The spectral localizer for semifinite spectral triples

Hermann Schulz-Baldes and Tom Stoiber

Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Abstract

The notion of spectral localizer is extended to pairings with semifinite spectral triples. By a spectral flow argument, any semifinite index pairing is shown to be equal to the signature of the spectral localizer. As an application, a formula for the weak invariants of topological insulators is derived. This provides a new approach to their numerical evaluation.

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1 Overview

In two recent works [11, 12], it was shown that the integer-valued pairing of a $K$-theory class with a Fredholm module is equal to the signature of a certain gapped self-adjoint operator called the spectral localizer. It is the object of this work to generalize this equality to semifinite index theory. Let us describe the result on integer-valued pairings more explicitly in the case of an odd index pairing. Hence let $a$ be an invertible operator on a Hilbert space and $D = D^*$ an unbounded selfadjoint operator with compact resolvent and bounded commutator $[D, a]$. If $P = \chi(D > 0)$ is the positive spectral projection of $D$ expressed in terms of the characteristic function $\chi$, then $PaP + (1 - P)$ is a Fredholm operator and the index pairing is $\langle [a], [D] \rangle = \text{Ind}(PaP + (1 - P))$. The main result of [11, 13] then states that this pairing can be computed with the spectral localizer

$$L_\kappa = \begin{pmatrix} \kappa D & a \\ a^* & -\kappa D \end{pmatrix},$$

namely if $L_{\kappa, \rho}$ is the finite-dimensional restriction of $L_\kappa$ to the range of $\chi(|D \oplus D| < \rho)$, then provided $\rho$ is sufficiently large and $\kappa$ sufficiently small,

$$\langle [a], [D] \rangle = \frac{1}{2} \text{Sig}(L_{\kappa, \rho}).$$

The proof of (1) contains an analytic argument showing that 0 is not in the spectrum of $L_{\kappa, \rho}$ so that the signature is well-defined, and then a topological homotopy argument leading to the equality (1). In the works [11, 12], this latter step was based on an evaluation of the index map and actually (1) then appeared merely as corollary to a $K$-theoretic statement. An alternative argument for (1) using spectral flow was provided in [13, 14]. The main interest
of (1) is that the r.h.s. is amenable to numerical evaluation and is of use in connection with topological insulators, see [9, 14]. In fact, in applications the finite volume restriction of \( L_\kappa \) is readily obtained and no further functional calculus is involved so that merely the signature of a finite-dimensional matrix has to computed, either by diagonalizing \( L_{\kappa,\rho} \) or by the block Cholesky decomposition.

For the generalization of (1), Section 2 first reviews the definitions and main results about Fredholm operators and spectral flow in semifinite von Neumann algebras, based mainly on [16, 15, 1, 6, 8]. It then introduces the semifinite signature and analyses its link to spectral flow. With this arsenal established, the main results can be stated in Section 3. Section 4 adapts the arguments of [12] to prove that the signature is well-defined and stable. Section 5 then generalizes the spectral flow proof of (1) given in [13] to the present setting, while Section 6 provides the proof that also the even index pairings can be calculated with the half-signature. The spectral flow argument in Section 6 is different from the one given in [14], and we feel that it is more direct. Moreover, it does not require the normality of the off-diagonal entries of the (even) Dirac operator as in [12, 14], but instead supposes that the spectral triple is QC\(^1\) in the notation of [5]. Finally Section 7 discusses applications to the numerical computation of weak invariants of topological insulators.

2 Semifinite spectral flow and signature

Let \( \mathcal{N} \) be a von Neumann algebra with a semifinite faithful normal trace \( \tau \). A projection \( P \in \mathcal{N} \) is called finite if \( \tau(P) < \infty \). Let \( \mathcal{K} \) denote the norm-closure of the smallest algebraic ideal in \( \mathcal{N} \) containing the finite projections. This C*-algebra is called the ideal of \( \tau \)-compact operators in \( \mathcal{N} \) and any projection in \( \mathcal{K} \) is finite. Associated to \( \mathcal{K} \) is a short exact sequence \( 0 \to \mathcal{K} \to \mathcal{N} \to \mathcal{N}/\mathcal{K} \to 0 \). The quotient \( \mathcal{N}/\mathcal{K} \) is called the Calkin algebra and the quotient map is denoted by \( \pi \). Associated to this exact sequence one has the standard notion of Breuer-Fredholm operator, but for the definition of spectral flow an extension to skew-corners is needed and will be described next. Associated to projections \( P, Q \in \mathcal{N} \), the skew-corner \( P \mathcal{N} Q \) consists of operators in \( \mathcal{N} \) mapping \( \text{Ran}(Q) \) to \( \text{Ran}(P) \) and then \( T \in P \mathcal{N} Q \) will be viewed as an operator between these subspaces. The most basic example is \( PQ \in P \mathcal{N} Q \). Note that \( P \mathcal{N} Q \) is an algebra only if \( P = Q \). For any \( T \in \mathcal{N} \) we will denote by \( \text{Ker}(T) \) both the kernel as a subspace as well as the projection onto it, which is an element of \( \mathcal{N} \). Furthermore, for any pair of projections \( P, Q \in \mathcal{N} \), let \( P \cap Q \) denote the projection onto the intersection of \( \text{Ran}(P) \) and \( \text{Ran}(Q) \).

**Definition 1** ([6, 1]) Let \( P, Q \in \mathcal{N} \) be projections and \( T \in \mathcal{N} \). Then \( T \) is called \( (P \cdot Q) \)-Fredholm if \( \text{Ker}(T) \cap Q \) and \( \text{Ker}(T^*) \cap P \) are \( \tau \)-finite projections and there exists a projection \( E \in \mathcal{N} \) such that \( P - E \) is finite and \( \text{Ran}(E) \subseteq \text{Ran}(T) \). Its (semifinite) index is then defined as

\[
\text{Ind}_{(P,Q)}(T) = \tau(\text{Ker}(T) \cap Q) - \tau(\text{Ker}(T^*) \cap P).
\]

If \( P = Q = 1 \), this is reduces to the standard semifinite index, which is sometimes denoted by \( \tau \)-Ind. There exists the following generalization of Atkinson’s theorem.
Theorem 2 ([6]) Let $P, Q, R \in \mathcal{N}$ be projections and $T \in P \cap R$.

(i) $T$ is $(P \cdot Q)$-Fredholm if and only if there exists $S \in Q \cap P$ with $TS - P \in PKP$ and $ST - Q \in QKQ$.

(ii) The set of $(P \cdot Q)$-Fredholm operators is open in $P \cap Q$.

(iii) If $T$ is $(P \cdot Q)$-Fredholm and $S \in Q \cap R$ is $(Q \cdot R)$-Fredholm, then $TS$ is $(P \cdot R)$-Fredholm

\[ \text{Ind}(P \cdot R)(TS) = \text{Ind}(P \cdot Q)(T) + \text{Ind}(Q \cdot R)(S). \]

The following criterion is crucial for the definition of the spectral flow [15, 1].

Proposition 3 If $P, Q \in \mathcal{N}$ are projections with $\|\pi(P - Q)\| < 1$, then $P \cap Q$ is $(P \cdot Q)$-Fredholm.

Definition 4 For projections $P, Q \in \mathcal{N}$ with $\|\pi(P - Q)\| < 1$, the essential codimension is

\[ \text{ec}(P, Q) := \text{Ind}(P \cdot Q)(P) = \tau((1 - P) \cap Q) - \tau((1 - Q) \cap P). \]

Denote by $\mathcal{F}^{sa} \subset \mathcal{N}$ the space of self-adjoint Fredholm operators.

Definition 5 Let $\{T_t\}_{t \in [0,1]}$ be a norm-continuous path in $\mathcal{F}^{sa}$ and $0 = t_0 < t_1 < \ldots < t_{n+1} = 1$ be a partition such that

$\|\pi(p_k+1 - p_k)\| \leq 1/2$,

for all $k = 0, \ldots, n$ with $p_k := \chi(T_{t_k} \geq 0)$. Then the spectral flow is defined as the real number

\[ \text{Sf}(\{T_t\}_{t \in [0,1]}) := \sum_{k=0}^{n} \text{ec}(p_k, p_{k+1}). \]

If it is clear from the context, the index $t \in [0,1]$ is dropped.

Let us quote the following basic properties of the spectral flow [1].

Proposition 6 Let $\{T_t\}$ and $\{T_t'\}$ be norm-continuous paths in $\mathcal{F}^{sa}$.

(i) The spectral flow is well defined and does not depend on the choice of partition.

(ii) (Homotopy invariance) If $\{T_t\}$ and $\{T_t'\}$ have the same endpoints and are connected by a norm-continuous homotopy within $\mathcal{F}^{sa}$, then

\[ \text{Sf}(\{T_t\}) = \text{Sf}(\{T_t'\}). \]

(iii) (Concatenation) If $T_1 = T_0'$, then

$\text{Sf}(\{T_t\} \ast \{T_t'\}) = \text{Sf}(\{T_t\}) + \text{Sf}(\{T_t'\}),$

with $\ast$ denoting concatenation of paths.
(iv) (Homomorphism) 
\[ \text{Sf}(\{T_t \oplus T'_t\}) = \text{Sf}(\{T_t\}) + \text{Sf}(\{T'_t\}). \]

If one has a path \( \{D_t\} \) of self-adjoint unbounded operators affiliated to \( \mathcal{N} \) (notably, each spectral projection of \( D_t \) lies in \( \mathcal{N} \)) such that its bounded transform 
\[ f(D_t) := D_t(1 + D_t^2)^{-1/2} \quad (2) \]
is a norm-continuous path in \( \mathcal{F}^{sa} \), then its spectral flow can be defined by 
\[ \text{Sf}(\{D_t\}) := \text{Sf}(\{f(D_t)\}). \]
It then satisfies all the properties of Proposition 6.

If \( T_0 \) and \( T_1 \) are in \( \mathcal{F}^{sa} \) and \( T_0 - T_1 \) is compact, one can always consider the spectral flow along the straight-line path which will be denoted by 
\[ \text{Sf}(T_0, T_1) := \text{Sf}(\{tT_1 + (1-t)T_0\}). \]
If \( T_0, T_1 \) and \( T_2 \) are in \( \mathcal{F}^{sa} \) and both \( T_0 - T_1 \) and \( T_1 - T_2 \) are compact, then one has 
\[ \text{Sf}(T_0, T_1) + \text{Sf}(T_1, T_2) = \text{Sf}(T_0, T_2). \]
This is also true for certain pairs of unbounded operators, e.g. if \( D = D^* \) is a selfadjoint operator affiliated to \( \mathcal{N} \) and with compact resolvent and \( u \) unitary operator such that \([D, u]\) extends to a bounded operator, then the straight-line path from \( D \) to \( uDu^* \) becomes under the bounded transform a norm-continuous path in \( \mathcal{F}^{sa} \) and one has 
\[ \text{Sf}(D, uDu^*) = \text{Sf}(\{D + tu[D, u^*]\}). \]

**Definition 7** Let \( T \in \mathcal{N} \) be self-adjoint with a \( \tau \)-finite (i.e. \( \tau \)-trace-class) support projection. Then the signature of \( T \) is defined by 
\[ \text{Sig}(T) := \tau(\text{sgn}(T)), \quad \text{sgn}(T) := \chi(T > 0) - \chi(T < 0). \]
Let us note that the two summands \( \chi(T > 0) \) and \( \chi(T < 0) \) are separately trace-class. The following generalizes Sylvester’s law of inertia.

**Proposition 8** Let \( T \in \mathcal{N} \) be self-adjoint with a \( \tau \)-finite support projection. Further let \( A \in \mathcal{N} \) be invertible. Then 
\[ \text{Sig}(A^*TA) = \text{Sig}(T). \]

**Proof.** Decomposing into positive and negative part \( T = T_+ - T_- \), it is enough to prove the statement for \( T \geq 0 \). In that case, one has 
\[ \text{Sig}(T) = \tau(\text{Supp}(T)), \quad \text{Sig}(A^*TA) = \tau(\text{Supp}(A^*TA)). \]
with the respective support projections. For $TA = v|TA|$ the unique polar decomposition with partial isometry $v$, one has $A^*T = |TA| v^* = v^* v|TA| v^*$ and thus

$$\text{Ran}(v) = \overline{\text{Ran}(TA)} = \overline{\text{Ran}(T)}, \quad \text{Ran}(v^*) = \overline{\text{Ran}(A^*T)} = \overline{\text{Ran}(A^*TA)}.$$ 

Since $T$ and $A^*TA$ are self-adjoint, their support projections are given by $\text{Supp}(T) = v^*v$ and $\text{Supp}(A^*TA) = vv^*$. Hence the claim follows from $\tau(v^*v) = \tau(vv^*)$.

Certain spectral flows can be computed in terms of the signature:

**Proposition 9** Let $\{T_t\}$ be a norm-continuous path of self-adjoints in $\mathcal{N}$ such that the support projections satisfy $\text{Supp}(T_t) \leq P$ and the range projections satisfy $\text{Ran}(T_t) \leq P$ for all $t$ and a single $\tau$-finite projection $P \in \mathcal{N}$. Then $\{F_t\} := \{PT_tP + 1 - P\}$ is a norm-continuous path in $\mathcal{F}^{sa}$ and

$$\text{Sf}(\{F_t\}) = \frac{1}{2} \left( \text{Sig}(T_1) - \text{Sig}(T_0) \right) + \frac{1}{2} \left( \tau(P\text{Ker}(T_1)) - \tau(P\text{Ker}(T_0)) \right). \quad (3)$$

In particular, if the endpoints $T_0$ and $T_1$ are invertible elements of the algebra $PNP$, then the second term vanishes.

**Proof.** The Fredholm-property and continuity are obvious. Since $\pi(F_t) = 1$, the two point partition is sufficiently fine and hence the spectral flow is given by

$$\text{Sf}(\{F_t\}) = \text{ec}(p_0, p_1),$$

with $p_t$ as in Defintion 5. Since $P$ and $T_t$ commute, one has

$$p_t = \chi(F_t > 0) = \chi\left(PT_tP \oplus (1 - P) > 0\right) = P\chi(T_t > 0)P \oplus (1 - P),$$

and hence

$$\tau((1 - p_0) \cap p_1) = \tau\left(\left(P(1 - \chi(T_0 > 0))P\right) \cap \left(P\chi(T_1 > 0)\right)\right)$$

$$= \tau\left(\left(P - P\chi(T_0 > 0)\right) \cap \chi(T_1 > 0)\right)$$

$$= \tau(P\chi(T_1 > 0)) - \tau(P\chi(T_0 > 0) \cap \chi(T_1 > 0)).$$

Switching 0 and 1 and taking the difference leads to

$$\text{Sf}(\{F_t\}) = \tau(P\chi(T_1 > 0) - P\chi(T_0 > 0)).$$

Finally, noting that $P\chi(T_t > 0) = \chi(T_t > 0)$ and thus

$$\tau(P\text{Ker}(T_t)) + \tau(\chi(T_t > 0)) = \tau\left(P\chi(T_t > 0)\right) = \tau(P) - \tau(\chi(T_t < 0)),$$

one obtains the formula (3). \qed

The signature also has an additional invariance property that is somewhat inconvenient to express in terms of the spectral flow:
Proposition 10 If \( \{T_t\}_{t \in [0,1]} \) is a continuous path of self-adjoints all of which have compact support projections and such that for every \( t \in [0,1] \) there is an open interval \( \Delta_t \) around 0 such that \( \Delta \cap \sigma(T_t) \subset \{0\} \), then for all \( t, t' \in [0,1] \)

\[
\text{Sig}(T_t) = \text{Sig}(T_{t'}). 
\]

Proof. As \([0,1]\) is compact, the spectra \( \sigma(T_t) \setminus \{0\} \) have a common gap \( \Delta \) and hence one can choose continuous functions \( f, g \) such that

\[
\chi(T_t > 0) = f(T_t), \quad \chi(T_t < 0) = g(T_t), \quad \forall \ t \in [0,1].
\]

Therefore the paths \( t \mapsto \chi(T_t > 0) \) and \( t \mapsto \chi(T_t < 0) \) are actually norm-continuous paths of projections. Since projections that are close in norm must be unitarily equivalent, this implies that the signature is constant along the path. \( \square \)

3 Spectral localizer

A semifinite spectral triple \([7]\) (also called an unbounded semifinite Fredholm module) is a tuple \((D, A, \mathcal{N})\) consisting of a semifinite von Neumann algebra \( \mathcal{N} \) with a trace \( \tau \), a \(*\)-subalgebra \( A \subset \mathcal{N} \) and a self-adjoint operator \( D \) affiliated to \( \mathcal{N} \) (namely, each spectral projection of \( D \) lies in \( \mathcal{N} \)), which satisfy the following conditions:

(i) For all \( a \in A \) the commutator \([D, a]\) is densely defined and extends to a bounded operator (which is then an element of \( \mathcal{N} \)).

(ii) For any \( a \in A \) and \( z \notin \mathbb{R} \), the products \( R(z)a = (D - z)^{-1}a \) are in \( \mathcal{K} \), \( i.e. \) are \( \tau \)-compact.

(iii) The triple is called even if there is a self-adjoint unitary \( \gamma \in \mathcal{N} \) that commutes with all \( a \in A \) and anticommutes with \( D \), otherwise it is called odd.

It will also be assumed that 0 is not an eigenvalue of \( D \). This can be done without loss of generality by adding a term proportional to \( \chi(D = 0) \) to \( D \), which is merely a \( \tau \)-compact perturbation. If \( A \) has no unit, let \( A^+ \subset \mathcal{N} \) be its minimal unitalization.

For an odd spectral triple and a unitary \( u \in A^+ \), one has the index pairing

\[
\langle [u], [D] \rangle := \text{Ind}_{(P, P)}(PuP),
\]

where \( P = \chi(D > 0) \). The index pairing is connected to a spectral flow (see \( e.g. \) \([8]\)):

\[
\langle [u], [D] \rangle = \text{Sf}(u^*Du, D),
\]

(4)

For an even spectral triple, the Dirac operator \( D \) anti-commutes with a self-adjoint unitary \( \gamma \) which is represented by the matrix diag(1, -1) in the grading provided by \( \gamma \). As \( D \) and all
odd functions of $D$ are off-diagonal in this representation, there is an unbounded operator $D_0$ and a unitary $F$ such that

$$D = \begin{pmatrix} 0 & D_0^* \\ D_0 & 0 \end{pmatrix}, \quad \text{sgn}(D) = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}. \tag{5}$$

In the same sense, any element $a \in \mathcal{A}^+$ decomposes with respect to the grading of $\gamma$ as

$$a = \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}.$$

For a projection $p \in \mathcal{A}^+$, the index pairing is defined by \cite{6}

$$\langle [p], [D] \rangle := \text{Ind}_{(p_+ p_-)}(p_+ F^* p_-), \tag{6}$$

where the properties of the spectral triple show that the skew-corner index is well-defined. It can be written as a spectral flow by

$$\langle [p], [D] \rangle = \text{ec}(p_+, F^* p_- F) = \text{Sf}(F^*(1 - 2p_-)F, 1 - 2p_+).$$

If $p$ is given by $p = \chi(h < 0)$ for a self-adjoint invertible operator $h \in \mathcal{A}^+$, then continuously deforming $(1 - 2p)$ to $h$ shows

$$\langle [p], [D] \rangle = \text{Sf}((F^*(1 - 2p_-)F, 1 - 2p_+) = \text{Sf}(F^* h_- F, h_+). \tag{7}$$

The straight-line paths (and the homotopy) are well-defined since $[a, \text{sgn}(D)]$ is $\tau$-compact for any $a \in \mathcal{A}$ and hence $F^* a_- - a_+ F^*$ is also $\tau$-compact. We now define the spectral localizer:

**Definition 11** Let $(\mathcal{N}, \mathcal{A}, D)$ be a spectral triple.

(i) If the triple is odd, assume that $a \in \mathcal{A}^+$ is invertible and define the spectral localizer by

$$L_\kappa := \begin{pmatrix} \kappa D & a \\ a^* & -\kappa D \end{pmatrix},$$

as an operator affiliated to $M_2(\mathbb{C}) \otimes \mathcal{N}$. With the abbreviations

$$h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}, \quad D' = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix},$$

the spectral localizer can also be written as $L_\kappa = h + \kappa D'$.

(ii) For an even spectral triple let us suppose that the commutators $[[D], a]$ are densely defined and extend to bounded operators for all $a \in \mathcal{A}$, notably the triple $(\mathcal{N}, \mathcal{A}, D)$ is $QC^1$ in the notation of \cite{5}. If $h \in \mathcal{A}^+$ is invertible and self-adjoint, the associated spectral localizer is defined by

$$L_\kappa := \kappa D + h \gamma = \begin{pmatrix} h_+ & \kappa D_0^* \\ \kappa D_0 & -h_- \end{pmatrix},$$

with the last expression again as a matrix w.r.t. the grading $\gamma$. 7
Further set $P_\rho := \chi((D')^2 < \rho^2)$ in the odd case and $P_\rho := \chi(D^2 < \rho^2)$ in the even case. Then in both cases the reduced spectral localizer is defined by

$$L_{\kappa,\rho} := P_\rho L_\kappa P_\rho .$$

The two cases of odd and even pairings are similar: there is a self-adjoint unitary that anti-commutes with $h$ and commutes with $D'$ in the odd case, respectively that anti-commutes with $D$ and commutes with $h$ in the even case. Both of the associated pairings can now be read off the spectral localizer, as shows the main result of the paper:

**Theorem 12** Let $(\mathcal{N}, \mathcal{A}, D)$ be a spectral triple and $h = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ with $a \in \mathcal{A}^+$ invertible in the odd case and, respectively, $h \in \mathcal{A}^+$ invertible in the even case. Further let $g = \|h^{-1}\|^{-1}$ be the size of the spectral gap of $h$. If $\kappa > 0$ is chosen so small that

$$\kappa \leq \frac{g^3}{12 \|[D, h]\| \|h\|}, \quad (8)$$

and $\rho$ so large that

$$\rho > \frac{2g}{\kappa}, \quad (9)$$

then the spectral localizer $L_{\kappa,\rho}$ as defined above is invertible and satisfies

$$\langle [a |a|^{-1}], [D] \rangle = \frac{1}{2} \operatorname{Sig}(L_{\kappa,\rho})$$

in the odd case and

$$\langle [\chi(h < 0)], [D] \rangle = \frac{1}{2} \operatorname{Sig}(L_{\kappa,\rho})$$

in the even case.

### 4 Constancy of the signature

For a given $\rho > 0$, let us introduce a smooth cut-off function $G \in C^\infty(\mathbb{R})$ such that

$$G_\rho(x) = \begin{cases} 1, & \text{for } |x| \leq \frac{\rho}{2} \\ 0, & \text{for } |x| > \rho \end{cases},$$

and whose Fourier transform satisfies $\|\mathcal{F}(G'_\rho)\|_1 \leq 8\rho^{-1}$. A standard estimate (e.g. [11]) shows

$$\|[G_\rho(D), a]\| \leq \frac{8}{\rho} \|[D, a]\| .$$

**Lemma 13** A pair $(\kappa, \rho)$ is called admissible if it satisfies the inequalities (8) and (9).

(i) If $(\kappa, \rho)$ is admissible, then

$$(L_{\kappa,\rho})^2 > \frac{g^2}{4} P_\rho .$$
(ii) If \((\kappa, \rho)\) and \((\kappa', \rho')\) are admissible, then

\[
\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho'}) .
\]

**Proof.** Let us focus on the proof in the odd case, following [12]. The even case can be dealt with by similar means as in [12] (it is enough to replace \(H \otimes \sigma_0\) and \(H \otimes \Gamma\) appearing in [12] with \(h\) and \(h\gamma\) respectively). For \(\rho \leq \rho'\), let us introduce the path

\[
\lambda \in [0, 1] \mapsto L_{\kappa, \rho, \rho'}(\lambda) := \kappa P_{\rho} D' P_{\rho'} + P_{\rho'} G_{\lambda, \rho} h G_{\lambda, \rho} P_{\rho'} ,
\]

with \(G_{\lambda, \rho} = (1 - \lambda) + \lambda G_{\rho}(D')\). By literally the same computation as in [13], one obtains

\[
(L_{\kappa, \rho, \rho'}(\lambda))^2 + (1 - P_{\rho'}) > 0 ,
\]

and

\[
(L_{\kappa, \rho, \rho'}(0))^2 > \frac{g^2}{4} P_{\rho'} ,
\]

provided that \((\kappa, \rho)\) is admissible. In particular, \((L_{\kappa, \rho})^2 = (L_{\kappa, \rho}(0))^2 > \frac{g^2}{4} P_{\rho'}\). Next let us show that \(\text{Sig}(L_{\kappa, \rho}) = \text{Sig}(L_{\kappa', \rho'})\) for any admissible pairs \((\kappa, \rho)\) and \((\kappa', \rho')\) with \(\rho \leq \rho'\). As \(L_{\kappa, \rho, \rho'}(1 + (1 - \lambda)\kappa, \rho')\) is gapped around 0 for any \(\lambda\), the signature remains unchanged by Proposition 10, hence one may assume \(\kappa = \kappa'\). Since the path (10) is continuous and also satisfies the conditions of Proposition 10, it is sufficient to prove \(\text{Sig}(L_{\kappa, \rho, \rho'}(1)) = \text{Sig}(L_{\kappa, \rho, \rho'}(1))\). As \(P_{\rho'} G_{\rho} = P_{\rho'} G_{\rho}\), one has

\[
L_{\kappa, \rho, \rho'}(1) = \kappa P_{\rho} D' P_{\rho} + P_{\rho'} G_{\rho} h G_{\rho} P_{\rho'} = L_{\kappa, \rho, \rho'}(1) + \kappa (P_{\rho'} - P_{\rho}) D'(P_{\rho'} - P_{\rho}) ,
\]

and, moreover, the last sum is direct. Hence

\[
\text{Sig}(L_{\kappa, \rho, \rho'}(1)) = \text{Sig}(L_{\kappa, \rho, \rho'}(1)) + \text{Sig}((P_{\rho'} - P_{\rho}) D'(P_{\rho'} - P_{\rho})) = \text{Sig}(L_{\kappa, \rho, \rho'}(1)) ,
\]

where the second equality holds obviously due to the definition of \(D'\). \(\square\)

For \(a\) affiliated to \(\mathcal{N}\) or matrices over \(\mathcal{N}\), let us write \(a_{\rho} = P_{\rho} a P_{\rho}\) and \(a_{\rho'} = (1 - P_{\rho}) a (1 - P_{\rho})\). Then \(D = D_{\rho} + D_{\rho'}\) and, in the odd and even case respectively,

\[
L_{\kappa} = L_{\kappa, \rho} \oplus L_{\kappa, \rho'} + P_{\rho} h P_{\rho} + P_{\rho'} h P_{\rho} ,
\]

\[
L_{\kappa} = L_{\kappa, \rho} \oplus L_{\kappa, \rho'} + P_{\rho} (h \gamma) P_{\rho'} + P_{\rho'} (h \gamma) P_{\rho} .
\]

**Lemma 14** If \((\kappa, \rho)\) is admissible and \(\rho\) is large enough, then \(L_{\kappa, \rho'}\) is invertible in \(P_{\rho'} \mathcal{N} P_{\rho'}\) and

\[
\text{Sf}(L_{\kappa}, L_{\kappa, \rho} \oplus L_{\kappa, \rho'}) = 0 .
\]

**Proof.** We will show that for \(\rho\) large enough the term of \(L_{\kappa}\) that is off-diagonal with respect to the decomposition \(1 = P_{\rho} \oplus P_{\rho'}\) can be shrunk to zero with a linear path \(t \in [0, 1] \mapsto L_{\kappa}(t)\) in the invertible operators given by

\[
L_{\kappa}(t) = L_{\kappa, \rho} \oplus L_{\kappa, \rho'} + t(P_{\rho} L_{\kappa} P_{\rho} + P_{\rho'} L_{\kappa} P_{\rho}) .
\]
Let us focus on the odd case as the even case follows by essentially the same argument. The diagonal part is invertible because $|L_{\kappa,\rho}| > \frac{g^2}{2} P_{\rho}$ by Lemma 13 and
\[
(L_{\kappa,\rho})^2 = \left( \frac{\kappa^2 D_0^2 + a_{\rho^c}^* a_{\rho^c}}{\kappa [D_0, a_{\rho^c}]} \right) \geq \frac{\kappa^2 \rho^2 P_{\rho^c} 1_2}{g^2} + \kappa \left( [D_0, a_{\rho^c}]^* \right) 
\]
\[
\geq (\frac{\kappa^2 \rho^2 - \kappa \|[D_0, a]\|}{g^2} P_{\rho^c} 1_2 \geq \frac{1}{2} \kappa^2 \rho^2 P_{\rho^c} 1_2,
\]
(11)
due to (8) and for $\rho$ sufficiently large. The inverse is again diagonal in the decomposition $1 = P_{\rho} \oplus P_{\rho^c}$ and given by
\[
|L_{\kappa,\rho} + L_{\kappa,\rho^c}|^{-\frac{1}{2}} = P_{\rho} |L_{\kappa,\rho}|^{-\frac{1}{2}} \oplus P_{\rho^c} |L_{\kappa,\rho^c}|^{-\frac{1}{2}},
\]
such that $L_{\kappa}(t)$ is equal to
\[
|L_{\kappa,\rho} + L_{\kappa,\rho^c}|^{-\frac{1}{2}} \left( S + t \left( |L_{\kappa,\rho}|^{-\frac{1}{2}} P_{\rho^c} h P_{\rho^c} |L_{\kappa,\rho^c}|^{-\frac{1}{2}} + |L_{\kappa,\rho^c}|^{-\frac{1}{2}} P_{\rho^c} h P_{\rho} |L_{\kappa,\rho^c}|^{-\frac{1}{2}} \right) \right) |L_{\kappa,\rho} + L_{\kappa,\rho^c}|^{-\frac{1}{2}},
\]
with $S$ being the unitary from the polar decomposition of $L_{\kappa,\rho} + L_{\kappa,\rho^c}$. By a Neumann series argument, this is invertible for $t \leq 1$ if the norm of the off-diagonal component is less than 1. Since
\[
\left\| |L_{\kappa,\rho}|^{-\frac{1}{2}} P_{\rho^c} h P_{\rho} |L_{\kappa,\rho^c}|^{-\frac{1}{2}} \right\| \leq \frac{2 \frac{1}{2} \|h\|}{g^2 (\rho^2 \kappa^2)^{\frac{1}{4}}},
\]
this is the case for $\rho$ large enough.

5 Proof of the odd index formula

This section provides the proof of the odd case of Theorem 12. Thus let us consider an odd spectral triple and suppose that $(\kappa, \rho)$ is admissible. By Lemma 13 (ii) it is enough to prove the signature formula for some admissible $(\kappa, \rho)$ and we assume $\rho$ to be large enough for Lemma 14 to hold. As $a$ is invertible, it is connected to its polar decomposition $u = a |a|^{-\frac{1}{2}}$ by a continuous path and hence the spectral flow formula (4) gives
\[
\langle [u], [D] \rangle = \text{Sf} \left( \left( \begin{array}{cc} u^* & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \kappa D & 0 \\ 0 & -\kappa D \end{array} \right) \left( \begin{array}{cc} u & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} \kappa D & 0 \\ 0 & -\kappa D \end{array} \right) \right) 
\]
\[
= \text{Sf} \left( \left( \begin{array}{cc} \kappa D & 0 \\ 0 & -\kappa D \end{array} \right), \left( \begin{array}{cc} u^* & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \kappa D & 0 \\ 0 & -\kappa D \end{array} \right) \left( \begin{array}{cc} u^* & 0 \\ 0 & 1 \end{array} \right) \right).
\]
Noting that the path is in the invertibles except at the left endpoint, which has by assumption a trivial kernel, one must have
\[
\text{Sf} \left( \left( \begin{array}{cc} \kappa D & 0 \\ 0 & -\kappa D \end{array} \right), \left( \begin{array}{cc} \kappa D & 1 \\ 1 & -\kappa D \end{array} \right) \right) = 0,
\]
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and hence also
\[
\text{Sf}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa D & 1 \\ 1 & -\kappa D \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.
\]
Using the concatenation property of the spectral flow and deforming the resulting path again to a straight-line path with a homotopy in the space of Fredholm operators implies
\[
\langle [u], [D] \rangle = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \kappa D & 1 \\ 1 & -\kappa D \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix}\right) = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D u u^* & u \\ u^* & -\kappa D \end{pmatrix}\right).
\]
Now \(\kappa u Du^* = \kappa D + \kappa u[D, u^*]\) and \(\kappa u[D, u^*]\) is a bounded summand that for \(\kappa\) sufficiently small does not alter the invertibility of the spectral localizer of \(u\) so that
\[
\text{Sf}\left(\begin{pmatrix} \kappa D u u^* & u \\ u^* & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D u u^* & u \\ u^* & -\kappa D \end{pmatrix}\right) = 0.
\]
Therefore again by concatenation and deforming the paths one obtains
\[
\langle [u], [D] \rangle = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & u \\ u^* & -\kappa D \end{pmatrix}\right).
\]
Finally one can connect \(u\) to \(a\) by the path \(t \in [0, 1] \mapsto a(t + (1 - t)|a|^{-1})\) leading to a path of invertible spectral localizers. Hence again by concatenation and deformation of the straight-line paths
\[
\langle [u], [D] \rangle = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, \begin{pmatrix} \kappa D & a \\ a^* & -\kappa D \end{pmatrix}\right) = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, L_\kappa\right).
\]
Now by Lemma 14 one can decouple \(L_\kappa\) to \(L_{\kappa,\rho} \oplus L_{\kappa,\rho^c}\) so that, once again by concatenation and deforming the linear paths,
\[
\langle [u], [D] \rangle = \text{Sf}\left(\begin{pmatrix} \kappa D & 0 \\ 0 & -\kappa D \end{pmatrix}, L_{\kappa,\rho} \oplus L_{\kappa,\rho^c}\right).
\]
Finally also the first entry is diagonal in the decomposition \(P_\rho \oplus P_{\rho^c}\) so that
\[
\langle [u], [D] \rangle = \text{Sf}\left(\begin{pmatrix} \kappa D_{\rho} & 0 \\ 0 & -\kappa D_{\rho} \end{pmatrix} \oplus \begin{pmatrix} \kappa D_{\rho^c} & 0 \\ 0 & -\kappa D_{\rho^c} \end{pmatrix}, L_{\kappa,\rho} \oplus L_{\kappa,\rho^c}\right)
\]
\[
= \text{Sf}\left(\begin{pmatrix} \kappa D_{\rho} & 0 \\ 0 & -\kappa D_{\rho} \end{pmatrix}, L_{\kappa,\rho}\right),
\]
because one can use the homomorphism property of Proposition 6(iv) and the fact that the spectral flow on \(P_{\rho^c}\) vanishes since the path is in the invertibles due to (11). Finally, the signature formula Proposition 9 completes the proof of the odd case of Theorem 12 by noting that the signature of the left endpoint vanishes.
6 Proof of the even index formula

In the even case not only \([D, a]\) is assumed to be bounded for every \(a \in \mathcal{A}\), but \([\|D\|, a]\) as well. This implies that the (in general inequivalent) representation of \(\mathcal{A}\) in \(\mathcal{N}\) given by

\[
a \in \mathcal{A} \mapsto \pi^+(a) = \begin{pmatrix} a^+ & 0 \\ 0 & F_{a^+}F^* \end{pmatrix}
\]

also defines an even spectral triple \((\mathcal{N}, \pi^+(\mathcal{A}), D)\). However, writing out (6) one finds that its index pairing vanishes. The spectral localizer with respect to this triple is still useful and is denoted by

\[
L^+_{\kappa} := \begin{pmatrix} h^+ & \kappa D_0^* \\ \kappa D_0 & -Fh^+F^* \end{pmatrix},
\]

and analogously \(L^+_{\kappa, \rho} = P_\rho L^+_{\kappa} P_\rho\). Note that in a spectral triple \([a, \text{sgn}(D)]\) is \(\tau\)-compact for all \(a \in \mathcal{A}\) and hence also \(a - \pi^+(a) \in \mathcal{K}\), so e.g. the spectral flow from \(L^+\) to \(L^+_{\kappa}\) is well-defined.

The starting point for the proof of the even signature formula is the spectral flow formula

\[
\langle [p], [D] \rangle = \text{Sf}(h^+, Fh^+F^*) \text{ given in (7).}
\]

The additivity of the spectral flow leads to

\[
\langle [p], [D] \rangle = \text{Sf}\left(\begin{pmatrix} h^+ & 0 \\ 0 & -Fh^+F^* \end{pmatrix}, \begin{pmatrix} h^+ & 0 \\ 0 & -h^- \end{pmatrix}\right) = \text{Sf}\left(\pi^+(h)\gamma, h\gamma\right).
\]

Again by Lemma 13(ii) it is enough to prove the signature formula for some admissible \((\kappa, \rho)\), so one can assume that \(\kappa\) is as small and \(\rho\) as large as necessary.

Lemma 15 For \(\kappa\) small enough, one has \(\text{Sf}(h\gamma, L^+_{\kappa}) = 0\) and \(\text{Sf}(\pi^+(h)\gamma, L^+_{\kappa}) = 0\).

Proof. Considering that

\[
((1 - t)(h\gamma) + tL^+_{\kappa})^2 = h^2 + t^2\kappa^2D^2 + t\kappa[h, D] \geq (g^2 - \kappa\|D, h\|)1,
\]

the straight-line path connecting \(h\gamma\) and \(L^+_{\kappa}\) is invertible for \(\kappa\|D, h\| < g^2\). The same holds for the other path since \((\mathcal{N}, \pi^+(\mathcal{A}), D)\) is also a spectral triple.

Since the intermediate paths have compact differences, we can concatenate and then deform back to a straight-line path, hence

\[
\langle [p], [D] \rangle = \text{Sf}\left(L^+_{\kappa}, L^+_{\kappa}\right).
\]

According to Lemma 14 the endpoints decouple, namely

\[
\text{Sf}(L^+_{\kappa}, L_{\kappa, \rho} \oplus L_{\kappa, \rho'}) = 0 = \text{Sf}(L^+_{\kappa}, L^+_{\kappa, \rho} \oplus L^+_{\kappa, \rho'})
\]

for sufficiently large \(\rho\). The paths that shrink the off-diagonal parts of the localizers to zero again have compact differences, so the spectral flow can be decomposed into two summands.
using the additivity. The contribution on $P_\rho$ can be expressed in terms of the signature due to Proposition 9:

$$Sf(L^+_{\kappa,\rho}, L_\kappa) = Sf(L^+_{\kappa,\rho}, L_{\kappa,\rho}) + Sf(L^+_{\kappa,\rho'}, L_{\kappa,\rho'})$$

$$= \frac{1}{2}(\text{Sig}(L_{\kappa,\rho}) - \text{Sig}(L^+_{\kappa,\rho})) + Sf(L^+_{\kappa,\rho}, L_{\kappa,\rho'}).$$

Considering that $L^+_{\kappa,\rho}$ is unitarily equivalent to its negative due to

$$-L^+_{\kappa,\rho} = \begin{pmatrix} 0 & F^* \\ -F & 0 \end{pmatrix} L^+_{\kappa,\rho} \begin{pmatrix} 0 & -F^* \\ F & 0 \end{pmatrix},$$

the signature $\text{Sig}(L^+_{\kappa,\rho})$ vanishes. It only remains to show that the last summand also vanishes:

**Lemma 16** For $\kappa \rho$ large enough, one has $Sf(L^+_{\kappa,\rho'}, L_{\kappa,\rho'}) = 0$.

**Proof.** Again let us consider the square

$$((1 - t)L^+_{\kappa,\rho'} + tL_{\kappa,\rho'})^2$$

$$= \kappa^2 D^2_{\rho'} + t\kappa[h_{\rho'}, D_{\rho'}] + (1 - t)\kappa[\pi^+(h)_{\rho'}, D_{\rho'}]$$

$$+ \left( \begin{array}{cc} (h_+)^2_{\rho'} & 0 \\ 0 & t^2(h_-)^2_{\rho'} + (1 - t)^2(Fh_+F^*)_{\rho'}^2 + t(1 - t)[(h_-)_{\rho'}, (Fh_+F^*)_{\rho'}] \end{array} \right)$$

$$\geq \kappa^2 \rho^2 - \kappa ||[D, h]|| - \kappa ||[D, \pi^+(h)]|| - ||h||^2$$

where $h_{\rho'} = P_{\rho'} h P_{\rho'}$ and $(h_+)^2_{\rho'} = P_{\rho'}^2 h_{\rho'}^2 P_{\rho'}$ with $P_{\rho'} = \text{diag}(P_{\rho'}^+, P_{\rho'}^-)$. This shows that the path lies within the invertibles for large $\kappa \rho$.\hfill $\square$

### 7 Application to topological insulators

If the semifinite von Neumann algebra $(\mathcal{N}, \tau)$ is of type I$_{\infty}$, namely given by a pair $(\mathcal{B}(\mathcal{H}), \text{Tr})$, then the finite volume spectral localizer $L_{\kappa,\rho}$ is a finite-dimensional matrix and it is immediately possible to use it for numerical computation of the index pairing based on Theorem 12, see [9, 10, 14]. In the setting of a type II$_{\infty}$-von Neumann algebra, the spectral localizer $L_{\kappa,\rho}$ is in general an operator of infinite rank, but its signature may still be well approximated by finite-dimensional quantities that are accessible to numerical computations. Here we sketch how this can be done for a large class of Schrödinger-type operators describing topological insulators. A typical example for an observable algebra is the disordered non-commutative torus $A = C(\Omega) \rtimes_\xi \mathbb{Z}^d$, constructed from an invariant ergodic probability space $(\Omega, \mathbb{Z}^d, \mathbb{P})$ describing homogeneous disorder and a twist $\xi$ provided by the magnetic field [18]. In the representation on $l^2(\mathbb{Z}^d)$, one considers for $n$ lattice directions $e_1, \ldots, e_n$ the Dirac operator $D = \sum_{k=1}^n \sigma_k \otimes X_k$ with $\sigma_k$ a representation of the complex Clifford algebra of $n$ generators and $X_1, \ldots, X_n$ the unbounded position operators. For $n < d$, the non-integer-valued index pairing with $D$ is a so-called weak Chern number and is localized by a spectral triple $(\mathcal{A}, \mathcal{N}, D)$ (cf. [2, 3]), where $\mathcal{N}$ is the von Neumann algebra generated by $\mathcal{A}$ and bounded functions of
$X_1, \ldots, X_n$ equipped with a trace $\tau$ that can be interpreted as an average trace per volume. A self-adjoint invertible $h \in M_N(A_d)$, assumed to take the form $h = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ for $n$ odd, describes a random family $(h_\omega)_{\omega \in \Omega}$ of Hamiltonians on $\ell^2(\mathbb{Z}^d, \mathbb{C}^N)$ modeling a topological insulator and the index pairings $\langle [\chi(h \leq 0)], [D] \rangle$, respectively $\langle [a \mid a^{-1}], [D] \rangle$, are related to linear response coefficients and the appearance of topologically protected boundary states [18].

The spectral localizer $L_{\kappa, \rho} = (L_{\kappa, \rho, \omega})_{\omega \in \Omega}$ can be considered a random family of Schrödinger-type Hamiltonians that describes the restriction of $h$ to a cylinder $B^n_\rho \times \mathbb{Z}^{d-n}$ with an additional potential $\kappa D_\rho$. For numerical computations, one further truncates to operators $L^{(V_\ell)}_{\kappa, \rho, \omega}$ acting on the finite-dimensional space $\ell^2(B^n_\rho \times V_\ell)$ with $V_\ell$ being a cube with sides $\ell$ and supplied with e.g. periodic or Dirichlet boundary conditions. For $h$ satisfying the usual smoothness conditions and $f \in C_0(\mathbb{R})$, one can show that almost surely with respect to $\mathbb{P}$

$$\tau(f(L_{\kappa, \rho})) \overset{a.s.}{=} \lim_{\ell \to \infty} \frac{|B^n_\rho|}{|V_\ell|} \text{Tr}(f(L^{(V_\ell)}_{\kappa, \rho, \omega})).$$

Since $L_{\kappa, \rho}$ has a spectral gap, one can replace $\text{Sig}(L_{\kappa, \rho}) = \tau(\text{sgn}(L_{\kappa, \rho})) = \tau(f(L_{\kappa, \rho}))$ for a suitable continuous function $f$ and hence the signature can be approximated using only finite-dimensional algebra. When choosing restrictions with periodic boundary conditions, one can adapt methods from [17] to show that the approximations $L^{(V_\ell)}_{\kappa, \rho, \omega}$ have a uniform spectral gap and

$$\text{Sig}(L_{\kappa, \rho}) \overset{a.s.}{=} \lim_{\ell \to \infty} \frac{|B^n_\rho|}{|V_\ell|} \left[ \#(\text{positive eigenvalues of } L^{(V_\ell)}_{\kappa, \rho, \omega}) - \#(\text{negative eigenvalues of } L^{(V_\ell)}_{\kappa, \rho, \omega}) \right],$$

where the deterministic component $|\text{Sig}(L_{\kappa, \rho}) - |B^n_\rho||V_\ell|^{-1} \mathbb{E} \text{Sig}(L^{(V_\ell)}_{\kappa, \rho, \omega})|$ of the finite volume error is exponentially small in $\ell$ for typical Hamiltonians. We expect that the method described here generalizes to compute weak invariants of other aperiodic quantum systems which are e.g. described by point patterns [2].

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