M-Theory Brane Deformations

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Abstract

Using the techniques developed by Lunin and Maldacena we calculate the supergravity solutions of membranes and fivebranes in the presence of a background C field. All the distinct possible C-field configurations are explored. Decoupling limits for these branes are then described that preserve the deformation leading to families of M-theory brane deformation duals. The decoupled geometry is then explored using probe brane techniques and brane thermodynamics.

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1 Introduction

Over the past few years there has been a renewed interest in the various deformations of the field theories that arise on brane world volumes. For the D3 brane, the undeformed world volume theory is $\mathcal{N} = 4$ Yang-Mills. Its deformations include: noncommutative Yang-Mills where the Neveu-Shwarz two form is present along the brane world volume [1–5]; the so called dipole theory where the Neveu-Schwarz two form is present with one leg along the brane world volume and one off it [6–10]; and finally the $\mathcal{N} = 1$ $\beta$-deformed Yang-Mills theory where the Neveu Schwarz two form is present entirely off the brane world volume [11]. Often these deformed theories have interesting properties such as the S-duality properties of the marginally deformed $\mathcal{N} = 4$ theory [12]. Other recent developments have been with so called puff field theories [13].

Many of the generalisations to M-theory have already been explored though of course even the undeformed case is somewhat of a mystery for more than one coincident brane. The cases that have been examined already are where the three form C-field is present with all legs along the five-brane world volume [14, 15] and the generalisation of the dipole theory the so called discpole theory, [8, 16]. In this paper we systematically go through all the distinct deformations for both the membrane and the five-brane with the background C-field in all the possible configurations with legs on and off the brane world volume. (This will, of course, recover some of the already known solutions). Essentially this will be a systematic application of the method of Lunin and Maldacena [11] to give all C-field deformations of M-theory brane geometries.

With these geometries at hand we describe various decoupling limits (where the deformation is preserved in the limit). We then explore these decoupled geometries, which will be the supergravity duals of a deformed M-theory brane world volume theory. Through use of probe brane techniques and an analysis of the thermodynamic properties we will attempt to learn about these dual world volume theories.

Part of the motivation will also be to explore the solution generating method and
decoupling procedures themselves. In particular, we will see how the solution generating technique and thus the deformations of the dual theory and the decoupling limit commute. We will also see how different embedding choices of the solution generating procedure become relevant. Finally, we will also see through thermodynamic calculations that some of the deformations will leave the entropy invariant. This is a surprise given that in the dual theory the deformation will alter the interactions.

Overall the calculations presented here are a laboratory to study how decoupling limits and deformations of branes in M-theory work along with the different aspects of hidden symmetries in string and M-theory.

There has also been the study of the dipole deformations of Yang-Mills theory [17] motivated by the idea that the deformation may provide an additional scale through which additional decouplings may take place. The M-theory analogue of this would also be interesting to explore especially in the context of $G_2$ compactification [18].

2 Deformation method

This is a brief review of a technique, which can be used to construct solutions to eleven dimensional supergravity with the C-field switched on. Essentially, it is the method, described by Lunin and Maldacena [11]. The origins of the method date back to the use of T-duality (perhaps combined with Lorentz transformations) as a solution generating symmetry [19]. The starting point is an eleven dimensional background in which there are at least three $U(1)$ isometries. These isometries will form a three torus. The metric and the potential may then be placed in the following adapted form:

$$\begin{align*}
  ds^2 &= \Delta^{1/3} M_{ab} D \varphi^a D \varphi^b + \Delta^{-1/6} g_{\mu\nu} dx^\mu dx^\nu \\
  C^{(3)} &= \frac{1}{2} (C_{a\mu\nu} D \varphi^a \wedge dx^\mu \wedge dx^\nu + C_{ab\mu} D \varphi^a \wedge D \varphi^b \wedge dx^\mu) \\
  &\quad+ \frac{1}{6} (C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda + C_{abc} D \varphi^a \wedge D \varphi^b \wedge D \varphi^c) \\
  D \varphi^a &= d \varphi^a + \mathcal{A}_\mu^a dx^\mu
\end{align*}$$

(1) (2) (3)
where \( a, b, c \) label the directions of the \( T^3 \) and \( \mu, \nu, \lambda \cdots \) are for the remaining coordinates. The determinant of \( M \) is one and so \( \sqrt{\Delta} \) is the volume of the three-torus. Performing an \( S^1 \) reduction along one of the \( U(1)'s \) (for illustrative purposes we here choose \( \varphi_3 \)) produces a type IIA solution.

\[
M_{ab} D\varphi^a D\varphi^b = e^{-2\phi/3} h_{mn} D\varphi^m D\varphi^n + e^{4\phi/3} (D\varphi^3 + N_m D\varphi^m)^2 \quad (4)
\]

where \( m \) and \( n \) run over the remaining \( U(1)'s \) of the \( T^3 \) and \( h, \) like \( M, \) has unit determinant. Then a T-Duality is performed on another of the \( T^3 \) cycles (\( \varphi_1 \) in what follows) and a toroidally compactified IIB solution is produced with \( \{ \varphi^1, \varphi^2 \} \) as the coordinates of the two torus. The IIB solution is written in terms of the original eleven dimensional fields as follows, with IIB fields on the left and eleven dimensional fields on the right:

\[
ds^2 = \frac{1}{h_{11}} \left[ \frac{1}{\sqrt{\Delta}} (D\varphi^1 - CD\varphi^2)^2 + \sqrt{\Delta} (D\varphi^2)^2 \right] + e^{2\phi/3} g_{\mu\nu} dx^\mu dx^\nu \quad (5)
\]

\[
B^{(2)} = \frac{h_{12}}{h_{11}} D\varphi^1 \wedge D\varphi^2 - C_{32\mu} D\varphi^2 \wedge dx^\mu + D\varphi^1 \wedge A^1 - \frac{1}{2} C_{3\mu\nu} dx^\mu \wedge dx^\nu + C_{31\mu} dx^\mu \wedge A^1 \quad (6)
\]

\[
e^{2\Phi} = \frac{e^{2\phi}}{h_{11}}, \quad C^{(0)} = N_1
\]

\[
C^{(2)} = -(N_2 - \frac{h_{12}}{h_{11}} N_1) D\varphi^1 \wedge D\varphi^2 - C_{12\mu} D\varphi^2 \wedge dx^\mu - D\varphi^1 \wedge A^3 - \frac{1}{2} C_{1\mu\nu} dx^\mu \wedge dx^\nu + C_{31\mu} dx^\mu \wedge A^3 \quad (7)
\]

\[
C^{(4)} = -\left( \frac{1}{2} C_{2\mu\nu} + 2 C_{32\mu} A^3 - \frac{h_{12}}{h_{11}} (C_{1\mu\nu} + 2 C_{31\mu} A^3) \right) D\varphi^2 \wedge dx^\mu \wedge dx^\nu + \frac{1}{6} (C_{\mu\nu\lambda} + 3 C_{3\mu\nu} A^3) dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge D\varphi^1 + \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} + \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge D\varphi^2. \quad (8)
\]

\( D\varphi_1 = d\varphi_1 - C_{31\mu} dx^\mu, \) and \( D\varphi_2 = d\varphi_2 + A_\mu^2 dx^\mu. \) \( C \) is the component \( C_{123} \) on the \( T^3. \) The last two terms in the Ramond-Ramond four-form can be determined using self-duality of the five-form field strength in ten dimensions. This reduction has an
$SL(2, \mathcal{R}) \times SL(2, \mathcal{R})$ symmetry which provide a means by which supergravity solutions can be generated. The specific solution generating transformation described in [11] involves a T-Duality, a coordinate transformation and then another T-Duality. It is realised through the use of one of the $SL(2, \mathcal{R})$ symmetries to produce a rotation in the $\varphi^1 - \varphi^2$ plane. The particular element of $SL(2, \mathcal{R})$ is chosen in such a way that the regularity of the solution is preserved so no new singular points are generated. In this way the symmetries of the eight dimensional theory are exploited to generate new solutions of the eleven dimensional supergravity. The $SL(2, \mathcal{R})$ element used to produce the rotation is

\[
\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}.
\]

Putting this rotation into effect in the toroidally compactified IIB theory allows an alternative interpretation of this rotation as shifts in the fields of 11 dimensional supergravity. Solutions obtained in this manner will be referred to as $\gamma$-deformed. It can be seen from the IIB metric that the effect of this rotation in the $\varphi^1 - \varphi^2$ plane is equivalently realised by making the following shifts in the eleven dimensional theory

\[
\Delta_\gamma = \frac{\Delta}{[(1 - \gamma C)^2 + \gamma^2 \Delta]^2}, \quad (9)
\]

\[
C_\gamma = \frac{C(1 - \gamma C) - \gamma \Delta}{[(1 - \gamma C)^2 + \gamma^2 \Delta]} \quad (10)
\]

This analysis is considerably simplified for the branes in M-theory, which we will be discussing because for every choice of the $T^3$ used to generate the new solution, $C$ may be set to zero in the original solution. For some simple cases, (9) and (10) are the only changes required to produce the new solution. There are, however, extra terms arising when, for example, the $T^3$ is completely in the space perpendicular to the membrane world volume.

\footnote{It is not that the C-field vanishes for these solutions just that the C field on the solution generating torus will vanish.}
3 The membrane

We begin with the membrane. The undeformed solution is given by:

\[
\begin{align*}
\frac{ds^2}{3} &= H^{-2/3}(-dt^2 + d\rho^2 + \rho^2 d\varphi_4^2) + H^{1/3}(dr^2 + r^2 d\Omega_7^2) \\
F_0^{(4)} &= \rho \partial_r H^{-1} dr \land dt \land d\rho \land d\varphi_4 ,
\end{align*}
\]  

(11)  
(12)

with

\[
H = 1 + \frac{25 \pi^2 N_6^6}{r^6} .
\]

(13)

We use the same coordinate system as Lunin and Maldacena to maintain contact with their work. The volume element for the unit seven sphere is given by

\[
\begin{align*}
d\Omega_7^2 &= d\theta^2 + s_\theta^2 (d\alpha^2 + s_\alpha^2 d\beta^2) + c_\theta^2 d\varphi_1^2 \\
&+ s_\theta^2 (c_\alpha^2 d\varphi_2^2 + s_\alpha^2 (c_\beta^2 d\varphi_3^2 + s_\beta^2 d\varphi_4^2)).
\end{align*}
\]  

(14)

One may then introduce the following coordinates [11]:

\[
\begin{align*}
\phi_1 &= \psi + \varphi_3 \\
\phi_2 &= \psi - \varphi_3 - \varphi_2 \\
\phi_3 &= \psi + \varphi_2 - \varphi_1 \\
\phi_4 &= \psi + \varphi_1.
\end{align*}
\]

The background, adapted to begin the deformation process described above, is:

\[
\begin{align*}
\frac{ds_0^2}{3} &= H^{-2/3}\left(-dt^2 + d\rho^2 + \rho^2 d\varphi_4^2\right) \\
&+ H^{1/3}\left(dr^2 + r^2(d\theta^2 + s_\theta^2 (d\alpha^2 + s_\alpha^2 d\beta^2)) + d\psi^2 + s_\theta^2 s_\alpha^2 d\varphi_1^2 \\
&+ 2s_\theta^2 s_\alpha^2 (s_\beta^2 - c_\beta^2)d\psi d\varphi_1 + s_\theta^2 s_\alpha^2 c_\beta^2 d\varphi_2 \\
&+ 2s_\theta^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\psi d\varphi_2 - 2c_\beta^2 s_\theta^2 s_\alpha^2 d\varphi_1 d\varphi_2 \\
&+ (c_\beta^2 + s_\beta^2 c_\alpha^2) d\varphi_3^2 + 2(c_\theta^2 - s_\theta^2 c_\alpha^2) d\psi d\varphi_3 + 2s_\theta^2 s_\alpha^2 d\varphi_2 d\varphi_3\right) \\
F_0^{(4)} &= \rho \left(\partial_\psi H^{-1}\right) dr \land dt \land d\rho \land d\varphi_4.
\end{align*}
\]  

(15)  
(16)

\[c_\theta = \cos(\theta), \ s_\theta = \sin(\theta)\]
There are two types of deformation to be considered for the membrane. The choice depends on whether we wish to turn the C-field on completely off the brane or with a single leg along the brane. In the former set-up the $T^3$ used for the solution generating transformation will be transverse to the brane and in the latter, two of the directions of the torus will be transverse and one will be longitudinal to the brane. There are four cases, which exhaust every possible choice for both types of deformation. These four choices of $T^3$ are given by:

\[
\{\varphi^1, \varphi^2, \varphi^3\} \quad \text{or} \quad \{\varphi^4, \varphi^i, \varphi^j\} \quad \text{with } i, j = 1, 2, 3.
\]

Once the $T^3$ has been fixed, the particular choice of $S^1$ reduction direction and T-duality direction are irrelevant in the eleven dimensional theory. However, the different embeddings of the specific $T^3$’s chosen will be shown to result in different solutions.

### 3.1 Deformation for $T^3$ on $\{\varphi^1, \varphi^2, \varphi^3\}$

The deformed M2 using $T^3$ given by $\{\varphi_1, \varphi_2, \varphi_3\}$ is

\[
\text{ds}^2 = (1 + \gamma^2 \Delta_{123})^{1/3} \left\{ H^{-2/3} \left( -dt^2 + d\rho^2 + \rho^2 d\varphi_1^2 \right) + H^{1/3} \left( dr^2 + r^2 \left( d\theta^2 + s_\theta^2 (d\alpha^2 + s_\alpha^2 d\beta^2) \right) \right) \right. \\
+ \frac{H^{1/3} r^2}{(1 + \gamma^2 \Delta_{123})^{2/3}} \left\{ s_\theta^2 s_\alpha^2 d\varphi_1^2 + s_\theta^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) d\varphi_2^2 \\
+ (c_\theta^2 + s_\theta^2 c_\alpha^2) d\varphi_3^2 - 2 c_\beta^2 s_\theta^2 s_\alpha^2 d\varphi_1 d\varphi_2 \\
+ 2 s_\theta^2 c_\alpha c_\beta d\varphi_2 d\varphi_3 + 2 s_\alpha^2 s_\theta^2 (s_\beta^2 - c_\beta^2) d\psi d\varphi_1 \\
+ 2 s_\theta^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\psi d\varphi_2 + 2 (c_\theta^2 - s_\theta^2 c_\alpha^2) d\psi d\varphi_3 \right\} \\
+ \frac{H^{1/3} r^2}{(1 + \gamma^2 \Delta_{123})^{2/3}} \left\{ \left( 1 + \gamma^2 \Delta_{123} f_1 (\alpha, \beta, \theta) \right) d\psi^2 \right\} \right. \\
\]

(17)
where the volume of the three-torus is

\[
\Delta_{123} = Hr^6 s_\alpha^2 s_\beta^2 s_\gamma^2 (c_\beta^2 s_\alpha^2 s_\beta^2 + c_\alpha^2 (c_\gamma^2 + c_\beta^2 s_\alpha^2 s_\beta^2))
\]

\[
= Hr^6 \Delta'_{123}(\alpha, \beta, \theta)
\]

There are two new contributions to the field strength. The first is the result of the three form potential components turned on along the \(T^3\) during the deformation process. This can be seen by looking at the effect of an \(SL(2, R)\) rotation on the \(T^2\) part of the IIB metric. Re-interpreting this transformation on the coordinates as a shift in the fields of eleven dimensional supergravity gives

\[
C^{(\gamma)}_{\varphi_1 \varphi_2 \varphi_3} = \frac{-\gamma \Delta}{1 + \gamma^2 \Delta}
\]

The second of the new contributions to the \(\gamma\)-deformed field strength originates with the non-zero \(C_{\mu_\rho \varphi \lambda}\). This enters the IIB theory through the Ramond-Ramond four form. Specifically for this \(T^3\) reduction, the RR field in the \(T^2\) reduced IIB theory is

\[
C^{(4)}_0 = -\frac{1}{6} C_{\mu \nu \lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge D\varphi_1 + d_{\mu_1 \mu_2 \mu_3 \mu_4} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4}
\]

\[
+ \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge D\varphi_2
\]

Using the self-duality of the associated five-form field strength and the isometries in the \(\varphi_i\) directions implies that

\[d_{\mu_1 \mu_2 \mu_3 \mu_4} = 0.\]

The \(SL(2, R)\) action produces a new RR four-form

\[
C^{(4)}_\gamma = C^{(4)}_0 + \gamma \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge D\varphi^1
\]

\[
f_1(\alpha, \beta, \theta) = \frac{64 \times 256 \Delta'_{123} s_2 s_3^2 s_2 s_2^2}{s_\alpha^2 s_\beta^4 f_2(\alpha, \beta, \theta)}
\]

\[
f_2(\alpha, \beta, \theta) = 66 - 2c_{4\beta} - 3c_{4\beta - 2\theta} - 4c_{4(\beta - \theta)} + 16c_{2\alpha} (7 + c_{4\beta}) c_\theta^2 + 70c_{2\theta}
\]

\[
+ 8c_{4\theta} - 4c_{4(\beta + \theta)} - 3c_{2(2\beta + \theta)} + 8c_{4\alpha} (7 + 8c_{2\theta}) s_2 s_2^2 s_\theta
\]
Re-interpreting the $SL(2, \mathcal{R})$ transformation as a shift in the fields of eleven dimensional supergravity leads to the second new contribution to the eleven dimensional four-form

\begin{equation}
\frac{1}{6} C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda - \frac{1}{6} C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda - \gamma \hat{d}_{\mu_1 \mu_2 \mu_3} dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3}.
\end{equation}

The field strength for the deformed background is

\begin{equation}
F^{(4)}_{\gamma} = F^{(4)}_0 - \partial_\kappa \left[ \frac{\gamma \Delta_{123}}{\kappa + \gamma^2 \Delta_{123}} \right] d\kappa \wedge d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 + \gamma \sqrt{\Delta_{123}} \star_8 F^{(4)}_0
\end{equation}

where $\kappa \in \{r, \alpha, \beta, \theta\}$. The deformed solution, $ds^2_{(\gamma)}$ is related to the undeformed solution, $ds^2_0$ by

\begin{equation}
\begin{aligned}
ds^2_{(\gamma)} &= \frac{1}{(1 + \gamma^2 \Delta)^{2/3}} \left\{ ds^2_0 + \gamma^2 \Delta \left( dx^2_|| + H^{1/3} \sum_{i=1}^4 d\mu^2_i + r^2 f_1 d\psi^2 \right) \right\}
\end{aligned}
\end{equation}

where $dx^2_||$ is the metric on the membrane world volume

\begin{equation}
dx^2_|| = H^{-2/3} \left( - dt^2 + d\rho^2 + \rho^2 d\varphi^2_4 \right)
\end{equation}

with the $\mu_i$ coordinates, related to the chosen coordinate system by

\begin{equation}
\sum_{i=1}^4 d\mu^2_i = dr^2 + r^2 \left( d\theta^2 + s^2_\theta (d\alpha^2 + s^2_\alpha d\beta^2) \right).
\end{equation}

This shows, in an intuitive way, the effect of an $SL(2, \mathcal{R})$ rotation in this $T^2$ reduced IIB theory on the eleven dimensional geometry. For this specific deformation procedure, this is the only M-theory analogue of the $\beta$-deformation studied for the D3 Brane and it’s world volume theory. In that case the deformation effect in the gauge theory was implemented using a modified product for the Chiral Superfields. This is similar to the way in which a Non-Commutative theory can be obtained from a Commutative one after replacing the regular product with the Moyle product. However for the $\beta$-deformed D3/$\mathcal{N} = 4$ system, the modified product produces an effect within the Chiral Superpotential alone. There appear multiplicative pure phase factors, which
are determined by the charges of the gauge theory fields under the globally symmetric field theoretic realisation of the bulk spacetime torus. While the $\beta$-deformation preserved the conformal nature of the world volume theory, the Supersymmetry was broken down to $\mathcal{N} = 1$. We investigate the effect of different $T^3$ embeddings on the Supersymmetry and thermodynamics in the M-theory set-ups.

For the membrane, a useful interpretation is to view the field theory dual as the infrared (or equivalently strong coupling) limit of D=2+1 Yang-Mills theory. The $\gamma$ deformation will then be the analogue of the $\beta$ deformation for the strongly coupled three dimensional theory. Deformations of 2+1 dimensional strongly coupled theories may have some condensed matter applications, see [20] for recent uses of holography to condensed matter systems.

3.2 Membrane analogue of Dipole deformations

For the second type of deformation of the membrane, the $T^3$ is chosen with one $U(1)$ on the brane world volume and two in the transverse space. For all possible choices within this class of deformation the IIB theory has Ramond-Ramond four form

$$C^{(4)} = 0.$$ 

The only new contribution to the eleven dimensional field strength comes from the potential turned on along the $T^3$. While there are other fields in the IIB theory, which contain non-vanishing components of $C^{(3)}$, these can never generate new eleven dimensional fields because they enter as $T^2$ zero-forms and two-forms only. The wedge product ensures that no new terms can be generated under the action of $SL(2,R)$ on the $T^2$ coordinates. Choosing to wrap the membrane on the $S^1$ producing a fundamental string in IIA appears to result in the generation of novel new contributions. However, the lack of the existence of a non-vanishing connection one-form for this choice of $U(1)$ results in this term always vanishing. Thus, all possible different IIB solutions originating from different choices of reduction and T-Duality angles are lifted to exactly the same solution in M-Theory. The deformed solutions are presented here in a form showing their relation to the undeformed membrane. The full solutions
can be found in Appendix A. In general, a deformation of this type on \( \{ \varphi_4, \varphi_i, \varphi_j \} \) produces a solution of the form

\[
\begin{align*}
\text{ds}^2 &= \frac{1}{(1 + \gamma^2 \Delta_{4ij})^{2/3}} \left\{ \text{ds}_0^2 + \gamma^2 \Delta_{4ij} \left( H^{-2/3}(-dt^2 + d\rho^2) ight) + H^{1/3} \left( \sum_{i=1}^{4} d\mu_i^2 + r^2(F_1 \, d\psi^2 + F_2 \, d\varphi_k^2 + F_3 \, d\psi d\varphi_k) \right) \right\} \\
F_\gamma^{(4)} &= F_0^{(4)} - \gamma \d_k \left\{ \frac{\Delta_{4ij}}{1 + \gamma^2 \Delta_{4ij}} \right\} \text{d}x^k \wedge \text{d}\varphi_4 \wedge \text{d}\varphi_i \wedge \text{d}\varphi_j
\end{align*}
\]

(29)

where \( \varphi_k \) labels the remaining \( U(1) \) of the \( T^3 \), \( \kappa \in \{ \rho, r, \alpha, \beta, \theta \} \) and \( F_1, F_2, F_3 \) are functions of \( \alpha, \beta, \theta \) only, depend on the specific \( T^3 \) chosen and are presented in Appendix A. For each choice however, the function \( \Delta_{4ij} \) has the same \( r \) dependence (allowing analysis of the decoupling limits to be carried out in parallel) but different angular dependence. This has consequences concerning the supersymmetry of the deformed solution. The functions \( \Delta_{4ij} \) are given by:

\[
\begin{align*}
\Delta_{412} &= \rho^2 r^4 s^4_\theta s^2_\alpha c^2_\beta + c^2_\alpha - c^2_\beta s^2_\alpha, \\
\Delta_{413} &= \rho^2 r^4 s^4_\theta s^2_\alpha c^2_\beta + c^2_\alpha s^2_\theta, \\
\Delta_{423} &= \rho^2 r^4 s^4_\theta c^2_\beta c^2_\alpha s^2_\theta + c^2_\alpha c^2_\theta.
\end{align*}
\]

(31)

(32)

(33)

4 The five-brane

The undeformed five-brane solution is given by:

\[
\begin{align*}
\text{ds}^2 &= H^{-1/3} \left( -dt^2 + \sum_{i=1}^{5} dx_i^2 \right) + H^{2/3} \sum_{i=6}^{10} dx_i^2 \\
F^{(4)} &= \frac{1}{4!} H^{2/3} \epsilon_{i_1 i_2 i_3 i_4 i_5} \partial^{i_5} \left( \partial^{i_1} H(x_6, \ldots, x_{10}) \right) \text{d}x^{i_1} \wedge \text{d}x^{i_2} \wedge \text{d}x^{i_3} \wedge \text{d}x^{i_4}.
\end{align*}
\]

(34)

(35)

In what follows we will Wick rotate to time on the five-brane world volume so that we may apply the usual solution generating technique. We will then Wick rotate back at the end to give a good solution. This is similar to the procedure that was used to find the supergravity duals to the noncommutative open string. The isometry group
on the M5 world volume after Wick rotation is $SO(6)$. In the space orthogonal to the brane we have $SU(2) \times SU(2) \approx SO(4) \subset SO(5)$ symmetry which may be used to identify two $U(1)$ isometries.

$$x_6 = r_1 \cos \theta_1, \quad x_7 = r_1 \sin \theta_1, \quad x_8 = r_2 \cos \theta_2, \quad x_9 = r_2 \sin \theta_2 \quad (36)$$

with

$$x_{10} = \sqrt{r^2 - r_1^2 - r_2^2}. \quad (37)$$

The rescaling,

$$\tilde{r}_1 = \frac{r_1}{r}, \quad \tilde{r}_2 = \frac{r_2}{r} \quad (38)$$

is then performed to ensure that $r$ is the only “radial” type coordinate. The appearance of additional coordinate singularities at

$$\tilde{r}_1^2 + \tilde{r}_2^2 = 1 \quad (39)$$

is actually the single point

$$x_{10} = 0 \quad (40)$$

and is the consequence of the lack of a globally well defined cover of the sphere. This is a coordinate singularity and hence begin. The M5 solution after Wick rotation is then

$$ds^2 = H^{-1/3} \sum_{i=1}^{3} \left( d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) + H^{2/3} \left\{ dr^2 + r^2 \left( \tilde{r}_1^2 d\theta_1^2 + \tilde{r}_2^2 d\theta_2^2 + \frac{(1 - \tilde{r}_2^2)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} d\tilde{r}_1^2 + \frac{(1 - \tilde{r}_1^2)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} d\tilde{r}_2^2 \right) \right\} \quad (41)$$

with the harmonic function

$$F_0^{(4)} = \tilde{r}_1 \tilde{r}_2 r^4 (\partial_r H) d\tilde{r}_1 \wedge d\theta_1 \wedge d\tilde{r}_2 \wedge d\theta_2 \quad (42)$$

and

$H = \left( 1 + \frac{N\pi l^3}{r^3} \right). \quad (43)$
We now use the Lunin/Maldacena parameterisation again for the $S^5$ and get

\[
\begin{align*}
\text{ds}^2 &= \ H^{-1/3} \left\{ d\rho^2 + \rho^2 \left( d\alpha^2 + s_\alpha^2 d\beta^2 + d\psi^2 + s_\alpha^2 d\varphi_1^2 \right) \\
&\quad + (c_\alpha^2 + s_\alpha^2 c_\beta^2) d\varphi_2^2 + 2 s_\alpha^2 (c_\beta^2 - s_\beta^2) d\psi d\varphi_1 \\
&\quad + 2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\psi d\varphi_2 + 2 s_\alpha^2 c_\beta^2 d\varphi_1 d\varphi_2 \right\} \\
&\quad + H^{2/3} \left\{ dr^2 + r^2 (\tilde{r}_1^2 d\theta_1^2 + \tilde{r}_2^2 d\theta_2^2 + \frac{(1 - \tilde{r}_1^2)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} d\tilde{r}_1^2 \\
&\quad + \frac{(1 - \tilde{r}_1^2)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} d\tilde{r}_2^2 + \frac{2 \tilde{r}_1 \tilde{r}_2}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} d\tilde{r}_1 d\tilde{r}_2 \right\}
\end{align*}
\]

(44)

At the level of the type IIB theory, choice of $S^1$-reduction and T-duality directions appear to produce different solutions. The eleven dimensional lift of these solutions however produce the same solution in M-theory. This is obviously a consequence of eleven dimensional covariance that essentially produces a hidden symmetry from the IIB point of view.

4.1 M-theory analogue of Non-Commutative Deformation

When the $T^3$ is chosen to lie completely on the Euclidean M5 world volume, the new contributions to the field strength originate with the same terms in the Ramond-Ramond four-form as for the membrane cases considered. For this case, in a simple diagonal coordinate system, the deformed solution is

\[
\begin{align*}
\text{ds}^2_\gamma &= (1 + \gamma^2 H^{-1} \mu_1^2 \mu_2^2 \mu_3^2)^{1/3} \left\{ H^{-1/3} \sum_{i=1}^{3} d\mu_i^2 + H^{2/3} \left( dr^2 + r^2 d\Omega_4^2 \right) \right\} \\
&\quad + \frac{1}{(1 + \gamma^2 H^{-1} \mu_1^2 \mu_2^2 \mu_3^2)^{2/3}} \left\{ H^{-1/3} \sum_{i=1}^{3} \mu_i^2 d\varphi_i^2 \right\}
\end{align*}
\]

(45)

\[
\begin{align*}
F^{(4)}_\gamma &= F^{(4)}_0 + \gamma \sqrt{\Delta} \star_8 F^{(4)}_0 \\
&\quad - \gamma \sum_{\kappa \in \{r, \mu_1, \mu_2, \mu_3\}} \partial_\kappa \frac{H^{-1} \mu_1^2 \mu_2^2 \mu_3^2}{(1 + \gamma^2 H^{-1} \mu_1^2 \mu_2^2 \mu_3^2)} d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3
\end{align*}
\]

(46)
where
\[
\Delta_{\phi_1\phi_2\phi_3} = H^{-1}\mu_1^2\mu_2^2\mu_3^2.
\] (47)

The last term contributing to the new eleven dimensional field strength is from the potential component, which is turned on along the \(T^3\) directions as a result of the deformation. The other new contribution originates with the non-vanishing \(C_{r_1\theta_1\theta_2}\) and \(C_{r_2\theta_1\theta_2}\). These live entirely in the 8 dimensional space perpendicular to the \(T^3\) and appear in the \(T^2\) reduced IIB theory as non-vanishing \(C_{\mu\nu\lambda}\). As for the membrane, this term is obtained by transforming the eleven dimensional fields in such a way that they reproduce the effect of an \(SL(2,\mathbb{R})\) transformation in the IIB theory.

\[
F_0^{(4)} \rightarrow F_0^{(4)} + \gamma\sqrt{\Delta} \ast_8 F_0^{(4)}
\] (48)

### 4.2 M-theory analogue of Dipole Deformations on the five brane; type 1

When two \(U(1)\)'s along the Brane are used (e.g. \(\phi_2, \phi_3\)) along with one \(U(1)\) from the perpendicular space (e.g. \(\theta_1\)), the deformed solution is given by

\[
\begin{align*}
\text{ds}_\gamma^2 &= \frac{1}{(1 + \gamma^2\mu_2^2\mu_3^2 r^2\bar{r}_1^2)^{2/3}} \left\{ H^{-1/3} \sum_{i=2}^{3} \mu_i^2 d\phi_i^2 + H^{2/3} r^2 \bar{r}_1^2 d\theta_1^2 \right\} \\
&+ (1 + \gamma^2\mu_2^2\mu_3^2 r^2\bar{r}_1^2)^{1/3} \left\{ H^{-1/3} \sum_{i=1}^{3} d\mu_i^2 + \mu_1^2 d\phi_1^2 \\
&+ H^{2/3} \left( dr^2 + r^2 \left( \bar{r}_2^2 d\theta_2^2 + \frac{(1 - \bar{r}_2^2)}{(1 - \bar{r}_1^2 - \bar{r}_2^2)} d\bar{r}_2^2 \right) + \frac{(1 - \bar{r}_1^2)}{(1 - \bar{r}_1^2 - \bar{r}_2^2)} d\bar{r}_1^2 d\bar{r}_2 \right) \right\}
\end{align*}
\] (49)

\[
F_\gamma^{(4)} = F_0^{(4)} - \gamma \sum_\kappa \partial_\kappa \frac{\mu_2^2\mu_3^2 r^2\bar{r}_1^2}{(1 + \gamma^2\mu_2^2\mu_3^2 r^2\bar{r}_1^2)} dx^\kappa \wedge d\phi_2 \wedge d\phi_3 \wedge d\theta_1
\] (50)

with \(\kappa \in \{\mu_2, \mu_3, r, \bar{r}_1\}\). From this solution, all others choices involving two \(U(1)\)'s ON and one \(U(1)\) OFF the five brane world volume can be obtained via a simple change of label amongst the \(\phi_i\), \(\mu_i\ i = 1, 2, 3\) and \(\theta_m\), \(\bar{r}_m\), \(m = 1, 2\). The lack of additional new terms (common to all the alternative five brane deformations) is due to the existence
of only a non-zero $C_{a\mu\nu}$ type term in the eleven dimensional solution. This appears only in $T^2$ reduced IIB theory as zero-forms and two-forms on the $T^2$. No new terms can be generated from these forms.

4.3 **M-theory analogue of Dipole Deformations on the five brane; type 2**

We now consider deforming the five brane using one world volume $U(1)$ and two from the transverse space. Unlike previous examples, it now matters whether the $S^1$ reduction and T-duality transformation are performed ON or OFF the brane world volume. We consider here three choices:

1. $S^1$ reduce along brane w/v and T-dualise along transverse $U(1)$
2. $S^1$ reduce along transverse space and T-dualise along w/v $U(1)$
3. $S^1$ reduce and T-dualise along transverse $U(1)'s$

Each of these choices produces a toroidally compactified IIB solution with different Neveu-Schwarz Neveu-Schwarz two-forms and Ramond-Ramond two and four-forms. For the membrane no changes to the eleven dimensional three-form were obtained as the result of an $SL(2,R)$ transformation on the IIB solutions and it was concluded that the choice of T-duality and reduction directions were irrelevant from an eleven dimensional perspective. However, for the five brane this is not the case.

The third in the above list of choices is the easiest to deal with. T-dualising and $S^1$ reducing both along transverse directions results in the vanishing of all terms in the IIB theory, which are capable of generating new eleven dimensional field strength components through an $SL(2,R)$ rotation. We find for $T^3$ on $\{\varphi_1, \theta_1, \theta_2\}$

\[
\begin{align*}
B_7^{(2)} &= B_0^{(2)} = 0 \\
C_7^{(2)} &= C_0^{(2)} = 0 \\
C_7^{(4)} &= C_0^{(4)} = 0
\end{align*}
\]  

(51)
and this is the same regardless of our choice of $S^1$-reduction and T-duality directions. The deformed solution obtained using choice 3 on $\{\varphi_1, \theta_1, \theta_2\}$ is then

$$ds_\gamma^2 = \frac{1}{(1 + \gamma^2 \Delta)^{2/3}} \left( ds_0^2 + (\gamma^2 \Delta) \Delta^{-1/6} g_{\mu\nu} dx^\mu dx^\nu \right)$$

$$F_\gamma^{(4)} = F_0^{(4)} - \gamma \sum_\kappa \frac{\partial_\kappa (H r^4 \rho^2 s_\alpha^2 r_1^2 r_2^2)}{(1 + \gamma^2 H r^4 \rho^2 s_\alpha^2 r_1^2 r_2^2)^2} dx^\kappa \wedge d\phi_1 \wedge d\phi_2 \wedge d\theta_1$$

where $\kappa \in \{\rho, \alpha, r, \tilde{r}_1, \tilde{r}_2\}$ and $\mu, \nu$ indices span the eight dimensional space. For the volume of the $T^3$

$$\Delta = \Delta_{\varphi_1 \theta_1 \theta_2} = H r^4 \tilde{r}_1 \tilde{r}_2 \rho^2 s_\alpha^2$$

and metric components

$$\Delta^{-1/6} g_{\mu\nu} = G_{\mu\nu} \text{ when } \mu, \nu \neq \{\varphi_2, \psi\}$$

$$\Delta^{-1/6} g_{\varphi_2 \varphi_2} = H^{-1/3} \rho^2 (c_\alpha^2 + s_\alpha^2 c_\beta^2 s_\beta^2)$$

$$\Delta^{-1/6} g_{\psi \psi} = H^{-1/3} \rho^2 \left( 1 - s_\alpha^2 (c_\beta^2 - s_\beta^2) \right)$$

$$\Delta^{-1/6} g_{\rho \rho} = 2 H^{-1/3} \rho^2 \left( 2 s_\alpha^2 c_\beta^2 s_\beta^2 - c_\alpha^2 \right).$$

(55)

For choices 1 and 2, however, there are more changes to be made. Whilst, in previous examples, the R-R four-form was responsible for newly generated terms, in these cases it produces no new effects. Instead, changes originate with certain non-vanishing terms in the NS-NS and R-R two forms of the toroidally compactified IIB theory.

When the $S^1$ reduction is carried out along a $U(1)$ perpendicular to the brane world volume ($\theta_2$) and the solution is then T-dualised along a $U(1)$ on the world-volume ($\varphi_1$) the deformation in the IIB solution looks like

$$B_\gamma^{(2)} = B_0^{(2)} + \gamma C_{\theta_1 \theta_2 \mu} D\varphi_1 \wedge dx^\mu$$

$$C_\gamma^{(2)} = C_0^{(2)}$$

$$C_\gamma^{(4)} = C_0^{(4)}.$$

(56)
This can be re-interpreted as the following shift in the connection one-form for the T-dualised $U(1)$ direction

$$A_0^{\varphi_1} \rightarrow A_\gamma^{\varphi_1} = A_0^{\varphi_1} + \gamma C_{\theta_1 \theta_2 \mu} dx^\mu. \quad (57)$$

The effect of choosing the other transverse $U(1)$ of the $T^3$ ($\theta_1$) as the $U(1)$ of the $S^1$ reduction is the introduction of a negative sign due to the anti-symmetric nature of the three-form and we get

$$A_0^{\varphi_1} \rightarrow A_\gamma^{\varphi_1} = A_0^{\varphi_1} - \gamma C_{\theta_1 \theta_2 \mu} dx^\mu. \quad (58)$$

When the $S^1$ reduction is carried out along a $U(1)$ on the brane ($\varphi_1$) and the solution is then T-dualised along a $U(1)$ perpendicular to the world volume ($\theta_1$) the deformation in the IIB solution looks like

$$B^{(2)}_\gamma = B^{(2)}_0$$
$$C^{(2)}_\gamma = C^{(2)}_0 - \gamma C_{\theta_1 \theta_2 \mu} D\theta_1 \wedge dx^\mu$$
$$C^{(4)}_\gamma = C^{(4)}_0. \quad (59)$$

This can be re-interpreted as the following shift in the connection one-form for the $U(1)$ of the $S^1$ reduction

$$A_0^{\varphi_1} \rightarrow A_\gamma^{\varphi_1} = A_0^{\varphi_1} + \gamma C_{\theta_1 \theta_2 \mu} dx^\mu. \quad (60)$$

Again, we obtain a minus sign with the additional $O(\gamma)$ terms when we choose to T-dualise along the other $U(1)$ ($\theta_2$) of the $T^3$, which is perpendicular to the brane world volume as follows

$$A_0^{\varphi_1} \rightarrow A_\gamma^{\varphi_1} = A_0^{\varphi_1} - \gamma C_{\theta_1 \theta_2 \mu} \quad (61)$$

Thus, we find a symmetry in the following pairs of solutions

Type A:

$S^1$ Reduction on $\theta_2$ with T – Duality along $\varphi_1$

$S^1$ Reduction on $\varphi_1$ with T – Duality along $\theta_1$
Type B:

\[ S^1 \text{ Reduction on } \theta_1 \text{ with } T - \text{Duality along } \varphi_1 \]
\[ S^1 \text{ Reduction on } \varphi_1 \text{ with } T - \text{Duality along } \theta_2 \]

(63)

with the only difference between the two pairs appearing as a relative minus sign in the \( \mathcal{O}(\gamma) \) correction to the connection one form associated with the world volume \( U(1) \). For all choices involving a \( T^3 \) on \( \{ \varphi_1, \theta_1, \theta_2 \} \) the deformed field strength is given by

\[ F^{(4)}_\gamma = F^{(4)}_\gamma - \gamma \sum_\kappa \frac{\partial_\kappa \Delta}{(1 + \gamma^2 \Delta)^2} dx^\kappa \wedge d\varphi_1 \wedge d\theta_1 \wedge d\theta_2 \]  

(64)

with \( \kappa \in \{ \rho, \alpha, r, \tilde{r}_1, \tilde{r}_2 \} \) and

\[ \Delta = \Delta_{\varphi_1 \theta_1 \theta_2} = H \rho^2 s_\alpha^2 r_1^2 \tilde{r}_2^2. \]  

(65)

The metric is given by

\[ ds^2_{\gamma} = \frac{1}{(1 + \gamma^2 \Delta)^{2/3}} \left\{ ds^2_0 + \Delta^{-1/6}(\gamma^2 \Delta)g_{\mu\nu}dx^\mu dx^\nu + \Delta^{1/3}M_{\varphi_1 \varphi_1}(\delta^\varphi_1)^2 + 2(-1)^P \Delta^{1/3}M_{\varphi_1 \varphi_1}D^1 \delta^\varphi_1 \right\} \]

(66)

where the last term is the only one for which there is a difference between the metrics produced by the two types of deformation described here. The difference is given by

\[ P = 1 \text{ for type A deformations} \]
\[ P = 2 \text{ for type B deformations} \]

Common to both types of deformation are

\[ \Delta^{-1/6}g_{\mu\nu} = \begin{cases} 
C^0_{\mu\nu} & \text{for } \mu, \nu \notin \{ \psi, \varphi_2 \} \\
H^{-1/3} \rho^2 \left( 1 - s_\alpha^2 (c_\beta^2 - s_\beta^2) \right) & \text{for } \psi \psi \\
H^{-1/3} \rho^2 \left( c_\alpha^2 + s_\alpha^2 c_\beta^2 s_\beta^2 \right) & \text{for } \varphi_2 \varphi_2 \\
H^{-1/3} \rho^2 \left( s_\alpha^2 c_\beta^2 - c_\alpha^2 - c_\beta^2 (s_\beta^2 - c_\beta^2) \right) & \text{for } \psi \varphi_2 \\
\end{cases} \]
The \((\delta \varphi)^2\) terms, which, along with the new off-diagonal components generated as a result of the shift in the connection one-form appear in terms of

\[
\delta \varphi = A_{\gamma}^{\varphi_1} - A_{0}^{\varphi_1} \\
= A_{r_1}^{\varphi_1} d\tilde{r}_1 + A_{r_2}^{\varphi_1} d\tilde{r}_2 \\
= \gamma C_{r_1} r_1 d\tilde{r}_1 - \gamma C_{r_2} r_2 d\tilde{r}_2 \\
= \frac{1}{4} \gamma r^4 (\partial_r H) \left( \tilde{r}_1 \tilde{r}_2^2 d\tilde{r}_1 - \tilde{r}_1^2 \tilde{r}_2 d\tilde{r}_2 \right),
\]

(67)

with

\[
D \varphi = d\varphi_1 + (c_\beta^2 - s_\beta^2) d\psi + c_\alpha^2 d\varphi_2.
\]

(68)

We could, of course, have chosen to use an \{ON,OFF,OFF\} \(T^3\) involving \(\varphi_2\) instead of \(\varphi_1\). The effect of this choice follows trivially from the solution in terms of \(\varphi_1\). The deformed field strength is given by direct exchange of \(\phi_2\) for \(\phi_1\) in components. The volume of the \(T^3\) changes only by an overall factor determined by the undeformed metric as follows:

\[
\Delta \varphi_2 \theta_1 \theta_2 = \frac{G_{\varphi_2 \varphi_2}}{G_{\varphi_1 \varphi_1}} \Delta \varphi_1 \theta_1 \theta_2.
\]

(69)

The changes to the metric are fully contained within the following interchanges

\[
D \varphi^1 \rightarrow D \varphi^2 \\
A_{\gamma}^{\varphi_1} \rightarrow A_{\gamma}^{\varphi_2}
\]

(70)

whilst, as for the \(\varphi_1\) choice

\[
A_{\gamma}^{\varphi_2} = A_{0}^{\varphi_2} + \delta \varphi_2.
\]

(71)

Obviously, as this is a different embedding

\[
A_{0}^{\varphi_2} = A_{\psi}^{\varphi_2} d\psi + A_{\varphi_2}^{\varphi_1} d\varphi_1 \\
= \frac{s_\alpha^2 c_\beta^2 - c_\alpha^2}{s_\alpha^2 c_\beta^2 + c_\alpha^2} d\psi + A_{\varphi_2}^{\varphi_1} d\varphi_1 \\
\neq A_{0}^{\varphi_1}
\]

(72)
but

$$\delta \varphi_2 = \delta \varphi_1.$$  \hspace{1cm} (73)

The result of these symmetries between the deformed solution is that the r-dependence of the metric and potential components are the same for all possible choices of deformation after we have specified two pieces of information. Firstly, the type of brane we are working with and secondly, the number of $U(1)$'s, which are parallel and transverse to the brane world volume. The analysis of the near horizon regions can therefore be carried out for many deformation cases simultaneously.

5 Decoupling

The Supergravity action tells us how to take a sensible decoupling limit. As the only parameter in eleven dimensional supergravity, the planck length is used to determine the low energy limit, by taking $l_p \to 0$ relative to some fixed energy scale. We then scale r appropriately so the supergravity action is finite in the low energy limit.

The Bosonic part of the action is

$$S = S_1 + S_2 + S_3$$

$$= \frac{1}{\kappa_{11}^2} \int d^{11}x \left\{ \sqrt{-g} (\mathcal{R} - \frac{1}{12} F^2) + \frac{2}{(72)^2} \epsilon^{M_1 \cdots M_11} F_{M_1 \cdots M_4} F_{M_5 \cdots M_8} C_{M_9 M_{10} M_{11}} \right\}$$

where the eleven dimensional gravitational coupling is

$$\kappa_{11}^2 \sim l_p^9.$$  \hspace{1cm} (75)

5.1 The membrane

For the membrane the decoupling limit is taken by

$$l_p \to 0$$  \hspace{1cm} (76)

with

$$u^{1/2} \equiv \frac{r}{l_p^{3/2}} = \text{fixed}.$$  \hspace{1cm} (77)
This is the near horizon limit defined by Maldacena. For the undeformed membrane an \( AdS_4 \times S^7 \) near horizon geometry is produced. By comparing the near horizon of the deformed M2 background with the deformed \( AdS_4 \times S^7 \) we can investigate the relationship between the deformation process and the near horizon limit. The three cases where the deformation is performed using a \( T^3 \) with one cycle on the brane world volume and two cycles in the transverse space have \( \Delta \) factors with the same \( r \)-dependence. They also have the same \( r \)-dependence in the non-vanishing \( C^{(3)} \) components. Therefore, their near horizon limits can be analyzed simultaneously. However, the case with all \( U(1)'s \) of the \( T^3 \) off the membrane world volume must be treated separately. Importantly we shall also need to scale \( \gamma \) so that the deformation is not washed out by the limit.

The deformed membranes to be considered explicitly are those with \( T^3 \) given by \( \{ \varphi_1, \varphi_2, \varphi_3 \} \) and \( \{ \varphi_4, \varphi_1, \varphi_3 \} \). In both cases, there appear several different \( r \)-dependent combinations in the metric. For \( \{ \varphi_1, \varphi_2, \varphi_3 \} \) these are

\[
H^{-2/3}(1 + \gamma^2 \Delta_{123})^{1/3}, \quad H^{1/3}r^2(1 + \gamma^2 \Delta_{123})^{1/3}, \quad \frac{H^{1/3}r^2}{(1 + \gamma^2 \Delta_{123})^{2/3}}
\]

where

\[
\Delta_{123} = Hr^6 \Delta_{123}(\alpha, \beta, \theta), \tag{78}
\]

with \( \delta' \) independent of \( r \). For the \( \{ \varphi_4, \varphi_1, \varphi_3 \} \) deformation the metric contains

\[
\frac{H^{-2/3}}{(1 + \gamma^2 \Delta_{413})^{2/3}}, \quad H^{2/3}(1 + \gamma^2 \Delta_{413})^{1/3}, \quad \frac{H^{1/3}r^2}{(1 + \gamma^2 \Delta_{413})^{2/3}}, \quad H^{1/3}r^2(1 + \gamma^2 \Delta_{413})^{1/3}
\]

with

\[
\Delta_{413} = \rho^2 r^4 \Delta'_{413}(\alpha, \beta, \theta).
\]

Finiteness of the Einstein-Hilbert term requires

\[
\frac{ds^2}{l_p^2} \equiv \text{finite}. \tag{79}
\]

For all possible deformations we must scale for \( \gamma \) in such a way as to preserve the effect of the deformation in the near horizon region. This implies we must keep

\[
(1 + \gamma^2 \Delta) \equiv \text{fixed} \tag{80}
\]
while

\[ H \sim \frac{(2^5\pi^2N)}{u^3l_p^3}. \]  

(81)

For \( T^3 \) given by \( \{\varphi_1, \varphi_2, \varphi_3\} \) we find

\[ \gamma^2 H r^6 \equiv \text{fixed} \]  

(82)

in the near horizon limit. Changing to the new \( u \) variable and allowing \( l_p \to 0 \) gives

\[ \gamma^2 l_p^6 \equiv \text{finite} \]  

(83)

and so the deformation parameter is scaled as

\[ \gamma \sim \tilde{\gamma} l_p^{-3} \]  

(84)

where \( \tilde{\gamma} \) is a scale independent deformation parameter. Similarly, for the deformation involving \( T^3 \) on \( \{\varphi_4, \varphi_1, \varphi_3\} \) the scale independent combination must be

\[ \gamma^2 r^4 \equiv \text{fixed}. \]  

(85)

In the near horizon limit \( \gamma \) is scaled like

\[ \gamma \sim \tilde{\gamma} l_p^{-3}. \]  

(86)

This particular scaling is common to both types of deformation and it keeps the Einstein-Hilbert term of the eleven dimensional supergravity action finite in the near horizon region as required. We must also check that the terms involving the C field are also finite in this limit. This is not ensured since the deformation switches on new values components of C and its associated field strength. since the deformation parameter must scale so will the field strength and so we must check the action remains finite.

From the undeformed membrane

\[ S_2 = \frac{1}{\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left| F^{(4)} \right|^2 \]  

(87)

\[ \sim \frac{1}{l_p^9} \int d^{11}x \sqrt{-g} \left[ F_{u \mu \rho \varphi_1} \underbrace{g_{tt} g^{\rho \rho} g^{\varphi_4 \varphi_4} g_{uu}}_{\sim (l_p^{-2})^4} F_{u \mu \rho \varphi_4} + \cdots \right] \]  

(88)
Looking at the undeformed field strength in the near horizon limit we have

\[ F_{\mu\nu\varphi_4} = \frac{dr}{dr} \rho \partial_r H^{-1} = \frac{l_p^{3/2}}{2 \sqrt{u}} \frac{\rho}{H^2} \left( \frac{6(25 \pi^2 N) l_p^6}{r^7} \right) \]

\[ \sim \left. \frac{l_p^3}{r} \left[ \frac{3 \rho u^2}{r} \left( \frac{25 \pi^2 N}{6(25 \pi^2 N)} \right) \right] \right|_{l_p \to 0} \]

(89)

where the factor in brackets is fixed in the near horizon limit. This \( l_p \) dependence is just the one required to ensure the action is finite in the decoupling limit provided. For deformation on \( T^3 \) given by \( \{ \varphi_1, \varphi_2, \varphi_3 \} \) non-zero \( F_{\kappa\varphi_1\varphi_2\varphi_3} \) is generated. These components have two possible \( r \)-dependencies. When \( \kappa = r \) we get

\[ F_{r\varphi_1\varphi_2\varphi_3} = -\frac{\gamma (6r^5) \Delta'_{123}(\alpha, \beta, \theta)}{(1 + \gamma^2 H r^6 \Delta'_{123}(\alpha, \beta, \theta))^2} \]

(90)

In the near horizon limit, the denominator is fixed and we see

\[ F_{u\varphi_1\varphi_2\varphi_3} \sim \left. \frac{l_p^6}{r} \left[ -\frac{3 \tilde{\gamma} \Delta'_{123} u^2}{(1 + \gamma^2 (25 \pi^2 N) \Delta'_{123})^2} \right] \right|_{l_p \to 0} \]

(91)

Since only contributions to the field strength, which are finite in units of \( l_p^3 \) affect the near horizon region the effects of \( F_{u\varphi_1\varphi_2\varphi_3} \) vanishes there. Alternatively, when \( \kappa \in \{ \alpha, \beta, \theta \} \)

\[ F_{\kappa\varphi_1\varphi_2\varphi_3} = -\frac{\gamma H r^6 \partial_{\kappa} \Delta'_{123}}{(1 + \gamma^2 \Delta_{123})^2} \]

(92)

which scales like

\[ \gamma H r^6 \sim \left. \frac{l_p^3}{r} \left[ \tilde{\gamma}(25 \pi^2 N) \right] \right|_{l_p \to 0} \]

(93)

as required for it to remain present in the near horizon limit. Another contribution to the action comes from

\[ S_2 = \cdots + \frac{1}{\kappa_{11}^2} \int d^{11}x \sqrt{-g} F_{\alpha\beta\theta\psi} g^{\alpha\alpha} g^{\beta\beta} g^{\theta\theta} g^{\psi\psi} F_{\alpha\beta\theta\psi} \]

(94)

where \( F_{\alpha\beta\theta\psi} \) is one of the components of the field generated in the Ramond-Ramond four-form of IIB when we performed an \( SL(2, \mathcal{R}) \) transformation. It was found to be

\[ F_{\alpha\beta\theta\psi} d\alpha \wedge d\beta \wedge d\theta \wedge d\psi = \gamma \sqrt{\Delta_{123}} \ast_8 F_0^{(4)} \]

(95)
where \( F_0^{(4)} \) is the field strength for the undeformed M2 and the hodge star is in eight dimensions. The symmetry in the appearance of \( F_{ut\varphi 4} \) and \( F_{\alpha\beta\varphi \psi} \) indicates that to keep the \( F_{\alpha\beta\varphi \psi} \) we require the same near horizon scalings

\[
F_0^{(4)} \sim \gamma \sqrt{\Delta_{123}} \ast 8 F_0^{(4)} \sim l_p^3
\]  

(96)

The eight dimensional hodge star involves the square root of the determinant of the eight dimensional metric along with the product of four inverse metrics. As a consequence of the finiteness of the Hilbert term in the action, the latter contributes a total of \( (l_p)^{-8} \) while the former contributes \( (l_p)^8 \). At the same time

\[
\sqrt{\Delta_{123}} \mid_{l_p \to 0} \sim l_p^3 \left[ (2^5 \pi^2 N)^{1/2} \sqrt{\Delta_{123}} \right]
\]  

(97)

This combines with the \( \gamma \) factor to produce a scale independent quantity giving in the near horizon

\[
\gamma \sqrt{\Delta_{123}} \mid_{l_p \to 0} \sim l_p^0 \left[ l_p^{-3} \right]
\]  

(98)

and thus

\[
S_2 = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \mid F^{(4)} \mid
\]  

(99)

is finite in the near horizon limit for

\[
l_p \to 0 \\
u = \text{fixed} \\
\gamma \sim \tilde{\gamma} l_p^{-3}.
\]  

(100)

Having defined a consistent near horizon limit with finite action for which the deformation is preserved we now present the solutions. The \( \gamma \)-deformed membrane (with
the $T^3$ in the transverse space) in the near horizon is

$$
\text{ds}_\gamma^2 = l_p^2 \left[ (1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{1/3} \left\{ \frac{u^2}{(2^5 \pi^2 N)^{2/3}} (-dt^2 + d\rho^2 + \rho^2 d\varphi_4^2) + (2^5 \pi^2 N)^{1/3} \left( \frac{du^2}{4u^2} + d\theta^2 + s_\rho^2 (d\alpha^2 + s_\alpha^2 d\beta^2) \right) \right\} + \frac{(2^5 \pi^2 N)^{1/3}}{(1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{2/3}} \left\{ s_\rho^2 s_\alpha^2 d\varphi_1^2 + s_\rho^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) d\varphi_2^2 + (c_\rho^2 + s_\rho^2 c_\alpha^2) d\varphi_3^2 - 2 c_\beta^2 s_\rho^2 s_\alpha^2 d\varphi_4 d\varphi_2 + 2 s_\rho^2 c_\alpha^2 d\varphi_2 d\varphi_3 + 2 s_\rho^2 s_\rho^2 (s_\beta^2 - c_\beta^2) d\varphi_1 d\psi + 2 s_\rho^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\varphi_2 d\psi + 2 (c_\rho^2 - s_\rho^2 c_\alpha^2) d\varphi_3 d\psi + \left( 1 + \gamma^2 (2^5 \pi^2 N) f_1 \Delta'_{123} \right) d\psi^2 \right\} \right]
$$

(101)

with

$$
F^{(4)}_\gamma = l_p^4 \left[ \frac{3 \rho u^2}{(2^5 \pi^2 N)} \ du \wedge dt \wedge d\rho \wedge d\varphi_4 + \frac{6 (2^5 \pi^2 N)^{1/3}}{\sqrt{\Delta'_{123}} s_\rho^2 s_\alpha} \ d\alpha \wedge d\beta \wedge d\theta \wedge d\psi - \sum_{i \in \{\alpha, \beta, \theta\}} \frac{\gamma (2^5 \pi^2 N) \partial_i \Delta'_{123}}{(1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{2/3}} \ dx_i \wedge d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \right]
$$

(102)

The near horizon region of the undeformed membrane is

$$
\text{ds}^2 = \text{ds}_{\text{AdS}_4} \left( \frac{1}{2} L \right) + d\Omega_7^2 (L)
$$

(103)

where

$$
L^2 = l_p^2 (2^5 \pi^2 N)^{1/3}
$$

(104)

To see the effect of the deformation, the near horizon solution can be expressed in terms of the undeformed near horizon solution

$$
\text{ds}_\gamma^2 = (1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{1/3} \text{ds}_{\text{AdS}_4} \left( \frac{1}{2} L \right) + \frac{1}{(1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{2/3}} d\Omega_7^2 (L) + \gamma^2 \frac{l_p^2 (2^5 \pi^2 N)^{1/3} \Delta'_{123}}{(1 + \gamma^2 (2^5 \pi^2 N) \Delta'_{123})^{2/3}} \left\{ d\theta^2 + s_\rho^2 (d\alpha^2 + s_\alpha^2 d\beta^2) + f d\psi^2 \right\}
$$

(105)
While the field strength for the undeformed brane in the near horizon limit is

$$F_0^{(4)} \sim l_p^3 \frac{3 \rho u^2}{(2^3 \pi^2 N^2)} du \wedge dt \wedge d\rho \wedge d\phi_4. \quad (106)$$

This is also the solution which is obtained when one starts with the membrane and takes the near horizon solution first and then performs a deformation of that solution. The deformation process and decoupling limit therefore commute. It is not clear whether this is inevitable. This result continues to hold true for the dipole-type deformations of the membrane. For completeness an example solution is included in Appendix B.

5.2 The M5 Brane

The near horizon limit of the M5 brane is found by taking

$$l_p \to 0 \quad (107)$$

with

$$u^2 \equiv \frac{r}{l_p^3} = \text{fixed}. \quad (108)$$

This limit allows us to explore the near horizon region while keeping the energies on the brane world volume fixed. As for the membrane, finiteness of the Einstein-Hilbert term requires $ds^2$ to be finite in units of the planck length squared. We also wish to preserve the deformation in the near horizon limit. Appearing in the deformed metric are terms like

$$H^{-1/3}(1 + \gamma^2 \Delta), \quad H^{2/3}(1 + \gamma^2 \Delta)^{1/3} dr^2, \quad H^{2/3}(1 + \gamma^2 \Delta)^{1/3} r^2, \quad H^{-1/3}(1 + \gamma^2 \Delta)^{-2/3}. \quad (109)$$

As for the membrane, the only condition that must be satisfied to ensure the Einstein-Hilbert term is finite in the near horizon limit is

$$1 + \gamma^2 \Delta = \text{fixed}. \quad (110)$$

Every possible deformation scenario leads to a scaling of the deformation parameter like

$$\gamma \sim \tilde{\gamma} l_p^{-3}. \quad (111)$$
The non-vanishing field strength components then scale like the planck length cubed as for the membrane and as required for a finite supergravity action. Firstly, we present the five brane analogue of the non-commutative deformation in the decoupling limit. Again for simplicity we work with the magnitudes and polar angles of the complexified coordinate system.

\[
\begin{align*}
\mathcal{ds}_g^2 &= l_p^2 \left\{ (1 + \tilde{\gamma}^2 (\frac{\mu_1^2 \mu_2^2 \mu_3^2}{N\pi^3})^{1/3} \sum_{i=1}^3 d\mu_i^2 + 4 \frac{(N\pi)^{2/3}}{u^2} du^2 + (N\pi)^{2/3} d\Omega_4^2 \right. \\
& \left. + \frac{1}{(1 + \tilde{\gamma}^2 (\frac{\mu_1^2 \mu_2^2 \mu_3^2}{N\pi^3})^{2/3}} u^2 (N\pi)^{1/3} \mu_i^2 d\phi_i^2 \right\}.
\end{align*}
\]  

(112)

All terms within the four form remain in the decoupling limit. We find that

\[
F_0^{(4)} = -3 \mu_1^3 \tilde{r}_1 \tilde{r}_2 (N\pi) d\tilde{r}_1 \wedge d\theta_1 \wedge d\tilde{r}_2 \wedge d\theta_2
\]  

(113)

whilst the terms originating with the three-form potential turned on along the $T^3$ can depend on $r$ in one of two possible ways, both of which scale in such a way that the resultant field strength components are finite in units of $l_p^3$ in the near horizon. That is:

\[
\begin{align*}
F_{\mu_1 \phi_2 \phi_3}^\gamma &= \partial_\mu H^{-1} \sim l_p^3 \\
F_{\kappa \phi_2 \phi_3}^\gamma &= H^{-1} \sim l_p^3
\end{align*}
\]

for $\kappa \in \{\mu_1, \mu_2, \mu_3\}$. At the same time, the term generated via the deformation procedure as a result of non-vanishing $C_{\mu\nu\lambda}$ in the IIB solution is

\[
\gamma \sqrt{\Delta} \star_8 F_0^{(4)}
\]

which scales in the same way as $F_0^{(4)}$ in the decoupling limit since the terms involved scale like

\[
\gamma \sqrt{H^{-1}} \sim l_p^0
\]

and

\[
\sqrt{\left| \det g_{(8)} \right|} G^{\tilde{r}_1 \tilde{r}_1} G^{\tilde{r}_2 \tilde{r}_2} \sim l_p^0
\]
giving
\[ \gamma \sqrt{\Delta} \star F_0^{(4)} \sim F_0^{(4)} \]  
which is finite in units of \( l_p^3 \). For the membrane analogue of the dipole deformations of type I using \( T^3 \) on \( \{ \phi_2, \phi_3, \theta_1 \} \) the decoupling limit gives

\[
\begin{align*}
\mathrm{d}s_\gamma^2 &= \ell_p^2 \left\{ \frac{1}{(1 + \gamma^2 u^4 \tilde{r}_1^2 \mu_2^2 \mu_3^2)^{2/3}} \left( \frac{u^2}{(N\pi)^{1/3}} \sum_{i=2}^{3} \mu_i^2 \mathrm{d}\phi_i^2 + (N\pi)^{2/3} \tilde{r}_1^2 \mathrm{d}\theta_1^2 \right) 
+ (1 + \gamma^2 u^4 \tilde{r}_1^2 \mu_2^2 \mu_3^2)^{1/3} \left( \frac{u^2}{(N\pi)^{1/3}} \sum_{i=1}^{3} \mu_i^2 \mathrm{d}\phi_i^2 + 4 \frac{(N\pi)^{2/3}}{u^2} \mathrm{d}u^2 
+ (N\pi)^{2/3} \left( \tilde{r}_2^2 \mathrm{d}\theta_2^2 + \frac{(1 - \tilde{r}_2^2)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} \mathrm{d}\tilde{r}_2^2 + \frac{(1 - \tilde{r}_1)}{(1 - \tilde{r}_1^2 - \tilde{r}_2^2)} \mathrm{d}\tilde{r}_1^2 \right) \right) \right\} 
\end{align*}
\]

\[
F_\gamma^{(4)} = \ell_p^3 \left\{ -3 \tilde{r}_1 \tilde{r}_2 (N\pi) \mathrm{d}\tilde{r}_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\tilde{r}_2 \wedge \mathrm{d}\theta_2 
- \frac{\gamma}{(1 + \gamma^2 u^4 \tilde{r}_1^2 \mu_2^2 \mu_3^2)^{2}} \left( 4u^3 \tilde{r}_1^2 \mu_2^2 \mu_3^2 \mathrm{d}u \wedge \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_3 \wedge \mathrm{d}\theta_1 
+ u^4 \sum_{\kappa} \partial_\kappa (\tilde{r}_1^2 \mu_2^2 \mu_3^2) \mathrm{d}x^\kappa \wedge \mathrm{d}\phi_2 \wedge \mathrm{d}\phi_3 \wedge \mathrm{d}\theta_1 \right) \right\}
\]

with \( \kappa \in \{ \tilde{r}_1, \mu_2, \mu_3 \} \). For the five brane dipole deformations of type II we have for all choices of reduction and T-duality

\[
F_\gamma^{(4)} = F_0^{(4)} - \gamma \sum_{\kappa} \frac{\partial_\kappa \Delta}{(1 + \gamma^2 \Delta)^2} \mathrm{d}x^\kappa \wedge \mathrm{d}\varphi_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 
\sim -\ell_p^3 \left\{ 3 (N\pi) \tilde{r}_1^2 \tilde{r}_2^2 \mathrm{d}\tilde{r}_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\tilde{r}_2 \wedge \mathrm{d}\theta_2 
+ \frac{\gamma}{(1 + \gamma^2 u^2 (N\pi) \Delta)^2} \left( 2 (N\pi) u \tilde{r}_1^2 \tilde{r}_2^2 \rho^2 s_0^2 \mathrm{d}u \wedge \mathrm{d}\varphi_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 
+ u^2 (N\pi) \sum_m \partial_\kappa \Delta \mathrm{d}x^\kappa \wedge \mathrm{d}\varphi_1 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \right) \right\}
\]
for \( m \in \{ \rho, \alpha, \tilde{r}_1, \tilde{r}_2 \} \). For the metric we have already seen

\[
\begin{align*}
\text{ds}_\gamma^2 &= \frac{1}{(1 + \gamma^2 \Delta)^{2/3}} \left\{ \begin{array}{c}
\text{(1)} \\
\text{(2)}
\end{array} \right\} \\
&= \frac{1}{(1 + \gamma^2 \Delta)^{2/3}} \left\{ ds_0^2 + \Delta^{-1/6} (\gamma^2 \Delta) g_{\mu \nu} dx^\mu dx^\nu \\
&\quad + \Delta^{1/3} M_{\varphi_1 \varphi_1} (\delta^{\varphi_1})^2 + 2 (-1)^P \Delta^{1/3} M_{\varphi_1 \varphi_1} D_{\varphi_1} \delta^{\varphi_1} \right\} \quad (118)
\end{align*}
\]

Given that the scaling of \( \gamma \) was chosen in such a way as to preserve the deformation in the near horizon limit

\[
\gamma^2 \Delta \sim l_p^0.
\]

The form in which this deformed metric has been presented allows us to quickly identify the troublesome terms by separating them out from the well behaved, familiar, terms of the undeformed metric. Namely for term (1) we have

\[
ds_0^2 \sim l_p^2
\]
as required. From term (2), after identifying the scale independent \((\gamma^2 \Delta)\) we see the familiar terms

\[
\Delta^{-1/6} g_{\mu \nu} dx^\mu dx^\nu \sim l_p^2.
\]

Thus (2) scales in such a way that the action remains finite as the decoupling limit is taken. After we have identified the undeformed metric term

\[
\Delta^{1/3} M_{\varphi_1 \varphi_1} \sim l_p^2
\]

we see that terms (3) and (4) will scale like \((\delta^{\varphi_1})^2\) and \((\delta^{\varphi_1})\) respectively, whilst, as the eleven dimensional planck length is taken to zero \(\delta^{\varphi_1}\) scales like

\[
- \left( \frac{3N\pi}{4} \right) \tilde{\gamma} \tilde{r}_1 \tilde{r}_2 \left( \tilde{r}_2 d\tilde{r}_1 - \tilde{r}_1 d\tilde{r}_2 \right)
\]

which produces the correct scaling of the metric in the near horizon limit.
6 Probe branes

When searching for supersymmetric configurations, a good place to start is with the zeros of the potential from the probe brane sigma model. This is also related to calculating the action of Wilson surfaces in the dual theory. The probe membrane action is given by:

\[
S_\sigma = \int d^3\xi \left\{ \sqrt{|\text{det} \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} g_{mn}|} - \frac{1}{6} \varepsilon^{ijk} \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} \frac{\partial x^p}{\partial x^k} C_{mnp} \right\} .
\] (120)

We first orient the probe brane to be parallel to the original brane stack. Then, after choosing static or Monge gauge we look for brane configurations that minimise the potential energy of the brane. This gives the following possibilities for each type of deformation. The potential vanishes for

- \( T^3 \) on \( \{\varphi_1, \varphi_2, \varphi_3\} \rightarrow \alpha = 0, \) or \( \theta = 0 \)
- \( T^3 \) on \( \{\varphi_4, \varphi_1, \varphi_2\} \rightarrow \alpha = 0, \) or \( \theta = 0 \)
- \( T^3 \) on \( \{\varphi_4, \varphi_1, \varphi_3\} \rightarrow \alpha = 0, \) or \( \theta = 0 \)
- \( T^3 \) on \( \{\varphi_4, \varphi_2, \varphi_3\} \rightarrow \theta = 0 \) only

The supersymmetric locus for the membrane deformed on \( \{\varphi_4, \varphi_2, \varphi_3\} \) is zero dimensional unlike each of the other three cases. The particular \( T^3 \) embedding chosen has physical consequences.

For the five-brane, when the deformation has \( C^{(3)} \) vanishing on the world volume, we can take the simple \( \sigma \)-model

\[
S_\sigma = \int d^6x \left\{ \sqrt{|\text{det} \frac{\partial x^m}{\partial x^i} \frac{\partial x^n}{\partial x^j} g_{mn}|} - f^*(C^{(6)}) \right\} ,
\] (121)

where \( f^* \) is the pull back operation. Again, looking for the vanishing of the sigma model potential after orienting the probe along the original stack reveals that the condition for minimising the potential is equivalent to the vanishing of the functions
\[ \Delta \varphi \varphi \varphi \] given below:

\[ \Delta \varphi \varphi \varphi \varphi = r^2 r^2 r^4 s^2 (c^2 + \frac{1}{4} s^2 s^2) \]

\[ \Delta \varphi \varphi \varphi \varphi = r^2 r^2 r^4 s^2 (c^2 + \frac{1}{4} s^2 s^2) \]

\[ \Delta \varphi \varphi \varphi \varphi = H r^2 r^2 r^2 s^2 \]

\[ \Delta \varphi \varphi \varphi \varphi = H r^2 r^2 r^2 s^2 \]

As \( \alpha, \beta \in (0, \frac{\pi}{2}) \), deformations on the first three \( T^3 \)'s listed leave only the point \( \alpha = 0 \) in the supersymmetric locus. However, for \( T^3 \) on \( \{ \varphi_2, \theta_1, \theta_2 \} \) there exist configurations, which are supersymmetric for \( \alpha = 0 \) with any value of \( \beta \) or \( \beta = 0 \) with any value of \( \alpha \).

7 Black Branes, \( \gamma \)-Deformations and Entropy

We now consider non-extremal versions of theses deformed solutions. That is we will thermalise the deformed branes and analyse their thermodynamics. The Bekenstein-Hawking Entropy of the non-extremal deformed solutions is given by [21]:

\[ S_{BH} = \frac{1}{4} \int \sqrt{\text{det} g'_{ij}} d^9 x \]  
(122)

where the prime indicates that it is the reduced determinant. That is the determinant taken over the nine-dimensional block of the metric where temporal and radial components are neglected.

We will examine the effects of the deformation on the black brane entropy for the various types of deformation.

Following, [21], we thermalise the deformed branes with the introduction of the following factor and its reciprocal into the temporal and radial components of the metric as follows:

\[ G_{tt} \rightarrow (1 - \frac{r_0}{r_c}) G_{tt} \]

\[ G_{rr} \rightarrow (1 - \frac{r_0}{r_c})^{-1} G_{rr} \]
where $c$ is the power of $r$ appearing in the brane harmonic function (this is obviously
dependent on the dimension of the brane. (One must also change the definition on
the brane charge [21]). As with the study of Probe branes and supersymmetry, the
particular embedding of the $T^3$ is important in this analysis.

For a $T^3$ trivially fibred over an 8 dimensional manifold, the metric may be placed
in block diagonal form (with one $3 \times 3$ block describing the $T^3$ and the $8 \times 8$ block
describing the perpendicular space). The $T^3$ volume is warped homogenously by the
deformation as is the eight dimensional space. This occurs in such a way as to
cancel the effect of the warping on the determinant of the reduced metric used in the
Bekenstein-Hawking formula. The deformed metric, $G^{(\gamma)}_{MN}$ may be written in terms of
the nondeformed metric $G^{(0)}_{MN}$ as follows:

$$G^{(\gamma)}_{MN} = \begin{pmatrix}
(1 + \gamma^2 \Delta)^{-2/3} G^{(0)}_{ab} & 0 \\
0 & (1 + \gamma^2 \Delta)^{1/3} G^{(0)}_{\mu\nu}
\end{pmatrix}
$$

(123)

where $a, b = 1, 2, 3$, $\mu, \nu = 4 \cdots 10$. This being the only change, the multiplicative
factors can be pulled outside to produce an overall multiplicative factor in the deter-
minant. Specifically, calculating the reduced determinant for the Bekenstein-Hawking
Entropy gives

$$\det' G^{(\gamma)}_{MN} = [(1 + \gamma^2 \Delta)^{-2/3}]^3 \times [(1 + \gamma^2 \Delta)^{1/3}]^{11-3-2} \times \det' G^{(0)}_{MN}
$$

(124)

And so for the M-theory analogues (where they apply) of the Non-Commutative,
Dipole and $\gamma$-deformations of the Membrane

$$\det' G^{(\gamma)}_{MN} = \det' G^{(0)}_{MN}
$$

(125)

and therefore

$$S^{(\gamma)}_{BH} = S^{(0)}_{BH}.
$$

(126)

Thus in fact there is no change in entropy. The deformed brane has the same
entropy as the undeformed brane.\footnote{This is provided that the temporal direction is not included in the deformation procedure (which
not always the case for our the five-brane).}
This invariance of the entropy under the deformation procedure for trivial embeddings is a consequence of the properties of this specific deformation in eleven dimensions. It is not the case for the analogous deformation of a ten dimensional Supergravity theory using a $U(1) \times U(1)$ background isometry.

This is quite unusual, since the deformation will introduce an interaction in the dual description and so one would imagine that the entropy of the theory would change. It is possible that this is a consequence of the large N limit, since the supergravity description is only good at large N. Perhaps it is possible that the deformations of the theory are somehow subleading at large N but this seems unlikely. The original Lunin Maldacena deformation corresponding to the beta deformed theory certainly had large N implications. Also, the solution is deformed, only the reduced determinant (which is relevant for the entropy) is invariant.

Of the deformations of these eleven dimensional backgrounds with trivial $T^3$ fibration, there is however one special example not sharing this cancellation. The “Non-Commutative” deformation of the five-brane world-volume produces a metric with a reduced determinant that retains a remnant of the deformation procedure.

The simplicity of the general cases of deformation of trivial fibrations and the observed invariance of the Bekenstein-Hawking entropy is in marked contrast to the effect of the deformation involving the choice of a non-trivial $T^3$ embedding. Geometrically along with the $T^3$ expansion/contraction, the eight dimensional base in these cases experiences a form of shearing. The deformation takes effect in a highly inhomogeneous manner. The result of this is the appearance of additional additive terms within the determinant of the metric. These are the source of changes in the Bekenstein-Hawking Entropy. As an example we consider the deformation of the membrane on $\{\varphi_1, \varphi_2, \varphi_3\}$ where

$$\sqrt{|\text{det}'G^{(7)}_{\gamma}|} = \frac{1}{(1 + \gamma^2 \Delta_{123})^{1/2}} \sqrt{|\text{det}'G^{(0)}_{\gamma} + \gamma^2 \rho^2 r^8 s_\theta^4 s_a^2 \Delta_{123}^2 f_1|}$$

(127)

A numerical evaluation was attempted but as yet, an explicit solution has not presented itself. For the deformations involving $T^3$'s with one-cycles both on and off the membrane world volume, the changes in the reduced determinant are far more
complicated. For each of these cases we have in general
\[ \sqrt{\left| \det G^{(\gamma)}_{MN} \right|} = \frac{1}{(1 + \gamma^2 \Delta)} \sqrt{\left| \det G^{(0)}_{MN} + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^4) \right|} \] (128)

Along with the larger power of the deformation pre-factor in the denominator, there are far more terms at \( \mathcal{O}(\gamma^2) \) inside the square root. The more complicated the embedding, the more non-trivial the functions making an appearance here. The \( \mathcal{O}(\gamma^4) \) are a new feature in the reduced determinant for the “dipole” type deformations of the membrane. For deformation on \( \{\varphi_4, \varphi_1, \varphi_2\} \)

\[
\det G^{(\gamma)}_{MN} = \frac{1}{(1 + \gamma^2 \Delta_{412})} \left\{ \det G^{(0)}_{MN} + \gamma^2 \left[ (\rho^2 r^6 s^4 r^2 a^2 \Delta_{412}) \left( g_1 G^{(0)}_{\varphi\varphi_2} + g_2 G^{(0)}_{\psi\psi} - 2 g_3 G^{(0)}_{\varphi_3\psi} \right) \right] \\
+ \gamma^4 \left[ H^{1/3} (r^2 \Delta_{412})^2 r^6 \rho^2 s^4 a^2 \left( g_1 g_2 - g_3^2 \right) \Delta_{412} \right] \right\} \] (129)

8 Conclusions

This paper has explored the result of applying the M-theory deformation process to branes. In particular, we have seen how to generate families of solutions beginning with the usual membrane and five-brane solutions. The deformed solutions have a well defined decoupling limit where the deformation is scaled so as to survive the decoupling limit. Importantly, the deformation process is shown to commute with taking decoupling limit.

The properties of these solutions were investigated. Perhaps the most intriguing result is that the entropy of a class of deformed branes is the same as that of the undeformed branes despite the deformed theory having different amounts of supersymmetry and new interactions.

Let us consider the membrane. From the dual theory perspective this is a statement that there is for \( N=8, 2+1 \) dimensional Yang-Mills theory in the large \( N \) limit, at strong coupling there is a deformation that breaks supersymmetry and yet preserves the entropy of the theory.
There are also some deeper questions about M-theory geometry. We know from other work how the so called “open string” metric is an invariant under a similar deformation process [22]. Later it was shown that one could form the equivalent invariant metric for M-theory [23]. It would be interesting to see if one could form a similar invariant combination of metric and C-field that captures the class of the solution and is invariant under the deformation process. This is similar in spirit to the generalised geometry program where T-dualities are just diffeomorphisms of a more general geometry of bigger space. Perhaps the entropy invariance mentioned above is a sign of such a deeper symmetry.

One future possibility would be to look at $G_2$ manifolds which have interesting phenomenological possibilities and study the deformations of these. This may allow the introduction of a new scale (coming from the deformation) that may allow additional decoupling.

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A “Dipole” deformations of the membrane

A.1 $T^3$ on $\{\varphi_4, \varphi_1, \varphi_2\}$

The deformed M2 using $T^3$ given by $\{\varphi_4, \varphi_1, \varphi_2\}$ is

\[
\begin{align*}
\csc \theta &= \cosec \theta, \\
\sec \theta &= \sec \theta
\end{align*}
\]

(130)
\[ ds_4^2 = (1 + \gamma^2 \Delta_{412})^{1/3} \left\{ H^{-2/3} \left( -dt^2 + d\rho^2 \right) + H^{1/3} \left( dr^2 + r^2(d\theta^2 + s_\theta^2(d\alpha^2 + s_\alpha^2 d\beta^2)) \right) \right\} \]

\[ + \frac{1}{(1 + \gamma^2 \Delta_{412})^{2/3}} \left\{ H^{-2/3} \rho^2 d\varphi_4^2 + H^{1/3} r^2 \left( \begin{array}{c} s_\theta^2 s_\alpha^2 d\varphi_1^2 + s_\theta^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) d\varphi_2^2 \\ -2s_\theta^2 s_\phi^2 s_\alpha^2 d\varphi_1 d\varphi_2 + 2s_\alpha^2 s_\theta^2 (s_\beta^2 - c_\beta^2) d\varphi_1 d\psi \\ + 2s_\theta^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\varphi_2 d\psi + 2s_\alpha^2 c_\phi^2 d\varphi_2 d\varphi_3 \end{array} \right) \right\} \]

\[ + \frac{H^{1/3} r^2}{(1 + \gamma^2 \Delta_{412})^{2/3}} \left\{ \left( 1 + \gamma^2 \Delta_{412} \ g_1(\alpha, \beta, \theta) \right) d\psi^2 \right. \\

\left. + \left( (c_\theta^2 - s_\theta^2 c_\alpha^2) + \gamma^2 \Delta_{412} \ g_2(\alpha, \beta, \theta) \right) d\varphi_3^2 \right. \\

\left. + \left( (c_\theta^2 + s_\theta^2 c_\alpha^2) + \frac{1}{2} \gamma^2 \Delta_{412} \ g_3(\alpha, \beta, \theta) \right) d\psi d\varphi_3 \right\} \] (131)

\[ F_\gamma^{(4)} = F_0^{(4)} - \partial^\kappa \left[ \frac{\gamma \Delta_{412}}{(1 + \gamma^2 \Delta_{412})} \right] dx^\kappa \wedge d\varphi_4^2 \wedge d\varphi_1^2 \wedge d\varphi_2^2 \] (132)

where \( \kappa \in \{ \rho, r, \alpha, \beta, \theta \} \) and

\[ \Delta_{412} = \rho^2 r^4 \left( s_\theta^2 s_\alpha^2 s_\beta^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) - s_\theta^2 c_\alpha^2 c_\beta^2 \right) \] (133)

with

\[ g_1(\alpha, \beta, \theta) = \frac{16c_\alpha^2 (1 - (c_\alpha^2 + c_\beta^2 s_\alpha^2) s_\theta^2)}{(9 + 7c_\alpha^2 + 2c_\beta^2 s_\alpha^2)} \]

\[ g_2(\alpha, \beta, \theta) = c_\theta^2 + \frac{s_\theta^2}{4s_\alpha^2 c_\alpha^2 c_\beta^2 + s_c^2} \]

\[ g_3(\alpha, \beta, \theta) = 2c_\theta^2 - \frac{c_\alpha^2 (3 + c_\beta^2)}{(c_\alpha^2 + c_\alpha^2)} \]
A.2  $T^3$ on $\{\varphi_4, \varphi_1, \varphi_3\}$

The deformed M2 using $T^3$ with $\{\varphi_4, \varphi_1, \varphi_3\}$ is

$$
\begin{align*}
\text{ds}^2 &= (1 + \gamma^2 \Delta_{413})^{1/3} \left\{ H^{-2/3} \left( - dt^2 + d\rho^2 \right) + H^{1/3} \left( dr^2 \\
&+ r^2 (d\theta^2 + s_\theta^2 (d\alpha^2 + s_\alpha^2 d\beta^2)) \right) \right\} \\
&+ \frac{1}{(1 + \gamma^2 \Delta_{413})^{2/3}} \left\{ H^{-2/3} \rho^2 d\varphi_4^2 + H^{1/3} \left( s_\theta^2 s_\alpha^2 d\varphi_1^2 \\
&+ (c_\theta^2 + s_\theta^2 c_\alpha^2) d\varphi_3^2 + s_\alpha^2 s_\theta^2 (s_\beta^2 - c_\beta^2) d\varphi_4 d\varphi_1 \\
&+ (c_\alpha^2 - s_\alpha^2 c_\beta^2) d\varphi_3 d\varphi_4 - c_\beta^2 s_\alpha^2 s_\theta^2 d\varphi_1 d\varphi_2 \\
&+ s_\theta^2 c_\alpha^2 d\varphi_2 d\varphi_3 \right) \right\} \\
&+ \frac{H^{1/3} r^2}{(1 + \gamma^2 \Delta_{413})^{2/3}} \left\{ (s_\theta^2 s_\alpha^2 c_\beta^2 + c_\alpha^2) + \gamma^2 \Delta_{413} h_1(\alpha, \beta, \theta) \right\} d\varphi_2^2 \\
&+ \left( 1 + \gamma^2 \Delta_{413} h_2(\alpha, \beta, \theta) \right) d\psi^2 \\
&+ \left( s_\theta^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) + \gamma^2 \Delta_{413} h_3(\alpha, \beta, \theta) \right) d\varphi_2 d\psi \right\}
\end{align*}
$$

$$(134)$$

$$F_4^{(4)} = F_0^{(4)} - \partial_\kappa \left[ \frac{\gamma \Delta_{412}}{(1 + \gamma^2 \Delta_{413})} \right] dx^\kappa \wedge d\varphi_4 \wedge d\varphi_1 \wedge d\varphi_3 \quad (135)$$

for $\kappa \in \{\rho, r, \alpha, \beta, \theta\}$ with

$$
\Delta_{413} = \rho^2 r^4 \left( s_\alpha^2 s_\theta^2 (c_\beta^2 + c_\alpha^2 s_\theta^2) \right) \quad (136)
$$

and

$$
\begin{align*}
h_1(\alpha, \beta, \theta) &= c_\theta^2 + c_\beta^2 s_\alpha^2 s_\theta^2 - \frac{c_\theta^4}{(c_\theta^2 + c_\alpha^2 s_\theta^2)} \\
h_2(\alpha, \beta, \theta) &= 1 + 3c_\theta^2 - (c_\alpha^2 + c_2^2 s_\alpha^2) s_\theta^2 - \frac{4c_\theta^4}{(c_\theta^2 + c_\alpha^2 s_\theta^2)} \\
h_3(\alpha, \beta, \theta) &= c_\beta^2 s_\alpha^2 s_\theta^2 - \frac{1}{(c_\theta^2 s_\alpha^2 + sec^2_\theta)}
\end{align*}
$$
A.3 $T^3$ on $\{\varphi_4, \varphi_2, \varphi_3\}$

The deformed M2 using $T^3$ with $\{\varphi_4, \varphi_2, \varphi_3\}$ is

$$\begin{align*}
ds^2 &= (1 + \gamma^2 \Delta_{423})^{1/3} \left\{ H^{-2/3} \left( -dt^2 + d\rho^2 \right) + H^{1/3} \left( dr^2 ight. \\
&\quad \left. + r^2 (d\theta^2 + s_\rho^2 (da^2 + s_\alpha^2 d\beta^2)) \right) \right\} \\
&\quad + \frac{1}{(1 + \gamma^2 \Delta_{423})^{2/3}} \left\{ H^{-2/3} \rho^2 d\varphi_4^2 + H^{1/3} r^2 \left( \\
&\quad + (c_\rho^2 + s_\rho^2 c_\alpha^2) d\varphi_3^2 + s_\rho^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) d\varphi_2^2 \\
&\quad + s_\rho^2 c_\alpha^2 d\varphi_2 d\varphi_3 + s_\rho^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) d\varphi_2 d\psi \\
&\quad + (c_\rho^2 - s_\rho^2 c_\alpha^2) d\varphi_3 d\psi \right) \right\} \\
&\quad + \frac{H^{1/3} r^2}{(1 + \gamma^2 \Delta_{423})^{2/3}} \left\{ \left( 1 + \gamma^2 \Delta_{423} h_1(\alpha, \beta, \theta) \right) d\psi^2 \\
&\quad + \left( s_\rho^2 s_\alpha^2 + \gamma^2 \Delta_{423} h_2(\alpha, \beta, \theta) \right) d\varphi_1^2 \\
&\quad + \left( s_\rho^2 s_\alpha^2 (s_\beta^2 - c_\beta^2) + \gamma^2 \Delta_{423} h_3(\alpha, \beta, \theta) \right) d\varphi_1 d\psi \right\} \quad (137)
\end{align*}$$

$$\begin{align*}
F^{(4)}_\gamma &= F^{(4)}_0 - \gamma \partial_\kappa \left[ \frac{\Delta_{423}}{(1 + \gamma^2 \Delta_{423})} \right] dx^\kappa \wedge d\varphi_4 \wedge d\varphi_3 \wedge d\varphi_2 \quad (138)
\end{align*}$$

where

$$\Delta_{423} = \rho^2 r^4 (s_\rho^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) (c_\rho^2 + s_\rho^2 c_\alpha^2) - s_\rho^4 c_\alpha^2) \quad (139)$$
and

\[ h_1(\alpha, \beta, \theta) = \frac{s_\theta^2 c_\beta^2 s_\alpha^2 (1 - c_\beta^2 s_\alpha^2)}{(c_\theta^2 c_\beta^2 s_\alpha^2 + c_\alpha^2 (c_\theta^2 + c_\beta^2 s_\alpha^2 s_\theta^2))} \]
\[ + s_\theta^2 \left( \frac{c_\alpha^2 (c_\theta^2 + c_\beta^2 s_\alpha^2 (4 + 3c_\theta^2 - c_\beta^2 s_\alpha s_\theta^2)) - c_\alpha^4 (c_\theta^2 + c_\beta^2 s_\alpha^2 s_\theta^2)}{(c_\theta^2 c_\beta^2 s_\alpha^2 + c_\alpha^2 (c_\theta^2 + c_\beta^2 s_\alpha^2 s_\theta^2))} \right) \]
\[ h_2(\alpha, \beta, \theta) = \frac{s_\theta^2 c_\beta^2 \left( -c_\alpha^2 c_\beta^2 (-4 + 3c_\theta^2) s_\alpha^2 - c_\alpha^2 c_\beta^2 (-4 + 3c_\theta^2) s_\alpha^4 + c_\beta^4 s_\alpha^2 s_\theta^2 \right)}{(c_\theta^2 + c_\beta^2 s_\alpha^2)^2 (c_\theta^2 + c_\beta^2 s_\alpha^2)} \]
\[ + \frac{s_\theta^2 c_\beta^2 \left( c_\alpha^2 c_\beta^2 (-4 + 3c_\theta^2) - c_\alpha^2 c_\beta^2 (-3 + c_\beta^2) s_\alpha^2 + c_\beta^4 s_\alpha^4 s_\beta^2 \right)}{(c_\theta^2 + c_\beta^2 s_\alpha^2)^2 (c_\theta^2 + c_\beta^2 s_\alpha^2)} \]
\[ h_3(\alpha, \beta, \theta) = \frac{s_\theta^2 c_\beta^2 \left( -c_\alpha^2 (1 + 5c_\beta) s_\alpha^2 + 2 c_\beta^2 s_\alpha^4 s_\beta^2 \right)}{2(c_\theta^2 + c_\beta^2 s_\alpha^2)^2 (c_\theta^2 + c_\beta^2 s_\alpha^2)} \]
\[ + \frac{s_\theta^2 c_\beta^2 c_\alpha^2 s_\alpha^2 (c_\alpha^2 (1 - 3c_\beta) + 2 c_\beta^2 s_\alpha^2 s_\beta^2) s_\theta^2)}{2(c_\theta^2 + c_\beta^2 s_\alpha^2)^2 (c_\theta^2 + c_\beta^2 s_\alpha^2)} \]  \hspace{1cm} (140)
B Near Horizon region for Membrane deformed on \( \{ \varphi_4, \varphi_1, \varphi_3 \} \)

For \( T^3 \) given by \( \{ \varphi_4, \varphi_1, \varphi_3 \} \) the near horizon solution is

\[
\begin{align*}
\text{ds}^2 & \sim \ell_p^2 \left[ (1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413})^{1/3} \left( \frac{u^2}{(2^5 \pi^2 N)^{2/3}} \left( -dt^2 + d\rho^2 \right) + (2^5 \pi^2 N)^{1/3} \left( \frac{du^2}{4u^2} + (d\theta^2 + s_\theta^2 (d\alpha^2 + s_\alpha^2 d\beta^2)) \right) \right) \right] \\
&+ \frac{1}{(1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413})^{2/3}} \left( \frac{u^2}{(2^5 \pi^2 N)^{2/3}} \rho^2 d\varphi_4^2 + (2^5 \pi^2 N)^{1/3} \left( s_\theta s_\alpha d\varphi_1^2 + (c_\theta^2 + s_\theta^2 c_\alpha^2) d\varphi_3^2 \right. \right. \\
&\left. \left. + 2s_\alpha^2 s_\theta^2 \left( s_\beta^2 - c_\beta^2 \right) d\varphi_1 d\psi + 2(c_\theta^2 - s_\theta^2 c_\alpha^2) d\varphi_3 d\psi \right. \right. \\
&\left. \left. - 2c_\beta^2 s_\theta^2 s_\alpha^2 d\varphi_1 d\varphi_2 + 2s_\theta^2 c_\alpha^2 d\varphi_2 d\varphi_3 \right) \right] \\
&+ \frac{(2^5 \pi^2 N)^{1/3}}{(1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413})^{2/3}} \left( \left( s_\theta^2 (s_\alpha^2 c_\beta^2 + c_\alpha^2) + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413} h_1 \right) d\varphi_2^2 \right. \\
&\left. + \left( 1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413} h_2 \right) d\psi^2 \right. \\
&\left. + 2 \left( s_\theta^2 (s_\alpha^2 c_\beta^2 - c_\alpha^2) + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413} h_3 \right) d\varphi_2 d\psi \right) \\
\end{align*}
\]

\[ F_{\gamma}^{(4)} \sim \ell_p^2 \left[ \frac{3 \rho u^2}{2^5 \pi^2 N} \text{d} u \wedge \text{d} t \wedge \text{d} \rho \wedge \text{d} \varphi_4 - \frac{\tilde{\gamma}}{(1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413})^2} \left( \frac{2 u \rho^2 \Delta'_{413} \text{d} u \wedge \text{d} \varphi_4 \wedge \text{d} \varphi_1 \wedge \text{d} \varphi_3}{(1 + \tilde{\gamma}^2 \rho^2 u^2 \Delta'_{413})^2} \right) \right. \\
\left. + \sum_{i \in \{\rho \alpha \theta\}} u^2 \partial_i (\rho^2 \Delta'_{413}) \text{d} x^i \wedge \text{d} \varphi_4 \wedge \text{d} \varphi_1 \wedge \text{d} \varphi_3 \right] \tag{141}
\]

Unlike the deformation using a \( T^3 \) embedded entirely in the transverse space, the \( F_{u \varphi_4 \varphi_1 \varphi_3} \) does not vanish in the decoupling limit.
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