ON THE BOUNDED COHOMOLOGY
OF ERGODIC GROUP ACTIONS

By

JON AARONSON∗ AND BENJAMIN WEISS

Dedicated, with admiration, to Larry Zalcman
upon concluding thirty years as editor of the Journal d’Analyse

Abstract. In this note we show existence of bounded, continuous, transitive
cocycles over a transitive action by homeomorphisms of any finitely generated
group on a Polish space, and bounded, measurable, ergodic cocycles over any
ergodic, probability-preserving action of \(\mathbb{Z}^d\).

0 Introduction

Cocycles and skew product actions. Let \(\Gamma\) be a countable group and
let \(X\) be a space. In the sequel, \(X\) will represent either a Polish, metric space
\(X = (X, d)\) or a standard probability space \(X = (X, \mathcal{B}, m)\). A \(\Gamma\)-action
on \(X\) is a homomorphism \(T : \Gamma \to \text{Aut}(X)\). In the topological case, \(\text{Aut}(X) = \text{Homeo}(X)\) in
and \(\text{Aut}(X) = \text{PPT}(X, \mathcal{B}, m)\) the group of probability-preserving transformations of
\((X, \mathcal{B}, m)\) in the probabilistic case.

Let \(G\) be an abelian topological group equipped with a norm \(\| \cdot \|_G\). To define
a \(\Gamma\)-skew product action on \(X \times G\), we need a \(T\)-cocycle, that is, a function
\(F : \Gamma \times X \to G\) satisfying

\[
F(nk, x) = F(k, x) + F(n, T_kx) \quad (n, k \in \Gamma).
\]

The cocycle \(F : \Gamma \times X \to G\) is assumed to be continuous in the topological case
and measurable in the probabilistic case.

The \(F\)-skew product transformations are then defined on \(X \times G\) by

\[
T_n^{(F)}(x, z) := (T_n(x), z + F(n, x)) \quad (n \in \Gamma).
\]

∗ Aaronson’s research was partially supported by ISF grant No. 1289/17.
The assumptions and (Ψ) ensure that $T^{(F)} : \Gamma \times (X \times G) \to \text{Aut}(X \times G)$ is a $\Gamma$-action, called the **skew product action**.

We will also consider cocycles which are bounded in the sense that

$$\sup_{x \in X} \| F(\gamma, x) \|_G < \infty \quad \forall \gamma \in \Gamma.$$ 

In case $\Gamma = \mathbb{Z}$, it is easy to exhibit cocycles. Let $\varphi : X \to G$ and define $F = F^{(\varphi)} : \mathbb{Z} \times X \to G$ by

$$F(n, x) = \begin{cases} \sum_{k=0}^{n-1} \varphi(T^k x), & n \geq 1, \\ 0, & n = 0, \\ - \sum_{k=1}^{\lfloor |n| \rfloor} \varphi(T^{-k} x), & n \leq -1. \end{cases}$$

This is a cocycle and indeed any cocycle is of this form. We sometimes write $T^{(F)}_n = T^n_\varphi$ where $T_\varphi : X \times G \to X \times G$ is the **skew product transformation** defined by $T_\varphi(x, z) := (T(x), z + \varphi(x))$.

Construction of cocycles for the actions of multidimensional groups (e.g., $\Gamma = \mathbb{Z}^2$) is more difficult.

Note that a constant cocycle for an action $T : \Gamma \to \text{Aut}(X)$ is given by a homomorphism $h : \Gamma \to G$ an in this case; the skew product action

$$T^{(h)} : \Gamma \to \text{Aut}(X \times G)$$

is given by the (direct) product action $T \times h$ where $(T \times h)_\gamma(x, y) := (T_\gamma(x), y + h(\gamma))$.

The simplest $G$-valued, non-constant $T$-cocycles for an action $T : \Gamma \to \text{Aut}(X)$ are given by a **coboundary**, that is, a function $h : \Gamma \times X \to G$ defined by

$$h(n, x) = c(x) - c(T_n x),$$

where $c : X \to G$ (the **transfer function**) is measurable or continuous in the probabilistic and topological cases respectively. It is not hard to see that a coboundary is a cocycle.

For full shifts of $\mathbb{Z}^d$, the only Hölder continuous $\mathbb{R}^k$-valued cocycles are sums of a coboundary and a constant cocycle (homomorphism). See §4. The dynamical properties of such cocycles are somewhat limited (see §4). However, for certain infinitely generated groups, constant cocycles can have robust dynamics (see §3).

**Topological cocycles.** The first topologically transitive, topological skew product $\mathbb{Z}$-actions on $\mathbb{T} \times \mathbb{R}$ were constructed in [Sni30] and [Bes37].

We prove:
**Theorem 1.** Let $X$ be a perfect, Polish space, $\Gamma$ be a countable, finitely generated group and let $(\mathcal{B}, \| \cdot \|_\mathcal{B})$ be a separable Banach space.

If $T : \Gamma \to \text{Homeo}(X)$ is a topologically transitive action, then there is a bounded, continuous cocycle $h : \Gamma \times X \to \mathcal{B}$ so that the skew product action $(X \times \mathcal{B}, T_h)$ is topologically transitive.

Theorem 1 was established for $\mathbb{Z}$-actions in [Sid73]. Note that the skew product action of a coboundary cannot be transitive. The cocycle in Theorem 1 is constructed as a limit of coboundaries. In §3, we consider versions of Theorem 1 for certain infinitely generated groups.

**Measurable cocycles.** As shown in [Her79], for $(X, \mathcal{B}, m, T)$ an ergodic, probability-preserving transformation and $G$ a locally compact, Polish, amenable group, there is a measurable function $F : X \to G$ so that the skew product $(X \times G, m \times m_G, T_F)$ is ergodic. In case, e.g., $G$ is a closed subgroup of $\mathbb{R}^d$, there is a bounded, measurable function $F : X \to G$ as above ([AW04]).

Now let $S : \Gamma \times Y \to \text{Aut}(Y)$ be an ergodic action of the countable, amenable group $\Gamma$ on the probability space $Y$; there is a cocycle $G : \Gamma \times Y \to G$ so that $S^{(G)}$ is ergodic.

By [CFW81], the actions

$$T : \mathbb{Z} \times X \to \text{Aut}(X) \& S : \Gamma \times Y \to \text{Aut}(Y)$$

are orbit equivalent. The orbit equivalence transports $F$ to a cocycle

$$G : \Gamma \times Y \to G$$

for $S$ so that the actions $T^{(F)} \& S^{(G)}$ are orbit equivalent. Thus $S^{(G)}$ is ergodic.

However, regularity properties of $F$ (e.g., boundedness) need not pass to $G$.

We prove:

**Theorem 2.** Let $d, D \geq 1$, let $G \leq \mathbb{R}^D$ be a closed subgroup of full dimension and let $S \subset G$ be finite, symmetric and generating in the sense that $(S) = G$.

Let $T$ be an ergodic $\mathbb{Z}^d$-action of the standard non-atomic probability space $(X, \mathcal{B}, m)$. There is a bounded, measurable cocycle $F : \mathbb{Z}^d \times X \to G$ with ergodic skew product action $T^{(F)}$ so that $F(e_k, \cdot) \in S \cup \{0\}$ $\forall 1 \leq k \leq d$.

Here and throughout, $e_k^{(d)} \in \mathbb{R}^d$ ($1 \leq k \leq d$) with $(e_k^{(d)})_j = 1_{[j = k]}$. In other words, $\{e_k^{(d)} : 1 \leq k \leq d\}$ is the usual orthonormal basis for $\mathbb{R}^d$. When the dimension $d$ is unambiguous, we suppress it and write $e_k^{(d)} = e_k$.

The proof of Theorem 1 is given in §2 and that of Theorem 2 is given in §3. In both proofs the advertised cocycles are limits of coboundaries satisfying finite essential value conditions.
1 Proof of Theorem 1

Suppose that \((X, T)\) is a free action of the countable \(\Gamma\) by homeomorphisms on a complete, separable, perfect metric space \((X, d)\) and that \((B, \| \cdot \|_B)\) is a separable Banach space.

**Bounded, continuous cocycles.** A function \(F : X \to B\) is **bounded** if
\[
\sup_{x \in X} \|F(x)\|_B < \infty.
\]

We denote the collection of bounded, continuous (abbr. BC) \(B\)-valued functions by \(C_B(X, B)\). It is a Banach space when equipped with the supremum norm
\[
\|F\|_{\text{sup}} := \sup_{x \in X} \|F(x)\|_B.
\]

We call the cocycle \(h : \Gamma \times X \to B\) **BC** if for each \(\gamma \in \Gamma\), \(x \mapsto h(\gamma, x)\) is a BC function \(X \to B\).

Denote the collection of BC, \(B\)-valued cocycles for \(T\) by \(\text{Coc}(X, T, B)\). Fixing a finite, symmetric set \(\Sigma\) of generators for \(\Gamma\), we may define
\[
\|h\|_{\text{Coc}} := \max_{\gamma \in \Sigma} \sup_{x \in X} \|h(\gamma, x)\|_B \quad (h \in \text{Coc}(X, T, B)).
\]

It is not hard to see that \((\text{Coc}(X, T, B), \| \cdot \|_{\text{Coc}})\) is a Banach space.

**Word metric on \(\Gamma\).** Define the **\(\Sigma\)-word length norm** on \(\Gamma\) by
\[
\|N\|_{\Sigma} := \min\{k \geq 1 : \exists \sigma_1, \sigma_2, \ldots, \sigma_k \in \Sigma, \ N = \sigma_1 \sigma_2 \cdots \sigma_k\}.
\]

It follows that \(\|NN'\|_{\Sigma} \leq \|N\|_{\Sigma} + \|N'\|_{\Sigma}\) and, since \(\Sigma\) is symmetric, \(\|N^{-1}\|_{\Sigma} = \|N\|_{\Sigma}\).

The left invariant **\(\Sigma\)-word metric** on \(\Gamma\) is the distance
\[
\rho_{\Sigma}(k, \ell) := \|k^{-1} \ell\|_{\Sigma}.
\]

It is easy to see that for \(N \in \Gamma\) & \(h \in \text{Coc}(X, T, B)\),
\[
|h(N, x)| \leq \|h\|_{\text{Coc}} \|N\|_{\Sigma}.
\]

**Coboundaries.** The cocycle \(h \in \text{Coc}(X, T, B)\) is a **BC coboundary** if for some \(c \in C_B(X, B)\), (the transfer function)
\[
\text{(cob)} \quad h(n, x) = c(x) - c(T_n x).
\]

In this case we denote \(h := \nabla c\).
We denote the collection of BC coboundaries by
\[ \partial C_B(X, T; \mathbb{B}) := \{ \nabla g : g \in C_B(X, T; \mathbb{B}) \}. \]

It is easy to see that \( \partial C_B(X; \mathbb{B}) \subset \text{Coc}(X, T; \mathbb{B}) \). The cocycle advertised in Theorem 1 will appear in \( \partial C_B(X; \mathbb{B}) \).

**Topological essential values at a transitive point.** Let \( h \in \text{Coc}(X, T; \mathbb{B}) \), fix a \( T \)-transitive point \( x_0 \in X \) and set
\[ \mathcal{T}_{x_0} := \{ \text{open nbds of } x_0 \}. \]

For \( U \in \mathcal{T}_{x_0}, \ t \in \mathbb{B}, \ n \in \Gamma \) and \( \epsilon > 0 \) we will need the “**essential value conditions**”
\[
\begin{align*}
\text{EVC}(x_0, U, t, \epsilon, n) : & \ T_nx_0 \in U \ \& \ |h(n, x_0) - t|_B < \epsilon; \\
\text{EVC}(x_0, U, t, 0, n) : & \ T_nx_0 \in U \ \& \ h(n, x_0) = t.
\end{align*}
\]

Let
\[ E(h, x_0) = \{ r \in \mathbb{B} : \forall \epsilon > 0 \ \& \ U \in \mathcal{T}_{x_0}, \ \exists n \in \Gamma \ \text{such that} \ \text{EVC}(x_0, U, t, \epsilon, n) \ \text{holds} \}. \]

**Proposition 1.** \( E(h, x_0) \) is a closed subsemigroup of \( \mathbb{B} \) and if \( E(h, x_0) = \mathbb{B} \), then
\[
\{ R_n(x_0, 0) : n \in \Gamma \} = X \times \mathbb{B},
\]
where \( R \) is the \( h \)-skew product action.

**Proof.** Let \( s, \ t \in E(h, x_0), \ U \in \mathcal{T}_{x_0} \) \& \( \epsilon > 0 \). We will show that \( \exists K \in \Gamma \) so that
\[
\| h(K, x_0) - (s + t) \|_B < \epsilon.
\]

By definition, \( \exists n \in \Gamma \) so that \( T_nx_0 \in U \ \& \ |h(n, x_0) - s|_B < \frac{\epsilon}{2} \).

Thus, \( V := U \cap T_n^{-1}U \cap \{ |h(n, \cdot) - s|_B < \frac{\epsilon}{2} \} \in \mathcal{T}_{x_0} \) and by definition \( \exists N \in \Gamma \) such that \( T_Nx_0 \in V \ \& \ |h(N, x_0) - t|_B < \frac{\epsilon}{2} \).

Now \( T_Nx_0 \in V \subset T_n^{-1}U \cap \{ |h(n, \cdot) - s|_B < \frac{\epsilon}{2} \} \) so
\[
T_Nx_0 = T_n \circ T_N(x_0) \in U \ \& \ |h(n, T_Nx_0) - s|_B < \frac{\epsilon}{2}.
\]

It follows that
\[
|h(nN, x_0) - (s + t)|_B \leq |h(N, x_0) - t|_B + |h(n, T_Nx_0) - s|_B < \epsilon. \quad \square\star
\]
It is immediate that $E(h, x_0)$ is closed and that
\[(x_0, t) \in \{ T^n_h(x_0, 0) : n = 1, 2, \ldots \} \forall t \in E(h, x_0). \]
Using this and $T_h$-invariance, we see (\$) when $E(h, x_0) = \mathbb{B}$. \qed

**Lemma 2.** Let $V \in \mathfrak{T}_{x_0}$, $s \in \mathbb{B}$, $\Delta > 0$ and let $F \in \partial \text{Coc}(X, T, \mathbb{B})$, then
\[\exists h \in \partial \text{Coc}(X, T, \mathbb{B}), \quad \|h\|_{\text{Coc}} < \Delta \text{ and } n \in \Gamma \]
so that $F + h$ satisfies $\text{EVC}(x_0, V, s, \Delta, n)$.

**Proof.** Suppose that $F_g = c - c \circ T_g$ where $c \in C(X)$, and find $W \in \mathfrak{T}_{x_0}$, $W \subset V$ so that
\[|c(x) - c(y)| < \frac{\Delta}{3} \quad \forall x, y \in W.\]
Fix $N \in \Gamma$ so that
\[\frac{\|s\|_{\mathbb{B}}}{\|N\|_{\Sigma}} < \Delta \quad \text{and} \quad T_N(x_0) \in W.\]
Next, fix $\varepsilon > 0$ so that the sets
\[\{ T_k B(x_0, 2\varepsilon) : \|k\| \leq 3\|N\| \}\]
are disjoint. Here, for $x \in X$, $r > 0$, $B(x, r) := \{ y \in X : d(y, x) \leq r \}$ is the closed $r$-ball centered at $x$.

Let
\[G(x) := \left( 1 - \frac{d(B(x_0, \varepsilon), x)}{\varepsilon} \right)_+ \quad (x \in X), \]
\[a(k) := \left( 1 - \frac{\rho_{\Sigma}(N, k)}{\|N\|} \right)_+ \quad (k \in \Gamma), \]
where $d(B, x) := \inf_{y \in B} d(y, x)$ and $a_+ := \max\{ a, 0 \}$ for $a \in \mathbb{R}$. It follows that
\[G \in C(X, [0, 1]), \quad G|_{\mathbb{B}(x_0, \varepsilon)} \equiv 1 \quad \text{and} \quad G|_{\mathbb{B}(x_0, 2\varepsilon)} \equiv 0\]
and that
\[|a(k) - a(k\gamma)| \leq \frac{1}{\|N\|} \quad \forall \gamma \in \Sigma \text{ and } k \in \Gamma.\]

Now define $g : X \to \mathbb{B}$, $g \geq 0$ by
\[g(x) = \begin{cases} sa(k)G(T_k^{-1}x), & x \in T_k B(x_0, 2\varepsilon), \quad \rho_{\Sigma}(N, k) \leq \|N\|, \\ 0, & x \in X \setminus \bigcup_{k \in \Gamma, \rho_{\Sigma}(N, k) \leq \|N\|} T_k B(x_0, 2\varepsilon). \end{cases}\]
It follows that $g \in C_{\mathbb{B}}(X, \mathbb{B})$. 

Define \( h(n, x) := g(T_n x) - g(x) \). It follows from the above that

\[
\| h(\gamma, x) \|_B \leq \frac{\| s \|_B}{\| N \|_{\Sigma}} < \frac{\Delta}{3} \quad \forall \gamma \in \Sigma.
\]

To see that \( h \) satisfies \( \text{EVC}(x_0, W, s, 0, N) \), we have that \( T_N(x_0) \in W \) and

\[
h(N, x_0) = g(T_Nx_0) - g(x_0) = sa(N) - sa(0) = s.
\]

Lastly,

\[
|F(N, x_0) + h(N, x_0) - s| \leq |F(N, x_0)| = |c(x_0) - c(T_N(x_0))| < \frac{\Delta}{3}
\]
since \( x_0, T_Nx_0 \in W \). Thus \( F + h \) satisfies \( \text{EVC}(x_0, V, s, \Delta, N) \). \( \square \)

**Categorical construction.** Let \( x_0 \in X \) be a \( T \)-transitive point. For each \( V \in \Sigma_{x_0} \), \( s \in B \), \( n \in \Gamma \) and \( \Delta > 0 \), let

\[
E(x_0, V, s, \Delta) := \{ h \in \partial \text{Coc}(X, T, B) : \exists n \in \Gamma \text{ such that } h \text{ satisfies } \text{EVC}(x_0, V, s, \Delta, n) \}.
\]

It follows from Lemma 2 that \( E(x_0, V, s, \Delta) \) is open and dense in \( \partial \text{Coc}(X, T, B) \). Fix

- a decreasing sequence \( (U_n : n \geq 1) \), \( U_n \in \Sigma_{x_0} \) so that \( \forall V \in \Sigma_{x_0} \exists N_V \) so that \( U_n \subset V \forall n \geq N_V \);
- \( (t_n : n \geq 1) \in B^\mathbb{N} \) dense, and taking each value i.o.; and
- \( \eta_n > \eta_{n+1} \uparrow 0 \).

By Baire’s theorem,

\[
E := \bigcap_{k=1}^{\infty} E(x_0, U_k, t_k, \eta_k)
\]
is residual in \( \partial \text{Coc}(X, T, B) \).

By Proposition 1, for each \( h \in E \) we have that \( E(h, x_0) = B \) and hence

\[
\{ R_n(x_0, 0) : n \in \Gamma \} = X \times B.
\]

This proves Theorem 1. \( \square \)

We note that sequential constructions are also possible; see \( \S 3.\)
2 Proof of Theorem 2

We adapt here from §3 of [ALV98] the essential value conditions or EVC’s, which give countably many conditions for the ergodicity of the skew product action $T^{(φ)} : \mathbb{Z}^d \to \text{MPT}(X \times G, \mathcal{B}(X \times G), m \times m_G)$.

These are best understood in terms of cocycles with respect to the orbit equivalence relation of $T$ and its groupoid as in [FM77] (see below).

**Orbit cocycles.** The orbit equivalence relation generated by the free $\mathbb{Z}^d$-action $T$ is

$$R = R_T := \{(x, T_n x) : x \in X, \ n \in \mathbb{Z}^d\}.$$

An $R$-cocycle is a measurable function $\tilde{ϕ} : R \to G$ such that if $(x, y), (y, z) \in R$, then

$$\tilde{ϕ}(x, z) = \tilde{ϕ}(x, y) + \tilde{ϕ}(y, z).$$

The $R$-cocycle $\tilde{ϕ} : R \to G$ corresponds to a $T$-cocycle $ϕ : \mathbb{Z}^d \times X \to G$ via

$$ϕ(n, x) := \tilde{ϕ}(x, T_n x).$$

**Groupoid.** A partial probability-preserving transformation of $X$ is a pair $(R, A)$, where $A \in \mathcal{B}$ and $R : A \to RA$ is measurable, invertible and $m|_{RA} \circ R^{-1} = m|_A$. The set $A$ is called the domain of $(R, A)$. We will sometimes abuse this notation by writing $R = (R, A)$ and $A = \mathcal{D}(R)$. Similarly, the image of $(R, A)$ is the set $\mathcal{I}(R) = RA$.

An $R$-holonomy is a partial probability-preserving transformation $R$ of $X$ with the additional property that

$$(x, R(x)) \in R \ \forall \ x \in \mathcal{D}(R).$$

The groupoid of $R$ (or of $T$) is the collection

$$[[R]] = [[T]] := \{\text{$R$-holonomies}\}.$$

The full group of $R$ is

$$[T] := \{R \in [[R]] : \mathcal{D}(R) = \mathcal{I}(R) = X \text{ mod } m\}.$$ 

For $R \in [[T]]$, the function $x \mapsto φ(R, x)$ ($\mathcal{D}(R) \to G$) is defined by

$$φ(R, x) = \tilde{ϕ}(x, Rx).$$
The cocycle property ensures that
\[ \varphi(R \circ S, x) = \varphi(S, x) + \varphi(R, Sx) \]
on \mathcal{D}(R \circ S) = \mathcal{D}(S) \cap S^{-1} \mathcal{D}(R) \text{ and for } \varphi_{T_a}(x) = \varphi(n, x).

An \mathcal{R}\text{-holonomy can be thought of as a random power of } T_. For A \in \mathcal{B}(X) and \chi : A \rightarrow \mathbb{Z}^d \text{ measurable, define } T^{(\chi)} : A \rightarrow X \text{ by } T^{(\chi)}(x) := T_{\chi(x)}x. \text{ Any } \mathcal{R}\text{-holonomy is of this form (but a "random power" need not be a partial probability-preserving transformation).}

The orbit equivalence relation of the skew product action \( T^{(\varphi)} \) is given by
\[ \mathcal{R}_{T^{(\varphi)}} = \{ ((x, y), (x', y')) \in (X \times \mathbb{G})^2 : (x, x') \in \mathcal{R}_T \text{ & } y' = y + \varphi(x, x') \}. \]

**Essential value conditions.** Let \( A \in \mathcal{B}, \ U \) a subset of \( \mathbb{G} \), and \( c > 0 \). We say that the measurable \( T \)-cocycle \( \varphi : X \rightarrow \mathbb{G} \) satisfies \( \text{EVC}_{T}(U, c, A) \) if \( \exists R \in \{ [T] \} \) such that
\[ \mathcal{D}(R), \ \mathcal{Z}(R) \subset A, \ \varphi_{R} \in U \text{ on } \mathcal{D}(R), \ m(\mathcal{D}(R))) > cm(A). \]
This is in honor of the collection of **essential values** introduced in [Sch77],
\[ E(T, \varphi) := \{ a \in \mathbb{G} : \forall A \in \mathcal{B}_+, \ U \in \mathcal{I}_a, \ \exists n \in \mathbb{Z}^d, \ m(A \cap T_{n}^{-1}A \cap [\varphi(n, \cdot) \in U]) > 0 \} \]
where \( \mathcal{I}_a := \{ U \ni a \text{ open in } \mathbb{G} \}. \)

It is shown in [Sch77] that \( E(T, \varphi) \) is a closed subgroup of \( \mathbb{G} \) and that \( T^{(\varphi)} \) is ergodic iff \( E(T, \varphi) = \mathbb{G} \). The following is a standard consequence of this; see [ALV98] or [AW04].

**Ergodicity Proposition.** The skew product action \( T^{(\varphi)} \) is ergodic with respect to the product measure \( m \times m_{\mathbb{G}} \) iff there exist
- a countable base \( \mathcal{U} \) for \( \mathcal{I}_0 \);
- a countable, dense collection \( A \subset \mathcal{B}; \)
- a countable collection \( S \subset \mathbb{G} \) so that \( \overline{(S)} = \mathbb{G} \) and a number \( 0 < c < 1 \) such that \( \varphi \) satisfies \( \text{EVC}_{T}(\sigma + U, c, A) \forall A \in \mathcal{A}, \ \sigma \in S, \ U \in \mathcal{U}. \)

Essential value conditions are impervious to small changes. The following is a standard modification of lemma 3.5 of [ALV98].

**Stability Lemma.** If \( \psi : \mathbb{Z}^d \times X \rightarrow \mathbb{G} \) is a measurable cocycle satisfying \( \text{EVC}_{T}(U, c, A) \) where \( A \in \mathcal{B}, \ c > 0, \ U \subset \mathbb{G}, \) then \( \exists \delta > 0 \) such that if \( \varphi : \mathbb{Z}^d \times X \rightarrow \mathbb{G} \) is another measurable cocycle, and
\[ m([\varphi(e_k, \cdot) \neq \psi(e_k, \cdot)]) < \delta \ \forall 1 \leq k \leq d, \]
then \( \varphi \) satisfies \( \text{EVC}_{T}(U, c, A). \)
Coboundaries. The advertised cocycle is constructed as a limit of coboundaries, a coboundary being a cocycle $\psi: \mathbb{Z}^d \times X \to G$ of the form

$$\psi(n, x) = F(x) - F(T_n x)$$

where $F: X \to G$ is a measurable function called the transfer function. We will denote the coboundary with transfer function $F$ by

$$\nabla F(n, x) := F(x) - F(T_n x).$$

Discrete distance. Let $\|\cdot\| = \|\cdot\|_1$ on $\mathbb{R}^d$. We will consider $\mathbb{Z}^d$ as a discrete metric space and write:

For $k \in \mathbb{Z}^d$, $R > 0$, $\Sigma(k, R) := \{j \in \mathbb{Z}^d : \|j - k\| \leq R\}$ and $\Sigma_R := \Sigma(0, R)$.

For $Q \subset \mathbb{Z}^d$, $R > 0$, $\Sigma(Q, R) := \bigcup_{k \in Q} \Sigma(k, R)$,

$Q^o := \{k \in Q : \Sigma(k, 1) \subset Q\}$ and $\partial Q := Q \setminus Q^o$.

Rokhlin towers. A Rokhlin tower is a collection

$$\mathcal{T} = \mathcal{T}_{N,B} = \{T_k B : k \in \Sigma_N\}$$

where $B \in \mathcal{B}$ is such that these sets are disjoint.

The breadth of $\mathcal{T} = \mathcal{T}_{N,B}$ is $N_{\mathcal{T}} = N$, the base is $B_{\mathcal{T}} = B$ and the error of the Rokhlin tower is $\epsilon_{\mathcal{T}} := m(X \setminus \bigcup_{k \in \Sigma_N} T_k B)$.

The Rokhlin Lemma for $\mathbb{Z}^d$ actions as in [Con73] and [KW72] (see also [OW87]) says that: Any free, ergodic, probability-preserving $\mathbb{Z}^d$ action has Rokhlin towers of any breadth and error (in $0, 1$).

Castles. A castle is an array of disjoint Rokhlin towers with the same breadth. A castle may be derived from a Rokhlin tower by partitioning its base.

The interior of $\mathcal{T}_{N,B}$ is

$$\mathcal{T}^o_{N,B} := \{T_k B : k \in \Sigma_N, k \pm e_i \in \Sigma_N, i = 1, 2, \ldots, d\} = \mathcal{T}_{N-1,B}$$

and the boundary of $\mathcal{T}_{N,B}$ is

$$\partial \mathcal{T}_{N,B} := \mathcal{T}_{N,B} \setminus \mathcal{T}^o_{N,B} = \{T_k B : \|k\| = N\}.$$

Let $Q \subset \Sigma_N$. We will write

$$T_Q B := \bigcup_{k \in Q} T_k B.$$
**Purifications.** Given a Rokhlin tower $\mathcal{T}_{N,B}$ and a partition $\alpha \subset B$, the $\alpha$-purification of $B$ is the partition

$$\beta := \left\{ B_k := B \cap \bigcap_{\|k\| \leq N} T_k^{-1} a_k : a \in \alpha^N \right\}$$

of $B$ and the $\alpha$-purification of $\mathcal{T}_{N,B}$ is the corresponding castle

$$\mathcal{P} = \mathcal{P}_{\mathcal{T},\alpha} := \{ T_k b : k \in \Sigma_N, \ b \in \beta \}.$$  

For the rest of this paper, we fix a finite, symmetric generator set $S \subset G \setminus \{0\}$.

**Step functions.** Let $\mathcal{P} = \{ T_k b : k \in \Sigma_N, \ b \in \beta \}$ be a purification of the Rokhlin tower $\mathcal{T} = \mathcal{T}_{N,B}$. A $\mathcal{P}$-step function $F : X \to G$ is one of the form

$$F = \sum_{k \in \Sigma_N, \ b \in \beta} a_{k,b} 1_{T_k b}.$$  

It is called $\mathcal{T}$-internal if $F|_{\mathcal{T} \Sigma_N B} \equiv 0$ and $S$-incremental

$$\nabla F(e_i, \cdot) \in S \cup \{0\} \ (i = 1, 2, \ldots, d).$$

**Inductive Lemma.** Let $\mathcal{P}_0$ be a purification of the Rokhlin tower $\mathcal{T}_0$ and let $F_0 : X \to G$ be a $\mathcal{T}_0$-internal, $S$-incremental $\mathcal{P}_0$-step function.

Fix $\epsilon > 0$, $0 < r < \frac{1}{2n}$, $\sigma \in S$ and $A \in \mathcal{B}_+$. There exist

- a Rokhlin tower $\mathcal{T}$ with $\epsilon_{\mathcal{T}} < \epsilon$ and a purification $\mathcal{P}$, and
- a $\mathcal{T}$-internal, $S$-incremental $\mathcal{P}$-step function $F : X \to G$ so that

(a) $\mu(\{ \nabla F(e_i, \cdot) = \nabla F_0(e_i, \cdot) \forall i = 1, 2, \ldots, d \}) > 1 - \epsilon$;

(b) $\exists a \mathcal{R}_{\mathcal{T}}$-holonomy $R$ with $\mathcal{D}(R), \mathcal{I}(R) \subset A$, $\mu(\mathcal{D}(R)) \geq r \mu(A)$ so that $F(R(x)) - F(x) = \sigma \forall x \in \mathcal{D}(R)$.

**Proof.** Since $F_0 : X \to \mathbb{Z}$ is bounded, $\exists K \in \mathbb{N}$, $A_1, A_2, \ldots, A_K \in \mathcal{B} \cap A$ so that $A = \bigcup_{k=1}^K A_k$ and $F_0$ is constant on each $A_k$.

Consider the measurable partition $\alpha := \{ A_1, A_2, \ldots, A_K, X \setminus A \}$ of $X$.

Fix $0 < \delta \ll \epsilon$ and, in particular, $\delta < \frac{1}{4}$. By the ergodic theorem,

$$\exists n \geq N_{\mathcal{T}_0} \text{ so that } m(a_n) > 1 - \delta^2,$$

where

$$a_n := \bigcap_{a \in a, n \geq n} \left[ \frac{1}{|\Sigma_n|} \sum_{k \in \Sigma_n} 1_{a} \circ T_k = m(a)(1 \pm \delta) \right].$$

Now let $\mathcal{T}$ be a Rokhlin tower with $N_{\mathcal{T}} \geq \frac{2n}{\delta}$ and $\epsilon_{\mathcal{T}} < \delta$.  

BOUNDED COHOMOLOGY

11
Let $\bar{\alpha} = \alpha \lor \{a_n, a^n \} \lor \{B_{\mathcal{P}}^0, B^c_{\mathcal{P}} \}$ and let $\beta$ be the $\bar{\alpha}$-purification of $B_{\mathcal{T}}$ with $\mathcal{P}$ the $\bar{\alpha}$-purification of $\mathcal{T}$.

Define

$$\beta_\odot := \{ b \in \beta : \#\{ k \in \Sigma_{N_{\mathcal{T}}} : T_k b \subset a_n \} > (1 - \delta) |\Sigma_{N_{\mathcal{T}}}| \} \cup \bigcup_{b \in \beta_\odot} b.$$ 

By the Chebyshev–Fubini theorem,

$$m(U_\odot) > \left(1 - \frac{\delta}{1 - \delta}\right) \cdot m(B_{\mathcal{T}}) > \frac{1}{2} \cdot m(B_{\mathcal{T}}).$$

To obtain $F$ satisfying the essential value condition, we make two changes to $F_0$. The first preparatory change is to ensure that $F$ will be $\mathcal{T}$-internal and $S$-incremental. Let

$$Q := \Sigma_{N_{\mathcal{T}}} \cap \Sigma(\partial \Sigma_{N_{\mathcal{T}}}, N_{\mathcal{T}_0}) \sqcup \Sigma(\partial \Sigma_{\lfloor N_{\mathcal{T}} \rfloor}, N_{\mathcal{T}_0})$$

and define

$$F_1(x) := \begin{cases} 0, & x \in T_k b \text{ where } b \in \beta, k \in Q \text{ and } \exists k_0 \in Q \cap \Sigma(k, N_{\mathcal{T}_0}), T_k b \subset B_{\mathcal{T}_0}, \\ F_0(x), & \text{else; } \end{cases}$$

then

$$m([F_0 \neq F_1]) \leq \#Q m(B_{\mathcal{T}}).$$

Now, for some constant $C = C_d > 0$,

$$\#Q \leq C N_{\mathcal{T}_0} N^{d-1}_{\mathcal{T}} \text{ and } m(B_{\mathcal{T}}) \leq \frac{C}{N^d_{\mathcal{T}}}$$

whence

$$m([F_0 \neq F_1]) \leq \frac{C^2 N_{\mathcal{T}_0}}{N_{\mathcal{T}}}.$$ 

The change was made on full $\mathcal{T}_0$ subtowers in the purification and so $F_1|_{T_0 B_{\mathcal{T}}} \equiv 0$ since $F_0$ is $\mathcal{T}_0$-internal. In particular, $F_1$ is $\mathcal{T}$-internal. Moreover, $F_1$ is $S$-incremental again because the change was made on full $\mathcal{T}_0$ subtowers in the purification and on the remaining $\mathcal{T}_0$ subtowers, $F_0$ is $S$-incremental.

We can now define $F : X \to \mathbb{G}$ by

$$F(x) = \begin{cases} F_1(x) + \sigma, & x \in T_{\Sigma_{\lfloor N_{\mathcal{T}} \rfloor}} B_{\mathcal{T}}, \\ F_1(x), & \text{else.} \end{cases}$$
The changes made were small enough for (a). It follows that $F$ is $T_0$-internal and $S$-incremental.

**Proof that $F$ satisfies (b).** For each $b \in \beta \cup \{1, 2, \ldots, K\}$, let

$$K_{b,j} := \{k \in \Sigma_N: T_k b \subset A_j\}.$$

By construction, for each $b \in \beta$ and $j \in \{1, 2, \ldots, K\}$,

$$\#K_{b,j} \cap \Sigma_{\mathcal{N}} \ll K_{b,j} \cap \Sigma_N \setminus \Sigma_{\mathcal{N}}$$

and there is an injection

$$t_{b,j}: K_{b,j} \cap \Sigma_{\mathcal{N}} \to K_{b,j} \cap \Sigma_N \setminus \Sigma_{\mathcal{N}}.$$

Define the $R_T$-holonomy by

$$R(x) = T_{t_{b,j}(j)}(x) \quad x \in T_j b, \ b \in \beta, \ j \in K_{b,j} \cap \Sigma_{\mathcal{N}};$$

then $F(R(x)) - F(x) = \sigma$ and

$$\mu(D(R)) \geq \sum_{b \in \beta} \sum_{j=1}^K \sum_{k \in K_{b,j} \cap \Sigma_N} \mu(A \cap T_k b) = \int_{U} \sum_{k \in \Sigma_{\mathcal{N}}} 1_A \circ T_k dm$$

$$> (1 - \delta) \sum_{\Sigma_{\mathcal{N}}} |m(U) m(A) \geq (1 - 2\delta) \sum_{\Sigma_{\mathcal{N}}} |m(B_T) m(A)$$

$$\geq (1 - 2\delta)(1 - \delta) \frac{|\Sigma_{\mathcal{N}}|}{|\Sigma_N|} m(A) \geq \frac{(1 - 2\delta)(1 - \delta)}{3 \cdot 2^d} m(A) > rm(A).$$

To finish the proof of Theorem 2, fix

- a countable, dense collection $\mathcal{A} \subset \mathcal{B}$;
- a finite, symmetric $S \subset G$ so that $\overline{S} = G$. Write down a sequence

$$((\sigma_n, A_n): n \in \mathbb{N}) \in (S \times \mathcal{A})^\mathbb{N}$$

so that for each $(\sigma, A) \in S \times \mathcal{A},$

$$\# \{n \in \mathbb{N}: (\sigma_n, A_n) = (\sigma, A)\} = \infty.$$

Next (below), we will apply the Inductive Lemma recursively with $r = \frac{1}{2^m}$ and $0 < \varepsilon_{n+1} < \varepsilon_n < \frac{1}{2^m}$ to obtain a sequence of $S$-incremental $F_n: X \to G$ so that for each $n \geq 1,$

(a$_n$) $\mu([\nabla F_n(e_i, \cdot) \neq \nabla F_{n-1}(e_i, \cdot) \forall i = 1, 2, \ldots, d]) < \varepsilon_n$;

(b$_n$) $\nabla F_n$ satisfies EVC($(\sigma_k), r, A_k) \forall 1 \leq k \leq n$;

(c$_n$) any measurable $G: X \to G,$ satisfying

$$\mu([\nabla G(e_i, \cdot) \neq \nabla F_n(e_i, \cdot) \forall i = 1, 2, \ldots, d]) < \varepsilon_{n+1},$$

also satisfies (b$_n$).
To see that constructing the sequence suffices to establish the theorem, we note that by the Borel–Cantelli lemma, \((a_n)\) entails that for \(m\)-a.e. \(x \in X, k \in \mathbb{Z}^d\),

\[ \exists \lim_{n \to \infty} F_n(k, x) =: F(k, x), \]

indeed \(F_n(x, k) = F(x, k)\) \(\forall\) large \(n\). Thus \(F : \mathbb{Z}^d \times X \to G\) is an \(S\)-incremental cocycle satisfying by \((b_n)\) and \((c_n)\)

\[ \text{EVC}_T(\{\sigma\}, r, A) \quad \forall A \in \mathcal{A}, \sigma \in S. \]

By the Ergodicity Proposition, the skew product action is ergodic.

The sequence is constructed recursively. Indeed, suppose that for \(1 \leq i \leq n\), \(F_1, F_2, \ldots, F_n : X \to G\) measurable and \(\epsilon_1 > \epsilon_2 > \cdots > \epsilon_{n+1} > 0\) are given so that each \(F_i\) satisfies \((a_i)\) wrt \(\epsilon_i\), \((b_i)\) and \((c_i)\) wrt \(\epsilon_{i+1}\).

Using the Inductive Lemma, construct \(F_{n+1} : X \to G\) measurable with \(\nabla F_{n+1}\) satisfying \((a_{n+1})\) wrt \(\epsilon_{n+1}\) and \(\text{EVC}(\{\sigma_{n+1}, r, A_{n+1}\})\). By \((c_n)\), \(\nabla F_{n+1}\) also satisfies \(\text{EVC}(\{\sigma_k, r, A_k\})\) for \(1 \leq k \leq n\), whence \((b_{n+1})\). Lastly, use the Stability Lemma to find \(\epsilon_{n+2} > 0\) so that \((c_{n+1})\) holds.

The sequence has been constructed recursively. \(\Box\)

3 Infinitely generated groups

We do not know if Theorem 1 holds for all infinite, countable groups. In this section we prove versions for certain examples of infinitely generated groups.

Locally finite groups. Say that \(\Gamma\) is a normally, locally finite group if it is the increasing union of finite normal subgroups

\[ G_1 \triangleleft G_2 \triangleleft G_3 \triangleleft \cdots \uparrow \Gamma. \]

Theorem 3. Let \(\Gamma\) be a normally, locally finite group, let \((X, d)\) be a perfect Polish space and let \(T : \Gamma \to \text{Homeo}(X)\) be a \(\tt\) action.

For any separable Banach space, \(\exists\) a continuous cocycle \(h : \Gamma \times X \to B\) so that the skew product action \(T^{(h)} : \Gamma \to \text{Homeo}(X \times B)\) is \(\tt\).

Proof. Let \(x_0 \in X\) be a properly recurrent point, i.e., \(x_0 \in \overline{T_{\Gamma \setminus \{e\}}(x_0)}\).

We claim that because of the finiteness of the groups \(G_n\) and the perfectness of \(X\),

\[ \forall\] Given \(U \in \mathcal{T}_{x_0}, k \geq 1, \exists N > k\) and \(\gamma U \in G_N \setminus G_k\) so that \(T_{\gamma U}(x_0) \in U \setminus \{x_0\}\).

We will need, in addition, the following lemma:
Let $F \in C(X, \mathbb{B})$, $s \in \mathbb{B}$, $\Delta > 0$. Then there exists $f \in C(X, \mathbb{B})$ so that

(i) $f \circ T_g \equiv f \forall g \in G_{N-1}$;
(ii) $\partial(F + f)$ satisfies EVC($x_0, U, s, \Delta, \gamma_U$).

**Proof of Theorem 2.** By possibly shrinking $U$ (and suitably adjusting $\gamma_U$ and $N$) we can ensure that

$$\sup_{y \in U} \|F(y) - F(x_0)\|_{\mathbb{B}} < \Delta.$$ 

Fix $\epsilon > 0$ so that the sets

$$\{T_g B(x_0, 2\epsilon) : g \in G_N\}$$

are disjoint. Here, for $x \in X$, $r > 0$, $B(x, r) := \{y \in X : d(y, x) \leq r\}$ is the closed $r$-ball centered at $x$.

For $x \in X$, $g \in G_N$, let

$$w(x) := \left(1 - \frac{d(B(T_g \gamma_U(x_0), \epsilon), x)}{\epsilon}\right)_+,$$

where $d(B, x) := \inf_{y \in B} d(y, x)$ and $a_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. It follows that

$$w \in C(X, [0, 1]), \ w|_{B(T_g \gamma_U(x_0), \epsilon)} \equiv 1 \ & \ w|_{B(T_g \gamma_U(x_0), 2\epsilon)} \equiv 0.$$

Now define $f : X \to \mathbb{B}$ by

$$g(x) = \begin{cases} sG(T_{g^{-1}} x), & x \in T_g B(T_g \gamma_U(x_0), 2\epsilon), \ g \in G_{N-1}, \\ 0, & x \in X \setminus \bigcup_{g \in G_{N-1}} T_g B(x_0, 2\epsilon). \end{cases}$$

This $f \in C_{B}(X, \mathbb{B})$ is as required since $\gamma_U G_{N-1} = G_{N-1} \gamma_U$.  

**Proof of Theorem 3.** Fix

- a decreasing sequence $(U_n : n \geq 1)$, $U_n \in \mathcal{I}_{x_0}$ so that $\forall V \in \mathcal{I}_{x_0} \exists N_V$ so that $U_n \subset V \forall n \geq N_V$;
- $(s_n : n \geq 1) \in \mathbb{B}^N$ dense, and taking each value i.o.;
- $\Delta_n \downarrow 0$.

Using **Theorem 2** iteratively, construct:

- $\kappa_n < \kappa_{n+1} \to \infty$ and $\gamma_U \in G_{\kappa_n} \setminus G_{\kappa_{n-1}}$ so that $T_{\gamma_U}(x_0) \in U_n$;
- $f_n \in C(X, \mathbb{B})$ so that
  (a) $f_n \circ T_g \equiv f_n \forall g \in G_{\kappa_n}$;
  (b) $\partial \sum_{k=1}^{n} f_k$ satisfies EVC($x_0, U_n, s, \Delta_n, \gamma_U$).
For each \( \gamma \in \Gamma, \ x \in X \), the sum

\[
  h(\gamma, x) := \sum_{n=1}^{\infty} (f_n(T_\gamma(x)) - f_n(x))
\]

converges (only finitely many elements being non-zero); \( h : \Gamma \times X \to \mathcal{B} \) is a continuous cocycle.

Now for \( \gamma \in G_{kN}, \ h(\gamma, x) = \sum_{n=1}^{N} (f_n(T_\gamma(x)) - f_n(x)) \), whence \( h \) satisfies EVC(\( x_0, U_n, s_n, 0, \gamma U_n \)) \( \forall n \geq 1 \).

By Proposition 1, \( E(h, x_0) = \mathcal{B} \) and hence \( T^h \) is \( \tt \).

\[ \square \]

**Actions of \( \mathbb{Z}^\infty \).** Here we consider \( \mathbb{Z}^\infty := \{ g = (g_1, g_2, \ldots) \in \mathbb{Z}^\mathbb{N} : g_n = 0 \ \forall \ n \ \text{large} \} \)

with coordinatewise addition. The multiplicative group \( \mathbb{Q}_+ \) is isomorphic with \( \mathbb{Z}^\infty \) by prime factorization.

For \( F \subset \mathbb{N} \) finite, define \( \pi_F : \mathbb{Z}^\infty \to \mathbb{Z}^F \) by \( \pi_F(\gamma) := \gamma|_F \).

**Proposition 4.** Let \((X, d)\) be a perfect Polish space, let \( T : \mathbb{Z}^\infty \to \text{Homeo}(X) \) be a free, \( \tt \) action, and let \( \mathcal{B} \) be a separable Banach space.

There is a cocycle \( h : \mathbb{Z}^\infty \times X \to \mathcal{B} \) so that the skew product action \( T^h : \mathbb{Z}^\infty \times (X \times \mathcal{B}) \to \text{Homeo}(X \times \mathcal{B}) \) is \( \tt \).

**Proof.** There are two cases covering the theorem.

**Case 1.** There is a \( T \)-transitive point \( x_0 \in X \) and \( N \geq 1, \ \gamma_k \in \mathbb{Z}^{[1,N]} \) so that \( T_{\gamma_k}(x_0) \to x_0 \).

**Proof of the proposition in Case 1.** Since \( \mathbb{Z}^{[1,N]} \) is finitely generated, by the proof of Theorem 1, there is a cocycle \( \eta : \mathbb{Z}^{[1,N]} \times X \to \mathcal{B} \) so that

\[
  \{ T^{(\eta)}_g(x_0, 0) : g \in \mathbb{Z}^{[1,N]} \} \supset \{ x_0 \} \times \mathcal{B}.
\]

Define \( h : \mathbb{Z}^\infty \times X \to \mathcal{B} \) by

\[
  h(\gamma, x) := \eta(\pi_{[1,N]}(\gamma), x);
\]

then \( h : \mathbb{Z}^\infty \times X \to \mathcal{B} \) is a cocycle and

\[
  \{ T^{(h)}_\gamma(x_0, 0) : \gamma \in \mathbb{Z}^\infty \} = \{ T^{(\eta)}_g(x_0, 0) : g \in \mathbb{Z}^{[1,N]} \} \supset \{ x_0 \} \times \mathcal{B}.
\]

By transitivity of \( x_0 \) for \( T \), we have

\[
  \{ T^{(h)}_\gamma(x_0, 0) : \gamma \in \mathbb{Z}^\infty \} = X \times \mathcal{B}.
\]

\[ \square \]
If Case 1 fails, then we are in 

**Case 2** There is a $T$-transitive point $x_0 \in X$ and $\gamma_k \in \mathbb{Z}^\infty$ so that $T_{\gamma_k}(x_0) \to x_0$ and $\forall k, K \geq 1$, there exists $L > K$ so that $(\gamma_k)_L \neq 0$.

**Proof of the proposition in Case 2.** We prove

\[ H : \mathbb{Z}^\infty \to \mathbb{B} \]

so that the direct product action $T \times H : \gamma \to \text{Homeo}(X \times \mathbb{B})$ is \texttt{tt}.

**Proof of \( \Box \).** In the absence of case 1, we have that

\[ \forall V \in \mathcal{I}_{x_0}, s \in \mathbb{B} \quad \text{and} \quad n_0 \geq 1, \]

there are $n > n_0$ and $\gamma \in \mathbb{Z}^{[1,n]}$ with $\gamma_n \neq 0$ and $T_\gamma(x_0) \in V$.

Given $n$, $\gamma$ and $s$ as above, let $h_{n,\gamma,s} : \mathbb{Z}^\infty \to \mathbb{B}$ be the homomorphism given by

\[ h_{n,\gamma,s}(g) = \pi_{n,\gamma}(g_n) \cdot s := \frac{g}{\gamma_n} \cdot s. \]

Fix

- a decreasing sequence $(U_n : n \geq 1), \quad U_n \in \mathcal{I}_{x_0}$ so that $\forall V \in \mathcal{I}_{x_0} \exists N_V$ so that $U_n \subset V \forall n \geq N_V$;

- $(s_n : n \geq 1) \in \mathbb{B}^\mathbb{N}$ dense, and taking each value i.o.

Using \( \Box \), iteratively construct:

\[ \kappa_n < \kappa_{n+1} \to \infty \quad \text{and} \quad \gamma^{(n)} \in \mathbb{Z}^{[1,\kappa_n]}, \quad \gamma^{(n)}_{\kappa_n} \neq e_{G_{\kappa_n}} \quad \text{so that} \quad T_{\gamma^{(n)}} \in U_n. \]

We have that for each $\gamma \in \mathbb{Z}^\infty$, the sum

\[ H(\gamma) := \sum_{n=1}^{\infty} h_{\kappa_n,\gamma^{(n)},s_n}(\gamma) \]

converges (only finitely many elements being non-zero); $H : \mathbb{Z}^\infty \to \mathbb{B}$ is a homomorphism.

Considering $H : \mathbb{Z}^\infty \times X \to \mathbb{B}$ as a (constant) cocycle, we have that $H$ satisfies EVC$(x_0, U_n, s_n, 0, e^{(n)}) \forall n \geq 1$.

By Proposition 1, we have that $E(H, x_0) = \mathbb{B}$ & hence $T(H) = T \times H$ is \texttt{tt}. \( \Box \)

### 4 Hölder continuous cocycles for $\mathbb{Z}^d$ shifts.

Let $S$ be a finite set, let $d \geq 1$ and let $X = S^{\mathbb{Z}^d}$.

The function $f : X \to \mathbb{R}^D$ is **Hölder continuous** if for some $\theta \in (0, 1)$ and $M > 0$,

\[ \|f(x) - f(y)\|_2 \leq M\theta^{\theta(x,y)} \quad \forall x, y \in X \]
where \( t(x, y) := \min\{ \| n \|_1 : n \in \mathbb{Z}^d, x_n \neq y_n \} \). Here and throughout, for \( N \geq 1 \) and \( p > 0 \), \( \| (x_1, \ldots, x_N) \|_p := \left( \sum_{k=1}^N |x_k|^p \right)^{\frac{1}{p}} \).

A Hölder continuous function taking finitely many values is aka a block function.

A cocycle \( F : \mathbb{Z}^d \times X \to \mathbb{R}^D \) is called Hölder continuous if \( x \mapsto F(n, x) \) is Hölder continuous \( \forall n \in \mathbb{Z}^d \).

**Example: Random walks.** Let \( D \geq 1 \) and let \((X, T) = (S^Z, \text{Shift})\) where \( S \) is a finite set, large enough so that \( \exists \phi : S \to \mathbb{R}^D \) with

\[
\text{Semigroup}(\phi(S)) = \mathbb{R}^D.
\]

Define \( \varphi : X \to \mathbb{R}^D \) by \( \varphi(x) := \phi(x_0) \). It is not hard to see that the skew product \( T_\varphi \) is t.t.

If \( \mu \in \mathcal{P}(S) \) satisfies \( \sum_{s \in S} \mu(s) \phi(s) = 0 \) then \((X \times \mathbb{R}^D, \mu^\times m_{\mathbb{R}^D}, T_\varphi)\) is a measure preserving transformation and is ergodic if \( D = 1, 2 \) (see [HR53]). For \( D \geq 3 \), \( T_\varphi \) is dissipative by the local limit theorem (see, e.g., [Bre68]) whence not ergodic.

These random walk constructions have no ergodic analogues for higher dimensional actions. The reason is basically

**Schmidt’s Theorem.** Let \((X, T) = (S^Z, \text{Shift})\) where \( d \geq 2 \) and \( S \) is a finite set and let \( F : \mathbb{Z}^d \times X \to \mathbb{R}^k \) be a Hölder continuous cocycle. Then

\[
F(n, x) = g(T_n x) - g(x) + H(n)
\]

where \( g : X \to \mathbb{R}^k \) is Hölder continuous and \( H : \mathbb{Z}^2 \to \mathbb{R}^k \) is a homomorphism.

This can be deduced from the more general Theorem 3.2 in [Sch95] which is a symbolic version of a similar result for multidimensional Anosov actions (Theorem 2.9 in [KS94]).

**Corollary.** Let \( d \geq 2 \) and \((X, T) = (S^Z, \text{Shift})\) where \( S \) is a finite set. For \( d' \geq d \), there is no Hölder continuous cocycle \( F : \mathbb{Z}^d \times X \to \mathbb{R}^{d'} \) with the skew product \((X \times \mathbb{R}^{d'}, T^{(F)})\) topologically transitive.

**Proof.** Let \( F : \mathbb{Z}^d \times X \to \mathbb{R}^d \) be a Hölder continuous cocycle. By Schmidt’s Theorem

\[
F(n, x) = g(T_n x) - g(x) + H(n),
\]

where \( g : X \to \mathbb{R}^d \) is Hölder continuous and \( H : \mathbb{Z}^d \to \mathbb{R}^d \) is a homomorphism, whence the skew product action \( T^{(F)} \) is continuously conjugate to the product action \( T \times H \) where \( H_n(z) := z + H(n) \).
Topological transitivity is impossible since either \( \dim \text{Span} H(\mathbb{Z}^d) < d' \) or \( H(\mathbb{Z}^d) \) is discrete. \( \square \)

On the other hand,\( \mathcal{S} \) There is a homomorphism
\[
H : \mathbb{Z}^d \to \mathbb{R}^{d-1}
\]
whose product action \((X, m \times m_{\mathbb{R}^{d-1}}, T \times H)\) is ergodic.

**Proof of \( \mathcal{S} \).** Let \( d \geq 1 \), let \((X, T) = (\{0, 1\}^{\mathbb{Z}^d}, \text{Shift})\) and let \( m \in \mathcal{P}(\{0, 1\}^{\mathbb{Z}^d})\) be symmetric product measure.

Define \( H : \mathbb{Z}^d \to \mathbb{R}^{d-1} \) by
\[
H((n_1, n_2, \ldots, n_d)) := \sum_{j=1}^{d-1} n_j e_j^{(d-1)} + n_d \bar{\alpha}
\]
with
\[
\bar{\alpha} := \sum_{k=1}^{d-1} a_k e_k^{(d-1)},
\]
where \( 1, \alpha_1, \alpha_2, \ldots, \alpha_{d-1} \) are linearly independent over \( \mathbb{Q} \) so that \((\mathbb{T}^{d-1}, m_{\mathbb{T}^{d-1}}, R_{\bar{\alpha}})\) is ergodic (\( R_{\bar{\alpha}}(x) := x + \bar{\alpha} \)).

The action \((X, T, m)\) is strongly mixing, whence **mildly mixing**. By [SW82], the product \( \mathbb{Z}^d \) action \((X \times \mathbb{R}^{d-1}, T \times H, m \times m_{\mathbb{R}^{d-1}})\) is ergodic if and only if the action \((\mathbb{R}^{d-1}, H, m_{\mathbb{R}^{d-1}})\) is conservative, ergodic.

We have that \((\mathbb{R}^{d-1}, m_{\mathbb{R}^{d-1}}, H) \cong (\mathbb{T}^{d-1} \times \mathbb{Z}^{d-1}, m_{\mathbb{T}^{d-1}} \times \# , J)\), where \( \cong \) denotes measure theoretic isomorphism of actions and

\[
J_{\alpha}(x, z) := \begin{cases} 
(x, z + e_k^{(d-1)}), & 1 \leq k \leq d - 1, \\
(x + \bar{\alpha}, z + \tau(x)), & k = d,
\end{cases}
\]
with
\[
\tau(x) := (\lfloor x_1 + \alpha_1 \rfloor, \lfloor x_2 + \alpha_2 \rfloor, \ldots, \lfloor x_{d-1} + \alpha_{d-1} \rfloor).
\]
If \( F : \mathbb{T}^{d-1} \times \mathbb{Z}^{d-1} \to \mathbb{C} \) is a measurable, \( J \)-invariant function \((F \circ J_n = F \forall n \in \mathbb{Z}^d)\), then
\[
F(x, n) = F \circ J_{n,0}(x, 0) = F(x, 0) \quad \text{and} \quad F(x, 0) = F \circ J_{\alpha}(x, 0) = F(R_{\bar{\alpha}}(x), 0).
\]

Thus, by ergodicity of \( R_{\bar{\alpha}} \), \( F \) is a.e. constant and \( J \) is ergodic. Conservativity follows as the underlying measure space is non-atomic. \( \square \)
**Concluding remarks.** We have been unable to establish versions of Theorem 1 for actions of $\mathbb{Q}$ or $\Sigma(\mathbb{N})$ (the group of finite permutations of $\mathbb{N}$). We have also been unable to prove Theorem 2 for cocycles with values in an arbitrary, countable amenable group (established for $\mathbb{Z}$-actions in [AW04]).

On the other hand, Theorem 2 can be generalized to free actions of countable amenable groups. The appropriate inductive lemma is also established using the ergodic theorem and an appropriate Rokhlin lemma for countable amenable group actions as in [OW87].

**REFERENCES**

[ALV98] J. Aaronson, M. Lemańczyk and D. Volny, *A cut salad of cocycles*, Fund. Math. **157** (1998), 99–119.

[AW04] J. Aaronson and B. Weiss, *On Herman’s theorem for ergodic, amenable group extensions of endomorphisms*, Ergodic Theory Dynam. Systems **24** (2004), 1283–1293.

[Bes37] A. S. Besicovitch, *A problem on topological transformation of the plane*, Fund. Math. **28** (1937), 61–65.

[Bre68] L. Breiman, *Probability*, Addison-Wesley, Reading, MA-London-Don Mills, ON, 1968.

[CFW81] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynam. Systems **1** (1982), 431–450.

[Con73] J. P. Conze, *Entropie d’un groupe abélien de transformations*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **25** (1972/73), 11–30.

[FM77] J. Feldman and C. C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. **234** (1977), 289–324.

[Her79] M. R. Herman, *Construction de difféomorphismes ergodiques*, unpublished, 1979.

[HR53] T. E. Harris and H. Robbins, *Ergodic theory of Markov chains admitting an infinite invariant measure*, Proc. Natl. Acad. Sci. USA **39** (1953), 860–864.

[KS94] A. Katok and R. J. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 131–156.

[KW72] Y. Katznelson and B. Weiss, *Commuting measure-preserving transformations*, Israel J. Math. **12** (1972), 161–173.

[OW87] D. S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Anal. Math. **48** (1987), 1–141.

[Sch77] K Schmidt, *Cocycles on Ergodic Transformation Groups*, Macmillan, Delhi, 1977.

[Sch95] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math. **170** (1995), 237–269.

[Sid73] E. A. Sidorov, *Topologically transitive cylindrical cascades*, Mat. Zametki **14** (1973), 441–452; English translation: Math. Notes **14** (1973), 810–816.

[Sni30] L. G. Snirelman, *Example of a transformation of the plane*, Izv. Donsk. Politekhn. Inst. **4** (1930), 64–74.

[SW82] K. Schmidt and P. Walters, *Mildly mixing actions of locally compact groups*, Proc. London Math. Soc. (3) **45** (1982), 506–518.
Jon Aaronson  
SCHOOL OF MATHEMATICAL SCIENCES  
TEL AVIV UNIVERSITY  
TEL AVIV 69978, ISRAEL  
email: aaro@tau.ac.il  

Benjamin Weiss  
INSTITUTE OF MATHEMATICS  
HEBREW UNIVERSITY OF JERUSALEM  
JERUSALEM 91904, ISRAEL  
email: weiss@math.huji.ac.il  

(Received January 8, 2019 and in revised form June 4, 2019)