THE $\chi_{-y}$-GENERA OF RELATIVE HILBERT SCHEMES FOR LINEAR SYSTEMS ON ABELIAN AND K3 SURFACES

LOTHAR GÖTSCHE AND VIVEK SHENDE

Abstract. For an ample line bundle on an Abelian or K3 surface, minimal with respect to the polarization, the relative Hilbert scheme of points on the complete linear system is known to be smooth. We give an explicit expression in quasi-Jacobi forms for the $\chi_{-y}$ genus of the restriction of the Hilbert scheme to a general linear subsystem. This generalizes a result of Yoshioka and Kawai for the complete linear system on the K3 surface, a result of Maulik, Pandharipande, and Thomas on the Euler characteristics of linear subsystems on the K3 surface, and a conjecture of the authors.

1. Introduction

Let $S$ be a smooth complex algebraic surface, and $L$ a line bundle over $S$. Consider a linear system $\mathbb{P}^{\delta} \subset |L|$. Let $C \to \mathbb{P}^{\delta}$ the universal family of curves over the linear system, and $C^{[n]}$ the relative Hilbert schemes of points on the fibres. Under suitable hypotheses the Euler numbers of the $C^{[n]}$ control the number $\delta$-nodal curves in $\mathbb{P}^{\delta}$ [KST, KS]. In these cases, the relative Hilbert schemes can be identified with the surface variant of the stable pairs spaces of Pandharipande and Thomas [PT1, PT3, KT]. When the spaces $C^{[n]}$ are smooth, their Euler numbers may be computed by integrating Chern classes. Writing these as “tautological” integrals over $S^{[n]}$ allows the fact [EGL] that all such integrals are determined by the Chern classes of $S$ and $L$ to be imported into enumerative geometry; one concludes that the number of $\delta$-nodal curves in a $\delta$-dimensional slice of a $\delta$-very-ample linear system is given by a universal formula. This result had been previously conjectured by the first author [Göt2] and previously proven by other methods [Tze].

When $S$ is a K3 surface, explicit formulas for the $\chi(C^{[n]})$ are known. The derivation of these is however rather indirect: one shows [MPT] an equivalence between the stable pairs and Gromov-Witten theories and calculates [BL] the latter. Similar methods may be expected to work for the Abelian surface; alternatively, the formula for the K3 surface determines the formula for the Abelian surface “by universality”.

In [GS], we replace the topological Euler characteristic $\chi$ with the Hirzebruch $\chi_{-y}$ genus ($\chi = \chi_{-1}$). Imitating the formulas of [KST] leads to putative refined curve counts which conjecturally are given by a universal formula in the Chern numbers. The refinement recovers at $y = 1$ the counts of complex curves, and conjecturally for a toric surface

\footnote{According to [MPT], this is in the spirit of but does not directly follow from the conjectural 3-fold equivalence of [MNOP].}
computes tropical refined Severi degrees [BG, IM]. The tropical refined Severi degrees are defined combinatorially, but carry two meaningful enumerative specializations: at $y = 1$ they count complex curves, and at $y = -1$ they count real curves [BG, M].

We moreover conjectured in [GS] an explicit formula for the refined invariants in the case of K3 or Abelian surfaces. Our goal here is to give a derivation of this formula, which in turn determines two of the four series involved in the (still conjectural) formula for a general surface.

The quantity
\[
X_y(x) := \frac{x(1 - ye^{x(y^{1/2} - y^{-1/2})})}{y^{1/2}(1 - e^{x(y^{1/2} - y^{-1/2})})}
\]
is the normalized power series that defines the genus $\chi_y(M) = y^{-\dim(M)/2} \chi_y(M)$. That is, for a vector bundle $E$ with Chern roots $e_1, \ldots, e_n$ we define $X_y(E) = \prod^n_{i=1} X_y(e_i)$. For a smooth projective variety $M$, we write $X_y(M) := X_y(T_M)$, and by the Riemann-Roch formula [Hi] we have
\[
\chi_y(M) := \int_M X_y(M) = y^{-\dim(M)/2} \sum_{p,q} (-1)^{p+q} y^p h^{p,q}(M).
\]

We collect the $\chi_y$ genera of relative Hilbert schemes over complete linear systems on the Abelian and K3 surfaces into generating series.

**Definition 1.** Throughout we write $L_g$ to indicate a line bundle with no higher cohomology whose sections have arithmetic genus $g$. Note for all $g \geq 2$ there is an Abelian surface $A_g$ carrying a line bundle $L_g$ such that the relative Hilbert schemes $C_g[n] \to |L_g|$ are smooth. We define
\[
\mathbb{A} := \sum_{g \geq 2} \sum_{n \geq 0} \chi_y(C_g[n]) t^{n+1-g} q^{g-1}.
\]
Similarly let $L_g$ be a linear system of genus $g$ curves on a K3 surface $K_g$ such that the relative Hilbert schemes $C_g[n] \to |L_g|$ are smooth. We define
\[
\mathbb{K} := \sum_{g \geq 0} \sum_{n \geq 0} \chi_y(C_g[n]) t^{n+1-g} q^{g-1}.
\]

We require two more generating series which contain the same information as $\mathbb{A}$. Writing $D = q^{1/2}$, we define
\[
\mathbb{H} := D^{-1} \mathbb{A} = \sum_{g \geq 2} \sum_{n \geq 0} \chi_y(C_g[n]) t^{n+1-g} q^{g-1}/(g-1)
\]
\[
\mathbb{X} := \frac{\mathbb{H}}{X_y(\mathbb{H})} = \frac{y^{1/2}(1 - e^{y^{1/2} - y^{-1/2})})}{1 - ye^{y^{1/2} - y^{-1/2})}}
\]

As we recall in Section 2, when the surface $S$, line bundle $L$, and linear system $\mathbb{P}^d \subset |L|$ are such that $L$ has no higher cohomology and the relative Hilbert scheme $C[n] \to \mathbb{P}^d$ has nonsingular total space, the Hirzebruch genus is given by some universal expression
(depending on $n, \delta$) in the Chern classes of $S, L$. Thus we may write $\chi_{-y}(C_{[S,L],\delta}^n)$ for the evaluation of this expression for any $S, L$, or indeed any specification of the Chern numbers $c_1(S)^2, c_1(S) L, L^2, c_2(S)$. We write $\chi_{-y}(C_{[S,L],\delta}^n) := \chi_{-y}(C_{[S,L],\chi(L)-1}^n)$ corresponding to the complete linear system. In speaking of $\chi_{-y}(C_{[S,L],\delta}^n)$, we mean the evaluation on the specified Chern numbers of the formulas which usually give these quantities. More generally in the same way we may 'integrate tautological classes over $C_{[S,L],\delta}^n$. Arguments similar to those of [Göt2, EGL, KST, GS] establish:

**Theorem 2.** There exist two more series $B_1, B_2 \in \mathbb{Q}[y^{\pm 1/2}, t^{-1}][[t,q]]$ such that the following hold:

\[
X^k B_1^2 B_2^{2k} L K S \chi(O_S)/2 A^{1-\chi(O_S)}/2 = \sum_g \sum_{n \geq 0} t^{n+g+1} q^{g-1} \chi_{-y}(C_{[S,L],\chi(L)-1-k}^n)
\]

\[
H^k B_1^2 B_2^{2k} L K S \chi(O_S)/2 A^{1-\chi(O_S)}/2 = \sum_g \sum_{n \geq 0} t^{n+g+1} q^{g-1} \left( \int_{C_{[S,L],\chi(L)}} X_{-y}(C_{[S,L],\chi(L)-1-k}^n) \cap H^k \right)
\]

The meaning of the sum is that we fix $c_1(S)^2, c_2(S), c_1(S) L$, and vary only $L^2$, which we track by $g = g(L)$.

In both formulas, the summand on the RHS vanishes unless $g \geq k + 2 + L K S - \chi(O_S)$. Indeed, this may be checked when $L$ is an actual line bundle with no higher cohomology on an actual surface $S$, where it amounts to $\dim |L| \geq k$.

The Hodge polynomials of the relative Hilbert schemes on $K3$ surfaces were computed by Kawai and Yoshioka; specializing these gives an explicit formula for $K$. In the present note we will compute $A$.

We introduce some notation in order to state the answer. Let $z$ be a complex variable and $\tau$ a variable from the complex upper half plane. We denote $y = e^{2\pi iz}, q := e^{2\pi i \tau}$. We denote one of the standard theta functions by

\[
\theta(z) = \theta(z, \tau) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n \frac{1}{2}(n+\frac{1}{2})^2} y^{n+\frac{1}{2}} = q^{1/8} (y^{1/2} - y^{-1/2}) \prod_{n>0} (1-q^n)(1-q^n y)(1-q^n/y),
\]

and the Eisenstein series of weight 2 by

\[
G_2(\tau) := -\frac{1}{24} + \sum_{n > 0} \left( \sum_{d|n} d \right) q^n.
\]

By abuse of notation we also write $\theta(y) := \theta(z), G_2(q) := G_2(\tau)$. Let $' denote $\frac{1}{2\pi i} \frac{\partial}{\partial \tau} = y \frac{\partial}{\partial y}$, and $D = \frac{1}{2\pi i} \frac{\partial}{\partial \tau} = y \frac{\partial}{\partial y}$.

**Definition 3.** We write

\[
A(y, q) := \sum_{nd>0} \text{sgn}(d) n^2 y^d q^{nd}.
\]
Here, as also a number of times below $\sum_{nd>0}$ denotes the sum over pairs $n, d$ of integers with $nd > 0$. Thus $A(y, q)$ can be viewed as a theta function for an indefinite lattice, as considered e.g. in [Z], [GZ].

**Remark 4.** It is elementary to show that $A(y, q)$ can be rewritten as follows.

$$A(y, q) = -\frac{1}{3} \theta''(y) - 2G_2(q) \frac{\theta'(y)}{\theta(y)}$$

$$= \frac{1}{\theta(y)} \left( -\frac{1}{6} D\theta'(y) - 2G_2(q) \theta'(y) \right).$$

The equality in the second line holds by the heat equation $\theta''(y) = \frac{1}{2} D\theta(y)$. We include a sketch of the proof of the equality in the first line (communicated to us by Don Zagier):

Denote $y_1 = e^{2\pi i z_1}$, $y_2 = e^{2\pi i z_2}$ for complex variables $z_1, z_2$. In [Z] it is proved that

$$\theta'(0) \theta(y_1 y_2) = \frac{y_1 y_2 - 1}{(y_1 - 1)(y_2 - 1)} - \sum_{nd>0} \text{sgn}(d)y_1^dy_2^qnd.$$

We apply $\left( \frac{1}{2\pi i} \frac{\partial}{\partial z_2} \right)^2$ to both sides of (1), take the coefficient of $z_2^0$ and put $y_1 := y$. On the right hand side this gives $A(y, q)$. On the left hand side after some computation we get

$$-\frac{1}{3} \theta''(y) - 2G_2(q) \frac{\theta'(y)}{\theta(y)}.$$

We abbreviate

$$[n]_y := \frac{y^{n/2} - y^{-n/2}}{y^{1/2} - y^{-1/2}} = y^{(n-1)/2} + y^{(n-3)/2} + \ldots + y^{-(n-1)/2}.$$

**Theorem 5.** Let $L_g$ be a linear system of genus $g$ curves on an Abelian surface $A_g$ such that the relative Hilbert schemes $\mathcal{H}_g \rightarrow \mathcal{L}_g$ are smooth. Then,

$$\mathcal{A} := \sum_{g \geq 2} \sum_{n \geq 0} \chi_{-g} \left( \mathcal{H}_g \right) t^{n+1}q^{n-1}$$

$$= (t + t^{-1} - y^{1/2} - y^{-1/2}) \sum_{n>0,d>0} n^2 [d]_y [d]_{ty^{1/2}} [d]_{ty^{-1/2}} q^{nd}$$

$$= \sum_{n>0,d>0} \frac{n^2 y^{d/2} - y^{-d/2}}{y^{1/2} - y^{-1/2}}(t^d + t^{-d} - y^{d/2} - y^{-d/2}) q^{nd}$$

$$= \frac{1}{y^{1/2} - y^{-1/2}} \left( A \left( y^{1/2} / t, q \right) + A \left( y^{1/2} / t, q \right) - A(y, q) \right)$$

The following is a specialization (and slight reformulation) of a result of Kawai and Yoshioka [KY].

**Theorem 6.** [KY] Let $L_g$ be a linear system of genus $g$ curves on a $K3$ surface $K_g$ such that the relative Hilbert schemes $\mathcal{H}_g \rightarrow \mathcal{L}_g$ are smooth.

$$\mathcal{K} := \sum_{g \geq 0} \sum_{n \geq 0} \chi_{-g} \left( \mathcal{H}_g \right) t^{n+1}q^{n-1} = \frac{y^{-1/2} - y^{1/2}}{\Delta(q)} \frac{\theta'(0)^3}{\theta(y^{1/2})^2 \theta(ty^{1/2})^2 \theta(y)}. $$
Finally, we write explicitly the specialization ($y = 1$) to Euler numbers. We denote $B_1(q, t) := B_1(q, 1, t)$, $B_2(q, t) := B_2(q, 1, t)$. In [GS] we have introduced the function

$$D\widetilde{G}_2(y, q) := \sum_{n,d>0} n[d]_y^2 q^{nd} = \frac{D \log \frac{\theta'(0)}{\theta(t)}}{y - 2 + y^{-1}},$$

(the second identity is elementary). From the third line in Theorem 5, we have

$$A(q, 1, t) = (t - 2 + t^{-1})D\widetilde{G}_2(t, q) = DD \log \frac{\theta'(0)}{\theta(t)}$$

$$c(q, 1, t) = (t - 2 + t^{-1})D\widetilde{G}_2(t, q) = D \log \frac{\theta'(0)}{\theta(t)}$$

From Theorem 5 it is easy to see that

$$\mathbb{K}(q, 1, t) = \theta'(0)^2 \Delta(q)\theta(t)^2 = \frac{1}{(t - 2 + t^{-1})q \prod_{n>0}(1 - q^n)^2(1 - q^n t)^2(1 - y^n/t)^2} = \frac{1}{\phi_{10,1}(t, q)}$$

where $\phi_{10,1}(t, q)$ is up to normalization the unique Jacobi cusp form on $SL(2, \mathbb{Z})$ of weight 10 and index 1. It is easy to see that $X_{-1}(x) = (1 + x)$, thus

$$\mathbb{X}(q, 1, t) = \frac{D \log \frac{\theta'(0)}{\theta(t)}}{1 + D \log \frac{\theta'(0)}{\theta(t)}}$$

Putting this together we get the following.

**Corollary 7.** The generating series of integrals against the hyperplane class is:

$$\sum_{g \geq k+2+2L_KS-\chi(\mathcal{O}_S)} \sum_{n \geq 0} \left( \int_{[S/L]} c(C_{/[S/L]}^n) \cap H^k \right) t^{n-g+1} q^{g-1}$$

$$= \left( D \log \frac{\theta'(0)}{\theta(t)} \right)^k B_1^{K_2} B_2^{L_KS} \left( \frac{\theta'(0)^2}{\Delta(q)\theta(t)^2} \right)^{\chi(\mathcal{O}_S)/2} \left( D \log \frac{\theta'(0)}{\theta(t)} \right)^{1-\chi(\mathcal{O}_S)/2}.$$

The generating series of Euler characteristics is:

$$\sum_{g \geq k+2+2L_KS-\chi(\mathcal{O}_S)} \sum_{n \geq 0} \chi(C_{/[S/L]}^n) t^{n-g+1} q^{g-1}$$

$$= \left( \frac{D \log \frac{\theta'(0)}{\theta(t)}}{1 + D \log \frac{\theta'(0)}{\theta(t)}} \right)^k B_1^{K_2} B_2^{L_KS} \left( \frac{\theta'(0)^2}{\Delta(q)\theta(t)^2} \right)^{\chi(\mathcal{O}_S)/2} \left( D \log \frac{\theta'(0)}{\theta(t)} \right)^{1-\chi(\mathcal{O}_S)/2}.$$
We return to the setting of [GS], where polynomials \( N_{[S,L],\delta}(y) \) were defined by the following formula, in which \( g = g(L) \).

\[
(2) \quad \sum_{n \geq 0} \chi_y \left( c_{[S,L],\delta}^{[n]} \right) t^{n+1-g} = \sum_{i=0}^{\infty} N_{[S,L],\delta}(y)(t + t^{-1} - y^{1/2} - y^{-1/2})^{g-i-1}.
\]

This formula refines the change of variable used to pass from Euler numbers of Hilbert schemes to enumerative information (of the sort sometimes called Gopakumar-Vafa or 'BPS' invariants). In the good situation where \([S,L] \) comes from a line bundle on a surface with no higher cohomology and the appropriate relative Hilbert schemes are nonsingular, \( N_{[L],\delta}(y) \) vanishes conjecturally in terms of two undetermined power series, \([GS, \text{Conj. 66}] \) gives a conjectural generating function for the highest order term \( N_{[S,L],\delta}^{\delta} \). To establish this formula, and to better understand the \( N_{[S,L],\delta}^{i} \), it remains to develop the series introduced here in the variable \( x = (t + t^{-1} - y^{1/2} - y^{-1/2}) \).

From Theorem 2 and Equation (2) we obtain the generating series:

**Corollary 8.** Let \( S, L \) be arbitrary, \( g \) the arithmetic genus of \( L \), then

\[
\sum_{i} N_{[S,L],\chi(L)-1-k}^{i}(y)x^{g-i-1} = \text{Coeff}_{q^{g-1}} \left[ x^{k} \prod_{s \geq 1} K^{2s} \mathbb{B}_{2}^{s} \mathbb{K}^{s} \mathbb{A}^{1-\chi(O_{S})/2} \right].
\]

We define

\[
\tilde{\Delta}(y, q) := \frac{\Delta(q)y^2}{(y - 2 + y^{-1})y'(0)^2} = q \prod_{n>0} (1 - q^n)^{20}(1 - q^n y^2) (1 - q^n y)^2.
\]

**Corollary 9.** [GS, Conj.66] If \( K_{S} \) is numerically trivial, then

\[
N_{\chi(L)-1-k}^{\chi(L)}(y)(y, x) = \text{Coeff}_{q^{g-1}} \left[ \frac{D G_{2}(y, q) D^{\chi(O_{S})/2} \Delta(y, q)^{1-\chi(O_{S})/2}}{\Delta(y, q)^{\chi(O_{S})/2}} \right].
\]

More generally, we want expressions for all the \( N_{i} \), or in other words, we want to expand \( \mathbb{A} \) and \( K \) in \( x \) rather than \( t \).

We define polynomials \( s_{n}(y) \) and their generating function \( S(y, x) \) by

\[
s_{n}(y) := \sum_{k=0}^{n} \binom{n}{k}^{2} y^{k-n/2} = \text{Coeff}_{t^{n}} \left( y^{-1/2}(1 + t)(1 + ty) \right)^{n},
\]

\[
S(y, x) := \sum_{n \geq 0} (-1)^n \frac{s_{n}(y)}{(y^{1/2} - y^{-1/2})^{2n+1}} x^{n+1}.
\]

Then we have
Remark 11. It is remarkable that the generating functions for Abelian and K3 surfaces 
\( \sum_{nd>0} \text{sgn}(d) e^{S(y,x)(d-n)} y^d q^{nd} \), on an Abelian surface we have for 
\[ h \]
The first author conjectured, on the basis of numerical evidence, that Equations (4), (5) 
are determined by the same polynomials \( P \) for the 
Max-Planck-Institut für Mathematik, Bonn.

To see explicitly the development of \( A, K \) in \( x \), we expand 
\[ e^{S(y,x)z} = \sum_{n \geq 0} P_n(y, z) \frac{x^n}{(y^{1/2} - y^{-1/2})^n} \] 
with \( P_n(y, x) \in \mathbb{Q}[\frac{1}{y-1}, z] \), e.g.

\[ P_0 = 1, \quad P_1 = z, \quad P_2 = \frac{z^2}{2} - \frac{z y + 1}{2 y - 1}, \quad P_3 = \frac{z^3}{6} - \frac{z^2 y + 1}{4 y - 1} + \frac{z y^2 + 4 y + 1}{3 (y - 1)^2}, \quad P_4 = \frac{z^4}{24} - \frac{z^3 y + 1}{2 y - 1} + \frac{z^2 11 y^2 + 38 y + 11}{24 (y - 1)^2} - \frac{z y^3 + 9 y^2 + 9 y + 1}{4 (y - 1)^3}. \]

Remark 11. It is remarkable that the generating functions for Abelian and K3 surfaces 
are determined by the same polynomials \( P \); Using Corollary [8] we have on a K3 surface 
\[ \sum_y N_{[S,L_g]}^g(y) q^{g-1} = \frac{1}{\Delta(y,q)}, \] 
and for \( h \geq 1 \):

\[ \sum_y N_{[S,L_g]}^{g-h}(y) q^{g-1} = \frac{1}{\Delta(y,q)} \sum_{nd>0} \text{sgn}(d) \frac{P_{h-1}(y, d-n)}{(y^{1/2} - y^{-1/2})^h} y^d q^{nd}. \]

On an Abelian surface we have for \( h \geq 2 \),

\[ \sum_y N_{[A,L_g]}^{g-h}(y) q^{g-1} = \sum_{nd>0} \text{sgn}(d) \frac{P_{h-1}(y, d)}{(y^{1/2} - y^{-1/2})^h} n^2 (y^d - 1) q^{nd}. \]

In fact, we first arrived at the formula asserted in Theorem 5 in the following manner. 
The first author conjectured, on the basis of numerical evidence, that Equations (1), (5) 
held for some undetermined coefficients \( P_i \). This suffices in principle to (conjecturally) 
determine \( A \) from \( K \). Don Zagier made this determination explicit, providing a formula 
for the \( P_i \) and for \( A \). Finally we have reversed the procedure, proving the formula for \( A \) 
geometrically and deriving Equations (1), (5) as consequences.

Acknowledgements. We thank Don Zagier for the contributions mentioned imme-
diately above, and Kôta Yoshioka for helpful correspondence about sheaves on Abelian 
surfaces. Part of this work was carried out while the first-named author was at the 
Max-Planck-Institut für Mathematik, Bonn.
2. Universality arguments

In this section we give the proof of Theorem 2.

**Definition 12.** Let $S$ be a surface, $L$ a line bundle on $S$, and $L^{[n]}$ the corresponding tautological vector bundle on $S^{[n]}$. Let $e^x$ denote a trivial line bundle with nontrivial $\mathbb{C}^*$ action with equivariant first Chern class $x$. Then we define

$$D^{S,L}(x, y, t) := \sum_{n \geq 0} t^n \int_{S^{[n]}} X_{-y}(T S^{[n]}) \frac{c_n(L^{[n]} \otimes e^x)}{X_{-y}(L^{[n]} \otimes e^x)}.$$

As explained in [GS, Prop. 47], for a linear subsystem $\mathbb{P}^\delta \subset |L|$ such that the relative Hilbert schemes $C^{[n]}_{S,L,\delta} \to \mathbb{P}^\delta$ are all smooth – e.g., a general $\delta$-dimensional linear subsystem when $L$ is $\delta$-very-ample [KST] – we may extract the $\chi - y$ genera by taking a residue:

$$\sum_{n \geq 0} \chi_{-y}(C^{[n]}_{S,L,\delta}) t^n = \text{res}_{x=0} \left[ D^{S,L}(x, y, t) \left( \frac{X_{-y}(x)}{x} \right)^{\delta + 1} \right] dx.$$

Since $D^{S,L}$ is defined by a tautological integral, by [EGL] it depends only on the Chern numbers $c_2(S), c_1(S)^2, c_1(S), c_1(L), c_1(L)^2$. Thus we may make sense of it for arbitrary values of these quantities. Thus we view Equation 6 as defining the quantities $\chi_{-y}(C^{[n]}_{S,L,\delta})$ in terms of $\delta, n$, and the Chern numbers of $S, L$, without any assumptions on even the existence of such a surface and line bundle.

The change of variable

$$q(x) = \frac{x}{X_{-y}(x)} = \frac{x^{1/2}(1 - e^x(y^{1/2} - y^{-1/2}))}{1 - ye^x(y^{1/2} - y^{-1/2})}$$

is inverse to

$$x(q) = \frac{\log(1 - y^{1/2}q) - \log(1 - y^{-1/2}q)}{y^{-1/2} - y^{1/2}} = \sum_{n > 0} [n]_y q^n n.$$

We find $\frac{dx}{dq} = \frac{1}{(1 - y^{-1/2}q)(1 - y^{1/2}q)}$. Plugging into the residue formula (6), and writing for convenience

$$\overline{D}^{S,L}(q, y, t) := D^{S,L}(x(q), y, t)$$

we find

$$\sum_{n \geq 0} \chi_{-y}(C^{[n]}_{S,L,\delta}) t^n = \text{res}_{q=0} \left[ \overline{D}^{S,L}(q, y, t) q^{-(\delta + 1)} \frac{1}{(1 - y^{-1/2}q)(1 - y^{1/2}q)} \right]$$

$$= \text{Coeff}_{q^1} \left[ \overline{D}^{S,L}(q, y, t) \frac{1}{(1 - y^{-1/2}q)(1 - y^{1/2}q)} \right].$$

Note this differs from [GS] by the normalization by $y^{-\dim/2}$.
As the term in square brackets is a power series, we may re-sum to obtain

\[
\sum_{\delta \geq 0} \sum_{n \geq 0} \chi_y(C^{[n]}_{S,L,d}) t^n q^\delta = \frac{D^{S,L}(q,y,t)}{(1-y^{-1/2}q)(1-y^{1/2}q)}.
\]

Since \(X_y\) is a genus, by [EGL] there exist power series \(a_0, a_1, a_2, a_3 \in \mathbb{Q}[y^{1/2}][[t, x]]\) such that \(D^{S,L}(x, y, t) = a_0^{(L)} l^2 A_1 l^2 A_2 l^{K_S} a_3^{(O_S)}\) (for a detailed argument, see [GS, Sec. 3.2]). Setting

\[ A_i(q, y, t) := a_i(x(q), y, t) \in \mathbb{Q}[y^{1/2}][[t, q]], \]

we get

\[
\sum_{\delta \geq 0} \sum_{n \geq 0} \chi_y(C^{[n]}_{S,L,d}) t^n q^\delta = A_0^{(L)} A_1 l^2 A_2 l^{K_S} A_3^{(O_S)} \frac{1}{(1-y^{-1/2}q)(1-y^{1/2}q)}.\]

Note that by [K], the coefficient of \(q^0\) in \(D^{S,L}(x, y, t)\) is \((1-y^{-1/2}t)(1-y^{1/2}t)t^{g-1}\) for \(g\) the arithmetic genus of a curve in \([L]\). Thus \(A_i(q, y, t) \in ((1-y^{-1/2}t)(1-y^{1/2}t))^{t_i} + q\mathbb{Q}[y^{1/2}][[t, q]]\) with \(t_0 = 1, t_1 = 0, t_2 = 1, t_3 = -1\).

If \(R\) is a commutative ring, and \(f \in R[[q]]\) is an invertible power series, we denote by \(f^{-1}\) its compositional inverse. Let

\[ X(q, y, t) := \left(\frac{qt}{A_0}\right)^{-1} \in \mathbb{Q}[y^{1/2}, t^{-1}][[t, q]]. \]

This is set up so that

\[ \frac{X(q, y, t)}{A_0(X, y, t)} = q/t \]

and hence \(A_0(X, y, t) = q^{-1}tX(q, y, t)\).

Denoting \(B_1(q, y, t) := A_1(X, y, t), B_2(q, y, t) := A_2(X, y, t)q/t, B_3(q, y, t) := A_3(X, y, t)t/q,\) the substitution \(q \mapsto X\) gives:

\[
\sum_{\delta \geq 0} \sum_{n \geq 0} \chi_y(C^{[n]}_{S,L,d}) t^{n+1-g} X^\delta = \frac{(X/q)^{\chi(L)} B_1^{K_S} (B_2/q) L^{K_S} B_3^{(O_S)}}{(1-y^{-1/2}X)(1-y^{1/2}X)}.\]

As in [Göt2] we use the residue formula. Let \(R\) be a commutative ring, and \(f \in R[[q]], g \in aq + q^2 R[[q]],\) with \(a\) invertible in \(R,\) then

\[ f = \sum_{k=0}^{\infty} g(q)^k \left[ \frac{f(q)Dg(q)}{g(q)^{k+1}} \right] \bigg|_{q=0}.\]

We apply this to Equation (10) with \(g(q) = X.\) On the one hand, \(\sum_{n \geq 0} \chi_y(C^{[n]}_{S,L,d}) t^{n+1-g}\) is the coefficient of \(X^\delta\) of the RHS. On the other, taking the coefficient by the residue
formulas above gives, with $g$ again the arithmetic genus of a curve in $|L|$, 
\[
\sum_{n\geq 0} \chi_y(C^{[n]}_{[S,L]}) t^{n+1-g} = \frac{D X \cdot X^{-\delta-1} \cdot (X/q)^{\chi(L)} \prod_{k=1}^{2} (1 - y^{1/2} X) (1 - y^{-1/2} X)^{\chi(O_{S})}}{(1 - y^{1/2} X) (1 - y^{-1/2} X)}
\]
\[
= \text{Coeff}_{q^{g-1}} \frac{D X \cdot (X)^{\chi(L) - \delta} \prod_{k=1}^{2} (1 - y^{1/2} X) (1 - y^{-1/2} X)}{(1 - y^{1/2} X) (1 - y^{-1/2} X)}.
\]

We collect terms with fixed $k = \chi(L) - 1 - \delta$, i.e. $k$ is the number of point conditions we impose to cut down to $\mathbb{P}^{k}$. We now explicitly note the genus of the line bundle appearing in its subscript.

**Corollary 13.** Fix $k \geq 0$, then
\[
\sum_{g \geq k+2 - L K S - \chi(O_{S})} \sum_{n \geq 0} \chi_y(C^{[n]}_{[S,L]}) t^{n+1-g} q^{g-1} = \frac{\sum_{g \geq k+2 - L K S - \chi(O_{S})} \chi_y(C^{[n]}_{[S,L]}) t^{n+1-g} q^{g-1}}{(1 - y^{1/2} X) (1 - y^{-1/2} X)}.
\]

In particular, when $S = A$ is an abelian surface,
\[
\frac{D X}{(1 - y^{1/2} X) (1 - y^{-1/2} X)} = \sum_{g \geq 2} \sum_{n \geq 0} \chi_y(C^{[n]}_{[A,L]}) t^{n+1-g} q^{g-1} =: \mathbb{A}.
\]

Note that
\[
\frac{D X}{(1 - y^{1/2} X) (1 - y^{-1/2} X)} = D \left( \frac{\log(1 - y^{1/2} X) - \log(1 - y^{-1/2} X)}{y^{-1/2} - y^{1/2}} \right).
\]

Thus, by $\mathbb{H} = D^{-1} \mathbb{A}$, we see
\[
(11) \quad \mathbb{H} = \log(1 - y^{1/2} X) - \log(1 - y^{-1/2} X) = x(X).
\]

We have already seen how to invert this function:
\[
(12) \quad X = q(\mathbb{H}) = \frac{\mathbb{H}}{X_y(\mathbb{H})} = \frac{y^{1/2}(1 - e^{H(y^{1/2} - y^{-1/2})})}{1 - y e^{H(y^{1/2} - y^{-1/2})}}.
\]

Similarly, when $S = K$ is a K3 surface,
\[
\mathbb{A} \mathbb{B}^{2} = \sum_{g \geq 0} \sum_{n \geq 0} \chi_y(C^{[n]}_{[K,L]}) t^{n+1-g} q^{g-1} =: \mathbb{K},
\]
and so $\mathbb{B}^{3} = (\mathbb{K}/\mathbb{A})^{1/2}$. Putting everything together, this proves the first formula of Theorem 2.

We now prove the second formula. The argument takes place for fixed $[S, L]$. Write $H$ for the pullback of the hyperplane class from $|L|$. Denote
\[
Z_{[S,L]}(x, y, t) := t^{1-g} D^{S,L}(x, y, t) q(x)^{-\chi(L)}.
\]
Equation (3) asserts that when the relevant spaces are smooth, we have

\[
\sum_{n \geq 0} X_{-g}(C_{[S,L],\chi(L)-1-k})t^{n+1-g} = \text{res}_{x=0} \left[ t^{1-g} D^{S,L}(x, y, t)q(x)^{-\delta-1} \right] dx
\]

\[
= \text{res}_{x=0} \left[ Z_{[S,L]}(x, y, t)q(x)^k \right] dx.
\]

By the same proof, if the \(C_{[S,L]}^{[n]} \) are smooth, we have

\[
\sum_{n \geq 0} \left( \int_{C_{[S,L]}^{[n]}} X_{-g}(C_{[S,L]}^{[n]}) \cap H^k \right) t^{n+1-g} = \text{res}_{x=0} \left[ Z_{[S,L]}(x, y, t)x^k \right] dx.
\]

Write \( f(q, y, t) := \mathbb{H}^{K_2}_1 \mathbb{H}^{LK_S}_2 \mathbb{H}^{\chi(O_S)/2}_1 A^{1-\chi(O_S)/2}_1 \). We have shown

\[
\text{res}_{x=0} \left[ Z_{[S,L]}(x, y, t)q(x)^k \right] dx = \text{Coeff} \left[ \mathbb{X}^k f(q, y, t) \right].
\]

Let again \( x(q) \) from (7) be the compositional inverse of \( q(x) \). Write \( x(q)^k := \sum_{l \geq k} a_l(y)q^l \), such that \( \sum_{l \geq k} a_l(y)q^l(x)^l = x^k \). Thus we get

\[
\text{res}_{x=0} \left[ Z_{[S,L]}(x, y, t)x^k \right] dx = \text{res}_{x=0} \left[ \sum_{l \geq k} a_l(y)Z_{[S,L]}(x, y, t)q(x)^l \right] dx
\]

\[
= \text{Coeff} \left[ \sum_{l \geq k} a_l(y)\mathbb{X}^l f(q, y, t) \right] = \text{Coeff} \left[ x(\mathbb{X})^k f(q, y, t) \right] = \text{Coeff} \left[ \mathbb{H}^k f(q, y, t) \right].
\]

The last equality is by (11).

3. Calculations for the Abelian surface

Kawai and Yoshioka determined \( \mathbb{K} \) by comparing various moduli spaces of stable sheaves and stable pairs on a K3 surface [KY]. A modification of their argument suffices to determine \( \mathbb{A} \) except for the coefficient of \( t^0 \), and a vanishing result in [GS] allows us to determine this coefficient from the rest.

3.1. Yoshioka’s lemma.

**Lemma 14. [Yos99, Lem. 2.1]** Let \( X \) be a smooth projective surface with polarization \( H \), and let \( C \) be a curve class minimizing \( C.H \). For a sheaf \( F \) with \( c_1(F) = dC \), we write \( \text{deg}(F) = d = (c_1(F).H)/(C.H) \). Let \((r, d)\) and \((r_1, d_1)\) be pairs of integers such that \( r_1d - d_1r = 1 \), with \( r \geq 0 \) and \( r_1 > 0 \). Let \((r_2, d_2) := (r, d) - (r_1, d_1) \). Below let \( E_i \) be of rank \( r_i \) and degree \( d_i \), and let \( E_1 \) always be a vector bundle.

- If \( E_1, E_2 \) are \( \mu \)-stable, then every nontrivial extension

\[
0 \to E_1 \to E \to E_2 \to 0
\]

is \( \mu \)-stable.
• If $E_1, E$ are $\mu$-stable, then for any vector subspace $V \subset \text{Hom}(E_1, E)$, if the evaluation map $V \otimes E_1 \to E$ is not injective, then it is surjective in codimension 1. Moreover
  - If $V \otimes E_1 \to E$ is injective, then the cokernel is $\mu$-stable.
  - If $V \otimes E_1 \to E$ is surjective in codimension 1, the kernel is $\mu$-stable.

Remark 15. In [Yos99], this lemma is proven under the assumption that $NS(X) = \mathbb{Z}$, but in [KY] it is pointed out that the proof only requires the assumption stated above.

Note that if $d = 1$, i.e. we are looking at sheaves with $c_1 = C$, then the condition $r_1 d - d_1 r = 1$ is always satisfied by $(r_1, d_1) = (1, 0)$, hence we may always take $E_1 = \mathcal{O}_X$. We now extract explicitly the special cases we will be concerned with.

Corollary 16. Let $X$ be a smooth projective surface with polarization $H$, and let $C$ be a curve class minimizing $C.H$.

• Assume $F$ is $\mu$-stable and $c_1(F) = C$. Then every nontrivial extension $0 \to \mathcal{O}_X \to E \to F \to 0$ is stable.
• Assume $E$ is $\mu$-stable of positive rank and $c_1(E) = C$. Then any non-zero section induces an exact sequence $0 \to \mathcal{O}_X \to E \to F \to 0$ and $F$ is stable.

Proof. The only thing which is not immediate from the lemma is to check is the possibility in the second case that $\mathcal{O}_X \to E$ is surjective in codimension 1 rather than being injective. But then in any case $E$ must either be torsion (which it is not by assumption) or the map from $\mathcal{O}_X$ must be an isomorphism in codimension 1, in which case the kernel must be a torsion subsheaf of $\mathcal{O}_X$, hence zero. \hfill \square

Let $\mathcal{M}(r, d, e)$ denote the moduli space of semistable sheaves of rank $r$, degree $d$, and Euler number $e$. We will below always assume that $\mathcal{M}(r, d, e)$ only consists of $\mu$-stable sheaves. Let $\mathcal{P}^1(r, d, e)$ be the space of “coherent systems” [LeP], i.e. it parameterizes a stable sheaf (of rank $r$, degree $d$, and Euler number $e$) plus a section, up to isomorphism. This corresponds to a special choice of the stability condition for pairs, which ensures that a pair of sheaf and section is stable, if and only if the sheaf is stable. There is a forgetful map $\mathcal{P}^1(r, d, e) \to \mathcal{M}(r, d, e)$ with fibre $\mathbb{P}\text{Ext}^1(F, \mathcal{O}_X)$ over a sheaf $E$.

The above corollary implies the existence of another map:

Corollary 17. For $r \geq 0$, there exists a morphism $\mathcal{P}^1(r + 1, d, e + \chi(\mathcal{O}_X)) \to \mathcal{M}(r, d, e)$ which takes $\mathcal{O}_X \to E$ to its cokernel. The fibre over a sheaf $F$ is $\mathbb{P}\text{Ext}^1(F, \mathcal{O}_X)$.

Let us consider the space $\mathcal{P}^1(0, 1, e)$. This by definition consists of a stable, rank zero sheaf $E$ together with a section $\mathcal{O}_X \to E$. By stability, $E$ is a pure sheaf supported on a curve (i.e. torsion free with rank one on its support). As explained in [PT3, Appendix B], dualizing gives an isomorphism between $\mathcal{P}^1(0, 1, e)$ and the relative Hilbert scheme of degree $e + g - 1$, where $g$ is the arithmetic genus of the support of $E$.\footnote{Note we are not indexing by the Mukai vector.}
3.2. A relation between moduli spaces. We now specialize to $K_X = \mathcal{O}_X$. Note in this case that if $E$ is any stable sheaf with zero rank or positive first Chern class, then

$$H^2(E) = \text{Hom}(E, K_X)^* = \text{Hom}(E, \mathcal{O}_X)^* = 0$$

In the zero rank case the last equality is obvious; for positive rank it is ensured by stability. Additionally we have $\text{Ext}^1(F, \mathcal{O}_X) = H^1(F)^*$ by Serre duality. Thus the dimensions of the fibres of the two maps to $\mathcal{M}(r, d, e)$ are related:

$$P^1(r, d, e) \xrightarrow{\text{PH}^0(F)} \mathcal{M}(r, d, e) \xleftarrow{\text{PH}^1(F)} P^1(r + 1, d, e + \chi(\mathcal{O}_X)).$$

We indicate throughout the first map by $P \to \mathcal{M}$ and the second by $\mathcal{M} \leftarrow P$.

We denote the Hodge polynomial of $V$ by $[V]$. We write $L$ for the Hodge polynomial of the affine line, and $[n] = [P^{n-1}]$ for $n \in \mathbb{Z}_{>0}$. We also write $[0] = 0$ and $[-n] = -L^{-n}[n]$.

Let $\mathcal{M}(r, d, e)$ denote the locus with $h^0 = s$. Since the map $P \to \mathcal{M}$ is given on the above strata as the projectivization of a vector bundle, we have:

$$[P^1(r, d, e)] = \sum_i [e + i][\mathcal{M}(r, d, e)_{e+i}]$$

Considering instead the map $\mathcal{M} \leftarrow P$ and using the vanishing of $h^2(F)$ to write $\chi(F) = h^0(F) - h^1(F)$, we have:

$$[P^1(r + 1, d, e + \chi(\mathcal{O}_X))] = \sum_i [i][\mathcal{M}(r, d, e)_{e+i}]$$

As observed in [KY], this establishes a recursion:

$$[P^1(r, d, e)] = \sum_i [e + i][\mathcal{M}(r, d, e)_{e+i}]$$

$$= [e][\mathcal{M}(r, d, e)] + L^e \sum_i [i][\mathcal{M}(r, d, e)_{e+i}]$$

$$= [e][\mathcal{M}(r, d, e)] + L^e \sum_i [P^1(r + 1, d, e + \chi(\mathcal{O}_X))].$$

Because the dimension of $P$ contains a term $-re$, iterating this leads to empty moduli spaces $P$ when either (1) we are working on the a K3 surface where $e$ is increased at each step by $\chi(\mathcal{O}_X) = 2$, or (2) when $e > 0$ and we are on the abelian surface. In these cases we may sum the recursion (which is to say, the following sum is really a finite sum):

$$(13) \quad [P^1(r, d, e)] = \sum_{b=0}^{\infty} [e + b\chi(\mathcal{O}_X)]L^{\sum_{j=0}^{b-1} e + j\chi(\mathcal{O}_X)}[\mathcal{M}(r + b, d, e + b\chi(\mathcal{O}_X))].$$

To evaluate this sum, it remains to (1) use the deformation equivalence of moduli of sheaves on K3 or Abelian surfaces and Hilbert schemes of points on these surfaces and then (2) plug in the formula for the Hodge polynomial of the Hilbert scheme [Göti]. For the K3 surfaces, this is done in [KY]. We proceed now to the case of the Abelian surface, where one must moreover deal with the $e \leq 0$ case in some other way.

---

4When $e = 0$ on the abelian surface, we learn that $[P^1(r, d, 0)]$ is independent of $r \geq 0$, but we have not found any use for this fact.
3.3. Abelian surfaces. Let $A$ be an Abelian surface.

We change notation slightly from the previous section, and write $\mathcal{M}_{\mathrm{num}}(r, d, e)$ for what was written there $\mathcal{M}(r, d, e)$: the moduli space of sheaves where $c_1 = d$ is fixed only in cohomology. We now denote $\mathcal{M}(r, C, e)$ the moduli space where $c_1 = C$ is fixed in Pic, and similarly for the spaces $\mathcal{P}$. Note the discussion there for $\mathcal{M}, \mathcal{P}$ and thus Equation \((13)\) holds for these spaces as well.

Twisting by line bundles gives an isomorphism $A^\vee / A^\vee[r] \times \mathcal{M}(r, C, e) \cong \mathcal{M}_{\mathrm{num}}(r, d, e)$. The space $\mathcal{M}(r, d, e)$ has dimension

$$\dim \mathrm{Ext}^1(F, F) = 2 - \chi(F, F) = 2 - \int \mathrm{ch}(F)^\vee \mathrm{ch}(F) = 2 + C^2 - 2re = 2g(C) - 2re$$

If this is greater than 2, then according to [Yos01, Thm. 0.1], $\mathcal{M}(r, d, e)$ is deformation equivalent to $A^\vee \times A^{[n]}$ for the appropriate $n$.

For $\chi_{-y}(\mathcal{M}(r, C, e)) = 0$. On the other hand, according to [Yos01 Lem. 4.19], when $r| (g - 1)$, then $\mathcal{M}(r, C, \frac{g-1}{r})$ is a finite set of $r^2$ points. Thus for $e > 0$, equation \((13)\) gives:

$$[\mathcal{P}^1(0, C, e)] = [e] \sum_{b=0}^\infty \mathbb{L}^{be}[\mathcal{M}(b, C, e)] = [e] \left( \sum_{b=0}^{b<(g-1)/e} \mathbb{L}^{be}[A^{g-1-be}] + \mathbb{L}^{g-1} \sum_{b=(g-1)/e} b^2 \right)$$

To treat the case of negative Euler characteristic, note by [GS Prop. 37],

$$[\mathcal{P}^1(0, C, e)] - \mathbb{L}^e[\mathcal{P}^1(0, C, -e)] = [e][\mathcal{M}(0, C, e)] = 0$$

We now pass from $[\cdot]$ to $\overline{\chi}_{-y}$; note that in addition to specializing parameters we must multiply by $y^{-\dim / 2}$; by the isomorphism with Hilbert schemes, we have $\dim \mathcal{P}^1(0, C, e) = 2g - 1 + e - 1$. All terms containing $[A^{[2]}]$ vanish, leaving:

$$\sum_{e\neq 0} t^e \overline{\chi}_{-y}(\mathcal{P}^1(0, C, e)) = y^{1/2} \sum_{e|(g(C)-1)} \frac{y^e - 1}{y - 1} \left( y^{-e/2} t^e + y^{e/2} y^{-e} t^{-e} \right) \left( \frac{g - 1}{e} \right)^2$$

$$= \sum_{e|(g(C)-1)} \frac{y^{e/2} - y^{-e/2}}{y^{1/2} - y^{-1/2}} \left( t^e + t^{-e} \right) \left( \frac{g - 1}{e} \right)^2$$

It remains to determine the contribution of sheaves with $e = 0$. Let $g = g(C)$, and

$$f_g(y, t) := \sum_{e \in \mathbb{Z}} \overline{\chi}_{-y}(\mathcal{P}^1(0, C, e)) t^e = \sum_{e \in \mathbb{Z}} \overline{\chi}_{-y}(C_{[\mathcal{A}, C], g-2}) t^e$$

Thus by \((2)\) the $N^i$ are defined by expanding

$$f_g(y, t) = \sum_{i \geq 0} N^i_{[\mathcal{A}, C], g-2}(y) \left( \frac{1 - y^{-1/2} t}{t} \right)^{g-1-i}.$$
We have shown in [GS] that, for surfaces with trivial canonical bundle, \( N_{[S,L],\delta}^i(y) = 0 \) for \( i > \delta \). Applying this here, we see that \( f_y(y, t) \) is a Laurent polynomial in \( t, y^{1/2} \), divisible by \( \frac{(1-y^{-1/2})(1-y^{1/2})}{t} \). In particular \( f_y(y, y^{1/2}) = 0 \), in other words

\[
\text{Coeff}(f_y(y, t)) = - \sum_{e \in (g(C)-1)} \frac{y^{e/2} - y^{-e/2}}{y^{1/2} - y^{-1/2}} \left( \frac{g-1}{e} \right)^2.
\]

and

\[
f_y(y, t) = \sum_{e \in (g(C)-1)} \frac{y^{e/2} - y^{-e/2}}{y^{1/2} - y^{-1/2}} \left( t^e + t^{-e} - y^{e/2} - y^{-e/2} \right) \left( \frac{g-1}{e} \right)^2.
\]

Putting \( d := e, n := (g-1)/e \), we see that \( A = \sum_{g \geq 2} f_y(y, t)q^{g-1} \) is given by the second line of Theorem 5.

**Remark 18.** For the Hodge polynomial \( \overline{h}(X) = (xy)^{-\dim(X)/2} \sum_{p,q} h^{p,q}(X)(-x)^p(-y)^q \), the above argument gives

\[
\sum_{e \neq 0} t^e \overline{h}(P^1(0, C, e)) = \sum_{e > 0} \left( \frac{(xy)^{e/2} - (xy)^{-e/2}}{(xy)^{1/2} - (xy)^{-1/2}} \right)^2 \left( \overline{h}(A[g-1])^e + (t^e + t^{-e}) \sum_{0 < b < (g-1)/e} \overline{h}(A^{g-1-be}) + \sum_{b = (g-1)/e} b^2 \right).
\]

### 4. The refined invariants for surfaces with \( K_S \) numerically trivial.

In this section we prove Corollary 9 and Theorem 10. Let \( S \) be a surface with \( K_S \) numerically trivial. Write

\[
x := t + t^{-1} - y^{1/2} - y^{-1/2} = \frac{1}{t}(1 - ty^{1/2})(1 - ty^{-1/2}) = -y^{-1/2}(1 - y^{1/2})t(1 - y^{-1/2}/t).
\]

Let \( L_g \) denote a line bundle on \( S \) with \( g(L_g) = g \). We know

\[
\sum_g \sum_i N_{[S,L_g],\chi(L_g)-1-k}^i(y)x^{g-i-1}q^{g-1} = \frac{X^kX^{\chi(O_S)/2}A^{1-\chi(O_S)/2}}{x^{k+1-\chi(O_S)}Q[x][q^{-1}][q]}.
\]

We know that \( N_{[S,L_g],\chi(L_g)-1-k}^i = 0 \) for \( i > \chi(L_g) - 1 - k = g - (k + 2 - \chi(O_S)) \).

This means

\[
\frac{X^kX^{\chi(O_S)/2}A^{1-\chi(O_S)/2}}{x^{k+1-\chi(O_S)}Q[x][q^{-1}][q]} \in x^{k+1-\chi(O_S)}Q[x][q^{-1}][q]
\]

and

\[
\sum_g N_{[S,L_g],\chi(L_g)-1-k}^i(y)q^{g-1} = \frac{X^kX^{\chi(O_S)/2}A^{1-\chi(O_S)/2}}{x^{k+1-\chi(O_S)}Q[x][q^{-1}][q]} \bigg|_{t = y^{1/2}}.
\]

By Theorem 5 and the fact that \( [d]_y |_{y=1} = d \), we get

\[
\frac{A}{x} \bigg|_{t = y^{1/2}} = \sum_{n>0,d>0} n^2d[d]_y^2q^{nd} = D\overline{DG}_2(y, q).
\]
As \( \mathbb{H} = D^{-1}A \), we also see that \( \frac{\mathbb{H}}{x} \big|_{t=y^{1/2}} = \overline{DG}_2(y, q) \). By Theorem 5 we get \( (xK) \big|_{t=y^{1/2}} = \tilde{A}(y, q) \). As \( \mathbb{K} \in \mathbb{H} \cdot (1 + \mathbb{Q}[y][[\mathbb{H}]] \), we find that
\[
\frac{\mathbb{K}}{x} \big|_{t=y^{1/2}} = \frac{\mathbb{H}}{x} \big|_{t=y^{1/2}} = \overline{DG}_2(y, q).
\]
Substituting, we have proven Corollary 9: 
\[
\sum_{g} \mathcal{N}_g \chi(L_g)^{-1-k} \Big|_{S, L_g} \chi(L_g)^{-1-k}(y)q^{g-1} = \frac{\overline{DG}_2(y, q)^k(D\overline{DG}_2(y, q))^{1-\chi(O_S)/2}}{\Delta(y, q)^{\chi(O_S)/2}}.
\]

Now we prove Theorem 11. This proof (except the easy (15)) is due to Don Zagier. As before let \( e^{2\pi iz} := y, e^{2\pi iz_1} := y_1 := ty^{1/2}, e^{2\pi iz_2} := y_2 = y^{1/2}/t \). Note that \( y_1y_2 = y \). We can rewrite the formula of Theorem 4 as follows:
\[
K = \frac{y^{-1/2} - y^{1/2}}{\Delta(q)} \frac{\theta'(0)^3}{\theta(y_1)\theta(y_2)\theta(y)} = \frac{1}{(y^{-1/2} - y^{1/2})\Delta(y, q)} \frac{\theta'(0)\theta(y_1y_2)}{\theta(y_1)\theta(y_2)}.
\]
Let \( \varepsilon := \frac{(y_1-1)(y_2-1)}{y_1} = \frac{x}{y^{1/2} - y^{1/2}} \). Then by (11) we have
\[
(x\tilde{\Delta}(y, q)K = 1 - \varepsilon \sum_{n \geq 0} \sum_{d \geq 0} \sum_{n \geq 0} \sum_{d \geq 0} \frac{\text{sgn}(d)e^{-(d-n)2\pi iz_2}y^dq^nd}{y^{1/2} - y^{1/2}} = 1 - \varepsilon \sum_{n \geq 0} \sum_{d \geq 0} \frac{\text{sgn}(d)e^{-(d-n)2\pi iz_2}y^dq^nd}{y^{1/2} - y^{1/2}} = 1 - \varepsilon \sum_{n \geq 0} \sum_{d \geq 0} \frac{\text{sgn}(d)e^{-(d-n)2\pi iz_2}y^dq^nd}{y^{1/2} - y^{1/2}}.
\]
Note that \( \varepsilon = \varepsilon(2\pi iz_2) = \frac{1}{y^{1/2} - y^{1/2}}(y^{1/2} - 1) = \frac{e^{2\pi iz_2} - 1}{y^{1/2} - y^{1/2}} \) is a power series in \( \mathbb{Q}[\frac{1}{y^{1/2}}][[2\pi iz_2]] \), starting with \( 2\pi iz_2 \). Let \( 2\pi iz_2(\varepsilon) \) be the inverse series. Then the formula for \( K \) of the Theorem follows from (14) together with the claim that
\[
2\pi iz_2(\varepsilon) = \sum_{n \geq 0} \frac{s_n(y)}{(y^{1/2} - y^{1/2})^n} n + 1 = -S(y, x).
\]
By the Lagrange inversion formula \( \text{Coeff}_{x^n} (1 + x)(1 + xy) \) of the \( K \) for \( \mathbb{K} \). To prove the formula for \( A \), note that by the definition \( A(y, q) = \sum_{n \geq 0} \sum_{d \geq 0} \text{sgn}(d)n^2y^dq^nd \). Thus we have by Theorem 5 that
\[
(y^{1/2} - y^{-1/2})A = A(y_1, q) + A(y_2, q) - A(y, q) = \sum_{n \geq 0} \sum_{d \geq 0} \text{sgn}(d)n^2y_1^dy_2^d - (y^d - 1)q^nd = \sum_{n \geq 0} \sum_{d \geq 0} \text{sgn}(d)n^2y_1^dy_2^d - (y^d - 1)q^nd.
\]
and use again (15). This finishes the proof of the formula for $A$. The formula for $H$ now follows directly from the definition $H = D^{-1}A$.

References

[BG] F. Block, L. Göttsche, Refined curve counting with tropical geometry, in preparation.
[BL] J. Bryan, C. Leung, The enumerative geometry of K3 surfaces and modular forms, JAMS 13.2 (2000) 371–410.
[EGL] G. Ellingsrud, L. Göttsche and M. Lehn, On the cobordism class of the Hilbert scheme of a surface, Jour. Alg. Geom. 10 (2001), 81–100.
[Göt] L. Göttsche, The Betti numbers of the Hilbert schemes of points on a smooth surface, Math. Ann. 286 (1990), 193–207.
[Göt2] L. Göttsche, A conjectural generating function for numbers of curves on surfaces, Comm. Math. Phys. 196 (1998), 523–533.
[GS] L. Göttsche and V. Shende, Refined curve counting on complex surfaces, arXiv:1208.1973.
[GZ] L. Göttsche, D. Zagier, Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b_2 = 1$, Selecta Math. (N.S.) 4 (1998), 69–115.
[Hi] F. Hirzebruch, Topological methods in algebraic geometry, (Springer-Verlag, Berlin, 1995).
[IM] I. Itenberg, G. Mikhalkin, On Block-Goettsche multiplicities for planar tropical curves, arXiv:1201.0451.
[KT] M. Kool and R. P. Thomas, A short proof of the Göttsche conjecture, arXiv:1001.3211.
[PT1] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), 407–447.
[PT3] R. Pandharipande and R. P. Thomas, Stable pairs and BPS-invariants, Jour. AMS. 23, (2010), 267–297.
[Tze] Y-j. Tzeng, Proof of the Göttsche-Yau-Zaslow Formula, J. Diff. Geom. 90.3 (2012), 439–472.