Infinite determinantal measures and the ergodic decomposition of infinite Pickrell measures. III.
The infinite Bessel process as the limit of the radial parts of finite-dimensional projections of infinite Pickrell measures

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Abstract. In the third paper of the series we complete the proof of our main result: a description of the ergodic decomposition of infinite Pickrell measures. We first prove that the scaling limit of the determinantal measures corresponding to the radial parts of Pickrell measures is precisely the infinite Bessel process introduced in the first paper of the series. We prove that the 'Gaussian parameter' for ergodic components vanishes almost surely. To do this, we associate a finite measure with each configuration and establish convergence to the scaling limit in the space of finite measures on the space of finite measures. We finally prove that the Pickrell measures corresponding to different values of the parameter are mutually singular.

Keywords: weak convergence, the Harish-Chandra–Itzykson–Zuber integral, infinite Bessel process, Jacobi polynomials.

§ 1. Introduction

This paper is the third and final in a series of three articles giving an explicit construction of the ergodic decomposition of infinite Pickrell measures. References to the other parts \[1\], \[2\] of the series are organized as follows: Proposition II.2.3 (for Proposition 2.3 in \[2\]), equation (I.9) (for equation (1.9) in \[1\]) and so on.

In § 2 we reconsider the radial parts of Pickrell measures. We start by recalling the determinantal representation for the radial parts of finite Pickrell measures and the convergence of the corresponding determinantal processes to the modified Bessel point process (the ordinary Bessel point process of Tracy and Widom \[3\] subject to the change of variables \(y = 4/x\)). Next, we represent the radial parts of infinite Pickrell measures as infinite determinantal measures corresponding to finite-rank perturbations of an orthogonal polynomial Jacobi ensemble. The main result of this section is Proposition 2.5 which shows that the scaling limit of the infinite determinantal measures corresponding to the radial parts of infinite Pickrell measures is precisely the modified infinite Bessel point process described in the first

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part (see [1]). Infinite determinantal measures are transformed into finite ones by taking the product with a suitable multiplicative functional, followed by establishing weak convergence both in the space of finite measures on the space of configurations and in the space of finite measures on the space of finite measures. This will be essential in proving (in § 3) that the ‘Gaussian parameter’ is equal to zero.

In § 3 we pass from the convergence of the rescaled radial parts of Pickrell measures (in the space of finite measures on the space of configurations and in the space of finite measures on the space of finite measures) to the convergence of finite-dimensional approximations of Pickrell measures on the space of finite measures. In particular, we establish that the ‘Gaussian parameter’ for the ergodic components of infinite Pickrell measures vanishes almost everywhere. Proposition 3.1 proved in this section enables us to complete the proof of Proposition I.1.16.

The final § 4 provides a proof of Lemma I.1.14, which relies on the well-known asymptotics of the Harish-Chandra–Itzykson–Zuber orbital integrals. Combining Lemma I.1.14 with Proposition I.1.16, we conclude the proof of Theorem I.1.11. Finally, using Kakutani’s theorem, we prove in the same way as Borodin and Olshanski [4] that Pickrell measures corresponding to distinct values of the parameter $s$ are mutually singular.

§ 2. Weak convergence of the rescaled radial parts of Pickrell measures

2.1. The case $s > -1$: finite Pickrell measures.

2.1.1. Determinantal representation of the radial parts of finite Pickrell measures.

We reconsider radial parts of the Pickrell measures $\mu^{(s)}_n$ and start with the case $s > -1$. Recall that $P^{(s)}_n$ stands for the Jacobi polynomials corresponding to the weight $(1 - u)^s$ on the interval $[-1, 1]$.

We start by giving a determinantal representation for the radial parts of finite Pickrell measures: in other words, we simply rewrite the formula (I.5) in the coordinates $\lambda_1, \ldots, \lambda_n$. Let

$$K^{(s)}_n(\lambda_1, \lambda_2) = \frac{n(n+s)}{2n+s} \frac{1}{(1+\lambda_1)^{s/2}(1+\lambda_2)^{s/2}} \times \frac{P^{(s)}_n(\lambda_{i-1})P^{(s)}_{n-1}(\lambda_{i+1}) - P^{(s)}_n(\lambda_{i+1})P^{(s)}_{n-1}(\lambda_{i-1})}{\lambda_1 - \lambda_2}.$$

The kernel $K^{(s)}_n$ is the image of the Christoffel–Darboux kernel $\widetilde{K}^{(s)}_n$ (see (I.34)) under the change of variables

$$u_i = \frac{\lambda_i - 1}{\lambda_i + 1}.$$

Another representation for the kernel $K^{(s)}_n$ is

$$K^{(s)}_n(\lambda_1, \lambda_2) = \frac{1}{(1+\lambda_1)^{s/2+1}(1+\lambda_2)^{s/2+1}} \times \sum_{l=0}^{n-1} (2l + s + 1)P^{(s)}_l \left(\frac{\lambda_1 - 1}{\lambda_1 + 1}\right).$$

P^{(s)}_l \left(\frac{\lambda_2 - 1}{\lambda_2 + 1}\right).$$
By definition, $\hat{K}_n^{(s)}$ is the kernel of the orthogonal projection operator in $L_2((0, +\infty), \text{Leb})$ onto the subspace

$$\hat{L}^{(s,n)} = \text{Span} \left( \frac{1}{(\lambda + 1)^{s/2 + 1}} P_l^{(s)} \left( \frac{\lambda - 1}{\lambda + 1} \right), l = 0, \ldots, n - 1 \right)$$

$$= \text{Span} \left( \frac{1}{(\lambda + 1)^{s/2 + 1}} \left( \frac{\lambda - 1}{\lambda + 1} \right)^l, l = 0, \ldots, n - 1 \right).$$

Proposition I.1.17 yields the following determinantal representation for the radial part of the Pickrell measure.

**Proposition 2.1.** For $s > -1$ we have

$$(\text{rad}_n)_{*} \mu_n^{(s)} = \frac{1}{n!} \det \hat{K}_n^{(s)}(\lambda_i, \lambda_j) \prod_{i=1}^{n} d\lambda_i.$$

2.1.2. **Scaling.** For every $\beta > 0$ let $\text{hom}_\beta : (0, +\infty) \to (0, +\infty)$ be the homothety map that sends $x$ into $\beta x$. We keep the same symbol for the induced transformation of the space of configurations $\text{Conf}((0, +\infty))$.

We now give an explicit determinantal representation for the measure

$$(\text{conf} \circ \text{hom}_{n^2} \circ \text{rad}_n)_{*} \mu_n^{(s)},$$

which is the image in the space of configurations of the rescaled radial part of the Pickrell measure $\mu_n^{(s)}$.

Consider the rescaled Christoffel–Darboux kernel

$$K_n^{(s)}(\lambda_1, \lambda_2) = n^2 \hat{K}_n^{(s)}(n^2\lambda_1, n^2\lambda_2)$$

corresponding to the orthogonal projection onto the rescaled subspace

$$L^{(s,n)} = \text{Span} \left( \frac{1}{(n^2\lambda + 1)^{s/2 + 1}} P_l^{(s)} \left( \frac{n^2\lambda - 1}{n^2\lambda + 1} \right) \right)$$

$$= \text{Span} \left( \frac{1}{(n^2\lambda + 1)^{s/2 + 1}} \left( \frac{n^2\lambda - 1}{n^2\lambda + 1} \right)^l, l = 0, \ldots, n - 1 \right).$$

The kernel $K_n^{(s)}$ induces a determinantal process $P_{K_n^{(s)}}$ on $\text{Conf}((0, +\infty))$.

**Proposition 2.2.** For $s > -1$ we have

$$(\text{hom}_{n^2} \circ \text{rad}_n)_{*} \mu_n^{(s)} = \frac{1}{n!} \det K_n^{(s)}(\lambda_i, \lambda_j) \prod_{i=1}^{n} d\lambda_i.$$

Equivalently,

$$(\text{conf} \circ \text{hom}_{n^2} \circ \text{rad}_n)_{*} \mu_n^{(s)} = P_{K_n^{(s)}}.$$
2.1.3. The scaling limit. Computation of the scaling limit for the radial parts of finite Pickrell measures is a variant of the well-known result of Tracy and Widom [3], who proved that the scaling limit of the Jacobi orthogonal polynomial ensembles is the Bessel point process.

Proposition 2.3. For any \( s > -1 \), the kernel \( K_n^{(s)} \) converges to the modified Bessel kernel \( J^{(s)} \) as \( n \to \infty \), uniformly in all variables on compact subsets of \((0, +\infty) \times (0, +\infty)\). We therefore have

\[
K_n^{(s)} \to J^{(s)} \quad \text{in} \quad \mathcal{S}_{1,\text{loc}}((0, +\infty), \text{Leb}),
\]

\[
P_{K_n^{(s)}} \to P_{J^{(s)}} \quad \text{in} \quad \mathcal{M}_{\text{fin}}\text{Conf}((0, +\infty)).
\]

Proof. This is an immediate corollary of the classical Heine–Mehler asymptotics for the Jacobi polynomials (see, for example, the monograph of Szegö [5]). \(\square\)

Remark. The Heine–Mehler asymptotics shows that uniform convergence actually holds on arbitrary simply connected compact subsets of \((\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0)\).

2.2. The case \( s \leq -1 \): infinite Pickrell measures.

2.2.1. Representation of the radial parts of infinite Pickrell measures as infinite determinantal measures. Our first aim is to show that the measure (I.17) is an infinite determinantal measure for \( s \leq -1 \). Similarly to the definitions in the introduction, we put

\[
\hat{V}^{(s,n)} = \text{Span} \left\{ \frac{1}{(\lambda + 1)^{s/2+1}}, \frac{1}{(\lambda + 1)^{s/2+1}} \frac{(\lambda - 1)}{\lambda + 1}, \ldots, \frac{1}{(\lambda + 1)^{s/2+1}} P_{n-n_s}^{(s+2n_s-1)} \left( \frac{\lambda - 1}{\lambda + 1} \right) \right\},
\]

\[
\hat{H}^{(s,n)} = \hat{V}^{(s,n)} \oplus \hat{L}^{(s+2n_s,n-n_s)}.
\]

Consider the rescaled subspaces

\[
V^{(s,n)} = \text{Span} \left\{ \frac{1}{(n^2\lambda + 1)^{s/2+1}}, \frac{1}{(n^2\lambda + 1)^{s/2+1}} \frac{n^2\lambda - 1}{n^2\lambda + 1}, \ldots, \frac{1}{(n^2\lambda + 1)^{s/2+1}} P_{n-n_s}^{(s+2n_s-1)} \left( \frac{n^2\lambda - 1}{n^2\lambda + 1} \right) \right\},
\]

\[
H^{(s,n)} = V^{(s,n)} \oplus L^{(s+2n_s,n-n_s)}.
\]

Proposition 2.4. For arbitrary \( s \leq -1 \) and \( R > 0 \), the radial part of the Pickrell measure is an infinite determinantal measure corresponding to the subspace \( H = \hat{H}^{(s,n)} \) and the subset \( E_0 = (0, R) \):

\[
(\text{rad}_n)_* \mu_n^{(s)} = \mathbb{B}(\hat{H}^{(s,n)}, (0, R)).
\]

For the rescaled radial part we have

\[
\text{conf}_* \iota^{(n)}(\mu^{(s)}) = (\text{conf} \circ \text{hom}_{n^2} \circ \text{rad}_n)_* \mu_n^{(s)} = \mathbb{B}(H^{(s,n)}, (0, R)).
\]
2.3. The modified Bessel point process as the scaling limit of the radial parts of infinite Pickrell measures: statement of Proposition 2.5. Let
\[ \mathbb{B}^{(s,n)} = \mathbb{B}(H^{(s,n)}, (0, R)). \]
We now describe the limit \( \mathbb{B}^{(s)} \) of the measures \( \mathbb{B}^{(s,n)} \). Namely, we multiply our sequence of infinite measures by a convergent multiplicative functional and establish the convergence of the resulting sequence of determinantal probability measures. It will be convenient to put \( g_\beta(x) = \exp(-\beta x) \) for every \( \beta > 0 \) and choose \( f \) to be a function of the form \( f(x) = \min(x, 1) \). Thus we put
\[ L^{(n,s,\beta)} = \exp\left(-\frac{\beta x^2}{2}\right) H^{(s,n)}. \]

It is clear from the definition that \( L^{(n,s,\beta)} \) is a closed subspace of \( L_2((0, +\infty), \text{Leb}) \).
We denote the corresponding orthogonal projection operator by \( \Pi^{(n,s,\beta)} \). Recall also that \( \Pi^{(s,\beta)} \) is the operator of orthogonal projection onto the subspace \( L^{(s,\beta)} = \exp(-\beta x/2) H^{(s)} \) (see (I.8), (I.9)).

**Proposition 2.5.**

1. For all \( \beta > 0 \) we have \( \Psi_{g_\beta} \in L_1(\text{Conf}(0, +\infty), \mathbb{B}^{(s)}) \), and for all \( n > -s + 1 \) we also have \( \Psi_{g_\beta} \in L_1(\text{Conf}(0, +\infty), \mathbb{B}^{(s,n)}) \).

2. We have
\[
\frac{\Psi_{g_\beta} \mathbb{B}^{(s,n)}}{\int \Psi_{g_\beta} d\mathbb{B}^{(s,n)}} = P_{\Pi^{(n,s,\beta)}},
\]
\[
\frac{\Psi_{g_\beta} \mathbb{B}^{(s)}}{\int \Psi_{g_\beta} d\mathbb{B}^{(s)}} = P_{\Pi^{(s,\beta)}}.
\]

3. We have
\[ \Pi^{(n,s,\beta)} \to \Pi^{(s,\beta)} \quad \text{in } \mathcal{I}_{1,\text{loc}}((0, +\infty), \text{Leb}) \quad \text{as } n \to \infty \]
and, therefore,
\[ P_{\Pi^{(n,s,\beta)}} \to P_{\Pi^{(s,\beta)}} \]
weakly in \( \mathcal{M}_\text{fin}(\text{Conf}((0, +\infty))) \) as \( n \to \infty \).

4. For \( f(x) = \min(x, 1) \) we have
\[
\sqrt{f} \Pi^{(n,s,\beta)} \sqrt{f}, \sqrt{f} \Pi^{(s,\beta)} \sqrt{f} \in \mathcal{I}_1((0, +\infty), \text{Leb}),
\]
\[
\sqrt{f} \Pi^{(n,s,\beta)} \sqrt{f} \to \sqrt{f} \Pi^{(s,\beta)} \sqrt{f} \quad \text{in } \mathcal{I}_1((0, +\infty), \text{Leb}) \quad \text{as } n \to \infty
\]
and, therefore,
\[ (\sigma_f)_* P_{\Pi^{(g,\beta,n)}} \to (\sigma_f)_* P_{\Pi^{(g,\beta)}} \]
weakly in \( \mathcal{M}_\text{fin}(\mathcal{M}_\text{fin}((0, +\infty))) \) as \( n \to \infty \).

The proof of Proposition 2.5 occupies the rest of this section.
2.4. Proof of Proposition 2.5.

2.4.1. Proof of the first three assertions. For \( s > -1 \) we write

\[
L^{(n,s,\beta)} = \exp\left(-\frac{\beta x}{2}\right)L_{Jac}^{(s,n)}, \quad L^{(s,\beta)} = \exp\left(-\frac{\beta x}{2}\right)L^{(s)}
\]

and keep the notation \( \Pi^{(n,s,\beta)} \), \( \Pi^{(s,\beta)} \) for the corresponding orthogonal projection operators. For \( s > -1 \), using Proposition II.2.3 on the convergence of induced processes, we clearly have

\[
\Psi g_{\beta} P_{K^{(s)}} n \int \Psi g_{\beta} dP_{K^{(s)}} = P_{\Pi^{(n,s,\beta)}},
\]

\[
\Psi g_{\beta} P_{J^{(s)}} n \int \Psi g_{\beta} dP_{J^{(s)}} = P_{\Pi^{(s,\beta)}}
\]

and also

\[
\Pi^{(n,s,\beta)} \rightarrow \Pi^{s,\beta} \quad \text{in} \quad I_{1,loc}((0, +\infty), \text{Leb}) \quad \text{as} \quad n \rightarrow \infty.
\]

If \( x_n \rightarrow x \) as \( n \rightarrow \infty \), then, of course,

\[
\lim_{n \rightarrow \infty} \frac{1}{n^2 \alpha} \left(n^2 x_n + 1\right)^\alpha = x^\alpha
\]

for every \( \alpha \in \mathbb{R} \) and, by the Heine–Mehler classical asymptotic formulae, we obtain

\[
\lim_{n \rightarrow \infty} \frac{1}{(n^2 x + 1)^{\alpha/2+1}} P_n^{(\alpha)} \left(\frac{n^2 x_n - 1}{n^2 x_n + 1}\right) = \frac{J_\alpha(2/\sqrt{x})}{\sqrt{x}}
\]

for every \( \alpha > -1 \). The following assertion on linear independence is an immediate corollary of Proposition I.2.21 after the change of variables \( y = 4/x \).

**Proposition 2.6.** For all \( s \leq -1 \) and for any \( R > 0 \), the functions

\[
x^{-s/2-1} \chi(0,R), \ldots, \frac{J_{s+2n_s-1}(2/\sqrt{x})}{\sqrt{x}} \chi(0,R)
\]

are linearly independent and, furthermore, independent of the subspace \( \chi(0,R)L^{s+2n_s} \).

The proof of Proposition I.2.21 clearly also implies that the functions

\[
e^{-\beta x/2} x^{-s/2-1}, \ldots, e^{-\beta x/2} \frac{J_{s+2n_s-1}(2/\sqrt{x})}{\sqrt{x}}
\]

are linearly independent and, furthermore, independent of the subspace \( e^{-\beta x/2} L^{s+2n_s} \). The first three assertions of Proposition 2.5 follow from their abstract counterparts established in the previous subsections: the first and second assertions follow from Corollary I.2.19, and the third one follows from Proposition II.2.6. We proceed to prove the fourth and last assertion of Proposition 2.5.
2.4.2. The asymptotics of \( J^{(s)} \) at 0 and at infinity. We shall need the asymptotics of the modified Bessel kernel \( J^{(s)} \) at 0 and at infinity.

Recall that the Bessel function is denoted by \( J_s \) and the ordinary Bessel kernel by \( \tilde{J}_s \). We start with a simple estimate of \( \tilde{J}_s \).

**Proposition 2.7.** For all \( s > -1 \) and any \( R > 0 \)

\[
\int_R^{+\infty} \frac{\tilde{J}_s(y, y)}{y} \, dy < +\infty.
\]  

**Proof.** Rewrite (1) in the form

\[
\int_R^{+\infty} \frac{1}{y} \int_0^1 (J_s(\sqrt{ty}))^2 \, dt \, dy = \int_0^1 dt \int_0^{+\infty} \frac{(J_s(\sqrt{y}))^2}{y} \, dy
\]

\[
= \int_0^{+\infty} \min \left( \frac{y}{R}, 1 \right) \cdot \frac{J_s(\sqrt{y})^2}{y} \, dy = \frac{1}{R} \int_0^R J_s(\sqrt{y})^2 \, dy + \int_R^{+\infty} \frac{J_s(\sqrt{y})^2}{y} \, dy.
\]

It follows immediately from the asymptotics of the Bessel functions at zero and at infinity that both integrals converge. □

By making the substitution \( y = 4/x \), we arrive at the following proposition.

**Proposition 2.8.** For all \( s > -1 \) and any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
\int_0^\delta x J^{(s)}(x, x) \, dx < \varepsilon.
\]

We also need the following result.

**Proposition 2.9.** For every \( R > 0 \)

\[
\int_0^R \tilde{J}_s(y, y) \, dy < \infty.
\]

**Proof.** First note that

\[
\int_0^R (J_s(\sqrt{y}))^2 \, dy < +\infty
\]

since for every fixed \( s > -1 \) and for all sufficiently small \( y > 0 \) we have

\[
(J_s(\sqrt{y}))^2 = O(y^s).
\]

We can now write

\[
\int_0^R \tilde{J}_s(y, y) \, dy = \int_0^1 \int_0^R (J_s(\sqrt{ty}))^2 \, dy \, dt \leq (R + 1) \int_0^R (J_s(\sqrt{y}))^2 \, dy < +\infty.
\]

□

By making the substitution \( y = 4/x \), we obtain the following proposition.

**Proposition 2.10.** For every \( R > 0 \)

\[
\int_R^{+\infty} J^{(s)}(x, x) \, dx < \infty.
\]
2.4.3. Asymptotics of the kernels $K^{(n,s)}$ at infinity, uniform in $n$. We turn to the uniform asymptotics of the kernels $K^{(n,s)}$ and the limiting kernel $J^{(s)}$ at infinity. This uniform asymptotics is needed to establish the last assertion of Proposition 2.5.

**Proposition 2.11.** For all $s > -1$ and for any $\varepsilon > 0$ there is an $R > 0$ such that

$$\sup_{n \in \mathbb{N}} \int_{R}^{+\infty} K^{(n,s)}(x,x) \, dx < \varepsilon. \quad (2)$$

**Proof.** We start by verifying (2) for $s > 0$. If $s > 0$, then the classical inequalities for the Bessel functions and Jacobi polynomials (see, for example, [5]) yield the existence of a constant $C > 0$ such that for all $x \geq 1$ we have

$$\sup_{n \in \mathbb{N}} K^{(n,s)}(x,x) < \frac{C}{x^2}.$$ 

This proves the proposition for $s > 0$.

To consider the remaining case $s \in (-1,0]$, we recall that the kernels $K^{(n,s)}$ are rank-one perturbations of the kernels $K^{(n-1,s+2)}$ and we state the following obvious general assertion.

**Proposition 2.12.** Let $K_n, K, \tilde{K}_n, \tilde{K} \in \mathcal{S}_{1,\text{loc}}((0,+\infty),\text{Leb})$ be locally trace-class projection operators acting in $L_2((0,+\infty),\text{Leb})$. Assume that the following conditions hold:

1. $K_n \to K$, $\tilde{K}_n \to \tilde{K}$ in $\mathcal{S}_{1,\text{loc}}((0,+\infty),\text{Leb})$ as $n \to \infty$;
2. For every $\varepsilon > 0$ there is an $R > 0$ such that

$$\sup_{n \to \infty} \text{tr} \chi((R,+\infty)) K_n \chi((R,+\infty)) < \varepsilon; \quad \text{tr} \chi((R,+\infty)) \tilde{K} \chi((R,+\infty)) < \varepsilon;$$

3. There is an $R_0 > 0$ such that

$$\text{tr} \chi((R_0,+\infty)) \tilde{K} \chi((R_0,+\infty)) < \varepsilon;$$

4. The projection operator $\tilde{K}_n$ is a rank-one perturbation of $K_n$.

Then for every $\varepsilon > 0$ there is an $R > 0$ such that

$$\sup_{n \to \infty} \text{tr} \chi((R,+\infty)) \tilde{K}_n \chi((R,+\infty)) < \varepsilon.$$ 

This completes the proof of Proposition 2.11. \qed

2.4.4. Asymptotics of the kernels $K^{(n,s)}$ at zero, uniform in $n$, and completion of the proof of Proposition 2.5. We now turn to the uniform asymptotics of the kernels $K^{(n,s)}$ and the limiting kernel $J^{(s)}$ at zero. Again, this uniform asymptotics is needed to establish the last assertion of Proposition 2.5.

**Proposition 2.13.** For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $n \in \mathbb{N}$

$$\int_{0}^{\delta} x K^{(n,s)}(x,x) \, dx < \varepsilon.$$ 

**Proof.** Going back to the variable $u$, we can restate this proposition as follows.
Proposition 2.14. For every \( \varepsilon > 0 \) one can find an \( R > 0 \) and \( n_0 \in \mathbb{N} \) such that for all \( n > n_0 \)

\[
\frac{1}{n^2} \int_{-1}^{1-R/n^2} \frac{1+u}{1-u} \tilde{K}^{(s)}_n(u,u) \, du < \varepsilon.
\]

First note that the function \( \frac{1+u}{1-u} \) is bounded above on \([-1,0]\), hence

\[
\frac{1}{n^2} \int_{-1}^{0} \frac{1+u}{1-u} \tilde{K}^{(s)}_n(u,u) \, du \leq \frac{2}{n^2} \int_{-1}^{1} \tilde{K}^{(s)}_n(u,u) \, du = \frac{2}{n}.
\]

We proceed to estimating the integral

\[
\frac{1}{n^2} \int_{0}^{1-R/n^2} \frac{1+u}{1-u} \tilde{K}^{(s)}_n(u,u) \, du.
\]

Fix some \( \kappa > 0 \) (the precise value of \( \kappa \) to be specified later). We write the kernel as a sum of two terms:

\[
\tilde{K}^{(s)}_n(u,u) = \left( \sum_{l \leq \kappa n} (2l + s + 1)P_l^{(s)}(u)^2 \right) (1-u)^s + \left( \sum_{l > \kappa n} (2l + s + 1)P_l^{(s)}(u)^2 \right) (1-u)^s.
\]

First we estimate the sum

\[
\frac{1}{n^2} \sum_{l \leq \kappa n} \int_{-1}^{1-R/n^2} \frac{1+u}{1-u} (2l + s + 1)P_l^{(s)}(u)^2 (1-u)^s \, du.
\]

Using the trivial estimate

\[
\max_{u \in [-1,1]} |P_l^{(s)}(u)| = O(l^2),
\]

we find that the integral (3) is bounded above by

\[
\text{const} \cdot \frac{1}{n^2} \sum_{l \leq \kappa n} l^{2s+1} \int_{-1}^{1-c/n^2} (1-u)^{s-1} \, du.
\]

We now consider three cases: \( s > 0 \), \( s = 0 \) and \( -1 < s < 0 \).

Case one. If \( s > 0 \), then the integral (4) is bounded above by

\[
\text{const} \cdot \frac{1}{n^2} \sum_{l \leq \kappa n} l^{2s+1} \frac{1}{l^{2s}} \leq \text{const} \cdot \kappa^2.
\]

Case two. If \( s = 0 \), then the integral (4) is bounded above by

\[
\text{const} \cdot \frac{1}{n^2} \sum_{l \leq \kappa n} l \cdot \log \frac{n}{l} \leq \text{const} \cdot \kappa^2.
\]
**Case three.** Finally, if \(-1 < s < 0\), then we arrive at the following upper bound for the integral (4):

\[
\text{const} \cdot \frac{1}{n^2} \left( \sum_{l \leq \kappa n} l^{2s+1} \right) \cdot R^s n^{-2s} \leq \text{const} \cdot R^s \kappa^{2+2s}.
\]

Note that in this case the upper bound decreases as \(R\) grows. In all three cases, the contribution of (4) can be made arbitrarily small by choosing \(\kappa\) to be sufficiently small.

Next, we estimate the expression

\[
\frac{1}{n^2} \sum_{\kappa n \leq l < n} \int_0^{1-1/l^2} \frac{1+u}{1-u} (2l + s + 1)(P_l^{(s)}(u))^2 (1-u)^s du.
\]  
(5)

Here we use the estimate (7.32.5) in [5]:

\[
|P_l^{(s)}(u)| \leq \text{const} \left( \frac{1-u}{{(1-u)}^{s/2+1/4}} \right),
\]

for \(u \in [0, 1-1/l^2]\). This yields the following upper bound for the integral (5):

\[
\text{const} \cdot \frac{1}{n^2} \sum_{\kappa n \leq l < n} l \leq \text{const} \cdot \kappa^2.
\]

This bound can also be made arbitrarily small by choosing a sufficiently small \(\kappa\).

It remains to estimate the integral

\[
\frac{1}{n^2} \sum_{\kappa n \leq l < n} \int_0^{1-R/n^2} \frac{1+u}{1-u} (2l + s + 1)(P_l^{(s)})^2 (1-u)^s du.
\]  
(6)

Here we again use the estimate (7.32.5) in [5]. Since the ratio \(l/n\) is bounded below, we arrive at the uniform estimate

\[
|P_l^{(s)}(u)| \leq \text{const} \cdot \left( \frac{1-u}{{(1-u)}^{s/2+1/4}} \right),
\]

which holds when \(\kappa n \leq l \leq n\) and \(u \in [0, 1-R/n^2]\). The constant in this estimate depends on \(\kappa\) and does not grow as \(R\) increases.

Thus we arrive at the following upper bound for the integral (6):

\[
\text{const} \cdot \frac{1}{n^3} \sum_{\kappa n \leq l < n} \int_0^{1-R/n^2} (1-u)^{-3/2} du \leq \text{const} \cdot \left( \frac{1-n}{\sqrt{R}} \right).
\]

By choosing \(\kappa\) to be sufficiently small (depending on \(\varepsilon\)) and then \(R\) to be sufficiently large (depending on \(\varepsilon\) and \(\kappa\)), we complete the proof of Proposition 2.13. □

The fourth assertion of Proposition 2.5 is now an immediate corollary of the uniform estimates in Propositions 2.11, 2.13 and the general fact stated in Proposition II.2.13.

This completes the proof of Proposition 2.5. □
§ 3. Convergence of approximating measures on the Pickrell set and proof of Propositions I.1.15, I.1.16

3.1. Proof of Proposition I.1.15. Proposition I.1.15 follows easily from what has already been established. Recall that we have a natural forgetful map conf: $\Omega_P \to \text{Conf}(0, +\infty)$ that sends every element $\omega = (\gamma, x)$, $x = (x_1, \ldots, x_n, \ldots)$ into the configuration $\omega(x) = (x_1, \ldots, x_n, \ldots)$. By definition, conf is an $\mathcal{r}(n)(\mu(s))-\text{almost surely bijective map}$. The characterization (given in Proposition 2.4) of the measure $\text{conf}^*\mathcal{r}(n)(\mu(s))$ as an infinite determinantal measure, along with the first assertion of Proposition 2.5, now yields Proposition I.1.15.

We proceed to prove Proposition I.1.16.

3.2. Proof of Proposition I.1.16. Recall that, by definition, we have $\text{conf}_s \nu(s,n,\beta) = \mathcal{P}_\Pi(s,n,\beta)$, and Proposition 2.5 implies that for all $s \in \mathbb{R}$ and $\beta > 0$,

$$\mathcal{P}_\Pi(s,n,\beta) \to \mathcal{P}_\Pi(s,\beta)$$

in $\mathcal{M}_{\text{fin}}(\text{Conf}((0, +\infty)))$ as $n \to \infty$. Furthermore, setting $f(x) = \min(x, 1)$, we have weak convergence

$$(\sigma_f)_s \mathcal{P}_\Pi(s,n,\beta) \to (\sigma_f)_s \mathcal{P}_\Pi(s,\beta)$$

in $\mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}((0, +\infty)))$. We now pass from the weak convergence (established in Proposition 2.5) of probability measures on the space of configurations to the weak convergence of probability measures on the Pickrell set.

Consider the natural map

$$s: \Omega_P \to \mathcal{M}_{\text{fin}}((0, +\infty))$$

defined by

$$s(\omega) = \sum_{i=1}^{\infty} \min(x_i(\omega), 1)\delta_{x_i(\omega)}.$$  

The restriction of $s$ to the subset $\Omega_P^0$ is bijective. We recall that $\Omega_P^0$ is defined as the set of all elements $\omega = (\gamma, x) \in \Omega_P$ such that $\gamma = \sum x_i(\omega)$.

Remark. The function $\min(x, 1)$ is chosen only as a concrete example: we could have chosen any other positive bounded function on $(0, +\infty)$ coinciding with $x$ on some interval $(0, \varepsilon)$ and bounded away from zero on its complement.

Consider the set

$$s\Omega_P = \left\{ \eta \in \mathcal{M}_{\text{fin}}((0, +\infty)): \eta = \sum_{i=1}^{\infty} \min(x_i, 1)\delta_{x_i} \text{ for some } x_i > 0 \right\}.$$  

Clearly, $s\Omega_P$ is closed in $\mathcal{M}_{\text{fin}}((0, +\infty))$.

Every measure $\eta$ in $s\Omega_P$ admits a unique representation $\eta = s\omega$ for a unique $\omega \in \Omega_P^0$. Consequently, to every finite Borel measure $\mathcal{P} \in \mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}((0, +\infty)))$ supported on $s\Omega_P$ there corresponds a unique measure $p\mathcal{P}$ on $\Omega_P$ such that $s_*p\mathcal{P} = \mathcal{P}$ and $p\mathcal{P}(\Omega_P \setminus \Omega_P^0) = 0$. 
3.3. Weak convergence in $M_{\text{fin}}(\Omega_P)$ and in $M_{\text{fin}}((0, +\infty))$. The following proposition describes the relationship between weak convergence in the space of finite measures on the space of finite measures on the half-line and weak convergence in the space of measures on the Pickrell set.

**Proposition 3.1.** Suppose that $\nu_n, \nu \in M_{\text{fin}}M_{\text{fin}}((0, +\infty))$ are supported on $s\Omega_P$ and $\nu_n \rightharpoonup \nu$ weakly in $M_{\text{fin}}M_{\text{fin}}((0, +\infty))$ as $n \to \infty$. Then $p\nu_n \rightharpoonup p\nu$ weakly in $M_{\text{fin}}(\Omega_P)$ as $n \to \infty$.

Of course, the map $s$ is discontinuous since the function

$$\omega \to \sum_{i=1}^{\infty} \min(x_i(\omega), 1)$$

is not continuous on the Pickrell set. Nonetheless, we have the following relationship between the tightness of measures on $\Omega_P$ and on $M_{\text{fin}}((0, +\infty))$.

**Lemma 3.2.** Let $P_\alpha \in M_{\text{fin}}M_{\text{fin}}((0, +\infty))$ be a tight family of measures. Then the family $pP_\alpha$ is also tight.

**Proof.** Take $R > 0$ and consider the subset

$$\Omega_P(R) = \left\{ \omega \in \Omega_P : \gamma(\omega) \leq R, \sum_{i=1}^{\infty} \min(x_i(\omega), 1) \leq R \right\}.$$ 

The subset $\Omega_P(R)$ is compact in $\Omega_P$, and every compact subset of $s\Omega_P$ is a subset of $s\Omega_P(R)$ for a sufficiently large $R$. Consequently, for every $\varepsilon > 0$ one can find a sufficiently large $R$ such that

$$P_\alpha(s(\Omega_P(R))) > 1 - \varepsilon \quad \text{for all } \alpha.$$ 

Since all measures $P_\alpha$ are supported on $\Omega_P^0$, it follows that

$$pP_\alpha(\Omega_P(R)) > 1 - \varepsilon \quad \text{for all } \alpha,$$

which proves tightness, as required. □

**Corollary 3.3.** Let

$$P_n \in M_{\text{fin}}(M_{\text{fin}}((0, +\infty))), \quad n \in \mathbb{N}, \quad P \in M_{\text{fin}}(M_{\text{fin}}((0, +\infty)))$$

be finite Borel measures. Suppose that the following assumptions hold:

1. The measures $P_n$ are supported on $s\Omega_P$ for all $n \in \mathbb{N}$;
2. $P_n \rightharpoonup P$ weakly in $M_{\text{fin}}M_{\text{fin}}((0, +\infty))$ as $n \to \infty$.

Then $P$ is also supported on $s\Omega_P$ and $pP_n \rightharpoonup pP$ weakly in $M_{\text{fin}}(\Omega_P)$ as $n \to \infty$.

**Proof.** The measure $P$ is of course supported on $s\Omega_P$ since $s\Omega_P$ is a closed set. The desired weak convergence in $M_{\text{fin}}(\Omega_P)$ can be established in three steps.
Step 1: The family $\mathbb{P}_n$ is tight.

The family $\mathbb{P}_n$ is tight by Lemma 3.2 and, therefore, it must have an accumulation point $\mathbb{P}' \in \mathcal{M}_{\text{fin}}(\Omega_P)$.

Step 2: The finite-dimensional distributions converge.

Let $l \in \mathbb{N}$ and let $\psi_l : (0, +\infty) \to \mathbb{R}$ be continuous compactly supported functions. We put $\varphi_l(x) = \min(x, 1) \psi_l(x)$, take $t_1, \ldots, t_l \in \mathbb{R}$ and observe that, by definition,

$$\exp\left(i \sum_{k=1}^l t_k \left(\sum_{r=1}^\infty \varphi_k(x_r(\omega))\right)\right) = \exp\left(i \sum_{k=1}^l t_k \int \psi_k (s \omega)\right)$$

for any $\omega \in \Omega_P$, hence

$$\int_{\Omega_P} \exp\left(i \sum_{k=1}^l t_k \left(\sum_{r=1}^\infty \varphi_k(x_r(\omega))\right)\right) d\mathbb{P}'(\omega) = \int_{\mathcal{M}_{\text{fin}}((0, +\infty))} \exp\left(i \sum_{k=1}^l t_k \int \psi_k (\eta)\right) d(s_* \mathbb{P}'(\eta)).$$

We now write

$$\int_{\Omega_P} \exp\left(i \sum_{k=1}^l t_k \left(\sum_{r=1}^\infty \varphi_k(x_r(\omega))\right)\right) d\mathbb{P}'(\omega) = \lim_{n \to \infty} \int_{\Omega_P} \exp\left(i \sum_{k=1}^l t_k \left(\sum_{r=1}^\infty \varphi_k(x_r(\omega))\right)\right) d\mathbb{P}_n(\omega).$$

On the other hand, since $\mathbb{P}_n \to \mathbb{P}$ weakly in $\mathcal{M}_{\text{fin}}(\mathcal{M}_{\text{fin}}((0, +\infty)))$, we have

$$\lim_{n \to \infty} \int_{\mathcal{M}_{\text{fin}}((0, +\infty))} \exp\left(i \sum_{k=1}^l t_k \int \psi_k (\eta)\right) d(s_* \mathbb{P}_n) = \int_{\mathcal{M}_{\text{fin}}((0, +\infty))} \exp\left(i \sum_{k=1}^l t_k \int \psi_k (\eta)\right) d\mathbb{P}.$$

It follows that

$$\int_{\Omega_P} \exp\left(i \sum_{k=1}^l t_k \left(\sum_{r=1}^\infty \varphi_k(x_r(\omega))\right)\right) d\mathbb{P}'(\omega) = \int_{\mathcal{M}_{\text{fin}}((0, +\infty))} \exp\left(i \sum_{k=1}^l t_k \int \psi_k (\eta)\right) d\mathbb{P}.$$

Since integrals of functions of the form $\exp(i \sum_{k=1}^l t_k \int \psi_k (\eta))$ determine a unique finite Borel measure on $\mathcal{M}_{\text{fin}}((0, +\infty))$, we have

$$s_* \mathbb{P}' = \mathbb{P}.$$
Step 3: The limit measure is supported on $\Omega_0^p$.
To verify that $P^\prime(\Omega_P \setminus \Omega_0^p) = 0$, we put

$$
\gamma'(\omega) = \gamma(\omega) + \sum_{k : x_k(\omega) \geq 1} (1 - x_k(\omega)).
$$

Since the sum on the right-hand side is finite and $\gamma'$ is continuous on $\Omega_P$, it follows that

$$
\int_{\Omega_P} \exp(-\gamma'(\omega)) \, dP'(\omega) = \lim_{n \to \infty} \int_{\Omega_P} \exp(-\gamma'(\omega)) \, dP_n(\omega).
$$

Moreover,

$$
\int_{\Omega_P} \exp\left(-\sum_{i=1}^\infty \min(1, x_i(\omega))\right) \, dP'(\omega)
= \lim_{n \to \infty} \int_{\Omega_P} \exp\left(-\sum_{i=1}^\infty \min(1, x_i(\omega))\right) \, dP_n(\omega).
$$

Since $pP_n(\Omega_P \setminus \Omega_0^p) = 0$ for all $n$, we have

$$
\int_{\Omega_P} \exp(-\gamma'(\omega)) \, dP_n(\omega) = \int_{\Omega_P} \exp\left(-\sum_{i=1}^\infty \min(1, x_i(\omega))\right) \, dP_n(\omega)
$$

for all $n$. It follows that

$$
\int_{\Omega_P} \exp(-\gamma'(\omega)) \, dP'(\omega) = \int_{\Omega_P} \exp\left(-\sum_{i=1}^\infty \min(1, x_i(\omega))\right) \, dP'(\omega),
$$

whence $\gamma'(\omega) = \sum_{i=1}^\infty \min(1, x_i(\omega))$. As a result, the equality $\gamma(\omega) = \sum_{i=1}^\infty x_i(\omega)$ holds $P'$-almost surely, and $P'(\Omega_P \setminus \Omega_0^p) = 0$. Thus $P' = pP$. $\square$

§ 4. Proof of Lemma I.1.14
and completion of the proof of Theorem I.1.11

4.1. Reduction of Lemma I.1.14 to Lemma 4.1. Recall that we have introduced a sequence of maps

$$
\mathbf{r}^{(n)} : \text{Mat}(n, \mathbb{C}) \to \Omega_0^p, \quad n \in \mathbb{N},
$$

sending every matrix $z \in \text{Mat}(n, \mathbb{C})$ into the point

$$
\mathbf{r}^{(n)}(z) = \left( \frac{\text{tr} z^* z}{n^2}, \frac{\lambda_1(z)}{n^2}, \ldots, \frac{\lambda_n(z)}{n^2}, 0, \ldots, 0, \ldots \right),
$$

where $\lambda_1(z) \geq \cdots \geq \lambda_n(z) \geq 0$ are the eigenvalues of $z^* z$ counted with multiplicities and arranged in non-increasing order. By definition,

$$
\gamma(\mathbf{r}^{(n)}(z)) = \frac{\text{tr} z^* z}{n^2}.
$$
Following Vershik [6], we now introduce on Mat($\mathbb{N}, \mathbb{C}$) a sequence of averaging operators over the compact groups $U(n) \times U(n)$:

$$(A_n f)(z) = \int_{U(n) \times U(n)} f(u_1 z u_2^{-1}) \, du_1 \, du_2,$$

where $du$ stands for the normalized Haar measure on the group $U(n)$. For any $(U(\infty) \times U(\infty))$-invariant probability measure on Mat($\mathbb{N}, \mathbb{C}$), the operator $A_n$ is the conditional expectation operator with respect to the sigma-algebra of $(U(n) \times U(n))$-invariant sets.

By definition, $(A_n f)(z)$ depends only on $r_n(z)$.

**Lemma 4.1.** Let $m \in \mathbb{N}$. There is a positive function $\varphi$ in the Schwartz space on Mat($m, \mathbb{C}$) as well as a positive continuous function $f$ on $\Omega_P$ such that for all $z \in \text{Mat}(\mathbb{N}, \mathbb{C})$ and for any $n \geq m$ we have

$$f(r_n(z)) \leq (A_n \varphi)(z).$$

**Remark.** The function $\varphi$, initially defined on Mat($m, \mathbb{C}$), can be extended to Mat($\mathbb{N}, \mathbb{C}$) in a natural way. Namely, the value of $\varphi$ at an infinite matrix $z$ is set to be simply the value of $\varphi$ applied to the upper-left $m \times m$-corner of $z$.

We postpone the proof of this lemma till the next subsection and proceed to prove Lemma I.1.14.

Refining the definition of the class $\mathcal{F}$ (given in the introduction to the first paper of the series), we take an $m \in \mathbb{N}$ and let $\mathcal{F}(m)$ be the family of all sigma-finite $(U(\infty) \times U(\infty))$-invariant Borel measures $\nu$ on Mat($\mathbb{N}, \mathbb{C}$) such that, for every $R > 0$,

$$\nu\left(\left\{z : \max_{i,j \leq m} |z_{ij}| < R\right\}\right) < +\infty.$$

In other words, the measure of a set of infinite matrices is finite if the corresponding set of upper-left $m \times m$-corners is compact. In particular, the projections $(\pi_n^\infty)_* \nu$ are well defined for all $n \geq m$. For example, if $s + m > 0$, then the Pickrell measure $\mu(s)$ belongs to $\mathcal{F}(m)$.

Recall furthermore that, by the results of [7], [8], every measure $\nu \in \mathcal{F}(m)$ admits a unique ergodic decomposition into finite ergodic components. In other words, for any such $\nu$ there is a unique sigma-finite Borel measure $\overline{\nu}$ on $\Omega_P$ such that

$$\nu = \int_{\Omega_P} \eta_\omega \, d\overline{\nu}(\omega).$$

Since the orbit of a unitary group is of course a compact set, the measures $(r_n)_* \nu$ are well defined for $n > m$ and may be thought of as finite-dimensional approximations of the spectral measure $\overline{\nu}$. Indeed, recall from the introduction to the first paper of the series that if $\nu$ is a finite measure, then $\overline{\nu}$ is a weak limit of the measures $(r_n)_* \nu$ as $n \to \infty$. The following proposition is a stronger and more precise version of Lemma I.1.14 from the introduction.

**Proposition 4.2.** Suppose that $m \in \mathbb{N}$, $\nu \in \mathcal{F}(m)$, $\varphi$ and $f$ are given by Lemma 4.1 and $\varphi \in L_1(\text{Mat}(\mathbb{N}, \mathbb{C}), \nu)$. 


Then
(1) \( f \in L_1(\Omega_P, (r^{(n)})*\nu) \) for all \( n > m \);
(2) \( f \in L_1(\Omega_P, \nu) \);
(3) \( f(r^{(n)})*\nu \to f\nu \) weakly in \( \mathcal{M}_{\text{fin}}(\Omega_P) \).

Proof. Step 1: The martingale convergence theorem and ergodic decomposition.

For every \( z \in \text{Mat}_{\text{reg}} \) and for any bounded continuous function \( \varphi \) on \( \text{Mat}(\mathbb{N}; \mathbb{C}) \) we have

\[
\lim_{n \to \infty} A_n \varphi(z) = \int_{\text{Mat}(\mathbb{N}; \mathbb{C})} f \, d\eta_{r^{(n)}(z)}.
\]  

(7)

(Here, as always, the symbol \( \eta_\omega \) stands for the ergodic probability measure corresponding to an \( \omega \in \Omega_P \).) One can regard (7) as a pointwise version of the equality (I.14) from the introduction to the first paper.

Indeed, (7) follows immediately from the definition of regular matrices, the Vershik–Olshanski characterization of the convergence of orbital measures \( [9] \) and the reverse martingale convergence theorem.

Step 2. Now let \( \varphi \) and \( f \) satisfy the conditions of Lemma 4.1 and assume that

\( \varphi \in L_1(\text{Mat}(\mathbb{N}, \mathbb{C}), \nu) \).

Lemma 4.3. For every \( \varepsilon > 0 \) there is a \((U(\infty) \times U(\infty))\)-invariant set \( Y_\varepsilon \subset \text{Mat}(\mathbb{N}, \mathbb{C}) \) such that

(1) \( \nu(Y_\varepsilon) < +\infty \);
(2) for all \( n > m \)

\[
\int_{\text{Mat}(\mathbb{N}, \mathbb{C}) \setminus Y_\varepsilon} f(r^{(n)}(z)) \, d\nu(z) < \varepsilon.
\]

Proof. Since \( \varphi \in L_1(\text{Mat}(\mathbb{N}, \mathbb{C}), \nu) \), we have

\[
\int_{\Omega_P} \left( \int_{\text{Mat}(\mathbb{N}, \mathbb{C})} \varphi \, d\eta_\omega \right) d\nu(\omega) < +\infty.
\]

We choose a Borel subset \( \tilde{Y}_\varepsilon \subset \Omega_P \) in such a way that \( \nu(\tilde{Y}_\varepsilon) < +\infty \) and

\[
\int_{\tilde{Y}_\varepsilon} \left( \int_{\text{Mat}(\mathbb{N}, \mathbb{C})} \varphi \, d\eta_\omega \right) d\nu(\omega) < \varepsilon.
\]

The pre-image of \( \tilde{Y}_\varepsilon \) under the map \( r^{(\infty)} \) or, more precisely, the set

\[
Y_\varepsilon = \{ z \in \text{Mat}_{\text{reg}} : r^{(\infty)}(z) \in \tilde{Y}_\varepsilon \},
\]

is by definition \((U(\infty) \times U(\infty))\)-invariant and has all the desired properties. \( \square \)

Step 3. Let \( \psi : \Omega_P \to \mathbb{R} \) be continuous and bounded. Take an \( \varepsilon > 0 \) and consider the corresponding set \( Y_\varepsilon \).

For every \( z \in \text{Mat}_{\text{reg}} \) we have

\[
\lim_{n \to \infty} \psi(r^{(n)}(z)) \cdot f(r^{(n)}(z)) = \psi(r^{(\infty)}(z)) \cdot f(r^{(\infty)}(z)).
\]
Since $\nu(Y_\varepsilon) < \infty$, the bounded convergence theorem gives
\[
\lim_{n \to \infty} \int_{Y_\varepsilon} \psi(r^{(n)}(z)) \cdot f(r^{(n)}(z)) \, d\nu(z) = \int_{Y_\varepsilon} \psi(r^{(\infty)}(z)) \cdot f(r^{(\infty)}(z)) \, d\nu(z).
\]
By the definition of $Y_\varepsilon$, we have
\[
\left| \int_{\text{Mat}(\mathbb{N}, \mathbb{C}) \setminus Y_\varepsilon} \psi(r^{(n)}(z)) \cdot f(r^{(n)}(z)) \, d\nu(z) \right| \leq \varepsilon \sup_{\Omega_P} |\psi|
\]
for all $n \in \mathbb{N}$, $n > m$. It follows that
\[
\lim_{n \to \infty} \int_{\text{Mat}(\mathbb{N}, \mathbb{C})} \psi(r^{(n)}(z)) \cdot f(r^{(n)}(z)) \, d\nu(z) = \int_{\text{Mat}(\mathbb{N}, \mathbb{C})} \psi(r^{(\infty)}(z)) \cdot f(r^{(\infty)}(z)) \, d\nu(z),
\]
which in turn implies that
\[
\lim_{n \to \infty} \int_{\Omega_P} \psi f(r^{(n)})(\nu) = \int_{\Omega_P} \psi f \, d\nu.
\]
This establishes weak convergence and thus completes the proof of Proposition 4.2 and Lemma I.1.14. \(\square\)

4.2. Proof of Lemma 4.1. We introduce an inner product $\langle \cdot, \cdot \rangle$ on $\text{Mat}(m, \mathbb{C})$ by $\langle z_1, z_2 \rangle = \text{Re} \text{tr}(z_1^* z_2)$. This inner product admits a natural extension to a pairing between the projective limit $\text{Mat}(\mathbb{N}, \mathbb{C})$ and the inductive limit
\[
\text{Mat}_0 = \bigcup_{m=1}^{\infty} \text{Mat}(m, \mathbb{C}).
\]
For a matrix $\zeta \in \text{Mat}_0$ we put
\[
\Xi_\zeta(z) = \exp(i \langle \zeta, z \rangle), \quad z \in \text{Mat}(\mathbb{N}, \mathbb{C}).
\]

We start with the following simple estimate for the Fourier transform of orbital measures.

Lemma 4.4. Suppose that $m \in \mathbb{N}$. For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $n > m$ and for any $\zeta \in \text{Mat}(m, \mathbb{C})$ and $z \in \text{Mat}(\mathbb{N}, \mathbb{C})$ satisfying
\[
\text{tr}(\zeta^* \zeta) \text{tr}((\pi_n^{\infty}(z))^* (\pi_n^{\infty}(z))) < \delta n^2
\]
we have
\[
|1 - A_n \Xi_\zeta(z)| < \varepsilon.
\]
Proof. This is a simple corollary of the power series representation of the Harish-Chandra–Itzykson–Zuber orbital integral (see, for example, [10], [11]). Indeed, let $\sigma_1, \ldots, \sigma_m$ be the eigenvalues of $\zeta^* \zeta$ and let $x_1^{(n)}, \ldots, x_n^{(n)}$ be the eigenvalues of $\pi_n^{\infty}(z)$.\]
The standard power series representation (see [10], [11]) of the orbital integral gives the expansions

\[ A_n \Xi(z) = 1 + \sum_{\lambda \in \mathbb{Y}_+} a(\lambda, n) s_\lambda(\sigma_1, \ldots, \sigma_m) \cdot s_\lambda\left(\frac{x_1^{(n)}}{n^2}, \ldots, \frac{x_n^{(n)}}{n^2}\right) \]

for any \( n \in \mathbb{N} \), where the summation is over the set \( \mathbb{Y}_+ \) of all non-empty Young diagrams \( \lambda \), the symbol \( s_\lambda \) stands for the Schur polynomial corresponding to \( \lambda \), and the coefficients \( a(\lambda, n) \) satisfy

\[ \sup_{\lambda \in \mathbb{Y}_+} |a(\lambda, n)| \leq 1. \]

The lemma follows immediately. \( \square \)

Corollary 4.5. For any \( m \in \mathbb{N} \), \( \varepsilon > 0 \) and \( R > 0 \) there is a positive function \( \psi : \text{Mat}(m, \mathbb{C}) \to (0, 1] \) in the Schwartz space such that for all \( n > m \) we have

\[ \mathcal{A}_n \psi(\pi_m^\infty(z)) \geq 1 - \varepsilon \quad (8) \]

for all \( z \) satisfying

\[ \text{tr}\left((\pi_m^\infty(z))^* (\pi_m^\infty(z))\right) < Rn^2. \]

Proof. Let \( \psi \) be a \((0, 1]\)-valued function in the Schwartz space. We assume additionally that \( \psi(0) = 1 \) and that the Fourier transform of \( \psi \) is supported in the ball of radius \( \varepsilon_0 \) centred at the origin. One can easily construct such a function. By Lemma 4.4, if \( \varepsilon_0 \) is small enough (depending on \( m, \varepsilon \) and \( R \)), then (8) holds for all \( n > m \). \( \square \)

We now conclude the proof of Lemma 4.1.

Take a sequence \( R_n \to \infty \) and let \( \psi_n \) be the corresponding sequence of functions in the Schwartz space that satisfy the conditions of Corollary 4.5. Take a sequence of positive numbers \( t_n \) of sufficiently rapid decay such that the function

\[ \varphi = \sum_{n=1}^{\infty} t_n \psi_n \]

belongs to the Schwartz space.

Let \( \tilde{f} \) be a positive continuous function on \((0, +\infty)\) such that for every \( n \), if \( t \leq R_n \), then \( \tilde{f}(t) < t_n/2 \). For any \( \omega \in \Omega_P \), \( \omega = (\gamma, x) \), we put

\[ f(\omega) = \tilde{f}(\gamma(\omega)). \]

The function \( f \) is by definition positive and continuous. By Corollary 4.5, the functions \( \varphi \) and \( f \) satisfy all the conditions of Lemma 4.1, which completes the proof of this lemma.
4.3. Completion of the proof of Theorem I.1.11.

Lemma 4.6. Let $E$ be a locally compact complete metric space. Let $\mathbb{B}_n, \mathbb{B}$ be sigma-finite measures on $E$, let $P$ be a probability measure on $E$, and let $f, g$ be positive bounded continuous functions on $E$. Assume that for all $n \in \mathbb{N}$

$$g \in L_1(E, \mathbb{B}_n)$$

and that the following conditions hold as $n \to \infty$:

1. $f \mathbb{B}_n \to f \mathbb{B}$ weakly in $\mathcal{M}_{\text{fin}}(E)$;
2. $\int_E g \, d\mathbb{B}_n \to P$ weakly in $\mathcal{M}_{\text{fin}}(E)$.

Then

$$g \in L_1(E, \mathbb{B}), \quad P = \frac{g \mathbb{B}}{\int_E g \, d\mathbb{B}}.$$

**Proof.** Let $\varphi$ be a non-negative bounded continuous function on $E$. On the one hand, as $n \to \infty$, we have

$$\int_E \varphi \cdot fg \, d\mathbb{B}_n \to \int_E \varphi \cdot fg \, d\mathbb{B}.$$ 

On the other hand,

$$\frac{\int_E \varphi \cdot fg \, d\mathbb{B}_n}{\int_E g \, d\mathbb{B}_n} \to \int_E \varphi \cdot f \, dP. \quad (9)$$

By choosing $\varphi = 1$, we obtain

$$\lim_{n \to \infty} \int_E g \, d\mathbb{B}_n = \frac{\int_E fg \, d\mathbb{B}}{\int_E f \, dP} > 0.$$ 

The sequence $\int_E g \, d\mathbb{B}_n$ is thus bounded away from zero and infinity. Furthermore, for any bounded continuous positive $\varphi$ we have

$$\lim_{n \to \infty} \int_E g \, d\mathbb{B}_n = \frac{\int_E \varphi fg \, d\mathbb{B}}{\int_E \varphi f \, dP}.$$ 

We now take an $R > 0$ and $\varphi(x) = \min(1/f(x), R)$. Letting $R$ tend to infinity, we obtain

$$\lim_{n \to \infty} \int_E g \, d\mathbb{B}_n = \int_E g \, d\mathbb{B}. \quad (10)$$
Substituting (10) back into (9), we arrive at the equality
\[
\int_E \varphi f \, dP = \frac{\int_E \varphi f g \, dB}{\int_E g \, dB}.
\]
Note that here, as in (9), \(\varphi\) may be an arbitrary non-negative bounded continuous function on \(E\). In particular, taking a compactly supported function \(\psi\) on \(E\) and setting \(\varphi = \psi/f\), we obtain
\[
\int_E \psi \, dP = \frac{\int_E \psi g \, dB}{\int_E g \, dB}.
\]
Since this equality holds for all compactly supported functions \(\psi\) on \(E\), we conclude that
\[
P = \frac{gB}{\int_E g \, dB}. \quad \square
\]
Combining Lemma 4.6 with Lemma I.1.14 and Proposition I.1.16, we can conclude the proof of Theorem I.1.11.

4.4. Proof of Proposition I.1.9. Using Kakutani’s theorem, we now conclude the proof of Proposition I.1.9. Take \(n\) large enough so that \(n + s > 1, n + s' > 1\) and compute the Hellinger integral
\[
\text{Hel}(n, s, s') = E\left(\sqrt{(P(n,n-1,s) \times P(n,n,s)) \cdot (P(n,n-1,s') \times P(n,n,s'))}\right)
\]
\[
= \sqrt{\frac{\Gamma(2n - 1 + s)}{\Gamma(n-1)\Gamma(n+s)} \cdot \frac{\Gamma(2n - 1 + s')}{{\Gamma(n-1)\Gamma(n+s')}} \cdot \frac{\Gamma(2n+s)}{\Gamma(n)\Gamma(n+s)} \cdot \frac{\Gamma(2n+s')}{\Gamma(n)\Gamma(n+s')}}
\]
\[
\times \int_0^\infty r^{n-1}(1+r)^{-2n-1-(s+s')/2} \, dr \cdot \int_0^\infty r^{n-1}(1+r)^{-2n-(s+s')/2} \, dr
\]
\[
= \frac{\sqrt{\Gamma(2n - 1 + s) \cdot \Gamma(2n - 1 + s')}}{\Gamma(2n - 1 + (s + s')/2)} \cdot \frac{\sqrt{\Gamma(2n + s) \cdot \Gamma(2n + s')}}{\Gamma(2n + (s + s')/2)}
\]
\[
\times \left(\frac{\Gamma(n + (s + s')/2)}{\Gamma(n+s) \cdot \Gamma(n+s')}\right)^2.
\]
We now recall a classical asymptotic formula: as \(t \to \infty\), we have
\[
\frac{\Gamma(t + a_1) \cdot \Gamma(t + a_2)}{(\Gamma(t + (a_1 + a_2)/2))^2} = 1 + \frac{(a_1 + a_2)^2}{4t} + O\left(\frac{1}{t^2}\right).
\]
It follows that
\[
\text{Hel}(n, s, s') = 1 - \frac{(s + s')^2}{8n} + O\left(\frac{1}{n^2}\right).
\]
Hence, by Kakutani’s theorem combined with (I.40), we can conclude that the Pickrell measures $\mu^{(s)}$ and $\mu^{(s')}$, finite or infinite, are mutually singular when $s \neq s'$. □

4.5. **Proof of Proposition I.1.4.** In view of Proposition I.1.10 and Theorem I.1.11, it suffices to prove that the ergodic decomposition measures $\overline{\mu}^{(s_1)}$, $\overline{\mu}^{(s_2)}$ are singular. Since the measures $\mu^{(s_1)}$ and $\mu^{(s_2)}$ are mutually singular (by Proposition I.1.9), there is a subset $D \subset \text{Mat}(\mathbb{N}, \mathbb{C})$ such that

$$\mu^{(s_1)}(D) = 0, \quad \mu^{(s_2)}(\text{Mat}(\mathbb{N}, \mathbb{C}) \setminus D) = 0.$$ 

Consider the set

$$\tilde{D} = \{ z \in \text{Mat}(\mathbb{N}, \mathbb{C}) : \lim_{n \to \infty} A_n \chi_D(z) = 1 \}.$$ 

By definition, $\tilde{D}$ is $(U(\infty) \times U(\infty))$-invariant, and we have

$$\mu^{(s_1)}(\tilde{D}) = 0, \quad \mu^{(s_2)}(\text{Mat}(\mathbb{N}, \mathbb{C}) \setminus \tilde{D}) = 0.$$ 

We now define a set $\overline{D} \subset \Omega_P$ as

$$\overline{D} = \{ \omega \in \Omega_P : \eta_\omega(\tilde{D}) = 1 \}.$$ 

Clearly,

$$\overline{\mu}^{(s_1)}(\overline{D}) = 0, \quad \overline{\mu}^{(s_2)}(\Omega_P \setminus \overline{D}) = 0. \quad \Box$$

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