Spin dynamics of the bilinear-biquadratic $S = 1$ Heisenberg model on the triangular lattice: a quantum Monte Carlo study

Annika Völl$^1$ and Stefan Wessel$^1$

$^1$Institut für Theoretische Festkörperphysik, JARA-FIT and JARA-HPC, RWTH Aachen University, 52056 Aachen, Germany

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We study thermodynamic properties as well as the dynamical spin and quadrupolar structure factors of the O(3)-symmetric spin-1 Heisenberg model with bilinear-biquadratic exchange interactions on the triangular lattice. Based on a sign-problem-free quantum Monte Carlo approach, we access both the ferromagnetic and the ferroquadrupolar ordered, spin nematic phase as well as the SU(3)-symmetric point which separates these phases. Signatures of Goldstone soft-modes in the dynamical spin and the quadrupolar structure factors are identified, and the properties of the low-energy excitations are compared to the thermodynamic behavior observed at finite temperatures as well as to Schwinger-boson flavor-wave theory.

I. INTRODUCTION

In recent years, the emergence of unconventional order in magnetic systems has been investigated both experimentally as well as from various theoretical perspectives. In contrast to quantum spin liquids, which are characterized by the absence of any symmetry-breaking long-range order, spin nematic states exhibit long-ranged quadrupolar correlations without conventional (dipolar) magnetic order, and thus also do not feature magnetic Bragg peaks. While time-reversal symmetry remains intact, the spontaneous breaking of the spin rotation symmetry in spin nematic phases leads to the emergence of low-energy Goldstone modes with a linear dispersion relation, providing an algebraic contribution to the low-temperature ($T$) specific heat. Spin nematic phases have indeed been identified in several model systems, such as in spin-1/2 Heisenberg systems with competing ferromagnetic and antiferromagnetic or ring-exchange interactions, as well as in spin-1 Heisenberg models with sizable biquadratic interaction terms in addition to the bilinear spin exchange interaction.

One particular motivation for various recent studies of the triangular lattice spin-1 Heisenberg model with additional biquadratic exchange terms was its initially proposed relevance to the unconventional magnetic properties observed in the Ni-based compound NiGa$_2$S$_4$. In this material, spin-1 carrying Ni$^{2+}$-ions reside on weakly coupled triangular lattice layers. While no signatures of low-temperature dipolar magnetic long-range order was detected, NiGa$_2$S$_4$ exhibits a $T^2$-dependent low-temperature specific heat, akin to linearly dispersing two-dimensional low-energy soft modes. Within the spin-1 bilinear-biquadratic Heisenberg model scenario, two distinct spin nematic states have been identified and were proposed as possible ground states for NiGa$_2$S$_4$. These states differ in the relative orientation of the local director, specifying the axis perpendicular to the plane of dominant residual local spin fluctuations. While the ferroquadrupolar state is characterized by a uniform alignment of the directors, akin to the uni-axial nematic liquid crystal state, the antiferroquadrupolar state exhibits a three-sublattice structure with mutually perpendicular directors on neighboring lattice sites. A variety of experimental techniques, applied to NiGa$_2$S$_4$, however revealed a dynamical freezing with incommensurate short-range order of local magnetic moments that form at low temperatures, and alternative scenarios have been put forward to explain the peculiar properties of this compound (cf. Ref. [12] for an overview). Even in the absence of true long-range quadrupolar order, the emergence of substantial quadrupolar correlations may still be employed to rationalize e.g. the two-peak structure observed in the specific heat of NiGa$_2$S$_4$.

It therefore appears relevant, also from a more general perspective, to provide further theoretical characterizations of quadrupolar order for future probes based, e.g., on inelastic neutron or light scattering experiments, nuclear magnetic resonance techniques, and possibly resonant X-ray scattering. Several studies of the dynamical spin and quadrupolar structure factors in triangular-lattice-based spin nematic phases have indeed been performed, based on approximate schemes to calculate these dynamical quantities, e.g. within Schwinger-boson flavor-wave theory. The quality of such approximate schemes may be assessed by employing alternative methods, applicable within appropriate model parameter regimes. While for an underlying square lattice geometry, the spin-1 bilinear-biquadratic Heisenberg model and its thermodynamic properties have been studied by unbiased, large-scale quantum Monte Carlo (QMC) simulations over a wide range of model parameters, the non-bipartition of the triangular lattice typically leads to a severe QMC sign-problem for Heisenberg-like spin models. Only recently was a sign-problem-free QMC representation of the partition function formulated for the pure biquadratic exchange model.

Here, we expand on the approach of Ref. and cover an extended region of parameter space of the spin-1 bilinear-biquadratic Heisenberg model on the triangular lattice, which covers both the ferroquadrupolar and the ferromagnetically ordered region as well as the SU(3)-symmetric point that separates both phases. Beyond an identification of the ground state order parameters and...
finite temperature thermodynamic properties, we in particu-
lar employ the QMC algorithm to measure the dy-
namical structure factors for both spin correlations as well as quadrupolar correlation functions. This allows us to
to identify features of these phases in the dynamical spin
structure factors related to, e.g., neutron scattering ex-
periments, as well as to quantitatively relate these spec-
tral properties to the thermodynamic behavior. We fur-
thermore compare our numerical findings to the results
from flavor-wave theory which in addition provides us
with exact results at the SU(3) point.

The remainder of this article is organized as follows:
In Sec. II, we present details on the considered model
and the employed numerical method. Then, in Sec. III,
we present our QMC results on the order parameters and
the employed numerical method. Then, in Sec. III,
and analyzed in terms of the low-energy modes and the corresponding thermo-
dynamic properties, with some final conclusions given in
Sec. V.

II. MODEL AND METHOD

We consider in the following the spin-1 bilinear-
biquadratic Heisenberg model on the triangular lattice,
described in terms of localized spin-1 degrees of freedom
\( S_i \) on each lattice site \( i \) by the Hamiltonian

\[
H = J \sum_{\langle i,j \rangle} \left[ \cos \theta S_i \cdot S_j + \sin \theta (S_i \cdot S_j)^2 \right],
\]

(1)

parametrized by the angular parameter \( \theta \), following con-
ventional notations. In the following, we set \( J = 1 \), and
concentrate on the parameter regime \( \theta \in [-\pi, -\pi/2] \),
which is accessible by sign-problem-free QMC simula-
tions, as detailed below. This parameter range includes
the SU(3)-symmetric point, \( \theta = \theta_{SU(3)} = -3\pi/4 \), that
separates (within the considered parameter range) a fer-
romagnetic phase for \( \theta \in [-\pi, \theta_{SU(3)}] \) from a spin ne-
matic phase with pure ferroquadrupolar order for \( \theta \in
(\theta_{SU(3)}, \pi/2) \) (see e.g. Ref. [9] for a detailed study of the
full quantum phase diagram of the Hamiltonian \( H \) and
the full extent of both phases).

Our QMC algorithm is based on the directed loop
stochastic series expansion approach. In order to ob-
tain a sign-problem-free representation of the quantum
partition function \( Z = \text{Tr} \exp(-\beta H) \), with the inverse
temperature \( \beta = 1/T \), we require all matrix elements of
\( (-H) \) to be non-negative. In the parameter region of the
ferromagnetic phase, \( \theta \in [-\pi, \theta_{SU(3)}] \), this is feasible in
the standard local-\( S^z \)-basis with eigenstates \( |1\rangle, |0\rangle, |1\rangle \)
of \( S^z \) for the eigenvalues 1, 0, -1, respectively. In this
parameter region, all off-diagonal matrix elements of \( H \)
are non-positive, and we merely require to subtract a con-
stant \( C = -\cos \theta + 2 \sin \theta \) from \( H \) in order to en-
sure the shifted \( -H' = -H + C \) to be non-negative. For
\( \theta \in (\theta_{SU(3)}, \pi/2) \), the Hamiltonian \( H \) exhibits positive
off-diagonal matrix elements. In order to avoid the sign-
problem in this parameter region on the non-bipartite
triangular lattice, we employ the idea of Ref. [23] and
consider the phase-rotated basis state \( |0'\rangle = i|0\rangle \). Fur-
thermore, we subtract a constant \( C \geq \sin \theta \) to ensure the shifted Hamiltonian \( H' \) to be non-positive in the rotated
basis (this basis rotation requires to be accounted for
when measuring observables). It is worth noticing that
for \( \theta = -\pi/2 \) the matrix elements of the shifted Hamil-
tonian \( H' \) are either 0 or -1 in the rotated basis. In fact,
the two-site bond Hamiltonian for \( \theta = -\pi/2 \) equals a pro-
jector,

\[
-H'|_{\theta=-\pi/2} = (|0'0\rangle + |1\bar{1}\rangle) (|0'0\rangle + |1\bar{1}\rangle)
\]

and the directed loop equations allow for a QMC sam-
ing in terms of a three-color closed-loop representa-
tion.

In our simulations, we considered finite systems with
\( N = L^2 \) lattice sites and periodic boundary condi-
tions in both lattice directions \( a_1 = (a,0)^T \), and \( a_2 = (a/2, \sqrt{3}a/2)^T \), where the lattice constant is set to \( a = 1 \)
in the following. In order to access ground state properti-
es, \( \beta \) must be chosen sufficiently large. We typically
require \( \beta \geq L \) for this purpose. Besides the energy
\( E = \langle H \rangle \) and the specific heat \( C_V = \langle dE/dT \rangle/N \), we
measured the uniform susceptibility,

\[
\chi_{\alpha} = \frac{\beta}{N} \left( \sum_{i=0}^{N} S_i^\alpha \right)^2,
\]

(2)
as well as the (equal-time) spin structure factor

\[
S_S(q) = \frac{1}{3N} \sum_{i,j} e^{-iq\cdot(r_i-r_j)} \langle S_i \cdot S_j \rangle,
\]

(3)
with \( r_i \) denoting the position of the \( i \)-th lattice site. A
prefactor \( 1/3 \) was inserted in Eq. (3) to comply with the
notation of Ref. [23]. Within the QMC simulations, we
employ the O(3) invariance of \( H \), to access \( S_S(q) \) through
the diagonal operator components \( S_i^z \) as

\[
S_S(q) = \frac{1}{N} \sum_{i,j} e^{-iq\cdot(r_i-r_j)} \langle S_i^z S_j^z \rangle
\]

(4)
The spin structure factor allows to detect the presence of (dipolar)
magnetic order, indicated in the ferromag-
netic phase by an extensive scaling of \( S_S(q) \) at the fer-
romagnetic Bragg peak position \( q = 0 \). In order to ac-
cess quadrupolar order, we also consider the correlation
function \( \langle \sum_{\alpha,\beta} Q_\alpha^\beta Q_\alpha^{-\beta} \rangle = \langle \text{Tr} (Q_i Q_j) \rangle \) of the magnetic
quadrupolar moments, given by the traceless, symmetric
\( 3 \times 3 \) matrix operator \( Q_i \) with elements

\[
Q_i^{\alpha\beta} = \frac{1}{2} \left( S_i^\alpha S_i^\beta + S_i^\beta S_i^\alpha \right) - \frac{2}{3} \delta_{\alpha\beta},
\]

(5)
with \( \delta_{\alpha\beta} \) the Kronecker delta, and \( \alpha,\beta \in \{x, y, z\} \). Due
to the explicit O(3) symmetry of the Hamiltonian \( H \),
we can probe for the O(3)-symmetric quadrupolar correlations upon measuring the trace of the quadrupolar structure factor

$$S_Q(q) = \frac{2}{5N} \sum_{i,j} e^{-i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle \text{Tr}(Q_i Q_j) \rangle$$

(6)
in the QMC simulations through the (in the computational basis) diagonal operator components $Q_{zz}^{ij}$, as

$$S_Q(q) = \frac{3}{N} \sum_{i,j} e^{-i\mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \langle Q_{zz}^{ij} Q_{zz}^{jj} \rangle,$$

(7) which allows for an efficient implementation. The prefactor $2/5$ in the definition of $S_Q(q)$ in Eq. (6) was chosen in order to comply with the notation of Ref. 23. The quadrupolar structure factor allows to detect the presence of quadrupolar order, which in the ferroquadrupolar case is signaled by an extensive scaling of $S_Q(q)$ at $q = 0$. Within the ferromagnetic phase, $S_Q(q)$ also exhibits extensive scaling at $q = 0$, with quadrupolar long-ranged correlations induced by the ferromagnetic ordered ground state, whereas $S_S(q)$ does not exhibit Bragg peaks in the ferroquadrupolar phase.

In order to access information on the excitations in these different regimes, we also extracted from the QMC simulations the dynamical spin and quadrupolar structure factors, which are defined through

$$S_S(\omega, q) = \frac{1}{N} \int dt \sum_{i,j} e^{i(\omega t - \mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j))} \langle S_i^z(t) S_j^z(0) \rangle,$$

(8) and

$$S_Q(\omega, q) = \frac{3}{N} \int dt \sum_{i,j} e^{i(\omega t - \mathbf{q} \cdot (\mathbf{r}_i - \mathbf{r}_j))} \langle Q_i^{zz}(t) Q_j^{zz}(0) \rangle,$$

(9) respectively. Within the QMC simulations, we can efficiently[23] measure the imaginary-time displaced correlation functions directly in Matsubara frequency representation, related via

$$S_X(i\omega_n, q) = \int_0^\infty d\omega \frac{\omega}{\pi} \frac{(1 - e^{-\beta\omega})}{\omega^2 + \omega_n^2} S_X(\omega, q),$$

(10) to the dynamical structure factors for $X = S$ and $Q$, respectively. Here, $\omega_n = 2\pi n/\beta$ for $n = 0, 1, 2, \ldots$ denote the Matsubara frequencies, where typically we require values of $n$ up to 160 to access the leading $1/\omega_n^2$ asymptotic behavior of the $S_X(i\omega_n, q)$. The numerical inversion of Eq. (10) to obtain $S_X(\omega, q)$ from the Matsubara frequency QMC data $S_X(i\omega_n, q)$ was performed using the stochastic analytic continuation method[23].

### III. THERMODYNAMIC PROPERTIES

Before considering these dynamical properties, we establish first the nature of the ground state of the bilinear-biquadratic spin-1 Heisenberg model, Eq. (1) on the triangular lattice within the considered parameter region $\theta \in [-\pi, -\pi/2]$. For this purpose, we show in Fig. 1 the values of the spin and quadrupolar structure factors at $q = 0$, obtained from linear extrapolations in $1/L$ of the finite system values to the thermodynamic limit. For several values of $\theta$, the finite system data and the extrapolations are shown in Fig. 2. For $\theta < \theta_{SU(3)}$, a fully polarized ferromagnetic state is exhibited by a saturated value $S_S(q = 0)/N = 1/3$ of the spin structure factor, which leads in effect also to a finite value for the ferroquadrupolar order parameter, indicated by the finite value of $S_Q(q = 0)/N = 1/15$ in this parameters range. At the $SU(3)$-symmetric point, $\theta = \theta_{SU(3)}$, both structure factors become degenerate, with $S_S(q = 0)/N = \ldots$
In addition to the Shottky-like peak at $T = O(J)$, the specific heat exhibits for the ferroquadrupolar region a super-linear suppression at low temperatures, while $C_V$ vanishes linearly with temperature within the ferromagnetic region. In fact, from the dispersions of the low-energy Goldstone modes, considered in more detail in Sec. IV, we expect the low-$T$ specific heat to exhibit a linear (quadratic) asymptotic low-$T$ scaling within the ferromagnetic (ferroquadrupolar) regime, which can be observed more directly from the appropriately rescaled low-$T$ specific heat data shown in Fig. 4. In all considered cases, do the rescaled quantities approach to constant low-$T$ values. Anticipating the further discussion in Sec. IV, we consider first the pure biquadratic model, $\theta = -\pi/2$, in more detail. For a single, linearly dispersing low-energy mode, with $\omega(q) = c|q|$, we obtain a leading contribution to the low-$T$ specific heat $C_V = 3 \zeta(3)/\pi T^2/c^2$, where $\zeta(3) \approx 1.202$. Employing as an estimate for the velocity a value of $c \approx 4.8$ (cf. the discussion in Sec. IV), we obtain a value of $C_V/T^2 \approx 0.05$. The data in Fig. 4 is then in accord with the presence of two independent such linear soft-mode contributions. Similarly, for $\theta = -0.875\pi$, we obtain $C_V/T^2 \approx 0.2$ for two linear soft-modes with $c \approx 3.4$, again in accord with the specific heat data in Fig. 4. For $\theta \leq \theta_{SU(3)}$, the system exhibits quadratic low-energy soft modes (cf. the discussion in Sec. IV). A single quadratically dispersing low-energy mode with $\omega(q) = b|q|^2$ provides a contribution $C_V = \pi/12 \times T/\theta b$ to the low-$T$ specific heat. From the exact result of the low-energy dispersion at the SU(3) point, with $b \approx 3\sqrt{3} \approx 5.196$ (cf. the discussion in Sec. IV), we obtain a low-$T$ contribution of $C_V/T \approx 0.24$, and the data in Fig. 4 exhibits the presence of two such quadratic soft-modes in the excitation spectrum at the SU(3) point. For $\theta = -0.875\pi$, within the ferromagnetic region, we estimate from the data discussed in Sec. IV a value of $b \approx 1.4$, which leads to a low-$T$ contribution of $C_V/T \approx 0.19$. The data in
that our QMC data at the SU(3)-symmetric point fall perfectly onto the flavor-wave theory result, which provides a useful quality check of our numerical procedure. Moving away from the SU(3) point into the quadrupolar phase, flavor-wave theory provides only approximate dispersions. Nevertheless, we find that flavor-wave theory still accounts for the dispersion relation from QMC rather accurately, with a largest deviation in excitation energy of about 15% for the pure biquadratic model.

Within the quadrupolar phase, the integrated spectral weight $I_S(q) = \int S_S(\omega, q) \, d\omega/(2\pi)$ vanishes linearly with $|q|$ near the $\Gamma$-point, in accord with the flavor-wave theory result\cite{23}, as seen in Fig. 6 where besides $I_S(q)$, we also show the corresponding quadrupolar quantity $I_Q(q) = \int S_Q(\omega, q) \, d\omega/(2\pi)$ for several values of $\theta$. Here, we employ the fact that $I_X(q) = S_X(q)$ is conveniently obtained from the QMC equal-time correlations for both $X = S$ and $Q$. On the other hand, $I_Q(q)$ is seen to grow upon approaching the quadrupolar Bragg-peak position $q = \Gamma = 0$, where it diverges (cf. Sec. III). The dispersing mode in $S_Q(\omega, q)$ thus provides direct access to the Goldstone mode in the excitation spectrum from the O(3) symmetry breaking in the quadrupolar phase. This low-energy quadrupolar wave exhibits a linear dispersion, and we estimate the corresponding velocity for the pure biquadratic model ($\theta = -\pi/2$) to be $c \approx 4.8$ from our data shown in Fig. 5, a value that falls slightly below the value of $c = 3 \times 1.869(4) \approx 5.6$ estimated in Ref. 23 (after accounting for the factor 1/3 in the value of $J$ actually employed in Ref. 23), while flavor-wave theory gives a slightly smaller value of $c = \sqrt{18} \approx 4.2$ for $\theta = -\pi/2$. The deviation from Ref. 23 is due to finite-size effects that restrict us from extracting the asymptotic value of $c$ from the observed dispersion near the $\Gamma$-point in Fig. 5, while the estimate in Ref. 23 was obtained from a finite-size analysis of winding number-based estimators.

Fig. 5 is then in accord with the presence of a single such quadratic soft-mode, reflecting the ferromagnetic ground state for this values of $\theta$.

IV. DYNAMICAL STRUCTURE FACTORS

In light of the results in the previous section, we next present our results for the dynamical structure factors. For this purpose, we collect in Fig. 5 both $S_S(\omega, q)$ and $S_Q(\omega, q)$ along a standard symmetry path through the triangular lattice Brillouin zone for different values of $\theta$. We included in Fig. 5 in addition the results for the quadrupolar wave dispersions obtained from the Schwinger-boson flavor-wave theory from Ref. 9. Within this approach, one finds two degenerate branches of quadrupolar waves, with dispersion relation $\omega(q) = \sqrt{A(q) - B(q)}$, where $A(q) = 6J(\cos \theta \gamma(q) - \sin \theta)$, $B(q) = 6J(\sin \theta - \cos \theta)\gamma(q)$ and $\gamma(q) = \sum_j \exp(i\delta_j \cdot q)/6$, with $\delta_j$, $j = 1, \ldots, 6$ the six nearest neighbor vectors on the triangular lattice. Here, a factor of 2 missing in the formula for $A(q)$ and $B(q)$ in Ref. 9 has been corrected for\cite{23}. From Fig. 5 we find  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{(Color online) Dynamical spin and quadrupolar structure factors $S_S(\omega, q)$ and $S_Q(\omega, q)$ for different values of $\theta$ along the path $\Gamma \to K \to M \to \Gamma$ through the Brillouin zone ($\Gamma = (0, 0)^T$, $K = (4\pi/3, 0)^T$, $M = (2\pi/3, \pi/\sqrt{3})^T$). The dashed lines indicate the quadrupolar wave dispersion relations obtained within flavor-wave theory.}
\end{figure}
Within flavor-wave theory as well as based on general Goldstone-mode counting considerations, we conclude for the presence of two such linear Goldstone modes, and indeed, this count is in accord with the quantitative behavior of the low-$T$ specific heat, as discussed in Sec. III. In Fig. 7, we show the dynamical quadrupolar structure factor $S_Q(\omega, \mathbf{q})$ at $\theta = -0.5\pi$ for different temperatures. While the spectral weight within the higher energy range of the quadrupolar wave dispersion relation ($\omega \approx 5 - 10$) dissolves beyond $T \approx 0.1J$, one still finds indications for low-energy soft-modes at temperatures $T \approx J$, i.e. of order the exchange constant. This quantity thus provides a robust probe for quadrupolar correlations in the system also at elevated temperature scales.

Within the ferromagnetic regime, the low-energy Goldstone mode from the $O(3)$ symmetry breaking is directly exhibited through the quadratically-dispersing soft-mode detected by $S_S(\omega, \mathbf{q})$, with a flat ($\mathbf{q}$-independent) integrated spectral weight $I_S(\mathbf{q})$, also shown in Fig. 6. Such a quadratic magnon dispersion relation is again in accord with general considerations, as well as the $T$-linear low-temperature specific heat observed in the ferromagnetic regime (cf. Sec. III).

In addition to the low-energy modes, that we discussed thus far, we also observe in $S_Q(\omega, \mathbf{q})$ additional, higher energy scattering weight, which is resolved most clearly when sufficiently separated from the low-energy mode, such as for $\theta < \theta_{SU(3)}$, within the ferromagnetic region. Additional spectral weight is also found at other values of $\theta$ than those shown in Fig. 5, cf. the data for $S_Q(\omega, \mathbf{q})$ shown in Fig. 8. Within the flavor-wave theory calculation of $S_Q(\omega, \mathbf{q})$ for the quadrupolar regime, a continuum of scattering states from two-magnon contributions was obtained. In contrast to the QMC spectral functions, which exhibit a dominant spectral weight at low energies, the results in Ref. [19] however exhibit also regions with a maximum in the spectral weight shifted towards the center of the continuum. On the other hand, our QMC spectral functions may be compared to the results in Ref. [15]. There, the model in Eq. (1) was considered within the antiferroquadrupolar regime, and calculations of the dynamical quadrupolar structure factor were performed both within quantum non-linear sigma-model as well as flavor-wave theory. In addition to a dominant low-energy mode, a narrow, high-energy mode was observed in the dynamical quadrupolar structure factor. Given the limited spectral resolution of the QMC spectral functions at elevated energies, our data would be consistent with a similar scenario also for the ferroquadrupolar parameter regime, accessed by our simulations.

V. CONCLUSIONS

Employing a sign-problem free presentation of the quantum partition function, we performed quantum Monte Carlo simulations of the spin-1 bilinear-biquadratic Heisenberg model on the triangular lattice within a restricted parameter regime, which includes both ferromagnetic as well as ferroquadrupolar ordered regions. In addition, we examined the SU(3)-symmetric point that separates these regions. We identified the corresponding order parameters and extracted the leading low-temperature algebraic scaling of the specific heat, providing direct probes of to the Goldstone soft-modes. From quantum Monte Carlo calculations, combined with numerical analytic continuation, we obtained the dynamical spin and quadrupolar structure factors. These provided us with direct access to the dispersion relations of the Goldstone modes. Our numerical results are found to be in excellent agreement with the exact flavor-wave-theory results at the SU(3)-symmetric point. Furthermore, we observed only weak quantitative deviations in the dispersion relations obtained within the ferroquadrupolar phase, such that flavor-wave-theory is seen to provide an reasonably accurate account of the low-energy modes. The dynamical quadrupolar structure factor was found to provide a useful probe for the ferroquadrupolar state also at elevated temperatures. In addition, we observed further spectral weight in the dynamical quadrupolar structure factor, which is shifted towards high energies within the ferromagnetic regime. It will be interesting to study this distinct contribution to the spectral function by means of other approaches, which can access more accurately the high energy scattering weight. For the future, it will also be interesting to study possible continuous quantum phase transitions between spin nematic phases and paramagnetic or spin liquid regions in systems with competing interactions on non-bipartite lattices.
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