Domain Theory for Modeling OOP: A Summary
(Domain Theory for The Construction of NOOP, and The Construction of COOP as a Step Towards Constructing NOOP)

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Abstract

Domain theory is ‘a mathematical theory that serves as a foundation for the semantics of programming languages’ [A94]. Domains form the basis of a theory of partial information, which extends the familiar notion of partial function to encompass a whole spectrum of “degrees of definedness”, so as to model incremental higher-order computation (i.e., computing with infinite data values, such as functions defined over an infinite domain like the domain of integers, infinite trees, and such as objects of object-oriented programming\(^1\)). General considerations from recursion theory dictate that partial functions are unavoidable in any discussion of computability. Domain theory provides an appropriately abstract setting in which the notion of a partial function can be lifted and used to give meaning to higher types, recursive types, etc.

NOOP is a domain-theoretic model of nominally-typed OOP [Abd12, Abd13, Abd14, CA13, AC14]. NOOP was used to prove the identification of inheritance and subtyping in mainstream nominally-typed OO programming languages and the validity of this identification. In this report we first present the definitions of basic domain theoretic notions and domain constructors used in the construction of NOOP, then we present the construction of a simple structural model of OOP called COOP as a step towards the construction of NOOP. Like the construction of NOOP, the construction of COOP uses earlier presented domain constructors.

\(^1\)Objects of OOP are typically infinite data values because they are usually recursively-defined via their definitions using the special self-referential variables “this” or “self”.
1 Basic Domain Theory Notions

Domain theory is a branch of mathematics that builds on set theory, order theory (i.e., the theory of partially-ordered sets, a.k.a., posets), and topology (i.e., the theory of topological spaces). It is relatively easy to digest the basic definitions of domain theory once the computational motivations behind these definitions are understood. Standard references on set theory include [Bre58, End77, Hal60]. Standard references on order theory include [DP90, Har05].

Gierz, et al, [GHK+03], present a detailed encyclopaedic account of domain theory, connecting domain theory to order theory and to topology. Otherwise, literature on domain theory is somewhat fractured. Terminology in domain theory is somewhat less standard than that of set theory and order theory. Accordingly, there is no standard formulation of domain theory. Literature on domain theory includes [Sco76, Sto77, Sco81, Sco83, Plo83, All86, CP88, GS90, KP93, AJ94]. Stoy’s book [Sto77] is a particularly detailed account of the motivations behind domain theoretic definitions (Stoy, following Scott’s original formulation [Sco76], uses complete lattices, rather than cpos, for domains.)

In this and the next section we present the definitions of basic domain theory notions used in constructing NOOP and COOP. In Section 3 we present the definitions of the domain constructors used in the constructions.

Definition 1.1 (Partial Order). A partial order (also called a partially-ordered set, or, for short, a poset) is a pair \((X, \sqsubseteq)\) consisting of a set \(X\) (called the universe of the ordering), and a binary relation \(\sqsubseteq\) on the set \(X\), such that

- \(\forall x \in X, x \sqsubseteq x\) (\(\sqsubseteq\) is reflexive)
- \(\forall x, y \in X, x \sqsubseteq y \land y \sqsubseteq x \implies x = y\) (\(\sqsubseteq\) is antisymmetric)
- \(\forall x, y, z \in X, x \sqsubseteq y \land y \sqsubseteq z \implies x \sqsubseteq z\) (\(\sqsubseteq\) is transitive)

where \(\implies\) is implication. The relation \(\sqsubseteq\) is usually called the ‘less than or equals’ relation when discussing general posets, and is called the ‘approximates’ relation in domain theory. Intuitively, \(x \sqsubseteq y\) means \(x\) is ‘no more informative than’ (i.e., approximates information contained in) \(y\). A poset \((X, \sqsubseteq)\) is usually referred to using the symbol for its universe, \(X\). We do so below. When we need to specifically refer to the universe, i.e., the set underlying a poset \(X\), we instead use the bar notation \(|X|\) to denote this universe.

Remark 1.2. In domain theory, the approximation ordering is defined on mathematical values used to denote computational data values. The approximation ordering has intuitive connections to information theory. A computational value whose denotation approximates the denotation of another computational value is considered no more informative than the second data value. The approximation ordering is a qualitative expression of the relative informational content.

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2Chapter 5 in [TGS08] presents an excellent introduction to fixed points—a central topic in order theory and domain theory—that is particularly suited for mathematically-inclined programmers.
of computational values (which are denoted by elements of the universe of the ordering). Computational values whose denotations are higher in the approximation ordering are more informative than ones whose denotations are lower in the ordering.

Remark 1.3. The least computational value is divergence (as in an ‘infinite loop’). It gives no information, and thus is the least informative computational value. Given that divergence gives no information, the abstract mathematical value denoting divergence is called ‘bottom’, is at the bottom of the approximation ordering (hence the name), and is usually denoted by the symbol ⊥.

Definition 1.4 (Induced Partial Order). Every subset \( S \) of the universe of a poset \( \mathcal{X} \) has an associated partial order called the induced partial order of \( S \). Members of the ordering relation of the induced order are those of the ordering of \( \mathcal{X} \) restricted to elements \( S \).

Remark 1.5. The induced partial order of a subset of a poset \( \mathcal{X} \) is sometimes called a subposet of \( \mathcal{X} \). In a usually-harmless and standard abuse of terminology and notation, we refer to induced partial orders as subsets instead, and we use \( S \) to denote both the subset and its induced partial order.

Definition 1.6 (Upper bound). Given a subset \( S \) of a poset \( \mathcal{X} \), an upper bound of \( S \) in \( \mathcal{X} \), is an element \( x \in \mathcal{X} \) such that \( \forall s \in S, s \sqsubseteq x \).

Definition 1.7 (Bounded). A subset \( S \) of a poset \( \mathcal{X} \) is bounded in \( \mathcal{X} \) iff \( S \) has an upper bound in \( \mathcal{X} \).

Definition 1.8 (Least Upper Bound). An upper bound of a subset \( S \) in a poset \( \mathcal{X} \) is a least upper bound (also called a lub, or LUB) of \( S \) iff this upper bound approximates all upper bounds of \( S \) in \( \mathcal{X} \). If it exists, the lub of \( S \) is denoted \( \bigsqcup S \).

Definition 1.9 (Downward-Closed). A subset \( S \) of a poset \( \mathcal{X} \) is a downward-closed set iff all elements \( x \) of \( \mathcal{X} \) that approximate some element in \( S \) belong to \( S \). Thus, \( S \) is downward-closed iff \( \forall x \in \mathcal{X}.((\exists s \in S, x \sqsubseteq s) \implies x \in S) \).

Definition 1.10 (Chain). A countable subset \( S \) of a poset \( \mathcal{X} \) with elements \( s_i \) is a chain if \( \forall i, j \in \mathbb{N}.i \leq j \rightarrow s_i \sqsubseteq s_j \).

Remark 1.11. Every finite chain includes its lub (the maximum element of the chain). Infinite chains (like set \( \mathbb{N} \) under the standard ordering) do not necessarily have maximal elements.

\[ A \text{ lub of a subset } S \text{ may not exist, either because } S \text{ has no upper bounds or because } S \text{ has more than one upper bound but there is no least element (i.e., a minimum) among them.} \]
Definition 1.12 (Anti-chain). A countable subset $S$ of a poset $X$ with elements $s_i$ is an anti-chain if $\forall i, j \in \mathbb{N} \cdot i \neq j \rightarrow s_i \nsubseteq s_j$.

Remark 1.13. A flat poset $R$ is an anti-chain $S$ with elements $s_i$ and an additional bottom element $\bot_R$, such that $\bot_R \subseteq s_i$ and $\bot_R \neq s_i$ for all $i$. A flat poset, thus, is said to be the lifting of the underlying anti-chain.

Definition 1.14 (Directed). A subset $S$ of a poset $X$ is directed iff every finite subset of $S$ is bounded in $S$.

Remark 1.15. Every chain is a directed set, but not necessarily vice versa.

Definition 1.16 (Consistent). A subset $S$ of a poset $X$ is consistent in $X$ iff every finite subset of $S$ is bounded in $X$.

Remark 1.17. In general posets, every bounded set is consistent, but not necessarily vice versa. Consistency requires the boundedness of finite subsets only. Thus, boundedness (where all subsets are bounded) is a stronger condition than consistency.

Remark 1.18. Because $S$ is a subset of $X$, boundedness in $S$ implies boundedness in $X$, and thus every directed set $S$ is a consistent set, but not necessarily vice versa. Directedness is thus also a stronger condition than consistency.

Definition 1.19 (Ideal). A subset $S$ of a poset $X$ is an ideal iff it is downward-closed and directed.

Definition 1.20 (Lower set). A subset $S_x$ of a poset $X$ is a lower set of an element $x \in |X|$ iff it contains all elements of $|X|$ that are less than or equal to $x$ (and nothing else). Thus, for $x \in |X|$, $S_x$ is the lower set of $x$ iff $S_x = \{s \in |X| | s \subseteq x\}$.

Definition 1.21 (Principal Ideal). A subset $S_x$ of a poset $X$ is a principal ideal (determined by $x$) iff it is the lower set of $x$.

Theorem 1.22. (Principal Ideals are Ideals) A subset $S$ of a poset $X$ is an ideal if it is a principal ideal.

Proof. Note that, by definition and using the transitivity of $\subseteq$, a lower set of an element $x \in X$ is downward-closed. The lower set of $x$ is also directed because it contains $x$ and $x$ is a bound for all (finite) subsets of the lower set. \qed

Definition 1.23 (Weak Ideal). A non-empty subset $S$ of a poset $X$ is a weak ideal iff it is downward-closed and is closed under lubs of its chains.

Remark 1.24. Every flat poset is a weak ideal. Chains in flat posets have two elements, the lower of which is always $\bot$. 

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**Definition 1.25** (Finitary Basis). A poset \( \mathcal{X} \) is a *finitary basis* iff its universe, \(|\mathcal{X}|\), is countable and every finite bounded subset \( \mathcal{S} \) of \( \mathcal{X} \) has a lub in \( \mathcal{X} \).

**Remark 1.26.** From the definition of finitary basis, the fact that a finite subset \( \mathcal{S} \) of a finitary basis \( \mathcal{X} \) is bounded is equivalent to \( \mathcal{S} \) having a lub. Generally, this statement is true only in one direction for an arbitrary poset (i.e., the trivial \( \leq \) direction, which asserts the boundedness of a set if it has a lub.) In a finitary basis, the opposite direction is true as well for all finite subsets of the finitary basis.

**Definition 1.27** (Complete Partial Order). A poset \( \mathcal{X} \) is a *complete partial order* (cpo, or, sometimes, dcpo) iff every directed subset \( \mathcal{S} \) of \( \mathcal{X} \) has a lub in \( \mathcal{X} \), i.e., a cpo is closed over lubs of its directed subsets.

**Theorem 1.28** (Ideals over a FB form a cpo). Given a finitary basis \( \mathcal{X} \), the set \( \mathcal{I}_X \) of ideals of \( \mathcal{X} \) is a cpo under the subset ordering \( \subseteq \).

**Proof.** Under the subset ordering, a directed set \( \mathcal{J} \) of ideals of \( \mathcal{X} \) is one in which each finite subset \( \mathcal{J}_f \) of \( \mathcal{J} \) has an element in \( \mathcal{J} \) (i.e., an ideal) that includes all elements in the elements of \( \mathcal{J}_f \). Every such directed set \( \mathcal{J} \) has a lub in \( \mathcal{X} \) under the subset ordering, namely the union of elements of \( \mathcal{J} \), \( \bigcup \mathcal{J} \). This union is always an ideal, and thus a member of \( \mathcal{I}_X \).

**Definition 1.29** (Constructed Domain). Given a finitary basis \( \mathcal{X} \), the set \( \mathcal{I}_X \), of ideals of \( \mathcal{X} \), forms a poset, \( (\mathcal{I}_X, \subseteq) \) is called the *domain determined by* \( \mathcal{X} \) or, sometimes, the *ideal completion* of \( \mathcal{X} \). \( \mathcal{I}_X \) is, thus, called a *constructed domain* (i.e., one that is defined by the finitary basis \( \mathcal{X} \)).

**Remark 1.30.** By Theorem 1.28, the ideal completion of (i.e., the domain determined by) every finitary basis is a cpo.

**Definition 1.31** (Finite Element of a CPO). An element \( d \) of a cpo \( \mathcal{D} \) is a *finite element* (or, equivalently, isolated or compact) iff \( d \) belongs to each directed subset \( \mathcal{S} \) that \( d \) is a lub of. The set of finite elements of a cpo \( \mathcal{D} \) is denoted by \( \mathcal{D}_0 \).

**Definition 1.32** (Isomorphic Partial Orders). Two posets are *isomorphic* iff there is an order-preserving one-to-one onto function between them.

**Definition 1.33** (Domain). A cpo \( \mathcal{D} \) is a *domain* iff its finite elements \( \mathcal{D}_0 \) form a finitary basis and \( \mathcal{D} \) is isomorphic to the domain determined by the finitary basis \( \mathcal{D}_0 \).

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4A Coq [BC04] development (i.e., a Coq proof script) with a proof of this theorem is available upon request.

5This definition of finite elements is weaker than the usual definition for cpos. In the context of domains, which are finitary-based, the two definitions are equivalent.
Definition 1.34 (Subdomain). As a counterpart to the notion of subset in set theory, and subposet in order theory, a domain \( D \) is a subdomain of a domain \( E \) iff (1) their universes are in the subset relation, \(|D| \subseteq |E|\), (2) they have the same bottom element, \( \perp_D = \perp_E \), (3) restricted to elements of their respective universes, they have the same approximation ordering, \( \forall d_1, d_2 \in D. d_1 \sqsubseteq_D d_2 \Leftrightarrow d_1 \sqsubseteq_E d_2 \) \( \text{(i.e., approximation ordering for } D \text{ is the approximation ordering of } E \text{ restricted to elements of } D) \), and (4) restricted to elements of their respective universes, they have the same lub relation, \( \forall d_1, d_2, d_3 \in D. (d_1 \sqcup_D d_2 = d_3) \Leftrightarrow (d_1 \sqcup_E d_2 = d_3) \) \( \text{(i.e., the lub relation for } D \text{ is the lub relation of } E \text{ restricted to elements of } D) \).

Remark 1.35. For a subdomain \( D \) of domain \( E \), the domain determined by \( D^0 \) is isomorphic to the domain determined by \( E^0 \cap D \) (which must be a finitary basis.)

Remark 1.36. In Definition 1.34, we use Scott’s definition of subdomains because we define NOOP and COOP domains as subdomains of Scott’s universal domain \( \mathcal{U} \). Scott \([\text{Sco81, CP88}]\) shows that every domain is isomorphic to a subdomain of \( \mathcal{U} \). Under the subdomain ordering, all the subdomains of \( \mathcal{U} \) form a domain (itself also a subdomain of \( \mathcal{U} \), by the universality of \( \mathcal{U} \)). All domains given in a domain equation and all recursively defined domains in the equation are elements of this space of domains (again, a domain that consists of all of the subdomains of \( \mathcal{U} \) as its elements). Thus, solutions of recursive domain equations (as elements of the domain of subdomains of \( \mathcal{U} \)) are defined in the same way \( \text{(e.g., as least fixed-points, or lfps of generating functions)} \) as solutions of recursive definitions specifying elements in any other computational domain (a subdomain of \( \mathcal{U} \)).

2 Notions for Functional Domains

To model computable functions, domain theory provides functional domains, whose elements are particular mathematical functions mapping elements from one computational domain to another. To define functional domains, we will introduce the domain theoretic notions of ‘approximable mappings’ (AMs), ‘finite-step mapping’, and ‘continuous functions’.

Definition 2.1 (Approximable Mapping). Given two finitary basis \( A \) and \( B \), with ordering relations \( \sqsubseteq_A \) and \( \sqsubseteq_B \), respectively, a relation \( f_{am} \subseteq |A| \times |B| \) is an approximable mapping (AM) iff

1. Condition 1: \( (\bot_A, \bot_B) \in f_{am} \) (pointedness)

2. Condition 2: \( \forall a \in A. \forall b_1, b_2 \in B. ((a, b_2) \in f_{am} \wedge b_1 \sqsubseteq_B b_2 \rightarrow (a, b_1) \in f_{am}) \) (downward-closure)

3. Condition 3: \( \forall a \in A. \forall b_1, b_2 \in B. ((a, b_1) \in f_{am} \wedge (a, b_2) \in f_{am} \rightarrow (a, b_1 \sqcup_B b_2) \in f_{am}) \) (directedness)
4. Condition 4: \( \forall a_1, a_2 \in A. \forall b \in B. ((a_1, b) \in f_{am} \land a_1 \subseteq A \rightarrow (a_2, b) \in f_{am}) \) (monotonicity)

**Definition 2.2** (Set Image under a Relation). Given sets \( A, B \) and a relation \( r \subseteq A \times B \), the set image of a subset \( S \) of \( A \) under \( r \), denoted by \( r(S) \), is the set of all \( b \in B \) related in \( r \) to some element in \( S \). Hence, relation \( r \) is viewed as a function over subsets of \( A \). For \( S \subseteq A \), we have \( r(S) = \{ b \in B | \exists a \in S. (a, b) \in r \} \). The set image of a relation \( r \) also allows viewing \( r \) as a function \( r : A \rightarrow \wp(B) \), where \( r(a) = r(\{a\}) \) for \( a \in A \). In other words, for \( a \in A \), function \( r \) returns the set of all \( b \in B \) related to \( a \) in \( r \) (viewed as a relation).

**Theorem 2.3** (AMs map ideals to ideals). Given finitary basis \( A \) and \( B \), if \( f_{am} \) is an approximable mapping from \( A \) to \( B \), and if \( I \) is an ideal in \( A \), then \( f_{am}(I) \), the set image of \( I \) under \( f_{am} \), is an ideal in \( B \).

**Proof.** From the definition of an ideal, and using AM Condition 2 (which guarantees the set image is downward-closed), and AM Condition 3 (which guarantees the set image is directed).

**Theorem 2.4** (AMs are monotonic). Given finitary basis \( A \) and \( B \), if \( f_{am} \) is an approximable mapping from \( A \) to \( B \), and if \( I_1 \) and \( I_2 \) are ideals in \( A \) such that \( I_1 \subseteq I_2 \), then \( f_{am}(I_1) \subseteq f_{am}(I_2) \) in \( B \).

**Proof.** By AM Condition 4.

**Definition 2.5** (Finite-Step Mapping). Given finitary basis \( A \) and \( B \), an approximable mapping \( f_{am} \) is a finite-step mapping iff it is the smallest approximable mapping containing some finite subset of \( |A| \times |B| \).

**Definition 2.6** (Continuous Function). Given domains \( A \) and \( B \), a function \( f : A \rightarrow B \) from domain \( A \) to \( B \) is a continuous function iff the value of \( f \) at the lub of a directed set of \( a \)'s in \( A \) is the lub, in \( B \), of the (directed) set of function values \( f(a) \).

**Remark 2.7.** Continuity of a function requires the value of the function at a (non-finite) limit point \( l \) to equal the limit of values of the function at the finite approximations to \( l \). Continuous functions are thus said to “have no surprises at the limit”.

**Remark 2.8.** Because of the four AM conditions, if finitary basis \( A \) and \( B \) determine domains \( A \) and \( B \), respectively, then every approximable mapping in \( |A| \times |B| \) determines a continuous function in \( A \rightarrow B \), and vice versa. Check Cartwright and Parsons’ ‘Domain Theory: An Introduction’ monograph [CP88] and other domain theory literature for proof and more details.
Remark 2.9. To motivate the preceding definitions, it should be noted that continuous functions capture the fact that computation is of a “finitely-based” nature. Only finite data values can have canonical representations inside a computing device. From a domain-theoretic perspective, an (infinite) function can be computable only if its value “at infinity” (i.e., at an infinite input data value) is the one we expect by only seeing (and extrapolating from) the values of the function at all finite inputs that approximate the infinite input data value (finite inputs are all that can be represented inside computers, and thus they are all that can be computed with). See Stoy’s book [Sto77] for more details on motivation and intuitions behind domain theoretic definitions.\(^6\)

Remark 2.10. Approximable mappings offer the means to accurately characterize and define continuous functions (which, as mentioned above, capture the finitely-based nature of computation). Finite-step mappings, as the “finite/representable parts” of AMs, offer the means by which continuous functions can be constructed from more elementary parts that can be represented in a computing device.

3 Domain Constructors

In this section we present the domain constructors used to define NOOP and COOP.

3.1 Coalesced Sum (+)

The first domain constructor we present is the coalesced sum domain constructor, +. The expression \( \mathcal{A} + \mathcal{B} \) denotes the coalesced sum of two domains \( \mathcal{A} \) and \( \mathcal{B} \), with approximation ordering relations \( \sqsubseteq_A \) and \( \sqsubseteq_B \), respectively. A coalesced sum is a domain-theoretic counterpart of the standard set-theoretic disjoint union operation.

If \( \mathcal{C} = \mathcal{A} + \mathcal{B} \) then

\[
|\mathcal{C}| = \{\perp_\mathcal{C}\} \cup \{(0, a)|a \in (|\mathcal{A}| \setminus \{\perp_\mathcal{A}\})\} \cup \{(1, b)|b \in (|\mathcal{B}| \setminus \{\perp_\mathcal{B}\})\}
\]

where 0 and 1 are used in \( \mathcal{C} \) to tag non-bottom elements from \( \mathcal{A} \) and \( \mathcal{B} \), respectively.

The ordering relation \( \sqsubseteq_\mathcal{C} \), on elements of \( \mathcal{C} \), is defined, for all \( c_1, c_2 \in \mathcal{C} \), by the predicate

\[
c_1 \sqsubseteq_\mathcal{C} c_2 \iff (c_1 = \perp_\mathcal{C}) \vee (c_1 = (0, a_1) \wedge c_2 = (0, a_2) \wedge a_1 \sqsubseteq_\mathcal{A} a_2) \vee (c_1 = (1, b_1) \wedge c_2 = (1, b_2) \wedge b_1 \sqsubseteq_\mathcal{B} b_2)
\]

\(^6\)Via Roger’s work, Dana Scott managed to connect the notion of continuous functions to the notion of computable functions in computability theory. Again, see Stoy’s book [Sto77] for more details.
3.2 Strict Product (×)\(^7\)

We use \(A \times B\) to denote the strict product of two domains, \(A\) and \(B\), with approximation ordering relations \(\sqsubseteq_A\) and \(\sqsubseteq_B\), respectively. A strict product is an order-theoretic counterpart of the standard set-theoretic cross-product operation.

If \(C = A \times B\) then

\[
|C| = (|A| \setminus \{\bot_A\}) \times (|B| \setminus \{\bot_B\}) \cup \{\bot_C\}
\]  

(2)

Strictness of \(\times\) means that in \(C\), \(\bot_C\) replaces all pairs \((a, b) \in A \times B\) where \(a = \bot_A\) or \(b = \bot_B\). Similar to the definition of the coalesced sum constructor, this strictness is achieved in the definition above by excluding \(\bot_A\) and \(\bot_B\) from the input sets of the set-theoretic cross product. Sometimes the strict product \(A \times B\) is called their ‘smash product’.

The ordering relation \(\sqsubseteq_C\), on elements of \(C\), is defined as follows. \(\forall c_1, c_2 \in C, \forall a_1, a_2 \in A \setminus \{\bot_A\}, \forall b_1, b_2 \in B \setminus \{\bot_B\}\) where \(c_1 = (a_1, b_1)\) or \(c_1 = \bot_C\), and \(c_2 = (a_2, b_2)\) or \(c_2 = \bot_C\)

\[
c_1 \sqsubseteq_C c_2 \iff (c_1 = \bot_C \lor (a_1 \sqsubseteq_A a_2 \land b_1 \sqsubseteq_B b_2)).
\]  

(3)

3.3 Continuous Functions (\(\to\))

Functional domains and functional domain constructors are necessary for accurately modeling OOP. Functional domains of \textbf{NOOP} (and \textbf{COOP}) are: (1) the auxiliary domain of methods whose members are strict continuous functions modeling object methods, and (2) the auxiliary domain of records, whose members are ‘record functions’ modeling record components of objects. (A record function, constructed using a new domain constructor \(--\circ\), called ‘rec’, is a function defined over a finite set of labels. See [Abd12, AC14] for the definition of the records domain constructor, \(--\circ\), and proofs of its properties. See [Abd14] for a summary.)

The symbol \(\to\) is used to denote the standard continuous functions domain constructor. Making use of the definitions of domain theoretic notions presented in Section 2, particularly approximable mappings and finite-step mappings, we refer the reader to Chapter 3 of Cartwright and Parsons’ monograph on Domain Theory [CP88] (which is an update of Scott’s lecture notes [Sco81]) for the details of the definition of the continuous functions domain constructor \(\to\). Since there is a one-to-one correspondence between domains and their finitary bases, and given that the latter are simpler and more intuitive notions, Cartwright and Parsons’ monograph describes how the domain \(A \to B\) of continuous functions from domain \(A\) to domain \(B\) is determined by constructing its finitary basis from the finitary basis of domains \(A\) and \(B\) (See Remark 2.8).

\(^7\)In agreement with the standard convention in domain theory literature, the symbol \(\times\) is overloaded in this report. The symbol \(\times\) is used to denote the strict product of ordered sets (including domains), and is also used to denote the standard set-theoretic cross product (which ignores any ordering on its input sets). It should always be clear from context which meaning is attributed to \(\times\).
In this report, we use the symbol $\rightarrow$ to denote the strict continuous functions domain constructor, which simply constructs a space like the space of continuous functions from domain $A$ to domain $B$ but where all so-called “one-step functions” of the form $\perp_A \mapsto b$ (for $b \in B \setminus \{\perp_B\}$) are eliminated (i.e., are mapped to the one-step function $\perp_A \mapsto \perp_B$, which is the bottom element of the constructed function space.) Strict continuous functions map $\perp_A$ only to $\perp_B$, thereby modeling strict computable functions (i.e., functions that have “call-by-value” semantics.)

A notable property of functional domain constructors is that the set of continuous functions between two domains itself forms a domain. This property (i.e., finding a mathematical space having this property) has been much behind the development of domain theory.

### 3.4 Strict Finite Sequences ($D^*$)

For the purpose of constructing methods of NOOP and COOP, one more domain constructor is needed: the constructor of the domain of strict finite sequences. This constructor is used to construct the finite sequences of objects that are passed as arguments to methods in NOOP and COOP. Sometimes the domain $D^*$ of finite sequences of elements of domain $D$ is called the Kleene closure of domain $D$.

The Kleene closure, $D^*$, constructs a domain of finite sequences of elements of its input domain, $D$, including the empty sequence. Our definition of $^*$ excludes constructing sequences of $D$ where a member of the sequence is $\perp_D$. Thus, $^*$ is said to construct strict finite sequences.

The Kleene closure is defined as a set of all $n$-tuples of elements of $D$ (where $n$ is a natural number). Thus

$$|D^*| = \{\perp_D\} \cup \bigcup_{n \in \mathbb{N}} \{< d_0, \ldots, d_i, \ldots, d_{n-1} > | d_i \in (|D| \setminus \{\perp_D\})\}$$

An element $u$ of $D^*$ approximates an element $v$ of $D^*$ iff $u = \perp_D$, or the lengths of both $u$ and $v$ are equal to a natural number $k$, and $u_i \subseteq_D v_i$ for all $0 \leq i < k$.

### 4 COOP: A Simple Structural Model of OOP

In this section we present the construction of COOP as a simple structural domain-theoretic model of OOP. The reasons for constructing a structural model of OOP, i.e., COOP, as a step towards constructing NOOP as a model of nominally-typed OOP are threefold. First, (1) earlier research on structural OOP needs to be put on a more rigorous footing. The literature on models of structural OOP glosses over important technical details like the construction of a domain of records, having methods of multiple arity, and objects being purely OO (i.e., not allowing functions and non-object values have first-class status in the constructed domain of “objects”), all of which we address in the construction of COOP. Second, (2) the construction of COOP is similar to but
simpler than the construction of NOOP (e.g., COOP "objects" do not include signatures, and thus constructing COOP does not need an extra filtering step to match signatures with record components of objects as is needed for NOOP construction). Understanding how COOP is constructed makes it easier to understand the construction of NOOP. Third, and most importantly, (3) the rigorous definition of COOP alongside the definition of NOOP clarifies the distinction between structural OOP and nominal OOP.

As mathematical models, COOP and NOOP are collections of semantics domains. In denotational semantics, domains are used to model computational constructs. Domains of COOP and NOOP correspond to the set of all possible object values, field values, and method values of structural and nominal OO programs, respectively. Similarly, specific subdomains of COOP and NOOP domains correspond to specific structural and nominal types definable in structurally-typed and nominally-typed OO languages. COOP and NOOP, thus, give an abstract mathematical meaning to the most fundamental concepts of structurally-typed and nominally-typed OOP.

Focusing on COOP, our presentation of COOP proceeds as follows. The domain of objects of COOP is the solution of a reflexive domain equation. In Section 4.1 we first present the COOP domain equation. In Section 4.2 we then show how COOP domains are constructed as the solution of the COOP domain equation. The domains of COOP are constructed using standard domain theoretic construction methods that make use of standard domain constructors as well as the records domain constructor, \(-\text{rec}\) (pronounced “rec”), described in [Abd12, Abd13, Abd14, AC14].

The view of objects in COOP is a very simple one. An object in COOP is a record of functions that map sequences of objects to objects. In other words, in COOP an object is ‘a finite collection of methods’, where a method is a labeled function mapping sequences of objects to objects. (In COOP, unlike NOOP, we encode fields as zero-ary methods.)

Given that it is a structural model of OOP, COOP closely resembles SOOP, the model of OOP Cardelli presented in [Car84, Car88]. Given that objects of COOP, like those of SOOP, miss nominality information, COOP is a structural model of OOP. Our presentation of the construction of COOP in the following sections shows how to rigorously construct a model like Cardelli’s.

COOP, however, differs from SOOP in five respects:

1. Unlike SOOP, but similar to many mainstream OO languages, the COOP domain equation does not allow functions as first-class values (thus, COOP does not support direct “currying” of functions). Only objects are first-class values in COOP.

2. Unlike SOOP, COOP uses the records domain constructor, \(-\text{rec}\), to construct records rather than using the standard continuous functions domain constructor (which is used in SOOP). The definition of \(-\text{rec}\) is presented in [Abd12, Abd13, Abd14, AC14].

3. Unlike SOOP, methods in COOP objects are multi-ary functions over
objects.\footnote{Since \texttt{SOOP} defines a domain for a simple functional language with objects based on \texttt{ML}, it is natural to force all functions to be unary (as in \texttt{ML}). In this context, a multi-ary function can be transparently curried.}

4. For simplicity, \texttt{COOP} objects have fields only modeled by (constant) 0-ary functions, not as a separate component in objects. Thus, names of fields and methods in \texttt{COOP} objects share the same namespace.

5. Since we do not use \texttt{COOP} to prove type safety results (even though it can be used), \texttt{COOP} does not need to have a counterpart to the $\mathcal{W} = \{\text{\textit{wrong}}\}$ domain that is used in \texttt{SOOP} to detect type errors.

When compared to \texttt{NOOP}, as presented in [Abd12, Abd13, Abd14, AC14], it is easy to see that \texttt{COOP}, and thus also \texttt{SOOP}, does not accurately capture the notion of inheritance as it has evolved in statically-typed nominal OO languages like \texttt{JAVA} [GJSB05], \texttt{C++ [CPP11]}, \texttt{C# [CSh07]}, \texttt{SCALA [Ode09]}, and \texttt{X10 [SBP+11]}.

\subsection*{4.1 COOP Domain Equation}

The domain equation that defines \texttt{COOP} makes use of two simple domains $\mathcal{B}$ and $\mathcal{L}$. Domain $\mathcal{B}$ is a domain of atomic “base objects”. $\mathcal{B}$ could be a domain that contains a single non-bottom value, \emph{e.g.}, \texttt{unit} or \texttt{null}, or the set of Boolean values \{\texttt{true}, \texttt{false}\}, the set of integers, or some more complex set of primitive values that is the union of Boolean values and various forms of numbers (\emph{e.g.}, whole numbers and floats) and other primitive objects, such as characters and strings, etc.

Domain $\mathcal{L}$ is a flat countable non-empty domain of labels. Elements of $\mathcal{L}$ (or, $|\mathcal{L}|$, more accurately) are proper labels used as names of record members (fields and methods), or the improper “bottom label”, $\bot_{\mathcal{L}}$, that is added to proper labels to make $\mathcal{L}$ a flat domain. Elements of $\mathcal{L}$ other than $\bot_{\mathcal{L}}$ (proper labels) will serve as method names in \texttt{COOP}.

The domain equation of \texttt{COOP} is

$$
\mathcal{O} = \mathcal{B} + \mathcal{L} \leadsto (\mathcal{O}^* \leadsto \mathcal{O})
$$

(4)

Domain $\mathcal{O}$ is a domain of simple objects, and it is the primary domain of \texttt{COOP}. Equation (4) states that a \texttt{COOP} object (an element of $\mathcal{O}$) is either (1) a base object (an element of domain $\mathcal{B}$); or is (2) a record of methods (\emph{i.e.}, a finite mapping from labels, functioning as method names, to functions), where, in turn, methods are functions from sequences of objects to objects.

\subsection*{4.2 COOP Construction}

The construction of domain $\mathcal{O}$, as the solution of domain equation (4), is done using standard techniques for solving recursive domain equations (we use the
‘least fixed point (lfp) construction’, which, according to Plotkin [Plo78], is equivalent to the ‘inverse limit’ construction).

Conceptually, the right-hand-side (RHS) of the COOP domain equation (Equation (4)) is interpreted as a function

\[ \lambda O_i, B + L \rightarrow (O_i^* \rightarrow O_i) \]

(5)

over domains, from a putative interpretation \( O_i \) for \( O \) to a better approximation \( O_{i+1} \) for \( O \). Each element in this sequence is a domain. The solution, \( O \), to the domain equation is the least upper bound (lub) of the sequence \( O_0, O_1, \ldots \).

Thus, the construction of \( O \) proceeds in iterations, numbered \( i + 1 \) for \( i \geq 0 \). We use the empty domain as the initial value, \( O_0 \), for domain \( O \), and for each iteration \( i + 1 \) we take the output domain produced by the domain constructions using the domains \( O_i, L \) and \( B \) (the values for the function given by Formula (5)) as the domain \( O_{i+1} \) introduced in iteration \( i + 1 \).

4.2.1 A General COOP Construction Iteration

For a general iteration \( i + 1 \) in the construction of COOP, the construction method thus proceeds by constructing

\[ M_{i+1} = O_i^* \rightarrow O_i \]

using the strict continuous functions domain constructor, \( \rightarrow \), and the sequences domain constructor, \( * \). Then, using the records domain constructor, \( \rightarrow \), we construct the domain of records

\[ R_{i+1} = L \rightarrow M_{i+1} \]

and, finally, using the coalesced sum domain constructor, \( + \), we construct

\[ O_{i+1} = B + R_{i+1} \]

4.2.2 The Solution of the COOP Domain Equation

Given the continuity of all domain constructors used in the function defined by the lambda expression (5), and given that composition of domain constructors preserves continuity, the function defined by the RHS of the COOP domain equation is a continuous function [CP88, Theorem 2.10 and Corollary 2.11]. The least upper bound (lub) of the sequence \( O_0, O_1, \ldots \) of domains constructed in the construction iterations is the least fixed point (lfp) of the function given by Formula (5). According to standard theorems of domain theory about the lfp of continuous functions, the lub of the domains \( O_i \) (i.e., their “limit” domain) is simply their union, and this lub is the solution of Equation (4).

To complete the construction of COOP, we thus construct the solution \( O \) of the COOP domain equation by constructing the union of all constructed domains \( O_i \), i.e., \( O \) will be given by the equation

\[ O = \bigcup_{i \geq 0} O_i \]
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