River channel networks created by Poisson Equation and Inhomogeneous Permeability Models

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Abstract. This paper presents original mathematical models to simulate formation of dendritic tree networks such as natural river channels. These are referred to as the Poisson Equation Model (PEM) and as the Inhomogeneous Permeability Model (IPM), respectively, where the two-dimensional Poisson equations are solved under the homogeneous or the inhomogeneous condition. Particularly important is the IPM, which assumes that permeability varies depending on the site, which reflects regional fluctuations of geographical properties such as soil, precipitation, and so on. Natural river basins are supposed to realize a kind of optimization principle such that total energy expenditure or flow resistance is minimal, as stated by the Optimal Channel Network (OCN) theory or the Constructal law. From a viewpoint of optimization, possible structures of area-to-point flow systems are explored by use of PEMs and IPMs. According to our numerical simulations, it is supposed that the origin of the fractal structure that characterizes natural fluvial landscapes is heterogeneity in geomorphic conditions. This study indicates that the permeability randomization method can be utilized not only as a practical but also as an instructive tool to reproduce naturally disturbed river network systems.

Keywords: Constructal law, Finite Difference Method (FDM), Fractal, Inhomogeneous Permeability Model (IPM), Optimal Channel Network (OCN) theory, Poisson Equation Model (PEM)
1. Introduction

The tree-shaped network is one of the most popular fractal structures that abound in nature, which ranges from biological systems such as neural dendrites and tree branches to physical systems such as frosts, lightning and river basins [1,2,3]. For decades, many mathematical modelers have pursued the way to reproduce these patterns. It has been particularly a fascinating subject for fluvial geomorphologists to elucidate the mechanisms of meandering streams with bifurcation branches in natural river channels. Various theoretical models have been constructed on the assumption that a kind of simple laws that optimize the configuration stands behind visible varieties. In this article, we investigate the mechanism and the origin of these universally observed structures from a viewpoint of the optimization principle.

One of the influential and reputable theories for optimization of river basin networks is the Optimal Channel Network (OCN) by Rodriguez-Iturbe, Rinaldo, et al. [3,4,5,6,7]. According to the OCN theory, the most important structural characteristics observed in natural drainage networks are derived from three principles of optimal energy expenditure: (1) the principle of minimum energy expenditure in any links (branches) of the network, (2) the principle of equal energy expenditure per unit area of channel anywhere in the network, and (3) the principle of minimum total energy expenditure in the whole network. The OCN theory states that their joint works, especially the principle (3), regulate the dynamics of optimal channel network formation [4].

In ordinary OCN models, a rectangular simulation area consists of a lot of square lattices with an equal interval. Then, tree network patterns are formed connecting all these lattice points. Among all configurations, the one in which total energy expenditure is minimal is chosen as the most feasible structure. In principle, all the possible network patterns should be examined to obtain an exact solution because these patterns are assumed stochastically to be equal. However, this procedure is not only difficult but practically impossible due to the similar problem as in the case of the travelling salesperson problem [5]. That is, too many cases should be tested if we wish to accomplish the aim perfectly. As is well known, the number of permutation reaches astronomical figures if the number of components is sufficiently large.

Various approximate methods have been developed to find seemingly real solutions. It is certain that these methods can reproduce fine and realistic structures of natural river basins if high-performance computer systems are available [6]. However, approximate solutions not only depend strongly on how the initial conditions are settled, but also it seems difficult to draw these minute images by means of usual PC systems. High-performance systems such as a supercomputer are unavailable for most researchers. We would like to develop an easy and convenient program that can work on reasonable price PCs.

Even if the above-mentioned problems are solved, the essential question remains. Namely,
the OCN models do not simulate developing processes of channel network systems. The OCN theory is indifferent to the mechanism of river network formation. What is important is whether the resultant configuration is optimal or not. I wonder that the absence of explanation for developing processes is the most serious defect and the reason why we are not satisfied with this model.

Another optimization theory expected to account for channel network formation has been proposed by Bejan et al. as a Constructal law. The Constructal law is summarized as follows: the flow system such as area-to-point or volume-to-point flow optimizes itself in such a way that global flow resistance is minimal [8,9,10,11,12]. In other words, every flow system that persists for a long time should evolve its configuration that provides easier access under given constraints. Bejan and Lorente also insist that the Constructal law is a self-standing law that is distinct from the second law of thermodynamics because the second law does not refer to configurations, i.e., “architecture” [9].

Compared with the OCN model, the simulation model based on the Constructal law can reproduce growing processes of river channel network systems, which is the most outstanding merit. That is, Errera and Bejan adopted a Poisson equation at the core of the two-dimensional area-to-point flow model [8]. Thus, we tentatively refer this basic model as a Poisson Equation Model (PEM). The advantage of the PEM is that this model deduces driving force of river channel formation from a Poisson equation system.

In the case of original PEMs, a rectangular simulation area, which is initially coated with homogeneous porous media, is divided into a lot of small square blocks. Then, blocks whose pressure gradient exceed a critical value are entirely removed, then, replaced by new media with different permeability and added to the already existing drainage network. Thus, at the last moment when the simulation is completely ended, all the blocks have changed to new media and any pattern is washed away because the channel covers the whole area. What we can see at the end is only a vast extent of water like a huge lake or an ocean. That is, it is in intermediate stages that we can see tree network structures. It is certain that river network patterns created by Constructal models seem coarse and not so fine as compared with those by OCN models [8]. The reason is probably related to the above-mentioned algorithm of the original PEM that the relevant block is entirely removed.

PEMs also have difficulties for simulation time. Different from the OCN model, however, elapsed time mainly depends on the method to solve Poisson equations. As time for calculations increases with the increase in the number of unknown variables, so the number of square blocks should be restricted to an appropriate range. Furthermore, mathematical modelers and programmers should master the techniques for approximate calculations such as a Finite Difference Method (FDM) and for solutions of simultaneous linear equations with a huge number of unknown variables such as a Cholesky decomposition.
As explained above, there exist both advantages and disadvantages in two optimization models, that is, the OCN model by Rodriguez-Iturbe, et al. and the PEM by Bejan, et al. Concerning the difference of simulation areas, it is possible to think that a simulation area of OCN models is also composed of a lot of square blocks as well as that of PEMs. In this case, river channels are ditches carved by runoff, which connect central points of two adjacent blocks on the surface of media. Because a certain part of media remains even after the channel formation, the tree network patterns survive and do not disappear when the total simulation process is finished. Hence, network patterns continue to exist to the end.

We would like to make two existing models compromise and integrate into a single new model that incorporates advantages of both models. In conclusion, our objective in this study is to construct a simulation model of optimal river channel formation that satisfies following three features: (1) clear mechanisms that drive tree network formation, (2) moderate performances that can reproduce fine and naturalistic structures, and (3) simple and plain algorithms that can work on usual PC systems.

Our mathematical model is mainly based on the PEM by Bejan et al. with a few modifications added to dissolve above-mentioned difficulties. In the original PEM, for example, those blocks whose pressure gradient is more than a threshold value are all removed in each calculation, while only one block with the maximal or the minimal value is removed for each calculation in our novel PEMs. Moreover, tree network patterns remain even after the whole simulations are completed because channels are carved on the surface as in OCN models.

However, further improvement is required to mimic natural fractal flow structures. We achieved this aim by introducing a newly devised variable permeability system. The model using this technique is, as it were, the PEM performed with inhomogeneous permeability, then, we name it the Inhomogeneous Permeability Model (IPM) in this article. The newly proposed IPM will be a simple and useful simulation tool for geophysical researchers to reconstruct naturally disturbed fractal river networks.

The scope of the IPM is not restricted to the simulation of river channel formation. It can be applied to other kinds of tree network systems such as dendrites, frosts, and so on. Considering that a lot of physical phenomena in the natural world are described by Poisson equations, we can expect that the application of IPMs is further extended to various areas in physics, chemistry and biology.

2. Poisson Equation Model (PEM)

2.1. PC performance, simulation area and resolution

The device used in this study is a usual laptop PC for domestic use (TOSHIBA: dynabook T75)
with CPU (Intel: Core i7-6500U), which was purchased in a nearby electric retail shop with a reasonable price. The programing language is Java, and all programs are compiled using a free software (Borland: Turbo JBuilder 2007).

The simulation area of mathematical models in this article, including both PEMs and IPMs, is a square lattice as illustrated in Fig. 1 (a). The square area is further divided into $N \times N$ small square cells, where $N$ denotes the number of cells along a side in the simulation area. We adopt $N = 37$, thus, the whole area is constituted of $37 \times 37 = 1369$ cells. The upper limit of $N$ will depend on the performance of the computer system in use. Our PC system requires about 52 to 53 minutes for drawing tree network patterns of $N = 37$. Cells are also called blocks, sites, elements, etc.

Both simulation time and resolution of images are closely related with the computer performances, particularly calculation speed. Our PC performance that requires about 52 to 53 minutes to draw a $37 \times 37$ image is no better than those of preceding studies. As for the resolution of images, for example, the lattice size is $128 \times 128$ in the OCN model [6], and $51 \times 51$ in the PEM [8], both of which are larger than that of our simulation area. Although not specified clearly, it is strongly suggested that the performances of computer systems used in these preceding studies are much higher than those in our study. We would like to stress that even domestic type PCs of reasonable prices are usable to draw sufficiently fine and naturalistic tree network images.

If the simulation area is filled with homogeneous media, permeability $K$ is also constant, which means that the area is occupied with the same quality of soil. The mathematical model in this case is selectively named the Poisson Equation Model (PEM), and distinguished from the Inhomogeneous Permeability Model (IPM), which is introduced later. If any distortion is not found in the simulation area, each block should be a perfect square and the length of the side $h$ is exactly constant. This precondition is effective for both the PEM and the IPM.

Figure 1 (a) shows the case in which the outlet is located at the corner (vertex) of the domain as an initial network cell, which is shown by a black square. There are two candidate cells that can be the second network cell in the next step, which are also shown by gray squares in Fig. 1 (a). The network cell means that consisting of the network.
Figure 1: (a) Simulation area of PEM and IPM. A black square located at the lower left corner of the square mesh indicates the outlet. Two gray squares adjacent to the black square exhibit the candidate cells. (b) Initial distributions of \( u \)-values in PEM. The abscissa stands for the central position of each cell along the \( x \)-axis. Solid lines connect \( u \)-values corresponding to the same \( y \)-coordinate.

2.2. Poisson equation and boundary conditions

As well as preceding studies by Errera and Bejan [8], we also begin with the following two-dimensional Poisson equation, where two variables \( (x,y) \) signify horizontal coordinates.

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\dot{m}'' \nu}{W K} = 0.
\]

The pressure field \( P(x,y) \) is induced by the uniform mass flow \( \dot{m}'' \) from the upper surface, that is, from the positive to the negative side along the vertical direction, as a constant rainfall. In the initial state of PEMs, the whole area is filled with the porous media of constant permeability \( K \). As for other parameters, \( \nu = \mu / \rho \) is the kinematic viscosity where \( \mu \) and \( \rho \) are the viscosity and the density of the fluid, respectively, and \( W \) is the depth of the media. Non-dimensionalization of (1) leads to the following simplified form (2), which is extensively used in the analyses of PEMs [13].

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y) = 0.
\]

In general, the two-dimensional function \( u(x,y) \) represents various kinds of static scalar fields such as gravitational, chemical and electric potential. Therefore, the vector field described by the first partial derivatives \( (\partial u / \partial x, \partial u / \partial y) \) stands for physical force such as gravitation, etc. Meanwhile, the function \( f(x,y) \) reflects the divergence of substances, such as mass or electric charge, that flow into or out of the system. In the current circumstance, a certain amount of water is continuously and uniformly poured into the simulation area from the
external environment, i.e., from the upper surface, which controls the amount of \( f(x,y) \) together with permeability. Thus, in the homogeneous condition, it is reasonable to assume that \( f(x,y) \) is constant at the initial moment all over the area except for the outlet.

After all, the above Poisson equation (1) or (2) expresses the equation of continuity, the meaning of which is that any imbalanced distribution of substances, such as clustering in one place, should not be realized in the natural world. In particular, (2) is referred to as the Laplace equation in the case of \( f(x,y)=0 \).

The PEM signified by the Poisson equation (2) is a conceptual model, so that the absolute values of non-dimensional variables make no sense. Only relative \( u \)-values are significant. Considering that (2) is linear and that absolute \( u \)-values are meaningless, we can give a finite value to \( f(x,y) \) arbitrarily. Then, we postulate that \( f(x,y)=1/N^2 \) at the starting point in all blocks except for the outlet. That is, the total amount of water pouring into the whole simulation area is \( 1 \) per unit time.

In the most part of boundaries, Neumann boundary conditions, \( \partial u/\partial n=0 \), are imposed, which means that neither inflow nor outflow of water takes place through boundaries. However, as a \( u \)-value of the outlet is fixed at 0 throughout the simulation, the Dirichlet boundary condition, \( u=0 \), is exceptionally imposed only in this site. The profile of \( u \)-values along the \( x \)-axis at the starting point is portrayed in Fig. 1 (b), where the outlet locates at the lower left corner. All of \( u \)-values are positive, although \( u=0 \) at the outlet.

Here, we would like to introduce some special terms restrictively used in this article. First, the “network cell” is defined as a cell that is already a component of the existing network. Next, the “candidate cell” means a cell adjacent to the network, which is expected to be a new “network cell” in the next step. Further, the “critical cell” is the newest “network cell” just generated in the latest step. Besides, the “critical \( u \)-value”, the “critical gradient”, and so on, are contextually used indicating those of the critical cell just changed from the candidate to the network cell.

### 2.3. Algorithm and selection rules

As is explained above, numerical simulations are performed within a square area, which is further divided into a great many small square blocks. In the case of PEMs, it is assumed that a total area is coated with homogeneous porous media with constant permeability \( K \) in the initial state of simulations except for the outlet from which water is drained. On the condition of a constant rainfall, the pressure field \( P(x,y) \) or \( u(x,y) \) represented by a solution of the Poisson equation (1) or (2) continues to increase uniformly and proportionally to precipitation, i.e., time. For the first step, two blocks that exist adjacent to the outlet are candidates that can be changed to the channel. Taking the symmetric configuration into account, the pressure gradients of two
candidate cells are equal. However, after one of two blocks is accidentally selected and connected with the outlet, symmetry will be broken.

After the first conversion, the block previously occupied by initial media is replaced by new media with higher permeability $K_p (K_p > K)$, meaning that water is easy to flow within new regions. These two blocks including the outlet constitute an initial form of the tree-shaped channel network, then, the Poisson equation is recalculated in new conditions. In this way, the number of blocks with permeability $K_p$ continues to increase one by one, joining to the existing network. This process is continuously repeated until all the site is changed to the network cell, and a channel network covers the whole area.

The candidates expected to be next network cells must be adjoining the existing river network. Here, two selection rules can be conceivable, which are the “Max selection rule” and the “Min selection rule”. In the case of the Max selection rule, one of the candidate cells whose slope to the adjacent network cell is maximal is selected and changed to a critical cell, while the cell with the minimal slope is selected in the case of the Min selection rule. Here, the slope, i.e., the gradient can be calculated from either the Poisson equation (1) as $\Delta P/h$ or (2) as $\Delta u/h$, where $h$ is the interval between adjacent cells. However, the distributions of two gradients are exactly proportional with each other. Then, whichever value is used, we can get the same result as for the locations of critical cells by both Max and Min selection rules.

From the optimal viewpoint, the Max selection rule is preferred to the Min selection rule because the former system reaches the final optimal state more quickly than the latter. Thus, the Max selection rule should be adopted in principle. Simulations by use of the Min selection rule will be tested and discussed in the section 4.5.

At the starting point of simulations, $t=0$, the number of network cells is one. That is, the outlet is the only network cell, which functions as a seed of the forthcoming network. Thereafter, one cell whose gradient $m=\Delta u/h$ is maximal is chosen among candidate cells and converted into the network cell when the Max selection rule is employed. At the same time, permeability in the new network cell is lifted to $K_p (K_p > K)$ due to lower flow resistance within these cells. Suppose that $k_p$ is the ratio of permeability in network cells to that in non-network cells including candidate cells, then, $k_p=K/K_0<1$. Thus, $f(x,y)=k_p/N^2$ for network cells on the condition of constant rainfall. Resultantly, the number of the network cells with $f(x,y)=k_p/N^2$ continues to increase one by one in every step of the simulation.

Errera and Bejan examined the dependence of tree network patterns on $k_p$ within the range from 0.001 to 0.95, where they found that the smaller $k_p$-value, the slenderer the tree network pattern [8]. However, their simulations are ended in middle stages because, as explained before, any pattern disappears if simulations are completed to the end. According to our numerical experiments, where simulations are accomplished, any serious differences are not recognized within the same range of $k_p$, i.e., $k_p=0.001–0.95$, as for final tree network patterns. Thus, $k_p=0.1$
is fixed for all cases in our study, which is also selected as a representative value in preceding studies [8,9].

Meanwhile, non-network cells continue to store rainfall in proportion to time because water is not drained in these areas. Hence, the values of \( f(x,y) \) in non-network cells also continue to increase until they are converted to network cells. In our PC algorithm, \( f(x,y) = \frac{1}{N^2} \times (t+1) \) is postulated for non-network cells, where \( t \) means elapsed time from the starting point.

It should be noted that the matter of importance is not the \( u \)-value but the gradient \( m = \frac{\Delta u}{h} \) to the neighboring cell, where \( \Delta u \) is the difference of \( u \)-values and \( h \) is the interval between adjacent cells. The path with the maximum gradient forms a new branch joining to the network. Some candidates could have more than two paths toward the network cells. In general, the number of candidate paths is larger than that of candidate cells for each simulation stage. We adopted the algorithm where priority is given to the path that is determined in the earliest stage. Thus, the gradient of the oldest path is calculated for each candidate cell and compared with each other among all candidate cells in every stage.

If the interval between cells \( h \) is constant and \( h = 1 \), the difference of \( u \)-values between a candidate cell and its neighboring network cell is exactly equal to the gradient. Thus, \( m = \frac{\Delta u}{h} \), then, the gradient can be signified as \( \Delta u \) with the denominator omitted. Afterward, we will deal with the case where permeability in each cell is not equal and varied depending on the site. Even in this Inhomogeneous Permeability Model (IPM), \( h = 1 \) continues to hold.

### 2.4. Finite Difference Method (FDM)

The techniques to solve Poisson equations are well known. However, it is not so easy to design practical and efficient PC programs because a lot of memories and long elapsed time are required. In Computer Aided Engineering (CAE), several methods such as the Finite Element Method (FEM), the Boundary Element Method (BEM) and the Finite Difference Method (FDM) are utilized to analyze mathematical models described by differential equations. In the simulations of this article, the area is a square mesh, and it is easy to divide the whole area into the small squares of the same size. Moreover, the shape of the boundary is not complicated. In these situations, the best choice is the FDM because not only the principle is plain, but the simulation speed is faster than those of the FEM and the BEM if the number of \( N \) is equal. Then, we adopt the usual FDM for the analyses of mathematical models in this study [13].

In any way, when solving Poisson equations, we must deal with the simultaneous linear equations with a great number of unknown variables. The number of variables rises to \( 37 \times 37 = 1369 \) in the simulations of this article. However, the square matrix describing simultaneous linear equations is symmetric, so that the Cholesky decomposition method can be implemented, which saves a lot of time and labor.
The procedures to apply the FDM to the Poisson equation are as follows. First, the Poisson equation (2) is discretized using following formulae.

\[
\frac{\partial^2 u}{\partial x^2} = \frac{u(x + h, y) + u(x - h, y) - 2u(x, y)}{h^2},
\]

\[
\frac{\partial^2 u}{\partial y^2} = \frac{u(x, y + h) + u(x, y - h) - 2u(x, y)}{h^2}.
\]

As a result, (2) is modified, such as

\[
\frac{u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y)}{h^2} + f(x, y) = 0.
\]

It is allowed that \(h = 1\) in PEMs because the absolute values of \(u\) are meaningless. Accordingly,

\[
-u(x, y - 1) - u(x - 1, y) + 4u(x, y) - u(x + 1, y) - u(x, y + 1) = f(x, y).
\]

Next, a total of \(N \times N\) cells constituting a simulation area are consecutively numbered in order from the lower left to the upper right, i.e., from \(n = 0\) to \((N+1)(N-1)\). In the case of Fig. 1 (a), \(n = 0\) at the outlet. Using the notations “\(\%\)” and “\(/\)” to denote the remainder and the quotient, the relations between the \(x\)- or the \(y\)-coordinate \(n_x, n_y\), and the consecutive number \(n\) are as follows.

\[
n_x = n \% N, \quad n_y = n / N, \quad n = n_x + N n_y.
\]

Then, (5) is rewritten using arrays, such as

\[
-u[n - N] - u[n - 1] + 4u[n] - u[n + 1] - u[n + N] = f[n].
\]

The coefficients for cells locating on the border are somewhat altered according to boundary conditions. Signifying \([u[n]\) and \([f[n]\) as \(u_n\) and \(f_n\), respectively, then, the Neumann boundary conditions are applied. Moreover, the Dirichlet boundary condition is employed at the outlet, i.e., the lower left corner of the domain, whose solution is always \(u = 0\), i.e., \(f_0 = 0\). As a result, the simultaneous linear equations are expressed as follows.

\[
\begin{align*}
&n = 0: \quad u_0 = f_0, \\
&n = 1: \quad -u_0 + 4u_1 - u_2 - 2u_{N+1} = f_1, \\
& \quad \ldots \\
&n = N - 2: \quad -u_{N-3} + 4u_{N-2} - u_{N-1} - 2u_{2N-2} = f_{N-2}, \\
&n = N - 1: \quad -2u_{N-2} + 4u_{N-1} - 2u_{2N-1} = f_{N-1}, \\
&n = N: \quad -u_0 + 4u_N - 2u_{N+1} - u_{2N} = f_N, \\
& \quad \ldots \\
&n = n: \quad -u_{n-N} - u_{n-1} + 4u_n - u_{n+1} - u_{n+N} = f_n, \\
&n = (N + 1)(N - 1): \quad -2u_{(N+1)(N-1)-N} - 2u_{(N+1)(N-1)-1} + 4u_{(N+1)(N-1)} = f_{(N+1)(N-1)}.
\end{align*}
\]

Here, the variable \(n\) is varied from 0 to \((N+1)(N-1)\), the number of which is equal to that of total cells.

However, the matrix of coefficients must be symmetric so that the Cholesky decomposition
method can be applied, which requires several equations to be multiplied or divided by 2.

\[ n = 0: \quad u_0 = f_0, \]

\[ n = 1: \quad -u_0 + 4u_1 - u_2 - 2u_{N+1} = f_1, \]

\[ n = N - 2: \quad -u_{N-3} + 4u_{N-2} - u_{N-1} - 2u_{2N-2} = f_{N-2}, \]

\[ n = N - 1: \quad -u_{N-2} + 2u_{N-1} - u_{2N-1} = 0.5f_{N-1}, \]

\[ n = N: \quad -u_0 + 4u_N - 2u_{N+1} - u_{2N} = f_N, \]

\[ n = n: \quad -2u_{n-N} - 2u_{n-1} + 8u_n - 2u_{n+1} - 2u_{n+N} = 2f_n, \]

\[ n = (N + 1)(N - 1): \quad -u_{(N+1)(N-1)-N} - u_{(N+1)(N-1)-1} + 2u_{(N+1)(N-1)} = 0.5f_{(N+1)(N-1)}. \]

In this way, we can reach the final form of linear simultaneous equations (9) consisting of \( N \times N \) equations, which could be solved using the FDM and the Cholesky decomposition method.

The simultaneous equations (9) are also represented by the matrix \( (A_{ij}) \), as follows.

\[
\begin{pmatrix}
A_{00} & A_{01} & \cdots & A_{0(N-1)(N+1)} \\
A_{10} & A_{11} & \cdots & A_{1(N-1)(N+1)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{(N-1)(N+1)0} & A_{(N-1)(N+1)1} & \cdots & A_{(N-1)(N-1)(N+1)}
\end{pmatrix}
\begin{pmatrix}
u_0 \\
u_1 \\
\vdots \\
u_{(N-1)(N+1)}
\end{pmatrix}
= \begin{pmatrix}
f_0 \\
f_1 \\
\vdots \\
f_{(N-1)(N+1)}
\end{pmatrix}.
\]

(10)

Both subscripts \( i, j \) denote the consecutive number \( n \), which ranges from 0 to \( (N+1)(N-1) \).

Comparing (9) and (10) with each other, for example, the following values are specified except for the boundary cells, i.e., \( n \neq 0 \).

\[
A_{n-n-N} = A_{n-Nn} = -2, \quad A_{n-n-1} = A_{n-1n} = -2, \quad A_{nn} = 8, \]

\[
A_{n-n+1} = A_{n+1n} = -2, \quad A_{n+n-N} = A_{n+Nn} = -2, \quad f_n = 2 / N^2. \]

(11)

As for other matrix elements than those relating with neighboring cells, the values are all zero. Thereafter, the additional alterations for some matrix values and an \( f \)-value are required at the outlet. That is, when \( n=0 \),

\[
A_{00} = 1, \quad A_{n0} = A_{0n} = 0 \quad (n = 1, 2, \cdots, (N-1)(N+1)), \quad f_0 = 0. \]

(12)

Including (12), the coefficient matrix \( (A_{ij}) \) is symmetric, and its elements are integers in the homogeneous PEMs. The non-zero elements are distributed mainly along the diagonal line. When inhomogeneity of permeability is introduced later, these elements will be changed to real numbers.

In every step, the whole \( u \)-values are recalculated and redistributed globally within the whole simulation area. As the \( f \)-values increase with time due to water storage effects, the \( u \)-values also increase for non-network cells. Then, one of the candidate cells adjacent to the current network is chosen as a critical cell according to the selection rule and turned to a new network cell. Because the location of the outlet is fixed throughout the whole simulation area.
process, the matrix elements $A_{ij}$ identified by (10) are not changed and only $f_n$ are altered from \(1/N^2 \times (t+1)\) to $k_p$.

As the calculation and distribution are restarted and repeated, the network cells whose $f$-value is $k_p$ continue to increase one after another. Then, the river channel network pattern is updated and developed gradually until all the cells are converted to network cells and the channel network spreads over the whole area.

### 2.5. Simulation results of PEMs

We start from the simulation of PEMs in homogeneous conditions with the Max selection rule. Figure 2 (a) and (b) show the temporal variations of critical and average $u$-values. The critical $u$-value means that of a new network cell just changed from a candidate cell in the latest step. The average $u$-value is calculated for all cells within the simulation area, the number of which is $37 \times 37 = 1369$.

In general, variations of $u$-values can be controlled by two opposite effects: one is the water storage by rainfall that contributes to the gradual increase in $f$-values, and the other is the conversion from the candidate (non-network) to the network cell that induces sudden drop of permeability. Naturally enough, the former effect gives rise to an increase in $u$-values while the latter does a decrease in $u$-values. Parabolic convex curves in Fig. 2 (a) and (b) are certainly due to a tradeoff of these two competing tendencies. However, the variation of critical $u$-values fluctuates through the simulation, as shown in (a), although average $u$-values exhibit a smooth variation. The reason why unevenness is levelled in (b) is possibly attributed to an averaging effect.

Regarding Fig. 2 (b), after an increase in the first half and a decrease in the second half, the average $u$-value at the final state ($t=1368$) should be decreased as compared with that at the initial state ($t=0$) because of optimization to minimize total flow resistance as predicted by the Constructal theory. Taking into consideration the linearity of Poisson equations (1) and (2), it is theoretically conjectured that the decreasing rate of the average $u$-value is $1/10$ from the start to the end, which is just the same amount as $k_p$.

On the other hand, the variation of the critical difference is composed of a sharp increase in the early stages and a gentle decrease from the middle to the ending stages, as shown in Fig. 2 (c). It should be noted that the critical difference $\Delta u$ is the same as the critical gradient $m=\Delta u/h$ because $h=1$, where $h$ denotes the distance between two adjacent cells. The effect of boundary conditions probably accounts for anomalies and disturbances at the last quarter of the simulation.
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Figure 2: Temporal variations of (a) critical $u$-values, (b) average $u$-values and (c) critical differences $\Delta u$ in PEM (Max selection rule).

The final stream pattern in the simulation of Fig. 2 is shown in Fig. 3 (a). The simulation pattern consists of a main diagonal stream, whose length is $2 \times (N-1)=72$, and a lot of regularly oriented comb-shaped tributaries that join the main stream almost directly from the upper or the right side. The structure does not seem natural but artificial under the such idealized conditions as constant permeability, i.e., homogeneous topography, soil and precipitation. Bifurcation or self-similarity is hardly seen, which characterizes fractal structures in natural river basins. Thus, it might be supposed that the realistic tree-shaped structures are difficult to appear in idealized homogeneous conditions as in Figs. 2 and 3 (a).

Most of the mathematical modelers might think that it is easy to properly disturb network patterns and get naturalistic images by using the random number. For example, suppose that precipitation is changed depending on the site. The randomization of precipitation is simply implemented by varying $f$-values of right sides of simultaneous equations (10). Contrary to our expectation, however, this procedure neither induce much disturbance nor reproduce realistic images even if a sufficiently large scale of disorder is incorporated. Figure 3 (b) shows the pattern when the initial $f$-values are scattered within the range from $0.1/N^2$ to $1.9/N^2$. Much difference is not observed between two images, (a) and (b).
3. Inhomogeneous Permeability Model (IPM)

3.1. Poisson equation in IPMs

Reconstruction of natural and realistic river channels and exploration of their origin are main themes in this study. As shown in the latest section, the disturbances restricted in $f$-values are no use of these subjects. For this reason, we introduce an alternative method where the initial distribution of permeability is randomized from the beginning, which are uniform in the previous section. The new model is named the Inhomogeneous Permeability Model (IPM). However, the IPM is not independent of the PEM, but founded on the previous PEM. In short, the IPM is the expanded version of the PEM that can extensively deal with inhomogeneous conditions.

Then, we newly employ the following two-dimensional Poisson equation corresponding to inhomogeneous permeability $K(x,y)$.

$$\frac{\partial (K \partial P/\partial x)}{\partial x} + \frac{\partial (K \partial P/\partial y)}{\partial y} + \frac{\hat{m}^* \nu}{W} = 0. \quad (13)$$

In the initial state, the whole area is covered with a media of non-uniformly distributed permeability $K(x,y)$. We modify (13) to the dimensionless form (14) for practical use.

Figure 3: Tree network patterns by PEM (Max selection rule). (a) Homogeneous condition. The situation is the same as in Fig. 2. (b) Inhomogeneous condition. Only $f$-values are randomized within the range from $0.1/N^2$ to $1.9/N^2$. 

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The dimensionless function \( k(x,y) \) defines the initial distribution of permeability, whose average value is 1. It should be noted that \( k \)-values are given as a scalar not in each branch but in each site, thus, the number of which is equal to that of the site, \( N \times N \).

In this way, constant permeability in PEMs are replaced by the two-dimensional inhomogeneous permeability function \( k(x,y) \) in IPMs, values of which are scattered in a simulation area with a proper random number table. Using the consecutive number \( n \), the random number \( r[n] \) and the one-dimensional array \( k[n] \) instead of \( k(x,y) \), \( k[n] \) is specified, such as

\[
\begin{align*}
  k[n] &= 1 + k_r r[n], & 0 < k_r < 1, \quad -1 < r[n] < 1, \\
  (n &= 0, 1, \cdots, N \times N). \tag{15}
\end{align*}
\]

The randomization parameter \( k_r \) indicates the width of randomization, hence, \( k[n] \) ranges between \( 1-k_r \) and \( 1+k_r \). The variable \( k[n] \) should be positive in any site, which requires \( 0 < k_r < 1 \).

The consecutive cell number \( n \) varies from 0 to \( N^2 - 1 = (N+1) \times (N-1) \), covering all the cells. In our numerical simulations, \( k_r \)-values are varied within a range of \( 0 < k_r < 0.15 \), however, fixed in each simulation.

In general, large permeability means the increase in easiness to flow through the corresponding cell. On the contrary, small permeability signifies the decrease in easiness, i.e., the increase in difficulty to pass through the cell. It is supposed that permeability comprehensively reflects geographic properties of each site such as roughness of the soil, unevenness of the surface, and so on. In IPMs, the distance between adjacent cells \( h \) is constant as well as in PEMs. However, we would like to stress the following. That is, even so, the newly proposed permeability randomization method can reproduce sufficiently natural and realistic river basin images, which is one of the main objectives in our study.

In IPMs, a total of \( (N+1)(N-1) \times (N+1)(N-1) \) matrix elements \( A_{ij} \) are altered instead of \( f_n \). Accordingly, \( A_{ij} \) are changed from integers to real numbers, however, symmetry of the matrix is preserved as before. After all, two relations (3) is rewritten using the permeability array \( k[n] \), such as
\[
\frac{\partial(k \partial u/\partial x)}{\partial x} = \frac{k_{n+1} - k_{n-1}}{2h} \times \frac{u(x+h,y) - u(x-h,y)}{2h} + k_n \frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2},
\]
\[
\frac{\partial(k \partial u/\partial y)}{\partial y} = \frac{k_{n+1} - k_{n-1}}{2h} \times \frac{u(x,y+h) - u(x,y-h)}{2h} + k_n \frac{u(x,y+h) + u(x,y-h) - 2u(x,y)}{h^2}. \tag{16}
\]
where \(k[n]\) is signified as \(k_n\) for simplicity. The substitution of (16) to the Poisson equation (14) leads to a following equation (17).
\[
\frac{k_{n+1} - k_{n-1}}{4h^2} + k_n \frac{u(x+h,y) + u(x-h,y) - 2u(x,y)}{h^2} + \frac{k_{n+1} - k_{n-1}}{4h^2} + k_n \frac{u(x,y+h) + u(x,y-h) - 2u(x,y)}{h^2} = 0.
\]
Then, we reach the final formula (18) as an expanded Poisson equation of IPMs.
\[
\frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2} u(x,y-h) - \frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2} u(x-h,y) + \frac{4k_n}{h^2} u(x,y) + f(x,y) = 0.
\]
\[
\frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2} u(x+h,y) - \frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2} u(x,y+h) = f(x,y). \tag{18}
\]
After all, the following relations are obtained for matrix elements except for the boundary.
\[
A_{n-N} = A_{n-N} = -\frac{k_{n-N} - k_{n-1} + 4k_n}{4h^2}, \quad A_{n-N} = A_{n-N} = -\frac{k_{n-1} - k_{n-N} + 4k_n}{4h^2}, \quad A_n = A_n = \frac{4k_n}{h^2}, \tag{19}
\]
\[
A_{n+1} = A_{n+1} = -\frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2}, \quad A_{n+1} = A_{n+1} = -\frac{k_{n+1} - k_{n-1} + 4k_n}{4h^2}.
\]
Matrix elements related to the boundary are changed similarly to the case of PEMs.

Initial distribution of \(u\)-values is shown in Fig. 4 for the randomization parameter \(k_r=0.1\). Boundary conditions are the same as in PEMs of the homogeneous condition. Because a constant interval is sustained even in the case of IPMs, the value compared with each other is also the difference between the candidate and the network cells \(Au\) as well as in PEMs. If \(h=1\) is assumed, comparable differences are following four, where the coordinate of the corresponding central cell is \((x,y)\).
\[
u(x,y) - u(x+h,y), \quad u(x,y) - u(x-h,y), \quad u(x,y) - u(x,y+h), \quad u(x,y) - u(x,y-h). \tag{20}
\]
Figure 4: Initial distributions of $u$-values in IPM. The randomization parameter $k_r=0.1$. The random number table $Rn\#0$ is in use. Boundary conditions are the same as in PEMs. Solid lines connect $u$-values corresponding to the same $y$-coordinate.

We performed numerical simulations of IPMs using three different random number tables, which are tentatively assigned as $Rn\#0$, $Rn\#1$ and $Rn\#2$, respectively. Simulation results by $Rn\#0$ are mainly cited, for example, in Figs. 4, 6, 7 and 11 except for Fig. 5, where those by $Rn\#0$, $Rn\#1$ and $Rn\#2$ are all exhibited. As for Figs. 10 and 12, on the other hand, simulation data are evaluated by the ensemble average of three randomization.

3.2. Simulation results of IPMs

In the first place, we would like to survey the dependence of IPMs on randomization by using different random number tables, $Rn\#0$, $Rn\#1$ and $Rn\#2$. Figure 5 shows three river network patterns by the IPM on the condition of representative parameter settings, i.e., the reduction ratio of permeability $k_r=0.1$ and the randomization parameter $k_r=0.1$. Although the appearances are quite different, three images seem equally natural, and no special difference is observed as for essential features such as complexity. A delicate sensitivity to parameters seen in appearances is characteristic of chaotic systems such as the IPM. After Fig. 5, simulation images by the random number table $Rn\#0$ are mainly exhibited.
Next, we examine the dependence of tree network patterns on the randomization parameter $k_r$ using the Rn#0 random number table. The simulation results of IPMs with the Max selection rule are shown in Fig. 6 for three randomization parameters $k_r$. Another parameter $k_p$ is fixed at 0.1, which is the ratio of permeability in network cells to that in non-network cells. Although every cell has different permeability in the non-network state, the value of the conversion ratio $k_p$ is constant for all cells. That is, the non-network cell with permeability $k(x,y)$ is indiscriminately converted to the network cell with permeability $k_p \times k(x,y)$.

The same random number table, Rn#0, are referred to for all figures. The outlet placed at the lower left corner is emphasized by the comparatively large blue square. It seems that river network patterns are sensitively changed depending on the value of $k_r$. The disturbance of channel networks is increased with the increase in the parameter $k_r$. 

Figure 6: Tree network patterns by IPM (Max selection rule, Rn#0). The Neumann boundary conditions ($\partial u/\partial n=0$) are imposed except for the outlet. The random number table in use is Rn#0 in all simulations. There is no difference between Fig. 6 (c) and Fig. 5 (a).
3.3. Concave ditch and convex bank that separate mountainous catchment and river basin areas

Figure 7 shows the temporal variations of key values in the IPM when the Max selection rule and the Rn≠0 random number table are adopted. As for the critical $u$-value (a), the variation seems more irregularly fluctuated compared with the case of the PEM, which is unquestionably due to the introduction of inhomogeneous permeability. Meanwhile, the variation in the average $u$-value (b) shows an even parabolic curve, the reason of which could be attributed to the compromise of two opposite effects, i.e., the water storage effect and the conversion effect from the candidate (non-network) to the network cell together with averaging effects as well as in Fig. 2 (b).

However, the more significant and essential difference emerges in the variation of the critical difference $Δu$ in the IPM. As shown in Fig. 7 (c), the permeability randomization process often induces the situation where the maximum $Δu$ is negative, which means the emergences of a “ditch” or a “moat” in the edge of the mountainous catchment area and a “bank” or a “ridge” in the edge of the river basin area. These situations could become outstanding as the randomization parameter $k_r$ is increased.

![Figure 7: Temporal variations of (a) critical $u$-values, (b) average $u$-values and (c) critical differences $Δu$ in IPM (Max selection rule, Rn≠0). Circumstances are equal to those in Fig. 6 (c), i.e., the randomization parameter $k_r=0.1$.](image)
The convex bank is a kind of “fence” that surrounds the river channel area and shields it from the mountainous catchment area. As a result, a concave ditch is inevitably formed just outside of the bank. As for the cells constituting the bank, \(u\)-values are larger than those of the ditch adjoining to the bank. The topography of this area would look like a “marshy belt”, which is soon filled with water under the constant rainfall condition.

However, these ditch and bank do not cause such a serious problem to the progress of simulations. Considering that the difference \(\Delta u\) is negative for the cells in the concave ditch, the absolute value of \(\Delta u\), i.e., \(-\Delta u\), should be the smallest when the maximum \(\Delta u\) is selected. In such a situation, a concave ditch is no more than a shallow marsh that accidentally appears in the mountainous region, where water is easy to overflow through the lowest ridge. Thus, it is allowable to assume as follows. That is, even if the concave ditch appears, as soon as it is filled with water, the lowest ridge is collapsed, and a canal is dug that connects the ditch with the existing channel network. Three river channel patterns in Fig. 6 are all drawn on the assumption of these mechanisms.

4. Discussion

4.1. Energy expenditure function

The Optimal Channel Network (OCN) is one of the most reputed theories for river channel analyses, which states that the flow configuration that minimizes the total rate of energy expenditure is realized as an optimal stream in the natural world. Total energy expenditure \(E\) is defined as follows.

\[
P_i = kQ_i^{0.5}L_i, \quad E = \sum_i P_i = k \sum_i Q_i^{0.5}L_i. \tag{21}
\]

where \(k\) is a constant, and \(P_i\), \(Q_i\) and \(L_i\) denote the rate of energy expenditure, the flow and the length of the \(i\)-th branch, respectively [3,4,5,6,7]. On the calculation of \(E\), these values are summed up for all \(i\), the number of which is \(2N\times(N-1)\). The optimization function \(E\) quantifies the stability of river network systems.

According to Rodriguez-Iturbe, Rinaldo, et al., the energy expenditure function (21) is identified as follows [4]. Consider a channel with a rectangular section, whose width \(w\), flow depth \(d\) and length \(L\). Accordingly, the cross-sectional flow area \(A_w = wd\), and the wetted perimeter section \(P_w = w + 2d\), respectively. In general, energy expenditure of an individual channel \(P\) is generated via two mechanisms: one is friction against side walls and the bottom, and the other is maintenance of the channel such as the removal and the transportation of the sediment.


\[
P = C_f \rho v^3 + KC_f^m \rho^m v^2mP_wL, \quad v = \frac{Q}{A_w}.
\] (22)

In the above formula (22), the first term means the energy loss due to friction, and the second term does the loss due to maintenance. As for parameters used in (22), \( C_f \) is the dimensionless resistance coefficient, \( \rho \) is the density of water, \( v \) is the velocity of the flow, \( K \) depends only on the soil and fluid properties, \( m \) is a constant and \( Q \) is the flow.

Meanwhile, according to the principle (2) of the OCN theory, which is found in the Introduction section, energy expenditure per unit area of a channel is equal anywhere in the network, thus, \( P/(P, L) = \text{const.} \) This equality requires the following relation.

\[
C_f \rho v^3 + KC_f^m \rho^m v^{2m} = C = \text{const.}
\] (23)

The relation (23) implies that the velocity \( v \) tends to be constant throughout the network, thereafter, the following equation is obtained.

\[
P = \frac{Q L C}{d v} + dL2C = (\text{const}) \frac{Q L}{d} + (\text{const}) dL.
\] (24)

Moreover, the OCN principle (1) insists the minimization of \( P \) in any link of the network. The application of this local optimization principle is substantiated, for example, by differentiation \( dP/d(d) = 0 \). As a result, we can reach following relations.

\[
d = (\text{const}) Q^{0.5}, \quad w = \frac{Q}{vd} = (\text{const}) Q^{0.5}, \quad P = (\text{const}) Q^{0.5}L.
\] (25)

Finally, rewrite \( P, Q \) and \( L \) to \( P_i, Q_i, \) and \( L_i \), respectively, and add together for all \( i \), which leads to the energy expenditure function (21).

4.2. Comparison of PEM with OCN theory

It is possible to search for optimal structures directly from (21) according to the OCN theory. Then, we provide one more supplementary PC program for this aim. We can obtain exact solutions using this program because all the possible configurations are examined. However, the mesh size \( N \) is restricted within very small values since enormously long time is required for the large number of \( N \). These difficulties are characteristic frequently encountered in this kind of problems such as the travelling salesperson problem.

Figure 8 shows some possible solutions for \( N=4, 5 \) and \( 6 \) in homogeneous conditions. Among four patterns, (a) and (b) are the exact solutions, that is, the most optimal states corresponding to \( N=4 \) and \( 5 \), where all the cases are entirely tested. The values of total energy expenditure calculated by (21) are (a) \( E=24.302 \) and (b) \( E=43.067 \), respectively. Then, we can naturally conjecture that the solution for \( N=6 \) is (c), where total energy expenditure is \( E=68.073 \). However, this supposition is wrong. Despite irregular and asymmetric with respect to the
diagonal line, the correct solution for \( N=6 \) is (d) whose total energy expenditure is \( E=67.757 \), which is certainly smaller than the value of (c). It is interesting that the simulation results of \( N=6 \) is quite different from that expected from those of \( N=4 \) and 5.

\[
\begin{align*}
(a) & \quad E=24.302 \ (N=4), \\
(b) & \quad E=43.067 \ (N=5), \\
(c) & \quad E=68.073 \ (N=6), \\
(d) & \quad E=67.757 \ (N=6).
\end{align*}
\]

Figure 8: Optimal streams derived from OCN theory. Among four, (a), (b) and (d) are the most optimal states corresponding to \( N=4, 5 \) and 6. (c) is not the exact solution. (a) \( E=24.302 \ (N=4) \), (b) \( E=43.067 \ (N=5) \), (c) \( E=68.073 \ (N=6) \), (d) \( E=67.757 \ (N=6) \).

The actual maximal number of \( N \) is no more than 6 at the best in usual PC systems. If the upper limit is \( N=6 \), there is nothing for \( N\geq7 \) but to infer from the results of Fig. 8. Hence, the next step is to conjecture and conceptually reconstruct the possible optimal streams in homogeneous conditions.

Two patterns shown in Fig. 9 are those configurations inferred from Fig. 8, which are different from the real simulation results. The first pattern (a) is proved to be the most optimal at least within the range \( N\leq5 \), as shown in Fig. 8 (a) and (b), while non-optimal for \( N>6 \).
Meanwhile, the second pattern with the comb-shaped branches (b) can be conjectured from Fig. 8 (d), especially from the lower two lines. When \(N=37\), the \(E\)-value of (b) is much smaller than that of (a). A dramatic reduction of the \(E\)-value suggests that (b) is much more optimal than (a) at least in the case of \(N=37\). However, this result does not necessarily guarantee that (b) is the most optimal.

Figure 9: Possible optimal tree network patterns in homogeneous condition. These two are inferred from the results of Fig. 8. \(N=37\). (a) \(E=5627.632\). (b) \(E=5007.155\).

The subject in this section is to confirm the reliability of the PEM in homogeneous conditions. This attempt succeeds by comparing the simulation result of Fig. 3 (a) with two patterns of Fig. 9. We can clearly recognize that Fig. 3 (a) resembles Fig. 9 (b) in appearance more closely than Fig. 9 (a). These facts that the real simulation result by the PEM remarkably resembles Fig. 9 (b), and that the \(E\)-value of Fig. 9 (b) is much smaller than that of Fig. 9 (a) conclude that Fig. 9 (b) is more optimal than Fig. 9 (a). This consistency might suggest that the PEM properly reproduce the optimal stream in perfectly homogeneous conditions.

However, it is difficult to declare that the simulation image in Fig. 3 (a) mimics natural river basins. The comb-like structure in Fig. 3 (a) is not only unnatural but does not show any fractal structure, which is characteristic of natural river channels. We infer that the reason resides in too idealized conditions in the PEM. Diversity and heterogeneity are the general in the natural world. Perfect homogeneity in circumstances is a fictitious event. The absence of reality could cause deviation from the proper solution.

### 4.3. Comparison of IPM with OCN theory

As explained in the previous section, it is impossible to find the exact solution even in the
homogeneous condition in the case of \( N=37 \). If so, why is it possible in the heterogeneous condition? Then, we must attempt an alternative approach to give a satisfactory explanation for consistency of IPMs with the OCN theory.

The values of total energy expenditure \( E \) in IPMs by use of three different random number tables, \( Rn\#0, Rn\#1 \) and \( Rn\#2 \), are listed in Table 1 for various randomization parameters \( k_r \), in which the Max selection rule are adopted. The \( E \)-value in the PEM (\( k_r=0 \)) is also included. The averaged \( E \)-values are plotted in Fig. 10 (a) with the upper and the under bars showing the maximum and the minimum value among three. Comparatively large fluctuations in \( E \)-values are observed depending on randomization tables, which is probably due to delicate sensitivities to parameters characteristic of chaotic systems such as the IPM. \( E \)-values are increased from about 4900 to 5600 within a range of \( 0.01 \leq k_r \leq 0.125 \), which could be due to the increase in randomization.

Not only total energy expenditure but mainstream lengths are also listed in Table 1 and plotted in Fig. 10 (b), where “the mainstream” means “the longest stream”. If there exists no reverse segment where the river flows from the left to the right cell or from the lower to the upper cell, the mainstream length should be \( 2 \times (N-1) = 72 \) for \( N=37 \) in the simulation setting of this article, which is the case in \( k_r=0 \). The increase in the mainstream length is similar to that in total energy expenditure \( E \) within the whole range, suggesting a close correlation between \( E \)- and \( L \)-values.

Table 1: Total energy expenditure and mainstream lengths for various randomization parameters \( k_r \) in IPM (Max selection rule).

| Randomization Parameter \( (k_r) \) | Total Energy Expenditure \( (E) \) | Mainstream Length \( (L) \) |
|-----------------------------------|-------------------------------|-----------------|
| \( Rn\#0 \) \( Rn\#1 \) \( Rn\#2 \) | \( \text{Av.} \) \( \text{Rn\#0} \) \( \text{Rn\#1} \) \( \text{Rn\#2} \) \( \text{Av.} \) |
| 0.0 | 5005.888 | ----- | ----- | 5005.888 | 72 | ----- | ----- | 72 |
| 0.01 | 4968.991 | 4892.941 | 4883.729 | 4915.220 | 74 | 75 | 77 | 75.3 |
| 0.025 | 5061.982 | 4916.279 | 5016.371 | 4998.211 | 82 | 80 | 85 | 82.3 |
| 0.05 | 5061.619 | 5212.102 | 5037.988 | 5103.903 | 85 | 104 | 99 | 96 |
| 0.075 | 5125.429 | 5324.377 | 5057.344 | 5169.050 | 84 | 107 | 101 | 97.3 |
| 0.1 | 5316.518 | 5587.487 | 5258.154 | 5387.386 | 94 | 105 | 112 | 103.7 |
| 0.125 | 5508.229 | 5553.031 | 5640.351 | 5567.204 | 106 | 106 | 117 | 109.7 |
| 0.15 | 5474.184 | 5155.754 | 5178.614 | 5269.517 | 102 | 87 | 86 | 91.7 |
Figure 10: Dependences of (a) total energy expenditure and (b) mainstream lengths on randomization parameters $k_r$ in IPM (Max selection rule). The ensemble averages of three kinds of randomization are represented by crosses ($\times$) with the vertical lines that show the maximum and the minimum.

In fact, unnatural structures begin to appear on simulation images with the increase in the randomization parameter $k_r$, especially for the large parameter region such as $k_r \geq 0.15$. For example, the parallel straight streams become outstanding little by little. Although the reason is not clear, we guess that it is attributed to the way of randomization. In our IPMs, random numbers are assigned to the individual cell. As a result, it happens that the difference of permeability between adjacent cells becomes extremely large with the increase in $k_r$. However, such a situation is difficult to happen in the natural world because it is usual that permeability changes continuously and gently over a wide area including a number of cells. It is probable that these inconsistencies with the real world cause the appearance of unnaturally disturbed patterns in comparatively large $k_r$ ranges.

4.4. Horton’s law, Hack’s law and Melton’s law

As for the configuration of natural river channels, some empirical laws have been known far before the emergence of the OCN theory. To begin with, it is necessary to introduce an appropriate ordering system for stream branches to describe these laws. The order of stream segments is usually measured with the Strahler ordering procedure: (1) channels that originate at a source are defined as first-order streams; (2) when two streams of the order $\omega$ join, a stream of the order $\omega + 1$ is created; and (3) when two streams of the different order join, the channel segment immediately downstream has the higher order of two combining streams [14]. The number of streams $N_\omega$ of the order $\omega$ is counted using this ordering system in the present study.
Horton introduced two famous laws with respect to the number $N_\omega$ and the average length $\bar{L}_\omega$ of the streams of the order $\omega$ in river channel networks. According to the Horton’s laws, two ratios, i.e., the bifurcation ratio $R_b$ and the length ratio $R_L$, which are mathematically expressed as follows, are constant \[3,10,15\].

$$\frac{N_\omega}{N_{\omega+1}} = R_b, \quad \frac{\bar{L}_\omega}{\bar{L}_{\omega+1}} = R_L. \quad (26)$$

In natural river basins, $R_b$ ranges between 3 and 5, typically 4, while $R_L$ ranges between 1.5 and 3.5, typically 2. Following relations are also derived from (26).

$$N_\omega = N_1 R_b^\omega, \quad \ln N_\omega = \ln N_1 - (\omega - 1) \ln R_b,$$
$$\bar{L}_\omega = \bar{L}_1 R_L^{\omega-1}, \quad \ln \bar{L}_\omega = \ln \bar{L}_1 + (\omega - 1) \ln R_L. \quad (27)$$

The Hack’s law expresses the relation between the mainstream length $L_\omega$ and the area $A_\omega$ of a river basin with the streams up to the order $\omega$ \[10,16\].

$$L_\omega = \alpha (A_\omega)^\beta. \quad (28)$$

In the calculation (28), $\alpha \sim 1.4$ and $\beta \sim 0.568$ are constant. As our simulation models, whether the PEM or the IPM, assume that every block within the whole simulation area is occupied by some stream or other in the final state, so the area corresponding to the mainstream is $A_\omega = A = 37 \times 37 - 1 = 1368$, where the outlet is excepted. Thus, the theoretical value of the mainstream length $L_{\omega_{\text{max}}}$ should be

$$L_{\omega_{\text{max}}} \sim 1.4 \times 1368^{0.585} \sim 84.611. \quad (29)$$

The Melton’s law, on the other hand, indicates the following relations.

$$D_\omega = \frac{L_F}{A}, \quad F_S = \frac{N_S}{A}, \quad F_S = 0.694(D_\omega)^2. \quad (30)$$

The drainage density $D_\omega$ and the stream frequency $F_S$ are defined by the total length of streams of all orders $L_F$, the number of streams of all orders $N_S$ and the total drainage area $A$ \[10,17\]. Considering that $L_F = A$ in our simulation settings, the drainage density $D_\omega = 1$ in the final state. Then,

$$F_S = 0.694. \quad (31)$$

This is the theoretical value of $F_S$ expected by the Melton’s law.

Comparison between the theoretical values derived from the Hack’s and the Melton’s laws and those obtained by the simulation results of the PEM and IPMs are shown in Table 2. The simulation data of mainstream lengths and stream frequencies are averaged among three randomization conditions, $Rn\#0$, $Rn\#1$ and $Rn\#2$. As Fig. 10 (b) shows in the previous section, mainstream lengths, i.e., $L$-values continue to increase with the increase in $k_r$ within the range
of $0 \leq k_r \leq 0.125$, and seem to cross the theoretical value, $L=84.611$, at $k_r=0.025$. On the other hand, stream frequencies derived from the simulation data are somewhat larger than the theoretical value, $F_S=0.694$, while differences tend to be diminished with the increase in the randomization parameter $k_r$.

For both mainstream lengths and stream frequencies, the simulation data and the theoretical values are not so different with each other. However, it seems difficult to insist that the simulation data show good accordance with the theoretical values. Although not clear, reasons of inconsistency could be attributed to the choice of the ordering system, the boundary condition, the simulation setting in which all the cells are finally occupied by channels, and so on.

The theoretical value of the mainstream length derived from Hack’s law is $L=84.611$, and that of the stream frequency from the Melton’s law is $F_S=0.694$, respectively. The drainage density $D_\omega=1.0$.

### 4.5. Simulation results by Min selection rule

So far, we have employed the Max selection rule for the numerical simulations drawing river channel networks. From the viewpoint of optimal principles such as the OCN theory or the Constructal law, the usage of the Max selection rule is rational because the fastest relaxation to the optimal stable state is expected. If so, it is impossible to create any tree network pattern by
means of the Min selection rule? This question deserves to be tested. Thus, the last subject remained to be explored in this article is to confirm whether the Min selection rule generates tree-shaped network patterns or not.

Interestingly, it is certified that the Min selection rule can promote the formation of unique river channel patterns as well as the Max selection rule. Figure 11 shows the simulation results of IPMs in the case of the Min selection rule for three randomization parameters \( k_r \). Other settings such as boundary conditions, the random number tables in use, etc., are the same as those in Fig. 6 by the Max selection rule. Three simulation images in Fig. 11 seem to be natural and realistic while those for \( k_r \sim 0 \) are unnaturally disturbed, which are not explicitly exhibited.

![Figure 11](image)

Figure 11: Tree network patterns by IPM (Min selection rule, Rn#0). The simulation settings are the same as in Fig. 6, except for the selection rule. \( k_r=0.1 \), (a) \( k_r=0.025 \), (b) \( k_r=0.05 \), (c) \( k_r=0.1 \).

It seems that appearances of simulation patterns are considerably different from those of Fig. 6, which could be summarized as follows. The river channel patterns by the Max selection rule seem to be composed of more than two main streams that branch off comparatively near the outlet, whose lengths are not so different with each other. Meanwhile, a meandering main stream is obviously specified in the case of the Min selection rule, whose lengths are distinguishably long compared with other tributaries. It is likely that both types of river channel patterns could be substantiated in the natural world. For example, the widespread tree-like patterns by the Max selection rule exist in comparatively flat plains, while the meandering slender patterns with a prolonged mainstream by the Min selection rule are frequently observed in steep mountainous regions.

The amounts of total energy expenditure \( E \) and mainstream lengths \( L \) for the Min selection rule are both listed in Table 3, which include individual values corresponding to three random number tables, Rn#0, Rn#1 and Rn#2, and those averaged among three. It seems that both values tend to be decreased with the increase in the randomization parameter \( k_r \) within the
whole range, especially $k_r>0.05$, indicating a gradual approach to the optimal stable state.

Table 3: Total energy expenditure and mainstream lengths for various randomization parameters $k_r$ in IPM (Min selection rule).

| Randomization Parameter ($k_r$) | Total Energy Expenditure ($E$) | Mainstream Length ($L$) |
|---------------------------------|-------------------------------|------------------------|
|                                 | Rn#0 | Rn#1 | Rn#2 | Av. | Rn#0 | Rn#1 | Rn#2 | Av. |
| 0.0                             | 6943.343 | ----- | ----- | 6943.343 | 126 | ----- | ----- | 126 |
| 0.01                            | 5956.806 | 6675.033 | 5419.558 | 6017.132 | 125 | 178 | 102 | 135 |
| 0.025                           | 6209.246 | 7751.226 | 5575.898 | 6512.123 | 141 | 216 | 123 | 160 |
| 0.05                            | 6177.158 | 5955.672 | 5693.208 | 5942.013 | 149 | 140 | 108 | 132.3 |
| 0.075                           | 5673.344 | 5911.467 | 5877.564 | 5820.792 | 111 | 147 | 126 | 128 |
| 0.1                             | 6003.809 | 5758.865 | 5855.369 | 5872.681 | 130 | 134 | 126 | 130 |
| 0.125                           | 5945.198 | 5806.885 | 5954.712 | 5902.265 | 115 | 134 | 118 | 122.3 |
| 0.15                            | 5864.379 | 5628.409 | 5368.520 | 5620.436 | 116 | 120 | 99 | 111.7 |

These tendencies are more precisely distinguished by plotting data on the diagram. In Fig. 12, the dependences of the rates of total energy expenditure $E$ and mainstream lengths $L$ on the randomization parameters $k_r$ are exhibited by use of averaged data. For comparison, the values of the Max selection rule are also plotted. Both $E$- and $L$-values are distinctively large in the case of the Min selection rule ($\bullet$) compared with the case of the Max selection rule ($\times$) for all randomization parameters. Considering that the $E$-value is the barometer of optimization, i.e., stabilization and that the $E$-values by the Max selection rule are lower than those by the Min selection rule for all $k_r$, as shown in Fig. 12 (a), it is certain that the Max selection rule is more favorable than the Min selection rule.

However, we should not overlook the following observation, that is, both $E$- and $L$-values by the Min selection rule tend to decrease with the increase in the randomization parameter $k_r$. On the contrary, these values by the Max selection rule tend to increase in the same $k_r$-region. As a result, the differences of these values between two selection rules are further diminished. A gradual approach of two data series might suggest that the Min selection rule is apt to be realized within the range of comparatively large randomization parameters, $0.075 \leq k_r \leq 0.15$. 

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Figure 12: Dependences of (a) total energy expenditure and (b) mainstream lengths on randomization parameters $k_r$ in IPM. In both figures, crosses ($\times$) and disks (●) show the values by the Max selection rule and by the Min selection rule, respectively, where the ensemble averages for Rn#0, Rn#1 and Rn#2 are employed.

4.6. Implications for Maximum and Minimum Entropy Production

According to the OCN theory, the Max selection rule should dominate channel network formation because the total rates of energy expenditure, i.e., $E$-values, are smaller, as shown in Table 3 and Fig. 12 (a). The system with the Max selection rule also could reach the optimal stable state more rapidly than any other system. However, the difference of $E$-values between two selection rules is not so large especially when the randomization parameter exists within the range of $0.075 \leq k_r \leq 0.15$. Taking these simulation results into consideration, it is not denied that the Min selection rule cannot be excluded and can coexist in the natural world as well as the Max selection rule. It is challenging to discuss these problems with reference to entropic theories.

Entropy is an index of homogeneity or heterogeneity. That is, “entropy of homogeneous states is high, while that of inhomogeneous states is low”. The fractal geometry such as natural river basins is a typical example of low entropy states.

There are three fundamental laws that control entropy production in physical systems. These are the second law of thermodynamics, the principle of Minimum Entropy Production (mEP) and the principle of Maximum Entropy Production (MEP) [13,18]. The second law of thermodynamics, which has been well-known as “the law of increasing entropy”, is the most fundamental that is effective in isolated systems. Meanwhile, the principles of mEP and MEP work on non-equilibrium open systems. We assume that the Max selection rule and the Min selection rule embody the MEP principle and the mEP principle, respectively.
Several decades ago, Prigogine proposed the mEP principle, which stated that the open systems existing in the state near thermodynamic equilibrium are stabilized where entropy production is minimal. The principle is effective in the states where the interaction between the system and the external environment is linear [19,20,21]. The mEP principle has been broadly recognized among natural scientists. Prigogine also stated that the entropic behaviors in the state far from equilibrium, where the interaction is non-linear, remained to be explored.

Later, Kleidon insisted that the MEP principle is valid in the open systems existing in the state far from equilibrium, that is, the open systems in this situation are stabilized where entropy production is maximal [22]. The MEP principle could reveal the creativity of entropy in universe because dissipative structures characterized by low entropy are created in such conditions. As tree-shaped networks such as the river channel system are typical examples of dissipative structures, the introduction of the MEP principle is essential for full explanation of their formation [23].

However, simulation results in this study seem to somewhat contradict these theories. According to our simulation results, not only the Max but the Min selection rule can invoke the formation of a kind of dissipative structures. Considering that the Min selection rule means invocation of the mEP principle, dissipative structures with low entropy can be created even in the systems not so far from equilibrium.

In his recent paper, Reis states that Maximum Entropy Production (MEP) occurs when all the forces in the system are kept constant. On the other hand, Minimum Entropy Production (mEP) gives rise to when all the currents that cross the system are kept constant [18]. In summary, many unsolved problems remain to be investigated for the interpretation of entropic theories such as mEP and MEP, which belong to future issues.

5. Conclusions

(1) The basic PEM and the elaborate IPM presented originally in this paper are practical and convenient tools to investigate optimal flow structures and to simulate real river basins, which are ubiquitously observed in the natural world. Both models show consistency with optimization principles advocated by the OCN theory and the Constructal law.

(2) River channel patterns created by PEMs in homogeneous conditions such as a constant rainfall and uniform topographical features show simple and artificial structures and do not exhibit fractal properties. Hence, the origin of fractal structures that characterize natural river channels is supposed to be inhomogeneity that exists everywhere on the Earth.

(3) The IPM can successfully reproduce tree network patterns by both the Max and the Min selection rules. Only the Max selection rule seems to dominate within the range of small randomization parameters \(0<k_r<0.05\), while both selection rules could coexist within the
range of comparatively large randomization parameters $0.05 < k_r < 0.15$.

(4) Our IPM is not perfect, thus, there exist some room for improvement. For example, IPMs hardly satisfy the Horton’s laws. As a result, it is difficult to identify two ratios, $R_b$ and $R_L$, which makes it difficult to calculate fractal dimensions. To overcome this challenge, we might reconsider and improve the ordering system. Moreover, the IPM should correspond with large randomization parameters such as $k_r \geq 0.15$ as well as small $k_r$-values. These difficulties are belong to future issues.

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