Towards Finite Generation of the Canonical Ring Without the MMP

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Abstract. This paper is the first of two steps in a project to prove finite generation of the log canonical ring without Mori theory.

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1. Introduction

In this paper I establish the first of two steps in a project to prove finite generation of the log canonical ring without the Minimal Model Program. I prove:

Theorem 1.1. Let \((X, \Delta)\) be a projective klt pair and assume Property \(\mathcal{L}^G_A\) in dimensions \(\leq \dim X\). Then the log canonical ring \(R(X, K_X + \Delta)\) is finitely generated.

Property \(\mathcal{L}^G_A\) is stated below. Let me sketch the strategy for the proof of finite generation and present difficulties that arise on the way. The natural idea is to pick a smooth divisor \(S\) on \(X\) and to restrict the algebra to it. If we are very lucky, the restricted algebra will be finitely generated and we might hope that the generators lift to generators on \(X\). There are several issues with this approach.

Firstly, in order to obtain something meaningful on \(S\), \(S\) should be a log canonical centre of some pair \((X, \Delta')\) such that \(R(X, K_X + \Delta)\) and
$R(X, K_X + \Delta')$ share a common truncation. This issue did not exist in the case of pl flips.

Secondly, even if the restricted algebra were finitely generated, the same might not be obvious for the kernel of the restriction map. Note that the "kernel issue" also did not exist in the case of pl flips, since the relative Picard number $= 1$ ensured that the kernel was a principal ideal, at least after shrinking the base and passing to a truncation. So far this seems to have been the greatest conceptual issue in attempts to prove the finite generation by the plan just outlined.

Thirdly, the natural choice is to use the Hacon-M$^c$Kernan extension theorem, see Theorem 5.1 below, and hence we must be able to ensure that $S$ does not belong to the stable base locus of $K_X + \Delta'$.

The idea to resolve the kernel issue is to view $R(X, K_X + \Delta)$ as a subalgebra of a larger algebra, which would a priori contain generators of the kernel. In practice this means that the new algebra will have higher rank grading. Namely, we will see that the rank corresponds to the number of components of $\Delta$ and of an effective divisor $D \sim_Q K_X + \Delta$.

Let me illustrate this on a basic example which will model the general lines of the proof in Section 6. Say we wanted to prove by induction that the ring $R(X, H)$ was finitely generated, where $H$ is an ample divisor. By passing to a truncation and by taking a general member of $|\kappa H|$ for $\kappa \gg 0$, we may assume that $H$ is smooth and very ample. By Serre's vanishing the restriction map $\rho_k : H^0(X, \mathcal{O}_X(kH)) \to H^0(H, \mathcal{O}_H(kH))$ is surjective for all $k$, and by induction $R(H, \mathcal{O}_H(H))$ is finitely generated. If $\sigma_H \in H^0(X, \mathcal{O}_X(H))$ is a section such that $\text{div} \sigma_H = H$ and $\mathcal{H}$ is a finite set of generators of the finite dimensional vector space $\bigoplus_{i=1}^d H^0(X, \mathcal{O}_X(iH))$, for some $d$, such that the set $\{s_H \in \mathcal{H} : s \in \mathcal{H}\}$ generates $R(H, \mathcal{O}_H(H))$, it is easy to see that $\mathcal{H} \cup \{\sigma_H\}$ is a set of generators of $R(X, H)$, since $\ker(\rho_k) = H^0(X, \mathcal{O}_X((k - 1)H)) \cdot \sigma_H$. A version of this idea applies to the case of pl flips and forms the basis of the construction of Shokurov and Hacon-M$^c$Kernan.

It is natural to try and restrict to a component of $\Delta$, the issue of course being that $(X, \Delta)$ does not have log canonical centres. Therefore I allow restrictions to components of some effective divisor $D \sim_Q K_X + \Delta$, and a tie-breaking-like technique allows to create log canonical centres. Algebras encountered this way are, in effect, plt algebras, and their restriction is handled in Section 5. This is technically the most involved part of the proof.

Since the algebras we consider are of higher rank, not all divisors will have the same log canonical centres. I therefore restrict to available
centres, and lift generators from algebras that live on different divisors. Since the restrictions will also be algebras of higher rank, the induction process must start from them. Thus, the main technical result of the paper is the following.

**Theorem 1.2.** Let $X$ be a smooth projective variety, and for $i = 1, \ldots, \ell$ let $D_i = k_i(K_X + \Delta_i + A)$, where $A$ is an ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth log canonical pair with $|D_i| \neq \emptyset$. Assume Property $\mathcal{L}_{A}^{G}$ in dimensions $\leq \dim X$. Then the Cox ring $R(X; D_1, \ldots, D_\ell)$ is finitely generated.

Property $\mathcal{L}_{A}^{G}$ in the statement of Theorems 1.1 and 1.2 describes the convex geometry of the set of log canonical pairs with big boundaries in terms of divisorial components of the stable base loci. More precisely:

**Property $\mathcal{L}_{A}^{G}$.** Let $X$ be a smooth variety projective over an affine variety $Z$, $B$ a simple normal crossings divisor on $X$ and $A$ a general ample $\mathbb{Q}$-divisor. Let $V \subset \text{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of $B$ and let $\mathcal{L}_{V} = \{\Theta \in V : (X, \Theta) \text{ is log canonical}\}$; this is a rational polytope in $V$. Then for any component $G$ of $B$, the set

$$\mathcal{L}_{A}^{G} = \{\Phi \in \mathcal{L}_{V} : G \not\subset B(K_X + \Phi + A)\}$$

is a rational polytope.

Precise definitions are given in Section 2. This property is a consequence of the MMP, see Proposition 3.9 below. Here I would like to comment on a possible strategy to prove Property $\mathcal{L}_{A}^{G}$ without using Mori theory. It seems likely that the method of the proof will be quite similar to techniques used in Section 5 below to handle finite generation of restricted algebras. The method involved is very recent, and has appeared in [Hac08] in order to handle the proof of Non-vanishing theorem; [Pău08] gives a proof of that statement without the MMP. The necessary technical tools were developed in [Laz07, Section 3] in order to prove that certain superlinear functions are in fact piecewise linear, and they might be of use if one wishes to prove that certain sets are polytopes.

Finally, it is my hope that the techniques of this paper could be adapted to handle finite generation in the case of log canonical singularities and the abundance conjecture.

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2. Notation and conventions

Unless stated otherwise, varieties in this paper are normal over $\mathbb{C}$ and projective over an affine variety $Z$. The group of Weil, respectively Cartier, divisors on a variety $X$ is denoted by $\text{WDiv}(X)$, respectively $\text{Div}(X)$. We denote $\text{WDiv}(X)_{\kappa \geq 0} = \{ D \in \text{WDiv}(X) : \kappa(X, D) \geq 0 \}$, and similarly for $\text{Div}(X)_{\kappa \geq 0}$, where $\kappa$ is the Iitaka dimension. Subscripts denote the rings in which the coefficients are taken.

We say an ample $\mathbb{Q}$-divisor $A$ on a variety $X$ is (very) general if there is a sufficiently divisible positive integer $k$ such that $kA$ is very ample and $kA$ is a (very) general section of $|kA|$. In particular we can assume that for some $k \gg 0$, $kA$ is a smooth divisor on $X$.

For any two divisors $P = \sum p_i E_i$ and $Q = \sum q_i E_i$ on $X$ set

$P \wedge Q = \sum \min\{p_i, q_i\} E_i$.

I use basic properties of b-divisors, see \cite{Cor07}. The cone of mobile b-divisors on $X$ is denoted by $\text{Mob}(X)$.

For the definition and basic properties of multiplier ideals used in this paper see \cite{HM08}.

The sets of non-negative (respectively non-positive) rational and real numbers are denoted by $\mathbb{Q}_+$ and $\mathbb{R}_+$ ($\mathbb{Q}_-\text{ and } \mathbb{R}_-$ respectively).

Convex geometry. If $S = \sum \mathbb{N} e_i$ is a submonoid of $\mathbb{N}^n$, I denote $S_{\mathbb{Q}} = \sum \mathbb{Q}_+ e_i$ and $S_{\mathbb{R}} = \sum \mathbb{R}_+ e_i$. A monoid $S \subset \mathbb{N}^n$ is saturated if $S = S_{\mathbb{R}} \cap \mathbb{N}^n$.

If $S = \sum_{i=1}^n \mathbb{N} e_i$ and $\kappa_1, \ldots, \kappa_n$ are positive integers, the submonoid $S' = \sum_{i=1}^n \mathbb{N} \kappa_i e_i$ is called a truncation of $S$. If $\kappa_1 = \cdots = \kappa_n = \kappa$, I denote $S^{(\kappa)} := \sum_{i=1}^n \mathbb{N} \kappa e_i$, and this truncation does not depend on a choice of generators of $S$.

A submonoid $S = \sum \mathbb{N} e_i$ of $\mathbb{N}^n$ (respectively a cone $C = \sum \mathbb{R}_+ e_i$ in $\mathbb{R}^n$) is called simplicial if its generators $e_i$ are linearly independent in $\mathbb{R}^n$, and the $e_i$ form a basis of $S$ (respectively $C$).

I often use Gordan’s lemma without explicit mention, see \cite[Lemma 2.4]{Laz07}, and also that if $\lambda : \mathcal{M} \rightarrow S$ is an additive surjective map...
between finitely generated saturated monoids, and if \( \mathcal{C} \) is a rational polyhedral cone in \( S_\mathbb{R} \), then \( \lambda^{-1}(\mathcal{S} \cap \mathcal{C}) = \mathcal{M} \cap \lambda^{-1}(\mathcal{C}) \). In particular, the inverse image of a saturated finitely generated submonoid of \( \mathcal{S} \) is a saturated finitely generated submonoid of \( \mathcal{M} \).

For a polytope \( \mathcal{P} \subset \mathbb{R}^n \), I denote \( \mathcal{P}_\mathbb{Q} = \mathcal{P} \cap \mathbb{Q}^n \). A polytope is \textit{rational} if it is the convex hull of finitely many rational points.

If \( \mathcal{B} \subset \mathbb{R}^n \) is a convex set, then \( \mathbb{R}_+ \mathcal{B} \) will denote the set \( \{rb : r \in \mathbb{R}_+, b \in \mathcal{B}\} \). In particular, if \( \mathcal{B} \) is a rational polytope, \( \mathbb{R}_+ \mathcal{B} \) is a rational polyhedral cone. The dimension of the rational polytope \( \mathcal{P} \), denoted \( \dim \mathcal{P} \), is the dimension of the smallest rational affine space containing \( \mathcal{P} \).

Let \( \mathcal{S} \subset \mathbb{N}^n \) be a finitely generated monoid, \( \mathcal{C} \in \{\mathcal{S}, \mathcal{S}_2, \mathcal{S}_\mathbb{R}\} \) and \( V \) an \( \mathbb{R} \)-vector space. A function \( f : \mathcal{C} \to V \) is: \textit{positively homogeneous} if \( f(\lambda x) = \lambda f(x) \) for \( x \in \mathcal{C} \), \( \lambda \geq 0 \); \textit{superadditive} if \( f(x) + f(y) \leq f(x+y) \) for \( x, y \in \mathcal{C} \); \textit{\( \mathbb{Q} \)-superadditive} if \( \lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y) \) for \( x, y \in \mathcal{C}, \lambda, \mu \in \mathbb{Q}_+ \); \textit{\( \mathbb{Q} \)-additive} if the previous inequality is an equality; and \textit{superlinear} if \( \lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y) \) for \( x, y \in \mathcal{S}_\mathbb{R} \), \( \lambda, \mu \in \mathbb{R}_+ \). Similarly for \textit{additive}, \textit{subadditive}, \textit{sublinear}. It is \textit{piecewise additive} if there is a finite polyhedral decomposition \( \mathcal{C} = \bigcup \mathcal{C}_i \) such that \( f|_{\mathcal{C}_i} \) is additive for every \( i \); additionally, if each \( \mathcal{C}_i \) is a rational cone, it is \textit{rationally piecewise additive}. Similarly for (rationally) piecewise linear.

Assume furthermore that \( f \) is linear on \( \mathcal{C} \) and \( \dim \mathcal{C} = n \). The \textit{linear extension of} \( f \) \textit{to} \( \mathbb{R}^n \) is the unique linear function \( \ell : \mathbb{R}^n \to V \) such that \( \ell|_{\mathcal{C}} = f \).

In this paper the \textit{relative interior} of a cone \( \mathcal{C} = \sum \mathbb{R}_+ e_i \subset \mathbb{R}^n \), denoted by relint \( \mathcal{C} \), is the topological interior of \( \mathcal{C} \) in the space \( \sum \mathbb{R} e_i \) union the origin. If \( \dim \mathcal{C} = n \), we instead call it the \textit{interior} of \( \mathcal{C} \) and denote it by \( \text{int} \mathcal{C} \). The boundary of a closed set \( \mathcal{C} \) is denoted by \( \partial \mathcal{C} \).

**Asymptotic invariants.** The standard references on asymptotic invariants arising from linear series are [Nak04, ELM+06].

**Definition 2.1.** Let \( X \) be a variety and \( D \in \text{WDiv}(X)_\mathbb{R} \). For \( k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\} \), define
\[
|D|_k = \{ C \in \text{WDiv}(X)_k : C \geq 0, C \sim_k D \}.
\]
If \( T \) is a prime divisor on \( X \) such that \( T \not\in \text{Fix}[D] \), then \( |D|_T \) denotes the image of the linear system \( |D| \) under restriction to \( T \). The \textit{stable base locus} of \( D \) is \( \mathcal{B}(D) = \bigcap_{C \in |D|_\mathbb{R}} \text{Supp} C \) if \( |D|_\mathbb{R} \neq \emptyset \), otherwise we define \( \mathcal{B}(D) = X \). The \textit{diminished base locus} is \( \mathcal{B}_-(D) = \bigcup_{\varepsilon > 0} \mathcal{B}(D + \varepsilon A) \) for an ample divisor \( A \); this definition does not depend on a choice of \( A \). In particular \( \mathcal{B}_-(D) \subset \mathcal{B}(D) \).
It is elementary that $B(D_1 + D_2) \subseteq B(D_1) \cup B(D_2)$ for $D_1, D_2 \in W\text{Div}(X)_\mathbb{R}$. In other words, the set $\{D \in W\text{Div}(X)_\mathbb{R} : x \notin B(D)\}$ is convex for every point $x \in X$. By [BCHM06, Lemma 3.5.3], $B(D) = \text{\textsf{Supp}} \cap D$. \text{Supp} \text{\textsf{Supp}} C when $D$ is a $\mathbb{Q}$-divisor, which is the standard definition of the stable base locus.

**Definition 2.2.** Let $Z$ be a closed subvariety of a smooth variety $X$ and let $D$ be an effective $\mathbb{Q}$-divisor on $X$. The asymptotic order of vanishing of $D$ along $Z$ is

$$\text{ord}_Z \parallel D \parallel = \inf \{\text{mult}_Z C : C \in |D|_\mathbb{Q}\}.$$ 

More generally, one can consider any discrete valuation $\nu$ of $k(X)$ and define $\nu \parallel D \parallel = \inf \{\nu(C) : C \in |D|_\mathbb{Q}\}$ for an effective $\mathbb{Q}$-divisor $D$. Then [ELM+06] shows that $\nu \parallel D \parallel = \nu \parallel E \parallel$ if $D$ and $E$ are numerically equivalent big divisors, and that $\nu$ extends to a sublinear function on $\text{Big}(X)_\mathbb{R}$.

**Remark 2.3.** When $X$ is projective, Nakayama in [Nak04] defines a function $\sigma_Z : \text{Big}(X) \to \mathbb{R}_+$ by

$$\sigma_Z(D) = \lim_{\varepsilon \to 0} \text{ord}_Z \parallel D + \varepsilon A \parallel$$

for any ample $\mathbb{R}$-divisor $A$, and shows that it agrees with $\text{ord}_Z \parallel \cdot \parallel$ on big classes. Analytic properties of these invariants were studied in [Bou04].

We can define the restricted version of the invariant introduced.

**Definition 2.4.** Let $S$ be a smooth divisor on a smooth variety $X$ and let $D \in \text{Div}(X)_{\mathbb{Q}\geq 0}$ be such that $S \not\subseteq B(D)$. Let $P$ be a closed subvariety of $S$. The restricted asymptotic order of vanishing of $|D|_S$ along $P$ is

$$\text{ord}_P \parallel D \parallel_S = \inf \{\text{mult}_P C : kC \in |kD|_S \text{ for some } k \geq 1\}.$$ 

**Remark 2.5.** Similarly as in Remark 2.3 [Hac08] introduces a function $\sigma_P \parallel \cdot \parallel_S : \mathcal{C}_- \to \mathbb{R}_+$ by

$$\sigma_P \parallel D \parallel_S = \lim_{\varepsilon \to 0} \text{ord}_P \parallel D + \varepsilon A \parallel_S$$

for any ample $\mathbb{R}$-divisor $A$, where $\mathcal{C}_- \subset \text{\textsf{Big}}(X)$ is the set of classes of divisors $D$ such that $S \not\subseteq B_-(D)$. Then one can define a formal sum $N_\sigma \parallel D \parallel_S = \sum \sigma_P \parallel D \parallel_S \cdot P$ over all prime divisors $P$ on $S$. If $S \not\subseteq B(D)$, then for every $\varepsilon_0 > 0$ we have $\lim_{\varepsilon \to \varepsilon_0} \sigma_P \parallel D + \varepsilon A \parallel_S = \text{ord}_P \parallel D + \varepsilon_0 A \parallel_S$ for any ample divisor $A$ on $X$ similarly as in [Nak04, Lemma 2.1.1], cf. [Hac08, Lemma 7.8].
In this paper I need a few basic properties cf. [Hac08, Lemma 7.14].

**Lemma 2.6.** Let $S$ be a smooth divisor on a smooth projective variety $X$, let $D \in \text{Div}(X)^{\kappa \geq 0}$ be such that $S \not\subset B(D)$ and let $P$ be a closed subvariety of $S$. If $A$ is an ample $\mathbb{Q}$-divisor on $X$, then $\text{ord}_P \|D + A\|_S \leq \text{ord}_P \|D\|_S$, and in particular $\sigma_P \|D\|_S = 0$, if $\sigma_P \|D\|_S = 0$, then there is a positive integer $l$ such that $\text{mult}_P \text{Fix} |l(D + A)|_S = 0$.

**Proof.** The first statement is trivial. For the second one, we have $\text{ord}_P \|D + \frac{1}{2}A\|_S = 0$. Set $n = \dim X$, let $H$ be a very ample divisor on $X$ and fix a positive integer $l$ such that $H' = \frac{1}{2}A - (K_X + S) - (n + 1)H$ is very ample. Let $\Delta \sim \mathbb{Q}D + \frac{1}{2}A$ be a $\mathbb{Q}$-divisor such that $S \not\subset \text{Supp} \Delta$ and $\text{mult}_P \Delta|_S < 1/l$. We have

$$H^1(X, J_{\Delta|_S}(K_S + H'|_S + (n + 1)H|_S + l\Delta|_S + mH|_S)) = 0$$

for $m \geq -n$ by Nadel vanishing. Since $l(D + A) \sim \mathbb{Q}K_X + S + H' + (n + 1)H + l\Delta$, the sheaf $J_{\Delta|_S}(l(D + A))$ is globally generated by $[HM08, \text{Lemma 5.7}]$ and its sections lift to $H^0(X, l(D + A))$ by $[HM08, \text{Lemma 4.4(3)}]$. Since $\text{mult}_P \text{Fix} |l(D + A)|_S = 0$, $J_{\Delta|_S}$ does not vanish along $P$ and so $\text{mult}_P \text{Fix} |l(D + A)|_S = 0$. \hfill $\square$

3. Higher rank algebras

In this section I adapt some of the definitions from [Laz07] to suit the context of this paper.

**Definition 3.1.** Let $X$ be a variety, $S$ a finitely generated submonoid of $\mathbb{N}^r$, and let $\mu : S \to \text{WDiv}(X)^{\kappa \geq 0}$ be an additive map. The algebra

$$R(X, \mu(S)) = \bigoplus_{s \in S} H^0(X, \mathcal{O}_X(\mu(s)))$$

is called the *divisorial $S$-graded algebra associated to $\mu$*. When $S = \bigoplus_{i=1}^t \mathbb{N}e_i$ is a simplicial cone, the algebra $R(X, \mu(S))$ is called the *Cox ring associated to $\mu$*, and is denoted also by $R(X; \mu(e_1), \ldots, \mu(e_t))$.

**Remark 3.2.** Algebras considered in this paper are *algebras of sections*. I will occasionally, and without explicit mention, view them as algebras of rational functions, in particular to be able to write $H^0(X, D) \simeq H^0(X, \text{Mob}(D)) \subset k(X)$.

Assume now that $X$ is smooth, $D \in \text{Div}(X)$ and that $\Gamma$ is a prime divisor on $X$. If $\sigma_\Gamma$ is the global section of $\mathcal{O}_X(\Gamma)$ such that $\text{div} \sigma_\Gamma = \Gamma$, from the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(D - \Gamma)) \xrightarrow{\sigma_\Gamma} H^0(X, \mathcal{O}_X(D)) \xrightarrow{\rho_D, \Gamma} H^0(\Gamma, \mathcal{O}_\Gamma(D))$$
we define \( \text{res}_\Gamma H^0(X, \mathcal{O}_X(D)) = \text{Im}(\rho_{D,\Gamma}) \). For \( \sigma \in H^0(X, \mathcal{O}_X(D)) \), I denote \( \sigma|_\Gamma := \rho_{D,\Gamma}(\sigma) \). Observe that

\[
(1) \quad \ker(\rho_{D,\Gamma}) = H^0(X, \mathcal{O}_X(D - \Gamma)) \cdot \sigma|_\Gamma,
\]

and that \( \text{res}_\Gamma H^0(X, \mathcal{O}_X(D)) = 0 \) if \( \Gamma \subset \text{Bs}[D] \). If \( D \sim D' \) such that the restriction \( D'|_\Gamma \) is defined, then

\[
\text{res}_\Gamma H^0(X, \mathcal{O}_X(D)) \simeq \text{res}_\Gamma H^0(X, \mathcal{O}_X(D')) \subset H^0(\Gamma, \mathcal{O}_\Gamma(D'|_\Gamma)).
\]

The restriction of \( R(X, \mu(S)) \) to \( \Gamma \) is defined as

\[
\text{res}_\Gamma R(X, \mu(S)) = \bigoplus_{s \in S} \text{res}_\Gamma H^0(X, \mathcal{O}_X(\mu(s))).
\]

This is an \( S \)-graded, not necessarily divisorial algebra.

**Remark 3.3.** Under assumptions from Definition 3.1 we define the map \( \text{Mob}_\mu : S \to \text{Mob}(X) \) by \( \text{Mob}_\mu(s) = \text{Mob}(\mu(s)) \) for every \( s \in S \). Then we have a b-divisorial algebra

\[
R(X, \text{Mob}_\mu(S)) \simeq R(X, \mu(S))
\]
as defined in [Laz07]. If \( S' \) is a finitely generated submonoid of \( S \), I use \( R(X, \mu(S')) \) to denote \( R(X, \mu|_{S'}(S')) \). If \( S \) is a submonoid of \( \text{WDiv}(X)^{\kappa \geq 0} \) and \( \iota : S \to S \) is the identity map, I use \( R(X, S) \) to denote \( R(X, \iota(S)) \).

The following lemma summarises the basic properties of higher rank finite generation.

**Lemma 3.4.** Let \( S \subset \mathbb{N}^n \) be a finitely generated monoid and let \( R = \bigoplus_{s \in S} R_s \) be an \( S \)-graded algebra.

1. Let \( S' \) be a truncation of \( S \). If the \( S' \)-graded algebra \( R' = \bigoplus_{s \in S'} R_s \) is finitely generated over \( R_0 \), then \( R \) is finitely generated over \( R_0 \).
2. Assume furthermore that \( S \) is saturated and let \( S'' \subset S \) be a finitely generated saturated submonoid. If \( R \) is finitely generated over \( R_0 \), then the \( S'' \)-graded algebra \( R'' = \bigoplus_{s \in S''} R_s \) is finitely generated over \( R_0 \).
3. Let \( X \) be a variety and let \( \mu : S \to \text{WDiv}(X)^{\kappa \geq 0} \) be an additive map. If there exists a rational polyhedral subdivision \( S_\mathbb{Q} = \bigcup_{i=1}^k \Delta_i \) such that, for each \( i \), \( \text{Mob}_{\mu|\Delta_i \cap S} \) is an additive map up to truncation, then the algebra \( R(X, \mu(S)) \) is finitely generated.

**Proof.** See [Laz07, Lemmas 5.1 and 5.2] and [ELM+06, Lemma 4.8]. \( \square \)
I will need the following result in the proof of Proposition 3.7 and in Section 5.

**Lemma 3.5.** Let $X$ be a variety, $\mathcal{S} \subset \mathbb{N}^r$ a finitely generated monoid and let $f: \mathcal{S} \to G$ be a superadditive map to a monoid $G$ which is a subset of $\text{WDiv}(X)$ or $\text{Mob}(X)$, such that for every $s \in \mathcal{S}$ there is a positive integer $\iota_s$ such that $f|_{\mathbb{N}_{\iota_s} \cdot s}$ is an additive map.

Then there is a unique $\mathbb{Q}$-superadditive function $f^\sharp: \mathcal{S}_{\mathbb{Q}} \to G_{\mathbb{Q}}$ such that for every $s \in \mathcal{S}$ there is a positive integer $\lambda_s$ with $f|_{\mathbb{N}_{\lambda_s} \cdot s} = f^\sharp|_{\mathbb{N}_{\lambda_s} \cdot s}$. Furthermore, let $\mathcal{C}$ be a rational polyhedral subcone of $\mathcal{S}_{\mathbb{R}}$. Then $f^\sharp|_{\mathcal{C} \cap \mathcal{S}}$ is additive up to truncation if and only if $f^\sharp|_{\mathcal{C} \cap \mathcal{S}_{\mathbb{Q}}}$ is $\mathbb{Q}$-additive.

If $\mu: \mathcal{S} \to \text{Div}(X)$ is an additive map and $m = \text{Mob}_\mu$ is such that for every $s \in \mathcal{S}$ there is a positive integer $\iota_s$ such that $m|_{\mathbb{N}_{\iota_s} \cdot s}$ is an additive map, then we have

$$m^\sharp(s) = \mu(s) - \sum (\text{ord}_E \| \mu(s) \|)E,$$

where the sum runs over all geometric valuations $E$ on $X$.

**Proof.** See the proof of [Laz07, Lemma 5.4]. Equation (2) is a restatement of the definition of $m^\sharp$ from that proof.

**Definition 3.6.** In the context of Lemma 3.5 the function $f^\sharp$ is called the straightening of $f$.

**Proposition 3.7.** Let $X$ be a variety, $\mathcal{S} \subset \mathbb{N}^r$ a finitely generated saturated monoid and $\mu: \mathcal{S} \to \text{WDiv}(X)^{\kappa \geq 0}$ an additive map. Let $\mathcal{L}$ be a finitely generated submonoid of $\mathcal{S}$ and assume $R(X, \mu(\mathcal{S}))$ is finitely generated. Then $R(X, \mu(\mathcal{L}))$ is finitely generated. Moreover, the map $m = \text{Mob}_\mu$ is piecewise additive up to truncation. In particular, there is a positive integer $p$ such that $\text{Mob}_\mu(ips) = i \text{Mob}_\mu(ps)$ for every $i \in \mathbb{N}$ and every $s \in \mathcal{L}$.

**Proof.** Denote $\mathcal{M} = \mathcal{L}_{\mathbb{R}} \cap \mathbb{N}^r$. By Lemma 3.4(2), $R(X, \mu(\mathcal{M}))$ is finitely generated, and by the proof of [ELM+06, Theorem 4.1], there is a finite rational polyhedral subdivision $\mathcal{M}_{\mathbb{R}} = \bigcup \Delta_i$ such that for every geometric valuation $E$ on $X$, the map $\text{ord}_E \| \cdot \|$ is $\mathbb{Q}$-additive on $\Delta_i \cap \mathcal{M}_{\mathbb{Q}}$ for every $i$. Since for every saturated rank 1 submonoid $\mathcal{R} \subset \mathcal{M}$ the algebra $R(X, \mu(\mathcal{R}))$ is finitely generated by Lemma 3.4(2), the map $m|_{\mathcal{R} \cap \mathcal{L}}$ is additive up to truncation by [Cor07, Lemma 2.3.53] and thus there is the well-defined straightening $m^\sharp: \mathcal{L}_{\mathbb{Q}} \to \text{Mob}(X)_{\mathbb{Q}}$ since $\mathcal{M}_{\mathbb{Q}} = \mathcal{L}_{\mathbb{Q}}$. Then (2) implies that the map $m^\sharp|_{\Delta_i \cap \mathcal{L}_{\mathbb{Q}}}$ is $\mathbb{Q}$-additive for every $i$, hence by Lemma 3.5 the map $m$ is piecewise additive up to truncation, and therefore $R(X, \mu(\mathcal{L}))$ is finitely generated by Lemma 3.4(3).
The following lemma shows that finite generation implies certain boundedness on the convex geometry of boundaries.

**Lemma 3.8.** Let \((X, \Delta = B + A)\) be a log smooth klt pair, where \(A\) is a general ample \(\mathbb{Q}\)-divisor, \(B\) is an effective \(\mathbb{R}\)-divisor, and assume that no component of \(B\) is in \(\mathcal{B}(K_X + \Delta)\). Assume Property \(\mathcal{L}_A^G\) and Theorem 1.2 in dimension \(\dim X\). Let \(V \subset \text{Div}(X)_{\mathbb{R}}\) be the vector space spanned by the components of \(B\) and \(W \subset V\) the smallest rational affine subspace containing \(B\). Then there is a constant \(\eta > 0\) and a positive integer \(r\) such that if \(\Phi \in W\) and \(k\) is a positive integer such that \(\|\Phi - B\| < \eta\) and \(k(K_X + \Phi + A)/r\) is Cartier, then no component of \(B\) is in \(\text{Fix}|k(K_X + \Phi + A)|\).

**Proof.** Let \(K_X\) be a divisor such that \(\mathcal{O}_X(K_X) \simeq \omega_X\) and \(\text{Supp} A \not\subset \text{Supp} K_X\), and let \(\Lambda \subset \text{Div}(X)\) be the monoid spanned by components of \(K_X, B\) and \(A\). Let \(G\) be a components of \(B\). By Property \(\mathcal{L}_A^G\) there is a rational polytope \(\mathcal{P} \subset W\) such that \(\Delta \in \text{relint} \mathcal{P}\) and \(G \not\subset \mathcal{B}(K_X + \Phi + A)\) for every \(\Phi \in \mathcal{P}\). Let \(D_1, \ldots, D_\ell\) be generators of \(\mathbb{R}_+(K_X + A + \mathcal{P}) \cap \Lambda\). By Theorem 1.2 the Cox ring \(R(X; D_1, \ldots, D_\ell)\) is finitely generated, and thus so is the algebra \(R(X, \Lambda)\) by projection. By Proposition 3.7 there is a rational polyhedral cone \(\mathcal{C} \subset \Lambda_{\mathbb{R}}\) such that \(\Delta \in \mathcal{C}\) and the map \(\text{Mob}_{i|\mathcal{C}\cap\Lambda(r)}\) is additive for some positive integer \(r\), where \(i: \Lambda \rightarrow \Lambda\) is the identity map. In particular, if \(\Phi \in \mathcal{C} \cap \mathcal{P}\) and \(k(K_X + \Phi + A)/r\) is Cartier, then \(G \not\subset \text{Fix}|k(K_X + \Phi + A)|\). Pick \(\eta\) such that \(\Phi \in \mathcal{C} \cap \mathcal{P}\) whenever \(\Phi \in W\) and \(\|\Phi - \Delta\| < \eta\). We can take \(\eta\) and \(r\) to work for all components of \(B\), and we are done. \(\square\)

To conclude this section, I show how results of [BCHM06] imply Property \(\mathcal{L}_A^G\). Of course, a hope is that this will be proved without Mori theory.

**Proposition 3.9.** Property \(\mathcal{L}_A^G\) follows from the MMP.

**Proof.** Let \(K_X\) be a divisor such that \(\mathcal{O}_X(K_X) \simeq \omega_X\) and \(\text{Supp} A \not\subset \text{Supp} K_X\), and let \(\Lambda\) be the monoid in \(\text{Div}(X)\) generated by the components of \(K_X, B\) and \(A\). Let \(i: \Lambda \rightarrow \Lambda\) be the identity map, and denote \(\mathcal{S} = \mathbb{R}_+(K_X + A + \mathcal{L}_V) \cap \Lambda\). Since \(\mathcal{L}_V\) is a rational polytope, \(\mathcal{S}\) is a finitely generated monoid and let \(D_i\) be generators of \(\mathcal{S}\). By [BCHM06] Corollary 1.1.9, the Cox ring \(R(X; D_1, \ldots, D_\ell)\) is finitely generated, thus so is the algebra \(R(X, \mathcal{S})\) by projection. The set \(\mathcal{M} = \{D \in \mathcal{S} : |D|_{\mathbb{Q}} \neq \emptyset\}\) is a convex cone, and therefore finitely generated since \(R(X, \mathcal{S})\) is finitely generated, so I can assume \(\mathcal{M} = \mathcal{S}\). By Proposition 3.7 the map \(\text{Mob}_i\) is piecewise additive up to truncation, which proves that the closure \(\mathcal{C}\) of the set \((\mathcal{L}_A^G)_{\mathbb{Q}}\) is a rational
polytope, and I claim it equals $L_A^G$. Otherwise there exists $\Phi \in L_A^G \setminus \mathcal{C}$, and therefore the convex hull of the set $\mathcal{C} \cup \{\Phi\}$, which is by convexity a subset of $L_A^G$, contains a rational point $\Phi' \in L_A^G \setminus \mathcal{C}$, a contradiction. \qed

4. Diophantine approximation

I need a few results from Diophantine approximation theory.

**Lemma 4.1.** Let $\Lambda \subset \mathbb{R}^n$ be a lattice spanned by rational vectors, and let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a vector $v \in V$ and denote $X = Nv + \Lambda$. Then the closure of $X$ is symmetric with respect to the origin. Moreover, if $\pi: V \to V/\Lambda$ is the quotient map, then the closure of $\pi(X)$ is a finite disjoint union of connected components. If $v$ is not contained in any proper rational affine subspace of $V$, then $X$ is dense in $V$.

**Proof.** Let $G$ be the closure of $\pi(X)$. Then $G$ is a closed infinite subgroup of the compact group $V/\Lambda$. The connected component $G_0$ of the identity in $G$ is a Lie subgroup of $V/\Lambda$ and so by [Bum04, Theorem 15.2], $G_0$ is a torus. Thus $G_0 = V_0/\Lambda_0$, where $V_0 = \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{R}$ is a rational subspace of $V$. Since $G/G_0$ is discrete and compact, it is finite, and it is straightforward that $X$ is symmetric with respect to the origin. Therefore a translate of $v$ by a rational vector is contained in $V_0$, and so if $v$ is not contained in any proper rational affine subspace of $V$, then $V_0 = V$. \qed

The next result is [BCHM06, Lemma 3.7.7].

**Lemma 4.2.** Let $x \in \mathbb{R}^n$ and let $W$ be the smallest rational affine space containing $x$. Fix a positive integer $k$ and a positive real number $\varepsilon$. Then there are $w_1, \ldots, w_p \in W \cap \mathbb{Q}^n$ and positive integers $k_1, \ldots, k_p$ divisible by $k$, such that $x = \sum_{i=1}^p r_i w_i$ with $r_i > 0$ and $\sum r_i = 1$, $\|x - w_i\| < \varepsilon/k_i$ and $k_i w_i/k$ is integral for every $i$.

I will need a refinement of this lemma when the smallest rational affine space containing a point is not necessarily of maximal dimension.

**Lemma 4.3.** Let $x \in \mathbb{R}^n$, let $0 < \varepsilon, \eta \ll 1$ be rational numbers and let $w_1 \in \mathbb{Q}^n$ and $k_1 \in \mathbb{N}$ be such that $\|x - w_1\| < \varepsilon/k_1$ and $k_1 w_1$ is integral. Then there are $w_2, \ldots, w_m \in \mathbb{Q}^n$, positive integers $k_2, \ldots, k_m$ such that $\|x - w_i\| < \varepsilon/k_i$ and $k_i w_i$ is integral for every $i$, and positive numbers $r_1, \ldots, r_m$ such that $x = \sum_{i=1}^m r_i w_i$ and $\sum r_i = 1$. Furthermore, we can assume that $w_3, \ldots, w_m$ belong to the smallest rational affine space containing $x$, and we can write

$$x = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} w_2 + \xi,$$

with $\|\xi\| < \eta/(k_1 + k_2)$. 

Proof. Let $W$ be the minimal rational affine subspace containing $x$, let \( \pi : \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n \) be the quotient map and let $G$ be the closure of the set $\pi(\mathbb{N} x + \mathbb{Z}^n)$. Then by Lemma 4.1 we have $\pi(-k_1 x) \in G$ and there is $k_2 \in \mathbb{N}$ such that $\pi(k_2 x)$ is in the connected component of $\pi(-k_1 x)$ in $G$ and $\|k_2 x - y\| < \eta$ for some $y \in \mathbb{R}^n$ with $\pi(y) = \pi(-k_1 x)$. Thus there is a point $w_2 \in \mathbb{Q}^n$ such that $k_2 w_2 \in \mathbb{Z}^n$, $\|k_2 x - k_2 w_2\| < \varepsilon$ and the open segment $(w_1, w_2)$ intersects $W$.

Pick $t \in (0, 1)$ such that $w_t = tw_1 + (1 - t)w_2 \in W$, and choose, by Lemma 4.2 rational points $w_3, \ldots, w_m \in W$ and positive integers $k_3, \ldots, k_m$ such that $k_i w_i \in \mathbb{Z}^n$, $\|x - w_i\| < \varepsilon/k_i$ and $x = \sum_{i=3}^{m} r_i w_i + r_t w_t$ with $r_t > 0$ and all $r_i > 0$, and $r_t + \sum_{i=3}^{m} r_i = 1$. Thus $x = \sum_{i=3}^{m} r_i w_i$ with $r_1 = tr_t$ and $r_2 = (1 - t)r_t$.

Finally, observe that the vector $y/k_2 - w_2$ is parallel to the vector $x - w_1$ and $\|y - k_2 w_2\| = \|k_1 x - k_1 w_1\|$. Denote $z = x - y/k_2$. Then

\[
\frac{x - w_1}{w_2 + z} - x = \frac{x - w_1}{w_2 - y/k_2} = \frac{k_2}{k_1},
\]

so

\[
x = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} (w_2 + z) = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} w_2 + \xi,
\]

where $\|\xi\| = \|k_2 z/(k_1 + k_2)\| < \eta/(k_1 + k_2)$. \(\square\)

Remark 4.4. Assuming notation from the previous proof, the connected components of $G$ are precisely the connected components of the set $\pi(\bigcup_{k=0}^{n} kW)$. Therefore $y/k_2 \in W$.

Remark 4.5. Assume $\lambda : V \to W$ is a linear map between vector spaces such that $\lambda(V_Q) \subset W_Q$. Let $x \in V$ and let $H \subset V$ be the smallest rational affine subspace containing $x$. Then $\lambda(H)$ is the smallest rational affine subspace of $W$ containing $\lambda(x)$. Otherwise, assume $H' \neq \lambda(H)$ is the smallest rational affine subspace containing $\lambda(x)$. Then $\lambda^{-1}(H')$ is a rational affine subspace containing $x$ and $H \nsubseteq \lambda^{-1}(H')$, a contradiction.

5. Restricting plt algebras

In this section I establish one of the technically most difficult steps in the proof of Theorem 1.2. Crucial results and techniques will be those used to prove Non-vanishing theorem in [Hac08] using methods developed in [HM08], and the techniques of [Laz07 Section 3].

The key result is the following Hacon-M^cKernan extension theorem [HM08, Theorem 6.2], whose proof relies on deep techniques initiated by [Siu98].
Theorem 5.1. Let \( \pi: X \to Z \) be a projective morphism to a normal affine variety \( Z \), where \( (X, \Delta = S + A + B) \) is a purely log terminal pair, \( S = \lfloor \Delta \rfloor \) is irreducible, \( (X, S) \) is log smooth, \( A \) is a general ample \( \mathbb{Q} \)-divisor and \( (S, \Omega + A|_S) \) is canonical, where \( \Omega = (\Delta - S)|_S \). Assume \( S \not\subset B(K_X + \Delta) \), and let

\[
F = \liminf_{m \to \infty} \frac{1}{m} \text{Fix} |m(K_X + \Delta)|_S.
\]

If \( \varepsilon > 0 \) is any rational number such that \( \varepsilon (K_X + \Delta) + A \) is ample and if \( \Phi \) is any \( \mathbb{Q} \)-divisor on \( S \) and \( k > 0 \) is any integer such that both \( k\Delta \) and \( k\Phi \) are Cartier, and \( \Omega \wedge (1 - \frac{\varepsilon}{k})F \leq \Phi \leq \Omega \), then

\[
|k(K_S + \Omega - \Phi)| + k\Phi \subset |k(K_X + \Delta)|_S.
\]

The immediate consequence is:

Corollary 5.2. Let \( \pi: X \to Z \) be a projective morphism to a normal affine variety \( Z \), where \( (X, \Delta = S + A + B) \) is a purely log terminal pair, \( S = \lfloor \Delta \rfloor \) is irreducible, \( (X, S) \) is log smooth, \( A \) is a general ample \( \mathbb{Q} \)-divisor and \( (S, \Omega + A|_S) \) is canonical, where \( \Omega = (\Delta - S)|_S \). Assume \( S \not\subset B(K_X + \Delta) \), and let \( \Phi_m = \Omega \wedge \frac{1}{m} \text{Fix} |m(K_X + \Delta)|_S \) for every \( m \) such that \( m\Delta \) is Cartier. Then

\[
|m(K_S + \Omega - \Phi_m)| + m\Phi_m = |m(K_X + \Delta)|_S.
\]

Definition-Lemma 5.3. Let \( (X, \Delta) \) be a log pair and let \( f: Y \to X \) be a proper birational morphism. We can write uniquely

\[
K_Y + B_Y = f^*(K_X + \Delta) + E_Y,
\]

where \( B_Y \) and \( E_Y \) are effective with no common components and \( E_Y \) is \( f \)-exceptional. There is a well-defined boundary \( b \)-divisor \( B(X, \Delta) \) given by \( B(X, \Delta)_Y = B_Y \) for every model \( Y \to X \).

Proof. Let \( h: Y' \to Y \) be a log resolution and denote \( g = f \circ h \). Pushing forward \( K_{Y'} + B_{Y'} = g^*(K_X + \Delta) + E_{Y'} \) via \( h_* \) yields

\[
K_Y + h_* B_{Y'} = f^*(K_X + \Delta) + h_* E_{Y'},
\]

and thus \( h_* B_{Y'} = B_Y \) since \( h_* B_{Y'} \) and \( h_* E_{Y'} \) have no common components. \( \square \)

Lemma 5.4. Let \( (X, \Delta) \) be a log canonical pair. There exists a log resolution \( Y \to X \) such that the components of \( \{ B(X, \Delta)_Y \} \) are disjoint.

Proof. See \cite{KM98} Proposition 2.36 or \cite{HM05} Lemma 6.7. \( \square \)

The main result of this section is the following.
**Theorem 5.5.** Let $X$ be a smooth variety, $S$ a smooth prime divisor and $A$ a very general ample $\mathbb{Q}$-divisor on $X$. For $i = 1, \ldots, \ell$ let $D_i = k_i(K_X + \Delta_i)$, where $(X, \Delta_i = S + B_i + A)$ is a log smooth plt pair with $|\Delta_i| = S$ and $|D_i| \neq \emptyset$. Assume Property $L_1$ in dimensions $\leq \dim X$ and Theorem L2 in dimension $\dim X - 1$. Then the algebra $\text{res}_S R(X; D_1, \ldots, D_{\ell})$ is finitely generated.

**Proof.** Step 1. I first show that we can assume $S \notin \text{Fix} |D_i|$ for all $i$.

To prove this, let $K_X$ be a divisor with $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\text{Supp } A \subset \text{Supp } K_X$, and let $\Lambda$ be the monoid in $\text{Div}(X)$ generated by the components of $K_X$ and all $\Delta_i$. Denote $C_S = \{P \in \Lambda_{\mathbb{R}} : S \notin B(P)\}$. By Property $L_1$, the set $\Lambda = \sum_i \mathbb{R}_+ D_i \cap C_S$ is a rational polyhedral cone.

The monoid $\sum_{i=1}^\ell \mathbb{R}_+ D_i \cap \Lambda$ is finitely generated and let $P_1, \ldots, P_q$ be its generators with $P_i = D_i$ for $i = 1, \ldots, \ell$. Let $\mu : \bigoplus_{i=1}^\ell \mathbb{N} e_i \rightarrow \text{Div}(X)$ be an additive map from a simplicial monoid such that $\mu(e_i) = P_i$. Therefore $\mathcal{S} = \mu^{-1}(A \cap \Lambda) \cap \bigoplus_{i=1}^\ell \mathbb{N} e_i$ is a finitely generated monoid and let $h_1, \ldots, h_m$ be generators of $\mathcal{S}$, and observe that $\mu(h_i)$ is a multiple of an adjoint bundle for every $i$.

Since $\text{res}_S H^0(X, \mu(s)) = 0$ for every $s \in \left( \bigoplus_{i=1}^\ell \mathbb{N} e_i \right) \setminus \mathcal{S}$, we have that the algebra $\text{res}_S R(X, \mu(\bigoplus_{i=1}^\ell \mathbb{N} e_i)) = \text{res}_S R(X; D_1, \ldots, D_{\ell})$ is finitely generated if and only if $\text{res}_S R(X, \mu(\mathcal{S}))$ is. Since we have the diagram

$$
\begin{array}{ccc}
R(X; \mu(h_1), \ldots, \mu(h_m)) & \longrightarrow & R(X, \mu(\mathcal{S})) \\
\downarrow & & \downarrow \\
\text{res}_S R(X; \mu(h_1), \ldots, \mu(h_m)) & \longrightarrow & \text{res}_S R(X, \mu(\mathcal{S}))
\end{array}
$$

where the horizontal maps are natural projections and the vertical maps are restrictions to $S$, it is enough to prove that the restricted algebra $\text{res}_S R(X; \mu(h_1), \ldots, \mu(h_m))$ is finitely generated. By passing to a truncation, I can assume further that $S \notin \text{Fix} |\mu(h_i)|$ for $i = 1, \ldots, m$.

Step 2. Therefore I can assume $\mathcal{S} = \bigoplus_{i=1}^\ell \mathbb{N} e_i$ and $\mu(e_i) = D_i$ for every $i$. For $s = \sum_{i=1}^\ell t_i e_i \in \mathcal{S}_\mathbb{Q}$ and $t_s = \sum_{i=1}^\ell t_i k_i$, denote $\Delta_s = \sum_{i=1}^\ell t_i k_i \Delta_i/t_s$ and $\Omega_s = (\Delta_s - S)_{|S}$. Observe that

$$
R(X; D_1, \ldots, D_{\ell}) = \bigoplus_{s \in \mathcal{S}} H^0(X, t_s(K_X + \Delta_s)).
$$

In this step I show that we can assume that $(S, \Omega_s + A_{|S})$ is terminal for every $s \in \mathcal{S}_\mathbb{Q}$. 
Let $\sum F_k = \bigcup_i \text{Supp } B_i$, and denote $B_i = B(X, \Delta_i)$ and $B = B(X, S + \nu \sum_k F_k + A)$, where $\nu = \max_{i,k}\{\text{mult } F_k B_i\}$. By Lemma 5.4 there is a log resolution $f : Y \to X$ such that the components of $\{B_Y\}$ do not intersect, and denote $D'_i = k_i(K_Y + B_{i'})$. Observe that
\begin{equation}
R(X; D_1, \ldots, D_t) \simeq R(Y; D'_1, \ldots, D'_t).
\end{equation}
Since $B_i \leq \nu \sum F_k$, by comparing discrepancies we see that the components of $\{B_Y\}$ do not intersect for every $i$, and notice that $f^*A = f_s^{-1}A = B_Y$ for every $i$ since $A$ is very general. For $s = \sum_i t_i k_i \in \mathbb{Q}$ and $t_s = \sum_i t_i k_i$, denote $\Delta'_s = \sum_i t_i k_i B_{i}/t_s$. Let $H$ be a small effective $f$-exceptional $\mathbb{Q}$-divisor such that $A' \sim f^*A - H$ is a general ample $\mathbb{Q}$-divisor, and let $T = f_s^{-1}S$. Then, setting $\Psi_s = \Delta'_s - f^*A - T + H \geq 0$ and $\Omega'_s = \Psi_s|_T + A'|_T$, the pair $(T, \Omega'_s + A'|_T)$ is terminal and $K_Y + T + \Psi_s + A' \sim_{\mathbb{Q}} K_Y + \Delta'_s$. Now replace $X$ by $Y$, $S$ by $T$, $\Delta_s$ by $T + \Psi_s + A'$ and $\Omega_s$ by $\Omega'_s$.

Step 3. For every $s \in S$, denote $F_s = \frac{1}{t_s} \text{Fix } |t_s(K_X + \Delta_s)||_S$ and $F_s^2 = \liminf_{m \to \infty} F_{ms}$. Define the maps $\Theta: S \to \text{Div}(S)_{\mathbb{Q}}$ and $\Theta^2: S \to \text{Div}(S)_{\mathbb{Q}}$ by
\begin{equation}
\Theta(s) = \Omega_s - \Omega_s \wedge F_s, \quad \Theta^2(s) = \Omega_s - \Omega_s \wedge F_s^2.
\end{equation}
Then, denoting $\Theta_s = \Theta(s)$ and $\Theta^2_s = \Theta^2(s)$, we have
\begin{equation}
\text{res } R(X; D_1, \ldots, D_t) \simeq \bigoplus_{s \in S} H^0(S, t_s(K_S + \Theta_s))
\end{equation}
by Corollary 7.2. Furthermore, for $s \in S$ let $\varepsilon > 0$ be a rational number such that $\varepsilon(K_X + \Delta_s) + A$ is ample. Then by Theorem 5.1 we have
\begin{equation}
|k_s(K_S + \Omega_s - \Phi_s)| + k_s \Phi_s \subset |k_s(K_X + \Delta_s)||_S
\end{equation}
for any $\Phi_s$ and $k_s$ such that $k_s \Delta_s, k_s \Phi_s \in \text{Div}(X)$ and $\Omega_s \wedge (1 - \frac{\varepsilon}{k_s})F_s \leq \Phi_s \leq \Omega_s$. Then similarly as in the proof of [HM08, Theorem 7.1], by Lemma 3.8 we have that $\Omega_s \wedge F_s^2$ is rational and
\begin{equation}
\text{res } R(X, k_s^2(K_X + \Delta_s)) \simeq R(S, k_s^2(K_S + \Theta^2_s)),
\end{equation}
where $k_s^2 \Theta^2_s$ and $k_s^2 \Delta_s$ are both Cartier. Note also, by the same proof, that $G \not\subset \text{B}(K_S + \Theta^2_s)$ for every component $G$ of $\Theta^2_s$. In particular, $\Theta_{k_s^{2p} \Theta^2_s} = \Theta_{k_s^{2p} \Theta^2_s}$ for every $p \in \mathbb{N}$.

Define maps $\lambda: S \to \text{Div}(S)_{\mathbb{Q}}$ and $\lambda^2: S \to \text{Div}(S)_{\mathbb{Q}}$ by
\begin{equation}
\lambda(s) = t_s(K_S + \Theta_s), \quad \lambda^2(s) = t_s(K_S + \Theta^2_s).
\end{equation}
By Theorem 5.8 below, there is a finite rational polyhedral subdivision $S_{\mathbb{R}} = \bigcup \mathcal{C}_i$ such that the map $\lambda^2$ is linear on each $\mathcal{C}_i$. In particular, there is a sufficiently divisible positive integer $\kappa$ such that $\kappa \lambda^2(s)$ is Cartier.
for every $s \in \mathcal{S}$, and thus $\kappa \lambda^e(s) = \lambda(\kappa s)$ for every $s \in \mathcal{S}$. Therefore the restriction of $\lambda$ to $\mathcal{S}_i^{(\kappa)}$ is additive, where $\mathcal{S}_i = \mathcal{S} \cap \mathcal{C}_i$. If $s_1^i, \ldots, s_x^i$ are generators of $\mathcal{S}_i^{(\kappa)}$, then the Cox ring $R(\mathcal{S}; \lambda(s_1^i), \ldots, \lambda(s_x^i))$ is finitely generated by Theorem 1.2, and so is the algebra $R(\mathcal{S}, \lambda(\mathcal{S}_i^{(\kappa)}))$ by projection. Hence the algebra $\bigoplus_{s \in \mathcal{S}} H^0(\mathcal{S}, \lambda(s))$ is finitely generated, and this together with (1) finishes the proof. □

It remains to prove that the map $\lambda^e$ is rationally piecewise linear. Firstly we have the following result, which can be viewed as a global version of Lemma 3.8. Recall that $\mathcal{S} = \bigoplus_{i=1}^t \mathbb{N}e_i$.

**Lemma 5.6.** There is a positive integer $r$ such that the following stands. If $\Psi \in \text{Div}(\mathcal{S}) \cap \mathcal{C}$ is such that $\text{Supp} \Psi \subset \bigcup_{i=1}^t \text{Supp}(\Omega_{\ell_i} - A_{i|S})$ and no component of $\Psi$ is in $\mathbf{B}(K_\mathcal{S} + \Psi + A_{i|S})$, then no component of $\Psi$ is in Fix $|k(K_\mathcal{S} + \Psi + A_{i|S})|$ for every $k$ with $k(\Psi + A_{i|S})/r$ Cartier.

**Proof.** Let $\sum_{j=1}^q G_j = \bigcup_{i=1}^t \text{Supp}(\Omega_{\ell_i} - A_{i|S})$, and for each $j$ let $\mathcal{P}_{G_j} = \{ \Xi \in \bigcup_{j=0}^1 G_j : G_j \not\subset \mathbf{B}(K_\mathcal{S} + \Xi + A_{i|S}) \}$. Each $\mathcal{P}_{G_j}$ is a rational polytope by Property $\mathcal{C}_\mathcal{A}$. Let $K_\mathcal{S}$ be a divisor such that $\mathcal{O}_\mathcal{S}(K_\mathcal{S}) \simeq \omega_\mathcal{S}$ and $\text{Supp } A \not\subset \text{Supp } K_\mathcal{S}$, let $\mathcal{P}$ be the convex hull of all rational polytopes $K_\mathcal{S} + A_{i|S} + \mathcal{P}_{G_j}$, and set $\mathcal{C} = \mathbb{R}_+ \mathcal{P}$. Observe that $K_\mathcal{S} + \Psi + A_{i|S} \in \mathcal{C}$. Let $G_{q+1}, \ldots, G_w$ be the components of $K_\mathcal{S} + A_{i|S}$ not equal to $G_j$ for $j = 1, \ldots, q$, and let $\Lambda = \bigoplus_{j=1}^w \mathbb{N}G_j$. Then by Theorem 1.2 in dimension $\dim S$ the algebra $R(S, \mathcal{C} \cap \Lambda)$ is finitely generated and the map $\text{Mob}_{\mathcal{C} \cap \Lambda(r)}$ is piecewise additive for some $r$ by Proposition 3.7, where $\iota : \Lambda \to \Lambda$ is the identity map. In particular, if $G_j \not\subset \mathbf{B}(K_\mathcal{S} + \Psi + A_{i|S})$ and $k(\Psi + A_{i|S})/r$ is Cartier, then $G_j \not\subset \text{Fix } |k(K_\mathcal{S} + \Psi + A_{i|S})|$.

**Theorem 5.7.** For any $s, t \in \mathcal{S}_\mathbb{R}$ we have

$$\lim_{\varepsilon \to 0} \Theta^e_{s + \varepsilon(t-s)} = \Theta^e_s.$$ 

**Proof. Step 1.** First we will prove that $\Theta^e_s = \Theta^e_s$, where

$$\Theta^e_s = \Omega_s - \Omega_s \wedge N_\mathcal{S} \| K_\mathcal{S} + \Delta_s \|,$$

cf. Remark 2.5. I am closely following the proof of [Hac08, Theorem 7.16]. Let $r$ be a positive integer as in Lemma 5.6, let $\phi < 1$ be the smallest positive coefficient of $\Omega_s - \Theta^e_s$ if it exists, and set $\phi = 1$ otherwise. Let $V \subset \text{Div}(X)_\mathbb{R}$ and $W \subset \text{Div}(S)_\mathbb{R}$ be the smallest rational affine spaces containing $\Delta_s$ and $\Theta^e_s$ respectively. Let $0 < \eta < 1$ be a rational number such that $\eta(K_\mathcal{S} + \Delta_s) + \frac{1}{2}A$ is ample, and if $\Delta' \in V$ with $\| \Delta' - \Delta_s \| < \eta$, then $\Delta' - \Delta_s + \frac{1}{2}A$ is ample. Then by Lemma 4.2 there are rational points $(\Delta_i, \Theta_i) \in V \times W$ and integers $k_i \gg 0$ such that:
(1) we may write \( \Delta_s = \sum r_i \Delta_i \) and \( \Theta_s = \sum r_i \Theta_i \), where \( r_i > 0 \) and \( \sum r_i = 1 \).

(2) \( k_i \Delta_i / r \) are integral and \( \| \Delta_s - \Delta_i \| < \phi \eta / 2 k_i \).

(3) \( k_i \Theta_i / k_s \) are integral, \( \| \Theta_s - \Theta_i \| < \phi \eta / 2 k_i \) and observe that \( \Theta_i \leq \Omega_i \) since \( k_i \gg 0 \) and \( (\Delta_i, \Theta_i) \in V \times W \).

**Step 2.** Set \( A_i = \Delta_i / k_i \) and \( \Omega_i = (\Delta_i - S)_i \). In this step I prove that for any component \( P \in \text{Supp} \Omega_s \), and for any \( l > 0 \) sufficiently divisible, we have

\[
\text{mult}_P(\Omega_i \wedge \frac{1}{l} \text{Fix} |l(\Delta_s + A_i)|_s) \leq \text{mult}_P(\Omega_i - \Theta_i).
\]

If \( \phi = 1 \), \((\text{2.5})\) follows immediately from Lemma \text{2.6}. Now assume \( 0 < \phi < 1 \). Since \( \| \Omega_s - \Omega_i \| < \phi \eta / 2 k_i \) and \( \| \Theta_s - \Theta_i \| < \phi \eta / 2 k_i \), it suffices to show that

\[
\text{mult}_P(\Omega_i \wedge \frac{1}{l} \text{Fix} |l(\Delta_s + A_i)|_s) \leq (1 - \frac{\eta}{k_i}) \text{mult}_P(\Omega_s - \Theta_s).
\]

Let \( \delta > \eta / k_i \) be a rational number such that \( \delta (\Delta_s + A_i) + \frac{1}{2} A_i \) is ample. Since

\[
\Delta_s + A_i = (1 - \delta)(\Delta_s + A_i) + \frac{1}{2} A_i + (\delta(\Delta_s + A_i) + \frac{1}{2} A_i),
\]

we have

\[
\text{ord}_P \| \Delta_s + A_i \|_s \leq (1 - \delta) \text{ord}_P \| \Delta_s + A_i \|_s,
\]

and thus

\[
\text{mult}_P \frac{1}{l} \text{Fix} |l(\Delta_s + A_i)|_s \leq (1 - \frac{\eta}{k_i}) \sigma_P \| \Delta_s + A_i \|_s
\]

for \( l \) sufficiently divisible, cf. Lemma \text{2.6}.

**Step 3.** In this step we prove that there exists an effective divisor \( H' \) on \( X \) not containing \( S \) such that for all sufficiently divisible positive integers \( m \) we have

\[
|m(\Delta_s + \Theta_i)| + m(\Omega_i - \Theta_i) + (mA_i + H')_s
\]

\[
\subset |m(\Delta_s) + mA_i + H'|_s.
\]

First observe that since \( S \not\subseteq B(K_X + \Delta_s) \) and \( \Delta_i - \Delta_s + A_i \) is ample, we have \( S \not\subseteq B_1(\Omega_i - \Theta_i) \) for \( m \) sufficiently divisible. Assume further that \( m \) is divisible by \( l \), for \( l \) as in Step 2. Let \( f : Y \to X \) be a log resolution of \( (X, \Delta_s + A_i) \) and of \( |m(\Delta_s + A_i)|_s \). Let \( \Gamma = \mathcal{B}(X, \Delta_i + A_i)_Y \) and \( E = K_Y + \Gamma - f^*(K_X + \Delta_i + A_i) \), and define

\[
\Xi = \Gamma - \Gamma \wedge \frac{1}{m} \text{Fix} |m(\Delta_s + \Theta_i)|.
\]

We have that \( m(\Delta_s + \Xi) \) is Cartier, \( \text{Fix} |m(\Delta_s + \Xi)| \wedge \Xi = 0 \) and \( \text{Mob}(m(\Delta_s + \Xi)) \) is free. Since \( \text{Fix} |m(\Delta_s + \Xi)| + \Xi \) has simple normal crossings support, it follows that \( \mathcal{B}(K_Y + \Xi) \) contains no log canonical
centres of \((Y, [\Xi])\). Let \(T = f^{-1}_* S, \Gamma_T = (\Gamma - T)_T\) and \(\Xi_T = (\Xi - T)_T\), and consider a section
\[
\sigma \in H^0(T, \mathcal{O}_T(m(K_T + \Xi_T))) = H^0(T, \mathcal{O}_T(m(K_T + \Xi_T))).
\]
By [HM08, Theorem 5.3], there is an ample divisor \(H\) on \(Y\) such that if \(\tau \in H^0(T, \mathcal{O}_T(H))\), then \(\sigma \cdot \tau\) is in the image of the homomorphism
\[
H^0(Y, \mathcal{O}_Y(m(K_Y + \Xi) + H)) \to H^0(T, \mathcal{O}_T(m(K_T + \Xi_T)))
\]
Therefore
\[
(8) \quad |m(K_T + \Xi_T)| + m(\Gamma_T - \Xi_T) + H|_T \subset |m(K_Y + \Gamma) + H|_T.
\]
We claim that
\[
(9) \quad \Omega_i + A_{i|S} \geq (f|_T)_* \Xi_T \geq \Theta_i + A_{i|S}
\]
and so, as \((S, \Omega_i + A_{i|S})\) is canonical, we have
\[
|m(K_S + \Theta_i)| + m((f|_T)_* \Xi_T - \Theta_i)
\]
\[
\subset |m(K_S + (f|_T)_* \Xi_T)| = (f|_T)_* |m(K_T + \Xi_T)|.
\]
Pushing forward the inclusion \((8)\), we obtain \((7)\) for \(H' = f_* H\).

We will now prove the inequality \((7)\) claimed above. We have \(\Xi_T \leq \Gamma_T\) and \((f|_T)_* \Gamma_T = \Omega_i + A_{i|S}\) and so the first inequality follows.

In order to prove the second inequality, let \(P\) be any prime divisor on \(S\) and let \(P' = (f|_T)_*^{-1} P\). Assume that \(P \subset \text{Supp} \, \Omega_s\), and thus \(P' \subset \text{Supp} \, \Gamma_T\). Then there is a component \(Q\) of the support of \(\Gamma\) such that
\[
\text{mult}_P \, \text{Fix} \, |m(K_Y + \Gamma)|_T = \text{mult}_Q \, \text{Fix} \, |m(K_Y + \Gamma)|
\]
and \(\text{mult}_{P'} \, \Gamma_T = \text{mult}_Q \Gamma\). Therefore
\[
\text{mult}_{P'} \, \Xi_T = \text{mult}_{P'} \, \Gamma_T - \text{min}\{\text{mult}_{P'} \, \Gamma_T, \text{mult}_{P'} \, \frac{1}{m} \text{Fix} \, |m(K_Y + \Gamma)|_T\}.
\]
Notice that \(\text{mult}_{P'} \, \Gamma_T = \text{mult}_P (\Omega_i + A_{i|S})\) and since \(E|_T\) is exceptional, we have that
\[
\text{mult}_{P'} \, \text{Fix} \, |m(K_Y + \Gamma)|_T = \text{mult}_P \, \text{Fix} \, |m(K_X + \Delta_i + A_i)|_S.
\]
Therefore \((f|_T)_* \Xi_T = \Omega_i + A_{i|S} - \Omega_i \land \frac{1}{m} \text{Fix} \, |m(K_X + \Delta_i + A_i)|_S\). The inequality now follows from Step 2.

Step 4. In this step we prove
\[
(10) \quad |k_i(K_S + \Theta_i)| + k_i(\Omega_i - \Theta_i) \subset |k_i(K_X + \Delta_i)|_S.
\]
For any \(\Sigma \in |k_i(K_S + \Theta_i)|\) and any \(m > 0\) sufficiently divisible, we may choose a divisor \(G \in |m(K_X + \Delta_i) + mA_i + H|\) such that \(G|_S = \frac{k}{m} \Sigma + m(\Omega_i - \Theta_i) + (mA_i + H)|_S\). If we define \(\Lambda = \frac{k-1}{m} G + \Delta_i - S - A\), then
\[
k_i(K_X + \Delta_i) \sim_Q K_X + S + \Lambda + A_i - \frac{k-1}{m} H,
\]
where $A_i - \frac{k_i - 1}{m} H$ is ample as $m \gg 0$. By [HM08, Lemma 4.4(3)], we have a surjective homomorphism
\[
H^0(X, J_{S, \Lambda}(k_i(K_X + \Delta_i))) \to H^0(S, J_{\Lambda_i|S}(k_i(K_X + \Delta_i))).
\]
Since $(S, \Omega_i)$ is canonical, $(S, \Omega_i + \frac{k_i - 1}{m} H|_S)$ is klt as $m \gg 0$, and therefore $J_{\Omega_i + \frac{k_i - 1}{m} H|_S} = \mathcal{O}_S$. Since
\[
\Lambda_i|S - (\Sigma + k_i(\Omega_i - \Theta_i)) = \frac{k_i - 1}{m} G_i|_S + \Omega_i - A_i|_S - (\Sigma + k_i(\Omega_i - \Theta_i)) \leq \Omega_i + \frac{k_i - 1}{m} H|_S,
\]
then by [HM08, Lemma 4.3(3)] we have $\mathcal{I}_{\Sigma + k_i(\Omega_i - \Theta_i)} \subset J_{\Lambda_i|S}$, and so
\[
\Sigma + k_i(\Omega_i - \Theta_i) \in |k_i(K_X + \Delta_i)|_S,
\]
which proves (10).

Step 5. There are ample divisors $A_n$ with $\text{Supp} A_n \subset \text{Supp}(\Delta_s - S)$ such that $\|A_n\| \to 0$ and $\Delta_s + A_n$ are $\mathbb{Q}$-divisors. Observe that $\Theta^s = \lim_{n \to \infty} \Theta^n$ with
\[
\Theta^n = \Omega_n - \Omega_n \wedge N_{\sigma}\|K_X + \Delta_n\|_S,
\]
where $\Delta_n = \Delta_s + A_n$ and $\Omega_n = (\Delta_n - S)|_S$. Note that
\[
N_{\sigma}\|K_X + \Delta_n\|_S = \sum \text{ord}_P (\|K_X + \Delta_n\|_S) \cdot P
\]
for all prime divisors $P$ on $S$ for all $n$, cf. Remark 2.5. But then as in Step 3 of the proof of Theorem 5.5, no component of $\Theta^n$ is in $\mathcal{B}(K_S + \Theta^n)$, and thus, by Property $\mathcal{L}_{\sigma}^\mathcal{B}$ and since $\Theta^n \geq \Theta^s$ for every $n$, no component of $\Theta^s$ is in $\mathcal{B}(K_S + \Theta^s)$. Since $k_i$ is divisible by $r$ and $\Theta_i \in W$, by (10) we have
\[
\Omega_i - \Theta_i \geq \Omega_i \wedge \frac{1}{k_i} \text{Fix} |k_i(K_X + \Delta_i)|_S \geq \Omega_i - \Theta^s_i,
\]
and so $\Theta^s_i \geq \Theta_i$, where
\[
\Theta^s_i = \Omega_i - \Theta_i \wedge \liminf_{m \to \infty} \frac{1}{m} \text{Fix} |m(K_X + \Delta_i)|_S.
\]
Let $P$ be a prime divisor on $S$. If $\text{mult}_P \Theta^s_i = 0$, then $\text{mult}_P \Theta^s_i = 0$ since $\Theta^s_i \geq \Theta^s_i$ by Lemma 2.6. Otherwise $\text{mult}_P \Theta_i > 0$ for all $i$ and thus $\text{mult}_P \Theta^s_i > 0$. Therefore by concavity we have
\[
\text{mult}_P \Theta^s_i \geq \sum r_i \text{mult}_P \Theta^s_i \geq \sum r_i \text{mult}_P \Theta_i = \text{mult}_P \Theta^s,
\]
proving the claim from Step 1.
Step 6. Now let $C$ be an ample $\mathbb{Q}$-divisor such that $\Delta_t - \Delta_s + C$ is ample. Then by the claim from Step 1 and by Lemma 2.6, $\Omega_s - \Theta^s_\ast = \Omega_s \wedge \lim_{\varepsilon \to 0} \left( \sum \text{ord}_P \|K_X + \Delta_s + \varepsilon(\Delta_t - \Delta_s + C)\|_s \cdot P \right)$

$$\leq \Omega_s \wedge \lim_{\varepsilon \to 0} \left( \sum \text{ord}_P \|K_X + \Delta_s + \varepsilon(\Delta_t - \Delta_s)\|_s \cdot P \right) \leq \Omega_s - \Theta^s_\ast,$$

where the last inequality follows from convexity. Therefore that inequality is an equality, and this completes the proof.

Now, let $Z$ be a prime divisor on $S$ and let $\mathcal{L}_Z$ be the closure in $S_\mathbb{R}$ of the set $\{s \in S_\mathbb{R} : \text{mult}_Z \Theta^s_\ast > 0\}$. Then $\mathcal{L}_Z$ is a closed cone. Let $\lambda^s_Z : S_\mathbb{R} \to \mathbb{R}$ be the function given by $\lambda^s_Z(s) = \text{mult}_Z \lambda^s(s)$, and similarly for $\Theta^s_\ast$.

**Theorem 5.8.** For every prime divisor $Z$ on $S$, the map $\lambda^s_Z$ is rationally piecewise linear. Therefore, $\lambda^s$ is rationally piecewise linear.

**Proof.** Let $G_1, \ldots, G_w$ be prime divisors on $X$ not equal to $S$ and $\text{Supp} A$ such that $\text{Supp}(\Delta_s - S - A) \subset \sum G_i$ for every $s \in S$. Let $\nu = \max\{\text{mult}_{G_i} \Delta_s : s \in S, i = 1, \ldots, w\} < 1$, and let $0 < \eta < 1 - \nu$ be a rational number such that $A - \eta \sum G_i$ is ample. Let $A' \sim_{\mathbb{Q}} A - \eta \sum G_i$ be a general ample $\mathbb{Q}$-divisor. Define $\Delta'_s = \Delta_s - A + \eta \sum G_i + A' \geq 0$, and observe that $\Delta'_s \sim_{\mathbb{Q}} \Delta_s$, $[\Delta'_s] = S$ and $(S, (\Delta'_s - S)_s)$ is terminal.

Define the map $\chi : S \to \text{Div}(X)$ by $\chi(s) = \kappa t_s(\Delta'_s)$, for $\kappa$ sufficiently divisible. Then as before, we can construct maps $\Theta^s : S_\mathbb{R} \to \text{Div}(S)_\mathbb{R}$, $\tilde{\lambda}^s : S_\mathbb{R} \to \text{Div}(S)_\mathbb{R}$ and $\tilde{\lambda}^s_Z : S_\mathbb{R} \to \mathbb{R}$ associated to $\chi$. By construction, $\text{ord}_E \|\tilde{\lambda}^s_E/\kappa t_s\|_s = \text{ord}_E \|\lambda^s_E/t_s\|_s$, and thus $\text{mult}_Z \tilde{\Theta}^s = \text{mult}_Z \Theta^s_\ast + \eta$ for every $s \in \mathcal{L}_Z$. Let $\tilde{\mathcal{L}}_Z$ be the closure in $S_\mathbb{R}$ of the set $\{s \in S_\mathbb{R} : \text{mult}_Z \Theta^s_\ast > 0\}$, and thus $\mathcal{L}_Z$ is the closure in $S_\mathbb{R}$ of the set $\{s \in S_\mathbb{R} : \text{mult}_Z \Theta^s_\ast > \eta\}$. Note that $\text{mult}_Z \Theta^s_\ast \geq \eta$ for every $s \in \mathcal{L}_Z$ by Theorem [5.7]. Now for every face $\mathcal{F}$ of $S_\mathbb{R}$, either $\mathcal{F} \cap \mathcal{L}_Z \subset \text{relint}(\mathcal{F} \cap \tilde{\mathcal{L}}_Z)$ or $\partial(\mathcal{F} \cap \mathcal{L}_Z) \cap \partial(\mathcal{F} \cap \tilde{\mathcal{L}}_Z) \subset \partial\mathcal{F}$. Therefore by compactness there is a rational polyhedral cone $\mathcal{M}_Z$ such that $\mathcal{L}_Z \subset \mathcal{M}_Z \subset \tilde{\mathcal{L}}_Z$, and so the map $\tilde{\lambda}^s_Z|_{\mathcal{M}_Z}$ is superlinear.

By Theorem [3.10] below, for any 2-plane $H \subset \mathbb{R}^\ell$ the map $\tilde{\lambda}^s_Z|_{\mathcal{M}_Z \cap H}$ is piecewise linear, and thus $\tilde{\lambda}^s_Z|_{\mathcal{M}_Z}$ is piecewise linear by [Laz07, Lemma 3.8].

To prove that $\tilde{\lambda}^s_Z|_{\mathcal{M}_Z}$ is rationally piecewise linear, let $k = \dim \mathcal{M}_Z$ and let $\mathcal{M}_Z = \bigcup \mathcal{C}_m$ be a finite polyhedral decomposition such that $\tilde{\lambda}^s_Z|_{\mathcal{C}_m}$ is linear for every $m$. Let $\mathcal{H}$ be a hyperplane which contains a common $(k - 1)$-dimensional face of cones $\mathcal{C}_i$ and $\mathcal{C}_j$ and assume $\mathcal{H}$ is
not rational. By Step 1 of the proof of [Laz07, Lemma 3.5] there is a point \( s \in \mathcal{C}_i \cap \mathcal{C}_j \) such that the minimal affine rational space containing \( s \) has dimension \( k - 1 \). Then as in Step 1 of the proof of Theorem 5.10 there is a \( k \)-dimensional cone \( \tilde{C} \) such that \( s \in \text{int} \tilde{C} \) and the map \( \lambda^d_z|_{\tilde{C}} \) is linear. But then the cones \( \tilde{C} \cap \mathcal{C}_i \) and \( \tilde{C} \cap \mathcal{C}_j \) are \( k \)-dimensional and linear extensions of \( \lambda^d_z|_{\mathcal{C}_i} \) and \( \lambda^d_z|_{\mathcal{C}_j} \) coincide since they are equal to the linear extension of \( \lambda^d_z|_{\tilde{C}} \), a contradiction. Therefore all \((k - 1)\)-dimensional faces of the cones \( \mathcal{C}_i \) belong to rational hyperplanes and thus \( \mathcal{C}_i \) are rational cones.

Therefore the map \( \lambda^d_z|_{\mathcal{M}_Z} \) is rationally piecewise linear, and since \( \mathcal{L}_Z \) is the closure of the set \( \{ s \in \mathcal{S}_R : \text{mult}_Z \Theta^d_s > \eta \} \), we have that \( \mathcal{L}_Z \) is a rational polyhedral cone, the map \( \lambda^d_z|_{\mathcal{L}_Z} \) is rationally piecewise linear, and therefore so is \( \lambda^d_z \). Now it is trivial that \( \lambda^d \) is a rationally piecewise linear map. \( \square \)

Thus it remains to prove that \( \lambda^d_z|_{\mathcal{M}_Z \cap \mathcal{H}} \) is piecewise linear for every 2-plane \( \mathcal{H} \subset \mathbb{R}^\ell \). As in Step 1 of the proof of Theorem 5.3 by replacing \( \mathcal{S}_R \) by \( \mathcal{M}_Z \) and \( \lambda^d_z \) by \( \tilde{\lambda}^d_z \), it is enough to assume, and I will until the end of the section, that \( \lambda^d_z \) is a superlinear function on \( \mathcal{S}_R \) for a fixed prime divisor \( Z \) on \( S \).

Let \( C_s \) be a local Lipschitz constant of \( \Theta^d \) around \( s \in \mathcal{S}_R \) in the smallest rational affine space containing \( s \). For every \( s \in \mathcal{S} \), let \( \phi_s \) be the smallest coefficient of \( \Omega_s - \Theta^d_s \).

**Theorem 5.9.** Fix \( s \in \mathcal{S}_R \) and let \( U \subset \mathbb{R}^\ell \) be the smallest rational affine subspace containing \( s \). If \( \phi_s > 0 \), let \( 0 < \delta \ll 1 \) be a rational number such that \( \phi_u > 0 \) for \( u \in U \) with \( \| u - s \| \leq \delta \), set \( \phi = \min \{ \phi_u : u \in U, \| u - s \| \leq \delta \} \) and let \( 0 < \varepsilon \ll \delta \) be a rational number such that \((C_s/\phi + 1)\varepsilon(K_X + \Delta_s) + A \) is ample. If \( \phi_s = 0 \) and \( \text{Supp} \Delta_s = \sum F_i \), let \( 0 < \varepsilon \ll 1 \) be a rational number such that \( \sum F_i + A \) is ample for any \( f_i \in (-\varepsilon, \varepsilon) \), and set \( \phi = 1 \). Let \( t \in U \cap \mathcal{S}_Q \) and \( k_t \gg 0 \) be an integer such that \( \| t - s \| < \varepsilon/k_t \), \( k_t \Delta_t/r \) is Cartier for \( r \) as in Lemma 5.7 and \( S \not\subset B(K_X + \Delta_t) \). Then for any divisor \( \Theta \) on \( S \) such that \( \Theta \leq \Omega_t \), \( \| \Theta - \Theta^d_s \| < \phi \varepsilon/k_t \) and \( k_t \Theta/r \) is Cartier we have

\[ |k_t(K_S + \Theta)| + k_t(\Omega_t - \Theta) \subset |k_t(K_X + \Delta_t)|_S. \]

**Proof.** Set \( A_t = A/k_t \). I first prove that for any component \( P \in \text{Supp} \Omega_s \), and for any \( l > 0 \) sufficiently divisible, we have

(11) \( \text{mult}_P(\Omega_t \wedge \frac{1}{l} \text{Fix} |l(K_X + \Delta_t + A_t)|_S) \leq \text{mult}_P(\Omega_t - \Theta). \)
Assume first that \( \phi_s = 0 \). Then in particular \( \text{ord}_P \| K_X + \Delta_s \|_S = 0 \) and \( \Delta_t - \Delta_s + A_t \) is ample since \( \| \Delta_t - \Delta_s \| < \varepsilon/k_t \), so

\[
\text{ord}_P \| K_X + \Delta_t + A_t \|_S = \text{ord}_P \| K_X + \Delta_t + (\Delta_t - \Delta_s + A_t) \|_S \\
\leq \text{ord}_P \| K_X + \Delta_s \|_S = 0.
\]

Since for \( l \) sufficiently divisible we have

\[
(12) \quad \text{mult}_P \frac{1}{t} \text{Fix} |l(K_X + \Delta_t + A_t)|_S = \text{ord}_P \| K_X + \Delta_t + A_t \|_S
\]
as in Step 3 of the proof of Theorem 5.5, we obtain (11).

Now assume that \( \phi_s \neq 0 \) and set \( C = C_s/\phi \). By Lipschitz continuity we have \( \| \Theta^\sharp_t - \Theta^\sharp_s \| < C\varepsilon/k_t \), so \( \| \Theta^\sharp_t - \Theta^\sharp \| < (C + 1)\varepsilon/k_t \). Therefore it suffices to show that

\[
\text{mult}_P(\Omega_t \wedge \frac{1}{t} \text{Fix} |l(K_X + \Delta_t + A_t)|_S) \leq (1 - \frac{C+1}{k_t}\varepsilon) \text{mult}_P(\Omega_t - \Theta^\sharp_t).
\]

Since \( k_t \gg 0 \), we can choose a rational number \( \eta > (C + 1)\varepsilon/k_t \) such that \( \eta(K_X + \Delta_t) + A_t \) is ample. From

\[
K_X + \Delta_t + A_t = (1 - \eta)(K_X + \Delta_t) + (\eta(K_X + \Delta_t) + A_t),
\]

we have

\[
\text{ord}_P \| K_X + \Delta_t + A_t \|_S \leq (1 - \eta) \text{ord}_P \| K_X + \Delta_t \|_S,
\]

and thus by (12),

\[
\text{mult}_P \frac{1}{t} \text{Fix} |l(K_X + \Delta_t + A_t)|_S \leq (1 - \frac{C+1}{k_t}\varepsilon) \text{ord}_P \| K_X + \Delta_t \|_S
\]

for \( l \) sufficiently divisible.

Now the theorem follows as in Steps 3 and 4 of the proof of Theorem 5.10. \( \square \)

Finally, we have

**Theorem 5.10.** Fix \( s \in S_R \) and let \( R \) be a ray in \( S_R \) not containing \( s \). Then there exists a ray \( R' \subset \mathbb{R}_+s + R \) not containing \( s \) such that the map \( \lambda^Z_{\sharp,s+R'}|_{S_R} \) is linear. In particular, for every 2-plane \( H \subset \mathbb{R}^\ell \), the map \( \lambda^Z_{\sharp}|_{S_R \cap H} \) is piecewise linear.

**Proof.** Step 1. Let \( U \subset \mathbb{R}^\ell \) be the smallest rational affine space containing \( s \). In this step I prove that the map \( \Theta^\sharp \) is linear in a neighbourhood of \( s \) contained in \( U \).

Let \( \varepsilon \) and \( \phi \) be as in Theorem 5.9. Let \( W \subset \mathbb{R}^\ell \) and \( V \subset \text{Div}(S)_\mathbb{R} \) be the smallest rational affine spaces containing \( s \) and \( \Theta^\sharp_s \) respectively, and let \( r \) be as in Lemma 5.6. By Lemma 4.2, there exist rational points \( (t_i, \Theta^\sharp_{t_i}) \in W \times V \) and integers \( k_{t_i} \gg 0 \) such that:

1. We may write \( s = \sum r_{t_i}t_i, \Delta_s = \sum r_{t_i}\Delta_{t_i} \) and \( \Theta^\sharp_s = \sum r_{t_i}\Theta^\sharp_{t_i} \), where \( r_{t_i} > 0 \) and \( \sum r_{t_i} = 1 \),
Observe that $S \not\subset B(K_X + \Delta_i)$ since $t_i \in W$ for every $i$ and $\varepsilon \ll 1$ by Property $L^G_A$. By local Lipschitz continuity and by Theorem 5.9 we have that

$$|k_{t_i}(K_S + \Theta_{t_i}^s)| + k_{t_i}(\Omega_{t_i} - \Theta_{t_i}^s) \subset |k_{t_i}(K_X + \Delta_i)|_{s}.$$ 

Since $\Theta_{t_i}^s \in V$ and $k_{t_i} \Theta_{t_i}^s/r$ is Cartier, no component of $\Theta_{t_i}^s$ is in $\text{Fix}|k_{t_i}(K_S + \Theta_{t_i}^s)|$ for every $i$ by Lemma 5.6. In particular,

$$\Omega_{t_i} - \Theta_{t_i}^s \geq \Omega_{t_i} \wedge \frac{1}{k_{t_i}} \text{Fix}|k_{t_i}(K_X + \Delta_i)|_{s} \geq \Omega_{t_i} - \Theta_{t_i}^s,$$

and so

$$\Theta_{t_i}^s \geq \Theta_{t_i}.$$ 

But by assumption (1) and since the map $\Theta_{t}^{s}$ is concave, we have

$$\Theta_{Z}^{s}(s) \geq \sum r_{t_i} \Theta_{Z}^{s}(t_i) \geq \sum r_{t_i} \text{mult}_{Z} \Theta_{t_i}^{s} = \Theta_{Z}^{s}(s),$$

which proves the statement by [Laz07, Lemma 2.6].

**Step 2.** Now assume $s \in S_{Q}$, $\phi_{s} = 0$ and fix $u \in R$ such that $s$ and $u$ belong to a rational affine subspace $P$ of $\mathbb{R}^\ell$. Let $\Delta: \bigoplus_{i=1}^{\ell} \mathbb{R}e_i \to \text{Div}(X)_R$ be a linear map given by $\Delta(p_i) = \Delta_{p_i}$ for linearly independent points $p_1, \ldots, p_\ell \in P \cap S_{Q}$, and then extended linearly. Observe that $\Delta(p) = \Delta_p$ for every $p \in P \cap S_R$.

Let $W$ be the smallest rational affine subspace containing $s$ and $u$. If there is a sequence $s_n \in (s, u]$ such that $\lim_{n \to \infty} s_n = s$ and $\phi_{s_n} = 0$, then $\lambda^s$ is linear on the cone $\mathbb{R}_{+}(s + s_1)$ by [Laz07, Lemma 2.6].

Therefore we can assume that there are rational numbers $0 < \varepsilon, \eta \ll 1$ such that for all $v \in [s, u]$ with $0 < \|v - s\| < 2\varepsilon$ we have $\phi_v > 0$, that for every prime divisor $P$ on $S$, we have either $\text{mult}_{P} \Omega_{v} > \text{mult}_{P} \Theta_{v}^{s}$ or $\text{mult}_{P} \Omega_{v} = \text{mult}_{P} \Theta_{v}^{s}$ and either $\text{mult}_{P} \Theta_{v}^{s} = 0$ or $\text{mult}_{P} \Theta_{v}^{s} > 0$ for all such $v$, and that $\Delta_v - \Delta_s + \Xi + A$ is ample for all such $v$ and for any divisor $\Xi$ such that $\text{Supp} \Xi \subset \text{Supp} \Delta_s \cup \text{Supp} \Delta_u$ and $\|\Xi\| < \eta$.

Pick $t \in (s, u]$ such that $\|s - t\| < \varepsilon/k_s$, $k_s s$ is integral and the smallest rational affine subspace containing $t$ is precisely $W$. Let $0 < \delta \ll 1$ be a rational number such that $\phi_v > 0$ for $v \in W$ with $\|v - t\| \leq \delta$, set $\phi = \min\{\phi_v : v \in W, \|v - t\| \leq \delta\}$ and let $0 < \xi \ll \min\{\delta, \varepsilon\}$ be a rational number such that $(C_t/\phi + 1)\xi(K_X + \Delta_t) + A$ is ample. Denote by $V \subset \text{Div}(S)_R$ the smallest rational affine space containing $\Theta_{t}^{s} = \Theta_{s}$ and $\Theta_{t}^{s}$, and let $r$ be as in Lemma 5.6. Then by Lemma 4.3 there exist rational points $(t_i, \Theta_{t_i}^s) \in W \times V$ and integers $k_{t_i} \gg 0$ such that:
(1) we may write $t = \sum r_i t_i$, $\Delta_t = \sum r_i \Delta_{t_i}$ and $\Theta^t_i = \sum r_i \Theta'_{t_i}$, where $r_i > 0$ and $\sum r_i t_i = 1$,

(2) $t_1 = s$, $\Theta'_t = \Theta^t_1 = \Omega t_1$, $k_t = k_s$,

(3) $k_t \Delta_{t_i}/r$ are integral and $\|t - t_i\| < \xi/k_t$ for $i = 2, \ldots, n - 1$,

(4) $\Theta'_t \leq \Omega t_i, k_t \Theta'_t/r$ are integral, $\|\Theta^t_i - \Theta'_{t_i}\| < \phi \xi/k_t$ and $(t_i, \Theta'_{t_i})$

belong to the smallest rational affine space containing $(t, \Theta^t_i)$ for $i = 2, \ldots, n - 1$,

(5) $\Delta_t = \frac{k_t}{k_t + k_{t_n}} \Delta_{t_1} + \frac{k_{t_n}}{k_t + k_{t_n}} \Delta_{t_n} + \Psi$, where $k_{t_n} \Delta_{t_n}/r$ is integral, $\|t - t_n\| < \varepsilon/k_{t_n}$ and $\|\Psi\| < \eta/(k_t + k_{t_n})$,

(6) $\Theta^t_i = \frac{k_t}{k_t + k_{t_n}} \Theta'_t + \frac{k_{t_n}}{k_t + k_{t_n}} \Theta'_{t_n} + \Phi$, where $\Theta'_{t_n} \leq \Omega t_n$, $k_t \Theta'_{t_n}/r$ is integral, $\|\Theta^t_i - \Theta'_{t_n}\| < \varepsilon/k_{t_n}$ and $\|\Phi\| < \eta/(k_t + k_{t_n})$.

Observe also that $\text{Supp} \Psi \subset \text{Supp} \Delta_t$ and $\text{Supp} \Phi \subset \text{Supp} \Theta^t_i$ by Remarks 4.4 and 4.5 applied to the linear map $\Delta$ defined at the beginning of Step 2. Then by Theorem 5.9

\[ |k_{t_i}(K_S + \Theta'_t)| + k_{t_i}(\Omega t_i - \Theta'_{t_i}) \subset |k_{t_i}(K_X + \Delta_{t_i})| \]

for $i = 2, \ldots, n - 1$. Let $P$ be a component in $\text{Supp} \Omega_t$ and denote $A_{t_n} = A/k_{t_n}$. I claim that

(13) $\text{mult}_P(\Omega_{t_n} \wedge \frac{1}{r} \text{Fix} |l(K_X + \Delta_{t_n} + A_{t_n})|s) \leq \text{mult}_P(\Omega_{t_n} - \Theta'_{t_n})$

for $l \gg 0$ sufficiently divisible. Assume first that $\text{mult}_P \Theta^t = 0$. Then $\text{mult}_P \Theta^t_i = 0$ by the choice of $\varepsilon$, and thus $\text{mult}_P \Theta'_{t_n} = 0$ since $\Theta'_{t_n} \in V$. Therefore

$\text{mult}_P(\Omega_{t_n} \wedge \frac{1}{r} \text{Fix} |l(K_X + \Delta_{t_n} + A_{t_n})|s)$

$\leq \text{mult}_P \Omega_{t_n} = \text{mult}_P(\Omega_{t_n} - \Theta'_{t_n})$.

Now assume that $\text{mult}_P \Theta^t_i > 0$. Then for $l$ sufficiently divisible we have

$\text{mult}_P \frac{1}{r} \text{Fix} |l(K_X + \Delta_{t_n} + A_{t_n})|s = \text{ord}_P \|K_X + \Delta_{t_n} + A_{t_n}\|s$

as in Step 3 of the proof of Theorem 5.9 and since $\Delta_t - \Delta_{t_1} = -\frac{k_{t_1} + k_{t_n}}{k_{t_1}} \Psi + A$ is ample by the choice of $\eta$,

$\text{mult}_P(\Omega_{t_n} \wedge \frac{1}{r} \text{Fix} |l(K_X + \Delta_{t_n} + A_{t_n})|s) \leq \text{ord}_P \|K_X + \Delta_{t_n} + A_{t_n}\|s$

$= \text{ord}_P \|K_X + \Delta_t + \frac{k_{t_1}}{k_{t_n}} (\Delta_t - \Delta_{t_1} - \frac{k_{t_1} + k_{t_n}}{k_{t_1}} \Psi + A)\|s$

$\leq \text{ord}_P \|K_X + \Delta_t\|s = \text{mult}_P(\Omega_t - \Theta^t_i)$.

Combining assumptions (5) and (6) above we have

$\Omega_t - \Theta^t_i \leq \Omega_t - \Theta^t_i + \frac{k_{t_1}}{k_{t_n}} (\Omega_t - \Theta^t_i - \frac{k_{t_1} + k_{t_n}}{k_{t_1}} (\Psi|S - \Phi)) = \Omega_{t_n} - \Theta'_{t_n}$,
and (13) is proved. Furthermore, we can choose \( \varepsilon \ll 1 \) and \( k_{t_n} \gg 0 \) such that \( S \not\subset \mathcal{B}(K_X + \Delta_{t_n}) \). Otherwise, if we denote \( Q = \{ p \in S_\mathbb{R} : S \not\subset \mathcal{B}(K_X + \Delta_p) \} \), \( Q \) is a rational polyhedral cone by Property \( \mathcal{L}^G_\mathcal{A} \), and \( t \in \partial Q \) for every \( t \in [s, u] \) with \( 0 < \| t - s \| \ll 1 \), and thus \( s \in \partial Q \). But then for \( 0 < \| t - s \| \ll 1 \), \( s \) and \( t \) belong to the same face of \( Q \), and so does \( t_n \), a contradiction. Therefore as in the proof of Theorem 5.7 we have

\[
|k_{t_n}(K_S + \Theta'_{t_n})| + k_{t_n}(\Omega_{t_n} - \Theta'_{t_n}) \subset |k_{t_n}(K_X + \Delta_{t_n})|s.
\]

Denote \( \sum G_j = \text{Supp}(\Omega_{s} - A_{|s|}) \cup \text{Supp}(\Omega_{u} - A_{|s|}) \), and let \( Q' = \{ \Xi \in \sum_j [0, 1]G_j : Z \not\subset \mathcal{B}(K_S + \Xi + A_{|s|}) \} \). Then by Property \( \mathcal{L}^G_\mathcal{A} \), \( Q' \) is a rational polytope and \( \Theta^g_p \in Q' \) for every \( p \in S_\mathbb{R} \). Therefore as above and by Theorem 5.7 if \( \varepsilon \ll 1 \) then \( Z \not\subset \mathcal{B}(K_S + \Theta'_{t_n}) \), and as in Step 1 we have that \( \lambda^g_s \) is linear on the cone \( \sum_{i=1}^p \mathbb{R}_+ t_i \), and in particular on the cone \( \mathbb{R}_+ s + \mathbb{R}_+ t \).

**Step 3.** Assume now that \( s \in S_\mathbb{Q} \), \( \phi_s > 0 \) and fix \( u \in \mathcal{R} \). Let again \( W \) be the smallest rational affine space containing \( s \) and \( u \). Let \( 0 < \xi \ll 1 \) be a rational number such that \( \phi_v > 0 \) for \( v \in [s, u] \) with \( \| v - s \| \leq 2\xi \), that for every prime divisor \( P \) on \( S \) we have either \( \text{mult}_P \Omega_v > \text{mult}_P \Theta^g_v \) or \( \text{mult}_P \Omega_v = \text{mult}_P \Theta^g_v \) for all such \( v \), and let \( \phi = \min \{ \phi_v : v \in [s, u] \} \). \( \| v - s \| \leq 2\xi \).

Let \( k_s \) be a positive integer such that \( k_s \Delta_s/r \) and \( k_s \Theta^g_s/r \) are integral, where \( r \) is as in Lemma 5.6. Let us first show that there is a real number \( 0 < \varepsilon \ll \xi \) such that \( (C_t/\phi + 1)\varepsilon(K_X + \Delta_v) + A \) is ample for all \( v \in S_\mathbb{R} \) such that \( \| v - s \| \ll 2\xi \), where \( \| t - s \| = \varepsilon/k_s \). If \( \Theta^g \) is locally Lipschitz around \( s \) this is straightforward. Otherwise, assume \( \Theta^g \) is not locally Lipschitz around \( s \) and assume we cannot find such \( \varepsilon \). But then there is a sequence \( s_n \in (s, u) \) such that \( \lim_{n \to \infty} s_n = s \) and \( C_{s_n} \| s_n - s \| \geq M \), where \( M \) is a constant and \( C_{s_n} \to \infty \). Since a local Lipschitz constant is the maximum of local slopes of the concave function \( \Theta^g_{[s, u]} \), we have that

\[
\frac{\Theta^g_{s_n} - \Theta^g_s}{\|s_n - s\|} > C_{s_n}.
\]

Therefore

\[
\Theta^g_{s_n} - \Theta^g_s > C_{s_n} \|s_n - s\| \geq M
\]

for all \( n \in \mathbb{N} \), which contradicts Theorem 5.7.

Increase \( \varepsilon \) a bit, and pick \( t \in (s, u) \) such that \( \| s - t \| < \varepsilon/k_s \), the smallest rational subspace containing \( t \) is precisely \( W \) and \( (C_t/\phi + 1)\varepsilon(K_X + \Delta_v) + A \) is ample for all \( v \in S_\mathbb{R} \) such that \( \| v - s \| < 2\varepsilon \). In particular, \( \Theta^g \) is locally Lipschitz in a neighbourhood of \( t \) contained in \( W \). Furthermore, by changing \( \phi \) slightly I can assume that \( \phi \leq
min{φ_v : v ∈ W, ||v - t|| < ε}. Denote by V the smallest rational affine space containing Θ'_s and Θ'_t, and let r be as in Lemma 5.6. Then by Lemma 4.3 there exist rational points (t_i, Θ'_t_i) ∈ W × V and integers k_{t_i} + 0 such that:

1. we may write t = ∑ r_{t_i}t_i, Δ_t = ∑ r_{t_i}Δ_{t_i} and Θ'_t = ∑ r_{t_i}Θ'_t_i, where r_{t_i} > 0 and ∑ r_{t_i} = 1,
2. t_i = s, Θ'_t_i = Θ'_t, k_{t_i} = k_s,
3. k_{t_i}Δ_{t_i}/r are integral and ∥t - t_i∥ < ε/k_{t_i} for all i,
4. Θ'_t_i ≤ Ω_{t_i}, k_{t_i}Θ'_t_i/r are integral and ∥Θ'_t_i - Θ'_t_i∥ < φε/k_{t_i}.

Observe that similarly as in Step 2 we have S ∉ B(K_X + Δ_{t_i}) for all i, and therefore by Theorem 5.9

‖k_{t_i}(K_S + Θ'_t_i)‖ + k_{t_i}(Ω_{t_i} - Θ'_t_i) ≤ ‖k_{t_i}(K_X + Δ_{t_i})‖}

for all i. Then we finish as in Step 2.

Step 4. Assume in this step that s ∈ S_R is a non-rational point and fix u ∈ R. By Step 1 there is a rational cone C = ∑_{i=1}^k R+g_i with g_i ∈ S_Q and k > 1 such that λ^Z_{s} is linear on C and s = ∑ α_i g_i with all α_i > 0. Consider the rational point g = ∑_{i=1}^k g_i. Then by Step 2 there is a point s' = α g + β u with α, β > 0 such that the map λ^Z_{s} is linear on the cone R+g + R+s'. Now we have

λ^Z_{s} (C) = λ^Z_{s} (g + s') = λ^Z_{s} (g) + λ^Z_{s} (s') = ∑ λ^Z_{s} (g_i) + λ^Z_{s} (s'),

so the map λ^Z_{s}|_{C+R+s'} is linear by [Laz07, Lemma 2.6]. Taking μ = max_i{α_i} and taking a point ̂u = μ s + u in the relative interior of R+s + R, it is easy to check that

̂u = ∑ (μ α_i - ̂u_i) t_i + ̂u s' ∈ C + R+s',

so the map λ^Z_{s}|_{R+s+R,s} is linear.

Step 5. Finally, let H be any 2-plane in R^ℓ. Then by the previous steps, for every ray R ⊂ S_R ∩ H there is a polyhedral cone C_R with R ⊂ C_R ⊂ S_R ∩ H such that there is a polyhedral decomposition C_R = C_{R,1} ∪ C_{R,2} with λ^Z_{s}|_{C_{R,1}} and λ^Z_{s}|_{C_{R,2}} being linear maps, and if R ⊂ relint(S_R ∩ H), then R ⊂ relint C_R.

Let S^{ℓ-1} be the unit sphere. Restricting to the compact set S^{ℓ-1} ∩ S_R ∩ H we see that λ^Z_{s}|_{S_R ∩ H} is piecewise linear.

6. Proof of the main result

Proof of Theorem 1.2
Step 1. I first show that it is enough to prove the theorem in the case when $A$ is a general ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth klt pair for every $i$.

Let $p$ and $k$ be sufficiently divisible positive integers such that all divisors $k(\Delta_i + pA)$ are very ample and $(p + 1)kA$ is very ample. Let $(p + 1)kA_i$ be a general section of $|k(\Delta_i + pA)|$ and let $(p + 1)kA'$ be a general section of $|(p + 1)kA|$. Set $\Delta_i' = \frac{p}{p + 1}\Delta_i + A_i$. Then the pairs $(X, \Delta_i' + A')$ are klt and

$$(p + 1)k(K_X + \Delta_i + A) \sim (p + 1)k(K_X + \Delta_i' + A') =: D_i'$$

for every $i$. Then a truncation of $R(X; D_1, \ldots, D_\ell)$ is isomorphic to $R(X; D_1', \ldots, D_\ell')$, so it is enough to prove the latter algebra is finitely generated.

Step 2. Therefore I can assume that $\Delta_i = \sum_{j=1}^N \delta_{ij} F_j$ with $\delta_{ij} \in \{0, 1\}$. Write $K_X + \Delta_i + A \sim_{\mathbb{Q}} \sum_{j=1}^N f_{ij} F_j \geq 0$, where $F_j \neq A$ since $A$ is general. By blowing up, and by possibly replacing the pair $(X, \Delta_i)$ by $(Y, \Delta_i')$ for some model $Y \to X$ as in Step 2 of the proof of Theorem [5.5] I can assume that the divisor $\sum_{j=1}^N F_j$ has simple normal crossings. Thus for every $i$,

$$K_X \sim_{\mathbb{Q}} -A + \sum_{j=1}^N f_{ij} F_j,$$

where $f_{ij} = f_{ij}' - \delta_{ij} > -1$.

Let $\Lambda = \bigoplus_{j=1}^N \mathbb{N}F_j \subset \text{Div}(X)$ be a simplicial monoid and denote $T = \{(t_1, \ldots, t_\ell) : t_i \geq 0, \sum t_i = 1\} \subset \mathbb{R}^\ell$. For each $\tau = (t_1, \ldots, t_\ell) \in T$, denote $\delta_{\tau} = \sum_i t_i \delta_{ij}$ and $f_{\tau j} = \sum_i t_i f_{ij}$, and observe that $K_X \sim_{\mathbb{R}} -A + \sum_{j=1}^N f_{\tau j} F_j$. Denote $\mathcal{B}_\tau = \sum_{j=1}^N (\delta_{\tau j} + f_{\tau j}) F_j \subset \Lambda_{\mathbb{R}}$ and let $\mathcal{B} = \bigcup_{\tau \in T} \mathcal{B}_\tau$. It is easy to see that $\mathcal{B}$ is a rational polytope: every point in $\mathcal{B}$ is a barycentric combination of the vertices of $\mathcal{B}_{\tau_1}, \ldots, \mathcal{B}_{\tau_\ell}$, where $\tau_i$ are the standard basis vectors of $\mathbb{R}^\ell$. Thus $\mathcal{C} = \mathbb{R}_+ \mathcal{B}$ is a rational polyhedral cone.

For each $j = 1, \ldots, N$ fix a section $\sigma_j \in H^0(X, F_j)$ such that $\text{div} \sigma_j = F_j$. Consider the $\Lambda$-graded algebra $\mathcal{R} = \bigoplus_{s \in \Lambda} \mathbb{R}_s \subset R(X; F_1, \ldots, F_N)$ generated by the elements of $R(X, \mathcal{C} \cap \Lambda)$ and all $\sigma_j$; observe that $\mathcal{R}_s = H^0(X, s)$ for every $s \in \mathcal{C} \cap \Lambda$. I claim that it is enough to show that $\mathcal{R}$ is finitely generated.

To see this, assume $\mathcal{R}$ is finitely generated and denote

$$\omega_i = rk_i \sum_{j} (\delta_{ij} + f_{ij}) F_j \in \Lambda$$

for $r$ sufficiently divisible and $i = 1, \ldots, \ell$. Set $\mathcal{G} = \sum_i \mathbb{R}_+ \omega_i \cap \Lambda$ and observe that $\omega_i \sim rD_i$. Then by Lemma [3.4.2] the algebra $R(X, \mathcal{C} \cap$
Λ) is finitely generated, and therefore by Proposition 3.7 there is a finite rational polyhedral subdivision $\mathcal{G}_R = \bigcup_k \mathcal{G}_k$ such that the map $\text{Mob}_{\delta}^{\mathcal{G}_k \cap \Lambda}$ is additive up to truncation for every $k$, where $\iota : \Lambda \to \Lambda$ is the identity map.

Let $\omega_1', \ldots, \omega_\ell'$ be generators of $\mathcal{G}$ such that $\omega_i' = \omega_i$ for $i = 1, \ldots, \ell$, and let $\pi : \bigoplus_{i=1}^\ell \mathbb{N} \omega_i' \to \mathcal{G}$ be the natural projection. Then the map $\text{Mob}_{\pi|\mathcal{G} \cap \Lambda}$ is additive up to truncation for every $k$, and thus $R(X, \pi(\bigoplus_{i=1}^\ell \mathbb{N} \omega_i'))$ is finitely generated by Lemma 3.4(3). Therefore $R(X, \pi(\bigoplus_{i=1}^\ell \mathbb{N} \omega_i)) \simeq R(X; rD_1, \ldots, rD_\ell)$ is finitely generated by Lemma 3.4(2), thus $R(X; D_1, \ldots, D_\ell)$ is finitely generated by Lemma 3.4(1).

Step 3. Therefore it suffices to prove that $\mathcal{R}$ is finitely generated. Take a point $\sum_j (f_{rj} + b_{rj})F_j \in \mathcal{B} \setminus \{0\}$; in particular $b_{rj} \in [\delta_{rj}, 1]$. Setting

$$r_\tau = \max_{j=1}^N \{ f_{rj} + b_{rj} \} \quad \text{and} \quad b'_{rj} = -f_{rj} + \frac{f_{rj} + b_{rj}}{r_\tau},$$

we have

$$\sum_j (f_{rj} + b_{rj})F_j = r_\tau \sum_j (f_{rj} + b'_{rj})F_j. \tag{14}$$

Observe that $r_\tau \in (0, 1]$, $b'_{rj} \in [\delta_{rj}, 1]$ and there exists $j_0$ such that $b'_{rj_0} = 1$. For every $j = 1, \ldots, N$, let

$$\mathcal{F}_{rj} = (1 + f_{rj})F_j + \sum_{k \neq j} [\delta_{rk} + f_{rk}, 1 + f_{rk}]F_k,$$

and set $\mathcal{F}_j = \bigcup_{r \in \mathcal{T}} \mathcal{F}_{rj}$, which is a rational polytope. Then $\mathcal{C}_j = \mathbb{R}_+ \mathcal{F}_j$ is a rational polyhedral cone, and (14) shows that $\mathcal{C} = \bigcup_j \mathcal{C}_j$. Furthermore, since $\sum_j (f_{rj} + b_{rj})F_j \sim_{\mathbb{R}} K_X + \sum_j b_{rj}F_j + A$ for $r \in \mathcal{T}$, for every $j$ and for every $s \in \mathcal{C}_j \cap \Lambda$ there is $r_s \in \mathbb{Q}_+$ such that $s \sim_{\mathbb{Q}} r_s (K_X + F_j + \Delta_s + A)$ where $\text{Supp} \Delta_s \subset \sum_{k \neq j} F_k$ and the pair $(X, F_j + \Delta_s + A)$ is log canonical.

Step 4. Assume that the restricted algebra $\text{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda)$ is finitely generated for every $j$. I will show that then $\mathcal{R}$ is finitely generated.

Let $V = \sum_{j=1}^N \mathbb{R}F_j \simeq \mathbb{R}^N$, and let $\| \cdot \|$ be the Euclidean norm on $V$. By compactness there is a constant $C$ such that every $\mathcal{F}_j \subset V$ is contained in the closed ball centred at the origin with radius $C$. Let $\deg$ denote the total degree function on $\Lambda$, i.e. $\deg(\sum_{j=1}^N \alpha_j g_j) = \sum_{j=1}^N \alpha_j$; it induces the degree function on elements of $\mathcal{R}$. Let $M$ be a positive integer such that, for each $j$, $\text{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda)$ is generated by $\{ \sigma|_{F_j} : \sigma \in R(X, \mathcal{C}_j \cap \Lambda), \deg \sigma \leq M \}$, and such that $M \geq C N^{1/2} \max_{i,j} \frac{1}{1 - \delta_{ij}}$.

By Hölder’s inequality we have $\|s\| \geq N^{-1/2} \deg s$ for all $s \in \mathcal{C} \cap \Lambda$, and
thus

\[ \|s\|/C \geq \max_{i,j} \left\{ \frac{1}{1 - \delta_{ij}} \right\} \]

for all \( s \in \mathcal{C} \cap \Lambda \) with \( \deg s \geq M \). Let \( \mathcal{H} \) be a finite set of generators of the finite dimensional vector space

\[ \bigoplus_{s \in \mathcal{C} \cap \Lambda, \deg s \leq M} H^0(X, s) \]

such that for every \( j \), the set \( \{ \sigma_{|F_j} : \sigma \in \mathcal{H} \} \) generates \( \text{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda) \). I claim that \( \mathfrak{R} \) is generated by \( \{ \sigma_1, \ldots, \sigma_N \} \cup \mathcal{H} \), with \( \sigma_j \) as in Step 2.

To that end, take any section \( \sigma \in \mathfrak{R} \) with \( \deg \sigma > M \). By definition, possibly by considering monomial parts of \( \sigma \) and dividing \( \sigma \) by a suitable product of sections \( \sigma_j \), I can assume that \( \sigma \in R(X, \mathcal{C} \cap \Lambda) \). Furthermore, by Step 3 there exists \( w \in \{1, \ldots, N\} \) such that \( \sigma \in R(X, \mathcal{C}_w \cap \Lambda) \), thus there is \( \tau \in \mathcal{T} \cap \mathbb{Q}^k \) such that \( \sigma \in H^0(X, r_{\sigma} \sum_j (f_{\tau j} + b_{\tau j})F_j) \) with \( b_{\tau w} = 1 \). Observe that \( r_{\sigma} \geq \max_{i,j} \frac{1}{1 - \delta_{ij}} \) since \( \| \sum_j (f_{\tau j} + b_{\tau j})F_j \| \leq C \), and in particular \( \frac{r_{\sigma}}{r_{\tau}} \geq \delta_{\tau w} \) for every \( \tau \in \mathcal{T} \).

Therefore by assumption there are elements \( \theta_1, \ldots, \theta_z \in \mathcal{H} \) and a polynomial \( \varphi \in \mathbb{C}[X_1, \ldots, X_z] \) such that \( \sigma_{|F_w} = \varphi(\theta_{1|F_w}, \ldots, \theta_{z|F_w}) \). Therefore by (1) in Remark 3.2

\[ (\sigma - \varphi(\theta_1, \ldots, \theta_z))/\sigma_w \in H^0(X, r_{\sigma} \sum_j (f_{\tau j} + b_{\tau j})F_j - F_w). \]

Since

\[ r_{\sigma} \sum_j (f_{\tau j} + b_{\tau j})F_j - F_w = r_{\sigma} \left( (f_{\tau w} + \frac{r_{\tau w}}{r_{\sigma}})F_w + \sum_{j \neq w} (f_{\tau j} + b_{\tau j})F_j \right), \]

we have \( r_{\sigma} \sum_j (f_{\tau j} + b_{\tau j})F_j - F_w \in \mathcal{C} \cap \Lambda \). We finish by descending induction on \( \deg \sigma \).

**Step 5.** Therefore it remains to show that for each \( j \), the restricted algebra \( \text{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda) \) is finitely generated.

To that end, choose a rational \( 0 < \varepsilon \ll 1 \) such that \( \varepsilon \sum_{k \in I} F_k + A \) is ample for every \( I \subset \{1, \ldots, N\} \), and let \( A_I \sim_{\mathbb{Q}} \varepsilon \sum_{k \in I} F_k + A \) be a very general ample \( \mathbb{Q} \)-divisor. Fix \( j \), and for \( I \subset \{1, \ldots, N\} \setminus \{j\} \) let

\[ \mathcal{F}_{\tau j}^I = (1 + f_{\tau j})F_j + \sum_{k \in I} [1 - \varepsilon + f_{\tau k}, 1 + f_{\tau k}]F_k \]

\[ + \sum_{k \notin I \cup \{j\}} [\delta_{\tau k} + f_{\tau k}, 1 - \varepsilon + f_{\tau k}]F_k. \]

Set \( \mathcal{F}_j^I = \bigcup_{\tau \in \mathcal{T}} \mathcal{F}_{\tau j}^I \); these are rational polytopes such that \( \mathcal{F}_j = \bigcup_{I \subset \{1, \ldots, N\} \setminus \{j\}} \mathcal{F}_j^I \), and therefore \( \mathcal{C}_j^I = \mathbb{R}_+ \mathcal{F}_j^I \) are rational polyhedral
cones such that $C_j = \bigcup_{I \subseteq \{1, \ldots, N\} \setminus \{j\}} C_j^I$. Furthermore, for every $s \in C_j \cap \Lambda$ we have $s \sim_Q r_s(K_X + F_j + \Delta_s + A) \sim_Q r_s(K_X + F_j + \Delta'_s + A_I)$, where $\Delta'_s = \Delta_s - \varepsilon \sum_{k \in j} F_k \geq 0$ and $[F_j + \Delta'_s + A_I] = F_j$.

Therefore it is enough to prove that $\text{res}_{F_j} R(X, C_j^I \cap \Lambda)$ is finitely generated for every $I$. Fix $I$ and let $h_1, \ldots, h_m$ be generators of $C_j^I \cap \Lambda$. Similarly as in Step 1 of the proof of Theorem 5.3 it is enough to prove that the restricted algebra $\text{res}_{F_j} R(X; h_1, \ldots, h_m)$ is finitely generated. For $p$ sufficiently divisible, by the argument above we have $p h_v \sim \rho_v(K_X + F_j + \nu_v + A_I) =: H_v$, where $[B_v] \subset \sum_{k \notin j} F_k$, $[B_v] = 0$, $\rho_v \in \mathbb{N}$ and $A_I$ is a very general ample $\mathbb{Q}$-divisor. Therefore it is enough to show that $\text{res}_{F_j} R(X; h_1, \ldots, h_m)$ is finitely generated by Lemma 3.4(1). But this follows from Theorem 5.3 and the proof is complete. 

Proof of Theorem 5.4. By [FM00, Theorem 5.2] and by induction on $\dim X$, we may assume $K_X + \Delta$ is big. Write $K_X + \Delta \sim_Q B + C$ with $B$ effective and $C$ ample. Let $f : Y \to X$ be a log resolution of $(X, \Delta + B + C)$ and let $H$ be an effective $f$-exceptional divisor such that $f^*C - H$ is ample. Then writing $K_Y + \Gamma = f^*(K_X + \Delta) + E$, where $\Gamma = B(X, \Delta)_Y$, we have that $R(Y, K_Y + \Gamma)$ and $R(X, K_X + \Delta)$ have isomorphic truncations. Since $K_Y + \Gamma \sim_Q (f^*B + H + E) + (f^*C - H)$, we may assume from the start that $\text{Supp}(\Delta + B + C)$ has simple normal crossings. Let $\varepsilon$ be a small positive rational number and set $\Delta' = (\Delta + \varepsilon B) + \varepsilon C$. Then $K_X + \Delta' \sim_Q (\varepsilon + 1)(K_X + \Delta)$, and $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ have isomorphic truncations, so the result follows from Theorem 5.4. 

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