On Analytic Solvability and Hypoellipticity
For $\bar{\partial}$ and $\bar{\partial}_b$

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1 Introduction

For each $0 \leq p, q \leq n$, we denote by $\Lambda^{p,q}(\mathbb{C}^n)$ the bundle of $(p, q)$-forms on $\mathbb{C}^n$. Let $U$ be any open set in $\mathbb{C}^n$. We denote by $E^{p,q}(U)$ the space of smooth sections of $\Lambda^{p,q}(\mathbb{C}^n)$ over $U$. For any smooth hypersurface $M$ in $\mathbb{C}^n$, we denote by $I^{p,q}(M)$ the ideal in $\Lambda^{p,q}(\mathbb{C}^n)$ generated by $\rho$ and $\bar{\partial}\rho$, where $\rho : \mathbb{C}^n \to \mathbb{R}$ is any smooth function that vanishes on $M$ and whose gradient is nowhere zero on $M$. Let $\Lambda^{p,q}(M)$ denote the orthogonal complement, with respect to the standard Hermitian metric on $\mathbb{C}^n$, of $I^{p,q}(M)$ in $\Lambda^{p,q}(\mathbb{C}^n)|_M$.

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary $\partial\Omega$, equipped with some defining function $\rho \in C^\omega$ in a neighborhood of $\partial\Omega$. Let $L^2(\partial\Omega)$ denote the Lebesgue space of measurable complex-valued functions on $\partial\Omega$ that are square integrable with respect to surface measure, and let $H^2 \subset L^2(\partial\Omega)$ be the subspace of all CR functions, that is, of functions annihilated by $\bar{\partial}_b$. Let $S : L^2(\partial\Omega) \to H^2$ be the Szegő projection. Let $L^2(\Omega)$ be the Lebesgue space over $\Omega$, with respect to Lebesgue measure on $\mathbb{C}^n$, and let $A^2(\Omega)$ be the subspace of all holomorphic functions. Let $P : L^2(\Omega) \to A^2(\Omega)$ be the Bergman projection. We denote by $C^\omega(\partial\Omega)$ the space of all real analytic functions over $\partial\Omega$, and by $C^\omega(\overline{\Omega})$ the space of all real analytic functions over $\overline{\Omega}$. Let $C^\omega_{(p,q)}(\Omega)$ denote the space of all $(p, q)$-forms with coefficients in $C^\omega(\overline{\Omega})$, and let $C^\omega_{(p,q)}(\partial\Omega)$ denote all forms in $\Lambda^{p,q}(\partial\Omega)$ with real analytic coefficients on $\partial\Omega$.

There exist [CH1] bounded pseudoconvex domains $\Omega \subset \mathbb{C}^2$ with real analytic boundaries, for which the Szegő projection does not map $C^\omega(\partial\Omega)$ to $C^\omega(\partial\Omega)$. Although it has not yet been proved, we think it highly likely that the Bergman projection fails to preserve $C^\omega(\overline{\Omega})$ for those same domains. An anal-

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ogous counterexample for the $C^\infty$ category has been established more recently
[CH2].

As is well known, the Szegö and Bergman projections are closely related to
the $\partial b$ and $\Box$ equations and to the $\overline{\partial}$-Neumann problem, respectively. Let $N_b$
denote the tangential Neumann operator and $N$ the Neumann operator. Thus
$\Box N_b = I$ on $\Lambda^{p,q}(\partial\Omega)$ and $\Box N = I$ on $\Lambda^{p,q}(\Omega)$. Then these are related to
the Szegö and Bergman projections respectively by the following formulas of Kohn
[FK]:

$$S = I - \overline{\partial}_b N_b \overline{\partial}_b, \quad P = I - \overline{\partial}^* N \overline{\partial}.$$ 

Here $\overline{\partial}_b$, $\overline{\partial}$ denote the operators adjoint to $\overline{\partial}_b$ and $\overline{\partial}$, respectively.  

The counterexample of [CH1] for the Szegö projection thus implies that for
bounded, pseudoconvex, real analytic domains $\Omega \subset \mathbb{C}^2$, for a general $(0,1)$ form
$f$ in $C^\omega(\partial\Omega)$ belonging to the range of $\overline{\partial}_b$, the canonical solution
$u = \overline{\partial}_b N_b f$ of $\overline{\partial}_b u = f$ need not always belong to $C^\omega(\partial\Omega)$. It is then a natural question,
posed to us by N. Sibony, whether there exists a better solution of $\overline{\partial}_b u = f$,
that does belong to $C^\omega(\partial\Omega)$. This is only an issue in the context of global
existence, since the Cauchy-Kowalevska theorem guarantees local $C^\omega$ solvability
given $C^\omega$ data. Since it is likely that analogous counterexamples to global real
analytic regularity will eventually be established in higher dimensions and for
the $\overline{\partial}$–Neumann problem, the corresponding questions for those contexts are
of interest as well. One aim of this paper is to answer these questions in the
affirmative.

**THEOREM 1.1** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with real
analytic boundary, and suppose that $0 \leq p \leq n$ and $0 < q \leq n$. Let $f \in C^\omega_{(p,q)}(\Omega)$
be $\overline{\partial}$-closed. Then there is $u \in C^\omega_{(p,q-1)}(\Omega)$ such that $\overline{\partial} u = f$ in $\Omega$.

**THEOREM 1.2** Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with real
analytic boundary. Then

(i) If $n \geq 3$ and $f \in C^\omega_{(0,1)}(\partial\Omega)$ is $\overline{\partial}_b$-closed, there exists $u \in C^\omega(\partial\Omega)$
satisfying $\overline{\partial}_b u = f$.

(ii) If $n = 2$ and $f \in C^\omega_{(0,1)}(\partial\Omega) \cap \text{Range}(\overline{\partial}_b)$, then there exists $u \in C^\omega(\partial\Omega)$
satisfying $\overline{\partial}_b u = f$.

Our second aim is to examine the connection between global regularity for
the Szegö and Bergman projections on the one hand, and the corresponding
Neumann operators on the other. In the $C^\infty$ category there is already known
rigorously to be a close connection between the two [BS]. Theorems 1.1 and 1.2
have the following direct consequences. Denote by $N_{p,q}$ the Neumann operator

1In the case of $\mathbb{C}^2$, we define the tangential Neumann operator by $N_b f = u$ where $\overline{\partial}_b \overline{\partial}_b u = \pi f$, $u$ is orthogonal to the nullspace of $\overline{\partial}_b$, and $\pi f$ is the orthogonal projection of $f$ onto the
range of $\overline{\partial}_b$. Thus $\overline{\partial}_b \overline{\partial}_b N_b = \pi$, rather than the identity operator.
and by $P_{p,q}$ the Bergman projection acting on $(p, q)$ forms on $\Omega$. Denote by $N_b$ the tangential Neumann operator acting on $(0, 1)$ forms on $\partial \Omega$, and by $S$ the Szegö projection, acting on functions.

Corollary 1.3 Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary. Then

(i) Suppose that $n \geq 2, 0 \leq p \leq n$, and $0 < q \leq n$. Then $P_{p,q-1}$ preserves $C^\omega_{(p,q-1)}(\Omega)$ if and only if $\overline{\partial} \circ N_{p,q}$ maps $C^\omega_{(p,q)}(\Omega) \cap \text{Kernel}(\overline{\partial})$ to $C^\omega_{(p,q)}(\Omega)$.

(ii) Suppose that $n = 2$. Then $S$ preserves $C^\omega(\partial \Omega)$ if and only if $\overline{\partial}_b \circ N_b$ maps $C^\omega_{(0,1)}(\partial \Omega) \cap \text{Range}(\overline{\partial}_b)$ to $C^\omega(\partial \Omega)$.

(iii) Suppose that $n > 2$. Then $S$ preserves $C^\omega(\partial \Omega)$ if and only if $\overline{\partial}_b \circ N_b$ maps $C^\omega_{(0,1)}(\partial \Omega) \cap \text{Kernel}(\overline{\partial}_b)$ to $C^\omega(\partial \Omega)$.

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2 Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. We require the following theorem of Diederich and Fornaess [DF1] and [DF2].

Lemma 2.1 Let $\Omega$ be a pseudoconvex bounded domain in $\mathbb{C}^n$ with real analytic boundary. Then there exists a decreasing sequence $\{\Omega_j\}$ of strictly pseudoconvex domains with smooth boundaries $\partial \Omega_j$ such that $\Omega = \bigcap_{j=1}^{\infty} \Omega_j$.

Proof of Theorem 1.1. Let $f = \sum_{I,J} f_{IJ} dz^I \wedge d\overline{z}^J$ be a $\overline{\partial}$-closed $(p, q)$ form in $C^\omega(\Omega)$, where $0 < q \leq n$. Then there exists $j$ such that each $f_{IJ}$ can be extended to a real analytic function on $\Omega_j$. Denoting the extension also by $f_{IJ}$, the form $f = \sum_{I,J} f_{IJ} dz^I \wedge d\overline{z}^J$ is still $\overline{\partial}$-closed, by analytic continuation.

Since $\Omega_j$ is a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, the $\overline{\partial}$-Neumann problem is solvable on $\Omega_j$. Thus [FK] there exists $g \in C^\omega_{(p,q-1)}(\Omega_j)$ satisfying $\Box g = f$ on $\Omega_j$, where $\Box$ equals the ordinary Laplacian, acting componentwise on $g$. $g$ also satisfies the $\overline{\partial}$-Neumann boundary conditions.

Since $\Box g \in C^\omega$, it follows (from ellipticity of $\Box$, or from the explicit expression for the fundamental solution for the Laplacian) that $g \in C^\omega$ on the interior of $\Omega_j$. In particular, $g \in C^\omega(\overline{\Omega})$, and thus $\overline{\partial} g$ is also real analytic on $\overline{\Omega}$. Since $\overline{\partial} f = 0$, it follows from the $\overline{\partial}$-Neumann formalism [FK] that $\overline{\partial}(\overline{\partial} g) = f$. \[\square\]
3 Bochner extension for forms

The strategy for the proof of Theorem 1.2, given a form defined on \( \partial \Omega \), will be to extend it to a \( \overline{\partial} \)-closed form defined in a neighborhood of \( \partial \Omega \), or perhaps even in a neighborhood of \( \Omega \), then to argue as before.

**Proposition 3.1** Let \( n \geq 2 \) and \( 0 \leq q < n \). Let \( \Omega \subset \mathbb{C}^n \) be a bounded domain with real analytic boundary. Let \( f \in C^{(0,q)}(\partial \Omega) \) be \( \overline{\partial}_b \)-closed. Then there exists a \( \overline{\partial} \)-closed form \( F \in C^{(0,q)} \), defined in some neighborhood of \( \partial \Omega \), whose restriction to \( \partial \Omega \), in the sense of forms, equals \( f \).

Some related results for the \( C^\infty \) category appear in [FK] and [S]. Note that for \( C^2 \), the hypothesis that \( \overline{\partial}_b f = 0 \) holds automatically for all \((0,1)\) forms \( f \) on \( \partial \Omega \).

Let \( \rho \) be a defining function for \( \Omega \) that is real analytic in a neighborhood of \( \partial \Omega \). In particular, \( \Omega = \{ z : \rho(z) < 0 \} \). Let \( N = \frac{4}{|\nabla \rho|^2} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial \rho}{\partial \bar{z}_j} \). Let \( \mathcal{J} \) be the contraction operator on differential forms. The proof of the next lemma is straightforward and is left to the reader.

**Lemma 3.2** The following identities hold in a neighborhood of \( \partial \Omega \) for any form \( \phi \).

\[
\begin{aligned}
(i) \quad \phi &= N \mathcal{J}(\overline{\partial} \rho \wedge \phi) + \overline{\partial} \rho \wedge (N \mathcal{J} \phi) \\
(ii) \quad N \mathcal{J}(N \mathcal{J} \phi) &= 0.
\end{aligned}
\]

**Proof of Proposition 3.1.**

To simplify notation we treat only the case \( q = 1 \) explicitly, but it will be apparent that the same reasoning applies to forms of arbitrary degree. Let a \( \overline{\partial}_b \)-closed \((0,1)\) form \( f \in C^{\omega}(\partial \Omega) \) be given. We claim that in a sufficiently small neighborhood of \( \partial \Omega \) there exists a unique \( C^\omega \) form \( F \) satisfying the differential equation

\[
(3.1) \quad N \mathcal{J} \overline{\partial} F = 0
\]

with the side condition

\[
(3.2) \quad N \mathcal{J} F = 0,
\]

whose restriction to \( \partial \Omega \) equals \( f \). Indeed, locally near any point in \( \partial \Omega \) we may fix \( C^\omega \) \((0,1)\) forms \( \{ \tilde{\omega}_j : 1 \leq j \leq n-1 \} \) that constitute an orthonormal basis for the orthocomplement of \( \overline{\partial} \rho \). The condition \( N \mathcal{J} F = 0 \) means that \( F = \sum_{j=1}^{n-1} F_j \tilde{\omega}_j \) for some coefficients \( F_j \). Then \( N \mathcal{J} \overline{\partial} F \) belongs also to the span of the \( \tilde{\omega}_j \) and has the form

\[
(3.3) \quad N \mathcal{J} \overline{\partial} F = \sum_{j=1}^{n-1} \pm (\bar{L} F_j) \tilde{\omega}_j
\]

with the side condition

\[
(3.2) \quad N \mathcal{J} F = 0,
\]
modulo an operator of order 0 acting on $F$, where $\bar{L}$ is a complex vector field that is nowhere tangent to $\partial \Omega$. Thus (3.3) is a first-order system for which $\partial \Omega$ is a noncharacteristic hypersurface. The coefficients $F_j$ are determined on $\partial \Omega$ by $f$. Therefore the Cauchy-Kowalevska theorem guarantees local existence and uniqueness of a solution $F$ of (3.1) and (3.2) that equals $f$ on $\partial \Omega$. Existence in a neighborhood of the full boundary follows from the local existence together with local uniqueness.

This form $F$ is $\bar{\partial}$-closed. Indeed, $\bar{\partial}F = N \bar{\partial}(\partial \rho \wedge \bar{\partial}F) + \partial \rho \wedge (N \bar{\partial}F)$, and the second term vanishes by (3.1). On the other hand, $G = N \bar{\partial} \rho \wedge \bar{\partial}F$ satisfies the differential equation

$$N \bar{\partial}G = N \bar{\partial} \rho = 0.$$  

But $G$ may be expressed as $\sum_{i<j} G_{ij} \bar{\omega}_i \wedge \bar{\omega}_j$ for certain $C^\infty$ coefficients $G_{ij}$. Just as above,

$$N \bar{\partial}G = \sum_{i<j} \pm (L G_{ij}) \bar{\omega}_i \wedge \bar{\omega}_j$$

modulo a term of order zero belonging to the span of the $\bar{\omega}_i \wedge \bar{\omega}_j$, where $\bar{L}$ is the same vector field as above. On $\partial \Omega$, $G = \bar{\partial}b f$ vanishes, so the uniqueness conclusion of the Cauchy-Kowalevska theorem guarantees that $G = N \bar{\partial} \rho \wedge \bar{\partial}F$ vanishes identically in a neighborhood of $\partial \Omega$.

4 Proof of Theorem 1.2

We will need the following well known existence theorem.

**Lemma 4.1** Suppose that $n \geq 3$ and that $\Omega_j$ ($j = 1, 2$) are bounded, strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries, with $\Omega_1 \subset \subset \Omega_2$. Then for each $\bar{\partial}$-closed $(0, 1)$ form $f \in C^\infty(\Omega_2 \setminus \Omega_1)$ there exists a function $u \in C^\infty(\Omega_2 \setminus \Omega_1)$ satisfying $\bar{\partial}u = f$.

**Proof.** Since $n \geq 3$, the Levi form has $n-1$ positive eigenvalues at each point of the outer boundary, and has $n-1 \geq 2$ negative eigenvalues at each point of the inner boundary. Thus $\Omega_2 \setminus \Omega_1$ satisfies the $Z(1)$ condition at each boundary point, and this condition is sufficient [FK] to guarantee the existence of a smooth solution.

It is now straightforward to deduce Theorem 1.2 in the easier case $n \geq 3$.

**Proof.** Let $f \in C^\infty_{(0,1)}(\partial \Omega)$ satisfy $\bar{\partial}b f = 0$. By Lemma 2.1 and Proposition 3.1, there are strictly pseudoconvex domains $\Omega_1$ and $\Omega_2$ such that $\Omega_1 \subset \subset \Omega \subset \subset \Omega_2$ and a $\bar{\partial}$-closed $(0, 1)$-form $\phi = \phi_1 d\bar{z}_1 + \cdots + \phi_n d\bar{z}_n \in C^\infty(U)$, where $U = (\Omega_2 \setminus \Omega_1)$, such that

$$N \bar{\partial}(\partial \rho \wedge \phi) = f \quad \text{on } \partial \Omega.$$
By Lemma 4.1, there is \( u \in C^\infty(\Omega_2 \setminus \Omega_1) \) such that \( \overline{\partial} u = \phi \). The ellipticity of \( \overline{\partial} \) implies that \( u \in C^\omega(\Omega_2 \setminus \Omega_1) \), and

\[
\overline{\partial} u = N \int (\overline{\partial} u) = N \int (\overline{\partial} \phi) = f \quad \text{on } \partial \Omega.
\]

The proof of Theorem 1.2 is thus complete when \( n \geq 3 \). □

Next we shall prove Theorem 1.2 for the case \( n = 2 \).

**Proposition 4.2** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^2 \) with real analytic boundary. Let \( F \) be a \( \overline{\partial} \)-closed \((0,1)\)-form with \( C^\infty \) coefficients, defined in a neighborhood of \( \partial \Omega \), whose restriction to \( \partial \Omega \) belongs to \( \overline{\partial}b(C^\infty) \). Then there exists a \( \overline{\partial} \)-closed \((0,1)\)-form \( \tilde{F} \) having Lipschitz continuous coefficients, defined on a neighborhood \( \Omega' \) of \( \overline{\Omega} \), that agrees with \( F \) on \( \Omega' \setminus \overline{\Omega} \).

**Proof.** Fix a \( C^\omega \) defining function \( \rho \) for \( \Omega \), and fix \( u_0 \in C^\infty(\partial \Omega) \) satisfying

\[
\overline{\partial} u_0 = N \int (\overline{\partial} \rho) = F \quad \text{on } \partial \Omega.
\]

Extend \( u_0 \) to a smooth function on \( \Omega' \). Then

\[
(4.1) \quad \overline{\partial} u_0 \wedge \overline{\partial} \rho - F \wedge \overline{\partial} \rho = O(\rho).
\]

Let

\[
h = N \int F - N \int (\overline{\partial} u_0),
\]

and define

\[
u(z) = u_0(z) + \rho(z)h(z).
\]

Then

\[
\overline{\partial} u = \overline{\partial} u_0(z) + h(z)\overline{\partial} \rho(z) + \rho(z)\overline{\partial} h(z)
\]

It is easy to see that

\[
(4.3) \quad \overline{\partial} u \wedge \overline{\partial} \rho = F \wedge \overline{\partial} \rho \quad \text{on } \partial \Omega,
\]

and

\[
(4.4) \quad N \int (\overline{\partial} u) - N \int F = \rho(z) N \int (\overline{\partial} h) = \rho(z) (N \int (\overline{\partial} \rho) (N \int (F - \overline{\partial} u_0)).
\]

We let

\[
\tilde{F} = \begin{cases} F & \text{on } \Omega' \setminus \Omega \\ \overline{\partial} u & \text{on } \Omega. \end{cases}
\]

By (4.1) and (4.4), \( \overline{\partial} u \) and \( F \) agree to first order \( \partial \Omega \). Thus \( \tilde{F} \) has Lipschitz coefficients. \( \overline{\partial} \tilde{F} \in L^\infty \) vanishes identically on both \( \Omega \) and \( \Omega' \setminus \overline{\Omega} \), hence vanishes on \( \Omega' \) in the sense of distributions. The proof of the proposition is complete. □

To complete the proof of Theorem 1.2, we need the following lemma.
Lemma 4.3 Let $\Omega \subset \mathbb{C}^2$ be bounded and pseudoconvex, with $C^\infty$ boundary. Suppose that $\Omega'$ is a neighborhood of $\overline{\Omega}$, that $F$ is a $\overline{\partial}$-closed $(0,1)$ form having $C^\infty$ coefficients in a neighborhood of $\partial \Omega$, and that $v \in C^1(\Omega' \setminus \Omega)$ satisfies $\overline{\partial} v = F$. Then $v$ extends to a function belonging to $C^\infty$ in a neighborhood of $\partial \Omega$.

Proof. $v \in C^\infty(\Omega' \setminus \Omega)$, since $\overline{\partial} v \in C^\infty$ there. For any point $p \in \partial \Omega$ there exist $\delta > 0$ and a real analytic function $u_p$ satisfying $\overline{\partial} u_p = F$ on the ball $B(p, \delta)$. Thus $\overline{\partial}(v - u_p) = 0$ on $B(p, \delta) \cap (\Omega' \setminus \Omega)$. Therefore $v - u_p$ is holomorphic in $B(p, \delta) \cap (\Omega' \setminus \Omega)$.

$p$ is a point of finite type in $\partial \Omega$, since $\Omega \subset \mathbb{C}^2$ is bounded and has real analytic boundary. $\overline{\partial}_b(v - u_p) = 0$ on $\partial \Omega$ near $p$, by continuity since $v$ was assumed to be $C^1$ up to the boundary. Since $\Omega$ is pseudoconvex, and $p$ is a boundary point of finite type, a theorem of Bedford and Fornaess [BF] asserts that for some small ball $B$ centered at $p$ there exists a holomorphic function $h$ on $B \cap \Omega$, continuous in $B \cap \overline{\Omega}$, that agrees with $v - u_p$ on $\partial \Omega$. Extend $h$ to $B$ by defining it to be $v - u_p$ in $B \setminus \Omega$.

Now $h$ is continuous, and is annihilated by $\overline{\partial}$ both inside and outside $\partial \Omega$, near $p$. Therefore $\overline{\partial} h = 0$ in the sense of distributions, so $h$ is a genuine holomorphic function near $p$. It agrees with $v - u_p$ in $B \setminus \Omega$, hence $v - u_p$ has been extended to a function holomorphic in a neighborhood of $p$. Since $u_p$ is real analytic, $v$ itself has thus been extended to a function real analytic in a neighborhood of $p$, which was an arbitrary point of $\partial \Omega$.

We are now in a position to complete the proof of Theorem 1.2 for the case $n = 2$.

Proof. Let $f \in C^\infty_{(0,1)}(\partial \Omega) \cap \text{Range}(\overline{\partial}_b)$. By Proposition 3.1, there exists a $\overline{\partial}$-closed $(0,1)$ form $F$, defined in a neighborhood of $\partial \Omega$, that extends $f$ in the sense of forms. By Proposition 4.2, there are a neighborhood $\Omega'$ of $\overline{\Omega}$ and a $\overline{\partial}$-closed $(0,1)$-form $\tilde{F}$ with Lipschitz continuous coefficients such that $\tilde{F} = F$ on $\Omega' \setminus \Omega$. By Lemma 2.1, we may assume $\Omega'$ to be strictly pseudoconvex. Therefore there exists a solution $v$ of $\overline{\partial} v = \tilde{F}$ in $\Omega'$, and elliptic regularity theorems guarantee that $v \in C^{1,\alpha}$, for every $\alpha < 1$. In particular, $\overline{\partial} v = F$ in $\Omega' \setminus \Omega$. By Lemma 4.3, $v \in C^\infty(\partial \Omega)$ and consequently $\overline{\partial} v = F$ in a neighborhood of $\partial \Omega$, by analytic continuation. Since $F$ extends $f$, we thus have $\overline{\partial} v = f$ on $\partial \Omega$, concluding the proof of Theorem 1.2 for the case $n = 2$.

5 Proof of Corollary 1.3

Proof. Write $P = P_{p,q-1}$, $N = N_{p,q-1}$. From the formula $P = I - \overline{\partial}^* N \overline{\partial}$ it follows directly that if $\overline{\partial}^* N$ maps $C^\infty(\overline{\Omega}) \cap \ker(\overline{\partial})$ to $C^\infty(\overline{\Omega})$, then $P$ preserves $C^\infty(\overline{\Omega})$.

Conversely, for any $\overline{\partial}$-closed $(p, q)$-form $f \in C^\infty(\overline{\Omega})$, Theorem 1.1 guarantees the existence of $u \in C^\infty_{(p,q-1)}(\overline{\Omega})$ satisfying $\overline{\partial} u = f$. Then $Pu \in C^\infty(\overline{\Omega})$, by the
global regularity hypothesis on $P$. Since $u - Pu$ is orthogonal to the $L^2$ nullspace of $\overline{\partial}$, it equals $\overline{\partial}^* Nf$.

(ii) and (iii) can be proved similarly by using Theorem 1.2 and the formula $S = I - \overline{\partial}_b \circ N_b \circ \overline{\partial}_b$. Existence of $N_b$ and subelliptic estimates follow from work of Kohn [K], since boundaries of bounded, pseudoconvex, real analytic domains are necessarily of finite ideal type as defined in [K].

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