NOTES ON KODAIRA ENERGIES OF POLARIZED VARIETIES

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In this note we propose a couple of conjectures concerning Kodaira energies of polarized varieties and give a few partial answers.

§1. Conjectures

Let \( V \) be a variety over \( \mathbb{C} \) and let \( B = \sum b_i B_i \) be an effective \( \mathbb{Q} \)-Weil divisor on \( V \) such that \( b_i \leq 1 \) for any \( i \). Such a pair \((V, B)\) will be called a log variety. It is said to be log terminal if it has only weak log terminal singularities in the sense of [KMM]. In this case, the \( \mathbb{Q} \)-bundle \( K_V + B \) is called the log canonical bundle of \( (V, B) \) and will be denoted by \( K(V, B) \).

A \( \mathbb{Q} \)-bundle \( L \) on a log terminal variety \((V, B)\) is said to be log ample if there is an effective \( \mathbb{Q} \)-divisor \( E \) such that \((V, B + E)\) is log terminal and \( L - \epsilon E \) is ample for any \( 0 < \epsilon \leq 1 \). Note that “log ample” implies “nef big”, and the converse is also true if \( b_i < 1 \) for all \( i \).

For a big \( \mathbb{Q} \)-bundle \( L \) on a log terminal variety \((V, B)\), we define

\[
\kappa(V, B, L) = -\inf \{ t \in \mathbb{Q} | \kappa(K(V, B) + tL) \geq 0 \},
\]

which will be called the Kodaira energy of \((V, B, L)\). When \( B = 0 \), we write simply \( \kappa(V, L) \).

Clearly \( \kappa(V, B, L) < 0 \) if and only if \( K(V, B) \) is not pseudo-effective. We conjecture that \( \kappa(V, B, L) \in \mathbb{Q} \) in this case (cf. [Ba]). This will be derived from the following

**Fibration Conjecture.** Let \( L \) be a big \( \mathbb{Q} \)-bundle on a log terminal variety \((V, B)\) such that \( k = \kappa(V, B, L) < 0 \) and \( K(V, B) + tL \) is log ample for many \( t > 0 \). Then there is a birational model \((V', B')\) of \((V, B)\) together with a morphism \( \Phi : V' \to W \) such that \( \dim W < \dim V \), \( \Phi_* \mathcal{O}_{V'} = \mathcal{O}_W \), the relative Picard number \( \rho(V'/W) = 1 \), and \( K(V', B') - kL' = \Phi^* A \) for some ample \( \mathbb{Q} \)-bundle \( A \) on \( W \), where \( L' \) is the proper transform (as a Weil divisor) of \( L \) on \( V' \).

The birational map \( V \to V' \) will be obtained by applying the Log Minimal Model Programm, and will be a composite of elementary divisorial contractions and flips. Note that \( L' \) may not be nef even if \( L \) is ample.

Any way, we have \( K(F, B'_F) = kL'_F \) for any general fiber \( F \) of \( \Phi \), and \( L'_F \) is ample since \( \rho(V'/W) = 1 \). Hence, by a certain conjectural boundedness of \( \mathbb{Q} \)-Fano varieties, we shall obtain the following

**Spectrum Conjecture.** Let \( S_n \) be the Kodaira spectrum of polarized \( n \)-folds, namely, the set of all the possible Kodaira energies of \((V, L)\), where \( V \) is a variety with \( \dim V = n \) having only terminal singularities and \( L \) is an ample line bundle on \( V \). Then \( \{ t \in S_n | t < -\delta \} \) is a finite subset of \( \mathbb{Q} \) for any \( \delta > 0 \).
If we allow $V$ to have log terminal singularities, the assertion of the Spectrum Conjecture is false.

To be precise, let $\text{Lim}(X)$ denote the set of limit points of $X$, namely, $p \in \text{Lim}(X)$ if and only if $U \cap X$ is an infinite set for any neighborhood $U$ of $p$. Let $\text{Lim}^k(X) = \text{Lim}(\text{Lim}^{k-1}(X))$, let $X \cup Y$ denote $(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$ and let $S'_n$ be the set of all the possible Kodaira energies of $(V, B, L)$ such that $(V, B)$ is log terminal, $\dim V = n$, $B$ is a Z-Weil divisor (or equivalently, $b_i = 1$ for any $i$; possibly $B = 0$), and $L$ is ample on $V$. Then we have the following

**Log Spectrum Conjecture.** For any $k \leq n$, let $S'_{n,k}$ be the set $\{t|t - k \in S'_n \text{ and } -1 < t \leq 0\}$. Then $\text{Lim}^{n-k}(S'_{n,k}) = \{0\}$ and $\text{Lim}^{n-k}(S'_{n,k} \cup S'_{n+1,k+1}) = \text{Lim}^{n-k}(S'_{n,k} \cup \text{Lim}(S'_{n,k-1})) = \emptyset$. Moreover, for any $s < 0$, there exists $\delta > 0$ such that $\{t \in S'_n |s < t < s + \delta\}$ is a finite set.

The conclusion cannot be simplified even if $(V, B)$ is assumed to be smooth, or if we assume $B = 0$ allowing $V$ to have log terminal singularities.

**§2. Results**

The preceding conjectures are verified if $\dim V \leq 3$ under mild additional assumptions.

**Theorem 1.** Let $(V, B, L)$ be as in the Fibration Conjecture. Suppose that $n = \dim V \leq 3$ and that $V$ is $\mathbb{Q}$-factorial, namely, every Weil divisor on $V$ is $\mathbb{Q}$-Cartier. Then the assertion of the conjecture is true.

**Theorem 2.** The Spectrum Conjecture is true for the Kodaira spectrum of polarized 3-folds such that $V$ is $\mathbb{Q}$-factorial.

Remark. The $\mathbb{Q}$-factoriality is needed to apply the theory [Sho], [Ka2], [Ka1].

Outline of proof of Theorem 1. Set $\tau = \inf \{t \in \mathbb{Q} | K(V, B) + tL \text{ is nef}\}$. Then we see $\tau \in \mathbb{Q}$ by using Cone Theorem (cf. [KMM]). Next, by using the Base-Point-Free Theorem, we get a fibration $f : V \rightarrow X$ and an ample $\mathbb{Q}$-bundle $A$ on $X$ such that $K(V, B) + \tau L = f^*A$.

Now we let the Log Minimal Model Programm (cf. [KMM], [Sho], [Ka2]) run over $X$. By several elementary divisorial contractions and log flips, $(V, B, L)$ is transformed to a pair $(V_1, B_1)$ satisfying one of the following conditions:

1. There is an extremal ray $R$ on $V_1$ over $X$ such that its contraction morphism $\rho : V_1 \rightarrow W$ is of fibration type.
2. $K(V_1, B_1)$ is relatively nef over $X$.

During the process, $L$ is transformed to a $\mathbb{Q}$-bundle $L_1$ on $V_1$ as Weil divisors. It is easy to see that the bigness is preserved, and that the Kodaira energy does not change. Thus, in case (1), we are done by setting $V' = V_1$.

In case (2), we can show that $A_{V_1}$ is log ample on $(V_1, B_1)$. This is easy to prove when $b_i < 1$ for all $i$, but the proof is a little complicated in general.

Thus, $(V_1, B_1, L_1)$ satisfies the same condition as $(V, B, L)$. Note also that $\tau_2 = \inf \{t \in \mathbb{Q} | K(V_1, B_1) + tL_1 \text{ is nef}\} < \tau$. By the same process as above we get another triple $(V_2, B_2, L_2)$, and continue as long as necessary. By the termination theorem, we reach the above situation (1) after finite steps.

For the proof of Theorem 2, [Ka1] is essential.

**§3. Classification**

By the same method as in [F2], we can classify smooth polarized 3-folds $(M, L)$ with $\kappa(M, L) < -\frac{1}{2}$ as follows.
(3.1) $K + 3L$ is nef and $\kappa\epsilon(M, L) \geq -3$ unless $(M, L) \cong (\mathbb{P}^3, \mathcal{O}(1))$.

(3.2) $K + 2L$ is nef and $\kappa\epsilon \geq -2$ unless $(M, L)$ is a smooth scroll over a curve or a hyperquadric in $\mathbb{P}^2$, $\kappa\epsilon = -3$ in these cases.

(3.3) From now on, $K + 2L$ is assumed to be nef. If there is a divisor $E$ such that $(E, E_0) \cong (\mathbb{P}^2, \mathcal{O}(1))$ and $[E]_E = \mathcal{O}(-1)$, we have $\pi : M -\to M_1$ be the blow down of $E$ to a smooth point. Then the push-down $L_1$ of $L$ is ample on $M_1$ and $\pi^*(K_1 + 2L_1) = K + 2L$ for the canonical bundle $K_1$ of $M_1$. If there is a similar divisor on $M_1$, we blow it down again. After several steps, we get a polarized manifold $(M', L')$ such that $(K' + 2L')_M = K + 2L$, $\kappa\epsilon(M', L') = \kappa\epsilon(M, L) \geq -2$ on which there is no divisor of the above type. This model $(M', L')$ is called the (first) reduction of $(M, L)$.

(3.4) $\kappa\epsilon(M, L) = -2$ if and only if $K + 2L$ is not big. In this case, according to the value of $\kappa(K + 2L)$, $(M, L)$ is classified as follows:

(3.4.0) $K + 2L = 0$, i.e., $(M, L)$ is a Del Pezzo 3-fold.

(3.4.1) $(M, L)$ is a hyperquadric fibration over a curve.

(3.4.2) $(M, L)$ is a scroll over a surface.

Remark: In the cases (3.4.1) and (3.4.2), we have $M = M'$ by the ampleness of $L$.

(3.5) $K + 2L$ is nef and big if and only if $K' + 2L'$ is ample. Moreover, in this case, $K' + L'$ is nef except the following cases:

(3.5.1) $M'$ is a $\mathbb{P}^2$-bundle over a curve and $L'_F = \mathcal{O}(2)$ for any fiber $F$, $\kappa\epsilon = -3/2$.

(3.5.2) $M'$ is a hyperquadric in $\mathbb{P}^4$ and $L' = \mathcal{O}(2)$. $\kappa\epsilon = -3/2$.

(3.5.3) $(M', L') \cong (\mathbb{P}^3, \mathcal{O}(3))$, $\kappa\epsilon = -4/3$.

(3.6) From now on, $K' + L'$ is assumed to be nef. Then it is not big if and only if $\kappa\epsilon(M, L) = -1$. These cases are classified as follows:

(3.6.0) $K' + L' = 0$.

(3.6.1) $(M', L')$ is a Del Pezzo fibration over a curve.

(3.6.2) $(M', L')$ is a conic bundle over a surface.

(3.7) To study the case in which $K' + L'$ is nef big, we use the theory of second reduction as in [BS]. We have a birational morphism $\varphi : M' -\to M''$, such that $K' + L' = \varphi^* A$ for some ample line bundle $A$ on $M''$, and this pair $(M'', A)$ is called the second reduction of $(M, L)$. However, unlike the case of first reduction, $M''$ may have singularities, $L'' = \varphi, L'$ may not be invertible, may not be nef. By a careful analysis of the map $\varphi$ using Mori theory, we see that the singularity of $M''$ is of very special type. It is a hypersurface singularity of the type $\{x^2 + y^2 + z^2 + u^2 = 0\}$ $(k = 2, 3)$, or the quotient singularity isomorphic to the vertex of the cone over the Veronese surface $(\mathbb{P}^2, \mathcal{O}(2))$. In particular $L''$ is invertible except at quotient singularities and $2L''$ is invertible everywhere.

The cases $-1 < \kappa\epsilon < -1/2$ can be classified according to the type of the second reduction $(M'', A)$ as follows.

(3.8) $\kappa\epsilon = -4/5$. $(M'', A) \cong (\mathbb{P}^3, \mathcal{O}(1))$ and $L'' = \mathcal{O}(5)$.

(3.9.0) $\kappa\epsilon = -3/4$. $M''$ is a hyperquadric in $\mathbb{P}^4$ and $L'' = \mathcal{O}(4)$.

(3.9.1) $\kappa\epsilon = -3/4$. $(M'', A)$ is a scroll over a curve, $L'_F = \mathcal{O}(4)$ for any fiber $F$.

(3.10) $\kappa\epsilon = -5/7$. $(M'', A)$ is a cone over $(\mathbb{P}^2, \mathcal{O}(2))$. Compare [F2; (4.8.0)].

(3.11) $\kappa\epsilon = -2/3$. $(M'', A)$ is a Del Pezzo 3-fold, i.e., $K'' = -2A$.

(3.11.1) $\kappa\epsilon = -2/3$. $(M'', A)$ is a hyperquadric fibration over a curve.

(3.11.2) $\kappa\epsilon = -2/3$. $(M'', A)$ is a scroll over a surface.

In the following cases, $(M'', A)$ may need to be further blow down to another model $(M', A^b)$. But this pair has no worse singularities than $(M', A)$. 


(3.12.0.0) $\kappa \epsilon = -3/5$. $M^p$ is a hyperquadric in $\mathbb{P}^4$ and $A^b = \mathcal{O}(2)$.

(3.12.0.1) $\kappa \epsilon = -3/5$. $M^p$ has exactly one quotient singularity, and the blow-up $M^\#$ at this point is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth plane cubic $C$. $A^b_{M^\#}$ is $3H - E_C$, where $H$ is the pull-back of $\mathcal{O}(1)$ of $\mathbb{P}^3$ and $E_C$ is the exceptional divisor over $C$. Compare \[\text{F2};\{4.6.0.1.0\}\].

(3.12.0.2) $\kappa \epsilon = -3/5$. $M^\#$ has exactly one quotient singularity, and the blow-up $M^\#$ at this point is isomorphic to the blow-up of $\mathbb{P}^3$ along a smooth plane cubic $C$. $A^b_{M^\#}$ is $3H - E_C$, where $H$ is the pull-back of $\mathcal{O}(1)$ of $\mathbb{P}^3$ and $E_C$ is the exceptional divisor over $C$. Compare \[\text{F2};\{4.6.0.2.1\}\].

(3.12.1) $\kappa \epsilon = -3/5$. $M^p$ is a $\mathbb{P}^2$-fibration over a curve and $A^b_F = \mathcal{O}(2)$ for any general fiber $F$.

(3.13) $\kappa \epsilon = -4/7$. $(M^p, A^p) \cong (\mathbb{P}^3, \mathcal{O}(3))$ and $L^b = \mathcal{O}(7)$.

(3.14) $\kappa \epsilon = -5/9$. $(M^p, B)$ is a cone over $(\mathbb{P}^2, \mathcal{O}(2))$ for some line bundle $B$ such that $A^b = 2B$.

(3.15) When $\kappa \epsilon \geq -1/2$, we can reduce the problem to the case in which $K^b + A^b$ is nef, where $K^b$ is the canonical $\mathbb{Q}$-bundle of $M^p$.

Note that $M^p$ is obtained from $M$ without using flips.

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