ALMOST CRITICAL LOCAL WELL-POSEDNESS FOR THE SPACE-TIME MONOPOLE EQUATION IN LORENZ GAUGE

ACHENEF TESFAHUN

Abstract. Recently, Candy and Bournaveas proved local well-posedness of the space-time monopole equation in Lorenz gauge for initial data in $H^s$ with $s > \frac{1}{4}$. The equation is $L^2$-critical, and hence a $\frac{1}{4}$ derivative gap is left between their result and the scaling prediction. In this paper, we consider initial data in the Fourier-Lebesgue space $\hat{H}^s_p$ for $1 < p \leq 2$ which coincides with $H^s$ when $p = 2$ but scales like lower regularity Sobolev spaces for $1 < p < 2$. In particular, we will see that as $p \to 1^+$, the critical exponent $s^c_p \to 1^-$, in which case $\hat{H}^1_{1+}$ is the critical space. We shall prove almost optimal local well-posedness to the space-time monopole equation in Lorenz gauge with initial data in the aforementioned spaces that correspond to $p$ close to 1.

1. Introduction

The space-time Monopole equation can be derived by a dimensional reduction from the Anti-Self-Dual Yang Mills equations, and is given by

\begin{equation}
F_A = *D_A \phi,
\end{equation}

where $F_A$ is the curvature of a one-form connection $A = A_\alpha dx^\alpha$ ($\alpha = 0, 1, 2$), $D_A \phi$ is a covariant derivative of the Higgs field $\phi$ and $*$ is the Hodge star operator with respect to the Minkowski metric diag$(-1, 1, 1)$ on $\mathbb{R}^{1+2}$. The space-time Monopole equation was first introduced by Ward [9] as a space-time analog of Bogomolny Equations or Magnetic Monopole equations. For more detailed survey of the equation see [6].

The unknowns are $A_\alpha$ and $\phi$ which are maps from $\mathbb{R}^{1+2}$ into a Lie algebra $\mathfrak{g}$ of a Lie group $G$, which for simplicity we assume to be the matrix group $SU(n)$:

$$A_\alpha, \phi : \mathbb{R}^{1+2} \to \mathfrak{g}.$$
The curvature of the connection, $F_A$, and the covariant derivative of the Higgs field, $D_A\phi$, are given by
\[
F_A = \frac{1}{2} (\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]) \, dx^\alpha \wedge dx^\beta,
\]
\[
D_A\phi = (\partial_\alpha \phi + [A_\alpha, \phi]) \, dx^\alpha,
\]
where $[\cdot, \cdot]$ is a Lie bracket. Using the definition of the Hodge star operator we write
\[
* D_A\phi = - (\partial_t \phi + [A_0, \phi]) \, dx^1 \wedge dx^2 - (\partial_1 \phi + [A_1, \phi]) \, dx^0 \wedge dx^2 + (\partial_2 \phi + [A_2, \phi]) \, dx^0 \wedge dx^1,
\]
which can be equated with $F_A$ to rewrite (1.1) as
\[
\begin{align*}
\partial_t \phi + \partial_1 A_2 - \partial_2 A_1 &= [A_2, A_1] + [\phi, A_0], \\
\partial_t A_1 - \partial_1 A_0 - \partial_2 \phi &= [A_1, A_0] + [A_2, \phi], \\
\partial_t A_2 - \partial_2 A_0 + \partial_1 \phi &= [A_2, A_0] + [\phi, A_1].
\end{align*}
\]
These equations are invariant under the gauge transformations
\[
A_\alpha \rightarrow A'_\alpha = O A O^{-1} + O \partial_\alpha O^{-1}, \quad \phi \rightarrow \phi' = O \phi O^{-1},
\]
where $O : \mathbb{R}^{1+2} \rightarrow G$ is a smooth and compactly supported map. The most popular gauges are (i) Temporal gauge: $A_0 = 0$, (ii) Coulomb gauge: $\partial^j A_j = 0$ and (iii) Lorenz gauge: $\partial^\alpha A_\alpha = 0$.

In Coulomb gauge, the system (1.2) can be written as a system of nonlinear wave equations for $(A_1, A_2, \phi)$ that contains null structure in the bilinear terms, coupled with elliptic equation for $A_0$. Czubak [2] proved local well-posedness of the space-time monopole equation (1.2) in the Coulumb gauge for small initial data (for both $A$ and $\phi$) in $H^s$ with $s > \frac{1}{4}$. Recently, Candy and Bournaveas [1] also observed that the space-time monopole equation in Lorenz gauge can be written as a system of nonlinear wave equations for $(A, \phi)$ and that the bilinear terms contained are null forms. They combined this new structure with bilinear estimates for the homogeneous wave equations in [5] to show that the Cauchy problem for (1.2) in Lorenz gauge is locally well-posed for large initial data in $H^s$ with $s > \frac{1}{4}$. This lifts the smallness assumption on the initial data by Czubak.
On the other hand, the space-time monopole equation is invariant under the scaling

\begin{align}
A^\lambda_\alpha(t, x) &= \lambda A_\alpha(\lambda t, \lambda x), \\
\phi^\lambda(t, x) &= \lambda \phi(\lambda t, \lambda x).
\end{align}

Then we have (similarly for $A_\alpha$, since both of the fields scale the same)

$$
\|\phi^\lambda(0, x)\|_{H^s} = \lambda^s \|\phi(0, x)\|_{\tilde{H}^s},
$$

which suggests that (1.2) is $L^2$-critical. So there is still a $\frac{1}{4}$ derivative gap between the scaling prediction and the result by Candy–Bournaveas.

In this paper, instead of the standard $H^s$ space we shall consider initial data in the more general space $\tilde{H}_p^s$, defined by the norm

$$
\|f\|_{\tilde{H}_p^s} = \|\langle \xi \rangle^s \hat{f}(\xi)\|_{L^p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p \leq 2,
$$

where $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. The homogeneous version $\hat{H}_p^s$ can be defined similarly. In the special case $p = 2$, we have $\hat{H}_p^s = H^s$. These spaces have been used by many authors to improve regularity results for a number of dispersive and wave equations. For instance, Grünrock [7] used these spaces to prove almost critical local well-posedness for 3D wave equations with quadratic nonlinearities. More recently, Grigoryan and Nahmod [8] used them to prove almost critical local well-posedness for 2D wave equations with quadratic null forms of Klainerman type.

For the scaling (1.3), it is easy to see

$$
\|\phi^\lambda(0, x)\|_{\tilde{H}_p^s} = \lambda^{s+\frac{2}{p}} \|\phi(0, x)\|_{\tilde{H}_p^s},
$$

which suggests that (1.2) is $\tilde{H}_p^{\frac{2}{p}-}$-critical, where

$$
s_c^p = \frac{2}{p} - 1.
$$

Again, for $p = 2$ we have the $L^2$-criticality but as $p \to 1^+$, $s_c^p \to 1^-$. Our goal in this paper is to prove local well-posedness of the space-time monopole equation (1.2) in Lorenz gauge for initial data in $\tilde{H}_p^s$ for $1 < p \leq 2$ and $s \geq \frac{1}{p}$. Note that $0 \leq s_c^p < \frac{1}{p}$ for $1 < p \leq 2$ but $s_c^p \to \frac{1}{p}$ as $p \to 1^+$. It means that on one end (when $p = 1+$) we have almost critical well-posedness in $\tilde{H}_1^{1-}$ and on the other end
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When \( p = 2 \) we have well-posedness in \( H^{\frac{1}{2}} \) (which is weaker than the result by Candy-Bournaveas).

To this end, we complement (1.2) with initial data

\[ A_\alpha(0) = a_\alpha \in \widehat{H}^{s}_p, \quad \phi(0) = \phi_0 \in \widehat{H}^{s}_p, \]

and state our main result as follows.

**Theorem 1.1.** Assume \( 1 < p \leq 2 \) and \( s \geq \frac{1}{p} \). Then given initial data (1.4), there exists \( T > 0 \) and a solution \((A_\alpha, \phi)\) to the space-time monopole equation (1.2) in Lorenz gauge with regularity

\[ (A_\alpha, \phi) \in C \left([-T, T], \widehat{H}^{s}_p\right). \]

Moreover, the solution is unique in a certain subspace of this regularity class, the solution depends continuously on the data, and higher regularity data persists in time.

The rest of the paper is organized as follows. In Section 2 we reformulate Theorem 1.1 to Theorem 2.1 after rewriting (1.2) in Lorenz gauge. In the same Section we introduce the spaces we shall work in together with some of their properties, and state a Lemma about an estimate in these spaces to the solution of the inhomogeneous linear wave equation. In Section 3 we show that Theorem 2.1 reduces to proving null form estimates. In Section 4 and 5 we prove these null form estimates. Finally, in Section 6 we give the proof of the Lemma mentioned.

**2. Rewriting (1.2) in Lorenz gauge and restating Theorem 1.1**

We follow [1] to rewrite (1.2) in Lorenz gauge. Define \( u, v : \mathbb{R}^{1+2} \to \mathfrak{g} \times \mathfrak{g} \) by

\[ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} A_0 + A_1 \\ \phi + A_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} A_0 - A_1 \\ \phi - A_2 \end{pmatrix}. \]

so that

\[ (A_0, A_1, A_2, \phi) = \frac{1}{2}(u_1 + v_1, u_1 - v_1, u_2 - v_2, u_2 + v_2). \]

The Lorenz gauge, \( \partial_t A_0 - \partial_1 A_1 - \partial_2 A_2 = 0 \), becomes

\[ \partial_t(u_1 + v_1) - \partial_1(u_1 - v_1) - \partial_2(u_2 - v_2) = 0. \]
Observe that

\[ [A_2, A_1] + [\phi, A_0] \pm ([A_2, A_0] + [\phi, A_1]) = [\phi \pm A_2, A_0 \pm A_1], \]

\[ [A_1, A_0] + [A_2, \phi] = \frac{1}{2} ([A_1 - A_0, A_1 + A_0] + [A_2 - \phi, A_2 + \phi]). \]

Using these identities and the Lorenz gauge, we can rewrite (1.2) as

\[
\begin{align*}
\partial_t u_1 - \partial_1 u_1 - \partial_2 u_2 &= \frac{1}{2}(u \cdot v - v \cdot u) \\
\partial_t u_2 + \partial_1 u_2 - \partial_2 u_1 &= [u_2, u_1] \\
\partial_t v_1 + \partial_1 v_1 + \partial_2 v_2 &= \frac{1}{2}(v \cdot u - u \cdot v) \\
\partial_t v_2 - \partial_1 v_2 + \partial_2 v_1 &= [v_2, v_1].
\end{align*}
\]

Furthermore, by introducing the matrices

\[
\alpha_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

we can rewrite the above equations as

\[
\begin{align*}
\partial_t u - \alpha \cdot \nabla u &= N(u, v) \\
\partial_t v + \alpha \cdot \nabla v &= N(v, u),
\end{align*}
\]

where \( \alpha = (\alpha_1, \alpha_2) \) and

\[
N(u, v) = \begin{pmatrix} \frac{1}{2}(u \cdot v - v \cdot u) \\ \beta u \cdot u \end{pmatrix}.
\]

The initial data for (2.1) become

\[
\begin{align*}
u(0) &= \begin{pmatrix} a_0 + a_1 \\ \phi_0 + a_2 \end{pmatrix}, \\
v(0) &= \begin{pmatrix} a_0 - a_1 \\ \phi_0 - a_2 \end{pmatrix}.
\end{align*}
\]

We diagonalize (2.1) by defining the projections

\[
P_{\pm}(\xi) = \frac{1}{2} \left( I \pm \alpha \cdot \hat{\xi} \right),
\]

where \( \hat{\xi} \equiv \frac{\xi}{|\xi|} \). Then \( u \) and \( v \) split into \( u = u_+ + u_- \), \( v = v_+ + v_- \), where \( u_{\pm} = P_{\pm}(D)u \) and \( v_{\pm} = P_{\pm}(D)v \). Here \( D = -i\nabla \) with symbol \( \xi \). Now, applying \( P_{\pm}(D) \)
to (2.1), and using the identities $\alpha \cdot D = |D|P_+(D) - |D|P_-(D)$, $P^2_\pm(D) = P_\pm(D)$ and $P_\pm(D)P^\mp_\pm(D) = 0$, (2.1) becomes

\begin{equation}
\begin{cases}
(i\partial_t + |D|)u_\pm = iP_\pm(D)N(u, v), \\
(i\partial_t + |D|)v_\pm = iP_\pm(D)N(v, u).
\end{cases}
\end{equation}

The initial data for (2.3) become

\begin{equation}
(u_\pm(0), v_\pm(0)) = (f_\pm, g_\pm) \in \hat{H}_p^s,
\end{equation}

where

\begin{align*}
f_\pm &= P_\pm(D) \begin{pmatrix} a_0 + a_1 \\ \phi_0 + a_2 \end{pmatrix}, \\
g_\pm &= P_\pm(D) \begin{pmatrix} a_0 - a_1 \\ \phi_0 - a_2 \end{pmatrix}.
\end{align*}

We now reformulate Theorem 1.1 as follows.

**Theorem 2.1.** Assume $1 < p \leq 2$ and $s \geq \frac{1}{p}$. Then given initial data (2.4), there exists $T > 0$ and a solution $(u_\pm, v_\pm)$ to (2.3) with regularity

\begin{equation}
(u_\pm, v_\pm) \in C([-T, T], \hat{H}_p^s).
\end{equation}

Moreover, the solution is unique in a certain subspace of this regularity class, the solution depends continuously on the data, and higher regularity data persists in time.

The rest of the paper is dedicated to the proof of Theorem 2.1.

Let us first fix some notation and introduce the spaces we shall work in together with some of their properties.

In estimates we use $A \lesssim B$ as shorthand for $A \leq CB$, where $C \gg 1$ is a positive constant. We use the shorthand $A \approx B$ for $A \lesssim B \lesssim A$. Throughout the paper, we use $p'$ as a conjugate exponent for $p$, and $\mathbb{1}_{\{\cdot\}}$ denotes the indicator function which is 1 if the condition in the bracket is satisfied and 0 otherwise.

Consider now the inhomogeneous linear equation

\begin{equation}
(i\partial_t + h(D))u = F(t, x), \quad u(0) = f(x),
\end{equation}

which has the representation formula

\begin{equation}
u(t) = W(t)f + \int_0^t W(t - t')F(t') \, dt',
\end{equation}

where

\begin{align*}
f_\pm &= P_\pm(D) \begin{pmatrix} a_0 + a_1 \\ \phi_0 + a_2 \end{pmatrix}, \\
g_\pm &= P_\pm(D) \begin{pmatrix} a_0 - a_1 \\ \phi_0 - a_2 \end{pmatrix}.
\end{align*}
where $\mathcal{W}(t) = e^{ith(D)}$ is the associated solution group. We define the associated $X_{p}^{s,b}$ spaces with the norm

$$
\|u\|_{X_{p}^{s,b}} = \|\langle \xi \rangle^{s}(-\tau + h(\xi))^{b}\tilde{u}(\tau, \xi)\|_{L_{\tau}^{p'}}.
$$

In the special cases, $h(\xi) = \pm|\xi|$, we use the notation $X_{p}^{s,b,\pm}$. We also define $\widetilde{L}_{\tau}^{p} = X_{p}^{0,0}$. The restriction of these spaces to a time slab $S_{T} = [-T, T] \times \mathbb{R}$ are denoted $X_{p}^{s,b}(S_{T})$, defined in the usual way by the norm

$$
\|u\|_{X_{p}^{s,b}(S_{T})} = \inf \left\{ \|u'\|_{X_{p}^{s,b}} : u' = u \text{ on } S_{T} \right\}.
$$

For $b > \frac{1}{p}$, we have the embedding

$$(2.7) \quad X_{p}^{s,b} \subset C\left(\mathbb{R}; \widetilde{H}_{p}^{s}\right)$$

where $C$ depends only on $b$. Indeed,

$$
\|u(t)\|_{\widetilde{H}_{p}^{s}}' = \int \langle \xi \rangle^{p'}|\tilde{u}(t, \xi)|^{p'}d\xi = \int \langle \xi \rangle^{p'}\int e^{it\tau}\tilde{u}(\tau, \xi)d\tau^{p'}d\xi,
$$

but by Hölder inequality

$$
\left| \int e^{it\tau}\tilde{u}(\tau, \xi)d\tau \right| \leq \left( \int \langle -\tau + h(\xi) \rangle^{-pb}d\tau \right)^{\frac{1}{p}} \left( \int \langle -\tau + h(\xi) \rangle^{p'b}|\tilde{u}(\tau, \xi)|^{p'}d\tau \right)^{\frac{1}{p'}},
$$

where the first integral on the right is bounded since $bp > 1$. A combination of these estimates will imply (2.7).

We also need the following estimate for the solution (2.6) of the inhomogeneous linear equation (2.5). The proof is included in the last section by modifying the proof of Lemma 5 in [4].

**Lemma 2.2.** Let $1 < p < \infty$, $\frac{1}{p} < b \leq 1$, $0 \leq \varepsilon \leq 1 - b$ and $0 < T < 1$. Assume $f \in \widetilde{H}_{p}^{s}$ and $F \in X_{p}^{s,b-1+\varepsilon}(S_{T})$. Then $u$ in (2.6) satisfies

$$
\|u\|_{X_{p}^{s,b}(S_{T})} \leq C \|f\|_{\widetilde{H}_{p}^{s}} + C T^{\varepsilon} \|F\|_{X_{p}^{s,b-1+\varepsilon}(S_{T})},
$$

where $C$ depends on $b$. 
3. Reduction of Theorem 2.1 to null form estimates

Set \( \frac{1}{p} = 1 - 2\varepsilon \) for \( 0 < \varepsilon \leq \frac{1}{4} \). By persistence of higher regularity argument it suffices to prove Theorem 2.1 for \( s = \frac{1}{p} \).

We shall then iterate the solutions to (2.3)–(2.4) in

\[
u_{\pm}, v_{\pm} \in X_{p, \pm}^{\frac{1}{p} + \varepsilon} (S_T).
\]

By a standard argument, using Lemma 2.2, the local existence problem of Theorem 2.1 reduces to proving the bilinear estimate (the estimate for \( N(v, u) \) is symmetrical)

\[
\|P_{\pm}(D)N(u, v)\|_{X_{p, \pm}^{\frac{1}{p}}} \lesssim \left(\|u_{\pm}\|_{X_{p, \pm}^{\frac{1}{p} + \varepsilon}} + \|u_{-}\|_{X_{p, -}^{\frac{1}{p} + \varepsilon}} + \|v_{+}\|_{X_{p, +}^{\frac{1}{p} + \varepsilon}} + \|v_{-}\|_{X_{p, -}^{\frac{1}{p} + \varepsilon}}\right)^2.
\]

Writing \( u = P_{+}(D)u_{+} + P_{-}(D)u_{-} \) and \( v = P_{+}(D)v_{+} + P_{-}(D)v_{-} \), we get

\[
N(u, v) = \sum_{\pm_1, \pm_2} \left( \frac{1}{2} \left( P_{\pm_1}(D)u_{\pm_1} \cdot P_{\pm_2}(D)v_{\pm_2} - P_{\pm_2}(D)v_{\pm_2} \cdot P_{\pm_1}(D)u_{\pm_1} \right) \right) P_{\mp_1}(D)\beta u_{\pm_1} \cdot P_{\mp_2}(D)u_{\pm_2},
\]

where in the second row we used the identity \( \beta P_{\pm}(D) = P_{\mp}(D)\beta \); the signs \( \pm_1 \) and \( \pm_2 \) are independent. Now, noting the property

\[
\|u(-t, x)\|_{X_{p, \pm}^{\frac{1}{p}}} = \|u(t, x)\|_{X_{p, \pm}^{\frac{1}{p}}}, \quad \|u(t, -x)\|_{X_{p, \pm}^{\frac{1}{p}}} = \|u(t, x)\|_{X_{p, \pm}^{\frac{1}{p}}},
\]

and that \( P_{\pm}(\xi) \) is bounded, (3.1) will reduce to proving

\[
\|P_{+}(D)w \cdot P_{\pm}(D)z\|_{X_{p, \pm}^{\frac{1}{p}}} \lesssim \|w\|_{X_{p, \pm}^{\frac{1}{p} + \varepsilon}} \|z\|_{X_{p, \pm}^{\frac{1}{p} + \varepsilon}},
\]

for \( w, z \in S(\mathbb{R}^{1+2}) \) taking values in \( \mathfrak{g} \times \mathfrak{g} \).

The key observation in the proof of (3.2) is the bilinear terms \( P_{+}(D)w \cdot P_{\pm}(D)z \) are null forms as shown in [1]. This can be seen by taking their space-time Fourier transform. Indeed,

\[
\langle P_{+}(D)w \cdot \tilde{P}_{\pm}(D)z(\tau, \xi) \rangle = \int_{\mathbb{R}^{1+2}} P_{\pm}(\xi - \eta)P_{+}(\eta)\tilde{w}(\lambda, \eta) \cdot \tilde{z}(\tau - \lambda, \xi - \eta) \, d\lambda \, d\eta,
\]

where the symbol satisfies the estimate (see Lemma 2.2. in [1])

\[
\|P_{\pm}(\xi - \eta)P_{+}(\eta)\| \lesssim \theta(\eta, \mp(\xi - \eta)).
\]
The angles on the right hand side quantifies the null structure (see Lemma 2 and Remark 2 in [4]).

So in view of (3.3) and (3.4), the proof of (3.2) essentially reduces to

\[
\|Q_{\pm}(\phi, \psi)\|_{X^{\frac{1}{p}, \theta}} \lesssim \|\phi\|_{X^{\frac{1}{p}+\epsilon, \pi}} \|\psi\|_{X^{\frac{1}{p}+\epsilon, \pi}},
\]

where

\[
Q_{\pm}(\phi, \psi)(\tau, \xi) = \int_{\mathbb{R}^{1+2}} \theta(\eta, \pm(\xi - \eta))\tilde{\phi}(\lambda, \eta)\tilde{\psi}(\tau - \lambda, \xi - \eta) \, d\lambda \, d\eta.
\]

for \(\phi, \psi : \mathbb{R}^{1+2} \to \mathbb{R}\) such that \(\tilde{\phi}, \tilde{\psi} > 0\).

So everything boils down to proving (3.5). We identify two cases where the product \(\phi\psi\) has Fourier support contained in either of the following sets:

(I) \(\{\xi : |\xi| < 1\}\), low frequency case

(II) \(\{\xi : |\xi| \geq 1\}\), high frequency case.

We give the proof of (3.5) in both of these cases in the following two Sections.

4. PROOF OF (3.5) IN THE CASE OF (I)

Let \(\chi \in C_c^\infty(\mathbb{R}^2)\) such that \(\chi(\xi) = 1\) on the set \(\{\xi : |\xi| < 1\}\). Then we estimate

\[
\text{l.h.s of (3.5)} \leq \left\|\hat{\phi}\psi\right\|_{L^{p'}_{\xi}} = \left\|\chi\hat{\phi}\tilde{\psi}\right\|_{L^{p'}_{\xi}} \leq \|\chi\|_{L^{\frac{1}{p}+\epsilon}_\xi} \left\|\hat{\phi}\tilde{\psi}\right\|_{L^{p'}_{\xi}} \lesssim \left\|\hat{\phi}\tilde{\psi}\right\|_{L^{p'}_{\xi}},
\]

where we used Hölder inequality in \(\xi\) and the assumption that \(\frac{1}{p'} = 2\epsilon\) (since \(\frac{1}{p} = 1 - 2\epsilon\)). Applying Young’s inequality to the convolution \(\hat{\phi}\tilde{\psi} = \hat{\phi} * \tilde{\psi}\), first in \(\tau\) and then in \(\xi\), we obtain

\[
\left\|\hat{\phi}\tilde{\psi}\right\|_{L^{p'}_{\xi}} \leq \left\|\hat{\phi}\right\|_{L^{\frac{1}{p}+\epsilon}_\xi} \|\tilde{\psi}\|_{L^{p'}_{\xi}} \leq \left\|\hat{\phi}\right\|_{L^{\frac{1}{p}+\epsilon}_\xi} \|\tilde{\psi}\|_{L^{p'}_{\xi}}.
\]

Now, by Hölder

\[
\left\|\hat{\phi}\right\|_{L^{\frac{1}{p}+\epsilon}_\xi} \leq \left\|\langle\xi\rangle^{\frac{1}{p}+\epsilon}\left(-\tau \pm |\xi|\right)^{\frac{1}{p}+\epsilon}\hat{\phi}\right\|_{L^{p'}_{\xi}} \left\|\langle\tau \pm |\xi|\rangle^{-\frac{1}{p}+\epsilon}\right\|_{L^{p'}_{\xi}} \left\|\langle\xi\rangle^{\frac{1}{p}+\epsilon}\hat{\phi}\right\|_{L^{p'}_{\xi}} \lesssim \left\|\langle\xi\rangle^{\frac{1}{p}+\epsilon}\hat{\phi}\right\|_{L^{p'}_{\xi}} = \|\phi\|_{X^{\frac{1}{p}+\epsilon, \pi}},
\]

\]
where we used the fact that
\[ \left\| (-\tau \pm |\xi|)^{-(\frac{1}{p}+\varepsilon)} \right\|_{L^p_{\xi}} \lesssim 1, \quad \left\| \xi \right\|_{L^\infty_{\xi}} \lesssim 1, \]
and since \( \frac{1}{p} + \varepsilon > 1\) and \( \frac{2}{1 - \frac{2}{p}} = 2\frac{1 - \frac{2}{p}}{1} > 2 \).

Similarly,
\[ \left\| (\xi + \xi) \frac{\hat{\psi}}{} \right\|_{L^{p+\varepsilon}_{\tau}} \lesssim \left\| (\xi + \xi) \frac{\hat{\psi}}{} \right\|_{L^{p+\varepsilon}_{\tau}} \lesssim \left\| \xi + \xi \right\|_{L^{\infty}_{\xi}} \cdot \]

5. Proof of (3.5) in the case of (II)

The angles in (3.6) satisfy the following estimates (see eg. [3], [1]):
\[ (5.1) \quad \theta(\eta, \xi - \eta) \approx \frac{r_{+}^\frac{1}{2}}{\min(|\eta|, |\xi - \eta|)^\frac{1}{2}}, \quad \theta(\eta, - (\xi - \eta)) \approx \frac{|\xi|^\frac{1}{2}r_{-}^\frac{1}{2}}{|\eta|^\frac{1}{2}|\xi - \eta|^\frac{1}{2}}, \]
where
\[ r_{+} = |\eta| + |\xi - \eta|, - |\xi|, \quad r_{-} = |\xi| - |\eta| - |\xi - \eta|. \]

So in view of (3.6) and (5.1), the estimate (3.5) reduces to
\[ (5.2) \quad \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{+} (\phi, \psi) \right\|_{L^{p+\varepsilon}_{\xi}} \lesssim \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{+} (\phi, \psi) \right\|_{X_{p+}^{0, \frac{1}{2}+\varepsilon}}, \]
\[ (5.3) \quad \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{-} (\phi, \psi) \right\|_{L^{p+\varepsilon}_{\xi}} \lesssim \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{-} (\phi, \psi) \right\|_{X_{p-}^{0, \frac{1}{2}+\varepsilon}}, \]
where
\[ \mathcal{R}^\frac{1}{2}_{\pm} (\phi, \psi) (\tau, \xi) = \int_{R^{1+2}} r_{+}^\frac{1}{2} \tilde{\phi}(\lambda, \eta) \tilde{\psi}(\tau - \lambda, \xi - \eta) d\lambda d\eta. \]

By transfer principle (see eg. Proposition A.2 in [8]), the estimates (5.2) and (5.3) further reduce to
\[ (5.4) \quad \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{+} (e^{it[D]} f, e^{it[D]} g) \right\|_{L^{p+\varepsilon}_{\xi}} \lesssim \left\| (D + \frac{1}{2}) f \right\|_{L^p_{\xi}} \left\| (D + \frac{1}{2}) g \right\|_{L^p_{\xi}}, \]
\[ (5.5) \quad \left\| (D + \frac{1}{2}) \mathcal{R}^\frac{1}{2}_{-} (e^{it[D]} f, e^{-it[D]} g) \right\|_{L^{p+\varepsilon}_{\xi}} \lesssim \left\| (D + \frac{1}{2}) f \right\|_{L^p_{\xi}} \left\| (D + \frac{1}{2}) g \right\|_{L^p_{\xi}}. \]

These reduce to estimating
\[ (5.6) \quad \left\| \mathcal{B}_{\pm, \pm} (f, g) \right\|_{L^{p+\varepsilon}_{\xi}} \lesssim \left\| f \right\|_{L^p_{\xi}} \left\| g \right\|_{L^p_{\xi}}, \]
where
\[
B_{+,+}(f,g)(\tau,\xi) = \int_{\mathbb{R}^2} \frac{|\xi|}{|\tau - |\xi||^{1+\frac{p}{2}}} \hat{f}(\eta)\hat{g}(\xi-\eta)\delta(\tau - |\eta| - |\xi - \eta|) d\eta \\
\]
\[
B_{+-}(f,g)(\tau,\xi) = \int_{\mathbb{R}^2} \frac{|\xi|}{|\tau - |\xi||^{1+\frac{p}{2}}} \hat{f}(\eta)\hat{g}(\xi-\eta)\delta(\tau - |\eta| + |\xi - \eta|) d\eta.
\]

5.1. Proof of (5.6) for \(B_{+,+}\). By Hölder inequality
\[
|B_{+,+}(f,g)(\tau,\xi)| \leq |I(\tau,\xi)|^{\frac{1}{p'}} \left\{ \int_{\mathbb{R}^2} |\hat{f}(\eta)|^{p'} |\hat{g}(\xi-\eta)|^{p'} \delta(\tau - |\eta| - |\xi - \eta|) d\eta \right\}^{\frac{1}{p'}}
\]
where
\[
I(\tau,\xi) = |\xi| |\tau - |\xi||^{\frac{1}{p}} \int_{\mathbb{R}^2} \frac{\delta(\tau - |\eta| - |\xi - \eta|)}{|\eta||\xi - \eta|^{1+\frac{p}{2}}} d\eta.
\]
By Proposition 4.3 in [5],
\[
\int_{\mathbb{R}^2} \frac{\delta(\tau - |\eta| - |\xi - \eta|)}{|\eta||\xi - \eta|^{1+\frac{p}{2}}} d\eta \approx \frac{1}{\tau |\tau - |\xi||^{\frac{p}{2}}} \leq \frac{1}{|\xi| |\tau - |\xi||^{\frac{p}{2}}},
\]
where we used the fact that \(p > 1\) (in the first inequality) and \(\tau > |\xi|\) (in the second inequality). Thus, we get
\[
I(\tau,\xi) \lesssim 1 \quad \text{for all } \tau, \xi,
\]
which in turn implies
\[
\int_{\mathbb{R}^{1+2}} |B_{+,+}(f,g)(\tau,\xi)|^{p'} d\tau d\xi \leq \int_{\mathbb{R}^{1+5}} |\hat{f}(\eta)|^{p'} |\hat{g}(\xi-\eta)|^{p'} \delta(\tau - |\eta| - |\xi - \eta|) d\tau d\eta d\xi d\eta = \int_{\mathbb{R}^2} |\hat{f}(\eta)|^{p'} d\eta \int_{\mathbb{R}^2} |\hat{g}(\eta)|^{p'} d\eta,
\]
where to get the equality we used the fact that \(\int_{\mathbb{R}} \delta(\tau - |\eta| - |\xi - \eta|) d\tau = 1\). This proves (5.6) in the \(B_{+,+}\) case.

5.2. Estimate for \(B_{+-}\). Again, by Hölder inequality
\[
|B_{+-}(f,g)(\tau,\xi)| \leq |J(\tau,\xi)|^{\frac{1}{p'}} \left\{ \int_{\mathbb{R}^2} |\hat{f}(\eta)|^{p'} |\hat{g}(\xi-\eta)|^{p'} \delta(\tau - |\eta| + |\xi - \eta|) d\eta \right\}^{\frac{1}{p'}}
\]
where
\[
J(\tau,\xi) = |\xi|^{1+\frac{p}{2}} ||\tau| - |\xi||^{\frac{1}{p}} \int_{\mathbb{R}^2} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^{1+\frac{p}{2}}|\xi - \eta|^{1+\frac{p}{2}}} d\eta.
\]
By a similar argument as in the preceding subsection, it then suffices to show
\[
J(\tau,\xi) \lesssim 1 \quad \text{for all } \tau, \xi.
\]
To do so, we split the integral
\[
\int_{\mathbb{R}^2} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^{1 + \frac{\alpha}{2}}|\xi - \eta|^{1 + \frac{\alpha}{2}}} \, d\eta = J_1 + J_2,
\]
where
\[
J_1 = \int_{|\eta| + |\xi - \eta| \leq 2|\xi|} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^{1 + \frac{\alpha}{2}}|\xi - \eta|^{1 + \frac{\alpha}{2}}} \, d\eta,
\]
\[
J_2 = \int_{|\eta| + |\xi - \eta| \geq 2|\xi|} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^{1 + \frac{\alpha}{2}}|\xi - \eta|^{1 + \frac{\alpha}{2}}} \, d\eta.
\]

For \( J_1 \), we apply Lemma 4.5 in [5] (using the fact that \( p > 1 \)) to get the estimate
\[
J_1 \approx \frac{1}{|\xi|^{1 + \frac{\alpha}{2}}|\xi| - |\tau|}.
\]
which in turn implies \( J(\tau, \xi) \lesssim 1 \) for all \( \tau, \xi \).

Next, we consider \( J_2 \). Applying Lemma 4.4 in [5] with
\[
F(|\eta|, |\xi - \eta|) = |\eta|^{-1 - \frac{\alpha}{2}}|\xi - \eta|^{-1 - \frac{\alpha}{2}} 1_{\{||\eta| + |\xi - \eta| \geq 2|\xi|\}}.
\]
we get
\[
J_2 \approx \int_1^\infty \left| x^2 - \tau^2 \right|^{-\frac{\alpha}{2}} \int_{x}^\infty \frac{x|\xi| + \tau}{2} \left| x^2 - \frac{\tau^2}{|\xi|^2} \right|^{-\frac{\alpha}{2}} \left| \xi^{2-\frac{\alpha}{2}} \right| (x^2 - 1)^{-\frac{\alpha}{2}} \, dx.
\]
Thus
\[
J_2 \approx \int_2^\infty \left| x^2 - \tau^2 \right|^{-\frac{\alpha}{2}} |\xi|^{-p} \int_{x^2}^\infty \frac{x^2 - \tau^2}{|\xi|^2} \left| x^2 - 1 \right|^{-\frac{\alpha}{2}} \, dx,
\]
\[
\lesssim \int_2^\infty \left| x^2 - \tau^2 \right|^{-\frac{\alpha}{2}} |\xi|^{-p} \int_{x^2}^\infty x^{-p-1} \, dx
\]
\[
\lesssim |\xi|^{-p-\frac{1}{2}} \left| |\xi| - |\tau| \right|^{-\frac{1}{2}},
\]
where we used the fact that \( \tau \leq |\xi| \) and \( \int_2^\infty x^{-p-1} \, dx \lesssim 1 \). Using the estimate for \( J_2 \) in \( J \), we get
\[
J(\tau, \xi) \lesssim \left\{ \left| \frac{|\xi| - |\tau|}{|\xi|} \right| \right\}^{\frac{\alpha}{2}} \leq 1 \quad \text{for all } \tau, \xi.
\]
Let \( \rho \in C_c^\infty([-1,1]) \) be a cut-off function such that \( \rho(t) = 1 \) for \( |t| \leq 1 \) and \( \rho(t) = 0 \) for \( |t| \geq 2 \). Set \( \rho_T(t) = \rho(t/T) \).

The homogeneous part of the solution satisfies the estimate
\[
\|W(t)f\|_{X_p^{s,b}(S_T)} \leq \|\rho_T(t)W(t)f\|_{X_p^{s,b}} = \|\rho_T\|_{\dot{H}_p^s} \|f\|_{\dot{H}_p^s} \leq C_p \|f\|_{\dot{H}_p^s}.
\]

Next, consider the inhomogeneous part of the solution. Write
\[
v(t) = \int_0^t W(t-t')F(t')dt'.
\]

Extend \( F \) by zero outside \( S_T \). Taking Fourier transform in space,
\[
\widehat{v}(t,\xi) = \int_0^t e^{i(t-t')h(\xi)} \widehat{F}(t',\xi) dt' \approx \int \frac{e^{ith(\xi)} - e^{ith(\xi)}}{i(\lambda - h(\xi))} \widehat{F}(\lambda,\xi) d\lambda
\]
and then also in time,
\[
\overline{v}(\tau,\xi) = \int e^{i(\tau - \lambda)} \int \frac{\delta(\tau - \lambda) - \delta(\tau - h(\xi))}{i(\lambda - h(\xi))} \widehat{F}(\lambda,\xi) d\lambda
\]
\[
= \int \frac{\overline{F}(\tau,\xi)}{i(\tau - h(\xi))} \delta(\tau - h(\xi)) \int \frac{\overline{F}(\lambda,\xi)}{i(\lambda - h(\xi))} d\lambda.
\]

Now split \( F = F_1 + F_2 \) corresponding to the Fourier domains \( \{T|\lambda - h(\xi)| \lesssim 1\} \) and \( \{T|\lambda - h(\xi)| \gtrsim 1\} \) respectively. Write \( v = v_1 + v_2 \) accordingly. Expand
\[
v_1(t) = \sum_{n=1}^{\infty} i^n \frac{t^n}{n!} W(t) f_n
\]
where
\[
\widehat{f}_n(\xi) = \int [i(\lambda - h(\xi))]^{n-1} 1_{\{T|\lambda - h(\xi)| \leq 1\}} \overline{\widehat{F}(\lambda,\xi)} d\lambda.
\]

Then
\[
\|v_1\|_{X_p^{s,b}(S_T)} \leq \|\rho_Tv_1\|_{X_p^{s,b}} \leq \sum_{n=1}^{\infty} \frac{T^n}{n!} \left( \frac{t}{T} \right)^n \rho_T(t) \|W(t)f_n\|_{X_p^{s,b}} \leq \sum_{n=1}^{\infty} \frac{T^n}{n!} \left( \frac{t}{T} \right)^n \rho_T(t) \|f_n\|_{\dot{H}_p^s}.
\]
Using Hölder inequality with respect to the variable $\lambda$, we get

\[
\|f_n\|_{\hat{H}^{b,-n+b}_p} \lesssim T^{b+\varepsilon-n-b} \|F\|_{X_p^{\varepsilon,b-1+\varepsilon}}.
\]

On the other hand, we can estimate

\[
\left\|\left(\frac{t}{T}\right)^n \rho_T\right\|_{\hat{H}^{b}_p} \leq CT^{b-\varepsilon} \|t^n \rho\|_{\hat{H}^{b}_p} \leq C T^{b-\varepsilon} n^{2n}.
\]

Indeed,

\[
\left\|\left(\frac{t}{T}\right)^n \rho_T\right\|_{\hat{H}^{b}_p} = T^{-np'} \int (\tau)^{p'} |\hat{\rho}^{(n)}(\tau)|^{p'} d\tau
\]

\[
= T^{p'} \int (\tau)^{p'} |\hat{\rho}^{(n)}(T\tau)|^{p'} d\tau
\]

\[
\leq CT^{p'-p'b-1} \int (\tau)^{p'} |\hat{\rho}^{(n)}(\tau)|^{p'} d\tau
\]

\[
= CT^{p'-p'b-1} \|t^n \rho\|_{\hat{H}^{b}_p},
\]

from which we get the first inequality in (6.2) by taking the $p'$-th root. Whereas the second inequality can be estimated using the support assumption of $\rho$ as

\[
\|t^n \rho\|_{\hat{H}^{b}_p} \leq C \|t^n \rho\|_{H^1} \leq C n^{2n} \|\rho\|_{H^1}.
\]

So in view of (6.1) and (6.2), we have

\[
\|v_1\|_{X_p^{\varepsilon,b}(S_T)} \leq CT^{\varepsilon} \left(\sum_{n=1}^{\infty} \frac{n^{2n}}{n!}\right) \|F\|_{X_p^{\varepsilon,b-1+\varepsilon}}
\]

\[
\leq CT^{\varepsilon} \|F\|_{X_p^{\varepsilon,b-1+\varepsilon}}.
\]

It remains to prove the estimate for $v_2$. We split $v_2 = w_1 - w_2$, where

\[
\tilde{w}_1(\tau, \xi) = \frac{1_{\{T|\lambda-h(\xi)|\geq 1\}} \tilde{F}(\tau, \xi)}{i(\tau - h(\xi))},
\]

\[
\tilde{w}_2(\tau, \xi) = \delta(\tau - h(\xi)) \tilde{g}(\xi),
\]

for

\[
\tilde{g}(\xi) = \int \frac{1_{\{T|\lambda-h(\xi)|\geq 1\}} \tilde{F}(\lambda, \xi)}{i(\lambda - h(\xi))} d\lambda.
\]

Obviously,

\[
\|w_1\|_{X_p^{\varepsilon,b}(S_T)} \lesssim T^{\varepsilon} \|F\|_{X_p^{\varepsilon,b-1+\varepsilon}}.
\]
On the other hand,
\[
\|w_2\|_{\mathcal{X}_{p,b}^{s,h}(S_T)} \leq \|\rho_T w_1\|_{\mathcal{X}_{p,b}^{s,h}} \leq \|\rho_T\|_{\widetilde{H}_{p}^{\frac{1}{2}}} \|g\|_{\widetilde{H}_{p}^{\frac{1}{2}}}.
\]
Now, one can easily show
\[
\|\rho_T\|_{\widetilde{H}_{p}^{\frac{1}{2}}} \leq C T^{1-p-b} \|\rho\|_{\widetilde{H}_{p}^{\frac{1}{2}}} \leq C \rho T^{1-p-b}
\]
and using Hölder inequality, we obtain
\[
\|g\|_{\widetilde{H}_{p}^{\frac{1}{2}}} \leq T^{b+\varepsilon-\frac{1}{p}} \|F\|_{\mathcal{X}_{s,b-1+\varepsilon}^0}.
\]
A combination of these estimates gives the desired estimate for \(w_2\).

REFERENCES

[1] T. Candy, N. Bournaveas, Local well-posedness for the space time Monopole equation in Lorenz gauge, NoDEA 19 (2012), 67-78.
[2] M. Czubak, Local well-posedness for the 2+1-dimensional Monopole equation, Anal. PDE 3(2), 151-174 (2010).
[3] P. D’Ancona, D. Foschi, and S. Selberg, Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions, Journal of Hyperbolic Differential Equations (2007), no. 2, 295-330.
[4] P. D’Ancona, D. Foschi, and S. Selberg, Null structure and almost optimal local regularity of the Dirac-Klein-Gordon system., Journal of the EMS 9 (2007) no. 4, 877-898.
[5] D. Foschi, S. Klainerman, Bilinear space-time estimates for homogeneous wave equations , Ann. Sci. Ecole. Sup. (4) 33 (2000), No. 2, 211-274.
[6] B. Dai, C. L. Terng, K. Uhlenbeck On the space-time Monopole equation, Surveys in Differential Geometry, vol. X, pp. 1–30. International Press, Somerville (2006).
[7] A. Grünrock, On the wave equation with quadratic nonlinearities in three space dimensions , Journal of Hyperbolic Differential Equations 8 (2011), No. 1–8, no. 2, 295-330.
[8] V. Grigoryan, A.R. Nahmod, Almost critical well-posedness for nonlinear wave equation with \(Q_{\mu\nu}\) null forms in 2d, http://arxiv.org.
[9] R., Ward, Twistors in 2 + 1 dimensions, J. Math. Phys. 30(10), 2246–2251 (1989).

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, ALFRED GETZ’ VEI 1, N-7491 TRONDHEIM, NORWAY

E-mail address: tesfahun@math.ntnu.no