Teissier’s problem on proportionality of nef and big classes over a compact Kähler manifold

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Abstract

We solve Teissier’s proportionality problem for transcendental nef classes over a compact Kähler manifold which says that the equality in the Khovanskii-Teissier inequalities hold for two nef and big classes if and only if the two classes are proportional. This result recovers the previous one of Boucksom-Favre-Jonsson for the case of nef and big line bundles over a (complex) projective algebraic manifold.

1 Introduction

Around the year 1979, inspired by the Aleksandrov-Fenchel inequalities in convex geometry, Khovanskii and Teissier discovered independently deep inequalities in algebraic geometry which now is called Khovanskii-Teissier inequalities. These inequalities present a nice relationship between the theory of mixed volumes and algebraic geometry. Their proofs are based on the usual Hodge-Riemann bilinear relations. A natural problem is how to characterize the equality case in these inequalities for two nef and big line bundles, which was considered by Teissier [Tei82, Tei88].

In their nice paper [BFJ09], Boucksom, Favre and Jonsson solved this problem and the answer is that the equality holds if and only if two line bundles are (numerically) proportional. In their paper, they proved an algebro-geometric version of the Diskant inequality in convex geometry following the same strategy of Diskant which is based on the differentiability of the volume function of convex bodies. To obtain their Diskant inequality, they develop an algebraic construction of the positive intersection products of pseudo-effective classes and use them to prove that the volume function on the Néron-Severi space of a projective variety is $C^1$-differentiable, expressing its differential as a positive intersection product. Note that their results hold on any complete algebraic variety over an algebraically closed field of characteristic zero. Later, Cutkosky [Cut13] extended these remarkable results to a complete variety over an arbitrary field.

On the other hand, Dinh and Nguyén generalized the Hodge-Riemann bilinear relations (and some other results) to compact Kähler manifolds in
the mixed situation. Using these relations, one can easily get the Khovanskii-
Teissier inequalities for transcendental nef classes. So a natural question is
how to characterize the equality case in this situation. In this note, we give
the same answer of this question as in the algebro-geometric case.

In [BFJ09] and [Cut13], a key ingredient, in the proof of the differentia-
bility theorem of the volume of big line bundles over a projective variety, and
thus in the proof of the algebro-geometric version of the Diskant inequality,
is the weak holomorphic Morse inequality
\[
\text{vol}(A - B) \geq A^n - nA^{n-1}B
\]
for any nef line bundles A and B. Hence, if one would like to use their
methods to extend their results to transcendental classes, the main missing
part is the weak transcendently holomorphic Morse inequality. However,
up to now, it is not fully proved yet (see [Xia13, Pop14]). In this note,
without using the transcendental version of Diskant inequality, we can still
solve Teissier’s proportionality problem for transcendental classes. Thus,
our result covers the previous one of Boucksom-Favre-Jonsson. Indeed, the
key idea in the proof of our main result has been hidden in our pre-
vious work [FX14]. For readers’ convenience, we will present it in details.

2 The main theorem

Let us first recall the definition of nefness and bigness for (1,1)-classes on a
compact Kähler manifold. Assume \( X \) is an \( n \)-dimensional compact Kähler
manifold with a Kähler metric \( \omega \). Let \( \alpha \in H_{BC}^{1,1}(X, \mathbb{R}) \) be a \( (1,1) \) Bott-
Chern class. Then \( \alpha \) is called \textit{nef} if for any \( \varepsilon > 0 \), there exists a smooth
representation \( \alpha_\varepsilon \in \alpha \) such that \( \alpha_\varepsilon > -\varepsilon \omega \). This definition is equivalent to
say that \( \alpha \) belongs to the closure of the Kähler cone of \( X \) which is denoted
as \( \overline{\mathcal{K}} \). And \( \alpha \) is called \textit{big} if there exist a positive number \( \delta \) and a positive
current \( T \in \alpha \) such that \( T > \delta \omega \) (such a current \( T \) is called a \textit{Kähler current}).
This is equivalent to say that \( \alpha \) belongs to the interior of pseudo-effective
cone which is denoted as \( \mathcal{E}^\circ \). For more notions, such as the movable cone
\( \overline{\mathcal{M}} \) in the following theorem, one can consult [BDPP13].

**Theorem 2.1.** Assume \( X \) is an \( n \)-dimensional compact Kähler manifold.
Let \( \alpha, \beta \in \overline{\mathcal{K}} \cap \mathcal{E}^\circ \) be two nef and big classes. Denote \( s_k := \alpha^k \cdot \beta^{n-k} \). Then
the following statements are equivalent:

1. \( s_k = s_{k-1} \cdot s_{k+1} \) for \( 1 \leq k \leq n - 1 \);
2. \( s_k = s_{0}^{n-k} \cdot s_{n}^{k} \) for \( 0 \leq k \leq n \);
3. \( s_{n-1} = s_{0} \cdot s_{n}^{n-1} \);
4. \( \text{vol}(\alpha + \beta)^{1/n} = \text{vol}(\alpha)^{1/n} + \text{vol}(\beta)^{1/n} \);
\((5)\) \(\alpha\) and \(\beta\) are proportional;
\(\(6)\) \(\alpha^{n-1}\) and \(\beta^{n-1}\) are proportional.

Moreover, all of the above statements are equivalent to the \((n-1)\)-th map \(\alpha \mapsto \alpha^{n-1}\) embedding the nef big cone \(\overline{K} \cap \mathcal{E}^0\) into the movable cone \(\overline{M}\).

Proof. For a projective algebraic manifold, the usual Khovanskii-Teissier inequalities imply
\[ s_k^2 \geq s_{k-1} \cdot s_{k+1}, \quad \text{for } 1 \leq k \leq n-1 \]  
(2.1)  
if \(\alpha\) and \(\beta\) are two nef divisors. We remark that it also holds if \(\alpha\) and \(\beta\) are two transcendental nef classes on a compact Kähler manifold (see [DN06]). Its proof is a consequence of Ninh and Nguyêñ’s result on mixed Hodge-Riemann bilinear relations for compact Kähler manifolds. For example, one can consult Proposition 6.2.1 in [Cao13]. In fact, let \(\omega_1, \cdots, \omega_{n-2}\) be \((n-2)\) Kähler classes of \(X\). Consider the following quadratic \(Q\) on \(H^{1,1}_{BC}(X; \mathbb{R})\):
\[ Q(\lambda, \mu) := \int_X \lambda \wedge \mu \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2}. \]

According to [DN06], \(Q\) is of signature \((1, h^{1,1})\). For any \(\alpha, \beta \in \overline{\mathcal{K}}\) and \(t \in \mathbb{R}\), consider \(Q(\alpha + t\beta, \alpha + t\beta)\), \(t \in \mathbb{R}\). As a function of \(t\), we claim that \(Q(\alpha + t\beta, \alpha + t\beta)\) has at least a real solution. We only need to consider the case when \(\alpha\) and \(\beta\) are linearly independent and thus, \(\alpha\) and \(\beta\) span a 2-dimensional subspace. In view of the signature of \(Q\), it can not be positive on this 2-dimensional subspace. Now our claim follows from this easily. The existence of real solution is equivalent to
\[ \left( \int_X \alpha \wedge \beta \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \right)^2 \geq \left( \int_X \alpha^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \right) \cdot \left( \int_X \beta^2 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \right). \]

Since \(\omega_1, \cdots, \omega_{n-2}\) are arbitrary, taking appropriate \(\omega_i\)’s and then taking limits, we obtain the inequalities \((2.1)\) for any two transcendental nef classes.

We commence to prove the main result. It is easy to see the equivalences of \((1)-(4)\) (see e.g. [Cut13]). Since \(\alpha, \beta \in \overline{\mathcal{K}} \cap \mathcal{E}^0\), it is clear that \(s_k > 0\) for \(0 \leq k \leq n\). We first prove \((1) \iff (3)\). It is trivial that \((1)\) implies \((3)\). On the other hand, the equalities in \((2.1)\) imply
\[ \frac{s_{n-1}}{s_0} = \frac{s_{n-1}}{s_{n-2}} \cdot \frac{s_{n-2}}{s_{n-3}} \cdots \frac{s_1}{s_0} \geq \frac{s_{n-1}}{s_{n-2}} \cdot \frac{s_{n-2}}{s_{n-3}} \cdots \left( \frac{s_2}{s_1} \right)^2 \geq \cdots \geq \left( \frac{s_n}{s_{n-1}} \right)^{n-1}. \]
(2.2)  
Thus if \((3)\) holds, then all inequalities in \((2.2)\) are equalities, and hence \((1)\) holds. Now let us prove \((1) \iff (2)\). This also follows from the equalities \((2.1)\), since we will have
\[ \left( \frac{s_k}{s_{k-1}} \right)^{n-k} \cdots \left( \frac{s_1}{s_0} \right)^{n-k} \geq \left( \frac{s_n}{s_{n-1}} \right)^k \cdots \left( \frac{s_{k+1}}{s_k} \right)^k. \]  
(2.3)
which clearly implies \( (1) \Leftrightarrow (2) \). Next we prove \( (2) \Leftrightarrow (4) \). Inequality (2.3) can be rewritten as

\[
 s_k^n \geq s_0^{n-k} \cdot s_k^n, \quad \text{for } 0 \leq k \leq n.
\]

These inequalities yield

\[
 \text{vol}(\alpha + \beta) = (\alpha + \beta)^n = \sum \frac{n!}{k!(n-k)!} s_k^n \\
 \geq \sum \frac{n!}{k!(n-k)!} s_0^{n-k/n} \cdot s_k^{k/n} \\
 = (\text{vol}(\alpha)^{1/n} + \text{vol}(\beta)^{1/n})^n.
\]

This implies \( (2) \Leftrightarrow (4) \).

The implication \( (5) \Rightarrow (3) \) is trivial. Now the real difficulty is to prove \( (3) \Rightarrow (5) \), but which can be finished following from the ideas in our previous work [FX14]. Without loss of generality, assume \( \text{vol}(\alpha) = \text{vol}(\beta) \). If (3) holds, we will construct two equal positive \((1,1)\)-currents in \( \alpha \) and \( \beta \) respectively. Hence this implies (5). To prove this, we first construct two positive \((1,1)\)-currents in \( \alpha \) and \( \beta \) respectively which are equal on a Zariski open set. The construction heavily depends on the main theorem in [BEGZ10] which solves Monge-Ampère equations in big cohomology classes. Then, by the support theorem of currents, the difference of these two currents can only be a combination of some prime divisors. By showing that all the coefficients in the combination vanish, we deduce that these two currents are equal. All is all, the key elements in the proof of \( (3) \Rightarrow (5) \) are to solve Monge-Ampère equations in nef and big cohomology classes and to use some basic facts in pluripotential theory. In the following, we will carry out all the details.

We will use the same symbol \( \alpha \) (resp. \( \beta \)) to denote a smooth representation in the cohomology class \( \alpha \) (resp. \( \beta \)). Fix a Kähler metric \( \omega \) and a smooth volume form \( \Phi \) with \( \int_X \Phi = 1 \). Since \( \alpha \) and \( \beta \) are nef and big, Theorem C of [BEGZ10] implies that we can solve the following two degenerate complex Monge-Ampère equations:

\[
\langle (\alpha + i\partial \bar{\partial} \varphi)^n \rangle = c_{\alpha,0} \Phi, \tag{2.4}
\]

\[
\langle (\beta + i\partial \bar{\partial} \psi)^n \rangle = c_{\beta,0} \Phi, \tag{2.5}
\]

where \( \langle \cdot \rangle \) denotes the non-pluripolar products of positive currents, and \( c_{\alpha,0} = \text{vol}(\alpha) = \text{vol}(\beta) = c_{\beta,0} \). Moreover, \( \varphi \) (resp. \( \psi \)) has minimal singularities and is smooth on the ample locus \( \text{Amp}(\alpha) \) (resp. \( \text{Amp}(\beta) \)), which is a Zariski open set of \( X \) depending only on the cohomology class of \( \alpha \) (resp. \( \beta \)). Let us first briefly recall how the solutions \( \varphi \) and \( \psi \) are obtained. Indeed, based on Yau’s seminal work [Yau78] on the Calabi conjecture, the above two degenerate complex Monge-Ampère equations can be solved by
approximation. By Yau’s theorem, for $0 < t < 1$, we can solve the following two families of Monge-Ampère equations:

\begin{align}
(\alpha + t\omega + i\partial \bar{\partial} \varphi_t)^n &= c_{\alpha,t} \Phi, \\
(\beta + t\omega + i\partial \bar{\partial} \psi_t)^n &= c_{\beta,t} \Phi,
\end{align}

where $c_{\alpha,t} = \int_X (\alpha + t\omega)^n$, $c_{\beta,t} = \int_X (\beta + t\omega)^n$ and $\sup_X \varphi_t = \sup_X \psi_t = 0$. Denote $\alpha_t = \alpha + t\omega + i\partial \bar{\partial} \varphi_t$ and $\beta_t = \beta + t\omega + i\partial \bar{\partial} \psi_t$.

We consider the limits of $\alpha_t$ and $\beta_t$ as $t$ tends to zero. By basic properties of plurisubharmonic functions, since $\sup_X \varphi_t = \sup_X \psi_t = 0$, the family of solutions $\{\varphi_t\}$ (resp. $\{\psi_t\}$) is compact in $L^1(X)$-topology. Thus there exists a convergent subsequence which we still denote it by the same symbol $\{\varphi_t\}$ (resp. $\{\psi_t\}$) and there exists an $\alpha$-psh function $\varphi$ (resp. a $\beta$-psh function $\psi$) such that, when $t$ tends to zero, we have the following limits in the sense of currents on $X$:

\begin{align}
\alpha_t &\to \alpha + i\partial \bar{\partial} \varphi, \\
\beta_t &\to \beta + i\partial \bar{\partial} \psi.
\end{align}

Furthermore, by the theory developed in [BEGZ10] and basic estimates in [Yau78], $\varphi_t$ (resp. $\psi_t$) is compact in $C^\infty_{\text{loc}}(\operatorname{Amp}(\alpha))$ (resp. $C^\infty_{\text{loc}}(\operatorname{Amp}(\beta))$). Therefore there exist convergent subsequences such that the convergences (2.8) and (2.9) is in the topology of $C^\infty_{\text{loc}}(\operatorname{Amp}(\alpha))$ and $C^\infty_{\text{loc}}(\operatorname{Amp}(\beta))$. Hence $\varphi$ (resp. $\psi$) is smooth on $\operatorname{Amp}(\alpha)$ (resp. $\operatorname{Amp}(\beta)$) respectively. Moreover, since $\Phi$ is a smooth volume form, $\alpha + i\partial \bar{\partial} \varphi$ (resp. $\beta + i\partial \bar{\partial} \psi$) must be a Kähler metric on $\operatorname{Amp}(\alpha)$ (resp. $\operatorname{Amp}(\beta)$).

Denote the Zariski open set $\operatorname{Amp}(\alpha) \cap \operatorname{Amp}(\beta)$ by $\operatorname{Amp}(\alpha, \beta)$, and denote $\alpha + i\partial \bar{\partial} \varphi$ (resp. $\beta + i\partial \bar{\partial} \psi$) by $\alpha_0$ (resp. $\beta_0$). We first claim that $\alpha_0 = \beta_0$ on $\operatorname{Amp}(\alpha, \beta)$. Let $c_t = c_{\alpha,t}/c_{\beta,t}$. By our assumption $\operatorname{vol}(\alpha) = \operatorname{vol}(\beta)$, it is clear that

\[ \lim_{t \to 0} c_t = 1. \]

Assume $\alpha^{n-1} = \beta^{n-1} + \Theta(\alpha, \beta)$ for some smooth $(n-1, n-1)$-form $\Theta(\alpha, \beta)$. Then

\[ \alpha_t^{n-1} = \beta_t^{n-1} + \Theta_t \]

for some smooth $(n-1, n-1)$-form $\Theta_t$. Pointwisely, $\alpha_t$, $\beta_t$, $\alpha_t^{n-1}$, $\beta_t^{n-1}$ and $\Theta_t$ can be viewed as matrices. In this sense, we have

\[ \frac{\det \alpha_t^{n-1}}{\det \beta_t^{n-1}} = \left( \frac{\det \alpha_t}{\det \beta_t} \right)^{n-1}. \]
Hence, we have

\[
c_t^{n-1/n} = \left( \frac{\det \alpha_t^{n-1}}{\det \beta_t^{n-1}} \right)^{1/n} = \left( \frac{\det (\beta_t^{n-1} + \Theta_t)}{\det \beta_t^{n-1}} \right)^{n-1} \leq 1 + \frac{1}{n} \sum (\beta_t^{n-1})_{ij} (\Theta_t)_{ij},
\]

where the matrix \( (\beta_t^{n-1})_{ij} \) is the inverse of \( \beta_t^{n-1} \). Equivalently, multiplying both sides of (2.13) by \( \beta_t^n \), we have

\[
c_t^{n-1/n} \beta_t^n \leq \beta_t^n + \beta_t \wedge \Theta_t.
\]

(2.14)

Note that \( \beta_t \wedge \Theta_t = \alpha_t^{n-1} \wedge \beta_t - \beta_t^n \). Consider \( \{ \alpha_t^{n-1} \wedge \beta_t \} \) (resp. \( \{ \beta_t^n \} \)) as a family of positive measures, then it is of bounded mass. Thus there exist convergent subsequences, which we still denote as \( \alpha_t^{n-1} \wedge \beta_t \) and \( \beta_t^n \), and positive measures \( \mu_1 \) and \( \mu_2 \) such that

\[
\alpha_t^{n-1} \wedge \beta_t \to \mu_1,
\]

(2.15)

\[
\beta_t^n \to \mu_2
\]

(2.16)

in the sense of measures. If denote \( \mu = \mu_1 - \mu_2 \), then \( \beta_t \wedge \Theta_t \to \mu \). We claim that \( \mu \) is a zero measure. It is not hard to see from (2.10) and (2.14) that \( \mu \) is a positive measure on \( X \): Let \( f \) be any positive smooth function over \( X \), we have

\[
\int_X f \mu = \lim_{t \to 0} \int_X f (\beta_t \wedge \Theta_t) \geq \liminf_{t \to 0} \int_X f (c_t^{n-1} \beta_t^n - \beta_t^n) = 0.
\]

Meanwhile, the assumption (3) implies

\[
\int_X \mu = \lim_{t \to 0} \int_X (\beta_t \wedge \alpha_t^{n-1} - \beta_t^n) = \int_X (\beta \wedge \alpha^{n-1} - \beta^n) = 0.
\]

Hence \( \mu \) must vanish identically. In particular, since \( \text{Amp}(\alpha, \beta) \) is a Zariski open set (thus a Borel measurable set), we have

\[
\beta_t \wedge \Theta_t \to 0
\]

(2.17)

in the sense of measures on \( \text{Amp}(\alpha, \beta) \). Using the convergence (2.8) and (2.9) in the topology of \( C_{\text{loc}}^\infty(\text{Amp}(\alpha)) \) and \( C_{\text{loc}}^\infty(\text{Amp}(\beta)) \), it is clear that there exists some smooth form \( \Theta_0 \) which is only defined on \( \text{Amp}(\alpha, \beta) \) such that \( \Theta_t \to \Theta_0 \) in the topology of \( C_{\text{loc}}^\infty(\text{Amp}(\alpha, \beta)) \). This implies in the same topology

\[
\beta_t \wedge \Theta_t \to \beta_0 \wedge \Theta_0.
\]

(2.18)

Combining (2.17) and (2.18), and using uniqueness of the limit, we obtain

\[
\beta_0 \wedge \Theta_0 = 0
\]

(2.19)
on $\text{Amp}(\alpha,\beta)$. The above equality (2.19) implies that if we take the limits on $\text{Amp}(\alpha,\beta)$ of both sides of (2.13), we have

$$
1 = \left(\frac{\det \alpha_0^{n-1}}{\det \beta_0^{n-1}}\right)^{\frac{1}{n}} = \left(\frac{\det (\beta_0^{n-1} + \Theta_0)}{\det \beta_0^{n-1}}\right)^{\frac{1}{n}} \leq 1 + \frac{1}{n} \sum_{i,j} (\beta_0^{n-1})_{ij} \nu \Theta_0^{ij}
$$

(2.21)

$$
= 1.
$$

(2.22)

This forces $\Theta_0 = 0$, and hence $\alpha_0^{n-1} = \beta_0^{n-1}$ on $\text{Amp}(\alpha,\beta)$. Since $\alpha_0$ and $\beta_0$ are Kähler metrics, we have $\alpha_0 = \beta_0$ on $\text{Amp}(\alpha,\beta)$. We claim $\alpha_0 = \beta_0$ on all $X$. Before going on, we need the following two lemmas.

**Lemma 2.1.** (see [Dem], pp. 142-143) Let $T$ be a $d$-closed $(p,p)$-current. Suppose $\text{supp} \ T$ is contained in an analytic subset $A$. If $\dim A < n - p$, then $T = 0$; if $T$ is of order zero and $A$ is of pure dimension $n - p$ with $(n - p)$-dimensional irreducible components $A_1, \ldots, A_k$, then $T = \sum c_j [A_j]$ with $c_j \in \mathbb{C}$.

**Lemma 2.2.** (see [Bon04], Proposition 3.2 and Proposition 3.6) Let $\alpha$ be a nef and big class, and let $T_{\text{min}}$ be a positive current in $\alpha$ with minimal singularities. Then the Lelong number $\nu(T_{\text{min}}, x) = 0$ for any point $x \in X$.

It is clear that $S := X \setminus \text{Amp}(\alpha,\beta)$ is a proper analytic subset of $X$. Let $T = \alpha_0 - \beta_0$, then $T$ is a real $d$-closed $(1,1)$-current and $\text{supp} \ T \subset S$. If $\text{codim} S \geq 2$, then $T = 0$ according to Lemma 2.1. This implies $\alpha_0 = \beta_0$ on $X$; if $\text{codim} S = 1$ and $S$ has only irreducible components $D_1, \ldots, D_k$ of pure dimension one, then Lemma 2.1 implies $\alpha_0 - \beta_0 = \sum c_j [D_j]$; if $\text{codim} S = 1$ and $S$ has also components of codimension more than one, we just repeat the proof of Lemma 2.1 in [Dem] (for more details, see [FX14]), and still get $\alpha_0 - \beta_0 = \sum c_j [D_j]$. Since $\alpha_0$ and $\beta_0$ are real, all $c_j$ can be chosen to be real numbers. If there exists at least one $c_j > 0$, we write this equality as

$$
\alpha_0 - \sum c_{j'} [D_{j'}] = \beta_0 + \sum c_{j''} [D_{j''}]
$$

(2.23)

with $c_{j'} \leq 0$ and $c_{j''} > 0$. Fix a $j''$ which we denote as $j''_0$. We take a generic point $x \in D_{j''_0}$, for example, we can take such a point $x$ with $\nu([D_{j''_0}], x) = 1$ and $x \notin \bigcup_{j \neq j''_0} D_j$. Then taking the Lelong number at the point $x$ on both sides of (2.23), we have

$$
\nu(\alpha_0, x) - \sum c_{j'} \nu([D_{j'}], x) = \nu(\beta_0, x) + \sum c_{j''} \nu([D_{j''}], x).
$$

Since $\alpha_0$ and $\beta_0$ are positive currents with minimal singularities in nef and big classes, Lemma 2.2 tells us that $\nu(\alpha_0, x) = 0$ and $\nu(\beta_0, x) = 0$. The property of $x$ also implies $\nu([D_{j'}], x) = 0$ and $\nu([D_{j''}], x) = 0$ for all $j'$ and
all $j'' \neq j_0''$. All these force $c_{j''_0} = 0$, which contradicts to our assumption $c_{j''_0} > 0$. Thus we have

$$\alpha_0 - \sum c_{j'}[D_{j'}] = \beta_0.$$ 

By the same reason, we can also prove $c_{j'} = 0$. Hence we finish the proof of $\alpha_0 = \beta_0$ over $X$ and of the implication (3) $\Rightarrow$ (5).

The implication (5) $\Rightarrow$ (6) is trivial, and it is clear (5) $\Rightarrow$ (3). For the implication of (6) $\Rightarrow$ (3), suppose $\alpha^{n-1} = c\beta^{n-1}$ for some $c > 0$, then we have $\alpha^n = c\beta^{n-1}\cdot \alpha \geq (\beta^n)^{n-1/n}(\alpha^n)^{1/n}$, and $\alpha^{n-1} \cdot \beta = c\beta^n \geq (\alpha^n)^{n-1/n}(\beta^n)^{1/n}$. This yields $(\alpha^n)^{n-1/n} = c(\beta^n)^{n-1/n}$, and as a consequence, we get $\alpha^{n-1} \cdot \beta = c\beta^n = (\alpha^n)^{n-1/n}(\beta^n)^{1/n}$ which is just (3). Summarizing all the above arguments, we have finished the proof of the equivalences of (1) $- (6)$.

Moreover, the statement that the $(n - 1)$-th exterior power map $\alpha \mapsto \alpha^{n-1}$ embeds the nef big cone $K\cap E^\circ$ into the movable cone $M$ is equivalent to the implication (6) $\Rightarrow$ (5), and hence to the equivalence (6) $\Leftrightarrow$ (5). Therefore we finish the proof of our theorem.

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