Values at non-positive integers of partially twisted multiple zeta-functions I

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Abstract.
We study the behavior of partially twisted multiple zeta-functions. We give new closed and explicit formulas for special values at non-positive integer points of such zeta-functions. Our method is based on a result of M. de Crisenoy on the fully twisted case and the Mellin-Barnes integral formula.

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1 Introduction

Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) be the sets of positive integers, non-negative integers, rational integers, real numbers, and complex numbers, respectively.

Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) and \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \) be two vectors of complex parameters such that \( \Re(\gamma_j) > 0 \) and \( \Re(b_j) > -\Re(\gamma_1) \) for all \( j = 1, \ldots, n \). The generalized Euler-Zagier multiple zeta-function is defined for \( n \)–tuples of complex variables \( s = (s_1, \ldots, s_n) \) by

\[
\zeta_n(s; \gamma; b) := \sum_{m_1 \geq 1}^{\text{m}_2, \ldots, m_n \geq 0} \frac{1}{\prod_{j=1}^{n} (\gamma_1 m_1 + \cdots + \gamma_j m_j + b_j)^{s_j}}. \tag{1}
\]

This series converges absolutely in the domain

\[
\mathcal{D}_n := \{ s = (s_1, \ldots, s_n) \in \mathbb{C}^n \mid \Re(s_j + \cdots + s_n) > n + 1 - j \text{ for all } j = 1, \ldots, n \} \tag{2}
\]

(see [11]), and has the meromorphic continuation to the whole complex space \( \mathbb{C}^n \) whose possible poles are located in the union of the hyperplanes

\[ s_j + \cdots + s_n = (n + 1 - j) - k_j \quad (1 \leq j \leq n, \, k_1, \ldots, k_n \in \mathbb{N}_0). \]
Moreover it is known that for \( n \geq 2 \), the points \( s = -N \), where \( N = (N_1, \ldots, N_n) \in \mathbb{N}^n \), lie in most cases on the singular locus above and are points of indeterminacy. The evaluation of (limit) values of multiple zeta-functions at those points was first considered by S. Akiyama, S. Egami and Y. Tanigawa [1], and then studied by several subsequent papers such as [10], [14], [13] and [5].

In [10], Y. Komori proved that for any \( N = (N_1, \ldots, N_n) \in \mathbb{N}^n \) and \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n \) such that \( \theta_j + \cdots + \theta_n \neq 0 \) for all \( j = 1, \ldots, n \), the limit

\[
\zeta_n^\theta(-N; \gamma; b) := \lim_{t \to 0} \zeta_n(-N + t\theta; \gamma; b)
\] (3)

exists, and expressed this limit in terms of \( N, \theta \) and generalized multiple Bernoulli numbers defined implicitly as coefficients of some multiple series.

In [5], we gave a closed explicit formula for \( \zeta_n^\theta(-N; \gamma; b) \) in terms of \( N, \theta \) and only classical Bernoulli numbers \( B_k \ (k \in \mathbb{N}_0) \) defined by

\[
\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}. \] (4)

Moreover in [6] we extended partially this result to the case of more general multiple zeta-functions defined by

\[
\zeta_n(s, P) = \sum_{m_1, \ldots, m_n \geq 1} \prod_{j=1}^{n} P_j(m_1, \ldots, m_j)^{-s_j}, \] (5)

where \( P = (P_1, \ldots, P_n) \) with certain polynomials \( P_j \in \mathbb{R}[X_1, \ldots, X_j] \). In this general case, instead of Bernoulli numbers, certain period integrals appear in the result.

Now we consider the twisted situation. Let \( \mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \} \), and let \( \mu_k = (\mu_1, \ldots, \mu_k) \in (\mathbb{T} \setminus \{1\})^k \), where \( k \in \{0, \ldots, n\} \).

The natural twisted version of (5) is

\[
\zeta_n(s, P, \mu_k) = \sum_{m_1, \ldots, m_n \geq 1} \prod_{j=1}^{n} P_j(m_1, \ldots, m_j)^{-s_j}. \] (6)

It follows from the method of [3] (see also [4]) that these series have meromorphic continuation to \( \mathbb{C}^n \) for fairly general class of polynomials \( P_j \).

When \( k = n \), that is the “fully twisted” case, this type of multiple series was studied by M. de Crisenoy [2]. Under certain conditions, he proved that \( \zeta_n(s, P, \mu_n) \) is entire, so its behavior is much simpler than the non-twisted case. He obtained an explicit formula for its values at non-positive integer points in terms of Lerch zeta-functions (see Proposition 1 below for the exact statement).

The aim of the present series of papers is to consider the case when \( k < n \). Then \( \zeta_n(s, P, \mu_k) \) is usually not entire, and the complexity of its set of singularities
and therefore the complexity of its special values increases when \( k \) decreases. Our strategy is to begin with the result of de Crisenoy \([2]\) in the case \( k = n \), and first consider the case \( k = n - 1 \) by using the Mellin-Barnes integral formula. Most of the results presented in this paper are actually restricted in this case. However we also try to consider the case \( k = n - 2 \). By the same method it is possible to treat the case \( k \leq n - 3 \) in principle, but the actual argument will become more and more complicated in practice.

In this paper we mainly study the special case when all \( P_j \) are linear polynomials. After reviewing the result of de Crisenoy briefly in the next section, we first state the main results in Section 3. In the case of twisted generalized Euler-Zagier multiple zeta-functions we will give the completely explicit formulas (Theorems 1 and 2), whose proofs are shown in Sections 4 and 5. We also prove the explicit formula in the simplest non-linear situation, the “power sum” case (Theorem 3), which will be proved in Section 6. The treatment of the general non-linear case is postponed to our next paper \([7]\).

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2 Review of de Crisenoy’s result

Here we recall the result of de Crisenoy \([2]\). Let \( P_1, \ldots, P_L, Q \in \mathbb{R}[X_1, \ldots, X_n], \mu_n = (\mu_1, \ldots, \mu_n) \in (\mathbb{T}\setminus\{1\})^n \), and \( s = (s_1, \ldots, s_L) \in \mathbb{C}^L \). We write \( P = (P_1, \ldots, P_L) \). He considered the general multiple series of the form

\[
Z_n(s, P, Q, \mu_n) = \sum_{m_1, \ldots, m_n \geq 1} \frac{(\prod_{j=1}^{n} \mu_j^{m_j}) \, Q(m_1, \ldots, m_n)}{\prod_{\ell=1}^{L} P_\ell(m_1, \ldots, m_n)^{s_\ell}}.
\]

He introduced the condition HDF. A polynomial \( P \in \mathbb{R}[X_1, \ldots, X_n] \) is called HDF if the following (i) and (ii) hold:

(i) \( P(x) > 0 \) for all \( x = (x_1, \ldots, x_n) \in [1, \infty)^n \),

(ii) there exists \( \varepsilon_0 > 0 \) such that

\[
\frac{\partial^\beta P}{P}(x) \ll \prod_{j=1}^{n} x_j^{-\varepsilon_0 \beta_j}
\]

for any \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n \) and \( x = (x_1, \ldots, x_n) \in [1, \infty)^n \). (Or equivalently, if \( \beta_j \geq 1 \) for some \( j \in \{1, \ldots, n\} \), then \( (\partial^\beta P/P)(x) \ll x_j^{-\varepsilon_0} \) for \( x = (x_1, \ldots, x_n) \in [1, \infty)^n \).)
For any $\alpha = (\alpha_1, \ldots, \alpha_L) \in \mathbb{N}_0^L$, we define $a_{k, \alpha} = a_{k, \alpha}(P, Q)$ as the coefficients of the expansion

$$Q(X_1, \ldots, X_n) \prod_{\ell=1}^LP_{\ell}(X_1, \ldots, X_n)^{\alpha_{\ell}} = \sum_{k \in \mathbb{N}_0^n} a_{k, \alpha} X_1^{k_1} \cdots X_n^{k_n}.$$  

Denote by $S(\alpha) = S(\alpha; P, Q)$ the finite subset of $\mathbb{N}_0^n$ consisting of all $k$ for which $a_{k, \alpha} \neq 0$. We write $|x| = |x_1| + \cdots + |x_n|$. Then, de Crisenoy proved the following results.

**Proposition 1.** (de Crisenoy [2]) Assume that the polynomials $P_1, \ldots, P_L$ satisfy the condition HDF, and that $\prod_{\ell=1}^LP_{\ell}(x)$ tends to $\infty$ as $|x| \to \infty$, $x \in [1, \infty)^n$. Then

(i) $Z_n(s, P, Q, \mu_n)$ can be continued to the whole space $\mathbb{C}^L$ as an entire function.

(ii) For any $\alpha = (\alpha_1, \ldots, \alpha_L) \in \mathbb{N}_0^L$, we have

$$Z_n(-\alpha, P, Q, \mu_n) = \sum_{k \in S(\alpha)} a_{k, \alpha} \prod_{j=1}^n \phi_{\mu_j}(-k_j),$$  

(8)

where $\phi_{\mu}(s) = \sum_{m=1}^\infty \mu^m m^{-s}$ is the twisted (Lerch) zeta-function.

It is to be noted here that, since the point $s = -\alpha$ is a regular point of $Z_n(s, P, Q, \mu_n)$ because of the assertion (i), we can evaluate the value at $s = -\alpha$ as a finite definite value in the assertion (ii).

### 3 Statement of results

Our main aim in this paper is to study the partially twisted multiple zeta-functions whose denominators are linear forms. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ be such that $\Re(\gamma_j) > 0$ and $\Re(b_j) > -\Re(\gamma_1)$ for all $j = 1, \ldots, n$.

Let $0 \leq k \leq n$. The partially twisted generalized Euler-Zagier multiple zeta-function is defined formally for $n$–tuples of complex variables $s = (s_1, \ldots, s_n)$ by

$$\zeta_{n,k}(s, \gamma, b, \mu_k) = \sum_{m_1 \geq 1} \cdots \sum_{m_n \geq 0} \prod_{j=1}^k \mu_j^{m_j} \prod_{j=1}^n (\gamma_1 m_1 + \cdots + \gamma_j m_j + b_j)^{s_j}$$  

(9)

(when $k = 0$, we understand that the numerator on the right-hand side is 1) which is absolutely convergent in the domain $\mathcal{D}_n$ (see [2]). The meromorphic continuation and the location of singularities of the function $\zeta_{n,k}(s, \gamma, b, \mu_k)$ are discussed in [2] (which are partly announced in [8]).

When $k = n$, this series is a special case of (7) studied by de Crisenoy [2], whose result implies that $\zeta_{n,n}(s, \gamma, b, \mu_n)$ is entire in $s$.  

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When $k < n$, $\zeta_{n,k}(s, \gamma, b, \mu_k)$ has meromorphic continuation to $\mathbb{C}^n$, but is not entire. Moreover, the complexity of its set of singularities and therefore the complexity of its special values, increases when $k$ decreases. In our previous article \cite{5} we handled the case $k = 0$ (that is, the non-twisted case) by a method different from that in \cite{2}.

In the present paper we develop another approach. Our following two theorems (i.e. Theorem 1 and Theorem 2 proved in Section 4 and Section 5 respectively) deal with the cases $k = n - 1$ and $k = n - 2$. In these cases we use, in addition to de Crisenoy’s result (Proposition 1 above), the Mellin-Barnes formula to determine the set of singularities and the values of $\zeta_{n,k}(s, \gamma, b, \mu_k)$ at non-positive integers.

We prepare some more notations.

1. For any $a \in \mathbb{C} \setminus (-N_0)$, let $\zeta(s, a) = \sum_{m=0}^{\infty} (m + a)^{-s}$ be the Hurwitz zeta-function (as for the definition of $\zeta(s, a)$ for any $a \in \mathbb{C} \setminus (-N_0)$, see \cite{12} Lemma 1);

2. For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, and $k < n$, denote $s_k = (s_1, \ldots, s_k)$. Similarly we use the notation $\gamma_k, b_k, \mu_k$ etc.

3. For any $N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$ and any $l \in \mathbb{Z}$, let
\[ N_{n-1}^*(l) = (N_1, \ldots, N_{n-2}, N_{n-1} + N_n + l). \]

4. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ we define the polynomial (in $b$) $c_n(b; \alpha, k)$ (where $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n, |k| \leq |\alpha|$) as the coefficients of the polynomial $\prod_{j=1}^{n} (\sum_{i=1}^{j} X_i + b_j)^{\alpha_j}$; that is
\[ \prod_{j=1}^{n} (\sum_{i=1}^{j} X_i + b_j)^{\alpha_j} = \sum_{k \in \mathbb{N}_0^n, \ |k| \leq |\alpha|} c_n(b; \alpha, k) X_1^{k_1} \ldots X_n^{k_n}. \quad (10) \]

5. Similarly, for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\tilde{c}_n(b; \alpha, k)$ are defined by
\[ \prod_{j=1}^{n} (\sum_{i=1}^{j} \gamma_i X_i + b_j)^{\alpha_j} = \sum_{k \in \mathbb{N}_0^n, \ |k| \leq |\alpha|} \tilde{c}_n(b; \alpha, k) X_1^{k_1} \ldots X_n^{k_n}. \quad (11) \]

Remark 1. The quantities $c_n(b; \alpha, k)$ and $\tilde{c}_n(b; \alpha, k)$ appear also in \cite{4}. Obviously $\tilde{c}_n(b; \alpha, k) = c_n(b; \alpha, k) \gamma_1^{k_1} \ldots \gamma_n^{k_n}$.

Theorem 1. Let $n \geq 2$. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ with $\Re(\gamma_j) > 0$ for all $j = 1, \ldots, n$. Let $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$, satisfying the conditions
\[ \begin{cases} \Re(b_j) > -\Re(\gamma_1) \text{ (for all } j = 1, \ldots, n), \\ b_n - b_{n-1} \notin (-\infty, 0], \ (b_n - b_{n-1})/\gamma_n \notin (-\infty, 0]. \end{cases} \quad (12) \]
Theorem 2. Let \( \mu_{n-1} = (\mu_1, \ldots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1} \). Then, the series \( \zeta_{n,n-1}(s, \gamma, b, \mu_{n-1}) \) has meromorphic continuation to the whole space \( \mathbb{C}^n \) and its possible poles are located only on the hyperplane \( s_n = 1 \). Furthermore, for any \( N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n \), we have

\[
\zeta_{n,n-1}(-N, \gamma, b, \mu_{n-1}) = -\frac{1}{N_n + 1} \sum_{k \in \mathbb{N}_0^n} c_{n-1}(b'_{n-1}; N_{n-1}^*)(1) \gamma_{n-1}^1 \prod_{j=1}^{n-1} \gamma_j^{k_j} \phi_{j}(-k_j) \\
+ \sum_{l=0}^{N_n} \binom{N_n}{l} \sum_{k \in \mathbb{N}_0^n \setminus \{(l)\}} c_{n-1}(b'_{n-1}; N_{n-1}^*(-l), k) \gamma_{n}^l \prod_{j=1}^{n-1} \gamma_j^{k_j} \phi_{j}(-k_j) \\
\times \zeta \left( -l; \frac{b_{n} - b_{n-1}}{\gamma_n} \right),
\]

where \( b'_{n-1} = (b_1, b_2, \ldots, b_{n-1}) \) with \( b_j' = b_j - (\gamma_2 + \cdots + \gamma_j) \) (2 \( \leq j \leq n-1 \)) and \( c_{n-1}(b'_{n-1}; N_{n-1}^*(-l), k) \) is defined as in \( \text{[10]} \).

Remark 2. Moreover, the special values of Hurwitz and Lerch zeta-functions appearing on the right-hand side can be written down more explicitly. In fact, it is well-known that \( \zeta(-n, a) = -B_{n+1}(a)/(n+1) \), where \( B_{n+1}(a) \) denotes the Bernoulli polynomial of order \( n + 1 \). As for \( \phi\mu(-n) \), we have

\[
\phi\mu(-n) = \frac{(-1)^n \mu}{1 - \mu} \sum_{\ell=0}^{n} \ell! S(n, \ell). 
\]

where \( S(n, \ell) \) denotes the Stirling number of the second kind attached to \( (n, \ell) \) (see de Crisenoy \[2 \] Lemma 5.7).

Theorem 2. Let \( n \geq 2 \). Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) with \( \Re(\gamma_j) > 0 \) for all \( j = 1, \ldots, n \), and and \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \) satisfying \( \text{[12]} \). Let \( \mu_{n-2} = (\mu_1, \ldots, \mu_{n-2}) \in (\mathbb{T} \setminus \{1\})^{n-2} \). Then, the series \( \zeta_{n,n-2}(s, \gamma, b, \mu_{n-2}) \) has meromorphic continuation to the whole space \( \mathbb{C}^n \) and its possible singularities are located only on the hyperplanes \( s_n = 1 \) and \( s_{n-1} + s_n = k \) \((k \in \mathbb{Z}, k \leq 2) \).

Furthermore, for any \( N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n \), as \( \delta = (\delta_1, \ldots, \delta_n) \) tends to \((0, \ldots, 0)\), we have

\[
\zeta_{n,n-2}(-N + \delta, \gamma, b, \mu_{n-2}) = -\frac{1}{N_n + 1} \zeta_{n-1,n-2}(-N_{n-1}^*(1), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_{n-1}^{-1} 
\]
+ \sum_{l=0}^{N_n} \left( \begin{array}{c} N_n \\ l \end{array} \right) \zeta_{n-1,n-2}((-N^*_n-1(-l)), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_l \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \\
+ \frac{(-1)^{N_n-1} N_n! N_{n-1}!}{(N_{n-1} + N_n + 1)!} \left( \frac{\delta_n + O(\delta_n^2)}{\delta_{n-1} + \delta_n} \right) \\
\times \zeta_{n-2,n-2}((-N_{n-2} + \delta_{n-2}), \gamma_{n-2}, b_{n-2}, \mu_{n-2}) \\
\times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_{n-1} \gamma_{n-1} + N_n + 1 \\
+ O \left( \max_{1 \leq j \leq n} |\delta_j| \right), \\
where, when \( n = 2 \), we understand that \( \zeta_{0,0} = 1 \).

Remark 3. When \( n = 2 \), Theorem 2 gives the result on the (non-twisted) double zeta-function, which coincides with Corollary 5.2.

As a corollary, we obtain the following result:

Corollary 1. Assume that the assumptions of Theorem 2 hold. Let \( N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n \) and \( \theta \in \mathbb{C} \). Then, the limit

\[
\zeta^\theta_{n,n-2}(-N, \gamma, b, \mu_{n-2}) := \lim_{\delta \to 0, \frac{b_n}{\delta_{n-1} + \delta_n} \to \theta} \zeta_{n,n-2}(-N + \delta, \gamma, b, \mu_{n-2})
\]

exists and is given by

\[
\zeta^\theta_{n,n-2}(-N, \gamma, b, \mu_{n-2}) = -\frac{1}{N_n + 1} \zeta_{n-1,n-2}((-N^*_n-1(1)), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_{n-1}^{-1} \\
+ \sum_{l=0}^{N_n} \left( \begin{array}{c} N_n \\ l \end{array} \right) \zeta_{n-1,n-2}((-N^*_n-1(-l)), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_l \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \\
+ \frac{(-1)^{N_n-1} N_n! N_{n-1}!}{(N_{n-1} + N_n + 1)!} \zeta_{n-2,n-2}((-N_{n-2} + \delta_{n-2}), \gamma_{n-2}, b_{n-2}, \mu_{n-2}) \gamma_{n-2}^{-1} \gamma_{n-1}^{-1} \gamma_{n-1} + N_n + 1 \\
\times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_{n-1} \gamma_{n-1} + N_n + 1.
\]

Moreover, on the right-hand side, we may apply Theorem 2 to the \( \zeta_{n-1,n-2} \) factors and Proposition 7 to the \( \zeta_{n-2,n-2} \) factor, to obtain a more explicit expression of \( \zeta^\theta_{n,n-2}(-N, \gamma, b, \mu_{n-2}) \).

The argument to prove Theorem 2 can be extended to the case \( k \leq n - 3 \), to obtain the same type of explicit formulas. However, for smaller values of \( k \), more and
more relevant singularities will appear, so the description of indeterminacy will be much more complicated.

By the method in the present paper, it is possible to study the behavior of multiple zeta-functions of more general form \( (6) \), whose denominators are not necessarily linear forms. The general treatment will be developed in our next paper \[7 \], but here, we discuss the following special type of non-linear forms.

Let \( h = (h_1, \ldots, h_n) \in \mathbb{N}^n \), and define

\[
\zeta_{n,k}(s, h, \gamma, b, \mu_k) = \sum_{m_1, \ldots, m_n \geq 1} \frac{\prod_{j=1}^{k} \mu_j^{m_j}}{\prod_{j=1}^{n}(\gamma_1 m_1^{h_1} + \cdots + \gamma_j m_j^{h_j} + b_j)^{s_j}}.
\]

(14)

Analogous to \( (11) \), we define \( \tilde{c}_n(b; h, \alpha, k) \) by

\[
\prod_{j=1}^{n} \left( \sum_{i=1}^{j} \gamma_i X_i^{h_i} + b_j \right)^{\alpha_j} = \sum_{k \in \mathbb{N}_0^n} \tilde{c}_n(b; h, \alpha, k) X_1^{k_1} \cdots X_n^{k_n}.
\]

(15)

Note that the sum on the right-hand side is actually a finite sum. As in Section 2, we denote by \( S(\alpha) \) the set of all \( k \) such that \( \tilde{c}_n(b; h, \alpha, k) \neq 0 \). Using this notation, we can formulate our third main result as follows.

**Theorem 3.** Under the same assumptions as in Theorem 1, we have

\[
\zeta_{n,n-1}(-N, h, \gamma, b, \mu_{n-1}) = -\frac{\delta_{1,h_n}}{N_n + 1} \sum_{k \in S(N_{n-1}^*)} \tilde{c}_{n-1}(b_{n-1}; h_{n-1}, N_{n-1}^*(1), k) \gamma_{n-1} \prod_{j=1}^{n-1} \phi_{\mu_j}(-k_j)
\]

\[
+ \sum_{l=0}^{N_n} \left( \frac{N_n}{l} \right) \sum_{k \in S(N_{n-1}^*(-l))} \tilde{c}_{n-1}(b_{n-1}; h_{n-1}, N_{n-1}^*(-l), k) \gamma_n^l \left( \prod_{j=1}^{n-1} \phi_{\mu_j}(-k_j) \right) \times \zeta\left(-l, h_n, \frac{b_n - b_{n-1}}{\gamma_n}\right),
\]

where \( \delta_{1,h_n} \) denotes the Kronecker delta.

4 Proof of Theorem 1

Now we start the proof of Theorem 1.

Let \( n \geq 2 \), and fix \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n \) and \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \) such that \( \Re(\gamma_j) > 0 \) and \( \Re(b_j) > -\Re(\gamma_1) \) for all \( j = 1, \ldots, n \) and \( (b_n - b_{n-1})/\gamma_n \notin (-\infty, 0] \). Fix also \( \mu_{n-1} = (\mu_1, \ldots, \mu_{n-1}) \in (\mathbb{T} \setminus \{1\})^{n-1} \).
The zeta function
\[
\zeta_{n, n}(s, \gamma, b, \mu_{n-1}) = \sum_{m_1 \geq 1}^{m_1 \geq 1, m_2, \ldots, m_n \geq 0} \frac{\prod_{j=1}^{n-1} \mu_j^{m_j}}{\prod_{j=1}^{n} (\gamma_1 m_1 + \cdots + \gamma_j m_j + b_j)^s},
\]
is absolutely convergent (see [11]) in the region \(D_n\), hence especially in its subregion

\[\mathcal{A}_n = \{ s \in \mathbb{C}^n | \Re s_j > 1 (1 \leq j \leq n) \}.\]

Recall the Mellin-Barnes integral formula:
\[
(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s + z)\Gamma(-z)}{\Gamma(s)} \lambda^z dz,
\]
where \(s, \lambda \in \mathbb{C}, \Re s > 0, \lambda \neq 0, |\arg \lambda| < \pi\) (the principal branch), \(-\Re s < c < 0\), and the path of the integral is the vertical line \(\Re z = c\) (see [15]).

Here we assume temporarily that \(s \in \mathcal{A}_n\) and
\[
\Re (b_n - b_{n-1}) > 0.
\]

Our starting point is the decomposition
\[
(\gamma_1 m_1 + \cdots + \gamma_n m_n + b_n)^{-s_n}
= (\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1})^{-s_n}
\times \left(1 + \frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1}}\right)^{-s_n}.
\]

Under the assumption (18) we see that
\[
|\arg \left(\frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1}}\right)| < \pi,
\]
hence the above decomposition (19) is valid, and using (17) we obtain
\[
(\gamma_1 m_1 + \cdots + \gamma_n m_n + b_n)^{-s_n}
= (\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1})^{-s_n}
\times \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \left(\frac{\gamma_n m_n + b_n - b_{n-1}}{\gamma_1 m_1 + \cdots + \gamma_{n-1} m_{n-1} + b_{n-1}}\right)^z dz,
\]
where \(-\Re s_n < c < 0\). But since \(s \in \mathcal{A}_n\), we have \(\Re s_n > 1\), so we may assume (more strongly)
\[
-\Re s_n < c < -1.
\]
Substituting (20) into (9) (with \( k = n - 1 \)) and changing the order of integration and summation, we have

\[
\zeta_{n-1}(s, \gamma, b, \mu_{n-1})
= \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \sum_{m_1 \geq 1} \frac{\prod_{l=1}^{n-1} \mu_l^m}{\prod_{j=1}^n (\gamma_1 m_1 + \cdots + \gamma_j m_j + b_j)^{s_j}} \times (\gamma_1 m_1 + \cdots + \gamma_n m_n + b_{n-1})^{-s_n - z} (\gamma_n m_n + b_n - b_{n-1})^{s_n} dz
\]

\[
\times \zeta(-z, b_n - b_{n-1})^{-s_n - z} \sum_{n \geq 1} (s_{n-1}(z), \gamma_{n-1}, b_{n-1}, \mu_{n-1})
\times \zeta(-z, b_n - b_{n-1})^{-s_n - z} \gamma_n^z dz,
\]

where \( s_{n-1}(z) = (s_1, \ldots, s_{n-2}, s_{n-1} + s_n + z) \). (Under the assumption (21), both of the above two zeta factors in the integrand are convergent.)

Let \( M \) be a positive integer, and now we shift the path of integration to \( \Re z = M + 1/2 \). We claim that this shifting is possible, and also we can remove the assumption (18). In fact, in the strip \( c \leq \Re z \leq M + 1/2 \), by Stirling’s formula we have

\[
\Gamma(s_n + z)\Gamma(-z) \ll e^{-\pi(|3s_n|/2+|3z|)}(|3s_n| + |3z| + 1)^{R_{s_n} + R_z - 1/2}(|3z| + 1)^{-R_z - 1/2}.
\]

The factor \( \zeta_{n-1,n-1}(s_{n-1}(z), \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \) is \( O(1) \) for any \( b_{n-1} \) satisfying (12), because it is in the domain of absolute convergence. As we mentioned in Section 3 the Hurwitz zeta-function \( \zeta(s, a) \) can be defined for any complex \( a \) except for the case when \( a = -l, l \in \mathbb{N}_0 \). Moreover it holds that

\[
\zeta(s, a/w) w^{-s} = O \left( |w|^{-(1-R_s)} (|3s| + 1)^{\max\{0,1-R_s\}+\varepsilon} \exp(|3s| \max\{|\arg a|, |\arg w|\}) \right)
\]

if \( a/w \notin (-\infty, 0] \) (see [12, Lemma 2]). Therefore, under the assumption \( (b_n - b_{n-1})/\gamma_n \notin (-\infty, 0] \), we have

\[
\zeta(-z, b_n - b_{n-1})^{-s_n - z} \ll (|3z| + 1)^{\max\{0,1+R_z\}+\varepsilon} \exp(|3z| \max\{|\arg(b_n - b_{n-1})|, |\arg \gamma_n|\}).
\]

These estimates imply that the integrand on the right-hand side of (22) is

\[
\ll (\text{The factor of polynomial order in } |3z|) \times \exp(|3z| \max\{|\arg(b_n - b_{n-1})|, |\arg \gamma_n|\} - \pi)
\]

(here, the implied constant may depend on \( s_n \)). Therefore, if we further assume \( b_n - b_{n-1} \notin (-\infty, 0] \), we see that the integrand is of exponential decay. This implies
that, only under the assumption (12), the integral is absolutely convergent, and the indicated shifting of the path of integral is possible. The assumption (18) is not necessary (or in other words, we can continue (22) with respect to $b$ to the wider region given by (12)). The proof of the claim is complete.

Carrying out this shifting, we find that the relevant poles are $z = -1$ (from the Hurwitz zeta factor) and $z = 0, 1, 2, \ldots, M$ (from $\Gamma(-z)$). Counting the residues, we obtain

$$
\zeta_{n,n-1}(s, \gamma, b, \mu_{n-1})
= \frac{1}{s_n - 1} \zeta_{n-1,n-1}(s_{n-1}^*(1), \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1}
+ \sum_{l=0}^{M} \binom{-s_n}{l} \zeta_{n-1,n-1}(s_{n-1}^*(l), \gamma_{n-1}, b_{n-1}, \mu_{n-1})
\times \zeta(-l, \frac{b_n - b_{n-1}}{\gamma_n}, \frac{b_{n-1}}{\gamma_n})
+ \frac{1}{2 \pi i} \int_{(M+1/2)} \frac{\Gamma(s_n + z) \Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(s_{n-1}^*(z), \gamma_{n-1}, b_{n-1}, \mu_{n-1})
\times \zeta(-z, \frac{b_n - b_{n-1}}{\gamma_n}, \frac{b_{n-1}}{\gamma_n}) \gamma_n dz.
$$

Since $\zeta_{n-1,n-1}$ is entire, the poles (in $z$) of the integrand of the above integral are $z = -1, 0, 1, 2, \ldots$ and $z = -s_n, -s_n - 1, -s_n - 2, \ldots$. Therefore the above integral can be continued holomorphically to the region satisfying $\Re(-s_n) < M + 1/2$, that is,

$$\{ s \in \mathbb{C}^n \mid \Re s_n > -M - 1/2 \}.$$ 

Since $M$ is arbitrary, we can show from (24) that $\zeta_{n,n-1}(s, \gamma, b, \mu_{n-1})$ can be continued meromorphically to the whole space $\mathbb{C}^n$. Moreover, again noting that $\zeta_{n-1,n-1}$ is entire, we find that the only possible singularity is the hyperplane $s_n = 1$.

Let $N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n$. Then $s = -N$ is a regular point of the function $\zeta_{n,n-1}(s, \gamma_n, b_n, \mu_{n-1})$.

Put $s = -N$ on (24). Then the integral is equal to 0, because of the factor $\Gamma(s_n)$ on the denominator. Also, when $l > N_n$, then the binomial coefficient $\binom{N_n}{l}$ is equal to 0. (We may assume that $M$ is sufficiently large, satisfying $M > N_n$.) Noting $s_{n-1}^*(l)|_{s=-N} = -N_{n-1}^*(-l)$, we obtain the following explicit formula:

$$
\zeta_{n,n-1}(-N, \gamma, b, \mu_{n-1})
= -\frac{1}{N_n + 1} \zeta_{n-1,n-1}(-N_{n-1}^*(1), \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1}
+ \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1,n-1}(-N_{n-1}^*(-l), \gamma_{n-1}, b_{n-1}, \mu_{n-1})
$$

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The special values \( \zeta_{n-1, n-1}( -k_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1} ) \) (where \( k_{n-1} = (k_1, \ldots, k_{n-1}) \in N_0^{n-1} \)) are evaluated explicitly by Proposition \( \text{[1]} \) in terms of special values of the Lerch zeta-function \( \phi_{\mu_j}(s) \). Since

\[
-N_{n-1} - N_n + l \leq -N_{n-1} \leq 0
\]

for \( l \leq N_n \), we can apply Proposition \( \text{[1]} \) to the factors \( \zeta_{n-1, n-1} \) appearing on the right-hand side of the above.

Let \( b'_1 = b_1, b'_j = b_j - (\gamma_2 + \cdots + \gamma_j) \) (\( 2 \leq j \leq n - 1 \)). Then we can write

\[
\zeta_{n-1, n-1}(s_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) = \sum_{m_1, \ldots, m_{n-1} \geq 1} \prod_{j=1}^{n-1} \mu_j^{m_j} \left( \gamma_1 m_1 + \cdots + \gamma_j m_j + b'_j s_j \right),
\]

which agrees with the notation of Proposition \( \text{[1]} \). Since Proposition \( \text{[1]} \) is proved for polynomials of real coefficients, here we temporarily assume that \( \gamma_j, b_j \in \mathbb{R} \) (\( 1 \leq j \leq n \)). Then the HDF condition is clearly satisfied, and by Proposition \( \text{[1]} \) we have

\[
\zeta_{n-1, n-1}( -N_{n-1}^{-s}( -l), \gamma_{n-1}, b_{n-1}, \mu_{n-1} ) = \sum_{k \in [0]^{n-1}_{\mathbb{N}}} \bar{\zeta}_{n-1}(b'_{n-1}; N_{n-1}^{-s}( -l), k) \prod_{j=1}^{n-1} \phi_{\mu_j}( -k_j ) \quad (l \geq -1),
\]

where \( \bar{\zeta}_{n-1}(b'_{n-1}; N_{n-1}^{-s}( -l), k) \) is that defined by \( \text{[1]} \). Applying this to the right-hand side of \( \text{(25)} \), and noting Remark \( \text{[1]} \) we obtain the assertion of Theorem \( \text{[1]} \). The restriction \( \gamma_j, b_j \in \mathbb{R} \) can be removed by the analytic continuation with respect to \( \gamma_j, b_j \).

**Remark 4.** For \( \mu \in (\mathbb{T} \setminus \{1\})^n \), we can apply the same argument as above to \( \zeta_{n, n}(s, \gamma, b, \mu) \).

The result is that the special values \( \zeta_{n, n}( -N, \gamma, b, \mu ) \) can be written in terms of \( \zeta_{n-1, n-1}( -N_{n-1}^{-s}( -l), \gamma_{n-1}, b_{n-1}, \mu_{n-1} ) \) and special values of the Hurwitz-Lerch zeta-function

\[
\phi \left( s, \frac{b_n - b_{n-1}}{\gamma_n}, \mu_n \right) = \sum_{m=0}^{\infty} \mu_n^m \left( m + \frac{b_n - b_{n-1}}{\gamma_n} \right)^{-s}.
\]

This gives another way of computing the special values by induction.

## 5 Proof of Theorem [2]

First, as a preparation, we consider the behavior of \( \zeta_{n, n-1} \) around its singularity \( s_n = 1 \). We will use in the sequel of this section the notations of Section \( \text{[1]} \).
Let \( n \geq 2 \). Let \( s_n = 1 + \delta_n \), where \( \delta_n \) is a small (non-zero) complex number. Then

\[
\frac{1}{s_n} \zeta_{n-1,n-1}(s_{n-1}^*, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1} = \frac{1}{\delta_n} \zeta_{n-1,n-1}(s_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1} + \frac{\partial}{\partial s_{n-1}} \zeta_{n-1,n-1}(s_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1} + O(\delta_n),
\]

so from (26) we have

\[
\zeta_{n-1,n-1}(s_1, \ldots, s_{n-1}, 1 + \delta_n, \gamma_n, b_n, \mu_{n-1}) = \frac{1}{\delta_n} \zeta_{n-1,n-1}(s_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1} + O(\delta_n),
\]

say. So far we have worked under the assumption \( n \geq 2 \). However when \( n = 1 \), we see that

\[
\zeta_{1,0}(1 + \delta_1, \gamma_1, b_1, \mu_0) = \sum_{m_1=1}^{\infty} (\gamma_1 m_1 + b_1)^{-1-\delta_1} = \gamma_1^{-1-\delta_1} \zeta(1 + \delta_1, \gamma_1/b_1) - b_1^{-1-\delta_1} = \frac{1}{\delta_1^{1-\delta_1}} + (\text{constant}) + O(|\delta_1|),
\]

so (26) is valid also for \( n = 1 \) with the convention \( \zeta_{0,0} = 1 \).
Now we start the proof of Theorem 2. Let $n \geq 2$. Fix $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$ such that $\Re(\gamma_j) > 0$ and $\Re(b_j) > -\Re(\gamma_1)$ for all $j = 1, \ldots, n$ and $(b_n - b_{n-1})/\gamma_n \notin (-\infty, 0]$. Fix also $\mu_{n-2} = (\mu_1, \ldots, \mu_{n-2}) \in (\mathbb{T} \setminus \{1\})^{n-2}$.

Assume $s \in A_n$. Analogous to (22), this time we obtain

$$\zeta_{n-2}(s, \gamma, b, \mu_{n-2}) = \frac{1}{2\pi i} \int \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1, n-2}(s_{n-1}^*(z), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \times \zeta\left(-z, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^s dz,$$

where $-\Re s_n < c < -1$. The factor

$$\zeta_{n-1, n-2}(s_{n-1}^*(z), \gamma_{n-1}, b_{n-1}, \mu_{n-2})$$

is not entire, but its pole $s_{n-1} + s_n + z = 1$, that is, $z = 1 - s_{n-1} - s_n$ is irrelevant when we shift the path from $\Re z = c$ to $\Re z = M + 1/2$, because $\Re(1 - s_{n-1} - s_n) < -\Re s_n < c$. Therefore, analogous to (24), we have

$$\zeta_{n-2}(s, \gamma, b, \mu_{n-2}) = \frac{1}{s_n - 1} \zeta_{n-1, n-2}(s_{n-1}^*(-1), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_n^{-1} + \sum_{l=0}^{M} \left(-s_n\right) \zeta_{n-1, n-2}(s_{n-1}^*(l), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \times \zeta\left(-l, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^l + \frac{1}{2\pi i} \int_{(M+1/2)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1, n-2}(s_{n-1}^*(z), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \times \zeta\left(-z, \frac{b_n - b_{n-1}}{\gamma_n}\right) \gamma_n^s dz.$$

Here, the (unique) singularity of $\zeta_{n-1, n-2}(s_{n-1}^*(l), \gamma_{n-1}, b_{n-1}, \mu_{n-2})$ is $s_{n-1} + s_n = 1 - l$ ($l = -1, 0, 1, 2, \ldots, M$). Letting $M \to \infty$ we obtain the meromorphic continuation of $\zeta_{n-2}(s, \gamma, b, \mu_{n-2})$, and its (possible) singularities are

$$\begin{cases} s_n = 1, \\ s_{n-1} + s_n = 2, 1, 0, -1, -2, \ldots \end{cases}$$

(29)

Now we want to evaluate the value of $\zeta_{n-2}(s, \gamma, b, \mu_{n-2})$ at $s = -N \in -\mathbb{N}_0^n$. The above (29) shows that $s = -N$ can be on a singular locus.
Let \( \delta = (\delta_1, \ldots, \delta_n) \), where \( \delta_j \)s are small (non-zero) complex numbers, and observe the right-hand side of (28) with \( s = -N + \delta \). Since

\[-N_{n-1}^*(-l) = (-N_1, \ldots, -N_{n-2}, -N_{n-1} - N_n + l),\]

the only relevant singularity of \( \zeta_{n-1,n-2} \) factor appears when \( l = N_{n-1} + N_n + 1 \). (We may assume \( M > N_{n-1} + N_n + 1 \).) Analogous to (25), we have

\[
\zeta_{n-1,n-2}(-N + \delta, \gamma, b_n, \mu_{n-2}) = \frac{1}{N_n + 1} \zeta_{n-1,n-2}(-N_{n-1}^*(1), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \gamma_n^{-1} + \sum_{l=0}^{N_n} \binom{N_n}{l} \zeta_{n-1,n-2}(-N_{n-1}^*(-l), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \times \zeta \left( -l, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l R(\delta) + O \left( \max_{1 \leq j \leq n} |\delta_j| \right),
\]

where \( R(\delta) \) denotes the contribution coming from the term \( l = N_{n-1} + N_n + 1 \). Using (26), we can evaluate \( R(\delta) \) as follows:

\[
R(\delta) = \left( \frac{N_n - \delta_n}{N_{n-1} + N_n + 1} \right) \times \zeta_{n-1,n-2}((-N_1 + \delta_1, \ldots, -N_{n-2} + \delta_{n-2}, 1 + \delta_{n-1} + \delta_n), \gamma_{n-1}, b_{n-1}, \mu_{n-2}) \times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^{N_n-1+1} = \frac{(N_n - \delta_n)(N_n - 1 - \delta_n) \cdots (-\delta_n) \cdots (N_{n-1} - \delta_n)}{(N_{n-1} + N_n + 1)!} \times \left\{ \frac{1}{\delta_{n-1} + \delta_n} \zeta_{n-2,n-2}(-N_{n-2} + \delta_{n-2}, \gamma_{n-2}, b_{n-2}, \mu_{n-2}) \gamma_{n-1}^{-1} + B(-N_{n-2} + \delta_{n-2}, \gamma_{n-2}, b_{n-2}, \mu_{n-2}) + O(|\delta_{n-1} + \delta_n|) \right\} \times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^{N_n-1+1} = \frac{(N_n - \delta_n)(N_n - 1 - \delta_n) \cdots (-\delta_n) \cdots (N_{n-1} - \delta_n)}{(N_{n-1} + N_n + 1)!(\delta_{n-1} + \delta_n)} \times \zeta_{n-2,n-2}(-N_{n-2} + \delta_{n-2}, \gamma_{n-2}, b_{n-2}, \mu_{n-2}) \times \zeta \left( -N_{n-1} - N_n - 1, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_{n-1}^{-1} \gamma_n^{N_n-1+N_n+1} + O(|\delta_n|),
\]

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where \( B(\cdot) \) is defined in (26). This formula describes the situation of indeterminacy. We may understand the behavior of \( \zeta_{n,n-2} \) around the point \( s = -N \) from (30) and (31). This ends the proof of Theorem 2.

6 The power sum case

In this final section we prove Theorem 3. The series \( \zeta_{n,k}(s, h, \gamma, b, \mu_k) \), defined by (14), is an obvious generalization of \( \zeta_{n,k}(s, \gamma, b, \mu_k) \) (with a slight change of the condition of the summation), and hence it can be treated in a quite similar way as in the linear case.

First assume \( s \in A_n \). The analogue of (22) is

\[
\zeta_{n-1,n-1}(s, h, \gamma, b, \mu_{n-1}) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(s_{n-1}(z), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \times \zeta(-z, h_n, \frac{b_n - b_{n-1}}{\gamma_n}) \gamma_n^z dz,
\]

where \( -\Re s_n < c < -1 \) and

\[
\zeta(s, h, b) = \sum_{m=0}^{\infty} \frac{1}{(m^h + b)^s} \quad (h \in \mathbb{N}, b \in \mathbb{C}, |\arg b| < \pi).
\]

The analytic properties of \( \zeta(s, h, b) \) can also be studied by using the Mellin-Barnes formula.

**Lemma 1.** The series \( \zeta(s, h, b) \) can be continued meromorphically to the whole complex plane. When \( h = 1 \) (the case of the Hurwitz zeta-function), it has only one pole at \( s = 1 \), while when \( h \geq 2 \), it has infinitely many poles \( s = -l + h^{-1} \) (\( l \in \mathbb{N}_0 \)).

**Proof.** First assume \( \Re s > 1 \). Using the Mellin-Barnes formula (17) we have

\[
\zeta(s, h, b) = b^{-s} + \sum_{m=1}^{\infty} m^{-hs}(1 + b/m^h)^{-s} = b^{-s} + \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s + z)\Gamma(-z)}{\Gamma(s)} \left( \frac{b}{m^h} \right)^z dz
\]

\((-\Re s < c_1 < 0), which is, after changing the order of integration and summation,

\[
= b^{-s} + \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s + z)\Gamma(-z)}{\Gamma(s)} \zeta(h(s + z))b^z dz.
\]
To assure the convergence of \( \zeta(h(s + z)) \), we have to choose \( c_1 \) satisfying \( h^{-1} - \Re s < c_1 < 0 \). Now, shift the path to \( \Re z = M + 1/2 \) (which is possible because \( |\arg b| < \pi \)), and count the residues of relevant poles at \( z = 0, 1, 2, \ldots, M \). We obtain

\[
\zeta(s, h, b) = b^{-s} + \sum_{l=0}^{M} \left( \frac{-s}{l} \right) \zeta(h(s + l)) b^l + \frac{1}{2\pi i} \int_{(M+1/2)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \zeta(h(s + z)) b^z dz.
\] (36)

Considering the situation \( M \to \infty \), we find that (36) gives the meromorphic continuation of \( \zeta(s, h, b) \) to the whole plane. The Riemann zeta factor in the sum on the right-hand side gives the poles (of order at most 1) at \( s = -l + h^{-1} \) \((l \in \mathbb{N}_0)\).

When \( h = 1 \), the poles \( s = -l + h^{-1} = -l + 1 \) are cancelled with the binomial factor for \( l \geq 1 \), so the only pole is \( s = 1 \). This is of course the case of the Hurwitz zeta-function. When \( h \geq 2 \), all of \( s = -l + h^{-1} \) \((l \in \mathbb{N}_0)\) are really poles. The residue at \( s = -l + h^{-1} \) is

\[
\frac{1}{h} \left( l - h^{-1} \right) b^l.
\] (37)

\( \square \)

Now, using (36) we evaluate \( \zeta(s, h, b) \) for any fixed \( s \in \mathbb{C} \). Choose \( M \) so large as \( \Re(h(s + z)) > 1 \) for \( \Re z = M + 1/2 \). Denote the integral on the right-hand side of (36) by \( J(M) \). Then, putting \( s = \sigma + it \) and \( z = M + 1/2 + iy \), we see that

\[
J(M) \ll e^{\pi |t|/2} (|t| + 1)^{1/2 - \sigma} \int_{-\infty}^{\infty} e^{-\pi |t+y|/2} (|t+y| + 1)^{\sigma + M} \times e^{-\pi |y|/2} (|y| + 1)^{-M-1} \times \exp (-\pi/2 |t+y| + (|\arg b| - \pi/2 |y|)) dy.
\]

Denote the integral on the right-hand side here by \( J_1(M) \), and apply [12, Lemma 4] to evaluate \( J_1(M) \). We find that

\[
J_1(M) \ll (1 + (|t| + 1)^{\sigma + M}) (|t| + 1)^{-M-1 + \delta(b)} e^{(\arg b - \pi/2)|t|} + (1 + (|t| + 1)^{\sigma + M}) e^{-\pi |t|/2},
\]

where \( \delta(b) = 1 \) if \( \arg b = 0 \) (that is, \( b \in \mathbb{R}_{>0} \)) and \( \delta(b) = 0 \) otherwise. Therefore we have

\[
J(M) \ll |b|^{M+1/2} A_1(|t|) e^{\pi |t|/2},
\] (38)
where \( A_1(|t|) \) (and \( A_2(|t|), A_3(|t|) \) hereafter) denotes a certain quantity which is of polynomial order in \(|t|\). Therefore from (36) we find that
\[
\zeta(s, h, b) \ll |b|^{\max{(-\sigma, M+1/2)}} A_2(|t|) e^{t \arg b}.
\] (Note that \( A_1(|t|), A_2(|t|) \) can be explicitly determined.) In particular, \( \zeta(s, h, 1) \) is of polynomial order in \(|t|\). We use this fact to prove the following lemma.

**Lemma 2.** Let \( s \) be in a fixed vertical strip in \( \mathbb{C} \), excluding a small neighborhood of \( s = 1 \). If \( a, w \in \mathbb{C} \) with \( a/w \notin (-\infty, 0] \), then
\[
\zeta(s, h, a/w)w^{-s} = O \left( |w|^{-\sigma} A_3(|t|) \exp(|t| \max{\{|\arg a|, |\arg w|\}}) \right)
\]
(the implied constant may depend on \( a/w \)).

**Proof.** This lemma is an analogue of [12, Lemma 2], and the proof is similar, so we just give a brief sketch. Let \( s \in \mathbb{C} \), and we choose \( N \) so large that \( \Re(s + N) > 1 \). As generalizations of [12, (2.6), (2.10)], we can show
\[
\zeta(s, h, b) = \sum_{n=0}^{N-1} \frac{(1-b)^n}{n!}(s)_n \zeta(s + n, h, 1) - (s)_N \int_1^b \frac{(\xi - b)^{N-1}}{(N-1)!} \zeta(s + N, h, \xi) d\xi
\]
(40) (where \((s)_n\) denotes the Pochhammer symbol) and
\[
\zeta(s + N, h, \xi)w^{-s} \ll |w|^{-\sigma} \exp(|t| \max{\{|\arg a|, |\arg w|\}})
\]
(41) (for any \( \xi \) on the segment joining 1 and \( a/w \)) by the same argument. Putting \( b = a/w \) in (10), and applying (11) and the fact mentioned just before the statement of the lemma, we obtain the assertion.

Now let us go back to (32), and shift the path to \( \Re z = M + 1/3 \). Here, if we choose \( \Re z = M + 1/2 \) as before, there appears a small problem when \( h_n = 2 \), so we choose \( \Re z = M + 1/3 \). The above Lemma ensures that this shifting is possible (similar to the argument in Section 1). The relevant poles are \( z = 0, 1, 2, \ldots, M \) (from \( \Gamma(-z) \)) and \( z = -1 \) (if \( h_n = 1 \)) or \( z = l - h_n^{-1} (0 \leq l \leq M, \text{if } h_n \geq 2) \). Analogous to (24), for \( h_n \geq 2 \), we obtain
\[
\zeta_{n-1}(s, h, \gamma, b, \mu_{n-1})
\]
\[
= \sum_{l=0}^{M} \frac{\Gamma(s_n + l - h_n^{-1}) \Gamma(-l + h_n^{-1})}{\Gamma(s_n)}
\]
\[
\times \zeta_{n-1,n-1}(s_n l - h_n^{-1}, h_n, \gamma_n, b_n, \mu_n)
\]
(42)
\[
\sum_{l=0}^{M} \left( -\frac{s_n}{l} \right) \zeta_{n-1,n-1}(s_{n-1}^*(l), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \\
\times \zeta \left( -l, h_n, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l \\
+ \frac{1}{2\pi i} \int_{(M+1/3)} \frac{\Gamma(s_n + z)\Gamma(-z)}{\Gamma(s_n)} \zeta_{n-1,n-1}(s_{n-1}^*(z), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \\
\times \zeta \left( -z, h_n, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^zdz.
\]

If \( h_n = 1 \), then only the term corresponding to \( l = 0 \) on the first sum appears, which is equal to

\[
\frac{1}{s_n - 1} \zeta_{n-1,n-1}(s_{n-1}^*(-1), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1}. \quad (43)
\]

Now put \( s = -N \) and obtain an explicit formula, similar to [23]. Because of the existence of the factor \( \Gamma(s_n) \) on the denominator, the integral term vanishes. If \( h_n \geq 2 \), the first sum also vanishes by the same reason. Therefore

\[
\zeta_{n,n-1}(-N, h, \gamma, b, \mu_{n-1}) \\
= \frac{-\delta_{l,h_n}}{N_n + 1} \zeta_{n-1,n-1}(-N_{n-1}^*(1), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \gamma_n^{-1} \\
+ \sum_{l=0}^{N_n} \left( \frac{N_n}{l} \right) \zeta_{n-1,n-1}(-N_{n-1}^*(-l), h_{n-1}, \gamma_{n-1}, b_{n-1}, \mu_{n-1}) \\
\times \zeta \left( -l, h_n, \frac{b_n - b_{n-1}}{\gamma_n} \right) \gamma_n^l.
\]

Finally, applying Proposition 1 to the right-hand side, we arrive at the assertion of Theorem 3.

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