LETTER

Self-similar motion for modeling anomalous diffusion and nonextensive statistical distributions

Zhifu Huang\textsuperscript{1}, Guozhen Su\textsuperscript{1}, Aziz El Kaabouchi\textsuperscript{2}, Qiuping A Wang\textsuperscript{2,3} and Jincan Chen\textsuperscript{1,2}

\textsuperscript{1} Department of Physics and Institute of Theoretical Physics and Astrophysics, Xiamen University, Xiamen 361005, People’s Republic of China
\textsuperscript{2} ISMANS, 44 Avenue Bartholdi, 72000 Le Mans, France
\textsuperscript{3} LPEC, UMR6087, Université du Maine, Avenue Olivier Messiaen, 72085 Le Mans, France
E-mail: zhuang@xmu.edu.cn, gzsu@xmu.edu.cn, aek@ismans.fr, awang@ismans.fr and jcchen@xmu.edu.cn (corresponding author)

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Abstract. We introduce a new universality class of one-dimensional iteration models giving rise to self-similar motion. The curves of the mean square displacement versus time show that the motion is a kind of anomalous diffusion with the diffusion coefficient depending on the self-similar rates. Moreover, it is found that the distribution of the displacement agrees to a reliable precision with the $q$-Gaussian type distribution in some cases and the bimodal distribution in some other cases. The results show that the self-similar motion may be used to investigate anomalous diffusion and nonextensive statistical distributions.

Keywords: stochastic particle dynamics (theory), connections between chaos and statistical physics, diffusion

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Diffusion is one of the most important types of motion in nature. Its thorough understanding from dynamical and thermodynamic points of view is still a big matter of investigation. A widely investigated diffusion, among many others, is the anomalous one which occurs in many physical and biological systems [1]–[4] often containing fractal and self-similar structures, long-range interaction and/or long-duration memory, and so on. Anomalous diffusion is characterized by a one-dimensional mean square displacement such as

$$\langle x^2(\Delta \tau) \rangle \propto \Delta \tau^\alpha,$$  \hspace{1cm} (1)

where $x(\Delta \tau)$ is the displacement in time interval $\Delta \tau$, $\alpha$ is called the diffusion coefficient characterizing the time behavior of the mean square displacement. The cases of $\alpha < 1$ and $\alpha > 1$ correspond to the subdiffusion and superdiffusion, respectively, while $\alpha = 1$ corresponds to the normal diffusion or Brownian motion.

The probability distribution of the displacement in anomalous diffusion usually does not agree with a Gaussian distribution coming from independent or nearly independent contributions, but may take the form of a non-Gaussian distribution, e.g., a Levy stable form [1, 5] or stretched Gaussian shape [1], [6]–[10], or the form of a $q$-Gaussian, given by

$$p(x) \propto [1 - (1 - q)\beta x^2]^{1/(1-q)},$$  \hspace{1cm} (2)

where $\beta$ is a parameter characterizing the width of the distribution and $q$ is the nonextensivity index [11]–[15]. Anomalous diffusion is sometimes associated with the $q$-Gaussian distribution, as in liquid with vortices [16] or in driven-dissipative dusty plasma [17, 18], and sometimes not [19]. In equation (2), $q \neq 1$ indicates a departure from the Gaussian shape while $q \to 1$ limit yields the normal Gaussian distribution.

As a family of stochastic motions, anomalous diffusion has been widely studied within various models and for various circumstances [20], such as fractional Brownian motions [21], Lévy motions [14], fractional stable Lévy motions [22, 23], etc. On the basis of the continuous time random walk, Scher and Montroll [24] observed that the measurements of the transient photocurrent in amorphous solids display subdiffusion transport properties, and Klafter et al [25] established a unified framework of anomalous diffusion with fractional differential equation models. On the other hand, the iterations of deterministic dynamical systems can never be completely independent from each other, since they are generated by a deterministic algorithm. Concerning this matter, much work has been done to find the properties of iteration that allow a classification of deterministic systems. May [26] proposed a simple mathematical model called a logistic map and established the theory of complicated dynamics. It is shown that there exists periodic motion or chaos in a deterministic dynamical system, which enables us to analyze the statistical properties from deterministic dynamics by using a simple iterative algorithm. Chaotic dynamics can be considered as a physical phenomenon that bridges between the regular evolution of systems and the random systems [27]–[29]. An intriguing aspect of the chaotic motion is that it exhibits self-similar structures [30] characterized by scale invariance and plays a central role in a large number of physics phenomena [31]. It is common knowledge that nonlinear maps show deterministic dynamics and often have self-similar structures. They exhibit various routes to chaos [32].

In particular, one-dimensional mapping is a tool widely used for studying the emergence of complexity in dynamical systems. Tirnakli et al [33, 34] showed that the
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Figure 1. The illustration of a self-similar motion for the parameters $r_1 = 1.2$ and $r_2 = 0.9$, where maps (a), (b), (c), and (d) correspond to the cases of $i = 1$, 2, 3, and 10, respectively.

probability distribution of the sums of iterates at the edge of chaos of the $z$-logistic map is numerically consistent with a $q$-Gaussian distribution given by equation (2) with $q = 1.63$. Moreover, Feigenbaum [35,36] proved that the self-similar rates are constants in all unimodal dissipative maps. Ruiz et al [37] obtained several Feigenbaum-like constants in a new universality class of one-dimensional dissipative maps and derived several values of $q$ by fitting the probability distributions to equation (2). Their iterations, however, can only yield one value of $q$ each time and cannot match the form of anomalous diffusion given by equation (1).

In this letter, we construct a universality class of self-similar motions with a new method of iteration. The velocity is iterated with self-similar structure on different scales over the whole time period. Furthermore, we will analyze the properties of diffusion through simulating the underlying deterministic dynamical system.

In previous works, Cantor [38] iterated a continuous line to get a self-similar line segment called the Cantor set. Similarly, we use a simple iteration method to get a $T$-periodic motion of which the time distribution of velocity is self-similar. Let the velocity of the motion be of square-wave form with constant $v_1$ in its first half-period and $-v_2$ in the second half-period ($v_1, v_2 > 0$), i.e., the velocity distribution at the first iteration is

\[
v(1, t) = \begin{cases} v_1, & 0 \leq t < \frac{T}{2} \\ -v_2, & \frac{T}{2} \leq t < T. \end{cases}
\]

as shown in figure 1(a), where $v_1$ and $v_2$ are equal to 1.2 and 0.9, respectively. We construct a universality class of motions with self-similar rates by iterating the first and
second half-periods (similarly to the first iteration for the whole period) with the rates $r_1$ and $-r_2$, respectively, that is

$$v(i + 1, t) = \begin{cases} v_1 + r_1 v(i, 2t), & 0 \leq t < \frac{T}{2} \\ -v_2 - r_2 v(i, 2t - T), & \frac{T}{2} \leq t < T \end{cases} \quad i = 1, 2, 3, \ldots \quad (4)$$

In order to make the motion similar to its original motion after iterations, the iteration rate is assumed to be proportional to the velocity, i.e., $r_1/r_2 = v_1/v_2$. The maximum absolute velocity $\max[|v(i, t)|]$, the average velocity $\overline{v(i)}$ and the mean square velocity $\overline{v^2(i)}$ at the $i$th iteration can be derived from equation (4) and given by, respectively,

$$\max[|v(i, t)|] = \begin{cases} v_1(1 - r_1^i) & r_1 \geq r_2 \\ -v_2(1 - r_2^i) & r_1 \leq r_2, \end{cases} \quad (5)$$

$$\overline{v(i)} = \frac{(v_1 - v_2)/2[1 - ((r_1 - r_2)/2)^i]}{1 - (r_1 - r_2)/2} \quad (6)$$

and

$$\overline{v^2(i)} = a \left( \frac{r_1^2 + r_2^2}{2} \right)^{i-1} + b \left( \frac{r_1 - r_2}{2} \right)^i + c \quad (7)$$

where

$$a = c_1(1 + c_2), \quad b = \frac{2c_1c_2}{r_2 - r_1}, \quad c = \frac{v_1^2 + v_2^2 - 2c_1}{r_1^2 + r_2^2}$$

$$c_1 = \frac{(v_1 r_1 + v_2 r_2)(v_1 - v_2)}{r_1 - r_2 - 2} \quad \text{and} \quad c_2 = \frac{(r_1^2 + r_2^2)(r_1 - r_2)}{r_1^2 + r_2^2 - (r_1 - r_2)}$$

Using equations (6) and (7), one can define a new scaled velocity as

$$u(i, t) = \frac{v(i, t) - \overline{v(i)}}{\sqrt{\overline{v^2(i)} - \overline{v(i)}^2}} \quad (8)$$

It can be seen from equation (4) that when the trajectory is iterated over the whole period, the self-similar rates $r_1$ and $r_2$ can be changed continuously. It is also obvious from equation (4) that the number of velocities is doubled by each iteration. Thus, the time interval of a single velocity in the $i$th iteration is

$$\Delta t(i) = \frac{T}{2^i} \quad (9)$$

According to equation (9), when $i$ is large enough, $\Delta t(i)$ will tend to zero. It should be pointed out that if $r_1 > 1$ and/or $r_2 > 1$, the velocity increases rapidly and diverges at many points as $i$ tends to infinity. However, the product of $\max[|u(i, t)|]$ and $\Delta t(i)$

$$\max[|u(i, t)|] \Delta t(i) \propto \left[ \frac{\max(r_1, r_2)}{\sqrt{2(r_1^2 + r_2^2)}} \right]^i \quad (10)$$
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Figure 2. The mean square displacement versus time interval curves, where \( r_1 = 1.5 \), \( \sigma^2(n) \) represents the mean square displacement in the time interval \( n\Delta t \) and \( n_0 \) is equal to \( 2^{10} \) in the simulation. The hollow and solid dots correspond to the cases of the 25th and 27th iterations, respectively.

Can be derived from equations (5)–(8) and tends to zero for \( i \to \infty \). In the \( i \)th iteration, the displacement in the time interval \( n\Delta t(i) \) beginning from a randomly chosen time \( t_{\text{ran}} \) can be calculated using

\[
x(i, t_{\text{ran}}, n) = \int_{t_{\text{ran}}}^{t_{\text{ran}}+n\Delta t(i)} u(i, t) \, dt,
\]

where \( n \) is a positive integer. In order to get a reasonable statistical property, \( n \) should be large enough, i.e., \( n \gg 1 \), and the period should be much longer than the calculation time interval, i.e., \( T \gg n\Delta t(i) \). From equation (11), one can calculate the displacement \( x(i, t_{\text{ran}}, n) \) for each randomly chosen \( t_{\text{ran}} \) and then find the mean square displacement as

\[
\sigma^2(i, n) = \left\langle x^2(i, t_{\text{ran}}, n) \right\rangle
\]

where each \( \sigma^2(i, n) \) is the average of \( x^2(i, t_{\text{ran}}, n) \) for different beginning times \( t_{\text{ran}} \).

Using equations (3)–(12), we can plot the curves of \( \ln[\sigma^2(i, n)/\sigma^2(i, n_0)] \) varying with \( \ln(n/n_0) \), as shown in figure 2 in which \( r_1 = 1.5 \), \( n_0 = 2^{10} \), and \( i = 25 \) and 27, respectively. It can be seen from figure 2 that the curves for \( i = 25 \) and 27 are overlapped. This means that when the number of iterations is large enough, \( \sigma^2(i, n)/\sigma^2(i, n_0) \) is independent of the iteration number. It is very interesting to note that if \( n\Delta t \) is chosen to be equal to \( \Delta\tau \) in equation (1), the slope of the curves in figure 2 is just equal to \( \alpha \) in equation (1). This illustrates clearly that the numerical simulations based on equations (3)–(12) are in excellent agreement with equation (1), and the different choices of \( r_1 \) and \( r_2 \) correspond to different values of the diffusion coefficient \( \alpha \).

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Figure 3. The $r_1$ versus $r_2$ curve for the diffusion coefficient $\alpha = 1$, where solid and dashed lines correspond to the cases of the 25th and 27th iterations, respectively. The curve divides the $r_1$–$r_2$ plane into two regions, which correspond to the cases of subdiffusion ($\alpha < 1$) and superdiffusion ($\alpha > 1$), respectively.

Using equations (3)–(12), we can also plot the curve of $r_1$ versus $r_2$ as shown in figure 3 for $\alpha = 1$. It is obvious that the regions above and below the curve in the $r_1$–$r_2$ plane correspond to the cases of subdiffusion ($\alpha < 1$) and superdiffusion ($\alpha > 1$), respectively. The curves for $i = 25$ and 27 are overlapped with each other, meaning once again that when the number of iterations is large enough, the curves are stable. This indicates that the self-similar motion can yield anomalous diffusion, where the superdiffusion and subdiffusion can be characterized by different self-similar rates.

Moreover, we can do a statistical analysis on the displacements of self-similar motion. Equations (3)–(12) make it possible to calculate the distributions of the displacements for different self-similar rates. The results are shown in figure 4, where $r_1 = 1.3$ and $r_2 = 0.8r_1$, $1.0r_1$ and $2.0r_1$, respectively. On the other hand, we can easily generate the $q$-Gaussian distribution curves for different values of $q$ by using equation (2), as shown by the solid lines in figure 4. We can say that the distribution of displacements can be nicely fitted with the $q$-Gaussian distributions. The different self-similar rates correspond to the different values of $q$. When $q > 1$ and $x^2 \gg 1/[(q - 1)\beta]$, equation (2) may be simplified into the power-law shape, i.e.,

$$p(x) \propto x^\gamma,$$

(13)

where $\gamma = 2/(1 - q)$. This means that self-similar motion may be used to analyze the asymptotic of Lévy stable forms [1, 5].

The above results imply that not only the anomalous diffusion but also nonextensive statistical distributions can be well simulated by the self-similar motion. It is worth mentioning that, compared with the previous works [33, 34, 37], [39]–[41] where only a single value or several discrete values of $q$ were obtained, the present work sheds further light on the matter by obtaining continuous values of $q$ in the ranges of $1.1 < r_1 < 1.5$ and $0.8r_1 < r_2 < 2.0r_1$. Figure 5 shows the corresponding relation between $q$ and $r_2/r_1$ for some given values of $r_1$. Further investigation reveals that in other ranges of $r_1$ and $r_2$, we can obtain other distribution types. For example, when $r_1 = r_2 = 1.8$ or $r_1 = r_2 = 2.0$, we
The distributions of displacements of self-similar motion for different self-similar rates, where $r_1 = 1.3$, $r_2 = 0.8r_1$, $1.0r_1$ and $2.0r_1$, respectively. The standard Gaussian curve and $q$-Gaussian curves with $q = 0.78$, 1.36 and 1.96 are represented by dashed and solid lines, respectively. The number of iterations $i = 25$.

The $q$ versus $r_2/r_1$ curves for some given values of $r_1$. Triangular, round and square dots correspond to the cases of $r_1 = 1.1, 1.3$ and 1.5, respectively. The number of iterations $i = 25$.

obtain the bimodal distributions of displacements as shown in figure 6. Similar bimodal distributions were observed in the models of fractional diffusion equations [42]–[44] and the distributions of epitaxial island growth [45] and neon nanobubbles in aluminum [46]. This result implies that the bimodal distributions in complex systems may be generated by the present self-similar motion as well.

To sum up, we have introduced a new model in order to generate a self-similar periodic motion which is iterated over the whole time period. It is found that this self-similar motion can be used to simulate anomalous diffusion with diffusion coefficients depending on self-similar rates. It is also found that this self-similar motion can be directly used to analyze the $q$-Gaussian and bimodal distributions in nonextensive statistical mechanics and the different self-similar rates can reveal different values of $q$. Although further precision is needed to elucidate the connection between anomalous diffusion and
nonextensive statistical distributions, the present results provide helpful elements for the
further understanding of the occurrence of anomalous diffusion, and $q$-Gaussian and
bimodal distributions in many natural, artificial, and social complex systems, and for
the correct interpretation of experimental results for certain complex dynamical systems
including, in particular, the ubiquitous dissipative systems. It may be expected that
further research in this direction will open new perspectives and shed new light on the
relationship between anomalous diffusion and nonextensive statistical distributions.

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