BLOW UP OF FRACTIONAL SCHRÖDINGER EQUATIONS ON MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

HUALI ZHANG AND SHILIANG ZHAO

Abstract. In this paper, the well-posedness of Cauchy’s problem of fractional Schrödinger equations with a power type nonlinearity on manifolds with nonnegative Ricci curvature is studied. Under suitable volume conditions, the smooth solution will blow up in finite time no matter how small the initial data is, which follows from a new weight function and ODE inequalities. Moreover, the upper-bound of the lifespan can be estimated.

1. Introduction and Main results

Let $0 < \alpha < 2$ and in this short article we study the Cauchy’s problem of nonlinear fractional Schrödinger equations

$$
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t u - (-\Delta)^{\frac{\alpha}{2}} u = F(u, \bar{u}), \quad (t, x) \in \mathbb{R}^+ \times M \\
u|_{t=0} = \varphi_0(x) + i\varphi_1(x),
\end{array} \right.
\end{aligned}
$$

where $\varphi_0(x), \varphi_1(x) \in C^\infty(M)$ are real valued functions and $M$ is a complete manifold with nonnegative Ricci curvature. $F$ is a nonlinear function of $u, \bar{u}$, satisfying

$$
|F(u, \bar{u})| \lesssim |u|^p,
$$

where $\bar{u}$ is the conjugate of $u$ and $p > 1$. Note that when $\alpha = 2$, (1.1) is the nonlinear Schrödinger equation. When $1 < \alpha < 2$, (1.1) was introduced by Laskin in [32]. Namely, the quantum mechanics path integral over Brownian trajectories leads to the well known Schrödinger equation ($\alpha = 2$), and the path integral over Lévy trajectories leads to the fractional Schrödinger equation ($1 < \alpha < 2$).

First we recall some known results about the nonlinear Schrödinger equations on $\mathbb{R}^n$. The global existence results for small data can be established when $p$ is large enough. If $p > \frac{\sqrt{n^2 + 2n + 4} + n + 2}{2n}$, it was shown by Strauss [42] that the global solutions exist for small initial data. For $F = |u|^{p-1}u$, the wave operators can be constructed in general for small data when $p > 1 + \frac{2}{n}$ [15], [37]. And hence the global existence is guaranteed by the conservation of mass and energy and the wave operators. When $p$ is small, the structure of the nonlinearity starts to play a crucial role. For example, consider $F = |u|^{p-1}u$ and $1 < p < 1 + \frac{2}{n}$. For $1 \leq n \leq 3$, asymptotically free solutions can not exist [21], [38]. For $F = |u|^p$, Ikeda-Inui [28] proved a small data blow-up result of $L^2$ when $1 < p < 1 + \frac{2}{n}$. For other nonlinearities such as $u^2, \bar{u}^2$, few examples of global existence for small data below the Strauss exponent are known. We refer the readers to [12], [17], [18], [21], [23].

\textit{Date}: November 5, 2019.

\textit{2010 Mathematics Subject Classification.} Primary 35A01.

\textit{Key words and phrases.} Fractional Schrödinger equations, blow-up, weight function, heat kernel.
and references therein. Furthermore, there are some Strichartz estimates for nonlinear Schrödinger equations on manifolds, for instances \cite{1 4 6 3 2 5 11 9 10 14 40 27 28}.

As for nonlinear fractional Schrödinger equations, the Strichartz estimates was established by Cho-Koh-Seo \cite{13} in the radial case. For $F = |u|^{p-1}u$, Guo-Sire-Wang-Zhao \cite{23} showed the global well-posedness of radial solutions in the energy critical case. Boulenger-Himmelsbach-Lenzmann \cite{6} derived a blow-up result with radial data for (1.1) in both $L^2$-supercritical and $L^2$-critical cases respectively, and Guo-Zhu \cite{22} further completed the blow-up result with radial data for general dimensions and nonlinearities. For cubic nonlinearity $|u|^2u$, Guo-Han-Xin \cite{20} showed that the period boundary value problem of (1.1) is globally well-posed. Ionescu-Pusateri \cite{29} proved that the global small, smooth solutions of (1.1) exists if $\alpha = \frac{1}{2}$.

Guo-Huo \cite{21} established the global well-posedness of (1.1) if $1 < \alpha < 2$, where the key tri-linear estimates in Bourgain space played a important role.

In this paper, we get the blow-up results for a class of Schrödinger equations on Riemannian manifolds with nonnegative Ricci curvature. To be precise, let $(M, g)$ be a complete manifold of dimension $n$ with nonnegative Ricci curvature. Denote by $d$ the geodesic distance and $\mu$ the Riemannian measure. By the Bishop-Gromov inequality, $M$ satisfies the doubling condition: there exists constant $C > 0$ such that $V(x, 2r) \leq CV(x, r)$, $\forall r > 0, x \in M$, where $V(x, r)$ is the volume of the geodesic ball centered at $x$ with radius $r$. In this paper, we will use the Einstein’s summation convention. Then in local coordinates, the Laplace-Beltrami operator can be expressed as

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_j} \left( \sqrt{\det g} g^{jk} \frac{\partial}{\partial x_k} \right)$$

where $(g^{jk})_{1 \leq j, k \leq n}$ is the inverse matrix of $(g_{jk})_{1 \leq j, k \leq n}$. For any $p \in M$, consider the normal coordinates in which the Riemannian metric can be written as

$$g = dr^2 + r^2 g_{\alpha \beta} (r, \theta) d\theta^\alpha d\theta^\beta.$$

Thus for any distance function $f(r) = f(\delta(x, p))$, we have by \cite{13}

$$\Delta f(r) = f'''(r) + \frac{n-1}{r} f'(r) + \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial r} f'(r)$$

where $G = \det(g_{\alpha \beta})$. Denote by $p_t(x, y)$ the Schwartz kernel of the heat semigroup $e^{-t\Delta}$. According to \cite{53}, the heat kernel $p_t(x, y)$ satisfies the Gaussian upper bound:

$$p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left( -\frac{c d^2(x, y)}{t} \right), \quad \forall t > 0, x, y \in M.$$

The main results of this paper is as follows:

**Theorem 1.1.** Consider the Cauchy’s problem for nonlinear Schrödinger equations on a complete manifold $(M, g)$ with nonnegative Ricci curvature

$$i \partial_t u - (\Delta)^{\frac{\alpha}{2}} u = F(u, \overline{u}),$$

$$u(0, x) = \varphi_0(x) + i \varphi_1(x),$$

where $u(x)$ is a complex-valued function on $\mathbb{R}^n \times M$ and $\varphi_0, \varphi_1$ are smooth functions on $M$. If $\alpha > \frac{1}{2}$, then the solution $u$ exists globally in time. If $\alpha < \frac{1}{2}$, then the solution $u$ blow-up in finite time.
where $0 < \alpha < 2$, $\varphi_0(x), \varphi_1(x) \in C_\infty(M)$. Assume that the following two conditions holds

\begin{align}
(1.7) & \quad (1) \left| \frac{r}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial r} \right| \leq C \quad \forall r > 0. \\
(1.8) & \quad (2) V(x, r) \sim r^n \quad \forall x \in M, \; r > 0.
\end{align}

If $\text{Im} F = |u|^p, 1 < p < 1 + \frac{\alpha}{n}$ and $\int_M \varphi_0(x) d\mu > 0$, then the smooth solution of (1.6) will blow up in a finite time no matter how small the initial data is.

**Remark 1.2.** For all $p > 1$, it’s easy to see that the local solution of (1.6) exists in $H^s(\mathbb{R}^n)$ ($s > \frac{n}{2}$) by bootstrap argument and contraction mapping principle.

The paper is organized as follows: In Section 2, we prove some lemmas which play an important role in our proof; Our main results will be shown in Section 3; In Section 4, we give an example which satisfies our assumptions on the Riemannian metric. Now we introduce some notations. If $f, g$ are two functions, we say $f \lesssim g$ if and only if there exists a constant $c > 0$ such that $f \leq cg$. We say $f \sim g$ if and only if there exists a constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$. Set $f \land g = \min\{f, g\}$.

The constant $C, c$ may change from line to line.

## 2. Preliminary Results

In this section, we will prove several lemmas which will be frequently used.

**Lemma 2.1.** Let $f(x) \geq 0$ and we have

\[ 1 \land f(x) \sim \frac{1}{1 + f^{-1}(x)}. \]

**Proof.** It is direct to check that

\[ \frac{1 \land f(x)}{2} \leq \frac{1}{1 + f^{-1}(x)} \leq 1 \land f(x). \]

\[ \square \]

**Lemma 2.2.** For $\alpha, \gamma > 0$, the following hold for every $t, R > 0, x \in M$:

\begin{align}
(1) & \quad \int_M [t^{\frac{\alpha}{2}} + d_2(x, y)]^{-\gamma} d\mu(y) \lesssim t^{\frac{n-2\gamma}{2}}, \quad \forall \gamma > n \\
(2) & \quad \int_{B(x, R)} d(x, y)^{-\gamma} d\mu(y) \lesssim R^{n-\gamma}, \quad \forall 0 \leq \gamma < n \\
(3) & \quad \int_{M \setminus B(x, R)} d(x, y)^{-\gamma} d\mu(y) \lesssim R^{n-\gamma}, \quad \forall \gamma > n
\end{align}

**Proof.** (1) When $\gamma > n$, we have

\[
\int_M [t^{\frac{\alpha}{2}} + d_2(x, y)]^{-\gamma} d\mu(y) = \sum_{j \in \mathbb{Z}} \int_{B(x, 2^j \frac{t^{\alpha}}{2}) \setminus B(x, 2^{j-1} \frac{t^{\alpha}}{2})} [t^{\frac{\alpha}{2}} + d_2(x, y)]^{-\gamma} d\mu(y) \\
\leq t^{\frac{\alpha}{2}} \sum_{j \in \mathbb{Z}} [1 + 2^{j-1}]^{-\gamma} 2^{(j-1)n} t^{\frac{\alpha}{2}} \lesssim t^{\frac{n-2\gamma}{2}}.
\]
Proof. To start with, we have by (1.5) Lemma 2.3. Now we give the proof on manifolds. Note that in local coordinate, we have by (1.4)

\[ \int_{M} d(x,y)^{-\gamma} d\mu(y) = \sum_{j \geq 0} \int_{B(x,2^{-j}R) \setminus B(x,2^{-j-1}R)} d(x,y)^{-\gamma} d\mu(y) \]

\[ \lesssim \sum_{j \geq 0} 2^{(j+1)\gamma} R^{-\gamma} 2^{-j-1} R^n \lesssim R^{n-\gamma}. \]

(3) Similarly we have for \( \gamma > n \)

\[ \int_{M \setminus B(x,R)} d(x,y)^{-\gamma} d\mu(y) = \sum_{j \geq 0} \int_{B(x,2^{j+1}R) \setminus B(x,2^j R)} d(x,y)^{-\gamma} d\mu(y) \]

\[ \lesssim \sum_{j \geq 0} 2^{-j\gamma} R^{-\gamma} 2^{(j+1)n} R^n \lesssim R^{n-\gamma}. \]

The following estimates about the weight functions play an important role in our proof. Note that when \( M = \mathbb{R}^n \), the following estimates are essentially known in [19], [35]. Now we give the proof on manifolds.

**Lemma 2.3.** Let \( h(t,x) = \frac{t^{1+\hat{\beta}}}{(d^2(x,y)+t^{2})^{n-\alpha}} \) and we have for \( 0 < \alpha < 2, \beta > 0 \)

\[ |(-\Delta)^{\frac{\hat{\beta}}{2}} h(t,x)| \lesssim t^{-\hat{\beta}} h(t,x), \quad \forall x \in M, t > 0. \]

**Proof.** To start with, we have by (1.5)

\[ |(-\Delta)^{\frac{\hat{\beta}}{2}} h(t,x)| = |(-\Delta)^{\frac{\hat{\beta}}{2} - 1} \Delta h(t,x)| \]

\[ \lesssim \int_{0}^{\infty} e^{-s\Delta} |\Delta h(t,x)| \frac{ds}{s^{n/2}} \]

\[ \lesssim \int_{0}^{\infty} \int_{M} \frac{1}{V(x,\sqrt{s})} \exp \left( -\frac{d^2(x,y)}{t} \right) |\Delta h(t,y)| d\mu(y) ds \]

\[ \lesssim \int_{M} |\Delta h(t,y)| d(x,y)^{2-n-\alpha} d\mu(y). \]

Note that in local coordinate, we have by (1.4)

\[ \Delta h(t,y) = \frac{\partial^2 h(t,r)}{\partial r^2} + \frac{n-1}{r} \frac{\partial h(t,r)}{\partial r} + \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial r} \frac{\partial h(t,r)}{\partial r} \]

where \( r = d(y,p) \). According to the definition of \( h \), we obtain

\[ \frac{\partial h(t,r)}{\partial r} = \frac{(n+\beta)t^{1+\frac{\hat{\beta}}{2}} r}{[t^{\frac{\hat{\beta}}{2}} + r^2] \frac{n-\alpha}{2} + 1} \]

\[ \frac{\partial^2 h(t,r)}{\partial r^2} = -\frac{(n+\beta)t^{1+\frac{\hat{\beta}}{2}}}{[t^{\frac{\hat{\beta}}{2}} + r^2] \frac{n-\alpha}{2} + 1} + \frac{(n+\beta)(n+\beta+2)t^{1+\frac{\hat{\beta}}{2}} r}{[t^{\frac{\hat{\beta}}{2}} + r^2] \frac{n-\alpha}{2} + 2}. \]

In turn, we have by (1.7)

\[ |\Delta h(t,y)| \lesssim \frac{t^{1+\frac{\hat{\beta}}{2}}}{[t^{\frac{\hat{\beta}}{2}} + r^2] \frac{n-\alpha}{2} + 1}. \]
By Lemma 2.1 we have

\[ \frac{t^{1+\frac{\alpha}{2}}}{[d(x, p)^2 + t^{\frac{\alpha}{2}}]^{\frac{n+\beta}{2}}} = \frac{1}{[1 + (\frac{d(x, p)}{t^{\frac{\alpha}{2}}})^2]^{\frac{n+\beta}{2}}} \sim \left[ 1 \wedge \left( \frac{t^{\frac{\alpha}{2}}}{d(x, p)} \right)^2 \right]^{\frac{n+\beta}{2}}. \]

Thus it is sufficient to prove (2.1)

\[ \int_M \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y) \lesssim t^{1-\frac{n+\alpha}{2}} \left[ 1 \wedge \left( \frac{t^{\frac{\alpha}{2}}}{d(x, p)} \right)^2 \right]^{\frac{n+\beta}{2}}. \]

Next we will show (2.1) and denote

\[ I = \int_M \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y). \]

**Case I:** \( d(x, p) \leq t^{\frac{\alpha}{2}} \).

Set

\[ I_1 = \int_{d(x, y) \leq t^{\frac{\alpha}{2}}} \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y). \]

\[ I_2 = \int_{d(x, y) > t^{\frac{\alpha}{2}}} \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y). \]

For \( I_1 \), we have by Lemma 2.2

\[ I_1 \lesssim t^{\frac{2-\alpha}{\beta}} \int_{d(x, y) \leq t^{\frac{\alpha}{2}}} d(x, y)^{2-\alpha} d\mu(y) \lesssim t^{1-\frac{n+\alpha}{2}}. \]

Note that we have used the assumption \( 0 < \alpha < 2 \).

For \( I_2 \), we obtain

\[ I_2 \lesssim t^{\frac{2-\alpha}{\beta}} \int_{d(x, y) > t^{\frac{\alpha}{2}}} \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y) \lesssim t^{\frac{2-\alpha}{\beta}} \int_M [t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}} d\mu(y) \lesssim t^{1-\frac{n+\alpha}{2}}. \]

We have used Lemma 2.2 in the last step.

**Case II:** \( d(x, p) > t^{\frac{\alpha}{2}} \).

In this case, set

\[ I'_1 = \int_{d(x, y) \leq t^{\frac{\alpha}{2}}} \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y). \]

\[ I'_2 = \int_{d(x, y) > t^{\frac{\alpha}{2}}} \frac{[t^{\frac{\alpha}{2}} + d(y, p)^2]^{-\frac{n+\beta}{2}}}{d(x, y)^{2-\alpha}} d\mu(y). \]

For \( I'_1 \), by Lemma 2.2 we obtain

\[ I'_1 \lesssim [t^{\frac{\alpha}{2}} + d(x, p)^2]^{-\frac{n+\beta}{2}} \int_{d(x, y) \leq t^{\frac{\alpha}{2}}} d(x, y)^{2-\alpha} d\mu(y) \lesssim d(x, p)^{-\alpha-\beta} \lesssim \frac{t^{\frac{\alpha}{2}}}{d(x, p)^{\alpha+\beta}}. \]

We have used the fact \( d(y, p) \geq d(x, p) - d(x, y) \geq \frac{\delta}{2}d(x, p) \) in the first inequality.
we have

\[ I_2' \lesssim d(x, p)^{-n-\alpha} \int_{m_{d(x,y)}} [t^{\frac{n}{\alpha}} + d(y, p)^2]^{-\frac{n+\beta}{\alpha}} d(x, y)^2 dm(y) \]

\[ \lesssim d(x, p)^{-n-\alpha} (J_1 + J_2), \]

where

\[ J_1 = \int_{d(y,p) \leq 2d(x,p) \cap d(x,y) > d(x,p)} [t^{\frac{n}{\alpha}} + d(y, p)^2]^{-\frac{n+\beta}{\alpha}} d(x, y)^2 dm(y), \]

\[ J_2 = \int_{d(y,p) > 2d(x,p) \cap d(x,y) > d(x,p)} [t^{\frac{n}{\alpha}} + d(y, p)^2]^{-\frac{n+\beta}{\alpha}} d(x, y)^2 dm(y). \]

Note that for \( J_1 \) the following holds

\[ J_1 \lesssim d(x, p)^2 \int_{B(p, 2d(x,p))} d(x, p)^{-n-\beta-2} dm(y) \]

\[ \lesssim d(x, p)^{-\beta}, \]

where we have used the facts \( d(x, y) \leq d(x, p) + d(y, p) \leq 3d(x, p) \) and \( d(y, p) \geq d(x, y) - d(x, p) \geq \frac{1}{2}d(x, p) \).

Moreover

\[ J_2 \lesssim \int_{M \setminus B(p, 2d(x,p))} d(y, p)^2 d(y, p)^{-n-\beta-2} dm(y) \]

\[ \lesssim d(x, p)^{-\beta}. \]

Thus we have

\[ I_2' \lesssim t^{-\frac{n}{\alpha}} \]

Combining these estimates, we have proved the desired results. \( \square \)

3. Proof of Theorem 1.1

Let

\[ u(t, x) = w(t, x) + iv(t, x), \]

and take the image part of the nonlinear fractional Schrödinger equation \((1.6)\), we have

\[
\begin{cases}
  w_t + (-\Delta)^{\frac{n}{2}} w = (w^2 + v^2)^{\frac{n}{2}} \\
  t = 0 : w = \varphi_0(x).
\end{cases}
\]

(3.1)

Now fix a point \( p \in M \). For \( 0 < \alpha < 2 \), let \( h(t, x) = \frac{t^{1+\frac{n}{2}}}{(d^2(x,p)+t^{\frac{n}{2}})^{\frac{n+\beta}{2}}} \) and consider the following function,

\[ \phi(t) = \int_M h(T, x)w(t, x)dm(x), \]

where \( T = t + N \) and \( N \) will be determined later. To start with, we have

\[
\phi'(t) = \int_M h(T, x)w_t dm(x) + \int_M \partial h(T, x)wdm(x)
\]

\[ = \int_M h(T, x)(-(\Delta)^{\frac{n}{2}} v + (w^2 + v^2)^{\frac{n}{2}})dm(x) + \int_M \partial h(T, x)wdm(x). \]
Note that \((-\Delta)^{\gamma}\) is self-adjoint. It follows
\[
\phi'(t) = \int_M \partial_t h(T, x)w - (-\Delta)^{\gamma} h(T, x)v \mu(x) + \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x).
\]
Now consider
\[
I = \int_M \partial_t h(T, x)w - (-\Delta)^{\gamma} h(T, x)v \mu(x).
\]
Note that
\[
|\partial_t h(T, x)| = |(1 + \frac{n}{\alpha}) \frac{T^{\frac{\gamma}{2}}}{(d^2(x, p) + T^{\frac{\gamma}{2}})^{\frac{\alpha}{2}}} - (1 + \frac{n}{\alpha}) \frac{T^{\frac{\alpha + \gamma}{2}}}{(d^2(x, p) + T^{\frac{\gamma}{2}})^{\frac{\alpha}{2} + 1}}| \lesssim \frac{1}{T} h(T, x).
\]
By Lemma 2.3, we have
\[
|I| \lesssim \int_M T^{-1} h(T, x)(|w| + |v|) d\mu(x)
\]
\[
\lesssim T^{-1} \left( \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x) \right)^{\frac{1}{p}} \left( \int_M h(T, x) d\mu(x) \right)^{\frac{1}{p}}
\]
\[
\lesssim T^{-\frac{\alpha + \gamma}{p'}} \left( \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x) \right)^{\frac{1}{p}},
\]
where we have used the Hölder inequality in the second step. As a result, we have
\[
|I| \leq \frac{1}{2} \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x) + CT^{\frac{\gamma}{p'} - v'}.
\]
In turn, we obtain
\[
\phi'(t) \geq \frac{1}{2} \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x) - CT^{\frac{\gamma}{p'} - v'}.
\]
On the other hand,
\[
|\phi(t)| \leq \left( \int_M h(T, x)|w|^p d\mu(x) \right)^{\frac{1}{p'}} \left( \int_M h(T, x) d\mu(x) \right)^{\frac{1}{p}}
\]
\[
\lesssim T^{\frac{\alpha}{p'}} \left( \int_M h(T, x)(w^2 + v^2)^{\frac{\gamma}{2}} d\mu(x) \right)^{\frac{1}{p}}.
\]
Then we obtain
\[
\phi'(t) \geq C \frac{|\phi(t)|^p}{T^{\frac{\gamma}{p}(p-1)}} - CT^{\frac{\gamma}{p'} - v'}.
\]
Therefore we have
\[
\phi(t) \geq \int_M h(N, x)\phi_0(x) d\mu(x) - CN^{\frac{\gamma}{p'} - v' + 1} + C \int_0^t \frac{|\phi(\tau)|^p}{(\tau + N)^{\frac{\gamma}{p}(p-1)}} d\tau.
\]
Note that \(\frac{\gamma}{p'} - v' + 1 < 0\) whenever \(p < 1 + \frac{\gamma}{n}\). By the dominated convergence theorem, the following holds,
\[
\lim_{N \to \infty} \int_M h(N, x)\phi_0(x) d\mu(x) = \int_M \phi_0(x) d\mu(x).
\]
Thus for \(N\) large enough, we conclude that
\[
\phi(t) \geq \frac{1}{2} \int_M \phi_0(x) d\mu(x) + C \int_0^t \frac{|\phi(\tau)|^p}{(\tau + N)^{\frac{\gamma}{p}(p-1)}} d\tau.
\]
Denote by $\varphi(t)$ the right side of the above inequality and it follows

$$\varphi'(t) \geq C \frac{\varphi^p(t)}{(t + N)^{\frac{n}{p}(p-1)}}.$$  

Finally we get

$$\varphi(t) \geq C \left[ \varphi^1 - p(0) + N^1 - n + \alpha(p-1)^{-1} - (t + N)^{1 - \frac{n}{p}(p-1)} - N \right].$$

Since $1 - \frac{n}{p}(p-1) > 0$ when $1 < p < 1 + \frac{2}{n}$, then $\varphi(t)$ tends to infinity if $t \to (N^1 - \frac{n}{p}(p-1) + C\varphi(0)^{1-p}1^{-\frac{n}{p}(p-1)} - N$. Hence we have proved the theorem.

4. Appendix

Note that when $M$ is a rotationally symmetric manifold with nonnegative Ricci curvature, (1.7) holds automatically. In fact, the Riemannian metric can be expressed as

$$g = dr^2 + \phi(r)^2 \tilde{g}_{\alpha\beta}(\theta)d\theta^\alpha d\theta^\beta,$$

where $\phi(0) = 0, \phi'(0) = 1, \phi(r) > 0$ for every $r > 0$ and $\tilde{g}$ is the standard metric on sphere. Thus we have $\sqrt{G} = (\frac{\phi(r)}{r})^{n-1}\sqrt{\det \tilde{g}}$. According to the nonnegative of Ricci curvature, we have $\frac{\phi''(r)}{\phi(r)} \leq 0$ (see §3.2.3 of [39]). Then the facts $\phi(r) > 0$ and $\phi'(0) = 1$ give that $\phi'(r)$ is a decreasing function with $0 \leq \phi'(r) \leq 1$ for $r > 0$. Thus we have

$$\phi'(r) \leq \frac{1}{r} \int_0^r \phi'(s)ds = \frac{\phi(r)}{r}.$$

It is direct to check that $\frac{r}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial r} = (n-1)\frac{r\phi'(r) - \phi(r)}{\phi(r)}$ and hence (1.7) holds.

Note also that by our proof the range of $p$ in Theorem 1.1 is determined by the volume growth of geodesic balls on manifolds and $\frac{n}{p} \to 0$ as $n \to \infty$. Thus it explains why the blow-up results of Schrödinger equations with polynomial non-linearities on hyperbolic spaces should not be expected. As is known, hyperbolic spaces have the Ricci curvature -1 and the volume of balls grow exponentially.

5. Acknowledgement

The first author would like to express the great gratitude to Prof. Yi Zhou for his helpful discussions and advice. The second author would also thank Prof. Changxing Miao and Prof. Jiqiang Zheng for valuable recommendations. The first author is supported by Education Department of Hunan Province, general Program(Grant No.17C0039); the State Scholarship Fund of China Scholarship Council and Hunan Provincial Key Laboratory of Intelligent Processing of Big Data on Transportation, Changsha University of Science and Technology, Changsha; 410114, China. The second author is supported by the National Natural Science Foundation of China under Grant No.11901407.

References

1. J. P. Anker, V. Pierfelice. Nonlinear Schrödinger equation on real hyperbolic spaces, Ann. Inst. H. Poincar Anal. NonLinéaire 26 (2009), 1853-1869.
2. V. Banica. The nonlinear Schrödinger equation on the hyperbolic space, Comm. Partial Differ. Equ. 32(10) (2007), 1643-1677.
3. V. Banica, R. Carles and G. Staffilani. Scattering theory for radial nonlinear Schrödinger equations on hyperbolic space, Geom. Funct. Anal. 18 (2008), 367-399.
4. V. Banica , T. Duyckaerts. Weighted Strichartz estimates for radial Schrödinger equations on noncompact manifolds, Dyn. Partial Differ. Equ. 4 (2007), 335-359.
5. M. D. Blair, H. F. Smith, C. D. Sogge. Strichartz estimates and the nonlinear Schrödinger equation on manifolds with boundary, Math. Ann. 354 (2012), 1397-1430.
6. T. Boulenger, D. Himmelsbach and E. Lenzmann. Blowup for fractional NLS, J. Funct. Anal., 271, (2016), 2569-2603.
7. J. Bourgain. Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3, 107-156 (1993)
8. J. Bourgain. Exponential sums and nonlinear Schrödinger equations, Geom. Funct. Anal. 3 (1993), 157-178.
9. N. Burq, P. Gérard and N. Tzvetkov. An instability property of the nonlinear Schrödinger equation on $S^d$, Math. Res. Lett. 9 (2002), 323-335.
10. N. Burq, P. Gérard and N. Tzvetkov. Strichartz inequalities and nonlinear Schrödinger equation on on compact manifolds, Amer. J. Math. 126 (2004), 569-605.
11. N. Burq, P. Gérard and N. Tzvetkov. Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces, Invent. Math. 159 (2005), 187-223.
12. T. Cazenave, F. Weisler. Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys. 147 (1) (1992), 75-100.
13. C.H. Cho, Y. Koh and I. Seo. On inhomogeneous Strichartz estimates for fractional Schrödinger equations and their applications. Discrete. Continuous Dynamical Systems. A 36(4) (2015), 1128-1157.
14. J. Fontaine. Une équation semi-linéaire des ondes sur $H^3$, C. R. Acad. Sci. Paris Sér. I Math. 319 (1994), 935-948.
15. J. Ginibre, T. Ozawa and G. Velo. On the existence of the wave operators for a class of nonlinear Schrödinger equations. Ann. Inst. H. Poincaré Phys. Thor. 60 (2) (1994), 211-239.
16. J. Ginibre, N. Hayashi. Almost global existence of small solutions to quadratic nonlinear Schrödinger equations in three space dimensions, Math. Z. 219 (1) (1995), 119-140.
17. P. Germain, N. Masmoudi and J. Shatah. Global solutions for 3D quadratic Schrödinger equations, Int. Math. Res. Not. 3 (2009), 414-432.
18. P. Germain, N. Masmoudi and J. Shatah. Global solutions for 2D quadratic Schrödinger equations, J. Math. Pures Appl. 97 (2012), 505-543.
19. L. Grafakos, S. Oh, The Kato-Ponce inequality. Commun. Part. Diff. Equ. 39(6)(2014), 1128-1157.
20. B. Guo, Y. Han and J. Xin. Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation, Appl. Math. Comput. 204 (2008), 468-477.
21. B. Guo, Z. Huo. Global Well-Posedness for the Fractional Nonlinear Schrödinger Equation, Comm. Partial Differ. Equ. 36 (2011), 247-255.
22. Q. Guo, S. H. Zhu. Sharp criteria of scattering for the fractional NLS, arXiv: 1706.02549
23. Z. Guo, Y. Sire, Y. Wang and L. Zhao. On the energy-critical fractional Schrödinger equation in the radial case, arXiv:1310.6816
24. N. Hayashi, P. Naumkin. Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations, Amer. J. Math 120 (2)(1998), 369-389.
25. N. Hayashi, P. Naumkin. On the quadratic nonlinear Schrödinger equation in three dimensions, Int. Math. Res. Not. 3 (2000), 115-132.
26. M. Ikeda, T. Inui. Small data blow-up of $L^2$ or $H^1$ solution for the semilinear Schrödinger equation without gauge invariance, J. Evol. Equ. 15 (2015), 571-581.
27. A. Ionescu, G. Staffilani. Semilinear Schrödinger flows on hyperbolic spaces: scattering in $H^1$, Math. Ann 345 (2009), 133-158.
28. A. Ionescu, B. Pausader and G.Staffilani. On the global well-posedness of energy-critical Schrödinger equations in curved spaces.
29. A. Ionescu, F. Pusateri. Nonlinear fractional Schrödinger equations in one dimension, J. Funct. Anal., 266 (2014), 139-176.
30. Y. Kawahara. Global existence and asymptotic behavior of small solutions to nonlinear Schrödinger equations in 3D, Differential Integral Equations 18(2)(2005), 169-194.
31. S. Klainerman, G. Ponce. Global small amplitude solutions to nonlinear evolution equations, Communications on Pure and Applied Mathematics 21(1) (1983), 133-141.
32. N. Laskin. Fractional Schrödinger equation. Phys. Rev. E 66(5) (2002), 249-264.
33. T. Li, Y. Chen. Global solutions for nonlinear evolution equations, Longman Scientific Technical Press, New York, 1991.
34. P. Li, S-T. Yau: On the parabolic kernel of the Schrödinger operator. Acta Math. 156(1986), 153-201.
35. C. Miao, B. Yuan, B. Zhang, Well-posedness of the Cauchy problem for the fractional power dissipative equations, Nonlinear Anal. 68(2008), 461-484.
36. H.P. McKean, J. Shatah. The nonlinear Schrödinger equation and the nonlinear heat equation reduction to linear form, Communications on Pure and Applied Mathematics, Vol. XLIV (1991), 1067-1080.
37. K. Nakanishi. Asymptotically-free solutions for the short-range nonlinear Schrödinger equation. SIAM J. Math. Anal. 32(6) (2001), 1265-1271.
38. T. Ozawa. Long range scattering for nonlinear Schrödinger equations in one space dimension. Comm. Math. Phys. 139(3) (1991), 479-493.
39. P. Petersen. Riemannian geometry, volume 171 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1994.
40. V. Pierfelice. Weighted Strichartz estimates for the Schrödinger and wave equations on Damek-Ricci spaces. Math.Z. 260(2) (2008), 377-392.
41. C. Sun, J. Zheng. Scattering below ground state of 3D focusing cubic fractional Schrödinger equation with radial data, arXiv: 1702.03148.
42. W.A. Strauss. Nonlinear scattering theory at low energy. Journal of Functional Analysis 41(1) (1981), 110-133.
43. Y. Tsutsumi, K. Yajima. The asymptotic behavior of nonlinear Schrödinger equations, Bull. Amer. Math. Soc. 11(1) (1984), 186-188.

School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha 410114, People’s Republic of China.

E-mail address: zhlmath@yahoo.com

School of Mathematical Sciences, Sichuan University, Chengdu 610064, People’s Republic of China.

E-mail address: zhaoshiliang@scu.edu.cn