Wave Function Renormalization in Heavy Baryon Chiral Perturbation Theory*

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Abstract

We establish exact relations between relativistic form factors and amplitudes for single–baryon processes and the corresponding quantities calculated in the framework of heavy baryon chiral perturbation theory. A crucial ingredient for the proper matching is the first complete treatment of baryon wave function renormalization in heavy baryon chiral perturbation theory.

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1. Heavy baryon chiral perturbation theory (HBCHPT) is a method to calculate Green functions and S–matrix elements for single–baryon processes in a systematic chiral expansion. Although it singles out a special reference frame characterized by a time–like unit four–vector $v$, Lorentz invariance of the underlying meson–baryon Lagrangian ensures that physical observables like S–matrix elements do not depend on the choice of this frame.

The purpose of this letter is to derive the relativistic form factors and amplitudes from the frame–dependent quantities of HBCHPT. Our main result is that baryon wave function renormalization in HBCHPT depends on the chosen frame through the baryon momentum. In coordinate space, the wave function renormalization “constant” $Z_N$ is in fact a non–trivial differential operator.

2. We first recall the main ingredients of HBCHPT. For definiteness and because the chiral pion–nucleon Lagrangian has a well–established form up to $O(q^3)$, we shall restrict the discussion to chiral $SU(2)$. With the appropriate changes of coupling constants, the discussion and our results can immediately be carried over to chiral $SU(3)$.

The starting point is the generating functional $Z[j, \eta, \bar{\eta}]$ of Green functions defined by the path integral

$$e^{iZ[j,\eta,\bar{\eta}]} = N \int [du d\Psi d\bar{\Psi}] \exp[i\{S_M + S_{MB} + \int d^4x (\bar{\eta}\Psi + \bar{\Psi}\eta)\}] .$$

(1)

The purely mesonic action $\tilde{S}_M$ is the usual one except that the low–energy constants are modified. The difference between $S_M$ and $\tilde{S}_M$ is due to the fermion determinant (closed nucleon loops), analogous to the difference between the mesonic constants for chiral $SU(2)$ and $SU(3)$, respectively. There, the contributions of kaon and eta loops are included in the $SU(2)$ low–energy constants just like the nucleon loops in the present case. The meson–baryon action $S_{MB}$ corresponds to the relativistic pion–nucleon Lagrangian

$$\mathcal{L}_{\pi N} = \bar{\Psi}(i \nabla - m + \frac{\tilde{g}_A}{2} \gamma_5)\Psi + \ldots$$

(2)

where $m, \tilde{g}_A$ are the nucleon mass and the neutron decay constant in the chiral limit. The nucleon doublet is denoted as $\Psi$ and $\eta, \bar{\eta}$ are fermionic sources. The covariant derivative $\nabla_\mu$ and the vielbein field $u_\mu$ are defined as usual: bosonic external fields are denoted collectively as $j$.

The functional $Z[j,\eta,\bar{\eta}]$ generates fully relativistic Green functions. In order to obtain a systematic chiral expansion, one performs a frame–dependent decomposition of the nucleon field $\Psi$ in the functional integral. In this way, the dependence on the nucleon mass $m$ is shifted from the nucleon propagator to the vertices of an effective Lagrangian, so that the integration over the new fermionic variables produces a systematic low–energy expansion.

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1 A similar situation arises in the nonrelativistic treatment of scalar field theories.

2 Our notation is the same as in Refs. 5, 6.
In terms of the velocity–dependent fields \( N_v, H_v \) defined as

\[
N_v(x) = \exp[imv \cdot x] P^+_v \Psi(x) \\
H_v(x) = \exp[imv \cdot x] P^-_v \Psi(x)
\]

with

\[
P^\pm_v = \frac{1}{2} (1 \pm \sigma_\nu v \nu), \quad v^2 = 1,
\]

the pion–nucleon action \( S_{MB} \) takes the form

\[
S_{MB} = \int d^4x \{ \bar{N}_v A N_v + \bar{H}_v B N_v + \bar{N}_v B' H_v - \bar{H}_v C H_v \} \tag{4}
\]

with the spin matrix \( S^\mu = i/2 \gamma_5 \sigma^{\mu\nu} v^\nu \). Rewriting also the source terms in (1) in terms of \( N_v, H_v \) and corresponding sources

\[
\rho_v = e^{imv \cdot x} P^+_v \eta, \quad R_v = e^{imv \cdot x} P^-_v \eta, \tag{5}
\]

one can integrate out the “heavy” components \( H_v \) to obtain a non–local action in the “light” fields \( N_v \) \([1, 2, 3]\). By expanding \( C^{-1} \) in a power series in \( 1/m \),

\[
C^{-1} = \frac{1}{2m} - \frac{iv \cdot \nabla + \frac{\gamma_5 A S \cdot u}{2}}{(2m)^2} + \ldots, \tag{6}
\]

this non–local action turns into a series of local actions of well–defined chiral dimensions. Integration over \( N_v \) leads to

\[
e^{iZ[j, \eta, \bar{\eta}]} = N \int [du] e^{i(S_M[u,j]+Z_{MB}[u,j,\eta,\bar{\eta}])} \tag{7}
\]

where

\[
Z_{MB}[u, j, \eta, \bar{\eta}] = - \int d^4x \{ \bar{\rho}_v(A + B'C^{-1}B)^{-1} \rho_v \\
+ \bar{R}_v C^{-1} B (A + B'C^{-1}B)^{-1} \rho_v + \bar{\rho}_v(A + B'C^{-1}B)^{-1} B'C^{-1} R_v \\
+ \bar{R}_v C^{-1} B (A + B'C^{-1}B)^{-1} B'C^{-1} R_v - \bar{R}_v C^{-1} R_v \}. \tag{8}
\]

Note that \( S_M \) rather than \( \tilde{S}_M \) appears in (7) due to an interchange of limits: \( C^{-1} \) has been expanded in (6) before functional integration over \( N_v \). This makes the fermion determinant trivial to any finite order in \( 1/m \) \([1, 11]\).

From here on, the standard procedure of CHPT \([7]\) can be applied: the action in the functional integral (7) is expanded around the classical solution \( u_{cl}[j] \) of the lowest–order
equation of motion. Integration over the quantum fluctuations generates a systematic low-energy expansion for $Z[j, \eta, \bar{\eta}]$. Although each term in this expansion of definite chiral order depends on the chosen frame, the functional $Z[j, \eta, \bar{\eta}]$ is Lorentz invariant giving rise to fully relativistic Green functions. Again due to the already mentioned interchange of limits ($1/m$ expansion before functional or loop integrations), the equivalence between Green functions calculated in the relativistic formalism and in HBCHPT is strictly true only at tree level, but requires a proper matching of low-energy constants beyond tree level.

3. We have written (8) on purpose as a functional of the original fermionic sources $\eta, \bar{\eta}$ to emphasize its relativistic character. The decomposition (5), on the other hand, is frame dependent. It has been common practice to neglect the “heavy” sources $R_v$ for processes involving baryons (rather than anti–baryons). We will show that already to $O(q^3)$ this omission is not justified.

Although the functional $Z_{MB}[u, j, \eta, \bar{\eta}]$ still depends on the meson fields $u(\phi)$, the nucleons have been integrated out. We can therefore extract the structure of Green functions from this functional as far as external nucleons are concerned. For non–trivial S–matrix elements, this functional must exhibit poles in momentum space in the in– and outgoing nucleon momenta. Since in HBCHPT $C^{-1}$ is expanded in a series of local operators, those poles can only be due to the operator

$$(A + B'C^{-1}B)^{-1}.$$ 

As a consequence, all terms in (8) can contribute to S–matrix elements except for the contact term $\bar{R_v} C^{-1} R_v$. However, the three terms involving the “heavy” source $R_v$ can only produce one–nucleon poles if there are no external lines coming from the factors $C^{-1}B$ or $B'C^{-1}$ next to $\bar{R_v}$ or $R_v$. Thus, the contributions of “heavy” sources appear only in external nucleon propagators. For our subsequent calculation of wave function renormalization to $O(q^3)$, only the field–independent differential operators

$$P_v^{-} C^{-1} B = \frac{1}{2m} P_v^{-} i \not \partial P_v^{+} + \ldots$$

$$B'C^{-1} P_v^{-} = \frac{1}{2m} P_v^{+} i \not \partial P_v^{-} + \ldots$$

will actually matter.

Let us first consider the two–point function. From (7) and (8), after functional integration over the meson fields, the nucleon propagator in momentum space has the general form

$$S_N(p) = P_v^{+} S_{++}(k) P_v^{+} + P_v^{-} S_{+-}(k) P_v^{-} + P_v^{+} S_{-+}(k) P_v^{+} + P_v^{-} S_{--}(k) P_v^{-},$$

with the off–shell momentum $p$ decomposed in the usual way as

$$p = m v + k.$$
with a residual momentum $k$.

In the same way, we deduce the structure of a general $n$–point function ($n \geq 3$):

$$
\left( P^+_v S_{+}(k_{\text{out}}) P^+_v + P^-_v S_{-}(k_{\text{out}}) P^-_v \right) P^+_v T[j] P^+_v \left( P^+_v S_{+}(k_{\text{in}}) P^+_v + P^-_v S_{-}(k_{\text{in}}) P^-_v \right)
$$

$$
= S_N(p_{\text{out}}) P^+_v T[j] P^+_v S_N(p_{\text{in}}),
$$

with an obvious notation for the nucleon momenta. From here on, we always neglect the contact term $\bar{R}_v C^{-1} R_v$ in (8) because it cannot contribute to S–matrix elements. The functional $T[j]$ is the quantity that one calculates with the usual HBCHPT Lagrangians [1, 4, 11], with the external nucleon propagators removed. It depends on the bosonic external fields $j$ and gives rise to Green functions and S–matrix elements along the well–known rules of chiral perturbation theory [7]. We emphasize once again that $S_N(p)$ in (10) and (11) is the fully relativistic propagator.

4. We now turn to the calculation of the nucleon propagator to $O(q^3)$. For this purpose, we recall the pion–nucleon Lagrangian of HBCHPT in the formulation of Ref. [5]:

$$
\tilde{\mathcal{L}}_{\pi N} = \tilde{\mathcal{L}}_{\pi N}^{(1)} + \tilde{\mathcal{L}}_{\pi N}^{(2)} + \tilde{\mathcal{L}}_{\pi N}^{(3)} + \ldots,
$$

$$
\tilde{\mathcal{L}}_{\pi N}^{(1)} = \tilde{N}_v (iv \cdot \nabla + \hat{g}_A S \cdot u) N_v,
$$

$$
\tilde{\mathcal{L}}_{\pi N}^{(2)} = \tilde{N}_v \left( -\frac{1}{2m} (\nabla \cdot \nabla + i \hat{g}_A S \cdot \nabla, v \cdot u) \right) + \frac{a_1}{m} \langle u \cdot u \rangle + \frac{a_2}{m} \langle (v \cdot u)^2 \rangle + \frac{a_3}{m} \langle \chi_+ \rangle + \frac{a_4}{m} \left( \chi_+ - \frac{1}{2} \langle \chi_+ \rangle \right)
$$

$$
+ \frac{1}{m} \varepsilon^{\mu\nu\rho\sigma} v_\rho S_\sigma [i a_5 u_\mu u_\nu + a_6 f_{+\mu\nu} + a_7 v_{j\mu\nu}] N_v
$$

and $\tilde{\mathcal{L}}_{\pi N}^{(3)}$ can be found in Ref. [5]. The term in (14) with coupling constant $a_3$ contributes to the nucleon mass:

$$
\langle \chi_+ \rangle = 4M^2 + \ldots,
$$

with $M$ the pion mass at lowest order, $O(q^2)$.

With the Lagrangian (12), the nucleon propagator is given to $O(q^3)$ by

$$
S_{++}(k) = \frac{1}{2m} P^+_v S_{++} k^+ P^+_v,
$$

$$
S_{+-}(k) = \frac{1}{2m} P^+_v S_{++} k^+ P^-_v,
$$

$$
S_{-+}(k) = \frac{1}{2m} P^-_v S_{++} k^+ P^+_v.
$$

(16)
\( S_{--}(k) \) does not contribute to \( O(q^3) \). The loop contribution of \( O(q^3) \) to the nucleon self-energy is [14]

\[
\Sigma_{\text{loop}}(x) = -\frac{3 g_A^2}{(4\pi F)^2} \left\{ \frac{3}{4} x \left( M^2 - \frac{2}{3} x^2 \right) \left( 32\pi^2 \Lambda(\mu) + \ln \frac{M^2}{\mu^2} \right) \right.
\]
\[
+ \left( M^2 - x^2 \right)^{3/2} \arccos \left( \frac{x}{M} \right) - \frac{x}{2} (M^2 - x^2) \right\} \quad (x^2 < M^2)
\]
\[
\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \ln 4\pi + 1 + \Gamma'(1) \right\} .
\]

The nucleon pole is determined entirely by \( S_{++}(k) \) which can also be written as

\[
S_{++}(k)^{-1} = \frac{1}{2m} \left\{ p^2 - m^2 + 8a_3 M^2 - 2m \Sigma_{\text{loop}}(v \cdot k) \right\} .
\]

We define the on–shell nucleon momentum \( p_N \) as

\[
p = mv + k \quad \text{on–shell} \quad p_N = m_N v + Q , \quad m_N = m + \Delta m ,
\]
\[
p_N^2 = m_N^2 \quad \implies \quad 2m_N v \cdot Q + Q^2 = 0 ,
\]

where \( Q \) is another residual momentum, \( m_N \) is the physical nucleon mass (in the isospin limit) and \( \Delta m \) is at least \( O(q^2) \). Of course, we can always choose a frame for, say, the incoming nucleon with \( Q = 0 \). But unless we are interested only in the forward direction, we can obviously not make the same choice for the outgoing nucleon as well. On–shell we have

\[
v \cdot k = \Delta m - \frac{Q^2}{2m_N} = O(q^2) .
\]

Therefore, to \( O(q^3) \) we find from [17], [18] and [21]

\[
m_N^2 = m^2 - 8a_3 M^2 + 2m \Sigma_{\text{loop}}(0) + O(q^4)
\]

implying

\[
\Delta m = -\frac{4a_3 M^2}{m_N} - \frac{3g_A^2 \pi M_\pi^2}{2(4\pi F_\pi)^2} + O(q^4)
\]

in agreement with Refs. [3, 11].

Wave function renormalization is more subtle. We introduce yet another (arbitrary) four–vector \( r \) to control the on–shell limit \( p \to p_N \) by letting the real parameter \( \lambda \) tend to zero:

\[
p = p_N + \lambda r \]
\[
k = \Delta m v + Q + \lambda r .
\]

Although the final result for \( Z_N \) must be independent of how we approach the nucleon pole, the actual calculation to a given order will profit from a clever choice of \( r \). Choosing \( r = v \),
we have
\[ v \cdot k = v \cdot p_N - m + \lambda \]
\[ \Sigma_{\text{loop}}(v \cdot k) = \Sigma_{\text{loop}}(v \cdot p_N - m) + \lambda \Sigma'_{\text{loop}}(v \cdot p_N - m) + O(\lambda^2) \] (25)
\[ \Sigma'_{\text{loop}}(v \cdot p_N - m) = \Sigma'_{\text{loop}}(0) + O(q^3) \]

and thus
\[ \Sigma'_{\text{loop}}(v \cdot p_N - m) = \frac{-9g_A^2M_\pi^2}{2(4\pi F_\pi)^2} \left[ (4\pi)^2 \Lambda(\mu) + \ln \frac{M_\pi}{\mu} + \frac{1}{3} \right] + O(q^3) \] (26)
where the prime stands for the derivative.

Near the nucleon mass shell, we can write
\[ S_{++}(k)^{-1} = \frac{\lambda}{m} \left[ v \cdot p_N - m \Sigma'_{\text{loop}}(v \cdot p_N - m) \right] + O(\lambda^2) \] (27)
\[ S_N(p) = \frac{m(P_v^+ + \frac{1}{2m} k^\perp)}{\lambda[v \cdot p_N - m \Sigma'_{\text{loop}}(v \cdot p_N - m)] + O(\lambda^2)} \] (28)

Referring to the general structure of Green functions in (11) and recalling the relation between Green functions and S–matrix elements for external fermions, we define the wave function renormalization “constant” \( Z_N(Q) \) in the usual way as
\[ Z_N(Q)u(p_N) = \lim_{p \to p_N} S_N(p)(\not{p} - m_N)u(p_N), \] (29)
implying
\[ Z_N(Q)u(p_N) = \frac{m(P_v^+ + \frac{1}{2m} k^\perp)}{v \cdot p_N - m \Sigma'_{\text{loop}}(v \cdot p_N - m)} \] (30)

Neglecting consistently higher–order terms, we find
\[ Z_N(Q) = \frac{m}{m_N} \frac{v \cdot p_N - \frac{1}{2} v \cdot Q}{v \cdot p_N - m \Sigma'_{\text{loop}}(v \cdot p_N - m)} \] (31)
\[ = \frac{m}{m_N} \frac{1 + v \cdot Q}{1 + v \cdot Q - \Sigma'_{\text{loop}}(v \cdot p_N - m)} \]
and therefore finally
\[ Z_N(Q) = 1 + \frac{4a_3 M_\pi^2}{m_N^2} + \frac{Q^2}{4m_N^2} - \frac{9g_A^2 M_\pi^2}{2(4\pi F_\pi)^2} \left[ (4\pi)^2 \Lambda(\mu) + \ln \frac{M_\pi}{\mu} + \frac{1}{3} \right] + O(q^3). \] (32)

The wave function renormalization “function” \( Z_N(Q) \) is our main result. For \( Q = 0 \), it agrees with the recent result of Fearing et al. \[12\]. However, as already emphasized, one cannot neglect the momentum dependence of \( Z_N(Q) \) altogether, except of course in the forward direction.
5. In principle, any choice of reference frame is equally acceptable due to Lorentz invariance of the theory. In practice, some choices will be more convenient than others for extracting amplitudes of a given order in the low–energy expansion. An especially convenient choice is the “initial–nucleon rest frame” (INRF) defined as

\[ \begin{align*}
p_{\text{in}} &= m_N v + k_1 \\
p_{\text{out}} &= m_N v + q = m_N v + k_2 \\
q &= p_{\text{out}} - p_{\text{in}} = k_2 - k_1 , \quad t = q^2 .
\end{align*} \]

In other words, the INRF corresponds to

\[ Q_{\text{in}} = 0 , \quad Q_{\text{out}} = q . \]

In this frame, wave function renormalization for single–nucleon processes assumes the form

\[ \sqrt{Z_{\text{in}}^N Z_{\text{out}}^N} = \sqrt{Z_N^0 Z_N(q)} \]

\[ = 1 + \frac{4a_3 M^2}{m_N^2} + \frac{t}{8m_N^2} - \frac{9g_A^2 M^2}{2(4\pi F)^2} \left[ (4\pi)^2 \Lambda(\mu) + \ln \frac{M}{\mu} + \frac{1}{3} \right] + O(q^3) . \]

The following relations are useful for actual calculations in the INRF:

\[ \begin{align*}
k_1^2 &= \Delta m^2 = O(q^4) , \\
k_2^2 &= \Delta m^2 + \left( 1 - \frac{\Delta m}{m_N} \right) t = t + O(q^4) \\
v \cdot k_1 &= \Delta m , \\
v \cdot k_2 &= \Delta m - \frac{t}{2m_N} , \\
k_1 \cdot k_2 &= \Delta m^2 - \frac{\Delta m}{2m_N} t = O(q^4) .
\end{align*} \]

In the INRF, it is straightforward to derive relations between HBCHPT amplitudes and their relativistic counterparts. The results are collected in Table 1. Note that, in contrast to (35), the relations in Table 1 are exact, i.e., they hold to all orders in the chiral expansion.

We can now summarize the procedure for obtaining relativistic S–matrix elements for a general one–nucleon process in HBCHPT. We concentrate on the fermionic part because the bosonic part is well–known [7].

- Calculate Green functions with the usual chiral Lagrangian of HBCHPT, e.g., in the form of Ref. [5]. This amounts to considering only the term \( \bar{\rho}_v (A + B' C^{-1} B)^{-1} \rho_v \)

in the generating functional (8). The relevant Green functions are contained in the functional \( T[j] \) in (11).

- Amputate the external nucleon propagators and multiply with a factor \( \sqrt{Z_{\text{in}}^N Z_{\text{out}}^N} \) to account for nucleon wave function renormalization. In the INRF to \( O(q^3) \), the relevant expression is given in (34).

- Relate the HBCHPT amplitudes to the relativistic ones with the help of Table 1. These relations are specific to the INRF.
Table 1: Relations between relativistic covariants and the corresponding quantities in the INRF with $\bar{u}(p_{\text{out}})\Gamma u(p_{\text{in}}) = \bar{u}(p_{\text{out}})P_v^+\tilde{\Gamma}P_v^+u(p_{\text{in}})$.

| $\Gamma$         | $\tilde{\Gamma}$          |
|------------------|----------------------------|
| 1                | 1                          |
| $\gamma_5$      | $\frac{q \cdot S}{m_N(1 - t/4m_N^2)}$ |
| $\gamma^\mu$    | $(1 - t/4m_N^2)^{-1}\left(v^\mu + \frac{q^\mu}{2m_N} + \frac{i}{m_N}\varepsilon^{\mu\nu\rho\sigma}q_\nu v_\rho S_\sigma\right)$ |
| $\gamma^\mu\gamma_5$ | $2S^\mu - \frac{q \cdot S}{m_N(1 - t/4m_N^2)}v^\mu$ |
| $\sigma^{\mu\nu}$ | $2\varepsilon^{\mu\nu\rho\sigma}v_\rho S_\sigma + \frac{1}{2m_N(1 - t/4m_N^2)}\left\{i(q^\mu v^\nu - q^\nu v^\mu) + 2(v^\mu\varepsilon^{\nu\lambda\rho\sigma} - v^\nu\varepsilon^{\mu\lambda\rho\sigma})q_\lambda v_\rho S_\sigma\right\}$ |

6. To demonstrate the effect of wave function renormalization for the matching between HBCHPT and relativistic amplitudes, we investigate some examples at tree level. Thus, for the following discussion we disregard all loop contributions including the one in (35).

For the relativistic Lagrangian, we take the leading–order Lagrangian (2) except for adding a term of $O(q^2)$ to keep track of the nucleon mass correction proportional to $a_3$:

$$L_{\text{rel}} = \bar{\Psi} \left( i \nabla - m + \frac{g_A}{2} \not{\gamma}_5 + \frac{a_3}{m} \langle \chi^+ \rangle \right) \Psi . \quad (37)$$

Our first example concerns the nucleon isovector vector form factors $F_{1}^{V}(t)$ ($i = 1, 2$). From Table 1 we obtain the following exact relation between the relativistic and the HBCHPT matrix elements in the INRF:

$$\langle p_{\text{out}}|\bar{q}\gamma^\mu \tau_a q|p_{\text{in}}\rangle = \bar{u}(p_{\text{out}})\tau_a\left[\gamma^\mu F_{1}^{V}(t) + \frac{i}{2m_N}\sigma^{\mu\nu}q_\nu F_{2}^{V}(t)\right]u(p_{\text{in}}) \quad (38)$$

$$= \left(1 - \frac{t}{4m_N^2}\right)^{-1}\bar{u}_+(p_{\text{out}})\tau_a \left\{ \left[ F_{1}^{V}(t) + \frac{t}{4m_N^2} F_{2}^{V}(t) \right] \left(v^\mu + \frac{q^\mu}{2m_N}\right) \right.$$  

$$+ \left[ F_{1}^{V}(t) + F_{2}^{V}(t) \right] \frac{i}{m_N} \varepsilon^{\mu\nu\rho\sigma}q_\nu v_\rho S_\sigma \right\} u_+(p_{\text{in}}) ,$$

with

$$u_+(p) = P_v^- u(p) .$$

8
The natural quantities emerging in the HBCHPT calculation \cite{11} are the Sachs form factors

\begin{align}
G_E(t) &= F^V_1(t) + \frac{t}{4m_N^2} F^V_2(t) \\
G_M(t) &= F^V_1(t) + F^V_2(t)
\end{align}

(39)

We concentrate here on the more illuminating case of $G_E(t)$. To determine $G_E(t)$ from the respective Lagrangians, we trace the external isovector vector field $V_\mu(x)$ in the covariant derivative

$$
\nabla_\mu = \partial_\mu - iV_\mu + \ldots
$$

(40)

The relativistic calculation with the Lagrangian (37) at tree level is then trivial:

$$
G_E(t) = 1
$$

(41)

The HBCHPT calculation is not as trivial because even with the simple Lagrangian (37) the corresponding Lagrangian

$$
\hat{\mathcal{L}}_{\text{HBCHPT}} = \bar{N}_v (A + B'C^{-1}B) N_v
$$

(42)

consists of a whole tower of terms with increasing chiral dimensions due to the expansion of $C^{-1}$. Since we have calculated wave function renormalization at $O(q^3)$, we can check the equivalence with the relativistic calculation up to the same order.

The relevant part of (42) for the calculation of $G_E(t)$ is\footnote{To avoid confusion with the unit vector $v$, we depart here from the standard notation for the external vector field.}

$$
\hat{\mathcal{L}}_{\text{HBCHPT}} = \bar{N}_v \left( iv \cdot \nabla - \frac{1}{2m} \nabla \cdot \nabla - \frac{1}{8m^2} [\nabla_\mu, [\nabla^\mu, iv \cdot \nabla]] \right) N_v + \ldots
$$

(43)

giving rise to the vertex

$$
\left(1 + \frac{t}{8m^2}\right) v \cdot V + \frac{1}{2m} (2k_1 + q) \cdot V \simeq \left(1 + \frac{t}{8m^2} + \frac{\Delta m}{m}\right) (v + \frac{q}{2m}) \cdot V.
$$

(44)

The difference between the two sides is of higher order. We have nicely reproduced the relevant Lorentz structure for $G_E(t)$ in (38) so that we can immediately read off

$$
\frac{G_E(t)}{1 - t/4m_N^2} = \left(1 + \frac{t}{8m^2} + \frac{\Delta m}{m}\right) \sqrt{Z_{\text{in}}^N Z_{\text{out}}^N}.
$$

(45)

Inserting (38), we find indeed (to $O(q^3)$ in the matrix element)

$$
G_E(t) = 1
$$

(46)

in agreement with the relativistic result \cite{11}.
Another instructive example is the isovector axial form factor $G_A(t)$ defined through the matrix element (neglecting second–class currents) for the isovector axial current

$$\langle p_{\text{out}} | \bar{q} \gamma^\mu \gamma_5 \tau_a q | p_{\text{in}} \rangle = \bar{u}(p_{\text{out}}) \tau_a \left[ \gamma^\mu \gamma_5 G_A(t) + \frac{q^\mu}{2m_N} \gamma_5 G_P(t) \right] u(p_{\text{in}})$$  \hfill (47)

$$= \left( 1 - \frac{t}{4m_N^2} \right)^{-1} \bar{u}_{\partial}(p_{\text{out}}) \tau_a \left[ 2 \left( 1 - \frac{t}{4m_N^2} \right) S^\mu - \frac{q \cdot S}{m_N} v^\mu \right] G_A(t)$$

$$+ \frac{q \cdot S}{2m_N} q^\mu G_P(t) \right] u_{\partial}(p_{\text{in}}).$$

To extract $G_A(t)$ from the Lagrangians (37) and (42), we trace the external isovector axial–vector field $a^\mu(x)$ in the vielbein field

$$u^\mu = 2a^\mu + \ldots$$  \hfill (48)

The relativistic calculation with (37) is again trivial:

$$G_A(t) = \mathcal{g}_A.$$

The HBCHPT calculation is more involved than in the previous case because one has to account for the field transformations used in Ref. [5] to eliminate equation–of–motion terms. We leave it as an exercise to verify that, after some algebra, the relevant piece of the Lagrangian (42) is of the form

$$\hat{\mathcal{L}}_{\text{HBCHPT}} \simeq \hat{N}_v \left\{ \frac{\mathcal{g}_A}{2m} \left( S \cdot \nabla, v \cdot u \right) + \frac{\mathcal{g}_A}{8m^2} \left( \nabla^\mu, \nabla^\nu, S \cdot u \right) \right\} N_v.$$

To $O(q^3)$, this Lagrangian produces of course the appropriate Lorentz structure for $G_A(t)$ in (47). From (47) and (50) we infer

$$G_A(t) = \mathcal{g}_A \left( 1 + \frac{\Delta m}{m} - \frac{t}{8m^2} \right) \sqrt{Z_{\text{in}}^N Z_{\text{out}}^N}$$

and thus to $O(q^3)$

$$G_A(t) = \mathcal{g}_A,$$

again in agreement with the relativistic result (49).

Finally, we list the relevant relations for elastic pion–nucleon scattering. With the kinematics defined by

$$\pi(q_1) + N(p_{\text{in}}) \rightarrow \pi(q_2) + N(p_{\text{out}}),$$

the usual definition of invariant amplitudes is (ignoring isospin)

$$\bar{u}(p_{\text{out}}) [A(\nu, t) + g_1 B(\nu, t)] u(p_{\text{in}}) = \bar{u}(p_{\text{out}}) \left[ D(\nu, t) + \frac{i}{2m_N} \sigma^{\mu \nu} q_2 q_1 B(\nu, t) \right] u(p_{\text{in}})$$  \hfill (54)
\[ D = A + \nu B, \quad \nu = \frac{s - u}{4m_N} = \frac{q_1(p_{in} + p_{out})}{2m_N}. \]

In HBCHPT, the same matrix element can be decomposed as

\[ \bar{u}_+(p_{out}) [\alpha(\nu, t) + i\varepsilon^{\mu\nu\rho\sigma} q_\mu q_\nu v_\rho S_\sigma \beta(\nu, t)] u_+(p_{in}) . \]  

Table 1 yields the translation between relativistic and HBCHPT amplitudes in the INRF:

\[
A(\nu, t) = \alpha(\nu, t) + m_N\nu\beta(\nu, t) \tag{56}
\]

\[
B(\nu, t) = -m_N \left( 1 - \frac{t}{4m_N^2} \right) \beta(\nu, t) . \tag{57}
\]

The complete calculation of \( \alpha \) and \( \beta \) to \( O(q^3) \), including the proper wave function renormalization \( (55) \), can be found in Ref. [13].

7. Most calculations in HBCHPT for chiral \( SU(2) \) have not been done with our favourite Lagrangian in Ref. [5], but with the original version of Ref. [11] that differs by equation–of–motion terms. Wave function renormalization depends on the form of the Lagrangian even though S–matrix elements are unchanged. It is therefore worthwhile to repeat the previous discussion of the nucleon propagator with the Lagrangian of Ref. [11]. Since the loop contribution is unchanged, we suppress it for the time being and reinsert it only in the final result for \( Z_N(Q) \). In the present framework, the only difference to the previous case is in the function

\[ S_{++}^{-1}(k) = \frac{v \cdot k}{2m} + \frac{1}{m} \left[ k^2 - (v \cdot k)^2 \right] + \frac{4a_3M^2}{m} + \text{loops}. \]  

Comparing with \( (16) \), we observe that the equation–of–motion terms in the second–order Lagrangian of Ref. [11] induce a term proportional to \( (v \cdot k)^2 \). Even though \( (57) \) is not of the simple form \( (18) \), the pole position is of course unchanged, giving the same mass correction \( (23) \) as before. On the other hand, wave function renormalization is simpler here because instead of \( (27) \) one now has

\[ S_{++}(k) = \lambda + O(\lambda^2) \]  

near the mass shell leading to

\[ Z_N(Q) = \frac{1}{m_N} \left( v \cdot p_N - \frac{v \cdot Q}{2} \right) = 1 + \frac{v \cdot Q}{2m_N} = 1 - \frac{Q^2}{4m_N^2} \]  

to the required accuracy. Reinserting the loop contribution, we compare the final result with the previous one in Table 2. As expected, the two functions \( Z_N(Q) \) differ due to the field transformation necessary to pass from one form of the Lagrangian to the other. Since that field transformation is accomplished by a non–trivial differential operator \( [5] \), the change of sign in the term proportional to \( Q^2 \) should not come as a surprise.
Table 2: Wave function renormalization $Z_N(Q)$ for the Lagrangians of Refs. [5] and [11], respectively, with $\Sigma_{\text{loop}}' = -\frac{9g_A^2M_{\pi}^2}{2(4\pi F_{\pi})^2} \left[ (4\pi)^2 \Lambda(\mu) + \ln \frac{M_{\pi}}{\mu} + \frac{1}{3} \right]$.

| Lagrangian | $Z_N(Q)$ |
|------------|----------|
| EM [5]     | $1 + \frac{Q^2}{4m_N^2} + \frac{4a_3M_{\pi}^2}{m_N^2} + \Sigma_{\text{loop}}' + O(q^3)$ |
| BKKM [11]  | $1 - \frac{Q^2}{4m_N^2} + \Sigma_{\text{loop}}' + O(q^3)$ |

8. We have analysed the structure of the generating functional of Green functions in HBCHPT. This analysis has led to the first complete treatment of baryon wave function renormalization to $O(q^3)$. We have shown that wave function renormalization in HBCHPT cannot be described by a constant. Instead, in momentum space it depends on the chosen frame via the baryon momentum.

We have also presented the general relations between HBCHPT and relativistic amplitudes in the convenient initial–nucleon rest frame. Those relations are exact, i.e., they hold to all orders in the low–energy expansion. We have checked the correspondence for the vector and axial–vector nucleon form factors at tree level. The correct expression for wave function renormalization is crucial for this correspondence.

With our results, the relativistic amplitudes for any single–baryon process can unambiguously be calculated in HBCHPT to $O(q^3)$, e.g., for elastic pion–nucleon scattering [13].

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