On Spacetimes Admitting Shear-free, Irrotational, Geodesic Timelike Congruences

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Abstract

A comprehensive analysis of general relativistic spacetimes which admit a shear-free, irrotational and geodesic timelike congruence is presented. The equations governing the models for a general energy-momentum tensor are written down. Coordinates in which the metric of such spacetimes takes on a simplified form are established. The general subcases of ‘zero anisotropic stress’, ‘zero heat flux vector’ and ‘two component fluids’ are investigated. In particular, perfect fluid Friedmann-Robertson-Walker models and spatially homogeneous models are discussed. Models with a variety of physically relevant energy-momentum tensors are considered. Anisotropic fluid models and viscous fluid models with heat conduction are examined. Also, models with a perfect fluid plus a magnetic field or with pure radiation, and models with two non-collinear perfect fluids (satisfying a variety of physical conditions) are investigated. In particular, models with a (single) perfect fluid which is tilting with respect to the shear-free, vorticity-free and acceleration-free timelike congruence are discussed.

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1. Introduction

In this paper, we shall consider general relativistic spacetimes which admit a shear-free, irrotational and geodesic (SIG) timelike congruence. It is well known \cite{1, 2} that if the stress-energy tensor is a perfect fluid whose flow lines form a SIG timelike congruence and whose density and pressure, as measured by a comoving observer, satisfy a barotropic equation of state, \( p = p(\mu) \) (with \( \mu + p \neq 0 \)), then the spacetime must be a Friedmann-Robertson-Walker (FRW) model. FRW spacetimes admit a six-parameter isometry group, \( G_6 \), whose orbits form spacelike hypersurfaces of constant curvature; coordinates can be chosen in such spacetimes so that the metric may be written as:

\[
ds^2 = -dt^2 + H^2(t) \frac{dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)}{(1 + \frac{\kappa}{2} r^2)^2},
\]

(1.1)

where the constant \( \kappa \) has been normalised to take the values \( \pm 1 \) and 0. It is also known \cite{1, 3} that if an equation of state \( p = p(\mu) \) is not assumed a priori then the resulting spacetime is still FRW. However, in the case of more general stress-energy tensors, there do exist solutions of the Einstein field equations that are not necessarily FRW models but where the spacetime admits a SIG timelike congruence.

It is our aim here to explore the richness of spacetimes admitting a SIG timelike congruence. For illustration, we begin by presenting an example of a non-FRW spacetime that contains a SIG timelike congruence. Consider the following Bianchi VI\(_0\) spacetime with “equal scale factors” (spatially homogeneous spacetimes will be studied in detail later):

\[
ds^2 = -dt^2 + X^2(t) \left( dx^2 + e^{-2x} dy^2 + e^{2x} dz^2 \right).
\]

(1.2)

Taking the source to be a fluid with four-velocity \( u^a = \delta^a_t \), we observe that the fluid flow lines form a SIG timelike congruence. However, the source cannot be a perfect fluid since the Bianchi VI\(_0\) models do not contain FRW models as special cases. Indeed, this model can be interpreted as an anisotropic fluid with

\[
\mu = \frac{3X^2}{X^2} - \frac{1}{X^2},
\]

(1.3)

\[
p_{\parallel} = -\left[ \frac{2X}{X^2} + \frac{X^2}{X^2} + \frac{1}{X^2} \right],
\]

(1.4)

\[
p_{\perp} = -\left[ \frac{2X}{X^2} + \frac{X^2}{X^2} - \frac{1}{X^2} \right] \neq p_{\parallel},
\]

(1.5)

where \( \mu \) is the energy density, and \( p_{\parallel} \) and \( p_{\perp} \) are, respectively, the pressures parallel to and perpendicular to \( n_a = X \delta_a^x \). Hence, there exists an anisotropic fluid solution whose flow lines are SIG. This example refutes the belief that spatially homogeneous spacetimes admitting a SIG timelike congruence must necessarily be FRW models.

We are primarily interested in cosmological models. In such models (and, in fact, in all models) other physical conditions must be satisfied. The precise conditions depend on the particular energy-momentum tensor considered; for example, for viscous fluid models with heat conduction a suitable set of thermodynamical laws must also be satisfied. All models should satisfy the (various) energy conditions \cite{4}. We note that there exist models in the above example in which the weak, dominant and strong energy conditions are all satisfied. Consequently, physical constraints (or, at least, the energy conditions alone) will not force the models to be FRW.

The remainder of the paper is organised as follows. In the next section the governing equations are displayed. For a spacetime in which there exists a shear-free, irrotational, geodesic timelike congruence,
coordinates can be chosen so that the metric takes on a simplified form [cf. equations (2.19) and (2.20)]; the governing equations are then given in these coordinates. In the following two sections general spacetimes are investigated with zero anisotropic stress and zero heat flux. At the end of section three, we show that in the special case of a perfect fluid whose flow forms a SIG timelike congruence that the density and pressure automatically satisfy a barotropic equation of state. In section 5, anisotropic fluid models are considered. In section 6, the two special physical subcases of a viscous fluid with heat conduction (satisfying a set of phenomenological equations) and a perfect fluid plus a magnetic field are investigated. In section 7, two component fluid models are considered, and particular attention is paid to the special physical subcases of a perfect fluid with pure radiation and two non-collinear perfect fluids (satisfying a variety of physical conditions such as, for example, separate energy conservation and linear equations of state). Finally, a single ‘tilting’ perfect fluid is considered. We conclude with a discussion.

2. General Equations

Einstein’s field equations are

\[ G_{ab} = T_{ab} \]  (2.1)

where the stress-energy tensor can formally be decomposed with respect to a timelike vector field \( u^a \) according to

\[ T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab} \]  ,  (2.2)

where

\[ q_a u^a = 0, \quad \pi^a_a = 0, \quad \pi_{ab} u^b = 0 \]  ,  (2.3)

and the projection tensor is defined by \( h_{ab} = g_{ab} + u_a u_b \). In this formal decomposition \( \mu, p, q_a \) and \( \pi_{ab} \) are given by

\[ \mu = T_{ab} u^a u^b \]  ,  (2.4)

\[ p = \frac{1}{3} h^{ab} T_{ab} \]  ,  (2.5)

\[ q_a = -h^c_a T_{cd} u^d \]  ,  (2.6)

\[ \pi_{ab} = h^c_a h^d_b T_{cd} - \frac{1}{3} (h^{cd} T_{cd}) h_{ab} \]  .  (2.7)

For a fluid with four-velocity \( u^a \), these quantities denote the energy density, pressure, heat conduction and anisotropic stress, respectively, as measured by an observer moving with the fluid.

In this paper, we consider spacetimes in which the shear, vorticity and acceleration (see \[ \] for definitions) of \( u_a \) are all zero, whence the covariant derivative of \( u_a \) can be written as

\[ u_{a;b} = \frac{1}{3} \theta h_{ab} \]  ,  (2.8)

where \( \theta \equiv u^a ;_a \) is the expansion. The relevant equations in the case of zero shear, zero vorticity and zero acceleration are \[ \] :

the conservation equations

\[ \dot{\mu} + (\mu + p) \theta + q^a ;_a = 0 \]  ,  (2.9)
\[
\dot{h}_{ab} (p_{,b} + \pi^c_{b;c} + \dot{q}_b + \frac{4}{3} \dot{\theta} q_b) = 0 ;
\] (2.10)

the Raychaudhuri equation
\[
\dot{\theta} + \frac{1}{3} \theta^2 + \frac{1}{2} (\mu + 3p) = 0 ;
\] (2.11)

the propagation equation
\[
\pi_{ab} = 2 E_{ab} ;
\] (2.12)

the constraint equations
\[
\frac{2}{3} \theta_{,b} h^b_a = q_a ,
\] (2.13)
\[
H_{ab} = 0 ;
\] (2.14)

and the Bianchi identities (using the above)
\[
\pi_{tb}^{,b} = \frac{1}{3} h_{tb}^{,b} \mu_{,b} - \frac{1}{3} \theta q_t ,
\] (2.15)
\[
\dot{\pi}_{tm} + \frac{2}{3} \theta \pi_{tm} = -\frac{1}{2} h_{tk}^{,a} h_{mc} q_{(ac)} + \frac{1}{6} q_a^{,a} h_{mt} .
\] (2.16)

The tensors \(E_{ab}\) and \(H_{ab}\) are the electric and the magnetic parts of the Weyl tensor, \(C^a_{abcd}\), and are given by
\[
E_{ab} = C_{acbd} u^c u^d ,
\] (2.17)
\[
H_{ab} = \frac{1}{2} C_{acst} u^c \eta^{st} u^d .
\] (2.18)

In the above, an overdot denotes differentiation along the fluid flow lines; for example, \(\dot{\theta} \equiv \theta_{,a} u^a\).

The conditions of zero acceleration and zero vorticity imply that it is possible to choose a comoving coordinate system such that \(u^a = \delta^a_t\) and such that the metric may be written as
\[
ds^2 = -dt^2 + g_{\alpha\beta}(t,x^\gamma) dx^\alpha dx^\beta .
\] (2.19)

The shear-free condition then implies that
\[
g_{\alpha\beta} = H^2(t,x^\gamma) h_{\alpha\beta}(x^\gamma) ,
\] (2.20)
and hence the expansion is \(\theta = 3\dot{H}/H\). We note that coordinates can also be chosen so that \(h_{\alpha\beta}\) is diagonal.

Also, the Gauss-Codazzi equations imply that the Ricci tensor of the three-space of constant \(t\) is
\[
R^*_{\alpha\beta} = \pi_{\alpha\beta} + \frac{1}{3} g_{\alpha\beta} (2\mu - \frac{2}{3} \theta^2)
\] (2.21)

which yields
\[
R^* = 2\mu - \frac{2}{3} \theta^2 .
\] (2.22)

In the coordinates \((t,x^\alpha)\) of (2.19), \(u^a = (1,0,0,0)\), \(q_0 = 0\), \(\pi_{0a} = 0\), \(\dot{\theta} = \partial_t \theta\), and \(h_{ab} \psi_{,b} = 0\) implies \(\psi_{,a} = 0\). We shall adopt the coordinates \((t,x^\alpha)\) for the remainder of the paper. The energy density,
the pressure, the heat flux and the anisotropic stress tensor are then given by

\[ \mu = \frac{1}{2} R^* + 3 \left( \frac{\dot{H}}{H} \right)^2, \tag{2.23} \]
\[ p = -\frac{1}{6} R^* - \frac{2H}{H^2} - \left( \frac{\dot{H}}{H} \right)^2, \tag{2.24} \]
\[ q_\alpha = 2 (\dot{H}/H)_{,\alpha} = 2 \partial_t [\partial_\alpha (\ln H)] = 2 \partial_\alpha [\partial_t (\ln H)] , \tag{2.25} \]
\[ \pi_{\alpha\beta} = R^*_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} R^*. \tag{2.26} \]

Equations (2.23) and (2.24) follow directly from (2.21) and (2.22). Equation (2.24) follows by inserting (2.23) into the Raychaudhuri equation (2.11). The conservation equations (2.9) and (2.10) become

\[ \dot{\mu} + 3 \frac{\dot{H}}{H} (\mu + p) + g^{\alpha\beta} q_{\alpha;\beta} = 0 , \tag{2.27} \]
\[ \partial_t (q_\alpha) + 3 \frac{\dot{H}}{H} q_\alpha + p_\alpha + g^{\gamma\delta} \pi_{\alpha\beta;\gamma} = 0 . \tag{2.28} \]

Now, the three-metric \( g_{\alpha\beta} \) is conformal to the metric \( h_{\alpha\beta} \), \( g_{\alpha\beta} = H^2 h_{\alpha\beta} \), and thus the Ricci tensor \( R^*_{\alpha\beta} \) is related to the Ricci tensor \( R_{\alpha\beta} \), of the metric \( h_{\alpha\beta} \), by

\[ R^*_{\alpha\beta} = 3 R_{\alpha\beta} - H^{-1} \nabla_\alpha \nabla_\beta H - H^{-1} (\nabla^2 H) h_{\alpha\beta} + 2 H^{-2} \nabla_\alpha H \nabla_\beta H \]
\[ \equiv 3 R_{\alpha\beta} - \nabla_\alpha \nabla_\beta (\ln H) + \nabla_\alpha (\ln H) \nabla_\beta (\ln H) \]
\[ -[\nabla^2 (\ln H) + \nabla (\ln H) \cdot \nabla (\ln H)] h_{\alpha\beta} \] \tag{2.29}

where \( \nabla^2 H \equiv h^{\alpha\beta} \nabla_\alpha \nabla_\beta H \), \( \nabla H \cdot \nabla H \equiv h^{\alpha\beta} \nabla_\alpha H \nabla_\beta H \), \( h^{\alpha\beta} h_{\alpha\gamma} = \delta^\alpha_\gamma \), and \( \nabla_\alpha \) is the covariant derivative with respect to the metric \( h_{\alpha\beta} \). Thus, re-expressing (2.23)–(2.28) in terms of the three-metric \( h_{\alpha\beta} \), we find

\[ \mu = \frac{1}{2} H^{-2} (\beta R) - 2 H^{-3} \nabla^2 H + H^{-4} \nabla H \cdot \nabla H + 3 H^{-2} \dot{H}^2 , \tag{2.31} \]
\[ p = -2 H^{-1} \dot{H} - \frac{1}{3} \mu , \tag{2.32} \]
\[ q_\alpha = 2 \partial_t [\nabla_\alpha (\ln H)] \tag{2.33} \]
\[ \pi_{\alpha\beta} = 3 R_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} 3 R - \nabla_\alpha \nabla_\beta (\ln H) + \nabla_\alpha (\ln H) \nabla_\beta (\ln H) \]
\[ + \frac{1}{3} [\nabla^2 (\ln H) - \nabla (\ln H) \cdot \nabla (\ln H)] h_{\alpha\beta} \] \tag{2.34}

and

\[ \dot{\mu} + 3 H^{-1} \dot{H} (\mu + p) + H^{-1} \nabla^\alpha q_\alpha = 0 , \tag{2.35} \]
\[ \partial_t q_\alpha + 3 H^{-1} \dot{H} q_\alpha + \nabla_\alpha p + H^{-2} \nabla^\beta \pi_{\alpha\beta} = 0 . \tag{2.36} \]

where \( \nabla^\alpha \equiv h^{\alpha\beta} \nabla_\beta \).

Finally, the Bianchi identities (2.15) and (2.16) reduce to

\[ H^{-1} \nabla^\beta \pi_{\alpha\beta} + H^{-3} (\nabla^\beta H) \pi_{\alpha\beta} = \frac{1}{3} \nabla_\alpha \mu - H^{-1} \dot{H} q_\alpha , \tag{2.37} \]
\[ 2 \partial_t \pi_{\alpha\beta} = -\nabla_\alpha q_\beta + 2 q_\alpha (\nabla_\beta (\ln H)) + \frac{1}{3} H^2 \nabla^\gamma (H^{-2} q_\gamma) h_{\alpha\beta} \] \tag{2.38}
Equation (2.37) is equivalent to the contracted Bianchi identity $\nabla^\alpha(3R_{\alpha\beta}) = \frac{1}{2}\nabla_\beta(3R)$, and (2.38) simply expresses the result that $\partial_t(3R_{\alpha\beta} - \frac{4}{3}h_{\alpha\beta}3R) = 0$.

This completes our general analysis and any further progress can only be made on a case by case basis.

3. Zero Anisotropic Stress

In this section, we take the anisotropic stress tensor to be zero, $\pi_{ab} = 0$. The equations (2.9)–(2.16) and (2.21)–(2.22) then become

\[
\begin{align*}
\dot{\mu} + (\mu + p)\dot{\theta} + q^\alpha_{\cdot\alpha} &= 0 \ , \\
p_{\cdot\alpha} + q_{\alpha} + \frac{4}{3}\theta q_{\alpha} &= 0 \ , \\
\dot{\theta} + \frac{1}{3}\theta^2 + \frac{1}{2}(\mu + 3p) &= 0 \ , \\
E_{ab} &= H_{ab} = 0 \ , \\
\frac{2}{3}\theta_{\cdot\alpha} &= q_{\alpha} \ , \\
\mu_{\cdot\alpha} &= \theta q_{\alpha} \ , \\
\frac{1}{2}(q_{\alpha\cdot\beta} + q_{\beta\cdot\alpha}) &= \frac{1}{3}q^\gamma_{\cdot\gamma}g_{\alpha\beta} \ , \\
R^*_{\alpha\beta} &= \frac{1}{3}g_{\alpha\beta}\left(2\mu - \frac{2}{3}\theta^2\right) \ , \\
R^* &= 2\mu - \frac{2}{3}\theta^2 .
\end{align*}
\]

Equation (3.4) implies that the Weyl tensor is zero and thus the spacetime is conformally flat. Equation (3.6) can be integrated with the aid of (3.5) to obtain

\[
\mu = \frac{1}{3}\theta^2 + \frac{1}{2}f(t) \ ,
\]

where $f$ is an arbitrary function of integration. Comparing (3.11) with (3.9), we find

\[
R^* = f(t) .
\]

Therefore, each three-space ($t = \text{const}$) is a space of constant curvature and thus, we may choose coordinates so that the metric has the form

\[
\begin{align*}
ds^2 &= -dt^2 + \Omega^{-2}(dx^2 + dy^2 + dz^2) \ , \\
\Omega(t,x^\alpha) &= a(t)(x^2 + y^2 + z^2) + b_\alpha(t)x^\alpha + c(t) \ ,
\end{align*}
\]

where $a$, $b_\alpha$ and $c$ are arbitrary functions of $t$. The Ricci scalar is then

\[
\begin{align*}
R^* &= 6(4ac - \delta^\alpha_{\beta}b_\alpha b_\beta) = R^*(t) \ , \\
&= 4\Omega\nabla^2\Omega - 6\nabla\Omega \cdot \nabla\Omega \ .
\end{align*}
\]

[$\nabla$ is the covariant derivative with respect to the three-metric $h_{\alpha\beta}(= \delta_{\alpha\beta})$]. The expansion is now given as

\[
\theta = -\frac{3\Omega}{\dot{\Omega}} .
\]
Equation (3.7) reduces to \( \dot{\Omega}_{\alpha \beta} = \frac{1}{3} (\nabla^2 \dot{\Omega}) \delta_{\alpha \beta} \) which is trivially satisfied by \( \Omega \) as given in (3.13). Also (3.2) follows directly from (3.3) and the spatial gradient of (3.3).

The density, \( \mu \), the pressure, \( p \), and the heat flux, \( q_{\alpha} \), can all be expressed in terms of \( \Omega \) and its derivatives:

\[
\mu = 2 \Omega \nabla^2 \Omega - 3 \nabla \Omega \cdot \nabla \Omega + 3 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 ,
\]

(3.17)

\[
p = 2 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - 3 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - 2 \Omega \nabla^2 \Omega + \nabla \Omega \cdot \nabla \Omega ,
\]

(3.18)

\[
q_{\alpha} = -2 \nabla_{\alpha} \left( \frac{\dot{\Omega}}{\Omega} \right) .
\]

(3.19)

The conservation equation (3.1) becomes

\[
\dot{\mu} - 3 (\mu + p) \frac{\dot{\Omega}}{\Omega} - \frac{1}{2} \Omega^2 (\Omega^{-2} R^*) = 0 ,
\]

(3.20)

which is automatically satisfied by \( \mu \) and \( p \) as given in (3.17) and (3.18).

The spherically symmetric solutions are found by setting \( b_{\alpha}(t) \) equal to zero in (3.13) and were originally discussed by Maiti [8]. The metric can then be written as

\[
ds^2 = -dt^2 + \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{[F(t)r^2 + G(t)]^2} ,
\]

(3.21)

where \( F \) and \( G \) are arbitrary functions. The model can then be interpreted as a perfect fluid with heat conduction, where the fluid four-velocity is \( u^a = \delta_t^a \). The density, pressure, and heat flux are now given by

\[
\mu = 3(\dot{F}r^2 + \dot{G})(Fr^2 + G)^{-2} + 12FG ,
\]

(3.22)

\[
p = 2(\dot{F}r^2 + \dot{G})(Fr^2 + G)^{-1} - 5(\dot{F}r^2 + \dot{G})(Fr^2 + G)^{-2} - 4FG ,
\]

(3.23)

\[
q_1 = -4r[\dot{F}G - \dot{G}F](Fr^2 + G)^{-2} , \quad q_2 = q_3 = 0 ,
\]

(3.24)

where \( q_1 \) is the heat conduction in the radial direction. We note that the model is conformally flat. Cosmological models of this type have been studied by Kolassis et al. [3] and Banerjee et al. [5].

**Perfect Fluid**

If both the anisotropic stress tensor and the heat flux vector are zero (\( \pi_{ab} = q_{a} = 0 \)) then the stress-energy tensor (2.2) is that of a perfect fluid and the metric is the FRW metric (1). Thus if the stress-energy tensor is a perfect fluid, whose flow lines form a SIG timelike congruence, then the solutions of the Einstein field equations must be the FRW models. Furthermore, equations (3.2) and (3.6) imply that both \( \mu \) and \( p \) are functions of the single variable \( t \), [that is, \( \mu = \mu(t) \), \( p = p(t) \)]. Hence \( \mu \) and \( p \) satisfy a barotropic equation of state \( p = p(\mu) \). Therefore, \( p = p(\mu) \) is a consequence of the assumptions and need not be specified a priori.

4. **Zero Heat Flux**

Here, we shall take the heat flux to be zero, \( q_{a} = 0 \). We immediately obtain from equation (2.13) that \( \theta = \theta(t) \), and hence we can set \( H = H(t) \) without loss of generality. The anisotropic stress tensor is now
The energy density and the isotropic pressure, as given by (2.31) and (2.32), are
\[
\mu = 3H^{-2} H^2 + \frac{1}{2} H^{-2} (\dot{3}R) ,
\]
\[
p = -2H^{-1} \ddot{H} - H^{-2} \dot{H}^2 - \frac{3}{6} R .
\]

The conservation equations
\[
\dot{\mu} + 3H^{-1} \dot{H} (\mu + p) = 0 ,
\]
\[
\nabla_a p + H^{-2} \nabla^\beta \pi_{\alpha\beta} = 0 ,
\]
and the Bianchi identities
\[
H^{-2} \nabla^\beta \pi_{\alpha\beta} = \frac{1}{3} \nabla_\alpha \mu ,
\]
\[
\partial_\tau \pi_{\alpha\beta} = 0 ,
\]
are automatically satisfied.

The zero heat flux models, under consideration in this section, can be subdivided into three distinct classes depending on the number of distinct eigenvalues of \(\pi_{\alpha\beta}\). If \(\pi_{\alpha\beta}\) has three distinct eigenvalues then the spacetime is of Petrov type I \([10]\). If \(\pi_{\alpha\beta}\) has exactly two distinct eigenvalues then the spacetime is of Petrov type D \([10, 11]\). Finally, if \(\pi_{\alpha\beta}\) has only one eigenvalue (the eigenvalue is zero since \(\pi_{\alpha\beta}\) is tracefree, and hence \(\pi_{\alpha\beta}\) is identically zero) then the spacetime is conformally flat and corresponds to a perfect fluid FRW model.

In comoving coordinates, the metric has the following form:
\[
ds^2 = -dt^2 + H^2(t) h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta .
\]
Thus any spatial three-metric will give rise to a shear-free, irrotational, geodesic model with zero heat flux. In particular, there exist examples of spatially homogeneous spacetimes of all the Bianchi types \([12]\), I–IX, and the Kantowski-Sachs solution \([13]\) that admit a SIG timelike congruence. However, only Bianchi types I, V, VII\(\alpha\), VII\(\beta\) and IX can give rise to FRW models. Indeed, Mimosa and Crawford \([14]\) recently drew attention to the same fact: “No anisotropic model can simultaneously exhibit a perfect fluid matter content and a shear-free timelike congruence . . . But the possibility still exists of considering an imperfect fluid for shear-free anisotropic models” (where presumably the authors mean a shear-free timelike fluid congruence, otherwise the statement is untrue — see section 6).

As an illustrative example, we shall consider the following Bianchi IV spacetime with “equal scale factors,” since this does not contain any FRW models as special cases:
\[
ds^2 = -dt^2 + H^2(t) \{dx^2 + e^{2x} (x^2 + 1) dy^2 + 2x e^{2x} dydz + e^{2x} dz^2\} .
\]
Taking the source to be an anisotropic fluid with the four-velocity of this fluid given by \(u^a = \delta_a^0\), we observe that the acceleration, shear, and vorticity of the fluid congruence all vanish, and that the heat flux as measured by a comoving observer is zero. Using \([4,1]\), we find that the non-zero components of the...
anisotropic stress tensor are

\[
\pi_{xx} = -\frac{1}{3}, \quad (4.10)
\]

\[
\pi_{yy} = e^{2x} \left( \frac{2}{3} x^2 - 2x - \frac{1}{3} \right), \quad (4.11)
\]

\[
\pi_{yz} = e^{2x} \left( \frac{2}{3} x - 1 \right), \quad (4.12)
\]

\[
\pi_{zz} = \frac{2}{3} e^{2x}. \quad (4.13)
\]

The eigenvalue of \( \pi_{\alpha\beta} \) can be found by solving the characteristic equation \( \det(\lambda h_{\alpha\beta} - \pi_{\alpha\beta}) = 0 \). There are three distinct eigenvalues; namely,

\[
\lambda = \frac{1}{3}, \quad \lambda = \frac{1}{6} \pm \frac{1}{2} \sqrt{5}. \quad (4.14)
\]

Thus, the spacetime is of Petrov type I. The energy density and the isotropic pressure are given by (4.2) and (4.3), respectively, with \( R = -13/2 \).

We note that for each three-metric \( h_{\alpha\beta} \) there exists an anisotropic stress tensor \( \pi_{\alpha\beta} \), but not all of the \( \pi_{\alpha\beta} \) are related to physical relevant matter. Therefore, it is of interest to study models with some extra physical conditions. Perhaps the most reasonable condition to assume here is an equation of state between the energy density, \( \mu \), and the isotropic pressure, \( p \).

**Equation of state**: \( p = p(\mu) \)

Here, we consider the case where the energy density and the isotropic pressure satisfy an equation of state, \( p = p(\mu) \), where both \( p \) and \( \mu \) are differentiable. This implies that the integrability condition

\[
(\nabla_\alpha p)\dot{\mu} = (\nabla_\alpha \mu)\dot{p} \quad (4.15)
\]

must be satisfied. Thus, inserting (4.2) and (4.3) into (4.15) yields

\[
(\nabla_\alpha \mu) \partial_t (H^{-1} \dot{H}) = 0. \quad (4.16)
\]

Two possible solutions exist: either (i) \( \nabla_\alpha \mu = 0 \) or (ii) \( \partial_t (H^{-1} \dot{H}) = 0 \).

(i) \( \nabla_\alpha \mu = 0 \)

Here \( \mu = \mu(t) \), and (4.2) implies that the Ricci scalar is constant, \( \frac{3}{R} = \text{const} \). The isotropic pressure is now also a function of \( t \), \( p = p(t) \). The anisotropic stress-tensor, as given by (4.1), is now subject to the constraint \( \nabla^\alpha \pi_{\alpha\beta} = 0 \). Therefore, models with zero heat flux where the energy density and the isotropic pressure satisfy a barotropic equation of state are characterised by a single scale factor, \( H(t) \), and a spatial three-metric whose Ricci scalar is constant (not to be confused with three-spaces of constant curvature).

We now make a brief comment on the consequences of the various energy conditions for the above models. The weak energy condition \( [15] \) — \( T_{ab}v^av^b \) is non-negative for all unit timelike vectors \( v^a \) — implies that the energy density, \( \mu \), is non-negative and hence, if \( \frac{3}{R} \) is negative then (4.2) implies that either \( \dot{H} > 0 \) or \( \dot{H} < 0 \) for all time, \( t \); thus, in the former case, the cosmology will be an open model. The strong energy condition — \( 2T_{ab}v^av^b + T^a_a \) is non-negative for all unit timelike vectors \( v^a \) — implies that \( \mu + 3p \geq 0 \), and thus the deceleration parameters, \( q \equiv -\dot{H}H/\dot{H}^2 \), is non-negative.
\( \partial_t \left( H^{-1} \ddot{H} \right) = 0 \)

Therefore, \( \ddot{H} = \lambda H \) where \( \lambda \) is a constant. We can integrate this equation to get

\[
\dot{H}^2 = \lambda H^2 + \nu , \tag{4.17}
\]

where \( \nu \) is a constant. The energy density and the isotropic pressure are then

\[
\mu = \frac{1}{2} \dot{H}^{-2} (3R + 6\nu) + 3\lambda , \tag{4.18}
\]

\[
p = -\frac{1}{3}\mu - 2\lambda . \tag{4.19}
\]

If the fluid satisfies the strong energy condition then we can set \( \lambda = -\alpha^2 \leq 0 \) and \( \nu = \alpha^2\beta^2 \geq 0 \) for some constants \( \alpha \) and \( \beta \). If \( \alpha \neq 0 \) then (4.17) integrates to yield

\[
H(t) = \beta \sin(\alpha t) \tag{4.20}
\]

which is a closed model. If \( \lambda = 0 \) then

\[
H(t) = \gamma t \quad (\gamma = \text{const}) \tag{4.21}
\]

which is an open model. (In both cases, \( \frac{dp}{d\mu} = -\frac{1}{3} \).)

Therefore, if the heat flux of the fluid is zero and the energy density and the isotropic pressure satisfy a barotropic equation of state then the hypersurfaces \( t = \text{constant} \) are for the most part hypersurfaces of constant Ricci scalar with the single exception of the equation of state \( \frac{dp}{d\mu} = -\frac{1}{3} \).

Orthogonal Spatially Homogeneous Models

The function \( H \) appearing in the metric (2.20) must be independent of the spatial coordinates for the metric to represent a spatially homogeneous model. Therefore, the expansion also only depends on \( t, \theta = \theta(t) \), and hence the heat flux is zero.

To make further progress, we assume that the matter associated with these spatially homogeneous models inherits the symmetries [3]; in particular, all scalars should be independent of the spatial coordinates. Hence, both the energy density, \( \mu \), and the isotropic pressure, \( p \), are now functions of only \( t \). Equation (4.1) then implies that the Ricci scalar, \( 3R \), is constant, which is case (i) for zero heat flux models with a barotropic equation of state. The eigenvalues of the anisotropic stress tensor, \( \pi_{\alpha\beta} \), should also be constants. Indeed, as might be expected, all of the Bianchi types (I–IX) and the Kantowski-Sach models admit solutions with constant Ricci scalar such that the eigenvalues of \( \pi_{\alpha\beta} \), as given by (4.1), are constant.

5. Anisotropic Fluid

The stress-energy tensor for an anisotropic fluid is

\[
T_{ab} = \mu u_a u_b + p_\parallel n_a n_b + p_\perp (u_a u_b - n_a n_b + g_{ab}) , \tag{5.1}
\]

where \( u^a \) is a unit timelike vector, \( u_a u^a = -1 \), and \( n^a \) is a unit spacelike vector, \( n_a n^a = 1 \), orthogonal to \( u^a \), \( u^a n_a = 0 \). The scalars \( p_\parallel \) and \( p_\perp \) are the pressure parallel and perpendicular to \( n^a \), respectively. Equation (2.4) implies that the heat flux in the case of an anisotropic fluid is zero. Thus, shear-free, irrotational,
geodesic anisotropic fluids are a subcase of the models discussed in section 4. Equations (2.3) and (2.7) yield expressions for the isotropic pressure and the anisotropic stress tensor:

\[ p = \frac{1}{3} (p_\parallel + 2p_\perp) \]

\[ \pi_{ab} = (p_\parallel - p_\perp) \left\{ n_a n_b - \frac{1}{3} (g_{ab} + u_a u_b) \right\} . \]

In terms of the comoving coordinates of (2.19), the spacelike vector \( n_a = (0, n_\alpha) \) and the anisotropic stress tensor is given by

\[ \pi_{\alpha\beta} = (p_\parallel - p_\perp) (n_\alpha n_\beta - \frac{1}{3} H^2 h_{\alpha\beta}) . \]

The anisotropic stress tensor (5.4) may be decomposed as \[5\]

\[ \pi_{\alpha\beta} = P (N_\alpha N_\beta - \frac{1}{3} h_{\alpha\beta}) , \]

where \( N_\alpha(x^\beta) = H^{-1}(t) n_\alpha \) is a unit vector with respect to the metric \( h_{\alpha\beta} \) and \( P \) is independent of \( t \), \( P = P(x^\alpha) \). The function \( P \) is a measure of the anisotropy of the fluid,

\[ P = H^2(t) (p_\parallel - p_\perp) , \]

and is subject to the constraint equation

\[ \frac{1}{6} \nabla_\alpha (3R) = \nabla^\beta (P N_\beta) N_\alpha - \frac{1}{3} \nabla_a P + P N_\beta \nabla_\beta N_\alpha . \]

The energy density and the isotropic pressure are given by expressions (4.2) and (4.3), respectively. The anisotropic pressures are related to \( p \) and \( P \) by

\[ p_\perp = p - \frac{P}{3H^2} , \]

\[ p_\parallel = p + \frac{2P}{3H^2} . \]

The anisotropic stress tensor (5.3) has only two distinct eigenvalues, \( \frac{2}{3} P \) and \(-\frac{1}{3} P \) (double), and hence the spacetime is of Petrov type D (if \( P \neq 0 \)). However, if \( P \) is identically zero then (5.4) implies that \( p_\parallel = p_\perp \) and thus, the spacetime is then a perfect fluid FRW model. Conversely, if the metric is of the form (5.8) and \( \pi_{\alpha\beta} \) has exactly two distinct eigenvalues then \( \pi_{\alpha\beta} \) must necessarily be of the form (5.3) and hence the stress-energy tensor will formally be that of an anisotropic fluid with four-velocity \( u^a = \delta^a_t \)

Furthermore, if \( p \) and \( \mu \) satisfy a barotropic equation of state, \( p = p(\mu) \), then either the Ricci scalar, \( 3R \), is constant or \( \frac{dp}{d\mu} = -\frac{1}{3} \) (see section 4). The case where the Ricci scalar, \( 3R \), is constant is the most interesting; in particular, there exists examples of spatially homogeneous anisotropic fluid models of Bianchi types II, III, VI and VIII, and a Kantowski-Sachs solution \[16\].

6. Special Cases

Viscous Fluid

For a viscous fluid with heat conduction the following phenomenological equations are satisfied:

\[ \pi_{ab} = -2\eta \sigma_{ab} = 0 , \]

\[ q_a = -\chi h_a^b (T_b + T \dot{u}_b) = -\chi h_a^b T_b , \]

\[ p = p_t - \zeta \theta , \]
where $\zeta$ is the bulk viscosity coefficient ($\eta$ is the shear viscosity coefficient), $p_t$ is the thermodynamic pressure, $\chi$ is the heat conductivity, and $T$ is the temperature. These quantities are restricted by equations of state of the form 
\[
\zeta = \zeta(\mu, T) \quad [\eta = \eta(\mu, T), \chi = \chi(\mu, T), p_t = p_t(\mu, T)],
\]
and the various energy conditions. In addition, the Gibb’s relation, the baryon conservation law and the second law of thermodynamics must be satisfied.

Since $\pi_{ab} = 0$, these models are a subcase of the models discussed in section 3, and hence there exists coordinates in which the metric is given by equations (3.12) and (3.13), and $\mu$, $p$ and $q_\alpha$ are given by (3.17)–(3.19).

As an illustration, let us consider the case in which $T = T(\mu)$, whence $\zeta = \zeta(\mu)$, $\chi = \chi(\mu)$ and $p_t = p_t(\mu)$. For example, at the high temperatures in the early universe when the energy density was dominated by relativistic species (that is, when the universe was radiation dominated), $\mu = \frac{1}{2}g a T^4 = \mu(T)$, where $g = g(T)$ is the number of effective degrees of freedom contributing to the universe at temperature $T$. From equations (6.2) and (3.6), we obtain
\[
\theta = \frac{1}{\theta} \frac{\mu}{\alpha} = -\chi T, \alpha = -\chi \frac{dT}{d\mu} \mu, \alpha, \quad (6.4)
\]
whence
\[
\theta = -\left(\chi \frac{dT}{d\mu}\right)^{-1} = \theta(\mu), \quad (6.5)
\]
since $\mu, \alpha \neq 0$ ($q_\alpha \neq 0$), else the spacetime is a perfect fluid FRW spacetime. This implies that $f(t) = \text{constant}$ in (3.10) and $p = p(\mu)$ from (6.3). Equation (3.3) then implies that $\dot{\mu}$ is also a function of $\mu$.

Now, the right hand-side of the Gibb’s relation, $\frac{1}{\theta} \frac{d\theta}{dt} \left(\frac{\mu}{\alpha}\right) + \frac{p}{\theta} \frac{d\theta}{dt} \left(\frac{1}{n}\right)$ (where $n$ is the baryon number density), is a perfect differential, which implies that either $n = n(\mu)$ (whence, from the baryon conservation law, $\dot{n} = -\dot{n}\theta$, $\dot{n}$ and hence $\dot{\mu}$ is a function of $\mu$ only), or we have the integrability condition
\[
T \frac{dp_t}{d\mu} = (\mu + p_t) \frac{dT}{d\mu}, \quad (6.6)
\]
whence from (6.3) we obtain
\[
\theta = -\frac{\mu + p_t}{\chi T} \left(\frac{dp_t}{d\mu}\right)^{-1}, \quad (6.7)
\]
Assuming that $\frac{dp_t}{d\mu} > 0$ (and noting that $(\mu + p_t)(\chi T)^{-1} > 0$ if the weak energy condition is satisfied) we thus have $\theta < 0$. Otherwise (6.7) can be regarded as an expression defining $\chi$.

Summarising, we have that
\[
\theta = f_1(\mu), \quad (6.8)
\]
\[
\dot{\mu} = f_2(\mu), \quad (6.9)
\]
in addition to $p = f_3(\mu)$. These imply the conditions
\[
\frac{\partial f_i}{\partial x^\alpha} \frac{\partial \mu}{\partial t} = \frac{\partial f_i}{\partial t} \frac{\partial \mu}{\partial x^\alpha}; \quad i = 1, 2; (3) \quad (6.10)
\]
which imply conditions on the (arbitrary functions in the) metric (3.12)–(3.13) using (3.16), (3.17) and (3.18).

For example, in the case of spherical symmetry, when the functions $b_\alpha(t) = 0$ in (3.13) whence the metric is given by (3.22), and $\mu$, $p$ and $q$, are given by equations (3.22), (3.22) and (3.24), respectively, (6.8) implies that
\[
F(t) G(t) = \alpha, \quad (6.11)
\]
where $\alpha$ is a constant, whence
\begin{equation}
\mu = \frac{3(F^2 r^2 - \alpha)^2 \dot{F}^2}{(F^2 r^2 + \alpha)^2 F^2} + 12 \alpha .
\end{equation}

Equations (6.9) and (6.10) ($\mu, \mu_t, \mu_r = \mu, \mu_t, \mu_r$) then imply that
\begin{equation}
\ddot{F} = \dot{F}^2 ,
\end{equation}
and hence
\begin{equation}
F = c e^{\beta t} ,
\end{equation}
where $\beta$ and $c$ integration constants. A constant rescaling of $r$ can then be used to set $c = \alpha$, whence the metric can be written as
\begin{equation}
ds^2 = -dt^2 + \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}{[\alpha r^2 e^{\beta t} + e^{-\beta t}]^2} .
\end{equation}

Spacetimes of this form have been studied before [17]. From equations (3.22)–(3.24) we can then write
\begin{align}
\mu &= 12 \alpha + \frac{1}{3} \theta^2 , \\
p &= 2 \beta^2 - 4 \alpha - \frac{5}{9} \theta^2 ,
\end{align}
where
\begin{equation}
\theta = 3 \beta (1 - \alpha e^{2\beta t} r^2) (1 + \alpha e^{2\beta t} r^2)^{-1} ,
\end{equation}
and
\begin{equation}
q_1 = -\frac{8 \alpha \beta r}{[\alpha r^2 e^{\beta t} + e^{-\beta t}]^2} .
\end{equation}

**Perfect Fluid plus a Magnetic Field**

If the source of the gravitational field is a combination of a perfect fluid and a pure magnetic field then the stress-energy tensor has the form
\begin{equation}
T_{ab} = \bar{\mu} u_a u_b + \bar{p} (g_{ab} + u_a u_b) + T_{MAG}^{ab} .
\end{equation}

If $h^a$ is the local magnetic field measured by $u^a$ (and the local electric field $e^a$ is zero) then the stress-energy tensor for a pure magnetic field is given by the Minkowski tensor [18]
\begin{equation}
T_{MAG}^{ab} = \lambda \left[ \left( \frac{1}{2} g_{ab} - u_a u_b \right) h^2 - h_a h_b \right] ,
\end{equation}
where $\lambda$ is the magnetic permeability (assumed constant) and $h^2 \equiv h_a h^a$ ($h^a$ is orthogonal to $u^a$, $h^a u_a = 0$). For instance, the above stress-energy tensor, (6.20), would be appropriate for a plasma in a strong magnetic field when the particle collision density is low [19, 20].

Equation (6.20) is formally of the form of an anisotropic fluid [21] with energy density and isotropic pressure
\begin{align}
\mu &= \bar{\mu} + \frac{\lambda}{2} h^2 , \\
p &= \bar{p} + \frac{\lambda}{6} h^2 ,
\end{align}

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and anisotropic stress tensor

\[ \pi_{ab} = \lambda h^2 \left\{ \frac{1}{3} (g_{ab} + u_a u_b) - n_a n_b \right\} , \tag{6.24} \]

where \( n_a = h_a / h \). The anisotropic pressures are formally

\[ p_{\perp} = \overline{p} + \frac{\lambda}{2} h^2 , \tag{6.25} \]
\[ p_{\parallel} = \overline{p} - \frac{\lambda}{2} h^2 . \tag{6.26} \]

The magnetic field must also satisfy Maxwell’s equations which reduce to

\[ (u^i h^j - u^j h^i)_{;i} = 0 \] for a pure magnetic field [18]. If the integral curves of \( u^i \) are shear-free, irrotational and geodesic then \([6.27]\) implies

\[ h^i_{;i} = 0 , \tag{6.28} \]
\[ \frac{2}{3} \theta h_i + h_{i;j} u^j = 0 . \tag{6.29} \]

Contracting \([6.29]\) with \( h^i \) implies that

\[ \frac{1}{2} \partial_i (h^2) + \frac{2}{3} \theta h^2 = 0 . \tag{6.30} \]

Now, equation \([5.6]\) implies that

\[ - \lambda^2 h^2 = P(x^\alpha) / H^2(t) . \tag{6.31} \]

Inserting the result \([6.31]\) into \([6.30]\) yields \( \dot{H} = 0 \), and thus the expansion is zero. Therefore, the magnetic field is static. In comoving coordinates, \([4.8]\), equation \([6.29]\) is trivially satisfied and \([6.28]\) reduces to

\[ \nabla^\alpha h_\alpha = 0 . \tag{6.32} \]

In addition, \( h_\alpha \) must also satisfy \( \nabla^\beta \pi_{\alpha\beta} = \frac{4}{3} \nabla_\alpha \mu \), which reduces to

\[ \frac{1}{3 \lambda} \nabla_\alpha \overline{\rho} = \frac{1}{3} h \nabla_\alpha h - h^\beta \nabla_\beta h_\alpha . \tag{6.33} \]

We note that there is no non-trivial solution if \( \overline{\rho} \) is given by the equation of state \( \overline{\rho} = (\gamma - 1) \overline{\rho} \) with \( \gamma \) constant and in which the weak energy condition holds.

### 7. Two Component Fluids

Here, we consider the stress-energy tensor associated with a mixture of two fluids. We take one of the fluid components to be a perfect fluid whose flow forms a SIG timelike congruence (with fluid four-velocity \( u^a = \delta^a_0 \)); the density and pressure of this perfect fluid are denoted by \( \mu_1 \) and \( p_1 \), respectively. We denote the contribution of this fluid to the stress-energy tensor by

\[ (1)T_{ab} \equiv (\mu_1 + p_1) u_a u_b + p_1 g_{ab} . \tag{7.1} \]
The second fluid will be taken to be either another perfect fluid with a velocity \( \nu^a \) that is tilted with respect to \( u^a \) or a pure radiation field with the following, respective, contributions to the stress-energy tensor

\[
(2) T_{ab} = (\mu_2 + p_2) u_a u_b + p_2 g_{ab} , \tag{7.2}
\]

\[
(2) T_{ab} = \epsilon k_a k_b . \tag{7.3}
\]

where \( \nu^a \) is a unit timelike vector \( (\nu^a \neq u^a) \), and \( k^a \) is a null vector, \( k_a k^a = 0 \). The total stress-energy tensor is simply

\[
T_{ab} = (1) T_{ab} + (2) T_{ab} . \tag{7.4}
\]

The stress-energy tensor (7.4) is equivalent to a single fluid with a non-zero heat flux and a non-zero anisotropic stress tensor. However, the anisotropic stress tensor associated with the fluid velocity \( u^a \) is directly related to the heat flux (relative to \( u^a \)) via an equation of the form:

\[
\pi_{ab} = \pi \left\{ q_a q_b - \frac{1}{3} (q_c q^c) (g_{ab} + u_a u_b) \right\} . \tag{7.5}
\]

In comoving coordinates \( (u^a = \delta^a_0) \), the non-zero components of the anisotropic stress tensor are

\[
\pi_{\alpha\beta} = \pi \left\{ q_\alpha q_\beta - \frac{1}{3} (h_{\gamma\delta} q_\gamma q_\delta) h_{\alpha\beta} \right\} . \tag{7.6}
\]

Letelier has examined a two-perfect-fluid model of an anisotropic fluid. In particular, he examined the case where the stress-energy tensor consists of the sum of two perfect fluids, and of one perfect fluid and a null fluid. He studied the two-perfect-fluid model in the instance where both the perfect fluid components were irrotational. Ferrando, Morales and Portilla have studied the two-perfect-fluid model where one of the fluid components was shear-free, irrotational and geodesic. They used an initial value formulation of general relativity to construct solutions. We essentially rederive their result using straightforward index notation, and indicate that the form of the metric they derive is appropriate for all shear-free, irrotational, geodesic timelike congruences when the anisotropic stress tensor has the form specified by equation (7.5). The results follow quite quickly from the general analysis of section 2.

Inserting the expression (7.6) for the anisotropic stress tensor into the left hand-side of (2.38) and then using (2.36) to get an expression for \( \partial_t q_\alpha \) yields

\[
\nabla_\beta q_\alpha = A_\alpha q_\beta + A_\beta q_\alpha + B \epsilon_\alpha q_\beta + C h_{\alpha\beta}
\]

(7.7)

(the precise expressions for \( A_\alpha, B \) and \( C \) are unnecessary for the remainder of the discussion). The unit spacelike vector

\[
w_\alpha = \frac{q_\alpha}{\sqrt{h^{\gamma\gamma} q_\gamma q_\gamma}}
\]

then satisfies

\[
\nabla_\beta w_\alpha = \frac{1}{2} \Theta (h_{\alpha\beta} - w_\alpha w_\beta) + a_\alpha w_\beta,
\]

(7.9)

where

\[
\Theta = \nabla^{\alpha} w_\alpha,
\]

(7.10)

\[
a_\alpha = w_\beta \nabla_\beta w_\alpha \left[ = A_\alpha - (h^{\gamma\gamma} A_\beta w_\gamma) w_\alpha \right].
\]

(7.11)

Therefore, the vector \( w_\alpha \) is shear-free and twist-free, in terms of the three-dimensional geometry. Thus, the three spaces admit an umbilical foliation, and hence, there exists a coordinate system such that the three-metric has the following form

\[
h_{\alpha\beta} dx^\alpha dx^\beta = a^2 (x^\alpha) dx^2 + b^2 (x^\alpha) (dy^2 + dz^2)
\]

(7.12)

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and the vector
\[ w_α = a δ^α_x \]  
Equations (7.13) and (2.34) imply that \( H = H(t, x) \). Equation (2.34) then implies that
\[ 3R_{xy} = \nabla_x \nabla_y (\ln H) = -\partial_x (\ln H) \partial_y (\ln a) \]  
\[ 3R_{xz} = \nabla_x \nabla_z (\ln H) = -\partial_x (\ln H) \partial_z (\ln a) \]  
But \( \partial_x (\ln H) \) is a function of both \( t \) and \( x \) (otherwise \( q_α \) is identically zero). Therefore, we must have \( 3R_{xy} = 3R_{xz} = 0 \) and hence, we can take \( a = 1 \) without loss of generality. Calculating \( 3R_{xy} \) and \( 3R_{xz} \) for the metric (7.12) with \( a = 1 \), we find
\[ 3R_{xy} = -b^{-2} \partial_{xy} b + b^{-3} \partial_x b \partial_y b \]  
\[ 3R_{xz} = -b^{-2} \partial_{xz} b + b^{-3} \partial_x b \partial_z b \]  
and hence
\[ b = f(x) \phi(y, z) \]  
for some, as yet unknown, functions \( f \) and \( \phi \). Thus, the metric for the two component fluid (7.4) may be written as
\[ ds^2 = -dt^2 + H^2(t, x) \left( dx^2 + f^2(x) \phi^2(y, z) (dy^2 + dz^2) \right) \]  
The only non-zero components of the anisotropic stress tensor, (2.34), for the above metric are
\[ \pi_{xx} = -2f^{-2} \phi^2 \pi_{yy} = -2f^{-2} \phi^{-2} \pi_{zz} \]  
\[ = \frac{2}{3} [-H^{-1} H_{xx} - 2H^{-2} H_x^2 - H^{-1} f^{-1} f_x - f^{-1} f_{xx} + f^{-2} f_x^2 \]  
\[ + f^{-2} \{ \phi^{-3} (\phi_{yy} + \phi_{zz}) - \phi^{-4} (\phi_y^2 + \phi_z^2) \}] \]  
We note that \( H \) is not a separable function, \( H \neq X(x)T(t) \), otherwise the energy flux and the anisotropic stress tensor would both be zero.

Finally, \( \pi_{xx} \) must also satisfy
\[ \pi_{xx} = \frac{2}{3} \pi q^2 \]  
where
\[ q = 2\partial_t \partial_x (\ln H) \]  
and the energy flux is simply \( q_α = q δ^α_x \). The functions \( q \) and \( \pi \) are therefore determined by (7.22) and (7.23).
No further progress can be made in general, without specifying some sort of equation of state, usually an equation relating \( q \) and \( \pi \). For illustrative purposes, we will consider (i) a perfect fluid and a null fluid mixture, and (ii) a two-perfect-fluid mixture.

**Perfect fluid plus pure radiation**

The stress-energy tensor for a mixture of a perfect fluid and pure radiation is
\[ T_{ab} = (\mu_1 + p_1) u_a u_b + p_1 g_{ab} + e k_a k_b \]  
where \( k_a \) is a null vector. We are free to rescale \( k_a \) and \( e \) such that \( k_a u^a = -1 \). Thus, in comoving coordinates we choose \( k_a = (-1, k_α) \) and \( k_α = H \delta^α_x \). Then, the various quantities \( (\mu, p, q_α, \pi_{αβ}) \) are
\[ \mu = \mu_1 + e \]  
\[ p = p_1 + \frac{1}{3} e \]  
\[ q_α = e H \delta^α_x \]  
\[ \pi_{αβ} = e H^2 \{ \delta^α_x δ^β_y - \frac{1}{3} h_{αβ} \} \]  

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Therefore,

\[ q = eH = 2\partial_t\partial_x(\ln H) \quad (7.29) \]

and

\[ \pi_{xx} = \frac{2}{3}eH^2 \quad . \quad (7.30) \]

Using (7.29), we obtain

\[ \pi_{xx} = \frac{4}{3}H\partial_t\partial_x(\ln H) \quad (7.31) \]

Comparing (7.31) to (7.21) we find

\[ \phi^{-3}(\phi_{yy} + \phi_{zz}) - \phi^{-4}(\phi_y^2 + \phi_z^2) = -k \quad (7.32) \]

for some constant \( k \), and thus

\[ \phi^{-1} = 1 + \frac{k}{4}(y^2 + z^2) \quad . \quad (7.33) \]

The functions \( H \) and \( f \) must also satisfy

\[ 2H\partial_t(H^{-1}H_x) + H^{-1}H_{xx} + 2H^{-2}H_x^2 + H^{-1}H_xf^{-1}f_x = -f^{-1}f_{xx} + f^{-2}f_x^2 - kf^{-2} \quad . \quad (7.34) \]

Equation (7.34) can be integrated to yield

\[ H'(\equiv \frac{dH}{d\tau}) = \frac{(2\alpha H - 1)}{cH^2} \quad , \quad (7.36) \]

where \( c \) is an arbitrary constant. Thus, the line element

\[ ds^2 = -\left(\frac{dH}{H'} - dx\right)^2 + H^2(dx^2 + dy^2 + dz^2) \quad , \quad (7.37) \]

where \( H' \) is given by (7.36) with \( \alpha = 1 \), is an example of a spacetime where the stress-energy tensor consists of a perfect fluid (with 4-velocity \( u^a = H'\delta^a_H \)) plus pure radiation such that the perfect fluid’s flow lines form a SIG timelike congruence.

Once \( \phi, f \) and \( H \) are known then the functions \( \mu_1, p_1, \) and \( e \) can be calculated from

\[ \mu_1 = \mu - e \quad , \quad (7.38) \]

\[ p_1 = p - \frac{1}{3}e \quad , \quad (7.39) \]

\[ e = 2H^{-1}\partial_t(H^{-1}H_x) \quad , \quad (7.40) \]

where \( \mu \) and \( p \) are given by (2.31) and (2.32).
Two-perfect-fluid mixture

The stress-energy tensor for a two-perfect-fluid mixture is

\[ T_{ab} = (\mu_1 + p_1) u_a u_b + (p_1 + p_2) g_{ab} + (\mu_2 + p_2) v_a v_b \]  \hspace{1cm} (7.41)

We shall assume that the two fluid four-velocities are not collinear, \( u_a \neq v_a \), else the stress-energy tensor would formally be that of a single fluid with energy density \( \mu_1 + \mu_2 \) and pressure \( p_1 + p_2 \) and hence the resulting spacetime would be FRW. According to (2.4) and (2.5) the energy density and the isotropic pressure of the mixture are

\[ \mu = \mu_1 + (\mu_2 + p_2) \cosh^2 \psi - p_2 \]  \hspace{1cm} (7.42)

\[ p = p_1 + p_2 + \frac{1}{3} (\mu_2 + p_2) \sinh^2 \psi \]  \hspace{1cm} (7.43)

where \( \cosh \psi \equiv -v_a u^a \) and \( \psi \) is called the tilt angle. The energy flux, (2.6), is

\[ q_a = (\mu_2 + p_2) \cosh \psi (v_a + \cosh \psi u_a) \]  \hspace{1cm} (7.44)

and the anisotropic stress tensor, (2.7), is

\[ \pi_{ab} = \frac{1}{\cosh^2 \psi (\mu_2 + p_2)} \left[ q_a q_b - \frac{1}{3} (g^{cd} q_c q_d) (g_{ab} + u_a u_b) \right] . \]  \hspace{1cm} (7.45)

In comoving coordinates, \( u_a = -\delta^0_a \), whence the four-velocity of the tilting perfect fluid is given by \( v_a = (-\cosh \psi, \sinh \psi \delta^x_a) \), and

\[ q_a = (\mu_2 + p_2) \cosh \psi \sinh \psi \delta^x_a \]  \hspace{1cm} (7.46)

and

\[ \pi_{\alpha\beta} = \frac{1}{\cosh^2 \psi (\mu_2 + p_2)} \left[ q_\alpha q_\beta - \frac{1}{3} (h^{\gamma\delta} q_\gamma q_\delta) h_{\alpha\beta} \right] . \]  \hspace{1cm} (7.47)

Therefore, the quantities \( q \) and \( \pi \) are given by

\[ q = (\mu_2 + p_2) \cosh \psi \sinh \psi \]  \hspace{1cm} (7.48)

\[ \pi = \frac{1}{(\mu_2 + p_2) \cosh^2 \psi} . \]  \hspace{1cm} (7.49)

From (7.48) and (7.49), we obtain the identity

\[ q\pi = \frac{\sinh \psi}{\cosh \psi} . \]  \hspace{1cm} (7.50)

Now, the four quantities \( \mu, p, q, \) and \( \pi \) are given, respectively, by equations (2.31), (2.32), (7.23) [see also (7.23)] and (7.22) [where \( \pi_{xx} \) is given by (7.21); see also (2.34)] in terms of the functions \( H, f \) and \( \phi \) [in metric (7.19)] and their derivatives. In addition, \( \mu, p, q \) and \( \pi \) are given in terms of the five (unknown) physical quantities \( \mu_1, p_1, \mu_2, p_2 \) and \( \psi \) through equations (7.42), (7.43), (7.48) and (7.49), respectively. At this point, therefore, the system of (four) equations is underdetermined, and no further progress can be made until physical conditions on \( \mu_1, p_1, \mu_2, p_2 \) and \( \psi \) are specified. If one such condition is specified, the remaining four physical quantities can then in principle be expressed on terms of \( H, f \) and \( \phi \) and their derivatives. Two conditions on \( \mu_1, p_1, \mu_2, p_2 \) and \( \psi \) would then [through equations (2.31), (2.32), (7.22) and (7.23)] give rise to a (differential) equation in terms of \( H, f \) and \( \phi \) that would need to be satisfied (further restricting the form of the metric).
Let us consider the following conditions: (i) separate energy conservation, (ii) linear equations of state for \( p_1 \) and \( p_2 \), (iii) a constant tilt angle \( \psi \), (iv) the second perfect fluid is due to a scalar field, and (v) a single tilting perfect fluid.

(i) **Separate energy conservation.** Here we assume that the energy momentum of each perfect fluid is separately conserved; that is, \((A)\ T_{ab}{}^{\mu} = 0 \ (A = 1, 2)\). Due to total stress-energy conservation, this leads to one extra constraint:

\[
\frac{\dot{\mu}_1}{\mu_1 + p_1} = \frac{3\dot{H}}{H} , \tag{7.51}
\]

where \( p_1 \) is a function of \( t \) only, \( p_1 = p_1(t) \). We note that if \( \mu_1 \) and \( p_1 \) satisfy an equation of state of the form \( \mu_1 = \mu_1(p_1) = \mu_1(t) \), we immediately have that \( q_a = \pi_{ab} = 0 \) and the resulting spacetime is FRW.

(ii) **Linear equations of state.** Here we assume that each fluid satisfies a linear equation of state, viz.,

\[
p_1 = (\gamma_1 - 1)\mu_1 \ , \tag{7.52}
p_2 = (\gamma_2 - 1)\mu_2 \ , \tag{7.53}
\]

where the \( \gamma_1 \) and \( \gamma_2 \) are constants. Equations \((7.42)\) and \((7.43)\) then yield

\[
p + (1 - \gamma_1)\mu = \mu_2 \left\{ \gamma_1 (\gamma_2 - 1) + \frac{1}{3} \gamma_2[-3\gamma_1 \cosh^2 \psi + 4 \cosh^2 \psi - 1] \right\} \ , \tag{7.54}
\]

and equations \((7.49)\) and \((7.50)\) imply

\[
\mu_2 = \frac{1}{\gamma_2\pi \cosh^2 \psi} \quad \text{and} \quad \frac{1}{\cosh^2 \psi} = 1 - q^2 \pi^2 \ , \tag{7.55}
\]

whence

\[
1 - \frac{\gamma_1}{\gamma_2} = \pi \left\{ p + (1 - \gamma_1)\mu - q^2 \pi \left[ \frac{1}{3} - \gamma_1 + \frac{\gamma_1}{\gamma_2} \right] \right\} \ . \tag{7.56}
\]

(iii) **Constant tilt angle.** If \( \psi = \psi_0 = \text{constant} \) then from \((7.50)\) we have that

\[
q\pi = \frac{\sinh \psi_0}{\cosh \psi_0} = \alpha \ ; \quad \alpha \ \text{const} \ , \tag{7.57}
\]

whence from \((7.22)\) and \((7.23)\) we have that

\[
\pi_{xx} = \frac{4\alpha}{3} \partial_t \partial_x (\ln H) \ , \tag{7.58}
\]

which is similar to the expression given by \((7.31)\) (with \( H \) ‘replaced by \( \alpha \')). In particular, \( \pi_{xx} \) is a function of \( t \) and \( x \) only. Therefore, \( \phi \) again satisfies the differential equation \((7.32)\) for a constant \( k \), and is consequently given by equation \((7.33)\) [that is, \( \phi^{-1} = 1 + \frac{k}{4}(y^2 + z^2) \)], and now \( H \) and \( f \) satisfy an equation identical to \((7.34)\) except that \( H \) is ‘replaced by \( \alpha \’ \) in the first term on the left hand-side [that is, the first term on the left hand-side of equation \((7.34)\) is now \( 2\alpha \partial_t (H^{-1}H_x) \)].

(iv) **Perfect fluid plus a scalar field.** The energy-momentum tensor of a scalar field is formally equivalent to that of a perfect fluid with

\[
\mu_\phi = \frac{1}{2} (\nabla \phi)^2 + V(\phi) \ , \tag{7.59}
p_\phi = \frac{1}{2} (\nabla \phi)^2 - V(\phi) \ , \tag{7.60}
\]
where $V$ is the potential of the scalar field. Therefore, a perfect fluid plus scalar field source can be formally treated as a two-perfect-fluid mixture (with separate energy conservation). The field $\phi$ satisfies a (Klein-Gordon) differential equation; therefore, for a specific $V(\phi)$ we will obtain a relation (effective ‘equation of state’) between $\mu_\phi$ and $p_\phi$.

(v) Single tilting perfect fluid. Here we assume that

$$\mu_1 = p_1 = 0 \ ,$$

so that the energy-momentum tensor represents a single perfect fluid whose four-velocity is tilting with respect to the shear-free, irrotational and geodesic timelike congruence (and hence the results at the end of section 3 do not apply).

From (7.42) and (7.43), using (7.61), we have that

$$\mu + p = \frac{1}{3}(\mu_2 + p_2)\left[3\cosh^2\psi + \sinh^2\psi\right] \ ,$$

whence from (7.48) and (7.49) we obtain

$$q = 3(\mu + p)\cosh\psi\sinh\psi$$

and hence, we have the identity

$$q^2 = \pi q^2 \left[\left(\mu + p\right) - \frac{1}{3}\pi q^2\right] \ .$$

Now, from (2.31) and (2.32) we obtain [for metric (7.13)]

$$\mu + p = -2H^{-1}H_t + 2H^{-2}H_t^2 + \frac{2}{3}H^{-2} \left\{ -2H^{-1}H_{xx} + H^{-2}H_x^2 \right\}$$

where

$$\Phi(y,z) \equiv -\phi^{-3}(\phi_{yy} + \phi_{zz}) + \phi^{-4}(\phi_y^2 + \phi_z^2) \ .$$

Using equations (7.20)–(7.23) and (7.66), we see that equation (7.63) is of the form

$$a(t,x)\Phi^2(y,z) + b(t,x)\Phi(y,z) + c(t,x) = 0 \ ,$$

and hence $\Phi(y,z) = k$, a constant; consequently $\phi$ again satisfies the differential equation (7.32) and is thus given by equation (7.33) [that is, $\phi^{-1} = 1 + \frac{k}{4}(y^2 + z^2)$].

Finally, using (7.22) and (7.23), (7.63) becomes

$$\left[\frac{H_{xt}H_x - H_x H_{tx}}{H^2}\right]^2 = \frac{3}{8}\pi_{xx} \left[\mu + p - \frac{1}{2}\pi_{xx}\right] \ ,$$

where $(\mu + p)$ is given by (7.66), $\pi_{xx}$ is given by (7.20) and (7.21), and where $\Phi(y,z) = k$. Equation (7.63) is a differential equation for $H(t,x)$ and $f(x)$.

Similar to the case of a perfect fluid with pure radiation, it is possible to find special solutions to the differential equation (7.63). If $f = 1, k = 0$ and $H = H(\tau)$, where $\tau = x + \alpha t$ ($\alpha$ constant), then (7.63)
becomes a quadratic equation in \( H''/(H')^2 \) (\( H' \equiv \frac{dH}{d\tau} \)):

\[
H^2 \left\{ (1 + 6\alpha^2) H^2 - 4 \right\} \left[ \frac{H''}{(H')^2} \right]^2 + 2H \left\{ (2 - 15\alpha^2) H^2 - 3 \right\} \left[ \frac{H''}{(H')^2} \right] + 4 \left\{ (1 + 6\alpha^2) H^2 + 1 \right\} = 0 .
\] (7.70)

This quadratic equation can be solved and the solution for \( H(\tau) \) can be formally obtained. As an illustration, for the specific choice of \( \alpha^2 = \frac{4}{3} \),

\[
\frac{H''}{(H')^2} = \frac{18 H^2 + 3 - \sqrt{216 H^2 + 25}}{H (9 H^2 - 4)}
\] (7.71)
is one of the solutions of the quadratic equation (7.70). Equation (7.71) can be integrated to yield

\[
H' = c \sqrt{\frac{|9 H^2 - 4|}{H^2}} \frac{5 \sqrt{216 H^2 + 25} - 25 - 108 H^2}{\sqrt{11 \sqrt{216 H^2 + 25} - 73 - 108 H^2}} ,
\] (7.72)

where \( c \) is an arbitrary constant. Thus, the line element

\[
ds^2 = \frac{3}{4} \left( \frac{dH}{H'} - dx \right)^2 + H^2 (dx^2 + dy^2 + dz^2) ,
\] (7.73)

where \( H' \) is given by (7.72), is an example of a model with a single tilting perfect fluid where the fluid is tilting with respect to a shear-free, irrotational, and geodesic timelike congruence.

Once \( \phi, f \) and \( H \) are known, the scalar \( \mu_2 \) and \( p_2 \) can be calculated from

\[
\mu_2 = \mu - \frac{3}{2} \pi_{xx} ,
\] (7.74)

\[
p_2 = \frac{2q^2}{3\pi_{xx}} - \mu ,
\] (7.75)

where \( \mu, \pi_{xx} \) and \( q \) are given by equations (2.31), (7.21) and (7.23), respectively.

8. Discussion

In this paper, we have considered spacetimes, with a general energy-momentum tensor [formally decomposed with respect to \( u^a \) according to (2.2)–(2.7)], in which the shear, vorticity and acceleration of a timelike congruence \( u^a \) are all zero. In such spacetimes, it is always possible to choose a synchronous coordinate system with \( u^a = \delta^a_t \) [see (2.19)–(2.20)] such that all dynamical information is encoded in a single scalar function \( H(t, x^a) \) and such that the geometry of each of the spacelike hypersurfaces \( t = \text{const} \) is encoded (up to a conformal factor) in the positive-definite three-metric \( h_{\alpha\beta}(x^a) \).

In section 3, spacetimes with zero anisotropic stress (with respect to \( u^a \)) were considered. Each three-space (of constant \( t \)) was shown to be a space of constant curvature. However, the curvature was not necessarily the same on all the \( t = \text{const} \) hypersurfaces. In section 4, spacetimes with zero heat flux vector (with respect to \( u^a \)) were considered. The scalar function \( H \) was demonstrated to be independent of the spatial coordinates [that is, \( H = H(t) \)], and the anisotropic stress tensor was shown to be given entirely in terms of \( h_{\alpha\beta}(x^a) \) through (4.1). When the energy density and isotropic pressure satisfy a barotropic equation of state, \( p = p(\mu) \), we saw that (except in the special case \( \frac{dp}{d\mu} = -\frac{1}{3} \)) the Ricci scalar of the
three-surface [with metric \( h_{\alpha\beta}(x^\gamma) \)] is constant, whence the anisotropic stress-tensor is ‘divergence-free’ (all of the Bianchi and Kantowski-Sachs models admit subcases which fall into this category). In section 5, an anisotropic fluid source, (7.1), was studied. These models were shown to be a subset of the models discussed in section 4. The fluid’s anisotropic stress tensor was found to possess two distinct eigenvalues (and hence the spacetime is of Petrov type D).

In section 6, two physically relevant energy-momentum tensors were considered, which were sub-cases of models dealt with in earlier sections. First, a viscous fluid with heat conduction satisfying the phenomenological equations (6.1)–(6.3) was considered, a subcase of the models with zero anisotropic stress tensor. In the case that the temperature depends only on the energy density, \( T = T(\mu) \), the various governing equations and thermodynamical laws were found to imply a number of constraints; in particular, we were able to ‘integrate’ these conditions for the spherical symmetric case, (6.13)–(6.19). Second, a perfect fluid plus a magnetic field with energy-momentum tensor given by (6.20) and (6.21), which is formally equivalent to an anisotropic fluid, was considered. For this case, the Maxwell equations were shown to imply that the expansion must be zero and hence, the magnetic field must be static.

Finally, in section 7, a number of physically relevant cases were investigated which can be considered (or grouped together) as two component fluids in which the stress-energy tensor is given by (7.4) with (7.1) (representing a perfect fluid whose flow forms a SIG timelike congruence) and one of (7.2) or (7.3) [representing a second (tilting) fluid]. The stress-energy tensor is formally equivalent to that of a single fluid with a non-zero heat flux and a non-zero anisotropic stress tensor that is directly related to the heat flux via (7.5). In this particular case, coordinates can be chosen such that the metric has the simplified form (7.14) and only depends on three arbitrary functions, \( H(t,x) \), \( f(x) \) and \( \phi(y,z) \). A number of special cases were then considered. First, the ‘second fluid’ was taken to be pure radiation, (7.3), so that the total stress-energy tensor is due to a mixture of a perfect fluid and pure radiation. In this case, the function \( \phi(y,z) \) is given by equation (7.33), and the functions \( H(t,x) \) and \( f(x) \) must satisfy the differential equation (7.34). In the second case, the source was taken to be two non-collinear perfect fluids, (7.41). The various quantities in the formal expression for the stress-energy tensor were then found to be given by equations (7.42), (7.43), (7.48) and (7.49). These four equations relate five physical quantities to the metric functions \( H, f \) and \( \phi \), and their derivatives. Consequently, no further progress could be made until additional conditions were imposed on the five physical quantities. A number of conditions were then considered. The case of a single perfect fluid tilting with respect to the SIG timelike congruence is of special interest. In particular, we note that a general relativistic spacetime that admits a SIG timelike congruence and whose stress-energy tensor is a (single) perfect fluid need not be FRW. The metric given by (7.73) represents a specific example of such a spacetime. However, the spacetime is an FRW model when the perfect fluid’s flow lines form a SIG timelike congruence.

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