Girsanov’s formula for $G$-Brownian motion

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Abstract

In this paper, we establish Girsanov’s formula for $G$-Brownian motion. Peng (2007, 2008) constructed $G$-Brownian motion on the space of continuous paths under a sublinear expectation called $G$-expectation; as obtained by Denis et al. (2011), $G$-expectation is represented as the supremum of linear expectations with respect to martingale measures of a certain class. Our argument is based on this representation with an enlargement of the associated class of martingale measures, and on Girsanov’s formula for martingales in the classical stochastic analysis. The methodology differs from that of Xu et al. (2011), and applies to the multi-dimensional $G$-Brownian motion.

1 Introduction

Motivated by risk measures and volatility uncertainty problems in finance, S. Peng introduced the notion of $G$-Brownian motion. Intuitively, $G$-Brownian motion is a Brownian motion whose variance is uncertain. While the classical Brownian motion is defined on a probability space, $G$-Brownian motion is defined on a sublinear expectation space, that is, the triple $(\Omega, \mathcal{H}, \mathbb{E})$, where $\Omega$ is a given set, $\mathcal{H}$ is a vector lattice of real valued functions on $\Omega$ containing 1, which is the domain of a sublinear expectation $\mathbb{E}$. $G$-Brownian motion is defined by using two notions concerning distributions on a sublinear expectation space: identical distributedness and independence. On a sublinear expectation space, the notion of distributions cannot be interpreted as that on a probability space; indeed, as introduced in [3], it also needs to be interpreted as a sublinear expectation on a class of test functions, whose class is suitably chosen according to the domain $\mathcal{H}$.

Peng [5, 6] constructed a sublinear expectation space on which the canonical process of the space $\Omega = C_0([0, \infty); \mathbb{R}^d)$ of continuous paths starting from 0 becomes a $G$-Brownian motion. The sublinear expectation in this space is called $G$-expectation. Itô’s integrals with respect to $G$-Brownian motion and the quadratic variation process of $G$-Brownian motion were also defined in [5, 6]. Recently, L. Denis, M. Hu and

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S. Peng proved in [1] that $G$-expectation can be represented as the supremum of linear expectations, referred to as the upper expectation, with respect to martingale measures of a certain class.

In this paper, we derive Girsanov’s formula for $G$-Brownian motion; when we are given a $G$-Brownian motion and a drift on the sublinear expectation space of Peng [5, 6], we construct a new sublinear expectation space on which the $G$-Brownian motion with the drift is a $G$-Brownian motion. Through the construction, $G$-expectation is transformed into a weighted $G$-expectation. The weight has the same form as that in the classical Girsanov’s formula, in which Itô’s integral for $G$-Brownian motion and the quadratic variation process are involved. A remarkable point of the construction is that not only $G$-expectation but also its domain is changed. As a sublinear expectation space is the notion including the domain of a sublinear expectation, in general some care about the choice of domains is needed when changing sublinear expectations. In the course of our discussion, it is also required that the notion of distributions is appropriately defined in the new sublinear expectation space. Those are main reasons why the domain of $G$-expectation is changed in order to formulate Girsanov’s formula for $G$-Brownian motion. Our method to prove the formula relies on the representation of the upper expectation for $G$-expectation due to Denis-Hu-Peng [1], with an enlargement of the associated class of martingale measures, and on Girsanov’s formula for martingales in the classical stochastic analysis.

During the preparation of the manuscript of this paper, we have noticed that, in [9], they give Girsanov’s formula for one-dimensional $G$-Brownian motion. We emphasize that our methodology is quite different from theirs; their proof relies on the martingale characterization of one-dimensional $G$-Brownian motion given in [10], which restricts their argument to one-dimension, whereas, as our Theorem 5.3 shows, the method we employ in this paper equally works for the multi-dimensional $G$-Brownian motion.

This paper is organized as follows. From Section 2 through Section 4, we introduce necessary notions and preliminaries: notion of distributions on a sublinear expectation space, the construction of $G$-expectation and related notions, and the upper expectation for $G$-expectation given by Denis-Hu-Peng [1]. In Section 5, we state and prove Girsanov’s formula for $G$-Brownian motion.

1.1 Notation

- $C_{b,Lip}(\mathbb{R}^n)$: the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^n$
- $C_{l,Lip}(\mathbb{R}^n)$: the space of all functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|$$

for all $x, y \in \mathbb{R}^n$ for some $C > 0, k \in \mathbb{N}$ depending on $\varphi$
- $\mathbb{R}^{d \times d}$: all $d \times d$ real matrices
- $I_d$: the $d \times d$ unit matrix
- $|x| := \sqrt{x \cdot x}$: the norm of $x \in \mathbb{R}^n$, where $\cdot$ is the inner product of $\mathbb{R}^n$
\[ \|A\| := \sqrt{\text{tr}(AA^*)} : \text{the norm of } A \in \mathbb{R}^{d \times d}, \text{where } A^* \text{ is the transposed matrix of } A \]

- For a probability measure \( P \), \( E_P \) denotes the expectation with respect to \( P \)

In the sequel, unless otherwise stated, probability spaces we deal with are all assumed to be completed.

## 2 Sublinear expectation spaces

Following Peng [3] we introduce the definition of sublinear expectations and related notions.

Let \( \Omega \) be a given set and \( \mathcal{H} \) a vector lattice of real functions on \( \Omega \) containing 1.

**Definition 2.1.** A functional \( E : \mathcal{H} \to \mathbb{R} \) is called a **sublinear expectation** if it satisfies

(i) \( E[X] \leq E[Y] \) if \( X \leq Y \),

(ii) \( E[c] = c \) for all \( c \in \mathbb{R} \),

(iii) \( E[X + Y] \leq E[X] + E[Y] \) for all \( X, Y \in \mathcal{H} \),

(iv) \( E[\lambda X] = \lambda E[X] \) for all \( \lambda \geq 0 \).

The triple \((\Omega, \mathcal{H}, E)\) is called a **sublinear expectation space**.

**Definition 2.2.** Let \((\Omega, \mathcal{H}, E)\) be a sublinear expectation space. \( X = (X^1, \ldots, X^n) \) is called an \( n \)-dimensional **random vector**, denoted by \( X \in \mathcal{H}^n \), if \( X^i \in \mathcal{H} \) for each \( i = 1, \ldots, n \). \( \{X_t; t \geq 0\} \) is called an \( n \)-dimensional **stochastic process** if for each \( t \geq 0, X_t \) is an \( n \)-dimensional random vector.

Next we introduce the notion of distributions of random variables under a sublinear expectation space. Let us consider the following sublinear expectation space: if \( X \in \mathcal{H}^n \), then \( \varphi(X) \in \mathcal{H} \) for each \( \varphi \in C_{l,Lip}(\mathbb{R}^n) \). Here the essential requirement for \( \mathcal{H} \) is that \( \mathcal{H} \) contains all constants and \( X \in \mathcal{H} \) implies \( |X| \in \mathcal{H} \). Therefore \( C_{l,Lip}(\mathbb{R}^n) \) can be replaced by another space of functions defined on \( \mathbb{R}^n \), such as \( C_{b,Lip}(\mathbb{R}^n) \), the space of all bounded Lipschitz continuous functions.

**Definition 2.3.** Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional random vectors, and \( X_3 \) an \( m \)-dimensional random vector defined on a sublinear expectation space \((\Omega, \mathcal{H}, E)\). \( X_1 \) and \( X_2 \) are called **identically distributed** if

\[ E[\varphi(X_1)] = E[\varphi(X_2)] \text{ for each } \varphi \in C_{l,Lip}(\mathbb{R}^n). \]

\( X_3 \) is said to be **independent** from \( X_1 \) if

\[ E[\varphi(X_1, X_3)] = E[E[\varphi(x, X_3)]|_{x=X_1}] \text{ for each } \varphi \in C_{l,Lip}(\mathbb{R}^{n+m}). \]
3 G-Brownian motion and G-expectation

Following Peng [5, 6], we introduce the construction of G-Brownian motion and related notions.

Fix $T > 0$, and let $\Theta$ be a given non-empty, bounded and closed subset of $\mathbb{R}^{d \times d}$. In the following, we let $\Omega := C_0([0,T]; \mathbb{R}^d)$ be the space of all $\mathbb{R}^d$-valued continuous functions $(\omega_t)_{t \in [0,T]}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \max_{t \in [0,T]} |\omega^1_t - \omega^2_t|.$$ 

For each $t \in [0,T]$, we also set $\Omega_t := \{\omega \wedge t : \omega \in \Omega\}$. We denote by $\mathcal{B}(\Omega)$ (resp. $\mathcal{B}(\Omega_t)$) the Borel $\sigma$-algebra on $\Omega$ (resp. $\Omega_t$).

### 3.1 G-Brownian motion and G-expectation

For $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$, let $u_\varphi \in C([0,T] \times \mathbb{R}^d)$ be the viscosity solution of G-heat equation with Cauchy condition $\varphi$:

$$
\begin{align*}
\frac{\partial u}{\partial t} - G(D^2 u) &= 0 & \text{in } (0,T) \times \mathbb{R}^d, \\
u|_{t=0} &= \varphi & \text{in } \mathbb{R}^d,
\end{align*}
$$

where $D^2 u$ is the Hessian matrix of $u$ and

$$G(A) := \sup_{\gamma \in \Theta} \left\{ \frac{1}{2} \text{tr}[\gamma \gamma^* A] \right\}$$

for $d \times d$ symmetric real matrix $A$.

**Remark 3.1.** For the existence and uniqueness of a viscosity solution of (3.1), refer to Appendix C, Section 3 in [3]. If there exists a constant $\sigma_0 > 0$ such that $\gamma \gamma^* \geq \sigma_0 I_d$ for all $\gamma \in \Theta$, then the solution of (3.1) becomes a $C^{1,2}$-solution.

Let $B$ be the canonical process of $\Omega$. For each $t \in [0,T]$, we denote by $C_{b,\text{Lip}}(\Omega_t)$ the set of all bounded Lipschitz cylinder functionals on $\Omega_t$:

$$C_{b,\text{Lip}}(\Omega_t) := \{\varphi(B_{t_1}, \ldots, B_{t_n}) : n \in \mathbb{N}, \ t_1, \ldots, t_n \in [0, t], \ \varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)\},$$

and we write $C_{b,\text{Lip}}(\Omega) \equiv C_{b,\text{Lip}}(\Omega_T)$ simply. We can construct a consistent sublinear expectation $\mathbb{E}$ on $C_{b,\text{Lip}}(\Omega)$ such that

- for all $0 \leq s < t \leq T$ and $\varphi \in C_{b,\text{Lip}}(\mathbb{R}^d)$,

$$\mathbb{E}[\varphi(B_t - B_s)] = \mathbb{E}[\varphi(B_{t-s})] = u_\varphi(t - s, 0),$$

- for all $n \in \mathbb{N}, \ 0 \leq t_1 < \cdots < t_n \leq T$ and $\varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n)$,

$$\mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_n})] = \mathbb{E}[\varphi_1(B_{t_1}, \ldots, B_{t_{n-1}})],$$

where $\varphi_1(x_1, \ldots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \ldots, x_{n-1}, B_{t_n}^{t_{n-1}} + x_{n-1})]$ with $B_t^s := B_t - B_s$ for $0 \leq s \leq t \leq T$. 
For $t_{k-1} \leq t < t_k$, the related conditional expectation of $\varphi(B_{t_1}, \ldots, B_{t_n})$ on $C_{b,Lip}(\Omega_t)$ is defined by

$$\mathbb{E}_t[\varphi(B_{t_1}, \ldots, B_{t_n})] := \varphi_{n-k}(B_{t_1}, \ldots, B_{t_{k-1}}, B_t),$$

where $\varphi_{n-k}(x_1, \ldots, x_{k-1}, x_k) = \mathbb{E}[\varphi(x_1, \ldots, x_{k-1}, B^t_{t_k} + x_k, \ldots, B^t_{t_n} + x_k)].$

Let $\mathcal{L}^1_G(\Omega_t)$ be the completion of $C_{b,Lip}(\Omega_t)$ under the norm $\mathbb{E}[|\cdot|]$, and we write $\mathcal{L}^1_G(\Omega) \equiv \mathcal{L}^1_G(\Omega_T)$ simply. We can extend $\mathbb{E}[]$ (resp. $\mathbb{E}_t[]$) to a unique sublinear expectation (resp. a conditional sublinear expectation) on $\mathcal{L}^1_G(\Omega)$. It is called $G$-expectation (resp. conditional $G$-expectation).

**Definition 3.2.** A stochastic process $B$ on $(\Omega, \mathcal{L}^1_G(\Omega), \mathbb{E})$ is called a **$G$-Brownian motion** if

(i) $B_0 = 0$,

(ii) for all $0 \leq s < t \leq T$ and $\varphi \in C_{b,Lip}(\mathbb{R}^d)$,

$$\mathbb{E}[\varphi(B_t - B_s)] = \mathbb{E}[\varphi(B_{t-s})] = u_\varphi(t-s,0),$$

(iii) for all $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n \leq T$ and $\varphi \in C_{b,Lip}((\mathbb{R}^d)^n)$,

$$\mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_n})] = \mathbb{E}[\varphi_1(B_{t_1}, \ldots, B_{t_{n-1}})],$$

where $\varphi_1(x_1, \ldots, x_{n-1}) := \mathbb{E}[\varphi(x_1, \ldots, x_{n-1}, B_{t_n} - B_{t_{n-1}} + x_{n-1})].$

Note that (ii) means $B_t - B_s$ and $B_{t-s}$ are identically distributed, and that (iii) means $B_{t_n} - B_{t_{n-1}}$ is independent from $(B_{t_1}, \ldots, B_{t_{n-1}})$. From the above definition, we can see that on the sublinear expectation space $(\Omega, \mathcal{L}^1_G(\Omega), \mathbb{E})$, the canonical process is a $G$-Brownian motion.

### 3.2 Itô’s integral for $G$-Brownian motion

For each $p \geq 1$, we denote by $\mathcal{L}^p_G(\Omega_t)$ the completion of $C_{b,Lip}(\Omega_t)$ under $\mathbb{E}[|\cdot|^p]^{1/p}$. Let

$$M^0_G(\Omega) := \left\{ \sum_{k=0}^{n-1} \xi_k \mathbb{1}_{[t_k,t_{k+1})} : n \in \mathbb{N}, \ 0 = t_0 < t_1 < \cdots < t_n = T, \ \xi_k \in \mathcal{L}^p_G(\Omega_{t_k}) \right\},$$

and let $M^p_G(\Omega)$ be the completion of $M^0_G(\Omega)$ under $(\int_0^T \mathbb{E}[|\cdot|^p]dt)^{1/p}$.

For every $h \in (M^2_G(\Omega))^d$, Itô’s integral for $G$-Brownian motion

$$\int_0^t h_s \cdot dB_s = \sum_{i=1}^d \int_0^t h^i_s dB^i_s$$

is defined as an element of $\mathcal{L}^2_G(\Omega_t)$. Moreover, since, for every $i = 1, \ldots, d$, the $i$-th coordinate $B^i$ of $B$ belongs to $M^2_G(\Omega)$, the mutual variation of $B^i$ and $B^j$

$$\langle B^i, B^j \rangle_t := B^i_t B^j_t - \int_0^t B^i_s dB^j_s - \int_0^t B^j_s dB^i_s$$
is defined. We denote by $\langle B \rangle_t := (\langle B^i, B^j \rangle_t)_{i,j=1}^d$, $0 \leq t \leq T$, the quadratic variation of $B$.

In addition, for each $\eta \in M^1_G(\Omega)$, we can define $\int_0^t \eta_s d\langle B^i, B^j \rangle_s$ as an element of $L^1_G(\Omega_t)$. Note that if $\eta^1, \eta^2 \in M^1_G(\Omega)$, then $\eta^1 \eta^2 \in M^1_G(\Omega)$. For each $h \in (M^2_G(\Omega))^d$, we write

$$\int_0^t h_s \cdot (d\langle B \rangle_s h_s) := \sum_{i,j=1}^d \int_0^t h^i_s h^j_s d\langle B^i, B^j \rangle_s.$$  

### 3.3 $G$-martingales

Now we introduce the notion of $G$-martingales.

**Definition 3.3.** A process $X = \{X_t; 0 \leq t \leq T\}$ is called a $\mathbf{G}$-martingale if for each $0 \leq s \leq t \leq T$, we have $X_t \in L^1_G(\Omega_t)$ and

$$\mathbb{E}_s[X_t] = X_s \quad \text{in } L^1_G(\Omega_s).$$

We call $X$ a symmetric $\mathbf{G}$-martingale if both $X$ and $-X$ are $\mathbf{G}$-martingales.

For $h \in (M^2_G(\Omega))^d$, for example, Itô’s integral process $\int_0^t h_s \cdot dB_s$ is a symmetric $\mathbf{G}$-martingale. $\int_0^t h_s \cdot (d\langle B \rangle_s h_s) - \int_0^t 2G(h_s h^*_s)ds$ is a $\mathbf{G}$-martingale, but in general, not a symmetric $\mathbf{G}$-martingale (see [6] Example 50).

### 4 An upper expectation for $G$-expectation

We introduce a representation of $G$-expectation as an upper expectation proved in Denis-Hu-Peng [1].

Let $W$ be a standard $d$-dimensional Brownian motion under a probability measure $P$ on $\Omega$, and let $\mathbb{F}^W$ be the filtration generated by $W$:

$$\mathcal{F}_t^W := \sigma(W_u; 0 \leq u \leq t) \vee \mathcal{N}, \quad \mathbb{F}^W := \{\mathcal{F}_t^W; t \geq 0\},$$

where $\mathcal{N}$ is the collection of all $P$-null subsets. For a given bounded and closed set $\Theta \subset \mathbb{R}^{d \times d}$, let

$$\mathcal{A}^{\Theta}_{0,T} := \{\text{all } \Theta\text{-valued } \mathbb{F}^W\text{-progressively measurable processes on the interval } [0,T]\}.$$  

We identify two elements $\theta, \theta' \in \mathcal{A}^{\Theta}_{0,T}$ if they are equivalent:

$$\theta_t(\omega) = \theta'_t(\omega) \quad dt \times P\text{-a.e. } (t, \omega) \in [0,T] \times \Omega.$$ 

The quotient set of $\mathcal{A}^{\Theta}_{0,T}$ by this equivalence relation is still denoted by the same symbol $\mathcal{A}^{\Theta}_{0,T}$. For each $\theta \in \mathcal{A}^{\Theta}_{0,T}$, let $P_\theta$ be the law of the process $\{\int_0^t \theta_s dW_s; 0 \leq t \leq T\}$. Now we define the **capacity** $c : \mathcal{B}(\Omega) \to [0,1]$ by

$$c(A) := \sup_{\theta \in \mathcal{A}^{\Theta}_{0,T}} P_\theta(A) \quad \text{for } A \in \mathcal{B}(\Omega).$$

We introduce capacity-related terminology.
• A property holds **quasi-surely** (q.s.) if it holds outside a set $A$ with $c(A) = 0$.

• A mapping $X : \Omega \to \mathbb{R}$ is said to be **quasi-continuous** (q.c.) if for all $\varepsilon > 0$, there exists an open set $O$ with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

• We say that $X : \Omega \to \mathbb{R}$ has a **q.c. version** if there exists a q.c. function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s.

For $t \in [0, T]$, we denote by $L^0(\Omega_t)$ the space of all $B(\Omega_t)$-measurable real-valued functions. For $t = T$, we simply write $L^0(\Omega)$. For each $X \in L^0(\Omega)$ such that $E_{P_\theta}[X]$ exists for all $\theta \in A^\Theta_{0,T}$, we set

$$\mathbb{E}[X] := \sup_{\theta \in A^\Theta_{0,T}} E_{P_\theta}[X].$$

The following theorem plays a key role in the formulation and proof of Girsanov’s formula.

**Theorem 4.1** ([1] Theorem 52). $L^1_G(\Omega_t)$ and $\mathbb{E}$ are characterized as follows:

$$L^1_G(\Omega_t) = \{X \in L^0(\Omega_t) : X has a q.c. version, \lim_{n \to \infty} \mathbb{E}[|X| \mathbb{1}_{(|X| > n)}] = 0\},$$

$$\mathbb{E}[X] = \mathbb{E}[X] \text{ for all } X \in L^1_G(\Omega).$$

### 5 Main result

In this section, we firstly characterize symmetric $G$-martingales. We then state and prove the main result of this paper, Girsanov’s formula for $G$-Brownian motion. We also explore a condition that plays a similar role to Novikov’s condition in the classical stochastic analysis.

#### 5.1 A characterization of symmetric $G$-martingales

We start with a lemma that characterizes conditional $G$-expectations. For each $\theta \in A^\Theta_{0,T}$ and $t \in [0, T]$, set

$$\mathcal{A}(t, \theta) := \{\theta' \in A^\Theta_{0,T} : \theta' = \theta \text{ on } [0, t]\},$$

where the identity between $\theta'$ and $\theta$ is to be understood as

$$\theta'_s(\omega) = \theta_s(\omega) \quad ds \times P\text{-a.e. } (s, \omega) \in [0, t] \times \Omega.$$

**Lemma 5.1.** For each $\theta \in A^\Theta_{0,T}$, $X \in L^1_G(\Omega)$ and $t \in [0, T]$, it holds that

$$\mathbb{E}_t[X] = \operatorname{esssup}_{\theta' \in \mathcal{A}(t, \theta)} E_{P_{\theta'}}[X|\mathcal{F}_t] \quad P_\theta\text{-a.s.}, \quad (5.1)$$

where $\{\mathcal{F}_t; 0 \leq t \leq T\}$ is the natural filtration of $B$. 

It is noted in the proof of Proposition 3.4 of Soner-Touzi-Zhang [8] that the validity of (5.1) for \( X \in C_{b,Lip}(\Omega) \) follows from [1]; the assertion for \( X \in \mathcal{L}_G^2(\Omega) \) is then seen to hold by approximations as is done in the proof of their proposition. However, since our setting for \( \Theta \) is different from theirs as will be mentioned just before Lemma 5.6 below, we give a proof of this lemma for the sake of self-containedness of the paper.

**Proof of Lemma 5.1.** By Lemma 44 of [1] and by the characterization of \( G \)-expectation (Theorem 4.1), we see that

\[
\mathbb{E}[\varphi(x, B^t_\tau)]|_{x = \zeta} = \text{ess sup}_{\theta \in A^\theta_{0,T}} E_P[\varphi(\zeta, B^{t,\theta}_T)|F^W_t] \quad \text{P-a.s.}
\]

for all \( t \in [0,T], m \in \mathbb{N}, \varphi \in C_{b,Lip}(\mathbb{R}^{m+d}) \) and \( \zeta \in L^2(\Omega, F^W_t, P; \mathbb{R}^m) \). Here and below we write

\[
B^{s,\theta}_t = \int_s^t \theta_u dW_u \quad \text{for } 0 \leq s \leq t \leq T.
\]

Then, repeating the same argument as in the proof of Theorem 45 of [1], we see inductively that

\[
\mathbb{E}[\varphi(x, B^{s_1}_t, B^{s_2}_t, \ldots, B^{s_k}_t)|x = \zeta] = \text{ess sup}_{\theta \in A^\theta_{0,T}} E_P[\varphi(\zeta, B^{s_1}_t, B^{s_2}_t, \ldots, B^{s_k}_t)|F^W_t] (5.2)
\]

P-a.s. for all \( t \in [0,T], k, m \in \mathbb{N}, t \leq s_1 < \cdots < s_k \leq T, \varphi \in C_{b,Lip}(\mathbb{R}^m \times (\mathbb{R}^d)^k) \) and \( \zeta \in L^2(\Omega, F^W_t, P; \mathbb{R}^m) \). Now we fix \( \theta \in A^\theta_{0,T} \) and \( t \in [0,T) \) arbitrarily. We take \( X = \varphi(B_{t_1}, \ldots, B_{t_n}) \in C_{b,Lip}(\Omega) \) with a partition \( 0 = t_0 \leq t_1 < \cdots < t_n = t_{n+1} = T \), and let \( i = 0, 1, \ldots, n \) be such that \( t \in [t_i, t_{i+1}) \). If we set

\[
\varphi_1(x_1, \ldots, x_i, x) := \mathbb{E}[\varphi(x_1, \ldots, x_i, B^{t_{i+1}}_{t_i} + x, \ldots, B^{t_{n}}_{t_{i+n}} + x)]
\]

for \((x_1, \ldots, x_i, x) \in (\mathbb{R}^d)^{i+1}\), then we have by (5.2)

\[
\varphi_1(B^{0,\theta}_{t_1}, \ldots, B^{0,\theta}_{t_i}, B^{0,\theta}_{t}) = \text{ess sup}_{\theta' \in A(t,\theta)} E_P[\varphi(B^{0,\theta'}_{t_1}, \ldots, B^{0,\theta'}_{t_{n+n}})|F^W_t] \quad \text{P-a.s.}
\]

Let \( U \in \mathcal{F}_t \) be arbitrary and set \( V = \{B^{0,\theta} \in U\} \in \mathcal{F}^W_t \). Then

\[
E_{P_U} [\mathbb{1}_V \varphi_1(B_{t_1}, \ldots, B_t, B_t)] = E_P[\mathbb{1}_V \varphi_1(B^{0,\theta}_{t_1}, \ldots, B^{0,\theta}_{t_i}, B^{0,\theta}_{t})]
\]

\[
= E_P[\mathbb{1}_V \text{ess sup}_{\theta' \in A(t,\theta)} E_P[\varphi(B^{0,\theta'}_{t_1}, \ldots, B^{0,\theta'}_{t_{n+n}})|F^W_t]]
\]

\[
= \sup_{\theta' \in A(t,\theta)} E_P[\mathbb{1}_V \varphi(B^{0,\theta'}_{t_1}, \ldots, B^{0,\theta'}_{t_{n+n}})]
\]

where we used Yan’s commutation theorem (see, e.g., [4] Theorem a3) for the third line. Using Yan’s commutation theorem again, and noting \( P_\theta = P_{\theta'} \) on \( \mathcal{F}_t \) for \( \theta' \in A(t,\theta) \), we see that this is further rewritten as

\[
E_{P_U} [\mathbb{1}_V \text{ess sup}_{\theta' \in A(t,\theta)} E_{P_{\theta'}}[\varphi(B_{t_1}, \ldots, B_{t_{n+n}})|\mathcal{F}_t]].
\]
As $\varphi_1(B_{t_1}, \ldots, B_{t_i}, B_t) = \mathbb{E}_t[\varphi(B_{t_1}, \ldots, B_{t_n})]$ by definition, it follows that

$$\mathbb{E}_t[\varphi(B_{t_1}, \ldots, B_{t_n})] = \operatorname{ess sup}_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[\varphi(B_{t_1}, \ldots, B_{t_n}) | \mathcal{F}_t] \quad P_\theta \text{-a.s.}$$

Therefore (5.1) is proved for $X \in C_{b, \text{Lip}}(\Omega)$.

Now for $X \in L^1_G(\Omega)$, we take a sequence $\{X_n\}_{n=1}^\infty \subset C_{b, \text{Lip}}(\Omega)$ such that

$$\mathbb{E}[|X - X_n|] \to 0 \quad \text{as } n \to \infty.$$ 

For each $\theta \in A^{\Theta}_{0,T}$,

$$E_{P_\theta}[||\mathbb{E}_t[X] - \operatorname{ess sup}_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[X | \mathcal{F}_t]|]$$

$$\leq E_{P_\theta}[||\mathbb{E}_t[X] - \mathbb{E}_t[X_n]|] + E_{P_\theta}[\operatorname{ess sup}_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[X | \mathcal{F}_t] - \operatorname{ess sup}_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[X_n | \mathcal{F}_t]|]$$

$$=: I_n + II_n.$$ 

It is easily seen that $I_n \leq \mathbb{E}[|X - X_n|]$. Also for $II_n$, we have

$$II_n \leq E_{P_\theta}[\operatorname{ess sup}_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[|X - X_n| | \mathcal{F}_t]]$$

$$= \sup_{\theta' \in A(t, \theta)} E_{P_{\theta'}}[|X - X_n|]$$

$$\leq \mathbb{E}[|X - X_n|],$$

where the equality follows from Yan’s commutation theorem and the identity $P_\theta = P_{\theta'}$ on $\mathcal{F}_t$ for $\theta' \in A(t, \theta)$. Therefore both $I_n$ and $II_n$ converge to 0 as $n \to \infty$, which yields (5.1) for $X \in L^1_G(\Omega)$. \qed

As a consequence of Lemma 5.1, we have the following characterization of symmetric $G$-martingales.

**Proposition 5.2.** $X = \{X_t; 0 \leq t \leq T\}$ is a symmetric $G$-martingale on $(\Omega, \mathcal{L}^1_G(\Omega), \mathbb{E})$ if and only if $X_t \in \mathcal{L}^1_G(\Omega_t)$ for all $t \in [0, T]$ and $X$ is a $P_\theta$-martingale for each $\theta \in A^{\Theta}_{0,T}$.

**Proof.** We start with the only if part. The condition that $X_t \in \mathcal{L}^1_G(\Omega_t)$, $t \in [0, T]$, means that $X$ is a process on $(\Omega, \mathcal{L}^1_G(\Omega), \mathbb{E})$. If $X$ is also a $P_\theta$-martingale for each $\theta \in A^{\Theta}_{0,T}$, we have, for $0 \leq s \leq t \leq T$,

$$X_s = \operatorname{ess sup}_{\theta' \in A(s, \theta)} E_{P_{\theta'}}[X_t | \mathcal{F}_s] \quad P_\theta \text{-a.s.}$$

By Lemma 5.1, it follows that $X_s = \mathbb{E}_s[X_t] \; P_\theta \text{-a.s.}$ and that

$$\mathbb{E}[|\mathbb{E}_s[X_t] - X_s|] = 0.$$ 

Similarly, we have $\mathbb{E}_s[-X_t] = -X_s$ in $\mathcal{L}^1_G(\Omega_s)$ and hence $X$ is a symmetric $G$-martingale.
Conversely, if $X$ is a symmetric $G$-martingale, then $X_t \in L^1_G(\Omega_t)$ for all $t \in [0,T]$. Since $X$ is a $G$-martingale,

$$0 = \mathbb{E}[|\mathbb{E}_s[X_t] - X_s|] = \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{P_\theta}[|\mathbb{E}_s[X_t] - X_s|].$$

Therefore, for every $\theta \in \mathcal{A}_{0,T}^\Theta$, we have by Lemma 5.1,

$$X_s = \mathbb{E}_s[X_t] = \operatorname{ess sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P_{\theta'}}[X_t|\mathcal{F}_s] \geq E_{P_{\theta}}[X_t|\mathcal{F}_s] \text{ $P_\theta$-a.s.}$$

Similarly, we deduce $X_s \leq E_{P_{\theta}}[X_t|\mathcal{F}_s]$ $P_\theta$-a.s. from that $-X$ is a $G$-martingale. Hence, $X$ is a $P_\theta$-martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$. □

### 5.2 Girsanov’s formula for $G$-Brownian motion

Let $h \in (M^2_G(\Omega))^d$. We define, for $0 \leq t \leq T$,

$$D_t := \exp \left( \int_0^t h_s \cdot dB_s - \frac{1}{2} \int_0^t h_s \cdot (d\langle B \rangle_s) \right),$$

$$\hat{B}_t := B_t - \int_0^t (d\langle B \rangle_s) h_s),$$

and we set

$$\hat{C}_{b,Lip}(\Omega) := \{ \varphi(\hat{B}_{t_1}, \ldots, \hat{B}_{t_n}) : n \in \mathbb{N}, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,Lip}(\mathbb{R}^n) \}.$$

Girsanov’s formula for $G$-Brownian motion is stated as follows.

**Theorem 5.3.** Assume that there exists $\sigma_0 > 0$ such that

$$\gamma^* \geq \sigma_0 I_d \quad \text{for all } \gamma \in \Theta,$$

and that $D$ is a symmetric $G$-martingale on $(\Omega, L^1_G(\Omega), \mathbb{E})$. Define a sublinear expectation $\hat{\mathbb{E}}$ by

$$\hat{\mathbb{E}}[X] := \mathbb{E}[XD_T] \quad \text{for } X \in \hat{C}_{b,Lip}(\Omega).$$

Let $\hat{\mathcal{H}}$ be the completion of $\hat{C}_{b,Lip}(\Omega)$ under the norm $\hat{\mathbb{E}}[|\cdot|]$, and extend $\hat{\mathbb{E}}$ to a unique sublinear expectation on $\hat{\mathcal{H}}$. Then the process $\{\hat{B}_t; 0 \leq t \leq T\}$ is a $G$-Brownian motion on the sublinear expectation space $(\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})$.

We remark that the uniform nondegeneracy of $\Theta$ is also assumed in [9].

We prove Theorem 5.3 in the next subsection. Before we proceed to the proof, there are several things we must verify. The first thing is that the right-hand side of (5.5) is well-defined, and that the functional $\hat{\mathbb{E}}$ defined by (5.5) is indeed a sublinear expectation. Those are immediate from the assumption that $D$ is a symmetric $G$-martingale. The
second is that \( \{ \hat{B}_t; 0 \leq t \leq T \} \) is a stochastic process on \((\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})\), which we will check in the next lemma. For a fixed \( \theta \in \mathcal{A}^\theta_{0,T} \), set
\[
Q_\theta(A) := E_{P_\theta}[1_A D_T] \quad \text{for} \ A \in \mathcal{B}(\Omega). \tag{5.6}
\]
Note that, by Theorem 4.1, we have
\[
\hat{\mathbb{E}}[X] = \sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[X] \tag{5.7}
\]
for all \( X \in \hat{\mathcal{C}}_{b,\text{Lip}}(\Omega) \).

**Lemma 5.4.** For all \( t \in [0, T] \), we have \( \hat{B}_t \in \hat{\mathcal{H}}^d \). Therefore \( \hat{B} \) is a stochastic process on \((\Omega, \hat{\mathcal{H}}, \hat{\mathbb{E}})\).

**Proof.** Fix \( i = 1, \ldots, d \). Note that the \( i \)-th coordinate \( B^i \) of the canonical process \( B \) and the process \( D \) are \( P_\theta \)-martingales (Proposition 5.2) and satisfy the relation
\[
\hat{B}^i_t = B^i_t - \int_0^t \frac{d\langle D, B^i \rangle_s}{Ds}.
\]

Therefore, by Girsanov’s formula, \( \hat{B}^i \) is a local martingale under \( Q_\theta \) and
\[
\langle \hat{B}^i \rangle_t = \langle B^i \rangle_t \quad \text{for all} \ t \in [0, T], \ P_\theta-\text{a.s. and} \ P_\theta-\text{a.s.}
\]

By definition, \( \langle B^i \rangle_T \) under \( P_\theta \) is identical in law with \( \int_0^T (\theta_s \theta^*_s)^{ii} ds \), where \((\theta_s \theta^*_s)^{ii}\) is the \((i, i)\)-entry of the matrix \( \theta_s \theta^*_s \). We thus deduce that, by the boundedness of \( \Theta \), there exists a constant \( C > 0 \) depending only on \( \Theta \) such that
\[
\langle \hat{B}^i \rangle_T \leq CT \quad P_\theta\text{-a.s.} \tag{5.8}
\]

Moreover, by the time-change formula due to Dambis-Dubins-Schwarz (see, e.g., [2] Theorem 3.4.6), there exists a standard Brownian motion \( \beta \) under \( Q_\theta \) such that
\[
\hat{B}^i_t = \beta_{\langle \hat{B}^i \rangle_t} \quad \text{for all} \ t \in [0, T], \ P_\theta\text{-a.s.}
\]

Combining these, we have, for some \( p > 1 \) (actually, for all \( p > 1 \)),
\[
\sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[|\hat{B}^i_t|^p] \leq \sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[\max_{0 \leq t \leq CT} |\beta_t|^p] = E_{P}[\max_{0 \leq t \leq CT} |W_t|^p] < \infty, \tag{5.9}
\]

where \( W \) is a one-dimensional Brownian motion under a probability measure \( P \).

Now define the sequence \( \{ \varphi_n(\hat{B}^i_t) \}_{n=1}^\infty \subset \hat{\mathcal{C}}_{b,\text{Lip}}(\Omega) \) through
\[
\varphi_n(x) := (x \lor n) \lor (-n) \quad \text{for} \ x \in \mathbb{R}.
\]

This approximates \( \hat{B}^i_t \) under the norm \( \hat{\mathbb{E}}[| \cdot |] \). Indeed, by (5.9)
\[
\sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[|\hat{B}^i_t - \varphi_n(\hat{B}^i_t)|] \leq \sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[|\hat{B}^i_t|^p 1_{|\hat{B}^i_t| > n}] \to 0 \ (n \to \infty). \tag{5.10}
\]

By noting that, from (5.7), \( \hat{\mathcal{H}} \) can be seen as the completion of \( \hat{\mathcal{C}}_{b,\text{Lip}}(\Omega) \) under the norm \( \sup_{\theta \in \mathcal{A}^\theta_{0,T}} E_{Q_\theta}[| \cdot |] \), (5.10) means \( \hat{B}^i_t \in \hat{\mathcal{H}} \). \( \square \)
In the proof of Theorem 5.3, it will also be required that $\hat{B}$ is a true martingale under $Q_\theta$, which follows immediately from (5.8) and Corollary IV.1.25 of [7]. We state it in the lemma.

**Lemma 5.5.** For each $\theta \in \mathcal{A}_0^\Theta$, the process $\{\hat{B}_t; 0 \leq t \leq T\}$ is a $Q_\theta$-martingale.

### 5.3 Proof of Theorem 5.3

A probability measure $P$ on $(\Omega, \mathcal{B}(\Omega))$ is called a **martingale measure** if the canonical process $B$ is a martingale with respect to $\mathbb{F}^B$ under $P$, where $\mathbb{F}^B$ is the filtration generated by $B$:

$$\mathbb{F}^B_t := \sigma(B_u; 0 \leq u \leq t) \vee \mathcal{N}, \quad \mathbb{F}^B := \{\mathbb{F}^B_t; 0 \leq t \leq T\},$$

where $\mathcal{N}$ is the collection of all $P$-null subsets. Let $\mathcal{P}$ be the family of all martingale measures $P$ satisfying

$$\frac{d\langle B \rangle_t^P}{dt} \in \{\gamma \gamma^*; \gamma \in \Theta\}, \quad \text{a.e. } t \in [0, T], \ P\text{-a.s.,}$$

where $\langle B \rangle^P$ is the quadratic variation process of $B$ under $P$.

If a given bounded closed set $\Theta \subset \mathbb{R}^{d \times d}$ particularly has the form $\{\gamma \in \mathbb{R}^{d \times d}; \sigma_0 I_d \leq \gamma \gamma^* \leq \sigma_1 I_d\}$ for some constants $0 < \sigma_0 < \sigma_1 < \infty$, the following lemma is a consequence of Proposition 3.4 in [8]. Although the setting of $\Theta$ in this paper is different from that in [8], we can prove the lemma similarly.

**Lemma 5.6.** For all $X \in C_{b,\text{Lip}}(\Omega)$,

$$\mathbb{E}[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X].$$

**Proof.** Since $\{P_\theta : \theta \in \mathcal{A}_0^\Theta\} \subset \mathcal{P}$, it is clear that $\mathbb{E}[X] = \mathbb{E}[X] \leq \sup_{P \in \mathcal{P}} \mathbb{E}_P[X]$. We check the reverse inequality

$$\mathbb{E}[X] \geq \sup_{P \in \mathcal{P}} \mathbb{E}_P[X]. \quad (5.11)$$

For each $n \in \mathbb{N}$, we set the statement $P(n)$ as follows:

$$P(n) : \text{For all } 0 \leq t_1 < \cdots < t_n \leq T, \text{ and } \varphi \in C_{b,\text{Lip}}((\mathbb{R}^d)^n),$$

$$\sup_{P \in \mathcal{P}} \mathbb{E}_P[\varphi(B_{t_1}, \ldots, B_{t_n})] \leq \mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_n})] \quad \text{holds.}$$

We show (5.11) by the induction with respect to $n$.

(i) First we let $n = 1$, and $v$ be the solution of the following $G$-heat equation:

$$\begin{cases}
- \frac{\partial v}{\partial t} - G(D^2 v) = 0 & \text{in } (0, t_1) \times \mathbb{R}^d, \\
v|_{t=t_1} = \varphi & \text{in } \mathbb{R}^d.
\end{cases}$$
Note that \( v \in C^{1,2}((0, t_1) \times \mathbb{R}^d) \) by assumption (5.4) (see Remark 3.1). For all \( P \in \mathcal{P} \), it follows from Itô’s formula that \( P \)-a.s.

\[
\varphi(B_{t_1}) = v(t_1, B_{t_1}) \\
= v(0, 0) + \int_0^{t_1} (Dv)(t, B_t) \cdot dB_t \\
+ \int_0^{t_1} \left( -G((D^2v)(t, B_t))dt + \frac{1}{2} \text{tr}[(D^2v)(t, B_t)d(B)^P_t]\right) \\
\leq v(0, 0) + \int_0^{t_1} (Dv)(t, B_t) \cdot dB_t.
\]

Taking the expectation under \( P \), we have \( E_P[\varphi(B_{t_1})] \leq v(0, 0) = \mathbb{E}[\varphi(B_{t_1})] \). Hence

\[
\sup_{P \in \mathcal{P}} E_P[\varphi(B_{t_1})] \leq \mathbb{E}[\varphi(B_{t_1})].
\]

(ii) We now assume that \( P(n) \) is true for some \( n \in \mathbb{N} \). Take \( 0 \leq t_1 < \cdots < t_n < t_{n+1} \leq T \) and \( \varphi \in C_{b,Lip}(\mathbb{R}^d)^{n+1} \) to be arbitrary. By the definition of conditional \( G \)-expectations, it holds that

\[
\mathbb{E}_n[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t_{n+1}})] = v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n}),
\]

where \( v(t, x; x_1, \ldots, x_n) \in C^{1,2}((t_n, t_{n+1}) \times \mathbb{R}^d) \) is the solution of the following \( G \)-heat equation:

\[
\begin{cases}
- \frac{\partial v}{\partial t} - G(D^2v) = 0 & \text{in } (t_n, t_{n+1}) \times \mathbb{R}^d, \\
v(t_{n+1}, x; x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_n, x), & x \in \mathbb{R}^d.
\end{cases}
\]

Under each \( P \in \mathcal{P} \), we apply Itô’s formula (see Remark 5.7) to \( v(t_{n+1}, B_{t_{n+1}}; B_{t_1}, \ldots, B_{t_n}) \) to obtain \( P \)-a.s.

\[
\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t_{n+1}}) = v(t_{n+1}, B_{t_{n+1}}; B_{t_1}, \ldots, B_{t_n}) \\
= v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n}) + \int_{t_n}^{t_{n+1}} (Dv)(t, B_t; B_{t_1}, \ldots, B_{t_n}) \cdot dB_t \\
+ \int_{t_n}^{t_{n+1}} \left( -G((D^2v)(t, B_t; B_{t_1}, \ldots, B_{t_n}))dt + \frac{1}{2} \text{tr}[(D^2v)(t, B_t; B_{t_1}, \ldots, B_{t_n})d(B)^P_t]\right) \\
\leq v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n}) + \int_{t_n}^{t_{n+1}} (Dv)(t, B_t; B_{t_1}, \ldots, B_{t_n}) \cdot dB_t.
\]

Taking the expectation under \( P \), we have

\[
E_P[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t_{n+1}})] \leq E_P[v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n})].
\]

Therefore

\[
\sup_{P \in \mathcal{P}} E_P[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t_{n+1}})] \leq \sup_{P \in \mathcal{P}} E_P[v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n})].
\]
Notice that the function \((x_1, \ldots, x_n) \mapsto v(t_n, x_n; x_1, \ldots, x_n)\) belongs to \(C_{b, \text{Lip}}((\mathbb{R}^d)^n)\).

By the assumption that \(P(n)\) is true, we have
\[
\sup_{P \in \mathcal{P}} \mathbb{E}[v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n})] \leq \mathbb{E}[v(t_n, B_{t_n}; B_{t_1}, \ldots, B_{t_n})] = \mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t_{n+1}})].
\]

So \(P(n + 1)\) is also true, and hence we complete the induction argument.

\(\square\)

\textbf{Remark 5.7.} The second equality in (5.12) may be seen in the following manner: for each \(i = 1, \ldots, n\), define the process \(M_i\) on \([t_n, t_{n+1}]\) by
\[
M_i^t := B_{t_i}, \quad t_n \leq t \leq t_{n+1},
\]
and set \(M_t := (t, B_t, M_1^t, \ldots, M_n^t)\). Clearly \(\{M_t; t_n \leq t \leq t_{n+1}\}\) is an \(\mathbb{F}^B\)-semimartingale. We may write \(v(M_t)\) for \(v(t, B_t; B_{t_1}, \ldots, B_{t_n})\), to which Itô’s formula applies to yield the desired equality.

Now we are in a position to prove Theorem 5.3.

\textbf{Proof of Theorem 5.3.} It is sufficient to show that for all \(k \in \mathbb{N}\), \(t_1, \ldots, t_k \in [0, T]\), and \(\varphi \in C_{b, \text{Lip}}((\mathbb{R}^d)^k)\),
\[
\mathbb{E}[\varphi(\hat{B}_{t_1}, \ldots, \hat{B}_{t_k})] = \mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_k})].
\]

Indeed, it is obvious that \(\hat{B}\) satisfies Definition 3.2 (i). If we obtain the above equation, it then follows from the right-hand side that \(\hat{B}\) satisfies Definition 3.2 (ii), (iii) under \(\mathbb{E}\).

Note that, since \(\hat{\mathcal{H}}\) is the completion of \(C_{b, \text{Lip}}(\Omega)\), identical distributedness and independence on \((\Omega, \mathcal{F}, \mathbb{E})\) can be, as those on \((\Omega, \mathcal{L}_c^B(\Omega), \mathbb{E})\) are, checked through test functions of the class consisting of bounded, Lipschitz cylinder functionals (see Definition 2.3 and the comment given just before it).

For simplicity, we write \(\varphi(B)\) and \(\varphi(\hat{B})\) for \(\varphi(B_{t_1}, \ldots, B_{t_k})\) and \(\varphi(\hat{B}_{t_1}, \ldots, \hat{B}_{t_k})\) respectively.

(i) First we show that \(\mathbb{E}[\varphi(B)] \leq \mathbb{E}[\varphi(\hat{B})]\).

It is enough to show the following:

for all \(\Theta\)-valued simple process \(\theta\) on \([0, T]\), \(E_{P_0}[\varphi(B)] \leq \mathbb{E}[\varphi(\hat{B})].\) \hspace{1cm} (5.13)

To see this, we fix \(\theta \in \mathcal{A}_0^\Theta_{0,T}\). Then, for all \(\varepsilon > 0\), there exists a \(\Theta\)-valued simple process \(\theta^\varepsilon\) on \([0, T]\) such that
\[
E_P[\int_0^T \|\theta^\varepsilon_s - \theta_s\|^2 \, ds] < \varepsilon^2
\]
(see, e.g., [2] Problem 3.2.5). Therefore, if (5.13) holds, we have
\[
E_{P_0}[\varphi(B)] \equiv E_{P_0}[\varphi(B_{t_1}, \ldots, B_{t_k})]
\leq E_{P_0}[\varphi(B_{t_1}, \ldots, B_{t_k})] + C_{\varphi}E_P\left[\left(\sum_{i=1}^k \sum_{j=1}^d \sum_{l=1}^d \int_0^{t_i} (\theta_s - \theta^\varepsilon_s)^j dW^l_s\right)^2\right]^{1/2}
\]
where $C_\varphi$ is a Lipschitz constant of $\varphi$ and $(\theta_s - \theta^\varepsilon_s)^{jl}$ is the $(j, l)$-entry of $\theta_s - \theta^\varepsilon_s$. Since $\varepsilon > 0$ is arbitrary, we get

$$E_{P_0}[\varphi(B)] \leq \overline{\mathbb{E}}[\varphi(\tilde{B})].$$

Now we show (5.13). Let $\theta$ be given in the form

$$\theta_t = \eta_0 \mathbb{1}_{[0,t_0]}(t) + \eta_1(W)_t \mathbb{1}_{(t_1,t_2]}(t) + \cdots + \eta_{n-1}(W)_t \mathbb{1}_{(t_{n-1},t_n]}(t)$$

for $0 \leq t \leq T$, where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0,T]$, $\eta_0 \in \Theta$, and $\eta_i(\omega) \equiv \eta_i(\omega_t)$, $t \leq t_i$, $\omega \in \Omega$, is a $\Theta$-valued measurable functional on $\Omega$ for $i = 1, \ldots, n - 1$. We now define the sequence of random variables $\{\bar{\eta}_i\}_{i=1}^{n-1}$ and the simple process $\bar{\theta} = \{\tilde{\theta}_t; 0 \leq t \leq T\}$ as follows:

$$\left\{
\begin{array}{l}
\bar{\eta}_0 := \eta_0,
\bar{\eta}_1 := \eta_1(W_t - \int_0^t \tilde{\theta}_s^* h_s(\bar{\theta}) ds, t \leq t_1), \\
\vdots \\
\bar{\eta}_{n-1} := \eta_{n-1}(W_t - \int_0^t \tilde{\theta}_s^* h_s(\bar{\theta}) ds, t \leq t_{n-1}),
\end{array}
\right.$$ 

where $h_s(\bar{\theta}) := h_s(\int_0^s \tilde{\theta}_u dW_u)$. As the right-hand side of (5.14) is given as a functional of $W$, we denote it by $\theta_t(W)$ with a slight abuse of notation. Then, from the above construction of $\bar{\theta}$, for all $0 \leq t \leq T$,

$$\bar{\theta}_t = \theta_t(W - \int_0^t \tilde{\theta}_s^* h_s(\bar{\theta}) ds).$$

Set

$$\bar{W}_t := W_t - \int_0^t \tilde{\theta}_s^* h_s(\bar{\theta}) ds, \quad 0 \leq t \leq T,$$

$$D_T^{\bar{\theta}} := \exp \left( \int_0^T \tilde{\theta}_t^* h_t(\bar{\theta}) \cdot dW_t - \frac{1}{2} \int_0^T h_t(\bar{\theta}) \cdot (\tilde{\theta}_t \tilde{\theta}_t^* h_t(\bar{\theta})) dt \right),$$

$$\bar{P}(A) := E\left[ \mathbb{1}_A D_T^{\bar{\theta}} \right], \quad A \in \mathcal{F}_T^W.$$ 

Since, by Girsanov’s formula, $\bar{W}$ is a Brownian motion under $\bar{P}$, we have

$$E_{P_0}[\varphi(B)] = E_{\bar{P}}[\varphi(\int_0^T \theta_s(\bar{W}) d\bar{W}_s)]$$
\[= E_P[\varphi(\int_0^T \tilde{\theta}_s dW_s - \int_0^T \tilde{\theta}_s \tilde{\theta}^*_s h_s(\tilde{\theta}) ds) D_T^{(\tilde{\theta})}]\]
\[= E_{P_\theta}[\varphi(B - \int_0^T (d\langle B \rangle_s h_s)) D_T]\]
\[\leq \mathbb{E}[\varphi(B - \int_0^T (d\langle B \rangle_s h_s)) D_T] = \hat{\mathbb{E}}[\varphi(\hat{B})],\]

which shows (5.13).

(ii) Next we show that \(\hat{\mathbb{E}}[\varphi(\hat{B})] \leq \mathbb{E}[\varphi(B)]\).

For each \(\theta \in \mathcal{A}_{0,T}^\theta\), let \(Q_\theta\) be the measure defined by (5.6). By Lemma 5.5, \(\hat{B}\) is a \(Q_\theta\)-martingale. Girsanov's formula also implies that
\[
\langle \hat{B} \rangle = \langle B \rangle, \quad P_\theta\text{-a.s. and } Q_\theta\text{-a.s.}
\]

Hence \(Q_\theta \circ \hat{B}^{-1} \in \mathcal{P}\), where \(Q_\theta \circ \hat{B}^{-1}(A) := Q_\theta(\hat{B} \in A)\) for each \(A \in \mathcal{B}(\Omega)\). Then, using Lemma 5.6, we have
\[
E_{P_\theta}[\varphi(\hat{B}) D_T] = E_{Q_\theta \circ \hat{B}^{-1}}[\varphi(B)] \leq \sup_{P \in \mathcal{P}} E_P[\varphi(B)] = \mathbb{E}[\varphi(B)].
\]

Therefore we get
\[
\hat{\mathbb{E}}[\varphi(\hat{B})] = \sup_{\theta \in \mathcal{A}_{0,T}^\theta} E_{P_\theta}[\varphi(\hat{B}) D_T] \leq \mathbb{E}[\varphi(B)],
\]

and complete the proof. \(\square\)

We conclude this subsection with a remark on the construction of \(\hat{\mathbb{E}}\).

**Remark 5.8.** The equation (5.5) holds on \(\hat{\mathcal{H}}\), namely
\[
XD_T \in \mathcal{L}^1_G(\Omega) \quad \text{for all } X \in \hat{\mathcal{H}}.
\]

To see this, it is sufficient to check the completeness of \(\mathcal{L} := \{X \in \mathcal{L}^1_G(\Omega) : XD_T \in \mathcal{L}^1_G(\Omega)\}\) with respect to the norm \(\hat{\mathbb{E}}[\| \cdot \|]\). Let \(\{X_n\}_{n=1}^\infty \subset \mathcal{L}\) be an \(\hat{\mathbb{E}}[\| \cdot \|]\)-Cauchy sequence, that is,
\[
\mathbb{E}[\| X_n - X_m \| D_T] \rightarrow 0 \quad (n, m \rightarrow \infty).
\]

This implies \(\{X_n D_T\}_{n=1}^\infty \subset \mathcal{L}^1_G(\Omega)\) is an \(\mathbb{E}[\| \cdot \|]\)-Cauchy sequence. Hence, from completeness of \(\mathcal{L}^1_G(\Omega)\), there exists a unique \(Y \in \mathcal{L}^1_G(\Omega)\) such that
\[
\mathbb{E}[\| X_n D_T - Y \|] \rightarrow 0 \quad (n \rightarrow \infty).
\]

As \(X := Y D_T^{-1}\) is in \(\mathcal{L}\), we get \(\hat{\mathbb{E}}[\| X_n - X \|] \rightarrow 0 \quad (n \rightarrow \infty)\). Therefore \(\mathcal{L}\) is complete under the norm \(\hat{\mathbb{E}}[\| \cdot \|]\).
5.4 \textit{G}-Novikov’s condition

For \( h \in (M^2_G(\Omega))^d \), consider the process \( D \) defined by (5.3). In this subsection, we give a sufficient condition for \( D \) to be a symmetric \( G \)-martingale, which reads as follows: there exists \( \varepsilon > 0 \) such that

\[
E \left[ \exp \left( \frac{1}{2} (1 + \varepsilon) \int_0^T h_s \cdot (d\langle B \rangle_s h_s) \right) \right] < \infty.
\] (5.15)

This condition may be regarded as a sublinear counterpart to the well-known Novikov’s condition in the classical stochastic analysis, and we refer to it as \( G \)-Novikov’s condition.

We remark that in the one-dimensional case, this condition is the same as that imposed in [9]

\textbf{Proposition 5.9.} If \( h \in (M^2_G(\Omega))^d \) satisfies \( G \)-Novikov’s condition (5.15), then the process \( D \) is a symmetric \( G \)-martingale.

\textit{Proof.} Note that under the condition (5.15), the usual Novikov’s condition is fulfilled for all \( \theta \in \mathcal{A}_{0,T}^\Theta \):

\[
E_{P_\theta} \left[ \exp \left( \frac{1}{2} \int_0^T h_s \cdot (d\langle B \rangle_s h_s) \right) \right] < \infty.
\]

Therefore \( D \) is a \( P_\theta \)-martingale for each \( \theta \in \mathcal{A}_{0,T}^\Theta \). In view of Proposition 5.2, it remains to prove that \( D_t \in L^1_G(\Omega_t) \) for each \( t \in [0, T] \).

Fix \( t \in [0, T] \) and let

\[
p = \frac{1 + \varepsilon}{2\sqrt{1 + \varepsilon} - 1}, \quad q = \frac{2\sqrt{1 + \varepsilon} - 1}{\sqrt{1 + \varepsilon}}.
\]

Note that \( p, q > 1 \) and

\[
p^2 q^2 = \frac{pq(pq - 1)}{q - 1} = 1 + \varepsilon.
\] (5.16)

Then, for all \( \theta \in \mathcal{A}_{0,T}^\Theta \),

\[
E_{P_\theta}[(D_t)^p]
= E_{P_\theta} \left[ \exp \left( \int_0^t ph_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q h_s \cdot (d\langle B \rangle_s h_s) \right) \right]
\times \exp \left( \frac{p(pq - 1)}{2} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right)
\leq E_{P_\theta} \left[ \exp \left( \int_0^t pq h_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q^2 h_s \cdot (d\langle B \rangle_s h_s) \right) \right]^{1/q}
\times E_{P_\theta} \left[ \exp \left( \frac{pq(pq - 1)}{2(q - 1)} \int_0^t h_s \cdot (d\langle B \rangle_s h_s) \right) \right]^{1-1/q}.
\]
By (5.15) and (5.16), we have
\[
\mathbb{E}\left[ \exp \left( \frac{pq(pq-1)}{2(q-1)} \int_0^t h_s \cdot (d\langle B\rangle_s h_s) \right) \right] < \infty,
\]
and Novikov’s condition implies that the process
\[
\left\{ \exp \left( \int_0^t pq h_s \cdot dB_s - \frac{1}{2} \int_0^t p^2 q^2 h_s \cdot (d\langle B\rangle_s h_s) \right) ; 0 \leq t \leq T \right\}
\]
is a $P_\theta$-martingale. Therefore $\mathbb{E}[D_t^p] < \infty$, and hence
\[
\lim_{N \to \infty} \mathbb{E}[D_t 1_{\{D_t > N\}}] = 0.
\]
Moreover, $D_t$ has a q.c. version and belongs to $L^0(\Omega_t)$ since $\int_0^t h_s \cdot dB_s$ and $\int_0^t h_s \cdot (d\langle B\rangle_s h_s)$ do by their definitions. Therefore, from the characterization of $\mathcal{L}^1_G(\Omega_t)$ (Theorem 4.1), we have $D_t \in \mathcal{L}^1_G(\Omega_t)$.

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