Fundamental Properties of Quaternion Spinors

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Abstract—The interior structure of arbitrary sets of quaternion units is analyzed using general methods of the theory of matrices. It is shown that the units are composed of quadratic combinations of fundamental objects having a dual mathematical meaning as spinor couples and dyads locally describing 2D surfaces. A detailed study of algebraic relationships between the spinor sets belonging to different quaternion units is suggested as an initial step aimed at producing a self-consistent geometric image of spinor-surface distribution on the physical 3D space background.

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1. INTRODUCTION: PHYSICAL DOMAINS IN QUATERNION MATHEMATICS

Studies of hypercomplex numbers reveal many mathematical relations resembling formulas of physical theories known from the experiment or heuristic assumptions. These coincidences are found in the associative (by multiplication) algebras of double (split-complex) numbers, dual numbers, quaternion numbers, and of the embracing all of them bi-quaternion algebra, so the basement of all these algebras are the four quaternion (Q-) units [1]. Hamilton, the inventor of quaternions, was the first to discover that the triad of imaginary Q-units behaves as a set of unit vectors forming an orthogonal coordinate system in 3D space, a geometric and physical fact [2]; Descartes introduced this important object heuristically more than 200 years before Hamilton’s quaternions. Maxwell also used quaternions to formulate the equations of electrodynamics, and 70 years later Fueter strikingly found that these equations (in vacuum) are equivalent to the differentiability conditions for a vector function of a Q-variable [3]. The spin term introduced by Pauli into the Schrödinger equation was later shown to reflect the Q-properties of the quantum-mechanical space [4]. A further development of the notion of Q-space exposed a formal equivalence of the space curvature and the Yang-Mills field intensity [5], while a Q-description of movable frames resulted in a specific (rotational) version of the theory of relativity [6, 7]. An important property of Q-numbers is linked to spinors. Rastall [8] identified spinors with Q-ideals, mathematical objects emerging in equations involving idempotent matrices, but this observation was in fact used only to offer a different description of the Dirac spinor theory.

It seems though that the Q-spinors deserve a more detailed study, since they not only can serve as material for an alternative reconstitution of known physical structures, but themselves form a powerful tool for revealing the space (and time) properties. This study is aimed at a wide analysis of the origin of quaternion spinors, the freedom they dispose, their deep links with the structure of Q-numbers and their distribution on the 3D space background. In Section 2, a sketch of quaternion algebra is given with samples of representations of the algebra units. In Section 3, a component-free procedure based on the theory of matrices is developed for detecting spinors in the structure of Q-units. In Section 4, relations linking spinors associated with different directions (dimensions) of a Q-space are deduced. Section 5 contains complete tables of mutual projections of spinor-vectors, and a discussion in Section 6 concludes the study.

2. QUATERNIONS IN SHORT

A quaternion \( a = x \cdot I + y_1 \cdot q_1 + y_2 \cdot q_2 + y_3 \cdot q_3 \) is a number built on the basis of one real (scalar) unit \( I \) and three imaginary (vector) units \( q_1, q_2, q_3 \), each unit having real coefficients \( \{x, y_1, y_2, y_3\} \in \mathbb{R} \). Let small Latin indices enumerate vector units \( k, m, n, \ldots = 1, 2, 3 \), then the quaternion is rewritten in a compact form as \( a = x + y_k q_k \), where summing in the repeated indices is implied, and the real unit symbol is traditionally omitted. Q-numbers admit

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1 This vector notations is related to Hamilton’s traditional notations (still used in literature) as \( q_1 = i, q_2 = j, q_3 = k \).
algebraic addition similar to that of complex numbers, multiplication is associative but no more commutative, the vector units commute with the real unit but do not commute between themselves; this is reflected in the multiplication table

\[ I q_k = q_k I = q_k, \quad q_k q_n = -\delta_{kn} + \varepsilon_{knm} q_m, \]

where \( \delta_{kn}, \varepsilon_{knm} \) are the 3D Kronecker and Levi-Civita symbols. Q-conjugation \( \bar{a} = x - y_k q_k \) helps one to define the modulus of a Q-number

\[ |a| = \sqrt{\bar{a}a} = \sqrt{x^2 + y_k y_k} \equiv \sqrt{x^2 + y^2} \in \mathbb{R} \]

with two consequences. First, the inverse quaternion is defined,

\[ a^{-1} = \bar{a}/|a|^2, \]

so that the quaternion division (right and left as multiplication) can be introduced:

\[ \left( \frac{a}{b} \right)_{\text{right}} = \frac{ab}{|b|^2}, \quad \left( \frac{a}{b} \right)_{\text{left}} = \frac{ba}{|b|^2}; \]

\( b \) being another quaternion. Second, there is an Euler-type formula for a quaternion:

\[ a = |a| \exp \left( \frac{y_k q_k \cdot \theta}{y} \right), \]

a series with \( y = \sqrt{y_k y_k} \), \( \theta \equiv \arccos(x/|a|) \). With these properties the quaternions are proved to compose the last (in dimension) associative division algebra over the real numbers (a finite-dimensional division ring). The simplest (canonical) representation of the Q-units is given by the 2 \( \times \) 2 Pauli matrices multiplied by \(-i\):

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ q_2 = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]

\[ q_3 = q_1 q_2 = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

but the representation (2) is not unique. Another one is obtained if the imaginary (scalar) unit \( i \) is replaced with the matrix \( q_2 \) of Eq. (2), also an imaginary (but vector) unit, and each matrix component equal to 1 is replaced with the unit 2 \( \times \) 2-matrix; then the quaternion units are described by 4 \( \times \) 4 matrices with real but still constant components. A drastic change occurs if the units acquire variable parameters. One straightforwardly shows [9, 10] that the multiplication table (1) keeps its form under two types of transformation of the units, each forming a group. The first one is the special orthogonal group \( SO(3, \mathbb{C}) \) represented by 3 \( \times \) 3-matrices \( O_{k'n} \) performing rotations of the vector units (at in general complex angles \( \Phi \)):

\[ q_{k'} = O_{k'n}(\Phi) q_n, \]

an irreducible representation of the group element is an “elementary rotation” at an angle \( \Phi \) about, e.g., the vector \( q_1 \):

\[ O_{k'n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{pmatrix}. \]

Another group, evidently leaving the table (1) form-invariant, is the special linear group represented by the matrices \( U \)

\[ q_{k'} = U q_k U^{-1}. \]

In the case of the 2 \( \times \) 2-matrix representation of the Q-units this group is the spinor group \( SL(2, \mathbb{C}) \) known to be 2 : 1 isomorphic to \( SO(3, \mathbb{C}) \). An \( SL(2, \mathbb{C}) \) transformation operator leading to the same rotation as the operator in Eq. (4) is

\[ U = \begin{pmatrix} \cos(\Phi/2) & -i \sin(\Phi/2) \\ -i \sin(\Phi/2) & \cos(\Phi/2) \end{pmatrix}, \]

\[ U^{-1} = \begin{pmatrix} \cos(\Phi/2) & i \sin(\Phi/2) \\ i \sin(\Phi/2) & \cos(\Phi/2) \end{pmatrix}. \]

One readily finds that any 2 \( \times \) 2-matrix \( U \in SL(2, \mathbb{C}) \) is a Q-number built on the basis (2), e.g., the matrices (6) can be written as

\[ U = \cos \frac{\Phi}{2} + q_1 \sin \frac{\Phi}{2}, \]

\[ U^{-1} = \cos \frac{\Phi}{2} - q_1 \sin \frac{\Phi}{2}, \]

the transformations (4), (6) leaving the vector \( q_1 \) intact. Both groups \( SO(3, \mathbb{C}), SL(2, \mathbb{C}) \) do not change the scalar unit. It is also known that the group \( SO(3, \mathbb{C}) \) is 1 : 1 isomorphic to the Lorentz group, and that is where the relativity theory mentioned in [6] appears. But the goal of this paper is the spinor structure lying in the basement of the quaternion algebra.

### 3. THEORY OF MATRICES AND THE SPINOR BASEMENT OF Q-NUMBERS

To reveal the mathematical basement of any set of the Q-unit triads (frequently associated with the three dimensions of physical space) make a useful exercise from the matrix algebra, namely, analyze the eigenvector-eigenvalue problem for an arbitrary
$2 \times 2$-matrix $A$ with (in general) complex-number elements.

It is straightforward to prove that $A$ can be represented as a sum of a diagonal matrix with non-zero trace and a traceless matrix, or as a (bi-)quaternion with the scalar and vector parts (e.g., [9])

$$ A = \frac{\text{Tr} A}{2} \cdot I + \sqrt{\det A - \frac{\text{Tr}^2 A}{4}} \cdot \mathbf{q}; \quad (8) $$

here $I$ is the unit matrix, $\mathbf{q}$ is a traceless $2 \times 2$-matrix with $\det \mathbf{q} = 1$, hence it is an imaginary Q-unit ($\mathbf{q}^2 = -1$) belonging to some Q-triad; the eigenvectors of $\mathbf{q}$ surely fit for $I$. Consider two complimentary cases:

**Case (i):** let $A$ be invertible, i.e., $\det A \neq 0$. Then Eq. (8) yields an expression for the eigenvalues $\alpha$ of $A$:

$$ \alpha = \sqrt{\det A} (\cos \Phi + \lambda \sin \Phi), $$$$ \cos \Phi \equiv \frac{\text{Tr} A}{2 \sqrt{\det A}}, $$

where $\det A$, $\text{Tr} A$, $\Phi \in \mathbb{C}$, the set of complex numbers; $\lambda$ are eigenvalues of $\mathbf{q}$. According to the theory of matrices, all matrices $\mathbf{q}$ are similar [due to Eq. (5)], so their eigenvalues coincide with those of the canonical matrix $\mathbf{q}_3$ [from Eq. (2)] that obviously has $\lambda = \pm i$. Therefore,

$$ \alpha = \sqrt{\det A} \exp(\pm i \Phi); \quad (9) $$

the same result of course follows from solving the characteristic equation for $A$. Thus any invertible $2 \times 2$-matrix $A$ has exactly two distinct eigenvalues, so it is simple, and due to the spectral theorem (e.g., [11]) it can be decomposed into orthogonal parts:

$$ A = \alpha^+ C^+ + \alpha^- C^-, \quad C^+ C^- = 0, \quad (10) $$

where $C^\pm$ are projectors formed as direct products of right ($2D$-column) $\psi^\pm$ and left ($2D$-row) $\varphi^\pm$ biorthonormal eigenvectors of $A$,

$$ C^\pm = \psi^\pm \varphi^\pm. \quad (11) $$

The projectors are known to be singular ($\det C^\pm = 0$) idempotent ($C^\pm C^\pm = C^\pm$) matrices with unit trace $\text{Tr} C^\pm = 1$ equal to the only eigenvalue of $C^\pm$.

**Case (ii):** let the matrix $A$ be non-invertible (singular: $\det A = 0$), and let $A \rightarrow C^\pm$, then Eq. (8) yields

$$ C^\pm = \frac{I \pm i \mathbf{q}}{2}. \quad (12) $$

From Eqs. (11) and (12) one immediately obtains equivalents of the decomposition (10) ($A = I$ or $A = \mathbf{q}$):

$$ I = C^+ + C^- = \psi^+ \varphi^+ + \psi^- \varphi^-, \quad (13a) $$

$$ q = iC^+ - iC^- = i(\psi^+ \varphi^+ - \psi^- \varphi^-). \quad (13b) $$

Equations (13) are fundamental relations demonstrating that units of any Q-triad consist of more elementary objects constituting a bi-orthonormal basis $\{\varphi^\pm, \psi^\pm\}$ of a $2D$ vector space, (in general) a complex-number-valued surface. On the one hand, a comparison of Eqs. (13) and (5) shows that the vectors of the basis are spinors whose transformations $\psi' = U \psi, \varphi' = U^{-1} \varphi, U \in SL(2, \mathbb{C})$ do not affect the quaternion multiplication rule (1). On the other hand, the vectors $\psi^\pm$ and co-vectors $\varphi^\pm$ obey the standard “metric requirements” $\varphi^\pm \psi^\mp = 1, \varphi^\pm \psi^\pm = 0$. In dyad notations (as Lamé coefficients linking the basic and tangent $2D$-surfaces)

$$ \psi^\pm \equiv h^A_{(M)}, \quad \varphi^\pm \equiv h^A_{(N)B}, \quad (14) $$

where $A, B, ... = 1, 2$ are contravariant and covariant component indices, and $(M), (N), ... = 1, 2$ are parity indices (always written at the bottom of a symbol), the orthogonality and normalization conditions shrink to the unique equation

$$ h^A_{(M)} h^A_{(N)B} = \delta^A_{MN}. \quad (15) $$

while Eq. (13a) acquires the form

$$ h^A_{(M)} h^A_{(N)B} = \delta^A_{MN}. \quad (16) $$

then the metric requirements $\delta^A_{MN}$ being $2D$ Kronecker symbols for different types of indices. Eq. (15) describes the metric of a $2D$ tangent plane, while expressions for the metric tensor of the $2D$ base and its reciprocal follow from Eq. (16):

$$ g^{AB} \equiv h^A_{(M)} h^B_{(N)A}, \quad g^{BC} \equiv h^B_{(M)} h^C_{(N)M}. \quad (17) $$

Let us now return to the simplest case of Q-units given by Eq. (2) and notice that the constant (“plane”) spinors of the form

$$ \psi^+ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \varphi^+ = (0, 1), \quad (18) $$

$$ \psi^- = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \varphi^- = (1, 0) $$

are the eigenvectors of $\mathbf{q}_3$ [of Eq. (2)], hence they constitute this unit according to Eq. (13b):

$$ \mathbf{q}_3 = i(\psi^+ \varphi^+ - \psi^- \varphi^-). \quad (19) $$

Eq. (13a) holding as well. However, one easily finds that the representation (18) enables one to build all other vector Q-units of the set (2) composing linear combinations of tensor products of the spinors with opposite parity,

$$ \mathbf{q}_2 = \psi^+ \varphi^- - \psi^- \varphi^+, \quad (20) $$

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An important observation must be made here. The Q-units given by Eqs. (13a), (19)–(21) obey the basic multiplication rule (1) irrespective of the specific form of the spinor elements entering into their structure. One directly verifies that by composing all possible products of the units. Moreover, it is shown in [12] that the representation of the Q-units by quadratic combinations of spinors is unique since there exist only two linear combinations of nilpotent matrices of the type \( \psi^\pm \varphi^\pm \) and only two linear combinations of idempotent matrices of the type \( \psi^\pm \varphi^\pm \); namely, these four combinations form four Q-units. This means that any pair of orthogonal normalized spinor functions, not only those given by Eqs. (22), form a set of Q-units, and the mathematical properties of the spinors completely determine the geometric behavior of the respective Q-triad (in particular, its behavior as a relativistic frame of reference, which may have useful physical consequences).

4. EIGENVECTORS OF DIFFERENT Q-UNITs AND CYCLIC RECURRENT FORMULAS

It is shown above that each Q-unit has its spinor eigenvectors (with the same eigenvalues \( \pm i \)), and any set of Q-units can be built using only one set of spinor-vectors as, e.g., in Eqs. (13a), (19)–(21). This fact means that all eigenvectors of any Q-triad are functionally dependent. A special example of the respective relations is adduced in [12]; below the relations are obtained in a universal procedure and in the general form.

Let the spinors \( \rho^\pm, \xi^\pm \) be left and right eigenvectors of \( q_1 \) and \( q^\pm \), \( \theta^\pm \) be left and right eigenvectors of \( q_2 \) similarly to \( \varphi^\pm, \psi^\pm \), an eigenvector of \( q_3 \). Then the scalar unit and a ring of vector Q-units have three equivalent representations:

\[
I = \xi^+ \rho^+ + \xi^- \rho^- = \theta^+ \eta^+ + \theta^- \eta^- = \psi^+ \varphi^+ + \psi^- \varphi^-,
\]

\[
q_1 = i(\xi^+ \rho^- - \xi^- \rho^+) = \theta^+ \eta^- - \theta^- \eta^+ = -i(\psi^+ \varphi^- + \psi^- \varphi^+),
\]

\[
q_2 = -i(\xi^+ \rho^- + \xi^- \rho^+) = i(\theta^+ \eta^- + \theta^- \eta^+) = \psi^+ \varphi^- - \psi^- \varphi^+,
\]

\[
q_3 = \xi^+ \rho^- - \xi^- \rho^+ = -i(\theta^+ \eta^- + \theta^- \eta^+) = i(\psi^+ \varphi^- - \psi^- \varphi^+).
\]

Altogether there are 12 nonlinear equations for 12 functions \( \rho^\pm, \xi^\pm, \eta^\pm, \theta^\pm, \varphi^\pm, \psi^\pm \). But according to the rule (1), the scalar unit (22a) equals minus square of any vector unit from the set (22), while each vector unit is a product of two others, e.g., Eq. (22d) is a product of (22b) and (22c). Moreover, the three spinor couples are normalized in 6 equations, and also they obey 6 orthogonality conditions, so a great degree of “algebraic symmetry” is expected in the relations linking the eigenvectors.

Avoiding a detailed derivation of all relations, let us demonstrate the routine for only one spinor. For instance, the spinor \( \xi^+ \) is found as a function of \( \theta^\pm \) from the first equalities of the set (22) after multiplication from the right of Eq. (22a) by \( \xi^+ \), Eq. (22b) by \( (-i\xi^+) \), Eq. (22c) by \( i\xi^- \), and Eq. (22d) by \( \xi^- \); this yields the four equations:

\[
\begin{align*}
\xi^+ &= (\eta^+ \xi^+) \theta^+ + (\eta^- \xi^+) \theta^-; \\
\xi^+ &= -i(\eta^- \xi^+) \theta^+ + i(\eta^+ \xi^+) \theta^-; \\
\xi^+ &= -(\eta^+ \xi^-) \theta^+ + (\eta^- \xi^-) \theta^-; \\
\xi^+ &= -i(\eta^- \xi^-) \theta^+ - i(\eta^+ \xi^-) \theta^-.
\end{align*}
\]

All coefficients of the basic spinor-vector \( \theta^+ \) must be equal, the coefficients of \( \theta^- \) must be equal too; one readily finds that both requirements result in the unique line of equalities:

\[
\eta^+ \xi^+ = -i\eta^- \xi^- = -\eta^+ \xi^- = -i\eta^- \xi^+,
\]

relating all possible contractions (scalar products) of the co-vectors \( \eta^\pm \) from the spinor surface No. 2 \( \{\eta^\pm, \theta^\pm\} \) with the vectors \( \xi^\pm \) from the spinor surface No. 1 \( \{\rho^\pm, \xi^\pm\} \); each scalar product in general being a complex number. Eqs. (23) and (24) allow for expressing \( \xi^+ \) through \( \theta^\pm \) up to a constant factor, chosen here as \( (\eta^+ \xi^+) \):

\[
\xi^+ = (\eta^+ \xi^+)(\theta^+ + i\theta^-).
\]

The co-vector \( \rho^+ \) for this spinor is found as a function of the co-vectors \( \eta^\pm \) in a similar procedure after multiplication of Eqs. (22) form the left by \( \pm \rho^\pm \) with or without the factor \( i \). There are analogous [but “inverse” to those of Eq. (24)] relations between all scalar products of the co-vectors \( \rho^\pm \) from surface No. 1 with the vectors \( \theta^\pm \) from surface No. 2:

\[
\rho^+ \theta^+ = -\rho^- \theta^- = i\rho^+ \theta^- = i\rho^- \theta^+.
\]

The sought-for function is found as

\[
\rho^+ = (\rho^+ \theta^+)(\eta^+ + i\eta^-);
\]

the constant factors of Eqs. (25) and (27) are interconnected by the normalization constraint

\[
\rho^+ \xi^+ = 2(\eta^+ \xi^+)(\rho^+ \theta^+) = 1.
\]

Similarly all other spinor relations are obtained. The full list is given below in the form: a spinor vector of the cycle \( \xi^\pm \rightarrow \theta^\pm \rightarrow \psi^\pm \rightarrow \xi^\pm \); (co-vector of the cycle \( \rho^\pm \rightarrow \eta^\pm \rightarrow \varphi^\pm \rightarrow \rho^\pm \)) is expressed through

\[
q_1 = -i(\psi^+ \varphi^- + \psi^- \varphi^+).
\]
the “previous” spinor (the left equality) and through the “following” vector (the right equality):

\[(\varphi^+ \xi^+)(\psi^+ - \psi^-) = \xi^+, \quad (\eta^+ \psi^+)(\varphi^+ \theta^+) = \rho^+, \quad (\rho^+ \theta^+)(\xi^+ - \xi^-) = \theta^+, \quad (\rho^+ \psi^+)(\xi^+ + i \xi^-) = \psi^- \quad (29a)\]

\[-i(\varphi^+ \xi^+)(\psi^+ + \psi^-) = \xi^- = (\eta^+ \xi^+)(-\theta^+ + i \theta^-), \quad (\rho^+ \theta^+)(\xi^+ + \xi^-) = \theta^+ = (\rho^+ \psi^+)(\xi^+ - i \xi^-), \quad (\varphi^+ \theta^+)(\eta^+ - \eta^-) = \varphi^+ = (\varphi^+ \xi^+)(\rho^+ - i \rho^-), \quad i(\rho^+ \psi^+)(\varphi^+ + \varphi^-) = \rho^- = (\rho^+ \psi^+)(\varphi^+ - \varphi^-) = \rho^-. \quad (29f)\]

while all orthogonality conditions of the type \(\rho^\pm \xi^\mp = 0\) are satisfied identically. Due to Eqs. (26), (28), (30), (31), only 3 scalar products can be now considered to be independent; we choose the three contractions and denote them as

\[\rho^+ \theta^+ \equiv x, \quad \eta^+ \psi^+ \equiv y, \quad \varphi^+ \xi^+ \equiv z. \quad (32)\]

As expected, the obtained equalities demonstrate a noticeable algebraic symmetry. Indeed, Eqs. (29a, 29b, 29c), expressing positive spinor-vectors through a previous one and a following one, have similar forms; the same property is possessed by other three groups of spinors. One also notes that the subsequent substitution of spinors

\[\xi^\pm(\psi^\pm) \rightarrow \theta^\pm[\xi^\pm(\psi^\pm)] \rightarrow \psi^\pm\{\theta^\pm[\xi^\pm(\psi^\pm)]\}\]

within Eqs. (29a−29f) yields a definite value of the cubic product

\[(\rho^+ \theta^+)(\eta^+ \psi^+)(\varphi^+ \xi^+) \equiv xyz = \frac{1 - i}{4}; \quad (33)\]

it can be easily verified that the rest of Eqs. (29) gives the same result. So only two contractions out of three from Eq. (32) are in fact independent, hence two free coefficients of two different spinor-vectors (or co-vectors) exhaustively determine all other spinors. The above observations hint to construct a recurrent formula for a “following” spinor from a “previous” one; it is convenient to use for this purpose the notation (14) of spinors as dyads, but those endowed with an extra index referring a spinor to one of the three spinor sets

\[n h^A_{(M)} \equiv (\xi^\pm, \theta^\pm, \psi^\pm), \quad n h^A_{(N)B} \equiv (\rho^\pm, \eta^\pm, \varphi^\pm), \quad (34)\]

\[n = 1, 2, 3, \text{ so that } 1 h^A_{(M)} \equiv \xi^\pm, \quad 2 h^A_{(M)} \equiv \theta^\pm, \quad 1 h^A_{(N)B} \equiv \rho^\pm, \text{ etc; then the left equalities of Eqs. (29b, 29c) are written as}\]

\[2 h^A_{(1)} = (1 h^A_{(1)B} 2 h^A_{(1)B})(1 h^A_{(1)} - 1 h^A_{(2)}), \quad 2 h^A_{(2)} = -i(1 h^A_{(1)B} 2 h^A_{(1)B})(1 h^A_{(1)} + 1 h^A_{(2)}).\]

These separate equations can be united into the following one (for \(n = 1\))

\[n+1 h^A_{(K)} = \frac{1 + i 2^{K-1}}{2} \left(n h^A_{(M)B} n+1 h^A_{(M)}\right) \times (\delta_{NK} + \varepsilon_{NK}) n h^A_{(M)}, \quad (35)\]

where \(\varepsilon_{NK} \equiv \delta^1_N \delta^2_K - \delta^2_N \delta^1_K\) is the 2D Levi-Civita symbol, summation is implied in all repeated 2D indices, and a relation of the type

\[\rho^+ \theta^+ = \frac{1 + i}{2} (\rho^+ \theta^+ + \rho^- \theta^-) \equiv \frac{1 + i}{2} \left(1 h^A_{(M)B} 2 h^A_{(M)}\right) \equiv x. \quad (36)\]
is used, showing the implied structure of a freely chosen coefficient \( x \). The recurrent formula (35) determines the functional dependence between any spinor vectors; its “cyclic” use\(^2\) leads to Eqs. (29a–29f). Recurrent relations for the spinor co-vectors corresponding to the rest of the set (29g–29l) are obtained from Eq. (35) by lowering the vector index with the help of the respective 2D metric tensor. Such a metric for, e.g., the spinor surface No. 2 \( \{ \eta^\pm, \varphi^\pm \} \) is defined as in Eq. (17):

\[
2g(\eta) \equiv \eta^+\eta^- + \eta^-\eta^+ \iff 2g_{AC} = 2h(M)A^2h(M)C.
\]

(37)

But if this metric is considered from the viewpoint of spin surface No. 1 \( \{ \rho^\pm, \xi^\pm \} \), the spinors \( \eta^\pm \) should be substituted by \( \rho^\pm \) with the help of Eqs. (29h, 29k), then Eq. (37) reads

\[
2g(\rho) \equiv -2(\eta^+\xi^+\rho^+ + \rho^-\rho^+).
\]

In the dyad notations [using Eq. (28), (36)], the metric has the form

\[
2g_{AC} = \frac{i}{\left(1+h(L)D^2h(D)_L\right)}\frac{\sigma_{PQ}^1h(P)A^1h(Q)C}{\sigma_{PQ}^2P^2Q^2 + \delta_1^2P^2Q^2},
\]

which helps us to suggest its “recurrent format” for any spinor surface:

\[
n+1g_{AC}(^n)h = \frac{i}{\left(1+h(L)D^{n+1}h(D)_L\right)}\frac{\sigma_{PQ}^n h(P)A^n h(Q)C}{\sigma_{PQ}^nP^nQ^n + \delta_1^nP^nQ^n}. \]

(38)

The metric (38) lowers the upper index of the spinor-vector (35), yielding the cyclic recurrent formula for the spinor-co-vector

\[
n+1h_{AC} = \frac{i(1 + i2K^{-1})}{2\left(1+h(L)D^{n+1}h(D)_L\right)}\left(\sigma_{PQ} - \tau_{PQ}\right)^n h(Q)C, \]

\[
\tau_{PQ} \equiv \delta_1^P\delta_1^Q - \delta_2^P\delta_2^Q; \]

(39)

Eqs. (39) lead to the relations (29g–29l). Thus all spinor-vector (co-vector) links (35), (39) are established, and generic expressions for the metric tensors of any spinor surface (38) as functions of any spinor families are deduced.

5. TABLE OF SPINOR SCALAR PRODUCTS

The basic spinors’ orthonormality conditions are “scalar products” on the dyad proper spin-surface. The correlations between all other scalar products

\[2\text{ If } n = 3 \text{, then } n + 1 \to 1.\]

Table 1 is obviously composed of four blocks of square \( 3 \times 3 \)-matrices of the type

\[
T \equiv \begin{pmatrix} S^+ & S^\pm \\ S^\mp & S^- \end{pmatrix},
\]

(40)

where elements of the blocks \( S^+, S^- \) are scalar products of spinors of the same parity, and elements of \( S^\pm \) are scalar products of spinors of opposite parity. Analyzing the properties of the matrix (40), one discovers that determinants of all its blocks vanish:

\[
\text{det } S^+ = \text{det } S^- = \text{det } S^\pm = \text{det } S^\mp = 0,
\]

(41)

though not identically, but due to interdependence of the free factors (33), so Eq. (33) represents a special solution of Eqs. (41). Table 1 represents a set of fundamental relations between quaternion spinors in the general case, but in applications particular values of the scalar products can be needed. Table 2 reflects a special case, where the explicitly shown spinor matrices are eigenvectors of the simplest vector \(Q\)-units from Eqs. (2), while the coefficients satisfying the condition (33) are \( x = (1-i)/2, \ y = z = 1/\sqrt{2} \).

If the scalar products in Table 2 are traditionally treated as mutual projections of involved unit vectors (i.e., as cosines of an angle between the vectors), then one has to conclude that all vectors from different spin-surfaces are inclined to each other at an angle proportional (up to a degree of the imaginary unit) to \( \pi/4 \); it is readily verified that any other choice of the \(Q\)-units’ eigenvectors and the coefficients (32) will lead to a similar result.
Table 2

| $\xi^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ | $\theta^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ | $\psi^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\xi^- = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\theta^- = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix}$ | $\psi^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
|---|---|---|---|---|---|
| $\rho^+ = \frac{1}{\sqrt{2}} (-1, 1)$ | 1 | $\frac{1 - i}{2}$ | 1 | 0 | $-\frac{1 + i}{2}$ | $-\frac{1}{\sqrt{2}}$ |
| $\eta^+ = \frac{1}{\sqrt{2}} (-i, 1)$ | $\frac{1 + i}{2}$ | 1 | $\frac{1}{\sqrt{2}}$ | $-\frac{1 + i}{2}$ | 0 | $-\frac{i}{\sqrt{2}}$ |
| $\varphi^+ = (0, 1)$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 | $-\frac{i}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | 0 |
| $\rho^- = \frac{i}{\sqrt{2}} (1, 1)$ | 0 | $\frac{1 - i}{2}$ | $\frac{i}{\sqrt{2}}$ | 1 | $-\frac{1 + i}{2}$ | $\frac{i}{\sqrt{2}}$ |
| $\eta^- = -\frac{1}{\sqrt{2}} (i, 1)$ | $-\frac{1 - i}{2}$ | 0 | $-\frac{1}{\sqrt{2}}$ | $-\frac{1 - i}{2}$ | 1 | $-iy$ |
| $\varphi^- = (1, 0)$ | $-\frac{1}{\sqrt{2}}$ | $\frac{i}{\sqrt{2}}$ | 0 | $-\frac{i}{\sqrt{2}}$ | $\frac{i}{\sqrt{2}}$ | 1 |

6. DISCUSSION

Thus, from the most general positions, a “structural content” of any sets of quaternion units is revealed, and the fundamental properties of the constituent functions are established. It is shown that the functions have a dual mathematical meaning: they are spinors (i.e., objects of 1/2 dimension of a vector) and, at the same time, born in couples, they form dyads, basic vectors of 2D surfaces (called here spin-surfaces). There are other characteristic properties of the spinors to be emphasized. The vector Q-units are interrelated by a nonlinear action, vector multiplication, while the respective spinor functions (“square roots” of the units) are linearly dependent, and a couple of spinors (2 rows and 2 columns of ‘±’ parity), eigenvectors of any Q-triad vector, is sufficient for building the whole set of Q-units. Moreover, each set of Q-units is expressed through quadratic combinations of one spinor couple uniquely. Hence altogether there are three variants of expressing all Q-units through spinor couples born as eigenvectors by three different vector units. With the help of these expressions, algebraic interrelations between spinor sets from the same family are found, and a table of all possible scalar products of the spinors is produced.

Table 2

| $\xi^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ | $\theta^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ | $\psi^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ | $\xi^- = -\frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | $\theta^- = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ -1 \end{pmatrix}$ | $\psi^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
|---|---|---|---|---|---|
| $\rho^+ = \frac{1}{\sqrt{2}} (-1, 1)$ | 1 | $\frac{1 - i}{2}$ | 1 | 0 | $-\frac{1 + i}{2}$ | $-\frac{1}{\sqrt{2}}$ |
| $\eta^+ = \frac{1}{\sqrt{2}} (-i, 1)$ | $\frac{1 + i}{2}$ | 1 | $\frac{1}{\sqrt{2}}$ | $-\frac{1 + i}{2}$ | 0 | $-\frac{i}{\sqrt{2}}$ |
| $\varphi^+ = (0, 1)$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ | 1 | $-\frac{i}{\sqrt{2}}$ | $-\frac{1}{\sqrt{2}}$ | 0 |
| $\rho^- = \frac{i}{\sqrt{2}} (1, 1)$ | 0 | $\frac{1 - i}{2}$ | $\frac{i}{\sqrt{2}}$ | 1 | $-\frac{1 + i}{2}$ | $\frac{i}{\sqrt{2}}$ |
| $\eta^- = -\frac{1}{\sqrt{2}} (i, 1)$ | $-\frac{1 - i}{2}$ | 0 | $-\frac{1}{\sqrt{2}}$ | $-\frac{1 - i}{2}$ | 1 | $-iy$ |
| $\varphi^- = (1, 0)$ | $-\frac{1}{\sqrt{2}}$ | $\frac{i}{\sqrt{2}}$ | 0 | $-\frac{i}{\sqrt{2}}$ | $\frac{i}{\sqrt{2}}$ | 1 |

A question arises: for which purpose is this boring maths routine? To the author’s understanding, the results of the study promise to reveal in the spinor algebra “no less geometry” than in a Q-triad naturally related to a Cartesian coordinate system in 3D space (locally quasi-Euclidean$^3$). The Q-spinors considered above, regarded as vectors and co-vectors on spin-surfaces, evidently form a certain mathematical entity that can be named “pre-geometry”$^4$ if the term “geometry” is reserved for the properties of the physical space admitting experimental observation and measurement (by vector Q-units in this context). But if the geometric perception of a Q-triad meets no obstacles, a visualization of the spinor pre-geometry constituting the basement of 3D space dimensions still remains vague. An advance in this area seems to be a challenging task because construction of a plausible geometric image of the “hardly conceivable” spinor could lead to a better understanding of the geometro-physical foundations of quantum mechanics. Indeed, Eqs. (9) and (10) evoke the existence of the spinor function

$$\Psi \equiv \sqrt{\det A} \exp(i\Phi)\psi,$$

that comprises all ingredients of a de Broglie-Pauli-type quantum-mechanical wave function: it contains a complex-number valued amplitude $\det A \in \mathbb{C}$, a pure wave part $\exp(\pm i\Phi)$ (if $\Phi \in \mathbb{R}$, which can always be achieved), and a spin term $\psi^+$ or $\psi^-$; each constituent may depend on the space-time coordinates.

At the same time, Eq. (42) describes the components of a 2D vector belonging to a pre-geometry spin-surface locally determined by the basic vector couple $h_\lambda^M \equiv \psi^\pm$. It is worthwhile to add that a picture of an abstract spin-surface given separately of its native Q-triad would be hardly informative, hence hardly helpful. The pre-geometry obviously “interacts” with the

$^3$This local (tangent) 3D space has the Euclidian plane metric tensor $g_{\alpha\beta} = \delta_{\alpha\beta}$, but it is not “completely” Euclidian since vector multiplication in it is not commutative.

$^4$In a sense, this model of pre-geometry corresponds to the famous notion suggested by J.A. Wheeler.
geometry, e.g., rotations of an $SU(2)$ spinor should cause double-angled rotations of its Q-triad, and this must even be reflected in the mutual image. The Q-spinor properties detailed in this study seem to be a good basement for an attempt to generate such an image; a variant of a consistent picture will be offered in a subsequent publication.

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