cdots X_r^{i_r}$ that form a basis of $S^{d_2-d_3}V^*$. We also choose the lexicographic ordering on the monomials giving a basis for $S^{d_1-d_3}V^*$. We write $W$ relative to these orderings and we notice that, if the entry $W_{ij}$ of $W$ is nonzero, then, for $i \leq k$, $l \leq j$, $(i, j) \neq (k, l)$, $W_{kl}$ is either zero or is larger than $W_{ij}$ in the lexicographic ordering. This shows two things:

(i) if $W_{ij} \neq 0$, then $W'_{ij}$ is equal to $W_{ij}$ plus a linear combination of monomials that are larger than $W_{ij}$ in the lexicographic ordering;

(ii) if $W_{ij} \neq 0$, then, for $i < k \leq a_{31}$, $W'_{kj}$ is either zero or is a combination of monomials larger than $W_{ij}$.
Performing on $W'_j$ row operations of the form $cR_k + R_i \rightarrow R_i$, $i < k$, $c$ being a scalar, we do not disturb properties (i) and (ii). Moreover, performing a certain sequence of such operations, we can arrive at $W_j$. This proves remark (8.3).

To show that $W''$ has $\omega$ linearly independent rows we proceed as follows. Performing, possibly, row operations on $W''$ of the kind mentioned above, we may assume that $W'_p = W'_p$. Now all we need to do is find $\omega - a_{21}$ linearly independent elements among those $W'_{iq}$ for which $W'_{iq} = 0$. But from (i) and (ii) we know that those $W'_{iq}$ for which $W_{iq} \neq 0$ are linearly independent. As there are at least $\omega - a_{21}$ indices $i$ for which $W_{iq} \neq 0$ but $W_{iq} = 0$, we are done.

(8.6) Remark: Assume that $\xi_2 \sim (\xi_2)_{ki}$ with $i \leq a_{32} - 1$. Then $k \leq m + a_{21} + a_{31} - \omega$.

Owing to the fact that $\xi_2$ has $a_{21} + a_{31}$ linearly independent rows, we may assume that $k \leq m$ in (iii). Owing to the fact that $\xi_3$ has $a_{32}$ linearly independent entries, we may assume that $i = 1$ in (ii) and, in view of (8.2), that $k \leq m + a_{21}$. According to (8.6), we may assume that $k \leq m + a_{21} + a_{31} - \omega$ in (i) if $2 \leq i \leq a_{32} - 1$. According to (8.4), we may assume that $k \leq m + a_{31}$ in (i) if $i = a_{32}$. Thus, in order to show that $(\xi_2, \xi_3, \gamma)$ is semistable, it is enough to show that $\gamma \sim \gamma_k$

with \( \frac{l}{n} > k\lambda_1 \) and $0 \leq k \leq m$,

or with \( \frac{l}{n} > k\lambda_1 + \alpha_2 \) and $m < k \leq m + a_{21}$,

or with \( \frac{l}{n} > k\lambda_1 + i\alpha_2 + \alpha_3 \) and $m + a_{21} < k \leq m + a_{21} + a_{31} - \omega$, $2 \leq i \leq a_{32} - 1$,

or with \( \frac{l}{n} > k\lambda_1 + a_{32}\alpha_2 + \alpha_3 \) and $m + a_{21} + a_{31} - \omega < k \leq m + a_{31}$,

or with \( \frac{l}{n} > k\lambda_1 + (a_{32} + 1)\alpha_2 + \alpha_3 \) and $m + a_{31} < k \leq m + a_{21} + a_{31} - 1$.

(8.7) Claim: Let $W$ be the space of morphisms of sheaves on $\mathbb{P}^r$ of the form

$$\varphi : mO(-d_1) \oplus O(-d_2) \oplus O(-d_3) \rightarrow nO.$$  

Assume that $m < \omega - a_{21}$, in other words assume that

$$m < \left( \frac{d_1 - d_2 + r - 1}{r - 1} \right).$$

Then for any nonsingular polarization $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \mu_1)$ satisfying

$$a_{21}\lambda_1 < \lambda_2 < \frac{1 - m\lambda_1 - a_{31}\lambda_1 + a_{32}a_{21}\lambda_1}{1 + a_{32}},$$

$$\lambda_2 < 1 - m\lambda_1,$$

$$\lambda_1 \leq \frac{1}{m + a_{21}} - \frac{1}{mn + a_{21}n} \left( \frac{d_3 + r}{r} \right),$$

the set of semistable morphisms admits a geometric quotient $W^{ss}(G, \Lambda)/G$, which is a quasiprojective variety.
Proof: Let \( \varphi \) be in \( W^{**}(G, \Lambda) \). According to (2.6), we need to show that \( (\xi_2, \xi_3, \gamma(f)) \) is semistable. We argue by contradiction. Assume that \( \gamma(\varphi) \sim \gamma_{lk} \) with \( l/n > k\lambda_1 + 0 \leq k \leq m \). Let \( \psi = (\psi^1, \psi^2, \psi^3) \) denote the truncated matrix consisting of the first \( l \) rows of \( \varphi \). By assumption \( \psi \) has kernel inside \( k^m \oplus S^{d_1-d_2}V^* \oplus S^{d_1-d_3}V^* \) of dimension at least \( m + a_{21} + a_{31} - k \), which is greater than \( m + a_{21} \) and \( m + a_{31} \) because, by hypothesis, \( m < a_{21} \) and \( m < a_{31} \). This shows that \( \text{Ker}(\psi) \) intersects \( S^{d_1-d_2}V^* \) and \( S^{d_1-d_3}V^* \) nontrivially, forcing \( \psi^2 = 0 \) and \( \psi^3 = 0 \). Moreover, replacing possibly \( \varphi \) with an equivalent morphism, we may assume that there are at least \( m - k \) linearly independent elements in the kernel of \( \psi^1 \), viewed as a subspace of \( k^m \). We get \( \varphi \sim \varphi_{k,0,0} \), contradicting the semistability of \( \varphi \).

Assume now that \( \gamma(\varphi) \sim \gamma_{lk} \) with \( l/n > k\lambda_1 + a_2 \) and \( m < k \leq m + a_{21} \). Note that automatically \( l \geq l_{m,0,0} \). This excludes those \( \gamma_{lk} \) with \( k < a_{21} \) because, as we saw above, the condition \( k < a_{21} \) forces \( \psi^1 = 0 \), \( \psi^2 = 0 \), so it yields \( \varphi \sim \varphi_{m,0,0} \), which is a contradiction. Thus \( a_{21} \leq k \leq a_{21} + m \). We have

\[
\text{Ker}(\psi) \geq a_{31} - k \geq a_{31} - a_{21} - m > a_{31} - \omega \geq 0,
\]
hence \( \psi^3 = 0 \). We cannot have \( \psi^2 = 0 \) because this would lead to the contradiction \( \varphi \sim \varphi_{m,0,0} \). Thus the elements from \( \text{Ker}(\psi) \) project onto \( m + a_{21} - k \) linearly independent elements in \( k^m \). As \( l/n > (k - a_{21})\lambda_1 + \lambda_2 \), we obtain the contradiction \( \varphi \sim \varphi_{k-a_{31},1,0} \).

Assume that \( \gamma(\varphi) \sim \gamma_{lk} \) with \( l/n > k\lambda_1 + a_2 + a_3, m + a_{21} < k \leq m + a_{21} + a_{31} - \omega \) and \( 2 \leq i \leq a_{32} - 1 \). Notice that automatically \( l \geq l_{m,1,0} \), so we cannot have \( \psi^3 = 0 \), because this would lead to the contradiction \( \varphi \sim \varphi_{m,1,0} \). Thus \( \text{Ker}(\psi) \) intersects \( S^{d_1-d_2}V^* \) trivially, forcing

\[
m + a_{21} + a_{31} - k \leq m + a_{21}, \quad \text{so} \quad a_{31} \leq k, \quad \text{so} \quad a_{31} \leq m + a_{21} + a_{31} - \omega.
\]
This contradicts the hypothesis on \( m \).

We next examine the case \( \gamma(\varphi) \sim \gamma_{lk} \) with \( l/n > k\lambda_1 + a_2 + a_3 = (k - a_{31})\lambda_1 + \lambda_2 \) and \( m + a_{21} + a_{31} - \omega < k \leq m + a_{31} \). As before, \( \psi^3 \neq 0 \). Thus

\[
m + a_{21} \geq \text{Ker}(\psi), \quad \psi^2 \geq m + a_{21} + a_{31} - k \geq a_{21} > \omega - a_{21} > m.
\]
This shows that \( k \geq a_{31} \) and that \( \text{Ker}(\psi) \) intersects \( S^{d_1-d_2}V^* \) nontrivially. Replacing possibly \( \varphi \) with an equivalent morphism, we may assume that \( \psi^2 = 0 \) and that \( \text{Ker}(\psi^1) \geq m + a_{21} + a_{31} - k - a_{21} = m + a_{31} - k \). We arrive at the contradiction \( \varphi \sim \varphi_{k-a_{31},0,1} \).

The last situation we need to examine is \( \gamma(\varphi) \sim \gamma_{lk} \) with

\[
\frac{l}{n} > k\lambda_1 + (a_2 + 1)\alpha_2 + \alpha_3 = (k - a_{21} - a_{31})\lambda_1 + \lambda_2 + \lambda_3
\]
and \( m + a_{31} < k \leq m + a_{21} + a_{31} - 1 \). Notice that automatically \( l \geq l_{m,0,1} \), so \( \text{Ker}(\psi) \) intersects \( S^{d_1-d_2}V^* \) and \( S^{d_1-d_3}V^* \) trivially, otherwise we would get the contradictions \( \varphi \sim \varphi_{m,1,0} \) or \( \varphi \sim \varphi_{m,0,1} \).

Assume that \((f, g)\) is a nonzero vector in \( \text{Ker}(\psi^2, \psi^3) \) viewed as a subspace of \( S^{d_1-d_2}V^* \oplus S^{d_1-d_3}V^* \). We saw above that both \( f \) and \( g \) must be nonzero, so we can write \( f = hf_1, \; g = hg_1 \) with \( f_1, \; g_1 \) relatively prime and

\[
\max\{0, \; d_1 - d_2 - d_3\} \leq d = \deg(h) \leq d_1 - d_2.
\]
The rows of \((\psi^2, \psi^3)\) are of the form \((-g_1u, f_1u)\), hence they are vectors in a vector space of dimension equal to \( \dim(S^{d_2+d_3-d_1}V^*) \). Had we had the inequality \( l \geq \)}
\( l_{m,0,0} + \dim(S^{d_2+d_3+d-d_1}V^*) \), we would arrive at the contradiction \( \varphi \sim \varphi_{m,0,0} \). But this inequality follows from the condition

\[
n(a_{31}\lambda_1 + (a_{32} + 1)\alpha_2 + \alpha_3) \geq \dim(S^{d_3}V^*)
\]

which is equivalent to the last condition from the statement of the claim.

In conclusion, \( \text{Ker}(\psi) \) intersects trivially \( S^{d_1-d_2}V^* \oplus S^{d_1-d_3}V^* \), hence its elements project onto linearly independent elements in \( k^m \). We get \( m \geq \text{ker}(\psi) \geq m + a_{21} + a_{31} - k \) forcing \( k \geq a_{21} + a_{31} \). We arrive at the contradiction \( \varphi \sim \varphi_{k-a_{21}-a_{31},1,1} \). This finishes the proof of the claim. The remaining conditions from the statement of the claim are there to ensure that \( \lambda_3 > 0 \) and \( \alpha_3 > 0 \).

According to corollary 7.2.2 in [3], if \( a_2 > 0, \alpha_3 > 0 \) and if \( \lambda_2 \geq a_{21}c_1(1,1)/n \), then the conclusion of (8.7) holds. The constant \( c_1(1,1) \) can be computed as in the proof of lemma 9.1.2 (loc. cit.) One has

\[
c_1(1,1) = \frac{\dim(S^{d_3}V)}{\dim(S^{d_1-d_3}V)}.
\]

Thus, according to Drézet and Trautmann, the conclusion of (8.7) holds under the hypotheses

\[
a_{21}\lambda_1 < \lambda_2 < \frac{1-m\lambda_1-a_{31}\lambda_1+a_{32}a_{21}\lambda_1}{1+a_{32}}, \quad \lambda_2 < 1-m\lambda_1, \quad \lambda_2 \geq \frac{a_{21}}{na_{31}} \left( \frac{r+d_3}{r} \right).
\]

Our result is not contained in Drézet and Trautmann’s result.

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