THE TRACE OF UNIFORM HYPERGRAPHS WITH APPLICATION TO ESTRADA INDEX

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Abstract. In this paper we investigate the traces of the adjacency tensor of hypergraphs (simply called the traces of hypergraphs). We give new expressions for the traces of hypertrees and linear unicyclic hypergraphs by the weight function assigned to their connected sub-hypergraphs, and provide some perturbation results for the traces of a hypergraph with cut vertices. As applications we determine the unique hypertree with maximum Estrada index among all hypertrees with fixed number of edges and perfect matchings, and the unique unicyclic hypergraph with maximum Estrada index among all unicyclic hypergraph with fixed number of edges and girth 3.

1. Introduction

A hypergraph $\mathcal{H} = (V, E)$ consists of a vertex set $V = \{v_1, v_2, \ldots, v_n\}$ denoted by $V(\mathcal{H})$ and an edge set $E = \{e_1, e_2, \ldots, e_k\}$ denoted by $E(\mathcal{H})$, where $e_i \subseteq V$ for $i \in [k]$. If there exist no different $i$ and $j$ such that $e_i \subseteq e_j$, then $\mathcal{H}$ is called simple. If $|e_i| = m$ for each $i \in [k]$ and $m \geq 2$, then $\mathcal{H}$ is called an $m$-uniform hypergraph. A simple graph is exactly a simple 2-uniform hypergraph.

For an $m$-uniform hypergraph $\mathcal{H}$ on vertices $v_1, \ldots, v_n$. Cooper and Dutle [7] introduced the adjacency tensor of $\mathcal{H}$ as follows.

Definition 1.1. ( [7]) Let $\mathcal{H}$ be an $m$-uniform hypergraph on $n$ vertices $v_1, v_2, \ldots, v_n$. The adjacency tensor of $\mathcal{H}$ is defined as $A(\mathcal{H}) = (a_{i_1 i_2 \ldots i_m})$, an $m$-th order $n$-dimensional tensor, where

$$a_{i_1 i_2 \ldots i_m} = \begin{cases} \frac{1}{(m-1)!}, & \text{if } \{v_{i_1}, \ldots, v_{i_m}\} \in E(\mathcal{H}); \\ 0, & \text{else.} \end{cases}$$

Note that if $m = 2$, then $A(\mathcal{H})$ is exactly the usual adjacency matrix of the simple graph $\mathcal{H}$. In this situation, the $d$-th trace of $A(\mathcal{H})$, namely the trace of $A(\mathcal{H})^d$, is exactly the number of closed walks of $\mathcal{H}$ with length $d$ starting from each vertex of $\mathcal{H}$.

To deal with the high order case, Morozov and Shakirov [33] introduced the traces of polynomial maps $f$ given by a system of homogeneous polynomials of arbitrary degrees. As a tensor $T = (t_{i_1 i_2 \ldots i_m})$ of order $m$ and dimension $n$ naturally induces a system of homogeneous polynomials

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\( \mathcal{T}_{x^{m-1}} \) defined by
\[
(\mathcal{T}_{x^{m-1}})_i = \sum_{i_2, \ldots, i_m \in [n]} t_{i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}, \quad i \in [n],
\]
where \( x = (x_1, \ldots, x_n)^\top \). Using the traces defined by Morozov and Shakirov \[33\], the \( d \)-th trace \( \text{Tr}_d(\mathcal{T}) \) of \( \mathcal{T} \) is expressed as follow:
\[
(1.1) \quad \text{Tr}_d(\mathcal{T}) = (m - 1)^{n-1} \sum_{d_1 + \cdots + d_n = d, i_1, \ldots, i_n \in [n]} \prod_{i=1}^{n} \frac{1}{(d_i(m - 1))!} \left( \sum_{y_i \in [n]^{m-1}} t_{i y_i} \frac{\partial}{\partial a_{i y_i}} \right)^{d_i} \text{Tr}(A^{d(m-1)}),
\]
where \( A = (a_{ij}) \) be an \( n \times n \) auxiliary matrix by taking all \( a_{ij} \)'s as variables, \( t_{i y_i} = t_{i_1 i_2 \ldots i_m} \) and \( \frac{\partial}{\partial a_{y_i}} = \frac{\partial}{\partial a_{i_1}} \cdots \frac{\partial}{\partial a_{i_m}} \) if \( y_i = (i_2, \ldots, i_m) \).

The traces of a tensor are closely related to its eigenvalues, which were introduced by Lim \[29\] and Qi \[35\] as follows, where \( I = (i_{11}, \ldots, i_{m}) \) is the identity tensor of order \( m \) and dimension \( n \), namely, \( i_{11} \ldots i_{m} = 1 \) if \( i_1 = i_2 = \cdots = i_m \in [n] \) and \( i_{11} \ldots i_{m} = 0 \) otherwise.

**Definition 1.2.** \([29,35]\) Let \( \mathcal{T} \) be an \( m \)-th order \( n \)-dimensional tensor. For some \( \lambda \in \mathbb{C} \), if the polynomial system \((\lambda I - \mathcal{T})x^{m-1} = 0\), or equivalently \( \mathcal{T}_{x^{m-1}} = \lambda x^{[m-1]} \), has a solution \( x \in \mathbb{C}^n \setminus \{0\} \), then \( \lambda \) is called an eigenvalue of \( \mathcal{T} \) and \( x \) is an eigenvector of \( \mathcal{T} \) associated with \( \lambda \), where \( x^{[m-1]} := (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}) \).

The determinant \( \det \mathcal{T} \) of \( \mathcal{T} \) is defined to be the resultant of the polynomials \( \mathcal{T}_{x^{m-1}} \) \[25\], and the characteristic polynomial of \( \mathcal{T} \) is defined to be \( \varphi_\mathcal{T}(\lambda) := \det(\lambda I - \mathcal{T}) \) \[4,35\]. It is known that \( \lambda \) is an eigenvalue of \( \mathcal{T} \) if and only if it is a root of \( \varphi_\mathcal{T}(\lambda) \). The spectrum of \( \mathcal{T} \) is the multi-set of the roots of \( \varphi_\mathcal{T}(\lambda) \).

Morozov and Shakirov proved that
\[
(1.2) \quad \det(\mathcal{I} - \mathcal{T}) = \exp \left( \sum_{d=1}^{\infty} -\frac{\text{Tr}_d(\mathcal{T})}{d} \right) = \sum_{d=0}^{\infty} P_d \left( -\frac{\text{Tr}_1(\mathcal{T})}{1}, \ldots, -\frac{\text{Tr}_d(\mathcal{T})}{d} \right),
\]
where \( P_d \) (the \( d \)th Schur polynomial) is defined as \( P_0 = 1 \), and for \( d > 0 \),
\[
P_d(t_1, \ldots, t_d) = \sum_{\ell=1}^{d} \sum_{d_1 + \cdots + d_\ell = d, d_i \in \mathbb{Z}^+} \frac{t_{d_1} \cdots t_{d_\ell}}{\ell!},
\]
From Eq.\;\((1.2)\), we can get the determinant \( \det \mathcal{T} \) and the characteristic polynomial \( \varphi_\mathcal{T}(\lambda) = \det(\lambda I - \mathcal{T}) \) in terms of traces by considering the degree of the resultant; see \[7,36\] for details.

Cooper and Dulte \[7\] gave explicit formulas for some low co-degree coefficients of the characteristic polynomial of \( \mathcal{A}(\mathcal{H}) \) of a uniform hypergraph \( \mathcal{H} \). Shao, Qi and Hu \[36\] provided a graph interpretation for the \( d \)-th trace of a general tensor, and proved that
\[
(1.3) \quad \text{Tr}_d(\mathcal{T}) = \sum_{i=1}^{N} \lambda_i^d,
\]
which is consistent with the matrix case, where $\lambda_1, \ldots, \lambda_N$ are all eigenvalues of $T$, and $N = n(m - 1)^{n-1}$.

The $d$-th trace of a uniform hypergraph $\mathcal{H}$, denoted by $\text{Tr}_d(\mathcal{H})$, is defined to be $\text{Tr}_d(A(\mathcal{H}))$.

Clark and Cooper [6] generalized the Harary-Sachs theorem of graphs to uniform hypergraphs by expressing the trace as a weighted sum over a family of Veblen hypergraphs. Chen, Bu and Zhou [5] gave a formula for the spectral moments (equivalently the traces) of a hypertree in terms of the number of sub-hypertrees.

The traces of a hypergraph are also related to the Estrada index of the hypergraph, which was recently introduced by Sun, Zhou and Bu [37].

**Definition 1.3.** (37) Let $\mathcal{H}$ be an $m$-uniform hypergraph on $n$ vertices, and let $\lambda_1, \ldots, \lambda_N$ be all eigenvalues of the adjacency tensor $A(\mathcal{H})$ of $\mathcal{H}$, where $N = n(m - 1)^{n-1}$. The Estrada index of $\mathcal{H}$ is defined to be

$$EE(\mathcal{H}) = \sum_{i=1}^{N} e^{\lambda_i}.$$  

By Eq. (1.3), it is easily seen that

$$EE(\mathcal{H}) = \sum_{d=0}^{\infty} \frac{\text{Tr}_d(\mathcal{H})}{d!}.$$  

When $m = 2$, the Estrada index in Definition 1.3 is exactly that of a graph, which was first introduced by Estrada [14] in 2000 and found useful in measuring the degree of protein folding [13] and the centrality of complex networks [12]. So the Estrada index of hypergraphs may have potential applications in networks modelled as hypergraphs. Peña, Gutman and Rada [34] conjectured that the path is the unique graph with the minimum Estrada index among all graphs (trees) with given order, and the star is the unique one with the maximum Estrada index among all trees with given order. The conjecture was partly proved by Das and Lee [8], and completely proved by Deng [9]. The other versions of Estrada index of hypergraphs were also investigated by Duan, Dam and Wang [11] via signless Laplacian tensor or Laplacian tensor, and Lu, Xue and Zhu [32] via signless Laplacian matrix. Recently, Fan et al. [21] proved that among all hypertrees with fixed number of edges, the hyperpath is the unique one with minimum Estrada index and the hyperstar is the unique one with maximum Estrada index, which provided a hypergraph version of the result of Peña-Gutman-Rada conjecture.

We shall note here the development of spectral hypergraph theory. Since the Perron-Frobenius theorem of nonnegative matrices was generalized to nonnegative tensors [3, 23, 39, 41], the spectral hypergraph theory develops rapidly on many topics, such as the spectral radius [2, 19, 24, 27, 28, 30, 31], the eigenvariety [15, 17, 20], the spectral symmetry [16, 18, 36, 43], the eigenvalues of hypertrees [42].

In this paper, we will give new expressions of the traces of uniform hypergraphs, especially for hypertrees and unicyclic hypergraphs, and provide some perturbation results for the traces of
a hypergraph when its structure is locally changed. As an application of the trace results, we determine the unique hypertree with maximum Estrada index among all hypertrees with fixed number of edges and perfect matching. We characterize the linear unicyclic hypergraphs with maximum Estrada index among all unicyclic hypergraphs with fixed number of edges and given girth, and particularly determine with maximum Estrada index among all unicyclic hypergraphs with fixed number of edges and girth 3.

2. Preliminaries

2.1. Tensors and hypergraphs. A tensor (also called hypermatrix) $T = (t_{i_1i_2\ldots i_m})$ of order $m$ and dimension $n$ over $\mathbb{C}$ refers to a multiarray of entries $t_{i_1i_2\ldots i_m} \in \mathbb{C}$ for all $i_j \in [n] := \{1, 2, \ldots, n\}$ and $j \in [m]$, which can be viewed to be the coordinates of the classical tensor (as a multilinear function) under a certain basis. Surely, if $m = 2$, then $T$ is a matrix of size $n \times n$.

Let $H$ be a hypergraph. $H$ is called trivial if it contains only one vertex; otherwise, it is called nontrivial. $H$ is called linear if any two different edges intersect into at most one vertex. Let $v \in V(H)$, and let $E_v(H)$ denote the set of edges of $H$ that contains the vertex $v$. The degree $d_v(H)$ of $v$ in $H$ is the cardinality of $E_v(H)$. A vertex $v$ of $H$ is called a cored vertex if it has degree one. An edge $e$ of $H$ is called a pendant edge if it contains $|e| - 1$ cored vertices. A walk $W$ in $H$ is a sequence of alternate vertices and edges: $v_0e_1v_1e_2\ldots e_lv_l$, where $v_i \neq v_{i+1}$ and $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 0, 1, \ldots, l-1$. If $v_0 = v_l$, then $W$ is called a circuit, and is called a cycle if no vertices or edges are repeated except $v_0 = v_l$. The hypergraph $H$ is said to be connected if every two vertices are connected by a walk; $H$ is called a hypertree if it is connected and acyclic, and is called unicyclic if it contains exactly one cycle.

A matching $M$ of $H$ is a set of vertex-disjoint edges of $H$, and $M$ is called a perfect matching of $H$ if it covers all vertices of $H$. A multi-hypergraph is a hypergraph allowed to have multiple edges, and is called $m$-valent if each vertex has degree of multiple of $m$. A Veblen hypergraph is an $m$-uniform and $m$-valent multi-hypergraph. Throughout this paper, all hypergraphs are consider simple unless stated somewhere.

Hu, Qi and Shao [26] introduced a class of hypergraphs which are constructed from simple graph.

Definition 2.1. (26) Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For an integer $m \geq 3$, the $m$-th power of $G$, denoted by $G^m := (V^m, E^m)$, is defined to be the $m$-uniform hypergraph with vertex set $V^m = V \cup \{i_{e,1}, \ldots, i_{e,m-2} : e \in E\}$ and edge set $E^m = \{e \cup \{i_{e,1}, \ldots, i_{e,m-2}\} : e \in E\}$, where $i_{e,1}, \ldots, i_{e,m-2}$ are new vertices inserted to each edge $e \in E(G)$.

By Definition 2.1 the power $T^m$ of a tree $T$ is a hypertree, and the power $U^m$ of a unicyclic graph $U$ is a linear unicyclic hypergraph. Denote $P_n, C_n, S_n$ respectively a path, a cycle and a star all with $n$ edges (as simple graphs). Then $C_n^m$ is a linear cycle as hypergraph, and $P_n^m$ is called
a hyperpath, and \( S_n^m \) is called a hyperstar. The center of \( S_n \) or \( S_n^m \) is the vertex with maximum degree.

2.2. **Traces of hypergraphs.** Shao, Qi and Hu \cite{36} gave a graph interpretation for the \( d \)-th trace \( \text{Tr}_d(T) \). Let

\[
F_d = \{ ((i_1, \alpha_1), \ldots , (i_d, \alpha_d)) : i_1 \leq \cdots \leq i_d, \alpha_j \in [n]^{m-1} \},
\]

where \( i_j \) is called the primary index (or root) of the \( m \)-tuple \( f_j := (i_j, \alpha_j) \) for \( j \in [d] \). Define an \( i_j \)-rooted directed star for the tuple \( f_j \):

\[
S_{f_j}(i_j) = (V_j, \{(i_j, u_k) : k = 1, \ldots , m - 1 \}),
\]

where \( V_j = \{ i_j, u_1, \ldots , u_{m-1} \} \) is considered as a set by omitting multiple indices if \( \alpha_j = (u_1, \ldots , u_{m-1}) \). So we get a multi-directed graph associated with \( F \), denoted and defined as

\[
R(F) = \bigcup_{j=1}^{d} S_{f_j}(i_j).
\]

Denote by \( b(F) \) the product of the factorial of the multiplicities of all arcs of \( R(F) \), \( c(F) \) the product of the factorial of the outdegree of the all vertices of \( R(F) \), and \( W(F) \) the set of vertex sequences of all Euler tours of \( R(F) \). Shao, Qi and Hu \cite{36} proved that

\[
(2.1) \quad \text{Tr}_d(T) = (m - 1)^{n-1} \sum_{F \in F_d} \frac{b(F)}{c(F)} |\pi_F(T)| |W(F)|,
\]

where \( \pi_F(T) = \prod_{i=1}^{d} t_{i, \alpha_i} \) if \( F = ((i_1, \alpha_1), \ldots , (i_d, \alpha_d)) \).

Let \( \mathcal{H} \) be an \( m \)-uniform hypergraph on \( n \) vertices and let \( A(\mathcal{H}) \) be the adjacency tensor of \( \mathcal{H} \). Given an ordering of the vertices of \( \mathcal{H} \), let

\[
F_d(\mathcal{H}) := \{(e_1(v_1), \ldots , e_d(v_d)) : e_i \in E(\mathcal{H}), v_1 \leq \cdots \leq v_d \},
\]

be the set of \( d \)-tuples of ordered rooted edges, where \( e_i(v_i) \) is an edge \( e_i \) with root of \( v_i \in e_i \) for \( i \in [d] \). Define a rooted directed star \( S_{e_i}(v_i) = (e_i, \{(v_i, u) : u \in e_i \setminus \{v_i\} \}) \) for each \( i \in [d] \), and multi-directed graph \( R(F) = \bigcup_{i=1}^{d} S_{e_i}(v_i) \) associated with \( F \in F_d(\mathcal{H}) \). Let

\[
F_d^+(\mathcal{H}) := \{ F \in F_d(\mathcal{H}) : R(F) \text{ is Eulerian} \}.
\]

For an \( F \in F_d^+(\mathcal{H}) \), denote \( V(F) := V(R(F)) \), \( r_v(F) \) the number of edges in \( F \) with \( v \) as the root, and \( d^+_v(F) := (m - 1)r_v(F) \) (namely, the outdegree of \( v \) in \( R(F) \)). Denote by \( \tau(F) := \tau_u(R(F)) \) the number of arborescences of \( R(F) \) with root \( u \) (namely, a directed \( u \)-rooted spanning tree such that all vertices except \( u \) has a directed path from itself to \( u \), which is equal to the principal minor of the Laplacian matrix \( L(R(F)) \) of \( R(F) \) by deleting the row and column indexed by \( u \) \cite{38}. As \( R(F) \) is Eulerian, \( \tau_u(R(F)) \) is independent of the choice of the root \( u \) so that the root \( u \) is omitted. Fan et al. \cite{21} give an expression of the \( d \)-th trace of \( \mathcal{H} \) as follows.

**Lemma 2.2** \cite{21}. For an \( m \)-uniform hypergraph \( \mathcal{H} \) on \( n \) vertices,

\[
\text{Tr}_d(\mathcal{H}) = d(m - 1)^n \sum_{F \in F_d^+(\mathcal{H})} \frac{\tau(F)}{\prod_{v \in V(F)} d^+_v(F)}.
\]
For each $F \in \mathcal{F}_d^+(H)$, we get a multi-hypergraph induced by the edges in $F$ by omitting the roots, denoted by $\mathcal{V}_F$, which is an $m$-uniform and $m$-valent multi-hypergraph called Veblen hypergraph [6]. On the other side, given a Veblen hypergraph $H$, a rooting of $H$ is an ordering $F = (e_1(v_1), \ldots, e_t(v_t))$ of all edges of $H$, where $v_i$ is the root of $e_i$ for $i \in [t]$, and $v_1 \leq \cdots \leq v_t$ under the given order of the vertices of $H$. If $R(F)$ is Eulerian, then $F$ is called an Euler rooting of $H$; in this case, $H$ is called Euler rooted with each edge rooted as in $F$ by omitting the order. Note that an Euler rooted hypergraph may have more than one Euler rooting as a vertex can occur as roots of different edges. Denote by $R(H)$ the set of Euler rooting of $H$.

Denote by $\mathcal{V}_d^+(H)$ the set of connected Veblen hypergraphs with $d$ edges associated with $H$ as follows:

$$\mathcal{V}_d^+(H) = \bigcup_{G \in \mathcal{C}(H)} \{ \mathcal{V}_F : F \in \mathcal{F}_d^+(H), \mathcal{V}_F = G \},$$

where $\mathcal{C}(H)$ denotes the set of connected sub-hypergraphs of $H$, and $H$ denotes the underlying hypergraph of a multi-hypergraph $H$ which is obtained removing duplicate edges of $H$.

For a hypergraph $G$, denote

$$\mathcal{W}_d^+(G) = \{ \mathcal{V}_F : F \in \mathcal{F}_d^+(G), \mathcal{V}_F = G \},$$

that is, the set of Veblen hypergraph with $G$ as underlying hypergraph which has an Euler rooting. So,

$$\mathcal{V}_d^+(H) = \bigcup_{G \in \mathcal{C}(H)} \mathcal{W}_d^+(G).$$

For a Veblen hypergraph $H$, denote

$$C_H = \sum_{F \in \mathcal{R}(H)} \frac{\tau(F)}{\prod_{v \in V(F)} d^+_v(F)},$$

and

(2.2) $$\text{tr}_d(G) = \sum_{H \in \mathcal{W}_d^+(G)} C_H.$$ 

By Lemma 2.2, we get another expression of $\text{Tr}_d(H)$ as follows:

$$\text{Tr}_d(H) = d(m - 1)^n \sum_{H \in \mathcal{V}_d^+(H)} \sum_{F \in \mathcal{R}(H)} \frac{\tau(F)}{\prod_{v \in V(F)} d^+_v(F)}$$

(2.3) $$= d(m - 1)^n \sum_{G \in \mathcal{C}(H)} \sum_{H \in \mathcal{W}_d^+(G)} C_H$$

$$= d(m - 1)^n \sum_{G \in \mathcal{C}(H)} \text{tr}_d(G).$$

Lemma 2.3. Let $H$ be an $m$-uniform hypergraph on $n$ vertices. Then

$$\text{Tr}_d(H) = d(m - 1)^n \sum_{G \in \mathcal{C}(H)} \text{tr}_d(G).$$

At the end of this section, we give an inequality involved with combinatorial numbers, which will be used in the later proofs.
Lemma 2.4. Let \( x, y \) be positive integers, and \( a, b \) be nonnegative integers. Then

\[
(2.4) \quad (x + y + a)!b! + (x + y + b)!a! > (x + a)!(y + b)! + (x + b)!(y + a)!.
\]

Proof. By symmetry we can assume that \( a \geq b \) and \( x \geq y \). Then

\[
\begin{align*}
(x + y + a)!b! - (x + a)!(y + b)! + (x + y + b)!a! - (x + b)!(y + a)!
&= (x + a)!b! ((x + a + 1) \cdots (x + a + y) - (b + 1) \cdots (b + y)) \\
& \quad + (x + b)!a! ((a + 1) \cdots (a + y) - (x + b + 1) \cdots (x + b + y)).
\end{align*}
\]

So, if \( x + b \geq a \), then Eq. \((2.4)\) holds. Otherwise, we have

\[
(x + y + a)!b! - (x + a)!(y + b)! + (x + y + b)!a! - (x + b)!(y + a)!
= (x + a)!b! ((x + a + 1) \cdots (x + a + y) - (b + 1) \cdots (b + y))
- (x + b)!a! ((a + 1) \cdots (a + y) - (x + b + 1) \cdots (x + b + y)).
\]

It is easily seen that \((x + a)!b! \geq (x + b)!a!\) and

\[
(x + a + 1) \cdot (x + a + y) - (b + 1) \cdot (b + y) > (a + 1) \cdots (a + y) - (x + b + 1) \cdots (x + b + y).
\]

The result also follows. \( \square \)

3. Traces of Hypertrees

The coalescence of two nontrivial connected hypergraphs \( H_1 \) and \( H_2 \) is the hypergraph obtained by identifying one vertex \( v_1 \) of \( H_1 \) and one vertex \( v_2 \) of \( H_2 \) to produce a new vertex \( u \), denoted by \( H_1(v_1) \odot H_2(v_2) \), also written as \( H_1(u) \odot H_2(u) \). Let \( v_1' \) be another vertex of \( H_1 \) different from \( v_1 \). The coalescence \( H_1(v_1') \odot H_2(v_2) \) is called obtained from \( H_1(v_1) \odot H_2(v_2) \) by relocating \( H_1 \) from \( v_1 \) to \( v_1' \). A vertex \( u \) of a connected hypergraph \( H \) is called a cut vertex of \( H \) if \( H \) can be written as \( H_1(u) \odot H_2(u) \), where \( H_1, H_2 \) are both nontrivial and connected, called the branches of \( H \).

The following lemma gives a characterization of Veblen hypergraphs associated with a hypertree.

Lemma 3.1 (\([21]\)). Let \( H \) be an \( m \)-uniform Veblen multi-hypergraph whose underlying hypergraph \( H \) is a hypertree. Then \( H \) is uniquely Euler rooted such that all vertices of each edge occur as roots of the edge in a same number of times, and hence every edge of \( H \) repeats in a multiple of \( m \) times.

Let \( H \) be an \( m \)-uniform Veblen hypergraph with \( d \) edges whose underlying hypergraph \( \overline{H} \) is a hypertree. By Lemma 3.1 \( m \mid d \), and \( H \) can be expressed as a weighted hypertree \( \overline{H}(\omega) \), where

\[
\omega : E(\overline{H}) \to \mathbb{Z}^+,
\]

such that the multiplicity of an edge \( e \in E(H) \) is \( m\omega(e) \), and \( \omega(\overline{H}) := \sum_{e \in E(\overline{H})} \omega(e) = d/m \). So,

\[
\mathcal{W}_d(\overline{H}) = \{ \overline{H}(\omega) : \omega(\overline{H}) = d/m \},
\]
and \( \mathcal{W}_d(\hat{T}) \neq \emptyset \) if and only if \( m \mid d \). Fan et al. \cite{22} proved that
\[
C_{T(\omega)} = (m - 1)^{-|V(\hat{T})|} m^{(m-2)|E(\hat{T})|} \left( \prod_{e \in E(\hat{T})} \omega(e) \right)^{m-1} \prod_{v \in V(\hat{T})} \frac{(d_v(\hat{T}(\omega)) - 1)!}{\prod_{e \in E(\hat{T}) \cap \{e\}} \omega(e)!},
\]
where \( d_v(\hat{T}(\omega)) = \sum_{e \in V(\hat{T})} \omega(e) \), the weighted degree of the vertex \( v \) in \( \hat{T}(\omega) \).

Let \( \mathcal{T} \) be an \( m \)-uniform hypertree. By Lemma \cite{3.1}, for \( d \in \mathbb{Z}^+ \), \( \mathcal{V}_d(\mathcal{T}) \neq \emptyset \) if and only if \( m \mid d \). By above discussion, we have
\[
\mathcal{V}_d(\mathcal{T}) = \bigcup_{\hat{T} \in \mathcal{E}(\mathcal{T})} \mathcal{W}_d(\hat{T}) = \bigcup_{\hat{T} \in \mathcal{E}(\mathcal{T})} \{ \hat{T}(\omega) : \omega(\hat{T}) = d/m \}.
\]
Note that \( \prod_{v \in V(\hat{T})} \prod_{e \in E(\hat{T})} \omega(e)! = \prod_{e \in E(\hat{T})} (\omega(e)!)^m \). By Lemma \cite{2.3} Eqs. \cite{3.1} and \cite{3.2}, we get an expression of the traces of the hypertrees.

**Theorem 3.2.** Let \( \mathcal{T} \) be an \( m \)-uniform hypertree. If \( m \mid d \), then
\[
(3.3) \quad Tr_d(\mathcal{T}) = d(m-1)^{|V(\hat{T})|} \sum_{\hat{T} \in \mathcal{E}(\mathcal{T})} \text{tr}_d(\hat{T}), \quad \text{tr}_d(\hat{T}) = \sum_{\omega(\hat{T}) = d/m} C_{\hat{T}(\omega)},
\]
where
\[
(3.4) \quad C_{\hat{T}(\omega)} = (m - 1)^{-|V(\hat{T})|} m^{(m-2)|E(\hat{T})|} \left( \prod_{v \in V(\hat{T})} \frac{(d_v(\hat{T}(\omega)) - 1)!}{(\omega(e)!)^m} \right) \prod_{e \in E(\hat{T})} \omega(e)^{m-1};
\]
otherwise, \( Tr_d(\mathcal{T}) = \text{tr}_d(\hat{T}) = 0 \).

Let \( \mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_p \) be pairwise disjoint connected hypergraphs, where \( p \geq 1 \). Let \( v_1, \ldots, v_p \in V(\mathcal{H}_0) \), and \( u_i \in V(\mathcal{H}_i) \) for \( i \in [p] \). Denote by \( \mathcal{H}_0(v_1, \ldots, v_p) \circ (\mathcal{H}_1(u_1), \ldots, \mathcal{H}_p(u_p)) \) the hypergraph obtained from \( \mathcal{H}_0 \) by attaching \( \mathcal{H}_1, \ldots, \mathcal{H}_p \) to \( \mathcal{H}_0 \) with \( u_i \) identified with \( v_i \) for each \( i \in [p] \), also written as \( \mathcal{H}_0(v_1, \ldots, v_p) \circ (\mathcal{H}_1(v_1), \ldots, \mathcal{H}_p(v_p)) \). Let \( \mathcal{H} \) be a hypergraph, and \( \mathcal{H}_i \) be a sub-hypergraph of \( \mathcal{H} \) for \( i \in [s+t] \). Denote by \( \mathcal{C}(\mathcal{H}; \mathcal{H}_1, \ldots, \mathcal{H}_s, \mathcal{H}_s^\times, \ldots, \mathcal{H}_{s+t}^\times) \) the set of connected sub-hypergraphs of \( \mathcal{H} \) which contain the edges of \( \mathcal{H}_1, \ldots, \mathcal{H}_s \) and contain no edges of \( \mathcal{H}_{s+1}, \ldots, \mathcal{H}_{s+t} \), where \( s, t \) are nonnegative integers. Let \( S \subseteq V(\mathcal{H}) \). Denote \( \mathcal{H} - S \) the sub-hypergraph of \( \mathcal{H} \) obtained by deleting the vertices \( S \) together with the edges containing the vertices of \( S \).

![Figure 3.1](image-url) Figure 3.1. The hypergraph \( \mathcal{H} \) in Lemma \cite{3.3}
Lemma 3.3. Let $P_3$ be a path on vertices $v_0, v_1, v_2, v_3$ with edges $\{v_{i-1}, v_i\}$ for $i = 1, 2, 3$. Let $P_3^m$ be the power of $P_3$ with edges $e_i = \{v_{i-1}, v_i, u_{i1}, \ldots, u_{im-2}\}$ for $i = 1, 2, 3$. Let $\mathcal{H} = P_3^m(v_1, u_{21}, \ldots, u_{m-2}) \circ (\mathcal{T}_1(v_1), \mathcal{T}_1(u_{21}), \ldots, \mathcal{T}_1(u_{m-2}))$, where $\mathcal{T}_1$ is a nontrivial $m$-uniform hypertree, and $\mathcal{T}_i$ is an $m$-uniform hypertree allowed to be trivial with only one vertex $u_{2i}$ for $i \in [m-2]$; see Fig. 3.1. Let $\mathcal{T}(w)$ be a nontrivial $m$-uniform hypertree with root $w$. Then

\[ (3.5) \quad \text{Tr}_d(\mathcal{H}(v_1) \circ \mathcal{T}(w)) \geq \text{Tr}_d(\mathcal{H}(v_2) \circ \mathcal{T}(w)), \]

with strict inequality if $m \mid d$ and $d/m \geq 2$.

Proof. Let $\mathcal{H}_1 := \mathcal{H}(v_1) \circ \mathcal{T}(w)$ and $\mathcal{H}_2 := \mathcal{H}(v_2) \circ \mathcal{T}(w)$. We first give decompositions of $\mathcal{C}(\mathcal{H}_1)$ and $\mathcal{C}(\mathcal{H}_2)$ as follows.

\[ \mathcal{C}(\mathcal{H}_1) = \mathcal{C}(\mathcal{H}_1; \mathcal{T}_1^*) \cup \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1^*) \cup \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1), i = 1, 2. \]

Furthermore, as all hypergraphs in $\mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1)$ are connected,

\[ \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1) = \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2) \cup \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2^*). \]

It is easily seen that $\mathcal{C}(\mathcal{H}_1; \mathcal{T}_1^*) = \mathcal{C}(\mathcal{H}_2; \mathcal{T}_1^*)$. There is an isomorphism $\phi$ between $\mathcal{H}_1 - (V(\mathcal{T}_1) \setminus \{v_1\})$ and $\mathcal{H}_2 - (V(\mathcal{T}_1) \setminus \{v_1\})$ which maps $v_0$ to $v_3$, $v_1$ to $v_2$. So, there is bijection from $\mathcal{C}(\mathcal{H}_1; \mathcal{T}_1^*)$ to $\mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1^*)$ such that $\mathcal{G}$ is mapped to $\phi|_{\mathcal{G}}(\mathcal{G})$, and $\mathcal{G}$ is isomorphic to $\phi|_{\mathcal{G}}(\mathcal{G})$.

By Lemma 2.3 and the above decompositions, we have

\[ (3.6) \quad \text{Tr}_d(\mathcal{H}_1) - \text{Tr}_d(\mathcal{H}_2) = d(m-1)^{\mathcal{C}(\mathcal{H}_1)} \left( \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2)} \text{tr}_d(\mathcal{G}) - \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1, e_2)} \text{tr}_d(\mathcal{G}) + \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2^*)} \text{tr}_d(\mathcal{G}) \right). \]

We will prove that

\[ (3.7) \quad \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2)} \text{tr}_d(\mathcal{G}) \geq \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1, e_2)} \text{tr}_d(\mathcal{G}), \]

with strict inequality if $m \mid d$ and $d/m \geq 3$.

By Lemma 3.2 we assume that $m \mid d$; otherwise $\text{tr}_d(\mathcal{G}) = 0$. Note that $\mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1, e_2)$ or $\mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2)$ is nonempty if and only if $d/m \geq 3$. So we also assume that $d/m \geq 3$. For each $\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1, e_2)$, we can write $\mathcal{G} = \mathcal{H}'(v_2) \circ \mathcal{T}(w)$, where $\mathcal{H}'$ is a sub-hypergraph of $\mathcal{H}$ which contains $e_2$ and a rooted sub-hypertree $\mathcal{T}_1'(v_1)$ of $\mathcal{T}_1$, and $\mathcal{T}(w)$ is a rooted sub-hypertree of $\mathcal{T}$. There is a bijection $\psi$ between $\mathcal{C}(\mathcal{H}_2; \mathcal{T}, \mathcal{T}_1, e_2)$ and $\mathcal{C}(\mathcal{H}_1; \mathcal{T}, \mathcal{T}_1, e_2)$ such that

\[ \psi(\mathcal{G}) = \psi(\mathcal{H}'(v_2) \circ \mathcal{T}(w)) = \mathcal{H}'(v_1) \circ \mathcal{T}(w) = \tilde{\mathcal{G}}. \]

Also, each weight function $\omega : E(\mathcal{G}) \to \mathbb{Z}^+$ is naturally associated with a weight function $\tilde{\omega} : E(\tilde{\mathcal{G}}) \to \mathbb{Z}^+$ such that $\omega|_{E(\mathcal{H})} = \tilde{\omega}|_{E(\mathcal{H}')} = \tilde{\omega}|_{E(\mathcal{T}(w))}$, and there is a bijection between $\mathcal{W}_d(\mathcal{G})$ and $\mathcal{W}_d(\tilde{\mathcal{G}})$ such that $\mathcal{G}(\omega)$ is mapped to $\tilde{\mathcal{G}}(\tilde{\omega})$.

Let $d_{v_1}(\mathcal{T}_1'(v_1)) = x$ and $d_{v_2}(\mathcal{T}(w)) = y$. We divide the discussion into cases.
Case 1: $e_1 \notin E(\mathcal{G})$ and $e_3 \notin E(\mathcal{G})$. By Theorem 3.2,

$$C_{\tilde{G}}(\omega) = (x + \omega(e_2) - 1)!(y + \omega(e_2) - 1)!f_{\tilde{G};v_1,v_2}(\omega),$$

$$C_{\tilde{G}}(\tilde{\omega}) = (x + y + \omega(e_2) - 1)!(\omega(e_2) - 1)!f_{\tilde{G};v_1,v_2}(\tilde{\omega}),$$

where $f_{\tilde{G};v_1,v_2}(\omega) = f_{\tilde{G};v_1,v_2}(\tilde{\omega})$ and

$$f_{\tilde{G};v_1,v_2}(\omega) := (m - 1)^{-|V(\mathcal{G})|}m^{(m-2)|E(\mathcal{G})|} \prod_{v \in V(\mathcal{G}) \setminus \{v_1,v_2\}} (d_v(\mathcal{G}(\omega)) - 1)! \prod_{e \in E(\mathcal{G})} \frac{\omega(e)^{m-1}}{(\omega(e))^{m}}.$$

So we have $C_{\tilde{G}}(\tilde{\omega}) > C_{\tilde{G}}(\omega)$, and $\text{tr}_d(\tilde{G}) > \text{tr}_d(\mathcal{G})$, which implies that

$$\sum_{\mathcal{G} \in \mathcal{C}'(H_2;T,T_1,e_1,e_2,e_3)} \text{tr}_d(\tilde{G}) > \sum_{\mathcal{G} \in \mathcal{C}'(H_2;T,T_1,e_1,e_2,e_3)} \text{tr}_d(\mathcal{G}).$$

Case 2: $e_1 \in E(\mathcal{G})$ and $e_3 \notin E(\mathcal{G})$, or $e_1 \notin E(\mathcal{G})$ and $e_3 \in E(\mathcal{G})$. Let $\mathcal{G}[e_1] \in \mathcal{C}'(H_2;T,T_1,e_1,e_2,e_3^x)$. Let $\mathcal{G}[e_3] \in \mathcal{C}'(H_2;T,T_1,e_2,e_3^x,e_3)$, which is obtained from $\mathcal{G}[e_1]$ by relocating the edge $e_1$ from $v_1$ to $v_2$ and labelling the edge $e_1$ as $e_3$. So, there is a bijection from $\mathcal{C}'(H_2;T,T_1,e_2,e_3^x)$ to $\mathcal{C}'(H_2;T,T_1,e_2,e_3^x,e_3)$ which maps $\mathcal{G}[e_1]$ to $\mathcal{G}[e_3]$. Also, by the map $\psi$ defined before, $\psi$ is a bijection between $\mathcal{C}'(H_2;T,T_1,e_2,e_3^x) \in \mathcal{C}'(H_1;T_1,e_2,e_3^x,e_3)$ and $\mathcal{C}'(H_2;T,T_1,e_2,e_3^x,e_3)$ which maps $\mathcal{G}[e_3]$ to $\tilde{\mathcal{G}}[e_3] := \psi(\mathcal{G}[e_3])$.

Each weight function $\omega : E(\mathcal{G}[e_1]) \to \mathbb{Z}^+$ induces a weight function $\omega' : E(\mathcal{G}[e_3]) \to \mathbb{Z}^+$ such that $\omega'(e_3) = \omega(e_1)$ and $\omega'|_{E(\mathcal{G}[e_3]) \setminus \{e_3\}} = \omega|_{E(\mathcal{G}[e_1]) \setminus \{e_1\}}$. As defined before, $\omega$ and $\omega'$ induce $\tilde{\omega}$ defined on $E(\tilde{\mathcal{G}}[e_1])$ and $\omega'$ defined on $E(\tilde{\mathcal{G}}[e_3])$. By Lemma 3.2, we have

$$C_{\tilde{G}[e_3]}(\omega') = (x + \omega(e_1) + \omega(e_2) - 1)!(y + \omega(e_2) - 1)!f_{\tilde{G}[e_1];v_1,v_2}(\omega'),$$

$$C_{\tilde{G}[e_1]}(\omega) = (x + \omega(e_2) - 1)!(y + \omega(e_1) + \omega(e_2) - 1)!f_{\tilde{G}[e_3];v_1,v_2}(\omega'),$$

$$C_{\tilde{G}[e_1]}(\omega') = (x + y + \omega(e_1) + \omega(e_2) - 1)!f_{\tilde{G}[e_3];v_1,v_2}(\omega'),$$

$$C_{\tilde{G}[e_3]}(\omega) = (x + y + \omega(e_2) - 1)!f_{\tilde{G}[e_1];v_1,v_2}(\omega'),$$

where $f_{\tilde{G}[e_1];v_1,v_2}(\omega') = f_{\tilde{G}[e_3];v_1,v_2}(\omega') = f_{\tilde{G}[e_1];v_1,v_2}(\tilde{\omega}') = f_{\tilde{G}[e_3];v_1,v_2}(\tilde{\omega}')$ as defined in Eq. (3.8). By Lemma 2.4

$$C_{\tilde{G}[e_1]}(\tilde{\omega}) + C_{\tilde{G}[e_3]}(\omega') > C_{\tilde{G}[e_1]}(\omega) + C_{\tilde{G}[e_3]}(\omega').$$

So

$$\text{tr}_d(\tilde{G}[e_1]) + \text{tr}_d(\tilde{G}[e_3]) > \text{tr}_d(\tilde{G}[e_1]) + \text{tr}_d(\tilde{G}[e_3]),$$

which implies that

$$\sum_{\tilde{\mathcal{G}} \in \mathcal{C}'(H_1;T,T_1,e_2,e_3^x,e_3)} \text{tr}_d(\tilde{\mathcal{G}}) > \sum_{\tilde{\mathcal{G}} \in \mathcal{C}'(H_2;T,T_1,e_2,e_3^x,e_3)} \text{tr}_d(\tilde{\mathcal{G}}).$$

Case 3: $e_1 \in E(\mathcal{G})$ and $e_3 \in E(\mathcal{G})$. As noted before, each hypergraph $\mathcal{G} \in \mathcal{C}'(H_2;T,T_1,e_2,e_1,e_3)$ is bijectively corresponding to $\tilde{\mathcal{G}} := \psi(\mathcal{G}) \in \mathcal{C}'(H_1;T,T_1,e_2,e_1,e_3)$ and each weight function $\omega$ on $E(\mathcal{G})$ induces a weight function $\tilde{\omega}$ on $E(\tilde{\mathcal{G}})$. Also, each $\omega$ (respectively, $\tilde{\omega}$) is associated with
another weight function $\omega'$ (respectively, $\tilde{\omega}'$) only by swapping the weight of $e_1$ and the weight of $e_3$. By Lemma 3.2 we have

$$C_{\tilde{G}(\omega)} = (x + \omega(e_1) + \omega(e_2) - 1)! (y + \omega(e_2) + \omega(e_3) - 1)! f_{\tilde{G};v_1,v_2}(\omega),$$

$$C_{\tilde{G}(\omega')} = (x + \omega(e_2) + \omega(e_3) - 1)! (y + \omega(e_1) + \omega(e_2) - 1)! f_{\tilde{G};v_1,v_2}(\omega'),$$

$$C_{\tilde{G}(\tilde{\omega})} = (x + y + \omega(e_1) + \omega(e_2) - 1)! (\omega(e_2) + \omega(e_3) - 1)! f_{\tilde{G};v_1,v_2}(\tilde{\omega}),$$

$$C_{\tilde{G}(\tilde{\omega}')} = (x + \omega(e_2) + \omega(e_3) - 1)! (y + \omega(e_1) + \omega(e_2) - 1)! f_{\tilde{G};v_1,v_2}(\tilde{\omega}'),$$

where $f_{\tilde{G};v_1,v_2}(\omega) = f_{\tilde{G};v_1,v_2}(\omega') = f_{\tilde{G};v_1,v_2}(\tilde{\omega}) = f_{\tilde{G};v_1,v_2}(\tilde{\omega}')$ as defined in Eq. (3.8). By Lemma 2.4,

$$C_{\tilde{G}(\tilde{\omega})} + C_{\tilde{G}(\tilde{\omega}') > C_{\tilde{G}(\omega)} + C_{\tilde{G}(\omega')}},$$

So

$$\text{tr}_d(\tilde{G}) - \text{tr}_d(G) = \sum_{\omega, \omega(G)=d/m} \left( C_{\tilde{G}(\tilde{\omega})} - C_{\tilde{G}(\omega)} \right)$$

$$(3.11) = \frac{1}{2} \sum_{\omega, \omega(G)=d/m} \left( (C_{\tilde{G}(\tilde{\omega})} + C_{\tilde{G}(\tilde{\omega}')} - (C_{\tilde{G}(\omega)} + C_{\tilde{G}(\omega')}) \right) > 0,$$

which implies that

$$\sum_{\tilde{G} \in \mathcal{E}(\mathcal{H}_1;\mathcal{T},T_1,e_1,e_3)} \text{tr}_d(\tilde{G}) > \sum_{\tilde{G} \in \mathcal{E}(\mathcal{H}_2;\mathcal{T},T_1,e_2,e_3)} \text{tr}_d(\tilde{G}).$$

Combining Eqs. (3.9), (3.10) and (3.12), we arrive at the inequality in Eq. (3.7). Note that $\text{tr}_d(\tilde{G}) \geq 0$ for each $\tilde{G} \in \mathcal{E}(\mathcal{H}_1;\mathcal{T},T_1,e_2)$, with strict inequality if and only if $m \mid d$ and $d/m \geq 2$ as in this case $\mathcal{E}(\mathcal{H}_1;\mathcal{T},T_1,e_2) \neq \emptyset$ by Lemma 3.1. So the result follows by Eq. (3.6). \qed

We note that the method to prove the inequality (3.11) may be called “symmetric sum”, which will be used in later discussion, e.g. Lemma 6.3 and Lemma 6.4.

4. Traces of Linear Unicyclic Hypergraphs

In this section we will discuss the traces of linear unicyclic hypergraphs. We need the following lemma to characterize the Veblen hypergraph containing cycles.

**Lemma 4.1** ([21]). Let $H$ be an $m$-uniform Veblen multi-hypergraph, and let $e$ be an edge of $H$ which contains a cored vertex. If $H$ has an Euler rooting, then $e$ repeats $k \cdot m$ times for some positive integer $k$, and all cored vertices in $e$ occur as a root of $e$ in $k$ times.

Let $U$ be an $m$-uniform linear unicyclic hypergraph. As $U$ is linear, $U$ contains a cycle $C_n^m$ of length $n \geq 3$, and $U$ is obtained from $C_n^m$ by some hypertrees to the vertices of $C_n^m$, namely $U = C_n^m(v_1, \ldots, v_p) \odot (T_1(v_1), \ldots, T_p(v_p))$, where $T_i$'s are hypertrees for $i \in [p]$ and $p$ is a nonnegative integer.
We have a decomposition of the Veblen hypergraph associated with $\mathcal{U}$:

\begin{equation}
\mathcal{V}_d(\mathcal{U}) = \mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\}) \cup \mathcal{V}_d(\mathcal{U}, \{C_n^m\}),
\end{equation}

where $\mathcal{V}_d(U^m; \{\hat{C}_n^m\})$ (respectively, $\mathcal{V}_d(U^m; \{C_n^m\})$) denotes the subset of $\mathcal{V}_d(U^m)$ which consists of Veblen hypergraphs that contain no $C_n^m$ (respectively, contain $\hat{C}_n^m$).

For each $H \in \mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\})$, $\mathcal{H}$ is a hypertree. By Lemma 3.1, $\mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\}) \neq \emptyset$ if and only if $m \mid d$; and in this case

\begin{equation}
\mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\}) = \{ \hat{T}(\omega) : \hat{T} \in \mathcal{C}^*_{tree}(\mathcal{U}), \omega(\hat{T}) = d/m \},
\end{equation}

where $\mathcal{C}^*_{tree}(\mathcal{U})$ denoted the connected sub-hypergraphs of $\mathcal{U}$ which are hypertrees.

For each $H \in \mathcal{V}_d(\mathcal{U}, \{C_n^m\})$, $\mathcal{H}$ contains $C_n^m$. We may assume that $H = C_n^m(v_1, \ldots, v_q) \odot (T_1^0(v_1), \ldots, T_q^0(v_q))$, where $T_i^0(v_i)$ is a subhypertree of $T_i(v_i)$ with root $v_i$ for $i \in [q]$ and $q \leq p$. As each $v_i$ is a cut vertex, $H|_{T_i^0(v_i)}$, the limitation of $H$ on the vertex set of $T_i^0(v_i)$, is still a Veblen hypergraph (hypertree). By Lemma 3.1, the number of edges of $H|_{T_i^0(v_i)}$, denoted by $d_i$, is a multiple of $m$, and $H|_{T_i^0(v_i)} = T_i^0(v_i)(\omega^i)$ with $\omega^i(T_i^0(v_i)) = d_i/m$. So, $H|_{C_n^m}$ is a Veblen hypergraph. By Lemma 4.1, as each edge of $\mathcal{C}^*_n$, contains cored vertices, the number of edges of $H|_{C_n^m}$, denoted by $d_0$, is a multiple of $m$, and $H|_{C_n^m} = C_n^m(\omega^0)$ with $\omega^0(C_n^m) = d_0/m$. So $H$ is a weighted hypergraph $\mathcal{H}(\omega)$ with weight

\[ \omega : E(\mathcal{H}) \rightarrow \mathbb{Z}^+ \]

such that each edge $e$ of $H$ has multiplicity $m\omega(e)$, and $\omega(H) = d/m$. By above discussion, we find that $\mathcal{V}_d(\mathcal{U}, \{C_n^m\}) \neq \emptyset$ if and only if $m \mid d$; and in this case

\begin{equation}
\mathcal{V}_d(\mathcal{U}, \{C_n^m\}) = \{ \mathcal{G}(\omega) : \mathcal{G} \in \mathcal{C}^*_n(\mathcal{U}), \omega(\mathcal{G}) = d/m \},
\end{equation}

and for $\mathcal{G} \in \mathcal{C}^*_n(\mathcal{U})$,

\begin{equation}
\mathcal{V}_d(\mathcal{G}) = \{ \mathcal{G}(\omega) : \omega(\mathcal{G}) = d/m \},
\end{equation}

where $\mathcal{C}^*_n(\mathcal{U})$ denotes the connected sub-hypergraphs of $\mathcal{U}$ which contains the cycle $C_n^m$.

**Corollary 4.2.** Let $\mathcal{U}$ be a linear unicyclic hypergraph which contains a cycle $C_n^m$. Then

\begin{equation}
\mathcal{V}_d(\mathcal{U}) = \mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\}) \cup \mathcal{V}_d(\mathcal{U}, \{C_n^m\}),
\end{equation}

and each of $\mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\})$ and $\mathcal{V}_d(\mathcal{U}, \{C_n^m\})$ is nonempty if and only if $m \mid d$, where $\mathcal{V}_d(\mathcal{U}, \{\hat{C}_n^m\})$ and $\mathcal{V}_d(\mathcal{U}, \{C_n^m\})$ are defined in Eqs. (4.2) and (4.3) respectively.

Let $\mathcal{G}(\omega) \in \mathcal{V}_d(\mathcal{U}, \{C_n^m\})$ with $\mathcal{G} = C_n^m(v_1, \ldots, v_q) \odot (T_1^0(v_1), \ldots, T_q^0(v_q))$. As discussed above,

\[ \mathcal{G}(\omega) = C_n^m(\omega^0)(v_1, \ldots, v_q) \odot (T_1^0(\omega^1)(v_1), \ldots, T_q^0(\omega^q)(v_q)) \]

Note that $\mathcal{G}(\omega)$ is Euler rooted if and only if $C_n^m(\omega^0)$ and all of $T_1^0(\omega^1), \ldots, T_q^0(\omega^q)$ are Euler rooted.
Suppose that $C_n$ has vertices $u_1, \ldots, u_n$ and edges $\{u_i, u_{i+1}\}$ for $i \in [n]$, $u_{n+1} = u_1$. We label the edges of $C^m_n$ as $e_i = \{u_i, u_{i+1}, w_1, \ldots, w_{i-m-2}\}$ for $i \in [n]$. Denote $\omega_0^i = \omega_0(e_i)$ for $i \in [n]$, and $\omega_{\min}^0 = \min_{i \in [n]} \omega_0(e_i)$. As shown in [22], the set of Euler rootings of $C^m_n(\omega^0)$ has a decomposition:

$$\mathcal{R}(C^m_n(\omega^0)) = \bigcup_{x=0}^{2\omega_{\min}^0} \mathcal{R}(C^m_n(\omega^0); x),$$

where $\mathcal{R}(C^m_n(\omega^0); x)$ consists of those rootings $F$ such that $u_i$ acts a root of $e_i$ (respectively, $e_{i-1}$) in $\omega_0^i - \omega_{\min}^0 + x$ (respectively, $\omega_0^i - \omega_{\min}^0 + x$) times for $i \in [n]$, and each vertex of $e_i \setminus \{u_i, u_{i+1}\}$ acts as a root of $e_i$ in $\omega(e_i)$ times. So, $r_v(F) = d_v(C^m_n(\omega^0))$ for each $F \in \mathcal{R}(C^m_n(\omega^0); x)$. Define $\Omega_{C_n}(\omega^0)$ as follows for the later discussion:

$$(4.6) \quad \Omega_{C_n}(\omega^0) = \sum_{x=0}^{2\omega_{\min}^0} \prod_{i=1}^{n} \left( \frac{(\omega_0^i)!^2}{(\omega_0^i - \omega_{\min}^0 + x)!} \right) \sum_{l=0}^{n-1} \prod_{i=1}^{l} (\omega_0^i + \omega_{\min}^0 - x) \prod_{i=l+2}^{n} (\omega_0^i - \omega_{\min}^0 + x).$$

Consequently, we have a decompositon

$$\mathcal{R}(\mathcal{G}(\omega)) = \bigcup_{x=0}^{2\omega_{\min}^0} \mathcal{R}(\mathcal{G}(\omega); x),$$

where

$$\mathcal{R}(\mathcal{G}(\omega); x) = \{(F_0, F_1, \ldots, F_q) : F_0 \in \mathcal{R}(C^m_n(\omega^0); x), F_i \in \mathcal{R}(\mathcal{T}_i'(\omega^i)), i \in [q]\}.$$

By the formula given in [22], for each $(F_0, F_1, \ldots, F_q) \in \mathcal{R}(\mathcal{G}(\omega); x),$

$$\tau(F_0) = 2m^{n(m-2)-1} \left( \prod_{i=1}^{n} \omega_0^i \right)^{m-2} \sum_{l=0}^{n-1} \prod_{i=1}^{l} (\omega_0^i + \omega_{\min}^0 - x) \prod_{i=l+2}^{n} (\omega_0^i - \omega_{\min}^0 + x),$$

$$\tau(F_i) = m^{(m-2)|E(\mathcal{T}_i')|} \left( \prod_{e \in E(\mathcal{T}_i')} \omega(e) \right)^{m-1}, \quad i \in [q],$$

and hence $\tau(F) = \tau(F_0) \prod_{i=1}^{q} \tau(F_i)$, which equals

$$(4.7) \quad \tau(F) = 2m^{(m-2)|E(\mathcal{G})|-1} \left( \prod_{e \in E(\mathcal{G})} \omega(e) \right)^{m-1} \sum_{l=0}^{n-1} \prod_{i=1}^{l} (\omega_0^i + \omega_{\min}^0 - x) \prod_{i=l+2}^{n} (\omega_0^i - \omega_{\min}^0 + x).$$

Note that by Lemma 3.1, for each $F_i \in \mathcal{R}(\mathcal{T}_i'(\omega^i))$, every vertex $v \in e \in \mathcal{T}_i'(\omega^i)$ acts as root of $e$ in $\omega^i(e)$ times. So, we have

$$|\mathcal{R}(\mathcal{G}(\omega); x)| = \prod_{v \in V(\mathcal{G})} \left( r_v(F) \right) = \prod_{v \in V(\mathcal{G})} \frac{r_v(F)!}{\prod_{e \in E_v(\mathcal{G})} r_v(e)!} = \prod_{v \in V(\mathcal{G})} \frac{d_v(\mathcal{G}(\omega))!}{\prod_{e \in E_v(\mathcal{G})} r_v(e)!}.$$
As

\[ \prod_{v \in V(G)} d_v(G(\omega))! = \prod_{e \in E(G)} \prod_{v \in e} r_v(e)! = \prod_{e \in U_i \in |E(G)|} \prod_{v \in e} r_v(e)! \cdot \prod_{e \in E(C_m)} \prod_{v \in e} r_v(e)! \]

\[ = \prod_{e \in U_i \in E(T')} (\omega(e))!^m \cdot \prod_{e \in E(C_m)} (\omega(e))^{m-2} \cdot \prod_{i=1}^n (\omega_i^0 + \omega_{\min}^0 - x)! (\omega_i^0 - \omega_{\min}^0 + x)! \]

\[ = \prod_{e \in E(G)} (\omega(e))!^m \cdot \prod_{i=1}^n (\omega_i^0 + \omega_{\min}^0 - x)! (\omega_i^0 - \omega_{\min}^0 + x)! \]

we have

\[ (4.8) \quad |\mathcal{A}(G(\omega); x)| = \frac{d_v(G(\omega))! \prod_{i=1}^n (\omega_i^0)!^2}{\prod_{e \in E(G)} (\omega(e))!^m \prod_{i=1}^n (\omega_i^0 + \omega_{\min}^0 - x)! (\omega_i^0 - \omega_{\min}^0 + x)!}. \]

Note that for each \( F \in \mathcal{A}(G(\omega); x) \),

\[ \prod_{v \in V(F)} d_v^+(F) = (m-1)^{|V(G)|} \prod_{v \in V(G)} d_v(G(\omega)). \]

By definition, we have

\[ C_{G(\omega)} = \sum_{F \in \mathcal{A}(G(\omega))} \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)} = \frac{2\omega_{\min}}{\sum_{x=0}^{2\omega_{\min}} \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)}} \]

\[ = \sum_{x=0}^{2\omega_{\min}} |\mathcal{A}(G(\omega); x)| \cdot \frac{\tau(F)}{\prod_{v \in V(F)} d_v^+(F)} \]

\[ = 2(m-1)^{|V(G)|} m (m-2)^{|E(G)|-1} \prod_{v \in V(G)} (d_v(G(\omega)) - 1)! \prod_{e \in E(G)} \frac{\omega(e)^{m-1}}{(\omega(e))!^m} \cdot \Omega C_m^m (\omega^0). \]

We are now giving an expression for the traces of linear unicyclic hypergraphs.

**Theorem 4.3.** Let \( \mathcal{U} \) be an \( m \)-uniform linear unicyclic hypergraph. If \( m \mid d \), then

\[ (4.10) \quad \text{Tr}_d(\mathcal{U}) = d(m-1)^{|V(\mathcal{U})|} \left( \sum_{\mathcal{T} \in \mathcal{E}_{\text{tree}}(\mathcal{U})} \text{tr}_d(\mathcal{T}) + \sum_{\mathcal{G} \in \mathcal{E}_{\text{cyclic}}(\mathcal{U})} \text{tr}_d(\mathcal{G}) \right), \]

and

\[ (4.11) \quad \text{tr}_d(\mathcal{T}) = \sum_{\omega: \omega(\mathcal{T}) = d/m} C_{\mathcal{T}(\omega)}; \quad \text{tr}_d(\mathcal{G}) = \sum_{\omega: \omega(\mathcal{G}) = d/m} C_{\mathcal{G}(\omega)}, \]

where \( C_{\mathcal{T}(\omega)} \) and \( C_{\mathcal{G}(\omega)} \) are defined in Eqs. (3.4) and (4.9) respectively; otherwise, \( \text{Tr}_d(\mathcal{U}) = 0 \).
**Lemma 4.4.** Let \( \mathcal{U} \) be an \( m \)-uniform linear unicyclic hypergraph which contains a cycle \( C^m_n \), and let \( e \) be an edge of \( C^m_n \) which contains a vertex \( v \) of degree 2 and a vertex \( u \) with degree 1. Let \( \mathcal{T} \) be an \( m \)-uniform nontrivial hypertree with root \( w \). Then

\[
\text{Tr}_d(\mathcal{U}(v) \circ \mathcal{T}(w)) \geq \text{Tr}_d(\mathcal{U}(u) \circ \mathcal{T}(w))
\]

with equality if \( d/m \) is an integer and \( d/m \geq 2 \).

**Proof.** Let \( \mathcal{H}_1 := \mathcal{U}(v) \circ \mathcal{T}(w) \) and \( \mathcal{H}_2 := \mathcal{U}(u) \circ \mathcal{T}(w) \). We have

\[
\mathcal{C}(\mathcal{H}_i) = \mathcal{C}(\mathcal{H}_i; \mathcal{T}^\times) \cup \mathcal{C}(\mathcal{H}_i; \mathcal{T}, (C^m_n)^\times) \cup \mathcal{C}(\mathcal{H}_i; \mathcal{T}, C^m_n), \ i = 1, 2.
\]

Furthermore, as all hypergraphs in \( \mathcal{C}(\mathcal{H}_i; \mathcal{T}, \mathcal{T}_i) \) are connected,

\[
\mathcal{C}(\mathcal{H}_1; \mathcal{T}, C^m_n) = \mathcal{C}(\mathcal{H}_1; \mathcal{T}, C^m_n, e) \cup \mathcal{C}(\mathcal{H}_1; \mathcal{T}, C^m_n, e^\times), \ \mathcal{C}(\mathcal{H}_2; \mathcal{T}, C^m_n) = \mathcal{C}(\mathcal{H}_2; \mathcal{T}, C^m_n, e).
\]
It is easily seen that $\mathcal{C}(\mathcal{H}_1; \mathcal{T}_k) = \mathcal{C}(\mathcal{H}_2; \mathcal{T}_k)$ and $\mathcal{C}(\mathcal{H}_1; (C^m_n)^\times) = \mathcal{C}(\mathcal{H}_2; (C^m_n)^\times)$. So,

$$
\text{Tr}_d(\mathcal{H}_1) - \text{Tr}_d(\mathcal{H}_2) = d(m-1)^n \left( \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; C^m_n, e)} \text{tr}_d(\mathcal{G}) - \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; C^m_n, e)} \text{tr}_d(\mathcal{G}) + \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; C^m_n, e^\times)} \text{tr}_d(\mathcal{G}) \right).
$$

We will prove that

$$
\sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; C^m_n, e)} \text{tr}_d(\mathcal{G}) \geq \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; C^m_n, e)} \text{tr}_d(\mathcal{G}),
$$

with strict inequality if $d/m$ is an integer and $d/m \geq 3$.

By Theorem 4.3 we assume that $m \mid d$; otherwise $\text{tr}_d(\mathcal{G}) = 0$. Note that $\mathcal{C}(\mathcal{H}_2; T, C^m_n, e)$ or $\mathcal{C}(\mathcal{H}_1; T, C^m_n, e)$ is nonempty if and only if $d/m \geq 2$. So we also assume that $d/m \geq 2$. For each $\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; T, C^m_n, e)$, we can write $\mathcal{G} = \mathcal{U}'(u) \circ \mathcal{T}'(w)$, where $\mathcal{U}'$ is a sub-hypergraph of $\mathcal{U}$ which contains the edge $e$, and $\mathcal{T}'(w)$ is a rooted sub-hypertree of $\mathcal{T}$. There is a bijection $\psi$ between $\mathcal{C}(\mathcal{H}_2; T, C^m_n, e)$ and $\mathcal{C}(\mathcal{H}_1; T, C^m_n, e)$ such that $\psi(\mathcal{G}) = \psi(\mathcal{U}'(u) \circ \mathcal{T}'(w)) = \mathcal{U}'(v) \circ \mathcal{T}'(w) = \mathcal{G}$. Also, each weight function $\omega : E(\mathcal{G}) \to \mathbb{Z}^+$ is naturally associated with a weight function $\tilde{\omega} : E(\tilde{\mathcal{G}}) \to \mathbb{Z}^+$ such that $\omega|_{E(\mathcal{U}')} = \tilde{\omega}|_{E(\mathcal{U}')}$. Let $d_w(\mathcal{T}'(\omega)|_{E(\mathcal{T}'_1)}) = x$. We divide the discussions into cases.

Case 1. $\mathcal{U}'$ is a hypertree. Then $\mathcal{U}'$ is obtained from $e$ by attaching some hypertrees to its vertices, especially attaching $\mathcal{T}_1$ to $v$. Let $d_u(\mathcal{T}_1|_{E(\mathcal{T}_1)}) = y$. If $\mathcal{T}_1$ is trivial (or equivalently $y = 0$), then $\mathcal{G}$ is isomorphic to $\tilde{\mathcal{G}}$. So we assume that $y > 0$. By Theorem 4.3 we have

$$
C_{\mathcal{G}(\omega)} = (x + \omega(e) - 1)!(y + \omega(e) - 1)! f_{\mathcal{G}; u, v}(\omega),
$$

$$
C_{\mathcal{G}(\tilde{\omega})} = (x + y + \omega(e) - 1)!(\omega(e) - 1)! f_{\mathcal{G}; u, v}(\tilde{\omega}),
$$

where $f_{\mathcal{G}; u, v}(\tilde{\omega}) = f_{\mathcal{G}; u, v}(\omega)$ as defined in (3.8). It is easily seen $C_{\mathcal{G}(\tilde{\omega})} > C_{\mathcal{G}(\omega)}$, and $\text{tr}_d(\mathcal{G}) > \text{tr}_d(\mathcal{\mathcal{G}})$ in this case. So we have

$$
\sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_1; T, C^m_n, e^\times) \cap \mathcal{E}(\mathcal{T}_1)} \text{tr}_d(\mathcal{G}) \geq \sum_{\mathcal{G} \in \mathcal{C}(\mathcal{H}_2; T, C^m_n, e^\times) \cap \mathcal{E}(\mathcal{T}_1)} \text{tr}_d(\mathcal{G}),
$$

with strictly inequality if $d/m$ is an integer and $d/m \geq 3$, as in this case $\mathcal{U}'$ can have another edge of $C^m_n$ that contains $v$.

Case 2. $\mathcal{U}'$ contains $C^m_n$. Let $d_v(\mathcal{U}' - \{e\})(\omega|_{E(\mathcal{U}' \setminus \{e\})}) = y$, where $\mathcal{U}' - \{e\}$ is obtained from $\mathcal{U}'$ by deleting the edge $e$. By Theorem 4.3 we have

$$
C_{\mathcal{G}(\omega)} = (x + \omega(e) - 1)!(y + \omega(e) - 1)! h_{\mathcal{G}; u, v}(\omega),
$$

$$
C_{\mathcal{G}(\tilde{\omega})} = (x + y + \omega(e) - 1)!(\omega(e) - 1)! h_{\mathcal{G}; u, v}(\tilde{\omega}),
$$

where $h_{\mathcal{G}; u, v}(\tilde{\omega}) = h_{\mathcal{G}; u, v}(\omega)$ defined as follows:

$$
h_{\mathcal{G}; u, v}(\omega) := 2(m-1)^{-|V(\mathcal{G})|} m^{m-2}|E(\mathcal{G})|-1 \prod_{v \in V(\mathcal{G}) \setminus \{u, v\}} (d_v(\mathcal{G}(\omega)) - 1)! \prod_{e \in E(\mathcal{G})} \frac{\omega(e)^{m-1}}{\omega(e)!} \Omega_{C^m_n}(\omega|_{E(C^m_n)}).
$$
Lemma 5.2. Let $T$ be an $m$-uniform hypertree with perfect matching. If $T$ has more than one edge, then

$$T = T'(u) \odot Comb_u(u),$$

where $u$ is an edge, then

$$(4.16) \quad \sum_{\tilde{G} \in \mathcal{C}(H_1; T, C_m, e^m) \cap \mathcal{C}_{cycle}(H_1)} \text{tr}_d(\tilde{G}) \geq \sum_{G \in \mathcal{C}(H_2; T, C_m, e^m) \cap \mathcal{C}_{cycle}(H_2)} \text{tr}_d(G),$$

with strictly inequality if $m \mid d$ is an integer and $d/m \geq n + 1$.

By Eqs. (6.2) and (4.16), we arrive the inequality (4.13). As $\mathcal{C}(H_1; T, C_m, e^m) \neq \emptyset$ if $m \mid d$ and $d/m \geq 2$, we get the desired result by Eq. (4.12). $\square$

5. Maximum Estrada index Of hypertrees with perfect matching

In this section, we will determine the hypertree(s) with maximum Estrada index among all hypertrees with given number of edges and perfect matchings. We need the following lemma.

Lemma 5.1 (21). Let $e = \{u, v_1, \ldots, v_{m-1}\}$ be an edge, and let $H_1 = e(v_1, \ldots, v_p) \odot (G_1(\tilde{v}_1), \ldots, G_p(\tilde{v}_p))$, where $1 \leq p \leq m - 1$. Let $H_2(w)$ be a nontrivial $m$-uniform rooted hypergraph. Then

$$\text{Tr}_d(H_1(v_1) \odot H_2(w)) \geq \text{Tr}_d(H_1(u) \odot H_2(w)),$$

with strict inequality if $m \mid d$ and $d/m \geq 2$, and hence

$$EE(H_1(v_1) \odot H_2(w)) > EE(H_1(u) \odot H_2(w)).$$

We first introduce a special $m$-uniform hypertree. Let $e_0, e_1, \ldots, e_{m-1}$ be $m$ disjoint edges ($m$-sets), where $e_0 = \{u, v_1, \ldots, v_{m-1}\}$, and $u_i \in e_i$ for $i \in [m - 1]$. Denote $Comb_u = e_0(v_1, \ldots, v_{m-1}) \odot (e_1(u_1), \ldots, e_{m-1}(u_{m-1}))$, called an $m$-comb with endpoint $u$; see Fig. 5.1. Let $e$ be an edge ($m$-set) contains a vertex $v$. Let $Comb_u, \ldots, Comb_u$ be $t$ pairwise disjoint $m$-combs. Denote $T_{m,t} = e(v) \odot (Comb_u(u_1), \ldots, Comb_u(u_t))$; see Fig. 5.2. It is easy to see that $T_{m,t}$ has a unique perfect matching of size $t(m - 1) + 1$.

Let $T$ be an $m$-uniform hypertree of order $n$. If $T$ has a perfect matching $M$ of size $k$, then $n = km$, and $|E(T)| = \frac{n - 2}{m - 1} = \frac{km - 1}{m - 1} = k + \frac{k - 1}{m - 1}$, which implies that $(k - 1) \mid (m - 1)$, and $T$ has $\frac{k - 1}{m - 1}$ edges outside $M$.

Lemma 5.2. Let $T$ be an $m$-uniform hypertree with perfect matching. If $T$ has more than one edge, then

$$T = T'(u) \odot Comb_u(u),$$
Figure 5.2. The hypergraph $T_{m,t}$

where $T'$ is a sub-hypertree of $T$ with perfect matching; consequently, $T$ has a unique perfect matching.

Proof. Let $P^m_d$ be a longest path of $T$ with consecutive edges $e_1, \ldots, e_d$, where $d \geq 2$. Let $e_{d-1} = \{u, v_1, \ldots, v_{m-1}\}$, where $v_{m-1} \in e_d$. Clearly, $e_d$ is a pendent edge of $T$ by the definition of $P^m_d$. Let $M$ be a perfect matching of $T$. Surely, $e_d \in M$, which implies that $v_{m-1}$ is covered by $M$. So $e_{d-1} \notin M$, and each vertex $v_i$ of $e_{d-1}$ will be covered by some edge $f_i \in M$ for $i \in [m-2]$. As the maximum length of the paths of $T$ is $d$, all edges $f_i$ for $i \in [m-2]$ are pendent edges. Similarly, except $e_{d-1}$ and $f_i$, no edges contain $v_i$ for $i \in [m-2]$, and except $e_{d-1}$ and $e_d$, no edges contain $v_{m-1}$. So we get a comb

\[ \text{Comb}_u = e_{d-1}(v_1, \ldots, v_{m-2}, v_{m-1}) \odot (f_1(v_1), \ldots, f_{m-2}(v_{m-2}), e_d(v_{m-1})), \]

and $T = T'(u) \odot \text{Comb}_u(u)$. As all vertices of $\text{Comb}_u(u)$ are covered by the edges of $M$ except $u$, $T'$ has a perfect matching. By induction, $M$ is the unique perfect matching of $T$. \qed

Theorem 5.3. Let $T$ be an $m$-uniform hypertree of order $mk$ with perfect matching. Then

\[ EE(T) \leq EE(T_{m, k-1, m-1}), \]

with equality if and only if $T = T_{m, k-1, m-1}$.

Proof. Let $T_0$ be a hypertree with maximum Estrada index among all hypertrees of order $mk$ with perfect matchings. If $k = 1$, then $T_0$ is an edge, and the result follows clearly. So we assume that $k > 1$. Let $M$ be the unique perfect matching of $T_0$. We assert that if $e \in M$, then $e$ is a pendent edge of $T_0$; otherwise, letting $e = \{u, v_1, \ldots, v_{m-1}\}$, then $T_0 = \mathcal{H}(u) \odot T'_0(u)$, where $\mathcal{H} = e(v_1, \ldots, v_p) \odot (T'_1, \ldots, T'_p)$, $1 \leq p \leq m - 1$. By Lemma 5.1, we have

\[ EE(T_0) < EE(\mathcal{H}(v_1) \odot T'_0(u)). \]
As $H(v) \cap T_0(u)$ also has a perfect matching, we get a contradiction to the definition of $T_0$.

Let $w$ be a vertex of $T_0$ with maximum degree $\Delta > 1$. Then $T_0$ contains a hyperstar $S_{\Delta}^m$ centered at $w$ as sub-hypertree. Let $e_1, \ldots, e_\Delta$ be edges of $S_{\Delta}^m$ which share a common vertex $u$. Suppose that $u$ is covered by $e_1 \in M$. Then neither of $e_2, \ldots, e_\Delta$ belongs to $M$. So every vertices of $e_i$ except $u$ are covered by the edges of $M$ for $i = 1, \ldots, \Delta$, implying that $T_0$ contains a comb $T_{m, \Delta-1}$.

If $\Delta = 2$, then $T_0 = T_{m,1}$, the result follow. Assume that $\Delta > 2$ and $T_0 \neq T_{m,\Delta-1}$. By the above assertion, all edges of $M$ are pendent, so $T_0 = T_{m,\Delta-1}(w_1, w_2, \ldots, w_l) \cup (T_{1}^1(w_1), T_{2}^1(w_2), \ldots, T_{p}^1(w_p))$, where $w_1, \ldots, w_p$ are the vertices of $S_{\Delta}^m$ except $w$, and $p \geq 1$. Let $\hat{T}_0 = T_{m,\Delta-1}(w_1, w_2, \ldots, w_l) \cup (T_{1}^1(w_1), T_{2}^1(w_2), \ldots, T_{p}^1(w_p))$, which also has a perfect matching. By Lemma 3.3, we have

$$Tr_d(\hat{T}_0) \geq Tr_d(T_0),$$

with strict inequality if $d/m$ is an integer and $d/m \geq 2$. So,

$$EE(\hat{T}_0) > EE(T_0),$$

a contradiction to the definition of $T_0$. So, $T_0 = T_{m,\Delta-1}$ and $\Delta = \frac{k-1}{m-1} + 1$. \hfill \qed

6. Maximum Estrada index of linear unicyclic hypergraphs with given girth

Recall the girth of a hypergraph $H$, denoted by $g(H)$, is the minimum length of the cycles of $H$. If $H$ contains no cycles, then we define $g(H) = +\infty$. Let $U_{z,g}^m$ denote the set of $m$-uniform linear unicyclic hypergraphs with $z$ edges and girth $g$. In this section we will characterize the unicyclic hypergraphs with maximum Estrada index among all hypergraphs in $U_{z,g}^m$.

**Lemma 6.1.** Let $U$ be $m$-uniform linear unicyclic hypergraph with maximum Estrada index among all hypergraphs in $U_{z,g}^m$. Then $U$ is obtained from $C_g^m$ by attaching some hyperstars to the vertices of $C_g^m$ of degree 2 with their centers identified with the vertices of $C_g^m$.

**Proof.** It is known $U$ is obtained from $C_g^m$ by attaching some hypertrees at the vertices of $C_g^m$. Let $S = \{v_1, \ldots, v_g\}$ be the set of vertices of $C_g^m$ with degree 2. We first assert that all hypertrees are attached to the vertices of $S$. Otherwise, let $T$ be a hypertree attached at a cored vertex $w$ of $C_g^m$, where $w \notin S$, and let $e$ be an edge of $C_g^m$ which contains $w$ and a vertex $v \in S$. Then $U = U'(w) \cup T(w)$, where $U'$ is a unicyclic sub-hypergraph of $U$. By Lemma 4.4,

$$Tr_d(U) \leq Tr_d(U'(v) \cup T(w)),$$

with strict inequality if $m \mid d$ and $d/m \geq 2$. So $EE(U) < EE(U'(v) \cup T(w))$, a contradiction to the definition of $U$.

By the above discussion, $U$ is obtained by attaching some hypertrees to the vertices of $S$. We next assert all hypertrees are hyperstars centered at the vertices of $S$. Otherwise, let $T$ be a hypertree attached at $v \in S$, which is not a hyperstar centered at $v$. Then there exists an edge $e$ outside $C_g^m$ which contains the vertex $v$ and a vertex $w$ to which a sub-hypertree $T'$ of $T$ is attached. Letting $e = \{v, w, u_1, \ldots, u_{m-2}\}$, we can write $U = e(v, w, u_1, \ldots, u_p) \cup (U'(v), T'(w), \hat{T}_1(u_1), \ldots, \hat{T}_p(u_p))$,
where $U'$ is a unicyclic sub-hypergraph of $U$, and $\hat{T}_i$ is a sub-hypertree of $T$ for $i = 1, \ldots, p$, $0 \leq p \leq m - 2$. (Note that $\hat{T}_i(u_i)$ may be trivial.) Let $\hat{U}$ be obtained from $U$ by relocating $T'$ from $w$ to $v$. By Lemma 5.1, we have $EE(U) < EE(U)$; a contradiction. The result follows.

We will determine the unique $m$-uniform linear unicyclic hypergraph of girth 3 with maximum Estrada index in $W^m_{0,3}$. We need some lemmas for preparation.

**Lemma 6.2.** Let $e = \{u, v_1, \ldots, v_{m-1}\}$ be an edge, and let $H = e(v_1, \ldots, v_p) \odot (T_1(v_1), \ldots, T_p(v_p))$, where $1 \leq p \leq m - 1$. Let $T$ be a nontrivial $m$-uniform hypertree with root $w$.

1. Let $H_1 = H(v_1) \odot T(w)$ and $H_2 = H(u) \odot T(w)$. Then
   
   $$tr_d(H_1) \geq tr_d(H_2),$$

   with strict inequality if $m|d$ and $d/m \geq |E(H_1)|$.

2. Let $H_{11} = H_1(v_1) \odot \hat{T}(\hat{w})$ and $H_{12} = H_1(u) \odot \hat{T}(\hat{w})$, $H_{21} = H_2(v_1) \odot \hat{T}(\hat{w})$ and $H_{12} = H_2(u) \odot \hat{T}(\hat{w})$. Then
   
   $$tr_d(H_{11}) + tr_d(H_{12}) \geq tr_d(H_{21}) + tr_d(H_{22}),$$

   with strict inequality if $m|d$ and $d/m \geq |E(H_{11})|$.

**Proof.** (1) By Theorem 3.2, if $m | d$, then

$$tr_d(H_i) = \sum_{\omega: \omega(H_i) = d/m} C_{H_i}(\omega), \ i = 1, 2;$$

otherwise, $tr_d(H_i) = 0$. So we assume that $m \nmid d$ in the following. Note that each weight function $\omega : E(H_1) \to \mathbb{Z}^+$ with $\omega(H_1) = d/m$ induces a weight function $\hat{\omega} : E(H_2) \to \mathbb{Z}^+$ such that $\hat{\omega}|E(H) = \omega|E(H)$ and $\hat{\omega}|E(T(w)) = \omega|E(T(w))$. Let $d_{v_1}(T(\omega|E(T_1))) = x$, $d_w(T(\omega|E(T(w)))) = y$. By Theorem 3.2

$$C_{H_1}(\omega) = (x + y + \omega(e) - 1)!/\omega(e) - 1)!f_{H_1;v_1,u}(\omega),$$

$$C_{H_2}(\hat{\omega}) = (x + \omega(e) - 1)!/\omega(e) - 1)!f_{H_2;v_1,u}(\hat{\omega}),$$

where $f_{H_1;v_1,u}(\omega) = f_{H_2;v_1,u}(\hat{\omega})$ as defined in (3.3). Surely, $C_{H_1}(\omega) > C_{H_2}(\hat{\omega})$ if $d/m \geq |E(H_1)|$ as in this case $H_i(\omega)$ exists for $i = 1, 2$.

(2) As discussed in (1), we assume that $m \nmid d$. Each weight function $\omega_1 : E(H_{11}) \to \mathbb{Z}^+$ with $\omega(H_{11}) = d/m$ induces a weight function $\omega_{ij} : E(H_{ij}) \to \mathbb{Z}^+$ such that $\omega_{ij}|E(H) = \omega|E(H)$ and $\omega_{ij}|E(T(w)) = \omega|E(T(w))$; and $\omega_{ij}|E(T'(w')) = \omega|E(T'(w'))$ for $i, j = 1, 2$. Let $d_{v_1}(T(\omega|E(T_1))) = x$, $d_w(T(\omega|E(T(w)))) = y$, and $d_{v_1}(T(\omega|E(T_1))) = z$. We have

$$C_{H_{11}(\omega_{11})} = (x + y + z + \omega_1(e) - 1)!/\omega_1(e) - 1)!f_{H_{11};v_1,u}(\omega_{11}),$$

$$C_{H_{12}(\omega_{12})} = (x + y + \omega_1(e) - 1)!/\omega_1(e) - 1)!f_{H_{12};v_1,u}(\omega_{12}),$$

$$C_{H_{21}(\omega_{21})} = (x + z + \omega_1(e) - 1)!/\omega_1(e) - 1)!f_{H_{21};v_1,u}(\omega_{21}),$$

$$C_{H_{22}(\omega_{22})} = (x + \omega_1(e) - 1)!/\omega_1(e) - 1)!f_{H_{22};v_1,u}(\omega_{22}),$$

where $f_{H_{11};v_1,u}(\omega) = f_{H_{21};v_1,u}(\omega)$ as defined in (3.8). Surely, $C_{H_{11}(\omega)} > C_{H_{22}(\omega)}$ if $d/m \geq |E(H_{11})|$ as in this case $H_i(\omega)$ exists for $i = 1, 2, 3$. 

Some lemmas for preparation.
where \( f_{H_{11};v_1,u}(\omega_{11}) = f_{H_{21};v_1,u}(\omega_{21}) = f_{H_{12};v_1,u}(\omega_{12}) = f_{H_{22};v_1,u}(\omega_{22}) \) as defined in \( (3.8) \). By Lemma \( 2.4 \)
\[
C_{H_{11}(\omega_{11})} + C_{H_{12}(\omega_{21})} > C_{H_{21}(\omega_{12})} + C_{H_{22}(\omega_{22})},
\]
which yields the result by Theorem \( 3.2 \) \( \square \)

\[\text{Figure 6.1. The hypergraph } H \text{ in Lemma 6.3}\]

**Lemma 6.3.** Let \( P_2 \) be a path on vertices \( v_0, v_1, v_2 \) with edges \( \{v_{i-1}, v_i\} \) for \( i = 1, 2 \). Let \( P_2^n \) be the power of \( P_2 \) with edges \( e_i = \{v_{i-1}, v_i, u_i, \ldots, u_i, m-2\} \) for \( i = 1, 2 \). Let \( H = P_2^n(v_0, v_1) \cap (\mathcal{T}_0(v_0), \mathcal{T}_1(v_1)) \), where \( \mathcal{T}_0 \) is a nontrivial \( m \)-uniform hypertree, and \( \mathcal{T}_1 \) is a \( m \)-uniform hypertree allowed to be trivial with only one vertex \( v_1 \); see Fig. 6.1. Let \( \mathcal{T}(w) \) be a \( m \)-uniform nontrivial hypertree with root \( w \). Then
\[\text{tr}_d(H(v_0) \circ \mathcal{T}(w)) \geq \text{tr}_d(H(v_2) \circ \mathcal{T}(w)),\]
with strict inequality if \( d/m \) is an integer and \( d/m \geq 4 \).

**Proof.** Let \( H_1 := H(v_0) \circ \mathcal{T}(w) \) and \( H_2 := H(v_2) \circ \mathcal{T}(w) \). By Lemma \( 3.2 \) we assume that \( m \mid d \); otherwise \( \text{tr}_d(H_i) = 0 \). Each weight function \( \omega : E(H_2) \rightarrow \mathbb{Z}^+ \) with \( \omega(H_2) = d/m \) induces a weight function \( \tilde{\omega} : E(H_1) \rightarrow \mathbb{Z}^+ \) such that \( \tilde{\omega}|_{E(H_1)} = \omega|_{E(H_1)} \) and \( \tilde{\omega}|_{E(T(w))} = \omega|_{E(T(w))} \). Let \( \omega' \) (respectively, \( \tilde{\omega}' \)) be obtained from \( \omega \) (respectively, \( \tilde{\omega} \)) only by swapping the weight of \( e_1 \) and the weight of \( e_2 \). Let \( d_{v_0}(\mathcal{T}_0(\omega|_{E(T_0)})) = x \) and \( d_{v_2}(\mathcal{T}(\omega|_{E(T_1)})) = y \). By Theorem \( 3.2 \) we have
\[
\begin{align*}
C_{H_2(\omega)} &= (x + \omega(e_1) - 1)(y + \omega(e_2) - 1)!f_{H_2;v_0,v_2}(\omega), \\
C_{H_2(\omega')} &= (x + \omega(e_2) - 1)(y + \omega(e_1) - 1)!f_{H_2;v_0,v_2}(\omega'), \\
C_{H_1(\tilde{\omega})} &= (x + y + \omega(e_1) - 1)(\omega(e_2) - 1)!f_{H_1;v_0,v_2}(\tilde{\omega}), \\
C_{H_1(\tilde{\omega}')} &= (x + y + \omega(e_2) - 1)(\omega(e_1) - 1)!f_{H_1;v_0,v_2}(\tilde{\omega}'),
\end{align*}
\]
where \( f_{H_1;v_0,v_2}(\omega) = f_{H_1;v_0,v_2}(\omega') = f_{H_1;v_0,v_2}(\tilde{\omega}) = f_{H_1;v_0,v_2}(\tilde{\omega}') \) as defined in Eq. \( (3.8) \). By Lemma \( 2.4 \)
\[
C_{H_1(\tilde{\omega})} + C_{H_1(\tilde{\omega}')} > C_{H_2(\omega)} + C_{H_2(\omega')}.
\]
So
\[
\text{tr}_d(\mathcal{H}_1) - \text{tr}_d(\mathcal{H}_2) = \sum_{\omega: \omega(\mathcal{H}_2) = d/m} (C_{\mathcal{H}_1}(\omega) - C_{\mathcal{H}_2}(\omega))
\]
\[
= \frac{1}{2} \sum_{\omega: \omega(\mathcal{H}_2) = d/m} ((C_{\mathcal{H}_1}(\omega) + C_{\mathcal{H}_1}(\omega')) - (C_{\mathcal{H}_2}(\omega) + C_{\mathcal{H}_2}(\omega')))
\]
\[> 0.\]

The result follows. \(\square\)

**Lemma 6.4.** Let \(C_m^3\) be the \(m\)-th power of \(C_3\) with edges \(e_i \supseteq \{v_i, v_{i+1}\}\) for \(i = 1, 2, 3\), where \(v_4 = v_1\). Let \(\mathcal{H} = C_m^3(v_1, v_3) \odot (T_1(v_1), \mathcal{H}_3(v_3))\), where \(T_1\) is nontrivial and \(\mathcal{H}_3\) may be trivial. Let \(T_2\) be a nontrivial \(m\)-uniform hypertree with root \(w\). Then
\[
\text{tr}_d(\mathcal{H}(v_1) \odot T_2(w)) \geq \text{tr}_d(\mathcal{H}(v_2) \odot T_2(w)),
\]
with strict inequality if \(m|d\) and \(d/m \geq |E(\mathcal{H}(v_1) \odot T_2(w))|\).

**Proof.** Let \(\mathcal{H}_1 = \mathcal{H}(v_1) \odot T_2(w)\) and \(\mathcal{H}_2 = \mathcal{H}(v_2) \odot T_2(w)\). By Theorem 4.3, we assume \(m|d\); otherwise \(\text{tr}_d(\mathcal{H}_i) = 0\) for \(i = 1, 2\). Each weight function \(\omega : E(\mathcal{H}_1) \to \mathbb{Z}^+\) induces a weight function \(\tilde{\omega} : E(\mathcal{H}_2) \to \mathbb{Z}^+\) such that \(\tilde{\omega}|_{E(\mathcal{H}_1)} = \omega|_{E(\mathcal{H}_1)}\) and \(\tilde{\omega}|_{E(T_2(w))} = \omega|_{E(T_2(w))}\). Let \(\omega'\) (respectively \(\tilde{\omega}'\)) be obtained from \(\omega\) (respectively \(\tilde{\omega}\)) by swapping the weight of \(e_2\) and the weight of \(e_3\). Let \(d_{v_1}(T_1(\omega|_{E(T_1)})) = x\), \(d_w(T_2(\omega|_{E(T_2)})) = y\), and let \(\omega_i := \omega(e_i)\) for \(i = 1, 2, 3\). By Theorem 4.3, we get
\[
\begin{align*}
C_{\mathcal{H}_1}(\omega) &= (x + y + \omega_1 + \omega_3 - 1)!(\omega_1 + \omega_2 - 1)!h_{\mathcal{H}_1:v_1,v_2}(\omega), \\
C_{\mathcal{H}_1}(\omega') &= (x + y + \omega_1 + \omega_2 - 1)!(\omega_1 + \omega_3 - 1)!h_{\mathcal{H}_1:v_1,v_2}(\omega'), \\
C_{\mathcal{H}_2}(\omega) &= (x + \omega_1 + \omega_3 - 1)!(y + \omega_1 + \omega_2 - 1)!h_{\mathcal{H}_2:v_1,v_2}(\tilde{\omega}), \\
C_{\mathcal{H}_2}(\omega') &= (x + \omega_1 + \omega_2 - 1)!(y + \omega_1 + \omega_3 - 1)!h_{\mathcal{H}_2:v_1,v_2}(\tilde{\omega}'),
\end{align*}
\]
where \(h_{\mathcal{H}_1:v_1,v_2}(\omega) = h_{\mathcal{H}_2:v_1,v_2}(\omega') = h_{\mathcal{H}_2:v_1,v_2}(\tilde{\omega}) = h_{\mathcal{H}_2:v_1,v_2}(\tilde{\omega}')\) as defined in (4.15). By verifying \(\Omega_{C_3^3}(\omega|_{E(C_3^m)}) = \Omega_{C_3^3}(\omega'|_{E(C_3^m)})\), we have
\[
C_{\mathcal{H}_1}(\omega) + C_{\mathcal{H}_1}(\omega') > C_{\mathcal{H}_2}(\tilde{\omega}) + C_{\mathcal{H}_2}(\tilde{\omega}').
\]
So
\[
\text{tr}_d(\mathcal{H}_1) - \text{tr}_d(\mathcal{H}_2) = \sum_{\omega: \omega(G) = d/m} (C_{\mathcal{H}_1}(\omega) - C_{\mathcal{H}_2}(\tilde{\omega}))
\]
\[
= \frac{1}{2} \sum_{\omega: \omega(G) = d/m} ((C_{\mathcal{H}_1}(\omega) + C_{\mathcal{H}_1}(\omega')) - (C_{\mathcal{H}_2}(\tilde{\omega}) + C_{\mathcal{H}_2}(\tilde{\omega}')))
\]
\[> 0.\]

The result follows. \(\square\)
Corollary 6.5. Let $C_3^m$ be the $m$-th power of $C_3$ with edges $e_i \supseteq \{v_i, v_{i+1}\}$ for $i = 1, 2, 3$, where $v_4 = v_1$. Let $H = C_3^m(v_1, v_3) \odot (T_1(v_1), T_3(v_3))$, where $T_1$ is nontrivial and $T_3$ may be trivial. Let $T_2$ be a nontrivial $m$-uniform hypertree with root $w$. Then

$$Tr_d(H(v_1) \odot T_2(w)) \geq Tr_d(H(v_2) \odot T_2(w)),$$

with strict inequality if $m|d$ and $d/m \geq 2$.

Proof. Let $H_1 := H(v_1) \odot T_2(w)$ and $H_2 := H(v_2) \odot T_2(w)$. We have

$$C(H_i) = C(H_i; T_2^x) \cup C(H_i; T_2; T_1^x) \cup C(H_i; T_2; T_1), i = 1, 2.$$

As all hypergraphs in $C(H_i; T_2, T_1)$ are connected, we also have

$$C(H_1; T_2, T_1) = C(H_1; T_2, T_1, e_1^m) \cup C(H_1; T_2, T_1, (e_1^m)^0), C(H_2; T_2, T_1) = C(H_2; T_2, T_1, e_1^m).$$

Obviously, $C(H_1; T_2^x) = C(H_2; T_2^x)$, and there is a bijection $\phi$ from $C(H_1; T_2, T_1^x)$ to $C(H_2; T_2, T_1^x)$ such that $G$ is isomorphic to $\phi(G)$ for each $G \in C(H_1; T_2, T_1^x)$. By Theorem 4.3

(6.1)

$$Tr_d(H_1) - Tr_d(H_2) = d(m-1)^N \left( \sum_{G \in C(H_1; T_2, T_1, e_1)} tr_d(G) - \sum_{G \in C(H_2; T_2, T_1, e_1)} tr_d(G) + \sum_{G \in C(H_1; T_2, T_1, e_1)} tr_d(G) \right),$$

where $N$ is the number of vertices of $H_1$ or $H_2$. We will prove that

(6.2)

$$\sum_{G \in C(H_1; T_2, T_1, e_1)} tr_d(G) \geq \sum_{G \in C(H_1; T_2, T_1, e_1)} tr_d(G),$$

with strict inequality if $d/m$ is an integer and $d/m \geq 3$.

By Theorem 4.3 we may assume $m | d$; otherwise $tr_d(G) = 0$. Note that $C(H_2; T_2, T_1, e_1^m)$ or $C(H_1; T_2, T_1, e_1^m)$ is nonempty if and only if $d/m \geq 3$. So we also assume that $d/m \geq 3$. For each $G \in C(H_1; T_2, T_1, e_1^m)$, we can write $G = H'(v_2) \odot T'(w)$, where $H'$ is a sub-hypergraph of $H$ which contains rooted sub-hypertree $T'_1(v_1)$ of $T_1(v_1)$ and the edge $e_1^m$, and $T'_2(w)$ is a rooted sub-hypertree of $T_2(w)$. Then there is a bijection $\psi$ from $C(H_2; T_2, T_1, e_1^m)$ to $C(H_1; T_2, T_1, e_1^m)$ such that $\psi(H'(v_2) \odot T'_2(w)) = H'(v_1) \odot T'_2(w)$. Now we let $G = H'(v_2) \odot T'_2(w)$ and $\tilde{G} := \psi(G) = H'(v_1) \odot T'_2(w)$, and divide the discussion into cases.

Case 1. $G$ contains neither $e_2$ nor $e_3$. Then $\tilde{G}, G$ are respectively corresponding the hypergraphs $H_1, H_2$ in Lemma 6.2(1), and hence $tr_d(\tilde{G}) \geq tr_d(G)$. So we have

(6.3)

$$\sum_{G \in C(H_1; T_2, T_1, e_1^m, e_2^x, e_3^x)} tr_d(G) \geq \sum_{G \in C(H_2; T_2, T_1, e_1^m, e_2^x, e_3^x)} tr_d(G),$$

with strict inequality if $d/m \geq 3$ as in this case $C(H_1; T_2, T_1, e_1^m, e_2^x, e_3^x)$ is nonempty.

Case 2. $G$ contains exactly one edge of $e_2$ and $e_3$. Let $G[e_3] \in C(H_2; T_2, T_1, e_1, e_2^x, e_3)$ which contains $e_3$ but no $e_2$. Let $T'$ be the (edge maximal) sub-hypertree of $G[e_3]$ which contains $e_3$ and contains no edges of $T_1, T_2$ or $e_1$. Let $G[e_2] \in C(H_2; T_2, T_1, e_1, e_2, e_3^x)$ which is obtained from $G[e_3]$ by relocating the sub-hypertree $T'$ attached at $v_1$ to $v_2$ and labeling the edge $e_3$ as $e_2$. By the bijection
ψ defined before, we have \( \tilde{G}[e_3] \in \mathcal{G}(H_1; T_2, T_1, e_1, e_z^*, e_3) \) and \( \tilde{G}[e_2] \in \mathcal{G}(H_1; T_2, T_1, e_1, e_2, e_z^*) \). Now \( \tilde{G}[e_3], \tilde{G}[e_2], G[e_3], G[e_2] \) are corresponding \( H_1, H_{12}, H_{21}, H_{22} \) in Lemma 6.2(2), and hence

\[
\text{tr}_d(\tilde{G}[e_3]) + \text{tr}_d(\tilde{G}[e_2]) \geq \text{tr}_d(G[e_3]) + \text{tr}_d(G[e_2]).
\]

So we have

\[
(6.4) \quad \sum_{\tilde{G} \in \mathcal{G}(H_1; T_2, T_1, e_1, e_z^*, e_3)} \text{tr}_d(\tilde{G}) \geq \sum_{G \in \mathcal{G}(H_2; T_2, T_1, e_1, e_2, e_z^*, e_3)} \text{tr}_d(G),
\]

with strict inequality if \( d/m \geq 4 \).

Case 3. \( \mathcal{G} \) contains both \( e_2 \) and \( e_3 \). Then \( \mathcal{G} \) contains the cycle \( C^m_3 \). Now \( \tilde{G}, G \) are respectively corresponding to \( H(v_1) \circ T_2(w) \) and \( H(v_2) \circ T_2(w) \) in Lemma 6.4 and

\[
\text{tr}_d(\tilde{G}) \geq \text{tr}_d(G).
\]

So we have

\[
(6.5) \quad \sum_{\tilde{G} \in \mathcal{G}(H_1; T_2, T_1, e_1, e_z, e_3)} \text{tr}_d(\tilde{G}) \geq \sum_{G \in \mathcal{G}(H_2; T_2, T_1, e_1, e_2, e_z, e_3)} \text{tr}_d(G),
\]

with strict inequality if \( d/m \geq 5 \).

Combining Eqs. (6.3), (6.4) and (6.5), we arrive at the inequality (6.2). As \( \mathcal{G}(H_1; T_2, T_1, e_1^*) \neq \emptyset \) if \( d/m \geq 2 \), we get the desired result by Eq. (6.1).

Denote \( S_{n,t} := C_n(v) \circ S_t(w) \), where \( v \) is a vertex of \( C_n \) and \( w \) is the center of \( S_t \). Then the \( m \)-th power of \( S_{n,t} \) is \( S_{n,t}^m := C_n^m(v) \circ S_t^m(w) \), namely, a hypergraph obtained from \( C_n^m \) by attaching \( S_t^m \) with its center identified with a vertex of \( C_n^m \) with degree 2.

**Theorem 6.6.** Let \( \mathcal{U} \) be an \( m \)-uniform linear unicyclic hypergraph with \( z \) edges and girth 3. Then

\[
EE(\mathcal{U}) \leq EE(S_{3,z-3}^m),
\]

with equality if and only if \( \mathcal{U} = S_{3,z-3}^m \).

**Proof.** Let \( \mathcal{U}_0 \) be an \( m \)-uniform linear unicyclic hypergraph with \( z \) edges and girth 3 with maximum Estrada index among all hypergraphs in \( \mathcal{H}_{3,z-3}^m \). By Lemma 6.1, we can write \( \mathcal{U}_0 \) as

\[
\mathcal{U}_0 = C_3^m(v_1, v_2, v_3) \circ (S_{z_1}^m(v_1), S_{z_2}^m(v_2), S_{z_3}^m(v_3)),
\]

where \( v_i \) is the vertex of \( C_3^m \) with degree 2 and also the center of \( S_{z_i}^m \) for \( i = 1, 2, 3 \), and some of the hyperstars \( S_{z_i}^m \) may be trivial. If \( \mathcal{U}_0 \) is not isomorphic to \( S_{3,z-3}^m \), then there exists two nontrivial hyperstars among \( S_{z_i}^m \) for \( i = 1, 2, 3 \). Without loss of generality, let \( S_{z_1}^m(v_1), S_{z_2}^m(v_2) \) be nontrivial. Now let \( \mathcal{U}_0 \) be obtained from \( \mathcal{U}_0 \) by relocating \( S_{z_2}^m \) from \( v_2 \) to \( v_1 \). By Corollary 6.5, we have

\[
\text{Tr}_d(\mathcal{U}_0) \leq \text{Tr}_d(\tilde{\mathcal{U}}_0),
\]

with strict inequality if \( m \mid d \) and \( d/m \geq 2 \). So

\[
EE(\mathcal{U}_0) < EE(\tilde{\mathcal{U}}_0),
\]
which yields a contradiction to the definition of $\mathcal{U}_0$. So $\mathcal{U}_0 = S_{3,z-3}^m$. The result follows. □

Du and Zhou [10] proved that $S_{3,z-3}$ is the unique unicyclic graph with maximum Estrada index among all unicyclic graphs with $z$ edges. We have the following problem:

**Problem 6.7.** Is $S_{3,z-3}^m$ the unique unicyclic hypergraph with maximum Estrada index among all linear unicyclic hypergraphs with $z$ edges?

**References**

[1] T. van Aardenne-Ehrenfest, N. G. de Bruijn, Circuits and trees in oriented linear graphs, In: I. Gessel, G. C. Rota (Eds), *Classic Papers in Combinatorics*, Birkhäuser Boston, Boston, 1987, pp. 149-163.

[2] S. Bai, L. Lu, A bound on the spectral radius of hypergraphs with $e$ edges. *Linear Algebra Appl.*, 549(2018), 203-218.

[3] K. C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, *Comm. Math. Sci.*, 6 (2008), 507-520.

[4] K. C. Chang, K. Pearson, T. Zhang, On eigenvalue problems of real symmetric tensors, *J. Math. Anal. Appl.*, 350 (2009), 416-422.

[5] L. Chen, C. Bu, J. Zhou, Spectral moments of hypertrees and their applications, *Linear Multilinear Algebra*, 2021, DOI: 10.1080/03081087.2021.1953431.

[6] G. Clark, J. Cooper, A Harary-Sachs theorem for hypergraphs, *J. Combin. Theory Ser. B*, 149 (2021), 1-15.

[7] J. Cooper, A. Dutle, Spectra of uniform hypergraphs, *Linear Algebra Appl.*, 436 (2012), 3268-3292.

[8] K. Das, S. Lee, On the Estrada index conjecture, *Linear Algebra Appl.*, 431 (2009), 1351-1359.

[9] H. Deng, A proof of a conjectures on the Estrada index, *MATCH Commun. Math. Comput. Chem.*, 62 (2009), 599-606.

[10] Z.-B. Du, B. Zhou, The Estrada index of unicyclic graph, *Linear Algebra Appl.*, 436 (2012), 3149-3159.

[11] C. Duan, E. R. van Dam, L. Wang, Signless Laplacian Estrada index and Laplacian Estrada index of uniform hypergraphs, Available at arXiv: 2206.00109.

[12] E. Estrada, J. A. Rodríguez-Valáquez, Subgraph centrality in complex networks, *Phys. Rev. E*, 71 (056103) (2005), 1-9.

[13] E. Estrada, Characterization of the folding degree of proteins, *Bioinformatics*, 18 (2002), 697-704.

[14] E. Estrada, Characterization of 3D molecular structure, *Chem. Phys. Lett.*, 319 (2000), 713-718.

[15] Y.-Z. Fan, Y.-H. Bao, T. Huang, Eigenvariety of nonnegative symmetric weakly irreducible tensors associated with spectral radius and its application to hypergraphs, *Linear Algebra Appl.*, 564 (2019), 72-94.

[16] Y.-Z. Fan, T. Huang, Y.-H. Bao, C. L. Zhu, and Y.-P. Li, The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs. *Trans. Amer. Math. Soc.*, 372(3) (2019), 2213-2233.

[17] Y.-Z. Fan, T. Huang, Y.-H. Bao, The dimension of eigenvariety of nonnegative tensors associated with spectral radius, *Proc. Amer. Math. Soc.*, 150 (2022), 2287-2299.

[18] Y.-Z. Fan, M. Li, Y. Wang, The cyclic index of adjacency tensor of generalized power hypergraphs, *Discrete Math.*, 344 (2021), 112329.

[19] Y.-Z. Fan, Y.-Y. Tan, X.-X. Peng, A.-H. Liu, Maximizing spectral radii of uniform hypergraphs with few edges, *Discuss. Math. Graph Theory*, 36(4)(2016), 845-856.

[20] Y.-Z. Fan, M.-Y. Tian, M. Li, Fan, The stabilizing index and cyclic index of the coalescence and Cartesian product of uniform hypergraphs, *J. Combin. Theory Ser. A*, 185 (2022), 105537.

[21] Y.-Z. Fan, Y. Yang, C.-M. She, J. Zheng, Y.-M. Song, H.-X. Yang, The trace and Estrada index of uniform hypergraphs with cut vertices, Available at arXiv: 2205.15502.
[22] Y.-Z. Fan, H.-X. Yang, J. Zheng, High-ordered spectral characterization of unicyclic graphs, Available at arXiv: 2208.13204.
[23] S. Friedland, S. Gaubert, L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, Linear Algebra Appl., 438 (2013), 738-749.
[24] G. Gao, A. Chang, Y. Hou, Spectral radius on linear r-graphs without expanded $K_{r+1}$, SIAM J. Discrete. Math., 36(2)(2022), 1000-1011.
[25] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977.
[26] S. Hu, L. Qi, J. Shao, Cored hypergraphs, power hypergraphs and their Laplacian eigenvalues, Linear Algebra Appl., 439(2013), 2980-2998.
[27] P. Keevash, J. Lenz, D. Mubayi, Spectral extremal problems for hypergraphs, SIAM J. Discrete Math., 28(4)(2014), 1838-1854.
[28] H. Li, J. Shao, L. Qi, The extremal spectral radii of k-uniform supertrees, J. Comb. Optim., 32(2016), 741-764.
[29] L.-H. Lim, Singular values and eigenvalues of tensors: a variational approach, in Computational Advances in Multi-Sensor Adaptive Processing, 2005 1st IEEE International Workshop, IEEE, Piscataway, NJ, 2005, pp. 129-132.
[30] L. Liu, L. Kang, E. Shan, On the irregularity of uniform hypergraphs, European J. Combin., 71(2018), 22-32.
[31] L. Lu, S. Man, Connected hypergraphs with small spectral radius. Linear Algebra Appl., 509 (2016), 206-227.
[32] H. Lu, N. Xue, Z. Zhu, On the signless Laplacian Estrada index of uniform hypergraphs, Int. J. Quantum. Chem., 121 (2021), e26579.
[33] A. Morozov, Sh. Shakirov, Analogue of the identity Log Det = Trace Log for resultants, J. Geom. Phys., 61(3) (2011), 708-726.
[34] J. A. Peña, I. Gutman, J. Rada, Estimating the Estrada index, Linear Algebra Appl., 427 (2007), 70-76.
[35] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput., 40 (2005), 1302-1324.
[36] J.-Y. Shao, L. Qi, S. Hu, Some new trace formulas of tensors with applications in spectral hypergraph theory, Linear Multilinear Algebra, 63 (2015), 971-992.
[37] L. Sun, H. Zhou, C. Bu, Estrada index of hypergraphs via eigenvalues of tensors, Available at arXiv: 2107.03837.
[38] W. T. Tutte, C. A. B. Smith, On unicursal paths in a network of degree 4, Amer. Math. Monthly, 48(4) (1941), 233-237.
[39] Y. Yang, Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J Matrix Anal. Appl., 31 (5) (2010), 2517-2530.
[40] Y. Yang, Q. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, SIAM J Matrix Anal. Appl., 32 (4) (2011), 1236-1250.
[41] Y. Yang, Q. Yang, On some properties of nonnegative weakly irreducible tensors, Available at arXiv: 1111.0713v2.
[42] W. Zhang, L. Kang, E. Shan, Y. Bai, The spectra of uniform hypertrees. Linear Algebra Appl., 533(2017), 84-94.
[43] Zhou, J., Sun, L., Wang, W., Bu, C.: Some spectral properties of uniform hypergraphs. Electron. J. Combin., 21 (2014), #P4.24.
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