NOVEL ALGEBRAIC CONSEQUENCES OF CHIRAL SYMMETRY

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Phenomenologically motivated Lie-algebraic sum rules determine the representations of unbroken $SU(2)_L \times SU(2)_R$ filled out by mesons containing a single heavy quark, in the limit that the heavy quark mass goes to infinity. This representation content determines the strong single-pion transition amplitudes of all heavy meson states in the chiral limit. Predictions are found to be in agreement with experiment, and provide insight into the spectroscopy of heavy mesons.

The dynamical consequences of broken chiral symmetry have proved remarkably successful in explaining and predicting low-energy strong interaction phenomena in a manner consistent with quantum chromodynamics (QCD). It is less well known that chiral symmetry has additional consequences beyond those explored using chiral perturbation theory ($\chi^P T$). The nonperturbative QCD effects which break chiral symmetry spontaneously also arrange hadrons into reducible representations of the unbroken chiral group. In principle any number of hadrons can participate in a given reducible chiral multiplet. Moreover, the angles which mix the various irreducible representations are not fixed by any QCD symmetry. One might therefore conclude that these algebraic consequences of chiral symmetry cannot yield much predictive power. However, it turns out that several phenomenologically inspired Lie-algebraic sum rules severely constrain both the permissible particle content and the mixing angles of these representations.

Here we concentrate on systems of mesons that transform in the $I = \frac{1}{2}$ representation of the diagonal isospin subgroup of $SU(2)_L \times SU(2)_R$. We assume that pion scattering processes involving these mesons are determined by the sum of all chiral tree graphs. This is certainly true in the large-$N_c$ limit. However, if the mass splitting between any two members of a given chiral multiplet is small compared to $\Lambda_\chi$, then pions can be considered soft and restriction to chiral tree graphs is automatic when one works to leading order in $\chi - PT$. We take this point of view since it allows us to express pion scattering amplitudes in terms of the undetermined constants that appear in $\chi - PT$. The algebraic content of chiral symmetry allows one to prove that, for a given helicity, only pairs of $I = \frac{1}{2}$ mesons communicate by single-pion emission and absorption. We consider the consequences of this result for

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mesons that carry a single heavy quark. We show that constraints on the heavy meson mass matrix that arise from QCD in the heavy quark expansion determine the reducible chiral multiplets of unbroken $SU(2)_L \times SU(2)_R$ filled out by the infinite tower of heavy meson doublets. This representation content predicts all strong single-pion transitions of these states. For example, the transition amplitude for the process $P^* \to P \pi$ is found to vanish identically. These purely group-theoretical predictions are consistent with observation, and yield interesting insight into the spectroscopy of heavy mesons.

First we review some essential formalism. In practice, the algebraic consequences of chiral symmetry arise from a need for cancellations among chiral tree graphs involving pions—constructed from the most general chiral invariant operators. In the chiral limit, a generalized Adler-Weisberger sum rule can be expressed as

$$[X_i, X_j] = i \epsilon_{ijk} T_k,$$

(1)

where $T_i$ is the isospin matrix, and $X_i$ is an axial-vector coupling matrix, related to the matrix element of the process $\alpha(p, \lambda) \to \beta(p', \lambda') + \pi(q, i)$ in any frame in which the momenta are collinear. Here $\alpha$ and $\beta$ are arbitrary single-hadron states and $\lambda$ is helicity—which is conserved in the collinear frame. Together with Eq. (1), the defining relations, $[T_i, T_j] = i \epsilon_{ijk} T_k$ and $[T_i, X_j] = i \epsilon_{ijk} X_k$, close the chiral algebra and we see that for each helicity, $\lambda$, hadrons fall into representations of $SU(2)_L \times SU(2)_R$, in spite of the fact that the group is spontaneously broken. However, $X_i$ does not commute with the mass-squared matrix, $\hat{m}^2$, and therefore, in general, these representations are reducible. More information about the mass-squared matrix follows from the sum rule

$$[X_j, [X_i, m^2]] \propto \delta_{ij}.$$

(2)

For a system of $I = \frac{1}{2}$ mesons, with no single-pion transitions to states of higher isospin, this sum rule requires no assumption beyond Eq. (1). This sum rule implies that the hadronic mass-squared matrix is the sum of a chiral invariant and the fourth component of a chiral 4-vector; i.e. $\hat{m}^2 = \hat{m}^2_0 + \hat{m}^2_4$. The only representations of $SU(2)_L \times SU(2)_R$ that contain only a single $I = \frac{1}{2}$ representation of the diagonal isospin subgroup are $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$. So, in general, $I = \frac{1}{2}$ states of definite helicity are linear combinations of any number of these irreducible representations with undetermined coefficients. Mass splitting can only occur as a consequence of mixing between these representations since $\hat{m}^2$ is a sum of $(0, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ contributions. In a basis in which all linear combinations of $(0, \frac{1}{2})$, and $(\frac{1}{2}, 0)$ irreducible representations appear in that order, the mass-squared matrix takes the supermatrix form...
\[ \hat{m}^2 = \begin{pmatrix} \hat{A} & 0 \\ 0 & \hat{B} \end{pmatrix} + \begin{pmatrix} 0 & \hat{G} \\ \hat{G}^\dagger & 0 \end{pmatrix}. \] 

The case of zero-helicity is of special interest. The \( SU(2) \times SU(2) \) algebra has an endomorphism, \( \Pi : \Pi X_i \Pi = -X_i \), and \( \Pi T_i \Pi = T_i \). The eigenvalues of \( \Pi \) are normality, \( \eta_\alpha \equiv P_\alpha(-)^J_\alpha \), where \( P_\alpha \) and \( J_\alpha \) are the intrinsic parity and spin of \( \alpha \), respectively. Only zero-helicity states of opposite normality communicate by single-pion emission and absorption. The effect of the normality operator is to change \((0, \frac{1}{2})\) representations into \((\frac{1}{2}, 0)\) representations and vice versa.

Since \( \Pi \) commutes with \( \hat{m}^2 \), it follows that \( \hat{A} = \hat{B} \) and \( \hat{G} = \hat{G}^\dagger \). Eigenstates with \( \eta = (\pm) \) are eigenvectors of \( \hat{A} \pm \hat{G} \). The matrix element of \( X_i \) between two arbitrary \( I = \frac{1}{2} \) states, \( \alpha \) and \( \beta \), of opposite normality can be written as \( \langle \beta | X_i | \alpha \rangle \equiv T_i g_{\beta\alpha\tau} \), where \( T_i = \tau_i / 2 \) and the \( \tau_i \) are the Pauli matrices. By building physical particle states out of fundamental states of definite normality, it is straightforward to show that \( |g_{\beta\alpha\tau}| \leq 1 \); an exact consequence of chiral symmetry.

We assume that no \( I = 0 \) Regge trajectories contribute to transitions in which there are different particles in the initial and final state. In other words, we assume that scattering becomes purely elastic at high energies. Phenomena suggest that this should be a very good approximation. For example, the cross sections for the processes \( \pi + N \to a_1(1260) + N \) and \( N + N \to N^*(1440) + N \) are less than 10\% of those for \( \pi + N \to \pi + N \) and \( N + N \to N + N \), respectively. Although this can be understood on the basis of simple “diffraction” arguments, it is certainly not clear—from the QCD point of view—why this is the case. Here we treat this constraint as experimental input. This assumption leads to a superconvergence relation, which can be expressed in Lie-algebraic form as

\[ [m^2, [X_i, [X_j, m^2]]] = 0. \] 

This sum rule is simply the statement that \( \hat{m}_0^2 \) and \( \hat{m}_4^2 \) commute, or using Eq. (5), \( \langle \alpha | \hat{m}_0^2 | \beta \rangle = 0 \) when \( \alpha \neq \beta \). We can immediately extract the general consequences of this sum rule for the helicity zero states of the \( I = \frac{1}{2} \) mesons. Below we specialize a general theorem, proved in Ref. 3, to zero helicity.

As shown above, physical eigenstates of \( \eta = (\pm) \) are eigenstates of \( \hat{A} \pm \hat{G} \). Eq. (5) implies that \( \hat{A} \) and \( \hat{G} \) commute. Suppose that the vector \( \vec{a} \) represents a physical state in the \( \eta = (+) \) basis. Since \( \hat{A} \) and \( \hat{G} \) commute, \( \vec{a} \) is a simultaneous eigenvector of \( \hat{A} \) and \( \hat{G} \), say with eigenvalues \( \mu^2 \) and \( \Delta \), respectively. Similarly, suppose that the vector \( \vec{b} \) represents a physical state in the \( \eta = (-) \) basis; \( \vec{b} \) is also a simultaneous eigenvector of \( \hat{A} \) and \( \hat{G} \). There are then two
states of the absorption. In terms of coupling constants, $|g| = 1$ if $\alpha$ and $\beta$ are paired, or $g_{\alpha\pi} = 0$, otherwise. This result is completely general and applies to all $\lambda = 0$ states of the $I = \frac{3}{2}$ mesons. The interactions of heavy mesons with pions are also constrained by heavy quark symmetry.

Heavy hadron $\chi - PT$ provides an expansion in powers of momenta divided by $\Lambda_{\chi} \sim m_r$, and in powers of $\Lambda_{QCD}$ divided by the heavy hadron mass $m_h$. Mesons containing a heavy quark can be classified by the spin $(s_t)$ and the parity $(\pi_t)$ of the light quark$^{13}$. Consequently, heavy mesons fall into degenerate doublets labelled by $s_t = \frac{1}{2}$ and $\pi_t = (-)$ and are denoted $P^0(0^-)$ and $P^+(1^-)$.$^4$ The first excited states have $s_t = \frac{1}{2}$ and $\pi_t = (+)$, and are denoted $P^0_1(0^+)$ and $P^+_1(1^+)$. At the next level we have $s_t = \frac{3}{2}$ and $\pi_t = (+)$, corresponding to $P^0_2(1^+)$ and $P^+_2(2^+)$. The $\frac{1}{2}^-$, $\frac{3}{2}^+$, and $\frac{5}{2}^+$ doublets can be assembled into the “superfields” $H_\alpha$, $S_\alpha$, and $T_\alpha$, respectively. The pion transitions within the heavy meson doublets are contained in the operators$^9$:

$$g \text{ Tr}[H_a H_b A_{ba} \gamma_5] + g' \text{ Tr}[S_a S_b A_{ba} \gamma_5] + g'' \text{ Tr}[T_a T_b A_{ba} \gamma_5]$$

where $A_{ba}$ is the usual axial-vector Goldstone boson matrix.

We are now in a position to consider the joint consequences of unbroken $SU(2)_L \times SU(2)_R$ and heavy quark symmetry. We first relate the coupling constants that appear in the effective lagrangian to matrix elements of the axial-vector matrix, $X_i$:

$$\langle P|X_i|P^*\rangle = g T_i \quad \langle P_1^0|X_i|P^*_0\rangle = g' T_i \quad \langle P_1|X_i|P^*_2\rangle = g'' T_i$$

$$\langle P_1^0|X_i|P^*_0\rangle = f' T_i \quad \langle P|X_i|P^*_1\rangle = f'' T_i \quad \langle P_1|X_i|P^*_3\rangle = h T_i.$$  

The overall phases have been fixed by convention as we will be concerned only with absolute values. In making this identification, we are clearly working to leading order in heavy hadron chiral perturbation theory; hence, the identification in Eq. (7) holds for each helicity, up to an overall phase. The

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$^a$ $P$ can be a $D$ or a $B$ meson, and the “$s$” superscript indicates positive normality $(\gamma = (+))$, since, by convention, “$s$” is assigned to states in the spin-parity series $J^P = 0^+, 1^-, 2^+, \ldots$.
orbital angular momentum of a single-pion transition is subject to the selection rule \( P_\alpha P_\beta = -(-)^\ell \) where \( P_\alpha \) is the intrinsic parity of \( \alpha \). If \( \alpha \) and \( \beta \) are nearly degenerate—a necessary condition for the applicability of \( \chi - PT \)—then the contribution to the transition of the \( \ell \)th partial wave will be of order \(( M_\alpha^2 - M_\beta^2)^\ell \) and so one can neglect higher partial waves. Hence to leading order in heavy hadron \( \chi - PT \) the single-pion transitions within the heavy meson doublets are P-wave and between adjacent doublets of opposite parity are S-wave (unless there are further selection rules). These transition matrix elements are bounded as a consequence of the participation of heavy mesons in multiplets of unbroken chiral symmetry: \( g, h, g', f' \) and \( g'' \) must all take values between \(-1 \) and \(1\).

Consider the low-lying heavy meson doublets. The general theorem deduced from the three chiral commutation relations allows two scenarios consistent with heavy quark symmetry: (i) \( P \) is paired with \( P_0^* \), and \( P^* \) is paired with \( P_1' \). This yields \(|h| = 1, g = g' = f' = 0\). (ii) \( P \) is paired with \( P^* \). This yields \(|g| = 1 \) and \( h = 0 \). Of course, here we assume that only adjacent heavy meson states participate in a given chiral multiplet. One might suppose that since \( P_0^* \) and \( P_1' \) are unobserved, \( P \) could be paired with \( P_2^* \). However, this transition is forbidden by heavy quark symmetry. So in the absence of the \( \frac{1}{2}^+ \) doublet, case (ii) would have to be realized.

It is straightforward to show that case (ii) is inconsistent with QCD in the heavy quark limit. Heavy quark symmetry constrains the heavy hadron mass matrix. These constraints are easily translated to the mass-squared matrix. In the heavy quark limit, the combination \( 3M_{P^*}^2 + M_P^2 \) is independent of the mass-squared difference: \( M_{P^*}^2 - M_P^2 \). This is a simple consequence of the fact that the leading mass splitting between \( P \) and \( P^* \) arises from the coupling of the spin of the light quark to the spin of the heavy quark. If \( P \) and \( P^* \) are paired in the sense of case (ii), then we know from the general theorem derived above that their squared-masses can be written as \( \mu^2 \pm \Delta \), where \( \Delta \neq 0 \) is a diagonal element of the matrix \( \hat{m}_2 \). However, it is then clearly impossible that \( 3M_{P^*}^2 + M_P^2 \) be independent of \( \Delta \) in the heavy quark limit. Therefore, case (ii) is inconsistent with QCD, whereas case (i) easily accommodates the QCD constraint. Since no two members of a heavy meson doublet have the same spin, there exist similar constraints on the mass-squared matrix of any arbitrary heavy meson doublet, and so the chiral pairing of the two lowest lying doublets should be realized by all heavy meson quartets labelled by the light quark spin. Therefore, in the heavy quark limit, the algebraic constraints uniquely determine the single-pion transition amplitudes of all heavy meson states. This is our main result. As regards the lowest lying doublets, we conclude that \( g = g' = g'' = f' = 0 \) and \(|h| = 1\). What
reducible representations of $SU(2)_L \times SU(2)_R$ does this solution correspond to? It is straightforward to show that the zero-helicity states of heavy mesons of a given light quark spin fall into quartets composed of reducible doublets of opposite normality. Other-helicity states fall into reducible doublets.

There is not a great deal of experimental data available for the strong pion transitions of the heavy meson states. Here we will see how our predictions compare with existing experimental results. Note that the solution to the chiral commutation relations is not intended to give a detailed fit to the heavy meson spectrum; in fact, many transitions forbidden by our selection rule occur in nature. Rather the purpose of this work is to understand general features using symmetries of QCD. Much effort has centered on determining the $P^* \to P\pi$ transition amplitude. The kinematically allowed decay $D^* \to D\pi$ is determined by

$$\Gamma(D^+ \to D^0\pi^+) = \frac{g^2}{12\pi f_\pi} |p_\pi|^3. \quad (8)$$

In our convention, $f_\pi = 93$ MeV. There is an experimental upper limit on the $D^*$ width: $g^2 < 0.5$. The radiative $D^*$ decays offer an indirect method of determining $g$, and lead to the lower bound $g^2 \gtrsim 0.1$.

There are two sorts of chiral symmetry breaking effects that we have to consider. In the decay width formula for $D^* \to D\pi$ we have used physical pions to compute kinematical factors, and yet the coupling constant $g$ was evaluated at zero pion mass; a non-zero pion mass interferes with the counting of powers of energy which is essential in deriving the Lie-algebraic sum rules. We know of no way of accounting for this small effect in a systematic fashion. The second type of breaking arises from chiral symmetry breaking operators in the effective lagrangian. These effects lead to an effective coupling constant of the form $g_{eff} = g(1 + O(m_q))$, where the corrections arise from non-analytic (in $m_q$) one loop graphs constructed from the leading order operators. This sort of correction is generically large; i.e., a 20% effect. However, these corrections are weighted by $g$, and so vanish along with the axial-vector source. Evidently, the prediction $g = 0$ is not subject to explicit chiral symmetry breaking effects of this type. Therefore, deviations of $g$ from 0 should be due entirely to chiral symmetry breaking effects of the kinematic type discussed above. In other words, our solution unambiguously predicts that the transition amplitude for the process $P^* \to P\pi$ should be very close to 0, and therefore the decay width for the process $D^* \to D\pi$ should be very close to the experimental lower bound implied by the radiative decays. The issue of $1/M$ corrections is discussed in Ref. 5.

The $\frac{1}{2}^+$ states have not been observed in the $D$ and $B$ meson systems. Of
course these states are expected to exist. If one takes $M_{D_1'} = M_{D_0^*} = 2.4$ GeV as suggested by the quark model, one finds $\Gamma(D_0^* \to D\pi^-) = |h|^2 \ [980 \text{ MeV}]$ and $\Gamma(D_1' \to D^\star\pi^-) = |h|^2 \ [400 \text{ MeV}]$ and so a priori it is not surprising that these states are unobserved. These results are quite sensitive to the choice of the heavy meson masses, and yet it is gratifying that our general solution, $|h| = 1$, implies that these widths should take the maximum values allowed by unbroken chiral symmetry. The alternative solution, $h = 0$, would unambiguously predict that these states are narrow.

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