Closed-form expressions for derivatives of Bessel functions with respect to the order

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Abstract
We have used recent integral representations of the derivatives of the Bessel functions with respect to the order to obtain closed-form expressions in terms of generalized hypergeometric functions and Meijer-G functions. Also, we have carried out similar calculations for the derivatives of the modified Bessel functions with respect to the order, obtaining closed-form expressions as well. For this purpose, we have obtained integral representations of the derivatives of the modified Bessel functions with respect to the order. As by-products, we have calculated two non-tabulated integrals.

Keywords: Bessel functions, modified Bessel functions, generalized hypergeometric functions, Meijer-G function

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1. Introduction

The Bessel functions have had many applications since F. W. Bessel (1784-1846) found this kind of functions in his studies of planetary motion. In Physics, these functions arise naturally in boundary value problems of potential theory for cylindrical domains [8, Chap.6]. In Mathematics, the Bessel functions are encountered in the theory of differential equations with turning points, and well as with poles [11, Sect. 10.72]. Thereby, the theory of Bessel functions has been studied extensively in many classical textbooks [14, 11].

Usually, the definition of the Bessel function of the first kind $J_\nu(z)$ and
the modified Bessel function \( I_\nu(z) \) are given in series form as follows:

\[
J_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k! \Gamma(\nu + k + 1)}, \quad (1)
\]

and

\[
I_\nu(z) = \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k! \Gamma(\nu + k + 1)}. \quad (2)
\]

The Bessel function of the second kind \( Y_\nu(z) \) is defined in terms of the Bessel function of the first kind as

\[
Y_\nu(z) = \frac{J_\nu(z) \cos \pi \nu - J_{-\nu}(z)}{\sin \pi \nu}, \quad \nu \notin \mathbb{Z}, \quad (3)
\]

and similarly, for the Macdonald function \( K_\nu(z) \), we have

\[
K_\nu(z) = \frac{\pi I_{-\nu}(z) - I_\nu(z)}{2 \sin \pi \nu}, \quad \nu \notin \mathbb{Z}. \quad (4)
\]

Despite the fact, that the literature about the Bessel functions is very large as mentioned before, the literature regarding the derivatives of \( J_\nu, Y_\nu, I_\nu \) and \( K_\nu \) with respect to the order \( \nu \) is relatively scarce. For instance, for \( \nu = \pm 1/2 \) we find expressions for the order derivatives in terms of the exponential integral \( \text{Ei}(z) \) and the sine and cosine integrals, \( \text{Ci}(z) \) and \( \text{Si}(z) \) \cite{10,3}. By using the recurrence relations of Bessel \cite{11} Eqn. 10.6.1] and modified Bessel functions \cite{8} Eqn. 5.7.9, we can derive expressions for half-integral order \( \nu = n \pm 1/2 \). Also, for integral order \( \nu = n \) we find some series representations in \cite{3}. For arbitrary order, we have the following series representations \cite{11}, Eqns. 10.15.1 & 10.38.1]

\[
\frac{\partial J_\nu(z)}{\partial \nu} = J_\nu(z) \log \left( \frac{z}{2} \right) - \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu + k + 1) (-1)^k (z/2)^{2k}}{k! \Gamma(\nu + k + 1)}, \quad (5)
\]

and

\[
\frac{\partial I_\nu(z)}{\partial \nu} = I_\nu(z) \log \left( \frac{z}{2} \right) - \left( \frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu + k + 1) (z/2)^{2k}}{k! \Gamma(\nu + k + 1)}, \quad (6)
\]

which are obtained directly from (1) and (2). Also, from (3) and (4), we can calculate the order derivative of \( Y_\nu \) and \( K_\nu \) as \cite{11} Eqns 10.15.2 & 10.38.2,

\[
\frac{\partial Y_\nu(z)}{\partial \nu} = \cot \pi \nu \left[ \frac{\partial J_\nu(z)}{\partial \nu} - \pi Y_\nu(z) \right] - \csc \pi \nu \frac{\partial J_{-\nu}(z)}{\partial \nu} - \pi J_\nu(z), \quad (7)
\]
and
\[ \frac{\partial K_\nu(z)}{\partial \nu} = \frac{\pi}{2} \csc \pi \nu \left[ \frac{\partial I_{-\nu}(z)}{\partial \nu} - \frac{\partial I_\nu(z)}{\partial \nu} \right] - \pi \cot \pi \nu K_\nu(z). \quad (8) \]

Although we can accelerate the convergence of the alternating series given in (5) by using Cohen-Villegas-Zagier algorithm [5], this series does not converge properly for large \( z \), and it is not useful from a numeric point of view. Also, the series given in (6) is not useful for large \( z \) as well.

Nonetheless, in the literature we find integral representations of \( J_\nu(z) \) and \( I_\nu(z) \) in [2], which read as,

\[ \frac{\partial J_\nu(z)}{\partial \nu} = \pi \nu \int_0^{\pi/2} \tan \theta \, Y_0(z \sin^2 \theta) \, J_\nu(z \cos^2 \theta) \, d\theta, \quad \text{Re} \, \nu > 0, \quad (9) \]

and

\[ \frac{\partial I_\nu(z)}{\partial \nu} = -2 \nu \int_0^{\pi/2} \tan \theta \, K_0(z \sin^2 \theta) \, I_\nu(z \cos^2 \theta) \, d\theta, \quad \text{Re} \, \nu > 0. \quad (10) \]

Recently, new integral representations of \( J_\nu(z) \) and \( Y_\nu(z) \) are given in [6] for \( \nu > 0, |\arg z| \leq \pi \), and \( z \neq 0 \):

\[ \frac{\partial J_\nu(z)}{\partial \nu} = \pi \nu \left[ Y_\nu(z) \int_0^z J_\nu^2(t) \, dt + J_\nu(z) \int_z^\infty Y_\nu(t) J_\nu(t) \, dt \right], \quad (11) \]

and

\[ \frac{\partial Y_\nu(z)}{\partial \nu} = \pi \nu \left[ J_\nu(z) \left( \int_z^\infty \frac{Y_\nu^2(t)}{t} \, dt - \frac{1}{2\nu} \right) - Y_\nu(z) \int_z^\infty \frac{J_\nu(t) Y_\nu(t)}{t} \, dt \right]. \quad (12) \]

It is worth noting that [6] does not state the following direct result from (11) and (12),

\[ \frac{\partial}{\partial \nu} (J_\nu(z) Y_\nu(z)) \]

\[ = \pi \nu \left[ Y_\nu^2(z) \int_0^z \frac{J_\nu^2(t)}{t} \, dt + J_\nu^2(z) \left( \int_z^\infty \frac{Y_\nu^2(t)}{t} \, dt - \frac{1}{2\nu} \right) \right]. \quad (13) \]
However, in [4], we found $\frac{\partial J_\nu}{\partial \nu}$ in closed-form as,

$$\frac{\partial J_\nu(z)}{\partial \nu} = \frac{\pi [Y_\nu(z) - \cot \pi \nu J_\nu(z)]}{2 \Gamma^2(\nu + 1)} \left( \frac{z}{2} \right)^{2\nu} _2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + \frac{1}{2}, \nu + 1 \end{array} \left| -z^2 \right. \right) + J_\nu(z) \left[ \frac{1}{2\nu} - \psi(\nu + 1) + \log \left( \frac{z}{2} \right) \right] + \frac{z^2}{4(\nu^2 - 1)} _3F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \left| -z^2 \right. \right).$$

Also, from (14), the derivatives of $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ with respect to the order are calculated. Nevertheless, in [4], the calculation of (14) relies mostly on symbolic computer algebra. Since this calculation is highly non-trivial, the aim of this paper is to provide such calculation. For this purpose, we calculate the integrals given in (11) and (12). Moreover, we derive integral representations similar to (11) and (12) for the modified Bessel functions $I_\nu$ and $K_\nu$, wherein the integrals are calculated in closed-form as well. Therefore, the scope of this paper is the calculation of all these integrals to justify the closed-form expressions of the order derivatives of the Bessel and modified Bessel functions found in the literature. Also, since these closed-form expressions cannot be applied for $\nu \in \mathbb{Z}$, alternative expressions in terms of Meijer-$G$ functions are calculated as well.

We organize this article as follows. In Section 2, we calculate the integrals that appear in (11) and (12). For this purpose, we introduce the generalized hypergeometric function and its asymptotic behavior to rewrite (11)-(13) in closed-form. In Section 3, we derive integral representations of $\frac{\partial I_\nu}{\partial \nu}$ and $\frac{\partial K_\nu}{\partial \nu}$ similar to (11) and (12). We calculate the integrals of these integral representations to obtain the derivatives of the modified Bessel functions with respect to the order in closed-form. Section 4 is devoted to these same calculations, but in terms of Meijer-$G$ functions. Finally, we collect the conclusions in Section 5.
2. Order derivatives for Bessel functions

To calculate the integrals given in (11) and (12), we have to introduce the generalized hypergeometric function:

\[ pFq \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) \mid z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{15} \]

where \((\alpha)_k\) denotes the Pochhammer symbol [11, Eqn. 5.2.5],

\[ (\alpha)_k = \frac{\Gamma (\alpha + k)}{\Gamma (\alpha)}. \tag{16} \]

Next, we present an equivalent way to define a hypergeometric function [1, Sect. 2.1]. Any series

\[ \sum_{k=0}^{\infty} c_k, \]

that satisfies

\[ \frac{c_{k+1}}{c_k} = \frac{(k + a_1) \cdots (k + a_p) z}{(k + 1) (k + b_1) \cdots (k + b_q)}, \tag{17} \]

defines a hypergeometric series

\[ \sum_{k=0}^{\infty} c_k = c_0 pFq \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) \bigg| z \right). \tag{18} \]

The first integral of (11) can be calculated straightforwardly from the following tabulated integral [12, Eqn. 1.8.3]:

\[ \int_0^x t^\lambda J_\nu (t) J_\mu (t) dt = \frac{x^{\lambda+\mu+\nu+1}}{2^{\mu+\nu} (\lambda + \mu + \nu + 1) \Gamma (\mu + 1) \Gamma (\nu + 1)} \times 3F_4 \left( \begin{array}{c} \frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}, \frac{\lambda+\mu+\nu+1}{2} \\ \mu + 1, \nu + 1, \mu + \nu + 1, \frac{\lambda+\mu+\nu+3}{2} \end{array} \right| - x^2 \right) \tag{19} \]

\[ \text{Re} (\lambda + \mu + \nu) > -1, \]

thus, if \( \nu > 0 \), we have

\[ \int_0^z J_\nu^2 (t) \frac{dt}{t} = \frac{(z/2)^{2\nu}}{2\nu \Gamma^2 (\nu + 1)} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right| - z^2 \right). \tag{20} \]

The second integral in (11) is calculated as follows.
Theorem 1. If \( z \neq 0 \), \( |\arg z| < \pi \) and \( \nu > 0, \nu \notin \mathbb{Z} \), the following integral holds true:

\[
\int_{z}^{\infty} \frac{J_{\nu} (t) Y_{\nu} (t)}{t} \, dt = \frac{-1}{\pi \nu} \left[ \log \left( \frac{2}{z} \right) + \psi (\nu) + \frac{1}{2 \nu} \right] + \frac{\pi}{2} \cot \pi \nu \left( \frac{z/2}{\nu + 1} \right) - \frac{1}{4 (1 - \nu^2)} 3 \binom{\nu, \nu + 1}{1, 1, 2 \nu + 1, -z^2}.
\]

Proof. First, let us calculate the corresponding indefinite integral of (21) applying the definition of the \( Y_{\nu} (z) \) function given in (3). Thereby, we have

\[
\int \frac{J_{\nu} (t) Y_{\nu} (t)}{t} \, dt = \cot \pi \nu \int \frac{J_{\nu}^2 (t)}{t} \, dt - \csc \pi \nu \int \frac{J_{-\nu} (t) J_{\nu} (t)}{t} \, dt. \tag{22}
\]

Notice that the first integral on the RHS of (22) has been calculated in (20). However, the general expression given in (19) fails for the second integral. Nonetheless, taking \( \mu = -\nu \) in the following expression [11, Eqn. 10.8.3]

\[
J_{\nu} (z) J_{\mu} (z) = \left( \frac{z}{2} \right)^{\mu + \nu} \sum_{n=0}^{\infty} \frac{(\nu + \mu + n + 1)^n (-1)^n (z/2)^{2n}}{n! \Gamma (n + \mu + 1) \Gamma (n + \nu + 1)},
\]

and separating the first term, we can integrate term by term, arriving at

\[
\int \frac{J_{-\nu} (t) J_{\nu} (t)}{t} \, dt = \frac{\log t}{\Gamma (1 + \nu) \Gamma (1 - \nu)} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{\Gamma (2k + 1) (-1)^k (t/2)^{2k+1}}{k! \Gamma (k + 1) \Gamma (k + \nu + 1) \Gamma (k - \nu + 1)},
\]

where we have used the definition of the Pochhammer symbol [11]. Using now the following properties of the gamma function [8, Eqn. 1.2.1&2]:

\[
\Gamma (z + 1) = z \Gamma (z), \tag{24}
\]

and

\[
\Gamma (z) \Gamma (1 - z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}, \tag{25}
\]
and expressing the sum given in (23) as a hypergeometric function, after some simplification, we arrive at

\[
\int J_{-\nu}(t) J_{\nu}(t) \, dt = \sin \frac{\pi \nu}{\pi \nu} \left\{ \log t - \frac{t^2}{4(1 - \nu^2)} \right. \right. 3F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \right| - t^2 \bigg) \right\}.
\]

Inserting now the results (20) and (26) in (22), we obtain

\[
\int J_{\nu}(t) Y_{\nu}(t) \, dt = \frac{1}{\pi \nu} \left[ -\log t + \left( \frac{t}{2} \right)^{2\nu} \frac{\pi \cot \pi \nu}{2 \Gamma^2 (\nu + 1)} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right| - t^2 \bigg) \right. \right. + \frac{t^2}{4(1 - \nu^2)} 3F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \right| - t^2 \bigg].
\]

To calculate (27) with the integration limits given in (21), we have to perform the following limits:

\[
limit_{t \to \infty} \frac{\cot \pi \nu}{2\nu \Gamma^2 (\nu + 1)} \left( \frac{t}{2} \right)^{2\nu} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right| - t^2 \bigg) = \cot \pi \nu \left( \frac{t}{2} \right)^{2\nu} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right| - t^2 \bigg),
\]
and

\[
lim_{t \to \infty} \frac{t^2}{4\pi \nu (1 - \nu^2)} 3F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \right| - t^2 \bigg).
\]

For this purpose, let us apply the following asymptotic formula for the \( pF_{p+1} \) hypergeometric function as \( |z| \to \infty \) (see [11, Sect. 16.11]):

\[
pF_{p+1} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_{p+1} \end{array} \right| z \right) = \frac{\prod_{j=1}^{p+1} \Gamma (b_j)}{\sqrt{\pi} \prod_{k=1}^{p} \Gamma (a_k)} (-z)^x \left\{ \cos (\pi \chi + 2\sqrt{-z}) \left[ 1 + O \left( \frac{1}{z} \right) \right] \right. \right. + \frac{c_1}{2\sqrt{-z}} \sin (\pi \chi + 2\sqrt{-z}) \left[ 1 + O \left( \frac{1}{z} \right) \right] \right\} \right. \right. + \frac{\prod_{j=1}^{p+1} \Gamma (b_j)}{\prod_{k=1}^{p} \Gamma (a_k)} \sum_{k=1}^{p} \frac{\Gamma (a_k) \prod_{j=1,j \neq k}^{p} \Gamma (a_j - a_k)}{\prod_{j=1}^{p+1} \Gamma (b_j - a_k)} (-z)^{-a_k} \left[ 1 + O \left( \frac{1}{z} \right) \right],
\]
wherein the case of simple poles (i.e. \( a_j - a_k \notin \mathbb{Z} \)) and the following definitions are considered:

\[
A_p = \sum_{k=1}^{p} a_k, \quad B_{p+1} = \sum_{k=1}^{p+1} b_k, \\
\chi = \frac{1}{2} \left( A_p - B_{p+1} + \frac{1}{2} \right), \\
A = \sum_{s=2}^{p} \sum_{j=1}^{s-1} a_s a_j, \quad B = \sum_{s=2}^{p+1} \sum_{j=1}^{s-1} b_s b_j, \\
c_1 = 2 \left( B - A + \frac{1}{4} (3A_p + B_{p+1} - 2) (A_p - B_{p+1}) - \frac{3}{16} \right).
\]

Therefore, after some long but simple calculations using the properties (24), (25) and [8, Eqn. 1.2.3]

\[
2^{z-1} \Gamma (z) \Gamma \left( z + \frac{1}{2} \right) = \sqrt{\pi} \Gamma (2z), \quad (31)
\]

of the gamma function, the asymptotic expansion of (28) reads as

\[
\frac{\cot \pi \nu}{2 \nu} \left( \frac{t}{2} \right)^{2 \nu} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right) - t^2 \quad (32)
\]

Notice that to calculate the limit given in (29), we cannot apply directly (30) since we have a double pole \( (a_1 = a_2 = 1) \). Nevertheless, we can still using (30), calculating the following asymptotic expansion:

\[
\frac{t^2}{4\pi \nu (1 - \nu^2)} 3F_4 \left( \begin{array}{c} 1, 1 + \epsilon, \frac{3}{2}, 2, 2 - \nu, 2 + \nu \\ \frac{3}{2}, 2, 2 - \nu, 2 + \nu \end{array} \right) - t^2 \\
= -\frac{\Gamma (\epsilon - \frac{1}{2}) \cot \pi \nu}{2\pi^{3/2} t \Gamma (1 + \epsilon)} + \frac{t^{-2+\epsilon} \cos (2t + \frac{\pi \epsilon}{2}) \csc \pi \nu}{2\pi \Gamma (1 + \epsilon)} \\
+ \frac{1}{2\pi \nu \epsilon} + \frac{t^{-2\epsilon} \Gamma (\frac{1}{2} - \epsilon) \csc \pi \nu}{2\sqrt{\pi} \epsilon^2 \Gamma (-\epsilon) \Gamma (1 - \nu - \epsilon) \Gamma (1 + \nu - \epsilon)} + O \left( \frac{1}{t^3} \right),
\]
and then calculating the limit $\epsilon \to 0$. For this purpose, consider the following first order Taylor-series expansions as $\epsilon \to 0$,

$$
\Gamma (a - \epsilon) \approx \Gamma (a) [1 - \psi (a) \epsilon], \quad (33)
$$

$$
\frac{1}{\Gamma (a - \epsilon)} \approx \frac{1}{\Gamma (a)} [1 + \psi (a) \epsilon], \quad (34)
$$

$$
a^\epsilon \approx 1 + \log (a) \epsilon, \quad (35)
$$

where $\psi (z) = \Gamma' (z) / \Gamma (z)$ denotes the digamma function \[11\, \text{Eqn. 5.2.2}\]. Also, consider the following approximation (see \[11\, \text{Eqn. 5.7.1}\]),

$$
\Gamma (\epsilon) \approx \frac{1}{\epsilon} - \gamma, \quad \epsilon \to 0, \quad (36)
$$

where $\gamma = 0.57721566\ldots$ denotes Euler’s constant. Therefore, taking into account (33)–(36), we have

$$
\lim_{\epsilon \to 0} \frac{t^2}{4 \pi \nu (1 - \nu^2)} 3F_4 \left( \begin{array}{c} 1, 1 + \epsilon, \frac{3}{2} \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \left| - t^2 \right. \right) \quad (37)
$$

$$
\sim \frac{1}{2 \pi \nu} \left[ \log \left( \frac{t^2}{4} \right) - \psi (1 + \nu) - \psi (1 - \nu) \right], \quad t \to \infty,
$$

where we have considered that \[8\, \text{Eqn. 1.3.8}\]

$$
\psi \left( \frac{1}{2} \right) = -\gamma - 2 \log 2.
$$

Taking into account (32) and (37), and applying the following properties of the digamma function \[8\, \text{Eqns. 1.3.3&4}\]

$$
\psi (z + 1) = \frac{1}{z} + \psi (z), \quad (38)
$$

$$
\psi (1 - z) - \psi (z) = \pi \cot \pi z, \quad (39)
$$

we arrive at

$$
\lim_{t \to \infty} \frac{1}{\pi \nu} \left[ \frac{\pi \cot \pi \nu} {2 \Gamma^2 (\nu + 1)} \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \left| - t^2 \right. \right) \right. \quad (40)
$$

$$
- \log t + \frac{t^2}{4 (1 - \nu^2)} 3F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ 2, 2 - \nu, 2 + \nu \end{array} \left| - t^2 \right. \right) \right]
$$

$$
= - \frac{1}{2 \nu} \left[ \frac{1}{\nu} + \psi (\nu) + \log 2 \right].
$$
Finally, according to (27) and (40), we conclude (21).

Next, we will calculate the integrals given in the integral representation of $\partial Y_\nu/\partial \nu$ given in (12).

**Theorem 2.** If $z \neq 0$, $|\arg z| < \pi$ and $\nu > 0$, $\nu \not\in \mathbb{Z}$, the following integral holds true:

$$
\int_0^\infty \frac{Y_\nu^2(t)}{t} dt = \frac{1}{2\pi^2 \nu} \left[ \left( \frac{z}{2} \right)^{-2\nu} \Gamma^2(\nu) \, _2F_3 \left( \begin{array}{c} -\nu, \frac{1}{2} - \nu \\ 1 - \nu, 1 - \nu, 1 - 2\nu \end{array} \right) - z^2 \right] 
- \left( \frac{z}{2} \right)^{2\nu} \Gamma^2(-\nu) \cos^2 \pi \nu \, _2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \right) - z^2 \right] 
- \frac{1 + 2 \cot^2 \pi \nu}{2\nu} - \frac{2 \cot \pi \nu}{\pi \nu} \left[ \frac{z^2}{4 (1 - \nu^2)} \, _4F_4 \left( \begin{array}{c} 1, 1, \frac{3}{2}, \nu - 2 \nu + 2 + \nu \\ 2, 2, 2 - \nu, 2 + \nu \end{array} \right) - z^2 \right] 
+ \log \left( \frac{2}{z} \right) + \frac{1}{2\nu} + \psi(\nu) \right].
$$

**Proof.** First, let us calculate the indefinite integral of (41). By using the definition given in (3) of the Bessel function of the second kind $Y_\nu(z)$, we have that

$$
\int \frac{Y_\nu^2(t)}{t} dt = \cot^2 \pi \nu \int \frac{J_\nu^2(t)}{t} dt + \csc^2 \pi \nu \int \frac{J_{-\nu}^2(t)}{t} dt 
- \frac{2 \cos \pi \nu}{\sin^2 \pi \nu} \int \frac{J_\nu(t) J_{-\nu}(t)}{t} dt.
$$

Notice that we have already calculated the first integral given on the RHS of (12) in (20), thus the second integral on the RHS of (12) is precisely (20) performing the change $\nu \to -\nu$. Also, we have already calculated the third
integral on the RHS of (42) in (26). Collecting all these results, we have

\[
\int \frac{Y^2(t)}{t} dt = \cot^2 \pi \nu \left( \frac{t}{2} \right)^{-2\nu} \left( \begin{array}{c}
\nu, \nu + \frac{1}{2}, 2
\nu + 1, \nu + 1, 2\nu + 1
\end{array} \right) - t^2
\]  

(43)

\[
- \frac{\cot^2 \pi \nu}{2\nu \Gamma^2 (1 - \nu)} \left( \begin{array}{c}
-\nu, \frac{1}{2} - \nu, 1 - \nu, 1 - 2\nu
\end{array} \right) - t^2
\]

- \frac{2 \cot \pi \nu}{\pi \nu} \left[ \log t - \frac{t^2}{4\nu (1 - \nu^2)} \right]

(44)

To calculate (42) with the integration limits given in (41), we have to consider the asymptotic expansion (32), replacing \( \nu \rightarrow \pm \nu \)

\[
\pm \frac{t/2}{2\nu \Gamma^2 (1 \pm \nu)} \left( \begin{array}{c}
\pm \nu, \frac{1}{2} \pm \nu, 1 \pm \nu, 1 \pm 2\nu
\nu + 1, \nu + 1, 2\nu + 1
\end{array} \right) - t^2
\]

(44)

\[
= \pm \frac{1}{2\nu} - \frac{1}{\pi t} + O \left( \frac{1}{t^2} \right), \quad t \rightarrow \infty.
\]

Also, considering the asymptotic expansion (37) and taking into account the properties of the digamma function given in (38) and (39), we have

\[
\frac{t^2}{4\pi \nu (1 - \nu^2)} \left( \begin{array}{c}
1, 1, \frac{3}{2}
2, 2 - \nu, 2 + \nu
\end{array} \right) - t^2
\]

(45)

\[
\sim \frac{1}{2\pi \nu} \left[ \log \left( \frac{t^2}{4} \right) - \frac{1}{\nu} - 2 \psi (\nu) - \pi \cot \pi \nu \right], \quad t \rightarrow \infty.
\]

Therefore, taking into account the indefinite integral (43) and the asymptotic expansions (44) and (45), after some simple calculations wherein we apply the reflection formula of the gamma function (25), we arrive at (41).

Finally, according to the integral representation given in (11), and the integrals calculated in (20) and (21), we express in closed-form the order
The derivative of the Bessel function as,

\[
\frac{\partial J_\nu(z)}{\partial \nu} = -\pi J_{-\nu}(z) \csc \nu \left( \frac{z}{2} \right)^{2\nu} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \left| -z^2 \right. \right) \\
- J_\nu(z) \left[ \frac{z^2}{4 (1 - \nu^2)} \right] 3F_4 \left( \begin{array}{c} 1, 1, 3 - \nu \\ 2, 2 - \nu, 2 + \nu \end{array} \left| -z^2 \right. \right) + \frac{\log \left( \frac{2}{z} \right)}{\nu} + \psi(\nu),
\]

where we have taken into account the definition of \( Y_\nu(z) \) given in (3). Note that (46) is equivalent to the result obtained by Brychov in (14).

Similarly, substituting (41) and (21) in (12), after some simplification, we arrive at,

\[
\frac{\partial Y_\nu(z)}{\partial \nu} = J_\nu(z) \left[ \frac{\Gamma^2(\nu)}{2\pi} \left( \frac{z}{2} \right)^{-2\nu} 2F_3 \left( \begin{array}{c} -\nu, \frac{1}{2} - \nu \\ 1 - \nu, 1 - \nu, 1 - 2\nu \end{array} \left| -z^2 \right. \right) - \pi \csc^2 \nu \right] \\
- \frac{\cos \pi \nu}{2} \Gamma^2(-\nu) J_{-\nu}(z) \left( \frac{z}{2} \right)^{2\nu} 2F_3 \left( \begin{array}{c} \nu, \nu + \frac{1}{2} \\ \nu + 1, \nu + 1, 2\nu + 1 \end{array} \left| -z^2 \right. \right) + \log \left( \frac{2}{z} \right) + \frac{1}{2\nu} + \psi(\nu)
\]

\times (Y_\nu(z) - 2 \cot \pi \nu J_\nu(z)),
\]

which is equivalent to the result given in [4].
Finally, according to (20) and (41), we rewrite (13) in closed-form as,

\begin{align*}
\frac{\partial}{\partial \nu} \left( J_\nu(z) Y_\nu(z) \right) &= 
\frac{J_{-\nu}(z)}{2\pi} \left( \frac{z}{2} \right)^{2\nu} \Gamma^2(-\nu) \\
&\times \left[ J_{-\nu}(z) - 2 \cos \pi \nu J_\nu(z) \right] \begin{pmatrix} \nu, \nu + \frac{1}{2} \\
\nu + 1, \nu + 1, 2\nu + 1 \end{pmatrix} - z^2 \\
&+ J_\nu^2(z) \left\{ \frac{(z/2)^{-2\nu}}{2\pi} \Gamma^2(\nu) \right. \\
&\times \begin{pmatrix} -\nu, \frac{1}{2} - \nu \\
1 - \nu, 1 - \nu, 1 - 2\nu \end{pmatrix} - z^2 \\
&- \pi \csc^2 \pi \nu - 2 \cot \pi \nu \\
&\times \left[ \frac{z^2}{4 (1 - \nu^2)} \right] \begin{pmatrix} 1, 1, \frac{3}{2} \\
2, 2, 2 - \nu, 2 + \nu \end{pmatrix} - z^2 \\
&+ \log \left( \frac{2}{z} \right) + \frac{1}{2\nu} + \psi(\nu) \right) .
\end{align*}

3. Order derivatives for modified Bessel functions

Similar integrals as in the previous Section can be calculated replacing Bessel functions by modified Bessel functions. Here we collect the results with a sketch of the proof.

**Theorem 3.** If \( \nu > 0 \), the following integral holds true:

\[
\int_0^z \frac{I_\nu^2(t)}{t} \, dt = \frac{(z/2)^{2\nu}}{2\nu \Gamma^2(\nu + 1)} \begin{pmatrix} \nu, \nu + \frac{1}{2} \\
\nu + 1, \nu + 1, 2\nu + 1 \end{pmatrix} z^2 .
\]  

**Proof.** Integrate term by term the following power series (Cauchy product) [11, Eqn. 10.31.3],

\[
I_\nu(z) I_\mu(z) = \left( \frac{z}{2} \right)^{\mu+\nu} \sum_{n=0}^{\infty} \frac{(\nu + \mu + n + 1)_n (z/2)^{2n}}{n! \Gamma(n + \mu + 1) \Gamma(n + \nu + 1)} ,
\]

taking \( \mu = \nu \), and recasting the result as a hypergeometric series. \( \blacksquare \)

**Remark 4.** If we take \( \mu = -\nu \) in (50), we will arrive at

\[
\int \frac{I_{-\nu}(t) I_\nu(t)}{t} \, dt = \frac{\sin \pi \nu}{\pi \nu} \left[ \log t + \frac{t^2}{4 (1 - \nu^2)} \right] \begin{pmatrix} 1, 1, \frac{3}{2} \\
2, 2, 2 + \nu, 2 - \nu \end{pmatrix} t^2 .
\]
Theorem 5. If $z \neq 0$, $|\arg z| < \pi$, and $\nu > 0$, $\nu \notin \mathbb{Z}$, the following integral holds true:

$$
\int_{z}^{\infty} \frac{I_{\nu}(t) K_{\nu}(t)}{t} dt = \frac{1}{2\nu} \left[ \frac{\pi \csc \pi \nu \ (z/2)^{2\nu}}{2 \Gamma^2(\nu + 1)} \right. 
\left. _2F_3 \left( \nu, \nu + \frac{1}{2} \left| \frac{z^2}{2\nu} \right. \right) \right.
\left. - \frac{z^2}{4 (1 - \nu^2)} _3F_4 \left( 1, 1, \frac{3}{2} \left| \frac{z^2}{2\nu} \right. \right) \right] + \log \left( \frac{2}{z} \right) + \psi(\nu) + \frac{1}{2\nu}.
$$

Proof. Expanding $K_{\nu}(z)$ in (52) and then using (49) and (51), we obtain the following result for the indefinite integral:

$$
\int \frac{I_{\nu}(t) K_{\nu}(t)}{t} dt = \frac{1}{2\nu} \left[ \log t - \frac{\pi \csc \pi \nu \ (t/2)^{2\nu}}{2 \Gamma^2(\nu + 1)} _2F_3 \left( \nu, \nu + \frac{1}{2} \left| t^2 \right. \right) \right.
\left. + \frac{t^2}{4 (1 - \nu^2)} _3F_4 \left( 1, 1, \frac{3}{2} \left| t^2 \right. \right) \right].
$$

To obtain (52), perform the asymptotic calculation of the hypergeometric functions given in (53), rewriting (30) as

$$
pF_{p+1} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_{p+1} \end{array} \right| z \right)
\sim \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{2\sqrt{\pi} \prod_{k=1}^{p} \Gamma(a_k)} z^{\chi} e^{\sqrt{\pi}} \left[ 1 + O \left( \frac{1}{\sqrt{z}} \right) \right]
\left. + \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{\prod_{k=1}^{p} \Gamma(b_k)} \sum_{k=1}^{p} \Gamma(a_k) \prod_{j=1,j\neq k}^{p} \Gamma(a_j - a_k) \prod_{j=1}^{p+1} \Gamma(b_j - a_k) \left( -z \right)^{-a_k} \left[ 1 + O \left( \frac{1}{z} \right) \right] \right].
$$

Theorem 6. If $z \neq 0$, $|\arg z| \leq \pi$, and $\nu \notin \mathbb{Z}$, $\nu \neq \pm 1/2, \pm 3/2$, the follow-
ing integral holds true:

\[
\int_{\infty}^{\infty} \frac{K_{\nu}^{2}(t)}{t} dt = \frac{1}{8\nu} \left\{ \left( \frac{z}{2} \right)^{-2\nu} \Gamma^{2}(\nu) \, _2F_3 \left( \begin{array}{c}
\nu, \frac{1}{2} + \nu \\
1 + \nu, 1 + \nu, 1 + 2\nu
\end{array} \right) \right. \\
- \left. \left( \frac{z}{2} \right)^{2\nu} \Gamma^{2}(-\nu) \, _2F_3 \left( \begin{array}{c}
-\nu, \frac{1}{2} - \nu \\
1 - \nu, 1 - \nu, 1 - 2\nu
\end{array} \right) \right. \\
+ 4\pi \csc \pi \nu \left[ \log \left( \frac{z}{2} \right) + \frac{z^2}{4(1 - \nu^2)} \, _3F_4 \left( \begin{array}{c}
1, 1, \frac{3}{2} \\
2, 2, 2 - \nu, 2 + \nu
\end{array} \right) \right. \\
- \frac{1}{2\nu} - \psi(\nu) - \frac{\pi}{2} \cot \pi \nu \right\}.
\]

Proof. Consider the definition given in (4) for \( K_{\nu}(z) \) to write the following.

\[
\int \frac{K_{\nu}^{2}(t)}{t} dt = \frac{\pi^2}{4} \csc^2 \pi \nu \left[ \int \frac{I_{\nu}^{2}(t)}{t} dt + \int \frac{I_{-\nu}^{2}(t)}{t} dt - 2 \int \frac{I_{-\nu}(t) I_{\nu}(t)}{t} dt \right].
\]

Taking into account the results given in (49) and (51), we obtain

\[
\int \frac{K_{\nu}^{2}(t)}{t} dt = \frac{\pi^2}{4} \csc^2 \pi \nu \left\{ \frac{(t/2)^{2\nu}}{2\nu \Gamma^2(\nu + 1)} \, _2F_3 \left( \begin{array}{c}
\nu, \nu + \frac{1}{2} \\
\nu + 1, \nu + 1, 2\nu + 1
\end{array} \right) \right. \\
- \frac{(t/2)^{-2\nu}}{2\nu \Gamma^2(1 - \nu)} \, _2F_3 \left( \begin{array}{c}
-\nu, \frac{1}{2} - \nu \\
1 - \nu, 1 - \nu, 1 - 2\nu
\end{array} \right) \\
- 2\sin \pi \nu \left[ \log t + \frac{t^2}{4(1 - \nu^2)} \, _3F_4 \left( \begin{array}{c}
1, 1, \frac{3}{2} \\
2, 2, 2 + \nu, 2 - \nu
\end{array} \right) \right] \}
\]

According to (54), we have the following asymptotic expansion as \( t \to \infty \)

\[
\pm \frac{(t/2)^{\pm 2\nu}}{2\nu \Gamma^2(\nu \pm 1)} \, _2F_3 \left( \begin{array}{c}
\pm \nu, \frac{1}{2} \pm \nu \\
1 \pm \nu, 1 \pm \nu, 1 \pm 2\nu
\end{array} \right) \left[ \right. \\
\approx \frac{e^{2t}}{4\pi t^2} + \frac{i (-1)^{\mp \nu} t}{\pi t} \pm \frac{(-1)^{\mp \nu}}{2\nu}.
\]

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Also,
\[
\lim_{\epsilon \to 0} \frac{t^2}{4 (1 - \nu^2)} \begin{pmatrix} 1, 1 + \epsilon, \frac{3}{2} \\ 2, 2 + \nu, 2 - \nu \end{pmatrix} 3F_4 \left( \frac{t^2}{4} \right) \approx \frac{i \nu \cot \pi \nu}{t} + \frac{\nu e^{2t} \csc \pi \nu}{4 t^2} + \frac{\psi (1 + \nu) + \psi (1 - \nu) - \log (-t^2)}{2} + \log 2.
\] (58)

Taking into account (57) and (58) in (56), after some simplification, we eventually arrive at (55).

Next, following a similar derivation as the one given in [6] for the integral representation of \( \partial J_\nu / \partial \nu \), we obtain an integral representation of \( \partial I_\nu / \partial \nu \).

\textbf{Theorem 7.} For \( \nu > 0 \) and \( z \neq 0 \), \( |\arg z| \leq \pi \), we have
\[
\frac{\partial I_\nu (z)}{\partial \nu} = -2 \nu \left[ I_\nu (z) \int_z^\infty \frac{K_\nu (t) I_\nu (t)}{t} \, dt + K_\nu (z) \int_0^z \frac{I_\nu^2 (t)}{t} \, dt \right].
\] (59)

\textbf{Proof.} Any linear combination of the modified Bessel functions \( I_\nu (z) \) and \( K_\nu (z) \) satisfies the following ODE [8, Eqn. 5.7.7],
\[
u'' (z) + \frac{1}{z} \nu' (z) - \left( 1 + \frac{\nu^2}{z^2} \right) \nu (z) = 0.
\] (60)

Consider now \( u (z) = I_\nu (z) \), and perform the derivative with respect to the order in (60), to obtain
\[
\frac{d^2}{dz^2} \left( \frac{\partial I_\nu (z)}{\partial \nu} \right) + \frac{1}{z} \frac{d}{dz} \left( \frac{\partial I_\nu (z)}{\partial \nu} \right) - \left( 1 + \frac{\nu^2}{z^2} \right) \frac{\partial I_\nu (z)}{\partial \nu} = \frac{2 \nu}{z^2} I_\nu (z).
\]

Applying now the method of variation of parameters [7, Sect. 16.516], taking into account the following Wronskian [8, Eqn. 5.9.5]
\[
W [I_\nu (z), K_\nu (z)] = -\frac{1}{z},
\]
the general solution of (60) is given by
\[
\frac{\partial I_\nu (z)}{\partial \nu} = -2 \nu \left[ I_\nu (z) \int_z^\infty \frac{K_\nu (t) I_\nu (t)}{t} \, dt + K_\nu (z) \int_0^z \frac{I_\nu^2 (t)}{t} \, dt \right] + a_\nu I_\nu (z) + b_\nu K_\nu (z),
\] (61)
where \( a_\nu \) and \( b_\nu \) are constants that can be determined as follows. First, notice that from the series representation (6), for \( \nu > 0 \) we have that

\[
\lim_{z \to 0} \nabla \nu \frac{\partial I_\nu(z)}{\partial \nu} = \lim_{z \to 0} I_\nu(z) \log \left( \frac{z}{2} \right) = 0,
\]  

(62)
since, according to [8, Eqn. 5.16.4],

\[
I_\nu(z) \approx \frac{(z/2)^\nu}{\Gamma(1 + \nu)}, \quad z \to 0.
\]  

(63)

Also, from (49), we have

\[
\int_0^z \frac{I_\nu^2(t)}{t} dt \approx \frac{(z/2)^{2\nu}}{2\nu \Gamma^2(\nu + 1)}, \quad z \to 0,
\]  

(64)

and from (52), we have as well

\[
\int_0^{\infty} \frac{I_\nu(t)}{t} K_\nu(t) dt \approx \frac{1}{2\nu} \log \left( \frac{2}{z} \right), \quad z \to 0.
\]  

(65)

Therefore, performing the limit \( z \to 0 \) on both sides of (61) and taking into account (62)-(65), we conclude that \( b_\nu = 0 \) since \( K_\nu(z) \) is divergent as \( z \to 0 \) [8, Eqn. 5.16.4]. Thereby, rewrite (61) as

\[
\frac{\partial I_\nu(z)}{\partial \nu} = -2\nu \left\{ I_\nu(z) \left[ a_\nu + \int_z^{\infty} \frac{K_\nu(t) I_\nu(t) dt}{t} \right] + K_\nu(z) \int_0^z \frac{I_\nu^2(t)}{t} dt \right\}.
\]  

(66)

Consider now the following asymptotic expansions [11, Eqns. 10.40.1-2] as \( z \to \infty \),

\[
I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[ 1 - \frac{4\nu^2 - 1}{8z} + O \left( \frac{1}{z^2} \right) \right],
\]  

(67)

\[
K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{4\nu^2 - 1}{8z} + O \left( \frac{1}{z^2} \right) \right].
\]  

(68)

On the one hand, performing the order derivative in (67), the asymptotic expansion on the LHS of (67) is

\[
\frac{\partial I_\nu(z)}{\partial \nu} \approx -\frac{\nu e^z}{\sqrt{2\pi z^{3/2}}}, \quad z \to \infty.
\]  

(69)
On the other hand, taking into account (67) and (68), we have
\[
\int_{z}^{\infty} \frac{K_{\nu}(t) I_{\nu}(t)}{t} dt \approx \frac{1}{2z}, \quad z \to \infty.
\] (70)

Also, from (49) and (54), we have,
\[
\int_{0}^{z} \frac{I_{\nu}^{2}(t)}{t} dt \approx \frac{e^{2z}}{4\pi z^{2}}, \quad z \to \infty.
\] (71)

Therefore, from (67), (68), (70), and (71), the asymptotic expansion on the RHS of (66) is
\[
\frac{\partial I_{\nu}(z)}{\partial \nu} \approx -2\nu \frac{e^{z}}{\sqrt{2\pi z}} \left( \frac{1}{2z} + a_{\nu} \right), \quad z \to \infty.
\] (72)

Comparing (69) to (72), we conclude that \(a_{\nu} = 0\), hence we obtain the integral representation given in (59).

Once we have set the integral representation of \(\frac{\partial I_{\nu}}{\partial \nu}\), applying the results given in (49) and (52), we can rewrite (59) in closed-form as follows:
\[
\frac{\partial I_{\nu}(z)}{\partial \nu} = I_{\nu}(z) \left[ \frac{z^{2}}{4(1-\nu^{2})} \right] _{3}F_{4} \left( \begin{array}{c} 1, \frac{3}{2} \, \frac{1}{2} \nu, 2, 2, 2 \nu, 2 + \nu \end{array} \right| z^{2} \right) + \log \left( \frac{z^{2}}{2} \right) - \psi(\nu) - \frac{1}{2\nu}, \quad \nu \to \infty.
\] (73)

which is equivalent to the result given in [4].

Also, according to (8) and the above result (73), after some simplification, we arrive at
\[
\frac{\partial K_{\nu}(z)}{\partial \nu} = \frac{\pi}{2} \csc \pi \nu \left\{ \pi \cot \pi \nu I_{\nu}(z) - [I_{\nu}(z) + I_{-\nu}(z)] \right\}
\]

\[
\left\{ \begin{array}{l}
\left[ \frac{z^{2}}{4(1-\nu^{2})} \right] _{3}F_{4} \left( \begin{array}{c} 1, \frac{3}{2} \, \frac{1}{2} \nu, 2, 2, 2 \nu, 2 + \nu \end{array} \right| z^{2} \right) + \log \left( \frac{z^{2}}{2} \right) - \psi(\nu) - \frac{1}{2\nu}, \\
\frac{1}{4} \left\{ I_{-\nu}(z) \Gamma^{2}(\nu) \left( \frac{z}{2} \right) ^{-2\nu} _{2}F_{3} \left( \begin{array}{c} \nu, \frac{1}{2} + \nu \end{array} \right| 1 + \nu, 1 + \nu, 1 + 2\nu \right| z^{2} \right) - I_{\nu}(z) \Gamma^{2}(\nu) \left( \frac{z}{2} \right) ^{-2\nu} _{2}F_{3} \left( \begin{array}{c} -\nu, \frac{1}{2} - \nu \end{array} \right| 1 - \nu, 1 - \nu, 1 - 2\nu \right| z^{2} \right)
\end{array} \right\},
\]
which is equivalent to the result given in [4].

Finally, taking into account the main results of this Section, we can derive an integral representation for the order derivative of the Macdonald function \( K_\nu(z) \).

**Theorem 8.** For \( \nu > 0 \) and \( z \neq 0 \), \( |\arg z| \leq \pi \), we have

\[
\frac{\partial K_\nu(z)}{\partial \nu} = 2\nu \left[ K_\nu(z) \int_z^\infty \frac{I_\nu(t)K_\nu(t)}{t}dt - I_\nu(z) \int_z^\infty \frac{K_\nu^2(t)}{t}dt \right]. \tag{75}
\]

**Proof.** Substituting (52) and (55) in (75), taking into account the definition given in (4) of the Macdonald function \( K_\nu(z) \), and the reflection formula (25), after some algebra we arrive at (75). \( \blacksquare \)

4. Alternative expressions for integral order

So far, we have obtained closed-form expressions for \( \partial J_\nu/\partial \nu \) and \( \partial Y_\nu/\partial \nu \) in (46) and (47), and for \( \partial I_\nu/\partial \nu \) and \( \partial K_\nu/\partial \nu \) in (73) and (74). However, these expressions cannot be applied for \( \nu \in \mathbb{Z} \). Nonetheless, we can derive alternative expressions that avoid this problem using Meijer-\( G \) functions. This function is usually defined by the following Mellin-Barnes integral representation [11, Eqn. 16.17.1]:

\[
G_{m,n}^{p,q} \left( z \left| \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array} \right. \right) = \frac{1}{2\pi i} \int_L \prod_{\ell=1}^m \Gamma (a_\ell - s) \prod_{\ell=1}^n \Gamma (1 - a_\ell + s) \prod_{k=1}^{q-1} \Gamma (1 - b_{k+1} + s) \prod_{k=0}^{p-1} \Gamma (a_{k+1} - s) z^s ds,
\]

where the integration path \( L \) separates the poles of the factors \( \Gamma (b_\ell - s) \) from those of the factors \( \Gamma (1 - a_\ell + s) \). Also, \( m \) and \( n \) are integers such that \( 0 \leq m \leq q \) and \( 0 \leq n \leq p \), and none of \( a_k - b_j \) is a positive integer when \( 1 \leq k \leq n \) and \( 1 \leq j \leq m \).

First, we introduce some properties of the Meijer-\( G \) function that will be used below. The Meijer-\( G \) function satisfies the following reduction formulas [13, Eqns. 8.2.2(8)-(9)]:

\[
G_{m,n}^{p,q} \left( z \left| \begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q, a_1
\end{array} \right. \right) = G_{m-1,n-1}^{p-1,q-1} \left( z \left| \begin{array}{c}
a_2, \ldots, a_p \\
b_1, \ldots, b_{q-1}
\end{array} \right. \right), \tag{77}
\]
and
\[
G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_{p-1}, b_1 \\ b_1, \ldots, b_q \end{array} \right. \right) = G_{p-1,q-1}^{m-1,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_{p-1} \\ b_2, \ldots, b_q \end{array} \right. \right). \tag{78}
\]

Also, it satisfies the following derivative formulas \([13, \text{Eqns. 8.2.2}(36)-(37)]\):
\[
\frac{d}{dz} \left[ z^{1-a_1} G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) \right] = z^{-a_1} G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1 - 1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right), \quad n \geq 1, \tag{79}
\]
and
\[
\frac{d}{dz} \left[ z^{1-a_p} G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) \right] = -z^{-a_p} G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_{p-1} \\ b_1, \ldots, b_q \end{array} \right. \right), \quad n \leq p - 1. \tag{80}
\]

The translation formula in the parameters reads as \([13, \text{Eqn. 8.2.2}(15)]\),
\[
z^\alpha G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right. \right) = G_{p,q}^{m,n} \left( z \left| \begin{array}{c} a_1 + \alpha, \ldots, a_p + \alpha \\ b_1 + \alpha, \ldots, b_q + \alpha \end{array} \right. \right). \tag{81}
\]

Also, the generalized hypergeometric function \(pF_q\) can be expressed in terms of the Meijer-\(G\) function as follows \([13, \text{Eqn. 8.4.51}(1)]\):
\[
pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right| -x \right) = \frac{\prod_{\ell=1}^p \Gamma (b_\ell)}{\prod_{\ell=1}^q \Gamma (a_\ell)} G_{p,q+1}^{1,0} \left( x \left| \begin{array}{c} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{array} \right. \right). \tag{82}
\]

Finally, for the asymptotic behavior of the Meijer-\(G\) function \((76)\), we
introduce the following notation:

\[ \mu = q - m - n, \quad \sigma = q - p, \]

\[ \Xi_1 = \sum_{h=1}^{q} b_h, \quad \Lambda_1 = \sum_{h=1}^{p} a_h, \]

\[ \theta = \frac{(1 - \sigma)/2 + \Xi_1 - \Lambda_1}{\sigma}, \]

\[ A_{m,n}^{q} = \left(-\frac{1}{2\pi i}\right)^{\mu} \exp \left(i\pi \left[ \sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j \right] \right), \]

\[ H_{p,q}(z) = \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} \exp \left(-\sigma z^{1/\sigma}\right) z^{\theta} \sum_{k=0}^{\infty} M_k z^{-k/\sigma}, \]

where the first coefficient in the last expansion is \( M_0 = 1 \). Thereby, according to [9, Eqn. 5.10(8)], the following result is satisfied:

**Theorem 9.** If \( 0 \leq n \leq p \leq q - 2 \), \( p + 1 \leq m + n \leq (p + q)/2 \), and \( \arg z = 0 \), then

\[ G_{p,q}^{m,n}(z) \sim A_{m,n}^{q} H_{p,q}(ze^{i\pi\mu}) + \bar{A}_{m,n}^{q} H_{p,q}(ze^{-i\pi\mu}), \quad z \to \infty. \]  

(83)

4.1. Order derivatives of Bessel functions

**Theorem 10.** \( \forall \nu \in \mathbb{R} \) and \( \text{Re} \ z > 0 \), the following integral holds true:

\[ \int_{z}^{\infty} \frac{J_{\nu}(t) Y_{\nu}(t)}{t} \, dt = \frac{-1}{2\sqrt{\pi}} G_{2,4}^{3,0} \left( z^2 \left| 1/2, 1 \right| 0, 0, \nu, -\nu \right). \]  

(84)

**Proof.** According to the representation [13, Eqn. 8.4.20(9)]

\[ J_{\nu}(\sqrt{x}) Y_{\nu}(\sqrt{x}) = -\frac{1}{\sqrt{\pi}} G_{1,3}^{2,0} \left( x \left| 1/2 \right| 0, \nu, -\nu \right), \]

we have the following indefinite integral,

\[ \int \frac{J_{\nu}(t) Y_{\nu}(t)}{t} \, dt = -\frac{1}{\sqrt{\pi}} \int G_{1,3}^{2,0} \left( t^2 \left| 1/2 \right| 0, \nu, -\nu \right) \frac{dt}{t}. \]

Performing the change of variables \( u = t^2 \) and applying the reduction formula (78), we obtain

\[ \int \frac{J_{\nu}(t) Y_{\nu}(t)}{t} \, dt = -\frac{1}{2\sqrt{\pi}} \int G_{2,4}^{3,0} \left( u \left| 1/2, 0 \right| 0, 0, \nu, -\nu \right) \frac{du}{u}. \]

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Taking now $a_p = 1$ in the derivative formula (80), we arrive at

$$\int J_\nu(t) Y_\nu(t) \frac{dt}{t} = \frac{1}{2\sqrt{\pi}} G^{3.0}_{2.4} \left( t^2 \left| \frac{1}{2}, 1, 0, 0, \nu, -\nu \right. \right).$$

Since, according to (83),

$$\lim_{z \to \infty} G^{3.0}_{2.4} \left( z^2 \left| \frac{1}{2}, 1, 0, 0, \nu, -\nu \right. \right) = \lim_{z \to \infty} \frac{\sin \pi \nu}{\sqrt{\pi} z^2} e^{-2iz} = 0,$$

we conclude (84), as we wanted to prove. \(\blacksquare\)

**Theorem 11.** \(\forall \nu > 0 \text{ and } \Re z > 0, \text{ the following integrals holds true:}\)

$$\int_{z}^{\infty} \frac{Y_\nu^2(t)}{t} \, dt = \frac{1}{2\nu} + \frac{1}{\sqrt{\pi}} G^{4.0}_{3.5} \left( z^2 \left| \frac{1}{2}, 1/2 - \nu, 1 \right. , 0, 0, \nu, -\nu, 1/2 - \nu \right)$$

$$- \frac{(z/2)^{2\nu}}{2\nu \Gamma^2(\nu + 1)} 2F_3 \left( \nu, 1/2 + \nu, 2\nu + 1, \nu + 1, \nu + 1 \left| -z^2 \right. \right).$$

**Proof.** According to the representation [13, Eqn. 8.4.20(7)]

$$Y_\nu^2(\sqrt{x}) = \frac{2}{\sqrt{\pi}} G^{3.0}_{2.4} \left( x \left| \frac{1}{2}, 1/2 - \nu, 1 \right. , 0, 0, \nu, -\nu, 1/2 - \nu \right)$$

$$+ \frac{1}{\sqrt{\pi}} G^{1.1}_{1.3} \left( x \left| \frac{1}{2}, 1 \right. , \nu, -\nu, 0 \right),$$

we have the following indefinite integral,

$$\int \frac{Y_\nu^2(t)}{t} \, dt = \frac{1}{\sqrt{\pi}} \int G^{3.0}_{2.4} \left( u \left| \frac{1}{2}, 1/2 - \nu, 1 \right. , 0, 0, \nu, -\nu, 1/2 - \nu \right) \frac{du}{u}$$

$$+ \frac{1}{2\sqrt{\pi}} \int G^{1.1}_{1.3} \left( u \left| \frac{1}{2}, 1 \right. , \nu, -\nu, 0 \right) \frac{du}{u},$$

where we have performed the change of variables $u = t^2$. The first integral on the RHS of (86) is calculated using the reduction formula (78), and then
applying the derivative formula (80) with $a_p = 1$,

\[
\frac{1}{\sqrt{\pi}} \int G_{2,4}^{3,0} \left( u \right| \begin{array}{c} 0, \nu, -\nu, 1/2 - \nu \\ 1/2, 1/2, 1/2 - \nu \end{array} \right) \frac{du}{u} = -\frac{1}{\sqrt{\pi}} G_{3,5}^{4,0} \left( t^2 \right| \begin{array}{c} 1/2, 1/2 - \nu, 1 \\ 0, 0, \nu, -\nu, 1/2 - \nu \end{array} \right).
\]  

(87)

The second integral on the RHS of (86) is calculated using the reduction formula (77), and then applying the derivative formula (79) with $a_1 = 1$,

\[
\frac{1}{2\sqrt{\pi}} \int G_{1,3}^{1,1} \left( u \right| \begin{array}{c} 1/2 \\ \nu, -\nu, 0 \end{array} \right) \frac{du}{u} = \frac{1}{2\sqrt{\pi}} G_{2,4}^{1,2} \left( t^2 \right| \begin{array}{c} 1, 1/2 \\ \nu, -\nu, 0, 0 \end{array} \right).
\]

(88)

Notice that applying the translation formula (81) and then the reduction formula (82), we can express the RHS of (88) as a hypergeometric function,

\[
\frac{1}{2\sqrt{\pi}} \int G_{1,3}^{1,1} \left( u \right| \begin{array}{c} 1/2 \\ \nu, -\nu, 0 \end{array} \right) \frac{du}{u} = \frac{t^{2\nu}}{2\sqrt{\pi}} G_{2,4}^{1,2} \left( t^2 \right| \begin{array}{c} 1 - \nu, 1/2 - \nu \\ 0, -2\nu, -\nu, -\nu \end{array} \right)
\]

\[
= \frac{(t/2)^{2\nu}}{2\Gamma^2 (\nu + 1)} \binom{\nu, 1/2 + \nu}{2\nu + 1, \nu + 1, \nu + 1} _2F_3 \left( -t^2 \right),
\]

where we have applied the duplication formula of the gamma function (31).

Therefore, substituting the results (87) and (89) in (86), we obtain

\[
\int \frac{Y_{\nu}^2 (t)}{t} dt = \frac{1}{2\nu \sqrt{\pi}} G_{2,4}^{1,2} \left( t^2 \right| \begin{array}{c} 1/2, 1/2 - \nu, 1 \\ 0, 0, \nu, -\nu, 1/2 - \nu \end{array} \right)
\]

\[
+ \frac{(t/2)^{2\nu}}{2\Gamma^2 (\nu + 1)} \binom{\nu, 1/2 + \nu}{2\nu + 1, \nu + 1, \nu + 1} _2F_3 \left( -t^2 \right),
\]

(90)

Finally, notice that the hypergeometric function obtained in (89) is the integral calculated in (49), thus

\[
\lim_{z \to \infty} \frac{(z/2)^{2\nu}}{2\Gamma^2 (\nu + 1)} \binom{\nu, 1/2 + \nu}{2\nu + 1, \nu + 1, \nu + 1} _2F_3 \left( -z^2 \right)
\]

\[
= \int_0^\infty \frac{J_{\nu}^2 (t)}{t} dt = \frac{1}{2\nu}.
\]

(91)
where we have applied \[14\), Eqn. 13.42(1). Also, according to (83), we have that

$$
\lim_{z \to \infty} G_{4,0}^{3,5}(z^2 \left| \begin{array}{c}
1/2, 1/2 - \nu, 1 \\
0, 0, \nu, -\nu, 1/2 - \nu
\end{array} \right.) = \lim_{z \to \infty} \frac{-\cos \pi \nu}{\sqrt{\pi} z^2} e^{-2iz} = 0.
$$

(92)

Therefore, from (90), (91) and (92), we conclude (85), as we wanted to prove.

According to the integral representation given in (11), and the results obtained in (20) and (84), we calculate the order derivative of the Bessel function of the first kind as,

$$
\frac{\partial J_{\nu}(z)}{\partial \nu} = \frac{\pi}{2} \left[ Y_{\nu}(z) \left( \frac{z}{2} \right)^{2\nu} \frac{\Gamma^2(\nu + 1)}{\nu} _2F_3 \left( \begin{array}{c}
\nu, 1/2 + \nu \\
2\nu + 1, \nu + 1, \nu + 1
\end{array} \left. -z^2 \right\} \right) - \nu J_{\nu}(z) \sqrt{\frac{\pi}{\nu}} G_{3,0}^{3,0}(z^2 \left| \begin{array}{c}
1/2, 1 \\
0, 0, \nu, -\nu
\end{array} \right.) \right], \quad \nu > 0, \Re z > 0.
$$

(93)

As by-product, from (9) and (93), we obtain the calculation of the following integral, which does not seem to be reported in the literature,

$$
\int_0^{\pi/2} \tan \theta Y_{\nu}(z \sin^2 \theta) J_{\nu}(z \cos^2 \theta) d\theta = \frac{Y_{\nu}(z) \left( \frac{z}{2} \right)^{2\nu} \frac{\Gamma^2(\nu + 1)}{\nu} _2F_3 \left( \begin{array}{c}
\nu, 1/2 + \nu \\
2\nu + 1, \nu + 1, \nu + 1
\end{array} \left. -t^2 \right\} \right) - \frac{\nu J_{\nu}(z)}{2\sqrt{\pi}} G_{3,0}^{3,0}(z^2 \left| \begin{array}{c}
1/2, 1 \\
0, 0, \nu, -\nu
\end{array} \right.) \right], \quad \nu > 0.
$$

(94)

Also, according to the integral representation given in (12), and the results obtained in (84) and (85), the order derivative of the Bessel function of the second kind is

$$
\frac{\partial Y_{\nu}(z)}{\partial \nu} = J_{\nu}(z) \left[ \sqrt{\frac{\pi}{\nu}} G_{3,5}^{4,0}(z^2 \left| \begin{array}{c}
1/2, 1/2 - \nu, 1 \\
0, 0, \nu, -\nu, 1/2 - \nu
\end{array} \right.) \right] - \frac{\pi \left( \frac{z}{2} \right)^{2\nu}}{2 \Gamma^2(\nu + 1)} _2F_3 \left( \begin{array}{c}
\nu, 1/2 + \nu \\
2\nu + 1, \nu + 1, \nu + 1
\end{array} \left. -z^2 \right\} \right) + \frac{\sqrt{\pi \nu} Y_{\nu}(z)}{2} G_{2,4}^{3,0}(z^2 \left| \begin{array}{c}
1/2, 1 \\
0, 0, \nu, -\nu
\end{array} \right.), \quad \nu > 0, \Re z > 0.
$$

(95)
4.2. Order derivatives of modified Bessel functions

**Theorem 12.** \( \forall \nu > 0 \) and \( \text{Re} \, z > 0 \), the following integral holds true:

\[
\int_{z}^{\infty} \frac{I_{\nu}(t) K_{\nu}(t)}{t} \, dt = \frac{1}{4\sqrt{\pi}} G_{2,4}^{3,1} \left( z^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array} \right). \tag{96}
\]

**Proof.** According to the representation [13, Eqn. 8.4.23(19)]

\[
I_{\nu}(\sqrt{x}) K_{\nu}(\sqrt{x}) = \frac{1}{2\sqrt{\pi}} G_{1,3}^{2,1} \left( x \middle| \begin{array}{c} 1/2 \\ 0, \nu, -\nu \end{array} \right),
\]

we have the following indefinite integral

\[
\int \frac{I_{\nu}(t) K_{\nu}(t)}{t} \, dt = \frac{1}{4\sqrt{\pi}} \int G_{1,3}^{2,1} \left( u \middle| \begin{array}{c} 1/2 \\ 0, \nu, -\nu \end{array} \right) \frac{du}{u},
\]

where we have performed the change of variables \( u = t^2 \). Now, applying the reduction formula (78) and the derivative formula (80) with \( a_p = 1 \), we arrive at

\[
\int \frac{I_{\nu}(t) K_{\nu}(t)}{t} \, dt = -\frac{1}{4\sqrt{\pi}} G_{2,4}^{3,1} \left( t^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array} \right). \tag{97}
\]

Finally, note that, according to (83), we have

\[
\lim_{z \to \infty} G_{2,4}^{3,1} \left( z^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array} \right) = -2\sqrt{\pi} \sin \pi \nu \lim_{z \to \infty} \frac{e^{-2z}}{z^2} = 0,
\]

thus we get (96) from (97), as we wanted to prove. \( \blacksquare \)

**Theorem 13.** \( \forall \nu \in \mathbb{R} \) and \( \text{Re} \, z > 0 \), the following integral holds true:

\[
\int_{z}^{\infty} \frac{K_{2\nu}(t)}{t} \, dt = \frac{\sqrt{\pi}}{4} G_{2,4}^{4,0} \left( z^2 \middle| \begin{array}{c} 1/2, 1 \\ 0, 0, \nu, -\nu \end{array} \right). \tag{98}
\]

**Proof.** Following the same steps as in the previous theorem, departing from the representation [13, Eqn. 8.4.23(27)]

\[
K_{2\nu}(\sqrt{x}) = \frac{\sqrt{\pi}}{2} G_{1,3}^{3,0} \left( x \middle| \begin{array}{c} 1/2 \\ 0, \nu, -\nu \end{array} \right),
\]

we obtain the desired result. \( \blacksquare \)
According to the integral representation given in (59), and the results obtained in (49) and (96), we calculate the order derivative of the modified Bessel function as,

$$
\frac{\partial I_\nu(z)}{\partial \nu} = -\nu I_\nu(z) \frac{\Gamma^3}{2\sqrt{\pi}} G_{3,4}^{3,1}
\begin{array}{c|c}
\frac{1}{2}, 1 & 0, 0, \nu, -\nu \\
\end{array}
\end{equation}

$$

$$
- \frac{K_\nu(z) (z/2)^{2\nu}}{\Gamma^2 (\nu + 1)} 2F_3
\begin{array}{c|c}
\nu, \nu + \frac{1}{2} & \nu + 1, \nu + 1, 2\nu + 1 \\
\end{array}
\left| z^2 \right|
\end{equation}

$$
\nu > 0, \Re z > 0.

As by-product, according to (10) and (99), we calculate the following integral, which does not seem to be reported in the literature,

$$
\int_0^{\pi/2} \tan \theta \ K_0(z \sin^2 \theta) I_\nu(z \cos^2 \theta) \, d\theta
= \frac{I_\nu(z)}{4\sqrt{\pi}} G_{3,4}^{3,1}
\begin{array}{c|c}
\frac{1}{2}, 1 & 0, 0, \nu, -\nu \\
\end{array}
\end{equation}

$$

$$
+ \frac{K_\nu(z) (z/2)^{2\nu}}{2\nu \Gamma^2 (\nu + 1)} 2F_3
\begin{array}{c|c}
\nu, \nu + \frac{1}{2} & \nu + 1, \nu + 1, 2\nu + 1 \\
\end{array}
\left| z^2 \right|, \quad \nu > 0.

$$
Finally, according to the integral representation given in (75), and the results obtained in (96) and (98), the order derivative of the Macdonald function is

$$
\frac{\partial K_\nu(z)}{\partial \nu} = \frac{\nu}{2} \left[ \frac{K_\nu(z)}{\sqrt{\pi}} G_{2,4}^{3,1}
\begin{array}{c|c}
\frac{1}{2}, 1 & 0, 0, \nu, -\nu \\
\end{array}
\end{equation}

$$

$$
- \sqrt{\pi} I_\nu(z) G_{2,4}^{4,0}
\begin{array}{c|c}
\frac{1}{2}, 1 & 0, 0, \nu, -\nu \\
\end{array}
\left| z^2 \right|
\end{equation}

$$
\nu > 0, \Re z > 0.

5. Conclusions

We have calculated some integrals involving Bessel functions in terms of generalized hypergeometric functions in (20), (21), and (41); and in terms of Meijer-\(G\) functions in (84) and (85). These integrals have been applied to express in closed-form the derivative of the Bessel functions with respect to the order from integral representations given in the literature, i.e. (11) and (12). We have expressed these results using hypergeometric functions,
namely (46) and (47), and Meijer-G functions, namely (93) and (95). We have carried out similar calculations to obtain closed-form expressions for the derivative of the modified Bessel functions with respect to the order, both in terms of hypergeometric functions, namely (73) and (74), as well as in terms of Meijer-G functions, namely (99) and (101). For this purpose, we have obtained integral representations for the order derivative of the modified Bessel functions in (59) and (75).

Despite the fact, the expressions given in (46) and (47) for $\partial J_\nu / \partial \nu$ and $\partial Y_\nu / \partial \nu$ as well as the expressions (73) and (74) for $\partial I_\nu / \partial \nu$ and $\partial K_\nu / \partial \nu$ have been already calculated in [4] with the aid of symbolic computer algebra, we have presented here a formal derivation which turns out to be highly non-trivial. However, we cannot use these expressions for integral order $\nu$. Nonetheless, this is not the case for the expressions using Meijer-G functions, namely (93), (95), (99) and (101). However, for non-integral order $\nu$, the former expressions are computed much more rapidly than these latter ones ($\approx 10$ times faster).

Finally, as by-products, we have calculated two integrals in (94) and (100), which do not seem to be reported in the literature.

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