SOME NEW RESULTS FOR SUBSEQUENCES OF NÖRLUND LOGARITHMIC MEANS OF WALSH-FOURIER SERIES

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Abstract. We prove that there exists a martingale \( f \in H_p \) such that the subsequence \( \{L_n f\} \) of Nörlund logarithmic means with respect to the Walsh system are not bounded in the Lebesgue space weak \( -L_p \) for 0 < \( p \) < 1. Moreover, we prove that for any \( f \in L_p(G), p \geq 1 \), \( L_n f \) converge to \( f \) at any Lebesgue point \( x \). Some new related inequalities are derived.

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1. Introduction

The terminology and notations used in this introduction can be found in Section 2.

It is well-known that Vilenkin systems do not form bases in the space \( L_1 \). Moreover, there is a function in the Hardy space \( H_1 \), such that the partial sums of \( f \) are not bounded in the \( L_1 \)-norm. Moreover, (see Tephnadze [28]) there exists a martingale \( f \in H_p \) (0 < \( p \) < 1), such that

\[
\sup_{n \in \mathbb{N}} \| S_{2^n+1} f \|_{\text{weak} - L_p} = \infty.
\]

The reason of the divergence of \( S_{2^n+1} f \) is that when 0 < \( p \) < 1 the Fourier coefficients of \( f \in H_p \) are not uniformly bounded (see Tephnadze [27]).

On the other hand, (for details see e.g. the books [23] and [33]) the subsequence \( \{S_{2^n}\} \) of partial sums is bounded from the martingale
Hardy space $H_p$ to the space $H_p$, for all $p > 0$, that is the following inequality holds:

(1) \[ \| S_{2^n} f \|_{H_p} \leq c_p \| f \|_{H_p}, \quad n \in \mathbb{N}, \quad p > 0. \]

It is also well-known that (see [23])

(2) \[ S_{2^n} f(x) \to f(x), \quad \text{for all Lebesgue points of } f \in L_p(G), \quad \text{where } p \geq 1. \]

Weisz [35] considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved that the inequality

(3) \[ \| \sigma_k f \|_p \leq c_p \| f \|_{H_p}, \quad p > 1/2 \quad \text{and} \quad f \in H_p, \]

holds. Moreover, Goginava [8] (see also [22] and [26]) proved that the assumption $p > 1/2$ in (3) is essential. In particular, he showed that there exists a martingale $f \in H_{1/2}$ such that $\sup_{n \in \mathbb{N}} \| \sigma_n f \|_{1/2} = +\infty$. However, Weisz [35] (see also [19]) proved that for every $f \in H_p$, there exists an absolute constant $c_p$, such that the following inequality holds:

(4) \[ \| \sigma_{2^n} f \|_{H_p} \leq c_p \| f \|_{H_p}, \quad n \in \mathbb{N}, \quad p > 0. \]

Móricz and Siddiqi [16] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of $L_p$ functions in norm. Similar results for the two-dimensional case can be found in Nagy [18, 17]. Approximation properties for general summability methods can be found in [4, 3]. Fridli, Manchanda and Siddiqi [6] improved and extended the results of Móricz and Siddiqi [16] to martingale Hardy spaces. The case when $\{q_k = 1/k : k \in \mathbb{N}\}$ was excluded, since the methods are not applicable to Nörlund logarithmic means. In [7] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the Lebesgue space $L_1$. In particular, they proved that there exists an function in the space $L_1$, such that $\sup_{n \in \mathbb{N}} \| L_n f \|_1 = \infty$.

In [20] (see also [5]) it was proved that there exists a martingale $f \in H_p$, $(0 < p < 1)$ such that $\sup_{n \in \mathbb{N}} \| L_n f \|_p = \infty$.

In [21] (see also [31]) it was proved that there exists a martingale $f \in H_1$ such that

(5) \[ \sup_{n \in \mathbb{N}} \| L_n f \|_1 = \infty. \]

However, Goginava [9] proved that

\[ \| L_{2^n} f \|_1 \leq c \| f \|_1, \quad f \in L_1, \quad n \in \mathbb{N}. \]
From this result it immediately follows that for every $f \in H_1$, there exists an absolute constant $c$, such that the inequality

$$
\|L_{2^n}f\|_1 \leq c \|f\|_{H_1}
$$

holds for all $n \in \mathbb{N}$. Goginava [9] also proved that for any $f \in L^1(G)$,

$$
L_{2^n}f(x) \to f(x), \quad \text{a.e., as } n \to \infty.
$$

According to (1), (4) and (6), the following question is quite natural:

**Question 1.** Is the subsequence $\{L_{2^n}\}$ also bounded on the martingale Hardy spaces $H_p(G)$ when $0 < p < 1$?

In Theorem 2 of this paper we give a negative answer to this question. In particular, we further develop some methods considered in [2, 11] and prove that for any $0 < p < 1$, there exists a martingale $f \in H_p$ such that $\sup_{n \in \mathbb{N}} \|L_{2^n}f\|_{weak-L_p} = \infty$. Moreover, in our Theorem 1 we generalize the result of Goginava [9] and prove that for any $f \in L^1(G)$ and for any Lebesgue point $x$,

$$
L_{2^n}f(x) \to f(x), \quad \text{as } n \to \infty.
$$

The main results in this paper are presented and proved in Section 4. Section 3 is used to present some auxiliary lemmas, where, in particular, Lemma 2 is new and of independent interest. In order not to disturb our discussions later on some definitions and notations are given in Section 4. Finally, Section 5 is reserved for some open questions we hope can be a source of inspiration for further research in this interesting area.

2. Definitions and Notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by $Z_2$ the discrete cyclic group of order 2, that is $Z_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $Z_2$ is given so that the measure of a singleton is $1/2$.

Define the group $G$ as the complete direct product of the group $Z_2$, with the product of the discrete topologies of $Z_2$'s. The elements of $G$ are represented by sequences $x := (x_0, x_1, ..., x_j, ...)$, where $x_k = 0 \lor 1$.

It is easy to give a base for the neighborhood of $x \in G$ namely:

$$
I_0(x) := G, \quad I_n(x) := \{y \in G : y_0 = x_0, ..., y_{n-1} = x_{n-1}\} \quad (n \in \mathbb{N}).
$$

Denote $I_n := I_n(0)$, $\overline{I}_n := G \setminus I_n$ and $e_n := (0, ..., 0, x_n = 1, 0, ...) \in G$, for $n \in \mathbb{N}$. It is easy to show that $\overline{I}_M = \bigcup_{s=0}^{M-1} I_s \setminus I_{s+1}$. 
If \( n \in \mathbb{N} \), then every \( n \) can be uniquely expressed as \( n = \sum_{k=0}^{\infty} n_j 2^j \), where \( n_j \in \mathbb{Z}_2 \) \( (j \in \mathbb{N}) \) and only a finite numbers of \( n_j \) differ from zero. Let \( |n| := \max\{k \in \mathbb{N} : n_k \neq 0\} \).

The norms (or quasi-norms) of the spaces \( L_p(G) \) and \( \text{weak} - L_p(G) \), \((0 < p < \infty)\) are, respectively, defined by

\[
\|f\|_p^p := \int_G |f|^p \, d\mu, \quad \|f\|_{\text{weak}-L_p}^p := \sup_{\lambda>0} \lambda^p \mu(f > \lambda).
\]

The \( k \)-th Rademacher function is defined by

\[
r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}).
\]

Now, define the Walsh system \( w := (w_n : n \in \mathbb{N}) \) on \( G \) as:

\[
w_n(x) := \prod_{k=0}^{n-1} r_{n_k}(x) = r_{|n|}(x) (-1)^{\sum_{k=0}^{n-1} n_k x_k} \quad (n \in \mathbb{N}).
\]

It is well-known that (see e.g. [23])

\[
w_n (x + y) = w_n (x) w_n (y).
\]

The Walsh system is orthonormal and complete in \( L_2(G) \) (see e.g. [23]).

If \( f \in L_1(G) \) we can establish Fourier coefficients, partial sums of the Fourier series, Dirichlet kernels with respect to the Walsh system in the usual manner:

\[
\hat{f}(k) := \int_G f w_k \, d\mu \quad (k \in \mathbb{N}),
\]

\[
S_nf := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \quad D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+).
\]

Recall that (for details see e.g. [1])

\[
D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases}
\]

and

\[
D_n = w_n \sum_{k=0}^{\infty} n_k r_k D_{2^k} = w_n \sum_{k=0}^{\infty} n_k (D_{2^{k+1}} - D_{2^k}), \quad \text{for } n = \sum_{i=0}^{\infty} n_i 2^i.
\]

Let \( \{q_k : k \geq 0\} \) be a sequence of nonnegative numbers. The Nörlund means for the Fourier series of \( f \) are defined by

\[
\frac{1}{l_n} \sum_{k=0}^{n} q_{n-k} S_k f.
\]
In the special case when \( \{q_k = 1 : k \in \mathbb{N}\} \), we get the Fejér means

\[ \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f. \]

If \( q_k = 1/(k+1) \), then we get the Nörlund logarithmic means:

\[ L_n f := \frac{1}{l_n} \sum_{k=0}^{n-1} \frac{S_k f}{n-k}, \quad l_n := \sum_{k=1}^{n} \frac{1}{k}. \]

The \( n \)-th Riesz logarithmic mean of the Fourier series of the integrable function \( f \) is defined by

\[ R_n f := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f}{k}, \quad l_n := \sum_{k=1}^{n} \frac{1}{k}. \]

We note that it is an inverse of the Nörlund logarithmic means.

The convolution of two functions \( f, g \in L_1(G) \) is defined by

\[ (f \ast g)(x) := \int_{G} f(x+t)g(t) \, d\mu(t) \quad (x \in G). \]

It is well-known that if \( f \in L_p(G), g \in L_1(G) \) and \( 1 \leq p < \infty \). Then \( f \ast g \in L_p(G) \) and the corresponding inequality holds:

\[ \|f \ast g\|_p \leq \|f\|_p \|g\|_1. \]

The representations

\[ L_n f(x) = \int_{G} f(t) P_n(x + t) \, d\mu(t) \quad \text{and} \quad R_n f(x) = \int_{G} f(t) Y_n(x + t) \, d\mu(t) \]

for \( n \in \mathbb{N} \) play a central role in the sequel, where

\[ P_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k \quad \text{and} \quad Y_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_k D_k \]

are called the kernels of the Nörlund logarithmic and the Riesz means, respectively.

It is well-known that (see e.g. Goginava [9] and Tephnadze [30]):

\[ P_{2n}(x) = D_{2n}(x) - \psi_{2n-1}(x)Y_{2n}(x). \]

Moreover, for all \( n \in \mathbb{N} \),

\[ \|P_{2n}\|_1 < c < \infty \quad \text{and} \quad \|Y_n\|_1 < c < \infty. \]

In the case \( f \in L_1(G) \) the maximal functions are given by

\[ M(f)(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \, d\mu(u) \right| = \sup_{n \in \mathbb{N}} 2^n \left| \int_{I_n(x)} f(u) \, d\mu(u) \right|. \]
It is well-known (for details see e.g. \[23\] and \[32\]) that if \(f \in L_1(G)\), then the inequality
\[
\|M(f)\|_{\text{weak}-L_1} \leq \|f\|_1.
\]
holds. According to a density argument of Calderon-Zygmund (see \[32\]) we obtain that if \(f \in L_1(G)\), then
\[
2^n \left| \int_{I_n(x)} f(u) \, d\mu(u) \right| \to 0, \quad \text{as } n \to \infty.
\]
A point \(x\) on the Walsh group is called a Lebesgue point of \(f \in L_1(G)\), if
\[
\lim_{n \to \infty} 2^n \int_{I_n(x)} f(t) \, d\mu(t) = f(x) \quad \text{a.e. } x \in G.
\]
According to \[2\] we find that if \(f \in L_1(G)\), then a.e point is a Lebesgue point.
The \(\sigma\)-algebra generated by the intervals \(\{I_n(x) : x \in G\}\) is denoted by \(F_n(n \in \mathbb{N})\). Let \(f := \{f^{(n)} : n \in \mathbb{N}\}\) be a martingale with respect to \(F_n(n \in \mathbb{N})\) (for details see e.g. \[33\]).
The maximal function of a martingale \(f\) is defined by
\[
f^* := \sup_{n \in \mathbb{N}} |f^{(n)}|.
\]
For \(0 < p < \infty\) the Hardy martingale spaces \(H_p(G)\) consist of all martingales for which
\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]
If \(f \in L_1(G)\), then it is easy to show that the sequence \(F = (S_{2^n} f : n \in \mathbb{N})\) is a martingale and \(F^* = M(f)\).
If \(f = \{f^{(n)} : n \in \mathbb{N}\}\) is a martingale, then the Walsh-Fourier coefficients must be defined in a slightly different manner:
\[
\hat{f}(i) := \lim_{k \to \infty} \int_G f^{(k)}(x) w_i(x) \, d\mu(x).
\]
The Walsh-Fourier coefficients of \(f \in L_1(G)\) are the same as those of the martingale \((S_{2^n} f : n \in \mathbb{N})\) obtained from \(f\).
A bounded measurable function \(a\) is a \(p\)-atom if there exists an interval \(I\) such that
\[
\int_I a \, d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.
\]
3. Auxiliary Results

The Hardy martingale space $H_p(G)$ has an atomic characterization (see Weisz [33, 34]):

**Lemma 1.** A martingale $f = (f^{(n)}, n \in \mathbb{N})$ is in $H_p(0 < p \leq 1)$ if and only if there exist a sequence $(a_k, k \in \mathbb{N})$ of $p$-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of real numbers such that for every $n \in \mathbb{N}$:

$$\sum_{k=0}^{\infty} \mu_k S_{2^n} a_k = f^{(n)}, \quad \text{where} \quad \sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$  

Moreover,

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p\right)^{1/p},$$

where the infimum is taken over all decompositions of $f$ of the form (14).

We also state and prove the following new lemma of independent interest:

**Lemma 2.** Let $n \in \mathbb{N}$ and $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$. Then

$$\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} \frac{D_j}{2^{2\alpha_k+1} - j} = \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k-1}} \frac{w_{2j+1}}{2^{2\alpha_k+1} - 2j - 1} = \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k-1}} \frac{w_j}{2^{2\alpha_k+1} - 2j - 1} \geq \frac{1}{3}.$$ 

**Proof.** Let $x \in I_2(e_0 + e_1) \in I_0 \setminus I_1$. According to (8) and (9) we can conclude that

$$D_j(x) = \begin{cases} w_j, & \text{if } j \text{ is odd number}, \\ 0, & \text{if } j \text{ is even number}, \end{cases}$$

and

$$\sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1} - 1} \frac{D_j}{2^{2\alpha_k+1} - j} = \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k-1}} \frac{w_{2j+1}}{2^{2\alpha_k+1} - 2j - 1} = w_1 \sum_{j=2^{2\alpha_k-1}+1}^{2^{2\alpha_k-1}} \frac{w_{2j}}{2^{2\alpha_k+1} - 2j - 1}.$$
Since
\[
\sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}-1} \left| \frac{1}{2^{2\alpha_k+1} - 4j + 3} - \frac{1}{2^{2\alpha_k+1} - 4j + 1} \right| ≤ \sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}-1} \frac{2}{(2^{2\alpha_k+1} - 4j + 3)(2^{2\alpha_k+1} - 4j + 1)} ≤ \frac{1}{8} \sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}-1} \frac{2}{(2^{2\alpha_k+1} - 4j)(2^{2\alpha_k+1} - 4j)} ≤ \frac{1}{8} \left( \frac{1}{k-1} - \frac{1}{k} \right) \leq \frac{1}{8} + \frac{1}{8} = \frac{1}{4},
\]
according to
\[
w_{4k+2} = w_2 w_{4k} = -w_{4k}, \quad \text{for} \quad x \in I_2(e_0 + e_1),
\]
we can conclude that
\[
\begin{align*}
&\left| w_{2^{2\alpha_k+1}-2} + \frac{w_{2^{2\alpha_k+1}-4}}{3} + \sum_{j=2^{2\alpha_k-1+1}}^{2^{2\alpha_k-1}} \frac{w_{2j}}{2^{2\alpha_k+1} - 2j - 1} \right| \\
&= \left| \frac{w_{2^{2\alpha_k+1}-4}}{3} - w_{2^{2\alpha_k+1}-4} + \sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}} \left( \frac{w_{4j-4}}{2^{2\alpha_k+1} - 4j + 3} + \frac{w_{4j-2}}{2^{2\alpha_k+1} - 4j + 1} \right) \right| \\
&= \left| \frac{2w_{2^{2\alpha_k+1}-4}}{3} - \sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}} \left( \frac{w_{4j-4}}{2^{2\alpha_k+1} - 4j + 3} - \frac{w_{4j-4}}{2^{2\alpha_k+1} - 4j + 1} \right) \right| \\
&≥ \frac{2}{3} - \sum_{j=2^{2\alpha_k-2+1}}^{2^{2\alpha_k-1}} \left| \frac{1}{2^{2\alpha_k+1} - 4j + 3} - \frac{1}{2^{2\alpha_k+1} - 4j + 1} \right| \\
&≥ \frac{2}{3} - \frac{1}{4} = \frac{5}{12} \geq \frac{1}{3},
\end{align*}
\]
The proof is complete. \(\square\)
4. Main results

Our first main result reads:

**Theorem 1.** Let $p \geq 1$ and $f \in L_p(G)$. Then

\begin{equation}
\|L_{2^n} f - f\|_p \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

Moreover,

\begin{equation}
\lim_{n \to \infty} L_{2^n} f(x) = f(x),
\end{equation}

for all Lebesgue points of $f$.

**Proof.** Let $n \in \mathbb{N}$. By combining (11) and (13) we immediately get

\[ \|L_{2^n} f\|_p \leq c_p \|f\|_p \quad \text{for all} \quad n \in \mathbb{N}, \]

which immediately implies (15).

To prove a.e convergence we use identity (12) to obtain that

\begin{equation}
L_{2^n} f(x) = \int_G f(t) P_{2^n}(x + t) d\mu(t)
\end{equation}

\[ = \int_G f(t) D_{2^n}(x + t) d\mu(t) \]

\[ - \int_G f(t) w_{2^n-1}(x + t) Y_{2^n}(x + t) \]

\[ := I - II. \]

By applying (2) we can conclude that

\begin{equation}
I = S_{2^n} f(x) \to f(x)
\end{equation}

for all Lebesgue points of $f \in L_p(G)$. Moreover, by using (17) we find that

\[ II = \psi_{2^n-1}(x) \int_G f(t) Y_{2^n}(x + t) \psi_{2^n-1}(t) d(t). \]

In view of (13) we see that

\[ f(t) Y_{2^n}(x + t) \in L_p \quad \text{where} \quad p \geq 1 \quad \text{for any} \quad x \in G, \]
and also note that $II$ describes the Fourier coefficients of an integrable function. Hence, according to the Riemann-Lebesgue Lemma it vanishes as $n \to \infty$, i.e.

$$II \to 0 \text{ for any } x \in G, \ n \to \infty.$$ \hfill (19)

The proof of (16) follows by just combining (17)-(19).

The proof is complete. \hfill \Box

Our next main result is the following answer of question 1:

**Theorem 2.** Let $0 < p < 1$. Then there exists a martingale $f \in H_p$ such that

$$\sup_{n \in \mathbb{N}} \|L_{2^n} f\|_{weak-L_p} = \infty.$$

**Proof.** Let $\{\alpha_k : k \in \mathbb{N}\}$ be an increasing sequence of the positive integers such that

$$\sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty,$$

$$\sum_{\eta=0}^{k-1} \frac{(2^{2\alpha_k})^{1/p}}{\sqrt{\alpha_\eta}} < \frac{(2^{2\alpha_k})^{1/p}}{\sqrt{\alpha_k}},$$

$$\frac{(2^{2\alpha_{k-1}})^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{2^{2\alpha_k-8}}{\alpha_k^{1/2}l_{2^{2\alpha_k+1}}}.$$ \hfill (20)-(22)

We note that such an increasing sequence $\{\alpha_k : k \in \mathbb{N}\}$ which satisfies conditions (20)-(22) can obviously be constructed.

Let

$$f^{(n)}(x) := \sum_{\{k; 2\alpha_k < n\}} \lambda_k \alpha_k,$$

where

$$\lambda_k = \frac{1}{\sqrt{\alpha_k}}$$

and

$$\alpha_k = 2^{2\alpha_k(1/p-1)} (D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}).$$

From (20) and Lemma \hfill \Box we can conclude that $f = (f^{(n)}, n \in \mathbb{N}) \in H_p$. 

It is easy to show that

\[
\hat{f}(j) = \begin{cases} 
\frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}}, & \text{if } j \in \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}, \ k \in \mathbb{N}, \\
0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{2^{2\alpha_k}, \ldots, 2^{2\alpha_k+1} - 1\}. 
\end{cases}
\]

Moreover,

\[
L_{2^{2\alpha_k+1}} f = \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=1}^{2^{2\alpha_k}-1} \frac{S_j f}{2^{2\alpha_k+1} - j} + \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{S_j f}{2^{2\alpha_k+1} - j} := I + II.
\]

Let \( j < 2^{2\alpha_k} \). Then from (21), (22) and (23) we have

\[
|S_j f(x)| \leq \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} |\hat{f}(v)| \leq \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \\
\leq \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_k}/p}{\sqrt{\alpha_k}} \leq \frac{2^{2\alpha_k-4}}{\alpha_k^{1/2} l_{2^{2\alpha_k+1}}}.
\]

Consequently,

\[
|I| \leq \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=1}^{2^{2\alpha_k}-1} \frac{|S_j f(x)|}{2^{2\alpha_k+1} - j} \\
\leq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k-4}}{\alpha_k^{1/2}} \sum_{j=1}^{2^{2\alpha_k-1}/p} \frac{1}{j} \leq \frac{2^{2\alpha_k-1}/p}{\sqrt{\alpha_k-1}}.
\]

Let \( 2^{2\alpha_k} \leq j \leq 2^{2\alpha_k+1} - 1 \). Then we have the following equality

\[
S_j f = \sum_{\eta=0}^{k-1} \sum_{v=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \hat{f}(v)w_v + \sum_{v=2^{2\alpha_k}}^{j-1} \hat{f}(v)w_v \\
= \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} (D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}) + \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} (D_j - D_{2^{2\alpha_k}}).
\]
This gives that

\[(26)\]
\[II = \frac{1}{l_{2^{2\alpha_k+1}}} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}} \frac{1}{2^{2\alpha_k+1} - j} \left( \sum_{\eta=0}^{k-1} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} (D_{2^{2\alpha_k+1}} - D_{2^{2\alpha_k}}(j)) \right) + \frac{1}{l_{2^{2\alpha_k+1}}} 2^{2\alpha_k(1/p-1)} \sum_{j=2^{2\alpha_k}}^{2^{2\alpha_k+1}-1} \frac{w_{2j+1}}{2^{2\alpha_k+1} - 2j - 1} := II_1 + II_2.\]

Let \(x \in I_2(e_0 + e_1) \in I_0 \setminus I_1\). Since \(\alpha_0 \geq 1\) we obtain that \(2\alpha_k \geq 2\), for all \(k \in \mathbb{N}\) and if we apply \(\text{(8)}\) we get that \(D_{2^{2\alpha_k}} = 0\).

\[(27)\] \[II_1 = 0\]

and

\[(28)\] \[|II_2| \geq \frac{1}{3} \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)}}{\sqrt{\alpha_k}} \geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-1}}{\sqrt{\alpha_k}}.\]

By combining \(\text{(22)}, \text{(24)}-(28)\) for \(x \in I_2(e_0 + e_1)\) and \(0 < p < 1\) we have that

\[|L_{2^{2\alpha_k+1}} f(x)| \geq II_2 - II_1 - \mathcal{I} \geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-2}}{\sqrt{\alpha_k}} - \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-3}}{\sqrt{\alpha_k}} \geq \frac{1}{l_{2^{2\alpha_k+1}}} \frac{2^{2\alpha_k(1/p-1)-3}}{2^{2\alpha_k(1/p-1)-1}} \geq \frac{\ln(2^{2\alpha_k+1} + 1)}{\sqrt{\alpha_k}} \frac{2^{2\alpha_k(1/p-1)-3}}{(4\alpha_k + 1)\sqrt{\alpha_k}} \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}}.\]
Hence, we can conclude that
\[
\left\| L_{\alpha_k} f \right\|_{\text{weak-}\, L_p} \\
\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \mu \left\{ x \in G : |L_{2^{2\alpha_k+1}} f| \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \right\}^{1/p} \\
\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \mu \left\{ x \in I_2(e_0 + e_1) : |L_{2^{2\alpha_k+1}} f| \geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \right\}^{1/p} \\
\geq \frac{2^{2\alpha_k(1/p-1)-6}}{\alpha_k^{3/2}} \left( \mu \left( x \in I_2(e_0 + e_1) \right) \right)^{1/p} \\
> \frac{c^{2\alpha_k(1/p-1)}}{\alpha_k^{3/2}} \to \infty, \quad \text{as } k \to \infty.
\]

The proof is complete. \qed

5. Open questions

It is known (for details see e.g. the books \cite{23} and \cite{33}) that the subsequence \( \{S_{2^n}\} \) of the partial sums is bounded from the martingale Hardy space \( H_p \) to the Lebesgue space \( L_p \), for all \( p > 0 \). On the other hand, (see Tephnadze \cite{28}) there exists a martingale \( f \in H_p \) \((0 < p < 1)\), such that
\[
\sup_{n \in \mathbb{N}} \|S_{2^{n+1}} f\|_{\text{weak-}\, L_p} = \infty.
\]

However, Simon \cite{24} proved that for all \( f \in H_p \), there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1).
\]

In \cite{31} it was proved that for all \( f \in H_p \), there exists an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \frac{\|L_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1).
\]

**Open Problem 1:** a) Let \( f \in H_p \), where \( 0 < p < 1 \). Does there exist an absolute constant \( c_p \), depending only on \( p \), such that the following inequality holds:
\[
\sum_{k=1}^{\infty} \frac{\log^p k \|L_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p, \quad (0 < p < 1)?
\]

b) For \(0 < p < 1/2\) and any non-decreasing function \(\Phi : \mathbb{N} \to [1, \infty)\) satisfying the conditions

\[
\lim_{n \to \infty} \Phi (n) = +\infty,
\]
is it possible to find a martingale \(f \in H_p (G_m)\) such that

\[
\sum_{n=1}^{\infty} \frac{\log^p n \|L_n f\|_p^p \Phi (n)}{n^{2-p}} = \infty?
\]

**Open Problem 2:** a) Let \(f \in H_p\) where \(0 < p \leq 1\) and

\[
\omega_{H_p} \left( \frac{1}{M_n}, f \right) = o \left( \frac{\log n}{2n(1/p-1) \log^2 \|p\| n} \right), \quad \text{as} \quad n \to \infty.
\]

Does the following convergence result hold:

\[
\|L_k f - f\|_{H_p} \to 0, \quad \text{as} \quad k \to \infty?
\]

b) Let \(0 < p \leq 1\). Does there exist a martingale \(f \in H_p\), for which

\[
\omega_{H_p} \left( \frac{1}{2^n}, f \right) = O \left( \frac{\log n}{2n(1/p-1) \log^2 \|p\| n} \right), \quad \text{as} \quad n \to \infty
\]

and

\[
\|L_k f - f\|_{weak-L_p} \to 0, \quad \text{as} \quad k \to \infty?
\]

Reisz logarithmic means have better approximation properties then the Fejér means and if it is converging in some sense then the Fejér means converge in the same sense. Moreover, it has similar boundedness properties as the Fejér means when we consider \((H_p, L_p)\) and \((H_p, weak - L_p)\) type inequalities for the maximal operators of Riesz logarithmic means for \(0 < p \leq 1\). In particular, it was proved in [29] that the maximal operator of the Riesz logarithmic means of Vilenkin-Fourier series is bounded from the Hardy space \(H_{1/2}(G_m)\) to the space \(weak - L_{1/2}(G_m)\). It follows that it is bounded from the martingale Hardy space \(H_p(G_m)\) to the space \(L_p(G_m)\) when \(p > 1/2\). On the other hand, (for details see [30]) boundedness does not hold from the martingale Hardy space \(H_p(G_m)\) to the space \(L_p(G_m)\) when \(0 < p \leq 1/2\). Moreover, (see [30]) there exists a martingale \(f \in H_p(G_m)\), where \(0 < p < 1/2\) such that
In the endpoint case $p = 1/2$ it is open problem to prove divergence of the Riesz logarithmic means:

**Open Problem 3:** Does there exist a martingale $f \in H_{1/2}(G_m)$, such that

$$\sup_{n \in \mathbb{N}} \|R_n f\|_{1/2} = \infty?$$

According to estimate (4) it is interesting to consider boundedness of $R_{M_n} f$ from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $0 < p \leq 1/2$, so we pose the following:

**Open Problem 4:** Does there exist a martingale $f \in H_p(G_m)$, where $0 < p < 1/2$, such that

$$\sup_{n \in \mathbb{N}} \|R_{M_n} f\|_p = \infty?$$

If we prove this result we immediately find that the maximal operator

$$\sup_{n \in \mathbb{N}} |R_{M_n} f|$$

is not bounded from the martingale Hardy space $H_p(G_m)$ to the space $L_p(G_m)$ when $0 < p < 1/2$.

**Availability of data and material**

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**Competing interests**

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