On the String-Theoretic Euler Number of a Class of Absolutely Isolated Singularities

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Abstract
An explicit computation of the so-called string-theoretic E-function of a normal complex variety $X$ with at most log-terminal singularities can be achieved by constructing one snc-desingularization of $X$, accompanied with the intersection graph of the exceptional prime divisors, and with the precise knowledge of their structure. In the present paper, it is shown that this is feasible for the case in which $X$ is the underlying space of a class of absolutely isolated singularities (including both usual $A_n$-singularities and Fermat singularities of arbitrary dimension). As byproduct of the exact evaluation of $e_{str}(X) = \lim_{u,v \to 1} E_{str}(X; u, v)$, for this class of singularities, one gets counterexamples to a conjecture of Batyrev concerning the boundedness of the string-theoretic index. Finally, the string-theoretic Euler number is also computed for global complete intersections in $\mathbb{P}^N_C$ with prescribed singularities of the above type.

1 Introduction

The so-called $E_{str}$-polynomials $E_{str}(X; u, v)$ of normal complex varieties $X$ with at most Gorenstein quotient or toroidal singularities were introduced in [5], and were used as main tools in [4,5,6] for the proof of several mirror-symmetry identities. More recently, Batyrev [1] generalized this notion also for $X$’s having at most log-terminal singularities, by introducing $E_{str}$-functions instead which may be not even rational. These new invariants have already found lots of applications in the study of log-flips and of cohomological McKay correspondence. (See [3,1.6,4.11 and 8.4] and [8, Thm. 5.1].)

In the present paper we give explicit formulae for the evaluation of the function $E_{str}(X; u, v)$ for those $X$’s which are the underlying spaces of two special series of $A_{n,\ell}^{(r)}$-singularities (see below (d) for the precise definition) by constructing an appropriate snc-resolution $\varphi : \tilde{X} \to X$, by examining the nature of the arising exceptional prime divisors and, finally, by computing their $E$-polynomials. (In [7] this was carried out for all three-dimensional $A$-$D$-$E$ singularities).

(a) Log-terminal singularities. Let $X$ be a normal complex variety. Suppose that $X$ is $\mathbb{Q}$-Gorenstein, i.e., that a positive integer multiple of its canonical Weil divisor

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$K_X$ is a Cartier divisor. $X$ is said to have at most log-terminal (respectively, canonical/terminal) singularities if there exists an snc-desingularization $\varphi : \tilde{X} \rightarrow X$, i.e., a desingularization of $X$ whose exceptional locus $\mathcal{E}(\varphi) = \cup_{i=1}^{m} D_i$ consists of smooth prime divisors $D_1, D_2, \ldots, D_m$ with only normal crossings, such that the “discrepancy” w.r.t. $\varphi$ is of the form $K_{\tilde{X}} - \varphi^*(K_X) = \sum_{i=1}^{m} a_i D_i$, with all the $a_i$’s $> -1$ ($\geq 0 / > 0$).

These inequalities do not depend on the particular choice of $\varphi$.

(b) $E$-polynomials. Deligne proved in [3, §8] that the cohomology groups $H^i(X, \mathbb{Q})$ of any complex variety $X$ are endowed with a natural mixed Hodge structure (MHS). The same remains true if one works with cohomologies $H^i_c(X, \mathbb{Q})$ with compact supports. There exist namely an increasing weight-filtration $W_\bullet$ and a decreasing Hodge-filtration of $H^i_c(X, \mathbb{Q})$ (resp. $H^i_c(X, \mathbb{C})$) which induces a natural filtration $\mathcal{F}_\bullet$ on the complexification of the corresponding graded pieces $Gr^W_k(H^i_c(X, \mathbb{Q}))$ (resp. $Gr^W_k(H^i_c(X, \mathbb{C}))$). Let

$$h^{p,q}(H^i(X, \mathbb{C})) := \dim_{\mathbb{C}} Gr^p_{\mathcal{F}} Gr^W_k (H^i(X, \mathbb{C}))$$

(respectively $h^{p,q}(H^i_c(X, \mathbb{C})) := \dim_{\mathbb{C}} Gr^p_{\mathcal{F}} Gr^W_k (H^i_c(X, \mathbb{C}))$)

denote hereafter the corresponding Hodge numbers. The so-called $E$-polynomial of $X$ is defined to be

$$E(X; u, v) := \sum_{p,q} e^{p,q}(X) \ u^p v^q \in \mathbb{Z}[u,v],$$

where $e^{p,q}(X) := \sum_{i\geq 0} (-1)^i h^{p,q}(H^i_c(X, \mathbb{C}))$. (If we set $u = v = 1$, then $E(X; 1, 1)$ equals the usual topological Euler characteristic $e(X)$ of $X$.)

(c) $E_{\text{str}}$-functions. To pass to string-theoretic invariants, one takes essentially into account the “discrepancy coefficients”.

**Definition 1.1** Let $X$ be a normal complex variety with at most log-terminal singularities, $\varphi : \tilde{X} \rightarrow X$ an snc-desingularization of $X$ as in (a), $D_1, D_2, \ldots, D_m$ the prime divisors of the exceptional locus, and $I := \{1, 2, \ldots, m\}$. For any subset $J \subseteq I$ define

$$D_J := \begin{cases} \tilde{X}, & \text{if } J = \emptyset \\ \cap_{j \in J} D_j, & \text{if } J \neq \emptyset \end{cases}$$

and $D_J^0 := D_J \setminus \bigcup_{j \in I \setminus J} D_j$.

The algebraic function

$$E_{\text{str}}(X; u, v) := \sum_{J \subseteq I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{q_j+1} - 1} \quad (1.1)$$

(under the convention for $\prod_{j \in J}$ to be 1, if $J = \emptyset$, and $E(\emptyset; u, v) := 0$) is called the string-theoretic $E$-function (or simply $E_{\text{str}}$-function) of $X$.

The major result of [1] says that:

**Theorem 1.2** $E_{\text{str}}(X; u, v)$ is independent of the choice of the snc-desingularization $\varphi : \tilde{X} \rightarrow X$. 
Remark 1.3 (i) Though the string-theoretic function \( E_{\text{str}}(X; u, v) \) enjoys this particularly important invariance property, to evaluate it by (1.1) one needs not only the existence of (at least one) snc-desingularization (which is guaranteed, e.g., by Hironaka’s main theorems [18]), but also the precise knowledge of what kind of exceptional prime divisors are available on the corresponding smooth model, and which are their intersections. In general, there are several ways to resolve log-terminal singularities, involving different choices for the centers of the modifications of \( X \) and, sometimes, necessary extra normalizations, blow-ups of non-reduced subschemes etc. For this reason, a first realistic attempt to understand the behaviour of (1.1), from the computational point of view, cannot overlook the class of absolutely isolated singularities, i.e., isolated singularities resolvable by a finite sequence of (usual) blow-ups of close d points, for which one may keep the needed details (strict transforms after each step of the resolution procedure, snc-condition etc.) under control.

(ii) It is also worth mentioning that the “first summand” in (1.1), i.e., for \( J = \emptyset \), equals
\[
E(X \setminus \bigcup_{j=1}^m D_j; u, v) = E(X \setminus \text{Sing}(X); u, v)
\]
(where \( \text{Sing}(X) \) denotes the singular locus of \( X \)). This means that it can be described exclusively by the study of topological properties of \( X \) “around” the singularities without involving any resolution data.

Definition 1.4 The rational number
\[
e_{\text{str}}(X) := \lim_{u,v \to 1} E_{\text{str}}(X; u, v) = \sum_{J \subseteq I} e(D^j_\circ J) \prod_{j \in J} \frac{1}{a_j + 1}
\]
is called the string-theoretic Euler number of \( X \). Moreover, the string-theoretic index \( \text{ind}_{\text{str}}(X) \) of \( X \) is defined to be the integer
\[
\text{ind}_{\text{str}}(X) := \min \left\{ l \in \mathbb{Z}_{\geq 1} \mid e_{\text{str}}(X) \in \frac{1}{l}\mathbb{Z} \right\}.
\]

Examples 1.5 (i) For \( \mathbb{Q} \)-Gorenstein toric varieties \( X \), \( \text{ind}_{\text{str}}(X) = 1 \), and \( e_{\text{str}}(X) \) is equal to the normalized volume of the defining fan. Moreover, for Gorenstein toric varieties \( X \), \( E_{\text{str}}(X; u, v) \) is a polynomial (cf. [1, 4.4 and 4.10]).

(ii) Normal algebraic surfaces \( X \) with at most log-terminal singularities have string-theoretic index \( \text{ind}_{\text{str}}(X) = 1 \). There exist, however, normal complex varieties \( X \) of dimension \( d \geq 3 \) with at most Gorenstein canonical singularities having \( \text{ind}_{\text{str}}(X) > 1 \).

Batyrev’s conjecture [1, 5.9], concerning the range of \( \text{ind}_{\text{str}}(X) \), can be stated as follows:

Conjecture 1.6 (Boundedness of the string-theoretic index) Let \( X \) be an \( r \)-dimensional normal complex variety having at most Gorenstein canonical singularities. Then \( \text{ind}_{\text{str}}(X) \) is bounded by a constant \( C(r) \) depending only on \( r \).

As it turns out (see below Remark [1.9]), and in contrast to initial expectations due to some classes of examples (see, e.g., [1, 5.1, 5.10] for the case of cones over certain smooth projective Fano varieties), conjecture [1.4] is not true in general. Nevertheless, the characterization of those classes of \( X \)’s, which admit bounded string-theoretic indices, remains an unsolved problem.
(d) The $A_{n,\ell}^{(r)}$'s. We define the $r$-dimensional $A_{n,\ell}^{(r)}$-singularities as those isolated hypersurface singularities which have underlying spaces of the form

$$X_{n,\ell}^{(r)} := \text{Spec } (\mathbb{C} [x_1, \ldots, x_{r+1}] / (f)),$$

where $r, n, \ell$ are integers, such that $r \geq \ell \geq 2$, $n + 1 \geq \ell$, and

$$f (x_1, \ldots, x_{r+1}) := x_1^{n+1} + x_2^\ell + x_3^\ell + \cdots + x_{r+1}^\ell.$$  \hspace{1cm} (1.3)

These are obviously singularities of Brieskorn-Pham type. In addition, by our assumptions about $r, \ell$ and $n$, they are canonical (see Reid \[22\] Prop. 4.3, p. 297). The notation is chosen in this manner to remind that they include, in particular, both subseries of usual $r$-dimensional $A_n$-singularities ($A_{n,2}^{(r)}$'s) and of Fermat singularities ($A_{\ell-1,\ell}^{(r)}$'s).

(e) Some auxiliary combinatorial functions. At first, for $p, q \in \mathbb{Z}_{\geq 0}$, let us denote Kronecker's symbol by

$$\delta_{p,q} = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{if } p \neq q. \end{cases}$$

Next, fixing $r, \ell$ and $n$, as in (d), we set $d := \text{lcm } (n+1, \ell)$, and define three functions $a, b$ and $c : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\mathbb{Z}_{\geq 0} \ni i \mapsto a(i) = \sum_{p=0}^{n-1} \delta_{i, \frac{p}{d}}.$$  \hspace{1cm} (1.4)

by the multinomial coefficients

$$\mathbb{Z}_{\geq 0} \ni j \mapsto b(j) = \begin{cases} \sum_{(\nu_1, \nu_2, \ldots, \nu_{r-1}) \in \mathcal{B}_j} \left( \begin{array}{c} \nu_1 \nu_2 \cdots \nu_{r-1} \\ r \end{array} \right), & \text{if } \mathcal{B}_j \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (1.5)

(with $(\nu_1, \nu_2, \ldots, \nu_{r-1}) := \frac{r!}{\nu_1 ! \nu_2 ! \cdots \nu_{r-1} !}$), and by the convolutional formula

$$\mathbb{Z}_{\geq 0} \ni k \mapsto c(k) = \sum_{(i,j) \in \mathcal{C}_k} a(i) b(j),$$  \hspace{1cm} (1.6)

where for each $j \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{B}_j := \left\{ (\nu_1, \nu_2, \ldots, \nu_{r-1}) \in (\mathbb{Z}_{\geq 0})^{\ell-1} \mid \nu_1 + \nu_2 + \cdots + \nu_{r-1} = r \right. \left. \text{and (whenever } \ell \geq 3 \text{)} \right. \left. d (\nu_2 + 2\nu_3 + \cdots + (\ell - 2) \nu_{\ell-1}) = j \ell \right\},$$

and for each $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}_k := \left\{ (i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid i + j = k \right\}$.

Finally, for any four-tuple $(\kappa, \nu, \xi) \in (\mathbb{Z}_{\geq 0})^4$ with $\kappa \geq \nu \geq \xi$, let us recall the definition of the non-central Eulerian numbers $\mathcal{G} (\kappa, \nu \mid \nu, \xi)$ of generalized factorials (with translation summand $\xi$). These are the coefficients which occur in the development of

$$(\nu^{\ell+\xi}) = \sum_{\lambda=0}^{\kappa} \mathcal{G} (\kappa, \nu \mid \nu, \xi) \left( \begin{array}{c} \ell+\kappa-\lambda \\ \kappa \end{array} \right).$$
and satisfy the recurrence relation

\[(\kappa + 1) \mathcal{S}(\kappa + 1, \lambda | \nu, \xi) = (\nu \lambda - \kappa + \xi) \mathcal{S}(\kappa, \lambda | \nu, \xi) + (\nu (\kappa - \lambda + 1) + \kappa - \xi) \mathcal{S}(\kappa, \lambda - 1 | \nu, \xi)\]

with initial conditions \(\mathcal{S}(0, 0 | \nu, \xi) = 1\) and \(\mathcal{S}(\kappa, 0 | \nu, \xi) = \left(\frac{\kappa}{\nu}\right)\). In fact, it can be shown that

\[\mathcal{S}(\kappa, \lambda | \nu, \xi) = \sum_{j=0}^{\lambda} (-1)^{j} \binom{\nu(j)}{\kappa} + 1\]

(f) Main results. We can now state the main results.

**Proposition 1.7** Let \(X = X_{n, \ell}^{(r)}\) be the underlying space of the \(A_{n, \ell}^{(r)}\) singularity. Then the \(E\)-polynomial \(E(X \setminus \{0\}; u, v)\) equals

\[
(uv - 1) \left[ 1 + (uv)^{\ell - 1} + \sum_{p=1}^{r-2} \left( (uv)^{p} + (-1)^{r-1} e(d(p + \frac{m}{n+1} - \frac{j}{2}))u^{p} v^{r-2} \right) \left( (uv)^{r} + (-1)^{r-2} \mathcal{S}(r - 1, p + 1 | \ell - 1, p) u v^{r-2} \right) \right]
\]

(1.7)

(with the \(e\)-function as defined in \([1.4]\)).

Formula (1.7) provides the “first summand” of the \(E_{\text{str}}\)-function of \(X\). On the other hand, if \(\ell\) divides either \(n\) or \(n + 1\), \(A_{n, \ell}^{(r)}\)’s are absolutely isolated (see below Proposition 1.3), and the \(E_{\text{str}}\)-function of \(X_{n, \ell}^{(r)}\)’s is computed as follows:

**Theorem 1.8** If the integer \(\ell\) divides either \(n\) or \(n + 1\), then the \(E_{\text{str}}\)-function of \(X = X_{n, \ell}^{(r)}\) is given by the formula

| Cases | \(E_{\text{str}}(X; u, v)\) |
|-------|------------------|
| \(\ell \mid n + 1\) | 
| \(E(X \setminus \{0\}; u, v) + (uv - 1) \left( \sum_{j=1}^{m-1} \frac{uv}{(uv)^{r-2} + (-1)^{r-1} \mathcal{S}(r - 1, p + 1 | \ell - 1, p) u v^{r-2} \right) \right)\) |
| \(\ell \mid n\) | 
| \(E(X \setminus \{0\}; u, v) + (uv - 1) \left( \sum_{j=1}^{m-1} \frac{uv}{(uv)^{r-2} + (-1)^{r-1} \mathcal{S}(r - 1, p + 1 | \ell - 1, p) u v^{r-2} \right) \right)\) |
In particular, for the string-theoretic Euler number we obtain:

| Cases | $e_{\text{str}}(X)$ |
|-------|---------------------|
| $\ell | n + 1$ | $\frac{m-r}{m(r-\ell)+1} \left[ \frac{1}{r} ((1-\ell)^r - 1) + r \right]$ + $\frac{1}{m(r-\ell)+1} \left[ \frac{1}{r} ((1-\ell)^{r+1} - 1) + r + 1 \right]$ |
| $\ell | n$ | $\frac{r}{(m-1)(r-\ell)+r} + \frac{(m-1)r}{(r-\ell)(m-1)} \left[ \frac{1}{r} ((1-\ell)^r - 1) + r \right]$ |

The above number $m$ is defined to be

$$m := \begin{cases} \frac{n+1}{\ell}, & \text{if } n + 1 \equiv 0 \pmod{\ell} \\ \frac{n}{\ell} + 1, & \text{if } n \equiv 0 \pmod{\ell} \end{cases}$$

Remark 1.9 Counterexamples to conjecture 1.6 occur already for $\ell = 2$, as we have:

$$e_{\text{str}}(X^{(r)}_{n,2}) = \begin{cases} \frac{m(r-1)+2}{m(r-2)+1} = \frac{n(r-1)+r+3}{n(r-2)+r}, & \text{if both } n \text{ and } r \text{ are odd} \\ \frac{m r}{m(r-2)+1} = \frac{r(n+1)}{(r-2)(n+1)+2}, & \text{if } n \text{ is odd and } r \text{ even} \\ \frac{2(m-1)(r-1)+r}{2(m-1)(r-2)+r} = \frac{(r-1)n+r}{(r-2)n+r}, & \text{if } n \text{ is even and } r \text{ odd} \\ \frac{2(m-1)r}{2(m-1)(r-2)+r} = \frac{r(n+1)}{(r-2)n+r}, & \text{if both } n \text{ and } r \text{ are even} \end{cases}$$

For instance, in dimension $r = 3$, we obtain:

$$\lim_{n \to \infty, n \text{ even}} \text{ind}_{\text{str}}(X^{(3)}_{n,2}) = \infty.$$  

On the other hand, for all odd $n$'s, $e_{\text{str}}(X^{(3)}_{n,2}) = 2$ and $\text{ind}_{\text{str}}(X^{(3)}_{n,2}) = 1$.

2 On the MHS of the cohomology groups of links

At first, we shall exploit the fact that $A^{(r)}_{n,\ell}$'s are quasihomogeneous singularities, and show that Proposition 1.7 is a byproduct of a more general result concerning isolated singularities of this sort (see 2.8).

(a) Links and Milnor fibers. Let $(W,0) \subseteq (\mathbb{C}^N,0)$ be the germ of a complex analytic set $W$ having pure dimension $r + 1$ and the origin as isolated singularity. Assume that $f : W \to \mathbb{C}$ is a holomorphic function, such that $f |_{W \setminus \{0\}}$ is non-singular. Obviously, $X := f^{-1}(0)$ is a complex analytic subset of $\mathbb{C}^N$ of pure dimension $r$ with the origin as isolated singularity. Let $L := L(X,0) := \mathcal{S}_e \cap X$ denote its link, where $\mathcal{S}_e := \{ z \in \mathbb{C}^N \mid \|z\| = \varepsilon \}, 0 < \varepsilon \ll 1$. $L$ is a differentiable, compact, oriented manifold of dimension $2r - 1$, and there are isomorphisms:

$$H^{i+1}(X, X \setminus \{0\}, \mathbb{Q}) \cong H^i(X \setminus \{0\}, \mathbb{Q}) \cong H^i(L, \mathbb{Q}).$$

(2.1)
If $B_{\varepsilon'}$ is the open ball with 0 as its center and $\varepsilon'$ as its radius, where $\varepsilon < \varepsilon' \ll 1$, it is known that the map
\[
f \big|_{B_{\varepsilon'} \cap f^{-1}(D^*_\alpha)} : B_{\varepsilon'} \cap f^{-1}(D^*_\alpha) \longrightarrow D^*_\alpha
\]
determines a differentiable fibre bundle, where $D^*_\alpha := \{ t \in \mathbb{C} \mid 0 < |t| < \alpha \}$ is a small punctured disc in $\mathbb{C}$ with $0 < \alpha < \varepsilon$. Let $F = F_1$ be the corresponding fiber, the so-called (open) Milnor fiber. The study of the relation between the MHS-structures of the cohomology groups of $L$ and $F$ relies on certain corollaries of a theorem of Steenbrink [27 (2.3)] and Hamm [17, Thm. 1.6.1]. (The coefficients of the cohomology groups are always taken from $\mathbb{C}$.)

**Theorem 2.1 (Steenbrink-Hamm)** For all i, there exists an exact MHS-sequence:
\[
\cdots \longrightarrow H^{i-1}(L) \longrightarrow H^i_c(F) \longrightarrow H^i(F) \longrightarrow H^i(L) \longrightarrow \cdots
\]

**Corollary 2.2** We have the following exact sequence and isomorphisms of MHS:

1. $0 \rightarrow H^{i-1}(F) \rightarrow H^{i-1}(L) \rightarrow H^i_c(F) \rightarrow H^i(F) \rightarrow H^i(L) \rightarrow H^{i+1}_c(F) \rightarrow 0$
2. $H^i(L) \cong H^i(F)$, for all $i < r - 1$.
3. $H^{i-1}(L) \cong H^i_c(F)$, for all $i > r - 1$.

**Proof.** Since $B_{\varepsilon'}$ is a complex Stein manifold, $F$ is a complex Stein manifold too. Hence, $F$ has the homotopy type of a CW-complex of real dimension $r$ (see [16]), which means that $H^i(F) \cong H^{2r-i}(F) = 0$ for all $i \geq r + 1$. The exactness in (i) and the existence of MHS-isomorphisms (ii) and (iii) follow from the long exact sequence of Theorem 2.1, combined with the vanishing of these cohomology groups. □

**Corollary 2.3** For all $p, q$, the Hodge numbers of the two “middle” cohomology groups of $F$ satisfy the equalities:
\[
h^{p,q}(H^r(F)) = h^{p,q}(H^{r-1}(F)) + h^{p,q}(H^r_c(F)) - h^{p,q}(H^{r+1}_c(F)) + h^{p,q}(H^r(L)) - h^{p,q}(H^{r-1}(L)) - h^{r-p,r-q}(H^r(F)) - h^{r-p,r-q}(H^{r-1}(F)) + h^{p,q}(H^r(L)) - h^{p,q}(H^{r-1}(L))
\]

**Proof.** The first equality is obvious by [2.2 (i)], and the second one follows from Poincaré duality. □

**Proposition 2.4** If $N = r + 1, W = \mathbb{C}^{r+1}$ and $(X, 0)$ is a purely r-dimensional isolated hypersurface singularity, with $r \geq 2$, then the only “non-trivial” Hodge numbers of the cohomology groups of its link $L = L(X, 0)$ are
\[
h^{p,q}(H^{r-1}(L)) = h^{r-p,r-q}(H^r(L)), \text{ with } p + q \leq r - 1,
\]
as we have:

1. $h^{p,q}(H^i(L)) = 0$, for all $p, q$ whenever $i \notin \{0, r - 1, r, 2r - 1\}$.
2. $h^{p,q}(H^0(L)) = 1$, for $p = q = 0$, and $= 0$, otherwise.
3. $h^{p,q}(H^{2r-1}(L)) = 1$, for $p = q = r$, and $= 0$, otherwise.
4. $h^{p,q}(H^{r-1}(L)) = h^{r-p,r-q}(H^r(L))$, for all $p, q$, and equals 0 whenever $p + q > r - 1$. 
(b) **Quasihomogeneous isolated singularities.** A polynomial

\[ f \in \mathbb{C} [x_1, x_2, \ldots, x_{r+1}] \]

is *quasihomogeneous* of degree \( d \) with respect to the *weights*

\[ w = (w_1, \ldots, w_{r+1}) \in (\mathbb{Z}_{\geq 1})^{r+1} \]

if

\[ f (\lambda^{w_1} x_1, \ldots, \lambda^{w_{r+1}} x_{r+1}) = \lambda^d f (x_1, \ldots, x_{r+1}), \quad \forall \lambda, \quad \lambda \in \mathbb{C}^* . \]

Hereafter we consider such an \( f \), assume that \( r \geq 2 \) and that

\[ X_f := \{ (x_1, \ldots, x_{r+1}) \in \mathbb{C}^{r+1} \mid f (x_1, \ldots, x_{r+1}) = 0 \} \]

has no other singularities than \( 0 \in \mathbb{C}^{r+1} \). Note that the Milnor algebra

\[ M (f) := \mathbb{C} [x_1, x_2, \ldots, x_{r+1}] / \left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{r+1}} \right) \]

associated to \( f \) is a graded \( \mathbb{C} \)-algebra of finite type (with \( \deg(x_i) = w_i, \quad i = 1, \ldots, r + 1 \)) whose Poincaré series \( P_{M(f)} (t) \) equals

\[ P_{M(f)} (t) = \sum_{k \geq 0} \dim_{\mathbb{C}} (M (f)_k) \cdot t^k = \frac{(1 - t^{d - w_1}) (1 - t^{d - w_2}) \cdots (1 - t^{d - w_{r+1}})}{(1 - t^{w_1}) (1 - t^{w_2}) \cdots (1 - t^{w_{r+1}})} \quad (2.2) \]

(cf. [10] (7.27), p. 112)]. Next, we define the *quasismooth* weighted projective hypersurfaces

\[ Z = \{ [x_0 : x_1 : \ldots : x_{r+1}] \in \mathbb{P}_{\mathbb{C}}^{r+1} (1, w) \mid \mathcal{F} (x_0, \ldots, x_{r+1}) = 0 \} \]

where \( \mathcal{F} (x_0, \ldots, x_{r+1}) := x_0^d - f (x_1, \ldots, x_{r+1}) \), and

\[ Z_\infty = \{ [x_0 : x_1 : \ldots : x_{r+1}] \in Z \mid x_0 = 0 \} \]

\[ \cong \{ [x_1 : \ldots : x_{r+1}] \in \mathbb{P}_{\mathbb{C}}^r (w) \mid f (x_1, \ldots, x_{r+1}) = 0 \} . \]

We have

\[ M (\mathcal{F}) = M (f) \otimes \mathbb{C} [x_0] / (x_0^{d-1}) \quad (2.3) \]

and the map \( (x_1, \ldots, x_{r+1}) \mapsto [1 : x_1 : \ldots : x_{r+1}] \) induces a diffeomorphism between

\[ F = \{ (x_1, \ldots, x_{r+1}) \in \mathbb{C}^{r+1} \mid f (x_1, \ldots, x_{r+1}) = 1 \} \]

and the complement \( Z \setminus Z_\infty \), where this \( F \) is diffeomorphic to the (usual) Milnor fiber of the singularity \( (X_f, 0) \) (see [11] (1.13), p. 72)). Moreover, \( F \) has the homotopy type of a bouquet of \( \mu (f) \) \( r \)-spheres, with

\[ \mu (f) = \lim_{t \to \infty} P_{M(f)} (t) = \prod_{i=1}^{r+1} \left( \frac{d}{w_i} - 1 \right) \quad (2.4) \]
denoting the corresponding Milnor number. The primitive cohomology groups of $Z_{\infty}$ are defined by the exact sequence

$$0 \rightarrow H^{r-1}(\mathbb{P}^r C(w), C) \rightarrow H^{r-1}(Z_{\infty}, C) \rightarrow H^{r-1}_{\text{prim}}(Z_{\infty}, C) \rightarrow 0.$$ 

Since both $\mathbb{P}^r C(w)$ and $Z_{\infty}$ are orbifolds, they are equipped with pure Hodge structure, and therefore both $H^{r-1}(Z_{\infty}, C)$ and $H^{r-1}_{\text{prim}}(Z_{\infty}, C)$ decompose, say as

$$H^{r-1}(Z_{\infty}, C) = \bigoplus_{p+q=r-1} H^{p,q}(Z_{\infty}), H^{r-1}_{\text{prim}}(Z_{\infty}, C) = \bigoplus_{p+q=r-1} H^{p,q}_{\text{prim}}(Z_{\infty}),$$

(The same is also valid for $H^{r}_{\text{prim}}(Z, C))$.

**Lemma 2.5** For the Milnor fiber $F$ of $(X_f, 0)$ we have

(i) $h^{p,q}(H^0(F, C)) = 1$, for $p = q = 0$, and $= 0$, otherwise.

(ii) $H^i(F, C) = 0$, for all $i \notin \{0, r\}$.

(iii) $h^{p,q}(H^r(F, C)) = 0$, for $p + q \notin \{r, r + 1\}$.

(iv) $h^{p,r-p}(H^r(F, C)) = H^{p,r-p}_{\text{prim}}(Z) = h^{p,r-p}(Z) - \delta_{p,r-p}$, for $0 \leq p \leq r$.

(v) $h^{p,r+1-p}(H^r(F, C)) = H^{p,r+1-p}_{\text{prim}}(Z_{\infty}) = h^{p,r+1-p}(Z_{\infty}) - \delta_{p-1,r-p}$, for $1 \leq p \leq r$.

**Proof.** (i) This follows from (2.2) (ii) and (2.4) (ii).

(ii)-(v). At first note that $H^{p,q}(\mathbb{P}^r C(w))$ (resp., $H^{p,q}(\mathbb{P}^{r+1} (1, w))$) is $\cong C$, whenever $p = q$, and $= 0$, otherwise. As Steenbrink points out in [28, p. 216], there is an exact MHS-sequence of Gysin-type:

$$\cdots \rightarrow H^i(Z, C) \rightarrow H^i(Z \setminus Z_{\infty}, C) \rightarrow H^{i-1}(Z_{\infty}, C) (-1) \rightarrow H^{i+1}(Z, C) \rightarrow \cdots$$

By the Weak Lefschetz Theorem [12, 4.2.2], the homomorphism

$$H^{p,q}(Z_{\infty}) \rightarrow H^{p,q+1}(\mathbb{P}^r C(w)) [\text{resp., } H^{p,q}(Z) \rightarrow H^{p,q+1}(\mathbb{P}^{r+1} (1, w))]$$

is an isomorphism for $p + q > r - 1$ (resp., $p + q > r$) and an epimorphism for $p + q = r - 1$ (resp., $p + q = r$). Thus, $\theta$ is an isomorphism for all $i \notin \{0, r\}$, proving (ii). Moreover, since

$$W_j(H^r(F, C)) = \begin{cases} 
0, & \text{if } j < r \\
H^r(F, C), & \text{if } j > r
\end{cases}$$

i.e., $Gr^W_j(H^r(F, C)) = 0$, for $j \notin \{r, r + 1\}$, (cf. [3 §8.2]), (iii) is obvious, and the above exact MHS-sequence gives the isomorphisms

$$W_r(H^r(F, C)) = \text{Im}(H^r(Z, C) \rightarrow H^r(Z \setminus Z_{\infty}, C))$$

$$\cong \text{CoKer}(H^{r-2}(Z_{\infty}, C) (-1) \rightarrow H^r(Z, C))$$

and

$$Gr^W_{r+1}(H^r(F, C))$$

$$= H^r(Z \setminus Z_{\infty}) / \text{Ker}(H^r(Z \setminus Z_{\infty}, C) \rightarrow H^{r-1}(Z_{\infty}, C) (-1))$$

$$\cong \text{Ker}(H^{r-1}(Z_{\infty}, C) (-1) \rightarrow H^{r+1}(Z, C))$$

$$\cong \text{CoKer}(H^{r-1}(\mathbb{P}^{r+1} (1, w), C)(-1) \rightarrow H^{r-1}(Z_{\infty}, C) (-1))$$

respectively, proving (iv) and (v). □
Theorem 2.6 (Griffiths-Steenbrink) If \((X_f, 0)\) is an \(r\)-dimensional isolated quasi-homogeneous hypersurface singularity of degree \(d\) w.r.t. the weights \(w_1, \ldots, w_{r+1}\), then
\[
H^{p-1-r-p}_{\text{prim}}(Z_{\infty}) \cong M(f)_{pd-(w_1+\ldots+w_{r+1})}.
\]
Hence,
\[
\begin{cases}
   h^{p-r}(H^r(F, \mathbb{C})) = \dim\left(M(f)_{pd-(w_1+\ldots+w_{r+1})+i}\right) \\
   h^{p+1-r}(H^r(F, \mathbb{C})) = \dim\left(M(f)_{(p+1)d-(w_1+\ldots+w_{r+1})}\right)
\end{cases}
\]
(2.5)

Proof. Extending Griffiths’ results \([4]\) to the case of weighted homogeneous hypersurfaces, the global sections of the sheaves
\[
\Omega^p_{\mathbb{P}^r_C(w)}(Z_{\infty}) = \Omega^p_{\mathbb{P}^r_C(w)} \otimes \mathcal{O}_{\mathbb{P}^r_C(w)}(Z_{\infty}),
\]
as well as the graded pieces of middle cohomology of \(\mathbb{P}^r_C(w) \setminus Z_{\infty}\), are described by means of special auxiliary differential forms with poles along \(Z_{\infty}\). In particular,
\[
H^0(\mathbb{P}^r_C(w), \Omega^r_{\mathbb{P}^r_C(w)}(Z_{\infty})) = \left\{ \frac{g \cdot \Omega_0}{f} \mid g \in \mathbb{C}[x_1, \ldots, x_{r+1}]_{d-(w_1+\ldots+w_{r+1})} \right\},
\]
where
\[
\Omega_0 := \sum_{i=1}^{r+1} (-1)^i w_i x_i \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_{r+1},
\]
and
\[
Gr^{p\bullet}(H^r(\mathbb{P}^r_C(w) \setminus Z_{\infty}, \mathbb{C})) \cong H^{r-p}(\mathbb{P}^r_C(w), \Omega^p_{\mathbb{P}^r_C(w)}(\log Z_{\infty}))
\]
\[
= \frac{H^0(\mathbb{P}^r_C(w), \Omega^r_{\mathbb{P}^r_C(w)}((r-p-1)Z_{\infty}))}{H^0(\mathbb{P}^r_C(w), \Omega^r_{\mathbb{P}^r_C(w)}((r-p)Z_{\infty})) + \partial(H^0(\mathbb{P}^r_C(w), \Omega^{r-1}_{\mathbb{P}^r_C(w)}((r-p)Z_{\infty}))}
\]
(\(\partial\) denotes the corresponding differential operator). Since the map
\[
H^0(\mathbb{P}^r_C(w), \Omega^r_{\mathbb{P}^r_C(w)}((r-p-1)Z_{\infty})) \xrightarrow{\Phi} \mathbb{C}[x_1, \ldots, x_{r+1}]_{(r-p+1)d-(w_1+\ldots+w_{r+1})}
\]
\[
\frac{g \cdot \Omega_0}{f^{r-p+1}} \quad \mapsto \quad g
\]
defines an isomorphism, and
\[
H^0(\mathbb{P}^r_C(w), \Omega^r_{\mathbb{P}^r_C(w)}((r-p)Z_{\infty})) + \partial(H^0(\mathbb{P}^r_C(w), \Omega^{r-1}_{\mathbb{P}^r_C(w)}((r-p)Z_{\infty}))
\]
has
\[
\left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{r+1}} \right)_{(r-p+1)d-(w_1+\ldots+w_{r+1})}
\]
as its image under \(\Phi\) (see \([3], \S 11]\)), we get
\[
Gr^{p\bullet}(H^r(\mathbb{P}^r_C(w) \setminus Z_{\infty}, \mathbb{C})) \cong M(f)_{(r-p+1)d-(w_1+\ldots+w_{r+1})}.
\]
Using Hard Lefschetz Theorem one deduces the exact MHS-sequence:
\[
0 \rightarrow H^{r-2}(\mathbb{P}^r_C(w), \mathbb{C}) \xrightarrow{\phi} H^r(\mathbb{P}^r_C(w), \mathbb{C}) \rightarrow H^r(\mathbb{P}^r_C(w) \setminus Z_{\infty}, \mathbb{C}) \rightarrow H^r_{\text{prim}}(Z_{\infty}, \mathbb{C}) \rightarrow 0,
\]
giving
\[
H^{p-1-r-p}_{\text{prim}}(Z_{\infty}) \cong M(f)_{(r-p)d-(w_1+\ldots+w_{r+1})} \cong M(f)_{(p+1)d-(w_1+\ldots+w_{r+1})}.
\]
Formulae (2.5) follow from Lemma 2.5 (iv), (v), and (2.3). □
Lemma 2.7 If \((X_f, 0)\) is an \(r\)-dimensional isolated quasihomogeneous hypersurface singularity with \(L\) as its link and \(F\) as its Milnor fiber, then

\[
h^{p,q} \left( H^{r-1}(L, \mathbb{C}) \right) = 0, \text{ whenever } p + q \neq r - 1,
\]

and the “non-trivial” Hodge numbers of the cohomology groups of its link \(L\) are

\[
h^{p,r-1-p} \left( H^{r-1}(L, \mathbb{C}) \right) = h^{r-p,p+1} \left( H^r(L, \mathbb{C}) \right) = h^{p+1,r-p} \left( H^r(F, \mathbb{C}) \right)
\]

(2.6)

for \(p = 0, 1, \ldots, r - 1\), and can be therefore read off from \((2.5)\).

**Proof.** If \(p + q \notin \{ r - 1, r + 1 \}\), then by \(2.3, 2.4\) (iv) and \(2.5\) (i), (iii), we obtain

\[
h^{r-p,r-q} \left( H^{r-1}(L, \mathbb{C}) \right) = h^{p,q} \left( H^r(L, \mathbb{C}) \right) = h^{p,q} \left( H^{r-1}(L, \mathbb{C}) \right) = 0,
\]

because the corresponding Hodge numbers of \(H^{r-1}(F, \mathbb{C})\) and \(H^r(F, \mathbb{C})\) vanish, and \(p + q < r - 1\) (resp., \(= r \) \(\iff\) \(r - p + (r - q) > r + 1\) (resp., \(= r \) \(\iff\) \(r - 1\)). On the other hand, if \(p + q \in \{ r - 1, r + 1 \}\), Cor. \(2.3\) gives:

\[
h^{p,q} \left( H^{r-1}(L, \mathbb{C}) \right) - h^{p,q} \left( H^r(L, \mathbb{C}) \right) = h^{r-p,r-q} \left( H^r(F, \mathbb{C}) \right) - h^{p,q} \left( H^r(F, \mathbb{C}) \right).
\]

(2.7)

If \(p + q = r - 1\), the Hodge numbers \(h^{p,q} \left( H^r(L, \mathbb{C}) \right) = h^{r-p,r-q} \left( H^{r-1}(L, \mathbb{C}) \right)\) vanish by \(2.4\) (iv). Analogously, \(h^{p,q} \left( H^{r-1}(L, \mathbb{C}) \right)\) vanishes whenever \(p + q = r + 1\). Finally, \((2.6)\) follows from Lemma \(2.3\) (iii) and \(2.7\).  

**Proposition 2.8** If \((X_f, 0)\) is an \(r\)-dimensional isolated quasihomogeneous hypersurface singularity of degree \(d\) w.r.t. the weights \(w_1, \ldots, w_{r+1}\), and \(L\) its link, then the \(E\)-polynomial \(E(X_f \setminus \{0\}; u, v)\) equals

\[
(uv - 1) \left[ \sum_{p=0}^{r-1} \left( (uv)^p + (-1)^{r-1} h^{p,r-1-p} \left( H^{r-1}(L, \mathbb{C}) \right) u^p v^{r-p-1} \right) \right]
\]

(2.8)

and its coefficients are therefore computable in terms of \(d\) and \(w_1, \ldots, w_{r+1}\) via \((2.6)\) and \((2.7)\).

**Proof.** Using \((2.1)\) and Poincaré duality, we obtain:

\[
h^{p,q} \left( H^i(L, \mathbb{C}) \right) = h^{p,q} \left( H^i(X_f \setminus \{0\}, \mathbb{C}) \right) = h^{d-p,d-q} \left( H^{2d-i}_{c}(X_f \setminus \{0\}, \mathbb{C}) \right).
\]

Hence,

\[
E(X_f \setminus \{0\}; u, v) = (uv)^r E(L; u^{-1}, v^{-1}).
\]

(2.9)
On the other hand, Proposition 2.4 gives

\[ E(L; u, v) = \sum_{0 \leq p, q \leq r} e^{p,q}(L) \ u^p v^q = \]

\[ = \sum_{0 \leq p, q \leq r} \left[ h^{p,q}(H^0(L)) - h^{p,q}(H^{2r-1}(L)) \right] u^p v^q + \]

\[ + (-1)^{r-1} \sum_{0 \leq p, q \leq r} \left[ h^{p,q}(H^{r-1}(L)) - h^{p,q}(H^r(L)) \right] u^p v^q = \]

\[ = \sum_{0 \leq p, q \leq r} \left[ h^{p,q}(H^0(L)) - h^{p,q}(H^{2r-1}(L)) \right] u^p v^q + \]

\[ + (-1)^{r-1} \sum_{0 \leq p, q \leq r} \left[ h^{p,q}(H^{r-1}(L)) - h^{r-p,r-q}(H^{r-1}(L)) \right] u^p v^q = \]

\[ = 1 - (uv)^r - (-1)^{r-1} \left[ \sum_{0 \leq p, q \leq r} h^{p,q}(H^{r-1}(L)) \ u^p v^q \right] + \]

\[ + (-1)^r \left[ \sum_{1 \leq p, q \leq r \atop p + q = r-1} h^{r-p,r-q}(H^{r-1}(L)) \ u^p v^q \right]. \]

(The terms containing coefficients \( h^{p,q}(H^{r-1}(L)) \), with \( p + q = r \), cancel out, as they occur in both summands). Since \((X, \Theta)\) is an isolated quasihomogeneous hypersurface singularity, we may use Lemma 2.7 to write

\[ E(L; u, v) = 1 - (uv)^r - (-1)^{r-1} \left[ \sum_{0 \leq p, q \leq r-1 \atop p + q = r-1} h^{p,q}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] + \]

\[ + (-1)^r \left[ \sum_{1 \leq p, q \leq r \atop p + q = r-1} h^{r-p,r-q}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] = \]

\[ = 1 - (uv)^r + (-1)^{r-1} \left[ \sum_{p = 0}^{r-1} h^{r-1-p}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] + \]

\[ + (-1)^r \left[ \sum_{p = 0}^{r-1} h^{r-1-p}(H^{r-1}(L)) \ u^{p+1} v^{r-p} \right] = \]

\[ = 1 - (uv)^r + (-1)^{r-1} \left[ \sum_{p = 0}^{r-1} h^{r-1-p}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] \left(1 - uv\right) = \]

\[ = (1 - uv) \sum_{p = 0}^{r-1} (uv)^p + (-1)^{r-1} \left(1 - uv\right) \left[ \sum_{p = 0}^{r-1} h^{r-1-p}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] = \]

\[ = (1 - uv) \left[ \sum_{p = 0}^{r-1} (uv)^p + (-1)^{r-1} h^{r-1-p}(H^{r-1}(L)) \ u^p v^{r-1-p} \right] \]

Combining the last equality with (2.3), using

\[ h^{p,r-1-p}(H^{r-1}(L, \mathbb{C})) = (-1)^{r-1} e^{p,r-1-p}(L) \]

\[ = (-1)^{r-1} e^{-r-1-p,p}(L) = h^{r-1-p,p}(H^{r-1}(L, \mathbb{C})) \]

and substituting \( r - p - 1 \) for \( p \), we deduce formula (2.8). \( \square \)
Remark 2.9  (i) By (2.8), \( e(X_f \setminus \{0\}) = E(X_f \setminus \{0\}) : 1, 1) = e(L) = 0 \), which is also obvious from the fact that \( L \) is an odd-dimensional differentiable manifold.

(ii) If the singularity \((X_f, 0) \) in (2.8) is, in addition, a rational singularity, then
\[
 h^{0, r-1}(H^{r-1}(L)) = h^{r-1, 0}(H^{r-1}(L)) = 0.
\]

(See the proof of Proposition 4.1 of [7].)

(iii) The defining polynomial \((1.3)\) of an \( A^{(r)}_{n, \ell} \)-singularity is quasihomogeneous of degree \( d = \text{lcm}(n + 1, \ell) \) w.r.t. the weights \( \left( \frac{d}{n+1}, \frac{d}{\ell}, \frac{d}{7}, \ldots, \frac{d}{7} \right) \), with Poincaré polynomial
\[
P_M(f)(t) = \left( 1 + \sum_{j=1}^{n-1} t^{\frac{d}{n+j}} \right) \left( 1 + \sum_{j=1}^{\ell-1} t^{\frac{d}{\ell+j}} \right)^r
\]
and Milnor number \( \mu(f) = n(\ell - 1)^r \) (see (2.3) and (2.4)). Moreover, since \( A^{(r)}_{n, \ell} \)'s are canonical, they are also rational singularities.

Proof of Proposition 1.7: To produce formula (1.7) for the \( E \)-polynomial of \( X^{(r)}_{n, \ell} \setminus \{0\} \), it suffices to evaluate (2.8) via (2.6), (2.5) and (2.9) (ii)-(iii) in terms of \( n, \ell \) and \( d \). Since the function \( a, \) defined in (1.4), can be expressed as
\[
a(i) = \begin{cases} 1, & \text{if } i \in \left\{ 0, \frac{d}{n+1}, \frac{2d}{n+1}, \ldots, \frac{(n-1)d}{n+1} \right\} \\ 0, & \text{otherwise}, \end{cases}
\]
and since \( b(j), \) as defined in (1.3), gives the coefficient of \( t^j \) in the multinomial expansion of the second factor of (2.10), we need the convolutional function (1.6) in order to write the required dimensions as
\[
h^{p, r-1-p}(H^{r-1}(L, \mathbb{C})) = h^{p+1, r-p}(H^r(F, \mathbb{C}))
\]
\[
= \text{dim}_{\mathbb{C}}(M(f)_{d(p+1) - \frac{1}{n+1} - r}, 0) = c(d(p + 1 - \frac{1}{n+1} - r)),
\]
and to end up to (1.7).

\[\square\]

3 Desingularization and Theorem’s Proof

Next, using blow-ups of closed points, we shall construct snc-resolutions for all \( A^{(r)}_{n, \ell} \)-singularities for which either \( \ell \mid n \) or \( \ell \mid n + 1 \). Let \( X = X^{(r)}_{n, \ell} \) be their underlying spaces and denote by
\[
Y^{(r-1)}_{\ell} := \left\{ [z_1 : z_2 : \ldots : z_{r+1}] \in \mathbb{P}_C^{r+1} \mid \sum_{j=1}^{r+1} z_j^\ell = 0 \right\}
\]
the \((r - 1)\)-dimensional Fermat hypersurface of degree \( \ell \geq 2 \) in the projective space \( \mathbb{P}_C^r \).

Proposition 3.1  (i) If \((n + 1) \equiv 0 \pmod{\ell}\), then there exists an snc-desingularization \( \varphi : \tilde{X} \rightarrow X \) with discrepancy
\[
K_{\tilde{X}} - \varphi^*(K_X) = \sum_{i=1}^{m} i (r - \ell) D_i
\]
where
\[
D_i \cong \mathbb{P}(O_{Y_{\ell}^{(r-2)}}(i)) \oplus O_{Y_{\ell}^{(r-2)}}(1), \quad \forall i, \quad 1 \leq i \leq m - 1, \quad \text{and} \quad D_m \cong Y^{(r-1)}_{\ell}.
\]
(ii) If \( n \equiv 0 (\text{mod } \ell) \), then there is an snc-desingularization \( \varphi : \tilde{X} \to X \) with discrepancy
\[
K_{\tilde{X}} - \varphi^* (K_X) = \sum_{i=1}^{m-1} i (r - \ell) \ D_i + [(m - 1) \ell (r - \ell) + (r - 1)] \ D_m
\]
where
\[
D_i \cong \mathbb{P}(\mathcal{O}_{X_{\ell}}(-2) + \mathcal{O}_{Y_{\ell}}(-2) (1)), \quad \forall i, \ 1 \leq i \leq m - 1, \quad \text{and} \quad D_m \cong \mathbb{P}^{r-1}.
\]
In both cases \( D_i \cap D_j \cong \mathbb{P}^{r-2} \) for all \( i, 1 \leq i \leq m - 1 \), and \( D_i \cap D_j = \emptyset \) for all \( i, j, 1 \leq i, j \leq m \), with \( |i - j| \neq 1 \). (The number \( m \) is defined as in (1.8)).

**Proof.** Let \( f \) be the polynomial \([L] \) and \( X = X_f = X^{(r)}_{n, \ell} \) the underlying space of the \( A^{(r)}_{n, \ell} \)-singularity.

**Construction of the desingularization.** Let \( \pi : \text{Bl}_0(\mathbb{C}^{r+1}) \to \mathbb{C}^{r+1} \) be the blow up of \( \mathbb{C}^{r+1} \) at the origin, with
\[
\text{Bl}_0(\mathbb{C}^{r+1}) = \left\{ ((x_1, \ldots, x_{r+1}), [t_1 : \ldots : t_{r+1}]) \in \mathbb{C}^{r+1} \times \mathbb{P}^r_{\mathbb{C}} \mid \begin{array}{c}
x_i t_j = x_j t_i, \\
\forall i, j, \ 1 \leq i, j \leq r + 1
\end{array} \right\}
\]
and \( \mathcal{E} := \pi^{-1}(0) = \{0\} \times \mathbb{P}^r_{\mathbb{C}} \), and let \( U_i \) denote the open set given by \( \{t_i \neq 0\} \). In terms of analytic coordinates,
\[
U_i = \left\{ ((x_1, \ldots, x_{r+1}), (\xi_1, \ldots, \xi_{r+1})) \in \mathbb{C}^{r+1} \times \mathbb{C}^r \mid \begin{array}{c}
x_j = x_i \xi_j, \forall j, \\
j \in \{1, \ldots, r \} \setminus \{i\}
\end{array} \right\},
\]
where \( \xi_j = \frac{t_j}{t_i} \). Identifying \( U_i \) with a copy of \( \mathbb{C}^{r+1} \) w.r.t. the coordinates \( x_1, \xi_1, \ldots, \xi_i, \ldots, \xi_{r+1} \), the restriction \( \pi|_{U_i} \) is given by mapping
\[
\begin{align*}
\mathbb{C}^{r+1} \ni (x_i, \xi_1, \ldots, \xi_{r+1}) &\xrightarrow{\cong} ((x_1, \ldots, x_{r+1}), (\xi_1, \ldots, \xi_{r+1}) \mid \xi_1 : \ldots : 1) \\
&\xrightarrow{\pi|_{U_i}} (x_1, \xi_1, \ldots, x_i \xi_{i-1}, x_i, x_i \xi_{i+1}, \ldots, x_i \xi_{r+1})
\end{align*}
\]
Further, \( \mathcal{E}_i := \mathcal{E} \cap U_i \) is described as the coordinate hyperplane \( (x_i = 0) \); i.e., the open cover \( \{U_j\}_{1 \leq i \leq r+1} \) of \( \text{Bl}_0(\mathbb{C}^{r+1}) \) restricts to \( \mathcal{E} \) to provide the standard open cover of \( \mathbb{P}^r_{\mathbb{C}} \) by affine spaces \( \mathbb{C}^{r+1} \), with \( \{\xi_j\}_{j \in \{1, \ldots, r+1\} \setminus \{i\}} \) being the analytic coordinates of \( \mathcal{E}_i \).

**Notation.** To work with a more convenient notation we define
\[
\text{Bl}_0(\mathbb{C}^{r+1}) = \bigcup_{i=1}^{r+1} U_i, \quad U_i = \text{Spec} (\mathbb{C}[y_{1,i}, \ldots, y_{r,i+1}] ),
\]
by setting as coordinates for \( U_i \)'s:
\[
y_{i,k} := \begin{cases} 
x_k, & \text{for } i = k \\
\xi_k, & \text{for } i \neq k
\end{cases}
\]

**The first blow-up.** Blowing up \( X \) at the origin, we take the diagram
\[
\begin{array}{ccc}
\mathcal{E} & \subset & \text{Bl}_0(\mathbb{C}^{r+1}) \\
& \cup & \cup \\
\mathcal{E}_X := \mathcal{E} \cap \text{Bl}_0(X) & \subset & \text{Bl}_0(X)
\end{array} \quad \xrightarrow{\pi} \quad \begin{array}{ccc}
\mathbb{C}^{r+1} & \cup & \cup \\
& & \\
& & X
\end{array}
\]
and consider the strict transform
\[ \text{Bl}_0(X) = \pi^{-1}(X \cap (\mathbb{C}^{r+1} \setminus \{0\})) = \pi^{-1}(X) \cap (\text{Bl}_0(\mathbb{C}^{r+1}) \setminus \mathcal{E}) \]
of X in $\mathbb{C}^{r+1}$ under $\pi$, and the corresponding exceptional divisor $\mathcal{E}_X$.

- **LOCAL DESCRIPTION of $\text{Bl}_0(X)$ AND $\mathcal{E}_X$.** Pulling back $f$, we get
  \[ \pi^*(f) \mid_{U_i} = x_i^t f_i = y_i^t, \]
  with $f_i(y_{i,1}, \ldots, y_{i,r+1}) = x_i^{n+1} - y_{i,2} + \cdots + y_{i,r+1}$.

  \[ \text{Bl}_0(X) \mid_{U_i} \cong \left\{(y_{i,1}, \ldots, y_{i,r+1}) \in \mathbb{C}^{r+1} \mid y_i = f_i(y_{i,1}, \ldots, y_{i,r+1}) = 0 \right\}, \]
  and the equations for $\mathcal{E}_X$ read as follows:
  \[ \mathcal{E}_X \mid_{U_i} \cong \left\{(y_{i,1}, \ldots, y_{i,r+1}) \in \mathbb{C}^{r+1} \mid y_i = f_i(y_{i,1}, \ldots, y_{i,r+1}) = 0 \right\}. \]

  Thus, the only singular affine patch is $U_1 = \text{Spec}(\mathbb{C}[y_{1,1}, \ldots, y_{1,r+1}])$ whenever $n > \ell$.

- **GLOBAL DESCRIPTION of $\text{Bl}_0(X)$ AND $\mathcal{E}_X$.** Passing to global coordinates, we can write
  \[ \text{Bl}_0(X) = \left\{(x_1, \ldots, x_{r+1}, [t_1 : \cdots : t_{r+1}]) \mid (x_1^{n+1} - t_1^r + \sum_{j=2}^{r+1} t_j^r = 0) \right\} \]
  and $\mathcal{E}_X$ equals
  \[ \left\{(0, [t_1 : \cdots : t_{r+1}]) \in (0) \times \mathbb{P}_{\mathbb{C}}^r \mid t_1^r + t_2^r + \cdots + t_{r+1}^r = 0 \right\}, \text{ if } n = \ell - 1 \]
  \[ \left\{(0, [t_1 : \cdots : t_{r+1}]) \in (0) \times \mathbb{P}_{\mathbb{C}}^r \mid t_1^r + t_2^r + \cdots + t_{r+1}^r = 0 \right\}, \text{ otherwise} \]

- **THE (FERMAT) SINGULARITY $A_{r-1,\ell}$ ($m = 1$).** Blowing up the origin once, we achieve immediately the required desingularization, having exceptional divisor $\mathcal{E}_X \cong Y^{(r-1)}_\ell$.

- **THE SINGULARITY $A_{r,\ell}$ ($m = 2$).** In this case, $\text{Bl}_0(X)$ is smooth, whereas $\mathcal{E}_X \subset \text{Bl}_0(X)$ has a singular, ordinary $\ell$-fold point at
  \[ Q = (0, [1 : 0 : \cdots : 0]) \in \mathcal{E}_X \mid_{U_1}. \]

  To obtain an snc-resolution of the original singularity, we blow up once more at $Q$, and consider $\varphi = \pi_1 \circ \pi_2$.

  \[ \widetilde{X} = \text{Bl}_Q(\text{Bl}_0(X)) \xrightarrow{\pi_2} \text{Bl}_0(X) \xrightarrow{\pi_1} X. \]

The new exceptional divisor $D_0$ is a $\mathbb{P}_{\mathbb{C}}^{r-1}$, and the strict transform $D_1$ of $\mathcal{E}_X$ is nothing but the $((r-1))$-dimensional blow-up of $\mathcal{E}_X$ at $Q$. Since $\mathcal{E}_X$ can be viewed as the projective cone $C^{pr}(Y^{(r-2)}_\ell) \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ over the Fermat hypersurface $Y^{(r-2)}_\ell \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ with $[1 : 0 : \cdots : 0]$ as its vertex, blowing up $[1 : 0 : \cdots : 0]$, the diagram

\[
\begin{array}{ccc}
\mathbb{P}(\mathbb{C}_{\mathbb{P}_{\mathbb{C}}^{r-1}} \oplus \mathbb{C}_{\mathbb{P}_{\mathbb{C}}^{r-1}}(1)) & \cong & \text{Bl}_{[1:0:0]}(\mathbb{P}_{\mathbb{C}}^r) \\
\cup & & \cup \\
\text{Bl}_{[1:0:0]}(C^{pr}(Y^{(r-2)}_\ell)) & \rightarrow & C^{pr}(Y^{(r-2)}_\ell) \cong \mathcal{E}_X
\end{array}
\]
yields the isomorphism
\[
\text{Bl}(Y_{\ell}^{(r-2)}) \cong \mathbb{P}(\mathcal{O}_{Y_{\ell}^{(r-2)}}(1)).
\]

Hence, \(D_1\) is a \(\mathbb{P}^1\)-bundle of rank 2 over \(Y_{\ell}^{(r-2)}\) meeting \(D_2\) along
\[
(D_1 \cdot D_2)|_{D_1} = \mathbb{P}(\mathcal{O}_{Y_{\ell}^{(r-2)}}(1)) \cong Y_{\ell}^{(r-2)}.
\]
(see Fig. 1).

\[\text{Fig. 1}\]

- **Singularities** \(A_{n,\ell}^{(r)}\) with \(n > \ell, m \geq 2\), and either \(\ell | n\) or \(\ell | n + 1\). Locally, these singularities can be reduced successively to one of the above types as follows:

\[
\begin{align*}
A_{n,\ell}^{(r)} & \rightsquigarrow A_{n-\ell,\ell}^{(r)} \rightsquigarrow A_{n-2\ell,\ell}^{(r)} \rightsquigarrow \cdots \rightsquigarrow A_{2\ell-1,\ell}^{(r)} \rightsquigarrow A_{\ell-1,\ell}^{(r)} \rightsquigarrow A_{1,\ell}^{(r)}
\end{align*}
\]

(if \(n + 1 \equiv 0 \mod \ell\))

\[
\begin{align*}
A_{n,\ell}^{(r)} & \rightsquigarrow A_{n-\ell,\ell}^{(r)} \rightsquigarrow A_{n-2\ell,\ell}^{(r)} \rightsquigarrow \cdots \rightsquigarrow A_{\ell,\ell}^{(r)} \rightsquigarrow A_{0,\ell}^{(r)} \rightsquigarrow A_{0,\ell}^{(r)}
\end{align*}
\]

(if \(n \equiv 0 \mod \ell\))

(Each \(\rightsquigarrow\) denotes the result of a local blow-up, and \(A_{n,\ell}^{(r)}, A_{0,\ell}^{(r)}\) stand for “smooth charts”). But also globally, \(\varphi : \tilde{X} \to X\) is decomposed into just \(m\) blow-ups
\[
\tilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_3} X_2 \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X
\]
(3.3)

\[
X_i := \text{Bl}_{Q_i}(\text{Bl}_{Q_{i-1}}(\cdots (\text{Bl}_{Q_1}(X)))\)), \quad \forall i, \ 1 \leq i \leq m,
\]

of \(m\) “separated” points \(Q_1 = 0, Q_2 = (0, [1 : 0 : 0 : \cdots : 0]), \ldots, Q_m,\) in the sense, that all the appearing exceptional divisors are prime (by construction) and, in addition,
if \( E_1 = \mathcal{E}_X, E_2, \ldots, E_m \) are the exceptional loci of \( \pi_1, \pi_2, \ldots, \pi_m \), respectively, the singular point \( Q_1 \) is resolved by \( \pi_1 \) and the (possibly existing) new singular point \( Q_{i+1} \) is not contained in the strict transforms of \( E_1, E_2, \ldots, E_{i-1} \) under \( \pi_i \). Thus, defining the divisor \( D_i \) to be the strict transform of \( E_i \) under \( \pi_i+1 \circ \pi_{i+2} \circ \cdots \circ \pi_{m-1} \circ \pi_m \) on \( \tilde{X} \), we obtain the intersection graph of Figure 2 with \( D_i \cap D_{i+1} \cong Y_{\ell}^{r-2} \).

\[ \text{Fig 2} \]

**Computation of the discrepancy coefficients.** Consider the Poincaré residue map

\[
\text{Res}_X : H^0(\mathbb{C}^{r+1}, \omega_{\mathbb{C}^{r+1}}(X)) \rightarrow H^0(X, \omega_X),
\]

where \( \omega_X = \mathcal{O}_X(K_X) = (\Omega_X)^{1/r} \subset \Omega_X^{\mathbb{C}(X)/\mathbb{C}}. \) The rational canonical differential

\[
\mathfrak{s} := \text{Res}_X \left( \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1}}{f} \right) = \frac{dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1}}{(\partial f / \partial x_1)}
\]

can be viewed as a (local) generator of \( H^0(X, \omega_X) \). Assume that \( n \neq \ell \) and that you have performed the first blow-up of \( X \) at \( 0 \). Then the new singularity (if any) on \( \text{Bl}_0(X) \) will belong to \( \mathcal{E}_X | U_1 \). For this reason, to find the discrepancy coefficient w.r.t. \( \pi : \text{Bl}_0(X) \rightarrow X \), it suffices to compare \( \mathfrak{s} \) with the rational canonical differential

\[
\mathfrak{f} := \frac{dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}}{(\partial f_1 / \partial y_{1,1})} \in \Omega_{\mathbb{C}(U_1)/\mathbb{C}}.
\]

\((U_1 \text{ is non-singular with local coordinates } y_{1,2}, \ldots, y_{1,r+1} \text{ at any point } P \text{ for which } \partial f_1(P) / \partial y_{1,1} \neq 0).\) In \( U_1 \) we have \( x_1 = y_{1,1} \) and

\[
x_j = x_1 \xi_j = y_{1,1} y_{1,j}, \quad \text{for all } j \in \{2, 3, \ldots, r+1\}.
\]

Hence,

\[
dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1}
\]

\[
= y_{1,1}^{-1} (y_{1,1} (dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}) +
\]

\[
+ \sum_{i=2}^{r+1} (-1)^i y_{1,i} \ dy_{1,2} \wedge \cdots \wedge dy_{1,i-1} \wedge dy_{1,r+1})
\]

and

\[
\partial f / \partial x_1 = (n+1) x_1^n = (n+1) y_{1,1}^n = \left( \frac{n+1}{n+1-\ell} \right) y_{1,1}^\ell (\partial f_1 / \partial y_{1,1}) \quad (3.5)
\]

On the other hand,

\[
df_1 = (n+1-\ell) y_{1,1}^{n-\ell} dy_{1,1} + \ell (y_{1,2}^{\ell-1} dy_{1,2} + \cdots + y_{1,r+1}^{\ell-1} dy_{1,r+1}) = 0
\]
if and only if
\[ dy_{1,1} = -\frac{\ell}{n+1-\ell} y_{1,1}^{n+1-\ell} (y_{1,2}^{\ell-1} dy_{1,2} + \cdots + y_{1,r+1}^{\ell-1} dy_{1,r+1}) \]  
(3.6)
Substituting the expression (3.6) for \( dy_{1,1} \) into the right-hand side of (3.4), we obtain
\[ dx_2 \wedge dx_3 \wedge \cdots \wedge dx_{r+1} \]
\[ = \left( -\frac{\ell}{n+1-\ell} y_{1,1}^{n+1-\ell-n} (y_{1,2}^{\ell-1} + \cdots + y_{1,r+1}^{\ell-1}) + y_{1,1}^{\ell} \right) dy_{1,2} \wedge \cdots \wedge dy_{1,r+1} \]  
(3.7)
Combining now (3.7) with \( y_{1,2}^{\ell-1} + \cdots + y_{1,r+1}^{\ell-1} = -y_{1,1}^{(n+1)-\ell} \) and (3.3), we get
\[ s = \frac{y_{1,1}^{r-\ell} dy_{1,2} \wedge dy_{1,3} \wedge \cdots \wedge dy_{1,r+1}}{y_{1,1}^{\ell} (\partial f_1 / \partial y_{1,1})} = y_{1,1}^{\ell-\ell} \]  
(3.8)
The equality (3.8) shows that the discrepancy coefficient of \( E_X \) with respect to \( \pi : B_\ell(X) \to X \) equals \( r-\ell \). Using the notation introduced in (3.3), one proves analogously that
\[ K_{X_i} - \pi^* (K_{X_{i-1}}) = (r-\ell) E_i, \quad \forall i, \; 1 \leq i \leq m-1. \]  
(3.9)
Moreover,
\[ K_{X_m} - \pi^* (K_{X_{m-1}}) = \begin{cases} (r-\ell) D_m, & \text{if } \ell \mid n+1 \\ (r-1) D_m, & \text{if } \ell \mid n \end{cases} \]  
(3.10)
Note that if \( \ell \mid n \), then we have to pass through \( A_{\ell,k}^{(r)} \). The additional blow-up which resolves the singularity of the exceptional locus (so that \( \varphi : \tilde{X} \to X \) fulfills the snc-condition) has a smooth point on the \( r \)-fold as its centre. Consequently, the discrepancy coefficient of \( D_m = D_{\varphi+1} \) equals \( r-1 \) (see [15, p. 187]). Now (3.9) gives:
\[ K_\tilde{X} - \varphi^* (K_X) = \sum_{i=1}^{m-1} (\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_m)^* ((r-\ell) E_i) + [K_{X_m} - \pi_m^* (K_{X_{m-1}})] \]  
(3.11)
Since
\[ (\pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_m)^* (E_i) = \begin{cases} \sum_{j=1}^{m} D_j, & \text{if } \ell \mid n+1 \\ \sum_{j=1}^{m-1} D_j + \ell D_m, & \text{if } \ell \mid n \end{cases} \]  
(3.12)
for all \( 1 \leq i \leq m-1 \), the formulae (3.1) and (3.2) follow from (3.10), (3.11) and (3.12).

**Remark 3.2**
(i) If \( n+1 \equiv 0 \pmod{\ell} \) and \( r = \ell \), then \( \varphi : \tilde{X} \to X \) is crepant.
(ii) Obviously,
\[ E(P_{\mathbb{C}}^{r-1}; u, v) = \sum_{p=0}^{r-1} (uv)^p. \]  
(3.13)
(iii) To complete the catalogue of the \( E \)-polynomials of our exceptional divisors, it suffices to find out those of \( Y_\ell^{(r-2)} \) (or, equivalently, of \( Y_\ell^{(r-1)} \)), as we have
\[
E(\mathcal{P}(\mathcal{O}_{Y_\ell^{(r-2)}} \oplus \mathcal{O}_{Y_\ell^{(r-2)}})) \equiv E(Y_\ell^{(r-2)}; u, v) \cdot (1 + uv)
\]
\[\tag{3.14}\]

(iv) According to the classical Lefschetz Hyperplane Theorem, the Fermat hypersurface \( Y_\ell^{(r-1)} \) has “non-trivial” Hodge \((p, q)\)-numbers only if \( p + q = r - 1 \). Next lemma expresses them by means of the non-central Eulerian numbers of generalized factorials (as defined in \S 1 (d)), and can be easily proven, e.g., by determining the \( \chi_y \)-characteristic of \( Y_\ell^{(r-1)} \) via Riemann-Roch Theorem (see \[19, \S 2\]), or, alternatively, by writing down the exact sequences involving the cohomology groups of \( \mathbb{P}^r_\ell \) via Gauss-Bonnet Theorem; see, e.g., \[11\), p. 152.)

**Lemma 3.3** The Hodge numbers of the \((r-1)\)-dimensional Fermat hypersurface \( Y_\ell^{(r-1)} \) of degree \( \ell \geq 2 \) are given by the formula
\[
h^{p, q}(Y_\ell^{(r-1)}) = \begin{cases} 
\mathcal{S}(r, p + 1 \mid \ell - 1, p) + \delta_{2p, r-1}, & \text{if } p + q = r - 1 \\
\delta_{p, q}, & \text{if } p + q \neq r - 1
\end{cases}
\]
Hence, \( Y_\ell^{(r-1)} \) has \( E \)-polynomial
\[
E(Y_\ell^{(r-1)}; u, v) = \sum_{0 \leq p, q \leq r-1} (-1)^{p+q} h^{p, q}(Y_\ell^{(r-1)}) u^p v^q
\]
\[
= \sum_{p=0}^{r-1} u^p \left[ v^p + (-1)^{r-1} \mathcal{S}(r, p + 1 \mid \ell - 1, p) v^{r-1-p} \right]
\]
and Euler number
\[
e(\mathcal{P}(\mathcal{O}_{Y_\ell^{(r-2)}} \oplus \mathcal{O}_{Y_\ell^{(r-2)}})) \equiv e(Y_\ell^{(r-2)}; u, v)
\]
\[
= \left[ \sum_{p=0}^{r-1} (-1)^{r-1} \mathcal{S}(r, p + 1 \mid \ell - 1, p) \right] + r = (1 - \ell)^{r+1} - 1 + r + 1
\]
\[\tag{3.16}\]

**Proof of Theorem 1.8.** (i) If \( n + 1 \equiv 0 \pmod{\ell} \), then Proposition 3.1 and (1.1) give:
\[
E_{\text{str}}(X; u, v) - E(X \setminus \{0\}; u, v) =
\]
\[
\sum_{i=1}^{m} \frac{(uv-1) E(D_i^{(r-1)}; u, v)}{(uv)(r-\ell+1)+1} + \sum_{i=1}^{m-1} \frac{(uv-1)^2 E(D_i^{(r-1)}; u, v)}{(uv)(r-\ell+1)(r-\ell+2)+1}
\]

But
\[
E(D_1^{(r-1)}; u, v) = \begin{cases} 
E(Y_\ell^{(r-1)}; u, v) & \text{if } \ell = n + 1 \ (\text{i.e., } m = 1) \\
E(Y_\ell^{(r-2)}; u, v) \cdot uv, & \text{otherwise}
\end{cases}
\]
\[
E(D_i^{(r-1)}; u, v) = E(Y_\ell^{(r-2)}; u, v) \cdot (uv - 1), \ \forall i, i \in \{2, \ldots, m-1\},
\]
by (3.14), and (for m ≥ 2) \( E(D_m^0; u, v) = E(Y_{\ell}^{(r-1)}; u, v) - E(Y_{\ell}^{(r-2)}; u, v) \),

\[ E(D_{i,i+1}^0; u, v) = E(Y_{\ell}^{(r-2)}; u, v), \forall i, i \in \{1, \ldots, m-1\}. \]

Consequently, for m ≥ 2, the difference \( E_{\text{str}}(X; u, v) - E(X \setminus \{0\}; u, v) \) equals

\[
(uv - 1) E(Y_{\ell}^{(r-2)}; u, v) \left[ \frac{uv}{(uv)^{r+1}} + \sum_{i=2}^{m-1} \frac{uv-1}{(uv)^{(r+1)i+1}} - \frac{uv-1}{(uv)^{(m+1)i+1}} \right]
\]

\[
+ \frac{(uv-1)E(Y_{\ell}^{(r-2)}; u, v)}{(uv)^{(m+1)i+1}} + \sum_{i=1}^{m-2} \frac{(uv-1)^2 E(Y_{\ell}^{(r-2)}; u, v)}{(uv)^{(r+1)i+1}}
\]

leading to the desired formula via (3.13). [For m = 1, the computation is straightforward.] Passing to the limit of \( E_{\text{str}}(X; u, v) \), for \( u, v \to 1 \), and taking (1.2) and (3.16) into account, one obtains the corresponding formula for the string-theoretic Euler number \( e_{\text{str}}(X) \).

(ii) If \( n \equiv 0 \pmod{\ell} \), then [3.3] (ii) and (1.1) give analogously

\[
E_{\text{str}}(X; u, v) - E(X \setminus \{0\}; u, v) =
\]

\[
\sum_{i=1}^{m-1} \frac{(uv-1)E(D_i^0; u, v)}{(uv)^{(r+1)i+1}} - \frac{uv-1}{(uv)^{(m+1)i+1}} E(D_m^0; u, v)
\]

\[
+ \left[ \sum_{i=1}^{m-2} \frac{(uv-1)^2 E(D_i^0; u, v)}{(uv)^{(r+1)i+1}} \right]
\]

\[
+ \frac{(uv-1)^2 E(D_m^0; u, v)}{(uv)^{(m+1)i+1}}
\]

Since \( D_1^0, D_2^0, \ldots, D_{m-1}^0 \) are as in (i), and

\[
E(D_m^0; u, v) = E(\mathbb{P}^{r-1}_C; u, v) - E(Y_{\ell}^{(r-2)}; u, v),
\]

\[
E(D_{i,i+1}^0; u, v) = E(Y_{\ell}^{(r-2)}; u, v), \text{ for all } i \in \{1, \ldots, m-1\},
\]

we obtain by (3.13) (3.14):

\[
E_{\text{str}}(X; u, v) - E(X \setminus \{0\}; u, v) =
\]

\[
(uv - 1) E(Y_{\ell}^{(r-2)}; u, v) \left[ \frac{uv}{(uv)^{r+1}} + \sum_{i=2}^{m-1} \frac{uv-1}{(uv)^{(r+1)i+1}} - \frac{uv-1}{(uv)^{(m+1)i+1}} \right]
\]

\[
+ \frac{(uv-1)(\sum_{i=1}^{m-1} (uv)^i)^2}{(uv)^{(m+1)i+1}} + \left[ \sum_{i=1}^{m-2} \frac{(uv-1)^2 E(Y_{\ell}^{(r-2)}; u, v)}{(uv)^{(r+1)i+1}} \right]
\]

\[
+ \frac{(uv-1)^2 E(Y_{\ell}^{(r-2)}; u, v)}{(uv)^{(m+1)i+1}}
\]

The string-theoretic Euler number is examined as in (i). \( \Box \)
4 Some global geometric examples

The $E_{str}$-function of a complex $r$-fold $V$ with only $k$ isolated log-terminal singularities $Q_1, Q_2, \ldots, Q_k$ equals:

$$E_{str}(V; u, v) = E(V; u, v) + \sum_{i=1}^{k} (E_{str}(V, Q_i; u, v) - 1) \quad (4.1)$$

In particular, a simple closed formula for the string-theoretic Euler number $e_{str}$ can be easily built whenever $V$ is a (global) complete intersection in a projective space, equipped with prescribed singularities belonging to the class under consideration.

**Proposition 4.1** Let $V = V(d_1, d_2, \ldots, d_{N-n})$ be an $r$-dimensional complete intersection of multidegree $(d_1, d_2, \ldots, d_{N-n})$ in $\mathbb{P}_c^N$ having only $k$ isolated singularities $Q_1, \ldots, Q_k$ of types $A_{n_1, \ell_1}, \ldots, A_{n_k, \ell_k}$ with either $\ell_i | n_i$ or $\ell_i | n_i + 1$, for all $i = 1, \ldots, k$. Then its string-theoretic Euler number equals

$$e_{str}(V) = \left(\binom{N+1}{r} + \sum_{\nu=1}^{r} \left(\sum_{1 \leq j_1 \leq \cdots \leq j_{\nu-r} \leq N-r} d_{j_1} \cdots d_{j_{\nu-r}}\right)\right) \left(\prod_{j=1}^{N-r} d_j\right) +$$

$$+ \sum_{i=1}^{k} \left[ e_{str}(V, Q_i) + (-1)^{r+1} n_i (\ell_i - 1)^{r-1} \right] \quad (4.2)$$

where $e_{str}(V, Q_i), i = 1, \ldots, k$, are computable via Theorem 1.8.

**Proof.** By a small deformation of $V$ one can always obtain a non-singular complete intersection $V'$ in $\mathbb{P}_c^N$ having multidegree $(d_1, d_2, \ldots, d_{N-n})$. Using a standard technique which involves the Mayer-Vietoris sequence (cf. [11]. Ch. 5, Cor. 4.4 (ii))), one shows easily that

$$e(V) = e(V') + (-1)^{r+1} \sum_{i=1}^{k} \left[ \text{Milnor number of } (V, Q_i) \right].$$

The Euler number of $V'$ can be computed again by evaluating the highest Chern class of $V'$ at its fundamental cycle (cf. Chen-Ogiue [5. Thm. 2.1]), and is expressible by the closed formula:

$$e(V') = \left(\binom{N+1}{r} + \sum_{\nu=1}^{r} (-1)^{\nu} \binom{N+1}{r-\nu} \left(\sum_{1 \leq j_1 \leq \cdots \leq j_{\nu-r} \leq N-r} d_{j_1} \cdots d_{j_{\nu-r}}\right)\right) \left(\prod_{j=1}^{N-r} d_j\right).$$

(4.2) follows clearly from (4.1). \qed

**Examples 4.2** Let us now apply (4.2) for some well-known hypersurfaces and complete intersections.

(i) Generalizing Hirzebruch’s method of constructing a singular quintic with 126 nodes ([21] p. 762), Werner defines in [28, pp. 216-217] a hypersurface $V \subset \mathbb{P}_c^4$ of degree 5 by homogenizing a three-dimensional affine complex variety of the form

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \lambda g_1(z_1, z_2) - g_2(z_3, z_4) = 0\}, \lambda \in \mathbb{C}^*,$$

where $\{g_i = 0\}, i = 1, 2$, are plane quintic curves having the three axes and a circumscribed conic (about the corresponding coordinate triangle) as their irreducible components (see Fig. 3). Since each of these curves has 3 $D_4$-singularities, $V$ (after homogenization) will have $3^2 = 9$ singularities of type $A_{2,3}^{(3)}$. This means that

$$e_{str}(V) = -200 + 9 \cdot (9 + 2^4 - 1) = 16$$
In fact, \( \varepsilon_{\text{str}}(V) = \varepsilon(\tilde{V}) = 16 \), where \( \tilde{V} \to V \) is the crepant desingularization of \( V \) arising from a single simultaneous blow-up of the 9 singularities (cf. Fig. 3). \( \tilde{V} \) is obviously a 3-dimensional Calabi-Yau manifold.

(ii) The \((N - 1)\)-dimensional Goryunov’s quartics \(\{ \}\):

\[
V_\kappa := \left\{ [z_1 : \ldots : z_{N + 1}] \in \mathbb{P}^N_k \mid 2(\kappa + 1) \sum_{1 \leq i < j \leq N + 1} z_i z_j + \kappa \left( \sum_{1 \leq j \leq N + 1} z_j^2 \right)^2 = 0 \right\}
\]

\((N \geq \kappa, N \geq 3, \kappa \geq 0)\) have \(2^\kappa \binom{N + 1}{\kappa + 1} A_3\)-singularities (\(A_{1,2}^{(N-1)}\)-singularities, in our notation), and string-theoretic Euler number

\[
\varepsilon_{\text{str}}(V_\kappa) = \frac{1}{4} \left( (-3)^{N+1} - 1 \right) + N + 1 + 2^\kappa \binom{N + 1}{\kappa + 1} \left[ \left( \frac{1}{N-2} \left( (-1)^{N} + N \right) \right) + (-1)^N - 1 \right],
\]

Note that, e.g., for \(N = 5\), the string-theoretic index of the underlying space of each of the singularities is \(3 > 1\), whereas the string-theoretic index \(\text{ind}_{\text{str}}(V_\kappa)\) of \(V_\kappa\) can be equal to 1 (for \(\kappa \in \{0, 1, 3, 4\}\)).

(iii) The \((n - 2)\)-dimensional Segre-Knôrrer complete intersection of two quadrics

\[
V := \{ z = [z_1 : z_2 : \ldots : z_{n+1}] \in \mathbb{P}^n_k \mid ^t z \ M \ z = 0 \}, \quad n \geq 4,
\]

where \(M\) and \(M’\) are the \((n + 1) \times (n + 1)\)-matrices:

\[
M = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 1 \\
0 & 0 & \cdots & \cdots & 1 & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0 \\
1 & 0 & \cdots & \cdots & 0 & 0
\end{pmatrix}, \quad M’ = \begin{pmatrix}
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 1 \\
\vdots & \vdots & \cdots & \cdots & 1 & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 1 & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0
\end{pmatrix},
\]

has \(Q = [1 : 0 : \cdots : 0 : 0]\) as single isolated point which is of type \(A_n\) (i.e., \(A_{n-2}\) in our notation, see [23, p. 48]). According to (1.2), the string-theoretic Euler number of \(V\) equals

\[
\varepsilon_{\text{str}}(V) = \sum_{\nu = 0}^{n-2} (-1)^\nu 2^{\nu+2} \binom{\nu+1}{\nu+3} (\nu + 1) + \varepsilon_{\text{str}}(V, Q) + (-1)^{n-1} n - 1
\]

\[
= n - 1 + \varepsilon_{\text{str}}(V, Q),
\]

with

\[
\varepsilon_{\text{str}}(V, Q) = \begin{cases}
\frac{(n-1)^2}{n^2 - 3n - 2}, & \text{if } n \text{ odd} \\
\frac{(n-2)(n+1)}{n(n-4)+(n-2)}, & \text{if } n \text{ even}
\end{cases}
\]
For \( n \leq 15 \), \( e_{\text{str}}(V) \) takes the following values:

| \( n \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( e_{\text{str}}(V) \) | 8 | 6 | 27/4 | 96/4 | 160/19 | 120/13 | 175/17 | 480/43 | 648/53 | 105/8 | 539/38 | 1344/89 |

(iv) Werner’s 3-dimensional complete intersection of a cubic and two quadrics

\[
V := \left\{ [z_1 : z_2 : \ldots : z_7] \in \mathbb{P}^6 \left| \begin{array}{c}
\sum_{i=1}^4 z_i^3 = \sum_{j=2}^7 z_j^2 \\
= \sum_{i=1}^3 z_{i+1}^2 + \sum_{j=4}^6 (j-3) z_{j+1}^2 = 0
\end{array} \right. \right\}
\]

has 4 singularities of type \( A^{(3)}_{2,3} \) at the points \([0 : 0 : 0 : \pm1 : \pm\sqrt{-1} : 1]\) and 18 singularities of type \( A^{(3)}_{1,2} \) (i.e., nodes) at the points

\[
\begin{align*}
[-\zeta_3^j : 1 : 0 : 0 : \pm\sqrt{-1} : 0 : 0], \\
[-\zeta_3^j : 0 : 1 : 0 : \pm\sqrt{-1} : 0 : 0], \\
[-\zeta_3^j : 0 : 0 : 1 : 0 : \pm\sqrt{-1} : 0],
\end{align*}
\]

\( j = 1, 2, 3 \), where \( \zeta_3 \) is a primitive third root of unity (see [28, pp. 221–222]). Its string-theoretic Euler number equals

\[
e_{\text{str}}(V) = -144 + 4 \cdot (9 + 2^4 - 1) + 18 \cdot 2 = -12 = e(\bar{V}),
\]

where \( \bar{V} \) is a Calabi–Yau threefold which arises after a crepant desingularization of \( V \) coming from the simultaneous (usual) blow-up of the 9 \( A^{(3)}_{2,3} \)-singularities and an appropriate small, projective resolution of the 18 nodes.

(v) Let \( V = V_1 \cap V_2 \cap \cdots \cap V_{N-r} \subset \mathbb{P}^N_\mathbb{C} \) be a complete intersection of Fermat hypersurfaces

\[
V_i = \left\{ [z_1 : \ldots : z_{N+1}] \in \mathbb{P}^N_\mathbb{C} \left| \sum_{j=1}^{N+1} b_{ij} z_j^d = 0 \right. \right\}, \quad 1 \leq i \leq N-r,
\]

of degree \( d \), \( 2 \leq d \leq r \), and assume that \( V \) is \( r \)-dimensional, i.e.,

\[
\text{rank} ((b_{ij})_{1 \leq i \leq N-r, 1 \leq j \leq N+1}) = N-r.
\]

Further, consider the map

\[
\Phi_d : \mathbb{P}^N_\mathbb{C} \to \mathbb{P}^N_\mathbb{C}, \quad [z_1 : \ldots : z_{N+1}] \mapsto [z_1^d : \ldots : z_{N+1}^d] = [\xi_1 : \ldots : \xi_{N+1}].
\]

\( \Phi_d \) displays \( \mathbb{P}^N_\mathbb{C} \) as a \( d \)-sheeted ramified covering of itself, branched along the coordinate axes \( \{\xi_j = 0\} \). On the other hand,

\[
\Phi_d(V_i) = \left\{ [\xi_1 : \ldots : \xi_{N+1}] \in \mathbb{P}^N_\mathbb{C} \left| \sum_{j=1}^{N+1} b_{ij} \xi_j = 0 \right. \right\}, \quad 1 \leq i \leq N-r,
\]

and \( \Phi_d(V) \cong \mathbb{P}^N_\mathbb{C} \subset \mathbb{P}^N_\mathbb{C} \). Now if

\[
\mathcal{L}_j := \{\xi_j = 0\} \cap \Phi_d(V) \subset \mathbb{P}^r_\mathbb{C}, \quad 1 \leq j \leq N+1,
\]

where \( \mathcal{L}_j \) is the pull-back to \( \mathbb{P}^r_\mathbb{C} \) of the \( j \)-dimensional linear system of curves on \( \mathbb{P}^N_\mathbb{C} \) obtained by \( \Phi_d \), it follows that

\[
\mathcal{L}_j = \{\xi_j = 0\} \cap \mathbb{P}^r_\mathbb{C}.
\]
denote by \( \mathcal{M}(\mathbb{P}_C^N) = \mathbb{C}(z_2/z_1, \ldots, z_{N+1}/z_1) \) the rational function field of \( \mathbb{P}_C^N \), and let

\[
\mathcal{M}(\mathbb{P}_C^N) \left( \sqrt[\psi_2]{\psi_1}, \ldots, \sqrt[\psi_{N+1}]{\psi_1} \right)
\]

be the Kummer extension of \( \mathcal{M}(\mathbb{P}_C^N) \) determined by adjoining \( \text{"d"-th roots of ratios} \), where \( \psi_j \) is the linear form defining the hyperplane \( L_j \). This is an abelian extension with Galois group \((\mathbb{Z}/d\mathbb{Z})^N \). The variety \( V \) can be thought of as the normalization of \( \mathbb{P}_C^N \) w.r.t. this field, as being the total space of the \( d^N \)-sheeted covering

\[
\Phi_d|V : V \longrightarrow \mathbb{P}_C^r
\]

of \( \mathbb{P}_C^r \), branched along the \( L_j \)'s. The hyperplane arrangement

\[
\mathcal{L} := \bigcup_{j=1}^{N+1} L_j = \left\{ \prod_{j=1}^{N+1} \psi_j = 0 \right\} \subset \mathbb{P}_C^r
\]

admits a natural stratification

\[
\mathcal{L} = \mathcal{L}^{(1)} \supset \mathcal{L}^{(2)} = \bigcup_{1 \leq j_1 < j_2 \leq N+1} L_{j_1,j_2} \supset \cdots \supset \mathcal{L}^{(r)} = \bigcup_{1 \leq j_1 < \cdots < j_r \leq N+1} L_{j_1,j_2,\ldots,j_r}
\]

where

\[
L_{j_1,j_2,\ldots,j_k} := L_{j_1} \cap L_{j_2} \cap \cdots \cap L_{j_k} \cong \mathbb{P}_C^{r-k} \subset \mathbb{P}_C^r, \quad 1 \leq k \leq r.
\]

(\( \mathcal{L}^{(r)} \) consists of the points of \( \mathcal{L} \), \( \mathcal{L}^{(r-1)} \) consists of the lines of \( \mathcal{L} \), etc). Let us now define

\[
t_i := t_i(0) := \# \left\{ \text{elements of } \mathcal{L}^{(r)} \text{ (i.e., points of } \mathcal{L} \text{) contained in exactly } i \text{ hyperplanes of } \mathcal{L} \right\}
\]

and, in general,

\[
t_i(\kappa) := \# \left\{ \text{elements of } \mathcal{L}^{(r-\kappa)} \text{ contained in exactly } i \text{ hyperplanes of } \mathcal{L} \right\}, \quad 0 \leq \kappa \leq r.
\]

\( \mathcal{L} \) is called a point arrangement if

\[
t_i(\kappa) = 0, \quad \text{for all } i > r - \kappa \quad \text{and for all } \kappa \in \{1, \ldots, r-2\}.
\]

The \( V \)'s defined by means of point arrangements have at most isolated singularities; more precisely, by analogy with the two-dimensional case (cf. \( \mathbb{R} \)), \( V \) inherits exactly \( d^{N-1} \) isolated singularities over each point of \( \mathcal{L} \) contained in \( i \geq r+1 \) hyperplanes. In particular, for point arrangements \( \mathcal{L} \) within \( \mathbb{P}_C^r \), for which

\[
t_i = 0, \quad \forall i, \quad i \geq r + 2,
\]

all singularities of \( V \) have to be \( \mathbb{A}^{(r)}_{d-1,d} \)-singularities. In this case, formula (4.3) reads as follows:

\[
e_{\text{str}}(V) = \sum_{\nu=0}^{r} (-1)^\nu \binom{N+1}{r-\nu} \binom{N-r+\nu-1}{\nu} d^{\nu+N-r} +
\]

\[
+ t_{r+1} \cdot d^{N-r-1} \left( \frac{1}{r-d+1} \left[ \frac{1}{d}((1-d)^{r+1} - 1) + r + 1 \right] \right).
\]

(4.3)
For $r = 3$, several combinatorial properties of hyperplane arrangements in $\mathbb{P}^3$, as well as properties of birational geometry of the resulting coverings, have been studied by Hunt [22]. As far as point arrangements are concerned (with $t_4 \geq 1$, $t_5 = t_6 = 0$) there are some interesting and aesthetically pleasing examples, given by the facet planes of certain regular (platonic) and semiregular (archimedean) solids (see Fig. 4). For these point arrangements, formula (4.3) gives:

\begin{align*}
\text{Solids} & \quad N & \quad t_3 & \quad t_4 = \frac{1}{4}[(N+1) - t_3] & \quad e_{\text{str}}(V) \quad \text{for } d = 2 & \quad e_{\text{str}}(V) \quad \text{for } d = 3 \\
A & 5 & 8 & 3 & 12 & 72 \\
B & 7 & 8 & 12 & 64 & -324 \\
C & 7 & 32 & 6 & -32 & -4212 \\
D, E & 13 & 256 & 27 & -111,616 & -68,496,840 \\
F & 13 & 208 & 39 & -99,328 & -62,828,136
\end{align*}

Examples A (with $d = 3$) and B (with $d = 2$) were first mentioned by Hirzebruch [21, pp. 764–765], who used them to construct 3-dimensional Calabi-Yau manifolds $\tilde{V}$ with Euler number 72 (resp., 64) by a “big” (resp. “small”, projective) crepant resolution of the 9 (resp., 96) singularities of $V$ (cf. the remarks in [28, p. 219]).

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References

[1] Batyrev V.V.: Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In: “Integrable Systems and Algebraic Geometry”, Proceedings of the Taniguchi Symposium 1997, Eds. M.-H. Saito, Y. Shimizu & K. Ueno; World Scientific, (1998), pp. 1-32.

[2] Batyrev V.V.: Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, Journal of the European Math. Soc. 1, (1999), 5-33.

[3] Batyrev V.V., Borisov L.A.: Mirror duality and string-theoretic Hodge numbers, Inventiones Math. 126, (1996), 183-203.

[4] Batyrev V.V., Cox D.: On the Hodge structure of projective hypersurfaces in toric varieties, Duke Math. J. 75, (1994), 293-338.

[5] Batyrev V.V., Dais D.I.: Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry, Topology 35, (1996), 901-929.

[6] Chen B.-Y., Ogiue K.: Some implications of the Euler-Poincaré characteristic for complete intersection manifolds, Proc. of the A.M.S. 44, (1974), 1-8.

[7] Dais D.I., Roczen M.: On the string-theoretic Euler number of 3-dimensional A-D-E singularities, preprint. [math.AG/0011117]

[8] Deligne P.: Théorie de Hodge III, Publ. Math. I.H.E.S. 44, (1975), 5-77.

[9] Denef J., Loeser F.: Motivic integration, quotient singularities and the McKay correspondence, preprint. [math.AG/9903187]

[10] Dimca A.: Topics on Real and Complex Singularities, Vieweg Advanced Lectures in Math., (1987).

[11] Dimca A.: Singularities and Topology of Hypersurfaces, Universitext, Springer-Verlag, (1992).

[12] Dolgachev I.: Weighted projective varieties. In: “Group actions and vector fields”, Proc. of Polish-North. Amer. Seminar, Vancouver 1981, Lecture Notes in Math., Vol. 956, (1982), Springer-Verlag, pp. 34-71.

[13] Goryunov V.V.: Symmetric quartics with many nodes. In: “Singularities and Bifurcations”, Advances in Soviet Math., Vol. 21, AMS, (1994), pp. 147-161.

[14] Griffiths P.: On the periods of certain rational integrals I, II, Ann. of Math. 90, (1969), 460-495 and 498-541.

[15] Griffiths P., Harris J.: Principles of Algebraic Geometry, J. Wiley & Sons, (1978).

[16] Hamm H.: Zum Homotopietyp Steinscher Räume, Jurnal für die reine und angew. Math. 338, (1983), 121-135.

[17] Hamm H.: Hodge numbers for isolated singularities of non-degenerate complete intersections. In: “Singularities (The Brieskorn Anniversary Volume)”, Eds. V. I. Arnold, G.-M. Greuel, J. H.M. Steenbrink, Progress in Math., Vol. 162, (1998), Birkhäuser, pp. 37-60.

[18] Hironaka H.: Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. 79, (1964), 109-326.
[19] Hirzebruch F.: Der Satz von Riemann-Roch in faisceau-theoretischer Formulierung. In: Proc. International Congress of Math. 1954, Vol. III, pp. 457-473. (See also: “Gesammelte Abhandlungen”, Bd. I, Springer-Verlag, 1987, pp. 128-144.)

[20] Hirzebruch F.: Arrangements of lines and algebraic surfaces. In “Arithmetic and Geometry II”, Progress in Math., Vol. 36, Birkhäuser, (1983), pp. 113-140. (See also: “Gesammelte Abhandlungen”, Bd. II, Springer-Verlag, (1987), pp. 679-706).

[21] Hirzebruch F.: Some examples of threefolds with trivial canonical bundle, “Gesammelte Abhandlungen”, Bd. II, Springer-Verlag, (1987), pp. 757-770.

[22] Hunt B.: Coverings and ball quotients with special emphasis on the 3-dimensional case, Bonner Mathematische Schriften, Bd. 174, (1986).

[23] Knörrer H.: Isolierte Singularitäten von Durchschnitten zweier Quadriken, Bonner Mathematische Schriften, Bd. 117, (1980).

[24] Milnor J.: Singular Points of Complex Hypersurfaces, Annals of Math. Studies, Vol. 61, Princeton University Press, (1968).

[25] Reid M.: Canonical threefolds. Journée de Géométrie Algébrique d’Angers, A. Beauville ed., Sijthoff and Noordhoff, Alphen aan den Rijn, (1980), pp. 273-310.

[26] Steenbrink J.H.M.: Intersection form for quasi-homogeneous singularities, Compositio Math., Vol. 34, (1977), 211-223.

[27] Steenbrink J.H.M.: Mixed Hodge structures associated with isolated singularities. In: Proc. of Symposia in Pure Math., Vol. 40, Part II, AMS, (1983), pp. 513-536.

[28] Werner J.: New examples of threefolds with $c_1 = 0$, (with an appendix by B. van Geemen), Math. Z. 203, (1990), 211-225.