Possible alternative mechanism to SUSY: conservative extensions of the Poincaré group

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Abstract  A group theoretical mechanism is outlined, which can indecomposably extend the Poincaré group by the compact internal (gauge) symmetries at the price of allowing some nilpotent (or, more precisely: solvable) internal symmetries in addition. Due to the presence of this nilpotent part, the prohibitive argument of the well known Coleman-Mandula and McGlenn no-go theorems do not go through. In contrast to SUSY or extended SUSY, in our construction the symmetries extending the Poincaré group will be all internal, i.e. they do not act on the spacetime, merely on some internal degrees of freedom — hence the name: conservative extensions of the Poincaré group. Using the Levi decomposition and O’Raifeartaigh theorem, the general structure of all possible conservative extensions of the Poincaré group is outlined, and a concrete example group is presented with U(1) being the compact gauge group component. It is argued that such nilpotent internal symmetries may be inapparent symmetries of some more fundamental field variables, and therefore do not carry an ab initio contradiction with the present experimental understanding in particle physics. The construction is compared to (extended) SUSY, since SUSY is somewhat analogous to the proposed mechanism. It is pointed out, however, that the proposed mechanism is less irregular in comparison to SUSY, in certain aspects. The only exoticity needed in comparison to a traditional gauge theory setting is that the full group of internal symmetries is not purely compact, but is a semi-direct product of a nilpotent and of a compact part.
1 Introduction

In Lagrangian field theories it is well understood that larger amount of symmetries of the Lagrangian gives less room for variants of the theory. In particular, the larger amount of direct-indecomposable (unified) symmetries reduce the number of possible free coupling parameters. This phenomenon motivated the search for unified symmetries in field theory, meaning that a plausible direct-indecomposable symmetry group was being searched for, which contained the known symmetry groups as subgroups. When it comes to building relativistic field theories to be applied in particle physics, the known symmetry groups are the Poincaré group and the compact internal (gauge) symmetries of the Standard Model, commuting with each other. Therefore, a rather plausible idea was to try to find a direct-indecomposable symmetry group, which contains Poincaré symmetries and compact internal symmetries, indecomposably. In 1964 it was realized by McGlinn [1] that whenever the compact internal symmetries are semi-simple, this is group theoretically impossible. This motivated the work of O'Raifeartaigh in 1965 [2] to try to understand all possible group extensions of the Poincaré symmetries. The pertinent O'Raifeartaigh theorem made it clear that the Lie group theoretical possibilities for a direct-indecomposable extension of the Poincaré group is rather limited. Historically, at the time of the publication of O'Raifeartaigh theorem, no constructive examples for the potentially allowed direct-indecomposable Poincaré group extensions were known. For instance supersymmetry (SUSY) was not known at the time, and the conformal Poincaré group, being a direct-indecomposable extension of the Poincaré group, was not in the physics folklore. Therefore, the potentially allowed Poincaré group extensions by means of the O'Raifeartaigh theorem were talked away by a little bit handwaving physics arguments. Not much later, in 1967 the famous Coleman-Mandula theorem [3] was published, stating that given some plausible assumptions, a unification of the Poincaré group with purely compact internal symmetries is not possible in the framework of quantum field theory. These attempts were historically reviewed in [4]. A few years later, the famous paper of Wess and Zumino was published [5], implicitly providing an example Lie group (the super-Poincaré group) which is an indecomposable extension of the Poincaré group, and thus providing an explicit example for one of the cases of O'Raifeartaigh theorem, allowing a direct-indecomposable extension of the Poincaré group. Motivated by this, Haag, Lopuszański and Sohnius [6] generalized the Coleman-Mandula theorem also allowing for super-Poincaré transformations. Since that work, the so called super-Lie algebra view of those transformations is the most popular in the literature, making it less obvious to see the underlying ordinary Lie group structure of the super-Poincaré transformations, and their relations to O’Raifeartaigh theorem. In the recent years it was re-understood that there do exist also other direct-indecomposable extensions of the Poincaré group. A rather well-understood example is the conformal Poincaré group, being
isomorphic to SO(2, 4), but also others have been found \cite{7, 8, 9, 10}, some of which can lead to field theories which may not be ab initio pathological. They bypass the Coleman-Mandula theorem by weakening some of its assumptions, for instance allowing for symmetry breaking.

In this paper a newly found direct-indecomposable Poincaré group extension \cite{11, 12} is discussed, which contains a Poincaré component, a compact internal group component, and a nilpotent internal group component. From the Lie group theoretical point of view, it resembles to the (extended) super-Poincaré group, since in its Levi decomposition its radical is a nilpotent Lie algebra. However, in contrast to SUSY, this group respects vector bundle structure of fields, i.e. all the non-Poincaré symmetries act spacetime pointwise on some internal degrees of freedom. This implies that symmetry breaking is not necessary in order to make this new symmetry concept to harmonize with a gauge-theory-like setting, where vector bundle structure of fundamental fields is essential to preserve. Hence, we call these constructions conservative extensions of the Poincaré group.

The outline of the paper is as follows. In Section 2 the general structure of Lie groups is recalled in the light of Levi decomposition theorem. In Section 3 the O’Raifeartaigh classification theorem on Poincaré group extensions is recalled. In Section 4 the structure of conservative extensions of the Poincaré group is outlined. In Section 5 the Lie algebra of the concrete conservative Poincaré group extension defined in \cite{11, 12} is presented.

2 General structure of Lie groups: Levi decomposition

In every finite dimensional real Lie algebra, one has the Killing form, being a real valued bilinear form defined by the formula \( x \cdot y := \text{Tr}(\text{ad}_x \text{ad}_y)\) for two elements \( x, y \) of the Lie algebra. The Levi decomposition theorem \cite{13, 14} states that the structure of a generic real finite dimensional connected and simply connected Lie group is as follows:

\[
\begin{align*}
\mathfrak{g} & = \mathfrak{r} \times (L_1 \times \cdots \times L_n) \\
\text{Lie group} & \text{degenerate directions of Killing form} & \text{non-degenerate directions of Killing form} \\
& \text{(called to be the radical)} & \text{(called to be the Levi factor)}
\end{align*}
\]

A subgroup spanned by the non-degenerate directions of the Killing form is called the Levi factor or semisimple part. It falls apart to direct product of subgroups which contain no proper normal subgroups, and are called the simple components. The normal (invariant) subgroup spanned by the degenerate directions of the Killing form is called the radical or solvable part. The radical \( R \) can also be equivalently characterized by the property that the Lie algebra \( r \) of \( R \) has terminating derived series. Namely, with the definition \( r^0 := r \), \( r^k := [r^{k-1}, r^{k-1}] \), there exists a finite \( k \) such that \( r^k = \{0\} \). A special case
is when $R$ is said to be *nilpotent*: in this case there exists a finite $k$ such that for all $x_1, \ldots, x_k \in r$ one has $\text{ad}_{x_1} \cdots \text{ad}_{x_k} = 0$. The extreme case is when $R$ is said to be *abelian*: in this case for all $x \in r$ one has $\text{ad}_x = 0$.

Whenever also non-simply connected or non-connected Lie groups are considered, their generic structure can be slightly more complex:

$$E_{\text{Lie group}} = \left( \left( \frac{R_{\text{radical}} \rtimes (L_1 \times \cdots \times L_n)}{I_{\text{discrete}}} \right) \times \frac{\mathcal{J}_{\text{discrete}}}{J_{\text{discrete}}} \right)$$

where $I$ is some discrete normal subgroup of $R \rtimes (L_1 \times \cdots \times L_n)$ and $J$ is some discrete subgroup of the outer automorphisms of the quotient group $(R \rtimes (L_1 \times \cdots \times L_n))/I$. It is not complicated to see that whenever a Lie group is injectively embedded into another, then its Lie algebra must be injectively embedded into the Lie algebra of the other. Thus, for studying necessary condition for injective embedding of Lie groups, one first needs to study the injective embeddings of Lie algebras, or equivalently, of connected and simply connected Lie groups. From now on, by Lie groups we shall always mean connected and simply connected ones, i.e. the universal covering groups.

Levi decomposition theorem can be illustrated with the Poincaré group:

$$\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$$

### 3 A classification of Poincaré group extensions

A classification scheme of Poincaré group extensions was outlined by O’Raifeartaigh [2], using the Levi decomposition theorem. It is based on the simple observation that when injectively embedding a finite dimensional real Lie algebra into another, then the Levi factor of the smaller Lie algebra cannot intersect with the radical of the larger one. This implies the following disjoint possibilities for a connected and simply connected extension $E = R \rtimes (L_1 \times \cdots \times L_n)$ of the Poincaré symmetries $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$.

A One has $E = \mathcal{P} \rtimes \{\text{some other Lie group}\}$, i.e. no unification occurs.

B One has not $A$ and $\mathcal{T} \subset R$ and $\mathcal{L} \subset L_1$, meaning that the translations $\mathcal{T}$ are injected into the radical $R$ and the homogeneous Lorentz group $\mathcal{L}$ is injected into one of the simple components $L_1$ of $E$.

C One has $(\mathcal{T} \rtimes \mathcal{L}) \subset L_1$, i.e. the entire Poincaré group is injected into one of the simple components $L_1$ of $E$.

Examples for case B are detailed in [12], namely the super-Poincaré group or the extended super-Poincaré group [3, 15, 16], as well as the extensions of the Poincaré group proposed by us [12]. Example for case C is the conformal
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Poincaré group, being isomorphic to SO(2, 4). However, also more complicated examples are being constructed [7, 8, 9] in the literature.

Knowing O’Raifeartaigh theorem, the argument of Coleman-Mandula theorem in case of a finite dimensional Poincaré group extension can be greatly simplified. First, Coleman-Mandula assumes implicitly that symmetry breaking is not present, which excludes case C. Secondly, it implicitly assumes that one has a positive definite invariant scalar product on the non-Poincaré directions of the Lie algebra, which excludes case B (along with SUSY, for instance). In case of SUSY or our Poincaré group extensions, the pertinent invariant scalar product is merely positive semidefinite, which provides a backdoor to the otherwise prohibitive argument.

4 Conservative extensions of the Poincaré group

As outlined in [12], the super-Poincaré group or extended super-Poincaré group cannot be considered as a vector bundle automorphism group with the spacetime being the base manifold. This implies that in a supersymmetric model a heavy symmetry breaking needs to be introduced in order to recover a gauge-theory-like setting, so characteristic to the Standard Model. Also in [12] the question is asked: what are those finite dimensional direct-indecomposable extensions $E$ of the Poincaré group $\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$, which respect the vector bundle structure of fundamental fields as well as the Lorentz metric of the spacetime? Technically, this means that one has $E = \mathcal{T} \rtimes \{ \text{some pointwise acting symmetries} \}$ with a surjective homomorphism $\{ \text{some pointwise acting symmetries} \} \rightarrow \mathcal{L}$ onto the Lorentz group. The answer [12] is a simple consequence of the Levi decomposition / O’Raifeartaigh theorem and of the definition of semidirect product:

$$E = (\underbrace{T}_{\text{translations}} \times \underbrace{\mathcal{N}}_{\text{solvable internal symmetries}}) \rtimes (\underbrace{G_1 \times \ldots \times G_m}_{\text{semisimple internal symmetries}} \times \underbrace{\mathcal{L}}_{\text{Lorentz symmetries}}) \quad (4)$$

must hold, where the semisimple internal symmetries $G_1 \times \ldots \times G_m$ commute with the translations $\mathcal{T}$, the Lorentz symmetries $\mathcal{L}$ have the canonical adjoint action on the translations $\mathcal{T}$, but the invariant subgroup of solvable internal symmetries $\mathcal{N}$ does not commute with the Lorentz symmetries nor with the semisimple internal symmetries. If one requires in addition that there exists a positive semidefinite invariant bilinear form on the Lie algebra of the non-Poincaré symmetries, then it also follows that $G_1 \times \ldots \times G_m$ is compact.

(Such a requirement is motivated by the positive energy condition for gauge fields.) With this requirement, the full internal symmetry group of such a Poincaré group extension shall have the structure $\{ \text{solvable} \} \rtimes \{ \text{compact} \}$. These kind of Poincaré group extensions we named conservative extensions, and are seen to have a number of rather favorable properties [12]: they are
direct-indecomposable, preserve causal structure of the spacetime, preserve vector bundle structure of fundamental fields, obey positive energy condition etc. Ideally, one could look for such a setting in which case the group of compact internal symmetries is identical to the Standard Model gauge group $U(1) \times SU(2) \times SU(3)$.

It is not difficult to see that conservative extensions of the Poincaré group do exist, i.e. that our definition is not empty. Take, for instance, the complexified Schrödinger Lie group, which is isomorphic to $H_3(\mathbb{C}) \rtimes SL(2, \mathbb{C})$. Here $H_3(\mathbb{C})$ denotes the complexified Heisenberg Lie group with three generators, being the lowest dimensional complex non-abelian nilpotent Lie group. Clearly, from this semi-direct product there exists a homomorphism onto $SL(2, \mathbb{C})$ and therefore also onto the homogeneous Lorentz group $L$, which acts canonically on the group of spacetime translations $\mathcal{T}$ in its adjoint representation. With these subgroup actions, the group $(\mathcal{T} \times H_3(\mathbb{C})) \rtimes L$ is uniquely well-defined and is direct-indecomposable. (Note that from the Lie algebra point of view, one has $SL(2, \mathbb{C}) \equiv L$). This provides the simplest conservative extension of the Poincaré group, and the non-Poincaré symmetries span a nilpotent Lie group $H_3(\mathbb{C})$, being part of the radical.

An other example is constructed in [11, 12], which is expected to be more interesting for physics. It contains a Poincaré component, a compact internal group component ($U(1)$ in the example), and unavoidably a nilpotent internal group component. In particular, it has the group structure $(\mathcal{T} \times N) \rtimes (U(1) \times L)$, where $N$ is a 20 dimensional real nilpotent Lie group, the Lorentz group $L$ acts with the canonical adjoint action on the translations $\mathcal{T}$, and both the compact $U(1)$ component and the Lorentz group component $L$ has non-vanishing adjoint action on $N$, which provides the direct-indecomposability. Clearly, it is essential in the construction that the radical $\mathcal{T}$ of the Poincaré group is extended by $N$, without which such a direct-indecomposability is not possible according to O’Raifeartaigh theorem. Also note, that the construction resembles to (extended) super-Poincaré group as outlined in [12], with the important difference that in case of the (extended) super-Poincaré group the translations are direct-indecomposably part of the nilpotent symmetries, called to be the group of supertranslations, forming a direct-indecomposable two-step nilpotent Lie group. In case of our construction, however, the translations are direct-decomposable from other symmetries within the radical, which makes it a conservative extension of the Poincaré group, in contrast to (extended) SUSY. It is also an important piece of information that the concrete conservative extension of the Poincaré group proposed in [11, 12] can be shown to have faithful unitary representations on some separable complex Hilbert space.

An important feature of the conservative extensions of the Poincaré group $\mathcal{P}$ is that there exists a homomorphism:
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\[
\mathcal{N} \cong \left( \mathcal{G}_1 \times \ldots \times \mathcal{G}_m \times \mathcal{P} \right)
\]

direct-indecomposable conservative extension of the Poincaré group, acting on fundamental field degrees of freedom

\[
\longrightarrow \mathcal{G}_1 \times \ldots \times \mathcal{G}_m \times \mathcal{P}
\]

observed direct-decomposable symmetries, acting on some derived field quantities which are function of fundamental degrees of freedom

and potentially can explain a Standard Model-like gauge theory setting from a direct-indecomposable fundamental symmetry, without a breaking of it.

5 Commutation relations of the concrete example

In this section the commutation relations of the generators of the Lie algebra of our concrete example group [11, 12] is outlined. The pertinent direct-indecomposable conservative extension of the Poincaré group is the automorphism group of some finite dimensional unital associative algebra valued classical fields over the four dimensional spacetime. Similar algebra valued field construction was tried by Anco and Wald in the end of '80-s [17], but they could not achieve the goal of direct-indecomposability due to the too simple structure of the algebra of fields which they applied.

In the followings \(S\) shall denote a complex two-dimensional vector space ("spinor space"), and \(S^*, \bar{S}, \bar{S}^*\) shall denote its dual, complex conjugate, complex conjugate dual vector space, respectively. Let us consider the complex unital associative algebra \(A(S^*) \otimes A(S^*)\), where \(A()\) denotes exterior algebra formation. Observe that this algebra also has an antilinear involution defined by the complex conjugation, which is compatible with the algebraic product in the sense that \(\bar{x}y = \bar{x} \bar{y}\) holds for any two algebra elements \(x, y\). We shall call a finite dimensional complex unital associative algebra \(A\) together with an antilinear involution \((\cdot)^+\) obeying \((xy)^+ = x^+ y^+\) a spin algebra whenever the pair \((A, (\cdot)^+)\) is isomorphic to \((A(S^*) \otimes A(S^*), (\cdot))\). The antilinear involution \((\cdot)^+\) (or, \((\cdot)\)) shall be referred to as charge conjugation. Thus, a spin algebra \(A\) is (not naturally) isomorphic to the concrete spin algebra \(A(S^*) \otimes A(S^*)\) with spinorial realization. In the followings, we shall often use a representation \(A \cong A(S^*) \otimes A(S^*)\) so that the simple formalism of traditional two-spinor calculus can be used.

For the sake of simplicity, we shall give our construction in the flat spacetime limit. Let \(\mathcal{M}\) denote a four real dimensional affine space, modeling a
(flat) spacetime manifold, and let \( T \) be its underlying vector space (“tangent space”). Take the trivial vector bundle \( A(M) := M \times A \). Our direct-
indecomposable conservative Poincaré group extension containing also \( U(1) \) shall be nothing but the automorphism group of the algebra of the sections of the \( A(M) \), i.e. of the spin algebra valued fields \([11, 12]\). In the follow-
ings Penrose abstract indices shall be used for the spacetime degrees of free-
dom and for the spinor degrees of freedom, as usual in the General Rela-
tivity literature \([13, 19]\). The symbol \( \nabla_a \) shall denote the affine covariant
derivation of the affine space \( M \). Also, given a point \( o \) (“origin”) of \( M \),
the symbol \( X_o \) shall denote the vectorization map against \( o \), which is the
vector field \( X_o : M \to T, x \mapsto (x-o) \). Let in the spinorial represen-
tation \( \sigma^{AA'}_a \) denote the usual Infeld-Van der Waerden symbol, also called
Pauli injection, or soldering form. It is some preferred injective linear map
\( T \to \text{Re} (\hat{S} \otimes \hat{S}) \), and is shown in \([11, 12]\) to be \( \text{Aut}(A) \)-invariant. Its inverse
map is denoted by \( \sigma^A_{aAA'} \). Let \( \omega_{[A'B'][C|D]} \) be a positive maximal form from
\( A \). Then, it is well-known that \( g(\sigma, \omega)_{ab} := \sigma_{aA}^{AA'} \sigma_{bB}^{BB'} \omega_{[A'B'][AB]} \) is a Lorentz
signature metric on \( T \), and its inverse metric is denoted by \( g(\sigma, \omega)^{ab} \). The symbol \( \Sigma(\sigma)_a^{bA} := \left( \sigma_{a}^{AC'} \sigma_{B}^{BC'} - g(\sigma, \omega)^{ab} g(\sigma, \omega)^{ac} \sigma_{c}^{AC'} \right) \) is called
the spin tensor in the literature, and can be considered as the gene-
rators of the \( \text{SL}(2, \mathbb{C}) \) group, as it is well-known. It can uniquely act on the full mixed
tensor algebra of \( S, S^*, \hat{S}, \hat{S}^* \) by requiring vanishing action on scalars, commutativity with duality form, realness of \( i \Sigma(\sigma)_a^{b} \), and Leibniz rule over tensor
product. Given a concrete spinorial representation \( A \equiv \Lambda(S^*) \otimes \Lambda(S^*) \), thus
the spin tensor can be uniquely extended to \( A \) as an algebra derivation valued
tensor \( \Sigma(\sigma)_a^{b} \), and it shall have vanishing action on scalars, shall obey Leibniz
rule against algebra multiplication of \( A \), and shall have realness of \( i \Sigma(\sigma)_a^{b} \)
against the charge conjugation within \( A \). The spin tensor \( \Sigma(\sigma)_a^{b} \), however,
is not invariant to the full action of \( \text{Aut}(A) \); the nilpotent normal subgroup
within \( \text{Aut}(A) \) which do not preserve the subspaces \( A_{pq} := \wedge^p \hat{S}^* \otimes \wedge^q \hat{S}^* \) of
pure \( p, q \)-forms do not preserve \( \Sigma(\sigma)_a^{b} \). That is, the definition of \( \Sigma(\sigma)_a^{b} \) is relat-
e to a concrete spinorial representation \( A \equiv \Lambda(S^*) \otimes \Lambda(S^*) \), which is also not preserved by the pertinent nilpotent normal subgroup.

Introduce the differential operators \( J_{o,ab} := \left( X_o^a i\nabla^b - X_o^b i\nabla^a \right) + \frac{1}{2} \Sigma_{ab} \) and \( P_o := i\nabla_o \) over the sections of the spin algebra bundle \( A(M) \), i.e.
over the spin algebra valued fields. They are called the \( o \)-angular momentum
and momentum operators, respectively, and are known to provide a faithful representation of the Poincaré Lie algebra in the Lie algebra of
differential operators on the sections of \( A(M) \). Given a concrete spinorial
representation \( A \equiv \Lambda(S^*) \otimes \Lambda(S^*) \), for each complex number \( c \) intro-
duce the unique algebra derivation operator which acts as \( \zeta_c : (\xi_{A'}) := c \xi_{A'} \) for all \( \xi_{A'} \in A_{1|0} = \wedge^1 \hat{S}^* \otimes \wedge^0 \hat{S}^* \). By construction, the map \( i \varphi \mapsto \zeta_{\varphi} \ (\varphi \in \mathbb{R}) \) provides a faithful representation of the Lie algebra of the \( U(1) \) group on the
algebra derivations of the spin algebra \( A \), and thus on the algebra derivations
of the spin algebra valued fields. Similarly to the spin tensor \( \Sigma_{ab} \), the defini-
tion of the operator $\zeta$ depends on a concrete chosen spinorial representation $A \equiv A(S^* \otimes A(S^*)$. By construction, the operators $P_a, J_{o\, cd}, \zeta$ provide a faithful representation of the Lie algebra of $\mathcal{P} \times U(1)$.

The direct-indecomposable unification of $\mathcal{P}$ and of $U(1)$ shall happen because $\text{Aut}(A)$ has a nilpotent normal subgroup on which both $\mathcal{P}$ and $U(1)$ has nonvanishing adjoint action. The generators of this nilpotent normal subgroup shall be detailed as follows. Take a concrete chosen spinorial representation $A \equiv A(S^* \otimes A(S^*)$. Take any element $\beta \in \text{Re} \left( A_{12} \otimes A_{10}^{\ast} \otimes A_{21} \otimes A_{11}^{\ast} \right) \subset \text{Re} \left( \text{Lin}(A) \right)$. Such an element, in the spinorial notation, can be represented as $(\beta_{B'[CD]}^{A'}, \bar{\beta}_{B'(CD')}^{A'})$, uniquely determined by the spinor tensor $\beta_{B'(CD)}^{A'}$.

Such an element $\beta$ defines a $A_{10} \rightarrow A_{12}$ linear operator via the formula $\bar{\xi}_A \mapsto \beta_{B'[CD]}^{A'} \bar{\xi}_A$. Direct verification shows that this can be uniquely extended as an algebra derivation operator $\nu_\beta$ of $A$, via requiring vanishing on scalars $A_{50}$, realness, and Leibniz rule. Also, for all elements $a \in \text{Re}(A)$, the linear map $\text{ad}_a : A \rightarrow A$ is an algebra derivation of $A$, called inner derivation. They can be uniquely parameterized by real elements not in the center of $A$, i.e. with elements $a \in \text{Re}(A_{10} \oplus A_{10}^{\ast} \oplus A_{11} \oplus A_{21} \oplus A_{12})$.

Let $\beta, \beta' \in \text{Re} \left( A_{12} \otimes A_{10}^{\ast} \oplus A_{21} \otimes A_{11}^{\ast} \right) \subset \text{Re} \left( \text{Lin}(A) \right)$ and take the elements $a, a' \in \text{Re}(A_{10} \oplus A_{10}^{\ast} \oplus A_{11} \oplus A_{21} \oplus A_{12})$ and $\varphi, \varphi' \in \mathbb{R}$, regarded as constant fields over the spacetime manifold $M$. Then the relations

\[
\begin{align*}
[\text{ad}_a, \text{ad}_{a'}] &= \text{ad}_{[a,a']}, \\
[\text{ad}_a, \nu_\beta] &= -\text{ad}_{\nu_\beta(a)}, \\
[\text{ad}_a, \zeta_{\varphi}] &= -\text{ad}_{\zeta_{\varphi}}(a), \\
[\text{ad}_a, J_{o\, cd}] &= -\text{ad}_{J_{o\, cd}(a)}, \\
[\text{ad}_a, P_e] &= 0, \\
[\nu_\beta, \nu_\beta'] &= 0, \\
[\nu_\beta, \zeta_{\varphi}] &= -\nu_{[\zeta_{\varphi}, \beta]}, \\
[\zeta_{\varphi}, J_{o\, cd}] &= -\nu_{J_{o\, cd}; \beta}, \\
[\zeta_{\varphi}, P_e] &= 0, \\
[\zeta_{\varphi}, \zeta_{\varphi'}] &= 0, \\
[J_{o\, cd}, \zeta_{\varphi}] &= 0, \\
[J_{o\, cd}, P_e] &= 0, \\
[J_{o\, cf}, J_{o\, df}] &= \pm i g_{de} J_{o\, cf} - \pm i g_{ce} J_{o\, df} + \pm i g_{cf} J_{o\, de} - \pm i g_{df} J_{o\, ce}, \\
[J_{o\, cd}, P_e] &= \pm i g_{ce} P_c - \pm i g_{de} P_d, \\
[P_c, P_d] &= 0
\end{align*}
\]

are seen to hold, where the operators $\text{ad}_a, \nu_\beta, \zeta_{\varphi}, J_{o\, cd}, P_e$ are regarded as acting on the smooth sections of $A(M)$, i.e. on spin algebra valued fields. These operators are algebra derivation valued on the algebra of smooth sections of $A(M)$, where in a concrete spinor representation $A \equiv A(S^* \otimes A(S^*)$, these fields can be regarded as a 9-tuple of spinor tensor fields.
\[
\left( \varphi, \xi^+(A')^1, \xi^-(A')^2, \xi^+(\bar{B}'C'), \xi^-(\bar{B}'C'), \chi(\bar{B}'C'A), \chi^-(\bar{B}'C'A), \omega(\bar{A}'\bar{B}'[CD]) \right) \tag{7}
\]

in the usual spinor index notation. The symmetry generators in Eq. (6) respect the vector bundle structure of \(A(M)\), the spin algebra structure of the fibers of \(A(M)\), as well as the soldering form \(\sigma^A_{AA'}\) viewed as a \(T^* \otimes \text{Re}(\Lambda^1)\) valued constant field over the affine space \(M\). They also happen to preserve the constant maximal forms \(\omega(\bar{A}'\bar{B}'[CD])\), i.e. constant sections of value in \(A_{22} \equiv \wedge^2 S^* \otimes \wedge^2 S^*\). If an additional generator, i.e. the operator \(\rho \mapsto \zeta_\rho\) (\(\rho \in \mathbb{R}\)) is also included among the Lie algebra generators of Eq. (6), then also the generators of the constant Weyl (conformal) rescalings of the flat space-time metric \(g_{ab}\) is included in the Lie algebra, and in that case the maximal forms are not preserved, but acted on with the Weyl rescalings.

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