Quantum multimeters: A programmable state discriminator

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(Dated: January 20, 2002)

We discuss a possibility to build a programmable quantum measurement device (a “quantum multimeter”). That is, a device that would be able to perform various desired generalized, positive operator value measure (POVM) measurements depending on a quantum state of a “program register”. As an example, we present a “universal state discriminator”. It serves for the unambiguous discrimination of a pair of known non-orthogonal states (from a certain set). If the two states are changed the apparatus can be switched via the choice of the program register to discriminate the new pair of states unambiguously. The proper POVM is determined by the state of an auxiliary quantum system. The probability of successful discrimination is not optimal for all pairs of non-orthogonal states from the given set. However, for some subsets it can be very close to the optimal value.

PACS numbers: 03.65.-w, 03.67.-a

I. INTRODUCTION

Quantum measurements are inevitable parts of all quantum devices. They represent the final step of any quantum computation [1]. The whole sets of quantum measurements are of vital importance in the quantum state estimation [2, 3]. Therefore, there is a natural question whether it is possible to construct a universal (multi-purpose) quantum measurement device. That is, an apparatus that could perform some large class of generalized measurements (POVM) in such a way that each member of this class could be selected by a particular quantum state of a “program register”.

The generalized measurement is defined by the fact that the probability of each of its result (the number of results may be, in general, larger than the dimension of the Hilbert space of the system under consideration) is given by the expression $p_{\mu} = \text{Tr}_S (A_{\mu} \rho_S)$, where $\rho_S$ is the state of the system and $A_{\mu}$ are positive operators that constitute the decomposition of the identity operator ($\sum_{\mu} A_{\mu} = 1$). This is the reason why it is called Positive Operator Valued Measure [2, 3, 4]. Each POVM can be implemented using an ancillary quantum system in a specific state and realizing a projective von Neumann measurement on the composite system [5]. In other words, if one has an “input” (measured) state $\rho_S$ in the Hilbert space $\mathcal{H}_S$ it is always possible to find some state $\rho_A$ in a space $\mathcal{H}_A$ and a set of orthogonal projectors $\{E_{\mu}\}$ acting on $\mathcal{H}_S \otimes \mathcal{H}_A$ ($\sum_{\mu} E_{\mu} = 1$) such that

$$A_{\mu} = \text{Tr}_A (E_{\mu} \rho_A) \quad (1)$$

are positive operators as discussed above.

In general, we can assume, that the initial state of the ancilla can be prepared with an arbitrary precision. The ancilla can be considered as a part of the “program register”. Further, we note that the general projection measurement on the composite system can be represented by a unitary transformation on the composite system followed by a fixed projection measurement (e.g., independent projective measurements on individual qubits). Therefore the problem of designing the programmable quantum multimeter reduces to the question whether an arbitrary unitary operation (on the Hilbert space with a given dimension) can be encoded in some quantum state of a program register of a finite dimension. It was shown that the answer to this question is “No”. Nielsen and Chuang proved that any two inequivalent operations require orthogonal program states [6]. Thus the number of encoded operations cannot be higher than the dimension of the Hilbert space of the program register. Since, in general, the set of all unitary operations can be infinite, the result of Nielsen and Chuang implies that no universal programmable gate array can be constructed using finite resources. They showed, however, that if the gate array is probabilistic, a universal gate array is possible. A probabilistic array is one that requires a measurement to be made at the output of the program register, and the output of the data register is only accepted if a particular result, or set of results, is obtained. This will happen with a probability, which is less than one. Vidal and Cirac [6] have proposed more general quantum processor that can perform probabilistically any operation (not only unitary) on a qubit. Another aspect of encoding quantum operations in states of a program register has been discussed by Huelga et al. [7]. They dealt with the so called teleportation of unitary operations. Unfortunately, the probabilistic realization of unitary operations cannot help to built a programmable quantum multimeter in the way mentioned above. The reason is that the probabilistic implementation of a given operation leads, at the end, to a different POVM than the deterministic implementation of the same operation would lead to.
In general, we can describe a “quantum multimeter” as a (fixed) unitary operation acting on the measured system (or a “data register”) and an ancillary system (“program register”) together and a (fixed) projective measurement realized afterwards on the same composite system. Clearly, such a device can perform only a restricted set of POVM’s. One can, therefore, ask what is the optimal unitary transformation that enables us to implement “the largest set of POVM’s” (in comparison with the set of POVM’s that would be obtainable when we allowed any unitary transformation on the same Hilbert space). One can also ask what unitary transformation can help to approximate all the POVM’s (generated by an arbitrary unitary transformation) with the highest precision (fidelity) on average. Clearly, the last task requires definition of the distance measure between two POVM’s. This is an interesting problem per se, however, it goes far beyond the scope of our considerations here. Both optimization problems mentioned above are rather non-trivial. Moreover, the introduced scheme is perhaps too general from a practical point of view. Therefore, in the present paper we will concentrate our attention on a more specific cases and especially on the problem of state discrimination.

II. IMPLEMENTATION

Let us suppose that a “customer” asks our “company” to produce a quantum measurement device that would be able to perform a finite number of specific POVM’s. Due the Neumark theorem \[5\] we know that each generalized measurement can be realized via a specific unitary operation utilizing a certain state of the ancilla. Therefore we can built an apparatus that is outlined in Fig. 1. We choose a large enough dimension of the ancillary system to cover the demands of all particular POVM’s and we built “processors” that provide a particular required unitary operations. The device is programmed by the state of the ancilla and by the proper choice of unitary transformation. The selection of the unitary transformation is, in fact, a classical procedure. Two inequivalent unitary operations correspond to two orthogonal states of a program register \[\text{[3]}\]. However, even if we restrict ourselves only to orthogonal program states addressing exactly desired unitary operations we still obtain a large variety of POVM’s related to various possible states of ancilla. This is a “side effect” that represents a “bonus” to our “customer”. The set of realizable POVM’s can be further enlarged if one allows general program states and a measurement on program register. (The question is only how can such additional generalized measurements be useful for a “customer”.

Now let us analyze the scheme given in Fig. 1 in more detail. The behavior of the device before a final measurement is performed can be described by a unitary operation

\[
\begin{align*}
\rho_D &\rightarrow U_i
\end{align*}
\]

\[
\begin{align*}
\rho_A &\rightarrow M
\end{align*}
\]

\[
\begin{align*}
\rho_P &\rightarrow i
\end{align*}
\]

and

\[
\begin{align*}
P & = \sum_{k=1}^{K} U_k \otimes |P_k⟩⟨P_k|,
\end{align*}
\]

where \(U_k\) are unitary operations acting on the measured and ancillary systems together (\(K\) is their total number) and \(|P_k⟩\) denote the orthonormal states of a program register (the dimension of corresponding Hilbert space is supposed to be \(K\)). At this point we focus our attention to the final measurement. We will assume independent projective measurements on three subsystems (data, ancilla, and program) described by the following set of projectors:

\[
\begin{align*}
E_{ijk} &= |D_i⟩⟨D_i| \otimes |A_j⟩⟨A_j| \otimes |P_k⟩⟨P_k|
\end{align*}
\]

where vectors \(|D_i⟩\) constitute an orthonormal basis in the Hilbert space corresponding to the data register and \(|A_j⟩\) and \(|P_k⟩\) are orthonormal bases in the spaces of ancilla and program, respectively. In general, \(|P_k⟩\) and \(|P'_k⟩\) represent two different bases. Direct application of Eq. (3) (we are tracing over both the ancilla and the program) gives us a POVM (a set of positive operators) that depends on the states of ancilla and program register:

\[
\begin{align*}
A_{ijk} &= \sum_{mnl} \langle A_m| U_n^† [D_i](D_i) \otimes |A_j⟩⟨A_j| \otimes |P_k⟩⟨P_k| U_l \rho_A |A_m⟩ \times ⟨P_l| P_m⟩⟨P'_l| P'_m⟩.
\end{align*}
\]

Here \(\rho_A\) is the state of the ancilla and \(\rho_P\) is the state of the program register.

If \(|P_k⟩ = |P'_k⟩\) for all \(k\), i.e., if the basis in Eq. (3) is the same as the basis of the measurement on the program register then Eq. (4) simplifies:

\[
\begin{align*}
A_{ijk} &= \sum_{m} \langle A_m| U_k^† [D_i](D_i) \otimes |A_j⟩⟨A_j| U_k \rho_A |A_m⟩ \times ⟨P_k| P_k⟩.
\end{align*}
\]
For the sake of simplicity, let us first analyze the case when there is no ancilla and unitary operations $U_k$ concerns only the data register (the measured system). Then Eq. (3) reads:

$$A_{ik} = U_k^\dagger |D_i\rangle \langle D_i| U_k \langle P_k| \rho_P|P_k\rangle.$$  

(6)

Let us note that it depends only on the diagonal elements of the density matrix describing the state of program register. The described situation is, in fact, equivalent to the one when we are “tossing a coin” and, depending on the random result, we are choosing one of several projective tests. In the given sense such an apparatus is equivalent to the one that is “programmed” classically. Anyway, its input is quantum and the program register can be set by the output of some other quantum processor. For instance, in the case when the data input is represented by a single spin-1/2 particle an apparatus of this kind can enable us to make a “software” selection of the von Neumann measurement in one of the three fixed orthogonal spatial axes (for this we would require a three dimensional Hilbert space of the program-register). [20]

Alternatively, we can perform any POVM that consists of a “probabilistic mixture” of these measurements in the sense explained above. But the set of POVM’s that can be realized using this scenario is rather restricted. For instance, it is impossible to realize any optimal unambiguous discrimination of two (known) non-orthogonal states of spin-1/2 particle no matter which and how many unitary operations are employed. To perform this task we need an ancilla.

However, before we leave this simple model let us look at the situation when the basis $\{|P_k\}\rangle_k$ is different from the measurement basis $\{|P'_k\}\rangle_k$. If we further assume the program register to be in a pure state, such that $\langle P_m| \rho_P|P_n\rangle = \xi_m \xi_n^*$, then

$$A_{ik} = X_k^\dagger |D_i\rangle \langle D_i| X_k,$$

(7)

where the operators

$$X_k = \sum_m U_m \xi_m (P_k'|P_m\rangle)$$

are neither unitary nor Hermitean in general (this feature is related to the “quantum processor” proposed in Ref. [8]).

III. DISCRIMINATION OF QUANTUM STATES

In the following we will study a particular example of a “quantum multimeter” serving for a programmable unambiguous state discrimination. So, it is in place to say a few words about quantum state discrimination now.

A general unknown quantum state cannot be determined completely by a measurement performed on a single copy of the system. But the situation is different if a prior knowledge is available — e.g., if one works only with states from a certain discrete set. Even quantum states that are mutually non-orthogonal can be distinguished with a certain probability provided they are linearly independent (for a review see Ref. [11]). There are, in fact, two different optimal strategies [11]: First, the strategy that determines the state with the minimum probability for the error [3] and, second, unambiguous or error-free discrimination (the measurement result never wrongly identify a state) that allows the possibility of an inconclusive result (with a minimal probability in the optimal case [2]. [3]. [4]. [5]. [6].) We will concentrate our attention to the unambiguous state discrimination. It has been first investigated by Ivanovic [12] for the case of two equally probable non-orthogonal states. Peres [4] solved the problem of discrimination of two states in a formulation with POVM measurement. Later Jaeger and Shimony [13] extended the solution to arbitrary a priori probabilities. Chefles and Barnett [16] have generalized Peres’s solution to an arbitrary number of equally probable states which are related by a symmetry transformation. Unambiguous state discrimination were already realized experimentally. The first experiment, designed for the discrimination of two linearly polarized states of light, were done by Huttner et al. [17]. There are also some newer proposals of optical implementations [15]. The interest in the quantum state discrimination is not only “academic” – unambiguous state discrimination can be used, e.g., as an efficient attack in quantum cryptography [19].

IV. “UNIVERSAL” DISCRIMINATOR

Let us suppose that our “customer” wants to discriminate unambiguously between two known non-orthogonal states. However, he/she would like to have a possibility to “switch” the apparatus in order to be able to work with several different pairs of states.

Let us have two (non-orthogonal) input states of a qubit. We can always choose such a basis that they read $\alpha_0 |0\rangle + \beta_0 |1\rangle$ with $\alpha_0 = \cos(\varphi_0/2)$ and $\beta_0 = \sin(\varphi_0/2)$; the value of $\varphi_0$ can be from 0 to $\pi/2$ ($\varphi_0$ is the angle between the two states). Let us have one additional ancillary qubit, initially in a state $|0\rangle$. On both the “data” and the ancilla we apply the following unitary transformation:

$$|0_D0_A\rangle \rightarrow \cos \theta |0_D0_A\rangle + \sin \theta |0_D1_A\rangle,$$

$$|1_D0_A\rangle \rightarrow |1_D0_A\rangle,$$

$$|0_D1_A\rangle \rightarrow -\sin \theta |0_D0_A\rangle + \cos \theta |0_D1_A\rangle,$$

$$|1_D1_A\rangle \rightarrow |1_D1_A\rangle,$$

(8)

where $\cos \theta = \tan(\varphi_0/2)$. If we then make a von Neumann measurement consisting of the projectors $P_+ = |+\rangle \langle +|$, $P_- = |-\rangle \langle -|$, and $P_0 = \mathbb{1} - P_+ - P_-$, we get

$$|\pm\rangle = (|0_D0_A\rangle \pm |1_D0_A\rangle) / \sqrt{2},$$

(9)
we can unambiguously determine the input state (with a certain probability of the success) \(\frac{14}{13}\). This measurement is optimal in the sense that the probability of inconclusive result is the lowest possible (and it is the same for both the states). The probability of the successful discrimination is \(2\sin^2(\varphi_0/2)\) \(\frac{14}{13}\).

Let us suppose now the set of pairs

\[
|\psi_1\rangle = \alpha |0_D\rangle + \beta |1_D\rangle, \\
|\psi_2\rangle = \alpha |0_D\rangle - \beta |1_D\rangle, \tag{10}
\]

where \(\alpha = \cos(\varphi/2)\) and \(\beta = \sin(\varphi/2)\), for all \(\varphi\) from the interval \((0, \pi)\). That is, we consider all pairs of states that lie on a real plane and that are located symmetrically around the state \(|0_D\rangle\); see Fig. 2. Further, let us suppose that the ancillary qubit is allowed to be in an arbitrary pure state

\[
a|0_A\rangle + b|1_A\rangle. \tag{11}
\]

Thus the total input state reads

\[
(a |0_D\rangle \pm \beta |1_D\rangle) \otimes (a |0_A\rangle \pm b |1_A\rangle) = \\
aa |0_D0_A\rangle + ab |0_D1_A\rangle \pm \beta a |1_D0_A\rangle \pm \beta b |1_D1_A\rangle. \tag{12}
\]

After the action of transformation \(\frac{6}{3}\) on this state one obtains the resulting state in the following form [the transformation is fixed for all \(\varphi\); still \(\cos \theta = \tan(\varphi_0/2)\)]

\[
(aa \cos \theta - ab \sin \theta) |0_D0_A\rangle + \\
(aa \sin \theta + ab \cos \theta) |0_D1_A\rangle \pm \\
\beta a |1_D0_A\rangle \pm \beta b |1_D1_A\rangle. \tag{13}
\]

If the coefficients \(a\) and \(b\) in the state of the ancilla are chosen in such a way that

\[
(aa \cos \theta - ab \sin \theta) = \beta a := q/\sqrt{2} \tag{14}
\]

then the expression \(\frac{13}{13}\) simplifies to the form

\[
q |\pm\rangle + \text{const}_1 |0_D1_A\rangle \pm \text{const}_2 |1_D1_A\rangle, \tag{15}
\]

where the states \(|\pm\rangle\) are defined by Eq. \(\frac{13}{13}\). Clearly, applying the projective measurement introduced above one is able to discriminate unambiguously states \(\frac{14}{13}\) for any given \(\varphi \in (0, \pi)\) provided he/she has prepared the proper state of the ancilla. The first term in Eq. \(\frac{15}{15}\) corresponds to the successful discrimination, while the last two terms correspond to inconclusive results. The probability of success is

\[
P_{\text{succ}} = |q|^2 = P_{\text{opt}} R(\varphi, \varphi_0) = 2\sin^2 \frac{\varphi}{2} R(\varphi, \varphi_0), \tag{16}
\]

where

\[
R(\varphi, \varphi_0) = \frac{\cos \varphi_0 (\cos \varphi + 1)}{1 + \cos \varphi_0 - \sin \varphi \sin \varphi_0} \tag{17}
\]

is the ratio between the actual value of the probability of successful discrimination and its optimal value. This expression is obtained from the condition \(\frac{14}{14}\) together with the normalization relation \(|a|^2 + |b|^2 = 1\).

From above it follows that it is possible to implement a “universal quantum multimeter” that is able to discriminate probabilistically but unambiguously (with no errors) between two non-orthogonal states for the large class of non-orthogonal pairs. The selection of the desired regime (i.e., the selection of the pair of states that should be unambiguously discriminate) is done by the change of the quantum state of the ancillary qubit. The probability of the successful discrimination can be optimal only for one such pair of states.

In the limit case when \(\varphi_0 = 0\), i.e. \(\theta = \pi/2\) (this is the fixed parameter of the employed unitary transformation), the probability of the successful discrimination for different \(\varphi\)’s (i.e., for different settings of the ancilla and different pairs of input states) is the same as in the “quasi-classical” case, \(P_{\text{succ}} = \frac{1}{2} \sin^2 \varphi\). By a quasi-classical approach we mean the probabilistic measurement when one randomly selects\(\frac{21}{21}\) the projective measurement in one of two orthogonal basis that both span the two-dimensional space containing both non-orthogonal states of interest \(\frac{10}{10}\). One basis consists of the state \(|\psi_1\rangle\) and its orthogonal complement \(|\psi_1^\perp\rangle\). If one finds the result corresponding to \(|\psi_1\rangle\) he/she can be sure that the state \(|\psi_1\rangle\) was not present. Analogously, the other basis consists of the state \(|\psi_2\rangle\) and its orthogonal complement.

On the other hand when \(\varphi_0 = \pi/2\), i.e. \(\theta = 0\), there is no way how to fulfill the condition \(\frac{14}{14}\) with \(a \neq 0\) (and \(P_{\text{succ}} \neq 0\)) unless \(\alpha = \beta = 1/\sqrt{2}\). That is, only two orthogonal states \(\frac{3}{3}\) can be unambiguously discriminated.

If the parameter \(\varphi_0\) is somewhere in between 0 and \(\pi/2\) the probability of success (as a function of \(\varphi\)) is very close to the optimal value in the relatively large vicinity of \(\varphi_0\); see Fig. 3. However, for small values of \(\varphi\) it goes below the success probability of the quasi-classical case and for \(\varphi = \pi/2\) (orthogonal states) the probability of successful discrimination is lower than unity.

One can ask for the optimal value of \(\varphi\) in the sense that the average probability of successful discrimination
We have proposed a programmable quantum measurement device for the error-free discrimination of two non-orthogonal states of qubit that works with a large set of pairs of states. The device can be set to discriminate unambiguously any two states that are symmetrically located around some fixed state [in the sense of Eq. (10)]. The setting is done through the state of a program register that is represented by another qubit. This means that the selection of the pair of states to be discriminated unambiguously depends on the state of a program register. Two possible input states (of the “data register”) that are in correspondence with the program setting are never wrongly identified but from time to time we can get an inconclusive result. The probability of successful discrimination is optimal only for one program setting. However, the device can be designed in such a way that the probability of successful discrimination is very close to the optimal value for a relatively large set of program settings.

We have also discussed some general questions concerning the possibilities to build multi-purpose quantum measurement devices (“quantum multimeters”) that could perform a required POVM depending on a quantum state of their program register. Most of these questions remain unanswered. For instance, let us suppose a set of all POVM’s that can be obtained if we combine the measured system with an ancilla of some fixed dimension in an arbitrary state and carry out an arbitrary projective (von Neumann) measurement on the composite system. This is equivalent to carrying out an arbitrary unitary operation followed by some fixed projective measurement. Imagine now that we can change only the state of the ancilla and carry out a unitary operation. Two possible input states of the composite system are unambiguously any two states that are symmetrically located around some fixed state. This construction with “switched operations” enables us to reach the optimal probability of successful discrimination for a large interval of angles \( \varphi \).

For pedagogical reasons till now we have only worked with the states from a particular real subspace of the Hilbert space of the data qubit. However, it should be stressed that the method works for any two “input” states that are symmetrically displaced with respect to \( |0_D \rangle \). In other words, the condition \( \| \theta \| \) can be fulfilled for any complex \( \alpha \) and \( \beta \). Simply,

\[
\frac{b}{a} = \frac{1}{\sin \theta} \left( \cos \theta - \frac{\beta}{\alpha} \right).
\]

The probability of the successful discrimination of states then reads

\[
P_{\text{succ}} = \frac{2 \sin \theta |\alpha \beta|^2}{1 - 2 \cos \theta \Re(\alpha \beta)},
\]

where \( \Re(\alpha \beta) \) denotes a real part of \( \alpha \beta \).

V. CONCLUSIONS

 Acknowledgments

This research was supported under the European Union project EQUIP (contract IST-1999-11053) and the project LN00A015 of the Ministry of Education of the Czech Republic.
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[20] If the final measurement is the projection to the z-axis the required operations are the identity and two "rotations", \((\sigma_x + \sigma_z)/\sqrt{2}\) and \((\sigma_y + \sigma_z)/\sqrt{2}\), where \(\sigma\)'s are the Pauli matrices.
[21] With the same probabilities provided that the frequencies of the occurrence of the input states are also the same.