Time endogeneity and an optimal weight function in pre-averaging covariance estimation

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Abstract

We establish a central limit theorem for a class of pre-averaging covariance estimators in a general endogenous time setting. In particular, we show that the time endogeneity has no impact on the asymptotic distribution in the first order. This contrasts with the case of the realized volatility in a pure diffusion setting. We also discuss an optimal choice of the weight function in the pre-averaging.

Keywords: Central limit theorem; Hitting times; Market microstructure noise; Nonsynchronous observations; Pre-averaging; Time endogeneity.

1 Introduction

In the past decade an improvement in the availability of financial high-frequency data has highlighted applications of the classic asymptotic theory for the quadratic covariation of a semimartingale to the inference for the covariance structure of asset returns. Empirical evidences, however, suggest that at ultra-high frequencies asset price processes follow a semimartingale contaminated by noise (called microstructure noise) rather than a pure semimartingale. In addition, at ultra-high frequencies financial data are possibly recorded at irregular times, and this causes the nonsynchronicity of observation times between multiple assets.

Recently various approaches have been proposed for estimating the quadratic covariation matrix of a semimartingale observed at a high frequency in a nonsynchronous manner with additive observation noise. Thus far the most prominent ones are the subsampling approach by [4] and [29], the realized kernel estimation by [3], the pre-averaging method by [6] and [7], the quasi maximum likelihood approach by [1] and [27], and the spectral method by [5]. In this paper we focus on the pre-averaging method, especially the modulated realized covariance (abbreviated MRC) introduced in [6].

Specifically, we consider the following model:

\[ Y_t = X_t + \epsilon_t, \quad t \geq 0, \]

where \( X = (X_t)_{t \geq 0} \) is a \( d \)-dimensional process (latent log-price) and \( \epsilon = (\epsilon_t)_{t \geq 0} \) is a \( d \)-dimensional error process (microstructure noise) which is, conditionally on the process \( X \), centered and independent. We assume that \( X \) is of the form

\[ X_t = X_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s, \]

where \( a = (a_s)_{s \geq 0} \) is an \( \mathbb{R}^d \)-valued càdlàg process, \( \sigma = (\sigma_s)_{s \geq 0} \) is an \( \mathbb{R}^d \otimes \mathbb{R}^{d'} \)-valued càdlàg volatility, and \( W \) is a \( d' \)-dimensional Wiener process. Our objective is the quadratic covariation matrix of \( X \) over some fixed interval \([0, T]\) (hereafter an asterisk denotes the transpose of a matrix):

\[ [X]_T = \int_0^T \Sigma_t \, dt, \quad \Sigma_t = \sigma_t \sigma_t^*. \]
Let us recall the definition of the MRC estimator in the synchronous sampling case. Suppose that we have observation data \((Y_t)_{t=0}^N\) with observation times \(0 \leq t_0 < t_1 < \cdots < t_{N-1} < t_N \leq T\). Then, we choose a weight function \(g\) on \([0,1]\) and a window size \(K\) with which we associate the variables called the pre-averaging of \(Y\):

\[
\overline{Y}_{t_i} = \sum_{j=1}^{K-1} g\left(\frac{j}{K}\right) \Delta_{t_{i+j}}, \quad \Delta_{t_i}Y = Y_{t_i} - Y_{t_{i-1}}.
\]

Since the observation errors are centered and independent, one can expect that \(\overline{Y}_{t_i}\)'s are close to the latent returns. Therefore, it is natural to consider the statistic \(\sum_{i=0}^{N-K+1} \overline{Y}_{t_i}(\overline{Y}_{t_i})^*\) as an estimator of \([X]\). In fact, after appropriate scaling \([6]\) showed that this estimator has the consistency and the asymptotic mixed normality as long as the observation times are equidistant, i.e. \(t_i = i/N\) and we consider the situation where \(N\) goes to infinity. This scaled estimator is called the MRC estimator.

Now, our main concern is the following two questions:

(a) **What happens when the observation times are endogenous?**

(b) **What is an optimal choice of the weight function \(g\)?**

By the term “endogenous” we mean that the observation times depend on the latent log-price process \(X\). Indeed, this issue is a relatively new subject in this area despite its importance for both theoretical and practical perspectives. In fact, in a pure one-dimensional diffusion setting, Fukasawa \([9]\) showed that the endogeneity of the observation times can cause a bias of the asymptotic distribution of the realized volatility \(\sum_{i=1}^{N}(\Delta t_i X)^2\), which is a natural estimator for \([X]_T\) in such a setting. This phenomenon was independently found by Li et al. \([20]\), and they also constructed a feasible central limit theorem as well as conducted empirical work that provides evidence that time endogeneity exists in financial data. In their analysis, the skewness and kurtosis of the returns \(\Delta t_i X\) play an important role. In particular, \([20]\) showed that the former quantity has a strong connection with the covariance between the returns \(\Delta t_i X\) and the durations \(t_i - t_{i-1}\) (see Remark 3 of that paper). Renault and Werker \([25]\) discussed the effect of this covariance on the volatility inference in a semi-parametric context. On the other hand, Li et al. \([21]\) derived a corresponding result to the one by \([20]\) in the presence of microstructure noise. More precisely, they considered the following estimator: choose two integers \(p\) and \(q\) such that \(p < q\), and set

\[
\hat{Y}_{t_i} = \frac{1}{p} \sum_{j=0}^{p-1} (Y_{t_{i+j+q}} - Y_{t_{i+j}}).
\]

They showed that after appropriate scaling, the estimator \(\sum_{i=0}^{N-(p+q)+1}(\hat{Y}_{t_i})^2\) is (possibly biased) asymptotic mixed normal under some regularity conditions; see Theorem 2 of \([21]\) for details. In particular, according to their theory the asymptotic distribution of the estimation error \(\sqrt{q} \sum_{i=0}^{N-(p+q)+1}(\hat{X}_{t_i})^2 - [X]_T\) due to the diffusion part is characterized by the probability limit of the processes given by

\[
\begin{align*}
\frac{N}{q} \sum_{q \geq q, t \leq t} \left( \sum_{j=1}^{q-j} \frac{q-j}{q} \Delta_{t_{i-j}} X \right)^2 (\Delta_{t_i} X)^2 & \quad \text{and} \quad \sqrt{N} \sum_{t_{i+p+q-1} \leq t} (\hat{X}_{t_i})^3 
\end{align*}
\]

for each \(t \in [0,T]\). Note that if \(p = q\) their estimator corresponds to the MRC estimator while \(g(x) = x \wedge (1 - x)\) and \(K = 2p\). In this paper we concentrate on the case where \(p = q\) because the estimator achieves the optimal rate of convergence under these circumstances.

Therefore, regarding question (a) one possible approach would be to find some counterparts of the quantities in Eq.\((1.1)\) in the multivariate and the general weight function setting. Unfortunately, we encounter some difficulties taking this approach. Namely, (i) it is not clear what the first quantity of \((1.1)\) corresponds to in the general weight function setting, and (ii) it is preferable to give an explicit relationship between the asymptotic distribution
of the estimator and the tuning parameters $g$ and $K$ in order to obtain information on the optimal choice. This is especially important for question (b). The characterization by the quantities in (1.1), however, is not adapted to this purpose because their limiting variables will depend on the tuning parameters in an unspecified way. For this reason we introduce another set of conditions, which is independent of the choice of the tuning parameters, for handling the time endogeneity. Those conditions seem to be reasonable for covering important models used in financial econometrics. Interestingly, it turns out that the time endogeneity has no impact on the asymptotic distribution of the MRC estimator under our condition. This is quite different from the case of the realized volatility in a pure diffusion setting and makes the derivation of feasible limit theorems easier.

On the other hand, regarding question (b) we try to find an optimal weight function in the sense that it minimizes the asymptotic variance of the MRC estimator in the univariate and parametric setting with equidistant observation times. To accomplish this, we need to extend the class of weight functions to those with unbounded supports. This is implemented in Section 2. After that, in Section 3.6 the double exponential density is shown to be an optimal weight function. In fact, it turns out that the double exponential density is a counterpart of the optimal kernel function for the flat-top realized kernel of Barndorff-Nielsen et al. [2]. Therefore, the MRC estimator with the double exponential density and the oracle window size $K$ achieves the parametric efficiency bound from [11]. We also point out that this optimal weight function has a computational advantage.

The proof of the main result, outlined in Section 3.5, is divided into two major parts: (A) constructing a martingale approximation of the estimation error process and, (B) applying Jacod [14]'s stable central limit theorem to the martingale approximation constructed in part (A). For part (A) a certain block splitting technique is commonly used in the literature (see [6], [7], [15] and [16]). However, the proof given in this paper does not rely on such a technique. Instead, we construct the desired martingale approximation directly using integration by parts for semimartingales. Our approach is therefore closer to those of Hayashi and Yoshida [13] and Koike [19]. As a consequence, the application of Jacod’s theory in part (B) is basically in line with them. The major (and essential) difference between their argument and ours is that we do not need a strong predictability type condition for the noise process $\epsilon$. Consequently, the application of Jacod’s theory in part (B) is basically in line with them. The major (and essential) difference between their argument and ours is that we do not need a strong predictability type condition for the noise process $\epsilon$.

This paper is organized as follows. Section 2 presents the mathematical model and the construction of the MRC estimator in a more general setting. Section 3 is devoted to the main result of this paper. Section 4 provides some illustrative examples of observation times which are possibly endogenous. Section 5 discusses a connection between Li et al. [21]’s result and ours as well as a feasible central limit theorem, while Section 6 provides a simulation study. All proofs are given in Section 7.

2 The setting

We begin by constructing a suitable stochastic basis on which our noisy process $Y$ is defined. Let $\mathcal{B}^{(0)} = (\Omega^{(0)},\mathcal{F}^{(0)},\mathbb{P}^{(0)})$ be a stochastic basis on which our latent process $X$ is defined, such that all the constituting processes $a, \sigma$ and $W$ are adapted. On the other hand, for any $t \geq 0$ we have a transition probability $Q_{t}(\omega^{(0)},dz)$ from $(\Omega^{(0)},\mathcal{F}^{(0)})$ into $\mathbb{R}^{d}$, which satisfies $\int zQ_{t}(\omega^{(0)},dz) = 0$ and will correspond to the conditional distribution of the noise at the time $t$ given $\mathcal{F}^{(0)}$. We endow the space $\Omega^{(1)} = (\mathbb{R}^{d})^{[0,\infty)}$ with the product Borel $\sigma$-field $\mathcal{F}^{(1)}$ and with the probability $Q(\omega^{(0)},d\omega^{(1)})$ which is the product $\otimes_{t \in \mathbb{R}^{+}}Q_{t}(\omega^{(0)},\cdot)$. Now the noise process $\epsilon = (\epsilon_{t})_{t \in \mathbb{R}^{+}}$ is realized as the canonical process on $(\Omega^{(1)},\mathcal{F}^{(1)})$. Finally, the stochastic basis $\mathcal{B} = (\Omega,\mathcal{F},\mathbb{P} = (\mathcal{F}_{t})_{t \in \mathbb{R}^{+}},\mathbb{P})$ on which we will work is defined as follows:

$$\Omega = \Omega^{(0)} \times \Omega^{(1)}, \quad \mathcal{F} = \mathcal{F}^{(0)} \otimes \mathcal{F}^{(1)}, \quad \mathcal{F}_{t} = \cap_{s > t} \mathcal{F}_{s}^{(0)} \otimes \mathcal{F}_{s}^{(1)}, \quad P(d\omega^{(0)},d\omega^{(1)}) = P^{(0)}(d\omega^{(0)})Q(\omega^{(0)},d\omega^{(1)}),$$

where $\mathcal{F}_{t}^{(1)} = \sigma(\epsilon_{s}; s \leq t)$. Any variable or process defined on either $\Omega^{(0)}$ or $\Omega^{(1)}$ can be considered in the usual way as a variable or a process on $\Omega$. 

3
We observe the components of the process \( Y = (Y^1, \ldots, Y^d) \) on the interval \([0, T]\) in a discrete and nonsynchronous manner. For each \( k = 1, \ldots, d \) the observation times for \( Y^k \) are denoted by \( t_0^k, t_1^k, \ldots \), i.e. the observation data \((Y^k_{t_i^k})_{t_i^k \leq T}\) are available. We assume that \((t_i^k)_{i=0}^\infty\) is a sequence of \( \mathbf{F}^{(0)}\)-stopping times which implicitly depend on a parameter \( n \in \mathbb{N} \) representing the observation frequency and satisfy that \( t_i^k \uparrow \infty \) as \( i \to \infty \) and \( \sup_{i \geq 0}(t_i^k \wedge t - t_i^{k-1} \wedge t) \to^p 0 \) as \( n \to \infty \) for any \( t \in \mathbb{R}_+ \), with setting \( t_{-1}^k = 0 \) for a notational convenience (hereafter we will refer to such a sequence as a sampling scheme for short).

Now we explain the construction of the MRC estimator in the nonsynchronous sampling setting. First, we need to synchronize the observation data. Following Barndorff-Nielsen et al. [3], we introduce the notion of refresh time:

**Definition 2.1 (Refresh time).** The refresh times \( T_0, T_1, \ldots \) of the sampling schemes \( \{(t_i^k)\}_{k=1}^d \) are defined sequentially by \( T_0 = \max\{t_0^1, \ldots, t_0^d\} \) and \( T_p = \max_{k=1}^d \min\{t_i^k | t_i^k > T_{p-1}\} \) for \( p = 1, 2, \ldots \).

We introduce synchronized observation times by interpolating the next-ticks into the grid \((T_p)_{p=0}^\infty\). That is, for each \( k = 1, \ldots, d \) define the synchronized observation times \((\tau_p^k)_{p=0}^\infty\) for \( Y^k \) by \( \tau_0^k = t_0^k \) and

\[
\tau_p^k = \min\{t_i^k | t_i^k > T_{p-1}\}, \quad p = 1, 2, \ldots.
\]

Here, we prefer the next-tick interpolation scheme to the previous-tick interpolation scheme because it automatically makes the resulting synchronized observation times stopping times. In fact, we have \( \tau_p^k = \inf_{i \geq 1} \left\{ t_i^k \mid \left\{ t_i^k > T_{p-1} \right\} \right\} \), where for an \( \mathbf{F}\)-stopping time \( \tau \) and a set \( A \in \mathcal{F}_\tau \), we define \( \tau_A \) by \( \tau_A(\omega) = \tau(\omega) \) if \( \omega \in A \); \( \tau_A(\omega) = \infty \) otherwise (see I-1.15 of [18]).

Based on the synchronized data constructed in the above, we introduce the pre-averaging as follows. We choose a sequence \( k_n \) of positive integers and a number \( \theta \in (0, \infty) \) such that

\[
k_n = \theta \sqrt{n} + o(n^{1/4})
\]

as \( n \to \infty \). We also choose a continuous function \( g : [0, 1] \to \mathbb{R} \) which is piecewise \( C^1 \) with a piecewise Lipschitz derivative \( g' \) and satisfies

\[
g(0) = g(1) = 0 \quad \text{and} \quad \int_0^1 g(x)^2 \, dx > 0.
\]

After that, for any \( d \)-dimensional stochastic process \( V = (V^1, \ldots, V^d) \) we define the quantity

\[
\nabla_i^k = \sum_{p=1}^{k_n-1} g \left( \frac{p}{k_n} \right) \left( V_{t_{i+p}^k}^k - V_{t_{i+p-1}^k}^k \right),
\]

and set \( \nabla_i = (\nabla_i^1, \ldots, \nabla_i^d)^* \). Now the MRC estimator in the nonsynchronous setting is defined as

\[
\text{MRC}[Y]^n_T = \frac{1}{\psi_2 k_n} \sum_{i=0}^{N^n_T} \nabla_i \left( Y_i \right)^* - \frac{\psi_1}{2 \psi_2 k_n^2} [Y]^n_T,
\]

where \( N^n_T = \max\{p | T_p \leq t\} \), \( \psi_1 = \int_0^1 g'(x)^2 \, dx \), \( \psi_2 = \int_0^1 g(x)^2 \, dx \) and

\[
[Y]^n_T = \sum_{p=1}^{N^n_T} \Delta_p Y (\Delta_p Y)^*, \quad \Delta_p Y = \left( Y_{\tau_p^1}^1 - Y_{\tau_{p-1}^1}^1, \ldots, Y_{\tau_p^d}^d - Y_{\tau_{p-1}^d}^d \right)^*.
\]

for each \( t \in [0, T] \).\footnote{We set \( \sum_{i=p}^q = 0 \) if \( p > q \) by convention.} In the synchronous and equidistant sampling case, a central limit theorem for the MRC estimator has been shown in [6]. One of our main purposes is to develop an asymptotic distribution theory for the MRC estimator in the situation where observation times are possibly nonsynchronous and endogenous.

Another main purpose is to find an optimal weight function \( g \), and to accomplish this we need to extend the definition of the MRC estimator for weight functions with unbounded supports. Specifically, we consider a function \( g \) on \( \mathbb{R} \) satisfying the following condition:
Then, a naïve extension of \((C_n)\) and \(\tilde{\psi} \) where the asymptotic theory developed in this paper to the original estimator \(MRC\) is as follows:

\[
\nabla_i^d = \sum_{p=1}^{N_n^k-i} g \left( \frac{p}{k_n} \right) \left( V^k_{\tau_i+p} - V^k_{\tau_i+p-1} \right).
\]

Unfortunately, this definition suffers from the end effect. In fact, summation by parts yields

\[
\tilde{\epsilon}^k_i = - \sum_{p=1}^{N_n^k-i} \left\{ g \left( \frac{p+1}{k_n} \right) - g \left( \frac{p}{k_n} \right) \right\} \epsilon^k_{\tau_i+p} + g \left( \frac{N_n^k-i}{k_n} \right) \epsilon^k_{\tau_i+k} - g \left( \frac{N_n^k-i+1}{k_n} \right) \epsilon^k_{\tau_i+0},
\]

hence the noise \(\epsilon^k_{\tau_i+0}\) and \(\epsilon^k_{\tau_i+N_n^k}\) at the end points will have some impact on the limiting variable of \(\tilde{\epsilon}^k_i\) unless \(g\) has a bounded support. To avoid this problem, we take the averages of the first and the last \(k_n\) distinct observations:

\[
\tilde{V}^k_0 = \frac{1}{k_n} \sum_{p=0}^{k_n-1} V^k_{\tau_p}, \quad \tilde{V}^k_T = \frac{1}{k_n} \sum_{p=N_n^k-k_n+1}^{N_n^k} V^k_{\tau_p}.
\]

This idea is commonly used in the literature of realized kernel estimators and called the jittering; see e.g. [2] and [3]. Now we define the adjusted returns \((\tilde{\Delta}_{\tau_p} V^k)_{p=k_n}^{N_n^k-k_n+1}\) based on the data \(\tilde{V}^k_0, V^k_{\tau_{k_n}}, V^k_{\tau_{k_n+1}}, \ldots, V^k_{\tau_{N_n^k-k_n-1}}, V^k_{\tau_{N_n^k-k_n}}, \tilde{V}^k_T\).

Namely, set \(\tilde{\Delta}_{\tau_p} V^k = V^k_{\tau_p} - V^k_{\tau_{k_n}}\) for \(p = k_n + 1, \ldots, N_n^k - k_n\) and

\[
\tilde{\Delta}_{\tau_p} V^k_{\tau_{k_n}} = V^k_{\tau_{k_n}} - \tilde{V}^k_0, \quad \tilde{\Delta}_{\tau_p} V^k_{\tau_{N_n^k-k_n+1}} = V^k_{\tau_{N_n^k-k_n}} - \tilde{V}^k_T.
\]

After that, our adjusted version of the pre-averaging is defined by

\[
\tilde{V}^k_{i,T} = \sum_{p=1}^{N_n^k-k_n+1-i} g \left( \frac{p}{k_n} \right) \tilde{\Delta}_{\tau_i+p} V^k = \sum_{p=k_n}^{N_n^k} g \left( \frac{p-i}{k_n} \right) \tilde{\Delta}_{\tau_p} V^k
\]

and \(\tilde{V}_{i,T} = (\tilde{V}^1_{i,T}, \ldots, \tilde{V}^d_{i,T})^*\). Consequently, our estimator takes the following form:

\[
\tilde{MRC}[Y]_T^n = \frac{1}{\psi_2 k_n} \sum_{i=k_n}^{N_n^k-k_n+1} \tilde{V}^k_{i,T} \left( \tilde{V}^k_{i,T} \right)^* - \frac{\psi_1}{\psi_2 k_n} [Y]_T^n,
\]

where \(\psi_1 = \int_{-\infty}^{\infty} g'(x)^2 dx\) and \(\psi_2 = \int_{-\infty}^{\infty} g(x)^2 dx\). Note that if \(g\) is a continuous function on \([0, 1]\) which is piecewise \(C^1\) with a piecewise Lipschitz derivative \(g'\) and satisfies (2.2), with extending \(g\) to the whole real line by setting \(g(x) = 0\) for \(x \notin [0, 1]\) we obtain a weight function \(g\) satisfying the condition \([W]\). In this case it can easily be shown that \(n^{1/4} \left( \tilde{MRC}[Y]_T^n - \tilde{MRC}[Y]_T^n \right) \rightarrow^p 0\) as \(n \rightarrow \infty\) under the assumptions of Theorem 3.1, so we can also apply the asymptotic theory developed in this paper to the original estimator \(MRC[Y]_T^n\).

3 Main result

3.1 Generalization of the framework of the synchronized observation times

We start with generalizing the framework of the grid \((T_p)\) and the synchronized observation times \((\tau^k_p)\) for a technical reason. In fact, this generalization will be useful for the localization procedure used in the proof.

In the remainder of this section we will suppose that the sequences \((T_p)_{p=0}^{\infty}\) and \((\tau^k_p)_{p=0}^{\infty} (k = 1, \ldots, d)\) are given \(a \text{ priori}\) and satisfies the following condition:
H1] (i) \((T_p)\) and \((\tau^k_p)\) \((k = 1, \ldots, d)\) are sampling schemes.
    (ii) \(\tau^k_0 \leq T_0\) and \(T_{p-1} < \tau^k_p \leq T_p\) for any \(p \geq 1\) and any \(k \in \{1, \ldots, d\}\).

Apparently, the sequence \((T_p)\) of the refresh times and the sequences \((\tau^k_p)\) \((k = 1, \ldots, d)\) of the next-ticks into \((\tau_p)\)
defined in the previous section constitute one example of such sequences.

After that, we define the quantities \(N^p_n\), (2.4) and \([Y]^n_t\) based on these schemes. Then, we define the process \(\widetilde{MRC}[Y]^n\) by

\[
\widetilde{MRC}[Y]^n_t = \frac{1}{\psi_2 k_n} \sum_{i=k_n}^{N^p_n - k_n + 1} \tilde{Y}_{i,T} \left( \tilde{Y}_{i,T} \right)^* - \frac{\psi_4}{2\psi_2 k_n^2} [Y]^n_t
\]

for each \(t \in [0, T]\). Here, we also extend the definition of the MRC estimator to a process for the later use. Note that the summands of the first term in the right hand side of the above definition are always defined by using all the returns on \([0, T]\). We will show a functional stable central limit theorem for the process \(\widetilde{MRC}[Y]^n\) in the following.

Remark 3.1. Apart from the theoretical necessity, the above generalization is meaningful in terms of applications. In fact, this allows us to use the Generalized Synchronization method, which was introduced by A¨ıt-Sahalia et al. [1], for the data synchronization instead of the method based on refresh times. Some advantages of such a generalization are explained in Section 3.3 of [1]. In particular, this generalization implies that the MRC estimator is robust to data misplacement error, as long as these misplaced data points are within the same sampling intervals are explained in Section 3.3.

3.2 Notations

In this subsection we introduce some notations in order to state our main result. First, we denote by \(\Upsilon_t\) the covariance matrix of \(\epsilon_t\), i.e. \(\Upsilon_t(\cdot) = \int zz^*Q_t(\cdot, dz)\) (we will assume the existence of the eighth moment of the noise later, so this matrix always exists).

Next, for each \(n\) we introduce an auxiliary filtration \(H^n = (H^n_t)_{t \geq 0}\) of \(\mathcal{F}^{(0)}\) satisfying the following condition:

\(H2\) (i) \(T_p\) and \(\tau^k_p\) are \(H^n\)-stopping times for every \(k, p\).

(ii) \(a, \sigma, W\) and \(\Upsilon\) are \(H^n\)-adapted for every \(k\).

A simple choice of \(H^n\) consists in taking \(H^n = F^{(0)}\), but other choices are possible. In the following we will consider conditional expectations of quantities related to the sequences \((T_p)\) and \((\tau^k_p)\) given \(H^n\), and \(H^n\) can be chosen appropriately to compute such ones.

We also introduce a random subset \(\mathcal{N}^n\) of \(\mathbb{N}\) such that \(\{ (\omega, p) \in \Omega \times \mathbb{N} \mid p \in \mathcal{N}^n(\omega) \}\) is a measurable set of \(\Omega \times \mathbb{N}\). In the following \(\mathcal{N}^n\) is used as an exceptional set in the computation of limiting variables appearing the asymptotic variance of the estimator. Again, this will be useful for the localization procedure used in the proof.

For each \(n\) and each \(\rho \geq 0\) we set

\[
D(\rho)^n_{p-1} = E \left[ (n[I_p])^p \mid H^n_{T_{p-1}} \right], \quad p = 1, 2, \ldots,
\]

where \(I_p = [T_{p-1}, T_p)\) and \(|\cdot|\) denotes the Lebesgue measure. This type of quantity often appears in the literature; see e.g. [3] and [12]. Moreover, for each \(n\) and each \(k, l = 1, \ldots, d\) we choose an \(H^n\)-optional [0, 1]-valued process \(\chi^{kl, n}\) such that

\[
\chi^{kl, n}_{T_{p-1}} = P \left( \tau^k_p = \tau^l_p \mid H^n_{T_{p-1}} \right)
\]

for every \(p \in \mathbb{N} - \mathcal{N}^n\). This quantity naturally appears when we compute the covariances between \((\tau^k_p)\) and \((\tau^l_p)\).

For a matrix \(A \in \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_2}\), we write the entries \(A^{kl}, 1 \leq k \leq d_1, 1 \leq l \leq d_2\), and the vector of its entries obtained by stacking its columns one below each other

\[
\text{vec}(A) = (A^{11}, A^{21}, \ldots, A^{d_1 1}, A^{12}, A^{22}, \ldots, A^{d_1 2}, \ldots, A^{d_1 (d_2 - 1)}, A^{d_1 d_2})^* \in \mathbb{R}^{d_1 d_2}.
\]
We denote by $\|A\|$ the Frobenius norm of $A$ i.e., $\|A\|^2 = \sum_{k=1}^{d_1} \sum_{l=1}^{d_2} (A_{kl})^2$.

For a (matrix-valued) function $x$ on $\mathbb{R}_+$, the modulus of continuity on $[0, T]$ is denoted by $w(x; h) = \sup\{|x(t) - x(s)|; t, s \in [0, T], |t - s| \leq h\}$ for $h > 0$. We write $X^n \overset{ucp}{\to} X$ for processes $X^n$ and $X$ to express that $\sup_{0 \leq t \leq T} |X^n_t - X_t| \to^p 0$.

$D([0, T], \mathbb{R}^{d^2})$ denotes the space of $\mathbb{R}^{d^2}$-valued càdlàg functions on $[0, T]$ equipped with the Skorokhod topology.

A sequence of random elements $X^n$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to converge stably in law to a random element $X$ defined on an appropriate extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ of $(\Omega, \mathcal{F}, P)$ if $E[Z f(X^n)] \to E[Z f(X)]$ for any $\mathcal{F}$-measurable bounded random variable $Z$ and any bounded continuous function $f$.

Finally, we introduce constants appearing in the representation of the asymptotic variance of the MRC estimator. For any real-valued bounded measurable functions $u, v$ on $\mathbb{R}$, we define the function $\phi_{u,v}(y) = \int_{-\infty}^{\infty} u(x - y)v(x)dx$. Then, we put

$$\Phi_{22} = \int_0^\infty \phi_{g,g}(y)^2dy, \quad \Phi_{12} = \int_0^\infty \phi_{g,g}(y)\phi_{g',g'}(y)dy, \quad \Phi_{11} = \int_0^\infty \phi_{g',g'}(y)^2dy.$$  

3.3 Conditions

Firstly, following [13] we impose continuity conditions on the drift and the volatility processes:

[A1] $w(a; h, T) = O_p(h^{1/4})$ as $h \to 0$.

[A2] $w(\Sigma; h, T) = O_p(h^{1/2-\lambda})$ as $h \to 0$ for any $\lambda > 0$.

Remark 3.2. [A1] and [A2] are satisfied by most practical stochastic volatility models, e.g. the Heston model. The continuity conditions on the coefficient processes are necessary due to the irregularity and the nonsynchronicity of the sampling schemes as [13].

Secondly, we impose the following regularity condition on the noise process:

[A3] $(\int \|z\|^8 Q_t(dz))_{t \geq 0}$ is a locally bounded process, and $w(T; h, T) = O_p(h^{1/2-\lambda})$ as $h \to 0$ for any $\lambda > 0$.

Remark 3.3. The locally boundedness of the process $(\int \|z\|^8 Q_t(dz))_{t \geq 0}$ is standard for proving central limit theorems of pre-averaging estimators; see e.g. [6] and [15]. It is essentially used for verifying a Lyapunov-type condition. The continuity of the covariance matrix process is necessary due to the same reason as for [A2].

Finally, we impose the following condition on the grid and the synchronized observation times:

[A4] (i) There exists a constant $\xi \in (\frac{5}{8}, 1)$ such that $r_n(T) = o_p(n^{-\xi})$ as $n \to \infty$, where $r_n(t) = \sup_{p \geq 0}(T_p \wedge t - T_{p-1} \wedge t)$ for each $t \geq 0$ (note that $T_{-1} = 0$ by convention).

(ii) For each $n$ we have an $\mathbf{H}^n$-optimal process $G^n$ such that $G^n_{T_p-1} = D(1)_{p-1}$ for every $p \in \mathbb{N} - \mathcal{N}^n$. Furthermore, there exist an $\mathbf{F}^{(0)}$-adapted positive-valued continuous process $G$ and a constant $\delta > 1 - \xi$ such that $n^\delta(G^n - G) \overset{ucp}{\to} 0$ as $n \to \infty$ and $w(G; h, T) = O_p(h^{1/2-\lambda})$ as $h \to 0$ for any $\lambda > 0$.

(iii) There exists a constant $\rho > 1/\xi$ such that $\sup_{p \in \mathbb{N} - \mathcal{N}^n} D(\rho)^p$ and $n^{1/\rho} \#\mathcal{N}^n$ are tight as $n \to \infty$.

(iv) For each $k, l$ there exist an $\mathbf{F}^{(0)}$-adapted continuous process $\chi^{kl}$ and a constant $\delta^{kl} > 1 - \xi$ such that $n^{\delta^{kl}}(\chi^{kl,n} - \chi^{kl}) \overset{ucp}{\to} 0$ as $n \to \infty$ and $w(\chi^{kl}; h, T) = O_p(h^{1/2-\lambda})$ as $h \to 0$ for any $\lambda > 0$.

(v) $\chi^{k,p}_p$ is an $\mathbf{F}^{(0)}$-predictable time for every $k, p$.

Remark 3.4. (i) The condition $\xi > \frac{5}{8}$ in [A4](i) is necessary for ensuring that the MRC estimator achieves the optimal convergence rate $n^{-1/4}$. In many cases $r_n(T) = o_p(n^{-\xi})$ holds for any $\xi \in (0, 1)$. For example, if $(T_p)$ is a sequence of Poisson arrival times with the intensity $\lambda n/\rho$ ($\lambda > 0$), then $r_n(T) = O_p(n^{-1/2} \log n)$ by Corollary 1 of [26].

(ii) [A4](ii)–(iv) ensure that quantities appearing in the asymptotic variance indeed converge. The continuity conditions imposed on the limiting processes are necessary for proving that we can ignore the impact of the time
endogeneity on the asymptotic distribution of the estimator. Note that these conditions themselves do not rule out any kind of time endogeneity.

(iii) Since we can always take \( \chi^{kk,n} \equiv 1 \), [A4](iv) is automatically satisfied for the case that \( k = l \) with \( \chi^{kl} \equiv 1 \).

(iv) [A4](v) is necessary due to the following technical reason: in the proof we will regard the noise process \( (\xi^k_{\tau^k}) \) as the differences of the purely discontinuous locally square-integrable martingale \( \sum_{p=1}^{\infty} \xi^k_{\tau^k p} 1_{\{\tau^k p \leq t\}} \) on \( \mathcal{B} \). Then we need to consider the predictable quadratic variation process (with respect to the filtration \( \mathcal{F} \)) of this process. Since \( Y^{kk} \) is \( \mathcal{F}_t \)-predictable under [A3], [A4](v) ensures that it is given by \( \sum_{p=1}^{\infty} Y^{kk}_{\tau^k p} 1_{\{\tau^k p \leq t\}} \). This condition is reasonable in the current framework because hitting times of continuous adapted processes are predictable.

(v) [A4] implies that \( n^{-1} N^n p \) converges to a non-zero random variable in probability (see Lemma 7.2). Therefore, the number of (synchronized) observations is of order \( n \).

(vi) Let us focus on our leading case, i.e., \( (T_p) \) is defined as the refresh times of \( \{(t^k_p)\}_{k=1}^d \) and \( \{(\tau^k_p)\}_{p=1}^d \) are defined as the next-tick interpolations to \( (T_p) \). In this case a Poisson sampling gives a simple but important example satisfying [A4]. More precisely, let \( (T^k_p) \) be a sequence of Poisson arrival times with the intensity \( np_k \) for each \( k \) and suppose that \( (t^k_p), \ldots, (t^k_d) \) are mutually independent and independent of \( X^{1}, \ldots, X^{d} \) and \( \mathcal{F}^{(1)} \). Then, without loss of generality we may assume that \( T^k_p \) is \( \mathcal{F}_0 \)-measurable for every \( k, i \), hence [A4](v) holds true. Furthermore, it is easy to show that [A4](iii) and (iv) hold true with \( \chi^{kl} \equiv 0 \) for \( k \neq l \), while [A4](i) follows from Corollary 1 of [26]. Finally, [A4](ii) is satisfied with

\[
G_s = \sum_{k=1}^{d} \sum_{1 \leq l_1 < \cdots < l_k \leq d} \frac{(-1)^{k-1}}{p^{l_1} \cdots p^{l_k}}.
\]

This can be proven as follows. Set \( p = \sum_{k=1}^{d} p_k \) and let \( \bar{N} \) be a Poisson process with the intensity \( np \). Let \( (\eta_j)_{j=1}^{d} \) be a sequence of i.i.d. random variables such that \( P(\eta_j = k) = p_k/p, k = 1, \ldots, d \). We assume that \( (\eta_j) \) is independent of \( \bar{N} \). For each \( k \in \{1, \ldots, d\} \) define the process \( N^{(k)} \) by \( N^{(k)} = \sum_{j=1}^{\bar{N}} 1_{\{\eta_j = k\}} \). A short calculation shows that \( N^{(k)} \) is a Poisson process with the intensity \( np_k \). Therefore, Theorem 6 of [8] implies that \( N^{(1)}, \ldots, N^{(d)} \) are independent. This fact yields \( D(1)^n_p = n^{-1} E[\min\{j|\{\eta_1, \ldots, \eta_j\} = \{1, \ldots, d\}] \). Now (3.1) follows from Eq.(6) of [28].

3.4 Result

Now we are ready to state the main theorem of this paper.

**Theorem 3.1.** Suppose that [W], [H1]–[H2] and [A1]–[A4] are satisfied. Then

\[
n^{1/4} \left\{ \text{vec} \left( \text{MRC}[Y^n_t] \right) - \text{vec} \left( [X] \right) \right\} \to \mathcal{D} \int_0^{d_s} w_s d\bar{W}_s \quad \text{in} \; \mathbb{D}(\{0, T\}; \mathbb{R}^{d^2})
\]

as \( n \to \infty \), where \( \bar{W} \) is a \( d^2 \)-dimensional standard Wiener process (defined on an extension of \( \mathcal{B} \)) independent of \( \mathcal{F} \) and \( w \) is the \( \mathbb{R}^d \otimes \mathbb{R}^{d^2} \)-valued \( \mathcal{F} \)-predictable process satisfying

\[
\sum_{j=1}^{d^2} u_s^{d(l-1)+k, j} w_s^{d(l'-1)+k', j} = \frac{2}{\sqrt{2}} \left[ \Phi_{22} \theta \left\{ \Sigma_s^{kk'} \Sigma_s^{ll'} + \Sigma_s^{kk'} \Sigma_s^l \Sigma_s^l \right\} G_s
+ \frac{\Phi_{12}}{\theta} \left\{ \Sigma_s^{kk'} \Sigma_s^l \Sigma_s^{ll'} + \Sigma_s^{kk'} \Sigma_s^{ll'} + \Sigma_s^{kk'} \Sigma_s^{ll'} + \Sigma_s^{kk'} \Sigma_s^l \Sigma_s^l \right\}
+ \frac{\Phi_{11}}{\theta^3} \left\{ \Sigma_s^{kk'} \Sigma_s^l \Sigma_s^{ll'} + \Sigma_s^{kk'} \Sigma_s^{ll'} \right\} \frac{1}{G_s} \right],
\]

where \( \Sigma_s^{kk'} = Y^{kk} \chi^{kl} \).

**Remark 3.5.** (i) The above theorem tells us that the observation times affect the asymptotic distribution of the MRC estimator only through the asymptotic conditional expected duration process \( G \) and the limiting process \( \chi^{kl} \) measuring the degree of the nonsynchronicity. In particular, the time endogeneity has no impact on the asymptotic
distribution. This contrasts with the case of the realized volatility in a pure diffusion setting, where the time endogeneity can cause a bias in the asymptotic distribution as demonstrated in [9] and [20].

(ii) It is also worth pointing out that the effect of the observation times is not through the Asymptotic Quadratic Variation of Time, unlike the case of the realized volatility as described in [23] for instance. Especially, even the randomness of the durations plays no role in the asymptotic distribution of the MRC estimator. This is again different from the case of the realized volatility, where the randomness of the durations inflates the asymptotic variance.

3.5 Outline of the proof

We start by introducing some notations. For processes $U$ and $V$, $U \bullet V$ denotes the integral (either stochastic or ordinary) of $U$ with respect to $V$. For a càdlàg process $V$, $V_-$ denotes the process $(V_\tau)_\tau \in \mathbb{R}_+$ and $\Delta V_\tau$ denotes the jump of $V$ at the time $\tau$, i.e. $\Delta V_\tau = V_\tau - V_{\tau-}$. For any semimartingale $V$ and any (random) interval $I$, we define the processes $V(I)_\tau$ and $I_\tau$ by $V(I)_\tau = \int_0^\tau 1_{I}(s-)dV_s$ and $I_\tau = I_{\tau}(t)$ respectively. For a function $u$ on $\mathbb{R}$ we write $u^n_p = u(p/k_n)$ for each $n \in \mathbb{N}$ and $p \in \mathbb{Z}$.

We define the processes $A$ and $M$ by $A_t = \int_0^t a_s ds$ and $M_t = \int_0^t \sigma_s dW_s$. We also define the $d$-dimensional process $\mathcal{E} = (\mathcal{E}^1, \ldots, \mathcal{E}^d)$ by

$$\mathcal{E}_t^k = -\frac{1}{k_n} \sum_{p=1}^{\infty} \epsilon_t^p 1_{(r^p \leq t)}, \quad k = 1, \ldots, d.$$ 

It can easily be checked that $\mathcal{E}_t^k$ is a purely discontinuous locally square-integrable martingale on $\mathcal{B}$ under $[A3]$. It is also not difficult to show that the predictable quadratic covariation of $\mathcal{E}^k$ and $\mathcal{E}^l$ is given by

$$\langle \mathcal{E}_t^k, \mathcal{E}_t^l \rangle_t = \frac{1}{k_n} \sum_{p=1}^{\infty} \tau_t^p 1_{(r^p \leq t)}$$

under $[A3]$ and $[A4]$.

We take a sufficiently small positive number $\gamma$. More precisely, $\gamma$ should satisfy

$$\gamma < \frac{1}{54} \wedge \frac{3}{2} \left( \frac{\xi - 1}{\rho} \right) \wedge \frac{1}{3} \left\{ \delta \wedge \min_{k,l} \delta_{kl} - (1 - \xi) \right\} \wedge \frac{3}{7} \left( \xi - \frac{5}{6} \right). \quad (3.4)$$

Then we set $d_n = [n^{1/4 + \gamma}]$. For each $U, V \in \{X, A, M, \mathcal{E}\}, u, v \in \{g, g'\}$ and $k, l \in \{1, \ldots, d\}$, we define the processes $M_{u,v}^{(k,l)}(U, V)_t$ and $L_{u,v}^{(k,l)}(U, V)_t$ by

$$M_{u,v}^{(k,l)}(U, V)_t = \sum_{q=k_n+2}^{q} \sum_{p=k_n,v(q-d_n)}^{q-2} c_{u,v}(p,q) U^{k_p}(I^l_q)_t \bullet V^{l_q}(I^p_q)_t,$$

$$L_{u,v}^{(k,l)}(U, V)_t = M_{u,v}^{(k,l)}(U, V)_t + M_{v,u}^{(l,k)}(V, U)_t,$$

where $I_p^k = \lfloor p - 1 \rfloor, \lceil p \rceil$ and $c_{u,v}(p,q) = \frac{1}{v_k} \sum_{i=k_n}^{q} u_{p-i}^n v_{q-i}^n$.

First, the following two lemmas give us a martingale approximation of the estimation error process. The proofs are given in Sections 7.2 and 7.3, respectively.

**Lemma 3.1.** Under the assumptions of Theorem 3.1, it holds that

$$n^{1/4} \left\{ \hat{\text{MRC}}[Y]^{n,k,l} - L^{(k,l)}[g]^{n} - [X^k, X^l] \right\} \overset{ucp}{\longrightarrow} 0$$

as $n \to \infty$ for every $k, l = 1, \ldots, d$, where

$$L^{(k,l)}[g]^{n} = L_{g,g'}^{(k,l)}(X, X)^n + L_{g,g'}^{(k,l)}(X, \mathcal{E})^n + L_{g,g'}^{(l,k)}(X, \mathcal{E})^n + L_{g,g'}^{(k,l)}(\mathcal{E}, \mathcal{E})^n.$$
Lemma 3.2. Under the assumptions of Theorem 3.1, it holds that $n^{1/4}M_{w,v}^{(k,l)}(V,A) \overset{ucp}{\to} 0$ and $n^{1/4}P_{w,v}^{(k,l)}(A,V) \overset{ucp}{\to} 0$ as $n \to \infty$ for every $k, l \in \{1, \ldots, d\}$, $V \in \{A, M, E\}$ and $u, v \in \{g, g'\}$.

The proof of Lemma 3.1 is based on some basic estimates and integration by parts for semimartingales. Lemma 3.2 can be considered as a counterpart of Lemma 13.1 from [13] (or [19]), but we do not need a strong predictability type condition for the proof, unlike [13] and [19].

Lemmas 3.1–3.2 tell us that we may consider the process $\bar{T}^{(k,l)}[g]^n = (\bar{T}^{(k,l)}[g]^n)_{1 \leq k, l \leq d}$ instead of the estimation error process of our estimator, where

$$\bar{T}^{(k,l)}[g]^n = \bar{T}_{g,g}^{(k,l)}(M,M)^n + \bar{T}_{g,g'}^{(k,l)}(M,E)^n + \bar{T}_{g',g}^{(l,k)}(E,M)^n + \bar{T}^{(l,k)}(E,E)^n.$$  

Since $\text{vec}(\bar{T}[g]^n)$ is obviously a $d^2$-dimensional locally square-integrable martingale on $B$, we can apply Theorem 2-2 of [14] for proving a stable central limit theorem for this quantity. For this purpose we need to check the following conditions for any $k, l, k', l' \in \{1, \ldots, d\}$ and any $t \in [0, T]$:

\begin{align*}
&n^{1/2} \langle \bar{T}^{(k,l)}[g]^n, \bar{T}^{(k',l')}[g]^n \rangle_t \to_p \int_0^t \varphi_{k,k'}^{(l,l')} ds, \\
&n \sum_{0 \leq s \leq t} |\Delta \bar{T}^{(k,l)}[g|^n_s|^4 \to_p 0, \\
&n^{1/4} \langle \bar{T}^{(k,l)}[g]^n, M[k'] \rangle_t \to_p 0, \\
&n^{1/4} \langle \bar{T}^{(k,l)}[g]^n, N \rangle_t \to_p 0,
\end{align*}

where $\varphi_{k,k'}^{(l,l')}$ is the quantity in the right hand side of (3.3) and Eq.(3.8) holds for any one-dimensional bounded martingale $N$ on $B$ being orthogonal to $M$.

Eq.(3.5) follows from the following lemma, which is proven in Section 7.4:

Lemma 3.3. Under the assumptions of Theorem 3.1, it holds that

\begin{align*}
&n^{1/2} \langle \bar{M}_{g,g}^{(k,l)}(M,M)^n, \bar{M}_{g,g'}^{(k',l')} (M,M)^n \rangle_t \to_p 2 \varphi^{22}_{\psi_2} \int_0^t \left\{ \sum_{s \in \Sigma} \sum_{l'} \sum_{l''} \sum_{l'''} \right\} G_s ds, \\
&n^{1/2} \langle \bar{M}_{g,g'}^{(k,l)} (M,E)^n, \bar{M}_{g,g'}^{(k',l')} (M,E)^n \rangle_t \to_p 2 \varphi^{12}_{\psi_2} \int_0^t \sum_{s \in \Sigma} \tilde{Y}_{l'} G_s ds, \\
&n^{1/2} \langle \bar{M}_{g',g}^{(k,l)} (E,M)^n, \bar{M}_{g',g'}^{(k',l')} (E,M)^n \rangle_t \to_p 2 \varphi^{11}_{\psi_2} \int_0^t \left\{ \tilde{Y}_{kk'} \bar{Y}_{ll'} + \bar{Y}_{kk'} \tilde{Y}_{ll'} \right\} G_s ds, \\
&n^{1/2} \langle \bar{M}_{g,g}^{(k,l)} (M,M)^n, \bar{M}_{g',g'}^{(k',l')} (E,E)^n \rangle_t \to_p 0, \\
&n^{1/2} \langle \bar{M}_{g,g}^{(k,l)} (M,M)^n, \bar{M}_{g',g'}^{(k',l')} (E,E)^n \rangle_t \to_p 0
\end{align*}

as $n \to \infty$ for every $k, l, k', l'$.

The proof of Lemma 3.3 is based on the same idea as the one used in Lemma 3.2. On the other hand, Eqs.(3.6)–(3.8) follow from the following lemma. The proof is given in Section 7.5.

Lemma 3.4. Let $k, l \in \{1, \ldots, d\}$, $u, v \in \{g, g'\}$, $U, V \in \{M, E\}$ and $t \in [0, T]$. Under the assumptions of Theorem 3.1, the following statements hold true:

(a) $n \sum_{0 \leq s \leq t} |\Delta M_{w,v}^{(k,l)}(U,V)|^4 \to_p 0$ as $n \to \infty$.
(b) $n^{1/4} \langle M_{w,v}^{(k,l)}(U,V)^n, M[k'] \rangle_t \to_p 0$ for every $k'$.
(c) $n^{1/4} \langle M_{w,v}^{(k,l)}(U,V)^n, N \rangle_t \to_p 0$ as $n \to \infty$ for any one-dimensional square-integrable martingale $N$ on $B$ orthogonal to $M$.

It is worth pointing out that Lemma 3.4(b) is a direct consequence of Lemma 3.2. This type of phenomenon is also found in [13] and [19].

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3.6 Optimal weight function

We turn to question (b). Noting that $\phi_{g,g}' = -\phi_{g',g}'$, in the univariate and equidistant sampling case our estimator has the same asymptotic variance as that of the flat-top realized kernel with the kernel function $\phi_{g,g}$ and the bandwidth $k_n$. Here, the flat-top realized kernel with the kernel function $K$ and the bandwidth $H$ is defined by

$$RK(Y) = \gamma_0(Y) + \sum_{h=1}^{N_T-1} K\left(\frac{h-1}{H}\right)\left\{\gamma_h(Y) + \gamma_{-h}(Y)\right\}, \quad \gamma_h(Y) = \sum_{j=h+1}^{N_T} \Delta_j Y \Delta_{j-h} Y.$$

According to Proposition 1 of [2], in the parametric setting, i.e. both $\sigma$ and $\Upsilon$ are constant, the asymptotic variance of $RK(Y)$ is minimized by the kernel $K_{\text{opt}}(x) = (1+x)e^{-x}$ with the oracle bandwidth $H = (\sqrt{\Upsilon}/\sigma)\sqrt{N_T}$. Therefore, if there exists a function $g$ on $\mathbb{R}$ satisfying $|W|$ and $\phi_{g,g} = K_{\text{opt}}$, such a function $g$ is an optimal weight function. Fortunately, we can find such a $g$ by a simple Fourier analysis and it is given by $g(x) = e^{-|x|}$. In other words, the (twice) double exponential density function is an optimal weight function for our estimator. In this case our estimator achieves the parametric efficiency bound $8\sigma^3\sqrt{\Upsilon}$ of the asymptotic variance from [11] with the oracle tuning parameter $\theta = \sqrt{\Upsilon}/\sigma$.

Despite its efficiency, the optimal kernel $K_{\text{opt}}$ is not preferable in practice due to its computational disadvantage. That is, since the support of $K_{\text{opt}}$ is unbounded, it requires $n$ (all) realized autocovariances $\gamma_h(Y)$ to be computed. As a consequence, the order of the computation for $RK(Y)$ becomes $O(n^2)$. In contrast, our optimal weight function has a nice feature in terms of the computation. Let us define the sequences $(y_{p}^{+})_{p=k_{n}}^{N_T-1}$ and $(y_{p}^{-})_{p=k_{n}}^{N_T-1}$ recursively by $y_{k_{n}}^{\pm} = \tilde{\Delta}_{k_{n}} Y$, $y_{k_{n}+1}^{\pm} = \tilde{\Delta}_{N_T-k_{n}+1} Y$ and

$$y_{p}^{+} = e^{-1/k_{n}}y_{p-1}^{+} + \tilde{\Delta}_{p} Y, \quad y_{p}^{-} = e^{-1/k_{n}}y_{p-1}^{-} + \tilde{\Delta}_{N_T-p+1} Y, \quad p = k_{n} + 1, \ldots, N_T - k_{n} + 1.$$

Then it can easily be seen that $\tilde{Y}_{i:T} = y_{i}^{+} + y_{N_T-k_{n}+1}^{+} - \tilde{\Delta}_{i} Y$, hence we can compute $(\tilde{Y}_{i:T})_{i=k_{n}}^{N_T}$ with the order $O(n)$. Consequently, the order of the computation of our estimator is $O(n)$, which is, in general, even less than that of the MRC estimator with a weight function with a bounded support.

4 Examples of the observation times

In this section we give two illustrative examples of the observation times that satisfy the condition $[A4]$ and are possibly endogenous. We shall start to discuss a univariate example, i.e. we assume $d = 1$. Note that in this case we have $T_i = t_i^1$.

Example 4.1 (Times generated by hitting barriers). This example was treated in Section 4.4 of [9] and Example 4 of [20]. Suppose that $[A2]$ is satisfied and $\Sigma_t > 0$ for every $t$. Define

$$t_0^1 = 0, \quad t_{i+1}^1 = \inf\left\{t > t_i^1 : |M_t - M_{t_i^1}| = -\alpha/\sqrt{n} \text{ or } |M_t - M_{t_i^1}| = \beta/\sqrt{n}\right\}$$

for positive constants $\alpha, \beta$. Then, using a representation of a continuous local martingale with Brownian motion, we have

$$P\left(M_{t_{i+1}^1} - M_{t_i^1} = -\alpha/\sqrt{n}, F_{t_i^1}^{(0)}\right) = \beta/(\alpha + \beta), \quad P\left(M_{t_{i+1}^1} - M_{t_i^1} = \beta/\sqrt{n}, F_{t_i^1}^{(0)}\right) = \alpha/(\alpha + \beta).$$

Combining the above formula with Proposition 2.1 of [24] (again using a representation of a continuous local martingale with Brownian motion), we obtain the following result: for each $r \geq 1$ there exists a positive constant $C_r$ such that $E\left[\int_{t_i^1}^{t_{i+1}^1} \sum_s ds\right]^{r} \leq C_r n^{-r}$ for every $n, i$. In particular, this inequality implies that $r_n(T) = o_p(n^{-\xi})$ as $n \to \infty$ for any $\xi \in (0, 1)$ and $[A4]$ holds true with $H^n = F^{(0)}$ because we have $\inf_{0 \leq t \leq T} \Sigma_t > 0$. Moreover, noting that $E\left[\left|M_{t_{i+1}^1} - M_{t_i^1}\right|^2, F_{t_i^1}^{(0)}\right] = \Sigma_t E\left[t_{i+1}^1 - t_i^1, |F_{t_i^1}^{(0)}|\right] + o_p(n^{-3\xi/2})$ as $n \to \infty$ uniformly in $i \leq N_T$ for any $\xi \in (0, 1)$ by $[A2]$, we also obtain $[A4]$ with $G_t = \alpha\beta/\Sigma_t$. $[A4]$ is automatically satisfied. Finally, $[A4]$ is also satisfied because $M$ is continuous.
Remark. In Example 4.1, the stable convergence result of Theorem 3.1 still holds true when we replace \( M \) in (4.1) by \( X \). This can be shown as follows. First, without loss of generality we may assume \( d' = 1 \). Define the process \( Z \) by \( Z_t = \exp \left( \int_0^t \frac{a_s}{\sigma_s} \, dW_s - \frac{1}{2} \int_0^t a_s^2 / \sigma_s^2 \, ds \right) \) for each \( t \geq 0 \). As is well known, \( Z_t \) is a positive continuous local martingale. Therefore, by a localization argument we may assume that both \( Z \) and \( 1/Z \) are bounded. In particular, \( Z \) is a martingale, so that we can define a probability measure \( \tilde{P}_T \) on \( (\Omega, \mathcal{F}) \) by \( \tilde{P}_T(E) = P(1_E Z_T) \). \( \tilde{P}_T \) is obviously equivalent to the probability measure \( P \). Set \( W'_t = W_t - \int_0^t a_s / \sigma_s \, ds \) for each \( t \). Then, by Girsanov’s theorem \((W'_t)_{0 \leq t \leq T}\) is a standard Wiener process on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \tilde{P}_T)\) and it holds that \( X_t = \int_0^t \sigma_s \, dW'_s \). Hence [A4] holds true under \( \tilde{P}_T \). Moreover, [A1]–[A3] are obviously satisfied under \( \tilde{P}_T \). Therefore, (3.2) holds true under \( \tilde{P}_T \). Since the stable convergence is stable under equivalent changes of probability measures, (3.2) also holds true under the original probability measure \( P \). It is worth mentioning that the continuity condition on the drift \( a \) is unnecessary in this case.

Next we turn to the multi-dimensional and nonsynchronous case.

Example 4.2 (Continuous-time analog of the Lo-MacKinlay model). In Lo and MacKinlay [22] they regarded the nonsynchronicity of observations as a kind of missing observation. That is, they first considered a series \( r_1, r_2, \ldots \) of completely synchronous latent returns. Here, the random vector \( r_t = (r^1_t, \ldots, r^d_t) \) represents the vector constituted of the latent returns of \( d \) assets for the \( t \)-th period. In each period \( t \) there is some chance that the transaction of the \( k \)-th asset does not occur with certain probability \( 1 - p^k \). If it does not occur, the observed return \((r^k_t)^o\) of the \( k \)-th asset for the \( t \)-th period is simply 0. On the other hand, if its transaction occurs in the \( t \)-th period, \((r^k_t)^o\) becomes the sum of its latent returns for all past conservative periods in which its transaction has not occurred. Mathematically speaking, we have \((r^k_t)^o = \sum_{i=0}^{\infty} X^k_{i}(t) r^k_{t-i}, \) where \( X^k(t) = (1 - \delta^k) \prod_{j=1}^{\infty} \delta^k_{t-j} \) and \((\delta^k_t)\) is an i.i.d. sequence of Bernoulli variables with probabilities \( p^k \) and \( 1 - p^k \) of taking values 0 and 1, which are independent of the latent returns. In the following we consider a continuous-time analog of this model.

Let \((\Omega', \mathcal{F}', (\mathcal{F}'_t), P')\) be a stochastic basis and suppose that \( X^1, \ldots, X^d \) are defined on this basis. Suppose also that \( \tilde{Y} = (\mathcal{F}'_t) \)-adapted. On the other hand, suppose that for each \( k = 1, \ldots, d \) we have a sequence \((\delta^k_m)_{m=0}^{\infty}\) of i.i.d. random variables on a probability space \((\Omega'', \mathcal{F}'', P'')\). Suppose also that \((\delta^1_m), \ldots, (\delta^d_m)\) are mutually independent and \( P(\delta^m_0 = 0) = 1 - P(\delta^m_1 = 1) = p^k \in [0, 1) \). Then we define the stochastic basis \( \mathcal{B}^{(0)} \) by

\[
\Omega^{(0)} = \Omega' \times \Omega'', \quad \mathcal{F}^{(0)} = \mathcal{F}' \otimes \mathcal{F}'', \quad \mathcal{F}'_t^{(0)} = \mathcal{F}'_t \otimes \mathcal{F}'', \quad P^{(0)} = P' \times P''.
\]

Let \((\tau_i)_{i=0}^{\infty}\) be a sampling scheme such that \( \tau_i \) is an \((\mathcal{F}'_t)\)-predictable time for every \( i \). Suppose also that \( \sup_{i \geq 0} (\tau_i \wedge T - \tau_{i-1} \wedge T) = o_p(n^{-E}) \) as \( n \to \infty \) for some \( \xi \in (0, 1) \). Now set \( M(i)^k = \min\{M | \sum_{m=0}^{M} (1 - \delta^k_m) = i\} \) and define \((t^k_i)\) by \( t^i_k = \tau_{M(i)^k} \). By construction \( t^i_0, t^i_1, \ldots \) are \( \mathcal{B}^{(0)} \)-predictable times and satisfy \( t^i_k \to \infty \) as \( i \to \infty \). Now let \( H^n \) be the filtration generated by the processes \( a_t, \sigma_t, W_t \) and \( \sum_i 1(t^i_k \leq t) \) \((k = 1, \ldots, d)\) and suppose that \([A4](ii)-(iii)\) are satisfied with replacing \((T_i)\) by \((\tau_i)\). Then, noting that \( M^k := (M(i + 1)^k - M(i)^k)_{i=0}^{\infty}\) is a sequence of independent and geometrically distributed random variables with the common success probability \( p^k \) and that \( M^1, \ldots, M^d \) are mutually independent, it can easily be shown that \([A4]\) holds true with

\[
G_s = \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} (-1)^{k-1} G^0_s \quad \text{and} \quad \chi^{kl}_s = \begin{cases} 1 & \text{if } k = l, \\ p^k p^l / (p^k + p^l - p^k p^l) & \text{otherwise}. \end{cases}
\]

Here, \( G^0 \) denotes the asymptotic conditional expected duration process corresponding to \((\tau_i)\). By taking an endogenous sampling scheme as the underlying sampling scheme \((\tau_i)\), we can obtain endogenous observation times.

5 Discussion and application

5.1 Connection with Li-Zhang-Zheng’s result

In this subsection we shall discuss how our conditions connect with the quantities (1.1) introduced by [21]. For simplicity we focus on the univariate case. First, in the light of the argument from Section 3.5 a counterpart of the
first quantity of Eq. (1.1) in our situation is given by
\[
    n^{1/2} \sum_{q=k_n+2}^{N_n} \left\{ \sum_{p=k_n \vee (q-d_n)}^{q-2} c_{u,v}(p,q)X(I_p) \right\}^2 X(I_q)^2.
\]
An easy computation shows that the above quantity has the same probability limit as that of \( n^{1/2} \langle M_{y,q}^{1,1} (M, M)^n \rangle_1 \). Therefore, Lemma 3.3 gives us the probability limit of the above quantity under our conditions. On the other hand, in our situation a counterpart of the second quantity of Eq. (3.1) in our situation a counterpart of the second quantity of Eq. (3.1) in our situation is given by
\[
    \left( \frac{X_{i,T}}{\sqrt{n}} \right) \rightarrow \mathcal{D} \end{array}
\]
We give a proof of Proposition 5.1 in Section 7.6. This result suggests that the tricity of the pre-averaging of \( X \) asymptotically vanishes even if the returns of \( X \) have non-zero skewness.

5.2 A feasible central limit theorem
The central limit theorem derived in Section 3 is infeasible in the sense that the asymptotic covariance matrix of the estimation error is unobservable. In order to derive a feasible central limit theorem, we therefore need an estimator for it. In this subsection we implement this with a (naïve) kernel-based approach as in Section 8.2 of [13] and the second estimator in Section 4 of [7].

We construct kernel estimators for the quantities appearing in the asymptotic variance (3.3) in the following way. Let \( (h_n)_{n=1}^{\infty} \) be a sequence of positive numbers tending to 0 as \( n \to \infty \), and define
\[
    \hat{\Sigma}_t^n = h_n^{-1} \left( \hat{\text{MRC}}[Y]_t^n - \hat{\text{MRC}}[Y]_{(t-h_n)+}^n \right), \quad \partial[Y]_t^n = h_n^{-1} \left( [Y]_t^n - [Y]_{(t-h_n)+}^n \right) / (2k_n^2)
\]
further for each \( t \in [0,T] \). The following lemma is shown in Section 7.7:

Lemma 5.1. Suppose that \([W], [H1]-[H2] and [A1]-[A4] are satisfied. Suppose also that the constant \( \rho \) from \([A4](ii)] satisfies \( \rho \geq 4/3 \) and that \( n^{1/4} h_n \to \infty \) as \( n \to \infty \). Then, \( \hat{\Sigma}_t^n \to_p \Sigma_t \) and \( \partial[Y]_t^n \to_p \theta^{-2} \hat{\Upsilon}_t / G_t \) as \( n \to \infty \) for any \( t \in (0,T) \). Furthermore, \( \sup_{0 \leq t \leq T} |\hat{\Sigma}_t^n| \) and \( \sup_{0 \leq t \leq T} |\partial[Y]_t^n| \) are tight as \( n \to \infty \).

According to the above lemma, we can construct a kernel-based estimator for the asymptotic covariance matrix as follows. Define
\[
    \hat{\Sigma}_t^{kl,k'l'} = \sqrt{n} h_n^{k'k'} \left( \frac{2}{\psi_n^2} \sum_{T_p} \left\{ \frac{\hat{\Sigma}^{n, kl}_{T_p}}{\hat{\Sigma}^{n, ll'}_{T_p}} + \frac{\hat{\Sigma}^{n, k'l'}_{T_p}}{\hat{\Sigma}^{n, ll'}_{T_p}} \right\} \right) \left( \frac{\partial[Y]_T^n}{\partial[Y]_{T_p}^{n, k'l'}} + \frac{\partial[Y]_T^n}{\partial[Y]_{T_p}^{n, k'}} + \frac{\partial[Y]_T^n}{\partial[Y]_{T_p}^{n, l'}} + \frac{\partial[Y]_T^n}{\partial[Y]_{T_p}^{n, ll'} + \partial[Y]_T^n / \partial[Y]_{T_p}^{n, k'l'}} \right) |I_{p+1}|
\]
for each \( p \geq 1 \) and every \( k, l, k', l' \), and set \( \hat{\text{avar}}_{t}^{kl,k'l'} = \sum_{p=1}^{N_n-1} \hat{\Sigma}_t^{kl,k'l'} |I_p| \). Then we obtain the following result:

Theorem 5.1. Under the assumptions of Lemma 5.1, it holds that \( \hat{\text{avar}}_{t}^{kl,k'l'} \to_p \int_0^T \hat{\Sigma}_s^{kl,k'l'} ds \) as \( n \to \infty \), where \( \hat{\Sigma}_s^{kl,k'l'} \) is the quantity in the right hand side of (3.3).

The proof is in Section 7.8. Combining Theorem 5.1 with Theorem 3.1, we obtain the following feasible central limit theorem:
Corollary 5.1. Define the $d^2 \times d^2$ random matrices $\hat{\Sigma}$ and $\Sigma$ by
\[
\hat{\Sigma}^{d(l-1)+k,d(l'-1)+k'} = n^{-\frac{1}{2}}\text{diag}(k,l,k',l') \quad \text{and} \quad \Sigma^{d(l-1)+k,d(l'-1)+k'} = \int_0^T \Omega_s^{k,l,k',l'} \, ds
\]
for $k,l,k',l' = 1, \ldots, d$ respectively. Under the assumptions of Theorem 5.1, we have
\[
\hat{\Sigma}^{-\frac{1}{2}} \left\{ \text{vec} \left( \text{MRC}\{Y\}_{1:T} \right) - \text{vec} \left( \{X\}_{1:T} \right) \right\} \rightarrow_{d^2} N_{d^2}(0, I_{d^2})
\]
as $n \to \infty$ whenever $d > 0$ a.s., where $N_{d^2}(0, I_{d^2})$ denotes the $d^2$-dimensional standard normal distribution.

6 Simulation study

In this section we supplement the asymptotic theory developed in this paper with a simulation analysis to illustrate the finite sample accuracy. We focus on the estimation of integrated covariances and assess the accuracy of the infeasible and feasible central limit theorems in finite samples.

6.1 Simulation design

We simulate data for one day, i.e. $T = 1$. Following [3] and [6], we consider the bivariate one factor stochastic model defined by the stochastic differential equation
\[
dX_t^k = \mu^k dt + \rho^k \sigma^k_i B_t^k + \sqrt{1 - (\rho^k)^2} \sigma^k_i W_t, \quad \sigma_t^k = \exp(\beta_0^k + \beta_1^k \lambda_t), \quad d\lambda_t = \alpha^k \eta^k dt + dB_t^k, \quad k = 1, 2,
\]
where $(B^1, B^2, W)$ is a 3-dimensional standard Wiener processes. The specification of the parameters in the model, which are assumed to be identical across the two volatility factors, is as follows: $(\mu^k, \beta_0^k, \beta_1^k, \alpha^k, \rho^k) = (0.03, -5/16, 1/8, -1/40, -0.3)$. The initial values $\lambda_0^k$ for the processes $\lambda_t^k$ at each simulation run are sampled independently from a normal distribution $N(0, (-2\alpha^k)^{-1})$, which is their stationary distribution.

To generate observation times, we consider Lo-MacKinlay type sampling schemes illustrated in Example 4.2. The underlying observation times $(\tau_i)_{i=0}^\infty$ are defined as follows:
\[
\tau_0 = 0, \quad \tau_{i+1} = \inf\{ t > \tau_i | W_t - W_{\tau_i} = 2\sqrt{n}(t - \tau_i) - 2/\sqrt{n} \}, \quad i = 0, 1, \ldots,
\]
where $n = 23,400$. This implies that the sequence $(\tau_{i+1} - \tau_i)_{i=0}^\infty$ is independent and identically distributed with the inverse Gaussian distribution $IG(2/\sqrt{n}, 2\sqrt{n})$, where the probability density function of the inverse Gaussian distribution $IG(\delta, \gamma)$ is given by
\[
p(z; \delta, \gamma) = \frac{\delta \gamma z^{-3/2}}{\sqrt{2\pi} \gamma} \exp \left\{ -\frac{1}{2} \left( \frac{\delta}{z} + \gamma \frac{z}{\gamma} \right) \right\}, \quad z > 0.
\]
In particular, the expected values of the number of observations and the durations for $(\tau_i)$ are given by $n$ and $1/n$, respectively. Moreover, the skewness and the kurtosis of $W_{\tau_{i+1}} - W_{\tau_i}$ are equal to $3/2$ and $27/4$, respectively. Therefore, the returns of the factor Wiener process $W$ have the same skewness as that of the model simulated in Section 5 of [20]. Here, unlike [20] we use a sampling scheme generated by hitting a line to avoid a numerical issue. In fact, in our model the distribution of observation times can exactly be simulated via generating inverse Gaussian random variables, so we do not suffer from any numerical error. The parameter $\lambda := (1/p^1, 1/p^2)$ from Example 4.2, which controls the probabilities of the occurrences of observations, is varied through \{(3, 6), (5, 10), (10, 20), (30, 60), (60, 120)\}.

In constructing noisy prices $Y$, we first generate a discretized path $X_{\tau_i}, X_{\tau_{i+1}}$ of $X$ using a standard Euler scheme. Here, it is worth mentioning that the Ornstein-Uhlenbeck process permits an exact discretization (see p.110 of [10] for instance). We use that fact here to avoid discretization errors in approximating the distribution of $d\theta_t^k$. After that, we add simulated microstructure noise $Y = X + \epsilon$ by taking
\[
\epsilon_{k,l|\{\sigma, X\}^{i,d}, \tau_i} \sim N(0, \Theta_{kk}) \quad \text{with} \quad \Theta_{kk} = \gamma^2 \sum_i \left( \sigma_{\tau_i}^k \right)^4 (\tau_i - \tau_{i-1}),
\]
where \( \epsilon^1 \) and \( \epsilon^2 \) are assumed to be mutually independent. The noise-to-signal ratio, \( \eta^2 \), takes the values 0.001 or 0.01. This choice again follows [3] and [6]. Simulation results are based on 10,000 Monte Carlo iterations for each scenario.

**Remark.** As was shown in [21], in finite samples the time endogeneity could have an impact when the model has a certain excessive feature (e.g., the drift is very large). Here we do not aim to investigate such an extreme case, so we simulate a standard model used in the literature as above.

### 6.2 Finite sample adjustments

It is well-known in the literature of pre-averaging estimators (see e.g. Section 4.2 of [15]) that some adjustments play important roles to avoid biases in small samples. Here, we explain such corrections used in this study.

First, we replace the constants \( \psi_1, \psi_2, \Phi_{11}, \Phi_{12} \) and \( \Phi_{22} \) by their Riemann approximations:

\[
\psi_1^{k_n} = k_n \sum_{p=-\infty}^{\infty} \left( e^{-\frac{i(p+1)}{k_n}} - e^{-\frac{ip}{k_n}} \right)^2 = 2k_n \frac{1-e^{-\frac{1}{k_n}}}{1+e^{-\frac{1}{k_n}}}, \\
\psi_2^{k_n} = \frac{1}{k_n} \sum_{p=-\infty}^{\infty} e^{-\frac{2ip}{k_n}} = \frac{1+e^{-\frac{2}{k_n}}}{k_n}, \\
\Phi_{11}^{k_n} = \frac{1}{k_n} \sum_{p=0}^{\infty} \left\{ \left( 1 - \frac{p}{k_n} \right) e^{-\frac{p}{k_n}} \right\}^2 = \alpha_n - 2\beta_n + \gamma_n, \\
\Phi_{12}^{k_n} = \frac{1}{k_n} \sum_{p=0}^{\infty} \left( \frac{p}{k_n} e^{-\frac{p}{k_n}} \right)^2 = \gamma_n, \\
\Phi_{22}^{k_n} = \frac{1}{k_n} \sum_{p=0}^{\infty} \left( 1 + \frac{p}{k_n} \right) e^{-\frac{p}{k_n}} = \alpha_n + 2\beta_n + \gamma_n.
\]

Here, we set \( \alpha_n = \{ (k_n - 1) e^{-\frac{2}{k_n}} \}^{-1} \), \( \beta_n = e^{-\frac{2}{k_n}} \alpha_n^2 \) and \( \gamma_n = (e^{-\frac{2}{k_n}} e^{-\frac{1}{k_n}}) \alpha_n^3 \). Next, following [15] and [6],

\[
\tilde{\text{MRC}}_{\text{adj}}[Y]_T^n = \left( 1 - \frac{\psi_1^{k_n}}{\psi_2^{k_n} k_n^2} \right)^{-1} \left\{ \frac{N_T^n}{N_T^n - 2k_n + 2} \frac{1}{\psi_2^{k_n} k_n} \sum_{i=k_n}^{N_T^n - k_n + 1} \tilde{Y}_{i,T} \tilde{Y}_{i,T}\right\} - \frac{\psi_1^{k_n}}{2 \psi_2^{k_n} k_n^2} |Y|_T^n.
\]

The factor \( N_T^n/(N_T^n - 2k_n + 2) \) of the first term from the right hand side of the above definition is a correction for the true number of its summands. The factor \( (1 - \psi_1^{k_n}/(\psi_2^{k_n} k_n^2))^{-1} \) corrects a finite sample bias due to the fact that \( [X]_T^n \) converges to \( [X]_T \) in the synchronous sampling case. Finally, we also correct the estimator \( \tilde{\text{avar}}_{kl,k'l'} \) of asymptotic covariance in a similar manner as follows. Set

\[
\tilde{\Sigma}_t^n = h_n^{-1} \left( \tilde{\text{MRC}}_{\text{adj}}[Y]_t^n - \tilde{\text{MRC}}_{\text{adj}}[Y]_{(t-h_n)^+} \right) \quad \text{and} \quad \tilde{\partial}[Y]_t^n = \partial[Y]_t^n - \tilde{\Sigma}_t^n.
\]

Then we define \( \tilde{\text{avar}}_{kl,k'l'}^{p,k,k'} \) by (5.1) with replacing \( \psi_2, \Phi_{11}, \Phi_{12}, \Phi_{22}, \tilde{\Sigma}^n \) and \( \partial[Y]_t^n \) by \( \psi_1^{k_n}, \Phi_{11}^{k_n}, \Phi_{12}^{k_n}, \Phi_{22}^{k_n}, \tilde{\Sigma}^n \) and \( \tilde{\partial}[Y]_t^n \), respectively. Now, noting that \( \tilde{\Sigma}_t^n = \partial[Y]_t^n = 0 \) if \( t \leq h_n \), we define

\[
\tilde{\text{avar}}_{kl,k'l'}^{p,k,k'} = \left( 1 - \frac{\psi_1^{k_n}}{\psi_2^{k_n} k_n^2} \right)^{-2} \frac{N_T^n}{N_T^n - k_n + 1 - N_{h_n}} \sum_{p=1}^{N_T^n - 1} \tilde{\Phi}_{kl,k'l'}^{p,k,k'} |I_p|.
\]

### 6.3 Results

In the following the MRC estimator is implemented with using the window size \( k_n = \lfloor 0.1 \sqrt{N_T^n} \rfloor \) and the weight function \( g(x) = e^{-x^2} \). First we assess the accuracy of the standard normal approximation of the infeasible standardized statistic

\[
\frac{n^{1/4} \tilde{\text{MRC}}[Y]_1^{n,12} - [X]_1^{22}}{\text{avar}_{12,12}},
\]

where

\[
\text{avar}_{12,12} = \frac{2}{\psi_2^{k_n}} \int_0^1 \left[ \theta_n \Phi_{22} (\Sigma_{11}^{12} + \Sigma_{22}^{12}) G + \frac{\Phi_{22}^{k_n}}{\theta_n} (\Sigma_{11}^{11} + \Sigma_{22}^{22} + \Sigma_{11}^{12} + \Sigma_{22}^{12} + \Sigma_{11}^{12} + \Sigma_{22}^{12}) \right] ds,
\]

and

\[
\text{avar}_{12,12} = \frac{2}{\psi_2^{k_n}} \int_0^1 \left[ \theta_n \Phi_{22} (\Sigma_{11}^{12} + \Sigma_{22}^{12}) G + \frac{\Phi_{22}^{k_n}}{\theta_n} (\Sigma_{11}^{11} + \Sigma_{22}^{22} + \Sigma_{11}^{12} + \Sigma_{22}^{12} + \Sigma_{11}^{12} + \Sigma_{22}^{12}) \right] ds.
\]
Table 1: Simulation results of the standardized estimates

| Infeasible | Feasible |
|------------|----------|
|            | Mean SD (95%) (99%) | Mean SD (95%) (99%) |
| \( \eta^2 = 0.001 \) | | |
| \( \lambda = (3, 6) \) | -.001 .990 95.34% 98.92% | -.096 .992 95.30% 98.61% |
| \( \lambda = (5, 10) \) | -.002 .992 95.28% 98.92% | -.116 .996 95.27% 98.52% |
| \( \lambda = (10, 20) \) | -.008 .992 95.40% 98.87% | -.155 1.011 95.30% 98.61% |
| \( \lambda = (30, 60) \) | -.025 .981 95.81% 98.84% | -.217 1.032 95.30% 98.61% |
| \( \lambda = (60, 120) \) | -.049 .984 96.05% 98.60% | -.331 1.099 95.12% 96.02% |

Note. We report the sample mean, standard deviation (SD) as well as the 95% and 99% coverages of the standardized statistics \((6.1)\) (left panel) and \((6.2)\) (right panel) included in the simulation study.

\[ G = \frac{1}{p^1} + \frac{1}{p^2} - \frac{1}{p^1}p^2 \] and \( \theta_n = k_n/\sqrt{n} \). Note that we replace the constant \( \theta \) in the theoretical asymptotic variance of \( \tilde{M}_{n,1}^{\alpha n,12} \) by its finite sample analog \( \theta_n \), again following [15]. We report the sample mean and standard deviation as well as 95% and 99% coverages of \((6.1)\) in the left panel of Table 1. As the table reveals, the central limit theorem for \((6.1)\) fairly works. Especially, it provides good approximations for relatively large sample sizes, i.e. for \( \lambda = (3, 6), (5, 10) \) and \( (10, 20) \).

Next, we turn to the accuracy of the asymptotic approximation of the feasible standardized statistic

\[ n^{1/4} \tilde{M}_{n,1}^{\alpha n,12} - [X]_{12}^{12} \sqrt{\tilde{\text{avar}}^{12,12}} \] \((6.2)\)

Here, \( h_n = (N^n)_{12}^{-1/5} \) is used as the bandwidth parameter. The results are reported in the right panel of Table 1. Again, the central limit theorem starts to work for relatively large sample sizes. Compared with the infeasible case, the sample mean of \((6.2)\) is remarkably downward biased. A similar observation is also found in [15]. According to that paper, this is explained by a (small) positive correlation between the estimator \( \tilde{M}_{n,1}^{\alpha n,12} \) and \( \tilde{\text{avar}}^{12,12} \). On the other hand, the standard deviation and the coverages of \((6.2)\) perform well at relatively high frequencies.

7 Proofs

7.1 Preliminaries

7.1.1 Asymptotic behavior of the point process \( N^n_t \)

The following lemma is a generalization of Lemma 2.3 of Fukasawa [9] and repeatedly used throughout this section.

Lemma 7.1. Consider a sequence \( F^\Omega = (\overline{\mathcal{F}}_j)_{j \in \mathbb{Z}_+} \) of filtrations and random variables \((\zeta^n_j)_{j \in \mathbb{N}}\) adapted to the filtration \( F^\Omega \) for each \( n \). Let \( \Lambda \) be a non-empty set and suppose that a non-negative integer-valued random variable \( N^n(\lambda) \) is given for each \( n \in \mathbb{N} \) and each \( \lambda \in \Lambda \). Suppose also that there exists an element \( \lambda_0 \in \Lambda \) satisfying (i) \( N^n(\lambda) \leq N^n(\lambda_0) \) a.s. for all \( \lambda \in \Lambda \) and (ii) \( N^n(\lambda_0) \) is an \( F^\Omega \)-stopping time. Then, the following statements hold.
true for any $\varpi \in [1, 2]$:

(a) if $\sum_{j=1}^{N_n(\lambda)} E \left[ |\zeta_j^n|^\varpi \right] \rightarrow 0$ as $n \rightarrow \infty$, then $\sup_{\lambda \in \Lambda} \left| \sum_{j=1}^{N_n(\lambda)} \left\{ \zeta_j^n - E \left[ \zeta_j^n \mathbb{1}_{F_{j-1}} \right] \right\} \right| \rightarrow 0$ as $n \rightarrow \infty$.

(b) if $\sum_{j=1}^{N_n(\lambda)} E \left[ |\zeta_j^n|^\varpi \right] = O_p(1)$ as $n \rightarrow \infty$, then $\sup_{\lambda \in \Lambda} \left| \sum_{j=1}^{N_n(\lambda)} \left\{ \zeta_j^n - E \left[ \zeta_j^n \mathbb{1}_{F_{j-1}} \right] \right\} \right| = O_p(1)$ as $n \rightarrow \infty$.

**Proof.** Note that

$$
\sup_{\lambda \in \Lambda} \left| \sum_{j=1}^{N_n(\lambda)} \left\{ \zeta_j^n - E \left[ \zeta_j^n \mathbb{1}_{F_{j-1}} \right] \right\} \right| \leq \sup_{1 \leq k \leq N_n(\lambda)} \left| \sum_{j=1}^{k} \eta_j^n \right|,
$$

where $\eta_j^n = \zeta_j^n - E \left[ \zeta_j^n \mathbb{1}_{F_{j-1}} \right]$.

Let $\tau$ be a bounded stopping time with respect the filtration $\mathbb{F}^n$. Then the Burkholder-Davis-Gundy (henceforth BDG) inequality and the $C_p$-inequality yield

$$
E \left[ \left| \sum_{j=1}^{\tau} \eta_j^n \right|^{\varpi} \right] \leq CE \left[ \sum_{j=1}^{\tau} \left\{ |\zeta_j^n|^\varpi + E \left[ |\zeta_j^n \mathbb{1}_{F_{j-1}}|^\varpi \right] \right\} \right]
$$

for some positive constant $C$ independent of $n$. Since $E \left[ \sum_{j=1}^{\tau} |\zeta_j^n|^\varpi \right] = E \left[ \sum_{j=1}^{\tau} E \left[ |\zeta_j^n|^\varpi \mathbb{1}_{F_{j-1}} \right] \right]$ by the optional stopping theorem and $E \left[ |\zeta_j^n \mathbb{1}_{F_{j-1}}|^\varpi \right] \leq E \left[ |\zeta_j^n|^\varpi \mathbb{1}_{F_{j-1}} \right]$ by the Lyapunov inequality, we obtain

$$
E \left[ \left| \sum_{j=1}^{\tau} \eta_j^n \right|^{\varpi} \right] \leq 2CE \left[ \sum_{k=1}^{\tau} E \left[ |\zeta_j^n|^\varpi \mathbb{1}_{F_{j-1}} \right] \right].
$$

Therefore, we obtain the desired result due to the Lenglart inequality.

The next lemma describes the asymptotic behavior of the point process $N_t^n$.

**Lemma 7.2.** Suppose that [H1]–[H2] and [A4] are satisfied. Let $H$ be a càdlàg process which is adapted to $\mathbb{H}^n$ for all $n$. Then we have

$$
\sup_{0 \leq s \leq t} \left| \frac{1}{n} \sum_{p=1}^{N_t^n} H_{T_{p,s}} - \frac{1}{n} \sum_{p=1}^{N_t^n} \frac{H_{T_{p,s}} - I_p}{G_{T_{p,s}}} \right| = O_p \left( n^{\frac{1}{2}} \right)
$$

(7.1)

as $n \rightarrow \infty$. Moreover, it holds that

$$
\frac{1}{n} N_t^n \rightarrow^p \int_0^t \frac{1}{G_s} ds
$$

(7.2)

as $n \rightarrow \infty$ for any $t \geq 0$.

**Proof.** First we show that $n^{-1} N_p^n = O_p(1)$ as $n \rightarrow \infty$. Take $L > 0$ arbitrarily and set

$$
S_L = \{ D(1)_p \leq L, G_{T_{p,s}} \geq L^{-1}, #N^n \leq Ln^{1/\rho} \}.
$$

Then we have

$$
E \left[ \sum_{p=1}^{\infty} 1_{\{T_p \leq T\} \cap S_L} \right] = E \left[ \sum_{p=1}^{\infty} \frac{D(1)_p}{G_{T_{p,s}}} 1_{\{T_p \leq T\} \cap S_L} \right] \leq E \left[ \sum_{p=1}^{\infty} \frac{D(1)_p}{G_{T_{p,s}}} 1_{\{T_p \leq T,D(1)_p \leq L,G_{T_{p,s}} \geq L^{-1}\}} \right] + Ln^{1/\rho}
$$

$$
\leq LnE \left[ \sum_{p=1}^{N_p^n} 1_{\{I_{p+1} \leq T,D(1)_p \leq L\}} \right] + Ln^{1/\rho} \leq Ln \left\{ E \left[ \sum_{p=1}^{N_p^n} |I_p| \right] + E \left[ \sum_{p=1}^{N_p^n} |I_{p+1}| 1_{\{D(1)_p \leq L\}} \right] \right\} + Ln^{1/\rho}
$$

$$
\leq LnT + L^2 + Ln^{1/\rho}.
$$
Therefore, noting that $N_T^n = \sum_{p=1}^\infty 1_{\{T_p \leq T\}}$, Chebyshev’s inequality yields

$$\limsup_{K \to \infty} \limsup_{n \to \infty} P(b_n N_T^n > K) \leq \limsup_{n \to \infty} P\left(\bigcup_{p=1}^\infty \{T_p \leq T\} \cap S_L^n\right).$$  \hspace{1cm} (7.3)

On the other hand, since it holds that

$$\bigcup_{p=1}^\infty \{T_p \leq T\} \cap S_L^n \subset \left\{ \sup_{0 \leq p \leq N_T^n} D(1)_p^n > L \right\} \cup \left\{ \inf_{0 \leq \omega \leq T} G^n_{1\omega} < L^{-1} \right\} \cup \{\#N^n > Ln^{1/\rho}\},$$

noting that $D(1)_p^n \leq \{D(\rho)_p^n\}^{1/\rho}$ by the Hölder inequality, [A4] implies that the quantity in the right hand side of Eq.(7.3) tends to 0 as $L \to \infty$. Consequently, we conclude that $n^{-1}N_T^n = O_p(1)$ as $n \to \infty$.

Next we prove (7.1). Set $\varpi = 2 \land \rho$. The assumptions and the fact that $n^{-1}N_T^n = O_p(1)$ yield

$$\sum_{p=1}^{N_T^n+1} E\left[n^{1-\frac{1}{1+\rho}} D(1)_{p-1} \right] \leq n^{-1} \sum_{p=1}^{N_T^n+1} H_{T_{p-1}}^n D(\varpi)^{n}_{p-1} = O_p(1),$$

hence Lemma 7.1(b) yields

$$\sup_{0 \leq t \leq T} n^{1-\frac{1}{1+\rho}} \sum_{p=1}^{N_T^n+1} \frac{H_{T_{p-1}}^n}{G_{T_{p-1}}^n} |I_p| - \frac{1}{n} \sum_{p=1}^{N_T^n+1} \frac{H_{T_{p-1}}^n}{G_{T_{p-1}}^n} D(1)^{n}_{p-1} = O_p(1).$$

Therefore, [A4](iii) implies that (7.1).

Finally, (7.2) immediately follows from (7.1) and [A4](ii). \hfill \Box

### 7.1.2 Localization

A standard localization procedure, described in detail in Lemma 4.4.9 of [17], for instance, allows us to systematically replace the conditions [A1]–[A3] by the following strengthened versions:

[SA1] We have [A1], and the process $a$ is bounded.

[SA2] We have [A2], and the process $\sigma$ is bounded.

[SA3] We have [A3], and the process $(\int \Vert z \Vert^8 Q_t(dz))_{t \geq 0}$ is bounded.

Next we introduce a strengthened version of [A4]. Set $\bar{\tau}_n = n^{-\xi}$.

[SA4] We have [A4], and for every $n$ it holds that

$$\sup_{p \geq 0} (T_p - T_{p-1}) \leq \bar{\tau}_n.$$ \hspace{1cm} (7.4)

**Lemma 7.3.** Assume [H1]–[H2] and [A4]. One can find sampling schemes $(\bar{T}_p)$ and $(\bar{\tau}_p^k)$ ($k = 1, \ldots, d$) satisfying the following conditions:

(i) $(\bar{T}_p)$ and $(\bar{\tau}_p^k)$ satisfy [H1] and [SA4] (with the limiting processes $\bar{G}$ and $\bar{\chi}^k$). Moreover, it holds that $G_t = \bar{G}_t$ and $\chi^k_t = \bar{\chi}^k_t$ for every $t \in [0, T]$ and for every $k, l$.

(ii) $\bar{T}_p$ and $\bar{\tau}_p^k$ are $H^n$-stopping times for every $k, p$.

(iii) There exists a subset $\Omega_n$ of $\Omega$ such that $\lim_n P(\Omega_n) = 0$. Moreover, on $\Omega_n$ we have $T_p \land T = \bar{T}_p \land T$ and $\tau_p^k(\omega) \land T = \bar{\tau}_p^k(\omega) \land T$ for every $k, p$.

**Proof.** Set $R_n = \inf\{t| r_n(t) > \bar{\tau}_n\}$. Since $(r_n(t))_{t \geq 0}$ is an $F(0)$-adapted continuous process, $R_n$ is an $F(0)$-predictable time. Moreover, $\Omega_n = \{R_n > T\}$ satisfies $\lim_n P(\Omega_n) = 0$ by [A4](i). Now we define $(\bar{T}_p)_{p=1}^\infty$ sequentially by $\bar{T}_{-1} = 0$ and

$$\bar{T}_p = \begin{cases} T_p \land R_n, & \text{if } T_{p-1} < R_n, \\ \bar{T}_{p-1} + n^{-1}, & \text{otherwise.} \end{cases}$$
Since we can rewrite $\tilde{T}_p$ as
\[ \tilde{T}_p = (T_p \wedge R_n) \{ T_{p-1} < R_n \} \wedge \left( \tilde{T}_{p-1} \vee R_n + n^{-1} \right) \{ T_{p-1} \geq R_n \}, \] (7.5)
$\tilde{T}_p$ is an $\mathcal{F}^{(0)}$-stopping time. Then it is obvious that $(\tilde{T}_p)$ is a sampling scheme and satisfies (7.4). After that, for each $k$ we define $(\tilde{\tau}_p^k)_{p=1}^{\infty}$ sequentially by $\tilde{\tau}_- = 0$ and
\[ \tilde{\tau}_p^k = \begin{cases} \tau_p^k \wedge R_n, & \text{if } T_{p-1} < R_n, \\ \tilde{T}_p, & \text{otherwise}. \end{cases} \]

Since $\tau_p^k$ has a similar representation to Eq. (7.5), it is an $\mathcal{F}^{(0)}$-predictable time by $[A4](v)$. Moreover, it is evident that $(\tilde{T}_p)$ and $(\tilde{\tau}_p^k)$ satisfy $[H1]$, (ii) and (iii).

Next, for each $n \geq 1$ and each $k, l = 1, \ldots, d$ we define the $\mathcal{H}^n$-optional processes $\tilde{G}^n$ and $\tilde{\chi}^{n,kl}$ by
\[ \tilde{G}^n_t = G^n_t \mathbf{1}_{[0,R_n]}(t) + 1_{[R_n,\infty)}(t), \quad \tilde{\chi}^{n,kl}_t = \chi^{n,kl}_t \mathbf{1}_{[0,R_n]}(t) + 1_{[R_n,\infty)}(t). \]
Set $\tilde{T} = [\tilde{T}_{p-1}, \tilde{T}_p)$. By the construction $|\tilde{T}|$ is equal to $|T|$ on the set $\{ T_p < R_n \}$, and to $n^{-1}$ on the set $\{ T_{p-1} \geq R_n \}$. Therefore, with setting $\tilde{N}^n = N^n \cap \{ p \in \mathbb{N} | T_{p-1} < R_n \leq T_p \}$, we have $\tilde{G}^{(\rho)}_{|\tilde{T}|} = E\left[ \left( n|\tilde{T}| \right)^\rho |H^{n}_{\tilde{T}-1} \right]$ for every $p \in \mathbb{N} - \tilde{N}^n$. Similarly, we also have $\tilde{\chi}^{n,kl}_{\tilde{T}-1} = P \left( \tilde{\tau}^k = \tilde{T} | H^{n}_{\tilde{T}-1} \right)$ for every $p \in \mathbb{N} - \tilde{N}^n$. Now we can easily see that $(\tilde{T}_p)$ and $(\tilde{\tau}_p^k)$ satisfy $[A4]$ and (i), and thus the proof is completed.

A standard localization argument, based upon the above lemma, allows us to replace $[A4]$ by $[SA4]$.

The last one is based on the following lemma:

**Lemma 7.4.** We have a.s.
\[ \limsup_{h \to 0} \sup_{s,u \in [0,t]} \frac{|M^s - M^u|}{\sqrt{2h \log h}} \leq \sup_{0 \leq s \leq t} |\Sigma^k_s| \]
for any $t > 0$ and every $k = 1, \ldots, d$.

**Proof.** Combining a representation of a continuous local martingale with Brownian motion and Lévy’s theorem on the uniform modulus of continuity of Brownian motion, we obtain the desired result.

Now suppose that there exists a positive constant $K$ such that $\sup_{0 \leq s \leq t} |\Sigma_t| \leq K$ for all $t > 0$. Set
\[ R_m = \inf \left\{ t \in (0, \infty) \left| \inf_{h \in (0, \frac{1}{m})} \max_{k=1,\ldots,d} \sup_{s,u \in [0,t]} \frac{|M^k_s - M^k_u|}{\sqrt{2h \log h}} > K + 1 \right\} \]
for each $m \in \mathbb{N}$. Then $R_m$ is a stopping time since $M^k$ is continuous and adapted, and $R_m \uparrow \infty$ a.s. by Lemma 7.4. This implies that under $[SA2]$ we can always assume that there exist positive constants $K$ and $h_0$ such that
\[ 0 < h < h_0 \Rightarrow \max_{k=1,\ldots,d} \sup_{s,u \in [0,t]} \frac{|M^k_s - M^k_u|}{\sqrt{2h \log h}} \leq K \] (7.6)
for all $\omega \in \Omega$, by a localization procedure based on $(R_m)_m=1$. In the remainder of this paper, we will always assume that we have positive constants $K$ and $h_0$ satisfying (7.6). Moreover, we only consider sufficiently large $n$ such that $d_n \tilde{r}_n < h_0$.

In the following we denote by $E_0$ the conditional expectation given $\mathcal{F}^{(0)}$ i.e., $E_0[\cdot] := E[\cdot | \mathcal{F}^{(0)}]$. Note that under the conditions $[SA1]$–$[SA4]$ and (7.6) there exists a positive constant $L$ such that
\[ E_0 \left( \sup_{0 \leq t \leq T} |V^k(I^k_p)_t|^2 \right) \leq L \tilde{r}_n \log n \] (7.7)
for every $V \in \{ A, M, \mathcal{C} \}$, $k \in \{1, \ldots, d\}$ and any $p \in \mathbb{N}$. 

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7.2 Proof of Lemma 3.1

Throughout the discussions, for (random) sequences \((x_n)\) and \((y_n)\), \(x_n \lesssim y_n\) means that there exists a (non-random) constant \(K \in [0, \infty)\) such that \(x_n \leq Ky_n\) for large \(n\). First we deal with the end effect. For each \(V, U \in \{X, A, M, \mathcal{E}\}\), \(u, v \in \{g, g'\}\) and \(k, l \in \{1, \ldots, d\}\), define the process \(\mathbb{H}^{(k,l)}(U, V)^n\) by

\[
\mathbb{H}^{(k,l)}(U, V)^n_t = \frac{1}{\psi^2_k n} \sum_{i=k_n}^{\infty} \bar{U}^k_{i-t}\bar{V}^l_i,
\]

where \(\bar{U}^k_{i-t} = \sum_{p=k_n}^{\infty} u^k_{p-t} U^k(p^k_t)\) and \(\bar{V}^l_i\) is defined similarly.

**Lemma 7.5.** Suppose that \([W], [H1]-[H2], [SA1]-[SA4]\) and \((7.6)\) are satisfied. Then

\[
n^{1/4} \left( \frac{\mathbb{M}[Y]^{n,k,l} - \mathbb{H}^{(k,l)}(g)[g]^{n,k,l}}{\psi^2_k n} + \frac{\psi_1}{\psi^2_2 k_n^2} [Y]^{n,k,l}_{\Delta}\right) \xrightarrow{\text{ucp}} 0
\]

as \(n \to \infty\), where \(\mathbb{H}^{(k,l)}(g)[g]^{n,k,l} = \mathbb{H}^{(k,l)}(X, X)^n + \mathbb{E}_{g,g'}(X, \mathcal{E})^n + \mathbb{M}[Y]^{n,k,l}_{\Delta}\).

**Proof.** We decompose the target quantity as

\[
\mathbb{H}^{(k,l)}(U, V)^n_t = \mathbb{H}^{(k,l)}(g)[g]^{n,k,l}_{\Delta} + \mathbb{E}_{g,g'}(X, \mathcal{E})^n + \mathbb{M}[Y]^{n,k,l}_{\Delta}\]

where \(\bar{Y}^k_{i-t} = \bar{X}^k(g)_{i-t} + \bar{X}^k(g')_{i-t}\) and \(\bar{Y}^l_{i-t}\) is defined similarly.

First consider \(\mathbb{H}^{(k,l)}(U, V)^n_t\). We further decompose it as

\[
\mathbb{H}^{(k,l)}(U, V)^n_t = \frac{1}{\psi^2_k n} \sum_{i=N_n^{k-1}}^{N_n^{k-1}+1} \bar{X}^k_{i-T_{t}} \bar{X}^l_{i-T_{t}} - \mathbb{H}^{(k,l)}(g)[g]^{n,k,l}_{\Delta} + \mathbb{E}_{g,g'}(X, \mathcal{E})^n + \mathbb{M}[Y]^{n,k,l}_{\Delta}.
\]

Set \(\Delta(g)_p^n = g_{p+1}^n - g_p^n\) for each \(p\). Since summation by parts yields

\[
\sum_{p=k_n+1}^{\infty} g_{p-i}^n \Delta(g)^n_{p+i} X^k = - \sum_{p=(k_n+1)\wedge(i-d_n)}^{N_n^{k-1}\wedge(i+d_n)-1} \Delta(g)^n_{p-i} \left( X_{p+i}^k - X_{p-i}^k \right) + g_{N_n^{k-1}-k_n}\wedge(i+d_n)-i \left( X_{N_n^{k-1}-(k_n+1)i-d_n}^k - X_{(N_n^{k-1}-(k_n+1)i-d_n)}^k \right),
\]

(7.6) and the Lipschitz continuity of \(g\) imply that

\[
\sum_{p=k_n+1}^{\infty} g_{p-i}^n \Delta(g)^n_{p+i} X^k \lesssim \frac{d_n}{k_n} \sqrt{d_n \log n} \lesssim n^{-\frac{1}{2}} \frac{(\xi-\frac{1}{2}-3\gamma) \sqrt{\log n}}{\log n}
\]

uniformly in \(i\). Combining this with (7.6) and \([W](ii)\), we obtain

\[
\frac{\bar{X}_{i-T_{t}}^k}{\mathbb{H}^{(k,l)}(U, V)^n_t} \lesssim \left( \sum_{p=k_n+1}^{\infty} g_{p-i}^n \Delta(g)^n_{p+i} X^k \right) + \left( \sum_{p=k_n+1}^{\infty} g_{p-i}^n \Delta(g)^n_{p+i} X^k \right) + \left( g_{N_n^{k-1}-k_n}^n \Delta(g)^n_{N_n^{k-1}-k_n} X^k \right)
\]

\[
\lesssim n^{-\frac{1}{2}} (\xi-\frac{1}{2}-3\gamma) \sqrt{\log n} \quad (7.8)
\]
uniformly in $i$. Therefore, it holds that
\[
\sup_{0 \leq t \leq T} \left| \mathbb{A}_{1,t}^{(1)} \right| \lesssim k_n^{-1} n^{-\frac{\xi}{2} + \frac{3\gamma}{4}} (\log n) d_n \lesssim n^{-\frac{\xi}{2} + \frac{3\gamma}{4}} \log n = o_p(n^{-1/4}).
\]
On the other hand, for each $r \in [1, 8]$ the BDG inequality, [SA3] and [W] yield
\[
E_0 \left[ \left| \mathbb{e}_{i,T}^r \right|^r \right] \lesssim E_0 \left[ \sum_{p=k_n}^{N_p^0-k_n} \Delta(g)^p \mathbb{e}_{i,T}^r \right] + E_0 \left[ \left| g_{N_p^0-k_n+1-i} \mathbb{e}_{N_p^0-k_n+1}^r \right|^r \right] + E_0 \left[ \left| g_{k_n-i} \mathbb{e}_{k_n-1}^r \right|^r \right] \lesssim k_n^{-r/2}
\]
uniformly in $i$. (7.8), (7.9), the Hölder inequality and the tightness of $n^{-1} N_T^r$ imply that
\[
E_0 \left[ \sup_{0 \leq t \leq T} \left| \mathbb{A}_{1,t}^{(2)} \right| \right] \lesssim k_n^{-1} n^{-\frac{\xi}{4} + \frac{3\gamma}{4} + \frac{3\gamma}{2}} \sqrt{\log n} \cdot d_n^{3/4} E_0 \left[ \sum_{k_n}^{N_p^0-k_n+1} \left| \mathbb{e}_{i,T}^r \right|^4 \right]^{1/4} \lesssim n^{-\frac{\xi}{4} + \frac{3\gamma}{4} + \frac{3\gamma}{2} \log n} = o_p(n^{-1/4}).
\]
Similarly we can prove $E_0 \left[ \sup_{0 \leq t \leq T} \left| \mathbb{A}_{1,t}^{(3)} \right| \right] = o_p(n^{-1/4})$ and $E_0 \left[ \sup_{0 \leq t \leq T} \left| \mathbb{A}_{1,t}^{(4)} \right| \right] \lesssim k_n^{-1} n^{3/4} n^{1/4} n^{-1} \lesssim n^{-\frac{\xi}{4} + \frac{3\gamma}{2} \log n} = o_p(n^{-1/4})$, hence we conclude that $n^{1/4} \mathbb{A}_{1,t} \xrightarrow{\text{ucp}} 0$.

Next, noting that $U(J)^k_{p,t} = 0$ if $\tau^k_{p-1} \geq t$, [W](ii) implies that $\mathbb{A}_{2,t} = - \sum_{i=N_p^0-k_n+1}^{N_p^0-k_n+1} \bar{\gamma}^k (g)^i \tilde{Y}^i (g)^i + o_p(n^{-1/4})$ uniformly in $t \in [0, T]$. Therefore, an argument similar to the above yields $n^{1/4} \mathbb{A}_{2,t} \xrightarrow{\text{ucp}} 0$.

Finally we consider $\mathbb{A}_{3,t}$. We decompose it as
\[
\mathbb{A}_{3,t} = \frac{1}{\psi^2 k_n} \sum_{i=k_n}^{N_p^0-d_n} \left[ \left\{ \left( \mathbb{T}_{i,T}^k - \bar{\gamma}^k (g)^i \right) \tilde{Y}_{i,T}^i + \bar{\gamma}^k (g)^i (\tilde{Y}_{i,T}^i - \bar{Y}^i (g)^i) \right\} \right] \mathbb{e}_{i,T}^r.
\]
Since we have
\[
\left| \mathbb{T}_{i,T}^k - \bar{\gamma}^k (g)^i \right| \left| \mathbb{T}_{i,T}^k - \bar{\gamma}^k (g)^i \right| \lesssim g_{k_n-i} \left| X_{k_n-k_n-i}^k (g)^i \right| + \sum_{p=N_p^0-k_n+1}^{\infty} \left| g_{p-i} \mathbb{\bar{\Delta}} X^k \right| + \sum_{p=N_p^0-k_n+1}^{\infty} \left| g_{p-i} X^k (J_p)^i \right| ,
\]
(7.6), [W](ii), (7.8) and (7.9) yield
\[
E_0 \left[ \sup_{0 \leq t \leq T} \left| \mathbb{T}_{i,T}^k - \bar{\gamma}^k (g)^i \right| \right] = \sqrt{k_n \log n} \cdot n^{-\frac{\xi}{4} + \frac{3\gamma}{4} + \frac{3\gamma}{2}} \sqrt{\log n} \cdot \frac{k_n}{k_n} \sum_{i=k_n}^{N_p^0-d_n} \left| g_{k_n-i} \right| + o_p(n^{-1/4}) = O_p \left( n^{-\frac{\xi}{4} + \frac{3\gamma}{4} + \frac{3\gamma}{2} \log n} \right) + o_p \left( n^{-1/4} \right).
\]
On the other hand, setting $\mathbb{\bar{\Delta}} (g)^n_p = \Delta (g)^n_p - k_n^{-1} (g)^n_p$, the Doob inequality, [SA3], the (piecewise) Lipschitz continuity of $g'$ and [W](ii) imply that
\[
E_0 \left[ \sup_{0 \leq t \leq T} \left| \sum_{p=k_n}^{N_p^0-k_n} \mathbb{\bar{\Delta}} (g)^n_p \frac{k_n}{k_n} \right| \right] \leq E_0 \left[ \max_{m \in \{k_n, \ldots, N_p^0-k_n\}} \sum_{p=k_n}^{m} \mathbb{\bar{\Delta}} (g)^n_p \frac{k_n}{k_n} \right] ^{2} \lesssim k_n^{-4} d_n
\]
\[
\lesssim k_n^{-4} d_n
\]
\[
\lesssim k_n^{-4} d_n
\]
\[
\lesssim k_n^{-4} d_n
\]
uniformly in \( i \). Since it holds that

\[
\left| e_{i,T}^k - \tilde{e}_{i,T}^k(g_j) \right| \leq \sum_{p=k}^{N^k_{n-k+1}} \Delta(g)^{n-1} \epsilon_{p}^k + g_{n-k+i}^{N_{n-k+1}-1} \epsilon_{p}^k + g_{N_{n-k+1}^k}^{n} \epsilon_{p}^k + \sum_{p=N_{n-k+1}^k}^{\infty} k^{-1} (g)^{n-1} \epsilon_{p}^k \chi_{\{t_p \leq t\}}
\]

[W](ii), (7.8), (7.9), the Schwarz inequality and the tightness of \( n^{-1} N_{n}^k \) imply that

\[
E_0 \left[ \sup_{0 \leq t \leq T} \left| \frac{1}{\psi_k n} \sum_{i=k}^{N_{n-k+1}} \left\{ e_{i,T}^k - \tilde{e}_{i,T}^k(g_j) \right\} \right| \right] = k^{-1} \cdot k^{-2} d_n^{1/2} \cdot n^{-1/2} (\xi^{1/2} - 3\gamma) \sqrt{\log n \cdot n + o_p \left( n^{-1/4} \right)}
\]

\[
= O_p \left( n^{-1/4} + 2\gamma \sqrt{\log n} \right) + o_p \left( n^{-1/4} \right) = o_p \left( n^{-1/4} \right).
\]

Consequently, we obtain \( \sup_{0 \leq t \leq T} \left| \tilde{\Lambda}_{3,t}^{(1)} \right| = o_p(n^{-1/4}) \). In a similar manner we can prove \( \sup_{0 \leq t \leq T} \left| \tilde{\Lambda}_{3,t}^{(2)} \right| = o_p(n^{-1/4}) \), hence we conclude \( n^{1/4} \tilde{\Lambda}_3 \) wcp 0. This completes the proof.

Next we give an approximation of \( \mathbb{H}_{u,v}^{(k,l)}(U,V)^n \) by \( \mathbb{L}_{u,v}^{(k,l)}(U,V)^n \) and a remainder term. The next result is a direct consequence of the condition [H1].

**Lemma 7.6.** Suppose that [H1] holds true. If \( I_p^k \cap I_q^l \neq \emptyset \), it holds that \( |p - q| \leq 1 \).

**Proof.** If \( I_p^k \cap I_q^l \neq \emptyset \), we have \( \tau_{p-1}^k < t_q^l \) and \( \tau_{q-1}^l < t_p^k \). Hence, by [H1] it holds that \( T_{p-2} < T_q \) and \( T_{q-2} < T_p \). Since \( (T_p) \) is increasing, we obtain \( p - 1 \leq q \) and \( q - 1 \leq p \).

**Lemma 7.7.** Under the assumptions of Lemma 7.5, it holds that

\[
n^{1/4} \left\{ \mathbb{H}_{u,v}^{(k,l)}(U,V)^n - \mathbb{L}_{u,v}^{(k,l)}(U,V)^n - \psi_2^{-1} \phi_{u,v}(0)[U^k, V^l] \right\} \ \text{wcp} \ 0
\]

as \( n \to \infty \) for any \( k, \ell \in \{1, \ldots, d\} \), \( U, V \in \{A, M, \mathcal{E}\} \) and \( u, v \in \{g, g'\} \).

**Proof.** By a direct computation we have \( \mathbb{H}_{u,v}^{(k,l)}(U,V)^n = \sum_{p,q=k_n}^\infty c_{u,v}(p,q)U^k(I_p^k)V^l(I_q^l) \), hence integration by parts yield

\[
\frac{1}{\psi_2 n} \sum_{i=k_n}^{\infty} \tilde{u}^k(u)_t \tilde{v}^l(v)_t = \sum_{p,q=k_n}^\infty c_{u,v}(p,q) \left\{ U^k(I_p^k)_t \cdot V^l(I_q^l)_t + V^l(I_q^l)_t \cdot U^k(I_p^k)_t + [U^k(I_p^k), V^l(I_q^l)]_t \right\}
\]

\[
=: \mathbb{B}_{1,t} + \mathbb{B}_{2,t} + \mathbb{B}_{3,t}.
\]

First we show that

\[
\mathbb{B}_{1,t} = \mathbb{M}_{u,v}^{(k,l)}(U,V)^n_t + o_p \left( n^{-1/4} \right)
\]

(7.10)

uniformly in \( t \in [0,T] \). Since \( U^k(I_p^k)_s = 0 \) if \( s \leq \tau_{p-1}^k \), we can rewrite \( \mathbb{B}_{1,t} \) as

\[
\mathbb{B}_{1,t} = \sum_{q=k_n}^{q+1} \sum_{p=(q-1)\vee k_n}^\infty c_{u,v}(p,q)U^k(I_p^k)_t \cdot V^l(I_q^l)_t.
\]

If \( V \in \{M, \mathcal{E}\} \), then \( \mathbb{B}_{1,t} \) is a locally square-integrable and

\[
E_0 \left[ \sum_{q=k_n}^{q+1} \sum_{p=(q-1)\vee k_n}^\infty c_{u,v}(p,q)U^k(I_p^k)_t \cdot V^l(I_q^l)_t \right]^2 \lesssim \bar{f}_n(\log n)(V^l)_T = o_p \left( n^{-1/2} \right)
\]

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by (7.7) and the tightness of \(\langle V^l \rangle_T\), hence we have

\[
B_{1,t} = \sum_{q=k_n+2}^{\infty} \sum_{p=k_n}^{q-2} c_{u,v}(p,q)U^k(I^k_p) - \bullet V^l(I^l_q)_t + o_p\left(n^{-1/4}\right)
\]

uniformly in \(t \in [0,T]\). It is not difficult to prove the above equation for \(V = A\). Therefore, noting that \([W](ii)\) yields \(\sup_{p,q:|p-q|\geq d_n} |c_{u,v}(p,q)| = o_p(n^{-\alpha})\) for any \(\alpha > 0\), we obtain (7.10).

By symmetry, it also holds that \(B_{2,t} = M_{v,u}^{(I^k_p)}(V,U)_t + o_p\left(n^{-1/4}\right)\) uniformly in \(t \in [0,T]\). Therefore, the proof is completed once we prove \(B_{3,t} = \psi_2^{-1} \phi_{u,v}(0)[U^k, V^l]_t + o_p\left(n^{-1/4}\right)\) uniformly in \(t \in [0,T]\). Since Lemma 7.6 yields

\[
B_{3,t} = \sum_{p,k_n}^{\infty} c_{u,v}(p,q)[U^k, V^l](I^k_p \cap I^l_q)_t,
\]

the (piecewise) Lipschitz continuity of \(u\) and \(v\), \([W](ii)\) and the tightness of \(n^{-1/2}\) imply that

\[
B_{3,t} = \sum_{p,k_n}^{\infty} c_{u,v}(p,p)[U^k, V^l](I^k_p)_s + O_p\left(k_n^{-1}\right) = \psi_2^{-1} \phi_{u,v}(0)[U^k, V^l]_t + o_p\left(n^{-1/4}\right)
\]

uniformly in \(t \in [0,T]\). Thus we complete the proof.

**Proof of Lemma 3.1.** Note that \(\phi_{g,g'}(0) = \phi_{g',g}(0) = 0\) due to integration by parts and \([W](ii)\). Therefore, in the light of Lemmas 7.5–7.7 the proof is completed once we show that

\[
n^{1/4} \left(\frac{1}{k^2} \mathbb{E}[|Y|^{n,k}] - \mathbb{E}[|\mathbf{e}^k, \mathbf{e}^l|]\right) \overset{a.s.}{\rightarrow} 0\tag{7.11}
\]

as \(n \to \infty\). First, it can easily be shown that

\[
\frac{1}{k^2} \mathbb{E}[|Y|^{n,k}] = \frac{1}{k^2} \sum_{p=1}^{N^\tau} \left(\epsilon_{1,k}^{p} \epsilon_{1,l}^{p} + \epsilon_{1,k}^{p} \epsilon_{2,l}^{p} - \epsilon_{1,k}^{p} \epsilon_{2,l}^{p} - \epsilon_{1,l}^{p} \epsilon_{2,k}^{p} + \epsilon_{2,l}^{p} \epsilon_{2,k}^{p}\right) + O_p(n^{-1/2})
\]

uniformly in \(t \in [0,T]\). On the other hand, we can write \(\mathbb{E}[|\mathbf{e}^k, \mathbf{e}^l|] = k_n^{-2} \sum_{p=1}^{N^\tau} \epsilon_{1,k}^{p} \epsilon_{1,l}^{p} 1_{\{\tau_k = \tau_l \leq t\}}\). Therefore, it holds that

\[
\sup_{0 \leq t \leq T} \left| \frac{1}{k^2} \sum_{p=1}^{N^\tau} \left(\epsilon_{1,k}^{p} \epsilon_{1,l}^{p} + \epsilon_{1,k}^{p} \epsilon_{2,l}^{p} - \epsilon_{1,k}^{p} \epsilon_{2,l}^{p} - \epsilon_{1,l}^{p} \epsilon_{2,k}^{p} + \epsilon_{2,l}^{p} \epsilon_{2,k}^{p}\right) + \frac{1}{k^2} \epsilon_{1,k}^{p} \epsilon_{1,l}^{p} 1_{\{\tau_k = \tau_l \leq t\}}\right| =: \Gamma_1 + \Gamma_2 + \Gamma_3.
\]

The Doob inequality yields \(\Gamma_1 = O_p(n^{-1/2})\), while we obviously have \(\Gamma_2 = O_p(n^{-1})\). Furthermore, for any \(\eta > 0\) it holds that

\[
P(\Gamma_3 > n^{-1} \eta | F(0)) \leq \frac{\eta^{1/2}}{\eta} E_0 \left[\frac{\eta^{1/2}}{\eta} \sum_{p=0}^{N^\tau} \frac{1}{k^2} \epsilon_{1,k}^{p} \epsilon_{1,l}^{p} 1_{\{\tau_k = \tau_l \leq t\}}\right] = O_p(1),\tag{7.12}
\]

hence we obtain \(\Gamma_3 = o_p(n^{-1/4})\). This yields Eq.\( (7.11)\). \(\square\)

7.3 **Proof of Lemma 3.2**

First, since \([SA2]\) and (7.4) yield \(\sup_{0 \leq t \leq T} |\mathbb{M}^{(I^k_p)}_{u,v}(A,A)_t| \lesssim d_n r_n = o_p(n^{-1/4})\), the lemma holds true for \(V = A\). Therefore, it suffices to consider the case that \(V \in \{M, \mathbf{e}\}\). In this case \(\mathbb{M}^{(I^k_p)}_{u,v}(A,V)_t\) is a locally square-integrable martingale and

\[
\mathbb{E}[\mathbb{M}^{(I^k_p)}_{u,v}(A,V)_T] = \sum_{q=k_n+2}^{\infty} \sum_{p=k_n}^{q-2} c_{u,v}(p,q)A^k(I^k_p) - \bullet V^l(I^l_q)_T + o_p(n^{-1/4})
\]

\[
\mathbb{E}[\mathbb{M}^{(I^k_p)}_{u,v}(A,V)_T] \lesssim (d_n r_n)^2 = o_p(n^{-1/2})
\]

hence we obtain \(\Gamma_3 = o_p(n^{-1/4})\). This yields Eq.\( (7.11)\). \(\square\)
by [SA1]–[SA3], (7.4) and the tightness of \(n^{-1}N_{\tau}^n\), hence we have \(n^{1/4} \rho_{u,v}(A,V)^n \xrightarrow{ucp} 0\) as \(n \to \infty\).

Now it remains to prove \(n^{1/4} \rho_{u,v}(V,A)^n \xrightarrow{ucp} 0\). To accomplish this, noting that \(V(I_p^k)_t = V(I_p^k)_t\), if \(I_p^k \leq \tau_q^0\), we rewrite \(\rho_{u,v}(V,A)^n\) as \(\rho_{u,v}(V,A)^n = \sum_{q=k_n+2}^{\infty} C_{u,v}(V)_{q}^k A(I_q^k)_t\), where

\[ C_{u,v}(V)^{k} = \sum_{p = k_n \vee (q-d_n)}^{q-2} c_{u,v}(p,q) V^k(I_q^k), \]

The following lemma is useful for obtaining various estimates in the proof.

**Lemma 7.8.** Under the assumptions of Lemma 7.5, for every \(r \in [1, 8]\) there exists a positive constant \(K_r\) such that \(E_0 \left[\left|\rho_{u,v}(V)^{k} \right|^r\right] \leq K_r \left(n^{-\xi + \frac{1}{2} + 3\gamma} \log n\right)^{r/2}\) for any \(q \geq k_n + 2\).

**Proof.** First suppose that \(V = M\). Then, summation by parts yields

\[ C_{u,v}(V)^{k} = - \sum_{p = k_n \vee (q-d_n)}^{q-3} \{c_{u,v}(p+1,q) - c_{u,v}(p,q)\} \left(V^k(I_p^k) - V^k(I_{p+1}^k)\right) \]

\[ + c(q-1,q) \left(V^k(I_{q-1}^k) - V^k(I_{k_n \vee (q-d_n)}^k)\right), \]

hence by (7.6) and (7.4) we have

\[ E_0 \left[\left|\rho_{u,v}(V)^{k} \right|^r\right] \lesssim \left(d_n k_n^{-1} \sqrt{d_n r_n \log n}\right)^{r/2} \lesssim \left(n^{-\xi + \frac{1}{2} + 3\gamma} \log n\right)^{r/2}. \]

Next suppose that \(V = \mathcal{E}\). Then, the BDG inequality yields

\[ E_0 \left[\left|\rho_{u,v}(V)^{k} \right|^r\right] \lesssim E_0 \left[ \left\{ \frac{1}{k_n^2} \sum_{p = k_n \vee (q-d_n)}^{q-2} \left|c_{u,v}(p,q)\right|^2 \right\}^r \right]. \]

If \(r \leq 2\), the Lyapunov inequality and [SA3] imply that

\[ E_0 \left[\left|\rho_{u,v}(V)^{k} \right|^r\right] \lesssim \left\{ E_0 \left[ \sum_{p = k_n \vee (q-d_n)}^{q-2} \left|c_{u,v}(p,q)\right|^2 \right] \right\}^{r/2} \lesssim (k_n^{-2} d_n)^{r/2}. \]

Otherwise, the Jensen inequality and [SA3] imply that

\[ E_0 \left[\left|\rho_{u,v}(V)^{k} \right|^r\right] \lesssim (k_n^{-2} d_n)^{r/2} E_0 \left[ \sum_{p = k_n \vee (q-d_n)}^{q-2} \left|c_{u,v}(p,q)\right|^r \right] \lesssim (k_n^{-2} d_n)^{r/2}. \]

This completes the proof.

**Proof of Lemma 3.2.** Due to the above argument, it suffices to prove \(n^{1/4} \rho_{u,v}(V,A)^n \xrightarrow{ucp} 0\) for \(V \in \{M, \mathcal{E}\}\).

First, since summation by parts yields

\[ \sum_{q = k_n + 2}^{\infty} C_{u,v}(V)_{q}^k \left\{ A^t(I_q^k)_t - A^t(I_q)_t \right\} \]

\[ = \sum_{q = k_n + 2}^{\infty} \sum_{p = k_n \vee (q-d_n+1)}^{q-2} \left(V^k(I_p^k) \left\{ c_{u,v}(p,q) - c_{u,v}(p,q+1) \right\} \left(A^t(I_q^k)_t - A^t(I_q^k)_t \right) \right) \]

\[ + \sum_{p = k_n}^{\infty} V^k(I_p^k) \left[c_{u,v}(p,p+d_n) \left(A^t(I_q^k)_t - A^t(I_q^k)_t \right) - c_{u,v}(p,p+2) \left(A^t(I_q^k)_t - A^t(I_q^k)_t \right) \right] , \]

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we have
\[ \sup_{0 \leq t \leq T} \left| \mathcal{M}_{u,v}^{(k,l)}(V,A)_t^n - \sum_{q=k_n+2}^{\infty} C_{u,v}(V)_q^k A^l(I_q)_t \right| = o_p(n^{-1/4}) \] (7.13)
due to (7.7) and the (piecewise) Lipschitz continuity of \( v \). Next, since Lemma 7.8 and [SA1] yield
\[ E_0 \left[ \sup_{0 \leq t \leq T} \sum_{q=k_n}^{\infty} C_{u,v}(V)_q^k A^l(I_q)_t \left| a^l_{T_{q-1}} |I_q| 1_{(T_{q-1} \leq t)} \right| \right] \leq \mathcal{O}_p \left( n^{-\frac{\xi}{2} + \frac{\gamma}{2} + 3\gamma} \log n \right)^{1/2} \{ w(a', 2\bar{r}_n, T) + \bar{r}_n \} = O_p \left( n^{-\frac{\xi}{2} + \frac{\gamma}{2} + 3\gamma} \right), \]
we obtain
\[ \sup_{0 \leq t \leq T} \left| \mathcal{M}_{u,v}^{(k,l)}(V,A)_t^n - \sum_{q=k_n}^{\infty} C_{u,v}(V)_q^k a^l_{T_{q-1}} |I_q| 1_{(T_{q-1} \leq t)} \right| = o_p(n^{-1/4}). \] (7.14)

Next we show that
\[ \sup_{0 \leq t \leq T} \left| \mathcal{M}_{u,v}^{(k,l)}(V,A)_t^n - n^{-1} \sum_{q=k_n}^{N_T^n + 1} C_{u,v}(V)_q^k a^l_{T_{q-1}} G_{T_{q-1}} \right| = o_p(n^{-1/4}). \] (7.15)

Set \( \varpi = \rho \wedge 2 \) and \( \mathcal{H}_n = \mathcal{H}_n^0 = \mathcal{H}_n^0 \vee \mathcal{F}^{(1)}_n \). Then, noting that [H2] implies that \( C_{u,v}(V)_q^k \) is \( \mathcal{H}_{T_{q-1}}^n \)-measurable, Lemma 7.8, [SA2], [A4](iii) and the tightness of \( n^{-1}N_T^n \) yield
\[ E_0 \left[ \sum_{q=k_n+2}^{N_T^n + 1} E \left[ \left| C_{u,v}(V)_q^k a^l_{T_{q-1}} |I_q| \right| \mathcal{H}_n^0 \vee \mathcal{F}^{(1)}_n \right] \right] \lesssim n^{-\frac{\varpi}{2}} \left( n^{-\frac{\xi}{2} + \frac{\gamma}{2} + 3\gamma} \log n \right)^{\varpi/2} \left( N_T^n + 1 \right) \sup_{0 \leq q \leq N_T^n} D(\varpi)_q \]
\[ = O_p \left( n^{-\frac{\varpi}{2} - \frac{\gamma}{2}} \log n \right)^{\varpi/2}. \]

Since \( \varpi \left( \frac{1+\xi}{2} - \frac{\gamma}{2} \right) > 1 \) by (3.4), Lemma 7.1 implies that
\[ \sup_{0 \leq t \leq T} \left| \mathcal{M}_{u,v}^{(k,l)}(V,A)_t^n - n^{-1} \sum_{q=k_n+2}^{N_T^n + 1} C_{u,v}(V)_q^k a^l_{T_{q-1}} G(1)_{T_{q-1}} \right| = o_p(n^{-1/4}). \]

Therefore, by Lemma 7.8, [SA1], [A4](ii)–(iii), (3.4) and the tightness of \( n^{-1}N_T^n \) we obtain (7.15).

Now we show that
\[ \sup_{0 \leq t \leq T} \left| \mathcal{M}_{u,v}^{(k,l)}(V,A)_t^n - n^{-1} \sum_{q=k_n+2}^{N_T^n + 1} C(V)_q^k F_{T_{q-1} - F_{T_{q-1} \wedge (d_n - d_n) - 2}} \right| = o_p(n^{-1/4}), \] (7.16)
where \( F = a^l G \). By Lemma 7.8, [SA2], [A4](ii), (7.4) and the tightness of \( n^{-1}N_T^n \) we have
\[ E_0 \left[ \sum_{q=k_n+2}^{N_T^n + 1} C_{u,v}(V)_q^k (F_{T_{q-1} - F_{T_{q-1} \wedge (d_n - d_n) - 2}}) \right] \lesssim \left( n^{-\frac{\xi}{2} + \frac{\gamma}{2} + 3\gamma} \log n \right)^{1/2} w(F; (d_n + 1)2\bar{r}_n, t) \cdot (n^{-1}N_T^n) = O_p \left( n^{-\frac{\xi}{2} + \frac{\gamma}{2} + 3\gamma} \log n \right)^{1/2}, \]
hence (3.4) yields (7.16).
After all, it is sufficient to show that $\sup_{0 \leq t \leq T} |A_t| \rightarrow^p 0$ as $n \rightarrow \infty$, where

$$A_t = n^{-3/4} \sum_{q=k_n+2}^{N_n + 1} C_{u,v} (V)^{k} F_{\mathcal{F}_{\tau_{p-1}}},$$

Set $H^p = \sum_{q=p+2}^{p+1} c_{u,v} (p,q) F_{\mathcal{F}_{\tau_{p-1}}}$, then by the construction $H^p$ is $\mathcal{F}_{\tau_{p-1}}$-measurable and we have $A_t = n^{-3/4} \sum_{p=k_n}^{N_n - 1} H^p V^k (I^k_p)$. Now let $\{SA1\} - \{SA4\}$ and the tightness of $(V^k)_T$ yield

$$n^{-3/2} \sum_{p=k_n}^{N_n - 1} E \left[ |H^p V^k (I^k_p)|^2 \right] \leq n^{-3/2} d_n^2 \sup_{0 \leq t \leq T} |F_t|^2 (V_T) = o_p(1),$$

hence Lemma 7.1(a) yields $A_t = n^{-3/4} \sum_{p=k_n}^{N_n - 1} H^p E \left[ V (I^k_p) | \mathcal{F}_{\tau_{p-1}} \right] + o_p(1) = o_p(1)$ uniformly in $t \in [0,T]$. \hfill \Box

### 7.4 Proof of Lemma 3.3

**Lemma 7.9.** Under the assumptions of Lemma 7.5, it holds that

$$\sum_{p=1}^{\infty} H_{\tau_{p-1}}^{k,k'} (I^k_p)_t \rightarrow^p \int_0^t H_s^k \tilde{T}_s^{kk'} G_s ds$$

as $n \rightarrow \infty$ for any $t \geq 0$ and any process $H$ satisfying the same assumption as that from Lemma 7.2.

**Proof.** Since $[e^k, e^{k'}] (I^k_p)_t = k_n^{-2} e^k_{\tau_{p-1}} e^{k'}_{\tau_{p-1}+1} \mathbb{1}_{\{\tau_{p-1} = \tau_{p-1}' \leq t\}}$, (7.12) yields

$$\sup_{0 \leq t \leq T} \left| \sum_{p=1}^{\infty} H_{\tau_{p-1}}^{k,k'} (I^k_p)_t - \frac{N_n + 1}{k_n^2} \sum_{p=1}^{N_n + 1} H_{\tau_{p-1}}^{k,k'} e^k_{\tau_{p-1}} e^{k'}_{\tau_{p-1}+1} \mathbb{1}_{\{\tau_{p-1} = \tau_{p-1}' \}} \right| = o_p(n^{-1/4}).$$

Therefore, by Lemma 7.1 we obtain

$$\sup_{0 \leq t \leq T} \left| \sum_{p=1}^{\infty} H_{\tau_{p-1}}^{k,k'} (I^k_p)_t - \frac{N_n + 1}{k_n^2} \sum_{p=1}^{N_n + 1} H_{\tau_{p-1}}^{k,k'} e^k_{\tau_{p-1}} e^{k'}_{\tau_{p-1}+1} \mathbb{1}_{\{\tau_{p-1} = \tau_{p-1}' \}} \right| = o_p(n^{-1/4}).$$

Then, by the use of the continuity of $T^{kk'}$ and Lemma 7.1 as well as [A4] we can prove

$$\sup_{0 \leq t \leq T} \left| \sum_{p=1}^{\infty} H_{\tau_{p-1}}^{k,k'} (I^k_p)_t - \frac{N_n + 1}{k_n^2} \sum_{p=1}^{N_n + 1} H_{\tau_{p-1}}^{k,k'} e^k_{\tau_{p-1}} e^{k'}_{\tau_{p-1}+1} \mathbb{1}_{\{\tau_{p-1} = \tau_{p-1}' \}} \right| = o_p(n^{-1/4}) + O_p \left( \frac{n^{-1/4}}{(n^{-1/4})} \right).$$

Now the desired result follows from Lemma 7.2, [A4] and (2.1). \hfill \Box

**Proof of Lemma 3.3.** Let $(U,u), (V,v), (\bar{U}, \bar{u}), (\bar{V}, \bar{v}) \in \{(M, g), (\mathcal{E}, g')\}$ and set

$$\mathfrak{W}^n_t := n^{1/2} \langle \mathcal{M}_{u,v}^{(k,j)} (U, V)^n, \mathcal{M}_{u,v}^{(k',j')} (\bar{U}, \bar{V})^n \rangle_t.$$

It suffices to compute the limiting variable of $\mathfrak{W}^n_t$ explicitly.

By the use of associativity and (bi-)linearity of integration and predictable quadratic covariation, we can rewrite

$$\mathfrak{W}^n_t = \sum_{q,q'=k_n+2}^{q,q'=k_n+2} H_{q,q'} (V^q, \bar{V}^{q'}) (I^q_t \cap I^{q'}_{t^*}),$$

where

$$H_{q,q'} = n^{1/2} \sum_{p=k_n \vee (q-d_n)}^{q-2} \sum_{p'=k_n \vee (q'-d_n)}^{q'-2} c_{u,v} (p,q) c_{u,v} (p',q') U^k (I^k_p) \bar{U}^{k'} (I^{k'}_{p'}).$$

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Note that for any \( r \in [1,4] \) there exists a positive constant \( K_r \) such that
\[
E_0 \left[ \sum_{p=k_n \vee (q-d_n)}^{q-2} c_{u,v}(p,q) U^k(I^k_p) \hat{U}^{k'}(I^k_{p'}) \right]^r \leq K_r \left( n^{-\xi+\frac{4}{\gamma}+\frac{2}{\gamma} \log n} \right)^r, \tag{7.18}
\]
\[
E_0 \left[ |H_{q,q'}|^r \right] \leq K_r \left( n^{-\xi+1+3\gamma \log n} \right)^r \tag{7.19}
\]
for any \( p', q \in \mathbb{N} \) by Lemma 7.8, [SA2]–[SA3], (7.7) and the Schwarz inequality. These estimates will often be used in the following.

Since Lemma 7.6 implies that \( I^l_q \cap I^l_{q'} = \emptyset \) if \( |q - q'| > 1 \), (7.18) and the Lipschitz continuity of \( \hat{v} \) yield
\[
E_0 \left[ |H_{q,q'} - H_{q,q''}| \right] \lesssim n^{-\xi+\frac{4}{\gamma}+\frac{2}{\gamma} \log n}
\]
uniformly in \( q, q' \), hence the tightness of \( \langle V^l, \hat{V}^l \rangle_t \) implies that \( \mathcal{V}^l_t = \sum_{q=k_n+2}^{\infty} H_{q,q}(V^l, \hat{V}^l)(I^l_q)_t + o_p(1) \).

Now we separately consider the following three cases:

Case 1: \( V = \hat{V} = M \). First, an argument similar to the proof of (7.13) (using (7.18) instead of (7.7)) yields \( \mathcal{V}^n_t = \sum_{q=k_n+2}^{\infty} H_{q,q}(M^l, M'^l)(I^l_q)_t + o_p(1) \). Next, by an argument similar to the proof of (7.14) (using (7.19) instead of Lemma 7.8) we can show that
\[
E_0 \left[ \sum_{q=k_n+2}^{\infty} H_{q,q} \left( \langle M^l, M'^l \rangle_t - \sum_{T_{q-1}}^{T_q} |I_q| [T_{q-1} \leq t] \right) \right] \lesssim (n^{-\xi+1+3\gamma \log n}) \left\{ \nu \left( \sum_{T_{q-1}}^{T_q} 2\bar{r}_n, t \right) + \bar{r}_n \right\}.
\]

Hence [SA2] yields
\[
\mathcal{V}^n_t = \sum_{q=k_n+2}^{N^n_q+1} H_{q,q} \sum_{T_{q-1}}^{T_q} |I_q| + o_p(1). \tag{7.20}
\]

Moreover, an argument similar to the proof of (7.15) (using (7.19) instead of Lemma 7.8) yields
\[
\mathcal{V}^n_t = n^{-1} \sum_{q=k_n+2}^{N^n_q+1} H_{q,q} \sum_{T_{q-1}}^{T_q} G_{T_{q-1}} + o_p(1).
\]

Now we show that
\[
\mathcal{V}^n_t = n^{-1} \sum_{q=k_n+2}^{N^n_q+1} H_{q,q} F_{T_{k_n \vee (q-d_n) - 2}} + o_p(1), \tag{7.21}
\]
where \( F = \Sigma^{l'} G \). By an argument similar to the proof of (7.16) (using (7.19) and [SA2] instead of Lemma 7.8 and [SA1] respectively), we can prove
\[
n^{-1} \sum_{q=k_n}^{N^n_q+1} H_{q,q} \left( F_{T_{q-1}} - F_{T_{k_n \vee (q-d_n) - 2}} \right) = O_p \left( n^{-\xi+1+3\gamma \log n} (d_n \bar{r}_n)^{1/2-\lambda} \right)
\]
\[
= O_p \left( n^{-\xi-\gamma/2+3\gamma + \lambda (\xi-\gamma/2)} \log n \right)
\]
for any \( \lambda > 0 \). Thus (3.4) yields (7.21).

(7.21) yields \( \mathcal{V}^n_t = n^{-1/2} \sum_{p,p' = k_n}^{N^n_k-1} \bar{H}_{p,p'} U^k(I^k_p) \hat{U}^{k'}(I^k_{p'}) + o_p(1) \), where
\[
\bar{H}_{p,p'} = \sum_{q=p\vee p'+2}^{p\vee p'+d_n} c_{u,v}(p,q) c_{u,v}(p',q) F_{T_{k_n \vee (q-d_n) - 2}}.
\]

Since \( \bar{H}_{p,p'} = 0 \) if \( |p - p'| \geq d_n - 1 \), (7.7) and the fact that \( \sup_{0 \leq t \leq T} |F_t| = O_p(1) \) imply that
\[
\mathcal{V}^n_t = n^{-1/2} \sum_{p,p' = k_n}^{\infty} \bar{H}_{p,p'} U^k(I^k_p) \hat{U}^{k'}(I^k_{p'}) + o_p(1),
\]
hence integration by parts yields

\[ \mathcal{G}_t^n = n^{-1/2} \sum_{p,p'=k_n}^{\infty} \mathcal{H}_{p,p'} \left\{ U^k(I_p^c) - \mathcal{U}^k(I_p^c)'t + \mathcal{U}^k(I_p^c)'t + [U^k, \mathcal{U}^k]((I_p^c) \cap I_p^c)'t \right\} + o_p(1) \]

\[ =: \Delta_{1,t} + \Delta_{2,t} + \Delta_{3,t} + o_p(1). \]

Consider \( \Delta_{1,t} \). Take a number \( L > 0 \) arbitrarily and set \( R^n_{\pi} = \text{inf}\{t|n^{-1}N^t_t > L\} \wedge \text{inf}\{t|G_t > L\} \). Clearly, \( R^n_{\pi} \) is an \( \mathcal{F}^{(\pi)} \)-stopping time and it holds that \( n^{-1}N^t_{n^{-1}R^n_{\pi}} \leq L + n^{-1} \) and \( G_{\pi,R^n_{\pi}} \leq L \) for every \( s \). Then, noting that \( \mathcal{H}_{p,p'} \) is \( \mathcal{F}^{(\pi)^{1/2} \wedge \pi} \)-measurable by the construction, \( [\mathcal{SA}2]-[\mathcal{SA}3], (7.4) \), the optional stopping theorem and the Doob inequality yield

\[ E \left[ |\Delta_{1,t} \wedge R^n_{\pi} |^2 \right] = n^{-1} E \left[ \sum_{p'=k_n}^{\infty} \left\{ \sum_{p=k_n}^{\infty} \mathcal{H}_{p,p'} U^k(I_p^c) \right\}^2 \right] \cdot (\mathcal{U}^k(I_p^c)'t) + o_p(1) \]

\[ \leq n^{-1} \sup_{p'=k_n}^{\infty} \sup_{0 \leq s \leq \wedge R^n_{\pi}} \left( \sum_{p=k_n}^{\infty} \mathcal{H}_{p,p'} U^k(I_p^c) \right)^2 \leq 4n^{-1} \sup_{p'=k_n}^{\infty} \sup_{0 \leq s \leq \wedge R^n_{\pi}} \left( \sum_{p=k_n}^{\infty} \mathcal{H}_{p,p'} \right)^2 \cdot (\mathcal{U}^k(I_p^c)'t) + o_p(1) \]

\[ \leq n^{-1} \mathcal{J}_{n,t} \cdot d^n = O \left( n^{-\frac{1}{2}+\frac{1}{4}+3\gamma} \right) = o(1). \]

Combining this estimate with the fact that \( \lim_{L \to \infty} \sup_{\pi} P(R^n_{\pi} \leq t) = 0 \), we obtain \( \Delta_{1,t} = o_p(1) \). By symmetry we also obtain \( \Delta_{2,t} = o_p(1) \). Finally, consider \( \Delta_{3,t} \). Since Lemma 7.6 implies that \( I_p^k \cap I_p^{k'} = \emptyset \) if \( |p - p'| > 1 \), the (piecewise) Lipschitz continuity of \( \mathcal{U} \) yields \( \Delta_{3,t} = n^{-1/2} \sum_{p=k_n}^{\infty} \mathcal{H}_{p,p'} U^k(I_p^c) + o_p(1) \). Then, by \( [\mathcal{SA}2] \) and \( \mathcal{SA}4 \) we obtain \( \Delta_{3,t} = n^{-1/2} \sum_{p=k_n}^{\infty} \sum_{q=p+2}^{p+d_n} c_{u,v}(p,q)c_{u,v}(p,q)F_{T_{p-1}} [U^k, \mathcal{U}^k]((I_p^c)'t) + o_p(1) \)

\[ = n^{-1/2} \sum_{p=k_n}^{\infty} \sum_{q=p+2}^{p+d_n} \sum_{p=k_n}^{\infty} F_{T_{p-1}} [U^k, \mathcal{U}^k]((I_p^c)'t) + o_p(1). \]

Consequently, we can compute the limiting variable of \( \Delta_{3,t} \) explicitly by \( (2.1) \), a Riemann approximation and Lemma 7.9, and thus we complete the proof of this case.

Case 2: \( V = \bar{V} = \mathcal{E} \). In this case we have \( \langle V^l, V^l' \rangle t = 1 \sum_{q,k_n}^{\infty} \mathcal{T}_{\bar{V}}^l (r_{\bar{V}}^l \leq t) \), hence it holds that

\[ \mathcal{G}_t^n = \psi_2^{-2} \sum_{p,k_n}^{p+d_n} \mathcal{H}_{p,p'} \sum_{p,k_n}^{p+d_n} \phi_{u,v}(q-p) \phi_{u,v}(p,q) F_{T_{p-1}} [U^k, \mathcal{U}^k]((I_p^c)'t) + o_p(1) \]

\[ = \psi_2^{-2} \sum_{p,k_n}^{p+d_n} \mathcal{H}_{p,p'} \phi_{u,v}(q-p) \phi_{u,v}(p,q) F_{T_{p-1}} [U^k, \mathcal{U}^k]((I_p^c)'t) + o_p(1). \]

Therefore, an argument similar to the proof of \( (7.20) \) yields \( \mathcal{G}_t^n = \psi_2^{-2} \sum_{q,k_n}^{\infty} \mathcal{H}_{q,q} \mathcal{T}_{\bar{V}}^l (r_{\bar{V}}^l \leq t) + o_p(1) \), and an argument similar to the proof of \( (7.15) \) (using \( (7.19) \) and \( [\mathcal{A}4](iv) \)) instead of Lemma 7.8 and \( [\mathcal{A}4](ii) \) respectively) implies that \( \mathcal{G}_t^n = \psi_2^{-2} \sum_{q,k_n}^{\infty} \mathcal{H}_{q,q} \mathcal{T}_{\bar{V}}^l (r_{\bar{V}}^l \leq t) + o_p(1) \). Now we can apply arguments similar to those of Case 1 after the equation \( (7.21) \), and thus we obtain

\[ \mathcal{G}_t^n = \psi_2^{-2} \sum_{q,k_n}^{\infty} \mathcal{H}_{q,q} \mathcal{T}_{\bar{V}}^l (r_{\bar{V}}^l \leq t) + o_p(1). \]

Now the proof is completed in a similar manner to the previous case.

Case 3: \( V \neq \bar{V} \). In this case we have \( \langle V^l, \bar{V}^l' \rangle = 0 \), hence it holds that \( \mathcal{G}_t^n \to 0 \).

Consequently, we complete the proof.
7.5 Proof of Lemma 3.4

(a) Since $\Delta M_{u,v}^{(k,l)}(U, V)^n \equiv 0$ if $V = M$, it suffices to consider the case that $V = \mathcal{E}$. In this case we have

$$\Delta M_{u,v}^{(k,l)}(U, V)^n_s = -\frac{1}{k_n} \sum_{q = k_n}^{\infty} C_{u,v}(U)_{q}^k \xi_{q}^{l} 1_{\{r_q^s = s\}},$$

hence it holds that $\left| \Delta M_{u,v}^{(k,l)}(U, V)^n_s \right|^4 = k_n^{-4} \sum_{q = k_n}^{\infty} \left| C_{u,v}(U)_{q}^k \xi_{q}^{l} \right|^4 1_{\{r_q^s = s\}}$. Therefore, the Schwarz inequality, Lemma 7.8 and [S3A] yield

$$E_0 \left[ n \sum_{0 \leq s \leq t} \left| \Delta M_{u,v}^{(k,l)}(U, V)^n_s \right|^4 \right] \leq n k_n^{-4} \left( n^{-\xi + \frac{1}{4} + 3\gamma} \log n \right)^2 \sum_{0 \leq s \leq t} \sum_{q = k_n}^{\infty} 1_{\{r_q^s = s\}}.$$  

Since $\sum_{0 \leq s \leq t} \sum_{q = k_n}^{\infty} 1_{\{r_q^s = s\}} \leq \sum_{q = k_n}^{\infty} 1_{\{r_q^t \leq t\}} = O_p(n)$, we conclude that

$$E_0 \left[ n \sum_{0 \leq s \leq t} \left| \Delta M_{u,v}^{(k,l)}(U, V)^n_s \right|^4 \right] = O_p \left( n^{-2\xi + 1 + 6\gamma} \log n \right)^2 = o_p(1).$$

(b) First, since we have

$$\langle M_{u,v}^{(k,l)}(U, V)^n, N \rangle_t = \sum_{q = k_n}^{\infty} C_{u,v}(U)_{q}^k (V^l, N)_{(q)}(l)_{t}$$

(7.22)

for any square-integrable martingale $N$, it is clear that the claim holds true if $V = \mathcal{E}$. On the other hand, if $V = M$, we define the $d$-dimensional process $B_t$ by $B_t^{i} = \int_{0}^{t} \sum_{k,l}^{d} \Sigma_{s}^{k,l} ds (l = 1, \ldots, d)$. Then, from (7.22) we obtain $\langle M_{u,v}^{(k,l)}(U, V)^n, N \rangle_t = M_{u,v}^{(k,l)}(U, B_t^{i})$, hence the claim follows from Lemma 3.2.

(c) First, from (7.22) the claim obviously holds true if $V = M$. Hence we concentrate the case that $V = \mathcal{E}^l$.

We will follow the strategy used in [15] and [16]. Fix a $t \geq 0$ and let $\mathcal{N}$ be the set of all square-integrable martingales orthogonal to $M$ and satisfying $n^{\frac{1}{2}} \langle M_{u,v}^{(k,l)}(U, V)^n, N \rangle_t \rightarrow 0$ as $n \rightarrow \infty$. Then, noting that $n^{\frac{1}{2}} \langle M_{u,v}^{(k,l)}(U, V)^n \rangle_t = O_0(1)$ by Lemma 3.3, the Kunita-Watanabe inequality implies that $\mathcal{N}$ is a closed subset of the Hilbert space $\mathcal{M}_{2}^{\frac{1}{2}}$ of all square-integrable martingales orthogonal to $M$.

Let $N$ be in the set $\mathcal{N}^0$ of all square-integrable martingales on $\mathcal{B}^{(0)}$ orthogonal to $M$. Then it is easy to check that $\langle \mathcal{E}^l, N \rangle_t = 0$, hence we have $\mathcal{N}^0 \subset \mathcal{N}$.

Let $N$ be in the set $\mathcal{N}^1$ of all square-integrable martingales having $N_\infty = f(\epsilon_1, \ldots, \epsilon_m)$, where $f$ is any bounded Borel function on $\mathbb{R}^{dm}$, $t_1 < \cdots < t_m$ and $m \geq 1$. Then it is easy to check that $N$ takes the following form (by convention $t_0 = 0$ and $t_{m+1} = \infty$): $t_j \leq t < t_{j+1} \Rightarrow N_t = M_j(\epsilon_1, \ldots, \epsilon_j)_t$ for $j = 0, \ldots, m$, and where $M_j(\xi, \ldots, \xi, \omega)$ is a version of the martingale

$$M_j(\xi, \ldots, \xi, \omega)_t = E(0) \left[ \int f(\xi, \ldots, \xi, \omega) Q_{t_{j+1}}(d \xi_{j+1} \cdots Q_{t_m}(d \xi_m) | \mathcal{F}_t^{(0)}) \right]$$

(with obvious conventions when $j = 0$ and $j = m$), which is measurable in $(\xi, \ldots, \xi, \omega)$. Here, $E(0)$ denotes the (conditional) expectation with respect to $P(0)$. Therefore, we have

$$[M_{u,v}^{(k,l)}(U, V)^n, N]_t = -\frac{1}{k_n} \sum_{j=1}^{m} \sum_{q = k_n}^{\infty} C_{u,v}(U)_{q}^k \xi_{q}^{l} \Delta N_{t_j} 1_{\{r_q^t = t_j\}},$$

hence the Schwarz inequality, Lemma 7.8, [S3A] and the boundedness of $N$ yield

$$E_0 \left[ n^{1/4}[M_{u,v}^{(k,l)}(U, V)^n, N]_t \right] \leq n^{1/4} \left( n^{-\xi + \frac{1}{4} + 3\gamma} \log n \right)^{1/2} \sum_{j=1}^{m} \sum_{q = k_n}^{\infty} 1_{\{r_q^t = t_j\}} \leq n^{-\xi + \frac{1}{4} + 3\gamma} \log n^{1/2} m = o_p(1).$$

29
Therefore, in order to prove \( N \in \mathcal{N} \) it suffices to show that \( n^{1/4} \left\{ [b_{u,v}^{(k,l)}(U,V)^n]_t - \langle M_{u,v}^{(k,l)}(U,V)^n \rangle_t \right\} \to P 0. \) Since \( \langle V, N \rangle - \langle V, N \rangle \) is an \( \mathbf{F}\)-martingale and

\[
n^{1/4} \left\{ [b_{u,v}^{(k,l)}(U,V)^n]_t - \langle M_{u,v}^{(k,l)}(U,V)^n \rangle_t \right\} = n^{1/4} \sum_{q=k_n}^{\infty} C_{u,v}(U)_{\bar{q}} \{ [V^t, N](I_p^i)_{\bar{t}} - \langle V^t, N \rangle(I_p^i)_{\bar{t}} \},
\]

noting that \( C_{u,v}(U)_{\bar{q}} \) is \( \mathcal{F}_{\bar{q}-1} \)-measurable, it suffices to show that

\[
n^{1/2} \sum_{q=k_n}^{\infty} |C_{u,v}(U)_{\bar{q}}|^2 E \left[ [V^t, N](I_p^i)_{\bar{t}} - \langle V^t, N \rangle(I_p^i)_{\bar{t}} \right]^2 |\mathcal{F}_{\bar{q}-1} \right] \to P 0 \tag{7.23}
\]

in the light of Lemma 7.1(a). The boundedness of \( N \), [SA3] and the tightness of \( n^{-1} N_{\mathbf{N}}^n \) yield

\[
E_0 \left[ \sum_{q=k_n}^{\infty} [V^t, N](I_p^i)_{\bar{t}}^2 \right] \leq \frac{1}{k_n^2} E_0 \left[ \sum_{q=k_n}^{\infty} \left| \epsilon_{\bar{q}} \right|^4 \left| 1_{\{\tau_{\bar{q}} \leq t\}} \right|^4 \right] = O_p(k_n^{-2}).
\]

Furthermore, noting that \( (V^t)_i = k_n^{-2} \sum_{q} \tau_{\bar{q}} 1_{\{\tau_{\bar{q}} \leq t\}} \), the Kunita-Watanabe inequality, [SA3] and the boundedness of \( N \) imply that

\[
E \left[ \sum_{q=k_n}^{\infty} (W, N)(I_p^i)_{\bar{t}}^2 \right] \leq E \left[ \sum_{q=k_n}^{\infty} (W)(I_p^i)_{\bar{t}}(N)(I_p^i)_{\bar{t}} \right] \lesssim k_n^{-2} E \left[ \sum_{q=k_n}^{\infty} (N)(I_p^i)_{\bar{t}} \right] \lesssim k_n^{-2} m.
\]

Combining these estimates with Lemma 7.8, we obtain (7.23), hence \( \mathcal{N} \subset \mathcal{N} \).

Since \( \mathcal{N}_0 \cup \mathcal{N}_1 \) is a total subset of \( \mathcal{M}_2 \), the above argument implies that \( \mathcal{N} = \mathcal{M}_2 \).

\[\square\]

### 7.6 Proof of Proposition 5.1

First, an argument similar to the proof of Lemma 7.5 yields

\[
\frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{n_{\mathbf{N}}^n - k_n + 1} (\bar{X}_i)_t^3 = \frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{n_{\mathbf{N}}^n} (\bar{X}(g)_i)_t^3 + o_p(1).
\]

Next, noting that an argument similar to the proof of (7.8) implies that \( |\bar{X}(g)_i| \leq n^{-\frac{1}{2}}(\bar{X}\frac{1}{2} + 3\gamma) \sqrt{\log n} \), we have

\[
\frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} (\bar{X}(g)_i)_t^3 \leq n^{-1/2} n^{-\frac{1}{2} + 3\gamma} \sqrt{\log n} \sum_{p=k_n}^{\infty} |g_{p-1}||I_p \cap [0, t]| \leq n^{-\frac{1}{2} + 3\gamma} \sqrt{\log n},
\]

hence it holds that

\[
\frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} (\bar{X}(g)_i)_t^3 = \frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} (\bar{M}(g)_i)_t^3 + o_p(1).
\]

Now, Itô's formula yields

\[
\frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} (\bar{M}(g)_i)_t^3 = 3 \frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} (\bar{M}(g)_i)_t^2 \cdot \bar{M}(g)_t + 3 \frac{\sqrt{n}}{k^{3/2}} \sum_{i=k_n}^{\infty} \bar{M}(g)_i \cdot [\bar{M}(g)]_t
\]

\[
= \mathbf{I}_t + \mathbf{II}_t.
\]

Since \( \mathbf{I} \) is a locally square-integrable martingale and its predictable quadratic variation satisfies

\[
(\mathbf{I})_t = \frac{1}{k^{3/2}} \sum_{i,j=k_n}^{\infty} (\bar{M}(g)_{\mathrm{\#}})_t^2 (\bar{M}(g)_{\mathrm{\#}})_t^2 \cdot (\bar{M}(g)_i, \bar{M}(g)_i)_t \leq \frac{1}{k^{3/2}} \sum_{i,j=k_n}^{\infty} g_{p-1} g_{p-1} (\bar{M})(I_p)_t \leq k^{-1/2} n^{-\frac{1}{2} + 3\gamma} \sqrt{\log n} = o_p(1),
\]

\[30\]
the Lenglart inequality yields \( \Pi_t = o_p(1) \). On the other hand, by the use of associativity and linearity we obtain

\[
\Pi_t = 3\psi_2 \sqrt{\frac{n}{k_n^2}} \sum_{p,q=k_n}^{\infty} c_{g,q}(p,q)M(I_p)_{\epsilon} \bullet [M](I_q)_{\epsilon},
\]

hence an argument similar to that from the proof of Lemma 7.7 implies that

\[
\Pi_t = 3\psi_2 \sqrt{\frac{n}{d_n}} \sum_{q=k_n+2}^{\infty} \sum_{p=k_n \vee (q-d_n)}^{q-2} c_{g,q}(p,q)M(I_p)_{\epsilon} \bullet [M](I_q)_{\epsilon} + o_p(1).
\]

Consequently, by Lemma 3.2 we obtain \( \Pi_t = o_p(1) \), and thus we complete the proof. □

7.7 Proof of Lemma 5.1

(a) It immediately follows from Theorem 3.1.

(b) First, Eqs. (7.11) and (7.17), Lemma 7.2 and the fact that \( n^{1/4}h_n^{-1} \rightarrow \infty \) as well as \( \rho \geq 4/3 \) yield

\[
\sup_{0 \leq t \leq T} \left| \partial[Y]_{t}^{n,k} - h_n^{-1} \sum_{p: (t-h_n)^+ < T_{p-1} \leq t} \frac{Y_{T_{p-1},1}X_{T_{p-1}}^{n,k}}{G_{T_{p-1}}^{n,k}} |I_p| \right| \rightarrow^p 0
\]

as \( n \rightarrow \infty \). Since it holds that \( \sup_{0 \leq t \leq T} \sum_{p: (t-h_n)^+ < T_{p-1} \leq t} |I_p| \leq h_n + \bar{r}n \). \[A4\] implies that \( \sup_{0 \leq t \leq T} |\partial[Y]_{t}^{n,k}| = O_P(1) \) and

\[
\partial[Y]_{t}^{n,k} = \frac{T_n^{k}}{\Theta^2 G_t} h_n^{-1} \sum_{p: (t-h_n)^+ < T_{p-1} \leq t} |I_p| + o_p(1).
\]

Since \( h_n^{-1} \sum_{p: (t-h_n)^+ < T_{p-1} \leq t} |I_p| \rightarrow^p 1 \) as \( n \rightarrow \infty \) for every \( t \in (0, T] \), we complete the proof. □

7.8 Proof of Theorem 5.1

Since \[A4\] and the Lenglart inequality yield \( \sum_{p=1}^{N_p} |I_p||I_{p+1}| = O_p(n^{-1}) \), by Lemma 5.1 and the dominated convergence theorem we obtain \( \tilde{\varphi}_{k,l,k',l'} = \sum_{p=1}^{N_p} \tilde{\varphi}_{k,l,k',l'} |I_p| \) as \( n \rightarrow \infty \), where \( \tilde{\varphi}_{k,l,k',l'} \) is defined by (5.1) with replacing \( \tilde{\varphi}_{k,l,k',l'} \) by the corresponding limiting variables given by Lemma 5.1. Then, we obtain the desired result by \[A4\] and Lemma 7.1 as well as (2.1). □

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