Exact expansions of Hankel transforms and related integrals

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Abstract
The Hankel transform \( H_\nu(q) = \int_0^\infty x f(x) J_\nu(qx) dx \) is studied for a positive parameter \( q \). Two particular cases \( \nu = 0 \) and \( \nu = 1 \) are investigated in much more detail. It is shown that the Hankel transforms are given by uniformly and absolutely convergent series in inverse powers of \( q \), provided special conditions on the function \( f(x) \) and its derivatives are imposed. It is necessary to underline that similar formulas obtained previously are in fact asymptotic expansions only valid when \( q \) tends to infinity. If one of the conditions is violated, our series become asymptotic series. The validity of the formulas is illustrated by a number of examples.

1 Introduction
The calculation of the Hankel transform

\[
\mathcal{H}_\nu(q) = \int_0^\infty x f(x) J_\nu(qx) dx
\]

is a problem which arises in mathematical physics (see, for instance, refs. [1], [2]), as well as in high energy nuclear and particle physics [3]-[5]. In particular, an asymptotic expansion of \( \mathcal{H}_\nu(q) \) is often needed, as \( q \to \infty \). For

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the first time asymptotic representations for $\mathcal{H}_0(q)$ and $\mathcal{H}_1(q)$ were given by Willis \cite{6} (see also \cite{1}) without investigating convergence conditions.

During the years that followed, asymptotics of $\mathcal{H}_\nu(q)$ and of more general transforms were derived by a number of authors using different conditions imposed on function $f(x)$ \cite{7}-\cite{15}. The aim of all these papers was to obtain an asymptotic expansion of $\mathcal{H}_\nu(q)$ valid when $q$ tends to infinity.

The goal of the present paper is to find such conditions on function $f(x)$ and its derivatives under which $\mathcal{H}_\nu(q)$ is presented by an uniformly and absolutely convergent series in powers of $q^{-1}$, for all $q > a > 0$. For greater clarity, we start from the Hankel transforms of order zero and one (see, correspondingly, Section 2 and Section 3). In Section 4 results obtained are generalized for the Hankel transform of an arbitrary order $\nu$ ($\Re \nu > -1$). Throughout the paper, definite integrals with the Bessel functions of the first kind are studied using series derived.

\section{Hankel transform of order zero}

Consider the Hankel transform of order zero
\begin{equation}
\mathcal{H}_0(q) = \int_0^\infty x f(x) J_0(qx) \, dx ,
\end{equation}
where $q$ is a positive parameter. We don’t assume that $q$ is large.

\textbf{Theorem 1.} The Hankel transform (2) can be presented by absolutely and uniformly convergent series for $q > a > 0$
\begin{equation}
\mathcal{H}_0(q) = \frac{1}{q^2} \sum_{m=0}^\infty (-1)^{m+1} \frac{\Gamma(2m+2)}{\Gamma^2(m+1)} \frac{f^{(2m+1)}(0)}{(2q)^{-2m}} ,
\end{equation}
provided that
\begin{enumerate}
\item $f(x)$ is a regular function at $x = 0$;
\item $f(x) = O(z^{-d})$, as $z \to \infty$, for $d > 1/2$;
\item $f^{(s)}(0)$ is a regular (i.e analytic and single-valued) function defined on a half-plane
\begin{equation}
H(\delta) = \{ s \in \mathbb{C} : \Re s \geq -\delta \}
\end{equation}
\end{enumerate}
\footnote{Methods of evaluating asymptotics of other integrals can be found in \cite{16}.}
for some $1/2 < \delta < 1$ and satisfies the growth condition

$$|f^{(s)}(0)| < Ca^s$$

(5)

for some $a > 0$ and all $s \in H(\delta)$.

**Proof 1.** According to the residue theorem, we have

$$H_0(q) = \int_0^\infty xJ_0(qx) \sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n \, dx$$

$$= - \int_0^\infty xJ_0(qx) \frac{1}{2i} \int_C \frac{(-1)^s}{\sin(\pi s)} \frac{f^{(s)}(0)}{\Gamma(s + 1)} x^s ds,$$

(6)

where the contour $C$ starts from $+\infty$ and returns to $+\infty$ after encircling clockwise the points $s = 0, 1, 2, \ldots$ which are poles of the integrand.

The integrand in (6) is a regular function in the half-plane $\Re s > -\delta$. We can deform the contour $C$ to the straight line path from $\alpha - i\infty$ to $\alpha + i\infty$, since the integral along an infinite-radius semicircle is zero for any fixed $x$.

As a result, we come to the expression

$$H_0(q) = -\frac{1}{2i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(-1)^s}{\sin(\pi s)} \frac{f^{(s)}(0)}{\Gamma(s + 1)} x^s ds \int_0^\infty x^{s+1} J_0(qx) \, dx,$$

(7)

where $-\delta < \alpha < -1/2$. The integral over $x$ converges, since $-1 < \Re s < -1/2$.

The Bessel function has the integral representation of the form [17]

$$J_0(qx) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left[ \frac{q}{2} \left( t - \frac{x^2}{t} \right) \right] \frac{dt}{t},$$

(8)

with $c > 0$. Now we replace $J_0(qx)$ in (7) with its integral representation (8).

After integration in $x$ with the use of formula 2.3.3.1 from [18], we find

$$H_0(q) = -\frac{1}{4i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{(-1)^s}{\sin(\pi s)} \frac{\Gamma(s/2 + 1)}{\Gamma(s + 1)} \frac{f^{(s)}(0)}{\Gamma(s + 1)} ds$$

$$\times \frac{1}{2\pi i} \left( \frac{q}{2} \right)^{-s/2-1} \int_{c-i\infty}^{c+i\infty} e^{qt/2} t^{s/2} dt.$$

(9)
Since $\Re s < 0$, we can apply formula 5.4.(1) in [19] for the inverse Laplace transform in (9). As a result, we get

$$H_0(q) = \left. -\frac{1}{4i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{(-1)^s}{\sin(\pi s)} \frac{\Gamma(s/2+1)}{\Gamma(s+1)\Gamma(-s/2)} f(s)(0) \left(\frac{q}{2}\right)^{-s-2} ds \right|_{\alpha = -1}$$

$$= \frac{1}{8\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{(-1)^s}{\cos(\pi s/2)} \frac{\Gamma(s/2+1)^2}{\Gamma(s+1)} f(s)(0) \left(\frac{q}{2}\right)^{-s-2} ds . \quad (10)$$

Using residue theorem, we come to formula (3). Q.E.D.

**Proof 2.** Now we prove that series (3) is equal to the Henkel transform (2). Let us rewrite the r.h.s. of eq. (3) as

$$S(q) = \frac{1}{q^3} \sum_{m=0}^{\infty} (-1)^{m+1} \frac{\Gamma(2m+2)}{\Gamma(m+1)} f(2m+1)(0) (2q)^{-(m+3/2)}$$

$$= \frac{1}{2} \sum_{m=0}^{\infty} \frac{f(2m+1)(0)}{\Gamma(2m+2)} \frac{\Gamma(m+3/2)}{\Gamma(-m-1/2)} \left(\frac{q^2}{4}\right)^{-(m+3/2)} . \quad (11)$$

Since $f^{(s)}(0)$ obeys conditions of Theorem 1, this series can be presented in the form

$$S(q) = \frac{1}{8i} \int_C \frac{(-1)^{(s+1)/2}}{\cos(\pi s/2)} \frac{f(s)(0)}{\Gamma(s+1)\Gamma(-s/2)} \left(\frac{q^2}{4}\right)^{-(s/2+1)} ds , \quad (12)$$

where the contour $C$ is defined after eq. (6). Let us introduce the integral

$$S_0(q) = \frac{1}{8i} \int_C \frac{(-1)^{s/2}}{\sin(\pi s/2)} \frac{f(s)(0)}{\Gamma(s+1)\Gamma(-s/2)} \left(\frac{q^2}{4}\right)^{-(s/2+1)} ds$$

$$= \frac{1}{8\pi i} \int_C \frac{(-1)^{s/2}}{f(s)(0)} \frac{\Gamma(s/2+1)^2}{\Gamma(s+1)} \left(\frac{q^2}{4}\right)^{-(s/2+1)} ds , \quad (13)$$

The integrand in (13) is a regular function in the half-plane $\Re s > -\delta$ for all $q > 0$. According to the Cauchy integral theorem, $S_0(q) = 0$. Thus, we
can safely add $S_0(q)$ to $S(q)$ and obtain

$$\mathcal{H}_0(q) = S(q) + S_0(q)$$

$$\frac{1}{4i} \int_C \frac{(-1)^s f^{(s)}(0)}{\sin(\pi s) \Gamma(s + 1) \Gamma(-s/2)} \left( \frac{q^2}{4} \right)^{-(s/2+1)} ds . \quad (14)$$

In deriving (14) we used the relation

$$\frac{(-1)^{(s+1)/2}}{\cos(\pi s/2)} + \frac{(-1)^{s/2}}{\sin(\pi s/2)} = \frac{2(-1)^{s}}{\sin(\pi s)} . \quad (15)$$

We have

$$\left. \frac{[\Gamma(s/2 + 1)]^2}{\Gamma(s + 1)} \right|_{s \to \infty} = 2^{-s} \sqrt{\frac{\pi s}{2}} [1 + O(s^{-1})] . \quad (16)$$

The integrand in (14) is a regular function and it decreases rapidly as $|s| \to \infty$ in the half-plane $\Re s > -\delta$ for $q > a$. It means that we can deform the contour $C$ to the straight line path from $c - i\infty$ to $c + i\infty$, since the integral along an infinite-radius semicircle is zero. As a result, we come to the expression

$$\mathcal{H}_0(q) = -\frac{1}{4i} \int_{c-i\infty}^{c+i\infty} \frac{(-1)^s f^{(s)}(0)}{\sin(\pi s) \Gamma(s + 1) \Gamma(-s/2)} \left( \frac{q^2}{4} \right)^{-(s/2+1)} ds , \quad (17)$$

where $-\delta < c < -1/2$. According to eq. 2.12.2.2 in [17], we have

$$\frac{\Gamma(s/2 + 1)}{\Gamma(-s/2)} \left( \frac{q^2}{4} \right)^{-(s/2+1)} = 2 \int_0^\infty dz z^{s+1} J_0(qz) . \quad (18)$$

Integral in (18) converges, since $-1 < \Re s < -1/2$. Correspondingly, we get

$$I(q) = -\int_0^\infty dz \, J_0(qz) \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{(-1)^s f^{(s)}(0)}{\sin(\pi s) \Gamma(s + 1)} z^s ds . \quad (19)$$

The residue theorem yields eq. (8) Q.E.D.
Because of the asymptotics
\[
\frac{\Gamma(2m+2)}{\Gamma^2(m+1)} \bigg|_{m \to \infty} = 2^{2m+1} \sqrt{\frac{m}{\pi}} \left[ 1 + O(m^{-1}) \right], \tag{20}
\]
the r.h.s. in (3) is the uniformly and absolutely convergent series for all \( q > a \). Consequently, it is the exact representation of the Hankel integral (2), contrary to the results of papers [7]-[12] in which similar expansions were derived as asymptotic series for large \( q \).

Let us see how does our formula (3) work? It is worthwhile considering several examples.

1. \( f(x) = e^{-ax}, a > 0 \). Then we come to the integral
\[
H_1(q) = \int_0^\infty x e^{-ax} J_0(qx) \, dx. \tag{21}
\]
On the other hand, \( f^{(2m+1)}(0) = -a^{2m+1} \), and we find from (3) for \( q > a \)
\[
H_1(q) = a \frac{2m}{(a^2 + q^2)^{3/2}}, \tag{22}
\]
in accordance with eq. 2.12.8.4. in [20]. To calculate this series, we used formula 5.2.13.2 from [18].

2. \( f(x) = x^{1+n} e^{-ax}, a > 0, \) integer \( n \geq 0 \). In such a case, \( f^{(p)}(0) = 0 \) for \( 0 \leq p \leq n \), and \( f^{(p)}(0) \neq 0 \) for \( p \geq n + 1 \). We get (see eq. 2.12.8.4. in [20])
\[
H_2(q) = \int_0^\infty x^{2+n} e^{-ax} J_0(qx) \, dx
\]
\[
= \frac{1}{a^{n+3}} \Gamma(n+3) \, _2F_1 \left( \frac{n}{2} + \frac{3}{2}, \frac{n}{2} + 2; 1; -\frac{q^2}{a^2} \right), \tag{23}
\]
where \( _2F_1(a, b; c; z) \) is the hypergeometric function [21]. It satisfies equation
\[
\begin{align*}
&_2F_1 \left( \frac{n}{2} + \frac{3}{2}, \frac{n}{2} + 2; 1; -\frac{q^2}{a^2} \right) = \sqrt{\pi} \left[ \frac{1}{\Gamma((n+4)/2) \Gamma(-(n+1)/2)} \right. \\
&\times \left. \left. \frac{a}{q} \right) \right. \frac{(n+3)}{2} \, _2F_1 \left( \frac{n}{2} + \frac{3}{2}, \frac{n}{2} + 2; \frac{3}{2}; -\frac{a^2}{q^2} \right) - \left. \frac{2}{\Gamma((n+3)/2) \Gamma(-(n+2)/2)} \right. \\
&\times \left. \left. \left. \frac{a}{q} \right) \right. \frac{(n+4)}{2} \, _2F_1 \left( \frac{n}{2} + 2, \frac{n}{2} + 2; \frac{3}{2}; -\frac{a^2}{q^2} \right) \right] \tag{24}
\end{align*}
\]
First, take \( n = 2p, \ p = 0, 1, 2, \ldots \), then the r.h.s. of (23) appears to be
\[
H_2(q) = \frac{(-1)^{p+1}}{2\pi} \left[ \Gamma(p + 3/2) \right]^2 \left( \frac{2}{q} \right)^{2p+3} 2F_1 \left( p + \frac{3}{2} ; p + \frac{3}{2} ; \frac{1}{2} ; -\frac{a^2}{q^2} \right). \quad (25)
\]

On the other hand, we have
\[
f^{(2m+1)}(0) = \begin{cases} 0, & m < p, \\ a^{2m-2p} \frac{\Gamma(2m + 2)}{\Gamma(2m - 2p + 1)}, & m \geq p. \end{cases} \quad (26)
\]

Then we obtain from (3)
\[
H_2(q) = \frac{(-1)^{p+1}}{2\pi} \left[ \Gamma(p + 3/2) \right]^2 \left( \frac{2}{q} \right)^{2p+3} \times \sum_{k=0}^{\infty} \frac{(p + 3/2)_k (p + 3/2)_k}{(1/2)_k k!} \left( -\frac{a^2}{q^2} \right)^k, \quad (27)
\]
where \((a)_n = \Gamma(a + n)/\Gamma(n)\) is the Pochhammer symbol [21]. We see that (27) coincides with (25).

Analogously, we analyze the case \( n = 2p + 1, \ p = 0, 1, 2, \ldots \), for which
\[
f^{(2m+1)}(0) = \begin{cases} 0, & m < p + 1, \\ -a^{2m-2p-1} \frac{\Gamma(2m + 2)}{\Gamma(2m - 2p)} , & m \geq p + 1. \end{cases} \quad (28)
\]

Then it is easy to demonstrate that both eqs. (24), (24) and our formula (3) give the same expression
\[
H_2(q) = \frac{(-1)^{p+1}}{2\pi} \left[ \Gamma(p + 5/2) \right]^2 a \left( \frac{2}{q} \right)^{2p+5} 2F_1 \left( p + \frac{5}{2} ; p + \frac{5}{2} ; \frac{3}{2} ; -\frac{a^2}{q^2} \right). \quad (29)
\]
As a result, we find the following asymptotics
\[
H_2(q) = \frac{(-1)^{p+1}}{2\pi} \left[ \Gamma(p + 3/2) \right]^2 a \left( \frac{2}{q} \right)^{2p+3} \left[ 1 + O(q^{-2}) \right], \quad (30)
\]
for \( n = 2p, \ p = 0, 1, 2, \ldots \), and
\[
H_2(q) = \frac{(-1)^{p+1}}{2\pi} \left[ \Gamma(p + 5/2) \right]^2 a \left( \frac{2}{q} \right)^{2p+5} \left[ 1 + O(q^{-2}) \right], \quad (31)
\]
for \( n = 2p + 1, \ p = 0, 1, 2, \ldots \).

3. \( f(x) = e^{-ax}I_0(cx), \ a > c > 0, \) then we consider the integral

\[
H_3(q) = \int_0^\infty xe^{-ax}I_0(cx)J_0(qx) \, dx , \tag{32}
\]

where \( I_\nu(z) \) is the modified Bessel function of the first kind \[17\]. According to eq. 2.15.16.2. in \[20\], this integral is equal to

\[
H_3(q) = \frac{2a}{\pi} \left( \frac{k}{qc} \right)^{3/2} (1 - k^2)^{3/4} [2E(k) - K(k)]. \tag{33}
\]

Here \( K(k) \) (\( E(k) \)) is the complete elliptic integral of the first (second) kind \[22\], and

\[
k = \frac{1}{\sqrt{2}} \left[ 1 - \frac{q^2 + a^2 - c^2}{\sqrt{(q^2 + a^2 - c^2)^2 + 4q^2c^2}} \right]^{1/2}. \tag{34}
\]

Note that \( k \approx c/q \) at large \( q \). The expression in the r.h.s. of \( (33) \) has the following asymptotics

\[
H_3(q) \bigg|_{q \to \infty} = \frac{a}{q^3} \left[ 1 - \frac{3(2a^2 + 3c^2)}{4q^2} + \frac{15(8a^4 + 40a^2c^2 + 15c^4)}{64q^4} \right] + O(q^{-9}) . \tag{35}
\]

At the same time, we find that\[2\]

\[
f^{(n)}(0) = (-1)^n a^n \sum_{k=0}^{[n/2]} \left( \begin{array}{c} n \\ 2k \end{array} \right) \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{c^2}{4a^2} \right)^k , \tag{36}
\]

where \( \left( \begin{array}{c} n \\ m \end{array} \right) \) is the binomial coefficient, and \([z]\) means the integer part of \( z \). For odd order derivatives, we obtain

\[
f^{(2m+1)}(0) = -a^{2m+1} \sum_{k=0}^{m} \left( \begin{array}{c} 2m + 1 \\ 2k \end{array} \right) \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{c^2}{4a^2} \right)^k. \tag{37}
\]

\[2\]For a special case \( a = c \), this sum is given by formula 4.2.7.37 in \[18\].
In particular,

\[ f^{(1)}(0) = -a, \]
\[ f^{(3)}(0) = -a \left[ a^2 + \frac{3c^2}{2} \right], \]
\[ f^{(5)}(0) = -a \left[ a^4 + 5a^2c^2 + \frac{15c^4}{8} \right]. \tag{38} \]

A compact analytical expression for \( H_4(q) \) valid for \( q > a + c \) is calculated in Appendix A to be

\[ H_3(q) = \frac{a}{q^3} F_4 \left( \frac{3}{2}, \frac{3}{2}, 1, -\frac{a^2}{q^2}, -\frac{c^2}{q^2} \right), \tag{39} \]

where \( F_4(\alpha, \beta, \gamma, \gamma', x, y) \) is the hypergeometric series of two variables [21]. It is the uniformly and absolutely convergence series for \( x, y < \infty \) [21]. Either from eqs. (3), (38) or from expansion (39) we reproduce the asymptotic expansion (35).

As a byproduct, we obtained the new analytical expression (39) for the improper integral (32), which is evidently much more appropriate for evaluating its asymptotics at \( q \gg a, c \) than tabulated expression (33).

4. Analogously, we can consider the integral with the function \( f(x) = J_1(ax)J_0(bx) \), where \( a, b > 0, q > a + b \). It is known that (see eq. 2.12.41.5 in [20])

\[ H_4(q) = \int_0^\infty xJ_1(ax)J_0(bx)J_0(qx) \, dx = -\frac{a}{2q^3} F_4 \left( \frac{3}{2}, \frac{3}{2}, 1, -\frac{a^2}{q^2}, -\frac{b^2}{q^2} \right). \tag{40} \]

On the other hand, we have

\[ f^{(2m+1)}(0) = (-1)^m 2^{2m-1} a b^{2m} \sum_{k=0}^m \binom{2m+1}{2k+1} \binom{2k+1}{k} \binom{2m-2k}{m-k} \left( \frac{a^2}{b^2} \right)^k. \tag{41} \]

As it is shown in Appendix A, formula (3) in combination with (41) results in expression in the r.h.s. of (40).

Let us underline that in all examples considered above, the series (3) converges uniformly and absolutely for \( q > a \), and, consequently, it is the exact representation of integral (2).
However, the series (3) may diverge if \( f^{(2m+1)}(0) \) rises rather quickly as \( m \to \infty \) and violates asymptotic condition (5) of Theorem 1. Suppose that \( f^{(n)}(0) = O(a^n \Gamma(n + b)) \), with \( a, b > 0 \), as \( n \) tends to infinity. Then in Proof 1 we are not allowed to deform the contour \( C \) in (6) and come to eq. (7). Analogously, we can’t come from (14) to (17) in Proof 2.

In such a case, the series in eq. (3) should be considered as an asymptotic series of \( \mathcal{H}_0(q) \), as \( q \to \infty \) (see also [6]-[9]). Let us illustrate this point by the following example.

5. \( f(x) = (x + a)^{-1}, a > 0 \). We have (see eq. 2.12.3.6. in [20])

\[
H_5(q) = \int_0^{\infty} \frac{x}{x + a} J_0(qx) \, dx = \frac{1}{q} - \frac{\pi a}{2} \left[ H_0(aq) - Y_0(aq) \right],
\]

(42)

where \( H_\nu(z) \) is the Struve function, and \( Y_\nu(z) \) is the Bessel function of the second kind [17]. We can use the asymptotic formula 10.42(2) from ref. [23] (\(|z| \to \infty, |\arg z| < \pi\))

\[
[H_\nu(z) - Y_\nu(z)] \sim \frac{1}{\pi} \sum_{m=0}^{p} \frac{\Gamma(m + 1/2)}{\Gamma(\nu + 1/2 - m)} \left( \frac{z}{2} \right)^{-(2m-\nu+1)} + O(z^{-(2p-\nu+3)}).
\]

(43)

Then we find the asymptotics of integral (42)

\[
H_5(q) \sim \frac{4}{\pi a^2 q^3} \left[ \sum_{m=0}^{p-1} (-1)^m 2^{2m} [\Gamma(m + 3/2)]^2 (aq)^{-2m} + O(q^{-2p}) \right].
\]

(44)

On the other hand, \( f^{(2m+1)}(0) = -a^{-2m-2} \Gamma(2m + 2) \), and we get from (3)

\[
H_5(q) \sim \frac{4}{\pi a^2 q^3} \sum_{m=0}^{\infty} (-1)^m 2^{2m} [\Gamma(m + 3/2)]^2 (aq)^{-2m} = \frac{1}{a^2 q^3} \, _3F_0 \left( \frac{3}{2}, \frac{3}{2}, 1; -\frac{4}{q^2 a^2} \right),
\]

(45)

which is just series (44). Here \( _pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) \) is the generalized hypergeometric series [21].
This series \((45)\) diverges, because \(\Gamma(m + 3/2)^2 \simeq m^{2m}e^{-2m}\) as \(m \to \infty\).

Let us define

\[
R_n(q) = \sum_{k=n+1}^{\infty} (-1)^m 2^{2m} \Gamma(m + 3/2)^2 (aq)^{-2m}
\]

\[
= (-1)^n 2^{2n+2} (aq)^{-2n-2} \sum_{p=0}^{\infty} (-1)^p 2^{2p} \Gamma(n + p + 5/2)^2 (aq)^{-2p}.
\] \((46)\)

Since

\[
\lim_{q \to \infty} q^{2n} R_n(q) = 0, \text{ for fixed } n,
\]

\[
\lim_{n \to \infty} q^{2n} R_n(q) = \infty, \text{ for fixed } q,
\] \((47)\)

we conclude that \((45)\) is by definition the asymptotic power series of the integral \(H_5(q)\) as \(q \to \infty\).

In all examples presented above, tabulated integrals were being considered which were defined in terms of algebraic or special functions. However, in some cases finding asymptotics of a tabulated expression may need additional calculations, while formula \((3)\) provides us with a desired asymptotic expansion immediately.

We can illustrate this statement by the following example.

6. \(f(x) = (x + a)^{-2}, a > 0\). It means that we consider the integral

\[
H_6(q) = \int_0^\infty \frac{x}{(x + a)^2} J_0(qx) \, dx.
\] \((48)\)

It is a tabulated integral (see eq. 2.12.3.9 in \([20]\)). But finding its asymptotics with the use of tabulated expression faces difficulties, since one should start from the integral

\[
H_6(q, \varepsilon) = \int_0^\infty \frac{x}{(x + a)^{2+\varepsilon}} J_0(qx) \, dx,
\] \((49)\)

calculate its asymptotics for \(\varepsilon \neq 0\), and only then take the limit \(\varepsilon \to 0\). The reason is that the tabulated expression for integral \((49)\) has three terms two
of which having simple poles at \( \varepsilon = 0 \) \[20\]. That is why, one cannot put \( \varepsilon = 0 \) from the very beginning.\footnote{In a sum of three terms, the poles cancel each other.}

On the other hand, formula \( 3 \) gives us the asymptotic expansion of \( H_6(q) \) \( 48 \) as \( q \to \infty \), without additional calculations. Indeed, from relation \( f^{(2m+1)}(0) = -a^{-2m-3}\Gamma(2m+3) \) we immediately find the desired expansion

\[
H_6(q) = \frac{8}{\pi(qa)^3} \sum_{m=0}^{\infty} (-1)^m 2^{2m} [\Gamma(m+3/2)]^2 (m+1)(aq)^{-2m}
\]

\[
= \frac{2}{(qa)^3} \sum_{m=0}^{\infty} \binom{-1/2}{m} \Gamma(3/2+2m) f(2m)(0) (a^2)^{-m}.
\]

\( 50 \)

Note that this series can be obtained by differentiating series \( 45 \) with respect to the parameter \( a \), since \( H_6 = -(\partial/\partial a)H_5 \).

\section{Hankel transform of order one}

\textbf{Theorem 2}. The Hankel transform

\[
\mathcal{H}_1(q) = \int_0^\infty dx x f(x) J_1(qx)
\]

\( 51 \)

can be presented by absolutely and uniformly convergent series for \( q > a > 0 \)

\[
\mathcal{H}_1(q) = \frac{1}{q^2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2m+2)}{\Gamma^2(m+1)} f^{(2m)}(0) (2q)^{-2m},
\]

\( 52 \)

provided that \( f(x) \) and its derivatives satisfy conditions of Theorem 1.

\textbf{Proof}. To prove this theorem, it is enough to use the well-known relation between Bessel functions \( 17 \)

\[
J_0(qx) = \frac{1}{q} \left[ \frac{1}{x} J_1(qx) + \frac{d}{dx} J_1(qx) \right].
\]

\( 53 \)

Then we have\footnote{We used the fact that \( x f(x)J_1(qx)|_{x=0} = x f(x)J_1(qx)|_{x=\infty} = 0 \).}

\[
\int_0^\infty x f^{(1)}(x) J_1(qx) dx = -q \int_0^\infty x f(x) J_0(qx) dx,
\]

\( 54 \)
Formula (52) follows immediately from eqs. (54), (3). Q.E.D.

Let us stress that the conditions of Theorem 2 guarantee that the r.h.s of eq. (52) is the uniformly convergent series, not only an asymptotic series.

Let us consider two examples.

7. \( f(x) = e^{-ax}, \ a > 0 \), then

\[
H_7(q) = \int_0^\infty xe^{-ax} J_1(qx) \, dx .
\] (55)

Taking into account that \( f^{(2m)}(0) = a^{2m} \), we obtain from (52) that

\[
H_7(q) = \frac{q}{(a^2 + q^2)^{3/2}} ,
\] (56)

for \( q > a \), in agreement with eq. 2.12.8.4. in [20].

8. \( f(x) = e^{-ax^2}, \ a > 0 \). We have (see eq. 2.12.9.3. in [17])

\[
H_8(q) = \int_0^\infty xe^{-ax^2} J_1(qx) \, dx = \frac{q\sqrt{\pi}}{8a^{3/2}} \, _1F_1 \left( \frac{3}{2}; 2; \frac{-q^2}{4a} \right) ,
\] (57)

where \( _1F_1(a; b; z) \) is the confluent hypergeometric function of the first kind [21]. Since \( f^{(2m)}(0) = (-1)^m (4a)^m \Gamma(m + 1/2)/\sqrt{\pi} \), series (52) should be regarded as an asymptotic series (see integral (42) and comments before it)

\[
H_8(q) \bigg|_{x \to -\infty} \overset{\text{as}}{=} \frac{2}{q^2 \pi} \sum_{m=0}^{p} \frac{\Gamma(m + 3/2) \Gamma(m + 1/2)}{m!} \left( \frac{4a}{q^2} \right)^m + O(q^{-2p-4}) .
\] (58)

The function \( _1F_1(a; b; x) \) has the following asymptotics [21]

\[
_1F_1(a; c; x) \bigg|_{x \to -\infty} \overset{\text{as}}{=} \frac{\Gamma(c)}{\Gamma(c - a)} (-x)^{-a} \sum_{m=0}^{p} \frac{(a)_m (a - c + 1)_m}{m!} (-x)^{-m}
\]

\[+ O(|x|^{-a-p-1}) .\] (59)

Using this formula, one can make sure that an asymptotic expansion of the r.h.s. of (57) coincides with the asymptotic power series (58).
4 Hankel transform of an arbitrary order

**Theorem 3.** The Hankel transform of the order \( \nu \), with \( \Re \nu > -1 \), can be presented by absolutely and uniformly convergent series for \( q > a > 0 \)

\[
\mathcal{H}_\nu(q) = \frac{2}{q^2} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \frac{\Gamma \left( \frac{\nu+k}{2} + 1 \right)}{\Gamma \left( \frac{\nu-k}{2} \right)} \left( \frac{q}{2} \right)^{-k}, \tag{60}
\]

provided that \( f(x) \) and its derivatives satisfy conditions of Theorem 1.

**Proof.** The proof proceeds as Proof 1 of Theorem 1, by using the integral representation \([17]\) \((c > 0, \Re \nu > -1)\)

\[
J_\nu(qx) = z^\nu \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp \left[ \frac{q}{2} \left( t - \frac{x^2}{t} \right) \right] \frac{dt}{t^{\nu+1}}, \tag{61}
\]

instead of integral representation \([8]\).

A formula analogous to \((60)\) was obtained in \([9]\) (see eq. (5.12) with \( \mu = 1 \) therein), but as asymptotic expansion of \( \mathcal{H}_\nu(q) \) as \( q \) tends to infinity. In our case, the r.h.s. of \((60)\) is the uniformly and absolutely convergent power series for all \( q > a > 0 \). Thus, it is the exact expansion of the Hankel transform \( \mathcal{H}_\nu(q) \) \((1)\).

One can easily check that for \( \nu = 0 \) and \( \nu = 1 \), \((3)\) and \((52)\) are reproduced from \((60)\).

9. As an example, we consider the integral\(^5\)

\[
H_9(q) = \int_{0}^{\infty} x e^{-ax} J_\nu(qx) \, dx. \tag{62}
\]

We find from \((62), (60)\)

\[
H_9(q) = \frac{2}{q^2} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma \left( \frac{\nu+k}{2} + 1 \right)}{\Gamma \left( \frac{\nu-k}{2} \right)} \left( -\frac{2a}{q} \right)^k
\]

\[
= \frac{1}{q^2} \left( \nu + \frac{\partial}{\partial \ln x} \right) \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma \left( \frac{\nu+k}{2} \right)}{\Gamma \left( \frac{\nu-k}{2} \right)} (-2x)^k. \tag{63}
\]

\(^5\)Two particular cases with \( \nu = 0 \) and \( \nu = 1 \) were considered above (see eqs. \((21), (55)\)).
where $x = a/q$. The series on the second line of eq. (63) is equal to (see eq. 5.2.14.29 in [18])

$$\sum_{k=0}^{\infty} \frac{1}{k!} \Gamma\left(\frac{\nu+k}{2}\right) (-2x)^k = \frac{(\sqrt{1+x^2} - x)^{\nu-1}}{\sqrt{1+x^2}}.$$  (64)

The series (64) converges for $|x| < 1$ (i.e. for $q > a$). As a result, we come to the expression

$$H_0(q) = \frac{1}{q^2} \left(\nu + \frac{\partial}{\partial \ln x}\right) \frac{(\sqrt{1+x^2} - x)^{\nu-1}}{\sqrt{1+x^2}} = -\frac{1}{q^2} \frac{\partial}{\partial x} \frac{(\sqrt{1+x^2} - x)^{\nu}}{\sqrt{1+x^2}}.$$  (65)

Formula 2.12.8.4 in [20] (with $\alpha = 2$) gives the same result for $H_0(q)$ (65). It demonstrates us that series (60) is really the exact representation of the Hankel transform (1) for all $q > a > 0$.

It is interesting that in a general case at large $q$ a leading term in the expansion of zero-order Hankel transform $H_0(q)$ is $O(q^{-3})$, while an expansion of $H_\nu(q)$, with $\text{Re}\nu > -1$, $\nu \neq 0$, starts from the term $O(q^{-2})$.

5 Discussions and conclusions

We have studied the Hankel transform (1), imposing several conditions on $f(x)$ and its derivatives. In particular, we demand that $f(s) = f(s)(0)$ is a regular function defined on a half-plane $\Re s > -\delta$ for some $1/2 < \delta < 1$, and that it grows as $s^\delta$ when $s \to \infty$ (see Theorem 1 for more details). The particular cases $\nu = 0$ and $\nu = 1$ have been investigated in greater detail.

It is shown that the Hankel transforms can be presented by the absolutely and uniformly convergent series in inverse powers of $q$ for $q > a > 0$ (see eqs. (3), (52) and (60)). Two proofs of this statement are given. If one of the conditions imposed on $f(s)$ is violated, series (3), (52) and (60) become asymptotic series as $q \to \infty$.

Our statements are illustrated by nine examples. As a byproduct, we have obtained the new analytical expression (39) for the definite integral of the Bessel functions $J_0(z)$ and $I_0(z)$ (32), which is much more appropriate than tabulated expression (33) for evaluating asymptotics of (32) at large $q$. In
some cases finding asymptotics of a tabulated integral may need complicated
calculations, while our series provides us with its asymptotics immediately,
as was shown for integral (48).

Previously, in a number of papers similar formulas were derived [6]-[12].
However, it is necessary to underline that they are in fact asymptotic expan-
sions valid only when $q$ tends to infinity.

In conclusion, one result of ref. [24] is worth discussing. The author of
[24] studied the integral

$$I(\lambda) = \int_0^{\infty} \Phi(\lambda t) f(t) \, dt ,$$

(66)

where $\lambda$ is a positive parameter, and $\Phi(t)$ is assumed to have a Laplace
transform $\Psi(s)$. $\Phi(t)$ can be one of familiar functions, e.g. the Bessel function
$J_\nu(t)$. Let $f(t) = \sum_{n=0}^{\infty} c_n t^n$ be an entire (i.e. analytic at all finite points of
the complex plane $\mathbb{C}$) function such that both $\Phi(\lambda t)f(t)$ and

$$|\Phi(\lambda t)| \sum_{n=0}^{\infty} |c_n| t^n$$

(67)

are integrable over $[0, \infty)$. Then Theorem 1 in [24] says that

$$I(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Psi^{(n)}(0) \, f^{(n)}(0) \, \lambda^{-n-1} .$$

(68)

However, (67) isn’t integrable for most integrals investigated in the present
paper. Consider, e.g. integral $H_1$ (21). In notations of ref. [24], $\Phi(t) = J_0(t)$,

$$f(t) = t e^{-at} = \sum_{n=0}^{\infty} c_n t^n ,$$

(69)

where $c_n = (-a)^{n-1}/\Gamma(n)$. We get

$$|\Phi(\lambda t)| \sum_{n=0}^{\infty} |c_n| t^n = |J_0(\lambda t)| |t e^{at}| .$$

(70)

---

6See integrals (21), (23), (52), (55) and (62).
It is evident that function (70) isn’t integrable over \([0, \infty)\) for \(\lambda, a > 0\).

So, one of conditions of Theorem 1 in [24] is violated. On the contrary, our Theorem 1 works, and it results in the exact expression for \(H_1(q)\) (22).

All said above allows us to conclude that our formulas (3), (52) and (60) are the new results which can be used to obtain analytic expressions for the Hankel transforms (1) with a positive parameter \(q\).

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**Appendix A**

Here we calculate a series which defines integral (32)

\[
H_3(q) = \frac{2a}{\sqrt{\pi}q^3} \sum_{m=0}^{\infty} \frac{\Gamma(m + 3/2)}{\Gamma(m + 1)} \left( -\frac{a^2}{q^2} \right)^m \sum_{k=0}^{m} \binom{2m + 1}{k} \binom{2k}{k} \left( \frac{c^2}{4a^2} \right)^k.
\]

where we used eq. (37). After putting \(m = n + k\), we find

\[
H_3(q) = \frac{2a}{\sqrt{\pi}q^3} \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( \frac{c^2}{4a^2} \right)^k \sum_{m=k}^{\infty} \frac{\Gamma(m + 3/2) \Gamma(2m + 2)}{\Gamma(m + 1) \Gamma(2m + 2 - 2k)} \left( -\frac{a^2}{q^2} \right)^m.
\] (A.1)

Using relation

\[
\frac{\Gamma(n + k + 3/2) \Gamma(2n + 2k + 2)}{\Gamma(n + k + 1) \Gamma(2n + 2)} = 2^{2k} \frac{\Gamma(n + k + 3/2)^2}{\Gamma(n + 3/2) n!}
\] (A.3)

\[
7\text{The same is true for } \Phi(t) = J_0(t), f(t) = t \cos t, t \sin t, tJ_\nu(t), \text{ etc.}
\]
we obtain \((q > a + c)\)

\[
H_3(q) = \frac{a}{q^3} \sum_{k=0}^{\infty} \frac{1}{(1)_k k!} \left(-\frac{c^2}{q^2}\right)^k \sum_{n=0}^{\infty} \left[\frac{(3/2)_{n+k}}{(3/2)_n n!}\right] \left(-\frac{a^2}{q^2}\right)^n
\]

\[
= \frac{a}{q^3} F_4\left(\frac{3}{2}, \frac{3}{2}, 1, -\frac{a^2}{q^2}, -\frac{c^2}{q^2}\right), \tag{A.4}
\]

where

\[
F_4(\alpha, \beta, \gamma, \gamma', x, y) = \sum_{n,m=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\gamma)_m (\gamma')_n}{n! m!} x^m y^n \tag{A.5}
\]

is the hypergeometric series of two variables \([21]\).

Analogously, we can calculate a series which defines integral \((40)\)

\[
H_4(q) = -\frac{a}{2q^3} \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{\Gamma^2(m+1)} 2^{-4m} \left(\frac{b^2}{q^2}\right)^m
\]

\[
\times \sum_{k=0}^{m} \frac{1}{\Gamma^2(m-k+1)\Gamma(k+2)k!} \left(\frac{a^2}{b^2}\right)^k = \frac{a}{2q^3} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+2)k!} 2^{-4k} \left(\frac{a^2}{q^2}\right)^k
\]

\[
\times \sum_{n=0}^{\infty} \left[\frac{\Gamma(2n+2k+2)}{\Gamma(n+k+1)}\right]^2 \frac{1}{\Gamma(n+1)n!} 2^{-4n} \left(\frac{b^2}{q^2}\right)^n. \tag{A.6}
\]

Since

\[
\left[\frac{\Gamma(2n+2k+2)}{\Gamma(n+k+1)}\right]^2 2^{-4n} = \frac{1}{\pi} \Gamma^2(n+k+3/2) 2^{4k+2}, \tag{A.7}
\]

we obtain \((q > a + b)\)

\[
H_4(q) = -\frac{a}{2q^3} \sum_{k=0}^{\infty} \frac{1}{(2)_k k!} \left(\frac{a^2}{q^2}\right)^k \sum_{n=0}^{\infty} \left[\frac{(3/2)_{n+k}}{(1)_n n!}\right] \left(\frac{b^2}{q^2}\right)^n
\]

\[
= -\frac{a}{2q^3} F_4\left(\frac{3}{2}, \frac{3}{2}, 2, 1, a^2 q^2, \frac{b^2}{q^2}\right). \tag{A.8}
\]

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