1 Introduction

The study of dissipative systems is of fundamental interest in many fields of physics, ranging from statistical mechanics to condensed matter, atomic physics etc.

It is well known that, classically, the action of a heat bath on a particle can be described (say, in one space dimension) by the "Langevin force":

\[ - \int_{-\infty}^{t} dt' \mu(t-t') \dot{x}(t') + F(t) , \]  

where \( \mu(t-t') \) is a friction coefficient, called "memory function", and \( F(t) \) is a random force. The memory function is a phenomenological quantity, depending on the detailed features of the coupling with the bath. Apart from the obvious requirement that it should be non-negative at all times, it is possible to show that the second law of thermodynamics requires its Fourier transform \( \tilde{\mu}(z) \) to have a real positive part on the real axis:

\[ \text{Re} \{ \tilde{\mu}(\omega + i0^+) \} \geq 0 \quad -\infty < \omega < \infty . \]  

1 Talk given by the author.
The random force $F(t)$ is usually assumed to have zero mean:

$$< F(t) > = 0 ,$$

and to satisfy the Gaussian property, according to which all odd correlators of $F(t)$ vanish, while the even ones can all be written as sums of products of the two-point function $< F(t)F(t') >$. The expression of the latter function, is determined by the fluctuation-dissipation theorem, in terms of the temperature $T$ of the bath, and of the memory function. For example, in the case of a memory function of the form $\mu(t-t') = f\delta(t-t')$ (which corresponds to the original form of the Langevin equation), one has:

$$< F(t)F(t') > = 2kTf\delta(t-t') ,$$

(1.3)

where $k$ is Boltzmann constant.

If the particle of mass $m$ is subject also to an external conservative force, with potential $V(x)$, its motion is then described by the (generalized) "Langevin equation":

$$m\ddot{x} + \int_{-\infty}^{t} dt' \mu(t-t') \dot{x}(t') + V'(x) = F(t) .$$

(1.4)

It is well known that this equation describes the approach to equilibrium of the particle (if one assumes of course that the heat bath is infinite in size and remains in equilibrium at all times), and indeed one can prove that, for any choice of the initial conditions, the probability density in the particle's phase space approaches, for large times, the canonical Maxwell-Boltzmann distribution.

It is natural to ask if an analogous theory of dissipation exists, for a quantum particle in interaction with a quantum-mechanical bath. In particular, is there a quantum Langevin equation, that one can use to describe the influence of the bath on the quantum behavior of the particle? At a first glance, it is not very clear what one should expect. Since the Langevin force describes the influence of the bath on the particle, and since the bath is itself a quantum system, it is quite possible that both the random force $F(t)$ and the memory function $\mu(t-t')$ become operators, in the quantum theory. The simplest possibility, which we shall assume, is to keep $\mu(t-t')$ a c-number. However, this is unlikely to be the case for $F(t)$, since the l.h.s. of the Langevin equation, which involves the particle's coordinate and momentum, is a-priori an operator. In the Quantum theory, besides the randomness already present in the classical theory (for $T > 0$), there will exist a further source of randomness due to intrinsic quantum fluctuations. The latter should be encoded by a set of commutators:

$$[\hat{F}(t), \hat{F}(t')] , \ [\hat{x}(t), \hat{F}(t')] , \ [\hat{p}(t), \hat{F}(t')] .$$

(1.5)

Consideration of unequal-time commutators appears to be necessary, because the random force $\hat{F}(t)$ does not obey any equation of motion, and so knowledge of equal-time commutators is not sufficient.
2 A microscopic model for dissipation

The standard approach to dissipation in Quantum Mechanics, is based on the physical picture that dissipation arises from coupling of the system of interest with a thermal bath. According to this picture, one considers a conservative microscopic model for the bath, usually consisting of a large number of oscillators, and postulates a certain form for the system-bath interaction. Elimination of the degrees of freedom of the bath gives rise to an effective equation of motion for the particle, including both the damping force, and the fluctuating force. An implementation of this philosophy, in the path-integral formalism, was developed long ago by Feynman and Vernon [2].

In a canonical framework, Ford et al. [1] proposed the following simple independent oscillators model for the particle-bath system:

\[ H = \frac{p^2}{2m} + V(x) + \sum_j \left[ \frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 (q_j - x)^2 \right] . \]  

(2.6)

One has the standard commutation rules:

\[ [x, p] = i\hbar , \quad [q_j, p_k] = i\hbar \delta_{jk} . \]  

(2.7)

The equations of motion implied by the above hamiltonian are:

\[ m\ddot{x} + V'(x) = \sum_j m_j \omega_j^2 (q_j - x) \]  

(2.8)

\[ \ddot{q}_j + \omega_j^2 q_j = \omega_j^2 x . \]  

(2.9)

The general solution of Eq.(2.9) is:

\[ q_j(t) = q_j^h(t) + x(t) - \int_{-\infty}^t dt' \cos[\omega_j(t - t')] \dot{x}(t') , \]  

(2.10)

where \( q_j^h(t) \) is the general solution of the homogeneous equation:

\[ q_j^h(t) = q_j \cos(\omega_j t) + p_j \frac{\sin(\omega_j t)}{m_j \omega_j} . \]  

(2.11)

The choice of the retarded solution in Eq.(2.10) explicitly breaks time-reversal. The picture one has in mind is that the particle is held fixed at \( x = 0 \) at \( t = -\infty \). Upon plugging Eq.(2.10) into Eq.(2.8), we see that the effective equation of motion for the particle of interest becomes a Langevin Equation, with a memory function and a fluctuating force, expressed in terms of the oscillator parameters. In detail, one finds:

\[ \mu(t) = \sum_j m_j \omega_j^2 \cos(\omega_j t) \Theta(t) , \]  

(2.12)
where \( \Theta(t) \) is the Heaviside function. It is shown in \[1\] that, by a suitable choice of parameters, one can reproduce the most general form of the memory function, compatible with the principles of thermodynamics. As for the random force, the model gives:

\[
F(t) = \sum_j m_j \omega_j^2 q_j^0(t) ,
\]

and accordingly one finds:

\[
\frac{1}{\hbar} [F(t), F(t')] = - \sum_j m_j \omega_j^3 \sin[\omega_j(t - t')] = \frac{d\mu}{dt}(t - t') + \frac{d\mu}{dt}(t' - t) .
\]

The question arises: how much of these commutation relations depend on the detailed microscopic model? A related question is whether it is possible to obtain the commutators for \( x(t), p(t) \) and \( F(t) \) directly from the Langevin Equation, without making recourse to microscopic models?

### 3 PB for dissipative systems

Our approach to the quantization of the Langevin’s equation proceeds in a way similar to that followed for conservative systems:

- **Step 1**: define first a set of Poisson Brackets (PB) \( \{ , \} \) on the classical ”phase space” \( \mathcal{P} \);
- **Step 2**: replace PB’s by commutators:

\[
\{ A, B \} \rightarrow \frac{1}{\hbar} [\hat{A}, \hat{B}] .
\]

The problems that arise in step 1, are twofold:

- **Problem 1**: what is the particle’s phase space \( \mathcal{P} \)? Since the particle is acted on by a random force, initial data do not determine its future evolution. In fact, since \( F(t) \) can a priori be anything, any path \( x(t) \) whatsoever is a possible trajectory for the particle. Thus, if we think of the phase space as the set of actual motions of the particle, we need to take for \( \mathcal{P} \) the infinite dimensional set of all paths \( x(t) \):

\[
\mathcal{P} = \{ x : \mathbb{R} \rightarrow \mathbb{R} \}
\]

- **Problem 2**: dissipative equations cannot be derived, in general, from an action principle. How does one get then a PB on \( \mathcal{P} \)?

In order to address both questions, we have elaborated \[3\] a definition of PB’s, that relies directly and exclusively on Langevin’s equation. To see how this can be done, let us look at the conservative limit, when \( \mu(t) = F(t) = 0 \):

\[
m\ddot{x} + V'(x) = 0 .
\]
Then, a canonical Poisson bracket exists, and we can use it to define a retarded two-point function:

\[ G^-(t, t') := \{ x(t), x(t') \} \text{, for } t \geq t' \ , \]

\[ G^-(t, t') \equiv 0 \text{, for } t < t' \ , \] (3.16)

We can easily derive the equation satisfied by \( G^-(t, t') \), for \( t > t' \). Indeed, linearity of the PB, and the Leibnitz rule imply:

\[
m \frac{d^2}{dt^2} G^-(t, t') = m \frac{d^2}{dt^2} \{ x(t), x(t') \} = m \{ \ddot{x}(t), x(t') \} = -\{ V'(x(t)), x(t') \} = -V''(x(t)) \{ x(t), x(t') \} = -V''(x(t)) G^-(t, t') \ . \] (3.17)

In view of the boundary condition:

\[
\lim_{t \to t'} m \frac{d}{dt} G^-(t, t') = \{ m\dot{x}(t), x(t) \} = -1 \ , \] (3.18)

we can write an equation valid for all times:

\[
\left( m \frac{d^2}{dt^2} + V''(x(t)) \right) G^-(t, t') = -\delta(t, t') \ , \] (3.19)

which shows that \( G^-(t, t') \) is the \textit{retarded} Green’s function for small perturbations of the classical motion \( x(t) \). One can similarly define an advanced two-point function \( G^+(t, t') \):

\[ G^+(t, t') := -\{ x(t), x(t') \} \text{, for } t \leq t' \ , \]

\[ G^+(t, t') \equiv 0 \text{, for } t > t' \ . \] (3.20)

It is easy to verify that \( G^+(t, t') \) is the advanced Green’s functions for small perturbations of classical motions. If we define the ”commutator function” \( \tilde{G}(t, t') \):

\[
\tilde{G}(t, t') \equiv G^-(t, t') - G^+(t, t') \ , \] (3.21)

we obviously have:

\[
\{ x(t), x(t') \} = \tilde{G}(t, t') \quad \forall \ t, t' . \] (3.22)

We see that antisymmetry of the PB follows from the \textit{reciprocity relation}:

\[ G^+(t, t') = G^-(t', t) \ . \] (3.23)

Our proposal [3] to define the PB for the Langevin equation, is to take Eq. (3.22) as the \textit{definition} of the PB. This way of defining PB, in terms of the Green’s functions associated with the operator that describes small perturbations of classical motions, was introduced, for systems admitting an action principle, by Peierls [4] (For a review of Peierls brackets, see Refs. [5, 6] .) Here, we are trying to extend this method to non-conservative systems, for which an
action principle cannot be found in general. So, we consider the retarded Green’s function for
the Langevin equation:
\[
\left( m \frac{d^2}{dt^2} + V''(x(t)) \right) G^-(t, t') + \int_{t'}^t d\tau \mu(t - \tau) \frac{d}{d\tau} G^-(\tau, t') = -\delta(t - t').
\]
(3.24)

Notice that the lower extremum of integration in the driving term is \( t' \), because of the retarded
boundary condition. Notice also that the "background" curve \( x(t) \) is an arbitrary
path, since any path is a possible motion for the particle, the random force being arbitrary.

We take \( G^+(t, t') \) as the Green’s function of the adjoint of Eq. (3.24):
\[
\left( m \frac{d^2}{dt^2} + V''(x(t)) \right) G^+ - \int_{t}^{t'} d\tau G^+-\tau, t' = -\delta(t - t').
\]
(3.25)

With this choice, the reciprocity relation Eq. (3.23) still holds. This is easily seen if we write
Eqs. (3.24) and (3.25) in the form of integral equations (with singular kernels):
\[
\int_{-\infty}^{\infty} d\tau L(t, \tau) G^- - \tau, t' = -\delta(t - t'),
\]
(3.26)
\[
\int_{-\infty}^{\infty} d\tau L^T(t, \tau) G^+ - \tau, t' = -\delta(t - t'),
\]
(3.27)
where the superscript \( T \) denotes transpose (transpose is the same as adjoint, because we are
in the real field). We switch to DeWitt [5] condensed notation, in which the continuous time
variable is treated as a discrete index and so linear integro-differential operators are written
as matrices. In this notation:
\[
L_{ij} G^{-jk} = -\delta_{ik}^l, \quad (L^T)_{ij} G^{+jk} = -\delta_{ik}^l.
\]
(3.28)

Multiplication of the second equation by \( G^{-il} \) gives:
\[
G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} \delta_{ik}^l = G^{-kl}.
\]
(3.29)

However, using the first of Eq. (3.28), the l.h.s. is also equal to:
\[
G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} L_{ji} G^{+jk} = \delta_i^l G^{+jk} = G^{-lk}.
\]
(3.30)

Comparison of the r.h.s. of the above two Eqs. proves the reciprocity relation.

The important issue is to check the Jacobi identity. A direct computation gives:
\[
\{\{x^i, x^j\}, x^k\} + \text{cyc.perm} = \tilde{G}^{il} \tilde{G}^{jk}_{..,l} + \tilde{G}^{jl} \tilde{G}^{ki}_{..,l} + \tilde{G}^{kl} \tilde{G}^{ij}_{..,l} \equiv T^{ijk},
\]
(3.31)

and so the Jacobi identity is fulfilled if the quantity \( T^{ijk} \) vanishes. Use of reciprocity relations
allows to write \( T^{ijk} \) only in terms of \( G^{-ij} \), and its functional derivatives. The latter are easily
obtained from Eq. (3.28):
\[
L_{ij,l} G^{-jk} + L_{ij} G^{-jk}_{..,l} = 0.
\]
(3.32)
Multiplication by $G^{+mi}$, and use of reciprocity relation, gives:

$$G^{-mk}_{,l} = -G^{-im}_{,l} L_{ij,l} G^{-jk} .$$  \hfill(3.33)

A simple computation then gives:

$$T^{ijk} = (G^{-li} G^{-mj} G^{-nk} + c.p.) (L_{mn} - L_{nm})_{,l}$$  \hfill(3.34)

This shows that the Jacobi identity holds if and only if the antisymmetric part of the operator for the perturbations, $L$, has vanishing functional derivatives (namely is independent on the unperturbed path $x(t)$). This is surely the case if the equations of motion derive from an action, because then $L_{nm}$ is the second functional derivative of the action $S$, and since derivatives commute, it is symmetric:

$$L_{mn} = S_{mn} = S_{nm} = L_{mn} .$$  \hfill(3.35)

Now, the operator $L$ for the Langevin equation is not symmetric, however its antisymmetric part has vanishing functional derivatives, because of the linear character of the friction term. Indeed, if we restore the plain notation for integral operators, we can write the antisymmetric part of $L$ (acting on a function $\psi(t)$) as:

$$(L - L^T)\psi(t) = \int_{-\infty}^{\infty} dt' \left[ \mu(t - t') + \mu(t' - t) \right] \dot{\psi}(t')$$  \hfill(3.36)

The kernel of $L - L^T$ is independent on the unperturbed path $x(t)$ and so it has null functional derivatives. Then, the friction term does not spoil the Jacobi identity.

By taking the Langevin Equation as a definition of $F(t)$, we can evaluate the PB satisfied by the external force:

$$\{F(t), F(t')\} = \frac{d\mu}{dt}(t - t') + \frac{d\mu}{dt}(t' - t) .$$  \hfill(3.37)

This is the same as $(1/(i\hbar))$ times the commutator that was obtained in \[1\], by using an explicit microscopic model. It is also possible to verify that the equal-time PB of the particle’s coordinate and velocity remain canonical:

$$\{x(t), x(t)\} = \{\dot{x}(t), \dot{x}(t)\} = 0 ,$$

$$m\{x(t), \dot{x}(t)\} = 1 .$$

It is important to observe that, when friction is present, $F(t)$ has non trivial brackets, and so it is inconsistent to set $F(t) = 0$. This means that a consistent PB can be written for the Langevin equation, only if the particle is acted on by an external force. However, if the friction term is zero, $F(t)$ has vanishing brackets with everything (including the particle’s coordinate); then, it is possible to take $F(t) = 0$ and to restrict the bracket onto the set of solutions of the equations of motion. In this way we recover back the (finite dimensional) phase space of the conservative system, and its canonical PB.
4 Conclusions

We have defined a set of PB for the generalized Langevin equation. Distinctive features of our approach are:

- the method does not require an action principle, and is based directly on the equations of motion;
- it can be applied to dissipative equations, for which an action principle does not exist, like the Langevin equation. In this way, we obtained the PB directly from the macroscopic description of dissipation, as provided by the Langevin equation, without making recourse to microscopic models;
- when dissipation is present, the relevant phase space is, a priori, the space of all paths, which is an infinite-dimensional space;
- in the absence of dissipation, when the system is conservative, it is possible to set the random force to zero and restrict the bracket onto the standard phase space, spanned by classical solutions of the e.o.m.. One recovers then the usual canonical PB.

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