AUTOMORPHISMS OF OT MANIFOLDS AND RAY CLASS NUMBERS

OLIVER BRAUNLING AND VICTOR VULETESCU

Abstract. We compute the automorphism group of OT manifolds of simple type. We show that the graded pieces under a natural filtration are related to a certain ray class group of the underlying number field. This does not solve the open question whether the geometry of the OT manifold sees the class number directly, but brings us a lot closer to a possible solution.

Let $K$ be a number field with $s \geq 1$ real places and $t \geq 1$ complex places. For suitable choices of a subgroup $U \subseteq \mathcal{O}_K^{\times,+}$ of the totally positive units, there is a properly discontinuous action of $\mathcal{O}_K \rtimes U$ on $\mathbb{H}^s \times \mathbb{C}^t$, essentially based on embedding $K$ via its infinite places and letting the group act by addition and multiplication. The key point is that

$$X(K,U) := (\mathbb{H}^s \times \mathbb{C}^t) / (\mathcal{O}_K \rtimes U)$$

becomes a compact complex manifold, a so-called Oeljeklaus–Toma manifold (or “OT manifold”) [OT05, OV13]. These manifolds are, in a way, higher-dimensional analogues of the type $S_0$ Inoue surfaces, one of the better understood types among the Class VII$_0$ surfaces in Kodaira’s classification.

If $t = 1$, then knowing $X := X(K,U)$, even just its fundamental group, suffices to reconstruct the number field $K$ uniquely. This can fail for $t > 1$. However, whenever $K$ is fully determined by $X$, it is natural to ask whether one can read off the arithmetic invariants of $K$ directly from the geometry of $X$.

So far, even for $t = 1$, it is not known how to read off the class group or even just the class number of $K$ from $X$. At the same time, several other invariants are readily accessible, e.g., if $X$ is an OT manifold of simple type:

| Geometry of $X$ | Arithmetic of $K$ |
|-----------------|-------------------|
| dimension       | $s + t$           |
| Betti number $b_1$ | $s$              |
| Betti number $b_2$ | $\frac{1}{2}s(s-1)$ |
| LCK rank        | $s$ (not CM) or $\frac{1}{2}$ (if $K$ is CM) |
| $h^{1,0}$       | 0                 |
| $h^{0,1}$       | $\geq s$         |
| normalized volume | $\sim q^s \sqrt{|\text{discriminant}| \cdot \text{regulator}}$ |
| admits LCK metric | if and only if $|\sigma_i(u)| = |\sigma_j(u)|$ for all $\sigma_i, \sigma_j, u$ |
| $H_1(X,\mathbb{Z})$ | $U \times (\mathcal{O}_K/J)$ for certain ideal $J$ |
| ?               | field automorphisms $\text{Aut}(K/\mathbb{Q})$ |
| ?               | class group |

(above, $\sigma_i, \sigma_j$ refers to genuine complex places, i.e. those complex embeddings whose image does not lie in the reals, and $u$ refers to any $u \in U$).

We refer to [OT05] or [OV13] for unexplained terminology.

We will not be able to solve this open problem, but we find invariants which are of the same arithmetic nature as class groups: ray class groups. This data turns out to be encoded in the holomorphic automorphism group of $X$.

O.B. was supported by the GK1821 “Cohomological Methods in Geometry” and a FRIAS Junior Fellowship.

V.V. was supported by a grant of Ministry of Research and Innovation, CNCS - UEFISCDI, project number PN-III-P4-ID-PCE-2016-0065, within PNCDI III.
Theorem 1. Suppose $X := X(K, U)$ is an OT manifold of simple type. Then the biholomorphism group $\text{Aut}(X)$ is canonically isomorphic to

$$
\left( \left( \frac{\mathcal{O}_K : J(U)}{\mathcal{O}_K} \right) \rtimes \left( \frac{\mathcal{O}_K^{\times,+}}{U} \right) \right) \rtimes A_U.
$$

(We define $A_U$ and $J(U)$ in the main body of the paper.) More concretely, it has a canonical ascending three step filtration $F_0 \subseteq F_1 \subseteq F_2$ whose graded pieces are

$$
gr_{i}F_{i} \text{Aut}(X) \simeq O_K / J(U); \quad gr_{i}F_{i} \rtimes \mathcal{O}_K^{\times,+} / U; \quad gr_{i}F_{i} \rtimes A_U.
$$

For $gr_{0}F_{i} \text{Aut}(X)$ the isomorphism is non-canonical, while for $gr_{i}F_{i} \text{Aut}(X)$ with $i = 1, 2$ it is.

See Theorem 4. These graded pieces may look innocuous, but let us point out that they are related to class-number like invariants of $K$.

Theorem 2. Let $K$ be a number field with $s \geq 1$ real places and precisely one complex place. Moreover, suppose $m$ is a modulus such that its finite part $m_0$ satisfies $J(U_{m,1}) = m_0$. Then the ray unit group $U_{m,1}$ is an admissible subgroup and let $X := X(K, U_{m,1})$ be the corresponding OT manifold. The graded Euler characteristic

$$
\chi^F(\text{Aut}(X)) = \prod_i |gr_i^F \text{Aut}(X)|^{-1}i
$$

satisfies

$$
\frac{h_m}{h} \leq \frac{\chi^F(\text{Aut}(X))}{|A_{U_{m,1}}|},
$$

where $h_m$ denotes the ray class number of $m$ and $h$ is the ordinary class number.

See Theorem 6. The statement of this result uses some definitions and jargon from number theory, which we shall review and summarize to the extent needed in Section 2.

Although not connected to the principal question of this paper, we also obtain the following result:

Theorem 3. Let $K$ be a number field with $s = t = 1$ embeddings and $u$ a (totally) positive fundamental unit, i.e. $\mathcal{O}_K^{\times,+} \simeq \mathbb{Z} \langle u \rangle$. Then all groups

$$
U_n := \mathbb{Z} \langle u^n \rangle
$$

are admissible, and for $X_n := X(K, U_n)$ we have

$$
\lim_{n \to \infty} \frac{\log |H_1(X_n, \mathbb{Z})_{\text{tor}}|}{n} = \log M(f),
$$

where $f$ is the minimal polynomial of the unit $u$, and $M(f)$ denotes the Mahler measure of $f$. In particular, $|H_1(X_n, \mathbb{Z})_{\text{tor}}|$ always grows asymptotically exponentially as $n \to +\infty$.

See Theorem 7. This essentially means that all Inoue surfaces of type $S^0$ sit inside a tree of covering spaces, which are itself Inoue surfaces of type $S^0$, and as we go further into the branches of the tree, the torsion in $H_1$ always grows exponentially, for example

$$
\cdots \to X_{2^3} \to X_{2^2} \to X_2 \to X_1,
$$

or similarly for all other primes, or other chains of numbers totally ordered by divisibility. This is not so relevant for our question around class numbers, but it explains why Inoue surfaces tend to have so much torsion homology.

Convention: The word ring always means a unital commutative and associative ring.
1. Biholomorphisms

Let $K$ be a number field with $s \geq 1$ real places and $t \geq 1$ complex places. Let $U \subseteq \mathcal{O}_K^\times$ be an admissible subgroup, i.e. a rank $s$ free abelian subgroup (see [OT05 §1]). We call $U$ of simple type if $K = \mathbb{Q}(u \mid u \in U)$, or equivalently if there is no proper subfield of $K$ which already contains $U$.

Definition 1. Suppose $U \subseteq \mathcal{O}_K^\times$ is an arbitrary subgroup. Then define

$$J(U) := \{ \text{ideal of } \mathcal{O}_K \text{ generated by } u - 1 \text{ for all } u \in U \}.$$

Definition 2. We define the fractional ideal

$$\mathcal{O}_K : J(U) := \{ \beta \in K \mid \forall u \in U : (1 - u)\beta \in \mathcal{O}_K \}.$$

This is also sometimes denoted by $J(U)^{-1}$.

(From a commutative algebra standpoint, this fractional ideal is the inverse of $J(U)$ in the sense of invertible $\mathcal{O}_K$-modules.)

Definition 3. We define $A_U := \{ g \in \text{Aut}(K/\mathbb{Q}) \mid gU = U \}$.

By $gU = U$ we mean that $g$ sends elements in $U$ to elements in $U$, and not that it would element-wise fix $U$, and $\text{Aut}(K/\mathbb{Q})$ denotes field automorphisms of $K$.

When we write $\mathcal{O}_K \rtimes U$, we mean the semi-direct product of the abelian groups $(\mathcal{O}_K, +)$ and $U$, where $U \subseteq \mathcal{O}_K^\times$ acts on $(\mathcal{O}_K, +)$ by multiplication with respect to the ring structure of $\mathcal{O}_K$. Each element of $\mathcal{O}_K \rtimes U$ can uniquely be written as a pair $(a, b)$ with $a \in \mathcal{O}_K$ and $b \in U$.

For a group $G$, let us write $G_{\text{ab}}$ for its abelianization, and if $G$ is abelian, write $G_{\text{tor}}$ for the subgroup of torsion elements, and $G_{\text{fr}} := G/G_{\text{tor}}$ for the torsion-free quotient.

Proposition 1. Let $K$ be a number field and $U \subseteq \mathcal{O}_K^\times$ an admissible subgroup. Then for $\pi := \mathcal{O}_K \rtimes U$, the kernel $\varpi$ in the short exact sequence

$$1 \rightarrow \varpi \rightarrow \pi \rightarrow \pi_{\text{ab},\text{fr}} \rightarrow 1$$

is precisely the subgroup $\mathcal{O}_K$ appearing in the definition of $\pi$ as a semi-direct product. The commutator subgroup is

$$[\pi, \pi] = \{ \text{pairs } (u, 1) \in \pi \mid u \in J(U) \}.$$

Moreover, we have a canonical short exact sequence

$$0 \rightarrow \mathcal{O}_K/J(U) \rightarrow H_1(X(K;U), \mathbb{Z}) \rightarrow U \rightarrow 0,$$

where $\mathcal{O}_K/J(U)$ is precisely the torsion subgroup of the middle term. In particular, this group needs at most $s + 2t$ generators.

Proof. A proof is given in [Bra16 Prop. 6], based on [PVT12 Thm. 4.2]. Loc. cit. requires $U$ to be a torsion-free subgroup, but since $K$ has at least one real place and $\sigma : K \rightarrow \mathbb{R}^\times$ is injective, either $\mathcal{O}_K^\times_{\text{tor}}$ is trivial or we have $\mathcal{O}_K^\times_{\text{tor}} = \langle -1 \rangle$. Either way, the subgroup of totally positive units $\mathcal{O}_K^\times_{\text{fr}}$ is necessarily torsion-free.

Let $K$ be a number field and $U \subseteq \mathcal{O}_K^\times$ an admissible subgroup. Write $X := X(K, U)$ as in Equation (1.11) to denote the corresponding Oeljeklaus–Toma manifold. Let us write

$$\mathfrak{g} : \mathcal{O}_K \hookrightarrow \mathbb{C}^s \times \mathbb{C}^t$$

for the map $\alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_{s+t}(\alpha))$,

where $\sigma_1, \ldots, \sigma_s : K \rightarrow \mathbb{R}$ denote the real embeddings, and $\sigma_{s+1}, \ldots, \sigma_{s+t} : K \rightarrow \mathbb{C}$ one representative for each complex conjugate pair of the genuinely complex embeddings.

Lemma 1. There are three constructions which naturally give biholomorphisms of $X$.

1. There is a canonical subgroup inclusion $\mathcal{O}_K : J(U) \rightarrow \text{Aut}(X)$, sending any $\beta \in (\mathcal{O}_K : J(U))$ to the biholomorphism

$$f : \mathbb{H}^s \times \mathbb{C}^t \rightarrow \mathbb{H}^s \times \mathbb{C}^t$$

$$z \mapsto z + \mathfrak{g}(\beta).$$
There is a canonical subgroup inclusion $A_U \hookrightarrow \text{Aut}(X)$.

The action of $O_K^{\times +}/U$.

Proof. (1) It is clear that this map is holomorphic and invertible on $\mathbb{H}^+ \times \mathbb{C}^\ell$. We need to show that it descends to the quotient modulo $O_K \times U$. To this end, we need to check that for all $u \in U$ and $\gamma \in O_K$ the identity

$$f(\sigma(u)\hat{z} + \sigma(\gamma)) \equiv f(\hat{z}) \mod O_K \times U$$

holds. Plugging in $f$ on the left hand side, we obtain $\sigma(u)\hat{z} + \sigma(\beta) \equiv \hat{z} + \sigma(u^{-1})\sigma(\gamma) + \sigma(u^{-1})\sigma(\beta)$ by letting $\sigma(U)$ act via $\sigma(u^{-1})$,

$$\equiv \hat{z} + \sigma(u^{-1})\gamma + \sigma(u^{-1})\beta \equiv \hat{z} + \sigma(u^{-1})\gamma + \sigma((u^{-1} - 1)\beta) + \sigma(\beta)$$

and since $u^{-1}\gamma \in O_K$ as $u$ is a unit, as well as $(u^{-1} - 1)\beta \in O_K$ by the very definition of the fractional ideal $(O_K: J(U))$, we can let $\sigma O_K$ act and obtain

$$\equiv \hat{z} + \sigma(\beta) = f(\hat{z}).$$

This is exactly what we had to show, namely Equation (1.4). Thus, $f$ descends to a biholomorphism $X \to X$. From deck transformation theory, it follows that $f$ acts trivially on this quotient if and only if $\sigma(\beta) \in \sigma O_K$, so in total we get a well-defined injection from the group $(O_K : J(U))$. This proves our first claim.

(2) A field automorphism $g \in A_U$ just maps elements to Galois conjugates, so at worst it permutes the embeddings, say $\pi$ is given by $\sigma_i(g\beta) = \sigma_{\pi(i)}(\beta)$. Correspondingly, define

$$f(z_1, \ldots, z_{s+t}) := f(z_{\pi(1)}, \ldots, z_{\pi(s+t)})$$

This is a biholomorphism. It descends modulo $O_K \times U$ since a field automorphism maps $O_K$ to itself, and by assumption we have $gU = U$, so $U$ is also preserved.

(3) Obvious. \qed

Consider a semi-direct product $G := A \times B$ with $A, B$ groups. Let $\text{Aut}(G; A) \subseteq \text{Aut}(G)$ denote the subgroup of automorphisms $\theta : G \to G$ such that $\theta(A) \subseteq A$, i.e. those automorphisms which map the subgroup $A$ into itself.

We recall a result from group theory due to J. Dietz \cite{Die12}: There is a canonical bijection between elements $\theta \in \text{Aut}(G; A)$ and triples $(\alpha, \beta, \delta)$, where

- $\alpha \in \text{Aut}(A)$,
- $\delta \in \text{Aut}(B)$,
- $\beta \in \text{Map}(B, A),$

such that the following conditions hold:

1. $\beta(b_1 b_2) = \beta(b_1)\beta(b_2)\delta(b_1)$ for all $b_1, b_2 \in B$,
2. $\alpha(a^b) = \alpha(a)^{\beta(\delta)}$ for all $a \in A, b \in B$.

We call such a triple $(\alpha, \beta, \delta)$ a Dietz triple. This is proven in \cite{Die12}, Lemma 2.1; we use the same notation as in the paper to make it particularly easy to use the statement loc. cit. directly. In her paper, Dietz writes a triple $(\alpha, \beta, \delta)$ as a matrix

$$\begin{bmatrix} \alpha & \beta \\ \delta & \end{bmatrix}.$$ 

To clarify notation, the superscripts in the conditions (1), (2) refer to the conjugation

$$g^h := h^{-1}gh$$

for arbitrary $g, h \in G$, and computed in $G$.

Remark 1. We recall a basic fact: If $a \in A$ and $b \in B$, then in the semi-direct product $A \times B$ the conjugation $a^b$ agrees with the action of $B$ on $A$ which underlies the semi-direct product structure.

We apply these general remarks to the fundamental group of an OT manifold.
Lemma 2. There is a bijection between elements of $\text{Aut}(\pi)$ and triples $(\alpha, \beta, \delta)$ with

- $\alpha \in \text{Aut}(O_K, +)$,
- $\delta \in \text{Aut}(U)$,
- $\beta \in \text{Map}(U, O_K)$

such that the following conditions hold:

1. $\beta(b_1 b_2) = \beta(b_1) + \beta(b_2) \delta(b_1)$ for all $b_1, b_2 \in U$,
2. $\alpha(ab) = \alpha(a) \delta(b)$ for all $a \in O_K$ and $b \in U$.

Here we use the notation “$\text{Aut}(O_K, +)$” to stress that we talk about the additive group $(O_K, +)$, and not, as one could misunderstand, automorphisms of $O_K$ as a ring.

Proof. We wish to apply the above group-theoretical facts to the semi-direct product $\pi := O_K \rtimes U$. This entails the following: (1) By Prop. 1 we have

$$1 \longrightarrow (O_K, +) \longrightarrow \pi \longrightarrow \pi_{ab, fr} \longrightarrow 1.$$}

Every group automorphism $\theta : \pi \to \pi$ induces an automorphism of the abelianization $\pi_{ab}$, and further on the torsion-free quotient $\pi_{ab, fr}$. Hence, by the above exact sequence $\theta$ maps $(O_K, +)$ to itself. Hence, $\text{Aut}(\pi; (O_K, +)) = \text{Aut}(\pi)$ is an equality of groups, i.e. we can describe arbitrary automorphisms using Dietz triples.

Working with the Dietz triples for $\pi$, conditions (1) and (2) unravel as follows:

1. $\beta(b_1 b_2) = \beta(b_1) + \beta(b_2) \delta(b_1)$ for all $b_1, b_2 \in U$,
2. $\alpha(ab) = \alpha(a) \delta(b)$ for all $a \in O_K$ and $b \in U$.

We justify this: For (1) we write $(O_K, +)$ additively, giving $\beta(b_1 b_2) = \beta(b_1) + \beta(b_2) \delta(b_1)$. Note that $\beta(b_2) \in O_K$ and $\delta(b_1) \in U$, so we may use Remark 1 to evaluate the conjugation $\beta(b_2) \delta(b_1)$ in $\pi$. Thus, $\beta(b_2) \delta(b_1)$ is the action of $\delta(b_1)$ on $\beta(b_2)$, but the semi-direct product $O_K \rtimes U$ is formed by letting $U$ act by multiplication on $O_K$, so this is simply the product $\beta(b_2) \delta(b_1)$ in the ring structure of $O_K$. For (2) the original condition is

$$\alpha(a^b) = \alpha(a)^{\beta(b) \delta(b)}.$$}

Now, on the left side again $a \in O_K$ while $b \in U$, so again by Remark 1 this is just the product $ab$ in the ring $O_K$. We have

$$\alpha(a)^{\beta(b) \delta(b)} = \left(\alpha(a)^{\beta(b)}\right)^{\delta(b)}.$$}

Here $\alpha(a) \in O_K$ and $\beta(b) \in O_K$, so we can compute the conjugation within the group $O_K$. Being abelian, the conjugation is necessarily trivial. Thus, the expression simplifies to $= \alpha(a)^{\beta(b)}$. Again, $\alpha(a) \in O_K$ and $\delta(b) \in U$, so by Remark 1 this is just $\alpha(a) \delta(b)$ in $O_K$. \qed

Lemma 3. Suppose we are in the situation of the previous lemma. Then the automorphisms corresponding to triples $(\alpha, \beta, \delta)$ with $\delta := \text{id}$ correspond to a subgroup of $\text{Aut}(\pi)$ which is canonically isomorphic to

$$\{\theta \in \text{Aut}(\pi) \mid \delta = \text{id}\} \cong (O_K : J(U)) \rtimes \text{Aut}_R(O_K).$$}

Here $R$ is the smallest subring of $O_K$ containing all $u \in U$, and $\text{Aut}_R(O_K)$ denotes the $R$-module automorphisms of $O_K$.

Proof. Assuming $\delta := \text{id}$ the Dietz conditions become

1. $\beta(b_1 b_2) = \beta(b_1) + b_1 \beta(b_2)$ for all $b_1, b_2 \in U$,
2. $\alpha(ab) = \alpha(a) b$ for all $a \in O_K$ and $b \in U$.

Condition (2) means that $\alpha \in \text{Aut}(O_K, +)$ is not just an automorphism of $(O_K, +)$ as an abelian group, but as an $R$-module over the subring $R \subseteq O_K$ which is defined by $R := \mathbb{Z}[u \mid u \in U]$, i.e. the smallest subring of $O_K$ containing all $u \in U$. We write $\alpha \in \text{Aut}_R(O_K)$. Next, we use that $U$ is abelian. From $\beta(b_2 b_1) = \beta(b_1) \beta(b_2)$ and (1) we get

$$\beta(b_2) + b_2 \beta(b_1) = \beta(b_1) + b_1 \beta(b_2)$$

$$(b_2 - 1) \beta(b_1) = (b_1 - 1) \beta(b_2)$$
in the ring $\mathcal{O}_K$. Pick $b_1 \in U \setminus \{1\}$ (exists!). Then for all $b_2 \in U \setminus \{1\}$ we obtain
\[
\frac{\beta(b_1)}{b_1 - 1} = \frac{\beta(b_2)}{b_2 - 1}
\]
in the fraction field $K$. Hence, this function is constant as $b_2$ varies over $U \setminus \{1\}$. Let $c_0 \in K$ be its value. Thus,
\[
\beta(b) = c_0(b - 1)
\]
holds for all $b \in U \setminus \{1\}$. Plugging in $b_1 = b_2 = 1$ in the Dietz condition (1), we also find $\beta(1) = 0$, so this formula is actually valid for all $b \in U$. Since $\beta(b) \in \mathcal{O}_K$ for all $b$ by assumption, we deduce $c_0 \in (\mathcal{O}_K : J(U))$, see Equation (1.1). Recall that by Lemma 4 for every $c_0 \in (\mathcal{O}_K : J(U))$ we in turn get an automorphism (in full detail: get an biholomorphism of the OT manifold, which canonically induces an automorphism of the fundamental group), so we have shown that there is a left exact sequence
\[
1 \rightarrow (\mathcal{O}_K : J(U)) \rightarrow \{ \theta \in \text{Aut}(\pi) \mid \delta = \text{id} \} \rightarrow \text{Aut}_R(\mathcal{O}_K),
\]
where we read the middle term as those automorphisms whose Dietz triple has $\delta = \text{id}$. The left map is $c_0 \mapsto (\text{id}, \beta, \text{id})$, where $\beta$ sends $b \mapsto c_0(b - 1)$, and the right map is $(\alpha, \beta, \text{id}) \mapsto \alpha$. Indeed, given any $\alpha \in \text{Aut}_R(\mathcal{O}_K)$ and defining $\beta(b) := 0$, we see that $(\alpha, \beta, \text{id})$ satisfies the Dietz conditions. It follows that the above sequence is also exact on the right and we leave it to the reader to check that this actually defines a right section, so this is a split exact sequence. We obtain the semi-direct product decomposition of our claim. \( \square \)

We obtain a left exact sequence
\[
(1.6) \quad 1 \rightarrow \{ \theta \in \text{Aut}(\pi) \mid \delta = \text{id} \} \rightarrow \text{Aut}(\pi) \rightarrow \text{Aut}(U),
\]
where the left group corresponds to the triples $(\alpha, \beta, \text{id})$ and the right arrow $T$ is the map $(\alpha, \beta, \delta) \mapsto \delta$.

**Lemma 4.** Suppose our OT manifold is of simple type. We have $\text{im} T = A_U$, where $A_U$ is as in Definition 5.

**Proof.** Let $(\alpha, \beta, \delta)$ be an arbitrary Dietz triple as in Lemma 2. Now, $\alpha \in \text{Aut}(\mathcal{O}_K, +)$. Pick some $a \in \mathcal{O}_K$ such that $\alpha(a) \neq 0$ (exists since $\alpha$ is a bijection). Define a function $\varphi: U \rightarrow K$ by
\[
(1.7) \quad \varphi(b) := \frac{\alpha(ba)}{\alpha(a)} \quad \text{for} \quad b \in U.
\]
By Dietz condition (2) we have $\alpha(ab) = \alpha(a)\delta(b)$, so this equals $\delta(b)$. We note that the choice of $a$ is irrelevant. We compute
\[
\varphi(b_1b_2) = \frac{\alpha(b_1b_2a)}{\alpha(a)} = \frac{\alpha(b_1(b_2a))}{\alpha(b_2a)} \frac{\alpha(b_2a)}{\alpha(a)},
\]
but $\frac{\alpha(b_1(b_2a))}{\alpha(b_2a)} = \varphi(b_1)$ since, as we had explained, the choice of $a$ is irrelevant, so we could also take $b_2a$ instead (moreover, $\alpha(b_2a) = \delta(b_2)a$ by condition (2) and since $\delta$ takes values in $U$, $\alpha(a) \neq 0$ implies that $\alpha(b_2a) \neq 0$, so the division above was fine). Thus, we find
\[
\varphi(b_1b_2) = \varphi(b_1) \cdot \varphi(b_2)
\]
for all $b_1, b_2 \in U$. Similarly, one checks that $\varphi(b_1 + b_2) = \varphi(b_1) + \varphi(b_2)$. Thus, by linear extension, we obtain that $\varphi: U \rightarrow K$ can be extended to a ring homomorphism
\[
\varphi: R \rightarrow K,
\]
where $R$ is the smallest subring of $\mathcal{O}_K$ containing all $a \in U$ as before. As $X$ is by assumption of simple type, there is no proper subfield of $K$ which already contains $U$. Thus, the field of fractions of $R$, which by $R \subseteq \mathcal{O}_K$ is contained in $K$, must be $K$ itself. Hence, $\varphi$, by extension to the field of fractions $\varphi(x/y) := \varphi(x)/\varphi(y)$ defines a field automorphism $\varphi: K \rightarrow K$. As we had remarked below Equation (1.7) $\varphi|_U = \delta$, but $\delta \in \text{Aut}(U)$, so $\varphi U \subseteq U$. It follows $\varphi \in A_U$. \( \square \)
Lemma 5. Suppose our OT manifold is of simple type. Then for $\pi := \pi_1(X)$, $\text{Aut}(\pi)$ is canonically isomorphic to

$$\{ \theta \in \text{Aut}(\pi) \mid \delta = \text{id} \} \rtimes A_U.$$  

Proof. By the previous lemma and Equation 1.6, we have the exact sequence

$$1 \rightarrow \{ \theta \in \text{Aut}(\pi) \mid \delta = \text{id} \} \rightarrow \text{Aut}(\pi) \xrightarrow{T} A_U.$$  

A right splitting is given by sending $\varphi \in A_U$ to $(\varphi |_{\mathcal{O}_K}, 0, \varphi |_U)$. The Dietz conditions are easily seen to hold. $\square$

Lemma 6. Suppose our OT manifold is of simple type. Then $\text{Aut}_R(\mathcal{O}_K) = \mathcal{O}_K^\times$, where $R$ is the smallest subring of $\mathcal{O}_K$ containing all $u \in U$.

Proof. Suppose $g \in \text{Aut}_R(\mathcal{O}_K)$. Let $\beta, \lambda \in \mathcal{O}_K$ be arbitrary. As $X$ is of simple type, we have $\mathbb{Q} \cdot R = K$, i.e. $\beta = \frac{1}{n} r$ for some $n \geq 1$ and $r \in R$. Then $g(\beta \lambda) = g(\frac{1}{n} r \lambda) = \frac{1}{n} r g(\lambda)$, as $g$ is an $R$-module homomorphism. Hence, $g(\beta \lambda) = \beta g(\lambda)$. It follows that $g$ is even an $\mathcal{O}_K$-module homomorphism. Thus, $g \in \text{Aut}_{\mathcal{O}_K}(\mathcal{O}_K)$ and since $\mathcal{O}_K$ is free of rank one over itself, $\text{Aut}_{\mathcal{O}_K}(\mathcal{O}_K) \cong \mathcal{O}_K^\times$; the converse inclusion is obvious. $\square$

Combining the previous lemmas, we obtain the following result.

Proposition 2. Suppose our OT manifold $X$ is of simple type. Then for $\pi := \pi_1(X)$, the automorphism group $\text{Aut}(\pi)$ is canonically isomorphic to

$$\left( (\mathcal{O}_K : J(U)) \rtimes \mathcal{O}_K^\times \right) \rtimes A_U.$$  

More concretely, it has a canonical ascending three step filtration $F_0 \subseteq F_1 \subseteq F_2$ whose graded pieces are

$$\text{gr}_0^F \text{Aut}(\pi) \cong (\mathcal{O}_K : J(U));$$  
$$\text{gr}_1^F \text{Aut}(\pi) \cong \mathcal{O}_K^\times;$$  
$$\text{gr}_2^F \text{Aut}(\pi) \cong A_U.$$  

These isomorphisms are all canonical.

Now we are ready to prove the key ingredient for our results.

Theorem 4. Suppose our OT manifold is of simple type. Then the biholomorphism group $\text{Aut}(X)$ is canonically isomorphic to

$$\left( \left( \mathcal{O}_K : J(U) \right) \rtimes \mathcal{O}_K^\times \right) \rtimes A_U.$$  

More concretely, it has a canonical ascending three step filtration $F_0 \subseteq F_1 \subseteq F_2$ whose graded pieces are

$$\text{gr}_0^F \text{Aut}(X) \cong \mathcal{O}_K / J(U);$$  
$$\text{gr}_1^F \text{Aut}(X) \cong \mathcal{O}_K^\times / U;$$  
$$\text{gr}_2^F \text{Aut}(X) \cong A_U.$$  

For $\text{gr}_0^F \text{Aut}(X)$ the isomorphism is non-canonical, while for $\text{gr}_i^F \text{Aut}(X)$ with $i = 1, 2$ it is.

Proof. By Lemma 2, all three groups in Equation 1.8 indeed induce biholomorphisms, but jointly they generate the entire iterated semi-direct product, so we just have to show that there are no other biholomorphisms. Let $\theta : X \rightarrow X$ be an arbitrary biholomorphism. It lifts to the universal covering space,

$$\tilde{\theta} : \mathbb{H}^s \times \mathbb{C}^t \rightarrow \mathbb{H}^s \times \mathbb{C}^t.$$  

Moreover, it induces a canonical map $\theta_* : \pi_1(X, \ast) \rightarrow \pi_1(X, \ast)$ on the fundamental group, and by Prop. 2 we get an element in

$$\left( \left( \mathcal{O}_K : J(U) \right) \rtimes \mathcal{O}_K^\times \right) \rtimes A_U.$$
We leave it to the reader to check that we can write \( \mathcal{O}_K^{x,+} \) instead of \( \mathcal{O}_K^x \), which amounts to the fact that \( \theta \) preserves being in the upper half plane. Now, by Lemma [1] we may associate a (possibly different) biholomorphism \( \theta' \) to this element. Thus, we have that \( f := \theta\theta'^{-1} \) is a biholomorphism of \( X \) which induces the identity on \( \pi_1(X,*) \). We are done once we prove that \( f = \text{id} \). Firstly, also \( f \) lifts to an automorphism \( \tilde{f} \) of the universal covering space. Since \( \tilde{f} \) descends modulo the action of \( \mathcal{O}_K \), we deduce that for any \( \gamma \in \mathcal{O}_K \) and \( \tilde{z} = (z_1, \ldots, z_{s+1}) \in \mathbb{H}^s \times \mathbb{C}^t \) there exists some \( \gamma'_s \in \mathcal{O}_K \) such that
\[
(1.9) \quad \tilde{f}(\tilde{z} + \sigma(\gamma)) - \tilde{f}(\tilde{z}) = \sigma(\gamma'_s).
\]
If we fix \( \gamma \) and let the point \( \tilde{z} \) vary, the value of \( \gamma'_s \) must vary continuously in \( \tilde{z} \). Since the image of \( \sigma \) is discrete, it follows that this function is locally constant and since \( \mathbb{H}^s \times \mathbb{C}^t \) is connected, it must be constant in \( \tilde{z} \). Then taking derivatives of Equation (1.9) yields
\[
\frac{\partial \tilde{f}}{\partial z_i}(\tilde{z} + \sigma(\gamma)) = \frac{\partial \tilde{f}}{\partial z_i}(\tilde{z}).
\]
It follows that the partial derivatives \( \frac{\partial \tilde{f}}{\partial z_i} \) descend to the quotient \( (\mathbb{H}^s \times \mathbb{C}^t)/\sigma(\mathcal{O}_K) \). However, \( (\mathbb{H}^s \times \mathbb{C}^t)/\sigma(\mathcal{O}_K) \) is an example of a Cousin group, as was proven by Oeljeklaus and Toma [OT05, Lemma 2.4] (this is also discussed in [BO15a, BO15b]), it carries no holomorphic functions except the constant ones. Thus, these partial derivatives are necessarily constant. It follows that
\[
(1.10) \quad \tilde{f}(\tilde{z}) = A\tilde{z} + B
\]
for a matrix \( A \). As \( \tilde{f} \) induces the identity on \( \pi_1 \), it follows that for any \( u \in U \) and any \( a \in \mathcal{O}_K \) we have
\[
\tilde{f}(\sigma(u)\tilde{z} + \sigma(a)) = \sigma(u)\tilde{f}(\tilde{z}) + \sigma(a).
\]
We hence get \( A\sigma(a) + B = \sigma(u)B + \sigma(a) \) for all \( u \in U, a \in \mathcal{O}_K \). But this plainly implies \( A = \text{id}, B = 0 \). □

2. Review of some class field theory

We briefly recall the (very few!) tools we need from class field theory. Let \( K \) be a number field. A modulus for \( K \) is a function
\[
m : \{\text{places of the number field } K\} \rightarrow \mathbb{Z}_{\geq 0}
\]
such that (1) for all but finitely many places \( P \) we have \( m(P) = 0 \), (2) if \( P \) is a real place, we only allow \( m(P) \in \{0, 1\} \), and (3) for complex places \( P \) we demand \( m(P) = 0 \). The algebro-geometrically inclined reader might prefer to think of a modulus as an effective Weil divisor on
\[
\text{Spec}(\mathcal{O}_K) \cup \{\text{real places}\},
\]
where real places are only allowed to have multiplicity zero or one. Fitting into this pattern, let \( m_0 \subseteq \mathcal{O}_K \) be the ideal defined by the prime factorization
\[
m_0 = \prod P^{\text{m}(P)},
\]
i.e. literally we take the possibly non-reduced closed subscheme cut out by the Weil divisor, ignoring the datum at the real places. One customarily also says that an ideal \( I \) divides \( m \) if we have \( m_0 \mid I \) as ideals in \( \mathcal{O}_K \).

There is the standard group homomorphism
\[
(2.1) \quad \text{div} : K^\times \rightarrow \prod_{P \in \{\text{maximal ideals of } \mathcal{O}_K\}} \mathbb{Z}, \quad a \mapsto (v_P(a))_P,
\]
which associates to any element \( a \in K^\times \) the exponents \( v_P(a) \) of its unique prime ideal factorization, \( P \) being one of the maximal primes. Equivalently, this is the map sending a rational function on \( \text{Spec} \mathcal{O}_K \) to its Weil divisor. There is a slight variation of this theme:

**Definition 4.** For \( K \) a number field and \( m \) a modulus, define
\[
(1) \quad I^S(m) := \prod_{P, v_P(a) = 0} \mathbb{Z}, \text{ where } P \text{ runs through the prime ideals of } \mathcal{O}_K; \text{ or equivalently this is the group of Weil divisors of } \text{Spec}(\mathcal{O}_K) - \{\text{primes dividing } m\}.
\]
(2) \( K_{m,1} := \{ a \in K^\times \mid v_P(a-1) \geq m(P) \text{ for all } P \mid \mathfrak{m}, \text{ and moreover } \sigma(a) > 0 \text{ for all real embeddings with } \mathfrak{m}(\sigma) = 1 \}; \)

(3) \( U_{m,1} := K_{m,1} \cap \mathcal{O}_K^\times. \)

Once we pick an arbitrary modulus \( \mathfrak{m} \), we can refine Equation 2.1 in the obvious way, to a group homomorphism

\[
K_{m,1} \longrightarrow I^S(\mathfrak{m}).
\]

If \( \mathfrak{m} = 1 \) is the zero modulus, i.e. \( m(P) = 0 \) for all places \( P \), this becomes Equation 2.1

**Definition 5.** For an arbitrary modulus \( \mathfrak{m} \) we call

\[
C_{\mathfrak{m}} := I^S(\mathfrak{m})/K_{m,1}
\]

the ray class group modulo \( \mathfrak{m} \).

**Theorem 5** (Global Class Field Theory). Let \( K \) be a number field.

1. For every modulus \( \mathfrak{m} \), the ray class group \( C_{\mathfrak{m}} \) is finite, and there exists a canonical finite abelian field extension \( L_\mathfrak{m}/K \) along with a canonical group isomorphism

\[
\psi_{\mathfrak{m}} : C_{\mathfrak{m}} \sim \text{Gal}(L_\mathfrak{m}/K).
\]

The field \( L_\mathfrak{m} \) is known as the ray class field of \( \mathfrak{m} \).

2. In fact, \( L_\mathfrak{m} \) can be characterized uniquely as the largest abelian field extension of \( K \) such that the ramification of \( L_\mathfrak{m} \) over \( K \) is bounded from above by the multiplicities of \( \mathfrak{m} \). The multiplicity 0 or 1 at the real places means whether we allow a real place to split into a pair of complex conjugate embeddings in \( L_\mathfrak{m} \) (multiplicity 1) or demand it to stay real (multiplicity 0).

3. If \( \mathfrak{m} \leq \mathfrak{m}' \) this induces an order-reversing correspondence \( L_\mathfrak{m} \subseteq L_{\mathfrak{m}'} \) and the diagram

\[
\begin{array}{ccc}
\psi_{\mathfrak{m}} & : & C_{\mathfrak{m}} \rightarrow \text{Gal}(L_{\mathfrak{m}}/K) \\
\downarrow & & \downarrow \\
\psi_{\mathfrak{m}'} & : & C_{\mathfrak{m}'} \rightarrow \text{Gal}(L_{\mathfrak{m}'}/K)
\end{array}
\]

commutes. Here the left-hand side downward arrow is the natural surjection from changing \( \mathfrak{m} \) in Definition 5, while the right-hand side downward arrow comes from the Galois tower

\[
\begin{array}{ccc}
L_{\mathfrak{m}'} & \rightarrow & L_{\mathfrak{m}} \\
\uparrow & & \uparrow \\
\mathfrak{m} & \rightarrow & K
\end{array}
\]

2.1. Exceptional moduli.

**Lemma 7.** Let \( \mathfrak{m} \) be a modulus. Then \( J(U_{m,1}) \subseteq \mathfrak{m}_0 \).

**Proof.** Every element in \( J(U_{m,1}) \) is of the shape \( a = \sum a_i(u_i - 1) \) for \( a_i \in \mathcal{O}_K \) and \( u_i \in U_{m,1} \). The unique prime ideal factorization of \( \mathfrak{m}_0 \) is (by the very definition of \( \mathfrak{m}_0 \)), \( \mathfrak{m}_0 = \prod P^{m(P)} \), and so it suffices to check that \( v_P(a) \geq m(P) \) for all prime ideals \( P \). If \( P \) divides \( \mathfrak{m} \), we have

\[
(2.2) v_P(u_i - 1) \geq m(P)
\]

for all \( u_i \in U_{m,1} \), just by Definition 2.1 so by the ultrametric inequality for valuations, we find

\[
v_P(a) \geq \min \{ v_P(a_i(u_i - 1)) \} \geq \min \{ v_P(u_i - 1) \} \geq m(P),
\]

so this is fine. If \( P \) does not divide \( \mathfrak{m} \), there is no counterpart of the condition of Equation 2.2 in the definition of \( U_{m,1} \), so we just get \( v_P(u_i - 1) \geq 0 \) since \( u_i \in \mathcal{O}_K^\times \) and therefore \( u_i - 1 \in \mathcal{O}_K \) is integral. On the other hand, then \( m(P) = 0 \), so actually Equation 2.2 holds simply for all prime ideals \( P \).

The following definition goes in the direction of a sufficient criterion to have equality:

**Definition 6.** Let \( K \) be a number field. We say that the modulus \( \mathfrak{m} \) is exceptional if

1. it has \( m(P) = 1 \) for all real places, and
(2) the ideal \( m_0 \) admits a set of generators \( g_1, \ldots, g_r \) such that each \( g_i + 1 \) is a totally positive unit, i.e. an element of \( \mathcal{O}_K^{\times+} \).

Lemma 8. If \( m \) is an exceptional modulus, we have equality of ideals \( J(U_{m,1}) = m_0 \).

Proof. The inclusion \( J(U_{m,1}) \subseteq m_0 \) is just Lemma \( \square \). We show the converse \( m_0 \subseteq J(U_{m,1}) \): Suppose \( g \in m_0 \). Then if \( g + 1 \) happens to be a totally positive unit, we get

\[
\nu_P((g + 1) - 1) = \nu_P(g) \geq m(P)
\]

for all prime ideals \( P \), and moreover \( \sigma(g + 1) > 0 \) for all the real places \( \sigma \). So in this case, we indeed have \( g + 1 \in U_{m,1} \). Thus, for an arbitrary \( a \in m_0 \), we expand it in terms of the ideal generators

\[
a = \sum a_i g_i = \sum a_i ((g_i + 1) - 1) \in J(U_{m,1}).
\]

Let us discuss a little how to work with exceptional moduli:

Example 1. Suppose \( m \) is a given modulus with \( m(P) = 1 \) for all real places and we want to check whether it is exceptional. To this end, compute the ray unit group \( U_{m,1} \). If \( J(U_{m,1}) \neq m_0 \), then \( m \) is not exceptional because otherwise this would contradict Lemma \( \square \). Conversely, if \( J(U_{m,1}) = m_0 \), then \( m \) is exceptional since the ideal \( J \) by its very definition is indeed generated from units \( g_i \) such that \( g_i + 1 \in U_{m,1} \) and \( U_{m,1} \subseteq \mathcal{O}_K^{\times+} \) by our condition on the real places.

Example 2. Of course, computing \( J(U_{m,1}) \) is costly, so for explicit example cases of exceptional moduli, the approach of the previous example is not to be recommended. Much better, one should simply pick a finite index subgroup \( U \subseteq \mathcal{O}_K^{\times+} \) and right away work with \( m_0 := J(U) \), and \( m(P) = 1 \) for all real places. Then \( m \) is an exceptional modulus by construction. We may consider this strategy for the following family \( \square \): Suppose \( m \geq 1 \). Then the polynomial

\[
f(T) = T^3 + mT - 1
\]

is irreducible, generates a cubic number field \( K \) with one real and one complex place, and the image of \( T \) in the number field, which we denote by \( u := \mathfrak{u} \), is a totally positive unit. Take \( U_l := \langle u^l \rangle \). Now, one needs to compute the fundamental unit \( v \) of \( K \) so that

\[
\mathcal{O}_K^\times = \langle -1 \rangle \times \langle v \rangle \quad \text{and} \quad \mathcal{O}_K^{\times+} = \langle 1 \rangle \times \langle v \rangle,
\]

i.e. \( \mathcal{O}_K^\times / \mathcal{O}_K^{\times+} \simeq \{ \pm 1 \} \). Define an exceptional modulus \( m \) via \( m_0 := J(U_l) \). It follows that \( J(U_{m,1}) = J(U_l) \). In a single computation, one finds the exponent \( e \) in \( u = v^e \), and then \( U/U_{m,1} = \langle \pm 1 \rangle \times \mathbb{Z}/(le\mathbb{Z}) \), so that \#U/U_{m,1} = 2le. We see that this produces an infinite family of exceptional moduli.

3. Torsion homology and ray class groups

Next, we need the following important computation from classical class field theory:

Proposition 3. For \( K \) an arbitrary number field and \( m \) an arbitrary modulus such that \( m(P) = 1 \) for all real places, there is an exact sequence of abelian groups

\[
1 \longrightarrow \frac{\mathcal{O}_K^{\times+}}{U_{m,1}} \longrightarrow \frac{(\mathcal{O}_K/m_0)^\times}{C_m} \longrightarrow C \longrightarrow 0.
\]

Here \( C \) denotes the ordinary ideal class group (\( = C_0 \), the ray class group for the trivial modulus).

Proof. This is \( \square \). This exercise follows directly from \( \square \).

The cardinalities \( h_m := |C_m| \) (and same for the trivial modulus, \( h := |C| \)) are known as the ray class number (resp. class number).
**Theorem 6.** Let $K$ be a number field with $s \geq 1$ real places and precisely one complex place. Moreover, suppose $m$ is an exceptional modulus. Then $U_{m,1}$ is an admissible subgroup in the sense of [OT03] §1. Let $X := X(K,U_{m,1})$ be the corresponding Oeljeklaus-Toma manifold. Then the graded Euler characteristic

$$
\chi^F(\text{Aut}(X)) = \prod_i |\text{gr}_i^F \text{Aut}(X)|^{(-1)^i}
$$

satisfies

$$
\frac{h_m}{h} \leq \frac{\chi^F(\text{Aut}(X))}{|A_{U_{m,1}}|},
$$

where $h_m$ denotes the ray class number of $m$ and $h$ is the ordinary class number.

**Proof.** We begin with the 4-term exact sequence of Prop. 3. Since $m$ is an exceptional modulus, by Lemma [we have $J(U_{m,1}) = m_0$, so this sequence specializes to

$$
1 \rightarrow \frac{\mathcal{O}_K^{\times,+}}{U_{m,1}} \rightarrow \left( \frac{\mathcal{O}_K}{J(U_{m,1})} \right)^{\times} \rightarrow \ker(C_m \rightarrow C) \rightarrow 0.
$$

Although there are much more direct ways to show this, note that this implies that $U/U_{m,1}$ is finite. In particular, the free rank of $U_{m,1}$ agrees with the one of $U = \mathcal{O}_K^{\times}$, and so is $s$ by Dirichlet’s Unit Theorem. Moreover, $U_{m,1} \subseteq \mathcal{O}_K^{\times,+}$ lies in the subgroup of totally positive units thanks to our condition on the real places in the modulus. It follows that $U_{m,1}$ is admissible in the sense of Oeljeklaus and Toma. Next, class field theory for the trivial modulus as well as $m$ produces the tower of class fields

$$
L_m \quad \text{ray class field for } m
$$

$$
\mid
$$

$$
H \quad \text{Hilbert class field}
$$

$$
\mid
$$

$$
K \quad \text{field}
$$

so that the Artin reciprocity symbol provides us with canonical and natural isomorphisms

$$
\text{Gal}(L_m/K) \cong C_m \quad \text{and} \quad \text{Gal}(H/K) \cong C.
$$

Thus, we have $\text{Gal}(L_m/H) \cong \ker(C_m \rightarrow C)$; and moreover by the tower law of field extension degrees,

$$
\frac{h_m}{h} = |\ker(C_m \rightarrow C)| = |\text{Gal}(L_m/H)| = \frac{\left| \left( \frac{\mathcal{O}_K}{J(U_{m,1})} \right)^{\times} \right|}{\left| \frac{\mathcal{O}_K^{\times,+}}{U_{m,1}} \right|}.
$$

By Theorem [we have a canonical filtration of the biholomorphism group,

$$
\text{gr}_0^F \text{Aut}(X) \simeq \frac{\mathcal{O}_K}{J(U_{m,1})};
$$

$$
\text{gr}_1^F \text{Aut}(X) \simeq \frac{\mathcal{O}_K^{\times,+}}{U_{m,1}};
$$

$$
\text{gr}_2^F \text{Aut}(X) \cong A_{U_{m,1}}.
$$

Thus, if we form a type of multiplicative Euler characteristic along the graded pieces

$$
\chi^F(\text{Aut}(X)) := \prod_i |\text{gr}_i^F \text{Aut}(X)|^{(-1)^i} = \frac{\left| \frac{\mathcal{O}_K}{J(U_{m,1})} \right| \cdot |A_{U_{m,1}}|}{|\mathcal{O}_K^{\times,+}/U_{m,1}|},
$$

we deduce from Equation [3.2] that

$$
\frac{h_m}{h} = \frac{\left| \left( \frac{\mathcal{O}_K}{J(U_{m,1})} \right)^{\times} \right|}{\left| \frac{\mathcal{O}_K^{\times,+}}{U_{m,1}} \right|} \leq \frac{\left| \frac{\mathcal{O}_K}{J(U_{m,1})} \right| \cdot |A_{U_{m,1}}|}{|\mathcal{O}_K^{\times,+}/U_{m,1}|} = \chi^F(\text{Aut}(X)).
$$

This finishes the proof. 

\[\Box\]
In a way, the principal point we wish to call attention to is that the so-called ray class group of a modulus \( \mathfrak{m} \), or the Galois group which is associated to it by class field theory, sits in a canonical exact sequence

\[
(3.4) \quad 1 \rightarrow \mathcal{O}_{K}^{\times,+}/U_{m,1} \rightarrow \left( \frac{\mathcal{O}_K}{J(U_{m,1})} \right)^{\times} \rightarrow \text{Gal}(L_m/H) \rightarrow 0,
\]

while (as we have shown) the automorphism group of \( X(K, U) \) possesses a canonical filtration \( F_\bullet \) whose graded pieces are (non-canonically) isomorphic to the groups in Equation \( (3.3) \). The group \( \mathcal{A}_{U_{m,1}} \) will frequently be trivial. Whenever this happens, note that the Sequence \( (3.4) \) could, albeit with quite some abuse of language, be rewritten as

\[
“1 \rightarrow \text{gr}_1^F \text{Aut}(X) \rightarrow \left( \text{gr}_0^F \text{Aut}(X) \right)^{\times} \rightarrow \text{Gal}(L_m/H) \rightarrow 0”.
\]

4. EXPOSITIVE TORSION ASYMPTOTICS

Finally, in the case of Oeljeklaus–Toma surfaces, the homology torsion growth can be related to the Mahler measure of a minimal polynomial.

**Theorem 7.** Let \( K \) be a number field with \( s = t = 1 \) embeddings and \( u \) a (totally) positive fundamental unit, i.e. \( \mathcal{O}_K^{\times,+} \simeq \mathbb{Z}\langle u \rangle \). Then all groups

\[
U_n := \mathbb{Z}\langle u^n \rangle
\]

are admissible, and for \( X_n := X(K, U_n) \) we have

\[
\lim_{n \rightarrow +\infty} \frac{\log |H_1(X_n, \mathbb{Z})_{\text{tor}}|}{n} = \log M(f),
\]

where \( f \) is the minimal polynomial of the unit \( u \), and \( M(f) \) denotes the Mahler measure of \( f \). Hence, \( |H_1(X_n, \mathbb{Z})_{\text{tor}}| \) always grows asymptotically exponentially as \( n \rightarrow +\infty \).

Note that in this case each \( X(K, U_n) \) is an Inoue surface \( X \) of type \( S^0 \). Further, by Dirichlet’s Unit Theorem, \( \mathcal{O}_K^{\times} \simeq \langle -1 \rangle \times \mathbb{Z}\langle u \rangle \) with \( u \) any fundamental unit. Thus, either \( u \) is totally positive so that \( \mathcal{O}_K^{\times,+} \simeq \mathbb{Z}\langle u \rangle \), or otherwise this is true after replacing \( u \) by \( -u \). Hence, once we have \( s = t = 1 \), a choice of \( u \) as in the statement of the theorem is always possible.

**Proof.** By Prop. [Bra16, Lemma 2] we have \( |H_1(X_n, \mathbb{Z})_{\text{tor}}| = |\mathcal{O}_K/J(\langle u^n \rangle)| \), where \( \langle u^n \rangle \) denotes the subgroup of \( \mathcal{O}_K^{\times,+} \) which is generated by \( u^n \); or equivalently the unique subgroup of \( \mathcal{O}_K^{\times,+} \) of index \( n \). By Lemma [Bra10, Lemma 2] we have \( J(\langle u^n \rangle) = (1 - u^n) \). Hence,

\[
|\mathcal{O}_K/J(\langle u^n \rangle)| = |N_{K/Q}(1 - u^n)| = |\sigma_i(1 - u^n)|,
\]

where \( \sigma_i \) for \( i = 1, 2, 3 \) denotes the three complex embeddings (one real, say \( \sigma_1 \), and one complex conjugate pair, say \( \sigma_2, \sigma_3 := \overline{\sigma_2} \)). We have

\[
1 = |N_{K/Q}(u^n)| = |\sigma_1(u^n)||\sigma_2(u^n)|^2
\]

since \( u \) is a unit. If \( |\sigma_i(u)| \leq 1 \) for all \( i \), then this equations forces that \( |\sigma_i(u)| = 1 \) for all \( i \), and then by Kronecker’s Theorem \( u \) must be a root of unity, which is impossible (by Dirichlet’s Unit Theorem \( u \) generates the non-torsion part of the unit group). Hence, we must have \( |\sigma_1(u)| > 1 \) and thus \( |\sigma_2(u)| < 1 \), or the other way round. We will now only handle the case \( |\sigma_1(u)| > 1 \) and leave the opposite case to the reader. We compute

\[
\frac{\log |H_1(X_n, \mathbb{Z})_{\text{tor}}|}{n} = \frac{\log |1 - \sigma_1(u^n)|}{n} + \frac{\log |1 - \sigma_2(u^n)|}{n}
\]

and therefore

\[
\frac{\log |H_1(X_n, \mathbb{Z})_{\text{tor}}|}{n} = \frac{\log |1 - \sigma_1(u^n)|}{n} + 2 \frac{\log |1 - \sigma_2(u^n)|}{n}
\]

\[
= \frac{\log |\sigma_1(u^n)(\sigma_1(u^n) - 1)|}{n} + 2 \frac{\log |1 - \sigma_2(u^n)|}{n}
\]

\[
= \log |\sigma_1(u)| + \frac{\log |1 - (\sigma_1(u)^{-1})^n|}{n} + 2 \frac{\log |1 - \sigma_2(u)^n|}{n}
\]
and since $|\sigma_1(u)|^{-1} < 1$ and $|\sigma_2(u)| < 1$, it follows that the second and third summand converge to zero as $n \to +\infty$. Next, since $|\sigma_1(u)| > 1$ and $|\sigma_2(u)| < 1$, the Mahler measure also satisfies $M(f) = |\sigma_1(u)|$, proving Equation 4.1 in this case. Furthermore, this means that

$$|H_1(X_n,\mathbb{Z})_{tor}| \approx |\sigma_1(u)|^n$$

for large $n$ with $|\sigma_1(u)| > 1$, so the torsion homology of $H_1$ grows strictly exponentially as an asymptotic. As explained, we leave the other case to the reader. The argument is entirely symmetric, just swapping the roles of $\sigma_1$ and $\sigma_2$. □

The previous proof explains the intense torsion growth which we had computationally observed in [Bra16], but which at that time had appeared somewhat mysterious.

This type of argument is not new, however, it might be new in the field of complex surfaces. It is a well-known type of behaviour in 3-manifold topology and knot invariants. In fact, it turns out that Inoue surfaces, by the general fact that their fundamental group has a canonical epimorphism to $\mathbb{Z}$,

$$\pi_1(X) \longrightarrow \mathbb{Z}$$

form an example of a space with an “augmented group” as fundamental group, in the sense of Silver and Williams [SW02]. One can rephrase the previous theorem in such a way that it becomes a special case of [SW02 Prop. 2.5]. To this end, note that $\prod_{n=1}^{\infty} \Delta(\zeta)$ in [SW02 Equation 2.2], can also be rewritten as a resultant, and the previous proof can alternatively be spelled out as a computation of exactly this resultant. We will not go into this in detail since the above proof is quicker than citing [SW02 Prop. 2.5]. Nonetheless, this elucidates the general picture.

References

[BO15a] L. Battisti and K. Oeljeklaus, A generalization of Sankaran and LVMB manifolds, Michigan Math. J. 64 (2015), no. 1, 203–222. MR 3265986

[BO15b] Holomorphic line bundles over domains in Cousin groups and the algebraic dimension of Oeljeklaus-Toma manifolds, Proc. Edinb. Math. Soc. (2) 58 (2015), no. 2, 273–285. MR 3341439

[Bra16] O. Braunling, Oeljeklaus-toma manifolds and arithmetic invariants, Math. Z. (2016), no. 2, 273–285. MR 3341439

[Die12] J. Dietz, Automorphism groups of semi-direct products, Comm. Algebra 40 (2012), no. 9, 3308–3316. MR 2981138

[Neu99] J. Neukirch, Algebraic number theory, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322, Springer-Verlag, Berlin, 1999, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. MR 1697859

[OT05] K. Oeljeklaus and M. Toma, Non-Kähler compact complex manifolds associated to number fields, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 1, 161–171. MR 2141693 (2006c:32020)

[OV13] L. Ornea and V. Vuletescu, Oeljeklaus-Toma manifolds and locally conformally Kähler metrics. A state of the art, Stud. Univ. Babeş-Bolyai Math. 58 (2013), no. 4, 459–468. MR 3195237 (document)

[PV12] M. Parton and V. Vuletescu, Examples of non-trivial rank in locally conformal Kähler geometry, Math. Z. 270 (2012), no. 1-2, 179–187. MR 2875828 (2012k:32022)

[SW02] D. S. Silver and S. Williams, Torsion numbers of augmented groups with applications to knots and links, Enseign. Math. (2) 48 (2002), no. 3-4, 317–343. MR 1959606

FRIAS, ALBERT LUDWIG UNIVERSITY OF FREIBURG, ALBERTSTRASSE 19, 79104 FREIBURG IM BREISGAU, GERMANY
E-mail address: oliver.braunling@math.uni-freiburg.de

VICTOR VULETESCU, UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS, 14 ACADEMIEI STR., 70109 BUCHAREST, ROMANIA
E-mail address: vuli@fmi.unibuc.ro