Hardy–Littlewood maximal operator on reflexive variable Lebesgue spaces over spaces of homogeneous type

by

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Abstract. We show that the Hardy–Littlewood maximal operator is bounded on a reflexive variable Lebesgue space $L^{p(\cdot)}$ over a space of homogeneous type $(X,d,\mu)$ if and only if it is bounded on its dual space $L^{p'(\cdot)}$, where $1/p(x) + 1/p'(x) = 1$ for $x \in X$. This result extends the corresponding result of Lars Diening from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces $(X,d,\mu)$ of homogeneous type.

1. Introduction. We begin with the definition of a space of homogeneous type (see, e.g., [C90a]). Given a set $X$ and a function $d : X \times X \to [0,\infty)$, one says that $(X,d)$ is a quasi-metric space if the following axioms hold:

(a) $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(c) for all $x,y,z \in X$ and some constant $\kappa \geq 1$,

\[
    d(x,y) \leq \kappa (d(x,y) + d(y,z)).
\]

For $x \in X$ and $r > 0$, consider the ball $B(x,r) = \{ y \in X : d(x,y) < r \}$. Given a quasi-metric space $(X,d)$ and a positive measure $\mu$ that is defined on the $\sigma$-algebra generated by quasi-metric balls, one says that $(X,d,\mu)$ is a space of homogeneous type if there exists a constant $C_\mu \geq 1$ such that for any $x \in X$ and any $r > 0$,

\[
    \mu(B(x,2r)) \leq C_\mu \mu(B(x,r)).
\]

To avoid trivial measures, we will always assume that $0 < \mu(B) < \infty$ for every ball $B$. Consequently, $\mu$ is a $\sigma$-finite measure.

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Given a complex-valued function \( f \in L^1_{\text{loc}}(X, d, \mu) \), we define its Hardy–Littlewood maximal function \( Mf \) by

\[
(Mf)(x) := \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(x)| \, d\mu(x), \quad x \in X,
\]

where the supremum is taken over all balls \( B \subset X \) containing \( x \in X \). The Hardy–Littlewood maximal operator \( M \) is a sublinear operator acting by the rule \( f \mapsto Mf \).

Let \( L^0(X, d, \mu) \) denote the set of all complex-valued measurable functions on \( X \) and let \( \mathcal{P}(X) \) denote the set of all measurable a.e. finite functions \( p : X \to [1, \infty] \). For a measurable set \( E \subset X \), put

\[
p_-(E) := \text{ess inf}_{x \in E} p(x), \quad p_+(E) := \text{ess sup}_{x \in E} p(x)
\]

and

\[
p_- := p_-(X), \quad p_+ := p_+(X).
\]

For \( f \in L^0(X, d, \mu) \) and \( p \in \mathcal{P}(X) \), consider the functional, which is called the modular, given by

\[
\varrho_p(f) := \int_X |f(x)|^{p(x)} \, d\mu(x).
\]

By definition, the variable Lebesgue space \( L^{p(\cdot)}(X, d, \mu) \) consists of all functions \( f \in L^0(X, d, \mu) \) such that \( \varrho_p(f/\lambda) < \infty \) for some \( \lambda > 0 \) depending on \( f \). It is a Banach space with respect to the Luxemburg–Nakano norm given by

\[
\|f\|_{L^{p(\cdot)}} := \inf\{\lambda > 0 : \varrho_p(f/\lambda) \leq 1\}.
\]

If \( p \in \mathcal{P}(X) \) is constant, then \( L^{p(\cdot)}(X, d, \mu) \) is nothing but the standard Lebesgue space \( L^p(X, d, \mu) \). Variable Lebesgue spaces are often called Nakano spaces. We refer to Maligranda’s paper \[M11\] for the role of Hidegoro Nakano in the study of variable Lebesgue spaces and to the monographs \[CF13\] \[DH^{+11}\] \[KM^{+16}\] for the basic properties of these spaces. We only mention that the space \( L^{p(\cdot)}(X, d, \mu) \) is reflexive if and only if \( 1 < p_-, p_+ < \infty \). In this case, the dual space \( [L^{p(\cdot)}(X, d, \mu)]^* \) is isomorphic to \( L^{p'(\cdot)}(X, d, \mu) \), where \( p' \in \mathcal{P}(X) \) is given by

\[
1/p(x) + 1/p'(x) = 1, \quad x \in X
\]

(see, e.g., \[CF13\] Proposition 2.79 and Corollary 2.81).

One of the central problems of harmonic analysis on variable Lebesgue spaces is the problem of boundedness of the Hardy–Littlewood maximal operator \( M \) on \( L^{p(\cdot)}(X, d, \mu) \). For a detailed history of this problem, we refer to the monographs \[CF13\] \[DH^{+11}\] \[KM^{+16}\]. We also mention that very recently Cruz-Uribe and Shukla \[CS18\] Theorem 1.1] proved a sufficient condition for
the boundedness of the fractional maximal operator $M_{\alpha}$, $0 \leq \alpha < 1$, on reflexive variable Lebesgue spaces $L^{p(\cdot)}(X, d, \mu)$ over spaces of homogeneous type, which includes the case of the Hardy–Littlewood maximal operator as a special case when $\alpha = 0$.

In 2005, Diening [D05, Theorem 8.1] (see also [DH+11, Theorem 5.7.2]) proved the following remarkable result: if $1 < p_-(\mathbb{R}^n), p_+ (\mathbb{R}^n) < 1$, then the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if it is bounded on its dual $L^{p'(\cdot)}(\mathbb{R}^n)$. Recently Lerner [L17, Theorem 1.1] generalized this result to the setting of weighted variable Lebesgue spaces $L^{p(\cdot)}_w(\mathbb{R}^n)$.

The aim of this paper is to present a self-contained proof of the following extension of Diening’s theorem to the setting of spaces of homogeneous type.

**Theorem 1.1 (Main result).** Let $(X, d, \mu)$ be a space of homogeneous type and $p \in \mathcal{P}(X)$ be such that $1 < p_-, p_+ < \infty$. The Hardy–Littlewood maximal operator $M$ is bounded on the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ if and only if it is bounded on its dual space $L^{p'(\cdot)}(X, d, \mu)$.

Our approach is based on the adaptation of Lerner’s proof [L17], which is heavily based on the Calderón–Zygmund decomposition and dyadic maximal functions in the Euclidean setting of $\mathbb{R}^n$, to the setting of spaces of homogeneous type. This becomes possible thanks to the recently developed techniques of dyadic decomposition of spaces of homogeneous type due to Hytönen and Kairema [HK12] (see also previous works by Christ [C90a, C90b]). Note that these techniques were successfully applied in [AHT17, AW18, CS18, K19] to study various problems on spaces of homogeneous type (this list is far from being exhaustive).

The paper is organized as follows. In Section 2 we describe the construction by Hytönen and Kairema [HK12] of a system of adjacent dyadic grids on a space of homogeneous type. Elements of this system are called dyadic cubes and have many important properties of usual dyadic cubes in $\mathbb{R}^n$.

In Section 3 we recall the definition of Banach function spaces and the main result of [K19] (see also [L17, Theorem 3.1]) saying that if the Hardy–Littlewood maximal operator $M$ is bounded on a Banach function space $\mathcal{E}(X, d, \mu)$, then its boundedness on the associate space $\mathcal{E}'(X, d, \mu)$ is equivalent to a certain condition $A_\infty$. Since the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ is a Banach function space, in order to prove Theorem 1.1 it is sufficient to verify that $L^{p(\cdot)}(X, d, \mu)$ satisfies the condition $A_\infty$.

In Section 4 we recall very useful relations between the norm and the modular in a variable Lebesgue space. This allows us to formulate a modular analogue of the condition $A_\infty$ and show that this modular analogue implies the (norm) condition $A_\infty$. The rest of the paper is devoted to the verification of the modular analogue of $A_\infty$ (see Lemma 4.4).
In Section 5, we prepare for the proof of the main result, extending [L17, Lemmas 5.1–5.3 and 4.1] with \( w \equiv 1 \) from the Euclidean setting of \( \mathbb{R}^n \) to the setting of spaces of homogeneous type. Finally, in Section 6, we complete the proof of Theorem 1.1 following the scheme of the proof of [L17, Theorem 1.1].

2. Dyadic decomposition of spaces of homogeneous type

2.1. Construction of Hytönen and Kairema. Let \((X, d, \mu)\) be a space of homogeneous type. The doubling property of \( \mu \) implies the following geometric doubling property of the quasi-metric \( d \): any ball \( B(x, r) \) can be covered by at most \( N := N(C_\mu, \kappa) \) balls of radius \( r/2 \). It is not difficult to show that \( N \leq C_\mu^{6+3\log_2 \kappa} \).

An important tool for our proofs is the concept of an adjacent system of dyadic grids \( D^t, t \in \{1, \ldots, K\} \), on a space of homogeneous type \((X, d, \mu)\). Christ [C90a, Theorem 11] (see also [C90b, Chap. VI, Theorem 14]) constructed a system of sets on \((X, d, \mu)\) which satisfy many of the properties of a system of dyadic cubes on the Euclidean space. His construction was further refined by Hytönen and Kairema [HK12, Theorem 2.2]. We will use the version from [AHT17, Theorem 4.1].

**Theorem 2.1.** Let \((X, d, \mu)\) be a space of homogeneous type with the constant \( \kappa \geq 1 \) in inequality (1.1) and the geometric doubling constant \( N \). Suppose the parameter \( \delta \in (0, 1) \) satisfies \( 96\kappa^2 \delta \leq 1 \). Then there exist an integer \( K = K(\kappa, N, \delta) \), a countable set \( \{z_{\alpha}^{k,t} : \alpha \in A_k\} \) of points with \( k \in \mathbb{Z} \) and \( t \in \{1, \ldots, K\} \), and a finite number of dyadic grids \( D^t := \{Q_{\alpha}^{k,t} : k \in \mathbb{Z}, \alpha \in A_k\} \), such that the following properties are fulfilled:

(a) for every \( t \in \{1, \ldots, K\} \) and \( k \in \mathbb{Z} \) one has

(i) \( X = \bigcup_{\alpha \in A_k} Q_{\alpha}^{k,t} \) (disjoint union);

(ii) if \( Q, P \in D^t \), then \( Q \cap P \in \{\emptyset, Q, P\} \);

(iii) if \( Q_{\alpha}^{k,t} \in D^t \), then

\[
B(z_{\alpha}^{k,t}, c_1 \delta^k) \subset Q_{\alpha}^{k,t} \subset B(z_{\alpha}^{k,t}, C_1 \delta^k),
\]

where \( c_1 = (12\kappa^4)^{-1} \) and \( C_1 := 4\kappa^2 \);

(b) for every \( t \in \{1, \ldots, K\} \) and every \( k \in \mathbb{Z} \), if \( Q_{\alpha}^{k,t} \in D^t \), then there exists at least one \( Q_{\beta}^{k+1,t} \in D^t \), called a child of \( Q_{\alpha}^{k,t} \), such that \( Q_{\beta}^{k+1,t} \subset Q_{\alpha}^{k,t} \), and there exists exactly one \( Q_{\gamma}^{k-1,t} \in D^t \), the parent of \( Q_{\alpha}^{k,t} \), such that \( Q_{\alpha}^{k,t} \subset Q_{\gamma}^{k-1,t} \).
(c) for every ball $B = B(x,r)$, there exists

$$Q_B \in \bigcup_{t=1}^{K} D^t$$

such that $B \subseteq Q_B$ and $Q_B = Q_{k-1,t}^{\alpha}$ for some indices $\alpha \in A_k$ and $t \in \{1, \ldots, K\}$, where $k$ is the unique integer such that $\delta^{k+1} < r \leq \delta^k$.

The collections $D^t$, $t \in \{1, \ldots, K\}$, are called dyadic grids on $X$. The sets $Q_{k,t}^{\alpha} \in D^t$ are referred to as dyadic cubes with center $z_{k,t}^{\alpha}$ and sidelength $\delta^k$; see (2.1). The sidelength of a cube $Q \in D^t$ will be denoted by $\ell(Q)$. We emphasize that these sets are not cubes in the standard sense even if the underlying space is $\mathbb{R}^n$. Parts (a) and (b) of the above theorem describe dyadic grids $D^t$, with $t \in \{1, \ldots, K\}$, individually. In particular, (2.1) permits a comparison between a dyadic cube and quasi-metric balls. Part (c) guarantees the existence of a finite family of dyadic grids such that an arbitrary quasi-metric ball is contained in a dyadic cube in one of these grids. Such a finite family of dyadic grids is referred to as an adjacent system of dyadic grids.

2.2. Dyadic maximal function. Let $D \in \bigcup_{t=1}^{K} D^t$ be a fixed dyadic grid. One can define the dyadic maximal function $M^D f$ of $f \in L^1_{\text{loc}}(X,d,\mu)$ by

$$(M^D f)(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(x)| \, d\mu(x), \quad x \in X,$$

where the supremum is taken over all dyadic cubes $Q \in D$ containing $x$.

The following important theorem is proved by Hytönen and Kairema [HK12, Proposition 7.9].

**Theorem 2.2.** Let $(X,d,\mu)$ be a space of homogeneous type and let $\bigcup_{t=1}^{K} D^t$ be the adjacent system of dyadic grids given by Theorem 2.1. There exists a constant $C_{HK}(X) \geq 1$ depending only on $(X,d,\mu)$ such that for every $f \in L^1_{\text{loc}}(X,d,\mu)$ and a.e. $x \in X$, one has

$$(M^{D^t} f)(x) \leq C_{HK}(X)(M f)(x), \quad t \in \{1, \ldots, K\},$$

$$(M f)(x) \leq C_{HK}(X) \sum_{t=1}^{K} (M^{D^t} f)(x).$$

2.3. Calderón–Zygmund decomposition of a cube. The following result is a consequence of Theorem 2.1.

**Lemma 2.3.** Suppose $(X,d,\mu)$ is a space of homogeneous type with the constants $\kappa \geq 1$ in inequality (1.1) and $C_\mu \geq 1$ in inequality (1.2). Let $(X,d,\mu)$
be equipped with an adjacent system of dyadic grids \( \{D^t, t=1, \ldots, K\} \) and let \( \delta \in (0, 1) \) be chosen as in Theorem 2.1. Then there is an \( \varepsilon = \varepsilon(\kappa, C_\mu, \delta) \in (0, 1) \) such that for every \( t \in \{1, \ldots, K\} \) and all \( Q, P \in D^t \), if \( Q \) is a child of \( P \), then
\[
\mu(Q) \geq \varepsilon \mu(P).
\]

**Proof.** See [AW18, Corollary 2.9] or [K19, Lemma 8]. ■

Let \( (X, d, \mu) \) be a space of homogeneous type and \( D = D^{t_0} = \bigcup_{t=1}^K D^t \) be a dyadic grid. Fix \( Q_0 \in D \). Then there exist \( k_0 \in \mathbb{Z} \) and \( \alpha_0 \in A_{k_0} \) such that
\[
Q_0 = Q_{k_0,t_0}^{\alpha_0}.
\]
Consider
\[
D(Q_0) := \{Q_{\alpha}^{k,t_0} : k \in \mathbb{Z}, k \geq k_0, \alpha \in A_k\} = \{Q' \in D^{t_0} : Q' \subset Q\},
\]
that is, the set of all dyadic cubes with respect to \( Q_0 \). The set \( D(Q_0) \) is formed by all dyadic descendants of the cube \( Q_0 \). For a measurable function \( f \) such that
\[
\int_{Q_0} |f(x)| \, d\mu(x) < \infty,
\]
consider the **local dyadic maximal function** of \( f \) defined by
\[
(M^{D(Q_0)}f)(x) := \sup_{Q \ni x, Q \in D(Q_0)} \frac{1}{\mu(Q)} \int_{Q} |f(x)| \, d\mu(x), \quad x \in Q_0.
\]

Given a dyadic grid \( D \in \bigcup_{t=1}^K D^t \), a **sparse family** \( S \subset D \) is a collection of dyadic cubes \( Q \in D \) for which there exists a collection of sets \( \{E(Q)\}_{Q \in S} \) such that the sets \( E(Q) \) are pairwise disjoint, \( E(Q) \subset Q \), and
\[
\mu(Q) \leq 2 \mu(E(Q)).
\]

We will need the following variation of the Calderón–Zygmund decomposition of the cube \( Q_0 \) (cf. [L17, Lemma 2.4]).

**Theorem 2.4.** Let \( (X, d, \mu) \) be a space of homogeneous type, \( D = D^{t_0} \in \bigcup_{t=1}^K D^t \) be a dyadic grid, \( Q_0 \in D \), and \( D(Q_0) \) be defined by (2.2). Suppose \( \varepsilon \in (0, 1) \) is as in Lemma 2.3. For a nonzero measurable function \( f \) on \( Q_0 \) satisfying (2.3) and \( k \in \mathbb{N} \), set
\[
\Omega_k(Q_0) := \left\{ x \in Q_0 : (M^{D(Q_0)}f)(x) > \left( \frac{2}{\varepsilon} \right)^k \frac{1}{\mu(Q_0)} \int_{Q_0} |f(x)| \, d\mu(x) \right\}.
\]

If \( \Omega_k(Q_0) \neq \emptyset \), then there exists a collection \( \{Q_j^k(Q_0)\}_{j \in J_k} \subset D(Q_0) \) that is pairwise disjoint, maximal with respect to inclusion, and such that
\[
\Omega_k(Q_0) = \bigcup_{j \in J_k} Q_j^k(Q_0).
\]
The collection of cubes
\[ S := \{ Q_j^k(Q_0) : Q_k(Q_0) \neq \emptyset, j \in J_k \} \]

is sparse, and for all \( j \) and \( k \), the sets
\[ E(Q_j^k(Q_0)) := Q_j^k(Q_0) \setminus Q_{k+1}(Q_0) \]
satisfy
\begin{equation}
\mu(Q_j^k(Q_0)) \leq 2\mu(E(Q_j^k(Q_0))).
\end{equation}

Proof. For each \( k \in \mathbb{N} \) satisfying \( Q_k(Q_0) \neq \emptyset \), the existence of a pairwise disjoint and inclusion-maximal collection \( \{ Q_j^k(Q_0) \}_{j \in J_k} \), such that \( (2.5) \) is fulfilled, follows from \([K19] \) Theorem 9(a)]. Moreover, in view of the same theorem, for every \( k \in \mathbb{N} \) satisfying \( Q_k(Q_0) \neq \emptyset \) and \( j \in J_k \), one has
\begin{equation}
\left( \frac{2}{\varepsilon} \right)^k \frac{1}{\mu(Q_0)} \int_{Q_0} |f(x)| \, d\mu(x) < \frac{1}{\mu(Q_j^k(Q_0))} \int_{Q_j^k(Q_0)} |f(x)| \, d\mu(x)
\end{equation}

It remains to prove \( (2.6) \). Since \( Q_{k+1}(Q_0) \subset Q_k(Q_0) \) and, for each fixed \( k \), the cubes \( Q_j^k(Q_0) \) are pairwise disjoint, it is clear that the sets \( E(Q_j^k(Q_0)) \) are pairwise disjoint for all \( j \) and \( k \). If \( Q_j^k(Q_0) \cap Q_{i+1}^k(Q_0) \neq \emptyset \), then by the maximality of the cubes in \( \{ Q_j^k(Q_0) \}_{j \in J_k} \) and the fact that \( 2/\varepsilon > 1 \), we have \( Q_{i+1}^k(Q_0) \subset Q_j^k(Q_0) \). In view of \( (2.5) \) and \( (2.7) \), we see that
\begin{align*}
\mu(Q_j^k(Q_0) \cap Q_{k+1}(Q_0)) &= \sum_{\{ i : Q_i^k(Q_0) \subset Q_j^k(Q_0) \}} \mu(Q_i^{k+1}(Q_0)) \\
&\leq \sum_{\{ i : Q_i^{k+1}(Q_0) \subset Q_j^k(Q_0) \}} \frac{\varepsilon/2}{\mu(Q_0)} \int_{Q_i^{k+1}(Q_0)} |f(x)| \, d\mu(x) \\
&\leq \frac{\varepsilon/2}{\mu(Q_0)} \int_{Q_j^k(Q_0)} |f(x)| \, d\mu(x) \\
&\leq \left( \frac{\varepsilon}{2} \right)^{k+1} \left( \frac{2}{\varepsilon} \right)^k \frac{\mu(Q_j^k(Q_0))}{\mu(Q_0)} = \frac{\mu(Q_j^k(Q_0))}{2}.
\end{align*}

Then
\begin{align*}
\mu(E(Q_j^k(Q_0))) &= \mu(Q_j^k(Q_0) \setminus Q_{k+1}(Q_0)) \\
&= \mu(Q_j^k(Q_0)) - \mu(Q_j^k(Q_0) \cap Q_{k+1}(Q_0)) \\
&\geq (1 - 1/2)\mu(Q_j^k(Q_0)),
\end{align*}
whence \( \mu(Q^k_0(0)) \leq 2\mu(E(Q^k_0(0))) \) for all \( j \) and \( k \), which completes the proof of (2.6). \( \blacksquare \)

3. Hardy–Littlewood maximal operator on the associate space of a Banach function space

3.1. Banach function spaces. Let us recall the definition of a Banach function space (see, e.g., [BS88, Chap. 1, Definition 1.1]). Let \( L^0_0(X, d, \mu) \) be the set of all nonnegative measurable functions on \( X \). The characteristic function of a set \( E \subset X \) is denoted by \( \chi_E \). A mapping \( \rho : L^0_0(X, d, \mu) \to [0, \infty] \) is called a Banach function norm if, for all functions \( f, g, f_n (n \in \mathbb{N}) \) in the set \( L^0_0(X, d, \mu) \), for all constants \( a \geq 0 \), and for all measurable subsets \( E \) of \( X \), the following properties hold:

(A1) \( \rho(f) = 0 \iff f = 0 \) a.e., \( \rho(a f) = a \rho(f) \), \( \rho(f + g) \leq \rho(f) + \rho(g) \),

(A2) \( 0 \leq g \leq f \) a.e. \( \Rightarrow \rho(g) \leq \rho(f) \) (the lattice property),

(A3) \( 0 \leq f_n \uparrow f \) a.e. \( \Rightarrow \rho(f_n) \uparrow \rho(f) \) (the Fatou property),

(A4) \( \mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty \),

(A5) \( \int_E f(x) d\mu(x) \leq C_E \rho(f) \)

with a constant \( C_E \in (0, \infty) \) that may depend on \( E \) and \( \rho \), but is independent of \( f \). When functions differing only on a set of measure zero are identified, the set \( \mathcal{E}(X, d, \mu) \) of all functions \( f \in L^0(X, d, \mu) \) for which \( \rho(|f|) < \infty \) is called a Banach function space. For each \( f \in \mathcal{E}(X, d, \mu) \), the norm of \( f \) is defined by

\[ \|f\|_\mathcal{E} := \rho(|f|). \]

The set \( \mathcal{E}(X, d, \mu) \) under the natural linear space operations and under this norm becomes a Banach space (see [BS88 Chap. 1, Theorems 1.4 and 1.6]). If \( \rho \) is a Banach function norm, its associate norm \( \rho' \) is defined on \( L^0_0(X, d, \mu) \) by

\[ \rho'(g) := \sup \left\{ \int_X f(x) g(x) d\mu(x) : f \in L^0_0(X, d, \mu), \rho(f) \leq 1 \right\}. \]

It is a Banach function norm itself [BS88, Chap. 1, Theorem 2.2]. The Banach function space \( \mathcal{E}'(X, d, \mu) \) determined by the Banach function norm \( \rho' \) is called the associate space (or Köthe dual) of \( \mathcal{E}(X, d, \mu) \).

3.2. The condition \( A_\infty \). Following [L17] and [K19, Definition 1], we say that a Banach function space \( \mathcal{E}(X, d, \mu) \) over a space \( (X, d, \mu) \) of homogeneous type satisfies the condition \( A_\infty \) if there exist constants \( \Phi, \theta > 0 \) such that for every dyadic grid \( D \in \bigcup_{t=1}^K D^t \), every finite sparse family \( S \subset D \), every collection \( \{\alpha_Q\}_{Q \in S} \) of nonnegative numbers, and every collection \( \{G_Q\}_{Q \in S} \) of pairwise disjoint measurable sets such that \( G_Q \subset Q \), one
has
\[
\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_\mathcal{E} \leq \Phi \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^\theta \left\| \sum_{Q \in S} \alpha_Q \chi_{Q} \right\|_\mathcal{E}.
\]

The following result is a generalization of [L17, Theorem 3.1] from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

**Theorem 3.1 ([K19, Theorem 2]).** Let $\mathcal{E}(X, d, \mu)$ be a Banach function space over a space of homogeneous type $(X, d, \mu)$ and let $\mathcal{E}'(X, d, \mu)$ be its associate space.

(a) If the Hardy–Littlewood maximal operator $M$ is bounded on $\mathcal{E}'(X, d, \mu)$, then $\mathcal{E}(X, d, \mu)$ satisfies the condition $A_\infty$.

(b) If the Hardy–Littlewood maximal operator $M$ is bounded on $\mathcal{E}(X, d, \mu)$, and $\mathcal{E}(X, d, \mu)$ satisfies the condition $A_\infty$, then the $M$ is bounded on $\mathcal{E}'(X, d, \mu)$.

Since the variable Lebesgue space $L^{p(\cdot)}(X, d, \mu)$ is a Banach function space and, under the condition $1 < p_-, p_+ < \infty$, its associate space $[L^{p(\cdot)}(X, d, \mu)]'$ is isomorphic to the variable Lebesgue space $L^{p'(\cdot)}(X, d, \mu)$ (see, e.g., [CF13, Section 2.10.3]), Theorem 3.1(b) immediately implies the following.

**Corollary 3.2.** Let $(X, d, \mu)$ be a space of homogeneous type and let $p \in \mathcal{P}(X)$ be such that $1 < p_-, p_+ < \infty$. If the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(X, d, \mu)$, and $L^{p(\cdot)}(X, d, \mu)$ satisfies the condition $A_\infty$, then $M$ is bounded on the dual space $L^{p'(\cdot)}(X, d, \mu)$.

It follows from Corollary 3.2 that in order to prove Theorem 1.1 it is sufficient to verify that $L^{p(\cdot)}(X, d, \mu)$ satisfies the condition $A_\infty$.

**4. Norm inequalities and modular inequalities**

**4.1. Norm-modular unit ball property.** In this subsection we formulate two very useful properties that relate norms and modulars in variable Lebesgue spaces.

**Lemma 4.1** (see, e.g., [DH+11, Lemma 3.2.4]). Let $(X, d, \mu)$ be a space of homogeneous type and $p \in \mathcal{P}(X)$. Then for every $f \in L^{p(\cdot)}(X, d, \mu)$ the inequalities $\|f\|_{L^{p(\cdot)}} \leq 1$ and $\varrho_{p(\cdot)}(f) \leq 1$ are equivalent.

**Lemma 4.2** (see, e.g., [DH+11, Lemma 3.2.5]). Let $(X, d, \mu)$ be a space of homogeneous type and $p \in \mathcal{P}(X)$ be such that $1 < p_-, p_+ < \infty$. Then for every $f \in L^{p(\cdot)}(X, d, \mu)$,
\[
\min\{\varrho_{p(\cdot)}(f)^{1/p_-}, \varrho_{p(\cdot)}(f)^{1/p_+}\} \leq \|f\|_{L^{p(\cdot)}} \leq \max\{\varrho_{p(\cdot)}(f)^{1/p_-}, \varrho_{p(\cdot)}(f)^{1/p_+}\}.
\]
4.2. Auxiliary lemma. The following auxiliary lemma illustrates the possibility of substitution of norm inequalities by modular inequalities.

**Lemma 4.3.** Let \((X, d, \mu)\) be a space of homogeneous type and let \(p \in P(X)\) satisfy \(1 < p_-, p_+ < \infty\). Suppose \(D \in \bigcup_{t=1}^{K} D_t\) is a dyadic grid. If \(S \in \mathcal{D}\) is a finite family and \(\{\alpha_Q\}_{Q \in S}\) is a family of nonnegative numbers such that
\[
\left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{L^p(\cdot)} \leq 1,
\]
then
\[
\sum_{Q \in S} \int_{Q} \alpha_Q^{p(x)} \mu(x) \leq 1.
\]

**Proof.** It is clear that
\[
\sum_{Q \in S} \int_{Q} \alpha_Q^{p(x)} d\mu(x) = \int_{X} \left( \sum_{Q \in S} \alpha_Q \chi_Q(x) \right)^{p(x)} d\mu(x).
\]
Since \(1 < p_- \leq p(x) \leq p_+ < \infty\) for a.e. \(x \in X\), one has
\[
\sum_{Q \in S} \left( \alpha_Q \chi_Q(x) \right)^{p(x)} \leq \left( \sum_{Q \in S} \alpha_Q \chi_Q(x) \right)^{p(x)}.
\]
Taking into account (4.1) and (4.2), it follows from Lemma 4.1 that
\[
\sum_{Q \in S} \int_{Q} \alpha_Q^{p(x)} \mu(x) \leq \int_{X} \left( \sum_{Q \in S} \alpha_Q \chi_Q(x) \right)^{p(x)} d\mu(x) \leq 1,
\]
which completes the proof. □

4.3. Modular version of the condition \(A_\infty\). In this subsection we formulate a modular analogue of the condition \(A_\infty\) and show that it implies the (norm) condition \(A_\infty\).

**Lemma 4.4.** Let \((X, d, \mu)\) be a space of homogeneous type and \(p \in P(X)\) satisfy \(1 < p_-, p_+ < \infty\). If there exist constants \(\Psi, \xi > 1\) such that for every dyadic grid \(D \in \bigcup_{t=1}^{K} D_t\), every finite sparse family \(S \subset \mathcal{D}\), every collection \(\{G_Q\}_{Q \in S}\) of pairwise disjoint measurable sets such that \(G_Q \subset Q\) and every collection \(\{\alpha_Q\}_{Q \in S}\) of nonnegative numbers such that
\[
\left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{L^p(\cdot)} = 1,
\]
one has
\[
\sum_{Q \in S} \int_{G_Q} \alpha_Q^{p(x)} d\mu(x) \leq \Psi \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^\xi,
\]
then the variable Lebesgue space \(L^{p(\cdot)}(X, d, \mu)\) satisfies the condition \(A_\infty\).
Proof. Fix a dyadic grid $D \in \bigcup_{t=1}^{K} D^t$, a finite sparse family $S \subset D$, and a collection $\{G_Q\}_{Q \in S}$ of pairwise disjoint measurable sets such that $G_Q \subset Q$. Let $\{\beta_Q\}_{Q \in S}$ be an arbitrary collection of nonnegative numbers. Put

$$\alpha_Q := \frac{\beta_Q}{\| \sum_{Q \in S} \beta_Q \chi_Q \|_{L^p(\cdot)}}.$$  

Then (4.3) is fulfilled. Since the sets $\{G_Q\}_{Q \in S}$ are pairwise disjoint, we have

$$\left( \sum_{Q \in S} \alpha_Q \chi_{G_Q}(x) \right)^{p(x)} = \sum_{Q \in S} \alpha_Q^{p(x)} \chi_{G_Q}(x), \quad x \in X.$$  

Hence

$$\sigma := \sum_{Q \in S} \int_{G_Q} \alpha_Q^{p(x)} d\mu(x) = \int_X \left( \sum_{Q \in S} \alpha_Q \chi_{G_Q}(x) \right)^{p(x)} d\mu(x).$$

By Lemma 4.2, (4.4), and (4.6), we have

$$\left\| \sum_{Q \in S} \alpha_Q \chi_{G_Q} \right\|_{L^p(\cdot)} \leq \max\{\sigma^{1/p-}, \sigma^{1/p+}\}$$

$$\leq \max\left\{ \Psi^{1/p-} \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\xi/p-}, \Psi^{1/p+} \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\xi/p+} \right\}$$

$$\leq \Psi^{1/p-} \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\xi/p+} \left\| \sum_{Q \in S} \beta_Q \chi_Q \right\|_{L^p(\cdot)},$$

because $\Psi > 1$, $\mu(G_Q) \leq \mu(Q)$ for all $Q \in S$ and $p_- \leq p_+$. It follows from (4.5) and (4.7) that

$$\left\| \sum_{Q \in S} \beta_Q \chi_{G_Q} \right\|_{L^p(\cdot)} \leq \Psi^{1/p-} \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{\xi/p+} \left\| \sum_{Q \in S} \beta_Q \chi_Q \right\|_{L^p(\cdot)},$$

that is, the space $L^p(\cdot)(X, d, \mu)$ satisfies the condition $A_\infty$ with $\Phi = \Psi^{1/p-}$ and $\theta = \xi/p_+$. \[\square\]

5. Preparations for the verification of the condition $A_\infty$

5.1. First lemma. Let $\|M\|_{B(L^p(\cdot))}$ denote the norm of the Hardy–Littlewood maximal operator on the variable Lebesgue space $L^p(\cdot)(X, d, \mu)$. As usual, for an exponent $r \in (1, \infty)$, let $r' = r/(r-1) \in (1, \infty)$ denote the conjugate exponent.

The preparation for the verification of the condition $A_\infty$ in the proof of Theorem 1.1 consists of four steps. The first step is the proof of the following extension of [L17, Lemma 5.1] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.
Lemma 5.1. Let \((X, d, \mu)\) be a space of homogeneous type and \(p \in \mathcal{P}(X)\) satisfy \(1 < p_-, p_+ < \infty\). Suppose the Hardy–Littlewood maximal operator \(M\) is bounded on \(L^{p(\cdot)}(X, d, \mu)\). There exist constants \(A, \lambda > 1\) such that for every dyadic grid \(D \in \bigcup_{t=1}^{K} D^t\), every family \(S_d \subset D\) of pairwise disjoint cubes, every family \(\{\alpha_Q\}_{Q \in S_d}\) of nonnegative numbers, if

\[
\sum_{Q \in S_d} \alpha_{Q}^{p(x)} d\mu(x) \leq 1,
\]

then

\[
\sum_{Q \in S_d} \mu(Q) \left( \frac{1}{\mu(Q)} \sum_{Q} \alpha_{Q}^{\lambda p(x)} d\mu(x) \right)^{1/\lambda} \leq A.
\]

Proof. Let \(\varepsilon \in (0, 1)\) be as in Lemma 2.3. Fix a dyadic grid \(D \in \bigcup_{t=1}^{K} D^t\) and a family of pairwise disjoint cubes \(S_d \subset D\). For \(k \in \mathbb{N}\) and \(Q \in S_d\), put

\[
\Omega_k(Q) := \left\{ x \in Q : (M^{D(Q)} \alpha_{Q}^{p(\cdot)})(x) > \left( \frac{2}{\varepsilon} \right)^k \frac{1}{\mu(Q)} \sum_{Q} \alpha_{Q}^{p(x)} d\mu(x) \right\}.
\]

By Theorem 2.4, if these sets are nonempty, then they can be written as

\[
\Omega_k(Q) = \bigcup_{j} Q_j^k(Q),
\]

where \(Q_j^k(Q) \in D(Q)\) are pairwise disjoint cubes for all \(j\) and \(k\), and

\[
\mu(Q_j^k(Q) \setminus \Omega_{k+1}(Q)) \geq \frac{1}{2} \mu(Q_j^k(Q)).
\]

Fix \(k \in \mathbb{N}\) and \(Q \in S_d\). If \(x \in \Omega_k(Q)\), then in view of (5.4) there exists \(j_0\) such that \(x \in Q_{j_0}^k(Q)\). It follows from (5.5) that

\[
\chi_{\Omega_k(Q)}(x) \leq \frac{2 \mu(Q_{j_0}^k(Q) \setminus \Omega_{k+1}(Q))}{\mu(Q_{j_0}^k(Q))} \leq \frac{2 \mu(Q_j^k(Q) \setminus \Omega_{k+1}(Q))}{\mu(Q_{j_0}^k(Q))} = \frac{2}{\mu(Q_{j_0}^k(Q))} \int_{Q_{j_0}^k(Q)} \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)}(y) d\mu(y)
\]

\[
\leq 2 \sup_{Q': \exists x, Q' \in D} \frac{1}{\mu(Q')} \int_{Q'} \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)}(y) d\mu(y),
\]

which implies that

\[
\chi_{\Omega_k(Q)}(x) \leq 2(M^{D} \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)})(x), \quad x \in X.
\]
Therefore, for all \( k \in \mathbb{N} \) and \( Q \in S_d \),
\[
\alpha_Q \chi_{\Omega_k(Q)}(x) \leq 2 \left( M^D (\alpha_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)}) \right)(x), \quad x \in X.
\]

Since the cubes in \( S_d \) are pairwise disjoint, the sets in \( \{ \Omega_k(Q) \}_{Q \in S_d} \) are also pairwise disjoint for every fixed \( k \in \mathbb{N} \). Hence, the above inequality implies that for \( k \in \mathbb{N} \),
\[
(5.6) \quad \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)}(x) \leq 2 \left( M^D \left( \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right) \right)(x), \quad x \in X.
\]

It follows from the boundedness of \( M \) on \( L^p(X, d\mu) \), Theorem 2.2, and (5.6) that
\[
(5.7) \quad \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \right\|_{L^p(.)} \leq 2 C_{HK}(X) \left\| M \right\|_{B(L^p(.))} \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^p(.)}.
\]

Set
\[
(5.8) \quad \tilde{\alpha}_Q := \alpha_Q \left( \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \right\|_{L^p(.)} \right)^{-1}.
\]

Then (5.7) can be rewritten as
\[
(5.9) \quad \frac{1}{2 C_{HK}(X) \left\| M \right\|_{B(L^p(.))}} \leq \left\| \sum_{Q \in S_d} \tilde{\alpha}_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^p(.)}.
\]

It follows from (5.8) that
\[
(5.10) \quad \left\| \sum_{Q \in S_d} \tilde{\alpha}_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^p(.)} \leq \left\| \sum_{Q \in S_d} \tilde{\alpha}_Q \chi_{\Omega_k(Q)} \right\|_{L^p(.)} = 1.
\]

Inequality (5.10) and Lemma 4.2 imply that
\[
(5.11) \quad \left\| \sum_{Q \in S_d} \tilde{\alpha}_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \right\|_{L^p(.)} \leq \left( \int_X \left( \sum_{Q \in S_d} \tilde{\alpha}_Q \chi_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)}(x) \right)^{p(x)} d\mu(x) \right)^{1/p_+}.
\]

Since the cubes in \( S_d \) are pairwise disjoint, so are the sets in the collection \( \{ \Omega_k(Q) \setminus \Omega_{k+1}(Q) \}_{Q \in S_d} \). Therefore, we deduce from (5.9) and (5.11) that
It follows from (5.12)–(5.14) that

\[(5.14)\]

\[Hence, in view of Lemma 4.2, we have\]

\[(5.13)\]

\[Again, taking into account that the cubes in \(S_d\) are pairwise disjoint, we deduce from (5.10) and Lemma 4.1 that\]

\[(5.15)\]

\[Since \(\Omega_{k+1}(Q) \subset \Omega_k(Q),\) it follows from (5.10) that\]

\[(5.16)\]

\[It follows from (5.12)–(5.14) that\]

\[\]

\[The above inequality and (5.8) imply that for \(k \in \mathbb{N},\)

\[(5.15)\]

\[It follows from (5.1) and Lemma 4.1 that\]

\[(5.16)\]

\[Since \(\Omega_1(Q) \subset Q,\) applying (5.16) and then applying (5.15) \(k - 1\) times, we\]
get

\[ 1 \geq \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_1(Q)} \right\|_{L^p(\cdot)} \geq \frac{1}{\beta} \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_2(Q)} \right\|_{L^p(\cdot)} \geq \ldots \]

\[ \geq \frac{1}{\beta^{k-1}} \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \right\|_{L^p(\cdot)}. \]

Thus

\[ \left\| \sum_{Q \in S_d} \alpha_Q \chi_{\Omega_k(Q)} \right\|_{L^p(\cdot)} \leq \beta^{k-1}, \quad k \in \mathbb{N}. \]

In view of Lemma 4.2, this inequality implies that

\[ (5.17) \quad \sum_{Q \in S_d} \int_{\Omega_k(Q)} \alpha_Q^{p(x)} d\mu(x) \leq \beta^{p-(k-1)}, \quad k \in \mathbb{N}. \]

Fix \( Q \in S_d \). Put \( \Omega_0(Q) := Q \). Then it follows from (5.3) that for \( k \in \mathbb{Z}_+ \) and \( x \in \Omega_k(Q) \setminus \Omega_{k+1}(Q) \), one has

\[ \alpha_Q^{p(x)} \leq (M^p(Q) \alpha_Q^{p(\cdot)})(x) \leq \left( \frac{2}{\epsilon} \right)^{k+1} \frac{1}{\mu(Q)} \int_{\Omega_k(Q)} \alpha_Q^{p(y)} d\mu(y). \]

From this inequality we obtain, for every \( \phi > 0 \),

\[ (5.18) \quad \int_{\Omega_k(Q)} \alpha_Q^{p(x)(1+\phi)} d\mu(x) = \sum_{k=0}^{\infty} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \alpha_Q^{p(x)(1+\phi)} d\mu(x) \]

\[ \leq \left( \frac{1}{\mu(Q)} \int_{\Omega_k(Q)} \alpha_Q^{p(y)} d\mu(y) \right)^{\phi} \sum_{k=0}^{\infty} \left( \frac{2}{\epsilon} \right)^{\phi(k+1)} \int_{\Omega_k(Q) \setminus \Omega_{k+1}(Q)} \alpha_Q^{p(x)} d\mu(x) \]

\[ \leq \left( \frac{1}{\mu(Q)} \int_{\Omega_k(Q)} \alpha_Q^{p(y)} d\mu(y) \right)^{\phi} \sum_{k=0}^{\infty} \left( \frac{2}{\epsilon} \right)^{\phi(k+1)} \int_{\Omega_k(Q)} \alpha_Q^{p(x)} d\mu(x). \]

It is easy to see that one can choose \( \phi > 0 \) such that

\[ 0 < (2/\epsilon)^{\phi} \beta^{p-} < 1. \]

Then

\[ (5.19) \quad \sum_{k=0}^{\infty} [(2/\epsilon)^{\phi} \beta^{p-}]^{k+1} < \infty. \]

Take \( \lambda := 1 + \phi \). By (5.18), we have
(5.20) \[
\sum_{Q \in S_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda} \\
\leq \sum_{Q \in S_d} \mu(Q) \left( \frac{1}{\mu(Q)} \left( \frac{1}{\mu(Q)} \int_Q \alpha^p_{Q} q(y) \, d\mu(y) \right)^{\phi} \right)^{1/\lambda} \\
\times \left( \sum_{k=0}^{\infty} (2/\epsilon)^{p(k+1)} \int_{\Omega_k(Q)} \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda}.
\]

Since \( \frac{1}{\lambda} = \frac{1}{1+\phi} \) and \( \frac{1}{\lambda'} = \frac{\phi}{1+\phi} \), we have

(5.21) \[
\mu(Q) \left( \frac{1}{\mu(Q)} \left( \frac{1}{\mu(Q)} \int_Q \alpha^p_{Q} q(y) \, d\mu(y) \right)^{\phi} \right)^{1/\lambda} \\
= \mu(Q) \left( \frac{1}{\mu(Q)^{1+\phi}} \right)^{1/\lambda} \left( \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{\phi/\lambda} = \left( \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda'}.
\]

Combining (5.20) with (5.21), applying Hölder’s inequality, and taking (5.1) into account, we obtain

(5.22) \[
\sum_{Q \in S_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda} \\
\leq \sum_{Q \in S_d} \left( \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda'} \left( \sum_{k=0}^{\infty} \left( \frac{2}{\epsilon} \right)^{p(k+1)} \int_{\Omega_k(Q)} \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda} \\
\leq \left( \sum_{Q \in S_d} \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda'} \left( \sum_{Q \in S_d} \sum_{k=0}^{\infty} \left( \frac{2}{\epsilon} \right)^{p(k+1)} \int_{\Omega_k(Q)} \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda} \\
= \left( \sum_{k=0}^{\infty} \left( \frac{2}{\epsilon} \right)^{p(k+1)} \sum_{Q \in S_d} \int_{\Omega_k(Q)} \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda}.
\]

It follows from (5.1), (5.17) and (5.22) that

\[
\sum_{Q \in S_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \alpha^p_{Q} q(x) \, d\mu(x) \right)^{1/\lambda} \\
\leq \left\{ \left( \frac{2}{\epsilon} \right)^{\phi} \sum_{Q \in S_d} \int_Q \alpha^p_{Q} q(x) \, d\mu(x) + \sum_{k=1}^{\infty} \left( \frac{2}{\epsilon} \right)^{p(k+1)} \beta^{-(k-1)} \right\}^{1/\lambda} \\
\leq \left\{ \left( \frac{2}{\epsilon} \right)^{\phi} + \sum_{k=1}^{\infty} \left( \frac{2}{\epsilon} \right)^{p(k+1)} \beta^{-(k-1)} \right\}^{1/\lambda} =: A.
\]
Combining $2/\varepsilon > 1$ and (5.19), we see that $A \in (1, \infty)$, which completes the proof of (5.2).

5.2. Second lemma. The next lemma generalizes [L17, Lemma 5.2] with $w \equiv 1$ from the Euclidean setting of $\mathbb{R}^n$ to the setting of spaces of homogeneous type.

**Lemma 5.2.** Let $(X,d,\mu)$ be a space of homogeneous type and $p \in \mathcal{P}(X)$ satisfy $1 < p_-, p_+ < \infty$. Suppose the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(X,d,\mu)$. There exist constants $B, \lambda > 1$ and a measure $\nu$ on $X$ such that for every dyadic grid $D \subseteq \bigcup_{t=1}^K D^t$ and every finite family $S_d \subseteq D$ of pairwise disjoint cubes, the following properties hold:

(i) if $Q \in D$ and $t \geq 0$ satisfy

$$
\int_Q t^{p(x)} \, d\mu(x) \leq 1,
$$

then

$$
\mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda} \leq B \int_Q t^{p(x)} \, d\mu(x) + \nu(Q);
$$

(ii) $\sum_{Q \in S_d} \nu(Q) \leq 2B$.

**Proof.** (i) Let $A, \lambda > 1$ be the constants from Lemma 5.1. Set

$$
B := 2^{p_+/p_-+1} A.
$$

Fix a dyadic grid $D \subseteq \bigcup_{t=1}^K D^t$. Given a cube $Q \in D$, consider the functions

$$
F_1(t) := \int_Q t^{p(x)} \, d\mu(x), \quad F_2(t) := \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda}, \quad t \geq 0,
$$

and the set

$$
A(Q) := \{ t > 0 : F_1(t) \leq 1, F_2(t) > BF_1(t) \}.
$$

Set

$$
t_Q := \begin{cases} 0 & \text{if } A(Q) = \emptyset, \\ \sup A(Q) & \text{if } A(Q) \neq \emptyset. \end{cases}
$$

We claim that

$$
F_1(t_Q) < 1.
$$

Indeed, if $F_1(t_Q) = 1$, then by the continuity of $F_1$ and $F_2$, we would have $F_2(t_Q) \geq B > A$, and this would contradict Lemma 5.1.

Further,

$$
F_2(t_Q) = BF_1(t_Q).
$$
Indeed, otherwise $F_2(t_Q) > BF_1(t_Q)$, which together with (5.26) and the continuity of $F_1$ and $F_2$ would imply that there exists $\varepsilon > 0$ such that

$$F_1(t_Q + \varepsilon) < 1, \quad F_2(t_Q + \varepsilon) > BF_1(t_Q + \varepsilon),$$

and these inequalities would contradict the definition of $t_Q$.

Set

$(5.28)$ \hspace{1cm} $\nu(Q) := F_2(t_Q)$

and suppose that (5.23) is fulfilled. Since $F_2$ is increasing, we see that

$(5.29)$ \hspace{1cm} $\mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda} \leq \nu(Q), \quad t \leq t_Q.$

On the other hand, if $t > t_Q$, then $t \notin A(Q)$, whence $F_2(t) \leq BF_1(t)$, that is,

$(5.30)$ \hspace{1cm} $\mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{p(x)} \, d\mu(x) \right)^{1/\lambda} \leq B \int_Q t^{p(x)} \, d\mu(x), \quad t > t_Q.$

Combining (5.29) and (5.30), we immediately arrive at (5.24), as desired.

(ii) Consider an arbitrary finite family $S_d \subset D$ of pairwise disjoint cubes. Among all subsets $\tilde{S}_d \subset S_d$ such that

$(5.31)$ \hspace{1cm} $\sum_{Q \in \tilde{S}_d} \int_Q t^{p(x)} \, d\mu(x) \leq 2,$

we choose a maximal subset $S'_d$ that is, a subset containing the largest number of cubes (it is not unique, in general).

We claim that $S'_d = S_d$. Indeed, assuming that $S'_d \subsetneq S_d$ and taking into account that $F_1$ is increasing, we have

$$\sum_{Q \in S'_d} \int_{Q} (t_Q / 2^{1/p_-})^{p(x)} \, d\mu(x) \leq \sum_{Q \in S'_d} \int_{Q} (t_Q^{p(x)}/2) \, d\mu(x) \leq 1.$$ 

By Lemma [5.1] this implies that

$$\sum_{Q \in S'_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{t_Q}{2^{1/p_-}} \right)^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda} \leq A.$$

Since $F_2$ is increasing, the above inequality yields

$$\sum_{Q \in S'_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda} \leq A,$$

whence

$$\sum_{Q \in S'_d} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} \, d\mu(x) \right)^{1/\lambda} \leq 2^{p_+/p_-} A.$$
This inequality and (5.25) and (5.27) imply that
\[
\sum_{Q \in S_d'} d_{Q} \frac{1}{B} \sum_{Q \in S_d'} \mu(Q) \left( \frac{1}{\mu(Q)} \int_{Q} t_{Q}^{p(x)} d\mu(x) \right)^{1/\lambda} \leq \frac{1}{2}.
\]

Now let \( P \in S_d \setminus S_d' \). Then, taking (5.26) into account, we get
\[
\sum_{Q \in S_d' \cup \{P\}} t_{Q}^{p(x)} d\mu(x) \leq \frac{1}{2} + \sum_{Q \in S_d} t_{Q}^{p(x)} d\mu(x) < \frac{3}{2}.
\]

This inequality, in view of (5.31), contradicts the maximality of \( S_d' \). This proves that \( S_d' = S_d \). It follows from (5.31), (5.27) and (5.28) that
\[
\sum_{Q \in S_d} \nu(Q) = B \sum_{Q \in S_d} t_{Q}^{p(x)} d\mu(x) \leq 2B,
\]
which completes the proof of (ii). ■

5.3. Third lemma. The next lemma is an extension of [L17, Lemma 5.3] with \( w \equiv 1 \) from the Euclidean setting of \( \mathbb{R}^n \) to the setting of spaces of homogeneous type.

**Lemma 5.3.** Let \((X, d, \mu)\) be a space of homogeneous type and \( p \in \mathcal{P}(X) \) satisfy \( 1 < p_- < p_+ < \infty \). Suppose the Hardy–Littlewood maximal operator \( M \) is bounded on \( L^{p(x)}(X, d, \mu) \). There exist constants \( D, \gamma > 1 \) and \( \zeta > 0 \) such that for every dyadic grid \( \mathcal{D} \in \bigcup_{K}^{K} \mathcal{D}^t \) and every cube \( Q \in \mathcal{D} \), if
\[
(5.32) \quad \min \left\{ 1, \frac{1}{\|\varphi_Q\|_{L^{p(x)}}^{1+\zeta}} \right\} \leq t \leq \max \left\{ 1, \frac{1}{\|\varphi_Q\|_{L^{p(x)}}^{1+\zeta}} \right\},
\]
then
\[
(5.33) \quad \left( \frac{1}{\mu(Q)} \int_{Q} t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} \leq D \frac{1}{\mu(Q)} \int_{Q} t^{p(x)} d\mu(x).
\]

**Proof.** Let \( A > 0 \) and \( \lambda > 1 \) be the constants of Lemma 5.1. Take any \( \gamma \) satisfying \( 1 < \gamma < \lambda \) and set
\[
\zeta := \frac{\lambda - \gamma}{\gamma(1+\lambda)} > 0.
\]

Fix a dyadic grid \( \mathcal{D} \in \bigcup_{K}^{K} \mathcal{D}^t \) and a cube \( Q \in \mathcal{D} \). For any \( \alpha > 0 \), we have
\[
(5.34) \quad \left( \frac{1}{\mu(Q)} \int_{Q} t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} = \left( \frac{1}{\mu(Q)} \int_{Q} t^{\gamma(p(x)-\alpha)} d\mu(x) \right)^{1/\gamma} t^\alpha.
\]

It follows from (5.32) that either
\[
(5.35) \quad 1 \leq t \leq \frac{1}{\|\varphi_Q\|_{L^{p(x)}}^{1+\zeta}},
\]
or
\[ (5.36) \quad \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \leq t \leq 1. \]

If \((5.35)\) is fulfilled and \(\gamma(p(x) - \alpha) \geq 0\), then
\[ (5.37) \quad t^{\gamma(p(x) - \alpha)} \leq \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x) - \alpha)} < 1 + \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x) - \alpha)} . \]

On the other hand, if \((5.35)\) is fulfilled and \(\gamma(p(x) - \alpha) < 0\), then
\[ (5.38) \quad t^{\gamma(p(x) - \alpha)} \leq 1 < 1 + \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x) - \alpha)} . \]

Analogously, if \((5.36)\) is fulfilled and \(\gamma(p(x) - \alpha) \geq 0\), then \((5.38)\) holds. On the other hand, if \((5.36)\) is fulfilled and \(\gamma(p(x) - \alpha) < 0\), then \((5.37)\) holds.

It follows from the above that if \((5.32)\) holds, then for all \(x \in X\) and all \(\alpha > 0\),
\[ t^{\gamma(p(x) - \alpha)} \leq 1 + \|xQ\|_{L^p(X)}^{\alpha(1+\xi)} \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x))} . \]

Integrating this inequality over the cube \(Q\) yields
\[ \int_Q t^{\gamma(p(x) - \alpha)} \, d\mu(x) \leq \mu(Q) + \|xQ\|_{L^p(X)}^{\alpha(1+\xi)} \int_Q \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x))} \, d\mu(x) . \]

Hence, since \((a^\gamma + b^\gamma)^{1/\gamma} \leq a + b\) for \(a, b \geq 0\) and \(\gamma > 1\), we see that
\[ (5.39) \quad \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma(p(x) - \alpha)} \, d\mu(x) \right)^{1/\gamma} \]
\[ \leq 1 + \left[ \|xQ\|_{L^p(X)}^{\alpha(1+\xi)} \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x))} \, d\mu(x) \right)^{1/\gamma} \right]^{1/\gamma} \]
\[ \leq 1 + \|xQ\|_{L^p(X)}^{\alpha(1+\xi)} \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma(p(x))} \, d\mu(x) \right)^{1/\gamma} . \]

Combining \((5.34)\) and \((5.39)\) we obtain, for \(\alpha > 0\),
\[ (5.40) \quad \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} \, d\mu(x) \right)^{1/\gamma} \]
\[ \leq t^\alpha + \|xQ\|_{L^p(X)}^{\alpha(1+\xi)} \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\|xQ\|_{L^p(X)}^{1+\xi}} \right)^{\gamma p(x)} \, d\mu(x) \right)^{1/\gamma} t^\alpha . \]
Let $\alpha = m_p(Q)$ be a median value of $p$ over $Q$, that is, a number satisfying

$$
\max\left\{ \frac{\mu(\{x \in Q : p(x) > m_p(Q)\})}{\mu(Q)}, \frac{\mu(\{x \in Q : p(x) < m_p(Q)\})}{\mu(Q)} \right\} \leq \frac{1}{2}.
$$

Set

$$
E_1(Q) := \{x \in Q : p(x) \leq m_p(Q)\}, \quad E_2(Q) := \{x \in Q : p(x) \geq m_p(Q)\}.
$$

It follows immediately from the definition of $m_p(Q)$ that

$$
\mu(E_j(Q)) \geq \frac{1}{2}\mu(Q), \quad j = 1, 2.
$$

Then, for $t \in (0, \infty)$, we have

$$
t^\alpha = t^{m_p(Q)} \leq \frac{2^{m_p(Q)}\mu(E_j(Q))}{\mu(Q)}
$$

$$
\leq \begin{cases} 
\frac{2}{\mu(Q)} \int_{E_1(Q)} t^{p(x)} d\mu(x) & \text{if } t \in (0, 1), \\
\frac{2}{\mu(Q)} \int_{E_2(Q)} t^{p(x)} d\mu(x) & \text{if } t \in [1, \infty)
\end{cases}
\leq \frac{2}{\mu(Q)} \int_Q t^{p(x)} d\mu(x).
$$

We claim that

$$
\chi_Q(x) \leq 2(M^D\chi_{E_j(Q)})(x), \quad x \in X, \ j = 1, 2.
$$

Indeed, if $x \notin Q$, then (5.43) is trivial. On the other hand, if $x \in Q$, then (5.41) implies that

$$
\chi_Q(x) = \frac{\mu(Q)}{\mu(Q)} \leq 2\frac{\mu(E_j(Q))}{\mu(Q)} = \frac{2}{\mu(Q)} \int_Q \chi_{E_j(Q)}(y) d\mu(y) \leq 2(M^D\chi_{E_j(Q)})(x),
$$

which completes the proof of (5.43).

It follows from (5.43), Theorem 2.2, and the boundedness of the Hardy–Littlewood maximal operator on $L^p(C)(X, d, \mu)$ that

$$
\|\chi_Q\|_{L^p(C)} \leq 2C_{HK}(X)\|M\|_{B(L^p(C))}\|\chi_{E_j(Q)}\|_{L^p(C)}, \quad j = 1, 2.
$$

In view of Lemma 4.2, we have

$$
\|\chi_{E_j(Q)}\|_{L^p(C)} \leq \max\left\{ (\mu(E_j(Q)))^{1/p_-(E_j(Q))}, (\mu(E_j(Q)))^{1/p_+(E_j(Q))} \right\}
\leq \max\left\{ (\mu(Q))^{1/p_-(E_j(Q))}, (\mu(Q))^{1/p_+(E_j(Q))} \right\}, \quad j = 1, 2.
$$

Taking into account the definition of the sets $E_j(Q)$, we see that

$$
p_-(E_1(Q)) \leq p_+(E_1(Q)) \leq m_p(Q) \leq p_-(E_2(Q)) \leq p_+(E_2(Q)).$$
Therefore, if \( \mu(Q) \leq 1 \), then
\[
\begin{align*}
(5.46) \quad \max \{ (\mu(Q))^{1/p-} (E_1(Q)), (\mu(Q))^{1/p+} (E_1(Q)) \} \leq (\mu(Q))^{1/m_p(Q)},
\end{align*}
\]
and if \( \mu(Q) > 1 \), then
\[
(5.47) \quad \max \{ (\mu(Q))^{1/p-} (E_2(Q)), (\mu(Q))^{1/p+} (E_2(Q)) \} \leq (\mu(Q))^{1/m_p(Q)}.
\]

If (5.35) is fulfilled, then \( \| \chi_Q \|_{L^p(\cdot)} \leq 1 \). Then, in view of Lemma 4.1, \( \mu(Q) \leq 1 \). On the other hand if (5.36) is fulfilled, then \( \| \chi_Q \|_{L^p(\cdot)} \geq 1 \). Therefore, by Lemma 4.1, \( \mu(Q) \geq 1 \). Thus, if (5.32) is fulfilled, then (5.44)–(5.47) imply that
\[
(5.48) \quad \| \chi_Q \|_{L^p(\cdot)} \leq 2C_{HK}(X) \| M \|_{B(L^p(\cdot))} (\mu(Q))^{1/m_p(Q)}.
\]

Set
\[
q := \frac{1 + \lambda}{1 + \gamma}, \quad q' := \frac{q}{q - 1}.
\]

Then
\[
(5.49) \quad \gamma q (1 + \zeta) = \gamma \frac{1 + \lambda}{1 + \gamma} \left( 1 + \frac{\lambda - \gamma}{\gamma (1 + \lambda)} \right) = \lambda
\]
and
\[
(5.50) \quad \gamma q' \zeta = \gamma \frac{1 + \lambda}{1 + \gamma} \left( \frac{1 + \lambda}{1 + \gamma} - 1 \right)^{-1} \frac{\lambda - \gamma}{\gamma (1 + \lambda)} = 1.
\]

Taking (5.49) and (5.50) into account, by Hölder’s inequality with exponents \( q, q' \in (1, \infty) \), we get
\[
(5.51) \quad \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{\gamma p(x)} d\mu(x)
\]
\[
\leq \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{\gamma q (1 + \zeta) p(x)} d\mu(x) \right)^{1/q}
\]
\[
= \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{\lambda p(x)} d\mu(x) \right)^{1/q}.
\]

Since
\[
\int_Q \left( \frac{1}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{p(x)} d\mu(x) = \int_X \left( \frac{\chi_Q(x)}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{p(x)} d\mu(x) \leq 1,
\]
applying Lemma 5.1 with \( S_d = \{Q\} \) and \( \alpha_Q = 1/\| \chi_Q \|_{L^p(\cdot)} \) we obtain
\[
\mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\| \chi_Q \|_{L^p(\cdot)}} \right)^{\lambda p(x)} d\mu(x) \right)^{1/\lambda} \leq A.
\]
Hence
\( (5.52) \int_Q \left( \frac{1}{\|\chi_Q\|_{L^p(Q)}} \right)^{\lambda p(x)} d\mu(x) \leq \left( \frac{A}{\mu(Q)} \right)^{\lambda/\gamma}. \)

Combining (5.51) and (5.52), we arrive at
\[ \frac{1}{\mu(Q)} \int_Q \left( \frac{1}{\|\chi_Q\|_{L^p(Q)}} \right)^{\gamma p(x)} d\mu(x) \leq \left( \frac{A}{\mu(Q)} \right)^{\lambda/\gamma}. \]

It follows from the above estimate and from (5.48) that
\[ \|\chi_Q\|_{L^p(Q)}^{\alpha(1+\zeta)} \leq (2C_{HK}(X)\|M\|_{B(L^p(\cdot))})^{\alpha(1+\zeta)}(\mu(Q))^\lambda/\gamma \cdot (\mu(Q))^{\gamma p(x)} \leq \left( \frac{A}{\mu(Q)} \right)^{\lambda/\gamma} \cdot \left( \frac{A}{\mu(Q)} \right)^{\lambda p(x)} \leq \frac{D}{\mu(Q)} \int_Q t^{p(x)} d\mu(x) \]
with
\[ D := 1 + (2C_{HK}(X)\|M\|_{B(L^p(\cdot))})^{p+(1+\zeta)} A^{\lambda/\gamma}. \]

Taking into account the definitions of \( \zeta \) and \( q \), we see that
\[ (5.54) \quad 1 + \zeta - \frac{\lambda}{\gamma q} = 1 + \frac{\lambda - \gamma}{\gamma(1+\lambda)} - 1 + \gamma = 0. \]

Combining (5.40), (5.42), (5.53) and (5.54), we get
\[ \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} \leq \frac{D}{2} t^{m_p(Q)} \leq \frac{D}{\mu(Q)} \int_Q t^{p(x)} d\mu(x) \]
which completes the proof of (5.33). \( \blacksquare \)

5.4. Fourth lemma. The next lemma is an extension of [L17 Lemma 4.1] with \( w \equiv 1 \) from the Euclidean setting of \( \mathbb{R}^n \) to the setting of spaces of homogeneous type.

**Lemma 5.4.** Let \( (X, d, \mu) \) be a space of homogeneous type and \( p \in \mathcal{P}(X) \) satisfy \( 1 < p_-, p_+ < \infty \). If the Hardy–Littlewood maximal operator \( M \) is bounded on \( L^p(\cdot)(X, d, \mu) \), then there exist constants \( C \), \( \gamma > 1 \) and \( \eta > 0 \) and a measure \( \nu \) on \( X \) such that for every dyadic grid \( \mathcal{D} \in \bigcup_{k=1}^K \mathcal{D}^k \) and every finite family \( \mathcal{S}_d \subset \mathcal{D} \) of pairwise disjoint cubes the following properties hold:

(i) if \( Q \in \mathcal{D} \) and \( t > 0 \) satisfies \( t\|\chi_Q\|_{L^p(\cdot)} \leq 1 \), then
\[ \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} \leq C \left( \int_Q t^{p(x)} d\mu(x) + 2t^\eta \nu(Q)(0,1)(t) \right); \]
(ii) \( \sum_{Q \in \mathcal{S}_d} \nu(Q) \leq C. \)
Proof. Let $B, D > 1$, and $1 < \gamma < \lambda$ be the constants given by Lemmas 5.2 and 5.3 and $\nu$ be the measure on $X$ given by Lemma 5.2. Put

$$\zeta := \frac{\lambda - \gamma}{\gamma(1 + \lambda)}$$

and take

(5.56) \quad $C := \max\{(2B)^{1+\zeta}, D\}$.\]

Then (ii) follows from Lemma 5.2(ii) because $C \geq 2B$.

Let us prove (i). If $t\|\chi_Q\|_{L^p(\cdot)} \leq 1$ and $t \geq 1$, then $\|\chi_Q\|_{L^p(\cdot)} \leq 1$ and therefore $\|\chi_Q\|_{L^p(\cdot)}^{1+\zeta} \leq \|\chi_Q\|_{L^p(\cdot)} \leq 1$. It is easy to check that (5.32) is fulfilled. Then it follows from Lemma 5.3 and (5.56) that

$$\left(\frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x)\right)^{1/\gamma} \leq \frac{C}{\mu(Q)} \int_Q t^{p(x)} d\mu(x),$$

which immediately implies (5.55) and completes the proof of part (i) for $t \geq 1$.

Assume that $t\|\chi_Q\|_{L^p(\cdot)} \leq 1$ and $0 < t < 1$. If

(5.57) \quad $\left(\frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x)\right)^{1/\gamma} \leq \frac{C}{\mu(Q)} \int_Q t^{p(x)} d\mu(x),$

then (5.55) is trivial.

Assume that (5.57) does not hold, that is,

(5.58) \quad $\frac{1}{\mu(Q)} \int_Q t^{p(x)} d\mu(x) < \frac{1}{C} \left(\frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x)\right)^{1/\gamma}.$

Set, as in the proof of Lemma 5.3,

$q := \frac{1 + \lambda}{1 + \gamma}, \quad q' := \frac{q}{q - 1}.$

By Hölder’s inequality, (5.58) and (5.49), we have

(5.59) \quad $\frac{1}{\mu(Q)} \int_Q t^{\frac{p(x)}{1+\zeta}} d\mu(x) \leq \left(\frac{1}{\mu(Q)} \int_Q t^{p(x)} d\mu(x)\right)^{\frac{1}{1+\zeta}}$

$$\leq \left(\frac{1}{C}\right)^{\frac{1}{1+\zeta}} \left(\frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x)\right)^{\frac{1}{q(1+\zeta)}}$$

$$= \left(\frac{1}{C}\right)^{\frac{1}{1+\zeta}} \left(\frac{1}{\mu(Q)} \int_Q t^{\frac{\lambda p(x)}{1+\zeta}} d\mu(x)\right)^{1/\lambda}.$$
It follows from (5.56) and (5.58) that
\[ \frac{D}{\mu(Q)} \int_{Q} t^{p(x)} d\mu(x) \leq \frac{C}{\mu(Q)} \int_{Q} t^{p(x)} d\mu(x) < \left( \frac{1}{\mu(Q)} \int_{Q} t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma}, \]
that is, (5.33) does not hold. Therefore, by Lemma 5.3, condition (5.32) is not fulfilled. Since \( 0 < t < 1 \), this means that
\[ 0 < t < \frac{1}{\|\chi_{Q}\|_{L_p(\cdot)}}, \]
whence \( \|t^{1/(1+\zeta)} \chi_{Q}\|_{L_p(\cdot)} \leq 1 \). Therefore, by Lemma 4.1,
\[ \int_{Q} t^{\frac{p(x)}{1+\zeta}} d\mu(x) = \int_{Q} (t^{\frac{1}{1+\zeta}} \chi_{Q}(x))^{p(x)} d\mu(x) \leq 1, \]
that is, (5.23) is fulfilled with \( t^{1/(1+\zeta)} \) in place of \( t \). Then, by Lemma 5.2, (5.24) holds with \( t^{1/(1+\zeta)} \) in place of \( t \), that is,
\[ \mu(Q) \left( \frac{1}{\mu(Q)} \int_{Q} \frac{\lambda p(x)}{1+\zeta} d\mu(x) \right)^{1/\lambda} \leq B \int_{Q} t^{\frac{p(x)}{1+\zeta}} d\mu(x) + \nu(Q). \]
It follows from (5.59), (5.60) and (5.66) that
\[ \mu(Q) \left( \frac{1}{\mu(Q)} \int_{Q} \frac{\lambda p(x)}{1+\zeta} d\mu(x) \right)^{1/\lambda} \leq B \left( \frac{1}{C} \right)^{\frac{1}{1+\zeta}} \left( \frac{1}{\mu(Q)} \int_{Q} \frac{\lambda p(x)}{1+\zeta} d\mu(x) \right)^{1/\lambda} + \nu(Q) \]
\[ \leq \frac{1}{2} \left( \frac{1}{\mu(Q)} \int_{Q} \frac{\lambda p(x)}{1+\zeta} d\mu(x) \right)^{1/\lambda} + \nu(Q). \]
Thus
\[ \left( \frac{1}{\mu(Q)} \int_{Q} \frac{\lambda p(x)}{1+\zeta} d\mu(x) \right)^{1/\lambda} \leq 2\nu(Q). \]
Since \( 0 < t < 1 \), we have
\[ t^{\frac{p(x)}{1+\zeta}} = t^{p(x)} t^{\frac{-\zeta p(x)}{1+\zeta}} \leq t^{p(x)} t^{\frac{-\zeta p}{1+\zeta}}. \]
Inequalities (5.61) and (5.62) imply that
\[ \mu(Q) \left( \frac{1}{\mu(Q)} \int_{Q} t^{\lambda p(x) - \frac{\zeta p}{1+\zeta}} d\mu(x) \right)^{1/\lambda} \leq 2\nu(Q), \]
whence
\[ \mu(Q) \left( \frac{1}{Q} \int_{Q} t^{\lambda p(x)} d\mu(x) \right)^{1/\lambda} \leq 2t^{\eta} \nu(Q) \]
with
\[ \eta := \frac{\zeta p}{1 + \zeta}. \]

Since \(1 < \gamma < \lambda\), by Hölder’s inequality we have
\[ (5.64) \quad \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} \leq \left( \frac{1}{\mu(Q)} \int_Q t^{\lambda p(x)} d\mu(x) \right)^{1/\lambda}. \]

Combining (5.63) and (5.64), we arrive at
\[ \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q t^{\gamma p(x)} d\mu(x) \right)^{1/\gamma} \leq 2t^{\eta} \nu(Q), \]
which implies (5.55) and completes the proof of (i) for \(0 < t < 1\). □

6. Proof of Theorem 1.1. It is sufficient to show that if the Hardy–Littlewood maximal operator \(M\) is bounded on the space \(L^p(\cdot)\), then it is also bounded on \(L^{p'}(\cdot)\). In turn, in view of Corollary 3.2 it is enough to verify the condition \(A_\infty\). To do this, we will apply Lemma 4.4.

Let \(D \in \bigcup_{t=1}^K D^t\) be a dyadic grid, \(S \subset D\) be a finite sparse family, and \(\{G_Q\}_{Q \in S}\) be a collection of nonnegative numbers such that
\[ \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{L^p(\cdot)} = 1. \]

Then for every \(Q \in S\),
\[ \alpha_Q \left\| \chi_Q \right\|_{L^p(\cdot)} \leq \left\| \sum_{Q \in S} \alpha_Q \chi_Q \right\|_{L^p(\cdot)} = 1. \]

Let \(C, \gamma > 1\) and \(\eta > 0\) be the constants and \(\nu\) be the measure from Lemma 5.4. Suppose \(Q \in S\) is such that \(\alpha_Q \geq 1\). Applying Hölder’s inequality and Lemma 5.4, we get
\[ \int_{G_Q} \alpha_Q^{p(x)} d\mu(x) \leq \left( \frac{\mu(G_Q)}{\mu(Q)} \right)^{1/\gamma'} \mu(Q) \left( \frac{1}{\mu(Q)} \int_Q \alpha_Q^{p(x)} d\mu(x) \right)^{1/\gamma} \]
\[ \leq C \left( \frac{\mu(G_Q)}{\mu(Q)} \right)^{1/\gamma'} \int_Q \alpha_Q^{p(x)} d\mu(x). \]
Combining this inequality with Lemma 4.3, we get
\begin{equation}
(6.1) \sum_{Q \in S : \alpha_Q \geq 1} \int_{Q} \alpha^p_Q(x) \, d\mu(x) \leq C \sum_{Q \in S : \alpha_Q \geq 1} \left( \frac{\mu(G_Q)}{\mu(Q)} \right)^{1/\gamma} \int_{Q} \alpha^p_Q(x) \, d\mu(x)
\end{equation}
\begin{equation}
\leq C \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{1/\gamma} \sum_{Q \in S} \alpha^p_Q(x) \, d\mu(x) \leq C \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right)^{1/\gamma} .
\end{equation}

For \( k \in \mathbb{N} \), put
\begin{equation}
(6.2) S_k := \{ Q \in S : 2^{-k} \leq \alpha_Q < 2^{-k+1} \}.
\end{equation}
If \( S_k \neq \emptyset \), then there exist \( i_k \in \mathbb{N} \) and cubes \( Q_1^k, \ldots, Q_{i_k}^k \) such that \( \bigcup_{Q \in S_k} Q = \bigcup_{j=1}^{i_k} Q_j^k \); if \( i, j \in \{ 1, \ldots, i_k \} \) and \( i \neq j \), then \( Q_i^k \cap Q_j^k = \emptyset \); and for every \( Q \in S_k \), there is \( j \in \{ 1, \ldots, i_k \} \) such that \( Q \subset Q_j^k \).

For \( k \in \mathbb{N} \) and \( S_k \neq \emptyset \), put
\begin{equation}
(6.3) \psi_{Q_j^k}(x) = \sum_{Q \in S_k : Q \subset Q_j^k} \chi_{G_Q}(x), \quad j \in \{ 1, \ldots, i_k \}.
\end{equation}
Then, taking (6.2) into account one has, for all \( j \in \{ 1, \ldots, i_k \} \),
\begin{equation}
(6.4) \sum_{Q \in S_k : Q \subset Q_j^k} \int_{G_Q} \alpha^p_Q(x) \, d\mu(x) = \sum_{Q \in S_k : Q \subset Q_j^k} \int_{Q_j^k} \alpha^p_Q(x) \chi_{G_Q}(x) \, d\mu(x)
\end{equation}
\begin{equation}
\leq \sum_{Q \in S_k : Q \subset Q_j^k} \int_{Q_j^k} (2^{-k+1})^p \chi_{G_Q}(x) \, d\mu(x) = \sum_{Q_j^k} \int_{Q_j^k} (2^{-k+1})^p \psi_{Q_j^k}(x) \, d\mu(x)
\end{equation}
\begin{equation}
\leq 2^p \sum_{Q_j^k} \int_{Q_j^k} 2^{-kp} \psi_{Q_j^k}(x) \, d\mu(x) \leq 2^p \sum_{Q_j^k} \alpha^{p(x)}_{Q_j^k} \psi_{Q_j^k}(x) \, d\mu(x).
\end{equation}

By Hölder’s inequality, for \( k \in \mathbb{N} \) with \( S_k \neq \emptyset \) and \( j \in \{ 1, \ldots, i_k \} \), one has
\begin{equation}
(6.5) \int_{Q_j^k} \alpha^p_{Q_j^k} \psi_{Q_j^k}(x) \, d\mu(x) \leq \mu(Q_j^k) \left( \frac{1}{\mu(Q_j^k)} \right) \int_{Q_j^k} \alpha^{p(x)}_{Q_j^k} \, d\mu(x)
\end{equation}
\begin{equation}
\times \left( \frac{1}{\mu(Q_j^k)} \right) \int_{Q_j^k} \psi^{\gamma'}_{Q_j^k}(x) \, d\mu(x) \right)^{1/\gamma'} .
\end{equation}
It follows from (6.2) and the hypothesis that the sets \( \{ Q_G \}_{Q \in S} \) are pairwise disjoint that
\begin{equation}
(6.6) \int_{Q_j^k} \psi^{\gamma'}_{Q_j^k}(x) \, d\mu(x) = \sum_{Q \in S_k : Q \subset Q_j^k} \mu(G_Q) \leq \left( \max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)} \right) \sum_{Q \in S_k : Q \subset Q_j^k} \mu(Q).
\end{equation}
Since \( S \) is a sparse family, there exists a collection of pairwise disjoint sets \( \{ E(Q) \}_{Q \in S} \) such that \( E(Q) \subset Q \) and \( \mu(Q) \leq 2\mu(E(Q)) \). Hence, for all \( k \in \mathbb{N} \)
such that $S_k \neq \emptyset$ and all $j \in \{1, \ldots, i_k\}$,

$$\sum_{Q \in S_k : Q \subset Q_j^k} \mu(Q) \leq 2 \sum_{Q \in S_k : Q \subset Q_j^k} \mu(E(Q)) = 2\mu \left( \bigcup_{Q \in S_k : Q \subset Q_j^k} E(Q) \right) \leq 2\mu(Q_j^k).$$

On the other hand, taking into account that $\alpha_{Q_j^k} < 1$, we deduce from Lemma 5.4 that

$$\mu(Q_j^k) \left( \frac{1}{\mu(Q_j^k)} \int_{Q_j^k} \alpha_{Q_j^k} \gamma p(x) \, d\mu(x) \right)^{1/\gamma} \leq C \int_{Q_j^k} \alpha_{Q_j^k} \gamma p(x) \, d\mu(x) + 2\alpha_{Q_j^k} \nu(Q_j^k).$$

Combining (6.4)–(6.8), we obtain for every $k \in \mathbb{N}$ such that $S_k \neq \emptyset$ and every $j \in \{1, \ldots, i_k\}$,

$$\sum_{Q \in S_k : Q \subset Q_j^k} \int_{G Q} \alpha_Q \gamma p(x) \, d\mu(x) \leq 2^{p+1/\gamma} \left( \frac{\mu(G Q)}{\mu(Q)} \right)^{1/\gamma'} \left( C \int_{Q_j^k} \alpha_{Q_j^k} \gamma p(x) \, d\mu(x) + 2\alpha_{Q_j^k} \nu(Q_j^k) \right).$$

Then

$$\sum_{Q \in S : \alpha_Q < 1} \int_{G Q} \alpha_Q \gamma p(x) \, d\mu(x) = \sum_{k \in \mathbb{N} : S_k \neq \emptyset} \sum_{Q \in S_k} \int_{G Q} \alpha_Q \gamma p(x) \, d\mu(x) \leq 2^{p+1/\gamma'} \left( \max_{Q \in S} \frac{\mu(G Q)}{\mu(Q)} \right)^{1/\gamma'} \times \sum_{k \in \mathbb{N} : S_k \neq \emptyset} \sum_{j=1}^{i_k} \left( C \int_{Q_j^k} \alpha_{Q_j^k} \gamma p(x) \, d\mu(x) + 2\alpha_{Q_j^k} \nu(Q_j^k) \right).$$

It follows from Lemma 4.3 that

$$\sum_{k \in \mathbb{N} : S_k \neq \emptyset} \sum_{j=1}^{i_k} \int_{Q_j^k} \alpha_{Q_j^k} \gamma p(x) \, d\mu(x) \leq \sum_{Q \in S} \int_{Q} \alpha_Q \gamma p(x) \, d\mu(x) \leq 1.$$

Since for every fixed $k$, the cubes $Q_1^k, \ldots, Q_{i_k}^k$ are pairwise disjoint, it follows
from Lemma 5.4(ii) and (6.2) that

\[
\sum_{k \in \mathbb{N} : S_k \neq \emptyset} \sum_{j=1}^{i_k} \alpha_{Q_j}^{\eta} \nu(Q_j) \leq \sum_{k \in \mathbb{N} : S_k \neq \emptyset} (2^{-k+1})^\eta \sum_{j=1}^{i_k} \nu(Q_j) \leq C \sum_{k \in \mathbb{N} : S_k \neq \emptyset} 2^{(-k+1)\eta} \leq C 2^\eta \sum_{k=1}^{\infty} \left(\frac{1}{2^\eta}\right)^k =: C_1.
\]

It follows from (6.9)–(6.11) that

\[
\sum_{Q \in S : \alpha_Q < 1} G_{Q} \alpha_{Q}^{p(x)} d\mu(x) \leq 2^{p+1/\gamma'} (C + C_1) \left(\max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^{1/\gamma'}.
\]

Combining (6.1) and (6.12), we see that

\[
\sum_{Q \in S} \int_{G_Q} \alpha_{Q}^{p(x)} d\mu(x) \leq \Psi \left(\max_{Q \in S} \frac{\mu(G_Q)}{\mu(Q)}\right)^{\xi}
\]

with \(\Psi := C + 2^{p+1/\gamma'} (C + 2C_1)\) and \(\xi = 1/\gamma'\). Hence (4.3) implies (4.4). By Lemma 4.4, the space \(L^{p(\cdot)}(X,d,\mu)\) satisfies the condition \(A_\infty\). Thus, the Hardy–Littlewood maximal operator \(M\) is bounded on the variable Lebesgue space \(L^{p(\cdot)}(X,d,\mu)\) in view of Corollary 3.2.

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