MEASURABLE SEQUENCES

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ABSTRACT
The paper deals with the distribution functions of sequences with respect to asymptotic density and measure density. Furthermore also polyadically continuous sequences and their extension to random variables are studied.

1 Introduction

In the first part we study sequences having an asymptotic distribution function in the sense of Schoenberg [Sch]. The connection between independence and statistical independence is established in case of continuous distribution functions.

Later we develop relations between distribution functions of sequences and distribution functions of random variables. We study statistical independence and independence in the sense of probability theory. In the last part we transfer some probabilistic limit laws to certain types of deterministic sequences. This makes heavily use of methods developed in [CQ].

The "general" notion of uniform distribution was introduced by Hermann Weyl (1916) in his famous paper [WEY]: a sequence \( \{v(n)\}, v(n) \in [0,1) \) is uniformly distributed if and only if for every \( x \in [0,1) \)

\[
\lim_{N \to \infty} \frac{1}{N} |\{n \leq N; v(n) < x\}| = x,
\]

where \( |A| \) denotes the cardinality of the set \( A \). This can be equivalently formulated using the notion of asymptotic density. Let \( \mathbb{N} \) be the set of positive integers. We say that a set \( A \subset \mathbb{N} \) has an asymptotic density if and only if the limit

\[
\lim_{N \to \infty} \frac{|A \cap [1,N]|}{N} := d(A)
\]

exists, and in this case the value \( d(A) \) is called the asymptotic density of \( A \). Let \( \mathcal{D} \) denote the system of all subsets of \( \mathbb{N} \) having an asymptotic density. Then a sequence \( \{v(n)\}, v(n) \in [0,1) \) is uniformly distributed if and only if for every \( x \in [0,1) \) the set \( \{n \leq N; v(n) < x\} \) belongs to \( \mathcal{D} \) and \( d(\{n \leq N; v(n) < x\}) = x \). Schoenberg [Sch] generalized this notion as follows:
we say that a sequence \( \{v(n)\}, v(n) \in [0, 1) \) has an asymptotic distribution function if and only if for each real number \( x \) the set \( \{n \leq N; v(n) < x\} \) belongs to \( D \). In this case the function \( F(x) = d(\{n \leq N; v(n) < x\}) \) is called the asymptotic distribution function of the sequence \( \{v(n)\} \).

Our aim is to study distribution functions of sequences. The following statement is useful in this context.

**Proposition 1.** If \( F \) is a non decreasing function defined on the real line then for each real numbers \( x_1, x_2 \) - the points of continuity of \( F \)-we have that for every \( \varepsilon > 0 \) there exist two continuous function \( g, g_1 \) such that

\[
g \leq \chi_{[x_1,x_2]} \leq g_1
\]

and

\[
\int_{-\infty}^{\infty} (g_1(x) - g(x)) \, dx < \varepsilon,
\]

\( \chi_{[x_1,x_2]} \) denoting the indicator function of the interval \([x_1, x_2]\).

The proof follows from a standard procedure, see [KN] page 54.

Another important notion of uniform distribution was introduce by Niven [NIV]. A sequence of positive integers \( k = \{k_n\} \) is called uniformly distributed in \( \mathbb{Z} \) if and only for each \( m \in \mathbb{N}, r \in \mathbb{Z} \) we have that \( \{n \in \mathbb{N}; k_n \equiv r \mod m\} \in D \) and \( d(\{n \in \mathbb{N}; k_n \equiv r \mod m\}) = \frac{1}{m} \). In sections 9 and 10 we will use this concept to prove structural properties concerning measurable sequences.

## 2 Mean value, dispersion and Buck measurability

Let \( v = \{v(n)\} \) be a sequence of real numbers. Set

\[
E_N(v) = \frac{1}{N} \sum_{n=1}^{N} v(n)
\]

for \( N = 1, 2, 3, \ldots \) and

\[
\underline{E}(v) = \liminf_{N \to \infty} E_N(v), \, \overline{E}(v) = \limsup_{N \to \infty} E_N(v).
\]

**Definition 1.** If \( \underline{E}(v) = \overline{E}(v) := E(v) \) we say that \( v \) has a mean value and the number \( E(v) \) will be called the mean value of \( v \).
Clearly we have

**Proposition 2.** If sequences $v, w$ have mean values then for all numbers $a, b$ the sequence $av + bw$ has a mean value and

$$E(av + bw) = aE(v) + bE(w).$$

**Proposition 3.** If $v$ is bounded sequence with elements in the interval $[a, b]$ and having an asymptotic distribution function $F$, then $v$ has a mean value and

$$E(v) = \int_{a}^{b} x dF(x).$$

**Definition 2.** We say that a sequence $v$ has a dispersion if $v$ has a mean value and the sequence $(v - E(v))^2$ has a mean value; in this case the number

$$D^2(v) = E((v - E(v))^2)$$

is called the dispersion of $v$.

If a bounded sequence $v$ has a dispersion then

$$D^2(v) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (v(n) - E(v))^2.$$

In the following we introduce Buck measurability, weak measurability and weak distribution functions.

R. C. Buck [BUC] constructed a measure density via covering of sets by arithmetic progressions. Denote

$$r + (m) = \{r + jm; j = 0, 1, 2, \ldots \}$$

for $r = 0, 1, 2, \ldots$ and $m \in \mathbb{N}$. Then $r + (m)$ belongs to $D$ and $d(r + (m)) = \frac{1}{m}$.

If $S \subset \mathbb{N}$ then the value

$$\mu^*(S) = \inf \left\{ \sum_{j=1}^{k} \frac{1}{m_k}; S \subset \bigcup_{j=1}^{k} r_j + (m_j) \right\}$$

is called Buck’s measure density of the set $S$.

The sets from the system

$$\mathcal{D}_\mu = \{ S \subset \mathbb{N}; \mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1 \}$$

are called Buck measurable.

The following trivial fact will be useful for us, (see [PAS3], page 39):
Proposition 4.  

a) $\mathcal{D}_\mu$ is an algebra of sets, the restriction $\mu = \mu^*|_{\mathcal{D}_\mu}$ is a finitely additive probability measure on $\mathcal{D}_\mu$.

b) $\mathcal{D}_\mu \subset \mathcal{D}$ and $d(S) = \mu(S)$ for every $S \in \mathcal{D}_\mu$.

c) A set $S \subset \mathbb{N}$ belongs to $\mathcal{D}_\mu$ if and only if for each $\varepsilon > 0$ sets $S_1, S_2 \in \mathcal{D}_\mu$ exist such that $S_1 \subset S \subset S_2$ and $\mu(S_2) - \mu(S_1) < \varepsilon$.

We say that a sequence of real numbers $\{v(n)\}$ is Buck measurable if and only if for every real number $x$ the set $\{n \in \mathbb{N}; v(n) < x\}$ belongs to $\mathcal{D}_\mu$. In this case the function

$$F(x) = \mu(\{n \in \mathbb{N}; v(n) < x\})$$

is called Buck’s distribution function (for short B-d.f.) of $\{v(n)\}$.

A Buck measurable sequence is called Buck uniformly distributed (for short B-u.d.) if and only if its Buck distribution function $F(x)$ satisfies

$$F(x) = 0, \text{ for } x < 0, \quad F(x) = x, \text{ for } x \in [0, 1], \quad F(x) = 1, \text{ for } x > 1. \quad (1)$$

Proposition 4 implies

Proposition 5. Each Buck measurable sequence of real numbers has an asymptotic distribution function which coincides with its Buck distribution function.

Definition 3. A real valued sequence $\{v(n)\}$ is called weakly Buck measurable if and only if the sets $\{n \in \mathbb{N}; v(n) < x\}$ are Buck measurable excluding at most a countable set of real numbers $x$. In this case the function

$$F(x) = \{n \in \mathbb{N}; v(n) < x\}$$

defined on the real line excluding at most a countable set is called a weak Buck distribution function of $\{v(n)\}$.

The standard procedure yields a variant of Chebyshev’s inequality:

Proposition 6. If a bounded sequence $v$ has weak distribution function then for each $\varepsilon > 0$ we have

$$\overline{d}(\{n \in \mathbb{N}; |v(n) - E(v)| > \varepsilon\}) \leq \frac{D^2(v)}{\varepsilon^2}.$$ 

Using these concepts we obtain the following

Proposition 7. If $v$ is a bounded sequence having a weak distribution function and $D^2(v) = 0$ then there exists a set $A \in \mathcal{D}$ such that $d(A) = 1$ and $\lim_A v(n) = E(v)$, were the limit is taken a along the set $A$. 

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Definition 4. Let \( v, w \) be sequences having weak asymptotic distribution functions. Suppose moreover that the sequence \( vw \) has a mean value. The value
\[
\rho(v, w) = \frac{|E(vw) - E(v)E(w)|}{D(v)D(w)}
\]
will be called the correlation coefficient of the sequences \( v, w \).

Definition 5. We say that the sequences \( v, w \) are correlated if and only if such values \( \alpha, \beta \) exist that
\[
\lim_{A} w(n) - \alpha v(n) - \beta = 0
\]
for some set \( A \) from \( \mathcal{D} \) such that \( d(A) = 1 \), where the limit is taken along the set \( A \).

In [P-T] the following result is proved

Proposition 8. The sequences \( v, w \) are correlated if and only if \( vw \) has a mean value and \( \rho(v, w) = 1 \). In this case for \( \alpha, \beta \) from Definition 5 we have
\[
\alpha = \frac{E(vw) - E(v)E(w)}{D^2(v)}, \beta = E(w) - \alpha E(v).
\]

This has the following implication:

Corollary 1. If \( v, w \) are sequences uniformly distributed modulo 1, then they are correlated if and only if \( E(vw) = \frac{1}{3} \) or \( E(vw) = \frac{1}{6} \).

3 Independent sequences

In the book [Ra] the following notion is defined:

Definition 6. Two bounded real valued sequences \( v, w \) are called statistically independent if and only if for every functions \( g, g_1 \), that are continuous on a closed interval containing the elements of both sequences
\[
\lim_{N \to \infty} E_N(g(v))E_N(g_1(w)) - E_N(g(v)g_1(w)) = 0.
\]

Properties of statistical independent sequences are studied in various papers. For a survey we refer to the monograph [SP].

Definition 7. Two sets \( S, S_1 \in \mathcal{D} \) are called independent if and only if \( S \cap S_1 \in \mathcal{D} \) and \( d(S \cap S_1) = d(S)d(S_1) \). Bounded sequences \( v, w \) having asymptotic distribution functions are called independent if and only if for arbitrary intervals \( I, I_1 \) the sets \( \{ n \in \mathbb{N}; v(n) \in I_1 \} \) and \( \{ n \in \mathbb{N}; w(n) \in I_1 \} \) are independent.
We shall prove

**Theorem 1.** Let $v, w$ be bounded sequences having continuous asymptotic distribution functions. Then these sequences are independent if and only if they are statistically independent.

We start with the following

**Proposition 9.** Let $v_k, w_k, (k \in \mathbb{N})$ be two systems of sequences of elements from a certain closed interval $[a, b]$. Suppose that for each $k \in \mathbb{N}$ the sequences $v_k, w_k$ are statistically independent. If $v_k$ converges uniformly to $v$ and $w_k$ converges uniformly to $w$ then the sequences $v, w$ are statistically independent.

**Proof.** Let $g, g_1$ be continuous functions defined on a closed interval containing the elements of both sequences. Then these functions are uniformly continuous. Thus $g(v_k)$ converges uniformly to $g(v)$, $g_1(w_k)$ converges uniformly to $g(w)$ and $g(v_k)g_1(w_k)$ converges uniformly to $g(v)g_1(w)$. Hence for given $\varepsilon > 0$ there exists $k$ with

$$|E_N(g(v_k)) - E_N(g(v))| < \varepsilon, |E_N(g(w_k)) - E_N(g(w))| < \varepsilon,$$

and

$$|E_N(g(v_k)g_1(w_k)) - E_N(g(v)g_1(w))| < \varepsilon.$$

Moreover there exists $N_0$ such that for $N \geq N_0$ we have

$$|E_N(g(v_k))E_N(w_k) - E_N(g(v_k)g_1(w_k))| < \varepsilon.$$

From the first inequalities we derive

$$|E_N(g(v_k))E_N(w_k) - E_N(g(v)E_N(g_1(w)))| < 2M\varepsilon,$$

where $M$ is an upper bound of $|v|, |w|$. This yields for $N \geq N_0$

$$|E_N(g(v)g_1(w)) - E_N(g(v))E_N(g_1(w))| < 2M\varepsilon + \varepsilon.$$

\[\square\]

**Definition 8.** If $S_1, \ldots, S_k$ are disjoint sets belonging to $\mathcal{D}$ and $c_1, \ldots, c_k \in \mathbb{R}$ then the sequence $s$ defined by

$$s(n) = \sum_{j=1}^{k} c_j \chi_{S_j}(n), n \in \mathbb{N}$$

is called a *simple sequence*. 
It is easy to check:

**Proposition 10.** If \( s \) is a simple sequence then \( s \) has a mean value and

\[
E(s) = \sum_{j=1}^{k} c_j d(S_j).
\]

This leads to the following consequence:

**Proposition 11.** Let \( s = \sum_{j=1}^{k} c_j \mathcal{X}_{S_j}, r = \sum_{j=1}^{\ell} r_j \mathcal{X}_{R_j} \) be such simple sequences that the sets \( S_j, R_m \), are independent for \( j = 1, \ldots, k, m = 1, \ldots, \ell \). Then they are statistically independent.

**Proposition 12.** If \( v, w \) are bounded independent sequences then they are statistically independent.

**Proof.** Let the values of \( v, w \) be contained in the interval \([a, b]\). Consider for \( k \in \mathbb{N} \) the partition of \([a, b]\) into disjoint subintervals \( I_j, j = 1, \ldots, m \) such that \( |I_j| < \frac{1}{k}, j = 1, \ldots, m \). Then the sets

\[
S_j = \{ n \in \mathbb{N}; v(n) \in I_j \}, R_i = \{ n \in \mathbb{N}; w(n) \in I_i \}, 1 \leq i, j \leq m
\]

are independent. Thus the simple sequences

\[
s_k = \sum_{j=1}^{m} c_j \mathcal{X}_{S_j}, r_k = \sum_{j=1}^{m} c_j \mathcal{X}_{R_j}, c_j \in I_j, j = 1, \ldots, m
\]

are statistically independent. Since \( |s_k(n) - v(n)| \leq \frac{1}{k} \) and \( |r_k(n) - w(n)| \leq \frac{1}{k} \) for \( n \in \mathbb{N} \) we obtain that \( s_k \) converges uniformly to \( v \) and \( r_k \) converges uniformly to \( w \). Thus due to Proposition 9 \( v \) and \( w \) are statistically independent. \( \square \)

**Proof of the second implication of Theorem 1.** Consider statistically independent bounded sequences \( v, w \) having continuous asymptotic distribution functions \( F, F_1 \) respectively. Let \( I_1 = [x_1, x_2], I_2 = [y_1, y_2] \). Since \( F, F_1 \), are continuous, Proposition 1 guarantees that for \( \varepsilon > 0 \) there exist positive continuous functions \( f, f_1, g, g_1 \) satisfying

\[
f \leq \mathcal{X}_{I_1} \leq f_1, g \leq \mathcal{X}_{I_2} \leq g_1
\]

and

\[
\int_a^b (f_1(x) - f(x))dF(x) < \varepsilon, \int_a^b (g_1(x) - g(x))dF_1(x) < \varepsilon.
\]

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From (2) we derive
\[ E_N(f(v)g(w)) \leq E_N(X_{I_1}(v)X_{I_2}(w)) \leq E_N(f_1(v)g_1(w)), \]
moreover
\[ E_N(f(v))E_N(g(w)) \leq E_N(X_{I_1}(v))E_N(X_{I_2}(w)) \leq E_N(f_1(v))E_N(g_1(w)). \]
If \( N \to \infty \) we obtain for \( \overline{E} = E(X_{I_1}(v)X_{I_2}(w)) \) and \( \underline{E} = E(X_{I_1}(v)X_{I_2}(w)) \) the inequalities
\[
\int_b^a f(x)dF(x) \int_b^a g(x)dF_1(x) \leq \overline{E} \leq \int_b^a f_1(x)dF(x) \int_b^a g_1(x)dF_1(x)
\]
and
\[
\int_b^a f(x)dF(x) \int_b^a g(x)dF_1(x) \leq \underline{E} \leq \int_b^a f_1(x)dF(x) \int_b^a g_1(x)dF_1(x).
\]
Set \( S_1 = \{ n \in \mathbb{N}; v(n) \in I_1 \}, S_2 = \{ n \in \mathbb{N}; w(n) \in I_2 \} \). Then
\[
\lim_{N \to \infty} E_N(X_{I_1}(v))E_N(X_{I_2}(w)) = d(S_1)d(S_2).
\]
This yields
\[
|\overline{E} - d(S_1)d(S_2)| \leq H\varepsilon, |\underline{E} - d(S_1)d(S_2)| \leq H\varepsilon,
\]
where \( H \) is suitable constant. Since \( \varepsilon > 0 \) is arbitrary we get \( \overline{E} = \underline{E} = d(S_1)d(S_2) \). If we consider that \( E = \overline{E} = \underline{E} = d(S_1 \cap S_2) \) the assertion follows.

An immediate consequence of the definition is the following:

**Proposition 13.** Let \( v, w \) be bounded sequences having continuous asymptotic distributions \( F, F_1 \), respectively. Suppose that these sequences are independent. Then for any intervals \( I_1 = [x_1, x_2], I_2 = [y_1, y_2] \) the set \( S = \{ n \in \mathbb{N}; (v(n), w(n)) \in I_1 \times I_2 \} \) belongs to \( \mathcal{D} \) and
\[
d(S) = (F(x_2) - F(x_1))(F_1(y_1) - F_1(y_2)).
\]

Using the above notation the standard method yields:

**Proposition 14.** Let \( A \) be a Riemann Stjeltjes measurable set with respect product measure \( F \times F_1 \) then the set \( R = \{ n \in \mathbb{N}; (v(n), w(n)) \in A \} \) belongs to \( \mathcal{D} \) and
\[
d(R) = \int \int_R dF(t_1)dF_1(t_2).
\]
Furthermore the following theorem holds (with the above notation).

**Theorem 2.** The sequence $v + w$ has an asymptotic distribution function $F_2$ given by

$$F_2(x) = \int \int_{\{(t_1,t_2); t_1 + t_2 \leq x\}} dF(t_1)dF(t_1).$$

This leads after some calculation to:

**Corollary 2.** If $v, w$ are two independent uniformly distributed sequences then the sequence $v + w$ has the distribution function $G$ where

- $G(x) = 0$, $x \leq 0$,
- $G(x) = \frac{x^2}{2}$, $x \in [0, 1]$,
- $G(x) = 2x - \frac{x^2}{2} - 1$, $x \in [1, 2]$,
- $G(x) = 1$, $x > 2$.

In following the more general notion of independence will be useful:

**Definition 9.** If $v_1, \ldots, v_k$ are bounded sequences having asymptotic distribution functions then they are called *independent* if and only if for all intervals $I_1, \ldots, I_k$ the set $S = \{n \in \mathbb{N}; v_j(n) \in I_j, j = 1, \ldots, k\}$ belongs to $\mathcal{D}$ and

$$d(S) = \prod_{j=1}^{k} d(\{n \in \mathbb{N}; v_j(n) \in I_j\}).$$

These sequences are called *statistically independent* if and only if

$$\lim_{N \to \infty} E_N(g_1(v_1) \ldots g_k(v_k)) - E_N(g_1(v_1)) \ldots E_N(g_k(v_k)) = 0$$

for any functions $g_1, \ldots, g_k$ continuous on closed intervals containing all elements of the given sequences.

**Theorem 3.** Let $v_1, \ldots, v_k$ be bounded sequences having continuous asymptotic distribution functions. Then they are independent if and only if they are statistically independent.

**Proposition 15.** Let $v_1, v_2, v_3$ be bounded sequences having asymptotic distribution functions. If these sequences are independent then $v_1 + v_2, v_3$ are independent, too.

From this we derive as above:

**Theorem 4.** If $v_1, \ldots, v_k$ are independent bounded sequences with continuous distribution functions, having the same mean value $E$ and the same dispersion $D^2$. Then

$$d\left(\left\{n \in \mathbb{N}; \left|\frac{v_1 + \cdots + v_k}{k} - E\right| \geq \varepsilon\right\}\right) \leq \frac{D^2}{n\varepsilon^2}.$$
4 Polyadicly continuous sequences

Denote by $\Omega$ the compact metric ring of polyadic integers, (see [N], [N1], [PAS5], which is the completion of $\mathbb{N}$ with respect to the polyadic metric

$$d(a, b) = \sum_{n=1}^{\infty} \frac{\psi_n(a - b)}{2^n},$$

where $\psi_n(x) = 0$ if $n$ divides $x$ and $\psi_n(x) = 1$ otherwise. For sequences $\{v(n)\}$ we shall use two synonymous expressions: sequences or arithmetical functions. A sequence $\{v(n)\}$ is called polyadically continuous (for short: p-continuous) if and only if for each $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\forall a, b \in \mathbb{N}; a \equiv b \pmod{m} \Rightarrow |v(a) - v(b)| < \varepsilon.$$

In [PAS2] it is proved:

**Proposition 16.** Let $\{v(n)\}$ be a p-continuous sequence of elements of $[0, 1]$. Suppose that $F$ is a continuous function defined on $[0, 1]$. Then $\{v(n)\}$ is Buck measurable with B-d.f. $F$ if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} h(v(n)) = \int_{0}^{1} h(x) dF(x)$$

for each continuous real valued function $h$ defined on $[0, 1]$.

This implies the following

**Proposition 17.** If a p-continuous sequence of elements in $[0, 1]$ has a continuous asymptotic distribution function then it is Buck measurable and its B-d.f. coincides with its asymptotic distribution function.

The next result is due to P. Erdős [Er], see also [PAS3], p.32. In the following we use the notation

$$A(v(n), I) = \{|n \in \mathbb{N}; v(n) \in I\},$$

for sequences $v(n) \in I \subset [0, 1]$.

**Theorem 5.** Suppose that $f$ is a non-negative additive arithmetic function such that for every prime $p$ we have $f(p) = f(p^k), k = 1, 2, 3, \ldots$, and for distinct primes $p_1, p_2$ we have $f(p_1) \neq f(p_2)$. Assume that the infinite series
\[ \sum_p \frac{f(p)}{p} \] (running over the primes) converges. Then for every interval \( I \), there holds \( A(\{ f(n) \}, I) \in \mathcal{D} \). Moreover, in this case, the function
\[ g(x) = d(A(\{ f(n) \}, [-\infty, x]) \]
is continuous on the real line.

**Corollary 3.** Let \( f \) be a non-negative additive arithmetic function such that for every prime \( p \) we have \( f(p) = f(p^k), k = 1, 2, 3, \ldots \), for different primes \( f(p_1) \neq f(p_2) \) and the series \( \sum_p f(p) \) converges. Then the sequence \( \{ f(n) \} \) is Buck measurable with continuous Buck distribution function.

**Proof.** Let \( N \in \mathbb{N} \). If \( n_1 \equiv n_2 \pmod{N!} \) then \( n_1, n_2 \) contain the same primes smaller than \( N \) in canonical decomposition and so in this case
\[ |f(n_1) - f(n_2)| \leq 2 \sum_{p>N} f(p). \]

Thus the convergence of \( \sum_p f(p) \) provides that \( \{ f(n) \} \) is a \( p \)-continuous sequence. This condition yields also the convergence of \( \sum_p \frac{f(p)}{p} \), and the assertion follows. \( \square \)

It is easy to check that each \( p \)-continuous sequence of real numbers is uniformly continuous with respect to the polyadic metric \( \mathfrak{d} \), and so each \( p \)-continuous sequence of real numbers \( \{ v(n) \} \) can be extended in the natural way to a real valued continuous function \( \tilde{v} \) defined on \( \Omega \) such that
\[ \tilde{v}(\alpha) = \lim_{j \to \infty} v(n_j), \]

where \( \{ n_j \} \) is a sequence of positive integers such that \( n_j \to \alpha \) for \( j \to \infty \) with respect the polyadic metric. The compact ring \( \Omega \) is equipped with Haar probability measure \( P \) and so the function \( \tilde{v} \) can be considered as random variable on the probability space \( (\Omega, P) \). As usually \( h \) is a random variable on \( \Omega \) and we denote \( E(h) = \int hdP \), the mean value of \( h \).

Let \( m \in \mathbb{N} \) and \( s = 0, 1, \ldots m - 1 \). Put
\[ s + m\Omega = \{ s + ma; \alpha \in \Omega \}. \]
The ring \( \Omega \) can be represented as disjoint union
\[ \Omega = \bigcup_{s=0}^{m-1} s + m\Omega. \]
(see [N], [N1]). Thus for the Haar probability measure $P$ we have
\[ P(s + m\Omega) = \frac{1}{m} \] (5)
for $m \in \mathbb{N}, s = 0, \ldots m - 1$.

In [PAS4] it is proven that
\[ \mu^*(S) = P(cl(S)) \] (6)
for each $S \subset \mathbb{N}$, where $cl(S)$ denote the topological closure of $S$ in $\Omega$.

**Example 1.** Let \( \{Q_k\} \) be an increasing sequence of integers such that $Q_0 = 1$ and $Q_k$ devides $Q_{k+1}, k = 1, 2, 3, \ldots$. Each positive integer $n$ can be uniquely represented in the form
\[ n = a_0 + a_1Q_1 + \cdots + a_kQ_k, \]
where $a_j < \frac{Q_{j+1}}{Q_j}, j = 1, \ldots, k$. To this $n$ we associate an element $\gamma(n)$ in the unit interval of the form
\[ \gamma(n) = \frac{a_0}{Q_1} + \cdots + \frac{a_k}{Q_{k+1}}. \]
The sequence \( \{\gamma(n)\} \) is known as van der Corput sequence in base \( \{Q_k\} \) and in [PAS2] it is proved that it is Buck uniformly distributed and $p$-continuous.

The following characterization allows us to apply results of probability theory to the distribution of $p$-continuous sequences:

**Theorem 6.** Let \( \{v(n)\} \) be a $p$-continuous sequence and $F$ a continuous real valued function defined on the real line. Then the following the statements are equivalent:

(i) $F$ is the distribution function of the random variable $\tilde{v}$.

(ii) \( \{v(n)\} \) is a Buck measurable sequence and $F$ is its B-d.f.

(iii) For each real number $x$ we have
\[ \mu^*(\{n \in \mathbb{N}; v(n) < x\}) = F(x). \]

**Proof.** (i) $\Rightarrow$ (ii). The continuity of $F$ yields
\[ P(\tilde{v} < x) = F(x) = P(\tilde{v} \leq x) \] (7)
for each real number, $x$. From the inclusion
\[ \{n \in \mathbb{N}; v(n) < x\} \subset \{\alpha \in \Omega; \tilde{v}(\alpha) \leq x\} \]
we obtain
\[ \text{cl} \left( \{ n \in \mathbb{N} ; v(n) < x \} \right) \subset \{ \alpha \in \Omega ; \tilde{v}(\alpha) \leq x \} . \]

Furthermore (7) yields
\[ \mu^* \left( \{ n \in \mathbb{N} ; v(n) < x \} \right) \leq F(x) \]
for every real number \( x \). On the other hand
\[ \mathbb{N} \setminus \{ n \in \mathbb{N} ; v(n) < x \} = \{ n \in \mathbb{N} ; v(n) \geq x \} , \]
therefore
\[ \text{cl} \left( \mathbb{N} \setminus \{ n \in \mathbb{N} ; v(n) < x \} \right) \subset \{ \alpha \in \Omega ; \tilde{v}(\alpha) \geq x \} . \]

Hence
\[ \mu^* \left( \mathbb{N} \setminus \{ n \in \mathbb{N} ; v(n) < x \} \right) \leq 1 - F(x) , \]
and so the set \( \{ n \in \mathbb{N} ; v(n) < x \} \) is Buck measurable and its measure density is \( F(x) \).

The implication (ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i).

Clearly
\[ \{ n \in \mathbb{N} ; v(n) < x \} \subset \{ \alpha \in \Omega ; \tilde{v}(\alpha) \leq x \} , \]
and so \( F(x) \leq P(\tilde{v} \leq x) \). On the other hand
\[ \{ \alpha \in \Omega ; \tilde{v}(\alpha) < x \} \subset \text{cl}(\{ n \in \mathbb{N} ; v(n) \leq x \}) \]
for \( \varepsilon > 0 \). This yields \( F(x) \leq P(\tilde{v} \leq x) \leq F(x + \varepsilon) \) for \( \varepsilon > 0 \). For \( \varepsilon \to 0^+ \) we obtain the assertion from the continuity of \( F \).

5 Independence and measurability

From our definitions in previous sections we immediately derive:

**Proposition 18.** The Buck measurable sequences \( \{ v_1(n) \} , \{ v_2(n) \} , \ldots , \{ v_r(n) \} \)
are independent if and only if for every \( x_1 , \ldots , x_r \in \mathbb{R} \) we have
\[ \mu \left( \bigcap_{j=1}^r \{ n \in \mathbb{N} ; v_j(n) < x_j \} \right) = \prod_{j=1}^r \mu(\{ n \in \mathbb{N} ; v_j(n) < x_j \}) . \]
Example 2. We come back to Example 1. Consider the sequences \( \{Q_k^{(j)}\} \) given such that \( Q_0^{(j)} = 1, j = 1, \ldots, r \) and \( Q_k^{(j)}/Q_{k+1}^{(j)} \) for \( j = 1, \ldots, r \) and \( k = 0, 1, 2, \ldots \). Let \( Q_k^{(j)} \), \( Q_k^{(j_1)} \) be relatively prime for \( j \neq j_1 \). Denote by \( \{\gamma_j(n)\} \) the van der Corput sequence with base \( Q_k^{(j)} \) for \( j = 1, \ldots, r \). Then these sequences are independent (see [IPT]).

Theorem 7. Let \( \{v_1(n)\}, \{v_2(n)\}, \ldots, \{v_k(n)\} \) be independent Buck measurable \( p \)-continuous sequences with continuous Buck distribution functions \( F_j, j = 1, \ldots, k \). Then the random variables \( \tilde{v}_1, \ldots, \tilde{v}_k \) are independent.

Proof. For \( x_1, \ldots, x_k \in \mathbb{R} \) we have
\[
\{\alpha \in \Omega; \tilde{v}_1(\alpha) < x_1, \ldots, \tilde{v}_k(\alpha) < x_k\} \subset \text{cl}\{\{n \in \mathbb{N}; v_1(n) \leq x_1, \ldots, v_k(n) \leq x_k\}\}.
\]
Thus \( P(\tilde{v}_1 < x_1, \ldots, \tilde{v}_k < x_k) \leq F_1(x) \ldots F_k(x_k) \), and so from the above theorem we get \( P(\tilde{v}_1 < x, \ldots, \tilde{v}_k < x) \leq P(\tilde{v}_1 < x) \ldots P(\tilde{v}_k < x) \).

On the other hand we have
\[
P(\tilde{v}_1 \leq x_1) \ldots P(\tilde{v}_k \leq x_k) =
\]
\[
= \mu(\{n \in \mathbb{N}; v_1(n) \leq x\}) \ldots \mu(\{n \in \mathbb{N}; v_k(n) \leq x\})
\]
\[
= P(\text{cl}\{\{n \in \mathbb{N}; v_1(n) \leq x_1, \ldots, v_k(n) \leq x_k\}\}) \leq P(\tilde{v}_1 \leq x_1, \ldots, \tilde{v}_k \leq x_k).
\]

Let \( F_1, \ldots, F_k \) be non-decreasing functions defined on \( \mathbb{R} \), \( (k \) is a fixed positive integer). A set \( B \subset \mathbb{R}^k \) is called Jordan Stieljes measurable with respect to the functions \( F_1, \ldots, F_k \) if and only the Riemann Stieltjes integral
\[
\int \int \ldots \int \chi_B \text{d}F_1 \ldots \text{d}F_k
\]
exists; \( \chi_B \) denoting the indicator function of \( B \).

Theorem 8. Let \( \{v_1(n)\}, \ldots, \{v_k(n)\} \) be independent Buck measurable \( p \)-continuous sequences with continuous Buck distribution functions \( F_1, \ldots, F_k \). Suppose that a set \( B \subset \mathbb{R}^k \) is Jordan Stieljes measurable with respect to the functions \( F_1, \ldots, F_k \). Then the set \( \{n \in \mathbb{N}; (v_1(n), \ldots, v_k(n)) \in B\} \) is Buck measurable and its Buck measure density is
\[
\int \int \ldots \int \chi_B \text{d}F_1 \ldots \text{d}F_k. \tag{8}
\]

Proof. If \( B = [a_1, b_1] \times \ldots \times [a_k, b_k] \) is a cylinder set then (8) follows directly from independence of \( \{v_1(n)\}, \ldots, \{v_k(n)\} \) and Theorem 6. Proposition 3 then implies the assertion. \( \square \)
6 Integral and mean value

Let \( h : \Omega \to (-\infty, \infty) \) be a continuous function. Since \( \Omega \) is a compact space, it is uniformly continuous. Consider \( m \in \mathbb{N} \). To the function \( h \) we can associate a periodic function \( h_m \) with period \( m \) in following way:

\[ \alpha \in s + m\Omega \iff h_m(\alpha) = h(s). \]

Clearly,

\[ \int h_m dP = \frac{1}{m} \sum_{s=0}^{m-1} h(s). \quad (9) \]

Clearly \( \lim_{N \to \infty} \vartheta(N!, 0) = 0 \), and so uniform continuity of \( h \) implies that \( h_{N!} \) converges uniformly to \( h \). From (9) we obtain

\[ \int h dP = \lim_{N \to \infty} \frac{1}{N!} \sum_{s=0}^{N!-1} h(s). \quad (10) \]

The function \( h \) restricted on \( \mathbb{N} \) is \( p \)-continuous. Thus there exists the limit \( \lim_{m \to \infty} \frac{1}{m} \sum_{s=0}^{m-1} h(s) \). From (10) we conclude

\[ \int h dP = \lim_{m \to \infty} \frac{1}{m} \sum_{s=0}^{m-1} h(s). \quad (11) \]

Remark 1. If the random variable \( \tilde{v} \) has a continuous distribution function \( F \) then

\[ \int \tilde{v} dP = \int_{-\infty}^{\infty} xdf(x) = E(v). \]

The central limit theorem immediately yields:

Proposition 19. Let \( \{v_k(n)\}, k = 1, 2, 3 \ldots \) be a sequence of \( p \)-continuous sequences such that for every \( k = 1, 2, 3, \ldots \) the sequences \( \{v_j(n)\}, j = 1 \ldots k \) are independent and have the same continuous Buck distribution function. Then for every \( x \in \mathbb{R} \) we have

\[ \lim_{k \to \infty} \mu\left( \left\{ n \in \mathbb{N}; \frac{v_1(n) + \cdots + v_k(n) - kE}{\sqrt{kD}} \leq x \right\} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt. \]

We conclude this section with the following metric result:

Theorem 9. Let \( v_k, k = 1, 2, 3, \ldots \) be a system of independent \( p \)-continuous uniformly distributed sequences. Then the sequence \( \{\tilde{v}_n(\alpha)\} \) is uniformly distributed for almost all \( \alpha \in \Omega \).
Proof. Denote

$$S_N(h, \alpha) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \tilde{v}_n(\alpha)}$$

for $h \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \Omega$. Put

$$A_h = \{ \alpha \in \Omega; \lim_{N \to \infty} S_N(h, \alpha) = 0 \}$$

for $h \neq 0$. For every $n \in \mathbb{N}$ we have $E(e^{2\pi i h \tilde{v}_n}) = 0$. Therefore the strong law of large numbers implies that $P(A_h) = 1$. Thus $P(\cap_{h \neq 0} A_h) = 1$ and the assertion follows.

7 Weak Buck measurability

Definition 10. Let $v = \{v(n)\}$ be a real valued sequence. We say that $v$ is weakly polyadicly continuous if and only if for each $\delta > 0$ there exists a set $A \in \mathcal{D}_{\mu}$ with $\mu(A) < \delta$ such that

$$n_1 \equiv n_2 \pmod{m} \Rightarrow |v(n_1) - v(n_2)| < \varepsilon$$

for all $n_1, n_2 \in \mathbb{N} \setminus A$ and a suitable $m \in \mathbb{N}$.

Our aim is to prove the following equivalence:

Theorem 10. A bounded sequence of real numbers is weakly Buck measurable if and only if it is weakly polyadicly continuous.

We start by the proof of the first implication. We recall the following notion:

Definition 11. A real valued sequence $v$ is called almost polyadicly continuous if and only if for each $\delta > 0$ there exists a set $A \in \mathcal{D}_{\mu}$ with $\mu(A) < \delta$ such that $v$ is polyadicly continuous on the set $\mathbb{N} \setminus A$.

Directly from the definition we get

Proposition 20. A set $S \subset \mathbb{N}$ is Buck measurable if and only if its indicator function $X_S$ is almost polyadicly continuous.

Proposition 21. If $v_1, v_2$ are two almost polyadicly continuous sequences and $c_1, c_2$ are real numbers then the sequence $c_1 v_1 + c_2 v_2$ is almost polyadicly continuous.
Proposition 22. If $v$ is a real valued sequence such that for each $\varepsilon > 0$ there exists an almost polyadicly sequence $v_0$ that $|v(n) - v_0(n)| < \varepsilon$ for $n \in \mathbb{N}$ then $v$ is weakly polyadicly continuous.

Proposition 23. Each bounded weakly Buck measurable sequence is weakly polyadicly continuous.

Proof. Let $v$ be a weakly Buck measurable sequence of elements in the interval $[a, b], a < b$. Consider $\varepsilon > 0$. Then there exists a partition $x_0, \ldots, x_k$ of $[a, b]$ such that the sets

$$S_i = \{ n \in \mathbb{N}; v(n) \in [x_i, x_{i+1}) \}, i = 0, \ldots, k - 2$$

and

$$S_{k-1} = \{ n \in \mathbb{N}; v(n) \in [x_{k-1}, b) \}$$

are Buck measurable and $x_{i+1} - x_i < \varepsilon$. Then the sequence

$$v_0(n) = \sum_{i=0}^{k-1} x_i \chi_{S_i}(n), n \in \mathbb{N}$$

is almost polyadicly continuous and $|v_0(n) - v(n)| < \varepsilon$. The assertion follows from Proposition 22.

Now we prove the second implication.

If $v = \{v(n)\}$ is a real valued sequence and $k = \{k_n\}$ is a sequence of positive integers then we shall denote $v(k) = \{v(k_n)\}$.

Proposition 24. A set $S \subset \mathbb{N}$ is Buck measurable if and only if for each sequence of positive integers $k$ the sequence $\chi_S(k)$ has a mean value and in this case

$$\mu(S) = E(\chi_S(k)).$$

This proposition is an easy reformulation of Theorem 7 in [PAS3] page 51 or Theorem 50 in [PAS5] page 113.

Proposition 25. If $v$ is a bounded weakly polyadicly continuous sequence then it has a mean value and for each sequence of positive integers $k$ which is uniformly distributed in $\mathbb{Z}$ we have

$$E(v(k)) = E(v).$$
Proof. Consider $\delta > 0, \varepsilon > 0$. Let $A, m$ be as in Definition 10. Suppose that $r_1, \ldots, r_s$ is the maximal finite sequence of elements of $\mathbb{N}\setminus A$ incongruent modulo $m$ and $r_{s+1}, \ldots, r_m$ its completion with respect to a complete residue system modulo $m$. Define the periodic sequence $v_m(n) = v(r_j)$ if and only if $n \equiv r_j \pmod{m}$ for $j = 1, \ldots, m$ and $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}\setminus A$ we have

$$|v_m(n) - v(n)| < \varepsilon. \quad (12)$$

For $N = 1, 2, 3, \ldots$ we obtain

$$E_N(v_m(k)) - E_N(v(k)) = \frac{1}{N} \sum_{n=1}^{N} (v_m(k_n) - v(k_n)) =$$

$$\frac{1}{N} \sum_{n \leq N, k_n \in A} (v_m(k_n) - v(k_n)) + \frac{1}{N} \sum_{n \leq N, k_n \notin A} (v_m(k_n) - v(k_n)).$$

And so from (12) we device

$$|E_N(v_m(k)) - E_N(v(k))| < 2HE_N(X_A(k)) + \varepsilon$$

where $H$ is upper bound of $\{|v(n)|\}$. If $k$ is uniformly distributed in $\mathbb{Z}$ we get for $N \to \infty$

$$|E(v_m) - E(v(k))| < 2H\delta + \varepsilon \quad (13)$$

and

$$|E(v_m) - E(v(k))| < 2H\delta + \varepsilon. \quad (14)$$

Therefore

$$E(v(k)) - E(v(k)) < 4H\delta + 2\varepsilon.$$

Since $\delta, \varepsilon$ are arbitrary we have $E(v(k)) = E(v(k)) = E(v(k))$. If in the inequalities (13) and (14) we substitute the sequence $\{n\}$ instead of $k$ we conclude $E(v) = E(v(k))$. \qed

Proposition 26. If $v$ is a weakly polyadicly continuous sequence of elements in $[a, b]$ and $f$ is a continuous real function defined on this interval then the sequence $f(v)$ is weakly polyadicly continuous, too.

Proof. The assertion follows immediately from the fact that a continuous function on a compact interval is uniformly continuous. \qed

Proposition 27. Each bounded weakly polyadic continuous real valued sequence is weakly Buck measurable.
Proof. Let \( v \) be a weakly polyadic continuous real valued sequence of elements in \([a, b]\). Then for every continuous function \( f \) defined on \([a, b]\) the sequence \( f(v) \) has a mean value and for every sequence of positive integers \( k \) uniformly distributed in \( \mathbb{Z} \) we have
\[ E(f(v)) = E(f(v(k)). \]

We define a positive linear functional
\[ \Phi(f) = E(f(v)) \]
on the linear space of all continuous real functions defined on \([a, b]\) such that \( \Phi(1) = 1 \).

Thus Riesz representation theorem provides that a non decreasing function \( F \) exists such \( F(a) = 0, F(b) = 1 \) and
\[ E(f(v(k)) = \Phi(f) = \int_a^b f(x)dF(x) \] (15)
holds for each sequence \( k \) uniformly distributed in \( \mathbb{Z} \). If the function \( F \) is continuous in \( x_0 \) then by Proposition [11] we can construct for every \( \varepsilon > 0 \) two continuous functions \( f_1, f_2 \) defined on \([a, b]\) satisfying
\[ \int_a^b (f_2(x) - f_1(x))dF(x) < \varepsilon \]
and
\[ f_1 \leq \mathcal{X}_{[0,x_0]} \leq f_2. \]
Hence for each sequence \( k \) uniformly distributed in \( \mathbb{Z} \) we have
\[ E(\mathcal{X}_{[0,x_0]}(v(k))) = F(x_0). \]

Proposition [24] implies that the set \( \{ n \in \mathbb{N}; v(n) < x_0 \} \) is Buck measurable. Since every non-decreasing function has at most a countable set of discontinuities, the proof is complete.

Let \( \mathcal{B}_\mu \) be the set of all bounded weakly measurable sequences. Theorem [3] implies

**Proposition 28.** Define the norm
\[ \|v\| = \sup\{\|v(n)\|; n \in \mathbb{N}\} \]
for \( v \in \mathcal{B}_\mu \). Then \((\mathcal{B}_\mu, +, \cdot, \| \cdot \|)\) is a Banach algebra.
8 Statistical independence

If $v = \{v(n)\}$ is a sequence and $g$ is a function defined on the set containing the elements of $v$ then we denote by $g(v)$ the sequence $\{g(v(n))\}$. The following theorem relates $p$-continuous independent sequences to the concept of uniform distribution in $\mathbb{Z}$.

**Theorem 11.** Let $\{v_1(n)\}, \ldots, \{v_k(n)\}$ be $p$-continuous independent sequences. Then for arbitrary functions $g_1, \ldots, g_k$ continuous on the real line we have

$$E\left(\prod_{j=1}^{k} g_j(v_j(k))\right) = \prod_{j=1}^{k} E\left(g_j(v_j(k))\right)$$

for each sequence $k = \{k_n\}$ which is uniformly distributed in $\mathbb{Z}$.

**Proof.** If $\{v(n)\}$ is a $p$-continuous function, then it is bounded. Every continuous function $g$ defined on the real line is uniformly continuous on the interval $[b_1, b_2]$ where $b_1$ is a lower bound of the sequence $\{v(n)\}$ and $b_2$ its upper bound. Thus the sequence $\{g(v(n))\}$ is $p$-continuous, too.

Let us consider $\{v_1(n)\}, \ldots, \{v_k(n)\}$ - polyadically continuous independent sequences. Then the random variables $\tilde{v}_1, \ldots, \tilde{v}_k$ are independent and so the random variables $g_1(\tilde{v}_1), \ldots, g_k(\tilde{v}_k)$ are independent, too. Thus

$$E(g_1(v_1) \ldots g_k(v_k)) = E(g_1(v_1)) \ldots E(g_k(v_k))$$

and the assertion follows from Proposition 25.

Theorem 3 and Proposition 25 imply

**Theorem 12.** Let $\{v_1(n)\}, \ldots, \{v_k(n)\}$ Buck measurable independent sequences having continuous Buck distribution functions. Then for arbitrary functions $g_1, \ldots, g_k$ continuous on the real line

$$E\left(\prod_{j=1}^{k} g_j(v_j(k))\right) = \prod_{j=1}^{k} E\left(g_j(v_j(k))\right)$$

for each sequence $k = \{k_n\}$ which is uniformly distributed in $\mathbb{Z}$.

**References**

[BUC] Buck, R., C., *The measure theoretic approach to density*, Amer. J. Math 68, 1946, 560–580
[CQ] Choimet, D., Queffelet, H., Analyse mathematique, grandes theoremes du vingtieme siecle Calvage Mounet, Paris, (2009)

[D-T] Drmota, M., Tichy, R. F., Sequences, Discrepancies and Applications, Springer, Berlin Heidelberg, Springer, Berlin Heidelberg, 1997

[Er] Erdős, P., On the density of some sequences of numbers II, Journal of the London Math. Soc. 12, (1937), 7 - 11

[G] Grekos, G., On various definitions of density (a survey), Tatra Mt. Math. Publ., 31, 2005, 17–27

[G1] Grekos, G., The density set (a survey), Tatra Mt. Math. Publ., 31, 2005, 103–111

[GST] Grabner, P. J., Strauch, O., Tichy, R. F., Lp-discrepancy and statistical independence of sequences, Czechoslovak Mathematical Journal, Vol.49(1999), No.1,97110

[GT] Grabner, P. J., Tichy, R. F., Remark on statistical independence of sequences, Mathematica Slovaca, Vol.44 (1994), No.1,91–94

[IPT] Iaco, M. R., Pasteka, M., Tichy R., F., Measure density for set decompositions and uniform distribution Rend. Circ. Math. Palermo (2) , 64, No. 2, 2015 , 323 – 339

[K-N] Kuipers, L., Niederreiter, H., Uniform distribution of Sequences, John Wiley and Sons, N.Y. London, Sydney Toronto, 1974

[N] Novoselov, E. V., Topological theory of polyadic numbers, Trudy Tbilis. Mat. Inst. 27, 1960, 61 – 69, (in russian)

[N1] Novoselov, E. V., New method in probabilistic number theory, Doklady akademii nauk. ser. matem. No. 2, 28, 1964, 307 – 364, in russian

[NAR] Narkiewicz, W. , Teoria liczb, (in polish) PWN, Warszawa, 1991

[NIV] Niven, I., Uniform distribution of sequences of integers, Trans. Amer. Math. Soc. 98, 52 – 61
[PAS] Paštėka, M., *Some properties of Buck’s measure density*, Math. Slovaca 42, no. 1, 1992, 15–32

[PAS2] Paštėka, M., *Remarks on one type of uniform distribution* Unif. Distrib. Theory 2, No. 1, 2007, 79–92

[PAS3] Paštėka, M., *On four approaches to density* Spectrum Slovakia 3. Frankfurt am Main: Peter Lang; Bratislava: VEDA, Publishing House of the Slovak Academy of Sciences, 2014

[PAS4] Paštėka, M., *Remarks on Buck’s measure density*. Tatra Mt. Math. Publ. 3, 1993, 191–200

[PAS5] Paštėka, M., *Density and related topics*, Veda, Bratislava, Academia, Praha, 2017.

[P-T] Paštėka, M., Tichy, R., *A note on the correlation coefficient of arithmetic functions*, Acta Acad. Paed. Agriensis, Sectio Mathematicae 30, 2003, 109–114

[Ra] Rauzy, G., *Propriétés statistiques de suites arithmétiques*. Le Mathematicien. No. li. Collection SUP, Presses Universitaires de France, Paris, 1976

[Sch] Schoenberg, I., *Über die asymptotische Verteilung reeller Zahlen mod 1.*, Math. Z., 1928, 171 – 199

[SP] Strauch, O., Porubský, S., *Distribution of Sequences a Sampler*, Peter Lang, SAV, Frankfurt am Main, Peter Lang, SAV, Frankfurt am Main, 2005

[WEY] Weyl, H., *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann, 77, 1916, 313–352

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