Gradual approximation of the domain of attraction
by gradual extension of the "embryo" of the
transformed optimal Lyapunov function

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Abstract In this paper an autonomous analytical system of ordinary differential
equations is considered. For an asymptotically stable steady state \(x^0\) of the system
a gradual approximation of the domain of attraction (\(DA\)) is presented in the case
when the matrix of the linearized system in \(x^0\) is diagonalizable. This technique is
based on the gradual extension of the "embryo" of an analytic function of several
complex variables. The analytic function is the transformed of a Lyapunov func-
tion whose natural domain of analyticity is the \(DA\) and which satisfies a linear
non-homogeneous partial differential equation. The equation permits to establish
an "embryo" of the transformed function and a first approximation of \(DA\). The
"embryo" is used for the determination of a new "embryo" and a new part of the
\(DA\). In this way, computing new "embryos" and new domains, the \(DA\) is grad-
ually approximated. Numerical examples are given for polynomial systems. For
systems considered recently in the literature the results are compared with those
obtained with other methods.

1 Introduction

The domain of attraction (\(DA\)) of a steady state of a dynamical system is the
set of initial states from which the system converges to the steady state itself. In
order to guarantee stable behavior of a dynamical system in a region of the state
parameters it is important to know the \(DA\) [1].

Theoretical research shows that the \(DA\) and its boundary are complicated sets
[2], [3], [4], [5]. In most cases, they do not admit an explicit elementary rep-
the DA with domains having a simpler shape. This practice became fundamental in the last 20 years [6]. The domain which approximates the DA is defined by a Lyapunov function, generally quadratic. For a given Lyapunov function the computation of the optimal estimate of the DA amounts to solving a non-convex distance problem [7], [8], [9], [10], [11], [12], [13].

In this paper we present a new technique for a gradual approximation of the DA in the case of autonomous analytical systems of ordinary differential equations. More precisely for the gradual approximation of the DA of an asymptotically stable steady state in which the matrix of the linearized system is diagonalizable. This technique is based on the theoretical results obtained in [14], [15], [16] and in the following we present a summary of this theory.

2 Theoretical summary

We consider the system of differential equations

$$\dot{x} = f(x)$$  \hspace{1cm} (1)

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is an analytical function with the following properties:

i. $f(0) = 0$ (i.e. zero is a steady state of (1))

ii. the real parts of the eigenvalues of $\frac{\partial f}{\partial x}(0)$ are negative (i.e. $x = 0$ is an asymptotically stable steady state)

The following theorem provides a tool of determining the DA of the zero steady state of (1).

**Theorem 1.** (see [14]) The DA of the null solution of (1) coincides with the natural domain of analyticity of the unique solution $V$ of the problem

$$\begin{cases}
\langle \nabla V, f \rangle = -\|x\|^2 \\
V(0) = 0
\end{cases}$$  \hspace{1cm} (2)

The function $V$ is positive on DA and $\lim_{x \to x_0} V(x) = \infty$ for any $x_0 \in \partial DA$.

Thus, the problem of finding the DA is reduced to the determination of the natural domain of analyticity of the solution $V$ of (2). This function will be called in the followings the optimal Lyapunov function.

To determine the optimal Lyapunov function $V$ and its domain of analyticity is not easy, but, in the diagonalizable case, we can determine the coefficients of the expansion of $W = V \circ S$ in 0, where $S$ reduces $\frac{\partial f}{\partial x}(0)$ to the diagonal form. Then, using a Cauchy-Hadamard type theorem, we can obtain the domain of convergence $D^0$ of the series of $W$, and $DA^0 = S(D^0)$ is a part of the domain of attraction.
2.1 The coefficients of the transformed optimal Lyapunov function in the diagonalizable case

For the system (1) the following theorem holds:

**Theorem 2.** (see [15]) For each isomorphism \( S : \mathbb{C}^n \to \mathbb{C}^n \) and \( g = S^{-1} \circ f \circ S \), the problem

\[
\begin{cases}
\langle \nabla W, g \rangle = -\|Sz\|^2 \\
W(0) = 0
\end{cases}
\]

has a unique analytical solution, namely \( W = V \circ S \) where \( V \) is the optimal Lyapunov function for (1).

The function \( W \) will be called the transformed optimal Lyapunov function.

In the followings, we will suppose that the matrix \( \frac{\partial f}{\partial x}(0) \) is diagonalizable.

Let be \( S : \mathbb{C}^n \to \mathbb{C}^n \) one isomorphism which reduces \( \frac{\partial f}{\partial x}(0) \) to the diagonal form \( S^{-1} \frac{\partial f}{\partial x}(0) S = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \) and \( g = S^{-1} \circ f \circ S \).

For this \( S \), we consider the expansion of \( W \) in 0:

\[
W(z_1, z_2, ..., z_n) = \sum_{m=2}^{\infty} \sum_{|j|=m} B_{j_1, j_2, ..., j_n} z_1^{j_1} z_2^{j_2} ..., z_n^{j_n}
\]

and the expansions in 0 of the scalar components \( g_i \) of \( g \):

\[
g_i(z_1, z_2, ..., z_n) = \lambda_i y_i + \sum_{m=2}^{\infty} \sum_{|j|=m} b_{j_1, j_2, ..., j_n}^i z_1^{j_1} z_2^{j_2} ..., z_n^{j_n}
\]

**Theorem 3.** (see [16]) The coefficients \( B_{j_1, j_2, ..., j_n} \) of the development (4) are given by the following relations:

\[
B_{j_1, j_2, ..., j_n} = \begin{cases}
-\frac{1}{2} \sum_{i=1}^{n} s_{i0}^2 & \text{if } |j| = j_0 = 2 \\
-\frac{2}{\lambda_p + \lambda_q} \sum_{i=1}^{n} s_{ip}s_{iq} & \text{if } |j| = 2 \text{ and } j_p = j_q = 1 \\
-\frac{1}{\sum_{j \leq j_i \lambda_i} \sum_{p=2}^{k_i} \sum_{k_i < k_i}^{n} (j_i - k_i + 1)} \sum_{i=1}^{n} b_{j_1, j_2, ..., j_n}^{i} B_{j_1 - k_1, ..., j_i - k_i + 1, ..., j_n - k_n} & \text{if } |j| \geq 3
\end{cases}
\]

According to (6), the coefficients of the terms of degree \( m \geq 3 \) are obtained in function of the coefficients of the terms of degree smaller than \( m \).
2.2 The domain of convergence of the series of $W$

We define the domain of convergence of the series (4) as the set of those $z \in \mathbb{C}^n$ which have the property that the series (4) is absolutely convergent in a neighborhood of $z$ [17]. We denote by $D^0$ this domain. Actually, the domain of convergence $D^0$ coincides with the interior of the set of all the points $z^0$ in which the series (4) is absolutely convergent (see [17]).

**Theorem 4.** (Cauchy-Hadamard see [17]) A point $z$ belongs to $D^0$ if and only if

$$\lim_{m \to \infty} \sqrt[m]{\sum |B_{j_1,j_2,\ldots, j_n} z_{j_1} z_{j_2} \ldots z_{j_n}|} < 1$$  \hspace{1cm} (7)

Using $D^0$ we can find a part of the $DA$.

**Theorem 5.** (see [15]) If $z$ belongs to $D^0$ then the point $x = Sz$ is in the domain of attraction $DA$.

Actually, by the previous theorem, we obtain the subdomain $S(D^0) \subset DA$, which has the property that $\partial S(D^0) \cap \partial DA \neq \emptyset$. We also observe that the subdomain $S(D^0)$ is symmetrical to the origin (actually, the domain of convergence $D^0$ is symmetrical to the origin and the axes of coordinates). This first approximation of the domain of attraction $DA$ will be denoted by $DA^0 = S(D^0)$.

Condition (7) represents an algorithmizable criterion for determining the region of convergence $D^0$.

2.3 A method of extending the estimate of DA

In practice, we can compute the coefficients $B_{j_1,j_2,\ldots, j_n}$ up to a finite degree $p$. This degree $p$ has to be big enough for assuming that the domain $D^0_p$ given by

$$D^0_p = \{ z \in \mathbb{C}^n / \sqrt[p]{\sum_{|j| = p} |B_{j_1,j_2,\ldots, j_n} z_{j_1} z_{j_2} \ldots z_{j_n}|} < 1 \}$$

approximates the region of convergence $D^0$ of the series of $W$. On this domain $D^0_p$ we approximate $W$ by the "embryo"

$$W^0_p(z_1, z_2, \ldots, z_n) = \sum_{m=2}^{p} \sum_{|j|=m} B_{j_1,j_2,\ldots, j_n} z_{j_1} z_{j_2} \ldots z_{j_n}$$  \hspace{1cm} (8)

The first estimate of the region of attraction $DA$ will be $DA^0_p = S(D^0_p)$.

In order to extend the first estimate $DA^0_p$, we will expand $W$ in a point $z^0$ close to $\partial D^0_p$ in which $|W^0_p(z^0)|$ is still small. That is because, according to Theorem 1, the points $z$ close to $\partial D^0_p$ for which $|W^0_p(z)|$ is extremely high are close to $\partial S^{-1}(DA)$.

To find the expansion of $W$ in $z^0$ close to $\partial D^0_p$, we will compute the expansion in $z^0$ of the "embryo" $W^0_p$ of $W$. We obtain:
We consider the set
\[ D^1_p = \{ z \in \mathbb{C}^n / \sqrt{\sum_{|j|=p} |B_{j_1,j_2,...,j_n} (z_1 - z_1^0)^{j_1} (z_2 - z_2^0)^{j_2} ... (z_n - z_n^0)^{j_n} | < 1} \} \]
which provides a new part \( DA^1_p = S(D^1_p) \) of \( DA \).

So, \( DA^0_p \cup DA^1_p \) gives a larger estimate of the \( DA \). We can continue this procedure for a few steps, till the values \( |W_p^k| \) become extremely large and we obtain the estimate \( DA^0_p \cup DA^1_p \cup ... \cup DA^k_p = S(D^0_p \cup D^1_p \cup ... \cup D^k_p) \) of \( DA \).

3 Numerical results

The computations were made using a program written in Mathematica 4, Wolfram Research. The data for the estimations (the degree up to which the approximation is made, the necessary timing for the estimations) are displayed in Table 1.

3.1 Systems with known domains of attraction

In this subsection, we will present some examples of systems of two or three differential equations, for which we can compute easily the \( DA \). We will apply our technique to these examples, and we will show how the real domains of attraction are gradually approximated. These examples are meant to validate our procedure.

3.1.1 Example 1

This is an example of a system for which the null solution has a bounded domain of attraction:

\[ \begin{cases} 
  \dot{x}_1 = -x_1[4 - (x_1 - 1)^2 - x_2^2] \\
  \dot{x}_2 = -x_2[4 - (x_1 - 1)^2 - x_2^2] 
\end{cases} \]  

The domain of attraction of the null solution of this system is the interior of the circle of radius 2 centered in \((1,0)\):

\[ DA = \{ (x_1, x_2) \in \mathbb{R}^2 / (x_1 - 1)^2 + x_2^2 < 4 \} \]

After three steps, we obtain the estimate shown in Figure 1.
The thick black line represents the true boundary of the domain of attraction, the dark grey set denotes the first estimate $DA^0_p$ of $DA$ and the further estimates $DA^k_p$ of $DA$ with $k \geq 1$ are colored in light grey.

Figure 1: The estimate of $DA$ obtained after three steps for system (10)

3.1.2 Example 2

The following system of three differential equations is considered:

\[
\begin{align*}
\dot{x}_1 &= -x_1(1 - x_1^2 - x_2^2 + x_3^2) \\
\dot{x}_2 &= -x_2(1 - x_1^2 - x_2^2 + x_3^2) \\
\dot{x}_3 &= -x_3(1 - x_1^2 - x_2^2 + x_3^2)
\end{align*}
\]

The boundary of the $DA$ of the null solution of this system is:

\[\partial DA = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / 1 - x_1^2 - x_2^2 + x_3^2 = 0\}\]

The first estimate $DA^0_p$ is shown in Figure 2.1. After 2 steps, we obtain the estimate shown in Figure 2.2.

Figure 2.1: The estimate of $DA$ obtained after 1 step for system (11)  Figure 2.2: The estimate of $DA$ obtained after 2 steps for system (11)
3.2 Systems for which we don’t know the DA

In this subsection, some systems of differential equations are presented for which we don’t know the DA. For these examples, we will apply the technique presented in Section 2, and we will show that the estimate obtained using this technique is better than the estimates obtained in [10].

3.2.1 Example 3

In [10], the following example is considered:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_1 - 3x_2 + x_1^2 - x_2^2 + x_1x_2
\end{align*}
\] (12)

In Figure 3.1, an estimate of the DA is shown obtained after two steps for three different points close to the boundary of \(DA_0^p\). We observe that this estimate covers the estimate presented in [10]. Figure 3.2 presents the estimate of DA obtained after four steps. The thick black lines plotted in the following figures represent a part of the approximated boundary of the DA. The black interrupted line represents the boundary of the estimate of the domain of attraction obtained in [10].

![Figure 3.1: The estimate of DA after 2 steps for three different points close to the boundary of DA_0^p for system (12).](image)

![Figure 3.2: The estimate of DA obtained after 4 steps for system (12).](image)

3.2.2 Example 4

In [10], the following system of three equations is considered:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -3x_1 - 3x_2 - 2x_3 + x_1^3 + x_2^3 + x_3^3
\end{align*}
\] (13)

Figure 4.1 shows the estimate of DA obtained after one step. The estimate of the DA obtained after two steps is presented in Fig. 4.2.
Table 1. Numerical data

| example | order of approximation | timing for 1\textsuperscript{st} step | timing for 2\textsuperscript{nd} step |
|---------|------------------------|---------------------------------------|---------------------------------------|
| 1       | 50                     | 1.7 min                               | 34.7 min                              |
| 2       | 30                     | 10.2 min                              | 14.7 h                                |
| 3       | 30                     | 0.9 min                               | 9.9 min                               |
| 4       | 30                     | 19.1 min                              | 16.2 h                                |

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