Abstract—Consider $N$ cooperative but non-communicating players where each plays one out of $M$ arms for $T$ turns. Players have different utilities for each arm, representable as an $N \times M$ matrix. These utilities are unknown to the players. In each turn players receive noisy observations of their utility for their selected arm. However, if any other players selected the same arm that turn, they will all receive zero utility due to the conflict. No other communication or coordination between the players is possible. Our goal is to design a distributed algorithm that learns the matching between players and arms that achieves max-min fairness while minimizing the regret. We present an algorithm and prove that it is regret optimal up to a $\log \log T$ factor. This is the first max-min fairness multi-player bandit algorithm with (near) order optimal regret.

I. INTRODUCTION

In online learning problems, an agent sequentially makes decisions and receives an associated reward. When this reward is stochastic, the problem takes the form of a stochastic multi-armed bandit problem [1]. However, stochastic bandits assume stationary reward processes that are rarely the case in practice; they are too optimistic about the environment. To deal with this shortcoming, one can model the rewards as being determined by an adversary, which leads to a formulation known as non-stochastic or adversarial bandits [1]. Naturally, the performance guarantees against such a powerful adversary are much weaker than in the stochastic case, as adversarial bandits are usually overly pessimistic about the environment. Is there an alternative that lies in the gap between the two?

Multi-player bandits is a promising answer, which has seen a surge of recent interest [2]–[13]. One primary reason for studying multi-player bandits is distributed resource allocation. Examples include channels in communication networks, computation resources on servers, consumers and items, etc. In most applications of interest, the reward of a player is a stochastic function of the decisions of other players that operate in the same environment. Thinking of arms as resources, we see that while these players are not adversaries, conflicts still arise due to the players’ preferences among the limited resources. To model that, we assign zero reward to players that choose the same arm. The goal of multi-player bandit algorithms is to provide a distributed way to learn how to share these resources optimally in an online manner. This is useful in applications where agents (players) follow a standard or protocol, like in wireless networks, autonomous vehicles, or with a team of robots.

A common network performance objective is the sum of rewards of the players over time. As such, maximizing the sum of rewards has received the vast majority of the attention in the multi-player bandit literature [14]–[19]. However, in the broader literature of network optimization, the sum of rewards is only one possible objective. One severe drawback of this objective is that it has no fairness guarantees. As such, the maximal sum of rewards assignment might starve some users. Often in applications, the designer wants to make sure that all users will enjoy at least minimal target Quality of Service.

Ensuring fairness has been recently recognized by the machine learning community as a problem of key importance. In addition to the interest in fair classifiers [20], fairness has been recognized as a major design parameter in reinforcement learning and single-player bandits as well [21]–[23]. Our work addresses this major concern for the emerging field of multi-player bandits.

In the context of maximizing the sum of rewards, many works on multi-player bandits have considered a model where all players have the same vector of expected rewards [24]–[27]. While being relevant in several applications, this model is not rich enough to study fairness since the worst off player is simply the one that was allocated the worst resource. To study fairness, a heterogeneous model is necessary, where players have different expected rewards for the arms (i.e., a matrix of expected rewards). In this case, a fair allocation may prevent some players from getting their best arm in order to significantly improve the allocation for less fortunate players.

Despite being a widely-applied objective in the broader resource allocation literature [28]–[30], max-min fairness between multi-player bandits has yet to be studied. Some bandit works have studied alternative objectives that can potentially exhibit some level of fairness [31], [32]. In the broader networking literature, a celebrated notion of fairness is $\alpha$-fairness [33] where $\alpha = 1$ yields proportional fairness and $\alpha = 2$ yields sum of utilities. While for constant $\alpha$, $\alpha$-fairness can be maximized in a similar manner to [17], the case of max-min fairness corresponds to $\alpha \rightarrow \infty$ and is fundamentally different.

Learning to play the max-min fairness allocation involves major technical challenges that do not arise in the case of maximizing the sum of rewards (or in the case of $\alpha$-fairness). The sum-rewards optimal allocation is unique for “almost all” scenarios (randomizing the expected rewards).
This is not the case with max-min fairness, as there will typically be multiple optimal allocations. This complicates the distributed learning process, since players will have to agree on a specific optimal allocation to play, which is difficult to do without communication. Specifically, this rules out using similar techniques to those used in [17] to solve the sum of rewards case under a similar multi-player bandit setting (i.e., matrix of expected rewards, no communication, and a collision model).

Trivially, any matching where all players achieve reward greater than \( \gamma \) has max-min value of at least \( \gamma \). We observe that finding these matchings, called \( \gamma \)-matchings in this paper, can be done via simple dynamics, where players keep their arm as long as they achieve more than what can be done via simple dynamics, where players keep their arm as long as they achieve more than \( \gamma \). This introduces an absorbing Markov chain with the \( \gamma \)-matchings as the absorbing states. This insight allows for a stronger analysis than in [17], that does not rely on ergodic Markov chains that have an exploration parameter \( \varepsilon \) that has to be tuned.

In this work we provide an algorithm that has provably order optimal regret up to a \( \log \log T \) factor (that can be arbitrarily improved to any factor that increases with the horizon \( T \)). We adopt the challenging model with heterogeneous arms (a matrix of expected rewards) and no communication between the players. The only information players receive regarding their peers is through the collisions that occur when two or more players pick the same arm.

A. Outline

In Section 2 we formulate our multi-player bandit problem of learning the max-min matching under the collision model with no communication between players. In Section 3 we present our distributed max-min fairness algorithm and state our regret bound (which is proved in Section 8). Section 4 analyzes the exploration phase of our algorithm. Section 5 analyzes the matching phase of our algorithm and concludes with bounding the probability of the exploitation error event. Section 6 presents simulation results that corroborate our theoretical findings and demonstrate that our algorithm learns the max-min optimal matching faster than our analytical bounds suggest. Finally, Section 7 concludes the paper.

II. PROBLEM FORMULATION

We consider a stochastic game with the set of players \( N = \{1, ..., N\} \) and a finite time horizon \( T \). The strategy space of each player is the set of \( M \) arms with indices denoted by \( i, j \in \{1, ..., M\} \). We assume that \( M \geq N \), since otherwise the max-min utility is trivially zero. The horizon \( T \) is not known by any of the players and is typically much larger than \( M \) and \( N \). We denote the discrete turn index by \( t \). At each turn \( t \), all players simultaneously pick one arm each. The arm that player \( n \) chooses at time \( t \) is \( a_n(t) \) and the strategy profile (vector of arms selected) at time \( t \) is \( a(t) \). Players do not know which arms the other players chose, and need not even know the number of players \( N \).

Define the no-collision indicator of arm \( i \) in strategy profile \( a \) to be

\[
\eta_n(a) = \begin{cases} 0 & \left| N_i(a) \right| > 1 \\ 1 & \text{otherwise.} \end{cases}
\]

where \( N_i(a) = \{n \mid a_n = i\} \) is the set of players that chose arm \( i \) in strategy profile \( a \). The instantaneous utility of player \( n \) at time \( t \) with strategy profile \( a(t) \) is

\[
v_n(a(t)) = r_{n,a_n(t)}(t) \eta_{a_n(t)}(a(t))
\]

where \( r_{n,a_n(t)}(t) \) is a random reward which is assumed to have a continuous distribution on \([0, 1]\). The sequence of rewards for arm \( i \) for player \( n \), \( \{r_{n,i}(t)\}_{t=1}^T \), is i.i.d. with expectation \( \mu_{n,i} \).

Next we define the total expected regret for this problem. This is the expected regret that the cooperating but non-communicating players accumulate over time from not playing the optimal max-min allocation.

**Definition 1.** The total expected regret is defined as

\[
R(T) = \sum_{t=1}^T \left( \gamma^* - \min_a \mathbb{E}\left\{v_n(a(t))\right\} \right)
\]

where \( \gamma^* = \max_n \min_a \mathbb{E}\left\{v_n(a)\right\} \). The expectation is over the randomness of the rewards \( \{r_{n,i}(t)\}_t \), that dictate the random actions \( \{a_n(t)\}_t \).

Note that replacing the minimum in (3) with a sum over the players yields the regret for the sum-reward objective case, after redefining \( \gamma^* \) to be the optimal sum-reward [16]–[19].

Rewards with a continuous distribution are natural in many applications (e.g., SNR in wireless networks). However, this assumption is only used to argue that since the probability for zero reward in a non-collision is zero, players can properly estimate their expected rewards. In the case where the probability of receiving zero reward is not zero, we can instead assume that each player can observe their no-collision indicator in addition to their reward. This alternative assumption requires no modifications to our algorithm or analysis. Observing one bit of feedback signifying whether other players chose the same arm is significantly less demanding than observing the actions of other players.

According to the seminal work in [35], the optimal regret of the single-player case is \( O(\log T) \). The next proposition shows that \( O(\log T) \) is a lower bound for our multi-player bandit case, since any multi-player bandit algorithm can be used as a single-player policy by simulating other players.

**Proposition 1.** The total expected regret as defined in (3) of any algorithm is at least \( \Omega(\log T) \).

**Proof.** For \( N = 1 \), the result directly follows from [35]. Assume that for \( N > 1 \) there is a policy that results in total expected regret better than \( \Omega(\log T) \). Then any single player, denoted player \( n \), can simulate \( N - 1 \) other players such that all their expected rewards are larger than her maximal expected reward. Player \( n \) can also generate the other players’ feedback signifying whether other players chose the same arm.
random rewards, that are independent of the actual rewards she receives. Player \( n \) also simulates the policies for other players, and even knows when a collision occurred for herself and can assign zero reward in that case. In this scenario, the expected reward of player \( n \) is the minimal expected reward among the non-colliding players. This implies that \( \gamma^* \) is the largest expected reward of player \( n \). Hence, in every turn \( t \) without a collision, the \( t \)-th term in (3) is equal to the \( t \)-th term of the single-player regret of player \( n \). If there is a collision in turn \( t \), then the \( t \)-th term in (3) is \( \gamma^* \), which bounds from above the \( t \)-th term of the single-player regret of player \( n \). Thus, the total expected regret upper bounds the single-player regret of player \( n \). Hence, simulating a valid single player multi-armed bandit policy that violates the \( \Omega \) bound, which is a contradiction. We conclude that this bound on the total expected regret is also valid for \( N > 1 \).

III. MY FAIR BANDIT ALGORITHM

In this section we describe our distributed multi-player bandit algorithm that achieves near order optimal regret for the max-min fairness problem. The key idea behind our algorithm is a global search parameter \( \gamma \) that all players track together (with no communication required). We define a \( \gamma \)-matching, which is a matching of players to arms such that the expected rewards of each player is at least \( \gamma \).

**Definition 2.** A choice of arms \( a \) is a \( \gamma \)-matching if and only if
\[
\min_n \mathbb{E}\{v_n(a)\} \geq \gamma.
\]

Essentially, the players want to find the maximal \( \gamma \) for which there still exists a \( \gamma \)-matching. However, even for a given \( \gamma \), distributedly converging to a \( \gamma \)-matching is a challenge. Players do not know their expected rewards and their coordination is extremely limited. To address these issues we divide the unknown horizon of \( T \) turns into epochs, one starting immediately after the other. Each epoch is further divided into four phases. In the \( k \)-th epoch:

1) **Exploration Phase** - this phase has a length of \([c_1 \log k]\) turns for some \( c_1 \geq 4 \). It is used for estimating the expectation of the arms. As shown in Section IV, the exploration phase contributes \( O(\log \log T \log T) \) to the total expected regret.

2) **Matching Phase** - this phase has a length of \([c_2 \log k]\) turns for some \( c_2 \geq 1 \). In this phase, players attempt to converge to a \( \gamma_k \)-matching, where each player plays an arm that is at least as good as \( \gamma_k \), up to the confidence intervals of the exploration phase. To find the matching, the players follow distributed dynamics that induce an absorbing Markov chain with the arm choices as states. The absorbing states of this chain are the desired matchings. When the matching phase is long enough, the probability that a matching exists but is not found is small enough. If a matching does not exist, the matching phase naturally does not converge. As shown in Section IV, the matching phase adds \( O(\log \log T \log T) \) to the total expected regret.

3) **Consensus Phase** - this phase has a length of \( M \) turns. The goal of this phase is to let all players know whether the matching phase ended with a matching, using the collisions for signaling. During this phase, every player who did not end the matching phase with a collision plays their matched arm, while the players that ended the matching phase with a collision sequentially play all the arms for the next \( M \) turns. If players deduce that they collectively converged to a matching, they note this for future reference in the matching indicator \( S_k \). If the matching phase succeeded, the search parameter is updated as \( \gamma_k = \gamma_{k-1} + \varepsilon_k \). The step size \( \varepsilon_k \) of \( \gamma_k \) is decreasing such that if even a slightly better matching exists, it will eventually be found. However, it might be that no \( \gamma_k \)-matching exists. Hence once in a while, with decreasing frequency, the players reset \( \gamma_k = 0 \) in order to allow themselves to keep finding new matchings. This phase adds \( O(M \log T) \) to the total expected regret.

4) **Exploitation Phase** - this phase has a length of \([c_3 (\frac{1}{2})^k]\) turns for some \( c_3 \geq 1 \). During this phase, players play the best matching that was “recently” found. This is the matching \( \bar{a}_{k^*} \), where \( k^* \) is the epoch within the last \( \frac{k}{2} \) epochs with the largest \( \gamma \) that resulted in a matching. This phase adds a vanishing term (with \( T \)) to the total expected regret since players eventually play an optimal matching with exponentially small error probability.

The Fair Bandit Algorithm is detailed in Algorithm 1. Our main result is given next, and is proved in Section 8.

**Theorem 1 (Main Theorem).** Assume that the rewards \( \{r_{n,i}(t)\} \), are independent in \( n \) and i.i.d. with \( t \), with continuous distributions on [0, 1] with positive expectations \( \{\mu_{n,i}\} \). Let \( T \) be the finite deterministic horizon of the game, which is unknown to the players. Let each player play according to Algorithm 1 with any constants \( c_1, c_2, c_3 \geq 4 \). Then, for large enough \( T \), the total expected regret is upper bounded by
\[
R(T) \leq 2 \left( (c_1 + c_2) \left( \log \log \frac{T}{c_3} \right) + M \right) \log \frac{T}{c_3} = O \left( \log \log T \log T \right)
\]
where the \( \log \log T \) factor can be made an arbitrarily slowly growing function of \( T \) by changing the length of the exploration and matching phases.

IV. EXPLORATION PHASE

Over time, players receive stochastic rewards from different arms and average them to estimate their expected reward for each arm. In each epoch, only \([c_1 \log k]\) turns are dedicated to exploration. However, the estimation uses all the previous exploration phases, so the number of samples for estimation grows linearly in the epoch number. Since players only have estimations for the expected rewards, they can never be sure if a matching is a \( \gamma \)-matching. The purpose of the exploration phase is to help the players become more confident over time.
that the matchings they converge to in the matching phase are indeed $\gamma$-matchings.

In our exploration phase each player picks an arm uniformly at random. This type of exploration phase is common in various multi-player bandit algorithms [17], [24]. However, the nature of what the players are trying to estimate is different.

**Algorithm 1 My Fair Bandit Algorithm**

**Initialization** - Set $V_{n,i} = 0$ and $s_{n,i} = 0$ for all $i$. Define $K$ as the index of the last epoch. Set reset counter $w = 0$ with expiration $e_w = 1$. Let $\varepsilon_0 > 0$.

For each epoch $k = 1, \ldots, K$:

1) **Exploration Phase:**
   a) For the next $[c_1 \log k]$ turns:
      i) Play an arm $i$ uniformly at random from all $M$ arms.
      ii) Receive $r_{n,i}(t)$ and set $\eta_i(a(t)) = 0$ if $r_{n,i}(t) = 0$ and $\eta_i(a(t)) = 1$ otherwise.
      iii) If $\eta_i(a(t)) = 1$ then update $V_{n,i} = V_{n,i} + 1$ and $s_{n,i} = s_{n,i} + r_{n,i}(t)$.
   b) Estimate the expectation of arm $i$ as $\mu_{n,i}^{k} = \frac{s_{n,i}}{V_{n,i}}$ for each $i = 1, \ldots, M$.
   c) Construct confidence intervals for each $\mu_{n,i}^{k}$ as $C_{n,i}^{k} = \frac{1}{\sqrt{\log V_{n,i}}}$.

2) **Matching Phase:**
   a) Update $w \leftarrow w + 1$. If $w = e_w$ then set $\gamma_k = 0$, $w = 0$, $e_w = \left\lceil \frac{3}{w} \right\rceil$ and update $\varepsilon_k = \frac{1}{1 + \log k}$. If $w < e_w$ then $\varepsilon_k = \varepsilon_{k-1}$.
   b) Let $E_{n,\gamma_k}^{k} = \{i | \mu_{n,i}^{k} \geq \gamma_k - C_{n,i}^{k}\}$.
   c) Pick $a_n(t)$ uniformly at random from $E_{n,\gamma_k}^{k}$.
   d) For the next $[c_2 \log k]$ turns:
      i) If $\eta_{a_n(t)}(a(t)) = 1$ keep playing the same arm, that is $a_n(t + 1) = a_n(t)$.
      ii) If $\eta_{a_n(t)}(a(t)) = 0$ then pick $a_n(t + 1)$ uniformly at random from $E_{n,\gamma_k}^{k}$.
   e) Set $\tilde{a}_{k,n} = a_n(t)$.

3) **Consensus Phase:**
   a) If $\eta_{\tilde{a}_{k,n}}(\tilde{a}_k) = 1$ then play $\tilde{a}_{k,n}$ for $M$ turns.
   b) If $\eta_{\tilde{a}_{k,n}}(\tilde{a}_k) = 0$ then play $a_n = 1, \ldots, M$ sequentially.

4) **Exploitation Phase:** For $[c_3 \left\lceil \frac{4}{\gamma} \right\rceil]$ turns, play $\tilde{a}_{k^*}n$ for the maximal $k^*$ such that $\frac{\gamma_k \gamma_{k^*}}{\log v} \leq e_k \leq k$.

End

With a sum of rewards objective, players just need to improve, over time, the accuracy of the estimation of the expected rewards. With max-min fairness each player needs to make a hard (binary) decision whether a certain arm gives more or less than $\gamma$. After the confidence intervals become small enough, if the estimations do fall within their confidence intervals, players can be confident about this hard decision. Under this success event, a matching $a$ is a $\gamma$-matching if all players observe that $\mu_{n,i}^{k} \geq \gamma - C_{n,i}^{k}$. The next lemma bounds the probability that this success event does not occur, so the estimation for epoch $k$ failed.

**Lemma 1 (Exploration Error Probability).** Let $\{\mu_{n,i}^{k}\}$ be the estimated reward expectations using all the exploration phases up to epoch $k$, with confidence intervals $\{C_{n,i}^{k}\}$. Define the minimal gap by

$$\Delta \triangleq \min_{i} \min_{r \neq j} |\mu_{n,i} - \mu_{n,j}|.$$

Define the $k$-th exploration error event as

$$E_{e,k} = \left\{ \exists n, i \mid |\mu_{n,i}^{k} - \mu_{n,i}| \geq C_{n,i}^{k} \text{ or } C_{n,i}^{k} \geq \frac{\Delta}{4} \right\}.$$

Then for all $k > k_0$ for a large enough constant $k_0$ we have

$$\mathbb{P}(E_{e,k}) \leq 3NMe^{-\frac{\Delta^2}{56}}.$$

**Proof.** After the $k$-th exploration phase, the estimation of the expected rewards is based on $T_e(k)$ samples, and

$$T_e(k) \geq c_1 \sum_{i=1}^{k} \log i \geq c_1 \frac{k}{2} \log \frac{k}{2}.$$
where (a) follows for $k \geq k_0$. We conclude that
\[
\mathbb{P}(E_{e,k}) = \mathbb{P}(E_{e,k} | V_m < \frac{T_e(k)}{5M}) \mathbb{P}(V_m < \frac{T_e(k)}{5M}) + \mathbb{P}(E_{e,k} | V_m \geq \frac{T_e(k)}{5M}) \mathbb{P}(V_m \geq \frac{T_e(k)}{5M}) \\
\leq \mathbb{P}(V_m < \frac{T_e(k)}{5M}) + \mathbb{P}(E_{e,k} | V_m \geq \frac{T_e(k)}{5M}) \\
\leq N M \epsilon_{\frac{T_e(k)}{5M}} + 2 N M \epsilon_{\frac{2T_e(k)}{5M}} \quad (11)
\]
where (a) follows for all $k > 0$ such that \( \frac{1}{\sqrt{\log \left( \frac{5M}{4} \right)}} < \frac{1}{k} \), since then \( V_m \geq \frac{T_e(k)}{5M} \) implies that the probability in (10). Inequality (a) also uses (9). Then, (7) follows by using (8) in (11) for a sufficiently large $k$.

V. Matching Phase

In this section, we analyze the matching phase, where the goal is to distributively find $\gamma$-matchings based on the estimated rewards from the exploration phase. We conclude by upper bounding the probability that an optimal matching (i.e., $\gamma^*$-matching) is not played during the exploitation phase. During the matching phase, the rewards of the arms are ignored, as only binary decisions of whether an arm is better or worse than $\gamma$ matters. These binary decisions induce the following bipartite graph:

**Definition 3.** Let $G_k$ be the bipartite graph of players and arms where edge $(n, i)$ exists if and only if $\mu_{n,i}^k \geq \gamma_k - C_{n,i}^k$.

During the $k$-th matching phase, players follow our dynamics to switch arms in order to find a $\gamma_k$-matching in $G_k$. These $\gamma_k$-matchings (up to confidence intervals) are absorbing states in the sense that players stop switching arms if they are all playing a $\gamma_k$-matching. The dynamics of the players induce the following Markov chain:

**Definition 4.** Define $\mathcal{E}_{n,\gamma_k}^k = \{i | \mu_{n,i}^k \geq \gamma_k - C_{n,i}^k\}$. The transition into $a(t+1)$ is dictated by the transition of each player $n$:

1. If $\eta_{n,t}(a(t)) = 1$ then $a_n(t+1) = a_n(t)$ with probability $1$.
2. If $\eta_{n,t}(a(t)) = 0$ then $a_n(t+1) = i$ with probability $\frac{1}{|\mathcal{E}_{n,\gamma_k}^k|}$ for all $i \in \mathcal{E}_{n,\gamma_k}^k$.

Note that the matchings in $G_k$ are $\gamma_k$-matchings only when the confidence intervals are small enough. Next we prove that if a matching exists in $G_k$, then the matching phase will find it with a probability that goes to one. However, we do not need this probability to converge to one, but simply to exceed a large enough constant.

**Lemma 2.** Let $\mathcal{G}_{N,M}$ be the set of all bipartite graphs with $N$ left vertices and $M$ right vertices that have a matching of size $N$. Define the random variable $T(G,a(0))$ as the first time the process of Definition 4, \{a(t)\}, constitutes a matching of size $N$, starting from $a(0)$. Define
\[
\overline{T} = \max_{G \in \mathcal{G}_{N,M}, a(0)} \mathbb{E} \{T(G,a(0))\}. \quad (12)
\]
If $G_k$ permits a matching then the $k$-th matching phase converges to a matching with probability $p \geq 1 - \frac{\overline{T}}{c_2 \log k}$.

**Proof.** We start by noting that the process $a(t)$ that evolves according to the dynamics in Definition 4 is a Markov chain. This follows since all transitions are a function of $a(t)$ alone, with no dependence on $a(t-1), ..., a(0)$ given $a(t)$. Let $\mathcal{M}$ be a matching in $G_k$. Define $\Phi_{\mathcal{M}}(a)$ to be the number of players that are playing in $a$ the arm they are matched to in $\mathcal{M}$. Observe the process $\Phi_{\mathcal{M}}(a(t))$. If there are no colliding players, then $a(t)$ is a matching (potentially different from $\mathcal{M}$) and no player will ever change their chosen arm. Otherwise, for every collision, at least one of the colliding players is not playing their arm in $\mathcal{M}$. There is a positive probability that this player will pick their arm in $\mathcal{M}$ at random and all other players will stay with the same arm. Hence, if $a(t)$ is not a matching, then there is a positive probability that $\Phi_{\mathcal{M}}(a(t+1)) = \Phi_{\mathcal{M}}(a(t)) + 1$. We conclude that every non-matching $a(t)$ has a positive probability path to a matching, making $a(t)$ an absorbing Markov chain with the matchings as the absorbing states. By Markov’s inequality
\[
\mathbb{P}(T(G_k,a_0) \geq c_2 \log k) \leq \frac{\mathbb{E} \{T(G_k,a_0)\}}{c_2 \log k} \quad \leq \frac{\overline{T}}{c_2 \log k}. \quad (13)
\]

Intriguingly, the Bernoulli trials stemming from trying to find a matching in $G_k$ are dependent, as after enough successes, there will no longer be a matching in $G_k$, yielding success probability 0. The next Lemma shows that Hoeffding’s inequality for binomial random variables still applies as long as there are few enough successes, such that there is still a matching in $G_k$.

**Lemma 3.** Consider a sequence of i.i.d. Bernoulli random variables $X_1, \ldots, X_L$ with success probability $p$. For $x < Lp$, consider $S_x = \sum_{i=1}^{x} X_i 1\{\sum_{j<i} X_j < x\}$. Then
\[
\mathbb{P}(S_x < x) \leq e^{-2L((Lp-x)^2)}. \quad (14)
\]

**Proof.** If $S_x = m < x$ then $\sum_{i=1}^{L} X_i < x$, as otherwise the indicators in $S_x$ of the first $x$ indices $i$ where $X_i = 1$ will be active, and so $S_x \geq x$, contradicting $S_k = m < x$. Observe that $\sum_{i=1}^{L} X_i \geq S_x = m$. Now assume for the sake of contradiction that $\sum_{i=1}^{L} X_i > m$. Examine the first $m + 1 \leq x$ indices $i$ where $X_i = 1$, and note that the indicators for these terms will be active in $S_x$, and so $S_x \geq m + 1$. This is a contradiction, hence for $m < x$, $S_x = m$ implies that $\sum_{i=1}^{L} X_i = m$, and so
\[
\mathbb{P}(S_x < x) \leq \mathbb{P} \left( \sum_{i=1}^{L} X_i < x \right) \leq e^{-2L((Lp-x)^2)}. \quad \Box
\]

We conclude this section by proving the main Lemma used to prove Theorem 1. The idea of the proof is to show that if
the past $\frac{k}{2}$ exploration phases succeeded, and enough matching trials succeeded, then a $\gamma^*$-matching was found within the last $\frac{k}{2}$ matching phases. This ensures that a $\gamma^*$-matching is played during the $k$-th exploitation phase.

**Lemma 4** (Exploitation Error Probability). Define the $k$-th exploitation error event $E_k$ as the event where the actions $\tilde{a}_k$ played in the $k$-th exploitation phase are not a $\gamma^*$-matching. Let $k_0$ be large enough such that for all $k > k_0$

$$\varepsilon_k < \frac{\Delta}{4} \text{ and } 1 - \frac{T}{c_2 \log k} - \frac{\log k}{k^6} \geq \frac{3}{4} \sqrt{\frac{T}{10}}. \quad (15)$$

Then for all $k > k_0$ we have

$$\mathbb{P}(E_k) \leq 7NM e^{-\frac{k}{10}} + e^{-\frac{k}{10}}. \quad (16)$$

**Proof.** Define $E_{c,\ell}$ as the event where a matching existed in $G_{c,\ell}$ and was not found in the $\ell$-th matching phase. From Lemma 2 we know that if there is a matching in $G_{c,\ell}$, then the $\ell$-th trial has success probability at least $1 - \frac{T}{c_2 \log k}$. Next we bound from below the number of trials. We define $k_w \geq \lceil \frac{k}{2} \rceil$ as the first epoch since $\lceil \frac{k}{2} \rceil$ where a reset occurred (so $\gamma_{k_w} = 0$). In the worst case the algorithm resets in epoch $\lceil \frac{k}{2} \rceil - 1$. Even still, the algorithm will reset again no later than $k_w \leq \lceil \frac{k}{2} \rceil - 1 + \left\lceil \frac{\log k}{3} \right\rceil \leq \frac{2}{3}k$. The subsequent reset will then happen at $k_{w+1}$, where $k_{w+1} \leq \frac{2}{3}k + \left\lceil \frac{\log k}{3} \right\rceil < k$ for $k > 9$. We conclude that during the past $\frac{k}{2}$ epochs, there exists at least one full period (from reset to reset) with length at least $k_0$. Recall the definition of the $\ell$-th exploration error event $E_{c,\ell}$ in (16). Define the event $A_k = \bigcap_{\ell=\lceil \frac{k}{2} \rceil}^{k_w} E_{c,\ell}$ for which $A'_k = \bigcup_{\ell=\lceil \frac{k}{2} \rceil}^{k_w} E_{c,\ell}$. We define $\beta_w$ as the number of successful trials needed after reset $w$ to reach $\gamma_k \geq \gamma^* - \frac{\Delta}{4}$. Note that $\beta_k < \log k$ since no more than $\log k$ steps of size $\varepsilon_{k_w}$ are needed. Then for all $k > k_0$

$$\mathbb{P}(E_k | A_k) \leq \mathbb{P}(\gamma_k < \gamma^* - \frac{\Delta}{4} | A_k) \leq \mathbb{P}\left(\sum_{\ell=k_w+1}^{k_{w+1}} \mathbb{1}\{E_{c,\ell}\} < \beta_k | A_k\right) \leq e^{-2 \left(1 - \frac{T}{c_2 \log k} - \frac{\log k}{k_{w+1} - k_w}\right)^\gamma (k_{w+1} - k_w)} \leq e^{-\frac{k}{10}} \quad (17)$$

where (a) follows since given $\bigcap_{\ell=\lceil \frac{k}{2} \rceil}^{k_w} E_{c,\ell}$, if $\gamma_k \geq \gamma^* - \frac{\Delta}{4}$ then a $\gamma^*$-matching was found before the $k$-th exploitation phase and $E_k$ did not occur. This follows since at the last success at $\ell \leq k$ we must have had for all $n$ that

$$\mu_{n,a_n} \geq \mu_{n,a_n}^\ell - C_{\epsilon_{k_w}}^\ell \geq \gamma_k - 2C_{\epsilon_{k_w}}^\ell \geq \gamma^* - \frac{\Delta}{4} - \varepsilon_{k_w} - 2C_{\epsilon_{k_w}}^\ell \geq \gamma^* - \Delta$$

which can only happen if $\mu_{n,a_n} \geq \gamma^*$. Inequality (b) in (17) follows by noting that the probability that $\max_{k/2 \leq \ell \leq k} \gamma\ell < \gamma^* - \frac{\Delta}{4}$ with a constant step size (per epoch) $\varepsilon_{k,w}$ implies fewer than $\frac{k - \varepsilon_{k,w}}{\varepsilon_{k,w}}$ successful trials between $k_w$ and $k_{w+1}$. In any trial $\ell \in [k_w, k_{w+1}]$ such that there have been fewer than $\frac{\gamma^* - \Delta}{\varepsilon_{k,w}}$ successes in $[k_w, \ell]$, at least one matching will exist in $G_{c,\ell}$ (an optimal matching $a^*_\ell$), since

$$\mu_{n,a_n} \geq \mu_{n,a_n}^\ell - C_{\epsilon_{k_w}}^\ell \geq \gamma^* - C_{\epsilon_{k_w}}^\ell \geq \gamma^* - \Delta$$

(19)

where (1) follows since for all $k > k_0$, $\varepsilon_k$ is sufficiently small such that $\gamma_k \leq \gamma^* - \frac{\Delta}{4} + \varepsilon_k \leq \gamma^*$. Inequality (c) in (17) follows from Lemma 3 with $\mathbb{P}(1 - \frac{T}{c_2 \log k})$. Inequality (d) follows from $k_{w+1} - k_w \geq \frac{k}{6}$ and (15). Finally, (16) is obtained by:

$$\mathbb{P}(E_k) = \mathbb{P}(E_k | A'_k) \mathbb{P}(A'_k) + \mathbb{P}(E_k | A_k) \mathbb{P}(A_k) \leq \mathbb{P}(E_k | A'_k) + \mathbb{P}(E_k | A_k) \leq 7NM e^{-\frac{k}{10}} + e^{-\frac{k}{10}} \quad (20)$$

where (a) is a union bound of $A'_k = \bigcup_{\ell=\lceil \frac{k}{2} \rceil}^{k_w} E_{c,\ell}$ over Lemma 1 together with (17), and (b) is a geometric sum. □

**VI. Numerical Simulations**

We simulated two multi-armed bandit games with the following expected rewards matrices:

$$U_1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$U_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Given expected rewards $[\mu_{n,i}]$, the rewards are generated as $r_{n,i}(t) = \mu_{n,i} + \varepsilon_{n,i}(t)$ where $\{\varepsilon_{n,i}(t)\}$ are independent and uniformly distributed on $[-0.05, 0.05]$ for each $n, i$. The chosen parameters were $c_1 = 1000$ and $c_2 = 2000$ and $c_3 = 4000$. At the beginning of the $k$-th matching phase, each player played her action from the last exploitation phase if it is in $\mathcal{E}_{n,\gamma_k}$, or a random action from $\mathcal{E}_{n,\gamma_k}$ otherwise. Although it has no effect on the theoretical bounds, it improved the performance in practice significantly. Another practical improvement was achieved by introducing a factor of 0.1 to the confidence intervals, which requires $c_1 > 400$ but does not
Regret

Regret

0

0

1

1

2

6

6

7

8

N

matrix (U

a novel fully distributed algorithm that achieves a near order
player only knows their own actions and rewards. We proposed
different expected rewards for the arms. We assume each
we employed the heterogeneous model where players have
by any player. To allow for a meaningful notion of fairness,
as resources, so as to maximize the minimal reward received
players cooperate to learn how to allocate arms, thought of

Fig. 2. Total regret as function of time, averaged over 100 experiments and
with \( U_2 \) as the expected reward matrix (\( N = 10 \)). The shaded area denotes one standard deviation around the mean. In this scenario only 136 matchings out of the 10! = 3628800 are optimal with minimal utility of 0.4. There are 3798 matchings with 0.3, 16180 with 0.25 62066 with 0.2, 785048 with 0.1 and 2761572 with 0.05. With more players and arms, \( k_0 \) is larger, but players still learn the max-min optimal matching by the sixth epoch. Again, the regret scales logarithmically as guaranteed by Theorem 1.

In Fig. 2 we present the total expected regret versus time, averaged over 100 realizations, with \( U_2 \) as the expected reward matrix (\( N = 10 \)). The shaded area denotes one standard deviation around the mean. In this scenario only 136 matchings out of the 10! = 3628800 are optimal with minimal utility of 0.4. There are 3798 matchings with 0.3, 16180 with 0.25 62066 with 0.2, 785048 with 0.1 and 2761572 with 0.05. With more players and arms, \( k_0 \) is larger, but players still learn the max-min optimal matching by the sixth epoch. Again, the regret scales logarithmically as guaranteed by Theorem 1.

\[ T \geq \sum_{k=1}^{K-1} \left( c_1 \log k + c_2 \log k + M + c_3 \left( \frac{4}{3} \right)^k \right) \]

\[ \geq 3c_3 \left( \left( \frac{4}{3} \right)^K - \frac{4}{3} \right) \]

\( K \) is upper bounded by \( K \leq \log_\frac{4}{3} \left( \frac{T}{3c_3} + \frac{4}{3} \right) \). Let \( k_0 \) be a constant epoch index that is large enough for the bounds of Lemma 1, Lemma 4 and inequality (c) in (22) to hold. Intuitively, this is the epoch after which the matching phase duration is long enough, the step size is small enough, and the confidence intervals are sufficiently tight. Define \( E_k \) as the event where a \( \gamma^* \)-matching is not played in the \( k \)-th exploitation phase. We now bound the total expected regret of epoch \( k > k_0 \), denoted by \( R_k \):

\[ R_k \leq M + (c_1 + c_2) \log k + \mathbb{P} (E_k) c_3 \left( \frac{4}{3} \right)^k + 3 \]

\[ \leq M + (c_1 + c_2) \log k + 3 \]

\[ + \left( 7NM e^{-\frac{c_1}{M}} k + e^{-\frac{2c_1}{M}} \right) c_3 \left( \frac{4}{3} \right)^k \]

\[ \leq M + (c_1 + c_2) \log k + 8NMC_3 \beta k \]

\[ \leq M + 2(c_1 + c_2) \log k \]

where (a) uses Lemma 4, (b) follows for some constant \( \beta < 1 \) since \( e^{-\frac{2c_1}{M}} \leq \frac{3}{2} \) and \( c_2 \geq 4 \) and (c) follows for \( k > k_0 \). We conclude that, for some additive constant \( C \),

\[ R(T) = \sum_{k=1}^{K} R_k \leq \sum_{k=1}^{k_0} \left( c_1 + c_2 \log k + c_3 \left( \frac{4}{3} \right)^k + 3 \right) \]

\[ + MK + 2 \sum_{k=k_0+1}^{K} (c_1 + c_2) \log k \]

\[ \leq C + MK + 2(c_1 + c_2) K \log K \]

where (a) follows by completing the last epoch to a full epoch which only increases \( R_K \), and by using (22). Then, we obtain (4) by upper bounding \( K \leq \log_\frac{4}{3} \left( \frac{T}{3c_3} + \frac{4}{3} \right) \).
