Disk counting statistics near hard edges of random normal matrices: the multi-component regime

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Abstract

We consider a two-dimensional point process whose points are separated into two disjoint components by a hard wall, and study the multivariate moment generating function of the corresponding disk counting statistics. We investigate the “hard edge regime” where all disk boundaries are a distance of order $\frac{1}{n}$ away from the hard wall, where $n$ is the number of points. We prove that as $n \to +\infty$, the asymptotics of the moment generating function are of the form

$$\exp\left(C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}} + O(n^{-\frac{3}{5}})\right),$$

and we determine the constants $C_1, \ldots, C_4$ explicitly. The oscillatory term $F_n$ is of order 1 and is given in terms of the Jacobi theta function. Our theorems allow us to derive various precise results on the disk counting function. For example, we prove that the asymptotic fluctuations of the number of points in one component are of order 1 and are given by an oscillatory discrete Gaussian. Furthermore, the variance of this random variable enjoys asymptotics described by the Weierstrass $\wp$-function.

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1 Introduction and statement of results

In recent years there have been a lot of works on counting statistics of two dimensional point processes, see e.g. [23, 42, 43, 53, 19, 24, 54, 3, 27, 18, 9] and references therein. The common feature of these works is that they all deal exclusively with models for which the points condensate on a single connected component ("the one-component regime"). In this paper we deviate from these earlier works in that we study the disk counting statistics of a Coulomb gas (at inverse temperature $\beta = 2$) whose points are separated into two disjoint components by a hard wall. Let us now introduce the Coulomb gas model investigated in this work.

The Mittag-Leffler ensemble is the following joint probability density function

$$\frac{1}{n!Z_n} \prod_{1 \leq j < k \leq n} |\zeta_k - \zeta_j|^2 \prod_{j=1}^{n} (|\zeta_j|^{2a}e^{-n|\zeta_j|^{2b}}, \quad \zeta_1, \ldots, \zeta_n \in \mathbb{C},$$

where $b > 0$ and $\alpha > -1$ are fixed parameters and $Z_n$ is the normalization constant. As $n \to +\infty$, with high probability the random points $\zeta_1, \ldots, \zeta_n$ accumulate on the disk centered at 0 of radius $b^{-\frac{1}{2\alpha}}$ according to the probability measure $\mu(d^2z) = b^{\alpha} |z|^{2\alpha-2} d^2z$ [39, 50]. This determinantal point process

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generates the complex Ginibre process (which corresponds to \((b, \alpha) = (1, 0)\)) and has attracted a lot of attention over the years, see e.g. [7, 10, 22].

In this paper we focus on the Mittag-Leffler ensemble with a hard wall that separates the random points into two disjoint components. To be precise, let \(0 < \rho_1 < \rho_2 < b^{-\frac{1}{2}}\). We consider the probability density

\[
\frac{1}{n! Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \prod_{j=1}^{n} e^{-nQ(z_j)}, \quad z_1, \ldots, z_n \in \mathbb{C},
\]

where \(Z_n\) is the normalization constant and

\[
Q(z) = \begin{cases} |z|^{2b} - \frac{2\pi}{n} \ln |z|, & \text{if } |z| \in [0, \rho_1] \cup [\rho_2, +\infty), \\ +\infty, & \text{otherwise.} \end{cases}
\]

Because \(\rho_1, \rho_2 < b^{-\frac{1}{2}}\), the macroscopic behavior of (1.2) is different from that of (1.1); it is described by a probability measure \(\mu_b\), which is supported on \(\{z \in \mathbb{C} : |z| \in [0, \rho_1] \cup [\rho_2, +\infty)\}\) and has a singular component on the circles of radii \(\rho_1\) and \(\rho_2\). This measure can be computed using standard balayage techniques [50] (we provide the details of this computation in Appendix A) and is given by

\[
\mu_b(d^2 z) = 2b^{2b-1} X_{[0,\rho_1] \cup \rho_2, b^{-\frac{1}{2}}} (r)dr d\theta \frac{d\theta}{2\pi} + \sigma_1 \delta_{\rho_1}(r)dr \frac{d\theta}{2\pi} + \sigma_2 \delta_{\rho_2}(r)dr \frac{d\theta}{2\pi},
\]

where \(z = re^{i\theta}, r > 0, \theta \in (-\pi, \pi]\) and

\[
\sigma_1 = \sigma_* - b\rho_1^{2b}, \quad \sigma_2 = b\rho_2^{2b} - \sigma_* , \quad \sigma_* := \frac{\rho_2^{2b} - \rho_1^{2b}}{2 \ln (\rho_2/\rho_1)}.
\]

The assumption \(0 < \rho_1 < \rho_2 < b^{-\frac{1}{2}}\) implies that \(b\rho_1^{2b} < \sigma_* < b\rho_2^{2b}\) and hence \(\sigma_1, \sigma_2 > 0\). The quantity \(\sigma_*\) is the mass of \(\mu_b\) on \(|z| \leq \rho_1\). Indeed, straightforward calculations show that

\[
\left. \begin{array}{l}
\int_{|z| \leq \rho_1} \mu_b(d^2 z) = \sigma_*, \\
\int_{|z| \geq \rho_2} \mu_b(d^2 z) = 1 - \sigma_*, 
\end{array} \right\} \quad \text{for large } n
\]

which means that for large \(n\), the number of \(z_j\)’s on \(|z| \leq \rho_1\) is roughly \(\sigma_* n\) with high probability (see also Corollary 1.6 and the asymptotics (1.29) and (1.32) below). The point process (1.2) is an example of a two-dimensional Coulomb gas that is rotation invariant (meaning that the density (1.2) remains unchanged if all \(z_j\)’s are multiplied by \(e^{i\beta}, \beta \in \mathbb{R}\)). This ensemble can be seen as a conditional process where the points from (1.1) are conditioned on the hole event \(\mathcal{H}\) that no \(\zeta_j\)’s lie in the annulus centered at \(0\) of radii \(\rho_1\) and \(\rho_2\). The partition function \(Z_n\) of (1.2) is precisely equal to \(Z_n(\mathcal{H})\), and its large \(n\) asymptotics were investigated in [1, 2, 25]; see also [38, 35, 40, 5, 6, 4, 36, 43] for related works on the hole event. The process (1.2) can also be realized as the eigenvalues of an \(n \times n\) random normal matrix \(M\) taken at random according to the probability density proportional to \(e^{-n\text{tr} Q(M)}dM\), where “\(\text{tr}\)” is the trace and \(dM\) is the measure on the set of \(n \times n\) normal matrices induced by the flat Euclidean metric of \(\mathbb{C}^{n \times n}\) [47, 28, 33]. Correlation kernels near hard edges have been studied in [60, 48, 51].

We will focus on the “the hard edge regime”, i.e. when all disk boundaries are a distance of order \(\frac{1}{n}\) away from the hard edges \(|z| = \rho_1\) and \(|z| = \rho_2\). (Further away from the hard edges, it turns out that there are no oscillations and the situation gets similar to the one-component semi-hard edge regime considered in [9].) Let us now be more specific. Let \(\text{N}(y) := \# \{z_j : |z_j| < y\} \) be the random
variable that counts the number of points of (1.2) in the disk centered at 0 of radius \( y \). Our main result is a precise asymptotic formula as \( n \to +\infty \) for the multivariate moment generating function (MGF)

\[
E \left[ \prod_{j=1}^{2m} e^{u_j N(r_j)} \right]
\]

where \( m \in \mathbb{N}_{>0} \) is arbitrary (but fixed), \( u_1, \ldots, u_{2m} \in \mathbb{R} \), and the radii \( r_1, \ldots, r_{2m} \) satisfy \( r_1 < \cdots < r_{2m} \) and are merging at a critical speed in the following way

\[
\begin{align*}
  r_\ell &= \rho_1 \left(1 - \frac{\ell t}{n}\right)^{\frac{1}{b}}, & \ell &\geq 0, \quad \ell = 1, \ldots, m, \\
  r_\ell &= \rho_2 \left(1 + \frac{\ell t}{n}\right)^{\frac{1}{b}}, & \ell &\geq 0, \quad \ell = m + 1, \ldots, 2m.
\end{align*}
\]

As \( n \to +\infty \), we prove that \( E \left[ \prod_{j=1}^{2m} e^{u_j N(r_j)} \right] \) enjoys asymptotics of the form

\[
\exp \left( C_1 n + C_2 \ln n + C_3 + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} + O(n^{-\frac{3}{5}}) \right),
\]

and we determine \( C_1, \ldots, C_4, \mathcal{F}_n \) explicitly. As corollaries of our various results on the generating function (1.7), we also provide a central limit theorem for the joint fluctuations of \( N(r_1), \ldots, N(r_{2m}) \), and precise asymptotic formulas for all cumulants of these random variables. Even for \( m = 1 \) our results are new.

We now introduce the necessary material to present our results. For \( j \geq 0 \), define

\[
\begin{align*}
  T_j(x; \vec{t}, \vec{u}) &= \sum_{\ell=1}^{m} \omega_\ell t_\ell e^{-\frac{1}{b} (e^{-b u_\ell^2} - x)} , & \tilde{T}_j(x; \vec{t}, \vec{u}) &= \sum_{\ell=m+1}^{2m} \omega_\ell t_\ell e^{-\frac{1}{b} (e^{-b u_\ell^2} - x)} ,
\end{align*}
\]

where

\[
\omega_\ell = \begin{cases} 
  e^{u_\ell + \cdots + u_{2m}} - e^{u_{\ell+1} + \cdots + u_{2m}}, & \text{if } \ell < 2m, \\
  e^{u_{2m}} - 1, & \text{if } \ell = 2m, \\
  1, & \text{if } \ell = 2m + 1,
\end{cases}
\]

Figure 1: Illustration of the point processes corresponding to (1.2) with \( n = 4096, \rho_1 = \frac{3}{5} b^{-\frac{1}{b}}, \rho_2 = \frac{4}{5} b^{-\frac{1}{b}}, \alpha = 0 \) and the indicated values of \( b \).
and \( \vec{r} = (t_1, \ldots, t_{2m}) \), \( \vec{u} = (u_1, \ldots, u_{2m}) \). Since the quantities \( T_0(b \rho_1^2; \vec{r}, \vec{u}) \) and \( \tilde{T}_0(b \rho_2^2; \vec{r}, \vec{u}) \) are independent of \( \vec{r} \), we will simply write \( T_0(b \rho_1^2; \vec{u}) \) and \( \tilde{T}_0(b \rho_2^2; \vec{u}) \) instead. Define

\[
\begin{align*}
    f(x; \vec{r}, \vec{u}) &= \frac{-\left(\frac{b \rho_1^2}{x} + \frac{a}{x} \right) T_1(x; \vec{r}, \vec{u}) - \frac{x}{2b} \tilde{T}_2(x; \vec{r}, \vec{u})}{1 + T_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})}, \\
    \hat{f}(x; \vec{r}, \vec{u}) &= \frac{\left(\frac{b \rho_2^2}{x} - \frac{a}{x} \right) \tilde{T}_1(x; \vec{r}, \vec{u}) + \frac{x}{2b} \tilde{T}_2(x; \vec{r}, \vec{u})}{1 - \tilde{T}_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})}.
\end{align*}
\]  

(1.13) 

(1.14)

Let \( \Omega := e^{u_1 + \cdots + u_{2m}} \) and let

\[
Q(\vec{r}, \vec{u}) := \frac{1 + T_0(\sigma_2; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})}{1 - \tilde{T}_0(\sigma_2; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})}.
\]  

(1.15)

Our main result involves \( \ln Q(\vec{r}, \vec{u}) \) and the next lemma, whose proof is given in Appendix B, shows that this logarithm is well-defined. It also shows that \( f \) and \( \hat{f} \) are smooth functions of \( x \in (b \rho_1^2, b \rho_2^2) \).

**Lemma 1.1.** Suppose \( \vec{u} \in \mathbb{R}^{2m} \), \( t_1 \cdots t_m \geq 0 \), and \( 0 \leq t_{m+1} < \cdots < t_{2m} \). For \( x \in [b \rho_1^2, b \rho_2^2] \), it holds that

\[
1 + T_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u}) > 0, \quad 1 - \tilde{T}_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u}) > 0.
\]

The Jacobi theta function \( \theta(z; \tau) \) is defined for \( z \in \mathbb{C} \) and \( \Im \tau > 0 \) by

\[
\theta(z; \tau) = \sum_{\ell = -\infty}^{\infty} e^{\pi i \ell^2 \tau + 2 \pi i \ell z}.
\]

This function satisfies

\[
\theta(z + 1; \tau) = \theta(z; \tau), \quad \theta(z + \tau; \tau) = e^{-\pi i \tau} e^{-\pi i \theta(z)}, \quad \theta(-z) = \theta(z), \quad \text{for all } z \in \mathbb{C},
\]

(1.16)

see also [49, Chapter 20] for further properties. We are now ready to state our main result.

**Theorem 1.2** (Merging radii at the hard edge: the multi-component regime). Let \( m \in \mathbb{N}_{>0} \), \( b > 0 \), \( 0 < \rho_1 < \rho_2 < b^{-\frac{1}{2}} \), \( t_1, \ldots, t_{2m} \geq 0 \), and \( \alpha > -1 \) be fixed parameters such that \( t_1 > \cdots > t_m \geq 0 \) and \( 0 \leq t_{m+1} < \cdots < t_{2m} \). For \( n \in \mathbb{N}_{>0} \), define

\[
r_\ell = \begin{cases} \rho_1 \left(1 - \frac{u_\ell}{n}\right)^{1/2}, & \ell = 1, \ldots, m, \\
\rho_2 \left(1 + \frac{u_\ell}{n}\right)^{1/2}, & \ell = m + 1, \ldots, 2m.\end{cases}
\]

(1.17)

For any fixed \( x_1, \ldots, x_{2m} \in \mathbb{R} \), there exists \( \delta > 0 \) such that

\[
\mathbb{E} \left[ \prod_{j=1}^{2m} e^{u_j N(r_j)} \right] = \exp \left( C_1 n + C_2 \ln n + C_3 + \mathcal{F}_n + \frac{C_4}{\sqrt{n}} + O(n^{-\frac{\delta}{2}}) \right), \quad \text{as } n \to +\infty
\]

(1.18)

uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta\} \), where

\[
C_1 = b \rho_1^{2m} \sum_{j=1}^{2m} u_j + \int_{b \rho_2^{2m}}^{\sigma_a} \ln(1 + T_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})) dx + \int_{a}^{b \rho_1^2} \ln(1 - \tilde{T}_0(x; \vec{r}, \vec{u}) + \tilde{T}_0(b \rho_2^2; \vec{u})) dx,
\]

\[
C_2 = -\frac{b \rho_2^{2m}}{2} T_1((b \rho_2^2); \vec{r}, \vec{u}) + \frac{b \rho_1^{2m}}{2} \tilde{T}_1(b \rho_2^2, \vec{r}, \vec{u}),
\]
\[ C_3 = -\frac{1}{2} \sum_{j=1}^{2m} u_j - \left( \alpha - \frac{2 \ln(\sigma_2/\sigma_1) + \ln Q(\vec{r}, \vec{u})}{4 \ln(p_2/p_1)} \right) \ln Q(\vec{r}, \vec{u}) \]

\[ + \int_{\partial F_2} f(x; \vec{r}, \vec{u}) + \frac{b_2^{2b} T_1(b_2^{2b}; \vec{r}, \vec{u})}{\Omega} dx + \int_{\partial F_2} f(x; \vec{r}, \vec{u}) - \frac{b_2^{2b} T_1(b_2^{2b}; \vec{r}, \vec{u})}{\Omega} \] \[ - \rho_1^{3d} T_1(b_2^{3d}; \vec{r}, \vec{u}) - \rho_2^{3d} \hat{T}_1(b_2^{3d}; \vec{r}, \vec{u}) - \rho_2^{3d} \hat{T}_1(b_2^{3d}; \vec{r}, \vec{u})^2 \] \[ = \sqrt{2} \pi i b \left( \frac{\rho_1^{3d} T_2(b_2^{3d}; \vec{r}, \vec{u}) - \rho_1^{3d} T_1(b_2^{3d}; \vec{r}, \vec{u})}{\Omega} - \rho_1^{3d} T_1(b_2^{3d}; \vec{r}, \vec{u})^2 \right), \]

\[ F_n = \frac{\ln(\sigma_n + 1/2 - \alpha - \ln(\sigma_2/\sigma_1) + \ln Q(\vec{r}, \vec{u}) + \frac{\pi i}{4 \ln(p_2/p_1)} }{\ln(\sigma_2/\sigma_1) + \ln Q(\vec{r}, \vec{u}) + \frac{\pi i}{4 \ln(p_2/p_1)} ), \] \[ (1.19) \]

and the constant \( \mathcal{I} \in \mathbb{R} \) is given by

\[ \mathcal{I} = \int_{-\infty}^{+\infty} \left\{ \frac{y e^{-y^2}}{\sqrt{\pi} \text{erfc}(y)} - \chi_{(0, +\infty)}(y) \left[ y^2 + \frac{1}{2} \right] \right\} dy \approx -0.81367. \] \[ (1.20) \]

In particular, since \( \mathbb{E}\left[ \prod_{j=1}^{2m} e^{u_j N(r)} \right] \) is analytic in \( u_1, \ldots, u_{2m} \in \mathbb{C} \) and is positive for \( u_1, \ldots, u_{2m} \in \mathbb{R} \), the asymptotic formula \( (1.18) \) together with Cauchy’s formula shows that

\[ \partial^{k_1} \cdots \partial^{k_{2m}} \left( \ln \mathbb{E}\left[ \prod_{j=1}^{2m} e^{u_j N(r)} \right] - \left( C_1 n + C_2 \ln n + C_3 + \frac{C_4}{\sqrt{n}} \right) \right) = O(n^{-2}), \quad \text{as } n \to +\infty, \]

\[ (1.21) \]

for any \( k_1, \ldots, k_{2m} \in \mathbb{N} \), and \( u_1, \ldots, u_{2m} \in \mathbb{R} \).

**Remark 1.3.** It is well-known that the theta function is a universal object of one-dimensional point processes in the multi-cut regime, see e.g. [31, 32, 16, 37, 52, 29, 17, 13, 14, 34, 15, 41, 26]. In dimension two, the emergence of this function was conjectured in [45, Section 1.5] and proved in [25] in the context of large gap problems (or equivalently, partition function asymptotics with hard edges). This function also appears in [8] in the study of microscopic correlations and smooth macroscopic statistics of two-dimensional rotation-invariant ensembles with soft edges. To our knowledge, Theorem 1.2 is the first result on counting statistics of a two-dimensional point process involving the \( \theta \)-function.

**Remark 1.4.** (Periodicity of \( F_n \)) In the multi-cut regime of one-dimensional point processes, asymptotic formulas are, in general, only quasiperiodic in \( n \), see e.g. [38, 32, 52, 17]. However, in the special case where the mass of the equilibrium measure on each interval of the support is a rational number, these asymptotic formulas become periodic, see e.g. [46] and [26, Corollary 2.2].

Interestingly, Theorem 1.2 shows that an analogous phenomenon holds in our two-dimensional setting. To see this, recall from (1.16) that \( \theta \) is periodic of period 1. Since \( n \) runs over the integers, the function \( n \to F_n \) is, in general, only quasiperiodic in \( n \). However, it follows from (1.19) and (1.6) that if the mass \( \sigma_2 = \int_{|z| \leq \rho_2} \mu_2(d^2z) \) is rational, then \( n \to F_n \) is periodic in \( n \).

More generally, asymptotic formulas related to a given two-dimensional point process are expected to be periodic in \( n \) whenever the masses of the components of the equilibrium measure are all rational. This holds true in the setting of this paper, as well as in the two-dimensional soft edge setting of [8].
Recent works \[30, 21\] on the so-called lemniscate ensemble also support this belief. This model has a \(d\)-fold rotational symmetry, where \(d\) is a parameter that determines the number of components. The mass of the equilibrium measure on each component is \(1/d\), and therefore one expects asymptotic formulas in this model to be periodic in \(n\) of period \(d\). The formulas in \[30, 21\] are consistent with this expectation: these formulas involve \(n = N d\) points and are not oscillatory as \(N \to +\infty\).

**Remark 1.5.** (Probabilistic interpretation of \(F_n\)) Using the well-known relation (see \[49, eq 21.5.8\])

\[
\theta(\tau^{-1} z; -\tau^{-1}) = e^{\pi z \tau^{-1}} \sqrt{-4\pi \theta(z; \tau)},
\]

we can rewrite \(F_n\) as

\[
e^{F_n} = \exp \left(-\frac{(\ln Q(\vec{t}, \vec{u}))^2}{4 \ln(\frac{p_1}{p_2})} \right) \sum_{\ell \in \mathbb{Z}} \left( \frac{p_2}{p_1} \right)^{(\ell - \langle \Lambda_n \rangle)^2} Q(\vec{t}, \vec{u})^{\ell - \langle \Lambda_n \rangle} \sum_{\ell \in \mathbb{Z}} \left( \frac{p_2}{p_1} \right)^{(\ell - \langle \Lambda_n \rangle)^2}, \tag{1.22}
\]

where \(\langle \Lambda_n \rangle := \Lambda_n - [\Lambda_n]\) is the fractional part of \(\Lambda_n\) and

\[
\Lambda_n := \sigma_n - \frac{1}{2} - \alpha + \frac{\ln(\sigma_2/\sigma_1)}{2 \ln(p_2/p_1)}.
\]

The right-most fraction in (1.22) is related to the random variable \(v_n(\vec{t}, \vec{u}) := (X_n - \langle \Lambda_n \rangle) \ln Q(\vec{t}, \vec{u})\) via

\[
\frac{\sum_{\ell \in \mathbb{Z}} (\frac{p_2}{p_1})^{(\ell - \langle \Lambda_n \rangle)^2} Q(\vec{t}, \vec{u})^{\ell - \langle \Lambda_n \rangle}}{\sum_{\ell \in \mathbb{Z}} (\frac{p_2}{p_1})^{(\ell - \langle \Lambda_n \rangle)^2}} = \mathbb{E}[e^{v_n(\vec{t}, \vec{u})}],
\]

where \(X_n\) is the discrete Gaussian random variable on \(\mathbb{Z}\) defined by

\[
\mathbb{P}(X_n = x) = (\frac{p_2}{p_1})^{(x - \langle \Lambda_n \rangle)^2}, \quad x \in \mathbb{Z}. \tag{1.24}
\]

In the multi-cut regime of one-dimensional point processes, fluctuation formulas analogous to (1.22) involving an oscillatory discrete Gaussian can be found in \[17, Section 8\] and \[26\].

The following corollary gives a probabilistic interpretation of \(X_n\) in terms of counting statistics and is analogous to the earlier result \[26, Corollary 1.4\] obtained in dimension one.

**Corollary 1.6.** Let \(x \in \mathbb{Z}\) be fixed. As \(n \to +\infty\),

\[
\mathbb{P}(N(p_1) = [\Lambda_n] + x) = \mathbb{P}(X_n = x) + o(1). \tag{1.25}
\]

**Proof.** The proof is inspired by the proof of \[26, Corollary 1.4\]. Using Theorem 1.2 with \(m = 1, u_2 = 0\) and \(t_1 = 0 = t_2\), we have \(C_1 = u_1\sigma_n, \ C_2 = 0, \ln Q(\vec{0}, \vec{u}) = u_1, \ C_3 = -\frac{u_1}{2} - u_1(x - 2\frac{\ln(\sigma_2/\sigma_1)}{2 \ln(p_2/p_1)}), \ C_4 = 0,\) and thus

\[
\mathbb{E}[e^{u_1 N(p_1)}] = e^{u_1 \Lambda_n} \mathbb{E}[e^{u_1 (X_n - \langle \Lambda_n \rangle)}(1 + \mathcal{O}(n^{-\frac{2}{3} \sigma}))], \quad \text{as } n \to +\infty , \tag{1.26}
\]

uniformly for \(u_1 \in K\), where \(K \subset \mathbb{R}\) is a compact subset containing 0. The rest of the proof proceeds by contradiction. Fix \(x \in \mathbb{Z}\) and suppose that (1.25) does not hold. Then there exists a sequence \(\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}\) such that \(\mathbb{P}(N(p_1) = [\Lambda_{n_k}] + x)\) remains bounded away from \(\mathbb{P}(X_{n_k} = x)\) for all \(k\). Since \(\langle \Lambda_{n_k} \rangle \in [0, 1)\) for all \(k\), there exists a subsequence \(\{n_k_j\}_{j=1}^{\infty}\) such that \(\langle \Lambda_{n_k_j} \rangle \to y_* \in [0, 1]\) as
$j \to +\infty$. Let $X_*$ be the random variable defined as in (1.24) but with $(\Lambda_n)$ replaced by $y_*$. Then (1.26) together with the fact that

$$E[e^{u_1X_{n,j}}] = E[e^{u_1X_*}] + o(1), \quad \text{as } j \to +\infty \text{ uniformly for } u_1 \in K$$

implies that

$$E[e^{u_1(N(\rho_1) - [\Lambda_{n,j}])] = E[e^{u_1X_*}] + o(1), \quad \text{as } j \to +\infty \text{ uniformly for } u_1 \in K.$$

By [12, top of page 415], this implies that $N(\rho_1) - [\Lambda_{n,j}]$ converges in distribution to $X_*$ as $j \to +\infty$. Since $P(X_{n,j} = x) = P(X_* = x) + o(1)$ as $j \to +\infty$, we thus have $P(N(\rho_1) = [\Lambda_{n,j}] + x) = P(X_{n,j} = x) + o(1)$ as $j \to +\infty$. We have obtained our contradiction.

For $\vec{j} \in (N^{2m})_{\geq 1} := \{\vec{j} = (j_1, \ldots, j_{2m}) \in N : j_1 + \cdots + j_{2m} \geq 1\}$, the joint cumulant $\kappa_{\vec{j}} = \kappa_{\vec{j}}(r_1, \ldots, r_{2m}; n, b, \alpha)$ of $N(r_1), \ldots, N(r_{2m})$ is defined by

$$\kappa_{\vec{j}} = \kappa_{j_1, \ldots, j_{2m}} := \partial_{u_1}^j \ln E[e^{u_1N(r_1) + \cdots + u_{2m}N(r_{2m})}]|_{\vec{u} = 0}, \quad (1.27)$$

where $\partial_{u_1}^j := \partial_{u_1}^{j_1} \cdots \partial_{u_2}^{j_2}$. As an immediate corollary of Theorem 1.2, we obtain the large $n$ behavior of any cumulant $\kappa_{\vec{j}}$.

**Corollary 1.7** (Asymptotics of cumulants). Let $m \in \mathbb{N}_{\geq 0}$, $b > 0$, $0 < \rho_1 < \rho_2 < b^{-1}$, $\vec{j} \in (N^{2m})_{\geq 1}$, $\alpha > -1$, $t_1 > \cdots > t_m \geq 0$ and $0 \leq t_{m+1} < \cdots < t_{2m}$ be fixed. For $n \in \mathbb{N}_{\geq 0}$, define $\{r_\ell\}_{\ell=1}^{2m}$ by (1.17). As $n \to +\infty$, the joint cumulant $\kappa_{\vec{j}}$ satisfies

$$\kappa_{\vec{j}} = \partial_{u_1}^j C_1|_{\vec{u} = 0} n + \partial_{u_2}^j C_2|_{\vec{u} = 0} \ln n + \partial_{u_3}^j C_3|_{\vec{u} = 0} + \partial_{u_4}^j F_n|_{\vec{u} = 0} + \frac{\partial_{u_5}^j C_1|_{\vec{u} = 0}}{\sqrt{n}} + O(n^{-1}), \quad (1.28)$$

where $C_1, C_2, F_n$ are as in Theorem 1.2.

Since $\mathbb{E}[N(r)] = \kappa_1(r)$, we obtain the following result after setting $\vec{j} = 1$ in Corollary 1.7 and performing some long but straightforward calculations.

**Corollary 1.8** (Asymptotics of $\mathbb{E}[N(r)]$). Under the assumptions of Corollary 1.7, the expectation value $\mathbb{E}[N(r)]$ obeys the following formula for any $1 \leq \ell \leq 2m$:

$$\mathbb{E}[N(r)] = b_1(t_\ell) n + c_1(t_\ell) \ln n + d_1(t_\ell) + f_1(n, t_\ell) + e_1(t_\ell)n^{-\frac{1}{2}} + O(n^{-\frac{1}{2}})$$

as $n \to +\infty$, where

$$b_1(t_\ell) = \begin{cases} b_1^{(2\ell)} + b_1^{(1\ell)} & \ell = m, m + 1, \ldots, t_m, \\
\frac{b_1^{(2\ell)} - b_1^{(1\ell)}}{t_\ell} & \ell = m, m + 1, \ldots, 2m, \ell > t_m. \end{cases}$$

$$c_1(t_\ell) = \begin{cases} -\frac{b_1^{(2\ell)} t_\ell}{2} & \ell = 1, \ldots, m, \\
\frac{b_1^{(2\ell)} t_\ell}{2} & \ell = m + 1, \ldots, 2m. \end{cases}$$

$$d_1(t_\ell) = -\frac{1}{2} e^{-\frac{t_\ell}{2}} \left(\alpha - \frac{\ln(\sigma_2/\sigma_1)}{2 \ln(\rho_2/\rho_1)}\right)^2 + b_1^{(2\ell)} t_\ell \left(\frac{b_1^{(2\ell)} t_\ell}{\sqrt{2\pi}(\sigma_2 - b_1^{(2\ell)})}\right)^2.$$
so we can express this second derivative in terms of the Weierstrass $\wp$-function. Indeed, these formulas involve the second derivative of $\ln \ell$.

We can also obtain asymptotic formulas for the covariance and variance of the disk counting function $m$. If $1 \leq \ell, k \leq m$, then $N(r_{\ell})$ and $N(r_{k})$ count the number of points in disks whose radii $r_{\ell}$ and $r_{k}$ are close to $\rho_{1}$.

If $m + 1 \leq \ell \leq k \leq 2m$, then $N(r_{\ell})$ and $N(r_{k})$ count the number of points in disks whose radii $r_{\ell}$ and $r_{k}$ are close to $\rho_{2}$.

If $1 \leq \ell \leq m$ and $m + 1 \leq k \leq 2m$, then $N(r_{\ell})$ counts the number of points in a disk of radius $r_{\ell} \approx \rho_{1}$, whereas $N(r_{k})$ counts the number of points in a disk of radius $r_{k} \approx \rho_{2}$.

The formulas for the covariance are naturally expressed in terms of the Weierstrass $\wp$-function. Indeed, these formulas involve the second derivative of $\ln \ell$. By [49, formula 23.6.14] $(\theta_{1}(\pi \tau)$ and $\theta_{3}(\tau)$ in [49] correspond here to $\theta_{1}(z)$ and $\theta(z)$, respectively).

$$\wp(z; \tau) = \frac{\theta'''_{1}(0; \tau)}{3\theta'_{1}(0; \tau)} - \frac{d^{2}}{dz^{2}} \ln \theta_{1}(z; \tau), \quad \text{where} \quad \theta_{1}(z; \tau) = e^{i \pi z + \frac{\pi \tau}{2}} \theta \left( z + \frac{1 + \tau}{2}; \tau \right),$$

so we can express this second derivative in terms of the Weierstrass $\wp$-function as

$$- \frac{d^{2}}{dz^{2}} \ln \theta(z; \tau) = \wp \left( z - \frac{1 + \tau}{2}; \tau \right) = c,$$

where $c \in \mathbb{C}$ is defined by

$$c := \frac{\theta'''_{1}(0; \tau)}{3\theta'_{1}(0; \tau)} \quad (1.30)$$
The function \( z \mapsto \varphi(z; \tau) \) is doubly periodic in the complex plane, of periods 1 and \( \tau \). In the following corollaries, we let \( c = c(\rho_1, \rho_2) \) be given by (1.30) with \( \tau \) defined by

\[
\tau := \frac{\pi i}{\ln(\rho_2/\rho_1)}.
\]

**Corollary 1.9** (Asymptotics of the covariance for \( 1 \leq \ell \leq k \leq m \)). Under the assumptions of Corollary 1.7, the covariance \( \text{Cov}(N(r_\ell), N(r_k)) \) obeys the following formula for any \( 1 \leq \ell \leq k \leq m \):

\[
\text{Cov}(N(r_\ell), N(r_k)) = b_{(1,1)}(t_\ell, t_k) n + c_{(1,1)}(t_\ell, t_k) \ln n + d_{(1,1)}(t_\ell, t_k) + f_{(1,1)}(n, t_\ell, t_k) + e_{(1,1)}(t_\ell, t_k)n^{-\frac{1}{2}} + O(n^{-\frac{3}{2}})
\]
as \( n \to +\infty \), where

\[
b_{(1,1)}(t_\ell, t_k) = \frac{1}{t_\ell} e^{-\frac{\ell}{2} \left( \ln(2\rho_2/\rho_1) \right)} - \frac{1}{t_\ell + t_k} e^{-\frac{\ell}{2} \left( \ln(2\rho_2/\rho_1) \right)} \quad \text{if } t_\ell > 0,
\]

\[
b_{(1,1)}(t_\ell, 0) = 0, \quad c_{(1,1)}(t_\ell, t_k) = \frac{b_{(1,2)}(t_k)}{2},
\]

\[
d_{(1,1)}(t_\ell, t_k) = -e^{-\frac{\ell}{2} (\ln(2\rho_2/\rho_1))} \left( \alpha - \frac{\ln(\rho_2/\rho_1)}{2\ln(2\rho_2/\rho_1)} \right) + e^{-\frac{\ell}{2} (\ln(2\rho_2/\rho_1))} \left( \alpha - \frac{\ln(\rho_2/\rho_1) - 1}{2\ln(2\rho_2/\rho_1)} \right)
\]

\[
+ \int_{b_{(1,2)}(t_k)} b_{(1,2)}(t_k) \frac{1 - e^{-\frac{\ell}{2} (x-b_{(1,2)}^2)}}{x - b_{(1,2)}^2} + \frac{x t_\ell^2 + 2\alpha t_k e^{-\frac{\ell}{2} (x-b_{(1,2)}^2)}}{2b} dx - b_{(1,2)}(t_k) \ln \left( \frac{b_{(1,2)}(t_k)}{\sqrt{2\pi (\ln(2\rho_2/\rho_1))}} \right),
\]

\[
f_{(1,1)}(n, t_\ell, t_k) = -\frac{e^{-\frac{\ell}{2} (\ln(2\rho_2/\rho_1)) - \frac{\ell}{2} (x-b_{(1,2)}^2)}}{4\ln(2\rho_2/\rho_1)^2} \left\{ \varphi \left( n\sigma_* + \frac{\tau}{2} - \alpha + \frac{\ln(\rho_2/\rho_1)}{2\ln(2\rho_2/\rho_1)} \right) \right\} - c
\]

\[
- 2(e^{\frac{\ell}{2} (\ln(2\rho_2/\rho_1))} - 1) \ln(\rho_2/\rho_1) (\ln 2) \left( n\sigma_* + \frac{\tau}{2} - \alpha + \frac{\ln(\rho_2/\rho_1)}{2\ln(2\rho_2/\rho_1)} \right),
\]

\[
c_{(1,1)}(t_\ell, t_k) = \sqrt{2} \mathbb{I} b_{(1,2)}(t_k) (1 - b_{(1,2)}^2(2t_\ell + t_k)).
\]

In particular, with \( m = 1, \ell = k = 1 \) and \( t_1 = 0 \), as \( n \to +\infty \) we obtain

\[
\text{Var}[N(\rho_1)] = \frac{1}{2\ln(\rho_2/\rho_1)} - \frac{\varphi(n\sigma_* + \frac{\tau}{2} - \alpha + \frac{\ln(\rho_2/\rho_1)}{2\ln(2\rho_2/\rho_1)} \frac{\tau}{2}) - c}{4\ln(\rho_2/\rho_1)^2} + O(n^{-\frac{3}{2}}).
\]

**Remark 1.10.** It should be emphasized that the case \( t_k > 0 \) of Corollary 1.9 above is drastically different from the case \( t_k = 0 \). Indeed, if \( t_k > 0 \), then \( \text{Cov}(N(r_\ell), N(r_k)) \) is of order \( n \), while if \( t_k = 0 \), then \( \text{Cov}(N(r_\ell), N(r_k)) = \text{Cov}(N(r_\ell), N(\rho_1)) \) is of order \( 1 \).

**Corollary 1.11.** (Asymptotics of the covariance for \( m+1 \leq \ell \leq k \leq 2m \)). Under the assumptions of Corollary 1.7, the covariance \( \text{Cov}(N(r_\ell), N(r_k)) \) obeys the following formula for any \( m+1 \leq \ell \leq k \leq 2m \):

\[
\text{Cov}(N(r_\ell), N(r_k)) = b_{(1,1)}(t_\ell, t_k) n + c_{(1,1)}(t_\ell, t_k) \ln n + d_{(1,1)}(t_\ell, t_k) + f_{(1,1)}(n, t_\ell, t_k) + e_{(1,1)}(t_\ell, t_k)n^{-\frac{1}{2}} + O(n^{-\frac{3}{2}})
\]
as \( n \to +\infty \), where

\[
b_{(1,1)}(t_\ell, t_k) = \frac{1}{t_\ell} e^{-\frac{\ell}{2} (\ln(2\rho_2/\rho_1) - \sigma_*)} - \frac{1}{t_\ell + t_k} e^{-\frac{\ell}{2} (\ln(2\rho_2/\rho_1) - \sigma_*)} \quad \text{if } t_\ell > 0,
\]
Corollary 1.13. Suppose that the assumptions of Corollary 1.7 hold and that $G$ is a Gaussian. This should be compared to Corollary 1.6 (corresponding to the case $\Sigma = 1$), where

$$
N(t_{1}, t_{2}) = b_{1}(t_{1})n_{1} + c_{1}(t_{1})ln(n_{1})
$$

As $n \to +\infty$, where

$$
d_{(1,1)}(t_{1}, t_{2}) = e^{-\frac{t_{1}^{2}(\mu^{2} - \sigma^{2})}{2}}
$$

It is crucial for the next corollary that $t_{m}, t_{m+1} > 0$, so that $b_{1}(t_{1})n_{1} + c_{1}(t_{1})ln(n_{1})$ are of order $\sqrt{n}$ and are continuous Gaussian. This should be compared to Corollary 1.6 (corresponding to the case $m = 1$ and $t_{m} = 0$), which shows that the asymptotic fluctuations of $N(r_{1})$ are of order 1 and are described by a discrete Gaussian.

Corollary 1.12 (Asymptotics of the covariance for $1 \leq \ell \leq m$ and $m + 1 \leq k \leq 2m$). Under the assumptions of Corollary 1.7, the covariance $\text{Cov}(N(r_{1}), N(r_{k}))$ obeys the following formula for any $1 \leq \ell \leq m$ and $m + 1 \leq k \leq 2m$:

$$
\text{Cov}(N(r_{1}), N(r_{k})) = d_{(1,1)}(t_{1}, t_{2}) + f_{(1,1)}(n, t_{1}, t_{2}) + O(n^{-\frac{3}{2}})
$$

as $n \to +\infty$, where

$$
d_{(1,1)}(t_{1}, t_{2}) = e^{-\frac{t_{1}^{2}(\mu^{2} - \sigma^{2})}{2}}
$$

$$
f_{(1,1)}(n, t_{1}, t_{2}) = e^{-\frac{n^{2}(\mu^{2} - \sigma^{2})}{2}}
$$

and $b_{1}(t_{1})n_{1} + c_{1}(t_{1})ln(n_{1})$ are of order $\sqrt{n}$ and are continuous Gaussian. This should be compared to Corollary 1.6 (corresponding to the case $m = 1$ and $t_{m} = 0$), which shows that the asymptotic fluctuations of $N(r_{1})$ are of order 1 and are described by a discrete Gaussian.

Corollary 1.13. Suppose that the assumptions of Corollary 1.7 hold and that $t_{m}, t_{m+1} > 0$. As $n \to +\infty$, the random variable $(\mathcal{N}_{1}, \ldots, \mathcal{N}_{2m})$, where

$$
\mathcal{N}_{\ell} := \frac{N(r_{1}) - (b_{1}(t_{1})n_{1} + c_{1}(t_{1})ln(n_{1}))}{\sqrt{b_{1}(t_{1})(t_{1}, t_{2})n_{1}}}
$$

converges in distribution to a multivariate normal random variable of mean $(0, \ldots, 0)$ whose covariance matrix $\Sigma$ is given by

$$
\Sigma_{k,\ell} = \begin{cases} 
\frac{b_{1}(t_{1})b_{1}(t_{2})}{\sqrt{b_{1}(t_{1})(t_{1}, t_{2})n_{1}}}, & 1 \leq \ell \leq k \leq m \text{ or } m + 1 \leq \ell \leq k \leq 2m, \\
0, & 1 \leq \ell \leq m \text{ and } m + 1 \leq k \leq 2m,
\end{cases}
$$

where $b_{1}(t_{1})b_{1}(t_{2})$ is given by (1.31) for $1 \leq \ell \leq k \leq m$ and by (1.33) for $m + 1 \leq \ell \leq k \leq 2m$. 
Proof. By Lévy’s continuity theorem, the assertion will follow if we can show that the characteristic function $E[e^{i \sum_{t=1}^{2m} v_t N_t}]$ converges pointwise to $e^{-\frac{1}{2} \sum_{t,k=1}^{2m} v_t \Sigma_{t,k} v_k}$ for every $v_t \in \mathbb{R}^{2m}$ as $n \to +\infty$. Letting $u_t = \sqrt{b(t)}(t_n n + N_n n) n$ (1.34) and (1.18) show that

$$E[e^{i \sum_{t=1}^{2m} v_t N_t}] = e^{i \sum_{t=1}^{2m} u_t N_t} e^{-\sum_{t=1}^{2m} u_t (b(t) + c(t)) \ln u_t} \cdot$$

$$= e^{C_1(u) + C_2(u) \ln u + C_3(u) + O(n^{-\delta})} e^{-\sum_{t=1}^{2m} u_t (\partial_{x_n} C_1 + \partial_{x_n} C_2) \ln u}$$

as $n \to +\infty$ for any fixed $v_t \in \mathbb{R}^{2m}$. Since $C_j(\bar{u}) = 0$ for $j = 1, 2, 3$, $F_n(u) = \sum_{t=1}^{2m} f_t(n, t) u_t + O(|u|^2) = O((\bar{u})^2)$, and $|\bar{u}| = O(n^{-1/2})$, we obtain

$$E[e^{i \sum_{t=1}^{2m} v_t N_t}] = e^{2 \sum_{t,k=1}^{2m} u_t u_k \partial_{x_n} \partial_{x_n} C_1 \ln u + O(n^{-\delta} \ln n + |\bar{u}|^2 \ln n + |\bar{u}| + n^{-1/2})}$$

$$= e^{\sum_{t,k=1}^{2m} u_t u_k \partial_{x_n} \partial_{x_n} C_1 \ln u + O(n^{-\delta} \ln n + |\bar{u}|^2 \ln n + |\bar{u}| + n^{-1/2})} + \sum_{t,k=m+1}^{2m} \ln n \times$$

$$\times e^{\sum_{t,k=m+1}^{2m} u_t u_k \partial_{x_n} \partial_{x_n} C_1 \ln u + O(n^{-\delta} \ln n + |\bar{u}|^2 \ln n + |\bar{u}| + n^{-1/2})} \to e^{-\frac{1}{2} \sum_{t,k=1}^{2m} v_t \Sigma_{t,k} v_k}$$

as $n \to +\infty$, which completes the proof. \qed

Outline of proof. Using that $\prod_{1 \leq j < k \leq n} |z_k - z_j|^2$ is the product of two Vandermonde determinants, we obtain after standard manipulations that

$$\mathcal{E}_n := E[\prod_{t=1}^{2m} e^{u_t N_t}] = \frac{1}{n!} \int \cdots \int |z_k - z_j|^2 \prod_{j=1}^{n} w(z_j) d^2 z_j$$

$$= \frac{1}{n!} \det \left( \int_{C} z_j^k w(z) d^2 z \right)_{j,k=0}^{n-1}$$

$$= \frac{1}{n!} (2\pi)^n \prod_{j=0}^{n-1} \left( \int_{C} z_j^k w(z) d^2 z \right)_{j,k=0}^{n-1}$$

where the weight $w$ is defined by

$$w(z) := e^{-nQ(z)} \omega(|z|), \quad \omega(x) := \prod_{t=1}^{2m} \left\{ \begin{array}{ll} e^{u_t}, & \text{if } x < r_t, \\ 1, & \text{if } x \geq r_t. \end{array} \right.$$ (1.38)

Formula (1.37) directly follows from (1.36) and the fact that $w$ is rotation-invariant. Indeed, since $w(z) = w(|z|)$, the integral $\int_C z_j^k w(z) d^2 z$ is 0 for $j \neq k$ and is $2\pi \int_{C} u^{2j+1} w(u) d\mu$ for $j = k$. So only the main diagonal contributes for the determinants in (1.36).

Shifting $j \to j - 1$ and then performing the change of variables $v = nu^{2h}$ in (1.37), we obtain

$$\mathcal{E}_n = \prod_{j=1}^{n} \left( \int_{0}^{\rho_1/n} + \int_{\rho_1/n}^{\rho_2/n} \right) u^{2j+1} e^{-nu^{2h}} \omega(u) d\mu$$

$$= \prod_{j=1}^{n} \left( \int_{0}^{\rho_1/n} + \int_{\rho_1/n}^{\rho_2/n} \right) \frac{v^{2j+1}}{\nu^{2h}} e^{-u^{2h}} \omega((v/n)^{2h}) d\mu$$

$$= \prod_{j=1}^{n} \left( \int_{0}^{\rho_1/n} + \int_{\rho_1/n}^{\rho_2/n} \right) \frac{v^{2j+1}}{\nu^{2h}} e^{-u^{2h}} d\mu.$$ (1.39)
At this stage it is convenient to write
\[ \omega(x) = \sum_{\ell=1}^{2m+1} \omega_\ell 1_{[0,r_\ell]}(x), \]  \tag{1.40}
where \( \{\omega_\ell\}_{1}^{2m+1} \) are given by (1.12) and \( r_{2m+1} := +\infty \). Using this representation for \( \omega(x) \) in (1.39), we arrive at the following representation for \( \ln \mathcal{E}_n \):
\[ \ln \mathcal{E}_n = n \sum_{j=1}^{n} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \]  \tag{1.41}
where \( \gamma(a, z) \) is the incomplete gamma function defined by
\[ \gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt. \]
We infer from (1.41) that the asymptotics of \( \gamma(a, z) \) as \( z \to +\infty \) uniformly for \( a \in \left[ \frac{1+\alpha}{b}, \frac{1+\alpha}{b} + \frac{\epsilon}{b} \right] \) are needed to obtain large \( n \) asymptotics for \( \mathcal{E}_n \) — we recall these asymptotics in Appendix C.

Note from (1.35) that \( \mathcal{E}_n \) can also be viewed as a partition function with discontinuities approaching hard edges; see also e.g. [59, 44, 11, 57, 30, 25, 18, 20] for other works on partition functions.

In (1.41) and below, \( \ln \) always denotes the principal branch of the logarithm.

## 2 Proof of Theorem 1.2

Our proof strategy uses ideas from [24, 25, 27, 9]. Define
\[ j_{k,-} := \left\lfloor \frac{bn_2^{2b}}{1+\epsilon} - \alpha \right\rfloor, \quad j_{k,+} := \left\lceil \frac{bn_2^{2b}}{1-\epsilon} - \alpha \right\rceil, \quad k = 1, 2, \]
where \( \epsilon > 0 \) is independent of \( n \) and sufficiently small such that
\[ \frac{bn_2^{2b}}{1-\epsilon} < \frac{bn_2^{2b}}{1+\epsilon}, \quad \frac{bn_2^{2b}}{1-\epsilon} < 1. \]  \tag{2.1}
It is convenient to split the sum (1.41) into six parts:
\[ \ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3 + S_4 + S_5, \]  \tag{2.2}
where
\[ S_0 = \sum_{j=1}^{M'} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \quad S_1 = \sum_{j=M'+1}^{j_{1,-} - 1} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \]  \tag{2.3}
\[ S_2 = \sum_{j=j_{1,-}}^{j_{1,+}} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \quad S_3 = \sum_{j=j_{1,+} + 1}^{2m} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right). \]  \tag{2.4}

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\[ S_4 = \sum_{j=j_2,-}^{j_2,+} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \quad S_5 = \sum_{j=j_2,+1}^{n} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell F_{n,j,\ell} \right), \] (2.5)

and where \( M' > 0 \) is an integer independent of \( n \). For \( j = 1, \ldots, n \), we define \( a_j := \frac{j+\alpha}{\beta} \) and

\[ \hat{\lambda}_{j,\ell} := \frac{bnr_{j}^2}{j + \alpha}, \quad \hat{\eta}_{j,\ell} := (\hat{\lambda}_{j,\ell} - 1) \sqrt{\frac{2(\hat{\lambda}_{j,\ell} - 1 - \ln \hat{\lambda}_{j,\ell})}{(\hat{\lambda}_{j,\ell} - 1)^2}}, \quad \ell = 1, \ldots, 2m, \] (2.6)

\[ \lambda_{j,k} := \frac{bnr_{j}^2}{j + \alpha}, \quad \eta_{j,k} := (\lambda_{j,k} - 1) \sqrt{\frac{2(\lambda_{j,k} - 1 - \ln \lambda_{j,k})}{(\lambda_{j,k} - 1)^2}}, \quad k = 1, 2. \] (2.7)

With this notation, we can write

\[ F_{n,j,\ell} = \left\{ \begin{array}{ll}
\frac{\gamma(a,j,\lambda_{j,\ell})}{\gamma(a,j,\lambda_{j,\ell}) - \gamma(a,j,\lambda_{j,\ell})} & \ell = 1, \ldots, m, \\
\frac{\gamma(a,j,\lambda_{j,\ell}) - \gamma(a,j,\lambda_{j,\ell}) + \gamma(a,j,\lambda_{j,\ell})}{\gamma(a,j,\lambda_{j,\ell}) - \gamma(a,j,\lambda_{j,\ell}) + \gamma(a,j,\lambda_{j,\ell})} & \ell = m + 1, \ldots, 2m.
\end{array} \right. \] (2.8)

**Lemma 2.1.** Let \( x_1, \ldots, x_{2m} \in \mathbb{R} \) be fixed. There exists \( \delta > 0 \) such that

\[ S_0 = M' \ln \Omega + O(e^{-cn}), \quad \text{as } n \to +\infty, \] (2.9)

uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \}. \)

**Proof.** Using (1.42) and Lemma C.1 we infer that

\[ F_{n,j,\ell} = 1 + O(e^{-cn}), \quad \text{as } n \to +\infty \]

uniformly for \( j \in \{1, \ldots, M'\} \) and \( \ell \in \{1, \ldots, 2m\} \). Hence, since \( 1 + \sum_{\ell=1}^{2m} \omega_\ell = e^{u_1 + \cdots + u_{2m}} = \Omega \),

\[ S_0 = \sum_{j=1}^{M'} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_\ell \Gamma(1 + O(e^{-cn})) \right) = \sum_{j=1}^{M'} \ln \Omega + O(e^{-cn}), \quad \text{as } n \to +\infty. \]

Since the above error terms on the left of the second equality are independent of \( u_1, \ldots, u_{2m} \), the claim follows.

**Lemma 2.2.** We can choose \( M' \) sufficiently large such that the following holds. For any fixed \( x_1, \ldots, x_{2m} \in \mathbb{R} \), there exists \( \delta > 0 \) such that

\[ S_1 = (j_{1,-} - M' - 1) \ln \Omega + O(e^{-cn}), \]

as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \}. \)

**Proof.** By Lemma C.2 (i), for any \( \delta > 0 \) there exist \( A = A(\delta), C = C(\delta) > 0 \) such that \( |\gamma(a,z) - 1| \leq C e^{-\frac{z^2}{2}} \) for all \( a \geq A \) and all \( \lambda = \frac{z}{n} \geq 1 + \delta \), where \( \eta \) is defined by (C.1). For large enough, we have \( \lambda_{j,k}, \hat{\lambda}_{j,\ell} \geq 1 + \frac{\delta}{2} \) for all \( j \in \{M'+1, \ldots, j_{1,-} - 1\} \) and all \( k \) and \( \ell \). Thus, let us take \( \delta = \frac{\delta}{2} \) and choose \( M' \) large enough so that \( a_j = \frac{j+\alpha}{\beta} \geq A(\frac{\delta}{2}) \) for all \( j \in \{M'+1, \ldots, j_{1,-} - 1\} \). From (2.8), we infer that, as \( n \to +\infty \),

\[ F_{n,j,\ell} = \frac{1 + O(e^{-\frac{c\eta_j^2}{n}})}{1 + O(e^{-\frac{c\eta_j^2}{n}} + e^{-\frac{c\eta_j^2}{n}})} = 1 + O(e^{-cn}), \quad \ell = 1, \ldots, m, \]
\[
F_{n,j,\ell} = \frac{1 + \mathcal{O}(e^{-\frac{a_j g_j^2 \ell}{2}} + e^{-\frac{a_j g_j^2}{2}} + e^{-\frac{a_j g_j^2}{2a_j}})}{1 + \mathcal{O}(e^{-\frac{a_j g_j^2}{2} + e^{-\frac{a_j g_j^2}{2a_j}}})} = 1 + \mathcal{O}(e^{-cn}), \quad \ell = m + 1, \ldots, 2m,
\]

uniformly for \(j \in \{M' + 1, \ldots, j_{1,\infty} - 1\}\), and the claim follows. \(\square\)

Lemma 2.3. For any fixed \(x_1, \ldots, x_{2m} \in \mathbb{R}\), there exists \(\delta > 0\) such that

\[
S_5 = \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty,
\]

uniformly for \(u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}\).

Proof. We infer from (1.42) and Lemma C.2 (ii) that

\[
F_{n,j,\ell} = \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty
\]

uniformly for \(j \in \{j_2, + 1, \ldots, n\}\) and \(\ell \in \{1, \ldots, 2m\}\). Hence

\[
S_5 = \sum_{j=j_2, + 1}^{n} \ln \left(1 + \sum_{\ell=1}^{2m} \omega_l \mathcal{O}(e^{-cn})\right) = \mathcal{O}(e^{-cn}), \quad \text{as } n \to +\infty.
\]

Since the above error terms on the left of the second equality are independent of \(u_1, \ldots, u_{2m}\), the claim follows. \(\square\)

The following lemma will be used to obtain the large \(n\) asymptotics of \(S_3\).

Lemma 2.4. [Taken from [25, Lemma 3.4]] Let \(A = A(n), a_0 = a_0(n), B = B(n), b_0 = b_0(n)\) be bounded functions of \(n \in \{1, 2, \ldots\}\), such that

\[
a_n := An + a_0 \quad \text{and} \quad b_n := Bn + b_0
\]

are integers. Assume also that \(B - A\) is positive and remains bounded away from 0. Let \(f\) be a function independent of \(n\), and which is \(C^4([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}])\) for all \(n \in \{1, 2, \ldots\}\). Then as \(n \to +\infty\), we have

\[
\sum_{j=a_n}^{b_n} f(\frac{x}{n}) = n \int_A^B f(x)dx + \frac{(1 - 2a_0)f(A) + (1 + 2b_0)f(B)}{2} + (-1 + 6a_0 - 6a_0^2)f'(A) + (1 + 6b_0 + 6b_0^2)f'(B) + \frac{(-a_0 + 3a_0^2 - 2a_0^3)f''(A) + (b_0 + 3b_0^2 + 2b_0^3)f''(B)}{12n^2} + \mathcal{O}\left(\frac{m_{A,n}(f''') + m_{B,n}(f''')}{n^3} + \sum_{j=a_n}^{b_n-1} m_{j,n}(f''')}
\]

where, for a given function \(g\) continuous on \([\min\{\frac{a_n}{n}, A\}, \max\{\frac{b_n}{n}, B\}]\),

\[
m_{A,n}(g) := \max_{x \in [\min\{\frac{a_n}{n}, A\}, \max\{\frac{a_n}{n}, A\}]} |g(x)|, \quad m_{B,n}(g) := \max_{x \in [\min\{\frac{b_n}{n}, B\}, \max\{\frac{b_n}{n}, B\}]} |g(x)|,
\]

and for \(j \in \{a_n, \ldots, b_n - 1\}\), \(m_{j,n}(g) := \max_{x \in [\frac{j}{n}, \frac{j+1}{n}]} |g(x)|\).
Following the method of [25], we define
\[
\theta^{(n,\epsilon)}_{k,+} = \left(\frac{b_m\rho^2_k}{1-\epsilon} - \alpha\right) - \left(\frac{b_m\rho^2_k}{1+\epsilon} - \alpha\right), \quad \theta^{(n,\epsilon)}_{k,-} = \left[\frac{b_m\rho^2_k}{1-\epsilon} - \alpha\right] - \left(\frac{b_m\rho^2_k}{1+\epsilon} - \alpha\right), \quad k = 1, 2, \tag{2.12}
\]
and we split \(S_3\) in two parts
\[
S_3 = S_3^{(1)} + S_3^{(2)}, \tag{2.13}
\]
where
\[
S_3^{(1)} = \sum_{j=j_1,++1}^{\lfloor j_{*}\rfloor} \ln \left(1 + \sum_{\ell=1}^{2m} \omega_{F_n,j,\ell}\right), \quad S_3^{(2)} = \sum_{j=j_{*}+1}^{j_{2,-1}} \ln \left(1 + \sum_{\ell=1}^{2m} \omega_{F_n,j,\ell}\right), \tag{2.14}
\]
with
\[
j_* := n\sigma_* - \alpha, \tag{2.15}\]
and where \(\sigma_*\) is defined in (1.5). Define also
\[
\theta_* = j_* - \lfloor j_*\rfloor. \tag{2.16}\]
The identity
\[
a_j(\eta_{j,2}^2 - \eta_{j,1}^2) = 2(j_* - j) \ln \left(\frac{\rho_2}{\rho_1}\right), \tag{2.17}\]
implies that \(\eta_{j,2}^2 - \eta_{j,1}^2\) is positive for \(j \in \{j_{1,+}, \ldots, j_*\}\) and negative for \(j \in \{j_*+1, \ldots, j_{2,-}\}\).

**Lemma 2.5.** We can choose \(M'\) sufficiently large such that the following holds. For any fixed \(x_1, \ldots, x_{2m} \in \mathbb{R}\), there exists \(\delta > 0\) such that
\[
S_3^{(1)} = n \int_{x_1}^{x_2} f_1(x) dx + \left(1 - \frac{1}{2} + \theta^{(n,\epsilon)}_{1,+}\right) f_1 \left(\frac{b_p}{n}\right) + \left(1 - \alpha - \theta_*\right) f_1(\sigma_*) + \int_{x_1}^{x_2} f(x) dx
\]
\[
+ \sum_{j=0}^{\infty} \ln \left\{1 + \frac{\sigma_* - b_p^2}{b_p^2 - \sigma_*} \cdot \left(\frac{\rho_1}{\rho_2}\right)^{-2\theta_+(j)}\right\} + \mathcal{O}\left(\frac{(\ln n)^2}{n}\right) \tag{2.18}
\]
as \(n \to \infty\) uniformly for \(u_1 \in \{z \in \mathbb{C} : |z-x_1| \leq \delta\}, \ldots, u_{2m} \in \{z \in \mathbb{C} : |z-x_{2m}| \leq \delta\}\), where \(f_1(x) := \ln \left(1 + T_0(x) + \bar{T}_0(b_p^2)\right)\) and \(T_0, \overline{T}_0\) are defined in (1.13) and (1.11).

**Proof.** Using (2.1), we see that for \(j \in \{j_{1,+}, \ldots, j_{2,-}\}\) and \(\ell \in \{1, \ldots, m\}\), \(-\lambda_j, \ell - 1\) and \(-\lambda_j, \ell - 1\) are positive and remain bounded away from 0, while for \(j \in \{j_{1,+}, \ldots, j_{2,-}\}\) and \(\ell \in \{m+1, \ldots, 2m\}\), \(-\lambda_j, \ell - 1\) and \(-\lambda_j, \ell - 1\) are positive and remain bounded away from 0.

Hence, using Lemma C.3 (i) and (ii), as \(n \to \infty\) we have
\[
\gamma(a_j, a_j, \lambda_j, \ell, k) = e^{\frac{-a_j^2\eta_{j,2}^2}{2\pi}} \left\{\frac{1}{\lambda_j, \ell - 1 - \sqrt{a_j}} + \frac{1}{12(\lambda_j, \ell - 1)^3} \frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right\}, \quad \ell = 1, \ldots, m,
\]
\[
\gamma(a_j, a_j, \lambda_j, \ell, k) = 1 + e^{\frac{-a_j^2\eta_{j,2}^2}{2\pi}} \left\{\left(-1 - \frac{1}{\lambda_j, \ell - 1 - \sqrt{a_j}} + \frac{1}{12(\lambda_j, \ell - 1)^3} \frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2})\right), \quad \ell = m+1, \ldots, 2m,
\]
\[
\gamma(a_j, a_j \lambda_{j,1}) = e^{-\frac{\alpha_j^2}{2}} \left\{ \frac{1}{1 - \lambda_{j,1} \sqrt{a_j}} + \frac{1 + 10 \lambda_{j,1} + \lambda_{j,1}^2}{12(\lambda_{j,1} - 1)^3} \frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2}) \right\},
\]
\[
\gamma(a_j, a_j \lambda_{j,2}) = 1 + e^{-\frac{\alpha_j^2}{2}} \left\{ \frac{-1}{1 - \lambda_{j,2} \sqrt{a_j}} + \frac{1 + 10 \lambda_{j,2} + \lambda_{j,2}^2}{12(\lambda_{j,2} - 1)^3} \frac{1}{a_j^{3/2}} + \mathcal{O}(n^{-5/2}) \right\},
\]
uniformly for \(j \in \{j_1, + 1, \ldots, j_2, - 1\}\). Moreover, as \(n \to +\infty\),
\[
e^{-\frac{\alpha_j^2}{2}} e^{-\frac{\alpha_j^2}{2}} \left\{ \frac{1}{1 - \lambda_{j,1} \sqrt{a_j}} + \frac{1 + 2\lambda_{j,1} + \lambda_{j,1}^2}{2bm} + \mathcal{O}(n^{-2}) \right\}, \quad \ell = 1, \ldots, m,
\]
uniformly for \(j \in \{j_1, + 1, \ldots, j_2, - 1\}\). Hence, after a long but direct computation, we obtain
\[
F_{n,j,t} = \frac{e^{-\frac{\alpha_j^2}{2}} (j/n - \rho_1^{2b})}{1 + \frac{j/n - \rho_1^{2b}}{\rho_2^{2b} - \rho_1^{2b}} \frac{(2m)}{n^{2(j,n-j)}}} + \mathcal{O}\left(\frac{(2m)^{-2(j,n-j)}}{n} + \frac{1}{n^2}\right), \quad \ell = 1, \ldots, m,
\]
as \(n \to +\infty\) uniformly for \(j \in \{j_1, + 1, \ldots, j_2\}\). Substituting the above into (2.14) yields
\[
S_3^{(1)} = \sum_{j=j_1, + 1}^{j_2} \ln \left\{ 1 + \frac{T_0(j/n)}{1 + \frac{j/n - \rho_1^{2b}}{\rho_2^{2b} - \rho_1^{2b}} \frac{(2m)}{n^{2(j,n-j)}}} + \tilde{T}_0(\rho_2^{2b}) - \frac{\tilde{T}_0(j/n)}{1 + \frac{j/n - \rho_1^{2b}}{\rho_2^{2b} - \rho_1^{2b}} \frac{(2m)}{n^{2(j,n-j)}}} \right\},
\]
\[
+ (j/n)^2 T_2(j/n) + 2b^2 \rho_2^{2b} T_1(j/n) - \rho_1^{2b} j/n - \rho_1^{2b} \frac{2\alpha(j/n - \rho_1^{2b})}{1 + \frac{j/n - \rho_1^{2b}}{\rho_2^{2b} - \rho_1^{2b}} \frac{(2m)}{n^{2(j,n-j)}}} - 2b(j/n - \rho_1^{2b}) / n
\]
\[
+ \sum_{\ell=1}^{2m} \omega_\ell \mathcal{O}\left(\frac{(2m)^{-2(j,n-j)}}{n} + \frac{1}{n^2}\right) \right\}
\]
as \(n \to +\infty\), where the above error terms are independent of \(u_1, \ldots, u_{2m}\) and uniform for \(j \in \{j_1, + 1, \ldots, j_2\}\). Expanding further, we obtain
\[
S_3^{(1)} = S_1 + S_2 + \frac{1}{n} S_3 + \sum_{j=j_1, + 1}^{j_2} \mathcal{O}\left(\frac{(2m)^{-2(j,n-j)}}{n} + \frac{1}{n^2}\right), \quad \text{as } n \to +\infty,
\]
where
\[
S_1 = \sum_{j=j_1, + 1}^{j_2} \ln \left\{ 1 + T_0(j/n) + \tilde{T}_0(\rho_2^{2b}) \right\},
\]
\[
S_2 = \sum_{j=j_1, + 1}^{j_2} \ln \left\{ 1 - \frac{T_0(j/n)}{1 + T_0(j/n) + \tilde{T}_0(\rho_2^{2b})} \right\}.
\]
\[ S_3 = \sum_{j=j_1+1}^{[j_1]} (j/n)^2 T_2(j/n) + 2b^2 \rho_1^2 b T_1(j/n) - b \rho_1^2 b/j/n T_2(j/n) + 2a(j/n - b \rho_1^2 b) T_1(j/n) - 2b(j/n - b \rho_1^2 b)(1 + T_0(j/n) + \hat{T}_0(b \rho_1^2 b^3)), \]

and we have used that

\[
\left(1 - \frac{T_0(j/n) + \hat{T}_0(j/n)}{1 + T_0(j/n) + \hat{T}_0(b \rho_1^2 b^3)} \right) = 1 + O\left(\left(\frac{\rho_1}{\rho_1}\right)^{-2(j_1,-j)}\right)
\]

uniformly for \( j \in \{j_1, + 1, \ldots, [j_1]\} \) to obtain the expression for \( S_3 \). Clearly, \( M' \) can be chosen large enough such that

\[
\sum_{j=j_1+1}^{[j_1]} O\left(\left(\frac{\rho_1}{\rho_1}\right)^{-2(j_1,-j)} + \frac{1}{n^2}\right) = O(n^{-1}) + \sum_{j=j_1+1}^{[j_1]} O\left(\left(\frac{\rho_1}{\rho_1}\right)^{-2(j_1,-j)}\right) = O(n^{-1}) + O(n^{-100}) + \sum_{j=([j_1] - M' \ln n)} O\left(\left(\frac{\rho_1}{\rho_1}\right)^{-2(j_1,-j)}\right) = O\left(\frac{\ln n}{n}\right).
\]

Also, we can express \( S_1 \) and \( S_3 \) in terms of the functions \( f_1 \) and \( f \) as follows:

\[
S_1 = \sum_{j=j_1+1}^{[j_1]} f_1(j/n), \quad S_3 = \sum_{j=j_1+1}^{[j_1]} f(j/n).
\]

These sums can be expanded using Lemma 2.4 (with \( A = b \rho_1^2 b^{17} \), \( a_0 = 1 - \alpha - \theta(n,r) \), \( B = \sigma_\epsilon \) and \( b_0 = -\alpha - \theta_* \)); this gives the terms involving \( f_1 \) and \( f \) on the right-hand side of (2.18). Thus it only remains to expand \( S_2 \) (the analysis required for \( S_2 \) is similar but different from [25, Lemma 3.6]). Let \( s_{2,j} \) be the summand of \( S_2 \). Since \( s_{2,j} = O\left(\left(\frac{\rho_1}{\rho_1}\right)^{-2(j_1,-j)}\right) \) as \( n \to +\infty \) uniformly for \( j \in \{j_1, + 1, \ldots, [j_1]\} \), we have

\[
S_2 = \sum_{j=([j_1] - M' \ln n)}^{[j_1]} s_{2,j} + O(n^{-100}), \quad \text{as } n \to +\infty,
\]

provided \( M' \) is chosen large enough. Furthermore, since \( T_0 \) and \( \hat{T}_0 \) are analytic at \( \sigma_* \),

\[
S_2 = \sum_{j=([j_1] - M' \ln n)}^{[j_1]} \left\{ \ln \left(1 - \frac{T_0(\sigma_\epsilon) + \hat{T}_0(\sigma_\epsilon)}{1 + T_0(\sigma_\epsilon) + \hat{T}_0(b \rho_1^2 b^3)} \left(\frac{\sigma_* - b \rho_1^2 b}{b \rho_1^2 b - \sigma_*} + \frac{\rho_1^2 b}{\rho_1^2 b + \sigma_* - b \rho_1^2 b}\right)^{-2(j_1,-j)}\right) \right\} + O(n^{-100}) + O\left(\frac{\ln n^2}{n}\right)
\]

as \( n \to +\infty \). Changing now indices, we find

\[
S_2 = \sum_{j=0}^{+\infty} \ln \left(1 - \frac{T_0(\sigma_\epsilon) + \hat{T}_0(\sigma_\epsilon)}{1 + T_0(\sigma_\epsilon) + \hat{T}_0(b \rho_1^2 b^3)} \left(\frac{\sigma_* - b \rho_1^2 b}{b \rho_1^2 b - \sigma_*} + \frac{\rho_1^2 b}{\rho_1^2 b + \sigma_* - b \rho_1^2 b}\right)^{-2(\theta_* + j)}\right) + O\left(\frac{\ln n^2}{n}\right) \quad \text{as } n \to +\infty.
\]

The claim follows after a computation combining the asymptotics of \( S_1, S_2, S_3 \).
Lemma 2.6. We can choose $M'$ sufficiently large such that the following holds. For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that

$$S_3^{(2)} = n \int_{\sigma_0}^{\sigma_2} \hat{f}_1(x)dx + \left( \alpha - \frac{1}{2} + \theta_x \right) \int_{\sigma_0}^{\sigma_2} \hat{f}_1(\sigma_x) + \left( \theta^{(n, e)}_{2, \infty} - \alpha - \frac{1}{2} \right) \int_{\sigma_0}^{\sigma_2} \hat{f}(x)dx + \sum_{\ell=0}^{\infty} \ln \left( 1 + \frac{T_0(\sigma_0) + \hat{T}_0(\sigma_0)}{1 - T_0(\sigma_0) + \hat{T}_0(b^{2b}_\ell / \ell^{b + \rho_2})} \right) + \mathcal{O} \left( \frac{(\ln n)^2}{n} \right)$$

as $n \to +\infty$ uniformly for $u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \}$, where $\hat{f}_1(x) := \ln \left( 1 - \hat{T}_0(x) + \hat{T}_0(b^{2b}_\ell) \right)$ and $\hat{f}$ and $T_j, \hat{T}_j$ are defined in (1.14) and (1.11).

Proof. After a long but direct computation using (2.19) and (2.20), we obtain

$$F_{n,j,l} = e^{-\frac{\ell}{n}(j/n + \rho_2 - \rho_1)} \frac{b^{2\rho_2 - j/n}_{j/n - \rho_2} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}}{1 + b^{2\rho_2 - j/n}_{j/n - \rho_2} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}} + \mathcal{O} \left( \frac{\left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}}{n} \right) \quad \ell = 1, \ldots, m,$n \to +\infty uniformly for $j \in \{ \lfloor j \rfloor + 1, \ldots, j_2 - 1 \}$. Hence,

$$S_3^{(2)} = \sum_{j=\lfloor j \rfloor + 1}^{j_2 - 1} \ln \left( 1 + T_0(j/n) \right) \frac{b^{2\rho_2 - j/n}_{j/n - \rho_2} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}}{1 + b^{2\rho_2 - j/n}_{j/n - \rho_2} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}} + \frac{\hat{T}_0(0/j)}{1 + b^{2\rho_2 - j/n}_{j/n - \rho_2} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}} \left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)} + \mathcal{O} \left( \frac{\left( \frac{\rho_2}{\rho_1} \right)^{2(\rho_2 - j/n)}}{n} \right) \right)$$

as $n \to +\infty$, where the above error terms are independent of $u_1, \ldots, u_{2m}$ and uniform for $j \in \{ \lfloor j \rfloor + 1, \ldots, j_2 - 1 \}$. Expanding further, we obtain

$$S_3^{(2)} = S_4 + S_5 + \frac{1}{n} S_6 + \mathcal{O} \left( \frac{\ln n}{n} \right), \quad \text{as} \ n \to +\infty,$n \to +\infty, where
We can express $S_4$ and $S_6$ in terms of the functions $\hat{f}_1$ and $\hat{f}$ as follows:

$$S_4 = \sum_{j=j_1+1}^{[j_s]} \hat{f}_1(j/n), \quad S_6 = \sum_{j=j_1+1}^{[j_s]} \hat{f}(j/n).$$

These sums can be expanded using Lemma 2.4 (with $A = \sigma_*$, $a_0 = 1 - \alpha - \theta_*$, $B = \frac{bq^{2n}}{1 + \epsilon}$ and $b_0 = \theta_*(n^{2\epsilon} - \alpha - 1)$; this gives the terms involving $\hat{f}_1$ and $\hat{f}$ on the right-hand side of (2.21)). We now turn to the analysis of $S_5$. Let $s_{5,j}$ be the summand of $S_5$. Since $s_{5,j} = \mathcal{O}(\frac{bq^n}{\rho_1})^{2(j-j_0)}$ as $n \to +\infty$ uniformly for $j \in \{[j_s] + 1, \ldots, j_{s*} - 1\}$, we have

$$S_5 = \sum_{j=[j_s]+1}^{[j_s+M'\ln n]} s_{5,j} + \mathcal{O}(n^{-100}), \quad \text{as } n \to +\infty,$$

provided $M'$ is chosen large enough. Furthermore, since $T_0$ and $\hat{T}_0$ are analytic at $\sigma_*$,

$$S_5 = \sum_{j=[j_s]+1}^{[j_s+M'\ln n]} \left[ \ln \left( 1 + \frac{T_0(\sigma_*) + \hat{T}_0(\sigma_*)}{1 - T_0(\sigma_*) + \hat{T}_0(\rho_2 b_0^{2n})} \frac{bq^{2n} - \sigma_*}{\sigma_* - \frac{bq^{2n}}{\rho_1}} \left( \frac{\rho_2 b_0^{2n}}{\rho_1} \right)^{-2(j-j_0)} \right) + \mathcal{O}(j/n - \sigma_*) \right] + \mathcal{O}(n^{-100})$$

$$= \sum_{j=[j_s]+1}^{[j_s+M'\ln n]} \ln \left( 1 + \frac{T_0(\sigma_*) + \hat{T}_0(\sigma_*)}{1 - T_0(\sigma_*) + \hat{T}_0(\rho_2 b_0^{2n})} \frac{bq^{2n} - \sigma_*}{\sigma_* - \frac{bq^{2n}}{\rho_1}} \left( \frac{\rho_2 b_0^{2n}}{\rho_1} \right)^{-2(j-j_0)} \right) + \mathcal{O}\left( \left( \ln n \right)^2 \frac{n}{n} \right)$$

as $n \to +\infty$. Changing indices, we get

$$S_5 = \sum_{j=0}^{+\infty} \ln \left( 1 + \frac{T_0(\sigma_*) + \hat{T}_0(\sigma_*)}{1 - T_0(\sigma_*) + \hat{T}_0(\rho_2 b_0^{2n})} \frac{bq^{2n} - \sigma_*}{\sigma_* - \frac{bq^{2n}}{\rho_1}} \left( \frac{\rho_2 b_0^{2n}}{\rho_1} \right)^{-2(j+1-j_0)} \right) + \mathcal{O}\left( \left( \ln n \right)^2 \frac{n}{n} \right), \quad \text{as } n \to +\infty.$$

The claim follows after a computation combining the asymptotics of $S_4, S_5, S_6$.

Recall that $\sigma_1, \sigma_2 > 0$ were defined in (1.5), $Q = Q(\bar{t}, \bar{u})$ was defined in (1.15), and $F_n$ was defined in terms of the Jacobi theta function $\theta(z; \tau)$ in (1.19).

**Lemma 2.7.** We can choose $M'$ sufficiently large such that the following holds. For any fixed $x_1, \ldots, x_{2m} \in \mathbb{R}$, there exists $\delta > 0$ such that

$$S_3 = n \left\{ \int_{\sigma_1}^{\sigma_*} f_1(x)dx + \int_{\sigma_*}^{\sigma_1} \hat{f}_1(x)dx \right\} + \left( \frac{1}{2} - \alpha - \theta_* \right) (f_1(\sigma_*) - \hat{f}_1(\sigma_*)) + \int_{\sigma_*}^{\sigma_*} (\hat{f}_1(x)dx + \int_{\sigma_*}^{\sigma_*} \hat{f}(x)dx$$

$$+ \theta_0 \ln Q - \frac{\ln Q}{2} \left( 1 - \frac{2 (\ln (2g_1 + 1))}{2 + (\ln (\rho_2 b_0^{2n}))} \right) + F_n + \mathcal{O}\left( \left( \ln n \right)^2 \frac{n}{n} \right)$$

as $n \to +\infty$ uniformly for $u_1 \in \{ z \in \mathbb{C} : \vert z - x_1 \vert \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : \vert z - x_{2m} \vert \leq \delta \}$, where $f_1(x) := \ln (1 + T_0(x) + \hat{T}_0(\rho_2 b_0^{2n}))$, $\hat{f}_1(x) := \ln (1 - \hat{T}_0(x) + \hat{T}_0(\rho_2 b_0^{2n}))$ and $T_j, \hat{T}_j, f$ and $\hat{f}$ are defined in (1.11)–(1.14).
The desired conclusion is a direct consequence of Lemmas 2.5 and 2.6 if we can show that

\[ J = \theta_1 \ln Q - \frac{2}{\ln v} \cdot \frac{\ln Q}{2} \left( 1 + \frac{2}{\ln v} \left( \frac{\ln Q}{\ln v} \right) \right) + F_n. \] (2.22)

To show (2.22), we introduce the short-hand notation \( v := \frac{\theta_j - \theta_1}{\theta_1 - \theta_1} = \sigma_2/\sigma_1 > 0 \) and \( w := \rho_1/\rho_2 \in (0,1) \). Then we can write

\[ \Sigma_n = \sum_{j=0}^{\infty} \ln \left\{ 1 - \frac{Q - 1}{w^2 - (\theta_1 + j)} \right\} + \sum_{j=0}^{\infty} \ln \left\{ 1 + \frac{Q - 1}{w^2 - (\theta_1 - j)} \right\}. \]

Combining the logarithms in the two sums and then letting \( j = \ell - 1 \), this becomes

\[ \Sigma_n = \sum_{j=0}^{\infty} \ln \left( 1 + w^{2j+2} \theta_j \theta_1 \right) \left( 1 + w^{2j} \frac{\ln Q}{\ln v} \right) = \sum_{\ell=1}^{\infty} \ln \left( 1 + w^{2\ell-1} \theta_\ell \theta_1 \right) \left( 1 + w^{2\ell-1} \frac{\ln Q}{\ln v} \right). \]

Using the Jacobi triple product identity

\[ \theta(z; \tau) = \prod_{\ell=1}^{\infty} (1 - e^{2\pi i \tau}) (1 + e^{2\pi i \tau}) (1 - e^{2\pi i \tau}) ^2, \]

we obtain

\[ \Sigma_n = \theta_1 \ln Q - \frac{\ln Q}{2} \left( 1 + \frac{2}{\ln v} \left( \frac{\ln Q}{\ln v} \right) \right) + \ln \left( \frac{\ln Q}{\ln v} - \theta_1 + \frac{1}{2} - \frac{\pi i}{\ln v} \right) - \ln \left( \frac{\ln Q}{\ln v} - \theta_1 - \frac{1}{2} + \frac{\pi i}{\ln v} \right). \]

This can be written as

\[ \Sigma_n = \theta_1 \ln Q - \frac{\ln Q}{2} \left( 1 + \frac{2}{\ln v} \left( \frac{\ln Q}{\ln v} \right) \right) + \ln \left( \frac{\ln Q}{\ln v} - \theta_1 + \frac{1}{2} - \frac{\pi i}{\ln v} \right) - \ln \left( \frac{\ln Q}{\ln v} - \theta_1 - \frac{1}{2} + \frac{\pi i}{\ln v} \right). \]

Using the periodicity property \( \theta(z+1; \tau) = \theta(z; \tau) \), we can replace \( \theta_1 = j_\gamma - j_\lambda \) by \( j_\lambda = n\sigma_2 - \alpha \) and +1 by -1/2 inside \( \theta \). Using also \( \theta(-z; \tau) = \theta(z; \tau) \), \( v = \sigma_2/\sigma_1 \) and \( w = \rho_1/\rho_2 \), we arrive at (2.22). \( \square \)

We now turn our attention to \( S_2 \) and \( S_4 \). Let \( M := n \hat{\pi} \). We split \( S_{2k} \), \( k = 1, 2 \), as follows:

\[ S_{2k} = S_{2k}^{(1)} + S_{2k}^{(2)} + S_{2k}^{(3)}, \quad S_{2k}^{(v)} := \sum_{j, \lambda, k \in I_v} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_{\ell} F_{n,j,\ell} \right), \quad v = 1, 2, 3, \] (2.23)
where
\[ I_1 = [1 - \epsilon, 1 - \frac{M}{\sqrt{n}}), \quad I_2 = [1 - \frac{M}{\sqrt{n}}, 1 + \frac{M}{\sqrt{n}}], \quad I_3 = (1 + \frac{M}{\sqrt{n}}, 1 + \epsilon]. \]

From (2.23), we see that the large \( n \) asymptotics of \( \{S_{2k}^{(v)}\}_{k=1,2,3} \) involve the asymptotics of \( \gamma(a, z) \) when \( a \to +\infty, z \to +\infty \) with \( \lambda = \frac{z}{a} \in [1 - \epsilon, 1 + \epsilon] \). These sums can also be rewritten using
\[
\sum_{j: \lambda_j \in I_3} \frac{g_{k,-1}}{j} = \sum_{j: \lambda_j \in I_2} g_{k,-} + \sum_{j: \lambda_j \in I_1} g_{k,+} + \sum_{j: \lambda_j \in I_3} g_{k,+}, \tag{2.24}
\]
where \( g_{k,-} := \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) \) and \( g_{k,+} := \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) \) for \( k = 1, 2 \). Let us also define
\[
\theta_{k,-}^{(n, M)} := g_{k,-} - \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) = \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) - \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right),
\]
\[
\theta_{k,+}^{(n, M)} := \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) - g_{k,+} = \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right) - \left( \frac{bn\rho}{1 + \frac{M}{\sqrt{n}}} - \alpha \right).
\]

Clearly, \( \theta_{k,-}^{(n, M)}, \theta_{k,+}^{(n, M)} \in [0, 1) \).

**Lemma 2.8.** For any fixed \( x_1, \ldots, x_{2m} \in \mathbb{R} \), there exists \( \delta > 0 \) such that
\[
S_{2}^{(3)} = \left( b_{\rho_1} \rho_2 n - j_{1,-} - bM\rho_1^{2b}\sqrt{n} + bM^2\rho_2^{2b} - \alpha + \theta_{1,-}^{(n, M)} - bM^3\rho_1^{2b}n^{-\frac{1}{2}} \right) \ln \Omega + O(M^4n^{-1}),
\]
as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \).

**Proof.** For all sufficiently large \( n \), and uniformly for \( j \in \{ j : \lambda_j \in I_3 \} \), we have that \( \eta_{j,2}, \hat{\eta}_{j,\ell}, \ell = m + 1, \ldots, 2m \) are positive and bounded away from 0, and furthermore
\[
\min_{\ell \in \{1, \ldots, m\}} \{ \eta_{j,1}, \hat{\eta}_{j,\ell} \} \geq \frac{M}{\sqrt{n}} + O(\frac{1}{\sqrt{n}}), \quad \sqrt{a_j / 2} \min_{\ell \in \{1, \ldots, m\}} \{ \eta_{j,1}, \hat{\eta}_{j,\ell} \} \leq -Mn^{1/2} + O(1).
\]

Using Lemma C.3 and the fact that \( M = n^{1/2} \), we infer that
\[
F_{n,j,\ell} = \frac{1 + O(e^{-n^{1/2}}} \frac{a_j}{\ell} + e^{-n^{1/2}}} {1 + O(e^{-n^{1/2}}} \frac{a_j}{\ell} + e^{-n^{1/2}}), \quad \ell = m + 1, \ldots, 2m,
\]
uniformly for \( j \in \{ j_{1,\ldots,1} \}, g_{1,-} - 1 \}. \) Hence, by (2.24),
\[
S_{2}^{(3)} = \sum_{j=1}^{g_{1,-}} \ln \left( 1 + \sum_{\ell=1}^{2m} \omega_{\ell} [1 + O(e^{-cn^{1/2}})] \right) = (g_{1,-} - j_{1,-}) \ln \Omega + O(ue^{cn^{1/2}}), \quad \text{as } n \to +\infty.
\]

In the above, the error terms before the second equality are independent of \( u_1, \ldots, u_{2m} \), so the expansion (see [24, Lemma 2.4])
\[
g_{1,-} - j_{1,-} = b_{\rho_1} \rho_2 n - j_{1,-} - bM\rho_1^{2b}\sqrt{n} + bM^2\rho_2^{2b} - \alpha + \theta_{1,-}^{(n, M)} - bM^3\rho_1^{2b}n^{-\frac{1}{2}} + O(M^4n^{-1})
\]
yields the claim. \( \square \)
Lemma 2.9. For any fixed \( x_1, \ldots, x_{2m} \in \mathbb{R} \), there exists \( \delta > 0 \) such that
\[
S_4^{(1)} = O(n^{-100}),
\]
as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \} \), \( \ldots \), \( u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \).

Proof. For all sufficiently large \( n \), and uniformly for \( j \in \{ j : \lambda_{j,2} \in I_1 \} \), we have that \( \eta_{j,1}, \hat{\eta}_{j,\ell}, \ell = 1, \ldots, m \) are negative and bounded away from 0, and furthermore
\[
\max_{\ell \in \{m+1, \ldots, 2m\}} \{ \eta_{j,2}, \hat{\eta}_{j,\ell} \} \leq -\frac{M}{\sqrt{n}} + O\left( \frac{1}{\sqrt{n}} \right), \quad -\sqrt{a_{j}/2} \max_{\ell \in \{m+1, \ldots, 2m\}} \{ \eta_{j,2}, \hat{\eta}_{j,\ell} \} \geq \frac{M \sqrt{a_{j}}}{\sqrt{2}} + O(1).
\]
Using Lemma C.3 and the fact that \( M = n^{1/2} \), we infer that
\[
F_{n,j,\ell} = O(e^{-cn^{1/5}}) = O(n^{-101}), \quad \ell = 1, \ldots, 2m,
\]
uniformly for \( j \in \{ g_{2,+} + 1, \ldots, j_{2,+} \} \). Hence, by (2.24),
\[
S_4^{(1)} = \sum_{j = g_{2,+} + 1}^{j_{2,+}} \ln \left( 1 + \sum_{\ell = 1}^{2m} \omega_{\ell} O(n^{-101}) \right) = O(n^{-100}), \quad \text{as } n \to +\infty,
\]
and the claim follows. \( \square \)

Lemma 2.10. For any fixed \( x_1, \ldots, x_{2m} \in \mathbb{R} \), there exists \( \delta > 0 \) such that
\[
S_4^{(1)} = D_1^{(n)} + D_2^{(M)} \sqrt{n} + D_3 \ln n + D_4^{(n,\varepsilon, M)} + D_5^{(n,M)} + O\left( \frac{M^4}{n} + \frac{1}{\sqrt{nM}} + \frac{1}{M^6} + M^{14} \right),
\]
as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \} \), \( \ldots \), \( u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \), where
\[
D_1^{(n)} = \int_{b_{20}^{(n)}}^{b_{20}^{(n+1)}} f_1(x) dx, \quad D_2^{(M)} = -b_{20}^{(n)} f_1(b_{20}^{(n)}) M, \quad D_3 = \frac{-b_{20}^{(n)} T_1(b_{20}^{(n)})}{2(1 + T_0(b_{20}^{(n)}) + T_0(b_{20}^{(n+1)}))},
\]
\[
D_4^{(n,\varepsilon, M)} = -b_{20}^{(n)} M^2 \left( f_1(b_{20}^{(n)}) \right) + \frac{b_{20}^{(n)}}{2} f_1^2(b_{20}^{(n)}) \right) - \frac{b_{20}^{(n)} T_1(b_{20}^{(n)})}{1 + T_0(b_{20}^{(n)}) + T_0(b_{20}^{(n+1)})} \ln \left( \frac{\varepsilon}{M(1 - \varepsilon)} \right)
\]
\[
+ \int_{b_{20}^{(n)}}^{b_{20}^{(n+1)}} \left\{ f(x) + \frac{b_{20}^{(n)} T_1(b_{20}^{(n)})}{1 + T_0(b_{20}^{(n)}) + T_0(b_{20}^{(n+1)})} (x - b_{20}^{(n)}) \right\} dx + \left( \alpha - \frac{1}{2} \right) f_1(b_{20}^{(n)}) f_1^2(b_{20}^{(n+1)}) \left( \alpha - \frac{1}{2} \right) + 5 b_{20}^{(n+1)} f_1^2(b_{20}^{(n+1)}) \left( \alpha - \frac{1}{2} \right) + 6 \delta_{1,+}^{(n)} M^2
\]
\[
D_5^{(n,M)} = -b_{20}^{(n)} M^2 \left( f_1(b_{20}^{(n)}) \right) + \frac{b_{20}^{(n)}}{2} f_1^2(b_{20}^{(n)}) \right) + \frac{b_{20}^{(n)} T_1(b_{20}^{(n)})}{1 + T_0(b_{20}^{(n)}) + T_0(b_{20}^{(n+1)})} \ln \left( \frac{\varepsilon}{M(1 - \varepsilon)} \right)
\]
\[
+ \int_{b_{20}^{(n)}}^{b_{20}^{(n+1)}} \left\{ f(x) + \frac{b_{20}^{(n)} T_1(b_{20}^{(n)})}{1 + T_0(b_{20}^{(n)}) + T_0(b_{20}^{(n+1)})} (x - b_{20}^{(n)}) \right\} dx + \left( \alpha - \frac{1}{2} \right) f_1(b_{20}^{(n)}) f_1^2(b_{20}^{(n+1)}) \left( \alpha - \frac{1}{2} \right) + 5 b_{20}^{(n+1)} f_1^2(b_{20}^{(n+1)}) \left( \alpha - \frac{1}{2} \right) + 6 \delta_{1,+}^{(n)} M^2
\]
where \( f_1 \) and \( f \) are as in the statement of Lemma 2.7.

Proof. Using (2.8) and Lemma C.2, we obtain
\[
S_2^{(1)} = \sum_{j = g_{1,+} + 1}^{j_{1,+}} \ln \left( 1 + \sum_{\ell = 1}^{m} \omega_{\ell} \left( \frac{1}{2} \text{erfc} \left( -\eta_{j,\ell} \sqrt{a_{j}/2} \right) - R_{a_{j}}(\eta_{j,\ell}) \right) \right) + O(e^{-cn^{1/5}}) + \sum_{\ell = m+1}^{2m} \omega_{\ell} (1 + O(e^{-cn^{1/5}}))
\]
Furthermore, by a direct analysis, we find
\[ S_2^{(1)} = \sum_{j=g_1+1}^{j_1+} \left( f_1(j/n) + \frac{1}{n} f(j/n) + \frac{2b^3 \rho_1^4 \rho_0 \mathcal{T}_1(j/n)}{n^2 (1 + \mathcal{T}_0(j/n) + \mathcal{T}_0(b_{2b}^2))(j/n - b_{1b}^2)^3} \right. \\
+ \frac{1}{n^3} \left( 1 + \mathcal{T}_0(j/n) + \mathcal{T}_0(b_{2b}^2) \right)(j/n - b_{2b}^2)^5 \\
+ \mathcal{O}\left( \frac{n^{-2}}{(j/n - b_{1b}^2)^2} + \frac{n^{-3}}{(j/n - b_{2b}^2)^4} + \frac{n^{-4}}{(j/n - b_{2b}^2)^7} + \frac{n^{-6}}{(j/n - b_{1b}^2)^{12}} \right) \right).
\]

Note that
\[ \sum_{j=g_1+1}^{j_1+} \mathcal{O}\left( \frac{n^{-2}}{(j/n - b_{1b}^2)^2} + \frac{n^{-3}}{(j/n - b_{2b}^2)^4} + \frac{n^{-4}}{(j/n - b_{2b}^2)^7} + \frac{n^{-6}}{(j/n - b_{1b}^2)^{12}} \right) = \mathcal{O}\left( \frac{1}{M \sqrt{n}} + \frac{1}{M^3 \sqrt{n}} + \frac{1}{M^4} + \frac{\sqrt{n}}{M^4} \right). \]

Also, using Lemma 2.4 (with \( A = \frac{b_{2b}^2}{1 - \varepsilon}, a_0 = 1 - \alpha - \theta_1^{(n,M)}, B = \frac{b_{2b}^2}{1 - \varepsilon}, b_0 = -\alpha - \theta_1^{(n,c)} \)), we get
\[ \sum_{j=g_1+1}^{j_1+} f_1(j/n) = \int_{b_{2b}^2}^{b_{2b}^2} f_1(x) dx + \left( 1 - \frac{1}{2} + \theta_1^{(n,M)} \right) f_1\left( \frac{b_{2b}^2}{1 - \varepsilon} \right) + \left( \frac{1}{2} - \alpha \right) f_1\left( \frac{b_{2b}^2}{1 - \varepsilon} \right) + \mathcal{O}(n^{-1}). \]

Furthermore, by a direct analysis,
\[ n \int_{b_{2b}^2}^{b_{2b}^2} f_1(x) dx = n \int_{b_{2b}^2}^{b_{2b}^2} f_1(x) dx - b_{1b}^2 f_1(b_{1b}^2) M \sqrt{n} - M^2 b_{1b}^2 f_1(b_{1b}^2) + \frac{b_{2b}^2}{2} f_1(b_{1b}^2) + \mathcal{O}\left( \frac{M^4}{n} \right). \]

For any fixed $f$, using Lemma C.2, we obtain

$$f_1(b_{p_{2b}}) = f_1(b_{p_{2b}^2}) + \frac{M}{\sqrt{n}} b_{p_{2b}} f'_1(b_{p_{2b}^2}) + O\left(\frac{M^2}{n}\right),$$

$$\int_{\frac{b_{p_{2b}^2}}{2}} f(x) dx = \int_{\frac{b_{p_{2b}^2}}{2}} \left( f(x) + \frac{b_{p_{2b}^2} T_1(b_{p_{2b}^2})}{1 + T_0(b_{p_{2b}^2}) + T_0(b_{p_{2b}^2})} (x - b_{p_{2b}^2}) \right) dx - \frac{b_{p_{2b}^2} T_1(b_{p_{2b}^2})}{2(1 + T_0(b_{p_{2b}^2}) + T_0(b_{p_{2b}^2})) \ln n}.$$
\[= \mathcal{O}(e^{-an}) + \sum_{j=j_2}^{g_2-1} \ln \left( \frac{n}{1 + \frac{\sqrt{2}}{2}} \right) + \sum_{\ell=m+1}^{2m} \omega_{\ell} \frac{1}{\pi} \text{erfc} \left( \frac{-\hat{\eta}_{1,\ell} \sqrt{\alpha_j/2}}{1 - \frac{1}{2} \text{erfc} \left( -\eta_{j,2} \sqrt{\alpha_j/2} - R_{a_j} (\eta_{j,2}) \right)} \right), \]

\text{(2.26)}

Using then (C.3), we find

\[S_3^{(3)} = \sum_{j=j_2}^{g_2-1} \left( \hat{f}_1(j/n) + \frac{1}{n} \hat{f}(j/n) + \frac{1}{n^2} \frac{-2b^3 \rho^2 \hat{\eta}_1 (j/n)}{1 - \hat{T}_0 (j/n) + \hat{T}_0 (b \rho^2 - j/n)^3} \right) \]

\[+ \frac{1}{n^3} \frac{10 b^5 \rho^2 \hat{\eta}_1 (j/n) (b \rho^2 - j/n)^{-5}}{1 - \hat{T}_0 (j/n) + \hat{T}_0 (b \rho^2)} \]

\[+ \mathcal{O} \left( \frac{n^{-2}}{b \rho^2 - j/n)^2} + \frac{n^{-4}}{b \rho^2 - j/n)^4} + \frac{n^{-6}}{b \rho^2 - j/n)^6} \right). \]

Note that

\[\sum_{j=j_2}^{g_2-1} \mathcal{O} \left( \frac{n^{-2}}{b \rho^2 - j/n)^2} + \frac{n^{-4}}{b \rho^2 - j/n)^4} + \frac{n^{-6}}{b \rho^2 - j/n)^6} \right) = \mathcal{O} \left( \frac{1}{M \sqrt{n}} + \frac{1}{M^3 \sqrt{n}} + \frac{1}{M^6 \sqrt{n}} \right). \]

Also, using Lemma 2.4 (with \( A = \frac{b \rho^2}{1 + \tau} \), \( a_0 = \theta_{2,\tau}^{(n,r)} - \alpha \), \( B = \frac{b \rho^2}{1 + \tau} \), \( b_0 = \theta_{2,\tau}^{(n,M)} - 1 - \alpha \)), we get

\[\sum_{j=j_2}^{g_2-1} \hat{f}_1(j/n) = n \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \hat{f}_1(x) dx + (\frac{1}{2} + \alpha - \theta_{2,\tau}^{(n,r)}) \hat{f}_1 \left( \frac{b \rho^2}{1 + \tau} \right) + (\theta_{2,\tau}^{(n,M)} - \frac{1}{2} - \alpha) \hat{f}_1 \left( \frac{b \rho^2}{1 + \tau} \right) + \mathcal{O}(n^{-1}), \]

\[\frac{1}{n} \sum_{j=j_2}^{g_2-1} \hat{f}(j/n) = \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \hat{f}(x) dx + \mathcal{O} \left( \frac{1}{M \sqrt{n}} \right), \]

\[\frac{1}{n^2} \sum_{j=j_2}^{g_2-1} \frac{-2b^3 \rho^2 \hat{\eta}_1 (j/n) (b \rho^2 - j/n)^{-3}}{1 - \hat{T}_0 (j/n) + \hat{T}_0 (b \rho^2)} = \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \frac{-2b^3 \rho^2 \hat{\eta}_1 (x) (b \rho^2 - x)^{-3}}{1 - \hat{T}_0 (x) + \hat{T}_0 (b \rho^2)} dx + \mathcal{O} \left( \frac{1}{M^3 \sqrt{n}} \right), \]

\[\frac{1}{n^3} \sum_{j=j_2}^{g_2-1} \frac{10 b^5 \rho^2 \hat{\eta}_1 (j/n) (b \rho^2 - j/n)^{-5}}{1 - \hat{T}_0 (j/n) + \hat{T}_0 (b \rho^2)} = \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \frac{10 b^5 \rho^2 \hat{\eta}_1 (x) (b \rho^2 - x)^{-5}}{1 - \hat{T}_0 (x) + \hat{T}_0 (b \rho^2)} dx + \mathcal{O} \left( \frac{1}{M^5 \sqrt{n}} \right). \]

Furthermore, by a direct analysis,

\[n \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \hat{f}_1(x) dx = n \int_{\frac{b \rho^2}{1 + \tau}}^{\frac{b \rho^2}{1 + \tau}} \hat{f}_1(x) dx + M^2 \frac{b^2 \rho_2^2}{2} \hat{f}_1 \left( b \rho^2 \right) \]

\[+ \frac{M^3}{\sqrt{n}} b \rho_2^2 \left( \hat{f}_1 \left( b \rho^2 \right) + \frac{b \rho^2}{6} \hat{f}_1^\prime \left( b \rho^2 \right) \right) + \mathcal{O} \left( \frac{M^4}{n} \right), \]

\[\hat{f}_1 \left( \frac{b \rho^2}{1 + \tau} \right) = \hat{f}_1 \left( b \rho^2 \right) + \frac{M^2}{\sqrt{n}} b \rho_2^2 \hat{f}_1 \left( b \rho^2 \right) + \mathcal{O} \left( \frac{M^2}{n} \right), \]

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Lemma 2.13. \( \tilde{\lambda} \) where, for 

\[
\int_{1 - \frac{b^2}{2}}^{1 + \frac{b^2}{2}} f(x) \, dx = \int_{-1 + \frac{b^2}{2}}^{1 + \frac{b^2}{2}} \left\{ \tilde{f}(x) - b \rho T_1(b \rho^2) \frac{\ln n - b \rho T_2(b \rho^2)}{2} \ln \frac{M(1 + \epsilon)}{\epsilon} \right\} \, dx + \frac{b \rho T_1(b \rho^2)}{2} \ln n - b \rho T_2(b \rho^2) \ln \frac{M(1 + \epsilon)}{\epsilon} \\
+ \frac{M}{\sqrt{n}} \left\{ \left( b + \alpha \right) \rho^2 T_1(b \rho^2) + \frac{b \rho T_2(b \rho^2)}{2} + b \rho T_1(b \rho^2)^2 \right\} + O \left( \frac{M^2}{n} \right),
\]

\[
\frac{1}{n} \int_{1 - \frac{b^2}{2}}^{1 + \frac{b^2}{2}} -2b^3 \rho^3 T_1(x)(b \rho^2 - x)^{-3} \, dx = - \frac{b}{M^2} T_1(b \rho^2) + O \left( \frac{1}{M^2} \right),
\]

\[
\frac{1}{n^2} \int_{1 - \frac{b^2}{2}}^{1 + \frac{b^2}{2}} \frac{10b^5 \rho^5 T_1(x)(b \rho^2 - x)^{-5}}{1 - T_0(x) + \hat{T}_0(b \rho^2)} \, dx = \frac{5b T_1(b \rho^2)}{2 \rho^2 M^4} + O \left( \frac{1}{M^3} \right).
\]

The claim now follows after a computation. 

For \( \ell \in \{1, \ldots, 2m\} \) and \( j \in \{1, \ldots, n\} \), we define \( \hat{M}_{\ell,j} := (\hat{\lambda}_{j,\ell} - 1) \), \( M_{j,1} := (\sqrt{n}(\lambda_{j,1} - 1) \) and \( M_{j,2} := (\sqrt{n}(\lambda_{j,2} - 1) \). For the large \( n \) asymptotics of \( S_{2}^{(2)} \), we will need the following lemma.

Lemma 2.12 (Taken from [25, Lemma 3.11]). Let \( h \in C^3(\mathbb{R}) \) and \( k \in \{1, 2\} \). As \( n \to +\infty \), we have

\[
\sum_{j=g_k-}^{g_{k+}} h(M_{j,k}) = b \rho k \int_{-M}^{M} h(t) \, dt \sqrt{n} - 2b \rho k \int_{-M}^{M} t h(t) \, dt + \left( \frac{1}{2} - \theta_{(n,M)}^{(n,M)} \right) h(M) + \left( \frac{1}{2} - \theta_{(n,M)}^{(n,M)} \right) h(-M)
\]

\[
+ \frac{1}{\sqrt{n}} \left[ 3b \rho k \int_{-M}^{M} t^2 h(t) \, dt + \left( \frac{1}{12} + \frac{\theta_{(n,M)}^{(n,M)}(\theta_{(n,M)}^{(n,M)} - 1)}{2} \right) h'(M) - \left( \frac{1}{12} + \frac{\theta_{(n,M)}^{(n,M)}(\theta_{(n,M)}^{(n,M)} - 1)}{2} \right) h'(-M) \right]
\]

\[
+ O \left( \frac{1}{n^{3/2}} \sum_{j=g_k-}^{g_{k+}} \frac{1}{3} \right) \left( 1 + |M_{j,k}|^3 \right) \hat{m}_{j,n,k}(h) + (1 + M_{j,k}^2) \hat{m}_{j,n,k}(h') + (1 + |M_{j,k}|) \hat{m}_{j,n,k}(h'') + \hat{m}_{j,n,k}(h''') \right),
\]

where, \( \hat{m}_{j,n,k}(\hat{h})(\cdot) \). \( \hat{m}_{j,n,k}(\hat{h}) \).

Lemma 2.13. Let \( x_1, \ldots, x_{2m} \in \mathbb{R} \) be fixed. There exists \( \delta > 0 \) such that

\[
S_{2}^{(2)} = E_{2}^{(M)} \sqrt{n} + E_{4}^{(M)} + E_{5}^{(M)} + O \left( \frac{M}{n} + \frac{M^4}{n^2} \right),
\]

\[
E_{2}^{(M)} = 2b \rho^2 M \ln(1 + T_0(b \rho^2)) + \hat{T}_0(b \rho^2),
\]

\[
E_{4}^{(M)} = \ln(1 + T_0(b \rho^2)) + \hat{T}_0(b \rho^2))(1 - \theta_{(n,M)}^{(n,M)} - \theta_{(n,M)}^{(n,M)}) + b \rho^2 \int_{-M}^{M} h_1(t) \, dt,
\]

\[
E_{5}^{(M)} = 2b \rho^2 M \ln(1 + T_0(b \rho^2)) + \hat{T}_0(b \rho^2)) \left( \frac{1}{2} - \theta_{(n,M)}^{(n,M)} \right) \ln(M) + \left( \frac{1}{2} - \theta_{(n,M)}^{(n,M)} \right) \ln(-M)
\]

\[
+ b \rho^2 \int_{-M}^{M} (h_2(t) - 2t h_1(t)) \, dt,
\]

as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \), where \( h_1, h_2 \) are given by

\[
h_1(x) = - \frac{2 \rho T_1(b \rho^2)}{1 + T_0(b \rho^2) + \hat{T}_0(b \rho^2)} \sqrt{2 \pi} \text{erfc}(\frac{-x}{\sqrt{2}}),
\]
\[ h_2(x) = -\frac{h_1(x)^2}{2} + \frac{1}{1 + T_0(b\rho_2^{2b}) + T_0(b\rho_2^{2b}) \sqrt{2\pi} \text{erfc}(-\frac{x\rho_2^{2b}}{\sqrt{2}})} \left\{ \left( \rho_1^b x - \frac{5}{3} \rho_1^b x^3 \right) T_1(b\rho_2^{2b}) \right. \\
\left. - \rho_1^b x T_2(b\rho_2^{2b}) + \frac{4 - 10\rho_1^b x^2}{3} T_1(b\rho_1^b) \right\} \sqrt{2\pi} \text{erfc}(-\frac{x\rho_1^b}{\sqrt{2}}). \]

**Proof.** In a similar way as in (2.25), we infer that there exists \( \delta > 0 \) such that

\[ S_2^{(2)} = \sum_{j, \lambda_{j,1} \in I_2} \ln \left( 1 + T_0(b\rho_2^{2b}) + \sum_{t=1}^m \omega_t \frac{1}{2} \text{erfc}(\frac{-\hat{\eta}_{j,t} \sqrt{a_j/2}}{2}) - R_{a_j}(\hat{\eta}_{j,t}) \right) \right) + O(e^{-cn^{1/3}}), \quad (2.28) \]

as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_2m \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \}. \) For \( j \in \{ j : \lambda_{j,1} \in I_2 \}, \) we have \( 1 - \frac{M}{\sqrt{n}} \leq \lambda_{j,1} = \frac{2m^{2b} \rho_2^{2b}}{3 \sqrt{2}} \leq 1 + \frac{M}{\sqrt{n}}, -M \leq M_{j,1} \leq M, \) and

\[ M_{j,1} - \frac{t_{\ell}}{n} - \frac{t_{\ell} M_{j,1}}{n}, \quad \ell = 1, \ldots, m. \]

Furthermore, for \( \ell \in \{1, \ldots, m\}, \) as \( n \to +\infty \) we have

\[-\hat{\eta}_{j,t} \sqrt{a_j/2} - \frac{M_{j,1} \rho_1^b}{\sqrt{2}} + \frac{7M_{j,1}^2 - 12t_{\ell} M_{j,1}}{36 \rho_1^b n^2} + O \left( \frac{1 + M_{j,1}^4}{n^2} \right), \]

\[-\hat{\eta}_{j,t} \sqrt{a_j/2} - \frac{M_{j,1} \rho_1^b}{\sqrt{2}} + \frac{5M_{j,1}^2 + 6t_{\ell}}{6 \sqrt{2} \rho_1^b n} - \frac{\rho_1^b M_{j,1} (53M_{j,1}^2 + 12t_{\ell})}{72 \sqrt{2} n} + O \left( \frac{1 + M_{j,1}^4}{n^{3/2}} \right), \]

uniformly for \( j \in \{ j : \lambda_{j,1} \in I_2 \}. \) Hence, for \( \ell \in \{1, \ldots, m\}, \) by (C.3) as \( n \to +\infty \) we have

\[ R_{a_j}(\hat{\eta}_{j,t}) = \frac{e^{-M_{j,1}^2 \rho_1^b}}{\sqrt{2\pi}} \left( -\frac{1}{3 \rho_1^b n} - \frac{M_{j,1} (3 + 10M_{j,1}^2 \rho_1^b + 12t_{\ell} \rho_1^b)}{36 \rho_1^b n} + O((1 + M_{j,1} n^{-\frac{1}{2}}) \right) \]

and

\[ \frac{1}{2} \text{erfc}(\frac{-\hat{\eta}_{j,t} \sqrt{a_j/2}}{2}) = \frac{1}{2} \text{erfc}(\frac{-M_{j,1} \rho_1^b}{\sqrt{2}}) - \frac{e^{-M_{j,1}^2 \rho_1^b}}{2 \sqrt{2\pi}} + \frac{e^{-M_{j,1}^2 \rho_1^b}}{2 \sqrt{2\pi}} \frac{53M_{j,1}^2 + 12t_{\ell} - 25M_{j,1}^4 \rho_1^b - 60M_{j,1}^2 \rho_1^b - 96t_{\ell} \rho_1^b}{72 \sqrt{2} \rho_1^b n} + O \left( \frac{1 + M_{j,1}^8}{n^{3/2}} \right), \]

uniformly for \( j \in \{ j : \lambda_{j,1} \in I_2 \}. \) Extending the above asymptotic formulas to higher order (see [9, Lemma 2.8] for details in a similar situation), we deduce that the argument of \( \ln \) in (2.28) enjoys the following asymptotics

\[ 1 + \tilde{T}_0(b\rho_2^{2b}) + \sum_{t=1}^m \omega_t \frac{1}{2} \text{erfc}(\frac{-\hat{\eta}_{j,t} \sqrt{a_j/2}}{2}) - R_{a_j}(\hat{\eta}_{j,t}) \]

\[ = g_1(M_{j,1}) + \frac{g_2(M_{j,1})}{\sqrt{n}} + \frac{g_3(M_{j,1})}{n} + \frac{g_4(M_{j,1})}{n^{3/2}} + \frac{g_5(M_{j,1})}{n^2} + O \left( \frac{1 + |M_{j,1}|^{13}}{n^{5/2}} \right), \quad (2.29) \]

as \( n \to +\infty, \) where

\[ g_1(x) = 1 + T_0(b\rho_1^b) + \tilde{T}_0(b\rho_2^{2b}), \quad g_2(x) = -\frac{e^{-\frac{1}{2} x^2 \rho_1^b}}{\sqrt{2\pi} \text{erfc}(-\frac{x \rho_1^b}{\sqrt{2}})}, \]

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\[ g_3(x) = \frac{e^{-\frac{1}{2}x^2\rho_b^h}}{3\sqrt{2\pi} \text{erfc}(\frac{x\rho_b^h}{\sqrt{2}})} \left\{ \frac{e^{-\frac{1}{2}x^2\rho_b^2}}{\sqrt{2\pi} \text{erfc}(\frac{x\rho_b^2}{\sqrt{2}})} (4 - 10x^2\rho_b^2) + T_1(b\rho_1^{2h}) (3x\rho_1^h - 5x^3\rho_1^h) \right\} \]

We remark that equation (2.29) coincides with [9, eq (2.29)] except that (2.29) has the extra term \( \hat{T}_0(b\rho_2^{2h}) \) on each side and \((\rho, M, \eta, \eta, \ell)\) in [9, eq (2.29)] are given by \((\rho_1, M_1, \eta_1, \eta_1, \ell_1)\) here. The functions \( g_4 \) and \( g_5 \) can also be computed explicitly, but we do not write them down. Since

\[ \sqrt{\frac{2\pi}{x^2 \hat{\rho}_b^h}} \sim -\frac{\hat{\rho}_b^h x}{2} \text{ as } x \to -\infty, \quad g_2(x) = O(x) \text{ as } x \to -\infty. \]

At first glance, it seems that \( g_3(x) = O(x^4) \) as \( x \to -\infty \). However, some cancellations take place and in fact \( g_3(x) = O(x^2) \) as \( x \to -\infty \). Likewise, the expressions for \( g_4 \) and \( g_5 \) suggest a priori that \( g_4(x) = O(x^7) \) and \( g_5(x) = O(x^{10}) \) as \( x \to -\infty \), but here too, cancellations take place and in fact \( g_4(x) = O(x^5) \) and \( g_5(x) = O(x^4) \) as \( x \to -\infty \). Using the above large \( x \) estimates for \( \{g_j(x)\}_{j=2}^{\infty} \) and (2.29), we obtain after a computation that

\[ S_2^{(2)} = \sum_{j=g_1-}^{g_1+} \left\{ \ln(1 + T_0(b\rho_1^{2h}) + \hat{T}_0(b\rho_2^{2h})) + \frac{h_1(M_{j,1})}{\sqrt{n}} + \frac{h_2(M_{j,1})}{n} + O \left( \frac{1 + |M_{j,1}|^3}{n^{3/2}} + \frac{1 + |M_{j,1}|^{13}}{n^{5/2}} \right) \right\}. \]

as \( n \to +\infty \). The above error can be estimated as follows:

\[ \sum_{j=g_1-}^{g_1+} O \left( \frac{1 + |M_{j,1}|^3}{n^{3/2}} + \frac{1 + |M_{j,1}|^{13}}{n^{5/2}} \right) = O \left( \frac{M^4}{n} + \frac{M^{14}}{n^2} \right), \quad \text{as } n \to +\infty. \]

Using then Lemma 2.12, we obtain the claim. \( \square \)

**Lemma 2.14.** Let \( x_1, \ldots, x_{2m} \in \mathbb{R} \) be fixed. There exists \( \delta > 0 \) such that

\[ S_4^{(2)} = \hat{E}_4^{(M)} \sqrt{n} + \hat{E}_4^{(M)} + \frac{\hat{E}_5^{(M)}}{\sqrt{n}} + O \left( \frac{M^4}{n} + \frac{M^{14}}{n^2} \right), \]

\[ \hat{E}_4^{(M)} = 0, \quad \hat{E}_4^{(M)} = b\rho_2^M \int_{-M}^{M} \hat{h}_1(t) dt, \]

\[ \hat{E}_5^{(M)} = \left( \frac{1}{2} - \theta_{2,-}^{(n,M)} \right) \hat{h}_1(M) + \left( \frac{1}{2} - \theta_{2,+}^{(n,M)} \right) \hat{h}_1(-M) + b\rho_2^M \int_{-M}^{M} \left( \hat{h}_2(t) - 2t \hat{h}_1(t) \right) dt, \]

as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \), where \( \hat{h}_1, \hat{h}_2 \) are given by

\[ \hat{h}_1(x) = 2b\rho_2^M \frac{e^{-\frac{1}{2}x^2\rho_b^h}}{2\sqrt{2\pi} \left( 2 - \text{erfc}(\frac{x\rho_b^h}{\sqrt{2}}) \right)}, \]

\[ \hat{h}_2(x) = -\frac{\hat{h}_1(x)^2}{2} + \frac{e^{-\frac{1}{2}x^2\rho_b^h}}{2\sqrt{2\pi} \left( 2 - \text{erfc}(\frac{x\rho_b^h}{\sqrt{2}}) \right)} \left\{ \left( -\rho_b^h x + \frac{5}{3} \rho_b^M x^3 \right) \hat{T}_1(b\rho_2^{2h}) \right\}. \]
Lemma 2.8, we deduce from (2.30) that

\[ S^{(2)}_1 = \mathcal{O}(e^{-c_1 n}) + \sum_{j: \lambda_j \in I_2} \ln \left( 1 + \sum_{\ell=m+1}^{2m} \omega_\ell \frac{1}{2} \text{erfc} \left( -\frac{n}{\sqrt{2}} a_j / 2 \right) - R_{\lambda_j}(n) \right), \]

as \( n \to +\infty \) uniformly for \( u_1 \in \{ z \in \mathbb{C} : |z - x_1| \leq \delta \}, \ldots, u_{2m} \in \{ z \in \mathbb{C} : |z - x_{2m}| \leq \delta \} \). For \( j \in \{ j: \lambda_j \in I_2 \} \), we have \( 1 - \frac{M}{\sqrt{n}} \leq \lambda_j \leq \frac{b_0 \rho_2}{\sqrt{n}} - 1 + \frac{M}{\sqrt{n}}, \) \( -M \leq \lambda_j \leq M \), and

\[ \dot{M}_{j,k} = M_{j,2} + \frac{t_k}{\sqrt{n}} + \frac{t_k M_{j,2}}{n}, \quad k = m + 1, \ldots, 2m. \]

Furthermore, for \( \ell \in \{ m + 1, \ldots, 2m \} \), as \( n \to +\infty \) we have

\[ \frac{\dot{\eta}_{j,\ell}}{\sqrt{2}} - \frac{M_{j,2}}{\sqrt{n}} = \frac{a_{j,2}}{\sqrt{2}} + \frac{7M_{j,2}^2 + 12t_{\ell} M_{j,2} + \mathcal{O} \left( \frac{1 + M_{j,2}^4}{n^2} \right)}{36n^{3/2}} \]

and

\[ \text{erfc} \left( -\frac{\dot{\eta}_{j,\ell} a_j / 2}{\sqrt{2}} \right) = \frac{1}{2} \text{erfc} \left( -\frac{\rho_2 M_{j,2}}{\sqrt{2}} \right) - \frac{e^{-\frac{M_{j,2}^2}{2}}}{6\sqrt{2n}} \rho_2 (5M_{j,2}^2 - 6t_{\ell}) \]

as \( n \to +\infty \) uniformly for \( j \in \{ j: \lambda_j \in I_2 \} \). Extending the above asymptotic formulas to higher order (see [9, Lemma 2.8] for details in a similar situation), we deduce from (2.30) that

\[ 1 + \sum_{\ell=m+1}^{2m} \omega_\ell \frac{1}{2} \text{erfc} \left( -\frac{\dot{\eta}_{j,\ell} a_j / 2}{\sqrt{2}} \right) - R_{\lambda_j}(n) \leq \mathcal{O} \left( \frac{1}{n^{5/2}} \right), \]

as \( n \to +\infty \), where

\[ \dot{g}_2(x) = \frac{e^{-\frac{1}{x^2} \rho_2^2} T_1(b_0 \rho_2^2)}{\sqrt{2\pi} (2 - \text{erfc} \left( -\frac{x \rho_2^2}{\sqrt{2}} \right))}; \]

\[ \dot{g}_3(x) = \frac{e^{-\frac{1}{x^2} \rho_2^2}}{\sqrt{2\pi} (2 - \text{erfc} \left( -\frac{x \rho_2^2}{\sqrt{2}} \right))} \left( -\rho_2^2 x + \frac{5}{3} \rho_2^3 x^3 \right) T_1(b_0 \rho_2^2). \]
The functions \( \hat{g}_4 \) and \( \hat{g}_5 \) can also be computed explicitly, but we do not write them down. Since 
\[
\frac{e^{-\frac{1}{2}x^2-x^2b}}{\sqrt{2\pi} \left( 2 - \text{erfc} \left( -\frac{x}{\sqrt{2}} \right) \right)} \sim \frac{e^{-\frac{1}{2}x^2-x^2b}}{2} \quad \text{as} \quad x \to +\infty, 
\]
\( \hat{g}_2(x) = \mathcal{O}(x) \) as \( x \to +\infty \). At first glance, it seems that 
\( \hat{g}_3(x) = \mathcal{O}(x^2) \) as \( x \to +\infty \). However, some cancellations take place and in fact 
\( \hat{g}_3(x) = \mathcal{O}(x^2) \) as \( x \to +\infty \). Likewise, the expressions for \( \hat{g}_4 \) and \( \hat{g}_5 \) suggest a priori that 
\( \hat{g}_4(x) = \mathcal{O}(x^7) \) and 
\( \hat{g}_5(x) = \mathcal{O}(x^4) \) as \( x \to +\infty \), but here too, cancellations take place and in fact 
\( \hat{g}_4(x) = \mathcal{O}(x^2) \) and 
\( \hat{g}_5(x) = \mathcal{O}(x^2) \) as \( x \to +\infty \). Using the above large \( x \) estimates for \( \{\hat{g}_i(x)\} \) and (2.33), we obtain after a computation that 
\[
S_4^{(2)} = \sum_{j=g_2}^{g_3} \left\{ \hat{h}_1(M_{j,2}) + \hat{h}_2(M_{j,2}) + \mathcal{O}\left( \frac{1 + |M_{j,2}|^3}{n^{3/2}} + \frac{1 + |M_{j,2}|^3}{n^{5/2}} \right) \right\}. 
\]
as \( n \to +\infty \), where \( \hat{h}_1 = \hat{g}_2 \) and \( \hat{h}_2 = -\hat{g}_2 + \hat{g}_3 \). Note that 
\[
\sum_{j=g_2}^{g_3} \mathcal{O}\left( \frac{1 + |M_{j,2}|^3}{n^{3/2}} + \frac{1 + |M_{j,2}|^3}{n^{5/2}} \right) = \mathcal{O}\left( \frac{M^4}{n} + \frac{M^4}{n^2} \right), \quad \text{as} \quad n \to +\infty. 
\]
Using then Lemma 2.12, we find the claim. \( \square \)

Define the real constants \( \{I_j\} \subset \mathbb{R} \) by
\[
I_1 = \int_{-\infty}^{+\infty} \left\{ \frac{e^{-y^2}}{\sqrt{\pi} \text{erfc}(y)} - \chi_{(0,+\infty)}(y) \left[ y + \frac{y}{2(1+y^2)} \right] \right\} dy, 
\quad (2.34)
\]
\[
I_2 = \int_{-\infty}^{+\infty} \left\{ \frac{y^2e^{-y^2}}{\sqrt{\pi} \text{erfc}(y)} - \chi_{(0,+\infty)}(y) \left[ y^4 + \frac{y^2}{2} - \frac{1}{2} \right] \right\} dy, 
\quad (2.35)
\]
\[
I_3 = \int_{-\infty}^{+\infty} \left\{ \left( \frac{e^{-y^2}}{\sqrt{\pi} \text{erfc}(y)} \right)^2 - \chi_{(0,+\infty)}(y) \left[ y^2 + 1 \right] \right\} dy, 
\quad (2.36)
\]
\[
I_4 = \int_{-\infty}^{+\infty} \left\{ \left( \frac{y^2e^{-y^2}}{\sqrt{\pi} \text{erfc}(y)} \right)^2 - \chi_{(0,+\infty)}(y) \left[ y^4 + y^2 - \frac{3}{4} \right] \right\} dy. 
\quad (2.37)
\]
Recall also that \( I \) is given by (1.20).

Lemma 2.15. For any fixed \( x_1, \ldots, x_2m \in \mathbb{R} \), there exists \( \delta > 0 \) such that
\[
S_2 = -j_{1,-} \ln \Omega + F_1^{(c)} n + F_2 \ln n + F_3^{(n,c)} + \frac{F_4}{\sqrt{n}} + \mathcal{O}\left( \frac{\sqrt{n}}{M^{1/1}} + \frac{1}{M^6} + \frac{1}{\sqrt{n}M} + \frac{M^4}{n} + \frac{M^4}{n^2} \right), 
\quad \text{as} \quad n \to +\infty \]
Lemma 2.16. By combining Lemmas 2.8, 2.10 and 2.13, we have

Proof. and as in the statement of Lemma 2.7.

Using that are as in the statement, and are as in the statement of Lemma 2.7.

as in the statement, and

Using that , we readily verify that . Furthermore, a long but direct computation shows that (see [9, Lemma 2.9] for a similar situation with more details provided)

and the claim follows.

Lemma 2.16. For any fixed , there exists such that

as uniformly for , where

and as in the statement of Lemma 2.7.
Proof. By combining Lemmas 2.9, 2.11 and 2.14, we have
\[
S_4 = \tilde{F}_1^{(c)} n + \tilde{F}_2 \sqrt{n} + \tilde{F}_2 \ln n + \tilde{F}_3^{(n,\epsilon, M)} + \tilde{F}_4^{(M)} \frac{\sqrt{n}}{n} + O\left(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^6} + \frac{1}{\sqrt{n} M} + \frac{M^4}{n} + \frac{M^{14}}{n^2}\right),
\]
as \(n \to +\infty\) uniformly for \(u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}\), \(\ldots\), \(u_{2m} \in \{z \in \mathbb{C} : |z - x_{2m}| \leq \delta\}\), where \(\tilde{F}_1^{(c)}\) and \(\tilde{F}_2\) are as in the statement, and
\[
\begin{align*}
\tilde{F}_2 &= \tilde{E}_2^{(M)}, \\
\tilde{F}_3^{(n,\epsilon, M)} &= \tilde{E}_3^{(n,\epsilon, M)} + \tilde{E}_4^{(M)}, \\
\tilde{F}_4^{(M)} &= \tilde{E}_5^{(n, M)} + \tilde{E}_5^{(M)}.
\end{align*}
\]
It is easy to check that \(\tilde{F}_2 = 0\). Furthermore, a long but direct computation shows that (see e.g. [9, Proof of Lemma 2.9] for a similar analysis with more details provided)
\[
\tilde{F}_3^{(n,\epsilon, M)} = \tilde{F}_3^{(n,\epsilon)} + O(M^{-6}), \quad \tilde{F}_4^{(M)} = \tilde{F}_4 + O(M^{-1}),
\]
and the claim follows.

End of the proof of Theorem 1.2. Let \(M' > 0\) be sufficiently large such that Lemmas 2.2 and 2.15 hold. Using (2.2) and Lemmas 2.1, 2.2, 2.15, 2.7, 2.16 and 2.3, we conclude that for any \(x_1, \ldots, x_m \in \mathbb{R}\), there exists \(\delta > 0\) such that
\[
\ln \mathcal{E}_n = S_0 + S_1 + S_2 + S_3 + S_4 + S_5
\]
\[
= M' \ln \Omega + (j_1 - M' - 1) \ln \Omega - j_1 \ln \Omega + C_1 n + C_2 \ln n + (C_3 + \ln \Omega) + \mathcal{F}_n + C_4 \frac{\sqrt{n}}{\sqrt{M}}
\]
\[
+ O\left(\frac{\sqrt{n}}{M^{11}} + \frac{1}{M^{6}} + \frac{1}{\sqrt{n} M} + \frac{M^{4}}{n} + \frac{M^{14}}{n^2}\right),
\]
as \(n \to +\infty\) uniformly for \(u_1 \in \{z \in \mathbb{C} : |z - x_1| \leq \delta\}\), \(\ldots\), \(u_{m} \in \{z \in \mathbb{C} : |z - x_{m}| \leq \delta\}\), where
\[
C_1 = F_1^{(c)} + \int_{\sigma_{*}}^{\sigma_{t}} f_1(x) dx + \int_{\sigma_{*}}^{\sigma_{t} + \epsilon} \hat{f}_1(x) dx + \tilde{F}_1^{(c)}, \quad C_2 = F_2 + \tilde{F}_2, \quad C_4 = F_4 + \tilde{F}_4,
\]
\[
C_3 + \ln \Omega = F_3^{(n, \epsilon)} + \left(\alpha - \frac{1}{2} + \theta(n, \epsilon)\right) f_1\left(\frac{b^2 p^{1/2} x}{1 - \epsilon}\right) + \left(\beta(n, \epsilon) - \alpha - \frac{1}{2}\right) \hat{f}_1\left(\frac{b^2 p^{1/2} x}{1 + \epsilon}\right)
\]
\[
+ \left(\frac{1}{2} - \alpha - \theta_{*}\right) (f_1(\sigma_{*}) - \hat{f}_1(\sigma_{*})) + \int_{\sigma_{*}}^{\sigma_{t} + \epsilon} f(x) dx + \int_{\sigma_{*}}^{\sigma_{t} - \epsilon} \hat{f}(x) dx
\]
\[
+ \theta_{*} \ln Q - \frac{\ln Q}{2} \left(1 - \frac{2 \ln(\sigma_{2}/\sigma_{1})}{2 \ln(p_2/p_1)}\right) + \tilde{F}_3^{(n, \epsilon)}.
\]
It only remains to show that the constants \(\{C_j\}_1^4\) can be expressed as in the statement of Theorem 1.2. This is easily verified for \(C_1\) and \(C_2\). Using the identities
\[
f_1(\sigma_{*}) - \hat{f}_1(\sigma_{*}) = \ln Q, \quad I_4 = \frac{\ln(2 \sqrt{\pi})}{2},
\]
\[
\int_{\sigma_{*}}^{\sigma_{t} + \epsilon} \frac{b^2 p^{1/2}}{\Omega(x - b^2 p^{1/2})^2} \Omega(x - b^2 p^{1/2}) \ln \left(\frac{\sigma_{*} - b^2 p^{1/2}}{b^2 p^{1/2}} - \ln \frac{\epsilon}{1 - \epsilon}\right),
\]
\[
\frac{\ln \Omega(\sigma_{*})}{2} - \frac{\ln \Omega(\sigma_{t} + \epsilon)}{2} - \frac{\ln \Omega(\sigma_{*} - \epsilon)}{2} - \frac{\ln \Omega(\sigma_{t})}{2} - \frac{\ln \Omega(\sigma_{*})}{2} + \frac{\ln \Omega(\sigma_{t} + \epsilon)}{2} + \frac{\ln \Omega(\sigma_{*} - \epsilon)}{2} + \frac{\ln \Omega(\sigma_{t})}{2} - \frac{\ln \Omega(\sigma_{*})}{2}.
\]
\[
- \int_{\sigma}^{\delta} p_2^{2b} \frac{b^2 T_1(b p_2^{2b}; \bar{t}, \bar{u})}{b p_2^{2b} - x} dx = b_2^{2b} \frac{b_{2b} \hat{T}_1(b p_2^{2b}; \bar{t}, \bar{u})}{b p_2^{2b} - x} \left( \ln \frac{b p_2^{2b}}{b p_2^{2b} - \sigma} + \ln \frac{\epsilon}{1 + \epsilon} \right).
\]

long but straightforward calculations show that \( C_3 \) also can be written as in Theorem 1.2. Moreover, substituting the expressions for \( F_4 \) and \( \hat{F}_4 \) obtained in Lemmas 2.15 and 2.16 into the relation \( C_4 = F_4 + \hat{F}_4 \), we infer that

\[
C_4 = \sqrt{2b} \left( \frac{\rho_1^b T_2(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} - 5 T_1(b p_1^{2b}; \bar{t}, \bar{u}) \right) \Omega
+ \frac{10 \sqrt{2b}}{3} \left( \frac{\rho_1^b T_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} + \frac{\rho_1^b \hat{T}_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} \right) (\mathcal{I}_2 - \mathcal{I}_4)
+ \sqrt{2b} \left[ \frac{\rho_1^b T_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} \left( \frac{2}{3} - \frac{\rho_1^b T_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} \right) + \frac{\rho_1^b \hat{T}_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} \left( \frac{2}{3} - \frac{\rho_1^b \hat{T}_1(b p_1^{2b}; \bar{t}, \bar{u})}{\Omega} \right) \right] \mathcal{I}_3.
\]

Using the identities \( \mathcal{I}_3 = \mathcal{I} \) and \( \mathcal{I}_2 - \mathcal{I}_4 = \mathcal{I} \), which can be obtained via integration by parts (see [9, Lemma 2.10] for details), we conclude that \( C_4 \) also can be expressed as in Theorem 1.2. This finishes the proof of Theorem 1.2. □

A Balayage of radially symmetric measures

The measure \( \mu_h \) is defined as the unique minimizer of the energy functional

\[
I[\nu] = \iint \ln \frac{1}{|z - w|} \nu(d^2 z) \nu(d^2 w) + \int Q(z) \nu(d^2 z), \quad Q(z) := \begin{cases} |z|^{2b}, & \text{if } |z| \in [0, \rho_1] \cup [\rho_2, +\infty), \\ +\infty, & \text{otherwise}, \end{cases}
\]

among all Borel probability measures \( \nu \) on \( \mathbb{C} \). In this appendix, we present a derivation of the expression (1.4) for \( \mu_h \).

Let \( C = \{ z \in \mathbb{C} : |z| \in [0, \rho_1] \cup [\rho_2, +\infty) \} \) and \( G = \mathbb{C} \setminus C = \{ z \in \mathbb{C} : |z| \in (\rho_1, \rho_2) \} \). The unique minimizer of

\[
\tilde{I}[\nu] = \iint \ln \frac{1}{|z - w|} \nu(d^2 z) \nu(d^2 w) + \int |z|^{2b} \nu(d^2 z)
\]

among all Borel probability measures \( \nu \) on \( \mathbb{C} \) is given by \( \mu(d^2 z) = \chi_{[0, b^{-\frac{1}{b}}]}(|z|) \frac{1}{4\pi} \Delta |z|^{2b} d^2 z = \chi_{[b^{-\frac{1}{b}}, \infty)}(|z|) \frac{2b}{\pi} |z|^{2b-2} d^2 z \) (see [50, Example IV.6.2]). We start with the following lemma.

**Lemma A.1.** We have

\[
\mu_h = \mu \cdot \chi_G + \hat{\mu},
\]

where \( \hat{\mu} := \text{Ball}(\mu \cdot \chi_G, \partial G) \) is the balayage of \( \mu \cdot \chi_G \) onto \( \partial G = \{|z| = \rho_1\} \cup \{|z| = \rho_2\} \).

**Proof.** Let us first introduce some notation and background from [50]. Given a Borel probability measure \( \nu \), define

\[
U^\nu(z) := \int_{\mathbb{C}} \ln \frac{1}{|z - w|} \nu(d^2 w), \quad z \in \mathbb{C}.
\]

By [50, Theorem I.1.3 (d) and (f)], \( \mu \) and \( \mu_h \) satisfy the following conditions: there exist \( F, F_h \in \mathbb{R} \) such that

\[
\begin{cases}
U^\mu(z) + \frac{1}{2} |z|^{2b} \geq F, \quad z \in \mathbb{C}, \\
U^{\mu_h}(z) + \frac{1}{2} |z|^{2b} = F_h, \quad |z| \leq b^{-\frac{1}{b}},
\end{cases}
\]

(A.2)
\[
\begin{aligned}
U^{\mu_h}(z) + \frac{1}{2} \hat{Q}(z) &\geq F_h, \quad z \in \mathbb{C}, \\
U^{\nu_h}(z) + \frac{1}{2} \hat{Q}(z) &= F_h, \quad z \in \text{supp}(\mu_h).
\end{aligned}
\]  

(A.3)

Furthermore, by [50, Theorem I.3.3], the conditions (A.3) uniquely characterize \( \mu_h \) in the sense that if \( \nu \) is a Borel probability measure with compact support which satisfies \( U^{\nu}(z) + \frac{1}{2} \hat{Q}(z) \geq F_\nu \) for some constant \( F_\nu \) and all \( z \in \mathbb{C} \) and if \( U^{\nu}(z) + \frac{1}{2} \hat{Q}(z) = F_\nu \) for all \( z \in \text{supp}(\nu) \), then \( \nu = \mu_h \).

Let \( \tilde{\mu}_h \) denote the right-hand side of (A.1). We will show that \( \tilde{\mu}_h = \mu_h \) using the aforementioned uniqueness theorem, namely [50, Theorem I.3.3]. Since \( \tilde{\mu} \) is the balayage of \( \mu \cdot \chi_G \) onto \( \partial G \), [50, Theorem II.4.1] shows that

\[
\begin{aligned}
\mu(z) \leq U^{\mu \chi_G}(z), \quad z \in \mathbb{C}, \\
U^{\mu}(z) = U^{\mu \chi_G}(z), \quad z \in C.
\end{aligned}
\]

Combining the above with (A.2) and using that \( U^{\mu} = U^{\mu \chi_C} + U^{\mu \chi_G} \), we obtain (recall that \( \hat{Q}(z) = +\infty \) for \( z \in \bar{G} \))

\[
\begin{aligned}
U^{\mu \chi_C}(z) + U^{\mu}(z) + \frac{1}{2} \hat{Q}(z) &\geq F, \quad |z| \notin [0, r_1] \cup [r_2, b^{-\frac{1}{4}}], \\
U^{\mu \chi_C}(z) + U^{\mu}(z) + \frac{1}{2} \hat{Q}(z) &= F, \quad |z| \in [0, r_1] \cup [r_2, b^{-\frac{1}{4}}].
\end{aligned}
\]

Since \( U^{\mu \chi_C} + U^{\mu} = U^{\tilde{\mu}_h} \), [50, Theorem I.3.3] applies and gives \( \tilde{\mu}_h = \mu_h \).

It only remains to compute \( \tilde{\mu} := \text{Bal}(\mu \cdot \chi_G, \partial G) \) explicitly. Recall that

\[
\mu(d^2z) = \frac{b^2}{\pi} |z|^{2b-2} r \chi_{(0, b^{-\frac{1}{4}})}(r)d^2z, \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in (-\pi, \pi],
\]

and that \( 0 < r_1 < r_2 < b^{-\frac{1}{4}} \). Hence \( \mu \cdot \chi_G \) is the radially symmetric measure given by

\[
(\mu \cdot \chi_G)(d^2z) = f(r) \chi_{(r_1, r_2)}(r) r dr d\theta
\]

where \( f(r) := b^2 r^{2b-2} \). Fix \( z \in \mathbb{C} \setminus \overline{G} \) and compute the logarithmic potential

\[
U^{\mu \chi_G}(z) = \int_\mathbb{C} \frac{1}{|z - w|} (\mu \cdot \chi_G)(d^2w) = 2 \int_{r_1}^{r_2} f(r) r dr \int_0^{2\pi} \frac{1}{|z - re^{i\theta}|} d\theta.
\]

Now we use the familiar integral (e.g. [50, Chapter 0, page 22])

\[
\frac{1}{2\pi} \int_0^{2\pi} \ln \frac{1}{|z - re^{i\theta}|} d\theta = \begin{cases} 
\ln \frac{1}{r}, & |z| \leq r, \\
\ln \frac{1}{|z|}, & |z| \geq r.
\end{cases}
\]

(A.4)

It follows that

\[
U^{\mu \chi_G}(z) = \begin{cases} 
C_1, & \text{if } |z| \leq r_1, \\
C_2 \ln \frac{1}{|z|}, & \text{if } |z| \geq r_2,
\end{cases}
\]

where \( C_1 = 2 \int_{r_1}^{r_2} f(r) r dr \ln \frac{1}{r}, \quad C_2 = 2 \int_{r_1}^{r_2} f(r) r dr \).

We make the ansatz that \( \tilde{\mu} \) has the form (see [50, Theorem II.4.1])

\[
\tilde{\mu}(d^2z) = \sigma_1 \delta_{r_1}(r) dr \frac{d\theta}{2\pi} + \sigma_2 \delta_{r_2}(r) dr \frac{d\theta}{2\pi}, \quad z = re^{i\theta}, \quad r \geq 0, \quad \theta \in (-\pi, \pi],
\]

(A.5)
for some constants $\sigma_1$ and $\sigma_2$. Using again (A.4) we obtain

$$U^\#(z) = \begin{cases} 
\sigma_1 \ln \frac{1}{\rho_1} + \sigma_2 \ln \frac{1}{\rho_2}, & \text{if } |z| \leq \rho_1, \\
(\sigma_1 + \sigma_2) \ln \frac{1}{\rho_1}, & \text{if } |z| \geq \rho_2.
\end{cases}$$

By definition of $\hat{\mu}$, we must have $U^{\mu \chi_G}(z) = U^\#(z)$ for all $z \notin \overline{G}$, so we have the system

$$\begin{cases}
\sigma_1 \ln \frac{1}{\rho_1} + \sigma_2 \ln \frac{1}{\rho_2} = C_1, \\
\sigma_1 + \sigma_2 = C_2.
\end{cases}$$

The solution is

$$\sigma_1 = \frac{C_1 - C_2 \ln \frac{1}{\rho_2}}{\ln \frac{\rho_2}{\rho_1}}, \quad \sigma_2 = \frac{C_2 \ln \frac{1}{\rho_1} - C_1}{\ln \frac{\rho_2}{\rho_1}}. \quad \text{(A.6)}$$

Recalling that $f(r) = b^2 r^{2b-2}$, we get

$$C_1 = 2 \int_{\rho_1}^{\rho_2} rf(r) \ln \frac{1}{r} \, dr = -b \rho_2^b \ln(\rho_2) + b \rho_1^b \ln(\rho_1) + \frac{1}{2}(\rho_2^b - \rho_1^b),$$

$$C_2 = 2 \int_{\rho_1}^{\rho_2} rf(r) \, dr = b(\rho_2^b - \rho_1^b).$$

Substituting the above expressions into (A.6) and recalling (A.1) and (A.5), we arrive at (1.4).

### B Proof of Lemma 1.1

Define $\phi_1$ and $\phi_2$ by

$$\phi_1(x; \vec{t}, \vec{u}) := 1 + T_0(x; \vec{t}, \vec{u}) + \hat{T}_0(b \rho_2^b; \vec{u}) = 1 + \sum_{\ell=1}^{m} \omega_{\ell} e^{-\frac{t_{\ell}}{2b}(x-b \rho_1^b)} + \sum_{\ell=m+1}^{2m} \omega_{\ell},$$

$$\phi_2(x; \vec{t}, \vec{u}) := 1 - \hat{T}_0(x; \vec{t}, \vec{u}) + \hat{T}_0(b \rho_2^b; \vec{u}) = 1 - \sum_{\ell=m+1}^{2m} \omega_{\ell} e^{-\frac{t_{\ell}}{2b}(b \rho_2^b-x)} + \sum_{\ell=m+1}^{2m} \omega_{\ell}.$$

A calculation gives

$$\partial_{u_{2m}} \phi_1 = e^{u_1 + \cdots + u_{2m}} e^{-\frac{t_1}{2b}(x-b \rho_1^b)} + \sum_{\ell=2}^{m} e^{u_1 + \cdots + u_{2m}} (e^{-\frac{t_\ell}{2b}(x-b \rho_1^b)} - e^{-\frac{t_{\ell-1}}{2b}(x-b \rho_1^b)}),$$

$$\partial_{u_{2m}} \phi_2 = e^{u_1 + \cdots + u_{2m}} (1 - e^{-\frac{t_1}{2b}(x-b \rho_1^b)}),$$

$$\partial_{u_{2m}} \phi_2 = e^{u_1 + \cdots + u_{2m}} (1 - e^{-\frac{t_{m+1}}{2b}(b \rho_2^b-x)}) + \sum_{\ell=m+2}^{2m} e^{u_1 + \cdots + u_{2m}} (e^{-\frac{t_\ell}{2b}(b \rho_2^b-x)} - e^{-\frac{t_{\ell-1}}{2b}(b \rho_2^b-x)}).$$

Using that $t_1 > \cdots > t_m \geq 0$, we see that $\partial_{u_{2m}} \phi_1(x; \vec{t}, \vec{u}) > 0$ for any $x \geq b \rho_1^b$, and using that $0 \leq t_{m+1} < \cdots < t_{2m}$, we see that $\partial_{u_{2m}} \phi_2(x; \vec{t}, \vec{u}) \geq 0$ for any $x \leq b \rho_2^b$. In view of the limits

$$\lim_{u_{2m} \to \infty} \phi_1(x; \vec{t}, \vec{u}) = 0, \quad \lim_{u_{2m} \to \infty} \phi_2(x; \vec{t}, \vec{u}) = e^{-\frac{t_{2m}}{2b}(b \rho_2^b-x)} > 0,$$

the desired inequalities follow.
Uniform asymptotics of the incomplete gamma function

Lemma C.1 (From [49, formula 8.11.2]). Let $a > 0$ be fixed. As $z \to +\infty$,

$$\gamma(a, z) = \Gamma(a) + O(e^{-\frac{z}{2}}).$$

Lemma C.2 (From [55, Section 11.2.4]). We have

$$\gamma(a, z) = \Gamma(a) = \frac{1}{2} \text{erfc}(-\eta \sqrt{a/2}) - R_a(\eta), \quad R_a(\eta) = \frac{e^{-\frac{1}{2} \eta^2}}{2 \pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2} a u^2} g(u) du,$$

where $\text{erfc}$ is the complementary error function,

$$\lambda = \frac{z}{a}, \quad \eta = (\lambda - 1) \sqrt{\frac{2(\lambda - 1 - \ln \lambda)}{(\lambda - 1)^2}}, \quad g(u) := \frac{dt}{du} \frac{1}{\lambda - t} + \frac{1}{u + i \eta}. \quad (C.1)$$

The variables $t$ and $u$ are related by the bijection $t \mapsto u$ from $\mathcal{L} := \{ \frac{\theta}{\pi} e^{i \theta} : -\pi < \theta < \pi \}$ to $\mathbb{R}$ given by

$$u = -i(t - 1) \sqrt{\frac{2(t - 1 - \ln t)}{(t - 1)^2}}, \quad t \in \mathcal{L}. \quad (C.2)$$

The principal branch is taken for the roots in (C.1) and (C.2). In addition,

$$R_a(\eta) \sim \frac{e^{-\frac{1}{2} \eta^2}}{\sqrt{2 \pi a}} \sum_{j=0}^{\infty} c_j(\eta) \frac{1}{a^j}, \quad \text{as } a \to +\infty \quad (C.3)$$

uniformly for $z \in [0, \infty)$ where all coefficients $c_j(\eta)$ are bounded functions of $\eta \in \mathbb{R}$ (i.e. bounded for $\lambda \in (0, +\infty)$). The first two coefficients are given by (see [55, p. 312])

$$c_0(\eta) = \frac{1}{\lambda - 1} - \frac{1}{\eta}, \quad c_1(\eta) = \frac{1}{\eta^3} - \frac{1}{(\lambda - 1)^2} - \frac{1}{12(\lambda - 1)}.$$

More generally, we have

$$c_j(\eta) = \frac{1}{\eta} \frac{d}{d\eta} c_{j-1}(\eta) + \frac{\gamma_j}{\lambda - 1}, \quad j \geq 1, \quad (C.4)$$

where the $\gamma_j$ are the Stirling coefficients

$$\gamma_j = \frac{(-1)^j}{2^j j!} \left[ \frac{d^j}{dx^j} \left( \frac{x^2}{2 - \ln(1 + x)} \right) \right]_{x=0}^{j+\frac{1}{2}}.$$

In particular, the following hold:

(i) As $a \to +\infty$, $\gamma(a, \lambda a) = \Gamma(a) \left( 1 + O(e^{-\frac{a \eta^2}{2}}) \right)$ uniformly for $\lambda \geq 1 + \delta$ for each fixed $\delta > 0$.

(ii) As $a \to +\infty$, $\gamma(a, \lambda a) = \Gamma(a) O(e^{-\frac{a \eta^2}{2}})$ uniformly for $\lambda$ in compact subsets of $(0, 1)$.

The following lemma is essentially a result of Tricomi [56]; however, the coefficients appearing in Lemma C.3 below are written in a non-recursive way using [9, Lemma A.4].
Lemma C.3 (From [9, Lemma A.4]). Let $N \geq 0$ be an integer, let $\eta$ be as in (C.1), let $\varphi_j(\lambda) := \frac{(-1)^{j+1}(2j-1)!}{2^{2j+1}j!}$, and let $S(\varphi_j(\lambda))$ denote the singular part of $\varphi_j(\lambda)$ at $\lambda = 1$, i.e., $S(\varphi_j(\lambda))$ is the sum of the singular terms in the Laurent expansion of $\varphi_j(\lambda)$ at $\lambda = 1$. The first $S(\varphi_j(\lambda))$ are given by

$$S(\varphi_0(\lambda)) = -\frac{1}{\lambda-1}, \quad S(\varphi_1(\lambda)) = \frac{1}{(\lambda-1)^3} + \frac{1}{(\lambda-1)^2} + \frac{12}{15} - \frac{1}{12}(\lambda-1),$$

$$S(\varphi_2(\lambda)) = -\frac{3}{(\lambda-1)^5} - \frac{5}{(\lambda-1)^4} + \frac{25}{55} - \frac{1}{12}(\lambda-1)^2 - \frac{1}{288}(\lambda-1).$$

(i) As $a \to +\infty$, uniformly for $\lambda \geq 1 + \frac{1}{\sqrt{a}}$,\n
$$\frac{\gamma(a, \lambda a)}{\Gamma(a)^{N}} = 1 + \frac{e^{-\frac{\pi a^2}{\sqrt{2}}}}{\sqrt{2\pi}} \left\{ \sum_{j=0}^{N-1} S(\varphi_j(\lambda)) \frac{O\left(1\right)}{a^{\frac{N+1}{2}}} + O\left(\frac{1}{\sqrt{\alpha a}}\right)^{N+\frac{1}{2}} \right\}. $$

(ii) As $a \to +\infty$, uniformly for $\lambda \in [\epsilon, 1 + \frac{1}{\sqrt{a}}]$ for any fixed $\epsilon > 0$,\n
$$\frac{\gamma(a, \lambda a)}{\Gamma(a)^{N}} = e^{-\frac{\pi a^2}{\sqrt{2}}} \left\{ \sum_{j=0}^{N-1} S(\varphi_j(\lambda)) \frac{O\left(1\right)}{a^{\frac{N+1}{2}}} + O\left(\frac{1}{\sqrt{\alpha a}}\right)^{N+\frac{1}{2}} \right\}. $$

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