ON PROPERTIES OF $\text{adari(pal)}$ AND $\text{ganit}_v(\text{pic})$

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Abstract. The paper discusses properties of $\text{adari(pal)}$ and $\text{ganit}_v(\text{pic})$ which are Ecalle’s maps among certain sets of moulds related to the double shuffle relations of MZVs. We give self-contained proof of their basic properties which are exhibited in Ecalle’s papers and partially proved in Schneps’ paper.

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0. Introduction

The multiple zeta value (MZV for short) is a power series defined by

$$\zeta(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}},$$

for $k_1, \ldots, k_r \in \mathbb{N}$ with $k_r \geq 2$. It is known that MZVs satisfy the double shuffle relations which consist of the shuffle relations and the harmonic relations. To deal with these relations systematically, the generating functions of MZVs are considered in [E03], [E11], [G01a], [G01b] and [IKZ]. In these papers, the shuffle relations (resp. the harmonic relations) are lifted to the relations between the generating functions which Ecalle ([E03], [E11]) calls symmetral (resp. symmetril) relations. To study
properties of MZVs, Ecalle introduces some groups and some Lie algebras of moulds related with these lifted relations, and he reveals various properties of them. Our final aim is to understand the following diagram (though which is not attained in this paper) which is displayed without proof in [E11 §4.7]:

\[ \begin{array}{ccc}
GARI(\Gamma)_{\text{as}} & \xrightarrow{\text{adari(pal)}} & GARI(\Gamma)_{\text{as,sin}} \\
\exp_p & & \exp_p \\
ARI(\Gamma)_{\text{al}} & \xrightarrow{\text{adari(pal)}} & ARI(\Gamma)_{\text{al,il}}
\end{array} \]

Related to the above diagram, in [SaSch] and [Sch15], the following property of the above map adari(pal) is discussed.

**Theorem 3.26** ([SaSch Theorem 7.2], [Sch15 Theorem 4.6.1]) The Lie algebra automorphism adari(pal) on ARI(\Gamma) induces a bijection \( \text{adari(pal)} : ARI(\Gamma)_{\text{al,il}} \rightarrow ARI(\Gamma)_{\text{al,il}} \).

In [Sch15], the above theorem is deduced from the following.

**Theorem 3.24** (cf. [SaSch Proposition 6.2], [Sch15 Lemma 4.4.2]) The map ganit\(_v\)(pic) induces a group isomorphism from \((GARI(\Gamma)_{\text{as}}, \times)\) to \((GARI(\Gamma)_{\text{is}}, \times)\) and induces a Lie algebra isomorphism from \((ARI(\Gamma)_{\text{al}}, [\cdot, \cdot])\) to \((ARI(\Gamma)_{\text{il}}, [\cdot, \cdot])\).

However, the proof of this statement is not fully presented\(^2\). This paper gives self-contained proof of this theorem by clarifying several compatibilities of the maps adari(pal) and ganit\(_v\)(pic) including Theorem 2.12, Corollary 3.8 and Theorem 3.24. By using these corollary and theorems, we obtain the following commutative diagram in [E11 §4.7].

**Corollary 3.25** ([E11 §4.7]) The following diagram commutes:

\[ \begin{array}{ccc}
GARI(\Gamma)_{\text{as}} & \xrightarrow{\text{ganit\(_v\)(pic)}} & GARI(\Gamma)_{\text{is}} \\
\exp_p & & \exp_p \\
ARI(\Gamma)_{\text{al}} & \xrightarrow{\text{ganit\(_v\)(pic)}} & ARI(\Gamma)_{\text{il}}
\end{array} \]

The construction of this paper is as follows. In §1 we review the definition (Definition 1.1) of moulds introduced in [E03] and [E11] and we explain several fundamental properties (Definition 1.5) of moulds. We also give some examples (Example 1.7) of moulds, which satisfies the above properties. In §2 we recall the definition (Definition 2.1) of dimoulds introduced in [Sau], and by using these, we will show Theorem 2.12, a correspondence result of certain sets of moulds. In §3

\(^2\)In precise, in [SaSch proof of Proposition 6.2] only the case of depth 4 is shown, and in [Sch15 proof of Lemma 4.4.2] “A straightforward calculation” (in Sch15 the end of page 55) is not straightforward at all at least for the author. We note that this issue of the calculation is justified in Remark 3.23.
we review the definition (Definition 3.1) of the map \( m_{\Gamma}(B) \) introduced in [E11] (2.36), and we prove Theorem 3.7 that, for \( m_{\Gamma}(B) \), is an automorphism on \( GARI(\Gamma) \) and \( ARI(\Gamma) \). In §3.2 we introduce elements \( g_{B}(v_{r}) \in K(V_{r}) \) (Definition 3.9) as an analogue of \( m_{\Gamma}(B) \). In §3.3 we prove the recurrence formula (Proposition 3.18) of \( g_{B}(v_{r}) \) for \( B = \text{pic} \). By using this recurrence formula, we prove Theorem 3.26 that \( m_{\Gamma}(\text{pic}) \) induces certain isomorphisms. As a corollary, we obtain compatibilities of \( m_{\Gamma}(\text{pic}) \) with exponential maps (Corollary 3.25) and obtain an automorphism \( m_{\Gamma}(\text{pic}) \) (Theorem 3.26). As an appendix, we prove \( \text{expari}(ARI(\Gamma)_{\text{al}}) = GARI(\Gamma)_{\text{al}} \) (Theorem A.7), which is used to show Theorem 3.26 (cf. [Sch15, Proposition 2.6.1]).

1. Moulds

In §1.1, we review the definition of moulds introduced in [E03] and [E11]. For our convenience, we recall the definition by following [FK]. We introduce the alternality, the symmetrality, the alternality and the symmetrality in §1.2. In §1.2, we give several examples of moulds, some of which will play an important role in later sections.

1.1. Definition of moulds. Let \( \Gamma \) be a finite abelian group. We set \( \mathcal{F} := \bigcup_{m \geq 1} \mathbb{Q}(x_{1}, \ldots, x_{m}) \).

**Definition 1.1 (FK Definition 1.1).** A mould on \( \mathbb{Z}_{\geq 0} \) with values in \( \mathcal{F} \) is a collection (a sequence) 

\[
M = (M(x_{1}, \ldots, x_{m}))_{m \in \mathbb{Z}_{\geq 0}} = (M(\emptyset), M(x_{1}), M(x_{1}, x_{2}), \ldots),
\]

with \( M(\emptyset) \in \mathbb{Q} \) and \( M(x_{1}, \ldots, x_{m}) \in \mathbb{Q}(x_{1}, \ldots, x_{m})^{\oplus m} \) for \( m \geq 1 \), which is described by a summation

\[
M(x_{1}, \ldots, x_{m}) = \bigoplus_{(\sigma_{1}, \ldots, \sigma_{m}) \in \Gamma^{\oplus m}} M_{\sigma_{1}, \ldots, \sigma_{m}}(x_{1}, \ldots, x_{m})
\]

where each \( M_{\sigma_{1}, \ldots, \sigma_{m}}(x_{1}, \ldots, x_{m}) \in \mathbb{Q}(x_{1}, \ldots, x_{m}) \). We denote the set of all moulds with values in \( \mathcal{F} \) by \( \mathcal{M}(\mathcal{F}; \Gamma) \). The set \( \mathcal{M}(\mathcal{F}; \Gamma) \) forms a \( \mathbb{Q} \)-linear space by

\[
A + B := (A(x_{1}, \ldots, x_{m}) + B(x_{1}, \ldots, x_{m}))_{m \in \mathbb{Z}_{\geq 0}},
\]

\[
cA := (cA(x_{1}, \ldots, x_{m}))_{m \in \mathbb{Z}_{\geq 0}},
\]

for \( A, B \in \mathcal{M}(\mathcal{F}; \Gamma) \) and \( c \in \mathbb{Q} \), namely the addition and the scalar are taken componentwise. We define a product on \( \mathcal{M}(\mathcal{F}; \Gamma) \) by

\[
(A \times B)_{\sigma_{1}, \ldots, \sigma_{m}}(x_{1}, \ldots, x_{m}) := \sum_{i=0}^{m} A_{\sigma_{1}, \ldots, \sigma_{i}}(x_{1}, \ldots, x_{i}) B_{\sigma_{i+1}, \ldots, \sigma_{m}}(x_{i+1}, \ldots, x_{m}),
\]

for \( A, B \in \mathcal{M}(\mathcal{F}; \Gamma) \) and for \( m \geq 0 \) and for \( (\sigma_{1}, \ldots, \sigma_{m}) \in \Gamma^{\oplus m} \). Then the pair \( (\mathcal{M}(\mathcal{F}; \Gamma), \times) \) is a non-commutative, associative, unital \( \mathbb{Q} \)-algebra. Here, the unit \( I \in \mathcal{M}(\mathcal{F}; \Gamma) \) is given by \( I := (1, 0, 0, \ldots) \).

We prepare \( BIMU(\Gamma) \) and \( BIMU(\Gamma) \) as copies of \( \mathcal{M}(\mathcal{F}; \Gamma) \).

---

3We may take \( \mathcal{F} \) as the field of all meromorphic functions or the Laurent series ring.
Remark 1.2. We often regard our moulds in Definition 1.1 as if a bimould introduced in [E03] and [E11], and we sometimes denote \( M_{x_1, \ldots, x_m} \) by \( M(\sigma_1, \ldots, \sigma_m) \) for \( M \in \text{BIMU}(\Gamma) \) and denote \( M(\sigma_1, \ldots, \sigma_m) \) for \( M \in \text{BIMU}(\Gamma) \). Moreover, we sometimes use \( u_i \) instead of \( x_i \) for \( \text{BIMU}(\Gamma) \) and use \( v_i \) instead of \( x_i \) for \( \text{BIMU}(\Gamma) \).

Notation 1.3. We often use the following algebraic formulation which is useful to describe several properties of moulds: Put \( U := \{ (\nu) \}_{i \in \mathbb{N}, \sigma \in \Gamma} \) and \( V := \{ (\omega) \}_{i \in \mathbb{N}, \sigma \in \Gamma} \). Let \( U_\mathbb{Z} \) and \( V_\mathbb{Z} \) be the sets such that
\[
U_\mathbb{Z} := \{ (\nu) \mid u = a_1 u_1 + \cdots + a_k u_k, \ k \in \mathbb{N}, \ a_j \in \mathbb{Z}, \ \sigma \in \Gamma \},
\]
\[
V_\mathbb{Z} := \{ (\omega) \mid v = a_1 v_1 + \cdots + a_k v_k, \ k \in \mathbb{N}, \ a_j \in \mathbb{Z}, \ \sigma \in \Gamma \}.
\]
and let \( U_\mathbb{Z}^\star \) (resp. \( V_\mathbb{Z}^\star \)) be the non-commutative free monoid generated by all elements of \( U_\mathbb{Z} \) (resp. \( V_\mathbb{Z} \)) with the empty word \( \emptyset \) as the unit. Occasionally we denote each element \( \omega = u_1 \cdots u_m \in U_\mathbb{Z}^\star \) with \( u_1, \ldots, u_m \in U_\mathbb{Z} \) by \( \omega = (u_1, \ldots, u_m) \) as a sequence. The length of \( \omega = (u_1, \ldots, u_m) \) is defined to be \( l(\omega) := m \). We define the length \( l(\omega) \) for any elements \( \omega \in V_\mathbb{Z}^\star \) in the same way.

For our simplicity we occasionally denote \( M \in \text{BIMU}(\Gamma) \) (resp. \( \text{BIMU}(\Gamma) \)) by
\[
M = (M(u_m))_{m \in \mathbb{Z}_+}, \quad \text{resp.} \quad M = (M(v_m))_{m \in \mathbb{Z}_+}
\]
where \( u_0 := \emptyset \) and \( u_m := (u_1, \ldots, u_m) \) (resp. \( v_0 := \emptyset \) and \( v_m := (\sigma_1, \ldots, \sigma_m) \)) for \( m \geq 1 \). Under the notations, the product of \( A, B \in \text{BIMU}(\Gamma) \) (resp. \( \text{BIMU}(\Gamma) \)) is expressed as
\[
A \times B = \left( \sum_{u_m = \alpha \beta} A(\alpha) B(\beta) \right)_{m \in \mathbb{Z}_+}
\]
where \( \alpha \) and \( \beta \) run over \( U_\mathbb{Z}^\star \) (resp. \( V_\mathbb{Z}^\star \)).

We put
\[
\text{ARI}(\Gamma) := \{ M \in \text{BIMU}(\Gamma) \mid M(\emptyset) = 0 \},
\]
\[
\text{GARI}(\Gamma) := \{ M \in \text{BIMU}(\Gamma) \mid M(\emptyset) = 1 \}.
\]
By replacing \( \text{BIMU}(\Gamma) \) to \( \overline{\text{BIMU}}(\Gamma) \), two sets \( \overline{\text{ARI}}(\Gamma) \) and \( \overline{\text{GARI}}(\Gamma) \) are also defined. We set
\[
[A, B] := A \times B - B \times A,
\]
for \( A, B \in \mathcal{M}(\mathcal{F}; \Gamma) \). Then we see that \( (\text{ARI}(\Gamma), [\cdot, \cdot]) \) and \( (\overline{\text{ARI}}(\Gamma), [\cdot, \cdot]) \) form Lie algebras and \( (\text{GARI}(\Gamma), \times) \) and \( (\overline{\text{GARI}}(\Gamma), \times) \) form groups.

1.2. Alternality and symmetrality, alternility and symmetrility. We put \( \mathcal{A}_U := \mathbb{Q}(U_\mathbb{Z}) \) to be the non-commutative polynomial \( \mathbb{Q} \)-algebra generated by \( U_\mathbb{Z} \) (i.e. \( \mathcal{A}_U \) is the \( \mathbb{Q} \)-linear space generated by \( U_\mathbb{Z}^\star \)). We equip \( \mathcal{A}_U \) a product \( \boxplus : \mathcal{A}_U^{\boxplus} \rightarrow \mathcal{A}_U \) which is linearly defined by \( \emptyset \boxplus \omega := \omega \boxplus \emptyset := \omega \) and
\[
(1.3) \quad \alpha \omega \boxplus b \eta := a(\omega \boxplus b \eta) + b(\alpha \omega \boxplus \eta),
\]
for \( a, b \in U_\mathbb{Z} \) and \( \omega, \eta \in U_\mathbb{Z}^\star \). Then the pair \( (\mathcal{A}_U, \boxplus) \) forms a commutative, associative, unital \( \mathbb{Q} \)-algebra.\(^4\) Let \( \{ \text{Sh}(\omega \eta) \}_{\omega, \eta, \alpha \in U_\mathbb{Z}^\star} \) be the family in \( \mathbb{Z} \) defined

\(^4\)See [K] for detail.
by

\[
\omega \omega \eta = \sum_{\alpha \in U^*_{\omega}} \text{Sh}_{\alpha} \left( \begin{array}{c} \omega; \\ \eta \\ \alpha \end{array} \right). \tag{1.4}
\]

We put \( K := \mathbb{Q}(v_i \mid i \in \mathbb{N}) \), that is, the commutative field generated by all \( v_i \) over \( \mathbb{Q} \). We define \( A_V := K[V_2] \) to be the non-commutative polynomial \( K \)-algebra generated by \( V_2 \) (i.e. \( A_V \) is the \( K \)-linear space generated by \( V_2^* \)). As with \( A_V \), the algebra \( A_V \) is equipped the product \( \omega \) and the pair \( (A_V, \omega) \) forms a commutative, associative, unital \( K \)-algebra. While, we also equip \( A_V \) a product \( \omega_* : A_V^{\otimes 2} \to A_V \) which is linearly defined by \( \emptyset \omega_* \emptyset := \omega \omega_* \emptyset := w \) and \( (\omega^v) \omega_* (\omega^v) \eta := 0 \) for \( (\omega^v), (\omega^v) \in V_2 \) with \( v = v' \) and \( \omega, \eta \in V_2^* \), and

\[
(\omega^v) \omega_* (\omega^v) \eta := (\omega^v) \left( \omega \omega_* (\omega^v) \eta \right) + (\omega^v) \left( (\omega^v) \omega_* \eta \right) + \frac{1}{v - v'} \left( (\omega^v) (\omega \omega_* \eta) - (\omega^v) (\omega \omega_* \eta) \right)
\]

for \( (\omega^v), (\omega^v) \in V_2 \) with \( v \neq v' \) and \( \omega, \eta \in V_2^* \). Then the pair \( (A_V, \omega_*) \) forms a commutative, non-associative, unital \( K \)-algebra. Let \( \{ \text{Sh}_{\alpha} (\omega^m) \}_{\omega, \eta, \alpha \in V_2^*} \) to be the family in \( K \) defined by

\[
\omega \omega_* \eta = \sum_{\alpha \in V_2^*} \text{Sh}_{\alpha} \left( \begin{array}{c} \omega; \\ \eta \\ \alpha \end{array} \right). \tag{1.5}
\]

For \( \omega = (u_1, \ldots, u_m) \in V_2^* \) with \( u_1, \ldots, u_m \in V_2 \), we denote \( L(\omega) := \{ u_1, \ldots, u_m \} \) to be the set of all letters appearing in \( \omega \). Then for \( \omega, \eta \in V_2^* \), two words \( \omega \) and \( \eta \) have no same letters if and only if \( L(\omega) \cap L(\eta) = \emptyset \) holds.

**Lemma 1.4 (cf. [FK] Lemma A.7).** Let \( r \geq 2 \). For \( \omega, \eta, \alpha_1, \ldots, \alpha_r \in U^*_{\omega} \), we have

\[
\text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega; \\ \eta \\ \alpha_1 \cdots \alpha_r \end{array} \right) = \sum_{\substack{\omega_1, \ldots, \omega_r \\ \eta_1, \ldots, \eta_r}} \text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega_1; \\ \eta_1 \\ \alpha_1 \end{array} \right) \cdots \text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega_r; \\ \eta_r \\ \alpha_r \end{array} \right), \tag{1.6}
\]

and for \( \omega, \eta \in V_2^* \) with \( L(\omega) \cap L(\eta) = \emptyset \) and for \( \alpha_1, \ldots, \alpha_r \in V_2^* \), we have

\[
\text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega; \\ \eta \\ \alpha_1 \cdots \alpha_r \end{array} \right) = \sum_{\substack{\omega_1, \ldots, \omega_r \\ \eta_1, \ldots, \eta_r}} \text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega_1; \\ \eta_1 \\ \alpha_1 \end{array} \right) \cdots \text{Sh}_{\alpha_1 \cdots \alpha_r} \left( \begin{array}{c} \omega_r; \\ \eta_r \\ \alpha_r \end{array} \right) \tag{1.7}
\]

where \( \omega_1, \ldots, \omega_r, \eta_1, \ldots, \eta_r \) run over \( U^*_{\omega} \).

**Proof.** The equation (1.6) is proved in [FK] Lemma A.7. By the same way, we obtain (1.7).

**Definition 1.5 (cf. [FK] Definition 1.4).** A mould \( M \in \text{ARI}(\Gamma) \) (resp. \( \in \text{GARI}(\Gamma) \)) is called **alternal** (resp. **symmetrical**) if we have

\[
\sum_{\alpha \in U^*_{\omega}} \text{Sh}_{\alpha} \left( \begin{array}{c} \omega_1; \\ \ldots \\ \omega_p; \\ \sigma_{p+1}; \\ \ldots \\ \sigma_{p+q}; \\ \omega_1; \\ \ldots \\ \omega_p; \\ \sigma_{p+1}; \\ \ldots \\ \sigma_{p+q} \end{array} \right) M(\alpha) = 0 \tag{1.8}
\]

(resp. \( = M(\omega_1; \ldots, \omega_p; \sigma_{p+1}; \ldots, \sigma_{p+q}) M(\omega_1; \ldots, \omega_p; \sigma_{p+1}; \ldots, \sigma_{p+q}) \))
for $p, q \geq 1$. We denote $\text{ARI}(\Gamma)_{al}$ (resp. $\text{GARI}(\Gamma)_{as}$) to be the subset consisting of alternal (resp. symmetral) moulds. While, a mould $M \in \text{ARI}(\Gamma)$ (resp. $\in \text{GARI}(\Gamma)$) is called alternal (resp. symmetral) if we have

\[
\sum_{\alpha \in V \_Z} \text{Sh}_{\alpha} \left( (\sigma_1, \ldots, \sigma_p) / (v_1, \ldots, v_{p+q}) \right) M(\alpha) = 0
\]

(1.9) (resp. $= M(\sigma_1, \ldots, \sigma_p)M(\sigma_{p+1}, \ldots, \sigma_{p+q})$)

for $p, q \geq 1$.

**Remark 1.6.** For any mould $M \in \text{ARI}(\Gamma)$ (resp. $\in \text{GARI}(\Gamma)$), the alternality (resp. symmetrality) is also defined, that is, $M$ is called alternal (resp. symmetral) if we have

\[
\sum_{\alpha \in V \_Z} \text{Sh}_{\alpha} \left( (\sigma_1, \ldots, \sigma_p) / (v_1, \ldots, v_{p+q}) \right) M(\alpha) = 0
\]

(1.10) (resp. $= M(\sigma_1, \ldots, \sigma_p)M(\sigma_{p+1}, \ldots, \sigma_{p+q})$)

for $p, q \geq 1$.

We denote $\text{ARI}(\Gamma)_{al}$ (resp. $\text{ARI}(\Gamma)_{as}$) to be the subset of $\text{ARI}(\Gamma)$ consisting of alternal (resp. alternil) moulds, and we denote $\text{GARI}(\Gamma)_{as}$ (resp. $\text{GARI}(\Gamma)_{ln}$) to be the subset of $\text{GARI}(\Gamma)$ consisting of symmetrical (resp. symmetril) moulds.

Our purpose in this paper is to give self-contained proof of various properties of these sets (see [2] and [3]).

**Examples 1.7.** We give examples of Definition 1.5.

(a). The mould $A \in \text{ARI}(\Gamma)$ is defined by $A(u_0) := A(u_1) := 0$ and for $m \geq 2$

\[
A(u_m) := \frac{1}{(u_2 - u_1) \cdots (u_m - u_{m-1})}.
\]

Then this mould $A$ is alternal.

(b). The mould $paj \in \text{GARI}(\Gamma)$ is defined by $paj(u_0) := 1$ and for $m \geq 1$

\[
paj(u_m) := \frac{1}{u_1(u_1 + u_2) \cdots (u_1 + \cdots + u_m)}.
\]

Then this mould $paj$ is symmetrical.

(c). The mould $C \in \text{ARI}(\Gamma)$ is defined by $C(v_0) := 0$ and $C(v_1) := 1$ and for $m \geq 2$

\[
C(v_m) := \frac{1}{(v_2 - v_1) \cdots (v_m - v_1)}.
\]

Then this mould $C$ is alternil.

(d). The mould $pic \in \text{GARI}(\Gamma)$ is defined by $pic(v_0) := 1$ and for $m \geq 1$

\[
pic(v_m) := \frac{1}{v_1v_2 \cdots v_m}.
\]

Then this mould $pic$ is symmetrical.

(e). The mould $pij \in \text{GARI}(\Gamma)$ is defined by $pij(v_0) := 1$ and $pij(v_1) := \frac{1}{v_1}$ and for $m \geq 2$

\[
pij(v_m) := \frac{1}{(v_1 - v_2)(v_2 - v_3) \cdots (v_{m-1} - v_m)v_m}.
\]

Then this mould $pij$ is symmetrical.
Remark 1.8. We mention two properties of the mould \( \text{pic} \in \mathbb{GARI}(\Gamma) \):

(a) We put \( \omega_1 = (\cdot \rangle \) and \( \eta_1 = (\cdot \rangle \). By using the mould \( \text{pic} \) and using flexions in [FK] Definition 1.8, we can rewrite (1.5) as

\[
\begin{aligned}
(\omega_1, \omega) \uplus_s (\eta_1, \eta) &:= (\omega_1) (\omega \uplus_s (\eta_1, \eta)) + (\eta_1) (\omega_1) (\omega \uplus_s \eta) - \text{pic}(\omega_1 | \eta_1) (\omega \uplus_s \eta) - \text{pic}(\eta_1 | \omega_1) (\eta \uplus_s \omega) \\
&= \text{pic}(\omega_1 | \eta_1) (\omega \uplus_s \eta) - \text{pic}(\eta_1 | \omega_1) (\eta \uplus_s \omega) - \text{pic}(\omega_1 | \eta_1) (\omega \uplus_s \eta) - \text{pic}(\eta_1 | \omega_1) (\eta \uplus_s \omega).
\end{aligned}
\]

(b) By the definition of the mould \( \text{pic} \), the following relations hold:

\[
\begin{aligned}
\text{pic}(\omega_1 | \omega_2) &= \text{pic}(\omega_1), \\
\text{pic}(\omega_1 | \omega_2) &= -\text{pic}(\omega_2 | \omega_1), \\
\text{pic}(\omega_1 | (\omega_2, \omega_3)) + \text{pic}(\omega_2 | (\omega_1, \omega_3)) + \text{pic}(\omega_3 | (\omega_1, \omega_2)) &= 0, \\
\text{pic}(\omega) \text{pic}(\eta) &= \text{pic}(\omega, \eta),
\end{aligned}
\]

for \( \omega_1, \omega_2, \omega_3 \in V_{\mathbb{Z}} \) and \( \omega, \eta \in V_{\mathbb{Z}}^* \).

We often use these properties to prove the alternility and the symmetrility.

2. Exponential Maps

In this section, we recall the definition of dimoulds introduced in [Sau], and by using dimoulds, we prove theorem on the exponential map. In §2.1, we review the definition of dimoulds, and we show equivalent conditions of the alternility, the symmetrality and the alternility, the symmetrility for moulds. In §2.2, by using these equivalent conditions, we prove Theorem 2.12 which is not presented in [E11], but is required in §3.

2.1. Dimoulds.

Definition 2.1 (cf. [Sau] Definition 5.2). A dimould\(^5\) \( M^{\bullet \bullet} \) with values in \( \mathcal{F} \) is a sequence:

\[
M^{\bullet \bullet} := (M^{\bullet \bullet}(x_1, \ldots, x_{r+s}) \mid r, s \in \mathbb{N}_0),
\]

with \( M^{\bullet \bullet}(0; 0) \in \mathbb{Q} \) and \( M^{\bullet \bullet}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) \in \mathbb{Q}(x_1, \ldots, x_{r+s}) \oplus \mathbb{Q}^{r+s} \) for \( r \geq 1 \) or \( s \geq 1 \), which is described by a summation

\[
M^{\bullet \bullet}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) = \bigoplus_{(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}) \in \Gamma^{r+s}} M^{\bullet \bullet}_{\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}),
\]

where each \( M^{\bullet \bullet}_{\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) \in \mathbb{Q}(x_1, \ldots, x_{r+s}) \). We denote the set of all dimoulds with values in \( \mathcal{F} \) by \( \mathcal{M}_2(\mathcal{F}; \Gamma) \). By the component-wise summation and the component-wise scalar multiple, the set \( \mathcal{M}_2(\mathcal{F}; \Gamma) \) forms a \( \mathbb{Q} \)-linear space. The product of \( \mathcal{M}_2(\mathcal{F}; \Gamma) \) which is denoted by the same symbol \( \times \) as the product of \( \mathcal{M}(\mathcal{F}; \Gamma) \) is defined by

\[
(A^{\bullet \bullet} \times B^{\bullet \bullet})_{\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_{r+s}}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s}) := \sum_{i=0}^r \sum_{j=0}^s A^{\bullet \bullet}_{\sigma_1, \ldots, \sigma_i, \sigma_{i+1}, \ldots, \sigma_{r+j}}(x_1, \ldots, x_i; x_{r+i+1}, \ldots, x_{r+s}) \times B^{\bullet \bullet}_{\sigma_{r+i+1}, \ldots, \sigma_{r+j+1}, \ldots, \sigma_{r+s}}(x_{i+1}, \ldots, x_r; x_{r+j+1}, \ldots, x_{r+s}),
\]

\(^5\)We note that the definition of the dimould is different from the definition of the bimould introduced in [E03] and [E11].
for $A^{\bullet \bullet}, B^{\bullet \bullet} \in \mathcal{M}_2(F; \Gamma)$ and for $r, s \geq 0$ and for $(\sigma_1, \ldots, \sigma_{r+s}) \in \Gamma^{r+s}$. Then $(\mathcal{M}_2(F; \Gamma), \times)$ is non-commutative, associative $\mathbb{Q}$-algebra. Here, the unit $I^{\bullet \bullet}$ of $(\mathcal{M}_2(F; \Gamma), \times)$ is defined as follows:

$$ I^{\bullet \bullet}(\omega; \eta) := \begin{cases} 1 & (\omega = \eta = \emptyset), \\ 0 & \text{(otherwise)}. \end{cases} $$

We prepare $\text{DIMU}(\Gamma)$ and $\overline{\text{DIMU}}(\Gamma)$ as copies of $\mathcal{M}_2(F; \Gamma)$.

**Remark 2.2.** Similar to the notation of Remark 1.2, we sometimes denote $M^{\bullet \bullet}_{\sigma_1, \ldots, \sigma_r; \sigma_{r+1}, \ldots, \sigma_{r+s}}(x_1, \ldots, x_r; x_{r+1}, \ldots, x_{r+s})$ by

$$ M^{\bullet \bullet}(\sigma_1, \ldots, \sigma_r; \sigma_{r+1}, \ldots, \sigma_{r+s}) \quad \text{(resp. } M^{\bullet \bullet}(\sigma_1, \ldots, \sigma_r; \sigma_{r+1}, \ldots, \sigma_{r+s}) \text{)} $$

for $M^{\bullet \bullet} \in \text{DIMU}(\Gamma)$ (resp. $\in \overline{\text{DIMU}}(\Gamma)$). As with the notation (1.1), we often denote $M^{\bullet \bullet} \in \text{DIMU}(\Gamma)$ (resp. $\in \overline{\text{DIMU}}(\Gamma)$) as

$$ M^{\bullet \bullet} = (M^{\bullet \bullet}(\omega; \eta))_{\omega, \eta} $$

by putting $\omega := (\sigma_1, \ldots, \sigma_r)$ and $\eta := (\sigma_{r+1}, \ldots, \sigma_{r+s})$ (resp. $\omega := (\sigma_1, \ldots, \sigma_r)$ and $\eta := (\sigma_{r+1}, \ldots, \sigma_{r+s})$). By using this notation, we can rewrite (2.1) by

$$ A^{\bullet \bullet} \times B^{\bullet \bullet} = (A^{\bullet \bullet} \times B^{\bullet \bullet}(\omega; \eta))_{\omega, \eta} = \left( \sum_{\eta = \eta_1 \eta_2} A^{\bullet \bullet}(\omega_1; \eta_1)B^{\bullet \bullet}(\omega_2; \eta_2) \right)_{\omega, \eta}, $$

where $\omega_1, \omega_2, \eta_1, \eta_2$ run over $U^*_2$ (resp. $V^*_2$).

We introduce two maps from the set $\overline{\text{BIMU}}(\Gamma)$ of moulds to the set $\overline{\text{DIMU}}(\Gamma)$ of dimoulds.

**Definition 2.3 (Sau, §5.2).** (i): The $\mathbb{Q}$-linear map $\mathcal{S}h: \overline{\text{BIMU}}(\Gamma) \to \overline{\text{DIMU}}(\Gamma)$ is defined by

$$ \mathcal{S}h(M) = (\mathcal{S}h(M)(\omega; \eta))_{\omega, \eta} := \left( \sum_{\alpha \in V^*_2} \text{Sh} \left( \frac{\omega}{\alpha}; \frac{\eta}{\alpha} \right) M(\alpha) \right)_{\omega, \eta} $$

for $M \in \overline{\text{BIMU}}(\Gamma)$.

(ii): The $\mathbb{Q}$-linear map $i: \overline{\text{BIMU}}(\Gamma) \otimes \mathbb{Q} \overline{\text{BIMU}}(\Gamma) \to \overline{\text{DIMU}}(\Gamma)$ is defined by

$$ i(M \otimes N) = (i(M \otimes N)(\omega; \eta))_{\omega, \eta} := (M(\omega)N(\eta))_{\omega, \eta} $$

for $M, N \in \overline{\text{BIMU}}(\Gamma)$. It is clear that $i$ is injective. For our simplicity, we denote $i(M \otimes N)$ by $M \otimes N$.

By using these two maps, we reformulate the symmetrical moulds and the alternal moulds in terms of dimoulds.

**Proposition 2.4 (Sau, Lemma 5.2).** For mould $M \in \overline{\text{BIMU}}(\Gamma)$, we have

(i) $M \in \overline{\text{ARI}}(\Gamma)_{\mathbb{M}} \iff \mathcal{S}h(M) = M \otimes I + I \otimes M$,

(ii) $M \in \overline{\text{GARI}}(\Gamma)_{\mathbb{M}} \iff \mathcal{S}h(M) = M \otimes M$. 
Proof. Because we can prove (i) as with (ii), we only prove (i).

(⇐): Let \( M \in BIMU(\Gamma) \). By the definition of the map \( S_h \), we have

\[
S_h(M)(\omega; \eta) = \begin{cases}
  M(\emptyset) & (\omega = \eta = \emptyset), \\
  M(\eta) & (\omega = \emptyset, \eta \neq \emptyset), \\
  M(\omega) & (\omega \neq \emptyset, \eta = \emptyset), \\
  \sum_{\alpha \in V_2} S_h\left(\frac{\omega; \eta}{\alpha}\right) M(\alpha) & (\omega, \eta \neq \emptyset).
\end{cases}
\]

While, we have

\[
(M \otimes I + I \otimes M)(\omega; \eta) = M(\omega)I(\eta) + I(\omega)M(\eta) = \begin{cases}
  2 \cdot M(\emptyset) & (\omega = \eta = \emptyset), \\
  M(\eta) & (\omega = \emptyset, \eta \neq \emptyset), \\
  M(\omega) & (\omega \neq \emptyset, \eta = \emptyset), \\
  0 & (\omega, \eta \neq \emptyset).
\end{cases}
\]

If we have \( S_h(M) = M \otimes I + I \otimes M \), then we get \( M(\emptyset) = 0 \) and

\[
\sum_{\alpha \in V_2} S_h\left(\frac{\omega; \eta}{\alpha}\right) M(\alpha) = 0,
\]

for \( \omega, \eta \neq \emptyset \). Therefore, we obtain \( M \in AR(\Gamma)_a \). By observing this discussion, we get the proof of (⇒).

\( \square \)

Corollary 2.5. We have

\( S_h(I) = I \otimes I = I^{\bullet \bullet} \).

Proof. The first equality is from \( I \in GARI(\Gamma) \) and Proposition 2.4, the second equality is from the definition of the both side. \( \square \)

We prove two important lemmas.

Lemma 2.6 ([Sau, Lemma 5.1]). The map \( S_h \) is an algebra homomorphism, that is, we have

\[
S_h(M \times N) = S_h(M) \times S_h(N),
\]

for \( M, N \in BIMU(\Gamma) \).

Proof. By using the definition of the map \( S_h \) and the product \( \times \), we have

\[
S_h(M \times N)(\omega; \eta) = \sum_{\alpha \in V_2} S_h\left(\frac{\omega; \eta}{\alpha}\right) \sum_{\alpha_1, \alpha_2} M(\alpha_1)N(\alpha_2)
\]

\[
= \sum_{\alpha_1, \alpha_2 \in V_2} S_h\left(\frac{\omega; \eta}{\alpha_1 \alpha_2}\right) M(\alpha_1)N(\alpha_2).
\]

By using (1.6) for \( r = 2 \), we get

\[
= \sum_{\alpha_1, \alpha_2 \in V_2} \sum_{\substack{\omega_1 = \omega \mid \omega_2 \\eta_1 = \eta \mid \eta_2}} S_h\left(\frac{\omega_1; \eta_1}{\alpha_1}\right) S_h\left(\frac{\omega_2; \eta_2}{\alpha_2}\right) M(\alpha_1)N(\alpha_2)
\]

\[
= \sum_{\substack{\omega = \omega_1 \mid \omega_2 \\eta = \eta_1 \mid \eta_2}} S_h(M)(\omega; \eta_1)S_h(N)(\omega_2; \eta_2)
\]

\[
= (S_h(M) \times S_h(N))(\omega; \eta).
\]

Hence, we finish the proof. \( \square \)
Remark 2.7. We define the \( \mathbb{Q} \)-linear map \( Sh_\ast : \text{BIMU}(\Gamma) \to \text{DIMU}(\Gamma) \) by
\[
(2.3) \quad Sh_\ast (M) = (Sh_\ast (M) (\omega; \eta))_{\omega, \eta} := \left( \sum_{\alpha \in V_2^*} Sh_\ast \left( \frac{\omega}{\alpha} \right) M(\alpha) \right)_{\omega, \eta},
\]
for \( M \in \text{BIMU}(\Gamma) \). Then we get the following two:
(i) \( M \in \text{ARI}(\Gamma)_{il} \iff Sh_\ast (M) = M \otimes I + I \otimes M \),
(ii) \( M \in \text{GARI}(\Gamma)_{il} \iff Sh_\ast (M) = M \otimes M \).
Moreover, the map \( Sh_\ast \) forms an algebra homomorphism.

Lemma 2.8 ([Sau (5.7)]). For \( M_1, M_2, N_1, N_2 \in \text{BIMU}(\Gamma) \), we have
\[
(2.4) \quad (M_1 \otimes M_2) \times (N_1 \otimes N_2) = (M_1 \times N_1) \otimes (M_2 \times N_2).
\]

Proof. By the definition of the product \( \times \), it is easy to prove this equation. \( \square \)

Proposition 2.9. The following two hold: \( \square \)
(a) The pairs \( \text{(GARI}(\Gamma)_{as}, \times) \) and \( \text{(GARI}(\Gamma)_{al}, \times) \) form subgroups of \( \text{(GARI}(\Gamma), \times) \).
(b) The pairs \( \text{(ARI}(\Gamma)_{al}, [\cdot, \cdot]) \) and \( \text{(ARI}(\Gamma)_{il}, [\cdot, \cdot]) \) form Lie subalgebras of \( \text{(ARI}(\Gamma), [\cdot, \cdot]) \).

Proof. Let \( M, N \in \text{GARI}(\Gamma)_{as} \). By using Lemma 2.6 and Lemma 2.8, we have
\[
Sh(M \times N) = Sh(M) \times Sh(N) = (M \times M) \otimes (N \times N) = (M \times N) \otimes (M \times N).
\]
By using Proposition 2.4(ii), we get \( M \times N \in \text{GARI}(\Gamma)_{as} \). Let \( M, N \in \text{ARI}(\Gamma)_{il} \).
Then we have
\[
Sh([M, N]) = Sh(M) \times Sh(N) - Sh(N) \times Sh(M) = (M \otimes I + I \otimes M) \times (N \otimes I + I \otimes N) - (N \otimes I + I \otimes N) \times (M \otimes I + I \otimes M) = (M \times N) \otimes I + M \otimes N + N \otimes M + I \otimes (M \times N) - (N \otimes M) \otimes I - N \otimes M - M \otimes N - I \otimes (N \times M) = [M, N] \otimes I + I \otimes [M, N].
\]
Hence, we get \([M, N] \in \text{ARI}(\Gamma)_{il} \). By using the map \( Sh_\ast \), we can prove that \( \text{(GARI}(\Gamma)_{as}, \times) \) forms a subgroup and \( \text{(ARI}(\Gamma)_{il}, [\cdot, \cdot]) \) forms a Lie subalgebra similarly. \( \square \)

2.2. Exponential map \( \exp_{\times} \).

Definition 2.10. We define the map \( \exp_{\times} : \text{ARI}(\Gamma) \to \text{GARI}(\Gamma) \) by
\[
(2.5) \quad \exp_{\times} (A) := \sum_{k \geq 0} \frac{1}{k!} A^{\times k},
\]
for \( A \in \text{ARI}(\Gamma) \). Here, we use the notation \( A^{\times k} := \underbrace{A \times \cdots \times A}_{k} \).

\(^6\)In [Sau Proposition 5.1], it is proved that the pair \( \text{(GARI}(\Gamma)_{as}, \times) \) forms a subgroup of \( \text{(GARI}(\Gamma), \times) \) and the pair \( \text{(ARI}(\Gamma)_{il}, [\cdot, \cdot]) \) forms a Lie subalgebra of \( \text{(ARI}(\Gamma), [\cdot, \cdot]) \).
There is the inverse map \( \log \times : \operatorname{GARI}(\Gamma) \to \operatorname{ARI}(\Gamma) \) of map \( \exp \times \) defined by

\[
\log \times (S) := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} (S-I)^{\times k},
\]

for \( S \in \operatorname{GARI}(\Gamma) \), and so we have \( \exp \times \circ \log \times = \operatorname{id}_{\operatorname{GARI}(\Gamma)} \) and \( \log \times \circ \exp \times = \operatorname{id}_{\operatorname{ARI}(\Gamma)} \).

We note that

\[
\exp \times (M+N) = \exp \times (M) \times \exp \times (N)
\]

for \( M, N \in \operatorname{ARI}(\Gamma) \) with \( M \times N = N \times M \).

**Remark 2.11.** We also have the following exponential maps:

(a). For \( \operatorname{ARI}(\Gamma) \) and \( \overline{\operatorname{GARI}(\Gamma)} \), the similar exponential and the similar logarithm maps are defined.

(b). By replacing the product \( \times \) of \( \operatorname{BIMU}(\Gamma) \) in (2.5) and (2.6) to the product of \( \operatorname{DIMU}(\Gamma) \), the exponential and the logarithm maps of dimoulds are defined.

For our simplicity, we denote these map by the same symbols \( \exp \times \) and \( \log \times \).

**Theorem 2.12.** The following three hold:

\[
\begin{align*}
(2.8) & \quad \exp \times (\operatorname{ARI}(\Gamma)_{\text{al}}) = \operatorname{GARI}(\Gamma)_{\text{as}}, \\
(2.9) & \quad \exp \times (\overline{\operatorname{ARI}(\Gamma)_{\text{al}}}) = \overline{\operatorname{GARI}(\Gamma)_{\text{as}}}, \\
(2.10) & \quad \exp \times (\operatorname{ARI}(\Gamma)_{\text{il}}) = \overline{\operatorname{GARI}(\Gamma)_{\text{is}}}. \\
\end{align*}
\]

**Proof.** Because the above three are similarly shown, we only prove \( \exp \times (\operatorname{ARI}(\Gamma)_{\text{al}}) = \operatorname{GARI}(\Gamma)_{\text{as}} \).

(\( \subset \)): Let \( A \in \operatorname{ARI}(\Gamma)_{\text{al}} \). Then we will show \( \exp \times (A) \in \operatorname{GARI}(\Gamma)_{\text{as}} \). By using Lemma 2.6 and Proposition 2.4, we have

\[
\begin{align*}
\SH(\exp \times (A)) &= \exp \times (\SH(A)) \\
&= \exp \times (A \otimes I + I \otimes A).
\end{align*}
\]

Because we have \( (A \otimes I) \times (I \otimes A) = (I \otimes A) \times (A \otimes I) \), by (2.7) we get

\[
\begin{align*}
&= \exp \times (A \otimes I) \times \exp (I \otimes A) \\
&= (\exp \times (A \otimes I) \times (I \otimes \exp \times (A))) \\
&= \exp \times (A) \otimes \exp \times (A).
\end{align*}
\]

Hence, by Proposition 2.4, we obtain \( \exp (A) \in \operatorname{GARI}(\Gamma)_{\text{as}} \).

(\( \supset \)): Let \( S \in \operatorname{GARI}(\Gamma)_{\text{as}} \). Similarly, we have

\[
\begin{align*}
\SH(\log \times (S)) &= \log \times (\SH(S)) \\
&= \log \times (S \otimes S) \\
&= \log \times ((I \otimes S) \times (S \otimes I)) \\
&= \log \times (I \otimes S) + \log \times (S \otimes I) \\
&= I \otimes \log \times (S) + \log \times (S) \otimes I.
\end{align*}
\]

Hence, we get \( \log \times (S) \in \operatorname{ARI}(\Gamma)_{\text{al}} \).

---

\footnote{The equation (2.8) is proved in [Sau, Proposition 5.1].}
3. Automorphisms

In this section, we recall the definition of the map $\text{ganit}_v(B)$ (for $B \in \overline{\text{GARI}}(\Gamma)$) introduced in \cite[(2.36)]{E11}, and we prove several properties of this map. In \[3.1\] we review its definition (Definition \[3.1\]) and we prove that the map $\text{ganit}_v(B)$ forms an algebra automorphism on $(\overline{\text{BIMU}}(\Gamma), \times)$ (Proposition \[3.3\]), and that this map induces a group automorphism on $(\overline{\text{ARI}}(\Gamma), \times)$ and induces a Lie algebra automorphism on $(\overline{\text{ARI}}(\Gamma), [\cdot, \cdot])$ (Theorem \[3.7\]). As a corollary, we get a commutativity of $\text{ganit}_v(\text{pic})$ and $\exp_v$ (Corollary \[3.8\]). In \[3.2\], we introduce elements $g_B(v_r) \in \mathcal{K}(V_2)$ (Definition \[3.9\]) as an analogue of $\text{ganit}_v(B)$. In \[3.3\], we prove the recurrence formula (Proposition \[3.18\]) of $\text{ganit}_v(B)(v_r)$ for $B = \text{pic}$. By using this recurrence formula, we prove Theorem \[3.24\] that is, the map $\text{ganit}_v(B)$ induces a group isomorphism from $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ to $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ and induces a Lie algebra isomorphism from $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ to $\overline{\text{ARI}}(\Gamma)_{\text{al}}$. As a corollary, we obtain its commutativity (Corollary \[3.26\]) with the exponential map $\exp_v$ and we obtain a bijection between $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ and $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ (Theorem \[3.29\]).

3.1. Definition of $\text{ganit}_v(B)$. Hereafter, for our simplicity, we denote $\omega_i := (\sigma_i^v)$ for $i \geq 1$ and we denote $v_r = (\omega_1, \ldots, \omega_r)$ for $r \geq 1$.

Definition 3.1 (\cite[(2.36)]{E11}, \cite[(2.7.2)]{Sch15}). Let $B \in \overline{\text{GARI}}(\Gamma)$. We define the $\mathbb{Q}$-linear map $\text{ganit}_v(B) : \overline{\text{BIMU}}(\Gamma) \rightarrow \overline{\text{BIMU}}(\Gamma)$ by $(\text{ganit}_v(B)(A))(v_0) := A(0)$ and

\[(\text{ganit}_v(B)(A))(v_r) := \sum_{s \geq 1} \sum_{v_{s} = a_1 b_1 \cdots a_s b_s, \ a_1, \ldots, a_s \neq \emptyset, \ b_1, \ldots, b_s \neq \emptyset} B(a_1 [b_1] \cdots B(a_s [b_s]) A(a_1 [b_1] \cdots a_s [b_s]),
\]

for $r \geq 1$ and for $A \in \overline{\text{BIMU}}(\Gamma)$.

Examples 3.2. Because the definition of $\text{ganit}_v(B)(A)$ is complicated, we explicitly give some examples of $\text{ganit}_v(B)(A)$.

\[
\begin{align*}
(\text{ganit}_v(B)(A))(v_1) &= A(\omega_1), \\
(\text{ganit}_v(B)(A))(v_2) &= A(\omega_1, \omega_2) + B(\omega_1 [\omega_2]) A(\omega_1 [\omega_2]), \\
(\text{ganit}_v(B)(A))(v_3) &= A(\omega_1, \omega_2, \omega_3) + B(\omega_2 [\omega_3]) A(\omega_1, \omega_2 [\omega_3]) \\
&\quad + B(\omega_1 [\omega_2]) A(\omega_1, \omega_2, \omega_3) + B(\omega_1 [\omega_2, \omega_3]) A(\omega_1 [\omega_2, \omega_3]), \\
(\text{ganit}_v(B)(A))(v_4) &= A(\omega_1, \omega_2, \omega_3, \omega_4) + B(\omega_3 [\omega_4]) A(\omega_1, \omega_2, \omega_3 [\omega_4]) \\
&\quad + B(\omega_2 [\omega_3]) A(\omega_1, \omega_2, \omega_3, \omega_4) + B(\omega_2 [\omega_3, \omega_4]) A(\omega_1, \omega_2, [\omega_3, \omega_4]) \\
&\quad + B(\omega_1 [\omega_2]) A(\omega_1, \omega_2, \omega_3, \omega_4) + B(\omega_1 [\omega_2, \omega_3]) A(\omega_1, \omega_2, \omega_3 [\omega_4]) + B(\omega_1 [\omega_2, \omega_3, \omega_4]) A(\omega_1 [\omega_2, \omega_3, \omega_4]).
\end{align*}
\]

Remark 3.3. Let $B \in \overline{\text{GARI}}(\Gamma)$. By Definition \[3.1\] we have

\[
\begin{align*}
\text{ganit}_v(B)(A)(0) &= A(0) = 0, \\
\text{ganit}_v(B)(S)(0) &= S(0) = 1,
\end{align*}
\]

for $A \in \overline{\text{ARI}}(\Gamma)$ and $S \in \overline{\text{GARI}}(\Gamma)$. Therefore, $\text{ganit}_v(B)$ induces the $\mathbb{Q}$-linear maps on $\overline{\text{ARI}}(\Gamma)$ and on $\overline{\text{GARI}}(\Gamma)$.
**Remark 3.4.** Let $B, C \in \overline{\text{GARI}}(\Gamma)$. Assume that we have $\text{ganit}_v(B)(\text{ganit}_v(C)(A)) = A$ for any $A \in \overline{\text{BIMU}}(\Gamma)$. Then by the above examples, we get

$$A(\omega_1, \omega_2) + \{B(\omega_1 | \omega_2) + C(\omega_1 | \omega_2)\} A(\omega_1 | \omega_2) = A(\omega_1, \omega_2).$$

Therefore, we obtain $C(\omega_1) = -B(\omega_1)$. In general, by computing

$$\text{ganit}_v(B)(\text{ganit}_v(C)(A)) = A,$$

we see that the component $C(\omega_1)$ can be algebraically expressed by using $B(\omega_1)$ (1 \leq m \leq r). Hence, for any $B \in \overline{\text{GARI}}(\Gamma)$, there is the inverse map of $\text{ganit}_v(B)$.

In [E11 §4.7], it is mentioned that the inverse map of $\text{ganit}_v(B)$ is given by

$$\text{ganit}_v(B)(\text{ganit}_v(C)(A)) = A,$$

where the map $\text{pari}$ and $\text{anti}$ are defined by

$$\text{pari}(M)^{(a_1; \ldots; a_r)} := (-1)^r M^{(a_1; \ldots; a_r)},$$
$$\text{anti}(M)^{(a_1; \ldots; a_r)} := M^{(a_1; \ldots; a_r)}.$$\(1)

In fact, by direct calculation, we can check that $\text{ganit}_v(\text{pari}(\text{anti}(p_{ij})))$ is the inverse map of $\text{ganit}_v(\text{pic})$. In [B] and [Sch15], the mould $\text{poc} \in \overline{\text{GARI}}(\Gamma)$ is introduced by

$$\text{poc} := \text{pari} \circ \text{anti}(p_{ij}),$$

that is, $\text{poc}(v_0) := 1$ and $\text{poc}(v_1) := -\frac{1}{v_1}$ and

$$\text{poc}(v_r) := -\frac{1}{v_1(v_1 - v_2)(v_2 - v_3)\cdots(v_{r-1} - v_r)}.$$\(2)

By using this mould $\text{poc}$, the inverse map of $\text{ganit}_v(\text{pic})$ is denoted by $\text{ganit}_v(\text{poc}).$

**Proposition 3.5.** For any $B \in \overline{\text{GARI}}(\Gamma)$, the map $\text{ganit}_v(B)$ forms an algebra automorphism on $(\overline{\text{BIMU}}(\Gamma), \times)$, that is, for any $A_1, A_2 \in \overline{\text{BIMU}}(\Gamma)$, we have

$$\text{ganit}_v(B)(A_1 \times A_2) = \text{ganit}_v(B)(A_1) \times \text{ganit}_v(B)(A_2).$$

We use the following lemma to prove this proposition.

**Lemma 3.6.** Let $A_1, A_2 \in \overline{\text{BIMU}}(\Gamma)$ and $s \geq 1$. For $a_1, \ldots, a_s \in V_2^* \setminus \{\emptyset\}$ and for $b_1, \ldots, b_{s-1} \in V_2^* \setminus \{\emptyset\}$ and for $b_s \in V_2^*$, we put $\omega = a_1 | b_1 \cdots a_s | b_s$. Then we have

$$(A_1 \times A_2)(\omega) = A_1(\omega)A_2(\emptyset) + A_1(\emptyset)A_2(\omega) + \sum_{p=1}^{s-1} A_1(a_1 | b_1 \cdots a_p | b_p)A_2(a_{p+1} | b_{p+1} \cdots a_s | b_s) + \sum_{p=1}^{s} \sum_{a_p | b_p = \emptyset} A_1(a_1 | b_1 \cdots a_p | b_p \cdots a_{p-1} | b_{p-1})A_2(\beta a_{p+1} | b_{p+1} \cdots a_s | b_s).$$
Proof. By definition of the product $\times$, we have

$$(A_1 \times A_2)(\omega) = \sum_{\omega = \alpha \beta} A_1(\alpha)A_2(\beta)$$

$$= A_1(\omega)A_2(\emptyset) + \sum_{p=1}^{s} \sum_{a_p \beta = \alpha \beta} A_1(a_1|b_1 \cdots a_{p-1}|b_{p-1} \alpha)A_2(\beta a_p b_{p+1} \cdots a_s b_s)$$

$$= A_1(\omega)A_2(\emptyset) + \sum_{p=1}^{s} A_1(a_1|b_1 \cdots a_{p-1}|b_{p-1})A_2(a_p|b_p a_{p+1}|b_{p+1} \cdots a_s b_s) + \sum_{p=1}^{s} A_1(a_1|b_1 \cdots a_{p-1}|b_{p-1} \alpha)A_2(\beta a_p b_{p+1} \cdots a_s b_s).$$

The case of $p = 1$ in the second term coincides with $A_1(\emptyset)A_2(\omega)$. Hence, we obtain the claim. 

By using Lemma 3.6, we prove Proposition 3.5.

Proof of Proposition 3.5. Let $B \in \text{GARI}(\Gamma)$ and $A_1, A_2 \in \text{BIMU}(\Gamma)$. When $r = 0, 1$, it is easy to get

$$(\text{ganit}_r(B)(A_1 \times A_2))(v_r) = (A_1 \times A_2)(v_r) = (\text{ganit}_r(B)(A_1) \times \text{ganit}_r(B)(A_2))(v_r).$$

For $r \geq 2$, by definition we have

$$(\text{ganit}_r(B)(A_1 \times A_2))(v_r) = \sum_{s \geq 1} \sum_{v_r = a_1b_1 \cdots a_s b_s \atop a_1, \ldots, a_s \neq \emptyset \atop b_1, \ldots, b_{s-1} \neq \emptyset} B(a_1|b_1) \cdots B(a_s|b_s) (A_1 \times A_2)(a_1|b_1 \cdots a_s b_s).$$

By using Lemma 3.6 we calculate

(3.3)

$$(\text{ganit}_r(B)(A_1 \times A_2))(v_r) = (\text{ganit}(B)(A_1))(v_r)A_2(\emptyset) + A_1(\emptyset)(\text{ganit}(B)(A_2))(v_r)$$

$$+ \sum_{s \geq 1} \sum_{v_r = a_1b_1 \cdots a_s b_s \atop a_1, \ldots, a_s \neq \emptyset \atop b_1, \ldots, b_{s-1} \neq \emptyset} B(a_1|b_1) \cdots B(a_s|b_s) \left\{ \sum_{p=1}^{s-1} A_1(a_1|b_1 \cdots a_p b_p)A_2(a_p a_{p+1} \cdots a_s b_s) \right\}$$

$$+ \sum_{p=1}^{s} A_1(a_1|b_1 \cdots a_{p-1}|b_{p-1} \alpha)A_2(\beta a_p b_{p+1} \cdots a_s b_s).$$
Here, by putting $q = s - p$, we calculate the third term of the right hand side of (3.3) as

$$
\sum_{s \geq 1} \sum_{v_r = a_1 b_1 \cdots a_s b_s} B(a_1 b_1 \cdots b_s) \sum_{p=1}^{s-1} A_1(a_1 b_1 \cdots a_p b_p) A_2(a_{p+1} b_{p+1} \cdots a_s b_s)
$$

$$
= \sum_{p, q \geq 1} \sum_{v_r = a_1 b_1 \cdots a_p b_p} B(a_1 b_1 \cdots B(a_{p+q} b_{p+q}) A_1(a_1 b_1 \cdots a_p b_p) A_2(a_{p+1} b_{p+1} \cdots a_{p+q} b_{p+q}).
$$

While, we put $q = s - p + 1$ and we replace

$$
a_i \rightarrow a_{i+1}, \quad (p + 1 \leq i \leq s)
$$

$$
b_j \rightarrow b_{j+1}, \quad (p \leq j \leq s)
$$

and replace $\alpha = a_p, \beta = a_{p+1} \mid b_{p+1}$. Then we calculate the fourth term of the right hand side of (3.3) as

$$
\sum_{s \geq 1} \sum_{v_r = a_1 b_1 \cdots a_s b_s} B(a_1 b_1 \cdots b_s) \sum_{p=1}^{s-1} A_1(a_1 b_1 \cdots a_{p-1} b_{p-1} \alpha) A_2(\beta a_{p+1} b_{p+1} \cdots a_s b_s)
$$

$$
= \sum_{p, q \geq 1} \sum_{v_r = a_1 b_1 \cdots a_{p+q} b_{p+q}} B(a_1 b_1 \cdots B(a_{p+q} b_{p+q}) A_1(a_1 b_1 \cdots a_p b_p) A_2(a_{p+1} b_{p+1} \cdots a_{p+q} b_{p+q}).
$$

Hence, we get

$$(\text{ganit}_r (B(A_1 \times A_2))(v_r)) = (\text{ganit}_r (B)(A_1))(v_r) A_2(\emptyset) + A_1(\emptyset)(\text{ganit}_r (B)(A_2))(v_r)
$$

$$
+ \sum_{p, q \geq 1} \sum_{v_r = a_1 b_1 \cdots a_{p+q} b_{p+q}} B(a_1 b_1 \cdots B(a_{p+q} b_{p+q})
$$

$$
\cdot A_1(a_1 b_1 \cdots a_p b_p) A_2(a_{p+1} b_{p+1} \cdots a_{p+q} b_{p+q}).
$$
By factoring the summation of the third term, we calculate
\[ (\text{ganit}(B)(A_1))(v_r)A_2(\emptyset) + A_1(\emptyset)(\text{ganit}(B)(A_2))(v_r) \]
\[ + \sum_{v_r = \alpha \beta \alpha', \beta' \neq \emptyset} \sum_{p \geq 1} \sum_{a_1 = a_1 b_1 \ldots \beta} \cdot B(a_1 [b_1] \ldots B(a_p [b_p] A_1(a_1 \beta \ldots \alpha p \beta p)} \]
\[ \cdot \sum_{q \geq 1} \sum_{\beta = a_{p+1} b_{p+1} \ldots \beta p} B(a_{p+1} [b_{p+1}] \ldots \beta_{p+q} [b_{p+q}] A_2(a_{p+1} \beta \ldots \alpha_{p+q} \beta_{p+q}) \]
\[ = \sum_{v_r = \alpha \beta} (\text{ganit}_v(B)(A_1))(\alpha)(\text{ganit}_v(B)(A_2))(\beta) \]
\[ = (\text{ganit}_v(B)(A_1) \times \text{ganit}_v(B)(A_2))(v_r), \]

Hence, we finish the proof.

**Theorem 3.7.** For \( B \in \overline{\text{GARI}}(\Gamma) \), the \( \mathbb{Q} \)-linear map \( \text{ganit}_v(B) \) induces a group automorphism on \( (\overline{\text{GARI}}(\Gamma), \times) \) and induces a Lie algebra automorphism on \( (\overline{\text{ARI}}(\Gamma), [,]). \)

**Proof.** By combining Remark 3.4 and Proposition 3.3, we see that the map \( \text{ganit}_v(B) \) induces a group homomorphism on \( (\overline{\text{GARI}}(\Gamma), \times) \) and induces a Lie algebra homomorphism on \( (\overline{\text{ARI}}(\Gamma), [,]). \) Because the map \( \text{ganit}_v(B) \) has the inverse map by Remark 3.3, we get the claim.

**Corollary 3.8.** For \( B \in \overline{\text{GARI}}(\Gamma) \), the following diagram commutes:

\[ \xymatrix{ \overline{\text{GARI}}(\Gamma) \ar[r]^{\text{ganit}_v(B)} & \overline{\text{GARI}}(\Gamma) \ar[d]^{\text{exp}_\times} \ar[u]_{\text{exp}_\times} \ar[l]_{\text{ganit}_v(B)} \overline{\text{ARI}}(\Gamma) } \]

**Proof.** Let \( B \in \overline{\text{GARI}}(\Gamma) \) and \( A \in \overline{\text{ARI}}(\Gamma) \). By using the definition (2.5) of the map \( \text{exp}_\times \) and Theorem 3.7, we have
\[ \text{ganit}_v(B)(\text{exp}_\times(A)) = \text{ganit}_v(B) \left( \sum_{k \geq 0} \frac{1}{k!} A^\times k \right) = \sum_{k \geq 0} \frac{1}{k!} \text{ganit}_v(B)(A^\times k) \]
\[ = \sum_{k \geq 0} \frac{1}{k!} \text{ganit}_v(B)(A)^\times k = \text{exp}_\times(\text{ganit}_v(B)(A)). \]

Hence, we obtain the above diagram.

3.2. **Reformulation of \( \text{ganit}_v(B) \).** In this subsection, we introduce elements \( g_B(v_r) \in K(V) \) (Definition 3.9) as an analogue of \( \text{ganit}_v(B) \). We prove Lemma 3.17 which is essential to prove several theorems in §3.3.
Definition 3.9. For $B \in \text{GARI}(\Gamma)$ and $r \geq 0$, we define $g_B(v_r) \in \mathcal{K}(V_2)$ by
\[ g_B(v_0) := \emptyset \quad \text{and for } r \geq 1 \]
\[ g_B(v_r) := \sum_{s \geq 1} \sum_{\substack{v_r = a_1 b_1 \cdots a_s b_s \\ a_1, \ldots, a_s \neq \emptyset \\ b_1, \ldots, b_{s-1} \neq \emptyset}} B(a_1|b_1) \cdots B(a_s|b_s) \left( a_1 b_1 \cdots a_s b_s \right). \]

Remark 3.10. The above definition (3.4) is an analogue of (3.1). In fact, for $r = 1, 2, 3$, we have
\[ g_B(v_1) = (\omega_1), \]
\[ g_B(v_2) = (\omega_1, \omega_2) + B(\omega_1|\omega_2)(\omega_1|\omega_2), \]
\[ g_B(v_3) = (\omega_1, \omega_2, \omega_3) + B(\omega_2|\omega_3)(\omega_1, \omega_2|\omega_3) \]
\[ + B(\omega_1|\omega_2)(\omega_1, \omega_2|\omega_3) + B(\omega_1|(\omega_2, \omega_3))(\omega_1|\omega_2, \omega_3). \]

This $g_B$ is useful to prove Theorem 3.22.

Remark 3.11. Assume that $u_1, \ldots, u_r \in \mathcal{F}$ are algebraically independent over $\mathbb{Q}$, and put $\alpha_i := \left( \frac{\sigma_i}{\mathbb{Q}} \right)$ for $1 \leq i \leq r$. We denote $g_B(\alpha_1, \ldots, \alpha_r)$ to be the image of $g_B(v_r)$ under the field embedding $\mathbb{Q}(v_1, \ldots, v_r) \hookrightarrow \mathcal{F}$ sending $v_i \mapsto u_i$. For $c_1, \ldots, c_m \in \mathcal{K}$ and for $\beta_1, \ldots, \beta_m \in V_2^*$, we also denote
\[ g_B(c_1 \beta_1 + \cdots + c_m \beta_m) := c_1 g_B(\beta_1) + \cdots + c_m g_B(\beta_m). \]

We introduce several notations which are used to prove claims of this subsection.

Notation 3.12. In Notation 1.3, we denote the element $u_1 \cdots u_m \in V_2^*$ by $(u_1, \ldots, u_m)$ for $u_1, \ldots, u_m \in V_2$. To avoid confusion, we sometimes denote the element $(c_1, \ldots, c_t) \in (V_2^*)^t$ by $(c_1; \ldots; c_t)$ for $c_1, \ldots, c_t \in V_2^*$. By using notation, we denote
\[ W_B(u) := W_B(a_1; b_1; \ldots; a_s; b_s) := B(a_1|b_1) \cdots B(a_s|b_s) \left( a_1 b_1 \cdots a_s b_s \right), \]
for $s \geq 1$ and $u = (a_1; b_1; \ldots; a_s; b_s) \in (V_2^*)^{2s}$ and $B \in \text{GARI}(\Gamma)$. Because we have
\[ B(u|\emptyset) = 1 \quad \text{and} \quad (u|\omega)_n = (u|\omega)_n \] for $u \in V_2$ and $\omega \in V_2^* \setminus \{\emptyset\}$, it is easy to show
\[ W_B(u; \emptyset; \omega) = W_B(u|\emptyset; \omega). \]

Definition 3.13. For any word $\omega \in V_2^*$, and for $t \in \mathbb{N}$, we define two sets $D_t(\omega)$ and $E_t(\omega)$ consisting of decompositions of $\omega$ by
\[ D_t(\omega) := \left\{ (c_1; \cdots; c_t) \in (V_2^*)^t \mid \omega = c_1 \cdots c_t, \; c_1, \ldots, c_{t-1} \neq \emptyset \right\}, \]
\[ E_t(\omega) := \left\{ (c_1; \cdots; c_t) \in (V_2^*)^t \mid \omega = c_1 \cdots c_t, \; c_2, \ldots, c_{t-1} \neq \emptyset \right\}. \]

For $t \geq 2$, we define two subsets $D_t^{22}(\omega)$ and $D_t^1(\omega)$ of $D_t(\omega)$ by
\[ D_t^{22}(\omega) := \left\{ (c_1; \cdots; c_t) \in D_t(\omega) \mid l(c_1) \geq 2 \right\}, \]
\[ D_t^1(\omega) := \left\{ (c_1; \cdots; c_t) \in D_t(\omega) \mid l(c_1) = 1 \right\}. \]

Note that $E_t(\omega) = D_t(\omega)$.

Remark 3.14. By the above notation and definition, we have
\[ g_B(v_r) = \sum_{s \geq 1} \sum_{u \in D_s(v_r)} W_B(u). \]

It is clear to show the following lemma.
Lemma 3.15. For $t \geq 2$, we have
\begin{equation}
D_t(\omega) = D_t^{22}(\omega) \sqcup D_t^1(\omega),
\end{equation}
that is, a family of sets \(\{D_t^{22}(\omega), D_t^1(\omega)\}\) is a partition of \(D_t(\omega)\), and we have
\begin{equation}
E_t(\omega) = (\{0\} \times D_{t-1}(\omega)) \sqcup D_t(\omega),
\end{equation}
that is, a family of sets \(\{\{0\} \times D_{t-1}(\omega), D_t(\omega)\}\) is a partition of \(E_t(\omega)\).

Proof. These partitions follow from the definitions. \(\square\)

For $r \geq 1$ and $\omega = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_1, \ldots, \alpha_r \in V_2$, we sometimes denote
\[
\omega' := \begin{cases} (\alpha_2, \ldots, \alpha_r) & (r \geq 2), \\ \emptyset & (r = 1). \end{cases}
\]
By using these symbols, we have
\[
D_t^{22}(\omega) = \{ (\alpha_1 c_1; c_2; \cdots; c_t) \mid \omega' = c_1 \cdots c_t, c_1, \ldots, c_{t-1} \neq \emptyset \}, \\
D_t^1(\omega) = \{ (\alpha_1 c_1; c_2; \cdots; c_t) \mid \omega' = c_2 \cdots c_t, c_2, \ldots, c_{t-1} \neq \emptyset \}.
\]

Lemma 3.16. Let $\omega = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_1, \ldots, \alpha_r \in V_2$. Then there exist two bijections
\begin{alignat}{2}
D_t^{22}(\omega) & \rightarrow D_t(\omega') : (\alpha_1 c_1; c_2; \cdots; c_t) & \mapsto (c_1; c_2; \cdots; c_t), \\
D_t^1(\omega) & \rightarrow D_{t-1}(\omega') : (\alpha_1 c_1; c_2; \cdots; c_t) & \mapsto (c_2; \cdots; c_t),
\end{alignat}
for $t \geq 2$, and there exists a bijection
\begin{equation}
E_t(\omega') \rightarrow D_t(\omega_1, \omega') = D_t(\omega)
(b_0; u) \mapsto (\alpha_1 b_0; u)
\end{equation}
for $t \geq 2$.

Proof. These three bijections follow from the definitions. \(\square\)

Lemma 3.17. For $r \geq 1$, we have
\[
g_B(v_r) = \sum_{s \geq 0} \sum_{(b_0; u) \in E_{2s+1}(v_r)} B(\omega_1|b_0; (\omega_1|b_0)) W_B(u).
\]

Proof. By using the bijection (3.11) for $t = 2s$ and $\omega = v_r$, we have
\[
g_B(v_r) = \sum_{s \geq 1} \sum_{(a_1; b_1; u) \in D_{2s}(v_r)} W_B(a_1; b_1; u) = \sum_{s \geq 1} \sum_{(a_1; b_1; u) \in E_{2s}(v_r)} W_B(\omega_1 a_1; b_1; u).
\]
By using the partition (3.3) for $t = 2s$ and $\omega = v_r'$, we get
\[
= \sum_{s \geq 1} \sum_{(b_1; u) \in D_{2s+1}(v_r')} W_B(\omega_1; b_1; u) + \sum_{s \geq 1} \sum_{(a_1; b_1; u) \in D_{2s}(v_r')} W_B(\omega_1 a_1; b_1; u).
\]
By changing variables of the first summation and by applying (3.5) to the second summation, we calculate
\[
= \sum_{s \geq 0} \sum_{(b_0; u) \in D_{2s+1}(v_r')} W_B(\omega_1; b_0; u) + \sum_{s \geq 1} \sum_{(a_1; b_1; u) \in D_{2s}(v_r')} W_B(\omega_1; b_0; u) + \sum_{s \geq 1} \sum_{(b_0; u) \in (\emptyset) \times D_{2s}(v_r')} W_B(\omega_1; b_0; u).
\]
By using the partition \(\text{(3.3)}\) again, we have

\[
= W_B(\omega_1; v'_r) + \sum_{s \geq 1} \sum_{(b_0,u) \in E_{2s+1}(v'_r)} W_B(\omega_1; b_0; u)
\]

\[
= \sum_{s \geq 0} \sum_{(b_0,u) \in E_{2s+1}(v'_r)} \text{pic}(\omega_1[b_0](\omega_1)_b)W_B(u).
\]

Hence, we obtain the claim. \qed

3.3. Automorphisms between \(\text{ARI}(\Gamma)_{al}\) and \(\text{ARI}(\Gamma)_{il}\) and between \(\text{GARI}(\Gamma)_{as}\) and \(\text{GARI}(\Gamma)_{im}\). In this subsection, we prove the recurrence formula (Proposition 3.18) of \(g_B(v_r)\) for \(B = \text{pic}\). By using this recurrence formula, we prove Theorem 3.24 that is, the map \(g_{\text{al}}(B)\) induces a group isomorphism from \(\text{ARI}(\Gamma)_{al}\) to \(\text{ARI}(\Gamma)_{il}\) and induces a Lie algebra isomorphism from \(\text{GARI}(\Gamma)_{as}\) to \(\text{GARI}(\Gamma)_{im}\).

As a corollary, we obtain a commutative diagram (Corollary 3.25).

We consider the map \(g_{\text{al}}(B)\) in the case of \(B = \text{pic}\) defined in Example 1.7(d). For our simplicity, we often denote \(g_{\text{pic}}\) by \(g\) and denote \(W_{\text{pic}}(u)\) by \(W(u)\).

We prove the following key recurrence formulas of \(g\), which is important to prove Theorem 3.24.

**Proposition 3.18.** For \(r \geq 2\), we have

\[
(3.12) \quad g(v_r) = (\omega_1)g(v'_r) + \text{pic}(\omega_1[\omega_2]g(\omega_1)_{\omega_2}, v''_r),
\]

that is, we have

\[
g(\omega_1, \omega_2, v''_r) = (\omega_1)g(\omega_2, v''_r) + \text{pic}(\omega_1[\omega_2]g(\omega_1)_{\omega_2}, v''_r).
\]

**Proof.** By using the equation \(\text{(3.6)}\) and by using the partition \(\text{(3.7)}\), we calculate

\[
g(v_r) = \sum_{s \geq 1} \sum_{(\omega_1a_1;b_1,u) \in D_{2s}(v_r)} \text{pic}(\omega_1a_1_1)b_1(\omega_1a_1)_b)W(u)
\]

\[
+ \sum_{s \geq 1} \sum_{(\omega_1b_1,u) \in D_{2s}(v_r)} \text{pic}(\omega_1[b_1](\omega_1)_b)W(u).
\]

Because \(a_1 \neq \emptyset\) in the first summation, we have \((\omega_1a_1)_1[b_1] = a_1[b_1] and \((\omega_1a_1)_1)_b) = (\omega_1)_a_1)_b\). By using two bijections \(\text{(3.9)}, \text{(3.10)}\) for \(t = 2s\) and for \(\omega = \nu_r\), we get

\[
(3.13) \quad g(v_r) = (\omega_1) \sum_{s \geq 1} \sum_{(a_1;b_1,u) \in D_{2s}(v'_r)} \text{pic}(a_1[b_1](a_1)_b)W(u)
\]

\[
+ \sum_{s \geq 1} \sum_{(b_1,u) \in D_{2s-1}(v'_r)} \text{pic}(\omega_1[b_1](\omega_1)_b)W(u).
\]

Here, by the equation \(\text{(3.6)}\), the first term of the right hand side of the equation \(\text{(3.13)}\) is equal to \((\omega_1)g(v'_r)\). We calculate the second summation as below:

\[
\sum_{s \geq 1} \sum_{(b_1,u) \in D_{2s-1}(v'_r)} \text{pic}(\omega_1[b_1](\omega_1)_b)W(u)
\]

\[
= \text{pic}(\omega_1[v'_r](\omega_1)_v) + \sum_{s \geq 2} \sum_{(\omega_2b_1,u) \in D_{2s-1}(v'_r)} \text{pic}(\omega_1[\omega_2b_1](\omega_1)_{\omega_2b_1)W(u)
\]

\[
+ \sum_{s \geq 2} \sum_{(\omega_2b_1,u) \in D_{2s-1}(v'_r)} \text{pic}(\omega_1[\omega_2](\omega_1)_\omega)W(u).
\]
Here, for the first term, we have \( \text{pic}(\omega_1 [v'_r]) = \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [v''_r]) = \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [\omega_2]) \text{pic}(v''_r) \) and \((\omega_1 [v'_r]) = ((\omega_1 [\omega_2]) [v'_r]).\) For the second term, we have \( \text{pic}(\omega_1 [\omega_2 b_1]) = \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [b_1]) = \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [\omega_2]) [b_1] \) and \(\omega_1 [\omega_2 b_1] = (\omega_1 [\omega_2]) [b_1].\) So by applying two bijections \((3.9), (3.10)\) for \( t = 2s - 1 \) and for \( \omega = v'_r \) to the second and the third term, we get

\[
= \text{pic}(\omega_1 [\omega_2]) \left\{ \begin{array}{l}
\text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [v'_r]) ((\omega_1 [\omega_2]) [v'_r]) \\
+ \sum_{s \geq 2} \sum_{(b_1, u) \in D_{2s-1}(v''_r)} \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [b_1]) (\omega_1 [\omega_2]) b_1 W(u) + \sum_{s \geq 2} \sum_{u \in D_{2s-2}(v''_r)} (\omega_1 [\omega_2]) W(u) \end{array} \right\}.
\]

By putting the first and the second term together and by changing the variable \( s \geq 2 \) to \( s \geq 1 \) in the third term, we have

\[
= \text{pic}(\omega_1 [\omega_2]) \left\{ \begin{array}{l}
\sum_{s \geq 1} \sum_{(b_1, u) \in D_{2s-1}(v''_r)} \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [b_1]) (\omega_1 [\omega_2]) b_1 W(u) + \sum_{s \geq 1} \sum_{u \in D_{2s-2}(v''_r)} (\omega_1 [\omega_2]) W(u) \end{array} \right\}.
\]

By applying two bijections \((3.9), (3.10)\) for \( t = 2s \) and for \( \omega = (\omega_1 [\omega_2], v''_r) \) to the first and the second term, we have

\[
= \text{pic}(\omega_1 [\omega_2]) \left\{ \begin{array}{l}
\sum_{s \geq 1} \sum_{(a_1, b_1, u) \in D_{2s}(\omega_1 [\omega_2], v''_r)} \text{pic}(\omega_1 [\omega_2]) \text{pic}(\omega_1 [a_1]) (\omega_1 [\omega_2]) a_1 W(u)
+ \sum_{s \geq 1} \sum_{u \in D_{2s}(\omega_1 [\omega_2], v''_r)} \text{pic}(\omega_1 [\omega_2]) W(u) \end{array} \right\}
= \text{pic}(\omega_1 [\omega_2]) \sum_{s \geq 1} \sum_{a_1, b_1, u \in D_{2s}(\omega_1 [\omega_2], v''_r)} \text{pic}(\omega_1 [\omega_2]) W(u)
= \text{pic}(\omega_1 [\omega_2]) \sum_{s \geq 1} u \in D_{2s}(\omega_1 [\omega_2], v''_r) W(u).
\]

By the equation \((3.6)\), we see that the last member is equal to \( \text{pic}(\omega_1 [\omega_2]) g(\omega_1 [\omega_2], v''_r), \) that is, the second term of the right hand side of the equation \((3.13)\) is equal to \( \text{pic}(\omega_1 [\omega_2]) g(\omega_1 [\omega_2], v''_r). \) Hence, we obtain the claim.

**Lemma 3.19.** Let \( r, s \geq 1 \) and put \( \alpha := (\omega_2, \ldots, \omega_r) \) and \( \beta := (\omega_{r+1}, \ldots, \omega_{r+s}). \) Then we have

\[
g(\omega_1, \alpha \uplus \beta) = \sum_{p, q \geq 0} \sum_{(b_0, u) \in E_{2p+1}(\alpha)} \sum_{(d_0, v) \in E_{2q+1}(\beta)} \text{pic}(\omega_1 [(b_0d_0)]) (\omega_1 [(b_0d_0)]) \{W(u) \uplus W(v)\}.
\]
Proof. By the definition of $E_t(\omega)$, we have

$$ E_{2p+1}(\emptyset) = \begin{cases} \{\emptyset\} & (p = 0), \\
\emptyset & (p \geq 1), \end{cases} $$

for $p \geq 0$, so the equation (3.14) for $\alpha = \emptyset$ (i.e., $r = 1$) is equal to

$$ g(\omega_1, \beta) = \sum_{q \geq 0} \sum_{(d_0, v) \in E_{2q+1}(\beta)} \text{pic}(\omega_1 | d_0) \cdot (\omega_1 | d_0) W(v). $$

Because this equation follows from Lemma 3.17, the equation (3.14) holds for $\alpha = \emptyset$. Therefore, in the following, we assume $\alpha \neq \emptyset$, that is, $r \geq 2$.

We prove (3.14) by induction on $r + s (\geq 3)$. When $r + s = 3$ (i.e., $r = 2$, $s = 1$), we have $\alpha = (\omega_2)$ and $\beta = (\omega_3)$ and

$$ \alpha \cup \omega_3 \beta = (\omega_2, \omega_3) + (\omega_3, \omega_2) - \text{pic}(\omega_2 | \omega_3) \cdot (\omega_2 | \omega_3) - \text{pic}(\omega_3 | \omega_2) \cdot (\omega_3 | \omega_2). $$

So the left hand side of (3.14) is equal to

$$ g(\omega_1, \alpha \cup \omega_3 \beta) = g(\omega_1, \omega_2, \omega_3) + g(\omega_1, \omega_3, \omega_2) - \text{pic}(\omega_2 | \omega_3) g(\omega_1, \omega_2 | \omega_3) - \text{pic}(\omega_3 | \omega_2) g(\omega_1, \omega_3 | \omega_2) $$

$$ = (\omega_1, \omega_2, \omega_3) + \text{pic}(\omega_2 | \omega_3) \cdot (\omega_1, \omega_2 | \omega_3) + \text{pic}(\omega_1 | \omega_2) \cdot (\omega_1 | \omega_2, \omega_3) + \text{pic}(\omega_1 | (\omega_2, \omega_3)) \cdot (\omega_1 | \omega_2, \omega_3) $$

$$+ \text{pic}(\omega_3 | \omega_2) \cdot (\omega_1, \omega_3 | \omega_2) + \text{pic}(\omega_1 | \omega_3) \cdot (\omega_1 | \omega_3, \omega_2) + \text{pic}(\omega_1 | (\omega_3, \omega_2)) \cdot (\omega_1 | \omega_3, \omega_2) $$

$$ - \text{pic}(\omega_2 | \omega_3) \cdot ((\omega_1, \omega_2 | \omega_3) + \text{pic}(\omega_1 | \omega_2 | \omega_3) \cdot (\omega_1 | \omega_2 | \omega_3)) $$

$$- \text{pic}(\omega_3 | \omega_2) \cdot ((\omega_1, \omega_3 | \omega_2) + \text{pic}(\omega_1 | \omega_3 | \omega_2) \cdot (\omega_1 | \omega_3 | \omega_2)).$$

By using $(\omega_1 | \omega_2 | \omega_3) = (\omega_1 | \omega_3 | \omega_2) = (\omega_1 | (\omega_2, \omega_3))$ and by using Remark 1.3(b), we get

$$ = (\omega_1, \omega_2, \omega_3) + \text{pic}(\omega_1 | \omega_2) \cdot (\omega_1 | \omega_2, \omega_3) + \text{pic}(\omega_1 | (\omega_2, \omega_3)) \cdot (\omega_1 | \omega_2, \omega_3) $$

$$+ (\omega_1, \omega_3, \omega_2) + \text{pic}(\omega_1 | \omega_3) \cdot (\omega_1 | \omega_3, \omega_2).$$

While, by the definition of $E_t(\omega)$, we have

$$ E_{2p+1}(\omega_2) = \begin{cases} \{\omega_2\} & (p = 0), \\
\{\emptyset, \omega_2, \emptyset\} & (p = 1), \\
\emptyset & (p \geq 2), \end{cases} $$

$$ E_{2q+1}(\omega_3) = \begin{cases} \{\omega_3\} & (q = 0), \\
\{\emptyset, \omega_3, \emptyset\} & (q = 1), \\
\emptyset & (q \geq 2), \end{cases} $$

8Here, it means that $E_t(\emptyset)$ is the set consisting of only the empty word $\emptyset$ when $t = 1$ and is the emptyset when $t \geq 2$. 


for \( p, q \geq 0 \). Therefore, the right hand side of (3.14) is equal to

\[
\sum_{p,q \in \{0,1\}} \sum_{(b_0, u) \in E_{2p+1}(\omega_2)} \text{pic}(\omega_1 \{b_0d_0\}) \ (\omega_1 \{b_0d_0\}) \ \{W(u) \cup W(v)\} \\
= \sum_{p \in \{0,1\}} \sum_{(b_0, u) \in E_{2p+1}(\omega_2)} \text{pic}(\omega_1 \{b_0\omega_3\}) \ (\omega_1 \{b_0\omega_3\}) \ \{W(u) \cup W(v)\} \\
+ \sum_{p \in \{0,1\}} \sum_{(b_0, u) \in E_{2p+1}(\omega_2)} \text{pic}(\omega_1 \{b_0\}) \ (\omega_1 \{b_0\}) \ \{W(u) \cup W(v)\}
\]

\[
\begin{aligned}
= & \text{pic}(\omega_1 \{\omega_2, \omega_3\}) \ (\omega_1 \{\omega_2, \omega_3\}) \ \{\emptyset \cup \emptyset\} + \text{pic}(\omega_1 \{\omega_3\}) \ (\omega_1 \{\omega_3\}) \ \{W(\omega_2; \emptyset) \cup \emptyset\} \\
& + \text{pic}(\omega_1 \{\omega_2\}) \ (\omega_1 \{\omega_2\}) \ \{\emptyset \cup W(\omega_3; \emptyset)\} + \text{pic}(\omega_1 \{\emptyset\}) \ (\omega_1 \{\emptyset\}) \ \{W(\omega_2; \emptyset) \cup W(\omega_3; \emptyset)\} \\
= & \text{pic}(\omega_1 \{\omega_2, \omega_3\}) \ (\omega_1 \{\omega_2, \omega_3\}) + \text{pic}(\omega_1 \{\omega_3\}) \ (\omega_1 \{\omega_3\}) + \text{pic}(\omega_1 \{\omega_2\}) \ (\omega_1 \{\omega_2\}) + \text{pic}(\omega_1 \{\emptyset\}) \ (\omega_1 \{\emptyset\})
\end{aligned}
\]

Hence, the equation (3.14) holds for \( r + s = 3 \).

Assume that (3.14) holds for \( r + s \leq t \) (\( t \geq 3 \)). When \( r + s = t + 1 \), by the equation (3.14) of the product \( \cup \), we get

\[
g(\omega_1, \alpha \cup \beta) \\
= g(\omega_1, \omega_2, \alpha' \cup \beta) + g(\omega_1, \omega_{r+1}, \alpha \cup \beta') \\
- \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1, \omega_2, \alpha' \cup \beta') - \text{pic}(\omega_{r+1} \{\omega_2\}) g(\omega_1, \omega_{r+1}, \alpha' \cup \beta')
\]

By applying Proposition (3.13) to each term, we get

\[
\begin{aligned}
= & (\omega_1)g(\omega_2, \alpha' \cup \beta) + \text{pic}(\omega_1 \{\omega_2\}) g(\omega_1 \{\omega_2\}, \alpha' \cup \beta) \\
& + (\omega_1)g(\omega_{r+1}, \alpha \cup \beta') + \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1 \{\omega_{r+1}\}, \alpha \cup \beta') \\
& - \text{pic}(\omega_1 \{\omega_{r+1}\}) \left\{ (\omega_1)g(\omega_2 \{\omega_{r+1}\}, \alpha' \cup \beta') + \text{pic}(\omega_1 \{\omega_2\}) g(\omega_1 \{\omega_2\}, \alpha' \cup \beta') \right\} \\
& - \text{pic}(\omega_{r+1} \{\omega_2\}) \left\{ (\omega_1)g(\omega_{r+1} \{\omega_2\}, \alpha' \cup \beta') + \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1 \{\omega_{r+1}\}, \alpha' \cup \beta') \right\}
\end{aligned}
\]

Because \( \text{pic}(\omega_1 \{\omega_2 \{\omega_{r+1}\}\}) = \text{pic}(\omega_1 \{\omega_2\}) \) and \( \omega_1 \{\omega_2 \{\omega_{r+1}\}\} = \omega_1 \{\omega_{2, \omega_{r+1}}\} \), we have

\[
\begin{aligned}
= & (\omega_1)g(\omega_2, \alpha' \cup \beta) + \text{pic}(\omega_1 \{\omega_2\}) g(\omega_1 \{\omega_2\}, \alpha' \cup \beta) \\
& + (\omega_1)g(\omega_{r+1}, \alpha \cup \beta') + \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1 \{\omega_{r+1}\}, \alpha \cup \beta') \\
& - \text{pic}(\omega_1 \{\omega_{r+1}\}) \left\{ (\omega_1)g(\omega_2 \{\omega_{r+1}\}, \alpha' \cup \beta') + \text{pic}(\omega_1 \{\omega_2\}) g(\omega_1 \{\omega_2\}, \alpha' \cup \beta') \right\} \\
& - \text{pic}(\omega_{r+1} \{\omega_2\}) \left\{ (\omega_1)g(\omega_{r+1} \{\omega_2\}, \alpha' \cup \beta') + \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1 \{\omega_{r+1}\}, \alpha' \cup \beta') \right\}
\end{aligned}
\]

By using Remark (3.8)(b) (i.e. \( \text{pic}(\omega_2 \{\omega_{r+1}\}) \text{pic}(\omega_1 \{\omega_2\}) + \text{pic}(\omega_{r+1} \{\omega_2\}) \text{pic}(\omega_1 \{\omega_{r+1}\}) = \text{pic}(\omega_1 \{\omega_{2, \omega_{r+1}}\}) \)) and by rearranging each term, we get

\[
(3.15) \\
g(\omega_1, \alpha \cup \beta) = \left\{ (\omega_1)g(\omega_2, \alpha' \cup \beta) - \text{pic}(\omega_2 \{\omega_{r+1}\}) g(\omega_2 \{\omega_{r+1}\}, \alpha' \cup \beta) \right\} \\
+ \left\{ (\omega_1)g(\omega_{r+1}, \alpha \cup \beta') - \text{pic}(\omega_{r+1} \{\omega_2\}) g(\omega_{r+1} \{\omega_2\}, \alpha' \cup \beta') \right\} \\
+ \left\{ \text{pic}(\omega_1 \{\omega_2\}) g(\omega_1 \{\omega_2\}, \alpha' \cup \beta) + \text{pic}(\omega_1 \{\omega_{r+1}\}) g(\omega_1 \{\omega_{r+1}\}, \alpha \cup \beta') \right\} \\
- \text{pic}(\omega_1 \{\omega_{2, \omega_{r+1}}\}) g(\omega_1 \{\omega_{2, \omega_{r+1}}\}, \alpha' \cup \beta')
\]
Similarly, for the second term of the right hand side of (3.11), we obtain

\[
(\omega_1)g(\omega_2, \alpha' \llcorner \beta) - \text{pic}(\omega_2[\omega_{r+1}](\omega_1)g(\omega_2[\omega_{r+1}, \alpha' \llcorner \beta'])
\]

\[
= (\omega_1) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2q+1}(\alpha')} \text{pic}(\omega_2[(b_0d_0)](\omega_2)[b_0d_0]) \{W(u) \sqcup W(v)\}
\]

\[
- \text{pic}(\omega_2[\omega_{r+1}](\omega_1) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2q+1}(\alpha')} \text{pic}(\omega_2[(b_0d_0)](\omega_2)[b_0d_0]) \{W(u) \sqcup W(v)\}.
\]
By taking the sum of (3.18) and (3.19) and by using the definition (1.3) of the product $\omega$, we get

$$\{ (\omega_1)g(\omega_2, \alpha' \sqcup \beta) - \text{pic}(\omega_2 \mid \alpha, \alpha' \sqcup \beta) \} + \{ (\omega_1)g(\omega_{r+1}, \alpha' \sqcup \beta) - \text{pic}(\omega_{r+1} \mid \omega_2, \alpha' \sqcup \beta) \} = (\omega_1) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2p+1}(\alpha')} \{ \text{pic}(\omega_2 \mid [d_0]) \} \{ (\omega_2 \mid [d_0]) \} W(u) \sqcup W(v).$$

Hence, by using Lemma 3.17, we obtain

$$\text{(3.20) } \{ (\omega_1)g(\omega_2, \alpha' \sqcup \beta) - \text{pic}(\omega_2 \mid \alpha' \sqcup \beta) \} + \{ (\omega_1)g(\omega_{r+1}, \alpha' \sqcup \beta) - \text{pic}(\omega_{r+1} \mid \omega_2, \alpha' \sqcup \beta) \} = (\omega_1) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2p+1}(\alpha')} \{ W(u) \sqcup W(v) \}.$$

Secondly, by induction hypothesis, we calculate the third term of the right hand side of (3.16) as below:

$$\text{pic}(\omega_1 \mid \omega_2) g(\omega_1 \mid \omega_2, \alpha' \sqcup \beta) + \text{pic}(\omega_1 \mid \omega_{r+1}) g(\omega_1 \mid \omega_{r+1}, \alpha' \sqcup \beta)$$

$$- \text{pic}(\omega_1 \mid \omega_{r+1} + 1) g(\omega_1 \mid \omega_{r+1} + 1, \alpha' \sqcup \beta) = \text{pic}(\omega_1 \mid \omega_2) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2p+1}(\alpha')} \{ \text{pic}(\omega_1 \mid [d_0]) \} \{ W(u) \sqcup W(v) \}$$

$$+ \text{pic}(\omega_1 \mid \omega_{r+1}) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2p+1}(\alpha')} \{ \text{pic}(\omega_1 \mid [d_0]) \} \{ W(u) \sqcup W(v) \}$$

$$- \text{pic}(\omega_1 \mid \omega_{r+1} + 1) \sum_{p,q \geq 0} \sum_{(b_0,u) \in E_{2p+1}(\alpha')} \{ \text{pic}(\omega_1 \mid [d_0]) \} \{ W(u) \sqcup W(v) \}.$$

By using (3.16) and (3.17) and using the bijection (3.11), we get

$$= \sum_{p,q \geq 0} \sum_{(b_0,u) \in D_{2p+1}(\alpha)} \text{pic}(\omega_1 \mid [d_0]) \{ W(u) \sqcup W(v) \}$$

$$+ \sum_{p,q \geq 0} \sum_{(b_0,u) \in D_{2p+1}(\alpha)} \text{pic}(\omega_1 \mid [d_0]) \{ W(u) \sqcup W(v) \}$$

$$- \sum_{p,q \geq 0} \sum_{(b_0,u) \in D_{2p+1}(\alpha)} \text{pic}(\omega_1 \mid [d_0]) \{ W(u) \sqcup W(v) \}.$$
Hence, by using the partition (3.8) and by using $E_1(\alpha) = D_1(\alpha)$, we obtain

\begin{align*}
(3.21) \quad & \text{pic}(\omega_1 | \omega_2) g(\omega_1 | \omega_2, \alpha' \cup_\star \beta) + \text{pic}(\omega_1 | \omega_{r+1}) g(\omega_1 | \omega_{r+1}, \alpha \cup_\star \beta') \\
& - \text{pic}(\omega_1 | (\omega_2, \omega_{r+1})) g(\omega_1 | (\omega_2, \omega_{r+1}), \alpha' \cup_\star \beta') \\
& = \sum_{p,q \geq 0} \sum_{(b_0; u) \in D_{2p+1}(\alpha)} \text{pic}(\omega_1 | (b_0 d_0)) \left( \omega_1 | (b_0 d_0) \right) \{ W(u) \cup W(v) \} \\
& + \sum_{p \geq 1} \sum_{(b_0; u) \in \emptyset} \text{pic}(\omega_1 | (b_0 d_0)) \left( \omega_1 | (b_0 d_0) \right) \{ W(u) \cup W(v) \}.
\end{align*}

Therefore, by taking the sum of (3.20) and (3.21) and by using the partition (3.8) and by using $E_1(\alpha) = D_1(\alpha)$, we have

\begin{align*}
g(\omega_1, \alpha \cup_\star \beta) &= \sum_{p,q \geq 0} \sum_{(b_0; u) \in D_{2p+1}(\alpha)} \text{pic}(\omega_1 | (b_0 d_0)) \left( \omega_1 | (b_0 d_0) \right) \{ W(u) \cup W(v) \} \\
& + \sum_{p \geq 1} \sum_{(b_0; u) \in \emptyset} \text{pic}(\omega_1 | (b_0 d_0)) \left( \omega_1 | (b_0 d_0) \right) \{ W(u) \cup W(v) \} \\
& = \sum_{p,q \geq 0} \sum_{(b_0; u) \in E_{2p+1}(\alpha)} \text{pic}(\omega_1 | (b_0 d_0)) \left( \omega_1 | (b_0 d_0) \right) \{ W(u) \cup W(v) \}.
\end{align*}

Hence, we finish the proof. \qEd

**Proposition 3.20.** For $r, s \geq 1$ and for $\alpha := (\omega_1, \ldots, \omega_r)$, $\beta := (\omega_{r+1}, \ldots, \omega_{r+s})$, we have

\begin{equation}
(3.22) \quad g(\alpha \cup_\star \beta) = g(\alpha) \cup g(\beta).
\end{equation}

**Proof.** By the definition (3.11) of the product $\cup_\star$, we have

\begin{align*}
g(\alpha \cup_\star \beta) &= g(\omega_1, \alpha' \cup_\star \beta) + g(\omega_{r+1}, \alpha \cup_\star \beta') \\
& - \text{pic}(\omega_1 | \omega_{r+1}) g(\omega_1 | \omega_{r+1}, \alpha' \cup_\star \beta') - \text{pic}(\omega_{r+1} | \omega_1) g(\omega_{r+1} | \omega_1, \alpha' \cup_\star \beta')
\end{align*}
By using Lemma [3.19], we have
\[
g(\omega_1, \alpha' | \omega_s, \beta) - \text{pic}(\omega_1 [\omega_{r+1}] g(\omega_1 [\omega_{r+1}], \alpha' | \omega_s, \beta')
\]
\[
= \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
- \text{pic}(\omega_1 [\omega_{r+1}] \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
= \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
- \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
= \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
- \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\].

By using the partition (3.8) for \( t = 2q + 1 \) and by using \( E_1(\beta) = D_1(\beta) \), we get
\[
= \sum_{p \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{q \geq 1} \sum_{(d_0;v) \in \{0\} \times D_{2q}(\beta)} \text{pic}(\omega_1 [(b_0d_0)] (\omega_1 | (b_0d_0)) \{W(u) \sqcup W(v)\}
\]
\[
= \sum_{p \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{q \geq 1} \sum_{v \in D_{2q}(\beta)} \text{pic}(\omega_1 [b_0] (\omega_1 | b_0) \{W(u) \sqcup W(v)\}
\]

By using Lemma [3.17] (change the variables \( v \cdot \) to \( \beta \)), we obtain
\[
(3.23) \quad g(\omega_1, \alpha' | \omega_s, \beta) - \text{pic}(\omega_1 [\omega_{r+1}] g(\omega_1 [\omega_{r+1}], \alpha' | \omega_s, \beta')
\]
\[
= \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [b_0] \text{pic}(\omega_{r+1} | d_0) (\omega_1 | b_0) \{W(u) \sqcup (\omega_{r+1} | d_0) W(v)\}
\].

Similarly, we obtain
\[
(3.24) \quad g(\omega_{r+1}, \alpha | \omega_s, \beta') - \text{pic}(\omega_{r+1} | \omega_1) g(\omega_{r+1} | \omega_1, \alpha' | \omega_s, \beta')
\]
\[
= \sum_{p,q \geq 0} \sum_{(b_0;u) \in E_{2p+1}(\alpha')} \sum_{(d_0;v) \in E_{2q+1}(\beta')} \text{pic}(\omega_1 [b_0] \text{pic}(\omega_{r+1} | d_0) (\omega_{r+1} | d_0) \{(\omega_1 | b_0) W(u) \sqcup W(v)\}
\].
Therefore, by taking the sum of (3.23) and (3.24) and by using the definition (1.3) of the product \( \omega \), we get
\[
g(\alpha \sqcup \beta)
= \sum_{p, q \geq 0} \sum_{(b_0; u) \in E_{2p+1}(\alpha')} \sum_{(d_0; v) \in E_{2q+1}(\beta')} \pic(\omega_1 [b_0]) \pic(\omega_{r+1} [d_0]) \{ (\omega_1)_{b_0} \} W(u) \sqcup (\omega_{r+1})_{d_0} W(v)
\]
\[
= \left\{ \sum_{p \geq 0} \sum_{(b_0; u) \in E_{2p+1}(\alpha')} \pic(\omega_1 [b_0]) (\omega_1)_{b_0} W(u) \right\} \sqcup \left\{ \sum_{q \geq 0} \sum_{(d_0; v) \in E_{2q+1}(\beta')} \pic(\omega_{r+1} [d_0]) (\omega_{r+1})_{d_0} W(v) \right\}.
\]
By using Lemma 3.17, we obtain
\[
g(\alpha) \sqcup g(\beta).
\]
Hence, we obtain the claim. \(\square\)

By the above theorem, we immediately obtain the following corollary.

**Corollary 3.21.** For \( M \in \text{BIMU}(\Gamma) \) and for \( r, s \geq 1 \) and for \( \alpha := (\omega_1, \ldots, \omega_r), \beta := (\omega_{r+1}, \ldots, \omega_{r+s}) \), we have

(3.25)
\[
\mathcal{S}_\alpha (\text{ganit}_v(\pic)(M)) (\alpha; \beta)
= \sum_{p, q \geq 1} \sum_{(a_1; b_1; \ldots; a_p; b_p) \in D_{2p}(\alpha)} \sum_{(c_1; d_1; \ldots; c_q; d_q) \in D_{2q}(\beta)} \pic(a_1 [b_1]) \cdots \pic(a_p [b_p]) \pic(c_1 [d_1]) \cdots \pic(c_q [d_q])
\cdot \mathcal{S}_h(M)(a_1 [b_1] \cdots a_p [b_p]; c_1 [d_1] \cdots c_q [d_q]).
\]

**Theorem 3.22** (cf. [SaSch] Proposition 6.2, [Sch15] Lemma 4.4.2). The following two hold:

(i): \( \text{ganit}_v(\pic)(\text{ARI}(\Gamma)_{al}) \subset \overline{\text{ARI}(\Gamma)_{I}} \).

(ii): \( \text{ganit}_v(\pic)(\text{GARI}(\Gamma)_{as}) \subset \overline{\text{GARI}(\Gamma)_{I}} \).

**Proof.** Let \( r, s \geq 1 \) and \( \alpha := (\omega_1, \ldots, \omega_r), \beta := (\omega_{r+1}, \ldots, \omega_{r+s}) \). Note that we have
\[
a_1 [b_1] \cdots a_p [b_p] \neq \emptyset \quad \text{(resp.} \quad c_1 [d_1] \cdots c_q [d_q] \neq \emptyset \text{)}
\]
for \( (a_1; b_1; \ldots; a_p; b_p) \in D_{2p}(\alpha) \) (resp. \( (c_1; d_1; \ldots; c_q; d_q) \in D_{2q}(\beta) \)).

(i). Let \( A \in \overline{\text{ARI}(\Gamma)_{al}} \). By the definition of the alternal mould, we have \( \mathcal{S}_h(A)(\omega; \eta) = 0 \) for \( \omega, \eta \in V_{2}^* \setminus \{\emptyset\} \). Because by Corollary 3.21 we have
\[
\mathcal{S}_\alpha (\text{ganit}_v(\pic)(A)) (\alpha; \beta) = 0,
\]
we get
\[
\mathcal{S}_\alpha (\text{ganit}_v(\pic)(A)) = \text{ganit}_v(\pic)(A) \otimes I + I \otimes \text{ganit}_v(\pic)(A),
\]
that is, by Remark 2.7(i), we see that \( \text{ganit}_v(\pic)(A) \) is alternal. Hence, we obtain \( \text{ganit}_v(\pic)(\overline{\text{ARI}(\Gamma)_{al}}) \subset \overline{\text{ARI}(\Gamma)_{I}} \).
(ii). Let $S \in \text{GARI}(\Gamma)_{\text{as}}$. By Proposition \[24\](ii), we have $S_h(S) = S \otimes S$. So by using Corollary \[3.21\] we get

$$\text{Sh}_i(\text{ganit}_v(\text{pic})(S))(\alpha; \beta) = \sum_{p,q \geq 1} \sum_{(a_1, b_1, \ldots, a_p, b_p) \in D_{2p}(\alpha)} \sum_{(c_1, d_1, \ldots, c_q, d_q) \in D_{2q}(\beta)} \text{pic}(a_1 | b_1) \cdot \text{pic}(a_p | b_p) \cdot \text{pic}(c_1 | d_1) \cdot \text{pic}(c_q | d_q) \cdot S(a_1 | b_1, \ldots, a_p | b_p) \cdot S(c_1 | d_1, \ldots, c_q | d_q)$$

$$= \left\{ \sum_{p \geq 1} \sum_{(a_1, b_1, \ldots, a_p, b_p) \in D_{2p}(\alpha)} \text{pic}(a_1 | b_1) \cdot \text{pic}(a_p | b_p) \cdot S(a_1 | b_1, \ldots, a_p | b_p) \right\} \cdot \left\{ \sum_{q \geq 1} \sum_{(c_1, d_1, \ldots, c_q, d_q) \in D_{2q}(\beta)} \text{pic}(c_1 | d_1) \cdot \text{pic}(c_q | d_q) \cdot S(c_1 | d_1, \ldots, c_q | d_q) \right\}$$

$$= (\text{ganit}_v(\text{pic})(S))(\alpha) (\text{ganit}_v(\text{pic})(S))(\beta)$$

$$= (\text{ganit}_v(\text{pic})(S) \otimes \text{ganit}_v(\text{pic})(S))(\alpha; \beta).$$

Hence, we obtain $\text{Sh}_i(\text{ganit}_v(\text{pic})(S)) = \text{ganit}_v(\text{pic})(S) \otimes \text{ganit}_v(\text{pic})(S)$, that is, we see that $\text{ganit}_v(\text{pic})(S)$ is symmetrizable. Hence, we obtain $\text{ganit}_v(\text{pic})(\text{GARI}(\Gamma)_{\text{as}}) \subset \text{GARI}(\Gamma)_{\text{as}}$. □

**Remark 3.23.** As an analogue of Proposition \[3.21\] we have

$$g_{\text{poc}}(V_r) = g_{\text{poc}}(V_{r-1})(\omega_r) + \text{poc}(\omega_{r-1}, \omega_r) g_{\text{poc}}(V_{r-2}, \omega_{r-1}, \omega_r),$$

for $r \geq 2$. Here, poc is defined in \[3.2\]. By using this equation \[3.26\], we get an analogue of Proposition \[3.24\] that is, we have

$$g_{\text{poc}}(\alpha \cup \beta) = g_{\text{poc}}(\alpha) \cup \text{Sh}_i(\text{ganit}_v(\text{pic})(M))(\alpha; \beta)$$

for $r, s \geq 1$ and for $\alpha := (\omega_1, \ldots, \omega_r)$, $\beta := (\omega_{r+1}, \ldots, \omega_{r+s})$. By using this equation, we obtain

$$S_h(\text{ganit}_v(\text{pic})(M))(\alpha; \beta)$$

as an analogue of Corollary \[3.21\]. It should be noted that the equation \[3.28\] is equivalent to

$$\sum_{w \in sh(u, v)} B(w) = \prod_{i \in J}(v_i - v_{i+1}) \prod_{j \in J}(v_j - v_{j+1}) A_{\nu_{i+1}, \nu_i}(v_{i+1}, v_i)$$

denoted in \[Sch15\] the end of page 55]. Hence, similarly to the proof of Theorem \[3.22\] we obtain

$$\text{ganit}_v(\text{pic})(\text{ARI}(\Gamma)_{\text{II}}) \subset \text{ARI}(\Gamma)_{\text{II}},$$

$$\text{ganit}_v(\text{pic})(\text{GARI}(\Gamma)_{\text{III}}) \subset \text{GARI}(\Gamma)_{\text{III}}.$$
Theorem 3.24 (cf. [SaSch], Proposition 6.2, [Sch15], Lemma 4.4.2). The map \( \text{ganit}_v(\text{pic}) \) induces a group isomorphism from \((\text{GARI}(\Gamma)_\text{as}, \times)\) to \((\text{GARI}(\Gamma)_\text{is}, \times)\) and induces a Lie algebra isomorphism from \((\text{ARI}(\Gamma)_\text{al}, [\cdot,\cdot])\) to \((\text{ARI}(\Gamma)_\text{il}, [\cdot,\cdot])\).

Proof. By combining Theorem 3.22 and Remark 3.23, we get

\[ \text{ganit}_v(\text{pic})(\text{ARI}(\Gamma)_\text{al}) = \text{ARI}(\Gamma)_\text{il}, \]
\[ \text{ganit}_v(\text{pic})(\text{GARI}(\Gamma)_\text{as}) = \text{GARI}(\Gamma)_\text{is}. \]

Hence, by using Theorem 3.7, we obtain the claim. \(\square\)

Corollary 3.25 ([E11], §4.7]). The following diagram commutes:

\[
\begin{array}{ccc}
\text{GARI}(\Gamma)_\text{as} & \xrightarrow{\text{ganit}_v(\text{pic})} & \text{GARI}(\Gamma)_\text{is} \\
\text{exp}_x & & \text{exp}_x \\
\text{ARI}(\Gamma)_\text{al} & \xrightarrow{\text{ganit}_v(\text{pic})} & \text{ARI}(\Gamma)_\text{il}
\end{array}
\]

Proof. By using Theorem 2.12, Corollary 3.8 and Theorem 3.24, we obtain this claim. \(\square\)

Theorem 3.24 is also used to prove the following.

Theorem 3.26 (cf. [SaSch], Theorem 7.2, [Sch15], Theorem 4.6.1]). The Lie algebra automorphism \( \text{adari}(\text{pal}) \) on \( \text{ARI}(\Gamma) \) induces a bijection \( \text{adari}(\text{pal}) : \text{ARI}(\Gamma)_\text{al} \to \text{ARI}(\Gamma)_\text{il} \).

Corollary 3.27. The vector space \( \text{ARI}(\Gamma)_\text{al} \) forms a Lie subalgebra of \( \text{ARI}(\Gamma) \) under the \( \text{ari}_\text{u} \)-bracket.

Remark 3.28. To finish the proof of Theorem 3.26, we actually need the proof of another claim, that is, Theorem A.7. We give this proof in Appendix A.

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APPENDIX A. On \( \text{expari}(\text{ARI}(\Gamma)_\text{al}) = \text{GARI}(\Gamma)_\text{as} \)

In this appendix, we give the proof of Theorem A.7 whose claim is that \( \text{expari}(\text{ARI}(\Gamma)_\text{al}) = \text{GARI}(\Gamma)_\text{as} \). In §A.1, we recall a claim in [FK], and we introduce a certain sequence and show the important property. These are useful to prove the above theorem. In §A.2, we actually give a proof of the above theorem.

A.1. Preparation. The following claim in [FK] is used to prove Theorem A.7.

Lemma A.1 ([FK], (A.3)] (cf. [FK], Proposition 1.13)). For \( A, B \in \text{ARI}(\Gamma)_\text{al} \), we have

\[ \text{arit}(B)(A) \in \text{ARI}(\Gamma)_\text{al}. \]

Here, \( \text{arit}(B) \) is a derivation on \( \text{BIMU}(\Gamma) \) (See [FK], Definition 1.9] for detailed definition of \( \text{arit} \)).

\[ \text{This diagram is not presented in [SaSch] and [Sch15].} \]
In §11 we introduced the non-commutative free monoid $U^*_{\mathbb{Z}}$ (Notation 1.3) generated by all element of the set $U_{\mathbb{Z}}$ with the empty word $\emptyset$ as the unit, and introduced the non-commutative polynomial $\mathbb{Q}$-algebra $A_U := \langle U_{\mathbb{Z}} \rangle$ (§1.2) generated by $U_{\mathbb{Z}}$. As an analogue of these, we consider the non-commutative free monoid $\mathbb{N}^*$ generated by all natural number with the empty word $\emptyset$ as the unit, that is, $\mathbb{N}^* = \bigcup_{k \geq 0} \mathbb{N}^k$, and consider the non-commutative polynomial $\mathbb{Q}$-algebra $\mathbb{Q}(\mathbb{N})$ generated by all natural number. We equip $\mathbb{Q}(\mathbb{N})$ the product $\sqcup$ defined in [13], and then the pair $(\mathbb{Q}(\mathbb{N}), \sqcup)$ forms a commutative, associative, unital $\mathbb{Q}$-algebra. Similarly to (1.4), we define the family $\langle \text{Sh}(^m_n)^k \rangle_{m, n, k \in \mathbb{N}^*}$ in $\mathbb{Z}$ by

$$m \sqcup n = \sum_{k \in \mathbb{N}^*} \text{Sh} \left( ^m_n \right) k,$$

for $m, n \in \mathbb{N}^*$.

**Definition A.2.** Let $f = \{ f(n_1, \ldots, n_r) \}_{r \in \mathbb{N}, n_i \in \mathbb{N}}$ be a family in $\mathbb{C}$. We call $f$ symmetrical if $f$ satisfies

$$\sum_{k \in \mathbb{N}^*} \text{Sh} \left( ^m_n \right) f(k) = f(m) f(n),$$

for $m, n \in \mathbb{N}^*$.

We define the family $\text{Ex} = \{ \text{Ex}(n_1, \ldots, n_r) \}_{r \in \mathbb{N}, n_i \in \mathbb{N}}$ in $\mathbb{Q}$ by

$$\text{Ex}(n_1, \ldots, n_r) := \frac{1}{(n_1 - 1)! \cdots (n_r - 1)!} \frac{1}{(n_1 + \cdots + n_r) \cdots (n_r - 1 + n_r) n_r}$$

for $r \in \mathbb{N}$ and $n_1, \ldots, n_r \in \mathbb{N}$.

**Lemma A.3.** The family $\text{Ex}$ is symmetrical.

**Proof.** Let $i, j \geq 1$ and $m = (m_1, \ldots, m_i) \in \mathbb{N}^*$ and $n = (m_{i+1}, \ldots, m_{i+j}) \in \mathbb{N}^*$. By the definition, it is sufficient to prove

$$\sum_{k \in \mathbb{N}^*} \text{Sh} \left( ^m_n \right) \text{Ex}(k) = \text{Ex}(m) \text{Ex}(n).$$

We prove this by induction on $k = i + j \geq 2$. When $k = 2$, i.e., $i = j = 1$, the left hand side of (A.2) is equal to

$$\text{Ex}(m_1, m_2) + \text{Ex}(m_2, m_1)$$

$$= \frac{1}{(m_1 - 1)!(m_2 - 1)!} \frac{1}{(m_1 + m_2)m_2} + \frac{1}{(m_1 - 1)!(m_2 - 1)!} \frac{1}{(m_2 + m_1)m_1}$$

$$= \frac{1}{(m_1 - 1)!(m_2 - 1)!} \frac{m_1 + m_2}{(m_1 + m_2)m_1 m_2}$$

$$= \text{Ex}(m_1) \text{Ex}(m_2).$$

Hence, the equation (A.2) holds for $k = 2$. Assume that the equation (A.2) holds for $2 \leq k \leq k_0 (\geq 2)$. When $k = k_0 + 1$, by putting $m' = (m_2, \ldots, m_i)$ and $n' = (n_2, \ldots, n_j)$, we have

$$\sum_{k \in \mathbb{N}^*} \text{Sh} \left( ^m_n \right) \text{Ex}(k) = \sum_{k \in \mathbb{N}^*} \left\{ \text{Sh} \left( ^{m'}_n \right) \text{Ex}(m_1, k) + \text{Sh} \left( ^m_{n'} \right) \text{Ex}(n_1, k) \right\}.$$ 

---

10This $\text{Ex}$ is denoted in [13] (2.52)].
Here, by the definition (A.1), we have $E_x(m) = \frac{1}{(m-1)!} |m|! E_x(m')$ (where $|m| := m_1 + \cdots + m_r$), so we get

$$Ex(m_1, k) = \frac{1}{(m_1 - 1)!} |m|! E_x(k), \quad Ex(n_1, k) = \frac{1}{(n_1 - 1)!} |n|! E_x(k).$$

Therefore, we calculate

$$\sum_{k \in \mathbb{N}} \text{Sh} \left( \binom{m; n}{k} \right) E_x(k) = \frac{1}{|m| + |n|} \sum_{k \in \mathbb{N}} \left\{ \text{Sh} \left( \binom{m'; n}{k} \right) \frac{1}{(m_1 - 1)!} E_x(k) + \text{Sh} \left( \binom{m; n'}{k} \right) \frac{1}{(n_1 - 1)!} E_x(k) \right\}.$$

By induction hypothesis, we have

$$= \frac{1}{|m| + |n|} \left\{ \frac{1}{(m_1 - 1)!} E_x(m') E_x(n) + \frac{1}{(n_1 - 1)!} E_x(m) E_x(n') \right\}$$

$$= \frac{|m|}{|m| + |n|} E_x(m) E_x(n) + \frac{|n|}{|m| + |n|} E_x(m) E_x(n)$$

$$= E_x(m) E_x(n).$$

Hence, we finish the proof. 

For $k \geq 0$ and $M \in \text{BIMU}(\Gamma)$, we define

$$\text{preari}_k(M) := \begin{cases} 1 & (k = 0), \\ \text{preari}(\text{preari}_{k-1}(M), M) & (k \geq 1). \end{cases}$$

We define the map $\text{expari} : \text{ARI}(\Gamma) \rightarrow \text{GARI}(\Gamma)$ by

$$\text{expari}(M) := \sum_{k \geq 0} \frac{1}{k!} \text{preari}_k(M),$$

for $M \in \text{ARI}(\Gamma)$. 

**A.2. Proof.** In this subsection, we first prove key formula (Proposition A.4) which is displayed in [E11, (2.51)]. By using this proposition, we show main claim (Theorem A.7) in this appendix.

**Proposition A.4** ([E11 (2.51)]). For any $A \in \text{ARI}(\Gamma)$, we have

$$\text{expari}(A) = I + \sum_{m = (m_r) \in \mathbb{N}^r} \sum_{r \geq 1} E_x(m) A_{m_1} \times \cdots \times A_{m_r},$$

where $A_m := \text{arit}(A)^{m-1}(A)$ for $m \in \mathbb{N}$.

To prove this proposition, we show the following two lemmas.

**Lemma A.5.** There exists a family $\{C(m_1, \ldots, m_r)\}_{r \in \mathbb{N}, (m_j) \in \mathbb{N}^r}$ in $\mathbb{Z}$ independent of the mould $A \in \text{BIMU}(\Gamma)$ such that

$$\text{preari}_k(A) = \sum_{m = (m_r) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k} C(m) A_{m_1} \times \cdots \times A_{m_r}.$$
for \( k \in \mathbb{N} \).

**Proof.** We prove the existence of \( C(m_1, \ldots, m_r) \) by induction on \( k = m_1 + \cdots + m_r \geq 1 \). When \( k = 1 \), the left hand side of (A.5) is equal to \( A \), and the right hand side of (A.5) is equal to \( (1)A_1 = C(1)A_1 \). So by putting \( C(1) := 1 \), we get the equation (A.5). Assume the equation (A.5) holds for \( k \leq k_0 (\in \mathbb{N}) \). When \( k = k_0 + 1 \), we have

\[
\text{preari}_{k_0+1}(A) = \text{preari}(\text{preari}_{k_0}(A), A).
\]

By induction hypothesis, we calculate

\[
\sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0} C(m)A_{m_1} \times \cdots \times A_{m_r}, A
\]

\[
= \text{arit}(A) \left( \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0} C(m)A_{m_1} \times \cdots \times A_{m_r} \right) + \left( \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0} C(m)A_{m_1} \times \cdots \times A_{m_r} \right) \times A.
\]

Because \( \text{arit}(A) \) is a derivation and \( A = A_1 \), we have

\[
= \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0} C(m) \sum_{i=1}^r A_{m_1} \times \cdots \times A_{m_{i-1}} \times A_{m_{i+1}} \times A_{m_{i+1}} \times \cdots \times A_{m_r}
\]

\[
+ \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0} C(m)A_{m_1} \times \cdots \times A_{m_r} \times A_1
\]

\[
= \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0+1} C(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_r)A_{m_1} \times \cdots \times A_{m_r}
\]

\[
+ \sum_{m = (m_p) \in \mathbb{N}^{r-1}} \sum_{m_1 + \cdots + m_{r-1} = k_0} C(m_1, \ldots, m_{r-1})A_{m_1} \times \cdots \times A_{m_{r-1}} \times A_1.
\]

Here, for the first summation, we regard \( C(m_1, \ldots, m_r) \) as 0 when there exists \( i \in \{1, \ldots, r\} \) such that \( m_i = 0 \). On the other hand, we have

\[
\text{preari}_{k_0+1}(A) = \sum_{m = (m_p) \in \mathbb{N}^r} \sum_{m_1 + \cdots + m_r = k_0+1} C(m)A_{m_1} \times \cdots \times A_{m_r}.
\]
Therefore, we get the following recurrence formulae

\begin{align*}
C(m_1) &= C(m_1 - 1) \
C(m) &= \sum_{i=1}^{r} C(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_r) + \delta_{m_r, 1} \cdot C(m_1, \ldots, m_{r-1})
\end{align*}

for \( m \in \mathbb{N}^r \) with \( k_0 + 1 = m_1 + \cdots + m_r \geq 2 \). Here, \( \delta_{m, 1} \) is the Kronecker delta and \( C(m_1, \ldots, m_r) := 0 \) if there exists \( i \in \{1, \ldots, r\} \) such that \( m_i = 0 \). Hence, by using these recurrence formulae, we obtain a family \( \{C(m_1, \ldots, m_r)\}_{r \in \mathbb{N}, (m_i) \in \mathbb{N}^r} \) in \( \mathbb{Z} \).

**Lemma A.6.** For \( r \geq 1 \) and \( m = (m_1, \ldots, m_r) \in \mathbb{N}^r \), we have

\[ C(m) = (m_1 + \cdots + m_r)! \cdot Ex(m). \]

**Proof.** By the definition (A.1), we have \( Ex(1) = 1 = C(1) \). So it is sufficient to prove that \( (m_1 + \cdots + m_r)! \cdot Ex(m) \) satisfies the recurrence formulae (A.6). When \( r = 1 \), we have \( Ex(m) = \frac{1}{m!} \), so we get

\[ m! \cdot Ex(m) = m! \cdot \frac{1}{m!} = 1 = (m - 1)! \cdot Ex(m - 1). \]

We prove the case of \( r \geq 2 \). Note that we have

\[ Ex(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_r) \]

\[ = \left\{ (m_i - 1) \prod_{k=1}^{r} \frac{1}{(m_k - 1)!} \right\} \cdot \prod_{j=1}^{i} \frac{m_j + \cdots + m_r}{m_j + \cdots + m_r - 1} \cdot \prod_{j=1}^{r} \frac{1}{m_j + \cdots + m_r}, \]

for \( 1 \leq i \leq r \). When \( m_r \neq 1 \), by using the expression (A.7), we calculate

\[ \sum_{i=1}^{r} (m_1 + \cdots + m_r - 1)! \cdot Ex(m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_r) \]

\[ = (m_1 + \cdots + m_r - 1)! \cdot \prod_{i=1}^{r} \left\{ \frac{1}{(m_i - 1)!} \right\} \cdot \prod_{j=1}^{i} \frac{m_j + \cdots + m_r}{m_j + \cdots + m_r - 1} \cdot \prod_{j=1}^{r} \frac{1}{m_j + \cdots + m_r} \]

\[ = (m_1 + \cdots + m_r - 1)! \cdot \prod_{k=1}^{r} \left\{ \frac{1}{(m_k - 1)! m_k + \cdots + m_r} \right\} \sum_{i=1}^{r} \left\{ (m_i - 1) \prod_{j=1}^{i} \frac{m_j + \cdots + m_r}{m_j + \cdots + m_r - 1} \right\}. \]

Here, we have

\[ m_1 + \cdots + m_r \]

\[ = \frac{m_1 + \cdots + m_r}{m_1 + \cdots + m_r - 1} \cdot \{(m_1 - 1) + (m_2 + \cdots + m_r)\} \]

\[ = \frac{m_1 + \cdots + m_r}{m_1 + \cdots + m_r - 1} \cdot \left\{(m_1 - 1) + \frac{m_2 + \cdots + m_r}{m_2 + \cdots + m_r - 1} \cdot \{(m_2 - 1) + (m_3 + \cdots + m_r)\}\right\} \]
By using this transformation repeatedly, we get

\[
\sum_{k=1}^{r} \left\{ (m_k - 1) \prod_{j=1}^{k} \frac{m_j + \cdots + m_r}{m_j + \cdots + m_r - 1} \right\}.
\]

Therefore, we have

\[
\sum_{k=1}^{r} (m_1 + \cdots + m_r - 1)! \cdot Ex(m_1, \ldots, m_k - 1, m_k + 1, \ldots, m_r)
\]

\[
= (m_1 + \cdots + m_r - 1)! \prod_{i=1}^{r} \left\{ \frac{1}{(m_i - 1)!} \frac{1}{m_i + \cdots + m_r} \right\} (m_1 + \cdots + m_r)
\]

\[
= (m_1 + \cdots + m_r)! \cdot Ex(m_1, \ldots, m_r),
\]

that is, we obtain the recurrence formulae (A.6) for \( m_r \neq 1 \). Similar to the case of \( m_r \neq 1 \), we obtain the case of \( m_r = 1 \) by using the following equation

\[
m_1 + \cdots + m_r + 1 = \sum_{k=1}^{r} \left\{ (m_k - 1) \prod_{j=1}^{k} \frac{m_j + \cdots + m_r - 1 + 1}{m_j + \cdots + m_r - 1} \right\} + \prod_{j=1}^{r-1} \frac{m_j + \cdots + m_r - 1 + 1}{m_j + \cdots + m_r - 1}.
\]

Hence, we get the claim. \( \square \)

By using the above two lemmas, we prove Proposition A.4.

**Proof of Proposition A.4** By using Lemma A.5 and Lemma A.6, we have

\[ \text{expari}(A) = I + \sum_{k \geq 1} \frac{1}{k!} \text{preari}_k(A) \]

By using Lemma A.5 we get

\[ = I + \sum_{k \geq 1} \frac{1}{k!} \sum_{m=(m_p) \in \mathbb{N}^r \atop m_1 + \cdots + m_r = k} C(m) A_{m_1} \times \cdots \times A_{m_r} \]

By using Lemma A.6 we calculate

\[ = I + \sum_{k \geq 1} \sum_{m=(m_p) \in \mathbb{N}^r \atop m_1 + \cdots + m_r = k} Ex(m) A_{m_1} \times \cdots \times A_{m_r} \]

\[ = I + \sum_{m=(m_p) \in \mathbb{N}^r \atop \sum_{r \geq 1} m_r} Ex(m) A_{m_1} \times \cdots \times A_{m_r}. \]

So we obtain the claim. \( \square \)

By using Proposition A.4, we prove the following theorem.

**Theorem A.7** ([Sch15, Proposition 2.6.1]). We have

\[ \text{expari}(\text{ARI}(\Gamma)_{al}) = \text{GARI}(\Gamma)_{as}. \]
Proof: We only prove \( \text{expari}(\text{ARI}(\Gamma)) \subset \text{GARI}(\Gamma) \). Let \( A \in \text{ARI}(\Gamma) \), that is, assume that \( A \) satisfies \( \text{Sh}(A) = A \otimes I + I \otimes A \). Then we show \( \text{expari}(A) \in \text{GARI}(\Gamma) \), that is,
\[
\text{Sh}(\text{expari}(A)) = \text{expari}(A) \otimes \text{expari}(A).
\]
By using Proposition \ref{prop:A.4}, we have
\[
\text{Sh}(\text{expari}(A)) = \text{Sh} \left( I + \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{r \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_r} \right)
\]
\[
= \text{Sh}(I) + \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{r \geq 1} \text{Ex}(m) \text{Sh} (A_{m_1} \times \cdots \times A_{m_r})
\]
\[
= \text{Sh}(I) + \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{r \geq 1} \text{Ex}(m) (A_{m_1} \otimes I + I \otimes A_{m_1}) \times \cdots \times (A_{m_r} \otimes I + I \otimes A_{m_r})
\]
By Lemma \ref{lem:A.1} and by the alternality of \( A \), we have \( A_m = \text{arit}(A)^{m-1}(A) \in \text{ARI}(\Gamma) \), that is, we get \( \text{Sh}(A_m) = A_m \otimes I + I \otimes A_m \). So we calculate
\[
= \text{Sh}(I) + \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{r \geq 1} \text{Ex}(m) \left( (A_{m_1} \times \cdots \times A_{m_r}) \otimes I + I \otimes (A_{m_1} \times \cdots \times A_{m_r}) \right)
\]
\[
+ \sum_{i+j=r} \sum_{\sigma \in \text{sh}(i,j)} \left( A_{m_{\sigma-1}(i)} \times \cdots \times A_{m_{\sigma-1}(i)} \right) \otimes \left( A_{m_{\sigma-1}(i+j)} \times \cdots \times A_{m_{\sigma-1}(i+j)} \right)
\]
\[
= \text{Sh}(I) + \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{i \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_i} \otimes I
\]
\[
+ I \otimes \sum_{m = (m_\sigma) \in \mathbb{N}^r} \sum_{j \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_j} + \sum_{m = (m_\sigma) \in \mathbb{N}^{r+j}} \sum_{i,j \geq 1} \text{Ex}(m) \left( A_{m_{\sigma-1}(i)} \times \cdots \times A_{m_{\sigma-1}(i)} \right) \otimes \left( A_{m_{\sigma-1}(i+j)} \times \cdots \times A_{m_{\sigma-1}(i+j)} \right)
\].
Because \( \{m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(i+j)}\} = \{m_1, \ldots, m_{i+j}\} \) for any \( \sigma \in \text{sh}(i,j) \), we get

\[
\text{Sh}(I) + \sum_{m=(m_p)\in \mathbb{N}^i} \sum_{j \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_i} \times A_{m_{i+1}} \times \cdots \times A_{m_{i+j}} \otimes I
\]

\[
+ I \otimes \sum_{m=(m_p)\in \mathbb{N}^j} \sum_{j \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_j} \times A_{m_{i+1}} \times \cdots \times A_{m_{i+j}} \cdot \left\{ \sum_{\sigma \in \text{sh}(i,j)} \text{Ex}(m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(i+j)}) (A_{m_1} \times \cdots \times A_{m_i}) \otimes (A_{m_{i+1}} \times \cdots \times A_{m_{i+j}}) \right\}
\]

Here, by Lemma A.3, \( \text{Ex} \) is symmetrical, so we have

\[
\sum_{\sigma \in \text{sh}(i,j)} \text{Ex}(m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(i+j)}) = \text{Ex}(m_1, \ldots, m_i) \text{Ex}(m_{i+1}, \ldots, m_{i+j}).
\]

Therefore, we calculate

\[
\text{Sh}(\text{expari}(A)) = \text{Sh}(I) + \left\{ \sum_{m=(m_p)\in \mathbb{N}^i} \sum_{j \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_i} \times A_{m_{i+1}} \times \cdots \times A_{m_{i+j}} \otimes I \right. \]

\[
\left. + I \otimes \sum_{m=(m_p)\in \mathbb{N}^j} \sum_{j \geq 1} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_j} \times A_{m_{i+1}} \times \cdots \times A_{m_{i+j}} \right\}
\]

\[
+ \left\{ \sum_{m=(m_p)\in \mathbb{N}^i} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_i} \otimes \sum_{m=(m_p)\in \mathbb{N}^j} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_j} \right\}
\]

Because \( \text{Sh}(I) = I \otimes I \), we get

\[
= \left\{ I + \sum_{m=(m_p)\in \mathbb{N}^i} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_i} \right\} \otimes \left\{ I + \sum_{m=(m_p)\in \mathbb{N}^j} \text{Ex}(m) A_{m_1} \times \cdots \times A_{m_j} \right\}
\]

\[
= \text{expari}(A) \otimes \text{expari}(A)
\]
Hence, we obtain \( \text{expari}(A) \in \text{GARI}(\Gamma)_{\text{ar}} \), that is, \( \text{expari}(\text{ARI}(\Gamma)_{\text{al}}) \subset \text{GARI}(\Gamma)_{\text{ar}} \).

\[\square\]

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