SOLUBILITY OF FERMAT EQUATIONS

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Abstract. The arithmetic of the equation \( a_1 x_1^d + a_2 x_2^d + a_3 x_3^d = 0 \) is considered for \( d \geq 2 \), with the outcome that the set of coefficients for which the equation admits a non-zero integer solution is shown to have density zero.

1. Introduction

Let \( d \in \mathbb{N} \). The purpose of this short note is to discuss the locus of rational points \( C_d(\mathbb{Q}) \) on the Fermat curves

\[ C_d : \ a_1 x_1^d + a_2 x_2^d + a_3 x_3^d = 0 \] (1)

in \( \mathbb{P}^2 \), for given \( a = (a_1, a_2, a_3) \in \mathbb{Z}^3 \). Our main goal is to show that a random such curve does not possess a rational point for \( d > 1 \). However, we will also discuss the number of solutions when \( C_d(\mathbb{Q}) \neq \emptyset \), and briefly consider the analogous problem in higher dimension.

Throughout our work we will allow \( i, j, k \) to denote distinct elements from the set \( \{1, 2, 3\} \). For any \( H \geq 1 \), let \( N_d(H) \) denote the number of \( a \in \mathbb{Z}^3 \) with \( |a_i| \leq H \), for which \( C_d(\mathbb{Q}) \neq \emptyset \). The following is our main result.

**Theorem 1.** We have

\[ N_d(H) \ll \frac{H^3}{(\log H)^{\psi(d)}}, \]

where if \( \phi \) denotes Euler’s totient function then

\[ \psi(d) := \frac{3}{\phi(d)} \left( 1 - \frac{1}{d} \right). \]

All of the implied constants in our work are allowed to depend at most upon \( d \). In the case \( d = 2 \) of conics, Theorem 1 yields

\[ N_2(H) \ll \frac{H^3}{(\log H)^2}. \]

This retrieves an earlier result of Serre [7]. In fact Guo [3] has established an asymptotic formula for the corresponding quantity in which the coefficients \( a_i \) are restricted to be odd, with \( a_1 a_2 a_3 \) square-free. In the case of additive cubics our result implies that

\[ N_3(H) \ll \frac{H^3}{\log H}. \]
This provides a partial answer to a question raised by Poonen and Voloch [6]: does a random cubic curve in $\mathbb{P}^2$ that is defined over $\mathbb{Q}$ possess a $\mathbb{Q}$-rational point? The proof of Theorem 1 will be established in §2 using the large sieve inequality, as pioneered by Serre [7] in the case $d = 2$.

It is natural to ask what happens for additive equations in more than three variables. For given $d \in \mathbb{N}$ and $H \geq 1$, let $M_d(H)$ denote the number of $a \in \mathbb{Z}^4$ with $|a_i| \leq H$, for which the equation

$$a_1 x_1^d + a_2 x_2^d + a_3 x_3^d + a_4 x_4^d = 0$$

is everywhere locally soluble. The following inequality will be proved in §3.

**Theorem 2.** We have

$$M_d(H) \gg H^4.$$

Theorem 2 provides an additive analogue of a result due to Poonen and Voloch [6, Theorem 3.6]. The latter establishes that a positive proportion of all hypersurfaces in $\mathbb{P}^{n-1}$ of degree $d$ that are defined over $\mathbb{Q}$ are everywhere locally soluble, provided that $n - 1, d \geq 2$ and $(n,d) \neq (3,2)$.

Returning to the setting of ternary forms, let us now consider the problem of describing $C_d(\mathbb{Q})$ when it is non-empty. When $d$ is sufficiently large, it has been conjectured by Granville [2] on the basis of a generalised version of the $abc$-conjecture, that the curve (1) never has any non-trivial rational points. While Faltings’ proof of the Mordell conjecture ensures that there are only finitely many rational points for each $d \geq 4$, it is notoriously difficult to achieve an effective bound for the total number of solutions in terms of the coefficients $a_1, a_2, a_3$. The following result deals with the much simpler scenario in which one restricts attention to rational points of bounded height on the curve.

**Theorem 3.** Let $d \geq 2$ and let $a \in \mathbb{Z}^3$ have pairwise coprime non-zero components. Then we have

$$\# \{ x \in C_d(\mathbb{Q}) : H(x) \leq B \} \ll \left( 1 + \frac{B^{\frac{d}{2}}}{|a_1 a_2 a_3|^{d/3}} \right) d^{\omega(a_1 a_2 a_3)},$$

where $H : \mathbb{P}^2(\mathbb{Q}) \to \mathbb{R}_{>0}$ is the usual exponential height function.

Theorem 3 will be established in §4. It generalises a result due to Heath-Brown [1, Theorem 2] which deals with the case $d = 2$. Theorem 3 is susceptible to improvement in a number of obvious directions. Firstly it would be easy to extend this result to counting rational points whose coordinates are constrained to lie in lopsided boxes, rather than in a cube. Secondly, at the expense of weakening the dependence of the estimate on $a_1, a_2, a_3$, the exponent of $B$ can be improved substantially. We will not pursue either of these lines of enquiry here, however. Theorem 3 can be used to provide some simple-minded evidence for the expected paucity of rational points on (1). Thus when $a_1, a_2, a_3$ are arbitrary integers, it follows from this result that there are only $O(d^{\omega(a_1 a_2 a_3)})$ points in $C_d(\mathbb{Q})$ with height at most $|a_1 a_2 a_3|^{2/(3d)}$. Thus if there are many rational points then their height must be large compared to the height of the defining form.

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2. The large sieve

For any \( H \geq 1 \) we let \( N^*_d(H) \) be defined as for \( N_d(H) \) but with the extra hypotheses that \( a_1a_2a_3 \neq 0 \) and \( \gcd(a_1, a_2, a_3) = 1 \). We will say that an integer \( a \in \mathbb{N} \) is \( d \)-free if \( \nu < d \) whenever \( p^\nu \mid a \). For \( H_i \geq 1 \) we let \( N^{**}_d(H) \) denote the number of \( d \)-free triples \( a \in \mathbb{Z}^3 \) with \( 0 < |a_i| \leq H_i \) and \( \gcd(a_1, a_2, a_3) = 1 \), such that \( C_d(\mathbb{Q}) \neq \emptyset \). We will use the large sieve inequality to show that

\[
N^{**}_d(H) \ll \frac{\prod_{i=1}^3(H_i + z^2)}{(\log z)^\psi(d)},
\]

for any \( z \geq 1 \), where \( \psi(d) \) is given in the statement of Theorem 1.

Let us begin by seeing how this suffices for the statement of Theorem 1. Now it is clear that

\[
N_d(H) = \sum_{k \leq H} N^*_d(k^{-1}H) + O(H^2),
\]

whence it certainly suffices to establish the theorem for \( N^*_d(H) \) in place of \( N_d(H) \). We now write \( a_i = u_i v_i^d \) in the definition of \( N^*_d(H) \), with each \( u_i \) being \( d \)-free and \( v_i > 0 \). It follows that

\[
N^*_d(H) = \sum_{v_i \leq H^{1/d}} N^{**}_d(H),
\]

where \( H \) has components \( H_i := H/v_i^d \). Let us break the summation over \( v \in \mathbb{N}^3 \) into two set \( S_1(H) \cup S_2(H) \), where \( S_1(H) \) denotes the set of vectors for which one of the components is bigger than \((\log H)^{3/(d\phi(d))})\), and \( S_2(H) \) denotes the remainder. It is trivial to see that

\[
\sum_{v \in S_1(H)} N^{**}_d(H) \ll \sum_{v \in S_1(H)} |\{u \in \mathbb{Z}^3 : |u_i| \leq H_i\}| \ll H^3 \sum_{v \in S_1(H)} \frac{1}{v_1^{d_1}v_2^{d_2}v_3^{d_3}}\]

\[
\ll \frac{H^3}{(\log H)^\psi(d)},
\]

which is satisfactory. Turning to the contribution from the set \( S_2(H) \), we deduce from (2) that

\[
\sum_{v \in S_2(H)} N^{**}_d(H) \ll \sum_{v \in S_2(H)} \frac{\prod_{i=1}^3(H_i + z^2)}{(\log z)^\psi(d)},
\]

for any \( z \geq 1 \). Since \( v_i \leq (\log H)^{3/(d\phi(d))} \), so it follows that

\[
H_i \geq \frac{H}{v_1^{d_1}v_2^{d_2}v_3^{d_3}} \gg 1,
\]

for \( H \gg 1 \). Taking \( z = H^{1/2}(v_1v_2v_3)^{-d/2} \) therefore yields

\[
\sum_{v \in S_2(H)} N^{**}_d(H) \ll \frac{H^3}{(\log H)^\psi(d)} \sum_{v \in S_2(H)} \frac{1}{v_1^{d_1}v_2^{d_2}v_3^{d_3}} \ll \frac{H^3}{(\log H)^\psi(d)}.
\]

This completes the deduction of Theorem 1 subject to (2).
We now proceed with the proof of (2). Let $p > 2$ and let $R_d(p)$ denote the number of $a \in \mathbb{F}_p^*$ for which there is a solution of the congruence

$$x^d \equiv a \pmod{p}.$$ 

It is an easy exercise in elementary number theory (see [3, §4.2], for example) to show that

$$R_d(p) = \frac{p - 1}{\gcd(d, p - 1}).$$  \hspace{1cm} (3)

We will be interested in the set of $a \in \mathbb{F}_p^3$ which arise as images of the points counted by $N_d^*(H)$. We denote the cardinality of such $a \in \mathbb{F}_p^3$ by $p^3 - \tau(p)$, where $\tau(p)$ denotes the number of vectors in $\mathbb{F}_p^3$ that are excluded.

We seek a good lower bound for $\tau(p)$, still under the assumption that $p > 2$. Let $a \in \mathbb{F}_p^3$ be such that $p \mid a_i$ and $p \nmid a_j a_k$. Then for fixed $a_j \in \mathbb{F}_p^*$ there are exactly $R_d(p)$ values of $a_k$ for which the congruence

$$a_j x_j^d + a_k x_k^d \equiv 0 \pmod{p}$$

has solutions with $p \nmid x_j x_k$. For the remaining vectors with $p \mid a_i$ and $p \nmid a_j a_k$, the only solution to the above congruence has $p \mid \gcd(x_j, x_k)$. But then the condition $C_d(Q) \neq \emptyset$ in $N_d^*(H)$ implies that

$$a_i x_i^d \equiv 0 \pmod{p^d},$$

for some $x_i \in \mathbb{Z}$ which is coprime to $p$. This is impossible since $a_i$ is $d$-free. Employing (3) this therefore establishes that

$$\tau(p) \geq \sum_i (p - 1)^2 \left(1 - \frac{1}{\gcd(d, p - 1)}\right) = 3(p - 1)^2 \left(1 - \frac{1}{\gcd(d, p - 1)}\right) \geq 3(p - 1)^2 \left(1 - \frac{1}{k}\right) = 3p^2 \left(1 - \frac{1}{k}\right) + O(p),$$

for any $k \mid \gcd(d, p - 1)$.

We are now ready for our application of the large sieve inequality in dimension three. Let $z, H_i \geq 1$. It easily follows from the arguments of [1] that

$$N_d^*(H) \ll \frac{\prod_{i=1}^3 (H_i + z^2)}{G(z)},$$  \hspace{1cm} (4)

where

$$G(z) := \sum_{n \leq z} |\mu(n)| \prod_{p|n} \frac{\tau(p)}{p^3 - \tau(p)}.$$ 

For any $k \in \mathbb{N}$, let $\mathcal{P}_k$ denote the set of primes congruent to 1 modulo $k$, and let $g_k$ be the non-negative multiplicative arithmetic function

$$g_k(n) := \frac{|\mu(n)| (3(1 - \frac{1}{k}))^{\omega(n)}}{n}.$$
Then we have
\[
G(z) \geq \sum_{k \mid d} \sum_{\substack{n \leq z \atop k \notdivides p \Rightarrow p \in \mathcal{P}_k}} \mu(n) \prod_{p \mid n} \left( \frac{3}{p} \left(1 - \frac{1}{k}\right) + O\left(\frac{1}{p^2}\right) \right) \geq \sum_{k \mid d} \sum_{\substack{n \leq z \atop k \notdivides p \Rightarrow p \in \mathcal{P}_k}} g_k(n) \geq \sum_{\substack{n \leq z \atop p \mid n \Rightarrow p \not\in \mathcal{P}_d}} g_d(n).
\]

It will be convenient to set \( \gamma := 3(1 - 1/d) \). Now it is easy to see that
\[
\sum_{n \leq z} g_d(n) \gg (\log z)^\gamma,
\]
and furthermore,
\[
\sum_{n \leq z \atop p \mid n \Rightarrow p \not\in \mathcal{P}_d} g_d(n) \leq \exp\left( \sum_{p \leq z} \frac{\gamma}{p} \right) \ll (\log z)^{\gamma(1 - 1/\phi(d))},
\]
by Dirichlet’s theorem on primes in arithmetic progression. Hence it follows that
\[
G(z) \gg \sum_{n \leq z \atop p \mid n \Rightarrow p \not\in \mathcal{P}_d} g_d(n) \gg (\log z)^\gamma \left( \sum_{n \leq z \atop p \mid n \Rightarrow p \not\in \mathcal{P}_d} g_d(n) \right)^{-1} \gg (\log z)^{\gamma/\phi(d)},
\]
in \((4)\). On noting that \( \gamma/\phi(d) = \psi(d) \), this therefore completes the proof of \((2)\).

3. A LIFTING ARGUMENT

In this section we prove Theorem \((2)\), which will be achieved via a simple lifting argument. As is well known, there exists a constant \( p_0 = p_0(d) > d \) such that for primes \( p \geq p_0 \) we will have \( p \notdivides d \), and furthermore, every congruence of the form
\[
b_1 x_1^d + \cdots + b_3 x_3^d \equiv 0 \pmod{p}
\]
will have a non-trivial solution when \( p \notdivides b_1 b_2 b_3 \). Any such solution can be lifted to a non-trivial solution in \( \mathbb{Q}_p^3 \). For each prime \( p < p_0 \) there exists a power of \( p \), which we denote by \( q_p \in \mathbb{N} \), and a residue class \( a_p \) modulo \( q_p \), such that the congruence
\[
a_1 x_1^d + \cdots + a_4 x_4^d \equiv 0 \pmod{q_p}
\]
has a solution which can be lifted to a non-trivial solution in \( \mathbb{Q}_p^4 \), if
\[
a \equiv a_p \pmod{q_p}.
\]
Let \( Q := \prod_{p < p_0} q_p \). By the Chinese remainder theorem there exists a residue class \( c \) mod \( Q \) such that \( a \equiv c \pmod{Q} \) implies that \( a \equiv a_p \pmod{q_p} \) for \( p < p_0 \). It follows that \( M_4(H) \gg M_d^4(H) \), where \( M_d^4(H) \) denotes the number of \( a \in \mathbb{Z}^4 \) such that \( |a_i| \leq H \), with \( a \equiv c \pmod{Q} \), and furthermore, any \( p \geq p_0 \) divides at most one of the coefficients \( a_i \). Assuming that \( p_0 \) is taken to be sufficiently large we deduce that \( M_d^4(H) \gg H^4 \), which therefore concludes the proof of Theorem \((2)\).
4. Geometry of numbers

In establishing Theorem 3 we will need to establish the existence of a small number of lattices, each of reasonably large determinant, that can be used to cover the integer solutions to the equation in (1). This is provided by the following result.

Lemma. Let $d \geq 2$ and let $a \in \mathbb{Z}^3$ such that $a := |a_1a_2a_3| \neq 0$ and $\gcd(a_i, a_j) = 1$. Let $x \in \mathbb{Z}^3$ be a solution of (1). Then there exist lattices $\Lambda_1, \ldots, \Lambda_J \subseteq \mathbb{Z}^3$ such that

(i) $J \leq 2d^{\omega(a)}$,
(ii) $\dim \Lambda_j = 3$ and $\det \Lambda_j \gg a^{2/d}$, for each $j \leq J$,
(iii) $x \in \bigcup_{j \leq J} \Lambda_j$.

Proof. Let $p \mid a$, and write $\alpha_i = v_p(a_i)$. Since the integers $a_1, a_2, a_3$ are pairwise coprime, we may assume without loss of generality that $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \geq 1$. For any prime $p$ let $\delta = v_p(d)$ be the $p$-adic order of $d$, and write

$$\gamma := \begin{cases} \delta + 1, & \text{if } p > 2 \text{ or } \delta = 0, \\ \delta + 2, & \text{if } p = 2 \text{ and } \delta \geq 1. \end{cases}$$

Let $x \in \mathbb{Z}^3$ be such that (1) holds. We claim that there exist sublattices $M_1, \ldots, M_K \subseteq \mathbb{Z}^3$ with $K \leq 2^{\gamma-\delta-1}d$, such that

$$\dim M_k = 3, \quad \det M_k \geq p^{2\alpha_3-\gamma+1},$$

for each $k \leq K$, and $x \in \bigcup_{k \leq K} M_k$. The Chinese remainder theorem will then produce at most $2d^{\omega(a)}$ integer sublattices overall, each of dimension 3 and determinant

$$\geq \prod_{p|\alpha_1\alpha_2\alpha_3} p^{2\nu_p(a)/d-\gamma+1} = a_{\alpha_3}^2 \prod_{p|\alpha_1\alpha_2\alpha_3} p^{-\gamma+1} \gg a_{\alpha_3}^2.$$

This completes the proof of the lemma subject to the construction of the lattices $M_1, \ldots, M_K$.

Turning to the claim, let $x \in \mathbb{Z}^3$ be such that (1) holds. Let us write $x_i = p^{\xi_i}x_i'$, for $i = 1, 2$, with $p \nmid x_i'x_2'$ and $\xi_1 \leq \xi_2$, say. Then we deduce that

$$a_1p^{d\xi_1}x_1'^d + a_2p^{d\xi_2}x_2'^d \equiv 0 \pmod{p^{\alpha_3}}.$$

There are now 3 possibilities to consider: either $\alpha_3 \leq d\xi_1$, or $d\xi_1 < \alpha_3 \leq d\xi_2$, or $d\xi_2 < \alpha_3$. The second case is plainly impossible. In the first case we may conclude that $x$ belongs to the set of $x \in \mathbb{Z}^3$ such that $p^{\lceil \alpha_3/d \rceil}$ divides $x_1$ and $x_2$. This defines an integer lattice of dimension 3 and determinant $\geq p^{2\alpha_3/d}$. Thus we may take $K = 1$ in this case.

Finally, in the third case, we must have $\xi_1 = \xi_2 = \xi$, say. But then it follows that

$$a_1x_1'^d + a_2x_2'^d \equiv 0 \pmod{p^{\alpha_3-d\xi}}.$$

Suppose first that $\alpha_3 - d\xi < \gamma$. Then we have

$$\frac{2\alpha_3}{\gamma} \leq p^{\xi} \leq p^{2(\gamma-1)/\gamma} \leq p^{2\xi+\gamma-1}.$$
Since \( x \) lies on the lattice of determinant \( p^{2\xi} \) that is determined by the conditions \( p^k \mid x_1 \) and \( p^k \mid x_2 \), we may clearly take \( K = 1 \) in this case also. Suppose now that \( \alpha_3 - d\xi \geq \gamma \). Then we have

\[
x^d + a_2 \overline{a}_1 \equiv 0 \pmod{p^{\alpha_3 - d\xi}},
\]

with \( x = x'_1 x'_2 \), and where \( \overline{b} \) denotes the multiplicative inverse of \( b \) modulo \( p^{\alpha_3 - d\xi} \). We now appeal to the well-known fact that for any \( b \in \mathbb{Z} \) coprime to \( p \), and any \( k \geq \gamma \), the number of solutions to the congruence

\[
x^d \equiv b \pmod{p^k}
\]
is either 0 or \( p^{\gamma - \delta - 1} \gcd(d, p^\delta(p - 1)) \). This therefore ensures the existence of \( K \leq 2^{\gamma - \delta - 1}d \) integers \( \lambda_1, \ldots, \lambda_K \) such that

\[
a_1 \lambda_k^d + a_2 \equiv 0 \pmod{p^{\alpha_3 - d\xi}},
\]

for \( 1 \leq k \leq K \). In particular, the point \( x \in \mathbb{Z}^3 \) in which we are interested must satisfy \( x_1 = p^d x'_1 \), \( x_2 = p^d x'_2 \) and

\[
x'_1 \equiv \lambda_k x'_2 \pmod{p^{\alpha_3 - d\xi}},
\]

for some \( 1 \leq k \leq K \). Assuming that \( d \geq 2 \), these conditions define a union of \( K \) lattices, each of dimension 3 and determinant \( p^{\alpha_3 + 2\xi - d\xi} \geq p^{\alpha_3/d} \). This completes the proof of the claim.

We are now ready to establish Theorem 3. Let \( a = \lvert a_1 a_2 a_3 \rvert \). In view of the lemma, the points that we are interested in belong to a union of \( J \leq 2d^{\omega(a)} \) lattices \( \Lambda_1, \ldots, \Lambda_J \subseteq \mathbb{Z}^3 \), each of dimension 3 and determinant \( \gg a^{2/d} \). Let us consider the overall contribution from the vectors belonging to one such lattice \( \Lambda_j \), say. We will work with a minimal basis \( b^{(1)}, b^{(2)}, b^{(3)} \) for \( \Lambda_j \), which satisfies the well-known bound

\[
|b^{(1)}||b^{(2)}||b^{(3)}| \gg a^{2/d},
\]

and furthermore, whenever \( x = \sum \lambda_i b^{(i)} \) for \( \lambda_i \in \mathbb{Z} \), so it follows that

\[
\lambda_i \ll \frac{|x|}{|b^{(i)}|} \ll \frac{B}{|b^{(i)}|} = B_i,
\]
say. Here \( \lvert z \rvert := \max_i |z_i| \) for any \( z \in \mathbb{R}^3 \). On carrying out this change of variables, \( (1) \) becomes \( G_j(\lambda_1, \lambda_2, \lambda_3) = 0 \), with \( G_j \) a ternary form of degree \( d \) that is defined over \( \mathbb{Z} \). We are now interested in counting integer solutions to this equation with \( \lambda_i \ll B_i \). It follows from a simple application of Siegel’s lemma that the number of such vectors is

\[
\ll 1 + \left( \frac{B^3}{|b^{(1)}||b^{(2)}||b^{(3)}|} \right)^{\frac{1}{2}} \ll 1 + a^{-\frac{3}{2}} B^0.
\]

On summing over the \( J \) lattices, this therefore establishes Theorem 3.

References

[1] P.X. Gallagher, The large sieve and probabilistic Galois theory. Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 91–101, Amer. Math. Soc., Providence, R.I., 1973.
[2] A. Granville, On the number of solutions to the generalized Fermat equation. Number theory (Halifax, NS, 1994), 197–207, CMS Conf. Proc. 15, Amer. Math. Soc., Providence, RI, 1995.
[3] C.R. Guo, On solvability of ternary quadratic forms. Proc. London Math. Soc. 70 (1995), 241–263.
[4] D.R. Heath-Brown, The density of rational points on cubic surfaces. *Acta Arith.* **79** (1997), 17–30.

[5] K. Ireland and M. Rosen, *A classical introduction to modern number theory*. 2nd ed., Springer-Verlag, 1990.

[6] B. Poonen and J.F. Voloch, Random Diophantine equations. *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, 175–184, Progr. Math. **226**, Birkhäuser, 2004.

[7] J.-P. Serre, Spécialisation des éléments de $\text{Br}_2(\mathbb{Q}(T_1, \ldots, T_n))$. *C. R. Acad. Sci. Paris* **311** (1990), 397–402.

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