MIXED TATE VOEVODSKY MOTIVE OF THE MODULI OF RATIONAL CURVES ON WEIGHTED PROJECTIVE STACKS

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Abstract. We consider the motive \( M_{\text{Hom}_{n}(\mathbb{P}^1, \mathcal{P}(a, b))} \), where \( \mathcal{P}(a, b) \) is the 1-dimensional \((a, b)\) weighted projective stack, over any field \( K \) with \( \text{char}(K) \) not dividing \( a \) or \( b \) in \( \text{DM}(K, \mathbb{Q}) \) the Voevodsky’s triangulated category of mixed motives of smooth separated tame Deligne–Mumford \( K \)-stacks of finite type with \( \mathbb{Q} \)-coefficients. We prove that the motive has the mixed Tate property as it lies in \( \text{DTM}(K, \mathbb{Q}) \) the Voevodsky’s full triangulated thick subcategory of effective geometric mixed Tate motives. In showing this, we prove that the morphism

\[ \Psi : \text{Poly}_{1}^{d_1-k, d_2-k} \times k^k \to R_{1, k}^{d_1, d_2} \setminus R_{1, k+1}^{d_1, d_2} \]

considered in [FW, HP] is indeed an isomorphism over \( \mathbb{Z} \). As a corollary, we acquire the Grothendieck virtual motive classes of the moduli stacks to be equal to

\[ L_{10}^{10n+1} - L_{10}^{10n-1} \]

in the Grothendieck ring of \( K \)-stacks over positive characteristics as well. In the end, we connect the arithmetic & étale topological invariants acquired in [HP, Park] regarding the moduli stack \( \mathcal{L}_{1, 12n} := \text{Hom}_{n}(\mathbb{P}^1, \overline{\mathcal{M}}_{1, 1}) \) of stable elliptic fibrations over \( \mathbb{P}^1 \), also known as stable elliptic surfaces, with 12n nodal singular fibers and a distinguished section through natural transformations under the universality of the Voevodsky’s motives as the reflection of the Tate motivic nature of the moduli stack.

1. Introduction

In this paper we consider the Voevodsky mixed motive associated to the Hom stack \( \text{Hom}_{a}(\mathbb{P}^1, \mathcal{P}(a, b)) \), where \( \mathcal{P}(a, b) \) is the 1-dimensional \((a, b)\) weighted projective stack, parameterizing the degree \( n \geq 1 \) morphisms \( f : \mathbb{P}^1 \to \mathcal{P}(a, b) \) with \( f^{*}\mathcal{O}_{\mathcal{P}(a, b)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n) \). The moduli \( \text{Hom}_{a}(\mathbb{P}^1, \mathcal{P}(a, b)) \) was formulated in [HP, Park] when the characteristic of base field \( K \) is not dividing \( a \) or \( b \) for the primary purpose of studying the arithmetic & the étale topology of the moduli stack \( \mathcal{L}_{1, 12n} := \text{Hom}_{n}(\mathbb{P}^1, \mathcal{P}(4, 6)) \) of stable elliptic fibrations over \( \mathbb{P}^1 \) as \( \mathcal{P}(4, 6) \) is isomorphic to \( \overline{\mathcal{M}}_{1, 1} \) the proper Deligne–Mumford stack of stable elliptic curves over base field \( K \) with \( \text{char}(K) \neq 2, 3 \).

With regard to the arithmetic, we have the following.

Corollary 1 (Corollary 1.2 of [HP]). If \( \text{char}(K) = 0 \), then

\[ [\mathcal{L}_{1, 12n}] = L_{10}^{10n+1} - L_{10}^{10n-1}. \]
If \( \text{char}(\mathbb{F}_q) \neq 2, 3 \),
\[
\#_q(L_{1,12n}) = q^{10n+1} - q^{10n-1}.
\]

With regard to the étale topology, we have the following.

**Corollary 2** (Corollary 1.6 of [Park]). The Hom stack \( L_{1,12n} := \text{Hom}_n([\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}] \)
for \( n \geq 1 \) over \( \overline{\mathbb{F}}_q \) with \( \text{char}(\mathbb{F}_q) \neq 2, 3 \) isomorphic to the Deligne–Mumford moduli stack of stable elliptic surfaces over \( \mathbb{P}^1 \) with \( 12n \) nodal singular fibers and a distinguished section has the following mixed Tate type compactly supported étale cohomology expressed in \( \ell \)-adic Galois representations

\[
H^i_{\text{ét},c}(L_{1,12n}/\mathbb{F}_q; \mathbb{Q}_\ell) \cong \begin{cases} 
\mathbb{Q}_\ell(-2) & i = 0 \\
\mathbb{Q}_\ell(-10n+1) & i = 20n+2 \\
\mathbb{Q}_\ell(-10n-1) & i = 20n-1 \\
0 & \text{else}
\end{cases}
\]

By the Poincaré duality, étale cohomology in dual \( \mathbb{Q}_\ell \)-vector spaces are

\[
H^i_{\text{ét}}(L_{1,12n}/\mathbb{F}_q; \mathbb{Q}_\ell) \cong \begin{cases} 
\mathbb{Q}_\ell(0) & i = 0 \\
\mathbb{Q}_\ell(-2) & i = 3 \\
0 & \text{else}
\end{cases}
\]

The single overarching geometric pattern of the moduli \( L_{1,12n} \) is that it is always of Tate type reflected by the various arithmetic topological invariants above. By the Voevodsky’s theory of motives which is the universal object for the Weil cohomology and the algebraic cycles, we can capture the Tate motivic nature of the Hom stack \( \text{Hom}_n([\mathbb{P}^1, \mathcal{P}(a,b)]) \).

**Theorem 3** (Tate motivic nature of the moduli). \textit{Compactly supported motive of the Hom stack} \( \mathcal{M}(\text{Hom}_n([\mathbb{P}^1, \mathcal{P}(a,b)]) \) over any base field \( K \) with \( \text{char}(K) \) not dividing \( a \) or \( b \) is an effective, geometric, and mixed Tate motive defining an object in \( \text{DTM}(K, \mathbb{Q}) \) the Voevodsky’s triangulated subcategory of effective geometric mixed Tate motives of smooth separated tame Deligne–Mumford \( K \)-stacks of finite type with \( \mathbb{Q} \)-coefficients.

\[
\mathcal{M}(\text{Hom}_n([\mathbb{P}^1, \mathcal{P}(a,b)]) \in \text{Obj}(\text{DTM}(K, \mathbb{Q})).
\]

When one is presented with an abelian category \( \mathcal{A} \), one can naturally consider an Euler characteristic of objects of \( \mathcal{A} \) through the Grothendieck group \( K_0(\mathcal{A}) \) defined as the group generated by the isomorphism classes in \( \mathcal{A} \) subject to the cut-and-paste relations \([B] = [A] + [C]\) of the scissors congruence type corresponding to the short exact sequences \( 0 \to A \to B \to C \to 0 \). Together with the cartesian product structure of \( \mathcal{A} \) inducing the multiplication \([A \times B] = [A][B]\) of classes, Grothendieck ring \( K_0(\mathcal{A}) \) can be
thought of as a **motivic Euler characteristic** being the universal object for the **additive invariants** of the category \(A\).

While the category of varieties \(\text{Var}_K\) or more generally the category of algebraic stacks \(\text{Stck}_K\) are not additive with no exact sequences, one can consider the exact sequences the Voevodsky’s triangulated category \(\text{DM}(K, \mathbb{Q})\) of effective geometric mixed motives. And this leads to \(K_0(\text{DM}(K, \mathbb{Q}))\) which is the free abelian group on the objects of \(\text{DM}(K, \mathbb{Q})\) subject to the relations \([Y] = [X] + [Z]\) corresponding to the exact triangles \(0 \to X \to Y \to Z \to X[1]\). The tensor product of the category \(\text{DM}(K, \mathbb{Q})\) makes \(K_0(\text{DM}(K, \mathbb{Q}))\) into a ring as in \([BD]\).

For us, the Grothendieck class \([X]\) of \(K_0(\text{DM}(K, \mathbb{Q}))\) measures the complexity of the motive \(\mathcal{M}(X)\) of the algebraic stack \(X\). And we note that the \(K_0(\text{Stck}_K)\) and \(K_0(\text{DM}(K, \mathbb{Q}))\) are isomorphic allowing us to work directly with \(K_0(\text{Stck}_K)\) to work with Grothendieck class of Voevodsky motive.

We begin by showing that the morphism considered in \([FW]\) Proposition 3.3 and \([HP]\) Proposition 15 is indeed an isomorphism over \(\mathbb{Z}\).

**Proposition 4.** The morphism

\[
\Psi : \text{Poly}_1^{(d_1-k, d_2-k)} \times A^k \to R^{(d_1, d_2)}_{1,k} \setminus R^{(d_1, d_2)}_{1,k+1}
\]

is an isomorphism over \(\mathbb{Z}\).

And this naturally leads to the extension of the \([HP]\) Theorem 1, Corollary 2 to \(\text{char}(K) \neq 2, 3\)

**Corollary 5** (Grothendieck virtual motive class over \(\mathbb{F}_q\)). If \(\text{char}(K) \neq 2, 3\), then

\[
[L_{1,12n}] = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}.
\]

The important point here is that we are able to directly acquire the weighted point count \(#_q(L_{1,12n}) = q^{10n+1} - q^{10n-1}\) over \(\mathbb{F}_q\) with \(\text{char}(\mathbb{F}_q) \neq 2, 3\) via the motivic arithmetic measure \(#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}\) as we have the Grothendieck virtual motive class not only over characteristic zero but also over positive characteristics as well.
We then connect the arithmetic & the étale topological invariants of $L_{1,12n}$ by a sequence of natural transformations showing the Tate motivic nature $\mathcal{M}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) \in \text{Obj}(\text{DTM}(K, \mathbb{Q}))$ manifests into the Grothendieck virtual motive class $[L_{1,12n}] = L^{10n+1} - L^{10n-1}$ being a polynomial in $L$ and the weighted point count $\#_q(L_{1,12n}) = q^{10n+1} - q^{10n-1}$ being a polynomial in $q$. Also the compactly supported étale cohomology with the eigenvalues of geometric Frobenius map $\text{Frob}^*_q$ expressed in isomorphisms of $\ell$-adic Galois representations is of mixed Tate type.

There are many other interests for showing the motive is mixed Tate with regard to the Chow Kûnnett property [Totaro] as well as the well-known result that over a number field $K$ one is able to extract the heart which is the $\mathbb{Q}$-linear abelian category $\text{TM}(K, \mathbb{Q})$ from the triangulated category $\text{DTM}(K, \mathbb{Q})$ through a canonical $t$-structure constructed by the work of [Levine].

**Theorem 6** (Levine). Over a number field $K$, there is a canonical $t$-structure on $\text{DTM}(K, \mathbb{Q})$ and one can therefore construct a $\mathbb{Q}$-linear abelian category $\text{TM}(K, \mathbb{Q})$ of mixed Tate motives.
2. Moduli stack of rational curves on weighted projective stacks

Let us first recall the definition of the target stack \( \mathcal{P}(a, b) \).

**Definition 7.** The 1-dimensional \( a, b \in \mathbb{N} \) weighted projective stack is defined as a quotient stack

\[
\mathcal{P}(a, b) := [(\mathbb{A}^2_{x,y} \setminus 0)/\mathbb{G}_m]
\]

Where \( \lambda \in \mathbb{G}_m \) acts by \( \lambda \cdot (x, y) = (\lambda^a x, \lambda^b y) \). In this case, \( x \) and \( y \) have degrees \( a \) and \( b \) respectively. A line bundle \( \mathcal{O}_{\mathcal{P}(a,b)}(m) \) is defined to be a line bundle associated to the sheaf of degree \( m \) homogeneous rational functions without poles on \( \mathbb{A}^2_{x,y} \setminus 0 \).

When the characteristic of the field \( K \) is not equal to 2 or 3, [Hassett] shows that \((\overline{M}_{1,1})_K \cong [(\text{Spec } K[a_4,a_6]-(0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)\) by using the Weierstrass equations, where \( \lambda \cdot a_i = \lambda^i a_i \) for \( \lambda \in \mathbb{G}_m \) and \( i = 4, 6 \). Thus, \( a_i \)'s have degree \( i \)'s respectively. Note that this is no longer true if characteristic of \( K \) is 2 or 3, as the Weierstrass equations are more complicated.

**Proposition 8.** For any \( a, b \in \mathbb{N} \) and over base field \( K \) with \( \text{char}(K) \) not dividing \( a \) or \( b \), the Hom stack \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \) parameterizing the morphisms \( f : \mathbb{P}^1 \to \mathcal{P}(a, b) \) with \( f^* \mathcal{O}_{\mathcal{P}(a,b)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n) \) for \( n \geq 1 \) is a smooth separated tame Deligne–Mumford stack of finite type.

**Proof.** These were established in [HP, Proposition 9, Proof of Theorem 1]. To recall the major points therein, \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \) is a smooth Deligne–Mumford stack by [Olsson] Theorem 1.1] isomorphic to the quotient stack \([T/\mathbb{G}_m] \), admitting \( T \) as a smooth schematic cover where \( T \subset H^0(\mathcal{O}_{\mathbb{P}^1}(an)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(bn)) \setminus 0 \) parameterizes the set of pairs \( (u, v) \) of non-monic polynomials with the degrees equal to either \( (\deg(u) = an \text{ and } 0 \leq \deg(v) \leq bn) \) or \( (\deg(v) = bn \text{ and } 0 \leq \deg(u) \leq an) \) but not both as they are mutually coprime. The quotient stack \([T/\mathbb{G}_m] \) parameterizes the equivalence class of pairs \( (u, v) \) and \( (u', v') \) that are equivalent when there exists \( \lambda \in \mathbb{G}_m \) so that \( u' = \lambda^a \cdot u \) and \( v' = \lambda^b \cdot v \). As \( \mathbb{G}_m \) acts on \( T \) properly with positive weights \( a, b > 0 \) the quotient stack \([T/\mathbb{G}_m] \) is separated. It is also tame as in [AOV, Theorem 3.2] since \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \) is over base field \( K \) with \( \text{char}(K) \) not dividing \( a \) or \( b \). \( \square \)

Denoting \( \deg u := k \) and \( \deg v := l \), then \( (u, v) \in T \) is whenever \( k = an \text{ or } l = bn \) (so that they do not simultaneously vanish at \( \infty \)) and \( u, v \) have no common roots. Since there are many possible degrees for a pair \( (u, v) \in T \), consider locally closed subsets \( T_{k,l} := \{(u, v) \in T : \deg u = k, \ \deg v = l \} \). Notice that \( T_{k-1, bn} \subset T_{k, bn} \) as for any \( (u, v) \in T_{k-1, bn} \), \( u(X, Y) \) has a description as \( Y^{an-k+1}u'(X, Y) \) which is \( u_{[1,0]}(X, Y) \) from a pencil polynomials \( u_{[t_0:t_1]}(X, Y) = Y^{an-k}(t_1Y-t_0X)u'(X, Y) \) where \( u_{[1, t_1]} \in T_{k, bn} \). Hence, we obtain the following stratification:
\[ T = T_{an, bn} \sqcup \left( \bigsqcup_{k=0}^{an-1} T_{k, bn} \right) \sqcup \left( \bigsqcup_{l=0}^{bn-1} T_{an, l} \right) \]

\[ T = T_{an, bn} \supseteq T_{an-1, bn} \supseteq \cdots \supseteq T_{0, bn} = T_{0, bn} \]

\[ T = T_{an, bn} \supseteq T_{an, bn-1} \supseteq \cdots \supseteq T_{an, 0} = T_{an, 0} \]

\[ T_{an-k, bn} \cap T_{an, bn-l} = \emptyset \quad \forall k, l > 0 \]

Define

\[ F_{k,l} := \{(u, v) \in T_{k,l} : u, v \text{ are monic}\} \]

Then, \( F_{k,l} \hookrightarrow T_{k,l} \) is a section of the projection morphism \( T_{k,l} \to F_{k,l} \) (induced by making \((u, v)\) to be a monic pair), which has \( \mathbb{G}_m \times \mathbb{G}_m \)-fibers. We can recognize the \( F_{k,l} \cong \text{Poly}^{(k,l)}_1 \) as below (inspired by [FW, HP]):

**Definition 9.** Fix a field \( K \) with algebraic closure \( \overline{K} \). Fix \( k, l \geq 0 \). Define \( \text{Poly}^{(k,l)}_1 \) to be the set of pairs \((u, v)\) of monic polynomials in \( K[z] \) so that:

1. \( \deg u = k \) and \( \deg v = l \).
2. \( u \) and \( v \) have no common root in \( \mathbb{K} \).

3. **Grothendieck virtual motive classes of the moduli stacks over \( \mathbb{F}_q \)**

the Grothendieck ring of algebraic stacks as the following.

**Definition 10.** [Ekedahl] [§1] Fix a field \( K \). Then the Grothendieck ring \( K_0(\text{Stck}_K) \) of algebraic stacks of finite type over \( K \) all of whose stabilizer group schemes are affine, is a group generated by isomorphism classes of \( K \)-stacks \([\mathcal{X}]\) of finite type, modulo relations:

- \([\mathcal{X}] = [\mathcal{X}] + [\mathcal{X} \setminus \mathcal{Z}]\) for \( \mathcal{Z} \subset \mathcal{X} \) a closed substack,
- \([\mathcal{E}] = [\mathcal{X} \times \mathbb{A}^n]\) for \( \mathcal{E} \) a vector bundle of rank \( n \) on \( \mathcal{X} \).

Multiplication on \( K_0(\text{Stck}_K) \) is induced by \([\mathcal{X}][\mathcal{Y}] := [\mathcal{X} \times_K \mathcal{Y}]\). There is a distinguished element \( \mathbb{L} := [\mathbb{A}^1] \in K_0(\text{Stck}_K) \), called the Lefschetz motive.

We first show that the morphism considered in [FW] Proposition 3.3 and [HP] Proposition 15 is indeed an isomorphism over \( \mathbb{Z} \) which allows us to extend the computation of \( [\mathcal{X}]_q \) ([FW2, HP] to fields of positive characteristic \( K = \mathbb{F}_q \) which has the advantage of allowing us to take the motivic arithmetic measure giving the weighted point count by the assignment \([\mathcal{X}] \mapsto \#_q(\mathcal{X})\) as it gives a well-defined ring homomorphism \( \#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \to \mathbb{Q} \) (c.f. [Ekedahl, §2]).
3.1. **Proof of Proposition 4**

*Proof.* We begin with the following lemma

**Lemma 11.** Let $f_1, \ldots, f_m, g$ be monic with indeterminate coefficients, and let $h \in \mathbb{Z}[f_1g, \ldots, f_mg]$ be such that for every maximal ideal $I \subset \mathbb{Z}[f_1g, \ldots, f_mg]$ with $h \not\in I$ we have $f_1g, \ldots, f_mg$ have only $g$ as a common factor in $\mathbb{Z}[f_1g, \ldots, f_mg]/I$. Then

$$\mathbb{Z}[f_1g, \ldots, f_mg, h^{-1}] = \mathbb{Z}[f_1, \ldots, f_m, g, h^{-1}].$$

*Proof.* Let $I \subset \mathbb{Z}[f_1g, \ldots, f_mg]$ be a maximal ideal with $h \not\in I$. By construction, in $\mathbb{Z}[f_1g, \ldots, f_mg]/I$ there exists a linear combination $h_1(f_1g) + \ldots + h_m(f_mg) \equiv (h_1f_1 + \ldots + h_mf_m)g \equiv g$. Multiplying through by product of the denominators $u_I \in \mathbb{Z}[f_1g, \ldots, f_mg]$, a unit in the local ring at $I$, and subtracting an element of $I$ if necessary, we obtain an identity

$$u_1g \in \mathbb{Z}[f_1g, \ldots, f_mg].$$

The ideal generated by the $u_I$ in $\mathbb{Z}[f_1g, \ldots, f_mg, h^{-1}]$ is contained in no maximal ideal, and hence must be $(1)$. This yields a linear combination of the $u_I$ adding to 1, which allows us to construct $g$ in $\mathbb{Z}[f_1g, \ldots, f_mg, h^{-1}]$. □

Let $X = \text{Spec}\mathbb{Z}[f_1, \ldots, f_m, g]$ and $Y = \text{Spec}\mathbb{Z}[f_1g, \ldots, f_mg]$. Then there is morphism

$$\phi : X \to Y$$

induced by the inclusion of rings

$$\mathbb{Z}[f_1g, \ldots, f_mg] \subset \mathbb{Z}[f_1, \ldots, f_m, g].$$

Let $S$ be the set of all $h \in \mathbb{Z}[f_1g, \ldots, f_mg]$ satisfying the hypothesis of the lemma above, and let $V \subset Y$ be the open subset

$$V = \bigcup_{h \in S} D(h) \subset Y.$$

Letting $U' = \phi^{-1}(V) \subset X$, the lemma then implies that

$$U' \to \phi(V)$$

is an isomorphism.

We will now show that $U' = U$, where $U$ is the locus where $f_1, \ldots, f_m$ have no common roots. Take a maximal ideal $J \subset U$, the kernel of a ring map $\mathbb{Z}[f_1, \ldots, f_m, g] \to F$ for some field $F$. Then the induced map $\mathbb{Z}[f_1g, \ldots, f_mg] \to F$ sends $f_1g, \ldots, f_mg$ to polynomials with no common root beyond $g$. In particular, the image of $J$ lies in $V$, so we can find a polynomial $h \in S$ such that $D(h)$ contains the image of $J$. Thus $J$ lies in $U'$, so $U = U'$ as desired. □
This has the following consequence for the motivic statements about the Poly_{lan,bn}^{1} to be true over any fields $K$ including the fields with positive characteristics. The expression of $[\text{Poly}_{\{d_{1},d_{2}\}}^{1}]$ was found in [HP, Proposition 18] for $\text{char}(K) = 0$ as a polynomial in $\mathbb{L}$ and we extend it to $\text{char}(K) > 0$.

**Proposition 12.** Fix $d_{1}, d_{2} \geq 0$.

$$[\text{Poly}_{\{d_{1},d_{2}\}}^{1}] = \begin{cases} \mathbb{L}^{d_{1}+d_{2}} - \mathbb{L}^{d_{1}+d_{2}-1}, & \text{if } d_{1}, d_{2} > 0, \\ \mathbb{L}^{d_{1}+d_{2}}, & \text{if } d_{1} = 0 \text{ or } d_{2} = 0. \end{cases}$$

**Proof.** The proof is identical to the [HP] Proposition 18 except that at Step 2, we have the morphism $\Psi$ to be an isomorphism by Proposition 4. □

And similarly this extends the motivic statements about the $\text{Hom}_{n}(\mathbb{P}^{1}, \mathbb{M}_{1,1})$ to the fields with positive characteristics.

### 3.2. Proof of Corollary [5]

**Proof.** The proof is identical to the proof of [HP] Theorem 1, Corollary 2 except that at Step 3, we have the Grothendieck virtual motive classes of $[\text{Poly}_{\{d_{1},d_{2}\}}^{1}]$ to be true over any fields $K$ with $\text{char}(K)$ not dividing $a$ or $b$ by Proposition 4. □

### 4. Voevodsky’s mixed motives and mixed Tate motives

Let $\text{DM}(K, R)$ be the Voevodsky’s triangulated category of mixed motives with compact support denoted by $C^{c}_{\text{gm}}(X)$ in [Voevodsky] for separated schemes $X$ of finite type over a base field $K$ in $R$–coefficients. When $K$ is a perfect field and admits a resolution of singularities the formulation and the properties of $\text{DM}(K, R)$ such as the Gysin distinguished triangle are worked out by [Voevodsky, MVW]. When $K$ is a perfect field and may not admit a resolution of singularities but the exponential characteristic of $K$ is invertible in $R$ similar works on $\text{DM}(K_{\text{perf}}, R)$ are done by [Kelly]. And for arbitrary field $K$, we now know that the pullback functor $\text{DM}(K, R) \to \text{DM}(K_{\text{perf}}, R)$ is an equivalence of categories by Proposition 8.1 of [CD].

We now extend the definition to the smooth but not necessarily proper Deligne–Mumford stacks based on the constructions of [Choudhury].

**Definition 13.** Let $\text{DM}(K, \mathbb{Q})$ be the Voevodsky’s triangulated category of effective geometric mixed motives for smooth separated tame Deligne–Mumford $K$–stacks of finite type in $\mathbb{Q}$–coefficients.

Note that as we work with smooth and not necessarily proper Deligne–Mumford stacks $X$ of finite type, we can identify the associated motive $\mathcal{M}(X)$ to be a direct summand of the motive of a quasi-projective variety by Theorem 4.6 of [Choudhury]. And this allows us to restrict ourselves to full
triangulated thick subcategory of effective geometric motives generated by the motives \(\mathcal{M}(V)\) for \(V \in \text{Sm}/K\) by Corollary 4.7 of Choudhury where \(\text{Sm}/K\) is the category smooth separated finite type \(K\)-schemes (see MVW, Definition 14.1).

We now recall the Gysin distinguished triangle which will aid us in showing mixed Tate property of motives by repeated application of the fullness of \(\text{DTM}(K, \mathbb{Q})\) (2-out-of-3 property) applied to the stratification.

**Proposition 14.** For a separated scheme \(X\) of finite type over an arbitrary field \(K\) and a closed subscheme \(Z\) of \(X\), there is an exact triangle in \(\text{DM}(K, \mathbb{Q})\) called the Gysin localization exact triangle:

\[
\mathcal{M}(Z) \to \mathcal{M}(X) \to \mathcal{M}(X - Z) \to \mathcal{M}(Z)[1]
\]

**Proof.** For the proof see [Voevodsky, Proposition 4.1.5, Theorem 4.3.7(3)] and [Kelly, Proposition 5.5.5, Theorem 5.5.14(3)] together with the equivalence [CD, Proposition 8.1]. □

Recall that given a triangulated category \(\mathcal{D}\), a full subcategory \(\mathcal{D}'\) is a triangulated subcategory if and only if it is invariant under the shift \(T\) of \(\mathcal{D}\) and for any distinguished triangle \(A \to B \to C \to A[1]\) for \(\mathcal{D}\) where \(A\) and \(B\) are in \(\mathcal{D}'\) there is an isomorphism \(C \cong C'\) with \(C'\) also in \(\mathcal{D}'\). A full triangulated subcategory \(\mathcal{D}' \subset \mathcal{D}\) is thick if it is closed under direct sums.

We now define the Voevodsky’s triangulated subcategory \(\text{DTM}(K, \mathbb{Q})\) of mixed Tate motives.

**Definition 15.** Let \(\text{DTM}(K, \mathbb{Q})\) be the Voevodsky’s full triangulated thick subcategory of effective geometric mixed Tate motives generated by the Tate objects \(\mathbb{Q}(n)\) for smooth separated tame Deligne–Mumford \(K\)-stacks of finite type in \(\mathbb{Q}\)-coefficients.

For example, \(\mathbb{P}^N\) and \(\mathbb{A}^N \setminus \{0\}\) have Tate motives:

\[
\mathcal{M}(\mathbb{P}^N) = \bigoplus_{i=0}^{N} \mathbb{Q}(i)[2i]
\]

\[
\mathcal{M}(\mathbb{A}^N \setminus \{0\}) = \mathbb{Q}(0) \oplus \mathbb{Q}(N)[(2N - 1)]
\]

The following result makes it easy to show that the motive of quotient stack \([X/G]\) is mixed Tate motive.

**Proposition 16** (Corollary 8.12 of Totaro). Let \(\mathfrak{X}\) be a quotient stack over \(K\). Let \(E\) be a principal \(GL(n)\)-bundle over \(\mathfrak{X}\) for some \(n\), viewed as a stack over \(K\). Then \(E\) is mixed Tate in \(\text{DM}(K, \mathbb{Q})\) if and only if \(\mathfrak{X}\) is mixed Tate.

Now we are ready to prove Theorem \(\mathfrak{X}\).
4.1. **Proof of Theorem** [3].

Proof. Showing that the motive of the Hom stack **M** (Hom\(_n(\mathbb{P}^1, \mathcal{P}(a, b))\)) is mixed Tate motive is equivalent to showing the motive of the quotient stack \( [T/G_m] \) is mixed Tate motive. By the Proposition [10] where \( T \) is the principal \( G_m \)-bundle over \( [T/G_m] \), we need to show that the motive of \( T \) is mixed Tate motive.

As we have the stratification of \( T \) by \( T_{k,l} \) as in equation (1), once we can show the motive of stratum \( \mathcal{M}(T_{k,l}) \) is mixed Tate motive then by the repeated application of the Gysin localization exact triangle and the fullness of **DTM**(\( K, \mathbb{Q} \)) (2-out-of-3 property) we have \( \mathcal{M}(T) \) having the mixed Tate property as desired.

\( T_{k,l} \) are the trivial \( G_m \times G_m \) bundle over Poly\(_1(d_1, d_2)\). By the Künneth property of \( G_m \) we can show \( \mathcal{M}(T_{k,l}) \) is mixed Tate motive by showing the motive \( \mathcal{M}(\text{Poly}_1(d_1, d_2)) \) is mixed Tate motive.

The proof that the motive \( \mathcal{M}(\text{Poly}_1(d_1, d_2)) \) is mixed Tate is analogous to [Horel, Lemma 6.2]. Here, we only state the differences to their work as we are working over arbitrary fields including the positive characteristics as well as the isomorphism of the filtration which is critical for the argument is provided at Proposition [1].

We first need to consider the lexicographic induction on the pair \((d_1, d_2)\). Since the order of \( d_1, d_2 \) does not matter for motive, we assume that \( d_1 \geq d_2 \). For the base cases, consider when \( d_2 = 0 \). Then the monic polynomial of degree 0 is nowhere vanishing, so any polynomial of degree \( d_1 \) constitutes a member of Poly\(_1(d_1, 0) \approx \mathbb{A}^{d_1} \) which is obviously mixed Tate. Similarly, \( d_1 = 0 \) is taken care of.

Then for \( d_1, d_2 > 0 \), we obtain the same filtration as in [HP] of \( \mathbb{A}^{d_1+d_2} \) by \( R_{1,k}^{(d_1, d_2)} \) where each \( R_{1,k}^{(d_1, d_2)} \) is a closed subscheme of \( \mathbb{A}^{d_1+d_2} \) which is the space of \((u, v)\) monic polynomials of degree \( d_1, d_2 \) respectively for which there exists a monic \( h \in K[z] \) with \( \deg(h) \geq k \) and monic polynomials \( g_i \in K[z] \) so that \( u = g_1 h \) and \( v = g_2 h \). Note that \( R_{1,0}^{(d_1, d_2)} = \mathbb{A}^{d_1+d_2} \).

By the Proposition [4] we have the isomorphism

\[ \Psi : \text{Poly}_1^{((d_1-k)_1, (d_2-k)_2)} \times \mathbb{A}^k \rightarrow R_{1,k}^{(d_1, d_2)} \setminus R_{1,k+1}^{(d_1, d_2)} \]

Using the induction hypothesis and [Horel Proposition 6.1], we have the motive \( \mathcal{M}(\mathbb{A}^{d_1+d_2} - R_{1,1}^{(d_1, d_2)}) \cong \mathcal{M}(\text{Poly}_1^{(d_1, d_2)}) \) is mixed Tate as desired.

\[ \square \]

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