EQUIVARIANT EMBEDDINGS OF MANIFOLDS INTO EUCLIDEAN SPACES

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Abstract. Suppose a finite group $G$ acts on a manifold $M$. By a theorem of Mostow, also Palais, there is a $G$-equivariant embedding of $M$ into the $m$-dimensional Euclidean space $\mathbb{R}^m$ for some $m$. We are interested in some explicit bounds of such $m$.

First we provide an upper bound: there exists a $G$-equivariant embedding of $M$ into $\mathbb{R}^{d(G) + 1}$, where $\vert G \vert$ is the order of $G$ and $M$ embeds into $\mathbb{R}^d$. Next we provide a lower bound for finite cyclic group action $G$: If there are $l$ points having pairwise co-prime lengths of $G$-orbits greater than 1 and there is a $G$-equivariant embedding of $M$ into $\mathbb{R}^m$, then $m \geq 2l$.

Some applications to surfaces are given.

1. Introduction

In this note, we assume that $M$ is a compact and connected polyhedron and $G$ is a finite group which acts faithfully on $M$. We often call such $M$ and $G$ a pair $(M, G)$. Recall $\mathbb{R}^m$ is the $m$-dimensional Euclidean space, and $SO(m)$ is the $m$-dimensional special orthogonal group, which acts on $\mathbb{R}^m$ canonically. We use $\vert G \vert$ to denote the order of a finite group $G$ and $F_g$ to denote the closed orientable surface of genus $g$.

Definitions: Let $(M, G)$ be a pair with the action of $G$ on $M$ given by a representation $\rho : G \to \text{Homeo}(M)$, where $\text{Homeo}(M)$ is the group of homeomorphisms on $M$. Call an embedding $e : M \to \mathbb{R}^m$ $G$-equivariant, if there is an orthogonal representation $\tilde{\rho} : G \to SO(m)$ such that

$$e \circ \rho(g) = \tilde{\rho}(g) \circ e$$

for any $g \in G$.

When $G$ is a finite cyclic group generated by a periodic map $f$ on $M$, we often call $G$-action as $f$-action and $G$-equivariant as $f$-equivariant.

The existence of $G$-equivariant embedding for pair $(M, G)$ follows from the work of Mostow, also Palais, see [Mos Theorem 6.1] and [Pal Theorem].
**Question 1.1.** For a given pair \((M, G)\), to find some concrete small \(n\), or stronger, the minimal \(n\), so that there is a \(G\)-equivariant embedding \(M \to \mathbb{R}^n\).

There are some systematic studies on \(G\)-equivariant embeddings for graphs and surfaces into \(\mathbb{R}^3\) and \(S^3\), see [Cos], [FNPT], [WWZZ1], [WWZZ2] and the references therein. Those studies rely on the geometry and topology of 3-manifolds developed in the last several decades, as well as on our 3-dimensional intuition. Once we know that there is a \(G\)-equivariant embedding \(M \to \mathbb{R}^3\), then the integer 3 is often the minimal \(n\) for the pair \((M, G)\) in Question 1.1, since usually graphs and surfaces themselves can not be embedded into \(\mathbb{R}^2\).

If there is no \(G\)-equivariant embedding \(M \to \mathbb{R}^3\) for a pair \((M, G)\), then Question 1.1 becomes more complicated. Some results for \(G\)-equivariant embeddings \(M \to \mathbb{R}^n\) to high dimensional Euclidean space appear recently, see [Zi], [Wa].

In this note, we try to give some general explicit bounds for Question 1.1.

First we give an upper bound in terms of \(|G|\), and the dimension of Euclidean space that \(M\) embeds. We state the smooth version. The topological version follows by just ignoring the smoothness in both the statement and its proof.

**Proposition 1.2.** Suppose \(M\) is a closed smooth manifold and there is a smooth finite group action \(G\) on \(M\). If there is a smooth embedding of \(M\) into \(\mathbb{R}^d\), then there exists a \(G\)-equivariant smooth embedding of \(M\) into \(\mathbb{R}^{d|G|+1}\).

Then we give a lower bound for finite cyclic group actions in terms of periods of periodic points for periodic maps. Suppose \(G = \mathbb{Z}_n\) is generated by a periodic map \(f\) of order \(n\) on \(M\). Under \(f\)-action each point of \(M\) has its \(f\)-orbit with length dividing \(n\).

**Proposition 1.3.** Suppose \(f\) is a periodic map on \(M\) and there are \(l\) points having pairwise co-prime lengths of \(f\)-orbits greater than 1. If there is an \(f\)-equivariant embedding \(e: M \to \mathbb{R}^m\), then \(m \geq 2l\).

Finite group actions on surfaces are keeping to be an active topic since the work of Hurwitz [Hu]. Some applications to finite group actions on surfaces are given below.

By Hurwitz theorem the order of any orientation-preserving finite group action on \(F_g\), \(g > 1\), is bounded by \(84(g - 1)\) [Hu]. Since \(F_g\) embeds into \(\mathbb{R}^3\) and \(84(g - 1) \times 3 = 252(g - 1)\), by Proposition 1.2 we have

**Corollary 1.4.** For any orientation-preserving finite group action \(G\) on \(F_g\), \(g > 1\), \(F_g\) can be \(G\)-equivariantly embedded into \(\mathbb{R}^{252(g - 1) + 1}\).

The next result is a corollary of Proposition 1.3.

**Corollary 1.5.** For any given integer \(m > 0\), there is a periodic map \(f\) on a closed orientable surface \(F\) such that there is no \(f\)-equivariant embedding \(e: F \to \mathbb{R}^m\).
Proposition 1.2, Proposition 1.3 and Corollary 1.5 will be proved in Sections 2, 3, 4 respectively.

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2. An upper bound for finite group actions

**Proof of Proposition 1.2** Suppose the smooth action of $G$ on $M$ is given by the representation

$$\rho : G \to \text{Diff}(M)$$

where $\text{Diff}(M)$ is the group of diffeomorphisms on $M$. and the smooth embedding of $M$ into $\mathbb{R}^d$ is given by

$$e : M \to \mathbb{R}^d.$$

We may assume $G = \{1, 2, ..., n\}$. With these notations we define a map

$$\tilde{e} : M \to \mathbb{R}^d | G$$

as follows: for any $x \in M$,

$$\tilde{e}(x) = (e(\rho(1)(x)), e(\rho(2)(x)), ..., e(\rho(n)(x))).$$

We will prove $\tilde{e}$ is a smooth embedding. Since $\rho(1)$ is a diffeomorphism of $M$ onto itself, the first component $\tilde{e}_1$ of the mapping $\tilde{e}$, where $\tilde{e}_1 = e \circ \rho(1)$, is a smooth embedding. By the same reason, all components $\tilde{e}_j$ of $\tilde{e}$, $\tilde{e}_j = e \circ \rho(j)$ are smooth. Hence $\tilde{e}$ is smooth and injective. The rank of the Jacobi matrix of $\tilde{e}$ at each point $x \in M$ is not less than the rank of the Jacobi matrix of $\tilde{e}_1$, the first component of $\tilde{e}$, which equals the dimension of $M$ since $\tilde{e}_1$ is an embedding. Thus $\tilde{e}$ is also an immersion, and is hence a smooth embedding.

We have smoothly embedded $M$ into $\mathbb{R}^d | G$ and we next prove that $\tilde{e}$ is a $G$-equivariant embedding. Recall that $O(m)$ is the $m$-dimensional orthogonal group. We construct a natural group action $\tilde{\rho}$ of $G$ on $\mathbb{R}^d | G$, that is to define an embedding

$$\tilde{\rho} : G \to O(d|G)$$

by

$$\tilde{\rho}(j)(y_1, y_2, ..., y_n) = (y_{1*}, y_{2*}, ..., y_{n*}),$$

where each element $\tilde{\rho}(j)$ acts as an orthogonal transformation. It suffices to show that the image of $M$ under the embedding $\tilde{e}$ is invariant under the group action $\tilde{\rho}$ on $\mathbb{R}^d | G$ and the actions $\tilde{\rho}$ and $\rho$ are commutative by the embedding $\tilde{e}$. If $y = \tilde{e}(x)$ where $x \in M$, then

$$\tilde{\rho}(j)(\tilde{e}(x)) = \tilde{\rho}(j)(e(\rho(1)(x)), e(\rho(2)(x)), ..., e(\rho(n)(x)))$$

$$= (e(\rho(1*j)(x)), e(\rho(2*j)(x)), ..., e(\rho(n*j)(x)))$$

$$= (e(\rho(1)(\rho(j)(x))), e(\rho(2)(\rho(j)(x))), ..., e(\rho(n)(\rho(j)(x))))$$

$$= \tilde{e}(\rho(j)(x)).$$
which implies \( \tilde{\rho}(j)(y) \in \tilde{e}(M) \) and thus \( \tilde{e}(M) \) is invariant under each \( \rho(j) \), and is hence invariant under \( \bar{\rho} \) of \( G \). We conclude

\[
\tilde{\rho}(g) \circ \tilde{e} = \tilde{e} \circ \rho(g)
\]

for all \( g \in G \). Moreover the restriction of the action \( \bar{\rho} \) on \( \tilde{e}(M) \) is the action \( \rho \).

If \( \bar{\rho}(G) \subset SO(d|G|) \), then we finish the proof. Otherwise let

\[
\bar{\rho}^* : G \to SO(d|G| + 1)
\]

be a group homomorphism defined as

\[
\bar{\rho}^*(g) = (\bar{\rho}(g), \det(\bar{\rho}(g))Id_{\mathbb{R}}),
\]

where \( \det(\bar{\rho}(g)) \) is 1 if \( \bar{\rho}(g) \) is orientation preserving and \(-1\) otherwise.

Now let

\[
\bar{e}^* : M \to \mathbb{R}^{d|G|+1} = \mathbb{R}^{d|G|} \times \mathbb{R},
\]

be defined as

\[
\bar{e}^*(x) = (\bar{e}(x), 0).
\]

Then \( \bar{e}^* \) is an embedding. Moreover

\[
\bar{\rho}^*(g) \circ \bar{e}^*(x) = \bar{\rho}^*(g)(\bar{e}(x), 0) = (\bar{\rho}(g) \circ \bar{e}(x), 0) = (\bar{e} \circ \rho(g)(x), 0) = \bar{e}^* \circ \rho(g)(x)
\]

for all \( x \in M \) and \( g \in G \). Proposition 1.2 is proved.

\[\square\]

3. A LOWER BOUND FOR FINITE CYCLIC GROUP ACTIONS

**Lemma 3.1.** Let \( A \) be an orientation-preserving isometry of \( \mathbb{R}^m \) with a fixed point (i.e., \( A \in SO(m) \)). Assume that there are \( s \) points in \( \mathbb{R}^m \) having pairwise coprime lengths of \( A \)-orbits greater than 1. Then \( m \geq 2s \).

**Proof.** Denote by \( w_1, \ldots, w_s > 1 \) the pairwise coprime lengths. Take any \( j = 1, \ldots, s \). Take \( x_j \in \mathbb{R}^m \) such that \( A^{w_j}x_j = x_j \). Define

\[
\overline{x}_j = \frac{x_j + Ax_j + \ldots + A^{w_j-1}x_j}{w_j}.
\]

Then \( A\overline{x}_j = \overline{x}_j \). So for \( u_j := x_j - \overline{x}_j \) we have \( u_j + Au_j + \ldots + A^{w_j-1}u_j = 0 \). Hence

\[
\det \left( id + A + \ldots + A^{w_j-1} \right) = \det \prod_{k=1}^{w_j-1} \left( A - e^{2\pi ik/w_j}Id \right) = 0.
\]

So \( A \) has an eigenvalue \( e^{2\pi ik_j/w_j} \) for some \( 1 \leq k_j \leq w_j - 1 \). Since \( w_1, \ldots, w_s \) are pairwise coprime, all these \( s \) eigenvalues are pairwise distinct.

Since \( A \) is a real operator, for odd \( w_j \) the conjugate \( e^{-2\pi ik_j/w_j} \) is also an eigenvalue of \( A \). Observe that \( n \) is not smaller than the number of eigenvalues of \( A \).

If every \( w_j \) is odd, then \( n \geq 2s \).
If some \( w_j \) is even, then such \( j \) is unique. If further \( n < 2s \), then \( n = 2s - 1 \), and the eigenvalues of \( A \) are \( e^{\pm 2\pi it/w_t} \) for \( t \neq j \), together with the eigenvalue \(-1\) corresponding to the even \( w_j \). This is impossible because the product of the eigenvalues is \( \det A > 0 \).

**Proof of Proposition 1.3.** Suppose \( f \) is a periodic map on \( M \) and there are \( l \) points having pairwise coprime lengths of \( f \)-orbits greater than 1. Suppose \( A : \mathbb{R}^m \to \mathbb{R}^m, A \in SO(m) \), extends \( f \) for some embedding \( e : M \to \mathbb{R}^m \). Then there are \( s \) points having pairwise coprime lengths of \( A \)-orbits greater than 1. By Lemma 3.1, \( m \geq 2l \).

4. Applications to surfaces

**Proof of Corollary 1.5.** It follows from Proposition 1.3 and the following Proposition 4.1.

**Proposition 4.1.** For each given integer \( l > 0 \), there is a periodic map \( f \) on a closed orientable surface \( F \) so that there are \( l \) points having pairwise coprime lengths of \( f \)-orbits greater than 1.

**Proof.** We will use the Hurwitz type construction to get such a periodic map \( f \). The theories of 2-orbifolds, especially those of fundamental groups and covering spaces, are parallel to those of 2-manifolds, see [Sc].

Let \( p_1, ..., p_l \) be the first \( l \) prime numbers and \( P_l = p_1p_2...p_l \) be their product, and \( \delta_j = \frac{P_l}{p_j} \). Let \( O_l \) be the 2-orbifold having underlying space the 2-sphere, and two singular points of index \( \delta_j \) for each \( j \in \{1, 2, ..., l\} \). Then we have its orbifold fundamental group presentation

\[
\pi_1(O_l) = \left\langle x_1, x'_1, x_2, x'_2, ..., x_l, x'_l | \prod_{j=1}^l x_j x'_j = 1, x^\delta_j = x'^\delta_j = 1, j \in \{1, 2, ..., l\} \right\rangle.
\]

Now define a homomorphism

\[
\phi : \pi_1(O_l) \to \mathbb{Z}_{P_l} = \langle f | f^{P_l} = 1 \rangle
\]

by

\[
\phi(x_j) = f^{p_j}, \quad \phi(x'_j) = f^{-p_j},
\]

where \( f \) is a generator of the cyclic group \( \mathbb{Z}_{P_l} \) for this moment.

Since \( p_1, p_2, ..., p_l \) are pairwise co-prime, \( \phi \) is surjective by Chinese Remainder Theorem.

Since all torsion subgroups in \( \pi_1(O_l) \) are conjugated to those finite cyclic groups \( \langle x_j \rangle \) and \( \langle x'_j \rangle \), \( j = 1, ..., l \) [Gr], and those \( \langle x_j \rangle \) and \( \langle x'_j \rangle \) inject into \( \mathbb{Z}_{P_l} \) under \( \phi \), we conclude that the kernel of \( \phi \) is torsion free [Ha], and hence the kernel of \( \phi \) is the fundamental group of a surface \( F \). Hence we have a short exact sequence

\[
1 \to \pi_1(F) \to \pi_1(O_l) \to \langle f | f^{P_l} = 1 \rangle \to 1,
\]
where $f$ acts on $F$ as a cyclic group action of order $P_l$, and we have a cyclic branched cover

$$p: F \to F/f = O_l.$$ 

Since the orbifold $O_l$ is closed and orientable, the surface $F$ is closed and orientable. There is a one-to-one correspondence between singular points of index $\delta_j$ on $O_l$ and the $f$-orbits on $F$ of length $p_j$. We conclude that there are $l$ points having the first $l$ primes as their lengths of $f$-orbits. Noting that the first $l$ primes are pairwise coprime and greater than 1, we have proved Proposition 4.1.

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