ON THE QUANTUM K-RING OF THE FLAG MANIFOLD

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ABSTRACT. We establish a finiteness property of the quantum K-ring of the complete flag manifold.

The main aim of this note is to prove a fundamental fact about the quantum K-ring of the complete flag manifold.

Theorem. The structure constants for (small) quantum multiplication of Schubert classes in \( QK_T(Fl_{r+1}) \) are polynomials in the equivariant and Novikov variables.

This is proved as Theorem 7 below. A priori, quantum structure constants are power series in the Novikov variables; our theorem says that in fact, only finitely many degrees appear. This property is sometimes referred to as finiteness of the quantum product.

Finiteness is known for Grassmannians, and more generally for cominuscule homogeneous spaces [1]. On the other hand, there are conjectural ring presentations for \( QK_T(Fl_{r+1}) \) which presume finiteness and which also include a precise connection with the K-homology of the affine Grassmannian [7, 10, 11, 13].

The proof of finiteness for cominuscule spaces relies on an understanding of the geometry of the moduli space of stable maps; in particular, certain subvarieties of the moduli space are shown to be rational [2]. Our proof, by contrast, is based on the reconstruction methods of Iritani-Milanov-Tonita (who also carried out computations for \( Fl_3 \) [8]; the argument consists of an analysis of the K-theoretic J-function.

Flag varieties. Let \( r \geq 1 \) be an integer. Let \( Fl_{r+1} \) be the variety of complete flags in \( \mathbb{C}^{r+1} \). Let

\[ P_1, \ldots, P_r \]

be the pull-backs via the Plücker embedding

\[ Fl_{r+1} \to \prod_{i=1}^{r} \mathbb{P}^{(r+1)_i-1} \]

of the line bundles \( \mathcal{O}(-1) \) on the projective space factors. Equivalently, if \( S_1 \subset \cdots \subset S_r \subset \mathbb{C}_{Fl}^{r+1} \) is the tautological flag of bundles on \( Fl_{r+1} \), then \( P_i = \bigwedge^i S_i \).

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The Novikov variables \( Q_1, \ldots, Q_r \) keep track of curve classes in \( H_2(Fl_{r+1}, \mathbb{Z}) \): to such a class \( d \), we assign the monomial \( \prod_{i=1}^r Q_i^{-f_d e_1(p_i)} \).

Let \( \Lambda_1, \ldots, \Lambda_{r+1} \) be characters of the torus \( T := (\mathbb{C}^*)^{r+1} \) for the standard action on \( \mathbb{C}^{r+1} \), inducing an action on \( Fl_{r+1} \). The \( T \)-equivariant \( K \)-ring of \( Fl_{r+1} \) has the following well-known presentation:

\[
K_T(Fl_{r+1}) \simeq \frac{\mathbb{C}[\Lambda_1^\pm, \ldots, \Lambda_{r+1}^\pm; P_1^\pm, \ldots, P_r^\pm]}{\langle H_k = e_k(\Lambda_1, \ldots, \Lambda_{r+1}) \rangle_{1 \leq k \leq r+1}}.
\]

Here

\[
H_k := \sum_{I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, r+1\}} \prod_{i \in I} P_i P_{i-1}^{-1}
\]

(setting \( P_0 \) and \( P_{r+1} \) equal to 1) and \( e_k \) is the \( k \)-th elementary symmetric polynomial in \( r + 1 \) variables. (See, e.g., [9] Chapter IV, Section 3.)

**Toda systems.** For \( k \geq 1 \), the \( q \)-difference Toda Hamiltonian is defined by

\[
H_k^q\text{-Toda}(p_i, \bar{z}_i) := \sum_{I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, r+1\}} \prod_{i=1}^k \left( 1 - \frac{\bar{z}_{i-1}}{\bar{z}_i} \right)^{1-\delta_{i-1, i-1, 1}} \prod_{i \in I} p_i.
\]

By convention, \( i_0 = 0 \). Note that \( H_k^q\text{-Toda} \) depends on \( \bar{z}_i \) through \( 1/\bar{z}_i \).

As an example,

\[
H_1^q\text{-Toda} = p_1 + \sum_{i=2}^{r+1} p_i \left( 1 - \frac{\bar{z}_{i-1}}{\bar{z}_i} \right).
\]

The above presentation is taken from [10] Section 5.2. To obtain the \( q \)-difference Toda operators, as in [5] Section 0, we make the substitutions:

\[
p_i \mapsto q^{\partial_i}, \quad \bar{z}_i \mapsto e^{t_i},
\]

and identify \( e^{t_i-t_{i+1}} \) with the Novikov variable \( Q_i \) for \( i = 1, \ldots, r \). Thus

\[
H_k^q\text{-Toda}(q^{\partial_i}, e^{t_i}) = \sum_{I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, r+1\}} \prod_{i=1}^k \left( 1 - \frac{e^{t_{i-1}}}{e^{t_i}} \right)^{1-\delta_{i-1, i-1, 1}} \prod_{i \in I} q^{\partial_i}
\]

\[
= \sum_{I = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, r+1\}} \prod_{i=1}^k \left( 1 - Q_{i-1} \right)^{1-\delta_{i-1, i-1, 1}} \prod_{i \in I} q^{\partial_i}.
\]

Note that

\[
\partial_i = Q_i \partial Q_i - Q_{i-1} \partial Q_{i-1} \quad \text{for} \quad 2 \leq i \leq r,
\]

\[
\partial_i = Q_i \partial Q_1, \quad \text{and}
\]

\[
\partial_{r+1} = -Q_r \partial Q_r.
\]

\( ^1 \)We use the variables \( t_1, \ldots, t_{r+1} \) while \( t_0, \ldots, t_r \) are used in [5].
Thus the Toda Hamiltonian is written in terms of $q$-shift operators $q^{Q_i} \partial_{Q_i}$, which act on power series by

$$q^{Q_i} \partial_{Q_i} f(Q_1, \ldots, Q_i, \ldots, Q_r) = f(Q_1, \ldots, qQ_i, \ldots, Q_r).$$

Similarly, the negative $q$-shift operator $q^{-Q_i} \partial_{Q_i}$ acts by replacing the variable $Q_i$ with $q^{-1} Q_i$.

It is proven in [5] that the $K$-theoretic $J$-function of $Fl_{r+1}$ is an eigenfunction of the $q$-difference Toda system. More precisely, write $J(Q, q)$ for the $K$-theoretic $J$-function of $Fl_{r+1}$, as defined in [5 Section 2.2]. Then we have

**Theorem 1** ([5], Corollary 2). For $1 \leq k \leq r + 1$,

$$H_k^{q^{-\text{Toda}}}(q^{\partial_{Q_i}}, e^{t_i}) P^{\log Q/\log q} J(Q, q) = e_k(\Lambda_1, \ldots, \Lambda_{r+1}) P^{\log Q/\log q} J(Q, q),$$

where $P^{\log Q/\log q} := \prod_{i=1}^r P_i^{\log Q_i/\log q}$.

**Relations.** The (small) quantum $K$-ring of $Fl_{r+1}$ is additively defined to be

$$QK_T(Fl_{r+1}) := K_T(Fl_{r+1}) \otimes \mathbb{C}[Q_1, \ldots, Q_r]$$

and is equipped with a quantum product $\ast$, deforming the tensor product on $K_T(Fl_{r+1})$. The structure constants of $\ast$ are defined in a rather involved way using $K$-theoretic Gromov-Witten invariants of $Fl_{r+1}$. See [3] and [12] for details.

**Theorem 1** yields relations in the small $T$-equivariant quantum $K$-ring of $Fl_{r+1}$ as follows. The theorem gives $q$-difference equations satisfied by $P^{\log Q/\log q} J(Q, q)$. The operators $H_k^{q^{-\text{Toda}}}(q^{\partial_{Q_i}}, e^{t_i})$ contain negative $q$-shift operators $q^{-Q_i} \partial_{Q_i}$, but the difference equations in Theorem 1 are equivalent to the difference equations

$$\left(\prod_{i=1}^r q^{Q_i} \partial_{Q_i}\right) H_k^{q^{-\text{Toda}}}(q^{\partial_{Q_i}}, e^{t_i}) P^{\log Q/\log q} J(Q, q)$$

$$= e_k(\Lambda_1, \ldots, \Lambda_{r+1}) \left(\prod_{i=1}^r q^{Q_i} \partial_{Q_i}\right) P^{\log Q/\log q} J(Q, q),$$

which do not contain negative $q$-shift operators.

In [8, §2], certain operators $A_{i,\text{com}}$ are defined, acting as endomorphisms of the $\mathbb{C}[[Q]]$-module $QK_T(Fl_{r+1})$. By [8 Proposition 2.6 and Corollary 2.9], these operators commute with one another and act as quantum multiplication by the class $A_{i,\text{com}}1$, where $1 \in QK_T(Fl_{r+1}) = K_T(Fl_{r+1}) \otimes \mathbb{C}[[Q]]$ is the identity element of the ring. If one replaces each $q^{Q_i} \partial_{Q_i}$ by $A_{i,\text{com}}$, the difference equations become relations

$$\left(\prod_{i=1}^r A_{i,\text{com}}\right) H_k^{q^{-\text{Toda}}}(A_{i,\text{com}} A_{i-1,\text{com}}^{-1}, e^{t_i}) 1$$

$$= e_k(\Lambda_1, \ldots, \Lambda_{r+1}) \left(\prod_{i=1}^r A_{i,\text{com}}\right) 1$$

in the ring $QK_T(Fl_{r+1})$ [8 Proposition 2.12]. (Here we set $A_{0,\text{com}} = A_{r+1,\text{com}} = 1$.)
Applying the operator $\prod_{i=1}^{r} A_{i,\text{com}}^{-1}$ to the above relation, we find

$$
H_{k}^{q-\text{To}}((A_{i,\text{com}}1) \ast (A_{i-1,\text{com}}^{-1}), e_{i}) = e_{k}(A_{1}, \ldots, A_{r+1}).
$$

In view of [8, Proposition 2.10], we have

$$
\sum_{i} Q_{i} \text{mod } Q.
$$

It follows that the relations (5) define the ring $QK_{r}(F_{r+1})$.

The $D_{q}$-module. Set $\tilde{J} := P^{\log Q/\log q} J$. The $D_{q}$-module structure established in [6] and elaborated in [8] implies the following. Let $f(Q, x) \in \mathbb{C}[Q_{1}, \ldots, Q_{r}, x_{1}, \ldots, x_{r}]$, then

$$
f(Q, q^{Q_{i}\partial_{Q_{i}}})(1 - q) \tilde{J} = \sum_{\beta}(1 - q)\tilde{T}(f_{\beta}\Phi_{\beta}),
$$

where on the right-hand side we have a finite sum with $f_{\beta} \in \mathbb{C}[1 - q][[Q]]$ and $\Phi_{\beta} \in K_{r}(F_{r+1})$. Here $\tilde{T} = P^{\log Q/\log q} T$, and $T$ is the fundamental solution considered in [8 Section 2.4]. The right-hand side can be computed from the leading terms in $q \to \infty$ limit of the left-hand side $f(Q, q^{Q_{i}\partial_{Q_{i}}})(1 - q) \tilde{J}$, namely the coefficients of $q^{\geq 0}$. Furthermore, this implies the following equation in $QK_{r}(F_{r+1})$:

$$
f(Q, A_{i,\text{com}}) \ast 1 = \sum_{\beta} f_{\beta} |_{q = 1} \Phi_{\beta}.
$$

See the proof of [8, Lemma 3.3].

Now let us write the $J$-function as a series $J = \sum_{d} Q^{d} J_{d}$, with $d = (d_{1}, \ldots, d_{r}) \in (\mathbb{Z}_{\geq 0})^{r}$ and $Q^{d} = \prod_{i=1}^{r} Q_{i}^{d_{i}}$. For each degree $d$, $J_{d}$ is a rational function in $q$ taking values in $K_{r}(F_{r+1})$. Therefore, as $q \to \infty$, we have

$$
J_{d} \sim C_{d}(P) q^{f(d)},
$$

for some $C_{d}(P) \in K_{r}(F_{r+1})$ and some function $f(d)$. Results of [5, Section 4] include an estimate on the function $f(d)$.

**Lemma 2** ([5 Eq. (7)]). We have $f(d) \leq -k_{d}$, where

$$
k_{d} := d_{1} + \cdots + d_{r} + \sum_{i=1}^{r+1} \frac{(d_{i} - d_{i-1})^{2}}{2}.
$$

Here by convention we set $d_{0} = d_{r+1} = 0$.

For a class $\Phi \in K_{r}(F_{r+1})$, we expand the fundamental solution by writing $T(\Phi) = \sum_{d} Q^{d} T_{d,\Phi}$. From the definition of $T$, the coefficient $T_{d,\Phi}$ encodes two-point Gromov-Witten invariants with one descendant insertion. This is a rational function in $q$, vanishing at $q = +\infty$; more precisely, as $q \to +\infty$, we have the asymptotics $T_{d,\Phi} \sim L_{d,\Phi} q^{v_{d,\Phi}}$.

**Lemma 3.** $v_{d,\Phi} \leq -k_{d}$, where $k_{d}$ is as in Lemma 2.

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2Our notation agrees with that of [5], but differs slightly from [8], where $P_{t}$ and $P_{t}^{-1}$ are interchanged.
Proof. Coefficients of \( T(\Phi) \) are obtained by \( \mathbb{C}^* \)-equivariant localization on the graph space with one marked point, \( \overline{\mathcal{M}}_{0,1}(F l_{r+1} \times \mathbb{P}^1, (d, 1)) \).

Consider the maps

\[
\overline{\mathcal{M}}_{0,1}(F l_{r+1} \times \mathbb{P}^1, (d, 1)) \xrightarrow{\mu \times ev} \Pi_d \times (F l_{r+1} \times \mathbb{P}^1) \leftarrow HQ_d \times \mathbb{P}^1.
\]

See [5] Section 0 for the definition of \( \Pi_d \) and [5] Section 3] for discussions on hyperquot schemes \( HQ_d \). The map \( \lambda \) is defined by \( \lambda(x, y) := (\lambda(x), ev(x, y), y) \). The maps \( \mu \) and \( \lambda \) are defined analogously to those in [5 Sections 2.1 and 3.1].

The varieties \( \overline{\mathcal{M}}_{0,1}(F l_{r+1} \times \mathbb{P}^1, (d, 1)) \) and \( HQ_d \times \mathbb{P}^1 \) are smooth stacks of expected dimension \( \dim (F l_{r+1}+2d_1+\cdots+2d_r+1) \), so their virtual structure sheaves coincide with their structure sheaves. Coefficients of \( T(\Phi) \) are therefore obtained by \( \mathbb{C}^* \)-equivariant localization applied to \( (\mu \times ev)_*(\mathcal{O}) \otimes (1 \otimes \Phi \otimes 1) \).

Also, \( \mu \times ev \) and \( \lambda \) are birational onto their common image, which has rational singularities. Hence coefficients of \( T(\Phi) \) are obtained by \( \mathbb{C}^* \)-equivariant localization applied to \( \lambda_*((\mathcal{O}_{HQ_d \times \mathbb{P}^1}) \otimes (1 \otimes \Phi \otimes 1)) \). The result then follows from the arguments of [5 Section 4.2]. \( \square \)

**The operator** \( A_{i, \text{com}} \). By studying the action of \( q \)-shift operators on the \( J \)-function, we can identify the operator \( A_{i, \text{com}} \in \text{End}(QK_T(F l_{r+1})) \).

**Lemma 4.** The operator \( A_{i, \text{com}} \) is the operator of (small) quantum product by \( P_i \).

**Proof.** By [8] Proposition 2.10], \( A_{i, \text{com}} \) is the operator of (small) quantum product by \( P_i + \sum_d c_d Q^d \), so it suffices to show that \( A_{i, \text{com}} 1 = P_i \). To this end we consider

\[
q^{Q_i \partial Q_i}(1 - q)\tilde{J}.
\]

Its \( q^{\geq 0} \) coefficients can come from two places:

1. \( d = 0 \): in this case the factor \( P_i^{\log Q/\log q} \) of \( \tilde{J} \) contributes \( P_i \).
2. \( d \neq 0 \): if such \( d \) exists, the effect of the difference operator \( q^{Q_i \partial Q_i} \) is \( q^{d_i} Q^d J_d \).

For this term to contribute to the \( q^{\geq 0} \) coefficient, we must have \( d_i + f(d) \geq 0 \).

By Lemma 2 we have

\[
0 \leq d_i + f(d) \leq d_i - k_d
\]

\[
= -\sum_{j \neq i} d_j - \sum_{i=1}^{r+1} \frac{(d_i - d_{i-1})^2}{2}.
\]

The last term is non-positive because \( d_j \geq 0 \) for all \( j \). And it is equal to 0 if and only if \( d_j = 0 \) for \( j \neq i \) and \( d_j - d_{j-1} = 0 \) for all \( j \). This implies \( d_1 = \cdots = d_r = 0 \), which is not the case.

Thus we find

\[
q^{Q_i \partial Q_i}(1 - q)\tilde{J} = (1 - q)\tilde{T} \bar{P}_i,
\]

and hence \( A_{i, \text{com}} 1 = P_i \). \( \square \)
Remark. The same argument shows that for distinct $i_1, \ldots, i_l \in \{1, \ldots, r\}$, we have
\[
\left( \prod_{k=1}^l q Q_{i_k}^k \partial Q_{i_k} \right) (1 - q) \tilde{J} = (1 - q) \tilde{T} \left( \prod_{k=1}^l P_{i_k} \right),
\]
and hence $P_{i_1} \cdots P_{i_l} = \prod_{k=1}^l P_{i_k}$. That is, for these elements, the quantum and classical product are the same.

Finiteness. The main ingredient in our theorem is a finiteness statement for products of the line bundle classes $P_i$.

**Proposition 5.** The (small) quantum product $P_{i_1} \cdots P_{i_l}$ is a finite linear combination of elements of $K_T(\text{Fl}_{r+1})$ whose coefficients are polynomials in $Q_1, \ldots, Q_r$.

**Proof.** Again we consider the $q \geq 0$ coefficients of $\prod_{k=1}^l q Q_{i_k}^k \partial Q_{i_k} (1 - q) \tilde{J}$. The $d = 0$ term of $\tilde{J}$ gives $\prod_{k=1}^l P_{i_k}$. For a $d \neq 0$ term of $\tilde{J}$ to contribute to the $q \geq 0$ coefficient, we must have
\[
\sum_{k=1}^l d_{i_k} + f(d) \geq 0.
\]
Each such term contributes $C'_d(P) Q^d$ to the $q \geq 0$ coefficients, where $C'_d(P)$ is a polynomial in the $P_i$'s. We need to show that there are only finitely many such terms.

If $\sum_{k=1}^l d_{i_k} + f(d) \geq 0$, then
\[
0 \leq \sum_{k=1}^l d_{i_k} + f(d) \leq \sum_{k=1}^l d_{i_k} - k_d
= \left( \sum_{k=1}^l d_{i_k} - \sum_{j=1}^r d_{j} \right) - \sum_{i=1}^{r+1} (d_i - d_{i-1})^2/2.
\]

The quadratic form $\sum_{i=1}^{r+1} (d_i - d_{i-1})^2/2$ in $d_1, \ldots, d_r$ is positive definite. Indeed it is nonnegative because it is a sum of squares. Also, if $\sum_{i=1}^{r+1} (d_i - d_{i-1})^2 = 0$, then $d_i - d_{i-1} = 0$ for all $i = 1, \ldots, r + 1$. Because $d_0 = d_{r+1} = 0$, we have $d_1 = \cdots = d_r = 0$. Therefore level sets of the function of $d_1, \ldots, d_r$
\[
\left( \sum_{k=1}^l d_{i_k} - \sum_{j=1}^r d_{j} \right) - \sum_{i=1}^{r+1} (d_i - d_{i-1})^2/2
\]
are ellipsoids. It follows that the set
\[
\left\{ (d_1, \ldots, d_r) \mid \left( \sum_{k=1}^l d_{i_k} - \sum_{j=1}^r d_{j} \right) - \sum_{i=1}^{r+1} (d_i - d_{i-1})^2/2 \geq 0 \right\}
\]
is a bounded subset of $\mathbb{R}^r$, so it can contain at most finitely many $(d_1, \ldots, d_r) \in (\mathbb{Z}_{\geq 0})^r$. 
The (finitely many) \( q^{\geq 0} \) terms of \( \prod_{k=1}^{l} q^{Q_{ik} \partial Q_{ik}} (1 - q) \tilde{J} \) can be ordered according to the exponents of \( q \). We then use terms
\[
q^n Q^d \tilde{T}(\Phi), \quad n \in \mathbb{Z}_{\geq 0}, d \in (\mathbb{Z}_{\geq 0})^r, \Phi \in K_T(Fl_{r+1})
\]
to inductively remove these \( q^{\geq 0} \) terms.

By Lemma 3, \( q^n Q^d \tilde{T}(\Phi) \) has only finitely many \( q^{\geq 0} \) terms, so the inductive process ends after finitely many steps. This means we can find a finite sum \( \sum_\beta (1 - q) \tilde{T}(f_\beta \Phi_\beta) \), with \( f_\beta \in \mathbb{C}[1 - q][Q] \) and \( \Phi_\beta \in K_T(Fl_{r+1}) \), such that
\[
\prod_{k=1}^{l} q^{Q_{ik} \partial Q_{ik}} (1 - q) \tilde{J} - \sum_\beta (1 - q) \tilde{T}(f_\beta \Phi_\beta)
\]
vanishes at \( q = +\infty \). The Proposition then follows from the fact that the expression of (6) equals zero, which is proved in Lemma 6 below. \( \square \)

**Lemma 6.** With notation as in (6) above, we have
\[
\prod_{k=1}^{l} q^{Q_{ik} \partial Q_{ik}} (1 - q) \tilde{J} = \sum_\beta (1 - q) \tilde{T}(f_\beta \Phi_\beta).
\]

**Proof.** We argue as in the proof of [8, Lemma 3.3]. Write
\[
M := (1 - q)^{-1} (P^{\log Q/\log q})^{-1} \left( \prod_{k=1}^{l} q^{Q_{ik} \partial Q_{ik}} (1 - q) \tilde{J} - \sum_\beta (1 - q) \tilde{T}(f_\beta \Phi_\beta) \right).
\]
Expand \( M \) as a series in \( Q \), we get \( M = \sum_d M_d Q^d \). Then we get \( M_0 = 0 \). \( M_d \) has poles only at \( q = \text{roots of unity} \), \( M_d \) is regular at both \( q = 0 \) and \( q = +\infty \) and vanishes at \( q = +\infty \).

By [8, Remark 2.11], we can write \( M \) as \( M = TU \) with \( T = \sum_d T_d Q^d \), \( U = \sum_d U_d Q^d \). Then \( T_0 = Id \) and \( U_0 = 0 \). \( T_d \) has only poles at \( q = \text{roots of unity} \). \( T_d \) is regular at \( q = 0 \), \( +\infty \), and vanishes at \( q = +\infty \). Also, \( U_d \) is a Laurent polynomials.

We want to show that \( U_d = 0 \) for all \( d > 0 \), by induction on \( d \) with respect to a partial order of \( d \) ample class. For \( d \) we have
\[
M_d = T_d + U_d + \sum_{\substack{d', d'' \leq d \\text{or } d', d'' \neq 0}} T_{d'} U_{d''}.
\]
\( M_d \) is known, so this equation and induction determine \( T_d + U_d \).

Since both \( T_d + U_d \) and \( T_d \) are regular at \( q = 0 \), so is \( U_d \). So the Laurent polynomial \( U_d \) has no \( q^{< 0} \) terms. Since both \( T_d + U_d \) and \( T_d \) are regular at \( q = +\infty \), so is \( U_d \). So the Laurent polynomial \( U_d \) has no \( q^{> 0} \) terms. Since \( T_d + U_d \) and \( T_d \) vanish at \( q = +\infty \), we have \( U_d \big|_{q=+\infty} = 0 \). Hence \( U_d = 0 \). \( \square \)
Finally, we turn to our main theorem. Let us write \( R(T) = \mathbb{C}[\Lambda^\pm_1, \ldots, \Lambda^\pm_{r+1}] \) for the representation ring of the torus, and \( R(T)[Q] = R(T)[Q_1, \ldots, Q_r] \) and \( R(T)[[Q]] = R(T)[[Q_1, \ldots, Q_r]] \). Then \( K_T(FL_{r+1}) \) is a free \( R(T) \)-module and \( QK_T(FL_{r+1}) \) is a free \( R(T)[[Q]] \)-module. Fix an \( R(T) \)-basis \( \{\sigma_w\} \) for \( K_T(FL_{r+1}) \), so \( \{\Phi_w = \sigma_w \otimes 1\} \) is an \( R(T)[[Q]] \)-basis for \( QK_T(FL_{r+1}) \).

**Theorem 7.** The structure constants of \( QK_T(FL_{r+1}) \) with respect to the basis \( \{\Phi_w\} \) are polynomials: they lie in the subring \( R(T)[Q] \subset R(T)[[Q]] \).

**Proof.** It is a basic fact that \( K_T(FL_{r+1}) \) is generated by \( P_1, \ldots, P_{r+1} \) as an \( R(T) \)-algebra; that is, there is a surjective homomorphism

\[
R(T)[P_1, \ldots, P_r] \twoheadrightarrow K_T(FL_{r+1}).
\]

In particular, each basis element \( \sigma_w \) can be written as a polynomial in \( P_i \)'s with coefficients in \( R(T) \). (One way to see this is as follows. The presentation \([1]\) establishes \( K_T(FL_{r+1}) \) as a quotient of \( R(T)[P_1^\pm, \ldots, P_r^\pm] \), so it suffices to write \( P_i^{-1} \) as a polynomial in \( P_i \) with coefficients in \( R(T) \). For each \( i \), one can find monomials \( \omega_{ij} \) in the variables \( \Lambda^\pm \) so that

\[
\prod_{i=1}^{r-1} (1 - \omega_{ij} P_i) = 0 \quad \text{in} \quad K_T(FL_{r+1}),
\]

and re-arranging this equation produces the desired expression. Alternatively, one can use the equivariant Riemann-Roch isomorphism together with the fact that the equivariant cohomology ring of \( FL_{r+1} \) is generated by divisor classes.)

The assignment \( P_1 P_2 \cdots P_k \mapsto P_1 \ast P_2 \ast \cdots \ast P_k \) defines a ring homomorphism

\[
\left(7\right) \quad R(T)[P_1, \ldots, P_r; Q_1, \ldots, Q_r] \to QK_T(FL_{r+1});
\]

let the kernel be \( I \). The resulting embedding of rings

\[
R(T)[P_1, \ldots, P_r; Q_1, \ldots, Q_r]/I \hookrightarrow QK_T(FL_{r+1})
\]

corresponds to the natural embedding of modules

\[
K_T(FL_{r+1}) \otimes \mathbb{C}[Q_1, \ldots, Q_r] \hookrightarrow K_T(FL_{r+1}) \otimes \mathbb{C}[[Q_1, \ldots, Q_r]].
\]

Since each basis element \( \sigma_w \) is a polynomial in \( \Lambda^\pm \) and \( P \), it follows that each basis element \( \Phi_w = \sigma_w \otimes 1 \) can be represented as a polynomial \( G_w = G_w(\Lambda^\pm, P, Q) \) in \( R(T)[P_1, \ldots, P_r; Q_1, \ldots, Q_r] \). The product of basis elements \( \Phi_u \ast \Phi_v \) is represented by \( G_u G_v \), and by Proposition 5 this product is a finite linear combination of classes in \( K_T(FL_{r+1}) \) with coefficients in \( \mathbb{C}[Q_1, \ldots, Q_r] \). \( \Box \)

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