ON THE DYNAMICAL FOUNDATIONS OF $\alpha$ DISKS

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ABSTRACT

The dynamical foundations of $\alpha$ disk models are described. At the heart of the viscous formalism of accretion disk models are correlations in the fluctuating components of the disk velocity, magnetic field, and gravitational potential. We relate these correlations to the large-scale mean flow dynamics used in phenomenological viscous disk models. MHD turbulence readily lends itself to the $\alpha$ formalism, but transport by self-gravity does not. Nonlocal transport is an intrinsic property of turbulent self-gravitating disks, which in general cannot be captured by an $\alpha$ model. Local energy dissipation and $\alpha$-like behavior can be reestablished if the pattern speeds associated with the amplitudes of an azimuthal Fourier decomposition of the turbulence are everywhere close to the local rotation frequency. In this situation, global wave transport must be absent. Shearing box simulations, which employ boundary conditions forcing local behavior, are probably not an adequate tool for modeling the behavior of self-gravitating disks. As a matter of principle, it is possible that disks that hover near the edge of gravitational stability may behave in accord with a local $\alpha$ model, but global simulations performed to date suggest matters are not this simple.

Subject headings: accretion, accretion disks — hydrodynamics — instabilities — turbulence

1. INTRODUCTION

For many years, the principal uncertainty and greatest impediment for the development of accretion disk theory was an understanding of the origin of turbulent transport. In their classic paper, Shakura & Sunyaev (1973) made the physically reasonable and enormously productive Ansatz that, whatever the underlying cause for its existence, the turbulent stress tensor $T_{ij}$ scaled with the local gas pressure $P$. They denoted the constant of proportionality as $\alpha$, and the "$\alpha$ disk" moniker has since become synonymous with the standard disk model. Despite the ongoing development of increasingly sophisticated large-scale numerical models, $\alpha$ disk modeling still remains the central link between theory and observations, the cornerstone of accretion disk phenomenology.

In the last several years, a promising candidate has emerged as the physical basis for $\alpha$ disk models. This is the magneto-rotational ("Balbus-Hawley") MHD instability (Balbus & Hawley 1991; 1998, and references therein). A large and ever-increasing body of numerical simulations leaves little doubt that this instability leads to the turbulent enhancement of angular momentum transport within an accretion disk. What has been lacking, however, is a systematic explanation of how turbulence may or may not lead to the phenomenological $\alpha$ disk equations that have been in use for many decades. Since this approach has been (and continues to be) the link between accretion disk theory and observations, a better understanding of the $\alpha$ formalism is clearly desirable. The present paper seeks to fill this role.

The dynamical foundations of viscous disk theory have always been somewhat fuzzy, and the benefits of sharpening our understanding are numerous. For example, it is desirable to clarify which results of $\alpha$ disk phenomenology are truly fundamental and which have more limited domains of applicability. The nature of time-dependent turbulent transport could be more fully elucidated: in what sense is it equivalent to a viscous stress? Most interestingly, by introducing the intermediate integral scales of turbulence into the investigation, an enormously richer class of physical problem emerges. Classical $\alpha$ disk theory addresses mean flow (macroscopic) dynamics, subsuming all integral scale structure into a viscous stress tensor. In this approach, macroscopic disk structure is coupled directly to the dissipative scales. One cannot begin to answer such questions as whether the turbulence is self-maintaining, or whether the transport is local or global; everything is simply prescribed. Finally, there are important questions facing disk modelers in nonmagnetized disks, such as protostellar disks on scales larger than a few AU. Is it sensible to model such regions with an $\alpha$ viscosity? Are disks that evolve under the influence of self-gravity $\alpha$ disks? What distinguishes turbulence well modeled by $\alpha$ viscosity from turbulence that is not? These and related questions form the focus of this paper.

An overview of this paper is as follows. In § 2, we present a review of classical viscous accretion theory. Although this material is well known, we revisit it with a renewed attention to how the viscous stress appears in the angular momentum and energy fluxes. This becomes a benchmark for the turbulent theories discussed in §§ 3 and 4. In § 3 we show that MHD turbulence acts very much along the lines of classical viscous theory, in both steady state as well as evolutionary disk models. In § 4, it is shown that the turbulent transport arising from self-gravity is not, in general, compatible with a viscous formalism. We discuss the physical basis for this behavior and show that there are limiting cases of restrictive generality, which are compatible with $\alpha$ disk theory. Finally § 5 summarizes our findings.
2. PRELIMINARIES

2.1. Classical Viscous Disk Theory

We begin with a brief review of classical viscous disk theory (Lynden-Bell & Pringle 1974; Pringle 1981). In Cartesian coordinates, with \( x, y, \) and \( z \) represented by dummy indices \( i, j, \) and \( k, \) the viscous stress tensor takes the form (Landau & Lifschitz 1959)

\[
\sigma_{ij} = \eta \left( \partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right),
\]

We use the standard notational convention of \( \partial_i \) denoting the partial derivative with respect to spatial coordinate \( i, \) and summation over repeated indices is implied unless stated otherwise. \( \delta_{ij} \) is the Kronecker delta function, \( v_i \) is the \( i \)th component of the velocity vector, and \( \eta \) is the dynamical viscosity. To the extent that the fluid behaves incompressibly, we may ignore the divergence term, a standard and generally well-justified procedure for the class of turbulence we wish to consider here. The idea of viscous disk theory is to regard the effects of turbulence as greatly enhancing the magnitude of \( \eta \) beyond its microscopic value, and doing nothing else.

The dynamical equations are mass conservation,

\[
\partial_t \rho + \partial_j (\rho v_j) = 0 ,
\]

and the equation of motion,

\[
\rho \partial_t v_j + \partial_j (\rho v_j v_i) = - \partial_j (P \delta_{ij} - \sigma_{ij}) - \rho \partial_i \Phi ,
\]

which may also be written

\[
\partial_t (\rho v_j) + \partial_i (\rho v_i v_j) + P \delta_{ij} - \sigma_{ij} = - \rho \partial_i \Phi ,
\]

an explicit statement of momentum conservation. (Our notation is again standard: \( \rho \) is the mass density, \( \Phi \) is the gravitational potential, and \( P \) is the gas pressure.) At this stage, we assume that \( \Phi \) is an imposed central disk potential; self-gravity is considered in § 4. In viscous disk theories, momentum transport—more usefully, angular momentum transport—is the task of \( \sigma_{ij}. \) If the differential rotation rate decreases with increasing radius, viscosity transports angular momentum outward.

Multiplying equation (3) by \( v_i, \) integrating terms by parts, using mass conservation, and finally summing over \( i \) leads to a mechanical energy equation

\[
\partial_t (\rho v^2/2) + \partial_i (\rho v_i v_j) = P \partial_i v_j - \partial_j (\rho v_i v_j) \Sigma_{ij} ,
\]

where

\[
v^2 = v_i v_i .
\]

The right-hand side of equation (5) represents work done on the fluid and heating of the fluid, respectively. The presence of work and heating terms links disk mechanics with thermodynamics. Here we wish to highlight the dual role of \( \sigma_{ij}: \) in equation (3), it is a term in the transport of mechanical energy flux; coupled in equation (5) to the strain \( \partial_j v_i, \) it is a mechanical energy loss term. In viscous disk models, the latter is, of course, the origin of accretion disk luminosity. Since \( \sigma_{ij} \) is a symmetric tensor,

\[
(\partial_j v_i) \sigma_{ij} = (1/2)(\partial_i v_j + \partial_j v_i) \sigma_{ij} = (1/2)\eta^{-1} \sigma_{ij} \sigma_{ij} > 0
\]

so the dissipated energy ultimately radiated is necessarily positive definite for incompressible turbulence (Landau & Lifschitz 1959).

In cylindrical coordinates \((R, \phi, z)\) the azimuthal equation of motion for an axisymmetric viscous disk expresses angular momentum conservation, and takes the explicit form

\[
\partial_t (\rho R v_\phi) + \nabla \cdot (\rho R v_\phi v_R - \eta R^2 \nabla \Omega) = 0 .
\]

The rotational velocity \( v_\phi \) is Keplerian,

\[
v_\phi = \frac{GM}{R} ,
\]

where \( M \) is the central mass, and

\[
\Omega = v_\phi .
\]

In the simplest form of viscous disk theory, we ignore the vertical structure, treating the disk as flat, set \( v_\phi = \Omega, \) and assume axisymmetry. That is, we use the height-integrated form of the mass, angular momentum, and energy equations. With \( \Sigma \) denoting the disk column density, mass conservation becomes

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial R \Sigma v_R}{\partial R} = 0 ,
\]

while angular momentum conservation follows immediately from equation (8):

\[
\frac{\partial}{\partial t} (\Sigma R^2 \Omega) + \frac{1}{R} \frac{\partial}{\partial R} \left( \Sigma R^2 \Omega v_R - \nabla \Sigma R^2 \frac{d \Omega}{d R} \right) = 0 ,
\]

where \( \nu \) is the kinematic viscosity, \( \eta \equiv \nu \rho. \) The energy dissipated per unit area of the disk \( Q_e \) ("emissivity") is found from equation (7) to be

\[
Q_e = (9/8)\nu \Sigma \Omega^2 .
\]

Under steady state conditions, the mass flux \( M \equiv -2\pi R \Sigma v_R \) and the angular momentum flux must both be constant. Assuming that the viscous stress vanishes at the inner edge of the disk \( R_0 \) leads to the relation

\[
M \left[ 1 - \left( \frac{R_0}{R} \right)^{3/2} \right] = 3\pi \nu \Sigma .
\]

The turbulent parameter \( \nu \Sigma \) is severely restricted by this relation and can be eliminated in favor of \( M \) in the expression for the emissivity, leading to (Pringle 1981)

\[
Q_e = \frac{3G\Sigma M}{8\pi R^3} \left[ 1 - \left( \frac{R_0}{R} \right)^{1/2} \right] .
\]

This is the classical \((Q_e, M)\) relationship, which leads to a surface emission temperature profile \( T_{\text{eff}}(R) \propto R^{-3/4}. \) Its utility lies in the absence of an explicit viscosity term, which has been eliminated by the requirements of constant angular momentum and mass flux.

We may drop the assumption of time-steady conditions, forgoing a functional restriction for \( \nu \Sigma \) in the process. Using equation (11) in equation (12) leads to a generalized mass accretion formula:

\[
\Sigma v_R R = -3R^{1/2} \frac{\partial}{\partial R} (\nu \Sigma R^2 \Omega) .
\]

This in turn may be used back in equation (11), yielding an equation for \( \Sigma \) in terms of \( \nu \) (Lynden-Bell & Pringle 1974):

\[
\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left( R^{1/2} \frac{\partial}{\partial R} (\nu R^{1/2} \Sigma) \right) .
\]
This is the classical evolutionary equation commonly used in accretion disk modeling. It requires an a priori specification of the functional dependence of \( v \) to be useful and leads to diffusive behavior (disk spreading) in eruptive systems. Ultimately mass is transported inward and angular momentum outward: all of the latter is in a vanishingly small component of the former.

3. MAGNETOHYDRODYNAMICAL TURBULENCE

The fundamental assumption underlying essentially all phenomenological modeling of turbulent disks is the following: it makes sense to use a two-scale approach to represent mathematically disk attributes of astrophysical interest. With the possible exception of “flickering” in CV systems (e.g., Welsh, Wood, & Horne 1996), observational data are assumed to involve length and timescales that are larger than the characteristic “eddy turnover” scales of the turbulence. One works with averages that are assumed to be well defined and to represent the large-scale properties of the disk, much as classical dynamo theory (Krause & Rädler 1980) represents mean fields and mean helicity. While this statement probably does not strike the reader as startling or controversial, it masks subtleties. The assumption has remarkably restrictive consequences.

The problem is that the power spectrum of almost all nondissipative quantities (including the stress tensor itself) is dominated by the largest scales of the turbulence, in this case, the disk scale height and rotation period. The implicit averaging must be on scales large compared to the scale height but small compared with the radius, and large compared with the orbital period but small compared with viscous and thermal time scales. There must be an asymptotic domain where these scales are cleanly separated, so that a computed radial disk profile is insensitive to the averaging procedure. Let us assume this is the case and pursue its consequences.

The dynamical equation of motion in the presence of a magnetic field \( B \) is

\[
\rho \frac{\partial \mathbf{v}}{\partial t} + (\rho \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla (\rho \Phi) + \left( \frac{B^2}{4\pi} \right) \mathbf{B} + \eta_v \nabla^2 \mathbf{v} .
\]

We have denoted the viscosity as \( \eta_v \) to distinguish it from the resistivity associated with the magnetic field, which we will denote as \( \eta_B \). We have dropped terms proportional to \( \mathbf{V} \cdot \mathbf{v} \) in the viscous term. The azimuthal component of this equation can be written in a form that expresses angular momentum conservation:

\[
\frac{\partial}{\partial t} (\rho \mathbf{v}_\phi) + \mathbf{V} \cdot \left[ \rho \mathbf{v}_\phi \mathbf{v} - \frac{B^2}{4\pi} \mathbf{B} \right]
+ \left( \frac{B^2}{8\pi} \right) \mathbf{v}_\phi - \eta_v \mathbf{v}_\phi \nabla^2 \mathbf{v} = 0 ,
\]

where the subscript \( \phi \) denotes a poloidal vector component.

We now separate the circular motion \( R\Omega \) from the non-circular motion \( u \), treating the latter as a fluctuating quantity, though not necessarily with vanishing mean. We have

\[
\mathbf{v} = R\Omega \mathbf{\hat{\phi}} + \mathbf{u} .
\]

Although mean drift velocities may be present (the disk must accrete), we assume that such motions are small compared with fluctuation amplitudes,

\[
|\langle \mathbf{u} \rangle|^2 \ll |\mathbf{u}^2| ,
\]

where the angle brackets denote a suitable average, discussed below. Furthermore, the direct contribution to the angular momentum flux from the microscopic viscosity \( \eta_v \) is generally negligible. (This is why turbulence is necessary!) We may drop this term.

Substituting for \( v \) in equation (19), and averaging over azimuth, denoting such means as \( \langle \cdot \rangle \) gives

\[
\frac{\partial}{\partial t} \langle \rho R^2 \Omega \mathbf{\hat{\phi}} \rangle + \mathbf{V} \cdot \mathbf{T} = 0 ,
\]

where \( \mathbf{T} \) is the poloidal stress tensor

\[
\mathbf{T} = \rho \mathbf{v}_\phi \mathbf{u}_\phi - \frac{1}{4\pi} \mathbf{B}\mathbf{B} .
\]

We have dropped the \( \rho \mathbf{v}_\phi \mathbf{u}_\phi \) term in comparison with \( \rho \Omega \) in the leading time derivative. Henceforth, we will use the Alfvén velocity

\[
\mathbf{u}_A \equiv \frac{B}{\sqrt{4\pi \rho}}
\]

in favor of the magnetic field vector \( \mathbf{B} \).

3.1. MHD Turbulence and Viscous Disk Theory

To make contact with the classical viscous disk theory of the previous section, we need to integrate equation (22) over \( z \) and to assume that surface terms can be dropped. Furthermore, we wish to regard height-integrated, azimuthal-averaged flow quantities as smooth functions of \( R \). As indicated above, this also implies some sort of radial smoothing—an average over a volume small compared with \( R \) but larger than a disk scale height. We follow the notation of Balbus & Hawley (1998) and define the density weighted mean of flow attribute \( X \) to be

\[
\langle X \rangle_\rho \equiv \frac{1}{2\pi \Sigma \Delta R} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \int_0^{2\pi} dR \rho X d\phi dz ,
\]

where \( \Sigma \) is the integrated and similarly radially averaged disk column density. The radial component of \( \mathbf{T}/\rho \) resulting from this operation will be denoted simply as \( W_{R\phi} \). Angular momentum conservation becomes

\[
\frac{\partial}{\partial t} \langle R^2 \Omega \rangle_\rho + \frac{1}{R} \frac{\partial}{\partial R} \left( R^3 \Sigma \langle u_R \rangle_\rho + R^2 \Sigma W_{R\phi} \right) = 0 .
\]

Mass conservation follows straightforwardly from integrating the fundamental equation and leads to a form essentially identical to equation (11):

\[
\frac{\partial}{\partial t} \langle \Sigma u_R \rangle_\rho + \frac{1}{R} \frac{\partial}{\partial R} \left( \Sigma R^2 W_{R\phi} \right) = 0 .
\]

Using this in equation (26) gives a general formula for the mass accretion rate

\[
\Sigma \langle u_R \rangle_\rho = -\frac{1}{R(R^2 \Omega)} \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) ,
\]

where the prime denotes differentiation with respect to \( R \). Combining equations (28) and (27) gives us the analog to
equation (17) in a turbulent disk, for any angular momentum profile:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{R} \frac{\partial}{\partial R} \left( \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) \right). \tag{29}$$

Since $W_{R\phi}$ is not known a priori, in practical terms, equation (29) represents only a marginal improvement on the phenomenological equation (17). But we may see that “viscous” evolution does not require the explicit adoption of a viscous stress tensor. Any disk in which $u_R$ and $u_\phi$ (and $u_{R\theta}$ and $u_{R\phi}$) are positively correlated must behave similarly, with the caveat that the correlation tensor must be a locally defined quantity.

We have discussed thus far only the dynamics of the turbulence. Once $W_{R\phi}$ is known, and it depends primarily on correlations on the largest turbulent scales, the disk evolution may be directly calculated by equation (29). Classical viscous disk theory also addresses the energetics. Since viscosity is the agent of transport, there must be dissipation as well. The energy is directly thermalized from its free source in the differential rotation, down to thermal scales.

In a turbulent disk, matters are more complex. Energy cascades from the differential rotation to the scales of the largest fluctuations, thence to the integral self-similar scales, and finally to the dissipative Kolmogorov scale (which may be set by resistivity rather than viscosity). In a steady state disk, we expect that the rate at which energy is extracted from the differential rotation, which may be easily calculated in terms of the stress tensor, to be equal to the rate at which it is thermalized, which would otherwise not be directly calculable. The upshot of this is that the steady state turbulent disks behave viscously in their energetics as well in their dynamics. But classical viscous theory makes a stronger assumption by its very nature: the rate of thermalization of the free energy of differential rotation is the same in both steady and evolutionary models. This may be true in an evolving turbulent disk, but it is not obviously true. It depends upon whether the cascade is efficient. Fortunately, we shall see that this question may be directly and quantitatively answered within the stress tensor formalism we are using.

The evolution of the magnetic field in a plasma with resistivity $\eta_b$ is given by

$$\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_b \nabla \times \mathbf{B}). \tag{30}$$

The energy of mechanical energy equation is obtained by dotting equation (18) with $\mathbf{v}$, dotting equation (30) with $\mathbf{B}$, and combining the two. After some simplification (e.g., Balbus & Hawley 1998), we arrive at

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \Phi + \frac{B^2}{8\pi} \right) + \nabla \cdot \left( \mathbf{v} \mathbf{B} \right) = PV \cdot \mathbf{v} - \eta_f (\partial_v v_j)(\partial_{v_j} v_i) - \frac{\eta_b}{4\pi} |\nabla \times \mathbf{B}|^2. \tag{31}$$

We have dropped the term $- \eta_f |\nabla \cdot \mathbf{v}|^2$ on the right-hand side, since in a turbulent disk it is generally small compared with the other viscous term. The right-hand pressure term, though also proportional to $\nabla \cdot \mathbf{v}$, cannot be dropped; as noted in $\S$ 2, it represents a link to the internal energy of the disk via the first law of thermodynamics. The unwritten energy flux in square brackets is

$$\left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \Phi + P \right) + \frac{B}{4\pi} \times (\mathbf{v} \times \mathbf{B}), \tag{32}$$

where, as before, we have not included the transport due to explicit viscosity or resistivity.

The energy flux is more complex than its angular momentum counterpart but can be greatly simplified by retaining only leading terms in a $u < R \Omega$ expansion. When averaged as before, the radial energy flux is

$$\Sigma \left( \frac{1}{2} R^2 \Omega^2 + \Phi \right)(u_R)_{\rho} + \Sigma R \Omega W_{R\phi}, \tag{33}$$

which may be compared with the radial angular momentum flux of equation (26),

$$\Sigma R^2 \Omega (u_R)_{\rho} + \Sigma R W_{R\phi}. \tag{34}$$

(Note that to effect the height integration, we have assumed that the magnetic field is force-free above the disk. This is a physically reasonable assumption, but one that is less than general. The energy dissipation rate is, however, indifferent to the presence or absence of surface terms resulting from vertical integrations, since the ignored vertical fluxes would contribute nothing to the local mechanical energy losses in the disk.) The key point is to observe that the only turbulence parameters entering into either the angular momentum or energy radial fluxes are $u_R$ and $W_{R\phi}$. This is the essence of an $a$ disk.

The issue is most clear for a steady model (Balbus & Hawley 1998). In this case the accretion rate

$$M \equiv -2\pi \Sigma R (u_R)_{\rho}, \tag{35}$$

is constant, and if the stress tensor at the inner edge of the disk $R_0$ is vanishingly small, then

$$W_{R\phi} = \frac{\dot{M} \Omega}{2\pi \Sigma} \left[ 1 - \left( \frac{R_0}{R} \right)^{1/2} \right]. \tag{36}$$

Now, one cannot calculate directly the thermalization losses, since the small-scale gradients are not known. But one can calculate the divergence of the large-scale flux, since the spatial dependence of $W_{R\phi}$ is determined by angular momentum conservation. This must be the small-scale dissipation rate. We find

$$Q_c = -\Sigma W_{R\phi} \frac{d\Omega}{d\ln R}, \tag{37}$$

which is precisely the analog of equation (13) if $W_{R\phi}$ is replaced by a large-scale viscous stress. (Note that $Q_c > 0$.) This result can also be obtained directly from the energy equation for the $u$ fluctuations themselves by demanding that sources and sinks balance in steady state (Balbus & Hawley 1998).

More is required, however. In viscous models, the thermalization rate is given by equation (13) whether steady conditions prevail or not. The question before us is whether the thermalization rate (37) is just as general: does it hold when $M$ is not constant and when the energy density of the disk changes with time? We now show that it does.

First, let us recall the fundamental relations for the angular and epicyclic frequencies,

$$\Omega^2 = \frac{1}{R} \frac{\partial \Phi}{\partial R}, \quad \kappa^2 = \frac{1}{R^3} \frac{\partial}{\partial R} \left( R^2 \Omega^2 \right) = \frac{1}{R^3} \frac{\partial}{\partial R} \left( R^3 \Phi \right), \tag{38}$$
as well as the specific energy,
\[ \frac{1}{2} R^2 \Omega^2 + \Phi = \frac{1}{2R} \frac{\partial}{\partial R} (R^2 \Phi). \]  
(39)

Thus, the energy flux of equation (33) becomes
\[ \frac{1}{2R} \langle u_R \rangle \frac{\partial}{\partial R} (R^2 \Phi) + \Sigma R \Omega W_{R\phi}. \]  
(40)

We may substitute for \( \langle u_R \rangle \), using equation (28). This gives an energy flux of
\[ -\Omega \left( \frac{R^2 \Phi}{(R^3 \Phi')} \right) \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) + \Sigma R \Omega W_{R\phi}. \]  
(41)

The quantity of interest is the heating rate \( Q_e \) due to turbulent dissipation per unit area, and it is given by vertically integrating the left-hand side of equation (31). Making use of the above, it may be written in the form
\[ -Q_e = \frac{1}{2R} \frac{\partial R^2 \Phi}{\partial R} \frac{\partial \Sigma}{\partial t} \]
\[ + \frac{1}{R} \frac{\partial}{\partial R} \left[ -\Omega \left( \frac{R^2 \Phi}{(R^3 \Phi')} \right) \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) + \Sigma R \Omega W_{R\phi} \right]. \]  
(42)

If we now use equation (29) for \( \partial \Sigma / \partial t \) in equation (42), we obtain
\[ -Q_e = \frac{1}{R^2} \frac{\partial (R^2 \Phi)}{\partial R} \frac{\partial}{\partial R} \left[ \frac{R^2 \Omega}{(R^3 \Phi')} \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) \right] \]
\[ - \frac{1}{R} \frac{\partial}{\partial R} \left[ \Omega \left( \frac{R^2 \Phi}{(R^3 \Phi')} \right) \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) - \Sigma R^2 \Omega W_{R\phi} \right]. \]  
(43)

This unwieldy formula immediately simplifies to
\[ -Q_e = -\frac{R \Omega}{(R^3 \Phi')} \left[ \frac{\partial}{\partial R} (\Sigma R^2 W_{R\phi}) \right] \frac{\partial}{\partial R} \left[ \frac{1}{R} \frac{\partial}{\partial R} (R^2 \Phi) \right], \]  
(44)

Furthermore, for any function \( \Phi \), the following identity is easily verified,
\[ (R^{-1} (R^2 \Phi))' = R^{-2} (R^3 \Phi') , \]  
(45)

leading to a complete collapse of our expression down to the single term
\[ -Q_e = \Sigma W_{R\phi} \frac{dQ}{d \ln R} , \]  
(46)

which is the desired result. To leading order in the turbulent fluctuation amplitudes, the thermalization rate per unit area of a magnetized disk is given by the above, whether the disk is evolving or in a steady state. This result is nicely compatible with classical viscous thin disk theory. That not all disk turbulence is so easily subsumed will be seen in the next section.

4. SELF-GRAVITY

Self-gravitational forces can be important for galactic and protostellar disks. In its most extreme manifestation, self-gravity can hold the disk together and cause substantial deviations from a Keplerian rotation law. But this requires a disk mass comparable to or in excess of the central compact mass, and we will not consider this limit. Instead we focus on a more common situation in which the local self-gravitating free-fall time \( (\frac{G \rho}{\Omega^2})^{1/2} \) is comparable to or smaller than the \( 1/\Omega \). There are several equivalent ways of expressing this condition, the classical Toomre (1964) \( Q \) criterion being the best known (Binney & Tremaine 1987). With \( c_s \) denoting the sound speed, if
\[ Q \equiv \frac{\kappa c_s}{\pi G \Sigma} < 1 , \]  
(47)

then local density perturbations are unstable to gravitational collapse in a thin disk. If we define the vertical scale height \( H \) by \( c_s H = \Omega \) and the disk density by \( \rho H = \Sigma \), then for a Keplerian disk, the \( Q \) criterion becomes
\[ \frac{\Omega^2}{\pi G \rho} < 1 , \]  
(48)
in rough agreement with our initial estimate, and we may avoid an explicit reference to the disk temperature.

Self-gravity is obviously important in the formation stages of a galaxy or a star, but it is also likely to be a key component in later evolutionary stages, especially in the outer regions of the disk where \( \Omega^2/\rho \) is likely to be small. We shall concentrate here on the latter case, assuming a well-defined Keplerian disk is present, with self-gravity causing small but critical departures from circular flow. The ratio of disk mass \( M_d \) to central mass \( M \) is found from equation (48) to be of order \( (H/QR) \), so our approximation is justified for thin disks.

Progress in the numerical modeling of disk systems has been impressive, and sophisticated simulations are now possible, although investigators understandably tend to want to explore the more dramatic behavior of very massive disks. One of the interesting questions these modelers are addressing is whether turbulence wrought by self-gravity is amenable to a viscous diffusion treatment (e.g., Laughlin & Royczyska 1996). We now examine this point.

4.1. Dynamical and Energy Fluxes

The self-gravity potential \( \Phi_S \) satisfies the Poisson equation, most conveniently written in the form
\[ \rho = \frac{1}{4 \pi G} \frac{\partial \Phi_S}{\partial t} , \]  
(49)

where the subscript \( i \) (or \( j, k \) below) denotes a Cartesian coordinate, and the summation convention on repeated subscripts is used unless otherwise stated. The connection between the gravitational force and its associated stress tensor was first made by Lynden-Bell & Kalnajs (1972):
\[ -\rho \frac{\partial \Phi_S}{\partial t} = -\frac{\partial \Phi_S}{\partial t} \frac{\partial}{\partial t} [\Phi_S] \]
\[ = \frac{1}{4 \pi G} \frac{\partial}{\partial t} \left[ -\frac{\partial \Phi_S}{\partial \hat{t}} [\hat{t} \Phi_S] + \frac{\delta}{2} (\frac{\partial \Phi_S}{\partial \hat{t}} [\hat{t} \Phi_S]) \right]. \]  
(50)

To keep both the gravitational and nongravitational components of the stress tensor on an equal footing, define the velocities
\[ u_{gi} = \frac{V \Phi_S}{\sqrt{4 \pi G \rho}} . \]  
(51)
Then, in the presence of self-gravity and magnetic fields, the $R\phi$ component of the stress tensor becomes

$$ W_{R\phi} = \langle u_R u_\phi + u_{GR} u_{G\phi} - u_{AR} u_{A\phi} \rangle_p . $$  

(52)

Equations (26) and (29), angular momentum conservation and the disk evolution equation, continue to hold in precisely the same form when self-gravity is present, if the stress tensor $W_{R\phi}$ is amended simply as above. Gravitational torques are calculated formally in exactly the same way as turbulent and magnetic torques. When $Q$ is of order unity, the kinetic $u$ terms and gravitational $u_\phi$ terms of $W_{R\phi}$ are comparable if $u \sim |\nabla \Phi_S|/2\Omega$—i.e., if the fluctuation velocities are due to self-gravity impulses on a rotation timescale.

Consider next the energetics of self-gravity. Is the volumetric dissipation rate still given by equation (46), with the gravitationally amended version of $W_{R\phi}$? The answer is not in general, but under some interesting conditions it is. Let us see how this emerges.

We seek to write the expression $\rho v \partial_t \Phi_S$ in conservation form: the time derivative of an energy density plus the divergence of a flux. We have

$$ \rho v \partial_t \Phi_S = \partial_i (\rho v_i \Phi_S) - \partial_i (\rho v) = \partial_i (\rho v_i \Phi_S) + \partial_i (\rho \phi S) - \rho \partial_i \Phi_S$$

(53)

where the last equality follows from mass conservation plus an integration by parts. There is no sign yet of the gravitational stress tensor putting in an appearance, but the final term remains dangling for the moment. This may be written

$$ \rho \partial_t \Phi_S = \frac{1}{4\pi G} (\partial_i \partial_i (\Phi_S, \Phi_S)) = \frac{1}{4\pi G} \partial_i [\partial_i (\Phi_S, \Phi_S)] - \frac{1}{8\pi G} \partial_i [\partial_i (\Phi_S, \Phi_S)].$$

(54)

Returning to vector-invariant notation,

$$ \rho v \cdot \nabla \Phi_S = \frac{\partial}{\partial t} \left( \rho \Phi_S + \frac{1}{8\pi G} |\nabla \Phi_S|^2 \right) + \nabla \cdot \left( \rho v \Phi_S - \frac{\nabla \Phi_S}{4\pi G} \frac{\partial \Phi_S}{\partial t} \right).$$

(55)

There is some ambiguity as to whether one assigns terms to the energy density or the flux. For example, an equivalent formulation of equation (55) is

$$\rho v \cdot \nabla \Phi_S = -\frac{1}{8\pi G} \frac{\partial}{\partial t} |\nabla \Phi_S|^2 + \nabla \cdot \left[ \rho v \Phi_S - \frac{\nabla \Phi_S}{4\pi G} \frac{\partial \Phi_S}{\partial t} + \frac{1}{8\pi G} \frac{\partial}{\partial t} \nabla \Phi_S^2 \right].$$

(56)

But there is no apportionment that of itself produces an $R\phi$ component of the gravitational stress tensor. Since the energy flux is most readily interpretable in equation (55), we shall use this form of energy conservation in our discussion below.

The combination $-\nabla \Phi_S \cdot \partial \Phi_S/\partial t$ will be familiar to students of acoustical theory (e.g., Lighthill 1978), where precisely this form of energy flux is associated not with a gravitational potential but with the velocity potential of irrotational sound waves. In the acoustic case, this emerges from the “$Pv$” term in the energy flux, a term that is third order in the fluctuation amplitudes for incompressible turbulence, and therefore negligible. Indeed, an important physical distinction between a disk in which there is a superposition of waves and a disk that is truly turbulent is the dominance of the $W_{R\phi}$ term over the $Pv$ in the latter’s energy flux.

If waves were present in a turbulent disk, would this change the relative dominance? Not in a thin non-self-gravitating Keplerian disk with good $R\phi$ correlations in the stress tensor. In a density wave, the pressure contribution to the energy flux will be of order $u^2 c_s$ in the velocities, whereas the stress tensor term is of order $\alpha R \Omega c_s^3$. Since $u^2 \sim \alpha c_s^2$, the stress tensor contribution will always be dominant (by a factor of $R/H$) in a thin disk.

The appearance of a second-order contribution of the potential in the energy flux suggests qualitatively new transport features in self-gravitating disks. In retrospect, the breakdown of the $\alpha$ formalism is perhaps not surprising. Turbulence in hydrodynamical shear flows or MHD disks arises because vorticity fields and magnetic fields are “ensnared” by shear and funnel this free energy into fluctuations. These fields may become ensnared because both are frozen into their respective fluids. Their evolution is entirely local, and the vorticity and magnetic fields are governed by essentially identical equations. Gravitational fields are not frozen into the fluid, and we should not expect local dissipation of its associated turbulence, which is the inevitable consequence of an energy flux depending upon the stress tensor and drift velocity, as may be seen in equation (33). Self-gravity is generally a global phenomenon (its field equation is elliptic), and one has no cause to expect a repetition of our earlier magnetic success with local theory.

4.2. The Local Limit

If instead of the combination $\rho v \Phi_S - \nabla \Phi_S \cdot \partial \Phi_S$ appearing in the energy flux, the combination $\Omega \nabla \Phi_S \cdot \partial \Phi_S$ emerged, we would be able to construct a local model of the dissipation. In this case, the gravitational component of the stress tensor would couple energetically precisely as magnetic and Reynolds stresses couple. It will turn out that the conversion of both these terms corresponds to

$$ \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) \Phi_S = 0 ,$$

(57)

when the disturbances are analyzed in terms of WKB waves. This is a very revealing requirement, for it is just this condition that defines the corotation resonance in linear density wave theory, and it is only at this location that waves couple directly the disk (e.g., Goldreich & Tremaine 1979, hereafter GT). It is quite natural, therefore, that this condition reemerges as the requirement for gravitationally driven energy stresses to be thermalized.

Let us examine the structure of the energy conservation equation further. We focus for simplicity upon an unmagnetized disks and assume that $R \Omega \gg u_\phi$. Denoting the volumetric mechanical energy losses as $-\epsilon$, the self-gravitational analog to equation (31) becomes

$$ \frac{\partial}{\partial t} \left[ \rho \left( \frac{v^2}{2} + \Phi + \Phi_S \right) + \frac{1}{8\pi G} |\nabla \Phi_S|^2 \right]$$

$$ + \nabla \cdot \left[ \rho v \left( \frac{v^2}{2} + \Phi + \Phi_S \right) - \nabla \Phi_S \left( \frac{\partial \Phi_S}{\partial t} \right) \right] = -\epsilon .$$

(58)

We have neglected the pressure contribution to the energy flux (but see below).
We rewrite the rate of production of mechanical work as given by the left-hand side of equation (58), separating the terms in a suggestive manner:

\[ \frac{\partial}{\partial t} \delta' + \nabla \cdot \mathbf{F} + \nabla \cdot \left( \rho \Phi \mathbf{v} - \frac{\nabla \Phi}{4\pi G} \right) = -\epsilon, \quad (59) \]

where the energy density \( \delta' \) is

\[ \delta' = \rho \left( \frac{v^2}{2} + \Phi + \Phi_S \right) + \frac{1}{8\pi G} |\nabla \Phi|^2, \quad (60) \]

the local flux \( \mathbf{F} \) is

\[ \mathbf{F} = \rho \mathbf{v} \left( \Phi + \frac{v^2}{2} + \frac{\Omega}{4\pi G} \nabla \Phi \right), \quad (61) \]

and

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \nabla \cdot \frac{\partial}{\partial \phi}. \quad (62) \]

The final terms on the left-hand side of equation (59) are “anomalous” from the point of view of a disk theory, and the flux will henceforth be denoted as \( F_k \). The radial component of \( \mathbf{F} \) is just given by equation (33), with \( W_{\phi \phi} \) amended as in equation (52). Were \( F_k \) negligible, we would be led directly to an \( \alpha \) disk model via the route we followed in §§ 2 and 3. However, since these terms may be of order \( R \Omega u_0^2 \), in the velocities, they cannot be neglected. Their physical interpretation is discussed in the next section.

4.3. Wave Fluxes in Self-gravitating Disks

To understand the role of the anomalous flux, it is helpful to study it in the context of the simplest possible fluctuating self-gravitating disk model: WKB waves in a thin, pressureless disk. (The inclusion of pressure terms leads to a more complicated calculation, but with precisely the same final conclusion.) The waves have the canonical form \( \exp \left( i \int k \, dz \right) \), where \( k \) is the local radial wavenumber, \( m \) the azimuthal wavenumber variable, and \( \omega \) the fixed wave frequency, and satisfy the dispersion relation

\[ (\omega - \omega)^2 = \kappa^2 - 2\pi G |k|. \quad (63) \]

The potential \( \Phi_S \) has the vertical spatial dependence \( e^{-|kz|} \) out of the disk midplane (Lin & Shu 1966; Binney & Tremaine 1987).

The radial anomalous energy flux, averaged over azimuth and integrated over height, is

\[ \langle F_k \cdot \hat{e}_\phi \rangle = \int_{-\infty}^{\infty} \langle \rho u_k \Phi_S - \frac{1}{4\pi G} \nabla \Phi \rangle dz \]

\[ \times \left( \frac{\partial \Phi_S}{\partial t} + \Omega \frac{\partial \Phi_S}{\partial \phi} \right) \phi dz - \frac{1}{4\pi G} \]

\[ \times \left( \Omega - \frac{\omega}{m} \right) \int_{-\infty}^{\infty} \langle \rho u_k \Phi_S \rangle \phi dz. \quad (64) \]

To do the first integral, we need to be able to express \( u_k \) in terms of \( \Phi_S \). This relation may be read off directly from equation (11) of GT:

\[ u_k = \frac{m \omega - \omega}{2\pi G \Sigma} \Phi_S(0) \text{ sgn} (k), \quad (65) \]

where \( \Phi_S(0) \) is the midplane \( (z = 0) \) value of the potential. Denoting the potential amplitude by \( \tilde{\Phi}_S \) and assuming \( \rho(z) = \Sigma \delta(z) \), the first integral is then

\[ \int_{-\infty}^{\infty} \langle \rho u_k \Phi_S \rangle dz = \frac{1}{4\pi G} (m \omega - \omega) \tilde{\Phi}_S^2 \text{ sgn} (k). \quad (66) \]

The second integral may be evaluated by noting that the integrand depends on \( z \) as \( \exp (-2|kz|) \). This gives

\[ -\frac{1}{4\pi G} \left( \Omega - \frac{\omega}{m} \right) \int_{-\infty}^{\infty} \langle \rho u_k \Phi_S \rangle dz \phi dz \]

\[ = -\frac{1}{8\pi G} (m \omega - \omega) \tilde{\Phi}_S^2 \text{ sgn} (k). \quad (67) \]

Thus,

\[ F_k \cdot \hat{e}_\phi = -\frac{1}{8\pi G} (\omega - m \Omega) \tilde{\Phi}_S^2 \text{ sgn} (k). \quad (68) \]

The angular momentum flux is also to be found in GT (eq. [30]; note their definition differs by a factor of \( 2\pi R \) from ours). It is simply

\[ F^J = -\frac{m}{8\pi G} \tilde{\Phi}_S^2 \text{ sgn} (k). \quad (69) \]

In other words, the anomalous radial energy flux is the product of this angular momentum flux and the Doppler-shifted wave pattern speed \( \omega/m - \Omega \). It therefore is identifiable as a true wave energy flux. (Turbulent energy and angular momentum fluxes, by way of contrast, are related by a factor \( \Omega \).) Its significance is that in a disk it will contribute to the total energy flux at a level comparable to the stress tensor \( W_{\phi \phi} \) if \( |\omega/m - \Omega|/\Omega \) is of order unity.

The effect is to prevent self-gravitating disks from behaving like \( \alpha \) disks; only if the anomalous flux vanishes can a self-gravitating disk behave like a local \( \alpha \) disk. The “forbidden zone” of wave propagation near the corotation point \( \omega = m \Omega \) will display properties similar to those of an \( \alpha \) disk. However, when a disk region undergoes forcing owing to a potential from a wave pattern rotating with a frequency very different from the local rotation frequency, it will not behave like an \( \alpha \) disk. Such a situation may occur, for example, when an exterior disk is forced by the potential caused by a developing central bar instability.

Energy can be exchanged between fluctuations and the differential rotation of the disk; unlike angular momentum, it need not be conserved in the noncircular motions. In contrast to a turbulent \( \alpha \) disk, a self-gravitating disk can evolve by extracting energy from the background shear and allocating it to the fluctuations (wave energy) without the need for mechanical energy dissipation. This allows for angular momentum transport with no associated local energy losses. Significant angular momentum transport of this type can occur if a global nonaxisymmetric mode develops in an initially gravitationally unstable disk. Such a construction need not be merely a transient initial condition. Such features are semipermanent, slowly evolving as the disk background parameters change (Papaloizou &
Savonije 1991; Papaloizou 1996; Laughlin & Royczyska 1996). More recently, careful analyses of self-gravitating disk simulations carried out by Laughlin, Korchagin, & Adams (1997, 1998) clearly show angular momentum transport produced by the onset of global nonaxisymmetric instability and subsequent generation of a global wave pattern that has extracted energy from the background shear. Transport of this type, which is seen to persist even after initial saturation, certainly does not have the character of that exhibited by an $\alpha$ disk (Laughlin & Royczyska 1996). These simulations, however, involve massive disks (comparable stellar and disk masses); similar studies of lower mass Keplerian disks have yet to be done.

4.4. Conditions under Which Self-Gravity Leads to an $\alpha$ Disk

There is a local limit in which the nonlocal energy flux terms vanish and equation (37) is recovered. It occurs when the shearing box limit is used to study self-gravitating disks.

The shearing box approximation is a standard approach to the dynamics of thin disks, both self-gravitating (Goldreich & Lynden-Bell 1965; Julian & Toomre 1966; Toomre 1981) and non-self-gravitating (Goldreich, Goodman, & Narayan 1986; Hawley, Gammie, & Balbus 1995). In this scheme, the disk is divided into local Cartesian patches with periodic boundary conditions being applied on their boundaries. Thus one considers a small box in the disk and sets up local corotating coordinates corotating with the patch center. Strictly periodic boundary conditions are applied in the azimuthal direction, so that $\phi$-averaging amounts to averaging over one azimuthal width of the box. However, in the radial direction, because of the presence of large-scale shear, periodicity is applied at boundary points that are azimuthally separating from one another. Related periodic points must shear apart with time. Thus, strict periodicity would hold only in comoving, shearing coordinates. Note that there is no preferred location in this description, so the box may be centered anywhere (except, of course, the origin).

The significance of these boundary conditions is that they force the divergence of $F^k_g$ to be zero when averaged over the box. If $\Delta R$ is the radial extent of the box, and $R \pm \Delta R/2$ represent the outer ($+$) and inner ($-$) boundaries, then the integrated box average leads to a term of the form

$$\left[ \left( \frac{\partial \Phi}{\partial t} \right)_* \right]^{R+\Delta R/2}_R \left( \frac{\Phi}{\partial R} \right)^{R-\Delta R/2}_R,$$

which must vanish. The square bracket notation denotes a difference to be taken between the upper and lower indicated locations. The boundary conditions force every fluid element on the inner edge to have a corresponding partner on the outer edge, and the appearance of $D/Dt$, rather than $\partial/\partial t$, ensures cancellation. Were the partial time derivative used, we would not be forced to this conclusion because the disk passes by “faster” at one of the boundaries compared with the other. If $F^k_g$ vanishes (in this averaged sense), we are lead to a standard $\alpha$ disk model. Clearly, however, this conclusion is entirely driven by the choice of boundary conditions. Energy loss or gain from an evolving wavelike flux would be quite incompatible with this type of periodicity.

There is no physical reason for the above boundary conditions to be satisfied in the neighborhood of an arbitrarily chosen disk location. Nevertheless, it is possible that there are circumstances under which $F^k_g$ may in effect vanish.

Disks evolving under the influence of their own self-gravity tend to hover near the critical $Q = 1$ level (Laughlin & Bodenheimer 1994). WKB waves with radial wavenumber $k_{crit} = \pi G \Sigma/c^2_0 \sim 1/(QH)$ are neutrally stable ($\omega - m\Omega = 0$); all other wavenumbers propagate. While ostensibly comparable in magnitude to $\mathcal{F}$, $F^k_g$ may be smaller in a $Q = 1$ disk. The most responsive (dominant?) local modes have $\omega - m\Omega \ll \Omega$, and this may be enough to suppress $F^k_g$. The effectiveness of this process depends both upon the disk’s alacrity in maintaining $Q$ near unity and upon the shape of the wave power spectra. Clearly, numerical simulations are needed to resolve the question of whether $Q = 1$ disks can be treated within the $\alpha$ formalism.

Values of $Q$ near unity are favored, of course, because dropping below this critical level results in vigorous dissipative shock heating, raising the temperature and stabilizing. Rising above $Q = 1$ allows the disk to cool and become destabilized (e.g., Sellwood & Carlberg 1984). The critical criterion also has some observational support through the work of Kennicutt (1989), who has found that the gaseous $Q$ value of active star-forming regions of disk galaxies is near critical. More generally, it is not yet well understood under what conditions heating and cooling will be able to regulate $Q$ efficiently in disks, thereby allowing the use of a simplifying $\alpha$ formalism.

5. SUMMARY

The dynamical foundations of viscous $\alpha$ disk models are rooted in the correlated fluctuations which create the underlying turbulent stresses. In this paper, we have shown that the mean flow dynamics of MHD turbulence follows the $\alpha$ prescription and in particular that the disk energy dissipation rate is always given by equation (37), even if the disk is evolving. The local character of MHD disturbances is itself rooted in the flux freezing equation, which forces local dissipation of the magnetic field in turbulent flow, analogous to vorticity dynamics in an unmagnetized shear layer.

The mean flow dynamics of a self-gravitating disk in general cannot be described so simply. Classical viscous disk theory requires a simple restrictive form for the mean momentum and energy fluxes (eqs. [34] and [33]); neither can depend upon transport properties other than $\langle u_R \rangle_\phi$ and $W_R$. The energy flux of self-gravitating disks is not reducible to a superposition of these quantities. Instead, what we refer to as anomalous flux terms are present. These terms allow self-gravitating disturbances (not necessarily of WKB form) to propagate nonlocally in the disk via the perturbed gravitational potential; a viscous disk cannot communicate with itself in a similar fashion. The angular momentum flux (strictly conserved) in a self-gravitating disk has the same canonical form it has in a non-self-gravitating disk, proportional to $W_R$; the energy flux is fundamentally different.

In a non-self-gravitating thin disk, wave energy transport depends upon terms in the flux that, while formally present, are small by order $H/R$. In a self-gravitating disk, the additional (nonpressure) terms that are present in the energy flux couple directly to the differential rotation of the disk, as does $W_R$. This additional coupling means in effect that transport becomes global on rotational timescales. Over similar times without self-gravity, the domain of wave influence is restricted to the vertical scale height $H$.

Shearing box simulations of self-gravitating disks employ
boundary conditions that force local behavior and inevitably must give rise to an $\alpha$ disk. Because self-gravity is intrinsically nonlocal in its manifestations, analyzing transport phenomena in self-gravitating disks within the shearing box formalism may be misleading. On the other hand, it is possible that critical $Q \approx 1$ disks will be dominated by wave-numbers for which $|\omega/m - \Omega|/\Omega$ is small, in which case $\alpha$ modeling might be a fair phenomenological description. To date however, global numerical simulations of massive self-gravitating disks do not seem to lend themselves readily to an $\alpha$ formalism. Whether the same is true for self-gravitating disks much less massive then their central stars is not yet known.

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