STATE BOUNDING FOR TIME-DELAY IMPULSIVE AND SWITCHING GENETIC REGULATORY NETWORKS WITH EXOGENOUS DISTURBANCE

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ABSTRACT. This paper focuses on the state bounding problem for the time-delay impulsive and switching genetic regulatory networks (ISGRNs) with exogenous disturbances. Firstly, a sufficient criterion for the state bounding is obtained such that all the trajectories of ISGRNs under consideration converge exponentially into a sphere on the basis of an average dwell time (ADT) switching. Besides, globally exponential stability conditions for the considered system are further stated when the exogenous disturbance vanishes. As a special case, the equivalent state bounding criteria are established by using the properties of some special matrices when there exist no impulses at the switching instants in ISGRNs. Finally, an illustrating example is given to demonstrate the derived results. Compared with the existing literatures, the considered genetic regulatory networks (GRNs) have more general structure and the approach adopted in the present paper is more simple than Lyapunov-Krasovskii functional (LKF) approach.

1. Introduction. Past years have witnessed the rapid development of complex science. As a kind of complex dynamical systems, GRNs including proteins, RNA, DNA, small molecules, and the biochemical processes have received considerable attention from several research fields such as biochemical science, mathematics, and physics [1, 6, 28, 29, 27]. There are some different models describing GRNs, among which the differential equation is an important studied object. Studying the differential equation model of GRNs not only can grasp the dynamical evolution of the products’ concentrations in genetic activities, but also contributes to analyzing the real process of genetic regulatory. In particular, many theoretical problems on
different differential equations of GRNs have been carried out, e.g., stability analysis [29], filtering [24], bifurcation [30], and reachable set estimation [26], etc. On other hand, time delay frequently appears in a practical dynamical system, so theoretical research for systems with time delay is significant. Taking a slow biochemical process into account, time delay that is an essential aspect for the evolution of the biosystem has been investigated in terms of different kinds of GRNs [1, 6, 30, 26, 35, 23, 34].

It is noted that hybrid impulsive and switching systems are widely used to describe the real-world processes that undergo switches and abrupt jumps in states [4, 12, 14, 15, 8, 5, 21]. Recently, the method of ADT is adopted to study the dynamical behaviors for such hybrid systems. For example, several sufficient conditions ensuring ISS for a switching system under several kinds of impulses were obtained in [12]. By introducing a novel ADT approach, an asynchronous filtering problem was investigated for a class of discrete-time impulsive switched systems in [14]. In [8], some synchronization conditions for an impulsive and switching dynamical network were developed by applying an average impulsive dwell-time method. Notably, some researchers have focused on the dynamic behavior for a time-delay GRN with impulsive effects or switching mechanisms because of the abrupt change of model parameters and system state [20, 29, 9, 6, 19, 10]. In particular, by employing a new LKF and the PDT method, some exponential stability criteria were presented for the discrete time-delay switched GRNs in [29]. Finite-time stability criteria for a time-delayed GRN with impulsive disturbances were derived with respect to several linear matrix inequalities (LMIs) by the method of LKF in [20]. Recently, some dynamical behaviors on the GRN with impulsive or switching effects can be found in [9].

Among the dynamical behaviors, the reachable set or state bounding for a system has attracted much attention. This problem not only is an essential aspect in the theory of robust control [2], but also is a precondition for some engineering applications in the controller design [16] and safety verification [25]. Nowadays, some researchers focus on time-delay systems with bounded exogenous disturbances. Under this situation, it is usually difficult to achieve the asymptotic stability. Thus, these researchers intend to investigate the reachable set or state bounding for a practical time-delay system. The main object is to obtain a closed bounded set such that all the system trajectories converge into it. In terms of the research methods, a commonly used approach is based on the Lyapunov technique [2, 31, 11]. For instance, a delay-dependent sufficient criterion involving a LMI was derived via Lyapunov-Razumikhin approach in [2]. To investigate the reachable state set of the distributed delay linear systems, an ellipsoid was derived such that the reachable set was contained via the delay partitioning method and LKF technique in [31]. Recent years, a different approach has been introduced to find a special sphere such that all the solutions exponentially converge within it for a linear time-varying system with time-varying delay [7, 22]. This method does not rely on the LKF-based technique, and meanwhile the properties of some special matrices including Metzler/nonnegative matrices are used to estimate the system state. Later, this approach is further developed to nonlinear systems [17], fuzzy systems [3], and hybrid systems [33, 32].

Based on the above observations, it can be seen that the time-delay GRNs with impulse or switching have been rather extensively investigated in terms of several aspects. However, the problem of state bounding for a time-delay ISGRN with exogenous disturbances has not been involved so far. From [26], one can observe
that the solution of a time-delay GRN with exogenous disturbances can converge exponentially into a sphere when the system datas satisfy some condition. But, it should be noted that, when the impulsive and switching effect is involved, the solution properties will be affected by the impulsive instant, the impulsive strength, and the switching rule. Thus, to ensure that a perturbed time-delay ISGRN will converge exponentially into a sphere, one should explore some effective conditions about the impulsive and switching function. On other hand, when the Lyapunov technique is used [2, 31, 11], constructing auxiliary function and LMIs is required. This is not a easy work and may lead to complex calculations. Thus, it is necessary to apply some alternative approach to discuss the state bounding of time-delay ISGRNs with exogenous disturbances. These motivate us to investigate some essential state bound conditions for the underlying system. It is worthy to highlight the following innovation points.

1) By using a direct method and the ADT technique, an explicit state bounding is derived such that all the trajectories of perturbed ISGRNs converge exponentially into a sphere. With the aid of the Metzler and nonnegative matrices, the equivalent state bounding criteria are further established when the impulses in ISGRNs vanish. In comparison with some existing literatures [20, 19, 10], the adopted method avoids the construct of LKF or auxiliary function, and hence reduces the computational complexity.

2) The state bounding problem is extended to ISGRNs as a time-delay hybrid nonlinear impulsive and switching system compared to [7, 33]. Different from some extensively studied time-delay GRN model [24, 26], the introduced ISGRN model is more general when the dynamics of mRNAs and proteins are simultaneously subjected to the impulsive effects, arbitrary switching, and time-varying delays. The obtained findings include the special cases for a GRN without impulse and switching, e.g., [26].

The remainder of this paper is comprised of four sections. In Section 2, the notations and ISGRNs are presented. In Section 3, the state bounding estimation of ISGRNs is studied and several sufficient conditions are obtained. A numerical example in Section 4 verifies the developed theoretical results. Finally, this paper is concluded in Section 5.

2. Notations and ISGRNs. Notations: Let \( \mathbb{Z}, \mathbb{R}, \mathbb{R}^n \), and \( \mathbb{R}^{n \times d} \) be respectively, the sets of integers, real numbers, \( n \)-dimensional column vectors, and \( (n \times d) \)-dimensional matrices. Denote \( \mathbb{R}_{\geq 0} \ (\mathbb{R}_{> 0}) \) the set of nonnegative real numbers (positive real numbers). With the nonnegative (positive) elements, the corresponding sets of \( n \)-dimensional column vectors and \( (n \times d) \)-dimensional matrices are denoted as \( \mathbb{R}^n_{\geq 0} \ (\mathbb{R}^n_{> 0}) \) and \( \mathbb{R}^{n \times d}_{\geq 0} \ (\mathbb{R}^{n \times d}_{> 0}) \). Denote \( \mathbb{Z}_i^d = \{i, i+1, i+2, i+3, \ldots, i+j-i\} \), where \( j \leq i \) and \( i, j \in \mathbb{Z} \). \( I_n \) represents the \( n \)-dimensional identity matrix. \( diag\{a_1, a_2, \ldots, a_n\} \) denotes an \( n \)-dimensional diagonal matrix. The superscript \( T \) is used to represent the transpose of a matrix. For \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^n_{\geq 0} \), denote \( |x| = (|x_1|, |x_2|, \ldots, |x_n|)^T \), \( (x)_i = x_i \), \( \|x\|_\infty = \max_{i \in \mathbb{Z}_i^d} |x_i| \), and \( \|x\|_w^w = \max_{i \in \mathbb{Z}_i^d} \frac{|x_i|}{u_i} \), where \( x_i \) and \( u_i \) are the \( i \)th components of \( x \) and \( u \), respectively. For a given matrix \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \), denote \( |A| = (|a_{ij}|)_{n \times n}, s(A) = \{\zeta \in \mathbb{C}: \det(\zeta I_n - A) = 0\}, \pi(A) = \max\{|\zeta|: \zeta \in s(A)\}, \) and \( \chi(A) = \max\{Re(\zeta): \zeta \in s(A)\} \). Furthermore, \( A >> B \) \( (A << B, A \leq B, A \geq B) \) if \( a_{ij} > b_{ij} \) \( (a_{ij} < b_{ij}, a_{ij} \leq b_{ij}, a_{ij} \geq b_{ij}) \) for all \( i \in \mathbb{Z}_i^d \) and \( j \in \mathbb{Z}_j^d \), where \( A, B \in \mathbb{R}^{n \times d} \). A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be Metzler suppose that its off-diagonal entries are all non-negative.
In many publications, a class of nonlinear time-delay GRNs with $n$ mRNAs and $n$ proteins is modeled as follows:

$$\begin{align*}
\dot{X}(t) &= -AX(t) + B\bar{G}(Y(t - \alpha(t))) + H, \\
\dot{Y}(t) &= -CY(t) + DX(t - \beta(t)), t \geq 0,
\end{align*}$$

(1)

where $M(t) = [m_1(t), m_2(t), \ldots, m_n(t)]^T$, $A = \text{diag}\{a_1, a_2, \ldots, a_n\}$, $B = (b_{ij})_{n \times n}$, $P(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T$, $G(P(t)) = [g_1(p_1(t)), g_2(p_2(t)), \ldots, g_n(p_n(t))]^T$, $H = [h_1, h_2, \ldots, h_n]^T$, $C = \text{diag}\{c_1, c_2, \ldots, c_n\}$, and $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$. $p_i(t)$ and $m_i(t)$ represent respectively, the concentrations of protein and mRNA of the $i$th gene; $a_j$ and $c_i$ are respectively, the $i$th mRNA and protein degradation rates; $d_i$ denotes the translation rate between the $i$th protein and $i$th mRNA; $b_{ij}$ is the dimensionless transcriptional rate from the $j$th transcription factor to $i$th mRNA, and satisfies that $b_{ij} > 0$ if gene $j$ is a activator of gene $i$, $b_{ij} = 0$ if gene $j$ does not influence gene $i$, and $b_{ij} < 0$ if gene $j$ is a repressor of gene $i$; $h_i$ is defined as $h_i = -\sum_{j \in S} b_{ij}$, where $S$ is the set with all repressors of gene $i$; $g_j(s)$ is a monotonic increasing function of the form $\frac{s^{\sigma_j}}{1+s^{\sigma_j}}$, where it is called a feedback regulation function with Hill form, and $\sigma_j > 0$ is called the Hill coefficient; $\alpha(t)$ and $\beta(t)$ are the time delays that occur in the processes of regulation and translation. One can learn more knowledge about GRNs with time delays from [35].

As studied in some existing literatures [26, 35, 34], we first introduce some assumptions about the nonlinear function and time delay throughout this paper.

Assumption 1. Assume that there are positive constants $l_j$ such that $|g_j(\epsilon_1) - g_j(\epsilon_2)| \leq l_j|\epsilon_1 - \epsilon_2|$, $\forall \epsilon_1, \epsilon_2 \in \mathbb{R}$, where $g_j(s)$ is the nonlinear regulation function in (1), $j = 1, 2, \ldots, n$.

Remark 1. Note that Assumption 1 is a global Lipschitz condition. Under this condition, the nonlinear function can be bounded by a linear function. Thus, a nonlinear problem can be converted to a linear one. Based on this, a series of vectors can be assumed to satisfy the matrix inequations with regard to the linear system data in the subsequent analysis. And then an exponential bounding is finally obtained by induction.

Assumption 2. The delays $\alpha(t)$ and $\beta(t)$ in system (1) satisfy $\alpha(t) \in [0, \bar{\alpha}]$ and $\beta(t) \in [0, \bar{\beta}]$, where $\bar{\alpha}$ and $\bar{\beta}$ are the known positive constants.

Remark 2. In this paper, a direct method without any auxiliary function or LKF, together with an ADT switching condition (see Definition 3.2), is adopted to obtain the state bounding. In the later proof, we can observe that the ADT switching condition and the size of the state bounding rely on the upper bound of the delay. Thus, the bounded delay is necessary.

In some early works, the existence of the equilibrium point of system (1) has been extensively studied. In this paper, it is assumed that $(M^*, P^*)$ is a equilibrium point of system (1). By letting $X(t) = M(t) - M^*$ and $Y(t) = P(t) - P^*$ for all $t \geq 0$, one obtains

$$\begin{align*}
\dot{X}(t) &= -AX(t) + B\bar{G}(Y(t - \alpha(t))), \\
\dot{Y}(t) &= -CY(t) + DX(t - \beta(t)), t \geq 0,
\end{align*}$$

(2)

where $X(t) = [X_1(t), X_2(t), \ldots, X_n(t)]^T$, $Y(t) = [Y_1(t), Y_2(t), \ldots, Y_n(t)]^T$, and $\bar{G}(Y) = G(Y + P^*) - G(P^*) = [\bar{g}_1(Y_1), \bar{g}_2(Y_2), \ldots, \bar{g}_n(Y_n)]^T$. Thus, the equilibrium point of system (1) is transferred to the origin.
As described in Introduction, the structure and state of system (2) may change abruptly, which will be modeled by a hybrid nonlinear impulsive and switching system. Furthermore, taking the exogenous disturbance input into a GRN, one thus obtains the following ISGRNs with exogenous disturbance:

\[
\begin{align*}
\dot{X}(t) &= -A_{r(t)}X(t) + B_{r(t)}\tilde{G}(Y(t) - \alpha(t)) + E_{r(t)}\Omega(t), \\
\dot{Y}(t) &= -C_{r(t)}Y(t) + D_{r(t)}X(t) - \beta(t) + F_{r(t)}\Omega(t), \\
\Delta X(t) &= I_{r(t)}X(t), \\
\Delta Y(t) &= J_{r(t)}Y(t), \\
\end{align*}
\]

where \(k = 1, 2, \ldots, \) and \(r(t) : [0, \infty) \to \mathbb{Z}_1^N \) is a piecewise continuous function representing a switching signal. \(\Delta X(t_k) = X(t_k^+) - X(t_k^-) \) and \(\Delta Y(t_k) = Y(t_k^+) - Y(t_k^-) \) denote the impulsive jumps of \(X(t)\) and \(Y(t)\) at \(t_k\). Without loss of generality, assume that \((X(t_k^+), Y(t_k^+)) = (X(t_k), Y(t_k))\), i.e., the trajectory \((X(t), Y(t))\) of system (3) is right continuous at \(t_k\). When \(t = t_k\), system (3) is switched from \(p\)th mode to \(q\)th mode \((p, q \in \mathbb{Z}_1^N\) and \(p \neq q\)). And \(I_{p,q} \in \mathbb{R}^{n \times n}\) and \(J_{p,q} \in \mathbb{R}^{n \times n}\) are the impulsive jump matrices at \(t_k\) with respect to the states \(X(t)\) and \(Y(t)\), respectively. Besides,

\[
\begin{align*}
A_{r(t)} &= \text{diag}\{a_{1}^{r(t)}, a_{2}^{r(t)}, \ldots, a_{n}^{r(t)}\},  \\
B_{r(t)} &= [b_{ij}^{r(t)}]_{n \times n}, I_{p,q}, J_{p,q} \in \mathbb{R}^{n \times n},  \\
C_{r(t)} &= \text{diag}\{c_{1}^{r(t)}, c_{2}^{r(t)}, \ldots, c_{n}^{r(t)}\},  \\
D_{r(t)} &= \text{diag}\{d_{1}^{r(t)}, d_{2}^{r(t)}, \ldots, d_{n}^{r(t)}\},  \\
E_{r(t)} &= \text{diag}\{e_{1}^{r(t)}, e_{2}^{r(t)}, \ldots, e_{n}^{r(t)}\},  \\
F_{r(t)} &= \text{diag}\{f_{1}^{r(t)}, f_{2}^{r(t)}, \ldots, f_{n}^{r(t)}\},
\end{align*}
\]

and \(\Omega(t) = [\omega_{1}(t), \omega_{2}(t), \ldots, \omega_{n}(t)]^T\) represents the exogenous perturbed input, which satisfies \(\max_{i \in \mathbb{Z}_1^N} \sup_{t \in [0, \infty)} |\omega_{i}(t)| \leq \Omega\). Besides, introduce some notations \(e_{\infty} = \max_{i \in \mathbb{Z}_1^N, p \in \mathbb{Z}_1^N} |e_{pi}^{p}|\) and \(f_{\infty} = \max_{i \in \mathbb{Z}_1^N, p \in \mathbb{Z}_1^N} |f_{i}^{p}|\) for convenience. In this paper, we assume that the exogenous disturbance is componentwise bounded. That is the disturbance vector satisfies \(\sup_{t \in [0, \infty)} |e_{pi}^{p}\omega_{i}(t)| \leq e_{\infty}\Omega\) and \(\sup_{t \in [0, \infty)} |f_{i}^{p}\omega_{i}(t)| \leq f_{\infty}\Omega\) for any \(i \in \mathbb{Z}_1^N\). In fact, in the subsequent proof, we do not need the precise information about the exogenous disturbance except the upper bound \(e_{\infty}\Omega\) and \(f_{\infty}\Omega\). Thus, our paper can be used to deal with the unknown exogenous disturbance if it is assumed to be bounded by given constants.

According to the representations of \(\tilde{G}(\cdot)\) and \(G(\cdot)\) in equations (1) and (2), it can be easily seen from Assumption 1 that \(G(\cdot)\) satisfies \(|\tilde{g}_{ij}(Y_{ij})| \leq l_{ij}Y_{ij}, j = 1, 2, \ldots, n\). We assume that \(\mu = \max_{j \in \mathbb{Z}_1^N} l_{ij}\). In the subsequent discussion, consider the following initial states related to system (3):

\[
X(t) = \phi(t), Y(t) = \psi(t),
\]

where \(t \in [-\gamma, 0], \gamma = \max\{\alpha, \beta\}\), and \(\phi(t), \psi(t) \in C([-\gamma, 0], \mathbb{R}^{n})\). Denote \(\|\phi, \psi\|_{u,v} = \max\{|\sup_{t \in [-\gamma, 0]} \|\phi(t)\|_{u,v}, |\sup_{t \in [-\gamma, 0]} \|\psi(t)\|_{u,v}\}| \) for \(u, v \in \mathbb{R}_{>0}\).

**Remark 3.** The publications [7, 26] also investigate the problem of state bounding for some systems. The considered model in [26] is a delayed GRN without impulsive and switching effects, and [7] focuses on a linear system with time-delaying delay.
However, system (3) is in fact a time-delay hybrid nonlinear impulsive and switching system. This leads to a completely different research model. In this paper, we not only consider the time-varying delay and nonlinear function but also need to cope with the effect of impulse and switching. On the other hand, the research problem and method in this paper are different from the [14], where the filtering problem is studied for discrete impulsive and switching systems by Lyapunov function and admissible edge-dependent ADT technique. Our paper will adopt a direct method and a distinct ADT switching to deal with the state bounding problem, not involving the construct of Lyapunov function.

3. State bounding estimation for ISGRNs. In this section, the state bounding of system (3) is studied. A sufficient criterion is obtained such that all trajectories of (3) converge exponentially into a sphere based on a method of ADT.

Definition 3.1. The solution \((X(t),Y(t))\) of (3) converges exponentially into a sphere \(B(\kappa_1)\), if there exist nonnegative constants \(\kappa_1,\kappa_2,\) and \(\kappa_3\) satisfying
\[
\|(X(t),Y(t))\|_\infty \leq \kappa_1 + \kappa_2 e^{-\kappa_3 t},
\]
where \(\|(X(t),Y(t))\|_\infty = \max\{\|X(t)\|_\infty,\|Y(t)\|_\infty\}\) and the sphere is defined as
\[
B(\kappa_1) = \{(X,Y) \in \mathbb{R}^n \times \mathbb{R}^n : \|X\|_\infty \leq \kappa_1,\|Y\|_\infty \leq \kappa_1\}.
\]

The main object in this paper is to find the nonnegative constants \(\kappa_1,\kappa_2,\) and \(\kappa_3\) such that system (3) converges exponentially within a sphere \(B(\kappa_1)\) with convergence rate \(\kappa_3\). It can be seen that the sphere \(B(\kappa_1)\) is an estimated ultimate state bounding. Before obtaining the main conclusions, the corresponding assumption, definition, and lemma are needed.

Assumption 3. For the system (3), there exist a series of vectors \(u_p \in \mathbb{R}_{>0}^n, v_p \in \mathbb{R}_{>0}^n (u_p, v_p \in \mathbb{R}^n, p \in \mathbb{Z}_1^N)\) and a constant \(\delta > \max_{p,q \in \mathbb{Z}_1^N} \{\frac{u_p}{v_p}, \frac{v_p}{u_p}\}\) such that
\[
\begin{align*}
-A_p u_p + \mu|B_p| v_p &< 0, \\
D_p b_p - C_p v_p &< 0, \\
|I_n + I_{p,q}| u_p &\leq \delta u_q, \\
|I_n + J_{p,q}| v_p &\leq \delta v_q, p, q \in \mathbb{Z}_1^N,
\end{align*}
\]
where \(u_p = [u_{p1}, \cdots, u_{pn}]^T\) and \(v_p = [v_{p1}, \cdots, v_{pn}]^T\).

Definition 3.2. [13] A positive constant \(\varpi\) is said to be the ADT with respect to the switching function \(s(t)\) when \(N_r(T_2, T_1) \leq N_0 + \frac{(T_2 - T_1)}{\varpi}\) is satisfied for \(N_0 \geq 0,\) where \(T_2 \geq T_1 \geq 0\) and \(N_r(T_2, T_1)\) represents the number of switches occurring in the interval \((T_1, T_2)\).

Remark 4. As studied in some existing literatures, \(N_0 = 0\) is chosen in the subsequent analysis.

Remark 5. The solution properties of ISGRNs will be affected by impulsive time, impulsive strength, and switching rule, thus the system trajectory may not converge exponentially into a sphere if the system generates impulse and switching. In this paper, to achieve the aim, the impulsive function is assumed to satisfy Assumption 3 and ADT switching satisfies \(\varpi > \frac{\theta \delta}{\varpi}\), where \(\theta\) is defined in equation (7). The first two inequalities in Assumption 3 ensures that the system state satisfies the estimation (12) during every impulsive interval, while the last two inequalities in Assumption 3 ensure that the instantaneous state undergoing impulsive jumps also
satisfies a similar estimation. On other hand, the switching function satisfying ADT switching and \( \omega > \frac{\ln \delta}{\omega} \) ensures an explicit bounding condition on the whole time line.

**Lemma 3.3.** [18] Assume that matrix \( M_1 \in \mathbb{R}^{n \times n} \) is Metzler and matrices \( M_2, M_3, M_4 \) belong to the set \( \mathbb{R}^{n \times n}_{\geq 0} \). Then one concludes the statements below are equivalent:

i) There exist two vectors \( u, v \in \mathbb{R}^n_0 \) satisfying \( M_1 u + M_2 v \leq 0, M_3 u + M_4 v \leq 0 \);

ii) \( \chi(M_1) < 1 \) and \( \chi(M_1 + M_2(I_n - M_4)^{-1}M_3) < 0 \);

iii) \( \chi(M_1) < 0 \) and \( \pi(M_3(-M_1)^{-1}M_2 + M_4) < 1 \).

**Theorem 3.4.** Consider the system (3) and let Assumptions 1, 2, and 3 hold. Then there exist nonnegative constants \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) such that all solutions of (3) under appropriate ADT switching converge exponentially into a sphere \( \mathbb{B}(\kappa_1) \), that is

\[
\| (X(t), Y(t)) \|_\infty \leq \kappa_1 + \kappa_2 e^{-\kappa_3 t},
\]

where \( \kappa_1, \kappa_2, \) and \( \kappa_3 \) are dependent on some factors among the initial conditions, system matrices, bounded disturbances, and switched function in system (3).

**Proof.** According to the system (3) and Assumption 3, some relevant notations are firstly introduced as follows:

\[
\rho_0 = \max \left\{ \frac{\epsilon \omega_0}{\min_{i \in \mathbb{Z}^n_1} \{ (A_p u_p - \mu|B_p|v_p)_i \}}, \frac{\omega_0}{\min_{i \in \mathbb{Z}^n_1} \{ (D_p u_p + C_p v_p)_i \}} \right\},
\]

\[
\omega_0 = \max \left\{ \max_{p \in \mathbb{Z}^n_1} \| (\phi, \psi) \|_{(u_p, v_p)}^\gamma, \rho_0 \right\},
\]

\[
\theta = \min_{p \in \mathbb{Z}^n_1} \{ \theta_{p1}, \theta_{p2}, \ldots, \theta_{pn}, \bar{\theta}_{p1}, \bar{\theta}_{p2}, \ldots, \bar{\theta}_{pn} \},
\]

where \( \theta_{pi} \) and \( \bar{\theta}_{pi} \) are given by

\[
\begin{align*}
- a^p_{\theta \bar{\theta}} u_{\bar{p}i} + \mu e^{\theta_{p \bar{p}} \alpha} \sum_{j=1}^n |b^j_{\bar{p}i} v_{pj} + \theta_{pji} u_{pji}| = 0, \\
- c^p_{\theta \bar{\theta}} v_{\bar{p}i} + a^p_{\theta \bar{\theta}} e^{\theta_{p \bar{p}} \beta} u_{pji} + \bar{\theta}_{pji} v_{pji} = 0.
\end{align*}
\]

Next, the task is divided into the following several steps.

**Step 1.** It is to demonstrate that the following holds:

\[
\begin{align*}
|X_i(t)|_{u_{pji}} &\leq \rho_0 + (\omega_0 - \rho_0) e^{-\theta(t-t_0)}, \\
|Y_i(t)|_{v_{pji}} &\leq \rho_0 + (\omega_0 - \rho_0) e^{-\theta(t-t_0)},
\end{align*}
\]

where \( p = r(t_0), t \in [\gamma, t_1] \). For a constant \( \lambda > 1 \), denote

\[
\dot{X}_i(t) = \frac{|X_i(t)|}{u_{pji}} - \lambda \rho_0 - \lambda (\omega_0 - \rho_0) e^{-\theta(t-t_0)},
\]

\[
\dot{Y}_i(t) = \frac{|Y_i(t)|}{v_{pji}} - \lambda \rho_0 - \lambda (\omega_0 - \rho_0) e^{-\theta(t-t_0)}.
\]

It can be easily found that \( \dot{X}_i(t) < 0 \) and \( \dot{Y}_i(t) < 0 \) based on the definition of \( \omega_0 \) in equation (6), where \( t \in [\gamma, t_0] \). Next, suppose that \( \dot{X}_i(t) < 0 \) and \( \dot{Y}_i(t) < 0 \) for all \( i \in \mathbb{Z}^n_1 \) and \( t \in (t_0, t_1) \). Otherwise, one can conclude that there exist \( t^* \in (t_0, t_1) \)
Thus, one derives the following:

\[ |X(t^*)| \leq \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)}] u_p, \]

\[ |X_i(t^*)| = \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)}] u_{pi}. \]

Thus, one derives the following:

\[
D^+|X_{i_1}(t)||_{t=t^*} = \text{sgn}(X_{i_1}(t^*)) \tilde{X}_{i_1}(t^*)
\]

\[
= \text{sgn}(X_{i_1}(t^*)) \left\{ -a_{i_1}^p X_{i_1}(t^*) + e_{i_1}^p \omega_{i_1}(t^*) + \left[ b_{i_1,1}^p, \ldots, b_{i_1,n}^p \right]
\right\}
\times \left[ \tilde{g}_1(Y_1(t^* - \alpha(t^*)), \ldots, \tilde{g}_n(Y_n(t^* - \alpha(t^*))) \right]^T
\]

\[
\leq -a_{i_1}^p |X_{i_1}(t^*)| + e_{i_1}^p \tilde{\Omega} + \sum_{j=1}^n |b_{i_1,j}^p| \left| \tilde{g}_j(Y_j(t^* - \alpha(t^*))) \right|
\]

\[
\leq -a_{i_1}^p |X_{i_1}(t^*)| + \sum_{j=1}^n \tilde{l}_j |b_{i_1,j}^p| |Y_j(t^* - \alpha(t^*))| + e_{\infty} \tilde{\Omega}
\]

\[
\leq -a_{i_1}^p \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)}] u_{pi}
\]

\[ + \mu \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - \alpha(t^*) - t_0)}] \sum_{j=1}^n |b_{i_1,j}^p| v_{pj}
\]

\[ + e_{\infty} \tilde{\Omega}. \]

Based on the definition of \( \rho_0 \), it follows that

\[
\lambda \rho_0 (-A_p u_p + \mu (B_p) v_i) + e_{\infty} \tilde{\Omega} < 0.
\]

It thus yields that

\[
D^+|X_{i_1}(t)||_{t=t^*} < -a_{i_1}^p \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)}] u_{pi}
\]

\[ + \mu \lambda [\rho_0 + (\varpi_0 - \rho_0) e^{-\theta(t^* - \alpha(t^*) - t_0)}] \sum_{j=1}^n |b_{i_1,j}^p| v_{pj}. \]

Hence, according to equation (8), we have

\[
D^n \tilde{X}_{i_1}(t)|_{t=t^*} = D^n|X_{i_1}(t)||_{t=t^*} + \theta \lambda (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)}
\]

\[
< \lambda (\varpi_0 - \rho_0) e^{-\theta(t^* - t_0)} \left[ -a_{i_1} + \frac{\mu e^{\theta \bar{a} t_0} \sum_{j=1}^n |b_{i_1,j}^p| v_{pj}}{u_{pi}} + \theta \right]
\]

\[ \leq 0, \]

which result in a contradiction. Therefore, one concludes that \( \tilde{X}_{i_1}(t) < 0 \) for all \( i \in \mathbb{Z}_1^n \) and \( t \in [t_0, t_1] \).
On the other hand, when case ii) is satisfied, the following condition holds

\[
|Y(t^*)| \leq \lambda \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \right] v_p,
\]

\[
|Y_i(t^*)| = \lambda \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \right] v_{pi}.
\]

Thus, it yields that

\[
D^+ |Y_i(t)|_{t=t^*} = \text{sgn}(Y_i(t^*)) \dot{Y}_i(t^*)
\]

\[
= \text{sgn}(Y_i(t^*))[-c_i^P Y_i(t^*)
+ f_i^P \omega_i(t^*) + d_i^P X_i(t^* - \beta(t^*))]
\]

\[
\leq -c_i^P |Y_i(t^*)| + f_i^P \bar{\Omega} + d_i^P |X_i(t^* - \beta(t^*))|
\]

\[
\leq -c_i^P |Y_i(t^*)| + d_i^P |X_i(t^* - \beta(t^*))| + f_{\infty} \bar{\Omega}
\]

\[
\leq -c_i^P \lambda \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \right] v_{pi}
+ d_i^P \lambda \rho_0 + \lambda(\varpi_0 - \rho_0)e^{-\theta(t^* - \beta(t^*) - t_0)} u_{pi} + f_{\infty} \bar{\Omega}.
\]

Based on the definition of \( \rho_0 \), it holds

\[
\lambda \rho_0 (D_p u_p - C_p v_p)_{i_1} + f_{\infty} \bar{\Omega} < 0.
\]

It follows that

\[
D^+ |Y_i(t)|_{t=t^*} \leq \lambda \rho_0 (-c_i^P v_{pi} + d_i^P u_{pi}) + f_{\infty} \bar{\Omega}
+ \lambda(\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \left[ -c_i^P v_{pi} + d_i^P e^{\theta \bar{\beta}} u_{pi} \right]
\]

\[
< \lambda(\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \left[ -c_i^P v_{pi} + d_i^P e^{\theta \bar{\beta}} u_{pi} \right].
\]

Hence

\[
D^+ \dot{Y}_i(t)_{t=t^*} \leq \frac{D^+ |Y_i(t)|_{t=t^*}}{v_{pi}} = \theta \lambda(\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)}
\]

\[
< \lambda(\varpi_0 - \rho_0)e^{-\theta(t^* - t_0)} \left[ -c_i^P + d_i^P e^{\theta \bar{\beta}} u_{pi} + \theta \right] \leq 0,
\]

which result in a contradiction. Therefore, \( \dot{Y}_i(t) < 0 \) holds for \( t \in [t_0, t_1) \) and \( 1 \leq i \leq n \).

Since \( \lambda > 1 \) is arbitrary, ones concludes that (9) holds when \( \lambda \) tends to 1. The step 1 thus is proved.

**Step 2.** It is to show that the following condition holds

\[
\begin{cases}
\frac{|X_i(t)|}{v_{pi}} \leq \rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t - t_{1})}, \\
\frac{|Y_i(t)|}{v_{pi}} \leq \rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t - t_{1})},
\end{cases}
\]

(10)
where \( \varpi_1 = \delta [\rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)}] \), \( t \in [-\gamma, t_2] \), \( q = r(t_1) \), \( i \in \mathbb{Z}_1^n \). For any \( \lambda > 1 \), define the following functions

\[
\tilde{X}_i(t) = \frac{|X_i(t)|}{u_{qi}} - \lambda \rho_0 - \lambda (\varpi_1 - \rho_0)e^{-\theta(t-t_1)},
\]

\[
\tilde{Y}_i(t) = \frac{|Y_i(t)|}{v_{qi}} - \lambda \rho_0 - \lambda (\varpi_1 - \rho_0)e^{-\theta(t-t_1)},
\]

for all \( i \in \mathbb{Z}_1^n \) and \( t \in [-\gamma, t_2] \).

Notice that

\[
\rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t-t_1)} = [\rho_0 + (\delta - 1)\rho_0e^{-\theta(t-t_0)}] + \delta (\varpi_0 - \rho_0)e^{-\theta(t-t_0)} \geq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t-t_0)} \right], \forall t \in [-\gamma, t_1].
\]

Thus, from equations (9) and (11), one gets

\[
\frac{|X_i(t)|}{u_{qi}} = \frac{|X_i(t)|}{u_{qi}} \frac{|X_i(t)|}{u_{qi}} \leq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t-t_0)} \right] \leq \rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t-t_1)}
\]

and

\[
\frac{|Y_i(t)|}{v_{qi}} \leq \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t-t_1)},
\]

where \( t \in [-\gamma, t_1] \). It can be easily seen that \( \tilde{Y}_i(t) < 0 \) and \( \tilde{X}_i(t) < 0 \), where \( t \in [-\gamma, t_1] \).

On other hand, from equation (9), one can derive

\[
\left\{ \begin{array}{l}
|X(t_1^-)| \leq \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] u_{r(t_0)}, \\
|Y(t_1^-)| \leq \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] v_{r(t_0)}.
\end{array} \right.
\]

Hence, it yields from Assumption 3 that

\[
|X(t_1^-)| \leq |I_n + I_{p,q}||X(t_1^-)| \leq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] u_q,
\]

and

\[
|Y(t_1^-)| \leq |I_n + J_{p,q}||Y(t_1^-)| \leq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] v_q.
\]

Then, it follows from equation (11) that

\[
\frac{|X_i(t_1^-)|}{u_{qi}} \leq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] \leq \rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t_1-t_1)},
\]

and

\[
\frac{|Y_i(t_1^-)|}{v_{qi}} \leq \delta \left[ \rho_0 + (\varpi_0 - \rho_0)e^{-\theta(t_1-t_0)} \right] \leq \rho_0 + (\varpi_1 - \rho_0)e^{-\theta(t_1-t_1)}.
\]
This implies that $\bar{X}_i(t) < 0$ and $\bar{Y}_i(t) < 0$ also holds when $t \in [-\gamma, t_1]$. Assume that $X_i(t) < 0$ and $\bar{Y}_i(t) < 0$ for all $i \in \mathbb{Z}_1^n$ and $t \in (t_1, t_2)$. Otherwise, one of the following cases is satisfied. That is, there exist $t^{**} \in (t_1, t_2)$ and $i_2 \in \mathbb{Z}_1^n$, such that i) $X_i(t) < 0$ for all $i \in \mathbb{Z}_1^n$ and $t \in (t_1, t^{**})$, $X_{i_2}(t^{**}) = 0$, and $D^+ X_{i_2}(t)|_{t=t^{**}} \geq 0$, and ii) $\bar{Y}_i(t) < 0$ for all $i \in \mathbb{Z}_1^n$ and $t \in (t_1, t^{**})$, $\bar{Y}_{i_2}(t^{**}) = 0$, and $D^+ \bar{Y}_{i_2}(t)|_{t=t^{**}} \geq 0$.

Consider the condition i). It yields that

$$|X(t)| \leq \lambda \left[ \rho_0 + (\varpi_1 - \rho_0) e^{-\theta(t-t_1)} \right] u_{q_1}, t \in (t_1, t^{**}),$$

$$|X_{i_2}(t^{**})| = \lambda \left[ \rho_0 + (\varpi_1 - \rho_0) e^{-\theta(t^{**}-t_1)} \right] u_{q_1},$$

and

$$D^+ |X_{i_2}(t)||_{t=t^{**}} = sgn(X_{i_2}(t^{**})) \dot{X}_{i_2}(t^{**})$$

$$\leq -a_{i_2}^2 |X_{i_2}(t^{**})|$$

$$+ \sum_{j=1}^n l_j |b_{i_2,j}^q| |Y_j(t^{**} - \alpha(t^{**}))| + e_\infty \tilde{\Omega}$$

$$\leq -a_{i_2}^2 \lambda \left[ \rho_0 + (\varpi_1 - \rho_0) e^{-\theta(t^{**}-t_1)} \right] u_{q_1}$$

$$+ \mu \lambda \left[ \rho_0 + (\varpi_1 - \rho_0) e^{-\theta(t^{**} - \alpha(t^{**}) - t_1)} \right] \sum_{j=1}^n |b_{i_2,j}^q| v_{q,j}$$

$$+ e_\infty \tilde{\Omega}.$$
Thus, for $t \in [-\gamma, t_k)$, $\omega_k = \delta \left[ \rho_0 + (\omega_{k-1} - \rho_0) e^{-\theta(t_k - t_{k-1})} \right]$, $k \in \mathbb{Z}_1^\infty$, and $i \in \mathbb{Z}_1^n$.

**Step 3.** Equation (4) will be proved based on the preceding analysis.

Notice that for $k = 1, 2, \cdots$,

$$
\omega_k - \rho_0
= (\delta - 1)\rho_0 + \delta(\omega_{k-1} - \rho_0)e^{-\theta(t_k - t_{k-1})}
= (\delta - 1)\rho_0 + \delta e^{-\theta(t_k - t_{k-1})}[(\delta - 1)\rho_0 + \delta(\omega_{k-2} - \rho_0)e^{-\theta(t_{k-1} - t_{k-2})}]
= (\delta - 1)\rho_0 \left[ 1 + \delta e^{-\theta(t_k - t_{k-1})} + \cdots + \delta^{k-2} e^{-\theta(t_{k-2} - t_{k-2})} + \delta^{k-1} e^{-\theta(t_{k-1} - t_{k-1})} \right]
+ (\omega_0 - \rho_0) \delta^k e^{-\theta(t_{k-1} - t_0)}.
$$

Thus, for $t \in [t_k, t_{k+1})$ and $1 \leq i \leq n$, it yields that

$$
\frac{|X_i(t)|}{\bar{v}_r(t_k) i}
\leq \rho_0 + (\delta - 1)\rho_0 \left[ 1 + \delta e^{-\theta(t_k - t_{k-1})} + \cdots + \delta^{k-2} e^{-\theta(t_{k-2} - t_{k-2})} + \delta^{k-1} e^{-\theta(t_{k-1} - t_{k-1})} \right]
+ (\omega_0 - \rho_0) \delta^k e^{-\theta(t_{k-1} - t_0)},
$$

and

$$
\frac{|Y_i(t)|}{\bar{v}_r(t_k) i}
\leq \rho_0 + (\delta - 1)\rho_0 \left[ 1 + \delta e^{-\theta(t_k - t_{k-1})} + \cdots + \delta^{k-2} e^{-\theta(t_{k-2} - t_{k-2})} + \delta^{k-1} e^{-\theta(t_{k-1} - t_{k-1})} \right]
+ (\omega_0 - \rho_0) \delta^k e^{-\theta(t_{k-1} - t_0)}.
$$

Using the ADT in Definition 3.2, one obtains

$$
\frac{|X_i(t)|}{\bar{v}_r(t_k) i}
\leq \rho_0 + (\delta - 1)\rho_0 \left[ 1 + \delta e^{-\theta \omega} + \cdots + (\delta e^{-\theta \omega})^{k-2} + (\delta e^{-\theta \omega})^{k-1} \right]
+ (\omega_0 - \rho_0) \delta^k e^{-\theta t},
$$

and

$$
\frac{|Y_i(t)|}{\bar{v}_r(t_k) i}
\leq \rho_0 + (\delta - 1)\rho_0 \left[ 1 + \delta e^{-\theta \omega} + \cdots + (\delta e^{-\theta \omega})^{k-2} + (\delta e^{-\theta \omega})^{k-1} \right]
+ (\omega_0 - \rho_0) \delta^k e^{-\theta t}.
$$

It can be seen that $0 < \delta e^{-\theta \omega} < 1$ if the ADT switching satisfies $\omega > \frac{\ln\delta}{\theta}$. On the other hand, it yields that $k \omega \leq t$ for $t \in [t_k, t_{k+1})$ according to Definition 3.2. Thus, from the above equations, we derive

$$
\begin{cases}
\frac{|X_i(t)|}{\bar{v}_r(t_k) i} \leq \rho_0 + (\delta - 1)\rho_0 \frac{1}{1 - \delta e^{-\theta \omega}} + (\omega_0 - \rho_0) e^{-\theta(t - \frac{\omega t_k}{\omega})} t \\
\frac{|Y_i(t)|}{\bar{v}_r(t_k) i} \leq \rho_0 + (\delta - 1)\rho_0 \frac{1}{1 - \delta e^{-\theta \omega}} + (\omega_0 - \rho_0) e^{-\theta(t - \frac{\omega t_k}{\omega})} t,
\end{cases}
$$

where $t \in [t_k, t_{k+1})$ and $i = 1, 2, \cdots, n$. Denote $\bar{u} = \max_{p \in \mathbb{Z}_1^n, i \in \mathbb{Z}_1^n} u_{pi}$ and $\bar{v} = \max_{p \in \mathbb{Z}_1^n, i \in \mathbb{Z}_1^n} v_{pi}$. Then
\[ \|(X(t), Y(t))\|_\infty \leq \max\{\bar{u}, \bar{v}\} \rho_0 \left( 1 + \frac{\delta - 1}{1 - \delta e^{-\theta \varpi}} \right) + \max\{\bar{u}, \bar{v}\} (\varpi_0 - \rho_0) e^{-\theta - \frac{\ln \kappa}{\varpi_0} t} \]

\[ = \kappa_1 + \kappa_2 e^{-\kappa_3 t}. \]

Thus, this proof is completed. \( \square \)

**Remark 6.** In the proof of Theorem 3.4, \( N_0 = 0 \) is used, where \( N_0 \) is defined in Definition 3.2. In fact, the state trajectory will converge exponentially into a sphere under appropriate ADT switching whether \( N_0 = 0 \) or \( N_0 \neq 0 \). Similar to the proof of Theorem 3.4, for any \( N_0 \), one can finally derive

\[ \|(X(t), Y(t))\|_\infty \leq \max\{\bar{u}, \bar{v}\} \rho_0 \left( 1 + \frac{\delta - 1}{1 - \delta e^{-\theta \varpi}} \right) + \max\{\bar{u}, \bar{v}\} (\varpi_0 - \rho_0) \delta e^{-\theta - \frac{\ln \kappa}{\varpi_0} t} \]

\[ = \kappa_1 + \kappa_2 e^{-\kappa_3 t}. \]

Thus, it can be seen that the converge rate does not rely on \( N_0 \) (\( \kappa_3 \) is irrelevant to \( N_0 \)). The main differences lie in the size of the state bounding (\( \kappa_1 \) and \( \kappa_2 \) will be affected by \( N_0 \)). In particular, \( \kappa_1 \) and \( \kappa_2 \) are smaller when \( N_0 = 0 \), which implies that the obtained state bounding is more precise.

**Remark 7.** The commonly adopted approach to deal with the problem of reachable set estimation for time-delay systems is the LKF method. It is in fact an indirect approach to obtain a bounding of the reachable set. The key of this method is to construct the LKF and estimate its derivative [2, 31, 11]. This leads to much complex computations since lots of variables and inequalities are involved. In this paper, a direct approach is used to obtain an explicit state bounding on every subinterval, and then the ADT switching satisfying \( \varpi > \frac{\ln \delta}{\kappa_1} \) is designed to obtain the explicit state bounding on the whole time line. Compared to the LKF approach, the method adopted in our paper does not require a series of linear matrix inequalities, and hence the result may be more easily verifiable than those bounding condition derived by LKF approach.

**Corollary 1.** In system (3), if \( \Omega(t) = 0 \) for \( t \geq 0 \) and the conditions in Theorem 3.4 hold, then system (3) is globally exponentially stable under appropriate ADT switching. On other hand, when system has zero initial functions (i.e. \( \phi(t) = \psi(t) = 0 \)) and the conditions in Theorem 3.4 hold, the system (3) is bounded within \( \max\{\bar{u}, \bar{v}\} \rho_0 (1 + \frac{\delta - 1}{1 - \delta e^{-\theta \varpi}}) \) under appropriate ADT switching.

**Proof.** It can be directly obtained from Theorem 3.4. \( \square \)

**Corollary 2.** If \( I_{pq} = J_{pq} = 0 \), i.e., there exists no impulse at switching instants in system (3), then all solutions of (3) with appropriate ADT switching converge exponentially into a sphere \( \mathbb{B}(\kappa_1) \), that is \( \|(X(t), Y(t))\|_\infty \leq \kappa_1 + \kappa_2 e^{-\kappa_3 t} \) under one of the conditions below:

i) There exists vectors \( u_p, v_p \in \mathbb{R}_{\geq 0}^{n_p} \) such that \( -A_p u_p + \mu |B_p| v_p << 0 \), \( D_p u_p - C_p v_p << 0 \) for any \( p \in \mathbb{Z}_1^n \);

ii) \( \chi(-A_p + \mu |B_p| C_p^{-1} D_p) < 0 \) for any \( p \in \mathbb{Z}_1^n \);

iii) \( \pi(C_p^{-1} D_p A_p^{-1} |B_p|) < \frac{1}{\mu} \) for any \( p \in \mathbb{Z}_1^n \),
where the relevant parameters are given in Theorem 3.4. In particular, when \( \Omega(t) = 0 \) \( (t \geq 0) \), system (3) with appropriate ADT switching is globally exponentially stable under one of the above criteria i), ii), and iii).

**Proof.** This result can be proved according to Lemma 3.3 and Theorem 3.4. \( \square \)

**Remark 8.** From Theorem 3.4, Corollary 1, and Corollary 2, one can see that system (3) converges within a sphere or is exponentially stable with convergence rate \( \kappa_3 = \theta - \frac{\ln \delta}{\varpi} \). Hence, the convergence rate relies on the vectors \( u_p, v_p \), the constant \( \delta \), and the average dwell time \( \varpi \) except the original system parameters. In particular, for a given system, there are many choices for \( u_p, v_p \), and \( \delta \). Thus, they can be chosen according to the practical requirements. Moreover, if all the parameters are chosen, then the convergence rate will become larger when the average dwell time increases.

4. **An illustrating example.** This section copes with one numerical example under two cases, which will be shown to demonstrate the effectiveness of the obtained theoretical results in Section 3.

Consider one ISG\textsuperscript{R}N:

\[
\begin{aligned}
    &\dot{X}(t) = -\Delta x(t)X(t) + B \tilde{G}(Y(t - \alpha(t))) + E \Omega(t), \\
    &\dot{Y}(t) = -C \tilde{G}(Y(t)) + F \Omega(t), t \geq 0, t \neq t_k, \\
    &\Delta X(t) = I_{r(t_k)} X(t), \\
    &\Delta Y(t) = J_{r(t_k)} Y(t), t = t_k,
\end{aligned}
\]

where the parameters are given below: \( n = 3, N = 2, t_0 = 0, \alpha(t) = \beta(t) = 0.2, \) \( \Omega(t) = 0.31 + 0.08 \sin(4t), 0.16 + 0.04 \cos(2t), 0.24 + 0.06 \cos(4t) \) \( t = 0.5, 0.6 \), \( C_1 = \text{diag}(0.6, 0.4, 0.5), D_1 = \text{diag}(0.5, 0.35, 0.4), A_2 = \text{diag}(1, 0.5, 0.8), \)
\( C_2 = \text{diag}(0.5, 0.5, 0.4), D_2 = \text{diag}(0.4, 0.4, 0.25), I_{1,2} = \text{diag}(0.25, 0.3, 0.2), I_{2,1} = \{0.3, 0.25, 0.8\}, J_{1,2} = \{0.2, 0.3, 0.3\}, J_{2,1} = \{0.2, 0.25, 0.3\}, E_1 = F_1 = E_2 = F_2 = \text{diag}(0.075, 0.15, 0.1), \) and
\[
B_1 = 0.2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B_2 = 0.2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Besides, the nonlinear feedback regulation function in (1) is assumed as \( g_i(s) = \frac{s^2}{1 + s^2}, i = 1, 2, 3, \) i.e., the Hill coefficient is 2. It can be seen that, in (1), \( H = 0 \) when all elements in matrix \( B \) are nonnegative. Thus, in this example, \( g_i(s) = \frac{s^2}{1 + s^2}, i = 1, 2, 3, \)

By simple computations, one can take \( \mu = \frac{3\sqrt{3}}{8}, \tilde{\Omega} = 0.3, u_1 = v_2 = [1, 1, 1]^T, \) \( v_1 = v_2 = [1.3, 1.5, 1.3]^T, \) and \( \delta = 1.3. \) Furthermore, it concludes that \( \theta = \text{min}\{0.4315, 0.2940, 0.2455, 0.7725, 0.1515, 0.5815, 0.2000, 0.1595, 0.1810, 0.1810, 0.2215, 0.1995\} = 0.1515 \) and \( \varpi > 1.7318. \)

**Case 1:** Set \( \varpi = 5. \) The switching sequence is described in (a) of Figure 1. It is assumed that \( \psi(t) = \phi(t) = [1.5 \quad 1.5 \quad 1.5]^T, \) where \( t \in [-0.2, 0]. \) Thus, one can obtain \( \kappa_1 = 0.4903, \kappa_2 = 1.9727, \) and \( \kappa_3 = 0.0990. \) According to Theorem 3.4, the state trajectories converge exponentially within a sphere \( \mathbb{B}(0, 0.4903) \), which can be shown in Figure 2.

**Case 2:** Set \( \varpi = 2.5 \) and assume that there exists no bounded exogenous input. The switching sequence is given in (b) of Figure 1. It is also assumed that \( \phi(t) = [4 \quad 4 \quad 4]^T \) and \( \psi(t) = [4 \quad 4 \quad 4]^T, \) where \( t \in [-0.2, 0]. \) Computations yield that
\[ \kappa_1 = 0, \kappa_2 = 5.7227, \text{ and } \kappa_3 = 0.0466. \] According to Corollary 1, the ISGRN is globally exponentially stable, which is illustrated in Figure 3.

5. **Conclusion.** In this paper, the problem of state bounding for the time-delay ISGRNs with bounded exogenous disturbance has been addressed. Firstly, a sufficient criterion is obtained such that all trajectories of the ISGRNs converge exponentially into a sphere via a method of ADT. Moreover, several easily verifiable conditions guaranteeing that ISGRN is bounded or globally exponentially stable are further precisely described when ISGRN satisfies some special conditions. Compared with the existing literatures, the considered model has more complex structure and the tool adopted in this paper is more simple than LKF approach. Finally, one numerical example is presented to demonstrate the theoretical findings in this paper. The state bounding problem for discrete-time switched GRNs can also be addressed by using the same method. In the future, the established results in this paper will be
Figure 3. (a) State trajectories $X_1, X_2, X_3$ in the case 2 and (b) State trajectories $Y_1, Y_2, Y_3$ in the case 2.

extended to the discrete-time case. Moreover, establishing the bounding conditions for an ISGRN with stochastic switched subsystems may be a meaningful topic.

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