Floquet Gauge Pumps as Sensors for Spectral Degeneracies Protected by Symmetry or Topology

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We introduce the concept of a Floquet gauge pump whereby a dynamically engineered Floquet Hamiltonian is employed to reveal the inherent degeneracy of the ground state in interacting systems. We demonstrate this concept in a one-dimensional XY model with periodically driven couplings and transverse field. In the high-frequency limit, we obtain the Floquet Hamiltonian consisting of the static XY and dynamically generated Dzyaloshinsky-Moriya interaction (DMI) terms. The dynamically generated magnetization current depends on the phases of complex coupling terms, with the XY interaction as the real and DMI as the imaginary part. As these phases are cycled, the current reveals the ground-state degeneracies that distinguish the ordered and disordered phases. We discuss experimental requirements needed to realize the Floquet gauge pump in a synthetic quantum spin system of interacting trapped ions.

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Introduction.—The nontrivial topology of gapped phases of matter is often manifested in states or modes localized at the boundary or defects of the system protected by symmetries and the bulk topological gap. This bulk-boundary correspondence has been rigorously proven in certain cases, especially in noninteracting systems [1–7]. Thus, the presence and properties of boundary modes is used as a telltale experimental signature of bulk topology [8]. However, boundary modes can arise in many other topologically trivial cases as well [9]. Moreover, the structure of boundary modes is far from clear in generic interacting many-body systems [10–13]. Therefore, more robust probes of bulk topology are highly sought after [14–19].

An example is provided by the fractional Josephson current $J_f$ between topological superconductors supporting Majorana bound states as the phase difference $\Delta \phi$ between the superconductors is cycled by changing the magnetic flux enclosed by the system [20–23]. Unlike the Cooper-pair-mediated conventional Josephson current between trivial superconductors $J_s \propto \sin \Delta \phi$, the fractional Josephson current is dominated by quasiparticle tunneling through the Majorana bound states, $J_f \propto \sin(\Delta \phi/2)$. In the presence of interactions, the fractional Josephson current probes the topological degeneracy of the interacting ground state in a given fermion parity sector [11].

In this Letter, we consider a general spatially resolved probe provided by the variations in the current flowing through the bridge between two gapped phases as a relevant gauge field is varied in a cycle. Ground-state degeneracies through this cycle produce an anomalous periodicity of the corresponding current on the cycle parameters \[24,25\].
A useful probe of symmetry-protected degeneracies of topological and ordered phases of quantum matter.

Floquet gauge pump.—A gauge pump is realized by a cyclic Hamiltonian $H(\phi)$ in the gauge parameters $\phi$ [31]. A constant $\phi$ can be gauged away as $U_{t}^{\dagger}(\phi)H(\phi)U_{t}(\phi) = H(0)$ with a unitary gauge transformation $U_{t}(\phi)$. The gauge pump is constructed by “bridging” two such Hamiltonians, $H_{L}$ and $H_{R}$, to form $H_{\text{gp}} = H_{L}(\phi_{L}) \otimes \mathbb{1}_{R} + H_{LR} + \mathbb{1}_{L} \otimes H_{R}(\phi_{R})$, where $\mathbb{1}_{R}(\mathbb{1}_{L})$ is the identity operator on the left (right) side of the bridge given by $H_{LR}$. After a gauge transformation $U_{g} = U_{gL}(\phi_{L}) \otimes U_{gR}(\phi_{R})$, we have the gauge-equivalent Hamiltonian $U_{g}^{\dagger}H_{\text{gp}}U_{g} = H_{L}(0) \otimes \mathbb{1}_{R} + H_{b}(\phi_{b}, \phi_{p}) + \mathbb{1}_{L} \otimes H_{R}(0)$, where $\phi_{b} \equiv \phi_{L} - \phi_{R}$, $\phi_{p} \equiv \phi_{L} + \phi_{R}$, and $H_{b} = U_{g}^{\dagger}H_{LR}U_{g}$. Then, the gauge currents on each side are [33]

$$
\begin{align*}
    j_{L} &= \frac{\partial H_{b}}{\partial \phi_{p}} = \left\langle \frac{\partial H_{b}}{\partial \phi_{p}} \right\rangle_{g} + \left\langle \frac{\partial H_{b}}{\partial \phi_{p}} \right\rangle_{g}, \\
    j_{R} &= \frac{\partial H_{b}}{\partial \phi_{R}} = \left\langle \frac{\partial H_{b}}{\partial \phi_{p}} \right\rangle_{g} - \left\langle \frac{\partial H_{b}}{\partial \phi_{p}} \right\rangle_{g},
\end{align*}
$$

where the expectation values $\langle \cdots \rangle_{g} = \langle U_{g} \cdots U_{g}^{\dagger} \rangle$. Note that the current flow is set with respect to the bridge, so the positive values have opposite directions on each side. If the bridge Hamiltonian $H_{b}$ is a function of $\phi_{b}$ only, the two currents $j_{L} = -j_{R}$ and no gauge charge is accumulated in the bridge itself. However, if the bridge Hamiltonian also depends on $\phi_{p}$, then $j_{b} = j_{L} + j_{R} = 2\left\langle \frac{\partial H_{b}}{\partial \phi_{p}} \right\rangle_{g}$ must be carried by the bridge itself. If the bridge is “grounded,” e.g., in a transport geometry of a mesoscopic device, this can flow through the bridge. Otherwise, the gauge charge will accumulate in the bridge.

The Floquet gauge pump provides two complementary functions. First, as we show below, periodic drive protocols with spatial variations may be used both to engineer the gauge parameters and to imitate the gauge pump geometry [28–30]. Second, drive parameters can be tuned to engineer Floquet topological phases of the system [34–46]. The signatures of both equilibrium and Floquet topological phases can then be probed by the dependence of the gauge current pumped through the system on tunable gauge parameters. To illustrate this, note that the stroboscopic dynamics of the driven Hamiltonian $H_{\text{gp}}(t) = H_{L}(t) \otimes 1_{R} + H_{LR}(t) + 1_{L} \otimes H_{R}(t) = H_{\text{gp}}(t + 2\pi/\Omega)$ with drive frequency $\Omega$ is governed by the Floquet Hamiltonian $H_{\text{F}} = i(\Omega/2\pi)\ln \text{Texp}[\mathcal{O}]$, where $\mathcal{O}$ is the quasieigenvalue and $\text{Texp}[\mathcal{O}] = \mathcal{O}$. However, at intermediate times that can be extremely long for sufficiently large systems, the Floquet state describes the dynamics of the system rather well. At high enough frequency, in particular, an initial equilibrium state of the average Hamiltonian $H_{\text{F}}$ is nearly the same as the Floquet state. In the following, we will assume that this is indeed the case and calculate gauge currents from the Floquet spectrum.

![FIG. 1. Sketch of the Floquet gauge pump and its ion-trap realization. (a) By periodically driving the left ($L$) and right ($R$) sides of the system with different gauge parameters, the pump geometry and currents $j_{L}$ and $j_{R}$ can be engineered and controlled. (b) Trapped ions can realize a Floquet gauge pump with effective spin degrees of freedom, in which magnetization currents $\dot{S}_{L,R}$ can be pumped by controlling the drive protocols on each side.](image)

where

\begin{equation}
    H_{LR}^{F} = H_{LR}^{(0)} + \sum_{n \in \mathbb{N}} \left\{ \frac{H_{LR}^{(n-1)} \cdot H_{LR}^{(n)}}{n\Omega} \right\},
\end{equation}

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where $O^{(n)} = (\Omega/2\pi) \int_{0}^{T_{\text{exp}}} O(t)e^{i\delta\omega t} dt$ are the Fourier components of operator $O$ and H.c. is the Hermitian conjugate. We denote the gauge parameters in this Floquet Hamiltonian as $\phi_{L}(\lambda)$ and $\phi_{R}(\lambda)$ with $\lambda$ denoting the drive parameters, such as frequency, harmonic amplitudes, and phases. The twofold function of the Floquet gauge pump is then to provide independent realizations and tuning of $\phi_{L}$ and $\phi_{R}$. Figure 1 sketches our setup.

The gauge currents $j_{a}(t) = \langle \frac{\partial H_{\text{gp}}(t)}{\partial \phi_{a}} \rangle$ are now time dependent. We show in the Supplemental Material [33] that, if the expectation value is calculated in Floquet modes $\langle \Psi(t) \rangle = e^{-i\epsilon(\Phi(t))}$, where $\epsilon$ is the quasieigenvalue and $\Phi(t + 2\pi/\Omega) = \Phi(t)$ is the periodic eigenstate satisfying $H_{\text{gp}}(t + i\partial/\partial t)\Phi(t) = e^{i\epsilon(\Phi(t))}$, then the average gauge current is

\begin{equation}
    j_{a}^{(0)} = \frac{\partial \epsilon}{\partial \phi_{a}}.
\end{equation}

The choice of physical state of the driven system requires care. Generic driven systems would, at infinitely long times, settle into a uniform mixed state by absorbing energy from the drive without bound [52,53], with the exception of integrable or many-body localized systems [54–58]. However, at intermediate times that can be extremely long for sufficiently large systems, the Floquet state describes the dynamics of the system rather well. At high enough frequency, in particular, an initial equilibrium state of the average Hamiltonian $H_{\text{F}}$ is nearly the same as the Floquet state. In the following, we will assume that this is indeed the case and calculate gauge currents from the Floquet spectrum.
Spin model.—To illustrate the concepts, here we consider a driven XY model in a transverse field $H_{XY}(t) = \sum_{j} J_j^x(t) S_j^x S_{j+1}^x + J_j^y(t) S_j^y S_{j+1}^y + J_{ij}(t) S_j^z S_i^z$, where $\{S_j^x, S_j^y, S_j^z\}$ are spin-$\frac{1}{2}$ operators, $\{J_j^x, J_j^y\}$ are nearest-neighbor couplings, and $h_i(t)$ is the transverse field at lattice site $j$. We take the periodic drive to be independent and uniform on each side with $J_j^x(t) = J_{x}^0 + \delta J_{x}^0 \cos(\Omega t - \theta_j)$, and $h_i(t) = h_{i}^0 + \delta h_{i}^0 \cos(\Omega t - \theta_i)$. With this choice, the high-frequency Floquet Hamiltonian takes the form [33] $H_{FY} = H_{L}^F + H_{LR}^F + H_{R}^F + O(\Omega^{-2})$.

$$H_{L}^F = \tilde{H}_a - \sum_{j=0}^{\infty} \zeta_a(j) S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)$$

$$H_{LR}^F = \tilde{H}_{LR} + \zeta_{LR}(S_j^x S_{j+1}^y + S_{j+1}^x S_j^y)$$

$$+ \zeta_{LR}(S_j^y S_{j+1}^x - S_{j+1}^y S_j^x).$$

Here $\tilde{H}_a$ is the average Hamiltonian on side $a$ and $\zeta_a = -\delta J_a^x \delta h_a^0 \sin(\theta_a^0 - \theta_j)/\Omega$ with $\delta J_a^x = (\delta J_a^x \pm \delta J_a^y)/2$. At the junction connecting the sites $\ell \in L$ and $r \in R$, $\tilde{H}_{LR}$ is the contribution from the averaged Hamiltonian, $\zeta_{LR} = -\delta J_{LR}^x \sin(\theta_j^R - \theta_j^L) \delta h_{LR}^0 \sin(\theta_j^L - \theta_j^R) \delta h_{LR}^0)/2(2\Omega)$, and $\zeta_{LR} = -\delta J_{LR}^y \sin(\theta_j^L - \theta_j^R) \delta h_{LR}^0 \sin(\theta_j^R - \theta_j^L) \delta h_{LR}^0)/2(2\Omega)$ is the DMI term dynamically generated by the drive. The $z$ component of the magnetization current on side $a$ is defined as $J_a = d<N_a>/dt$, where $N_a = \sum_{j=0}^{\infty} S_j^z$.

It is mathematically convenient to analyze the spin gauge pump in the dual fermionic language. To this end, we employ the Jordan-Wigner transformation [26] $S_j^x + iS_j^y = P_j c_j^\dagger$, with number operator $n_j = c_j^\dagger c_j$ (fermionic operators) at site $j$, and $P_j = \prod_{l<j} c_l e^{i\pi n_l}$ as the fermion parity to the left of site $j$. This is followed by the gauge transformation $e^{i\phi_j} c_j^\dagger \rightarrow c_j^\dagger$ for $j \in a$, to find, up to a constant, the equivalent fermion Hamiltonian $\tilde{H}_a^F + H_b + H_{LR}^F$.

$$\tilde{H}_a^F = \sum_{j=0}^{\infty} [w_a c_j^\dagger c_{j+1} + \Delta_j c_j^\dagger c_j + \mu_a n_j] + \text{H.c.},$$

$$H_b = w_b e^{i\phi} c_j^\dagger c_{j+1} + \Delta_b e^{i\phi} c_j^\dagger c_j + \text{H.c.},$$

where chemical potential $\mu_a = \frac{1}{2}h_a^0$, hopping amplitudes $w_a = \frac{1}{2}J_a^x$ and $w_b = \frac{1}{2}|(J_{a}^x + i\zeta_{LR})|/4$, and pairing amplitudes $\Delta_a = \frac{1}{2}|(J_a^x + i\zeta_a)|$ and $\Delta_b = \frac{1}{2}|(J_a^x + i\zeta_{LR})|$, $J_a^x \approx (J_a^x \pm J_{a}^y)/2$ are all real, and

$$\phi_{b} = \phi_{bb} + (\phi_{L} - \phi_{R}),$$

$$\phi_{p} = \phi_{pb} + (\phi_{L} + \phi_{R}),$$

with $\phi_{a} = \frac{1}{2}\arg(J_a^x + i\zeta_a)$ half of the pairing phase on each side, $\phi_{bb} = \arg(J_a^x + i\zeta_{LR})$ and $\phi_{pb} = \arg(J_a^x + i\zeta_{LR})$.

This fermionic Hamiltonian is composed of a Kitaev chain [20] on each side and a bridge Hamiltonian with both hopping and pairing terms at the junction, thus realizing an unconventional Josephson junction that can be controlled by the original drive parameters. In terms of fermions, the currents $J_a = d<N_a>/dt$, where $N_a = \sum_{j=0}^{\infty} n_j$ is the fermion number operator on side $a$.

The Kitaev chain has two phases. For $|\mu_a| < 2w_a$, there are unpaired Majorana bound states at the junction and the spectrum is doubly degenerate. In terms of spins, this corresponds to the ordered phase of the XY model. By contrast, when $|\mu_a| > 2w_a$, the spectrum is nondegenerate and corresponds to a disordered spin chain. The presence or absence of Majorana fermions at the junction affects the current across the junction.

Floquet gauge current.—The average Floquet gauge current is given by Eq. (4), which requires the calculation of the quasienergies of $H_{FY}$. We will do so perturbatively in the bridge Hamiltonian, where the unperturbed system (without the bridge) has a Floquet spectrum $\{\Omega_0\}$ with quasienergies $\epsilon_\alpha$. The quasienergy of a state $|\Phi_{\alpha,a=0}\rangle$ in second-order perturbation theory is $\epsilon = \epsilon_0 + \epsilon_b$, where $\epsilon_b = \langle \Phi_0|H_b|\Phi_0\rangle + \sum_{\alpha \neq 0} |\langle \Phi_0|H_b|\Phi_\alpha\rangle|^2/|\epsilon_\alpha - \epsilon_0|$. We present the details of this calculation in the Supplemental Material [33] and quote the main results here.

In the trivial (disordered) phase, the first-order contribution vanishes since $H_b$ projects out of the unperturbed ground state. Then the current takes the form

$$j_L^{(0)} = F_c \sin(2\phi_b) + F_L \sin(\phi_b + \phi_h) + F_r \sin(2\phi_h),$$

$$j_R^{(0)} = F_c \sin(2\phi_b) + F_R \sin(\phi_b - \phi_h) - F_c \sin(2\phi_h),$$

where $F_c \propto \Delta_b^2$, $F_{LR} \propto w_b \Delta_b$, and $F_c \propto w_b^2$ are second order in bridge tunneling amplitudes. The parity of the nondegenerate state is fixed over the entire range of gauge parameters.
We illustrate the processes contributing to gauge currents in this case in Figs. 2(a)–2(d). For \( \Delta_\phi = 0, w_\phi \neq 0 \) we have the conventional Josephson junction and the current \( j_b = -j_R = F_c \sin(2\phi_R) \) mediated by Cooper pair tunneling across the bridge. When \( \Delta_\phi \neq 0, w_\phi = 0 \), the currents \( j_L = j_R = F_c \sin(2\phi_p) \) are mediated by Cooper pair cotunneling to or from the bridge. In the general case, \( \Delta_\phi, w_\phi \neq 0 \), the current \( j_\text{ch} \) contain cross terms \( \sim \sin(\phi_p \pm \phi_h) \), contributed by Cooper pair condensation at the bridge. Remarkably, in this case, the period of the Josephson current in \( \phi_p \) and \( \phi_h \) is doubled, similar to the junction with Majorana fermions and conserved fermion parity (see below) [64].

In the topological (ordered) phase, the first-order contribution dominates since \( H_\text{ch} \) can now connect the unperturbed degenerate states. Then [33]

\[
\begin{align*}
  j_{L}^{(0)} &= P_0(F_c \sin \phi_p + F_q \sin \phi_h), \\
  j_{R}^{(0)} &= P_0(F_c \sin \phi_p - F_q \sin \phi_h),
\end{align*}
\]

where \( F_c \propto \Delta_\phi \) and \( F_q \propto w_\phi \) are linear in the bridge tunneling amplitudes, and \( P_0 = \langle \Phi_0 | P | \Phi_0 \rangle \), where \( P = \prod \text{e}^{i \phi_h} \), is the parity of the state \( | \Phi_0 \rangle \) in the doubly degenerate manifold. Note that \( \epsilon_b/P_0 = -(F_c \cos \phi_p + F_q \cos \phi_h) \) switches sign in the cycle [33]. The term \( \sim \sin \phi_h \) is the fractional Josephson current mediated by Majorana quasiparticle tunneling across the bridge [22]. The term \( \sim \sin \phi_p \) is an unconventional Josephson current mediated by Cooper pair splitting at the bridge [65]; see Figs. 2(e) and 2(f).

The two phases of the spin system can thus be distinguished by the Floquet gauge current in two ways: (i) the linear (ordered) vs quadratic (disordered) dependence of the gauge current on the bridge tunneling amplitudes, and (ii) the dependence of the current on \( \phi_h \) and \( \phi_p \). The latter is often formulated as doubling of the periodicity of the current in the topological vs trivial phase of fermions. This, in turn, relies on the conservation of the parity \( P_0 \) of the fermionic ground state. If, instead, the system is prepared with the same sign of \( \epsilon_b \), the parity of the ground state exhibits switches accompanied by discontinuities in the gauge current at topologically protected degeneracies in the topological phase, which are absent in the trivial phase.

In terms of spins, the parity operator \( P = \exp[i \pi \sum (S^z_j + 1/2)] = \prod \langle -2S^z_j \rangle \). Therefore, the total fermion parity is the maximal multipoint spin-z correlator. If the spin state is independently prepared for each value of \( \phi_h \) and \( \phi_p \), we should expect the state with the lowest energy is chosen. Therefore, the gauge current would show the same periodicity in both phases, while in the ordered phase it will show discontinuities accompanied with parity switches reflected in the sign reversals of the maximal multipoint spin-z correlator.

**Numerical results.**—To demonstrate these effects concretely, we have calculated the gauge current by exact diagonalization of the Floquet Hamiltonian in (7) and (8). In Fig. 3, we plot the average current \( j_b^{(0)} = j_L^{(0)} + j_R^{(0)} \) through the bridge for a representative set of parameters realizing the trivial and topological phases of fermions as a function of gauge parameters \( \phi_h \) and \( \phi_p \). Here, we have chosen \( | \Phi_0 \rangle \) as the lowest energy state of the whole system. In the trivial phase, the current scales with \( w_\phi^2 \) and since both \( w_\phi, \Delta_\phi \neq 0 \), compared to the conventional Josephson junction with \( \Delta_\phi = 0 \), the periodicity in \( \phi_p \) and \( \phi_h \) is doubled. In the topological phase, the current scales with \( w_\phi \) and its discontinuities coincide with parity switches, as expected. Note that except for discontinuous jumps due to parity switches in \( P_0 \), \( j_b = 2P_0F_c \sin \phi_p \) has no other dependence on \( \phi_h \).

The magnetization current of the Floquet gauge pump realized in the driven XY model is shown in Fig. 4 as a
function of phase shifts $\theta^h_a - \theta^l$ across the bridge. Note that, as these drive parameters are varied, both phases $\phi_h$ and $\phi_l$ as well as the amplitudes $w_h, \Delta_h, w_o,$ and $\Delta_c$ change in a range determined by the drive amplitudes. We discuss this dependence in the Supplemental Material [33]. The difference between disordered [Fig. 4(a)] and ordered [Fig. 4(b)] phases is that there is a true discontinuous change in the current in Fig. 4(b), while the change in Fig. 4(a) is gradual. The sign of the maximal multipoint spin-$z$ correlator provides a complementary signature of ground-state degeneracy.

Experimental realization.—The Floquet gauge pump can be realized using trapped atomic ions, which are a well-established system for simulating the time evolution of spin-lattice Hamiltonians [66]. Ions form defect-free lattices and can support quantum coherence times of longer than 10 min [67]. Interactions between ions—which map to interactions between effective quantum spins—can be fully controlled and reprogrammed using laser light [68]. These features have made trapped ions the leading platform for establishing atomic frequency standards [69] and for performing quantum simulations of 1D spin chains.

All necessary components for implementing the Floquet gauge pump have been previously demonstrated in trapped-ion systems. Transverse-field Ising and XY models are routinely generated by driving stimulated Raman transitions between the effective spin states [70,71]. The resulting spin model depends upon the specific amplitude, frequency, and phase characteristics of the Raman laser beams, which can be controlled by passing the beams through an acousto-optic modulator (AOM) [72]. Periodically driving the amplitude of the rf signal applied to the AOMs will imprint itself as a periodic drive on the spin-spin couplings $\{J^P(t), J^Y(t)\}$ and transverse fields $\vec{h}(t)$; indeed, this technique has already been used to realize Floquet engineering of a trapped-ion crystal [73]. The one key challenge will be to implement asymmetric drives on the left and right halves of the chain. This may be solved by either (i) adding a second pair of Raman beams so that both halves can be independently addressed, or by (ii) adding a second pair of frequency components to the AOM rf drive to split a single Raman beam into two parts, each with its own amplitude, frequency, phase, and deflection angle.

Topological degeneracies are detected by discontinuities in $d\langle S^z(t)\rangle/dt$. At high frequencies $\Omega \sim \text{MHz}$ in Fig. 4, $d\langle S^z(t)\rangle/dt \sim 100$ Hz (see [33]). Full contrast is obtained over $\sim 10$ ms. High resolution of 1 part in 1000 in $\theta^h$ is also easily achieved. A strength of trapped-ion systems is the ability to perform site-resolved spin-dependent fluorescence, which acts as a projective measurement along the $z$ direction and can discriminate between spin states with $> 99.9\%$ fidelity [74]. Since all effective spins are detected simultaneously during the measurement, all possible spin correlators (including the $N$-body correlator $(S_1^z S_2^z \ldots S_N^z)$) can be reconstructed when repeated trials are averaged.

Concluding remarks.—Dynamical probes to interrogate and uncover the emergent discrete symmetries that give rise to ground-state degeneracies are key to discovering topological or broken-symmetry phases of matter. The Floquet gauge pump we have introduced in this Letter is a unique experimental tool conceiving this goal.

Although our proposed proof-of-principle implementation in an ion-trap platform involved simple many-body systems, we expect that the Floquet gauge pump becomes a routine technique even in complex interacting systems. The required ingredient is the existence of the gauge group. A sufficient condition in the spin models is the absence of couplings between the directions parallel and perpendicular to the field. The gauge group is then the rotations around the field, which relabels the axes in the perpendicular direction and leaves the spectrum unchanged. In the XY model, for example, we can see immediately that the gauge pump works just as well in the presence of $S^z$-dependent interactions for spins [75,76], since these operators commute with the gauge current [33]. In practical setups of ion-trap simulators, tunable, variable-range interactions between effective spin degrees of freedom are realized, also in two-dimensional geometries [77], adding the possibility of magnetic frustration and potentially leading to exotic quantum spin liquid phases. Moreover, unveiling discrete symmetries can help in understanding the mechanisms leading to the formation of localized many-body boundary modes [11,13,24]. This is of fundamental importance for practical applications, in particular, if the vacuum is topological and those modes represent quasiparticles with non-Abelian braiding statistics.

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