Fuzzy Estimators for Parameters and Hazard Rate Function of Lindley Distribution: Simulation Study

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Abstract. The quantitative description of data is very essential for statistics. In standard statistics data are assumed to be numbers, vectors or classical functions. But in applications, real data are frequently not precise numbers or vectors, but often more or less imprecise. All kinds of data which cannot be presented as precise numbers or cannot be precisely classified are called non-precise or fuzzy. The present paper focused on obtaining the maximum likelihood and approximate Bayesian estimators of the parameters and hazard rate function for Lindley distribution when the data are available in fuzzy form. Since there are no exact forms, maximum likelihood estimators have been derived according to the algorithm of Newton-Raphson as an iterative technique and Bayes estimators have been derived based on informative gamma priors relative to squared error and precautionary loss functions according to a numerical approximate Lindley's technique. The obtained estimators have been compared through Monte-Carlo simulation study.

Keywords. Lindley Distribution, Fuzzy Estimators, Hazard Rate Function, Maximum Likelihood Estimation, Bayes Estimation.

1. Introduction
Statistics is concerned with data analysis and assessment of estimation of probability distributions. The quantitative analysis of data is therefore important for statistics. In standard statistics, numbers, vectors or classical functions are considered to be the data. However, real data in applications are often not accurate numbers or vectors but often more or less imprecise. All types of data that can't be described as precise numbers or can't be categorized specifically are considered non-precise (or fuzzy). Examples are verbal representations of data such as high temperature, low flexibility, and high blood pressure. Therefore, the effects of the precision calculation of variables are not precise/accurate numbers, but always more or less fuzzy. It is important to note that this form of uncertainty or fuzziness varies from errors, it is the inaccuracy of individual observations or measurements [1]. Zadeh (1965) proposed the fuzzy set theory [2]. Since 1965 Zadeh himself and some other scholars have considerably developed this theory. Most of the early research in fuzzy set theory concerned describing ambiguity in cognitive processes in humans. Fuzzy set theory is now applied to many problems in engineering, business, statistics, economics, medical and related health sciences, and the other natural sciences [3].

In statistics especially with estimation theory, several researches have been working with probability distributions on the estimation of parameters and hazard rate function based on complete and censored samples. Traditionally it is assumed that the available data are performed in exact/precise numbers. But, some lifetime data collected may be imprecise in real-world situations and are described in the form of fuzzy data. The scholars have thus noted the importance of generalizing statistical estimation methods of real numbers to fuzzy numbers. In this area, many applications of fuzzy techniques in statistical analysis can be found in, among others, Coppi et al. (2006)[4], Colubi et al. (2007)[5], Pak et al. (2013)[6,7], Pak et al. (2014)[8,9], Shafiq and Viertl (2014)[10], Pak and...
Mahmoudi (2015)[11], Pak (2016)[12], Shafiq (2017)[13], Al-Noor and Al-Sultany (2017)[14], Pak A. (2017)[15], Al-Noor and Hussein (2017)[16], and Al-Noor and Subhi (2018)[17]. In this paper we consider the two-parameter Lindley distribution (TPLD) to estimate its parameters and hazard rate function based on fuzzy data. The estimation methods adopted here are the maximum likelihood "as a non-Bayesian" estimation method and Bayesian estimation method with prior distributions defined as independent gamma distributions relative to symmetric and asymmetric loss functions. The remainder of this paper is arranged as follows: A brief detail about the TPLD provides in section 2. Sections 3 and 4 address respectively the maximum likelihood and Bayes estimators "based on fuzzy data" for the two unknown parameters and hazard rate function. Section 5 focuses on the results of Monte-Carlo simulation study and compares the performance of the obtained estimates according to that numerical results for different cases and sample sizes. Section 6 covers the conclusions drawn from the results of simulation study.

2. Lindley distribution

Originally, in the framework of Bayesian statistics, Lindley distribution was proposed with one parameter by Lindley (1958)[18]. However, many scholars attributed the real interest to this distribution began after the most important mathematical properties introduced by Ghitany et al. (2008)[19] and showed that this distribution, in many applications, provides a better model than the exponential distribution. As a generalization, Shanker et al. (2013) [20] introduced a two-parameter Lindley distribution (TPLD). The cumulative distribution function (CDF) and probability density function (PDF) of a TPLD are given, respectively, by:

\[ F(x; \alpha, \theta) = 1 - \frac{\theta + \alpha + \alpha \theta x}{\theta + \alpha} - e^{-\theta x}; \quad x \geq 0, \theta > 0, \alpha > -\theta \]

(1)

\[ f(x; \alpha, \theta) = \frac{\theta^2}{\theta + \alpha} (1 + \alpha x) e^{-\theta x}; \quad x \geq 0, \theta > 0, \alpha > -\theta \]

(2)

The TPLD's hazard rate function at a specified time, say \( t (t \geq 0) \) is defined by,

\[ h(t; \alpha, \theta) = \frac{f(t; \alpha, \theta)}{R(t; \alpha, \theta)} = \frac{\theta^2 (1 + \alpha t)}{\theta + \alpha + \alpha \theta t}; \quad \theta > 0, \alpha > -\theta \]

(3)

When \( \alpha = 1 \) and \( \alpha = 0 \) a TPLD reduces to one parameter Lindley distribution and to exponential distribution respectively.

3. Maximum likelihood estimation

Assume \( x = (x_1, x_2, \ldots, x_n) \) be an i.i.d. random vector of a random sample of size \( n \) chosen from a TPLD. The likelihood function with complete-data, say \( L(\alpha, \theta|x) \), is given by

\[ L(\alpha, \theta|x) = \prod_{i=1}^{n} f_X(x_i; \alpha, \theta) = \prod_{i=1}^{n} \frac{\theta^2}{\theta + \alpha} (1 + \alpha x_i) e^{-\theta x_i} \]

\[ = L(\alpha, \theta|x) = \frac{\theta^{2n}}{(\theta + \alpha)^n} \prod_{i=1}^{n} (1 + \alpha x_i) e^{-\theta \sum_{i=1}^{n} x_i} \]

(4)

Now, if \( x \) is not observed precisely and only partial information about it is accessible in the form of a fuzzy subset \( \tilde{x} \) with \( \mu_{\tilde{x}}(x) \) as a membership function (see [6]). For this case, according to Zadeh’s interpretation of the probability of a fuzzy event, the observed-data likelihood function can be found as,

\[ L(\alpha, \theta|\tilde{x}) = \prod_{i=1}^{n} \int f_X(x; \alpha, \theta) \mu_{\tilde{x}_i}(x) \, dx = \frac{\theta^{2n}}{(\theta + \alpha)^n} \prod_{i=1}^{n} \int (1 + \alpha x) e^{-\theta x} \mu_{\tilde{x}_i}(x) \, dx \]

(5)
The observed-data natural log-likelihood function, \( \varphi = \varphi(\alpha, \theta | \tilde{x}) = \ln L(\alpha, \theta | \tilde{x}) \), can be attained as,

\[
\varphi = 2n \ln \theta - n \ln(\theta + \alpha) + \sum_{i=1}^{n} \ln \left(1 + \alpha x\right) e^{-\theta x} \mu_{x_i}(x) dx
\]

(6)

The likelihood equations to \( \alpha \) and \( \theta \) respectively are given by,

\[
\frac{\partial \varphi}{\partial \alpha} = -\frac{n}{\theta + \alpha} + \sum_{i=1}^{n} \frac{\int x e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx} = 0
\]

(7)

\[
\frac{\partial \varphi}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + \alpha} - \sum_{i=1}^{n} \frac{\int x(1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx} = 0
\]

(8)

Since the solutions of the above two equations do not have closed forms, it is possible to use a Newton-Raphson (NR) algorithm that summarizes in three steps as

**Step 1.** Set iteration \( l = 0 \) with assumed starting values of \( \alpha \) and \( \theta \), say \( \alpha^{(0)} \) and \( \theta^{(0)} \).

**Step 2.** At a new iteration \( (l + 1) \), estimate the new values of \( \alpha \) and \( \theta \) as,

\[
\begin{bmatrix}
\hat{\alpha}^{(l+1)} \\
\hat{\theta}^{(l+1)}
\end{bmatrix} = \begin{bmatrix}
\hat{\alpha}^{(l)} \\
\hat{\theta}^{(l)}
\end{bmatrix} - \begin{bmatrix}
\frac{\partial^2 \varphi}{\partial \alpha^2} & \frac{\partial^2 \varphi}{\partial \alpha \partial \theta} \\
\frac{\partial^2 \varphi}{\partial \theta \partial \alpha} & \frac{\partial^2 \varphi}{\partial \theta^2}
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial \varphi}{\partial \alpha} \\
\frac{\partial \varphi}{\partial \theta}
\end{bmatrix}
\]

\[
\left[ \begin{array}{c}
\hat{\alpha}^{(l+1)} \\
\hat{\theta}^{(l+1)}
\end{array} \right] = \left[ \begin{array}{c}
\hat{\alpha}^{(l)} \\
\hat{\theta}^{(l)}
\end{array} \right] - \left[ \begin{array}{cc}
\frac{\partial^2 \varphi}{\partial \alpha^2} & \frac{\partial^2 \varphi}{\partial \alpha \partial \theta} \\
\frac{\partial^2 \varphi}{\partial \theta \partial \alpha} & \frac{\partial^2 \varphi}{\partial \theta^2}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\frac{\partial \varphi}{\partial \alpha} \\
\frac{\partial \varphi}{\partial \theta}
\end{array} \right]
\]

where \( \frac{\partial^2 \varphi}{\partial \alpha} \) and \( \frac{\partial^2 \varphi}{\partial \theta} \) correspondingly as in equations (7) and (8), and

\[
\frac{\partial^2 \varphi}{\partial \alpha^2} = -\frac{n}{(\theta + \alpha)^2} - \sum_{i=1}^{n} \left( \frac{\int x e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx} \right)^2
\]

(10)

\[
\frac{\partial^2 \varphi}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n}{(\theta + \alpha)^2}
\]

\[
+ \sum_{i=1}^{n} \left[ \frac{\int x^2 (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx} - \left( \frac{\int x (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx} \right)^2 \right]
\]

(11)

\[
\frac{\partial^2 \varphi}{\partial \alpha \partial \theta} = \frac{\partial^2 \varphi}{\partial \theta \partial \alpha} = -\frac{n}{(\theta + \alpha)^2}
\]

\[
- \sum_{i=1}^{n} \frac{\int x^2 e^{-\theta x} \mu_{x_i}(x) dx}{\int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx}
\]

\[
+ \sum_{i=1}^{n} \frac{\int x e^{-\theta x} \mu_{x_i}(x) dx \int x (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx}{\left( \int (1 + \alpha x) e^{-\theta x} \mu_{x_i}(x) dx \right)^2}
\]

(12)

**Step 3.** Repeat step 2 till convergence occurs, i.e. the absolute difference between two sequential iterations is less than pre-specified error tolerance, say \( \varepsilon = 0.0001 \). Once the convergence occurs, the present estimation of \( \alpha \) and \( \theta \) at \( (l + 1) \) be the ML estimates of that parameters and we referred to as \( (\hat{\alpha}_{ML}, \hat{\theta}_{ML}) \).

The estimate of hazard rate function of TPLD at time \( (t) \) can be attained through an invariant property of ML estimates, by changing the parameters in equation (3) by their ML estimates as,
\[ h_{ML}(t) = \frac{\hat{\theta}_{ML}^2 (1 + \hat{\alpha}_{ML} t)}{\hat{\theta}_{ML} + \hat{\alpha}_{ML} + \hat{\alpha}_{ML} \hat{\theta}_{ML} t}; \quad t \geq 0 \]  

4. Bayes estimation

Consider the prior distributions of \( \alpha \) and \( \theta \) respectively being selected as independent Gamma \((a, b)\) and Gamma \((c, d)\). A joint prior distribution of these parameters with the form \( P(\alpha, \theta) = P(\alpha) P(\theta) \) will be,

\[ P(\alpha, \theta) = \frac{b^a d^c}{\Gamma(a) \Gamma(c)} \alpha^{a-1} \theta^{c-1} e^{-(b \alpha + d \theta)} \]  

Merging equation (5) with (14), the joint posterior density function (JPDF) of \( \alpha \) and \( \theta \) given \( \bar{x} \) can be attained by,

\[ \Pi(\alpha, \theta | \bar{x}) = \frac{\int_0^\infty \int_0^\infty L(\alpha, \theta | \bar{x}) P(\alpha, \theta) \, d\alpha \, d\theta}{\int_0^\infty \int_0^\infty P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta} = \frac{P(\alpha, \theta | \bar{x})}{\int_0^\infty \int_0^\infty P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta} \]  

where,

\[ P(\alpha, \theta | \bar{x}) = \frac{b^a d^c \alpha^{a-1} \theta^{c-1} e^{-(b \alpha + d \theta)}}{\int_0^\infty \int_0^\infty L(\alpha, \theta | \bar{x}) P(\alpha, \theta) \, d\alpha \, d\theta} \]

Now, Bayes estimation of any function of the parameters, say \( u(\alpha, \theta) \), relative to squared error loss function (SELF) and precautionary loss function (PLF), say \( \hat{u}_{BS}(\alpha, \theta) \) and \( \hat{u}_{BP}(\alpha, \theta) \), can be respectively attained as,

\[ \hat{u}_{BS}(\alpha, \theta) = E[u(\alpha, \theta) | \bar{x}] = \frac{\int_0^\infty \int_0^\infty u(\alpha, \theta) P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta}{\int_0^\infty \int_0^\infty P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta} \]  

\[ \hat{u}_{BP}(\alpha, \theta) = \left[ E[u(\alpha, \theta) | \bar{x}] \right]^2 = \left[ \frac{\int_0^\infty \int_0^\infty u(\alpha, \theta) P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta}{\int_0^\infty \int_0^\infty P(\alpha, \theta | \bar{x}) \, d\alpha \, d\theta} \right]^2 \]

Equations (16) and (17) are of the form of ratio of two integrals which cannot be reduced into a closed form. Though, we can use the approximation method of Lindley (see [21]) to calculate the two integrals ratio, say \( I(\bar{x}) \), as,

\[ I(\bar{x}) = u(\hat{\alpha}, \hat{\theta}) + \frac{1}{2} \left[ (\hat{\alpha}_{aa} + 2\hat{\alpha}_a \hat{\alpha}_\theta) \hat{\alpha}_{\alpha \theta} + (\hat{\alpha}_{a \theta} + 2\hat{\alpha}_a \hat{\alpha}_\theta) \hat{\alpha}_{\alpha \theta} + (\hat{\alpha}_{\alpha \alpha} + 2\hat{\alpha}_a \hat{\alpha}_\theta) \hat{\alpha}_{\alpha \theta} \right] \]

\[ + \frac{1}{2} \left[ (\hat{\alpha}_{a a} \hat{\alpha}_{a \theta} + \hat{\alpha}_{a \theta} \hat{\alpha}_{a \theta}) \hat{\alpha}_{\alpha \theta} + \hat{\alpha}_{\alpha \theta} \hat{\alpha}_{a \theta} + \hat{\alpha}_{\alpha \theta} \hat{\alpha}_{a \theta} + \hat{\alpha}_{a a} \hat{\alpha}_{a \theta} \right] \]

\[ + \hat{\alpha}_{a a} \hat{\alpha}_{a \theta} \hat{\alpha}_{a \theta} \right] \]

where, \( \hat{\alpha} \) and \( \hat{\theta} \) represents respectively the ML estimators of \( \alpha \) and \( \theta \). \( \sigma_{ij} \) represents the \((i, j)^{th}\) elements of matrix \( \frac{\partial^2 \varphi}{\partial \alpha \partial \theta} \), sub-scripts \((i, j)\) refer to \( \alpha, \theta \) respectively. \( \hat{u}_a \) and \( \hat{u}_\theta \) represents the first derivative of \( u(\alpha, \theta) \) w.r.t. \( \alpha \) and \( \theta \) evaluated at \( \hat{\alpha} \) and \( \hat{\theta} \). \( \hat{u}_{aa a} \) represents the second derivative of \( u(\alpha, \theta) \) w.r.t. \( \alpha \) evaluated at \( \hat{\alpha} \) and \( \hat{\theta} \). Other expressions can be deduced in almost the same way.

\[ \hat{\alpha}_a = \frac{\partial \ln P(\alpha, \theta)}{\partial \alpha} \bigg|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}} = \frac{a - 1}{\hat{\alpha}} - b \]
\begin{align}
\hat{\theta}_\theta &= \left. \frac{\partial \ln P(\alpha, \theta)}{\partial \theta} \right|_{\theta = \hat{\theta}} = \frac{c - 1}{\hat{\theta}} - d \tag{20}
\end{align}

From (10), (11) and (12), we get,

\begin{align}
\hat{\phi}_{\alpha\alpha} &= \left. \frac{\partial^3 \phi}{\partial \alpha^3} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = - \frac{2n}{(\hat{\theta} + \hat{\alpha})^3} + 2 \sum_{i=1}^{n} \left( \frac{\int x \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right)^3 \\
\hat{\phi}_{\theta\theta} &= \left. \frac{\partial^3 \phi}{\partial \theta^3} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = \frac{4n}{(\hat{\theta} + \hat{\alpha})^3} - \frac{2n}{(\hat{\theta} + \hat{\alpha})^3} - \sum_{i=1}^{n} \left( \frac{\int x^3 (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right) \\
&+ 3 \sum_{i=1}^{n} \left( \frac{\int x \, (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x^2 (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right)^2 \\
&- 2 \sum_{i=1}^{n} \left( \frac{\int x \, (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x \, (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right)^3 \tag{22}
\end{align}

\begin{align}
\hat{\phi}_{\alpha\theta} &= \left. \frac{\partial^3 \phi}{\partial \alpha \partial \theta} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = \hat{\phi}_{\alpha\theta} = \left. \frac{\partial^3 \phi}{\partial \alpha \partial \theta} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = \hat{\phi}_{\beta\alpha} = \left. \frac{\partial^3 \phi}{\partial \beta \partial \alpha} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = - \frac{2n}{(\hat{\theta} + \hat{\alpha})^3} \\
&+ 2 \sum_{i=1}^{n} \left[ \frac{\int x \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x^2 e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x \, (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right] \tag{23}
\end{align}

\begin{align}
\hat{\phi}_{\theta\alpha} &= \left. \frac{\partial^3 \phi}{\partial \theta \partial \alpha} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = \hat{\phi}_{\alpha\theta} = \left. \frac{\partial^3 \phi}{\partial \alpha \partial \theta} \right|_{\alpha = \hat{\alpha}, \theta = \hat{\theta}} = - \frac{2n}{(\hat{\theta} + \hat{\alpha})^3} \\
&+ \sum_{i=1}^{n} \left[ \frac{\int x^3 e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} - \frac{\int x^2 (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right] \\
&+ 2 \sum_{i=1}^{n} \left[ \frac{\int x (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} - \frac{\int x (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx \int x^2 \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx}{\int (1 + \hat{\alpha} x) \, e^{-\hat{\theta} x} \mu_{i}(x) \, dx} \right] \tag{24}
\end{align}

Relative to SELF, the approximate Bayes estimators would be as follows:

- For parameter $\alpha$: Assume that $u(\hat{\alpha}, \hat{\theta}) = \alpha$ and then,
  \[ u_\alpha = 1, \ u_{\alpha\alpha} = u_{\theta\theta} = u_{\alpha\theta} = u_{\beta\alpha} = 0. \]
\[ \hat{a}_{BS} = E(\alpha | \tilde{x}) = \hat{a} + \hat{\beta}_a \hat{\sigma}_{aa} + \hat{\beta}_b \hat{\sigma}_{ab} \\
+ \frac{1}{2} \left[ \hat{\sigma}_{aa} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \right. \\
\left. + \hat{\sigma}_{ab} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \right] \tag{25} \]

- For parameter \( \theta \): Assume that \( u(\alpha, \theta) = \theta \) and then,
\[ u_\theta = 1, u_{\theta \theta} = u_{\alpha a} = u_{\alpha \theta} = u_{\theta a} = 0. \]
\begin{align*}
\tilde{\theta}_{BS} = E(\theta | \tilde{x}) &= \hat{\theta} + \hat{\beta}_\theta \hat{\sigma}_{\theta \theta} + \hat{\beta}_{aa} \hat{\sigma}_{aa} \\
+ \frac{1}{2} \left[ \hat{\sigma}_{\theta \theta} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \right. \\
\left. + \hat{\sigma}_{a \theta} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \right] \tag{26} \end{align*}

- For hazard rate function: Assume that \( u(\alpha, \theta) = h(t) = \frac{\theta^2(1+\alpha t)}{\theta^2 + \alpha + \alpha^2 t} \) and then,
\[ u_\alpha = - \frac{\theta^2}{(\theta^2 + \alpha + \alpha^2 t)}, \quad u_\theta = \frac{\theta^3(1+\alpha t)}{(\theta^2 + \alpha + \alpha^2 t)^2}, \quad u_{\alpha a} = u_{\alpha \theta} = u_{\theta a} = u_{\theta \theta} = 0. \]
\[ \tilde{h}_{BS}(t) = E(h(t) | \tilde{x}) = \frac{\hat{\theta}^2(1 + \hat{\alpha} t)}{\hat{\beta} + \hat{\alpha} + \hat{\alpha} \hat{\theta} t} \\
+ \frac{1}{2} \left[ (\hat{\alpha} \hat{\theta} + \hat{\alpha} \hat{\theta} \hat{\sigma}_{\theta \theta}) \hat{\sigma}_{\theta \theta} \right. \\
\left. + (\hat{\alpha} \hat{\theta} \hat{\sigma}_{\theta \theta} + \hat{\alpha} \hat{\theta} \hat{\sigma}_{a \theta} + \hat{\alpha} \hat{\sigma}_{a \theta} \hat{\sigma}_{\theta \theta}) \hat{\sigma}_{a \theta} \right. \\
\left. + (\hat{\alpha} \hat{\theta} \hat{\sigma}_{a \theta} + \hat{\alpha} \hat{\theta} \hat{\sigma}_{\theta \theta} + \hat{\alpha} \hat{\sigma}_{a \theta} \hat{\sigma}_{\theta \theta}) \hat{\sigma}_{aa} \right] \tag{27} \]

Relative to PLF, with the same argument, the approximate Bayes estimations would be as follows,
- For parameter \( \alpha \): Assume that \( u(\alpha, \theta) = \alpha^2 \) and then,
\[ u_\alpha = 2\alpha, \quad u_{\alpha a} = 2, \quad u_{\alpha \theta} = u_{\alpha \theta} = u_{\alpha a} = 0. \]
\[ \hat{a}_{BP} = [E(\alpha^2 | \tilde{x})]^{\frac{1}{2}} \tag{28} \]

where,
\[ E(\alpha^2 | \tilde{x}) = \hat{\alpha}^2 + 2\hat{\beta}_a \hat{\sigma}_{\theta \theta} + (1 + 2\hat{\beta}_a) \hat{\sigma}_{aa} \]
\[ + \hat{\sigma}_{aa} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \]
\[ + \hat{\sigma}_{ab} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \] \tag{29}

- For parameter \( \theta \): Assume that \( u(\alpha, \theta) = \theta^2 \) and then,
\[ u_\theta = 2\theta, \quad u_{\theta \theta} = 2, \quad u_{\alpha a} = u_{\alpha \theta} = u_{\theta a} = 0. \]
\[ \hat{\theta}_{BP} = [E(\theta^2 | \tilde{x})]^{\frac{1}{2}} \tag{30} \]

where,
\[ E(\theta^2 | \tilde{x}) = \hat{\theta}^2 + (1 + 2\hat{\beta}_a) \hat{\sigma}_{\theta \theta} + 2\hat{\beta}_a \hat{\sigma}_{\theta \theta} \]
\[ + \hat{\sigma}_{\theta \theta} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \]
\[ + \hat{\sigma}_{a \theta} (\hat{\varphi}_{b\theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta \theta} \hat{\sigma}_{\theta \theta} + \hat{\varphi}_{\theta \theta a} \hat{\sigma}_{a \theta} + \hat{\varphi}_{\theta a a} \hat{\sigma}_{aa}) \] \tag{31}

- For hazard rate function: Assume that \( u(\alpha, \theta) = h^2(t) = \left( \frac{\theta^2(1+\alpha t)}{\theta^2 + \alpha + \alpha^2 t} \right)^2 \) and then,
\[ u_{\theta a} = - \frac{2\theta^2(1+\alpha t)}{(\theta^2 + \alpha + \alpha^2 t)^3}, \quad u_{\theta \theta} = \frac{2\theta^3(1+\alpha t)}{(\theta^2 + \alpha + \alpha^2 t)^3}, \quad u_{\theta a} = \frac{2\theta^2(1+\alpha t)^2(\theta^2 + 2\alpha^2 t)}{(\theta^2 + \alpha + \alpha^2 t)^4}, \quad u_{\theta \theta} = \frac{2\theta^2(1+\alpha t)^2(\theta^2 + 2\alpha^2 t + 3\alpha^2 t^2)}{(\theta^2 + \alpha + \alpha^2 t)^4}, \tag{32} \]
\[ u_{\alpha \theta} = u_{\beta \alpha} = \frac{2\theta^3(1+\alpha t)(\theta+4\alpha+\alpha t)}{(\theta+\alpha+\alpha t)^4} \]

\[ \hat{h}_{PF}(t) = \left[ E(h^2(t) | \tilde{x}) \right]^\frac{1}{2} \]  

where,

\[ E(h^2(t) | \tilde{x}) = \left( \frac{\beta^2(1+\tilde{\alpha} t)}{\beta + \tilde{\alpha} + \tilde{\alpha} \beta t} \right)^2 + \frac{1}{2} \left[ (\tilde{u}_{\beta \theta} + 2\tilde{u}_{\alpha \beta} \beta \tilde{\alpha}) \tilde{\delta}_{\beta \theta} + (\tilde{u}_{\alpha \theta} + 2\tilde{u}_{\alpha \beta} \beta \tilde{\alpha}) \tilde{\delta}_{\alpha \theta} + (\tilde{u}_{\beta \alpha} + 2\tilde{u}_{\alpha \beta} \beta \tilde{\alpha}) \tilde{\delta}_{\beta \alpha} + (\tilde{u}_{\alpha \alpha} + 2\tilde{u}_{\alpha \beta} \beta \tilde{\alpha}) \tilde{\delta}_{\alpha \alpha} \right] \]

\[ + \left( \tilde{u}_{\beta \theta} \tilde{\delta}_{\beta \theta} + \tilde{u}_{\alpha \beta} \tilde{\delta}_{\beta \alpha} + \tilde{u}_{\alpha \alpha} \tilde{\delta}_{\alpha \alpha} + \tilde{u}_{\beta \alpha} \tilde{\delta}_{\alpha \beta} + \tilde{u}_{\beta \alpha} \tilde{\delta}_{\beta \alpha} + \tilde{u}_{\alpha \alpha} \tilde{\delta}_{\alpha \alpha} \right) \]

\[ + \tilde{u}_{\beta \alpha} \tilde{\delta}_{\beta \alpha} + \tilde{u}_{\alpha \alpha} \tilde{\delta}_{\alpha \alpha} \right] \]  

\[ (33) \]

5. **Simulation study**

The steps of simulation may be summarized by,

**Step 1.** Generate a random sample \( \chi \) of size \( n \) (\( n = 20, 30, 50 \)) from a TPLD with \( \alpha = 0.5 \) and \( \theta = 1.5 \) and 3. Since the explicit inverse form of the TPLD function cannot be achieved, it is possible to generate random samples based on the fact that Lindley distribution is a mixture of exponential distribution with parameter \( \theta \) and Gamma distribution with parameters \((2, \theta)\), and proportional mixing \( \frac{2}{\theta + \alpha} \), as

i. Generate \( s_i, v_i, w_i \) respectively as standard Uniform, Exponential with \( \theta \), and Gamma with \( (2, \theta) \) and \( i = 1, 2, ..., n \).

ii. If \( s_i \leq p = \frac{\theta}{\theta + \alpha} \), set \( x_i = v_i \), otherwise, set \( x_i = w_i \), \( i = 1, 2, ..., n \).

**Step 2.** Based on the FIS "fuzzy information system" (as in figure 1), encode the generated sample where each observation will be fuzzy based on a appropriate selected membership function of the eight membership functions (as in Table 1).

**Step 3.** Calculate the estimates of parameters and hazard rate function based on the obtained formulas. With NR, the starting values of \( \alpha \) and \( \theta \), have been chosen to be the moment estimators \([20]\) i.e.,

\[ \hat{\alpha}^{(0)} = \frac{M+2}{M(M+1)x} \]  

\[ \hat{\theta}^{(0)} = \frac{M+2}{(M+1)x} \]  

provided that \( \theta = M \alpha ; \tilde{x} = \frac{M+2}{aM(M+1)} \).

With Bayesian estimation prior 1 with \( a = b = c = d = 0.0001 \) and prior 2 with \( a = 3, b = 2 \) and \( c = 2, d = 2 \).

**Step 3.** Compare the obtained estimates for the parameters based on the average mean square error (MSE) and compare the obtained estimates of hazard rate function with different times based on the average integrated mean square error (IMSE).

\[ MSE(\hat{P}) = \frac{\sum_{j=1}^{R} (\hat{P}_j - P)^2}{R} \]  

\[ IMSE(\hat{h}(t)) = \frac{1}{R} \sum_{j=1}^{R} \left( \frac{1}{n_t} \sum_{i=1}^{n_t} \left( \hat{h}_j(t_i) - h(t_i) \right)^2 \right) \]  

where,

\( \hat{P}_j \) is the estimate of parameter respectively at the \( j^{th} \) replicate, \( \hat{h}_j(t_i) \) is the estimates of \( h(t) \) at the \( j^{th} \) replicate and \( i^{th} \) time, \( R \) is the number of sample replicates chosen to be 1000, and \( n_t \) is the number of times chosen to be (4), as \( t = 1, 2, 3, 4 \).

The results of simulation study have been summarized in Tables 2-7.
Table 1. The membership functions.

\[ \mu_{x_1}(x) = \begin{cases} 1 & ; \quad x \leq 0.05, \\ \frac{2.25 - x}{0.2} & ; \quad 0.05 \leq x \leq 0.25, \\ 0 & ; \quad \text{otherwise}, \end{cases} \]

\[ \mu_{x_2}(x) = \begin{cases} \frac{x - 0.25}{0.25} & ; \quad 0.25 \leq x \leq 0.5, \\ \frac{0.75 - x}{0.25} & ; \quad 0.5 \leq x \leq 0.75, \\ 0 & ; \quad \text{otherwise}, \end{cases} \]

\[ \mu_{x_3}(x) = \begin{cases} \frac{x - 0.75}{0.25} & ; \quad 0.75 \leq x \leq 1, \\ \frac{1.5 - x}{0.5} & ; \quad 1 \leq x \leq 1.5, \\ 0 & ; \quad \text{otherwise}, \end{cases} \]

\[ \mu_{x_4}(x) = \begin{cases} \frac{x - 1.5}{0.5} & ; \quad 1.5 \leq x \leq 2, \\ \frac{3 - x}{0} & ; \quad 2 \leq x \leq 3, \\ 0 & ; \quad \text{otherwise}. \end{cases} \]

Figure 1. Encode the simulated data through the FIS [6].

From Table 2 to Table 5 it is observed that the MSE values associated with ML and Bayes estimates are decreasing as the sample size increases. The performance of Bayes estimates with prior 2 is better than that with prior 1. The performance of Bayes estimates relative to PLF is better than that relative to SELF. Increase the value of \( \theta \), increasing the values of MSE that associated with ML and Bayes estimates. For more illustrative see figures 2 and 3.

From Tables 6 and 7, it is observed that the IMSE values associated with ML and Bayes estimators are decreasing as the sample size increases. The performance of Bayes estimates with prior 2 is better than that with prior 1 for all sample sizes. The performance of Bayes estimates according to PLF is better than that according to SELF. Increase the value of \( \theta \), increasing the values of IMSE associated with ML and Bayes estimators of hazard rate function with all sample sizes. For more illustrative see figure 4.
Table 2. MSE values for ML estimators of $\alpha$ with different cases.

| $n$ | $\alpha = 0.5$, $\theta = 1.5$ | $\alpha = 0.5$, $\theta = 3$ | $\alpha = 5$, $\theta = 1.5$ | $\alpha = 5$, $\theta = 3$ |
|-----|--------------------------------|--------------------------------|-------------------------------|-------------------------------|
| 20  | 0.3211176                      | 1.3050108                      | 0.7876371                     | 0.8571912                     |
| 30  | 0.1720963                      | 0.8694877                      | 0.4625790                     | 0.7170763                     |
| 50  | 0.0912434                      | 0.6542933                      | 0.0961845                     | 0.3581640                     |

Table 3. MSE values for Bayes estimators of $\alpha$ with different loss functions, cases and prior distributions.

| $n$ | $\alpha = 0.5$, $\theta = 1.5$ | $\alpha = 0.5$, $\theta = 3$ | $\alpha = 5$, $\theta = 1.5$ | $\alpha = 5$, $\theta = 3$ |
|-----|--------------------------------|--------------------------------|-------------------------------|-------------------------------|
| 20  | Prior 1: 0.6995432 0.5300141   | Prior 1: 2.8567707 1.0349632  | Prior 1: 1.8029032 1.2217908  | Prior 1: 1.9047525 1.3605992  |
| 30  | Prior 2: 0.3655968 0.3004507   | Prior 1: 1.8340797 1.0007966  | Prior 1: 0.8039590 0.6343615  | Prior 1: 0.5907594 0.8746735  |
| 50  | Prior 1: 0.1651508 0.1080571   | Prior 1: 1.1275026 0.6789923  | Prior 1: 0.1813647 0.1460007  | Prior 1: 0.5363792 0.2905413  |

Best Prior

| $n$ | $\alpha = 0.5$, $\theta = 1.5$ | $\alpha = 0.5$, $\theta = 3$ | $\alpha = 5$, $\theta = 1.5$ | $\alpha = 5$, $\theta = 3$ |
|-----|--------------------------------|--------------------------------|-------------------------------|-------------------------------|
| 20  | Prior 2: 0.6524728 0.3256671  | Prior 2: 1.9987220 0.9801375  | Prior 2: 1.6953178 0.8730152  | Prior 2: 1.7800966 1.0346305  |
| 30  | Prior 2: 0.3126281 0.2011370  | Prior 2: 1.7692416 0.6001381  | Prior 2: 0.7619186 0.5394407  | Prior 2: 1.5132128 0.806701  |
| 50  | Prior 2: 0.1607289 0.1078938  | Prior 2: 1.1121180 0.5818348  | Prior 2: 0.1794604 0.1297502  | Prior 2: 0.5232846 0.280633  |
Table 4. MSE values for ML estimators of $\theta$ with different cases.

| $n$ | $\alpha = 0.5, \theta = 1.5$ | $\alpha = 0.5, \theta = 3$ | $\alpha = 5, \theta = 1.5$ | $\alpha = 5, \theta = 3$ |
|-----|----------------|----------------|----------------|----------------|
| 20  | 0.0151593      | 0.1197633      | 0.0147007      | 0.1200117      |
| 30  | 0.0129988      | 0.0845984      | 0.0118765      | 0.0847590      |
| 50  | 0.0098780      | 0.0504194      | 0.0087671      | 0.0521394      |

Table 5. MSE values for Bayes estimators of $\theta$ with different loss functions, cases and prior distributions.

### Squared Error Loss Functions

| $n$ | $\alpha = 0.5, \theta = 1.5$ | $\alpha = 0.5, \theta = 3$ | $\alpha = 5, \theta = 1.5$ | $\alpha = 5, \theta = 3$ |
|-----|----------------|----------------|----------------|----------------|
|     | Prior 1 | Prior 2 | Prior 1 | Prior 2 | Prior 1 | Prior 2 | Prior 1 | Prior 2 |
| 20  | 0.0162091 | 0.0138401 | 0.1300329 | 0.0749008 | 0.0156786 | 0.0134277 | 0.1300817 | 0.0747637 |
| 30  | 0.0137409 | 0.0121173 | 0.0896891 | 0.0668250 | 0.0129176 | 0.0114659 | 0.0908345 | 0.0616623 |
| 50  | 0.0091387 | 0.0086015 | 0.0524818 | 0.0415006 | 0.0089869 | 0.0084751 | 0.0542186 | 0.0428208 |
| Best Prior | Prior 2 | Prior 2 | Prior 2 | Prior 2 |

### Precautionary Loss Function

| $n$ | $\alpha = 0.5, \theta = 1.5$ | $\alpha = 0.5, \theta = 3$ | $\alpha = 5, \theta = 1.5$ | $\alpha = 5, \theta = 3$ |
|-----|----------------|----------------|----------------|----------------|
|     | Prior 1 | Prior 2 | Prior 1 | Prior 2 | Prior 1 | Prior 2 | Prior 1 | Prior 2 |
| 20  | 0.0151593 | 0.0130168 | 0.1197633 | 0.0699754 | 0.0147007 | 0.0126623 | 0.1200117 | 0.0697027 |
| 30  | 0.0129988 | 0.0115104 | 0.0845984 | 0.0623981 | 0.0122765 | 0.0109430 | 0.0847590 | 0.0577549 |
| 50  | 0.0087870 | 0.0083701 | 0.0504194 | 0.0399718 | 0.0087671 | 0.0082811 | 0.0521394 | 0.0413004 |
| Best Prior | Prior 2 | Prior 2 | Prior 2 | Prior 2 |
Table 6. IMSE values for ML estimators of \( h(t) \) with different cases.

| \( n \) | \( \alpha = 0.5, \theta = 1.5 \) | \( \alpha = 0.5, \theta = 3 \) | \( \alpha = 5, \theta = 1.5 \) | \( \alpha = 5, \theta = 3 \) |
|---|---|---|---|---|
| 20 | 0.0181784 | 0.0290979 | 0.0174754 | 0.0182533 |
| 30 | 0.0170279 | 0.0270266 | 0.0165264 | 0.0175399 |
| 50 | 0.0094962 | 0.0256837 | 0.0075647 | 0.0115396 |

Table 7. IMSE values for Bayes estimators of \( h(t) \) with different loss functions, cases and prior distributions.

### Squared Error Loss Function

| \( n \) | \( \alpha = 0.5, \theta = 1.5 \) | \( \alpha = 0.5, \theta = 3 \) | \( \alpha = 5, \theta = 1.5 \) | \( \alpha = 5, \theta = 3 \) |
|---|---|---|---|---|
| 20 | Prior 1: 0.0085584 Prior 2: 0.0069321 | Prior 1: 0.0180638 Prior 2: 0.0103889 | Prior 1: 0.0087240 Prior 2: 0.0079527 | Prior 1: 0.0099715 Prior 2: 0.0099038 |
| 30 | Prior 1: 0.0078025 Prior 2: 0.0061307 | Prior 1: 0.0163629 Prior 2: 0.0093715 | Prior 1: 0.0079409 Prior 2: 0.0064098 | Prior 1: 0.0083291 Prior 2: 0.0080661 |
| 50 | Prior 1: 0.0069354 Prior 2: 0.0055561 | Prior 1: 0.0154582 Prior 2: 0.0089999 | Prior 1: 0.0075423 Prior 2: 0.0055831 | Prior 1: 0.0080843 Prior 2: 0.0079146 |
| Best Prior | Prior 1: 0.0069354 Prior 2: 0.0055561 | Prior 1: 0.0154582 Prior 2: 0.0089999 | Prior 1: 0.0075423 Prior 2: 0.0055831 | Prior 1: 0.0080843 Prior 2: 0.0079146 |

### Precautionary Loss Function

| \( n \) | \( \alpha = 0.5, \theta = 1.5 \) | \( \alpha = 0.5, \theta = 3 \) | \( \alpha = 5, \theta = 1.5 \) | \( \alpha = 5, \theta = 3 \) |
|---|---|---|---|---|
| 20 | Prior 1: 0.0082622 Prior 2: 0.0059913 | Prior 1: 0.0122623 Prior 2: 0.0090311 | Prior 1: 0.0081624 Prior 2: 0.0076321 | Prior 1: 0.0089341 Prior 2: 0.0079503 |
| 30 | Prior 1: 0.0075743 Prior 2: 0.0053311 | Prior 1: 0.0161299 Prior 2: 0.0088715 | Prior 1: 0.0074785 Prior 2: 0.0059100 | Prior 1: 0.0080271 Prior 2: 0.0071662 |
| 50 | Prior 1: 0.0068158 Prior 2: 0.0051188 | Prior 1: 0.0100228 Prior 2: 0.0081773 | Prior 1: 0.0064963 Prior 2: 0.0053917 | Prior 1: 0.0073587 Prior 2: 0.0058177 |
| Best Prior | Prior 1: 0.0068158 Prior 2: 0.0051188 | Prior 1: 0.0100228 Prior 2: 0.0081773 | Prior 1: 0.0064963 Prior 2: 0.0053917 | Prior 1: 0.0073587 Prior 2: 0.0058177 |
Figure 2. The MSE values for different estimators of $\alpha$

Figure 3. The MSE values for different estimators of $\theta$
6. Conclusion
In statistics, numbers, vectors or classical functions are considered to be the data. But, in real applications, data are often not accurate numbers or vectors but often more or less imprecise. All kinds of data that can’t be presented as precise numbers or can’t be classified precisely are titled as non-precise or fuzzy. This paper focused on obtaining the ML and approximate Bayesian estimators of the parameters and hazard rate function of Lindley distribution when the data is available in fuzzy form. Based on different cases under comparison, we conclude that the performance of Bayes estimators according to Lindley's approximation under precautionary loss function with informative gamma prior.
and hyper-parameters greater than one is better than that under squared error loss function. Also, Increase the value of $\theta$, increasing the values of MSE and IMSE that associated with ML and Bayes estimates of parameters and hazard rate function.

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