A no-short scalar hair theorem for rotating Kerr black holes

Shahar Hod

The Ruppin Academic Center, Emeq Hefer 40250, Israel
The Hadassah Institute, Jerusalem 91010, Israel

E-mail: shaharhod@gmail.com

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Abstract
If a black hole has hair, how short can this hair be? A partial answer to this intriguing question was recently provided by the ‘no-short hair’ theorem which asserts that the external fields of a spherically symmetric electrically neutral hairy black-hole configuration must extend beyond the null circular geodesic which characterizes the corresponding black-hole spacetime. One naturally wonders whether the no-short hair inequality \( r_{\text{hair}} > r_{\text{null}} \) is a generic property of all electrically neutral hairy black-hole spacetimes. In this paper we provide evidence that the answer to this interesting question may be positive. In particular, we prove that the recently discovered cloudy Kerr black-hole spacetimes—non-spherically symmetric non-static black holes which support linearized massive scalar fields in their exterior regions—also respect this no-short hair lower bound. Specifically, we \textit{analytically} derive the lower bound \( r_{\text{field}} / r_{\pm} > r_+ / r_- \) on the effective lengths of the external bound-state massive scalar clouds (here \( r_{\text{field}} \) is the peak location of the stationary bound-state scalar fields and \( r_{\pm} \) are the horizon radii of the black hole). Remarkably, this lower bound is universal in the sense that it is independent of the physical parameters (proper mass and angular harmonic indices) of the exterior scalar fields. Our results suggest that the lower bound \( r_{\text{hair}} > r_{\text{null}} \) may be a general property of asymptotically flat electrically neutral hairy black-hole configurations.

Keywords: black holes, scalar fields, null circular geodesics

1. Introduction

The elegant uniqueness theorems [1–4] have established the fact that all stationary black-hole solutions of the Einstein-vacuum equations are uniquely described by the Kerr [5, 6] spacetime metric. In his celebrated ‘no-hair’ conjecture, Wheeler [7, 8] went even one step
further by suggesting that the Kerr spacetime is the only stationary black-hole solution of the coupled Einstein-matter field equations.

Matter fields in an asymptotically flat black-hole spacetime are expected, according to the no-hair conjecture [7, 8], to be scattered away to infinity or to be absorbed into the black hole. Early analytical studies [9–11] of the coupled Einstein-matter field equations have provided support for the validity of this conjecture. In particular, these studies [9–11] have shown that static scalar fields, static spinor fields, and static massive vector fields cannot be supported in the extremal spacetime regions of asymptotically flat regular black holes.

However, subsequent numerical studies of the nonlinear Einstein-matter field equations have revealed that other matter models may be characterized by non-trivial (that is, non-Kerr like) hairy black-hole solutions. In particular, the first convincing counterexample to the no-hair conjecture was provided by the ‘colored’ hairy black-hole spacetimes [12]. These non-trivial solutions of the coupled Einstein–Yang–Mills equations describe non-vacuum black holes which support regular Yang–Mills fields in their exterior regions.

The intriguing discovery of the Einstein–Yang–Mills hairy black-hole solutions [12] resulted in an intense research effort to understand the physical properties of these (and other [12–23]) hairy black-hole spacetimes. These studies [12–23] have established the fact that various types of nonlinear matter models [12–23], when coupled to the Einstein field equations, may produce hairy black-hole solutions that violate the original formulation [7, 8] of the no-hair conjecture.

A generic feature of these hairy black-hole solutions [12–23] was revealed in [24], where it was proved that if a static spherically symmetric black hole has hair, then this hair cannot be short in the sense that the corresponding external matter fields must extend beyond the null circular geodesic which characterizes the black-hole spacetime [24, 25]:

\[
 r_{\text{field}} > r_{\text{null}}. \tag{1}
\]

It is important to emphasize the fact that this ‘no-short hair’ theorem was proved in [24] under the following three assumptions:

1. The hairy black-hole spacetime is static.
2. The hairy black-hole spacetime is spherically symmetric.
3. The energy density \( \rho = T^t_t \) outside the black-hole horizon approaches zero asymptotically faster than \( r \). In particular, the hairy black-hole spacetime is assumed to be electrically neutral.

It is worth noting that, under the above-mentioned assumptions, the no-short hair relation (1) is universal in the sense that it is independent of the physical parameters of the external matter fields [24, 25] (see footnote 4). The no-short hair lower bound (1) may therefore be regarded as a more modest alternative (and possibly a more robust alternative) to the original [7, 8] no-hair conjecture.

1. We refer here to electrically neutral asymptotically flat black-hole spacetimes.
2. The term ‘regular black holes’ is used in this paper to describe black-hole spacetimes which are characterized by regular event horizons.
3. It is interesting to note that most of these hairy black-hole solutions of the coupled Einstein-matter field equations are known to be dynamically unstable.
4. It is worth mentioning that an earlier (and very elegant) ‘no-short hair’ theorem for static spherically symmetric hairy black-hole configurations was established by Núñez et al [26]. It should be emphasized, however, that in [25] we have explicitly proved that non-spherically symmetric Kerr-black-hole-massive-scalar-field configurations can violate this earlier version of the no-short hair relation.
5. This requirement implies that the black-hole spacetime has no extra globally conserved charges (besides the ADM mass) defined at asymptotic infinity.
One naturally wonders whether the no-short hair property (1) is a generic feature of all asymptotically flat electrically neutral hairy black-hole spacetimes? The main goal of the present paper is to test the validity of the no-short hair lower bound (1) beyond the restricted regime of static spherically symmetric black-hole spacetimes. To that end, we shall study analytically the physical properties of (non-static, non-spherically symmetric) rotating Kerr black holes linearly coupled to stationary bound-state configurations of massive scalar fields.

Before proceeding, it is worth noting that former analytical studies [27] have shown that stationary bound-state configurations of massive scalar fields linearly coupled to near-extremal\(^7\) Kerr black holes conform to the lower bound (1). Moreover, numerical studies [28] of this non-static non-spherically symmetric physical system have provided further compelling evidence that the composed Kerr-black-hole-massive-scalar-field configurations respect the lower bound (1). In the present paper we shall provide a rigorous analytical proof for the validity of this no-short hair property for generic\(^8\) Kerr-black-hole-massive-scalar-field configurations.

2. Composed Kerr-black-hole-massive-scalar-field configurations

Recent analytical [27] and numerical [28] studies of the coupled Einstein–Klein–Gordon field equations have revealed that, due to the intriguing physical phenomenon of superradiant scattering of bosonic (integer-spin) fields in Kerr black-hole spacetimes [29, 30], these rotating black holes can support stationary (that is, non-decaying) scalar field configurations in their exterior spacetime regions.

These stationary bound-state regular field configurations mark the boundary between stable and unstable resonances of the composed Kerr-black-hole-massive-scalar-field system. In particular, for a given value of the azimuthal harmonic index \(m\), these orbiting field configurations are characterized by azimuthal frequencies \(\omega_{\text{field}}\) which coincide with the threshold (critical) frequency\(^9\)

\[
\omega_{\text{field}} = \omega_c \equiv m\Omega_H
\]

for superradiant amplification of bosonic fields in the rotating Kerr black-hole spacetime. Here \([5, 6]\)

\[
\Omega_H = \frac{a}{r_h^2 + a^2}
\]

is the Kerr black-hole angular velocity, where \(r_h\) and\(^10\) \(a\) are the horizon-radius and angular momentum per unit mass of the rotating Kerr black hole.

\(^6\) It is worth emphasizing that the composed Kerr-black-hole-massive-scalar-field configurations that we consider here respect the assumption made in the original no-short hair theorem [24] that the energy density \(T_t^t\) outside the black-hole horizon approaches zero asymptotically faster than \(r^{-4}\). The case of charged Kerr–Newman black holes linearly coupled to charged massive scalar fields was considered in [25]. In this case the energy density outside the black-hole horizon is characterized by the asymptotic behavior \(T_t^t \sim Q^2 / r^4\), where \(Q\) is the electric charge of the black-hole spacetime. Thus, charged Kerr–Newman black holes do not respect the assumption made in the original no-short hair theorem [24] that the energy density \(T_t^t\) outside the black-hole horizon approaches zero asymptotically faster than \(r^{-4}\).

\(^7\) That is, rapidly rotating black holes with the property \(J/M^2 \approx 1\) (Here \(J\) and \(M\) are respectively the angular momentum and the mass of the spinning black hole).

\(^8\) It is worth emphasizing that our analytical proof (to be given below) would be valid for generic values (that is, \(J/M^2 \in (0, 1)\)) of the dimensionless black-hole rotation parameter.

\(^9\) We shall use natural units in which \(G = c = h = 1\).

\(^10\) We shall assume \(a > 0\) and \(m > 0\) without loss of generality.
The resonance condition (2), which characterizes the stationary scalar field configurations in the rotating Kerr black-hole spacetime, guarantees that there is no net energy flux into the black hole [27, 28, 31]. In addition, for a scalar field of mass \( \mu \), the mutual gravitational attraction between the central black hole and the external massive field provides a natural confinement mechanism which prevents the orbiting scalar configuration from radiating its energy to infinity. In particular, the external bound-state massive scalar configurations are characterized by trapped field modes in the bounded frequency regime\(^{11}\) (see equation (11) below)

\[
0 < \omega_{\text{field}}^2 < \mu^2.
\] (4)

Before proceeding, we would like to emphasize that the bound-state Kerr-black-hole-massive-scalar-field configurations that we shall study in this paper should not be regarded as genuine hairy black-hole spacetimes. In particular, we shall treat the external stationary scalar field configurations at the linear level. Hence, throughout the paper we shall use the term ‘scalar clouds’ to describe these linearized bound-state field configurations\(^{12}\). While the fully nonlinear Einstein-scalar field equations can only be studied numerically\(^{28}\), below we shall explicitly demonstrate that the physical properties of the linearly coupled Kerr-black-hole-massive-scalar-field configurations can be studied analytically\(^{13}\).

3. Description of the system

We shall explore the physical properties of a rotating Kerr black hole of mass \( M \) and angular-momentum \( J = Ma \) which is linearly coupled to a scalar field \( \Psi \) of mass \( \mu \) (see footnote 11).

In terms of the Boyer–Lindquist coordinates \( (t, r, \theta, \phi) \), the black-hole spacetime metric is described by the line element\(^{[5, 6]}\)

\[
ds^2 = -\frac{\Delta}{\rho^2} dt^2 - a \sin^2 \theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [a dt - (r^2 + a^2) d\phi],
\] (5)

where \( \Delta \equiv r^2 - 2Mr + a^2 \) and \( \rho \equiv r^2 + a^2 \cos^2 \theta \). The zeros of \( \Delta \) determine the black-hole horizon radii:

\[
r_g = M \pm \sqrt{M^2 - a^2}.
\] (6)

The Klein–Gordon field equation

\[
(\nabla^{\mu} \nabla_{\mu} - \mu^2) \Psi = 0
\] (7)

determines the dynamics of the linearized massive scalar field \( \Psi \) in the curved black-hole spacetime. It is convenient to write the scalar eigenfunction \( \Psi \) in the form\(^{14}\)

\[
\Psi(t, r, \theta, \phi) = \sum_{l,m} e^{i\omega t} S_m(\theta; m, a) a^{i\omega \rho/r} R_{lm}(r; M, a, \mu, \omega_r) e^{-i\omega_r t},
\] (8)

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\(^{11}\) Note that the mass parameters \( \mu \) of the scalar field stands for \( \mu / \hbar \). This field parameter therefore has the dimensions of (length)\(^{-1}\).

\(^{12}\) It is worth noting that the term ‘hair’ is usually used in the black-hole physical literature to describe the nonlinear external matter fields of a non-vacuum black-hole spacetime.

\(^{13}\) It is important to stress the fact that our present analytical study is not complete. In particular, we believe that, using numerical techniques\(^{[28]}\), it would be highly interesting to further test the validity of the no-short hair property (1) in the nonlinear regime of the Kerr-massive-scalar-field hairy black-hole configurations.

\(^{14}\) The integer parameters \( m \) and \( l \gg |m| \) are the harmonic indices (azimuthal and spheroidal) of the scalar field mode (see equation (9) below).
in which case one finds that the Klein–Gordon wave equation (7) can be expressed as a set of two coupled ordinary differential equations: the first equation (see equation (9) below) determines the angular part $S_{lm}$ of the scalar eigenfunction $\Psi$, whereas the second equation (see equation (10) below) determines the radial part $R_{lm}$ of the scalar eigenfunction $\Psi$.

The characteristic angular equation (also known as the spheroidal wave equation) is given by [32–37]
\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_{lm}}{d\theta} \right) + \left[ K_{lm} + a^2 (\mu^2 - \omega^2) \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{lm} = 0. \tag{9}
\]
The angular solutions $S_{lm}(\theta)$ of (9) are required to be regular at the two boundaries, $\theta = 0$ and $\theta = \pi$. These boundary conditions single out a discrete family $\{K_{lm}\}$ of angular eigenvalues which characterize the massive scalar fields (see [38–40] and references therein).

The radial Klein–Gordon equation (also known as the radial Teukolsky equation) is given by [32, 33]
\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR_{lm}}{dr} \right) + \left\{ [\omega_c (r^2 + a^2) - ma]^2 + \Delta [2ma \omega_c - \mu^2 (r^2 + a^2) - K_{lm}] \right\} R_{lm} = 0. \tag{10}
\]
Note that the radial equation (10) is coupled to the angular equation (9). The stationary bound-state massive scalar clouds, which are supported in the Kerr black-hole spacetime, are characterized by exponentially decaying (bounded) radial eigenfunctions at spatial infinity [27, 28, 41]:
\[
R(r \to \infty) \sim e^{-\sqrt{\mu^2 - \omega_c^2} r}. \tag{11}
\]
In addition, regular field configurations in the black-hole spacetime are characterized by finite radial eigenfunctions. In particular [18]
\[
0 \leq R(r = r_c) < \infty. \tag{12}
\]

4. The effective radial potential of the composed Kerr-massive-scalar-field configurations

In order to analyze the spatial properties of the linearized bound-state massive scalar configurations (the stationary scalar clouds) in the Kerr black-hole spacetime, we shall first write the radial Teukolsky equation (10) in the form of a Schrödinger-like wave equation. Defining the new radial function
\[
\psi' = r R, \tag{13}
\]
and using the new radial coordinate $y$ which is defined by the relation [19]
\[
\frac{dy}{dr} = \frac{r^2}{\Delta}, \tag{14}
\]
\[15 These characteristic angular eigenfunctions are known as the spheroidal harmonic functions.

\[16 In particular, the characteristic eigenvalues $\{K_{lm}\}$ of the angular equation (9) also appear in the effective potential of the radial equation (10).

\[17 We shall henceforth omit the angular harmonic indices $m$ and $l$ for brevity.

\[18 One can assume $R(r = r_c) \geq 0$ without loss of generality.

\[19 Note that the transformation (14) maps the radial coordinate $r \in [0, \infty]$ into $y \in [-\infty, +\infty]$.\]
one can write the radial Teukolsky equation (10) in the compact form
\[ \frac{d^2\psi}{dy^2} - V(y)\psi = 0. \]  
(15)

The effective radial potential in the Schrödinger-like wave equation (15) is given by
\[ V = V(r; M, a, \mu, l, m) = \frac{2\Delta}{r_0^6}(Mr - a^2) + \frac{\Delta}{r^4}[K_{lm} - 2ma\omega_c + \mu^2(r^2 + a^2)] \]
\[ - \frac{1}{r^4}[(\omega_c)^2(r^2 + a^2) - ma]^2. \]  
(16)

In the next section we shall explore the near-horizon behavior of the effective radial potential (16) and the corresponding spatial properties of the associated radial eigenfunction \( \psi \).

5. The near-horizon behavior of the radial eigenfunctions

In the present section we shall analyze the near-horizon properties of the radial eigenfunction \( \psi \) which characterizes the stationary bound-state resonances of the massive scalar fields in the rotating Kerr black-hole spacetime. In particular, we shall prove that \( \psi \) is a positive (see footnote 18), increasing, and convex function in the near-horizon \( r - r_+ \ll r_+ \rightarrow r_+ \) region. To that end, we shall first analyze the spatial behavior of the effective radial potential (16) in the near-horizon region.

Substituting into (16) the resonant frequency (2) of the stationary scalar field, one finds the near-horizon behavior
\[ r_+^2 V(x \rightarrow 0) = F\tau \cdot x + O(x^2) \]  
(17)
of the effective radial potential, where we have used here the dimensionless variables
\[ x \equiv \frac{r - r_+}{r_+}; \quad \tau \equiv \frac{r_+ - r_+}{r_+}. \]  
(18)
The expansion coefficient in (17) is given by
\[ F \equiv K_{lm} - \frac{2(ma)^2}{r_+^2 + a^2} + \mu^2(r_+^2 + a^2) + \tau. \]  
(19)

Taking cognizance of the inequality (4) and using the lower bound [40] \( 20 \)
\[ K_{lm} \geq m^2 - a^2(\mu^2 - \omega_c^2) \]  
(20)
on the angular eigenvalues, one finds the characteristic inequality
\[ F > m^2 \cdot \frac{r_+^2}{r_+^2 + a^2} + \tau > 0. \]  
(21)

Taking cognizance of equations (17) and (21), one deduces that, in the near-horizon \( x \ll \tau \) region, the radial potential (16) takes the form of an effective potential barrier with \( V \gg 0. \)

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Using equations (14) and (18), one finds the relation
\[ y = \frac{r_+}{\tau} \ln(x) + O(x) \] (22)
in the near-horizon region
\[ x \ll \tau. \] (23)
This relation can also be written in the form\(^{21}\)
\[ x = e^{\gamma/\tau}[1 + O(e^{\gamma/\tau})]. \] (24)
Using equations (17) and (24), one finds that, in the near-horizon region (23), the Schrödinger-like radial equation (15) can be written in the form
\[ \frac{d^2\psi}{d\tilde{y}^2} = \frac{4F}{\tau} e^{2\tilde{y}}\psi = 0, \] (25)
where
\[ \tilde{y} \equiv \frac{\tau}{2r_+} y. \] (26)

The near-horizon radial equation (25) can be solved analytically. In particular, the solution of (25) which respects the boundary condition (12) can be expressed in terms of the modified Bessel function of the first kind\(^{22,23}\).
\[ \psi(y) = I_0 \left( \frac{2}{\sqrt{\tau}} e^{\gamma/2r_+} \right). \] (27)
Using the well-known properties of the modified Bessel function \(I_0\) \(^{36}\), one deduces from (27) that, in the near-horizon \( x \ll \tau \) region, the radial eigenfunction \(\psi\) is a positive, increasing, and convex function. That is
\[ \left\{ \psi > 0; \frac{d\psi}{dy} > 0; \frac{d^2\psi}{dy^2} > 0 \right\} \text{ for } 0 < x \ll \tau. \] (28)

Taking cognizance of the near-horizon behavior (28)\(^{24}\) and the far-region asymptotic behavior (11)\(^{25}\) of the radial eigenfunction \(\psi\), one arrives at the important conclusion that this function, which characterizes the stationary bound-state scalar configurations in the Kerr black-hole spacetime, must have (at least) one maximum point, \(x = x_{\text{max}}\), in the black-hole exterior region.

\(^{21}\) Note that the near-horizon region (23) corresponds to \(y \to -\infty\) (see equation (22)), which implies \(e^{\gamma/\tau} \to 0\).

\(^{22}\) Here we have used equations (9.1.54) and (9.6.3) of [36].

\(^{23}\) The second mathematical solution of (25) is given by the modified Bessel function of the second kind: \(K_0(y) = K_0(2\sqrt{\tau/\gamma} e^{\gamma/2r_+})\). However, using equation (9.6.13) of [36], one finds that this solution does not respect the boundary condition (12) at the black-hole horizon. (In particular, one finds \(\psi_2(y \to -\infty) \to -\gamma/2r_+ \to -\infty\) at the black-hole horizon.)

\(^{24}\) In particular, it is worth emphasizing again that (28) implies that \(\psi\) is a positive increasing function in the near-horizon \(x \ll \tau\) region.

\(^{25}\) In particular, it is worth emphasizing again that the stationary bound-state configurations of the massive scalar fields are characterized by exponentially decaying radial eigenfunctions at spatial infinity (see equation (11)).
6. A lower bound on the effective lengths of the stationary bound-state Kerr scalar clouds

We have seen that the radial eigenfunction $\psi$, which characterizes the stationary bound-state configurations of the massive scalar fields in the rotating Kerr black-hole spacetime, is a non-monotonic function. In particular, we have proved that $\psi$ must have (at least) one maximum point outside the black-hole horizon. In the present section we shall obtain a generic lower bound on the peak location, $r_{\text{max}}$, of these bound-state massive scalar configurations.

We first point out that the radial function $\psi$ must have an inflection point, $r = r_0$, somewhere in the interval $(r_-, r_{\text{max}})$. That is

$$r_- < r_0 < r_{\text{max}}. \quad (29)$$

Taking cognizance of the radial equation (15), one deduces that this inflection point (with $d^2\psi/dr^2 = 0$ at $y_0 = y_0(r_0)$) is a turning point of the effective radial potential (16). That is

$$V(r = r_0) = 0. \quad (30)$$

We shall now derive a lower bound on the radial location of this inflection point.

Substituting into (16) the resonant frequency (2) of the stationary scalar field, and using the characteristic inequalities (4) and (20), one finds the lower bound

$$V(r) > m^2 \cdot \frac{(r - r_+)(r_+ - r_-)}{r^4(r_+^2 + a^2)} + \frac{2\Delta}{r^6}(Mr - a^2) \quad (31)$$

on the effective radial potential. Furthermore, using the inequality $Mr - a^2 \geq M r_+ - a^2 = r_+(M - r_-) \geq 0$, one obtains from (31) the characteristic inequality

$$V(r) > m^2 \cdot \frac{r - r_+}{r^3(r_+^2 + a^2)} \times (r_+^2 - r_-). \quad (32)$$

Taking cognizance of equations (30) and (32), one finds the lower bound

$$r_0 > \frac{r_+^2}{r_-}. \quad (33)$$

on the radial location of the inflection point $r = r_0$ which characterizes the radial eigenfunction $\psi$.

Finally, taking cognizance of the inequalities (29) and (33), one obtains the lower bound

$$r_{\text{max}} > \frac{r_+^2}{r_-}. \quad (34)$$

on the peak location $r_{\text{max}}$ of the radial eigenfunction $\psi$ which characterizes the stationary bound-state massive scalar configurations in the rotating Kerr black-hole spacetime. Remarkably, this lower bound is universal in the sense that it is independent of the physical parameters (proper mass and angular harmonic indices) of the external massive scalar fields.

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Note that $d^2\psi/dr^2 > 0$ in the near-horizon $r \to r_+$ limit (see equation (28)), whereas $d^2\psi/dr^2 < 0$ at the maximum point $r = r_{\text{max}}$. Thus, the characteristic radial eigenfunction $\psi$ must have an inflection point (with $d^2\psi/dr^2 = 0$) somewhere between the horizon ($r = r_-$) and the maximum point ($r = r_{\text{max}}$).
7. Stationary bound-state Kerr scalar clouds and null circular geodesics

In the present section we shall show that the composed Kerr-black-hole-massive-scalar-field configurations respect the no-short hair lower bound (1). In particular, we shall prove that the peak location \( r_{\text{max}} \) of the radial eigenfunction \( \psi \), which characterizes the external bound-state scalar clouds, is located beyond the equatorial null circular geodesic of the corresponding black-hole spacetime.

The equatorial null circular geodesics of the rotating Kerr black-hole spacetimes are characterized by the relation

\[
\begin{align*}
\rho_\text{null} = 2M \left\{ 1 + \cos \left[ \frac{2}{3} \cos^{-1} \left( -\frac{a}{M} \right) \right] \right\}.
\end{align*}
\]

(35)

Taking cognizance of (6) and (35), one finds that the ratio \( \rho_\text{null} / r_c^2 \) is a monotonic increasing function of the black-hole rotation parameter \( a \). Specifically, this dimensionless ratio increases from 0 to 1 as \( a / M \) increases from 0 to 1. That is

\[
\frac{\rho_\text{null} r_c}{r_c^2} \leq 1.
\]

(36)

On the other hand, from equation (34) one finds the characteristic inequality

\[
\frac{r_{\text{max}} r_c}{r_c^2} > 1
\]

(37)

for the external bound-state scalar clouds. Taking cognizance of (36) and (37), one concludes that the stationary bound-state scalar configurations of the rotating Kerr black-hole spacetime are characterized by the relation

\[
r_{\text{max}} > \rho_\text{null}.
\]

(38)

8. Summary

The 'no-short hair' theorem \([24]\) asserts that the external matter fields of a static spherically symmetric electrically neutral hairy black-hole configuration must extend beyond the null circular geodesic which characterizes the corresponding black-hole spacetime.

The main goal of the present paper was to test the validity of the no-short hair lower bound (1) beyond the restricted regime of static spherically symmetric hairy black-hole spacetimes. To that end, we have studied analytically the physical properties of the recently discovered cloudy Kerr black-hole spacetimes. These are non-static non-spherically symmetric Kerr black holes (see footnote 6) which support linearized massive scalar fields in their exterior regions.

Using analytical techniques, we have established the fact that the stationary bound-state massive scalar configurations which characterize the rotating Kerr black-hole spacetime cannot be made arbitrarily compact. In particular, we have derived the lower bound (see equation (34))

\[\text{27 It is worth emphasizing again that the formal proof provided in [24] for the validity of the no-short hair lower bound (1) is restricted to the special (and most simple) case of static spherically symmetric hairy black-hole spacetimes. On the other hand, in the present study we consider non-static non-spherically symmetric composed Kerr-black-hole-massive-scalar-field configurations.}\]

\[\text{28 It is worth emphasizing that this inequality is valid for all values of the dimensionless black-hole rotation parameter } a/M \text{ (see equation (34)).}\]
on the peak location $r_{\text{max}}$ of the radial eigenfunction $\psi$ which characterizes the external bound-state massive scalar configurations of the rotating Kerr black-hole spacetime. Interestingly, the characteristic lower bound (39) is universal in the sense that it is independent of the physical parameters of the external massive scalar fields.

Furthermore, we have explicitly shown that the inequality (34), which characterizes the composed Kerr-black-hole-massive-scalar-field configurations, implies that these non-static non-spherically symmetric (see footnote 27) configurations respect the no-short hair lower bound (1). Our results, together with the results presented in [24], may therefore suggest that the lower bound $r_{\text{hair}} > r_{\text{null}}$ may be a general property of asymptotically flat electrically neutral hairy black-hole spacetimes.

It is important to emphasize the fact that the composed Kerr-black-hole-massive-scalar-field configurations that we have studied in the present paper respect the assumption made in the original no-short hair theorem [24] that the energy density $T_t^t$ outside the black-hole horizon approaches zero asymptotically faster than $r^{-4}$. On the other hand, in the charged (Kerr–Newman) black-hole case studied in [25] the electromagnetic energy density outside the black-hole horizon is characterized by the asymptotic behavior $T_t^t \sim Q^2/r^4$, where $Q$ is the electric charge of the black-hole spacetime. Thus, charged Kerr–Newman black holes do not respect the assumption made in the original no-short hair theorem [24] that the energy density $T_t^t$ outside the black-hole horizon approaches zero asymptotically faster than $r^{-4}$. The different asymptotic behavior of the energy density in the charged case studied in [25] (as compared with the neutral case considered in the present work) allows charged Kerr–Newman-black-hole-charged-massive-scalar-field configurations to violate the no-short hair bounds [25]. It is therefore important to emphasize that the results of the present paper are restricted to the neutral (Kerr) black-hole case.

Finally, we would like to emphasize again that in this paper the external scalar clouds (the stationary bound-state massive scalar configurations) were treated at the linear level. As we explicitly demonstrated, the main advantage of this approach lies in the fact that the physical properties of the composed Kerr-black-hole-linearized-massive-scalar-field configurations can be studied analytically. We believe that, using numerical techniques [28], it would be highly interesting to further test the validity of the no-short hair lower bound (1) in the nonlinear regime of hairy Kerr-massive-scalar-field black holes.

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