THE ALGEBRAIC BRAUER GROUP OF A PINCHED VARIETY

CRISTIAN D. GONZÁLEZ-AVILÉS

To Timothy J. Ford, for his book [SA].

Abstract. Let \( k \) be any field, let \( \tilde{X} \) be a projective and geometrically integral \( k \)-scheme and let \( \tilde{Y} \) be a finite closed subscheme of \( \tilde{X} \). If \( \psi: \tilde{Y} \to Y \) is a schematically dominant morphism between finite \( k \)-schemes and \( X \) is obtained by pinching \( \tilde{X} \) along \( \tilde{Y} \) via \( \psi \), we describe the kernel (and, in certain cases, the cokernel) of the induced pullback map \( \text{Br}_1 X \to \text{Br}_1 \tilde{X} \) between the corresponding algebraic (cohomological) Brauer groups of \( X \) and \( \tilde{X} \) solely in terms of the Brauer groups of the residue fields of \( Y \) and \( \tilde{Y} \) and the Amitsur subgroups of \( X \) and \( \tilde{X} \) in \( \text{Br}_k \).

As an application, we compute the algebraic Brauer group of a projective and geometrically integral \( k \)-scheme \( X \) with a finite non-normal locus whose normalization \( X^N \) is \( k \)-isomorphic to \( \mathbb{P}^{\dim X}_k \). If \( k \) is a local field and \( X \) is a projective and geometrically integral \( k \)-curve such that \( X^N \) is smooth, then we show that the order of the Amitsur subgroup of \( X \) in \( \text{Br}_k \) is the index of \( X \), i.e., the least positive degree of a 0-cycle on \( X \). This statement generalizes a well-known theorem of Roquette and Lichtenbaum, who obtained the above conclusion when \( k \) is a \( p \)-adic field and \( X \) is smooth over \( k \).

0. Introduction

The search for rational (or integral) solutions of concrete polynomial equations often involves singular varieties. For example, in their treatment of the Erdős-Straus conjecture via the Brauer-Manin obstruction [BL20], Bright and Loughran considered a certain singular \( \mathbb{Q} \)-surface \( X \) that is naturally associated to the conjecture and, in order to compute the indicated obstruction, determined its Brauer group \( \text{Br}_X \). In general, computing the (cohomological) Brauer group of a singular variety is a difficult problem. Consequently, the literature on this subject is rather sparse, especially over an imperfect field. See [CTSK, §§8.1-8.4] for a discussion of various examples over a field which is either algebraically closed or of characteristic 0 and [op.cit., §8.5] for a study of the Brauer group of a reduced and separated curve over a field of characteristic 0. A natural method for studying the Brauer group of a singular variety \( X \) in terms of a desingularization \( \tilde{X} \to X \) (when one is available) is

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to first compute the kernel of the pullback map \( Br_X \to Br\tilde{X} \) induced by \( \tilde{X} \to X \) and then apply Grothendieck's purity theorem \([\text{Ces19}]\) to the smooth variety \( \tilde{X} \) in order to obtain information about \( Br\tilde{X} \). This is, essentially, the approach adopted in \([\text{BL20}]\). If no desingularization of \( X \) is available, the above method will retain its value if a covering variety \( \tilde{X} \) of \( X \) can be found such that the structure of \( Br\tilde{X} \) is more accessible than that of \( BrX \). In this paper we implement this idea to study the algebraic Brauer group of a projective and geometrically integral \( k \)-scheme with a finite non-normal locus over an arbitrary field \( k \). To be precise, let \( f: X \to \text{Spec} \ k \) be a proper and geometrically integral \( k \)-scheme, let \( k^s \) be a fixed separable closure of \( k \) and set \( X^s = X \times_k k^s \). The structure of the algebraic Brauer group \( Br_1X = \ker[BrX \to BrX^s] \) of \( X \) is closely related to that of the Picard scheme \( \text{Pic}_{X/k} \) via a canonical exact sequence of abelian groups

\[
0 \to \text{Pic}_X \to (\text{Pic}_X^s)^\Gamma \to \text{Br} k \xrightarrow{\nu^*} Br_1X \to H^1_{\text{ét}}(k, \text{Pic}_{X/k}) \xrightarrow{d} H^3_{\text{ét}}(k, \mathbb{G}_m),
\]

where \( \Gamma = \text{Gal}(k^s/k) \). Unfortunately, the structure of \( \text{Pic}_{X/k} \) is unknown in general and determining this structure, especially over an imperfect field \( k \), is a difficult unsolved problem. However, two important advances on this problem have been made in the past few years under certain restrictions. Firstly, Geisser \([\text{Gei09}]\) advanced our understanding of the structure of the Picard variety of an arbitrary reduced and proper scheme \( X \) over a perfect field \( k \). His results imply, \textit{inter alia}, that if \( \nu = \nu_X: X^N \to X \) is the normalization morphism of \( X \), then the structure of \( \ker[\nu^*: Br_1X \to Br_1X^N] \) is closely related to the \( \Gamma \)-module structure of \( H^1_{\text{ét}}(X^s, \mathbb{Z}) \).

Secondly, if \( k \) is an arbitrary field and \( \nu \) is as above, Brion \([\text{Bri15}]\) established the surjectivity of the pullback morphism \( \nu^*: \text{Pic}_{X/k} \to \text{Pic}_{X^N/k} \) and described its kernel for any projective\(^2\) and geometrically integral \( k \)-scheme \( X \) with a finite non-normal locus. In fact, Brion worked in a much more general setting that we now (partially) describe over a field. Let \( \tilde{X} \) be a projective and geometrically integral \( k \)-scheme, let \( \tilde{Y} \) be a finite closed subscheme of \( \tilde{X} \) with corresponding inclusion morphism \( \tilde{\iota}: \tilde{Y} \to \tilde{X} \) and let \( \psi: \tilde{Y} \to Y \) be a schematically dominant morphism between finite \( k \)-schemes. By \([\text{Fer}, \S 5]\), there exists a pushout \( X \) of \( (\tilde{\iota}, \psi) \) in the category of \( k \)-schemes, i.e., a cocartesian diagram

\[
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{\iota}} & \tilde{X} \\
\psi \downarrow & & \downarrow \varphi \\
Y & \xrightarrow{\iota} & X,
\end{array}
\]

where \( \iota \) is a closed immersion, \( \varphi \) is finite and the morphism \( \tilde{X} \setminus \tilde{Y} \to X \setminus Y \) (induced by \( \varphi \)) is an isomorphism. The \( k \)-scheme \( X \) is projective and geometrically integral

\(^1\)We will not discuss this structure here.

\(^2\)A weaker hypothesis suffices but we assume projectivity here to simplify the exposition.
and the above diagram is cartesian as well. We say that $X$ is obtained by pinching $\tilde{X}$ along $\tilde{Y}$ via $\psi$ and that (0.2) is the pinching diagram associated with $(\tilde{X}, \tilde{Y}, \psi)$. Conversely, every projective and geometrically integral $k$-scheme $X$ with a finite non-normal locus can be obtained via the above construction. Indeed, if $Y$ is the closed subscheme of $X$ defined by the ideal $\text{Ann}_{\mathcal{O}_X}(\nu^*O_{X^N}/\mathcal{O}_X)$, then $X$ can be recovered by pinching $\tilde{X} = X^N$ along $\tilde{Y} := Y \times_X \tilde{X}$ via the first projection $\psi: \tilde{Y} \to Y$. See [Lan §3.1]. Now, by Brion’s key insight [Bri15 Lemma 2.2], the pinching diagram (0.2) induces an exact sequence of locally algebraic $k$-group schemes

\begin{equation}
0 \to \mu^{\tilde{Y}/\tilde{Y}} \to \text{Pic}_{X/k} \xrightarrow{\varphi^*} \text{Pic}_{\tilde{X}/k} \to 0,
\end{equation}

where $\mu^{\tilde{Y}/\tilde{Y}}$ is smooth, connected, affine and algebraic. See [Bri15, Remark 2.6]. In this paper we combine, in a rather elaborate way, the sequences (0.1) and (0.3) to prove the triviality of $\text{Coker} \varphi^*$. Further, we bound (and, in some cases, prove the triviality of) $\text{Coker} \varphi^*$ in terms of Brauer groups of fields and analogs, in cohomological degree 3, of relative Brauer groups of finite field extensions. In order to state our results, we introduce the following definitions. If $f: X \to S$ is a morphism of schemes, we write $\text{Br}(X/S)$ for the kernel of the pullback map $\varphi^*_f: \text{Br}_1 X \to \text{Br}_1 \tilde{X}$ solely in terms of the Brauer groups of the residue fields of $Y$ and $\tilde{Y}$ and the Amitsur subgroups of $X$ and $\tilde{X}$ in $\text{Br} k$, i.e., the images of the maps $(\text{Pic } X^s)^* \to \text{Br } k$ and $(\text{Pic } \tilde{X}^s)^* \to \text{Br } k$ in the sequence (0.1) for $X$ and $\tilde{X}$. Further, we bound (and, in some cases, prove the triviality of) $\text{Coker} \varphi^*$ in terms of Brauer groups of fields and analogs, in cohomological degree 3, of relative Brauer groups of finite field extensions. In order to state our results, we introduce the following definitions. If $f: X \to S$ is a morphism of schemes, we write $\text{Br}(X/S)$ for the kernel of the pullback map $\text{Br}(X/S) \to \text{Br}(X/k)$ induced by $f$. If $S = \text{Spec } k$, where $k$ is a field, $\text{Br}(X/S)$ will be denoted by $\text{Br}(X/k)$. We have $\text{Br}(X/k) = \text{Ker} [f^*: \text{Br } k \to \text{Br}_1 X]$, which is an abelian group of finite exponent annihilated by the index $I(X)$ of $X$ over $k$, i.e., the least positive degree of a 0-cycle on $X$. If $L/k$ is a field extension, then $\text{Br}(\text{Spec } L/k) = H^3(L/k) = \text{Ker}[\text{Br } k \to \text{Br } L]$, which is the \textit{relative Brauer group} of $L/k$. If $X$ is $k$-proper and geometrically integral then, by the exactness of (0.1), $\text{Br}(X/k)$ agrees with the Amitsur subgroup of $X$ in $\text{Br} k$. Now set $\text{Br}_a X = \text{Coker} [f^*: \text{Br } k \to \text{Br}_1 X]$. Then (0.1) induces isomorphisms of abelian groups

\begin{equation}
\text{Br}(X/k) = \text{Coker }[\text{Pic } X \leftrightarrow (\text{Pic } X^s)^*] \end{equation}

and

\begin{equation}
\text{Br}_a X = \text{Ker }[H^1_{\text{et}}(k, \text{Pic } X/k) \xrightarrow{d} H^3_{\text{et}}(k, \mathbb{G}_m)],
\end{equation}

where $d$ is the differential in (0.1). We will also need the following analog of $\text{Br}(X/k)$ in cohomological degree 3:

\begin{equation}
H^3(X/k) = \text{Ker }[H^3_{\text{et}}(k, \mathbb{G}_m) \to H^3_{\text{et}}(X, \mathbb{G}_m)].
\end{equation}

If $\varphi: \tilde{X} \to X$ is an arbitrary morphism of $k$-schemes, $\varphi^*_a: \text{Br}_a X \to \text{Br}_a \tilde{X}$ will denote the natural map induced by $\varphi^*_a: \text{Br}_1 X \to \text{Br}_1 \tilde{X}$. Note that

\begin{equation}
\text{Br}(X/k) = \text{Ker }[\text{Br } k \xrightarrow{f^*} \text{Br}_1 X] \subseteq \text{Ker }[\text{Br } k \xrightarrow{\varphi^*_a f^*} \text{Br}_1 \tilde{X}] = \text{Br}(\tilde{X}/k) \subseteq \text{Br } k.
\end{equation}
The subquotient $\text{Br}(\bar{X}/k)/\text{Br}(X/k)$ of $\text{Br} k$ intervenes frequently in our considerations. The main results of this paper are summarized in the following statement.

**Theorem 0.1.** Let $k$ be an arbitrary field and let $(\bar{X}, \bar{Y}, \psi)$ be as above, i.e., $\bar{X}$ is a projective and geometrically integral $k$-scheme, $\bar{Y}$ is a finite closed subscheme of $\bar{X}$ and $\psi: \bar{Y} \to Y$ is a schematically dominant morphism between finite $k$-schemes. If $X$ is obtained by pinching $\bar{X}$ along $\bar{Y}$ via $\psi$, then the pinching diagram \((0.2)\) induces

(i) an injection of abelian groups

$$\frac{\text{Br}(\bar{X}/k)}{\text{Br}(X/k)} \hookrightarrow \prod_{y \in Y, \bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y)),$$

(ii) a canonical exact sequence of abelian groups

$$0 \to \text{Coker} \left[ \frac{\text{Br}(\bar{X}/k)}{\text{Br}(X/k)} \to \prod_{y \in Y, \bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y)) \right] \to \text{Br}_a \bar{X} \xrightarrow{\varphi} \text{Br}_a X \to H^2_{\text{et}}(k, \mu_{\bar{Y}/Y}),$$

where $H^2_{\text{et}}(k, \mu_{\bar{Y}/Y})$ is an extension

$$0 \to \prod_{y \in Y} \text{Coker} \left[ \text{Br}(\kappa(y)) \to \prod_{\bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y)) \right] \to H^2_{\text{et}}(k, \mu_{\bar{Y}/Y}) \to \prod_{y \in Y, \bar{y} \in \bar{Y}_y} H^3(\kappa(\bar{y})/\kappa(y)) \to 0,$$

and

(iii) an exact sequence of abelian groups

$$0 \to \text{Ker} \varphi^*_1 \to \text{Br}_1 X \xrightarrow{\varphi^*_1} \text{Br}_1 \bar{X} \to \text{Coker} \varphi^*_a \to 0,$$

where $\text{Ker} \varphi^*_1$ is an extension

$$0 \to \frac{\text{Br}(\bar{X}/k)}{\text{Br}(X/k)} \to \text{Ker} \varphi^*_1 \to \text{Coker} \left[ \frac{\text{Br}(\bar{X}/k)}{\text{Br}(X/k)} \to \prod_{y \in Y, \bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y)) \right] \to 0.$$

The group $\prod_{y \in Y, \bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y))$ which appears in the above statement, and therefore also $\text{Br}(\bar{X}/k)/\text{Br}(X/k)$, is an abelian group of finite exponent annihilated by $\text{lcm}\{I(\bar{Y}_y) : y \in Y\}$. See Remark 3.5. Here $\bar{Y}_y = \psi^{-1}(y)$ is the fiber of $\psi$ at $y \in Y$.

**Corollary 0.2.** Let $k$ be any field and let $X$ be a projective and geometrically integral $k$-scheme with a finite non-normal locus $Y$ such that $X^N$ is $k$-isomorphic to $\mathbb{P}^{\dim X}_k$. Then there exists a canonical isomorphism of abelian groups

$$\text{Br}_1 X = \text{Br} k \oplus \prod_{y \in Y, \bar{y} \in \bar{Y}_y} \text{Br}(\kappa(\bar{y})/\kappa(y)),$$
where $\widetilde{Y}$ is the ramification locus of $X^N \to X$.

See Example 4.1 for a more general statement.

The proof of Theorem 0.1 also yields the following generalization of a theorem established by Ballico and Kollár [BK] when $S$ is the spectrum of a field of characteristic zero.

**Theorem 0.3.** Let $\widetilde{X}$ be a reduced scheme, let $S$ be a connected locally noetherian scheme and let $\widetilde{X} \to S$ be a projective and flat morphism with integral geometric fibers. Further, let $\widetilde{Y}$ be a closed subscheme of $\widetilde{X}$ which is finite and faithfully flat over $S$ and let $\psi: \widetilde{Y} \to Y$ be a schematically dominant morphism between finite and faithfully flat $S$-schemes such that the pullback map $\psi^*: \operatorname{Pic} Y \to \operatorname{Pic} \widetilde{Y}$ is surjective. If $X$ is obtained by pinching $\widetilde{X}$ along $\widetilde{Y}$ via $\psi$, then

$$\operatorname{Br}(X/S) = \operatorname{Br}(\widetilde{X}/S) \cap \operatorname{Br}(Y/S),$$

where the intersection takes place in $\operatorname{Br} S$.

If $S = \operatorname{Spec} k$, where $k$ is a field, the preceding theorem describes the Amitsur subgroup in $\operatorname{Br} k$ of the pinched variety $X$ in terms of the corresponding group for $\widetilde{X}$ and the group $\operatorname{Br}(Y/k) = \bigcap_{y \in Y} \operatorname{Br}(\kappa(y)/k)$.

If $k$ is a local field, $\operatorname{Br} k$ is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$. Consequently, if $X$ is $k$-proper and geometrically integral, then $\operatorname{Br}(X/k) \simeq \operatorname{Coker}[\operatorname{Pic} X \hookrightarrow (\operatorname{Pic} X^s)^\Gamma]$ is a finite subgroup of $\operatorname{Br} k$. If $X$ is smooth and $k$ is a $p$-adic field (i.e., a finite extension of $\mathbb{Q}_p$), Roquette and Lichtenbaum showed that the order of $\operatorname{Br}(X/k)$ is equal to the index of $X$ over $k$. See [Roq, Theorem 1] and [Lich, Theorem, p. 120]. In section 5 we apply Theorem 0.3 above with $S = \operatorname{Spec} k$, where $k$ is any local field, and extend the Roquette-Lichtenbaum theorem to a class of (not necessarily smooth) proper and geometrically integral $k$-curves that contains all proper and geometrically integral $k$-curves if $k$ is a $p$-adic field. The precise statement is the following.

**Corollary 0.4.** Let $X$ be a proper and geometrically integral curve over a local field $k$. If $X^N$ is smooth, then

$$[\operatorname{(Pic} X^s)^\Gamma: \operatorname{Pic} X] = I(X),$$

where $I(X)$ is the index of $X$ over $k$.

If $k$ is a local function field, i.e., a finite extension of $\mathbb{F}_p((t))$, then the above corollary applies, in particular, to (proper and geometrically integral) local complete intersection $k$-curves whose Jacobian numbers are less than $p$. See [LL20, Theorem 4.10]. Regarding the above corollary, we call the reader’s attention to the fact that, over an imperfect field such as $\mathbb{F}_p((t))$, there exist many curves $X$ such that the normal (i.e., regular) curve $X^N$ is not smooth. See Examples 4.3 and 4.5 below.
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1. Preliminaries

We say that a scheme $X$ is an FA scheme if every finite subset of $X$ is contained in an affine open subset of $X$. If $X$ admits an ample invertible sheaf, then $X$ is an FA scheme \cite[II, Corollary 4.5.4]{EGA}. In particular, every proper scheme of dimension 1 over a field is an FA scheme \cite[Lemmas 0A26 and 0B45]{SP}.

If $S$ is a scheme, $S_{\text{\acute e t}}$ will denote the category of abelian sheaves on the small \acute e tale site over $S$. If $f: X \to S$ is a morphism of schemes and $r \geq 0$ is an integer, we will write

$$H^r(X/S) := \text{Ker}[H^r_{\text{\acute e t}}(S, \mathbb{G}_m) \xrightarrow{f^*} H^r_{\text{\acute e t}}(X, \mathbb{G}_m)],$$

where $f^*$ is the pullback homomorphism induced by $f$. If $f: \text{Spec } K \to \text{Spec } k$ is the canonical morphism induced by a field extension $K/k$, then the abelian group (1.1) will be denoted by $H^r(K/k)$.

We henceforth assume that $S$ is locally noetherian, so that a morphism $q: T \to S$ is finite and locally free if, and only if, $q$ is finite and flat \cite[02K9, Lemma 29.48.2]{SP}. A finite and flat quasi-section of $f$ is a finite and faithfully flat morphism $q: T \to S$ such that $q = f \circ h$ for some morphism $h: T \to X$. If this is the case, then the induced morphism $(h, 1_T)_S: T \to X_T$ is a section of $f_T: X_T \to T$. If $f$ admits a finite and flat quasi-section of constant degree, then the index $I(X/S)$ of $f$ is the greatest common divisor of the degrees of all finite and flat quasi-sections of $f$ of constant degree. Since the proof of \cite[Proposition 3.8.1, p. 96]{CTSk} is valid in arbitrary cohomological degree, \cite[Remark 3.1]{GA17} remains valid if the word \acute e tale in [loc.cit.] is replaced with the word flat. Consequently, if $I(X/S)$ is defined, then

$$I(X/S) \cdot H^r(X/S) = 0 \quad (1.2)$$

for every $r \geq 1$ by (the flat analog of) \cite[Remark 3.1(d)]{GA17}. If $\tilde{f}: \tilde{X} \to S$ is another morphism of schemes and $\varphi: \tilde{X} \to X$ is an $S$-morphism, then a finite and flat quasi-section of $\tilde{f}$ is also a finite and flat quasi-section of $f$. Consequently, if $I(\tilde{X}/S)$ is defined, then $I(X/S)$ is also defined and divides $I(\tilde{X}/S)$. We also note that, if $f$ is finite and faithfully flat of constant degree, then $I(X/S)$ is defined since we may take $q = f$ and $h = \text{Id}_X$ above.

If $X$ is a scheme over a field $k$, we will write $I(X)$ for $I(X/\text{Spec } k)$. If $X$ is algebraic, i.e., of finite type over $k$, then $I(X)$ is the greatest common divisor of the degrees $[\kappa(x): k]$ of all closed points $x$ of $X$. Equivalently, $I(X)$ is the least positive degree of a 0-cycle on $X$. If $K/k$ is a finite field extension and $X = \text{Spec } K$, then
(1.2) yields

\[ [K:k] \cdot H^r(K/k) = 0 \]

for every \( r \geq 1 \).

We now recall that the (étale) relative Picard functor of \( X \) over \( S \) is the étale sheaf \( \text{Pic}_{X/S} \) on \( S \) associated to the presheaf \( \text{Sch}/S \to \text{Ab} \), \( (T \to S) \mapsto \text{Pic}_T \). We have \( \text{Pic}_{X/S} = R^1_{\text{ ét}} f_* \mathbb{G}_m, X \) by [Klei, §2]. Now, for any scheme \( Y \), let \( \text{Br}_Y = H^2(\text{ét}, G_m) \) be the cohomological Brauer group of \( Y \) and let \( \text{Br}_{X/S} = R^2_{\text{ ét}} f_* \mathbb{G}_m, X \) be the sheaf on \( S_{\text{ ét}} \) associated to the presheaf \( (T \to S) \mapsto \text{Br}_X(T) \). Set

\[ \text{Br}_1 X = \text{Ker}[\text{Br}_X \to \text{Br}_{X/S}(S)] \] (1.4)

where the indicated map is an instance of the canonical adjoint homomorphism \( P(S) \to P^\#(S) \), where \( P \) is a presheaf of abelian groups on \( \text{Sch}/S \) and \( P^\# \) is its associated (étale) sheaf [11, Remark, p. 46]. The pullback map \( f^*: \text{Br}_S \to \text{Br}_X \) factors through \( \text{Br}_1 X \subseteq \text{Br}_X \) [GA18, p. 2754]. Set

\[ \text{Br}(X/S) = \text{Ker}[\text{Br}_S \to \text{Br}_X] \] (1.5)

and

\[ \text{Br}_a X = \text{Coker}[\text{Br}_S \to \text{Br}_1 X] \] (1.6)

Since \( \text{Br}(X/S) = H^2(X/S) \) by (1.1), the formula (1.2) yields \( I(X/S) \cdot \text{Br}(X/S) = 0 \) when \( I(X/S) \) is defined.

**Remarks 1.1.** Let \( k \) be a field.

(a) If \( f: X \to \text{Spec} k \) is quasi-compact and quasi-separated, then \( \text{Pic}_{X/k}(k) = \text{Pic}(X^s)^f \) and \( \text{Br}_{X/k}(k) = \text{Br}(X^s)^f \) by [11, II, Corollary 2.2(ii), p. 94, and Theorem 6.4.1, p. 128]. Therefore \( \text{Br}_1 X = \text{Ker}[\text{Br}_X \to \text{Br}_{X^s}] \) (1.4) is the algebraic Brauer group of \( X \) over \( k \).

(b) If \( X \) is a proper algebraic curve over \( k \), then \( \text{Br}_a X = 0 \) by [GB, Corollary 5.8, p. 132], whence \( \text{Br}_1 X = \text{Br}_X \).

(c) By (b) and [CTSk, Theorem 5.6.1(vii), p. 148], \( \text{Br}_1 \mathbb{P}_k^1 = \text{Br} \mathbb{P}_k^1 = \text{Br} k \).

**Lemma 1.2.** If \( \mathcal{A} \) is an abelian category and \( f \) and \( g \) are morphisms in \( \mathcal{A} \) such that \( g \circ f \) is defined, then there exists a canonical exact sequence in \( \mathcal{A} \)

\[ 0 \to \text{Ker } f \to \text{Ker } (g \circ f) \to \text{Ker } g \to \text{Coker } f \to \text{Coker } (g \circ f) \to \text{Coker } g \to 0. \]

**Proof.** See, for example, [Bey, 1.2]. \( \square \)

If \( k \) is a field, a \( k \)-split unipotent \( k \)-group is an algebraic \( k \)-group that admits a composition series whose successive quotients are each \( k \)-isomorphic to \( \mathbb{G}_{a,k} \).

**Lemma 1.3.** Let \( k \) be a field and let \( U \) be a commutative, smooth, connected and unipotent algebraic \( k \)-group.

(i) If \( U \) is \( k \)-split, then \( H^r_{\text{ ét}}(k, U) = 0 \) for every \( r \geq 1 \).
(ii) \( H^r_{\text{ét}}(k, U) = 0 \) for every \( r \geq 2 \).

Proof. Assertion (i) follows from the case \( U = \mathbb{G}_{a,k} \) [Ser2, X, Proposition 1, p. 150] by induction on the length of a composition series for \( U \). In (ii), if \( \text{char } k = 0 \) (or, more generally, if \( k \) is perfect), then \( U \) is \( k \)-split by [Bo, Corollary 15.5(ii), p. 205], whence (ii) follows from (i). Thus in (ii) we may assume that \( \text{char } k = p > 0 \).

Since \( U \) is annihilated by some power of \( p \) [SGA3 new, XVII, Theorem 3.5] and the \( p \)-cohomological dimension of \( \text{Gal}(k^s/k) \) is \( \leq 1 \) [CG, II, §2.2, Proposition 3, p. 86], we have \( H^r_{\text{ét}}(k, U) = H^r(\text{Gal}(k^s/k), U(k^s)) = 0 \) for every \( r \geq 2 \), as asserted. \( \square \)

The following lemma and its proof were provided by Michel Brion.

Lemma 1.4. Let \( k \) be a field and let \( X \) be a geometrically integral \( k \)-scheme. Then the normalization \( X^N \) of \( X \) is geometrically integral.

Proof. We may assume that \( X \) is affine. Then \( X^N = \text{Spec } R \) is also affine. Since \( R \) is a subring of the function field \( k(X^N) \), which coincides with \( k(X) \) by [SP, 035E, Lemma 0BXC], \( R \otimes_k L \) is a subring of \( k(X) \otimes_k L \) for any field extension \( L/k \). Since \( k(X) \otimes_k L \) is integral, the lemma follows. \( \square \)

2. Pinched schemes

Recall that \( S \) is a locally noetherian scheme. Let \( \tilde{X} \) be a reduced FA scheme, let \( \tilde{f}: \tilde{X} \to S \) be a proper and flat morphism with integral geometric fibers and let \( \tilde{Y} \) be a closed subscheme of \( \tilde{X} \) which is finite, faithfully flat and of constant degree over \( S \). Further, let \( Y \) be a finite and faithfully flat \( S \)-scheme and let \( \psi: \tilde{Y} \to Y \) be a schematically dominant morphism of \( S \)-schemes, i.e, the induced morphism of Zariski sheaves \( \mathcal{O}_Y \to \psi_* \mathcal{O}_Y \) is injective [EGA I new, Proposition 5.4.1 and Definition 5.4.2, pp. 283-284]. By [Bri15, §2.1], there exists a cartesian and cocartesian diagram of \( S \)-schemes

\[
\begin{array}{ccc}
\tilde{Y} & \hookrightarrow & \tilde{X} \\
\psi \downarrow & & \downarrow \varphi \\
Y & \hookrightarrow & X,
\end{array}
\]

where: \( \iota \) is the inclusion, \( \iota \) is a closed immersion, \( \varphi \) is finite and surjective, the morphism \( \tilde{X} \setminus \tilde{Y} \to X \setminus Y \) induced by \( \varphi \) is an isomorphism, \( X \) is an FA scheme and the structural morphism \( f: X \to S \) is proper and flat with integral geometric fibers. See [Bri15] comment after the proof of Lemma 2.1. We say that \( X \) is obtained by pinching \( \tilde{X} \) along \( \tilde{Y} \) via \( \psi \). Note that, since \( I(\tilde{Y}/S) \) is defined, \( I(\tilde{X}/S), I(Y/S) \) and \( I(X/S) \) are also defined and the following holds

\[
I(X/S) \mid \gcd\{I(\tilde{X}/S), I(Y/S)\}.
\]
The pinching diagram (2.1) induces the right-hand square of the following commutative diagram of abelian groups

\[
\begin{array}{cccccc}
\text{Br} S & \xrightarrow{f^*} & \text{Br} X & \xrightarrow{\varphi^*} & \text{Br} \tilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Br} S & \xrightarrow{(f \circ \iota)^*} & \text{Br} Y & \xrightarrow{\psi^*} & \text{Br} \tilde{Y}.
\end{array}
\]

Applying Lemma 1.2 to the rows of the above diagram, we obtain an exact and commutative diagram of abelian groups

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Br}(X/S) & \xrightarrow{\subseteq} & \text{Br}(\tilde{X}/S) & \xrightarrow{f^*} & \text{Br}(\tilde{X}/X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Br}(Y/S) & \xrightarrow{\subseteq} & \text{Br}(\tilde{Y}/S) & \xrightarrow{(f \circ \iota)^*} & \text{Br}(\tilde{Y}/Y),
\end{array}
\]

where all groups are instances of (1.5) and all arrows on the left-hand square are inclusion maps between subgroups of \( \text{Br} S \).

In the next statement, \( R_{Y/S} \) (respectively, \( R_{\tilde{Y}/S} \)) denotes the Weil restriction functor associated to the finite and locally free morphisms \( Y \rightarrow S \) (respectively, \( \tilde{Y} \rightarrow S \)). See [BLR, §7.6].

**Theorem 2.1.** (Brion) The pinching diagram (2.1) induces an exact sequence in \( S_{\tilde{\alpha}} \)

\[
0 \rightarrow R_{Y/S}(\mathbb{G}_{m,Y}) \xrightarrow{\psi^*} R_{\tilde{Y}/S}(\mathbb{G}_{m,\tilde{Y}}) \rightarrow \text{Pic}_{X/S} \xrightarrow{\varphi^*} \text{Pic}_{\tilde{X}/S} \rightarrow 0.
\]

**Proof.** See [Bri15, Corollary 2.3].

The exact sequence of the theorem induces the following exact sequences in \( S_{\tilde{\alpha}} \):

\[
(2.4) \quad 0 \rightarrow R_{Y/S}(\mathbb{G}_{m,Y}) \xrightarrow{\psi^*} R_{\tilde{Y}/S}(\mathbb{G}_{m,\tilde{Y}}) \rightarrow \mu^{\tilde{Y}/Y} \rightarrow 0
\]

and

\[
(2.5) \quad 0 \rightarrow \mu^{\tilde{Y}/Y} \rightarrow \text{Pic}_{X/S} \xrightarrow{\varphi^*} \text{Pic}_{\tilde{X}/S} \rightarrow 0,
\]

where

\[
(2.6) \quad \mu^{\tilde{Y}/Y} := R_{\tilde{Y}/S}(\mathbb{G}_{m,\tilde{Y}})/\psi^*(R_{Y/S}(\mathbb{G}_{m,Y})) \in S_{\tilde{\alpha}}.
\]

If \( (Y, \tilde{Y}) = (\text{Spec } A, \text{Spec } B) \), we will write \( \mu^{B/A} \) for \( \mu^{\tilde{Y}/Y} \).

By [T] Theorem 6.4.2(ii), p. 128, \( H^r_{\tilde{\alpha}}(S, R_{Y/S}(\mathbb{G}_{m,Y})) = H^r_{\tilde{\alpha}}(Y, \mathbb{G}_m) \) for every integer \( r \geq 0 \), and similarly for \( \tilde{Y} \). Thus (2.4) and (2.5) induce the following exact sequences of abelian groups:

\[
(2.7) \quad 0 \rightarrow \text{Pic} \tilde{Y}/\psi^*(\text{Pic } Y) \rightarrow H^1_{\tilde{\alpha}}(S, \mu^{\tilde{Y}/Y}) \xrightarrow{\partial^1} \text{Br } Y \xrightarrow{\psi^*} \text{Br } \tilde{Y} \\
\quad \rightarrow H^2_{\tilde{\alpha}}(S, \mu^{\tilde{Y}/Y}) \rightarrow H^3_{\tilde{\alpha}}(Y, \mathbb{G}_m) \xrightarrow{\psi^*} H^3_{\tilde{\alpha}}(\tilde{Y}, \mathbb{G}_m) \rightarrow \ldots
\]
and

\[
\text{Pic}_X(S) \to \text{Pic}_{\widetilde{X}/S}(S) \xrightarrow{\partial^0} H^1_{\text{ét}}(S, \mu^Y) \to H^1_{\text{ét}}(S, \text{Pic}_X)
\]

where the maps \(\partial^1\) and \(\partial^0\) are connecting homomorphisms in étale cohomology.

Next, since \(\widetilde{\mathcal{X}} \to S\) is proper and flat with integral geometric fibers, the canonical map \(\mathcal{O}_T \to (\tilde{\mathcal{f}}_T)_*\mathcal{O}_{\tilde{X}}\) is an isomorphism of Zariski sheaves on \(T\) for every \(S\)-scheme \(T\) \([\text{Klei}, \text{Exercise 9.3.11, pp. 260 and 303}]\). Consequently, the Cartan-Leray spectral sequence \(H^r_{\text{ét}}(S, R^s\tilde{\mathcal{f}}_*\mathbb{G}_m, \tilde{X}) \Rightarrow H^{r+s}(\tilde{X}, \mathbb{G}_m)\) induces an exact sequence of abelian groups

\[
0 \to \text{Pic}_S \to \text{Pic}_{\tilde{X}} \to \text{Pic}_{\tilde{X}/S}(S) \xrightarrow{d^{0,1}} \text{Br} \to H^1_{\text{ét}}(S, \text{Pic}_{\tilde{X}/S}) \to H^3_{\text{ét}}(\tilde{X}/S),
\]

where \(H^3_{\text{ét}}(\tilde{X}/S)\) and \(\text{Br}_{\tilde{X}}\) are the groups \([1.1]\) and \([1.4]\), respectively (cf. \([\text{CTSk}, \text{Proposition 5.4.2, p. 138}]\)).

When \(S = \text{Spec } k\), where \(k\) is a field, we have \(\text{Pic}_S = 0\) and \(\text{Pic}_{\tilde{X}/k}(k) = (\text{Pic}_{\tilde{X}})^F\) by Remark \([1.1](a)\), whence the third map in \((2.9)\) is an injection \(\text{Pic}_{\tilde{X}} \to (\text{Pic}_{\tilde{X}})^F\). Identifying \(\text{Pic}_{\tilde{X}}\) with its image in \((\text{Pic}_{\tilde{X}})^F\) under the preceding injection, the map \(d^{0,1}_{2,X}\) in \((2.9)\) for \(S = \text{Spec } k\) induces an isomorphism of abelian groups

\[
(\text{Pic}_{\tilde{X}})^F/\text{Pic}_{\tilde{X}} \cong \text{Br}(\tilde{X}/k).
\]

**Lemma 2.2.** The following diagram commutes (up to sign)

\[
\begin{array}{ccc}
\text{Pic}_{\tilde{X}/S}(S) & \xrightarrow{\partial^0} & H^1_{\text{ét}}(S, \mu^Y) \\
\downarrow \pi^{0,1}_{2,X} & & \downarrow \pi^{1}_{2,X} \\
\text{Br}(\tilde{X}/S) & \xrightarrow{(\text{f}_0)_*} & \text{Br}(\tilde{Y}/S)
\end{array}
\]

where \(\pi^{0,1}_{2,X}\) is induced by the first differential in \((2.9)\), the bottom row comes from diagram \((2.3)\), \(\partial^0\) is the second map in \((2.8)\) and \(\pi^{1}_{2,X}\) is induced by the third map in \((2.7)\).

**Proof.** Since \(\tilde{X}\) is reduced, there exists a canonical exact sequence in \(\tilde{X}_{\text{ét}}\)

\[
0 \to \mathcal{G}_{m, \tilde{X}} \to \mathcal{R}^*_X \to \mathcal{D} \text{iv}_{\tilde{X}} \to 0,
\]

where \(\mathcal{R}^*_X\) (respectively, \(\mathcal{D} \text{iv}_{\tilde{X}}\)) is the étale sheaf of invertible rational functions (respectively, Cartier divisors) on \(\tilde{X}\) \([\text{GB}, \text{p. 71}]\). Since \(\tilde{f}_*\mathcal{G}_{m, \tilde{X}} = \mathcal{G}_{m,S}\) and \(R^1\tilde{f}_*\mathcal{R}^*_X = 0\),
by [GB Lemma 1.6] and [MiEt Proposition III.1.13, p. 88], the sequence (2.12) induces an exact sequence in $\mathcal{S}_\mathcal{E}^\sim$

\begin{equation}
0 \to \mathbb{G}_{m,S} \to \tilde{f}_*\mathcal{R}_X^* \xrightarrow{\tilde{q}} \tilde{f}_*\text{Div}_X \to \text{Pic}_{\tilde{X}/S} \to 0,
\end{equation}

which gives rise to two short exact sequences

\begin{equation}
0 \to \mathbb{G}_{m,S} \to \tilde{f}_*\mathcal{R}_X^* \to \text{Im} \tilde{q} \to 0
\end{equation}

and

\begin{equation}
0 \to \text{Im} \tilde{q} \to \tilde{f}_*\text{Div}_X \to \text{Pic}_{\tilde{X}/S} \to 0.
\end{equation}

The sequences (2.13), (2.14) and (2.15) are of the form [CTSS7 (1.A.8), (1.A.9) and (1.A.10) (respectively), pp. 398-399]. Thus, by the commutativity (up to sign) of [CTSS7 diagram (1.A.11), p. 399], the differential $d_{2,X}^{0,1} : \text{Pic}_{\tilde{X}/S}(S) \to \text{Br}S$ in the sequence (2.9) is (up to sign) the composition

$$\text{Pic}_{\tilde{X}/S}(S) \xrightarrow{\delta^0} H^1_{\mathcal{E}^\text{et}}(S, \text{Im} \tilde{q}) \xrightarrow{\delta^1} \text{Br}S,$$

where the first (respectively, second) map is the connecting homomorphism induced by the sequence (2.15) (respectively, (2.14)). The lemma then follows from the commutativity of the diagram

\begin{equation}
\begin{array}{ccc}
\text{Pic}_{\tilde{X}/S}(S) & \xrightarrow{\delta^0} & H^1_{\mathcal{E}^\text{et}}(S, \mu_{\tilde{Y}/Y}) \\
\downarrow \delta^0 & & \downarrow \delta^1 \\
H^1_{\mathcal{E}^\text{et}}(S, \text{Im} \tilde{q}) & \xrightarrow{\delta^3} & \text{Br}S \xrightarrow{(f_{\text{et}})^*} \text{Br}Y
\end{array}
\end{equation}

which, in turn, follows from the functoriality of étale cohomology. □

The sequence (2.9) induces the following exact sequences of abelian groups:

\begin{equation}
0 \to \text{Pic} \tilde{X}/\text{Pic} S \to \text{Pic}_{\tilde{X}/S}(S) \xrightarrow{d_{2,X}^{0,1}} \text{Br}(\tilde{X}/S) \to 0,
\end{equation}

\begin{equation}
0 \to \text{Br}S / \text{Br}(\tilde{X}/S) \to \text{Br}_{1}\tilde{X} \to \text{Br}_{a}\tilde{X} \to 0
\end{equation}

and

\begin{equation}
0 \to \text{Br}_{a}\tilde{X} \to H^1_{\mathcal{E}^\text{et}}(S, \text{Pic}_{\tilde{X}/S}) \to H^3(X/S),
\end{equation}

where $d_{2,X}^{0,1}$ is the left-hand vertical map in diagram (2.11) and $\text{Br}_{a}\tilde{X}$ is the group (1.6). The sequences (2.16)–(2.18) and their analogs for $f : X \to S$ yield the following
exact and commutative diagrams of abelian groups:

\[ (2.19) \begin{array}{cccccc}
0 & \rightarrow & \text{Pic} \bar{X}/\text{Pic} S & \rightarrow & \text{Pic}_{\bar{X}/S}(S) & \rightarrow & \text{Br}(X/S) & \rightarrow & 0 \\
& & \downarrow & & \downarrow \varphi & & \downarrow \zeta & & \\
0 & \rightarrow & \text{Pic} \tilde{X}/\text{Pic} S & \rightarrow & \text{Pic}_{\tilde{X}/S}(S) & \rightarrow & \text{Br}(\tilde{X}/S) & \rightarrow & 0,
\end{array} \]

\[ (2.20) \begin{array}{cccccc}
0 & \rightarrow & \text{Br} S/\text{Br}(X/S) & \rightarrow & \text{Br}_1 X & \rightarrow & \text{Br}_a X & \rightarrow & 0 \\
& & \downarrow & & \varphi^*_1 & & \varphi^*_2 & & \\
0 & \rightarrow & \text{Br} S/\text{Br}(\tilde{X}/S) & \rightarrow & \text{Br}_1 \tilde{X} & \rightarrow & \text{Br}_a \tilde{X} & \rightarrow & 0,
\end{array} \]

and

\[ (2.21) \begin{array}{cccccc}
0 & \rightarrow & \text{Br}_a X & \rightarrow & H^1_{\text{ét}}(S, \text{Pic}_{\bar{X}/S}) & \rightarrow & H^3_{\text{ét}}(S, \mathbb{G}_m) \\
& & \varphi^*_1 & & \varphi^* & & \\
0 & \rightarrow & \text{Br}_a \tilde{X} & \rightarrow & H^1_{\text{ét}}(S, \text{Pic}_{\tilde{X}/S}) & \rightarrow & H^3_{\text{ét}}(S, \mathbb{G}_m).
\end{array} \]

The first vertical map in diagram (2.19) is surjective by \[ \text{EGA IV}_4, \text{Proposition 21.8.5(ii)} \], whose hypotheses are satisfied since \( X \setminus Y \) is schematically dense in \( X \) by \[ \text{EGA I}_\text{new}, \text{Proposition 5.4.3, p. 284} \]. Consequently, the map \( \tilde{d}_{2, \tilde{X}}^{0,1} \) in diagram (2.19) induces an isomorphism of abelian groups

\[ (2.22) \tilde{d}_{2, \tilde{X}}^{0,1} : \text{Coker}[\text{Pic}_{X/S}(S) \rightarrow \text{Pic}_{\bar{X}/S}(S)] \cong \text{Br}(\tilde{X}/S)/\text{Br}(X/S). \]

Let

\[ (2.23) \alpha : \text{Br}(\tilde{X}/S)/\text{Br}(X/S) \hookrightarrow H^1_{\text{ét}}(S, \mu_{\tilde{Y}/Y}) \]

be the composition

\[ \text{Br}(\tilde{X}/S)/\text{Br}(X/S) \xrightarrow{\left(\tilde{d}_{2, \tilde{X}}^{0,1}\right)^{-1}} \text{Coker}[\text{Pic}_{X/S}(S) \rightarrow \text{Pic}_{\bar{X}/S}(S)] \xrightarrow{\partial^0} H^1_{\text{ét}}(S, \mu_{\tilde{Y}/Y}), \]

where \( \tilde{d}_{2, \tilde{X}}^{0,1} \) is the isomorphism (2.22) and \( \partial^0 \) is the second map in (2.8).

**Theorem 2.3.** There exists a canonical exact sequence of abelian groups

\[ 0 \rightarrow \text{Coker}\left[ \frac{\text{Br}(\tilde{X}/S)}{\text{Br}(X/S)} \right] \xrightarrow{\alpha} H^1_{\text{ét}}(S, \mu_{\tilde{Y}/Y}) \rightarrow \text{Br}_a X \xrightarrow{\varphi^*_1} \text{Br}_a \tilde{X} \rightarrow H^2_{\text{ét}}(S, \mu_{\tilde{Y}/Y}), \]

where \( \alpha \) is the map (2.23).
Proof. The sequence \((2.8)\) induces an exact sequence of abelian groups

\[
\frac{\text{Br}(\tilde{X}/S)}{\text{Br}(X/S)} \to H^1_{\text{ét}}(S, \mu^{\tilde{Y}/Y}) \to H^1_{\text{ét}}(S, \text{Pic}_X/S) \to \text{Br}(X/S),
\]

where \(\alpha\) is the composition \((2.23)\). On the other hand, diagram \((2.21)\) yields a canonical isomorphism \(\text{Ker}\varphi^* \simeq \text{Ker}\varphi^*\) and an injection \(\text{Coker}\varphi^* \to \text{Coker}\varphi^*\). Thus there exist a canonical isomorphism

\[
\text{Ker}\varphi^* \simeq \text{Coker}\left[\frac{\text{Br}(\tilde{X}/S)}{\text{Br}(X/S)} \to H^1_{\text{ét}}(S, \mu^{\tilde{Y}/Y})\right]
\]

and a canonical injection \(\text{Coker}\varphi^* \to H^2_{\text{ét}}(S, \mu^{\tilde{Y}/Y})\). The theorem is now clear. \(\square\)

Corollary 2.4. There exists a canonical exact sequence of abelian groups

\[
0 \to \text{Ker} \varphi^*_1 \to \text{Br}_1 X \xrightarrow{\varphi^*_1} \text{Br}_1 \tilde{X} \to \text{Coker} \varphi^*_1 \to 0,
\]

where \(\text{Ker} \varphi^*_1\) is an extension

\[
0 \to \frac{\text{Br}(\tilde{X}/S)}{\text{Br}(X/S)} \to \text{Ker} \varphi^*_1 \to \text{Ker} \varphi^*_1 \to 0.
\]

Proof. An application of the snake lemma to diagram \((2.20)\) yields an exact sequence of abelian groups

\[
0 \to \frac{\text{Br}(\tilde{X}/S)}{\text{Br}(X/S)} \to \text{Ker} \varphi^*_1 \to \text{Ker} \varphi^*_1 \to 0
\]

and an isomorphism \(\text{Coker} \varphi^*_1 = \text{Coker} \varphi^*_1\). The corollary is now immediate. \(\square\)

The above arguments also yield the following generalization of \([\text{BK}, \text{Theorem 1}]\) (see Remark 3.2 below).

Theorem 2.5. If \(\text{Pic} \tilde{Y} = \psi^*(\text{Pic} Y)\), then

\[
\text{Br}(X/S) = \text{Br}(\tilde{X}/S) \cap \text{Br}(Y/S),
\]

where the intersection takes place in \(\text{Br} S\).

Proof. Diagram \((2.3)\) shows that \(\text{Br}(X/S) \subseteq \text{Br}(\tilde{X}/S) \cap \text{Br}(Y/S)\). Now, by Lemma 2.2, the following diagram commutes (up to sign)

\[
\begin{array}{ccc}
\text{Br}(\tilde{X}/S)/\text{Br}(X/S) & \xrightarrow{\alpha} & H^1_{\text{ét}}(S, \mu^{\tilde{Y}/Y}) \\
\downarrow{\beta} & & \simeq \downarrow{\varphi^1} \\
\text{Br}(Y/S)/\text{Br}(Y/S) & \xrightarrow{(f \circ i)^*} & \text{Br}(\tilde{Y}/Y),
\end{array}
\]

where \(\beta\) is induced by the inclusion \(\text{Br}(\tilde{X}/S) \subseteq \text{Br}(Y/S)\), \(\alpha\) is the map \((2.23)\), \(\varphi^1\) is induced by the map \(\partial^1\) in the sequence \((2.7)\) and \((f \circ i)^*\) is induced by the
map \((f \circ i)^* : \text{Br}(\tilde{Y}/S) \to \text{Br}(\tilde{Y}/Y)\) in diagram (2.3). The map \(\tau_i^1\) above is an isomorphism by the exactness of (2.7) since \(\text{Pic} \tilde{Y}/\psi^*(\text{Pic} Y) = 0\). It follows that the map \(\beta\) in (2.24) is injective, i.e., \(\text{Br}(\tilde{X}/S) \cap \text{Br}(Y/S) \subseteq \text{Br}(X/S)\).

\[\Box\]

3. Pinched varieties

Henceforth, we assume that \(S = \text{Spec} \, k\), where \(k\) is a field. We will write \(p\) for the characteristic exponent of \(k\), i.e., \(p = 1\) if \(\text{char} \, k = 0\) and \(p = \text{char} \, k\) otherwise. In the pinching diagram (2.1), \(X\) and \(\tilde{X}\) are proper and geometrically integral FA \(k\)-schemes and \(\psi : \tilde{Y} \to Y\) is a morphism between finite \(k\)-schemes such that the induced morphism of Zariski sheaves \(\mathcal{O}_Y \to \psi_* \mathcal{O}_{\tilde{Y}}\) is injective. Then \(Y = \text{Spec} \, A\) and \(\tilde{Y} = \text{Spec} \, \tilde{A}\), where \(A = \Gamma(Y, \mathcal{O}_Y) \hookrightarrow \tilde{A} = \Gamma(\tilde{Y}, \mathcal{O}_{\tilde{Y}})\) are artinian \(k\)-algebras \([\text{EGA I}_{\text{new}}, \text{Proposition } 6.5.4, \text{p. } 309]\). We can write \(A = \prod_{y \in Y} \mathcal{O}_{Y, y}\) and \(\tilde{A} = \prod_{\tilde{y} \in \tilde{Y}} \mathcal{O}_{\tilde{Y}, \tilde{y}}\), where \(\mathcal{O}_{Y, y}\) and \(\mathcal{O}_{\tilde{Y}, \tilde{y}}\) are local artinian \(k\)-algebras \([\text{GW}, (12.5), \text{p. } 331]\). Since a local artinian ring is henselian (see, e.g., \([\text{Eis}, \text{Corollary 9.1, p. } 227]\) and \([\text{SP, 04GE, Lemma 10.153.10}]\)), the canonical map \(H^r_{\text{et}}(\mathcal{O}_{Y, y}, \mathbb{G}_m) \to H^r_{\text{et}}(\kappa(y), \mathbb{G}_m)\) is an isomorphism of abelian groups for every \(r \geq 1\) and every \(y \in Y\), and similarly for \(\mathcal{O}_{\tilde{Y}, \tilde{y}}\) \([\text{GB, Theorem 11.7(2), p. } 181]\). Thus the horizontal map in the following canonical commutative diagram of abelian groups is an isomorphism:

\[
\begin{align*}
H^r_{\text{et}}(k, \mathbb{G}_m) &\xrightarrow{\sim} \prod_{y \in Y} H^r_{\text{et}}(\kappa(y), \mathbb{G}_m). \\
\end{align*}
\]

For each \(y \in Y\), the \(y\)-th component of the oblique map above is the pullback homomorphism \(H^r_{\text{et}}(k, \mathbb{G}_m) \to H^r_{\text{et}}(\kappa(y), \mathbb{G}_m)\) induced by the inclusion \(k \subseteq \kappa(y)\). Setting \(r = 1\) in (3.1) (and in its analog for \(\tilde{Y}\)), we obtain \(\text{Pic} Y = \text{Pic} \tilde{Y} = 0\), whence

\[
H^1(k_{\text{et}}, \mu^{\tilde{Y}/Y}) \cong \text{Br}(\tilde{Y}/Y)
\]

by the exactness of the sequence (2.7). Further, the vertical and oblique maps in (3.1) have the same kernel. In particular, setting \(r = 2\) in (3.1), we obtain the equality

\[
\text{Br}(Y/k) = \bigcap_{y \in Y} \text{Br}(\kappa(y)/k),
\]

where the intersection takes place in \(\text{Br} k\).
Theorem 3.1. The following equality between subgroups of $\text{Br} k$ holds:

$$\text{Br}(X/k) = \text{Br}(\tilde{X}/k) \cap \bigcap_{y \in Y} \text{Br}(\kappa(y)/k).$$

Proof. Since $\text{Pic} \tilde{Y} = \psi^*(\text{Pic} Y) = 0$, the theorem follows at once from (3.3) and Theorem 2.5. \hfill \Box

Remark 3.2. The above statement generalizes [BK, Theorem 1], where the equality of Theorem 3.1 is obtained under the following assumptions: $\text{char} k = 0$, $X$ is a $k$-curve with normalization $\tilde{X}$ and $Y$ is the non-normal locus of $X$ (note that in [BK, Theorem 1] the groups $(\text{Pic} \tilde{X}s)/\Gamma$ and $\text{Br} \tilde{X}/k$ have been identified via (2.10), and similarly for $X$).

We now observe that, for every $r \geq 1$, there exists a canonical commutative diagram of abelian groups

\[
\begin{array}{ccc}
H^r_{\text{ét}}(Y, \mathbb{G}_m) & \xrightarrow{\sim} & \prod_{y \in Y} H^r_{\text{ét}}(\kappa(y), \mathbb{G}_m) \\
\downarrow \psi^r & & \downarrow \prod_{y \in Y} \rho_y^r \\
H^r_{\text{ét}}(\tilde{Y}, \mathbb{G}_m) & \xrightarrow{\sim} & \prod_{y \in Y} \prod_{\tilde{y} \in \tilde{Y}_y} H^r_{\text{ét}}(\kappa(\tilde{y}), \mathbb{G}_m)
\end{array}
\]

where, for each $y \in Y$, $\rho_y^*: H^r_{\text{ét}}(\kappa(y), \mathbb{G}_m) \rightarrow \prod_{\tilde{y} \in \tilde{Y}_y} H^r_{\text{ét}}(\kappa(\tilde{y}), \mathbb{G}_m)$ is the map whose $\tilde{y}$-component, for each $\tilde{y} \in \tilde{Y}_y$, is the pullback homomorphism $H^r_{\text{ét}}(\kappa(y), \mathbb{G}_m) \rightarrow H^r_{\text{ét}}(\kappa(\tilde{y}), \mathbb{G}_m)$ induced by the inclusion $\kappa(y) \subseteq \kappa(\tilde{y})$. Clearly, (3.4) induces isomorphisms of abelian groups

\[
H^r(\tilde{Y}/Y) \xrightarrow{\sim} \prod_{y \in Y} \cap_{\tilde{y} \in \tilde{Y}_y} H^r(\kappa(\tilde{y})/\kappa(y))
\]

(3.5)

where, for each $y \in Y$, the intersection takes place in $H^r_{\text{ét}}(\kappa(y), \mathbb{G}_m)$ and

\[
\begin{aligned}
\frac{H^r_{\text{ét}}(\tilde{Y}, \mathbb{G}_m)}{\psi^r(H^r_{\text{ét}}(Y, \mathbb{G}_m))} & \xrightarrow{\sim} \prod_{y \in Y} \text{Coker} \left[ H^r_{\text{ét}}(\kappa(y), \mathbb{G}_m) \rightarrow \prod_{\tilde{y} \in \tilde{Y}_y} H^r_{\text{ét}}(\kappa(\tilde{y}), \mathbb{G}_m) \right].
\end{aligned}
\]

Setting $r = 2$ in (3.5) and (3.6), we obtain canonical isomorphisms

\[
\text{Br}(\tilde{Y}/Y) \xrightarrow{\sim} \prod_{y \in Y} \cap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y))
\]

and

\[
\frac{\text{Br} \tilde{Y}}{\psi^*(\text{Br} Y)} \xrightarrow{\sim} \prod_{y \in Y} \text{Coker} \left[ \text{Br} \kappa(y) \rightarrow \prod_{\tilde{y} \in \tilde{Y}_y} \text{Br} \kappa(\tilde{y}) \right].
\]
Theorem 3.3. The pinching diagram (2.1) induces an exact sequence of abelian groups

\[ 0 \rightarrow \text{Coker} \left[ \frac{\text{Br}(\tilde{X}/k)}{\text{Br}(X/k)} \rightarrow \prod_{y \in Y, \tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)) \right] \rightarrow \text{Br}_a X \xrightarrow{\varphi_a^*} \text{Br}_a \tilde{X} \rightarrow H^2_{\text{ét}}(k, \mu^{\tilde{Y}/Y}), \]

where \( H^2_{\text{ét}}(k, \mu^{\tilde{Y}/Y}) \) is an extension

\[ 0 \rightarrow \prod_{y \in Y} \text{Coker} \left[ \frac{\text{Br} \kappa(y)}{\prod_{\tilde{y} \in \tilde{Y}_y} \text{Br} \kappa(\tilde{y})} \right] \rightarrow H^2_{\text{ét}}(k, \mu^{\tilde{Y}/Y}) \rightarrow \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(\rho)) \rightarrow 0. \]

Proof. By (3.2) and (3.7), there exist canonical isomorphisms

\[ (3.9) \quad H^1_{\text{ét}}(k, \mu^{\tilde{Y}/Y}) \xrightarrow{\sim} \text{Br}(\tilde{Y}/Y) \xrightarrow{\sim} \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)). \]

On the other hand, the exactness of (2.7) yields an extension

\[ 0 \rightarrow \frac{\text{Br}(\tilde{Y})}{\psi^*(\text{Br}(Y))} \rightarrow H^2_{\text{ét}}(k, \mu^{\tilde{Y}/Y}) \rightarrow H^3(\tilde{Y}/Y) \rightarrow 0. \]

The theorem is now immediate from Theorem 2.4 via the isomorphisms (3.3) (for \( r = 3 \)) and (3.8).

Corollary 3.4. There exists a canonical exact sequence of abelian groups

\[ 0 \rightarrow \text{Ker} \varphi_1^* \rightarrow \text{Br}_1 X \xrightarrow{\varphi_1^*} \text{Br}_1 \tilde{X} \rightarrow \text{Coker} \varphi_a^* \rightarrow 0, \]

where \( \text{Ker} \varphi_1^* \) is an extension

\[ 0 \rightarrow \frac{\text{Br}(\tilde{X}/k)}{\text{Br}(X/k)} \rightarrow \text{Ker} \varphi_1^* \rightarrow \text{Coker} \left[ \frac{\text{Br}(\tilde{X}/k)}{\text{Br}(X/k)} \rightarrow \prod_{y \in Y, \tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)) \right] \rightarrow 0. \]

Proof. The corollary follows from Theorem 3.3 in the same way that Corollary 2.4 follows from Theorem 2.3.

Remark 3.5. For each \( y \in Y \), we have \( I(\tilde{Y}_y) = \gcd\{[\kappa(\tilde{y})] : \tilde{y} \in \tilde{Y}_y\} \), whence \( I(\tilde{Y}_y) \cdot \bigcap_{\tilde{y} \in \tilde{Y}_y} H^r(\kappa(\tilde{y})/\kappa(\rho)) = 0 \) for all \( r \geq 1 \) by (1.3). Consequently, if \( m(\tilde{Y}/Y) := \text{lcm}\{I(\tilde{Y}_y) : y \in Y\} \), then \( \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} H^r(\kappa(\tilde{y})/\kappa(\rho)) \) is an \( m(\tilde{Y}/Y) \)-torsion group for every \( r \geq 1 \). In particular, \( \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(\rho)) \), and therefore also \( \text{Br}(\tilde{X}/k)/\text{Br}(X/k) \) and \( H^1_{\text{ét}}(k, \mu^{\tilde{Y}/Y}) \) (3.9), are \( m(\tilde{Y}/Y) \)-torsion groups.
We will now derive some concrete consequences of the preceding general considerations.

If $\psi: \tilde{Y} \to Y$ is a universal homeomorphism then, for every $y \in Y$, $\psi^{-1}(y)$ is a single (closed) point and $\kappa(\psi^{-1}(y))/\kappa(y)$ is a finite and purely inseparable extension [SP, 04DC, Lemma 29.45.5 and 01S2, Lemma 29.10.2]. Set $[\kappa(\psi^{-1}(y)) : \kappa(y)] = p^{n_y}$, where $n_y \geq 0$ is an integer. Then, by [SA, Corollary 11.4.2, p. 437], we have $\text{Br}(\kappa(\psi^{-1}(y))/\kappa(y)) = \text{Br}(\kappa(y))_{p^{n_y}}$, whence

$$\prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)) = \prod_{y \in Y} \text{Br}(\kappa(y))_{p^{n_y}}.$$  

(3.10)

**Proposition 3.6.** Assume that $\psi: \tilde{Y} \to Y$ is a universal homeomorphism. Then there exists a canonical exact sequence of abelian groups

$$0 \to \text{Ker} \varphi_1^* \to \text{Br}_1 X \xrightarrow{\varphi_1^*} \text{Br}_1 \tilde{X} \to 0.$$  

The group $\text{Ker} \varphi_1^*$ is an extension

$$0 \to \frac{\text{Br}(\tilde{X}/k)}{\text{Br}(X/k)} \to \text{Ker} \varphi_1^* \to \text{Coker} \left[ \frac{\text{Br}(\tilde{X}/k)}{\text{Br}(X/k)} \hookrightarrow \prod_{y \in Y} \text{Br}(\kappa(y))_{p^{n_y}} \right] \to 0$$

where, for each $y \in Y$, $p^{n_y} = [\kappa(\psi^{-1}(y)) : \kappa(y)]$.

**Proof.** By [Bri15, Proposition 4.11 and Remark 4.12], $\mu^{\tilde{Y}/Y}$ is a commutative, smooth and connected unipotent algebraic $k$-group. Consequently, $H^2(k_{\text{et}}, \mu^{\tilde{Y}/Y}) = 0$ by Lemma 1.3(ii). The proposition is now immediate from Theorem 3.3 and Corollary (3.4) using (3.10). $\square$

**Corollary 3.7.** Assume that $\psi: \tilde{Y} \to Y$ is a universal homeomorphism and that at least one of the following conditions holds:

(i) $k$ is perfect, or

(ii) $\psi$ induces isomorphisms on residue fields.

Then $\varphi_1^*: \text{Br}_1 X \to \text{Br}_1 \tilde{X}$ is an isomorphism of abelian groups.

**Proof.** If $k$ is perfect, then $\kappa(y)$ is perfect for every $y \in Y$ by [Bou, Corollary 1, p. A.V.43]. Consequently, $\text{Br}(y)$ contains no nontrivial $p$-torsion elements by [SA, Corollary 11.4.5, p. 439], whence $\text{Br}(\kappa(y))_{p^{n_y}} = 0$ for every $y \in Y$. In case (ii) we have $p^{n_y} = [\kappa(\psi^{-1}(y)) : \kappa(y)] = 1$ for all $y \in Y$, whence $\text{Br}(\kappa(y))_{p^{n_y}} = 0$ as well for every $y \in Y$. The corollary is now immediate from the proposition. $\square$

In order to state the final result of this section, we need the following definitions.

A finite morphism of $k$-schemes $X' \to X$ is called a partial seminormalization (of $X$) if (a) $X'$ is reduced, and (b) for every $x \in X$, $(X' \times_X \text{Spec} \kappa(x))_{\text{red}} = \text{Spec} \kappa(x')$ is a one-point scheme and the induced map $\kappa(x) \to \kappa(x')$ is an isomorphism [KK]
Definition 10.11. By [KK, p. 307], there exists a unique largest partial seminormalization of $X$ that dominates every other partial seminormalization of $X$. It is denoted by $\sigma: X^{\text{SN}} \to X$ and called the seminormalization morphism of $X$. If $Y$ is the non-normal locus of $X$, then $X^{\text{SN}}$ can be obtained from $X^N$ via a pinching diagram of the form

\[
(X^N \times_X Y)_{\text{red}} \xrightarrow{\sigma} X^N \\
\downarrow \downarrow \quad \downarrow \\
Z \quad \quad \quad \quad X^{\text{SN}}
\]

for a certain reduced $Y$-scheme $Z$. See [KK, §10.18, p. 310] for more details. In particular, seminormal curves are obtained from their normalizations by pinching points only (to obtain an arbitrary curve $X$ from $X^N$ via pinching, one also needs to pinch tangent vectors on $X^N$ to recover any “cuspidal” singularities that $X$ may have. See [SS, §1]).

**Proposition 3.8.** Let $k$ be any field and let $X$ be a proper and geometrically integral FA $k$-scheme with a finite non-normal locus. If $\sigma: X^{\text{SN}} \to X$ is the seminormalization morphism of $X$, then the pullback map $\sigma^*: \text{Br}_1 X \to \text{Br}_1 X^{\text{SN}}$ is an isomorphism of abelian groups.

**Proof.** By [SP, 0BXQ, Lemma 33.27.1(1), and 035E, Lemma 29.54.5(2)], the normalization morphism $\nu: X^N \to X$ is finite and surjective. Thus, by [EGA, II, Proposition 5.4.2(ii), Theorem 5.5.3(i) and Corollary 6.1.11] and [GLL, §2.2(3)], $X^N$ is a proper FA $k$-scheme. Further, by Lemma [14], $X^N$ is geometrically integral. Thus $X^N$ is a proper and geometrically integral FA $k$-scheme. Consequently, (3.11) is a pinching diagram of the form (2.1), whence $X^{\text{SN}}$ is also a proper and geometrically integral FA $k$-scheme (see the beginning of section 2). We now observe that Overkamp [Ov21, Theorem 2.24(i)] has partially generalized [SP, Lemma 0C1L] by showing that, if $k$ is any field and $\dim X = 1$, then the seminormalization morphism $\sigma: X^{\text{SN}} \to X$ factors as

\[
X^{\text{SN}} = X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \ldots \xrightarrow{\sigma_n} X_{n+1} = X
\]

for some $n \in \mathbb{N}$ where, for each $i = 1, \ldots, n$, $X_{i+1}$ is obtained from $X_i$ via a pinching diagram of the form

\[
\text{Spec} \left( \kappa(x_{i+1})[\epsilon] \right) \xrightarrow{\sigma_i} X_i \\
\downarrow \quad \downarrow \\
\text{Spec} \kappa(x_{i+1}) \xrightarrow{\psi_i} X_{i+1}
\]

for some closed point $x_{i+1} \in X_{i+1}$. A close examination of the proofs of [Ov21, Proposition 2.22, Corollary 2.23 and Theorem 2.24(i)] reveals that these proofs (and
therefore the factorization (3.12) and the associated diagrams (3.13) above) remain valid if the hypothesis \( \dim X = 1 \) is replaced with the condition that the cokernel of the canonical morphism \( O_X \hookrightarrow \nu_* O_{X^N} \) be of finite length, which certainly holds since \( X \) has a finite non-normal locus. Next we observe that, since \( X_1 = X^{SN} \) is a proper and geometrically integral FA \( k \)-scheme, diagram (3.13) with \( i = 1 \) is a pinching diagram of the form (2.1). It then follows that each of the diagrams (3.13) is of the form (2.1). Finally, since the map \( \psi_i \) in (3.13) is a universal homeomorphism that induces isomorphisms on residue fields, Corollary 3.7(ii) shows that the pullback map \( \sigma^*_i : Br_1 X_{i+1} \to Br_1 X_i \) is an isomorphism for every \( i = 1, \ldots, n \). Since \( \sigma^* = \sigma_1^* \circ \cdots \circ \sigma_n^* \), the proposition follows. \( \square \)

4. Examples

Example 4.1. In the pinching diagram (2.1) with \( S = \text{Spec} k \), assume that \( \tilde{X} \) is smooth and universally CH\(_0\)-trivial, i.e., the degree map \( \text{CH}_0(\tilde{X}_F) \to \mathbb{Z} \) is an isomorphism for every field extension \( F/k \). Examples include retract \( k \)-rational varieties, i.e., those \( k \)-schemes \( \tilde{X} \) for which there exists a dominant rational map \( \mathbb{P}^n_k \to \tilde{X} \) with a right inverse \( \tilde{X} \to \mathbb{P}^n_k \) for some \( n \geq 1 \). See [ACTP, §1.2, pp. 34-35]. By [ABBB, Theorem 1.1], the pullback map \( Br F \to Br \tilde{X}_F \) is an isomorphism for every field extension \( F/k \), whence \( Br(\tilde{X}/k) = Br_k \tilde{X} = 0 \). Thus Corollary 3.4 yields a canonical exact sequence of abelian groups

\[
0 \to \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)) \to \text{Br}_1 X \to \text{Br} k \to 0.
\]

The third map in the above sequence admits a canonical homomorphic section, namely the pullback map \( f^* : Br k \to Br_1 X \). Thus there exists a canonical isomorphism of abelian groups

\[
\text{Br}_1 X = Br k \oplus \prod_{y \in Y} \bigcap_{\tilde{y} \in \tilde{Y}_y} \text{Br}(\kappa(\tilde{y})/\kappa(y)).
\]

Example 4.2. In the pinching diagram (2.1) with \( S = \text{Spec} k \), assume that \( \tilde{X} \) is a Severi-Brauer \( k \)-variety, i.e., \( \tilde{X}^s \simeq P^{\dim \tilde{X}}_k \). Then the Amitsur subgroup of \( \tilde{X} \) in \( Br k \) is the finite cyclic subgroup of \( Br k \) generated by the class of \( \tilde{X} \), i.e., \( Br(\tilde{X}/k) = \langle \tilde{X} \rangle \). See [GS, Theorem 5.4.1] and [CTSk, Theorem 3.5.5]. Further, the pullback map \( \tilde{f}^* : Br k \to Br_1 \tilde{X} \) is surjective, i.e., \( Br_1 \tilde{X} = 0 \) [CTSk, p. 183], whence \( \tilde{f}^* \) induces an isomorphism of abelian groups \( \tilde{g} : Br k/\langle \tilde{X} \rangle \to Br_1 \tilde{X} \). Let \( g = \tilde{g}^{-1} \circ \varphi_1^* : Br_1 X \to Br k/\langle \tilde{X} \rangle \), where \( X \) is as in diagram (2.1). Then, by Corollary 3.4, the algebraic Brauer group of the pinched Severi-Brauer variety \( X \) can be described as follows: there exists a canonical exact sequence of abelian groups

\[
0 \to \text{Ker} g \to Br_1 X \xrightarrow{g} Br k/\langle \tilde{X} \rangle \to 0,
\]
where $\text{Ker } g$ is an extension

$$0 \to \frac{(\tilde{X})}{\text{Br}(X/k)} \to \text{Ker } g \to \text{Coker } \left[ \frac{(\tilde{X})}{\text{Br}(X/k)} \hookrightarrow \prod_{y \in Y \subseteq \bar{Y}} \text{Br}(\kappa(\bar{y})/\kappa(y)) \right] \to 0.$$ 

Example 4.3. (cf. [Lau, Example 3.23]) Let $k = F_2(t_1,t_2)$ and let $G$ be the nontrivial form of $G_{a,k}$ given by the equation $y^4 = x + t_1 x^2 + t_2^2 x^4$. The $k$-group $G$ is a 1-dimensional $k$-wound group [CGP, Definition B.2.1, p. 568]. Let $\tilde{X}$ be the normalization of the schematic closure of $G$ in $\mathbb{P}_k^2$. Clearly, $\tilde{X}(k) \neq \emptyset$, whence $\text{Br}(\tilde{X}/k) = 0$.

Now, since $(\tilde{X})^N \simeq \mathbb{P}_k^1$ by [Rus70, proof of Lemma 1.1] and $\tilde{X}$ is obtained by pinching $(\tilde{X})^N$, $\tilde{X}$ is integral, i.e., $\tilde{X}$ is geometrically integral. Further, $\tilde{X} - G = P_\infty$ is a non-smooth closed point with residue field $\kappa(P_\infty) = F_2(t_1^{1/2},t_2^{1/2})$ [Lau, Example 3.23]. For every $c \in k$, set $K_c = k(t_1^{1/2} + ct_2^{1/2}) \subset \kappa(P_\infty)$ and let $X_c$ be obtained by pinching $\tilde{X}$ along $\text{Spec } \kappa(P_\infty)$ via the canonical morphism $\text{Spec } \kappa(P_\infty) \to \text{Spec } K_c$, i.e., $X_c$ is given by the pushout diagram

$$\begin{array}{ccc}
\text{Spec } \kappa(P_\infty) & \xrightarrow{c} & \tilde{X} \\
\downarrow & & \downarrow \nu \\
\text{Spec } K_c & \xrightarrow{\nu} & X_c
\end{array}$$

The preceding diagram is of the form (2.1) since the FA $k$-scheme $\tilde{X}$ is proper and geometrically integral. Now, by [Lau, Proposition 3.2], the map $\nu$ above is the normalization morphism of $X_c$. Further, by [Lau, Lemmas 2.3(iv) and 3.17 and Theorem 3.21(a)], $X_c$ is a seminormal $k$-curve which is equipped with an almost homogeneous $G$-action, i.e., $X_c$ contains a homogeneous $G$-stable open subscheme.

Now we observe that, since $\kappa(P_\infty) = K_c(t_2^{1/2})$, $\kappa(P_\infty)/K_c$ is a purely inseparable quadratic extension. Consequently, Proposition 3.6 yields a canonical exact sequence of abelian groups

$$0 \to \text{Br}(K_c)_2 \to \text{Br}X_c \xrightarrow{\nu^*} \text{Br} \tilde{X} \to 0.$$ 

Note that $\tilde{X}$ is a regular non-smooth $k$-curve. For reasons that are explained in the next remark, and further illustrated in Example 4.5 below, the methods of this paper are useless for determining the structure of $\text{Br} \tilde{X}$.

Remark 4.4. Let $k$ be any imperfect field and let $X$ be a proper and geometrically integral $k$-curve with non-normal locus $Y$. If $X^N$ is geometrically normal, i.e., smooth [EGA II, Corollary 7.4.5, and IV$_4$, Corollary 17.5.2], Corollary 3.4 applied to the normalization morphism $\nu_X : X^N \to X$ relates the Brauer group of $X$ to the (more accessible) Brauer group of the smooth curve $X^N$. However, there exist many proper and geometrically integral $k$-curves whose normalization is not smooth. See
Example 4.3 above and also [KMT, Examples 6.8.2 and 6.12.3(2)], [Ach, Example 1.22] and [To, Example 3.1]. In general, there exists a finite and purely inseparable extension \( K/k \) such that the \( K \)-scheme \( (X_K)^N \) is smooth [SP, 0BXQ, Lemma 33.27.4]. Corollary 3.4 can then be applied to the triple \( ((X_K)^N, X_K, \nu_{X_K} \times_{X_K} Y_K) \) to relate the structure of \( \text{Br} X_K \) to that of \( \text{Br}(X_K)^N \). However, the methods of this paper yield no information about \( \text{Br} X \) itself, as we show in the next example.

Example 4.5. Let \( C \) be any nontrivial form of \( \mathbb{A}^1_k \), i.e., \( C_k' \cong \mathbb{A}^1_{k'} \) for some nontrivial, finite and purely inseparable extension \( k'/k \). The indicated class of affine \( k \)-curves contains the underlying scheme of every 1-dimensional \( k \)-wound group (e.g., the \( k \)-group \( G \) discussed in Example 4.3 above). See [KMT, Proposition 4.7, p. 45]. Now let \( X \) be the regular completion of \( C \) [SP, 0BXX]. Then \( P_\infty = X - C \) is a single (closed) point such that \( k \subseteq \kappa(P_\infty) \subseteq k' \). See [Rus70, Lemma 1.1] and [Ach, Remark 1.17]. The point \( P_\infty \) is often non-smooth, in which case \( X \) is a regular non-smooth \( k \)-curve. Next, let \( k/k' \) be any subextension of \( k'/k \) such that the regular completion of \( C_K \), i.e., \( (X_K)^N \) [SP, 0BXX], Lemma 53.2.9], is \( \mathbb{P}^1_K \) (e.g., \( K = k' \)). Further, let \( P \) (respectively, \( Q \)) be the unique point of \( X_K \) (respectively, \( \mathbb{P}^1_K \)) lying above \( P_\infty \). Then \( X_K \) fits into a pinching diagram

\[
\begin{array}{ccc}
\mathbb{P}^1_K \times_{X_K} Y_K & \rightarrow & \mathbb{P}^1_K \\
\nu_{X_K} \times_{X_K} Y_K & \downarrow & \nu_{X_K} \\
Y_K & \rightarrow & X_K,
\end{array}
\]

where \( \nu_{X_K} : \mathbb{P}^1_K \rightarrow X_K \) is the normalization morphism of \( X_K \). The schemes \( Y_K \) and \( \mathbb{P}^1_K \times_{X_K} Y_K \) are one-point schemes with underlying sets \( P \) and \( Q \), respectively [Ach, §4.1]. Since \( K \subseteq \kappa(P) \subseteq \kappa(Q) \subseteq k' \), the extension \( \kappa(Q)/\kappa(P) \) is purely inseparable, whence \( \psi \) is a universal homeomorphism [Bri15, Proposition 4.11 and Remark 4.12]. Thus, by (3.10) and Corollary 0.2 (see also Example 4.1), there exists a canonical isomorphism of abelian groups

\[
\text{Br} X_K = \text{Br} K \oplus \text{Br}(\kappa(P))_{p^d},
\]

where \( p^d := [\kappa(Q):\kappa(P)] \) divides \( [k':K] \). The preceding isomorphism essentially determines the structure of \( \text{Br} X_K \). However, in order to determine the structure of \( \text{Br} X \), one needs to understand the structure of \( \text{Br}(X_K/X) = \text{Ker}[\text{Br} X \rightarrow \text{Br} X_K] \), which requires methods (to be discussed elsewhere) that are essentially different from those of this paper.

5. The Roquette-Lichtenbaum theorem for a class of singular curves

Unless stated otherwise, in this section \( k \) denotes a non-archimedean local field, i.e., a finite extension of either \( \mathbb{Q}_p \) or \( \mathbb{F}_p((t)) \) for some prime number \( p \). We exclude from our discussion the archimedean local fields since the results of this section
over such fields are either trivial (when $k \simeq \mathbb{C}$) or already contained in $[\text{BK}]$ (when $k \simeq \mathbb{R}$).

The invariant map of local class field theory

\[
\text{inv} : \text{Br } k \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}
\]
is an isomorphism of abelian groups. In particular, every subgroup of finite exponent of $\text{Br } k$ is finite and cyclic, i.e., its order is equal to its exponent. If $X$ is an algebraic $k$-scheme, then $\text{Br}(X/k)$ (1.5) is a subgroup of $\text{Br } k$ that is annihilated by the index $I(X)$ of $X$. Consequently, $\text{Br}(X/k)$ is finite and cyclic and its order divides $I(X)$.

If $K/k$ is a finite field extension, then (5.1) induces an isomorphism of abelian groups

\[
\text{Br}(K/k) \xrightarrow{\sim} [K:k]^{-1}\mathbb{Z}/\mathbb{Z}.
\]

See [Sh, Theorem 37, p. 155].

**Theorem 5.1.** (Roquette-Lichtenbaum) *Let $X$ be a smooth, proper and geometrically irreducible curve over a non-archimedean local field $k$. Then the invariant map (5.1) induces an isomorphism of abelian groups

\[
\text{Br}(X/k) \xrightarrow{\sim} I(X)^{-1}\mathbb{Z}/\mathbb{Z}.
\]

Equivalently, $\text{Br}(X/k)$ is a group of order $I(X)$.*

**Proof.** If $k$ is a $p$-adic field (i.e., a finite extension of $\mathbb{Q}_p$), then the above statement is equivalent to [Lich, Theorem 3, p. 130] (see Remark 5.2 below). If $k$ is a local function field (i.e., a finite extension of $\mathbb{F}_p((t))$), then the proof is formally the same as that of [Lich, Theorem 3, p. 130] using Milne’s extension to the local function field case of Tate’s duality theorem for abelian varieties over $p$-adic fields [ADT, Theorem III.7.8, p. 285]. Here the relevant abelian variety is $\text{Pic}_X^{0,\text{red}}$, which is indeed an abelian variety by the smoothness of $X$ [G, Corollary 3.2].

**Remark 5.2.** Regarding the statement of Theorem 5.1, in [Lich, Theorem 3, p. 130] the term *connected* and the group $\text{Br}(k(X)/k)$, where $k(X)$ is the function field of $X$, appear in place of the term *irreducible* and the group $\text{Br}(X/k)$, respectively. Since $X$ is smooth, connectedness and irreducibility are equivalent concepts for $X$ [GW, Remark 6.37, p. 165]. Further, the pullback map $\text{Br } X \to \text{Br } k(X)$ is injective by [GB, Corollary 1.8, p. 170]. Thus Theorem 5.1 above (over a $p$-adic field) is indeed equivalent to [Lich, Theorem 3, p. 130].

**Theorem 5.3.** *Let $k$ be a non-archimedean local field and let $X$ be a proper and geometrically integral $k$-curve such that $X^N$ is smooth over $k$. Then $\text{Br}(X/k)$ is a group of order $I(X)$.*
Proof. Recall the normalization morphism $\nu : X^N \to X$ and let $Y$ be the non-normal locus of $X$. Since $X^N$ is a proper and geometrically integral FA $k$-scheme (see the proof of Theorem 3.8), the associated conductor square

$$
\begin{array}{ccc}
X^N \times_X Y & \xleftarrow{\nu} & X^N \\
\downarrow \nu & & \downarrow \nu \\
Y & \xleftarrow{\nu} & X
\end{array}
$$

is a pinching diagram of type (2.1). See [Lau, Lemma 3.1 and its preamble] for more details. Now Theorems 3.1 and 5.1 and the isomorphism (5.2) show that the invariant map (5.1) induces an isomorphism of abelian groups

$$\text{Br}(X/k) \cong \left( I(X^N) \cap \prod_{y \in Y} \left( \mathbb{Z}/ \mathbb{Z} \right) \right),$$

where the intersections take place in $\mathbb{Q}/\mathbb{Z}$. Thus $\#\text{Br}(X/k) = \gcd\{ I(X^N), I(Y) \}$, which is divisible by $I(X)$ (2.2). Since $\#\text{Br}(X/k)$ divides $I(X)$, we conclude that $\#\text{Br}(X/k) = \gcd\{ I(X^N), I(Y) \} = I(X)$, as claimed. \qed

Remark 5.4. Let $k$ be any field of characteristic exponent $p$ and let $X$ be an algebraic $k$-scheme. Then $X^N$ is geometrically normal if, and only if, $(X^N)_{k^{1/p}}$ is normal (see [EGA] IV, Proposition 6.7.7 and [Tan, Proposition 2.10(3)]). Thus the hypothesis of Theorem 5.3 certainly holds if $k$ is a $p$-adic field but not, in general, if $k$ is a local function field (see Example 4.3). Thus the methods of this paper are insufficient for obtaining a precise formula for $\#\text{Br}(X/k)$ (if such a formula exists) if $X$ is an arbitrary proper and geometrically integral curve over a local function field $k$.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE LA SERENA, CISTERNAS 1200, LA SERENA 1700000, CHILE

*Email address: cgonzalez@userena.cl*