AN INFORMAL OVERVIEW OF TRIPLES AND SYSTEMS

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Abstract. We describe triples and systems, expounded as an axiomatic algebraic umbrella theory for classical algebra, tropical algebra, hyperfields, and fuzzy rings.

1. Introduction

The goal of this overview is to present an axiomatic algebraic theory which unifies, simplifies, and “explains” aspects of tropical algebra [31, 38, 39, 36], hyperfields [5, 25, 45, 67], and fuzzy rings [16, 18, 25] in terms of familiar algebraic concepts. It was motivated by an attempt to understand whether or not it is incidental that basic algebraic theorems are mirrored in supertropical algebra, and was spurred by the realization that some of the same results are obtained in parallel research on hyperfields and fuzzy rings. Our objective is to hone in on the precise axioms that include these various examples, formulate the axiomatic structure, describe its uses, and review five papers [64, 4, 50, 48, 49] in which the theory is developed. The bulk of this survey concerns [64], in which the axiomatic framework is laid out, since the other papers build on it.

Other treatments can be found in [13, 27, 54]. Although we deal with general categorical issues, ours is largely a “hands on” approach, emphasizing a “negation map” which exists in all of the above-mentioned examples, and which often is obtained by means of a “symmetrization” functor. The other key ingredient is a surpassing relation ≤, to replace equality in our theorems. (In classical mathematics, ≤ is just equality.)

The quadruple (A, T, (−), ≤) is called a T-system. Although the investigation of systems has centered on semirings, having grown out of tropical considerations, it also could be used to develop a parallel Lie semi-algebra theory (and more generally Hopf semi-algebra theory).

1.1. Acquaintance with basic notions.

One starts with a set T that we want to study, called the set of tangible elements, endowed with a partial additive algebraic structure which however is not defined on all of T; this is resolved by embedding T in a larger set A with a fuller algebraic structure. Often T is a multiplicative monoid, a situation developed by Lorscheid [54, 55] when A is a semiring. However, there also are examples (such as Lie algebras) lacking associative multiplication. We usually denote a typical element of T as a, and a typical element of A as b.

Definition 1.1. A T-module over a set T is an additive monoid (A, +, 0) together with scalar multiplication T × A → A satisfying distributivity over T in the sense that

\[ a(b_1 + b_2) = ab_1 + ab_2 \]
for \( a \in \mathcal{T}, b_i \in \mathcal{A}, \) also stipulating that \( a0_\mathcal{A} = 0_\mathcal{A}. \)

A \textit{\( \mathcal{T} \)-monoid module} over a multiplicative monoid \( \mathcal{T} \) is a \( \mathcal{T} \)-module \( \mathcal{A} \) satisfying the extra conditions

\[
1_\mathcal{T}b = b, \quad (a_1a_2)b = a_1, \quad \forall a_i \in \mathcal{T}, \ b \in \mathcal{A}.
\]

For the sake of this exposition, we assume that \( \mathcal{A} \) is a \( \mathcal{T} \)-module and \( \mathcal{T} \subseteq \mathcal{A}. \)

A semigroup \( (\mathcal{A}, \cdot) \) has characteristic \( k > 0 \) if \( (k + 1)a = a \) for all \( a \in \mathcal{A}, \) with \( k \geq 1 \) minimal. \( \mathcal{A} \) has characteristic \( 0 \) if \( \mathcal{A} \) does not have characteristic \( k \) for any \( k \geq 1. \)

Properties of the characteristic are described in \[63, \S 6.4. \] Most of our major examples have characteristic \( 0, \) but some interesting examples have characteristic \( 2 \) or more.

A \textit{semiring\(^1 \)} satisfies all the axioms of ring except the existence of a \( 0 \) element and of negatives. A \textit{semiring\(^2 \)} is a semiring\(^1 \) with \( 0. \)

1.1.1. \textit{Brief overview.}

We introduce a formal negation map \((-)\), which we describe in \[1.3\] after some introductory examples, such that \( \mathcal{T} \) generates \( \mathcal{A} \) additively, creating a \textit{\( \mathcal{T} \)-triple} \( (\mathcal{A}, \mathcal{T}, (-)). \)

When a formal negation map is not available at the outset, we can introduce it in two ways, to be elaborated shortly:

- Declare the negation map to be the identity, as in the supertropical case, cf. \[1.2.2\]
- Apply symmetrization, to get the switch map, of second kind, cf. \[1.2.5\] Often \[3\] is applicable, where \( \mathcal{T} \) could take the role of the “thin elements.”

The element \( b^\circ := b(-)b \) is called a \textit{quasi-zero}. We write \( \mathcal{A}^\circ \) for \( \{b^\circ : b \in \mathcal{A}\}, \) and usually require that \( \mathcal{T} \cap \mathcal{A}^\circ = \emptyset, \) i.e., a quasi-zero cannot be tangible.

In classical algebra, the only quasi-zero is \( 0 \) itself, and \( a_1 - a_1 = 0 = a_2 - a_2 \) for all \( a_1, a_2. \) Accordingly, we call a triple “\( \mathcal{T} \)-classical” \[63, \text{Definition 2.45}\], when \( a_1(-)a_1 = a_2(-)a_2 \) for some \( a_1 \neq (\pm) a_2 \) in \( \mathcal{T}. \)

Examples from classical mathematics might provide some general intuition about employing \( \mathcal{A} \) to study \( \mathcal{T}. \) A rather trivial example: \( \mathcal{T} \) is the multiplicative subgroup of a field \( \mathcal{A}. \) Or \( \mathcal{A} \) could be a graded associative algebra, with \( \mathcal{T} \) its multiplicative submonoid of homogeneous elements.

But we are more interested in the non-classical situation, involving semirings which are not rings. Some motivating examples: The supertropical semiring, where \( \mathcal{T} \) is the set of tangible elements, the symmetrized semiring, and the power set of a hyperfield, where \( \mathcal{T} \) is the hyperfield itself. Since hyperfields are so varied, they provide a good test for this theory. Semirings in general, without negation maps, are too broad to yield as decisive results as we would like, which is the reason that negation maps and triples are introduced in the first place.

Since we need to correlate two structures (\( \mathcal{T} \) and \( \mathcal{A} \)), as well as the negation map (which could be viewed as a unary operator), it is convenient to work in the context of universal algebra, which was designed precisely for the purpose of discussing diverse structures together. (More recently these have been generalized to Lawvere’s theories and operads, but we do not delve into these aspects.)

To round things out, given a triple, we introduce the \textit{surpassing relation} \( \preceq, \) to replace equality in our theorems. (In classical mathematics, \( \preceq \) is just equality.) The quadruple \( (\mathcal{A}, \mathcal{T}, (-), \preceq) \) is called a \textit{\( \mathcal{T} \)-system}, cf. Definition \[1.5\]

1.2. \textit{Motivating examples.}

We elaborate the main non-classical examples motivating this theory.

1.2.1. \textit{Idempotent semirings.}

Tropical geometry has assumed a prominent position in mathematics because of its ability to simplify algebraic geometry while not changing certain invariants (often involving intersection numbers of varieties), thereby simplifying difficult computations. Outstanding applications abound, including \[1, 29, 43, 58, 62. \]

The main original idea, as expounded in \[36, 57, \] was to take the limit of the logarithm of the absolute values of the coordinates of an affine variety as the base of the logarithm goes to \( \infty. \) The underlying algebraic structure reverted from \( \mathbb{C} \) to the max-plus algebra \( \mathbb{R}_{\text{max}}, \) an ordered multiplicative monoid in which one defines \( a + b \) to be \( \max\{a, b\}. \) This is a semiring\(^1 \) and is clearly additively \textit{bipotent} in the sense that \( a + b \in \{a, b\}. \) Such algebras have been studied extensively some time ago, cf. \[12, 24. \]
Idempotent (in particular bipotent) semirings have characteristic 1, and their geometry has been studied intensively as “$F_1$-geometry,” cf. [9, 14]. But logarithms cannot be taken over the complex numbers, and the algebraic structure of bipotent semirings is often without direct interpretation in tropical geometry, so attention of tropicalists passed to the field of Puiseux series, which in characteristic 0 also is an algebraically closed field, but now with a natural valuation, thereby making available tools of valuation theory, cf. [8]. The collection [7] presents such a valuation theoretic approach. Thus one looks for an alternative to the max-plus algebra.

1.2.2. Supertropical semirings.

Izhakian [31] overcame many of the structural deficiencies of a max-plus algebra $T$ by adjoining an extra copy of $T$, called the ghost copy $G$ in [38, Definition 3.4], as well as $0$, and modifying addition. More generally, a supertropical semiring is a semiring with ghosts $(R, G, \nu) := T \cup G_0$, where $G_0 = G \cup \{0\}$, together with a projection $\nu : R \to G_0$ satisfying the extra properties:

(a) ($\nu$-Bipotence) $a + b \in \{a, b\}$, $\forall a, b \in R$ such that $\nu(a) \neq \nu(b)$;

(b) (Supertropicality) $a + b = \nu(a)$ if $\nu(a) = \nu(b)$.

The supertropical semiring is standard if $\nu|_T$ is 1:1. The supertropical semiring is called the standard supertropical semifield when $G_0$ is a semifield.

Mysteriously, although lacking negation, the supertropical semiring provides affine geometry and linear algebra quite parallel to the classical theory, by taking the negation map ($-$) to be the identity, so that $a^\circ = a + a$, and where the ghost ideal $G = \{a^\circ : a \in T\}$ takes the place of the 0 element. In every instance, the classical theorem involving equality $f = g$ is replaced by an assertion that $f = g + \text{ghost}$, called ghost surpassing. In particular when $g = 0$ this means that $f$ itself is a ghost.

For example, an irreducible affine variety should be the set of points which when evaluated at a given set of polynomials is ghost (not necessarily 0), leading to:

- a version of the Nullstellensatz in [38, Theorem 7.17],
- a link between decomposition of affine varieties and (non-unique) factorization of polynomials, illustrated in one indeterminate in [38, Remark 8.42 and Theorem 8.46],
- a version of the resultant of polynomials that can be computed by the classical Sylvester matrix and [41, Theorem 4.12 and Theorem 4.19].

Matrix theory also can be developed along supertropical lines. The supertropical Cayley-Hamilton theorem [39, Theorem 5.2] says that the characteristic polynomial evaluated on a matrix is a ghost. A matrix is called singular when its permanent (the tropical replacement of determinant) is a ghost; in [39, Theorem 6.5] the row rank, column rank, and submatrix rank of a matrix (in this sense) are seen to be equal. Solution of tropical equations is given in [2, 18, 22, 40]. Supertropical singularity also gives rise to semigroup versions of the classical algebraic group SL, as illustrated in [37].

Supertropical valuation theory is handled in a series of papers starting with [32], also cf. [47], generalized further in [34] and [35].

Note that standard supertropical semirings “almost” are bipotent, in the sense that $a_1 + a_2 \in \{a_1, a_2\}$ for any $a_1 \neq a_2$ in $T$. This turns out to be an important feature in triples.

1.2.3. Hyperfields and other related constructions.

Another algebraic construction is hyperfields [67], which are multiplicative groups in which sets replace elements when one takes sums. Hyperfields have received considerable attention recently [5, 28, 15] in part because of their diversity, and in fact Viro’s “tropical hyperfield” matches Izhakian’s construction. But there are important nontropical hyperfields (such as the hyperfield of signs, the phase hyperfield, and the “triangle” hyperfield) whose theories we also want to understand along similar lines. In hyperfield theory, one can replace “zero” by the property that a given set contains 0.

An intriguing phenomenon is that linear algebra over some classes of hyperfields follows classical lines as in the supertropical case, but the hyperfield of signs provides easy counterexamples to others, as discussed in [4].

1.2.4. Fuzzy rings.

Dress [16] introduced “fuzzy rings” a while ago in connection with matroids, and these also have been seen recently to be related to hypergroups in [5, 17, 18, 29, 56].
1.2.5. Symmetrization.

This construction uses Gaubert’s “symmetrized algebras” [22, 61, 2, 23] (which he designed for linear algebra) as a prototype. We start with $\mathcal{T}$, take $A = \hat{T} := \mathcal{T} \times \mathcal{T}$, and define the “switch map” $(-)$ by $(-)(a_0, a_1) = (a_1, a_0)$. The reader might already recognize this as the first step in constructing the integers from the natural numbers, where one identifies $(a_0, a_1)$ with $(a_0', a_1')$ if $a_0 + a_1' = a_0' + a_1$, but the trick here is to recognize the equivalence relation without modding it out, since everything could degenerate in the nonclassical applications. Equality $(a_0, a_1) = (b_0, b_1)$ often is replaced by the assertion $(a_0, a_1) = (b_0, b_1) + (c, c)$ for some $c \in \mathcal{T}$. The “symmetrized” $\mathcal{T}$-module also can be viewed as a $\hat{T}$-supermodule (i.e., 2-graded), via the twist action

$(a_0, a_1) \cdot_{\omega} (b_0, b_1) = (a_0 b_0 + a_1 b_1, a_0 b_1 + a_1 b_0)$, 

(1.1)

utilized in [44] to define and study the prime spectrum.

1.2.6. Functions.

Given some structure $\mathcal{A}$ and a set $S$, we can define the set of functions $\text{Fun}(S, \mathcal{A})$ from $S$ to $\mathcal{A}$, with operators defined elementwise, i.e.,

$\omega(f_1, \ldots, f_m)(s) = (\omega(f_1(s), \ldots, f_m(s))$.

For example, taking $S = \{1, \ldots, n\}$, a polynomial $f(\lambda_1, \ldots, \lambda_n)$ can be viewed as a function from $\text{Fun}(S, \mathcal{A})$ to $\mathcal{A}$. In other words, we take the substitution $\lambda_i \mapsto a_i$ and then send $(a_1, \ldots, a_n)$ to $f(a_1, \ldots, a_n)$.

One often identifies polynomials in terms of their values as functions on their set of definition. Then $\lambda^2$ and $\lambda$ would be identified as polynomials over the finite field $\mathbb{F}_2$.

1.3. Negation maps, triples, and systems.

These varied examples and their theories, which often mimic classical algebra, lead one to wonder whether the parallels among them are happenstance, or whether there is some straightforward axiomatic framework within which they can all be gathered and simplified. Unfortunately semirings may lack negation, so we also implement a formal negation map $(-)$ to serve as a partial replacement for negation.

**Definition 1.2.** A **negation map** on a $\mathcal{T}$-module $(\mathcal{A}, \cdot, +)$ is a map $(-) : (\mathcal{T}, +) \to (\mathcal{T}, +)$ together with a semigroup isomorphism

$(-) : (\mathcal{A}, +) \to (\mathcal{A}, +)$,

both of order $\leq 2$, written $a \mapsto (-)a$, satisfying

$(-)(ab) = ((-)a)b = a((-)b), \quad \forall a \in \mathcal{T}, \ b \in \mathcal{A}$. 

(1.2)

Obvious examples of negation maps are the identity map, which might seem trivial but in fact is the one used in supertropical algebra, the switch map $((-)(a_0, a_1) = (a_1, a_0)$ in the symmetrized algebra, the usual negation map $(-)a = -a$ in classical algebra, and the hypernegation in the definition of hypergroups. Accordingly, we say that the negation map $(-)$ is of the **first kind** if $(-)a = a$ for all $a \in \mathcal{T}$, and of the **second kind** if $(-)a \neq a$ for all $a \in \mathcal{T}$.

As indicated earlier, the quasi-zeros take the role customarily assigned to the zero element. In the supertropical theory the quasi-zeros are the “ghost” elements. In [3] Definition 2.6] the quasi-zeros are called “balanced elements” and have the form $(a, a)$.

When $1 \in \mathcal{T}$, the element $(-)1$ determines the negation map, since $(-)b = (-)(1b) = ((-)1)b$. When $\mathcal{T} \subseteq \mathcal{A}$, several important elements of $\mathcal{A}$ then are:

$e = 1^\circ = 1(-)1, \quad e' = e + 1, \quad e^\omega = e(-)e = e + e = 2e$. 

(1.3)

The most important quasi-zero for us is $e$. (For fuzzy rings, $e = 1 + \epsilon$.) But $e$ need not absorb in multiplication; rather, in any semiring with negation, Definition 1.1 implies

$ae = a(-)a = a^\circ$. 

(1.4)

**Definition 1.3.** A **pseudo-triple** is a collection $(\mathcal{A}, \mathcal{T}, (\cdot))$, where $\mathcal{A}$ is a $\mathcal{T}$-module with $\mathcal{T} \subset \mathcal{A}$, and $(-)$ is a negation map. A $\mathcal{T}$-**pseudo-triple** is a pseudo-triple in which $\mathcal{T} \subseteq \mathcal{A}$, where the negation map on $\mathcal{A}$ restricts to the negation map on $\mathcal{T}$.

A $\mathcal{T}$-**triple** is a $\mathcal{T}$-pseudo-triple, in which $\mathcal{T} \cap \mathcal{A}^\circ = \emptyset$ and $\mathcal{T}$ generates $(\mathcal{A}, +)$. 
Example 1.4. The main non-classical examples are:

- (The standard supertropical triple) $(\mathcal{A}, \mathcal{T}, (-))$ where $\mathcal{A} = \mathcal{T} \cup \mathcal{G}$ as before and $(-)$ is the identity map.
- (The symmetrized triple) $(\mathcal{A}, \mathcal{T}, (-))$ where $\mathcal{A} = \mathcal{A} \times \mathcal{A}$ with componentwise addition, and $\mathcal{T} = (\mathcal{T} \times \{0\}) \cup (\{0\} \times \mathcal{T})$ with multiplication $\mathcal{T} \times \mathcal{A} \to \mathcal{A}$ given by $(a_0, a_1)(b_0, b_1) = (a_0b_0 + a_1b_1, a_0b_1 + a_0b_1)$.

Here we take $(-)$ to be the switch map $((-)(a_0, a_1) = (a_1, a_0))$, which is of second kind.

- (The hyperfield pseudo-triple) \([64] \text{§2.4.1}\) $(\mathcal{P}(\mathcal{T}), \mathcal{T}, (-))$ where $\mathcal{T}$ is the original hyperfield, $\mathcal{P}(\mathcal{T})$ is its power set (with componentwise operations), and $(-)$ on the power set is induced from the hypernegation. Here $\preceq$ is $\subseteq$.

- (The fuzzy triple) \([64] \text{Appendix A}\) For any $\mathcal{T}$-monoid module $\mathcal{A}$ with an element $1' \in \mathcal{T}$ satisfying $(1')^2 = 1$, we can define a negation map $(-)$ on $\mathcal{T}$ and $\mathcal{A}$ given by $a \mapsto 1'a$. In particular, $(-)1 = 1'$.

- (The polynomial triple) Since any polynomial is a finite sum of monomials, we can take any triple $(\mathcal{A}, \mathcal{T}, (-))$ and form $(\mathcal{A}[\Lambda], \mathcal{T}_{\mathcal{A}[\Lambda]}, (-))$ where $\mathcal{T}_{\mathcal{A}[\Lambda]}$ is the set of monomials. Negation is taken elementwise. This can all be done formally, but from a geometric perspective it is useful to view polynomials as functions on varieties.

Although we introduced pseudo-triples since $\mathcal{T}$ need not generate $(\mathcal{A}, +)$ (for example, taking $\mathcal{A} = \mathcal{P}(\mathcal{T})$ for the phase hypergroup), we are more concerned with triples, and furthermore in a pseudo-triple one can take the sub-triple generated by $\mathcal{T}$. More triples related to tropical algebra are presented in \([64] \text{§3.2}\).

Structures other than monoids also are amenable to such an approach. This can all be formulated axiomatically in the context of universal algebra, as treated for example in \([62]\). Once the natural categorical setting is established, it provides the context in which tropicalization (described below) becomes a functor, thereby providing guidance to understand tropical versions of an assortment of mathematical structures.

Definition 1.5. Our structure of choice, a $\mathcal{T}$-system, is a quadruple $(\mathcal{A}, \mathcal{T}, (-), \preceq)$, where $(\mathcal{A}, \mathcal{T}, (-))$ is a $\mathcal{T}$-triple and $\preceq$ is a $\mathcal{T}$-surpassing relation (\([64] \text{Definition 2.65}\)) satisfying the crucial property that if $a + b \geq 0$ for $a, b \in \mathcal{T}$ then $b = (-)a$.

The main $\mathcal{T}$-surpassing relations are:

- (for supertropical, symmetrized, and fuzzy rings) $\preceq_o$, defined by $a \preceq_o b$ if $b = a + c^\circ$ for some $c$.
- (on sets) $\preceq \subseteq \subseteq$.

The relation $\preceq$ has an important theoretical role, replacing “=” and enabling us to define a broader category than one would obtain directly from universal algebra, cf. \([64] \text{§6}\). One major reason why $\preceq$ can formally replace equality in much of the theory is found in the “transfer principle” of \([5]\), given in the context of systems in \([64] \text{Theorem 6.17}\).

1.3.1. Ground triples versus module triples.

Classical structure theory involves the investigation of an algebraic structure as a small category (for example, viewing a monoid as a category with a single object whose morphisms are its elements), and homomorphisms then are functors between two of these small categories. On the other hand, one obtains classical representation theory via an abelian category, such as the class of modules over a given ring.

Analogously, there are two aspects of triples. We call a triple (resp. system) a ground triple (resp. ground system) when we study it as a small category with a single object in its own right, usually a semidomain. Ground triples have the same flavor as Lorscheid’s blueprints (albeit slightly more general, and with a negation map), whereas representation theory leads us to module systems, described below in \([47] \text{and} [6]\).

This situation leads to a fork in the road: The first path takes us to a structure theory based on functors of ground systems, translating into homomorphic images of systems via congruences in \([50] \text{§6}\) (especially prime systems, in which the product of non-trivial congruences is nontrivial \([50] \text{§6.2}\)). Ground systems often are designated in terms of the structure of $\mathcal{A}$ or $\mathcal{T}$, such as “semiring systems” or “Hopf systems” or “hyperfield systems.”
The paper [4] has a different flavor, dealing with matrices and linear algebra over ground systems, and focusing on subtleties concerning Cramer's rule and the equality of row rank, column rank, and submatrix rank.

The second path takes us to categories of module systems. In [50] we also bring in tensor products and Hom functors. In [49] we develop the homological theory, relying on work done already by Grandis [27] under the name of $N$-category and homological category (without the negation map), and there is a parallel approach of Connes and Consani in [44].

2. Contents of [64]: meta-tangible systems

The emphasis in [64] is on ground systems. One can apply the familiar constructions and concepts of classical algebra (direct sums [64] Definition 2.10], matrices [64] §4.5], involutions [64] §4.6], polynomials [64] §4.7], localization [64] §4.8], and tensor products [64] Remark 6.34]) to produce new triples and systems. The simple tensors $a \otimes b$ where $a, b \in \mathcal{T}$ comprise the tangible elements of the tensor product. The properties of tensors and Hom are treated in much greater depth in [50].

2.0.1. Basic properties of triples and systems.

Let us turn to important properties which could hold in triples. One basic axiom for this theory, holding in all tropical situations and many related theories, is:

**Definition 2.1.** A uniquely negated $\mathcal{T}$-triple $(\mathcal{A}, \mathcal{T}, (-))$ is **meta-tangible**, if the sum of two tangible elements is tangible unless they are quasi-negatives of each other.

A special case: $(\mathcal{A}, \mathcal{T}, (-))$ is **(−)**-bipotent if $a + b \in \{a, b\}$ whenever $a, b \in \mathcal{T}$ with $b \neq (-)a$. In other words, $a + b \in \{a, b, a^\circ\}$ for all $a, b \in \mathcal{T}$.

The stipulation in the definition that $b \neq (-)a$ is of utmost importance, since otherwise all of our main examples would fail. [64] Proposition 5.64 shows how to view a meta-tangible triple as a hypergroup, thereby enhancing the motivation of transferring hyperfield notions to triples and systems in general.

Any meta-tangible triple satisfying $\mathcal{T} \cap \mathcal{A}^0 = \emptyset$ is uniquely negated. The supertropical triple is (−)-bipotent, as is the modification of the symmetrized triple described in [64] Example 2.53.

The Krasner hyperfield triple (which is just the supertropicalization of the Boolean semifield $\mathbb{B}$) and the triple arising from the hyperfield of signs (which is just the symmetrization of $\mathbb{B}$) are (−)-bipotent, but the phase hyperfield triple and the triangle hyperfield triple are not even metatangible (although the latter is idempotent). But as seen in [64] Theorem 4.46], hyperfield triples satisfy another different property of independent interest:

**Definition 2.2.** [64] Definition 4.13] A surpassing relation $\preceq$ in a system is called $\mathcal{T}$-**reversible** if $a \preceq b + c$ implies $b \preceq a(-)c$ for $a, b \in \mathcal{T}$.

The category of hyperfields as given in [55] can be embedded into the category of uniquely negated $\mathcal{T}$-reversible systems ([64] Theorem 6.7]). Reversibility enables one to apply systems to matroid theory, although we have not yet embarked on that endeavor in earnest.

The **height** of an element $c \in \mathcal{A}$ (sometimes called “width” in the group-theoretic literature) is the minimal $t$ such that $c = \sum_{i=1}^t a_i$ with each $a_i \in \mathcal{T}$. (We say that $\emptyset$ has height 0.) By definition, every element of a triple has finite height. The **height** of $\mathcal{A}$ is the maximal height of its elements, when these heights are bounded. For example, the supertropical semiring has height 2, as does the symmetrized semiring of an idempotent semifield $\mathcal{T}$.

Some unexpected examples of meta-tangible systems sneak in when the triple has height $\geq 3$, as described in [64] Examples 2.73].

In [64] §5] the case is presented that meta-tangibility could be the major axiom in the theory of ground systems over a group $\mathcal{T}$, leading to a bevy of structure theorems on meta-tangible systems, starting with the observation [64] Lemma 5.5] that for $a, b \in \mathcal{T}$ either $a = (-)b$, $a + b = a$ (and thus $a^\circ + b = a^\circ$), or $a^\circ + b = b$.

In [64] Proposition 5.17] the following assertions are seen to be equivalent for a triple $(\mathcal{A}, \mathcal{T}, (-))$ containing $\mathbb{1}$:

(i) $\mathcal{T} \cup \mathcal{T}^\circ = \mathcal{A}$,

(ii) $\mathcal{A}$ is meta-tangible of height $\leq 2$. 
(iii) $\mathcal{A}$ is meta-tangible with $\epsilon' \in \{1, e\}$.

We obtain the following results for a meta-tangible system $(\mathcal{A}, \mathcal{T}, (-), \preceq)$:

- [64, Theorem 5.20] If $\mathcal{A}$ is not $(-)$-bipotent then $\epsilon' = 1$, with $\mathcal{A}$ of characteristic 2 when $(-)$ is of the first kind.
- [64, Theorem 5.27] Every element has the form $c^\epsilon$ or $mc$ for $c \in \mathcal{T}$ and $m \neq 2$. The extent to which this presentation is unique is described in [64, Theorem 5.31].
- [64, Theorem 5.33] Distributivity follows from the other axioms.
- [64, Theorem 5.34] The surpassing relation $\preceq$ must “almost” be $\preceq_c$.
- [64, Theorem 5.37] A key property of fuzzy rings holds.
- [64, Theorem 5.43] Reversibility holds, except in one pathological situation.
- [64, Theorem 5.55] A criterion is given in terms of sums of squares for $(\mathcal{A}, \mathcal{T}, (-))$ to be isomorphic to a symmetrized triple.

One would want a classification theorem of meta-tangible systems that reduces to classical algebras, the standard supertropical semiring, the symmetrized semiring, layered semirings, power sets of various hyperfields, or fuzzy rings, but there are several exceptional cases. Nonetheless, [64, Theorem 5.56] comes close. Namely, if $(-)$ is of the first kind then either $\mathcal{A}$ has characteristic 2 and height 2 with $\mathcal{A}^\circ$ bipotent, or $(\mathcal{A}, \mathcal{T}, (-))$ is isomorphic to a layered system. If $(-)$ is of the second kind then either $\mathcal{T}$ is $(-)$-bipotent, with $\mathcal{A}$ of height 2 (except for an exceptional case), $\mathcal{A}$ is isometric to a symmetrized semiring when $\mathcal{A}$ is real, or $\mathcal{A}$ is classical. More information about the exceptions are given in [64, Remark 5.57].

[64, §7] continues with some rudiments of linear algebra over a ground triple, to be discussed shortly.

In [64, §8], tropicalization is cast in terms of a functor on systems (from the classical to the nonclassical). This principle enables one in [64, §9] to define the “right” tropical versions of classical algebraic structures, including exterior algebras, Lie algebras, Lie superalgebras, and Poisson algebras.

### 3. Contents of [4]: Linear algebra over systems

The paper [4] was written with the objective of understanding some of the diverse theorems in linear algebra over semiring systems. We define a set of vectors $\{v_i \in \mathcal{A}^n : i \in I\}$ to be $\mathcal{T}$-dependent if $\sum_{i \in I} \alpha_i v_i \in (\mathcal{A}^\circ)^n$ for some nonempty subset $I' \subseteq I$ and $\alpha_i \in \mathcal{T}$, and the row rank of a matrix to be the maximal number of $\mathcal{T}$-independent rows.

A tangible vector is a vector all of whose elements are in $\mathcal{T}_0$. A tangible matrix is a matrix all of whose rows are tangible vectors.

The $(-)$-determinant $|A|$ of an $n \times n$ matrix $A = a_{i,j}$ is

$$\sum_{\pi \in S_n} (-)^{\pi} a_{\pi},$$

where $a_{\pi} := \prod_{i=1}^n a_{i,\pi(i)}$.

**Definition 3.1.** Write $a_{i,j}'$ for the $(-)$-determinant of the $j,i$ minor of a matrix $A$. The $(-)$-adjoint matrix $\text{adj}(A)$ is $(a_{i,j}')$.

**Theorem 3.2** (Generalized Laplace identity, [4, Theorem 1.56]). Suppose we fix $I = \{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$, and for any set $J \subset \{1, 2, \ldots, n\}$, with $|J| = m$, write $(a_{i,J})$ for the $m \times m$ minor $(a_{i,j}) : i \in I, j \in J$ and $(a_{i,J}')$ for the $(-)$-determinant of the $(n-m) \times (n-m)$ minor obtained by deleting all rows from $I$ and all columns from $J$. For $J = \{j_1, \ldots, j_m\}$ write $(-)^d$ for $(-)^{j_1+\cdots+j_m}$. Then $|A| = \sum_{J:|J|=m} (-)^d (a_{i,J}') (a_{i,\pi})$.

**Proof.** One can copy the proof of [33, Theorem 1, §2.4]. Namely, there are $\binom{n}{m}$ summands, each of which contribute $m!(n-m)!$ different terms to [33] with the proper sign, so altogether we get $m!(n-m)!\binom{n}{m} = n!$ terms from [33], in other words all the terms.

A matrix $A$ is nonsingular if $|A| \in \mathcal{T}$. Vectors are defined to be independent if and only if no tangible linear combination is in $\mathcal{A}^\circ(n)$. The row rank of $A$ is the largest number of independent rows of $A$.

The submatrix rank of $A$ is the largest size of a nonsingular square submatrix of $A$. 


In [64 Corollary 7.4] we see that the submatrix rank of a matrix over a \(T\)-cancellative meta-tangible triple is less than or equal to both the row rank and the column rank. This is improved in [4]:

**Theorem 3.3** ([4 Theorem 4.7(i)]). Let \((A, T, (-), \preceq_o)\) be a system. For any vector \(v\), the vector \(y = \frac{1}{\det(A)}\text{adj}(A)v\) satisfies \(\det(A)v \preceq_o Ay\). In particular, if \(\det(A)\) is invertible in \(T\), then \(x := \frac{1}{\det(A)}\text{adj}(A)v\) satisfies \(v \preceq_o Ax\).

The existence of a tangible such \(x\) is subtler. One considers valuations of systems \(\nu : (A, +) \to (\mathcal{G}, +)\), and their fibers \(\{a \in T : \nu(a) = g\}\) for \(g \in \mathcal{G}\); we call the system \(T\)-Noetherian if any ascending chain of fibers stabilizes.

**Theorem 3.4** ([4 Corollary 4.26]). In a \(T\)-Noetherian \(\mathcal{G}\)-valued system \(A\), if \(\det(A)\) is invertible, then for any vector \(v\), there is a tangible vector \(x\) with \(|x| = \frac{1}{\det(A)}\text{adj}(A)v\), such that \(Ax + v \in A^o\).

One obtains uniqueness of \(x\) [4 Theorem 4.7(ii)] using a property called “strong balance elimination.” After translating some more of the concepts of [3] into the language of systems, we turn to the question, raised privately for hyperfields by Baker:

**Question A.** When does the submatrix rank equal the row rank?

Our initial hope was that this would always be the case, in analogy to the supertropical situation. However, Gaubert observed that a (nonsquare) counterexample to Question A already can be found in [2], and the underlying system even is meta-tangible. Here the kind of negation map is critical: A rather general counterexample for triples of the second kind is given in [4, Proposition 3.3]; the essence of the example already exists in the “sign hyperfield.” Although the counterexample as given is a nonsquare \((3 \times 4)\) matrix, it can be modified to an \(n \times n\) matrix for any \(n \geq 4\). This counterexample is minimal in the sense that Question A has a positive answer for \(n \leq 2\) and for \(3 \times 3\) matrices under a mild assumption, cf. [4, Theorem 5.7].

Nevertheless, positive results are available. A positive answer for Question A along the lines of [3 Theorems 5.11, 5.20, 6.9] is given in [4, Theorem 5.8] for systems satisfying certain technical conditions. In [4, Theorem 5.11], we show that Question A has a positive answer for square matrices over meta-tangible triples of first kind of height 2, and this seems to be the “correct” framework in which we can lift theorems from classical algebra. A positive answer for all rectangular matrices is given in [4, Theorem 5.19], but with restrictive hypotheses that essentially reduce to the supertropical situation.

### 4. Contents of [20]: Grassmann Semialgebras

This paper unifies classical and tropical theory. Ironically, even though a vector space over a semifield need not contain a negation map, the elements of degree \(\geq 2\) does have a negation map given by \((-v \otimes w) = w \otimes v\), so these can be viewed in terms of triples and systems. In the process we investigate Hasse-Schmidt derivations on Grassmann exterior systems and use these results to provide a generalization of the Cayley-Hamilton theorem in [20, Theorem 3.18].

But the version given in [20, Theorem 2.6 and Definition 2.12] (over a free module \(V\) over an arbitrary semifield) is the construction which seems to “work.” This is obtained by taking a given base \(\{b_0, b_1, \ldots, b_{n-1}\}\) of \(V\), defining \((-b_i \wedge b_j = b_j \wedge b_i\) and \(b_i \wedge b_i = 0\) for each \(0 \leq i < j \leq n - 1\), and extending to all of the tensor algebra \(T(V)\). This does not imply \(v \wedge v = 0\) for arbitrary \(v \in V\); for example, taking \(v = b_0 + b_1\) yields \(v \wedge v = b_0b_1 + b_1b_0\) which need not be 0, but it acts like 0.

Thus the Grassmann semialgebra of a free module \(V\) has a natural negation map on all homogeneous vectors, with the ironic exception of \(V\) itself, obtained by switching two tensor components. This provides us “enough” negation, coupled with the relation \(\preceq_o\) to carry out the theory (But we do not mod out all elements \(v \otimes v\) for arbitrary \(v \in V\)!)

Our main theorem, [20 Theorem 3.17], describes the relation between a “Hasse-Schmidt derivation” \(D\{z\}\) and its “quasi-inverse” \(\overline{D}\{z\}\) defined in such a way to yield:

**Theorem 4.1.** \(\overline{D}\{z\}(D\{z\}u \wedge v) \geq u \wedge \overline{D}\{z\}v\).

In the classical case, one gets equality as shown in [20, Remark 3.19], so we recover [21]. Our main application is a generalization of the Cayley-Hamilton theorem to semi-algebras.
Theorem 4.2 ([20 Theorem 3.17]). \((D_n u + e_1 D_{n-1} u + \cdots + e_n u) \land v) (-) \geq 0\) for all \(u \in \bigwedge^n V_n\), which we also relate to super-semialgebras.

5. Contents of §50: Basic categorical considerations

The paper §50 elaborates on the categorical aspects of systems, with emphasis on important functors. The convolution triple \(C^S\) §50 Proposition 3.7] embraces important constructions including the symmetrized triple and polynomial triples (via the convolution product given before §50 Definition 3.4]).

In order to pave the way towards geometry, we consider “prime” systems and congruences, proving some basic results about polynomial systems:

Theorem 5.1 ([50 Proposition 6.19]). For every \(\mathcal{T}\)-congruence \(\Phi\) on a commutative \(\mathcal{T}\)-semiring system, \(\sqrt{\Phi}\) is an intersection of prime \(\mathcal{T}\)-congruences.

Theorem 5.2 ([50 Theorem 6.29]). Over a commutative prime triple \(\mathcal{A}\), any nonzero polynomial \(f \in \mathcal{T}[\lambda]\) of degree \(n\) cannot have \(n + 1\) distinct \(\mathcal{T}\)-roots in \(\mathcal{T}\).

Theorem 5.3 ([50 Corollary 6.30]). If \((\mathcal{A}, \mathcal{T}, (-))\) is a prime commutative triple with \(\mathcal{T}\) infinite, then so is \((\mathcal{A}[\lambda], \mathcal{T}, (-))\).

As in §54, the emphasis on §50 is for \(\mathcal{T}\) to be a cancellative multiplicative monoid (even a group), which encompasses many major applications. This slights the Lie theory, and indeed one could consider Hopf systems. Motivation can be found in §55.

An issue that must be confronted is the proper definition of morphism, cf. §50 Definitions 4.1, 4.3. In categories arising from universal algebra, one’s intuition would be to take the homomorphisms, i.e., those maps which preserve equality in the operators. We call these morphisms “strict.” However, this approach loses some major examples of hypergroups. Applications in tropical mathematics and hypergroups (cf. §45 Definition 2.3) tend to depend on the “surpassing relation” \(\preceq\) ([64, Definition 2.65]), so we are led to a broader definition called \(\preceq\)-category in §50 §4, Definition 4.34]. \(\preceq\)-morphisms often provide the correct venue for studying ground systems. On the other hand, §50 Proposition 4.37ff.] gives a way of verifying that some morphisms automatically are strict.

The situation is stricter for module systems §50 §5, 7, 8] over ground triples. The sticky point here is that the semigroups of morphisms \(\text{Mor}(\mathcal{A}, \mathcal{B})\) in our \(\mathcal{T}\)-module categories are not necessarily groups, so the traditional notion of abelian category has to be replaced by “semi-abelian,” §50 Theorem 7.16], and these lack some of the fundamental properties of abelian categories. The tensor product is only functorial when we restrict our attention to the stricter definition of morphisms, §50 Proposition 5.8!:

5.1. Module systems.

In both cases, in the theory of semirings and their modules, homomorphisms are described in terms of congruences, so congruences should be a focus of the theory. The null congruences contain the diagonal, and not necessarily zero, and lead us to null morphisms, §50 Definition 3.2]. An alternate way of viewing congruences in terms of “transitive” modules of \(\hat{\mathcal{M}}\) is given in §50 §8.4. In §50 §9], “Hom” is studied together with its dual, and again one only gets all the desired categorical properties (such as the adjoint isomorphism §50 Lemma 9.8)] when considering strict morphisms. In this way, the categories comprised of strict morphisms should be amenable to a categorical view, to be carried out in §49 for homology, again at times with a Hopfian flavor.

The functors between the various categories arising in this theory are described in §50 §10], also with an eye towards valuations of triples.

As in classical algebra, the “prime” systems §64 Definitions 2.11, 2.25] play an important role in affine geometry, via the Zariski topology §50 §5.3.1], so it is significant that we have a version of the fundamental theorem of algebra in §50 Theorem 7.29], which implies that a polynomial system over a prime system is prime §50 Corollary 7.30].

6. Contents of §48: Projective module systems

We start with a \(\preceq\) version of split epics (weaker than the classical definition):
Definition 6.1. An epic \( \pi : \mathcal{M} \to \mathcal{N} \) \( \preceq \)-splits if there is an \( \mathcal{N} \)-monic \( \nu : \mathcal{N} \to \mathcal{M} \) such that \( 1_{\mathcal{N}} \preceq \pi \nu \). In this case, we also say that \((\pi, \nu)\) \( \preceq \)-splits, and \( \mathcal{N} \) is a \( \preceq \)-retract of \( \mathcal{M} \).

A module system \( \mathcal{M} = (\mathcal{M}, T_{\mathcal{M}_2}, (-), \preceq) \) is the \( T \)-direct sum of subsystems \((\mathcal{M}_1, T_{\mathcal{M}_1}, (-), \preceq \) and \((\mathcal{M}_2, T_{\mathcal{M}_2}, (-), \preceq) \) if \( T_{\mathcal{M}_1} \cap T_{\mathcal{M}_2} = \emptyset \) and every \( a \in \mathcal{M} \) can be written \( a \preceq a_1 + a_2 \) for \( a_i \in \mathcal{M}_i \).

This leads to projective module systems.

Definition 6.2. A \( T \)-module system \( \mathcal{P} \) is projective if for any strict epic \( h : \mathcal{M} \to \mathcal{M}' \) of \( T \)-module systems, every morphism \( f : \mathcal{P} \to \mathcal{M}' \) lifts to a morphism \( \bar{f} : \mathcal{P} \to \mathcal{M} \), in the sense that \( h \bar{f} = f \).

\( \mathcal{P} \) is \( \preceq \)-projective if for any \( (T \text{-module system}) \) strict epic \( h : \mathcal{M} \to \mathcal{M}' \), every morphism \( f : \mathcal{P} \to \mathcal{M}' \) \( \preceq \)-lifts to a morphism \( \bar{f} : \mathcal{P} \to \mathcal{M} \), in the sense that \( f \preceq h \bar{f} \).

Their fundamental properties are then obtained, including the \( \preceq \)-Dual Basis Lemma [48 Proposition 5.14], leading to \( \preceq \)-projective resolutions and \( \preceq \)-projective dimension.

Building on projective modules, homology is work in progress [49], as is parallel work in geometry. One obtains a homology theory in the context of homological categories [27] and derived functors, in connection to the recent work of Connes and Consani [13].

7. Interface between systems and tropical mathematics

We conclude by relating systems to other approaches taken in tropical mathematics, [30, 57, 8, 9].

7.1. Tropical versus supertropical.

First we consider briefly some of the basic tools in affine tropical geometry, to see how they relate to the supertropical setting.

7.1.1. The “standard” tropical approach.

One often works in the polynomial semiring \( \mathbb{R}_{\max}[\Lambda] \), although here we replace \( \mathbb{R}_{\max} \) by any ordered semigroup \( (\Gamma, \cdot) \), with \( \Gamma_0 := \Gamma \cup \{0\} \) where \( \forall a = 0 \) for all \( a \in \Gamma \). For \( \mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n \), we write \( \Lambda^\mathbf{i} = \lambda_1^{i_1} \cdots \lambda_n^{i_n} \). A tropical hypersurface of a tropical polynomial \( f = \sum_i a_i \Lambda^i \in \Gamma[\Lambda] \) is defined as the set of points in which two monomials take on the same dominant value, which is the same thing as the supertropical value of \( f \) being a ghost.

Definition 7.1 ([24 Definition 5.1.1]). Given a polynomial \( f = \sum_i a_i \Lambda^i \), define \( \text{supp}(f) \) to be all the tuples \( \mathbf{i} = (i_1 \cdots i_n) \) for which the monomial in \( f \) has nonzero coefficient \( a_i \), and for any such monomial \( h \), write \( f_h \) for the polynomial obtained from deleting \( h \) from the summation.

The bend relation of \( f \) (with respect to a tropical hypersurface \( V \)) is generated by all

\[ \{ f \equiv_{\text{bend}} f_{h=a_i \Lambda^i} : i \in \text{supp}(f) \} \]

The point of this definition is that the variety \( V \) defined by a tropical polynomial is defined by two monomials (not necessarily the same throughout) taking equal dominant values at each point of \( V \), and then the bend relation reflects the equality of these values on \( V \), thence the relation.

7.1.2. Tropicalization and tropical ideals.

Finally, one needs to relate tropical algebra to Puiseux series via the following tropicalization map.

Definition 7.2. For any additive group \( \mathcal{M} \), one can define the group \( \mathcal{M}\{\{t\}\} \) of Puiseux series on the variable \( t \), which is the set of formal series of the form \( p = \sum_{k=\ell}^{\infty} c_k t^{k/N} \) where \( N \in \mathbb{N}, \ell \in \mathbb{Z}, \) and \( c_k \in \mathcal{A} \).

Customarily, \( \mathcal{M} = \mathbb{C} \).

Remark 7.3. If \( \mathcal{M} \) is an algebra, then \( \mathcal{M}\{\{t\}\} \) is also an algebra under the usual convolution product.

Definition 7.4. One has the Puiseux valuation \( \text{val} : \mathcal{M}\{\{t\}\} \setminus \{0\} \to \Gamma \) defined by

\[ \text{val}(p) = \min_{c_k \neq 0} (k/N) \]

sending a Puiseux series to its value under the Puiseux valuation \( \nu \). This induces a map

\[ \text{trop} : \mathcal{M}\{\{t\}\}[\Lambda] \to \Gamma[\Lambda], \]

called tropicalization, sending \( p(\lambda_1 \cdots \lambda_n) := \sum p_1 \lambda_1 \cdots \lambda_n \) to \( \sum \text{val}(p_1) \lambda_1 \cdots \lambda_n \).

Suppose \( I \triangleleft \mathcal{M}\{\{t\}\}[\Lambda] \). The bend congruence on \( \{\text{trop}(f) : f \in I\} \) is denoted as \( \text{Trop}(I) \).
Suppose $\mathcal{M}$ is a field. We can normalize a Puiseux series $p = \sum p_i \lambda_i^{j_i}$ at any given $i \in \text{supp}(p)$ by dividing through by $p_i$; then the normalized coefficient is $1$. Given two Puiseux series $p = \sum p_i \lambda_i^{j_i}$, $q = \sum q_i \lambda_i^{j_i}$ having a common monomial $\Lambda^\alpha = \lambda_1^{\beta_1} \cdots \lambda_n^{\beta_n}$ in their support, one can normalize both and assume that $p_i = q_i = 1$, and remove this monomial from their difference $p - q$, i.e., the coefficient of $\Lambda^\alpha$ in $\text{val}(p - q)$ is $0(= -\infty)$.

Accordingly, a tropical ideal of $\Gamma[\Lambda]$ is an ideal $\mathcal{I}$ such that for any two polynomials $f = \sum f_i \lambda_i^{j_i}$, $g = \sum g_i \lambda_i^{j_i} \in \mathcal{I}$ having a common monomial $\Lambda^\alpha$ there is $h = \sum h_i \lambda_i^{j_i} \in \mathcal{I}$ whose coefficient of $\Lambda^\alpha$ is $0$, for which
\[
  h_i \geq \min\{a_f f_i, b_g g_i\}, \forall i
\]
for suitable $a_f, b_g \in \mathcal{M}$.

For any tropical ideal $\mathcal{I}$, the sets of minimal indices of supports constitutes the set of circuits of a matroid. This can be formulated in terms of valuated matroids, defined in [17, Definition 1.1] as follows:

A valuated matroid of rank $m$ is a pair $(E,v)$ where $E$ is a set and $v : E^{(m)} \to \Gamma$ is a map satisfying the following properties:

(i) There exist $e_1, \ldots, e_m \in E$ with $v(e_1, \ldots, e_m) \neq 0$.

(ii) $v(e_1, \ldots, e_m) = v(e_\pi(1), \ldots, e_\pi(m))$ for each $\pi \in \mathfrak{S}$, the symmetric group on $\{1, \ldots, m\}$, and every permutation $\pi$. Furthermore, $v(e_1, \ldots, e_m) = 0$ in case some $e_i = e_j$.

(iii) For $(e_0, \ldots, e_m, e'_2, \ldots, e'_m) \in E$ there exists some $i$ with $1 \leq i \leq m$ and
\[
  v(e_1, \ldots, e_m)v(e_0, e'_2, \ldots, e'_m) \leq v(e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_m)v(e_i, e'_2, \ldots, e'_m).
\]

This information is encapsulated in the following result. [56, Theorem 1.1] Let $K$ be a field with a valuation $\text{val} : K \to \Gamma$, and let $Y$ be a closed subvariety of $K^{\times (n)}$ defined by an ideal $\mathcal{I} \triangleleft K[\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}]$. Then any of the following three objects determines the others:

(i) The congruence $\text{Trop}(\mathcal{I})$ on the semiring $\mathcal{S} := \Gamma[\lambda_1^{\pm 1}, \ldots, \lambda_n^{\pm 1}]$ of tropical Laurent polynomials;

(ii) The ideal trop$(\mathcal{I})$ in $\mathcal{S}$;

(iii) The set of valuated matroids of the vector spaces $\mathcal{I}_h$, where $\mathcal{I}_h$ is the degree $d$ part of the homogenization of the tropical ideal $\mathcal{I}$.

7.1.3. The supertropical approach.

In supertropical mathematics the definitions run somewhat more smoothly. $W$ was defined in terms of $\mathcal{T}^\circ$ so for $f, g \in \mathcal{T}[\Lambda]$ we define the $\circ$-equivalence $f \equiv_\circ g$ on $\mathcal{T}[\Lambda]$ if and only if $f^\circ = g^\circ$, i.e., $f(a)^\circ = g(a)^\circ$ for each $a \in \mathcal{T}$.

Proposition 7.5. The bend relation is the same as the $\circ$-equivalence, in the sense that $f \equiv_{\text{bend}} g$ iff $f \equiv_\circ g$, for any polynomials in $f, g \in \mathcal{T}[\Lambda]$.

Proof. The bend relation is obtained from a sequence of steps, each removing or adding on a monomial which takes on the same value of some polynomial $f$ on $V$. Thus the defining relations of the bend relation are all $\circ$ relations. Conversely, given a $\circ$ relation $f^\circ = g^\circ$, where $f = \sum f_i$ and $g = \sum g_j$ for monomials $f_i, g_j$, we have
\[
f \equiv_{\text{bend}} g \iff f_1 + f \equiv_{\text{bend}} g_1 + g_2 + f \equiv_{\text{bend}} \cdots \equiv_{\text{bend}} g + f \equiv_{\text{bend}} g + f_k \equiv_{\text{bend}} \cdots \equiv_{\text{bend}} g,
\]

implying $f \equiv_{\text{bend}} g$. (The same sort of argument is given in the proof of [60, Proposition 2.6].) \qed

In supertropical algebra, given a polynomial $f = \sum \alpha_i \Lambda^i$, define $\nu$-supp$(f)$ to be all the tuples $i = (i_1, \ldots, i_n)$ for which the monomial in $f$ has coefficient $\alpha_i \in \mathcal{T}$.

The supertropical version of tropical ideal is that if $f, g \in \mathcal{I}$ and $i \in \nu$-supp$(f) \cap \nu$-supp$(g)$, then, by normalizing, there are $a_f, b_g$ such that $i \notin \nu$-supp$(a_f f + b_g g)$. This is somewhat stronger than the claim of the previous paragraph, since it specifies the desired element.

Supertropical “$d$-bases” over a super-semifield are treated in [33], where vectors are defined to be independent iff no tangible linear combination is a ghost. If $V$ is defined as the set $v \in \mathcal{F}^{(n)} : f_j(v) \in \nu(F)$, $\forall j \in J$ for a set $\{f_j : j \in J\}$ of homogeneous polynomials of degree $m$, then taking $\mathcal{I} = \{f : f(V) \in \nu(F)\}$ and $\mathcal{I}_m$ to be its polynomials of degree $m$, one sees that the $d$-bases of $\mathcal{I}_m$ of cardinality $m$ comprise
a matroid (whose circuits are those polynomials of minimal support), by [33, Lemma 4.10]. On the other hand, submodules of free modules can fail to satisfy Steinitz’ exchange property ([33 Examples 4.18,4.9]), so there is room for considerable further investigation.

“Supertropicalization” then is the same tropicalization map as trop, now taken to the standard supertropical semifield strp : Γ ∪ G (where T = Γ). In view of Proposition [7.6], the analogous proof of [66, Theorem 1.1] yields the corresponding result:

Theorem 7.6. Let K be a field with a valuation : K → Γ, and let Y be a closed subvariety of K\( \times (n) \) defined by an ideal I ⊆ K[\( \lambda^\pm_1, \ldots, \lambda^\pm_n \)]. Then any of the following objects determines the others:

(i) The congruence given by ε-equivalence on I = strp(I) in the supertropical semiring\( S := \Gamma[\lambda^\pm_1, \ldots, \lambda^\pm_n] \) of tropical Laurent polynomials;

(ii) The ideal trop(I) in S;

(iii) The set of valuated matroids of the vector spaces I^n, where I^n is the degree d part of the homogenization of the tropical ideal I.

7.1.4. The systemic approach.

The supertropical approach can be generalized directly to the systemic approach, which also includes hyperfields and fuzzy rings. We assume (\( A, T, (\cdot), \leq \)) is a system.

Definition 7.7. The ε-equivalence on Fun(\( S, A \)) is defined by, f \( \equiv_{\varepsilon} \) g if and only if \( f^o \equiv g^o \), i.e. \( f(s)(b)^o = g(s)(b)^o \) for each \( s \in S \) and \( b \in A \).

This matches the supertropical definition.

Definition 7.8. Given \( f \in Fun(\( S, A \)) \) define \( \varepsilon\)-supp(f) = \( \{ s \in S : f(s) \in T \} \).

The systemic version of tropical ideal is that if \( f, g \in I \) and \( s \in \varepsilon\)-supp(f) \( \cap \varepsilon\)-supp(g), then there are \( a_f, b_g \in T \) such that \( s \notin \varepsilon\)-supp(\( a_f \cdot \varepsilon(b_g) \)).

Now one can view tropicalization as a functor as in [64, §8].

8. Areas for further research

8.1. Geometry. A \( \leq \)-root of a polynomial \( f \in A[\lambda_1, \ldots, \lambda_n] \) is some \( n \)-tuple \( (a_1, \ldots, a_n) \) such that \( f(a_1, \ldots, a_n) \geq 0 \). This leads naturally to affine \( \leq \)-varieties (as common \( \leq \)-roots of a set of polynomials), and algebraic geometry. An alternative approach is through Hopf semi-algebras.

8.2. Valuated matroids over systems. Viewing tropicalization as a functor, define the appropriate valuated matroid. Then one can address the recent work on matroids and valuated matroids, and formulate them over systems in analogy to [15]. Presumably, as in [1], in the presence of various assumptions, one could carry out the proofs of many of these assertions.

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