The Higgs Field As The Cheshire Cat
And His Yang-Mills ”Smiles”

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Abstract

The well-known Bogomol’nyi-Prasad-Sommerfeld (BPS) monopole is considered in the limit of the infinite mass of the Higgs field as a basis of the Yang-Mills field vacuum with the finite energy density. In this limit the Higgs field disappears, but it leaves its trace as the BPS monopole transformed into the Wu-Yang monopole obtained in the pure Yang-Mills theory by a spontaneous scale symmetry breaking in the class of functions with the zero value of the topological charge. The topological degeneration of the BPS monopole manifests itself as Gribov copies of the covariant Coulomb gauge in the form of the time integral of the Gauss constraint. We also show that, in the considered theory, there is a zero mode of the Gauss constraint leading to an electric monopole and an additional mass of \( \eta’ \)-meson in QCD. The consequences of the monopole vacuum in the form of a rising potential and topological confinement are studied in the framework of the perturbation theory. An estimation of the vacuum expectation value of the square of the magnetic tension is given through the \( \eta’ \)-meson mass, and arguments in favour of the stability of the monopole vacuum are considered. We also discuss why all these ”smiles” of the Cheshire cat are kept by the Dirac fundamental quantization, but not by the conventional Faddeev-Popov integral.
1 Introduction.

The nature of the vacuum of the Yang-Mills (YM) theory in the Minkowski space is an open problem at the present time. There were a lot of attempts to solve this problem. A typical feature of these attempts was the construction of the nontrivial physical vacuum in the Minkowski space on the basis of nonzero values of vacuum expectations coinciding with statistical averages (classical fields). As an example of these attempts we should like to point out the work [1] stimulated by the asymptotic freedom formula as a criterion for instability of the naive perturbation theory [2]. However, these attempts did not take account of the topological structure of vacuum.

This structure of the vacuum of the Yang-Mills theory was discovered in the Euclidean space [3], and it means that there exist classical in-, out-vacuum states corresponding to different topological indices \(|n|\) with zero values of energy, and tunnel transitions \(|n| \rightarrow |n + 1|\) occur between them. These transitions are described by instantons, i.e. Yang-Mills fields with fixed topological numbers \(\nu = n_{\text{out}} - n_{\text{in}}\) satisfying the minimum of the Yang-Mills action. A defect of this vacuum is non-physical status of a zero value of energy in quantum theory. However, the topological degeneration of initial data for Yang-Mills fields does not depend on the space where these fields are considered. The initial data of any classical solution in the Minkowski space-time are also topologically degenerated. It is worth to investigate topologically degenerated vacuum solutions in the Minkowski space-time in the class of functions with physical values of finite energy densities.

The present paper is devoted to just this investigation of the nontrivial topological vacuum of the Yang-Mills theory in the Minkowski space-time. This vacuum is stipulated by the fact that the homotopy groups of all the 3-dimensional paths (loops) on the \(SU(2)\) group manifold are nontrivial (p.325 in [3]):

\[
\pi_3(SU(2)) = \mathbb{Z}. \tag{1.1}
\]

The Yang-Mills vacuum should take account of this topology. We investigate the topological degeneration of the initial data using the well-known Bogomol’nyi-Prasad-Sommerfeld (BPS) monopole as an example. This monopole is a result of the spontaneous break-down of the \(SU(2)\) symmetry on the basis of the classical Higgs field \(\phi_0\) (i.e. the \(\lambda \phi^4\) theory) in the limit \(\lambda \rightarrow 0\). This means that we consider the ideal Bose gas of scalar Higgs particles. This Bose gas is called the Bose condensate. The Higgs Bose condensate has a direct analogy with Bose condensate in the theory of superfluid helium [4].

Thus, there is the possibility to construct the YM vacuum using the well-known Bose condensate of free scalar particles in the limit of their infinite mass when these particles disappear from the spectrum of elementary excitations of the theory leaving their ”traces” in the form of monopoles. The study of these ”traces” is the aim of the present paper.

One of these ”traces” is the topological degeneration of the BPS monopole perturbation theory that manifests itself as Gribov copies of the covariant Coulomb gauge treated as initial data of the Gauss constraint in the lowest order of the perturbation theory with the new monopole vacuum. The Gribov copies mean that there is a zero mode of the
Gauss law constraint expressed through the global variable $N(t)$ that describes a topological motion of the Yang-Mills Bose condensate as a whole system with the real momentum spectrum.

We construct the generating functional for weak perturbation excitations over this vacuum in the form of the Feynman path integral.

The paper is organized as follows:

Section 2 is devoted to a brief review of the problem of vacuum in quantum-field theories. We discuss what the vacuum is in such theories and the ways of symmetry break-down, which were very fruitful in modern physics.

Further we give the general picture of the symmetry break-down and its connection with the nonzero vacuum expectation value $\langle 0|\phi|0 \rangle \neq 0$ with the example of the $\lambda\phi^4$ theory.

In the conclusion of Section 2 we show that the Higgs sector of the Yang-Mills theory in the BPS limit $\lambda \to 0$ leads to nontrivial monopole solutions of the equations of motion with a finite energy density corresponding to the $SU(2) \to U(1)$ spontaneous break-down. The Bogomol’nyi equation, defining the lowest level of the monopole energy, determines the direct connection between Yang-Mills and Higgs multiplets. This will be a starting point for the construction of a consistent theory of the Yang-Mills vacuum in Section 4.

Section 3 is devoted to the construction of the Dirac variables in the Yang-Mills theory in the form of solutions of the Gauss constraint-shell equation. This will be the base of all our further consideration.

The topological degeneration of the initial data is the subject of Section 4. We argue in favour of that the vacuum in the ”old” instanton approach is not the physical one. Instead, we construct the monopole $\Phi_i^{(0)}(x)$ in the form of the stationary Bose condensate with the topological number 0 and the nonzero ”magnetic” tension $B(\Phi_i^{(0)})$ corresponding this monopole. All this is a result of the $SU(2)$ spontaneous break-down, describing by the classical equations of the non-Abelian theory in the class of fields with the topological number 0. These equations have nontrivial solutions in the form of Wu-Yang monopoles: $\Phi_i^{(0)}(x)$. Our construction of the YM vacuum is only a presentation of such solutions as BPS monopoles in the theory with Higgs fields in the limit of their infinite masses, but with the finite energy density; so that the BPS ”magnetic” tension $B(\Phi_i^{(0)})$ (in this limit) has itself a crucial importance. We show that in the considered limit the Gibbs expectation value $\langle B^2 \rangle$ (defined as the averaging $B^2$ over the volume) can be different from zero; it is a direct analogy with the Meisner effect in a superconductor. In the language of the group theory, it means the spontaneous break-down of the $U(1)$ group. This, in turn, is a precondition for the correct consideration of the $\eta'$-meson problem in QCD. A nonzero value of $\langle B^2 \rangle$ allows also us to regularize our theory by the introducing of an infrared cut-off parameter $\varepsilon(\langle B^2 \rangle)$ that plays the role of the size of the BPS monopole.

The goal of Section 4 is to show the nature of the topological degeneration of the monopole $\Phi_i^{(0)}(x)$ as the Gribov ambiguity of the covariant Coulomb gauge (in the form of the time integral of the Gauss constraint). This topological degeneration is defined by the non-perturbation factor $\exp(n\Phi_0^{(0)}(x))$, where $\Phi_0^{(0)}(x)$ is a solution of the Gribov ambiguity equation that coincides with the Higgs field in the form of a BPS monopole.
As the covariant Coulomb gauge is the time integral of the Gauss law constraint, the Gribov ambiguity signals that there are zero modes of the Gauss constraint considered as the equation for the time component $A_0^c(t, x)$ of the Yang-Mills field.

The in the main new step in our investigation is the introduction of the non-integer, continuous topological variable $N(t)$ for the definition of the zero mode of the solution of the Gauss law constraint for $A_0^c(t, x)$ in the form of the product $\dot{N}(t)\Phi_0^c(x)$. This zero mode induces the "electric" tension ("electric monopole") as a dynamic degree of freedom that cannot be removed by fixing of any gauge. This "electric" tension, in turn, generates the action of a free rotator describing the global rotation of the Yang-Mills vacuum as a whole system. The corresponding Schrödinger equation for vacuum has the real spectrum of momenta in contrast with the instanton case. The dependence of the rotator action on the Gibbs expectation value $<B^2>$, which, in turn, depends on the Higgs mass (through the Bogomol’nyi equation), confirms our suggestion about the Yang-Mills vacuum as a Bose condensate.

The topic of Section 5 is a more detailed analysis of zero modes of the covariant Coulomb gauge and the Yang-Mills (constraint-shell) action; also we decompose the "electric" tension into the transverse and the longitudinal parts with respect to the constraint-shell equation.

Section 6 is devoted to the calculation of the instantaneous potential of the current-current interaction in the presence of the Wu-Yang monopole. Instead of the Coulomb potential in QED, the corresponding Yang-Mills Green function take the form of the sum of the two potentials: the Coulomb potential and the rising one; it is of great importance for the analysis of the hadronization, in particular of the $\eta'$-meson problem.

The analysis of the Feynman and FP path integrals is the subject of Section 7.

The last two Sections, 8 and 9, are devoted to the analysis of the topological confinement and the quark confinement in QCD as direct consequences of the average over the topological degeneration.

The theory considered in Sections 8 allow us to contend that only the colourless ("hadronic") states form a complete set of physical states in QCD. We prove that the topological confinement leads to the quarks confinement in QCD, that the complete set of hadronic states ensures that QCD is an unitary theory. In Section 10 we estimate the value of the vacuum chromomagnetic field in QCD$_{(3+1)}$.

## 2 Vacuum as a result of spontaneous symmetry breakdown.

### 2.1 A physical vacuum as a Bose condensate.

All quantum-field theories are considered in the Hilbert-Fock space of second quantization (see §7.3 in [6] and [7]). It is quite logical to begin our consideration with a suitable abstract mathematical model (p.40 in [3]). So, let some algebra with involution $U$ be given (the creation - annihilation operators are examples of an algebra of that sort). We
denote this algebra by $\mathcal{C}^*$. One constructs the $\mathcal{C}^*$-homomorphism $\pi$ of the $\mathcal{C}^*$-algebra $U$ into the algebra $\mathcal{B}(\mathcal{H})$ of all the linear restricted operators defined in the Hilbert space $\mathcal{H}$. The homomorphism $\pi$ is called the representation of the $\mathcal{C}^*$-algebra $U$ in the Hilbert space $\mathcal{H}$. The representation $\pi$ is called the irreducible one, if every closed subspace in $\mathcal{H}$, invariant with respect to all the operators $\pi(A)(A \in U)$, is $\emptyset$ or the whole Hilbert space $\mathcal{H}$. The vector $\Phi \in \mathcal{H}$ is called the cyclic vector for the representation $\pi$, if all the vectors of the form $\pi(A)\Phi$, where $A \in U$, form a complete set (a linear shell) in $\mathcal{H}$. This representation with the cyclic vector is called the cyclic one.

If $\Phi$ is a vector in $\mathcal{H}$, then it generates the positive functional

$$F_\Phi = \langle \Phi, \pi(A)\Phi \rangle$$

(2.1)
on $U$ (in terms of the probability theory it is the mathematical expectation of the value $\pi(A)$ in the state $\Phi$). This functional is called the vector functional associated with the representation $\pi$ and the vector $\Phi$.

In these terms the Gelfand-Naimark- Sigal (GNS) construction of a vacuum (p.42 in [6]) consists in the following: one can determine some (cyclic) representation $\pi_F$ of the algebra $U$ in the given Hilbert space with the cyclic vector $\Phi_F$ for the given positive functional $F$ such that

$$F(A) = \langle \Phi_F, \pi_F(A)\Phi_F \rangle.$$ 

(2.2)
The representation $\pi_F$ is determined with these conditions as unique to within the unitary equivalence. This construction of the cyclic vector $\Phi_F$ as the vacuum vector (or simply the vacuum) allowed us to write down the theory of second quantization.

Let us consider the free (i.e. without an interaction) theory of one particle (boson). This boson has its fixed integer spin $s$ and mass $m$ (therefore also the fixed square of the Pauli-Lubanski vector $W^2 = m^2 s(s + 1)$, and its mass-shell equation is $p^2 = m^2$). As usual in quantum-field theories, we place our particle in a large enough closed box. Then, according to the Hilbert-Schmidt theorem (p.231 in [8]), the momentum spectrum of the particle is discrete.

Associating the ortho-normalized vectors of some infinite-dimensional Hilbert space with the quantum numbers of the momentum and spin (helicity), we obtain the one-particle Hilbert (Fock) space (at the level of special relativity), which we denote as $\Sigma^{[m,s]}$.

Let some algebra with involution (depending on the momentum $p$ and spin $s$) be given in the form of the Bose commutation relations of the operators $a(p), a^\dagger(p)$ (the formula (4.16) in [9]):

$$[a(p), a^\dagger(p)] = (2\pi)^3 2\omega_p \delta^3(p - p'),$$

(2.3)
where $\omega_p$ is the frequency corresponding to the momentum $p$, and

$$[a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0.$$ 

(2.4)
We suppose that the operator $a^\dagger(p)$ acts on the vacuum vector (which we shall denote sometimes as $|0>$) as

$$a^\dagger(p)|0> = \Phi(p)$$

(2.5)
for the eigenvalue $p$ of the momentum operator. All such vectors, depending on $p$ (and the value of helicity), form a linear shell of our $\mathcal{H} = \Sigma^{[m,s]}$. It is said that the boson with the momentum $p$ and spin $s$ (at a fixed helicity) \textit{was created from the vacuum} $|0\rangle$.

Operator $a(p)$ has a contrary action on the vacuum $|0\rangle$. This action is expressed as

$$a(p)|0\rangle = 0 ,$$

and we say that the operator $a(p)$ \textit{annihilates} the vacuum $|0\rangle$.

The considered simple example shows us how one can introduce the vacuum into some quantum-field theory. This is a purely mathematical construction, and this vacuum is called the \textit{mathematical one}. But, to construct a consistent relativistic quantum-field theory, we should impose some conditions on its vacuum.

First of all, this is the condition \textit{that the vacuum exists and is unique} to within a phase factor. The mentioned phase factor should preserve the (unit) norm of the vacuum vector. Thus, this factor has the form $\exp(i\phi)$, and we already have some \textit{degeneration} of the vacuum with respect to the Abelian $U(1)$ group.

Then we demand that our vacuum should be invariant with respect to pure Poincare translations $U(a,1)$ (p.251 in [6]). This leads, as a final result, to the conservation of the momentum-energy tensor of the theory. We associate always the \textit{minimum value of energy} with the vacuum vector $\Phi_0 \in \mathcal{H}$. We call the state with the minimum value of energy the \textit{ground physical state}. Note that, in the free quantum-field theory, the spectrum of the momentum-energy operator $P$ belongs to the set $V^+_m \cup \{0\}$, where $V^+_m = \{p \in M : p^2 \geq m^2, p^0 > 0\}$ at $m \geq 0$ and $M$ is the Minkowski space (the so-called \textit{strict condition of spectrality}).

All the states are considered as \textit{(perturbation) excitations over the vacuum}. We shall utilize this fact in our present work.

The condition that vacuum is invariant with respect to pure Poincare translations is satisfied when $P = 0$. All the states with such $P$ are invariant under pure Poincare translations. All the unitary representations of this class, except for the unit representation, the vacuum $(U(a, \hat{\Lambda}) \equiv 1$, where $\hat{\Lambda} \in SL(2,C))$ are \textit{infinite-dimensional} with respect to the 3-dimensional $SO(3)$ rotations.

We say that the \textit{cluster feature} (or the feature of the asymptotic factorisation) is fulfilled in the physical Hilbert space $\mathcal{H}$ if there exists the vector of unit norm $\Psi_0 \in \mathcal{H}$ such that

$$< \Phi, U(\lambda a,1)\Psi > \rightarrow < \Phi, \Psi_0 > < \Psi_0, \Psi > , \quad \lambda \rightarrow \infty ,$$

(2.7)

where $a$ is an arbitrary space-like vector in the Minkowski space $M$; $\Phi, \Psi \in \mathcal{H}$.

It turns out that the condition that the vacuum exists and is unique \textit{is equivalent to the cluster feature} (2.7).

Let us consider (p.294 in [6]) some quantum field $\phi$ and let us supply it with the index $\kappa$ ($\phi^{(\kappa)}$) which defines the type of this field (for example its spin). Thus, every $\phi^{(\kappa)}$ is a tensor or a spin-tensor with a finite number of its Lorentz components: $\phi_{l}^{(\kappa)}(l = 1,\ldots,r_{\kappa})$ and with a definite transformation features with respect to the eigen Lorentz group $L_+^\kappa$ or its covering $SL(2,C)$. 

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In these terms we can construct a consistent relativistic quantum field theory if its vacuum is cyclic in the following sense. The set $D_0$ of finite linear combinations of (spin-)tensor fields of the form $\phi^{(\kappa_1)}(f_1)\ldots \phi^{(\kappa_n)}(f_n)|0>\text{ is dense in }\mathcal{H}$.

A very important role in quantum-field theories plays the vacuum expectation value of some quantum field $\phi$ over the vacuum $<0|\phi|0>$. But this value is zero at the level of the algebra of creation-annihilation operators $a(p), a^\dagger(p')$ in the Fock-Hilbert space of second quantization: because of the relation (2.4).

Now, with the help of simple arguments, we shall show the connection of the relation $<0|\phi|0>=0$ with the question of (global) symmetry and its break-down (see for example §5.3 in [10]).

Let $U$ be an element of the (global) unitary realized symmetry group with respect to which the Hamiltonian $H_0$ of some quantum-field theory is invariant. Then we can write down this condition of invariance of the Hamiltonian $H_0$ as

$$UH_0U^\dagger = H_0.$$  \hspace{1cm} (2.8)

If the considered quantum field theory is realized in the physical Hilbert space $\mathcal{H}$, and we constructed already the irreducible representation $\pi(U) = U'$ in $\mathcal{H}$, then some transformation of the group $U$ induces the corresponding transformation on $\mathcal{H}$:

$$U'\Phi = \Psi.$$  \hspace{1cm} (2.9)

If $E_\Phi$ and $E_\Psi$ are the expectation values of the Hamiltonian $H_0$ in the states $\Phi$ and $\Psi$, respectively, then we can rewrite the condition of invariance of the Hamiltonian $H_0$ as

$$E_\Phi = <\Phi|H_0|\Phi> = <\Psi|H_0|\Psi> = E_\Psi.$$  \hspace{1cm} (2.10)

Thus, the symmetry of the Hamiltonian $H_0$ means the degeneration of eigenstates of the energy operator corresponding to the irreducible representation of the symmetry group. However, the relations (2.9),(2.10) stay implicit at our suggestion that vacuum exists and is unique. Really, since the states $\Phi$ and $\Psi$ should be connected with the ground state $|0>$ by the relations (see (2.5)):

$$\Phi = \phi^\dagger|0>, \quad \Psi = \psi^\dagger|0>.$$  \hspace{1cm} (2.11)

and

$$U'\phi^\dagger U' = \psi^\dagger,$$  \hspace{1cm} (2.12)

then the relation (2.13) is true if and only if

$$U'|0>=0.$$  \hspace{1cm} (2.13)

If the condition (2.13) is not fulfilled, then the condition (2.10) is also broken, and, together with this fact, the conclusion about the symmetry of degenerated levels of energy is also broken. This situation is called the spontaneous break-down of the symmetry $U$. 

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Thus, we can write down the condition of the spontaneous break-down of some symmetry as

$$U^\prime |0 \rangle \neq 0$$

(the vacuum becomes not invariant with respect to the group $U$).

If $U = \exp(ie^a Q_a)$, where $e^a$ are continuous group parameters (Euler angles), and $Q_a$ are generators (charges) of this group, then (2.14) is equivalent to that charges $Q_a$ do not annihilate the vacuum $|0\rangle$:

$$Q_a |0 \rangle \neq 0.$$  (2.15)

The statement equivalent to (2.14),(2.15) is that the field operators $\phi_i$ (considered as degrees of freedom in the Lagrangian formalism; one can interpret them as components of the group multiplet) have nonzero vacuum expectation values:

$$<0|\phi_i|0\rangle \neq 0.$$  (2.16)

The symmetry transformation (2.12) is equivalent to

$$\phi_i(x) \to \phi'_i(x) = \phi_i(x) + \delta\phi_i(x),$$  (2.17)

where

$$\delta\phi_i(x) = ie^a t^a_{ij} \phi_j(x),$$  (2.18)

and $t^a$ are the matrices of the adjoint representation of the Lie algebra of the group $U$.

The application of the Lie algebra and its adjoint representation guarantees the fulfilment of the Nöther theorem and the existence of conserved currents in the theory.

As it follows from the Nöther theory, the conserved charges have the form

$$Q^a = \int d^3 x J^a_0(x)$$  (2.19)

with

$$J^a_0 = -i \frac{\delta \mathcal{L}}{\delta \partial_0^a \phi_i} t^a_{ij} \phi_j$$  (2.20)

($\mathcal{L}$ is the Lagrangian of the theory).

Then, it is easy to see that

$$[Q^a, \phi_i] = it^a_{ij} \phi^j.$$  (2.21)

Thus, (2.15) means that at least some matrix elements of $<0|\phi_i|0\rangle$ are different from zero.

One can show (§10.3 B in [3]) that the break-down of the global symmetry is accompanied with the appearance of Goldstone massless and spinless bosons (the Goldstone theorem).

Note, and this remark has a great importance for our statement, that the conservation of the charges in the Nöther theory means that the Lagrangian is invariant under transformations of a (global) symmetry. The latter means, in turn, that we should define always the minima of the Lagrangian: in fact the minima of the potential $V(\phi_i)$. As we noted
above, the minimum state of the potential energy corresponds to the vacuum vector $|0\rangle$. It is obvious that the potential $V(\phi_i)$ has its (global) minimum at $\phi_i = <0|\phi_i|0\rangle \equiv a_i$. If the potential $V$ is a function of several quantum fields: $\phi_i, i = 1, \ldots, n$, we should solve a set of equations of the first order to define its (global) minimum. These equations describe the minimum surface for the potential $V$ (for example, it is the circle $\sigma^2 + \pi^2 = a^2$ in the $\sigma, \pi$ model with the Abelian $U(1)$ symmetry described in the monograph [10], p. 147-149), providing invariance of the Lagrangian with respect to the symmetry translations. The vacuum vector $|0\rangle$ remains invariant at such translations: according to (2.13). However, this degenerated construction is not steady in general; since the vacuum $|0\rangle$ is unique (because of the GNS theorem), the symmetry would be broken down inevitably.

If the potential of the considered theory has, for example, the two minima differing in sign, we should choose one of them.

The next example is the Lagrangian of the simplest and nevertheless very important $\lambda \phi^4$ theory having the form (p. 10 in [11])

$$L = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4. \quad (2.22)$$

The points of global minimum of this Lagrangian are $\phi_0 = \pm m/\sqrt{\lambda}$, and we choose the positive sign. Note also that if the initial value of $\phi$ is zero, the symmetry will be broken down spontaneous by during the time $t \sim m^{-1}$. Note also, and it will be very important for our further consideration, that the vacuum configuration $\phi_0 = m/\sqrt{\lambda}$ describes some sphere $S^2$ in the field configuration space.

Considered in the present work the $SU(2)$ group is one of the examples of local gauge symmetries, which differs from global symmetries by a dependence of the group parameters on coordinates. The general sketch of the proof remains the same as it was for the case of global symmetries. The key point here is also the nonzero vacuum expectation value $<0|\phi_i|0\rangle$.

It is naturally to ask: if $<0|\phi_i|0\rangle \neq 0$ in the case of a symmetry break-down, what is the physical nature of this value? Let us consider this question with an example of the most simple $\lambda \phi^4$ theory.

First of all, note that the expectation value of the number of particles in some physical state (in our consideration we are interested in the ground state) is

$$n = <a^\dagger(p) a(p) >; \quad (2.23)$$

it is the direct consequence of the CCR (2.3), (2.4).

Let us consider now a semi-classical system of Higgs scalar particles $\phi$ (this means that $n$ is very large according to the Bohr-Sommerfeld theory (§48 in [12]) in a state of thermodynamic equilibrium with the temperature $T$ (p. 78 in [12]).

Since the scalar particles have no conserved charges, and the creation-annihilation processes have the equal probabilities in the free dynamics theory, the chemical potential $\mu$ of the Higgs scalar is zero, and we can write down the small Bose distribution for the Higgs scalar theory as

$$n_p = \frac{1}{\exp(p_o/T) - 1}, \quad (2.24)$$
where \( p_0 = \sqrt{p^2 + m^2} \) is the energy of a particle with the momentum \( p \) and mass \( m \). As \( T \to 0 \), \( n_p \to 0 \). If \( T \neq 0 \), all the physical important quantities (thermodynamic potentials, Green functions, etc.) in the considered system are determined by the Gibbs expectation values:

\[
< \phi > = \frac{\text{Tr}[\exp(-H/T\phi)]}{\text{Tr}[\exp(-H/T)]},
\]

(2.25)

where \( H \) is the Hamiltonian of the considered system.

We can consider, in the limit \( \lambda \to 0 \), the ideal Bose gas of Higgs particles, \( \phi_0 \), which we shall call henceforth as the Bose-condensate. The collective motion of the above ideal Bose gas is described with the help of the stationary (at the zero momentum) theory, alike the situation with superfluid helium in the Bogoliubov theory \[10\, [13].

As a direct consequence of the CCR (2.3),(2.4)(see p.63 in \[11\]), we write down for the stationary Bose condensate \( \phi_0 \):

\[
n_p = (2\pi)^3 \phi_0^2 m \delta(p).
\]

(2.26)

Then we decompose the Higgs field, described by the Lagrangian (2.22), into the Bose condensate (which we call the classical field, since it is described by the Bohr-Sommerfeld-Gibbs theory, i.e. statistic physics) and the perturbation excitations over the Bose condensate, which we identify with the scalar particles. The alike way was utilized in the work \[13\].

Thus, the symmetry break-down parameter \( < 0|\phi_i|0 > \) coincides in our case with the value of the classical field \( \phi_0 \).

We can draw the following very important conclusions from the said above. Firstly, the presence of the Higgs Bose condensate is a principal sign of the symmetry break-down; secondly, the vacuum, in the form of the Bose condensate, has a nontrivial structure and can be considered as a real physical vacuum. It is true, since the Bose distribution of momenta, (2.24), is real. All this is correct not only for the \( \lambda \phi^4 \) theory, but also for the Yang-Mills theory, the aim of our present investigation.

### 2.2 Gauge Higgs effect.

Our idea, basing onto the above consideration, is to construct the consistent Yang-Mills vacuum, using the Higgs Bose condensate in the theory with monopoles \[14, \[15, \[16\] in the well-known Bogomol’nyi-Prasad-Sommerfeld (BPS) limit of the zero self-interaction: \( \lambda \to 0 \) (at \( m \to 0 \)), in the Higgs sector of the YM action (see \[14, \[17\]):

\[
S = -\frac{1}{4g^2} \int d^4xF_{\mu\nu}^bF_{\mu\nu}^b + \frac{1}{2} \int d^4x(D_\mu\phi, D^\mu\phi) - \frac{\lambda}{4} \int d^4x \left[ (\phi^b)^2 - \frac{m^2}{\lambda} \right]^2,
\]

(2.27)

where \( D_\mu\phi = \partial_\mu\phi + g[A_\mu, \phi] \) is the covariant derivative, \( g \) is the coupling constant.

We suppose that the initial data of all the fields are given to within stationary gauge transformations, the manifold of these transformations has a nontrivial structure of 3-dimensional paths in the group space of the non-Abelian \( SU(2) \) gauge group:

\[
\pi_3(SU(2)) = \mathbb{Z},
\]

(2.28)
where $\mathbf{Z}$ is the group of integers: $n = 0, \pm 1, \pm 2, \ldots$

In the case of the $SU(2)$ gauge theory the Yang-Mills fields $A^{\mu b}$ and Higgs fields $\phi^b$ take their values in the Lie algebra of the $SU(2)$ group.

If we want to obtain the fields corresponding to the action with finite values of energy, we should demand that the field $\phi(r)$ is finite as $r \to \infty$ in the Bogomol’nyi-Prasad-Sommerfeld (BPS) limit $\lambda \to 0$. This means that $\phi^a$ should go to the minimum of the potential $V$:

$$\phi^{a\infty}(n) \in M_0, \quad n = \frac{r}{r},$$

(2.29)

where $M_0$ is the manifold of the minimum of the potential $V$ (the vacuum manifold):

$$M_0 = \{\phi = a; \quad a^2 = m^2/\lambda\}$$

(2.30)

as $r \to \infty$. Thus, $M_0$ consists of the points of the sphere $S^2$ in the 3-dimensional space of the $SU(2)$ gauge symmetry, which is broken down spontaneously to the $U(1)$ group. We can choose the vector $\vec{\phi}$ along the axis $z$ in the Cartesian coordinates:

$$\vec{\phi} = (0, 0, m/\sqrt{\lambda})$$

(2.31)

as a configuration of the ground state. Thus, this vector remains invariant with respect to rotations around the axis $z$ (the $U(1)$ transformations).

Note, however, that the choice (2.31) in the whole space is topological trivial. Really ([4], §Φ4), the gauge condition $\phi_i = 0, i = 1, 2; \phi_3 = |\vec{\phi}|$ is impossible for the nontrivial topologies $n \neq 0$. For the field satisfying such condition its asymptotic at the spatial infinity is trivial: the solutions of the form $\vec{\phi}^{a\infty}(n) = V(n)\vec{\phi}$, where $V(n)$ is a continuous function of $n$ with its values in the $SU(2)$ group in the case of the Yang-Mills theory, are topological equivalent to $\vec{\phi} = (0, 0, a)$. On the other hand, $V(n)$, considered as the map $\pi_2(SU(2))$, is equal to zero.

We should define the topological structure of the manifold (2.30). First of all, because of the GNS construction, $\nabla \phi^a = 0$. In the case of some discrete group $G$, $\phi^{a\infty}$ should be constant, since it is a continuous function (from the topological point of view, we deal in this case with the group $\pi_0$ of the connection components, which is trivial in the case of a connected manifold; the sphere $S^2 := \{n = 1$ as $r \to \infty\}$ is namely such case). In this case $\phi^{a\infty}$ has a trivial topology.

If $dim(M_0) \neq 0$, $M_0$ has a nontrivial topology. Therefore, the group of symmetry $G$ should be continuous. One can show (p.p. 465-466 in [10]) that the covariant derivatives $D_i \phi$, taking part in the action (2.27), decreases as $r^{-2}$; thus the integral (2.27) is finite. This guarantees nontrivial topological features of the theory.

Issuing from the formula (2.30), which defines the manifold $M_0$ of the minimum of the potential $V$, and demanding that $\vec{\phi}(r)$ goes to some value of $M_0$, we see that the sphere $S^2 \simeq M_0$ maps into the sphere $S^2 := \{n = 1\}$ as $r \to \infty$. This map has the nontrivial homotopy group of 2-dimensional loops:

$$\pi_2 S^2 = \pi_3(SU(2)) = \pi_1(U(1)) = \pi_1 S^1 = \mathbf{Z}.$$
Namely this nontrivial topology determines magnetic charges connected with the remaining \(U(1)\) symmetry (these charges alone point to some electromagnetic theory). The presence of magnetic charges means that there exist solutions of the motion equations for the action (2.27) in the class of magnetic monopoles, i.e. the stationary vacuum solutions at the spatial infinity corresponding to the quantum-field configuration of the minimum energy \(E_{\text{min}}\) (according to our definition of the vacuum as a ground state of the minimum energy). We can write down these monopole solutions.

For example, the Higgs isovector should be proportional to \(n\) as \(r \to \infty\): in the light of the above said about the map \(S^2 \simeq M_0 \to S^2 := \{n = 1\}\) as \(r \to \infty\). Thus, its form should be
\[
\phi^a \sim \frac{x^a}{r} f(r, a)
\]
as \(r \to \infty\); \(f(r, a)\) is some continuous function which does not change the topology (2.32).

This solution for \(\phi^a\) appears for the first time in the work \([15]\) and it is called the hedgehog. A good analysis of hedgehogs is conducted in the monograph \([11]\) (p. 114-116).

One can show (§Φ11 in \([4]\)) that there exists the solution of the motion equations (regular in a finite spatial volume) in the form \([4, 17]\)
\[
\phi^a = x^a \text{gr} f_{\text{BPS}}(r), \\
f_{\text{BPS}}(r) = \left[ \frac{1}{\epsilon \tanh(r/\epsilon)} - \frac{1}{r} \right], \\
A^a_i(t, \vec{x}) \equiv \Phi^a_{\text{BPS}}(\vec{x}) = \epsilon\text{ak} \frac{x^a}{gr^2} f_{\text{BPS}}(r), \\
f_{\text{BPS}}^1 = \left[ 1 - \frac{r}{\epsilon \sinh(r/\epsilon)} \right]
\]
obtained in the BPS limit
\[
\lambda \to 0, \quad m \to 0: \quad \frac{1}{\epsilon} \equiv \frac{gm}{\sqrt{\lambda}} \neq 0.
\]
This solution satisfies the potentiality condition:
\[
B = \pm D\phi^a,
\]
where \(B\) is the magnetic tension in the theory (2.27). This equation (called the Bogomol'nyi equation) is obtained by the evaluation of the lowest bound of energy:
\[
E = 4\pi m a, \quad a = \frac{m}{\sqrt{\lambda}}
\]
(where \(m\) is the magnetic charge), for the monopole solutions.

\[1\] The statement that these solutions are regular in a finite spatial volume means that we should consider the topology (2.32) and the manifold \(M_0\), (2.30), also with account of this finite spatial volume. If we wish to adapt our theory to the needs of QCD (we shall see how to do this in Sections 8,9), the spatial volume determined by the typical hadronic size, \(\sim 1\) fm (\(\sim 5\) GeV\(^{-1}\)), is quite sufficient for our consideration.
The outlines of the proof of the formula (2.37) are the following (see §Φ11 in [1]). Following to the ’t Hooft-Polyakov model [14, 15], let us introduce the ”electromagnetic tension” as the scalar product

\[
F_{\mu\nu} = <F_{\mu\nu}^a, \phi_a^b >. \tag{2.39}
\]

The magnetic tension corresponding to this tensor is

\[
H^a = \frac{1}{2} \epsilon^{ijk} < F^b_{jk}, \phi^b > a^{-1}. \tag{2.40}
\]

We can write down the magnetic charge \( m \) as a stream of the magnetic tension \( H \) through an infinite removed sphere (multiplied on \((4\pi)^{-1}\)):

\[
m = \frac{1}{4\pi} \int dS \frac{H^a}{4\pi} \int d^3x \partial_i \epsilon^{ijk} < F^b_{jk}, \phi^b > a^{-1}. \tag{2.41}
\]

Note also that

\[
\epsilon^{ijk} \partial_i < F^b_{jk}, \phi^b > = \epsilon^{ijk} \nabla_i < F^b_{jk}, \phi^b > = \epsilon^{ijk}(< \nabla_i F^b_{jk}, \phi^b > + < F^b_{jk}, \nabla_i \phi^b >) = \epsilon^{ijk} < F^b_{jk}, \nabla_i \phi^b >
\]

(we utilized here the fact that the usual derivative \( \partial_i \) coincides with the covariant derivative \( \nabla_i \) for the gauge invariant value \( < F^b_{jk}, \phi^b > \); we took account also of the Bianchi identity \( \epsilon^{ijk} \nabla_i F^b_{jk} = 0 \)). Therefore,

\[
m = \frac{1}{8\pi} \int d^3x \epsilon^{ijk} < F^b_{jk}, \nabla_i \phi^b > a^{-1}. \tag{2.42}
\]

Then we consider the inequality

\[
\int dx < c, b > \leq \frac{1}{2} \int dx ( < c, c > + < b, b >) \tag{2.43}
\]

which follows from the relation \( \int < c - b, c - b > dx \geq 0 \); the equality is reached in the case \( c(x) = b(x) \) only (here \( c(x) \) and \( b(x) \) take theirs values in \( \mathbb{R}^n \)). Applying the inequality (2.43) to the tensors \( \frac{1}{2g} \epsilon^{ijk} F^b_{jk} \) and \( \nabla_i \phi^b \), we obtain that

\[
\int d^3x \frac{\epsilon^{ijk}}{2g} < F^b_{jk}, \nabla_i \phi^b > \leq \frac{1}{2} \int d^3x \{ \frac{1}{4g^2} < F^b_{jk}, F^b_{jk} > + < \nabla_i \phi^b, \nabla_i \phi^b > \}. \tag{2.44}
\]

The integral in the right-hand side of (2.44) differs from the energy \( E \) of the configuration \((\phi, A^a_\mu)\) in the absence of the potential term:

\[
E_1 = \frac{1}{4} \lambda \int d^3x [\phi^2 - a^2], \tag{2.45}
\]
only. Since the left-hand side of (2.45) differs in the factor only from the magnetic charge, we obtain the estimation
\[ m \leq \frac{g}{4\pi a} (E - E_1). \] (2.46)

In other words,
\[ E \geq 4\pi m \frac{a}{g} + E_1, \] (2.47)
and the equality is reached in the case
\[ \frac{1}{g} \epsilon^{ijk} F^b_{jk} = \nabla^i \phi^b. \] (2.48)

But this is the Bogomol’nyi equation (2.37) written down in the index form (at the plus sign).

Since \( E_1 \geq 0 \), and the magnetic charge \( m \) takes the integers only, the energy \( E \) of the configuration \( (\phi, A^a_\mu) \) allows the estimation
\[ E \geq 4\pi \frac{a}{g} \] (2.49)
in the general case.

In the BPS limit \( \lambda \to 0 \), if other parameters \((a, g)\) remain invariable, this estimation becomes exact:
\[ E = \frac{1}{4g^2} \int d^3x \ < F^b_{jk}, F^b_{jk} > + \frac{1}{2} \int d^3x \ < \nabla^i \phi^b, \nabla^i \phi^b >. \] (2.50)

If the fields \((\phi, A^a_\mu)\) satisfy the Bogomol’nyi equations (2.37),(2.48), the functional (2.50) reaches its minimum (2.38).

The solutions of the Bogomol’nyi equation (2.37) are definite formulas for the Yang-Mills fields \( A^a_\mu \) and Higgs multiplet \( \phi^a \) (depending only on \( x \)) [4, 17]. The resembling formulas will appear in our work on the basis of a similar theory. We shall discuss them in Section 4.

Thus, we see that the Bogomol’nyi equation (2.37) allows us to obtain a consistent theory involving the Yang-Mills and Higgs multiplets and yielding the solutions of the BPS monopole type. The Higgs sector of that theory defines the \( U(1) \) group of symmetry with a nontrivial topology, i.e. with magnetic charges and with radial magnetic fields. All this can be a base for the construction of a similar theory for the Yang-Mills vacuum in the Minkowski space.

In contrast to the "old" approach to the Yang-Mills vacuum, our conception of the Yang-Mills vacuum as a stationary Bose condensate yields the real spectrum of momentum. We shall show that the stationary vacuum fields have the winding number \( n = 0 \) and they have the form of BPS (Wu-Yang) monopoles. The "electric" and "magnetic" tensions corresponding to these vacuum fields will also be constructed. The topological degeneration \( (n \neq 0) \) in our theory is realized due to Gribov copies of the covariant Coulomb gauge imposed on the vacuum potentials. The Yang-Mills (gluon) fields are considered as weak perturbation excitations (multipoles) over this vacuum. These excitations have the asymptotic \( O(\frac{1}{r^{l+1}}), l > 1 \) at the spatial infinity.
3 The Dirac quantization of the Yang-Mills theory.

Let us consider the Yang-Mills theory with the local $SU(2)$ group in the four-dimensional Minkowski space-time. The action of this theory is given by the formula

$$W[A_\mu] = -\frac{1}{4} \int d^4 x F_\mu^a F^{\mu\nu}_a = \frac{1}{2} \int d^4 x (F_{\alpha i}^a B_i^{a2}) ,$$

(3.1)

where the standard definitions of the non-Abelian "electric" tension $F_{\alpha i}^a$:

$$F_{\alpha i}^a = \partial_\alpha A_i^a - D(A)^{ab} A_{0b}, \quad D_i^{ab} = (\delta^{ab} \partial_i + g \epsilon^{acb} A_{ci}) ,$$

(3.2)

and the "magnetic" one, $B_i^a$:

$$B_i^a = \epsilon_{ijk} (\partial^j A_{ak} + g \frac{1}{2} \epsilon^{abc} A_{bj} A_{ck}) ,$$

(3.3)

are used. The action (3.1) is invariant with respect to the gauge transformations $u(t; x)$:

$$\hat{A}_i^a = u(t; x)(A_i^a + \partial_i) u^{-1}(t; x), \quad \psi^u := u(t; x) \psi,$$

(3.4)

where $\hat{A}_\mu = g \frac{1}{2} \tau^a A_{\mu a}$, and $\psi^u$ is a spinor field.

Solutions of the non-Abelian constraint equation (the Gauss law constraint):

$$\frac{\delta W}{\delta A_0^a} = 0, \iff [D^2(A)]^{ac} A_{0c} = D_i^{ac} (A) \partial_0 A_i^c ,$$

(3.5)

and the motion equation:

$$\frac{\delta W}{\delta A_i^a} = 0, \iff [\delta_{ij} D_j^2 (A) - D_j (A) D_i (A)]^{ac} A_i^c = D_0^{ac} (A) [\partial^0 A_{ci} - D(A)_{ci} A^{ob}] ,$$

(3.6)

are determined by boundary conditions and initial data. They generalize the corresponding equations in the Maxwell electrodynamics (see the formulas (7),(8) in [19]).

The Gauss law constraint (3.5) connects the initial data of $A_0^a$ with the one of the spatial components $A_i^a$. To remove the non-physical variables, we can solve this constraint in the form of the naive perturbation series:

$$A_0^a = \frac{1}{\Delta} \partial_0 \partial_i A_i^a + ...,\quad (3.7)$$

where $\Delta$ is the Laplacian. As we remember from mathematical physics (see for example p.203 in [20]), the fundamental solution of the Laplace equation:

$$\Delta \mathcal{E}_3 = \delta(x),$$

(3.8)

is

$$\mathcal{E}_3 = -\frac{1}{4\pi x} ,$$

(3.9)
This defines the action of the operator $\Delta^{-1}$ on some continuous function $f(x)$:

$$\Delta^{-1} f(x) = -\frac{1}{4\pi} \int d^3 y \frac{f(y)}{x-y},$$

(3.10)

that is the Coulomb kernel of the non-local distribution (see also (12) in [19]).

Thus, the resolving of the constraint and the substitution of this solution into the equations of motion distinguishes the gauge-invariant non-local (radiation) variables. After the substitution of this solution into the equation (3.6) the lowest order of this equation in the coupling constant $g$ contains only transverse fields (this level coincides mathematically, as a linearized Yang-Mills theory [19]:

$$[\partial^2 - \Delta] A^T_{ik} + \ldots = 0, \quad A^T_{i} = [\delta_{ik} - \partial_i \Delta^{-1} \partial_k] A^{ck} + \ldots$$

(3.11)

This perturbation theory is well-known as the radiation [21] or Coulomb [22, 23] gauge with the generating functional of Green functions in the form of a Feynman integral in the rest frame of reference $l^{(0)} = (1,0,0,0)$:

$$Z_F[l^{(0)},J^{aT}] = \int \prod_{c=1}^{c=3} [d^2 A^{cT} d^2 E^{cT}]$$

$$\times \exp \left\{ iW^{T}_{l^{(0)}}[A^T, E^T] - i \int d^4 x [ J^{cT}_k \cdot A^c_k ] \right\},$$

(3.12)

with the constraint-shell action:

$$W^{T}_{l^{(0)}}[A^T, E^T] = W^I|_{\delta W^I_{0}=0},$$

(3.13)

given in the first order formalism ([24],p.83):

$$W^I = \int dt \int d^3 x \{ F_{0c} E_c^i - \frac{1}{2} [E^{cT}_i \sigma^c + B^c_i B^c_i] \} = \int dt \int d^3 x (E^c_i \partial_0 A_c^i + A_0^c D^c - H),$$

(3.14)

where

$$D^c = \partial_k E^{kc} - g [A^b_k, E^{kd}] e^c_{bd},$$

(3.15)

and

$$H = \frac{1}{2} (E^{c2}_c + B^{c2}_k) = \frac{1}{2} [(E^{Tc})^2 + (\partial_i \sigma^c)^2 + B^{c2}_k]$$

(3.16)

is the Hamiltonian of the Yang-Mills theory. We decompose here the ”electric tension” $E^{kc}$ into the transverse and longitudinal parts:

$$E^c_i = E^{Tc}_i + \partial_i \sigma^c, \quad \partial_i E^{Tc}_i = 0.$$ 

(3.17)

The constraint

$$\frac{\delta W^I}{\delta A_0^c} = 0 \iff D^c_{i} (A) E^c_i = 0$$

(3.18)
can be solved in terms of these (radiation) variables. The function $\sigma^a$ has the form \[22\]

$$
\sigma^a[A^T, E^T] = \left( \frac{1}{D_i(A)\partial^i} \right)^{ac} \epsilon_{cbd} A^T_k E^{Tkd} \equiv \left( \frac{1}{\Delta} \right)^{ac} \epsilon_{cbd} A^T_k E^{Tkd}.
$$

(3.19)

Note (see (16.24) in \[23\]) that $\text{Det} [D_i(A)\partial^i]$ in (3.19) is the Faddeev-Popov (FP) determinant in the YM Hamiltonian formalism:

$$
\hat{\Delta}^b_a A_{0b} - \partial_i E^i_a = 0, \quad \hat{\Delta}^b_a \equiv D^b_a \partial^j,
$$

(3.20)

where

$$
E_{ia} = \frac{\partial L}{\partial \dot{A}_{ia}} = F^a_{0i}
$$

(3.21)

is the canonical momentum (3.2).

A complete proof that $\text{det} \Delta^b_a$ is the FP determinant of the YM theory is given in the monograph \[23\], where it was shown that the radiation gauge in the YM theory is equivalent to the FP determinant $\text{det} \Delta^b_a$ (see (16.30) in \[23\]).

The operator quantization of the Yang-Mills theory in terms of the radiation variables belongs to Schwinger \[21\], who proved the relativistic covariance of the radiation variables (3.11). This means that the radiation fields are transformed as non-local functionals (Dirac variables \[19\]),

$$
A^T_k[A] = v^T[A](\hat{A}_k + \partial_k)(v^T[A])^{-1}, \quad \hat{A}^T = g A^T a \tau^a
$$

(3.22)

where the matrix $v^T[A]$ is defined from the condition of transversality: $\partial_k A^{kT} = 0$. At the level of the Feynman integral, as we have seen in QED, the relativistic covariance means the relativistic transformation of sources (this led to the change of the variables (29) in the work \[19\]).

The definition (3.22) can be interpreted as a transition to new variables, allowing us to rewrite the Feynman integral in the form of the FP integral \[22, 23, 26\]:

$$
Z_F[(0), J^{aT}] = \int \int \prod_{c=1}^{c=3} \left\{ [d^4 A^c] \delta(\partial_i A^{ci}) \text{Det}[D_i(A)\partial^i] \right\}
$$

\times \exp \left\{ iW[A] - i \int d^4 x (J^T_k \cdot A^{Tkc}[A]) \right\}.

(3.23)

It was proved in \[22, 25, 26\] that, on mass-shells of radiation fields, the scattering amplitudes do not depend on the factor $v^T[A]$. But the following question is quite reasonable: why we can not observe these scattering amplitudes? There are a few answers to this question: the infrared instability of the naive perturbation theory \[1, 27\], the Gribov ambiguity, or the zero value of the FP determinant \[28\], the topological degeneration of the physical states \[29, 30, 31\].
4 Topological degeneration of initial data.

4.1 Instanton theory.

One can find a lot of solutions of equations of classical electrodynamics. The nature chooses the two types of functions: the monopole (the electric charge) that determines non-local electrostatic phenomena (including instantaneous bound states) and the multipoles that determine the spatial components of gauge fields with the nonzero magnetic tensions.

The spatial components of non-Abelian fields, considered above as the radiation variables (3.11) in the naive perturbation theory (3.7), are also defined as multipoles. In the non-Abelian theory, however, it is a reason, as we saw this in Section 2, to assume that the spatial components of non-Abelian fields belong to the monopole class of functions like the time components of the Abelian fields (as the Coulomb potential for example).

This fact was revealed by the authors of the instanton theory \cite{3}. Instantons satisfy the duality equation in the Euclidean space (where the Hodge duality operator $\ast$ has the $\pm 1$ eigenvalues for external 2-forms defining the Yang-Mills tension tensor); thus, the instanton action coincides with the Chern-Simons functional (the Pontryagin index) (see the formula (10.104) in \cite{9}):

$$\nu[A] = \frac{g^2}{16\pi^2} \int_{t_{in}}^{t_{out}} dt \int d^3 x F^a_{\mu\nu} F^{a\mu\nu}_\ast = X[A_{out}] - X[A_{in}] = n(t_{out}) - n(t_{in}), \quad (4.1)$$

where (\cite{10} in \cite{9})

$$X[A] = -\frac{1}{8\pi^2} \int d^3 x \epsilon^{ijk} Tr[\hat{A}_i \partial_j \hat{A}_k - \frac{2}{3} \hat{A}_i \hat{A}_j \hat{A}_k], \quad A_{in, out} = A(t_{in, out}, x) \quad (4.2)$$

is the topological winding number functional of gauge fields, and $n$ is the value of this functional for the classical vacuum:

$$\hat{A}_i = L^n_i = v^{(n)}(x) \partial_i v^{(n)}(x)^{-1}. \quad (4.3)$$

The manifold of all the classical vacua in a non-Abelian theory represents the group of three-dimensional paths lying in the three-dimensional $SU(2)$-manifold with the homotopy group $\pi_3(SU(2)) = \mathbb{Z}$. The whole group of stationary matrices is split into the topological classes marked by the integer numbers (the degrees of the map) defined by the expression ((10.106) in \cite{9})

$$\mathcal{N}[n] = -\frac{1}{24\pi^2} \int d^3 x \epsilon^{ijk} Tr[L^n_i L^n_j L^n_k] \quad (4.4)$$

which shows how many times the three-dimensional path $v(x)$ turns around the $SU(2)$-manifold when the co-ordinate $x_i$ runs over the space where it is defined.

Gribov, in 1976, proposed to consider the instantons as Euclidean solutions interpolating between the classical vacua with different degrees of the map (as tunnel transitions between these classical vacua).
The degree of the map (4.4) can be considered as a condition for normalization that determines the class of functions with given classical vacua (4.3). In particular, to obtain the equation (4.3), we should choose the classical vacuum in the form

\[ v(n)(x) = \exp(n\dot{\Phi}_0(x)), \quad \dot{\Phi}_0 = -\frac{n^a x_a}{r} f_0(r) \quad (r = |r|) \]  

(compare with (16.34) in [10]; we should set \( x_0 = 0 \) in this formula for the stationary gauge transformations which we discuss now). The function \( f_0(r) \) satisfies the boundary conditions

\[ f_0(0) = 0, \quad f_0(\infty) = 1. \]  

(4.6)

Note a direct parallel between this solution and the formula (2.33). The common between the monopole and instantons theories is the same nature of topologies. In the case of the Yang-Mills instanton theory we deal with the map (2.28): \( S^3 \rightarrow SU(2) \) as \( x \rightarrow \infty \). This induces the homotopy group \( \pi_3(SU(2)) = \pi_3 S^3 = \mathbb{Z} \) (\( S^3 \) is the bound of the Euclidian space \( E_4 \)) coinciding with \( \pi_2 S^2 = \mathbb{Z} \) (see (2.32)) in the theory (2.27)-(2.37). This generates similar theories. But there exists also the principal distinction of the both theories. As a consequence of the relation \( \pi_3(SU(2)) = \pi_3 S^3 = \mathbb{Z} \), the instantons can exist in the YM theory without any spontaneous \( SU(2) \) break-down. This break-down is not the necessary thing in this case, and we consider \( SU(2) \) as an exact symmetry in the instanton theory.

Thus, we obtain the solution of the monopole type in (4.5) as \( x \rightarrow \infty \).

To show that these classical values are not sufficient to describe physical vacuum in the non-Abelian theory, we consider the quantum instanton, i.e. the corresponding zero vacuum solution of the Schrödinger equation

\[ \hat{H} \Psi_0[A] = 0, \]  

where \( \hat{H} = \int d^3x [E^2 + B^2] \), \( E = \frac{\delta}{\delta A} \) are operators of the Hamiltonian and field momentum respectively. This solution can be constructed by using the winding number functional (4.2) and its derivative,

\[ \frac{\delta}{\delta A_i} X[A] = \frac{g^2}{16\pi^2} B_i^c(A). \]  

(4.8)

The vacuum wave functional, in terms of the winding number functional (4.2), has the form of a plane wave [13]:

\[ \Psi_0[A] = \exp(iP_N X[A]) \]  

(4.9)

for non-physical imaginary values of the topological momentum \( P_N = \pm 8\pi i/g^2 \) [13, 32].

We would like to note that in QED this type of the wave functional belongs to the non-physical part of the spectrum like the wave function of the oscillator \( (p^2 + q^2) \phi_0 = 0 \). The value of this non-physical plane wave functional [7] for the classical vacuum (4.3) coincides with the semi-classical instanton wave function

\[ \exp(iW[A_{\text{instanton}}]) = \Psi_0[A = L_{\text{out}}] \times \Psi_0[A = L_{\text{in}}] = \exp(-\frac{8\pi^2}{g^2} [n_{\text{out}} - n_{\text{in}}]). \]  

(4.10)

\footnote{The wave function (4.9) is not normalized, the imaginary topological momentum \( P_N = \pm 8\pi i/g^2 \) turns it in a function with the non-integrable square.}
This exact relation between the semi-classical instanton and its quantum version (4.7) points out that classical instantons are also non-physical solutions; they tunnel permanently in the Euclidean space-time between the classical vacua with zero energies that do not belong to the physical spectrum.

4.2 Physical vacuum and gauge Higgs effect.

Our next step is the assertion [33] about the topological degeneration of initial data not only of the classical vacuum but also of all the physical fields with respect to the stationary gauge transformations

\[ \hat{A}^{(n)}_i(t_0, x) = v^{(n)}(x)A^{(0)}_i(t_0, x)v^{(n)}(x)^{-1} + L^n_i, \quad L^n_i = v^{(n)}(x)\partial_i v^{(n)}(x)^{-1}. \]  

(4.11)

The stationary transformations \( v^{(n)}(x) \) with \( n = 0 \) are called the small one; and those with \( n \neq 0 \), the large ones [33].

The group of transformations (4.11) means that the spatial components of the non-Abelian fields with nonzero magnetic tensions \( B(A) \neq 0 \) belong to the monopole class of functions like the time components of the Abelian fields. In this case the non-Abelian fields with nonzero magnetic tensions contain the non-perturbative monopole-type term, and the spatial components can be decomposed into sums of vacuum monopoles \( \Phi^{(0)}_i(x) \) and multipoles \( \hat{A}_i \):

\[ A^{(0)}_i(t_0, x) = \Phi^{(0)}_i(x) + \hat{A}_i(t_0, x). \]  

(4.12)

Each multipole is considered as a weak perturbation part with the following asymptotic at the spatial infinity:

\[ \hat{A}_i(t_0, x)|_{\text{asymptotic}} = O\left(\frac{1}{r^{l+1}}\right) \quad (l > 1). \]  

(4.13)

Nielsen and Olesen [27] and Matinyan and Savidy [1] introduced the vacuum magnetic tension, using the fact that all the asymptotically free theories are unstable, and the perturbation vacuum is not the lowest stable state.

The extension of the topological classification of classical vacua to all the initial data of the spatial components helps us to choose the vacuum monopole with the zero value of the winding number functional (4.1):

\[ X[A = \Phi^{(0)}_i] = 0, \quad \frac{\delta X[A]}{\delta A^a_i} \bigg|_{A = \Phi^{(0)}} \neq 0. \]  

(4.14)

The zero value of the winding number, transverseness and spherical symmetry (as a monopole) fix the class of initial data for spatial components:

\[ \hat{\Phi}_i = -i \frac{\tau^a}{2} \epsilon_{iak} \frac{r^k}{r^2} f(r). \]  

(4.15)

They contain only one function \( f(r) \). The classical equation for this function has the form

\[ D_k^{ab}(\Phi_i)F_{a}^{bk}(\Phi_i) = 0 \implies \frac{d^2 f}{dr^2} + \frac{f(f^2 - 1)}{r^2} = 0. \]  

(4.16)
We can see the three solutions of this equation:

\[ f_{1}^{PT} = 0, \quad f_{1}^{WY} = \pm 1 \quad (r \neq 0). \] (4.17)

The first solution corresponds to the naive unstable perturbation theory with the asymptotic freedom formula.

The two nontrivial solutions are well-known. They are the Wu-Yang monopoles, applied for the construction of physical variables in the work [34] (the hedgehog and the antihedgehog in terminology of [11, 15]). As it was shown in the paper [35], the Wu-Yang monopole leads to the rising potential of the instantaneous interaction with the quasiparticle current. This interaction rearranges the perturbation series, leads to the gluon constituent mass and removes the asymptotic freedom formula [36, 37] as an origin of instability.

The Wu-Yang monopole [38] is a solution of the classical equations everywhere besides the origin of coordinates, \( r = 0 \). The corresponding magnetic field is

\[ B_{a}^{i}(\Phi_{k}) = \frac{x_{a}x_{i}}{gr^{4}}. \] (4.18)

To remove the singularity at the origin of coordinates and regularize its energy, the Wu-Yang monopole is considered as a limit of the Bogomol’nyi-Prasad-Sommerfeld (BPS) monopole (2.35):

\[ f_{1}^{BPS} = [1 - \frac{r}{\epsilon \sinh r/\epsilon}] \Rightarrow f_{1}^{WY}, \] (4.19)

when the mass of the Higgs field goes to infinity in the limit of the infinite volume \( V \):

\[ \frac{1}{\epsilon} = \frac{gm}{\sqrt{\lambda}} = \frac{g^{2} < B^{2} > V}{4\pi} \rightarrow \infty. \] (4.20)

The BPS monopole has the finite energy density:

\[ \int_{\epsilon}^{\infty} d^{3}x [B_{a}^{i}(\Phi_{k})]^{2} \equiv V < B^{2} >= 4\pi \frac{gm}{g^{2}\sqrt{\lambda}} = \frac{4\pi}{g^{2}\epsilon} = \frac{1}{\alpha_{s}\epsilon}. \] (4.21)

(see also [39]). The infra-red cut-off parameter \( \epsilon \) disappears in the limit \( V \rightarrow \infty \), i.e. when the mass of the Higgs field goes to infinity and the Wu-Yang monopole turns, in a continuous way, into the BPS monopole. In this case the BPS-regularization of the Wu-Yang monopole is similar to the infrared regularization in QED by the introduction of the “photon mass” \( \lambda \) (see for example [40], p.413) that also violates the initial equations of motion [41].

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3It is necessary to note here that in the quantum-field theory the limit \( V \rightarrow \infty \) is carried out after the calculation of physical observable values: scattering sections, probabilities of decays and so on, which are normalized on time and volume units.
The vacuum density of energy of the monopole solution:

\[ \sim < B^2 > \equiv \frac{1}{\alpha_s \epsilon V} = \frac{4\pi m}{g\sqrt{\lambda} V}, \quad (4.22) \]

is removed by the finite counter-term in the Lagrangian [39]:

\[ \tilde{\mathcal{L}} = \mathcal{L} - \frac{< B^2 >}{2}. \quad (4.23) \]

This nonzero vacuum magnetic tension is a crucial difference of the topological degeneration of fields in the Minkowski space from the topological degeneration of the classical vacua in the instanton theory in the Euclidean one (where the normalization of the vacuum, \( F^{\mu\nu} = 0 \), is a precondition of the formula (4.3); see p.p.482-483 in [10]). The Bogol’nyi equation (2.37), applied for the vacuum topology (2.32), provides this nonzero vacuum magnetic tension.

The problem is to formulate the Dirac quantization of weak perturbations of non-Abelian fields in the presence of the non-perturbation monopole, taking account of the topological degeneration of all the initial data.

### 4.3 Dirac method and Gribov copies.

Instead of the artificial equations of the gauge-fixation method [11]:

\[ F(A_\mu) = 0, \quad F(A^u_\mu) = M_F u \neq 0 \implies Z^{FP} = \int \prod_\mu DA_\mu \det M_F \delta(F(A)) e^{i W}, \quad (4.24) \]

we repeat the Dirac constraint-shell formulation resolving the constraint (3.3) with nonzero initial data:

\[ \partial_0 A^i_c = 0 \implies A^i_c(t, x) = \Phi^c(0)(x). \quad (4.25) \]

The vacuum magneto-static field \( \Phi^c(0) \) has its zero value of the winding number (4.2), \( X[\Phi^c(0)] = 0 \), and satisfies the classical equations everywhere besides of a small region near the origin of coordinates of the size

\[ \epsilon \sim \frac{1}{\int d^3 x B^2(\Phi)} \equiv \frac{1}{< B^2 > V} \quad (4.26) \]

that disappears in the infinite volume limit.

The second step is the consideration of the perturbation theory (4.12), where the constraint (3.3) acquires the form

\[ [D^2(\Phi(0))]^{ac} A^0_c = \partial_0 [D^{ac}(\Phi(0)) A^i_c(0)]. \quad (4.27) \]

Dirac proposed [12] to remove the time component \( A_0 \) (the quantization of which contradicts the quantum principles as a non-dynamic degree of freedom); then the constraint (4.27) acquires the form

\[ \partial_0 [D^{ac}(\Phi(0)) A^i_c(0)] = 0. \quad (4.28) \]
We define the constraint-shell gauge
\[ [D^a_{ic}(\Phi^{(0)}) A^i_{c}(0)] = 0 \] (4.29)
as zero initial data of this constraint.

It is easy to see that the expression in the square brackets in (4.29) can be treated as an equal to zero, in the initial time instant, longitudinal component of a YM field (4.25). We shall denote it as \( A^a_{\parallel} \):
\[ A^a_{\parallel} \equiv [D^a_{ic}(\Phi^{(0)}) A^i_{c}(0)] = 0 \big|_{t=0}. \] (4.30)

Let us call the latter (initial) condition as the covariant Coulomb gauge. Then the constraint (4.27) means that the time derivatives of longitudinal fields equal to zero.

The topological degeneration of initial data means that not only the classical vacua, but also all the fields \( A_i^{(0)} = \Phi_i^{(0)} + \bar{A}_i^{(0)} \) in the gauge (4.29) are degenerated:
\[ \hat{A}_i^{(n)} = v^{(n)}(x)(\hat{A}_i^{(0)} + \partial_i v^{(n)}(x))^{-1}, \quad v^{(n)}(x) = \exp[n\Phi_0(x)]. \] (4.31)

The winding number functional (4.2), after the transformation (4.3), takes the form (see the formula (3.36) in [39])
\[ X[A_i^{(n)}] = X[A_i^{(0)}] + \mathcal{N}(n) + \frac{1}{8\pi^2} \int d^3x \epsilon^{ijk} Tr[\partial_i(\hat{A}_j^{(0)} L_k^n)], \] (4.32)
where \( \mathcal{N}(n) = n \) is given by the eq. (4.4).

The constraint-shell gauge (4.29), (4.30) keeps its form in each topological class:
\[ D^a_{ic}(\Phi_k^{(n)}) \hat{A}_{b}^{i(n)} = 0 \] (4.33)
if the phase \( \Phi_0(x) \) satisfies the equation of the Gribov ambiguity for the constraint-shell gauge (4.29), (4.30) (see also §T.26 in [4]):
\[ [D^a_{ic}(\Phi_k^{(0)})]^{ab} \Phi_{(0)}^{bc} = 0; \] (4.34)
this leads to the zero FP determinant \( \det (\hat{\Delta}) \) in (3.20) (it has the countable set of eigenvalues \( \lambda_i \) corresponding to the zero solution (4.34)). Note that the Gribov equation (4.34), written down in terms of the Higgs isoscalar \( \Phi_{(0)b} \), is the direct consequence of the Bogomol’nyj equation in the form (2.48) and the Bianchi identity \( \epsilon^{ijk} \nabla_i F_{jk} = 0 \).

Note also that (although indirectly) the equations (4.28), (4.30) are Cauchy conditions for the Gribov ambiguity eq. (4.34).

The Gribov ambiguity equation has a very interesting geometric interpretation. We should recall, to begin with, that (§T.22 in [4]) every gauge field \( A_\mu \), as an element of the adjoin representation of the given Lie algebra, sets some element \( b_\gamma \in G \), where \( G \) is the considered gauge Lie group. These elements \( b_\gamma \) are defined as
\[ b_\gamma = P \exp(- \int_T T \cdot A_\mu dx^\mu), \] (4.35)
where $P$ is the symbol of the parallel transfer along the curve $\gamma$ in the coordinate (for example, Minkowski) space; $T$ are the matrices of the adjoin representation of the Lie algebra. It is obvious that $b_\gamma = b_\gamma_1 b_\gamma_2$ as the end of the curve $\Gamma_1$ coincides with the beginning of the curve $\Gamma_2$ and the curve $\Gamma$ is formed from these curves. Thus, the group operation (the multiplication) is associative; there exists always the unit element and element inverse to the given one. This follows from the usual features of curves and the exponential function.

We see that elements $b_\gamma$ are defined on the set of external 1-forms in the Lie algebra. The cohomology classes of these external 1-forms are the elements of the cohomology group $H^1$ (see for example §T.7 in [4]). The Pontryagin formula for a degree of a map (see the lecture (26) in [43]):

$$\int_X f^* \omega = \text{deg } f$$

(4.36)

(where the map $f : X \to Y$ is smooth and maps the compact space $X$ into the compact space $Y$: the so-called eigen map; $f^*$ is the homomorphism of the cohomology groups: $H^1_X \to H^1_Y$, induced by the map $f$), sets an one-to-one correspondence between the homotopy and cohomology groups in considered theory [4]. In particular, the topological charge $n = 0$ corresponds to exact 1-forms, i.e to those which can be represented as a differential, $d\sigma$, of some 1-form $\sigma$. Because of the Poincare lemma, $d \cdot d\sigma = 0$, i.e. every exact form is closed. Note that the scalar field $\Phi_{(0)\delta}$ in the Gribov equation (4.34) has the topological charge $n = 0$. This charge, through the smooth Bogomol’nyj equation, is told to the magnetic tension tensor $F^b_{jk}$ (i.e to corresponding 2-forms) and to the corresponding YM fields (1-forms).

Let us consider now those curves $\Gamma$ which begin and finish at the same point $x_1$ of the Minkowski space. Such closed curves are called the 1-dimensional cycles, and the corresponding elements $b_\gamma$ can be written down as cyclic integrals:

$$b_\gamma = P \exp\left(- \oint \Sigma T \cdot A_\mu dx^\mu \right),$$

(4.37)

over some 1-dimensional cycle $\Sigma$. According to the De Rham theorem (see p.276 in [4]), if some external form $\omega$ is exact, its integral over every cycle defined on the considered manifold $M$ is equal to zero (this confirms also the formula (4.36) for the zero topological class). This means that the integral in (4.37) is equal to zero ($b_\gamma = 1$) for every exact form (corresponding to the zero topological charge according (4.36)).

The elements (4.37) form the holonomy subgroup $H$ in the initial gauge group. Those of these elements which correspond to the exact 1-forms form, in turn, the restricted holonomy subgroup, which we shall denote as $\Phi^0$. The action $b_\gamma$ of the gauge group $G$ on the manifold $M$ is described in terms of the principal fibre bundle $P(M,G)$ over the manifold $M$; thus, we shall write henceforth $\Phi^0(u)$ ($u \in P(M,G)$ is a fixed point of the contour $\Sigma$) for the elements of $\Phi^0$.

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4 We should apply the formula (4.36) to the topology (2.32) in our consideration.
One can say that the unit element of the holonomy group $H$ is *degenerated* with respect to all the exact forms corresponding to the zero topological charge. In our case of the vacuum Higgs fields $\Phi(0)_b$ namely these fields determine the class of exact forms and the group $\Phi^0(u)$.

As it was shown in the monographs [4], if $A_\mu$ and $A'_\mu$ are some two gauge fields, the gauge equivalence between which is realized by the function $g(x) \in G$, then

$$b'_\gamma = g(x_1) b_\gamma g(x_1)^{-1}$$  \hspace{1cm} (4.38)

as the curve $\Gamma$ begins and ends in the point $x_1$ (*the holonomy elements of some two gauge equivalent fields are conjugate*).

Let $g(x)$ have the spatial asymptotic $g(x) \to 1$ as $x \to \infty$ (indeed, $g(x)$ would have such asymptotic already on distances $\sim 1$ fm.; this is connected with needs of the quark confinement as we shall see this in Sections 8,9). Let also $b_\gamma$ and $b'_\gamma$ be elements of $H$ constructed by external forms belonging to some one class of cohomologies. In conclusion, the gauge fields $A_\mu$ and $A'_\mu$, forming the elements $b_\gamma$ and $b'_\gamma$ respectively, and connected by the gauge transformation $g(x) \to 1$, have the Coulomb gauge (4.30). Every such class is obtained from the zero class of exact forms as its Gribov copy (therefore as a Gribov copy of the ambiguity equation (4.34) also). As $g(x) \to 1$, $x \to \infty$, we can rewrite (4.38) as

$$b'_\gamma = b_\gamma \cdot 1.$$  \hspace{1cm} (4.39)

The latter equality reflects the structure of the cohomology group $H^1$: *some two 1-forms belonging to one class of cohomologies are equivalent to within some exact form* ($\S$T6 in [4]). In terms of the holonomy group $H$ this means that *some two elements of $H$ corresponding to the 1-forms belonging to one class of cohomologies are equivalent to within some element of the restricted holonomy group $\Phi^0$*.

Thus, the Gribov equation (4.34) and the Gribov transformations (4.31) describe correct the cohomologic structure of the transverse vacuum YM fields (satisfied the Coulomb gauge (4.30)) as elements of the connection of the principal fibre bundle $P(M, G)$, with the structural non-Abelian group $SU(2)$ broken down spontaneous to the $U(1)$ group and the Minkowski space $M$ as a base of this fibre bundle, at the spatial infinity.

Let us prove that the holonomy group $H$ has a nontrivial structure in the non-Abelian case only ($\S$T.26 in [4]).

Let us consider the principal trivial fibre bundle $\zeta = (M \times G, M, G, p)$ of topological trivial gauge fields on some manifold $M$; for example, it is the Minkowski space as in our theory (a more detailed discussion of this class of fibre bundles is given in the monograph [14]). The total space $\mathcal{E}_0 = M \times G$ of this fibre bundle can be deformed at a point. It follows from the fact that trivial fibre bundles are described (see the example 1 on p.107 in [14]) with the help of the unique chart map (the *trivialisation*):

$$M \times \mathbb{R}^n \to M \times \mathcal{V},$$  \hspace{1cm} (4.40)

where $\mathcal{V}$ is some linear space over some field $K$. 

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In this case all the homotopy groups are trivial: \( \pi_n E_0 = 0 \); this means that \( E_0 = 0 \) can be deformed at a point.

The nontrivial topology appears after the identification of all the gauge equivalent fields. Let us mark out the subgroup \( G_0^\infty \subset G^\infty \) consisting of transformations defined with functions \( g(x) \) such that \( g(x_0) = 1 \), \( x_0 \in M \). This group acts free on \( E_0 \). Really, the transformation law of the element \( b_\gamma \) under the gauge transformation corresponding to \( g(x) \) has the form

\[
b'_\gamma = g(x_2) b_\gamma g(x_1)^{-1},
\]

(compare with (4.38)), where \( x_1 \) and \( x_2 \) are the beginning and end of the curve \( \gamma \). This law means that the group \( G_0^\infty \) acts free on \( E_0 \). Really, if a gauge transformation keeps a gauge field \( A_\mu \) on its place, then

\[
b_\gamma = g(x_2) b_\gamma g(x_1)^{-1}.
\]

Choosing the curve to begin at \( x_0 \) and to end at an arbitrary point \( x \), we see that \( b_\gamma = g(x) b_\gamma \), i.e. \( g(x) \equiv 1 \). This is the definition of a free action of a group on a manifold (see p.16 in [44]).

But the group \( G^\infty \) acts not free on \( E_0 \) in the general case, and this is, as we shall show now, a ground cause of the Gribov ambiguities in the non-Abelian case. For example, if some gauge field \( A_\mu \) takes its values in the Lie algebra \( H' \subset G \), and \( g \in G \) is an element commuting with all the elements of \( H' \), then such gauge transformations are induced with \( g(x) \equiv g \) (the global gauge transformations).

The holonomy group \( H \) is an example of such groups: there exists always such gauge transformation \( g(x) \) that maps an element \( b_\gamma \in H \) to itself.

The space \( B_0 \) of orbits of \( G_0^\infty \) in \( E_0 \) is, obviously, the base of the fibre bundle with the fibre \( G_0^\infty \) and total space \( E_0 \). The task of the choice of the unique gauge field from every orbit (the gauge fixation in the language of the FP theory of path integrals) is equivalent to the task of the construction of a section of the above fibre bundle. In the case if the fibre bundle has a section every element of the homotopy group of the base is obtained by a natural homomorphism from some element of the homotopy group of the total space \( E_0 \) (more precisely, some element \( \alpha \) is obtained from the element \( q\alpha \), where \( q \) is the section).

Since, as we already emphasized, \( \pi_i E_0 = 0 \) at every \( i \), \( \pi_i B_0 = 0 \) at every \( i \) (this follows from the definition of homomorphism between the Hausdorff spaces \( E_0 \) and \( B_0 \)).

On the other hand (see p.322 in [44]), if

\[
\pi_k E_0 = \pi_{k-1} E_0 = 0,
\]

then

\[
\pi_k B_0 = \pi_{k-1} G_0^\infty.
\]

This is the result of the statement that (p.454 in [44]) the homotopic sequence of some fibre bundle with the linear-connected total space \( E \) and the base \( B \):

\[
\pi_1 E = \pi_1 B = 0,
\]
having the form
\[ \cdots \to \pi_{n+1}B \xrightarrow{\partial} \pi_nF \xrightarrow{i_*} \pi_nE \xrightarrow{p_*} \pi_nB \to \cdots \] (4.46)

(where \( F \) is the fibre of the considered fibre bundle), is exact.

It follows from (4.44) that the section exists if and only if all the homotopy groups of the space \( G_0^\infty \) are trivial. If the manifold \( M \) is topological equivalent to the sphere \( S^n \), the homotopy group \( \pi_kG_0^\infty \) is isomorphic to the group \( \pi_{k+n}G \). Really, in this case we can identify \( G_0^\infty \) with the space of maps of the cube \( I^n \) in \( G \) transferred all the bound \( \dot{I}^n \) of the cube \( I^n \) in the unit element of the group \( G \) (it follows from the remark that the sphere \( S^n \) is obtained from \( I^n \) by the deformation of the whole bound at a point). The elements of the homotopy group \( \pi_kG_0^\infty \) can be represented as homotopic classes of maps of the cube \( I^k \) in \( G_0^\infty \) transferred its bound \( \dot{I}^k \) in the unit element of the group \( G_\infty \). This map associates a function \( g_\nu(x) \) with the values in \( G \) defined on \( I^k \) and satisfying the condition \( g_\nu = 1 \) if \( \nu \in \dot{I}^k; x \in I^n \), or \( \nu \in I^k; x \in \dot{I}^n \). Considering the pair \( (x, \nu) \in I^n \times I^k \) as a point of the cube \( I^{n+k} \), we see that the maps \( I^k \to G_0^\infty \) is in an one-to-one correspondence with the maps \( I^{n+k} \to G \) transferring all its bound,

\[ \dot{I}^{n+k} = \dot{I}^n \times I^k \bigcup I^n \times \dot{I}^k, \] (4.47)
in the unit element of the group \( G \). The homotopic classes of maps of \( I^{n+k} \to G \), by definition, form the group \( \pi_{n+k}G \). This means that \( \pi_kG_0^\infty \) is isomorphic to \( \pi_{n+k}G \).

If \( G \) is a compact non-Abelian group, we can prove that it is impossible to choose, in a continuous way, the one gauge field from every orbit of \( G_0^\infty \) on \( S^n, n > 0 \), i.e. it is impossible to fix a gauge removing completely the gauge freedom.

In the case of \( S^3 \) and if \( G = SU(2) \) this follows immediately from the relation

\[ \pi_0G_0^\infty = \pi_3G = \pi_3SU(2) = \mathbb{Z} \] (4.48)

(see (2.28)). The general proof (for an arbitrary compact non-Abelian group) is to check that a compact non-Abelian group \( G \) has nonzero homotopy subgroups in arbitrary dimensions.

The proved results are connected with the problem of the continuous gauge fixation on the orbits of \( G_0^\infty \). However, it is easy to study the question about the continuous gauge fixation on the orbits of \( G^\infty \), utilising the same methods. The groups \( G^\infty \) act not free on \( \mathcal{E}_0 \) in the general case. But, removing the fields with the holonomy group \( H \) which do not coincide with \( G \) we obtain the subspace of fields of a common position \( \mathcal{E}_0' \), on which \( G^\infty \) acts free. The removal of the submanifold given by the infinite number of the equations does not influence homotopic groups, therefore

\[ \pi_i\mathcal{E}_0' = 0, \quad i \geq 0. \] (4.49)

This allows us to ascertain the absence of any section of the fibre bundle with the total space \( \mathcal{E}_0' \) in the case when gauge fields are defined on \( S^n, n > 0 \), and take theirs values in the Lie algebra of the compact non-Abelian group \( G \).

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The gauge fields on the Minkowski space, by the condition of the spatial asymptotic of the (4.13) type, can be considered as those defined on $S^n, n > 0$. This is the cause of the Gribov ambiguities in the non-Abelian case.

We considered above the case of topological trivial gauge fields defining the connection of the trivial principal fibre bundle $\zeta$. All these results can be extended to the case of topological nontrivial gauge fields (i.e. to connections of principal fibre bundles). For example, one can consider, instead of $\mathcal{E}_0$, the space $\mathcal{E}_n$ of gauge fields with the topological number $n$ on the sphere $S^4$. The space $\mathcal{E}_n$ can also be deformed at a point (this deformation is defined as $A_t = (1 - t)A + tA^{(0)}$).

Such is the origin of the Gribov ambiguities.

One can show [35, 39] that the Gribov equation (4.34), together with the topological condition

$$X[\Phi^{(n)}] = n,$$

are compatible with the unique solution of the classical equations. It is just the Wu-Yang monopole considered before. The nontrivial solution of the equation for the Gribov phase (4.34) in this case is well-known:

$$\hat{\Phi}_0^{(n)} = -i\pi \frac{\tau^a x_a}{r} f_B^{BPS}(r), \quad f_B^{BPS}(r) = \left[ \frac{1}{\tanh(r/\epsilon)} - \frac{1}{r} \right].$$

It is the Bogomol’nyi-Prasad-Sommerfeld (BPS) monopole [16]. Note here the very important detail. The Gribov phase (4.51) is nothing else than the $SU(2)$ Higgs isoscalar (compare with the formula (23) in the paper [35]).

Thus, instead of the topological degenerated classical vacuum for the instanton theory (in the physically unattainable region), we have the topological degenerated Wu-Yang monopole:

$$\hat{\Phi}_i^{(n)} := v^{(n)}(x) [\hat{\Phi}_0^{(0)} + \partial_i] v^{(n)}(x)^{-1}, \quad v^{(n)}(x) = \exp[n\hat{\Phi}_0(x)],$$

and the topological degenerated multipoles:

$$\hat{A}^{(n)} := v^{(n)}(x) \hat{A}^{(0)} v^{(n)}(x)^{-1}.$$
But the solution (4.55) cannot be removed from the constraint-shell action $W^* = \int dt \tilde{N}^2 I/2 + ...$ and from the winding number $X[A^{(N)}] = N + X[A^{(0)}]$. Finally, we obtain the Feynman path integral

$$Z_F = \int DN \prod_{i,c} [DE_i^{(0)} DA_i^{(0)}] e^{iW^*}$$

(4.57)

that contains the additional topological variable.

We consider the derivation of the integral (4.57) in the next sections.

### 4.4 Topological dynamics and chromo-electric monopole.

The repetition of the Dirac definition of observable variables in QED allowed us to determine vacuum fields and phases of their topological degeneration in the form of Gribov copies of the constraint-shell gauge.

The degeneration of initial data is an evidence of the zero mode of the Gauss law constraint. In the lowest order of the considered perturbation theory the constraint (4.54) has the solution (4.55) with the electric monopole

$$F_{b0} = \tilde{N}(t) D_{i}^{(0)} (\Phi_k(0)) \Phi_{0c}(x).$$

(4.58)

We call this new variable $N(t)$ - the winding number variable, and it is defined by the vacuum Chern-Simons functional, which is equal to the difference of the in and out values of this variable:

$$\nu[A_0, \Phi^{(0)}] = \frac{g^2}{16 \pi^2} \int_{t_{in}}^{t_{out}} dt \int d^3 x F_{\mu \nu} \tilde{F}^{\mu \nu} = \frac{\alpha_s}{2\pi} \int d^3 x F_{b0} B_{b}^{(0)}(\Phi_0) [N(t_{out}) - N(t_{in})]$$

$$= N(t_{out}) - N(t_{in}).$$

(4.59)

The winding number functional admits its generalization to the noninteger degrees of the map [34]:

$$X[\Phi^{(N)}] \neq n(n \in \mathbb{Z}), \quad (\tilde{\Phi}^{(N)} = e^{N\Phi_0} [\tilde{\Phi}_k^{(0)} + \partial_i] e^{-N\Phi_0}).$$

(4.60)

Thus, we can identify the global variable $N(t)$ with the winding number of degrees of freedom in the Minkowski space described by the free rotator action

$$W_N = \int d^4 x \frac{1}{2} (F_{b0}^c)^2 = \int dt \tilde{N}^2 I/2,$$

(4.61)

where the momentum of rotator (see also (4.6) in [39]):

$$I = \int_V d^3 x (D_{i}^{ac} (\Phi_k(0)) \Phi_{0c})^2 = \frac{4 \pi^2 \epsilon}{\alpha_s} = \frac{4 \pi^2}{\alpha_s} \frac{1}{V < B^2 >},$$

(4.62)

does not contribute to the local equations of motion. The free rotator action disappears in the limit $V \to \infty$. 

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Now, with account of the evaluation of the magnetic energy, (4.21), we can write down the action of the YM theory in the Minkowski space in the lowest order. This action contains as an "electric" as a "magnetic" BPS monopoles:

\[
W_{Z}[N, \Phi^{0BPS}] = \int dtd^{3}x \frac{1}{2}\{[F_{\mu}^{b}]^{2} - [B_{\mu}^{b}(\Phi^{0BPS})]^{2}\} = \int dt \frac{1}{2}\{IN^{2} - \frac{4\pi}{g^{2}\epsilon}\}. \quad (4.63)
\]

The topological degeneration of all the fields reduces to the degeneration of only one global topological variable \(N(t)\) with respect to the shift of this variable on integers: \(N \mapsto N + n, n = \pm 1, \pm 2, \ldots; 0 \leq N(t) \leq 1\). Thus, the topological variable \(N(t)\) determines the free rotator with the instanton-type wave function (4.9) of the topological motion in the Minkowski space-time:

\[
\Psi_{N} = \exp(iP_{N}N), \quad P_{N} = \dot{N}I = 2\pi k + \theta, \quad (4.64)
\]

where \(k\) is the number of the Brillouin zone, and \(\theta\) is the \(\theta\)-angle (or the Bloch quasi-momentum) \([13]\). The action (4.63) of the YM theory in the Minkowski space in the lowest order induces the corresponding Hamiltonian (in terms of the canonical momentum \(P_{N} = NI\)):

\[
H = \frac{2\pi}{g^{2}\epsilon}[P_{N}^{2}(\frac{g^{2}}{8\pi^{2}})^{2} + 1]. \quad (4.65)
\]

In contrast to the instanton wave function (4.9), the spectrum of the topological momentum is real and belongs to the physical values. Finally, the equations (4.61), (4.64) determine the countable spectrum of the global electric tension (4.58):

\[
F_{i0}^{b} = \dot{N}[D_{i}(\Phi^{0})A_{0}]^{b} = \alpha_{s}(\frac{\theta}{2\pi} + k)B_{i}^{b}(\Phi^{0}). \quad (4.66)
\]

It is an analogue of the Coleman spectrum of the electric tension in the \(QED_{(1+1)}\) \([13]\):

\[
G_{10} = N\frac{2\pi}{e} = e(\frac{\theta}{2\pi} + k). \quad (4.67)
\]

The application of the Dirac quantization to the 1-dimensional electrodynamics \(QED_{(1+1)}\) in the paper \([46]\) demonstrates the universality of the Dirac variables and their adequacy to the description of the topological dynamics in terms of a nontrivial homotopy group.

5 Zero mode of Gauss Law.

5.1 Dirac variables and zero mode of Gauss Law.

The constraint-shell theory is obtained by the explicit resolution of the Gauss law constraint (3.5), and our next step is connected with the initial action on the surface of these solutions:

\[
W^{*} = W[A\mu]_{A\mu = 0}. \quad (5.1)
\]
The results of a similar solution in QED are electrostatic and Coulomb-like atoms. In the non-Abelian case the topological degeneration in the form of Gribov copies means that the general solution of the Gauss law constraint (3.5) contains the zero mode $Z$. The general solution of the inhomogeneous equation (3.5) is the sum of the zero-mode solution $Z^a$ of the homogeneous equation,

$$(D^2(A))^{ab}Z_b = 0,$$  
and a particular solution, $\tilde{A}_0^a$, of the inhomogeneous one:

$$A_0^a = Z^a + \tilde{A}_0^a.$$  

The zero-mode $Z^a$, at the spatial infinity, can be represented as a form of the sum of the product of the new topological variable $\dot{N}(t)$, Gribov phase $\Phi(0)(x)$ and weak multipole corrections:

$$\hat{Z}(t, x)|_{\text{asymptotic}} = \dot{N}(t)\hat{\Phi}(0)(x) + O\left(\frac{1}{r^{l+1}}\right), \quad (l > 1).$$  

In this case the single one-parametric variable $N(t)$ reproduces the topological degeneration of all the field variables, if the Dirac variables are defined by the gauge transformations

$$0 = U_Z(\hat{Z} + \partial_0)U_Z^{-1}$$  

((39, the formula (2.9)),

$$\hat{A}_i = U_Z(\hat{A}_i^0 + \partial_i)U_Z^{-1}, \quad \hat{A}_i^{(0)} = \Phi^{(0)}_i + \bar{A}_i^{(0)},$$  

where the spatial asymptotic of $U_Z$ is ((2.14) in [39])

$$U_Z = T \exp\left[\int^t dt' \hat{Z}(t', x)\right]|_{\text{asymptotic}} = \exp[N(t)\hat{\Phi}(0)(x)].$$  

The topological degeneration of all the fields reduces to the degeneration of only one global topological variable $N(t)$ with respect to the shift of this variable on integers: $(N \implies N + n, n = \pm 1, \pm 2, ...)$.

### 5.2 Constraining with zero mode.

Let us formulate the equivalent unconstrained system for the YM theory in the monopole class of functions in the presence of the zero mode $Z^b$ of the Gauss law constraint:

$$A_0^a = Z^a + \tilde{A}_0^a, \quad F_{0k}^a = -D_k^{ab}(A)Z_b + \tilde{F}_{0k}^a \quad ((D^2(A))^{ab}Z_b = 0).$$  

To obtain the constraint-shell action:

$$W_{YM}(\text{constraint}) = W_{YM}[Z] + \tilde{W}_{YM}[\tilde{F}],$$  

we use the obvious decomposition:

$$F^2 = (-D\hat{Z} + \tilde{F})^2 = (D\hat{Z})^2 - 2D\hat{Z}\tilde{F} + \tilde{F}^2 = \partial(\hat{Z} \cdot D\hat{Z}) - 2\partial(\hat{Z}\tilde{F}) + (\tilde{F})^2.$$

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The latter relation is true because of the Bianchi identity $D\tilde{F} = 0$, the Gribov equation $D^2Z = 0$ and the explicit expression for the derivative $DZ$: $DZ = (\partial Z + gA \times Z)$. This shows that the zero mode part, $W_{YM}[Z]$, of the constraint-shell action (5.9) is the sum of the two integrals:

$$W_{YM}[Z] = \int dt \int d^3x \left[ \frac{1}{2} \partial_i (Z^a D_{ab}^i (A) Z^b) - \partial_i (\tilde{F}_{0i}^a Z_a) \right] = W^0 + W',$$ (5.11)

where the first term, $W^0$, is the action of a free rotator, and the second one, $W'$, describes the coupling of the zero-mode to local excitations. These terms are determined by the asymptotic of fields $(Z^a, \Pi^a)$ at the spatial infinity: (5.4), (4.13). We denote them as $\tilde{N}(t)\Phi_a^0(x), \Phi^a(x)$. The fluctuations $\tilde{F}_{0i}^a$ belong to the class of multipoles. Since the integral over monopole-multipole couplings vanishes (the Gauss-Ostrogradsky theorem and the asymptotic (5.4)), the fluctuation part of the second term drops out. The substitution of the solution with the asymptotic (5.4)) into the first term of the eq. (5.11) leads to the zero-mode action (4.61).

The action for the equivalent unconstrained system of local excitations (compare with the formula (21) in [19]):

$$\tilde{W}_{YM}[\tilde{F}] = \int d^4x \left\{ E^a_i \tilde{A}^{(0)}_a - \frac{1}{2} \left\{ E^2_k + B^2_k(A^{(0)}) + [D^a_k(\Phi^{(0)})\tilde{\sigma}_b]^2 \right\} \right\},$$ (5.12)

is obtained in terms of variables with the zero degree of the map:

$$\hat{F}_{0k} = UZ \tilde{F}_{0k}^0 U^{-1}_Z, \quad \hat{A}_i = UZ (\tilde{A}_i^{(0)} + \partial_i) U^{-1}_Z, \quad \hat{A}_i^{(0)}(t, x) = \Phi_i^{(0)}(t, x) + \hat{\tilde{A}}_i^{(0)}(t, x),$$ (5.13)

by the decomposing of the electrical components of the field strength tensor $F_{0i}^{(0)}$ into their transverse: $E^a_i$, and longitudinal: $F_{0i}^{(0)} = -D^a_i(\Phi^{(0)})\tilde{\sigma}_b$, parts, so that

$$F_{0i}^{(0)} = E^a_i - D^a_i(\Phi^{(0)})\tilde{\sigma}_b.$$ (5.14)

Here the function $\tilde{\sigma}_b$ is determined from the Gauss equation

$$((D^2(\Phi^{(0)}))^{ab} + g\epsilon^{abc} \tilde{A}_{ac}^{(0)} D^b(\Phi^{(0)}))\tilde{\sigma}_b = -g\epsilon^{abc} \tilde{A}_b^{(0)} E^i_c$$ (5.15)

(we can recommend our reader the monograph [24], p.88, where the formulas (5.14),(5.13) were derived in the Hamiltonian formalism of the YM theory).

If we introduce the current $j$ of independent non-Abelian variables:

$$j^a_0 = g\epsilon^{abc} [A_{ib} - \Phi^{(0)}_i] E^i_c,$$ (5.16)

the eq. (5.13) can be rewritten as

$$D^a_i(A) D^b_{ij}(\Phi^{(0)})\tilde{\sigma}^{ab} = j_0^c.$$ (5.17)

The latter equation depends in fact on the zero mode $\hat{Z}$ described in the previous subsection.

Due to the gauge-invariance, the dependence of the action for local excitations on the zero mode disappears, and we get the ordinary generalization of the covariant Coulomb gauge [21, 22, 23] in the presence of the Wu-Yang monopole.
6 Rising potential induced by monopole.

Now we can calculate the Green function of the Gauss equation (4.34) (see §4.C):

\[ D^2((\Phi^{(0)})^{ab}(x)G^c_{cb}(x,y)) = \delta^{ac}\delta^3(x-y) \]  

(6.1)

(it is the Green function of the equation (5.17) simultaneously), that forms the potential of the current-current instantaneous interaction:

\[-\frac{1}{2} \int_{V_0} d^3x d^3y j^b_0(x)G_{bc}(x,y)j^c_0(y). \]  

(6.2)

In the presence of the Wu-Yang monopole we have

\[ D^2((\Phi^{(0)})^{ab}(x) = \delta^{ab}\Delta - \frac{n^a n^b + \delta^{ab}}{r^2} + 2\left(\frac{n^a}{r}\partial^b - \frac{n^b}{r}\partial^a\right), \]  

(6.3)

where \( n_a(x) = x_a/r; \ r = |x| \). Let us decompose \( G^{ab} \) into the complete set of orthogonal vectors in the colour space:

\[ G^{ab}(x, y) = [n^a(x)n^b(y)V_0(z) + \sum_{\alpha=1,2} e^a_{\alpha}(x)e^{b\alpha}(y)V_1(z)]; \ (z = |x - y|). \]  

(6.4)

Substituting the latter into the first equation, we get the Euler equation (see §4.C, the equation (2.160)):

\[ \frac{d^2}{dz^2}V_n + \frac{2}{z}\frac{d}{dz}V_n - \frac{n}{z^2}V_n = 0, \quad n = 0, 1. \]  

(6.5)

The general solution for the latter equation is

\[ V_n(|x - y|) = d_n|x - y|^{l_1^n} + c_n|x - y|^{l_2^n}, \quad n = 0, 1, \]  

(6.6)

where \( d_n, c_n \) are constants, and \( l_1^n, l_2^n \) can be found as roots of the equation \( l^{n2} + l^n = n \), i.e.

\[ l_1^n = -\frac{1 + \sqrt{1 + 4n}}{2}; \quad l_2^n = -\frac{1 + \sqrt{1 + 4n}}{2}. \]  

(6.7)

It is easy to see that for \( n = 0 \) at \( d_0 = -1/4\pi \) we get the Coulomb-type potential:

\[ l_1^0 = -\frac{1 + \sqrt{1}}{2} = -1; \quad l_2^0 = -\frac{1 + \sqrt{1}}{2} = 0, \]  

(6.8)

\[ V_0(|x - y|) = -1/4\pi|x - y|^{-1} + c_0; \]  

(6.9)

and for \( n = 1 \), the ”golden section” potential with

\[ l_1^1 = -\frac{1 + \sqrt{5}}{2} \approx -1.618; \quad l_2^1 = -\frac{1 + \sqrt{5}}{2} \approx 0.618, \]  

(6.10)

\[ V_1(|x - y|) = -d_1|x - y|^{-1.618} + c_1|x - y|^{0.618}. \]  

(6.11)

The latter potential (in contrast with the Coulomb-type one) can lead to the rearrangement of the naive perturbation series and to the spontaneous chiral symmetry break-down. This potential can be considered as the origin of ”hadronization” of quarks and gluons in QCD [39, 48].
7 Feynman and FP path integrals.

The Feynman path integral over independent variables includes the integration over the topological variable $N(t)$:

$$Z_F[J] = \int \prod_t dN(t) \tilde{Z}[J^U], \quad (7.1)$$

where

$$\tilde{Z}[J^U] = \int \prod_{t,x} \prod_{a=1}^3 \left\{ \frac{d^2 A_a^{(0)}(t) d^2 E_a^{(0)}(t)}{2\pi} \right\} \exp \left\{ i \left\{ \mathcal{W}_{YM}(Z) + \tilde{W}_{YM}(A_a^{(0)}) + S[J^U] \right\} \right\}. \quad (7.2)$$

As we have seen above, functionals $\tilde{W}$, $S$ are given in terms of variables containing non-perturbation phase factors $U = U_Z$, (5.7), of the topological degeneration of initial data. These factors disappear in the action $\tilde{W}$, but not in the source term:

$$S[J^U] = \int d^4x J_i^a \bar{A}_i^a, \quad \bar{A}_i = U(\hat{A}^{(0)}) U^{-1}, \quad (7.3)$$

which reflects the fact of the topological degeneration of physical fields. In general, the phase factors $U_Z$, as a relic of the fundamental quantization, remember all the information about the frame of reference, monopoles, rising potential of the instant interaction and other initial data, including their topological degeneration and confinement (see farther).

The constraint-shell formulation distinguishes the bare" gluon", as a weak deviation of the monopole with the index $n = 0$, and the observable (physical) "gluon" averaged over the topological degeneration (i.e., the Gribov copies) [29]:

$$\bar{A}_{phys} = \lim_{L \to \infty} \frac{1}{2L} \sum_{n=-L}^{n=L} A^{(n)}(x) \sim \delta_{r,0}; \quad (7.4)$$

whereas in QED the constraint-shell field is a transverse photon. A more detailed analysis of the latter formula will be conducted in Section 8.

We can say that the Dirac variables with the topological degeneration of initial states in the non-Abelian theory determine the physical origin of hadronization and confinement as non-local monopole effects. The Dirac variables distinguish the unique gauge. In QED it is the Coulomb gauge; whereas in the YM theory it is the covariant generalization of the covariant Coulomb gauge in the presence of the monopole.

If we pass to other gauges of physical sources at the level of the FP integral in relativistic gauges, all the monopole effects of the degeneration and rising potential can be lost (as the Coulomb potential is lost in QED in relativistic invariant gauges). Recall that to prove the equivalence of the Feynman integral to the Faddeev-Popov integral in an arbitrary gauge, we ([19], § 2.5) change variables and concentrate all the monopole effects in phase factors ([19], the formula (40)) before physical sources. The change of sources removes all these effects (see [19], § 2.3).
The change of sources was possible in the Abelian theory only for scattering amplitudes \[22\] in neighbourhoods of poles of their Green functions when all the particle-like excitations of fields are on theirs mass-shells (we recommend our reader to understand this fact with the example of electron propagators). However, for the cases of non-local bound states and other phenomena where these fields are \textit{off their mass-shell}, the Faddeev theorem about the equivalence of the different \textquotedblleft gauges\textquotedblright{} (see for example (7.23) in \[23\]) is not valid.

\section{Free rotator: topological confinement.}

The topology can be an origin of the colour confinement as the complete destructive interference of the phase factors of the topological degeneration of initial data.

The mechanical analogy of the topological degeneration of initial data is the free rotator \(N(t)\) with the action of a free particle (compare with \(1.61\))

\[ W(N_{\text{out}}, N_{\text{in}}|t_1) = \int_0^{t_1} dt \frac{\dot{N}^2}{2} I, \quad p = \dot{N} I, \quad H_0 = \frac{p^2}{2I} \tag{8.1} \]

given on the ring, where the points \(N + n (n \in \mathbb{Z})\) are physically equivalent (see \(4.64\)). Instead of initial data \(N(t = 0) = N_{\text{in}}\) in mechanics in the space with a trivial topology, the observer of the rotator has a manifold of initial data \(N^{(n)}(t = 0) = N_{\text{in}} + n; \quad n = 0, \pm 1, \pm 2, ...\)

The observer does not know where is the rotator. It can be at points \(N_{\text{in}}, \quad N_{\text{in}} \pm 1, N_{\text{in}} \pm 2, ...\). Therefore, he should \textit{average} the wave function (compare with \(7.4\)):

\[ \Psi(N) = e^{ipN}, \tag{8.2} \]

over all the values of the topological degeneration with the \(\theta\)-angle measure: \(\exp(in\theta)\). As a result, we obtain the wave function

\[ \Psi(N)_{\text{observable}} = \lim_{L \to \infty} \frac{1}{2L} \sum_{n=-L}^{n=L} e^{i\theta} \Psi(N + n) = \exp\{i(2\pi k + \theta)N\}, \quad k \in \mathbb{Z}. \tag{8.3} \]

In the opposite case, \(p \neq 2\pi k + \theta\), the corresponding wave function (i.e. \textit{the probability amplitude}) disappears: \(\Psi(N)_{\text{observable}} = 0, \text{ due to the complete destructive interference.}\)

The consequence of this topological degeneration is that the part of values of momentum spectrum becomes \textit{unobservable} in comparison with a trivial topology.

This fact can be treated as \textit{confinement} of those values which do not coincide with

\[ p_k = 2\pi k + \theta, \quad 0 \leq \theta \leq \pi. \tag{8.4} \]

The observable spectrum follows also from the constraint of equivalence of the points \(N\) and \(N + 1:\)

\[ \Psi(N) = e^{-i\theta} \Psi(N + 1), \quad \Psi(N) = e^{i\pi N}. \tag{8.5} \]
(the $\theta$-angle is an eigenvector of the gauge transformation $T_1|\theta> = e^{i\theta}|\theta>$ corresponding to the rise of the topological number on unit: $T_1|n> = |n+1>$. This theory is valid both for the Euclidean and Minkowski spaces).

As a result, we obtain the spectral decomposition of the Green function of the free rotator (8.1) (as the probability amplitude of the transition from the point $N_{in}$ to $N_{out}$) over the observable values of spectrum (8.4):

$$G(N_{out}, N_{in}|t_1) = \langle N_{out} | \exp(-i\hat{H}t_1)|N_{in} > = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \exp \left[ -\frac{p_k^2}{2t_1} + ip_k(N_{out} - N_{in}) \right].$$

Using the connection with the Jacobian theta-functions [49]:

$$\Theta_3(Z|\tau) = \sum_{k=-\infty}^{+\infty} \exp[i\pi k^2 \tau + 2ikZ] = (-i\tau)^{-1/2} \exp[\frac{Z^2}{i\pi \tau}] \Theta_3(\frac{Z}{\tau} - \frac{1}{\tau}),$$

we can represent the expression (8.6) as the sum over all the paths:

$$G(N_{out}, N_{in}|t_1) = \sqrt{\frac{I}{4\pi it_1}} \sum_{n=-\infty}^{n=\infty} \exp[i\theta n] \exp[+iW(N_{out}, N_{in}|t_1)],$$

where

$$W(N_{out} + n, N_{in}|t_1) = \frac{(N_{out} + n - N_{in})^2 I}{2t_1}$$

is the rotator action (8.1).

9 Confinement as a destructive interference.

The topological confinement similar to the complete destructive interference of the phase factors of the topological degeneration (i.e., to a pure quantum effect) can be in the "classical non-Abelian field theory". Recall that, at the time of the first paper of Dirac [42], the so-called "classical relativistic field theories" were found in the papers of Schrödinger, Fock, Klein, Weyl [50, 51] as types of relativistic quantum mechanics, i.e., as results of the primary quantization. The phases of the gauge transformations were introduced by Weyl [51] as pure quantum magnitudes.

The free rotator theory shows that the topological degeneration can be removed if all the Green functions are averaged over the values of the topological variable and all the possible angles of orientation of the monopole unit vector $n$ in the group space (instead of the instanton averaging over interpolations between different vacua in the Euclidean space).

The averaging over all the parameters of degeneration can lead to the complete destructive interference of all the colour amplitudes [29, 30, 31]. In this case only colourless ("hadronic") states form the complete set of physical states. Using the example of a free
rotator, we see that the disappearance of the part of physical states due to the confinement does not violate the composition law for a Green function:

\[ G_{ij}(t_1, t_3) = \sum_h G_{ih}(t_1, t_2) G_{hj}(t_2, t_3), \]  

(9.1)

defined as the probability amplitude to find the system with the Hamiltonian \( H \) in the state \( j \) at the time \( t_3 \) if, at the time \( t_1 \), this system was in the state \( i \), where \( (i; j) \) belong to the complete set of all the states \( \{h\} \):

\[ G_{ij}(t_1, t_3) = |i\rangle \exp(-i \int_{t_1}^{t_3} H \rangle |j \rangle. \]  

(9.2)

A particular case of this composition law (9.1) is unitarity of the S-matrix:

\[ SS^+ = I \implies \sum_h <i|S|h><h|S^+|j> = <i|j>, \]  

(9.3)

known as the law of the probability conservation for the S-matrix elements \((S = I + iT)\), where

\[ \sum_h <i|T|h><h|T^*|j> = 2 \text{Im} <i|T|j> \]  

(9.4)

(compare with (64.2), (71.2) in [52]). The left-hand side of this law is similar to the spectral series of the free rotator (8.6).

The destructive interference keeps only the colourless "hadronic" states. Whereas the right-hand side of this law, far from resonances, can be represented by the perturbation series over the Feynman diagrams that follow from the Hamiltonian. Due to the gauge invariance, \( H[A^{(n)}, q^{(n)}] = H[A^{(0)}, q^{(0)}] \), where \( q \) are the fermion (quark) degrees of freedom. This means that the Hamiltonian \( H[A^{(0)}, q^{(0)}] \) depends on the Gribov phase in its BPS monopole form (4.51) (the Gribov phase (4.51) is a colour scalar), but it does not depend on the Gribov phase factors (4.52). The considered above holonomy theory (4.33)-(4.39) allow us to draw the conclusion that the colour confinement in the considered YM theory, with the Gribov equation (4.34) and its vacuum solution (4.51), is determined by the restricted holonomy group \( \Phi^0 \) generated by the zero topological sector of this YM theory (more precisely, by the YM fields of this sector satisfied the Coulomb gauge (4.30)). We can interpret this as a confinement criterion in QCD (which is also true for the gluonic theory with the \( SU(3)_{col} \rightarrow SU(2) \) spontaneous breakdown).

Thus, Hamiltonian \( H \) contains the perturbation series in terms only of zero degree of the map fields (i.e., in terms of constituent colour particles) that can be identified with Feynman partons. The Feynman path integral as the generating functional of this perturbation series is an analogue of the sum over all the paths of the free rotator (8.8).

Therefore, confinement, in the spirit of the complete destructive interference of colour amplitudes \([19, 23, 31]\), and the law of the probability conservation for the S-matrix elements, (9.4), lead to the Feynman quark-hadronic duality that is the base of all the partonic models \([53]\) and the QCD applications \([54]\). The quark-hadronic duality gives a
method of the direct experimental measurement of quark and gluon quantum numbers from a deep inelastic scattering cross-section \[53\]. For example, according to Particle Data Group, the ratio of the sum of the probabilities of the \(\tau\)-decay hadronic modes to the probability of the \(\tau\)-decay muonic mode is

\[
\sum_{h} w_{\tau \to h} / w_{\tau \to \mu} = 3.3 \pm 0.3. \tag{9.5}
\]

This is the left-hand side of the Eq.(9.4) normalized to the value of the leptonic mode probability of the \(\tau\)-decay. On the right-hand side of the Eq.(9.4) we have the ratio of the imaginary part of the sum over the quark-gluonic diagrams (in terms of constituent fields free from any Gribov phase factor) to the one of the leptonic diagrams. In the lowest order of QCD perturbation on the right-hand side we get the number of colours \(N_c\), therefore

\[
3.3 \pm 0.3 = N_c. \tag{9.6}
\]

Thus, the degeneration of initial data can explain us not only ”why we do not see quarks”, but also ”why we can measure their quantum numbers”. This mechanism of confinement, due to a quantum interference of the phase factors of the topological degeneration, disappears after a change of the ”physical” sources: \(A^* J^* \rightarrow AJ\), called the transition to another gauge in the gauge-fixing method. Then, for example, the Coulomb gauge \((4.30)\) is not valid. The Gribov ambiguity equation \((4.34)\), which describes the ambiguity in the choice of the YM fields satisfied the Coulomb gauge \((4.30)\) (having the Gribov phase \((4.51)\) as its solution), turns then in some formal differential equation, without of any physical sense. The restricted holonomy group \(\Phi^0\), constructed on the transverse YM fields satisfied the Coulomb gauge \((4.30)\), becomes trivial in this case. This means, in turn, that the confinement criterion, considered above, is not valid also.

Instead of the hadronization and confinement, we obtain then the scattering amplitudes of the free partons only. But these amplitudes do not exist as physical observable in the Dirac quantization scheme, which depends on initial data.

10 The U(1)-problem.

The value of the vacuum chromo-magnetic field \(< B^2 > \) can be estimated by the description of a process with an anomaly. The simplest process of such type is the of a pseudo-scalar bound state with an anomaly. In gauge theories there is the universal effective action for the description of this interaction:

\[
W_{eff} = \int dt \left\{ \frac{1}{2} \left( \eta_M^2 - M_P^2 \eta_M^2 \right) V + C_M \eta_P \dot{X} [A^{(N)}] \right\}, \tag{10.1}
\]

where \(\eta_M\) is a bound state with the mass \(M_P\) in its rest frame of reference, and \(X[A^{(N)}]\) is the topological ”winding number” functional. In 3-dimensional QED\(_{(3+1)}\) this action,
with the constant $\[19\]

$$C_M = C_{\text{positronium}} = \frac{\sqrt{2}}{m_e} \frac{8\pi^2}{\left(\frac{\psi_{\text{Sch}}(0)}{m_e^{3/2}}\right)} ,$$ \hspace{1cm} (10.2)$$

describes the decay of a positronium $\eta_M = \eta_P$ into two photons that are in the "winding number" functional

$$\hat{X}_{\text{QED}}[A] = \frac{e^2}{16\pi^2} \int d^3xF_{\mu\nu}^* F^{\mu\nu} \equiv \frac{e^2}{8\pi^2} \int d^3x \varepsilon_{ijk} A^i (\partial^k A^j - \partial^j A^k) .$$ \hspace{1cm} (10.3)$$

In 1-dimensional QED$_{(1+1)}$ this action (10.1), with the constant $C_M = 2\sqrt{\pi}$ and the "winding number" functional

$$\hat{X}_{\text{QED}}(A^{(N)}) = \frac{e}{4\pi} \int_{-V/2}^{V/2} dx F_{\mu\nu} e^{\mu\nu} = \hat{N}(t) \Rightarrow F_{01} = \frac{2\pi \hat{N}}{eV} ,$$ \hspace{1cm} (10.4)$$
describes the mass of the Schwinger bound state $\eta_P = \eta_{\text{Sch}}$, if the action (10.1) is added by the action of the Coleman electric field [30, 46]:

$$W_{\text{QED}} = \frac{1}{2} \int d^4x F_{\mu\nu}^2 = \frac{1}{2} \int dt \hat{N}^2 I_{\text{QED}} / 2 ,$$ \hspace{1cm} (10.5)$$

where

$$I_{\text{QED}} = \left(\frac{2\pi}{e}\right)^2 \frac{1}{V} .$$ \hspace{1cm} (10.6)$$

It is easy to see that the diagonalization of the total Lagrangian of the

$$L = [\frac{\hat{N}^2 I}{2} + C_M \eta_M \hat{N}] = [\frac{(\hat{N} + C_M \eta_M / I)^2 I}{2} - \frac{C_M^2 \eta_M^2 V}{2IV}]$$ \hspace{1cm} (10.7)$$
type leads to the mass of a pseudo-scalar meson in QED$_{(1+1)}$:

$$\Delta M^2 = \frac{C_M^2}{IV} = \frac{e^2}{\pi} .$$ \hspace{1cm} (10.8)$$

In QCD$_{(3+1)}$ a similar action for a pseudo-scalar meson $\eta_M = \eta_0$ was proposed in [54], where the "winding number" functional was given by

$$\hat{X}_{\text{QCD}}[A^{(N)}] = \frac{g^2}{16\pi^2} \int d^3xF_{\mu\nu}^a F_{\mu\nu}^a = \hat{N}(t) + \hat{X}[A^{(0)}]$$ \hspace{1cm} (10.9)$$

and

$$C_M = C_\eta = \frac{N_f}{F_\pi} \sqrt{\frac{2}{\pi}} ; \hspace{0.5cm} (N_f = 3) .$$ \hspace{1cm} (10.10)$$
As we have seen, QCD\textsubscript{(3+1)} has the chromo-electric monopole:

\[ F_{0i} = \dot{N} D_i^a(\Phi) \Phi_0 = \dot{N} B_i^a(\Phi) \frac{2\pi}{\alpha_s V} < B^2 >, \]

with the normalization

\[ \frac{g^2}{8\pi^2} \int d^3 D_i^{ab}(\Phi) \Phi_0 B^b_i(\Phi) = 1. \]

The action (10.11) should be added by the action of the topological dynamics of the zero mode \( \dot{N} \):

\[ W_{QCD} = \frac{1}{2} \int dt \int_V d^3 x F_{0i}^2 = \int dt \dot{N}^2 I_{QCD}^2, \]

with the mass \( I_{QCD} \) determined by the vacuum magnetic field:

\[ I_{QCD} = \left( \frac{2\pi}{\alpha_s} \right)^2 \frac{1}{V < B^2 >}. \]

In QCD\textsubscript{(3+1)} the equation for the diagonalization, (10.7), leads to an additional mass of the \( \eta_0 \) meson:

\[ L_{eff} = \frac{1}{2} [\dot{\eta_0}^2 - \eta_0^2(t)(m_0^2 + \Delta m_0^2)] V, \]

\[ \Delta m_0^2 = \frac{C_\eta^2}{I_{QCD} V} = \frac{N_f^2 \alpha_s^2}{2\pi} < B^2 >. \]

This result allows us to estimate the value of the vacuum chromomagnetic field in QCD\textsubscript{(3+1)}:

\[ < B^2 > = \frac{2\pi^3 F_\pi^2 \Delta m_\eta^2}{N_f^2 \alpha_s^2} = \frac{0.06 GeV^4}{\alpha_s^2}. \]

(see also [56]). After the calculation we can remove the infrared regularization \( V \rightarrow \infty \).

## 11 Conclusion.

The main problems of the discussion of stable vacuum states in some non-Abelian theory are the classes of functions and singularities. These problems exist in all models of the QCD vacuum, including instantons described by the \( \delta \)-type singularities in the Euclidean space.

Mysteries of the nature are not only the actions and symmetries, but also the class of functions with finite energy densities used in quantum field theories (including QED) for the description of physical processes. If we explain any effect by these singularities, choosing a model of the nontrivial QCD vacuum, we should answer the questions: "Where are singularities of this vacuum from?" and "What is a physical origin of these singularities?".

We presented here the model of the vacuum in the Yang-Mills (YM) theory in the monopole class of functions with the finite energy density without any singularity in a
finite volume, as a consequence ("smile") of the scalar Higgs field that disappears (like the Cheshire cat) from the spectrum of physical excitations of the theory in the limit of the infinite spatial volume. In other words, we proved that there exists a mathematically correct model of the YM vacuum, with the finite physical energy-momentum spectrum in the Minkowski space, constructed from the well-known Bose condensate of the Higgs scalar field in the limit of its infinite mass.

The $SU(2)$ symmetry of the YM vacuum is broken down spontaneously. The breakdown $SU(2) \rightarrow U(1)$ is realized in the presence of the Higgs $SU(2)$ isovector. If the Higgs field goes to the statistical (vacuum) expectation value at the spatial infinity, this leads to the nontrivial topological structure of the remaining group of symmetry, $U(1)$, induced by the Higgs vacuum expectation value. This nontrivial topological structure means the presence of topological (magnetic) charges in this theory, i.e. the inevitability of the monopole configurations of the YM vacuum with finite energies.

We have considered our theory in the BPS limit when the self-interaction between the Higgs particles goes to zero. This allows us to consider the Higgs particles (in the limit of their infinite number, i.e. at the level of statistical physics) as an ideal gas. We imposed an additional condition of stationarity of this ideal gas (Bose condensate). This choice influences the stationary nature of the monopole configurations of the YM vacuum.

The Bogomol’nyi equation obtained issuing from the evaluation of the lowest bound of energy for the monopole solutions (the latter one depends on the vacuum expectation value $m/\sqrt{\lambda}$) allowed us to find the monopole configurations of the YM vacuum as Bogomol’nyi-Prasad-Sommerfeld (BPS) or Wu-Yang monopoles (obtained as infinite volume limits of BPS monopoles).

We described the topological degeneration of initial data for monopole solutions at nonzero values of the topological charge. This topological degeneration manifests itself as Gribov copies of the covariant Coulomb gauge considered as zero initial data for the Gauss law constraint. These Gribov copies are defined by a solution of the Gribov ambiguity equation (4.34) in the class of functions of the BPS monopole type for the Higgs vacuum field.

The Gribov equation (4.34) describes the correct cohomological structure of the YM vacuum at the spatial infinity. There exists an one-to-one correspondence between the set of cohomology classes of YM fields and the set of Gribov copies of the Coulomb gauge (4.30). This cohomological structure corresponds to the elements of the holonomy group $H$ constructed on the transverse YM fields. The unit element of the holonomy group $H$ is degenerated with respect to the class of exact 1-forms (with the zero topological charge) induced by the Coulomb gauge (4.30) and the Bogomol’nyi equation (2.37).

The Yang-Mills fields are considered as sums of vacuum fields (monopoles) and weak perturbation excitations over this vacuum (multipoles). We suppose that these excitations have the same topological numbers as the vacuum components.

The important point of our investigations is that the square of the Gibbs expectation value of the magnetic tension, $<B^2>$, is different from zero. This is a direct analogy of the *Meisner effect* in a superconductor. In the language of the group theory it means the spontaneous break-down of the $U(1)$ symmetry.
We proved that there is the continuous topological variable $N(t)$ defining the zero mode of the Gauss equation and depending on the time $t$; it plays the role of the non-integer degree of the map. The calculations led to the action of a free rotator with the rotation momentum $I$ depending on $<B^2>$ and real spectrum of momentum. This spectrum describes the rotation of the Yang-Mills vacuum (as a Bose condensate, depending on the vacuum expectation value $<B^2>$) as a whole system.

The considered nontrivial topological structure of the vacuum in the YM theory can be in other non-Abelian theories. For example, there is the spontaneous $SU(3)_{col} \rightarrow SU(2)$ break-down with the antisymmetric choice of the Gell-Mann matrices $\lambda_2, \lambda_5, \lambda_7$, which leads to the Wu-Yang monopole (see the formulas (3.24-3.25) in [39]). The essential point of the theory [39] was the mix of world and group indices in the construction of the Wu-Yang monopole. One can consider the behaviour of quarks in the Wu-Yang monopole field and to write down the Green function of a quark (see (4.9)-(4.13) in [39]).

Physical arguments in the favour of the considered theory of the physical vacuum are an additional mass of the $\eta'$-meson in QCD, the rising potential and topological confinement [19].

We calculated explicitly the hadronization potential $V_1$ as one of the components in the decomposition of the Green function of the Gauss (Gribov) equation in the presence of the Wu-Yang monopole by the complete set of orthogonal vectors in the colour space. This is a non-local monopole effect.

We proved that the topological confinement can lead to the colour confinement in QCD in the form of the complete destructive interference of the phase factors of the topological degeneration. This means that only the colourless ("hadronic") states can be treated as physical states.

The Hamiltonian of QCD depends only on the Gribov phase $l^0(\mu)$ as a colour isoscalar. As a result, the criterion of the colour confinement in QCD, in the Coulomb gauge $l^0(\mu)$, is the existence of the nontrivial restricted holonomy group $\Phi_0$ constructed on the transverse YM fields of the zero topological sector.

The Lorentz covariance can be carried out by the Lorentz rotation of the time axis $l^0(\mu)$ along the complete momentum of each of the physical states, i.e. by the transition to the frame of reference where the initial data and the spectrum of these states are measured [39].

All these "smiles" of the Higgs scalar field disappear if we replace the fundamental Dirac variables [42, 44, 58] and change the gauge of their physical sources in order to obtain the conventional Faddeev-Popov integral [11] as a realization of the Feynman heuristic quantization [24]. This change removes all the time axes of the physical states, all the initial data, with their degeneration and destructive interference, and all monopole effects, including instantaneous interactions forming non-local bound states of the types of atoms in QED or hadrons in QCD. In other words, the "smiles" of the Higgs field show us the limitedness of the Faddeev-Popov heuristic path integral. The generalization of the Faddeev theorem of equivalence [22] (that is valid for local scattering processes) to the region of non-local processes removes both the initial data and the Laplace possibility of explaining (by these data) the non-local physical effects of the type of hadronization and
confinement in this world.

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