Importance of convective boundary layer flows with inhomogeneous material properties under linear and quadratic Boussinesq approximations around a horizontal cylinder

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A R T I C L E   I N F O

Keywords:
Quadratic Boussinesq approximation
Horizontal cylinders
Inhomogeneous material property

A B S T R A C T

This study investigates boundary layer flows with inhomogeneous material properties driven by natural convection using linear and quadratic Boussinesq approximations around a horizontal cylinder. The cylinder's surface was kept at a uniform temperature. The governing equations for the setup were formulated from the principles of mass continuity, momentum and energy under realistic assumptions. Four coupled partial differential equations were obtained and reduced using stream function to two. Using perturbation techniques with one spatial coordinate as the perturbation parameter, the partial differential equations were further reduced to a set of nonlinear coupled ordinary differential equations. The fluid's velocity, as well as the temperature distributions, were computed and analyzed using the Maple 17 platform. The results obtained were consistent with existing results from reference literature. Further analysis of the embedded flow parameters was also carried out and were compared with and graphical illustrations for the linear and quadratic Boussinesq approximations. The results of the study show a fundamental difference between the linear and quadratic Boussinesq approximation alongside an interconnection between constant and variable thermophysical properties.

1. Introduction

Several industrial systems (e.g. nuclear reactor cores, polymer processing, solar collectors, petroleum reservoirs, etc.) operate on the principle of natural convection flow driven by buoyancy (Umavathi et al., 2016). Most of these systems are known to function optimally at moderate to high temperature differences. While, a linear relationship between density and temperature accounts for the buoyancy when the temperature difference in the system is minimal, it however, fails to accurately describe and justify the nonlinearity in the heat transfer behavior when the temperature variation becomes significantly higher. Several researchers have studied this classical model problem under constant thermophysical properties (that is, constant viscosity and thermal conductivity with negligible heat generation) in different contexts. Some of the important contributions to these model problems where the flow is two-dimensional are listed in Prasad et al., 2014, Zainuddin et al., 2015, Mohamed et al., 2016, Prasad et al., 2016, Zokri et al., 2018, Roy and Gorla, 2019, Roy et al., 2019, Rath and Dah, 2020. There has also been some interest in flow with variable thermophysical properties which is more suggestive of realization under linear Boussinesq approximation. Several papers on natural convection models have been published particularly in the assessment of temperature-dependent thermal conductivity and/or viscosity around a horizontal cylinder under rational assumptions. Some studies conducted, examined variable viscosity models which were inversely related to temperature. However, these studies did not consider the heat source. Various researchers have examined natural convection flow with a temperature-dependent viscosity (Molla et al., 2005, Cheng, 2006, Ahmad et al., 2009, Molla et al., 2012 and Nabil et al., 2014). These studies showed that viscosity-variation parameter influences both the velocity profiles and temperature distributions, which contains limiting cases to some of the earlier fluid models. In addition, a significant amount of studies have been conducted in temperature-dependent viscosity alongside linear temperature variations of thermal conductivity under a negligible heat source (see, Uddin and Kumar, 2009, 2011, Roy et al., 2018).

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https://doi.org/10.1016/j.heliyon.2021.e07074
Received 3 November 2020; Received in revised form 5 February 2021; Accepted 12 May 2021

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Appreciable research has been carried out in the modeling and simulation of problems that describe the flow of Newtonian fluids under both linear heat generation and Boussinesq approximation in several geometries. Natural convection studies with linear heat generation in fluid flow are of significant importance in solving several technological and physical problems. These studies have also attracted a significant research audience (Molla et al., 2006, 2009, Cheng, 2009, Kasim et al., 2011, 2012, Azim and Chowdhury, 2013, Zainuddin et al., 2016, Adeniyan et al., 2021, Opadiran et al., 2021). However, there are limited studies on the natural convection boundary layer flow with constant or inhomogeneous material properties around a horizontal cylinder. At this layer, the density may cause a nonlinear variation with temperature due to temperature changes, a difference in concentration, etc. This nonlinear variation is usually referred to as nonlinear convection or nonlinear Boussinesq approximation. The nonlinear convection strongly influences the fluid flow fields and heat transfer attributes when the temperature difference becomes significantly higher. Analytical or numerical studies have been carried out for different forms of geometries (Goren, 1966, Sinha, 1969, Vajravelu and Sastri, 1977, Rakesh and Shilpa, 2016, Bassant and Bello, 2019, Mahanthesh et al., 2019, Kunnegowda et al., 2019, Thriveni and Mahanthesh, 2020a, b, Mahanthesh et al., 2021).

By adapting the nonlinear Boussinesq approximation to the natural convection flow of an inhomogeneous material properties around a horizontal cylinder with linear heat generation, relevant insight into the flow model can be obtained. However, such a mathematical model has its attendant problem. For instance, there is an increased level of non-linearity. Nonetheless, there has been increasing interest in the study of nonlinear convection due to the increased application of the concept in modern technology e.g. modern heat exchangers with variable viscosity and thermal conductivity.

The important contributions mentioned above study the problem of natural convection flow having a linear density-temperature relationship with constant or inhomogeneous material properties around a horizontal cylinder with/without heat generation. Studies on quadratic Boussinesq approximation have only been considered for other geometries and different flow conditions. Thus, this creates a need for further studies on a horizontal cylinder with quadratic Boussinesq approximation.

So far, no attempt has been made to investigate the heat and mass transfer behavior of nonlinear natural convection flow that has the properties of an inhomogeneous material with linear heat generation around the boundary layer of a heated horizontal cylinder. This study seeks to fill this gap. The stream function has been employed to reduce the four governing partial differential equations into two coupled partial differential equations. The two equations were solved using the perturbation technique and numerically using different numerical tools implemented on the Maple 17 platform. The convergence of the solution is discussed and comparisons are done with existing studies. Numerical results are compared with similar data in the literature and both are found to be similar. Results of the velocity and temperature distributions together with the coefficients of skin friction and heat transfer, the streamlines, isotherms and average Nusselt number are graphically examined for linear and quadratic Boussinesq approximations and variable thermophysical properties. The results, further reveal that the proposed technique is very effective and modest.

2. Model problem statement

Consider a steady natural convective flow; the flow is laminar, two-dimensional, and has incompressible Newtonian fluid in a gravitational field about a horizontal cylinder which is uniformly heated and has radius $R$. The fluid's viscosity, as well as the thermal conductivity, are considered to be dependent on temperature. Fig. 1 illustrates the problem's geometry and coordinates, where the fluid's mainstream temperature is $T_{\infty}$ and the constant surface temperature of the cylinder is $T_w$, where $T_w > T_{\infty}$.

Also, it is assumed that the conventional Prandtl boundary-layer assumptions are applicable to describe the transport phenomena. The flow is assumed to be fully developed and the quadratic Boussinesq approximation is adopted to account for temperature-induced buoyancy effects in the system (see Thriveni and Mahanthesh, 2020a, b, Mahanthesh et al., 2021; for details). The coupled partial differential equations arising from the continuity, balance of linear momentum and conservation of energy with incorporated heat source term, in usual notation, are written as

\[ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \]  
\[ \frac{\rho}{\delta} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \rho \left[ \beta_0 (T - T_{\infty}) + \beta_1 (T - T_{\infty})^2 \right] \sin \left( \frac{\pi x}{R} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right), \]  
\[ \rho C_p \left( \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) + \dot{Q} (T - T_{\infty}). \]  

where the fluid's tangential distance, normal distance, tangential velocity and the normal velocity are $\bar{x}$, $\bar{y}$, $\bar{v}$, and $\bar{w}$, respectively. Here, $\rho$, $T$, $\beta_0$ and $\beta_1$ are the fluid density, fluid temperature, acceleration due to gravity, the coefficients of thermal expansion while $\mu(T)$ and $\kappa(T)$ are the fluid's temperature-dependent model viscosity and thermal conductivity.

The boundary conditions associated with Equations (1)–(3) are:

\[ \bar{u} = 0, \quad \bar{v} = 0, \quad T = T_w \text{ at } \bar{y} = 0, \]  
\[ \bar{u} \to 0, \quad T \to T_{\infty} \text{ as } \bar{y} \to \infty. \]  

According to Uddin and Kumar, 2009, 2011, Akinbobola and Okoya, 2015, Opadiran, 2021, Fatunmbi et al., 2020, Fatunmbi and Okoya, 2020 the fluid’s temperature-related viscosity and thermal conductivity are

\[ \mu = \frac{\mu_0}{1 + a_0 (T - T_{\infty})}, \]  
and
\[ \kappa = \kappa_a \left[ 1 + e \left( \frac{T - T_w}{T - T_\infty} \right) \right] \]  

(7)

Here \( \mu_0 \) is the fluid ambient dynamic viscosity, \( \kappa_a \) is the reference thermal conductivity, \( a_0 \) is the reference viscosity which is a constant and \( e \) is the variable thermal conductivity parameter.

Defining non-dimensional variables

\[
\begin{align*}
    x &= \frac{x}{R}, \quad y = \frac{y}{R}, \\
    \bar{u} &= \frac{uG^{1/2} v}{R}, \quad \bar{v} = \frac{vG^{1/2} v}{R}, \\
    \theta &= \frac{T - T_w}{T_\infty - T_w}, \\
    Gr &= \frac{g\beta(T_w - T_\infty)R^3}{\nu^2},
\end{align*}
\]

(8)

and substituting in Equations (1)–(3), (6)–(7), give

\[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \]  

(9)

\[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \frac{1}{(1 + \omega \theta)^2} \left( \frac{\partial^2 u}{\partial y^2} \right) - \frac{\omega}{(1 + \omega \theta)^2} \left( \frac{\partial \theta}{\partial y} \right) \left( \frac{\partial u}{\partial y} \right) \]  

(10)

\[ \left( \frac{\partial \theta}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \left( 1 + \epsilon \theta \right) \left( \frac{\partial^2 \theta}{\partial y^2} \right) + \frac{1}{P_r(1 + \epsilon \theta)^2} \left( \frac{\partial \theta}{\partial y} \right)^2 + \gamma \theta, \]  

(11)

where \( Gr \) is the Grashof number, the Prandtl number is \( Pr = \mu_0 C_p / \kappa_a \), while the variable convection parameter \( \omega = a_0(T_w - T_\infty) \), the quadratic convection parameter \( \lambda = \frac{a_0^2}{\mu}(T_w - T_\infty) \) and \( y = QR^2 T_w Gr^{1/2} \) is the heat generation parameter, with the associated boundary conditions:

\[ u = 0, \; \nu = 0, \; \theta = 0 \quad \text{at} \quad y = 0, \]  

(12)

\[ u \rightarrow 0, \; \theta \rightarrow 1 \quad \text{as} \quad y \rightarrow \infty. \]  

(13)

Introducing the stream function \( \phi(x, y) \) defined as

\[ u = \frac{\partial \phi}{\partial y}, \quad \nu = -\frac{\partial \phi}{\partial x}, \]  

(14)

such that the prescribed stream function \( \phi \) identically satisfies the continuity Equation (9). Defining the transformations

\[ \phi = x f(x, y) \quad \text{and} \quad \theta = \Theta(x, y), \]  

(15)

give the following coupled nonlinear partial differential equations

\[
\begin{align*}
    \frac{1}{(1 + \omega \theta)^2} \left( \frac{\partial^2 f}{\partial y^2} \right) - \frac{\omega}{(1 + \omega \theta)^2} \left( \frac{\partial \theta}{\partial y} \right) \left( \frac{\partial f}{\partial y} \right)^2 \\
    + f \left( \frac{\partial^2 f}{\partial y^2} \right) + \left( \theta + \lambda \theta \right) \sin x \\
    \quad = x \left( \frac{\partial f}{\partial y} \left( \frac{\partial^2 f}{\partial x \partial y} \right) \right) - \frac{\partial f}{\partial x} \left( \frac{\partial^2 f}{\partial y^2} \right),
\end{align*}
\]

(16)

\[ \left( \frac{\partial \Theta}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \left( \frac{\partial^2 \Theta}{\partial y^2} \right) + \frac{1}{P_r(1 + \epsilon \theta)^2} \left( \frac{\partial \Theta}{\partial y} \right)^2 + f \left( \frac{\partial \Theta}{\partial y} \right) + \gamma \Theta \\
\quad = x \left( \frac{\partial f}{\partial y} \frac{\partial \theta}{\partial y} - \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial y^2} \right). \]  

(17)

And the associated boundary conditions written as

\[ f(x, y) = \frac{\partial f}{\partial y} = 0, \quad \Theta(x, y) = 1 \quad \text{at} \quad y = 0, \]  

(18)

\[ \frac{\partial f}{\partial y} \rightarrow 0, \quad \Theta(x, y) \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty. \]  

(19)

### 2.1. Particular scenarios

Examining the special cases of the resulting coupled system of equations describing the flow setup, the limiting process of constant thermophysical properties (that is, \( \epsilon = \omega = 0 \)) and linear convection parameter (\( \lambda = 0 \)) has been studied by Molla et al. (2006) and Azim and Chowdhury (2013). Also, the limiting case of constant thermal conductivity and linear convection parameter term (that is, \( \epsilon = \lambda = 0 \)), was investigated by Azim (2014) under negligible variations in the fluid viscosity. The same limit \( \epsilon = \lambda = 0 \) is discussed under varied viscosity by Molla et al. (2005).

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**Fig. 2.** Framework for the adaptive numerical procedure using the Mid-point method with Richardson extrapolation technique.

### 3. Method of solutions

The solutions of the nonlinear boundary value problem consisting of Equations (16)–(19) were obtained using the procedure in the program flow chart (see Fig. 2). The flow fields are written as

\[ f(x, y) = \sum_{n=0}^{\infty} x^n f_n(y), \quad \Theta(x, y) = \sum_{n=0}^{\infty} x^n \Theta_n(y), \]  

(20)

where \( x \) is the perturbation parameter.

This particular perturbation method was selected because it is derived from rational systematic steps. It has a small parameter built-in and it offers quick and good approximation whilst also offering an easy computation of streamlines and isotherms.

Substituting the expansions \( f \) and \( \Theta \), and collecting terms of the same order, the following nonlinear coupled ordinary differential equations are obtained (see the supplementary file for details). Considering the appropriate boundary conditions, the equations are written as

\[ \Theta^{(0)} \]  

\[ f^{(0)} + \omega f^{(0)} \Theta^{(0)} + \omega f^{(0)} \Theta^{(0)} = 0, \]  

(21)

\[ 1 \left( \Theta^{(0)} + \epsilon \Theta^{(0)} \right)^2 + f \Theta^{(0)} + \gamma \Theta = 0, \]  

(22)

\[ f^{(0)}(0) = f^{(0)}(0) = 0, \quad \Theta^{(0)}(0) = 1, \quad f^{(0)}(\infty) = \Theta^{(0)}(\infty) = 0, \]  

(23)

\[ \Theta^{(1)} \]  

\[ f^{(1)} + \omega f^{(1)} \Theta^{(1)} + f^{(1)} \Theta^{(0)} - f^{(1)} \Theta^{(0)} = 0, \]  

(24)

\[ 1 \frac{1}{P_r} \left( \Theta^{(1)} + \epsilon \Theta^{(1)} \right)^2 + f \Theta^{(1)} + \gamma \Theta = 0, \]  

(25)
\[ \nu \frac{f''(0)}{f'(0)} = 0, \quad \frac{\partial f}{\partial y}(\infty) = \frac{\partial f_1}{\partial y}(\infty) = 0, \quad (26) \]

\[ \mathcal{O}(x^2): \quad f'''' + \alpha (f'''' \theta_1 + f''' \theta_0 + f'' \theta_2 - f'' \theta_0' - f' \theta_0'' - f' \theta_0''') + 2 \alpha \theta_0(1 + \alpha \theta_0)(2 \alpha f_0 + f_1 + f_2 f_1') + f_2 f_1 - 3 f_1 f_2' = 0, \quad (27) \]

\[ f_1(0) = f_1''(0) = 0, \quad \theta_1(0) = 0, \quad f_1'(\infty) = \theta_1(\infty) = 0. \]

\[ \mathcal{O}(x^3): \quad f'''''' + \alpha (f'''''' \theta_1 + f''''' \theta_0 + f'''' \theta_2 - f'''' \theta_0' - f'' \theta_0'' - f'' \theta_0'''') + 2 \alpha \theta_0(1 + \alpha \theta_0)(2 \alpha f_0 + f_1 + f_2 f_1' + 3 f_1 f_2') + f_2 f_1' + 4 f_2 f_1' - 4 f_1 f_2' = 0, \quad (28) \]

\[ f_2(0) = f_2''(0) = 0, \quad \theta_2(0) = 0, \quad f_2'(\infty) = \theta_2(\infty) = 0, \quad (29) \]

\[ \mathcal{O}(x^4): \quad \ldots \]

\[ \mathcal{O}(x^5): \quad \ldots \]

\[ \mathcal{O}(x^6): \quad \ldots \]

\[ \mathcal{O}(x^7): \quad \ldots \]

\[ \mathcal{O}(x^8): \quad \ldots \]

\[ \mathcal{O}(x^9): \quad \ldots \]

Here, primes signify differentiation with respect to the dimensionless variable \( y \).

In practice, quantities that are of major interest are the wall shear stress and the heat transfer coefficients expressed as coefficient of skin-friction, local Nusselt number \( Nu \) on the surface of the cylinder and the average Nusselt number \( Nu_{L} \). The average Nusselt number is obtained by integrating \( Nu \) over the surface of the cylinder, stated as:

\[ C_f = \frac{\tau_w}{\frac{1}{2} \rho U_w^2}, \quad Nu = \frac{Re_d}{k(T_w - T_\infty)} \quad \text{and} \quad Nu_{L} = \frac{1}{\frac{1}{2} \int_0^x Nu \, dx}, \quad (30) \]

where

\[ \tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} \quad \text{and} \quad q_w = -k \left( \frac{\partial \theta}{\partial y} \right)_{y=0}, \quad (31) \]

\( \tau_w \) is prescribed as shear stress whereas \( q_w \) is the heat flux and \( U_w \) is the dimensionless mainstream velocity, which for natural convection is given as:

\[ U_w^2 = g(\beta(T_w - T_\infty))R. \quad (32) \]

To this end, Equations (30)–(32) with the dimensionless variables and parameters yield

\[ C_f \, Gr^{1/4} = \frac{x}{1 + \alpha} f''(x, 0), \quad Nu \, Gr^{-1/4} = -(1 + \epsilon) \theta'(x, 0). \quad (33) \]

Applying the series expansion in Equations (20) to Equation (33) results in

\[ C_f \, Gr^{1/4} = \frac{x}{1 + \alpha} f''(x, 0) = \frac{x}{1 + \alpha} \left( f''(0) + x f''(0) + x^2 f''(0) + x^3 f''(0) + \ldots \right), \quad (34) \]

and

\[ Nu \, Gr^{-1/4} = -(1 + \epsilon) \theta'(x, 0) = -(1 + \epsilon) \theta'(0) + x \theta'(0) + x^2 \theta'(0) + x^3 \theta'(0) + \ldots, \quad (35) \]

respectively. The exact solutions to the coupled Equations (21)–(29) are not feasible, therefore numerical techniques are adopted as the preferred solution approach.

### 3.1. Implementation, convergence and validation of solutions

In this section, Equations (21)–(29) were solved numerically on the Maple 17 computing platform. Maple uses the well-known Midpoint method with the Richardson extrapolation technique to obtain numerical solutions. Since numerical methods are used, it is expected that errors from rounding off and truncation errors will occur. The numerical solutions start from the lower stagnation point of the cylinder, that is, where the curvature parameter \( x = 0 \), round the cylinder to the upper stagnation point where \( x \approx x \). In terms of \( y \), it is easily demonstrated that \( y \rightarrow \infty \) is truncated at 5 (or 6).

Tables 1 and 2 describe the nature of the nth partial sum of the series (that is, n-term approximation (App.)) representing the flow velocity and temperature, respectively. Extensive computation reveals that the solution converges and this is quite clear from the number of decimal places in the tables. Following the tables provided, it is evident that the sums of the series generated for both velocity and temperature converge to at least 6 decimal places for a 5-term approximation when compared with a 7-term approximation. Therefore, the subsequent analysis is based on the 5-term approximation and \( y \rightarrow \infty \) is replaced with a finite value of \( y = 5 \) or \( y = 6 \). It is confirmed here that further increase beyond \( y = 5 \) or \( y = 6 \) does not significantly change the solution of velocity and temperature profiles.
Table 3. Skin friction and heat transfer coefficients for \( P_r = 1.0, \gamma = \epsilon = \omega = 0 \) and \( \lambda = 0 \) for linear case.

| \( x \) | Skin friction coefficient | Heat transfer coefficient |
|-------|--------------------------|--------------------------|
|       | Present | Molla et al., 2006 | Error | Present | Molla et al., 2006 | Error |
| 0     | 0.0000 | 0.0000 | - | 0.4220297 | 0.4216 | 0.10% |
| \( x/6 \) | 0.4118749 | 0.4144 | 0.61% | 0.4168703 | 0.4164 | 0.11% |
| \( x/3 \) | 0.7494455 | 0.7544 | 0.66% | 0.4012814 | 0.4009 | 0.095% |
| \( x/2 \) | 0.9495175 | 0.9550 | 0.58% | 0.3749311 | 0.3751 | 0.045% |
| \( 2x/3 \) | 0.9711171 | 0.9701 | 0.10% | 0.3372663 | 0.3389 | 0.48% |
| \( 5x/6 \) | 0.8066012 | 0.7824 | 3.00% | 0.2875123 | 0.2923 | 1.64% |
| \( x \) | 0.4927671 | 0.4125 | 16.28% | 0.2246733 | 0.2354 | 4.56% |

Table 4. Skin friction and heat transfer coefficients for \( P_r = 1.0, \gamma = \epsilon = \omega = 0 \) and \( \lambda = 0 \) for linear case.

| \( x \) | Skin friction coefficient | Heat transfer coefficient |
|-------|--------------------------|--------------------------|
|       | Present | Kasim et al., 2011 | Error | Present | Kasim et al., 2011 | Error |
| 0     | 0.0000 | 0.0000 | - | 0.4220297 | 0.421411 | 0.14% |
| \( x/6 \) | 0.4118749 | 0.4102366 | 0.12% | 0.4168703 | 0.416371 | 0.12% |
| \( x/3 \) | 0.7494455 | 0.751733 | 0.30% | 0.4012814 | 0.401111 | 0.04% |
| \( x/2 \) | 0.9495175 | 0.955196 | 0.59% | 0.3749311 | 0.375214 | 0.07% |
| \( 2x/3 \) | 0.9711171 | 0.977381 | 0.64% | 0.3372663 | 0.337486 | 0.07% |
| \( 5x/6 \) | 0.8066012 | 0.786216 | 0.03% | 0.2875123 | 0.283199 | 1.5% |

Fig. 3. The variation of velocity for (a) linear and (b) quadratic models.

To assess the efficiency of the present numerical method, results of the coefficients of skin friction and that of the heat transfer are compared with those produced by Molla et al. (2006) and Kasim et al. (2011). Tables 3 and 4 present the solutions of skin friction \( (C_fGr^{1/4}) \) together with that of heat transfer coefficient \( (NuGr^{-1/4}) \) and the error margin is less than 5%. In Table 3, it was observed that the error margin is higher at the last entry of the skin friction at \( x = \pi \). This gives us concern for future work.

3.2. Discussion of results

Each computational run has at least four unknowns: velocity \( f' \), temperature \( \theta \), skin friction coefficient \( (C_fGr^{1/4}) \) and heat transfer coefficient \( (NuGr^{-1/4}) \). All numerical computations are executed on HP Pavilion personal computer of Intel core(TM) i7-5500U CPU with 2.40 GHz 2.39 GHz processor and 12 GB RAM. Extensive computations for Tables 1–3 and case 1 are run for an average of 109 (or 120) seconds (simulation time) to obtain solutions of the coupled nonlinear ordinary differential equations, plot the graphs and generate the numerical values for comparisons, while the real time taken to run the computations is about \( 2^\frac{1}{4} \) (or 3) minutes for the linear (or quadratic) model, respectively. Similarly, further computations of Tables 1–3 and case 2 follow the same simulation time and real time.

The results of the numerics are displayed in Figs. 3, 4, 5, 6 and 7, 8, 9, 10, for the plots of non-dimensional velocity \( (f') \), dimensionless temperature \( (\theta) \), skin friction \( (C_fGr^{1/4}) \) and heat transfer \( (NuGr^{-1/4}) \) coefficients against \( y \) and \( x \), respectively highlighting the various values of \( \epsilon, \omega \) and \( \lambda \in [0, 1.5] \) under linear and quadratic Boussinesq approximations. The following constant values are assumed for the pertinent parameters except where stated as varying parameters; \( \omega = \epsilon = \gamma = 0.5 \) and \( P_r = 0.71 \).

The plots below are also computed using subscript (a) with the corresponding (b) for each figure which solved the linear and quadratic Boussinesq approximation problems under variable thermophysical properties. The responses associated with the linear and quadratic Boussinesq approximations were found to be qualitatively similar with the quadratic case showing significant increase across the \( x \) and \( y \) axes.

3.2.1. Case 1: Influence of variable viscosity

Variation of \( f', \theta \) and \( C_fGr^{1/4}, NuGr^{-1/4} \) with \( y \) and \( x \), respectively generated by the program executed for the study are shown in Figs. 3–6. Also, the influence of linear and quadratic Boussinesq approximations,
Fig. 4. The plots of temperature for (a) linear and (b) quadratic models.

Fig. 5. The shape of skin friction coefficient for (a) linear and (b) quadratic models.

Fig. 6. The heat transfer coefficient for (a) linear and (b) quadratic models.
the viscosity variation parameter $\omega$ on the fluid variables, and emerging physical quantities in the boundary layer are plotted in Figs. 3–6.

Fig. 3 presents a comparison of the dimensionless velocity against $y$ for both linear and quadratic Boussinesq approximations. As seen from Fig. 3, the velocity varies from zero, increases rapidly to a maximum positive value and then slightly decreases maintaining zero value up to infinity. Also, it can be seen that, the more the viscosity effect, the higher the peak value of the velocity in the boundary layer. The maximum velocity increases as the quadratic convection parameter $\lambda$ increases and the linear case ($\lambda = 0$) serves as the lower bound. This feature is common to viscosity that is proportional to the inverse of temperature (for example, see Fig. 2a in Uddin and Kumar, 2009).

Fig. 4 presents the numerical solution of the dimensionless temperature against $y$ for both linear and quadratic Boussinesq approximations. From Fig. 4, it is seen that more heat is generated in the linear convection parameter $\lambda = 0$ than the quadratic convection parameter $\lambda \neq 0$ in the thermal boundary layer. Comparing $\lambda = 0$ and 1.5 indicate that $\theta$ seems to be far less sensitive to a change in $\omega$. Further, the fluid temperature is seen to slightly decrease as the viscosity variation parameter $\omega$ increases while the constant viscosity case serves as the upper bound.

The skin friction coefficient as seen in Fig. 5 starts building up as the dimensionless coordinate $x$ increases within the interval $0 \leq x \leq 2$ to attain a maximum value at $x = 2$ and then decreases when $x > 2$ for both linear and quadratic convection terms. The effect of the increase in $\omega$ suppresses the coefficient of skin friction while the constant viscosity case serves as the upper bound for both linear and quadratic convection parameters. The results show a marked increase in the skin friction ($C_fGr^{1/4}$) with an increase in $\lambda$.

The effect of varying both $\omega$ and $\lambda$ is shown in Fig. 6 for the variation of $NuGr^{-1/4}$ versus $x$. Increasing $\lambda$ and $\omega$ significantly enhances the heat transfer coefficient. In addition, the constant viscosity case serves as the lower bound for both linear and quadratic convection terms. These results agree with the results from a similar study by Molla et al. (2006) and Azim and Chowdhury (2013) only for the case of constant viscosity ($\omega = 0$) and linear heat source.

### 3.2.2. Case 2: Effects of thermal conductivity

To further understand the thermal influence on the flow profile, it is imperative to examine the variation of the thermal conductivity parameter. With this in mind, variations of $f'$, $\theta$, and $C_fGr^{1/4}$, $NuGr^{-1/4}$ as a function of $y$ and $x$, respectively generated by the program executed for the study are shown in Figs. 7–10. Figs. 7–10 exemplify the combined effect of increased thermal conductivity and Boussinesq approximation parameters on the fluid variables and emerging physical quantities. The set of Figs. 7a, 8a, 9a, 10a and 7b, 8b, 9b, 10b illustrate Boussinesq plots using the linear model, $\lambda = 0$, and the quadratic model, $\lambda \neq 0$, respectively.

The velocity increases with the increase in thermal conductivity parameter while the constant thermal conductivity case serves as the lower-bound as depicted in Fig. 7 for both linear and quadratic Boussi-
nesq types. This feature is common to linear temperature-dependent thermal conductivity (for example, see Fig. 5a of Uddin and Kumar, 2009). As the Boussinesq approximation parameter increases, it causes the maximum velocity of the fluid to increase for a fixed thermal conductivity parameter. Therefore, the linear Boussinesq approximation serves as the lower bound.

The fluid’s thermal conductivity also causes the dimensionless temperature as a function of \( y \) to increase, as shown in Fig. 8, which is evident for both linear and quadratic Boussinesq terms. This is because thermal conductivity describes the ease of heat flow out of the fluid. Therefore, it is expected that the heat from the cylinder will be transported fast into the main-stream with an increase in the thermal conductivity. This makes the system reach thermal equilibrium fast (see Fig. 5 which mirrors that depicted in Fig. 5b of Uddin and Kumar, 2009). In the event of a constant thermal conductivity, this acts as the lower bound of the linear and quadratic Boussinesq sources. It is observed that the heat at the boundary layer is higher for the linear model as compared to the quadratic model Boussinesq approximation.

The plots of the profiles for the coefficient of skin friction against \( x \) are illustrated in Fig. 9 for both the linear and quadratic Boussinesq approximation models. The profiles of the two mathematical models are identical. The only notable difference is that the quadratic Boussinesq approximation produces a significant increase compared to the linear case. Also, as the fluid’s thermal conductivity parameter increases, the maximum coefficient of skin friction increases. Variation of thermal conductivity parameter is marginal for the linear model compared with the quadratic model. The constant thermal conductivity parameter case serves as the lower bound. For both models, the different maximum value of the coefficient of skin friction occurs at \( x = 2 \).

The heat transfer coefficient versus \( x \) for linear and quadratic Boussinesq approximation models are displayed in Fig. 10. It is noted that when the fluid’s thermal conductivity rises, the heat transfer coefficient increases for both linear and quadratic Boussinesq approximation models. Here, the heat transfer coefficient rises abruptly as the heat generation increases.

The next section determines the streamlines, isotherms, and average Nusselt number in terms of the inhomogeneous material properties of linear and quadratic Boussinesq models, respectively, to showcase the fluid properties.

4. Numerical model of streamlines, isotherms, and average Nusselt number

To study the streamlines and isotherms of the problem model, the initial model results allowed for progressive refinement of the numerical scheme contained in Fig. 2 to handle Equation (15) using several values of stream and temperature functions. This was implemented on Maple 17 which was used for finding roots of a polynomial in \( x \) for a fixed value of \( y \) when Equation (20) is substituted into Equation (15). Each step involved increasing the complexity and tackling with ingenu-
ity. The accuracy of these numerical experiments agrees qualitatively with that of Molla et al. (2006) under constant thermophysical properties and linear Boussinesq approximation.

4.1. Further discussion of model results

The streamlines and Isotherms properties for the linear and quadratic Boussinesq approximations for various thermophysical properties are shown in Figs. 11–16.

4.1.1. Case 3: Effects of \( \omega \) and \( \epsilon \) on the development of the streamlines

Figs. 11–13 depict the effects of convection parameter \( \lambda \), variable viscosity parameter \( \omega \) and variable thermal conductivity \( \epsilon \) on the behavior of the streamlines. The dimensionless value of the maximum streamline (\( \psi_{\text{max}} \)) within the regime of computation for Figs. 11–13 is located at the upper stagnation point (\( x = \pi \)) of the cylinder. Furthermore, it should be pointed out that the profiles in Figs. 11–13 clearly show that \( \psi_{\text{max}} \) started to increase from linear model (lower bound) to quadratic model. As is observed, Fig. 11 (constant viscosity) and Fig. 12 (variable viscosity) for a fixed value of \( \epsilon \) successfully depict the trends of the variation of streamlines for both linear and quadratic models. The \( \psi_{\text{max}} \) for the constant (or variable) viscosity case serves as the upper (or upper) bound for the linear (or quadratic) model. It is clear from the variation of viscosity that \( \psi_{\text{max}} \) changes marginally for linear Boussinesq approximation. In addition, it should be noted that Fig. 13 (constant thermal conductivity) and Fig. 12 (variable thermal conductivity) for a fixed value of \( \omega \) successfully portray the streamlines evolution for both linear and quadratic models. The \( \psi_{\text{max}} \) for both constant and variable thermal conductivity serves as the lower bound for both linear and quadratic Boussinesq approximations. This can be confirmed from the variation of thermal conductivity that \( \psi_{\text{max}} \) is enhanced (compare Fig. 13 and 12 for fixed \( \omega \)).

4.1.2. Case 4: Effects of \( \omega \) and \( \epsilon \) on the development of the isotherms

Figs. 14–16 show the effects of convection parameter \( \lambda \), variable viscosity parameter \( \omega \) and variable thermal conductivity \( \epsilon \) on the development of the isotherms. The isotherm patterns in Figs. 14–16 reveal that the development of the thermal boundary layer across the cylinder’s surface is significant for both zero and nonzero physical parameters \( \omega \), \( \epsilon \) and \( \lambda \) (which follows the variation of heat generation parameter previously reported for constant thermophysical properties fluid flow by Molla et al., 2006). It is important to note that all the linear models reveal that as \( x \) increases from the lower stagnation point (\( x = 0 \)), the heated fluid is observed to rise upward owing to the effect of buoyancy.
Fig. 13. Streamlines involved in the (a) linear and (b) quadratic models if $\omega = 0.5$ and $\epsilon = 0$.

Fig. 14. Isotherm rendering of the system for (a) linear and (b) quadratic models when $\omega = 0$ and $\epsilon = 0.7$.

Fig. 15. Typical Isotherm curves for (a) linear and (b) quadratic models when $\omega = 0.5$ and $\epsilon = 0.7$. 
4.1.3. **Case 5: Effects of \( \omega \) and \( \epsilon \) on the average Nusselt number**

In Figs. 17 and 18, the effects of the variable viscosity \( \alpha \) and variable thermal conductivity \( \epsilon \) parameters on the average Nusselt number are presented. Fig. 17 depicts the plots of average Nusselt number defined in Equation (30) for linear \( (\lambda = 0) \) and quadratic \( (\lambda \neq 0) \) convection parameters. While the plots show the computed results, the average Nusselt number trajectory as a function of viscosity parameter \( \omega \) shown in Fig. 17 is observed to increase monotonically. Finally, Fig. 18 shows the value of the model average Nusselt number versus thermal conductivity parameter for linear and quadratic models. As is observed, the linear model successfully depicts the trends of the average Nusselt number which increases with the thermal conductivity parameter while the reverse is the case for the corresponding quadratic model.

5. **Conclusion**

The natural convective flow with inhomogeneous material properties for linear and quadratic Boussinesq approximations was analyzed using series and numerical methods. The sum of the first five terms of the two-state variables provided reliable results which were noted to converge rapidly. Validation of the numerical results obtained was carried out using direct comparison with obtainable data from previous studies on both skin friction and heat transfer coefficients. The numerical results obtained agreed with previous studies showing a difference of less than 5\%. Numerical solutions of the two Boussinesq approximation types (linear and quadratic) are provided. The results obtained in this study show that the variations of viscosity, thermal conductivity, as well as the quadratic convection parameter, play a fundamental role in the heat transfer process and the following significant conclusions were made:

1. The qualitative behavior of linear and quadratic Boussinesq approximations are similar for the variation of physical parameters.
2. Numerical results obtained indicate that the flow velocity and heat transfer coefficient (or the temperature as well as the skin friction coefficient) rise (or decrease) with increasing \( \omega \).
3. The variation of \( \omega \) indicates that an increase in the quadratic convection parameter, \( \lambda \) enhances the flow velocity, skin friction coefficient, heat transfer coefficient while the temperature decreases.
Increasing the thermal conductivity parameter leads to an increase in the flow velocity, temperature, skin friction coefficient, and heat transfer coefficient for both linear and quadratic Boussinesq models.

The variation of thermal conductivity shows that increasing $\lambda$ enhances the flow velocity, skin friction coefficient, heat transfer coefficient, while the temperature reduces.

Under the influence of both variable viscosity and thermal conductivity, the different maximum value of the skin friction coefficients occurs at $x = 2$.

The analysis showed that the effect of thermal conductivity is to enhance $\psi_{\text{max}}$ for both linear and quadratic models.

The effect of variation of viscosity is to marginally decrease $\psi_{\text{max}}$ for the linear model while the reverse is the case for the quadratic model, that is, $\psi_{\text{max}}$ rises appreciably.

The influence of viscosity and thermal conductivity reduces the thickness of the thermal boundary layer for both linear and quadratic models.

Increasing the viscosity (or thermal conductivity) parameter leads to an increase in the average Nusselt number for both the linear and quadratic models (or linear models), respectively.

Finally, while this investigation and the supporting numerical results have been primarily theoretical, the result obtained have a few industrial applications. The temperature being lower for the quadratic Boussinesq model than the linear model indicate application in the production of better lubricants to reduce the wear of machinery. Furthermore, the present numerical results may be used in the industry by thermal design engineers since the variable thermal conductivity attain optimal efficiency than constant thermal conductivity.

Future modification of the study may be carried out considering heat and mass transfer of natural convective boundary layer flows with or without inhomogeneous material properties using the density variation to be a quadratic function of both temperature and concentration.

**Funding statement**

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

**Data availability statement**

Data will be made available on request.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

Supplementary content related to this article has been published online at https://doi.org/10.1016/j.heliyon.2021.e07074.

**Acknowledgements**

The authors take great pleasure in thanking the referees for valuable comments and suggestions which were very helpful in revising this work.

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