Magic squares and the symmetric group

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Abstract

Diaconis and Gamburd computed moments of secular coefficients in the CUE ensemble. We use the characteristic map to give a new combinatorial proof of their result. We also extend their computation to moments of traces of symmetric powers, where the same result holds but in a wider range.

1 Introduction

Consider the complex unitary group $U(N)$ endowed with the probability Haar measure. The $n$th secular coefficient of $U \in U(N)$ is defined through the expansion

$$\det(zI + U) = \sum_{n=0}^{N} z^{N-n} \text{Sc}_n(U).$$

If $A = (a_{i,j})$ is an $m \times n$ matrix with non-negative integer entries, Diaconis and Gamburd [DG06] define the row-sum vector $\text{row}(A) \in \mathbb{Z}^m$ and column-sum vector $\text{col}(A) \in \mathbb{Z}^n$ by

$$\text{row}(A)_i = \sum_{j=1}^{n} a_{i,j}, \quad \text{col}(A)_j = \sum_{i=1}^{m} a_{i,j}.$$ 

Given two partitions $\mu = (\mu_1, \ldots, \mu_m)$ and $\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n)$ they denote by $N_{\mu, \tilde{\mu}}$ the number of non-negative integer matrices $A$ with $\text{row}(A) = \mu$ and $\text{col}(A) = \tilde{\mu}$. Given sequences $(a_1, \ldots, a_\ell)$ and $(b_1, \ldots, b_\ell)$ of non-negative integers, they proved the following equality [DG06, Thm. 2]:

$$\int_{U(N)} \prod_{j=1}^{\ell} \text{Sc}_{a_j}(U)^{a_j} \text{Sc}_{b_j}(U)^{b_j} \, dU = N_{\mu, \tilde{\mu}}$$

(1.1)

as long as $\sum_{j=1}^{\ell} j a_j, \sum_{j=1}^{\ell} j b_j \leq N$. Here $\mu$ and $\tilde{\mu}$ are the partitions with $a_j$ and $b_j$ parts of size $j$, respectively.

Identity (1.1) answered a question raised in [HKS+96, SHW98], where the first two moments were computed. The results in [DG06] inspired the study of pseudomoments of the Riemann zeta function [CG06] and were used in [KRRGR18] to study the variance of divisor functions in short intervals. Recently, Najnudel, Paquette and Simm studied the distribution of $\text{Sc}_n$ with $n$ growing with $N$ [NPS20].

We give a new combinatorial proof of (1.1), which makes use of the characteristic map. This is in the spirit of Bump’s derivation [Bum04, Prop. 40.4] of the Diaconis-Shahshahani moment computation [DS94].

In §5 we show that a result similar to (1.1) holds for traces of symmetric powers in place of secular coefficients, with substantially relaxed conditions: $\min\{\sum a_j, \sum b_j\} \leq N$. These traces are also the complete homogeneous symmetric polynomials $h_n$ evaluated on the eigenvalues of the matrix.

2 The symmetric group

For a permutation $\pi$ we say that $S$ is an invariant set for $\pi$ if $\pi(S) = S$. Equivalently, $S$ is a union of cycles of $\pi$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$, we define the following function on the symmetric group $S_n$ acting on $[n] := \{1, 2, \ldots, n\}$:

$$d_\lambda(\pi) = \#\{(A_1, \ldots, A_\ell) : \cup A_i = [n], \text{ each } A_i \text{ is an invariant set with } |A_i| = \lambda_i\},$$
where \( \cup \) means disjoint union. We use the letter \( d \) here as short for divisor, as these functions are analogous to divisor functions over the integers. Given \( \mu, \tilde{\mu} \vdash n \), let us define

\[
N_{\mu, \tilde{\mu}}(n) := \frac{1}{|S_n|} \sum_{\pi \in S_n} d_\pi(\mu) d_{\tilde{\mu}}(\pi).
\]

**Proposition 2.1.** Suppose \( \mu, \tilde{\mu} \vdash n \). We have

\[
N_{\mu, \tilde{\mu}}(n) = N_{\mu, \tilde{\mu}}.
\]

**Proof.** By definition, given a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n \) we may express \( d_\lambda(\pi) \) as a sum over ordered set partitions:

\[
d_\lambda(\pi) = \sum_{(A_1, \ldots, A_\ell) : \cup A_i = [n]} \alpha_{A_1, \ldots, A_\ell}(\pi)
\]

where \( \alpha_{A_1, \ldots, A_\ell} \) is the indicator function of permutations \( \pi \in S_n \) with \( \pi(A_i) = A_i \) for all \( i \). Applying (2.1) with \( \lambda = \mu \) and multiplying by \( d_\mu(\pi) \) with \( \lambda = \tilde{\mu} \) we obtain

\[
d_\mu(\pi) d_{\tilde{\mu}}(\pi) = \sum_{(A_1, \ldots, A_\ell) : \cup A_i = [n]} \sum_{B_1, \ldots, B_\ell} \alpha_{A_1, \ldots, A_\ell}(\pi) \alpha_{B_1, \ldots, B_\ell}(\pi)
\]

where \( \ell(\lambda) \) is the number of parts in a partition. Averaging this over \( S_n \) and interchanging the order of summation, we find

\[
N_{\mu, \tilde{\mu}}(n) = \frac{1}{n!} \sum_{(A_1, \ldots, A_\ell) : \cup A_i = [n]} \sum_{B_1, \ldots, B_\ell} \sum_{\pi \in S_n} \alpha_{A_1, \ldots, A_\ell}(\pi) \alpha_{B_1, \ldots, B_\ell}(\pi).
\]

The inner sum in the right-hand side of (2.2) counts permutations \( \pi \in S_n \) for which \( A_i \) are invariant sets, as well as the \( B_j \). In particular \( \pi(A_i \cap B_j) \leq A_i, B_j \), forcing \( \pi(A_i \cap B_j) = A_i \cap B_j \). Conversely, given a permutation such that \( \pi(A_i \cap B_j) = A_i \cap B_j \) for all \( i, j \), it necessarily satisfies \( \pi(A_i) = A_i \) and \( \pi(B_j) = B_j \) for all \( i, j \). Thus, the inner sum counts \( \pi \) with \( \pi(A_i \cap B_j) = A_i \cap B_j \). The sets \( \{ A_i \cap B_j \}_{i,j} \) are disjoint and their union is \([n]\), and so such \( \pi \)s are determined uniquely by their restrictions to \( A_i, B_j \), which may be arbitrary, proving that the inner sum is \( \prod_{i,j=1}^n |A_i \cap B_j|! \). Hence,

\[
N_{\mu, \tilde{\mu}}(n) = \frac{1}{n!} \sum_{(A_1, \ldots, A_\ell) : \cup A_i = [n]} \sum_{B_1, \ldots, B_\ell} \prod_{i,j} |A_i \cap B_j|!.
\]

Observe that the \( n \times m \) matrix \( C = (|A_i \cap B_j|) \) has row \( C = \mu \) and col \( C = \tilde{\mu} \). Hence

\[
N_{\mu, \tilde{\mu}}(n) = \frac{1}{n!} \prod_{C=(c_{i,j})} \text{a matrix counted by } N_{\mu, \tilde{\mu}} \prod_{i,j} c_{i,j}! \cdot \# \{ [n] = \cup_{i,j} \cup C_{i,j}, |C_{i,j}| = c_{i,j} \}.
\]

The inner expression in the right-hand side is the number of ordered set partitions of \([n]\) into subsets \( C_{i,j} \) of size \( c_{i,j} \) (these sets correspond to \( A_i \cap B_j \) and one reconstructs \( A_i \) by \( A_i = \cup j C_{i,j} \) and similarly \( B_j = \cup i C_{i,j} \)). This is just the multinomial

\[
\left( c_{i,j} : 1 \leq i \leq \ell(\mu), 1 \leq j \leq \ell(\tilde{\mu}) \right) = \frac{n!}{\prod \{ c_{i,j} ! \}},
\]

so that (2.2) simplifies to

\[
N_{\mu, \tilde{\mu}}(n) = \sum_{X \text{ a matrix counted by } N_{\mu, \tilde{\mu}}} 1 = N_{\mu, \tilde{\mu}}
\]

as claimed. \( \square \)
3 The characteristic map

Endow $S_n$ with the uniform probability measure. The characteristic (or Frobenius) map $\text{Ch}^{(N)}$ is a linear map from class functions on $S_n$ to class functions on $U(N)$, with the property that if $n \leq N$ it is an isometry with respect to the $L_2$-norm, see [Bum04 Thm. 40.1]. It may be given by

$$\text{Ch}^{(N)}(f) = \frac{1}{n!} \sum_{\pi \in S_n} f(\pi)p_{\lambda(\pi)},$$

see [Bum04 Thm. 39.1]. Here $\lambda(\pi)$ is the partition associated with $\pi$, and $p_{\lambda}$ is the power sum symmetric polynomial associated with $\lambda$, evaluated at the eigenvalues of $U \in U(N)$.

Lemma 3.1. Suppose $\lambda \vdash n$. We have

$$\text{Ch}^{(N)}(\text{sgn} \cdot d_{\lambda}) = e_{\lambda},$$

where $\text{sgn}$ is the sign representation and $e_{\lambda}$ is the elementary symmetric polynomial associated with the partition $\lambda$.

Proof. Given $\pi \in S_n$, we set $p_{\pi} = p_{\lambda(\pi)}$ for convenience. We then have, by plugging (2.1) in the definition of $\text{Ch}^{(N)}(\text{sgn} \cdot d_{\lambda})$ and interchanging order of summation,

$$\text{Ch}^{(N)}(\text{sgn} \cdot d_{\lambda}) = \frac{1}{n!} \sum_{(A_1, \ldots, A_{\ell(\lambda)}) \subseteq [n]} \sum_{\pi \in S_n} \text{sgn}(\pi)p_{\pi}.$$

We claim that the inner sum is $e_{\lambda}$. Indeed, since $\pi$ is determined by the restrictions $\pi|_{A_i}$, and since $p_{\lambda} = \prod_i p_{\lambda_i}$, we have

$$\sum_{\pi \in S_n} \text{sgn}(\pi)p_{\pi} = \prod_{i=1}^{\ell(\lambda)} \left( \sum_{\pi \in S_{A_i}} \text{sgn}(\pi)p_{\pi_i} \right) = \prod_{i=1}^{\ell(\lambda)} \lambda_i!e_{\lambda_i},$$

where the last equality follows from the Newton-Girard identity $\sum_{\pi \in S_n} \text{sgn}(\pi)p_{\pi}/m! = e_m$. To finish, note that the number of ordered set partitions of $[n]$ into $\ell(\lambda)$ sets of sizes $\lambda_i$ is exactly the binomial coefficient $\binom{n}{\lambda_1, \ldots, \lambda_{\ell(\lambda)}}$. \qed

4 Conclusion of proof

Here we establish (1.1). Let $(a_1, \ldots, a_{\ell})$ and $(b_1, \ldots, b_{\ell})$ be sequences of non-negative integers with $\sum_{j=1}^{\ell} ja_j$, $\sum_{j=1}^{\ell} jb_j \leq N$. Let $\mu$ and $\tilde{\mu}$ be the partitions with $a_j$ and $b_j$ parts of size $j$, respectively.

If $\sum_{j} ja_j \neq \sum_{j} jb_j$, it is easy to see that both sides of (1.1) vanish. Indeed, for the right-hand side, note that the integrand is an homogeneous polynomial in the eigenvalues of $U$, whose degree is non-zero, so its integral must vanish by translation-invariance of the Haar measure. On the other hand, if $N_{\mu, \tilde{\mu}}$ is non-zero, we must have that $\mu$ and $\tilde{\mu}$ sum to the same number (if $A = (a_{i,j})$ is a matrix counted by $N$ then both $\mu$ and $\tilde{\mu}$ sum to $\sum_{i,j} a_{i,j}$).

Now assume $\sum_{j} ja_j = \sum_{j} jb_j = n \leq N$. As $\text{Sc}_j(U)^{\mu}\overline{\text{Sc}_j(U)^{\tilde{\mu}}} = e_{\mu}e_{\tilde{\mu}}$ by definition, the fact that $\text{Ch}^{(N)}$ is an isometry if $n \leq N$ shows, through Lemma 3.1 that the integral in (1.1) is equal to

$$\frac{1}{|S_n|} \sum_{\pi \in S_n} (\text{sgn} \cdot d_{\mu})(\pi)\text{sgn} \cdot d_{\tilde{\mu}}(\pi) = \frac{1}{|S_n|} \sum_{\pi \in S_n} d_{\mu}(\pi)d_{\tilde{\mu}}(\pi) = N_{\mu, \tilde{\mu}}(n),$$

and the proof is concluded by applying Proposition 2.1.
5 Symmetric powers

Let $\text{TrSym}^n(U)$ be the trace of the $n$th symmetric power of $U \in U(N)$. This is also the $n$th complete homogeneous symmetric polynomial $h_n$ evaluated on the eigenvalues of $U$.

**Theorem 5.1.** Let $\{a_j\}_{j=1}^\ell, \{b_j\}_{j=1}^\ell$ be sequences of non-negative integers. We have

$$\int_{U(N)} \prod_{j=1}^\ell \left(\text{TrSym}^j(U)^{a_j} \text{TrSym}^j(U)^{b_j}\right) dU = N_{\mu,\bar{\mu}}$$

as long as $\min\{\sum_{j=1}^\ell a_j, \sum_{j=1}^\ell b_j\} \leq N$. Here $\mu$ and $\bar{\mu}$ are the partitions with $a_j$ and $b_j$ parts of size $j$, respectively.

We start with the following corollary of Lemma 3.1

**Corollary 5.2.** Suppose $\lambda \vdash n$. We have $\text{Ch}^n(\lambda) = h_\lambda$.

This follows from Lemma 3.1 through the existence of an involution $\iota$ on the space of symmetric polynomials, with the properties $\text{Ch}^n(\text{sgn}: f) = \iota(\text{Ch}^n(f))$ [Bum04, Thm. 39.3] and $\iota(e_\lambda) = h_\lambda$ [Bum04, Thm. 36.3]. Alternatively, one may repeat the proof of Lemma 3.1 with the Newton-Girard identity $\sum_{\pi \in S_n} p_\pi/m! = h_m$.

Next we prove the following well-known identity, often proved as a consequence of the RSK correspondence.

**Lemma 5.3.** Given $\mu, \bar{\mu} \vdash n$ we have

$$\sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,\bar{\mu}} = N_{\mu,\bar{\mu}}$$

where $K_{\lambda,\mu}$ are the Kostka numbers.

*Proof.* We may expand $e_\lambda$ in the Schur basis, see [Sta99, p. 335]:

$$e_\mu = \sum_{\lambda \vdash n} K_{\lambda',\mu} s_\lambda$$

where $\lambda'$ is the conjugate of $\lambda$. Orthogonality of Schur functions [DG06, Eq. (22)] implies that

$$\int_{U(n)} \prod_{j=1}^\ell \left(\text{Sc}_j(U)^{a_j} \text{Sc}_j(U)^{b_j}\right) dU = \sum_{\lambda \vdash n} K_{\lambda',\mu} K_{\lambda',\bar{\mu}} = \sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,\bar{\mu}}.$$

On the other hand, this integral was shown to equal $N_{\mu,\bar{\mu}}$. \hfill \Box

We now prove Theorem 5.1. The case $\sum_{j=1}^\ell j a_j \neq \sum_{j=1}^\ell j b_j$ is treated as in the secular coefficients case. Next, assume that $\sum j a_j = \sum j b_j = n$ and $\min\{\sum_{j=1}^\ell a_j, \sum_{j=1}^\ell b_j\} \leq N$. The multiset of eigenvalues of $\text{TrSym}^j U$ consists of products of $j$ eigenvalues of $U$, and so the integrand in the left-hand side of (5.1) is $h_\mu h_{\bar{\mu}}$. We may expand $h_\lambda$ in the Schur basis, see Stanley [Sta99, Cor. 7.12.4]:

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda.$$

Orthogonality of Schur functions implies that the left-hand side of (5.1) is

$$\sum_{\ell(\lambda) \leq N} K_{\lambda,\mu} K_{\lambda,\bar{\mu}}.$$

As $K_{\lambda,\mu} \neq 0$ implies $\ell(\lambda) \leq \ell(\mu)$ [Sta99, Prop. 7.10.5], and $\min\{\ell(\mu), \ell(\mu')\} = \min\{\sum a_j, \sum b_j\} \leq N$ by assumption, this sum is equal to the full sum $\sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,\bar{\mu}}$, and the proof is concluded by (5.2).

A version of Theorem 5.1 with max in place of min, may also be derived from formulas for averages of ratios of characteristic polynomials [CTZ05, BG06].
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References

[BG06] Daniel Bump and Alex Gamburd. On the averages of characteristic polynomials from classical groups. Comm. Math. Phys., 265(1):227–274, 2006.

[Bum04] Daniel Bump. Lie groups, volume 225 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2004.

[CFZ05] JB Conrey, DW Farmer, and MR Zirnbauer. Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups U(N). arXiv preprint math-ph/0511024, 2005.

[CG06] Brian Conrey and Alex Gamburd. Pseudomoments of the Riemann zeta-function and pseudomagic squares. J. Number Theory, 117(2):263–278, 2006.

[DG06] Persi Diaconis and Alex Gamburd. Random matrices, magic squares and matching polynomials. Electron. J. Combin., 11(2):Research Paper 2, 26, 2004/06.

[DS94] Persi Diaconis and Mehrdad Shahshahani. On the eigenvalues of random matrices. volume 31A, pages 49–62. 1994. Studies in applied probability.

[HKS+96] Fritz Haake, Marek Kuś, Hans-Jürgen Sommers, Henning Schomerus, and Karol Życzkowski. Secular determinants of random unitary matrices. J. Phys. A, 29(13):3641–3658, 1996.

[KRRGR18] J. P. Keating, B. Rodgers, E. Roditty-Gershon, and Z. Rudnick. Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals. Math. Z., 288(1-2):167–198, 2018.

[NPS20] Joseph Najnudel, Elliot Paquette, and Nick Simm. Secular coefficients and the holomorphic multiplicative chaos. arXiv preprint arXiv:2011.01823, 2020.

[SHW98] Hans-Jürgen Sommers, Fritz Haake, and Joachim Weber. Joint densities of secular coefficients for unitary matrices. J. Phys. A, 31(19):4395–4401, 1998.

[Sta99] Richard P. Stanley. Enumerative combinatorics. Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

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