Some Characterizations on the Normalized Lommel, Struve and Bessel Functions of the First Kind

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Abstract

In this paper, we introduce new technique for determining some necessary and sufficient conditions of the normalized Bessel functions $j_\nu$, normalized Struve functions $h_\nu$ and normalized Lommel functions $s_{\mu,\nu}$ of the first kind, to be in the subclasses of starlike and convex functions of order $\alpha$ and type $\beta$.

Keywords: analytic function; starlike function; convex function; Bessel functions; Struve functions; Lommel functions.

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1. Introduction

Let $\mathcal{A}$ denotes the class of analytic functions of the form

\begin{equation}
 f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
 \end{equation}

which are defined on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$, and let $T$ be the subclass of $\mathcal{A}$ consisting of functions of the form

\begin{equation}
 f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0).
 \end{equation}
Definition 1. For $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, let $S^*(\alpha, \beta)$ denote the subclass of $A$ consisting of functions $f(z)$ of the form (1.1) and satisfy

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{1 - 2\alpha} \right| < \beta; \ z \in U,$$

and $f \in K(\alpha, \beta)$ denotes the subclass of $A$ consisting of functions $f(z)$ of the form (1.1) such that

$$\left| \frac{zf''(z)}{f'(z)} + 2 \frac{1}{1 - \alpha} \right| < \beta; \ z \in U.$$

The subclasses $S^*(\alpha, \beta)$ and $K(\alpha, \beta)$ are the well-known subclasses of starlike and convex functions of order $\alpha$ and type $\beta$, respectively, introduced by Gupta and Jain [17]. Moreover, let $T^*(\alpha, \beta)$ and $C(\alpha, \beta)$ be two subclasses of $T$ defined by

$$T^*(\alpha, \beta) = S^*(\alpha, \beta) \cap T \text{ and } C(\alpha, \beta) = K(\alpha, \beta) \cap T.$$ 

It is known that:

$$f(z) \in C(\alpha, \beta) \iff zf'(z) \in T^*(\alpha, \beta),$$

We note that $S^*(\alpha, 1) = S^*(\alpha)$ and $K(\alpha, 1) = K(\alpha)$ the subclasses of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$, respectively, which were introduced by Robertson [25], Schild [26] and MacGregor [20]. Also, $T^*(\alpha, 1) = T^*(\alpha)$ and $C(\alpha, 1) = C(\alpha)$, are the subclasses of starlike and convex functions of order $\alpha$ with negative coefficients introduced by see Silverman [27].

Recently, several researchers studied some subclasses of analytic functions, $F \subset A$, involving special functions, to find different conditions such that the members of $F$ have certain geometric properties such as univalency, starlikeness or convexity in $U$. In this context many results are available in the literature regarding the generalized hypergeometric functions (see [21] and [28]). Special functions, like Bessel, Struve and Lommel functions of the first kind have many attractive geometric properties. Recently, the geometric properties of these special functions were investigated motivated by some earlier results. In the sixties Brown [10]-[12], Kreyszig and Todd [18] and in Wilf [32] the univalence and starlikeness of Bessel functions of the first kind was considered, while in the recent years the radii of univalence, starlikeness and convexity for the normalized forms of Bessel, Struve and Lommel functions of the first kind were obtained, see the papers [1]-[9], [29], [30], [13]-[16] and the references cited therein. In the above works the authors used intensively some properties of the positive zeros of Bessel, Struve and Lommel functions of the first kind, under some conditions. In this paper our aim is to give some new results for the starlikeness and convexity of the normalized Bessel, Struve and Lommel functions of the first kind.

In this paper, we consider three classical special functions, the Bessel function of the first kind $J_\nu$, ...
the Struve function of the first kind $H_\nu$ and the Lommel function of the first kind $S_{\mu,\nu}$. It is known that the Bessel functions has the infinite series representation

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1) \Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu}, \quad (1.5)$$

where $z, \nu \in \mathbb{C}$ such that $\nu \notin \mathbb{Z}^- := \{-1, -2, \ldots\}$. Also, the Struve and Lommel functions can be represented as the infinite series

$$H_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\frac{3}{2}) \Gamma(k+\nu+\frac{3}{2})} \left(\frac{z}{2}\right)^{2k+\nu+1} ; \nu + \frac{1}{2} \notin \mathbb{Z}^- , \quad (1.6)$$

and

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right)}{\Gamma(k+\frac{3}{2}) \Gamma(k+\frac{\mu+\nu+3}{2})} \left(\frac{z}{2}\right)^{2k} ; \mu \pm \nu + 1 \notin \mathbb{Z}^- , \quad (1.7)$$

for $\mu, \nu, z \in \mathbb{C}$. In addition, we know that the Bessel function $J_\nu(z)$ is a solution of the homogeneous Bessel differential equation (see [23, p. 217])

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2) w(z) = 0,$$

and the Struve function $H_\nu(z)$ and Lommel function $S_{\mu,\nu}(z)$ are a particular solutions of the following non-homogeneous Bessel differential equations (see [23, p. 288 and p. 294]), respectively

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2) w(z) = \frac{(\frac{z}{2})^{\nu-1}}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)},$$

and

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2) w(z) = z^{\mu+1}.$$

It is worthy to notice that the functions $J_\nu(z)$, $H_\nu(z)$ and $S_{\mu,\nu}(z)$ are explicitly defined in terms of the hypergeometric function $1F_2$ by the following:

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \; 1F_2\left(1; 1, \nu+1; -\frac{z^2}{4}\right) ; \nu \notin \mathbb{Z}^- ,$$

$$H_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu+1}}{\sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right)} \; 1F_2\left(1; 3, \nu + \frac{3}{2}; -\frac{z^2}{4}\right) ; \nu + \frac{1}{2} \notin \mathbb{Z}^- ,$$

and

$$S_{\mu,\nu}(z) = \frac{z^{\mu+1}}{\left(\mu - \nu + 1\right) \left(\mu + \nu + 1\right)} \; 1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right) ; \frac{\mu+\nu+1}{2} \notin \mathbb{Z}^- .$$

We refer to Watson’s treatise [31] for comprehensive information about these functions.
In this paper, we are mainly interested in the normalized Bessel function of the first kind \( j_\nu : U \to C \), normalized Struve function of the first kind \( h_\nu : U \to C \), and normalized Lommel functions of the first kind \( s_{\mu,\nu} : U \to C \), which are defined as follows

\[
j_\nu(z) := \Gamma (\nu + 1) z^{1-\frac{\nu}{2}} J_\nu(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(1)_k (\nu + 1)_k} z^{k+1}; \quad \nu \notin \mathbb{Z}^- , \quad (1.8)
\]

\[
h_\nu(z) := \Gamma \left(\frac{3}{2}\right) \Gamma (\nu + \frac{3}{2}) z^{1-\frac{\nu}{2}} H_\nu(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{3}{2}\right)_k (\nu + \frac{3}{2})_k} z^{k+1}; \quad \nu + \frac{1}{2} \notin \mathbb{Z}^- , \quad (1.9)
\]

and

\[
s_{\mu,\nu}(z) := (\mu-\nu+1) (\mu+\nu+1) z^{-\mu} S_{\mu,\nu}(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(\mu-\nu+3)_k (\mu+\nu+3)_k} z^{k+1}; \quad \frac{\mu+\nu+1}{2} \notin \mathbb{Z}^- . \quad (1.10)
\]

Observe that

\[
s_{\nu,\nu}(z) = h_\nu(z) \quad \text{and} \quad s_{\nu-1,\nu}(z) = j_\nu(z) . \quad (1.11)
\]

Very recently, Cho et al. [19] and Murugusundaramoorthy and Janani [22] (see also Porwal and Dixit [24]) introduced some characterization of generalized Bessel functions of first kind to be in certain subclasses of uniformly starlike and uniformly convex functions. In the present paper, we determine necessary and sufficient conditions for normalized Bessel function of the first kind, normalized Struve function of the first kind and normalized Lommel functions of the first kind to be in certain Subclasses of analytic functions.

2. Characterizations on Lommel Functions

Unless otherwise mentioned, we assume in the reminder of this paper that, \( 0 \leq \alpha < 1 \), \( 0 < \beta \leq 1 \), and \( z \in U \).

To establish our main results, we shall require the following lemmas:

**Lemma 1 ([17])**. (i) A sufficient condition for a function \( f \) of the form (1.1) to be in the class \( S^*(\alpha, \beta) \) is that

\[
\sum_{k=2}^{\infty} [k - 1 + \beta (k + 1 - 2\alpha)] |a_k| \leq 2\beta (1 - \alpha) . \quad (2.1)
\]

(ii) A necessary and sufficient condition for a function \( f \) of the form (1.2) to be in the \( T^*(\alpha, \beta) \) is that

\[
\sum_{k=2}^{\infty} [k - 1 + \beta (k + 1 - 2\alpha)] a_k \leq 2\beta (1 - \alpha) . \quad (2.2)
\]

**Lemma 2 ([17])**. (i) A sufficient condition for a function \( f \) of the form (1.1) to be in the class \( K(\alpha, \beta) \) is that

\[
\sum_{k=2}^{\infty} k [k - 1 + \beta (k + 1 - 2\alpha)] |a_k| \leq 2\beta (1 - \alpha) . \quad (2.3)
\]
(ii) A necessary and sufficient condition for a function \( f \) of the form (1.2) to be in the \( C(\alpha, \beta) \) is that

\[
\sum_{k=2}^{\infty} k [k - 1 + \beta (k + 1 - 2\alpha)] a_k \leq 2 \beta (1 - \alpha) .
\] (2.4)

If

\[
f_0(z) = \frac{z}{1 + z} = \sum_{k=0}^{\infty} (-1)^k z^{k+1} \quad (z \in U)
\] (2.5)

and using the convolution principle, then let us define the function

\[
s_{\mu, \nu}(z) := s_{\mu, \nu}(z) * f_0(z),
\]

then we have the first result as follows.

**Theorem 1.** If \( \mu > \nu - 3 \), then the condition

\[
(1 + \beta) s'_{\mu + 2, \nu}(1) + 2 \beta (1 - \alpha) s_{\mu + 2, \nu}(1) \leq \frac{8 \beta (1 - \alpha)}{(\mu - \nu + 3) (\mu + \nu + 3)}
\] (2.6)

suffices that \( s_{\mu, \nu}(z) \in S^*(\alpha, \beta) \).

**Proof.** Since

\[
s_{\mu, \nu}(z) = z + \sum_{k=2}^{\infty} \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k-1} \left(\frac{\mu+\nu+3}{2}\right)_{k-1}} z^k ; \quad \frac{\mu+\nu+1}{2} \notin \mathbb{Z}_-
\]

By virtue of (i) in Lemma 1, it is suffices to show that

\[
\sum_{k=2}^{\infty} [k - 1 + \beta (k + 1 - 2\alpha)] \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k-1} \left(\frac{\mu+\nu+3}{2}\right)_{k-1}} \leq 2 \beta (1 - \alpha).
\]

By simple computation, we have

\[
L(\alpha, \beta; \mu, \nu) = \sum_{k=2}^{\infty} [k - 1 + \beta (k + 1 - 2\alpha)] \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k-1} \left(\frac{\mu+\nu+3}{2}\right)_{k-1}}
\]

\[
= \sum_{k=0}^{\infty} [k + 1 + \beta (k + 3 - 2\alpha)] \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k+1} \left(\frac{\mu+\nu+3}{2}\right)_{k+1}}
\]

\[
= \sum_{k=0}^{\infty} [(k+1) (1 + \beta) + 2 \beta (1 - \alpha)] \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k+1} \left(\frac{\mu+\nu+3}{2}\right)_{k+1}}
\]

\[
= (1 + \beta) \sum_{k=0}^{\infty} (k+1) \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k+1} \left(\frac{\mu+\nu+3}{2}\right)_{k+1}} + 2 \beta (1 - \alpha) \sum_{k=0}^{\infty} \frac{1}{\left(\frac{\mu-\nu+3}{2}\right)_{k+1} \left(\frac{\mu+\nu+3}{2}\right)_{k+1}}
\]
Proof. Since the condition (2.6) is satisfied. This completes the proof of Theorem 1.

Therefore, we see that the last expression (2.7) is bounded above by $2\beta(1 - \alpha)$ if condition (2.6) is satisfied. This completes the proof of Theorem 1.

If

$$f_1(z) = z \left(2 - \frac{1}{1 + z}\right) = z + \sum_{k=1}^{\infty} (-1)^{k+1} z^{k+1} \quad (z \in U),$$

and the function

$$t_{\mu, \nu}(z) := s_{\mu, \nu}(z) * f_1(z),$$

then we have the following result.

**Theorem 2.** For $\mu > \nu - 3$, then the condition (2.6) is the necessary and sufficient condition for $t_{\mu, \nu}(z)$ to be in the class $T^*(\alpha, \beta)$.

**Proof.** Since

$$t_{\mu, \nu}(z) = z - \sum_{k=2}^{\infty} \frac{1}{(\mu - \nu + 3) k_{-1} (\mu + \nu + 3) k_{-1}} z^k; \quad \frac{\mu + \nu + 1}{2} \notin \mathbb{Z}^-,$$

then by using Lemma 1 together with the same techniques given in the proof of Theorem 1, we have immediately Theorem 2.

**Theorem 3.** If $\mu > \nu - 3$, then the condition

$$(1 + \beta) s''_{\mu, \nu+2}(1) + 2 (1 + \beta) s'_{\mu, \nu+2}(1) + 2 (1 - \alpha) (2\beta - 1) s_{\mu, \nu+2}(1) \leq \frac{8\beta(1 - \alpha)}{\mu + \nu + 3}$$

suffices that $s_{\mu, \nu}(z) \in K(\alpha, \beta)$.

**Proof.** By virtue of Lemma 2, it is suffices to show that

$$\sum_{k=2}^{\infty} k \left[ k - 1 + \beta (k + 1 - 2\alpha) \right] \frac{1}{(\mu - \nu + 3) k_{-1} (\mu + \nu + 3) k_{-1}} \leq 2\beta(1 - \alpha).$$
By simple computation, we have
\[ F(\alpha, \beta; \mu, \nu) = \sum_{k=2}^{\infty} k [k - 1 + \beta (k + 1 - 2\alpha)] \left( \frac{\mu + \nu + 1}{2} \right)^{k-1} \left( \frac{\mu + \nu + 1}{2} \right) \]
\[ = \sum_{k=0}^{\infty} (k + 2) [k + 1 + \beta (k + 1 + 2 (1 - \alpha))] \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ = \sum_{k=0}^{\infty} \left[ (1 + \beta) (k + 1) (k + 2 (2 + \beta - \alpha) (k + 1) + 2 (1 - \alpha) (2 \beta - 1) \right] \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ = \left( 1 + \beta \right) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} + 2 (2 + \beta - \alpha) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ + 2 (1 - \alpha) (2 \beta - 1) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ = \left( 1 + \beta \right) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} + 2 (2 + \beta - \alpha) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ + 2 (1 - \alpha) (2 \beta - 1) \sum_{k=0}^{\infty} \left( \frac{\mu + \nu + 1}{2} \right)^{k+1} \]
\[ = \left( 1 + \beta \right) s_{\mu+2,\nu} ^{(1)} + 2 (1 + \beta) s_{\mu+2,\nu} ^{(1)} + 2 (1 - \alpha) (2 \beta - 1) s_{\mu+2,\nu} ^{(1)} \right] \]
\[ = (2.11) \]

Therefore, we see that the expression (2.11) is bounded above by $2\beta (1 - \alpha)$ if (2.10) is satisfied. Hence, the proof is completed.

**Theorem 4.** For $\mu > \nu - 3$, then the condition (2.10) is the necessary and sufficient condition for $t_{\mu,\nu}(z)$ to be in the class $C(\alpha, \beta)$.

Putting $\beta = 1$ in Theorems 2, 4, we obtain the following corollaries.

**Corollary 1.** The function $t_{\mu,\nu}(z)$ is a starlike function of order $\alpha$ $(0 \leq \alpha < 1)$, if and only if
\[ s_{\mu+2,\nu} ^{(1)} + (1 - \alpha) s_{\mu+2,\nu} ^{(1)} \leq \frac{4(1 - \alpha)}{(\mu - \nu + 3) (\mu + \nu + 3)} \]

**Corollary 2.** The function $t_{\mu,\nu}(z)$ is a convex function of order $\alpha$ $(0 \leq \alpha < 1)$, if and only if
\[ s_{\mu+2,\nu} ^{(2)} + 2s_{\mu+2,\nu} ^{(1)} + (1 - \alpha) s_{\mu+2,\nu} ^{(1)} \leq \frac{4(1 - \alpha)}{(\mu - \nu + 3) (\mu + \nu + 3)} \]

**3. Characterizations on Struve Functions**

Taking $\mu = \nu$ in Theorems 1-4, then we obtain the corresponding results of Struve function $h_{\nu}$, as following:
Theorem 5. If \( \nu > -\frac{1}{2} \), then the condition

\[
(1 + \beta) h'_{\nu+2}(1) + 2 \beta (1 - \alpha) h_{\nu+2}(1) \leq \frac{8 \beta (1 - \alpha)}{3 (2 \nu + 3)}
\]  

(3.1)
suffices that \( h_\nu(z) \in S^*(\alpha, \beta) \), where

\[
h_\nu(z) := h_\nu(z) \ast f_0(z) = z + \sum_{k=2}^{\infty} \frac{1}{(\nu + \frac{3}{2})_{k-1}} z^{k+1} \left( \nu > -\frac{1}{2} \right).
\]  

(3.2)

Theorem 6. If \( \nu > -\frac{3}{2} \), then the condition (3.1) is the necessary and sufficient condition for \( h_\nu(z) \) to be in the class \( T^*(\alpha, \beta) \), where

\[
h_\nu(z) := (h_\nu(z) \ast f_1(z)) = z - \sum_{k=2}^{\infty} \frac{1}{(\nu + \frac{3}{2})_{k-1}} z^{k+1} \left( \nu > -\frac{1}{2} \right).
\]  

(3.3)

Theorem 7. If \( \nu > -\frac{3}{2} \), then the condition

\[
(1 + \beta) h''_{\nu+2}(1) + 2 (1 + \beta) h'_{\nu+2}(1) + 2 (1 - \alpha) (2\beta - 1) h_{\nu+2}(1) \leq \frac{8 \beta (1 - \alpha)}{3 (2 \nu + 3)}
\]  

(3.2)
suffices that \( h_\nu(z) \in K(\alpha, \beta) \).

Theorem 8. If \( \nu > -\frac{3}{2} \), then the condition (3.2) is the necessary and sufficient condition for \( h_\nu(z) \) to be in the class \( C(\alpha, \beta) \).

Putting \( \beta = 1 \) in Theorems 6, 8, we have the following corollaries.

Corollary 3. The function \( h_\nu(z) \) is a starlike function of order \( \alpha \) \((0 \leq \alpha < 1)\), if and only if

\[
h''_{\nu+2}(1) + (1 - \alpha) h'_{\nu+2}(1) \leq \frac{4 (1 - \alpha)}{3 (2 \nu + 3)} \quad \left( 0 \leq \alpha < 1, \nu > -\frac{3}{2} \right).
\]

Corollary 4. The function \( h_\nu(z) \) is a convex function of order \( \alpha \) \((0 \leq \alpha < 1)\), if and only if

\[
h''_{\nu+2}(1) + 2 h'_{\nu+2}(1) + (1 - \alpha) h_{\nu+2}(1) \leq \frac{4 (1 - \alpha)}{3 (2 \nu + 3)} \quad \left( 0 \leq \alpha < 1, \nu > -\frac{3}{2} \right).
\]

### 4. Characterizations on Bessel Functions

Taking \( \mu = \nu - 1 \) in Theorems 1-4, then we obtain the corresponding results of Bessel function \( j_\nu \), as following:

Theorem 9. If \( \nu > -1 \), then the condition

\[
(1 + \beta) j'_{\nu+1}(1) + 2 \beta (1 - \alpha) j_{\nu+1}(1) \leq \frac{2 \beta (1 - \alpha)}{\nu + 1},
\]  

(4.1)
suffices that $j_\nu(z) \in \mathcal{S}^*(\alpha, \beta)$, where

$$j_\nu(z) := j_\nu(z) * f_0(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1)_{k-1}(\nu+1)_{k-1}} z^k \ (\nu > -1). \quad (4.2)$$

**Theorem 10.** If $\nu > -1$, then the condition $(4.1)$ is the necessary and sufficient condition for $j_\nu(z)$ to be in the class $T^*(\alpha, \beta)$, where

$$j_\nu(z) := j_\nu(z) * f_1(z) = z - \sum_{k=2}^{\infty} \frac{1}{(1)_{k-1}(\nu+1)_{k-1}} z^k \ (\nu > -1). \quad (4.3)$$

**Theorem 11.** If $\nu > -1$, then the condition

$$(1 + \beta) j''_{\nu+1}(1) + 2 (1 + \beta) j'_{\nu+1}(1) + 2 (1 - \alpha) (2\beta - 1) j_{\nu+1}(1) \leq \frac{2\beta(1-\alpha)}{\nu + 1}, \quad (4.4)$$

suffices that $j_\nu(z) \in K(\alpha, \beta)$.

**Theorem 12.** If $\nu > -1$, then the condition $(4.4)$ is the necessary and sufficient condition for $j_\nu(z)$ to be in the class $C(\alpha, \beta)$.

Putting $\beta = 1$ in Theorems 10, 12, we get the following corollaries.

**Corollary 5.** The function $j_\nu(z)$ is a starlike function of order $\alpha \ (0 \leq \alpha < 1)$, if and only if

$$j'_{\nu+1}(1) + (1 - \alpha) j_{\nu+1}(1) \leq \frac{1 - \alpha}{\nu + 1}. \quad (5.5)$$

**Corollary 6.** The function $j_\nu(z)$ is a convex function of order $\alpha \ (0 \leq \alpha < 1)$, if and only if

$$j''_{\nu+1}(1) + 2j'_{\nu+1}(1) + (1 - \alpha) j_{\nu+1}(1) \leq \frac{1 - \alpha}{\nu + 1}. \quad (5.6)$$

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