Free boundary regularity for a degenerate fully non-linear elliptic problem with right hand side

by
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Abstract
We consider an one-phase free boundary problem for a degenerate fully non-linear elliptic operators with non-zero right hand side. We use the approach present in [DeS] to prove that flat free boundaries and Lipschitz free boundaries are $C^{1,\gamma}$.

KEYWORDS: free boundary problems, degenerate fully non-linear elliptic operators, regularity theory.

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1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^n$ and $\mu \geq 0$ we consider the degenerate fully non-linear elliptic problem

\begin{equation}
\begin{cases}
\mathcal{L}_\mu u = f, & \text{in } \Omega_+(u), \\
|\nabla u| = Q, & \text{on } \mathcal{F}(u),
\end{cases}
\end{equation}

where $\mathcal{L}_\mu u := |\nabla u|^{\mu} \Delta u$, $Q \geq 0$ is a $C^{0,\alpha}$-continuous function, $f \in L^\infty(\Omega) \cap C(\Omega)$ and

$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$ and $\mathcal{F}(u) := \partial \Omega^+(u) \cap \Omega$.

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The study of the regularity of the free boundary $\mathcal{F}(u)$, to the problem \ref{1.1} has a large literature:

1. **Non-degenerate.** The case $\mu = f = 0$, was studied in the seminal works of Caffarelli: \cite{AC, C1, C2}. In the context of fully non-linear elliptic equations, the homogeneous problem $f = 0$ was addressed in \cite{FE1, FE2, W1, W2, FEL1, FEL2}. The non-homogeneous case $f \neq 0$ was studied in \cite{DeS} and \cite{DFS1}.

2. **Degenerate.** For $\mu > 0$, there are not results about problem \ref{1.1}.

In this paper we will develop the regularity theory of $\mathcal{F}(u)$. Precisely, we will apply the technique presented in \cite{DeS} to prove that flat free boundaries are $C^{1,\gamma}$ (see section 2 for the definition of viscosity solutions):

**Theorem 1.1.** Let $u$ be a viscosity solution to \ref{1.1} in ball $B_1(0)$. Suppose that $0 \in \mathcal{F}(u)$ and $Q(0) = 1$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if the graph of $u$ is $\bar{\varepsilon}$-flat in $B_1(0)$, i.e.

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+ \text{ for } x \in B_1(0),$$

and

$$\|f\|_{L^\infty(B_1(0))} \leq \bar{\varepsilon}, \quad |Q|_{C^{0,\alpha}(B_1(0))} \leq \bar{\varepsilon},$$

then $\mathcal{F}(u)$ is $C^{1,\beta}$ in $B_2(0)$.

As in \cite{DeS}, the strategy of the proof of Theorem 1.1 is to obtain the improvement of flatness property for the graph of a solution $u$: if the graph of $u$ oscillates away $\varepsilon$ from a hyperplane in $B_1$ then in $B_{2\varepsilon}$ it oscillates $\frac{\varepsilon}{\bar{\varepsilon}}$ away from possibly a different hyperplane. The fundamental steps to achieve this property are: Harnack type Inequality and Limiting solution. In our problem, the structure of the operator $\mathcal{V}_\mu$ requires some changes. In next section, we comment on the main difficulties we came across and how to overcome them.

Moreover, through a blow-up from Theorem 1.1 and the approach used in \cite{C1}, we obtain the our second main result:

**Theorem 1.2** (Lipschitz implies $C^{1,\beta}$). Let $u$ be a viscosity solution for the free boundary problem

$$\begin{cases}
\mathcal{V}_\mu u = f, & \text{in } \Omega_+(u), \\
|\nabla u| = Q, & \text{on } \mathcal{F}(u).
\end{cases}$$

Assume that $0 \in \mathcal{F}(u)$, $f \in L^\infty(B_1)$ is continuous in $B_1^+(u)$ and $Q(0) > 0$. If $\mathcal{F}(u)$ is a Lipschitz graph in a neighborhood of 0, then $\mathcal{F}(u)$ is $C^{1,\beta}$ in a (smaller) neighborhood of 0.

In Theorem 1.2 the size of the neighborhood where $\mathcal{F}(u)$ is $C^{1,\beta}$ depends on the radius $r$ of the ball $B_r$ where $\mathcal{F}(u)$ is Lipschitz, on the Lipschitz norm of $\mathcal{F}(u)$, on $n$, $\alpha$ and $\|f\|_{L^\infty}$. We also emphasize that to obtain the Theorem 1.2 via the improvement of flatness property for the graph of $u$, we will need Lipschitz regularity and non-degeneracy for $u$. As in \cite{DeS}, we will use Harnack Inequality and Maximum Principle for solutions of the equation $\mathcal{V}_\mu v = f$ in balls to establish Lipschitz regularity and non-degeneracy for $u$. Since we do not have Harnack Inequality available for $n > 2$, see \cite{BD} for $n = 2$, Theorem 1.2 will be proved for the case $n = 2$.

Finally, we believe that Theorems 1.1 and 1.2 can be established to the more general operator $\mathcal{V}_\mu u = |\nabla u|^r F(D^2u)$, where $F$ is uniformly elliptic and satisfies homogeneity property:

1. (Ellipticity condition) There exist constants $0 < \lambda \leq \Lambda$ such that for any $M, N \in \text{Sym}(n)$, with $M \geq 0$ there holds

$$\lambda\|M\| \leq F(N + M) - F(N) \leq \Lambda\|M\|.$$
2. (Homogeneity condition) For all $t \in \mathbb{R} - \{0\}$ and $M \in \text{Sym}(n)$,

$$F(tM) = |t|F(M).$$

The paper is organized as follows. In Section 2 we define the notion of viscosity solution to the free boundary problem (1.1) and gather few tools that we shall use in the proofs of Theorem 1.1 and Theorem 1.2. In Section 3 we present the proof of Harnack type inequality. Section 4 is devoted to the proof of improvement of flatness and in Section 5 we establish the regularity of the free boundary $\mathcal{F}(u)$.

2 Preliminaries

2.1 Notation and Definitions

Let us move towards the hypotheses, set-up and main notations used in this article. For $B_1$ we denote the open unit ball in the Euclidean space $\mathbb{R}^n$. Furthermore, if $x \in \mathbb{R}^n$ we denote $x = (x_1, \ldots, x_n)$. We start by gathering some basic information of the limiting configuration. We shall use viscosity solution setting to access the free boundary regularity theory.

**Definition 2.1.** Given two continuous functions $u$ and $\phi$ defined in an open $\Omega$ and a point $x_0 \in \Omega$, we say that $\phi$ touches $u$ by below (resp. above) at $x_0$ whenever $u(x_0) = \phi(x_0)$

$$u(x) \geq \phi(x) \text{ (resp. } u(x) \leq \phi(x) \text{)} \text{ in a neighborhood } \mathcal{O} \text{ of } x_0.$$

If this inequality is strict in $\mathcal{O} \setminus \{x_0\}$, we say that $\phi$ touches $u$ strictly by below (resp. above).

**Definition 2.2.** Let $u \in C(\Omega)$ nonnegative. We say that $u$ is a viscosity solution to

$$\begin{cases} \mathcal{L}_\mu u = f, & \text{in } \Omega_+(u), \\ |\nabla u| = Q, & \text{on } \mathcal{F}(u). \end{cases} \tag{2.1}$$

if and only if the following conditions are satisfied:

(F1) If $\phi \in C^2(\Omega^+(u))$ touches $u$ by below (resp. above) at $x_0 \in \Omega^+(u)$ then

$$\mathcal{L}_\mu \phi(x_0) \leq f(x_0) \quad \text{(resp. } \mathcal{L}_\mu \phi(x_0) \geq f(x_0))\).$$

(F2) If $\phi \in C^2(\Omega)$ and $\phi^+$ touches $u$ below (resp. above) at $x_0 \in \mathcal{F}(u)$ and $|\nabla \phi|(x_0) \neq 0$ then

$$|\nabla \phi|(x_0) \leq Q(x_0) \quad \text{(resp. } |\nabla \phi|(x_0) < Q(x_0))\).$$

We refer to the usual definition of subsolution, supersolution and solution of a degenerate PDE. Let us introduce the notion of comparison subsolution/supersolution.

**Definition 2.3.** We say $u \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1.1) in $\Omega$, if only if $u \in C^2(\Omega^+(u))$ and the following conditions are satisfied:

(G1) If $\mathcal{L}_\mu u > f(x)$ (resp. $< f$) in $\Omega^+(u)$;

(G2) If $x_0 \in \mathcal{F}(u)$, then

$$|\nabla u|(x_0) > Q(x_0) \quad \text{(resp. } 0 < |\nabla u|(x_0) < Q(x_0))\).$$

Next lemma provides a basic comparison principle for solutions to the free boundary problem (1.1). The Lemma below yields the crucial tool in the proof of main result.
Lemma 2.4. The following remark is an consequence of the definitions above: Let \( u, v \) be respectively a solution and a strict subsolution to (1.1) in \( \Omega \). If \( u \geq v^+ \) in \( \Omega \) then \( u > v^+ \) in \( \Omega^+(v) \cup \mathcal{F}(v) \).

As in [DeS], another fundamental tool in the proof of Theorem (1.1) is the regularity of solutions to the classical Neumann problem for the constant coefficient linear equation

\[
\begin{aligned}
\Delta u_\infty &= 0, \quad \text{in } B^+_\rho, \\
\frac{\partial u_\infty}{\partial \nu} &= 0, \quad \text{on } \Upsilon_\rho,
\end{aligned}
\]

where \( \nu := (0, 0, \ldots, 0, 1) \) and we denote by

\[
\begin{aligned}
B^+_\rho &:= \{ x \in \mathbb{R}^n : |x| < \rho, x_n > 0 \} \\
\Upsilon_\rho &:= \{ x \in \mathbb{R}^n : |x| < \rho, x_n = 0 \}.
\end{aligned}
\]

We use the notion of viscosity solution to (2.2):

**Definition 2.5.** Let \( u_\infty \in C(B_\rho \cap \{ x_n \geq 0 \}) \). We say that \( u_\infty \) is a viscosity solution to (2.2) if given \( P(x) \) a quadratic polynomial touching \( u_\infty \) by below (resp. above) at \( x_0 \in B_\rho \cap \{ x_n \geq 0 \} \), then

(i) if \( x_0 \in B^+_\rho \) then \( \Delta P(x_0) \leq 0 \) (resp. \( \Delta P(x_0) \geq 0 \));

(ii) if \( x_0 \in \Upsilon_\rho \) then \( \partial P(x_0) \frac{\partial u_\infty}{\partial \nu} \leq 0 \) (resp. \( \partial P(x_0) \frac{\partial u_\infty}{\partial \nu} \geq 0 \))

**Remark 2.6.** Notice that, in the definition above we can choose polynomials \( P \) that touch \( u_\infty \) strictly by below/above. Also, it suffices to verify that (ii) holds for polynomials \( \tilde{P} \) with \( \Delta \tilde{P} > 0 \) (see [DeS]).

The proof of \( C^2 \)-regularity of solutions to the classical Neumann problem is classical and will be omitted (see for example [DeS]).

**Lemma 2.7.** Let \( u_\infty \) be a viscosity solution to

\[
\begin{aligned}
\Delta u_\infty &= 0, \quad \text{in } B^+_\rho \\
\frac{\partial u_\infty}{\partial \nu} &= 0, \quad \text{on } \Upsilon_\rho
\end{aligned}
\]

with \( \| u_\infty \|_{L^\infty} \leq 1 \). There exists a universal constant \( C_0 > 0 \) such that

\[
|u_\infty(x) - u_\infty(0) - \nabla u_\infty(0) \cdot x| \leq C_0 \rho^2 \quad \text{in } B_\rho \cap \{ x_n \geq 0 \}.
\]

### 2.2 Difficulties and Changes

In this section, we comment on the main difficulties we came across to obtain the improvement of flatness property for the graph of a solution \( u \) of (1.1) and how to overcome them.

1. **Harnack type Inequality.** When we consider the problem (1.1) for \( \mu > 0 \), the first difficulty we find lies in the following fact: in general, if \( p \) is an affine function and \( u \) is a solution to the problem

\[
|\nabla v|^\mu \Delta v = f, \quad \text{in } B_r(x_0),
\]

we can not conclude that \( u + p \) is a solution to the equation (2.6). For \( \mu = 0 \) we know \( u + p \) is still solution for (2.6). In [DeS], this fact is important because it allows us to apply Harnack Inequality for \( v(x) = u(x) - x_n \), which is crucial to reach an improvement of flatness for the graph of \( u \). We overcome this difficulty as follows:
Step 1. We notice that the function $v(x) = u(x) - x^\mu$ is a solution to the problem
\begin{equation}
|\nabla v + e|^\mu \Delta v = f, \quad \text{in } B_r(x_0),
\end{equation}
where $e \in \mathbb{R}^n$ with $|e| = 1$. Then, we know from [I] that $v$ satisfies the following Harnack Inequality
\begin{equation}
\sup_{B_{r/2}(x_0)} v \leq C \left\{ \inf_{B_{r/2}(x_0)} v + \max(2, \|f\|_\infty) \right\},
\end{equation}
where $C > 0$ is a constant depending only on $n$ and $\mu$.

Step 2. Since we will use a blow-up argument to prove our main results (Theorem (1.1) and Theorem (1.2)), we can assume that $\|f\|_\infty$ is small. Using the homogeneity property of $\Delta$ we consider the scaling function $v_r(x) = \frac{v(rx + x_0)}{r}$ and apply (2.8) to obtain
\begin{equation}
\sup_{B_{r/2}(x_0)} v \leq C \left\{ \inf_{B_{r/2}(x_0)} v + 2r \right\}.
\end{equation}
Precisely, we use the following result:

**Lemma 2.8.** Let $u$ be a non-negative viscosity solution to
\begin{equation}
|\nabla v + e|^\mu \Delta v = f, \quad \text{in } B_\delta,
\end{equation}
where $0 < \delta < 1$, $\|f\|_\infty \leq 1$ and $|e| = 1$. Then, there exists a constant $C$ depending only on $n$ and $\mu$ such that
\begin{equation}
\sup_{B_{\delta/2}} v \leq C \left\{ \inf_{B_{\delta/2}} v + 2\delta \right\}.
\end{equation}

**Proof.** Define
\begin{equation}
u(x) = \frac{v(\delta x)}{\delta}
\end{equation}
for all $x \in B_1$. Notice that $v$ is a solution to
\begin{equation}
|\nabla u + e|^\mu \Delta u = \delta f(\delta x), \quad \text{in } B_1.
\end{equation}
Notice that, if $F(M) := \text{Trace } M$, the equation can be written as $G(Du, D^2u) = f$ with
\begin{equation}
G(\tilde{\mathbf{q}}, M) := |\tilde{\mathbf{e}} + \tilde{\mathbf{q}}|^\mu F(M).
\end{equation}
In particular, if $|\tilde{\mathbf{q}}| \geq 2$ then $|\tilde{\mathbf{e}} + \tilde{\mathbf{q}}| \geq 1$ and
\begin{equation}
\begin{cases}
G(\tilde{\mathbf{q}}, M) &= 0 \\
|\tilde{\mathbf{q}}| &\geq 2 \Rightarrow \begin{cases} \mathcal{A}^+(D^2u) + |f| &\geq 0 \\
\mathcal{A}^- (D^2u) - |f| &\leq 0 \end{cases}
\end{cases}
\end{equation}
Thus, from [I], we can apply Harnack inequality to obtain
\begin{equation}
\sup_{B_{1/2}} u \leq C \left\{ \inf_{B_{1/2}} u + \max(2, \delta \|f\|_\infty) \right\} \leq C \left\{ \inf_{B_{1/2}} u + 2 \right\},
\end{equation}
where $C = C(n, \mu)$ is a positive constant. The Lemma 2.8 is concluded. \qed
Step 3. The Harnack Inequality (2.9) is different from the Harnack Inequality used in [DeS]. In fact, for $0 < \varepsilon < 1$, DeSilva used the inequality

\begin{equation}
\sup_{B_{r/2}(x_0)} v \leq C \left\{ \inf_{B_{r/2}(x_0)} v + \|f\|_\infty \right\}
\end{equation}

to prove that if $\|f\|_\infty$ satisfies the smallness condition $\|f\|_\infty \leq \varepsilon^2$ we can build radial barriers $w_{r,x_0}$ and apply comparison techniques to achieve an appropriate Harnack type Inequality (see Theorem 3.1 and Lemma 3.3 in [DeS]) to establish the improvement of flatness. A carefully analysis the behavior of $v = u - x_n$ (or $v = x_n - u$) in a ball $B_r(x_0)$ with

$|\nabla u| < \frac{1}{2}$ in $B_r(x_0)$,

and $r_1 = r_1(n, \mu) > 0$, reveals that if we consider radial barriers $w_{r,x_0 - r_2 e_n}$ (or $w_{r,x_0 + r_2 e_n}$) the condition $\|f\|_\infty \leq \varepsilon^2$ used in (2.14) can be replaced by an adequate smallness condition of the radius $r = r(r_2)$ in (2.9) to obtain a Harnack type Inequality, where $r_2 = r_2(r_1)$.

2. Limiting solution. In the more general case $L^\mu u = |\nabla u|^\gamma F(D^2 u)$, where $F$ is uniformly elliptic and satisfies homogeneity property, our Limiting solution is given by a classical Neumann problem for the constant coefficient linear equation

\begin{equation}
\begin{cases}
F_0(D^2u_\infty) = 0, \quad &\text{in } B^+_r; \\
\frac{\partial u_\infty}{\partial \mu} = 0, \quad &\text{in } \Gamma_r,
\end{cases}
\end{equation}

where $\mu := (0, 0, \ldots, 0, 1)$. In [MS], Emmanouil Milakin and Luis E. Silvestre studied the regularity for viscosity solutions of fully nonlinear uniformly elliptic second order equations with Neumann boundary data. More precisely, they showed that viscosity solutions of the homogeneous problem with Neumann boundary data (2.15) are class $C^{1,\alpha}_0(B^+_r)$ for some $\alpha \in (0, 1)$. We point out that the regularity $C^{1,\alpha}_0(B^+_r)$ for $u_\infty$ is sufficient to obtain the improvement of flatness.

3. Harnack Type Inequality

In this section, based on comparison principle granted in Lemma 2.4 we prove a Harnack type inequality for a solution $u$ to the problem (1.1) with the following conditions:

\begin{align}
\|f\|_{L^\infty(\Omega)} &\leq \varepsilon^2, \\
\|Q - 1\|_{L^\infty(\Omega)} &\leq \varepsilon^2,
\end{align}

for $0 < \varepsilon < 1$.

**Lemma 3.1.** Let $u$ be a viscosity solution to (1.1) in $\Omega$, under assumptions (3.1)–(3.2). There exist a universal constant $\tilde{\varepsilon} > 0$ such that if $0 < \varepsilon \leq \tilde{\varepsilon}$ and $u$ satisfies

\begin{align}
p^+(x) &\leq u(x) \leq (p(x) + \varepsilon)^+, \quad |\sigma| < \frac{1}{20} \quad &\text{in } B_1(0), \\
p(x) &\leq x_n + \sigma, \\
then &\text{if at } x_0 = \frac{1}{10} e_n
\end{align}

\begin{align}
u(x_0) &\geq \left( p(x_0) + \frac{\varepsilon}{2} \right)^+, \\
then &\text{if } u \geq (p + c\varepsilon)^+ \quad &\text{in } \overline{B^+_1}(0),
\end{align}
for some \( 0 < c < 1 \). Analogously, if

\[
(3.6) \quad u(x_0) \leq \left(p(x_0) + \frac{\varepsilon}{2}\right)^+,
\]

then

\[
(3.7) \quad u \leq (p + (1 - c)\varepsilon)^+ \quad \text{in} \quad \overline{B}_{\frac{1}{c}}(0).
\]

Proof. We verify \((3.5)\). The proof of \((3.7)\) is analogous. Notice that

\[
(3.8) \quad B_{\frac{1}{c}}(x_0) \subset B_{\frac{1}{c}}(u).
\]

From [IS] we know that \( u \) is \( C^{1,\alpha} \) in \( B_{\frac{1}{c}}(x_0) \) with

\[
[u]_{1+\alpha,B_{1/40}(x_0)} \leq C \left(\|u\|_\infty + \|f\|_\infty^{\frac{1}{\alpha}}\right) \leq 3C,
\]

where \( \alpha = \alpha(n,\mu) \in (0,1) \) and \( C = C(n,\mu) > 1 \). Now we consider two cases:

Case 1 : \( |\nabla u(x_0)| < \frac{1}{4} \).

Choose \( r_1 = r_1(n,\mu) > 0 \) such that

\[
(3.9) \quad |\nabla u| \leq \frac{1}{2} \quad \text{in} \quad B_{r_1}(x_0).
\]

There exists a constant \( r_2 = r_2(n,\mu) > 0 \) that satisfies

\[
(x - r_2e_n) \in B_{r_1}(x_0), \quad \text{for all} \quad x \in B_{\frac{1}{c}}(x_0).
\]

For \( r_3 = \min \{\frac{r_1}{8}, \frac{r_2}{4}\} \) we apply the Lemma \( 2.8 \) in \( B_{2r_3}(x_0) \) and we obtain

\[
(3.10) \quad u(x) - p(x) \geq c_0(u(x_0) - p(x_0)) - 4r_3 > \frac{c_0\varepsilon}{2} - 4r_3
\]

for all \( x \in B_{r_3}(x_0) \). From \((3.9)\) and \((3.10)\) we can write

\[
\frac{c_0\varepsilon}{2} - 4r_3 \leq u(x) - p(x) = u(x - r_2e_n) + r_2e_n - p(x - r_2e_n) + r_2e_n = u(x - r_2e_n) + r_2e_n - p(x - r_2e_n) - r_2 \leq u(x - r_2e_n) - p(x - r_2e_n) + \frac{r_2}{2} - r_2,
\]

for all \( x \in B_{r_3}(x_0) \). Thus, we find

\[
(3.11) \quad \frac{c_0\varepsilon}{2} - 4r_3 \leq \frac{c_0\varepsilon}{2} - 4r_3 + \frac{r_2}{2} \leq u(x) - p(x),
\]

for all \( x \in B_{r_3}(x_0) \), where \( x = x_0 - r_2e_n \).

Let \( w: \overline{D} \rightarrow \mathbb{R} \) be defined by

\[
(3.12) \quad w(x) = c \left(|x - \overline{x}_0|^{-\gamma} - \left(\frac{4}{5}\right)^{-\gamma}\right).
\]
where \( D := B_{\frac{4}{5}}(\overline{x}_0) \setminus \overline{B}_{\frac{3}{5}}(\overline{x}_0) \). We choose \( c = c(n, \mu) > 0 \) such that

\[
    w = \begin{cases} 
        0, & \text{on } \partial B_{\frac{4}{5}}(\overline{x}_0), \\
        1, & \text{on } \partial B_{\frac{3}{5}}(\overline{x}_0). 
    \end{cases}
\]

Now define

\[
    v(x) = p(x) + \frac{c_0\varepsilon}{2} (w(x) - 1), \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0),
\]

and for \( t \geq 0 \),

\[
    v_t(x) = v(x) + t, \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0).
\]

By choice of \( c \) we have \( w \leq 1 \) in \( D \). Then, extending \( w \) to 1 in \( B_{\frac{3}{5}}(\overline{x}_0) \) we find

\[
    v_0(x) = v(x) \leq p(x) \leq u(x), \quad x \in \overline{B}_{\frac{4}{5}}(\overline{x}_0).
\]

Consider

\[
    t_0 = \sup \left\{ t \geq 0 : v_t \leq u \text{ in } \overline{B}_{\frac{4}{5}}(\overline{x}_0) \right\}.
\]

Assume, for the moment, that we have already verified \( t_0 \geq \frac{c_0\varepsilon}{2} \). From definition of \( v \) we have

\[
    u(x) \geq v(x) + t_0 \geq p(x) + \frac{c_0\varepsilon}{2} w(x), \quad \forall x \in B_{\frac{4}{5}}(\overline{x}_0).
\]

Notice that \( B_{\frac{4}{5}}(0) \subset B_{\frac{4}{5}}(\overline{x}_0) \) and

\[
    w(x) \geq \begin{cases} 
        c \left[ \left( \frac{4}{5} \right)^{-\gamma} - \left( \frac{1}{5} \right)^{-\gamma} \right], & \text{in } B_{\frac{4}{5}}(\overline{x}_0) \setminus B_{\frac{3}{5}}(\overline{x}_0), \\
        1, & \text{on } B_{\frac{3}{5}}(\overline{x}_0).
    \end{cases}
\]

Hence, we conclude (\( \varepsilon \) small) that

\[
    u(x) - p(x) \geq c_1 \varepsilon, \quad \text{in } B_{1/2}(0),
\]

and the result is proved. Let us now prove that indeed \( t_0 \geq \frac{c_0\varepsilon}{2} \). For that, we suppose for the sake of contradiction that \( t_0 < \frac{c_0\varepsilon}{2} \). Then there would exist \( y_0 \in B_{\frac{4}{5}}(\overline{x}_0) \) such that

\[
    v_t(y_0) = u(y_0).
\]

In the sequel, we show that \( y_0 \in B_{\frac{3}{5}}(\overline{x}_0) \). From definition of \( v_t \) and by the fact that \( w \) has zero boundary data on \( \partial B_{4/5}(\overline{x}_0) \) we have

\[
    v_t = p - \frac{c_0\varepsilon}{2} + t_0 < u \quad \text{in } \partial B_{4/5}(\overline{x}_0),
\]

where we have used that \( u \geq p \) and \( t_0 < \frac{c_0\varepsilon}{2} \). We compute directly,

\[
    \partial_t w = -\gamma (x_1 - \overline{x}_0^i) |x - \overline{x}_0|^{-\gamma - 2}
\]

and

\[
    \partial_{ij} w = \gamma |x - \overline{x}_0|^{-\gamma - 2} \left\{ (\gamma + 2) (x_i - \overline{x}_0^i) (x_j - \overline{x}_0^j) |x - \overline{x}_0|^{-2} - \delta_{ij} \right\}.
\]
Moreover, there exists $C = C(n, \mu, \gamma) > 1$ such that $|\nabla w| \leq C$ in $D$. Then, if $\varepsilon > 0$ is small we have in $D$

$$|\varepsilon_n + (c_0/2)\varepsilon \nabla w|^\mu \geq \left(\frac{1}{2}\right)^\mu. \quad (3.19)$$

Thus, if $\gamma = \gamma(n) > 1$ is large, from (3.18) and (3.19) we find

$$|\nabla v_t|^\kappa \Delta v_t \geq c_0 \varepsilon \gamma^\mu c^{-2} \left(\frac{3}{4}\right)^\mu \left(\frac{1}{2}\right)^\mu \psi \mu \geq \mu \geq 1,$$

where $\delta_0 = \delta_0(n, \mu) > 0$. On the other hand, we have

$$|\nabla v_t| \geq |\partial_n v_t| = |1 + (c_0/2)\varepsilon \partial_n w|, \text{ in } D. \quad (3.20)$$

By radial symmetry of $w$, we have

$$\partial_n w(x) = |\nabla w(x)| \langle \nu_x, e_n \rangle, \quad x \in D,$$

where $\nu_x$ is the unit vector in the direction of $x - \bar{x}_0$. From (3.17) we have

$$|\nabla w| = c\gamma |x - \bar{x}_0|^{-\gamma+2} |x - \bar{x}_0| \geq c_0 > 0, \quad \text{in } D.$$

Also we have $\langle \nu_x, e_n \rangle \geq c$ in $\{v_0 \leq 0\} \cap D$ (for $\varepsilon$ small enough). In fact, if $\varepsilon$ is small enough

$$\{v_0 \leq 0\} \cap D \subset \{p \leq \frac{c_0 \varepsilon}{2}\} = \left\{x_n \leq \frac{c_0 \varepsilon}{2} - \sigma\right\} \subset \{x_n < 1/20\}.$$

We therefore conclude that

$$\langle \nu_x, e_n \rangle = \frac{1}{|x_0 - x|} |\bar{x}_0 - x, e_n| \geq \frac{5}{4} |\bar{x}_0 - x, e_n| = \frac{5}{4} \left(\frac{1}{10} - r_2 - x_0 + \frac{1}{20} - \frac{1}{20}\right) = c_7, \quad \text{in } \{v_t \leq 0\} \cap D.$$

From (3.20) and (3.21) we obtain

$$|\nabla v_t|^2 \geq |\partial_n v_t|^2 \geq 1 + 2\varepsilon^2 |\nabla w|^2 \geq 1 + c_0 \varepsilon + c_10 \varepsilon^2 \geq 1 + \varepsilon^2.$$

Hence

$$|\nabla v_t|^2 \geq 1 + \varepsilon^2 > Q^2 \text{ in } D \cap \mathcal{F}(v_t).$$
in \( \{v_{t_0} \leq 0\} \cap D \). In particular, we have
\[
|\nabla v_{t_0}| > Q \quad \text{in} \quad D \cap \mathcal{F}(v_{t_0}).
\]
Thus, \( v_{t_0} \) is a strict subsolution in \( D \) and by Lemma 2.4 \((u \text{ is a viscosity solution of problem (1.1)) we conclude that } y_0 \in B_{r_2}(x_0). This \ is \ a \ contradiction. \ In \ fact, \ we \ would \ get
\[
u(y_0) = v_{t_0}(y_0) = v(y_0) + t_0 \leq p(y_0) + t_0 < p(y_0) + \frac{c_0\varepsilon}{2}.
\]
which drives us to a contradiction to (3.11). The Lemma 3.1 is concluded.

Case 2: \(|\nabla u(x_0)| \geq \frac{1}{4}\).

Since \( u \) is \( C^{1, \alpha} \) in \( B_{\frac{1}{40}}(x_0) \), there exists a constant \( r_0 = r_0(n, \mu) > 0 \) such that
\[
|\nabla u| \geq \frac{1}{8} \quad \text{in} \quad B_{r_0}(x_0).
\]
Then, \( u \) satisfies
\[
\Delta u = g \quad \text{in} \quad B_{r_0}(x_0),
\]
where \( g = \frac{f}{|\nabla u|} \) with \( \|g\|_{\infty} \leq \varepsilon^2 8 \mu \). Thus, by classical Harnack Inequality we obtain
\[
u(x) - p(x) \geq c_0(u(x) - p(x)) - C\|f\|_{\infty} \geq \frac{c_0\varepsilon}{2} - C_1\varepsilon^2 \geq c_1\varepsilon,
\]
for all \( x \in B_{1/40}(x_0) \), if \( \varepsilon > 0 \) is sufficiently small. Now, we consider the barrie
\[
w(x) = \begin{cases} 
\frac{c}{1 - \gamma - \left(\frac{1}{8}\right)^{-\gamma}}, & \text{in} \quad B_{\frac{1}{2}}(x_0) \setminus B_{1/40}(x_0), \\
1, & \text{on} \quad B_{1/40}(x_0),
\end{cases}
\]
and the Lemma 3.1 follows as in Case 1.

Now we establish the main tool in the proof of Theorem 1.1.

**Theorem 3.2.** Let \( u \) be a viscosity solution to (1.1) in \( \Omega \) under assumptions (2.1)–(2.2). There exists a universal constant \( \bar{\varepsilon} > 0 \) such that, if \( u \) satisfies at some \( x_0 \in \Omega^+(u) \cup F(u) \),
\[
(x_n + a_0)^+ \leq u(x) \leq (x_n + d_0)^+ \quad \text{in} \quad B_r(x_0) \subset \Omega,
\]
with \( |a_0| < \frac{1}{20} \) and
\[
d_0 - a_0 \leq \varepsilon r, \quad \varepsilon \leq \bar{\varepsilon}
\]
then
\[
(x_n + a_1)^+ \leq u(x) \leq (x_n + d_1)^+ \quad \text{in} \quad B_{\frac{1}{20}}(x_0)
\]
with
\[
a_0 \leq a_1 \leq d_1 \leq d_0, \quad d_1 - a_1 \leq (1 - c)\varepsilon r,
\]
and \( 0 < c < 1 \) universal.
Proof. With no loss of generality, we can assume $x_0 = 0$ and $r = 1$. We put $p(x) = x_n + a_0$ and by (3.24)

$$p^+ (x) \leq u(x) \leq (p(x) + \varepsilon)^+ \quad (d_0 \leq a_0 + \varepsilon).$$

Then, since

$$u \left( \frac{1}{10} e_n \right) \geq \left( p \left( \frac{1}{10} e_n \right) + \frac{\varepsilon}{2} \right)^+ \quad \text{or} \quad u \left( \frac{1}{10} e_n \right) < \left( p \left( \frac{1}{10} e_n \right) + \frac{\varepsilon}{2} \right)^+$$

we can apply Lemma 3.1 to obtain the result.

\[\square\]

From Harnack inequality, Theorem 3.2, precisely as in [DeS], we obtain the following key estimate for flatness improvement.

Corollary 3.3. Let $u$ be a viscosity solution to (1.1) in $\Omega$ under assumptions (3.1)–(3.2). If $u$ satisfies (3.24) then in $B_1(x_0)$ the function $\tilde{u}_\varepsilon := \frac{u - u_{0}}{\varepsilon}$ has a Hölder modulus of continuity at $X_0$ outside of ball of radius $\varepsilon/\tilde{\varepsilon}$, i.e. for all $x \in (\Omega^+ (u) \cup \mathfrak{F} (u)) \cap B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\tilde{\varepsilon}$

$$|\tilde{u}_\varepsilon (x) - \tilde{u}_\varepsilon (x_0)| \leq C|x - x_0|^\gamma.$$

4. Improvement of Flatness

In this section we prove the improvement of flatness lemma, from which the proof of main Theorem 1.1 will follow via an interactive argument. Next we present the basic induction step towards $C^{1, \gamma}$ regularity at 0.

Theorem 4.1 (Improvement of flatness). Let $u \in C(\Omega)$ be a viscosity solution to

$$\begin{cases}
L_{m}u = f, & \text{in} \quad \Omega_+(u), \\
|\nabla u| = Q, & \text{on} \quad \mathfrak{F}(u).
\end{cases}$$

with $0 < \varepsilon < 1$.

\begin{equation}
\max \left\{ \|f\|_{L^\infty(\Omega)}, \|Q - 1\|_{L^\infty(\Omega)} \right\} \leq \varepsilon^2.
\end{equation}

Suppose that $u$ satisfies

\begin{equation}
(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \text{for} \quad x \in B_1
\end{equation}

with $0 \in \mathfrak{F}(u)$. If $0 < r \leq r_0$ for $r_0$ a universal constant and $0 < \varepsilon \leq \varepsilon_0$ for some $\varepsilon_0$ depending on $r$, then

\begin{equation}
\left( x, \nu \right) - r \frac{\varepsilon}{2} \right)^+ \leq u(x) \leq \left( x, \nu \right) + r \frac{\varepsilon}{2} \right)^+ \quad x \in B_r,
\end{equation}

with $|\nu| = 1$, and $|\nu - e_n| \leq Ce^2$ for a universal constant $C > 0$.

Proof. We divide the proof of this Lemma into 3 steps. We use the following notation:

$$\Omega_\rho(u) := (B_1^+(u) \cup \mathfrak{F}(u)) \cap B_\rho.$$
Step 1 - Compactness Lemma: Fix \( r \leq r_0 \) with \( r_0 \) universal (the precise \( r_0 \) will be given in Step 3). Assume by contradiction that we can find a sequence \( \epsilon_k \to 0 \) and a sequence \( \{u_k\}_{k \geq 1} \subset C(\Omega) \) be a sequence of viscosity solution to

\[
\begin{cases}
L_{\mu} u_k = f_k & \text{in } \Omega_+^1(u_k) \\
|\nabla u_k| = Q_k(x) & \text{on } F(u_k)
\end{cases}
\]

with

\[
\max \{ \|f_k\|_{L^\infty}, \|Q_k - 1\|_{L^\infty} \} \leq \epsilon_k^2,
\]
as \( k \to \infty \), such that

\[
(x_n - \epsilon_k)^+ \leq u_k(x) \leq (x_n + \epsilon_k)^+ \quad \text{for } x \in B_1(0), \ 0 \in F(u_k)
\]

but it does not satisfy the conclusion \((4.4)\) of the Lemma. Let \( v_k : \Omega_1(u_k) \to \mathbb{R} \) defined by

\[
v_k(x) := \frac{u_k(x) - x_n}{\epsilon_k}.
\]

Then \((4.7)\) gives,

\[-1 \leq v_k(x) \leq 1 \quad \text{for } x \in \Omega_1(u_k).
\]

From Corollary \(3.3\) it follows that the function \( v_k \) satisfies

\[
|v_k(x) - v_k(y)| \leq C|x - y|^\gamma,
\]

for \( C \) universal and

\[
|x - y| \geq \epsilon_k/\bar{\epsilon}, \ \text{for } x, y \in \Omega_{1/2}(u_k).
\]

From \((4.7)\) it clearly follows that \( F(u_k) \to B_1 \cap \{x_n = 0\} \) in the Hausdorff distance. This fact and \((4.9)\) together with Ascoli-Arzela give that as \( \epsilon_k \to 0 \) the graphs of the \( v_k \) over \( \Omega_{1/2}(u_k) \) converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \( u_\infty \) over \( B_{1/2} \cap \{x_n \geq 0\} \).

Step 2 - Limiting Solution: We claim that \( \tilde{u} \) is a solution of the problem

\[
\begin{cases}
\Delta u_\infty = 0 & \text{in } B_{1/2}^+ \\
\partial_n u_\infty = 0 & \text{on } \bar{Y}_{1/2}\n\end{cases}
\]
in viscosity sense. In fact, given a quadratic polynomial \( P(x) \) touching \( \tilde{u} \) at \( x_0 \in B_{1/2}(0) \cap \{x_n \geq 0\} \) strictly by below we need to prove that

(i) \( x_0 \in B_{1/2}(0) \cap \{x_n > 0\} \) then \( \Delta P \leq 0 \);

(ii) \( x_0 \in B_{1/2}(0) \cap \{x_n = 0\} \) then \( \partial_n P(x_0) \leq 0 \).

As in \[DeS\], there exist points \( x_j \in \Omega_{1/2}(u_j) \), \( x_j \to x_0 \), and constants \( c_j \to 0 \) such that

\[
u_j(x_j) = \tilde{P}(x_j)
\]

and

\[
u_j(x) \geq \tilde{P}(x) \quad \text{in a neighborhood of } x_j
\]

where

\[
\tilde{P}(x) = \epsilon_j (P(x) + c_j) + x_n.
\]
We have two possibilities:

(a) If \( x_0 \in B_{\frac{1}{2}} \cap \{ x_n > 0 \} \) then, since \( P \) touches \( u_j \) by below at \( x_j \), we estimate

\[
\varepsilon_j^2 \geq f_j(x_j) \geq \mathcal{L}_\mu \hat{P} = \varepsilon_j |\nabla \hat{P}| \Delta \hat{P}.
\]

Using that \( \nabla \hat{P} = \varepsilon_j \nabla P + \varepsilon \) and taking \( \varepsilon_j \to 0 \) we obtain

\[ \Delta P \leq 0. \]

(b) If \( x_0 \in B_{\frac{1}{2}} \cap \{ x_n = 0 \} \) we can assume, see [DeS], that

\[ \Delta P > 0 \]

Notice that for \( j \) sufficiently large we have \( x_j \in \mathcal{F}(u_j) \). In fact, suppose by contradiction that there exists a subsequence \( x_{j_n} \in B_{\frac{1}{2}}^+ (u_j) \) such that \( x_{j_n} \to x_0 \). Then arguing as in (i) we obtain

\[ \Delta P \leq C \varepsilon_j, \]

which contradicts (4.11) as \( j_n \to \infty \). Therefore, there exists \( j_0 \in \mathbb{N} \) such that \( x_j \in \mathcal{F}(u_j) \) for \( j \geq j_0 \). Moreover,

\[ |\nabla \hat{P}| \geq 1 - \varepsilon_j |\nabla P| > 0, \]

for \( j \) sufficiently large (we can assume that \( j \geq j_0 \)). Since that \( \hat{P}^+ \) touches \( u_j \) by below we have

\[ |\nabla \hat{P}|^2 \leq Q_j(x_j) \leq (1 + \varepsilon_j^2). \]

Then, we obtain

\[ |\nabla \hat{P}|^2 \leq (1 + \varepsilon_j^2). \]

Moreover,

\[ |\nabla \hat{P}|^2 = \varepsilon_j^2 |\nabla P(x_j)|^2 + 1 + 2 \varepsilon_j \partial_n P(x_j), \]

where we have used \( |\nabla \hat{P}|^2 \leq C \). In conclusion, we obtain

\[ (4.12) \]

\[ \varepsilon_j^2 |\nabla P(x_j)|^2 + 1 + 2 \varepsilon_j \partial_n P(x_j) \leq 1 + \varepsilon_j^2. \]

Hence, dividing (4.12) by \( \varepsilon_j \) and taking \( j \to \infty \) we obtain \( \partial_n P(x_0) \leq 0 \).

The choice of \( r_0 \) and the conclusion of the Theorem 1.1 follows from the regularity of \( \hat{u} \):

**Step 3 - Improvement of flatness:** From the previous step, \( u_\infty \) solve \( 4.10 \) and from \( 4.8 \),

\[
-1 \leq u_\infty \leq 1 \quad \text{in} \quad B_{1/2} \cap \{ x_n \geq 0 \}.
\]

From Lemma 2.1 and the bound above we obtain that, for the given \( r \),

\[
|u_\infty(x) - u_\infty(0) - \langle \nabla u_\infty(0), x \rangle| \leq C_0 r^2 \quad \text{in} \quad B_r \cap \{ x_n \geq 0 \},
\]

for a universal constant \( C_0 \). In particular, since \( 0 \in \mathcal{F}(u_\infty) \) and \( \frac{\partial u_\infty(0)}{\partial n} = 0 \), we estimate

\[ \langle \hat{x}, \hat{v} \rangle - C_1 r^2 \leq u_\infty(x) \leq \langle \hat{x}, \hat{v} \rangle + C_0 r^2 \quad \text{in} \quad B_r \cap \{ x_n \geq 0 \}, \]
where \( \tilde{v}_i = \langle \nabla u_\infty(0), e_i \rangle, i = 1, \ldots, n - 1, |\tilde{v}| \leq \tilde{C} \) and \( \tilde{C} \) is a universal constant. Therefore, for \( k \) large enough we get,

\[
\langle \tilde{x}, \tilde{v} \rangle - C_1 r^2 \leq v_k(x) \leq \langle \tilde{x}, \tilde{v} \rangle + C_1 r^2 \quad \text{in } \Omega_r(u_k).
\]

From the definition of \( v_k \) the inequality above reads

\[
(4.13) \quad \epsilon_k \tilde{x} \cdot \tilde{v} + x_n - \epsilon_k C_1 r^2 \leq v_k \leq \epsilon_k \langle \tilde{x}, \tilde{v} \rangle + x_n + \epsilon_k C_1 r^2 \quad \text{in } \Omega_r(u_k).
\]

Define

\[
\nu := \frac{1}{\sqrt{1 + \epsilon_k^2}}(\epsilon_k \tilde{v}, 1).
\]

Since, for \( k \) large,

\[
1 \leq \sqrt{1 + \epsilon_k^2} \leq 1 + \frac{\epsilon_k^2}{2},
\]

we conclude from (4.13) that

\[
\langle x, \nu \rangle - \frac{\epsilon_k^2}{2} r \leq v_k \leq \langle x, \nu \rangle + \frac{\epsilon_k^2}{2} r \quad \text{in } \Omega_r(u_k).
\]

In particular, if \( r_0 \) is such that \( C_1 r_0 \leq \frac{1}{4} \) and also \( k \) is large enough so that \( \epsilon_k \leq \frac{1}{2} \) we find

\[
\langle x, \nu \rangle - \frac{\epsilon_k}{2} r \leq v_k \leq \langle x, \nu \rangle + \frac{\epsilon_k}{2} r \quad \text{in } \Omega_r(u_k),
\]

which together with (4.17) implies that

\[
\left( \langle x, \nu \rangle - \frac{\epsilon_k}{2} r \right) \leq v_k \leq \left( \langle x, \nu \rangle + \frac{\epsilon_k}{2} r \right) \quad \text{in } B_r.
\]

Thus the \( u_k \) satisfy the conclusion of the Lemma, and we reached a contradiction. \( \square \)

5 Regularity of the free boundary

In this section we will prove the Theorem 1.1 and via a blow-up from Theorem 1.1 we will present the proof of Theorem 1.2. The proof of Theorem 1.1 is based on flatness improvement coming from Harnack type estimates and it follows closely the work of [DeS]. Hereafter, we will assume

\[
(5.1) \quad |Q(x) - Q(y)| \leq \tau(|x - y|) \quad \text{for } x, y \in B_1,
\]

where the modulus of continuity \( \tau \) satisfies

\[
(5.2) \quad \tau(t) \lesssim C t^\beta,
\]

for some \( 0 < \beta < 1 \) and \( C > 0 \).

Proof of Theorem 1.1. The idea of proof is to iterate the Theorem 4.1 in the appropriate geometric scaling. Let \( u \) be a viscosity solution to the free boundary problem

\[
(5.3) \quad \begin{cases} \Sigma_p u = f, & \text{in } B_1^+(u), \\ |\nabla u| = Q, & \text{on } \mathfrak{F}(u).
\end{cases}
\]

where \( B_1^+(u) = \{ x \in B_1^+ : u(x) > 0 \} \) and \( \mathfrak{F}^+(u) := \partial B_1^+(u) \cap B_1 \). Let us fix \( \tilde{r} > 0 \) to be a universal constant such that

\[
(5.4) \quad \tilde{r} \leq \min \left\{ \left( \frac{1}{2} \right)^2, r_0 \right\},
\]

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with \( r_0 \) the universal constant in Theorem 4.1. For the choice \( \tau \), let \( \epsilon_0 := \epsilon_0(\tau) \) give by Theorem 4.1. Now, let

\[
\tau := \epsilon_0^2 \quad \text{and} \quad \epsilon = \epsilon_k := 2^{-k} \epsilon_0.
\]

Our choice of \( \tau \) guarantees that

\[
(x_n - \epsilon_0)^+ \leq u(x) \leq (x_n + \epsilon_0)^+ \quad \text{in} \ B_1.
\]

Thus by Theorem 4.1

\[
\left( (x, \nu_1) - \bar{r} \frac{\epsilon_0}{2} \right)^+ \leq u(x) \leq \left( (x, \nu_1) + \bar{r} \frac{\epsilon_0}{2} \right)^+ \quad \text{in} \ B_{\bar{r}},
\]

with \(|\nu_1| = 1\) and \(|\nu_1 - \nu_0| \leq C \epsilon_0^2\) (where \( \nu_0 = e_n \)).

**Smallness regime:** Consider the sequence of rescalings \( u_k : B_1 \to \mathbb{R} \)

\[
u_k(x) := \frac{u(\lambda_k x)}{\lambda_k}
\]

with \( \lambda_k = \tau^k, \ k = 0, 1, 2, \ldots \), for a fixed \( \tau \) as in (5.4). Then each \( u_k \) satisfies in the following free boundary problem

\[
\begin{cases}
\mathcal{L}_{\tau} u_k = f_k, & \text{in} \ B_1^+(u_k), \\
|\nabla u_k| = Q_k, & \text{on} \ \mathfrak{S}(u_k).
\end{cases}
\]

\[
f_k(x) := \lambda_k f(\lambda_k x) \quad \text{and} \quad Q_k(x) := Q(\lambda_k x).
\]

We claim that for the choices made in (5.5) the assumption (4.2) are held. Indeed, in \( B_1 \)

\[
|f_k(x)| \leq \|f\|_{\mathcal{L}^\infty} \lambda_k \leq \epsilon_0^2 2^{-2k} = (\epsilon_0 2^{-k})^2 = \epsilon_k^2,
\]

\[
|Q_k(x) - 1| = |Q(\lambda_k x) - Q_k(0)| \leq \tau(1) \lambda_k^\beta \leq \epsilon_0^2 2^{-2k} = \epsilon_k^2.
\]

Therefore, we can iterate the argument above and obtain that

\[
(x, \nu_k) - \epsilon_k^+ \leq u_k(x) \leq (x, \nu_k) + \epsilon_k^+ \quad \text{in} \ B_1,
\]

with \(|\nu_k| = 1, |\nu_k - \nu_{k+1}| \leq C \epsilon_k (\nu_0 = e_n)\), where \( C \) is a universal constant. Thus, we have

\[
\left( (x, \nu_k) - \frac{\epsilon_0}{2^k \tau} \right)^+ \leq u(x) \leq \left( (x, \nu_k) + \frac{\epsilon_0}{2^k \tau^k} \right)^+ \quad \text{in} \ B_{\tau^k},
\]

with

\[
|\nu_{k+1} - \nu_k| \leq C \frac{\epsilon_0}{2^k}.
\]

Which (5.5) implies that

\[
\partial \{ u > 0 \} \cap B_{\tau^k} \subset \left\{ |(x, \nu_k)| \leq \frac{\epsilon_0}{2^k \tau^k} \right\}
\]

This implies that \( B_{3/4} \cap \mathfrak{S}(u) \) is a \( C^{1,\gamma} \) graph. In fact, by (5.10) we have that \( \{ \nu_k \}_{k \geq 1} \) is a Cauchy sequence, therefore the limit

\[
\nu(0) := \lim_{k \to \infty} \nu_k
\]

exists. Yet from (5.10) we conclude

\[
|\nu_k - \nu(0)| \leq C \frac{\epsilon_0}{2^k}.
\]
From (5.11) we have

\[ |\langle x, \nu_k \rangle| \leq \frac{\epsilon_0}{2^k r^k}. \]  

(5.12)

Fix \( x \in B_{3/4} \cap \partial \{u > 0\} \) and choose \( k \) such that

\[ r^{k+1} \leq |x| \leq r^k. \]

Then

\[
|\langle x, \nu(0) \rangle| \leq |\langle x, \nu(0) - \nu_k \rangle| + |\langle x, \nu_k \rangle| \\
\leq |\nu(0) - \nu_k||x| + \frac{\epsilon_0}{2^k r^k} \\
\leq C \frac{\epsilon_0}{2^k} |x| + \frac{\epsilon_0}{2^k r^k} \\
\leq C \frac{\epsilon_0}{2^k} (|x| + r^k) \\
\leq C \frac{\epsilon_0}{2^k} (|x| + \frac{r^{k+1}}{r}) \\
\leq C \frac{\epsilon_0}{2^k} (1 + \frac{1}{r}) |x|.
\]

From the convenient choice of \( k \), we have \( |x| \geq r^{k+1} \). Hence, if we define \( 0 < \gamma < 1 \) such that

\[ \frac{1}{2} = \frac{\ln(2)}{\ln(r^{-1})}. \]

Thus, we have

\[
|\langle x, \nu(0) \rangle| \leq C \frac{1}{2} (1 + r^{-1}) |x| \\
= C \frac{1}{2} (k+1)(1 + r^{-1}) |x| \\
\leq C (1 + r^{-1})\epsilon_0 |x|^{1+\gamma} \leq C \epsilon_0 |x|^{1+\gamma}.
\]

Finally, we obtain

\[ \partial \{u > 0\} \cap B_r \subset \left\{ \langle x, \nu(0) \rangle \leq C \epsilon_0 r^{(1+\gamma)} \right\}, \]

which implies that \( \partial \{u > 0\} \) is a differentiable surface at \( 0 \) with normal \( \nu(0) \). Applying this argument at all points in \( \partial \{u > 0\} \cap B_{3/4} \) we see that \( \partial \{u > 0\} \cap B_{3/4} \) is in fact a \( C^{1,\gamma} \) surface. \( \square \)

The next lemma proof of a standard result that is Lipschitz continuity and non-degeneracy of a solution \( u \) to

\[
\begin{cases}
\mathcal{L}_\omega u = f, & \text{in } \Omega_+(u), \\
|\nabla u| = Q, & \text{on } \mathcal{H}(u).
\end{cases}
\]

Lemma 5.1. Let \( u \in C(\Omega) \) be a viscosity solution to (5.13). Given \( \epsilon \in (0,1) \), we can find a universal constant \( \tilde{\epsilon} \) such that if \( \epsilon \in (0,\tilde{\epsilon}] \), \( \mathcal{H}(u) \cap B_1 \neq \emptyset \), \( \mathcal{H}(u) \) is a Lipschitz graph in \( B_2 \)

\[ \max \left\{ \|f\|_{L^\infty(\Omega)}, \|Q - 1\|_{L^\infty(\Omega)} \right\} \leq \epsilon^2, \]

then \( u \) is Lipschitz and non-degenerate in \( B_1^+(u) \) i.e. there exists universal conconstants \( c_0, c_1 > 0 \)

\[ c_0 \text{dist}(z, \mathcal{H}(u)) \leq u(z) \leq c_1 \text{dist}(z, \mathcal{H}(u)) \quad \text{for all } z \in B_1^+(u). \]
Lemma 5.2 (Compactness). Let $u_k$ be a sequence of (Lipschitz) viscosity solutions to
\[
\begin{align*}
\Delta u_k &= f_k \quad \text{in } \Omega^+(u_k), \\
|\nabla u_k| &= Q_k \quad \text{on } \mathcal{F}(u_k)
\end{align*}
\]
where $f_k$ and $Q_k$ satisfies the assumption \((5.14)\). Assume that
(i) $u_k \to u_\infty$ uniformly on compacts;
(ii) $\partial\{u_k > 0\} \to \partial\{u_\infty > 0\}$ locally in the Hausdorff distance;
(v) $\|f_k\|_{L^\infty} + \|Q_k - 1\|_{L^\infty} = o(1)$, as $k \to \infty$.
Then $u_\infty$ be a viscosity solution of
\[
\begin{align*}
\Delta u_\infty &= 0, \quad \text{in } \Omega^+(u_\infty), \\
|\nabla u_\infty| &= 1, \quad \text{on } \mathcal{F}(u_\infty),
\end{align*}
\]
in the viscosity sense.

Proof. The proof that follow the same scheme of the model Lemma 4.1 (see also \([DeS]\) Lemma 7.3).

Although not strictly necessary, we use the following Liouville type result for global viscosity solutions to a one-phase homogeneous free boundary problem, that could be of independent interest. The result is more general, but we will only show the result for a one-phase problems..

Lemma 5.3. Let $v : \mathbb{R}^n \to \mathbb{R}$ be a non-negative viscosity solution to
\[
\begin{align*}
\Delta v &= 0, \quad \text{in } \{v > 0\}, \\
\langle \nabla v, \nu \rangle &= 1, \quad \text{on } \mathcal{F}(v) := \partial\{v > 0\}.
\end{align*}
\]
Assume that $\mathcal{F}(v) = \{x_n = g(x'), x' \in \mathbb{R}^{n-1}\}$ with $\text{Lip}(g) \leq M$. Then $g$ is linear and
\[ v(x) = x_n^+\]
Proof. Let’s follow the ideas of \([DeS]\). Initially, assume for simplicity, $0 \in \mathcal{F}(v)$. Also, balls (of radius $\rho$ center at 0) in $\mathbb{R}^{n-1}$ are denote by $B'_\rho$. By the regularity theory in \([C1]\), since $v$ is a solution in $B_2$, the free boundary $\mathcal{F}(v)$ is $C^{1,\gamma}$ in $B_1$ with a bound depending only on $n$ and on $M$. Thus,
\[ |g(x') - g(0) - \nabla g(0) \cdot x'| \leq C|x'|^{1+\gamma} \quad \text{for } x' \in B'_1 \]
with $C$ depending only on $n, M$. Moreover, since $v$ us a global solution, the rescaling
\[ g_\lambda(x') := \frac{1}{\lambda} g(\lambda x'), \quad x' \in B'_2 \]
which preserves the same Lipschitz constant as $g$, satisfies the same inequality as above, i.e.
\[ |g_\lambda(x') - g_\lambda(0) - \nabla g_\lambda(0) \cdot x'| \leq C|x'|^{1+\gamma} \quad \text{for } x' \in B'_1. \]
Thus,
\[ |g(y') - g(0) - \nabla g(0) \cdot y'| \leq C \frac{1}{R^\gamma} |y'|^{1+\gamma}, \quad y' \in B'_R. \]
Passing to the limit as $R \to \infty$ we obtain the desired claim.

Finally we can prove Theorem 1.2. In this section we finally the proof of our second main theorem.
Proof of Theorem 1.2. Let $\overline{\epsilon} > 0$ be the universal constant in Theorem 1.1 and $u$. Without loss of generality, assume $Q(0) = 1$. Consider the re-scaled function

$$u_k := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},$$

with $\delta_k \to 0$ as $k \to \infty$. Each $u_k$ solves

$$\left\{ \begin{array}{ll} \mathcal{L}_\mu u_k &= f_k \text{ in } B_1^+(u_k), \\
|\nabla u_k| &= Q_k \text{ on } \mathfrak{F}(u_k), \end{array} \right.$$  

with

$$f_k(x) := \delta_k f(\delta_k x) \quad \text{and} \quad Q_k(x) := Q(\delta_k x).$$

Furthermore, for $k$ large, the assumption (5.14) are satisfied for the universal constant $\overline{\epsilon}$. In fact, in $B_1$ we have

$$|f_k(x)| = \delta_k |f(\delta_k x)| \leq \delta_k \|f\|_{L^\infty} \leq \overline{\epsilon}^2$$

$$|Q_k(x) - 1| = |Q_k(x) - Q_k(0)| \leq \tau(1) \delta_k^\beta \leq \overline{\epsilon}^2$$

for $k$ large enough. Therefore, using non-degeneracy (see Lemma 5.1) and uniform Lipschitz continuity of the $u_k'$s (see Lemma 5.2), standard arguments imply that (up to a subsequence)

(i) There exists $u_\infty \in C(\Omega)$ such that $u_k \to u_\infty$ uniformly on compacts;

(ii) $\partial\{u_k > 0\} \to \partial\{u_\infty > 0\}$ locally in the Hausdorff distance;

(iii) $\|f_k\|_{L^\infty} + \|Q_k - 1\|_{L^\infty} = o(1)$, as $k \to \infty$

and, as in Lemma 5.2, the blow-up limit $u_\infty$ solves the global homogeneous one-phase free boundary problem

$$\left\{ \begin{array}{ll} \Delta u_\infty &= 0, \text{ in } \{u_\infty > 0\}, \\
|\nabla u_\infty| &= 1, \text{ on } \mathfrak{F}(u_\infty). \end{array} \right.$$  

Since $\mathfrak{F}(u)$ is a Lipschitz graph in a neighborhood of $0$ we also have from have (i) - (iii) that $\mathfrak{F}(u_\infty)$ is Lipschitz continuous. Thus, follows the Lemma 5.3 that $u_\infty$ is a so-called one-phase solution, i.e. (up to rotations)

$$u_\infty = x_n^+.$$  

Thus, for $k$ large enough we have

$$\|u_k - u_\infty\|_{L^\infty} \leq \overline{\epsilon}$$

and the facts that $u_k$ is $\overline{\epsilon}$-flat say in $B_1$ i.e.

$$(x_n - \overline{\epsilon})^+ \leq u_k(x) \leq (x_n + \overline{\epsilon})^+, \ x \in B_1.$$  

Therefore, we can apply our flatness Theorem 4.1 and conclude that $\mathfrak{F}(u_k)$ and hence $\mathfrak{F}(u)$ is $C^{1,\gamma}$, for some $\gamma \in (0, 1)$.

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References

[AC] Alt H.W., Caffarelli L.A., Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math 325, (1981), 105-144.

[AF] Argiolas R., Ferrari F., Flat free boundaries regularity in two-phase problems for a class of fully nonlinear elliptic operators with variables coefficients, Interfaces Free Bound. 11(2009), no 2, 177-199.

[BD] Birindelli, I and Demengel, F. Nonlinear Differ. Equ. Appl. 17 (2010), 697714.

[C1] Caffarelli L.A. A Harnack inequality approach to the regularity of free boundaries. Parte I: Lipschitz free boundaries are $C^{1,\alpha}$. Rev. Mat. Iberoamericana 3 (1987) no. 2, 139-162.

[C2] Caffarelli L.A. A Harnack inequality approach to the regularity of free boundaries. Parte II: Flat free boundaries are Lipschitz. Comm. Pure Appl. Math. 42 (1989) no. 1, 55-78.

[CC] Caffarelli L. A., Cabre X., Fully Nonlinear Elliptic Equations. Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995.

[DeS] De Silva, D. Free boundary regularity for a problem with right hand side. Interfaces Free Bound. 13 (2011), no. 2, 223-238.

[DFS] De Silva, D., Ferrari, F., Salsa, S. Two-Phase problems with distributed source: regularity of the free boundary. Preprint(2012).

[DFS1] De Silva, D., Ferrari, F., Salsa, S. Free boundary regularity for fully nonlinear non-homogeneous two-phase problems. Preprint(2013).

[FEL1] Feldman M., Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations, Indiana Univ. Math. J., 50(2001), no.3 1171-1200.

[FEL2] Feldman M., Regularity for non isotropic two-phase problems with Lipschitz free boundaries, Differential Integral Equations, 10(1997), no.6 1171-1179.

[FE1] Ferrari F., Two-phase problems for a class of fully nonlinear elliptic operators, Lipschitz free boundaries are $C^{1,\gamma}$, Amer. J. Math., 128(2006), 541-571.

[FE2] Ferrari F., Salsa S., Regularity of the free boundary in two-phase problems for elliptic operators, Adv. Math. 214(2007), 288-322.

[I] Imbert, C. Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully non-linear elliptic equations. Journal of Differential Equations. 250 (2011) 1553-1574.

[LR] Leitão, R. and Ricarte, G. Free boundary regularity for a degenerate problem with right hand side. To appear in Interfaces Free Bound. (2018).

[LT] Leitão, R. and Teixeira, E. Regularity and geometric estimates for minima of discontinuous functionals. Rev. Mat. Iberoam. 31 (2015), no. 1, 69–108.

[IS] C. Imbert and L. Silvestre, $C^{1,\alpha}$regularity of solutions of some degenerate fully non-linear elliptic equations, Adv. Math. 229 (2012), pp. 181-211.

[RT] Ricarte G.; Teixeira, E. Fully nonlinear singularly perturbed equations and asymptotic free boundary. Journal of functional analysis 261 (2011) 1624-1673

[MS] Milakes, Emmanouil.; Silvestre, Luis E. Regularity for Fully Nonlinear Elliptic Equations with Neumann Boundary Data, Communications in Partial Differential Equations, 31: 1227–1252, 2006.

[W1] Wang P.Y., Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. I. Lipschitz free boundary are $C^{1,\gamma}$, Comm. Pure Appl. Math. 53(2000), 799-810.
[W2] Wang P.Y., *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. II Flat free boundary are Lipschitz*, Comm. Partial Differential Equations 27(2002), 1497-1514.