MANIFOLDS WITH NONNEGATIVE ISOTROPIC CURVATURE

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Abstract. We prove that if \((M, g)\) is a compact orientable locally irreducible Riemannian manifold with nonnegative isotropic curvature, then one of the following possibilities hold:

(i) \(M\) admits a metric with positive isotropic curvature
(ii) \((M, g)\) is isometric to a locally symmetric space
(iii) \((M, g)\) is Kähler and biholomorphic to \(\mathbb{C}P^n\).

This is implied by the following two results:

(i) Let \((M, g)\) be a compact, locally irreducible Kähler manifold with no nonnegative isotropic curvature. Then either \(M\) is biholomorphic to \(\mathbb{C}P^n\) or isometric to a compact irreducible Hermitian symmetric space. This answers a question of Micallef and Wang in the affirmative.

(ii) Let \((M, g)\) be a compact quaternionic-Kähler manifold with nonnegative isotropic curvature. Then \((M, g)\) is a symmetric space.

The proof is based on the recent work of S. Brendle and R. Schoen on the Ricci flow.

1. Introduction

A Riemannian manifold \((M, g)\) is said to have nonnegative isotropic curvature, which we denote by \(K_{iso} \geq 0\), if

\[
R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0
\]

for every orthonormal 4-frame \(\{e_1, e_2, e_3, e_4\}\).

This notion was introduced by M. Micallef and J. Moore in \([8]\) where they proved that every compact simply-connected manifold with \(K_{iso} > 0\) is homeomorphic to a sphere. In \([9]\), M. Micallef and M. Wang make a detailed study of manifolds with \(K_{iso} \geq 0\). In this paper, we show that the study of compact manifolds with \(K_{iso} \geq 0\) reduces to the study of manifolds with \(K_{iso} > 0\). Our main result is the following theorem:

**Theorem 1.1.** Let \((M, g)\) be a compact orientable locally irreducible Riemannian manifold with nonnegative isotropic curvature, then one of the following holds:

(i) \(M\) admits a metric with positive isotropic curvature
(ii) \((M, g)\) is locally symmetric
(iii) \(M\) is Kähler and biholomorphic to \(\mathbb{C}P^n\).

This result is based on the classification of Kähler and quaternionic-Kähler manifolds with \(K_{iso} \geq 0\). In the Kähler case we prove the following result which was conjectured by Micallef and Wang in the aforementioned paper.

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1.2. Let \((M, g)\) be a compact locally irreducible Kähler manifold with nonnegative isotropic curvature. Then either \(M\) is biholomorphic to \(\mathbb{C}P^n\) or isometric to a compact irreducible Hermitian symmetric space.

In the quaternionic-Kähler case we prove

1.3. Let \((M, g)\) be a compact quaternionic-Kähler manifold with nonnegative isotropic curvature. Then \((M, g)\) is a symmetric space.

Theorem 1.2 along with the uniqueness (up to scaling) of the Kähler-Einstein metric on \(\mathbb{C}P^n\) implies the following:

1.4. Let \((M, g)\) be a compact Kähler-Einstein manifold with nonnegative isotropic curvature. Then \((M, g)\) is isometric to \((\mathbb{C}P^n, g_0)\) where \(g_0\) is a scalar multiple of the Fubini-Study metric.

Remarks:

(i) If the dimension of \(M\) is odd (which is precisely the case not treated in [9]), then either \(M\) should admit a metric with \(K_{iso} > 0\) or \((M, g)\) should be locally symmetric.

(ii) In Theorem 1.2 the three cases are not mutually exclusive but the following remark, the proof of which is in Section 4, is true:

If \((M, g)\) is a locally symmetric space of nonconstant sectional curvature or if \((M, g)\) is Kähler then \(M\) does not admit a metric of positive isotropic curvature.

(iii) The LeBrun-Salamon conjecture [7] states that a compact quaternionic-Kähler manifold with positive scalar curvature must be locally symmetric. Our result confirms this under the much stronger assumption of nonnegative isotropic curvature. Very recently, R. Kobayashi and K. Onda have announced a proof [6] of the full LeBrun-Salamon conjecture.

(iv) In light of Corollary 1.4 one can raise the question whether every compact Einstein manifold with \(K_{iso} \geq 0\) is locally symmetric. This is known to be true in dimension 4 [10].

(v) The following result is an easy corollary of the Brendle-Schoen theorems mentioned below. The proof of this remark is at the end of this paper.

Let \((M, g)\) be a compact Riemannian manifold with \(K_{iso} \geq 0\) everywhere and \(K_{iso} > 0\) at some point. Then \(M\) admits a metric with \(K_{iso} > 0\) everywhere.

The proofs of Theorems 1.1, 1.2, and 1.3 are based on the fundamental papers [3, 4] of S. Brendle and R. Schoen. According to their work, if \(g(t)\) is the solution to Ricci flow beginning at a metric \(g\) with \(K_{iso} \geq 0\), then \(g(t)\) has \(K_{iso} \geq 0\) for all \(t\). Moreover, if \(F(t)\) denotes the set of orthonormal 4-frames on which \(K_{iso} = 0\), then, for \(t > 0\), \(F(t)\) is invariant under parallel translation by the Levi-Civita connection of \(g(t)\). From this one quickly sees that \(g(t)\) either has \(K_{iso} > 0\) or \(g(t)\) has holonomy in \(U(m)\) or \(Sp(m)Sp(1)\), i.e., \((M, g(t))\) is Kähler or quaternionic-Kähler. One then studies the latter two cases.

Suppose now that \((M, g(t))\) is Kähler. If \(J\) is the almost complex structure, then one knows that \(F\) contains all orthonormal frames of the form \(\{e, Je, f, Jf\}\), where \(e, f\) are real tangent vectors. If these are the only elements of \(F\), then a version of
Frankel’s Conjecture due to W. Seaman implies that $M$ is biholomorphic to $\mathbb{C}P^n$. On the other hand, if $F$ has other elements, then we will show that the holonomy is a proper subgroup of $U(n)$. Then Berger’s theorem will imply that $(M, g(t))$ is symmetric for all small $t$. Hence so is $(M, g(\varepsilon))$.

If $(M, g(t))$ is quaternionic-Kähler, then one again considers two cases depending on the kind of elements in $F$. In the first case one shows that the second homotopy group $\pi_2(M) = \{0\}$. By a theorem of Salamon this will imply that $M$ is isometric to quaternionic projective space. In the second case one again shows that the holonomy group is a proper subgroup of $Sp(n)Sp(1)$ and hence $(M, g)$ is a symmetric space of rank $\geq 2$.

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2. Kähler manifolds with nonnegative isotropic curvature

This section is devoted to the proof of Theorem 1.2. Let $(M, h)$ be a compact Kähler manifold with $K_{iso} \geq 0$. Let $g(t), t \in [0, \varepsilon)$ be the solution to the Ricci flow equation with $g(0) = h$. For each $t > 0$, let

$$F_t = \{(e_1, e_2, e_3, e_4) | R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} = 0\},$$

where $F_t$ consists of all $g_t$-orthonormal 4-frames $(e_1, e_2, e_3, e_4)$ at all points of $M$ with the above property and $R$ denotes the curvature tensor of $g(t)$. It follows from the work of Brendle and Schoen that, for $t \in (0, \varepsilon)$,

(i) $g_t$ has $K_{iso} \geq 0$ (Section 2 in [3]).
(ii) $F_t$ is invariant under parallel transport by the Levi-Civita connection of $g_t$ (Proposition 5 in [4]).

We fix a $t > 0$ and denote $g_t$ by $g$ and $F_t$ by $F$. Note that by choosing $t$ sufficiently small we can assume that $g(t)$ is locally irreducible. Otherwise, $g = \lim_{t \to 0} g(t)$ would be locally reducible.

Moreover, since $(M, h)$ is Kähler, so is $(M, g)$. It can be easily checked that for a Kähler metric, 4-frames of the form $(v, J(v), w, J(w)) \in F$, for any $v, w \in T_pM$ satisfying $g(v, w) = g(v, Jw) = 0$.

The analysis of the Ricci flow splits into two cases, depending on the set $F$.

Case I: $F$ does not contain any frame of the form $(u, J(u), J(v), v)$.

For any $p \in M$, let $u, v \in T_pM$ with $g(u, v) = g(u, J(v)) = 0$. As mentioned above, by the symmetries of the curvature tensor of a Kähler manifold, we have $(u, J(u), v, J(v)) \in F$, i.e.,

\begin{align}
0 &= R(u, v, u, v) + R(u, J(v), u, J(v)) + R(J(u), v, J(u), v) \\
&\quad + R(J(u), J(v), J(u), J(v)) - 2R(u, J(u), v, J(v)) \\
&= 2 \left( R(u, v, u, v) + R(u, J(v), u, J(v)) - R(u, J(u), v, J(v)). \right)
\end{align}
Since \((u, J(u), J(v), v) \notin F\), we have
\[
0 < R(u, J(v), u, J(v)) + R(u, v, u, v) + R(J(u), J(v), J(u), J(v)) + R(J(u), v, J(u), v) - 2R(u, J(u), J(v), v).
\]
\[
= 2\left( R(u, v, u, v) + R(u, J(v), u, J(v)) + R(J(u), v, J(u), v) \right) \tag{2.2}
\]
\[
+ R(J(u), v, J(u), v) - 2R(u, J(u), J(v), v).
\]
Combining (2.1) and (2.2), we get
\[
(**) \quad R(u, v, u, v) + R(u, J(v), u, J(v)) > 0
\]
for every \(p \in M\) and \(u, v \in T_pM\) with \(g(u, v) = g(u, J(v)) = 0\). This condition, sometimes referred to as \textit{orthogonal bisectional curvature} in the literature, is precisely the condition (**) on Page 846 of [12].

We now state a version of the Frankel Conjecture, following [12] and [9]:

Suppose \((M^{2n}, g)\) is a compact locally irreducible Kähler manifold with nonnegative isotropic curvature and satisfying (**). Then \(M\) is biholomorphic to \(\mathbb{C}P^n\).

\[\textbf{Proof:}\] The proof is the same as in the paper of W. Seaman, [12], except for the following changes:

1. Since \(M\) is Kähler, \(b_2(M) \neq 0\). Hence we can apply a result of Micallef-Wang, Theorem 2.1 (b) of [9], to conclude that \(M\) is simply-connected and \(b_2(M) = 1\).

2. W. Seaman’s version of the Frankel Conjecture, Theorem B of [12], asserts the following:

\[\text{Suppose } (M^{2n}, g) \text{ is a compact } \text{Kähler manifold with nonnegative isotropic curvature and satisfying } (*) \quad \text{where } (*) \quad \text{is the condition that}
\]
\[R_{ikik} + R_{iil} + R_{jikj} + R_{jijl} > 0
\]
for all orthonormal vectors \(e_i, e_j, e_k, e_l\). A reading of the proof of this result shows that (*) is used only to guarantee that \(b_2(M) = 1\). The Micallef-Wang theorem replaces the condition (*) with the condition of local irreducibility to get \(b_2(M) = 1\).

In the rest of the proof of Seaman’s theorem above, only (**) is used: First, this is used to establish the holomorphicity of certain harmonic maps (Lemma A on Page 847 of [12]. We use a variant of this argument in Lemma 3.5). Second, this is used in Page 853 of [12] to establish the positivity of certain line bundles.

\[\square\]

Hence we conclude that \(M\) is biholomorphic to \(\mathbb{C}P^n\) under the assumptions of Case I.

\textbf{Case II:} \(F\) contains an orthonormal frame of the form \((u, J(u), J(v), v)\).

We claim that the holonomy group of \((M, g)\) cannot be the whole group \(U(n)\), where \(n = \dim_{\mathbb{C}} M\). This will complete the proof of Theorem 1.2 by Berger’s Holonomy Theorem.

Suppose \(\text{Hol}(M) = U(n)\). First, complete \(\{u = e_1, v = e_2\}\) to an orthonormal basis \(\{u, J(u), v, J(v), e_2, Je_2, ..., e_n, Je_n\}\) of \(T_pM\).
The fact that \((u, J(u), v, J(v))\) and \((u, J(u), J(v), v)\) both belong to \(F\) gives
\[
R(u, J(u), v, J(v)) = R(u, J(v), u, J(v)) + R(u, v, u, v) + R(J(u), J(v), J(u), J(v)) + R(J(u), v, J(u), v) = 0
\]
which, by the symmetries of Kähler curvature, gives
\[
R(u, J(u), v, J(v)) = R(u, J(v), u, J(v)) + R(u, v, u, v) = 0 \tag{2.3}
\]
We now use the holonomy action of \(U(n)\) repeatedly to get a contradiction. First, there is an element of \(U(n)\) under which
\[
u \to \frac{1}{\sqrt{2}}(u - J(v)), \quad v \to \frac{1}{\sqrt{2}}(u + J(v)). \tag{2.4}
\]
The equation corresponding to \(\tag{2.3}\) is
\[
R\left(\frac{1}{\sqrt{2}}(u - J(v)), \frac{1}{\sqrt{2}}(u + J(v)), \frac{1}{\sqrt{2}}(u + J(v)), \frac{1}{\sqrt{2}}(J(u) - v)\right) = 0.
\]
This equation gives
\[
R(u - J(v), J(u) + v, u + J(v), J(u) - v) = R(u, J(u), u, J(u)) + R(u, J(u), u, -v) + R(u, J(u), J(v), J(u)) + R(u, v, u, J(u)) + R(u, v, J(v), -v) + R(-J(v), J(u), u, J(u)) + R(-J(v), J(u), v, J(v)) + R(-J(v), v, u, J(u)) + R(-J(v), v, -v) = 0 \tag{2.5}
\]
Next, consider the element of \(U(n)\) which takes
\[
u \to \frac{1}{\sqrt{2}}(u - v), \quad v \to \frac{1}{\sqrt{2}}(u + v). \tag{2.6}
\]
Now the simplification of the equation corresponding to \(\tag{2.3}\) gives
\[
R(u, J(u), u, J(u)) + R(v, J(v), v, J(v)) - 4R(u, v, u, v) = 0. \tag{2.7}
\]
Combining \(\tag{2.7} \), \(\tag{2.5}\) and \(\tag{2.3}\) gives
\[
R(u, v, u, v) = R(u, J(v), u, J(v)) = R(u, J(u), u, J(u)) + R(v, J(v), v, J(v)) = 0. \tag{2.8}
\]
Extend \(\{e_1 = u, e_2 = v\}\) to an orthonormal basis \(\{e_1, J(e_1), e_2, J(e_2), \ldots, e_n, J(e_n)\}\). By considering the element of \(U(n)\) which interchanges \(u\) and \(e_i\), \(J(u)\) and \(J(e_i)\) and keeps the other elements of the basis fixed, we see that \((e_i, J(e_i), J(v), v) \in F\). A similar operation on \(v\) shows that \((e_i, J(e_i), e_j, J(e_j)) \in F\) for all \(1 \leq i \leq n\), \(1 \leq j \leq n\), \(i \neq j\). The equations corresponding to \(\tag{2.8}\) are
\[
R(e_i, e_j, e_j, e_j) = R(e_i, J(e_j), e_i, J(e_j)) = R(e_i, J(e_i), e_i, J(e_i)) + R(e_j, J(e_j), e_j, J(e_j)) = 0. \tag{2.9}
\]
The equations $R(e_i, J(e_i), e_i, J(e_i)) + R(e_j, J(e_j), e_j, J(e_j)) = 0$ for all $i \neq j$ clearly imply that $R(e_i, J(e_i), e_i, J(e_i)) = 0$ for all $i$. Combining this with (2.9) gives

$$R = \sum_{i,j=1}^{n} R(e_i, e_j, e_i, e_j) + \sum_{i,j=1}^{n} R(e_i, J(e_j), e_i, J(e_j)) = 0,$$

where $R$ is the scalar curvature. This is impossible since a scalar flat metric of nonnegative isotropic curvature has to be conformally flat. Since $g$ is Kähler, this in turn would imply that $(M, g)$ is flat and hence locally reducible.

This completes the proof of Theorem 1.2. □

3. QUATERNIONIC-KÄHLER MANIFOLDS WITH NONNEGATIVE ISOTROPIC CURVATURE

We first outline the basics of quaternionic-Kähler geometry, following the exposition in [5]. Let $\mathbb{H}$ denote the quaternions and identify $\mathbb{R}^{4n} = \mathbb{H}^n$ via

$$(v_1, v_2, v_3, v_4) \to v_1 + iv_2 + jv_3 + kv_4.$$  

On this space, $\mathbb{H}$ acts on the right and this defines an antihomomorphism

$$\lambda: \{\text{unit quaternions}\} \to SO(4n),$$

where $SO(4n)$ acts on the left. Denote the image by $Sp(1)$ and define

$$(3.1) \quad Sp(n) = \{A \in SO(4n) \mid AB = BA, \ B \in Sp(1)\},$$

and $Sp(n)Sp(1)$ to be the product of the two groups in $SO(4n)$.

Now let $(M, g)$ be a compact quaternionic-Kähler manifold of real dimension $4n$, i.e., a Riemannian manifold whose holonomy group is contained in $Sp(n)Sp(1)$. It is a fact that $M$ is quaternionic-Kähler if and only if its oriented orthonormal frame bundle $FM$ can be reduced to a principal $Sp(n)Sp(1)$ bundle $P$ so that the Levi-Civita connection on $FM$ drops down to $P$.

Given a frame $(e_A) \in P_x, \ x \in M$, there is a right action of $\mathbb{H}$ on $T_xM$ after identifying $T_xM$ with $\mathbb{H}^n$ via

$$\sum_A c_A e_A \to v_1 + iv_2 + jv_3 + kv_4,$$

where $v_1 = (c_1, ..., c_n), \ v_2 = (c_{n+1}, ..., c_{2n})$, etc. Let $I, J, K: T_xM \to T_xM$ denote the elements in $Sp(1) \subset SO(4n)$ which correspond to the right actions of $i, j, k \in \mathbb{H}$. Hence, locally, we have almost complex structures $I, J$ and $K$ satisfying $IJ = -JI = K, \ JK = -KJ = I$ and $KI = -IK = J$ for which the Riemannian metric is Hermitian.

For any curvature tensor $R$, define

$$R^{iso}(X, Y, Z, W) := R(X, Z, X, Z) + R(X, W, X, W) + R(Y, Z, Y, Z) + R(Y, W, Y, W) - 2R(X, Y, Z, W).$$

As before, consider Ricci flow beginning at $g$. Of course, since $g$ is Einstein the flow is just given by scaling the metric (This raises the question whether one can
derive (ii) on Page 2 directly without using the Ricci flow and the results of Brendle and Schoen).

Fix $t > 0$ and $F$ denote the corresponding set of 4-frames where the isotropic curvature vanishes.

Continuing with the analysis of $F$, two cases arise:

**Case I:** $F$ does not contain an element of the $\{X, T(X), T(Y), Y\}$ for $T = I, J$ or $K$ whenever $g(X,Y) = g(X, T(Y)) = 0$. Hence $R^{iso}(X, T(X), T(Y), Y) > 0$ for any such pair $\{X,Y\}$.

We will prove that $\pi_2(M) = \{0\}$. By a result of Salamon [11], this will imply that $M$ is isometric to the quaternionic projective space $\mathbb{H}P^n$.

Suppose $\pi_2(M) \neq \{0\}$. Then by the theorem of Sacks and Uhlenbeck, there is a stable minimal 2-sphere in $M$, represented by a conformal harmonic map $f : S^2 \to M$.

We recall the setting for the second variation formula for area, as formulated by Micallef and Moore [8]. Let $(M, g)$ be a Riemannian manifold and let $E = f^*TM \otimes \mathbb{C}$. The Riemannian metric of $f^*TM$ extends to a symmetric bilinear form $(\cdot, \cdot)$ and a Hermitian inner product $\langle \cdot, \cdot \rangle$ on $E$. Also, $E$ carries the pullback of the Levi-Civita connection on $M$, extended complex linearly. This connection is Hermitian with respect to $\langle \cdot, \cdot \rangle$. It is well-known that a Hermitian bundle $V$ on a Riemann surface $\Sigma$ with a Hermitian connection $\nabla$ can be endowed with a holomorphic structure $\overline{\nabla}$ in which a section $s$ of $V$ is holomorphic is and only if $\nabla_{\overline{\partial} z} s = 0$ in any local holomorphic coordinate $z$ on $\Sigma$.

Let $W$ be a vector space over $\mathbb{R}$ with an inner product $\langle \cdot, \cdot \rangle$. As above, extend $\langle \cdot, \cdot \rangle$ to a symmetric bilinear form $(\cdot, \cdot)$ and a Hermitian form $\langle \cdot, \cdot \rangle$ on $W \otimes \mathbb{C}$. An element $v \in W \otimes \mathbb{C}$ is said to be isotropic if $(v,v) = 0$. A 2-plane $P \subset W$ is isotropic if every element of $P$ is isotropic. It can be checked that $\{v,w\}$ spans an isotropic 2-plane if and only if $v$ and $w$ are linearly independent and $(v,v) = (w,w) = (v,w) = 0$.

Also, $\text{Span}_\mathbb{C}\{v, w\}$ is an isotropic 2-plane if and only if there exist orthonormal vectors $e_1, e_2, e_3, e_4$ in $W$ such that $v = \frac{\|v\|}{\sqrt{2}}(e_1 + \sqrt{-1}e_2), \quad w = \frac{\|w\|}{\sqrt{2}}(e_3 + \sqrt{-1}e_4)$, where the norm is with respect to $\langle \cdot, \cdot \rangle$.

In our case $W$ will be $T_p M$ or $f^*T_p M$.

Let

$$\mathcal{R} : \bigwedge^2 E_p \to \bigwedge^2 E_p$$

denote the complex linear extension of the curvature operator. If $v, w, e_1, .., e_4$ are as above, then it can be checked that

$$\langle \langle \mathcal{R}(v \wedge w), \ v \wedge w \rangle \rangle = \|v\|^2 \|w\|^2 R^{iso}(e_1, e_2, e_3, e_4).$$
Consider $S^2$ as $\mathbb{C} \cup \{\infty\}$ and let $Z$ be the holomorphic vector field which is $\frac{\partial}{\partial z}$ on $\mathbb{C}$ and 0 at $\infty$. Then $f_*Z$ is a holomorphic isotropic section of $E$, where $f_* : TS^2 \otimes \mathbb{C} \to E$. It follows from the stability inequality of Micallef and Moore that if $M$ has nonnegative isotropic curvature and $f$ is stable then
\[(3.2) \quad \langle \langle R(s \wedge f_*Z), s \wedge f_*Z \rangle \rangle = 0,\]
for any holomorphic section $s$ of $E$ such that $s$ and $f_*Z$ span an isotropic 2-plane.

Micallef and Moore also prove that if $M$ is even-dimensional, compact and has nonnegative isotropic curvature, then any stable minimal 2-sphere $f : S^2 \to M$ is isotropic, i.e., $f^*TM$ possesses an orthogonal complex structure $T$ such that
(i) $T$ is preserved by parallel translation and
(ii) If $I_0$ is the complex structure on $TS^2$ and $f_* : TS^2 \to f^*TM$, then
\[(3.3) \quad f_* \circ I_0 = T \circ f_*.
\]

We now assume that $(M, g)$ is quaternionic-Kähler of dimension $4n$ and has nonnegative isotropic curvature. $f : S^2 \to M$ will be a stable minimal surface and the assumption on the set $F$ (Case I above) will hold.

**Lemma 3.1.** (i) There is a 3-dimensional sub-bundle $W_0$ of $\text{End}(f^*TM)$ which is invariant under parallel transport by the Riemannian connection on $\text{End}(f^*TM)$. The complex structure $T$ above is a section of $W_0$.

(ii) Let $\xi_0$ be the line sub-bundle of $W_0$ generated by $T$. There is a 2-dimensional sub-bundle $\eta_0$ of $W_0$ which is also preserved by parallel transport and such that $W_0 = \xi_0 \oplus \eta_0$.

(iii) There is a covering of $S^2$ by open sets $\{U_i\}$ such that on each $U_i$, $\eta_0$ is generated by local sections $J, K$ satisfying
\[(1) \quad J, K \text{ are orthogonal complex structures on } f^*TM|_{U_i},\]
\[(2) \quad TJ = -JT = K, \quad JK = -KJ = T, \quad KT = -TK = J.\]

**Proof.** Let $p \in S^2$ and $U \ni p$ an open set on which there is an orthonormal frame field of the form
\[\{e_1, \ldots, e_n, T(e_1), \ldots, T(e_n), e_{n+1}, \ldots, e_{2n}, -T(e_{n+1}), \ldots, -T(e_{2n})\}.
\]
If we identify $f^*TM|_U$ with $U \times \mathbb{R}^{4n} = U \times \mathbb{H}^n$ via this frame, then $T$ becomes the element $I$ of $SO(4n)$ which corresponds to the action (multiplication on the right) of $i \in Sp(1)$ on $\mathbb{H}^n$. Under this identification, the metric on $f^*TM$ and the chosen frame get identified with the standard metric and the standard basis of $\mathbb{R}^{4n} = \mathbb{H}^n$, respectively.

(i) follows from the hypothesis that the holonomy of $f^*TM$ is a subgroup of $Sp(n)Sp(1)$ and the fact that $Sp(n)Sp(1)$ preserves the 3-dimensional vector space of endomorphisms of $\mathbb{R}^{4n} = \mathbb{H}^n$ generated by $I, J$ or $K$.

To see (ii) and (iii), we observe that the subspace of $f^*TM$ generated by $\{J, K\}$ is also preserved by parallel transport: Let $q \in U$ and $X \in f^*T_qM$. There are $c_1, c_2$ and $c_3$ such that
\[(3.4) \quad (\nabla_X J)(Y) = c_1 T(Y) + c_2 J(Y) + c_3 K(Y),\]
for all $Y \in T_qM$. Extend $Y$ to a neighborhood of $q$ so that

$$(\nabla_X Y)_q = 0.$$ 

Then

$$(\nabla_X J)_q(Y) = (\nabla_X (J(Y)))_q.$$ 

Now (3.3) gives

$$c_1 g(J(Y), J(Y)) = g(\nabla_X (J(Y)), T(Y)) = X(g(J(Y), T(Y))) = 0,$$

where we have used the fact that $T$ is parallel in the second equality. Therefore $c_1 = 0$ and $\nabla_X J$ is a linear combination of $J$ and $K$. Similarly, so is $\nabla_X K$. □

The endomorphisms $T, J, K$ extend to sections (local, in the case of $J$ and $K$) of the complexification $\text{End}(E) = \text{End}(f^*TM \otimes \mathbb{C})$. If $L$ denotes any of these extensions, then

$$(3.5) \quad L^2 = -\text{Id}, \quad (L(v), L(w)) = (v, w), \quad \langle \langle L(v), L(w) \rangle \rangle = \langle \langle v, w \rangle \rangle, \quad (L(v), v) = 0$$

for all $v, w \in E_p$.

Let $W = W_0 \otimes \mathbb{C}$, $\xi = \xi_0 \otimes \mathbb{C}$ and $\eta = \eta_0 \otimes \mathbb{C}$.

**Lemma 3.2.** $W, \xi$ and $\eta$ are holomorphic sub-bundles of the holomorphic bundle $E$.

**Proof.** If $s$ is a smooth section of $W, \xi$ or $\eta$, then $\nabla_s s$ is again a section of $W, \xi$ or $\eta$ respectively. □

**Lemma 3.3.** Let $S$ be a section of $\eta$. If $f_\ast Z_p$ is not an eigenvector of $S$, then \{$(S(f_\ast Z)\big|_p, f_\ast Z_p)$\} spans an isotropic 2-plane for each $p \in S^2$.

**Proof.** Locally

$$S = aJ + bK,$$

for some complex-valued functions $a$ and $b$. Let $v \in E_p$ be an isotropic vector. Then

$$(S(v), S(v)) = a^2(J(v), J(v)) + b^2(K(v), K(v)) + 2ab(J(v), K(v)) = 0,$$

where we have used (3.3) Similarly

$$(S(v), v) = a(J(v), v) + b(K(v), v) = 0.$$

Hence \{$(v, J(v))\}$ spans an isotropic 2-plane.

Since $f_\ast Z$ is an isotropic vector as noted earlier, the lemma is proved by taking $v = f_\ast Z$ above. □

**Lemma 3.4.** There is a holomorphic section $S$ of $\eta$ such that $S_p = K_p$ at some point $p \in S^2$.

**Proof.** Since $\eta$ is the complexification of the real bundle $\eta_0$, the first Chern class $c_1(\eta) = 0$. Hence $\eta$ has a nontrivial holomorphic section $S$. Let $U$ be an open set on which we can write $S = aJ + bK$ for some smooth complex functions $a$ and $b$. Note that since

$$S^2 = -(a^2 + b^2)\text{Id},$$

...
we cannot have $a^2 + b^2 \equiv 0$ on $U$. Otherwise $S^2 \equiv 0$ on $U$. But if $v \in E_p$ is a real vector, then

$$||S(v)||^2 = |a|^2||J(v)||^2 + |b|^2||K(v)||^2 + ab\langle(J(v), K(v))\rangle + \bar{a}\bar{b}\langle(K(v), J(v))\rangle$$

where we have used $\langle(K(v), J(v))\rangle = \langle(J(v), K(v))\rangle = g(J(v), K(v)) = 0$.

Hence

$$0 = ||S^2(v)||^2 = (|a|^2 + |b|^2)^2||v||^2,$$

which implies that $a = b = 0$ and hence $S \equiv 0$. Since $S$ is holomorphic, this would imply that $S = 0$ on $S^2$. So suppose that $a(p)^2 + b(p)^2 \neq 0$ for some $p \in U$. Since $T$ is a parallel section of $\text{End}(E)$, $TS$ is also a holomorphic section of $\text{End}(E)$. In fact, it is a section of $\eta$ since

$$TS = T(aJ + bK) = aK - bJ.$$

Define a new holomorphic section $\tilde{S}$ of $\eta$ by

$$\tilde{S} := \frac{1}{a(p)^2 + b(p)^2} (b(p)S + a(p)TS).$$

It is easily checked that $\tilde{S}_p = K_p$. 

\[\square\]

The following lemma asserts that stable minimal spheres in $M$ are holomorphic and anti-holomorphic under the curvature conditions of Case (i). The argument is essentially from the last paragraph of Page 846 and Lemma A of [12].

**Lemma 3.5.** Suppose that $R^{iso}(X, L(X), L(Y), Y) > 0$ at all points $p \in S^2$ and all $X, Y \in f^* T_p M$ satisfying $g(X, Y) = g(X, L(Y)) = 0$ whenever $L, J$ or $K$. Then there is a point $p$ at which

$$L(f_* Z)_p = \pm \sqrt{-1} f_* Z_p.$$ 

**Proof.** By [82],

$$\langle(R(s \wedge f_* Z), s \wedge f_* Z)\rangle = 0,$$

if $s$ is any holomorphic section for which $s$ and $f_* Z$ span an isotropic 2-plane. Hence we can apply Lemma 3.3 to conclude that the above equation holds for $s = S(f_* Z)$, for any holomorphic section $S$ of $\eta$.

Now consider the section $S$ and point $p$ of Lemma 3.4. Let $w = (f_* Z)_p$ and $E_p = f^* T_p M \otimes \mathbb{C}$. We can write $E_p = E_p^{(1,0)} \oplus E_p^{(0,1)}$, where $E_p^{(1,0)}$, $E_p^{(0,1)}$ are the eigenspaces of $K$ for the eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$ respectively.

Suppose that $w$ is not an eigenvector of $K$. We write $w = w' + w''$ as the sum of non-zero $(1,0)$ and $(0,1)$ parts. Since $w$ is also isotropic, we can find orthonormal $e_1, e_2 \in f^* T_p M$ such that $g(e_1, K(e_2)) = 0$ and

$$\sqrt{2} \frac{w'}{||w'||} = e_1 - \sqrt{-1} K(e_1), \quad \sqrt{2} \frac{w''}{||w''||} = e_2 + \sqrt{-1} K(e_2).$$

Since $w \wedge K(w) = -2\sqrt{-1} w' \wedge w''$, we have

$$\langle\langle R(w \wedge K(w)), w \wedge K(w)\rangle\rangle = 4\langle\langle R(w' \wedge w''), w' \wedge w''\rangle\rangle$$

$$= ||w'||^2||w''||^2 (R^{iso}(e_1, -K(e_1), e_2, K(e_2)))$$

$$= ||w'||^2||w''||^2 (R^{iso}(e_1, K(e_1), K(e_2), e_2)) > 0.$$
This contradiction with (3.2) completes the proof.

The final step in the proof that there cannot be a stable minimal 2-sphere in $M$ is to observe that the conclusion of Lemma 3.5 contradicts (3.3). Noting that $I_0 Z_p = \sqrt{-1} Z_p$ and using (3.3), we have $w = f_* Z_p = -\sqrt{-1} f_* (I_0 Z_p) = -\sqrt{-1} T(f_* Z_p) = -\sqrt{-1} T(w)$. Hence $w$ is eigenvector for $T$ as well as $K$ and we get

$$T(w) = \pm K(w) \Rightarrow J(w) = \mp w,$$

which contradicts $J^2 = -Id$. This completes the proof.

**Case II:** There is a point $p \in M$, $X, Y \in T_p M$ and $L \in \{I, J, K\}$ such that $(X, L(X), L(Y), Y) \in F$.

The following two facts will be needed in our computations, the second of which is a theorem of D. V. Alekseevskii [1]:

(i) The curvature tensor of the canonical metric on quaternionic projective space $(\mathbb{H}P^n, g_0)$, normalized to have sectional curvatures lying between 1 and 1/4, is given (cf. [2]) by

$$4R_0(X, Y)Z = g_0(X, Z)Y - g_0(Y, Z)X + 2g_0(I(X), Y)I(Z) + g_0(I(X), Z)I(Y) - g_0(I(Y), Z)I(X) + 2g_0(J(X), Y)J(Z) + g_0(J(X), Z)J(Y) - g_0(J(Y), Z)J(X) + 2g_0(K(X), Y)K(Z) + g_0(K(X), Z)K(Y) - g_0(K(Y), Z)K(X)$$

(ii) The curvature tensor of any quaternionic-Kähler metric can be decomposed as

$$R = \left(\frac{S}{S_0}\right) R_0 + R_1,$$

where $S$ and $S_0$ are the scalar curvatures of $R$ and $R_0$ respectively, $R_0$ is the curvature tensor in (3.6) and $R_1$ has the formal properties of a curvature tensor whose holonomy lies in $Sp(n)$, i.e., is hyper-Kähler. Note that $S$ has to be constant since $g$ is Einstein.

For convenience rescale $g$ so that $S/S_0 = 1$.

**Lemma 3.6.** Let $\{X, I(X), Y, I(Y)\} \subset T_p M$ be an orthonormal set and

$$e(X, Y) := g(J(X), Y)^2 + g(K(X), Y)^2.$$

We then have

(i) $R^{iso}(X, I(X), Y, I(Y)) = 4e(X, Y)$.

(ii) If $R^{iso}(X, I(X), I(Y), Y) = 0$, then

$$R^{iso}_1(X, I(X), I(Y), Y) = -R^{iso}_0(X, I(X), I(Y), Y) = -2(1 + e(X, Y)).$$

(iii) If $R^{iso}(X, I(X), I(Y), Y) = 0$, then

$$R_1(X, I(X), Y, I(Y)) = -\frac{1}{2}(1 + e(X, Y)).$$
Proof: (i) This follows from the decomposition (3.7) after noting that $R_1$ behaves like the curvature tensor of a Kähler manifold with respect to the almost complex structures $I, J, K$ and hence

\[(3.8) \quad R_1^{iso}(X, I(X), Y, I(Y)) = 0.\]

On the other hand (3.6) gives

\[(3.9) \quad R_0(X, Y, X, Y) = R_0(I(X), I(Y), I(X), I(Y)) = R_0(I(X), I(Y), I(X), I(Y)) = \frac{1}{4} (1 + 3e(X, Y)) \]

and

\[(3.10) \quad R_0(X, I(X), Y, I(Y)) = \frac{1}{2} (1 - e(X, Y)).\]

Combining these, we get

\[R_1^{iso}(X, I(X), Y, I(Y)) = R_0^{iso}(X, I(X), Y, I(Y)) = 4e(X, Y).\]

(ii) Since $R_1^{iso}(X, I(X), I(Y), Y) = 0$, we have

\[(3.11) \quad R_0^{iso}(X, I(X), I(Y), Y) + R_1^{iso}(X, I(X), I(Y), Y) = 0.\]

(3.11), (3.9) and (3.10) combine to give

\[(3.12) \quad R_1^{iso}(X, I(X), I(Y), Y) = -R_0^{iso}(X, I(X), I(Y), Y) = -2e(X, Y).\]

(iii) Noting that $R_1$ is a Kähler curvature tensor, (3.8) and (3.12) give

\[(3.13) \quad R_1(X, I(X), Y, I(Y)) = \frac{1}{4} (R_1^{iso}(X, I(X), Y, I(Y)) + R_1^{iso}(X, I(X), I(Y), Y))
\]

\[= -\frac{1}{2} (1 + e(X, Y)).\]

We continue with the analysis of Case II. The following facts will be repeatedly used:

(*) $Sp(n)$ acts transitively on the unit sphere in $\mathbb{R}^{4n}$. This action commutes with the action of $I, J$ and $K$.

(*) In the Alekseevskii decomposition $R = R_0 + R_1$, $R_1$ is a Kähler curvature tensor for $I, J$ and $K$. In particular, $R_1$ is Ricci-flat.

We claim that the holonomy group of $M$ has to be a proper subgroup of $Sp(n)Sp(1)$. Suppose that it is the whole group $Sp(n)Sp(1)$.

Assume, without loss of generality, that $L = I$ in the condition of Case II. For any $v \in T_pM$, denote the underlying real subspace of the quaternionic line containing $v$ by

\[L_v := \text{Span}_\mathbb{R} \{v, I(v), J(v), K(v)\}.\]
First we consider the case $Y \in V_X^\perp$. Let $Z$ be any unit vector in $V_X^\perp$. We can find a linear transformation $L : T_pM \to T_pM$ satisfying

\[
L(Y) = Z
\]

\[
L(u) = u \quad \text{on} \quad V_X.
\]

which belongs to $Sp(n)$. Hence $(X, I(X), I(Z), Z) \in F$ and (iii) of Lemma 3.6 implies that $R_1(X, I(X), Z, I(Z)) = -\frac{1}{2}$ for all $Z \in V_X^\perp$. Taking $Z = J(Y)$ and using the fact that $R_1$ is invariant under $J$ as well, we get

\[
R_1(X, I(X), Y, I(Y)) = R_1(X, I(X), J(Y), K(Y))
\]

(3.14)

\[
= R_1(X, I(X), -Y, I(Y)) = -R_1(X, I(X), Y, I(Y)).
\]

Hence we get $R_1(X, I(X), Y, I(Y)) = 0$, which contradicts $R_1(X, I(X), Y, I(Y)) = -\frac{1}{2}$.

Therefore we can assume that $Y = v + w$ with $0 \neq v \in V_X$ and $w \in V_X^\perp$. We will show that $R_1$ will then have constant positive holomorphic sectional curvature, which contradicts the fact that $R_1$ is Ricci-flat.

We have

\[
0 = R^{iso}(X, I(X), I(Y), Y) = R^{iso}(X, I(X), I(v) + I(w), v + w)
\]

(3.15)

\[
= R^{iso}(X, I(X), I(v), v) + R^{iso}(X, I(X), I(w), w) + f(w),
\]

where $f : T_pM \to \mathbb{R}$ is a linear functional depending on $R$ and $X$. We can find an element $L \in Sp(n)$ satisfying

\[
L(w) = -w
\]

\[
L(u) = u \quad \text{on} \quad V_X.
\]

From the fact that $(X, I(X), LI(Y), L(Y)) \in F$, we get the following equation corresponding to (3.15):

\[
R^{iso}(X, I(X), I(v), v) + R^{iso}(X, I(X), I(w), w) = f(w) = 0.
\]

Adding (3.16) and (3.15) we get

\[
R^{iso}(X, I(X), I(v), v) + R^{iso}(X, I(X), I(w), w) = 0.
\]

More generally if $z \in L_X^\perp$, we can find an element $L \in Sp(n)$ with $L(w) = z$ and $L(X) = X$. By repeating the above argument with $z$ in place of $w$, we get

\[
R^{iso}(X, I(X), I(v), v) + R^{iso}(X, I(X), I(z), z) = 0.
\]

Since $R_1^{iso}(X, I(X), I(z), z) = 0$ for any $z \in L_X^\perp$, the above equation along with (iii) of Lemma 3.6 implies that

\[
4R_1(X, I(X), z, I(z)) = R_1^{iso}(X, I(X), I(z), z) + R^{iso}(X, I(X), I(v), v)
\]

(3.17)

is independent of $z$. This implies, as in (3.14), that

\[
R_1(X, I(X), z, I(z)) = 0
\]

(3.18)

for any $z \in V_X^\perp$. Combining this with (3.17), we get

\[
R^{iso}(X, I(X), I(v), v) = 0.
\]

(3.19)
Rescale $v$ so that $\|v\| = 1$. Since $g(X, v) = g(I(X), v) = 0$ and $v \in V_X$, we then have $e(X, v) = g(J(X), v)^2 + g(K(X), v)^2 = \|v\|^2 = 1$. Hence (3.10) and (iii) of Lemma 3.6 imply

$$R_1(X, I(X), v, I(v)) = -1.$$  

This implies the following: Since $v = aJ(X) + bK(X)$ for some $a, b \in \mathbb{R}$ (with $a^2 + b^2 = 1$),

$$-1 = R_1(X, I(X), v, I(v)) = -(a^2 + b^2)R_1(X, I(X), X, I(X))$$

4. The classification theorem

We now outline the proof of Theorem 2.2. Let $(M, g)$ be a compact manifold of nonnegative isotropic curvature. Let $g(t)$, $t \in [0, \delta)$ denote the solution to Ricci flow with $g(0) = g$. By 3.3, $g(t)$ has nonnegative isotropic curvature. As before, for sufficiently small $t$, $g(t)$ will be locally irreducible. We fix such a $t$ and denote $g(t)$ by $g$ for now. Suppose $g$ does not have strictly positive isotropic curvature. Then $F \neq \emptyset$. This implies that $Hol^0(M, g) \neq SO(n)$. Suppose $Hol^0(M, g) = SO(n)$.

Let $(e_1, e_2, e_3, e_4) \in F$. If $(f_1, f_2, f_3, f_4)$ is any orthonormal frame, then we can find an element of $SO(n)$ which takes $e_i$ to $f_i$, $i = 1$ to 4. Hence by the invariance of $F$ under parallel transport, every orthonormal 4-frame would be in $F$. Since the scalar curvature can be expressed as the sum of isotropic curvatures, this implies that $R = 0$ and also that $g$ is conformally flat. We can assume that this holds for all sufficiently small $t$. From the evolution equation for $R$ under Ricci flow,

$$\frac{\partial R}{\partial t} = \Delta R + |Ric|^2,$$

we can assume that $Ric = 0$, as well. Hence $g$ would be flat, contradicting local irreducibility.

If $(M, g)$ is not locally symmetric, then $Hol^0$ would have to be either $U(m)$ or $Sp(m)Sp(1)$, with $m$ equal to $\frac{n}{2}$ or $\frac{n}{4}$, respectively. The other possibilities for $Hol^0$ in the Berger Holonomy theorem can be ruled out since they would again lead to scalar flatness or local symmetry.

In case $Hol^0 = U(m)$, the full holonomy group $Hol = U(m)$ or $Hol = U(m) \rtimes \mathbb{Z}_2$. This is because $Hol$ is contained in the normalizer of $Hol^0$ in $O(n)$. If $Hol = U(m) \rtimes \mathbb{Z}_2$ then a 2-fold cover $\tilde{M}$ equipped with the pull-back metric $\tilde{g}$ is Kähler. Theorem 1.2 applied to $\tilde{M}$ implies that either $\tilde{M}$ is biholomorphic to $\mathbb{C}P^n$ or $(\tilde{M}, \tilde{g})$ is symmetric. The former case is ruled out since $\mathbb{C}P^n$ can cover only a nonorientable manifold. In the latter case, since $(\tilde{M}, \tilde{g})$ is locally isometric to $(M, g)$, we can conclude that $(M, g)$ is locally symmetric. If $Hol = U(m)$, then $(M, g)$ is Kähler and again Theorem 1.2 can be applied.
If $Hol^0 = Sp(m)Sp(1)$, then again $Hol = Sp(m)Sp(1)$ or $Hol = Sp(m)Sp(1) \rtimes \mathbb{Z}_2$. The same argument as above (after invoking Theorem 1.3) yields that $(M, g)$ is locally symmetric. This completes the proof of Theorem 2.2.

Next, we prove Remark (ii) of Page 2: First, by the results of [9] or [12], if $(M, h)$ has positive isotropic curvature, then $H^2(M, \mathbb{R}) = \{0\}$. Hence $M$ cannot admit a Kähler metric. Suppose that there is a locally symmetric metric $g$ on $M$. Note that $M$ admits a metric of positive scalar curvature (namely $h$), $(M, g)$ will have to be a symmetric space of compact type. In particular $(M, g)$ is Einstein with positive scalar curvature. By the Bonnet-Myers theorem $M$ has finite fundamental group and hence the universal cover $	ilde{M}$ is a compact manifold. The Micallef-Moore theorem applied to $(M, \tilde{h})$ implies that $\tilde{M}$ is homeomorphic to a sphere. Hence the locally symmetric metric $g$ would have to be of constant positive sectional curvature.

Finally we prove Remark (iv) on Page 2: Suppose $K^{iso} \geq 0$ on $M$ and $K^{iso}(p) > 0$ for some $p \in M$. With notation as above, if $F(g(t))$ is not empty, i.e. if $g(t)$ does not have $K^{iso} > 0$, then by the invariance under parallel transport of $F(g(t))$, there is a 4-frame at every point of $M$ for which $K^{iso} = 0$. Hence we get a time dependent 4-frame $(e_1(t), e_2(t), e_3(t), e_4(t))$ at $p$ for which $K^{iso} = 0$. We can choose a sequence of times $t_i \to 0$ as $i \to \infty$ for which the corresponding sequence of frames converges to an orthonormal frame on $(M, g)$. This frame will satisfy $K^{iso} = 0$ contradicting $K^{iso} > 0$ at $p$. □.

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