Deformation Quantization and Reduction

Alberto S. Cattaneo

Abstract. This note is an overview of the Poisson sigma model (PSM) and its applications in deformation quantization. Reduction of coisotropic and pre-Poisson submanifolds, their appearance as branes of the PSM, quantization in terms of $L_\infty$- and $A_\infty$-algebras, and bimodule structures are recalled. As an application, an “almost” functorial quantization of Poisson maps is presented if no anomalies occur. This leads in principle to a novel approach for the quantization of Poisson–Lie groups.

1. Introduction

The Poisson sigma model (PSM) is a two-dimensional topological field theory with target a Poisson manifold $(M, \pi)$. It is defined by the action functional

$S = \int_{\Sigma} \eta dX + \frac{1}{2} \pi(X) \eta \eta,$

where the pair $(X, \eta)$ is a bundle map $T\Sigma \to T^*M$, and $\Sigma$ is a two-manifold.

The Hamiltonian study of the PSM on $\Sigma = [0, 1] \times \mathbb{R}$ leads to the construction of the symplectic groupoid of $M$ whenever $T^*M$ is an integrable Lie algebroid. The functional-integral perturbative quantization yields Kontsevich’s deformation quantization if $\Sigma$ is the disk and one chooses vanishing boundary conditions for $\eta$.

General boundary conditions (“branes”) compatible with symmetries and the perturbative expansion were studied in and turned out to correspond to coisotropic submanifolds. In (see also references therein) general boundary conditions compatible just with symmetries were classified; they correspond to what we now call pre-Poisson submanifolds (a generalization of the familiar notion of pre-symplectic submanifolds in the symplectic case).

The perturbative functional-integral quantization with branes gives rise to deformation quantization of the reduced spaces . The many-brane

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case leads to the construction of bimodules and morphisms between them [CF04] and potentially to a method for quantizing Poisson maps in an almost functorial way (see Section 8). This problem is also discussed in [B04] where results (and obstructions) for the symplectic case are given. The linear case (i.e., the case when the Poisson manifold is the dual of a Lie algebra and the coisotropic submanifold is an affine subspace) leads to many interesting results in Lie theory [CF04, CT].

A more general approach leads to a cohomological description of coisotropic submanifolds by $L_\infty$-algebras naturally associated to them [OP] and to a deformation quantization in terms of (possibly nonflat) $A_\infty$-algebras. A natural way to reinterpret these results is by associating a graded manifold to the coisotropic submanifold and use a graded version of Kontsevich’s $L_\infty$-quasi-isomorphism [CF07, LS]. This may also be regarded as a duality for the PSM with target a graded manifold [CF07, C06].

In the absence of the so-called anomaly—i.e., when the $A_\infty$-algebra can be made flat—one can go down to cohomology.

This paper is an overview of all these themes. It contains a more extended discussion, Section 7, of deformation quantization for graded manifolds and an introduction, Section 8, to methods for an almost functorial quantization of morphisms whenever the anomaly is not present.

Plan of the paper. In Section 2 we recall properties and reduction of coisotropic, presymplectic and pre-Poisson submanifolds collecting results in [Ca, CFa05, CZ, CZbis]. In Section 3 we give a simple derivation of the $L_\infty$-structure associated [OP] to a coisotropic submanifold; we also recall the BFV (Batalin–Fradkin–Vilkovisky) formalism [BF, BV] and its relation [S] to the $L_\infty$-structure. In Section 4 we recall the Hamiltonian description of the PSM and how coisotropic [CF04] and pre-Poisson [Ca, CFa05] submanifolds show up as boundary conditions (branes) for the PSM. Perturbative functional-integral quantization of the PSM [CF00] is recalled in Section 5 with results for coisotropic [CF04] and pre-Poisson [CFa05] branes. Section 6 deals with the reinterpretation of these results in terms of a duality for the PSM with target a graded manifold [CF07, C06]. In Section 7 we recall the Formality Theorem for graded manifolds [K, CF07] and its applications in deformation quantization [CF07, LS]; the potential anomaly [CF04, CF07] is discussed; subsection 7.3 contains a description of the induced properties in cohomology and is original; subsection 7.5 contains a new description of methods for quantizing the inclusion map of a Poisson submanifold; in particular, it contains a proof of a conjecture by [CR] that the deformation quantization of a Poisson submanifold determined by central constraints may be obtained by quotienting Kontsevich’s deformation quantization by the ideal generated by the image of the constraints under the Duflo–Kirillov–Kontsevich map. Finally, Section 8 recalls the many-brane case [CF04] and, in the absence of anomalies, presents a novel method for the quantization of Poisson maps compatible with compositions; the application to the quantization of Poisson–Lie groups is briefly described.

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2. Reduction

We recall reduction in the context of symplectic and Poisson manifolds. We will essentially follow [CZ] (see also [CZbis]). Let us first recall the linear case.

A symplectic space is a (possibly infinite dimensional) linear space \( V \) equipped with a nondegenerate skew-symmetric bilinear form \( \omega \), called a symplectic form. By nondegenerate we mean here that \( \omega(v, w) = 0 \) \( \forall v \in V \Rightarrow w = 0 \) (this is sometimes called a weak symplectic form). We denote by \( \omega^\sharp \) the induced linear map \( V \to V^\ast : v \mapsto \omega(v) \). With this notation, \( \omega \) is nondegenerate iff \( \omega^\sharp \) is injective.

The restriction of \( \omega \) to a subspace \( W \) is in general degenerate, the kernel of \( (\omega|_W)^\sharp \) being \( W^\perp \cap W \), where
\[
W^\perp := \{ v \in V : \omega(v, w) = 0 \ \forall w \in W \}
\]
is the symplectic orthogonal to \( W \). Thus, the reduced space \( \overline{W} := W/(W \cap W^\perp) \) is automatically endowed with a symplectic form.

An extreme case is when \( W \cap W^\perp = 0 \); i.e., when \( (\omega|_W)^\sharp \) is nondegenerate. In this case, \( W \) is called a symplectic subspace and \( \overline{W} = W \). In finite dimensions, one has the additional properties that \( W^\perp \) is also symplectic and that
\[
(2.1) \quad W \oplus W^\perp = V.
\]
That is, in finite dimensions a symplectic subspace may be equivalently described as a subspace which admits a symplectic complement. For this reason, it may as well be called a cosymplectic subspace.

The other nontrivial extreme case is when \( W^\perp \subset W \). In this case \( W \) is called a coisotropic subspace and \( \overline{W} = W/W^\perp \). The coisotropic case is in some sense the only case one has to consider, as we have the following simple

**Lemma 2.1.** Let \( W \) be a subspace of a finite dimensional symplectic space \( V \). Then it is always possible to find a symplectic subspace \( W' \) of \( V \) which contains \( W \) as a coisotropic subspace.

The Lemma is proved by adding \( W \) what is missing to make it symplectic. Any complement to \( W + W^\perp \) will do (see Lemma 3.1 in [CZ] for details).

A symplectic manifold is a smooth manifold endowed with a closed, nondegenerate 2-form. Here nondegenerate means that the bundle map \( \omega^\sharp: TM \to T^\ast M \) induced by the 2-form \( \omega \) on \( M \) is injective. The restriction of \( \omega \) to a submanifold \( C \) is in general degenerate. The kernel of \( (\omega|_C)^\sharp \) is the characteristic distribution \( T^\perp C \cap TC \) which in general is not smooth. Here \( T^\perp C \) is the bundle of symplectic orthogonal spaces to \( TC \) in the restriction \( T_C M \) of \( TM \) to \( C \). The submanifold \( N \) is called presymplectic if its characteristic distribution is smooth (i.e., \( T^\perp C \cap TC \) is a subbundle of \( TC \)); by the closedness of \( \omega \) it is automatically integrable. The corresponding leaf space \( L \) is called the reduced space. If it is a manifold, it is endowed with a unique symplectic form whose pullback to \( C \) is equal to the restriction of \( \omega \). The extreme nontrivial examples of presymplectic submanifolds are the symplectic and coisotropic submanifolds. A submanifold \( C \) is called symplectic (coisotropic) if \( T_x C \) is symplectic (coisotropic) in \( T_x M \) for every \( x \in C \). The reduction of a symplectic submanifold is the manifold itself. If \( C \) is a submanifold of a finite dimensional manifold \( M \), by Lemma 2.1 we may find a symplectic subspace of \( T_x M \) containing \( T_x C \) as a coisotropic subspace for every \( x \in C \). These subspaces may be chosen to be glued together smoothly if the submanifold is presymplectic. Namely, we have the following.
Proposition 2.2 (CZ). Let $C$ be a presymplectic submanifold of a symplectic manifold $M$. Then it is always possible to find a symplectic submanifold which contains $C$ as a coisotropic submanifold.

Moreover, one can show that these symplectic extensions are neighborhood equivalent. Namely, given two such extensions $C'$ and $C''$, there exists a tubular neighborhood $U$ of $C$ such that $U \cap C'$ and $U \cap C''$ are related by a symplectomorphisms (i.e., a diffeomorphism compatible with the symplectic form) of $U$ which fixes $C$.

We now move to the Poisson case. We will only work in finite dimensions. A Poisson space is a linear space $V$ endowed with a bivector $\pi$ (i.e., an element of $\Lambda^2 V$). We denote by $\pi^0$ the induced linear map $V^* \to V$, $\alpha \mapsto \pi(\alpha, \ )$. A finite dimensional symplectic space $(V, \omega)$ is automatically Poisson with $\pi^0 = (\omega^* )^{-1}$. The Poisson generalization of the notion of symplectic orthogonal of a subspace $W$ is the image under $\pi^0$ of its annihilator $W^0 := \{ \alpha \in V^* : \alpha(w) = 0 \ \forall w \in W \}$. In the symplectic case, $\pi^0(W^0) = W^\perp$. Condition (2.1) may be replaced by $W \oplus \pi^2(W^0) = V$; a subspace $W$ satisfying it is called coisotropic: it may equivalently be characterized by the condition that the projection of $\pi$ to $\Lambda^2(V/W)$ is symplectic.

A coisotropic subspace is analogously defined as a subspace $W$ with $\pi^2(W^0) \subset W$.

A Poisson manifold is a manifold $M$ endowed with a bivector $\pi$ (i.e., a section of $\Lambda^2 TM$) such that the bracket $\{ f, g \} := \pi(df, dg)$, $f, g \in C^\infty(M)$ satisfies the Jacobi identity. Equivalently, $[\pi, \pi] = 0$ with the Schouten–Nijenhuis bracket. This also amounts to saying that $(C^\infty(M), \{ \ , \ \})$ is a Poisson algebra.

We denote by $\pi^0 : T^* M \to TM$ the corresponding bundle map. A finite dimensional symplectic manifold $(M, \omega)$ is Poisson with $\pi^0 = \omega^*$. A submanifold $C$ is called coisotropic (coisotropic) if $T_x C$ is so in $T_x M \ \forall x \in C$. We will denote by $N^* C$—the conormal bundle—the vector bundle over $C$ with fiber at $x$ given by $N^*_x C := (T_x C)^0 \subset T^*_x M$. Observe that $\pi^0(N^* C)$ is a (singular) distribution on $C$ if $C$ is coisotropic. Invariant functions form naturally a Poisson algebra. If the leaf space $C$ is a manifold, it is naturally a Poisson manifold. A coisotropic manifold $C$ is also automatically a Poisson manifold (not a Poisson submanifold though). One way to see this is to think of a Poisson manifold as a manifold foliated by symplectic leaves. A coisotropic submanifold intersects symplectic leaves cleanly, and each intersection is a symplectic submanifold of the symplectic leaf; thus, a coisotropic submanifold is foliated in symplectic leaves, which makes it into a Poisson manifold. The induced Poisson bracket may also be obtained by Dirac’s procedure.

We now wish to describe a Poisson generalization of the notion of presymplectic submanifold leading to a generalization of Proposition 2.2. Namely, we call a submanifold $C$ pre-Poisson if $\pi^0(N^* C) + TC$ has constant rank along $C$. (In and references therein, this was called a “submanifold with strong regularity constraints”.) In the symplectic case, this is equivalent to $C$ being pre-Poisson.

An equivalent definition of a pre-Poisson submanifold $C$ amounts to asking that the bundle map $\phi : N^* C \to NC$ obtained by composing the injection $N^* C \to T^*_x M$ with $\pi^0$ and finally with the projection $T^*_C M \to NC$ should have constant rank. As special cases we recognize coisotropic submanifolds ($\phi = 0$) and coisotropic submanifolds ($\phi$ surjective). We have the following

Proposition 2.3 (CZ). Let $C$ be a pre-Poisson submanifold of a Poisson manifold $M$. Then it is always possible to find a cosymplectic submanifold $M'$ which
contains $C$ as a coisotropic submanifold. Moreover, this extension $M'$ is unique up to neighborhood equivalence.

In the following we will also need a description in terms of Lie algebroids. Recall that a Lie algebroid is a vector bundle $E$ over a smooth manifold $M$ endowed with a bundle map $\rho: E \to TM$ (the anchor map) and a Lie algebra structure on $\Gamma(E)$ with the property $[a, fb] = f[a, b] + \rho(a)fb$ for every $a, b \in \Gamma(E)$, $f \in C^\infty(M)$. The tangent bundle itself is a Lie algebroid. The cotangent bundle of Poisson manifold is also a Lie algebroid with $\rho = \pi^\sharp$ and $[df, dg] := d\{f, g\}$. If $C$ is a pre-Poisson submanifold, then $AC := N^*C \cap \pi^\sharp T^*C$ is a Lie subalgebroid \cite{CFa05}. Its anchor is defined just by restriction; as for the bracket, one has to extend sections of $AC$ to sections of $T^*M$ in a neighborhood of $C$, use the bracket on $T^*M$ and finally restrict; it is not difficult to check that the result is independent of the extension. Observe that for $C$ coisotropic, one simply has $AC = N^*C$.

3. Cohomological descriptions

The complex $\Gamma(\Lambda(AM))$ of multivector fields on a smooth manifold $M$ is naturally endowed with a Lie bracket of degree $-1$, the Schouten–Nijenhuis bracket. Let $\mathcal{V}(M) := \Gamma(\Lambda(AM))[1]$ be the corresponding graded Lie algebra (GLA). A Poisson bivector field is now the same as an element $\pi$ of $\mathcal{V}(M)$ of degree one which satisfies $[\pi, \pi] = 0$. In general, a self-commuting element of degree 1 in a GLA is called a Maurer–Cartan (MC) element. Observe that $[\pi, \ ]$ is a differential; actually, it is the Lie algebroid differential corresponding to the Lie algebroid structure on $T^*M$.

Let $C$ be a submanifold and $NC$ its normal bundle, canonically defined as the quotient by $TC$ of the restriction $T_C M$ of $TM$ to $C$. It is also the dual bundle of the conormal bundle $N^*C$ introduced above. Let $A$ be the graded commutative algebra $\Gamma(ANC)$. Denote by $P: \Gamma(AM) \to A$ the composition of the restriction to $C$ with the projection $T_CM \to NC$. It is not difficult to check that Ker $P$ is a Lie subalgebra. Moreover, $C$ is coisotropic iff $\pi \in \text{Ker } P$. In this case, there is a well-defined differential $\delta$ on $A$ acting on $X \in A$ by

$$
\delta X := P\left[\pi, X\right], \quad X \in P^{-1}(X).
$$

This is the differential corresponding to the Lie algebroid structure on $N^*C$. Observe that $H^0_\delta$ is the algebra of invariant functions on $C$.

The projection $P$ admits a section if $M$ is a vector bundle over $C$. In particular, this is true if $M$ is the normal bundle of $C$. The section $i$ for $P: \Gamma(AM(\mathcal{NC})) \to A$ is given as follows: If $f$ is a function on $C$, one defines $if$ as the pullback of $f$ by the projection $NC \to C$. If $X$ is a section of $NC$, one defines $iX$ as the unique vertical vector field $\tilde{X}$ on $NC$ which is constant along the fibers and such that $P\tilde{X} = X$. Finally, $i$ is extended to sections of $\Lambda NC$ so that it defines an algebra morphism. It turns out that $i(\Gamma(\mathcal{NC}))$ is an abelian subalgebra.

Let us now choose an embedding of $NC$ into our Poisson manifold $(M, \pi)$. By restriction $\pi$ defines a Poisson structure on $NC$, so a MC element in $\mathcal{V}(NC)$. Following Voronov \cite{Vor04}, we define the derived brackets

$$
\lambda_k: A^\otimes k \to A,
\lambda_k(a_1, \ldots, a_k) := P\left(\left[[\ldots\left[[\pi, ia_1], ia_2\right], \ldots\right], ia_k\right),
$$

(3.1)
and \( \lambda_0 := P(\pi) \in A \). Observe that \( \lambda_0 = 0 \) iff \( C \) is coisotropic; in this case, \( \delta = \lambda_1 \).

In general Voronov proved the following

**Theorem 3.1.** Let \( g \) be a GLA, \( \mathfrak{h} \) an abelian subalgebra, \( i \) the inclusion map. Let \( P \) be a projection to \( \mathfrak{h} \) such that \( P \circ i = id \) and that \( \ker P \) is a Lie subalgebra. Then the derived brackets of every MC element of \( L \) vanish, we simply speak of an \( L_\infty \)-structure on \( \mathfrak{h} \).

This means that the operations \( \lambda_k \) satisfy certain quadratic relations which in particular for \( \lambda_0 = 0 \) imply that \( \lambda_1 \) is a differential for \( \lambda_2 \) and that \( \lambda_2 \) satisfies the Jacobi identity up to \( \lambda_1 \)-homotopy. So the \( \lambda_1 \)-cohomology inherits the structure of a GLA.

In our case \( A \) is also a graded commutative algebra and the multibrackets are multiderivations. In this case we say that we have a \( P_\infty \)-algebra (\( P \) for Poisson).

In particular, when \( C \) is coisotropic, \( H_{\lambda_1} = H_\delta \) is a graded Poisson algebra. The Poisson structure in degree zero—i.e., on invariants functions—is the same as the one described in the previous Section.

Observe that the \( P_\infty \)-structure depends on a choice of embedding \( NC \hookrightarrow M \).

It is possible to show that different choices lead to \( L_\infty \)-isomorphic algebras. We will return on this in a forthcoming paper [CS].

The \( P_\infty \)-structure appeared first in \([OP]\) as a tool to describe deformations of a coisotropic submanifold. In \([CF07]\) it was rediscovered as the semiclassical limit of the quantization of coisotropic submanifolds. We will return on quantization in Section 7.

### 3.1. The BFV method.

The Batalin–Fradkin–Vilkovisky (BFV) method is an older method to describe coisotropic submanifold in terms of a differential graded Poisson algebra (DGPA) \([BF] \) \([BV]\). Its advantage is that DGPA are more manageable than \( P_\infty \)-algebras. The disadvantage is that it works only under certain assumptions. We will essentially follow Stasheff’s presentation \([S97]\).

Suppose first that the coisotropic submanifold \( C \) is given by global constraints, or, equivalently, that the normal bundle of \( C \) is trivial. Let \( \{ y^\mu \}_{\mu=1,...,n}, n = \text{codim} \ C \), be a basis of constraints (equivalently, a basis of sections of \( NC \) or a basis of transverse coordinates). Let \( B \) be the free graded commutative algebra generated by a set \( \{ b^\mu \} \) of odd variables of degree 1 (the “ghost momenta”). On \( C^\infty (NC) \otimes B \) one defines the Koszul differential \( \delta_0 := y^\mu \frac{\partial}{\partial y^\mu} \). The \( \delta_0 \)-cohomology is then concentrated in degree 0 and is \( C^\infty (C) \).

Let us also introduce a set \( \{ c_\nu \} \) of odd variables of degree \(-1\) (the “ghosts”) and the free graded commutative algebra \( C \) they generate. On \( C^\infty (NC) \otimes B \otimes C \) one has a unique Poisson structure with \( \{ b^\mu, c_\nu \} = \{ c_\nu, b^\mu \} = \delta_0^\mu_\nu \). Then \( \delta_0 \) is a Hamiltonian vector field with Hamiltonian function \( \Omega_0 = y^\mu c_\mu \). On this enlarged algebra the \( \delta_0 \)-cohomology is \( C^\infty (NC) \otimes C \simeq \Gamma (\Lambda NC) \).

By choosing an embedding \( NC \hookrightarrow M \), we get a Poisson structure on \( C^\infty (NC) \) which we can extend to \( C^\infty (NC) \otimes B \otimes C \). We now consider the sum of the two Poisson structure. Let \( F_0 := \frac{1}{2} \{ \Omega_0, \Omega_1 \} = \frac{1}{2} \{ y^\mu, y^\rho \} c_\mu c_\rho \). Observe that \( \{ F_0, \delta_0, \{ , , \} \} \) defines a nonflat \( L_\infty \)-structure \([S]\) on \( C^\infty (NC) \otimes B \otimes C \). Moreover, \( F_0 \)

\(^1\)Originally \( L_\infty \)-algebras were always assumed to miss the 0th term. When \( L_\infty \)-algebras with the 0th term turned out to be interesting, various terminologies were introduced. An \( L_\infty \)-algebra without the 0th term is called nowadays flat (resp. strict), while one where the 0th term is there is called curved (resp. weak). If no assumption on the 0th term is made (i.e., it may or may not vanish), we simply speak of an \( L_\infty \)-algebra.
is $\delta_0$-exact iff $C$ is coisotropic. In this case, one can use cohomological perturbation theory to kill $F_0$ as follows.

Let $h := b^{\mu} \frac{\partial}{\partial y^\mu}$. Then $[\delta_0, h] = E := b^{\mu} \frac{\partial}{\partial y^\mu} + y^\mu \frac{\partial}{\partial y^\mu}$. One then has a homotopy $s$ for $\delta_0$ (i.e. $s\delta_0 + \delta_0 s = id - pr$, with $pr$ the projection to a subspace isomorphic to the $\delta_0$-cohomology) defined by $s\alpha = ha|\alpha|$ if $E\alpha = |\alpha|\alpha$ and $s\alpha = 0$ if $E\alpha = 0$. Let us then define $\Omega := \sum_{i=0}^\infty \delta_i$ by induction as follows: Let $R_k := \sum_{i=0}^k \Omega_i$. Then define $\Omega_{k+1} = -\frac{1}{2} s \{ \{ R_k, R_k \} \}$. If $C$ is coisotropic, the following hold:

1. $\{ \Omega, \Omega \} = 0$, so $\{ D, \{ \ , \ \} \}$ defines a DGPA structure on $C^\infty(NC) \otimes \mathcal{B} \otimes \mathcal{C}$, where $D := \{ \Omega, \} = \delta_0 + \cdots$.
2. The $D$-cohomology $H_D$ is isomorphic to the Lie algebroid cohomology $H_\delta$ of $N^\ast C$.
3. This way $H_\delta$ gets the structure of a GPA.

It is natural to ask whether the GPA structure on $H_\delta$ is the same as the one induced before. The answer is affirmative. Actually, there is a stronger result:

**Theorem 3.2** (Schätz [S]). $\{ C^\infty(NC) \otimes \mathcal{B} \otimes \mathcal{C}, D, \{ \ , \ \} \}$ is $L_\infty$-quasi-isomorphic to $(A, \{ \lambda_k \})$.

Observe that the $L_\infty$-structure $(A, \{ \lambda_k \})$ can be defined for every submanifold, while the BFV formalism as presented above requires that the normal bundle should be trivial, or at least flat. A more general version of the BFV formalism allows for linearly dependent global constraints and uses the Koszul–Tate resolution. There is however a generalization by Bordemann and Herbig [B00] of the construction presented above which needs no assumption on the normal bundle and just requires the choice of a connection. The above Theorem holds also in the general case [S].

### 4. The reduced space of the Poisson sigma model

Given a finite dimensional manifold $M$, we denote by $PM$ the Banach manifold of $C^1$-paths in $M$; viz., $PM := C^1(I, M)$, $I = [0, 1]$. We denote by $T^*PM$ the vector bundle over $PM$ with fiber at $X \in PM$ the space of $C^0$ sections of $T^*I \otimes X^*T^*M$. Using integration over $I$ and the canonical pairing between $TM$ and $T^*M$, one can endow $T^*PM$ with a symplectic structure. Let now $\pi$ be Poisson bivector field on $M$. We define

$$C(M, \pi) := \{ (X, \zeta) \in T^*PM : dX + \pi^\sharp(X)\zeta = 0 \}.$$

These infinite-dimensional manifolds naturally appear when considering the PSM [14] on $\Sigma = I \times \mathbb{R}$. Namely, the map $X : \Sigma \to M$ may be regarded as a path in $PM$. On the other hand, introducing the coordinate $t$ on $\mathbb{R}$, we may write $\eta = \zeta + \lambda dt$, and reinterpret the pair $(X, \zeta)$ as a path in $T^*PM$. The bilinear form $X$ and $\zeta$ in the action functional (11) yields the canonical symplectic structure on $T^*PM$, while $\lambda$ appears linearly as a Lagrange multiplier leading to the constraints that define $C(M, \pi)$.

**Theorem 4.1** ([CF01]). $C(M, \pi)$ is a coisotropic submanifold of $T^*PM$. Its reduced space $\mathcal{C}(M, \pi)$ is the source simply connected symplectic groupoid of $M$, whenever $M$ is integrable (i.e., $T^*M$ is an integrable Lie algebroid).

We now want to discuss boundary conditions. Given two submanifolds $C_0$ and $C_1$ of $M$, let

$$C(M, \pi; C_0, C_1) := \{ (X, \zeta) \in C(M, \pi) : X(0) \in C_0, \ X(1) \in C_1 \}.$$
Observe that $C(M, \pi; C_0, C_1)$ has natural maps to $C_0$ and $C_1$ (evaluation of $X$ at 0 and 1). We call points in the image in $C_0 \times C_1$ connectable.

Its symplectic orthogonal bundle may be explicitly computed. For simplicity we choose local coordinates on $M$. (The correct, but more cumbersome, description involves choosing a torsion-free connection for $TM$.) One gets $\text{Ca}$

\begin{equation}
(4.1) \quad T_{(X, \iota)}^1 C(M, \pi; C_0, C_1) = \{(-\pi^\sharp(X)\beta, d\beta_i + \partial_i \pi^{jk}(X)\zeta_j \beta_k) : \\
\beta \in \Gamma(X^*T^*M), \beta(0) \in N_{X(i)}^\perp C_i, i = 0, 1\}.
\end{equation}

Intersection with $TC(M, \pi; C_0, C_1)$ just forces the boundary conditions for $\beta$ to be such that the $X$-variations are tangent to the submanifolds: viz., $\pi^\sharp(X(i))\beta(i) \in T_{X(i)}^\perp C_i, i = 0, 1$. Thus, $C(M, \pi; C_0, C_1)$ is presymplectic iff the pre-Poisson condition is satisfied at all connectable points $\text{Ca} \text{Fa05} \text{ Ca}$, and it is coisotropic iff the coisotropcity condition is satisfied at all connectable points $\text{CF04}$. The sections of $\delta X \oplus \delta \zeta$ of the characteristic distribution are parametrized by a section $\beta$ of $X^*T^*M$ with $\beta(i) \in A_{X(i)}^\perp C_i, i = 0, 1$.

\begin{equation}
(4.2) \quad \delta X = -\pi^\sharp(X)\beta,
\end{equation}

\begin{equation}
(4.3) \quad \delta \zeta_i = d\beta_i + \partial_i \pi^{jk}(X)\zeta_j \beta_k,
\end{equation}

where for simplicity we have written the second equation using local coordinates on $M$. (A more invariant, but more cumbersome, possibility is to write it upon choosing a torsion-free connection for $TM$ and observing that the distribution does not depend on this choice.)

To remove the condition on connectable points, one can e.g. take the second submanifold to be the whole manifold $M$. So we have

**Theorem 4.2.**

1. $C(M, \pi; C, M)$ is coisotropic in $T^*PM$ iff $C$ is coisotropic in $M$. ($\text{CF04}$).
2. $C(M, \pi; C, M)$ is presymplectic in $T^*PM$ iff $C$ is pre-Poisson in $M$. ($\text{CFa05} \text{ Ca}$).

Point 2) says in particular that pre-Poisson submanifolds are the most general boundary conditions for the Poisson sigma model compatible with symmetries. A submanifold chosen as a boundary condition is usually called a brane. Point 1) is important as it describes boundary conditions compatible with symmetries and perturbation theory around $\pi = 0$. Namely, consider the family of Poisson structures $\pi_\epsilon := \epsilon \pi, \epsilon \geq 0$. Let $C$ be a pre-Poisson submanifold for $\pi$. Then it is pre-Poisson for $\pi_\epsilon, \forall \epsilon \geq 0$. In particular, it is coisotropic for $\epsilon = 0$. One may indeed check that the codimension of the characteristic distribution of $C(M, \pi_\epsilon; C, M)$ is the same for all $\epsilon > 0$ but it may change for $\epsilon = 0$. It stays the same iff $C$ is coisotropic for $\pi_\epsilon$ (and hence coisotropic for $\pi_\epsilon$ for all $\epsilon$).

**5. Expectation values in perturbation theory**

We now consider the perturbative functional-integral quantization of the Poisson sigma model. Namely, we consider integrals of the form

\begin{equation}
(5.1) \quad \langle \mathcal{O} \rangle := \int e^{\pi S} \mathcal{O},
\end{equation}

\footnote{Actually, this is the image of the anchor of an infinite-dimensional Lie algebroid over $C(M, \pi; C, M)$. Explicitly this is described in $\text{BC}$ and in $\text{BCZ}$.}
where $\mathcal{O}$ is a function of the fields and the integration is over all fields $(X, \eta)$, $X \in \text{Map}(\Sigma, M)$, $\eta \in \Gamma(T\Sigma \otimes X^*T^*M)$. If $\Sigma$ has a boundary, the discussion in Section 4 shows that we have to choose a pre-Poisson submanifold $C$ of $M$ (a brane) and impose the conditions $X(\partial \Sigma) \subset C$, $\iota_\partial \eta \in \Gamma(T\Sigma \otimes X^*AC)$, where $\iota_\partial$ is the inclusion map $\partial \Sigma \hookrightarrow \Sigma$.

Also observe that the Poisson sigma model has symmetries (see (4.2)). This makes the integrand $e^{i\hbar S}$ invariant under the Lagrangian extension of the symmetries which has the form:

$$\delta X = -\pi^\sharp(X)\beta,$$

$$\delta \eta_i = d\beta_i + \partial_i \pi^{jk}(X)\eta_j \beta_k,$$

with $\beta \in \Gamma(X^*T^*M)$, $\iota_\partial \beta \in \Gamma(X^*AC)$. As a consequence the integral (5.1) cannot possibly converge (it would not even if we were in finite dimensions). The trick is to select representatives of the fields modulo symmetries in a consistent way. This is done using the BV formalism [BV] which in particular guarantees that the result is independent of the choice involved (“gauge fixing”). Its application to the PSM is described in [CF00]. An important issue is that an expectation value as in (5.1) is also gauge-fixing independent if $\mathcal{O}$ is invariant under symmetries. We will consider only boundary observables. Namely, let $f$ be a function on $C$. To it we associate the observable $\mathcal{O}_{f,u}(X, \eta) := f(X(u))$, where $u$ is some point of $\partial \Sigma$. The observable is invariant iff $f$ is invariant constant along the characteristic distribution of $C$. It will be interesting for us to take two such functions $f$ and $g$ and consider $\mathcal{O} = \mathcal{O}_{f,u}\mathcal{O}_{g,v}$.

We compute integrals as in (5.1) in perturbation theory, i.e., by expanding $S$ around its critical points and then computing the integral formally in terms of the momenta of the Gaussian distribution associated to the Hessian of $S$ (observe that after gauge fixing this is no longer degenerate). We will pick only the trivial critical point; viz., the class of solutions $\{X \text{ constant, } \eta = 0\}$. We fix the constant by imposing $X$ to be equal to some $x \in C$ at some boundary point which we denote by $\infty$. Finally, we select $\Sigma$ to be a disk and compute

$$f \ast g(x; u, v) := \int_{X(\infty) = x} e^{kS} \mathcal{O}_{f,u}\mathcal{O}_{g,v},$$

where the points $u, v, \infty$ are cyclically ordered. It turns out that such an integral may be explicitly computed in perturbation theory. The results are as follows.

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**Figure 1.** An example of a Kontsevich graph

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3Higher genera would be interesting to consider, but this has not been done so far, the main difficulty being the appearance of nontrivial cohomological classes in the space of solutions.
The case \( C = M \) has been studied in [CF00]. The result turns out to be independent of \( u \) and \( v \) and to define a star product \([BFPLS]\) on \( M \): viz., an associative product \( \star \) on \( C^\infty(M) \) \( \{ x \} \) (in the above computation \( \epsilon = \frac{i}{\hbar} \)) deforming the pointwise product in the direction of the Poisson bracket:

\[
(5.5) \quad f \star g = fg + \epsilon \{ f, g \} + \sum_{n=2}^{\infty} B_n(f, g),
\]

where the \( B_n \)s are bidifferential operator with the property \( B_n(1, \bullet) = B_n(\bullet, 1) = 0 \). Moreover, this star product is exactly the one defined by Kontsevich in [K].

The case \( C \) pre-Poisson has been considered in [CFa05]. It turns out that the \( \text{PSM} \) on \( M \) with boundary conditions on \( C \) is equivalent to the \( \text{PSM} \) on any \( M' \) as in Prop. 2.3 with the same boundary conditions. As a result, it is enough to consider coisotropic submanifolds.

The case \( C \) coisotropic has been considered in [CF04]. A few interesting, but unpleasant phenomena appear. First, it might happen that the result of (5.4) depends on \( u \) and \( v \). Second, the result regarded as a function on \( C^\infty(C) \) may not be invariant. Third, associativity may not be guaranteed. Under suitable conditions, one may ensure that everything works. We return on this in subsection 7.4, where we show that the vanishing of the first and second Lie algebroid cohomologies is a sufficient condition.

6. Super PSM and duality

To deal with symmetries in the PSM consistently, one has to resort to the BV formalism. In the case at hand the recipe (following from the AKSZ formalism [AKSZ] adapted to the PSM with boundary [CF01b]) consists just of formally replacing the fields \((X, \eta)\) in the action by “superfields” \((X, \xi)\). To simplify the discussion, let us choose a coordinate chart on \( M \). Then we may regard \( X \) as an element of \( \Omega^0(\Sigma) \otimes \mathbb{R}^m, \quad m = \text{dim} \, \Sigma, \)

\[\eta \in \Omega^1(\Sigma) \otimes (\mathbb{R}^m)^*,\]

The action in local coordinate reads

\[
S = \int_\Sigma \eta_i dX^i + \frac{1}{2} \pi^{ij}(X) \eta_i \eta_j,
\]

where we use Einstein’s convention on repeated indices.

The superfields are then \( X \in V := \Omega(\Sigma) \otimes \mathbb{R}^m, \quad \eta \in W := \Omega(\Sigma) \otimes (\mathbb{R}^m)^* \) and one just puts them in the action instead of the classical fields. The integral selects by definition the two-form component\(\text{II}\). The superaction is a function on \( V \oplus W \) regarded as a graded vector space. Observe that for consistency, we have to consider elements of \( V \) as even and those of \( W \) as odd. It is also useful to introduce a \( \mathbb{Z} \)-grading and assign \( V \) degree 0 and \( W \) degree 1, while integration over \( \Sigma \) is given degree \(-2\).

If we only consider the “kinetic term” \( S_0 = \int_\Sigma \eta_i dX^i \), it is quite natural to observe that there is a duality obtained by exchanging \( \eta \) and \( X \). The only problem is that we want to think of the zero form in \( X \) as a map. However, the zero form in \( \eta \) has to be considered as an odd element. The problem is solved if we allow the target \( M \) to be a graded manifold itself. The duality then exchanges \( M \) with a

\[\text{An invariant description requires introducing graded manifolds and regarding the space of superfields as the graded infinite-dimensional manifold Map(T[1]\Sigma, T^*[1]M).}\]
dual manifold $\tilde{M}$. If we now take into account the term involving $\pi$, we see that it corresponds to a similar term for a multivector field on $\tilde{M}$.

If $\Sigma$ has a boundary, the superfields have to be assigned boundary conditions. We choose a coisotropic submanifold $C$ of $M$ and require $X$ on the boundary to take values in $C$, while $\eta$ has to take values in the pullback of the conormal bundle of $C$. As we work in perturbation theory, we may actually consider just a formal neighborhood of $C$ into $M$ and regard $NC$ as the Poisson manifold. The dual manifold turns out to be the graded manifold $N^*\{1\}C$. To $\pi$ there corresponds a multivector field $\tilde{\pi}$ of total degree 2. The dual theory can be mathematically understood in terms of a formality theorem for graded manifolds.

7. Formality Theorem

Kontsevich’s star product is an application of his formality theorem stating that the DGLA of multidifferential operators on a smooth manifold is formal [K]. His explicit local formula for the $L_\infty$-quasi-isomorphism may also be obtained in terms of expectation values of boundary and bulk observables in the PSM with zero Poisson structure [CF00]. This construction may be generalized to smooth graded manifolds [CF07] (see also [C06] for the PSM version). An application [CF07, LS] is the deformation quantization of coisotropic submanifolds [CF04].

Let $\mathcal{M}$ be a smooth graded manifold $A = \bigoplus_{k \in \mathbb{Z}} A^k$ its graded algebra of functions, $\mathcal{V}(\mathcal{M})$ its complex of graded multivector fields and $\mathcal{D}(\mathcal{M})$ its complex of graded multidifferential operators (regarded as a subcomplex of the Hochschild complex of $A$). We consider $\mathcal{V}(\mathcal{M})$ as DGLAs by taking the total degree (shifted by one); viz.:

$\mathcal{V}^k(\mathcal{M}) := \{r$-multivector fields of homogeneous degree $s$ with $r + s = k + 1\}$,

$\mathcal{D}^k(\mathcal{M}) := \{r$-multidiff. operators of homogeneous degree $s$ with $r + s = k + 1\}$.

**Theorem 7.1** (Formality Theorem). There exists an $L_\infty$-quasi-isomorphism $U : \mathcal{V}(\mathcal{M}) \leadsto \mathcal{D}(\mathcal{M})$, hence the DGLA $\mathcal{D}(\mathcal{M})$ is formal. Moreover, $U$ may be chosen such that the degree-one component $U_1 : \mathcal{V}(\mathcal{M}) \leadsto \mathcal{D}(\mathcal{M})$ is the Hochschild–Kostant–Rosenberg (HKR) map.

See [K] for the proof in the ordinary case and [CF07] for the case of graded manifolds.

One may also extend $U$ to formal power series in a parameter $\epsilon$ and get an $L_\infty$-quasi-isomorphism $\mathcal{V}_\epsilon(\mathcal{M}) \leadsto \mathcal{D}_\epsilon(\mathcal{M})$, where

$\mathcal{V}_\epsilon(\mathcal{M}) := \epsilon \mathcal{V}(\mathcal{M})[[\epsilon]]$, \hspace{1em} $\mathcal{D}_\epsilon(\mathcal{M}) := \epsilon \mathcal{D}(\mathcal{M})[[\epsilon]]$.

In particular, since MC elements are mapped to MC elements by an $L_\infty$-morphism (with no convergence problems in the setting of formal power series), the Formality Theorem also shows that every MC element of $\mathcal{V}(\mathcal{M})$ gives rise to a MC element of $\mathcal{D}_\epsilon(\mathcal{M})$. In the case of ordinary manifolds, the former MC element is the same as a Poisson structure, while the latter defines a deformation quantization of the former. (A classification theorem also follows by the fact that an $L_\infty$-quasi-isomorphism actually yields an isomorphism of the sets of MC elements modulo gauge transformations).
The case of graded manifold is important for coisotropic submanifolds. As we recalled in Section 3, the algebra $A := \Gamma(NC)$ naturally appears in the cohomological description of a coisotropic submanifold $C$ of a Poisson manifold $(M, \pi)$. Now one can reinterpret $A$ as the algebra of functions on the graded manifold $N^*[1]C$ (where $[1]$ denotes a shift by 1 in the fiber coordinates). Moreover, the restriction of $\pi$ to $NC$ (obtained upon choosing an embedding of $NC$ into $M$) may be regarded as a MC element on $NC$ but also on $N^*[1]C$.

One way to see this is via Roytenberg’s Legendre mapping theorem [R] which states the existence of a canonical antisymplectomorphism between $T^*[n]E$ and $T^*[n](E^*[n])$ for every integer $n$ and for every graded vector bundle $E$. In particular, for $n = 1$ and observing that multivector fields on a graded manifold may be reinterpreted as functions on its tangent bundle shifted by 1, one gets an anti-isomorphism between the GLAs $\mathcal{V}(E)$ and $\mathcal{V}(E[1])$; in particular, this yields an isomorphism of the sets of MC elements. Specializing to $E = NC$ (as a graded vector bundle concentrated in degree zero) yields the sought for result.

A more direct way is just to observe that each derived bracket $\lambda_k$ in (3.1) is a multiderivation on the algebra $A$ and so a $k$-vector field on $N^*[1]C$. Their linear combination is the desired MC element. We now consider a more general situation.

### 7.1. Maurer–Cartan elements for graded manifolds

In general, given a graded manifold $\mathcal{M}$ and a MC element $\pi$ in $\mathcal{V}(\mathcal{M})$, by Theorem 3.1 (with $g = \mathcal{V}(\mathcal{M})$, $h = C^\infty(\mathcal{M})$, $P$ and $i$ the natural projection and inclusion), we may construct a $P_\infty$-algebra on $A := C^\infty(\mathcal{M})$ with multibrackets as in (3.1). Taking care of degrees, one sees that $\lambda_k$ is a multiderivation of degree $2 - i$, i.e.:

\[
\begin{align*}
\lambda_0 &= P_0 \in A^2, \\
\lambda_1 : A^i &\to A^{i+1}, \\
\lambda_2 : A^{j_1} \otimes A^{j_2} &\to A^{j_1+j_2}, \\
&\ldots \\
\lambda_i : A^{j_1} \otimes \cdots \otimes A^{j_i} &\to A^{j_1+\cdots+j_i+2-i}, \\
&\ldots
\end{align*}
\]

If $P_0 = 0$, then $\lambda_0 = 0$ and we have a flat $L_\infty$-algebra. In this case, $\lambda_1$ is a differential and its cohomology

\[
A^\bullet := H_{\lambda_1}^\bullet(A)
\]

is a graded Poisson algebra. In the following we will denote by $\{ , \}$ the induced Poisson bracket on $A$. In particular, $A^0$ is a Poisson algebra (but in general it is not the algebra of functions on a smooth manifold) whose Poisson bracket we will simply denote by $\{ , \}$.

**Definition 7.2.** A graded manifold $\mathcal{M}$ with a (flat) $P_\infty$-structure on $A = C^\infty(\mathcal{M})$ is called a (flat) $P_\infty$-manifold.
We now turn to multidifferential operators. An element \( m \in D^1_\epsilon(M) \) consists of a sequence \( m_i \), where \( m_i \) is an \( i \)-multidifferential operators of degree \( 2 - i \), i.e.,

\[
m_0 \in \epsilon A^2_\epsilon, \\
m_1 : A^1_\epsilon \to \epsilon A^{i+1}_\epsilon, \\
m_2 : A^{i_1}_\epsilon \otimes A^{i_2}_\epsilon \to \epsilon A^{i_1 + i_2}_\epsilon, \\
... \\
m_i : A^{j_1}_\epsilon \otimes \cdots \otimes A^{j_i}_\epsilon \to \epsilon A^{j_1 + \cdots + j_i + 2 - i}_\epsilon,
\]

Let us denote by \( \chi \) the (extension to \( A_\epsilon \)) of the multiplication on \( A \) and define \( \mu = \chi + m \); viz.:

\[
\mu_i = \begin{cases} 
  m_i & i \neq 2, \\
  \chi + m_2 & i = 2.
\end{cases}
\]

The element \( m \) is MC, i.e.,

\[
\delta m + \frac{1}{2} [m, m] = 0, \quad (\delta = [\chi, \ ]),
\]

iff the operations \( \mu_i \) define an \( A_\infty \)-structure on \( A_\epsilon \). If in addition \( \mu_0 \) vanishes, one says that the \( A_\infty \)-algebra is flat. In this case, \( H_{\mu_i}^* \) is an associative algebra.

Let \( \tilde{\mu} \) denote the skew-symmetrization of \( \mu \). Then \( (A_\epsilon, \tilde{\mu}) \) is an \( L_\infty \)-algebra. Since the multiplication \( \chi \) is graded commutative, the \( \tilde{\mu} \) will take values in \( \epsilon A_\epsilon \). So, dividing by \( \epsilon \) and working modulo \( \epsilon \), they define operations \( \tilde{\mu}_i \) on \( A \) which make it into an \( L_\infty \)-algebra. It may be shown, see [CF07], that the \( \tilde{\mu}_i \)s are actually multiderivations, so we have a \( P_\infty \)-algebra structure on \( A \).

### 7.2. Deformation quantization of \( P_\infty \)-manifolds.

Let \( A \) be the algebra of functions on a \( P_\infty \)-manifold \( M \). We denote by \( \chi \) the graded commutative product and by \( \lambda \) the multibrackets.

**Definition 7.3.** A deformation quantization of \( M \) consists of an \( A_\infty \)-structure on \( (A_\epsilon, \mu) \) with \( \mu = \chi + m \), such that:

1. \( m \) is of order \( \epsilon \);
2. \( m \) consists of multidifferential operators;
3. \( m \) vanishes when one of its arguments is the unit in \( A \); (i.e., \( m_i(a_1, \ldots, a_i) = 0 \) if \( a_j = 1 \) for some \( j \in \{1, \ldots, i\} \), \( i > 0 \));
4. the induced \( P_\infty \)-structure \( \tilde{\mu} \) is equal to \( \lambda \).

If \( M \) is flat, by deformation quantization in the flat sense we mean that in addition the condition \( \mu_0 = 0 \) is fulfilled.

The formality theorem for graded manifolds then implies the following

**Theorem 7.4.** Every \( P_\infty \)-manifold \( M \) admits a deformation quantization.

**Proof.** Let \( P \in V_\epsilon(M) \) be the MC element with derived brackets \( \lambda \). Define \( m \) by applying the \( L_\infty \)-quasi-isomorphism \( U \) to \( \epsilon P \):

\[
m = \sum_{n=0}^{\infty} \frac{1}{n!} U_n(P, \ldots, P).
\]

This is a MC element in \( \epsilon D_\epsilon(M) \) with the desired properties. \( \square \)
Observe however that the problem of quantizing flat $P_{\infty}$-manifolds in the flat sense is on the other hand not solved. In fact, we are not able to conclude that $P_0 = 0$ implies $m_0 = 0$ but only that $m_0 = O(\epsilon^2)$. This is the deformation quantization version of a (potential) anomaly (see [CF07]).

There are some cases when $m_0$ actually happens to vanish (see [CF04]). There are also cases [CF07] where $m_0$ can be killed by shifting the $A_{\infty}$-structure without changing its classical limit. Namely, for $\gamma \in \epsilon A^1[[\epsilon]]$ define

\[
\hat{m}_0 = m_0 + m_1(\gamma) + \frac{1}{2} m_2(\gamma, \gamma) + \cdots, \\
\hat{m}_1(a) = m_1(a) + m_2(a, \gamma) \pm m_2(\gamma, a) + \cdots.
\]

Then, $\forall \gamma, \mu + \hat{m}$ is again an $A_{\infty}$-structure on $A[[\epsilon]]$ that induces the same $P_{\infty}$-structure on $A$. One may then look for a $\gamma$ such that $\hat{m}_0 = 0$. It is not difficult to prove by induction, see [CF07], that a sufficient condition for this to happen is that $A^2 = \{0\}$. So we have the following

**Theorem 7.5.** Every flat $P_{\infty}$-manifold $M$ with $A^2 = \{0\}$ admits a deformation quantization in the flat sense.

### 7.3. Quantization of cohomology

Assume now that the flat $P_{\infty}$-manifold $M$ has been quantized in the flat $A_{\infty}$ sense. We denote by $\chi$ the graded commutative multiplication on the algebra of functions $A$, by $\lambda_i$ the multibrackets defining the flat $P_{\infty}$-structure on $A$ and by $\mu_i$ the multibrackets defining the flat $A_{\infty}$-structure on $A[[\epsilon]]$. We will also denote by $d$ the differential $\lambda_1$ and by $A$ the $d$-cohomology of $A$.

The $\mu_1$-cohomology is an associative algebra. However, in view of quantizing $A^\bullet$, or at least $A^0$, this is not the algebra we are in general interested in since the projection modulo $\epsilon$ is not a chain map. We have instead to proceed as follows. Observe that by appropriately rescaling the multibrackets, we get a new $A_{\infty}$-structure; viz., for given integers $s_i$, define $\tau_i = \epsilon^{s_i} \mu_i$, where we assume $\mu_i$ to be divisible by $\epsilon^{-s_i}$ whenever $s_j < 0$. If moreover $s_i + s_j$ is a function of $i + j$, the multibrackets $\tau_i$ also define an $A_{\infty}$-algebra.

In particular, we may take $s_i = i - 2$ since $\mu_1 = O(\epsilon)$. Namely, we define the new $A_{\infty}$-algebra structure

\[
\tau_i := \epsilon^{i-2} \mu_i.
\]

Observe that

\[
\tau_1 = d + O(\epsilon), \quad \tau_2 = \chi + O(\epsilon), \quad \tau_3 = O(\epsilon^{i-1}), \quad i > 2.
\]

By construction, the graded skew-symmetrization $\tilde{\tau}_2$ of $\tau_2$ is divisible by $\epsilon$; so there is a unique operation $\psi$ with $\tilde{\tau}_2 = \epsilon \psi$. The bracket $\lambda_2$ on $A$ is then $\psi$ modulo $\epsilon$.

We will also denote by $\delta = \sum_{n=0}^{\infty} \epsilon d_n$, $d_0 = d$, the differential $\tau_1$ and by $B$ the $\delta$-cohomology of $A[[\epsilon]]$. Observe that the differential $\delta$ is a deformation of the differential $d$.

Observe that by construction $\tau_2$, $\tilde{\tau}_2$ and $\psi$ are chain maps and we will denote by $[\tau_2]$, $[\tilde{\tau}_2]$ and $[\psi]$ the induced operations in cohomology. The operation $[\tau_2]$ is an associative product which we will also denote by $\star$ (and by $\star$ its restriction to $B^0$).

Whenever needed we denote $\delta$-cohomology classes by $[\ ]_{\delta}$ and $d$-cohomology classes by $[\ ]_d$. With these notations we have

\[
[a]_{\delta} \star [b]_{\delta} = [\tau_2]([a]_{\delta}, [b]_{\delta}) = [\tau_2(a, b)]_{\delta},
\]
where \([\tau_2]\) is the map induced by \(\tau_2\) in cohomology, while \(a\) and \(b\) are representatives of the classes \([a]_\delta\) and \([b]_\delta\). By construction \([\tilde{\tau}_2]\) defines the graded \(*\)-commutator \([\ ,\ ]\) on \(B\):
\[
[a]_\delta, [b]_\delta = [\tilde{\tau}_2]([a]_\delta, [b]_\delta) = [\tilde{\tau}_2(a, b)]_\delta.
\]
We will denote simply by \([\ ,\ ]\) its restriction to \(B^0\). Observe that multiplication by \(\epsilon\) commutes with \(\delta\). So \(B\) is also an \(\mathbb{R}[[\epsilon]]\)-module with \(\epsilon[a]_\delta = [\epsilon a]_\delta\). Moreover, we have
\[
[a]_\delta, [b]_\delta = [\psi]([a]_\delta, [b]_\delta).
\]
If \(B\) \((B^0)\) is \(\epsilon\)-torsion free—i.e., multiplication by \(\epsilon\) is injective—, then \([\psi]\) is the unique operation with the above property. Consider now the \(\mathbb{R}\)-linear projection
\[
\varpi : A[[\epsilon]] \rightarrow A, \quad \sum_{n=0}^{\infty} \epsilon^n a_n \mapsto a_0.
\]
Observe that \(\varpi : (A[[\epsilon]], \delta) \rightarrow (A, d)\) is a chain map. We will denote by \([\varpi]\) : \(B \rightarrow A\) the induced map in cohomology. Since \(\varpi\) is an algebra homomorphism, so is \([\varpi]\). The relation with the Poisson structure on \(A\) is clarified by the now obvious formula
\[
[\varpi](\psi)((a)_{\delta}, (b)_{\delta})) = \{ [\varpi](a)_{\delta}, [\varpi](b)_{\delta} \}, \quad \forall [a]_{\delta}, [b]_{\delta} \in B
\]
which immediately implies the

**Proposition 7.6.** The algebra homomorphism \([\varpi]\) has the following additional properties:

1. The image \(C \ (C^0)\) of \([\varpi]\) is a Poisson subalgebra of \(A \ (A^0)\).
2. If \(B\) is \(\epsilon\)-torsion free, then
\[
[\varpi] \left( \frac{[a]_\delta, [b]_\delta}{\epsilon} \right) = \{ [\varpi](a)_{\delta}, [\varpi](b)_{\delta} \}, \quad \forall [a]_{\delta}, [b]_{\delta} \in B.
\]
3. If \(B^0\) is \(\epsilon\)-torsion free, then the above formula holds for all \([a]_{\delta}, [b]_{\delta} \in B^0\).

Since obviously \(\text{Ker} \varpi = \epsilon A[[\epsilon]]\), we have the short exact sequence
\[
0 \rightarrow A[[\epsilon]] \xrightarrow{\epsilon} A[[\epsilon]] \xrightarrow{\varpi} A \rightarrow 0
\]
which induces the long exact sequence
\[
\cdots \rightarrow A^{i-1} \xrightarrow{\partial} B^i \xrightarrow{\varpi} B^i \xrightarrow{[\varpi]} A^i \xrightarrow{\partial} B^{i+1} \rightarrow \cdots
\]
in cohomology. Immediately we then get the

**Proposition 7.7.** The following statements are equivalent:

1. \([\varpi]\) is surjective;
2. \(B\) is \(\epsilon\)-torsion free;
3. \(\partial\) is trivial.

To continue our study of the problem, we now need Lemma A.1 of [CFT], which we state in a slightly modified version:

**Lemma 7.8.** Let \(k\) be a field and \(M\) a \(k[[\epsilon]]\)-module endowed with the \(k[[\epsilon]]\)-adic topology. Then \(M \simeq_k M_0[[\epsilon]]\) for some \(k\)-vector space \(M_0\) iff \(M\) is Hausdorff, complete and \(\epsilon\)-torsion free. Moreover, \(M_0 \simeq_k M/\epsilon M\).

We finally have the
Theorem 7.9. If $B$ is Hausdorff and complete in the $\epsilon$-adic topology, then $B$ is a deformation quantization of $A$ iff any (and so all) of the statements in Proposition 7.7 holds.

Proof. If $B$ is a deformation quantization of $A$, then in particular $B \cong_R A[[\epsilon]]$. So by Lemma 7.8 $B$ is $\epsilon$-torsion free. On the other hand, the statements in Proposition 7.7 imply that $A \cong_R B/\ker[\pi] = B/\ker \epsilon B$; so $B \cong_R A[[\epsilon]]$ by Lemma 7.8. Statement (2) in Proposition 7.6 completes the proof. □

We are not able to show that $\text{Im} \delta$ is closed in the $\epsilon$-adic topology, so we must put the extra condition in the Theorem. We wish however to make the following

Conjecture 7.10. The $\mathbb{R}[[\epsilon]]$-module $B$ is Hausdorff and complete in the $\epsilon$-adic topology.

In general we do not expect $B$ to be a deformation quantization of $A$. However, it is often enough to have $B^0$ as a deformation quantization of $A^0$. We end this Section by exploring some sufficient conditions for this to happen.

Lemma 7.11. If $B^i$ is Hausdorff and complete, then it is isomorphic to $C^i[[\epsilon]]$ as an $\mathbb{R}$-vector space iff it is $\epsilon$-torsion free.

Proof. The “only if” implication is obvious. As for the “if” part, by Lemma 7.8 we conclude that $B^i$ is isomorphic to $(B^i/\epsilon B^i)[[\epsilon]]$. On the other hand the long exact sequence gives $0 \to B^i \to B^i \to C^i \to 0$ which completes the proof. □

By statement (3) in Proposition 7.6 we then get

Corollary 7.12. If $B^0$ is Hausdorff, complete and $\epsilon$-torsion free, then it is a deformation quantization of $C^0$.

The long exact sequence yields the

Lemma 7.13. If $A^{-1} = \{0\}$, then $B^i$ is $\epsilon$-torsion free.

Thus, $A^{-1} = \{0\}$ is a sufficient condition for $B^0$ to be a deformation quantization of $C^0$ provided it is Hausdorff and complete. Finally, we have the

Lemma 7.14. If $A^{i+1} = \{0\}$, then $[\pi]^i : B^i \to A^i$ is surjective.

Proof. This is a standard proof in cohomological perturbation theory. Let $[a_0]_d \in A^i$. Choose one of its representatives $a_d \in A^i$. We look for a $\delta$-closed $a = \sum_{n=0}^{\infty} \epsilon^n a_n \in A^i[[\epsilon]]$. The equations we have to solve have the form

\begin{equation}
(7.2) \quad da_n = - \sum_{r+s=n+1}^{r>0} d_r a_s,
\end{equation}

and we may solve them by induction. Namely:

1. For $n = 0$, the equation is just $da_0 = 0$ which is satisfied by assumption.
2. Assume now that all equations for $a_k$, $k < n$, have been solved. This implies that that the r.h.s. of (7.2) is d-closed. Since we assume that $A^{-1} = \{0\}$, we may then find $a_n$ satisfying the equation. □

5Observe that the Conjecture is general false if we drop the conditions that $A$ is the algebra of functions on a graded manifold and that the components $d_n$ of $\delta$ are differential operators.
Remark 7.15. Observe that, if we knew that $B^{i+1}$ were Hausdorff (e.g., if Conjecture 7.10 were true), then we could derive Lemma 7.14 directly from the long exact sequence (this is just a variant of Nakayama’s Lemma). In fact, $A^{i-1} = \{0\}$ implies that multiplication by $\epsilon$ on $B^{i+1}$ is surjective. So every element $a \in B^{i+1}$ may be written as $a = \epsilon a_1$. Continuing this process, we get a sequence $a_n$ with $a = \epsilon^n a_n$. The sequence $\epsilon^n a_n$ converges to zero, so $a = 0$ since $B^{i+1}$ is Hausdorff. Thus, $B^{i+1} = \{0\}$ and by the long exact sequence again we get the result.

Putting together Theorem 7.5, Corollary 7.12, Lemmata 7.13 and 7.14, we get the following

**Theorem 7.16.** If $A^2 = A^1 = A^{-1} = \{0\}$, then $B^0$ is a deformation quantization of $A^0$ provided it is Hausdorff and complete.

If Conjecture 7.10 holds, we get the sufficient conditions in a much nicer way which makes reference only to the d-cohomology:

“**Theorem**”. A sufficient condition for a deformation quantization of $A^0$ to exist is $A^2 = A^1 = A^{-1} = \{0\}$.

Observe that the sufficient condition is by no means necessary. For example, in the case $\lambda_i = 0 \ \forall i$, we have $A^* = A^*$. On the other hand, $\delta = 0$ and $B^* = A^*[[\epsilon]]$ is a deformation quantization of $A^*$.

### 7.4. Deformation quantization of coisotropic submanifolds.

We now return to the case when $C$ is a coisotropic submanifold of $M$ (which is Poisson at least in a neighborhood of $C$). We take our graded manifold $M$ to be $N^*[1]|C$ with MC element the restriction of $\pi$ to $V(NC) = V(N^*[1]|C)$. Now $A^{-1} = \{0\}$ since we do not have negative degrees, while $A^0$ is the Poisson algebra of functions on $C$ that are invariant under the canonical distribution on $C$. If we denote by $C$ the leaf space, we also write $C^\infty(C) = A^0$. Observe that in general $C$ is not a manifold, so the above is just a definition. Using the notations of Section 7.3, we have $B^0 = \text{Ker} \delta$. So $B^0$ is Hausdorff and complete, and by Theorem 7.16 we have the

**Theorem 7.17 (CF07).** If the first and second Lie algebroid cohomology of $N^*C$ vanish, the zeroth cohomology $C^\infty(C)$ has a deformation quantization.

The second Lie algebroid cohomology is also the space where the usual BRST/BFV anomaly lives; see [B04] for this in the context of deformation quantization. The advantage of the present approach is that one has formulae for $m_0$, so one can in principle check whether it may be canceled.

Observe that the conditions of Theorem 7.17 are very strong and by no means necessary. At the moment we actually do not know a single example where the anomaly shows up, at least locally. There are examples where the Lie algebroid cohomology class of $m_0$ is nontrivial, see [W], but in these examples $C$ is a point, so there is no problem in quantizing its reduced space (the problem arises however in the many-brane setting of Section 8).

As $C^\infty(C)$ might be rather poor (e.g., just constant functions), it might be interesting to quantize the whole Lie algebroid cohomology. Assuming that the $A_\infty$-structure is flat, by Theorem 7.9 the deformed cohomology $B$ yields a deformation quantization of $A$ iff it is Hausdorff, complete and $\epsilon$-torsion free.
7.5. Poisson submanifolds. A particular case of a coisotropic submanifold is a Poisson submanifold, i.e., a submanifold $C$ of $(M, \pi)$ such that the inclusion map is a Poisson map. Equivalently, a Poisson submanifold is a coisotropic submanifold with trivial characteristic distribution. In this case, $C = C$, so the quantization of the reduction is not a problem. The interesting question is whether one can deform the pullback of the inclusion map to a morphism of associative algebras. To approach this problem, we associate to $C$ a different graded manifold as in the coisotropic case.

If the submanifold $C$ is determined by constraints, i.e., $C = \Phi^{-1}(0)$ with $\Phi: M \to V$ a given map to a vector space $V$, we set $\mathcal{M} := M \times V[-1]$ and reinterpret $\Phi$ as an element of $C^\infty(M) \otimes V$ and so as a vector field $X$ of degree 1 on $\mathcal{M}$. If we introduce coordinates $\{\mu^\alpha\}_{\alpha=1}^{\dim V}$ of degree $-1$, we have $X = \Phi^\alpha(x) \frac{\partial}{\partial \mu^\alpha}$, where the $\Phi^\alpha$s are the components of $\Phi$ w.r.t. the same basis. Observe that $[X, X] = 0$ and that $[\pi, X]$ vanishes on $C$. Moreover, the cohomology of $C^\infty(M)$ w.r.t the differential $\delta := [X, ]$ is $C^\infty(C)$. If $C$ is not determined by constraints, we take $\mathcal{M} := N[-1]C$ and $X$ the vector field of degree 1 corresponding to $\mathcal{M}$.

The crucial observation now is that $A := C^\infty(\mathcal{M}) = \bigoplus_{j \leq 0} A^j$ is a nonpositively graded commutative algebra. As a consequence, $A^i = 0$, $i > 0$, with the differential $\lambda_1 = \delta$. Observe moreover that $A^0$ is just $C^\infty(M)$ and that $A^0 = A^0/I = C^\infty(C)$, with $I = \delta A^{-1}$ the vanishing ideal of $C$.

7.5.1. Casimir functions. A particularly simple case is when the components of $\Phi$ are Casimir functions: viz., $\{\Phi^\alpha, f\} = 0 \forall \alpha \forall f \in C^\infty(M)$. Equivalently $[\pi, \Phi^\alpha] = 0 \forall \alpha$, i.e., $[\pi, X] = 0$. Thus $\tau := \pi + X$ is a MC element. The induced $P_\infty$-structure has $\lambda_1 = \delta$. Quantization has the following easy to check remarkable properties: i) $(A^0[[\epsilon]], \tau_2)$ is Kontsevich’s deformation quantization $A_M$ of $C^\infty(M)$; ii) $\tau_2(\mu^\alpha, f) = \tau_2(f, \mu^\alpha) = \mu^\alpha f, \forall \alpha$. As a consequence $1 := \tau_1(A^{-1}[[\epsilon]])$ is the two-sided ideal generated by $\{\tau_1(\mu^\alpha)\}$. A further easy computation, using Stokes’ theorem in the explicit expression of the coefficients by Kontsevich’s graphs, shows that $\tau_1(\mu^\alpha) = D(\Phi^\alpha)$, where $D$ is the Duflo–Kirillov–Kontsevich map

$$D: C^\infty(M) \to C^\infty(M)[[\epsilon]],$$

$$f \mapsto \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} U_{n+1}(f, \pi, \ldots, \pi).$$

So we get a deformation quantization of $C$ as $B^0 = A_M/1$, and the projection $A_M \to B^0$ is a quantization of the inclusion map. This proves a conjecture in [CR] (see also [C06, Sect. 5] for a previous sketch of this proof).

![Figure 2](image)

Figure 2. $\tau_1(\mu^\alpha) = D(\Phi^\alpha)$

7.5.2. The general case. In general $\pi + X$ is not a MC element. However, the fact that $[\pi, X]$ vanishes on $C$, makes $\pi + X$ a MC element up to $\delta$-exact terms. As a consequence, one may use cohomological perturbation theory as in [LS] and
find a MC element \( \hat{\pi} \) of the form \( \pi + X + \text{corrections} \) in the ideal of multivector fields of total degree 2 generated by \( \{ \mu^\alpha \} \). Observe that the only vector field in \( \hat{\pi} \) is \( X \), so \( \lambda_1 = \delta \). The vanishing of the cohomologies in positive degrees implies that we have an \( A_\infty \)-structure on \( A[[\epsilon]] \) and a surjective map \( B^0 \to A^0/I \). Moreover, the restriction of \( \tau_2 \) in degree 0 makes \( A^0[[\epsilon]] \) into an algebra with an algebra morphism to \( B^0 = A^0[[\epsilon]]/I, I := \tau_1(A^{-1}[[\epsilon]]) \). The two problems of this constructions are the following: i) \( (A^0[[\epsilon]], \tau_2) \) might not be associative; ii) \( B^0 \) might not be isomorphic to \( A^0/I[[\epsilon]] \).

It may be shown that \( (A^0[[\epsilon]], \tau_2) \) is associative iff \( \hat{\pi} \) has the form \( \pi + X + \pi' \) with \( \pi' \) a bivector field. If in addition one chooses the constraints to be linear (e.g., if one works with \( \mathcal{M} = N[-1]C \)), then one can also easily see that \( B^0 \) turns out to be a deformation quantization of \( C^\infty(C) \). However, finding a \( \pi' \) as above is a highly nontrivial problem, and it is not clear under which conditions a solution may exist.

A very simple case is when \( C \) consists of a point \( x \) (a zero of the Poisson structure). A quantization of the inclusion map can then be reinterpreted as a character (i.e., an algebra morphism to the ground ring) of the deformed algebra (deforming the evaluation at \( x \)). Even in such a simple situation, the existence of a \( \pi' \) is guaranteed only for \( \dim M = 2 \). In higher dimensions, it is an open problem.

8. Many branes

We now turn to the case when more than one coisotropic submanifold is chosen as a boundary condition. Namely, as in Section 5 we take \( \Sigma \) to be a disk. However, we now subdivide the boundary into closed, cyclically ordered intervals \( I_1, \ldots, I_n \) with exactly one intersection point between subsequent intervals. To the interval \( I_i \) we associate boundary conditions corresponding to a coisotropic submanifold \( C_i \).

Assuming clean intersections, in \cite{CF04} it is shown that the case \( i = 2 \) leads to the construction of a bimodule for the deformation quantizations of \( C_1 \) and \( C_2 \) if no anomaly appears.

In this Section we assume that no anomalies show up. If the submanifolds are pairwise transverse this amounts to asking that for each of them the anomaly discussed in subsection [7.4] vanishes. The vanishing of the second Lie algebroid cohomology for each submanifold is a sufficient condition by Theorem 7.17. Notice however that this is only a sufficient condition by no means necessary. A very simple example is when \( M \) is symplectic and \( C_1 \) is Lagrangian. In this case \( \hat{\pi}_1 \) is a point (assume \( C_1 \) to be connected), so there is no problem in quantizing it. The bimodule structure associated to \( C_2 = M \) can always be found (upon choosing the star product appropriately). On the other hand the relevant Lie algebroid cohomology is the de Rham cohomology of \( C_1 \). Another example is the linear case \( C_1 = \mathfrak{h}^0 \subset \mathfrak{g}^* \), where \( \mathfrak{h} \) is a Lie subalgebra of \( \mathfrak{g} \). It is shown in \cite{CF04} that there is no anomaly in this case even if the cohomology may be very complicated. Observe however that an example of nonvanishing anomaly has recently been found in \cite{W}.

We also use the notation \( A_C \) for the PSM deformation quantization of the Poisson algebra \( C^\infty(C) \), where \( C \) is a coisotropic submanifold of \( M \). Whenever a distribution is trivial, we omit underlining the submanifold. Observe that \( A_{\mathfrak{g}_L} \) is Kontsevich’s deformation quantization. With \( B_{C_1 \cap C_2} \) we then denote the \( A_{C_1} \)-module-\( A_{C_2} \) corresponding to the picture on the left in fig. 8 (where the big black dot
denotes the point at infinity which has to be sent to the point in the intersection where evaluation takes place).

\[ C_1 \cap C_2 \otimes A_{C_2} \rightarrow B_{C_1 \cap C_3} \]

**Figure 3. Two and three branes**

The situation with three coisotropic branes \( C_1, C_2 \) and \( C_3 \)—see the picture on the right in fig. 3—leads to a morphism of bimodules \[ B_{C_1 \cap C_2} \otimes A_{C_2} \rightarrow B_{C_1 \cap C_3} \] in a neighborhood of a triple intersection. The construction of the corresponding Kontsevich graphs has been analyzed in [CT]. In subsection 6.2 we discuss a special case.

One may also consider many branes and, instead of just invariant functions, sections of the corresponding complexes of exterior algebras of normal bundles. One may hope that this would lead to \( A_\infty \)-bimodules and morphisms thereof, but problems seem to arise with four and more branes.

In the rest of this Section we will concentrate on the special case when one of the branes is the whole manifold; we will consider only functions and no more than three branes.

**8.1. Quantization of morphisms.** Let \( \phi \) be a Poisson map \( M \rightarrow N \). Then its graph \( \text{Graph} \phi \) is a coisotropic submanifold of \( M \times N \), where \( M \) denotes \( M \) with opposite Poisson structure. We select the two branes \( C_1 = \text{Graph} \phi \) and \( C_2 = M \times N \) in \( M \times N \). Consider the right \( A_{M \times N} \)-module structure on \( B_{\text{Graph} \phi} \), forgetting about its left \( A_{\text{Graph} \phi} \)-module structure. Since \( A_{M \times N} \) is a subalgebra of \( A_{M \times N} \) and Kontsevich’s star product has the property \( A_{M \times N} = A_{M \times N}^{\text{opp}} \), we may regard \( B_{\text{Graph} \phi} \) as an \( A_M \)-bimodule-\( A_N \). This bimodule has a distinguished element, the constant function 1, and the map \( A_M \rightarrow B_{\text{Graph} \phi} \), \( f \mapsto f \cdot 1 \) is an isomorphism of \( \mathbb{R}[[\varepsilon]] \)-modules (since it is a deformation of the pullback of the diffeomorphism \( p_1: \text{Graph} \Phi \rightarrow M \) defined by projection). Thus, to every \( f \in A_N \) we associate a unique element \( \hat{\phi}(f) \) of \( A_M \) by the equation

\[ 1 \cdot f = \hat{\phi}(f) \cdot 1. \]

Observe that \( \hat{\phi} \) is a deformation of the pullback \( \phi^* \). It is not difficult to see that it is a morphism of associative algebras. This way, with the assumption that there is no anomaly, we have found a quantization procedure for Poisson maps.

Observe that one can exchange the role of \( C_1 \) and \( C_2 \) using the symmetry corresponding to reflecting the disk through the line joining the two intersections of the intervals \( I_1 \) and \( I_2 \) (● and | in fig. 3). In terms of Kontsevich graphs one gets the same formulae upon changing the sign of the Poisson structure; viz., we get the same result as before if we take \( M \times \overline{N} \) as the ambient Poisson manifold.
8.2. Compositions. Now suppose we have two Poisson maps $\phi: M \to N$ and $\psi: N \to K$. With no anomalies, we may quantize $\phi$, $\psi$, and $\psi \circ \phi$. The next problem we wish to address concerns the relation between $\psi \circ \phi$ and $\hat{\phi} \circ \hat{\psi}$.

To deal with it we consider the three branes $C_1 = \text{Graph} \phi \times K$, $C_2 = M \times N \times K$ and $C_3 = M \times \text{Graph} \psi$ in $\mathcal{M} \times N \times \mathcal{K}$ as in fig[3]. Let $p_1: M \times N \to M$ and $\tilde{p}_2: N \times K \to K$ be the canonical projections. To $f \in C^\infty (M)[\epsilon]$ and $g \in C^\infty (K)[\epsilon]$, we associate the element $D(f, g) \in C^\infty (\text{Graph} \psi \circ \phi)[\epsilon]$ obtained by putting $p_1^* f \otimes 1$ and $1 \otimes \tilde{p}_2^* g$ at the two intersection points (here 1 is the constant function on $C^\infty (N)$). It is possible to check that $D$ defines a morphism of $A_M$-bimodules-$A_K$ $B_{\text{Graph} \phi \otimes A_N} \otimes B_{\text{Graph} \psi} \to B_{\text{Graph} \psi \circ \phi}$. So it is enough to compute $\hat{\sigma}_{\phi, \psi} := D(1, 1) = 1 + O(\epsilon^2)$ in $B_{\text{Graph} \psi \circ \phi}$. Observe again that there is a unique element $\hat{\sigma}_{\phi, \psi} \in A_M$ such that $\hat{\sigma}_{\phi, \psi} \cdot 1 = \sigma_{\phi, \psi}$. Moreover, since $\hat{\sigma}_{\phi, \psi}$ is of the form $1 + O(\epsilon^2)$, it is invertible. Now for $h \in A_K$ we have the identities

$$\hat{\sigma}_{\phi, \psi} \cdot h = D(1, 1 \cdot h) = D(1, \hat{\psi}(h) \cdot 1) = D(1 \cdot \hat{\psi}(h), 1) = D(\hat{\phi}(\hat{\psi}(h)), 1, 1) = \hat{\phi} \circ \hat{\psi} \cdot \sigma_{\phi, \psi}.$$ 

Using $1 \cdot h = \hat{\psi} \circ \hat{\phi}(h) \cdot 1$ and the definition of $\hat{\sigma}_{\phi, \psi}$, we get

$$\hat{\sigma}_{\phi, \psi} \cdot h = \hat{\sigma}_{\phi, \psi} \cdot (\hat{\psi} \circ \hat{\phi}(h) \cdot 1) = (\hat{\sigma}_{\phi, \psi} * M \hat{\psi} \circ \hat{\phi}(h)) \cdot 1,$$

$$\hat{\phi} \circ \hat{\psi}(h) \cdot \sigma_{\psi, \phi} = \hat{\phi} \circ \hat{\psi}(h) \cdot (\hat{\sigma}_{\phi, \psi} \cdot 1) = (\hat{\phi} \circ \hat{\psi}(h) * M \hat{\sigma}_{\phi, \psi}) \cdot 1,$$

where $*_M$ denotes the star product on $M$. Finally,

$$\hat{\psi} \circ \hat{\phi}(h) = \hat{\sigma}_{\phi, \psi}^{-1} * M (\hat{\phi} \circ \hat{\psi}(h)) * M \hat{\sigma}_{\phi, \psi},$$

where $\hat{\sigma}_{\phi, \psi}^{-1}$ is the inverse of $\hat{\sigma}_{\phi, \psi}$ w.r.t. $*_M$.

Thus, upon conjugation (by an element which depends on the given Poisson maps), we see that the composition of quantizations is the quantization of compositions.

8.3. Quantum groups. We now want to apply the results of the last subsection to the case of a Poisson–Lie group $G$; i.e., a Poisson manifold and a Lie group such that the product $m: G \times G \to G$ and the inverse $^{-1}: \overline{G} \to G$ are Poisson maps. By Kontsevich we have an associative algebra $A_G$. On the other hand, if no anomaly is present, we get a morphism of algebras $\Delta := \hat{m}: A_G \to A_G \otimes A_G := A_{G \times G}$. The associativity equation $m \circ (m \otimes id) = m \circ (id \otimes m)$ then yields

$$(\Delta \otimes id) \circ \Delta = \Phi^{-1} *_{G^3} (id \otimes \Delta) \circ \Delta *_{G^3} \Phi,$$

with $\Phi = \hat{\sigma}_{id \otimes m, m}^{-1} *_{G^3} \hat{\sigma}_{m \otimes id, m} \in A_{G^3}$. In some lucky cases, $\Phi$ might turn out to be 1 (or at least central). In this case we would have quantized the Poisson–Lie group as a bialgebra. Otherwise, one may hope to get the right properties for the “associator” $\Phi$ in order to get a bialgebra out of it.

Finally, observe that we may also quantize the inverse and get a candidate for the antipode. If relations are preserved, we should get a Hopf algebra structure on $A_G$, i.e., the corresponding quantum group.

Just assuming that anomalies are absent is not enough to get a Hopf algebra as relations are preserved only up to conjugation by certain elements. If one cannot get rid of them, the resulting structure is probably that of a hopfish algebra as defined in [BW].
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Institut für Mathematik, Universität Zürich–Irchel, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

E-mail address: alberto.cattaneo@math.unizh.ch