Abstract

We take a Hamiltonian-based perspective to generalize Nesterov’s accelerated gradient descent and Polyak’s heavy ball method to a broad class of momentum methods in the setting of (possibly) constrained minimization in Banach spaces. Our perspective leads to a generic and unifying non-asymptotic analysis of convergence of these methods in both the function value (in the setting of convex optimization) and in the norm of the gradient (in the setting of unconstrained, possibly nonconvex, optimization). The convergence analysis is intuitive and based on the conserved quantities of the time-dependent Hamiltonian that we introduce and that produces generalized momentum methods as its equations of motion.

1 Introduction

Accelerated, momentum-based, methods enjoy optimal iteration complexity for the minimization of smooth convex functions over convex sets, which has led to their broad acceptance as algorithmic primitives in many machine learning applications. Further, decades of empirical experience suggest that momentum methods are capable of exploring multiple local minima [5], which makes them favorable in comparison to gradient flows. The latter have optimal worst-case complexity for convergence to stationary points, but are strongly attracted to local minima. Moreover, recent theoretical results have established that momentum methods escape saddle points faster than standard gradient descent [19,31], providing further evidence of their value in nonconvex optimization.

The first (locally) accelerated method for smooth and strongly convex minimization is from the 1960s and is due to Polyak [32]. Working with continuous-time dynamics, Polyak introduced the following (momentum-based) second-order ordinary differential equation (ODE)¹:

\[
\ddot{x}_t = \alpha_1 \dot{x}_t + \alpha_2 \nabla f(x_t),
\]

(HBD)

where \( f \) is the function being minimized and \( \alpha_1, \alpha_2 \) are constants. He also studied its two-step discretization, which can be written as:

\[
x_{k+1} = -\alpha \nabla f(x_k) + \beta (x_k - x_{k-1}),
\]

(HB)

¹Note that this ODE is often incorrectly attributed to the much later work of Su, Boyd, and Candes [35], who introduced a similar, though critically different, ODE of the form \( \ddot{x}_t = -\gamma \dot{x}_t - \nabla f(x_t) \), which they used to study Nesterov’s method for smooth (non-strongly convex) minimization.
where $\alpha, \beta$ are constants. In particular, under a suitable choice of $\alpha, \beta$, [32] showed that, when initialized “sufficiently close” to the optimal solution $x^*$, the method converges at rate $(1 - \sqrt{\kappa})^k$ where $\kappa$ denotes the condition number of the objective function $f$. This convergence rate was later proved to be optimal and globally achievable in [26].

For the setting of smooth minimization, Nesterov [27] provided an optimal method with convergence rate $1/k^2$. Nesterov [27] also showed that when this method is applied to strongly convex functions with a known strong convexity parameter and coupled with scheduled restart, it leads to the optimal rate for the class of smooth and strongly convex minimization problems. In later work (see, e.g., the textbook [29] and references therein), Nesterov introduced a separate, more direct method for the minimization of smooth and strongly convex functions that enjoys the same optimal rate as (HB) but that holds globally and with a better constant, and has the same continuous-time limit (HBD).

A flurry of research has followed these seminal papers on momentum-based methods [2–5, 7, 8, 11, 12, 14, 18, 22–25, 33–39], with many of these works [4, 7, 11, 12, 22, 33–35, 37–39] seeking to interpret Nesterov acceleration as a discretization of a continuous-time dynamical system. Further, some of these works led to physical interpretations of Polyak’s [3, 5] and Nesterov’s methods [7, 37] in the Lagrangian and Hamiltonian formalism. For the setting of nonconvex optimization, however, and, more broadly, convergence to points with a small norm of the gradient, the continuous-time perspective has been much less explored [5, 19, 34].

In this paper, we take a Hamiltonian-based perspective to derive a broad class of momentum methods that yield Nesterov’s and Polyak’s methods as special cases. Specifically, we generalize a Hamiltonian that was recently introduced for Nesterov-type methods [13] to cover a broad class of momentum-based methods. As a specific example, a class of methods obtained from this Hamiltonian and parametrized by $\lambda \in [0,1]$ interpolates between Nesterov’s method for smooth minimization [27] (when $\lambda = 1$) and a generalization of the heavy ball method [32] (when $\lambda = 0$). We show that because the methods are obtained as the equations of motion of this Hamiltonian, we can deduce invariants (conserved quantities of the Hamiltonian) that can be used to argue about convergence in function value (for convex optimization) and convergence to stationary points (for possibly nonconvex optimization). The techniques are general and lead to results in Banach spaces.

In terms of the convergence to stationary points, we consider the unconstrained case and focus on finding points with a small norm of the gradient, in either Hilbert or Banach spaces. We show that when $f$ is convex, any method from the class satisfies

$$\min_{0 \leq i \leq k} \|\nabla f(x_k)\|^2 \leq O\left(\frac{L(f(x_0-x^*))}{k}\right).$$

Note that any of these methods, when run for $k/2$ iterations after running Nesterov’s method for $k/2$ iterations, satisfies

$$\min_{k/2 \leq i \leq k} \|\nabla f(x_k)\|^2 \leq O\left(\frac{k}{k^2}\right).$$

While this is suboptimal for the case of convex functions—the optimal rate is $\min_{0 \leq i \leq k} \|\nabla f(x_k)\|^2 \leq O\left(\frac{L(f(x_0-x^*))}{k}\right)$ [9, 21, 28] and it is achieved by [21]—we conjecture that it is tight. In particular, [20] demonstrated that the convergence of the form $\min_{k/2 \leq i \leq k} \|\nabla f(x_k)\|^2 \leq O\left(\frac{L(x_0-x^*)^2}{k}\right)$ is tight for Nesterov’s method.

For the case of nonconvex functions, we show that methods that are instantiations of the heavy-ball method converge at the optimal [9] rate of $\min_{0 \leq i \leq k} \|\nabla f(x_k)\|^2 \leq O\left(\frac{L(f(x_0-x^*))}{k}\right)$. While a similar result exists for the case of Hilbert spaces [17], we are not aware of any other results for the more general Banach spaces in which the method does not lose its favorable properties. For example, it is possible to establish similar rates for modifications of Nesterov’s method that turn it into a descent method that makes at least as much progress as gradient descent, as in, e.g., [28, 30].

In this case, the analysis of convergence in the norm of the gradient boils down to the analysis of gradient descent. Unfortunately, because such methods monotonically decrease the function value, they lose the property of utilizing the momentum to escape shallow local minima. Note that, as mentioned at the beginning, global exploration of local minima is one of the primary reasons for
considering momentum-based methods in nonconvex optimization [5].

While the focus of our paper is the convergence to stationary points, in Appendix B we provide analysis of the methods in terms of convergence in function value in the setting of (possibly) constrained convex optimization in Banach spaces. We show that the entire class of methods parametrized by λ (as mentioned above) converges at rate $1/k^2$ as long as λ is bounded away from zero. When $\lambda = 0$, the convergence slows down to $1/k$. This agrees with previously obtained results for the heavy-ball method in the setting of smooth (non-strongly convex) minimization [16] (our case $\lambda = 0$). As a byproduct of this approach, we obtain a generalization of the heavy-ball method to constrained convex optimization in Banach spaces and show that it converges at rate $1/k$. Such a result was previously known only for unconstrained convex optimization in Euclidean spaces [16].

### 1.1 Related Work

In addition to the work already mentioned above, we provide a few more remarks regarding related work. First, for convex minimization in (function value), there are several approaches that apply to constrained minimization and general Banach spaces [7, 11, 12, 22, 37], with a subset of them being directly motivated by Lagrangian [37] and Hamiltonian [7] mechanics. The latter make use of Lyapunov functions to characterize convergence rates, and their applicability to convergence to stationary points is unclear. In contrast, our work is not based on Lyapunov functions; rather, our analysis of convergence rates stems from the analysis of conserved quantities of the Hamiltonian.

A significant body of recent work in nonconvex optimization focuses on convergence to approximate local minima, with many of the methods having (near-)optimal iteration complexities (see, e.g., [1, 9, 19]). The only work that we are aware of that has used a Hamiltonian perspective on convergence to stationary points and not in providing a general Hamiltonian perspective on nonconvex optimization.

### 1.2 Preliminaries

The primal, $n$-dimensional real vector space is denoted by $E$. The space $E$ is normed, endowed with a norm $\| \cdot \|$. Its dual space, consisting of all linear functions on $E$, is denoted by $E^*$. For $z \in E^*$ and $x \in E$, we denote by $\langle z, x \rangle$ the value of $z$ at $x$. The dual norm (associated with space $E^*$) is defined in the standard way as $\|z\|_* = \max_{x \in E} \frac{\langle z, x \rangle}{\|x\|}$. For Euclidean spaces, $\langle \cdot, \cdot \rangle$ is the standard inner product and $\|\cdot\| = \|\cdot\|_* = \|\cdot\|_2$.

We assume that $f : X \to \mathbb{R}$ is a (possibly non-convex) continuously-differentiable function, and $X \subseteq E$ is closed and convex. $x^* \in \text{argmin}_{x \in X} f(x)$ denotes any fixed minimizer of $f$. To avoid making vacuous statements, we will assume that $f(x^*) > -\infty$.

For all the methods, $x_t \in X$ will be the running solution, and $z_t \in Z \equiv \text{Lin}\{\nabla f(x) : x \in X\}$ (the space of all linear combinations of $\nabla f(x)$, for $x \in X$) will be some linear combination of the gradients $\nabla f(x_\tau)$ for $\tau \in [0, t]$. In the Hamiltonian formalism, $z_t$ will correspond to the momentum, $f(x)$ will correspond to the potential energy, and $\psi^*(z)$ will correspond to the kinetic energy, where $\psi^* : Z \to \mathbb{R}$ is a convex conjugate (defined below) of some strictly convex function $\psi : X \to \mathbb{R}$ (e.g., $\psi^*(z) = \frac{1}{2}\|z\|_2^2$ if $\|\cdot\|_2 = \|\cdot\|_2$ and $\psi(x) = \frac{1}{2}\|x\|_2^2$).

We now outline some useful definitions and facts that are used in the paper.

\footnote{Here we use the notation $\psi^*$ to emphasize that $x$ and $z$ do not, in general, belong to the same vector space.}
The equations of motion corresponding to a Hamiltonian $\mathcal{H}(x, z, t)$ are given by $\dot{x}_t = \nabla_x \mathcal{H}(x, z, t)$ and $\dot{z}_t = -\nabla_z \mathcal{H}(x, z, t)$, where $\nabla_x \mathcal{H}(x, z, t)$ (respectively, $\nabla_z \mathcal{H}(x, z, t)$) denotes the partial gradient of $\mathcal{H}(x, z, t)$ w.r.t. $x$ (respectively, $z$). Differentiating $\mathcal{H}(x, z, t)$ with respect to time $t$, if $x, z$ evolve according to the equations of motion of $\mathcal{H}(x, z, t)$, it follows that $\frac{d}{dt} \mathcal{H}(x, z, t) = \frac{\partial}{\partial t} \mathcal{H}(x, z, t)$. This will be used to derive the conserved quantities (invariants) of the generalized momentum Hamiltonian in the proof of Lemma 2.3 from Section 2.

To carry out the analysis of the cases of convex and nonconvex objectives in a unified way, we introduce the following notion of weak nonconvexity, similar to [1].

**Definition 1.1.** We say that a continuously-differentiable function $f : \mathcal{X} \to \mathbb{R}$ is $\epsilon_H$-weakly nonconvex for some $\epsilon_H \in \mathbb{R}_+$ if, $\forall x, y \in \mathcal{X}$, $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle - \frac{\epsilon_H}{2} \|y - x\|^2$.

We will mainly be concerned with cases $\epsilon_H = 0$ and $\epsilon_H = L$. Observe that a 0-weakly non-convex function is convex, by the standard first-order definition of convexity for continuously-differentiable functions that can be stated as $\forall x, y \in \mathcal{X}$, $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$.

**Definition 1.2.** A continuously-differentiable function $f : \mathcal{X} \to \mathbb{R}$ is $L$-smooth for $L \in \mathbb{R}_+$, if $\forall x, y \in \mathcal{X}$, $\|\nabla f(x) - \nabla f(y)\|_* \leq L \|x - y\|$.

Recall that $L$-smoothness of a function implies that $\forall x, y \in \mathcal{X}$, $f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle - \frac{L}{2} \|y - x\|^2$. It is not hard to show that an $L$-smooth function is also $L$-weakly non-convex.

**Definition 1.3.** A continuously-differentiable function $f : \mathcal{X} \to \mathbb{R}$ is $\mu$-strongly convex for $\mu \in \mathbb{R}_+$, if $\forall x, y \in \mathcal{X}$, $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2$.

**Definition 1.4.** The convex conjugate of $\psi : \mathcal{X} \to \mathbb{R}$ is defined as $\psi^*(z) = \sup_{x \in \mathcal{X}} \{\langle z, x \rangle - \psi(x)\}$.

Since throughout this paper we will be assuming that $\mathcal{X}$ is closed, sup in the previous definition can be replaced with max.

The following standard fact is a corollary of Danskin’s Theorem (see, e.g., [6]).

**Fact 1.5.** Let $\psi : \mathcal{X} \to \mathbb{R}$ be a strictly convex function. Then $\psi^*$ is continuously differentiable and $\nabla \psi^*(z) = \arg\max_{x \in \mathcal{X}} \{\langle z, x \rangle - \psi(x)\}$.

Another useful property of convex conjugacy is the duality between smoothness and strong convexity, which can be seen as a strengthening of Fact 1.5.

**Fact 1.6.** Let $\psi : \mathcal{X} \to \mathbb{R}$ be a $\mu$-strongly convex function. Then $\psi^*$ is $\frac{1}{2\mu}$-smooth.

For a strictly convex function $\psi : x \to \mathbb{R}$ that is continuously differentiable on $\mathcal{X}$, the Bregman divergence is defined in a usual way as $D_\psi(y, x) = \psi(y) - \psi(x) - \langle \nabla \psi(x), y - x \rangle$, where $y \in \mathcal{X}$ and $x \in \mathcal{X}$. Some useful properties of Bregman divergence are stated below.

**Fact 1.7.** (Properties of Bregman Divergence.) Let $\psi : \mathcal{X} \to \mathbb{R}$ be strictly convex and continuously differentiable on $\mathcal{X}$. Then:

(i) For any $u, v, w \in \mathcal{X}$, $D_\psi(u, v) = D_\psi(w, v) + \langle \nabla \psi(w) - \nabla \psi(v), u - w \rangle + D_\psi(u, w)$.

(ii) Let $\psi$ be $\mu$-strongly convex w.r.t. $\|\cdot\|$. Then, $\forall z, z', D_{\psi^*}(z, z') \geq \frac{\mu}{2} \|\nabla \psi^*(z) - \nabla \psi^*(z')\|^2$. 

4
2 Continuous-Time Methods and their Conserved Quantities

We start by describing the simplest Hamiltonian dynamics obtained from a time-invariant Hamiltonian. Such a dynamics is known to be non-convergent—because the total energy of the system is conserved, the dynamics is oscillatory. The oscillatory (instead of a divergent) behavior can be inferred by observing the norm of an average gradient.

We show how to transform this oscillatory Hamiltonian dynamics into convergent dynamics by introducing a time-variant Hamiltonian that dissipates energy. We then show that conserved quantities of this Hamiltonian imply convergence to stationary points. In later sections, we use the discrete versions of these conserved quantities to argue about convergence in function value (for convex optimization) and convergence to stationary points (for potentially nonconvex optimization).

2.1 A Simple Hamiltonian Dynamics

Figure 1: Non-convergence of the standard Hamiltonian dynamics for (a) a simple function with a saddle point and two local minima and (b) Styblinski-Tang function multiplied by 2.

Perhaps the simplest (time-invariant) Hamiltonian that one can formulate is: $H(x_t, z_t) = f(x_t) + \psi^*(z_t)$. Here, $f(x_t)$ can be viewed as the potential energy of a particle at position $x_t$, while $\psi^*(z_t)$ is its kinetic energy. The corresponding continuous-time dynamics is:

\[
\begin{align*}
\dot{x}_t &= \nabla_{x_t} H(x_t, z_t) = \nabla \psi^*(z_t), \\
\dot{z}_t &= -\nabla_{z_t} H(x_t, z_t) = -\nabla f(x_t).
\end{align*}
\] (HD)

This dynamics is meaningful only in the unconstrained regime, since otherwise we cannot guarantee that $x_t \in \mathcal{X}$. Hence, we assume here that $\mathcal{X} = \mathbb{R}^n$. As $H(x_t, z_t)$ does not explicitly depend on time, we have $\frac{d}{dt} H(x_t, z_t) = 0$. Equivalently, $H(x_t, z_t)$ is conserved with time. An immediate implication is that the norm of the averaged gradient decays as $1/t$, as stated in the following lemma.

**Lemma 2.1.** Let $x_t, z_t$ evolve according to (HD), for $z_0 = 0$ and arbitrary (but fixed) $x_0 \in \mathbb{R}^n$, and let $\psi^*(0) = 0$. If $\psi^*$ is $\mu$-strongly convex, then, $\forall t \geq 0$:

\[
\left\| \frac{1}{t} \int_0^t \nabla f(x_r) dr \right\|_* \leq \frac{\sqrt{2(f(x_0) - f(x^*))}}{\mu}.
\]

**Proof.** Since the Hamiltonian is conserved, $\psi^*(z_t) = f(x_0) - f(x_t) + \psi^*(z_0) = f(x_0) - f(x_t)$. As $f(x_t) \geq f(x^*)$, it follows that $\psi^*(z_t) \leq f(x_0) - f(x^*)$. By the $\mu$-strong convexity of $\psi^*$,
\[ \frac{1}{2} \| z_t \|_* \leq f(x_0) - f(x^*). \]

Finally, integrating the second equation from (HD), we have that
\[ z_t = z_0 - \int_0^t \nabla f(x_{\tau}) \, d\tau = -\int_0^t \nabla f(x_{\tau}) \, d\tau, \]
which, combined with the last inequality (after dividing both sides by \( t^2 \mu/2 \) and taking the square root of both sides), gives the claimed bound.

While Lemma 2.1 shows that the average of the gradients converges in norm \( \| \cdot \|_* \) to zero at a sublinear rate, it does not guarantee convergence of the dynamics to any stationary point of \( f(\cdot) \). Indeed, since the energy (equal to \( H(x_t, z_t) \)) is conserved with time, the dynamics is well-known to be oscillatory. Hence, the fact that the average gradient converges in the dual norm with time only implies that the path of the dynamics consists of cycle-like segments over which the gradients cancel out. This is illustrated by the videos shown in Fig. 13. In fact, as shown in Fig. 1(b), the dynamics may be oscillating around multiple stationary points.

### 2.2 Generalized Momentum Dynamics

![Accelerated dynamics](image1) ![Heavy ball dynamics](image2)

Figure 2: Example continuous-time paths of the (a) accelerated dynamics and (b) heavy ball dynamics in the unconstrained Euclidean setting for \( f(x) = \frac{x_1^4}{20} - 20x_1^2 + 10x_2^2 \) and \( \psi^*(z) = \frac{1}{2} \| z \|_2^2 \).

The standard Hamiltonian dynamics from the previous subsection is overly aggressive as a function of the history of the gradients (i.e., momentum \( z_t \)). As a consequence of energy conservation, the energy is exchanged between the potential and kinetic energy, which makes the dynamics exhibit oscillatory behavior. For the dynamics to be attracted to stationary points, it needs to be dampened as the gradients become smaller. This can be achieved by the accelerated dynamics (see, e.g., [12, 22]):

\[
\begin{align*}
\dot{x}_t &= \frac{\dot{\alpha}_t (\nabla \psi^*(z_t) - x_t)}{\alpha_t}, \\
\dot{z}_t &= -\dot{\alpha}_t \nabla f(x_t).
\end{align*}
\]

This dynamics can be shown to correspond to the following (time-dependent) Hamiltonian [13]:

\[ H(x_t, z_t, \alpha_t) = \alpha_t f(x_t/\alpha_t) + \psi^*(z_t), \]

where \( \alpha_t \) is a strictly increasing function of time \( t \) and \( \bar{x}_t = \alpha_t x_t \). To see that the dynamics from (AD) corresponds to the equations of motion of the Hamiltonian \( H_{\text{AD}} \), observe that

\[ \frac{d}{dt} \bar{x}_t = \frac{d}{dt}(\alpha_t x_t) = \dot{\alpha}_t \nabla_x H(x_t, z_t, \alpha_t) = \dot{\alpha}_t \nabla \psi^*(z_t), \]

\footnote{The videos can be played by clicking on them if the pdf is opened in Adobe Reader.}
which, after using the product rule and rearranging the terms, is exactly the first equation from (AD). Similarly, the second equation of motion for the Hamiltonian \( (\mathcal{H}_{\text{AD}}) \) \( \dot{z}_t = -\dot{\alpha}_t \nabla_x \mathcal{H}(\xi_t, z_t, \alpha_t) = -\dot{\alpha}_t \nabla f(\xi_t/\alpha_t) \) is exactly the second equation from (AD), as \( \xi_t = \alpha_t x_t \).

We show that it is possible to generalize the Hamiltonian \( (\mathcal{H}_{\text{AD}}) \) and its resulting equations of motion to capture a much broader class of convergent momentum-based methods that contains a generalization of Polyak’s heavy ball method \([32]\). In particular, consider:

\[
\mathcal{H}_M(\xi_t, z_t, \alpha_t) = h(\alpha_t)f(\xi_t/\alpha_t) + \psi^*(z_t),
\]

where, as before \( \xi_t = \alpha_t x_t \) and \( h(\alpha_t) \) is a positive function of \( \alpha_t \). We will mainly be considering the case \( h(\alpha_t) = \alpha_t^\lambda \) for \( \lambda \in [0,2] \) (see Appendix B and Section 3).

The resulting equations of motion of this Hamiltonian are:

\[
\dot{x}_t = \frac{\dot{\alpha}_t(\nabla\psi^*(z_t) - x_t)}{\alpha_t}, \quad (\text{MoD})
\]

\[
\dot{z}_t = -h(\alpha_t)\frac{\dot{\alpha}_t}{\alpha_t} \nabla f(x_t).
\]

Clearly, Hamiltonian \( (\mathcal{H}_M) \) and its equations of motion \( (\text{MoD}) \) generalize the accelerated dynamics: \( (\mathcal{H}_{\text{AD}}) \) and \( (\text{AD}) \) correspond to the case \( h(\alpha_t) = \alpha_t \). It is possible to show that the class of methods encapsulated by the equations of motion of \( (\mathcal{H}_M) \) also contains a generalization of Polyak’s heavy ball method, as shown in the following proposition. Example evolutions of the dynamics are illustrated in Fig. 2.

**Proposition 2.2.** Polyak’s heavy ball method is equivalent to \( (\text{MoD}) \) when \( \mathcal{X} = \mathbb{R}^d, \| \cdot \| = \| \cdot \|_2, \psi^*(z) = \frac{1}{2\eta} \| z \|_2^2, h(\alpha_t) = \alpha_t^0 = 1 \), and \( \frac{\dot{\alpha}_t}{\alpha_t} = \eta > 0 \).

**Proof.** Under the assumptions of the proposition, \( \dot{x}_t = \eta(\frac{1}{\mu}z_t - x_t) \) and \( z_t = -\eta \nabla f(x_t) \). Hence, \( \dot{x}_t = -\eta \dot{x}_t - \eta^2 \mu \nabla f(x_t) \). For suitable choices of \( \eta, \mu \), this is equivalent to \( (\text{HBD}) \) from \([32]\). \( \square \)

The main usefulness of Hamiltonian \( (\mathcal{H}_M) \) is that it can be used to argue about convergence in both the function value (for convex optimization problems) and convergence to stationary points (for potentially nonconvex problems). We show in the lemma below that it is possible to deduce two different conserved quantities (or invariants) of \( (\mathcal{H}_M) \) that can be used towards this goal.

**Lemma 2.3.** Let \( x_t, z_t \) evolve according to \( (\text{MoD}) \) for an arbitrary initial point \( x_0 = \nabla\psi^*(z_0) \in \mathcal{X} \) and some differentiable \( \psi^*(\cdot) \). Then, \( \forall t \geq 0, \frac{d}{dt} \mathcal{C}_t = 0 \) and \( \frac{d}{dt} \mathcal{C}_t' = 0 \), where:

\[
\mathcal{C}_t' \overset{\text{def}}{=} h(\alpha_t)f(x_t) - \int_0^t f(x_\tau) \frac{d(h(\alpha_\tau))}{d\tau} d\tau + \int_0^t h(\alpha_\tau) \frac{\dot{\alpha}_\tau}{\alpha_\tau} \langle \nabla f(x_\tau), x_\tau \rangle + \psi^*(z_\tau), \tag{2.1}
\]

and

\[
\mathcal{C}_t \overset{\text{def}}{=} h(\alpha_t)\alpha_t f(x_t) - h(\alpha_0)\alpha_0 f(x_0) - \int_0^t \frac{d(h(\alpha_\tau)\alpha_\tau)}{d\tau} f(x_\tau) d\tau + \alpha_0 D_{\psi^*}(z_t, z_0) + \int_0^t D_{\psi^*}(z_t, z_\sigma) d\sigma. \tag{2.2}
\]

The proof of Lemma 2.3 is provided in Appendix A in the supplementary material.

Let us now provide some context for how conserved quantities \( \mathcal{C}_t' \) and \( \mathcal{C}_t \) lead to convergence in function value and convergence in the norm of the gradient, respectively. First, when \( h(\alpha_t) = \alpha_t \) (in which case \( (\text{MoD}) \) is equivalent to \( (\text{AD}) \)), conservation of \( \mathcal{C}_t' \) can be shown to be equivalent to
the conservation of the scaled approximate duality gap from [11, 12] (see [13]). More generally, we show in Appendix B that the conservation of $C_t$ can be used to upper bound the optimality gap $f(\hat{x}_t) - f(x^*)$ for some $\hat{x}_t \in X$ that is constructed as a convex combination of $(x_\tau)_{\tau \in [0, t]}$.

Consider now $C_t$. If $\frac{d}{dt} C_t = 0$, then it is not hard to check that it must be $C_t = 0, \forall t$. Equivalently:

$$h(\alpha_t)_{\alpha_t} f(x_t) - h(\alpha_0)_{\alpha_0} f(x_0) - \int_0^t \frac{d(h(\alpha_\tau)_{\alpha_\tau})}{d\tau} f(x_\tau) d\tau$$

$$= -\alpha_0 D_{\psi^*}(z_t, z_0) - \int_0^t D_{\psi^*}(z_t, z_\sigma) \dot{\alpha}_\sigma d\sigma, \quad \forall t. \quad (2.3)$$

Observe that the right-hand side of (2.3) is always non-positive, as $\psi^*$ is assumed to be convex. Suppose for now that the right-hand side of (2.3) is strictly negative. Then, dividing both sides of (2.3) by $h(\alpha_t)_{\alpha_t}$, we would have: $f(x_t) - \frac{h(\alpha_0)_{\alpha_0}}{h(\alpha_t)_{\alpha_t}} f(x_0) - \frac{1}{h(\alpha_t)_{\alpha_t}} \int_0^t \frac{d(h(\alpha_\tau)_{\alpha_\tau})}{d\tau} f(x_\tau) d\tau < 0.$ In other words, the function value at the last seen point $x_t$ is strictly smaller than a weighted average of function values at points $x_\tau$ for $\tau \in [0, t]$. This means that the average function value must be strictly decreasing with time, which implies that the dynamics must be converging to a point.

To characterize to what type of a point the dynamics is converging to, assume now that the right-hand side of (2.3) is equal to zero. Assume further that $\psi^*$ is 1-strongly convex. Then, $D_{\psi^*}(z, w) \geq \frac{1}{2} \|z - w\|_2^2, \forall z, w,$ and, as $z_t = z_0 - \int_0^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau$, we have:

$$\left\| \int_0^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau \right\|_2^2 + \int_0^t \dot{\alpha}_\sigma d\sigma \left\| \int_\sigma^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau \right\|_2^2 d\sigma = 0.$$

Because all of $\dot{\alpha}_t$, $\alpha_t$, and $h(\alpha_t)$ are assumed to be positive for all $t$ and $f$ is continuously differentiable, the only way we can have the last equality is if $\|\nabla f(x_\tau)\|_\ast = 0, \forall \tau \in [0, t].$ For otherwise, there would exist $\sigma \in [0, t]$ such that $\left\| \int_\sigma^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau \right\|_\ast > 0$, which would violate the equality.

To argue that the dynamics converges to a point with a small norm of the gradient, one needs to further argue that when $\frac{1}{h(\alpha_t)_{\alpha_t}} \left\| \int_0^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau \right\|_\ast^2 + \frac{1}{h(\alpha_t)_{\alpha_t}} \int_0^t \dot{\alpha}_\sigma \left\| \int_\sigma^t h(\alpha_\tau)_{\alpha_\tau} \nabla f(x_\tau) d\tau \right\|_\ast^2 d\sigma \to 0,$ it must also be the case that $\|\nabla f(x_t)\|_\ast \to 0$. This is possible to argue in a similar way, under suitable regularity conditions on $\alpha_0$ and $h(\alpha_t)$. The details are omitted, while similar results are shown directly for the discrete case in Appendix C.2.

3 Convergence in the Norm of the Gradient

In this section, we provide an overview of our approach to the analysis of the convergence of the gradient norm and state our main results. All of the technical details are deferred to Appendix C.

3.1 An Overview of the Approach and a Structural Lemma

We consider a counterpart to $C_t$ that was derived for general momentum methods and defined in Lemma 2.3. This counterpart is defined as:

$$C_k = B_k f(y_k) - \sum_{i=0}^k b_i f(y_i) + \sum_{i=0}^k a_i D_{\psi^*}(z_k, z_i), \quad (3.1)$$

where $A_k = \sum_{i=0}^k a_i$, $B_k = \sum_{i=0}^k b_i$, and $a_i, b_i > 0, \forall i \geq 0$. Going from continuous to the discrete time, $\alpha_t$ translates into $A_k$, $h(\alpha_t)$ translates into $H_k$, and $\alpha_t h(\alpha_t)$ translates into $B_k$. 

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To characterize convergence to stationary points, we denote by \( E_k \overset{\text{def}}{=} C_k - C_{k-1} \) the discretization error between the iterations \( k - 1 \) and \( k \) (recall that in the continuous time domain, \( C_t \) was conserved). As \( C_0 = 0 \), we clearly have \( C_k = \sum_{i=1}^{k} E_i \). Given specific assumptions about the objective function (e.g., if it is convex or nonconvex, its degree of smoothness, etc.), to draw conclusions about the algorithm’s convergence to a stationary point, we will need to argue that the total discretization error \( \sum_{i=1}^{k} E_i \) is “sufficiently small” (possibly zero, or even negative). In general, the magnitude of the discretization error will be determined by the step sizes \( a_i, b_i \), which will be one of the constraining factors determining the rate of convergence.

**Decrease in the Average Function Value** The following (algorithm-independent) lemma implies that if \( C_k \) is non-increasing with \( k \) (namely, if \( E_k \leq 0 \), \( \forall k \)) then the average function value taken at all points constructed by the algorithm is decreasing with the iteration count \( k \). This implies that the algorithm stabilizes, i.e., it converges to a point. The lemma will be crucial in obtaining the results for the convergence in the norm of the gradient.

**Lemma 3.1.** Let \( C_k = B_k f(y_k) - \sum_{i=0}^{k} b_i f(y_i) + \sum_{i=0}^{k} a_i D_{\psi^*}(z_k, z_i) \) and \( E_k = C_k - C_{k-1} \), where \( \forall i, a_i, b_i > 0 \) and \( \forall k, A_k = \sum_{i=0}^{k} a_i, B_k = \sum_{i=0}^{k} b_i \). Then:

\[
\frac{1}{B_k} \sum_{i=0}^{k} b_i f(y_i) = f(y_0) - \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) E_i.
\]

**3.2 Convergence to Stationary Points in Hilbert Spaces**

In this subsection, we assume that \( (E, \| \cdot \|) \) is a Hilbert space, with the inner-product-induced norm. This implies that the norm is self-dual \( \| \cdot \|_* = \| \cdot \| \). Further, we take \( \psi(x) = \frac{\mu}{2} \| x \|^2 \), so that \( D_{\psi^*}(z, w) = \frac{1}{2\mu} \| z - w \|^2 \). For this setting, we consider the following discretization

\[
\begin{align*}
x_k &= \frac{A_{k-1}}{A_k} y_{k-1} + \frac{a_k}{A_k} \nabla \psi^* (z_{k-1}), \\
z_k &= z_{k-1} - \frac{a_k}{A_k} H_k \nabla f(x_k), \quad \text{(GMD)}
\end{align*}
\]

\[
y_k = x_k + \frac{a_k}{A_k} (\nabla \psi^* (z_k) - \nabla \psi^* (z_{k-1})).
\]

Note that when \( H_k = A_k \), (GMD) is equivalent to AGD+ from [10]. We have the following result.

**Theorem 3.2.** Let \( x_k, y_k, z_k \) evolve according to (GMD), for arbitrary \( x_0 = y_0 \in \mathbb{R}^n \) such that \( x_0 = \nabla \psi^* (z_0) \), where \( \psi(x) = \frac{\mu}{2} \| x \|^2 \). Let \( \frac{a_k}{A_k^2} = c \frac{\mu}{L H_k} \) for some \( c \in [0, 1] \) and \( B_{k-1} = A_{k-1} H_k \). If, for some \( c' \in [0, 1] \) and all \( i \leq k \): \( (1 - c') \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \geq c' H \left( \frac{1}{B_i} - \frac{1}{B_{i-1}} \right) \), then:

\[
\sum_{i=1}^{k} c' \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \frac{c(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) ||\nabla f(x_i)||^2 \leq f(x_0) - f(y_k) \leq f(x_0) - f(x^*).
\]

To demonstrate the usefulness of Theorem 3.2, consider \( H_k = A_k^\lambda \) for \( \lambda \in [0, 2] \). Then, we have the following corollaries, dealing with the convex and the nonconvex cases, respectively.
3.3 Convergence to Stationary Points in Banach Spaces

We now show that it is possible to obtain results for convergence to stationary points even in Banach spaces \((E, \| \cdot \|)\). However, we are only able to show such a result for a different discretization of (MoD). To obtain the result, we will only require that \(\psi(\cdot)\) is \(\mu\)-strongly convex with respect to the norm \(\| \cdot \|\). By Fact 1.6, this implies that \(\psi^*\) is \(\frac{1}{\mu}\)-smooth with respect to the dual norm \(\| \cdot \|_*\).

The alternative discretization uses a gradient descent step for \(y_k\) to ensure the decrease in \(C_k\) that depends on \(\| \nabla f(y_k) \|^2_*\). However, to ensure the right change in \(C_k\) over iterations \(k\), such a choice of \(y_k\) requires changes to the extrapolation step \(x_k\). In particular, the discrete-time algorithm is:

\[
\begin{align*}
    x_k &= y_{k-1} + \frac{a_k}{A_k} \nabla \psi^*(z_{k-1}) - \frac{a_k}{A_k A_{k-1}} \sum_{i=0}^{k-1} a_i \nabla \psi^*(z_i), \\
    z_k &= z_{k-1} - \frac{a_k}{A_k} H_1 \nabla f(x_k), \quad \text{(GMD)} \\
    y_k &= \arg\min_u \left\{ (\nabla f(x_k), u - x_k) + \frac{L}{2} \| u - x_k \|^2 \right\}.
\end{align*}
\]

Note that in Hilbert spaces, when \(\psi^*(z) = \frac{L}{2} \| z \|^2\), (GMD) is equivalent to (GMD). Hence, (GMD) can be seen as a generalization of (GMD) to Banach spaces.

Similarly as for the Hilbert spaces from previous subsection, we obtain an equivalent of Theorem 3.2 (see Theorem D.2 in Appendix D). Its corollary leads to the following convergence result.

Corollary 3.5. Let \(x_k, y_k, z_k\) evolve according to (GMD), for some \(\mu\)-strongly convex \(\psi\), where \(\mu > 0\), and where \(\frac{a_k^2}{A_k^2} = c \frac{\mu}{\| H_i \|_*}\), \(c \in (0, 1)\).

(i) If \(\frac{\mu}{\| H_i \|_*} = 1\), then \(\forall k \geq 1\):

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{c(1 - c)k}.
\]

In particular, for \(c = \frac{1}{2}\):

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{k}.
\]

(ii) If \(f\) is convex and \(H_i = A_i^2\), then:

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{c(1 - c)k}.
\]

In particular, for \(c = \frac{1}{2}\):

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{k}.
\]
4 Discussion

We presented a generic Hamiltonian-based framework for the analysis of general momentum methods in Banach spaces and in the settings of both convex and nonconvex optimization. Several questions that merit further investigation remain. For example, while convergence to stationary points in Euclidean spaces is well-understood [9, 21], much less is known in terms of both upper and lower bounds in general Banach spaces. Another interesting direction for future research is a rigorous characterization of the use of momentum to escape shallow local minima.

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A Omitted Proofs from Section 2

Lemma 2.3. Let $x_t, z_t$ evolve according to (MoD) for an arbitrary initial point $x_0 = \nabla \psi^*(z_0) \in X$ and some differentiable $\psi^*(\cdot)$. Then, $\forall t \geq 0$, $\frac{d}{dt}C_t^{\ell} = 0$ and $\frac{d}{dt}C_t = 0$, where:

$$C_t^{\ell} \overset{\text{def}}{=} h(\alpha_t)f(x_t) - \int_0^t f(x_\tau) \frac{d(h(\alpha_\tau))}{d\tau} d\tau + \int_0^t h(\alpha_t) \frac{\dot{\alpha}_\tau}{\alpha_\tau} \langle \nabla f(x_\tau), x_\tau \rangle + \psi^*(z_t), \quad (2.1)$$

and

$$C_t \overset{\text{def}}{=} h(\alpha_t)\alpha_t f(x_t) - h(\alpha_0)\alpha_0 f(x_0) - \int_0^t \frac{d(h(\alpha_\tau)\alpha_\tau)}{d\tau} f(x_\tau) d\tau$$

$$+ \alpha_0 D\psi^*(z_t, z_0) + \int_0^t D\dot{\psi}^*(z_t, z_\sigma) d\sigma. \quad (2.2)$$

Proof. The simplest way of proving the lemma is by directly computing $\frac{d}{dt}C_t^{\ell}$ and $\frac{d}{dt}C_t$, and showing that (MoD) implies that both are equal to zero. Here, we provide a longer, but more constructive proof that highlights how $C_t^{\ell}, C_t$ arise as invariants of $(H_M)$.

As $\bar{x}_t, z_t$ evolve according to the equations of motion of $(H_M)$, we have that $\frac{d}{dt}H_M(\bar{x}_t, z_t, \alpha_t) = \frac{\partial}{\partial \alpha_t}H_M(\bar{x}_t, z_t, \alpha_t) = \frac{d}{d\alpha_t}H_M(\bar{x}_t, z_t, \alpha_t) \cdot \dot{\alpha}_t$. Observe that:

$$\frac{d}{d\alpha_t}H_M(\bar{x}_t, z_t, \alpha_t) = h'(\alpha_t)f(\bar{x}_t/\alpha_t) - h(\alpha_t) \left\langle \nabla f(\bar{x}_t/\alpha_t), \frac{\bar{x}_t}{\alpha^2_t} \right\rangle$$

$$= h'(\alpha_t)f(x_t) - \frac{h(\alpha_t)}{\alpha_t} \langle \nabla f(x_t), x_t \rangle.$$

Hence, we have that:

$$\frac{d}{dt} (h(\alpha_t)f(x_t) + \psi^*(z_t)) = \frac{dh(\alpha_t)}{dt}f(x_t) - h(\alpha_t) \dot{\alpha}_t \langle \nabla f(x_t), x_t \rangle.$$

Equivalently, using the product rule of differentiation:

$$h(\alpha_t) \frac{d}{dt}f(x_t) + \frac{d}{dt} \psi^*(z_t) = -h(\alpha_t) \frac{\dot{\alpha}_t}{\alpha_t} \langle \nabla f(x_t), x_t \rangle. \quad (A.1)$$

Integrating both sides of (A.1) from 0 to $t$ and using using integration by parts leads to $C_t^{\ell}$.

To obtain $C_t$, observe (from (MoD)) that $z_t = -h(\alpha_t) \frac{\dot{\alpha}_t}{\alpha_t} \nabla f(x_t)$. Multiplying both sides of (A.1) by $\alpha_t$, we thus have:

$$\alpha_t h(\alpha_t) \frac{d}{dt}f(x_t) + \alpha_t \frac{d}{dt} \psi^*(z_t) = \alpha_t \langle \dot{z}_t, x_t \rangle. \quad (A.2)$$

As for $C_t^{\ell}$, to obtain $C_t$, we integrate both sides of the last equation from 0 to $t$. Integrating the left-hand side and applying integration by parts gives:

$$\int_0^t \left( \alpha_t h(\alpha_t) \frac{d}{d\tau}f(x_\tau) + \alpha_t \frac{d}{d\tau} \psi^*(z_\tau) \right) d\tau$$

$$= h(\alpha_t)\alpha_t f(x_t) - h(\alpha_0)\alpha_0 f(x_0) - \int_0^t \frac{d(h(\alpha_\tau)\alpha_\tau)}{d\tau} f(x_\tau) d\tau$$

$$+ \alpha_t \psi^*(z_t) - \alpha_0 \psi^*(z_0) - \int_0^t \dot{\alpha}_\tau \psi^*(z_\tau) d\tau. \quad (A.3)$$
On the other hand, by the definition of $x_t$ from (MoD), $x_t = \frac{\alpha_t}{\alpha_0} x_0 + \frac{1}{\alpha_0} \int_0^t \alpha_{\sigma} \nabla \psi^*(z_\sigma) d\sigma$. Thus, integrating the right-hand side of (A.2), we have:

$$
\int_0^t \alpha_\tau \langle \dot{z}_\tau, x_\tau \rangle d\tau = \int_0^t \left( \dot{z}_\tau, \alpha_0 x_0 + \int_0^\tau \nabla \psi^*(z_\sigma) \alpha_\sigma d\sigma \right) d\tau
$$

$$
\quad = \alpha_0 \langle z_t - z_0, \nabla \psi^*(z_0) \rangle + \int_0^t \int_0^\tau \langle \dot{z}_\tau, \nabla \psi^*(z_\sigma) \rangle \alpha_\sigma d\sigma d\tau,
$$

(A.4)

where we have used $x_0 = \nabla \psi^*(z_0)$. By elementary calculus, it is possible to exchange the order of integration on the right-hand side of (A.4), which leads to:

$$
\int_0^t \alpha_\tau \langle \dot{z}_\tau, x_\tau \rangle d\tau = \int_0^t \left( \dot{z}_\tau, \alpha_0 x_0 + \int_0^\tau \nabla \psi^*(z_\sigma) \alpha_\sigma d\sigma \right) d\tau
$$

$$
\quad = \alpha_0 \langle z_t - z_0, \nabla \psi^*(z_0) \rangle + \int_0^t \langle z_t - z_\sigma, \nabla \psi^*(z_\sigma) \rangle \alpha_\sigma d\sigma.
$$

(A.5)

By the definition of Bregman divergence, $D_{\psi}(z, w) = \psi^*(z) - \psi^*(w) - \nabla \psi^*(w, z - w)$. Thus, combining Eqs. (A.3) and (A.5) leads to $C_t = 0$. As this holds for an arbitrary $t$, the proof is complete. 

\[ \square \]

**B Convergence in Function Value**

In this section, we show that the invariants implied by the Hamiltonian generating the momentum-based methods can be used to argue about convergence in function value. We start by arguing about the continuous-time case, and then show how the same invariant in the discrete time setting can be used to argue about convergence of the discretized versions of (MoD).

All the results will be obtained for the following choice of $h(\alpha_t)$:

$$
h(\alpha_t) = \alpha_t^\lambda, \quad \text{where } \lambda \in [0, 1],
$$

with the same relationship holding between their corresponding discrete-time counterparts ($A_k$ and $H_k$). This choice of $h(\alpha_t)$ interpolates between the accelerated method (AD) (obtained for $\lambda = 1$) and the generalized heavy ball method (obtained for $\lambda = 0$).

**B.1 Convergence of the Continuous-Time Dynamics**

We have already shown in the proof of Lemma 2.3 that because (MoD) represents the equations of motion of the Hamiltonian $H_M(x_t, z_t, \alpha_t) = h(\alpha_t) f(x_t/\alpha_t) + \psi^*(z_t)$, we have the following conserved quantity $\forall t \geq 0$:

$$
C_t^f = h(\alpha_t) f(x_t) - \int_0^t f(x_\tau) \frac{d(h(\alpha_\tau))}{d\tau} d\tau + \int_0^t h(\alpha_\tau) \dot{\alpha}_\tau \langle \nabla f(x_\tau), x_\tau \rangle + \psi^*(z_\tau),
$$

(B.1)

This form of a conserved quantity will be used to argue about convergence in function value for objectives that are continuously-differentiable, but not necessarily strongly convex. We now show how Lemma 2.3 can be used to argue about the convergence in function value of (MoD).

**Lemma B.1.** Let $x_t, z_t$ evolve according to (MoD) for $h(\alpha_t) = \alpha_t^\lambda, \lambda \in [0, 1]$ and $x_0 = \nabla \psi^*(z_0) \in \text{relint}(\mathcal{X})$. If $\lambda = 0$, assume that $\frac{\alpha_t}{\alpha_0} = \eta > 0$. Denote:

$$
\hat{x}_t = \begin{cases} 
\frac{\lambda}{\alpha_t} x_0 + (1-\lambda) \int_0^t \frac{\alpha_\tau}{\alpha_0} \beta^\tau \alpha^\tau x_\tau \nabla \psi^*(z_\tau) d\tau + \frac{1-\lambda}{\lambda} x_0, & \text{if } \lambda \in (0, 1], \\
\frac{x_t + \int_0^t x_\tau d\tau}{1+t}, & \text{if } \lambda = 0.
\end{cases}
$$

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Then, \( \forall t \geq 0 \):
\[
f(\tilde{x}_t) - f(x^*) \leq \begin{cases} 
\frac{\alpha t}{t+1}(f(x_0) - f(x^*)) + \lambda D_{\psi}(x^*,x_0), & \text{if } \lambda \in (0, 1], \\
\frac{\alpha t}{t+1}(f(x_0) - f(x^*)) + \lambda D_{\psi}(x^*,x_0), & \text{if } \lambda = 0.
\end{cases}
\]

**Proof.** Lemma 2.3 implies that \( C_t^f = C_0^f, \forall t \geq 0 \). Hence, as \( h(\alpha_t) = \alpha_t^{\lambda} \):
\[
\alpha_t^{\lambda} f(x_t) - \alpha_0^{\lambda} f(x_0) - \lambda \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} f(x_\tau) \, d\tau = \psi^*(z_t) - \psi^*(z_t) - \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x_\tau \rangle \, d\tau.
\]
Write \(-\int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x_\tau \rangle \, d\tau \) as:
\[
-\int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x_\tau \rangle \, d\tau = \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x^* - x_\tau \rangle \, d\tau - \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x^* \rangle \, d\tau.
\]
Observe that, by convexity of \( f \):
\[
\int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x^* - x_\tau \rangle \, d\tau \leq \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} (f(x^*) - f(x_\tau)) \, d\tau.
\]
By the definition of \( z_t \) from (MoD),
\[
-\int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} \langle \nabla f(x_\tau), x^* \rangle \, d\tau = \int_0^t \langle z_\tau, x^* \rangle = \langle z_t - z_0, x^* \rangle.
\]
The next step is to combine \( \langle z_t - z_0, x^* \rangle \) with \( \psi^*(z_0) - \psi^*(z_t) \) to write them in the form of Bregman divergences. In particular, define \( z^* \) so that \( \nabla \psi^*(z^*) = x^* \). Then:
\[
\psi^*(z_0) - \psi^*(z_t) + \langle z_t - z_0, x^* \rangle = D_{\psi^*}(z_0, z^*) - D_{\psi^*}(z_t, z^*) \leq D_{\psi^*}(z_0, z^*).
\]
Using Fact 1.5 (which implies \( \psi^*(z) = \langle \nabla z, \nabla \psi^*(z) \rangle - \psi(\nabla \psi^*(z)) \)) and \( z_0 = \nabla \psi(x_0) \) (which follows from the assumption that \( x_0 \) is from the relative interior of \( \lambda \)), it is not hard to show that:
\[
D_{\psi^*}(z_0, z^*) = D_{\psi}(\nabla \psi^*(z^*), \nabla \psi^*(z_0)) = D_{\psi}(x^*, x_0).
\]
Combining with (B.2)-(B.5):
\[
\alpha_t^{\lambda} f(x_t) - \alpha_0^{\lambda} f(x_0) - \lambda \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} f(x_\tau) \, d\tau \leq \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} (f(x^*) - f(x_\tau)) \, d\tau + D_{\psi}(x^*, x_0).
\]
Assume first that \( \lambda > 0 \). Integrating and rearranging the terms in the last inequality:
\[
\alpha_t^{\lambda} f(x_t) + (1 - \lambda) \int_0^t \dot{\alpha}_t \alpha_t^{\lambda-1} f(x_\tau) \, d\tau + \frac{1 - \lambda}{\lambda} f(x_0) - \frac{\alpha_0}{\lambda} f(x^*)
\]
\[
\leq \frac{\alpha_0}{\lambda} (f(x_0) - f(x^*)) + D_{\psi}(x^*, x_0).
\]
It remains to divide both sides of the last inequality by $\frac{a_i^\lambda}{x_i}$ and apply Jensen’s inequality.

If $\lambda = 0$, then, assuming $\frac{a_i}{\alpha_i} = \eta$:

$$f(x_i) + \int_0^t f(x_t)\,dt - (1 + t)f(x^*) \leq f(x_0) - f(x^*) + D_\psi(x^*, x_0).$$

Similarly as for $\lambda > 0$, it remains to divide both sides by $1 + t$ and apply Jensen’s inequality.

Observe that when $\lambda = 1$ (that is, when (MoD) is equivalent to (AD)), $\hat{x}_t = x_t$, and we recover the standard guarantee on the last iterate of the accelerated dynamics [11, 12, 22]. When $\lambda = 0$, we obtain a $1/t$ convergence for the generalization of the heavy ball method. The result applies to constrained optimization and Banach spaces. We note that a generalization of the heavy ball method to constrained convex optimization was previously considered in [3]. However, the result from [3] applies only to Hilbert spaces and provides weak (asymptotic) convergence results. The second-order ODE considered in [3] seems to correspond to a different continuous-time dynamics than (MoD) with $h(\alpha_t) = 1$ and it is unclear how to compare it to (MoD).

**B.2 Discrete-Time Convergence**

Define the counterpart to the continuous-time conserved quantity $C_t^f$, $C_k^f$, as:

$$C_k^f = H_k f(y_k) - \sum_{i=1}^k h_i f(x_i) + \sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x_i \rangle + \psi^*(z_k),$$

where $A_k = \sum_{i=0}^k a_i$, $H_k = \sum_{i=0}^k h_i$, and $H_k = A_k^\lambda$.

The discretization of the continuous-time dynamics that we will use is:

$$x_k = \frac{H_{k-1}/H_k}{H_{k-1}/H_k + a_k/A_k} y_{k-1} + \frac{a_k/A_k}{H_{k-1}/H_k + a_k/A_k} \nabla \psi^*(z_{k-1}),$$

$$z_k = z_{k-1} - H_k \frac{a_k}{A_k} \nabla f(x_k),$$

$$y_k = x_k + \frac{a_k}{A_k} (\nabla \psi^*(z_k) - \nabla \psi^*(z_{k-1})).$$

This particular choice of the discretization will become clear from Proposition B.3. Namely, the discretization was chosen to ensure that $C_k^f \leq C_{k-1}^f$. Note that when $H_k = A_k (\lambda = 1)$, the method is precisely the AGD+ method from [10].

For (GMDt) to apply to constrained minimization, we need to show that the iterates $y_k$ remain in the feasible set. This is established by the following proposition.

**Proposition B.2.** Let $x_k, y_k, z_k$ evolve according to (GMDt), where $y_0 = \nabla \psi^*(z_0) \in \text{rel int } \mathcal{X}$ and $a_k, A_k, H_k > 0$ satisfy: $A_k = \sum_{i=0}^k a_i$ and $\frac{a_k^2}{A_k^2} \leq \frac{\mu}{\mathcal{L} H_k}$, $H_k = A_k^\lambda$ for $\lambda \in [0, 1]$. Then $y_k \in \mathcal{X}$, $\forall k \geq 0$.

**Proof.** The claim clearly holds for $k = 0$, by the choice of initialization. To simplify the notation, denote $\theta_k = \frac{a_k}{A_k}$, $\theta'_k = \frac{a_k/A_k}{H_{k-1}/H_k + a_k/A_k} = \frac{\theta_k}{H_{k-1}/H_k + \theta_k}$, $v_k = \nabla \psi^*(z_k)$. By the definition of a convex conjugate (Definition 1.4) and Fact 1.5, $v_k \in \mathcal{X}$, $\forall k$.

Under the assumptions of the proposition, it is not hard to show (see Section C.2) that either $\frac{a_k}{A_k} \sim \frac{1}{1}$ (if $\lambda > 0$) or $\frac{a_k}{A_k} = \text{const.}$ (if $\lambda = 0$). Hence, $\theta_k \leq \theta_{k-1}$, $\forall k \geq 1$. 

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To prove the proposition, we will first show that \( y_k \) can be expressed as a non-negative linear combination of \( \{v_i\}_{i=0}^{k} \). To complete the proof, we will show by induction on \( k \) that the coefficients of that linear combination must sum up to 1.

Using (GMD), we can write \( y_k \) in the following recursive form:

\[
y_k = (1 - \theta_k')y_{k-1} + \theta'_k v_{k-1} + \theta_k (v_k - v_{k-1}).
\]

Applying this definition recursively over \( i = 0, 1, \ldots, k \) and using that \( y_0 = v_0 \), we get that:

\[
y_k = \sum_{i=0}^{k} \gamma_{i,k} v_i,
\]

where:

\[
\gamma_{i,k} = \begin{cases} \theta_k, & \text{if } i = k; \\ \prod_{j=i+2}^{k} (1 - \theta'_j) \left[ \theta'_{i+1} (1 - \theta_i) + \theta_i - \theta_{i+1} \right], & \text{if } 1 \leq i < k; \\ \prod_{j=1}^{k-1} (1 - \theta'_j), & \text{if } i = 0,
\end{cases}
\]

where, by convention, we take \( \prod_{i=0}^{0} (\cdot) = 1 \) whenever \( j < i \).

As, for all \( i \geq 0, \theta_i, \theta'_i \in [0, 1] \) and \( \theta_{i+1} \leq \theta_i \), it immediately follows that \( \gamma_{i,k} \geq 0, \forall i \in \{0, 1, \ldots, k\} \).

We now show by induction on \( k \) that it must be \( \sum_{i=0}^{k} \gamma_{i,k} = 1 \). This clearly holds for \( k = 0 \). Suppose that it holds for some \( k - 1 \geq 0 \). Then \( y_{k-1} \in \mathcal{X} \). As \( x_k = (1 - \theta'_k) y_{k-1} + \theta'_k v_{k-1} \), it follows that \( x_k \in \mathcal{X} \), and, moreover, \( x_k \) is a convex combination of \( \{v_i\}_{i=0}^{k-1} \). Now, observe from the definition of \( x_k \) that \( \theta'_k y_{k-1} = x_k - (1 - \theta'_k) y_{k-1} \). Hence, we can express \( y_k \) as:

\[
y_k = (1 - \theta_k'/\theta_k') x_k + \theta_k (1 - \theta'_k) y_{k-1} + \theta_k v_k.
\]

As \( 1 - \frac{\theta_k}{\theta_k'} + \frac{\theta_k}{\theta_k'} (1 - \theta'_k) + \theta_k = 1 \) and each \( x_k, y_{k-1}, v_k \) are convex combinations of \( \{v_i\}_{i=0}^{k-1} \), it follows that \( \sum_{i=0}^{k} \gamma_{i,k} = 1 \), which, together with argued \( \gamma_{i,k} \geq 0, \forall i \), completes the proof. \( \square \)

**Proposition B.3.** Let \( x_k, y_k, z_k \) evolve according to (GMD), where \( \psi : \mathbb{X} \to \mathbb{R} \) is a \( \mu \)-strongly convex function, \( y_0 = \nabla \psi^*(z_0) \in \text{rel int} \mathcal{X} \) and \( a_k, A_k, H_k > 0 \) satisfy: \( A_k = \sum_{i=0}^{k} a_i \) and \( \frac{a_k^2}{A_k} \leq \frac{\mu}{LH_k} \). Then \( C_k^I \leq C_{k-1}^I, \forall k \geq 1 \).

**Proof.** By the definition of \( C_k^I \), we have:

\[
C_k^I - C_{k-1}^I = H_k f(y_k) - H_{k-1} f(y_{k-1}) - h_k f(x_k) + H_k \frac{a_k}{A_k} \langle \nabla f(x_k), x_k \rangle + \psi^*(z_k) - \psi^*(z_{k-1})
\]

Observe first, by smoothness and convexity of \( f \):

\[
H_k f(y_k) - H_{k-1} f(y_{k-1}) - h_k f(x_k) = H_k (f(y_k) - f(x_k)) + H_{k-1} (f(x_k) - f(y_{k-1})) \leq \langle \nabla f(x_k), H_k y_k - H_{k-1} y_{k-1} - h_k x_k \rangle + H_k \frac{L}{2} \| y_k - x_k \|^2.
\]

On the other hand, by the definitions of a Bregman divergence and \( z_k \):

\[
\psi^*(z_k) - \psi^*(z_{k-1}) = -D_{\psi^*}(z_{k-1}, z_k) - \langle \nabla \psi^*(z_{k-1}), z_{k-1} - z_k \rangle = -D_{\psi^*}(z_{k-1}, z_k) - H_k \frac{a_k}{A_k} \langle \nabla f(x_k), \nabla \psi^*(z_{k-1}) \rangle.
\]
As $\psi$ is $\mu$-strongly convex, we have (by Fact 1.7): $D_\psi(z_{k-1},z_k) \geq \frac{\mu}{2}\|\nabla\psi^*(z_k) - \nabla\psi^*(z_{k-1})\|^2$. Hence, combining (B.6)-(B.8):

$$C_k^f - C_{k-1}^f \leq H_k \frac{L}{2} \|y_k - x_k\|^2 - \frac{\mu}{2} \|\nabla\psi^*(z_k) - \nabla\psi^*(z_{k-1})\|^2$$

$$+ \left\langle \nabla f(x_k), H_k y_k - H_{k-1} y_{k-1} - h_k x_k + H_k \frac{a_k}{A_k} (x_k - \nabla\psi^*(z_k)) \right\rangle.$$

Note that we want to make the right-hand side of the last inequality non-positive. To do so, we can make the last term equal to zero by setting: $H_k y_k - H_{k-1} y_{k-1} - h_k x_k + H_k \frac{a_k}{A_k} (x_k - \nabla\psi^*(z_k)) = 0$. To make the first two terms non-positive, we require $y_k - x_k = \frac{a_k}{A_k} (\nabla\psi^*(z_k) - \nabla\psi^*(z_{k-1}))^4$. Solving these last two equations for $x_k$ gives (GMD$_f$). It remains to use that $H_k \frac{a_k^2}{A_k^2} \leq \frac{\mu}{\lambda}$.

To obtain a convergence rate for (GMD$_f$), it remains to show that $C_k^f \leq C_0^f$ implies a convergence in function value for (GMD$_f$), similarly as was done for the continuous-time case in Lemma B.1.

**Theorem B.4.** Let $x_k, y_k, z_k$ evolve according to (GMD$_f$), where $x_0 = y_0 = \nabla\psi^*(z_0) \in \text{rel int} \mathcal{X}$, $\psi : \mathcal{X} \to \mathbb{R}$ is $\mu$-strongly convex, $H_k = A_k^\lambda$, $\lambda \in [0,1]$, $A_k = \sum_{i=0}^k a_k$, $a_0 = 1$, and $\frac{a_k^2}{A_k^2} = c \frac{\mu}{H_k}$, for $c \in (0,1]$, $k \geq 1$. Define:

$$\tilde{x}_k = \frac{H_k y_k + \sum_{i=1}^k \frac{a_i}{A_i} H_i - h_i) x_i}{H_0 + \sum_{i=1}^k \frac{a_i}{A_i} H_i}.$$

Then, $\forall k \geq 1$, $\tilde{x}_k \in \mathcal{X}$ and:

$$f(\tilde{x}_k) - f(x^*) \leq \begin{cases} \frac{f(x_0) - f(x^*) + D_\psi(x^*,x_0)}{1 + \sqrt{\frac{\mu}{L} k}}, & \text{if } \lambda = 0, \\ \Theta(1) \frac{L}{\epsilon_\mu} \frac{f(x_0) - f(x^*) + D_\psi(x^*,x_0)}{k^2}, & \text{if } \lambda \in (0,1], \lambda = \Omega(1). \end{cases}$$

**Proof.** By Proposition B.2, $y_k \in \mathcal{X}$, $\forall k$. As $x_k \in \mathcal{X}$ (as a convex combination of $y_{k-1}$, $\nabla\psi^*(z_{k-1}) \in \mathcal{X}$), we have that $\tilde{x}_k$ is a convex combination of points from the feasible space $\mathcal{X}$, and, thus, it must be $\tilde{x}_k \in \mathcal{X}$, $\forall k$.

By Proposition B.3, $C_k^f \leq C_0^f$. Hence:

$$H_k f(y_k) - H_0 f(y_0) - \sum_{i=1}^k h_i f(x_i) \leq \psi^*(z_0) - \psi^*(z_k) - \sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x_i \rangle. \quad (B.9)$$

As in Lemma B.1, write $-\sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x_i \rangle$ as:

$$- \sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x_i \rangle = \sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x^* - x_i \rangle - \sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x^* \rangle. \quad (B.10)$$

By convexity of $f$:

$$\sum_{i=1}^k H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x^* - x_i \rangle \leq \sum_{i=1}^k H_i \frac{a_i}{A_i} (f(x^*) - f(x_i)). \quad (B.11)$$

\footnote{Note that the multiplier $\frac{a_k}{A_k}$ on the right-hand side is necessary here for $x_k$ to be explicitly defined. Any other factor would make $x_k$ depend on $\nabla\psi^*(z_k)$, which is a function of $x_k$ (as $z_k = z_{k-1} - H_k \frac{a_k}{A_k} \nabla f(x_k)$), and would thus make $x_k$ be only implicitly defined.}
Let $z^*$ be such that $\nabla \psi^*(z^*) = x^*$. Then, by the same arguments as in the proof of Lemma B.1:

$$\psi^*(z_0) - \psi^*(z_k) - \sum_{i=1}^{k} H_i \frac{a_i}{A_i} \langle \nabla f(x_i), x^* \rangle = D_{\psi^*}(z_0, z^*) - D_{\psi^*}(z_t, z^*) \leq D_{\psi}(x^*, x_0).$$  \hfill (B.12)

Combining (B.9)-(B.12):

$$H_k f(y_k) - H_0 f(y_0) - \sum_{i=1}^{k} h_i f(x_i) \leq \sum_{i=1}^{k} H_i \frac{a_i}{A_i} (f(x^*) - f(x_i)) + D_{\psi}(x^*, x_0).$$

To complete the proof, it remains to rearrange the terms in the last equation. Notice that $\sum_{i=1}^{k} h_i = H_k - H_0$, and, thus, the coefficients multiplying $f(\cdot)$ sum up to zero. Notice also that, as $H_i = A_i^\lambda$, $a_i = A_i - A_i - 1$, and $h_i = H_i - H_i - 1$, it must be $H_i \frac{a_i}{A_i} - h_i \geq 0$. We have:

$$H_k f(y_k) + \sum_{i=1}^{k} \left( H_i \frac{a_i}{A_i} - h_i \right) f(x_i) - \left( H_0 + \sum_{i=1}^{k} H_i \frac{a_i}{A_i} \right) f(x^*) \leq H_0 (f(x_0) - f(x^*)) + D_{\psi}(x^*, x_0).$$

Dividing both sides of the last equation by $\left( H_0 + \sum_{i=1}^{k} H_i \frac{a_i}{A_i} \right)$ and applying Jensen’s inequality:

$$f(\hat{x}_i) - f(x^*) \leq \frac{H_0 (f(y_0) - f(x^*)) + D_{\psi}(x^*, x_0)}{\left( H_0 + \sum_{i=1}^{k} H_i \frac{a_i}{A_i} \right)}$$

Recall that, as $a_0 = 1$, it must be $H_0 = 1$. To bound $H_0 + \sum_{i=1}^{k} H_i \frac{a_i}{A_i}$, we need to argue about the growth of $\frac{a_i}{A_i} H_i = \frac{a_i}{A_i^{1-x}}$. This was already done in Section C.2. Namely, when $\lambda = 0$, $\frac{a_i}{A_i^{1-x}} = \sqrt{\frac{c_i}{L}}$. For $\lambda > 0$ and constant-bounded-away from zero, $\frac{a_i}{A_i^{1-x}} = \Theta(1) \frac{c_i}{L}$. Hence, $\sum_{i=1}^{k} \frac{a_i}{A_i} H_i$ scales as $\Theta(k) \sqrt{\frac{c_i}{L}}$ for $\lambda = 0$ and $\Theta(k) \frac{c_i}{L}$ for $\lambda$ bounded away from zero, as claimed.

Observe that a generalization of the heavy ball method (obtained from (GMD)) for $H_k = A_k^0 = 1$ converges at rate $1/k$ for smooth convex functions. The result applies to constrained minimization and general normed vector spaces. Note that such a (nonasymptotic) result was previously known only for the setting of unconstrained minimization in Euclidean spaces [16].

C Proofs for Convergence in the Norm of the Gradient

C.1 Overview of the Approach and a Structural Lemma

Lemma 3.1. Let $C_k = B_k f(y_k) - \sum_{i=0}^{k} b_i f(y_i) + \sum_{i=0}^{k} a_i D_{\psi^*}(z_k, z_i)$ and $E_k = C_k - C_{k-1}$, where $\forall i, a_i, b_i > 0$ and $\forall k, A_k = \sum_{i=0}^{k} a_i$, $B_k = \sum_{i=0}^{k} b_i$. Then:

$$\frac{1}{B_k} \sum_{i=0}^{k} b_i f(y_i) = f(y_0) - \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) E_i.$$
Proof. Denote $S_k = \sum_{i=0}^{k} b_i f(y_i)$. The proof of the lemma is by induction on $k$. The base case is immediate, as, by the definition of $S_k$ and $B_k$, $S_0 = b_0 f(y_0) = B_0 f(y_0)$. Now assume that the statement is true for $k \geq 0$. Then, by the definition of $S_k$,

$$S_{k+1} = S_k + b_{k+1} f(y_{k+1}). \quad (C.1)$$

On the other hand, by the definition of $C_k$ and $E_k$, we have that $C_k = \sum_{i=1}^{k} E_i$, and, hence:

$$f(y_{k+1}) = \frac{1}{B_k} \left( S_k - \sum_{j=0}^{k} a_j D_{\psi^*}(z_{k+1}, z_j) + \sum_{i=1}^{k+1} E_i \right). \quad (C.2)$$

Combining Equations (C.1) and (C.2):

$$S_{k+1} = \left( 1 + \frac{b_{k+1}}{B_k} \right) S_k - \frac{b_{k+1}}{B_k} \sum_{j=0}^{k} a_j D_{\psi^*}(z_{k+1}, z_j) + \frac{b_{k+1}}{B_k} \sum_{i=1}^{k+1} E_i$$

$$= \frac{B_{k+1}}{B_k} S_k - B_{k+1} \sum_{j=0}^{k} \left( \frac{1}{B_k} - \frac{1}{B_{k+1}} \right) a_j D_{\psi^*}(z_{k+1}, z_j) + B_{k+1} \sum_{i=1}^{k+1} \left( \frac{1}{B_k} - \frac{1}{B_{k+1}} \right) E_i,$$

where we have used $B_{k+1} = B_k + b_{k+1}$. Applying the inductive hypothesis and grouping the terms into appropriate summations completes the proof. \qed

C.2 Convergence to Stationary Points in Hilbert Spaces

The following simple claim is useful for passage from Bregman divergences to gradient norms.

Claim C.1. Let $a, b > 0$. Then $a \| z + \Delta z \|^2 + b \| z \|^2 \geq \frac{ab}{a+b} \| \Delta z \|^2$.

Proof. As $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$, we have that $a \| z + \Delta z \|^2 + b \| z \|^2 = (a + b) \| z \|^2 + 2a \langle z, \Delta z \rangle + a \| \Delta z \|^2$. By Cauchy-Schwartz Inequality, $\langle z, \Delta z \rangle \leq \| z \| \| \Delta z \|$. Since the claim trivially holds for $\| \Delta z \| = 0$, assume $\| \Delta z \| \neq 0$ and let $c = \| \Delta z \|$. Then $a \| z + \Delta z \|^2 + b \| z \|^2 \geq \| \Delta z \|^2 ((a+b)c^2 - 2ac + a)$. As $(a+b)c^2 - 2ac + a$ is minimized for $c = \frac{a}{a+b}$, the claim follows. \qed

To relate Bregman divergences from the definition of $C_k$ to norms of the gradients, we will use the following application of Claim C.1.

Proposition C.2. Let $\psi(x) = \frac{\mu}{2} \| x \|^2$ (so that $D_{\psi^*}(w, z) = \frac{1}{2\mu} \| w - z \|^2$). Then:

$$\sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) \geq \frac{1}{2\mu} \sum_{j=1}^{i} \nu_j^i \| \nabla f(x_j) \|^2,$$

where $\nu_j^i = \frac{a_j^2}{A_i^2} H_k^2 (\frac{a_j}{2} + \frac{a_j a_{j-1}}{a_j + a_{j-1}})$ and $\nu_j^i = \frac{a_{j-1} a_j}{a_{j-1} + a_j} \frac{H_k^2}{A_i^2}$ for $1 \leq j \leq i-1$.

Proof. By the choice of function $\psi$ and by $\| \cdot \|$ being an inner-product norm, we have that $D_{\psi^*}(w, z) = \frac{1}{2\mu} \| w - z \|^2$. By the definition of $z_k$, we have that $\forall u > j \geq 0 : D_{\psi^*}(z_i, z_j) = \frac{1}{2\mu} \| \sum_{k=j+1}^{i} \frac{a_k}{A_k} H_k \nabla f(x_k) \|^2$. By Claim C.1, $\forall i > j + 1 > 0$:

$$a_j D_{\psi^*}(z_i, z_j) + a_{j+1} D_{\psi^*}(z_i, z_{j+1}) \geq \frac{1}{2\mu} \frac{a_j a_{j+1}}{a_j + a_{j+1}} \frac{H_j^2}{A_{j+1}} \| \nabla f(x_{j+1}) \|^2. \quad (C.3)$$
Write $\sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j)$ as:

$$
\sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) = \frac{a_0}{2} D_{\psi^*}(z_i, z_0) + \frac{a_i-1}{2} D_{\psi^*}(z_i, z_{i-1}) + \frac{1}{2} \sum_{j=0}^{i-2} \left( a_j D_{\psi^*}(z_i, z_j) + a_{j+1} D_{\psi^*}(z_i, z_{j+1}) \right).
$$

Combining the last equation with Eq. (C.3) and $\frac{a_0}{2} D_{\psi^*}(z_i, z_0) \geq 0$ completes the proof. \qed

**Discrete-Time Methods** Recall that the particular version of predictor-corrector discretization we consider for general momentum dynamics (MoD) is:

$$
x_k = \frac{A_{k-1}}{A_k} y_{k-1} + \frac{a_k}{A_k} \nabla \psi^*(z_{k-1}),
$$

$$
z_k = z_{k-1} - \frac{a_k}{A_k} H_k \nabla f(x_k),
$$

$$
y_k = x_k + \frac{a_k}{A_k} \left( \nabla \psi^*(z_k) - \nabla \psi^*(z_{k-1}) \right).
$$

When $H_k = A_k$, (GMD) is equivalent to AGD+ [10] and the method of similar triangles [15], which generalize Nesterov’s accelerated method [27] and accelerated extra-gradient method [11]. When $H_k = 1$, (GMD) is a slightly different discretization of the generalization of the heavy-ball dynamics (GMD) with $H_k = 1$, where the difference lies only in the size of the predictor step $x_k$. Working with the general momentum method (GMD) will allow us to obtain results for AGD+ and generalized heavy-ball method as special cases, and it will also allow us to analyze how the different choices of $H_k$ affect the convergence in the norms of the gradients.

**Discretization Error** Characterization of the discretization error is what crucially determines the convergence of the methods in the norm of the gradients, as well as in function value. Here, we show how the discretization error is affected by the choice of the step size and assumptions about the objective function, such as smoothness and convexity. It is characterized as follows.

**Lemma C.3.** Let $E_k \overset{\text{def}}{=} C_k - C_{k-1}$, where $C_k$ was defined in (3.1), with $a_i, b_i > 0$, $\forall i$, $A_k = \sum_{i=0}^{k} a_i$, $B_k = \sum_{i=0}^{k} b_i$, and $B_{k-1} = H_k A_{k-1}$. Let $x_k, y_k, z_k$ evolve according to (GMD), where $x_0 = y_0 \in \mathcal{X}$ is an arbitrary initial point such that $\nabla \psi^*(z_0) = x_0$, $\psi : \mathcal{X} \to \mathbb{R}$ is a $H$-strongly convex function w.r.t. $\| \cdot \|$, $\mathcal{X} = \mathbb{R}^n$, and $f$ is an $L$-smooth function w.r.t. $\| \cdot \|$. If $f$ is $\epsilon_H$-weakly non-convex for $c \in [0, 1]$, then

$$
E_k \leq - (1 - c) A_{k-1} D_{\psi^*}(z_{k-1}, z_k) + B_{k-1} \frac{\epsilon_H}{2} \|x_k - y_{k-1}\|^2.
$$

**Proof.** By the definitions of $E_k$ and $C_k$ and $D_{\psi^*}(z, \varepsilon) = 0$, we have:

$$
E_k = B_{k-1} \left[ f(y_k) - f(y_{k-1}) \right] + \sum_{i=0}^{k-1} a_i \left[ D_{\psi^*}(z_i, z_{i-1}) - D_{\psi^*}(z_{i-1}, z_i) \right]. \tag{C.4}
$$
As $f$ is $L$-smooth and $\epsilon_H$-weakly non-convex, we have:

\[
\begin{align*}
    f(y_k) - f(y_{k-1}) &= f(y_k) - f(x_k) + f(x_k) - f(y_{k-1}) \\
    &\leq \langle \nabla f(x_k), y_k - y_{k-1} \rangle + \frac{L}{2} \|y_k - x_k\|^2 + \frac{\epsilon_H}{2} \|x_k - y_{k-1}\|^2 \\
    &= \langle \nabla f(x_k), y_k - y_{k-1} \rangle + \frac{L a_k^2}{2 A_k^2} \|\nabla \psi^*(z_k) - \nabla \psi^*(z_{k-1})\|^2 \\
    &\quad + \frac{\epsilon_H}{2} \|x_k - y_{k-1}\|^2. \quad \text{(C.5)}
\end{align*}
\]

By the first part of Fact 1.7 and the definition of $z_k$, $\forall i \leq k - 1$:

\[
D_\psi^*(z_{k-1}, z_i) = D_\psi^*(z_k, z_i) + \langle \nabla \psi^*(z_k) - \nabla \psi^*(z_i), z_{k-1} - z_k \rangle + D_\psi^*(z_{k-1}, z_k)
\]

Hence, we have:

\[
\begin{align*}
    \sum_{i=0}^{k-1} a_i \left[ D_\psi^*(z_k, z_i) - D_\psi^*(z_{k-1}, z_i) \right] \\
    &= - A_k - D_\psi^*(z_{k-1}, z_k) - a_k H_k \left\langle \nabla f(x_k), A_k \nabla \psi^*(z_k) - \sum_{i=0}^{k-1} \nabla \psi^*(z_i) \right\rangle \\
    &= - A_k - D_\psi^*(z_{k-1}, z_k) - a_k H_k \left\langle \nabla f(x_k), \nabla \psi^*(z_k) - \frac{1}{A_k} \sum_{i=0}^{k-1} \nabla \psi^*(z_i) \right\rangle \\
    &= - A_k - D_\psi^*(z_{k-1}, z_k) - a_k H_k \langle \nabla f(x_k), \nabla \psi^*(z_k) - y_k \rangle, \quad \text{(C.6)}
\end{align*}
\]

where the last equality is by $y_k = \frac{1}{A_k} \sum_{i=0}^{k} a_i \nabla \psi^*(z_i)$, which follows by recursively applying the definition of $y_k$ from (GMD) and using that $y_0 = \nabla \psi^*(z_0)$.

By Fact 1.7, $D_\psi^*(z_{k-1}, z_k) \geq \frac{\mu}{L} \|\nabla \psi^*(z_k) - \nabla \psi^*(z_{k-1})\|^2$. Hence, combining Eqs.(C.4)–(C.6):

\[
E_k \leq B_{k-1} \left\langle \nabla f(x_k), y_k - y_{k-1} - \frac{a_k}{A_k} H_k \nabla \psi^*(z_k) - y_k \right\rangle + \frac{\epsilon_H B_{k-1}}{2} \|x_k - y_{k-1}\|^2 \\
+ \left( B_{k-1} - \frac{L a_k^2}{A_k^2} \right) \|\nabla \psi^*(z_k) - \nabla \psi^*(z_{k-1})\|^2 - (1 - c) A_k - D_\psi^*(z_{k-1}, z_k) \\
\leq \frac{\epsilon_H B_{k-1}}{2} \|x_k - y_{k-1}\|^2 - (1 - c) A_k - D_\psi^*(z_{k-1}, z_k),
\]

where we have used that $B_{k-1} = H_k A_k$, $y_k = \frac{A_k}{A_k} y_{k-1} + \frac{a_k}{A_k} \nabla \psi^*(z_k)$ (which implies $y_k - y_{k-1} - \frac{a_k}{A_k} H_k (\nabla \psi^*(z_k) - y_k) = 0$), and $\frac{a_k^2}{A_k^2} = c \cdot \frac{\mu}{L H_k}$. \hfill \Box

**Final Convergence Bound**  To be able to bound the non-negative term in the discretization error $E_k$ (that comes from $\epsilon_H$-weak non-convexity), we will make use of the following proposition.

**Proposition C.4.** Let $x_k, y_k, z_k$ evolve according to (GMD), for $x_0 = y_0 = \nabla \psi^*(z_0)$ and $\mu$-strongly convex $\psi$. Then:

\[
\frac{1}{2} \|x_k - y_{k-1}\|^2 \leq \frac{1}{\mu A_k^2 A_k} \sum_{i=0}^{k-2} a_i D_\psi^*(z_{k-1}, z_i).
\]
Proof. By recursively applying the definition of $y_k$ in (GMD), we have that $y_k = \frac{1}{\lambda_k} \sum_{i=0}^{k} a_i \nabla \psi^*(z_i)$, $\forall k \geq 0$. Further, by the definition of $x_k$ in (GMD), we have $x_k = \frac{A_{k-1}}{A_k} y_{k-1} + \frac{a_k}{A_k} \nabla \psi^*(z_{k-1})$, and it follows that:

$$x_k - y_{k-1} = \frac{a_k}{A_k} \left( \nabla \psi^*(z_{k-1}) - y_{k-1} \right) = \frac{a_k}{A_k A_{k-1}} \sum_{i=0}^{k-1} a_i \left( \nabla \psi^*(z_{k-1}) - \nabla \psi^*(z_i) \right).$$

Applying Jensen’s Inequality:

$$\|x_k - y_{k-1}\|^2 \leq \frac{a_k^2}{A_k^2 A_{k-1}} \sum_{i=0}^{k-1} a_i \|\nabla \psi^*(z_{k-1}) - \nabla \psi^*(z_i)\|^2.$$

The rest of the proof follows by using that (by Fact 1.7), $D_{\psi^*}(z, w) \geq \frac{\gamma}{2} \|\nabla \psi^*(z) - \nabla \psi^*(w)\|^2$, $\forall z, w$.

Using Lemmas 3.1 and C.3, we now show how to bound the minimum norm of the gradient, under suitable step sizes, so that Lemma C.3 applies.

**Theorem 3.2.** Let $x_k$, $y_k$, $z_k$ evolve according to (GMD), for arbitrary $x_0 = y_0 \in \mathbb{R}^n$ such that $x_0 = \nabla \psi^*(z_0)$, where $\psi(x) = \|x\|^2$. Let $\frac{a_k^2}{A_k^2} = c \frac{iH}{\mu}$, for some $c \in [0, 1]$ and $B_{k-1} = A_{k-1}H_k$. If, for some $c' \in [0, 1]$ and all $i \leq k$: $(1 - c') \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \geq \frac{cH}{L} \left( \frac{1}{B_i} - \frac{1}{B_{i-1}} \right)$, then:

$$\sum_{i=1}^{k} c' \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \frac{c(1 - c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) \|\nabla f(x_i)\|^2 \leq f(x_0) - f(y_k) \leq f(x_0) - f(x^*) \leq f(x_0) - f(x^*).$$

**Proof.** Using Lemma 3.1, we have:

$$\sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) - \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) E_i = f(y_0) - f(x_0) - \sum_{i=0}^{k} b_i f(y_i) \quad (C.7)$$

as $y_0 = x_0$ and $x^*$ minimizes $f$.

To prove the theorem, it suffices to bound below the left-hand side of Eq. (C.7).

Let us first bound the discretization error. Using Proposition C.4, we have that, $\forall i$:

$$E_i \leq \frac{eH B_{i-1}}{\mu} \frac{a_i^2}{A_i^2 A_{i-1}} \sum_{j=0}^{i-2} a_j D_{\psi^*}(z_{i-1}, z_j) - (1 - c) A_{i-1} D_{\psi^*}(z_{i-1}, z_i)$$

$$= \frac{eH}{L} \sum_{j=0}^{i-2} a_j D_{\psi^*}(z_{i-1}, z_j) - (1 - c) A_{i-1} D_{\psi^*}(z_{i-1}, z_i)$$

Therefore:

$$\sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) E_i \leq \frac{eH}{L} \sum_{i=2}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) \sum_{j=0}^{i-2} a_j D_{\psi^*}(z_{i-1}, z_j)$$

$$- (1 - c) \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_k} \right) A_{i-1} D_{\psi^*}(z_{i-1}, z_i). \quad (C.8)$$
As \( D_{\psi^*}(z_{i-1} z_i) = \frac{1}{2\mu} \| z_i - z_{i-1} \|^2 = \frac{1}{2\mu} A_i^{-2} H_i^2 \| \nabla f(x_i) \|^2 \), we further have:

\[
\sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) A_{i-1} D_{\psi^*}(z_{i-1}, z_i) = \sum_{i=1}^{k} \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) A_{i-1} \frac{1}{2\mu} A_i^{-2} H_i^2 \| \nabla f(x_i) \|^2 \\
= \frac{c}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_i} \right) \| \nabla f(x_i) \|^2,
\]

where we have used \( B_{i-1} = A_{i-1} H_i \) and \( \frac{a_i^2}{A_i^2} = \frac{c\mu}{H_i L} \), both from the statement of the theorem.

Combining Eqs. (C.7)–(C.9), we have:

\[
\sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_i} \right] \frac{c\mu H_i}{L} \left( \frac{1}{B_i} - \frac{1}{B_k} \right) \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \frac{c(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_i} \right) \| \nabla f(x_i) \|^2 \\
\leq f(x_0) - f(x^*). \]

To complete the proof, it remains to use that: \( (1-c')\left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \geq \frac{c\mu}{H_i L} \left( \frac{1}{B_i} - \frac{1}{B_k} \right) \).

To obtain useful convergence bounds, we need to show that it is possible to satisfy the assumptions of Theorem 3.2. We start by providing examples for the case of convex objectives, and then discuss the non-convex case.

**The Convex Case** When \( f \) is convex, \( \epsilon_H = 0 \), and Theorem 3.2 can be applied with \( c' = 0 \). Further, once \( c \in [0, 1] \), \( \mu \), and \( H_i \) are specified, all other parameters are set, since \( a_i \) and \( A_i \) can be computed from \( \frac{a_i^2}{A_i^2} = \frac{c\mu}{H_i L} \), and \( A_k = \sum_{i=0}^{k} a_i \), and, finally, we have that \( B_i = A_{i-1} H_i \). The only restriction that Theorem 3.2 imposes is that \( B_i \) is a non-decreasing sequence.

To illustrate the results, we take \( H_i = A_i^\lambda \), for \( \lambda \in [0, 2] \). When \( \lambda = 0 \), the algorithm becomes the generalized heavy-ball method. When \( \lambda = 1 \), the algorithm corresponds to AGD + [10]. Other values of \( \lambda \) can be seen as interpolating between these two methods. For this choice of \( H_i \), we have that \( \frac{a_i^2}{A_i^{2-\lambda}} = \frac{c\mu}{A_i^\lambda L} \), or, equivalently:

\[
\frac{a_i^2}{A_i^{2-\lambda}} = \frac{c\mu}{L}.
\]

It is not hard to verify that a sequence \( \{ a_i \} \), that satisfies this condition will grow as:

\[
a_i \sim \begin{cases} \left( \frac{c\mu}{L} \right)^{(2-\lambda)/\lambda}, & \text{if } \lambda > 0, \\ \sqrt{c\mu} \left( 1 - \sqrt{c\mu} / L \right)^{-(i-1)}, & \text{if } \lambda = 0. \end{cases}
\] (C.10)

The following corollary (of Theorem 3.2) shows that any generalized momentum method (GMD) with \( H_i = A_i^\lambda \), \( \lambda \in [0, 2] \), converges to a point with a small gradient norm at rate \( 1/k \).

**Corollary 3.3.** Let \( x_k, y_k, z_k \) evolve according to (GMD), for convex \( f \), \( \psi^*(z) = \frac{1}{2\mu} \| z \|^2 \), \( \mu > 0 \), and \( \frac{a_i^2}{A_i^{2-\lambda}} = \frac{c\mu}{L} \) for some \( \lambda \in [0, 2] \), \( c \in (0, 1] \), and \( c\mu L < 1 \). Then, \( \forall k \geq 1 \):

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{c(1-c)k + c\min\{\log k, k\sqrt{c\mu L}\}}.
\]

In particular, for \( c = \frac{1}{2} \), \( \min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{k} \).

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Proof. Applying Theorem 3.2, we have:
\[
\sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) + \frac{c(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) \|\nabla f(x_i)\|^2 \leq f(x_0) - f(x^*). \quad (C.11)
\]

Consider first the case \( \lambda \in (0, 2] \). Then, from Eq. (C.10), we have \( a_i \sim \left( \frac{c \mu}{L} \right)^{1/\lambda_i} \). When \( \frac{2-\lambda}{\lambda} \) is an integer, Faulhaber’s formula\(^5\) implies that \( A_i \sim \frac{\lambda}{2} \left( \frac{c \mu}{L} \right)^{1/\lambda_i} \). As the power sum \( \sum_{i=1}^{k} i^p \) is increasing in the power \( p \), we have that: \( A_i = \Omega \left( \lambda \left( \frac{c \mu}{L} \right)^{1/\lambda_i} \right) \). As \( B_i = A_{i-1} A_i \), we further have:
\[
B_i = \Omega \left( \lambda_{1+\lambda} \left( \frac{c \mu}{L} \right)^{1/\lambda_i} \right).
\]

When \( \lambda = 0 \), we have:
\[
B_i = A_{i-1} \sim \left( 1 - \sqrt{\frac{c \mu}{L}} \right)^{-i(i-1)}.
\]

In either case:
\[
\frac{a_{j-1}}{A_{j-1}} \sim \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\}.
\]

As \( B_i \) grows as a function of \( i \) at least as \( i^3 \), we have that:
\[
\frac{c(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) = \frac{c(1-c)}{2L} \Theta(k).
\]

It remains to bound \( \sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) \). By Proposition C.2 and \( a_j \geq a_{j-1} \),
\[
\sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) \geq \frac{1}{2 \mu} \sum_{j=1}^{i} \frac{a_{j-1} a_j^2}{2} H_j \|\nabla f(x_j)\|^2 \geq \frac{c}{4L} \sum_{j=1}^{i} \frac{a_{j-1}}{A_{j-1}} B_{j-1} \|\nabla f(x_j)\|^2.
\]

As \( \frac{a_{j-1}}{A_{j-1}} \sim \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} \), we have that:
\[
\sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] \sum_{j=0}^{i-1} a_j D_{\psi^*}(z_i, z_j) = \Omega \left( \frac{c \mu}{L} \right) \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} \sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] \sum_{j=1}^{i} \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} B_{j-1}.
\]

Finally, observe that:
\[
\sum_{i=1}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] \sum_{j=1}^{i} \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} B_{j-1} = \sum_{j=1}^{k} \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} B_{j-1} \sum_{i=j}^{k} \left[ \frac{1}{B_{i-1}} - \frac{1}{B_{i}} \right] = \sum_{j=1}^{k} \min \left\{ \sqrt{\frac{c \mu}{L}, \frac{1}{\lambda_j}} \right\} \left[ 1 - \frac{B_{j-1}}{B_k} \right] = \Omega \left( \min \left\{ \log(k)/\lambda, k \sqrt{\frac{c \mu}{L}} \right\} \right).
\]

\(^5\)An asymptotic form of Faulhaber’s formula gives \( \sum_{i=1}^{k} i^p = \frac{k^{p+1}}{p+1} + \frac{k^p}{p} + O(k^{p-1}) \).
Hence, we have that:

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{c(1 - c)k + c \min \{ \log k / \lambda, k \sqrt{c\mu / L} \}}.
\]

as claimed. \(
\square
\)

A few remarks are in place here. It is not hard to see that for an arbitrary positive sequence of numbers \(a_i, H_i, B_i\) that satisfy \(\frac{a_i^2}{A_i^2} = c \frac{\mu H_i}{L H_i}\), \(A_i = \sum_{j=0}^{i} a_j\), and \(B_{i-1} = A_{i-1} H_i\), it is not possible to get a better than \(1/k\) rate for the convergence to stationary points, as long as Proposition C.2 is used. This rate is known to be suboptimal – the optimal rate for smooth convex functions is \(1/k^2\) and it is achieved by the OGM-G algorithm from [21]. The rate \(1/k\) for the generalized momentum methods is not surprising, and in the case of \(\lambda = 1\) \((H_i = A_i\), in which case the method is essentially equivalent to Nesterov’s accelerated method in Euclidean spaces), this rate is known to be tight [20]. We expect that the same is true for an arbitrary (but fixed) value of \(\lambda\), though this may be possible to show only numerically [20].

**The Nonconvex Case** The main restriction for obtaining the results in the nonconvex case is ensuring that:

\[
(1 - c')(\frac{1}{B_i} - \frac{1}{B_{i-1}}) \geq \frac{c\epsilon_H}{L} \left(\frac{1}{B_i} - \frac{1}{B_k}\right),
\]

for some \(c, c'\). Let us first observe what kind of a constraint on the sequence \(B_i\) such a condition imposes. Rearranging the terms in the last expression, we have that it must be:

\[
\frac{B_i}{B_{i-1}} \geq 1 + \frac{c\epsilon_H}{(1 - c')L} \left(1 - \frac{B_i}{B_k}\right).
\]  
(C.12)

Since the last expression needs to be satisfied for all \(i\), when \(B_i\) grows polynomially with \(i\), it is not hard to verify that to ensure \(c' \geq 0\), we would need to have \(\frac{c\epsilon_H}{L} = O(\frac{1}{i})\), (in other words, for a fixed number of iterations \(k\), we would need \(\frac{c\epsilon_H}{L} = O(\frac{1}{k})\)), which leads to uninformative convergence bounds, unless \(\epsilon_H = O(L/k)\) (in which case one can show that \(\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(\frac{L(f(x_0) - f(x^*))}{c(1 - c)k})\)).

When \(\epsilon_H = \omega(L/k)\), we are unable to show the convergence rate of \(1/k\) for polynomially growing sequences \(a_i\) (and, consequently, polynomially growing \(A_i, B_i\)). Instead, we can only show such a convergence rate for constant \(H_i\). In particular, let us choose \(\mu\) and \(H_i\) so that \(\frac{\mu H_i}{L H_i} = 1\). Then \(\frac{a_i}{A_i} = \sqrt{c}\), and it follows that \(\frac{B_i}{B_{i-1}} = (1 - \sqrt{c})^{-1}\). As \(\epsilon_H \leq L\) and \(1 - \frac{B_i}{B_k} \leq 1\), to satisfy the condition from Eq. (C.12), it suffices to have \(\sqrt{c} \geq \frac{c'}{1 - c'}\). Equivalently, Eq. (C.12) is satisfied with:

\[
c' \leq 1 - \sqrt{c}.
\]

By the same arguments as in the proof of Corollary 3.3, we immediately have:

**Corollary 3.4.** Let \(x_k, y_k, z_k\) evolve according to (GMD), for \(\psi^*(\mathbf{z}) = \frac{1}{2\mu} \| \mathbf{z} \|^2\), \(\mu > 0\), \(\frac{\mu}{\sqrt{H_i}} = 1\), \(\frac{a_i^2}{A_i^2} = c\), and \(c \in (0, 1)\). Then, \(\forall k \geq 1:\n\)

\[
\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{c(1 - c)k + \sqrt{c(1 - \sqrt{c})ck}}.
\]

In particular, for \(c = \frac{1}{2}\): \(\min_{1 \leq i \leq k} \| \nabla f(x_i) \|^2 = O(1) \frac{L(f(x_0) - f(x^*))}{k}\).
D Convergence to Stationary Points in Banach Spaces

Discretization Error Similar as in previous section, we start by bounding the discretization error $E_k \overset{\text{def}}{=} C_k - C_{k-1}$.

**Lemma D.1.** Let $E_k \overset{\text{def}}{=} C_k - C_{k-1}$, where $C_k$ was defined in (3.1), with $a_i, b_i > 0$, $\forall i$, $A_k = \sum_{i=0}^{k} a_i$, $B_k = \sum_{i=0}^{k} b_i$, and $B_{k-1} = H_k A_{k-1}$. Let $x_k, y_k, z_k$ evolve according to (GMDR), where $x_0 = y_0 \in \mathbb{E}$ is an arbitrary initial point such that $B_k$ and $B_{k-1}$ where the last equality is by (GMDR), where $x_0 = y_0 \in \mathbb{E}$ is an arbitrary initial point such that $\nabla \psi^*(z_0) = x_0, \psi : \mathbf{X} \to \mathbb{R}$ is a $\mu$-strongly convex function w.r.t. $\| \cdot \|$, and $f$ is an $L$-smooth function w.r.t. $\| \cdot \|$. If $f$ is $\epsilon_H$-weakly non-convex for $\epsilon_H \in [0, L]$ and $\frac{a_k^2}{A_k} = c \cdot \frac{\mu}{\epsilon_H}$ for $c \in [0, 1]$, then
\[
E_k \leq (1 - c) \frac{B_{k-1}}{2L} \| \nabla f(x_k) \|^2 + B_{k-1} \frac{\epsilon_H}{2} \| x_k - y_{k-1} \|^2.
\]

**Proof.** The proof uses similar arguments as the proof of Lemma C.3. By the definitions of $E_k$ and $C_k$ and $D_{\psi^*}(z, z) = 0$, we have:
\[
E_k = B_{k-1} [f(y_k) - f(y_{k-1})] + \sum_{i=0}^{k-1} a_i [D_{\psi^*}(z_k, z_i) - D_{\psi^*}(z_{k-1}, z_i)].
\] (D.1)

As $f$ is $L$-smooth and $\epsilon_H$-weakly non-convex, we have:
\[
f(y_k) - f(y_{k-1}) = f(y_k) - f(x_k) + f(x_k) - f(y_{k-1})
\leq \langle \nabla f(x_k), x_k - y_{k-1} \rangle - \frac{1}{2L} \| \nabla f(x_k) \|^2 + \frac{\epsilon_H}{2} \| x_k - y_{k-1} \|^2.
\] (D.2)

By the first part of Fact 1.7 and the definition of $z_k, \forall i \leq k - 1$:
\[
D_{\psi^*}(z_k, z_i) = D_{\psi^*}(z_{k-1}, z_i) + \langle \nabla \psi^*(z_{k-1}) - \nabla \psi^*(z_i), z_k - z_{k-1} \rangle + D_{\psi^*}(z_k, z_{k-1})
= D_{\psi^*}(z_{k-1}, z_i) - \frac{a_k}{A_k} H_k \langle \nabla f(x_k), \nabla \psi^*(z_{k-1}) - \nabla \psi^*(z_i) \rangle + D_{\psi^*}(z_k, z_{k-1}).
\]

Hence, we have:
\[
\sum_{i=0}^{k-1} a_i [D_{\psi^*}(z_k, z_i) - D_{\psi^*}(z_{k-1}, z_i)] = A_{k-1} D_{\psi^*}(z_k, z_{k-1})
- \frac{a_k}{A_k} H_k \left( \nabla f(x_k), A_{k-1} \nabla \psi^*(z_{k-1}) - \sum_{i=0}^{k-1} \nabla \psi^*(z_i) \right)
= A_{k-1} D_{\psi^*}(z_k, z_{k-1}) - B_{k-1} \langle \nabla f(x_k), x_k - y_{k-1} \rangle,
\] (D.3)

where the last equality is by $B_k = A_{k-1} H_k$ and the definition of $x_k$, which is equivalent to
\[
x_k - y_{k-1} = \frac{a_k}{A_k} H_k \left( A_{k-1} \nabla \psi^*(z_{k-1}) - \sum_{i=0}^{k-1} a_i \nabla \psi^*(z_i) \right).
\]

By Fact 1.6,
\[
D_{\psi^*}(z_k, z_{k-1}) \leq \frac{1}{2\mu} \| z_k - z_{k-1} \|^2 + \frac{1}{2\mu} \frac{a_k^2 H_k^2}{A_k^2} \| \nabla f(x_k) \|^2.
\]

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Hence, combining Eqs. (D.1)–(D.3):

\[
E_k \leq B_{k-1} \left( -\frac{1}{2L} + \frac{a_k^2}{2\mu A_k^2} H_k \right) \| \nabla f(x_k) \|_*^2 + B_{k-1} \frac{\epsilon H}{2} \| x_k - y_{k-1} \|^2
\]

\[
= - \frac{(1-c)B_{k-1}}{2L} \| \nabla f(x_k) \|_*^2 + B_{k-1} \frac{\epsilon H}{2} \| x_k - y_{k-1} \|^2,
\]

where we have used that \( B_{k-1} = H_k A_{k-1} \) and \( \frac{a_k^2}{A_k^2} = c \cdot \frac{\mu}{L H_k} \).

**Final Convergence Bound** Since the proof of Proposition C.4 only required that \( \psi \) be \( \mu \) strongly convex and that \( x_k - y_{k-1} = \frac{a_k}{A_k} (\nabla \psi^*(z_{k-1}) - \frac{1}{A_k} \sum_{i=0}^{k-1} a_i \nabla \psi^*(z_i)) \), the same claim holds for the iterates of (GMD\(_B\)). Thus, we can draw a similar conclusion as for (GMD).

**Theorem D.2.** Let \( x_k, y_k, z_k \) evolve according to (GMD\(_B\)), for arbitrary \( x_0 = y_0 \in E \) such that \( x_0 = \nabla \psi^*(z_0) \), where \( \psi(x) \) is strongly convex w.r.t. \( \| \cdot \| \). Let \( \frac{a_k^2}{A_k^2} = c \cdot \frac{\mu}{L H_k} \) for some \( c \in [0, 1] \) and \( B_{k-1} = A_{k-1} H_k \). If for some \( c' \in [0, 1] \) and all \( i \leq k \):

\[
(1-c) \left( \frac{1}{B_{i-1}} - \frac{1}{B_i} \right) \geq \frac{c\epsilon H}{L} \left( \frac{1}{B_i} - \frac{1}{B_k} \right),
\]

then:

\[
\sum_{i=1}^{k} c' \left[ \frac{1}{B_{i-1}} - \frac{1}{B_i} \right] \sum_{j=0}^{i-1} a_j D \psi^*(z_i, z_j) + c \frac{(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) \| \nabla f(x_i) \|_*^2
\]

\[
\leq f(x_0) - f(y_k) \leq f(x_0) - f(x^*).
\]

The proof is the same as the proof of Theorem 3.2 and is thus omitted.

The main difference between (GMD) and (GMD\(_B\)) in terms of the conclusions about the convergence to stationary points is that, because we are no longer assuming strong convexity of \( \psi^* \), we can no longer lower bound \( \sum_{j=0}^{i-1} a_j D \psi^*(z_i, z_j) \) as a function of the norms of the gradients. However, the term \( \frac{(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) \| \nabla f(x_i) \|_*^2 \) from the theorem statement is still sufficient for obtaining \( 1/k \) asymptotic convergence, as shown in the following corollary.

**Corollary 3.5.** Let \( x_k, y_k, z_k \) evolve according to (GMD\(_B\)), for some \( \mu \)-strongly convex \( \psi \), where \( \mu > 0 \), and where \( \frac{a_k^2}{A_k^2} = c \cdot \frac{\mu}{L H_k} \), \( c \in (0, 1) \).

(i) If \( \frac{\mu}{LH_k} = 1 \), then \( \forall k \geq 1: \min_{1 \leq i \leq k} \| \nabla f(x_i) \|_*^2 = O(1)(\frac{L(f(x_0) - f(x^*)))}{c(1-c)k}) \).

In particular, for \( c = \frac{1}{2} \): \( \min_{1 \leq i \leq k} \| \nabla f(x_i) \|_*^2 = O(1)(\frac{L(f(x_0) - f(x^*)))}{c(1-c)k}) \).

(ii) If \( f \) is convex and \( H_i = A_i^\lambda \), then: \( \min_{1 \leq i \leq k} \| \nabla f(x_i) \|_*^2 = O(1)(\frac{L(f(x_0) - f(x^*)))}{c(1-c)k}) \).

In particular, for \( c = \frac{1}{2} \): \( \min_{1 \leq i \leq k} \| \nabla f(x_i) \|_*^2 = O(1)(\frac{L(f(x_0) - f(x^*)))}{c(1-c)k}) \).

*Proof.* The first part of the corollary follows because under the assumption that \( \frac{\mu L}{H_k} = 1 \), the condition of Theorem D.2 can be satisfied with \( c' = 1 - \sqrt{c} \geq 0 \), as discussed in the previous subsection. Further, in this case \( B_i \) grows exponentially fast and \( \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) = \Omega(k) \). As \( B_i \) is increasing and Bregman divergences are non-negative, we have:

\[
\frac{c(1-c)}{2L} \sum_{i=1}^{k} \left( 1 - \frac{B_{i-1}}{B_k} \right) \| \nabla f(x_i) \|_*^2 \leq f(x_0) - f(x^*),
\]

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which implies the claimed statement.

For the second part of the corollary, convexity of $f$ implies that the condition from Theorem D.2 can be satisfied with $c' = 1$. As discussed in the proof of Corollary 3.3, for $H_i = A_i^\lambda$, $B_i$ grows at least as fast as $i^3$, which, again, implies that $\sum_{i=1}^{k} \left(1 - \frac{B_{i-1}}{B_k}\right) = \Omega(k)$, and leads to the same conclusion as in the first part of the proof. \qed