On biorthogonal ensembles with two particle interaction

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February 25, 2015

Abstract

We prove asymptotics of one-point correlation functions and derive a large deviation principle for biorthogonal ensembles associated to probability density functions $\text{Prob}_k$ of the form

$$\frac{1}{Z_k} \prod_{i<j} |z_i - z_j| \cdot \prod_{i<j} |z_i^\theta - z_j^\theta| \cdot \exp \left( - k [Q(z_0) + \cdots + Q(z_k)] \right) \, d\nu(z_0) \cdots d\nu(z_k)$$

on $K^{k+1}$ where $K \subset [0, \infty)$. Here $\theta > 0$; $Q$ is a weight function on $K$; $\nu$ is a measure on $K$ satisfying a Bernstein-Markov property; and $Z_k$ is a normalizing constant.

1 Introduction

Let $K$ be a closed subset of the complex plane $\mathbb{C}$ and $\nu$ a measure on $K$. For $k = 1, 2, \ldots$, we will be concerned with the following ensemble of probability measures $\text{Prob}_k$ on $K^{k+1}$:

$$\frac{1}{Z_k} |VDM(z_0, \ldots, z_k)| e^{-k[Q(z_0)+\cdots+Q(z_k)]} |VDM(f(z_0), \ldots, f(z_k))| \, d\nu(z_0) \cdots d\nu(z_k). \quad (1.1)$$

Here

- $Z_k$ is a normalizing constant;
- $VDM(z_0, \ldots, z_k) = \prod_{0 \leq i < j \leq k} (z_j - z_i)$ is the Vandermonde determinant;
- $Q : K \rightarrow (-\infty, +\infty]$ is a lower semicontinuous function with $\{z \in K : Q(z) < \infty\}$ not polar; and
- $f : K \rightarrow \mathbb{C}$ is continuous.
The standard examples in the literature occur if $K = [0, \infty)$ and $f(x) = x^\theta$ for $\theta$ positive and rational; or $f(x) = e^x$. In this setting some restrictions should be imposed on the growth of $Q$ at infinity. We mention that in all cited literature in the introduction, the only measure $\nu$ considered was Lebesgue measure.

For $K \subset [0, \infty)$ and $f(x) = x^\theta$ for $\theta > 0$, such ensembles were first studied in the context of disordered conductors [20] (see also [16]). In [10], Cheliotis constructs a family of random matrices so that the joint probability distribution of the eigenvalues is of the form (1.1) where $f(x) = x^\theta$. In [7], such ensembles were termed biorthogonal referring to the fact that the probability distributions involve products of two determinants. They are also determinantal ensembles (see [8], [9], [13]).

In [7] Borodin found scaling limits of correlation kernels in certain special cases. He used explicit calculations involving associated biorthogonal Jacobi, Laguerre, and Hermite polynomials. In a recent paper in which $K = [0, \infty)$ and $\theta \geq 1$, Claeys and Romano [11] use Riemann-Hilbert methods on the biorthogonal polynomials and obtain explicit information on the equilibrium measure associated to (1.1) when $f(x) = x^\theta$. The equilibrium measure is defined as the minimizer among probability measures on $K$ of the functional

$$E^Q(\mu) = -\int_K \int_K \log |x - y|d\mu(x)d\mu(y)$$

$$- \int_K \int_K \log |x^\theta - y^\theta|d\mu(x)d\mu(y) + 2 \int_K Q(x)d\mu(x). \quad (1.2)$$

Under appropriate hypotheses (cf., Proposition 2.2 below), this minimizer is unique; we denote it by $\mu_{K,Q}$.

Of course if $\theta = 1$, then (1.1) is the joint probability distribution of the eigenvalues of a Hermitian Unitary Ensemble and these have been extensively studied [1]. For $\theta = 2$ they arose in the study of disordered bosons [18]. For $\theta > 1$ an integer, the ensembles were studied in [14] and a large deviation principle was established (with $Q$ satisfying a Lipshitz condition and $d\nu = \text{Lebesgue measure}$). For $\theta \geq 1$ an integer, the ensembles can be considered on any compact set $K \subset \mathbb{C}$.

In this paper we will be concerned with the global behavior as $k \to \infty$, i.e., macroscopic properties. We make no restrictions on $\theta > 0$ and we allow more general measures $\nu$ than Lebesgue measure. We separately deal with the cases $K \subset [0, \infty)$ compact in Section 4 and $K = [0, \infty)$ in Sections 5 and 6. In the case of $K$ compact, we will show (Corollary 4.8) that the empirical measure of a random point

$$\frac{1}{k+1} \sum_{i=0}^{k} \delta_{z_i}$$

converges almost surely weak* in

$$P = \prod_{k=1}^{\infty} (K^{k+1}, \text{Prob}_k)$$

to the equilibrium measure $\mu_{K,Q}$.
We also establish a large deviation property (LDP) for the empirical measures as follows. Let $\mathcal{M}(K)$ denote the probability measures on $K$ with the weak* topology and $j_k : K^{k+1} \to \mathcal{M}(K)$ be the map

$$j_k(z_0, \ldots, z_k) = \frac{1}{k+1} \sum_{i=0}^{k} \delta_{z_i}$$

Then (Theorem 4.11) the sequence of measures $\sigma_k = j_k(\text{Prob}_k)$ satisfy a LDP with speed $k^2/2$ and good rate function $E^Q(\mu) - E^Q(\mu_{K,Q})$ where $E^Q(\mu)$ is as in (1.2). These results hold, e.g., for $Q$ continuous and any measure $\nu$ which satisfies a mass-density condition (see Remark 4.4). The rate function in the LDP is then independent of the particular measure $\nu$. In the case $K = [0, \infty)$ we establish the analogous results (Corollary 5.10 for the a.s. convergence and Theorem 6.1 for the LDP). For $\nu$ Lebesgue measure, this answers a question raised in [11]. We also show (Theorem 5.11) that the one-point correlation function converges weak* to the equilibrium measure.

We use potential theory to prove these results. Indeed, sections 2 and 3 provide general potential-theoretic results for $K \subset \mathbb{C}$ and a continuous $f : K \to \mathbb{C}$. From section 4 onward we specialize to $K \subset [0, \infty)$ and $f(x) = x^\theta$. In this case, holding $k$ of the variables fixed in (1.1) we have a function of the form

$$z \to p_k(z)q_k(z^\theta) \exp(-kQ(z))$$

where $p_k, q_k$ are polynomials of degree $k$. We establish a Bernstein-Markov inequality for such functions (Theorem 4.3) as well as a Bernstein-Walsh type inequality (Proposition 4.1 and Theorem 5.2). With these inequalities established we obtain asymptotics of the normalizing constants $Z_k$ (Proposition 4.5 and Proposition 5.8). This yields a Johansson-type large deviation result (Corollary 4.7) and an LDP for $K$ compact. The case of $K$ unbounded requires an additional estimate (Lemma 5.6) on the integral of functions of the form (1.3) over $K$ by their integral over a compact set. In the case of $K$ unbounded, the Johansson-type estimate (Corollary 5.9) yields the result (Theorem 5.11) on the asymptotics of the one-point correlation function. The proof of the large deviation result in the unbounded case (Theorem 6.2) is different from the compact case but utilizes this latter setting to prove a lower bound.

### 2 General potential theory results

In this section we consider the general setting where $K$ is a closed subset of the complex plane $\mathbb{C}$ and $\nu$ is a positive measure on $K$. Recall a set $E \subset \mathbb{C}$ is polar if there exists $u \neq -\infty$ defined and subharmonic on a neighborhood of $E$ with $E \subset \{u = -\infty\}$ (cf., [21]). In (1.1), we let $Q : K \to [-\infty, +\infty]$ be a lower semicontinuous function with $\{z \in K : Q(z) < \infty\}$ not polar. We use the terminology that a property holds q.e. (quasi-everywhere) on a set $S \subset \mathbb{C}$ if it holds on $S \setminus P$ where $P$ is a polar set. Finally, we let $f : K \to \mathbb{C}$ be any continuous function. The natural assumption on $Q$ in case $K$ is unbounded is the following.
Definition 2.1. We call a lower semicontinuous function $Q$ on a closed, unbounded set $K \subset \mathbb{C}$ with $\{z \in K : Q(z) < \infty\}$ not polar $f-$admissible for $K$ if

$$\psi(x) := Q(x) - \frac{1}{2} \log [(1 + |x|^2)(1 + |f(x)|^2)]$$

satisfies

1. $\psi(x) \geq c = c(Q) > -\infty$ for all $x \in K$ and
2. $\liminf_{|x| \to \infty, x \in K} \psi(x) = \infty$.

Note that 2. implies 1.; also, since $1 + |f(x)|^2 \geq 1$, we have $\psi(x) \leq Q(x) - \frac{1}{2} \log (1 + |x|^2)$ so that 2. implies $Q$ is admissible in the usual potential-theoretic sense of \cite{21}:

$$\liminf_{|x| \to \infty, x \in K} [Q(x) - \frac{1}{2} \log (1 + |x|^2)] = \infty. \quad (2.1)$$

The hypothesized growth of $Q$ depends heavily on $f$: if $K \subset [0, \infty)$ and $f(x) = x^\theta$ for $\theta > 0$ then $Q(x) = (1 + \epsilon) \log |x|^{1+\theta}$ is $f-$admissible; if $f(x) = e^x$ then $Q(x) = (1 + \epsilon) |x|$ is $f-$admissible. Indeed, these are examples of strongly $f-$admissible weights for $K$: we say $Q$ is strongly $f-$admissible for $K$ if there exists $\delta > 0$ such that $(1 - \delta)Q$ is $f-$admissible for $K$.

The relevant weighted potential theory problem associated to $(1.1)$, consists in minimizing the following energy

$$E^Q_f(\mu) = E^Q(\mu) := \int_K \int_K \log \frac{1}{|x - y||f(x) - f(y)|w(x)w(y)}d\mu(x)d\mu(y) \quad (2.2)$$

over $\mu \in \mathcal{M}(K)$, the set of probability measures on $K$. Here $w = e^{-Q}$. Note that the double integral in $(2.2)$ is well-defined and different from $-\infty$. Indeed, let

$$k(x, y) := -\log |(x - y||f(x) - f(y)|w(x)w(y))|. \quad (2.3)$$

Using the inequality $|u - v| \leq \sqrt{1 + |u|^2} \sqrt{1 + |v|^2}$, we have

$$\log |x - y| + \log |f(x) - f(y)|$$

$$\leq \frac{1}{2} \log (1 + |x|^2) + \frac{1}{2} \log (1 + |y|^2) + \frac{1}{2} \log (1 + |f(x)|^2) + \frac{1}{2} \log (1 + |f(y)|^2).$$

Hence, by 1. in Definition 2.1

$$k(x, y) \geq \psi(x) + \psi(y) \geq 2c \text{ on } K \times K, \quad (2.4)$$

and the integrand of the double integral is bounded below by $2c$.

We also recall the definition of the logarithmic energy of $\mu$,

$$I(\mu) := \int_K \int_K \log \frac{1}{|x - y|} d\mu(x)d\mu(y) =: \int_K p_\mu(y)d\mu(y)$$
and the weighted logarithmic energy of $\mu$,

$$I^Q(\mu) := \int_K \int_K \log \frac{1}{|x-y|w(x)w(y)} d\mu(x) d\mu(y).$$  \hfill (2.5)

Since $1 + |f(x)|^2 \geq 1$, the double integral in (2.5) is also well-defined and different from $-\infty$. When $I(\mu) \neq -\infty$ or $\int Q d\mu < \infty$, we can rewrite $I^Q(\mu)$ as

$$I^Q(\mu) = I(\mu) + 2 \int_K Q d\mu.$$

For the push-forward measure $f_\ast \mu$ of $\mu$ on $f(K)$, we have

$$I(f_\ast \mu) = \int_{f(K)} \int_{f(K)} \log \frac{1}{|f(x) - f(y)|} d\mu(x) d\mu(y) = \int_{f(K)} \int_{f(K)} \log \frac{1}{|a-b|} f_\ast d\mu(a) f_\ast d\mu(b)$$

$$= \int_{f(K)} p_{f,\ast \mu}(a) f_\ast d\mu(b) = \int_K p_{f,\ast \mu}(f(z)) d\mu(z).$$

When $I^Q(\mu) \neq +\infty$ or $I(f_\ast \mu) \neq -\infty$, the energy $E^Q(\mu)$ can be rewritten as

$$E^Q(\mu) = I^Q(\mu) + I(f_\ast \mu).$$

**Proposition 2.2.** Let $K \subset \mathbb{C}$ be closed and let $Q$ be $f$-admissible $Q$ for $K$. Suppose there exists $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. Let $V_w := \inf\{E^Q(\mu), \mu \in \mathcal{M}(K)\}$. Then

1. $V_w$ is finite.
2. Setting $K_M := \{z, Q(z) \leq M\}$, we have, for sufficiently large $M < \infty$,

$$V_w = \inf\{E^Q(\mu), \mu \in \mathcal{M}(K_M)\}.$$

3. We have existence and uniqueness of $\mu_{K,Q}$ minimizing $E^Q$. The measure $\mu_{K,Q}$ has compact support and the logarithmic energies $I(\mu_{K,Q})$ and $I(f_\ast \mu_{K,Q})$ are finite.

4. The following Frostman-type inequalities hold true:

$$p_{\mu_{K,Q}}(z) + p_{f_\ast \mu_{K,Q}}(f(z)) + Q(z) \geq F_w \text{ q.e. on } K, \quad (2.6)$$

$$p_{\mu_{K,Q}}(z) + p_{f_\ast \mu_{K,Q}}(f(z)) + Q(z) \leq F_w \text{ on } \text{supp}(\mu_{K,Q}), \quad (2.7)$$

where $F_w := I(\mu_{K,Q}) + I(f_\ast \mu_{K,Q}) + \int Q d\mu_{K,Q} = V_w - \int Q d\mu_{K,Q}$.

5. If a measure $\mu \in \mathcal{M}(K)$ with compact support and $E^Q(\mu) < \infty$, satisfies

$$p_{\mu}(z) + p_{f_\ast \mu}(f(z)) + Q(z) \geq C \text{ q.e. on } K,$$

$$p_{\mu}(z) + p_{f_\ast \mu}(f(z)) + Q(z) \leq C \text{ on } \text{supp}(\mu),$$

for some constant $C$, then $\mu = \mu_{K,Q}$.  

5
Proof. For 1., we have $V_w < \infty$ by assumption. The other inequality $-\infty < V_w$ follows from the fact that the double integral in (2.2) is bounded below by $2c$. The proof of 2. follows the lines of [21] p. 29-30, namely one first proves that, for $M$ sufficiently large, \[ k(x, y) > V_w + 1 \quad \text{if} \quad (x, y) \notin K_M \times K_M, \] from which one derives that $E^Q(\mu) = V_w$ is possible only for measures with support in $K_M$.

We next prove 3. From 2., there is a sequence $\{\mu_n\} \subset \mathcal{M}(K_M)$ with \[ E^Q(\mu_n) \to V_w \quad \text{as} \quad n \to \infty. \]

The set $K_M$ is compact, hence, by Helly’s theorem, we get a subsequence of these measures converging weakly to a probability measure $\mu$ supported on $K_M$; and it is easy to see this $\mu := \mu_{K,Q}$ satisfies $E^Q(\mu) = V_w$. For the logarithmic energy of $\mu_{K,Q}$, we have $I(\mu_{K,Q}) > -\infty$ because $\mu_{K,Q}$ has compact support. Since $f$ is continuous and $f_\ast \mu_{K,Q}$ has its support in $f(K_M)$, we also have $I(f_\ast \mu_{K,Q}) > -\infty$. Now, recalling that $Q$ is bounded below, we may write $I(\mu_{K,Q})$ as the well-defined expression \[ I(\mu_{K,Q}) = V_w - I(f_\ast \mu_{K,Q}) - 2 \int_K Qd\mu_{K,Q}, \] from which follows that $I(\mu_{K,Q}) < \infty$ and then also $I(f_\ast \mu_{K,Q}) < \infty$.

The uniqueness follows from the fact that $\mu \to I(\mu)$ is strictly convex and $\mu \to I(f_\ast \mu)$ is convex on the subsets of $\mathcal{M}(K)$ where they are finite. To be precise, it is well-known that for $\mu_1$ and $\mu_2$ two measures with finite energies and $\mu_1(K) = \mu_2(K)$, we have $I(\mu_1 - \mu_2) \geq 0$ and $I(\mu_1 - \mu_2) = 0$ if and only if $\mu_1 = \mu_2$. Though we will not need it here, we note that the previous assertion holds true more generally with measures having unbounded supports and whenever $I(\mu_1 - \mu_2)$ is well-defined, cf. [13] Example 3.3.

Now if $\tilde{\mu} \in \mathcal{M}(K)$ is another measure which minimizes $E^Q$, we know from the proof of 2. that $\tilde{\mu} \in \mathcal{M}(K_M)$. Consequently, $I(\tilde{\mu}), I(f_\ast \tilde{\mu}) > -\infty$ and then also $I(\tilde{\mu}), I(f_\ast \tilde{\mu}) < \infty$. We have \[ E^Q(\frac{1}{2}(\mu_{K,Q} + \tilde{\mu})) + I(\frac{1}{2}(\mu_{K,Q} - \tilde{\mu})) + I(f_\ast(\frac{1}{2}(\mu_{K,Q} - \tilde{\mu}))) = \frac{1}{2}[E^Q(\mu_{K,Q}) + E^Q(\tilde{\mu})] = V_w. \]

The sum $I(\frac{1}{2}(\mu_{K,Q} - \tilde{\mu})) + I(f_\ast(\frac{1}{2}(\mu_{K,Q} - \tilde{\mu}))) \geq 0$ with equality if and only if $\mu_{K,Q} = \tilde{\mu}$; hence the result.

We next prove the first inequality in 4. Let $\mu \in \mathcal{M}(K)$ with compact support and consider the measure $\tilde{\mu} = t\mu + (1 - t)\mu_{K,Q}, \ t \in [0, 1]$. The inequality $E^Q(\mu_{K,Q}) \leq E^Q(\tilde{\mu})$ can be rewritten as \[ E^Q(\mu_{K,Q}) \leq t^2(I(\mu) + I(f_\ast \mu)) + (1 - t)^2(I(\mu_{K,Q}) + I(f_\ast \mu_{K,Q})) \]

\[ + 2t(1 - t)(I(\mu, \mu_{K,Q}) + I(f_\ast \mu, f_\ast \mu_{K,Q})) + 2 \int Qd(t\mu + (1 - t)\mu_{K,Q}), \]
where, for two measures $\mu$ and $\nu$, we denote by $I(\mu, \nu)$ the mutual logarithmic energy

$$ I(\mu, \nu) = -\int \int \log |x - y| d\mu(x)d\nu(y). $$

Note that the right-hand side of the above inequality is well-defined since the assumption that $\mu$ has a compact support implies that all terms in the sum are larger than $-\infty$. Letting $t$ tend to 0, we obtain

$$ F_w = I(\mu_{K,Q}) + I(f_*\mu_{K,Q}) + \int Q d\mu_{K,Q} \leq I(\mu, \mu_{K,Q}) + I(f_*\mu, f_*\mu_{K,Q}) + \int Q d\mu. \quad (2.8) $$

Now, we proceed by contradiction, assuming that there exists a nonpolar compact subset $K$ of $K$ such that

$$ \forall z \in K, \ p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z) < F_w. $$

Integrating this inequality with respect to a probability measure $\mu$ supported on $K$, we obtain

$$ I(\mu, \mu_{K,Q}) + I(f_*\mu, f_*\mu_{K,Q}) + \int Q d\mu < F_w, $$

which contradicts (2.8).

The proof of the second inequality in 4. is also by contradiction. Assume that

$$ \exists x_0 \in \text{supp}(\mu_{K,Q}), \ p_{\mu_{K,Q}}(x_0) + p_{f_*\mu_{K,Q}}(f(x_0)) + Q(x_0) > F_w. $$

By lowersemicontinuity, the inequality is satisfied in a neighborhood $V_{x_0}$ of $x_0$. Moreover $\mu_{K,Q}(V_{x_0}) > 0$ since $x_0 \in \text{supp}(\mu_{K,Q})$. Using the first inequality (2.6) on $\text{supp}(\mu_{K,Q}) \setminus V_{x_0}$ and the fact that $\mu_{K,Q}(E) = 0$ for $E$ a polar set (since $\mu_{K,Q}$ has finite logarithmic energy $I(\mu_{K,Q})$), we obtain

$$ F_w = \int (p_{\mu_{K,Q}}(z) + p_{f_*\mu_{K,Q}}(f(z)) + Q(z)) d\mu_{K,Q}(z) $$

$$ > F_w \mu_{K,Q}(V_{x_0}) + F_w \mu_{K,Q}(\text{supp}(\mu_{K,Q}) \setminus V_{x_0}) = F_w, $$

which is a contradiction.

Finally, we prove 5. We write

$$ \mu_{K,Q} = \mu + (\mu_{K,Q} - \mu). $$

Then

$$ E^Q(\mu) \geq E^Q(\mu_{K,Q}) = E^Q(\mu) + I(\mu_{K,Q} - \mu) + I(f_* (\mu_{K,Q} - \mu)) + 2R $$

7
with

\[ R := \int_K \left[ \int_K - \log |x - y| d\mu(y) + Q(x) \right] d(\mu_{K,Q} - \mu)(x) \\
- \int_K \int_K \log |f(x) - f(y)| d\mu(y)d(\mu_{K,Q} - \mu)(x) \\
= \int_K (p_\mu(x) + Q(x)) d(\mu_{K,Q} - \mu)(x) + \int_K p_{f,\mu}(f(x)) d(\mu_{K,Q} - \mu)(x) \\
= \int_K (p_\mu(x) + p_{f,\mu}(f(x)) + Q(x)) d(\mu_{K,Q} - \mu)(x). \]

Note that the above computation is justified. Indeed, from the assumptions \( E^Q(\mu) < \infty \) and \( \mu \) has compact support, the quantities \( E^Q(\mu) \), \( I^Q(\mu) \), \( I(f,\mu) \), \( I(\mu) \), \( \int Q d\mu \), and \( I(\mu,\mu_{K,Q}) \) are all finite. Making use of the two Frostman inequalities, we derive

\[ R \geq C \int_K d\mu_{K,Q} - C \int_K d\mu = 0. \]

Now, recall that \( I(\mu_{K,Q} - \mu) + I(f_\mu(\mu_{K,Q} - \mu)) \geq 0 \) with equality if and only if \( \mu_{K,Q} = \mu \). Thus

\[ E^Q(\mu) \geq E^Q(\mu_{K,Q}) \geq E^Q(\mu) \]

so that equality holds throughout, and \( E^Q(\mu) = E^Q(\mu_{K,Q}) \), from which follows \( \mu = \mu_{K,Q} \).

The condition that there exist \( \nu \in \mathcal{M}(K) \) with \( E^Q(\nu) < \infty \) is not automatic. For example, if \( f \) is a constant function, then trivially all measures \( \nu \) have \( I(f,\nu) = \infty \). We give a sufficient condition on \( f \) ensuring the hypothesis of Proposition 2.2 and as an immediate corollary we give an easily checkable condition (which holds in many cases occurring in the literature).

**Proposition 2.3.** Let \( D := \{(z, z) : z \in K\} \). If \( f : K \to \mathbb{C} \) is continuous and

\[ \Sigma := \left\{ z \in K : Q(z) < \infty \text{ and } \liminf_{(z_1, z_2) \to (z, z)} \frac{|f(z_1) - f(z_2)|}{z_1 - z_2} > 0 \right\} \]

is not polar, then there exist \( \nu \in \mathcal{M}(K) \) with \( E^Q(\nu) < \infty \).

**Proof.** Define

\[ \phi(z_1, z_2) := \left| \frac{f(z_1) - f(z_2)}{z_1 - z_2} \right|; \]

this is continuous on \((K \times K) \setminus D\). Extend \( \phi \) to \( D \) by defining

\[ \phi(z, z) := \liminf_{(z_1, z_2) \to (z, z)} \frac{|f(z_1) - f(z_2)|}{z_1 - z_2}. \]
Then \( \phi : K \times K \to \mathbb{C} \) is lower semicontinuous and we can write \( \Sigma = \bigcup_{n=1}^{\infty} \Sigma_n \) where
\[
\Sigma_n := \{ z \in K : Q(z) < n \; \text{and} \; \phi(z, z) > 1/n \}.
\]
This is an increasing union so for all sufficiently large \( n \), \( \Sigma_n \) is not polar. Fix such an \( n \).
Since polarity is a local property, see e.g. [17, Remark 4.2.13], there exists \( z \in \Sigma_n \) such that, for any neighborhood \( V_z \) of \( z \), \( \Sigma_n \cap V_z \) is not polar.

Now, the function \( \varphi \) is lower semicontinuous on \( K^2 \), hence there exists a neighborhood \( V_z \) of \( z \) such that \( \varphi(z_1, z_2) > 1/n \) on \( (\Sigma_n \cap V_z)^2 \) and by the preceding remark, \( \Sigma_n \cap V_z \) is not polar. Being not polar, \( \Sigma_n \cap V_z \) supports a measure \( \nu \) of finite logarithmic energy which is also of finite weighted logarithmic energy since \( Q(z) < n \) for \( z \in \Sigma_n \). It remains to prove that \( f_*\nu \) is also of finite logarithmic energy. This follows from
\[
I(f_*\nu) = \int_{\Sigma_n \cap V_z} \int_{\Sigma_n \cap V_z} \log \frac{1}{|f(z_1) - f(z_2)|} d\nu(z_1) d\nu(z_2)
\leq \log n + \int_{\Sigma_n \cap V_z} \int_{\Sigma_n \cap V_z} \log \frac{1}{|z_1 - z_2|} d\nu(z_1) d\nu(z_2) < \infty.
\]

As a consequence of Proposition 2.3 and the mean value theorem, we have the following.

**Corollary 2.4.** Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function which is differentiable for \( x > 0 \) and let \( K \subset [0, \infty) \). Assume the subset \( \{ z \in K : f'(z) \neq 0 \; \text{and} \; Q(z) < \infty \} \) is non polar. Then there exist \( \nu \in \mathcal{M}(K) \) with \( E^Q(\nu) < \infty \).

In particular, if \( f(x) = x^q \) and \( K \subset [0, \infty) \) with \( \{ z \in K : Q(z) < \infty \} \) nonpolar (the situation in this paper from section 4 onwards), then Proposition 2.2 is valid.

We turn briefly to an approximation property that we will use in Section 6 to prove our large deviation result in the unbounded case.

**Lemma 2.5.** Let \( K \) be a closed and nonpolar subset of \( \mathbb{C} \) and let \( Q \) be \( f \)-admissible on \( K \). Given \( \mu \in \mathcal{M}(K) \), there exist an increasing sequence of compact sets \( K_m \) in \( K \) and a sequence of measures \( \mu_m \in \mathcal{M}(K_m) \) such that

1. the measures \( \mu_m \) tend weakly to \( \mu \) as \( m \to \infty \);
2. the energies \( E^{Q_m}(\mu_m) \) tend to \( E^Q(\mu) \) as \( m \to \infty \), where \( Q_m := Q|_{K_m} \).

**Proof.** Since the measure \( \mu \) has finite mass, there exist an increasing sequence of compact subsets \( K_m \) of \( K \) with \( \mu(K \setminus K_m) \leq 1/m \). Then, the measures \( \bar{\mu}_m := \mu|_{K_m} \) are increasing and tend weakly to \( \mu \). Denoting as usual by \( k^+(x, y) \) and \( k^-(x, y) \) the positive and negative parts of the function \( k(x, y) \) that was defined in (2.3), we have, as \( m \to \infty \),
\[
\chi_m(x, y) k^+(x, y) \uparrow k^+(x, y) \quad \text{and} \quad \chi_m(x, y) k^-(x, y) \uparrow k^-(x, y),
\]
\((\mu \times \mu)\)-almost everywhere on \( K \times K \) where \( \chi_m(x, y) \) is the characteristic function of \( K_m \times K_m \) and we agree that the left-hand sides vanish when \( x = y \notin K_m \). By monotone convergence, we deduce that \( E^{Q_m}(\bar{\mu}_m) \) tend to \( E^Q(\mu) \) (possibly equal to \( +\infty \)) as \( m \to \infty \), where we recall that the energy \( E^Q(\mu) \), given by the double integral in (2.2), is always well defined since \( Q \) is \( f \)-admissible. Setting \( \mu_m := \bar{\mu}_m/\mu(K_m) \) gives the result. \( \Box \)
3 Additional results for $K$ compact

In this section, we restrict to the case where $K$ is compact. Let $Q$ be a lower semicontinuous function on $K$ with $\{z \in K : Q(z) < \infty\}$ not polar; we write $Q \in \mathcal{A}(K)$ and define $w(x) := e^{-Q(x)}$. In this compact setting, the class $\mathcal{A}(K)$ is universal; i.e., the same for all $f$. Here we naturally assume $f$ is such that there exists $\nu \in \mathcal{M}(K)$ with $E^Q(\nu) < \infty$. The proofs of Propositions 3.1-3.3 of [5] carry over in this setting. Let

$$|VDM^Q_k(z_0, ..., z_k)| = \text{weighted Vandermonde of order } k \quad (3.1)$$

and

$$\left(\delta^Q_k(f)\right)(K) = \delta^Q_k(K) := \max_{z_0, ..., z_k \in K} |VDM^Q_k(z_0, ..., z_k)|^{2/(k+1)}.$$  

We caution the reader that we will use terminology such as weighted Fekete points, etc., for notions defined relative to weighted Vandermondes as defined in (3.1).

**Theorem 3.1.** Given $K \subset \mathbb{C}$ compact and not polar, and $Q \in \mathcal{A}(K)$,

1. if $\{\mu_k = \frac{1}{k+1} \sum_{j=0}^{k} \delta z_{j(k)}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then
   $$\limsup_{k \to \infty} |VDM^Q_k(z_{0(k)}, ..., z_{k(k)})|^{2/(k+1)} \leq \exp (-E^Q(\mu)); \quad (3.2)$$

2. we have
   $$\delta^Q(K) := \lim_{k \to \infty} \delta^Q_k(K) = \exp (-E^Q(\mu_{K,Q}));$$

3. if $\{z_{j(k)}\}_{j=0, ..., k; k=2,3,...} \subset K$ and
   $$\lim_{k \to \infty} |VDM^Q_k(z_{0(k)}, ..., z_{k(k)})|^{2/(k+1)} = \exp (-E^Q(\mu_{K,Q})) \quad (3.3)$$

then

$$\mu_k = \frac{1}{k+1} \sum_{j=0}^{k} \delta z_{j(k)} \to \mu_{K,Q} \text{ weakly.}$$

**Proof.** We indicate the main ingredients. To prove the analogue of Proposition 3.1 of [5], which is 1. above, we simply observe that for any $M$,

$$h_M(x,y) := \min(M, -\log |x-y| - \log |f(x) - f(y)|)$$

$$\leq -\log |x-y| - \log |f(x) - f(y)| := h(x,y)$$

and $h(x,y)$ is lsc if $f$ is continuous. For 2., the analogue of Proposition 3.2 of [5], by usc of

$$(z_0, ..., z_k) \to |VDM^Q_k(z_0, ..., z_k)|,$$

maximizing $(k+1)$-tuples for $\delta^Q_k(K)$ (weighted Fekete points) exist. Finally, 3., the analogue of Proposition 3.3 of [5], uses the uniqueness of the measure $\mu_{K,Q}$ which minimizes $E^Q$. \qed
Remark 3.2. Arrays \( \{x^{(k)}_j\}_{j=0, \ldots, k; \; k=2,3, \ldots} \subset K \) satisfying (3.3) will be called asymptotic weighted Fekete arrays for \( K, Q, f \).

As a last result in this section, we give a refined version of Lemma 2.5 when \( K \) is a compact subset of \( \mathbb{C} \). This is an analogue of results in [6, Section 5] and will be used in a similar fashion to prove our large deviation result in the compact case, see Section 4. Here \( C(K) \) denotes the class of continuous, real-valued functions on \( K \).

Lemma 3.3. Let \( K \subset \mathbb{C} \) be compact and nonpolar and let \( \mu \in \mathcal{M}(K) \) with \( E^Q(\mu) < \infty \). There exist an increasing sequence of compact sets \( K_m \) in \( K \), a sequence of functions \( \{Q_m\} \subset C(K) \), and a sequence of measures \( \mu_m \in \mathcal{M}(K_m) \) satisfying

1. the measures \( \mu_m \) tend weakly to \( \mu \), as \( m \to \infty \);
2. the energies \( I(\mu_m) \) tend to \( I(\mu) \) as \( m \to \infty \);
3. the energies \( I(f_\mu \mu_m) \) tend to \( I(f_\mu \mu) \) as \( m \to \infty \);
4. the measures \( \mu_m \) are equal to the weighted equilibrium measures \( \mu_{K,Q_m} \).

Proof. By Lusin’s continuity theorem applied in \( K \) and \( f(K) \), it is easy to verify that, for every integer \( m \geq 1 \), there exists a compact subset \( K_m \) of \( K \) such that \( \mu(K \setminus K_m) \leq 1/m \), \( p_\mu \) is continuous on \( K_m \), and \( p_{f_\mu} \) is continuous on \( f(K_m) \), respectively considered as functions on \( K_m \) and \( f(K_m) \) only. We may assume that \( K_m \) is increasing as \( m \) tends to infinity. Then, the measures \( \tilde{\mu}_m := \mu_{K_m} \) are increasing and tend weakly to \( \mu \); similarly the measures \( f_\mu \tilde{\mu}_m = f_\mu (\mu_{K_m}) \) are increasing and tend weakly to \( f_\mu \mu \). As in the proof of Lemma 2.5, we have

\[
\chi_m(z, t) \log^+ |z - t| \uparrow \log^+ |z - t| \quad \text{and} \quad \chi_m(z, t) \log^+ |f(z) - f(t)| \uparrow \log^+ |f(z) - f(t)|, 
\]

as \( m \to \infty \), \( (\mu \times \mu) \)-almost everywhere on \( K \times K \) where \( \chi_m(z, t) \) is the characteristic function of \( K_m \times K_m \) and we agree that the left-hand sides vanish when \( z = t \notin K_m \). Similar pointwise convergence holds true for the negative parts of the log functions. Hence, by monotone convergence we have

\[
I(\tilde{\mu}_m) \to I(\mu), \quad I(f_\mu \tilde{\mu}_m) \to I(f_\mu \mu), \quad \text{as} \quad m \to \infty,
\]

where we observe that the compactness of \( K \) implies that the energies \( I(\mu) \) and \( I(f_\mu \mu) \) are well defined. Indeed, because of the assumption \( E^Q(\mu) < \infty \), the energies \( I(\mu) \) and \( I(f_\mu \mu) \) are finite but this is not used here.

Next, define \( \mu_m := \tilde{\mu}_m/\mu(K_m) \) and for \( z \in K \),

\[
Q_m(z) := -p_{\mu_m}(z) - p_{f_\mu \mu_m}(f(z)).
\]

Then, \( Q_m \) is continuous on \( K_m \) because

\[
p_{\mu_m}(z) = p_\mu(z) - p_{\mu - \mu_m}(z) \quad \text{and} \quad p_{f_\mu \mu_m}(f(z)) = p_{f_\mu}(f(z)) - p_{f_\mu - f_\mu \mu_m}(f(z)), \quad z \in K_m,
\]
and \( p_\mu(z) \) and \( p_{f,\mu}(f(z)) \) are continuous on \( K_m \). By the continuity principle for logarithmic potentials, \( p_{\mu_m}(z) \) and \( p_{f,\mu_m}(f(z)) \) are continuous on \( \mathbb{C} \), hence \( Q_m \) is continuous on \( K \).

Item 4. follows from the fact that \( \mu_m \) has compact support with \( E^{Q_m}(\mu_m) < \infty \) (because \( E^Q(\mu) < \infty \)), and it clearly satisfies the Frostman-type inequalities of Proposition \( \ref{frostman} \) for \( K \) and the weight \( Q_n \); hence we have \( \mu_m = \mu_{K,Q_m} \). We note that the assumption \( E^{Q}(\mu) < \infty \) has only been used to prove 4.

\[ \square \]

4 Bernstein-Markov property, \( Z_k \)-asymptotics, LDP for \( f(x) = x^\theta \) on \( K \subset [0, \infty) \) compact

In sections 4, 5 and 6, we always take \( f(x) = x^\theta \) for \( \theta > 0 \) (rational or irrational). In this section, we specialize to the situation of a compact subset \( K \subset [0, \infty) \). We begin by proving that a natural mass density condition on a measure \( \mu \) supported on a regular compact set \( K \subset [0, \infty) \) implies a weighted Bernstein-Markov property for weighted M"untz-type polynomials of the form \( p_n(x)q_n(x^\theta)e^{-n\theta(x)} \) for \( Q \in C(K) \). We remark that Markov-type inequalities on derivatives of (products of) actual M"untz polynomials were studied by Erd"elyi [15]. Our first step is a Bernstein-Walsh type growth estimate on M"untz-type polynomials \( p_n(x)q_n(x^\theta) \). Recall that for \( K \subset \mathbb{C} \) compact and nonpolar, the Green function for \( K \) is \( V^*_K(z) = \limsup_{\zeta \to z} V_K(\zeta) \) where

\[
V_K(z) := \sup\{ \frac{1}{\text{deg}(p)} \log |p(z)| : p \in \bigcup_n \mathcal{P}_n, \ |\max_{z \in K} |p(z)| \leq 1 \}
\]

and \( \mathcal{P}_n \) are the polynomials of degree at most \( n \). The standard Bernstein-Walsh estimate comes directly from [11]: for \( p \in \mathcal{P}_n \),

\[
|p(z)| \leq |\max_{z \in K} |p(z)| e^{nV_K(z)}, \ z \in \mathbb{C}.
\]

We say \( K \) is regular if \( V^*_K \) is continuous; equivalently, \( V_K = V^*_K \). Given an admissible weight \( Q \) on \( K \), the weighted Green function for the pair \( K,Q \) is \( V^*_{K,Q}(z) = \limsup_{\zeta \to z} V_{K,Q}(\zeta) \)

\[
V_{K,Q}(z) := \sup\{ \frac{1}{\text{deg}(p)} \log |p(z)| : p \in \bigcup_n \mathcal{P}_n, \ |\max_{z \in K} |pe^{-\text{deg}(p)}Q(z)| \leq 1 \}
\]

\[
= \sup\{ u(z) : u \in L(\mathbb{C}), \ u \leq Q \text{ on } K \}
\]

and

\[
L(\mathbb{C}) := \{ u \text{ subharmonic in } \mathbb{C} : \exists c \text{ with } u(z) \leq c + \log^+ |z| \}.
\]

Note that \( V^*_{K}, V^*_{K,Q} \in L^+(\mathbb{C}) \) where

\[
L^+(\mathbb{C}) := \{ u \text{ subharmonic in } \mathbb{C} : \exists C_1, C_2 \text{ with } C_1 + \log^+ |z| \leq u(z) \leq C_2 + \log^+ |z| \}.
\]

**Proposition 4.1.** Let \( K \subset [0, \infty) \) be compact and nonpolar, and let \( \theta > 0 \). Then for polynomials \( p_n, q_n \) of degree at most \( n \),

\[
|p_n(x)q_n(x^\theta)| \leq (\max_{s \in K} |p_n(s)q_n(s^\theta)|) \cdot e^{n(1+\theta)V_K(x)}
\]
for $x \in [0, \infty)$.

**Proof.** We first consider the case $\theta = a/b$ rational. Letting $h(t) = t^b$, for any $K \subset \mathbb{C}$ compact, it is easy to see that

$$V_{h^{-1}(K)}(z) = \frac{1}{b}V_K(h(z)).$$

Given polynomials $p_n, q_n$ of degree at most $n$, letting $z = t^b$,

$$t \to p_n(h(t)) \cdot q_n(h(t)^{a/b}) = p_n(t^b)q_n(t^a) = p_n(z)q_n(z^{a/b})$$

is a polynomial in $t$ of degree at most $n(a + b)$ and

$$\max_{z \in K} |p_n(z)q_n(z^{a/b})| = \max_{t \in h^{-1}(K)} |p_n(t^b)q_n(t^a)|.$$

Applying the standard Bernstein-Walsh inequality (4.2) to $p_n(t^b)q_n(t^a)$ we have

$$|p_n(t^b)q_n(t^a)| \leq \max_{t \in h^{-1}(K)} |p_n(t^b)q_n(t^a)| \cdot e^{n(a+b)V_{h^{-1}(K)}(t)}$$

which gives (4.4) in this case.

For $\theta > 0$ irrational, take a sequence of rational numbers $\{a_k/b_k\}$ with $\lim_{k \to \infty} a_k/b_k = \theta$. Given polynomials $p_n$ and $q_n$ of degree at most $n$, we fix $x \in K$ and apply (4.4) with $\theta$ replaced by $a_k/b_k$. Thus

$$|p_n(x)q_n(x^{a_k/b_k})| \leq \max_{s \in K} |p_n(s)q_n(s^{a_k/b_k})| \cdot e^{n(1+a_k/b_k)V_K(x)}.$$

Since $K$ is compact,

$$\lim_{k \to \infty} \left( \max_{s \in K} |p_n(s)q_n(s^{a_k/b_k})| \right) \cdot e^{n(1+a_k/b_k)V_K(x)} = \left( \max_{s \in K} |p_n(s)q_n(s)^\theta| \right) \cdot e^{n(1+\theta)V_K(x)}.$$

Letting $k \to \infty$ in (4.5) gives the result.

**Remark 4.2.** We will generalize Proposition 4.1 in the next section; see Theorem 5.2.

**Theorem 4.3.** Let $f(x) = x^\theta$ with $\theta > 0$ and $K \subset [0, \infty)$ compact and regular. Suppose that $\nu$ is a measure on $K$ with $\nu(K) < \infty$ satisfying the following mass density condition: there exists $t > 0$ such that for each $x_0 \in K$ we have $\nu(x_0 - r, x_0 + r) \geq r^t$ for $r = r(x_0) > 0$ sufficiently small. Then we get a Bernstein-Markov type inequality

$$\max_{s \in K} |p_n(s)q_n(s^\theta)| \leq M_n \int_K |p_n(x)q_n(x^\theta)|d\nu(x)$$

for polynomials $p_n, q_n$ of degree at most $n$ where $M_n^{1/n} \to 1$. Furthermore, if $Q \in C(K)$, we get a weighted Bernstein-Markov type inequality

$$\max_{s \in K} |p_n(s)q_n(s^\theta)| e^{-nQ(s)} \leq \widetilde{M}_n \int_K |p_n(x)q_n(x^\theta)| e^{-nQ(x)}d\nu(x)$$

where $\widetilde{M}_n^{1/n} \to 1$. 
Proof. The proof of (4.6) is modeled on that given on pp. 4764+ of [3]. Given \( \epsilon > 0 \), we can choose \( \delta > 0 \) so that \( V_K(z) \leq \frac{e^{\epsilon}}{1 + \theta} \) if \( \text{dist}(z,K) \leq \delta \). This is where regularity of \( K \) is used. By Proposition 4.1 for polynomials \( p_n,q_n \) of degree at most \( n \), for \( z \in [0,\infty) \) with \( \text{dist}(z,K) \leq \delta \),

\[
|p_n(z)q_n(z^\theta)| \leq \left( \max_{z \in K} |p_n(z)q_n(z^\theta)| \right) \cdot e^{\epsilon n}.
\]

Fix \( w \in K \) with \( |p_n(w)q_n(w^\theta)| \leq \max_{z \in K} |p_n(z)q_n(z^\theta)| \). We show that

\[
|p_n(z)q_n(z^\theta)| \geq \frac{1}{2} |p_n(w)q_n(w^\theta)| \quad \text{for} \quad z \in [0,\infty) \quad \text{with} \quad |z - w| < \frac{\delta}{4} \cdot e^{-\epsilon n}.
\]

(4.8)

Suppose we have proved (4.8). Then to prove (4.6), we define \( r_n := \frac{\delta}{4} \cdot e^{-\epsilon n} \) and observe that for \( n \) sufficiently large, we have

\[
\nu(w - r_n, w + r_n) \geq r_n^t \cdot \left( \frac{\delta}{4} \cdot e^{-\epsilon n} \right)^t.
\]

Then

\[
\int_K |p_n(x)q_n(x^\theta)|d\nu(x) \geq \int_{(w-r_n,w+r_n)} |p_n(x)q_n(x^\theta)|d\nu(x)
\]

\[
\geq \min_{z \in (w-r_n,w+r_n)} |p_n(z)q_n(z^\theta)| \cdot \nu(w - r_n, w + r_n) \geq \frac{1}{2} |p_n(w)q_n(w^\theta)| \cdot \left( \frac{\delta}{4} \cdot e^{-\epsilon n} \right)^t
\]

which proves (4.6).

To prove (4.8), for fixed \( z \in [0,\infty) \) with \( |z - w| < \frac{\delta}{4} \cdot e^{-\epsilon n} \) consider

\[
U(t) := p_n(w + at)q_n((w + at)^\theta) \quad \text{where} \quad a = \frac{z - w}{|z - w|} = \pm 1.
\]

Then \( U(0) = p_n(w)q_n(w^\theta) \) and \( U(|z - w|) = p_n(z)q_n(z^\theta) \). Furthermore, if \( |t| < \delta \), then \( \text{dist}(w + at, K) \leq \delta \) so that \( V_K(w + at) \leq \frac{e^{\epsilon}}{1 + \theta} \); in particular, for \( t \in (-\delta,\delta) \) real,

\[
|U(t)| = |p_n(w + at)q_n((w + at)^\theta)| \leq |p_n(w)q_n(w^\theta)| \cdot e^{\epsilon n}.
\]

(4.9)

Now assuming either that \( w \neq 0 \) or that \( 0 \not\in K \), we can take \( \delta > 0 \) small from the onset so that \( w + at \) stays in a wedge \( \{ z \in \mathbb{C} : -\theta/\pi < \text{Arg} z < \theta/\pi \} \) and \( t \rightarrow (w + at)^\theta \) is holomorphic if \( |t| < \delta \). We can then apply the Cauchy estimates to \( U'(t) \) in \( |t| < \delta/2 \) to obtain

\[
||U'||_{|t|<\delta/2} \leq \frac{2}{\delta} ||U||_{|t|<\delta}.
\]

Using

\[
U(|z - w|) - U(0) = \int_0^{|z - w|} U'(t)dt
\]

we find that for \( |z - w| < \frac{\delta}{4} \cdot e^{-\epsilon n} < \delta/2 \) with \( z \in [0,\infty) \) (using (4.9))

\[
|p_n(z)q_n(z^\theta) - p_n(w)q_n(w^\theta)| \leq |z - w| \cdot ||U'||_{|t|<\delta/2} \leq \frac{\delta}{4} \cdot e^{-\epsilon n} \cdot \frac{2}{\delta} ||U||_{|t|<\delta}
\]

14
which proves (4.8). If \( w = 0 \) we simply repeat the argument starting with \( \tilde{w} > 0 \) for which, e.g., \( |p_n(\tilde{w})q_n(\tilde{w}^\theta)| \geq \frac{1}{2} \max_{z \in K} |p_n(z)q_n(z^\theta)| \).

Given (4.6), to verify (4.7), fix \( Q \in C(K) \). Write \( w := e^{-Q} \). Given \( \epsilon > 0 \), we can find a holomorphic polynomial \( H = H(z) \) with \( \deg(H) =: h \) such that

\[
1 - 2\epsilon \leq w(x)/H(x) \leq 1 + 2\epsilon \quad \text{for } x \in K.
\]

Given polynomials \( p_n, q_n \) of degree at most \( n \) consider the function

\[
J(s) := p_n(s)q_n(s^\theta)H(s)^n.
\]

This is of the form \( \tilde{p}_m(s)\tilde{q}_m(s^\theta) \) where the degree \( m \leq (h + 1)n \). Thus from (4.6), given \( \delta_1 > 0 \), we can find \( C_1 \) with

\[
\max_{s \in K} |J(s)| \leq C_1(1 + \delta_1)^{(h+1)n} \int_K |J(x)|d\nu(x).
\]

Now using

\[
e^{-nQ(s)}|p_n(s)q_n(s^\theta)| = |J(s)|(w(s)/H(s))^n,
\]

for \( s \in K \) we have

\[
|J(s)|(1 - 2\epsilon)^n \leq |p_n(s)q_n(s^\theta)|e^{-nQ(s)} \leq |J(s)|(1 + 2\epsilon)^n.
\]

Thus

\[
\max_{s \in K} |p_n(s)q_n(s^\theta)|e^{-nQ(s)} \leq \max_{s \in K} |J(s)|(1 + 2\epsilon)^n
\]

\[
\leq (1 + 2\epsilon)^n \cdot C_1(1 + \delta_1)^{(h+1)n} \int_K |J(x)|d\nu(x)
\]

\[
\leq C_1(1 + \delta_1)^{(h+1)n} \frac{(1 + 2\epsilon)^n}{(1 - 2\epsilon)^n} \int_K |p_n(x)q_n(x^\theta)|e^{-nQ(x)}d\nu(x).
\]

For this \( \epsilon > 0 \), since \( h \) is fixed, we can choose \( \delta_1 > 0 \) so that \( (1 + \delta_1)^{(h+1)} \leq 1 + \epsilon \) and (4.7) follows.

\[\square\]

**Remark 4.4.** As an example, for any \( \alpha, M > 0 \), the measure \( d\nu(x) = x^\alpha dx \) on \([0, M]\) satisfies the mass density condition. Also, as in [3], one can weaken the mass density assumption that \( \nu(x_0 - r, x_0 + r) \geq r^t \) for \( r = r(x_0) > 0 \) sufficiently small for all \( x_0 \in K \) to have this condition on a subset of \( K \) of full capacity or even just in an “average” sense (see Theorems 2.1 and 2.2 of [3]).

A finite measure \( \nu \) on \( K \) is called a **strong Bernstein-Markov measure** for \( K \) if for any continuous weight \( Q \) on \( K \), the triple \((K, \nu, Q)\) satisfies a weighted Bernstein-Markov property (4.7). Thus Theorem 4.3 shows that a mass density assumption on \( \nu \) implies that \( \nu \) is a strong Bernstein-Markov measure for a regular compact set \( K \).
The weighted Bernstein-Markov property \((4.7)\) is sufficient to prove “free energy” asymptotics for admissible weights. Given a measure \(\nu\) on \(K\) with \(\nu(K) < \infty\), and \(Q \in \mathcal{A}(K)\), define

\[
Z_k := \int_{K^{k+1}} \prod_{i<j} |z_i - z_j| \cdot \prod_{i<j} |z_i^\theta - z_j^\theta| \cdot \exp \left( -k[Q(z_0) + \cdots + Q(z_k)] \right) \, d\nu(z_0) \cdots d\nu(z_k). \tag{4.10}
\]

**Proposition 4.5.** Let \(f(x) = x^\theta\) with \(\theta > 0\) and \(K \subset [0, \infty)\) compact and nonpolar. Suppose \(Q \in \mathcal{A}(K)\) and \(\nu\) is a measure on \(K\) with \(\nu(K) < \infty\) satisfying \((4.7)\). Then

\[
\lim_{k \to \infty} Z_k^{2^k/(k+1)} = \delta^Q(K) = \exp(-E^Q(\mu_{K,Q})). \tag{4.11}
\]

**Proof.** The proof of Proposition 4.8 of \([6]\) proves the upper bound

\[
\limsup_{k \to \infty} Z_k^{2^k/(k+1)} \leq \delta^Q(K) = \exp(-E^Q(\mu_{K,Q})),
\]

using part 2. of Theorem 3.1. To prove the lower bound

\[
\liminf_{k \to \infty} Z_k^{2^k/(k+1)} \geq \delta^Q(K) = \exp(-E^Q(\mu_{K,Q})),
\]

we start with a set of weighted Fekete points \((f_0, \ldots, f_k)\) of order \(k\) for \(K, Q\). Writing

\[
|VDM^Q_k(f_0, \ldots, f_k)| = \prod_{i<j} |f_i - f_j| \cdot \prod_{i<j} |f_i^\theta - f_j^\theta| \cdot \exp \left( -k[Q(f_0) + \cdots + Q(f_k)] \right),
\]

we observe that

\[
p(t) := VDM_k(t, f_1, \ldots, f_k) \cdot VDM_k(t^\theta, f_1^\theta, \ldots, f_k^\theta) \cdot \exp \left( -k[Q(f_1) + \cdots + Q(f_k)] \right) = p_k(t)q_k(t^\theta)
\]

where \(p_k, q_k\) are polynomials of degree at most \(k\). The weighted Müntz-type polynomial

\[
p(t) \exp \left( -kQ(t) \right)
\]

attains its maximum modulus on \(K\) at \(t = f_0\). Applying the weighted Bernstein-Markov type inequality \((4.7)\) gives

\[
|VDM^Q_k(f_0, \ldots, f_k)| \leq \tilde{M}_k \int_K |VDM^Q_k(t, f_1, \ldots, f_k)| \, d\nu(t).
\]

Continuing this process in each variable, we obtain the lower bound.

\[\square\]

**Remark 4.6.** In particular, if \(K\) is regular and \(\nu\) is a measure on \(K\) with \(\nu(K) < \infty\) satisfying a mass density condition as in Theorem 4.3, then for any \(Q \in C(K)\), \((4.11)\) holds. However, as in Proposition 4.9 of \([6]\), for any nonpolar \(K\), and any \(Q \in \mathcal{A}(K)\), we can always find a measure \(\nu\) on \(K\) such that \((4.11)\) is satisfied.
Now we define a probability measure $\text{Prob}_k$ on $K^{k+1}$: for a Borel set $A \subset K^{k+1}$,

$$\text{Prob}_k(A) := \frac{1}{Z_k} \cdot \int_A |VDM_k^Q(X_k)| d\nu(X_k)$$

(4.12)

where $X_k = (x_0, \ldots, x_k)$ and $d\nu(X_k) = d\nu(x_0) \cdots d\nu(x_k)$. Directly from (4.11) and (4.12) we obtain the following estimate.

**Corollary 4.7.** Let $K$ be a nonpolar compact set, $Q \in \mathcal{A}(K)$ and $\nu$ a finite measure on $K$ satisfying (4.11). Given $\eta > 0$, define

$$A_{k,\eta} := \{X_k \in K^{k+1} : |VDM_k^Q(X_k)| \geq (\delta^Q(K) - \eta)^{k(k+1)/2}\}.$$  

(4.13)

Then there exists $k^* = k^*(\eta)$ such that for all $k > k^*$,

$$\text{Prob}_k(K^{k+1} \setminus A_{k,\eta}) \leq \left(1 - \eta / (2\delta^Q(K))\right)^{k(k+1)} \nu(K^{k+1}).$$

We get the induced product probability measure $P$ on the space of arrays on $K$,

$$\chi := \{X = \{X_k \in K^k\}_{k \geq 1}\},$$

namely,

$$(\chi, P) := \prod_{k=1}^{\infty} (K^{k+1}, \text{Prob}_k).$$

As an immediate consequence of Corollary 4.7, the Borel-Cantelli lemma, and 3. of Theorem 3.1, we obtain:

**Corollary 4.8.** Let $Q \in \mathcal{A}(K)$ and $\nu$ a finite measure on a nonpolar compact set $K$ satisfying (4.11). For $P$-a.e. array $X = \{x_j^{(k)}\}_{j = 0, \ldots, k; k = 2, 3, \ldots} \in \chi$,

$$\frac{1}{k + 1} \sum_{j=0}^{k} \delta_{x_j^{(k)}} \to \mu_{K,Q} \text{ weakly as } k \to \infty.$$  

We have all of the ingredients needed to follow the arguments of section 6 of [6] to prove the analogue of Theorem 6.6 there and hence a large deviation principle. Given $G \subset \mathcal{M}(K)$, for each $k = 1, 2, \ldots$ we let

$$\tilde{G}_k := \{a = (a_0, \ldots, a_k) \in K^{k+1} : \frac{1}{k + 1} \sum_{j=0}^{k} \delta_{a_j} \in G\},$$

(4.14)

and set

$$J_k^Q(G) := \left[\int_{\tilde{G}_k} |VDM_k^Q(a)| d\nu(a)\right]^{2/(k(k+1))}.$$  

(4.15)
Definition 4.9. For \( \mu \in \mathcal{M}(K) \) we define
\[
J^Q(\mu) := \inf_{G \ni \mu} J^Q(G) \quad \text{where} \quad J^Q(G) := \limsup_{k \to \infty} J_k^Q(G);
\]
and if \( Q = 0 \) we simply write \( J(\mu), J_\mu(\mu) \).

Following the steps in section 6 of [6] with Corollary 5.3 there replaced by our approximation result, Lemma 3.3, we obtain equality of the \( J \) and \( J^Q \) functionals for any admissible weight \( Q \) provided \( \nu \) is a strong Bernstein-Markov measure for \( K \) (see Theorem 6.6 in [6]). Note that the definition of \( |VDM_k^Q| \) in (3.1) clearly differs from that in [6] and, in our weighted Bernstein-Markov property (4.7), we use an \( L^1 \)–norm.

Theorem 4.10. Let \( K \) be a nonpolar compact subset of \([0, \infty)\) and let \( \nu \in \mathcal{M}(K) \) be a strong Bernstein-Markov measure for \( K \) (e.g., if \( \nu \) satisfies a mass density condition and \( K \) is regular).

(i) For any \( \mu \in \mathcal{M}(K) \),
\[ \log J(\mu) = \log J^Q(\mu) = -I(\mu) - I(f_*\mu). \]

(ii) Let \( Q \in \mathcal{A}(K) \). Then
\[ J^Q(\mu) = J(\mu) \cdot e^{-2\int_K Q d\mu} \]
(and with the \( J, J^Q \) functionals as well) so that,
\[ \log J^Q(\mu) = \log J^Q(\mu) = -E^Q(\mu). \] (4.16)

Thus we simply write \( J, J^Q \) without an underline or overline.

Define \( j_k : K^{k+1} \to \mathcal{M}(K) \) via
\[ j_k(x_0, ..., x_k) = \frac{1}{k+1} \sum_{j=0}^k \delta_{x_j}. \] (4.17)

The push-forward \( \sigma_k := (j_k)_*(\text{Prob}_k) \) is a probability measure on \( \mathcal{M}(K) \): for a Borel set \( G \subset \mathcal{M}(K) \),
\[ \sigma_k(G) = \frac{1}{Z_k} \int_{G_k} |VDM_k^Q(x_0, ..., x_k)|d\nu(x_0) \cdots d\nu(x_k). \] (4.18)

Following section 7 of [6], Theorem 4.10 immediately yields a large deviation principle:

Theorem 4.11. Assume \( \nu \) is a strong Bernstein-Markov measure on a nonpolar compact set \( K, Q \in \mathcal{A}(K) \), and \( \nu \) satisfies (4.11). The sequence \( \{\sigma_k = (j_k)_*(\text{Prob}_k)\} \) of probability measures on \( \mathcal{M}(K) \) satisfies a large deviation principle with speed \( k^2/2 \) and good rate function \( I := I_{K,Q} \) where, for \( \mu \in \mathcal{M}(K) \),
\[ I(\mu) := \log J^Q(\mu, K, Q) - \log J^Q(\mu) = E^Q(\mu) - E^Q(\mu, K, Q). \]
This means that $\mathcal{I} : \mathcal{M}(K) \to [0, \infty]$ is a lower semicontinuous mapping such that the sublevel sets $\{\mu \in \mathcal{M}(K) : \mathcal{I}(\mu) \leq \alpha\}$ are compact in the weak topology on $\mathcal{M}(K)$ for all $\alpha \geq 0$ ($\mathcal{I}$ is “good”) satisfying: for all measurable sets $\Gamma \subset \mathcal{M}(K)$,

$$- \inf_{\mu \in \Gamma} \mathcal{I}(\mu) \leq \liminf_{k \to \infty} \frac{2}{k^2} \log \mu_k(\Gamma)$$

and

$$\limsup_{k \to \infty} \frac{2}{k^2} \log \mu_k(\Gamma) \leq - \inf_{\mu \in \Gamma} \mathcal{I}(\mu).$$

The lower semicontinuity of $\mu \to E^Q(\mu)$ implies the lower semicontinuity of $\mathcal{I}$.

We note that if $Q \in C(K)$, the strong Bernstein-Markov hypothesis on $\nu$ implies (4.11). Also, the rate function $\mathcal{I}$ is independent of the measure $\nu$.

5 Weighted Bernstein-Walsh type estimates for M"untz-type polynomials $p_n(x)q_n(x^{\theta})$ on $[0, \infty)$

In this section, we take $K = [0, \infty)$ and $f(x) = x^{\theta}$ with $\theta > 0$. We let $Q$ be an $f$-admissible weight on $K$ as in Definition 2.1: the function

$$\psi(x) := Q(x) - \frac{1}{2} \log [(1 + |x|^2)(1 + |f(x)|^2)]$$

satisfies

1. $\psi(x) \geq c = c(Q) > -\infty$ for all $x \in K$ and
2. $\liminf_{|x| \to \infty, x \in K} \psi(x) = \infty$.

We observe that the first part of Theorem 3.1

1. if $\{\mu_k = 1 \sum_{j=0}^{k} \delta_{z_j^{(k)}}\} \subset \mathcal{M}(K)$ converge weakly to $\mu \in \mathcal{M}(K)$, then

$$\limsup_{k \to \infty} |VDM_k^Q(z_0^{(k)}, ..., z_k^{(k)})|^{2/k(k+1)} \leq \exp (-E^Q(\mu)) \quad (5.1)$$

follows as in the case where $K$ is compact. In order to verify the validity of the rest of Theorem 3.1 in this situation, we will need to analyze the behavior of M"untz-type polynomials $p_n(x)q_n(x^{\theta})$ for $x$ large.

With $\theta > 0$ and $Q$ an $f$-admissible weight on $K = [0, \infty)$, let

$$R(x) := Q(x)/(1 + \theta).$$

Then $R(x)$ is admissible in the usual sense on $K = [0, \infty)$ and

$$V_{K,R}(z) := \sup\{v(z) : v \in L(\mathbb{C}), v \leq R \text{ on } K\}$$

satisfies $V_{K,R}(z) := \limsup_{\zeta \to z} V_{K,R}(\zeta) \in L(\mathbb{C})$ with $\Delta V_{K,R}^*$ having compact support (cf., [21]). We begin with a lemma.
Lemma 5.1. Let $b$ be a positive integer and $h : \mathbb{C} \to \mathbb{C}$ be the power map $h(t) = t^b$. Let $\tilde{K} := h^{-1}(K)$ and

$$\tilde{R}(t) := \frac{1}{b} R(t^b) = \frac{Q(t^b)}{b(1 + \theta)}. \quad (5.2)$$

Then

$$V_{\tilde{K},\tilde{R}}(t) = \frac{1}{b} V_{K,R}(t^b).$$

Proof. Since $v \in L(\mathbb{C})$ with $v \leq R$ on $K$ implies $u(t) := \frac{1}{b} v(t^b) \in L(\mathbb{C})$ with $u \leq \tilde{R}$ on $\tilde{K}$, we have $\frac{1}{b} V_{K,R}(t^b) \leq V_{\tilde{K},\tilde{R}}(t)$. For the reverse inequality, if we let $\omega := e^{2\pi i/b}$ be a primitive $b$-th root of unity, then for a function $u \in L(\mathbb{C})$, the function

$$v(t) := \frac{1}{b} [u(t) + u(\omega t) + \cdots + u(\omega^{b-1}t)]$$

is in $L(\mathbb{C})$ and satisfies $v(\omega t) = v(t)$. Thus we can write $v(t) = \frac{1}{b} w(t^b)$ where $w \in L(\mathbb{C})$. Since $\tilde{K}$ and $\tilde{R}$ are invariant under the rotation $t \to \omega t$, the upper envelope defining $V_{\tilde{K},\tilde{R}}$ can be taken over functions $v \in L(\mathbb{C})$ having the same invariance property with $v \leq \tilde{R}$ on $\tilde{K}$; for each such $v$, we can find $w \in L(\mathbb{C})$ with $v(t) = \frac{1}{b} w(t^b)$. Since $v \leq \tilde{R}$ on $\tilde{K}$, we have $w \leq R$ on $K$. This gives $V_{\tilde{K},\tilde{R}}(t) \leq \frac{1}{b} V_{K,R}(t^b)$.

Theorem 5.2. We have

$$|p_k(x)q_k(x^\theta)| e^{-kQ(x)} \leq \max_{s \in S_b} (|p_k(s)q_k(s^\theta)| e^{-kQ(s)}) \cdot e^{k[(1+\theta)\hat{V}_{K,R}(x) - Q(x)]}, \quad x \in K \quad (5.3)$$

for polynomials $p_k, q_k$ of degree at most $k$ where

$$S_b := \{x \in K : (1 + \theta)\hat{V}_{K,R}(x) \geq Q(x)\}. \quad (5.4)$$

Proof. For positive integers $a, b$, consider

$$p_k(t^b)q_k(t^a)e^{-k(a+b)\tilde{R}(t)}.$$

This is a weighted polynomial in $t$ of degree at most $k(a + b)$. From [21], we have the weighted Bernstein-Walsh inequality

$$|p_k(t^b)q_k(t^a)| e^{-k(a+b)\tilde{R}(t)} \leq \max_{t \in T} ([p_k(t^b)q_k(t^a)]|e^{-k(a+b)\tilde{R}(t)}) \cdot e^{k(a+b)[V_{\tilde{K},\tilde{R}}(t) - \tilde{R}(t)]} \quad (5.5)$$

for $t \in \tilde{K}$ where $T = \{t \in \tilde{K} : V_{\tilde{K},\tilde{R}}(t) \geq \tilde{R}(t)\}$ Using (5.2) this becomes, for $x = t^b \in K$,

$$|p_k(x)q_k(x^{a/b})| e^{-k\left(\frac{1+b}{1+\theta}\right)Q(x)} \leq \max_{x \in S_b} [|p_k(x)q_k(x^{a/b})| e^{-k\left(\frac{1+b}{1+\theta}\right)Q(x)}) \cdot e^{k\left(\frac{1+b}{1+\theta}\right)[(1+\theta)\hat{V}_{K,R}(x) - Q(x)]} \quad (5.6)$$
with $S_\theta$ as in (5.4). Note that $S_\theta$ is compact since $V_{K,R}^* \in L(\C)$ and $Q$ is $f$–admissible. Also note that $R$ and $S_\theta$ are determined by $\theta, Q$ and $K$, which are fixed.

Take a sequence of rational numbers $\{a_n/b_n\}$ with $\lim_{n \to \infty} a_n/b_n = \theta$. Given a pair of polynomials $p_k$ and $q_k$, we fix $x \in K$ and apply (5.6) with $a/b = a_n/b_n$. Thus

$$\left|p_k(x)q_k(x_a/n/b_n)\right|e^{-k\left(\frac{1+\alpha}{1+\theta}\right)Q(x)} \leq \max_{x \in S_\theta} |h_n(x)| \cdot e^{k\left(\frac{1+\alpha}{1+\theta}\right)\left((1+\theta)V_{K,R}^*(x) - Q(x)\right)} \tag{5.7}$$

where $h_n(x) := p_k(x)q_k(x_a/n/b_n)e^{-k\left(\frac{1+\alpha}{1+\theta}\right)Q(x)}$. Since $Q$ is fixed and $S_\theta$ is compact, $\lim_{n \to \infty} \max_{x \in S_\theta} |h_n(x)| = \max_{x \in S_\theta} (|p_k(x)|q_k(x_0^{(k)})|e^{-kQ(x)})$. Letting $n \to \infty$ in (5.7) gives the result.

We will now use these estimates to show that the sequence of probability measures $\mu_k := \frac{1}{k+1} \sum_{j=0}^k \delta_{z_j^{(k)}}$ associated to an array $\{z_j^{(k)}\}_{j=0,\ldots,k}$ of asymptotically weighted Fekete points for $K, R$ (see (3.3) and Remark 3.2) are uniformly tight. Indeed, we prove a stronger statement.

**Proposition 5.3.** Given an array of asymptotically weighted Fekete points $\{z_j^{(k)}\}_{j=0,\ldots,k}$ for $K, Q$ and $f(x) = x^\theta$ where $Q$ is $f$–admissible, for any $M > 0$ with $S_\theta \subseteq [0, M]$ and any $\delta > 0$, there exists $k_0$ such that for all $k > k_0$,

$$\frac{\# \left\{ j : z_j^{(k)} \in [0, M] \right\}}{k} > 1 - \delta.$$ 

**Proof.** Fix $M > 0$ with $S_\theta \subseteq [0, M]$ and $k$. Suppose $z_0^{(k)} > M$. Now

$$z_0^{(k)} \to VDM^Q_k(z_0^{(k)}, \ldots, z_k^{(k)}) = p_k(z_0^{(k)})q_k(z_k^{(k)}e^{-kQ(z_0^{(k)})})$$

for polynomials $p_k, q_k$ of degree $k$. By (5.3),

$$\left|p_k(z_0^{(k)})q_k(z_0^{(k)}e^{-kQ(z_0^{(k)})})\right|e^{-kQ(z_0^{(k)})} \leq \left(\max_{w \in S_\theta} |p_k(w)|q_k(w_0^{(k)})|e^{-kQ(w)}\right) \cdot e^{k[(1+\theta)V_{K,R}^*(z_0^{(k)}) - Q(z_0^{(k)})]}$$

where

$$\rho = \exp[\sup\{(1 + \theta)V_{K,R}^*(x) - Q(x) : x > M\}] < 1$$

since $V_{K,R}^* \in L^+(\C)$ and $Q$ is $f$–admissible (hence $Q(x) - \log|x|^{1+\theta} \to \infty$ as $x \to \infty$). Thus we can find $z_0^{(k)} \in S_\theta$ with

$$|VDM^Q_k(z_0^{(k)}, \ldots, z_k^{(k)})| = |p_k(z_0^{(k)})q_k(z_0^{(k)}e^{-kQ(z_0^{(k)})})|e^{-kQ(z_0^{(k)})} = \max_{w \in S_\theta} |p_k(w)|q_k(w_0^{(k)})|e^{-kQ(w)}$$

so that

$$|VDM^Q_k(z_0^{(k)}, \ldots, z_k^{(k)})| \geq |VDM^Q_k(z_0^{(k)}, \ldots, z_k^{(k)})|/\rho^k.$$
If \(|j: z^{(k)}_j > M|/k > \delta\), by applying the same reasoning for each such point \(z^{(k)}_j\), we obtain a set of \(k\) points \(\tilde{z}^{(k)}_0, \ldots, \tilde{z}^{(k)}_k \in K\) where \([\delta k]\) of the “tilde” points are new and lie in \(S_\theta\) with

\[
|VDM^Q_k(\tilde{z}^{(k)}_0, \ldots, \tilde{z}^{(k)}_k)| \geq |VDM^Q_k(z^{(k)}_0, \ldots, z^{(k)}_k)|/\rho^{[\delta k]-k}.
\]

Taking \(k(k+1)/2\) roots, we get that

\[
\liminf_{k \to \infty} \frac{|VDM^Q_k(\tilde{z}^{(k)}_0, \ldots, \tilde{z}^{(k)}_k)|^{2/(k+1)}}{\rho^{2\delta}} = \delta^Q(K)/\rho^{2\delta} > \delta^Q(K),
\]

a contradiction.

The importance of Proposition 5.3 is that the rest of Theorem 3.1; i.e., parts 2. and 3., follows for this setting of \(K = [0, \infty); f(x) = x^\theta\) with \(Q\) an \(f\)-admissible weight on \(K\). The uniform tightness allows one to extract a subsequence converging in the weak topology on \(\mathcal{M}(K)\); we omit the details.

**Corollary 5.4.** For \(K = [0, \infty), f(x) = x^\theta\) with \(\theta > 0\), and \(Q\) an \(f\)-admissible weight on \(K\),

1. we have

\[
\delta^Q(K) := \lim_{k \to \infty} \delta^Q_k(K) = \exp(-E^Q(\mu_{K,Q}));
\]

2. if \(\{z^{(k)}_j\}_{j=0, \ldots, k; \ k=2, 3, \ldots} \subset K\) and

\[
\lim_{k \to \infty} |VDM^Q_k(z^{(k)}_0, \ldots, z^{(k)}_k)|^{2/(k+1)} = \exp(-E^Q(\mu_{K,Q})),
\]

then

\[
\mu_k = \frac{1}{k+1} \sum_{j=0}^k \delta_{z^{(k)}_j} \to \mu_{K,Q} \text{ weakly.}
\]

In this situation of the unbounded set \(K = [0, \infty)\), we need some restriction on allowable measures \(\nu\) related to our weight \(Q\). Given a \(\sigma\)-finite measure \(\nu\) on \(K\) which puts no mass at 0 (but we may have 0 \(\in\) supp(\(\nu\))) we impose the condition that

\[
\exists \alpha > 0, \quad \int_K \epsilon(x)^\alpha d\nu(x) < \infty \tag{5.8}
\]

where \(\epsilon(x)\) is some nonnegative continuous function that tends to 0 as \(x\) tends to \(\infty\) chosen to insure that the integrals defining the free energies \(\{Z_k\}\) relative to \(\nu\) are finite. For simplicity, we take

\[
- \log \epsilon(x) \leq Q(x) - \log |x|^{1+\theta} \tag{5.9}
\]
where the right-hand side goes to infinity as \( x \to \infty \) by the \( f \)-admissibility of \( Q \).

For the (unbounded) set \( K = [0, \infty) \), we say that \((K, Q, \nu)\) satisfies a weighted Bernstein-Markov type inequality for a given \( \sigma \)-finite measure \( \nu \) and an \( f \)-admissible weight \( Q \) on \( K \) if

\[
\max_{s \in K} |p_k(s)q_k(s^\theta)|e^{-kQ(s)} \leq \tilde{M}_k \int_K |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x) \tag{5.10}
\]

for polynomials \( p_k, q_k \) of degree at most \( k \) where \( \tilde{M}_k^{1/k} \to 1 \).

**Remark 5.5.** From the estimate (5.3) in Theorem 5.2 it follows that for a given measure \( \nu \) and weight \( Q \) if (5.10) holds with \( K \) replaced by \( S_\theta \) and the restriction of \( \nu \) to \( S_\theta \) (or any subset of \( K \) containing \( S_\theta \)) then (5.10) holds for \( K \). In particular, this observation, in conjunction with Theorem 4.3, shows that Lebesgue measure \( dx \) on \( K \) satisfies (5.10) for any \( f \)-admissible weight \( Q \) on \( K \).

Using (5.3) in Theorem 5.2, we prove the analogue of Lemma 8.2 from [6] in our setting.

**Lemma 5.6.** Let \( Q \) be \( f \)-admissible and let \( \nu \) be a \( \sigma \)-finite measure such that \((K, Q, \nu)\) satisfies (5.8) and a weighted Bernstein-Markov type inequality (5.10). We can find a closed neighborhood \( N \) of \( S_\theta \) (see (5.4)) and a constant \( c > 0 \) independent of \( k \) such that, for all polynomials \( p_k, q_k \) of degree at most \( k \),

\[
\int_K |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x) \leq (1 + O(e^{-ck})) \int_N |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x). \tag{5.11}
\]

**Proof.** Multiplying by a constant, we may assume \( \max_{x \in S_\theta} |p_k(x)q_k(x^\theta)|e^{-kQ(x)} = 1 \). We show:

1. for \( \epsilon > 0 \) and for sufficiently large \( k \),

\[
\int_K |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x) \geq e^{-ck}; \quad \text{and}
\]

2. there exists a closed neighborhood \( N \) of \( S_\theta \) and \( c > 0 \) so that for sufficiently large \( k \),

\[
\int_{K \setminus N} |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x) \leq e^{-ck}.
\]

Indeed, 1. follows immediately from the weighted Bernstein-Markov type inequality (5.10):

\[
\int_K |p_k(x)q_k(x^\theta)|e^{-kQ(x)}d\nu(x) \geq M_k \cdot \max_{x \in K} |p_k(x)q_k(x^\theta)|e^{-kQ(x)}
\]

\[
\geq M_k \cdot \max_{x \in S_\theta} |p_k(x)q_k(x^\theta)|e^{-kQ(x)} = M_k.
\]

To prove 2., we appeal to Theorem 5.2 and (5.3). This gives, for \( x \in K \),

\[
|p_k(x)q_k(x^\theta)|e^{-kQ(x)} \leq \max_{s \in S_\theta} \left( |p_k(s)q_k(s^\theta)|e^{-kQ(s)} \right) \cdot e^{k(1 + c)\nu_{K,R}(x) - Q(x)}; \quad \text{i.e.,}
\]

23
\[ |p_k(x)q_k(x^\theta)| e^{-kQ(x)} \leq e^{k[(1+\theta)V_{K,R}^*(x) - Q(x)]}. \]

Now recall that the \( f \)-admissibility of \( Q \) implies the admissibility of \( R = Q/(1+\theta) \). Thus (see (2.1))

\[
\limsup_{x \to \infty} (1 + \theta) V_{K,R}^*(x) - Q(x) = -\infty.
\]

Using (5.9), we can find \( N \) containing \( S_\theta \) so that for a constant \( a \) with \( 0 < a < 1 \),

\[
(1 + \theta) V_{K,R}^*(x) - Q(x) \leq \log \epsilon(x) < \log a < 0 \text{ for } x \in K \setminus N.
\]

Then

\[
\int_{K \setminus N} |p_k(x)q_k(x^\theta)| e^{-kQ(x)} d\nu(x) \leq \int_{K \setminus N} \epsilon(x)^k d\nu(x)
\]

\[
\leq \sup_{x \in K \setminus N} \epsilon(x)^{k/2} \cdot \int_{K} \epsilon(x)^{\alpha} d\nu(x) \leq a^{k/2} \cdot \int_{K} \epsilon(x)^{\alpha} d\nu(x)
\]

for \( k \) sufficiently large. This yields 2. from (5.8).

**Remark 5.7.** Note that any \( N' \) containing \( N \) will also satisfy the conclusion of the lemma. Moreover, \( N \) can be chosen with \( \nu(N) < \infty \). The estimate (5.11) shows that the measure \( \nu \) restricted to \( N \) satisfies a weighted Bernstein-Markov type inequality with weight \( Q \) restricted to \( N \); i.e., we have the converse statement to Remark 5.5.

With this lemma, one obtains free energy asymptotics (Proposition 5.8) and a Johansson-type lemma (Corollary 5.9).

**Proposition 5.8.** Let \( Q \) be \( f \)-admissible and let \( \nu \) be a \( \sigma \)-finite measure such that \((K, Q, \nu)\) satisfies (5.8) and a weighted Bernstein-Markov type inequality (5.10). Then the \( L^2 \) normalization constants

\[
Z_k := \int_{K^{k+1}} \prod_{i<j} |z_i - z_j| \cdot \prod_{i<j} |z_i^\theta - z_j^\theta| \cdot \exp \left( -k[Q(z_0) + \cdots + Q(z_k)] \right) d\nu(z_0) \cdots d\nu(z_k)
\]

are finite and

\[
\lim_{k \to \infty} Z_k^{2/k(k+1)} = \delta^Q(K).
\]

**Proof.** The finiteness of \( Z_k \) for \( k \) sufficiently large is guaranteed by (5.8) and (5.9). The lower bound

\[
\delta^Q(K) \leq \liminf_{k \to \infty} Z_k^{2/k(k+1)}
\]

follows exactly as in the proof of Proposition 4.5 using the weighted Bernstein-Markov property (5.10). For the upper bound,

\[
\limsup_{k \to \infty} Z_k^{2/k(k+1)} \leq \delta^Q(K),
\]
we note that in each variable \( z_j \) for \( j = 0, \ldots, k \),
\[
z_j \rightarrow VDM_k(z_0, \ldots, z_k) \cdot VDM_k(z_0^\theta, \ldots, z_k^\theta) =: p_j(z_j)q_j(z_j^\theta)
\]
where \( p_j \) and \( q_j \) are polynomials of degree \( k \). Thus the integrand
\[
VDM_k(z_0, \ldots, z_k) \cdot VDM_k(z_0^\theta, \ldots, z_k^\theta) \cdot \exp\left(-k[Q(z_0) + \cdots + Q(z_k)]\right)
\]
in the definition of \( Z_k \) is, in each variable \( z_j \) for \( j = 0, \ldots, k \), of the form
\[
p_j(z_j)q_j(z_j^\theta) \cdot \exp(-kQ(z_j)).
\]
We can apply Lemma 5.6 for each of the variables \( z_0, \ldots, z_j \). Taking a neighborhood \( N \) of \( S_{\theta} \) with \( \nu(N) < \infty \), we obtain, with the notation introduced in (3.1),
\[
Z_k \leq (1 + O(e^{-ck}))^{k+1} \int_{N_{k+1}} |VDM_k^Q(z_0, \ldots, z_k)| d\nu(z_0) \cdots d\nu(z_k)
\]
\[
\leq (1 + O(e^{-ck}))^{k+1} (\delta^Q(K))^{k(k+1)/2} \nu(N)^{k+1}
\]
and the estimate (5.12) follows.

In order to prove a result on one-point correlation, we quantify the Johansson-type estimate from Corollary 4.7 in the unbounded setting, using Lemma 5.6.

**Corollary 5.9.** Let \( Q \) be \( f \)-admissible and let \( \nu \) be a \( \sigma \)-finite measure such that \((K, Q, \nu)\) satisfies (5.8) and a weighted Bernstein-Markov type inequality (5.10). Given \( \eta > 0 \), define the set \( A_{k,\eta} \) as in (4.13). Then there exist a constant \( c > 0 \) and \( k^* = k^*(\eta) \) such that for all \( k > k^* \),
\[
\text{Prob}_k(K^{k+1} \setminus A_{k,\eta}) \leq \left(1 - \frac{\eta}{2\delta^Q(K)}\right)^{k(k+1)/2} \nu(N)^{k+1} + O(e^{-ck}) \tag{5.13}
\]
where \( N \subset K \) is a closed neighborhood of \( S_{\theta} \) as in Lemma 5.6 with \( \nu(N) < \infty \).

**Proof.** We follow the proof of Corollary 8.4 of [4]. Thus we define \( B_{k,\eta} := K^{k+1} \setminus A_{k,\eta} \) and we decompose the integral in
\[
\text{Prob}_k(B_{k,\eta}) = \frac{1}{Z_k} \int_{B_{k,\eta}} |VDM_k^Q(X_k)| d\nu(X_k)
\]
as a sum of two integrals over \( B_{k,\eta} \cap N^{k+1} \) and \( B_{k,\eta} \cap (K^{k+1} \setminus N^{k+1}) \). From the definition of \( A_{k,\eta} \), the first term is less than
\[
\left(1 - \frac{\eta}{2\delta^Q(K)}\right)^{k(k+1)/2} \nu(N)^{k+1}
\]
for $k$ large. The second term is less than

$$\frac{1}{Z_k} \int_{K_k+1 \setminus N_k+1} |VDM^Q_k(X_k)| d\nu(X_k) \leq \sum_{j=0}^k \frac{1}{Z_k} \int_{U_j} |VDM^Q_k(X_k)| d\nu(X_k)$$

where $U_j = K \times \cdots \times (K \setminus N) \times \cdots K$ and the subset $K \setminus N$ is in $j$-th position. Again, since the integrand $|VDM^Q_k(X_k)|$ in each variable $x_j$ is of the form

$$p_j(x_j)q_j(x_j) \exp(-kQ(x_j))$$

where $p_j$ and $q_j$ are polynomials of degree $k$, we can apply Lemma 5.6 for the $j$-th variable to the integral over $U_j$. This gives the upper bound

$$O(e^{-ck}) \sum_{j=0}^k \frac{1}{Z_k} \int_{V_j} |VDM^Q_k(X_k)| d\nu(X_k)$$

where $V_j = K \times \cdots \times N \times \cdots K$. Replacing $N$ with $K$ we finally get the upper bound $O(k e^{-ck})$, which implies (5.13) with a different $c$.

As in section 3, we get the induced product probability measure $P$ on the space of arrays on $K$:

$$(\chi, P) := \prod_{k=1}^{\infty} (K^{k+1}, Prob_k).$$

The Borel-Cantelli lemma, Corollary 5.9, and 3. of Theorem 3.1 give us $P$-a.e. convergence to $\mu_{K,Q}$:

**Corollary 5.10.** Let $Q$ be $f$-admissible and let $\nu$ be a $\sigma$-finite measure such that $(K, Q, \nu)$ satisfies (5.5) and a weighted Bernstein-Markov type inequality (5.10). For $P$-a.e. array $X = \{X_k\}_{k=2,3,\ldots} = \{x_j^{(k)}\}_{j=0,\ldots,k; k=2,3,\ldots} \in \chi$,

$$\frac{1}{k+1} \sum_{j=0}^k \delta_{x_j^{(k)}} \rightarrow \mu_{K,Q} \text{ weakly as } k \rightarrow \infty.$$  

**Proof.** From 3. of Theorem 3.1 (which we observed follows from Proposition 5.3 in this setting), it suffices to verify for $P$-a.e. array $X = \{X_k\}_k \in \chi$,

$$\liminf_{k \rightarrow \infty} \left( |VDM^Q_k(X_k)| \right)^{2/(k+1)} = \delta^Q(K). \quad (5.14)$$

Given $\eta > 0$, the condition that for a given array $X = \{X_k\}_k$ we have

$$\liminf_{k \rightarrow \infty} \left( |VDM^Q_k(X_k)| \right)^{2/(k+1)} \leq \delta^Q(K) - \eta$$
means that $X_k \in K^{k+1} \setminus A_{k,\eta}$ for infinitely many $k$. Thus setting

$$E_k := \{ X \in \chi : X_k \in K^{k+1} \setminus A_{k,\eta} \},$$

we have, using Corollary 5.9

$$P(E_k) \leq \text{Prob}_k(K^{k+1} \setminus A_{k,\eta}) \leq \left(1 - \frac{\eta}{2\delta Q(K)}\right)^{k+1/2} \nu(N)^{k+1} + \mathcal{O}(e^{-ck}).$$

Since $\nu(N) < \infty$, for $k$ sufficiently large

$$\left(1 - \frac{\eta}{2\delta Q(K)}\right)^{k+1/2} \nu(N) + \mathcal{O}(e^{-c}) \leq r < 1$$

whence $\sum_{k=1}^{\infty} P(E_k) < +\infty$. By the Borel-Cantelli lemma,

$$P(\limsup_{k \to \infty} E_k) = 0,$$

where $\limsup_{k \to \infty} E_k = \cap_{k=1}^{\infty} \cup_{j=k}^{\infty} E_j$.

Thus, with probability one, only finitely many $E_k$ occur, and (5.14) follows. $\square$

Define the one-point correlation functions

$$R^{(k)}(x) := \int_{K^k} |VM_k(x, z_1, ..., z_k)|d\nu(z_1)...d\nu(z_k).$$

Then for each $k$

$$\mu_k := \frac{1}{Z_k} R^{(k)}(x)e^{-kQ(x)}d\nu(x)$$

is a probability measure on $K$. Using Corollary 5.9, we modify the arguments in [4] to prove the following.

**Theorem 5.11.** Let $Q$ be $f$–admissible and let $\nu$ be a $\sigma$–finite measure such that $(K, Q, \nu)$ satisfies (5.8) and a weighted Bernstein-Markov type inequality (5.10). With $\mu_k$ as in (5.15), $\mu_k \to \mu_{K,Q}$ weakly.

**Proof.** To prove Theorem 5.11 we fix a bounded $\phi \in C(K)$. For each $k$ we have

$$\int_K \phi(z)d\mu_k(z)$$

$$= \frac{1}{Z_k} \int_K \phi(x)\left[ \int_{K^k} |VM_k^Q(x, z_1, ..., z_k)|d\nu(z_1)...d\nu(z_k) \right] e^{-kQ(x)}d\nu(x)$$

$$= \frac{1}{Z_k} \int_{K^{k+1}} \phi(x)|VM_k^Q(x, z_1, ..., z_k)|d\nu(x)d\nu(z_1)...d\nu(z_k)$$

$$= \frac{1}{Z_k} \int_{K^{k+1}} \sum_{j=0}^{k} \frac{\phi(z_j)}{k+1} |VM_k^Q(z_0, z_1, ..., z_k)|d\nu(z_0)d\nu(z_1)...d\nu(z_k)$$

(here $x \to z_0$)
Choose points
\[ \tilde{z}_{k_0}^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)} \in A_{k_j, \eta_j} \]
with
\[ \psi_{k_j}(\tilde{z}_{k_0}^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)}) \geq \sup_{(w_0, \ldots, w_{k_j}) \in A_{k_j, \eta_j}} \psi_{k_j}(w_0, \ldots, w_{k_j}) - \epsilon_j \] for \( \epsilon_j \to 0 \).

If \( |\phi| \leq M \) on \( K \), then \( |\psi_{k_j}| \leq M \) on \( K^{k_j+1} \); using the above and (5.16),
\[
\limsup_{j \to \infty} \int_K \phi(z) d\mu_{k_j}(z) = \limsup_{j \to \infty} \int_{K^{k_j+1}} \psi_{k_j} d\Prob_{k_j}
\]
\[
= \limsup_{j \to \infty} \left[ \int_{A_{k_j, \eta_j}} \psi_{k_j} d\Prob_{k_j} + \int_{K^{k_j+1} \setminus A_{k_j, \eta_j}} \psi_{k_j} d\Prob_{k_j} \right]
\]
\[
\leq \limsup_{j \to \infty} \left( \frac{1}{k_j + 1} \sum_{k=0}^{k_j} \phi(\tilde{z}_k^{(k_j)}) + \epsilon_j + M \left[ (1 - \frac{\eta_j}{2\delta^Q(K)})^{k_j(k_j+1)/2} \cdot \mu(N)^{k_j+1} + 0(e^{-k_j\epsilon}) \right] \right)
\]
\[
= \limsup_{j \to \infty} \frac{1}{k_j + 1} \sum_{k=0}^{k_j} \phi(\tilde{z}_k^{(k_j)}),
\]
where \( \tilde{z}_0^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)} \in A_{k_j, \eta_j} \).

Now since \( \tilde{z}_0^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)} \in A_{k_j, \eta_j} \),
\[
|VD_{\mu_{k_j}}^Q(\tilde{z}_0^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)})| \geq \left( \delta^Q(K) - \eta_j \right)^{k_j(k_j+1)/2}
\]
so that
\[
\lim_{j \to \infty} |VD_{\mu_{k_j}}^Q(\tilde{z}_0^{(k_j)}, \ldots, \tilde{z}_{k_j}^{(k_j)})|^{2/k_j(k_j+1)} = \delta^Q(K).
\]
By Corollary 5.4,
\[ \frac{1}{k_j + 1} \sum_{k=0}^{k_j} \delta_{\tilde{z}_k^{(k_j)}} \to d\mu_{K,Q}. \]
Thus
\[ \frac{1}{k_j + 1} \sum_{k=0}^{k_j} \phi(\tilde{z}_k^{(k_j)}) \to \int_K \phi(z) d\mu_{K,Q}(z) \]
and hence
\[ \limsup_{j \to \infty} \int_K \phi(z) d\mu_{k_j}(z) \leq \int_K \phi(z) d\mu_{K,Q}(z). \]  
(5.17)

Applying (5.17) to \(-\phi\) we obtain
\[ \limsup_{j \to \infty} \int_K (-\phi(z)) d\mu_{k_j}(z) \leq \int_K (-\phi(z)) d\mu_{K,Q}(z); \]
i.e.,
\[ \liminf_{j \to \infty} \int_K \phi(z) d\mu_{k_j}(z) \geq \int_K \phi(z) d\mu_{K,Q}(z), \]
so that
\[ \lim_{j \to \infty} \int_K \phi(z) d\mu_{k_j}(z) = \int_K \phi(z) d\mu_{K,Q}(z). \]  
(5.18)

Thus for any sequence of positive integers increasing to infinity we can choose a subsequence \(\{k_j\}\) satisfying (5.16) for some \(\eta_j \downarrow 0\) so that (5.18) holds; hence \(\mu_k \to \mu_{K,Q}\) weakly.

\[ \square \]

6  LDP for \(K = [0, \infty)\) and \(Q\) strongly \(f\)–admissible

In this section, we prove a large deviation principle for \(K = [0, \infty)\). Given a \(\sigma\)-finite measure \(\nu\) and a weight \(Q\) on \(K = [0, \infty)\), recall for \(G \subset \mathcal{M}(K)\),
\[ \tilde{G}_n := \left\{ a = (a_0, \ldots, a_n) \in K^{n+1} : \frac{1}{n+1} \sum_{j=0}^{n} \delta_{a_j} \in G \right\} \]
and
\[ \sigma_n(G) = \frac{1}{Z_n} \int_{\tilde{G}_n} |VDM_n^Q(z_0, \ldots, z_n)| d\nu(z_0) \ldots d\nu(z_n) \]
where
\[ \lim_{n \to \infty} Z_n^{2/n(n+1)} = \delta^Q(K). \]  
(6.1)

We also recall that \(Q\) is strongly \(f\)–admissible for \(K\) if there exists \(\delta > 0\) such that \((1 - \delta)Q\) is \(f\)–admissible for \(K\).
Theorem 6.1. Let $Q$ be strongly $f$–admissible and let $\nu$ be a $\sigma$–finite measure such that $(K, Q, \nu)$ satisfies (5.8) with $\epsilon(x)$ as in (5.9), as well as a weighted Bernstein-Markov type inequality (5.10). Then the sequence $\{\sigma_k = (j_k)_*(\Prob_k)\}$ of probability measures on $\M(K)$ satisfies a large deviation principle with speed $k^2/2$ and good rate function $I := I_{K, Q}$ where, for $\mu \in \M(K)$, 
\[ I(\mu) = E^Q(\mu) - E^Q(\mu_{K, Q}). \]

Remark 6.2. The rate function is independent of $\nu$ satisfying the hypothesis and the result is independent of $\epsilon(x)$. Furthermore, if $\nu$ is Lebesgue measure we can take $\epsilon(x) = x^{-\epsilon}$ for $x \to \infty$ where $\epsilon > 0$ is sufficiently small.

Proof. The proof will be different from the one we gave in the compact case. The main steps are the following:

1. We show the sequence of measures $\{\sigma_n\} \subset \M(K)$ is exponentially tight.
2. We prove a weak large deviation upper bound: for $\mu \in \M(K)$,
\[ \limsup_{\epsilon \to 0} \left( \limsup_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(B(\mu, \epsilon)) \right) \leq -I(\mu) \]
where $B(\mu, \epsilon)$ is a ball of radius $\epsilon$ in the (metrizable) topology of $\M(K)$.
3. We prove a large deviation lower bound: for $\mu \in \M(K)$ and $G \subset \M(K)$ open with $\mu \in G$,
\[ \liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(G) \geq -I(\mu). \]

This general strategy 1. - 3. is fairly common in the literature but our approach to 3. is different. Here we deduce 3. from the corresponding result in the compact setting.

We recall that if
\[ k(x, y) := -\log |x - y| - \log |f(x) - f(y)| + Q(x) + Q(y), \]
where $Q$ is an $f$–admissible weight, we have
\[ k(x, y) \geq \psi(x) + \psi(y) \geq 2c \text{ on } K \times K. \]
If, moreover, $Q$ is strongly $f$–admissible, we have the estimate
\[ k(x, y) \geq \delta[Q(x) + Q(y)] + 2c \text{ on } K \times K \quad (6.2) \]
for $\delta > 0$.

Also, letting $\mu_n = \frac{1}{n+1} \sum_{j=0}^{n} \delta_{z_j}$,
\[ |VDM_n^Q(z_0, \ldots, z_n)| = \exp \left( -\frac{(n+1)^2}{2} \int_{x \neq y} k(x, y) d\mu_n(x) d\mu_n(y) \right). \quad (6.3) \]
Step 1. We prove exponential tightness of the sequence \( \{ \sigma_n \} \); i.e., for \( 0 < M < \infty \), there exists an open subset \( \mathcal{O}_M \) of \( \mathcal{M}(K) \) such that \( \mathcal{M}(K) \setminus \mathcal{O}_M \) is compact and

\[
\limsup_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(\mathcal{O}_M) < -M. \tag{6.4}
\]

**Proof.** For a real number \( \widetilde{M} \) (depending on \( M \)) to be specified later, define

\[
\mathcal{O}_M := \{ \mu \in \mathcal{M}(K) : \int_K Q d\mu > \widetilde{M} \}.
\]

We first prove that the set \( \mathcal{M}(K) \setminus \mathcal{O}_M \) is a compact subset of \( \mathcal{M}(K) \). If \( \mu_n \) is a sequence in the complement of \( \mathcal{O}_M \) that tends weakly to \( \mu \), first of all, \( \mu \in \mathcal{M}(K) \) by weak convergence; moreover, by lowersemicontinuity of \( Q \), for \( R < \infty \)

\[
\int_K \chi_{(0,R)} Q d\mu \leq \liminf_{n \to \infty} \int_K \chi_{(0,R)} Q d\mu_n.
\]

Since \( Q(x) \to \infty \) as \( x \to \infty \) clearly \( Q(x) \geq 0 \) for \( x \geq R \) if \( R \) is sufficiently large. Thus

\[
\int_K \chi_{(0,R)} Q d\mu \leq \liminf_{n \to \infty} \int_K Q d\mu_n.
\]

This is valid for each \( R \) hence

\[
\int_K Q d\mu \leq \liminf_{n \to \infty} \int_K Q d\mu_n.
\]

Thus \( \mu \) belongs to the set \( \mathcal{M}(K) \setminus \mathcal{O}_M \) which is closed. Moreover, the measures in the complement of \( \mathcal{O}_M \) are uniformly tight. Indeed, since \( Q(x) \) is bounded below on \( K \) and \( Q(x) \to +\infty \) as \( x \to \infty \), it is easy to check that, for any \( \epsilon \), there exist \( R > 0 \) such that \( \mu([-R, \infty)) < \epsilon \) for any \( \mu \in \mathcal{M}(K) \setminus \mathcal{O}_M \). By Prokhorov’s theorem, we may conclude that the complement of \( \mathcal{O}_M \) is a compact subset of \( \mathcal{M}(K) \).

Next, using (6.3), then (6.2), and finally the estimates

\[
\int \int_{x \neq y} Q(x) d\mu_n(x) d\mu_n(y) \geq \widetilde{M} - \frac{1}{(n+1)^2} \sum_{j=0}^n Q(z_j),
\]

\[
\int \int_{x \neq y} Q(y) d\mu_n(x) d\mu_n(y) = \frac{n}{(n+1)^2} \sum_{j=0}^n Q(z_j),
\]

we have

\[
\sigma_n(\mathcal{O}_M) = \frac{1}{Z_n} \int_{(\mathcal{O}_M)_n} |VDM_n^Q(z_0, \ldots, z_n)| \prod_{j=0}^n d\nu(z_j)
\]
\[ \leq \frac{1}{Z_n} \int_{(\mathcal{M})_n} \exp \left( -\frac{(n+1)^2}{2} \int \int_{x \neq y} \left[ \delta [Q(x) + Q(y)] + 2c \right] d\mu_n(x) d\mu_n(y) \right) \prod_{j=0}^{n} d\nu(z_j) \]

\[ \leq \frac{1}{Z_n} e^{-\delta \bar{M}(n+1)^2/2 - cn(n+1)} \int_{(\mathcal{M})_n} \exp \left( -\frac{(n-1)\delta}{2} \sum_{j=0}^{n} Q(z_j) \right) \prod_{j=0}^{n} d\nu(z_j) \]

\[ \leq \frac{1}{Z_n} e^{-\delta \bar{M}(n+1)^2/2 - cn(n+1)} \int_{K^{n+1}} \exp \left( -\frac{(n-1)\delta}{2} \sum_{j=0}^{n} Q(z_j) \right) \prod_{j=0}^{n} d\nu(z_j) \]

\[ = \frac{1}{Z_n} e^{-\delta \bar{M}(n+1)^2/2 - cn(n+1)} \prod_{j=0}^{n} \int_{K} e^{-(n-1)\delta Q(z_j)/2} d\nu(z_j). \]

If \( m = \min_{K} Q \geq 0 \) the above integrals are bounded, recall assumptions (5.8) and (5.9). If \( m < 0 \) then

\[ \int_{K} e^{-(n-1)\delta Q(z_j)/2} d\nu(z_j) \leq c_1 e^{-(n-1)\delta m/2} + c_2, \]

where \( c_1, c_2 \) are constants. Choosing \( \bar{M} \) so that

\[ \delta \bar{M} > M - 2c + E^Q(\mu_K, Q) - \delta m, \]

and in view of (6.1), we obtain (6.4).

\[ \square \]

**Remark 6.3.** Step 1. is the only place where strong \( f \)-admissibility is used.

**Step 2.** We prove a weak large deviation upper bound: for \( \mu \in \mathcal{M}(K) \),

\[ \limsup_{\epsilon \to 0} (\limsup_{n \to \infty} 2 \frac{\log \sigma_n(B(\mu, \epsilon))}{n(n+1)}) \leq -\mathcal{I}(\mu). \]

**Proof.** Observe that for \( \mu_n := \frac{1}{n+1} \sum_{j=0}^{n} \delta_{z_j} \) and

\[ k_M(x, y) := \min (M, -\log(||x - y||f(x) - f(y)||) + Q(x) + Q(y), \]

\[ \int \int_{x \neq y} k(x, y) d\mu_n(x) d\mu_n(y) \geq \int \int_{x \neq y} k_M(x, y) d\mu_n(x) d\mu_n(y) \]

\[ = E^Q_M(\mu_n) - \frac{M}{n+1} - \frac{2}{(n+1)^2} \sum_{j=0}^{n} Q(z_j), \]

where for \( \mu \in \mathcal{M}(K) \)

\[ E^Q_M(\mu) := \int_{K} \int_{K} k_M(x, y) d\mu(x) d\mu(y). \]
Thus from (6.3), we have for $\mu_n \in B(\mu, \epsilon)$,
\[
|\text{VDM}_n^Q(z_0, \ldots, z_n)| \leq \exp \left( -\frac{(n+1)^2}{2} \left( \int_{x \neq y} k_M(x, y) d\mu_n(x) d\mu_n(y) \right) \right) = \exp \left( -\frac{(n+1)^2}{2} \left( E_M^Q(\mu_n) - \frac{M}{n+1} - \frac{2}{(n+1)^2} \sum_{j=0}^{n} Q(z_j) \right) \right).
\]

Now we write $Q(z) = Q_M(z) + (Q^M(z) - M)$ where
\[
Q_M(z) := \min(M, Q(z)), \quad Q^M(z) := \max(M, Q(z)),
\]
and observe that
\[
E_M^Q(\mu_n) = E_M^Q(\mu_n) + \frac{2}{n+1} \sum_{j=0}^{n} (Q^M(z_j) - M).
\]

Thus, we get
\[
|\text{VDM}_n^Q(z_0, \ldots, z_n)| \leq \exp \left( -\frac{(n+1)^2}{2} \left( \inf_{\nu \in B_\epsilon} E_M^Q(\nu) + \frac{2n}{(n+1)^2} \sum_{j=0}^{n} (Q^M(z_j) - M) - \frac{(n+3)M}{(n+1)^2} \right) \right),
\]
where in the second sum we have replaced $Q$ with $Q^M$, which is possible since $Q \leq Q^M$.

Next, replacing $E_M^Q(\mu_n)$ by the infimum taken over the ball $B_\epsilon := B(\mu, \epsilon)$, and grouping the two sums, we obtain
\[
|\text{VDM}_n^Q(z_0, \ldots, z_n)| \leq \exp \left( -\frac{(n+1)^2}{2} \left( \inf_{\nu \in B_\epsilon} E_M^Q(\nu) + \frac{2n}{(n+1)^2} \sum_{j=0}^{n} (Q^M(z_j) - M) - \frac{(n+3)M}{(n+1)^2} \right) \right).
\]

Since $\lim_{n \to \infty} Z_n^{2/(n+1)} = \delta^Q(K) = e^{-E^Q(\mu_K, Q)}$, this gives
\[
\limsup_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(B_\epsilon) \leq E^Q(\mu_K, Q) - \inf_{\nu \in B_\epsilon} E_M^Q(\nu) + \limsup_{n \to \infty} \frac{2}{n} \log \int_K e^{-n(Q^M(z) - M)} d\nu(z).
\]

Because of assumptions (5.8) and (5.9), the integrals $\int_K e^{-n(Q^M(z) - M)} d\nu(z)$ are bounded and the limsup on the right-hand side vanishes. Consequently,
\[
\limsup_{\epsilon \to 0} \left( \limsup_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(B_\epsilon) \right) \leq E^Q(\mu_K, Q) - E_M^Q(\mu).
\]

This gives the weak large deviation upper bound upon letting $M \uparrow \infty$. $\square$
Step 3. We prove a large deviation lower bound: for $\mu \in \mathcal{M}(K)$ and $G \subset \mathcal{M}(K)$ open with $\mu \in G$,

$$\liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(G) \geq -\mathcal{I}(\mu).$$

(6.5)

**Proof.** Let $S_\theta$ be defined as in (5.4) and for $R > 0$ let $K_R := [0, R]$. Recall $Z_n$ and $\sigma_n$ have been defined in (4.10) and after (4.17). We also set

$$Z_n^R := \int_{K_R^{n+1}} |VDM_n^Q(z_0, \ldots, z_n)|d\nu(z_0)\ldots d\nu(z_n);$$

and, correspondingly, $\sigma_n^R$. Recall that with $Q_R := Q|_{K_R}$ we know from the compact case that for each such $R$,

$$\lim_{n \to \infty} (Z_n^R)^{2/n(n+1)} = \delta^Q(R)$$

and that

$$\delta^Q(R) = \delta^Q(K)$$

(6.6)

for $R$ sufficiently large, because $\mu_{K,Q}$ has compact support. Indeed, this was true with an $f-$admissible weight $Q$.

To verify (6.5), we first consider measures $\mu \in G$ of compact support. To this end, fix $\mu \in \mathcal{M}(K_R)$ for some $R$ with $S_\theta \subset K_R$. It is easy to see that if $G \subset \mathcal{M}(K)$ is open, then $G \cap \mathcal{M}(K_R) \subset \mathcal{M}(K_R)$ is open in $\mathcal{M}(K_R)$. Thus if $G \subset \mathcal{M}(K)$ is an open neighborhood of $\mu$ in $\mathcal{M}(K)$, we have

$$\sigma_n(G) \geq \sigma_n(G \cap \mathcal{M}(K_R)) = \frac{Z_n^R}{Z_n} \sigma_n^R(G \cap \mathcal{M}(K_R)) \geq (1 + O(e^{-nc}))^{-(n+1)} \sigma_n^R(G \cap \mathcal{M}(K_R))$$

using (5.11) from Lemma 5.6. From the compact case,

$$\liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n^R(G \cap \mathcal{M}(K_R)) \geq E^Q(\mu_{K_R,Q_R}) - E^Q_R(\mu) = E^Q(\mu_{K,Q}) - E^Q_R(\mu)$$

for $R$ sufficiently large using (6.6). Thus

$$\liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(G) \geq E^Q(\mu_{K,Q}) - E^Q_R(\mu).$$

Now $Q(x) \geq 0$ for $x \geq R$ sufficiently large, so $E^Q_R(\mu) \leq E^Q(\mu)$ for such $R$ and we have

$$\liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(G) \geq E^Q(\mu_{K,Q}) - E^Q(\mu).$$

We next verify the lower bound for measures $\mu \in G$ with unbounded support. From Lemma 2.5, we get $\mu_m \to \mu$ weakly and $E^{Q_m}(\mu_m) \to E^Q(\mu)$ where $\mu_m \in \mathcal{M}(K_{R_m})$. Thus given $G \subset \mathcal{M}(K)$ open with $\mu \in G$ for $m$ sufficiently large we have $\mu_m \in G$ and

$$\liminf_{n \to \infty} \frac{2}{n(n+1)} \log \sigma_n(G) \geq E^Q(\mu_{K,Q}) - E^Q(\mu_m).$$

Since $\mu_m \in \mathcal{M}(K_{R_m})$, $E^{Q_m}(\mu_m) = E^Q(\mu_m)$. Letting $m \to \infty$ gives the result. \qed

34
Steps 2. and 3. combined with Corollary D.6 of [1] give that \( \{\sigma_k\} \) satisfy a weak LDP with rate function \( I \) (cf., p. 7 of [12]). Together with 1., Lemma 1.2.18 of [12] then shows that \( \{\sigma_k\} \) satisfies the LDP with good rate function \( I \) (see also Theorem D.4 [1]).

7 Further extensions and remarks

Using the methods of this paper, many results can be extended to ensembles of probability distributions involving products of three or more Vandermonde factors (such distributions are not biorthogonal). Specifically, for \( K \subset \mathbb{C} \) and \( \nu \) a measure on \( K \) we consider probability measures \( \text{Prob}_k \) on \( K^{k+1} \) of the form:

\[
\frac{1}{Z_k} \prod_{s=1}^{m} \prod_{0 \leq i < j \leq k} |z_i^{\theta_s} - z_j^{\theta_s}| e^{-k(Q(z_0) + \cdots + Q(z_k))} d\nu(z_0) \cdots d\nu(z_k) \tag{7.1}
\]

where

1. \( \theta_1, \ldots, \theta_m > 0; \)
2. \( Q \) is an admissible function on \( K; \)
3. \( Z_k \) is a normalizing constant.

The associated energy functional is given by

\[
E^Q(\mu) := -\sum_{s=1}^{m} \int_K \int_K \log |z^{\theta_s} - w^{\theta_s}| d\mu(z) d\mu(w) + 2 \int_K Q(z) d\mu(z).
\]

For \( \theta_1, \ldots, \theta_m \) integers, \( K \) can be any compact subset of the plane. Holding \( k \) of the variables fixed in (7.1) we have an expression of the form

\[
z \to \prod_{i=1}^{s} p_i(z^{\theta_i}) e^{-kQ(z)} \tag{7.2}
\]

where the \( p_i \) are polynomials of degree \( k \). Proposition 4.1 can be replaced by the standard Bernstein-Walsh inequality for polynomials. With appropriate restrictions on \( Q \) and \( \nu \), the results analogous to Corollary 4.8 and Theorem 4.10 hold. For \( K = \mathbb{C} \), again with appropriate restrictions on \( Q \) and \( \nu \), results analogous to Corollary 5.10 and Theorems 5.11 and 6.1 hold. If not all the \( \theta_i \) are integers we restrict to \( K \subset [0, +\infty) \). Holding \( k \) of the variables fixed in the weighted Vandermonde product in (7.1) we have an expression of the form (7.2). Versions of Proposition 4.1 and Theorem 5.2 hold for these functions and results analogous to those of sections 4, 5 and 6 hold with appropriate restrictions on \( Q \) and \( \nu \).
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