New Models for Generating Hard
Random Boolean Formulas and
Disjunctive Logic Programs*

Giovanni Amendola¹, Francesco Ricca¹, Miroslaw Truszczynski²
¹University of Calabria, Italy, {amendola,ricca}@mat.unical.it
²University of Kentucky, USA, mirek@cs.uky.edu

Abstract
We propose two models of random quantified boolean formulas and their natural
random disjunctive logic program counterparts. The models extend the standard
models of random $k$-CNF formulas and the Chen-Interian model of random 2QBFs.
The first model controls the generation of programs and QSAT formulas by imposing
a specific structure on rules and clauses, respectively. The second model is based
on a family of QSAT formulas in a non-clausal form. We provide theoretical
bounds for the phase transition region in our models, and show experimentally
the presence of the easy-hard-easy pattern and its alignment with the location of
the phase transition. We show that boolean formulas and logic programs from our
models are significantly harder than those obtained from the standard $k$-CNF and
Chen-Interian models, and that their combination yields formulas and programs that
are “super-hard” to evaluate. We also provide evidence suggesting that formulas
from one of our models are well suited for assessing solvers tuned to real-world
instances. Finally, it is noteworthy that, to the best of our knowledge, our models
and results on random disjunctive logic programs are the first of their kind.

1 Introduction
Models for generating random instances of search problems have received much at-
tention from the artificial intelligence community in the last twenty years. The results
obtained for boolean satisfiability (SAT) [1,39] and constraint satisfaction (CP) [32]
have had a major impact on the development of fast and robust solvers, significantly
expanding their range of effectiveness as general purpose tools for solving hard search
and optimization problems arising in AI, and scientific and engineering applications.
They also revealed an intriguing phase-transition phenomenon often associated with the
inherent hardness of instances, and provided theoretical and experimental basis for a
good understanding of the “region” where the phase-transition occurs.

Models of random propositional formulas and QBFs that can reliably generate
large numbers of instances of a desired hardness are important [23]. Inherently hard
instances for SAT and QBF solvers are essential for designing and testing search
methods employed by solvers [11], and are used to assess their performance in solver

*Some of the results were presented in preliminary form at IJCAI 2017 [4].
competitions [27, 35, 11]. On the flip side, large collections of easy instances support the so-called \textit{fuzz} testing, used to reveal problems in solver implementation, as well as defects in solver design [10].

Previous work on models of random formulas focused on random CNF formulas and random prenex-form QBFs with the matrix in CNF or DNF (depending on the quantifier sequence). The fixed-length clause model of \( k \)-CNF formulas and its 2QBF extension have been especially well studied. Formulas in the fixed-length clause model consist of \( m \) clauses over a (fixed) set of \( n \) variables, each clause with \( k \) non-complementary literals. All formulas are assumed to be equally likely. For that model it is known that there are reals \( p_l(k) \) and \( p_u(k) \) such that if \( m/n < p_l(k) \), a formula from the model is almost surely satisfiable (SAT), and if \( m/n > p_u(k) \), almost surely unsatisfiable (UNSAT).

It is conjectured that \( p_l(k) = p_u(k) \). That conjecture is still open. However, it holds asymptotically, i.e., the two bounds converge to each other with \( k \to \infty \) [2]. For the best studied case of \( k = 3 \), we have \( p_l(3) \geq 3.52 \) [28] and \( p_u(3) \leq 4.49 \) [17], and experiments show that the phase transition ratio \( m/n \) is close to 4.26 [14]. Important for the solver design and testing is that instances from the phase transition region are hard and those from regions on both sides of the phase transition are easy, a property called the easy-hard-easy pattern [33] or, more accurately, the “easy-hard-less hard” pattern [13]. Empirical studies suggest that SAT solvers devised for solving random formulas are usually not effective with real world instances; vice versa solvers for industrial instances are less efficient on random formulas [27]. This is often attributed to some form of (hidden) structure present in industrial problems that solvers designed for industrial applications can exploit [6]. Finding models to generate random formulas with “structure” that behave similarly to those arising in practice is an important challenge [29].

Ansotegui et al. [5] presented the first model that may have this property: despite the “randomness” of its instances, they are better solved by solvers tuned to industrial applications. More recently, Giraldez-Cru and Levy [24] proposed a model of random SAT based on the notion of modularity, and showed that formulas with high modularity behave similarly to industrial ones.

The fixed-length clause model was extended to QBFs by Chen and Interian [12]. In addition to \( n \) and \( m \) (understood as above), their model includes parameters controlling the structure of formulas. Once these parameters are fixed, similar properties as in the case of the \( k \)-CNF model emerge. There is a phase transition region associated with a specific value of the ratio \( m/n \) (that does not depend on \( n \)) and the easy-hard-easy pattern can be experimentally verified.

These two models are based on formulas in normal forms. However, many applications give rise to formulas in non-normal forms motivating studies of solvers of non-normal form formulas and QBFs, and raising the need of models of random non-normal form formulas. The fixed-shape model proposed by Navarro and Voronkov [36], and studied by Creignou et al. [15], is a response to that challenge. The model is similar to that of the \( k \)-CNF one (or its extensions to QBFs), but fixed shape (and size) non-normal form formulas are used in place of \( k \)-clauses as the key building blocks. Experimental studies again show the phase-transition and the easy-hard-easy pattern.

Motivated by the work on random SAT and QBF models, researchers proposed models of random logic programs, and obtained empirical and theoretical results concerning their properties [44, 43, 34, 41, 42]. Those results are limited to non-disjunctive logic programs. No models for disjunctive logic programs have been proposed so far. Such results would be of substantial interest to answer set programming (ASP) [9], a popular
computational formalism based on disjunctive logic programs.

In this paper we propose two models of random QBF formulas and the corresponding models of disjunctive logic programs. First, we propose a controlled version of the Chen-Interian model in which CNF formulas that are used as matrices are subject to additional conditions restricting their structure. Second, we propose multi-component versions of the earlier models. In the multi-component models, propositional formulas and matrices of QBFs are disjunctions of \( t \)-\( k \)-CNF formulas (either standard or “controlled”). They are not formulas from the fixed-shape model of Navarro and Voronkov, as their building blocks (CNF or DNF formulas) do not have a fixed size. In each case, the standard translation from QBFs to disjunctive programs suggests random models for the latter.

For the new models, we present theoretical bounds on the region where the phase transition is located, and study experimentally their behavior. In our experiments, we consider several ASP, SAT and QBF solvers to exclude any possible bias that could be an artifact of a particular solver. We study the regions of hardness for the models and show empirically that they lie within their phase transition regions. We compare the hardness of the controlled model with the corresponding Chen-Interian model and find that the former can generate formulas that are significantly harder. For the multi-component versions of the standard random CNF and the Chen-Interian models we study hardness as a function of the ratio \( \frac{m}{n} \) and of the number of components \( t \). The latter was of main interest to us. The results show that the multi-component model allows for controlling hardness of formulas and programs in such a way that, even when the number of variables is fixed, raising \( t \) may result in instances that are orders of magnitude harder to evaluate. Moreover, we show that the combination of controlled and multi-component model allows to generate instances that are “super-hard” to evaluate.

As Ansotegui et al. [5], we compare SAT/QBF solvers designed for random instances with those designed for real-world ones. We find that for \( t \geq 2 \) our models generate instances better solved by solvers for real-world instances, and that the difference becomes more pronounced as \( t \) grows. For disjunctive logic programs, we measure the effect of \( t \) on processing them and show that \( t \) allows us to control the amount of computation dedicated to stable model checking [31].

Our results provide new ways to generate hard and easy instances of propositional formulas, QBFs and disjunctive programs. Our models can generate instances of increasing hardness with properties affecting solver performance in a similar way real-world instances do. The results are particularly important to the development of disjunctive ASP solvers, as no models for generating random disjunctive programs of desired hardness have been known before.

2 Preliminaries

A clause is a set of literals that contains no pair of complementary literals. By a CNF formula we mean an (ordered) tuple of clauses with repetitions of clauses allowed. Disjunctions of CNF formulas are also assumed to be (ordered) tuples and they also allow repetitions. The dual concepts (such as DNF formulas) are defined similarly. In other models, CNF formulas are viewed as sets of clauses, and disjunctions of CNF formulas are viewed as sets of CNF formulas. However, assuming some reasonable limit on the number of clauses in a formula, and assuming in each case the uniform

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2When we use the term “multi-component model,” we understand that the notion is parameterized by the underlying “standard,” or single-component, model.
distribution, the two probabilistic models are asymptotically equivalent for properties that do not depend on the order (such as satisfiability). Specifically, as the number of atoms tends to infinity, the probability that such a property holds in one model and the corresponding probability for the other model converge to each other. (We offer a technical justification for this claim in Appendix B.) Thus, there is no essential difference between the two models and we use them interchangeably.

By \( C(k,n,m) \) we denote the set of all \( k \)-CNF formulas consisting of \( m \) clauses over (some fixed) set of \( n \) propositional variables. Similarly, \( D(k,n,m) \) stands for the set of all \( k \)-DNF formulas of \( m \) products (conjunctions of non-complementary literals) over an \( n \)-element set of atoms.

### 2.1 The fixed-length clause model

The model is given by the set \( C(k,n,m) \) of CNF formulas, with all formulas assumed equally likely. Formulas from the model can be generated by selecting \( m \) \( k \)-literal clauses over a set of \( n \) variables uniformly, independently and with replacement. As we noted, the model is well understood. In particular, let us denote by \( p(k,n,m) \) the probability that a random formula in \( C(k,n,m) \) is SAT. We define \( \rho_l(k) \) to be the supremum over all real numbers \( \rho \) such that \( \lim_{n \to \infty} p(k,n,[\rho n]) = 1 \). Similarly, we define \( \rho_u(k) \) to be the infimum over all real numbers \( \rho \) such that \( \lim_{n \to \infty} p(k,n,[\rho n]) = 0 \). As we mentioned, \( \rho_l(k) \) and \( \rho_u(k) \) are well defined. Moreover, \( \rho_l(k) \leq \rho_u(k) \) and, it is conjectured that \( \rho_l(k) = \rho_u(k) \). Experimental results agree with these theoretical predictions.

### 2.2 The Chen-Interian model

The model generates QBFs of the form \( \forall X \exists Y F \). Sets \( X \) and \( Y \) are disjoint and contain all propositional variables that may appear in \( F \). The sizes of \( X \) and \( Y \) are prescribed to some specific integers \( A \) and \( E \), respectively. Moreover, each clause in \( F \) contains \( a \) literals over \( X \) and \( e \) literals over \( Y \) for some specific values \( a \) and \( e \). We denote the set of all such CNF formulas \( F \) with \( m \) clauses by \( C(a,e,A,E;m) \). Clearly, \( C(a,e,A,E;m) \subseteq C(a + e + A + E,m) \). We write \( Q(a,e,A,E;m) \) for the set of all QBFs \( \forall X \exists Y F \), where \( F \in C(a,e,A,E;m) \). The Chen-Interian model generates QBFs from \( Q(a,e,A,E;m) \), with all formulas equally likely.

Chen and Interian [12] presented a comprehensive experimental study of the model. Let us denote by \( q(a,e,A,E;m) \) the probability that a random QBF from \( Q(a,e,A,E;m) \) is true. Let \( r > 0 \) be fixed real. We set \( v_l(a,e;r) \) to be the supremum over all real numbers \( v \) such that \( \lim_{n \to \infty} q(a,e,A,E;[vn]) = 1 \), where \( A = [rE] \) and \( n = A + E \). Similarly, we set \( v_u(a,e;r) \) to be the infimum over all real numbers \( v \) such that \( \lim_{n \to \infty} q(a,e,A,E;[vn]) = 0 \), again with \( A = [rE] \) and \( n = A + E \). Chen and Interian proved the following result.

**Theorem 2.1.** \( v_l(a,e;r) \) and \( v_u(a,e;r) \) are well defined.

Clearly, \( v_l(a,e;r) \leq v_u(a,e;r) \). Whether \( v_l(a,e;r) = v_u(a,e;r) \) is an open problem. The quantities \( v_l(a,e;r) \) and \( v_u(a,e;r) \) delinate the phase-transition region. For QBFs generated from the model \( Q(a,e;[rE],E;[vn]) \) (with fixed \( n \) and \( r \)), Chen and Interian experimentally observed the easy-hard-easy pattern as \( v \) grows. They showed that the hard region is aligned with the phase transition, and that the same behavior emerges no matter what concrete \( r \) is fixed as the ratio \( A/E \).
3 New models of random formulas and QBFs

We propose several variations of the models described above. They are based on two ideas. First, we impose an additional structure on clauses in CNF formulas that serve as matrices of QBFs. Second, we consider disjunctions of CNF formulas both in the SAT and QBF setting.

3.1 The controlled model

To describe the model, we define first a version of a model of a random CNF formula. In this model, clauses are built of variables in a set \( X \cup Y \), where \( X \cap Y = \emptyset \); we set \( |X| = A \) and \( |Y| = E \). A formula in the model consists of \( 2A \) \( k \)-literal clauses. Each clause consists of a single literal over \( X \) and \( k - 1 \) literals over \( Y \), and for each literal over \( X \) there is a single clause in the formula that contains it. A formula in this model is generated taking \( 2A \) \((k-1)\)-literal clauses over \( Y \) and extending each of them by a literal over \( X \) (following some fixed one-to-one mapping between the clauses and the literals over \( X \)). We denote this model (and the corresponding set of formulas) by \( C^{\text{ctd}}(k,A,E) \). We write \( Q^{\text{ctd}}(k,A,E) \) for the model (and the set) of QBFs whose matrix is a formula from \( C^{\text{ctd}}(k,A,E) \). We refer to both models as controlled. In our work we are primarily interested in the controlled model for QBFs.

Clearly, \( Q^{\text{ctd}}(k,A,E) \subseteq Q(1,k-1;A,E;2A) \). Thus, the controlled model is related to the Chen-Interian model. The main difference is that the clauses, while random with respect to existential variables are not random with respect to universal variables. For each \( x \in X \) there is exactly one clause involving \( x \) and exactly one clause involving \( \neg x \). Consequently, the number of clauses is \( 2A \) and, moreover, for every truth assignment to \( X \), once we simplify the matrix accordingly, we are left with exactly (hence, the term “controlled”) \( A \) \((k-1)\)-literal clauses over \( E \) variables. In contrast, in the case of the Chen-Interian model \( Q(1,k-1;A,E;2A) \), similar simplifications leave us with \((k-1)\)-CNF formulas with varying number of clauses, with the average number being \( A \).

Let \( q^{\text{ctd}}(k,A,E) \) denote the probability that a random formula in \( Q^{\text{ctd}}(k,A,E) \) is true. As before, we define \( \mu^{\text{ctd}}(k) \) to be the supremum over all positive real numbers \( \rho \) such that \( \lim_{E \to \infty} q^{\text{ctd}}(k,|pE|,E) = 1 \), and \( \mu^{\text{ctd}}_u(k) \) to be the infimum over all positive real numbers \( \rho \) such that \( \lim_{E \to \infty} q^{\text{ctd}}(k,|pE|,E) = 0 \).

We will now derive bounds on \( \mu^{\text{ctd}}(k) \) and \( \mu^{\text{ctd}}_u(k) \) by exploiting results on random \((k-1)\)-CNF formulas.

**Theorem 3.1.** For every \( k \geq 2 \), \( \mu^{\text{ctd}}(k) \geq \frac{p(k-1)}{2} \) and \( \mu^{\text{ctd}}_u(k) \leq \rho_u(k-1) \).

**Proof.** Let \( \Phi \in Q^{\text{ctd}}(k,A,E) \), \( X = \{x_1,\ldots,x_A\} \), and \( Y = \{y_1,\ldots,y_E\} \). By the definition, \( \Phi = \forall X \exists Y F \), where \( F = C_1 \land \ldots \land C_{2A} \) is a \( k \)-CNF formula of \( 2A \) clauses \( C_i = l_{i1} \lor \ldots \lor l_{ik} \), and where \( l_{i1} \) is a literal over \( X \) and \( l_{i2},\ldots,l_{ik} \) are literals over \( Y \). We define \( C^Y_i = l_{i2} \lor \ldots \lor l_{ik} \) and \( F^Y = C^Y_1 \land \ldots \land C^Y_{2A} \). Moreover, for every interpretation \( I \) of \( X \) we define \( F^I = \land \{C_i^Y \mid C_i \in F \text{ and } I \models l_{i1}\} \).

Let us assume that \( \Phi \) is selected from \( Q^{\text{ctd}}(k,A,E) \) uniformly at random. By the definition of the model \( Q^{\text{ctd}}(k,A,E) \), \( F^Y \) can be regarded as selected from \( C(k-1,2A,E) \) uniformly at random and, for each interpretation \( I \) of \( X \), \( F^I \) can be regarded as selected uniformly at random from \( C(k-1,A,E) \).

To derive an upper bound on \( \mu^{\text{ctd}}_u(k) \), let us fix an interpretation \( I \) of \( X \). Clearly, if \( F^I \) is unsatisfiable, then \( \Phi \) is false. Let us choose any real \( \rho > \rho_u(k-1) \). If \( A/E \geq \rho \), the
probability that $F|t$ is unsatisfiable tends to 1 with $E$ and, consequently, the probability that $\Phi$ is false tends to 0 with $E$, too. It follows that if $\rho > \rho_u(k-1)$ and $A/E \geq \rho$, the probability that $\Phi$ is true tends to 0 with $E$. Since $\rho$ is an arbitrary real such that $\rho > \rho_u(k-1)$, $\mu(k) \leq \rho_u(k-1)$ follows.

To prove the lower bound, we observe that if the formula $F^Y$ is satisfiable, then for every interpretation $I$ of $X$, the formula $F|t$ is satisfiable or, equivalently, $\Phi$ is true. Let $\rho$ be a positive real number such that $\rho < \frac{\rho_l(k-1)}{2}$. By the definition of $\rho_l(k-1)$, if we assume that $A/E \leq \rho$, that is, $2A/E \leq 2\rho < \rho_l(k-1)$, the probability that $F^Y$ is satisfiable tends to 1 with $E$. Thus, the probability that $\Phi$ is true tends to 1 with $E$. It follows that $\mu_{ctd}(k) \geq \frac{b(k-1)}{2}$.

It follows that as $\rho$ grows, the properties of $Q^{ctd}(k, |pE|, E)$ change. For small values of $\rho$, randomly selected QBFs are almost surely true. As $\rho$ grows beyond $\mu_{ctd}(k)$ the proportion of false formulas grows until, eventually, when $\rho$ grows beyond $\mu_{ctd}(k)$, the formulas in the model are almost surely false. Clearly, $\mu_{ctd}(k) \leq \mu_{ctd}(k)$. As in the other cases, the question whether $\mu_{ctd}(k) = \mu_{ctd}(k)$ is open.

### 3.2 The multi-component models

Let $F$ be a class of propositional formulas (or a model of a random formula). By $t$-$F$ we denote the class of all disjunctions of $t$ formulas from $F$ (or a model generating disjunctions of random formulas from $F$). Similarly, if $Q$ is a class (model) of QBFs of the form $\forall X \exists Y F$, where $F \in F$, we write $t$-$Q$ for the class (model) of all QBFs of the form $\forall X \exists Y F$, where $F \in t$-$F$. We refer to models $t$-$F$ and $t$-$Q$ as multi-component. For QBFs we also consider the dual model to $t$-$Q$, based on conjunctions of $t$ DNF formulas. It gives rise to a multi-component model of disjunctive logic programs via the Eiter-Gottlob translation. In all cases, we assume that formulas (QBFs, respectively) are equally likely.

We first observe that the multi-component model $t$-$C(k,n,m)$ has similar satisfiability properties as $C(k,n,m)$, and that the phase transition regions in the two models are closely related. Let $p_l(k,n,m)$ be the probability that a random formula in $t$-$C(k,n,m)$ is SAT. Clearly, $p_l(k,n,m) = p(k,n,m)$.

**Theorem 3.2.** Let $t \geq 1$ be a fixed integer. Then, for every $\rho < p_l(k)$, $\lim_{n \to \infty} p_l(k,n, |pn|) = 1$, and for every $\rho > p_u(k)$, $\lim_{n \to \infty} p_l(k,n, |pn|) = 0$.

**Proof.** As we discussed earlier, we can assume that our model actually generates ordered $t$-tuples of $C(k,n,m)$ formulas (they represent disjunctions of $t$ formulas from the model $C(k,n,m)$, where repetitions of disjuncts are allowed, and disjunctions differing in the order of disjuncts are viewed as different). Thus, it is clear that

$$p_l(k,n,m) = 1 - (1 - p(k,n,m))^t. \tag{1}$$

It follows that for every fixed $t$, and every $\rho$,

$$\lim_{n \to \infty} p_l(k,n, |pn|) = 0 \quad \text{if and only if} \quad \lim_{n \to \infty} p(k,n, |pn|) = 0$$

and

$$\lim_{n \to \infty} p_l(k,n, |pn|) = 1 \quad \text{if and only if} \quad \lim_{n \to \infty} p(k,n, |pn|) = 1.$$

Thus, the assertion follows.
Theorem 3.2 implies that if the phase transition conjecture holds for the single component model $C(k, n, m)$, it also holds for the multi-component model $t$-$C(k, n, m)$, and the threshold value is the same for every $t$.

Theorem 3.2 describes the situation when $t$ is fixed and $n$ is large. When $n$ is fixed and $t$ grows, the identity \[^1\] shows that the region of the transition from SAT to UNSAT shifts to the right. (Of course, by Theorem 3.2 once we stop growing $t$ and start increasing $n$ again, the phase transition region will move back to the left.) Our experimental study discussed later provides results consistent with this theoretical analysis. Moreover, our experiments also show that the phase transition region is where the hard formulas are located, and that hardness depends significantly on $t$.

We also considered the multi-component model $t$-$Q(a, e; A, E; m)$ of QBFs, with the Chen-Interian model as its single-component specialization. Let $q_t(a, e; A, E; m)$ be the probability that a random QBF from $t$-$Q(a, e; A, E; m)$ is true (in particular, $q_1(a, e; A, E; m) = q(a, e; A, E; m)$). Using Theorem 2.1 and reasoning as above, we can prove that the phase transition regions for different values of $t$ coincide (and coincide with the phase transition region in the Chen-Interian model).

**Theorem 3.3.** For every integer $t \geq 1$ and real $r > 0$, if $\nu < \nu_t(a, e; r)$, then $\lim_{n \to \infty} q_t(a, e; A, E; \lfloor vn \rfloor) = 1$, and if $\nu > \nu_t(a, e; r)$, $\lim_{n \to \infty} q_t(a, e; A, E; \lfloor vn \rfloor) = 0$ (where $A = |rE|$ and $n = A + E$).

**Proof.** For the proof, we will assume that the model $t$-$Q(a, e; A, E; m)$ generates QBFs with matrices that are ordered $t$-tuples of formulas generated from the model $C(a, e; A, E; n)$. As before, we have that for each fixed positive integer $t$, \[ q_t(a, e; A, E; \lfloor vn \rfloor) = 1 - (1 - q(a, e; A, E; \lfloor vn \rfloor))^t. \] This identity implies the claim in the same way as \[^1\] implied the assertion of Theorem 3.2.

The experimental results on satisfiability of QBFs from $t$-$Q(a, e; A, E; m)$, which we present in Section 5, agree with our theoretical analysis; we will also see there the easy-hard-easy pattern and a strong dependence of hardness on $t$.

Finally, we considered the multi-component model $t$-$Q^{cd}(k, A, E)$, which incorporates both ideas we proposed in the paper. As in the other two cases, it is easy to derive the existence of the phase transition region and its invariance with respect to $t$ from the results on the underlying single-component model which, for the controlled model are given in Theorem 3.1. Let $q_t^{cd}(k, A, E)$ denote the probability that a random formula in $t$-$Q^{cd}(k, A, E)$ is true.

**Theorem 3.4.** For every integer $t \geq 1$, if $\rho < \mu_t^{cd}(k)$, then $\lim_{E \to \infty} q_t^{cd}(k, |PE|, E) = 1$, and if $\rho > \mu_u^{cd}(k)$, $\lim_{E \to \infty} q_t^{cd}(k, |PE|, E) = 0$.

**Proof.** For the proof, we will assume that the model $t$-$Q^{cd}(k, |PE|, E)$ generates QBFs with matrices that are ordered $t$-tuples of formulas generated from the model $C^{cd}(k, |PE|, E)$ (disjunctions of $t$ formulas from the model, where repetitions of disjuncts are allowed and the order matters). As in the two classes of multi-component models we considered above, we have that for each fixed positive integer $t$, \[ q_t^{cd}(k, |PE|, E) = 1 - (1 - q^{cd}(k, |PE|, E))^t. \] This identity, when combined with Theorem 3.1, implies the assertion.
We also studied the model $t$-$Q^{ctd}(k,A,E)$ experimentally. The results are reported in Section 5. As in other cases, they agree with the predictions of the theoretical analysis above. Importantly, they show that formulas from the model $t$-$Q^{ctd}(k,A,E)$ can be “super-hard.” That is, using “controlled form” CNF formulas in the disjunctions of a multi-component model, yields a way to generate formulas that are much harder than those generated from any other model considered before.

4 Random Disjunctive Programs

Our results on QBFs imply models of random disjunctive logic programs. This is important as disjunctive logic programs increase the expressive power of answer set programming, posing, at the same time, a computational challenge [9, 22].

Our approach to design models of random disjunctive programs is based on the translation from QBFs to programs due to Eiter and Gottlob [18]. The Eiter-Gottlob translation works on QBFs $\Phi = \exists X \forall Y G$, where $G$ is a DNF formula.

To describe the translation, let us assume that $X = \{x_1, \ldots, x_E\}$, $Y = \{y_1, \ldots, y_A\}$ and $G = D_1 \lor \cdots \lor D_m$, where $D_i = \bigwedge_{j=1}^{k_i} L_{i,j}$ and $L_{i,j}$ are literals over $X \cup Y$. For every atom $z \in X \cup Y$ we introduce a fresh atom $z'$. For every $z \in X \cup Y$, we set $\sigma(z) = z$ and $\sigma(\neg z) = z'$. Finally, we introduce one more fresh atom, say $w$, and define a disjunctive logic program $P_\Phi$ to consist of the following rules:

- $z \lor z'$ for each $z \in X \cup Y$
- $y \leftarrow w$ and $y' \leftarrow w$ for each $y \in Y$
- $w \leftarrow \sigma(L_{i,1}), \ldots, \sigma(L_{i,k_i})$ for each $D_i, i = 1, \ldots, m$
- $w \leftarrow \neg w$

**Theorem 4.1** (Eiter and Gottlob [18]). Let $\Phi$ be a QBF $\exists X \forall Y G$, where $G$ is a DNF formula over $X \cup Y$. Then $\Phi$ is true if and only if $P_\Phi$ has an answer set.

We will use this result to derive models of disjunctive logic programs from the models of QBFs that we considered above. We recall that these models consist of formulas of the form $\forall X \exists Y F$, where $F$ is a CNF formula. Before we can apply the Eiter-Gottlob translation, we have to transform these models (their formulas) into their dual counterparts.

To this end, for a CNF formula $F$, we denote by $\overline{F}$ the formula obtained from $\neg F$ by applying the De Morgan laws (thus, transforming $\neg F$ into DNF). Extending the notation, for each QBF $\Phi = \forall X \exists Y F$, where $F$ is a CNF formula, we write $\overline{\Phi}$ for the QBF $\exists X \forall Y \overline{F}$. Clearly, $\Phi$ is true if and only if $\overline{\Phi}$ is false (or equivalently, $\Phi$ is false if and only if $\overline{\Phi}$ is true).

**Corollary 4.1.** Let $\Phi$ be a QBF $\forall X \exists Y G$, where $G$ is a CNF formula over $X \cup Y$. Then $\Phi$ is false if and only if $P_{\Phi}$ has an answer set.

Given a model (set) of QBFs of the form $\forall X \exists Y F$, where $F$ is a CNF formula, the mapping $\Phi \mapsto \overline{\Phi}$ transforms the model into its dual, consisting of QBFs with a DNF formula in the matrix. To these formulas we can apply the Eiter-Gottlob translation, thus obtaining a model (set) of disjunctive logic programs. By Corollary 4.1, this model has the same satisfiability properties as the original QBF model modulo the switch between true and false.
We now define \( Q(e, a; E, A; m) = \{ \Phi : \Phi \in Q(e, a; E, A; m) \} \). The model (set) \( Q(e, a; E, A; m) \) is the dual to the Chen-Interian model \( Q(e, a; E, A; m) \). Applying the Eiter-Gottlob translation \( \Psi \mapsto P_{0} \) to QBFs \( \Psi \in Q(e, a; E, A; m) \), yields a model (set) of disjunctive logic programs, which we denote by \( D_{dip}(e, a; E, A; m) \). It follows from our comments after Corollary 4.1 that the theoretical results we obtained for the Chen-Interian model \( Q(e, a; E, A; m) \) apply directly to the model \( D_{dip}(e, a; E, A; m) \) (modulo the switch between true and false).

Next, we define \( Q^{ad}(k, E, A) = \{ \Phi : \Phi \in Q^{ad}(k, E, A) \} \). The model \( Q^{ad}(k, E, A) \) is dual to our controlled model of QBFs. By applying the Gottlob-Eiter translation to QBFs in \( Q^{ad}(k, E, A) \), we obtain the model (set) of disjunctive logic programs, which we denote by \( D_{dip}^{ad}(k, E, A) \). As before, by our comments following Corollary 4.1, the models \( Q^{ad}(k, E, A) \) and \( D_{dip}^{ad}(k, E, A) \) have the same satisfiability properties (modulo the switch between true and false).

### 4.1 Multi-component models of disjunctive logic programs

The translation proposed by Eiter and Gottlob can be extended to QBFs of the form \( \Phi = \exists X \forall Y G \), where \( G = G_{1} \wedge \ldots \wedge G_{t} \) and each \( G_{i} \) is a DNF formula. The translation is similar, except that we need \( t \) additional variables \( w_{1}, \ldots, w_{t} \) to represent DNF formulas \( G_{i} \). The translation consists of rules

\[
\begin{align*}
    z \lor z' & \quad \text{for each } z \in Z \\
y \leftarrow w \text{ and } y' \leftarrow w & \quad \text{for each } y \in Y \\
w \leftarrow w_{1}, \ldots, w_{t} \text{ and } w \leftarrow \neg w
\end{align*}
\]

that form the fixed part of the translation, and its core consisting of Horn rules

\[
w_{h} \leftarrow z_{1}, \ldots, z_{\ell}
\]

where \( h = 1, \ldots, t \), and the rules with the head \( w_{h} \) are obtained from the formula \( G_{h} \) just as in the original Eiter-Gottlob translation (except that \( w_{h} \) is now used as the head and not \( w \)). In fact, in the case when \( t = 1 \) the program above coincides with the result of the Eit-Gottlob translation modulo a rewriting, in which we eliminate the rule \( w \leftarrow w_{1} \) and replace \( w_{1} \) in the head of each rule in the core with \( w \).

Extending the earlier notation, we denote the program described above by \( P_{0} \). The following result can be derived by an argument similar to that Eiter and Gottlob used to prove their theorem.

**Theorem 4.2.** Let \( \Phi = \exists X \forall Y (G_{1} \wedge \ldots \wedge G_{t}) \), where each \( G_{i} \) is a DNF formula. Then \( \Phi \) is true if and only if \( P_{0} \) has an answer set.

We can now derive multi-component models of disjunctive logic programs from the multicomponent models of QBFs. The basic idea is the same as before. A multi-component model of QBFs gives rise to its dual via a transformation \( \Phi \mapsto \Phi \) (it consists of negating \( \Phi \) and applying De Morgan laws). Next, the translation above transforms QBFs from that dual model into disjunctive programs, yielding the corresponding multi-component model of programs. We apply this approach to two multi-component models of QBFs we considered in this paper: \( t-Q(e, a; E, A; m) \) and \( t-Q^{ad}(k, E, A) \). We denote the corresponding models of disjunctive logic programs by \( t-D_{dip}(e, a; E, A; m) \) and \( t-D_{dip}^{ad}(k, E, A) \).

**Corollary 4.2.** Let \( \Phi = \exists X \forall Y F \), where \( F \in t-Q(e, a; E, A; m) \) or \( F \in t-Q^{ad}(k, E, A) \). Then, \( \Phi \) is false (\( \Phi \) is true) if and only if \( P_{\Phi} \) has an answer set.
We now describe an experimental analysis of the behavior of our models and discuss their properties.

5 Empirical analysis

We now describe an experimental analysis of the behavior of our models and discuss their properties.

5.1 Experiment Setup

To claim that properties and patterns are inherent to a model and not an artifact of a solver used, we performed our experiments with several well-known SAT, QBF and ASP solvers. The SAT solvers included GLUCOSE 4.0 [1], LINGELING, version of 2015 [8]; and KCNFs, version of SAT’07 competition [16]. The QBF solvers included BO-CETAR (a combination of blogger preprocessor [23] and ghostqv [30] solver from QBF gallery 2014); AIGSOLVE [33]; RAREQS [26], version 1.2 from QBF competition 2016; and AQUA-S2V. Finally, the two ASP solvers we used in experiments were CLASP (a combination of gringo 4.5.3 [19]. All solvers were run in their default configurations. We stress that we did not aim at comparing solver performance, instead our goal was to identify solver-independent properties inherent to a model.

To support experiments, we developed a tool in Java to generate random CNF formulas from \( C(k, n, m) \), QBFs from \( Q(a, e; A, E; m) \) and \( Q^{id}(k, A, E) \), and programs from \( D_{dlp}(e, a; E, A; m) \) and \( D_{dlp}(k, E, A) \) (“dual” to QBFs from \( Q(e, a; E, A; m) \) and \( Q^{id}(k, E, A) \)). For each class \( C \) of formulas and programs listed, our tool generates also formulas (programs) from the corresponding multicomponent model \( t-C \).

Formulas and QBFs generated according to the multi-component models \( t-C(k, n, m) \), \( t-Q(a, e; A, E; m) \) and \( t-Q^{id}(k, A, E) \), where \( t > 1 \), are non-clausal or have non-clausal matrices (in the case of QBFs). As they do not adhere to the (Q)DIMACS format required by SAT/QBF solvers, the generator transforms non-clausal formulas to CNF using the Tseitin transformation [40]. That transformation introduces fresh auxiliary variables (while replacing binary subformulas) and new clauses (modeling the equivalence of each replacement) to obtain a CNF formula that is equisatisfiable to the original one. The Tseitin transformation is efficient, since it only causes a linear growth in size (whereas doing the same normalization via distributivity laws may lead to an exponential blow-up). Interestingly, the logic programs in the models \( t-D_{dlp}(e, a; E, A; m) \) and \( t-D_{dlp}(k, E, A) \) have a much simpler structure than the corresponding Tseitin-transformed formulas from the “dual” models \( t-Q(e, a; E, A; m) \) and \( t-Q^{id}(k, E, A) \). As can be seen from the translation, these programs need new variables only to represent each of the \( t \) components (disjuncts) of the matrix formula.

Once a formula \( \Phi \) is generated, it is stored in two files: one with an encoding of \( \Phi \) in the (Q)DIMACS numeric format of (Q)SAT solvers [22, 35], and the other one with

\[ \text{www.qbflib.org/DESCRIPTIONS/aqua16.pdf} \]

For this reason the Tseitin transformation is employed very often in real-world applications of SAT/QBF. Actually, many formulas used in SAT and QBF competitions [27, 35] come from applying it to non-normal form inputs suggested by problem statements.
the disjunctive logic program corresponding to $\Phi$ in the ASPCore 2.0 syntax \cite{11}. As discussed in the previous section, since the programs are generated from the negations of the QBFs in our random QBF models, they have answer sets if and only if the original formulas are false. Thus, when we analyze satisfiability we plot only the curves obtained by evaluating either the formulas or the corresponding logic programs (the plots are

Figure 1: Behavior of controlled model: phase transition and hardness.
symmetric to each other). In all the experiments the results are averaged over 128 samples of the same size.

Experiments were run on a Debian Linux with 2.30GHz Intel Xeon E5-4610 v2 CPUs and 128GB of RAM. Each execution was constrained to one single core by using the taskset command. Time measurements were performed by using the runlim tool.
The generator used in the experiments is publicly available at [https://www.mat.unical.it/ricca/RandomLogicProgramGenerator](https://www.mat.unical.it/ricca/RandomLogicProgramGenerator).
5.2 Behavior of the controlled model

We first study the satisfiability and hardness of formulas and corresponding programs generated according to the controlled model. We generated QBF instances from the model $Q_{ctd}^d(4, A, E)$ and program instances from the dual model $D_{dlp}(4, A, E)$ for the parameters $E$ and $A$ ranging over $[10..60]$ and $[20..180]$, respectively (consequently, the number of clauses ranges from 40 to 360).

Figure 4(a) shows the satisfiability results for the model $Q_{ctd}^d(4, A, E)$. The picture for $D_{dlp}(4, A, E)$ is dual (symmetric with respect to the plane given by the frequency of satisfiability equal to $1/2$); the results we show were in fact obtained by running CLASP on programs from $D_{dlp}(4, A, E)$ and adapted to the case of $Q_{ctd}^d(4, A, E))$. The gradient of colors ranging from yellow (QBF true) to black (QBF false) helps to identify the phase transition region, which is also projected on the $A$-$E$ plane below. We observe that phase transitions occur for a specific value of the ratio between universal and existential variables, specifically, for $A/E \simeq 2.37$. A different perspective on the same data is presented in Figure 2a, where the frequency of satisfiability is depicted with respect to the ratio $A/E$, and where the two straight lines show the bounds predicted by Theorem 3.1, assuming the bounds for satisfiability and unsatisfiability of 3-CNF formulas $[28, 17]$ (i.e., $\mu_{ctd}(4) \geq 3.52^2 = 1.76$ and $\mu_{ctd}^u(4) \leq 4.49$). We observe that the transition sharpens when the number of variables grows, and the transition occurs within the bounds predicted by the theoretical results.

To study the hardness of formulas, the average running times are plotted in Figure 4(b). Here the gradient of colors ranging from black (basically instantaneous execution) to yellow (the maximum average running time) helps to identify the hardness region. As before the region is also projected on the $A$-$E$ plane below. As expected hardness arises around the phase transition region and grows with the number of variables. To provide evidence that the hardness of the controlled model is independent of the solver used, Figure 2b plots the average execution times when running two QBF solvers (RAREQS and AQUA-S2V) and two ASP solvers (CLASP and WASP) on formulas/programs implied by formulas from $Q_{ctd}^d(4, A, E)$ with 48 existential variables, and the QBF solver BQ-CEGAR on formulas with 24 existential variables (for that solver, we had to decrease the size of formulas to ensure termination within a reasonable time). We note that all
solvers find hard formulas in the same region, and the maximum hardness coincides with
the transition zone marked by the red vertical strip. No data is reported in Figure 2b for
\texttt{AIGSOLVE} because it terminated abruptly in some instances (throwing \texttt{std::bad alloc})
and in some other we had to kill the process after 15 days of execution. (This behavior
is probably due to a memory access problem.)

5.3 Controlled vs Chen-Interian model

We now compare the controlled and the Chen-Interian models with respect to the
hardness of formulas having the same number of variables.

We start by presenting results on the behavior of the Chen-Interian model \(Q(a,e;A,E;m)\),
where we set \(a = 1, e = 3, \) and \(E = 70,\) and vary the number \(A\) of universal variables over
the range \([2..300]\) and the number \(m\) of clauses over the range \([200..700]\). These results
are shown in Figure 3. They confirm and extend the findings by Chen and Interian [12].

As before the gradient of colors in Figure 3, ranging from black to yellow, outlines
the phase transition and the easy-hard-easy pattern. The surface is also projected onto
the \(A\)-\(m\) plane for an alternative visualization. For every value of \(A\) (in fact, for every
value of the ratio \(A/E\); indeed, we recall that in our experiment the we fixed the value
of \(E\) to 70), as we grow \(m\) we observe the phase transition. The place where this phase
transition occurs depends on \(A\) (more generally, on the ratio \(A/E\); but in our experiments
\(E\) is fixed). For each value of \(A\) (more precisely, for each value of \(A/E\)), the hardest
formulas are located around the phase transition area, as evidenced by Figure 3(b). The
behavior presents there only for the values of \(A\) of up to about 85; for higher values of \(A,
the running times even on the formulas from the phase transition region are very small.

Figure 3(b) also shows that the overall peak of hardness occurs in the phase transition
region for a specific value of \(A\) or, as explained earlier, for a specific value of the ratio
\(A/E\).

Next, we compare the hardness properties of the controlled and the Chen-Interian
models with the same number of existential variables, which can be viewed as a measure
of the hardness of individual SAT instances that arise while solving a QBF of the form
\(\forall \exists F\). The graphs in Figure 4 capture the behavior of the hardness for the two models
under this constraint. For the controlled model, for each value of \(A\), the value on the
hardness graph (the blue line) is obtained by averaging the solve times on
formulas generated from the model \(Q_{\text{ctd}}(4,A,70)\). The matrices of these formulas are
4-CNF formulas over \(A+70\) variables and with \(2A\) clauses. The corresponding point
on the hardness graph for the Chen-Interian model is obtained by averaging the solve
times on formulas generated from the model \(Q_{\text{id}}(1,3;A,70;\text{max})\), where for each \(A\) (and
\(E = 70\)), \text{max} is selected to maximize the solve times (in particular, \text{max} falls in the
phase transition region for the combination of the values \(A\) and \(E = 70\)). The matrices
of these formulas are 4-CNF formulas over \(A+70\) variables and \text{max} clauses.

The results show that the peak hardness regions for the two models are not aligned.
The hardest formulas over 70 existential variables from the Chen-Interian models have
\(A \approx 50-55\) universal variables and \(m \approx 350\) clauses. The hardest formulas over 70
existential variables from the controlled model have \(A \approx 170\) and \(m \approx 340\). Our results
show that the hardest formulas from the controlled model are almost two orders of
magnitude harder than the hardest formulas from the Chen-Interian model. On the other
hand, while the hardest formulas (for a fixed value of \(E\), here \(E = 70\)) in the two models
have similar numbers of clauses (about 340-350), the Chen-Interian model formulas
have fewer universal variables (about 50-55 versus 170 in the controlled model).

It is also useful to look at the point where the hardness of one model meets the
other. It happens for $A \approx 150$. At this point, the CNF formulas that are the matrices of QBFs from the controlled model have 70 existential and about 150 universal variables, and about 300 clauses. The corresponding parameters for the formulas from the Chen-Interian model have very similar values. Indeed, the hardest formulas for the Chen-Interian model when $E = 70$ and $A = 150$ have about 300 clauses (cf. Figure 3).

To summarize, a direct comparison for the hardness of the two models is not clear cut. On the one hand, our results show that if we make the comparison for models with the same number of existential variables the points, in terms of $A$, in which the two model generate their hardest instances are very different. On the other hand, there is a setting (corresponding to the phase transition for the controlled model) in which the controlled model generates much harder formulas than any other setting (corresponding to a phase transition) for the Chen-Interian model.

For the sake of completeness, we report that we obtained results consistent with those discussed above experimenting with other settings of existential variables and clause lengths.

### 5.4 Behavior of Multi-component Model

To study the satisfiability of multi-component model instances (the location of the phase transition), we considered the setting with the number of variables (propositional atoms) fixed. Figure 5a shows the results for the $t$ component model $t$-$C(3,200,m)$, with $t \in \{1,3,5,7,9,11\}$. The $x$-axis gives the ratio of the numbers of clauses and variables ($m/200$), the $y$-axis shows the frequency of SAT. Consistently with our theoretical results, the phase transition shifts from left to right, and it sharpens for growing values of $t$. The same can be observed in Figure 6a showing the frequency of QBFs from $t$-$Q(1,3;24,12;m)$ that are true, for $t \in \{1,3,5,7,9,11\}$. The satisfiability plots obtained for logic programs from the corresponding models $t$-$D_{dp}(1,3;24,12;m)$ are symmetric with respect to the line $y = 0.5$ and are not reported.

To study the hardness of the multi-component model we computed the average solver running times. The results (on the same instances as before) for the GLUCOSE SAT solver and the BQ-CEGAR QBF solver are in Figures 5b and 6b. The plots show a strong dependency of the hardness on the number of components: the peak of hardness moves right and grows visibly with $t$. In more detail, the CNF formulas (one component) are solved by GLUCOSE in less than 0.42s, whereas instances with 11 components require about 7 minutes, i.e., they are more than 3 orders of magnitude harder. Analogous behavior is observed when running BQ-CEGAR on QBF formulas. Those from the one-component model are solved instantaneously (average time $\leq 0.01s$), those from the 11-component model require about one minute. The experiments with other solvers gave similar results.

To underline the dependency of the hardness on the number of components, for each solver we compute the average time over samples of the same size and plot its maximum (for simplicity maximum execution time) for several values of $t$ in Figures 5c (SAT) and 6c (QBF, programs). In particular, Figure 5c reports the results obtained by running GLUCOSE and LINGELING, and Figure 6c — the results obtained by running BQ-CEGAR, AIGSOLVE, AQUA-S2V, RAREQS and the results obtained by running CLASP and WASP on the corresponding programs. The picture shows that the peak of difficulty grows with the number of components no matter the implementation or the
Figure 5: Behavior of multi-component model (SAT): phase transition and hardness.

representation roughly, at a rate that is more than quadratic with \( t \) (y-axis in logarithmic scale).

Next, we discuss the behavior of formulas when both the number of variables and the number of components grow. Figure 7a reports on the behavior of CNF formulas with \( n \in \{100, 200\} \) and \( t \in \{1, 10\} \). Formulas with 100 variables are plotted in red,
and those with 200 variables in blue. We use squares to identify graphs for formulas with one component and stars for graphs concerning formulas with ten components. Figure 7a shows that when the number of variables grows the phase transition moves to the left, and the transition becomes sharper. By Theorem 3.2, we expect that the bounds on (un)satisfiability do not depend on \( t \), indeed when the number of variables grows the
right shift due to an increase in the number of components is compensated, and becomes negligible. Our experiments also confirm that hardness grows with both the number of components and the number of variables. This is seen in Figure 7a in the bottom, which plots the average number of choices taken by CLASP (we consider choices since
Figure 8: Comparing Chen-Interian and Controlled model: effect of components.

5.5 Combination of Controlled and Multi-component model

We now present the results obtained by combining the two models presented in this paper. We focus on the effect of the combination of models on the hardness of formulas. The results are summarized in Figure 8 where a bar plot depicts the maximum average execution times (i.e., the average execution times measured evaluating the hardest instances at the phase transition) obtained by running ASP and QBF solvers on instances of models \( t-Q(1,3;A,E;m) \) (multi-component with Chen-Interian) and \( t-Q^{ctd}(4,A,E) \) (multi-component with controlled) varying the number of components \( t \in \{1,3,5,7,9,11\} \). To obtain comparable execution times with both ASP and QBF solvers, CLASP and WASP were run on instances with \( E = 24 \), RAREQS and AquaS2V on instances with \( E = 18 \), whereas BQ-Cegar and AigSolve on instances with \( E = 12 \), and \( A \in [2,120] \) and \( m \in [2,300] \). Figure 8 shows histograms for each solver. The results obtained for each setting of \( t \) in \( t-Q(1,3;A,E;m) \) and \( t-Q^{ctd}(4,A,E) \) are reported side by side in blue and orange bars, respectively. The red horizontal line helps identifying a timeout of 24 hours, and a red bar ending with an arrow indicates that some execution required more than 24 hours. A red exclamation mark identifies abrupt termination of a solver.

We observe that, no matter the solver, the hardest instances of multi-component with controlled are at least one order of magnitude harder than the Chen-Interian-based counterparts for all settings of \( t \). Notably, the combination of the two new models allows to generate instance that are “super-hard”; indeed instances with one component are solved in less than 0.9s and it was sufficient to set \( t = 11 \) to obtain instances that are more than six orders of magnitude harder to evaluate (some “controlled” instances with \( E \geq 18 \) could not even be solved in 24 hours).
5.6 Impact on SAT Solving

A desirable property of a random model is to generate instances that behave similarly to real-world ones [29, 5]. This similarity has been measured empirically by comparing the performance of solvers for random and industrial instances. Following Ansótegui et al. [5], we measure the ratio of the execution times of solvers. We compared KCNFs (a well-known SAT solver specialized in random instances) with GLUCOSE and LINGELING (both specialized in real-world instances) to assess whether our model allows to generate instances that are better solved by solvers for real-world instances. Figure 9 shows the results for the model $t$-C$(3, 100, m)$, while varying the number of components $t \in \{1, 2, 3, 4, 5\}$. In particular, the x-axis gives the ratio of the numbers of clauses and variables ($m/100$), and the y-axis shows GLUCOSE versus KCNFs (in Figure 9a) and LINGELING versus KCNFs (in Figure 9b).

We observe that, KCNFs is faster (ratios $> 1$) than both GLUCOSE and LINGELING when $t = 1$, i.e., when our model coincides with the classical one for random formulas. Once we increase the number of components the result is reverted, GLUCOSE and LINGELING are faster than KCNFs (ratios $< 1$), and the difference grows significantly with $t$. This is independent of the clauses/variables ratio.

The difference between random and real-world instances is often attributed to the presence of some hidden structure in the latter [6]. We observed that multi-component models yield instances that are solved faster by solvers designed for real-world instances. We conjecture this is due to the component structure introduced by the model. This structure can be controlled by varying the number of components, yielding instances of varying hardness.

5.7 Impact on QBF and ASP Solving

An analysis distinguishing the behavior of random and industrial instances is not possible for ASP and QBF solvers. Indeed, no QBF/ASP solvers have ever been designated (or known) as specialized to random instances in ASP and QBF Evaluations so far (cf. [11, 35] and http://www.qbflib.org). Nonetheless our models has other interesting implications for QBF and ASP solvers.

Impact on QBF Solving  To assess the validity of our multicomponent Chen-Interian model for QBF, we submitted several instances to the QBF Evaluation 2016. All our instances (with $n = 100$ only, and $t \leq 6$) were classified as hard by the organizers, and helped identify a bug in one of the participating solvers, demonstrating the efficacy of our model in performance analysis and in correctness testing.

Impact on ASP Solving  For ASP solvers, Figure 10a outlines the impact of our model on answer set search for programs corresponding to QBF formulas $t$-$Q(1, 3, 24, 12, m)$ with $t \in \{1, 3, 5, 7, 9, 11\}$. ASP solvers evaluate disjunctive programs by first computing a candidate model, and then checking its stability (the latter task is co-NP complete). Thus, we plot (i) the ratio between the number of choices made during the search phase and the number of stable model checks performed by WASP and CLASP, and (ii) the ratio between the time spent in stable model checking and the total execution time for the solver WASP (results for CLASP are analogous) both for growing $t$. The ratio between the numbers of choices and model checks decreases when the number of components grows, following a similar behavior for both solvers. This is a machine-independent
measure of the impact of the two activities, and we observe that the role of the model checker grows with $t$. Specifically, the impact of the model checking on the total solving time grows from about about 3% ($t = 1$) to 88% ($t = 11$). Analogous considerations are supported by Figure 10b which outlines the impact of the combination of controlled
and multi-component model on answer set search for programs corresponding to QBF formulas $t$-$Q^{ctd}(4,32,16,m)$ with $t \in \{1,3,5,7,9,11\}$. Also in this case, and increase of $t$ causes both (i) a decrease of the ratio between the number of choices made during the search phase and the number of stable model checks, and (ii) an increase of the the time spent in stable model checking and the total execution time for the solver WASP (results for CLASP are analogous). Specifically, the impact of the model checking on the total solving time grows from about about 0.05% ($t = 1$) to 51% ($t = 11$).

It is known that on usual benchmarks ASP solvers spend more time in the model search phase than in the final model checking phase [37] (this also happens on benchmarks we generated for $t = 1$). However, our multi-component models allow us to generate in a controllable way instances that put emphasis on the model checking phase.

Finally, we report some other observations that point to a potential impact of our models in detecting areas of improvement for solvers. Let us recall that the two ASP solvers we studied, CLASP and WASP, employ different strategies for stable model checking. CLASP searches for unfounded sets [21], while WASP searches for a minimal model of the program reduct [3]. Both solvers are able to check partial interpretations, but they employ different heuristics for enabling this search space pruning technique. Figure 11 compares WASP and CLASP by plotting the ratio between the time required by the two solvers for finding an answer set (labeled WASP/CLASP) and the ratio between the number of partial and total checks performed by WASP (labeled pc/c(WASP)) and CLASP (labeled pc/c(CLASP)) for the two multi-component models we studied. The results on the multi-component Chen-Interian model are in Figure 11a. The results on the multi-component controlled model are in Figure 11b. One can see that WASP is faster than CLASP when the number of components is small. When the number of components grows CLASP becomes faster and takes over. Interestingly, the deterioration in the performance of WASP corresponds to the point in which the ratio pc/c starts increasing. In contrast, CLASP maintains consistently the ratio of about 70% of the numbers of partial and total checks, and this seems to pay off for larger values of $t$.

The results suggest that partial checking in WASP was implemented in a less efficient way then in CLASP, and it hinders WASP when the number of components grows. It seems also that for easier instances better performance could be obtained by disabling or reducing the number of partial checks as they do not seem to be essential for the performance. These observation suggest that there is space for solver developers to devise smarter heuristics for improving the usage of partial checking.

6 Conclusions

In this paper we proposed the controlled and multi-component models for random propositional formulas, and disjunctive logic programs. The models extend the well-known fixed clause length model for k-SAT, and the Chen-Interian model for QBF.

We provide theoretical bounds that predict the location of the region where the phase-transition occurs, and we present the results of an experimental analysis that confirms our theoretical findings in practice. Our experiments also show that the hardest instances are located in the phase transition region. Moreover, in the multi-component model the hardness of formulas depends significantly on the number of components.

Comparing models, we observed that the controlled model allows one to generate random instances that are much harder than those obtained with the Chen-Interian
model with the same number of existential variables. Further, multi-component model allows one to generate random instances with few components that are several orders of magnitude harder than those generated with the same number of variables from the underlying “single-component” model. Finally, a combination of the two new models

(a) Answer set computation: Multi-component

(b) Answer set computation: Multi-component Controlled Model

Figure 10: Impact on ASP solving: answer set computation.
results in the generation of programs and formulas that are “super-hard” to evaluate.

Our experiments with different solvers and encodings gave consistent results. This supports our claim that the phenomena we observed are inherent properties of the models rather than an artifact of the solver used.
Despite their simple structure the models have theoretical and empirical properties that make them important for further advancement of the SAT, QBF and ASP solvers.

First, the hardness of formulas/programs can be controlled and, unlike in the earlier models, not only in terms of the ratio of clauses to variables. Our experiments showed that the hardness *strongly* depends on the number of components. Thus, it can also be controlled by varying that parameter, and even a small number of components can lead to extremely hard instances. Further, in our experiments (as well as in the QBF Competition 2016) instances generated according to our models helped identifying bugs in existing solvers. Moreover, the multi-component model generates formulas that in at least one aspect are similar to instances arising in practice: they are solved better by SAT solvers specialized in industrial benchmarks than by SAT solvers specialized in random ones. This makes them useful for development and testing of solvers intended for practical applications. Finally, our models of random disjunctive programs are the first such models for that class of logic objects. This and the fact that it allows us to control the role of the stable model checking phase point to its potential for the development of ASP solvers.

Our work raises an interesting open question. The controlled model we proposed and studied stipulates that clauses in the matrix of QBFs contain exactly one universal variable. It is possible to lift this requirement. We discuss some natural extensions in 6. It turns out that when the number of universal variables per clause is greater than one, the generalized model generates instances that exhibit a qualitatively different behavior. Arguably, they are easier than formulas from the corresponding Chen-Interian model.

However, a comparison to the Chen-Interian model is not clear cut, a problem we already noted for the one universal variable case. In particular, we chose to compare for hardness formulas from the two models by fixing in each model the number of existential variables to the same value. Under this constraint, the hardest formulas in the basic controlled model contain more universal variables than it is the case for the hardest formulas from the Chen-Interian model. However, for the generalized controlled model and its *smooth* version, both discussed in 6, this relationship reverses. The hardest formulas from the (smooth) generalized controlled model have many fewer universal variables than the hardest ones from the Chen-Interian model. Developing alternative perspectives on formulas from the two models might provide a better understanding of the relative hardness. This is an important avenue to explore and it requires further studies.

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Appendix

Generalized Controlled Model

The controlled model $Q^{cd}(k, A, E)$ introduced in Section 3.1 stipulates that every clause in the matrix of a QBF from the model contains exactly one universal variable, and that each universal variable occurs in exactly two clauses, in one of them as a positive literal (not negated) and in the other one as the negative literal (negated). It follows that $Q^{cd}(k, A, E) \subseteq Q(1, k - 1; A, E; 2A)$, that is, the controlled model $Q^{cd}(k, A, E)$ is a restriction of the Chen-Interian model $Q(1, k - 1; A, E; 2A)$. Moreover, the key property of the controlled model $Q^{cd}(k, A, E)$ is that, for every truth assignment to the universal variables in $X$, once we simplify the matrix accordingly we are left with exactly $A$ $(k - 1)$-literal clauses over $E$ variables, whereas in the case of the Chen-Interian model $Q(1, k - 1; A, E; 2A)$, similar simplifications leave us with with varying number of $(k - 1)$-clauses, with the average number being $A$.

We now generalize the model to allow clauses with exactly $h$ occurrences of universal variables, where $h$ is a fixed integer satisfying $1 \leq h \leq k$. More precisely, the model consists of QBFs $\forall X \exists Y F$, where $F$ consists of $\binom{d}{h} 2^h$ $k$-literal clauses, each clause consists of $h$ literals over $X$ and $k - h$ literals over $Y$ (with no repetitions of variables), and where for every consistent set of $h$ literals over $X$ there is a single clause in the formula that contains them. A QBF in this model (to be precise, its matrix) is obtained by generating $\binom{d}{h} 2^h$ $h$-literal clauses over $X$ and extending each of them by a randomly generated consistent $(k - h)$-element set of literals over $Y$. We denote the set of QBFs obtained in this way by $Q^{cd}(h, k - h; A, E)$ and call it the generalized controlled model.

Clearly, $Q^{cd}(k, A, E) = Q^{cd}(1, k - 1; A, E)$. Thus, the controlled model we discussed in the paper is a special case of the model described here. We also note that $Q^{cd}(h, k - h; A, E) \subseteq Q(h, k - h; A, E; \binom{d}{h} 2^h)$. Thus, the generalized controlled model is a restriction of the appropriate Chen-Interian model — while random with respect to variables in $Y$ (existentially quantified variables), the way variables in $X$ (universally quantified variables) are treated is fully deterministic. In particular, for every truth assignment on $X$, once we simplify the matrix accordingly, we are left with exactly $\binom{d}{h}$ $(k - h)$-literal clauses over $E$ variables in $Y$, while in the case of the Chen-Interian model $Q(h, k - h; A, E; \binom{d}{h} 2^h)$, similar simplifications leave us with $(k - h)$-CNF formulas with varying number of clauses, with the average number being $\binom{d}{h}$.

Example 6.1. Consider a set $X = \{x_1, x_2, x_3\}$ of $A = 3$ universal variables and a set $Y = \{y_1, y_2, y_3, y_4\}$ of $E = 4$ existential variables. We are interested in “generalized controlled formulas” having $k = 5$ literals with $h = 2$ of them over $X$. That is, we are interested in the model $Q^{cd}(2, 3, 3, 4)$. According to the definition, we have to build $\binom{5}{2} 2^3 = 12$ clauses of length 5. For an appropriate enumeration $C_i$, $i = 1, \ldots, 12$, these clauses will satisfy:

$$
\begin{align*}
\{x_1, x_2\} & \subseteq C_1 & \{x_1, x_3\} & \subseteq C_5 & \{x_2, x_3\} & \subseteq C_9 \\
\{\neg x_1, x_2\} & \subseteq C_2 & \{\neg x_1, x_3\} & \subseteq C_6 & \{\neg x_2, x_3\} & \subseteq C_{10} \\
\{x_1, \neg x_2\} & \subseteq C_3 & \{x_1, \neg x_3\} & \subseteq C_7 & \{x_2, \neg x_3\} & \subseteq C_{11} \\
\{\neg x_1, \neg x_2\} & \subseteq C_4 & \{\neg x_1, \neg x_3\} & \subseteq C_8 & \{\neg x_2, \neg x_3\} & \subseteq C_{12}
\end{align*}
$$

Let us choose any truth assignment $\sigma$ on $X$, for instance, $\sigma(x_1) = x_1$, $\sigma(x_2) = \neg x_2$, and $\sigma(x_3) = \neg x_3$. Once we simplify the clauses with respect to this assignment, exactly $\binom{3}{2} = 3$ clauses $C_2 \setminus \{\neg x_1, x_2\}$, $C_6 \setminus \{\neg x_1, x_3\}$ and $C_9 \setminus \{x_2, x_3\}$ remain (all other clauses after the simplifications become tautologies and can be dropped).
Let \( q^{\text{gcd}}(h,k-h,A,E) \) denote the probability that a random formula in \( Q^{\text{gcd}}(h,k-h,A,E) \) is true. We define \( \mu_1^{\text{gcd}}(h,k-h) \) to be the supremum over all positive real numbers \( \rho \) such that

\[
\lim_{E \to \infty} q^{\text{gcd}}(h,k-h,\lfloor \rho E^{1/h} \rfloor,E) = 1,
\]

and \( \mu_0^{\text{gcd}}(h,k-h) \) to be the infimum over all positive real numbers \( \rho \) such that

\[
\lim_{E \to \infty} q^{\text{gcd}}(h,k-h,\lfloor \rho E^{1/h} \rfloor,E) = 0.
\]

We will now derive bounds on \( \mu_1^{\text{gcd}}(h,k-h) \) and \( \mu_0^{\text{gcd}}(h,k-h) \) by exploiting results on random \((k-h)\)-CNF formulas. The proof is an adaptation of the proof of Theorem 5.1.

**Theorem 6.1.** For every integers \( k \) and \( h \) such that \( k \geq 2 \) and \( 1 \leq h < k \), \( \mu_1^{\text{gcd}}(h,k-h) \) and \( \mu_0^{\text{gcd}}(h,k-h) \) are well defined.

**Proof.** Let \( \Phi \in \Phi^{\text{gcd}}(h,k-h,A,E) \), \( X = \{x_1,\ldots,x_h\} \), and \( Y = \{y_1,\ldots,y_E\} \). By the definition, \( \Phi = \forall X \exists Y F \), where \( F = C_1 \land \cdots \land C_N \) is a \( k\)-CNF formula of \( N = \binom{A}{h} \) clauses \( C_i = l_{i,1} \lor \cdots \lor l_{i,h} \) are literals over \( X \) and \( l_{i,h+1},\ldots,l_{i,k} \) are literals over \( Y \). We define \( C_i^Y = l_{i,h+1} \lor \cdots \lor l_{i,k} \) and \( F^Y = C_1^Y \land \cdots \land C_N^Y \). Moreover, for every interpretation \( I \) of \( X \) we define \( F|_I = \bigwedge \{ C_i^Y \mid C_i \in F \text{ and } I \not| l_{i,1} \lor \cdots \lor l_{i,h} \} \).

Let us assume that \( \Phi \) is selected from \( \Phi^{\text{gcd}}(h,k-h,A,E) \) uniformly at random. By the definition of the model \( \Phi^{\text{gcd}}(h,k-h,A,E) \), \( F^Y \) can be regarded as selected from \( C(k-h,N,E) \) uniformly at random and, for each truth assignment \( I \) of \( X \), \( F|_I \) can be regarded as selected uniformly at random from \( C(k-h,M,E) \), where \( M = \binom{A}{h} \).

To show that \( \mu_0^{\text{gcd}}(h,k-h) \) are well defined, it is enough to show that there are \( r \) and \( s \) such that

\[
\lim_{E \to \infty} q^{\text{gcd}}(h,k-h,\lfloor rE^{1/h} \rfloor,E) = 0 \quad \text{and} \quad \lim_{E \to \infty} q^{\text{gcd}}(h,k-h,\lfloor sE^{1/h} \rfloor,E) = 1.
\]

The proof relies on an obvious property that for every fixed positive integer \( h \), there are positive constants \( \alpha_0 \) and \( \beta_0 \) such that for every sufficiently large positive integer \( A \),

\[
\beta_0 A^h \geq \binom{A}{h} \geq \alpha_0 A^h.
\]

To prove the existence of \( r \), let us fix any real \( r \) such that \( \alpha_0 (r/2)^h > \rho_0(k-h) \), and let \( A = \lfloor rE^{1/h} \rfloor \). Next, let \( \Phi = \forall X \exists Y F \) be a QBF selected uniformly at random from \( \Phi^{\text{gcd}}(h,k-h,A,E) \) and \( I \) be a truth assignment on \( X \). Clearly, if \( F|_I \) is unsatisfiable, then \( \Phi \) is false.

For all sufficiently large \( E \), we have \( A \geq (r/2)E^{1/h} \). Consequently, \( A^h \geq (r/2)^h E \) and

\[
\binom{A}{h} \geq \alpha_0 (r/2)^h E.
\]

Since \( \alpha_0 (r/2)^h > \rho_0(k-h) \), it follows that the probability that \( F|_I \) is unsatisfiable tends to 1 with \( E \). Thus, the probability that \( \Phi \) is false tends to 0 with \( E \), too. In other words,

\[
\lim_{E \to \infty} q^{\text{gcd}}(h,k-h,\lfloor rE^{1/h} \rfloor,E) = 0.
\]
To prove the existence of $s$ we proceed similarly. Let $s$ be any positive real such that $2^h \beta^h \leq \rho_i(k - h)$ and let $A = \lfloor sE^{1/h} \rfloor$. Further, as before, let $\Phi = \forall X \exists Y F$ be a QBF selected uniformly at random from $Q^{gcd}(h,k - h,A,E)$.

Clearly, if the formula $F^Y$ is satisfiable, then for every interpretation $I$ of $X$, the formula $F^Y|I$ is satisfiable or, equivalently, $\Phi$ is true. In our case, we have that $A \leq sE^{1/h}$. Thus, $A^h \leq s^h E$. It follows that $2^h (\frac{A}{h}) \leq 2^h \beta^h \leq 2^h \beta^h E$. Thus, $2^h (\frac{A}{h}) / E \leq 2^h \beta^h s^h < \rho_i(k - h)$.

It follows that the probability that $F^Y$ is satisfiable tends to 1 with $E$ and so, the probability that $\Phi$ is true tends to 1 with $E$. In other words,

$$\lim_{E \to \infty} q^{gcd}(h,k - h, \lfloor sE^{1/h} \rfloor, E) = 1.$$ 

---

**Empirical Behavior.** We now discuss properties of the generalized controlled model presented above. In particular we compare the generalized controlled and the Chen-Interian models with respect to the hardness of formulas having the same number of variables, and comment on one possible extension of the generalized model.

We consider formulas from the Chen-Interian model $Q(a,e;A,E;m)$, where we set $a = 2$, $e = 3$, and $E = 32$, and vary the number $A$ of universal variables over the range $[2..80]$ and the number $m$ of clauses over the range $[10..1200]$. As we did in Section 5.3 we compare the hardness properties of the generalized controlled and the Chen-Interian models with the same number of existential variables. These results are shown in Figure 12. For the controlled model, for each value of $A$, the value on the corresponding hardness graph (the blue line) is obtained by averaging the solve times on formulas generated from the model $Q^{gcd}(2,3,A,32)$. The matrices of these formulas are 5-CNF formulas over $A + 32$ variables and with $4(\frac{A}{3})$ clauses. The corresponding point on the hardness graph for the Chen-Interian model is obtained by averaging the solve times on formulas generated from the model $Q(2,3;A,32;\text{max})$, where for each $A$ and $E = 32$, $\text{max}$ is selected to maximize the solve times (in particular, it falls in the phase transition.
region for the combination of the values $A$ and $E = 32$). The matrices of these formulas are 5-CNF formulas over $A + 32$ variables and $m$ clauses. The results show that the peak hardness regions for the two models are not aligned. Comparing this results with the one in Figure 4 we note that the generalized controlled model instances are much easier to solve than Chen-Interian ones, almost in every setting. The peak hardness from the generalized controlled model instances happens before the maximum hardness the peak hardness region for the Chen-Interian model. This is the opposite of what happens for (basic) controlled model instances as shown in Figure 4.

One possible weakness of the generalized controlled model is that the number of clauses, $m = 4\binom{A}{2}$, grows quadratically with the number $A$ of universal variables. Informally, this growth creates “long jumps” in terms of the number of clauses in a formula as we increment $A$ and so, also the corresponding jumps in the ratio of the number of clauses to the number of existential variables. That may cause the model to miss the “sweet spot” of maximum hardness. For example, already in our experiment with $h = 2$, formulas with $A = 14$ feature 364 clauses, and formulas with $A = 15$ feature 420 clauses. We established experimentally that formulas with $A = 14$ are satisfied with the frequency 0.1, whereas the frequency of a satisfiable instance for $A = 15$ is 1.

In order to verify whether the “jumps” contribute to the generation of easier formulas, we further extended the generalized controlled model to fill the gaps. Specifically, the smooth generalized controlled model, denoted by $Q^{\text{gcd}}(h, k - h, E; m)$, where we specify the number of existential variables and the number of clauses in the matrix, and where the number of universal variables is determined by the constraint $2^h \binom{A}{h} + 1 \leq m \leq 2^h \binom{A}{h}$. In particular, if $m = 2^h \binom{A}{h}$, $Q^{\text{gcd}}(h, k - h, E; m)$ is defined to coincide with the generalized controlled model $Q^{\text{gcd}}(h, k - h, A, E)$. Formulas for $m$ satisfying $2^h \binom{A}{h} + 1 \leq m < 2^h \binom{A}{h}$ are obtained by generating an instance of $Q^{\text{gcd}}(h, k - h, A, E)$ and randomly choosing $m$ among its clauses.

A phase transition result holds also for the smooth generalized controlled model. Let $q^{\text{gcd}}(h, k - h, E; m)$ denote the probability that a random formula in $Q^{\text{gcd}}(h, k - h, E; m)$ is true. We define $\mu^+_q^{\text{gcd}}(h, k - h)$ to be the supremum over all positive real numbers $\rho$ such that

$$\lim_{E \to \infty} q^{\text{gcd}}(h, k - h, E, \lfloor \rho E \rfloor) = 1,$$

and $\mu^-_q^{\text{gcd}}(h, k - h)$ to be the infimum over all positive real numbers $\rho$ such that

$$\lim_{E \to \infty} q^{\text{gcd}}(h, k - h, E, \lfloor \rho m \rfloor) = 0.$$

Theorem 6.1 implies the following result.

**Corollary 6.1.** For every integers $k$ and $h$ such that $k \geq 2$ and $1 \leq h < k$, $\mu^+_q^{\text{gcd}}(h, k - h)$ and $\mu^-_q^{\text{gcd}}(h, k - h)$ are well defined.

We experimented with the smooth generalized controlled model on the same setting as before but focusing on the phase transition region, that is, on values of $A$ that are close to 14. The results reported in Figure 13 were, thus, obtained varying $A$ from 10 to 18 (so $150 \leq m \leq 612$). It can be noted that the smooth model allows us to generate formulas that are precisely in the phase transition zone, moreover we can obtain harder formulas. Nonetheless, the smooth generalized controlled model remains less hard than the Chen-Interian model, if we compare the hardest formulas that can be generated with the same number of existential variables, disregarding the number of universal variables.
Figure 13: Comparing Generalized Controlled model with Smooth Generalized Controlled model.

As we noted in the main part of the paper, alternative ways to compare the hardness of the models may exist and finding them is an important open research question.

Additional notes on the generation of formulas

Let $X$ be a set consisting of $N$ elements. We will consider the following method to generate random elements of $X^t$ (the set of all $t$-tuples over $X$):

for each position $i$, $1 \leq i \leq t$, select an element from $X$ uniformly at random.

Clearly, every element of $X^t$ is equally likely to appear as the result of this method. Thus, the method generates $t$-tuples over $X$ uniformly at random.

Let $S$ be a property of $t$-tuples over $X$ and let $p_N$ be the probability that a $t$-tuple generated by the method described above has the property $S$. It follows that $p_N$ is the probability that an $t$-tuple selected from $X^t$ uniformly at random has the property $S$.

Next, let us define

$$D^t(X) = \{\langle x_1, \ldots, x_t \rangle \in X^t : x_i \neq x_j, \text{ for } i \neq j \}$$

In other words, $D^t(X)$ is the set of all tuples in $X^t$ with no repeating elements.

Let $S$ be a property of tuples in $X^t$. We will denote by $p'_N$ the probability that a tuple selected from $D^t(X)$ uniformly at random has the property $S$. Then, if $t$ is sufficiently smaller than $N$, $p'_N$ can be closely estimated by $p_N$. To show that, let us define

$$R^t(X) = X^t \setminus D^t(X).$$

Clearly,

$$p_N = \frac{|S|}{|X^t|} \quad \text{and} \quad p'_N = \frac{|D^t(X) \cap S|}{|D^t(X)|}.$$

It follows that

$$p_N = \frac{|R^t(X) \cap S|}{|X^t|} \leq p'_N \leq p_N + \frac{|R^t(X) \setminus S|}{|X^t|}.$$
and so,
\[ p_N - \left| \frac{R(X)}{|X|^t} \right| \leq p'_N \leq p_N + \left| \frac{R(X)}{|X|^t} \right| \]
or, more explicitly,
\[ p_N - \left( 1 - \frac{(N - t + 1) \cdots (N - 1)N}{N^t} \right) \leq p'_N \leq p_N + \left( 1 - \frac{(N - t + 1) \cdots (N - 1)N}{N^t} \right) \).

**Lemma 6.1.** If \( \lim_{N \to \infty} \frac{t^2}{N} = 0 \), then
\[ \lim_{N \to \infty} \frac{(N - t + 1) \cdots (N - 1)N}{N^t} = 1 \]

**Proof.** Clearly,
\[ \left( \frac{N - t}{N} \right)^t \leq \frac{(N - t + 1) \cdots (N - 1)N}{N^t} \leq 1. \]
Moreover,
\[ \left( \frac{N - t}{N} \right)^t = \left[ \left( 1 - \frac{1}{N/t} \right)^{N/t} \right]^{t^2/N} \]
Since \( \lim_{N \to \infty} \frac{t^2}{N} = 0 \) and \( t \) is a positive integer, \( \lim_{N \to \infty} N/t = \infty \). Thus,
\[ \lim_{N \to \infty} \left( 1 - \frac{1}{N/t} \right)^{N/t} = 1/e \]
and, consequently,
\[ \lim_{N \to \infty} \left[ \left( 1 - \frac{1}{N/t} \right)^{N/t} \right]^{t^2/N} = 1. \]

**Corollary 6.2.** If \( \lim_{N \to \infty} \frac{t^2}{N} = 0 \), then there is a sequence \( \varepsilon_N \) such that \( \lim_{N \to \infty} \varepsilon_N = 0 \) and
\[ p_N - \varepsilon_N \leq p'_N \leq p_N + \varepsilon_N. \]

Next, we observe that if the property \( S \) does not depend on the order of the elements in a tuple in \( D'(X) \), that is, the probability that a tuple in \( D'(X) \) has the property \( S \) is the same for every permutation of the elements in the tuple), then the probability that a set of \( t \) elements from \( X \) has a property \( S \) (its “set version” to be precise) is given by \( p'_N \).

Our earlier discussion shows that to estimate the probability that a \( t \)-element subset of \( X \) selected uniformly at random has a property \( S \), it is sufficient to estimate the probability that a \( t \)-tuple over \( X \) (an element of \( X' \)) selected uniformly at random has the property \( S \).

In this paper, we take advantage of this observation in the case when \( X \) consists of formulas and \( S \) is the property that a set (tuple) of formulas is satisfiable (SAT), and unsatisfiable (UNSAT).

In particular, we consider in the paper the case when \( X \) is the set of all non-tautological \( k \)-literal clauses over the set of \( n \) propositional variables. We note that \(|X| = 2^{k \binom{n}{k}}\). It follows that when studying the probability that a \( k \)-CNF formula with \( m \) clauses is satisfiable, where \( m = O(n) \), the results above apply and the probability, in the limit, is the same no matter whether we view formulas as sets or ordered tuples of clauses.
We also consider the case, when $X$ is the set of all $k$-CNF formulas with $m$ clauses over a set of $n$ variables, that is, the set $C(k,n,m)$. Also here, it makes no difference whether a disjunction of such formulas is considered a set of those formulas or an ordered tuple of such formulas. Since we consider disjunctions of $t$ CNF formulas, where $t$ is fixed, the probability of such a disjunction being satisfiable is, in the limit, not affected by how we interpret the disjunction — as a set or an ordered tuple.