Spin-field Interaction Effects and Loop Dynamics in AdS/CFT Duality

A. I. Karanikas and C. N. Ktorides

University of Athens, Physics Department
Nuclear & Particle Physics Section
Panepistimiopolis, Ilisia GR 157-71, Athens, Greece

Abstract

The spin-field interaction is considered, in the context of the gauge fields/string correspondence, in the large 't Hooft coupling limit. The latter can be viewed as a WKB-type approximation to the AdS/CFT duality conjecture. Basic theoretical objects entering the present study are (a) the Wilson loop functional, on the gauge field side and (b) the sigma model action for the string propagating in AdS_5. Spin effects are introduced in a worldline setting, via the spin factor for a particle entity propagating on a Wilson loop contour. The computational tools employed for conducting the relevant analysis, follow the methodological guidelines introduced in two papers by Polyakov and Rychkov. The main result is expressed in terms of the modification of the spin factor brought about by dynamical effects, both perturbative and non-perturbative, according to AdS/CFT in the considered limit.
1. Introduction

The connection between gauge field theories and strings has been posed, as a fundamental problem in theoretical physics, over two and a half decades ago by Polyakov [1]. On the field theoretical side and in the context of ’t Hooft’s [2] $\lambda \equiv g^2YM_N \gg 1$ limit, it was proposed in [3] that the quantity upon which such a relation can be pursued is the Wilson loop functional

$$W[C] = \frac{1}{N} \langle Tr P \exp \oint_C A_\mu dx_\mu \rangle$$

whose (closed) contour should provide a ‘base’ on which the two ends of an (open) string, propagating in five dimensions, are to be attached. The working assumption for quantifying this proposal is that, in the large $\lambda$ limit, the Wilson loop functional is expected to behave as

$$W[C] \propto e^{-\sqrt{\lambda}A_{\text{min}}(C)},$$

where $A_{\text{min}}$ is the minimal area swept by the string and is bounded by the contour $C$. This statement constitutes a zeroth, WKB-type, approximation to the problem.

As is well known, in a virtual simultaneity with [3], the AdS/CFT conjecture [5], followed by a number of key papers, most notably [6,7], which further elucidated its content, placed the gauge field/string duality issue on very concrete grounds, albeit ones that favor conformally symmetric gauge field theories (in particular the $\mathcal{N}=4$ supersymmetric YM system). Within this context, direct studies addressing themselves to the calculation of expectation values of a Wilson loop operator whose contour is traversed by heavy quarks, were first conducted by Maldacena in [8], followed by a more extensive investigation in [9], as well as by Rey and Yee [10]. In these approaches, the relevant Wilson loop functional takes the form

$$W[C] = \frac{1}{N} \left( Tr P \exp \oint_C (A_\mu dx_\mu + \Phi_i dy_i) \right) i = 1, \ldots, 6,$$

where the $\Phi_i$ comprise a set of massive Higgs scalars, simulating heavy quarks, in the adjoint representation of the SU(N) group. Such considerations have direct relevance to studies of the static potential problem in QCD, albeit in its $\mathcal{N}=4$ supersymmetric version.

On the other hand and taking as point of reference work of Polchinski and Strassler [11], the AdS/CFT framework can facilitate analyses which pertain to contributions, both perturbative and non-perturbative, associated with dynamical, hard scattering QCD processes. To the extent, then, that a Wilson loop configuration could enter a given description referring to a dynamical process, it becomes of interest to study the effects on the corresponding
Moving, now, away from the heavy quark limit inevitably brings into play the spin-field interaction term. It is this particular issue that the present work intends to address, always in a context wherein Eq. (1) is assumed to be valid, \textit{i.e.} in the limit of very large \(\lambda\). The way by which we propose to attack the problem is the following. First, we consider a particular casting, one we happen to be more familiar with [12], pertaining to the worldline description of the propagation, in spacetime, of a (matter) particle entity which enters a generic gauge field theoretical system. The specific feature of this casting is that employs a quantity known as \textit{spin factor} [13] which accounts for the particle’s spin \(j\). The relevant expression furnishes the probability amplitude associated with the propagation of the particle mode along a Wilson contour. The latter will be placed in 4-dimensional space -to be viewed as the boundary of AdS\(_5\) with the enclosed surface bulging, in general, into AdS\(_5\). It will not be required for the contour to be everywhere differentiable, simply piecewise continuous. On the other hand, we shall not consider either loop self intersections or retracings. The basic tool at our disposal for bringing spin effects into play is the area derivative operator [14], which facilitates the emergence of the spin factor through the employment of a second order formalism procedure. This task will be presented in section 2.

In section 3, we shall direct our considerations towards the string side of the story. In particular, we shall deal with the string action functional which furnishes the area element associated with the surface swept by the ‘chromoelectric’ string propagating in the, curved, AdS\(_5\) space. The minimization of the generated area, which is bounded by a Wilson loop, will be discussed within the framework of the analysis devised in two remarkable papers by Polyakov and Rychkov [15, 16]. The systematic, variational procedure developed in these works for describing the minimum area \(A_{\text{min}}\), leads to the emergence of the all-important \(\bar{g}(\sigma)\)-function which, in principle, carries all the dynamics (perturbative and nonperturbative) of the system. For our particular purposes, this setting becomes quintessential in connecting the worldline formalism, that was displayed in the previous section, with the string description. The basic computational guidelines pertaining to our investigation will be exhibited in this section; essentially, they concern the action of the area derivative operator on \(A_{\text{min}}\). The relevant discussion will be kept at a general, ‘semiheuristic’, level, leaving
the display of all the technical details to an Appendix.

Our central result is exhibited in section 4, where the effect on the loop functional, induced by the spin-field interaction dynamics, is formulated in terms of the $\vec{g}$-function. Some general suggestions pertaining to applications of this result are made in the concluding section.

2. The spin factor in the worldline formalism

Consider a particle entity of a given spin $j$ and with finite mass propagating on a closed worldline contour $\vec{c}(\sigma)$ (in Euclidean space) while interacting with a dynamical field $\vec{A}$. Following Ref. [12], the basic quantum mechanical quantity associated with the process can be written in the form (all indices suppressed)

$$K(T) = \text{Tr} \int_{\vec{c}(0)=\vec{c}(T)} D\vec{c}(\sigma) \exp \left( -\frac{1}{4} \int_{0}^{T} d\sigma \vec{c}^2(\sigma) \right) \left\langle P \exp \left( i \int_{0}^{T} d\vec{c} \cdot \vec{A} + \int_{0}^{T} d\sigma F \cdot J \right) \right\rangle_A. \quad (2)$$

This expression determines the probability amplitude for the system to evolve from $\vec{c}(0)$ to $\vec{c}(T)$. The matrices $J_{\mu\nu}$ stand for the Lorentz generators\(^1\) corresponding to the spin of the propagating particle entity, so the last term represents the spin-field interaction.

The above equation can be recast into the form

$$K(T) = \text{Tr} \int_{\vec{c}(0)=\vec{c}(T)} D\vec{c}(\sigma) \exp \left( -\frac{1}{4} \int_{0}^{T} d\sigma \vec{c}^2(\sigma) \right) \left\langle P \exp \left( i \int_{0}^{T} d\vec{c} (\sigma) \cdot \vec{A} \right) \right\rangle_A,$$

$$(3)$$

where

$$\frac{\delta}{\delta s_{\mu\nu}(\sigma)} = \lim_{\eta \to 0} \eta \int_{-\eta}^{\eta} dh \frac{\delta^2}{\delta c_{\mu}(\sigma + \frac{h}{2}) \delta c_{\nu}(\sigma - \frac{h}{2})}$$

defines the (regularized) expression for the area derivative operator [14].

Strictly speaking, expression (3) has well defined meaning only for smooth loops. On the other hand, when expressions like (2) are used to describe physically interesting, scattering processes the contour is forced to pass through points $x_i$ where momentum is imparted by

\(^1\)The term 'Lorentz' is used by abuse of language, given that we are working in the Euclidean formalism.
an external agent (field). Such a situation is realized by inserting a chain of delta functions \( \delta[\vec{c}(\sigma_i) - \vec{x}_i] \) in the integrand which produce a loop with cusps. Accordingly, the area derivative operator entering (3) must be understood piecewise, \( i.e. \),

\[
P \exp \left( \frac{i}{2} \int_0^T d\sigma \mathbf{J} \cdot \frac{\delta}{\delta s} \right) \rightarrow \cdots P \exp \left( \frac{i}{2} \int_{s_1}^{s_2} d\sigma \mathbf{J} \cdot \frac{\delta}{\delta s} \right) P \exp \left( \frac{i}{2} \int_0^{s_1} d\sigma \mathbf{J} \cdot \frac{\delta}{\delta s} \right). \quad (5)
\]

An integration by parts can now be performed to reformulate the amplitude (3) as follows

\[
K(T) = \text{Tr} \int_{c(0)=c(T)} \mathcal{D}c(\sigma) \exp \left( -\frac{1}{4} \int_0^T d\sigma \vec{c}'^2(\sigma) \right) P \exp \left( \frac{i}{2} \int_0^T d\sigma \mathbf{J} \cdot \omega(c) \right) \\
\times \left\langle P \exp \left( i \int_0^T d\vec{c} \cdot \mathbf{A} \right) \right\rangle_A,
\]

where

\[
\int_0^T d\sigma \omega_{\mu\nu}(c) = \lim_{\eta \to 0} \frac{1}{2} \int_{-\eta}^\eta dh \int_0^T d\sigma c''_{\mu}(\sigma - \frac{h}{2}) c''_{\nu}(\sigma + \frac{h}{2}) \\
= \frac{1}{4} \lim_{\eta \to 0} \int_{-\eta}^\eta dh \int_0^T d\sigma_1 \int_0^T d\sigma_2 c'_{[\mu}(c_2)c''_{\nu]}(\sigma_1) \delta(\sigma_2 - \sigma_1 - h). \quad (7)
\]

The quantity \( \omega_{\mu\nu}(c) \) defines the spin factor [12] associated with the particle mode propagating on the closed contour and has a geometrical/topological content. Thus, for example, in two dimensions it serves to distinguish free bosons from free fermions by the fact that a member of the latter species carries a factor \( (-1)^\nu \) when traversing a closed worldline contour \( \nu + 1 \) times. Our goal is to identify dynamical consequences associated with its presence which, by definition, are attributable to the spin of a particle mode propagating on the Wilson loop contour. As already mentioned, our attention will be restricted, throughout, to single traversals of, non self-intersecting, loops.

A general observation to make at this point is that Eq (7) leads to a vanishing result for the spin factor if, in the absence of the Wilson loop, the worldline contour is everywhere differentiable, since the presence of a factor \( \delta'(h) \) is required for the opposite to be the case. Non-trivial, dynamically induced, effects attributed to spin thereby demand the presence of Wilson loop configurations which include points of interaction of the propagating entity with the dynamical field. In a perturbative treatment of such a situation one expands the Wilson loop in a power series which produces the familiar vertices. Explicitly, the presence of the
spin factor gives rise to correlators of the form
\[
\lim_{\eta \to 0} \int_0^\eta d\eta \int_0^T d\sigma \left\langle c'\mu (\sigma - \frac{h}{2}) c''\nu (\sigma + \frac{h}{2}) c'(\sigma_1) c'(\sigma_2) \right\rangle_C \sim \delta(\sigma_2 - \sigma_1)(\delta_{\mu\kappa}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\kappa}),
\]
where \(\langle \cdots \rangle_C\) signifies averaging over paths.

It is evident from the above analysis that the spin factor incorporates, in a geometrical manner, the spin-field dynamics. The challenge, now, can be described as follows: Determine the expression for the minimal area on the string side and, upon doing that, use the operation in (3) to assess the nonperturbative, dynamical impact on the spin factor.

3. The area derivative of the Wilson loop functional

In Refs [15,16] a mathematical machinery was developed for the purpose of studying loop dynamics in the framework of the AdS/CFT correspondence (in the WKB approximation). We shall adopt the strategy introduced in these works with the eventual aim being the determination of the action of the area derivative, as given by (4), on a, piecewise continuous, Wilson loop functional – as demanded by Eq. (3). The dynamics of the chromo-electric flux lines is described, according to [3,8,16], by a relativistic string propagating in a five-dimensional (AdS) curved background. The relevant action functional (Euclidean formalism adopted throughout our analysis) is given by [15]
\[
S[\vec{x}(\xi), y(\xi)] = \frac{1}{2}\sqrt{\lambda} \int_D G_{MN}(x(\xi)) \partial_a x^M(\xi) \partial_a x^N(\xi)
\]
\[
= \frac{1}{2}\sqrt{\lambda} \int_D \frac{d^2\xi}{y^2(\xi)} [(\partial_a \vec{x}(\xi))^2 + (\partial_a y(\xi))^2],
\]
where \(x^M = (y, \vec{x}) = (y, x^\mu), M, N = 0, 1, \cdots, 4; \mu = 1, \cdots, 4\), with the \(y\)-coordinate taking a zero value at the boundary and growing toward infinity as one moves deeper into the interior of the AdS\(_5\) space.

The above functional is to be minimized under the boundary conditions \(\vec{x}|_{\partial D} = \vec{c}(\alpha(\sigma))\) and \(y|_{\partial D} = 0\), with the parametrization chosen so that
\[
A_{\text{min}}[c(\sigma)] = \min\{\alpha(\sigma)\} \min\{\vec{x}, y\} S[\vec{x}(\xi), y(\xi)].
\]
The functional \(A_{\text{min}}\) is invariant under reparametrizations of the boundary, a property that
can be easily deduced from the minimization condition (10):

$$c'_\mu(\sigma)\frac{\delta A_{\text{min}}}{\delta c_\mu(\sigma)} = 0.$$  

(11)

Following Refs [15,16], we adopt the static gauge $y(t, \sigma) = t$ and place the loop on the boundary of the AdS space, i.e. set $t = 0$. One, accordingly, writes

$$\vec{x}(t, \sigma) = \vec{c}(\sigma) + \frac{1}{2} \vec{f}(\sigma)t^2 + \frac{1}{3} \vec{g}(\sigma)t^3 + \cdots$$  

(12)

where, for now, the curve $\vec{c}(\sigma)$ is assumed to be everywhere differentiable. If there are cusps on the loop contour (i.e., discontinuities in the first derivative) the above expansion must be understood piecewise. Surface minimization eliminates the linear term in the expansion and determines its next coefficient:

$$\vec{f} = \vec{c}'^2 \frac{d}{d\sigma} \frac{\vec{c}'}{\vec{c}'^2}.$$  

(13)

The coefficient $\vec{g}(\sigma)$ is, at this point, unspecified. Employment of the Virasoro constraints leads to

$$\vec{c} \cdot \vec{g} = 0.$$  

(14)

It turns out that the latter relation simply expresses the reparametrization invariance of the minimal area (10) and, hence, the quantity $\vec{g}(\sigma)$, to be referred to as $\vec{g}$-function from hereon, remains undetermined. More illuminating, for our purposes, is an interim result through which (14) is derived and reads as follows

$$\frac{\delta A_{\text{min}}}{\delta \vec{c}(\sigma)} = -\sqrt{\vec{c}'^2} \vec{g}(\sigma).$$  

(15)

The above relation underlines the dynamical significance of the $\vec{g}$-function: It provides a measure of the change of $A_{\text{min}}$ when the Wilson loop contour is altered as a result of some interaction which reshapes its geometrical profile.

Consider now the action of the area derivative on the Wilson loop functional:

$$\frac{\delta}{\delta s_{\mu\nu}(s)} W[C] = \lim_{\eta \to 0} \int_{-\eta}^{\eta} dh h \left[ -\sqrt{\lambda} \frac{\delta^2 A_{\text{min}}}{\delta c_\mu \left( \sigma + \frac{h}{2} \right) \delta c_\nu \left( \sigma - \frac{h}{2} \right)} + \lambda \frac{\delta A_{\text{min}}}{\delta c_\mu \left( \sigma + \frac{h}{2} \right)} \frac{\delta A_{\text{min}}}{\delta c_\nu \left( \sigma - \frac{h}{2} \right)} \right] W[C].$$  

(16)

As it is known [17], the area derivative is a well defined operation only for smooth contours, i.e. everywhere differentiable. In such a case the last term in the above equation gives zero
contribution. If the loop under consideration has cusps (as happens in the framework of non-trivial applications of the worldline formalism) the operation must be understood piecewise [18].

In order to facilitate our considerations we follow Ref(s) [15, 16] by choosing the coordinate $\sigma$ on the minimal surface such that

$$\vec{c}'^2(\sigma) = 1, \quad \vec{x}(t, \sigma) \cdot \vec{c}'(\sigma) = 0.$$ 

We also introduce an orthonormal basis, which adjusts itself along the tangential ($\vec{t}$) and normal ($\vec{n}^a, a = 1, \cdots, D - 1$) directions defined by the contour, as follows

$$\{\vec{t}, \vec{n}^a\}, \ a = 1, \cdots, D - 1$$

$$\vec{t} = \frac{\vec{c}'}{\sqrt{\vec{c}'^2}}, \quad \vec{n}^a \cdot \vec{t} = 0, \quad \vec{n}^a \cdot \vec{n}^b = \delta^{ab}. \quad (17)$$

We now write

$$\frac{\delta}{\delta c_\mu} = n^a_\mu \left(\frac{\delta}{\delta c^a} + t_\mu \frac{\delta}{\delta \vec{t}}\right) \equiv n^a_\mu \delta_{\vec{c}^a} + t_\mu \frac{\delta}{\delta \vec{t}} \quad (18)$$

and upon using relations (14) and (15), as well as setting $s = \sigma + h/2$ and $s' = \sigma - h/2$, we determine

$$\frac{\delta^2 A_{\text{min}}}{\delta c_\mu(s) \delta c_\nu(s')} = -\frac{\delta g^a(s)}{\delta \vec{n}^a(s')} n^a_\mu(s)n^b_\nu(s') + R_{\mu\nu}(s, s') \delta'(s - s'), \quad (19)$$

where

$$R_{\mu\nu}(s, s') = 2\tilde{g}(s) \cdot \vec{n}^a(s')t_\mu(s)n^a_\nu(s') + \tilde{g}(s) \cdot \vec{t}(s')t_\mu(s)t_\nu(s') - \vec{t}(s) \cdot \vec{n}^a(s')g_{\mu\nu}(s)n^a_\nu(s'). \quad (20)$$

Given the defining expression for the area derivative, cf. Eq (4), one immediately realizes that only terms $\sim \delta'(s - s')$ in an antisymmetric combination $R_{[\mu\nu]}$ will give non-zero contributions to the area derivative. It, thus, becomes obvious that the last term in Eq (19) produces the result

$$R_{[\mu\nu]}(\sigma, \sigma) = t_{[\mu}(\sigma)g_{\nu]}(\sigma). \quad (21)$$

Turning our attention to the first term on the rhs of (20) we note that non-vanishing contributions should have the form

$$(r^aq^b - r^bq^a)n^a_\mu n^b_\nu \delta'(s - s'), \quad (22)$$
where \( r^a = \vec{n}^a \cdot \vec{r} \) and \( q^a = \vec{n}^a \cdot \vec{q} \). These functions must be determined from the coefficients of the expansion (12); otherwise the above contribution would be contour independent, having no impact on a calculation associated with non-trivial dynamics. In conclusion, a simple qualitative analysis, based on the scale invariance of \( A_{\text{min}} \), indicates that a contribution of the type (22) does not exist. This qualitative observation can be further substantiated through a straightforward argument based on dimensional grounds. Indeed, from Eq. (12) it can be observed that under a change of scale of the form \( \vec{c} \rightarrow \lambda \vec{c}, (t, \sigma) \rightarrow (\lambda t, \lambda \sigma) \) one has

\[
\vec{c}' \rightarrow \vec{c}', \quad \vec{f} \rightarrow \frac{1}{\lambda} \vec{f}, \quad \vec{g} \rightarrow \frac{1}{\lambda^2} \vec{g}, \ldots.
\]

On the other hand the area derivative, being of second order, should scale \( \sim \frac{1}{\lambda^2} \). In turn, this means that one of the quantities \( \vec{r}' \) or \( \vec{q}' \), which must arise through transverse variations of \( \vec{g} \), should be aligned with the tangential vector \( \vec{t} \), which, by definition, has zero transverse components. An explicit verification of the result prompted by the preceding, heuristic arguments, is presented in the Appendix. In the course of that computation the following relation is obtained (the \( \tilde{\cdot} \) denotes a Fourier transformed quantity, to be defined below)

\[
\delta \tilde{g}^a(p) = \left[ |p|^3 \delta^{ab} - |p|(f^a f^b - 3 f^a f^b) \right] n^b_\mu \delta \tilde{c}_\mu(p) \\
\quad - \left[ \vec{f} \cdot \vec{g} \delta^{ab} - \frac{3}{2} (f^a g^b + f^b g^a) - \frac{25}{12} (f^a g^b - f^b g^a) \right] n^b_\mu \delta \tilde{c}_\mu(p) + \cdots, \tag{23}
\]

where the \( p \) variable enters through a Fourier transform specified by

\[
F(s) = F(s' + h) = \int \frac{dp}{2\pi} e^{iph} \tilde{F}(s', p). \tag{24}
\]

Since \( |h| < \eta \rightarrow 0 \), what we have examined is the variation of \( \tilde{g}^a \) for \( |p| \rightarrow \infty \). The dots in (23), accordingly, represent terms that vanish as \( |p|^{-1} \). It also follows from (24) that all the functions on the rhs of (23) are calculated at \( s' \). We thereby deduce that

\[
\frac{\delta \tilde{g}^a(s)}{\delta \tilde{n}^b(s')} = \int \frac{dp}{2\pi} \left[ |p|^3 \delta^{ab} - |p|(f^a f^b - 3 f^a f^b) \right] e^{iph} \\
\quad - \left[ \vec{f} \cdot \vec{g} \delta^{ab} - \frac{3}{2} (f^a g^b + f^b g^a) - \frac{25}{12} (f^a g^b - f^b g^a) \right] \delta(h) + O(h). \tag{25}
\]

Referring to the formula for the area derivative, we immediately surmise that the first term on the rhs of Eq. (20) gives null contribution since the antisymmetric term in (25) is
proportional to \( \delta(s - s') \), and not \( \delta'(s - s') \). We have, therefore, determined that

\[
\lim_{\eta \to 0} \int_{-\eta}^{\eta} dh \frac{\delta^2 A_{\text{min}}}{\delta c_\mu \left( \sigma + \frac{h}{2} \right) \delta c_\nu \left( \sigma - \frac{h}{2} \right)} = t_{[\mu} \sigma) g_{\nu]}(\sigma) \tag{26}
\]

Ending this section we find it useful to apply the above result for the purpose of verifying the Makkenko-Migdal equation [19], see Refs. [20] for review expositions, for a differentiable, non-selfintersecting Wilson loop which is traversed only once, namely

\[
\tilde{\Delta} W[C] \approx 0, \tag{27}
\]

where the symbol \( \approx \) means that the finite part on the rhs is zero and the MM loop operator is defined [20] as

\[
\tilde{\Delta} = \int dc_\mu \frac{\delta}{\delta c_\mu(\sigma)} = \lim_{\eta \to 0} \lim_{\eta' \to 0} \int d\sigma c_\nu(\sigma) \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta}{\delta c_\mu(\sigma')} \int_{-\eta'}^{\eta'} dh \frac{\delta^2}{\delta c_\mu(\sigma + h) \delta c_\nu(\sigma)}. \tag{28}
\]

It can, now, be easily determined that

\[
\tilde{\Delta} A_{\text{min}} = 2 \lim_{\eta \to 0} \int d\sigma c_\nu(\sigma) \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta}{\delta c_\mu(\sigma')} [t_{\nu}(\sigma) g_{\mu}(\sigma)] = 2 \lim_{\eta \to 0} \int d\sigma \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta g_{\mu}(\sigma)}{\delta c_\mu(\sigma')} . \tag{29}
\]

But

\[
\frac{\delta^3 a(\sigma)}{\delta n^b(\sigma')} n^a(\sigma) \cdot n^b(\sigma') = -(D - 4) \vec{f} \cdot \vec{g} \delta(\sigma - \sigma') + \\
+ \frac{3!}{\pi (\sigma - \sigma')^4} \frac{1}{\pi (\sigma - \sigma')^2} \left[ \vec{f}^2 \delta^{ab} - 3 f^a f^b \right] \tilde{n}^a(\sigma) \cdot \tilde{n}^b(\sigma') + O(\sigma - \sigma') \tag{30}
\]

and so

\[
\tilde{\Delta} A_{\text{min}} = 2 \lim_{\eta \to 0} \int d\sigma \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta^3 a(\sigma)}{\delta n^b(\sigma')} \tilde{n}^a(\sigma) \cdot \tilde{n}^b(\sigma') \approx 0. \tag{31}
\]

4. The spin factor contribution and the role of the \( \vec{g} \)-function

Using the fact that the area derivative operator obeys the Leibnitz rule, we obtain, after applying it twice on the Wilson loop functional and using Eq. (26),

\[
\frac{\delta \delta}{\delta s_{\kappa \lambda}(\sigma') \delta s_{\mu \nu}(\sigma)} W = -\sqrt{\lambda} t_{[\mu}(\sigma) g_{\nu]}(\sigma) \frac{\delta}{\delta s_{\kappa \lambda}(\sigma')} W = \lambda t_{[\mu}(\sigma) g_{\nu]}(\sigma) t_{[\kappa}(\sigma') g_{\lambda]}(\sigma') W. \tag{32}
\]
Its action, as is evident from Eq (3), appears in an exponentiated form:

\[
P \exp \left( \frac{i}{2} \int d\sigma \, J \cdot \frac{\delta}{\delta s} \right) W[C] \propto P \exp \left( \frac{i}{2} \int d\sigma \, J \cdot \frac{\delta}{\delta s} \right) e^{-\sqrt{\lambda} \min[C]}
\]

\[
\propto P \exp \left[ \frac{i}{2} \sqrt{\lambda} \int d\sigma \, J_{\mu\nu}(\sigma) g_{\nu\lambda}(\sigma) \right] e^{-\sqrt{\lambda} \min[C]}.
\]

(33)

Once again, the operation of the area derivative must be understood piecewise if the Wilson loop configuration has cusps. In such a case the spin factor contribution factorizes into pieces, each one of which represents ‘spin factor’ expressions associated with the corresponding smooth component [14] of the piecewise connected contour.

Further progress can be made if one imposes, in the \( \sqrt{\lambda} \gg 1 \) limit at least, the Bianchi identity at every point at which the Wilson contour is smooth (differentiable). The latter reads

\[
\epsilon_{\kappa\lambda\mu\nu} \frac{\partial g^{(\sigma)}}{\partial \kappa \mu \lambda \sigma} W = \lim_{\eta \to 0} \epsilon_{\kappa\lambda\mu\nu} \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta}{\delta c_\lambda(\sigma')} \frac{\delta}{\delta s_{\mu \sigma}(\sigma')} W = 0.
\]

(34)

Substitution of the result (26) for the area derivative into the Bianchi identity, leads to

\[
\epsilon_{\kappa\lambda\mu\nu} \frac{\partial g^{(\sigma)}}{\partial \kappa \mu \lambda \sigma} W = \lim_{\eta \to 0} \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta}{\delta c_\lambda(\sigma')} \left[ t_{\mu}(\sigma) g_{\nu}(\sigma) \right] = 0.
\]

(35)

Referring, now, to Eq. (20), one finds

\[
t_{\mu}(\sigma) \frac{\delta g_{\nu}(\sigma)}{\delta c_\lambda(\sigma')} - (\mu \leftrightarrow \nu) = \frac{\delta g^{a}(\sigma)}{\delta \bar{n}^{b}(\sigma')} n^{b}_\lambda(\sigma) t_{\mu}(\sigma) n^{a}_\nu(\sigma)
\]

\[\quad - \delta'(\sigma - \sigma') \bar{n}^{a}(\sigma') n^{b}_\lambda(\sigma) t_{\mu}(\sigma) g_{\nu}(\sigma),
\]

(36)

which finally gives

\[
\epsilon_{\kappa\lambda\mu\nu} \lim_{\eta \to 0} \int_{\sigma - \eta}^{\sigma + \eta} d\sigma' \frac{\delta g^{a}(\sigma)}{\delta \bar{n}^{b}(\sigma')} n^{b}_\lambda(\sigma') t_{\mu}(\sigma) n^{a}_\nu(\sigma) = \epsilon_{\kappa\lambda\mu\nu} c''_\lambda(\sigma) t_{\mu}(\sigma) g_{\nu}(\sigma).
\]

(37)

Substituting into the above equation the result expressed by Eq (25), one concludes that

\[
\epsilon_{\kappa\lambda\mu\nu} c''_\lambda(\sigma) t_{\mu}(\sigma) g_{\nu}(\sigma) = 0.
\]

(38)

Given, now, that \( \vec{g} \cdot \vec{c}' = 0, \vec{c}' \cdot \vec{c}' = 0 \), we surmise that vectors \( \vec{g} \) and \( \vec{c}' \) are parallel to each other. Accordingly, we write \( \vec{g}(\sigma) = \phi(\sigma) \vec{c}'(\sigma) \), with \( \phi = \frac{|\vec{n}|}{|\vec{c}'|} \). In turn, this leads to the deduction that, for a smooth Wilson contour, the following holds true

\[
P \exp \left( \frac{i}{2} \int d\sigma J_{\mu\nu} \frac{\delta}{\delta s_{\mu \nu}} \right) W[C] \propto P \exp \left( \frac{i}{2} \int d\sigma \phi(\sigma) c''_{\mu}(\sigma) c''_{\nu}(\sigma) J_{\mu\nu} \right) W[C].
\]

(39)
The above relation constitutes the central result of this paper. It exhibits a ‘deformed’ spin factor expression which incorporates dynamical effects induced via the AdS/CFT duality conjecture, in the $\sqrt{\lambda} \gg 1$ limit. Comments and/or speculations surrounding this result will be presented in the concluding section which follows.

5. Assessments and concluding remarks

To initiate an evaluation, from a physics point of view, of implications of the analysis conducted in this paper, let us start by making some general comments in reference to the issue of perturbative vs. nonperturbative aspects of QCD, as a quantum field theory. For our starting point, we adopt the universally accepted belief that lattice QCD constitutes the most effective approach for the study of nonperturbative phenomena in the theory. As other, non-lattice, examples\(^2\) of, credible, attempts for non-perturbative, field theoretical investigations of the theory one could mention: (a) The loop equation approach [17,20] and (b) the Stochastic Vacuum Model [21]. In all these cases, the Wilson loop, which enters either directly or via the non-abelian Stoke’s theorem, constitutes a fundamental element of the corresponding formulations. Perturbation theory, on the other hand, bases its description strictly on locality premises.

With the above in place, consider a dynamical process involving fundamental particle entities, e.g. quarks, whose propagation is described in terms of worldline contours, in line, for example, with Eq. (3). Such a description mode adopts, just like string theory does, first quantization methods, as opposed to field theoretical formulations which adhere to a second quantization methodology. Suppose, now, that the worldline description of a given (QCD) process of interest involves closed worldline paths. Then, the interactions of the propagating particle entity with gauge fields, generate Wilson contours. If, now, (local) interaction(s) with some external agent(s) take place, then the contour will be deformed through the formation of cusps, i.e. points at which a four-momentum is imparted. Our working premise is that perturbative contributions to the process correspond to, local gluon exchanges, as well as their emission and/or absorption, with point of reference a corresponding cusp vertex.

\(^2\)By no means does this exhaust the list of all, relevant, theoretical proposals.
Non-perturbative, dynamical effects, on the other hand, should be associated with area deformations.

The preceding, intuitive, comments are very general and pertain to QCD as a quantum field theory, in a wider sense. The AdS/CFT setting, on the other hand, corresponds to a theory which is characterized as ‘holographic QCD’. Setting aside the issue regarding the precise connection between the quantum field theoretical and the holographic version of QCD, equivalently, the precise connection between gauge fields and strings, let us adopt as working hypothesis that the WKB-type approximation adopted in this study constitutes a zeroth approximation to QCD whose basic merit is that it contains both perturbative and non-perturbative contributions to dynamical physical processes. From such a perspective, the primary issue of relevance is to gain a concrete, quantitative perspective on the $\vec{g}$-function. So, let us assume that our main result, as expressed in general form by Eq. (33), gives the leading contribution to a given physical process, which involves an integral over Wilson contours with certain characteristics. As it stands, it tells us that the dominant, piecewise continuous contours are those for which $\vec{g} = 0$ and, consequently, see Eqs. (33) and (39), the spin factor becomes unity. By itself, this occurrence constitutes a consistency check with existing applications of the worldline formalism to situations where the eikonal approximation is valid [22], i.e. when the dominant contour is formed by straight line segments. On the other hand, a piecewise continuous contour does not have to be of a polygonal type. Distortions, induced by interactions which keep the path segments smooth while changing $A_{\text{min}}$ could very well give the dominant contribution. In other words, dynamical, non-perturbative information may very well reside in solutions of equation $\vec{g} = 0$. Reversing the argument, suppose one is in position to surmise the worldline, geometrical profile of a Wilson contour associated with a given dynamical process on purely physical grounds. Then, one could be in position to a priori determine $A_{\text{min}}$. Such scenarios are realistic and have, in fact, been studied, perturbatively, in the worldline context, see [22] and references therein. Non-perturbative dynamical contributions stemming from the WKB approximation to holographic QCD should enter as further correction terms to well established perturbative expressions based on resummation procedures, once some input for the $g$-function, phenomenological or, model-dependent, theoretical is provided.
As a final note of interest we consider the so called wavy line approximation [16], according to which the Wilson contour, in four dimensions, is parametrized as \((\sigma, \psi_i), i = 1, 2, 3\), with the \(\psi_i\) small transverse deviations. One finds, on account of reparametrization invariance and the Hamilton-Jacobi equations for the minimal area [16], that

\[
\frac{\delta A_{\text{min}}}{\delta \psi_i} \bigg|_{\psi_i=0} = 0.
\]

The above result gives another perspective on why for heavy quarks -as well as other situations for which the eikonal approximation is valid- (piecewise) straight Wilson paths play the dominant role: The \(\vec{g}\)-function vanishes in this case. Correction terms will arise, on the other hand, only if transverse fluctuations are taken into account. As alluded to already, any distortion of a given path segment, which keeps it smooth, while the \(\vec{g}\)-function remains zero, could have, in principle, non-trivial significance which will be reflected in the expression for \(A_{\text{min}}\).
Appendix

Here we shall give a proof of Eq. (23) in the text following closely the methodology of Ref [15,16].

As is obvious from Eq. (18), in order to compute the area derivative we need the normal variation of the $\vec{g}$-function. As a first step in this direction one defines, at every point of the surface, a basis $\{n^a_M(t, s)\}$ of $D - 1$ orthonormal vectors which satisfy the conditions

$$n^a_M(t, s) \dot{x}_M(t, s) = n^a_M(t, s)x'_M(t, s) = 0,$$

where $G_{MN}n^b_M = \delta^{ab}$ and $n^a_\mu(0, s) = n^a_\mu(s)$ are the vectors used in Eq. (19) of the text.

Under the normal variation

$$x_M(t, s) \rightarrow x_M(t, s) + \psi_M(t, s), \quad \psi_M(t, s) = \phi^a(t, s)n^a_M(t, s)$$

the change of the minimal surface to second order in $\phi^a$ reads

$$S^{(2)} = \int d^2 \xi \left[ \sqrt{g}(g^{\alpha\beta} \partial_\alpha \psi^a \partial_\beta \psi^a + 2g^{ab} \omega^{[ab]}_{\alpha} \partial_\beta \psi^a \psi^b + 2\psi^a \psi^a) + O(t^2 \psi^2) \right]$$

where we have written $\psi^a \equiv t\phi^a$ and have introduced $g_{\alpha\beta} = G_{MN}\partial_\alpha x_M \partial_\beta x_N$, while the, antisymmetric, quantities $\omega^{[ab]}_{\alpha}$ are spin connection coefficients and are given by

$$\omega^{[ab]}_{\alpha} = \partial_\alpha n^a_M \cdot n^a_M$$

The exact form of this result can be found in [16]. Here, all we need is the third order term in an expansion of $\psi_M$ in powers of $t$. Taking into account that $\phi$ is regular as $t \rightarrow 0$, we have omitted terms $\sim t^4$ in (A.3) which do not contribute to the normal variation of the $\vec{g}$-function.

Using the expansion (12) one easily determines that

$$g_{\alpha\beta} = \frac{1}{t^2} \left( 1 + \tilde{f}^2 t^2 + 2\tilde{f} \cdot \tilde{g} t^3 + \frac{1}{2} \tilde{f} \cdot \tilde{g}^2 t^3 \right)$$

and

$$\sqrt{g} = \frac{1}{t^2} \left( 1 + \frac{2}{3} \tilde{f} \cdot \tilde{g} t^3 \right) + O(t^2).$$
Now, the area derivative receives contributions from antisymmetric terms. We, therefore, have to find the behavior of the spin connection as \( t \to 0 \). This cannot be done in a unique way if \( D > 2 \). What one can do is to expand the basis vectors \( n^a_M(t,s) \) as a power series in \( t \):

\[
n^a_0(t,s) = t k^a_0(s) + \frac{1}{2} t^2 l^a_0(s) + \frac{1}{3} t^3 m^a_0(s) + \cdots
\]

\[
\vec{n}^a(t,s) = \frac{1}{2} t^2 \vec{l}^a(s) + \frac{1}{3} t^3 \vec{m}^a(s) + \cdots
\]

(A.7)

Combining these relations with (A.1) and using the expansion (12) we can determine that

\[
k^a_0 = f^a, \quad l^a_0 = -2(\vec{k}^a \cdot \vec{f} + g^a), \quad m^a_0 = -3(\frac{1}{2} \vec{t}^a \cdot \vec{f} + \vec{k}^a \cdot \vec{g} + h^a)
\]

(A.8)

and

\[
\vec{k}^a \cdot \vec{c}' = 0, \quad \vec{l}^a \cdot \vec{c}' + f^a = 0, \quad \vec{m}^a \cdot \vec{c}' + g^a + 3 \frac{1}{2} \vec{t}^a \cdot \vec{f} = 0.
\]

(A.9)

From the orthonormality condition we find that

\[
\vec{k}^a \cdot \vec{n}^b(s) + \vec{k}^b \cdot \vec{n}^a(s) = 0, \quad 2k^a_M \cdot k^b_M + \vec{t}^a \cdot \vec{n}^b(s) + \vec{l}^b \cdot \vec{n}^a(s) = 0
\]

\[
\frac{3}{2} t^a_M \cdot t^b_M + \vec{m}^a \cdot \vec{n}^b(s) + \vec{m}^b \cdot \vec{n}^a(s) = 0.
\]

(A.10)

With the above in place we return to our central objective and, to start with, assume that

\[
\vec{k}^a \cdot \vec{c}' = 0 \to \vec{k}^a = 0,
\]

(A.11)

which means that

\[
\vec{t}^a \cdot \vec{c}' = -f^a
\]

\[
\vec{l}^a \cdot \vec{n}^b(s) + \vec{l}^b \cdot \vec{n}^a(s) = -2k^a_0 k^b_0 = -2f^a f^b.
\]

(A.12)

From these relations we conclude that

\[
\vec{l}^a = -f^a \vec{c}' - f^a \vec{f} + \Lambda^{ab} \vec{n}^b(s).
\]

(A.13)

with \( \Lambda^{ab} \) antisymmetric, but otherwise arbitrary. The observation here is the following: On the one hand \( \Lambda^{ab} \) enters the second order term of the expansion (A.6) and consequently contributes to the normal variation of the \( \vec{g} \)-function, to the area derivative and to the spin factor. On the other hand, it does not depend on the functions \( \vec{c}', \vec{f}, \vec{g}, \cdots \) which determine
\(A_{\text{min}}\). This can be deduced, through scaling properties as follows: Under a change of scale \(\vec{c} \rightarrow \lambda \vec{c}, (t, s) \rightarrow \lambda (t, s)\), it must behave as \(\Lambda \rightarrow \frac{1}{\lambda^2}\), as can be seen from Eq. (A.6). Taking, now, into account that \(\vec{c} \rightarrow \vec{c}', \vec{f} \rightarrow \frac{1}{\lambda} \vec{f}, \vec{g} \rightarrow \frac{1}{\lambda^2} \vec{g}, \cdots\) and that \(\vec{n}^a(s) \cdot \vec{c} = 0 \rightarrow c^a = 0\), it becomes obvious that it is impossible to find an antisymmetric combination of the coefficient functions to construct \(\Lambda^{ab} = -\Lambda^{ba}\). Thus, this arbitrary function does not depend on the contour and consequently can be chosen at will. We shall take it to be zero. With the same reasoning assumption (A.10) can be justified and so we can determine the basis vectors:

\[
\begin{align*}
n_0^a(t, s) &= -tf^a - t^2g^a - t^3(h^a - f^a\vec{f}^2) + O(t^4) \\
n^a(t, s) &= \vec{n}^a(s) - \frac{1}{2}t^2(f^a\vec{f} + f^a\vec{c}') - \frac{1}{2}t^3(g^a\vec{f} + f^a\vec{g} + 2g^a\vec{c}') + O(t^4)
\end{align*}
\]

For the behavior of the spin connection we also need the derivative \(\vec{n}'^a(s)\). What we do know about it comes from the orthonormality condition \(\vec{n}^a(s) \cdot \vec{c}' = 0 \rightarrow \vec{n}'^a(s) \cdot \vec{c}' = -\vec{n}^a(s) \cdot \vec{c}''\) (A.15)

Adopting the same arguments as before we conclude from the preceding relation that

\[
\vec{n}'^a(s) = -(\vec{n}^a(s) \cdot \vec{c}'') \vec{c}' = -c''^a \vec{c}'
\] (A.16)

In conclusion, through the above analysis we have determined that

\[
\omega^{[ab]}_t = \frac{1}{2}t^2(g^a f^b - g^b f^a) \equiv \frac{1}{2}t^2r^{ab}, \quad \omega^{[ab]}_s = O(t^3).
\] (A.17)

Knowing the behavior of all the terms we now return to (A.3) and demand the perturbed surface also to be minimal. This leads to the equation

\[
\partial_\beta(\sqrt{g}g^{\alpha\beta}\partial_\alpha \psi^a) - 2\sqrt{g}\psi^a + 2\sqrt{g}g^{\alpha\beta}\omega^{[ab]}_\alpha\partial_\beta \psi^b = O(t^2 \psi)
\] (A.18)

To solve this equation we start from its asymptotic form as \(t \rightarrow 0\), treating the other terms as small perturbations. At this point it becomes very convenient to introduce, following Refs [17,18], the Fourier transform

\[
\phi^a(t, s) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip\sigma} \tilde{\phi}^a(t, p),
\] (A.19)

with \(\sigma = s - \frac{h}{2} = s' + \frac{h}{2}\), the point at which the area derivative is applied. The relevant observation here is that one is interested in large values for the variable \(p \sim \frac{1}{h}\), since the variable \(h\) is integrated in the vicinity of zero, cf. Eq (4) in the text.
On the other hand, one can be convinced, by appealing to (A.19), that the values of $t$ which are involved in our analysis are $t \sim |p| \sim h$. With these estimations (A.18) can be rewritten by retaining only those terms that are relevant to the normal variation of the $\vec{g}$-function. To accomplish this task the coefficient functions must be expanded around the point $s'$. The general form of such an expansion can be read from

$$F(s) = F(s') + (s - s')F'(s') + ... = F(s') + hF'(s') + ... \quad (A.20)$$

$$h\phi^a(t, s) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{iph} h\phi^a(t, p) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{iph} i\partial_p \phi^a(t, p)$$

Given the above, Eq. (A.18) reads, in Fourier space,

$$\hat{L}_{4}^{ab}(t, p) \tilde{\phi}^b(t, p) = \hat{L}_2^{ab}(t, p) \tilde{\phi}^b(t, p) + \hat{L}_1^{ab}(t, p) \tilde{\phi}^b(t, p) + ... \quad (A.21)$$

where we have written

$$\hat{L}_4^{ab} \equiv \left( \frac{1}{t} \partial_t^2 - \frac{3}{2} \partial_t - \frac{\vec{f}^2}{t} \right) \delta^{ab}, \quad \hat{L}_2^{ab} \equiv \vec{f}^2 (\partial_t^2 + p^2) \delta^{ab}.$$

$$\hat{L}_1^{ab} \equiv \left\{ \left[ 2 \vec{f} \cdot \vec{f} i\partial_p + \frac{4}{3} t (\vec{f} \cdot \vec{g}) \right] (\partial_t^2 + p^2) + \frac{3}{2} \vec{f} \cdot \vec{g} \partial_t - \frac{3}{2} \vec{f} \cdot \vec{f} i\partial_p + t \vec{f} \cdot \vec{f} i\partial_t \right\} \delta^{ab} + r_{ab}(\frac{1}{t} - \partial_t) \quad (A.22)$$

The subscripts labeling the operators in the above relation serve to signify their asymptotic behavior as $|p| \to \infty$:

$$\hat{L}_4^{ab} \tilde{\phi}^b \sim O(p^4), \quad \hat{L}_2^{ab} \tilde{\phi}^b \sim O(p^2), \quad \hat{L}_1^{ab} \tilde{\phi}^b \sim O(p). \quad (A.23)$$

The neglected terms in (A.21) are of order $O(p)$ so that their contribution will be four times weaker than the strongest one and thus irrelevant as far as the normal variation of the $\vec{g}$-function.

The solution of (A.21) can be written as

$$\tilde{\phi}^a(t, p) = \tilde{\phi}^a_{(0)}(t, p) + \int_0^\infty dt' G_p(t, t') \left[ \hat{L}_2^{ab}(t', p) + \hat{L}_1^{ab}(t', p) \right] \tilde{\phi}^a(t', p) \quad (A.24)$$

Here $\tilde{\phi}^a_{(0)}$ is the solution of the homogeneous equation

$$\hat{L}_4^{ab}(t, p) \tilde{\phi}^b(t, p) = 0 \quad (A.25)$$

$$\tilde{\phi}^a_{(0)}(t, p) = (1 + t |p|) e^{-|p| t} \tilde{\phi}^a_{(0)}(p)$$

The Green’s function

$$\hat{L}_4^{ab}(t, p) G_p(t, t') = \delta(t - t') \quad (A.26)$$
can be easily found:

\[ G_p(t, t') = \frac{1}{2|p|^3} \phi_-(t' | p|) [\phi_+(t' | p|) - \phi_-(t' | p|)] \theta(t - t') + (t \leftrightarrow t'), \tag{A.27} \]

with

\[ \phi_-(x) = (1 + x)e^{-x}, \phi_+(x) = (1 - x)e^x. \tag{A.28} \]

The solution of the integral equation (A.24) can be approached through an iterative procedure:

\[
\tilde{\phi}^a(t, p) = \tilde{\phi}^a_{(0)}(t, p) + \int_0^{\infty} dt' G_p(t, t') \left[ \hat{L}_2^{ab}(t', p) + \hat{L}_1^{ab}(t', p) \right] \tilde{\phi}^a_{(0)}(t', p) + \text{negligible terms} \tag{A.29}
\]

Expanding, now the result in a \( t \) power series one can see that the neglected terms in the above equation are of order \( O(t^4) \) and thus irrelevant for our purposes. The symmetric part of the solution (A.29) is easily determined to be

\[
\left[ 1 - \frac{1}{2} |p|^2 t^2 - \frac{1}{3} t^3 (\vec{f}^2 | p| + i \vec{f} \cdot \vec{f} \text{sign} p + \vec{f} \cdot \vec{g}) \right] \tilde{\phi}^a_{(0)}(p) + O(t^4), \tag{A.30}
\]

while the contribution to the antisymmetric part is

\[
\int_0^{\infty} dt' G_p(t, t') \left( \frac{1}{t'} - \partial_{t'} \right) e^{-|p| t'} (1 + |p| t') r^{ab} \tilde{\phi}^a = -\frac{1}{3} t^3 [\Gamma(0, 2 |p| t) + \frac{25}{12}] r^{ab} \tilde{\phi}^a + O(t^4). \tag{A.31}
\]

The next step is to integrate the ‘annoying’ incomplete gamma function:

\[
\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ip \Gamma(0, 2t |p|)} = 2 \text{Re} \lim_{\epsilon \to 0} \int_0^{\infty} dpe^{ip \Gamma(\epsilon, 2t |p|)} = 2 \text{Re} \lim_{\epsilon \to 0} \frac{t}{2\epsilon h} \Gamma(\epsilon) [1 - \frac{1}{(1 + \frac{4h}{27})^\epsilon}] = \frac{1}{t} + O(h) \tag{A.32}
\]

and thus the \( O(t^3) \) antisymmetric contribution to the solution can be taken to be just

\[
-\frac{1}{3} t^3 \frac{25}{12} r^{ab} \tilde{\phi}^a. \tag{A.33}
\]

To obtain the final result one must take into account that normal variations do not preserve the static gauge and, therefore, a redefinition of the \( t \) variable is needed. Repeating the relevant calculation of Ref [16] we arrive at Eq. (23) of the text.
Acknowledgement

The authors wish to acknowledge financial supports through the research program “Pythagoras” (grant 016) and by the General Secretariat of Research and Technology of the University of Athens.

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