Counterexamples to the topological Tverberg conjecture

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Abstract

The “topological Tverberg conjecture” by Bárány, Shlosman and Szücs (1981) states that any continuous map of a simplex of dimension \((r-1)(d+1)\) to \(\mathbb{R}^d\) maps points from \(r\) disjoint faces of the simplex to the same point in \(\mathbb{R}^d\). This was established for affine maps by Tverberg (1966), for the case when \(r\) is a prime by Bárány et al., and for prime power \(r\) by Özaydin (1987). We combine the generalized van Kampen theorem announced by Mabillard and Wagner (2014) with the constraint method of Blagojević, Ziegler and the author (2014), and thus prove the existence of counterexamples to the topological Tverberg conjecture for any number \(r\) of faces that is not a prime power. However, these counterexamples require that the dimension \(d\) of the codomain is sufficiently high: the smallest counterexample we obtain is for a map of the 100-dimensional simplex to \(\mathbb{R}^{19}\), for \(r = 6\).

The “topological Tverberg conjecture” states that for given integers \(r \geq 2\), \(d \geq 1\), \(N = (r-1)(d+1)\), and for any continuous map \(f : \Delta_N \to \mathbb{R}^d\) from the \(N\)-simplex \(\Delta_N\) into \(\mathbb{R}^d\) there are \(r\) pairwise disjoint faces \(\sigma_1, \ldots, \sigma_r\) of \(\Delta_N\) such that \(f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset\). This holds if \(f\) is an affine map: this is a reformulation of Tverberg’s original theorem [7]. The conjecture for continuous \(f\) was introduced, and proven for \(r\) a prime, by Bárány, Shlosman and Szücs [1], and later extended to the case when \(r\) is a prime power by Özaydin [6]. The conjecture is trivial for \(d = 1\). All other cases have remained open. According to Matoušek [5, p. 154], the validity of the conjecture for general \(r\) is one of the most challenging problems in topological combinatorics.

Here we prove the existence of counterexamples to the topological Tverberg conjecture for any \(r\) that is not a power of a prime and dimensions \(d \geq 3r+1\). Our construction builds on recent work of Mabillard and Wagner [4], from which we first obtain counterexamples to \(r\)-fold versions of the van Kampen–Flores theorem. Counterexamples to the topological Tverberg conjecture are then obtained by an additional application of the constraint method of Blagojević, Ziegler and the author [2].

In the conference proceedings version [4] Mabillard and Wagner announced the generalized van Kampen theorem together with an extended sketch of its proof; a full version of the paper is forthcoming. To state the generalized van Kampen theorem, we first need to fix some notation. We refer to Matoušek [5] for further explanations. For a simplicial complex \(K\) denote by

\[ K_{\Delta(2)}^{x_1, \ldots, x_r} = \{ (x_1, \ldots, x_r) \in \sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \text{ face of } K, \sigma_i \cap \sigma_j = \emptyset \forall i \neq j \} \]

the 2-wise deleted product of \(K\) and by \(K^{(d)}\) the \(d\)-skeleton of \(K\). The space \(K_{\Delta(2)}^{x_1, \ldots, x_r}\) is a polytopal cell complex in a natural way (its faces are products of simplices). Denote by \(W_r\) the vector space \(\{ (x_1, \ldots, x_r) \in \mathbb{R}^r \mid \sum x_i = 0 \}\) with the action by the symmetric group \(S_r\) that permutes coordinates.

**Theorem 1** (Mabillard & Wagner [4] Theorem 3). Suppose that \(r \geq 2\), \(k \geq 3\), and let \(K\) be a simplicial complex of dimension \((r-1)k\). Then the following statements are equivalent:

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(i) There exists an $\mathcal{S}_r$-equivariant map $K^{\times r}_{\Delta(2)} \to S(W^r_{\oplus r k})$.

(ii) There exists a continuous map $f: K \to \mathbb{R}^r$ such that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $K$ we have $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$.

An important result on the existence of equivariant maps was shown by Özaydin.

**Lemma 2** (Özaydin [6 Lemma 4.1]). Let $d \geq 3$ and $G$ be a finite group. Let $X$ be a $d$-dimensional free $G$-CW complex and let $Y$ be a $(d - 2)$-connected $G$-CW complex. There is a $G$-map $X \to Y$ if and only if there are $G_r$-maps $X \to Y$ for every Sylow $p$-subgroup $G_p$, $p$ prime.

Özaydin uses this result to prove the existence of $\mathcal{S}_r$-equivariant maps $(\Delta_{(r-1)(d+1)})^{\times r}_{\Delta(2)} \to S(W^r_{\oplus r k})$ for $r$ not a prime power. An initial motivation for Mabillard and Wagner was to use such a map to construct counterexamples to the topological Tverberg conjecture via $r$-fold versions of the Whitney trick. However, for their approach to work they need codimension $k \geq 3$. Here we first derive counterexamples to $r$-fold versions of the van Kampen–Flores theorem, which is a Tverberg-type statement with a bound on the dimension of faces, see Corollary 3 from the result of Mabillard and Wagner and Özaydin’s work, and eventually obtain counterexamples to the topological Tverberg conjecture by a combinatorial reduction.

**Corollary 3.** Let $r \geq 6$ be an integer that is not a prime power and $k \geq 3$ an integer. Then for any $N$ there exists a continuous map $f: \Delta_N \to \mathbb{R}^r$ such that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ with $\dim \sigma_i \leq (r-1)k$ we have $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$.

**Proof.** Let $K = \Delta_N^{((r-1)k)}$ denote the $((r-1)k)$-dimensional skeleton of the simplex $\Delta_N$ on $N + 1$ vertices. We only need to construct $f$ on $K$ and extend continuously to $\Delta_N$ in an arbitrary way. By Theorem 1 we need to show that there exists an $\mathcal{S}_r$-equivariant map $K^{\times r}_{\Delta(2)} \to S(W^r_{\oplus r k})$. The reasoning is the same as in [6 Proof of Theorem 4.2]: the free $\mathcal{S}_r$-space $K^{\times r}_{\Delta(2)}$ has dimension at most $d = r(r-1)k$, and $S(W^r_{\oplus r k}) \cong S(r-1)k-1$ is $(d-2)$-connected. By Lemma 2 the existence of an $\mathcal{S}_r$-map $K^{\times r}_{\Delta(2)} \to S(W^r_{\oplus r k})$ reduces to the existence of equivariant maps for Sylow $p$-subgroups, but $p$-groups have fixed points in $S(W^r_{\oplus r k})$ for $r$ not a prime power by [6 Lemma 2.1], so a constant map will do.

Any $r$ generic affine subspaces of dimension $(r-1)k$ in $\mathbb{R}^r$ intersect in a point by codimension reasons. Here we see that a continuous map $\Delta_N^{((r-1)k)} \to \mathbb{R}^r$ can avoid this intersection, and indeed a map without any such intersection exists for any $N$, but only if $r$ is not a prime power. Volovikov [8] proved that a map as postulated by Corollary 3 does not exist if $r$ is a prime power; see [2] for more general results with significantly simplified proofs.

The map $f$ in Corollary 3 could not be constructed if the topological Tverberg conjecture were true, since the validity of the topological Tverberg conjecture would imply such an intersection result for faces of bounded dimension by the constraint method. For the sake of completeness we will present a construction that does not rely on [2].

**Theorem 4** (The topological Tverberg conjecture fails). Let $r \geq 6$ be an integer that is not a prime power, and let $k \geq 3$ be an integer. Let $N = (r-1)(rk + 2)$. Then there exists a continuous map $F: \Delta_N \to \mathbb{R}^{rk+1}$ such that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ we have $F(\sigma_1) \cap \cdots \cap F(\sigma_r) = \emptyset$.

**Proof.** Let $f: \Delta_N \to \mathbb{R}^r$ be a continuous map as constructed in Corollary 3 that is, such that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ with $\dim \sigma_i \leq (r-1)k$ we have $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$. Define $F: \Delta_N \to \mathbb{R}^{rk+1}$, $x \mapsto (f(x), \text{dist}(x, \Delta_N^{((r-1)k)}))$. Suppose there were $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that there are points $x_i \in \sigma_i$ with $F(x_1) = \cdots = F(x_r)$. By restricting to subfaces if necessary we can assume that $x_i$ is in the relative interior of $\sigma_i$. Then all the $x_i$ have the same distance to the $(r-1)k$-skeleton of $\Delta_N$. 

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Suppose all $\sigma_i$ had dimension at least $(r - 1)k + 1$. Then these faces would involve at least $r((r - 1)k + 2) = (r - 1)(rk + 2) + 2 > N + 1$ vertices. Thus, one face $\sigma_j$ has dimension at most $(r - 1)k$ and $\text{dist}(x_j, \Delta^{((r-1)k)}_N) = 0$. But then we have $\text{dist}(x_i, \Delta^{((r-1)k)}_N) - \text{dist}(x_j, \Delta^{((r-1)k)}_N) = 0$ for all $i$, so $x_i \in \Delta^{((r-1)k)}_N$ and thus $\sigma_i \subseteq \Delta^{((r-1)k)}_N$ for all $i$. This contradicts our assumption on $f$.

If the topological Tverberg conjecture holds for $r$ pairwise disjoint faces and dimension $d + 1$, then it also holds for dimension $d$ and the same number of faces. Thus, we are only interested in low-dimensional counterexamples. If $r$ is not a prime power then the topological Tverberg conjecture fails for dimensions $3r + 1$ and above. Hence, the smallest counterexample this construction yields is a continuous map $\Delta^{100} \to \mathbb{R}^{19}$ such that any six pairwise disjoint faces have images that do not intersect in a common point.

This and further applications of these methods will be presented in [3].

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