Origin of the anomalous heat current in collisional granular fluids

D. Candela
Physics Department, University of Massachusetts, Amherst, MA 01003

R. L. Walsworth
Harvard-Smithsonian Center for Astrophysics, Cambridge, MA 02138
(Dated: 7 October 2005)

We present a heuristic explanation for the anomalous (density-gradient-dependent) heat current in collisional granular fluids. Inelastic grain collisions lead to highly non-equilibrium states which are characterized by large spatial gradients and/or temporal variations in the granular temperature. It is argued that the heat current in such non-equilibrium states is driven by the temperature gradient averaged over the typical collision time and/or mean free path. Due to the density dependence of the inelastic energy loss, the nonlocal averaging of the temperature gradient leads to an effective dependence of the heat current upon the density gradient.

PACS numbers: 45.70.Mg, 81.05.Rm, 44.10.+i

When energy is supplied to a system of inelastic grains, a particularly simple collisional state can occur in which only binary collisions between the grains are important. The collisional state occurs for hard grains at any density below jamming, and for more realistic soft grains at densities sufficiently low that enduring and multiparticle contacts are rare [1, 2]. In recent decades collisional granular fluids have served as a paradigm for simulations and for the application of statistical ideas to granular systems [3, 4]. One perplexing phenomenon long predicted [5] for collisional granular fluids is a violation of Fourier’s law, which states that the heat current is proportional to the temperature gradient and always flows from hot to cold. In contrast, the heat current predicted by kinetic theory [3, 4, 5, 6] for a granular fluid is

$$ q = -\kappa \nabla T - \mu \nabla n $$

where $T = m (v^2) / d$ is the granular temperature in $d$ space dimensions for particles of mass $m$ and mean squared velocity ($v^2$), and $n$ is the grain number density. Here $\kappa$ is the granular thermal conductivity and $\mu$ is a new transport coefficient that is nonzero for inelastic systems. Due to the last term in Eq. 1, the anomalous heat current is proportional to the temperature gradient and always flows from hot to cold. Although counterintuitive the anomalous heat current does not violate the second law of thermodynamics, as fluidized granular media are highly non-equilibrium systems.

In the presence of gravity the anomalous heat current is manifested by a nonzero heat current at the height at which the temperature gradient is zero [7]. More strikingly, for a system with a free upper surface (confined by gravity and excited from below), the anomalous heat current can lead to a temperature inversion: despite the strictly upward heat current, the temperature first falls as a function of height and then rises again in the vicinity of the free upper surface [7]. In a recent experiment using NMR to probe vibrofluidized mustard seeds, Huan et al. [11] observed such a temperature inversion and quantitatively fit the heat current (including the anomalous term) to theoretical predictions. Fig. 1 shows example data from this study along with the hydrodynamic fit. A temperature inversion may also have been observed in much earlier experiments on vibrofluidized systems [12, 13], but the inversion was not associated at the time with the anomalous heat current.

Although a density-gradient-dependent heat current has been demonstrated in numerical simulations of granular fluids [3, 11], and $\mu$ has been calculated using sophisticated statistical-physics methods [3, 4, 5, 6], a simple heuristic explanation of this phenomenon has not been presented to date. In this paper we present such an explanation, tracing the anomalous heat current directly to the highly non-equilibrium nature of collisional granular fluids. Due to inelastic energy losses, such fluids necessarily have large time derivatives and/or spatial gradients of the granular temperature. As the temperature may vary significantly over a mean-free-path distance or grain collision time, the inherent nonlocality of the grain collision process must be taken into account. By considering the consequences of this nonlocality, we show that the density dependence of the inelastic energy loss leads to the anomalous heat current term. We argue that the total heat current is fundamentally driven by temperature gradients, not density gradients, and that the apparent dependence of the heat current upon density gradients arises from re-expressing nonlocal quantities in terms of local ones.

The existing technical calculations [3, 4, 5, 6] of the anomalous transport coefficient $\mu$ are valid for wide ranges of the dimensionless density $n \sigma^3$ and the inelasticity parameter $1 - \alpha$ (here $\sigma$ is the grain diameter and $\alpha$ is the normal restitution coefficient). To validate our explanation for the anomalous heat current, we show that it predicts a value for $\mu$ that agrees up to factors of order unity with these more rigorous calculations in the dilute, nearly elastic limit $n \sigma^3 \ll 1$, $(1 - \alpha) \ll 1$. The aim of the present paper is not to reproduce precisely the earlier results, but rather to provide a simple, physically-motivated explanation for the last term in Eq. 1.
We begin with the commonly-used relaxation-time approximation for the evolution of the heat current:

$$\frac{\partial \mathbf{q}}{\partial t} = -\frac{b_q n T}{m} \nabla T - \nu_q \mathbf{q},$$  \hspace{1cm} (2)

where $\nu_q = b_q n \sigma^{d-1}(T/m)^{1/2}$ is the relaxation rate for heat currents, roughly equal to the grain collision frequency. Here and below, $b_q, b_\nu, \ldots$ are dimensionless constants of order unity that depend only upon the dimension of space $d$; values are given in Table I. Note that $(T/m)^{1/2}$ is a typical particle speed and $\sigma^{d-1}$ is roughly the collision cross section.

The first term on the RHS of Eq. 2 gives the buildup of the heat current due to the free streaming of particles from nearby regions with differing temperatures, while the second term expresses the relaxation of $\mathbf{q}$ towards zero due to collisions, which randomize the particle velocities. In an elastic system ($\alpha = 1$) perturbed by weak temperature and density gradients, the temperature varies slowly in time due to energy conservation. Hence a steady state is reached with $\partial \mathbf{q}/\partial t \approx 0$, giving Fourier’s law: $\mathbf{q} = -\kappa_0 \nabla T$ with $\kappa_0 = b_q n T/m \nu_q \sim (T/m)^{1/2}/\sigma^{d-1}$.

Implicit in this derivation of Fourier’s law are two related “near-equilibrium” assumptions: (i) dynamics affecting $n$ and $T$ occur very slowly relative to the heat current relaxation rate $\nu_q$, and (ii) spatial variations of $n$ and $T$ occur on characteristic length scales much greater than the mean free path $\lambda_q \sim 1/\sigma^{d-1}$. In an inelastic system ($\alpha < 1$) the near-equilibrium assumptions are not generally valid because the system continually loses energy, which causes temperature changes at rates comparable to $\nu_q$. If energy is supplied at the boundary of the system to maintain a steady state, this leads to density and temperature variations on length scales comparable to $\lambda_q$. To account for these non-equilibrium effects the derivation of Fourier’s law must be generalized to be non-local in space and time. We show here how this leads to an effective dependence of the heat current upon $\nabla n$.

As a first example, we consider a freely-cooling granular fluid (i.e., no energy input) with small gradients in $T$ and $n$. Locally, one has $\partial T/\partial t = -\zeta T$ with the cooling rate $\zeta$ given by leading order in the inelasticity $(1 - \alpha)$ by:

$$\zeta = b_\zeta n \sigma^{d-1}(T/m)^{1/2}[(1 - \alpha) + \mathcal{O}(1 - \alpha)^2].$$  \hspace{1cm} (3)

For typical values of the restitution coefficient $\alpha \approx 0.9$, the ratio of the cooling rate to the heat current relaxation rate, $\zeta/\nu_q = (b_\zeta/b_q)(1 - \alpha)$, is a small but not infinitesimal quantity ($\approx 0.1$ in 3D and 0.3 in 2D). Therefore $T$ changes non-negligibly during a typical grain collision time $1/\nu_q$, which violates the first near-equilibrium assumption above.

We use the following simple argument to estimate the heat current $\mathbf{q}(t)$ in this situation. According to Eq. 2 $\mathbf{q}(t)$ continuously relaxes towards the Fourier-law value $-\kappa_0 \nabla T$, which however is time-dependent. Thus, $\mathbf{q}(t)$

---

**TABLE I: Dimensionless prefactors used in kinetic theory**

| Prefactor | Quantity       | $d = 2$ | $d = 3$ |
|-----------|----------------|---------|---------|
| $b_q$     | Heat current   | $4/3$   | $5/2$   |
| $b_\nu$   | Relaxation frequency | $2\sqrt{\pi}/3$ | $32\sqrt{\pi}/15$ |
| $b_\zeta$ | Cooling rate   | $2\sqrt{\pi}$ | $8\sqrt{\pi}/3$ |
| $b_\lambda$ | Mean free path | $2$     | $\sqrt{\pi}$ |

---

FIG. 1: Measured granular density (top) and temperature (bottom) as functions of height for a dilute granular fluid that is vibrofluidized from below, from Ref. [14]. The symbols show NMR measurements for mustard seeds vibrated at 15g and 50 Hz, while the curves show a fit to the hydrodynamic theory of Ref. [15]. Dimensionless units $z^* = z/\sigma$, $n^* = n\sigma^3$, $T^* = T/m\sigma$ are used. The fitted restitution coefficient was $\alpha = 0.87$. The sample was confined by gravity to the bottom surface, so there was no vibrational energy input from above. Nevertheless, the experimental data show a clear minimum in the granular temperature $T^*$ as a function of height $z^*$. At heights above the temperature minimum, the heat current (always upward) is from cold to hot.
lags behind the Fourier-law value by approximately one
relaxation time $1/\nu_q$:

$$q(t) \approx -\kappa_0 \nabla T + (1/\nu_q) \partial(\kappa_0 \nabla T)/\partial t. \tag{4}$$

Equation 4 expresses the idea that the gradient that
drives the heat current is nonlocal in time: The heat
current at time $t$ reflects the weighted average of $-\kappa_0 \nabla T$
over earlier times, with an typical lag of one collision
$1/\nu_q$. The second term in Eq. 4 is readily calculated,

$$\partial(\kappa_0 \nabla T)/\partial t = \frac{\partial \kappa_0}{\partial t} \nabla T + \kappa_0 \nabla (-\zeta T)$$
$$= -\frac{3}{2} \kappa_0 \zeta \nabla T - \kappa_0 \zeta \left[ \frac{\partial \zeta}{\partial T} \nabla T + \frac{\partial \zeta}{\partial n} \nabla n \right]$$
$$= -2\kappa_0 \zeta \nabla T - \frac{\kappa_0 \zeta T}{\nu_q} \nabla n. \tag{5}$$

where $\zeta \propto nT^{1/2}$ and $\kappa_0 \propto T^{1/2}$ have been used to
simply this equation. Thus:

$$q \approx \left( 1 + 2 \frac{\zeta}{\nu_q} \right) \kappa_0 \nabla T - \frac{\kappa_0 \zeta T}{\nu_q} \nabla n. \tag{6}$$

From Eq. 5 we can read off the anomalous transport co-
efficient $\mu = \kappa_0 \zeta \nabla n/\nu_q$, proportional to the semi-small
parameter $\zeta/\nu_q$. The ordinary thermal conductivity also
acquires a correction proportional to the same parameter.
Both of these results are close to the results of more sophis-
cicated calculations \cite{5, 6, 7, 8}, for example the
value of $\mu$ is 5/4 times the result calculated in Ref. 8.

The origin of the heat current’s effective density-
gradient-dependence is the dependence of the cooling rate
on density $\partial \zeta/\partial n \neq 0$ in Eq. 4. For example, if at time
$t = 0$ there is a density gradient but no temperature
gradient, then at earlier times the denser regions must
have been hotter since they are cooling faster. Therefore,
averaging $\kappa_0 \nabla T$ over times $t < 0$ yields a non-zero
heat current in the absence of a temperature gradient at
$t = 0$.

As a second example we consider a boundary-
driven steady-state — e.g., a vertically-vibrated granular
medium such as that represented in the data of Fig. 1
and described in Ref. 11. Due to inelastic energy losses
and gravity, a boundary-driven steady-state necessarily
has strong spatial variations of $n$ and $T$, in violation of
the second near-equilibrium assumption listed above.

A rigorous treatment of transport in the presence of
strong gradients is complex, but the following simple
argument should give a reasonable approximation.
The typical distance over which a particle drifts before hav-
ing its velocity randomized by a collision is the mean free
path $\lambda_q = b_3(T/m)^{1/2}/\nu_q \sim 1/\sigma d^{-1}$, where $b_3$ is an
other dimensionless constant of order one. Therefore we
should replace $\nabla T$ in Eq. 2 by $\langle \nabla T \rangle$, which denotes the
spatial average of $\nabla T$ over the mean free path $\lambda_q$. Thus,
in the steady state the temperature gradient is effectively
nonlocal in space, just as it is effectively nonlocal in time
for the cooling state.

In the steady state $\langle \nabla T \rangle$, differs from $\nabla T$ by an
amount proportional to the curvature of $\nabla T$ times $\lambda_q^2$,
as shown schematically in Fig. 2. To simplify the calcu-
lation, we assume that $T$ and $n$ depend only upon the $z$
coordinate (typically in the direction opposite gravity). Then

$$\langle \partial T/\partial z \rangle = \frac{1}{2\lambda_q} \int_{-\lambda_q}^{\lambda_q} \frac{\partial T}{\partial z} (z + u) du$$
$$\approx \partial T/\partial z + (\lambda_q^2/6) \partial T^3/\partial z^3. \tag{7}$$

To lowest order in $\zeta/\nu_q$ we have $\partial T/\partial z = -q_z/\kappa_0$, where
$q_z$ is the $z$-component of the heat current $q$. Hence

$$\partial T^3/\partial z^3 \approx -\kappa_0^{-1} \partial^2 q_z/\partial z^2. \tag{8}$$

This omits corrections proportional to derivatives of $\kappa_0$, which considerably complicate the calculation but do not
affect the leading-order estimate for $\mu$.

To complete the argument we note that in a steady
state the energy loss rate per unit volume due to inelastic
collisions, $P_c$, is balanced by the negative divergence of
the heat current: $P_c = -\nabla \cdot q = -\partial q_z/\partial z$. The energy
loss rate is proportional to the cooling rate $\zeta$: $P_c = c \zeta T =
dn\zeta T/2$ where $c = dn/2$ is the specific heat 10. Using

FIG. 2: Schematic illustration of the mechanism for anom-
alous heat currents in a steady-state situation similar to that
shown in Fig. 1. At the height $z$ of the temperature mini-
num, the gradient $\partial T/\partial z$ has non-zero curvature due to the
density gradient and the density dependence of the inelasti-
c power loss. As a result of the curvature, the average of the
temperature gradient over the mean free path $\lambda_q$ is non-zero
(filled dot on graph). This non-zero effective gradient in turn
drives a heat current even though $\partial T/\partial z$ is zero at this height.
(Although it might appear from this diagram that an anom-
alous heat current can only occur over a small range of $z$ near
the minimum, this is not the case due to the increase of $\lambda_q$
with $z$.)
Eqs. 7 and 8 we have

\[
\frac{\partial T}{\partial z} - \frac{\partial T}{\partial z} \approx -\left(\frac{\lambda^2}{6\kappa_0}\right) \frac{\partial^2 q_z}{\partial z^2} \\
= \left(\frac{\lambda^2 d}{12\kappa_0}\right) \frac{\partial}{\partial z} \left(\n n \zeta\right) \\
= \frac{\lambda^2 d\kappa_0}{6\kappa_0} \left[ \frac{3}{4} \frac{\partial T}{\partial z} + \frac{T}{\partial n} \frac{\partial n}{\partial z}\right]
\]

(9)

where \( \zeta \propto nT^{1/2} \) has again been used. Thus we find for the steady-state heat current:

\[
q_z = -\kappa_0 \left(\frac{\partial T}{\partial z}\right) \lambda \\
= -\kappa_0 \left(1 + \frac{\kappa_2}{8b_q\nu_q}\right) \frac{\partial T}{\partial z} - \frac{\kappa_2}{6b_q\nu_q} \frac{n}{\partial z} \frac{\partial T}{\partial n} \frac{n}{\partial z} \\
\]

(10)

giving \( \mu = \frac{\kappa_2}{6b_q\nu_q} \frac{n}{\partial z} \) for the anomalous heat transport coefficient in a boundary-driven steady-state. This value for \( \mu \) in the steady state agrees with the cooling-state estimate of \( \mu \) derived above for \( b_\lambda = (6b_q/d)^{1/2} \), which gives reasonable values for the dimensionless constant \( b_\lambda \) (Table I).

For the boundary-driven steady-state, the origin of the anomalous heat current is again the density dependence of the collisional energy loss \( (P_e = c\zeta T \sim n^2) \), which forces the heat current and hence the temperature gradient to have a large, non-zero second spatial derivative (curvature). The curvature of the temperature gradient over distances comparable to the mean free path in turn implies that the effective (nonlocal) temperature gradient \( \langle \nabla T \rangle \), is different from the local gradient, leading to an extra term in the heat current that is proportional to the density gradient (Fig. 2).

In summary, although the anomalous heat current is nominally proportional to the density gradient, an analysis of the underlying physical effects shows that the driving term is actually the temperature gradient. Due to the large time and/or space derivatives of the temperature that necessarily occur in dilute granular fluids, the driving temperature gradient must be averaged over a typical collision time \( 1/\nu_q \) and mean free path \( \lambda_q \). Thus the effective gradient is nonlocal in time and space; when expressed in terms of local gradients an apparent density-driven heat current appears.

Finally, we emphasize that it is the strong space or time derivative of the temperature, rather than the inelasticity per se, that is the source of the anomalous heat current. Therefore, similar heat currents should in principle be observable in elastic systems if suitably strong temperature gradients could be imposed. From this point of view, the role of inelasticity in a granular fluid is primarily to create a highly non-equilibrium state, and some novel physical properties of granular fluids (like the anomalous heat current) are due fundamentally to the large deviation from equilibrium rather than the inelastic energy loss.

We thank A. Santos for useful communications including a derivation of \( \mu \) in the cooling state which motivated the heuristic argument given in the first part of this paper. We also thank W. M. Mullin for calculation of the \( b_q \) values in Table I. This work was supported by NSF Grant No. CTS-0310006.

[1] Collisional states are often called rapid granular flows. They can occur, however, in the absence of bulk flow as in the situations analyzed here.
[2] C. S. Campbell, J. Fluid Mech. 465, 261 (2002).
[3] P. K. Haff, J. Fluid Mech. 134, 401 (1983).
[4] J. T. Jenkins and S. B. Savage, J. Fluid Mech. 130, 187 (1983).
[5] C. K. K. Lun, S. B. Savage, D. J. Jeffrey, and N. Chepurny, J. Fluid Mech. 140, 223 (1984).
[6] N. Sela and I. Goldhirsch, J. Fluid Mech. 361, 41 (1998).
[7] V. Garzó and J. W. Dufty, Phys. Rev. E 59, 5895 (1999).
[8] J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, Phys. Rev. E 58, 4638 (1998).
[9] P. Soto, M. Mareschal, and D. Risso, Phys. Rev. Lett. 83, 5003 (1999).
[10] R. Ramírez and R. Soto, Physica A 322, 73 (2003).
[11] C. Huan, X. Yang, D. Candela, R. W. Mair, and R. L. Walsworth, Phys. Rev. E 69, 041302 (2004).
[12] E. Clément and J. Rajchenbach, Europhys. Lett. 16, 133 (1991).
[13] S. Warr, J. M. Huntley, and G. T. H. Jacques, Phys. Rev. E 52, 5583 (1995).
[14] J. J. Brey, D. Cubero, and M. J. Ruiz-Montero, Phys. Rev. E 59, 1256 (1999).
[15] J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E 63, 061305 (2001).
[16] It might be appear that if grain rotations were coupled to translational motion, however weakly, the specific heat would increase from \( dn/2 \) to \( 2d - 1 \) \( n/2 \) thus changing the value of \( \mu_\lambda \). However, if the power loss \( P_e \) remained constant the cooling rate \( \zeta \) would decrease by the same factor, leaving \( \mu \) unchanged.