INVARIANT TUBULAR NEIGHBORHOODS IN INFINITE-DIMENSIONAL RIEMANNIAN GEOMETRY, WITH APPLICATIONS TO YANG–MILLS THEORY

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Abstract. We show that if $G$ is an arbitrary group acting isometrically on an a (possibly infinite dimensional) Riemannian manifold, then every $G$–invariant submanifold with locally trivial normal bundle admits a $G$–invariant total tubular neighborhood. These results apply, in particular, to the Morse strata of the Yang-Mills functional over a closed surface. The resulting neighborhoods play an important role in calculations of gauge-equivariant cohomology for moduli spaces of flat connections over non-orientable surfaces (Baird [2], Ho–Liu [13], Ho–Liu–Ramras [14]).

1. Introduction

Consider a (possibly infinite-dimensional) smooth manifold $X$ acted on by a group $G$, and a smooth, $G$–invariant submanifold $Y \subset X$, which need not be closed. The question we address is: does $Y$ have a $G$–invariant tubular neighborhood inside $X$? For finite-dimensional manifolds and compact groups, it is well-known that the answer is yes (see Bredon [4], for example). The proof for compact groups involves averaging over the group, and the problem appears to be much more difficult for non-compact groups. A number of authors have considered specific infinite-dimensional versions of this problem: Ebin [8] constructed invariant tubular neighborhoods for orbits of the diffeomorphism group acting on the space of Riemannian metrics on a manifold; Kondracki and Rogulski [15] handled the case of orbits of the gauge group acting on the space of connections on a principal bundle; and recently A. Teleman [23] gave an explicit construction of a gauge-invariant tubular neighborhood for the space of connections $A$ on a Euclidean bundle satisfying $\dim(\ker(d_A)) = 1$.

We will show that invariant tubular neighborhoods exist whenever $X$ is a Riemannian manifold, $G$ acts by isometries, and the orthogonal complement of $TY$ in $TX|_Y$ is a smooth subbundle (Proposition 2.4 and Theorem 2.6). This last condition is automatically satisfied in the finite dimensional setting, but it is important to note that it does not come for free in the infinite-dimensional setting. Our motivation for this work comes from Yang–Mills theory, and we will show that these conditions are all satisfied in the case of the Morse strata of the Yang-Mills functional over a (possibly non-orientable) surface.

Mathematical applications of Yang–Mills theory began with Atiyah and Bott [1], who established a recursive formula for the gauge-equivariant cohomology of
the space of semi-stable holomorphic structures on a Hermitian bundle $E$ over a Riemann surface. They achieved this by treating the Yang–Mills functional as a gauge-equivariant Morse function on the infinite-dimensional space $\mathcal{A}(E)$ of connections on $E$. In ordinary Morse theory on a manifold $M$, each critical point corresponds to a cell in $M$, whose dimension is the index of that critical point. The zero-dimensional (co)homology of the critical point contributes to the (co)homology of $M$, with a shift in dimension corresponding to the index. Atiyah and Bott showed that the Morse strata of the Yang–Mills functional can be ordered, starting with the central (or semi-stable) stratum, so that each initial segment of the ordering forms an open set in $\mathcal{A}(E)$ (see Ramras [20] for a detailed discussion). Adding the strata $C_1, C_2, \ldots$ in order, the Thom isomorphism theorem shows that each stratum contributes its cohomology with a shift in dimension, given by the codimension of the stratum: letting $C_n = \bigcup_{i=1}^n C_i$, we have a long exact sequence

$$
\cdots \to H^*_{\mathcal{G}(E)}(C_n, C_{n-1}) \cong H^{*-\text{codim}(C_n)}_{\mathcal{G}(E)}(C_n) \to H^*_{\mathcal{G}(E)}(C_n) \to H^*_{\mathcal{G}(E)}(C_{n-1}) \to \cdots
$$

As discussed in Section 3, the $\mathcal{G}(E)$–invariant tubular neighborhoods constructed in this paper provide one method for establishing these Thom isomorphisms. (Atiyah and Bott proved that the Yang–Mills functional is “equivariantly perfect,” meaning that the boundary maps in the sequence (1) are all zero. This leads to the recursive formula mentioned above.)

Ho and Liu [12] extended Yang–Mills theory to bundles $E$ over non-orientable surfaces, where the Yang–Mills stratification is no longer “equivariantly perfect.” However, the Thom isomorphisms (1), together with the orientability result of Ho, Liu, and Ramras [14], establish equivariant Morse inequalities for the space of flat connections on $E$ (see the introduction to [14]). Moreover, these Thom isomorphisms have been used by Tom Baird [2] to produce formulas for the $U(3)$–equivariant Poincaré series of $\text{Hom}(\pi_1 \Sigma, U(3))$ (where $\Sigma$ is a non-orientable surface), establishing the “anti-perfection” conjecture of Ho and Liu [13].

We note that over a Riemann surface, an alternate approach to the Thom isomorphism in (1) would be to use the fact that after modding out the (based, complex) gauge group, one obtains a smooth algebraic variety, and results of Shatz [21] show that the image of each stratum is a smooth subvariety. This algebraic approach involves various technicalities, which we will not attempt to resolve here.

This paper is organized as follows. In Section 2 we present our construction of invariant tubular neighborhoods, and in Section 3 we use results of Daskalopoulos to show that our construction applies to the Morse strata of the Yang–Mills functional over a surface. In the last section, we compare our construction of tubular neighborhoods to other constructions in the literature.

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2. Invariant Tubular Neighborhoods

For basic definitions and terminology regarding infinite dimensional manifolds, as well as for many of the necessary constructions in this section, we follow Lang [16]. By smooth we will always mean $C^\infty$.

Throughout this section, $X$ will denote a (possibly infinite-dimensional) Riemannian manifold, and $G$ will denote a group acting on $X$ by smooth isometries. In more detail, this means $X$ is modeled on a Hilbert space $H$ (real or complex), and the tangent bundle $T(X)$ has a smoothly varying fiber-wise inner product (which we assume to be Hermitian in the complex case). We will assume these inner products induce the correct topology on $T_x(X)$, or in other words, we assume that the Riemannian metric on $X$ is strong. We will view the action of $G$ as a homomorphism $G \rightarrow \text{Isom}(X)$, where $\text{Isom}(X)$ denotes the group of smooth isometries. For the purposes of our arguments, both $G$ and $\text{Isom}(X)$ can be viewed as discrete groups.

Let $Y \subset X$ be a (locally closed), $G$–invariant submanifold; then the tangent space $T_Y(Y)$ is a closed subspace of the Hilbert space $T_Y(X)$ and hence has an orthogonal complement $N_Y(Y)$. The normal bundle $N(Y) \subset T(X)|_Y$ is then a well-defined subset of $T(X)|_Y$, and since $G$ preserves the metric, it acts on $N(Y)$. However, $N(Y)$ need not be a smooth subbundle in general. For the construction of tubular neighborhoods given by Lang [16, IV.5], this is irrelevant: Lang obtains a tubular neighborhood diffeomorphic to the quotient bundle $T(X)|_Y/T(Y)$ by using a splitting of the projection $T(X)|_Y/T(Y)$, which always exists if $X$ admits partitions of unity [16, III, Proposition 5.2].

In this section, we show that if $N(Y)$ is a smooth subbundle of $T(X)|_Y$, then $Y$ admits a $G$–invariant total tubular neighborhood $\tau(Y)$ (Theorem 2.40). The neighborhoods $\tau(Y)$ are homeomorphic to the normal bundle $N(Y)$, but may not be diffeomorphic to $N(Y)$. Since we are mainly interested in Thom isomorphisms, a homeomorphism is sufficient. (If $G = \{1\}$, or if $Y$ consists of a single orbit of $G$, then we do obtain a diffeomorphism $\tau(Y) \cong N(Y)$.)

Remark 2.1. The condition that $N(Y) \subset T(X)|_Y$ is a smooth subbundle could be replaced by the condition that there exists a splitting $s$ of the map $\pi$ in the exact sequence

$$0 \rightarrow T(Y) \rightarrow T(X)|_Y \xrightarrow{s} T(X)|_Y/T(Y) \rightarrow 0$$

whose image in $T(X)|_Y$ is invariant under $G$; our proof goes through equally well in this case, with $\text{Im}(s)$ playing the role of $N(Y)$.

2.1. The tangent bundle as a Riemannian manifold. Associated to the metric on $X$ we have the metric spray $F : TX \rightarrow T(TX)$, which is equivariant with respect to the natural actions of the isometry group $\text{Isom}(X)$. The spray $F$ induces an exponential map $f : D \rightarrow X$ (Lang [16, VII.7]), whose domain $D$ is an open subset of $T(X)$ containing the zero section. It follows from the construction of $f$ that $D$ is invariant under $\text{Isom}(X)$ and $f$ is $\text{Isom}(X)$–equivariant.

In addition to the exponential map $f$, we will also need a $\text{Isom}(X)$–invariant Riemannian metric on the tangent bundle $TX$, considered as a smooth manifold in its own right. This Riemannian metric on $T(TX)$ can be defined by explicit local formulae as in Do Carmo [7, Chapter 3, Exercise 2]. Here we give a more global perspective, from which it is clear that the action of $\text{Isom}(X)$ on $T(TX)$ preserves this metric.
Recall that there is a natural exact sequence of vector bundles over $TX$ of the form
\[
0 \to \pi^*_{TX}(TX) \to T(TX) \xrightarrow{\alpha} (\pi TX)^*TX \to 0,
\]
where $\pi_{TX}: TX \to X$ is the structure map for the tangent bundle $TX$, and $\alpha$ is defined via the commutative diagram
\[
\begin{array}{ccc}
T(TX) & \xrightarrow{\alpha} & TX \\
\downarrow{\pi_{T(TX)}} & & \downarrow{\pi_{TX}} \\
(TX) & \xrightarrow{\pi_{TX}} & X.
\end{array}
\]

The map $\alpha$ is surjective, because in a local chart $U \subset X$, this diagram becomes (see Lang [16, p. 96])
\[
(U \times H) \times (H \times H) \xrightarrow{(\pi_1,\pi_3)} U \times H \xrightarrow{(\pi_1,\pi_2)} U \times (H \times H) \xrightarrow{(\pi_1,\pi_2,\pi_3)} U \times H \xrightarrow{\pi_1} U.
\]

To identify the kernel of $\alpha$, consider the map
\[
\pi^*_{TX}(TX) = TX \oplus TX \xrightarrow{\phi} T(TX)
\]
sending a pair of vectors $(v, w) \in T_x X \oplus T_x X$ to the vector in $T_v(T_x X)$ represented by the curve $t \mapsto v + tw$ in $T_x X \subset TX$. This defines a map of vector bundles over $TX$, and from the local representation given above, one sees that $\phi$ is a smooth, injective bundle map whose image is precisely $\ker(\alpha)$.

By naturality, the maps in the exact sequence (2) are Isom($X$)-equivariant. The next step will be to construct an Isom($X$)-equivariant splitting of $\alpha$.

Locally, the spray $F$ has the form
\[
(x, v) \mapsto (x, v, v, f_x(v))
\]
where $f_x: T_x(X) \to H$ is homogeneous of degree two, meaning that $f_x(tv) = t^2f_x(tv)$ for any $v \in T_x(X)$ and any scalar $t$ [16, IV, Proposition 3.2]. We now recall a well-known lemma (see, for example, [16, I.3]), whose proof we include for completeness.

**Lemma 2.2.** If $q: E \to F$ is a smooth map between Banach spaces and $q$ is homogeneous of degree $p > 0$ (meaning that $q(tx) = t^p q(x)$ for all $x \in E$ and all scalars $t$), then $q(x) = \frac{1}{p!} (D^p q(0))(x, x, \ldots, x)$, where $D^p q$ is the $p^{th}$ derivative of $q$, considered as a map from $E$ in the space of multilinear functions $E^p \to F$.

**Proof.** The proof is by induction on $p$. Differentiating the equation $q(tx) = t^p q(x)$ with respect to $t$ gives
\[
(Dq(tx)) (x) = pt^{p-1}q(x).
\]
Note that when $p = 1$, the result follows by setting $t = 0$, completing the base case.

Setting $t = 1$ in (3), we have

$$q(x) = \frac{1}{p} (Dq(x)) (x).$$

Now, $Dq: E \to L(E, F)$, is homogeneous of degree $p - 1$: indeed, for any vector $v \in E$ we can differentiate the equation

$$q(t(x + rv)) = t^p q(x + rv)$$

with respect to $r$, obtaining

$$(Dq(t(x + rv))) (tv) = t^p (Dq(x + rv)) (v);$$

setting $r = 0$ and using linearity of $Dq(tx)$, one finds that

$$s(x, v, w) = (x, v, w, B(x, v, w)).$$

Since the spray $F$ is Isom($X$)--equivariant and (by polarization) $B(x, -, -)$ is the unique symmetric bilinear form satisfying $B(x, v, v) = f_x(v)$, it follows that $s$ is itself Isom($X$)--equivariant. Note also that $s$ is a map of vector bundles over $TX$.

The splitting $s$ gives us an explicit, Isom($X$)--equivariant direct sum decomposition of vector bundles over $TX$:

$$T_TX \cong \pi_{TX}^*TX \oplus \pi_{TX}^*(TX).$$

The Riemannian metric on $X$ pulls back to an inner product on $\pi_{TX}^*(TX)$ (which is again Isom($X$)--invariant) and now the direct sum decomposition endows $T_TX$ with an Isom($X$)--invariant inner product as well, making $TX$ into a Riemannian manifold with an isometric action of the group Isom($X$).

We note that there is another description of this splitting $s$ in terms of the Levi-Civita covariant derivative $D$ [16, VIII, Theorem 4.2]: given $(v, w) \in T_x(X) \oplus T_x(X)$ with $w = \gamma(0)$ for some curve $\gamma: (-\epsilon, \epsilon) \to X$, there is a unique lift of $\gamma$ to a curve $\gamma_v: (-\epsilon, \epsilon) \to TX$ such that $\gamma(0) = v$ and $\gamma_v$ is parallel to $\gamma$ with respect to $D$. The splitting $s$ now sends $(v, w)$ to $(\gamma_v)'(0) \in T_v(TX)$. This splitting is the same as the one given above, as can be seen by examining the definition of $D$--parallel curves.
2.2. Construction of the tubular neighborhood.

**Definition 2.3.** A tubular neighborhood of $Y$ in $X$ is an open neighborhood $\tau(Y)$ of $Y$ in $X$ together with a homeomorphism $\phi: U \to \tau(Y)$ from an open neighborhood $U \subset N(Y)$ containing the zero-section $Y \subset N(Y)$, such that $\phi$ restricts to the identity from $Y \subset N(Y)$ to $Y \subset X$. If $\phi$ is a diffeomorphism, we call $\tau(Y)$ smooth, and if $U = N(X)$, we call $\tau$ total.

**Proposition 2.4.** Let $X$ be a Riemannian manifold and let $Y \subset X$ be a (locally closed) submanifold whose normal bundle $N(Y) \subset T(X)|_Y$ is a smooth subbundle. Assume $G$ acts on $X$ by smooth isometries, leaving $Y$ invariant. Then there exists a $G$–invariant open neighborhood $Z$ of the zero section of $N(Y)$ with $Z \subset D \cap N(Y)$, and the exponential map restricts to an equivariant diffeomorphism $f: Z \to Y$, where $V$ is an open neighborhood of $Y$ in $X$. Hence $V$ is a $G$–invariant tubular neighborhood of $Y$.

**Proof.** We identify $Y$ with the zero section of $N(Y)$, and for $g \in G$ we will let $g$ denote both the self-map it induces on $X$ and the derivative of this map.

We begin by noting that the geodesic distance $d$ associated to the natural Isom–invariant Riemannian metric on $TX$ constructed above yields an $\text{Isom}(X)$–invariant distance function (topological metric) on $TX$ (see [16] VII.6 for a discussion of the geodesic distance, and the fact that the associated metric topology is the usual topology on $TX$). Since $G$ acts by isometries, the distance function $d$ is $G$–invariant.

For any $W \subset T(X)$ and any $w \in W$, we write $B_\epsilon(w, W) = \{w' \in W | d(w, w') < \epsilon\}$. Note that we identify $X$ with the zero section of $T(X)$, so $B_\epsilon(x, X)$ is defined.

Now fix $y \in Y$. Since the exponential map $f$ restricts to a local diffeomorphism $D \cap N(Y) \to X$ [16] p. 109, we know that for some $\epsilon_y > 0$, $f$ restricts to a diffeomorphism $B_{\epsilon_y}(y, N(Y)) \to f(B_{\epsilon_y}(y, N(Y)))$ (with $f(B_{\epsilon_y}(y, N(Y)))$ open in $X$). Let $\psi: f(B_{\epsilon_y}(y, N(Y))) \to B_{\epsilon_y}(y, N(Y))$ denote the inverse map. Now, $f(B_{\epsilon_y/2}(y, N(Y)))$ is an open neighborhood of $y$ in $X$, hence contains $B_{\epsilon_y}(y, X)$ for some $\epsilon_y' < \epsilon_y/4$. Set $Z_y = \psi(B_{\epsilon_y'}(y, X)) \subset N(Y)$. Then $Z_y$ is open in $N(Y)$ and we have

\[
Z_y \subset B_{\epsilon_y/2}(y, N(Y))
\]

and $f(Z_y) = B_{\epsilon_y}(y, X)$. In fact, for any $g \in G$,

\[
f(g(Z_y)) = B_{\epsilon_y}(g \cdot y, X).
\]

Define $Z_{y(O)}$ as above for one point $y(O)$ from each $G$–orbit $O \subset Y$. Then we claim that $f$ is injective, and hence restricts to a diffeomorphism, on the open set

\[
Z = \bigcup_{O \in Y/G, g \in G} g(Z_{y(O)}).
\]

Note here that in order to conclude that $f|_Z$ is a diffeomorphism onto its image, we are using the fact that $f(B_{\epsilon_y}(y, N(Y)))$ is open in $X$; this also shows that $f(Z)$ is open in $X$.

Say $x = f(g_1 \cdot z_1) = f(g_2 \cdot z_2)$ with $z_1 \in Z_{y_1}$, $z_2 \in Z_{y_2}$ and $g_1, g_2 \in G$ (here the $y_i$ are the chosen representatives for some two orbits). Then by [16] we have

\[
d(g_1 \cdot y_1, g_2 \cdot y_2) \leq d(g_1 \cdot y_1, x) + d(x, g_2 \cdot y_2) < \epsilon_y' + \epsilon_y' \leq \epsilon_y/4 + \epsilon_{y_2}/4.
\]
We may assume that \( \epsilon_{y_1} \geq \epsilon_{y_2} \). Then by (3) and (7) we have
\[
d(g_1 \cdot y_1, g_2 \cdot z_2) \leq d(g_1 \cdot y_1, g_2 \cdot y_2) + d(g_2 \cdot y_2, g_2 \cdot z_2)
\]
(8)
\[
< (\epsilon_{y_1}/4 + \epsilon_{y_2}/4) + \epsilon_{y_2}/2 \leq \epsilon_{y_1}.
\]
The fact that \( f \) is injective on \( B_{\epsilon_{y_1}}(y_1, N(Y)) \) implies that it is also injective on \( B_{\epsilon_{y_1}}(g_1 \cdot y_1, N(Y)) \). But \( g_1 \cdot z_1 \in B_{\epsilon_{y_1}}(g_1 \cdot y_1, N(Y)) \) by (3), and \( g_2 \cdot z_2 \in B_{\epsilon_{y_1}}(g_1 \cdot y_1, N(Y)) \) by (8), so we have a contradiction. \( \square \)

Before constructing a \( G \)-invariant total tubular neighborhood, we need a lemma which follows from the argument in Lang [16, VII, Proposition 4.1].

**Lemma 2.5.** Let \( X, Y, Z, \) and \( G \) be as in Proposition 2.4 and let \( \sigma: Y \to \mathbb{R} \) be a continuous \( G \)-invariant function such that for all \( y \in Y \), \( \sigma(y) > 0 \) and \( \{ v \in N_y(Y) \mid |v| < \sigma(y) \} \) lies inside \( Z \).

Then there is a homeomorphism \( N(Y) \to N(Y)_\sigma \), where
\[
N(Y)_\sigma = \{ v \in N(Y) \mid |v| < \sigma(\pi v) \}
\]
(here \( \pi: N(Y) \to Y \) is the projection, and \( |\cdot| \) is the norm on \( T_x(X) \)). This homeomorphism restricts to the identity on \( Y \subseteq N(Y) \).

We can now prove our main theorem.

**Theorem 2.6.** Let \( X \) be a Riemannian manifold, and let \( G \) act on \( X \) by isometries, leaving the submanifold \( Y \subseteq X \) invariant. Assume that \( N(X) \) is a smooth subbundle of \( T(X)|_Y \). Then there exists a \( G \)-invariant total tubular neighborhood \( \tau(Y) \) of \( Y \) in \( X \). The homeomorphism \( N(Y) \cong \tau(Y) \) is \( G \)-equivariant, but need not be smooth.

Proof. We apply Lemma 2.5. Set \( \sigma(y) = \sup \{ \epsilon > 0 : |v| < \epsilon \implies v \in Z \} \). Then \( \sigma(y) > 0 \) for any \( y \in Y \), \( \sigma \) is \( G \)-invariant, and we have \( N(Y)_\sigma \subseteq Z \). Continuity of \( \sigma \) follows from the assumption that the inner products defining \( |\cdot| \) vary continuously over \( X \), together with the fact that \( Z \) is open. \( \square \)

**Remark 2.7.** If \( G = \{ 1 \} \) and \( X \) admits partitions of unity, then Lang [15, VII, 4, Corollary 4.2] shows that one may choose \( \sigma \) to be smooth, and then the homeomorphism \( N(Y) \to \tau(Y) \) will be smooth as well. If \( Y \) consists of a single \( G \)-orbit, then \( \sigma \) can be chosen to be constant, and we obtain the same conclusion.

3. Applications to Yang–Mills Theory

We now apply Theorem 2.6 to the Morse strata of the Yang–Mills functional. Let \( P \to M \) be a principal \( U(n) \)-bundle over a surface, and let \( A^{k-1}(P) \) denote the affine space of connections on \( P \) lying in the Sobolev space \( L^2_{k-1} \) \( (k \geq 21) \). Similarly, let \( G^k(P) \) denote the \( L^2_{k-1} \) Sobolev completion of the unitary gauge group of \( P \), which acts smoothly on \( A^{k-1}(P) \) (Wehrheim [24]). Connections on \( P \) correspond precisely to Hermitian connections on the associated Hermitian bundle \( P \times_{U(n)} \mathbb{C}^n \), and \( G^k(P) \) is isomorphic to the Sobolev gauge group \( G^k(E) \). The following lemma will allow us to apply Theorem 2.6. A proof of this lemma may be found in Kondracki–Rogulski [15, Section 2.3].

**Lemma 3.1.** Let \( K \) be a compact connected Lie group and let \( P \) be a principal \( K \)-bundle over a Riemann surface \( M \) equipped with a Riemannian metric. The metric on \( M \) and a \( K \)-invariant inner product on the Lie algebra \( \mathfrak{t} \) of \( K \) induce
a Riemannian metric on $\mathcal{A}^{k-1}(P)$, the space of $L^2_{k-1}$ $K$–connections on $P$. This Riemannian metric is $\mathcal{G}^k(P)$–invariant.

As discussed in the introduction, the Morse strata of the Yang–Mills functional can be ordered $C_1 \leq C_2 \leq \cdots$, starting with the central stratum, in such a way that the union $C_n = \bigcup_{i=1}^n C_i$ of each initial segment of the ordering is open. Such an ordering was first discussed by Atiyah and Bott [1], and is described in more detail in Ramras [20]. We can now give our main application of Theorem 2.6.

**Corollary 3.2.** Let $P \to M$ be a smooth principal $U(n)$–bundle over a surface $M$, and let $\leq$ denote a linear order on the set of Morse strata as above. Then each stratum $S$ has a $\mathcal{G}(E)$–invariant tubular neighborhood lying inside the open set $C_n$, and there are Thom isomorphisms in gauge equivariant cohomology

$$H^*_\mathcal{G}(E)(C_n, C_{n-1}) \cong H^{*-\text{codim}(C_n)}_{\mathcal{G}(E)}(C_n).$$

If $M$ is orientable, or a non-orientable surface whose double cover has genus greater than 1, these isomorphisms hold with integer coefficients; in general they hold with $\mathbb{Z}/2$–coefficients.

**Proof.** Since $\mathcal{A}^{k-1}(P)$ is an affine space modeled on the vector space of $L^2_{k-1}$ sections of $\text{ad}P \otimes T^*M$, this space is a Hilbert manifold, as are the open subsets $\bigcup_{i \leq j} C_i$. When $M$ is orientable, the Morse strata for the Yang–Mills functional on $P$ are locally closed submanifolds of $\mathcal{A}^{k-1}(P)$ (Daskalopoulos [20]) and are invariant under $\mathcal{G}^k(P)$. So existence of the desired neighborhoods will follow by applying Theorem 2.6 and Lemma 3.1 to the submanifolds $C_j \subset \bigcup_{i \leq j} C_i$, once we show that the normal bundle $N(C_j) \subset T(\mathcal{A}^{k-1}(P))|_{C_j}$ is a smooth subbundle. Since the kernel of a smooth, surjective map of vector bundles is a smooth subbundle of the domain (see, for example, Lang [16, III.3]), it will suffice to show that the orthogonal projections $T_A(\mathcal{A}^{k-1}(P)) \to T_A(C_j)$ vary smoothly with $A \in \mathcal{C}_j$.

To understand these orthogonal projections, we follow Daskalopoulos’ proof that the strata are locally closed submanifolds [23 Proposition 3.5]. Recall [1] Section 5) that connections on $P$ correspond bijectively with holomorphic structures on the associated vector bundle $\text{Ad}P = P \times_{U(n)} \mathbb{C}^n$, where we view a holomorphic structure as an operator on sections $d' : L^2_{k-1}(\text{Ad}P) \to L^2_{k-1}(T^*M'' \otimes \text{Ad}P)$.

Here $T^*M''$ is the space of $(0, 1)$–forms on the complex manifold $M$. Sections in the kernel of $d'$ are then holomorphic. We will call $\text{Ad}P$, together with a $d''$ operator, a holomorphic bundle, which we denote by $E$. Note that by adjointness, $d''$ induces a corresponding operator on the endomorphism bundle $\text{End}(\text{Ad}P) = \text{End}E$.

Given a critical holomorphic bundle $E \in C_j$ let $S$ denote the orthogonal complement of the space of $L^2_{k-1}$ holomorphic endomorphisms of $E$ inside the space of all smooth endomorphisms (that is, $S$ is the kernel of the $d''$ operator associated to $\text{End}E$). Daskalopoulos constructs a map

$$f : S \times H^1(M, \text{End}E) \to \mathcal{A}^{k-1}(P)$$

and shows that $f$ is an isomorphism near zero, meaning that $f : U_0 \to U_E$ is an isomorphism for some neighborhood $U_0$ of 0 in the Hilbert space $S \times H^1(M, \text{End}E)$.
and some neighborhood $U_E$ of $E$. Daskalopoulos then shows that
\[ f^{-1}(U_E \cap C_j) = U_0 \cap (S \times H^1(M, \text{End'} E)), \]
so that $f$ provides a chart for $C_j$ near $E$. Here $\text{End'} E$ is the sheaf of holomorphic endomorphisms preserving the Harder-Narasimhan filtration on $E$. (The notation $\text{End'} E$ comes from Atiyah–Bott [1; Section 7]; Daskalopoulos uses the notation $UT(E, \ast) = \text{End'} E$, since these endomorphisms correspond, in a local ordered basis respecting the Harder-Narasimhan filtration, to upper triangular matrices.)

This chart provides an isomorphism between the tangent bundle to $U_E \cap C_j$ and the trivial bundle
\[ (U_0 \cap (S \times H^1(M, \text{End'} E))) \times (S \times H^1(M, \text{End'} E)). \]
We need to understand the orthogonal projections
\[ S \times H^1(M, \text{End} E) \longrightarrow S \times H^1(M, \text{End'} E) \]
with respect to the pullback, along $f$, of the metric on $\mathcal{A}^{k-1}(P)$ to the various tangent spaces $S \times H^1(M, \text{End} E)$ (although these tangent spaces are all isomorphic to one another, the pullback metric will vary depending on the behaviour of the map $f$). Observe that this orthogonal projection is simply the identity on $S$, plus an orthogonal projection $H^1(M, \text{End} E) \rightarrow H^1(M, \text{End'} E)$. These cohomology groups are finite dimensional, and since the metrics vary smoothly, so do the orthogonal projections.

Daskalopoulos produces the manifold structure at a general point $E \in C_j$ (as opposed to critical points) using a gauge transformation sending $E$ into a neighborhood of the critical set on which the manifold structure has already been established. So local triviality of $N(C_j)$ at arbitrary points of $E$ follows from the result near the critical set. This completes the proof of local triviality, in the case of an orientable surface.

When $M$ is non-orientable, we let $\widetilde{P} \rightarrow \widetilde{M}$ denote the pullback of $P$ to the orientable double cover $\widetilde{M} \rightarrow M$. There is a natural embedding $\mathcal{G}^k(P) \hookrightarrow \mathcal{G}^k(\widetilde{P})$, making the natural embedding $\mathcal{A}^{k-1}(P) \hookrightarrow \mathcal{A}^{k-1}(\widetilde{P})$ equivariant. In both cases, the image of the embedding is the set of fixed points of an involution $\tau$ arising from the deck transformation on $\widetilde{M}$ (see Ho–Liu [12]). The Morse strata in $\mathcal{A}^{k-1}(\widetilde{P})$ are, by definition, just the intersections of Morse strata in $\mathcal{A}^{k-1}(P)$ with $\mathcal{A}^{k-1}(\widetilde{P})$, and hence are locally closed submanifolds invariant under $\mathcal{G}^k(P) = \mathcal{G}^k(\widetilde{P})^\tau$. By Lemma 3.1, we obtain a gauge-invariant metric on $\mathcal{A}^{k-1}(\widetilde{P})$ from a choice of a metric on $\widetilde{M}$ and a $U(n)$–invariant inner product on $u(n)$. We will choose our metric on $\widetilde{M}$ to be invariant under the deck transformation, so that the resulting metric on $\mathcal{A}^{k-1}(\widetilde{P})$ is invariant under $\tau$. Now, the $\mathcal{G}^k(\widetilde{P})$–invariant metric on $\mathcal{A}^{k-1}(\widetilde{P})$ restricts to a $\mathcal{G}^k(P) = \mathcal{G}^k(\widetilde{P})^\tau$–invariant metric on $\mathcal{A}^{k-1}(\widetilde{P}) = \mathcal{A}^{k-1}(P)^\tau$, and it remains to check that the normal bundles are smooth subbundles $T(\mathcal{A}^{k-1}(P))$.

Given a Morse stratum $C \subset \mathcal{A}^{k-1}(P)$, it will suffice to show that the orthogonal projection $Q_C$: $T(\mathcal{A}^{k-1}(P)) \rightarrow TC$ is a smooth bundle map, since the kernel of this projection is precisely the normal bundle to $C$. By definition, $C = \widetilde{C} \cap \mathcal{A}^{k-1}(P)$ for some Morse stratum $C \subset \mathcal{A}^{k-1}(P)$. Let $Q_C$: $T(\mathcal{A}^{k-1}(P)) \rightarrow TC$ denote the orthogonal projection, which we have shown is a smooth bundle map. Since the involution $\tau$ is an isometry, it follows that $Q_C$ is simply the restriction of the $Q_C$ to $TC = (T\widetilde{C})^\tau \subset T\widetilde{C}$ (in general, one may check that if $W$ is a closed subspace of a
Hilbert space $H$ and $\tau: H \to H$ is a linear isometry, then the orthogonal projection $H^\perp \to W^\perp$ is the restriction of the orthogonal projection $H \to W$). Hence $Q_C$ is a smooth bundle map as well.

To produce the desired Thom isomorphisms, note that over a Riemann surface, the normal bundles to the Yang–Mills strata are naturally complex vector bundles (Atiyah–Bott [1, Section 7]). For any $G$–equivariant complex vector bundle $V \to X$, the homotopy orbit bundle $V_{hG} \to X_{hG}$ is still a complex vector bundle, hence orientable. Letting $\nu_n$ denote a $G$–invariant tubular neighborhood of $C_n$ inside $C_n$, we may excise the complement of $(\nu_n)_{hG(E)}$ in $(C_n)_{hG(E)}$ and apply the ordinary Thom isomorphism for the bundle $(\nu_n)_{hG(E)} \to (C_n)_{hG(E)}$.

The same argument works over a non-orientable surface, since it is proven in Ho–Liu–Ramras [14] that the normal bundles to the Yang–Mills strata (and the corresponding homotopy orbit bundles) are orientable (real) vector bundles (so long as the double cover of the surface has genus greater than 1). □

4. OTHER CONSTRUCTIONS OF TUBULAR NEIGHBORHOODS

4.1. General constructions. In order to put our construction in context, we briefly discuss existing general constructions of tubular neighborhoods for for submanifolds $Y \subset X$. There are many treatments of this topic in the literature, and the basic tool is always the exponential map, which is a local diffeomorphism from an open neighborhood $D \subset N(Y)$ containing the zero section $Y \subset N(Y)$ to an open neighborhood of $Y$ in $X$. Since the proof of Godement’s lemma [9] first appeared, most arguments [4, 5, 10, 11, 16, 17, 19] rely on Godement’s idea produce a smaller neighborhood $Z \subset D$ on which the exponential map is injective, hence a diffeomorphism (in [3, 18, 22], $Y$ is assumed to be compact, and a simplified version of Godement’s lemma is used). After this point, one argues that $N(Y)$ can be compressed into $Z$. In order to construct invariant neighborhoods for compact groups, Bredon uses integration over the group to make this compression $G$–equivariant. Surprisingly, we have not found the non-equivariant version of our argument, making use of a distance function on $T(X)$, elsewhere in the literature.

4.2. Relationship with the work of Kondracki–Rogulski, Ebin, and Teleman. Kondracki and Rogulski [15] provided a construction of gauge-invariant tubular neighborhoods of the gauge orbits inside the space $A^{k-1}(P)$ of (Sobolev) connections on a smooth principle $K$–bundle $P$ over a compact manifold $M$, where $K$ is a compact Lie group. Following Ebin [8], Kondracki and Rogulski utilize a weak gauge-invariant metric on the space of connections, and show that with respect to this weak metric, the orthogonal complement to $T(G^k(P) \cdot A)$ inside $T(A^{k-1}(P))$ is a smooth subbundle ([15, Lemma 3.3.1]). However, their argument also shows that the orthogonal complement with respect to the strong metric is a smooth subbundle. Hence Theorem 2.6 applies to this situation.

Ebin [8] considers the space of Riemannian metrics on a fixed finite-dimensional manifold $M$, equipped with the action of the diffeomorphism group of $M$. After passing to appropriate Sobolev completions of these objects, Ebin constructs diffeomorphism-invariant tubular neighborhoods for the orbits of the action. His arguments make use of a weak Riemannian metric on the space of metrics, and it is only with respect to this weak metric that Ebin proves the normal bundles to
orbits are smooth subbundles. Ebin’s arguments are somewhat subtler than those of Kondracki–Rogulski, and they do not directly apply to the strong Riemannian metric. Hence our results do not immediately apply in this situation. We note that Ebin uses his results in the Sobolev setting to derive results in the smooth setting.

A. Teleman [23] gives an explicit construction of a gauge-invariant tubular neighborhood for the space of connections $A$ on a Euclidean bundle satisfying the condition $\dim(\ker(d_A)) = 1$. Teleman’s methods are quite different from ours, and it is unclear whether our results apply to this situation.

References

[1] M. F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. Philos. Trans. Roy. Soc. London Ser. A, 308(1505):523–615, 1983.
[2] Thomas Baird. Antiperfection of Yang–Mills Morse theory over a nonorientable surface in rank three. [arXiv:0902.4581] 2009.
[3] Marcel Berger and Bernard Gostiaux. Differential geometry: manifolds, curves, and surfaces, volume 115 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1988. Translated from the French by Silvio Levy.
[4] Glen E. Bredon. Introduction to compact transformation groups. Academic Press, New York, 1972. Pure and Applied Mathematics, Vol. 46.
[5] Theodor Bröcker and Klaus Jänich. Introduction to differential topology. Cambridge University Press, Cambridge, 1982. Translated from the German by C. B. Thomas and M. J. Thomas.
[6] Georgios D. Daskalopoulos. The topology of the space of stable bundles on a compact Riemann surface. J. Differential Geom., 36(3):699–746, 1992.
[7] Manfredo Perdigão do Carmo. Riemannian geometry. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
[8] David G. Ebin. On the space of Riemannian metrics. Bull. Amer. Math. Soc., 74:1001–1003, 1968.
[9] Roger Godement. Topologie algébrique et théorie des faisceaux. Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13. Hermann, Paris, 1958.
[10] Victor Guillemin and Alan Pollack. Differential topology. Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.
[11] Morris W. Hirsch. Differential topology, volume 33 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
[12] Nan-Kuo Ho and Chiu-Chu Melissa Liu. Yang–Mills connections on nonorientable surfaces. Comm. Anal. Geom., 16(3):617–679, 2008.
[13] Nan-Kuo Ho and Chiu-Chu Melissa Liu. Anti-perfect Morse stratification. [arXiv:0808.3974] 2009.
[14] Nan-Kuo Ho, Chiu-Chu Melissa Liu, and Daniel A. Ramras. Orientability in Yang–Mills theory over nonorientable surfaces. Submitted for publication. [arXiv:0810.4882]
[15] Witold Kordacki and Jan Rogulski. On the stratification of the orbit space for the action of automorphisms on connections. Dissertationes Math. (Rozprawy Mat.), 250:67, 1986.
[16] Serge Lang. Differential and Riemannian manifolds, volume 160 of Graduate Texts in Mathematics. Springer-Verlag, New York, third edition, 1995.
[17] Juan Margalif Roig and Enrique Oterelo Domínguez. Differential topology, volume 173 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1992. With a preface by Peter W. Michor.
[18] John Milnor. Collected papers of John Milnor. III. American Mathematical Society, Providence, RI, 2007. Differential topology.
[19] Amiya Mukherjee. Topics in differential topology, volume 34 of Texts and Readings in Mathematics. Hindustan Book Agency, New Delhi, 2005.
[20] Daniel A. Ramras. On the Yang–Mills stratification for surfaces. Submitted. [arXiv:0805.2587] 2010.
[21] Stephen S. Shatz. The decomposition and specialization of algebraic families of vector bundles. Compositio Math., 55(2):163–187, 1977.
[22] Michael Spivak. *A comprehensive introduction to differential geometry. Vol. I.* Publish or Perish Inc., Wilmington, Del., second edition, 1979.

[23] Andrei Teleman. Harmonic sections in sphere bundles, normal neighborhoods of reduction loci, and instanton moduli spaces on definite 4-manifolds. *Geom. Topol.*, 11:1681–1730, 2007.

[24] Katrin Wehrheim. *Uhlenbeck compactness.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2004.

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INVARIANT TUBULAR NEIGHBORHOODS IN INFINITE-DIMENSIONAL RIEMANNIAN GEOMETRY, WITH APPLICATIONS TO YANG–MILLS THEORY

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Abstract. We present a new construction of tubular neighborhoods in (possibly infinite dimensional) Riemannian manifolds $M$, which allows us to show that if $G$ is an arbitrary group acting isometrically on $M$, then every $G$–invariant submanifold with locally trivial normal bundle has a $G$–invariant total tubular neighborhood. We apply this result to the Morse strata of the Yang-Mills functional over a closed surface. The resulting neighborhoods play an important role in calculations of gauge-equivariant cohomology for moduli spaces of flat connections over non-orientable surfaces (Baird [2], Ho–Liu [12], Ho–Liu–Ramras [14]).

1. Introduction

Consider a (possibly infinite-dimensional) smooth Banach manifold $X$ acted on by a group $G$, and a smooth, $G$–invariant submanifold $Y \subset X$ (which need not be closed). Does $Y$ have a $G$–invariant tubular neighborhood inside $X$? For finite-dimensional manifolds and compact groups, it is well-known that the answer is yes (see Bredon [4], for example). The proof involves averaging over the group, and the problem appears much more difficult for non-compact groups. Infinite-dimensional versions of this problem have been considered: Ebin [8] constructed invariant tubular neighborhoods for orbits of Diff($X$) on the space of Riemannian metrics on $X$; Kondracki and Rogulski [15] handled the case of orbits of the gauge group in the space of connections on a principal bundle; and recently A. Teleman [26] gave an explicit construction of a gauge-invariant tubular neighborhood for the space of connections $A$ on a Euclidean bundle satisfying $\dim(\ker(dA)) = 1$.

We show that invariant tubular neighborhoods exist whenever $X$ is Riemannian, $G$ acts by isometries, and the orthogonal complement of $TY$ in $TX|_Y$ is a smooth subbundle (Proposition 2.3 and Theorem 2.5). This last condition is automatically satisfied in the finite dimensional setting, but does not come for free in general. Our motivation comes from Yang–Mills theory, and we will show that these conditions are satisfied by the Morse strata of the Yang-Mills functional over a (possibly non-orientable) surface. The resulting neighborhoods play a role in cohomology computations for spaces of flat connections over non-orientable surfaces.

Mathematical applications of Yang–Mills theory began with Atiyah and Bott’s study [1] of the space $\mathcal{C}_{ss}(E)$ of semi-stable holomorphic structures on a Hermitian

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bundle over a Riemann surface. By treating the Yang–Mills $L$ functional as a gauge-equivariant Morse function on the space $C(E)$ of holomorphic structures, Atiyah and Bott established recursive formulae for the gauge-equivariant cohomology of the stable manifold $C_{ss}(E)$. In ordinary Morse theory, a critical point $c$ of index $n$ has stable manifold an $n$–cell, so the zero-dimensional cohomology of $c$ contributes to the cohomology of the manifold with a shift in dimension corresponding to the index. Atiyah and Bott showed that the Morse strata of $L$ can be ordered $C_{ss}(E) = C_1, C_2, \cdots \subset C(E)$ (see Ramras [22] for the case of non-orientable surfaces) such that when added in order, each $C_i$ contributes its cohomology with a shift in dimension: if $C_n = \cup_{i=1}^n C_i$, the Thom Isomorphism yields a long exact sequence in gauge–equivariant cohomology:

\begin{align}
\cdots \to H^*_G(C_n, C_{n-1}) \cong H^*_{G}^{\text{codim}(C_n)} (C_n) \to H^*_G(C_n) \to H^*_G(C_{n-1}) \to \cdots
\end{align}

The boundary maps in (1) are all zero [1, §7], which leads to the formulae for $H^*_G(C_{ss})$. The gauge–equivariant invariant tubular neighborhoods constructed here provide one method for establishing these Thom isomorphisms (Theorem 3.2).

Ho and Liu [13] extended Yang–Mills theory to bundles $E$ over non-orientable surfaces $\Sigma$. In this context, the boundary maps in (1) are no longer zero, but (1) together with the orientability result of Ho, Liu, and Ramras [14] does establish equivariant Morse inequalities for the space of flat connections on $E$ (see the introduction to [14]). These Thom isomorphisms have also been used by T. Baird [2] to produce formulas for the $U(3)$–equivariant Poincaré series of $\text{Hom}(\pi_1 \Sigma, U(3))$, establishing a conjecture of Ho and Liu [12].

Over a Riemann surface, an alternate approach to the Thom isomorphism in (1) would be to use the fact that after modding out the (based, complex) gauge group, one obtains a smooth algebraic variety, and results of Shatz [24, Section 4] show that the image of each stratum is a smooth subvariety. This algebraic approach involves various technicalities, which we will not attempt to resolve here.

This paper is organized as follows: Section 2 contains our construction of invariant tubular neighborhoods, and Section 3 applies this construction to the Morse strata of the Yang–Mills functional over a surface (following Daskalopoulos [7]). In Section 4 we compare our construction of tubular neighborhoods to the existing literature.

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2. Invariant Tubular Neighborhoods

For terminology and constructions regarding Banach manifolds, we follow Lang [16]. By smooth we will always mean $C^\infty$. Throughout this section, $X$ will denote a (possibly infinite-dimensional) Riemannian manifold and $G$ a group acting on $X$ by smooth isometries. This means $X$ is modeled on a Hilbert space $\mathbf{H}$ (real or complex), and the tangent bundle $TX$ has a smoothly varying fiberwise inner product (Hermitian in the complex case), which we refer to as a Riemannian metric. Throughout the paper all Riemannian metrics are assumed to be strong, meaning we assume that the fiberwise inner products induce the given topology on $T_x(X)$. Without this condition, the geodesic distance associated to a fiberwise inner product
(which will play a key role in our construction) can be very badly behaved: for an example in the setting of Fréchet manifolds, see Michor and Mumford [18].

Let \( Y \subset X \) be a (locally closed) \( G \)-invariant submanifold; then the tangent space \( T_yY \) is a closed subspace of \( T_pX \) and has an orthogonal complement \( N_yY \). Since \( G \) acts by isometries, it preserves the normal bundle \( NY = \bigcup_{y \in Y} N_yY \).

Note, however, that \( NY \) need not be locally trivial when \( X \) is infinite-dimensional. For the construction of tubular neighborhoods given by Lang [16, IV.5], this is irrelevant: Lang obtains a tubular neighborhood diffeomorphic to the quotient bundle \( TX|_Y/TY \), which is always a smooth vector bundle, by using a smooth splitting \( TX|_Y \cong TY \oplus (TX|_Y/TY) \).

We will show that if \( NY \) is a smooth subbundle of \( TX|_Y \), then \( Y \) admits a \( G \)-invariant total tubular neighborhood \( \tau Y \) (Theorem 2.5). The neighborhood \( \tau Y \) is homeomorphic to the normal bundle \( NY \), but may not be diffeomorphic to \( NY \). Since we are mainly interested in the Thom isomorphisms 1, a homeomorphism is sufficient. (If \( G = \{1\} \), or if \( Y \) consists of a single orbit of \( G \), then we do obtain a diffeomorphism \( \tau Y \cong NY \).)

**Remark 2.1.** The condition that \( NY \subset TX|_Y \) is a smooth subbundle could be replaced by the condition that there exists a smooth splitting of the map \( \pi \) in the exact sequence

\[
0 \longrightarrow TY \longrightarrow TX|_Y \overset{\pi}{\longrightarrow} TX|_Y/TY \longrightarrow 0,
\]

such that the image of the splitting in \( TX|_Y \) is invariant under \( G \). Our proof goes through equally well in this case, with \( \text{Im}(s) \) playing the role of \( NY \) (where \( s \) is the chosen splitting). Note that \( \text{Im}(s) \) is a smooth subbundle of \( TX|_Y \) by [16, III, Proposition 3.1].

2.1. The tangent bundle as a Riemannian manifold. Associated to the metric on \( X \) we have the metric spray \( F : TX \to T(TX) \), which is equivariant with respect to the natural action of the isometry group \( \text{Isom}(X) \). The spray induces an exponential map \( f : D \to X \) [16, VII.7], whose domain is an open subset of \( TX \) containing the zero section. It follows from the construction of \( f \) that \( D \) is invariant under \( \text{Isom}(X) \) and \( f \) is \( \text{Isom}(X) \)-equivariant. We also need an \( \text{Isom}(X) \)-invariant Riemannian metric on the tangent bundle \( TX \). This metric was first introduced by Sasaki [23], via an explicit local description (see also Do Carmo [6, Chapter 3, Exercise 2]). Here we give a more global perspective, from which it is clear that the action of \( \text{Isom}(X) \) on \( T(TX) \) preserves this metric.

There is a natural exact sequence of vector bundles over \( TX \)

\[
0 \longrightarrow (\pi_{TX})^*TX \overset{i}{\longrightarrow} T(TX) \overset{\alpha}{\longrightarrow} (\pi_{TX})^*TX \longrightarrow 0,
\]

whose terms we will now explain. Let \( \pi_{TX} : TX \to X \) denote the projection, and let \((\pi_{TX})^*TX\) be the pullback of the bundle \( TX \) along the map \( \pi_{TX} \). Note that, by definition, \((\pi_{TX})^*TX\) is isomorphic to the Whitney sum \( TX \oplus TX \) of the bundle \( TX \) with itself. The map \( i \) in (2) is defined by sending a pair of vectors

1Here locally closed means that \( Y = U \cap C \), where \( U \) is open in \( X \) and \( C \) is closed in \( X \). Since submanifolds are locally modeled on closed subspaces of \( H \), they are automatically locally closed, and we include the term locally closed only to emphasize that \( Y \) need not be closed.

2The usual proof in the finite dimensional setting, e.g. Milnor–Stasheff [20, Theorem 3.3], makes essential use of finite-dimensionality. Local triviality has been established in certain concrete infinite-dimensional settings (see Section 3). While it seems unlikely that local triviality holds in general, the author does not know of a counterexample.
\((v, w) \in T_x X \oplus T_x X\) to the vector in \(T_0(T_x X)\) represented by the curve \(t \mapsto v + tw\) in \(T_x X \subset TX\). We will see below that \(i\) is a smooth, injective bundle map. The map \(\alpha\) is defined by applying the universal property of the pullback \((\pi_{TX})^*TX\) to the diagram

\[
\begin{array}{ccc}
T(TX) & \xrightarrow{\alpha} & TX \\
\pi_{T(TX)} & & \pi_{TX}
\end{array}
\]

\[
\begin{array}{ccc}
TX & \xrightarrow{\pi_{TX}} & X.
\end{array}
\]

To see that \(\alpha\) is surjective and to identify its kernel, one uses the representation in local charts provided by [16, p. 96]:

\[
(U \times H) \times (H \times H) \xrightarrow{\pi_1, \pi_3} U \times H \times H \xrightarrow{\pi_1, \pi_2} U \times H \xrightarrow{\pi_1} U.
\]

From this local representation, one sees that \(i\) is a smooth, injective bundle map whose image is precisely \(\ker(\alpha)\).

By naturality, the maps in (2) are Isom\((X)\)–equivariant. The Levi–Civita connection, viewed as a covariant derivative [10 VIII, Theorem 4.2], can now be used to define an Isom\((X)\)–equivariant splitting

\[s: (\pi_{TX})^*TX \rightarrow T(TX)\]

of the map \(\alpha\) in (2). Specifically, given \((v, w) \in T_x X \oplus T_x X\) with \(w = \gamma'(0)\) for some curve \(\gamma: (-\epsilon, \epsilon) \rightarrow X\), there is a unique lift of \(\gamma\) to a curve \(\gamma_v: (-\epsilon, \epsilon) \rightarrow TX\) such that \(\gamma(0) = v\) and \(\gamma_v\) is parallel to \(\gamma\) with respect to \(D\). We define

\[s(v, w) = (\gamma_v)'(0) \in T_v(TX)\]

The splitting \(s\) gives rise to an Isom\((X)\)–equivariant isomorphism

\[T(TX) \cong (\pi_{TX})^*(TX) \oplus (\pi_{TX})^*(TX)\]

of vector bundles over \(TX\), and \(T(TX)\) now inherits a fiberwise inner product from that on \((\pi_{TX})^*(TX)\). This (strong) Riemannian metric on \(TX\) will play a key role in our construction of tubular neighborhoods.

2.2. Construction of the tubular neighborhood.

**Definition 2.2.** Assume that \(NY\) is a smooth subbundle of \(TX|_Y\). A tubular neighborhood of \(Y\) in \(X\) is an open neighborhood \(\tau Y\) of \(Y\) in \(X\) together with a homeomorphism \(\phi: U \rightarrow \tau Y\) from an open neighborhood \(U \subset NY\) containing the zero-section \(Y \subset NY\), such that \(\phi\) restricts to the identity from \(Y \subset NY\) to \(Y \subset X\). If \(\phi\) is a diffeomorphism, we call \(\tau Y\) smooth, and if \(U = NY\), we call \(\tau\) total.
Proposition 2.3. Let $X$ be a Riemannian manifold and let $Y \subset X$ be a (locally closed) submanifold whose normal bundle $NY \subset TX|_Y$ is a smooth subbundle. Assume $G$ acts on $X$ by smooth isometries, leaving $Y$ invariant, and let $D$ denote the domain of the exponential map associated to the metric spray. Then there exists a $G$–invariant open neighborhood $Z \subset D \cap NY$ of the zero section of $NY$ on which the exponential map restricts to a diffeomorphism onto an open neighborhood $V$ of $Y$ in $X$. Hence $V$ is a $G$–invariant tubular neighborhood of $Y$.

Proof. We identify $Y$ with the zero section of $NY$, and for $g \in G$ we let $g$ denote both the self-map it induces on $X$ and the derivative of this map.

The geodesic distance $d$ associated to the $\Isom(X)$–invariant Riemannian metric on $TX$ constructed above yields an $\Isom(X)$–invariant distance function on $TX$. We refer to [16, VII.6] for a discussion of the geodesic distance, and the fact that the associated metric topology is the usual topology on $TX$. For any $W \subset TX$ and any $w \in W$, we write $B_\epsilon(w, W) = \{w' \in W | d(w, w') < \epsilon\}$. We identify $X$ with the zero section of $TX$, so $B_\epsilon(x, X)$ is defined. Note that although the intrinsic geodesic distance in $X$ (induced by the inner product on $TX$) may be different from the restriction of $d$ to $X$, both induce the given topology on $X$.

The exponential map $f$ restricts to a local diffeomorphism $D \cap NY \to X$ [16, p. 109], so for each $y \in Y$ there exists $\epsilon_y > 0$ such that $f$ restricts to a diffeomorphism $B_{\epsilon_y}(y, NY) \to f(B_{\epsilon_y}(y, NY))$ (with $f(B_{\epsilon_y}(y, NY))$ open in $X$). Let $\psi: f(B_{\epsilon_y}(y, NY)) \to B_{\epsilon_y}(y, NY)$ denote the inverse map. Now, $f(B_{\epsilon_y/2}(y, NY))$ is an open neighborhood of $y$ in $X$, hence contains $B_{\epsilon_y/2}(y, X)$ for some $\epsilon_y < \epsilon_y/4$. Set $Z_y = \psi(B_{\epsilon_y/2}(y, X)) \subset NY$. Then $Z_y$ is open in $NY$ and we have

$$Z_y \subset B_{\epsilon_y/2}(y, NY)$$
and $f(Z_y) = B_{\epsilon_y}(g, X)$. In fact, for any $g \in G$,

$$f(g(Z_y)) = B_{\epsilon_y}(g \cdot y, X).$$

Using the Axiom of Choice, select one point $y(\mathcal{O})$ from each $G$–orbit $\mathcal{O} \subset Y$ and define $Z_{y(\mathcal{O})}$ as in [3]. We claim that $f$ is injective, and hence restricts to a diffeomorphism, on the open set

$$Z = \bigcup_{\mathcal{O} \in Y/G} g(Z_{y(\mathcal{O})}).$$

(To conclude that $f|_Z$ is a diffeomorphism onto its image, we use the fact that $f(B_{\epsilon_y}(y, NY))$ is open in $X$; this also shows that $f(Z)$ is open in $X$.)

Say $x = f(g_1 \cdot y_1) = f(g_2 \cdot y_2)$ with $y_1, y_2 \in Z_{y_1}, y_2 \in Z_{y_2}$ and $g_1, g_2 \in G$ (here the $y_i$ are the chosen representatives for some two orbits). Then by [4] we have

$$d(g_1 \cdot y_1, y_2) \leq d(g_1 \cdot y_1, x) + d(x, y_2) < \epsilon_{y_1} + \epsilon_{y_2} < \epsilon_{y_1}/4 + \epsilon_{y_2}/4.$$  

We may assume that $\epsilon_{y_1} \geq \epsilon_{y_2}$. Then [3] and [5] yield

$$d(g_1 \cdot y_1, g_2 \cdot y_2) \leq d(g_1 \cdot y_1, g_2 \cdot y_2) + d(g_2 \cdot y_2, g_2 \cdot z_2)$$

$$< (\epsilon_{y_1}/4 + \epsilon_{y_2}/4) + \epsilon_{y_2}/2 \leq \epsilon_{y_1}.$$

The fact that $f$ is injective on $B_{\epsilon_{y_1}}(y_1, NY)$ implies that it is also injective on $B_{\epsilon_{y_1}}(g_1 \cdot y_1, NY)$. But $g_1 \cdot z_2 \in B_{\epsilon_{y_1}}(g_1 \cdot y_1, NY)$ by [3], and $g_2 \cdot z_2 \in B_{\epsilon_{y_1}}(g_1 \cdot y_1, NY)$ by [6], so $g_1 \cdot y_1 = g_2 \cdot y_2$ and we conclude that $f$ is injective on $Z$. □

To build total neighborhoods, we use [16, VII, Proposition 4.1].
Lemma 2.4. Let $X$, $Y$, $Z$, and $G$ be as in Proposition 2.3 and let $\sigma: Y \to \mathbb{R}_{>0}$ be a $G$–equivariant map such that for all $y \in Y$, 
\[ \{v \in N_y(Y) : ||v|| < \sigma(y)\} \subset Z. \]
Then there is a homeomorphism $NY \to NY_\sigma = \{v \in NY : ||v|| < \sigma(\pi v)\}$, where $\pi: NY \to Y$ is the projection, and $|| \cdot ||$ is the norm on $T_x X$. This homeomorphism restricts to the identity on $Y \subset NY$.

We can now prove our main theorem.

Theorem 2.5. Let $X$ be a Riemannian manifold, and let $G$ act on $X$ by isometries, leaving the submanifold $Y \subset X$ invariant. Assume that $NY$ is a smooth subbundle of $TX|_Y$. Then there exists a $G$–equivariant total tubular neighborhood $\tau Y$ of $Y$ in $X$. The homeomorphism $NY \xrightarrow{\sim} \tau Y$ is $G$–equivariant, but need not be smooth.

Proof. Apply Lemma 2.4 with $\sigma(y) = \sup\{\epsilon > 0 : ||v|| < \epsilon \implies v \in Z\}$. Continuity of $\sigma$ follows from the assumption that the inner products defining $|| \cdot ||$ vary continuously over $X$, together with the fact that $Z$ is open.

If $G = \{1\}$ and $X$ admits partitions of unity, then [10 VII.4, Corollary 4.2] lets us make $\sigma$ smooth, and then the homeomorphism $NY \xrightarrow{\sim} \tau Y$ is smooth as well.

If $Y = G \cdot x$, then $\sigma$ can be made constant, to the same effect.

3. Applications to Yang–Mills Theory

We now apply Theorem 2.5 to the Morse strata of the Yang–Mills functional. Let $P \to M$ be a principal $U(n)$–bundle over a possibly non-orientable surface, and let $A^{k-1}(P)$ be the affine space of connections on $P$ of Sobolev type $L^2_{k-1}$ ($k \geq 2$). Similarly, let $G^k(P)$ be the $L^2_{k-1}$ Sobolev completion of the unitary gauge group of $P$, which acts smoothly on $A^{k-1}(P)$ (Wehrheim [27]). The following lemma allows us to apply Theorem 2.5. For a proof, see Kondracki–Rogulski [15 Section 2.3].

Lemma 3.1. Let $K$ be a compact connected Lie group and let $P$ be a principal $K$–bundle over a Riemann surface $M$ equipped with a Riemannian metric. The metric on $M$ and a $K$–invariant inner product on the Lie algebra $\mathfrak{k}$ of $K$ induce a $G^k(P)$–invariant Riemannian metric on $A^{k-1}(P)$.

As discussed above, the Morse strata of the Yang–Mills functional can be ordered $C_1 \leq C_2 \leq \cdots$, starting with the central stratum, in such a way that the union $C_n = \bigcup_{i=1}^n C_i$ of each initial segment of the ordering is open.

Theorem 3.2. There are Thom isomorphisms in equivariant cohomology
\[ H^*_G(P)(C_n, C_{n-1}; \mathbb{Z}) \cong H^*_{G^k(P)}((C_n; \mathbb{Z}). \]
(If $M$ is the Klein bottle, we must use $\mathbb{Z}/2\mathbb{Z}$–coefficients.)

Proof. First we show that each $C_n$ has a $G(E)$–invariant tubular neighborhood inside $C_n$. The affine space $A^{k-1}(P)$ and the open subsets $C_n$ are Hilbert manifolds. When $M$ is orientable, the Morse strata for the Yang–Mills functional on $P$ are locally closed, $G^k(P)$–invariant under submanifolds of $A^{k-1}(P)$ (Daskalopoulos [7]). Once we show that the normal bundle $N(C_n) \subset T(A^{k-1}(P)|_{C_n}$ is a
smooth subbundle, Theorem 2.5 and Lemma 3.1 produce the desired neighborhood. The kernel of a smooth surjection of bundles is a smooth subbundle of the domain (Lang [16, III.3]), so it suffices to prove that the orthogonal projections $T_A(A^{k-1}(P)) \to T_A(C_n)$ vary smoothly with $A \in C_n$.

We follow Daskalopoulos’ proof that the strata $C_n$ are locally closed submanifolds of $C(E)$ [7, Proposition 3.5]. Connections on $P$ correspond to Hermitian connections on the associated Hermitian bundle $Ad P = P \times_{U(n)} \mathbb{C}^n$ [1, Section 5], and $\mathcal{G}^k(P)$ is isomorphic to the Sobolev gauge group $\mathcal{G}^k(Ad P)$. We work with the space of holomorphic structures on $Ad P$, viewed as operators on sections

$$d'' : L^2_{k-1}(Ad P) \to L^2_{k-1}(T^*M'' \otimes Ad P)$$

where $T^*M''$ is the space of $(0, 1)$-forms on the complex manifold $M$. We call $Ad P$, together with a $d''$ operator, a holomorphic bundle, which we denote by $E$. Note that by adjointness, $d''$ induces a corresponding operator on the endomorphism bundle $\text{End}(Ad P) = \text{End} E$.

Given a critical point $E \in C_j$, let $S$ denote the orthogonal complement of the $L^2_{k-1}$ holomorphic endomorphisms of $E$ inside the smooth endomorphisms (that is, $S$ is the kernel of the $d''$ operator associated to $\text{End} E$). Daskalopoulos constructs a map

$$f : S \times H^1(M, \text{End} E) \to \mathcal{A}^{k-1}(P)$$

and shows that $f$ is an isomorphism near zero, meaning that $f : U_0 \to U_E$ is an isomorphism for some neighborhood $U_0$ of $0$ in $S \times H^1(M, \text{End} E)$ and some neighborhood $U_E$ of $E$. He then shows that $f^{-1}(U_E \cap C_j) = U_0 \cap (S \times H^1(M, \text{End'} E))$, so that $f$ provides a chart for $C_j$ near $E$. Here $\text{End'} E$ is the sheaf of holomorphic endomorphisms preserving the Harder-Narasimhan filtration on $E$. (The notation $\text{End'} E$ comes from Atiyah–Bott [1, Section 7]; Daskalopoulos uses the notation $UT(E, \ast) = \text{End'} E$, since these endomorphisms correspond, in a local ordered basis respecting the Harder-Narasimhan filtration, to upper triangular matrices.) This chart provides an isomorphism between the tangent bundle to $U_E \cap C_j$ and the trivial bundle

$$(U_0 \cap (S \times H^1(M, \text{End'} E))) \times (S \times H^1(M, \text{End'} E)).$$

We need to understand the orthogonal projections

$$S \times H^1(M, \text{End} E) \xrightarrow{\pi_E} S \times H^1(M, \text{End'} E)$$

with respect to the pullback, along $f$, of the metric on $\mathcal{A}^{k-1}(P)$ to the tangent spaces $S \times H^1(M, \text{End} E)$. Now, $\pi_E$ is the identity on $S$ plus an orthogonal projection $H^1(M, \text{End} E) \to H^1(M, \text{End'} E)$ between finite dimensional cohomology groups. Since the metrics vary smoothly, so do these projections.

Daskalopoulos produces the manifold structure at a general point $E \in C_j$ (as opposed to critical points) using a gauge transformation sending $E$ into a neighborhood of the critical set on which the manifold structure has already been established. So local triviality of $N(C_j)$ at arbitrary points of $E$ follows from the result near the critical set. This completes the proof of local triviality when $M$ is orientable.

When $M$ is non-orientable, let $\tilde{P} \to \tilde{M}$ denote the pullback of $P$ to the orientable double cover $\tilde{M} \to M$. There is a natural embedding $\mathcal{G}^k(P) \to \mathcal{G}^k(\tilde{P})$, making the natural embedding $\mathcal{A}^{k-1}(P) \to \mathcal{A}^{k-1}(\tilde{P})$ equivariant. In both cases, the image of the embedding is the set of fixed points of an involution $\tau$ arising from the
deck transformation on $\tilde{M}$ (see Ho–Liu \[13\]). The Morse strata in $A^{k-1}(\tilde{P})$ are, by definition, just the intersections of Morse strata in $A^{k-1}(\tilde{P})$ with $A^{k-1}(P)$, and hence are locally closed submanifolds invariant under $\mathcal{G}^k(P) = \mathcal{G}^k(\tilde{P})^\tau$. By Lemma \[3.1\] we obtain a gauge-invariant metric on $A^{k-1}(\tilde{P})$ from a choice of a metric on $\tilde{M}$ and a $U(n)$–invariant inner product on $u(n)$. We will choose our metric on $\tilde{M}$ to be invariant under the deck transformation, so that the resulting metric on $A^{k-1}(\tilde{P})$ is invariant under $\tau$. Now, the $\mathcal{G}^k(\tilde{P})$–invariant metric on $A^{k-1}(\tilde{P})$ restricts to a $\mathcal{G}^k(P) = \mathcal{G}^k(\tilde{P})^\tau$–invariant metric on $A^{k-1}(\tilde{P}) = A^{k-1}(P)^\tau$, and it remains to check that the normal bundles are smooth subbundles of $T(A^{k-1}(P))$.

Given a Morse stratum $C_n \subset A^{k-1}(P)$, it will suffice to show that the orthogonal projection $Q_{C_n} : TA^{k-1}(P) \to TC_n$ is a smooth bundle map, since its kernel is the normal bundle to $C_n$. By definition, $C_n = \tilde{C}_n \cap A^{k-1}(P)$ for some Morse stratum $\tilde{C}_n \subset A^{k-1}(\tilde{P})$. Let $Q_{\tilde{C}_n} : T\tilde{A}^{k-1}(\tilde{P}) \to T\tilde{C}_n$ denote the orthogonal projection, which we have shown is a smooth bundle map. The involution $\tau$ is an isometry, so $Q_{C_n}$ is simply the restriction of $Q_{\tilde{C}_n}$ to $TC_n = (T\tilde{C}_n)^\tau \subset T\tilde{C}_n$ (in general, one may check that if $W$ is a closed subspace of a Hilbert space $H$ and $\tau : H \to H$ is a linear isometry, then the orthogonal projection $H^\tau \to W^\tau$ is the restriction of the orthogonal projection $H \to W$). Hence $Q_{C_n}$ is a smooth bundle map as well.

To produce the desired Thom isomorphisms, note that over a Riemann surface the normal bundles to the Yang–Mills strata are naturally complex vector bundles (see Atiyah–Bott \[1\] Section 7). For any $G$–equivariant complex vector bundle $V \to X$, the homotopy orbit bundle $V_{hG} \to X_{hG}$ is again a complex vector bundle, hence orientable. Letting $\nu_n$ denote a $\mathcal{G}(E)$–invariant tubular neighborhood of $C_n$ inside $\tilde{C}_n$, we may excise the complement of $(\nu_n)_{h\mathcal{G}(E)}$ in $(\tilde{C}_n)_{h\mathcal{G}(E)}$ and apply the ordinary Thom isomorphism in integral cohomology to the bundle $(\nu_n)_{h\mathcal{G}(E)} \to (C_n)_{h\mathcal{G}(E)}$. The same argument works over a non-orientable surface, since the normal bundles to the Yang–Mills strata (and the corresponding homotopy orbit bundles) are orientable (real) vector bundles, except possible in the case of the Klein bottle (Ho–Liu–Ramras \[14\]).

\[ \square \]

4. Other constructions of tubular neighborhoods

Even in the non-equivariant case, we have not found our construction (making use of a distance function on $TX$) in the literature. The basic tool in all constructions of tubular neighborhoods is the exponential map $f$. Ever since the proof of Godement’s lemma \[9\] p. 150, most arguments \[4\ 5\ 10\ 11\ 16\ 17\ 21\] rely on \[9\] to produce a neighborhood on which $f$ is a diffeomorphism (in \[8\ 19\ 25\], the submanifold is compact, and a simplified version of \[9\] is used). After this point, one compresses $NY$ into $Z$. To construct invariant neighborhoods for compact groups, Bredon \[4\] averages over the group to make the compression equivariant.

Kondracki and Rogulski \[15\] constructed of gauge-invariant tubular neighborhoods of the gauge orbits in the space of (Sobolev) connections on a smooth principle $K$–bundle $P$ over a compact manifold $M$, where $K$ is a compact Lie group. Following Ebin \[8\], they utilize a weak gauge-invariant metric, and show that with respect to this metric, the normal bundle to an orbit is a smooth subbundle (\[15\ Lemma 3.3.1]). Their argument also shows that the strong normal bundle is a smooth subbundle, so Theorem \[25\] applies. On the other hand, our methods do
not readily apply to the situations (discussed in the introduction) considered by Ebin [8] and A. Teleman [26].

REFERENCES

[1] M. F. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A, 308, 523–615 (1983).
[2] T. Baird, Antiperfection of Yang–Mills Morse theory over a nonorientable surface in rank three, arXiv:0902.4581 (2009).
[3] M. Berger and B. Gostiaux, Differential geometry: manifolds, curves, and surfaces, Graduate Texts in Mathematics, volume 115, Springer-Verlag, New York, 1988, translated from the French by Silvio Levy.
[4] G. E. Bredon, Introduction to compact transformation groups, Academic Press, New York, 1972, pure and Applied Mathematics, Vol. 46.
[5] T. Bröcker and K. Jänich, Introduction to differential topology, Cambridge University Press, Cambridge, 1982, translated from the German by C. B. Thomas and M. J. Thomas.
[6] M. P. do Carmo, Riemannian geometry, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1992, translated from the second Portuguese edition by Francis Flaherty.
[7] G. D. Daskalopoulos, The topology of the space of stable bundles on a compact Riemann surface, J. Differential Geom., 36, 699–746 (1992).
[8] D. G. Ebin, On the space of Riemannian metrics, Bull. Amer. Math. Soc., 74, 1001–1003 (1968).
[9] R. Godement, Topologie algébrique et théorie des faisceaux, Actualités Sci. Ind. No. 1252. Publ. Math. Univ. Strasbourg. No. 13, Hermann, Paris, 1958.
[10] V. Guillemin and A. Pollack, Differential topology, Prentice-Hall Inc., Englewood Cliffs, N.J., 1974.
[11] M. W. Hirsch, Differential topology, Graduate Texts in Mathematics, volume 33, Springer-Verlag, New York, 1994, corrected reprint of the 1976 original.
[12] N.-K. Ho and C.-C. M. Liu, Anti-perfect Morse stratification, To appear in Selecta Math. (N.S.), arXiv:0808.3974.
[13] N.-K. Ho and C.-C. M. Liu, Yang–Mills connections on nonorientable surfaces, Comm. Anal. Geom., 16, 617–679 (2008).
[14] N.-K. Ho, C.-C. M. Liu, and D. Ramras, Orientability in Yang-Mills theory over nonorientable surfaces, Comm. Anal. Geom., 17, 903–953 (2009).
[15] W. Kondracki and J. Rogulski, On the stratification of the orbit space for the action of automorphisms on connections, Dissertationes Math. (Rozprawy Mat.), 250, 67 (1986).
[16] S. Lang, Differential and Riemannian manifolds, Graduate Texts in Mathematics, volume 100, 3rd edition, Springer-Verlag, New York, 1995.
[17] J. MARGALEF ROIG and E. OUTERelo DOMÍNGUEZ, Differential topology, North-Holland Mathematics Studies, volume 173, North-Holland Publishing Co., Amsterdam, 1992, with a preface by Peter W. Michor.
[18] P. W. Michor and D. Mumford, Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms, Doc. Math., 10, 217–245 (2005).
[19] J. Milnor, Collected papers of John Milnor. III, American Mathematical Society, Providence, RI, 2007, differential topology.
[20] J. W. Milnor and J. D. Stasheff, Characteristic classes, Princeton University Press, Princeton, N. J., 1974, annals of Mathematics Studies, No. 76.
[21] A. Mukherjee, Topics in differential topology, Texts and Readings in Mathematics, volume 34, Hindustan Book Agency, New Delhi, 2005.
[22] D. A. Ramras, On the Yang–Mills stratification for surfaces, Proc. Amer. Math. Soc., 139 no. 5., 1851–1863 (2011), arXiv:0805.2587.
[23] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tôhoku Math. J. (2), 10, 338–354 (1958).
[24] S. S. Shatz, The decomposition and specialization of algebraic families of vector bundles, Compositio Math., 35, 163–187 (1977).
[25] M. Spivak, A comprehensive introduction to differential geometry. Vol. I, 2nd edition, Publish or Perish Inc., Wilmington, Del., 1979.
[26] A. Teleman, Harmonic sections in sphere bundles, normal neighborhoods of reduction loci, and instanton moduli spaces on definite 4-manifolds, Geom. Topol., 11, 1681–1730 (2007).

[27] K. Wehrheim, Uhlenbeck compactness, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2004.

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