Dynamics of the anisotropic Kantowsky–Sachs geometries in $R^n$ gravity

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Abstract
We construct general anisotropic cosmological scenarios governed by an $f(R)$ gravitational sector. Focusing then on Kantowski–Sachs geometries in the case of $R^n$-gravity, and modeling the matter content as a perfect fluid, we perform a detailed phase-space analysis. We find that at late times the universe can result in a state of accelerating expansion, and additionally, for a particular $n$-range ($2 < n < 3$), it exhibits phantom behavior. Furthermore, isotropization has been achieved independently of the initial anisotropy degree, showing in a natural way why the observable universe is so homogeneous and isotropic, without relying on a cosmic no-hair theorem. Moreover, contracting solutions also have a large probability to be the late-time states of the universe. Finally, we can also obtain the realization of the cosmological bounce and turnaround, as well as of cyclic cosmology. These features indicate that anisotropic geometries in modified gravitational frameworks present radically different cosmological behaviors compared to the simple isotropic scenarios.

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1. Introduction
The observable universe is homogeneous and isotropic at a great accuracy [1], leading the large majority of cosmological works to focus on homogeneous and isotropic geometries. The explanation of these features, along with the horizon problem, was the main reason of the inflation paradigm construction [2]. Although the last subject is well explained, the homogeneity and the isotropy problems are, strictly speaking, not fully solved, since usually one starts straightaway with a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) metric and examines the evolution of fluctuations. On the other hand, the robust approach on the subject should be to start with an arbitrary metric and show that inflation is
indeed realized and that the universe evolves toward an FRW geometry, in agreement with observations. However, the complex structure of such an approach allows only for a numerical elaboration [3], and therefore in order to extract analytical solutions many authors start with one more assumption, that is, investigating anisotropic but homogeneous cosmologies. This class of geometries was known a long time ago [4], and it can exhibit very interesting cosmological behavior, either in inflationary or in post-inflationary cosmology [5]. Finally, note that such geometries may be relevant for the description of the black-hole interior [6].

The most well-studied homogeneous but anisotropic geometries are the Bianchi type (see [7, 8] and references therein) and the Kantowski–Sachs metrics [9–11], either in conventional or in higher dimensional framework. Although some Bianchi models (for instance the Bianchi IX one) are more realistic, their complicated phase-space behavior led many authors to investigate the simpler but very interesting Bianchi I, Bianchi III and Kantowski–Sachs geometries. In these types of metrics one can analytically examine the rich behavior and also incorporate the matter content [11–15], obtaining a very good picture of homogeneous but anisotropic cosmology.

On the other hand the observable universe is now known to be accelerating [16], and this feature led physicists to follow two directions in order to explain it. The first is to introduce the concept of dark energy (see [17] and references therein) on the right-hand side of the field equations, which could either be the simple cosmological constant or various new exotic ingredients [18–20]. The second direction is to modify the left-hand side of the field equations, that is, to modify the gravitational theory itself, with the extended gravitational theories known as $f(R)$-gravity (see [21, 22] and references therein) being the most examined case. Such an approach can still be in the spirit of general relativity since the only request is that the Hilbert–Einstein action should be generalized (replacing the Ricci scalar $R$ by its functions) asking for a gravitational interaction acting, in principle, in different ways at different scales. Such extended theories can present very interesting behavior and the corresponding cosmologies have been investigated in detail [23–27].

In this work we are interested in investigating homogeneous but anisotropic cosmologies, focusing on Kantowski–Sachs type, in a universe governed by $f(R)$-gravity. In particular, we perform a phase-space and stability analysis of such a scenario, examining in a systematic way the possible cosmological behaviors, focusing on the late-time stable solutions. Such an approach allows us to bypass the high nonlinearities of the cosmological equations, which prevent any complete analytical treatment, obtaining a (qualitative) description of the global dynamics of these models. Furthermore, in these asymptotic solutions we calculate various observable quantities, such as the deceleration parameter, the effective (total) equation-of-state parameter and the various density parameters. We stress that the results of anisotropic $f(R)$ cosmology are expected to be different than the corresponding ones of $f(R)$-gravity in isotropic geometries, similar to the differences between isotropic [28] and anisotropic [29] considerations in general relativity. Additionally, the results are expected to be different from anisotropic general relativity, too. As we see, anisotropic $f(R)$ cosmology can be consistent with observations.

The paper is organized as follows. In section 2 we construct the cosmological scenario of anisotropic $f(R)$-gravity, presenting the kinematical and dynamical variables, particularizing on the $f(R) = R^n$ ansatz in the case of Kantowski–Sachs geometry. Having extracted the cosmological equations, in section 3 we perform a systematic phase-space and stability analysis of the system. Thus, in section 4 we analyze the physical implications of the obtained results, and we discuss the cosmological behaviors of the scenario at hand. Finally, our results are summarized in section 5.
2. Anisotropic $f(R)$ cosmology

In this section we present the basic features of anisotropic Bianchi I, Bianchi III and Kantowski–Sachs geometries. After presenting the kinematical and dynamical variables in the first two subsections, we focus on the Kantowski–Sachs geometrical background and on the $R^n$-gravity in the last two subsections.

2.1. The geometry and kinematical variables

In order to investigate anisotropic cosmologies, it is usual to assume an anisotropic metric of the form \[12, 30\]
\[
ds^2 = -N(t)^2 dt^2 + \left[\frac{1}{e_1^1(t)}\right]^2 dr^2 + \left[\frac{1}{e_2^2(t)}\right]^2 \left[d\theta^2 + S(\theta)^2 d\varphi^2\right],
\]
where $1/e_1^1(t)$ and $1/e_2^2(t)$ are the expansion scale factors. The frame vectors in coordinate form are written as
\[
e_0 = N^{-1} \partial_t, \quad e_1 = e_1^1 \partial_r
\]
\[
e_2 = e_2^2 \partial_\theta, \quad e_3 = e_2^2 / S(\theta) \partial_\varphi.
\]

The metric (1) can describe three geometric families, that is,
\[
S(\theta) = \begin{cases} 
sin \theta & \text{for } k = +1, \\
\theta & \text{for } k = 0, \\
\sinh \theta & \text{for } k = -1,
\end{cases}
\]
known respectively as Kantowski–Sachs, Bianchi I and Bianchi III models.

In the following, we will focus on the Kantowski–Sachs geometry [10], since it is the most popular anisotropic model, and since all solutions are known analytically in the case of general relativity, even if some particular types of matter are coupled to gravity [11, 12]. These are spatially homogeneous spherically symmetric models [30, 31], with four Killing vectors $\partial_\varphi$, $\cos \varphi \partial_\vartheta - \sin \varphi \cot \vartheta \partial_\varphi$, $\sin \varphi \partial_\vartheta + \cos \varphi \cot \vartheta \partial_\varphi$, $\partial_x$ [32].

Let us now consider relativistic fluid dynamics (see, e.g., [33]) in such a geometry. For any given fluid 4-velocity vector field $u^\mu$, the projection tensor\(^3\)
\[
h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu
\]
projects into the instantaneous rest space of a comoving observer. It is standard to decompose the first covariant derivative $\nabla_\mu u_\nu$ into its irreducible parts
\[
\nabla_\mu u_\nu = -u_\mu u_\nu + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu} - \omega_{\mu\nu},
\]
where $\sigma_{\mu\nu}$ is the symmetric and trace-free shear tensor ($\sigma_{\mu\nu} = \sigma_{(\mu\nu)}$, $\sigma_{\mu\nu} u^\nu = 0$, $\sigma^\mu_{\nu\mu} = 0$), $\omega_{\mu\nu}$ is the antisymmetric vorticity tensor ($\omega_{\mu\nu} = \omega_{(\mu\nu)}$, $\omega_{\mu\nu} u^\nu = 0$) and $\dot{u}_\mu$ is the acceleration vector defined as $\dot{u}_\mu = u^\nu \nabla_\nu u_\mu$ (a dot denotes the derivative with respect to $t$). In the above expression we have also introduced the volume expansion scalar. In particular,
\[
\Theta = \nabla_\mu u^\mu,
\]
which defines a length scale $\ell$ along the flow lines, describing the volume expansion (contraction) behavior of the congruence completely, via the standard relation, namely
\[
\Theta \equiv \frac{3 \dot{\ell}}{\ell}.
\]
\(^3\) Covariant spacetime indices are denoted by letters from the second half of the Greek alphabet.
In cosmological contexts it is customary to use the Hubble scalar $H = \Theta / 3$ [34]. From these it becomes obvious that in FRW geometries, $\ell$ coincides with the scale factor. Finally, it can be shown that these kinematical fields are related through [33, 34]

$$\sigma_{\mu\nu} := \dot{u}(\mu)u(\nu) + \nabla(\mu)u(\nu) - \frac{1}{3} \Theta h_{\mu\nu}$$

(6)

$$\omega_{\mu\nu} := -u(\mu)\dot{u}(\nu) - \nabla(u(\mu)u(\nu))$$

(7)

Transiting to the synchronous temporal gauge, we can set $N$ to any positive function of $t$ or simply $N = 1$. Thus, we extract the following restrictions on kinematical variables:

$$\sigma_{\mu} = \text{diag}(0, -2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\mu\nu} = 0,$$

(8)

where

$$\sigma_+ = \frac{1}{3} \frac{d}{dt} \left[ \ln \frac{e_1}{e_2^2} \right].$$

(9)

Finally, note that the Hubble scalar can be expressed in terms of $e_1^1$ and $e_2^2$ as

$$H = -\frac{1}{3} \frac{d}{dt} \left[ \ln \frac{e_1}{e_2^2} \right].$$

(10)

### 2.2. The action and dynamical variables

Let us now construct $f(R)$ cosmology in such a geometrical background. In the metric formalism the action for $f(R)$-gravity is given by [21, 22]

$$S_{\text{metric}} = \int d^4x \sqrt{-g} \left[ f(R) - 2\Lambda + \mathcal{L}^{(m)} \right],$$

(11)

where $f(R)$ is a function of the Ricci scalar $R$, and $\mathcal{L}^{(m)}$ accounts for the matter content of the universe. Additionally, we use the metric signature $(-1, 1, 1, 1)$, Greek indices run from 0 to 3, and we impose the standard units in which $c = 8\pi G = 1$. Finally, in the following, and without loss of generality, we set the usual cosmological constant $\Lambda = 0$.

The fourth-order equations obtained by varying action (11) with respect to the metric write

$$G_{\mu\nu} = \frac{T_{\mu\nu}^{(m)}}{f(R)} + T_{\mu\nu}^R,$$

(12)

where the prime denotes differentiation with respect to $R$. In this expression $T_{\mu\nu}^{(m)}$ denotes the matter energy–momentum tensor, which is assumed to correspond to a perfect fluid with energy density $\rho_m$ and pressure $p_m$. Additionally, $T_{\mu\nu}^R$ denotes a correction term describing a ‘curvature-fluid’ energy–momentum tensor of geometric origin [35, 36]:

$$T_{\mu\nu}^R = \frac{1}{f'(R)} \left[ \frac{1}{2} g_{\mu\nu} (f(R) - R f'(R)) + \nabla(\mu) \nabla(\nu) f'(R) - g_{\mu\nu} \Box f'(R) \right],$$

(13)

where $\nabla(\mu)$ is the covariant derivative associated with the Levi-Civita connection of the metric and $\Box \equiv \nabla(\mu) \nabla(\mu)$. Note that in the last two terms of the right-hand side there appear fourth-order metric derivatives, justifying the name ‘fourth-order gravity’ used for this class of theories [22]. By taking the trace of equation (12) and re-ordering terms, one obtains the ‘trace equation’ (equation (5) in section 2.1 of [37])

$$3 \Box f'(R) + R f'(R) - 2 f(R) = T,$$

(14)

where $T = T_\mu^\mu$ is the trace of the energy–momentum tensor of ordinary matter.
In the phenomenological fluid description of a general matter source, the standard decomposition of the energy–momentum tensor $T_{\mu\nu}$ with respect to a timelike vector field $u^\mu$ is given by

$$T_{\mu\nu} = \mu u_\mu u_\nu + 2q_\mu u_\nu + P h_{\mu\nu} + \pi_{\mu\nu},$$

(15)

where $\mu$ denotes the energy density scalar, $P$ is the isotropic pressure scalar, $q_\mu$ is the energy current density vector ($q_\mu u^\mu = 0$) and $\pi_{\mu\nu}$ is the trace-free anisotropic pressure tensor ($\pi_{\mu\nu} u^\nu = 0, \pi^\mu_\mu = 0, \pi_{\mu\nu} = \pi_{\nu\mu}$).

The matter fields need to be related through an appropriate thermodynamical equation of state in order to provide a coherent picture of the physics underlying the fluid spacetime scenario. Applying this covariant decomposition to the ‘curvature-fluid’ energy–momentum tensor (13) we obtain

$$\mu = -\frac{1}{2} \left[ f(R) - R f'(R) + 6H \frac{df'}{dR} f'(R) \right],$$

$$P = -\frac{1}{2} \left[ -f(R) + R f'(R) - 4H \frac{df'}{dR} f'(R) - 2 \frac{d^2 f'}{dR^2} f'(R) \right].$$

(16)

Finally, the anisotropic pressure tensor is given by $\pi^\mu_\nu = \text{diag}(0, -2\pi^+, \pi^+, \pi^+)$, where

$$\pi^+ = \frac{\frac{df'}{dR}}{f'(R)} \sigma_+, \quad \sigma_+ = \frac{\frac{d^2 f'}{dR^2}}{f'(R)} \sigma_+.$$

(17)

2.3. The cosmological equations

Let us now present the cosmological equations in the homogeneous and anisotropic Kantowski–Sachs metric. In particular, Einstein’s equations (12) in the Kantowski–Sachs metric read

$$3\sigma_+^2 - 3H^2 - 2K = -\mu - \frac{\rho_m}{f'(R)},$$

$$-3(\sigma_+ + H)^2 - 2\sigma_+ - 2H - 2K = \frac{p_m}{f'(R)} + P - 2\pi^+, \quad (18)$$

In these expressions $\rho_m$ and $p_m$ are the energy density and pressure of the matter perfect fluid, and their ratio gives the matter equation-of-state parameter

$$w = \frac{p_m}{\rho_m}. \quad (19)$$

Furthermore, $2K$ is the Gauss curvature of the 3-spheres [34] given by

$$2K = \left( e_1^2 \right)^2, \quad (20)$$

and its evolution equation is written as

$$\dot{2K} = -2(\sigma_+ + H) \left(\dot{2K}\right). \quad (21)$$

Additionally, the evolution equation for $e_1^1$ reads (see equation (42) in section 4.1 of [30])

$$\dot{e}_1^1 = -(H - 2\sigma_+) e_1^1. \quad (22)$$

Finally, observing the form of the first of equations (18) one can define the various density parameters of the scenario at hand, namely [38] the curvature one

$$\Omega_k = \frac{2K}{3H^2}. \quad (23)$$
the matter one
\[ \Omega_m \equiv \frac{\rho_m}{3H^2 f'(R)}, \tag{24} \]

the ‘curvature-fluid’ one
\[ \Omega_{\text{curv.fl}} \equiv \frac{\mu}{3H^2} \tag{25} \]

and the shear one
\[ \Omega_\sigma \equiv \left(\frac{\sigma_+}{H}\right)^2, \tag{26} \]
satisfying \( \Omega_k + \Omega_m + \Omega_{\text{curv.fl}} + \Omega_\sigma = 1. \)

Now, the trace equation (14) in the Kantowski–Sachs metric reads
\[ -3 \frac{d^2}{dt^2} f'(R) - 9H \frac{d}{dt} f'(R) + R f'(R) - 2f(R) = -\rho_m + 3p_m, \tag{27} \]
and the Ricci scalar is written
\[ R = 12H^2 + 6\sigma_+^2 + 6H + 2^2K. \tag{28} \]

At this stage we can reduce equations (18) along with (27), with respect to \( \dot{H}, \dot{\sigma}_+ \) and \( \dot{2}K \) [39], acquiring the Raychaudhuri equation
\[ \dot{H} = -H^2 - 2\sigma_+^2 - \frac{1}{6f'(R)}[\rho_m + 3p_m] - \frac{H}{2f'(R)} \frac{d}{dt} f'(R) \]
\[ + \frac{1}{6} \left[ R - \frac{f(R)}{f'(R)} - \frac{3}{f'(R)} \frac{d^2}{dt^2} f'(R) \right], \tag{29} \]
the shear evolution
\[ \dot{\sigma}_+ = -\sigma_+^2 - 3H\sigma_+ + H^2 - \frac{\rho_m}{3f'(R)} - \frac{1}{6} \left[ R - \frac{f(R)}{f'(R)} \right] + \frac{(H - \sigma_+)}{f'(R)} \frac{d}{dt} f'(R) \tag{30} \]
and the Gauss constraint
\[ 2^2K = 3\sigma_+^2 - 3H^2 + \frac{\rho_m}{3f'(R)} + \frac{1}{2} \left[ R - \frac{f(R)}{f'(R)} \right] - \frac{3H}{f'(R)} \frac{d}{dt} f'(R). \tag{31} \]

It proves convenient to use the trace equation (27) to eliminate the derivative \( \frac{d^2}{dt^2} f'(R) \) in (29), obtaining a simpler form of the Raychaudhuri equation, namely
\[ \dot{H} = -H^2 - 2\sigma_+^2 - \frac{\rho_m}{3f'(R)} + \frac{f(R)}{6f'(R)} + \frac{1}{6} \frac{H}{f'(R)} \frac{d}{dt} f'(R). \tag{32} \]

Furthermore, the Gauss constraint can alternatively be expressed as
\[ \left[ H + \frac{1}{2} \frac{d}{dt} \frac{f'(R)}{f(R)} \right]^2 + \frac{1}{3} 2^2K = \sigma_+^2 + \frac{\rho_m}{3f'(R)} + \frac{1}{6} \left[ R - \frac{f(R)}{f'(R)} \right] + \frac{1}{4} \left( \frac{df'(R)}{dt} \right)^2. \tag{33} \]

In summary, the cosmological equations of \( f(R) \)-gravity in the Kantowski–Sachs background are the ‘Raychaudhuri equation’ (32), the shear evolution (30), the trace equation (27), the Gauss constraint (33), the evolution equation for the 2-curvature \( ^2K \) (21) and the evolution equation for \( e_1 \) (22). Finally, these equations should be completed by considering the evolution equations for matter sources.
2.4. \( \mathcal{R}^n \)-gravity

In this subsection we specify the above general cosmological system. In particular, in order to continue we have to make an assumption concerning the function \( f(R) \). We impose an ansatz of the form \( f(R) = R^n \) \([21, 22, 31, 40]\), since such an ansatz does not alter the characteristic length scale (and general relativity is recovered when \( n = 1 \)), and it leads to simple exact solutions which allow for comparison with observations \([41, 42]\). Additionally, following \([35, 36]\) we consider that the parameter \( n \) is related to the matter equation-of-state parameter through

\[
\frac{3}{2}(1 + w) = \frac{3}{2} \left(1 + \frac{3}{2} w\right).
\]  

(34)

Such a choice is imposed by the requirement of the existence of an Einstein static universe in FRW backgrounds, which leads to a severe constraint in \( f(R) \), namely

\[
f(R) = \frac{2 \Lambda_1}{n^2} + \frac{R}{n^2} (3 + w).
\]  

(35)

with \( w \neq -1 \). Inversely, it can be seen as the equation-of-state parameter of ordinary matter is a fixed function of \( n \), if we desire Einstein static solutions to exist. The existence of such a static solution (in practice as a saddle-unstable one) is of great importance in every cosmological theory, since it connects the Friedmann matter-dominated phase with the late-time accelerating phase, as required by observations \([35, 36]\). Thus, if we relax the condition \( n = \frac{3}{2}(1 + w) \) in \( \mathcal{R}^n \)-gravity, in general we cannot obtain the epoch sequence of the universe. Finally, note that the constraint \(-1/3 \leq w \leq 1\) that arises from the satisfaction of all the energy conditions for standard matter imposes bounds on \( n \), namely \( n \in [1, 3] \), with the most interesting cases being those of dust \((w = 0, n = 3/2)\) and radiation fluid \((w = 1/3, n = 2)\).

The cosmological equations for \( \mathcal{R}^n \)-gravity, in the homogeneous and anisotropic Kantowski–Sachs metric \((1)\), are obtained by specializing equations \((32), (30), (33), (27), (21), (22)\) and consider the conservation equation for standard matter. Equations \((32), (30), (33)\) and \((27)\) become respectively

\[
\dot{H} + H^2 = -2\sigma^2 - \frac{\rho_m}{3nR^{n-1}} - \frac{1}{6n} R + (n - 1) H \frac{\dot{R}}{R},
\]  

(35)

\[
\dot{\sigma} = -\sigma^2 - 3H\sigma + H^2 - \frac{\rho_m}{3nR^{n-1}} - \frac{n - 1}{6n} R - (n - 1) (\sigma_H - H) \frac{\dot{R}}{R},
\]  

(36)

\[
\left(H + \frac{n - 1}{2} \frac{\dot{R}}{R}\right)^2 + \frac{1}{3} \frac{\dot{K}}{K} = \sigma^2 + \frac{\rho_m}{3nR^{n-1}} + \frac{n - 1}{6n} R + \frac{(n - 1)^2}{4} \frac{\dot{R}^2}{R^2},
\]  

(37)

\[
\frac{\dot{R}}{R} = \frac{n - 2}{3n(n - 1)} \frac{\dot{R}}{R} + \frac{2(2 - n)}{3n(n - 1)} \frac{\rho_m}{R^{n-1}} - 3H \frac{\dot{R}}{R} - \frac{n - 2}{R^2} \frac{\dot{R}^2}{R^2}.
\]  

(38)

Additionally, we consider the evolution equation for the 2-curvature \( \dot{K} \) given by \((21)\), as well as the matter conservation equation

\[
\dot{\rho}_m = -2nH \rho_m.
\]  

(39)

Finally, in order to close the equation system we also consider the propagation equation \((22)\).

3. Phase-space analysis

In the previous section we formulated \( f(R) \)-gravity in the case of the homogeneous and anisotropic Kantowski–Sachs geometry, focusing on the \( f(R) = R^n \) ansatz. Having extracted the cosmological equations we can investigate the possible cosmological behaviors and discuss the corresponding physical implications by performing a phase-space analysis. Such a procedure bypasses the complexity of the cosmological equations and provides us the understanding of the dynamics of these scenarios.
3.1. The dynamical system

In order to perform the phase-space and stability analysis of the models at hand, we have to transform the cosmological equations into an autonomous dynamical system [43]. However, since the present system is more complicated, in order to avoid ambiguities related to the non-compactness at infinity we define compact variables that can describe both expanding and collapsing models [31, 44]. This will be achieved by introducing the auxiliary variables [45, 46]

\[ Q = \frac{H}{D}, \quad \Sigma = \frac{\sigma_+}{D}, \]

\[ x = \frac{(n - 1)\dot{R}}{2nD}, \quad y = \frac{(n - 1)R}{6nD^2}, \quad z = \frac{\rho_m}{3nR^{n-1}D^2}, \]

where we have defined

\[ D = \left( \frac{H + \frac{n - 1}{2} \dot{R}}{R} \right)^2 + \frac{1}{3}K. \]  

Furthermore, we introduce the time variable \( \tau \) through

\[ d\tau = \left( \frac{D}{n-1} \right) dt, \]

and from now on primes will denote derivatives with respect to \( \tau \). Finally, note that in the following we focus on the general case \( n \neq 1 \). The dynamical investigation of the Kantowsky–Sachs model in general relativity (that is for \( n = 1 \)) has been performed in [47].

In terms of these auxiliary variables the Gauss constraint (37) becomes

\[ x^2 + y + z + \Sigma^2 = 1. \]  

Moreover, the \( D \)-definition (41) becomes

\[ (Q + x)^2 + K = 1. \]

Finally, from definition (40) we obtain the bounds \( y \geq 0, \ z \geq 0 \) and \( K \geq 0 \). Therefore, we conclude that the auxiliary variables must be compact and lie at the intervals

\[ Q \in [-2, 2], \quad \Sigma \in [-1, 1], \]

\[ x \in [-1, 1], \quad y \in [0, 1], \quad z \in [0, 1] \]

while \( E_1^{1} \) is unconstrained.

In summary, using the dimensionless auxiliary variables (40), along with the two constraints (43) and (44), we reduce the complete cosmological system to a five-dimensional one given by

\[ \begin{align*}
Q' &= (1 - n)\Sigma Q^3 - (n - 1) \left[ -3x^2 + 2\Sigma x + (n - 3)\Sigma^2 + n(x^2 + y - 1) + 1 \right] Q^2 \\
&\quad - (n - 1)[x(-3x^2 + (x - 3\Sigma)\Sigma + n(x^2 + \Sigma^2 + y - 1) - 1) - \Sigma]Q \\
&\quad - x^2 + \Sigma^2 + n(x^2 - \Sigma^2 + y - 1) + 1, \\
\end{align*} \]

\[ \begin{align*}
\Sigma' &= -(n - 3)(n - 1)(Q + x)\Sigma^3 - (n - 1)(Q + x - 1)(Q + x + 1)\Sigma^2 \\
&\quad - (n - 1)(Q + x)[(n - 3)(x^2 - 1) + ny]\Sigma + (n - 1)([Q + x]^2 - 1). \\
\end{align*} \]
\[ x' = -(n - 3)(n - 1)x^3 - (n - 1)[(n - 3)Q + \Sigma]x^3 - (n - 1)[-3\Sigma^2 + 2Q\Sigma + n(\Sigma^2 + y + 2) + 5|x|^2 - (n - 1)(Q[(Q - 3\Sigma)\Sigma + n(\Sigma^2 + y - 1) + 1]] \]  

\[ y' = -2(n - 1)n(Q + x)^2 + y[-2(n - 4)n + 3](Q + x)\Sigma^2 - 2(n - 1)(Q + x - 1)(Q + x + 1)\Sigma - 2(Q + x)(x^2 - 1)n^2 + (-4x^3 - 4Qx^2 + 2x + Q)n + x[3x(Q + x) - 2]]) \]

\[ \begin{pmatrix} E_1 \end{pmatrix}' = E_1^{(n - 1)}[\{ -(n - 3)(Q + x)\Sigma^2 - [(Q + x)^2 - 3]\Sigma} + (-Q - x)[\Sigma^2 + n(x^2 + y + 1) + 1] \]

Proceeding forward, as we observe, equation (50) decouples from (46) to (49). Thus, if one is only interested in the dynamics of Kantowsky–Sachs analysis to the system (46)–(49), defined in the compact phase space

\[ \Psi = \{ x^2 + y + \Sigma^2 < 1, |Q + x| < 1, x \in [-1, 1], y \in [0, 1], \Sigma \in [-1, 1], Q \in [-2, 2] \].

Finally, while for the purpose of this work it is adequate to investigate the system (46)–(49), leaving outside the decoupled equation (50), for completeness we study equation (50) in appendix B.

### 3.2. Invariant sets and critical points

Using the auxiliary variables (40) the cosmological equations of motion (35)–(39) were transformed into the autonomous form (46)–(50), that is, in a form \( \mathbf{X}' = \mathbf{f}(\mathbf{X}) \), where \( \mathbf{X} \) is the column vector constituted by the auxiliary variables and \( \mathbf{f}(\mathbf{X}) \) the corresponding column vector of the autonomous equations. The critical points \( \mathbf{X}_c \) are extracted satisfying \( \mathbf{X}' = 0 \), and in order to determine the stability properties of these critical points we expand around \( \mathbf{X}_c \), setting \( \mathbf{X} = \mathbf{X}_c + \mathbf{U} \) with \( \mathbf{U} \) the perturbations of the variables considered as a column vector. Thus, up to first order we acquire \( \mathbf{U}' = \Xi \cdot \mathbf{U} \), where the matrix \( \Xi \) contains the coefficients of the perturbation equations. Therefore, for each critical point, the eigenvalues of \( \Xi \) determine its type and stability. In particular, eigenvalues with negative (positive) real parts correspond to a stable (unstable) point, while eigenvalues with real parts of different signs correspond to a saddle point. Lastly, when at least one eigenvalue has a zero real part, the corresponding point is a non-hyperbolic one.

Now, there are several invariant sets, that is, areas of the phase space that evolve to themselves under the dynamics, for the dynamical system (46)–(49). There are two invariant sets given by \( Q + x = \pm 1 \), corresponding to \( K = 0 \), that is, \( e_2^2 / D = 0 \). However, the solutions of the evolution equations inside this invariant set do not correspond to exact solutions of the field equations, since the frame variables has to satisfy \( \det(e_2) \neq 0 \). Nevertheless, it appears that this invariant set plays a fundamental role in describing the asymptotic behavior of cosmological models.

Another invariant set is \( y = 0 \), that is \( \rho_m = 0 \), \( R \equiv 12H^2 + 6H + 6\sigma^2 + 2^2K = 0 \), provided \( D \) is finite or \( \rho_m = 0 \), \( R = R_0 \), with \( R_0 \) a constant, provided \( D \to \infty \). This invariant set contains vacuum (Minkowski) and static models.
Another invariant set appears in the case of radiation background \( (n = 2, w = 1/3) \), namely that of \( x = 0 \).

Finally, we have identified the invariant set \( x^2 + y + \Sigma^2 = 1 \), which contains cosmological solutions without standard matter.

### 3.3. Local analysis of the dynamical system

The system (46)-(49) admits two circles of critical points, given by \( C_\epsilon : \Sigma^2 + Q^2 - 2Q = 0, \ x + Q = \epsilon, \ y = 0 \) (we use the notation \( \epsilon = \pm 1 \)). They exist for \( n \in [1, 3] \), they are located in the boundary of \( \Psi \) and they correspond to solutions in the full phase space, satisfying \( K = y = z = 0 \). In order to be more transparent, let us consider the parametrization

\[
C_\epsilon := \begin{cases}
Q = \epsilon + \sin u \\
\Sigma = \cos u \\
x = -\sin u.
\end{cases}
\quad u \in [0, 2\pi]
\] (52)

For all the critical points we calculate the eigenvalues of the perturbation matrix \( \Sigma \), which will determine their stability. These eigenvalues, evaluated at \( C_\epsilon \), are

\[
\{0, \ -2(n-1)(\cos u - 2\epsilon), \ -2(n-1)\left\{(n-2)\sin u - \epsilon(3-n)\right\}, \ 2(n-2)\sin u + 6\epsilon(n-1)\}.
\] (53)

For the values of \( u \) such that there exists only one zero eigenvalue, the curves are actually \('normally hyperbolic\)\(^4\), and thus we can analyze the stability by analyzing the sign of the real parts of the non-null eigenvalues [48]. Therefore, we deduce that

- **No part of \( C_+ (C_-) \) is a stable (unstable).** Thus, any part of \( C_+ (C_-) \) is either unstable (stable) or saddle.
- **For** \( -\frac{1}{3} < w \leq -\frac{1}{6} \), one branch of \( C_- \left( \frac{3(u-1)}{2n-1} < \sin(u) \leq 1 \right) \) is stable, and one branch of \( C_+ \left( \frac{3(u-1)}{2n-1} < \sin(u) \leq 1 \right) \) is unstable.
- **For** \( -\frac{1}{6} < w < \frac{1}{3} \), the whole \( C_- \) is stable and the whole \( C_+ \) is unstable.
- **For** \( \frac{1}{3} \leq w \leq 1 \), one branch of \( C_- \left( \frac{3(u-1)}{2n-1} < \sin(u) \leq 1 \right) \) is stable, and one branch of \( C_+ \left( \frac{3(u-1)}{2n-1} < \sin(u) \leq 1 \right) \) is unstable.

Note that amongst the curves \( C_\epsilon \), we can select the representative critical points labeled by \( N_\epsilon := (Q = 0, \ \Sigma = 0, \ x = \epsilon, \ y = 0) \) and \( L_\epsilon := (Q = 2\epsilon, \ \Sigma = 0, x = -\epsilon, \ y = 0) \) described in [36]. In our notation: \( L_- = C_{-|\epsilon=\pi/2}, N_+ = C_{+|\epsilon=\pi/2} \) and \( N_- = C_{-|\epsilon=\pi/2}, L_+ = C_{+|\epsilon=\pi/2} \). Moreover, and contrary to the investigation of [36], we obtain four new critical points, which are a pure result of the anisotropy. They are labeled by \( P_1^+ := (Q = \epsilon, \ \Sigma = -\epsilon, \ x = 0, \ y = 0) \) and \( P_2^+ := (Q = \epsilon, \ \Sigma = \epsilon, \ x = 0, \ y = 0) \), and obviously \( P_1^- := C_{-|\epsilon=0}, P_2^- := C_{+|\epsilon=\pi} \) and \( P_1^+ := C_{+|\epsilon=0}, P_2^+ := C_{-|\epsilon=\pi} \). Thus, it is easy to see that

- **For** \( -\frac{1}{3} < w < \frac{1}{3} \), \( L_\epsilon \) is unstable and \( L_\epsilon \) is stable. The critical point \( N_\epsilon (N_-) \) is always unstable (stable) except in the case \( w = -\frac{1}{3} \). These results match with the results obtained in [36].
- **The critical points** \( P_1^\epsilon \) and \( P_2^\epsilon \) always exist. They are non-hyperbolic, presenting a one-dimensional center manifold tangent to the line \( x + Q = 0 \). \( P_{1,2}(P_{1,2}^-) \) have a 3D stable (unstable) manifold provided \( -\frac{1}{3} < w < 1 \).

\(^4\) Since we are dealing with curves of critical points, every such point has necessarily at least one eigenvalue with zero real part. ‘Normally hyperbolic’ means that the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the curve [48].
Strictly speaking, the system (46)–(49) admits two more critical points with coordinates $C_+$, $C_-$, $L_+$, $L_-$, $P^+_1$, and $P^-_1$ of the cosmological system (we use $\epsilon = \pm 1$). $u$ varies in $[0, 2\pi]$. For more details on the stability of $C_+$, see the text.

| Critical point/curve | $Q$ | $\Sigma$ | $x$ | $y$ | Existence | Stability |
|----------------------|-----|---------|-----|-----|-----------|----------|
| $N^+_1$ | 0 | 0 | 1 | 0 | Always Unstable |
| $N^-_1$ | 0 | 0 | $-1$ | 0 | Always Stable |
| $L_+$ | 2 | 0 | $-1$ | 0 | Always Unstable for $1.25 \leq n \leq 2.5$ |
| $L_-$ | $-2$ | 0 | 1 | 0 | Always Stable for $1.25 \leq n \leq 2.5$ |
| $P^+_1$ | 1 | 1 | 0 | 0 | $1 < n \leq 3$ Non-hyperbolic with 3D unstable manifold |
| $P^-_1$ | $-1$ | $-1$ | 0 | 0 | $1 < n \leq 3$ Non-hyperbolic with 3D stable manifold |
| $P^+_2$ | 1 | $-1$ | 0 | 0 | $1 < n \leq 3$ Non-hyperbolic with 3D unstable manifold |
| $P^-_2$ | $-1$ | 1 | 0 | 0 | $1 < n \leq 3$ Non-hyperbolic with 3D stable manifold |
| $C_+$ | $1 + \sin u$ | $\cos u - \sin u$ | 0 | Always Unstable for $1.25 \leq n \leq 2.5$ |
| $C_-$ | $-1 + \sin u$ | $\cos u - \sin u$ | 0 | Always Stable for $1.25 \leq n \leq 2.5$ |

The curves of critical points $C_+$, and the representative critical points $N_1$, $L_1$, $P^+_1$ and $P^-_1$, are enumerated in table 1.

Until now we have extracted and analyzed the stability of the curves of critical points $C_+$ of the system (46)–(49), and their representative critical points. However, the system (46)–(49) admits additionally ten isolated critical points enumerated in table 2, where we also present the necessary conditions for their existence.\(^5\)

The eigenvalues of the Jacobian matrix are presented in appendix A and thus here we provide each point’s type.

- The two critical points $N_1$ exist for $-\frac{1}{3} \leq w \leq 1$. The critical point $A_1$ ($A_-$) is a sink, that is, a stable one (source, that is, an unstable one) provided $w_+ < w \leq 1$, where $w_+ = \frac{1}{3}(-2 + \sqrt{3}) \approx -0.09$. Otherwise they are saddle points. Equivalently, we can express the stability intervals in terms of $n$, using relation (34). Thus, the aforementioned interval becomes $n_+ < n \leq 3$, where $n_+ = 3(w_+ + 1)/2 \approx 1.37$.
- The critical points $B_1$ exist for $-\frac{1}{4} \leq w \leq \frac{1}{4}$. Thus, they are always saddle points having a two-dimensional stable manifold and a two-dimensional unstable manifold if $-\frac{1}{4} < w < \frac{1}{4}$.
- The critical points $P^+_1$ exist for $-\frac{1}{4} \leq w \leq w_+$. The are non-hyperbolic and due to the one-dimensional center manifold presented in appendix A, the stable (unstable) manifold of $P^+_1$ ($P^-_1$) is 3D for $-\frac{1}{3} < w < w_+$.
- The critical points $P^+_2$ exist for $w_+ \leq w \leq 1$. $P^+_2$ ($P^-_2$) has a 3D stable (unstable) manifold and a 1D unstable (stable) manifold provided $w_+ < w \leq 1$. Thus, they are always saddle points.
- The critical points $P^+_3$ exist for $\frac{2}{3} \leq w \leq 1$. They are non-hyperbolic, there exists a 2D center manifold and $P^+_3$ ($P^-_3$) has a 2D unstable (stable) manifold.

\(^5\) Strictly speaking, the system (46)–(49) admits two more critical points with coordinates $Q = \pm 2$, $\Sigma = \pm 1$, $x = 0$, $y = 0$. However, they are unphysical since they satisfy $|Q + x| > 1$ and thus $\square K = (\epsilon^2)^3 < 0$. 

11
Lastly, as we have mentioned, in this subsection we have investigated the system (46)–(49), leaving outside the decoupled equation (50). However, for completeness, we study equation (50), and especially the stability across the direction $E_1^1$, in appendix B.

3.4. Physical description of the solutions and connection with observables

Let us now present the formalism of obtaining the physical description of a critical point and also connecting with the basic observables relevant for a physical discussion. These will allow us to describe the cosmological behavior of each critical point, in the next section.

Firstly, around a critical point we obtain first-order expansions for $e_1^1$, $K^*$, $\rho_m^*$ and $R$ in terms of $r$, considering the versions of equations (22), (21) and (39), respectively, given by

\[
(e_1^1)' = -(n - 1)[Q^* - 2\Sigma^*]e_1^1,
\]

\[
(3^2 K)' = -2(n - 1)[Q^* + \Sigma^*]^2 K^*,
\]

\[
\rho_m' = -2n(n - 1)Q^*\rho_m,
\]

\[
R' = 2\Sigma^* R^*.
\]

### Table 2. The isolated critical points of the cosmological system.

We use the notation $n_s = \frac{1 + \sqrt{5}}{2} \approx 1.37$.

| Critical points | $Q$ | $\Sigma$ | $x$ | $y$ | Existence | Stability |
|----------------|-----|---------|-----|-----|-----------|-----------|
| $A_+$          | $\frac{2n-1}{3(n-1)}$ | 0 | $\frac{n-2}{3(n-1)}$ | $\frac{(2n-1)(4n-5)}{9(n-1)^2}$ | $1.25 \leq n \leq 3$ | Stable for $n_s < n < 3$ saddle for $1.25 \leq n \leq n_s$ |
| $A_-$          | $-\frac{2n-1}{3(n-1)}$ | 0 | $\frac{n-2}{3(n-1)}$ | $\frac{(2n-1)(4n-5)}{9(n-1)^2}$ | $1.25 \leq n \leq 3$ | Unstable for $n_s < n < 3$ saddle for $1.25 \leq n \leq n_s$ |
| $B_+$          | $\frac{1}{n^2}$ | 0 | $\frac{n-2}{n^2}$ | 0 | $1 < n \leq 2.5$ | Saddle |
| $B_-$          | $-\frac{1}{n^2}$ | 0 | $\frac{n-2}{n^2}$ | 0 | $1 < n \leq 2.5$ | Saddle |
| $P_s^+$        | $\frac{1}{n^2}$ | 0 | $\frac{n-2}{n^2}$ | $\frac{n-1}{n(n-2)^2}$ | $1 < n \leq n_s$ | Non-hyperbolic with 3D stable manifold |
| $P_s^-$        | $-\frac{1}{n^2}$ | 0 | $\frac{n-2}{n^2}$ | $\frac{n-1}{n(n-2)^2}$ | $1 < n \leq n_s$ | Non-hyperbolic with 3D unstable manifold |
| $P_s^+$        | $\frac{2n^2-3n+5}{3n^2-16n+10}$ | $-\frac{2n^2-2n-1}{3n^2-16n+10}$ | $\frac{3n-1}{3n^2-16n+10}$ | $\frac{9(n^2-16n+10)(n^2-8n+1)}{(7n^2-16n+10)^2}$ | $n_s \leq n \leq 3$ | Saddle with 3D stable manifold |
| $P_s^-$        | $-\frac{2n^2-3n+5}{3n^2-16n+10}$ | $\frac{2n^2-2n-1}{3n^2-16n+10}$ | $\frac{3n-1}{3n^2-16n+10}$ | $\frac{9(n^2-16n+10)(n^2-8n+1)}{(7n^2-16n+10)^2}$ | $n_s \leq n \leq 3$ | Saddle with 3D unstable manifold |
| $P_s^+$        | $\frac{1}{n^2}$ | $\frac{2n^2-3n+5}{n^2}$ | $\frac{n-2}{n^2}$ | 0 | $2.5 \leq n \leq 3$ | Non-hyperbolic, with a 2D center manifold |
| $P_s^-$        | $-\frac{1}{n^2}$ | $\frac{2n^2-3n+5}{n^2}$ | $\frac{n-2}{n^2}$ | 0 | $2.5 \leq n \leq 3$ | Non-hyperbolic, with a 2D center manifold |
where the star superscript denotes the evaluation at a specific critical point, and the prime denotes the derivative with respect to $\tau$. The last equation follows from the definition of $x$ given by (40). In general we can consider the case $x^* \neq 0$ and $y^* \neq 0$, since in the simple case of $x^* = 0$ and $y^* \neq 0$ we obtain $R = R_0$, with $R_0$ a constant, while in the case $y^* = 0$ we acquire $R = 0$.

In order to express the above-determined functions of $\tau$ in terms of the comoving time variable $t$, we invert the solution of

$$\frac{dt}{d\tau} = n - 1 \frac{D^*}{D^*}$$  \hspace{1cm} (55)

with $D^*$ being the first-order solution of

$$D' = D(n - 1) \Upsilon^*,$$  \hspace{1cm} (56)

where

$$\Upsilon^* = x^* - \Sigma^* + (Q^* + x^*)[-3x^{*2} + \Sigma^* x^* + (Q^* - 3 \Sigma^*)\Sigma^* + n(x^{*2} + \Sigma^{*2} + y^* - 1)].$$  \hspace{1cm} (57)

Solving equations (55), (56) (with initial conditions $D(0) = D_0$ and $t(0) = t_0$) we obtain

$$t(\tau) = \frac{1 - e^{(1 - n)\tau \Upsilon^*}}{D_0 \Upsilon^*} + t_0.$$  \hspace{1cm} (58)

Thus, inverting the last equation for $\tau$ and substituting in the solution of (54) with initial conditions $e_1(0) = e_1^{11}(0)$, $2K(0) = 2K_0$, $\rho_m(0) = \rho_{m0}$, $R(0) = R_0$, we acquire

$$e_1(t) = e_1^{11}(0)(D_0(t_0 - t) \Upsilon^* + 1) \frac{\Sigma^{*2}}{\Sigma^*},$$  

$$2K(t) = 2K_0(D_0(t_0 - t) \Upsilon^* + 1) \frac{\Sigma^{*2}}{\Sigma^*},$$  

$$\rho_m(t) = \rho_{m0}(D_0(t_0 - t) \Upsilon^* + 1) \frac{\Sigma^{*2}}{\Sigma^*},$$  

$$R(t) = R_0(D_0(t_0 - t) \Upsilon^* + 1) \frac{\Sigma^{*2}}{\Sigma^*}.\hspace{1cm} (59)$$

Finally, it can be shown that the length scale $\ell$ along the flow lines, defined in (5), can be expressed as [34]

$$\ell = \ell_0(\ell_1 - t \Upsilon^*)^{-\frac{Q^*}{\Sigma^*}},$$  \hspace{1cm} (60)

where $\ell_0 = \left[\left(e_1^{11}(0) \Sigma_0 \right)^{-1}\right]$ and $\ell_1 = D_0 \Upsilon^* + 1$. In summary, expressions (59) and (60) determine the solution, that is, the evolution of various quantities, at a critical point.

Let us now come to the observables. Using the above expressions, we can calculate the deceleration parameter $q$ defined as usual as [34]

$$q = -\frac{\ddot{\ell}}{(\dot{\ell})^2}. \hspace{1cm} (61)$$

Additionally, we can calculate the effective (total) equation-of-state parameter of the universe $w_{\text{eff}}$, defined conventionally as

$$w_{\text{eff}} \equiv \frac{P_{\text{tot}}}{\rho_{\text{tot}}} \equiv \frac{\frac{\rho_m}{\rho_{\text{tot}}} + P}{\frac{\rho_m}{\rho_{\text{tot}}} + \mu},$$  \hspace{1cm} (62)

where $P_{\text{tot}}$ and $\mu_{\text{tot}}$ are respectively the total isotropic pressure and the total energy density as they can be read from equation (18), where $P$ and $\mu$ are given by (16). Therefore, in terms of the auxiliary variables we have

$$q = \frac{\Sigma^2 - x(2Q + x) + 1}{Q^2} - \frac{ny}{(n - 1)Q^2} \hspace{1cm} (63)$$
Finally, the various density parameters defined in (23)–(26) in terms of the auxiliary variables straightforwardly read

\[
\begin{align*}
\Omega_k &= \frac{(Q + x)^2 - 1}{Q^2} \\
\Omega_m &= \frac{1 - x^2 - y - \Sigma^2}{Q^2} \\
\Omega_{\text{curv. fl}} &= \frac{y - 2xQ}{Q^2} \\
\Omega_\sigma &= \left(\frac{\Sigma}{Q}\right)^2.
\end{align*}
\] (65)

In conclusion, at each critical point we can calculate the values of the basic observables \(q\), \(w_{\text{eff}}\) and the various density parameters, and also calculate the specific physical solution, that is, obtain the behavior of \(\ell(t)\), \(\rho_m(t)\) and \(R(t)\).

Finally, it is interesting to note that in Kantowski–Sachs geometry one can easily handle isotropization. In particular, the geometry becomes isotropic if \(\sigma_x\) becomes zero, as can be seen from (1) and (9). Thus, critical points with \(\Sigma = 0\) (or more physically \(\Omega_\sigma = 0\)) correspond to Friedmann points, that is, to isotropic universes, and when such an isotropic point is an attractor we obtain asymptotic isotropization in the future [31].

4. Cosmological implications

In the previous sections we formulated \(f(R)\)-gravity in the case of the homogeneous and anisotropic Kantowski–Sachs geometry, focusing on the \(f(R) = R^n\) ansatz, and we performed a detailed phase-space analysis. Thus, in this section we discuss the physical implications of the mathematical results, focusing on the physical behavior and on observable quantities.

First of all, for each critical point of tables 1 and 2 we calculate the effective (total) equation-of-state parameter of the universe \(w_{\text{eff}}\) using (64), the deceleration parameter \(q\) using (63), and the values of the density parameters \(\Omega_k\), \(\Omega_m\), \(\Omega_{\text{curv. fl}}\) and \(\Omega_\sigma\) using (65). These results are presented in table 3.

Furthermore, for each critical point we use (59) and (60), in order to extract the behavior of the physically important quantities \(\ell(t)\), \(\rho_m(t)\) and \(R(t)\) at this critical point. As we have mentioned \(\ell(t)\) is the length scale along the flow lines and in the case of zero anisotropy (for instance in FRW cosmology) it is just the usual scale factor. Additionally, \(\rho_m(t)\) is the matter energy density and \(R(t)\) is the Ricci scalar. These solutions are presented in the last column of tables 4 and 5.

Finally, in the last column of tables 4 and 5 we also present the physical description of the corresponding solution, taking into account all the above information. In particular, since the auxiliary variable \(Q\) defined in (40) is the Hubble scalar divided by a positive constant, \(Q > 0\) corresponds to an expanding universe, while \(Q < 0\) to a contracting one. Furthermore, as usual, for an expanding universe \(q < 0\) corresponds to the accelerating expansion and \(q > 0\) to the decelerating expansion, while for a contracting universe \(q < 0\) corresponds to the decelerating contraction and \(q > 0\) to the accelerating contraction. Additionally, if \(w_{\text{eff}} < -1\) then the total equation-of-state parameter of the universe exhibits phantom behavior. Lastly, critical points with \(\Sigma = 0\) correspond to isotropic universes.
In the phase-space analysis of FRW $R^n$-gravity too, the authors have focused only on acceleration, without examining the total equation-of-state parameter of the universe, it is easy to see that if ones defines it and examines its features he will find phantom behavior in that case too. This is also the case in Bianchi I and Bianchi III $R^n$-gravity [31, 49], where the authors would have found the phantom behavior if they had calculated the total equation-of-state parameter. Therefore, we conclude that the phantom behavior is a result of the modification of gravity, as it has been discussed in detail in the literature [21–23]. Finally, in the case where the matter is dust ($w = 0$, that is, $n = 3/2$), $\rho_m(t)$ behaves like $\ell(t)^{-3}$ as expected, and this acts as a self-consistency test for our analysis. Additionally, note that in the special case $n = 2$, that is, when $w = 1/3$, i.e. when radiation dominates the universe, the aforementioned stable solution corresponds to a de Sitter expansion (this is not a new feature since de Sitter solutions are known to exist in Bianchi I and Bianchi III $R^n$-gravity [31]). This is of great significance since such a behavior can describe the inflationary epoch of the universe.

In the above critical point the isotropization has been achieved. Such late-time isotropic solutions, than can attract an initially anisotropic universe, are of significant cosmological interest and have been obtained and discussed in the literature [14, 31]. The acquisition of such a solution was one of the motives of this work, as of many works on anisotropic cosmologies, since, as we discussed in the introduction, it is the only robust approach in confronting isotropy of standard cosmology. The fact that this solution is accompanied by acceleration or phantom behavior makes it a very good candidate for the description of the observable universe.

| Critical points | $w_{eff}$ | q | $\Omega_k$ | $\Omega_m$ | $\Omega_{curv}B$ | $\Omega_\sigma$ |
|-----------------|-----------|---|------------|-------------|----------------|-------------|
| $A_0^\pm$      | $\frac{6}{5} - \frac{2n - 1}{5(2n - 1)}$ | 1 | 0 | 0 | 1 | 0 |
| $B_{1^n}$      | $\frac{2}{5}$ | 1 | 0 | 5 - 2n | 2(n - 2) | 0 |
| $P_{1^n}$      | $\frac{2}{5}$ | 2 | 0 | 0 | 0 | 1 |
| $P_{2^n}$      | $\frac{1}{5}$ | 0 | 0 | $2n + 2 + \frac{1}{2}$ | $2n - 1 - \frac{1}{2}$ | 0 |
| $C_{1^n}$      | $\frac{1}{2}$ | 2(n - 2) | 0 | 0 | 2(3 - n) | 2n - 5 |

Table 3. The values of the basic observables, namely the effective (total) equation-of-state parameter $w_{eff}$, the curvature density parameter $q$, the matter density parameter $\Omega_m$, the ‘curvature-fluid’ density parameter $\Omega_{curv}B$ and the shear density parameter $\Omega_\sigma$, defined in (63)–(65), at the isolated critical points and curves of critical points of the cosmological system. We display the information about the non-isolated critical points $P_{1^n}$, $P_{2^n}$ to emphasize they are solutions dominated by shear.
The behavior of $\ell(t)$ (length scale along the flow lines), of $\rho_m(t)$ (matter energy density) and of $R(t)$ (Ricci scalar) at the critical points of the cosmological system. We use the notations $s_1 = \frac{1}{\sqrt{10}} - \frac{3\sqrt{2}}{5}$, $s_2 = \frac{1}{\sqrt{10}} - \frac{3\sqrt{2}}{5}$, $s_3 = \frac{1}{\sqrt{10}} - \frac{3\sqrt{2}}{5}$, $a_n(t) = e^{-n\omega_0 \ell(t)}$, $n_a = \frac{1}{\sqrt{10}} \approx 1.37$ and $M_0 = \frac{1}{2} (5 + \sqrt{2}) \approx 2.40$.

| Critical points | $T^*$ | Solution/description |
|-----------------|-------|----------------------|
| $A_+$ | $\frac{n-1}{3(n-1)^2}$ | $\ell(t) = \begin{cases} \ell_0(\ell_1 - t^Y)^n & n \neq 2 \\ \ell_0(\ell_1 - t^Y)^{n-2} & n = 2 \end{cases}$, $\rho_m(t) = \rho_{m0} \left[ \frac{\ell}{\ell_0} \right]^{-2n}$, $R(t) = \begin{cases} \frac{R_0}{(\ell_1 - t^Y)^n} & n \neq 1.25 \\ R_0 & n = 1.25 \end{cases}$ |
| $A_-$ | $\frac{n-1}{3(n-1)^2}$ | $\ell(t) = \begin{cases} \ell_0(\ell_1 - t^Y)^n & n \neq 2 \\ \ell_0(\ell_1 - t^Y)^{n-2} & n = 2 \end{cases}$, $\rho_m(t) = \rho_{m0} \left[ \frac{\ell}{\ell_0} \right]^{-2n}$, $R(t) = \begin{cases} \frac{R_0}{(\ell_1 - t^Y)^n} & n \neq 1.25 \\ R_0 & n = 1.25 \end{cases}$ |
| $B_+$ | $\frac{2}{n-3}$ | $\ell(t) = \ell_0 \sqrt{\ell_1 - t^Y}$, $\rho_m(t) = \rho_{m0} (\ell_1 - t^Y)^{-n}$, $R(t) = 0$ |
| $B_-$ | $\frac{2}{n-3}$ | $\ell(t) = \ell_0 \sqrt{\ell_1 - t^Y}$, $\rho_m(t) = \rho_{m0} (\ell_1 - t^Y)^{-n}$, $R(t) = 0$ |
| $P_0^+$ | $\frac{1}{n-2}$ | $\ell(t) = a_1 + a_1 t$, $\rho_m(t) = \rho_{m0} (\ell_1 - t^Y)^{-n}$, $R(t) = \frac{R_0}{(\ell_1 - t^Y)^n}$ |
| $P_0^-$ | $\frac{1}{n-2}$ | $\ell(t) = a_1 + a_1 t$, $\rho_m(t) = \rho_{m0} (\ell_1 - t^Y)^{-n}$, $R(t) = \frac{R_0}{(\ell_1 - t^Y)^n}$ |
| $P_4^+$ | $\frac{3(n-2)}{5(n-1)^2 + 6(n-10)}$ | $\ell(t) = \begin{cases} \ell_0(\ell_1 - t^Y)^n & n \neq 2 \\ \ell_0(\ell_1 - t^Y)^{n-2} & n = 2 \end{cases}$, $\rho_m(t) = \rho_{m0} \left[ \frac{\ell}{\ell_0} \right]^{-2n}$, $R(t) = \begin{cases} \frac{R_0}{(\ell_1 - t^Y)^n} & n \neq 1.25 \\ R_0 & n = 1.25 \end{cases}$ |
| $P_4^-$ | $\frac{3(n-2)}{5(n-1)^2 + 6(n-10)}$ | $\ell(t) = \begin{cases} \ell_0(\ell_1 - t^Y)^n & n \neq 2 \\ \ell_0(\ell_1 - t^Y)^{n-2} & n = 2 \end{cases}$, $\rho_m(t) = \rho_{m0} \left[ \frac{\ell}{\ell_0} \right]^{-2n}$, $R(t) = \begin{cases} \frac{R_0}{(\ell_1 - t^Y)^n} & n \neq 1.25 \\ R_0 & n = 1.25 \end{cases}$ |
| $P_5^+$ | $\frac{(3n-2)}{n-2}$ | $\ell(t) = \ell_0 (\ell_1 - t^Y)^3$, $\rho_m(t) = \rho_{m0} \left( \frac{\ell}{\ell_0} \right)^{-2n}$, $R(t) = 0$ |
| $P_5^-$ | $\frac{(3n-2)}{n-2}$ | $\ell(t) = \ell_0 (\ell_1 - t^Y)^3$, $\rho_m(t) = \rho_{m0} \left( \frac{\ell}{\ell_0} \right)^{-2n}$, $R(t) = 0$ |
Table 5. Physical behavior of the solutions at the curves of critical points of the cosmological system. We use the notation $p_{\epsilon}(u) = \epsilon + \sin(u)$.  

| Critical points | $T^*$ | Solution/description |
|-----------------|-------|----------------------|
| $C_+$           | $[-3 + \sin(u)]$ | $\ell(t) = \ell_0(\ell_1 - t^{Y^*})^{\epsilon/\sin(u)}, \rho_m(t) = \rho_{m0} \left(\frac{\ell}{\ell_0}\right)^{-2\epsilon}, R(t) = 0$ Expanding. Decelerating. Total matter/energy mimics radiation. |
| $C_-$           | $[-3 + \sin(u)]$ | $\ell(t) = \ell_0(\ell_1 - t^{Y^*})^{\epsilon/\sin(u)}, \rho_m(t) = \rho_{m0} \left(\frac{\ell}{\ell_0}\right)^{-2\epsilon}, R(t) = 0$ Contracting. Accelerating. Total matter/energy mimics radiation. |

The critical point $A_-$ which corresponds to an isotropic contracting universe is not stable and thus it cannot be the late-time state of the universe. Similarly, the points $B_+$ and $B_-$, which correspond to isotropic expanding and contracting universes, respectively, and in which the total matter/energy mimics radiation, are saddle points and thus they cannot be the late-time solution for the universe too. The points $P_{5\epsilon}$, which correspond to decelerating expansion (for $\epsilon = +1$) and to accelerating contraction (for $\epsilon = -1$), are non-hyperbolic with a 2D center manifold, and thus, generically, the universe cannot be led to them.

The critical points $P_{1\epsilon}^-$ and $P_{2\epsilon}^-$ which correspond to accelerating contraction, in which the total matter/energy behaves like radiation, are non-hyperbolic critical points, but they do possess a 3D stable manifold. By an explicit and straightforward computation of their center manifolds [50] we deduce that for $2 < n < 3$ each center manifold is locally asymptotically stable (the equation governing the dynamics on the center manifold is a gradient-type differential equation with the potential function having a degenerate local minimum at the origin), and thus $P_{1\epsilon}^-$ and $P_{2\epsilon}^-$ are locally asymptotically stable [50] and can attract the universe at late times. In contrast, for $1 < n < 2$, $P_{1\epsilon}^-$ and $P_{2\epsilon}^-$ are locally asymptotically unstable (of saddle type). In summary, although $P_{1\epsilon}^-$ and $P_{2\epsilon}^-$ are not stable, they do have a significant probability to be a late-time state for the universe (this is realized for initial conditions on its stable manifold), or at least the universe can stay near these solutions for a long time before the dynamics remove it from them. On the other hand, the points $P_{1\epsilon}^+$ and $P_{2\epsilon}^+$, which correspond to decelerating expansion, possess a 3D unstable manifold and thus they cannot be a late-time solution of the universe.

The non-hyperbolic critical point with a 3D stable manifold is also the critical point $P_{4\epsilon}^+$, which corresponds to an asymptotically flat isotropic expansion with zero acceleration, and thus it also has a large probability to be a late-time state of the universe. However, note that this solution corresponds to zero acceleration, and thus $\ell(t)$ is a linear function of $t$. On the other hand, the critical point $P_{5\epsilon}^-$ (isotropic, contracting with zero acceleration) possesses a 3D unstable manifold and thus it cannot attract the universe at late times.

The critical point $P_{1\epsilon}^-$ is saddle with a 3D stable manifold, and thus it has a large probability to be a late-time state of the universe. It corresponds to a non-flat, accelerating expansion for $1.37 \lesssim n < 3$, and furthermore in the case $2 < n < 3$ it exhibits phantom behavior. Finally, for $n = 2$, that is, for $w = 1/3$, it corresponds to a de Sitter expansion. On the other hand $P_{4\epsilon}^-$ (decelerating contraction) is highly unstable and therefore it cannot attract the universe at late times.

We mention here that the critical points $P_{1\epsilon}^+, P_{2\epsilon}^+, P_{3\epsilon}^+, P_{4\epsilon}^+$ and $P_{5\epsilon}^-$ are not present in isotropic (FRW) $R^n$-gravity as compared with [36]. They arise as a pure result of the anisotropy, and this
Figure 1. Projection of the phase space on the invariant set $\gamma = 0$, in the case of dust matter ($w = 0, n = 3/2$). The critical points $Q_{\pm}$ have coordinates $Q_{\pm} := (Q = e/\sqrt{3}, \Sigma = 2\sqrt{2}e/3, -e/3)$ and $Q_{\pm} := (Q = e/3, \Sigma = \sqrt{5}e/3, 2e/3)$. All the points in the circle $C_+$ are unstable whereas all the points in the circle $C_-$ are stable. The heteroclinic sequences reveal the possible transition from expansion to contraction and vice versa (see the text).

shows that the much more complicated structure of anisotropic geometries leads to radically different cosmological behaviors compared to the simple isotropic scenarios.

Additionally, we have to analyze the behavior of the curves of critical points $C_c$. The points $C_-$, which correspond to accelerating contraction, are stable if $1.25 \leq n \leq 2.5$, and thus they can be late-time solutions of the universe. On the other hand, $C_+$, which correspond to decelerating expansion, are unstable and thus they cannot attract the universe at late times.

Let us end the physical discussion by referring to static solutions, in order to compare to the FRW case of [36]. From the cosmological point of view, static solutions possess $\dot{\ell}(t) = \text{const}$, that is, $\dot{\ell}(t) = 0$ and $\ddot{\ell}(t) = 0$, in order to obtain $H(t) = 0$ and $\dot{H}(t) = 0$ (note that one needs both conditions, since in a cosmological bounce or turnaround, that is, when a universe changes from contracting to expanding or vice versa, $\dot{\ell}(t)$ is zero instantly, but then it again becomes positive or negative). Thus, using our auxiliary variable $Q$ defined in (40), static solutions should have $Q = 0$ and $\dot{Q} = 0$. In conclusion, since in all the aforementioned critical points $\dot{Q} = 0$ by definition, static solutions are just those with $Q = 0$. At this point there is another important difference compared to the isotropic case of [36]. In particular, in FRW geometry, the Ricci scalar is $R = 6(H + 2H^2)$, and therefore static solutions have $R = 0$ (however, the inverse is not true, that is, not all solutions with $R = 0$ correspond to static ones, since one can have $R = 0$ but with $H$ and $\dot{H}$ non-zero). On the other hand, in the anisotropic case $R$ is given by (28), i.e. $R = 12H^2 + 6\sigma^2 + 6H + 2\dot{K}$, that is, we also have the presence of additional terms, and therefore static solutions do not correspond to $R = 0$ unless $\sigma$ and $\dot{K}$ are also zero, which is not fulfilled in general. Thus, in the anisotropic case one cannot use $R$, or equivalently the auxiliary variable $\gamma$ defined in (40), in order to straightway determine the static solutions, in contrast to the isotropic case [36] where the authors use $\gamma = 0$ for such a determination.

Now, in order to present the aforementioned results in a more transparent way, we perform a numerical elaboration of our cosmological system, using a seventh- to eighth-order
continuous Runge–Kutta method with absolute error $10^{-8}$ and relative error $10^{-4}$ [51]. In figure 1 we depict some orbits in the invariant set $y = 0$, in the case of dust matter ($w = 0, n = 3/2$). As we observe, we have the appearance of four critical points with coordinates $Q_{1}^{\epsilon} := (Q = 4\epsilon/3, \Sigma = 2\sqrt{2}\epsilon/3, -\epsilon/3)$ and $Q_{2}^{\epsilon} := (Q = \epsilon/3, \Sigma = \sqrt{2}\epsilon/3, 2\epsilon/3)$, located in the invariant curves $C^{\epsilon}$. All the points in the circle $C_{\epsilon}$ are unstable whereas all the points in the circle $C_{-\epsilon}$ are stable. Note that the critical points $P_{1}^{\epsilon}$ coincide with $N_{\epsilon}$. In the figure we also display heteroclinic sequences [38] of types

$$Q_{1}^{+} \rightarrow \begin{cases} B^{-} \rightarrow L_{-} \\ B^{+} \rightarrow P_{1}^{-} \\
Q_{1}^{-} \end{cases} \quad Q_{2}^{+} \rightarrow Q_{1}^{-} \quad Q_{2}^{-} \rightarrow P_{2}^{-} \quad P_{1}^{+} \rightarrow P_{1}^{-} \quad P_{2}^{+} \rightarrow P_{2}^{-} \quad Q_{2}^{-} \rightarrow L_{-}. \quad (66)$$

Thus, we can have evolutions in which the + branch and – branch are connected, that is, we can have the transition from expansion to contraction and vice versa. This is just a cosmological turnaround and a cosmological bounce, and their realization in the present scenario reveals the capabilities of the model. It is interesting to note that such behaviors can be realized in FRW $R^{n}$-gravity [40, 52]; however, they are impossible in general relativity Kantowsky–Sachs cosmology [45]. Therefore, we conclude that they are a result of the $R^{n}$ gravitational sector and not of the anisotropy.

In figure 2 we display some orbits in the case of radiation ($w = 1/3, n = 2$), where as we mentioned in subsection 3.2 the invariant set $x = 0$ appears. Particularly, we observe the existence of heteroclinic sequences of types

$$A_{-} \rightarrow \begin{cases} P_{2}^{+} \rightarrow P_{4}^{+} \rightarrow A_{+} \\
B^{-} \rightarrow A_{+} \end{cases} \quad B_{+} \rightarrow A_{+} \quad P_{1}^{-} \rightarrow P_{4}^{+} \rightarrow P_{2}^{+} \quad P_{2}^{-} \rightarrow P_{4}^{+} \rightarrow P_{1}^{+} \quad P_{1}^{+} \rightarrow P_{2}^{-} \quad P_{2}^{+} \rightarrow P_{1}^{-}. \quad (67)$$

revealing the realization of a cosmological bounce or a cosmological turnaround. Similarly to the isotropic case (see figure 5 of [36]) there is one orbit of type $B_{+} \rightarrow A_{+}$ and one of type $A_{-} \rightarrow B_{-}$. However, in the present case we have the additional existence of a heteroclinic sequence of type $A_{-} \rightarrow P_{1}^{-} \rightarrow P_{4}^{+} \rightarrow P_{1}^{-} \rightarrow A_{+}$, corresponding to the transition from the collapsing AdS to expanding dS phase, that is, we obtain a cosmological bounce followed by a de Sitter expansion, which could describe the inflationary stage. This significant behavior is a pure result of the anisotropy and reveals the capabilities of the scenario. Lastly, note the very interesting possibility of the eternal transition $P_{1}^{-} \rightarrow P_{2}^{+} \rightarrow P_{1}^{-} \rightarrow P_{2}^{+} \cdots$ which is just the realization of cyclic cosmology [53]. Bouncing solutions are found to exist both in FRW $R^{n}$-gravity [54] and in the Bianchi I and Bianchi III $R^{n}$-gravity [31] (see also [55]), and
Figure 2. Projection of the phase space on the invariant set $x = 0$, in the case of radiation ($w = 1/3, n = 2$). There is one orbit of type $B_+ \rightarrow A_+$ and one of type $A_- \rightarrow B_-$. The existence of a heteroclinic sequence of type $A_- \rightarrow P_{-4} \rightarrow P_{+4} \rightarrow A_+$ corresponds to a cosmological bounce.

Figure 3. Invariant set $x = 0, y = 0$ in the case of radiation ($w = 1/3, n = 2$). The shaded region corresponds to the unphysical portion of the phase plane. $P_{-2}$ ($P_{+2}$) is the local future (past) attractor, while $B_+$, $B_-$ and $P_{-1}, P_{+1}$ are saddle points.

thus they arise from the $R^n$ gravitational sector. However, in the present Kantowsky–Sachs geometry cyclicity seems to be realized relatively easily (however without being accompanied by isotropization), that is, without fine-tuning the model parameters, which is an advantage of the scenario.

Finally, in figure 3 we have extracted orbits located at $x = y = 0$, in the case of radiation. The shaded region corresponds to the unphysical portion of the phase plane (note however that
this region is not invariant, since an open set of orbits enter/abandon the unphysical boundary, and thus such evolutions have to be excluded too). As we observe, in this figure, the last two heteroclinic sequences of (67) are displayed.

Let us make a comment here by referring to the cosmological epoch sequence. As we mentioned in subsection 2.4, in the scenario at hand we can obtain the transition from the matter-dominated era to the accelerated one. However, note that the dynamical system analysis can only give analytical results relating to the late-time states of the universe. The precise evolution of the universe toward such late-time attractors depends on the initial conditions, and it can only be investigated through a detailed numerical elaboration similar to the partial one that we performed in order to produce the aforementioned three figures. Thus, by suitably determining the initial conditions, one can obtain universes that start with a de Sitter expansion, then transit to the matter-dominated era, and finally falling into the late time accelerating solution. The detailed examination of such behaviors, along with the investigation of the basins of attraction of the various evolutions in terms of the initial conditions, and the estimation of the measures of the corresponding sub-spaces of the phase space lie outside of this work and are left for a future project.

We conclude this section by discussing the cosmological evolution in the special case where $E_{11}^1 = 0$ or $K = 0$, since in this case the Kantowski–Sachs metric (in a comoving frame)

$$\text{d}s^2 = \left(\frac{n-1}{D^2}\right)\text{d}r^2 + \frac{1}{D^2(E_{11}^1)}\text{d}r^2 + \frac{1}{3D^2K}[\text{d}\vartheta^2 + \sin^2\varphi\text{d}\varphi^2]$$  \hspace{1cm} (68)

is singular. Although these points are unphysical their neighboring solutions could have a physical meaning, which can be extracted by obtaining first-order evolution rates valid in a small neighborhood of the critical points of the system. In particular, we first evaluate the perturbation matrix $\Xi$ at the critical point of interest, diagonalize it and obtain orders of magnitude for linear combinations of the vector components $(E_{11}^1, Q, \Sigma, x, y)^T$. Thus, the desired expansion can be obtained by taking the inverse linear transformation, and finally we preserve only the leading-order terms as $\tau \to -\infty$ or as $\tau \to \infty$. Although this procedure is straightforward in the case of dust matter ($w = 0, n = 3/2$) and radiation ($w = 1/3, n = 2$) we do not present it explicitly since we desire to remain in the general case of $E_{11}^1 \neq 0$ and $K \neq 0$.

5. Conclusions

In this work we constructed general anisotropic cosmological scenarios where the gravitational sector belongs to the extended $f(R)$ type, and we focused on Kantowski–Sachs geometries in the case of $R^n$-gravity. We performed a detailed phase-space analysis, extracting the late-time solutions, and we connected the mathematical results with physical behaviors and observables. As we saw, the universe at late times can result into a state of accelerating expansion, and additionally, for a particular $n$-range ($2 < n < 3$) it exhibits phantom behavior. Additionally, the universe has been isotropized, independently of the anisotropy degree of the initial conditions, and it asymptotically becomes flat. The fact that such features are in agreement with observations [1, 16] is a significant advantage of the model. Moreover, in the case of radiation ($n = 2, w = 1/3$) the aforementioned stable solution corresponds to a de Sitter expansion, and it can also describe the inflationary epoch of the universe.

Note that at first sight the above behavior could be ascribed to the cosmic no-hair theorem [56], which states that a solution of the cosmological equations, with a positive cosmological constant and under the perfect-fluid assumption for matter, converges to the de Sitter solution.
at late times. However, we mention that such a theorem holds for matter fluids less stiff than radiation but more importantly it has been elaborated for general relativity \cite{57}, without a robust extension to higher order gravitational theories \cite{58}. In our work we extracted our results without relying at all on the cosmic no-hair theorem, which is a significant advantage of the analysis.

Apart from the above behavior, in the scenario at hand the universe has a large probability to remain in a phase of (isotropic or anisotropic) decelerating expansion for a long time, before it will be attracted by the above global attractor at late times, and this acts as an additional advantage of the model, since it is in agreement with the observed cosmological behavior. However, the precise duration of such a transient phase, unlike the attractor behavior, does depend on the initial conditions of the universe, which have to be suitably determined in order to lead to a deceleration duration of the order of $10^9–10^{10}$ years, before the universe passes through the accelerating phase. Such an analysis can only arise through an explicit numerical elaboration, that is, beyond the analytical, dynamical-system treatment which is where this work focuses.

The Kantowski–Sachs anisotropic $R^n$-gravity can also lead to contracting solutions, either accelerating or decelerating, which are not globally stable. Thus, the universe can remain near these states for a long time, before the dynamics remove it toward the above expanding, accelerating, late-time attractors. The duration of these transient phases depends on the specific initial conditions.

One of the most interesting behaviors is the possibility of the realization of the transition between expanding and contracting solutions during the evolution. That is, the scenario at hand can exhibit the cosmological bounce or turnaround. Additionally, there can also appear an eternal transition between expanding and contracting phases, that is, we can obtain cyclic cosmology. These features can be of great significance for cosmology, since they are desirable in order for a model to be free of past or future singularities.

Before closing, let us make some comments concerning the use of observational data in order to constrain the present scenario. In particular, equation \eqref{31}, along with the $R^n$-ansatz, relation \eqref{34} and the definitions of the density parameters \eqref{23}–\eqref{26}, can be straightforwardly written in the form used for observational fitting \cite{59}. Thus, one can use observational data from Type Ia Supernovae (SNIa), baryon acoustic oscillations (BAO), and cosmic microwave background (CMB), along with requirements of Big Bang nucleosynthesis (BBN), to constrain the model parameter $n$. Additionally, one can constrain the initial allowed anisotropy value $\sigma_+$ (or $\Omega_{\sigma}$) through its present value $\Omega_{\sigma 0}$ and the shear evolution \eqref{30}. Such a procedure is necessary for every cosmological paradigm and can significantly enlighten the scenario at hand. However, since in this work we focused on the dynamical features, the detailed observational elaboration is left for a separate project.

In summary, anisotropic $R^n$-gravity has a very rich cosmological behavior and a large variety of evolutions and late-time solutions, compatible with observations. The much more complicated structure of anisotropic geometries leads to radically different implications compared to the simple isotropic scenarios. These features indicate that anisotropic universes governed by modified gravity can be a candidate for the description of nature, and deserve further investigation.

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Appendix A. Eigenvalues of the perturbation matrix $\Xi$ for the critical points

The system (46)–(49) admits ten isolated critical points presented in table 2. Here we provide the eigenvalues of the perturbation matrix $\Xi$ calculated at each critical point. We also provide the eigenvalues of the perturbation matrix at the special points $P_1^\epsilon$ and $P_2^\epsilon$ located at the invariant circles $C_\epsilon$.

For the two critical points $A_\epsilon$ the associated eigenvalues read

$\{-\sqrt{\frac{2(3w+1)}{3(w+1)}}, -\sqrt{\frac{(3w-2)(3w+1)}{3(w-1)}}[\times 2], -\sqrt{\frac{2(w+1)}{w-1}}\},$

with $[\times 2]$ denoting multiplicity 2.

For the critical points $B_\epsilon$ the associated eigenvalues are

$\{-\sqrt{\frac{2(3w+1)}{3(w+1)}}, \sqrt{\frac{(3w-2)(3w+1)}{3(w-1)}}[\times 2], -\sqrt{\frac{2(w+1)}{w-1}}\},$

For the critical points $P_1^\epsilon$ the associated eigenvalues read

$\{0, \epsilon(3w+1), 3\epsilon(3w+1), -\frac{3}{2}\epsilon(w-1)(3w+1)\},$

while for $P_2^\epsilon$ they write

$\{0, 3\epsilon(3w+1)[\times 2], -\frac{3}{2}(w-1)(3w+1)\}.$

For the critical points $P_3^\epsilon$ the associated eigenvalues are

$\{0, -\epsilon(3w+1),$$\quad -\frac{9w^2 + \sqrt{3}3w + 1\sqrt{9w^2 + 31w + 3} - 1}{2(3w - 1)}$$\quad -\frac{9w^2 - \sqrt{3}3w + 1\sqrt{9w^2 + 31w + 3} - 1}{2(3w - 1)}\},$

and thus there exists a one-dimensional center manifold tangent to the line

$x = -\frac{1}{2}Q(3w + 1), \quad y = \epsilon \frac{Q(w-1)(3w+1)^2}{(w+1)(3w-1)},$

$\Sigma = -\frac{1}{2}Q(3w - 1), \quad Q \in [-2, 2].$

For the critical points $P_4^\epsilon$ the associated eigenvalues are

$\{-\frac{6(3w+1)(9w^2 + 3w + 1)}{63w^2 + 30w + 7},$$\quad -\frac{3(3w+1)(9w^2 + 3w + \sqrt{9w^2 + 3w + 1}\sqrt{45w^2 + 51w + 5} + 1)}{63w^2 + 30w + 7},$$\quad -\frac{3(3w+1)(9w^2 + 12w + 1)}{63w^2 + 30w + 7},$$\quad -\frac{3(3w+1)(9w^2 + 3w - \sqrt{9w^2 + 3w + 1}\sqrt{45w^2 + 51w + 5} + 1)}{63w^2 + 30w + 7}\}.$
Finally, for the critical points $P^c_5$ the associated eigenvalues read
\[ \left\{ 0, 0, -\epsilon \frac{2(3w + 1)(-3w + \sqrt{3w^2 + 2} + 1)}{3w - 1}, 6\epsilon(w + 1) \right \}. \]

Appendix B. The direction $E^1_1$

In subsection 3.3 we have investigated the system (46)–(49), leaving outside the decoupled equation (50). However, this equation provides information of how Kantowsky–Sachs $f(R)$ models are related to more general anisotropic geometries. For instance from (50) we see that $E^1_1 = 0$ is an invariant set of (46)–(50). This set is closely related with the so-called silent boundary one [60–62].

Note that, as current investigations suggest, in $G_0$-cosmologies, that is, in cosmologies where there is no symmetry, both past and future attractors belong to the silent boundary [61, 64].

In this work we do not focus on how Kantowsky–Sachs $f(R)$ are related to more general anisotropic models, and thus, we do not elaborate completely equation (50). However, we can still obtain the corresponding significant physical information, since the stability along the $E^1_1$ direction is determined by calculating the sign of $\partial (E^1_1)/\partial E^1_1$, which coincides with the eigenvalue associated with the eigenvector $E^1_1$. Thus, if it is negative then the small perturbations in the $E^1_1$ direction decay, while if it is positive, they enhance.

From the sign of $\partial (E^1_1)/\partial E^1_1$ it follows that $A_+$ ($A_-$) is stable (unstable) to small perturbations in the $E^1_1$ direction provided $w_+ < w \leq 1$, otherwise it is unstable (stable).

From the sign of $\partial (E^1_1)/\partial E^1_1$ it follows that $B_+$ ($B_-$) is unstable (stable) to small perturbations in $E^1_1$.

From the signs of $\partial (E^1_1)/\partial E^1_1$ it follows that $P^e_1$ ($P^e_1$) is stable (unstable) to small perturbations in the $E^1_1$ direction provided $-\frac{3}{2} < w < 1$, otherwise it is unstable (stable). Since $\partial (E^1_1)/\partial E^1_1$ vanishes at $P^e_2$ and at $P^e_3$, it follows that they are neutrally stable to small perturbations in the $E^1_1$ direction.

By analyzing the sign of $\partial (E^1_1)/\partial E^1_1$ we deduce that $P^e_4$ ($P^e_5$) is stable (unstable) to perturbations in the $E^1_1$ direction if $w_+ < w \leq 1$.

From the sign of $\partial (E^1_1)/\partial E^1_1$ it follows that $P^e_4$ ($P^e_5$), whenever exists, is always unstable (stable) to small perturbations in the $E^1_1$ direction.

Finally, note that since
\[ \frac{\partial (E^1_1)}{\partial E^1_1}|_{C_*} = 2(n - 1)[\epsilon + \cos(u)], \]
for $1 < n \leq 3$, $C_*$ is always unstable (except for $u = \pi$) to small perturbations in the $E^1_1$ direction, while $C_+$ is always stable (except for $u \in [0, 2\pi]$).

Lastly, it is interesting to mention that all the critical points of the reduced system, except $P^e_{2,3}$, belong to the ‘silent boundary’ in the full phase space, that is each one has a representative
in $\Psi \times \mathbb{R}$ with $E_i^1 = 0$. In particular, the critical points $A_\epsilon$, $B_\epsilon$ correspond to isotropic silent singularities of the full five-dimensional phase space and $P_5^\epsilon$ correspond to an anisotropic one.

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