The Quantum Arnold Transformation for the
damped harmonic oscillator: from the
Caldirola-Kanai model toward the Bateman model

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Abstract. Using a quantum version of the Arnold transformation of classical mechanics, all quantum dynamical systems whose classical equations of motion are non-homogeneous linear second-order ordinary differential equations (LSODE), including systems with friction linear in velocity such as the damped harmonic oscillator, can be related to the quantum free-particle dynamical system. This implies that symmetries and simple computations in the free particle can be exported to the LSODE-system. The quantum Arnold transformation is given explicitly for the damped harmonic oscillator, and an algebraic connection between the Caldirola-Kanai model for the damped harmonic oscillator and the Bateman system will be sketched out.

1. Introduction
The interest in dissipative systems at the quantum level has remained constant since the early days of Quantum Mechanics. The difficulties in describing damping, which intuitively could be understood as a mesoscopic property, within the fundamental quantum framework, have motivated a huge amount of papers. In particular, the quantum damped harmonic oscillator, frequently described by the Caldirola-Kanai equation [1, 2], has attracted much attention, as it could be considered one of the simplest and paradigmatic examples of dissipative system.

The model for the quantum damped harmonic oscillator by Caldirola and Kanai (C-K), which includes a time-dependent Hamiltonian, has been considered to have some flaws. For instance, it is claimed that uncertainty relations are not preserved under time evolution and could eventually be violated [3, 4]. However, this inconsistency seems to be associated with a confusion between canonical momentum and “physical” momentum [5]. It should be understood that the C-K model accounts for a “canonical-level” description of the damped system, which in fact can be mapped into a “physical-level” one [6].

Coherent states for a generalized C-K model were calculated in [7] by finding creation and annihilation operators, built out of operators which commute with the Schrödinger equation. The corresponding number operator turns out to be an auxiliary, conserved operator, obviously different from the time-dependent Hamiltonian. This paper also defined the so-called loss energy

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states for the damped harmonic oscillator. The report by Dekker [8] provides a historical overview of some relevant results.

In a purely classical context, the symmetries of the equation of the damped harmonic oscillator with time-dependent parameters were found in [9]. Two comprehensive articles, [10, 11], are of special interest. In those papers the authors found, for the damped harmonic oscillator, finite-dimensional point symmetry groups for the corresponding Lagrangian (the un-extended Schrödinger group [12, 13, 14, 15]) and the equations of motion (\(SL(3, \mathbb{R})\)) respectively, and an infinite contact one for the set of trajectories of the classical equation. They singled out a “non-conventional” Hamiltonian from those generators of the symmetry, recovering some results from [7]. Then, they concluded that the damped harmonic oscillator should not be claimed to be dissipative at all at the quantum level, as this true, “non-conventional” Hamiltonian is conserved, and should be related to an oscillator with variable frequency.

There exists another interesting approach to the study of the classical damped harmonic oscillator, based on the observation that its classical equation of motion is a special case of the set of linear second-order ordinary differential equations (LSODE for short). In Classical Mechanics the family of solutions of a second-order differential equation corresponding to the motion of a given physical problem is sometimes related to that of a simpler system, considered as a toy model, in order to import from it simple general properties which could be hidden in the real problem. Both physical systems should share global properties of the solution manifold, such as topology and symplectic structure. The paradigmatic example is the transformation described by Arnold in [16], which brings any LSODE to the simplest form of the free Galilean particle equation. This transformation turns out to be extremely useful. In particular, it is possible to obtain the symmetry group of a particular instance of LSODE [9], in which the symmetries of the action of the corresponding system can be found as a subgroup [17].

Therefore, it is rather natural to generalize the Arnold transformation to the quantum level, to be denoted as Quantum Arnold Transformation (QAT), as much insight can be gained in the study of any system classically described by a LSODE and, in particular, the damped harmonic oscillator. This generalization was presented in [18] by the authors, and we shall provide here a short review on the subject\(^2\).

Besides the Caldirola-Kanai model, the Bateman dual system appears as an alternative description of dissipation in the damped harmonic oscillator. In his original paper [19], Bateman looked for a variational principle for equations of motion with a friction term linear in velocity, but he allowed the presence of extra equations. This trick effectively doubles the number of degrees of freedom, introducing a time-reversed version of the original damped harmonic oscillator, which acts as an energy reservoir and could be considered as an effective description of a thermal bath. The Hamiltonian that describes this system was rediscovered by Feschbach and Tikochinsky [20, 21, 22, 8] and the corresponding quantum theory was immediately analyzed.

Some issues regarding Bateman’s system arose. The Hamiltonian presents a set of complex eigenvalues of the energy (see [23] and references therein), and the vacuum of the theory seems to decay with time. This last feature was treated in [24], where Celeghini et al. suggested that the quantum theory of the dual system could find a more natural framework in quantum field theory. However, we understand that the ultimate reason is nevertheless the lack of a vacuum representation of the relevant group: there is no vacuum vanishing at all as there is no conventional vacuum [25, 26]: defining such a vacuum is an artifact of the usual Feschbach-Tikochinski construction [20, 21, 8]. On the other hand, in [23] the generalized eigenvectors corresponding to the complex eigenvalues are interpreted as resonant states, and the real spectrum is obtained (see also [25, 26]).

Bateman’s dual system is still frequently discussed [27]. Many authors have considered this

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\(^2\) Several partial generalizations of the classical Arnold transformation to the quantum theory can in fact be found in the literature (see [18] for references).
model as a good starting point for the formulation of the quantum theory of dissipation. One of the aims of this paper will be to show that the study of the symmetries of the Caldirola-Kanai model leads to the Bateman dual system, thus to be considered as a natural starting point for the study of quantum dissipation.

The paper is organized as follows. In Section 2 we review the classical Arnold transformation (CAT) as well as its quantum counterpart (QAT), and provide the particular expressions for the case of the Caldirola-Kanai model for the damped harmonic oscillator. Section 3 is devoted to the algebraic derivation of the Bateman dual system closing a Lie algebra for the conserved operators obtained through the QAT for the Caldirola-Kanai model together with the (non-conserved) Hamiltonian.

2. The Arnold transformation for the damped harmonic oscillator

The Caldirola-Kanai equation [1, 2], which attempts to describe the damped harmonic oscillator in one spatial dimension (with classical equation $\ddot{x} + \gamma \dot{x} + \omega^2 x = 0$), at the canonical level, can be written:

$$i\hbar \frac{\partial \phi}{\partial t} = \hat{H}_{\text{DHO}} \phi \equiv \frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 e^{\gamma t} \phi,$$

where $\omega$ is the given, constant frequency and $\gamma$ is the friction constant. Let $\mathcal{H}_{\text{FP}}$ be the Hilbert space of solutions of the free particle, time-dependent Schrödinger equation, and $\mathcal{H}_{\text{DHO}}$ the one corresponding to the damped harmonic oscillator. We shall denote by $\psi(\kappa, \tau) \in \mathcal{H}_{\text{FP}}$ the free particle solutions ($\kappa$ and $\tau$ denote the spatial and time variable for the free particle, respectively) and by $\phi(x, t) \in \mathcal{H}_{\text{DHO}}$ the damped harmonic oscillator ones. Then, the QAT [18], relating solutions of time-dependent Schrödinger equations for the free particle and the damped harmonic oscillator, or rather, its inverse, is given by:

$$\phi(x, t) = \hat{A}^{-1} \psi(\kappa, \tau) = \frac{1}{\sqrt{u_2(t)}} e^{\frac{i m}{\hbar} \int_0^t \frac{1}{u_2(t)} \frac{u_1(t)}{u_2(t)} x^2} \psi\left(\frac{x}{u_2(t)}, \frac{u_1(t)}{u_2(t)}\right).$$

This transformation is unitary, and its form is quite general [18]. For a general, homogeneous LSODE, the classical Arnold transformation (CAT) [16], present in (2), is explicitly

$$A: \mathbb{R} \times T \rightarrow \mathbb{R} \times \mathcal{T}$$

$$(x, t) \mapsto (\kappa, \tau) = A((x, t)) = \left(\frac{\dot{x}}{u_2(t)} \frac{u_1(t)}{u_2(t)}\right).$$

$\mathcal{T}$ and $T$ are open intervals of the real line containing $\tau = 0$ and $t = 0$, respectively, $u_1(t)$ and $u_2(t)$ are two independent solutions of the LSODE (here dots mean derivation with respect to $t$):

$$\ddot{x} + f(t) \dot{x} + \omega(t)^2 x = 0,$$

and $W(t)$ is the Wronskian $W(t) = u_1(t) u_2(t) - u_1(t) \dot{u}_2(t)$ of the two solutions. Applying the change of variables $A$ to a solution $x(t)$ of (4) on a given interval of time $T$, a solution $\kappa(\tau)$ of the classical equation of motion $\ddot{k} = 0$ on a interval $\mathcal{T}$ is obtained, where now dots mean derivation with respect to $\tau$.

We impose on $u_1$ and $u_2$ the condition that they preserve the identity of $\tau$ and $\kappa$, i.e., that $(\kappa, \tau)$ coincide with $(x, t)$ at an initial point $t_0$, arbitrarily taken to be $t_0 = 0$:

$$u_1(0) = 0, \quad u_2(0) = 1, \quad \dot{u}_1(0) = 1, \quad \dot{u}_2(0) = 0.$$

This fixes a unique form of $A$ for a given “target” LSODE-type physical system. However, the quantum Arnold transformation is still valid if solutions $u_1$ and $u_2$ do not satisfy (5) (see [18] for details).
For the case of the damped harmonic oscillator $\dot{f} = \gamma$ and $\omega(t) = \omega$, and the two independent solutions can be chosen (see [18] for details) as

$$u_1(t) = \frac{1}{\Omega} e^{-\frac{\gamma}{2} t} \sin \Omega t, \quad u_2(t) = e^{-\frac{\gamma}{2} t} \cos \Omega t + \frac{\gamma}{2\Omega} e^{-\frac{\gamma}{2} t} \sin \Omega t,$$

(6)

where, computing, $W(t) \equiv \dot{u}_1(t)u_2(t) - u_1(t)\dot{u}_2(t) = e^{-\gamma t}$, and

$$\Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}.$$  

(7)

Note that these solutions have good limit in the case of critical damping $\omega = \frac{\gamma}{2}$.

Then, with the change of variables given by (3), the classical equation of the free particle $\ddot{\kappa}(\tau) = 0$ transforms into the classical equation of the damped harmonic oscillator $\ddot{x}(t) + \gamma \dot{x}(t) + \omega^2 x(t) = 0$ up to a time-dependent global factor.

Coming back to the quantum level, the QAT can be diagrammatically represented as:

$$\begin{array}{ccc}
\mathcal{H}_{FP} & \xrightarrow{\hat{A}} & \mathcal{H}_{DHO} \\
\hat{U}^{FP} & & \hat{U}^{DHO} \\
\mathcal{H}_0 & \xleftarrow{\hat{U}^{DHO}} &
\end{array}$$

where $\hat{U}^{FP}$ and $\hat{U}^{DHO}$ stand for the evolution operators of the free particle and the damped harmonic oscillator, respectively, while $\mathcal{H}_0$ is the Hilbert space, either for the free particle or the damped harmonic oscillator, of solutions of their respective Schrödinger equations at $t = 0$. The fact that $\mathcal{H}_0$ is common to both systems is a consequence of imposing the conditions (5) (see [18] for the required modifications when these conditions are not satisfied).

Thanks to the commutativity of this diagram, and the unitarity of the operators appearing in it, we can map objects (wave functions, operators, expectation values, uncertainties) from one system to the other. In [18] we benefited from this fact for transporting the simplicity of the free particle to more involved systems, finding, for instance, an analytic expression for the evolution operator of the complicated systems (even with time-dependent Hamiltonians) in terms of the free particle evolution operator.

As already remarked, the basic symmetries of the free system are inherited by the LSODE-type system, as we are now able to transform the infinitesimal generators of translations (the Galilean momentum operator $\hat{\pi}$, corresponding to the classical conserved quantity ‘momentum’) and non-relativistic boosts (the position operator $\hat{\kappa}$, corresponding to the classical conserved quantity ‘initial position’). They are, explicitly,

$$\hat{\pi} = -\frac{i\hbar}{\partial \kappa}, \quad \hat{\kappa} = \kappa + \frac{i\hbar}{m\tau} \frac{\partial}{\partial \kappa},$$

(8)

(9)

that is, those basic, canonically commuting operators with constant expectation values, that respect the solutions of the free Schrödinger equation, have constant matrix elements (and constant expectation values in particular) and fall down to well defined, time-independent
operators in the Hilbert space of the free particle \(L^2(\mathbb{R})\). In general, these properties are satisfied whenever an operator \(\hat{O}(t)\) can be written as

\[
\hat{O}(t) = \hat{U}(t, t_0) \hat{O} \hat{U}^{-1}(t, t_0),
\]

where \(\hat{O}\) is \(\hat{O}(t_0)\) and \(\hat{U}(t, t_0)\) is the evolution operator satisfying the Schrödinger equation\(^3\). If the Hamiltonian is time-independent, as in the free particle case, time-evolution is a one-parameter group and then \(\hat{U}(\tau, \tau_0) = \hat{U}(\tau - \tau_0)\).

Let us apply the QAT (2) to (8) and (9). For the operator \(\hat{\pi}\) acting on \(\mathcal{H}_{FP}\) we have the corresponding operator \(\hat{P} = \hat{A}^{-1} \hat{\pi} \hat{A}\) on \(\mathcal{H}_{DHO}\). The action on functions \(\phi(x, t)\) can then be obtained as follows:

\[
\hat{P}\phi(x, t) = \hat{A}^{-1} \hat{\pi} \hat{A}\phi(x, t) = \hat{A}^{-1} \hat{\pi} A^* \left( \sqrt{\frac{m}{2\hbar}} e^{-\frac{\gamma t}{2\hbar} \frac{\hbar^2}{m} \frac{\hbar^2}{m} x^2} \phi(x, t) \right) = \\
= \frac{1}{\sqrt{\omega_2}} e^{\frac{\gamma t}{\hbar} \frac{\hbar^2}{m} \frac{\hbar^2}{m} x^2} A^{*-1} \left( -i\hbar^2 \frac{\partial}{\partial x} \right) \left( \sqrt{\frac{m}{2\hbar}} e^{-\frac{\gamma t}{2\hbar} \frac{\hbar^2}{m} \frac{\hbar^2}{m} x^2} \phi(x, t) \right) = \\
= \frac{1}{\sqrt{\omega_2}} e^{\frac{\gamma t}{\hbar} \frac{\hbar^2}{m} \frac{\hbar^2}{m} x^2} \left( -i\hbar \frac{\partial}{\partial x} \right) \left( \sqrt{\frac{m}{2\hbar}} e^{-\frac{\gamma t}{2\hbar} \frac{\hbar^2}{m} \frac{\hbar^2}{m} x^2} \phi(x, t) \right) = \\
= \left( -i\hbar \frac{\partial}{\partial x} - m \frac{\hbar^2}{2\hbar} x \right) \phi(x, t).
\]

We can perform the same computation for the position operator and then we have, explicitly:

\[
\hat{P} = -i\hbar e^{-\frac{\gamma t}{\hbar}} (\cos \Omega t + \frac{\gamma}{2\Omega} \sin \Omega t) \frac{\partial}{\partial x} + m \frac{\omega^2}{\Omega} e^{\frac{\gamma t}{\hbar}} \sin \Omega t \cdot x, \tag{12}
\]

\[
\hat{X} = e^{\frac{\gamma t}{\hbar}} (\cos \Omega t - \frac{\gamma}{2\Omega} \sin \Omega t) x + i\hbar e^{\frac{-\gamma t}{\hbar}} \sin \Omega t \frac{\partial}{\partial x}, \tag{13}
\]

thus providing the generators of the realization of the (centrally-extended) Heisenberg-Weyl symmetry on the damped harmonic oscillator.

One key observation is that \(\hat{H}_{DHO}\) does not make sense as an operator acting on the space of solutions of (1), while \(\hat{P}\) and \(\hat{X}\) do respect solutions. This can be proved by direct calculation or even obtaining the evolution operator \(\hat{U}(t, t_0)\) and using it to de-evolve these operators to check that they are actually conserved: they fall to well defined operators on the space \(\mathcal{H}_0\).

The fact that \(\hat{H}_{DHO}\) does not commute at different times \([\hat{H}_{DHO}(t_1), \hat{H}_{DHO}(t_2)] \neq 0\) makes its calculation trickier using conventional methods, but QAT can be used to obtain a simple, exact form [18]. The action of \(\hat{U}(t)\) on \(\hat{H}_{DHO}(t)\) shows that it does not fall down to the quotient by the time evolution generated by itself.

Let us stress that QAT can be useful to quickly perform some calculations [28, 29], avoiding tedious, direct evaluations which can become extremely involved in the system under study. For example, it can be used to compute the quantum propagator for any LSODE-type quantum system, following the idea of Takagi in [30] for the simple case of the harmonic oscillator, or even, as already mentioned, the evolution operator \(\hat{U}(t)\), which becomes very difficult to evaluate exactly when the Hamiltonian is time-dependent and does not commute with itself at different times, which is the case of the C-K Hamiltonian.

Actually, the evolution operator of the C-K damped harmonic oscillator can be related to the free evolution operator. Having in mind the diagram above, we write:

\[
\hat{A}(\hat{U}_{DHO}^\tau(t)\phi(x)) = \hat{U}_{FP}^\tau(\kappa)\phi(\kappa). \tag{14}
\]

\(^3\) Note that \(\hat{O}(t)\) is not the usual Heisenberg picture version \(\hat{O}_H(t) = \hat{U}(t, t_0)^\dagger \hat{O} \hat{U}(t, t_0)\) of its associated operator in Schrödinger picture \(\hat{O}\), although their relation is very simple when the Hamiltonian is time-independent.
In the Caldirola-Kanai model of the damped harmonic oscillator, neither the operator \(i\mathcal{A}\) nor the Hilbert space of solutions of the DHO Schrödinger equation. One may wonder what happens if time translation to be a symmetry. But will do it in an elegant way, trying to close an algebra of (constant, symmetry generating)

\[ \phi(x,t) = \psi(x) = e^{\log(1/u_2) x \frac{\partial}{\partial x}} \psi(x), \]

where \(e^{\log(1/u_2) x \frac{\partial}{\partial x}}\) is a dilation operator which is not unitary. To unitarize this operator, the generator must be shifted from \(x \frac{\partial}{\partial x}\) to \(x \frac{\partial}{\partial x} + \frac{1}{2}\), so that the true unitary operator is then

\[ \hat{U}_D(\frac{1}{u_2}) = e^{\log(1/u_2)(x \frac{\partial}{\partial x} + \frac{1}{2})} = \frac{1}{\sqrt{u_2}} e^{\log(1/u_2) x \frac{\partial}{\partial x}}. \]

But the factor \(\frac{1}{\sqrt{u_2}}\) is already present in the previous expression of \(\hat{U}^{\text{DHO}}(t)\). Therefore, it now reads

\[ \hat{U}^{\text{DHO}}(t) = e^{\frac{1}{2} i \frac{\hbar}{2 m} \frac{u_2}{u_1} x^2} A^{-1}(\hat{U}^{\text{FP}}(t)) \hat{U}_D(\frac{1}{u_2}) = \frac{1}{\sqrt{u_2}} e^{\frac{1}{2} i \frac{\hbar}{2 m} \frac{u_2}{u_1} x^2} e^{\frac{1}{2} \frac{\hbar}{2 m} \frac{u_2}{u_1} x^2} e^{\log(1/u_2) x \frac{\partial}{\partial x}}. \]

Its inverse is given by

\[ \hat{U}^{\text{DHO}}(t)^{-1} = \hat{U}^{\text{DHO}}(t)^\dagger = \sqrt{u_2} e^{\log(1/u_2) x \frac{\partial}{\partial x}} e^{-\frac{1}{2} \frac{\hbar}{2 m} \frac{u_2}{u_1} x^2} e^{-\frac{1}{2} i \frac{\hbar}{2 m} \frac{u_2}{u_1} x^2}. \]

Interestingly, we have been able to obtain an exact expression for the evolution operator as a product of operators. No perturbative approximation method, which could become cumbersome in some cases, is needed for any LSODE-related quantum system to obtain the evolution operator.

3. Deriving dissipative forces from a symmetry

Even though it is possible to set up a clear framework to deal with the quantum Caldirola-Kanai system for the DHO by employing the QAT, this does not provide by itself a well-defined operator (on solutions) associated with the actual time evolution. As mentioned previously, this is rooted in the fact that the conventional time evolution is not included in the symmetry group that can be imported from the free system: the Hamiltonian does not preserve the Hilbert space of solutions of the DHO Schrödinger equation. One may wonder what happens if time evolution symmetry is forced. We shall pursue this issue for the damped harmonic oscillator in this Section.

3.1. Including time symmetry

In the Caldirola-Kanai model of the damped harmonic oscillator, neither the operator \(i\hbar \frac{\partial}{\partial t}\), nor \(\hat{H}_{\text{DHO}}\) (which coincides with the former on solutions) close a Lie algebra under commutation with \(\hat{X}\) and \(\hat{P}\). We shall impose the condition for the time translation to be a symmetry. But will do it in an elegant way, trying to close an algebra of (constant, symmetry generating)
observables, taking advantage of the expressions obtained for the basic operators. Implicitly, we are forcing the system to become conservative, but this is the only information we are going to provide. The implementation is as follows: we wonder whether it is possible to incorporate $i\hbar \frac{\partial}{\partial t}$ into the basic Lie algebra of operators, trying to close an enlarged Lie algebra acting on the (possibly enlarged) Hilbert space. The answer to this question is in the affirmative, but it requires a delicate analysis.

The resulting enlarged algebra will include $\hat{X}, \hat{P}, \hat{H} \equiv \hat{H}_{DHO} = i\hbar \frac{\partial}{\partial t}$ and four more generators (plus the central one $\hat{I}$), denoted by $\hat{Q}, \hat{\Pi}, \hat{G}_1$ and $\hat{G}_2$. Then, together with the generators $\hat{X}$ and $\hat{P}$ and the Hamiltonian, let us introduce the following operators:

$$\hat{\Pi} = i\hbar e^{-\frac{\gamma}{2} (\cos \Omega t - \frac{\gamma}{2\Omega} \sin \Omega t) \frac{\partial}{\partial x} - m\frac{\omega^2}{\Omega} e^{\frac{\gamma}{2} \sin \Omega t} x}$$

$$\hat{\varrho} = e^{\frac{-\omega^2 t}{4\Omega^2}} (\cos \Omega t - \frac{3\gamma}{2\Omega} \sin \Omega t) x + i\hbar \frac{e^{-\frac{\gamma}{2} t}}{m\Omega} \sin \Omega t \frac{\partial}{\partial x}$$

$$\hat{G}_1 = \frac{1}{4\Omega^2} (-4\omega^2 + \gamma^2 \cos 2\Omega t + 2\gamma \Omega \sin 2\Omega t),$$

$$\hat{G}_2 = -\frac{\gamma}{\Omega^2} \sin^2 \Omega t,$$

so that they close the seven-dimensional algebra:

$$\begin{align*}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= \frac{i\hbar}{m} \hat{G}_2 \\
[\hat{Q}, \hat{P}] &= -i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 \\
[\hat{H}, \hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} \\
[\hat{H}, \hat{Q}] &= -2i\hbar \gamma \hat{X} - \frac{i\hbar}{m} \hat{P} + i\hbar \gamma \hat{Q} \\
[\hat{H}, \hat{G}_1] &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2
\end{align*}$$

$$\begin{align*}
[\hat{Q}, \hat{\Pi}] &= 2i\hbar \hat{G}_1 + i\hbar \hat{I} \\
[\hat{X}, \hat{\Pi}] &= i\hbar \hat{G}_1 \\
[\hat{P}, \hat{\Pi}] &= -i\hbar m\omega^2 \hat{G}_2 \\
[\hat{H}, \hat{\varrho}] &= 2i\hbar m\omega^2 \hat{X} - i\hbar m\omega^2 \hat{\varrho} \\
[\hat{H}, \hat{G}_1] &= -3i\hbar m\omega^2 \hat{X} + 2i\hbar m\omega^2 \hat{\varrho} - i\hbar \gamma \hat{\Pi} \\
[\hat{H}, \hat{G}_2] &= -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 - 2i\hbar \hat{I}.
\end{align*}$$

The outcome of this process depends on the realization, which in fact contains crucial information: we have started with a reducible representation of the Heisenberg-Weyl algebra with details about the time evolution of the system and we have arrived at an algebra whose physical content, if any, must be unveiled.

We see that this algebra corresponds to a centrally extended algebra. The central extensions determine the actual basic conjugated pairs and classify possible quantizations. The operators $\hat{\varrho}$ and $\hat{\Pi}$ (plus $\hat{I}$) expand a Heisenberg-Weyl subalgebra, and $\hat{H}, \hat{G}_1$ and $\hat{G}_2$ expand a 2-D affine algebra (with $\hat{H}$ acting as dilations). However, in this realization $\hat{\varrho}$ and $\hat{\Pi}$ are not basic, while $\hat{H}$ and $\hat{G}_2$ are basic, resulting in time being a canonical variable. This result is puzzling: time is not a coordinate or momentum of any degree of freedom and therefore the corresponding generator should not appear as an element of a conjugate pair. Is it possible to take advantage of the information encoded in this algebra to describe a physical system in which we have an evolution with respect to an external time variable? In fact, it is.

Our strategy here is to consider other possible quantizations of the un-extended algebra. A detailed study of the (projective) representations of the enlarged $(7+1)$ dimensional Lie algebra (that is, the possible central extensions) is going to show that there are three relevant kinds of representations, describing systems with different degrees of freedom. The underlying
“classical symmetries” shared by this class of physical realizations are the same, and contain the information about the time evolution of the system. And we shall see that there is in fact a unique quantization in which time is a non-canonical variable. This is the reason why the whole process is unambiguously defined.

Thinking of the algebra above as an abstract Lie algebra, it can be shown that a parameter \( k \) controls the central extensions which are allowed by the Jacobi identity of Lie algebras:

\[
\begin{align*}
\{\hat{X}, \hat{P}\} &= i\hbar \hat{I} \\
\{\hat{X}, \hat{Q}\} &= \frac{i\hbar}{m} \hat{G}_2 \\
\{\hat{Q}, \hat{P}\} &= -i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2 + i\hbar (1 - k) \hat{I} \\
\{\hat{H}, \hat{X}\} &= \frac{i\hbar}{m} \hat{\Pi} \\
\{\hat{H}, \hat{Q}\} &= -2i\hbar \gamma \hat{X} - \frac{i\hbar}{m} \hat{P} + i\hbar \hat{\dot{Q}} \\
\{\hat{H}, \hat{G}_1\} &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2 \\
\{\hat{H}, \hat{\Pi}\} &= 2i\hbar \omega^2 \hat{X} - i\hbar \omega^2 \hat{\dot{Q}} \\
\{\hat{P}, \hat{\Pi}\} &= -2i\hbar \omega^2 \hat{\dot{X}} + 2i\hbar \omega^2 \hat{\dot{Q}} - i\hbar \gamma \hat{\Pi}
\end{align*}
\]

(5)

It is convenient to perform the linear shift:

\[
\hat{Q} \equiv -\hat{\dot{Q}} + (1 - k) \hat{X},
\]

(26)

so that the actual degrees of freedom diagonalize:

\[
\begin{align*}
\{\hat{X}, \hat{P}\} &= i\hbar \hat{I} \\
\{\hat{X}, \hat{Q}\} &= \frac{-i\hbar}{m} \hat{G}_2 \\
\{\hat{Q}, \hat{P}\} &= i\hbar \gamma \hat{G}_1 - i\hbar \gamma \hat{G}_2 \\
\{\hat{H}, \hat{X}\} &= \frac{i\hbar}{m} \hat{\Pi} \\
\{\hat{H}, \hat{Q}\} &= i\hbar \gamma (1 + k) \hat{X} + \frac{i\hbar}{m} \hat{P} + i\hbar \gamma \hat{\dot{Q}} + \frac{i\hbar}{m} (1 - k) \hat{\Pi} \\
\{\hat{H}, \hat{G}_1\} &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2 \\
\{\hat{H}, \hat{\Pi}\} &= 2i\hbar \omega^2 \hat{X} - i\hbar \omega^2 \hat{\dot{Q}} - i\hbar \gamma \hat{\Pi} \\
\{\hat{P}, \hat{\Pi}\} &= -2i\hbar \omega^2 \hat{\dot{X}} + 2i\hbar \omega^2 \hat{\dot{Q}} - i\hbar \gamma \hat{\Pi}
\end{align*}
\]

(27)

Noticing the appearance of the central generator \( \hat{\Pi} \) on the right hand side of these commutation relations, we can see that the representations of this algebra include:

- For arbitrary \( k \), a generic family with 3 degrees of freedom: \((\hat{X}, \hat{P})\), \((\hat{Q}, \hat{\Pi})\) and \((\hat{H}, \hat{G}_2)\), then time being a canonical variable.
- For \( k = 1 \), already described, an anomalous family with 2 degrees of freedom: \((\hat{X}, \hat{P})\) and \((\hat{H}, \hat{G}_2)\), then time being a canonical variable.
- For \( k = -1 \), a family with 2 degrees of freedom: \((\hat{X}, \hat{P})\) and \((\hat{Q}, \hat{\Pi})\).

The third case is what we are looking for: it contains two degrees of freedom and time is not a canonical variable. Explicitly, for \( k = -1 \) we have:
As we have selected the representation in which \( \hat{\mathcal{G}}_1 \) and \( \hat{\mathcal{G}}_2 \) commute with the basic couples \((\hat{X},\hat{P})\) and \((\hat{Q},\hat{\Pi})\), and therefore will be represented as constants (times the identity operator) in any irreducible representation. Even more, their commutation rules with \( \hat{H} \) determine that the value of these constants must be zero \( (\hat{H},\hat{\mathcal{G}}_1) = (\hat{H},\hat{\mathcal{G}}_2) = 0 \), as \( \hat{\mathcal{G}}_1 \) and \( \hat{\mathcal{G}}_2 \) are proportional to the identity \( \hat{I} \); then, solve for \( \hat{\mathcal{G}}_1 \) and \( \hat{\mathcal{G}}_2 \) using the expressions above). Technically, the operators \( \hat{\mathcal{G}}_1 \) and \( \hat{\mathcal{G}}_2 \) are gauge, in the sense that, in the resulting physical system, their symmetry transformation does not produce any change in the parameters associated with basic operators (to be intuitive, they do not change coordinate or momentum of any degree of freedom of the system, so that their action is not physically observable, in analogy with a gauge transformation in electrodynamics). In consequence, the effective dimension of the algebra is \( 5 + 1 \): \((\hat{X},\hat{P}), (\hat{Q},\hat{\Pi}), \hat{H} \) and \( \hat{I} \).

\[
\begin{align*}
[\hat{X},\hat{P}] &= i\hbar \hat{I} & [\hat{Q},\hat{\Pi}] &= i\hbar \hat{I} \\
[\hat{X},\hat{Q}] &= 0 & [\hat{X},\hat{\Pi}] &= 0 \\
[\hat{Q},\hat{P}] &= 0 & [\hat{P},\hat{\Pi}] &= 0 \\
[\hat{H},\hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} & [\hat{H},\hat{\Pi}] &= i\hbar \omega^2 \hat{Q} \\
[\hat{H},\hat{Q}] &= \frac{i\hbar}{m} (\hat{P} + 2\hat{\Pi}) + i\hbar \gamma \hat{Q} & [\hat{H},\hat{\Pi}] &= i\hbar \omega^2 (\hat{X} - 2\hat{Q}) - i\hbar \gamma \hat{\Pi}.
\end{align*}
\]

This way, we end up with two pairs of independent, canonical operators, plus a Hamiltonian. As we have selected the representation in which \( \hat{H} \) is not a basic operator, it can be written in terms of the basic ones in an irreducible representation:

\[
\hat{H} = -\frac{1}{m} \hat{\Pi} \hat{P} - \frac{\gamma}{2} (\hat{Q} \hat{\Pi} + \hat{\Pi} \hat{Q}) - \frac{\hat{\Pi}^2}{m} + m\omega^2 \hat{X} \hat{Q} - m\omega^2 \hat{Q}^2.
\] (30)

Note that the normal ordering has been chosen to solve the ordering ambiguity, as customary in canonical quantization. The corresponding classical version of the Hamiltonian is\(^4\):

\[
H = -\frac{1}{m} \Pi P - \gamma Q \Pi - \frac{\Pi^2}{m} + m\omega^2 X Q - m\omega^2 Q^2.
\] (31)

\(^4\) This expression can be obtained directly regarding the previous algebra as a Poisson subalgebra and determining what function satisfies those Poisson brackets.
The physical content of this Hamiltonian can be analyzed classically by solving Hamilton’s equations of motion. After that, a canonical quantization would also be possible. However, it is interesting to establish the relationship of the physical system at which we have arrived with a known one: the new system with two degrees of freedom is, actually, Bateman’s dual system.

3.2. The Bateman dual system

The classical Hamiltonian (31) can be transformed, using the linear, constant, canonical transformation:

\[
\begin{align*}
X &= y + \frac{1}{m\gamma} p_y + \frac{1}{2} x \\
P &= p_x - m\frac{\gamma}{2} y - m\frac{\omega^2}{\gamma} x \\
Q &= y + \frac{1}{m\gamma} p_y - \frac{1}{2} x \\
\Pi &= -p_x - m\frac{\gamma}{2} y + m\frac{\omega^2}{\gamma} x,
\end{align*}
\]

into Bateman’s dual Hamiltonian [19]

\[
H_B = \frac{\hat{p}_x \hat{p}_y}{m} + \gamma \left( \hat{y} \partial \hat{y} - \hat{x} \partial \hat{x} \right) + m\Omega^2 \hat{x} \hat{y}.
\]

Let us remark the fact that this canonical transformation is time-independent and, therefore, it is just a change of variables for \(H\), so that \(H = H_B\). Then, it can be checked that \(H\) (or \(H_B\)) describes a damped particle \((x, p_x)\) and its time reversal \((y, p_y)\) computing the second order classical equations out of Hamilton’s equations:

\[
\ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \quad \ddot{y} - \gamma \dot{y} + \omega^2 y = 0.
\]

The system is conservative, so our objective of including time evolution among the symmetries has been accomplished, at the cost of including a new degree of freedom.

Performing the usual canonical quantization, the quantum Bateman Hamiltonian can be written:

\[
\hat{H}_B = \frac{\hat{p}_x \hat{p}_y}{m} + \gamma \left( \hat{y} \partial \hat{y} - \hat{x} \partial \hat{x} \right) + m\Omega^2 \hat{x} \hat{y},
\]

and the Schrödinger equation for the Bateman system is given by:

\[
\frac{i\hbar}{\partial t} \frac{\partial \phi(x, y, t)}{\partial t} = \left[ \frac{\hbar^2}{m} \frac{\partial^2}{\partial x \partial y} - i\hbar \frac{\gamma}{2} (y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x}) + m\Omega^2 x y \right] \phi(x, y, t).
\]

However, it has been argued that the quantum Bateman system possesses inconsistencies, such as complex eigenvalues and non-normalizable eigenstates. Chruściński & Jurkowski [23] showed that \(\hat{H}_B\) has real, continuous spectrum, and that the complex eigenvalues are associated with resonances, which in last instance are the responsible of dissipation.

The Bateman system also admits an equivalent description in terms of a first-order Schrödinger equation, which can be derived from the symmetry group associated with the Bateman model. In this framework, the questions of unitarity and the lack of a proper definition of a vacuum, which are poorly treated in the literature, are much better addressed. This matter is under study [25, 31] and will be published elsewhere [26].
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