Multimode circuit QED with hybrid metamaterial transmission lines

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Quantum transmission lines are central to superconducting and hybrid quantum computing. In this work we show how coupling them to a left-handed transmission line allows circuit QED to reach a new regime: multi-mode ultra-strong coupling. Out of the many potential applications of this novel device, we discuss the preparation of multipartite entangled states and the simulation of the spin-boson model where a quantum phase transition is reached up to finite size effects.

Quantum optics addresses the interaction of quanta of matter — atoms — with quanta of electromagnetic fields — photons. This is beautifully realized in cavity quantum electrodynamics (QED) [1], where the interaction between those units is made strong by confining the field into a small mode volume [2]. Circuit QED takes this further by confining microwave photons in a quasi 1D strip-line cavity and using superconducting qubits as artificial atoms with a large dipole moment [2,3]. Next to being a promising architecture for quantum computing, a multitude of basic quantum optical effects has been demonstrated [4]. Going beyond what can be reached in atomic systems, an ultrastrong coupling regime — where the coupling strength becomes comparable to the atomic energy scales — has been proposed [5] and achieved [6,7]. Furthermore in the circuit QED approach, elements are entirely human-made and can hence be flexibly engineered. This can lead to coupling to multiple modes [8–12] either in the same or distinct cavities. There is a wealth of proposals exploiting these features to create complex photonic states [13–15] involving a large number of cavities. Parallel to these developments are those of left-handed meta-materials. They have a wide variety of applications in photonics from the microwave to the visible range such as invisibility cloaks and perfect flat lenses [16,17]. For classical guided microwaves, left-handed transmission lines have been proposed [18] and studied [19] on the macroscopic scale. In the following we show how a hybrid transmission line, made of left and right-handed media, coupled to a flux qubit gives rise to ultrastrong multimode coupling.

The System: In one-dimension, left-handedness is defined as the wave vector $k_f$ and the Poynting vector having opposite orientation; the phase and group-velocity are opposite corresponding to a falling dispersion relation $\partial \omega(k)/\partial k < 0$. This can be achieved [18] by a discrete array of series capacitors and parallel inductors to ground, see Fig. $\blacksquare$ A low loss left-handed transmission line (LHTL) can be realized with superconductors [19–21]. This is the dual (inductors and capacitors interchanged) of the usual [21] discrete representation of the right-handed transmission line (RHTL). In practice, the LHTL remains a metamaterial composed of discrete elements, whereas the RHTL is a metal strip represented as the continuum limit of a ladder network [21]. We can understand the physics of this line as follows: For any ladder network with discrete time-translation symmetry, the eigenmodes are (propagating or decaying) plane waves with a dispersion relation derived from the solutions of [21]

$$\sin \left( \frac{k \Delta x}{2} \right) = \pm \frac{1}{2} \frac{Z_x}{Z_p}$$

(1)

For a RHTL, substituting impedances of the series elements $Z_s = i \omega L_r$ and parallel elements $Z_p = (i \omega C_r)^{-1}$ gives the usual dispersion

$$\omega_r (k_r) = \frac{2}{\sqrt{C_r L_r}} \sin \left( \frac{k_r \Delta x}{2} \right) \text{continuum} \rightarrow \frac{k_r}{\sqrt{C_r L_r}}$$

(2)

For the LHTL, we interchange the roles of inductors and capacitors and obtain from equation $\blacksquare$ propagating modes (real-valued $k$ for real $\omega$) with the opposite dispersion relation

$$\omega_l (k_l) = \frac{1}{2 \sqrt{C_l L_l}} \sin \left( \frac{k_l \Delta x}{2} \right)$$

(3)

Here, $C_{l/r}$ and $L_{l/r}$ are capacitances and inductances as defined in Fig. $\blacksquare$ and $c_r$ and $l_r$ are the capacitance and inductance per unit length in the RHTL. $\Delta x$ is the size of a unit cell. More details are in the supplementary material.

Unusual physics arises when right- and left-handed media are interfaced $\blacksquare$. We realize this with a coupled transmission line (CTL) shown in Fig. $\blacksquare$ a a discrete LHTL coupled to a RHTL to be taken into the continuum limit. A key unusual feature of the LHTL, as compared to a regular RHTL, is the divergence of the density of modes (DoM) at a low-frequency bound $\omega_{IR} = 1/2 \sqrt{C_l L_l}$, seen in Fig. $\blacksquare$ a, implying the existence of a quasi-continuous band even in a cavity. In the LHTL, low frequencies correspond to short wavelengths due to the falling dispersion relation $\omega(k)$. Thus, by only a small change in frequency, a new orthogonal mode can be found that is different by one node in the left-handed component. As the wavelength approaches the lattice constant, the dispersion relation in equation $\blacksquare$ becomes flat due to Bragg
reflection \cite{18} — the DoM develops a van-Hove-type singularity setting the aforementioned divergence at ωIR. Due to the hybrid nature of this new CTL, the closely spaced frequencies at this lower band-edge have nearly identical spatial structures in the RHTL. The fast oscillation in the LHTL ensure orthogonality between modes. Figure (b) shows three consecutive low frequency modes obtained from the full solution. Close to ωIR the RHTL provides a mere constant contribution thus the DoM is dominated by the divergence due to the LHTL and vice versa. In consequence, the DoM can be approximated by the sum of the densities in the uncoupled lines

\[
D(\omega) = \frac{4N_l\sqrt{C_l}L_l}{\pi}\tan\phi_L\sin\phi_L, \quad \phi_L = \frac{k_l(\omega)\Delta x}{2}.
\]

N_l is the number of cells in the LHTL. The agreement between this prediction and the numerically obtained modes of the full model is excellent up to small oscillations, see Figure 2. To engineer the DOM, one can control ωIR by the mesh size and independently N_l by the length of the LHTL.

To provide good coupling between both components one would like to have an impedance Z_0 (typically 50 Ω) requiring C_l = (2ωIRZ_0)^{-1} and L_l = Z_0/2ωIR. Furthermore for coupling to qubits ωIR should be chosen to lie around qubit frequencies (e.g. 4 GHz). The capacitances could be realized with interdigitated as well as with overlap capacitors and the parallel inductors could be realized with Josephson Junctions in the linear regime since they provide sufficient inductance in a small footprint. Furthermore it is shown in the supplementary material that disorder in these parameters has little effect.

The quantum behavior of the CTL is obtained through canonical quantization of the circuit in Fig. 1. This leads to a system of uncoupled quantum harmonic oscillators, each described by operators \( \hat{a}_n^\dagger \hat{a}_n \) acting on modes with frequencies \( \omega_n \). A qubit described in its energy eigenbasis by Pauli matrices \( \hat{\sigma}_x/z \) placed close to the CTL will couple to mode \( n \) with strength \( g_n \).

\[
\hat{H} = \frac{\Delta_0}{2} \hat{\sigma}_z + \sum_n g_n \hat{\sigma}_x (\hat{a}_n^\dagger + \hat{a}_n) + \sum_n \omega_n \hat{a}_n^\dagger \hat{a}_n. \quad (4)
\]

If the qubit is coupled to the RHTL, \( g_n \approx g_{n+1} \) for low frequency modes since they have similar spatial profiles in the RHTL. For a flux qubit \cite{23, 24}, the mode dependent part of the coupling strength is given by \( D(\omega_n)/I_n(x)/max\{I_n(x)\} \). The current \( I_n \) is averaged over the spatial extent of the qubit. Figure 2 shows that the qubit can be coupled to a wide range of modes. For frequencies sufficiently above ωIR the wavelength in the RHTL also starts to change away from the antinode towards a node, creating a deep minimum in coupling strength. This mode structure allows the qubit to simultaneously couple to multiple modes when \( N > 1 \) modes fall within a frequency interval of \( 2g_n \). We refer to this regime as multi-mode strong-coupling. It can be reached with other superconducting qubits, notably transmons \cite{23}, which should be placed at a charge antinode. Flux qubits on the other hand allow us to reach multi-mode ultrastrong coupling \cite{23, 24} — \( g_n/\omega_n > 0.1 \). This regime offers many new possibilities for circuit QED.

Applications: The multimode Rabi Hamiltonian, equation (4), allows us to prepare multimode entangled states. Within the rotating wave approximation it conserves the number of excitations. Exciting the qubit and placing its 0 ↔ 1 resonance frequency slightly above ωIR allows the qubit excitation to distribute itself over many modes, i.e., produce arbitrary superpositions of the form \( \sum_n c_n |0; n\rangle \). |0; n\rangle indicates the qubit in the ground state, a single photon in mode n and none in the other modes. For |1; 0\rangle only the qubit is excited. These states are in general entangled as seen from their Von Neumann entropy \cite{26}. Figure 3 shows the entropy, indicating multimode entanglement, of a single excitation, starting in |1; 0\rangle, that spread out over all modes for a time \( t \).

The Spin-Boson model \cite{27} is a fundamental model of quantum dissipation which allows to understand the transition between coherent and incoherent behaviour as well as a quantum phase transition suppressing quantum tunneling. It is described by the Hamiltonian in equation (4) in the limit where the modes form a continuum. The dense modes at the low-frequency end provide a generic
and realizable quantum simulator for this model. Our unusual density of modes provides a novel regime of subsubhonic models with a low-frequency cutoff, i.e., a spectral density of the form

\[ J(\omega) = \sum_n g_n^2 \delta(\omega - \omega_n) \approx \frac{N_l}{\pi \sqrt{2\omega_{\text{IR}}}} \Theta(\omega - \omega_{\text{IR}}). \]

The multimode cat state in equation (5) involves, according to the principle of adiabatic renormalization, all fast modes, those with \( \omega_n > \Delta_{\text{eff}} \), as they can adiabatically follow the qubit. Slow modes remain unaffected. Thus

\[ \lambda_n = \frac{g^2}{2} \Theta(\omega_n - \Delta_{\text{eff}}) \]

which leads to a self-consistency relation for \( \Delta_{\text{eff}} \). The ratio \( \Delta_{\text{eff}}/\Delta_0 \) measures the accumulated phase space distance of the dressing clouds, i.e. the total cat size, by taking the logarithm of equation (5). Thus, the low-energy states of the system are strongly renormalized as are their effective energies.

A true dissipative quantum phase transition has \( \Delta_{\text{eff}} = 0 \) in the localised phase. This limit would be reached if the modes were infinitely close (hence arbitrarily close to \( \omega_{\text{IR}} \)) as would result from an infinitely long LHTL or if \( \omega_{\text{IR}} \to 0 \) as in the case of infinitely dense LHTL unit cells. Note that in the usual sub-Ohmic spin-boson model, the latter is assumed. We thus conclude that our system approaches a quantum phase transition in the infinite sample limit.
FIG. 4: (color online). Plot of the renormalized tunnelling element $\Delta_{eff} \propto \langle I \rangle$ as function of the global coupling strength $g$. The individual mode coupling $g_n$ is the product of $g$ and the mode-dependent spatial contribution bounded by unity. The thin dashed lines correspond to the case when this effect is neglected i.e., $\langle I_n(x) \rangle / \max_n \{ \langle I_n(x) \rangle \} = 1$. This allows simultaneous tuning of the coupling to all modes. The discontinuous drop in $\Delta_{eff}$ distinguishes the regions of weakly and strongly renormalized energy splitting.

To corroborate the finite size-behaviour, we have studied the ground and first excited state of the qubit-CTL model using its actual modes in the adiabatic renormalization approach. We identify multiple regimes: for weak coupling or large $\Delta_0/\omega_{IR}$, there is only weak dressing manifest by a small shift of $\Delta_0$. At stronger coupling, we observe the quasi-localized phase, with $\Delta_{eff} \ll \Delta_0$. Remarkably, even at finite length, the two regimes are separated by a discontinuous transition as indicated by Figure 3. Figure 5 shows the corresponding finite-size phase diagram highlighting the need for ultrastrong coupling. We see that by tuning the bare qubit frequency slightly above the cutoff, $\Delta_0 > \omega_{IR}$ we can tune the system through the phase transition in situ, or by employing a tunable coupler. The phase transition is manifest by a discontinuous drop in the energy splitting (as measured through spectroscopy) of the qubit that is inconsistent with the tuning of the circuit alone, see Fig. 4.

Engineering the transmission line to have dense enough modes and an appropriate $\omega_{IR}$ can be accomplished using equation (4) and $\omega_{IR} = 1/2\sqrt{C_J L_j}$. On the level of partition functions, this model is equivalent to a one-dimensional Ising chain $[28, 31]$. This is discussed in more detail in the supplementary material. In the present case, this would be an Ising chain with an interaction that decays as $\propto |i-j|^{-1/2}$, where $i$ and $j$ are site indices, up to a range $r \propto \omega_{IR}^{-1}$, after which it decays exponentially. Thus, when cooled from high-temperatures the system is well described by mean-field theory, which predicts a ferromagnetic phase transition, until the correlation length reaches $r$. At that point, the system follows short-range physics and remains paramagnetic between magnetized blocks of size $r$ — in analogy to the tunnel coupling in the spin-boson model falling deeply, but not to zero.

In conclusion, we have proposed an engineered hybrid transmission line that allows to reach a new multimode strong coupling regime of circuit QED by combining a regular line with a metamaterial. This will open the way for novel applications in microwave photonics and strongly correlated photon states, out of which we have outlined the generation of multimode entanglement, multimode Schrödinger cat states and quantum phase transitions.

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