An exact solution of the five-dimensional Einstein equations with four-dimensional de Sitter-like expansion

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Abstract

We present an exact solution to the Einstein field equations which is Ricci and Riemann flat in five dimensions, but in four dimensions is a good model for the early vacuum-dominated universe.

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1 Introduction

There has recently been an uprising in interest in finding exact solutions of the Kaluza-Klein field equations in five dimensions (5D) which reproduce and extend known solutions of the Einstein field equations in four dimensions (4D) [1–5]. Particular interest revolves around solutions which are not only Ricci flat \( R_{AB} = 0; A, B, \ldots \in \{0, 1, 2, 3, 4\} \) where \( R_{AB} \) is the 5D Ricci tensor), but also Riemann flat \( R_{ABCD} = 0, \) where the vanishing of the Riemann-Christoffel tensor means that we are considering the analog of the Minkowski metric in 5D) [6–11]. This is because it is possible to have a flat 5D manifold which contains a curved 4D sub-manifold, as implied by Campbell’s embedding theorem [12–18]. So, the universe may be “empty” and simple in 5D, but contain matter of complicated forms in 4D [19, 20]. (This idea has been extended to higher-dimensional manifolds that are not Ricci-flat, in particular manifolds with non-zero cosmological constant [21, 22], scalar field sources [23], as well as manifolds with an arbitrary non-degenerate Ricci tensor [24]. In addition, the Campbell-Magaard theorem has been used to study the
embedding of Randall-Sundrum-type branes in 5D manifolds [25], suggesting that the curvature of any given brane is not necessarily determined by its stress-energy content.)

Despite the physical appeal of this idea, it is mathematically non-trivial to realize. Solutions of the flat and empty Einstein equations in 5D which correspond to solutions of \( G_{\alpha\beta} = T_{\alpha\beta} \) \((\alpha, \beta, \ldots \in \{0, 1, 2, 3\})\) in 4D with acceptable physics, are rare. (Here \( G_{\alpha\beta} \) is the 4D Einstein tensor and \( T_{\alpha\beta} \) is the induced stress-energy tensor obtained via the standard reduction of the 5D equations to their 4D counterparts; see reference [20]. We use units throughout which render the speed of light and Newton’s gravitational constant invisible via \( c = 1, 8\pi G = 1 \).) In what follows, we present and derive the properties of an exact 5D solution which provides a good 4D model for the vacuum-dominated early universe.

## 2 A New Solution and its Properties

Consider the five-dimensional line element with coordinates \( t, r, \theta, \phi, \ell \) such that

\[
\begin{align*}
    ds^2 &= \frac{\ell^2}{L^2} dt^2 - \left[ \ell \sinh \left( \frac{t}{L} \right) \right]^2 d\sigma_3^2 - d\ell^2 \\
    d\sigma_3^2 &= \left( 1 + \frac{k r^2}{4} \right)^{-2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \quad \text{and} \quad k = -1.
\end{align*}
\]

In five dimensions this defines a manifold \((\mathcal{M}, g_{AB})\) that is indeed both Ricci-flat and Riemann-flat, thus giving Minkowski space \( \mathbb{M}^5 \) in a different coordinate system. That (1) satisfies the Ricci-flat equations \( R_{AB} = 0 \) may be shown by tedious algebra (e.g. using the equations of reference [20]), and confirmed by computer (e.g. using the program GRTensor of reference [26]). The only humanly-practical way to show that (1) also satisfies the Riemann-flat equations \( R_{ABCD} = 0 \) is by computer, as may be verified.

The physical properties of the matter associated with (1) may, again, be derived either analytically or computationally. The basic procedure, in either approach, is to separate the purely 4D terms in \( R_{AB} = 0 \) from the other ones, compare with \( G_{\alpha\beta} = T_{\alpha\beta} \), and thereby obtain \( T_{\alpha\beta} = T_{\alpha\beta} (x^A, \partial g_{AB}/\partial x^C) \). Since the Einstein equations \( G_{AB} = 0 \) in empty 5D are equivalent to \( R_{AB} = 0 \) by straight algebra, what we are doing here is simply solving in effect the 5D Einstein equations, comparing the results to the 4D Einstein equations, and thereby evaluating the stress-energy tensor \( T_{\alpha\beta} \) necessary to balance the latter set of equations.

This procedure has in recent years been much used. A review of the algebraic technique and a list of applications is available [20]. Here we note that the procedure has been applied to cosmologies of the Friedmann-Robertson-Walker (FRW) type [27], 3D spherically symmetric solutions [28], solutions with off-diagonal metrics [29], Gödel-type spacetimes [30], and solutions containing a big bounce [31–35].
General theorems have also been proven, having to do with the field equations [36], dynamics [37] and the algebraic classification of 5D solutions with their associated 4D stress-energy tensors [38]. However, there is the constraint that the $T_{\alpha\beta}$ given by algebra should correspond to the properties of matter indicated by observational cosmology. For the early universe, this means that the equation of state for the matter should be close to that of the "classical vacuum". Here, the sum of the density $\rho$ and pressure $p$ is zero, as in inflationary cosmology [39]. We now proceed to this and other consequences of metric (1), to investigate its physical acceptability.

The line element (1) can be written in the useful "canonical" form [40] such that
\[ dS^2 = \frac{\ell^2}{L^2} \left[ dt^2 - \left( L \sinh \left( \frac{t}{L} \right) \right)^2 \, d\sigma^2 - d\ell^2. \]  
(2)

So with the 4D spacetime metric
\[ g_{\alpha\beta} = \text{diag} \left[ 1, -F_k(t, r), -r^2 F_k(t, r), -r^2 \sin^2 \theta F_k(t, r) \right] \]
and
\[ F_k(t, r) = L \sinh (t/L)^2 \left( 1 + kr^2 / 4 \right)^{-2}, \]  
(3)
we find the components of the stress-energy tensor $T_{\alpha\beta} = G_{\alpha\beta}$ to be
\[ T^0_0 = \frac{3}{L^2}, \quad T^1_1 = T^2_2 = T^3_3 = \frac{3}{L^2}. \]

In comoving coordinates this defines an energy density $\rho = T^0_0 = 3/L^2$ and pressure $p = -(T^1_1 + T^2_2 + T^3_3)/3 = -3/L^2$ for a vacuum with cosmological constant $\Lambda = 3/L^2$ and equation of state $\rho + p = 0$. The 4D Ricci scalar is $R = R_{\alpha\beta}g^{\alpha\beta} = 12/L^2$, and the 4D curvature scalar is $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 24/L^4$. This latter scalar implies that there are no singularities in the manifold because the constant $L \neq 0$.

Let us now look at (1), viewing its 4D part as describing an FRW model. The 4D hypersurfaces $\ell = \text{constant}$ therefore describe cosmologies with scale factor given by $S = S(t) = L \sinh(t/L)$. Here the Hubble parameter $H \equiv \dot{S}/S$ and deceleration parameter $q \equiv -\ddot{S}/\dot{S}^2$ (with $\dot{S} = dS/dt$) are found to be
\[ H = \frac{1}{L \tanh \left( \frac{t}{L} \right)} \quad \text{and} \quad q = -\tanh^2 \left( \frac{t}{L} \right). \]  
(4)

We note that $H$ is infinite at $t = 0$ and goes to $1/L = \sqrt{\Lambda/3}$ as $t \rightarrow \infty$, which is the Hubble parameter for de Sitter spacetime. Also, $q$ starts at zero when $t = 0$ and goes to $-1$ for $t \rightarrow \infty$. This is in line with astrophysical data which currently constrain the deceleration parameter to $-1 \leq q \leq 1$. Thus we conclude that our solution (1) describes an inflationary spacetime on $\ell = \text{constant}$ hypersurfaces, where the vacuum has repulsive properties.

The preceding paragraphs show that (1) has physical properties consistent with those of inflationary cosmology. However, the motivating factor for the latter approach to cosmology is that the (4D) horizon should grow fast enough to resolve certain problems of astrophysical nature, primarily to do with the 3 Kelvin microwave background [41, 42]. First, we recall that the horizon distance at time $t$ for
any FRW model can be defined [43] such that
\[
\int_0^r dr \left(1 + \frac{k r^2}{4}\right)^{-1} = \int_0^t \frac{dt'}{S(t')}. \tag{5}
\]
Multiplying both sides by the scale factor \(S(t)\) then gives
\[
d_{PH} = S(t) \int_0^r dr \left(1 + \frac{k r^2}{4}\right)^{-1} = S(t) \int_0^t \frac{dt'}{S(t')}, \tag{6}
\]
which defines the proper distance to the particle horizon at time \(t\). For the spacetime (3) this is
\[
d_{PH} = 2L \sinh \left(\frac{t}{L}\right) \left[\tanh(1) - \tanh \left(\frac{t}{L}\right)\right]. \tag{7}
\]
Here we see that \(d_{PH}\) is infinite because \(\tanh(1)\) is infinite. This means that during the inflationary period that the solution (1) describes on \(\ell = \) constant hypersurfaces, the entire universe is in causal contact. This is in line with the apparent isotropy of the microwave background.

Finally, we would like to point out an interesting coordinate transformation of the solution (1). Recall that \(\sinh(t) = (\exp(t) - \exp(-t))/2\), which with the coordinate change \(t \rightarrow tL\) in (1) gives
\[
dS^2 = \ell^2 dt^2 - \frac{1}{4} \ell^2 (e^{t} + ke^{-t})^2 d\sigma_3^2 - d\ell^2 \quad \text{with} \quad k = -1. \tag{8}
\]
This form of the metric resembles a solution noted by McManus [9]. For the solution (8), all 4D physical quantities are the same as those calculated for the solution (1), but with the replacement \(L \rightarrow \ell\). The 4D spacetime contained in (8) therefore still describes an inflationary vacuum with equation of state \(\rho + p = 0\). An important difference is that the 4D curvature scalar for (8) is \(K = 24/\ell^4\), which implies that the spacetime in (8) has a singularity at the point where \(\ell = 0\). This is in contrast to the solution (1), for which all physical quantities of spacetime were calculated to be (finite) constants. Evidently the simple coordinate transformation \(t \rightarrow tL\) casts (1) into a form where the vacuum evolves in accordance with how \(x^4 = \ell\) is determined by the extra component of the geodesic equation. This issue from the mathematical side has to do with whether we take the whole of the 4D part of the 5D manifold as defining the geometry of spacetime, or whether we take the 4D part of the 5D manifold without its prefactor. This 5D issue resembles the 4D one in scalar-tensor theory, where it manifests itself as a choice between what are commonly called the Jordan and Einstein frames. From the physical side, the choice has to do with how we define spacetime as a 4D slice of a 5D manifold; and we suggest that since the two choices only become differentiated over cosmological timescales, that it is essentially one of observation to decide.
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