BOUNDS FOR THE NUMBER OF GALOIS POINTS FOR PLANE CURVES

SATORU FUKASAWA

Abstract. A smooth point on a plane curve is said to be Galois (for the curve) if the projection from the point as a map from the curve to a line induces a Galois extension of function fields. It is known that the number of Galois points is finite except for a certain explicit example. We establish an upper bound for the number of Galois points of non-reflexive plane curves (with respect to the projective dual) other than the example in terms of the genus, degree and the inseparable degree of the dual map, and settle curves attaining the bound.

1. Introduction

In the 1980s, S. Kleiman introduced the notion of the reflexivity and established the theory of projective duality in positive characteristic ([9, 14]). The reflexivity is a significant condition for avoiding pathological phenomena in positive characteristic. On the contrary, it is known that some non-reflexive curves have wonderful properties, also in the theory of Galois points (for example, Homma’s result on Hermitian curves below).

In 1996, H. Yoshihara introduced the notion of Galois point ([17, 23]). Let \( C \subset \mathbb{P}^2 \) be an irreducible plane curve of degree \( d \geq 4 \) over an algebraically closed field \( K \) of characteristic \( p \geq 0 \) and let \( K(C) \) be its function field. A smooth point \( P \in C \) is said to be (inner) Galois for \( C \), if the function field extension \( K(C)/\pi_P^*K(\mathbb{P}^1) \) induced by the projection \( \pi_P : C \dashrightarrow \mathbb{P}^1 \) from \( P \) is Galois. We denote by \( \delta(C) \) the number of Galois points. It would be interesting to settle \( \delta(C) \). For example, there are applications of the distribution of Galois points to finite geometry (see [7, 18]).

When \( p = 0 \), Yoshihara and K. Miura [17, 23] showed that \( \delta(C) = 0, 1 \) or 4 for smooth curves. In \( p > 0 \), for the Fermat curve \( H \) of degree \( p^e + 1 \) (which is sometimes called Hermitian), M. Homma [12] proved that \( \delta(H) = (p^e)^3 + 1 \). Recently, the
The present author showed that \( \delta(C) = 0 \) or \( \delta \) for any other smooth curve \( C \). As a next step, it would be nice to give an upper bound for \( \delta(C) \) for all irreducible plane curves \( C \). Miura gave a certain inequality related to \( \delta(C) \) if \( p = 0 \) and \( d - 1 \) is prime. The present author and T. Hasegawa settled the case \( \delta(C) = \infty \). We call this case (FH). In [5], the present author gave the upper bound \( (d - 1)^3 + 1 \) for all irreducible plane curves \( C \) if \( p \neq 2 \) and \( C \) is not in the case (FH).

Nowadays six types of plane curves \( C \) with \( \delta(C) \geq 2 \) are known (Table in [24]). It is remarkable that, if the generic order of contact \( M(C) \) (i.e. \( I_P(C, T_P C) = M(C) \) for a general point \( P \in C \)) is at least three, there are three types of plane curves with many Galois points (Hermitian, (FH) and curves in Theorem 1.1(2) below, see [4]). If \( M(C) \geq 3 \), then \( p > 0 \), \( M(C) \) is a power of \( p \) and \( C \) is a non-reflexive curve (see Section 2). In this paper, we establish a sharper bound for non-reflexive curves, as follows.

**Theorem 1.1.** Let \( C \subset \mathbb{P}^2 \) be an irreducible plane curve of degree \( d \geq 4 \) and let \( g \) be the geometric genus. Assume that \( M(C) \geq 3 \) and \( C \) is not in the case (FH). Then,

\[
\delta(C) \leq (M(C) + 1)(2g - 2) + 3d.
\]

Furthermore, the equality holds if and only if \( d = p^e + 1 \) for some \( e > 0 \) and \( C \) is projectively equivalent to one of the following plane curves:

1. Fermat curve.
2. The image of the morphism

\[
\mathbb{P}^1 \to \mathbb{P}^2; \quad (s : t) \mapsto (s^{p^e+1} : (s + t)^{p^e+1} : t^{p^e+1}).
\]

Our result can be considered as an application of projective geometry in positive characteristic (including the theories of projective duality and Weierstrass points, Plücker formula and Kaji’s theorem).

Another purpose of this paper is to show that [5, Theorem 1.1] holds also in \( p = 2 \). We give the proof in the final section.

2. Preliminaries

Let \( (X : Y : Z) \) be a system of homogeneous coordinates of the projective plane \( \mathbb{P}^2 \) and let \( C_{sm} \) be the smooth locus of a plane curve \( C \). When \( P \in C_{sm} \), we denote by \( T_P C \subset \mathbb{P}^2 \) the (projective) tangent line at \( P \). For a projective line \( \ell \subset \mathbb{P}^2 \) and a point \( P \in C \cap \ell \), \( I_P(C, \ell) \) means the intersection multiplicity of \( C \) and \( \ell \) at \( P \). We
denote by $\overline{PR}$ the line passing through points $P$ and $R$ when $R \neq P$, and by $\pi_P$ the projection from a point $P \in \mathbb{P}^2$. The projection $\pi_P$ is represented by $R \mapsto \overline{PR}$. Let $r : \hat{C} \to C$ be the normalization and let $g$ be the genus of $\hat{C}$. We write $\hat{\pi}_P = \pi_P \circ r$. We denote by $e_{\hat{R}}$ the ramification index of $\hat{\pi}_P$ at $\hat{R} \in \hat{C}$. If $R = r(\hat{R}) \in C_{sm}$, then we denote $e_{\hat{R}}$ also by $e_R$. It is not difficult to check the following.

**Lemma 2.1.** Let $P \in \mathbb{P}^2$ and let $\hat{R} \in \hat{C}$ with $r(\hat{R}) = R \neq P$. Then for $\hat{\pi}_P$ we have the following.

1. If $P \in C_{sm}$, then $e_{\pi} = I_P(C, TPC) - 1$.
2. If $h$ is a linear polynomial defining $\overline{PR}$, then $e_{\hat{R}} = ord_{\hat{R}} r^* h$. In particular, if $R$ is smooth, then $e_R = I_R(C, \overline{PR})$.

Let $\hat{\mathbb{P}}^2$ be the dual projective plane. The dual map $\gamma : C_{sm} \to \hat{\mathbb{P}}^2$ is given by $P \mapsto T_P C$ and the dual curve $C^* \subset \hat{\mathbb{P}}^2$ is the closure of the image of $\gamma$. We denote by $q(\gamma)$ (resp. $s(\gamma)$) the inseparable (resp. separable) degree of the field extension induced by the dual map $\gamma$ of $C$ onto $C^*$. It is known that $q(\gamma) \geq 2$ if $M(C) \geq 3$ (see, for example, [19 Proposition 1.5]). If $q(\gamma) \geq 2$, then it follows from a theorem of Hefez and Kleiman ([9 (3.4)]) that $M(C) = q(\gamma)$. When $q(\gamma) \geq 2$, $C$ is said to be non-reflexive. For strange curves, see [2, 14]. Note that the projection $\pi_Q$ from a point $Q$ is not separable if and only if $C$ is strange and $Q$ is the strange center.

The order sequence of the morphism $r : \hat{C} \to \mathbb{P}^2$ is $\{0, 1, M(C)\}$ (see [10 Ch. 7], [22]). If $\hat{R} \in \hat{C}$ is a non-singular branch, i.e. there exists a line defined by $h = 0$ with $ord_{\hat{R}} r^* h = 1$, then there exists a unique tangent line at $R = r(\hat{R})$ defined by $h_R = 0$ such that $ord_{\hat{R}} r^* h_R \geq M(C)$. We denote by $T_R C \subset \mathbb{P}^2$ this tangent line, and by $\nu_R$ the order $ord_{\hat{R}} r^* h_R$ of the tangent line $h_R = 0$ at $\hat{R}$. If $\nu_R - M(C) > 0$, then we call the point $\hat{R}$ (or $R = r(\hat{R})$ if $R \in C_{sm}$) a flex. We denote by $\hat{C}_0 \subset \hat{C}$ the set of all non-singular branches and by $F(\hat{C}) \subset \hat{C}_0$ the set of all flexes. We recall the following two facts (see [22, Theorem 1.5], [20]).

**Fact 2.2** (Count of flexes). We have

$$\sum_{R \in \hat{C}_0} (\nu_R - M(C)) \leq (M(C) + 1)(2g - 2) + 3d.$$

**Fact 2.3** (Plücker formula). Let $d^*$ be the degree of the dual curve $C^*$. If $\hat{C}_0 = \hat{C}$ (i.e. $r : \hat{C} \to \mathbb{P}^2$ is unramified), then

$$s(\gamma)q(\gamma)d^* = 2g - 2 + 2d.$$
To settle curves attaining our bound, we also use an important theorem of Kaji [13, 15] on non-reflexive curves.

On a Galois covering of curves, we have the following two facts (see [21, III. 7.2], [10, Theorem 11.91]).

**Fact 2.4.** Let $\theta : C \to C'$ be a Galois covering of degree $d$. We denote by $e_P$ the ramification index at a point $P \in C$. Then we have the following.

1. If $\theta(P) = \theta(Q)$, then $e_P = e_Q$.
2. The index $e_P$ divides the degree $d$.

**Fact 2.5.** Let $p > 0$ and let $\theta : \mathbb{P}^1 \to \mathbb{P}^1$ be a Galois covering of degree $d$. If the index $e_P$ is equal to $p^e \geq 3$ for some $e > 0$ at a point $P$, then $d$ is divisible by $p^e$. Further, if $d > p^e$, then $d/p^e$ is not divisible by $p$ and there exists a point $Q$ such that $e_Q \geq 2$ and $e_Q$ divides $d/p^e$.

We mention two properties of Galois points. We denote by $\Delta \subset \hat{C}$ the set of all points $\hat{P} \in \hat{C}$ such that $r(\hat{P}) \in C$ is smooth and Galois for $C$.

**Lemma 2.6.** [5] Let $P_1, P_2 \in C_{	ext{sm}}$ be two distinct Galois points and let $h$ be a defining polynomial of the line $P_1P_2$. Then, $\ord_{\hat{R}} r^* h = 1$ for any $\hat{R} \in \hat{C}$ with $R = r(\hat{R}) \in P_1P_2$ (maybe $R = P_1$ or $P_2$).

**Proposition 2.7.** Assume that $(d + 2)/2 \leq M(C) \leq d - 1$. Then, $\Delta \subset F(\hat{C})$ and $\delta(C)(d - M(C)) \leq (M(C) + 1)(2g - 2) + 3d$. In particular, if $g = 0$, then $\delta(C) \leq d$.

**Proof.** Let $\hat{P} \in \Delta(C)$ and let $P = r(\hat{P}) \in C_{\text{sm}}$. We prove that $C \cap T_P C = \{ P \}$. Assume by contradiction that $Q \in C \cap T_P C$ and $Q \neq P$. It follows from Lemma 2.1 and Fact 2.3(1) that

$$d \geq M(C) + (M(C) - 1) \geq d + 1.$$

This is a contradiction. Therefore, $I_P(C, T_P C) = d$ and $\hat{P} \in F(\hat{C})$. It follows from Fact 2.2 that

$$\delta(C)(d - M(C)) \leq (M(C) + 1)(2g - 2) + 3d.$$

Let $g = 0$. Since

$$d(d - M(C)) - ((M(C) + 1)(-2) + 3d) = d^2 - 3d + 2 + M(C)(2 - d) = (d - 2)((d - 1) - M(C)) \geq 0,$$
we have $\delta(C) \leq d$.

\section*{3. Proof of Theorem 1.1}

Throughout this section, we assume that $(M(C) + 1)(2g - 2) + 3d \leq \delta(C) < \infty$. If $M(C) = d$, then the present author showed that $\delta(C) = 0$ or $\infty$ in [11] (Classification of curves with $M(C) = d$ by Homma [11] Theorem 3.4) is crucial. Therefore, we have $M(C) < d$. We prove that (i) $\hat{C}_0 = \hat{C}$ if $g \geq 1$, and (ii) $\Delta \subset F(\hat{C})$.

We consider the case where there exists a singular point $Q$ with multiplicity $d - 1$. Then, $\hat{C}$ is rational and $Q$ is a unique singular point. It follows from Bézout’s theorem that $Q \notin T_{p}C$ for any point $P \in C_{sm}$. Since $d > M(C)$, there exists a point $R \in C_{sm} \setminus \{P\}$ with $R \in T_{p}C$ if $I_{p}(C, T_{p}C) = M(C)$. If there exists a point $P \in r(\Delta)$ with $I_{p}(C, T_{p}C) = M(C)$, then it follows from Lemma 2.1 and Fact 2.4(1) that $I_{R}(C, T_{R}C) = M(C) - 1 \geq 2$. This is a contradiction to the order sequence $\{0, 1, M(C)\}$. Therefore, $\Delta \subset F(\hat{C})$.

We consider the case where there exists no singular point with multiplicity $d - 1$. By Proposition 2.7, if $g = 0$ and $M(C) \geq \frac{d+2}{2}$, then $\Delta \subset F(\hat{C})$. Assume that $g \geq 1$ or $M(C) < \frac{d+2}{2}$. Then, we prove that $\hat{C}_0 = \hat{C}$. Let $Q$ be a singular point with multiplicity $m \leq d - 2$. Note that the number of tangent directions at $Q$ is at most $m$. Assume that $Q$ is not a strange center. We prove that any point $\hat{R} \in \hat{C}$ with $r(\hat{R}) = Q$ is a non-singular branch. If there exists a line containing $Q$ and two Galois points, then we have this assertion by Lemma 2.6. Therefore, we consider the case where any line containing $Q$ has at most one inner Galois point.

We consider the projection $\hat{\pi}_{Q}$ from $Q$. Then, the number of ramification points is at most $2g - 2 + 2(d - m)$. If $2g - 2 + 2(d - m) + m \geq (M(C) + 1)(2g - 2) + 3d$, then $g = 0$ and $M(C) \geq \frac{d+2}{2}$. We have $2g - 2 + 2(d - m) + m < (M(C) + 1)(2g - 2) + 3d$. Then, there exist a line $\ell \ni Q$ such that (the point of $\mathbb{P}^{1}$ corresponding to) $\ell$ is not a branch point of $\hat{\pi}_{Q}$, the fiber $r^{-1}(C \cap \ell \setminus \{Q\})$ consists of $d - m$ points, and $\ell$ contains a Galois point $P \in C_{sm}$. In this case, there exists a point $\hat{R} \in \hat{C}$ with $R = r(\hat{R}) \in \overline{PQ} \setminus \{P, Q\}$ such that $\text{ord}_{\hat{R}}r^{*}h = 1$, where $h$ is a defining polynomial of the line $\overline{PR}$, by Lemma 2.1. It follows from Fact 2.4(1) that any point $\hat{R} \in r^{-1}(Q)$ is a non-singular branch.

We prove that $Q$ is not a strange center (under the assumption that any point $\hat{R}$ with $r(\hat{R}) \neq Q$ is a non-singular branch). Assume by contradiction that $Q$ is a strange center. Since the projection $\hat{\pi}_{Q}$ from $Q$ is not separable, $e_{P} = p^{e}m$ for some
integers $e > 0$ and $m > 0$ for each $P \in r(\Delta)$. By Lemma \ref{lemma1}(2), $I_P(C, T_P C) = p^e m$. We consider the projection $\hat{\pi}_P$ from a point $P \in r(\Delta)$. By Lemma \ref{lemma1}(1), $e_P = I_P(C, T_P C) - 1 = p^e m - 1$. On the other hand, $Q \in T_R C$ for any point $\hat{R}$ with $r(\hat{R}) \neq Q$, since any point $\hat{R}$ with $r(\hat{R}) \neq Q$ is a non-singular branch by the above discussion. Therefore, the projection $\hat{\pi}_P$ is ramified only at points in the line $\overline{PQ}$. There exist only tame ramification points for $\hat{\pi}_P$, by Fact \ref{fact2}(1). By Riemann-Hurwitz formula, this is a contradiction. We have $\hat{C}_0 = \hat{C}$ (if $g \geq 1$ or $M(C) < \frac{d+2}{2}$).

Since $d > M(C)$, if there exists a Galois point $P \in C_{sm}$ such that $I_P(C, T_P C) = M(C)$, then there exists a point $R \in C \cap T_P C$. Then, the ramification index $e_R = M(C) - 1 \geq 2$ at $\hat{R}$ for $\hat{\pi}_P$, where $\hat{R} \in \hat{C}$ with $r(\hat{R}) = R$. By $\hat{C}_0 = \hat{C}$ as above, this is a contradiction to the order sequence $\{0, 1, M(C)\}$. We have assertions (i)(ii). By the assumption $\delta(C) \geq (M(C) + 1)(2g - 2) + 3d$ and Fact \ref{fact2}, $\Delta = F(\hat{C})$ and $\delta(C) = (M(C) + 1)(2g - 2) + 3d$.

We consider curves with $\Delta = F(\hat{C})$ and $\delta(C) = (M(C) + 1)(2g - 2) + 3d$. For each $P \in r(\Delta)$, $I_P(C, T_P C) = M(C) + 1$. By Lemma \ref{lemma1}(1) and Fact \ref{fact2}(2), $M(C)$ divides $d - 1$. If $g \geq 1$, then $\hat{C}_0 = \hat{C}$. By Fact \ref{fact3}, $M(C)$ divides $2g - 2 + 2d$. Then, $2g \equiv 0$ modulo $M(C)$. When $g = 1$, $M(C) \geq 3$ does not divide $2g = 2$. We have $g \neq 1$.

We consider the case $g \geq 2$. Assume that $d - 1 > M(C)$. According to Kaji’s theorem \cite{Kaji} Theorem 4.1, Corollary 4.4, the geometric genus of $C^*$ is equal to that of $C$ and $s(\gamma) = 1$. For each $P \in r(\Delta)$, $\gamma(P) \in C^*$ is a singular point of $C^*$ with multiplicity at least $(d - 1)/M(C)$, since the set $r^{-1}(C \cap T_P C)$ consists of $(d - 1)/M(C)$ points and the tangent line at each point of $r^{-1}(C \cap T_P C)$ coincides with $T_P C$. We take $M := M(C)$. By genus formula for $C^*$ \cite{Hartshorne} V. Example 3.9.2, \cite{Kaji} p.135)) and Fact \ref{fact2} we have

$$g \leq \frac{1}{2} \left( \frac{2g - 2 + 2d}{M} - 1 \right) \left( \frac{2g - 2 + 2d}{M} - 2 \right) - \left( (M + 1)(2g - 2) + 3d \right) \times \frac{1}{2} \frac{d - 1}{M} \left( \frac{d - 1}{M} - 1 \right).$$

Then,

$$0 \leq (2g) \{ 2g + (4(d - 1) - 3M) - (M + 1)(d - 1)(d - 1 - M) - M^2 \} + (d - 1 - M) \{ 4(d - 1) - 2M - (3d - 2(M + 1))(d - 1) \}. $$
Since \((d - 1 - M)\{4(d - 1) - 2M - (3d - 2(M + 1))(d - 1)\} < 0\), we have
\[
2g \geq -(4(d - 1) - 3M) + (M + 1)(d - 1)(d - 1 - M) + M^2.
\]
By using the inequality \(2g \leq (d - 1)(d - 2)\),
\[
Md^2 - (M^2 + 3M + 3)d + 2M^2 + 5M + 3 \leq 0.
\]
We have \(d < M + 3 + 3/M \leq M + 4\). Since \(d \geq 2M + 1\), we have \(M < 3\). This is a contradiction. We have \(d - 1 = M(C)\). According to classification of curves with \(M(C) = d - 1\) by Ballico and Hefez [1], \(C\) is projectively equivalent to the Fermat curve.

We consider the case \(g = 0\). If \(d - 1 > M(C)\), it follows from Fact 2.5 and Lemma 2.6 that there exists a point \(\hat{Q} \notin \Delta = F(\hat{C})\) such that \(e_{\hat{Q}}\) divides \((d - 1)/M(C)\) which is not divisible by \(p\). By Lemma 2.1(2), \(M(C) = e_{\hat{Q}}\). This is a contradiction. Therefore, we have \(d - 1 = M(C)\). According to [4] (Classification of curves with \(M(C) = d - 1\) by Ballico and Hefez [1] is crucial), \(C\) is projectively equivalent to the image of the morphism
\[
\mathbb{P}^1 \to \mathbb{P}^2; \ (s : t) \mapsto (s^p + 1 : (s + t)^p + 1 : t^{p+1}).
\]

**Remark 3.1.** The condition \(\Delta \subset F(\hat{C})\) does not hold in general (see [16, Example 1]).

**4. Upper bound in \(p = 2\)**

We can drop the assumption “\(p \neq 2\)” in [5, Theorem 1.1]. Here we give the proof.
Assume that \(p = 2\), \(\delta(C) \geq (d - 1)^3 + 1\), \(C\) is not in the case (FH) and \(C\) is not a Fermat curve. We use [5, Proposition 3.1]. Then, \(d\) is odd, \(M(C) = q(\gamma) = 2\), \(s(\gamma) = 1\) and \(\hat{C}_0 = \hat{C}\). By Fact 2.3 \(d^* = (2g - 2 + 2d)/2\). By Fact 2.2 and Lemma 2.6 we have at least \((d - 1)^3 + 1 - (3(2g - 2) + 3d)\) points \(\hat{P} \in \Delta - r^{-1}(\bigcup_{\hat{R} \in F(\hat{C})} T_{\hat{R}}C)\).
For such a point \(\hat{P}\), there exists a tangent line \(T_{\hat{R}}C\) at some point \(\hat{R} \in \hat{C}\) such that \(T_{\hat{R}}C \ni r(\hat{P})\) coincides with the tangent lines at \((d - 1)/2\) points in \(r^{-1}(C \cap T_{\hat{R}}C)\), by Lemma 2.1(2) and Fact 2.4(1). Therefore, there exists a singular point of \(C^*\) with multiplicity at least \((d - 1)/2\) for each \(\hat{P}\). By genus formula for \(C^*\), we have
\[
g \leq \frac{1}{2} \left( \frac{2g - 2 + 2d}{2} - 1 \right) \left( \frac{2g - 2 + 2d}{2} - 2 \right)
- \{(d - 1)^3 + 1 - (3(2g - 2) + 3d)\} \times \frac{1}{2} \frac{d - 1}{2} \left( \frac{d - 1}{2} - 1 \right).
\]
Substituting $\frac{(d-1)(d-2)}{2}$ for $g$, we have a weak inequality

$$0 \leq g \leq \frac{1}{2} \left( \frac{d^2 - d}{2} - 1 \right) \left( \frac{d^2 - d}{2} - 2 \right) - (d^2 - 6d + 9)d \times \frac{1}{2} \left( \frac{d - 1}{2} - 1 \right).$$

Since $\frac{d^2 - d}{2} - 1 \leq d \times \frac{d-1}{2}$, we have

$$0 \leq \frac{1}{2} \left( \frac{d^2 - d}{2} - 2 \right) - (d^2 - 6d + 9) \times \frac{1}{2} \left( \frac{d - 1}{2} - 1 \right).$$

This implies $d \leq 6$. We have $d = 5$, because $d$ is odd. Coming back to the first inequality for $d = 5$, we have

$$g \leq \frac{1}{2} \left( \frac{2g - 2 + 10}{2} - 1 \right) \left( \frac{2g - 2 + 10}{2} - 2 \right) - \{4^3 + 1 - (3(2g - 2) + 15)\} \times 1.$$

Then, $g \geq 6$. Since $g \leq \frac{(5-1)(5-2)}{2} = 6$, $g = 6$ and $C$ is smooth. It follows from [3, Theorem 3] that this is a contradiction.

As a consequence, we have the following.

**Theorem 4.1.** Let $C \subset \mathbb{P}^2$ be an irreducible plane curve of degree $d \geq 4$ over an algebraically closed field $K$ of characteristic $p \geq 0$. Assume that $C$ is not in the case (FH). Then, $\delta(C) \leq (d - 1)^3 + 1$. Furthermore, $\delta(C) = (d - 1)^3 + 1$ if and only if $p > 0$, $d - 1$ is a power of $p$, and $C$ is projectively equivalent to the Fermat curve.

**Remark 4.2.** For $M(C) = 2$, we can prove that

$$\delta(C) < 3(2g - 2) + 3d$$

holds. Therefore, we can drop the assumption “$M(C) \geq 3$” in Theorem [11] at present, the author’s proof is not concise (in particular, for curves of low genus). It would be nice to give a short proof or a sharp bound.

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