Path Spaces and $\mathcal{W}$-Fusion in Minimal Models

by

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Abstract

Product forms of characters of Virasoro minimal models are obtained which factorize into $(2, \text{odd}) \times (3, \text{even})$ characters. These are related by generalized Rogers-Ramanujan identities to sum forms allowing for a quasiparticle interpretation. The corresponding dilogarithm identities are given and the factorization is used to analyse the related path space structure as well as the fusion of the maximally extended chiral algebra.

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1. Introduction

Rogers-Ramanujan type identities already appeared in the Eight-Vertex-SOS model [1], whose local height probabilities can be identified [2] with the characters of minimal models in CFT [3]. New developments emphasize the deep meaning of the occurrence of these identities for characters in statistical models as well as in CFTs. The sum sides of the identities lead to a quasiparticle interpretation [4][5][6] as well as to dilogarithm identities [7][8][9] which already appeared in integrable quantum field theories and statistical models [10][11][12][13]. Their product sides encode the structure of the path spaces describing the combinatorics (embedding structure) of the corresponding highest weight representations (HWRs). In this article we observe that product forms of characters of certain sectors of the Virasoro minimal models factorize into characters of (2, odd) and (3, even)-models. In fact, this applies exactly to those sectors which are invariant under the simple current, therefore belonging to the maximally extended chiral algebra. This is used to construct a path space generalizing [14][15] where the characters of the (2, p′)-models were given on a fusion graph. Moreover the fusion of the maximally extended chiral algebra, a W-algebra with one extra generator, can be obtained following the ideas in Ref. [16].

The paper is organized as follows: In the second section we derive a factorization of modular characters of minimal (even, odd)-models which implies the factorization of adjacency matrices defining the corresponding path Hilbert space in the third section. The fourth section will elucidate that this factorization property carries over to the fusion rules of the corresponding W-algebra.

2. Factorizing Characters

For certain HWRs of the Virasoro minimal models we develop expressions of the modular characters (modular forms) which are of a product form or of a sum form, corresponding to the two sides of a generalized Rogers-Ramanujan identity. It turns out that there exists a standard factorization into characters of (2, odd) and (3, even)-models.

We write $c(p, p') = 1 - 6\frac{(p-p')^2}{pp'}$ for the central charge and $h_{n,m}^{(p,p')}$ for the conformal weights for the $(n, m)$-sector of a $(p, p')$-model. According to Rocha-Caridi [17], the embedding structure of $(p, p')$-Virasoro minimal models implies that the modular characters $\chi(q) = q^{-c/24}$ Tr $q^{L_0}$, the trace running over a HWR, are given by

$$\chi_{n,m}^{(p,p')}(q) = \frac{q^{-c/24}}{\prod_{l \geq 1} (1 - q^l)} \sum_{k \in \mathbb{Z}} \left( q^{a(k)} - q^{b(k)} \right)$$

(2.1)

with $a(k) = h_{n+2pk,m}^{(p,p')}$ and $b(k) = h_{n+2pk,-m}^{(p,p')}$. Accordingly we get

$$q^{c/24} \chi_{n,m}^{(p,p')}(q) \prod_{l \geq 1} (1 - q^l) = q^{h_{n,m}^{(p,p')}} \sum_{k \in \mathbb{Z}} q^{pp'k^2} \left( q^{k(pm-p'n)} - q^{k(pm+p'n)+nm} \right).$$

(2.2)

In the following, we will also make use of the shorthand notation

$$\text{ch}_{n,m}^{(p,p')}(q) = q^{c(p,p')/24} h_{n,m}^{(p,p')} \chi_{n,m}^{(p,p')}(q).$$
\((2\nu, p')\)-models: For \(p = 2\nu\) and \(n = \nu\) one obtains from (2.2)
\[
\text{ch}_{\nu,m}^{(2\nu,p')} (q) = \prod_{l \geq 1} (1 - q^l)^{-1} \sum_{k \in \mathbb{Z}} (-)^k q^{\nu p' k^2 + \frac{(2\nu m - p') k}{2}},
\tag{2.3}
\]
which by Jacobi’s triple product identity
\[
\prod_{l \geq 1} (1 - v^l)(1 + v^{l-\frac{1}{2}} w)(1 + v^{l-\frac{1}{2}} w^{-1}) = \sum_{k \in \mathbb{Z}} v^{\frac{1}{2} k^2} w^k
\]
for \(v = q^{\nu p'}\) and \(w = -q^{(2m-p')/2}\) implies
\[
\text{ch}_{\nu,m}^{(2\nu,p')} (q) = \prod_{l \neq 0, \pm \nu m \mod \nu p', l \geq 1} (1 - q^l)^{-1}
\]
\[
= \sum_{n_1 \cdots n_K \geq 0} \frac{q^{N_1^2 + \cdots + N_K^2 + \nu m + \cdots + \nu K}}{(q)_{n_1} \cdots (q)_{n_K-1} (q^{\nu \gamma})_{N_K}}.
\tag{2.5}
\]
where \((q)_n = (1 - q) \cdots (1 - q^n)\), \(N_i = \sum_{j=i}^{K} n_j\), \(K = \frac{\nu p' + \gamma \nu}{2} - 2\) and \(\gamma_{\nu} = \{ 2, 1 \}\) for \(\nu\) even otherwise.
The equality of the product form (2.4) and the sum form (2.5) follows from Ref. [18]. For odd \(\nu\) these characters coincide with characters of some \((2, \text{odd})\)-model, and \(\sum_j N_j = n \mathcal{C}^{-1}_K n\), \(\mathcal{C}_K = (2 - T_{K})\), \(T_{K}\) being the adjacency matrix of the tadpole graph \(A_{2K}/\mathbb{Z}_2\).

It is noteworthy that the characters we find in this way are exactly those being invariant under the simple current [19]. Therefore they are also characters of the \(\mathcal{W}\)-algebra \(\mathcal{W}(2, \delta)\) with \(\delta = \frac{(\nu-1)(p'-2)}{2}\) at the central charge given by \(c(2\nu, p')\) [20][21][22]:
\[
\chi^{\mathcal{W}(2,\delta)}_{\nu,m} (q) = \chi^{(2\nu,p')}_{\nu,m} (q) \quad \text{and} \quad \text{ch}^{\mathcal{W}(2,\delta)}_{\nu,m} (q) = \text{ch}^{(2\nu,p')}_{\nu,m} (q).
\]
The above form of the characters allows us to prove the factorization of the \((2\nu, p')\)-characters: By (2.4) we get
\[
\text{ch}_{\nu,1}^{(2\nu,3)} (q) = \prod_{k \neq 0, \pm \nu \mod 3\nu} (1 - q^k)^{-1} \quad \text{and} \quad \text{ch}_{1,m}^{(2\nu,p')} (q) = \prod_{k \neq 0, \pm m \mod p'} (1 - q^k)^{-1}
\]
and therefore
\[
\text{ch}_{\nu,1}^{(2\nu,3)} (q) \text{ ch}_{1,m}^{(2\nu,p')} (q^\nu) = \prod_{k \neq 0, \pm \nu \mod 3\nu} (1 - q^{\nu k})^{-1} \prod_{k \neq 0, \pm m \mod p'} (1 - q^{\nu k})^{-1}
\]
\[
= \prod_{k \geq 1} (1 - q^{\nu k}) \prod_{k \equiv 0, \pm \nu m \mod \nu p'} (1 - q^{\nu k}) = \prod_{k \equiv 0, \pm m \mod \nu p'} (1 - q^{\nu k}) \prod_{k \geq 1} (1 - q^{\nu k}) \prod_{k \equiv 0, \pm \nu m \mod \nu p'} (1 - q^{\nu k})
\tag{2.6}
\]
\[
= \text{ch}_{\nu,m}^{(2\nu,p')} (q).
\]
By uniqueness of the prefactor \( q^{c/24-h} \) making the characters given as product expressions into modular forms [23], this relation carries over to the modular characters
\[
\chi_{\nu,1}^{(2\nu,3)}(q)\chi_{1,m}^{(2,\nu)}(q') = \chi_{\nu,m}^{(2\nu,p')}(q).
\]
The sum form of the characters also allows to derive dilogarithm identities for the effective central charge of the \((2,\text{odd})\)-model, following the calculations in Refs. [7][8]: The leading divergence of the characters is given by \( c_{\text{eff}} \) or can be calculated by a saddle point approximation. The positive solutions of the saddle point condition
\[
1 - \xi_i^{\beta_i} = \prod_{j \leq i} \xi_j^{2j} \prod_{j > i} \xi_j^{2i} \quad i = 1, \ldots, K \tag{2.7}
\]
with \( \beta_i = 1 + \delta_{iK}(\gamma_\nu - 1) \) are given by
\[
1 - \xi_i^{\beta_i} = \left( \frac{\sin \frac{\pi}{\nu p'}}{\sin \frac{(i+1)\pi}{\nu p'}} \right)^2.
\]
Inserting into the Rogers dilogarithm function
\[
L(z) := \frac{1}{2} \log(z) \log(1 - z) - \int_0^z \frac{\log(1-w)}{w} dw,
\]
one obtains
\[
c_{\text{eff}} = \frac{1}{L(1)} \sum_{i=1}^{K} \frac{1}{\beta_i} L(1 - \xi_i^{\beta_i}) = \frac{1}{2L(1)} \sum_{i=2}^{\nu p' - 2} L \left( \left( \frac{\sin \frac{\pi}{\nu p'}}{\sin \frac{\pi}{\nu p'}} \right)^2 \right) \tag{2.8}
\]
for the effective central charge of the \( c(2\nu, p') \)-theory under consideration. This identity already appeared in Refs. [24][25] in the context of \( SU(2) \)-WZW-models.

\((3\nu, p')\)-models: For \( p = 3\nu \), using the MacDonald identity for the Cartan matrix
\[
N = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}
\] [23] (Watson identity)
\[
\prod_{l \geq 1} (1 - u^{2l}v^l)(1 - u^{2l-1}v^{l-1})(1 - u^{2l-1}v^l)(1 - u^{4l-4}v^{2l-1})(1 - u^{4l}v^{2l-1})
\]
\[
= \sum_{k \in \mathbb{Z}} u^{3k^2-2k}\frac{(3k^2+k)}{2} - u^{3k^2-4k+1}\frac{(3k^2-k)}{2}
\]
with \( u = q^{\nu m} \) and \( v = q^{2\nu(p'-m)} \), we obtain from (2.2) with \( n = \nu \)
\[
\chi_{\nu,m}^{(3\nu,p')}(q) = \prod_{l \neq 0, \pm \nu m, \pm 2\nu p', \pm (2p'+m)\nu, \pm 2\nu(p'+m) \mod 4\nu p', l \geq 1} (1 - q^l)^{-1}, \tag{2.9}
\]
proving a conjecture in Ref. [26]. In a similar way as for (2.4), a sum form of (2.9) can be written as a product of sums of type (2.5). However, the factorization now involves more factors than in the \((2\nu, p')\)-case above.
\(\mathcal{W}\)-characters: According to an argument given by P. Christe [26], the above characters are a complete list of Virasoro characters of minimal models factorizing in the form \(\prod_{l \geq 1} (1 - q^l)^{n_l}\) with exponents \(n_l\) restricted to \(\{0, -1\}\). However, similar product forms can be obtained for \(\mathcal{W}\)-algebra characters which arise from summing up Virasoro characters within a simple current orbit \([20][21][22]\). This might be exemplified by the case \(\nu = 2\mu\) in the context of \((2\nu, p')\)-models. Starting again from Rocha-Caridi -

\[
q^{c/24 - h_{p_0,p'}^{(p_0,p')} (\chi_{p_0,p'}^{(p_0,p')} (q) - \chi_{p_0-p_0,m}^{(p_0,p')} (q)) / (1-q^j)} = \sum_{k \in \frac{1}{2} \mathbb{Z}} q^{pp'k^2} (q^{(pm-p'n)k} - q^{(pm+p'n)k+nm})
\]

- for \(p = 4\mu\), \(n = \mu\), application of the Jacobi triple product identity yields

\[
q^{c/24 - h_{4\mu,p'}^{(4\mu,p')} (\chi_{4\mu,p'}^{(4\mu,p')} (q) - \chi_{4\mu,m}^{(4\mu,p')} (q))} = \prod_{l \geq 1} (1 - q^l)^{-1} \prod_{l=0, \pm m \text{ mod } p'} (1 - q^{\mu l/2}). \tag{2.10}
\]

For \(\mu\) odd the corresponding \(\mathcal{W}\)-algebra \(\mathcal{W}(2, \delta)\) with \(\delta = \frac{(2\mu-1)(p'-2)}{2}\) is fermionic and the \(\mathcal{W}\)-character can be obtained as a sum of Virasoro characters by making use of the symmetry properties of (2.10) under \(q^{1/2} \rightarrow -q^{1/2}\):

\[
q^{c/24 - h_{4\mu,m}^{(4\mu,p')} (\chi_{4\mu,m}^{(4\mu,p')} (q))} = q^{c/24 - h_{4\mu,m}^{(4\mu,p')} (\chi_{4\mu,m}^{(4\mu,p')} (q) + \chi_{3\mu,m}^{(4\mu,p')} (q))} = \prod_{l \neq 0, \pm \mu \text{ mod } 3\mu} (1 - q^l)^{-1} \prod_{l \neq 0, \pm m \text{ mod } p'} (1 - q^{\mu l})^{-1} \prod_{l \equiv 0, \pm 2m \text{ mod } p'} (1 + q^{\mu l/2}).
\]

According to (2.6) this can be rewritten as

\[
\text{ch}^{\mathcal{W}(2, \delta)}_{4\mu,m} (q) = \text{ch}^{(2\mu,p')}_{4\mu,m} (q) \prod_{l \equiv 0, \pm 2m \text{ mod } p'} (1 + q^{\mu l/2}) \tag{2.11}
\]

As before, (2.10) and (2.11) imply multiple sum forms, now for the \(\mathcal{W}\)-characters, from which one again might derive dilogarithm expressions describing the asymptotics. Most presumably, similar sum and product forms do exist for more general \(\mathcal{W}\)-algebra characters. We also point out that the multiple-sum form (2.5) is by no means unique, and that there even exist simple-sum expressions in some cases [27].

3. Factorization of Path Spaces

The above factorization of the characters carries over to the path space description of the corresponding CFT. This shall be exemplified for the \((2\nu, p')\)-models.

Consider the set of sequences \((m_j)_{j \geq 0}\) taking values in \(\{0, \cdots, n - 1\}\) which are constrained by a matrix \(\mathcal{C} \in M_n(\{0, 1\})\) and have initial value \(m\)

\[
\mathcal{S}(\mathcal{C}, m) = \{(m_j)_{j \geq 0} \in \{0, \cdots, n - 1\}^\mathbb{N} \mid m_j = 0 \text{ for } j \gg 0; m_0 = m, \mathcal{C}_{m_{j+1}} = 1\}.
\]
\( \mathcal{C} \) shall be interpreted as adjacency matrix of a (labelled) graph which we call \( \mathcal{C} \)-graph. \( \mathcal{S}(\mathcal{C}, m) \) may then be understood as the set of all finite paths over this graph that start at node \( m \). \( \mathcal{H}_\mathcal{C} := \langle \mathcal{S}(\mathcal{C}, m) \rangle > \mathcal{Q} \) is the \( \mathcal{Q} \)-vector space freely generated by \( \mathcal{S} \) and becomes a Hilbert space by introducing the inner product in which different paths are orthonormal. On this we consider the endomorphism \( (h \in \mathcal{Q}) \)

\[
L_0^{(h)}(m_j)_{j \geq 0} = (h + \sum_{k \geq 0} km_k) (m_j)_{j \geq 0}
\]

which is also motivated by the form of the corner transfer matrix of the corresponding statistical model [28]. Define \( \mathcal{A}(r), \mathcal{B}(r) \in M_r(\{0,1\}) \) by \( (\mathcal{B}(r))_{ij} = 1 \) and \( (\mathcal{A}(r))_{ij} = \begin{cases} 1, & \text{if } i + j \leq r + 1; \\ 0, & \text{otherwise} \end{cases} \)

are the adjacency matrices of the graphs in figure 1. In the following, \( \mathcal{C} \) is a tensor product of two matrices of the above type, and the labelling of the corresponding graph has to be fixed by the order of the factors (see e.g. figure 2). We rewrite (hence defining this order)

\[
\mathcal{S} \left( \mathcal{A}(\frac{p'-1}{2}) \otimes \mathcal{B}(\nu), \nu(\frac{p'-1}{2} - m) \right) = \{(l_j)_{j \geq 0} \in \mathbb{N}^\mathcal{N} | l_j = \nu m_j + n_j, \\
m_0 = \frac{p'-1}{2} - m, m_j + m_{j+1} \leq \frac{p'-1}{2} - 1, 0 \leq m_j, 0 \leq n_j \leq \nu - 1\},
\]

and use

\[
\mathcal{H}_{\mathcal{A}B}(p', m, \nu) := \langle \mathcal{S}(\mathcal{A}(\frac{p'-1}{2}) \otimes \mathcal{B}(\nu), \nu(\frac{p'-1}{2} - m)) \rangle > \mathcal{Q}
\]
as well as \( \mathcal{H}_{\mathcal{A}}(p', m) := \langle \mathcal{S}(\mathcal{A}(\frac{p'-1}{2}), \frac{p'-1}{2} - m) \rangle > \mathcal{Q} \) and \( \mathcal{H}_{\mathcal{B}}(\nu) := \langle \mathcal{S}(\mathcal{B}(\nu), 0) \rangle > \mathcal{Q} \) as a shorter notation for the path Hilbert spaces. Using the notation in the definition of \( \mathcal{S} \),

\[
q^{L_0}(l_j) = q^{\sum_i i l_i(l_j)} = q^{\sum_i i(\nu m_i + n_i)(l_j)} = q^{\nu \sum_i i m_i q^{\sum_i i n_i}(l_j)}
\]

implies the factorization property

\[
\text{Tr}_{\mathcal{H}_{\mathcal{A}B}(p', m, \nu)} q^{L_0^{(0)}} = \text{Tr}_{\mathcal{H}_{\mathcal{A}}(p', m)} q^{\nu L_0^{(0)}} \text{Tr}_{\mathcal{H}_{\mathcal{B}}(\nu)} q^{L_0^{(0)}}.
\]

Due to the constraints of the sequences over the \( \mathcal{B} \)-graph and by use of (2.4),

\[
\text{Tr}_{\mathcal{H}_{\mathcal{B}}(\nu)} q^{L_0^{(0)}} = \prod_{i \geq 1} \left( \sum_{j=0}^{\nu-1} q^{ij} \right) = \prod_{k \neq 0, \pm \nu \mod 3\nu} (1 - q^k)^{-1} = \chi_{\nu,1}^{(2\nu,3)}(q).
\]
For the sequences over $A$-graphs which appear as fusion graphs of $(2,\text{odd})$-models it is known \[14\][15] that

\[
\text{Tr}_{\mathcal{H}_A}(p',m) \ q^{L_0(0)} = \prod_{k \neq 0, \pm m \mod p'} (1 - q^k)^{-1} = \chi_{1,m}^{(2,p')} (q).
\]  

Equations (2.6) and (3.2) therefore imply the following result:

Assume $2\nu$, $p'$ coprime and $1 \leq m \leq \frac{p' - 1}{2}$. Then the modular characters of the corresponding sectors of Virasoro minimal models are

\[
\chi_{2\nu,m}^{(2,p')} (q) = q^{-e(2\nu,p')/24} \text{Tr}_{\mathcal{H}_{AB}(p',m,\nu)} \ q^{L_0^{(h_{2\nu,p'})}}.
\]  

We briefly comment on a quasiparticle-like structure of the path spaces. The sum side of Rogers-Ramanujan identities has an interpretation in terms of quasiparticles occurring in statistical models in the sense of R. Kedem and B.M. McCoy \[4\]. In Refs. \[5\][6] the notion of quasiparticles was used for characters in a more general context \[9\], and we will use this term for a similar structure on the path spaces. Whereas the product side (2.4) always gives a direct description of the structure of the path space by the constraints, the sum side (2.5) can be interpreted as the partition function of $K$ different types of quasiparticles, which here are equivalence classes of elementary parts of the sequence. In this picture, a quasiparticle is excited by moving the corresponding patterns to the right along the sequence according to the constraints of the path space. Particles of different types can be excited independently, whereas particles of one type have minimal distance two. The latter condition originates from the structure of the annihilating ideal \[14\] and might be regarded as a generalized Pauli principle. In (2.5) the value of the summation variable $n_j$ is the number of the quasiparticles of type $j$, the factor $1/(q)^{n_j}$ generating all its excitations.

The ground state of the $(n_1, \cdots, n_K)$-particle sector is ordered with the energy weight (the number of the particle type) of the particles decreasing from the left to the right and therefore is given by the $q$-exponent in the denominator of (2.5) where the missing linear terms encode the path initial conditions.

In the case $\nu = 1$, all $\frac{p' - 3}{2}$ quasiparticles occurring are of the form $(m_l) = (\cdots k, j - k \cdots)$, $j = 1, \cdots, \frac{p' - 3}{2}$ being the energy weight (and denoting the particle type), and $k = 1, \cdots, j$ together with the position counting different energy configurations within one type (pattern equivalence class). In general, there are $N := \frac{p' - 1}{2} - 1$ (the number of allowed nonzero values on the paths) such simple quasiparticles as $(\cdots k, j - k \cdots)$ ($j = 1, \cdots, N$) and $K - N = \left\lfloor \frac{p'}{2} \right\rfloor$ additional ones.

In the simplest case, $\nu = 2$, the one additional quasiparticle is of the form $(\cdots N - k, 1 + k \cdots)$, where according to the constraints by the adjacency matrix only even values of $k = 0, 2, \cdots, N - 1$ are allowed. Hence particles of this type can only be moved in energy steps of two. This is the origin of the $\gamma_\nu = 2$ exponent in the $K^{th}$ $q$-bracket in (2.5).
It is evident that the product expression of the fermionic $\mathcal{W}$-characters in (2.11) implies similar path spaces. In fact, for $\mu$ odd the energy operator may be taken to be

$$L_0^{(h)}(l_j)_{j \geq 0} = (h + \sum_{k \geq 0} \frac{k}{2} l_k) (l_j)_{j \geq 0}$$ (3.7)

where the sequences $(l_i)_{i \geq 1}$ are subject to the conditions $(l_{2i})_{i \geq 1} \in \mathcal{S}(A(\frac{\nu - 1}{2}) \otimes B(\mu))$ and $l_i \in \{0, \mu\}$ for $i$ odd with $i \equiv 0, \pm 2m \mod p'$, finally $l_i = 0$ for the remaining positions. The $\mathcal{W}$-algebra characters which are already contained in the result (3.6) might be described in a way close to above, if one again uses $L_0$ of the form (3.7), restricting to the even positions of the sequence; the rules of $A \otimes B$ then describe the matching of sequence positions of distance two. The odd positions are simply set to zero in this case. In this picture, we see that the corresponding path spaces on Bratteli-like diagrams (figure 3) do not have the same structure for all sectors, but a projection to paths over graphs is not obvious in the case of summed characters.

4. Factorization of $\mathcal{W}$-Fusion

Recall that the Virasoro characters which factorize according to (2.6) are exactly those which are also characters of the maximally extended chiral algebra (the corresponding $\mathcal{W}$-algebra). This suggests that the path space structure examined above is related to the extended model rather than to the Virasoro minimal model. In particular, we expect this structure to contain the fusion of the corresponding $\mathcal{W}$-algebra, since it governs the relative size of the sectors, i.e. the quantum dimensions.

In the following we refer to methods developed by A. Recknagel [16]. This approach is motivated by algebraic field theory and relates the fusion to the $K$-theory of the algebra of observables or - simplifying - to some "good" approximation of it. Recall that one can describe the $K_0$-group of an associative algebra as the abelian group associated to the semi-group of stable equivalence classes of idempotents in the matrices having entries in that algebra [29].

Having established a grading preserving isomorphism between the HWRs of a $(2\nu, p')$-model which correspond to $\mathcal{W}$-sectors and the path spaces of the $A(\frac{\nu - 1}{2}) \otimes B(\nu)$-graph, one might expect that the $AF$-algebra which is defined by the path space (the path space being understood as the Bratteli diagram of this algebra) already carries much of the structure of the CFT. Of course the general situation is not as convincing as for $\nu = 1$ [14][15], since first, we do not have the same graph for all sectors, and second, we cannot relate the path space directly to an annihilating ideal. Note also that for $\nu > 1$ the $A(\frac{\nu - 1}{2}) \otimes B(\nu)$-graph cannot be a fusion graph. But still, at least for $\nu = 2$ the corresponding $AF$-algebra contains the fusion of the $\mathcal{W}(2, \frac{\nu - 2}{2})$ in the sense expressed in Ref. [16]: Taken for granted that it is a good enough approximation of the observable algebra, the fusion ring should be a maximal commutative subring of the ring of positive endomorphisms of its ordered $K_0$-group [29]. The question whether, given such a subring, any set of generators (generating the subring as a $\mathbb{Z}$-module) satisfying the fusion rule axioms determines the same fusion rule, is not yet completely analysed, but in many cases the choice of one generator seems to be sufficient to fix it.
The directed system of $K_0$-groups determined by the path space $S(A(2^{\frac{1}{2}}) \otimes B(\nu))$ (the starting node only influences its beginning but not the algebraic limit) is

$$\cdots \xrightarrow{A(2^{\frac{1}{2}}) \otimes B(\nu)} \mathbb{Z}^{2^{\nu-1}} \otimes \mathbb{Z}^\nu \xrightarrow{A(2^{\frac{1}{2}}) \otimes B(\nu)} \mathbb{Z}^{2^{\nu-1}} \otimes \mathbb{Z}^\nu \xrightarrow{A(2^{\frac{1}{2}}) \otimes B(\nu)} \cdots \hspace{1cm} (4.1)$$

The factorization of the graph suggests to first consider each graph separately and then to tensorize the resulting endomorphisms. The first graph already occurred in Ref. [32] as the fusion graph of the $h_{min}$-field of the $(2, p')$-Virasoro model. Here the adjacency matrix is bijective so that it may be used to identify the $K_0$-group of the algebraic limit of (4.1) with $\mathbb{Z}^{2^{\nu-1}}$. A maximal commuting subset of all positive $K_0$-endomorphisms is linearly generated by $p_k(T_{\nu-1}^{\nu}), k = 0, \ldots, \frac{p'-3}{2}$, the Chebychev polynomials of the second kind (defined by $p_0 = 1$, $p_1(x) = x$, $p_{n+2}(x) =xp_{n+1}(x) - p_n(x)$) evaluated at the adjacency matrix $T_{\nu-1}^{\nu}$ of the tadpole graph $A_{p'-1}/\mathbb{Z}2$ [30]. Moreover these polynomials are generators of the fusion ring of the $(2, p')$-Virasoro minimal model. Questions of uniqueness parallel those for the $SU(2)$-WZW models which are treated in Ref. [16] as an example.

Concerning the second graph let us first consider the case $\nu = 2$. The $B(2)$-graph is well known to describe the embedding of the Majorana algebra (the even part of the Clifford algebra) of $2n$ generators into the one with $2(n+1)$ generators. The $AF$-algebra described by the corresponding Bratteli diagram (the path space of the $B(2)$-graph which may also be obtained by the tower construction of V. Jones [31]) already appeared in Ref. [32], a suitable closure of it being the Neveu-Schwarz component of the observable algebra of the Ising model. Its $K_0$-group equals the dyads $\mathbb{Z}[\frac{1}{2}]$ with the usual order relation (from $\mathbb{R}$). To obtain the fusion ring here, it is necessary to use the refined version of the above method, as any endomorphism of $\mathbb{Z}[\frac{1}{2}]$ is given by multiplication with a dyadic number. These difficulties arise from the degeneracy of the adjacency matrix $B(2)$. For this reason it is proposed in Ref. [16] to consider stationary systems of filtration preserving endomorphisms of the directed system itself (which also covers the result for bijective adjacency matrices). These are endomorphisms $\{\rho_n\}$

$$\cdots \xrightarrow{B(2)} \mathbb{Z}^2 \xrightarrow{B(2)} \mathbb{Z}^2 \xrightarrow{B(2)} \cdots$$

where $\rho_n = \rho$ for all $n$, the $\rho$’s being endomorphisms of $\mathbb{Z}^2$ that commute with $B(2)$ and preserve the cone defined by $z_1 + z_2 > 0$. In fact, since the energy grading may not be well defined for equivalence classes of Bratteli diagrams, but only for distinguished representatives (the path spaces), not only the direct limit but the directed system itself should encode the fusion. It is clear that a commutative subring is now generated (as a $\mathbb{Z}$-module) by two elements. Next to the $2 \times 2$ unit matrix, the only possible choice for the second generator which is compatible with the axioms for a fusion ring is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the $B(2)$-graph therefore describing a $\mathbb{Z}_2$-fusion.

For arbitrary $\nu$ a maximal commuting subring of endomorphisms of $\mathbb{Z}^\nu$ commuting with $B(\nu)$ is generated by $\omega^l$, $0 \leq l < \nu$ ($\omega^\nu = id$), where $\omega$ is a cyclic permutation of the base.
Hence, neglecting questions of uniqueness of the above mentioned type, the $B(\nu)$-graph yields a $\mathbb{Z}_\nu$-fusion.

Although the above arguments are not yet at a rigorous level they encourage us to conjecture

the generators of the fusion ring of $W(2, \delta)$, $\delta = \frac{(\nu-1)(p'-2)}{2}$ at $c(2\nu, p')$ to be represented by

\[ \phi_{kn} := p_{k-1} \left( \mathcal{T}_{\nu}^{(T_{p'}/2)} \right) \otimes \omega^{n-1} \quad k = 1, \ldots, \frac{p' - 1}{2} \quad n = 1, \ldots, \nu \]

where $\omega$ is a generator of $\mathbb{Z}_\nu$, i.e. by “$(2, p')$-fusion $\otimes \mathbb{Z}_\nu$-fusion”.

In fact, for the fermionic $W$-algebras $W(2, \frac{p'-2}{2})$, $p'$ odd, for which some of the $W$-characters already appear as Virasoro characters in $(4, p')$-models ($\nu = 2$) the conjecture is supported for the Neveu-Schwarz sector by earlier results of Ref. [22] and in total by very recent results of W. Eholzer [33] who derived them starting from the representation theory of the modular group. An explicit example is given by the $W(2, 3)$-fusion at $c(4,5) = 7/10$ in figure 4. Furthermore, by comparison with further examples [34] the decomposition of the $B(\nu)$-matrix into a $\mathbb{Z}_\nu$ fusion is also suggested in the bosonic case. In fact, the fusion of the untwisted sector of $W(2, 3)$ at $c(5,6) = 4/5$, where the sectors $h = 2/3$ and $h = 1/15$ are doubled due to non vanishing $W_0$ eigenvalue, is given by $(2, 5)$-fusion $\otimes \mathbb{Z}_3$.

5. Conclusions and Outlook

For all Virasoro minimal models of type $(2\nu, p')$ we found factorizing product expressions for the characters of the simple current invariant sectors. In particular all unitary models are covered. For the corresponding fermionic $W(2, \delta)$ even more characters could be shown to factorize, in fact for $\nu = 2$, all of them. These are all chiral extensions of the Virasoro minimal models by one generator of vanishing self-coupling.

We derived a path space realization of the corresponding HWRs, in which the factorization became manifest in the graph adjacency. $K$-theoretic arguments were applied to demonstrate how the factorization carries over to the $W$-fusion. The case of $\nu = 1$ is exceptional in the sense that, by lack of a simple current, the Virasoro fusion is reproduced. For the $\nu = 2$ case, the results are established and in total coincidence with those of Ref. [33]. By investigating a bosonic example, we checked that this result most presumably generalizes to all $W$-algebras extending Virasoro minimal models.

It remains to find product and sum forms for the other characters of these models, and to understand in more detail the deviations in the structure of the path spaces, which seem to be necessary to avoid quantum dimensions smaller than one. For a complete description of minimal models the (odd, odd)-models should also be included. They in fact show a similar factorization structure involving more factors but again of the types $A$ and $B$.

We emphasize that by linear combination of $W$-characters, one can also derive sum forms of the Virasoro characters of minimal models, leading to dilogarithm identities. Furthermore the path space structure in these cases should give a direct access to the annihilating ideal, paving the way to generalize Ref. [14].
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Figure 1: A,T,B-graphs
Figure 2: Product graph for the (4,5)-model
Figure 3a: Path space for the (2,m)-sectors of the (4,5)-model
(branchings are allowed at the circles)
Figure 3b: Path spaces for the (1,m)-sectors of the (4,5)-model
(energies indicated in the central column)
Figure 4: Graph representation of the W(2,3/2)-fusion at c(4,5)