A GENERALIZATION OF ESCOBAR-RIEMANN MAPPING TYPE PROBLEM ON SMOOTH METRIC MEASURE SPACES

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ABSTRACT. In this article, we introduce an analogous problem to Yamabe type problem considered by Case in [4], which generalizes the Escobar-Riemann mapping problem for smooth metric measure spaces with boundary. The last problem will be called Escobar-Riemann mapping type problem. For this purpose, we consider the generalization of Sobolev Trace Inequality deduced by Bolley at. al. in [3]. This trace inequality allows us to introduce an Escobar quotient and its infimum. This infimum we call the Escobar weighted constant. The Escobar-Riemann mapping type problem for smooth metric measure spaces in manifolds with boundary consists of finding a function which attains the Escobar weighted constant. Furthermore, we resolve the later problem when Escobar weighted constant is negative. Finally, we get an Aubin type inequality connecting the weighted Escobar constant for compact smooth metric measure space and the optimal constant for the trace inequality in [3].

1. Introduction

When \((M^n, g)\) is a Riemannian manifold with boundary, we denote by \(\partial M\) the boundary of \(M\) and by \(H_g\) the trace of the second fundamental form of \(\partial M\). The Escobar-Riemann mapping problem for manifolds with boundary is concerned with finding a metric \(g\) with scalar curvature \(R_g \equiv 0\) in \(M\) and \(H_g\) constant on \(\partial M\), in the conformal class of the initial metric \(g\). Since this problem in the Euclidean half-space reduces to finding the minimizers in the sharp Trace Sobolev inequality, we consider a particular case of the Trace Gagliardo-Nirenberg-Sobolev inequality in [3].

To present the Trace Gagliardo-Nirenberg-Sobolev inequality, let \(\mathbb{R}^n_+ = \{(x, t) : x \in \mathbb{R}^{n-1}, t \geq 0\}\) denote the half-space and its boundary by \(\partial \mathbb{R}^n_+ = \{(x, 0) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}\}\). We identify \(\partial \mathbb{R}^n_+\) with \(\mathbb{R}^{n-1}\) whenever necessary.

Theorem 1. [3] Fix \(m \geq 0\). For all \(w \in W^{1,2}(\mathbb{R}^n_+) \cap L^2(m+n-2)(\mathbb{R}^n_+)\) it holds that

\[\int_{\mathbb{R}^n_+} |\nabla w|^2 \, dx + \int_{\partial \mathbb{R}^n_+} w^2 \, d\sigma \leq C \left( \int_{\mathbb{R}^n_+} |w|^2 \, dx \right)^{\frac{n}{2m+n-2}}.\]

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is defined by \( R \) Sobolev inequalities and following similar ideas in [4], we will introduce an Escobar quotient boundary defined by a five-tuple \((g, m)\) smooth metric measure space. Thus, it is necessary to consider the notion of Riemann mapping type problem for smooth metric measure spaces in manifolds with boundary. Then, using Theorem 1 instead of Gagliardo-Nirenberg-Sobolev inequalities, we will prove the following.

\[
\Lambda_{m,n} \left( \int_{\partial M^n} |w|^{2(m+n-1)} \right)^{\frac{2m+n-2}{m+n-1}} \leq \left( \int_{M^n} \|
abla w\|^2 \right)^{\frac{2m+n-2}{m+n-1}} \left( \int_{M^n} |w|^{2(m+n-1)} \right)^{\frac{m}{m+n-1}}
\]

where the constant \( \Lambda_{n,m} \) is given by

\[
\Lambda_{m,n} = (m + n - 2)^2 \left( \frac{\text{Vol}(S^{2m+n-1})}{2(2m + n - 2)} \right)^{\frac{2m+n-1}{m+n-1}} \left( \frac{\Gamma(2m + n - 1)}{\pi^m \Gamma(m + n - 1)} \right)^{\frac{1}{m+n-1}}
\]

and \( \text{Vol}(S^{2m+n-1}) \) is the volume of the \( 2m + n - 1 \) dimensional unit sphere. Moreover, equality holds if and only if \( w \) is a constant multiple of the function \( w_{\epsilon, x_0} \) defined on \( \mathbb{R}^n_+ \) by

\[
w_{\epsilon, x_0}(x, t) := \left( \frac{2\epsilon}{(\epsilon + t)^2 + |x - x_0|^2} \right)^{\frac{m+n-2}{2}}
\]

where \( \epsilon > 0 \) and \( x_0 \in \mathbb{R}^{n-1} \).

Del Pino and Dolbeaut studied the sharp Gagliardo-Nirenberg-Sobolev inequalities. Based on Del Pino and Dolbeaut’s result, Case in [1] considered a Yamabe type problem for smooth metric measure spaces in manifolds without boundary, which generalizes the Yamabe problem when \( m = 0 \). Then, using Theorem 1 instead of Gagliardo-Nirenberg-Sobolev inequalities and following similar ideas in [4], we will introduce an Escobar-Riemann mapping type problem for smooth metric measure spaces in manifolds with boundary. Thus, it is necessary to consider the notion of smooth metric measure space with boundary defined by a five-tuple \((M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)\) where \( dV_g \) and \( d\sigma_g \) are the volume form induced by the metric \( g \) in \( M \) and on the boundary \( \partial M \), respectively; a function \( \phi \) such that \( \phi \in C^\infty(M) \); and a parameter \( m \in [0, \infty) \). In addition, if \( m = 0 \), we require \( \phi = 0 \).

Let us denote the scalar curvature, the Laplacian and the Gradient associated to the metric \( g \) by \( R_g, \Delta_g, \) and \( \nabla_g \), respectively. The weighted scalar curvature \( R^m_\phi \) of a smooth metric measure space for \( m = 0 \) is \( R^0_\phi = R_g \) and for \( m \neq 0 \) is the function \( R^m_\phi := R_g + 2\Delta_g \phi - \frac{m+1}{m}|
abla_g \phi|^2 \). The weighted Escobar quotient for this smooth metric measure is defined by

\[
Q(w) = \frac{\int_M (|
abla w|^2 + \frac{m+n-2}{2(m+n-1)} R^m_\phi w^2) e^{-\phi}dV_g + \int_{\partial M} \frac{m+n-2}{2(m+n-1)} H^m_\phi w^2 e^{-\phi} d\sigma_g}{(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} e^{-\phi} d\sigma_g) \left( \int_M |w|^{\frac{2(m+n-1)}{m+n-2}} e^{-\phi} dV_g \right)^{\frac{m}{m+n-1}}}
\]
where we denote by \( H_\phi^m = H_g + \frac{\partial \phi}{\partial \eta} \) the Gromov mean curvature and \( \frac{\partial}{\partial \eta} \) is the outer normal derivative.

The \textit{weighted Escobar constant} \( \Lambda[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m] \in \mathbb{R} \cup \{ -\infty \} \) is defined by

\[
(5) \quad \Lambda := \Lambda[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m] = \inf \{ Q(w) : w \in H^1(M, e^{-\phi} dV_g) \}.
\]

If \( m = 0 \), the quotient coincides with the Sobolev quotient considered by Escobar in the Escobar-Riemann mapping problem. We prove the existence of a minimizer of the weighted Escobar constant when this constant is negative. The exact statement is

**Theorem A.** Let \((M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)\) be a compact smooth metric measure space with boundary, \( m \geq 0 \) and negative weighted Escobar constant. Then there exists a positive function \( w \in C^\infty(M) \) such that

\[
Q(w) = \Lambda[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m].
\]

Using Theorem A, we prove that the weighted Escobar constant for a compact smooth measure space with boundary is always less or equal than the weighted Escobar constant of the model case \((\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)\).

**Theorem B.** Let \((M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)\) be a compact smooth metric measure space with boundary such that \( m \geq 0 \). Then

\[
(6) \quad \Lambda[M^n, g, e^{-\phi} dV_g, m] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}.
\]

We recall that in the Escobar-Riemann mapping problem \((m = 0)\) if the inequality \((6)\) is strict, it follows the existence of the minimizer. The same result is expected for the Escobar-Riemann type problem. For that reason we conjecture that

**Conjecture.** Let \((M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m)\) be a compact smooth metric measure space with boundary such that \( m \geq 0 \) and

\[
(7) \quad \Lambda[M^n, g, e^{-\phi} dV_g, m] < \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m] = \Lambda_{m,n}.
\]

Then there exists a positive function \( w \in C^\infty(M) \) such that

\[
Q(w) = \Lambda[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m].
\]
This paper is organized as follows. In section 2, we give a different proof for Theorem 1 given in [3], for the particular case \( m \in \mathbb{N} \cup \{0\} \). In sections 3 and 4, we consider our notion smooth metric measure spaces with boundary and other concepts to introduce Escobar-Riemann type problem. In sections 5 and 6, we prove Theorem A and B, respectively.

2. General Trace Inequality

In this section, we give a proof for Theorem 1 in the case \( m \in \mathbb{N} \cup \{0\} \) different to the proof in [3]. As we mentioned in the introduction, the Trace Gagliardo-Nirenberg-Sobolev inequality prepares the way to introduce our Escobar-Riemann type problem. The proof that we present depends on the Sobolev Trace Inequality in \( \mathbb{R}^{n+2m} \) and its minimizers. This kind of ideas are due to Bakry et al. (see [2]).

Remark 1. In the case \( m = 0 \) in the inequality (1) we recover the Sobolev trace inequality (see [1], [7])

\[
\Lambda_{0,n} \left( \int_{\partial \mathbb{R}^n_+} |w|^{\frac{2(n-1)}{n-2}} \right)^{\frac{n-2}{n-4}} \leq \left( \int_{\mathbb{R}^n_+} |\nabla w|^2 \right)^{\frac{n-2}{n-4}},
\]

where \( \Lambda_{0,n} = \frac{n-2}{2} (\text{vol}(S^{n-1}))^{\frac{1}{n-4}} \). Equality in (8) holds if and only if \( w \) is a positive constant multiple of the functions of the form

\[
w = \left( \frac{\epsilon}{(\epsilon + t)^2 + |x-x_0|^2} \right)^{\frac{n-2}{2}}.
\]

Lemma 1. Let \( p, q, B, C \) be positive numbers and define \( h(\tau) = B\tau^p + C\tau^{-q} \) for \( \tau > 0 \). Then \( h \) attains the infimum in \( \tau_0 = (\frac{qB}{pA})^{\frac{1}{p+q}} \) and

\[
\inf_{\tau > 0} h(\tau) = h(\tau_0) = B^{\frac{q}{p+q}} C^{\frac{p}{p+q}} \left( \frac{q}{p} \right)^{\frac{p}{p+q}} \left( \frac{q+p}{p} \right).
\]

Proof. Since \( h \) is a positive continuous function for \( \tau > 0 \) and

\[
\lim_{\tau \to 0^+} h(\tau) = \lim_{\tau \to \infty} h(\tau) = \infty,
\]

it follows that \( h \) attains the infimum for some \( \tau_0 > 0 \). A direct computation shows that \( h'(\tau) = \tau^{p-1}(pB - qC\tau^{-q-p}) \). Therefore \( \tau_0 = (\frac{qC}{pB})^{\frac{1}{p+q}} \) and

\[1\text{After we posted on arXiv the previous version of this paper, we were informed by Nguyen Van Hoang that Theorem 1 is a particular case of Theorem 18 in [3].}
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(10)

\[ h\left(\frac{qC}{pB}\right)^{\frac{1}{p+q}} = B\left(\frac{qC}{pB}\right)^{\frac{p}{p+q}} + C\left(\frac{qC}{pB}\right)^{\frac{q}{p+q}} \]

\[ = B\frac{p}{p+q} C^{\frac{p}{p+q}} (\frac{q}{p})^{\frac{p}{p+q}} + B\frac{q}{p+q} C^{\frac{q}{p+q}} (\frac{q}{p})^{\frac{q}{p+q}} \]

\[ = B\frac{p}{p+q} C^{\frac{p}{p+q}} (\frac{q}{p})^{\frac{p}{p+q}} (1 + \frac{p}{q}) \]

\[ = B\frac{p}{p+q} C^{\frac{p}{p+q}} (\frac{q}{p})^{\frac{p+q}{q}}. \]

\[ \square \]

Remark 2. If \( m \to \infty \), the inequality (1) takes the form

(11)

\[ \Lambda_{\infty,n} \left( \int_{\partial R^n_+} |w|^2 \right)^2 \leq \left( \int_{R^n_+} |\nabla w|^2 \right) \left( \int_{R^n_+} |w|^2 \right) \]

where \( \lim_{m \to \infty} \Lambda_{m,n} = \Lambda_{\infty,n} \).

The inequality (11) is equivalent to the trace inequality \( H^1(M) \to L^2(\partial M) \)

(12)

\[ 2(\Lambda_{\infty,n})^{\frac{1}{2}} \left( \int_{\partial R^n_+} |w|^2 dx \right) \leq \int_{R^n_+} |\nabla w|^2 dx dt + \int_{R^n_+} |w|^2 dx dt. \]

In fact, suppose inequality (12) holds. For \( \tau > 0 \) define the function \( w_\tau(x,t) = w\left(\frac{1}{\tau}(x,t)\right) \).

The change of variable \((y,s) = \frac{1}{\tau}(x,t)\) implies

\[ \int_{\partial R^n_+} |w_\tau|^2(x,0) dx = \tau^{n-1} \int_{\partial R^n_+} |w|^2(y,0) dy, \]

\[ \int_{R^n_+} |\nabla w_\tau|^2(x,t) dx dt = \tau^{n-2} \int_{R^n_+} |\nabla w|^2(y,s) dy ds \]

and

\[ \int_{R^n_+} |w_\tau|^2(x,t) dx dt = \tau^n \int_{R^n_+} |w|^2(y,s) dy ds. \]

Then, using \( w_\tau \) and the equalities above in inequality (12) we get

(13)

\[ 2(\Lambda_{\infty,n})^{\frac{1}{2}} \left( \int_{\partial R^n_+} |w|^2(y,0) dy \right) \leq \tau B + \tau^{-1} C, \]

where \( B = \int_{R^n_+} |w|^2(y,s) dy ds \) and \( C = \int_{R^n_+} |\nabla w|^2(y,s) dy ds \). Lemma 1 yields that for \( \tau_0 = \left(\frac{C}{B}\right)^{\frac{1}{2}} \), it holds.
\[ \tau_0 B + \tau_0^{-1} C = 2 \frac{B^{1/2}}{C^{1/2}} = 2 \left( \int_{\mathbb{R}_+^n} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}_+^n} |w|^2 \, dx \right)^{1/2}. \]

Since inequality (13) is true for every \( \tau > 0 \), in particular it is true for \( \tau_0 = (\frac{C}{B})^{1/2} \) and by (14), we have

\[ 2(\Lambda_{\infty,n})^{1/2} \left( \int_{\partial \mathbb{R}_+^{n+2m}} |w|^2 \, dx \right)^{1/2} \leq 2 \left( \int_{\mathbb{R}_+^n} |\nabla w|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}_+^n} |w|^2 \, dx \right)^{1/2}, \]

which is equivalent to (11).

Now, suppose that inequality (11) holds, then inequality (15) holds. In addition, inequality (12) is a consequence of inequality \( 2ab \leq a^2 + b^2 \).

In our proof for the Theorem 1 we use the following Lemma, which was taken from [4].

**Lemma 2.** Fix \( k, l \geq 0, 2m \in \mathbb{N} \), and constants \( a, \tau > 0 \). Then

\[ \int_{\mathbb{R}^{2m}} \frac{|y|^{2l} \, dy}{(a + \frac{|y|^2}{\tau})^{2m+k}} = \pi^m \frac{\Gamma(m+l)\Gamma(m+k-l)\tau^{m+l}}{\Gamma(m)\Gamma(2m+k)a^{m+k-l}}. \]

**Proof of Theorem.** We are able to prove inequality (11) only for \( m \in \mathbb{N} \). For this purpose, consider the inequality (8) for \( \mathbb{R}_+^{n+2m} \). The idea of the proof consists of using this inequality for the special function

\[ f(y, x, t) := \left( w^\frac{m-n-2}{2} (x, t) + \frac{|y|^2}{\tau} \right)^{\frac{2m+n-2}{2}} \in C^\infty(\mathbb{R}_+^{n+2m}), \]

where \((x, t) \in \mathbb{R}_+^n, y \in \mathbb{R}^{2m}\) and \( \tau > 0 \).

Suppose \( f \) is of the form (16). First, we analyze the term on the left hand side of inequality (8). Fixing \((x, t)\) we note that \( \int_{\mathbb{R}_+^{2m+n}} f^{\frac{2(2m+n-1)}{2m+n-2}} \) takes the form of the function considered in Lemma 2 with \( a = w^{\frac{m-n-2}{2}} (x, t) \). Fubini’s Theorem, Lemma 2 with \( k = n - 1 \) and \( l = 0 \), and some calculation yield

\[ \int_{\partial \mathbb{R}_+^{2m+n}} f^{\frac{2(2m+n-1)}{2m+n-2}} \, dxdy = \pi^m \frac{\Gamma(m+n-1)\tau^m}{\Gamma(2m+n-1)} \int_{\partial \mathbb{R}_+^{2m+n}} w^{\frac{2(2m+n-1)}{2m+n-2}} \, dx. \]

In order to analyze the term on the right hand side of inequality (8), we compute
Lemma 2 leads to

\[ \left(\frac{2m+n-2}{2}\right)^2 \left(\frac{2}{m+n-2}\right)^2 w^{-\frac{2(m+n)}{m+n-2}} |\nabla w|^2 + \frac{4|y|^2}{\tau^2} \right) \]

\[ \left( w^{-\frac{2}{m+n-2}} + \frac{|y|^2}{\tau^2} \right)^{2m+n} \].

Using equalities (17) and (18) in inequality (8), we get that

\[ \frac{\Lambda}{\tau^{2m+n-1}} \left( \frac{\pi^m \Gamma(m+n-1)}{\Gamma(2m+n-1)} \int_{\partial \Omega} w^{\frac{2(m+n-1)}{m+n-2}} dx \right)^{\frac{2m+n-2}{2m+n-1}} \]

\[ \leq \left( \frac{2m+n-2}{m+n-2} \right)^2 \left( \frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)} \right) \int_{\Omega} |\nabla w|^2 dx + \left( \frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)} \right) \int_{\Omega} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt. \]

Rewriting (19), we obtain

\[ \Lambda_{2m+n,0} \left( \frac{\pi^m \Gamma(m+n-1)}{\Gamma(2m+n-1)} \int_{\partial \Omega} w^{\frac{2(m+n-1)}{m+n-2}} dx \right)^{\frac{2m+n-2}{2m+n-1}} \]

\[ \leq \left( \frac{2m+n-2}{m+n-2} \right)^2 \left( \frac{\pi^m \tau^m \Gamma(m+n)}{\Gamma(2m+n)} \right) \int_{\Omega} |\nabla w|^2 dx + \left( \frac{m(2m+n-2)^2 \pi^m \tau^{m-1} \Gamma(m+n)}{(m+n-1)\Gamma(2m+n)} \right) \int_{\Omega} |w|^{\frac{2(m+n-1)}{m+n-2}} dxdt. \]

where

\[ A = \frac{\Gamma(2m+n)}{(2m+n-2)^2 \pi^m \Gamma(m+n)}. \]

\[ h(\tau) = B \tau^{\frac{m}{2m+n-1}} + C \tau^{-\frac{m+n-1}{2m+n-1}}, \]

\[ B = \frac{1}{(m+n-2)^2} \int_{\Omega} |\nabla w|^2 dxdt, \]

and
\[ C = \frac{m}{m + n - 1} \int_{\mathbb{R}^n_+} |w|^{\frac{2(m+n-1)}{m+n-2}} \, dx \, dt. \]

Lemma 1 implies that the function \( h \) minimizes for \( \tau_0 = \left(\frac{(m+n-1)C}{mB}\right)^{\frac{m+n-2}{2m+n-1}} \) and

\[
\Lambda_{2m+n,0} \left( \frac{\pi^m \Gamma(m + n - 1)}{\Gamma(2m + n - 1)} \int_{\partial \mathbb{R}^n_+} w^{2(m+n-1)/(m+n-2)} \, dx \right)^{\frac{2m+n-2}{2(m+n-1)}} A \leq h(\tau_0).
\]

Inequality (21) proves inequality (1) with \( \Lambda_{m,n} \) as in (2). Next, we characterize the functions that achieve equality in (1). Note that for \( \mathbb{R}^{n+2m}_+ \) and \( f \) defined in (16), the equality in (8) holds if and only if

\[
f(y, x, t) = \left(\frac{(t + \epsilon)^2 + |x - x_0|^2 + |y|^2}{\tau}\right)^{-\frac{2m+n-2}{2}}, \quad \text{for} \quad \tau > 0,
\]

i.e.

\[
w^{\frac{2}{m+n-2}}(x, t) = \tau((t + \epsilon)^2 + |x - x_0|^2)
\]

(see Escobar [7] and Beckner [1]). Then, the family of functions \( \{w_{\epsilon, x_0}\} \) in (8) is the only one that satisfies the equality in (1).

3. Smooth metric measure spaces with boundary and the conformal Laplacian

Our approach is based on [4] and [5]. The first step is to introduce the definition of a smooth metric measure space with boundary

\[ \text{Definition 1. Let } (M^n, g) \text{ be a Riemannian manifold and let us denote by } dV_g \text{ and } d\sigma_g \text{ the volume form induced by } g \text{ in } M \text{ and } \partial M, \text{ respectively. Set a function } \phi \text{ such that } \phi \in C^\infty(M) \text{ and } m \in [0, \infty) \text{ be a dimensional parameter. In the case } m = 0, \text{ we require that } \phi = 0 \text{. A smooth metric measure space with boundary is the five-tuple} \ (M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m). \]

As in [4], sometimes we denote by the four-tuple \((M^n, g, v^m dV_g, v^m d\sigma_g)\) a smooth metric measure space where \( v \) and \( \phi \) are related by \( v^m = e^{-\phi} \). We denote by \( R_g \) the scalar curvature of \((M, g)\) and \( Ric \) and the Ricci tensor of \((M, g)\), \( \eta \) the outer normal on \( \partial M \) and \( \frac{\partial}{\partial \eta} \) the normal derivative. Also, we denote the second fundamental form, the trace of the second fundamental form, and the mean curvature on the boundary \( \partial M \), by \( h_{ij} \), \( H_g := g^{ij}h_{ij} \), and \( h_g = \frac{H_g}{n-1} \); respectively.
**Definition 2.** Given a smooth metric measure space \((M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)\). The weighted scalar curvature \(R^m_\phi\) and the Bakry-Émery Ricci curvature \(\text{Ric}^m_\phi\) are the tensors

\[
R^m_\phi := R_g + 2\Delta \phi - \frac{m+1}{m}|\nabla \phi|^2
\]

and

\[
\text{Ric}^m_\phi := \text{Ric} + \nabla^2 \phi - \frac{1}{m}d\phi \otimes d\phi.
\]

**Definition 3.** Let \((M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, \hat{m})\) be smooth metric measure spaces with boundary. We say they are pointwise conformally equivalent if there is a function \(\sigma \in C^\infty(M)\) such that

\[
(M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, \hat{m}) = (M^n, e^{\frac{2\sigma}{m+n-2}}g, e^{\frac{m+n-2}{m+n-2}}e^{-\phi}dV_g, e^{\frac{m+n-2}{m+n-2}}e^{-\phi}d\sigma_g, m).
\]

Let \((M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, e^{-\hat{\phi}}d\sigma_{\hat{g}}, \hat{m})\) be two pointwise conformally equivalent smooth metric measure space such that \(\hat{g} = e^{\frac{2\sigma}{m+n-2}}g\) and \(\hat{\phi} = \frac{m-\sigma}{m+n-2} + \phi\). Let us denote by \(L^m_\phi\) and \(\hat{L}^m_\phi\) their respective weighted conformal Laplacians. Similarly, we
denote with hat all quantities computed with respect to the smooth metric measure space \((M^n, \hat{g}, e^{-\hat{\phi}} dV_g, e^{-\hat{\phi}} d\sigma_g, m)\). Then we have \(\hat{v} = e^{\frac{m}{m+n-2}} v\) and the following transformation rules

\[
\hat{L}_m^\phi(w) = e^{-\frac{m+n+2}{2(m+n-2)}} L_m^\phi(e^{\frac{m}{m+n-2}} w), \quad \hat{B}_m^\phi(w) = e^{-\frac{m+n}{2(m+n-2)}} B_m^\phi(e^{\frac{m}{m+n-2}} w).
\]

We mention that the identity in the left hand size of (26) appears in [4]. On the other hand, we denote by \((w, \varphi)_M = \int_M w \varphi v^m dV_g\) the inner product in \(L^2(M, v^m dV_g)\). Also, we denote by \(||.||_{2, M}\) the norm in the space \(L^2(M, v^m dV_g)\), in some case we use the notation \(||.||\) for this norm. \(H^1(M, v^m dV_g)\) denotes the closure of \(C^\infty(M)\) with respect to the norm \(\int_M |\nabla w|^2 + |w|^2\).

Here and subsequently the integrals are computed using the measure \(v^m dV_g\).

4. Preliminaries for Escobar-Riemann type problem

In this section, we define the weighted Escobar quotient which generalizes the quotient considered by Escobar in [8] and we consider a suitable \(W\)-functional. In general, the weighted Escobar quotient is not necessarily finite. Similarly to [4], we define the energies of these functionals and we give some of their properties.

4.1. The weighted Escobar quotient. We start with the definition of the weighted Escobar quotient

**Definition 6.** Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary. The weighted Escobar quotient \(Q : C^\infty(M) \to \mathbb{R}\) is defined by

\[
Q(w) = \frac{((L_m^\phi w, w)_M + (B_m^\phi w, w)_{\partial M})(\int_M |w|^{2(m+n-1)} v^{-1})^{\frac{m}{m+n-1}}}{(\int_{\partial M} |w|^{2(m+n-1)} v^{-1})^{\frac{m}{2m+n-2}}}.\]

The weighted Escobar constant \(\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R}\) of the smooth metric measure space \((M^n, g, v^m dV_g, v^m d\sigma_g, m)\) is

\[
\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g, m] = \inf\{Q(w) : w \in H^1(M, v^m dV_g, v^m d\sigma_g)\}.
\]

**Remark 3.** In some cases, when the context is clear, we will not write the dependence of the smooth metric measure space with boundary, for example we write \(Q\) and \(\Lambda\) instead of \(Q[M^n, g, v^m dV_g, v^m d\sigma_g]\) and \(\Lambda[M^n, g, v^m dV_g, v^m d\sigma_g]\), respectively. We note that since \(C^\infty(M)\) is dense in \(H^1(M, v^m dV_g)\) and \(Q(||w||) = Q(w)\), it is sufficient to consider the
weighted Escobar constant by minimizing over the space of non-negative smooth functions on $M$, subsequently we will do this assumption without further comment.

Now, note that the weighted Escobar quotient is conformal in the sense of Definition 3. On the other hand, the weighted Escobar quotient satisfies similar properties to the weighted Yamabe quotient introduced by Case in [4], for example we observe that the weighted Escobar quotient is continuous in $m \in [0, \infty)$ and it is conformal in the sense of the Definition 3.

**Proposition 2.** Let $(M^n, g)$ be a compact Riemannian manifold with boundary. Fix $\phi \in C^\infty(M)$ and $m \in [0, \infty)$. Given any $w \in C^\infty(M)$, it holds that

$$
\lim_{k \to m} Q[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, k](w) = Q[M^n, g, e^{-\phi}dV_g, e^{-\phi}d\sigma_g, m](w).
$$

**Proposition 3.** Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. For any $\sigma, w \in C^\infty(M)$ it holds that

$$
Q[M^n, e^{{m+2\sigma \over m+n-2}} g, e^{m+n-2\sigma \over m+n-2} v^m dV_g, e^{m+n-2\sigma \over m+n-2} v^m d\sigma_g](w)
= Q[M^n, g, v^m dV_g, v^m d\sigma_g](e^{2\sigma \over m} w).
$$

**Proof.** A straightforward computation shows that the integrals

$$
\int_M |w|^{2(m+n-1) \over m+n-2} v^{m-1} dV_g \quad \text{and} \quad \int_{\partial M} |w|^{2(m+n-1) \over m+n-2} v^m d\sigma_g
$$

are invariant under the conformal transformation

$$
(g, v^m dV_g, v^m d\sigma_g, w) \to (e^{m+n-2\sigma \over m+n-2} g, e^{m+n-2\sigma \over m+n-2} v^m dV_g, e^{m+n-2\sigma \over m+n-2} v^m d\sigma_g, e^{-\sigma \over 2} w).
$$

By Proposition 1 the term $(L^m_{\phi} w, w) + (B^m_{\phi} w, w)$ is invariant under (32).

Similar to the smooth metric measure spaces we have some behavior for the boundary volume. Note that in the boundary the integral $\int_{\partial M} |w|^{2(m+n-1) \over m+n-2} v^m d\sigma_g$ measure the boundary volume $\int_{\partial M} \hat{v}^m d\sigma_g$ of

$$
(M^n, \hat{g}, \hat{v}^m dV_{\hat{g}}, \hat{v}^m d\sigma_{\hat{g}}, m) = (M^n, w^{4 \over m+n-2} g, w^{2(m+n) \over m+n-2} v^m dV_g, w^{2(m+n-1) \over m+n-2} v^m d\sigma_g, m).
$$

Also with the same purpose, to simplify calculus and to avoid the trivial non-compactness of the weighted Escobar-Riemann type problem, we give the next definition of the volume-normalized on the boundary.
Definition 7. Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary. We say that a positive function \(w \in C^\infty(M)\) is volume-normalized on the boundary if
\[
\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}} v^m d\sigma_g = 1.
\]

4.2. \(W\)-functional. We introduce a \(W\)-functional with similar properties as the \(W\)-functional considered by Case in [4] and Perelman in [12].

Definition 8. Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary. The \(W\)-functional, \(W : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}\), is defined by
\[
W(w, \tau) = W[M^n, g, v^m dV_g, v^m d\sigma_g](w, \tau)
= \tau^m \left( (L^m_\phi w, w) + (B^m_\phi w, w) \right) + \int_M \tau^{-\frac{1}{2}} w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} - \int_{\partial M} w^{\frac{2(m+n-1)}{m+n-2}}
\]
when \(m \in [0, \infty)\).

As the weighted Escobar quotient and the \(W\)-functional considered by Case in [4], the \(W\)-functional defined before is continuous in \(m\) and conformally invariant. Additionally, we have one scale invariant in the variable \(\tau\).

Proposition 4. Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary. Then
\[
\lim_{k \to m} W[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, k](w, \tau) = W[M^n, g, e^{-\phi} dV_g, e^{-\phi} d\sigma_g, m](w, \tau).
\]

Proposition 5. Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary. The \(W\)-functional is conformally invariant in its first component:
\[
W[M^n, e^{2\sigma} g, e^{(m+n)\sigma} v^m dV_g, e^{(m+n-1)\sigma} v^m d\sigma_g](w, \tau)
= W[M^n, g, v^m dV_g, v^m d\sigma_g](e^{\frac{(m+n-2)}{2}} w, \tau)
\]
for all \(\sigma, w \in C^\infty(M)\) and \(\tau > 0\). It is scale invariant in its second component:
\[
W[M^n, cg, v^m dV_{cg}, v^m d\sigma_{cg}](w, \tau)
= W[M^n, g, v^m dV_g, v^m d\sigma_g](c^{\frac{(n-1)(m+n-2)}{4(m+n-1)}} w, c^{-1} \tau).
\]

Proof. The equality (35) follows as in Proposition 3 and the equality (36) follows by a direct computation. \(\square\)
Since we are interested in minimizing the weighted Escobar quotient it is natural to define the following energies as infima using the $W$-functional and relating one of these energies with the weighted Escobar constant.

**Definition 9.** Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. Given $\tau > 0$, the $\tau$-energy $\nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau)$ is the number defined by

$$
\nu(\tau) = \nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau) = \inf \left\{ W(w, \tau) : w \in H^1(M, v^m dV_g, v^m d\sigma_g), \int_{\partial M} w^{2(m+n-1)} = 1 \right\}.
$$

The energy $\nu[M^n, g, v^m dV_g, v^m d\sigma_g] \in \mathbb{R} \cup \{-\infty\}$ is defined by

$$
\nu = \nu[M^n, g, v^m dV_g, v^m d\sigma_g] = \inf_{\tau > 0} \nu[g, v^m dV_g, v^m d\sigma_g](\tau).
$$

The conformal invariance in the $W$-functional is transferred to the energies.

**Proposition 6.** Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary. Then

$$
\nu[M^n, c e^{2\sigma} g, e^{(m+n)\sigma} v^m dV_{cg}, e^{(m+n-1)\sigma} v^m d\sigma_{cg}](c\tau) = \nu[M^n, g, v^m dV_g, v^m d\sigma_g](\tau),
$$

$$
\nu[M^n, c e^{2\sigma} g, e^{(m+n)\sigma} v^m dV_{cg}, e^{(m+n-1)\sigma} v^m d\sigma_{cg}] = \nu[M^n, g, v^m dV_g, v^m d\sigma_g]
$$

for all $\sigma \in C^\infty(M)$ and $c > 0$.

The following proposition shows that it is equivalent to considering the energy instead of the weighted Escobar constant when the latter is positive.

**Proposition 7.** Let $(M^n, g, v^m dV_g, v^m d\sigma_g)$ be a compact smooth metric measure space with boundary and denote by $\Lambda$ and $\nu$ the weighted Escobar constant and the energy, respectively.

- $\Lambda \in [-\infty, 0)$ if and only if $\nu = -\infty$;
- $\Lambda = 0$ if and only if $\nu = -1$; and
- $\Lambda > 0$ if and only if $\nu > -1$. Moreover, in this case we have

$$
\nu = \frac{2m + n - 1}{m} \left( \frac{m\Lambda}{m + n - 1} \right)^{\frac{m+n-1}{m+n-1}} - 1
$$

and $w$ is a volume-normalized minimizer of $\Lambda$ if and only if $(w, \tau)$ is a volume-normalized minimizer of $\nu$ for
\[
\tau = \left[ \frac{m \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1}}{(m+n-1)((L^m_{\phi}w, w) + (B^m_{\phi}w, w))} \right]^{\frac{m+n-1}{2(2m+n-1)}}.
\]

**Proof.** If \( \Lambda \in [-\infty, 0) \) then there is a volume-normalized function \( w \in C^\infty(M) \) such that \((L^m_{\phi}w, w) + (B^m_{\phi}w, w) < 0\). Then, it is clear that \( \mathcal{W}(w, \tau) \to -\infty \) as \( \tau \to \infty \) and it follows that \( \nu = -\infty \). Reciprocally, if \( \nu = -\infty \) there exist a volume-normalized function \( w > 0 \) such that \((L^m_{\phi}w, w) + (B^m_{\phi}w, w) < 0 \) and \( \Lambda \in [-\infty, 0) \).

Suppose \( \Lambda \geq 0 \). Lemma 1 shows that if \( A, B > 0 \), then

\[
\inf_{x>0} \{ Ax^{\frac{m}{m+n-1}} + Bx^{-1} \} = \frac{2m+n-1}{2m+n-1} \left[ \frac{m}{m+n-1} \right]^{\frac{m+n-1}{2m+n-1}}
\]

for all \( x > 0 \), with equality if and only if

\[
x = \left[ \frac{mB}{(m+n-1)A} \right]^{\frac{m+n-1}{2m+n-1}}.
\]

Notes that equality (40) is achieved in the case \( A = 0 \). Then, from equality (41), the definitions of \( \Lambda \) and \( \nu \) and taking minimizing sequences of these infima we get the remain equivalences. When \( \Lambda > 0 \) using (40) and (41) we get that (38) and (39) holds. \( \square \)

### 4.3. Variational formulae for the weighted energy functionals.

The next proposition contains the computation of the Euler-Lagrange equations of the minimizing of weighted Escobar quotient. We will use it in the proof of Theorem A on the regularity part.

**Proposition 8.** Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary and suppose that \( 0 \leq w \in H^1(M) \) is a volume-normalized minimizer of the weighted Escobar constant \( \Lambda \). Then \( w \) is a weak solution of

\[
\begin{align*}
L^m_{\phi}w + c_1 w^{\frac{m+n}{m+n-2}} v^{-1} &= 0, & \text{in } M, \\
B^m_{\phi}w &= c_2 w^{\frac{m+n}{m+n-2}}, & \text{on } \partial M
\end{align*}
\]

where

\[
c_1 = \frac{m\Lambda}{m+n-2} \left( \int_M w^{\frac{2(m+n-1)}{m+n-2}} v^{-1} \right)^{-\frac{2m+n-1}{m+n-1}}
\]

and
\[ c_2 = \frac{(2m + n - 2)\Lambda}{m + n - 2} \left( \int_M w^\frac{2(m+n-1)}{m+n-2} v^{-1} \right)^{\frac{m}{m+n-2}}. \]

Proof. This proposition follows immediately from the fact that the conformal Laplacian is self-adjoint, and the definition of the weighted Escobar constant. \(\square\)

Remark 4. If \(\Lambda = 0\) then we have in the proposition above that \(c_1 = 0, c_2 = 0\). In this case, it follows that the equations in (42) coincide with the equations for finding a new conformal smooth metric measure space such that \(\bar{R}_\phi^m \equiv 0\) and \(\bar{H}_\phi^m \equiv 0\). Moreover, the problem to find a conformal smooth metric measure space with \(\bar{R}_\phi^m \equiv 0\) and \(\bar{H}_\phi^m \equiv C\) is solved by a direct compact argument on the functional

\[ \bar{Q}(w) = \frac{(L_\phi^m w, w)_M + (B_\phi^m w, w)_{\partial M}}{(\int_{\partial M} |w|^{\frac{2(m+n-1)}{m+n-2}})^{\frac{m+n-2}{m+n-1}}} \]

due to \(\frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}\) for \(m > 0\).

Next, we consider the Euler Lagrange equation on the \(W\)-functional and we will use it in the proof of Theorem B.

Proposition 9. Let \((M^n, g, v^m dV_g, v^m d\sigma_g)\) be a compact smooth metric measure space with boundary, fix \(\tau > 0\), and suppose that \(w \in H^1(M)\) is a non-negative critical point of the map \(\xi \rightarrow W(\xi, \tau)\) acting on the space of volume-normalized elements of \(H^1(M)\). Then \(w\) is a weak solution of

\[ \begin{align*}
\tau \frac{m}{2(m+n-1)} L_\phi^m w + \frac{m+n-1}{m+n-2} \tau^{-\frac{1}{2}} w^\frac{m+n+2}{m+n-2} v^{-1} &= 0 \quad \text{in} \ M, \\
\tau \frac{m}{2(m+n-1)} B_\phi^m w &= c_3 w^\frac{m+n}{m+n-2} \quad \text{on} \ \partial M,
\end{align*} \]

where

\[ c_3 = (\nu(\tau) + 1) + \frac{\tau^{-\frac{1}{2}}}{m+n-2} \int_M w^\frac{2(m+n-1)}{m+n-2} v^{-1}. \]

If additionally \((w, \tau)\) is a minimizer of the energy, then

\[ c_3 = \frac{(m+n-1)(2m+n-2)}{(m+n-2)(2m+n-1)}(\nu + 1). \]

Proof. The equality (43) follows immediately from the definition of \(W\). If \((w, \tau)\) is a critical point of the map \((w, \tau) \rightarrow W(w, \tau)\), then
\[(45) \quad \frac{m}{m+n-1} \tau^{\frac{m}{2(m+n-1)}} \left( (L^m_{\phi} w, w) + (B^m_{\phi} w, w) \right) = \tau^{-\frac{1}{2}} \int_M w \frac{2(m+n-1)}{m+n-2} v^{-1}. \]

Using this identity we can express \( \nu \) and \( c_3 \) in terms of \((L^m_{\phi} w, w) + (B^m_{\phi} w, w)\) and these expressions yields (44).

4.4. Euclidean half-space as the model space weighted Escobar problem. Theorem \( \Pi \) gives a complete classification of the minimizers for the weighted Escobar quotient in the model space \((\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m)\) for \( m \) non-negative integer. In this subsection we take a new \((\tau, x_0)\)-parametric family of functions as in (3) with \( \tau > 0 \) and \( x_0 \in \mathbb{R}^{n-1} \).

To define the \((\tau, x_0)\)-parametric family of functions fix \( n \geq 3 \) and \( m > 0 \). Given any \( x_0 \in \mathbb{R}^{n-1} \) and \( \tau > 0 \), define the function \( w_{x_0, \tau} \in C^\infty(\mathbb{R}_n^+) \) by

\[(46) \quad w_{x_0, \tau}(t, x) = \tau^{\frac{(n-1)(m+n-2)}{4(m+n-1)}} \left[ \left( 1 + \left( \frac{c(m,n)}{\tau} \right)^{\frac{1}{2}} t \right)^2 + \frac{c(m,n)|x - x_0|^2}{\tau} \right]^{-\frac{m+n-2}{2}} \]

where \( c(m,n) = \frac{m+n-1}{m(m+n-2)^2} \). By change of variables we get

\[(47) \quad V = \int_{\partial \mathbb{R}^n_+} w_{x_0, \tau}^\frac{2(m+n-1)}{m+n-2} 1^m d\sigma = \int_{\partial \mathbb{R}^n_+} w_{0,1, \tau}^\frac{2(m+n-1)}{m+n-2} 1^m d\sigma. \]

A straightforward computation shows that

\[(48) \quad -\frac{m}{m+n-1} \Delta w_{x_0, \tau} + \frac{m+n-1}{m+n-2} \tau^{-\frac{1}{2}} w_{x_0, \tau}^\frac{m+n}{m+n-2} = 0 \quad \text{in} \quad \mathbb{R}_n^+, \]

\[ \quad \tau^{\frac{m}{2(m+n-1)}} \frac{\partial w_{x_0, \tau}}{\partial \eta} = \left( \frac{m+n-1}{m} \frac{1}{2} \right)^{\frac{1}{2}} w_{x_0, \tau}^{\frac{m+n}{m+n-2}} \quad \text{on} \quad \partial \mathbb{R}_n^+, \]

\[(49) \quad \sup_{(x,t) \in \mathbb{R}_n^+} w_{x_0, \tau}(x,t) = w_{x_0, \tau}(x_0,0) = \tau^{-\frac{(n-1)(m+n-2)}{4(m+n-1)}}, \]

and for any \( x \neq x_0 \),

\[(50) \quad \lim_{\tau \to 0^+} w_{x_0, \tau}(x,t) = 0. \]

Define \( \tilde{w}_{x_0, \tau} = V^{-\frac{m+n-2}{2}} w_{x_0, \tau} \); with \( V \) as in (47). Since \( \tilde{w}_{x_0, \tau} \) achieves the weighted Escobar quotient, by Proposition 7 there exits \( \tilde{\tau} > 0 \) such that
\[
\nu(\mathbb{R}^n, dt^2 + dx^2, dV, d\sigma, m) + 1 = \mathcal{W}(\mathbb{R}^n, dt^2 + dx^2, dV, d\sigma, m)(\tilde{w}_{x_0, \tau}, \tilde{\tau}) + 1
\]

\[
= \tilde{\tau}^{\frac{2(m+n-1)}{m+n-2}} \int_{\mathbb{R}^n} |\nabla w_{x_0, \tau}|^2 dV + \tilde{\tau}^{-\frac{1}{2}} V^{-1} \int_{\mathbb{R}^n} \frac{w_{x_0, \tau}^2}{w_{x_0, \tau}^2 + 2} dV.
\]

Then Proposition 9 yields \( \tilde{\tau} = \tau V^{-2(m+n-1)}. \)

5. **The Escobar type problem for negative weighted Escobar constant**

In this section, we prove Theorem A by a direct compactness argument. For this purpose, we develop some estimative for below for the Laplacian term in the Escobar quotient and some properties of Dirichlet eigenvalues and eigenfunctions. In this section, \( C \) is a real constant that depends only on the smooth metric measure space \( (M^n, g, v^m dV, v^m d\sigma_g) \) and possibly changing from line to line.

5.1. **A below bound for conformal Laplacian term.** All functions in the family \( \{w_{\epsilon, 0}\} \) as in (3) are minimizers of the weighted Escobar problem. Note that these functions are not uniformly bounded in \( H^1(M) \) as \( \epsilon \to 0. \) That shows that in general there is no reason to find a minimizing function by direct arguments in the weighted Escobar quotient. It is possible that if the weighted Escobar quotient is finite and we try to minimize it with a sequence of functions normalized, then the terms involved in the numerator of the weighted Escobar quotient evaluated in these functions are not bounded uniformly. The next lemma deals with the control of one of those terms from below.

**Lemma 3.** Let \( (M^n, g, v^m dV, v^m d\sigma_g) \) be a compact smooth metric measure space with boundary and suppose that \( \Lambda \) is finite, then there exists a real constant \( C \) such that any volume-normalized function \( \varphi \in H^1(M) \) satisfies

\[
(L^m_{\varphi} \varphi, \varphi) + (B^m_{\varphi} \varphi, \varphi) > C.
\]

**Proof.** Suppose that there exists a sequence of functions \( \{\varphi_i\}_{i=1}^{\infty} \) such that

\[
\lim_{i \to \infty} (L^m_{\varphi_i} \varphi_i, \varphi_i) + (B^m_{\varphi_i} \varphi_i, \varphi_i) = -\infty \quad \text{and} \quad \int_{\partial M} \varphi_i^{2(m+n-1)/m+n-2} = 1.
\]

Since \( \Lambda \) is finite there exists a real constant \( C \) such that every volume-normalized \( \varphi \) satisfies

\[
C \leq \Lambda(\varphi) = (L^m_{\varphi} \varphi, \varphi) + (B^m_{\varphi} \varphi, \varphi) \left( \int_M \varphi^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{m}{m+n-1}}.
\]
From the last inequality it follows that \( \lim_{i \to \infty} \int_M \varphi_i^{2(m+n-1)} = 0 \) and by the Hölder inequality it follows that \( \int_M \varphi_i^2 < C \) for any \( i \). Similarly, using that \( \varphi_i \) are volume normalized and the Hölder inequality we get \( \int_{\partial M} \varphi_i^2 < C \). Using these \( L^2 \) estimate we obtain that

\[
(L^m_{\phi} \varphi_i, \varphi_i) + (B^m_{\phi} \varphi_i, \varphi_i) > C
\]

contradicting the assumption (53).

5.2. **Dirichlet eigenvalues for the Conformal Laplacian.** In order to state the following lemma, we say that a real number \( \rho \) is an eigenvalue type Dirichlet on \( H^1_0(M) = \{ \varphi | \varphi \in H^1(M), \varphi \equiv 0 \text{ on } \partial M \} \) if \( \rho \) satisfies for some \( \varphi \in H^1_0(M) \)

\[
L^m_{\phi} \varphi = \rho \varphi \quad \text{in } M, \quad \varphi \equiv 0 \quad \text{on } \partial M.
\]

We also call \( \varphi \) an eigenfunction if it satisfies (54). Let us denote by \( \rho_1 \) the first eigenvalue type Dirichlet on \( H^1_{0,2}(M) \), then \( \rho_1 \) admits a variational characterization as

\[
\rho_1 = \inf_{\varphi \in H^1_0(M)} \frac{\int_M |\nabla \varphi|^2 + \frac{m+n-2}{4(m+n-1)} R^m_{\phi} \varphi^2}{\int_M \varphi^2}.
\]

We have \( \rho_1 \) is finite and we can choose an eigenfunction \( \varphi \) associated to this eigenvalue such that \( \varphi \geq 0 \). Moreover, using the maximum principle we can take \( \varphi > 0 \) in \( M \setminus \partial M \).

**Lemma 4.** Let \( (M^n, g, v^m dV_g, v^m d\sigma_g) \) be a compact smooth metric measure space with boundary and \( m > 0 \). Then \( \Lambda = -\infty \) if and only if \( \rho_1 \leq 0 \).

**Proof.** First, let us assume \( \rho_1 \leq 0 \). Let \( \varphi \) be a first eigenfunction of the problem (54) such that \( \varphi > 0 \) in \( M \setminus \partial M \). Let us define

\[
\psi_t = \frac{t\varphi + 1}{\sqrt{D}} \quad \text{where} \quad D = \left( \int_{\partial M} e^{-\phi} d\sigma_g \right)^{m+n-2/(m+n-1)}
\]

and observe that for some constant \( C > 0 \) we have

\[
\int_{\partial M} \psi_t^{2(m+n-1)} = 1 \quad \text{and} \quad \int_M \psi_t^{2(m+n-1)} \geq C > 0.
\]

**Claim 1.**

\[
(L^m_{\phi} \psi_t, \psi_t) + (B^m_{\phi} \psi_t, \psi_t) \to -\infty \quad \text{when} \quad t \to \infty.
\]
To prove this claim, we argue as Garcia and Muñoz in [9, Proposition 1]. First, we consider the case \( \rho_1 < 0 \), using that \( \psi_t \equiv 0 \) on \( \partial M \) we get

\[
(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) = \frac{1}{D} \left[ t^2 \left( \rho_1 \int_M \varphi^2 \right) + t \left( \frac{m+n-2}{2(m+n-1)} \int_M \varphi R^m_\phi \right) + E \right]
\]

where

\[
E = \frac{m+n-2}{4(m+n-1)} \int_M R^m_\phi + \frac{m+n-2}{2(m+n-1)} \int_M H^m_\phi.
\]

Since \( \rho_1 < 0 \), the quadratic term for \( t \) on the right hand side of (58) is negative. Letting \( t \to \infty \) it follows our claim in this case.

Now, we suppose that \( \rho_1 = 0 \), then

\[
(L^m_\phi \psi_t, \psi_t) + (B^m_\phi \psi_t, \psi_t) = \frac{1}{D} \left[ t \left( \frac{m+n-2}{2(m+n-1)} \int_M \varphi R^m_\phi \right) + E \right]
\]

where \( E \) is defined as in the previous case. Since \( \varphi \equiv 0 \) on \( \partial M \), by Hopf’s Lemma, \( \frac{\partial \varphi}{\partial \eta} < 0 \). Then, integrating by parts yields

\[
\frac{m+n-2}{4(m+n-1)} \int_M \varphi R^m_\phi = \int_M \Delta \varphi = \int_{\partial M} \frac{\partial \varphi}{\partial \eta} < 0.
\]

Then, the linear term for \( t \) on the right hand side of (59) is negative. Taking \( t \to \infty \) we get the conclusion in this case and we finish the claim’s proof.

Finally, from the estimates (56) and (57) we get that \( Q(\psi_t) \to -\infty \) as \( t \to \infty \), therefore we conclude \( \Lambda = -\infty \).

Next, we assume that \( \Lambda = -\infty \) and we prove that \( \rho_1 \leq 0 \). This assumption implies that \( R^m_\phi \) is not identically zero. Let us take a minimizing sequence of functions \( \{ \varphi_i \}_{i=1}^\infty \) of \( \Lambda \) such that

\[
\int_{\partial M} \varphi_i^{2(m+n-1)} = 1, \quad (L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) \leq 0 \quad \text{and} \quad \lim_{i \to \infty} Q(\varphi_i) = -\infty.
\]

Claim 2. \( \int_M \varphi_i^{2(m+n-2)} \to \infty \) when \( i \to \infty \).

Arguing by contradiction, we assume that there exists a constant \( C > 0 \) such that

\[
\int_M \varphi_i^{2(m+n-1)} < C,
\]

then by the Hölder inequality we get that \( \int_M \varphi_i^2 < C \) for every \( i \). On the other hand, we have that \( (L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) \to -\infty \) when \( i \to \infty \) since \( \lim_{i \to \infty} Q(\varphi_i) = -\infty \). Using this limit, the fact that \( R^m_\phi \) is a non-zero function and that \( \varphi_i \)
is normalized we get $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$, which is a contradiction with the initial assumption. Hence $\int_M \varphi_i^{\frac{2(m+n-1)}{m+n-2}} \to \infty$.

**Claim 3.** $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$.

Arguing by contradiction, suppose that there exists a constant $C > 0$ such that $\int_M \varphi_i^2 < C$. Then

$$\int_M |\nabla \varphi_i|^2 \leq (L^m_\phi \varphi_i, \varphi_i) + (B^m_\phi \varphi_i, \varphi_i) + C(\|\varphi_i\|_{2,M}^2 + \|\varphi_i\|_{2,\partial M}^2) < C. \quad (60)$$

On the other hand, by the Sobolev inequality we get that there exists a constant $C$ such that

$$\int_M \frac{2(m+n-1)}{m+n-2} \varphi_i^2 \to \infty \quad \text{when} \quad i \to \infty. \quad (61)$$

Then inequalities (60) and (61) yield $\int_M \varphi_i^2 \leq C$. This is a contradiction with the Claim 2 and we conclude that $\int_M \varphi_i^2 \to \infty$ when $i \to \infty$.

Now we are able to conclude the proof of the lemma. For this purpose let us define the functions $\psi_i = \frac{\varphi_i}{\|\varphi_i\|_{2,M}}$. Arguing as in the last part of Proposition 1 in Garcia and Muñoz \[9\], we get that a sub-sequence $\psi_i$ converges weakly to a function $\psi$ in $H^1_0(M)$ such that $\|\psi\|_{2,M} = 1$ and

$$\rho_1 \leq \int_M |\nabla \psi|^2 + \frac{m+n-2}{4(m+n-1)} R_\phi^m \psi^2 \leq \liminf_{i \to \infty} (L^m_\phi \psi_i, \psi_i) + (B^m_\phi \psi_i, \psi_i) \leq 0. \quad \square$$

5.3. **Proof of Theorem A.** In this subsection we prove Theorem A using the before Lemmas presented in this section.

**Proof of Theorem A.** Let $\{w_i\}_{i=1}^\infty$ be a sequence of positive functions such that $\int_{\partial M} w_i^{\frac{2(m+n-1)}{m+n-2}} = 1$, $Q(w_i) \leq 0$ and $Q(w_i) \to \Lambda$ when $i \to \infty$. Then

$$0 \geq (L^m_\phi w_i, w_i) + (B^m_\phi w_i, w_i) \geq \|\nabla w_i\|_{2,M}^2 - C(\|w_i\|_{2,M}^2 + \|w_i\|_{2,\partial M}^2). \quad (62)$$

First, we consider the case $\|w_i\|_{2,M} < C$, then the last inequality yields that $\{w_i\}_{i=1}^\infty$ are uniformly bounded in $H^1(M)$. Recall that $m > 0$, then $1 < \frac{2(m+n-1)}{m+n-2} < \frac{2(n-1)}{n-2}$, i.e. $\frac{2(m+n-1)}{m+n-2}$ is less than the critical Trace’s inequality exponent. By Sobolev’s and Trace’s embedding Theorems, there exists a function $w$ and a sub-sequence $\{w_i\}_{i=1}^\infty$ which
converges to $w$ in $L^2(M)$, $L^\frac{2(m+n-1)}{m+n-2}(M)$ and $L^\frac{2(m+n-1)}{m+n-2}(\partial M)$ and also $\{w_i\}_{i=1}^\infty$ converges weakly to $w$ in $H^1(M)$. It follows that there exist a constant $C$ such that
\[
\int_M w^\frac{2(m+n-1)}{m+n-2} v^{-1} \geq C \quad \text{and} \quad \|w\|^\frac{2(m+n-1)}{m+n-2} \partial M = 1.
\]

Then by construction, $w$ minimizes the weighted Escobar quotient and by Proposition 8, $w$ is a non-negative weak solution of
\[
L^m_\phi w + c_1 w^\frac{m+n}{m+n-2} v^{-1} = 0 \quad \text{in} \quad M,
\]
\[
B^m_\phi w = c_2 w^\frac{m+n}{m+n-2} \quad \text{on} \quad \partial M.
\]

Since $1 < \frac{m+n-1}{m+n-2} < \frac{n-1}{n-2}$, the usual elliptic regularity argument for sub-critical equations allows us to conclude that $w$ is in fact smooth and positive, as we desired.

Following, we prove that we do not have the case when $\|w_i\|_{2,M} \to \infty$ is unbounded. Arguing by contradiction, we assume that $\|w_i\|_{2,M} \to \infty$ when $i \to \infty$. Consider the $L^2$ re-normalized sequence of functions $\tilde{w}_i = \frac{w_i}{\|w_i\|_{2,M}}$. It follows that $\|\tilde{w}_i\|^\frac{2(m+n-1)}{m+n-2} \partial M \to 0$ when $i \to \infty$. Since $\tilde{w}_i$ satisfy the inequality (62) for every $i$ we know that $\{\tilde{w}_i\}_{i=1}^\infty$ is uniformly bounded in $H^{1,2}(M)$.

By Sobolev’s and Trace’s embedding Theorems, there exists a function $w$ and a subsequence $\{\tilde{w}_i\}_{i=1}^\infty$ which converges to $w$ in $L^2(M)$, $L^\frac{2(m+n-1)}{m+n-2}(M)$ and $L^\frac{2(m+n-1)}{m+n-2}(\partial M)$ and also weakly in $H^1(M)$. In consequence, $\|w\|_{2,M} = 1$ and using again that $\|\tilde{w}_i\|^\frac{2(m+n-1)}{m+n-2} \partial M \to 0$ when $i \to \infty$, we get that $w \equiv 0$ in $\partial M$.

On the other hand, Lemma 3 yields
\[
0 > (L^m_\phi w_i, w_i) + (B^m_\phi w_i, w_i) > -C.
\]
Therefore $(L^m_\phi \tilde{w}_i, \tilde{w}_i) + (B^m_\phi \tilde{w}_i, \tilde{w}_i) \to 0$ when $i \to \infty$. Using $w$ as a test function in (55), we conclude that
\[
\rho_1 \leq \int_M |\nabla w|^2 + \frac{m+n-2}{4(m+n-1)} R^m \phi w \leq \liminf_{i \to \infty} (L^m_\phi \tilde{w}_i, \tilde{w}_i) + (B^m_\phi \tilde{w}_i, \tilde{w}_i) = 0.
\]

But $\rho_1 \leq 0$ contradicts Lemma 4 because $\Lambda$ is finite by hypothesis.

6. Aubin type inequality for weighted Escobar constants

In this section, we find an upper bound for the $\tau$-energy as $\tau$ goes to zero, Theorem B is a consequence of this estimate. To prove this estimate, we use Theorem 1 and the family $\{w_{0,\tau}\}$ in (46) as test functions in the $W$-functional. Actually, Theorem 1 is the reason for which the weighted Escobar constant for the Euclidean half-space appears on the right hand side of the inequality (6). Similar ideas to prove Theorem B appeared in
As in the previous section, $C$ is a real constant that depends only on the smooth metric measure space $(M^n, g, v^m dV_g, v^m dσ_g)$ and possibly changing from line to line or in the same line.

Lemma 5. Let $(M^n, g, v^m dV_g, v^m dσ_g)$ be a compact smooth metric measure space with boundary and $m ≥ 0$, then

$$\limsup_{τ → 0} ν(τ) ≤ ν[\mathbb{R}^n_+, dt^2 + dx^2, dV, dσ, m].$$

Proof. First define $\tilde{w}_{x_0, τ} = V^{-\frac{m+n-2}{2(m+n-1)}} w_{x_0, τ};$ with $V$ as in (47). By Theorem 1 we know that $\tilde{w}_{x_0, τ}$ achieves the weighted Escobar quotient, hence by Proposition 7, there exits $\tilde{τ} > 0$ such that

$$\nu[\mathbb{R}^n_+, dt^2 + dx^2, dV, dσ, m](\tilde{w}_{x_0, τ}, \tilde{τ}) = \tilde{τ}^2 \frac{V}{V_{m+n-2}} \int_{\mathbb{R}^n_+} |\nabla w_{x_0, τ}|^2 dV + \tilde{τ}^{-1} V^{-1} \int_{\mathbb{R}^n_+} w_{x_0, τ} \frac{2(m+n-1)}{m+n-2} dV.$$

Then Proposition 9 yields $\tilde{τ} = τV^{-\frac{2}{m+n-1}}$. On the other hand, fix a point $p ∈ ∂M$ and let $(x_i, t)$ be the Fermi coordinates in some fixed neighborhood $U$ of $p = (0, ..., 0)$. Let $1 > ε > 0$ be such that $B(p, 2ε) ⊂ U$. Let $η : M → [0, 1]$ be a cutoff function such that $η ≡ 1$ on $B^+_{ε}, supp(η) ⊂ B^+_{2ε}$ and $|∇η|^2 < Cε^{-1}$ in $A^+_ε = B^+_{2ε} \setminus B^+_{ε}$. For each $0 < τ < 1$, define $f_τ : M → \mathbb{R}$ by $f_τ(x_1, ..., x_{n-1}, t) = ηw_{0, τ}(x_1, ..., x_{n-1}, t)$, and set $\tilde{f}_τ = V^{-\frac{2(m+n-1)}{m+n-2}} f_τ$ for

$$V_τ = \int_{∂M} \tilde{f}_τ^{\frac{2(m+n-1)}{m+n-2}}.$$

Proposition 5 implies that if $w$ is a normalized function with the metric $v^{-2}g$, then

$$\mathcal{W}(M^n, v^{-2}g, dV_{v^{-2}g}, dσ_{v^{-2}g}, m)(w, τ) = \mathcal{W}(M^n, g, v^m dV_g, v^m dσ_g)(v^{\frac{m+n-2}{2}} w, τ),$$

this equality allows us to consider without loss generality that $v ≡ 1$. Computing as in Lemma 3.4, and using that $dV_g = (1 + O(r)) dx dt$ and $dσ_g = (1 + O(r)) dx$ we obtain
\[ W[M^n, g, dV_g, d\sigma_g, m](f_\tau, \tilde{\tau}) + 1 \]
\[ = \frac{\tau^{m+n}}{V_{\tau}^{m+n-1}} \left( \int_{B_2^\tau} |f_\tau|^2 g + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dV_g \right) \]
\[ + \int_{B_2^\tau \cap \partial M} \frac{m+n-1}{2(2m+n-2)} H_g f_\tau^2 d\sigma_g \right) + \tilde{\tau}^{-\frac{1}{2}} V_{\tau}^{-\frac{1}{2}} \int_{B_2^\tau} 2^{(m+n-1)} f_\tau^2 dV_g \]
\[ \leq (1 + C\epsilon) \left\{ \frac{\tilde{\tau}^{\frac{m}{2}}}{V_{\tau}^{m+n-2}} \left( \int_{B_2^\tau} |f_\tau|^2 g + \frac{m+n-1}{4(m+n-2)} R_g f_\tau^2 dx \right) \right\}. \]

Let us recall that \( c(m, n) = \frac{m+n-1}{m(m+n-2)^m} \). Fixing \( \epsilon < 1 \) and after taking \( \sqrt{\tau} \leq \sqrt{c(m, n)} \) we obtain

\[ \int_{B_2^\tau} R_g f_\tau^2 dx dt \leq C \int_{B_2^\tau} u_{0, \tau}^2 dx dt \]
\[ = C\tau^{\frac{(n-1)(m+n-2)}{2(m+n-1)}} \int_{B_2^\tau} \left( \left(1 + \left( \frac{c(m,n)}{\tau} \right)^{\frac{n}{2}} \right)^2 + \frac{c(m,n)}{\tau} |x|^2 \right)^{m+n-2} \]
\[ = C\tau^{\frac{n-1}{2(m+n-1)}} \int_{B_2^\tau} \frac{dy dt}{((1 + s)^2 + |y|^2)^{m+n-2}}. \]

Similar as in \[\text{[10]}\] Lemma 3.5] we get

\[ \int_{B_2^\tau \cap \partial M} \frac{dy dt}{((1 + s)^2 + |y|^2)^{m+n-2}} = \begin{cases} 
C & \text{if } 4 - n - 2m < 0, \\
O(\tau^{m-\frac{n}{2}}) & \text{if } n = 3, \frac{1}{2} - m > 0 \text{ and } O(\log(\tau)) & \text{if } n = 3, \frac{1}{2} - m = 0.
\end{cases} \]

Then

\[ \int_{B_2^\tau} R_g f_\tau^2 dx dt = E_1 = \begin{cases} 
O(\tau^{\frac{n+1}{2(m+n-1)} + \frac{1}{2}}) & \text{if } 4 - n - 2m < 0, \\
O(\tau^{\frac{n}{2(m+n-1)} + m}) & \text{if } n = 3, m < \frac{1}{2} \text{ and } O(\tau^{\frac{n-1}{2(m+n-1)} + \frac{1}{2}} \log(\tau)) & \text{if } n = 3, \frac{1}{2} - m = 0.
\end{cases} \]

Now, we estimate the integrals on the right hand side in the second inequality of (65)
\[
\int_{B^{n-1}_{2\varepsilon}} H_g f^2_t dx \leq C \int_{B^{n-1}_{2\varepsilon}} w^2_{0,\varepsilon} dx = C \tau^{\frac{n-1}{2(m+n-1)}} \int_{B^{n-1}_{2\varepsilon}} (1 + |y|^2)^{-(m+n-2)} dx \\
\leq C \tau^{\frac{n-1}{2(m+n-1)}}
\]

(69)

\[
\int_{B^{n-1}_{2\varepsilon}} f^2_t \frac{\partial}{\partial t} w_{m+n-2} \, dx dt \leq \int_{B^{n}_{0,\varepsilon}} w^2_{m+n-2} \, dx dt.
\]

(70)

Let us estimate the gradient integral in \( A^+ = B^{+} \setminus B^+_\varepsilon \). Observe that

\[
|\nabla f|^2_t \leq C|\nabla f|^2 \leq C(\eta^2|\nabla w_{0,\varepsilon}|^2 + |\nabla \eta|^2 w^2_{0,\varepsilon}).
\]

(71)

Now, we get

\[
\int_{A^+} |\nabla \eta|^2 w^2_{0,\varepsilon} \, dx dt \leq C \epsilon^{-2} \int_{A^+} w^2_{0,\varepsilon} \, dx dt
\]

(72)

\[
\leq C \epsilon^{-2} \tau^{\frac{(n-1)(m+n-2)}{2(m+n-1)}} \frac{n}{2} \int_{A^+} \frac{1}{\sqrt{s^2 + |y|^2}} dx dt
\]

\[
\leq C \epsilon^{2-n-2m} \tau^{\frac{n-1}{2(m+n-1)} + m + \frac{n-3}{2}}
\]

and

\[
\int_{A^+} \eta^2 |\nabla w_{0,\varepsilon}|^2 \, dx dt \leq C \tau^{\frac{(n-1)(m+n-2)}{2(m+n-1)}} \frac{n}{2} \int_{A^+} \frac{1}{\sqrt{s^2 + |y|^2}} dx dt
\]

(73)

\[
\leq C \epsilon^{2-n-2m} \tau^{\frac{n-1}{2(m+n-1)} + m + \frac{n-3}{2}}.
\]

Then

\[
\int_{A^+} |\nabla f|^2 g_t \, dx dt \leq C \epsilon^{2-n-2m} \tau^{\frac{n-1}{2(m+n-1)} + m + \frac{n-3}{2}}.
\]

(74)

Since for the Fermi coordinates around \( p \) we obtain \( g^{tt} = 1, g^{ti} = 0 \) and \( g^{ij} = \delta_{ij} + O(|x, t|) \) where \( 1 \leq i, j \leq n - 1 \), it follows
\[
\begin{align*}
\int_{B_r} |\nabla f_\tau|^2 dx dt & \leq \int_{B_r} |\nabla w_{0,\tau}|^2 dx dt + C \int_{B_r} |x, t|(w_{0,\tau})_i(w_{0,\tau})_j dx dt \\
& \leq \int_{B_r} |\nabla w_{0,\tau}|^2 dx dt + C r^{-(n-1)/(2(m+n-1))}.
\end{align*}
\]

We already have the second inequality of (75) because

\[
\int_{B_r} |x, t|(w_{0,\tau})_i(w_{0,\tau})_j \leq C r^{-(n-1)/(2(m+n-1)) - 2} \int_{B_r} \frac{|x, t|x_i x_j dx dt}{((1 + \frac{c(m,n)}{r})^2 + \frac{c(m,n)}{r}|x|^2)^{m+n}} \\
\leq C r^{(n-1)/(2(m+n-1)) - 1} \int_{B_r^{+}} \frac{|y, s|^3 dy dt}{((1 + s)^2 + |y|^2)^{m+n}} \\
\leq C r^{(n-1)/(2(m+n-1))}.
\]

Using the inequalities (68), (69), (74) and (75) in the inequality (65) we get that

\[
\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](f_\tau, \tau) + 1 \\
\leq (1 + Ce) \left\{ V^{m/(m+n-1)}_{\tau} \left( \int_{\mathbb{R}^+_n} |\nabla w_{0,\tau}|^2 dx dt + C r^{(n-1)/(2(m+n-1))} \right) + \tau^{-\frac{1}{2}} V^{-1}_{\tau} \int_{\mathbb{R}^+_n} w_{0,\tau}^{2(m+n-1)/m+n-2} dx dt \right\}.
\]

Now using the inequality (64) we conclude

\[
\mathcal{W}[M^n, g, dV_g, d\sigma_g, m](f_\tau, \tau) + 1 \\
\leq (1 + Ce) \left\{ V^{m/(m+n-1)}_{\tau} \left( C r^{(n-1)/(2(m+n-1))} + C r^{(n-1)/(2(m+n-1)) + m + n-3/2} e^{2-n-2m} + E_1 \right) \right. \\
+ \frac{m}{2(m+n-1)} \left( V^{m+n-2/m+n-1}_{\tau} - V^{m+n-2/m+n-1}_{-1} \right) \int_{\mathbb{R}^+_n} |\nabla w_{0,\tau}|^2 dx dt \\
+ \tau^{-\frac{1}{2}} (V^{-1}_{\tau} - V^{-1}_{-1}) \int_{\mathbb{R}^+_n} w_{0,\tau}^{2(m+n-1)/m+n-2} dx dt \right\}.
\]

On the other hand, we obtain
\[ V - V_\tau \leq \int_{\mathbb{R}^{n-1}\backslash B_{\tau}} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} \, dx \]

\[ = \tau^{-\frac{m+n-2}{m+n-1}} \int_{\partial \mathbb{R}^{n}\backslash B_{\tau}} \left( 1 + \frac{c(m, n)}{\tau} |x|^2 \right)^{-(m+n-1)} \, dx \]

\[ = C \int_{\partial \mathbb{R}^{n}\backslash B_{\tau}} \left( 1 + |y|^2 \right)^{-(m+n-1)} \, dy \]

\[ \leq C \epsilon^{1-n-2m} \tau^{-\frac{m+n-2}{2}}. \]

In particular, we get that the constants \( V_\tau \) are uniformly bounded away from zero. Using estimate (79) and the Taylor expansion for the functions \( x^{-\frac{m+n-2}{m+n-1}} \) and \( x^{-1} \) we obtain

\[ V^{-\frac{m+n-2}{m+n-1}} - V^{-\frac{m+n-2}{m+n-1}} \leq C \epsilon^{1-n-2m} \tau^{-\frac{m+n-2}{2}}. \]

Additionally, the equality (64) implies the following estimates

\[ \tilde{\tau}^{\frac{m}{2(m+n-1)}} \int_{\mathbb{R}^{m+n-1}_{+}} |\nabla w_{0,\tau}|^2 \, dx \, dt \leq C \]

and

\[ \tilde{\tau}^{-\frac{1}{2}} \int_{\mathbb{R}^{m+n-1}_{+}} w_{0,\tau}^{\frac{2(m+n-1)}{m+n-2}} \, dx \, dt \leq C. \]

The substitution \( \tilde{\tau} = \tau V^{-\frac{1}{m+n-1}} \), the inequalities (80), (81), (82) and (78) yield

\[ W[M^n, g, dV_y, d\sigma_y, m] (\tilde{f}_\tau, \tilde{\tau}) + 1 \]

\[ \leq (1 + C\epsilon) \nu[\mathbb{R}^m_+, dt^2 + dx^2, 1^m dV_y, 1^m d\sigma_y] \]

\[ + (1 + C\epsilon) \left\{ \int_{\mathbb{R}^{m+n-1}_{+}} V^{-\frac{m+n-2}{m+n-1}} \left( C \tau^{\frac{1}{2}} + C \tau^{\frac{1}{2} + \frac{m+n-3}{2} \epsilon^2 - n - 2m} + \tau^{\frac{m}{2(m+n-1)}} E_1 \right) + C \epsilon^{1-n-2m} \tau^{-\frac{m+n-2}{2}} \right\}. \]

Finally, taking \( \tau \to 0 \) and after \( \epsilon \to 0 \) in (83) the conclusion follows. \qed

**Proof of Theorem B.** By the definition of \( \nu \) and Lemma 5 we obtain that

\[ \nu[M^n, g, v^m dV_y, v^m d\sigma_y] \leq \nu[\mathbb{R}^m_+, dt^2 + dx^2, dV, d\sigma, m]. \]

By Proposition 7 we conclude
\( \Lambda[M^n, g, v^m dV, v^m d\sigma] \leq \Lambda[\mathbb{R}^n_+, dt^2 + dx^2, dV, d\sigma, m]. \)

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