THE SOLUTION GAP OF THE BREZIS–NIRENBERG PROBLEM ON THE HYPERBOLIC SPACE

SOLEDAD BENGURIA

ABSTRACT. We consider the positive solutions of the nonlinear eigenvalue problem $-\Delta u = \lambda u + u^p$, with $p = \frac{n+2}{n-2}$ and $u \in H_0^1(\Omega)$, where $\Omega$ is a geodesic ball of radius $\theta_1$ on $\mathbb{H}^n$. For radial solutions, this equation can be written as an ODE having $n$ as a parameter. In this setting, the problem can be extended to consider real values of $n$. We show that if $2 < n < 4$ this problem has a unique positive solution if and only if $\lambda \in \left(\frac{n(n-2)}{4} + L^*, \lambda_1\right)$. Here $L^*$ is the first positive value of $L = -\ell(\ell+1)$ for which a suitably defined associated Legendre function $P_{\ell}^{-\alpha}(\cosh \theta) > 0$ if $0 < \theta < \theta_1$ and $P_{\ell}^{-\alpha}(\cosh \theta_1) = 0$, with $\alpha = (2-n)/2$.

1. Introduction

Given a bounded domain $\Omega$ in $\mathbb{R}^n$, Brezis and Nirenberg [5] considered the problem of existence of a function $u \in H_0^1(\Omega)$ satisfying

$$-\Delta u = \lambda u + u^p \quad \text{on } \Omega$$
$$u > 0 \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \partial \Omega,$$

where $p = (n+2)/(n-2)$ is the critical Sobolev exponent. If $\lambda \geq \lambda_1$, where $\lambda_1$ is the first Dirichlet eigenvalue, this problem has no solutions. Moreover, if the domain is star-shaped, there is no solution if $\lambda \leq 0$. Thus, when $\Omega$ is a ball, for any given value of $n$ there may exist a solution only if $\lambda \in (0, \lambda_1)$. It was shown in [5] that in dimension $n \geq 4$, there exists a solution for all $\lambda$ in this range. However, in dimension $n = 3$ Brezis and Nirenberg showed there is an additional interval where there is no solution, which we will refer to in this article as the solution gap of the Brezis-Nirenberg problem. When the domain is the unit ball, the solution gap when $n = 3$ is the interval $(0, \lambda_1^4]$.

The dimensions for which semilinear second order elliptic problems with a nonlinear term of critical exponent (of which (1) is an example) have a solution gap are referred to in the literature as critical dimensions. This definition was first introduced by Pucci and Serrin in [13]. In [9], Jannelli studies a general class of such problems, and the associated critical dimensions. He gives an alternative proof to the existence results obtained in [5] for problem (1). When $\Omega$ is a ball, and $n = 3$, Jannelli shows that (1) has no solution if $\lambda \leq j_{\alpha,1}^2$, where $\alpha = (2-n)/2$ and $j_{\alpha,1}$ denotes the first positive zero of the Bessel function $J_\alpha$.

If $u$ is radial, problem (1) can be written as an ordinary differential equation,

$$-u'' - \frac{(n-1)}{r}u' = \lambda u + u^p,$$
where $n$ can be thought of as a parameter in the equation, rather than the dimension of the space. By doing so one can study the behavior of the solution gap with respect to $n$ by taking $n$ to be a real number instead of a natural number. Jannelli’s methods in [9] can be easily extended to the case $2 < n < 4$, thus concluding that the solution gap of the Brezis-Nirenberg problem defined in the unit ball is the interval $\left( 0, j_{2,1}^2 \right]$. In particular, it follows that $n = 4$ is the first value of $n$ for which there is no solution gap.

The solution gap of the Brezis-Nirenberg problem can also be studied in non-Euclidean spaces. On the sphere $S^n$, for a fixed $n$, the solution gap is the subinterval of $(-n(n-2)/4, \lambda_1)$ for which [1] has no solution. As in the Euclidean case, $n = 3$ is a critical dimension, whereas $n \geq 4$ are not. It was shown in [1] that if $\Omega$ is a geodesic cap of radius $\theta_1$ in $S^3$ the solution gap is the interval $(-n(n-2)/4, (\pi^2 - 4\theta_1^2)/4\theta_1^2]$. If $u$ is radial, then [1] can be written as an ordinary differential equation that still makes sense when $n$ is a real number. It was shown in [3] that if $2 < n < 4$, the solution gap is the interval $(-n(n-2)/4, ((2\ell^* + 1)^2 - (n-1)^2)/4]$, where $\ell^*$ is the first positive value of $\ell$ for which the associated Legendre function $P_{\ell}^{\alpha}(\cos \theta_1)$ vanishes. Here $\alpha = (2 - n)/2$.

In this article we consider the Brezis-Nirenberg problem on the hyperbolic space $H^n$. That is, we consider the problem

$$
- \Delta_{H^n} u = \lambda u + u^p \quad \text{on} \quad \Omega
$$

$$
u > 0 \quad \text{on} \quad \Omega
$$

$$
u = 0 \quad \text{on} \quad \partial \Omega,
$$

where $p = (n+2)/(n-2)$, $\Omega$ is a geodesic ball on $H^n$ of radius $\theta_1 \in (0, \infty)$, and $\Delta_{H^n}$ is the Laplace-Beltrami operator.

It is not hard to show (see, e.g., page 285 in [15]) that there can be no solutions for $\lambda \not\in (n(n-2)/4, \lambda_1)$. Stapelkamp [15] showed that if $n \geq 4$ there is no solution gap, that is, that there is a solution for all values of $\lambda$ in this interval. When $n = 3$, however, she showed there is no solution if $\lambda \in (n(n-2)/4, \lambda^*)$. Here

$$
\lambda^* = 1 + \frac{\pi^2}{16 \text{arctanh}^2 R},
$$

where $R$ is the radius of the ball that results by taking the stereographic projection of the geodesic ball onto $\mathbb{R}^3$. Moreover, Stapelkamp shows that for each $\lambda \in (\lambda^*, \lambda_1)$, there exists a unique solution, and this solution is radial. A full characterization of the solutions to this problem in dimension $n \in \mathbb{N}$ (and any $p > 1$) is given in [2]. After the results of Stapelkamp and Bandle, there has been a vast literature on Brezis-Nirenberg type equations on hyperbolic spaces (see, e.g., [11], [7], [8], [4]).

For radial functions $u$, problem [2] can be written as an ordinary differential equation, with $n$ now simply representing a parameter in the equation rather than the dimension of the space. Our main result is that the solution gap of the Brezis-Nirenberg problem on the hyperbolic space has width $L^*$, where $L^*$ is the first positive value of $L = -\ell(\ell + 1)$ for which a suitably defined (see equation [6]) associated Legendre function $P_{\ell}^{-\alpha}(\cosh \theta)$ is positive if $0 < \theta < \theta_1$ and $P_{\ell}^{-\alpha}(\cosh \theta_1) = 0$. Here, as before, $\alpha = (2 - n)/2$. 
More precisely, we show the following:

**Theorem 1.1.** For any $2 < n < 4$ and $\theta_1 \in (0, \infty)$, the boundary value problem

$$- u''(\theta) - (n - 1) \coth \theta u'(\theta) = \lambda u + u^{\frac{n+2}{n-2}}$$

(3)

with $u \in H^1_0(\Omega)$, $u'(0) = u(\theta_1) = 0$, and $\theta \in [0, \theta_1]$ has a unique positive solution if and only if

$$\lambda \in \left(\frac{n(n-2)}{4} + L^*, \lambda_1\right).$$

(4)

In Figure 1 the graph $\lambda(n)$ illustrates the results of Theorem 1.1 when $\theta_1 = 1$. The shaded region represents the solution gap, and the region between the dotted and the solid lines corresponds to the region of existence of solutions given by (4).

In Section 2 we derive an expression for the first Dirichlet eigenvalue in terms of the parameter $\ell$ of an associated Legendre function, and use this expression to show that the interval of existence given by (4) is non-empty if $2 < n < 4$. In Section 3 we use a classical Lieb lemma to show the existence of solutions for $\lambda$ as in (4). In Section 4 we use a Pohozaev type argument to show that if $2 < n < 4$ there is a solution gap of the Brezis-Nirenberg problem. That is, we show there are no solutions if $\lambda \in (n(n-2)/4, n(n-2)/4 + L^*)$. Finally, in Section 5 we show that the uniqueness of solutions follows directly from [10].

2. Preliminaries

The associated Legendre functions $P^\alpha_\ell(\cosh \theta)$ and $P^{-\alpha}_\ell(\cosh \theta)$ are solutions of the Legendre equation

$$y''(\theta) + \coth \theta y'(\theta) + \left(-\ell(\ell + 1) - \frac{\alpha^2}{\sinh^2 \theta}\right) y(\theta) = 0.$$  

(5)
We will adopt the following convention for the associated Legendre functions:

\[ P_\ell^\alpha(cosh \theta) = \frac{1}{\Gamma(1-\alpha)} \coth^\alpha \left( \frac{\theta}{2} \right) \, _2F_1 \left[ -\ell, \ell + 1, 1 - \alpha; -\sinh^2 \left( \frac{\theta}{2} \right) \right], \tag{6} \]

where for complex numbers \( a, b, \) and \( c, \) the hypergeometric function \( _2F_1 \) is given by

\[ _2F_1 \left[ a, b, c; z \right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{7} \]

where \( (\beta)_n := \prod_{j=0}^{n-1} (\beta + j), \) for \( \beta \in \mathbb{C}. \)

**Remark 2.1.** Notice that the associated Legendre functions \( P_\ell^\alpha(cosh \theta) \) depend on \( \ell \) through the product \( \ell(\ell+1), \) rather than from \( \ell \) and \( \ell + 1 \) independently.

The associated Legendre functions given by (5) satisfy the following raising and lowering relations (see, e.g., [14], page 55, equations (20.11-1) and (20.11-2) with \( x = \cosh \theta \)):

\[ \dot{P}_\ell^\alpha(cosh \theta) = \frac{1}{\sinh \theta} P_{\ell+1}^\alpha(cosh \theta) + \frac{\alpha \cosh \theta}{\sinh^2 \theta} P_{\ell}^\alpha(cosh \theta), \tag{8} \]

and

\[ \dot{P}_{\ell+1}^\alpha(cosh \theta) = \frac{\ell(\ell+1) - \alpha(\alpha+1)}{\sinh \theta} P_{\ell}^\alpha(cosh \theta) - \frac{(\alpha+1) \cosh \theta}{\sinh^2 \theta} P_{\ell+1}^\alpha(cosh \theta). \tag{9} \]

Here \( \dot{P}_\ell^\alpha \) means the derivative of \( P_\ell^\alpha \) with respect to its argument. That is,

\[ \frac{d}{d\theta} P_\ell^\alpha(cosh \theta) = \sinh \theta \dot{P}_\ell^\alpha(cosh \theta). \]

Equations (8) and (9) are used in the proof of the non-existence result on Section 4.

**Definition 1.** Let \( L = -\ell(\ell+1). \) For \( 2 < n < 4, \) \( \alpha = (2 - n)/2, \) and \( \theta_1 \in (0, \infty), \) let \( L_1 \) be the smallest positive value of \( L \) such that \( P_\ell^\alpha(\cosh \theta) > 0 \) if \( 0 < \theta < \theta_1 \) and \( P_\ell^\alpha(\cosh \theta_1) = 0. \) Similarly, let \( L^* \) be the smallest positive value of \( L \) such that \( P_\ell^{-\alpha}(\cosh \theta) > 0 \) if \( 0 < \theta < \theta_1 \) and \( P_\ell^{-\alpha}(\cosh \theta_1) = 0. \)

In the next lemma we derive an expression for the first Dirichlet eigenvalue of \( -\Delta_{\mathbb{S}^n} u = \lambda u \) on a geodesic ball in terms of \( L_1. \) In Lemma 2.4 we use the expression for \( \lambda_1 \) obtained in Lemma 2.2 to show that the interval of existence given in equation (4) is non-empty if \( 2 < n < 4. \)

**Lemma 2.2.** The first Dirichlet eigenvalue of equation

\[ -u'' - (n-1) \coth \theta u' = \lambda_1 u. \tag{10} \]

is given by

\[ \lambda_1 = \frac{n(n-2)}{4} + L_1. \]
Lemma 2.4. Let $\ell$ be as in Definition 1. Then $L^* < L_1$.

Proof. Let $y_1(\theta) = P_{\ell_1}^\alpha(\cosh \theta)$ and $y_2(\theta) = P_{\ell^*}^{-\alpha}(\cosh \theta)$. Then $y_j$, $j \in \{1, 2\}$, satisfy

$$y_j'' + \coth \theta y_j' + k_j y_j = 0,$$  \hspace{1cm} (11)

where

$$k_1 = L_1 - \frac{\alpha^2}{\sinh^2 \theta}.$$  

and

$$k_2 = L^* - \frac{\alpha^2}{\sinh^2 \theta}.$$  

Let $W = y_1'y_2 - y_2'y_1$ and $W'' = y_1''y_2 - y_2''y_1$. Then it follows from equation (11) that

$$W' + \coth \theta W = (k_2 - k_1)y_1y_2.$$  

Multiplying by $\sinh \theta$ and integrating one has that

$$\int_0^{\theta_1} (W \sinh \theta)' \, d\theta = [L^* - L_1] \int_0^{\theta_1} y_1y_2 \sinh \theta \, d\theta.$$  

By choice of $L_1$ and $L^*$ it follows that $y_1$ and $y_2$ are positive on $[0, \theta_1)$ and vanish at $\theta_1$, so that

$$\int_0^{\theta_1} y_1y_2 \sinh \theta \, d\theta$$  

is positive and $W(\theta_1) = 0$. Thus, it suffices to show that $\lim_{\theta \to 0} W(\theta) \sinh \theta$ is negative.
It follows from equation (6) that the behavior of $y_1$ and $y_2$ near zero is

$$y_1 \approx \frac{1}{\Gamma(1-\alpha)} \coth^\alpha \left( \frac{\theta}{2} \right),$$

and

$$y_2 \approx \frac{1}{\Gamma(1+\alpha)} \coth^{-\alpha} \left( \frac{\theta}{2} \right).$$

Therefore,

$$\lim_{\theta \to 0} W(\theta) \sinh \theta = \frac{-\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \lim_{\theta \to 0} \sinh \theta \left( \frac{\tanh \left( \frac{\theta}{2} \right)}{\sinh^2 \left( \frac{\theta}{2} \right)} \right)$$

$$= \frac{-2\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)}.$$

Finally, since $\Gamma(1+\alpha) = \alpha \Gamma(\alpha)$, $\Gamma(\alpha)(1-\alpha) = \pi \sin^{-1}(\pi \alpha)$, and $0 < \alpha < 1$, we conclude that

$$\lim_{\theta \to 0} W(\theta) \sinh \theta = \frac{-2\sin(\pi \alpha)}{\pi} < 0.$$

From Lemmas 2.2 and 2.4 it follows that the interval of existence given by (4), that is, $(n(n-2)/4 + L^*, n(n-2)/4 + L_1)$, is nonempty if $2 < n < 4$.

3. Existence of solutions

In this section we present the proof of the following theorem:

**Theorem 3.1.** For any $2 < n < 4$ and $\theta_1 \in (0, \infty)$, the boundary value problem

$$-u''(\theta) - (n-1) \coth \theta \ u'(\theta) = \lambda u + u^{n+\frac{2}{n}},$$

with $u \in H_0^1(\Omega)$, $u'(0) = u(\theta_1) = 0$, and $\theta \in [0, \theta_1]$ has a positive solution if

$$\lambda \in \left( \frac{n(n-2)}{4} + L^*, \lambda_1 \right).$$

Here $L^*$ is as in Definition 4.

For natural values of $n$, the positive solutions of

$$-\Delta_{H^n} u = \lambda u + u^p,$$

on a geodesic ball with Dirichlet boundary conditions correspond to minimizers of

$$Q_\lambda(u) = \frac{\int |\nabla u|^2 \rho^{n-2} \, dx - \lambda \int u^2 \rho^n \, dx}{\left( \int u^{2n-n-2} \rho^n \, dx \right)^{n-2/n}}.$$
Here $\rho(x) = \frac{2}{1 - |x|^2}$ is such that $ds = \rho dx$.

If $u$ is radial, we can write
\[
Q_\lambda(u) = \frac{\omega_n \int_0^R u^2 \rho^{n-2} r^{n-1} dr - \lambda \omega_n \int_0^R u^2 \rho^n r^{n-1} dr}{\left( \omega_n \int_0^R u^{\frac{2n}{n-2}} \rho^n r^{n-1} dr \right)^{\frac{n-2}{n}}}.
\] (13)

Here $r = \tanh(\theta/2)$, $R = \tanh(\theta_1/2) < 1$, and $\omega_n$ represents the surface area of the unit sphere in $n$-dimensions, and is explicitly given by $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}$. This quotient is well defined if $n$ is a real number instead of a natural number.

**Lemma 3.2.** There exists a function $u \in H^1_0(\Omega)$, with $u'(0) = u(\theta_1) = 0$, such that $Q_\lambda(u) < S_n$ for all $\lambda > \frac{n(n-2)}{4} + L^*$. Here $S_n$ is the Sobolev constant.

**Proof.** Let $\varphi$ be an arbitrary cutoff function such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(R) = 0$, and let
\[
v_\epsilon(r) = \frac{\varphi(r)}{(\epsilon + r^2)^{\frac{n-2}{2}}}.
\]

As in [15], let
\[
u_\epsilon(r) = \rho^{\frac{2-n}{2}}(r)v_\epsilon(r).
\]

With this choice of $u_\epsilon$, and after integrating by parts, we have
\[
\int_0^R u^2 \rho^{n-2} r^{n-1} dr = \frac{n(n-2)}{4} \int_0^R \rho^2 v_\epsilon^2 r^{n+1} dr + \frac{n(n-2)}{2} \int_0^R v_\epsilon^2 \rho r^{n-1} dr
\]
\[+ \int_0^R v_\epsilon^2 r^{n-1} dr.
\] (14)

Using the fact that $r^2 + \frac{2}{\rho} = 1$ to combine the first two terms of equation (14), it follows that,
\[
Q_\lambda(u_\epsilon) = \frac{\omega_n \left( \frac{n(n-2)}{4} - \lambda \right) \int_0^R v_\epsilon^2 \rho^2 r^{n-1} dr + \omega_n \int_0^R v_\epsilon^2 r^{n-1} dr}{\left( \omega_n \int_0^R v_\epsilon^2 \rho^2 r^{n-1} dr \right)^{\frac{n-2}{n}}}.
\] (15)

**Claim 3.3.**
\[
\omega_n \left( \frac{n(n-2)}{4} - \lambda \right) \int_0^R v_\epsilon^2 \rho^2 r^{n-1} dr = \omega_n \left( \frac{n(n-2)}{4} - \lambda \right) \int_0^R \varphi^2 r^{3-n} \rho^2 dr
\]
\[+ O\left( \epsilon^{\frac{4-n}{2}} \right).
\]
Proof. Let

\[ I(\epsilon) = \int_0^R v_\epsilon^2 \rho^2 r^{n-1} \, dr = \int_0^R \frac{\phi^2}{(\epsilon + r^2)^{n-2}} \rho^2 r^{n-1} \, dr. \]

Then \( I(0) = \int_0^R \phi^2 \rho^2 r^{3-n} \, dr. \) Thus, it suffices to show that \( |I(\epsilon) - I(0)| = O\left(\epsilon^{\frac{4-n}{2}}\right). \)

If \( 0 < r < R < 1, \) then \( \rho(r) = \frac{2}{1 - r^2} < \frac{2}{1 - R^2}. \) Thus,

\[
|I(\epsilon) - I(0)| \leq \frac{4}{(1 - R^2)^2} \left| \int_0^R \phi^2 r^{n-1} \left( \frac{1}{(\epsilon + r^2)^{n-2}} - \frac{1}{r^{2(n-2)}} \right) \, dr \right| \\
= \frac{4(n-2)}{(1 - R^2)^2} \left| \int_0^R \int_0^\epsilon \frac{\phi^2 - 1 + 1}{(a + r^2)^n} \, da \, dr \right|.
\]

Let

\[ L_1(\epsilon) = \int_0^\epsilon \left( \int_0^R \frac{r^{n-1}}{(a + r^2)^n} \, dr \right) \, da, \]

and

\[ L_2(\epsilon) = \int_0^R (\phi^2 - 1) r^{n-1} \int_0^\epsilon \frac{1}{(a + r^2)^n} \, da \, dr. \]

Making the change of variables \( r = u \sqrt{a} \) in the inner integral of \( L_1(\epsilon), \) we have

\[
\int_0^R \frac{r^{n-1}}{(a + r^2)^n} \, dr = a^{\frac{2-n}{2}} \int_0^\frac{R}{\sqrt{a}} \frac{u^{n-1}}{(1 + u^2)^n} \, du \leq a^{\frac{2-n}{2}} \int_0^\infty \frac{u^{n-1}}{(1 + u^2)^n} \, du.
\]

Since we are considering \( n > 2, \) this last integral converges. Thus, and since \( n < 4, \)

\[ L_1(\epsilon) \leq C \int_0^\epsilon a^{\frac{2-n}{2}} \, da = O\left(\epsilon^{\frac{4-n}{2}}\right). \]

On the other hand, since \( \phi(0) = 1 \) and \( \phi'(0) = 0, \) for \( 0 \leq r < 1 \) we have that \( \phi^2 - 1 \leq Cr^2 \) for some \( C > 0. \) Thus,

\[
L_2(\epsilon) \leq C \int_0^R r^{n+1} \int_0^\epsilon \frac{1}{(a + r^2)^n} \, da \, dr \\
\leq C \int_0^R r^{n+1} \int_0^\epsilon \frac{1}{r^{2n-2}} \, da \, dr = C \epsilon \int_0^R r^{3-n} \, dr.
\]

Since \( n < 4, \) this last integral converges. Thus \( L_2(\epsilon) = O(\epsilon), \) and in particular \( O(\epsilon^{\frac{4-n}{2}}). \)

Claim 3.4.

\[ \omega_n \int_0^R v_\epsilon^2 r^{n-1} \, dr = \omega_n \int_0^R \phi'(r)^2 r^{3-n} \, dr + K_1 \epsilon^{\frac{2-n}{2}} + O(\epsilon^{\frac{4-n}{2}}), \]

where
Thus, since

\[ K_1 = \frac{\pi^\frac{n}{2} n(n-2) \Gamma \left( \frac{n}{2} \right)}{\Gamma(n)}. \]

**Proof.** Let

\[ J = \omega_n \int_0^R v_\epsilon^2 r^{n-1} \, dr. \]

Then we can write

\[ J = \omega_n \int_0^R \varphi'^2 r^{n-1} \left[ \frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} - \frac{2(n-2)r \varphi' r^2}{(\epsilon + r^2)^{n-1}} + \frac{r^2 \varphi'^2 (n-2)^2}{(\epsilon + r^2)^n} \right] \, dr. \]

Integrating by parts the second term, and since by hypothesis \( \varphi(R) = 0 \), we have

\[ J = \omega_n \int_0^R \varphi'^2 r^{n-1} \left[ \frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} + \frac{2(n-2)r \varphi' r^2}{(\epsilon + r^2)^{n-1}} + \frac{r^2 \varphi'^2 (n-2)^2}{(\epsilon + r^2)^n} \right] \, dr. \]

Thus, since \( (n-2)^2 - 2(n-2)(n-1) = -n(n-2) \), combining the last three terms we have

\[ J = \omega_n \int_0^R \varphi'^2 r^{n-1} \left[ \frac{\varphi'^2}{(\epsilon + r^2)^{n-2}} + \frac{2(n-2)r \varphi' r^2}{(\epsilon + r^2)^{n-1}} + \frac{r^2 \varphi'^2 (n-2)^2}{(\epsilon + r^2)^n} \right] \, dr. \]

Let us now estimate

\[ J_1(\epsilon) = \int_0^R \varphi'^2 (\epsilon + r^2)^{2-n} r^{n-1} \, dr. \]

Notice that

\[ J_1(0) = \int_0^R \varphi'^2 r^{3-n} \, dr. \]

In what follows we estimate the difference, i.e., \( \Delta(\epsilon) \equiv J_1(\epsilon) - J_1(0) \). We write,

\[ \Delta(\epsilon) = \int_0^1 \varphi'(r)^2 r^{3-n}(A) \, dr, \]

where

\[ A = 1 - (\epsilon + r^2)^{2-n} r^{2n-4} = 1 - (1 + \epsilon r^{-2})^{2-n} > 0, \]

since \( n > 2 \). Using the fact that

\[ (1 + x)^{-m} > 1 - mx \]

for \( x = \epsilon/r^2 \geq 0 \) and \( m = n - 2 > 0 \), we conclude that

\[ A < (n-2) \epsilon r^{-2}. \]

Thus,

\[ |\Delta(\epsilon)| < \epsilon(n-2) \int_0^R \varphi'(r)^2 r^{1-n} \, dr. \]

Notice that the integral on equation (17) converges. In fact, since \( \varphi(0) = 1 \) and \( \varphi'(0) = 0 \), for \( 0 \leq r < 1 \) one has \( \varphi'(r)^2 \leq C^2 r^2 \) for some positive constant \( C \); thus \( \varphi'(r)^2 r^{1-n} \leq C r^{3-n} \), which is integrable near 0 for all \( 2 < n < 4 \). Hence \( |\Delta(\epsilon)| = \mathcal{O}(\epsilon) \). Thus, from equation (16) we have
\[ J = \omega_n \int_0^R \varphi^2 r^{3-n} \, dr + \omega_n n(n-2)\epsilon \int_0^R \frac{\varphi^2 r^{n-1}}{(\epsilon + r^2)^n} \, dr + \mathcal{O}(\epsilon). \] (18)

Now let

\[ J_2(\epsilon) \equiv \int_0^R \frac{(\varphi^2 - 1) r^{n-1} + r^{n-1}}{(\epsilon + r^2)^n} \, dr. \]

Making the change of variables \( r = s \sqrt{\epsilon} \), we have

\[ \int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n} \, dr = \epsilon^{-\frac{n}{2}} \left( \int_0^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds - \int_0^{\sqrt[4]{\epsilon}} \frac{s^{n-1}}{(1 + s^2)^n} \, ds \right). \]

But

\[ \int_0^{\sqrt[4]{\epsilon}} \frac{s^{n-1}}{(1 + s^2)^n} \, ds \leq \int_0^\infty s^{n-1} \, ds = \frac{\epsilon^{n/2}}{nR^n}. \]

Notice that making the change of variables \( u = s^2 \), we can write

\[ \int_0^\infty \frac{s^{n-1}}{(1 + s^2)^n} \, ds = \frac{1}{2} \int_0^\infty \frac{u^{n-1}}{(1 + u)^n} \, du = \frac{1}{2} \frac{\Gamma \left( \frac{n}{2} \right)^2}{\Gamma(n)}. \]

Here we have used the standard integral

\[ \int_0^\infty \frac{x^{k-1}}{(1 + x)^{k+m}} \, dx = \frac{\Gamma(k) \Gamma(m)}{\Gamma(k + m)} \]

(see, e.g., [6], equation 856.11, page 213), which holds for all \( m, k > 0 \). Thus,

\[ \int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n} \, dr = \frac{\Gamma \left( \frac{n}{2} \right)^2}{2\Gamma(n)} \epsilon^{-\frac{n}{2}} + \mathcal{O}(1). \] (19)

On the other hand, since \( \varphi^2(r) \leq 1 + Cr^2 \), and setting once more \( r = s \sqrt{\epsilon} \), we have that

\[ \int_0^R \frac{(\varphi^2 - 1) r^{n-1}}{(\epsilon + r^2)^n} \, dr \leq C \epsilon^{-\frac{n}{2}} \left( \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} \, ds - \int_0^{\sqrt[4]{\epsilon}} \frac{s^{n+1}}{(1 + s^2)^n} \, ds \right). \]

But

\[ \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} \, ds \leq \int_0^\infty s^{1-n} \, ds = \mathcal{O} \left( \epsilon^{\frac{n-2}{4}} \right), \]

and \( \int_0^\infty \frac{s^{n+1}}{(1 + s^2)^n} \, ds \) is finite. Thus, and since \( 2 < n < 4 \),

\[ \int_0^R \frac{(\varphi^2 - 1) r^{n-1}}{(\epsilon + r^2)^n} \, dr \leq C \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr = \mathcal{O}(\epsilon^{\frac{2n}{4}}). \] (20)

Therefore, from equations (19) and (20) it follows that

\[ J_2(\epsilon) = \frac{\Gamma \left( \frac{n}{2} \right)^2}{2\Gamma(n)} \epsilon^{-\frac{n}{2}} + \mathcal{O}(\epsilon^{\frac{2n}{4}}). \]
Finally, from equation [18] it follows that

\[ J = \omega_n \int_0^1 \varphi'^2 r^{3-n} \, dr + \omega_n n(n-2) \epsilon^{2-n} \left( \frac{\Gamma \left( \frac{n}{2} \right)^2}{2 \Gamma(n)} \right) + \mathcal{O}(\epsilon^{4-n}). \]

But we are taking \( \omega_n = \frac{2\pi^2}{\Gamma \left( \frac{n}{2} \right)} \). Thus,

\[ J = \omega_n \int_0^1 \varphi'^2 r^{3-n} \, dr + \epsilon^{2-n} \left( \frac{n(n-2)\pi^2 \Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right) + \mathcal{O}(\epsilon^{4-n}). \]

\[ \square \]

Claim 3.5.

\[ \left( \omega_n \int_0^R \frac{2\pi}{n} r^{n-1} \, dr \right)^{\frac{n-2}{n}} = \epsilon^{2-n} K_2 + \mathcal{O}(\epsilon^{4-n}), \]

where

\[ K_2 = \left( \frac{n^{n/2} \Gamma(n/2)}{\Gamma(n)} \right)^{\frac{n-2}{n}}. \]

Proof. Let

\[ H(\epsilon) \equiv \omega_n \int_0^R \frac{2\pi}{n} r^{n-1} \, dr = \omega_n \int_0^R \frac{\varphi(r)^{2n/(n-2)}}{(\epsilon + r^2)^n} r^{n-1} \, dr. \]

Since \( \varphi(0) = 1 \), this integral diverges when \( \epsilon \to 0 \). Denote by \( H_1 \) the leading behavior of \( H(\epsilon) \), that is,

\[ H_1(\epsilon) = \omega_n \int_0^R \frac{r^{n-1}}{(\epsilon + r^2)^n} \, dr. \]

As in equation [19], we have

\[ H_1(\epsilon) = c_n \epsilon^{-n/2} + O(1), \quad (21) \]

where

\[ c_n = \frac{\omega_n \Gamma(n/2)}{2 \Gamma(n)} = \frac{\pi^{n/2} \Gamma(n/2)}{\Gamma(n)}. \quad (22) \]

It suffices now to show that

\[ H(\epsilon) - H_1(\epsilon) = \omega_n \int_0^R \frac{\varphi(r)^{2n/(n-2)} - 1}{(\epsilon + r^2)^n} r^{n-1} \, dr = \mathcal{O}(\epsilon^{2-n}). \]

But since \( \varphi(r) \leq 1 + C r^2 \) for some positive constant \( C \), then

\[ |H(\epsilon) - H_1(\epsilon)| \leq C_n \int_0^R \frac{r^{n+1}}{(\epsilon + r^2)^n} \, dr = \mathcal{O}(\epsilon^{2-n}), \quad (23) \]

where the last equality follows from equation [20]. Thus, from [21] and [23], we conclude that

\[ H(\epsilon) = \epsilon^{-n/2}[c_n + O(\epsilon)], \]

where \( c_n \) is given by [22].

\[ \square \]
Replacing the estimates obtained in the three previous claims in the definition of \( Q_\lambda(u_\epsilon) \) given in equation \([15]\), we obtain

\[
Q_\lambda(u_\epsilon) = \frac{K_1}{K_2} + \frac{\epsilon \frac{n-2}{2} \omega_n}{K_2} \left( \left( \frac{n(n-2)}{4} - \lambda \right) \int_0^R \varphi^2 r^{3-n} \rho^2 \, dr + \int_0^R \varphi^2 r^{3-n} \, dr \right) + O(\epsilon).
\]

Here

\[
K_1 = \frac{\pi \frac{n}{2} n(n-2) \Gamma \left( \frac{n}{2} \right)}{\Gamma(n)},
\]

and

\[
K_2 = \left( \frac{\pi^{n/2} \Gamma(n/2)}{\Gamma(n)} \right)^{\frac{n-2}{n}}.
\]

But

\[
\frac{K_1}{K_2} = \pi n(n-2) \left( \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma(n)} \right)^2
\]

which is precisely the Sobolev critical constant \( S_n \) (see, e.g., [16], with \( p = 2, m = n \) and \( q = \frac{2n}{n-2} \)). Therefore, to conclude that \( Q_\lambda(u_\epsilon) < S_n \), it suffices to show that for \( \lambda > \frac{n(n-2)}{4} + L^* \), there exists a choice of \( \varphi \) such that

\[
F(\varphi) \equiv \left( \frac{n(n-2)}{4} - \lambda \right) \int_0^R \varphi^2 r^{3-n} \rho^2 \, dr + \int_0^R \varphi^2 r^{3-n} \, dr
\]

is negative.

Let

\[
M(\varphi) = \int_0^R \varphi^2 r^{3-n} \, dr,
\]

and let \( \varphi_1 \) be the minimizer of \( M(\varphi) \) subject to the constraint \( \int_0^R \varphi^2 r^{3-n} \rho^2 \, dr = 1 \). Then \( \varphi_1 \) satisfies the Euler equation

\[
- \left( \varphi_1^{3-n} \right)' = \mu \varphi_1 r^{3-n} \rho^2.
\]

Here \( \mu \) is the Lagrange multiplier. Multiplying equation \([24]\) by \( \varphi_1 \) and integrating this equation by parts, and since \( \int_0^R \varphi_1^{2r^{3-n}} \rho^2 \, dr = 1 \), we obtain

\[
\int_0^R \varphi_1^{2r^{3-n}} \, dr = \mu.
\]

It follows that \( F(\varphi_1) = \frac{n(n-2)}{4} - \lambda + \mu \). Thus, \( F(\varphi_1) \) is negative as long as \( \lambda > \frac{n(n-2)}{4} + \mu \).

Notice that from \([25]\) one has that \( \mu \) is positive.
It suffices now to show that \( \mu = L^* \). Multiplying equation (24) by \(-r^{n-3}\), we obtain
\[
\varphi'' + \left(\frac{3-n}{r}\right)\varphi' + \mu \varphi r = 0. \tag{26}
\]
Making the change of variables \( \varphi(r) = r^{\frac{n-2}{2}} v(r) \), and after some rearrangement of terms, we can write equation (26) as
\[
v'' + \frac{v'}{r} + \left(\mu r^2 - \frac{(n-2)^2}{4r^2}\right) v = 0. \tag{27}
\]
Changing back to geodesic coordinates, and since \( r = \tanh \frac{\theta}{2} \), we can rewrite equation (27) as
\[
v'' + \coth \theta v' + \left(\mu - \frac{\alpha^2 \sinh^2 \theta}{2}\right) v = 0, \tag{28}
\]
where \( \alpha = \frac{2-n}{2} \). Equation (28) is a Legendre equation, whose solutions are \( P^\alpha_\ell \) and \( P^{-\alpha}_\ell \), where \( -\ell(\ell + 1) = \mu \).

It follows from equation (6) that the regular solution to equation (26) is
\[
\varphi(\theta) = \tanh^{-\alpha} \left(\frac{\theta}{2}\right) P^{-\alpha}_\ell (\cosh \theta).
\]
Since the solution must vanish at the boundary, it follows that \( L = L^* \). Thus, \( \mu = L^* \). This finishes the proof of Lemma 3.2.

\[\square\]

The proof of Theorem 3.1 now follows easily from a result by Lieb. In fact, by Lemma 1.2 in [5], it follows that if there exists some \( u \) such that \( Q_\lambda(u) < S_n \), then there exists a minimizer of \( Q_\lambda \). Given any constant \( \eta > 0 \), the quotient \( Q_\lambda(u) \) is invariant under the transformation \( u \to \eta u \). In order to compute the corresponding Euler equation, we minimize the numerator of equation (13) subject to the constraint \( \omega_n \int_0^R u^{\frac{2n}{n-2}} r^{n-1} dr = 1 \). We obtain
\[
\left(u'^\prime \rho^{n-2} r^{n-1}\right)' + \lambda u \rho^{n} r^{n-1} + \eta u^p \rho^{n} r^{n-1} = 0, \tag{29}
\]
where \( \eta \) is a Lagrange multiplier. Multiplying through by \( \omega_n u \), integrating between 0 and \( R \), and integrating by parts, we obtain
\[
\eta = \omega_n \left(\int_0^R u'^\prime \rho^{n-2} r^{n-1} dr - \lambda \int_0^R u^2 \rho^{n} r^{n-1} dr\right) \geq (\lambda_1 - \lambda) \omega_n \int_0^R u^2 \rho^{n} r^{n-1} dr.
\]
This last inequality follows from the variational characterization of \( \lambda_1 \). It follows that \( \eta > 0 \) provided that \( \lambda < \lambda_1 \). Setting \( u = \eta^{\frac{1}{p-1}} v \) in (29) one has that \( v \) satisfies
\[
\left(u'^' \rho^{n-2} r^{n-1}\right)' + \lambda u \rho^{n} r^{n-1} + u^p \rho^{n} r^{n-1} = 0. \tag{30}
\]
Finally, setting \( r = \tanh \frac{\theta}{2} \), equation (30) becomes (12). This finishes the proof of Theorem 3.1.
4. Nonexistence of solutions

In this section we use a Pohozaev type argument to show that if $2 < n < 4$ then problem (3) has a solution gap.

**Theorem 4.1.** For any $2 < n < 4$ and $\theta_1 \in (0, \infty)$, the boundary value problem

$$-u''(\theta) - (n - 1) \coth \theta u'(\theta) = \lambda u + u^{n+\frac{2}{n-2}}$$

with $u \in H^1_0(\Omega)$, $u'(0) = u(\theta_1) = 0$, and $\theta \in [0, \theta_1]$, has no solution if

$$\lambda \in \left(\frac{n(n - 2)}{4}, \frac{n(n - 2)}{4} + L^*\right).$$

Here $L^*$ is as in Definition 1.

**Proof.** Let $g$ be a smooth nonnegative function such that $g(0) = g'(0) = 0$. Writing equation (31) as

$$-\left(\sinh^{n-1} \theta u'\right)' \frac{\sinh^{n-1} \theta}{u} = \lambda u + u^p,$$

multiplying through by $g(\theta)u'(\theta) \sinh^{2n-2} \theta$, and integrating, we obtain

$$-\int_0^{\theta_1} \left(\frac{\sinh^{n-1} \theta u'}{2}\right)' g \, d\theta = \lambda \int_0^{\theta_1} \left(\frac{u^2}{2}\right)' g \sinh^{2n-2} \theta \, d\theta + \int_0^{\theta_1} \left(\frac{u^{p+1}}{p+1}\right)' g \sinh^{2n-2} \theta \, d\theta.$$

Integrating by parts, and since $u(\theta_1) = 0$, we obtain

$$\frac{1}{2} \int_0^{\theta_1} u^2 g' \sinh^{2n-2} \, d\theta + \frac{\lambda}{2} \int_0^{\theta_1} u^2 (g \sinh^{2n-2} \theta)' \, d\theta$$

$$+ \int_0^{\theta_1} \frac{u^{p+1}}{p+1} (g \sinh^{2n-2} \theta)' \, d\theta = \frac{\sinh^{2n-2} \theta_1 (u'(\theta_1))^2 g(\theta_1)}{2}.$$  \hspace{1cm} (34)

Let $f(\theta) = \frac{1}{2} g' \sinh^{n-1} \theta$. Multiplying equation (33) by $f(\theta)u(\theta) \sinh^{n-1} \theta$ and integrating, we obtain

$$-\int_0^{\theta_1} (\sinh^{n-1} \theta u')' f u \, d\theta = \lambda \int_0^{\theta_1} f \sinh^{n-1} \theta u^2 \, d\theta + \int_0^{\theta_1} u^{p+1} f \sinh^{n-1} \theta \, d\theta.$$  \hspace{1cm} (35)

After integrating by parts, this last equation can be written as

$$\int_0^{\theta_1} u^2 f \sinh^{n-1} \theta \, d\theta = \int_0^{\theta_1} u^2 \left(\lambda f \sinh^{n-1} \theta + \frac{1}{2} (f' \sinh^{n-1} \theta)'\right) \, d\theta$$

$$+ \int_0^{\theta_1} u^{p+1} f \sinh^{n-1} \theta \, d\theta.$$  \hspace{1cm} (35)

By subtracting equation (34) from equation (35) we obtain

$$\int_0^{\theta_1} A(\theta) u(\theta)^2 \, d\theta + \int_0^{\theta_1} B(\theta) u(\theta)^{p+1} \, d\theta = \frac{\sinh^{2n-2} \theta_1 (u'(\theta_1))^2 g(\theta_1)}{2},$$  \hspace{1cm} (36)
where
\[ A(\theta) \equiv \frac{1}{2} \left( f'(\theta) \sinh^{n-1} \theta \right)' + \lambda f(\theta) \sinh^{n-1} \theta + \frac{\lambda}{2} \left( g(\theta) \sinh^{2n-2} \theta \right)' \]
and
\[ B(\theta) = f(\theta) \sinh^{n-1} \theta + \frac{\left( g(\theta) \sinh^{2n-2}(\theta) \right)'}{p + 1} \]

Notice that the right-hand side of equation (36) is nonnegative. We will show that the left-hand side of (36) is negative, thus arriving at a contradiction.

Using the definition of \( f \) and simplifying, we can write
\[
A(\theta) = \sinh^{2n-2} \theta \left[ \frac{g'''}{4} + \frac{3}{4} (n-1) \coth \theta g'' + \left( \lambda + \frac{n-1}{4} + \frac{(n-1)(2n-3)}{4} \coth^2 \theta \right) g' + \lambda (n-1) \coth \theta g \right].
\]

Finally, making the change of variables \( T(\theta) = g(\theta) \sinh^2 \theta \), we obtain
\[
A(\theta) = \sinh^{2n-4} \theta \left[ \frac{T'''}{4} + \frac{3}{4} (n-3) \coth \theta T'' + \left( \frac{1}{4} \coth^2 \theta (n-3)(2n-11) + \lambda + \frac{1}{4} (n-7) \right) T' + (n-3) \left( \coth \theta (\lambda - 2) - \coth^3 \theta (n-4) \right) T \right].
\]

Simplifying \( B \), we obtain
\[
B(\theta) = \frac{(n-1) \sinh^{2n-2} \theta}{n} \left( g'(\theta) + (n-2) \coth \theta g \right).
\]

As before, we make the change of variables \( T(\theta) = g(\theta) \sinh^2 \theta \), to obtain
\[
B(\theta) = \frac{(n-1)}{n} \sinh^{2n-4} \theta \left( T' + (n-4) \coth \theta T \right).
\]

We will show that there is a choice of \( T \) for which \( A(\theta) \equiv 0 \). We will then show that for this choice of \( T \), \( B(\theta) \) is negative as long as
\[
\lambda \in \left( \frac{n(n-2)}{4}, \frac{n(n-2)}{4} + L^* \right). \tag{37}
\]

**Lemma 4.2.** Consider the equation
\[
\frac{T'''}{4} + \frac{3}{4} (n-3) \coth \theta T'' + \left( \frac{1}{4} \coth^2 \theta (n-3)(2n-11) + \lambda + \frac{1}{4} (n-7) \right) T' + (n-3) \left( \coth \theta (\lambda - 2) - \coth^3 \theta (n-4) \right) T = 0. \tag{38}
\]

Then
\[
T(\theta) = \sinh^{4-n} \theta P_\ell^\alpha(cosh \theta) P_{\ell-\alpha}(cosh \theta)
\]
is a solution of (38), where \( \alpha = (2-n)/2 \) and \( \ell(\ell+1) = \alpha(\alpha-1) - \lambda \).
Proof. Let \( v(\theta) = y_1(\theta)y_2(\theta) \), where \( y_1(\theta) = P_\ell^\alpha(\cosh \theta) \) and \( y_2(\theta) = P_\ell^{-\alpha}(\cosh \theta) \). Then \( y_1 \) and \( y_2 \) are solutions of
\[
y''(\theta) + \coth \theta y'(\theta) + k(\theta)y(\theta) = 0, \tag{39}
\]
where
\[
k(\theta) = -\ell(\ell + 1) - \frac{v^2}{\sinh^2 \theta}.
\]
It follows from equation (39) that
\[
y''_1 y_2 + y''_2 y_1 = -\coth \theta v' - 2kv,
\]
and from the above that
\[
v'' = 2y'_1 y'_2 - \coth \theta v' - 2kv.
\]
Similarly, we can write
\[
y'''_1 y'_2 + y'_1 y''_2 = -2 \coth \theta y'_1 y'_2 - kv',
\]
from which it follows that
\[
v''' = -\coth \theta v'' + \left( \frac{1}{\sinh^2 \theta} - 4k \right) v' - 2k'v - 4 \coth \theta y'_1 y'_2.
\]
Using the fact that \( y'_1 y'_2 = \frac{1}{2} (v'' + \coth \theta v' + 2kv) \), we obtain
\[
v''' + 3 \coth \theta v'' + \left( 2 \coth^2 \theta + 4k - \frac{1}{\sinh^2 \theta} \right) v' + (2k' + 4k \coth \theta) v = 0. \tag{40}
\]
Finally, replacing \( v(\theta) = T(\theta) \sinh^{n-4} \theta \) in equation (40), using the fact that \( \ell(\ell + 1) = \alpha(\alpha - 1) - \lambda \), and after significant simplification and rearrangement of terms, we obtain precisely equation (38).

It suffices now to show that for \( T \) as in the previous lemma, \( B \) is negative. We do so in the following lemma.

**Lemma 4.3.** Let
\[
T(\theta) = \sinh^{4-n} \theta P_\ell^\alpha(\cosh \theta)P_\ell^{-\alpha}(\cosh \theta)
\]
where \( \alpha = (2 - n)/2, \theta \in (0, \theta_1) \), and \( L = -\ell(\ell + 1) = \lambda - \alpha(\alpha - 1) \). Then
\[
B(\theta) = \frac{(n - 1)}{n} \sinh^{2n-4} \theta (T' + (n - 4) \coth \theta T) \tag{41}
\]
is negative if \( 0 < L \leq L^* \).

**Proof.** Notice that the condition \( 0 < L \leq L^* \) is precisely the same as (37). Substituting \( T(\theta) = \sinh^{4-n} \theta P_\ell^\alpha(\cosh \theta)P_\ell^{-\alpha}(\cosh \theta) \) in equation (41), we obtain
\[
B(\theta) = \frac{(n - 1)}{n} \sinh^{n+1} \theta \left( \dot{P}_\ell^\alpha P_\ell^{-\alpha} + P_\ell^\alpha \ddot{P}_\ell^{-\alpha} \right).
\]
Since \( \sinh \theta \) is positive for \( \theta > 0 \), and since \( P_\ell^\alpha P_\ell^{-\alpha} > 0 \) if \( 0 < L \leq L^* \), it suffices to show that
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\[
\frac{\dot{P}_\alpha}{P_\alpha} + \frac{\dot{P}_{-\alpha}}{P_{-\alpha}} < 0.
\]

Let

\[
y_\nu(\theta) = \frac{1}{\sinh \theta} \frac{P_\nu^{\nu+1}}{P_\nu^\nu} + \frac{\nu}{2 \sinh^2 \frac{\theta}{2}}.
\]

(42)

Then, by the raising relation given by equation (8) it follows that

\[
\frac{\dot{P}_\alpha}{P_\alpha} + \frac{\dot{P}_{-\alpha}}{P_{-\alpha}} = \frac{1}{\sinh \theta} \left( \frac{P_\alpha^{\alpha+1}}{P_\alpha^\alpha} + \frac{P_{-\alpha}^{\alpha+1}}{P_{-\alpha}^{-\alpha}} \right) = y_\alpha + y_{-\alpha}.
\]

We will show that for \( \theta \in (0, \theta_1) \), and if \(-1 < \nu < 1\), then \( y_\nu(\theta) < 0 \). This will imply that \( y_\alpha(\theta) + y_{-\alpha}(\theta) < 0 \), and therefore that \( B \) is negative.

From equations (6) and (7) it follows that

\[
P_\nu^\nu = \frac{1}{\Gamma(1 - \nu)} \coth^\nu \left( \frac{\theta}{2} \right) \left( 1 + \frac{\ell(\ell + 1)}{1 - \nu} \sinh^2 \left( \frac{\theta}{2} \right) + O \left( \sinh^4 \left( \frac{\theta}{2} \right) \right) \right).
\]

Then, and since \( \Gamma(1 - \nu) = -\nu \Gamma(-\nu) \), we can write

\[
\frac{P_\nu^{\nu+1}}{P_\nu^\nu} = -\nu \coth \left( \frac{\theta}{2} \right) \left( 1 - \frac{\ell(\ell + 1)}{\nu(1 - \nu)} \sinh^2 \left( \frac{\theta}{2} \right) + O \left( \sinh^4 \left( \frac{\theta}{2} \right) \right) \right).
\]

Therefore, and since \( \coth \left( \frac{\theta}{2} \right) / \sinh \theta = \left( 2 \sinh^2 \left( \frac{\theta}{2} \right) \right)^{-1} \), we have

\[
y_\nu = \frac{\ell(\ell + 1)}{2(1 - \nu)} + O \left( \sinh^2 \left( \frac{\theta}{2} \right) \right).
\]

Thus, if \(-1 < \nu < 1\), and since \( \ell(\ell + 1) < 0 \),

\[
\lim_{\theta \to 0} y_\nu(\theta) = \frac{\ell(\ell + 1)}{2(1 - \nu)} < 0.
\]

We will show by contradiction that there is no point at which \( y_\nu \) changes sign, thus concluding that \( y_\nu(\theta) \) is negative for all \( \theta > 0 \).

Taking the derivative of equation (42), we obtain

\[
y_\nu' = \frac{-\cosh \theta \ P_\nu^{\nu+1}}{\sinh^2 \theta \ P_\nu^\nu} + \frac{\dot{P}_\nu^{\nu+1}}{P_\nu^\nu} - \frac{\dot{P}_\nu^{\nu+1}}{P_\nu^\nu} - \frac{\nu \cosh \left( \frac{\theta}{2} \right)}{2 \sinh^3 \left( \frac{\theta}{2} \right)}.
\]

Using the raising and lowering relations given in equations (8) and (9), we can write

\[
y_\nu' = \frac{-1}{\sinh \theta} \left( \frac{P_\nu^{\nu+1}}{P_\nu^\nu} \right)^2 + \frac{(-2\nu - 2) \cosh \theta}{\sinh^2 \theta} \left( \frac{P_\nu^{\nu+1}}{P_\nu^\nu} \right) + \frac{\ell(\ell + 1) - \nu(\nu + 1)}{\sinh \theta} - \frac{\nu \cosh \left( \frac{\theta}{2} \right)}{2 \sinh^3 \left( \frac{\theta}{2} \right)}.
\]
Solving for \( \frac{P_{\nu+1}}{P_\nu} \) from equation (42), and after rearranging terms, we obtain

\[
y'_\nu = -\sinh \theta y^2_\nu + \frac{2(\nu - \cosh \theta)}{\sinh \theta} y_\nu + \frac{\ell(\ell + 1)}{\sinh \theta}.
\] (43)

Now suppose there was a point \( \theta^* \) at which \( y_\nu(\theta^*) \) crossed the \( \theta \)-axis. At this point, we would have \( y_\nu(\theta^*) = 0 \) and \( y'_\nu(\theta^*) > 0 \). But evaluating equation (43) at \( \theta^* \), we obtain

\[
y'_\nu(\theta^*) = \frac{\ell(\ell + 1)}{\sinh \theta^*} < 0,
\]

arriving at a contradiction.

This completes the proof of Theorem 4.1.

5. Uniqueness

Lemma 5.1. The problem

\[
u''(\theta) + (n - 1) \coth(\theta) u'(\theta) + \lambda u(\theta) + u(\theta)^p = 0
\] (44)

with \( u'(0) = u(\theta_1) = 0 \), \( 2 < n < 4 \), and \( \lambda > \frac{n(n-2)}{4} \), has at most one positive solution.

Proof. The proof of this lemma follows directly from [10]. In fact, making the change of variables \( u \to v \) given by \( u(\theta) = \sinh^\alpha(\theta)v(\theta) \), where \( \alpha = \frac{2-n}{2} \), equation (44) can be written as

\[
\sinh^2(\theta)v''(\theta) + \sinh \theta \cosh \theta v'(\theta) + G_\lambda(\theta)v(\theta) + v(\theta)^p = 0,
\] (45)

where

\[
G_\lambda(\theta) = -\alpha^2 + \left[ \lambda - \frac{n(n-2)}{4} \right] \sinh^2 \theta.
\]

We define the energy function

\[
E[v] \equiv \sinh^2 \theta v'(\theta)^2 + \frac{2}{p+1} v(\theta)^{p+1} + G_\lambda(\theta)v(\theta)^2 = 0.
\]

Then if \( v(\theta) \) is a solution of (45),

\[
\frac{dE}{d\theta} = G'_\lambda(\theta)v(\theta)^2.
\]

The function \( G_\lambda(\theta) \) is increasing as long as \( \lambda > \frac{n(n-2)}{4} \). That is, \( G_\lambda(\theta) \) is a \( \Lambda - function \) and it follows from [10] that \( v \) (and therefore \( u \)) is unique.

Remark 5.2. Uniqueness of solutions to this problem for \( \lambda \in (n(n-2)/4, (n-1)^2/4] \) was obtained by Mancini and Sandeep (see Proposition 4.4 in [11]). Notice that \( \lambda = (n-1)^2/4 \) corresponds to the first eigenvalue in the limiting case \( \theta_1 = \infty \). The interval considered in [11] is a strict subinterval of the interval we consider here.
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1 DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN - MADISON