GROUND RING FOR THE 2D NSR STRING

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Abstract

We discuss the BSRT quantization of 2D $N = 1$ supergravity coupled to superconformal matter with $\hat{c} \leq 1$ in the conformal gauge. The physical states are computed as BRST cohomology. In particular, we consider the ring structure and associated symmetry algebra for the 2D superstring ($\hat{c} = 1$).

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1. Introduction

One success of the continuum approach to 2D gravity [1,2] (for a review, see [3–5] and references therein) has been the computation of the physical states as BRST cohomology classes [6–8], at least for physical conditions under which the worldsheet cosmological constant may be set to zero. In particular, for the $c = 1$ matter coupling, the BRST analysis constructs in detail the infinite set of “discrete states” discovered by matrix model calculations and continuum calculations of tachyon scattering amplitudes. The emergence of the interpretation of the Liouville mode as the “time” coordinate of the embedding space – and thus the identification of the model as two dimensional string theory – is particularly evident in the continuum approach. More recently there has been some suggestion [9,10] that the model should be interpreted in terms of a three dimensional theory.

In [9] this suggestion was based on the observation that the discrete states at ghost number $-1$ provide a ring of operators, in cohomology, which is characteristic of the model. This ring can be identified as the polynomial ring generated by two elements. Further, the cohomology partners at ghost number 0 give rise to spin 1 currents which act as derivations of this ring. The algebra of charges is just $\mathcal{W}_\infty$ for one chiral sector. When left and right moving sectors are combined, the total ring – the “ground ring” – is generated by four elements with one constraint, and thus defines a three dimensional space. All this was discussed in [9], together with a concrete identification with matrix model results.

Soon after the conformal gauge quantization of 2D gravity models was understood [11,12], the same ansatz was applied to 2D supergravity coupled to superconformal matter with $\tilde{c} \leq 1$ ($\tilde{c} = \frac{2}{3}c$) [13,14]. As models of string theory, the 2D NSR strings which arise for $\tilde{c} = 1$ are naturally of great importance. There is apparently no matrix model formulation available, which makes application of the continuum approach extremely relevant in this case. The BRST analysis of the physical spectrum was recently carried out [15], and also discussed in [16] (which contains the further projection onto $\tilde{c} < 1$ minimal models as well).

For a given $N = 1$ super Virasoro module $\mathcal{V}$, the constraints $T(z) \sim 0, G(z) \sim 0$ (from coupling to supergravity) can be implemented by the BRST operator

$$d = \oint \frac{dz}{2\pi i} : (c(z)(T(z) + \frac{1}{2}T^G(z)) + \gamma(z)(G(z) + \frac{1}{2}G^G(z))) : , \quad (1.1)$$

acting on the tensor product module $\mathcal{V} \otimes \mathcal{F}^G$. The operators $T^G(z)$ and $G^G(z)$ generate the $N = 1$ superconformal algebra on $\mathcal{F}^G$, which is the tensor product of the Fock space of the spin $(2,-1)$ $bc$-ghosts with that of the spin $(\frac{3}{2},-\frac{1}{2}) \beta\gamma$-ghosts. The BRST operator
is nilpotent provided the central charge of $\mathcal{V}$ is equal to 10 [17–19]. For the 2D NSR string the super Liouville system consists of a free fermion together with a free scalar with background charge, and $\mathcal{V}$ is the product of the corresponding Fock spaces with those of the free matter system at $\hat{c} = 1$. More generally, the computation can be done for the case that both systems have a background charge. The minimal model matter coupling is then obtained by projection.

In this paper we continue this program by discussing the structure of the cohomology in the 2D NSR strings. That is, we determine the ring structure, and the corresponding symmetry algebra of currents, for the “critical” case in which the cosmological constant vanishes. We follow the approach in [9], and in fact the chiral structure turns out to be almost precisely the same as the bosonic case discussed there. Although, as we show, this could be anticipated by a “kinematic analysis,” a detailed calculation is actually required to establish the result. The ground ring structure depends on how the left and right moving sectors are joined. For the NSR string we can simply proceed with the full theory, and thus produce a 2D NSR closed string with the same ground ring structure and symmetry as the 2D bosonic string. Alternatively we may enforce a consistent GSO-type projection of the model [20], which gives a restriction of the ground ring and its symmetries.

These results are presented as follows. In Section 2 we summarize and further discuss the BRST cohomology computed for 2D world sheet $N = 1$ supergravity coupled to a free supermultiplet with background charge. This contains directly the results required for the 2D NSR string, which are then applied in Section 3 to obtain the ring structure and the algebra of charges for one chiral sector – both for the 2D NSR string, and its GSO projection. Putting together both left and right movers in Section 4, we derive the ground rings of these models and discuss their corresponding symmetry algebras. We have gathered all conventions and notations used herein into Appendix A. Further, we review in Appendix B the details of the computation of the relevant BRST cohomology, both the relative and absolute cohomologies for both NS and R sectors. For convenience, Appendix C contains various details about the bosonization required for writing the explicit representatives used in the text. Finally, in Appendix D, we present the generalization to the superconformal case of results on the structure of the $c = 1$ (now $\hat{c} = 1$) Fock space – in particular we introduce “super-Schur polynomials,” in terms of which expressions for the singular vectors may be given.
2. Summary of the BRST cohomology calculation

In this section we summarize the cohomology of the BRST operator \(d\) given in (1.1), where the module \(\mathcal{V} = \mathcal{F}(p^M, p^L)\) is taken as the tensor product of Fock spaces \(\mathcal{F}(p^M, Q^M)\) and \(\mathcal{F}(p^L, Q^L)\) corresponding to two scalar supermultiplets \((\phi^M, \psi^M)\) and \((\phi^L, \psi^L)\) with background charges \(Q^M\) and \(Q^L\), respectively. The condition of nilpotency then implies that \(\frac{1}{2}(Q^M)^2 + (Q^L)^2 = -1\). (2.1)

The resulting cohomology spaces will be denoted \(H^{(n)}_{\text{abs}}(\mathcal{F}(p^M, p^L), d)\). A summary of notations and conventions is given in Appendix A, and for completeness we include technical details of the computations in Appendix B.

The BRST charge \(d\) decomposes under ghost zero modes in the two sectors as

\[
\begin{align*}
\text{NS :} & \quad d = L_0 c_0 - M b_0 + \hat{d}, \quad (2.2) \\
\text{R :} & \quad d = L_0 c_0 - (M + \gamma_0\gamma_0) b_0 + \overline{d} \\
& \quad = L_0 c_0 - (M + \gamma_0\gamma_0) b_0 + (F\gamma_0 + N\beta_0 + \hat{d}). \quad (2.3)
\end{align*}
\]

where none of the operators (besides \(\overline{d}\)) in the expansion contains ghost zero modes. We have \(\{d, b_0\} = L_0 = L_0^M + L_0^L + L_0^G\) and \([d, \beta_0] = F - 2\gamma_0 b_0 = G_0 = G_0^M + G_0^L + G_0^G\).

The oscillator expressions for these operators can be read off from the formulae given in Appendix A, and the nontrivial (anti-) commutators between them, following from \(d^2 = 0\), are

\[
\begin{align*}
\text{NS :} & \quad \hat{d}^2 = ML_0, \quad (2.4) \\
\text{R :} & \quad FF = L_0, \quad [F, M] = 2N, \quad \hat{d}^2 = ML_0 + NF. \quad (2.5)
\end{align*}
\]

The computation of the BRST cohomology proceeds in two steps. First one determines the relative cohomology, denoted \(H^{(n)}_{\text{rel}}(\mathcal{F}(p^M, p^L), d)\), which is the cohomology of \(d\) on the subspace for which \(d\) reduces to \(\hat{d}\). For NS this subspace is (as for the bosonic case)

\[
\mathcal{F}_{\text{rel}}(p^M, p^L) \equiv \mathcal{F}(p^M, p^L) \cap \text{Ker } L_0 \cap \text{Ker } b_0, \quad (2.6)
\]

whilst for R we must restrict to

\[
\mathcal{F}_{\text{rel}}(p^M, p^L) \equiv \mathcal{F}(p^M, p^L) \cap \text{Ker } G_0 \cap \text{Ker } b_0 \cap \text{Ker } \beta_0. \quad (2.7)
\]

\footnote{Note that in our conventions both \(p^L\) and \(Q^L\) are purely imaginary.}
In the R sector we will also distinguish additional subspaces of $\mathcal{F}(p^M, p^L)$, namely

$$\mathcal{K}(p^M, p^L) \equiv \mathcal{F}(p^M, p^L) \cap \text{Ker } L_0 \cap \text{Ker } b_0,$$

$$\mathcal{F}^L_0(p^M, p^L) \equiv \mathcal{K}(p^M, p^L) \cap \text{Ker } \beta_0.$$  \hspace{1cm} (2.8)

(2.9)

Note that $\overrightarrow{d}$ (defined in (2.3) ) is nilpotent on $\mathcal{K}(p^M, p^L)$.  

The resulting relative cohomology is summarized as follows: Parametrize $(p^M, p^L)$ by $r, s \in \mathcal{C}$

$$p^M - Q^M = \sqrt{\frac{1}{2}}(r \alpha_+ + s \alpha_-),$$

$$i(p^L - Q^L) = \sqrt{\frac{1}{2}}(r \alpha_+ - s \alpha_-),$$

where

$$\alpha_\pm = \sqrt{\frac{1}{2}}(Q^M \pm iQ^L).$$  \hspace{1cm} (2.10)

Then $H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d)$ is nontrivial only in the following four cases $[15,16]$

(i) If both $r = s = 0$ (i.e. $p^M = Q^M$ and $ip^L = iQ^L$), then

$$H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d) = \begin{cases} \mathcal{C} \oplus \mathcal{C} & \text{if } n = 0 \text{ and } \kappa = 0, \\ \mathcal{C} & \text{if } n = 0 \text{ and } \kappa = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If either $r = 0$ or $s = 0$ (i.e. $\Delta(p^M) + \Delta(p^L) = \frac{1}{2}$), then

$$H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d) = \begin{cases} \mathcal{C} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $r, s \in \mathbb{Z}_+, r - s \in 2\mathbb{Z} + (1 - 2\kappa)$, then

$$H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d) = \begin{cases} \mathcal{C} & \text{if } n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) If $r, s \in \mathbb{Z}_-, r - s \in 2\mathbb{Z} + (1 - 2\kappa)$, then

$$H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d) = \begin{cases} \mathcal{C} & \text{if } n = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the “discrete states” arise only if both $\mathcal{F}(p^M, Q^M)$ and $\mathcal{F}(p^L, Q^L)$ are reducible. They occur at the same level, namely $\frac{r-s}{2}$, as the null vectors in these modules.

Given this result for $H_{rel}^{(n)}(\mathcal{F}(p^M, p^L), d)$, the full cohomology may be determined. Indeed, for the NS case, arguments as in [8] show that the cohomology is simply doubled
due to the ghost zero mode $c_0$. More precisely, each relative cohomology state $\psi$ gives rise to two absolute cohomology states,

$$\psi^{(1)} = \psi \quad \psi^{(2)} = c_0 \psi + \phi,$$

where $\phi$ is a solution (which always exists) to the equation $M\psi = \hat{d}\phi$.

In the R sector, as in the ten dimensional NSR string [21–23], the situation is more complicated but the final result is the same – we just have a doubling of states. For the exceptional case, when $p^+ = p^- = 0$, the most convenient approach is simply to enumerate all the states annihilated by $L_0$ and find the explicit form of the BRST operator acting on them. In the NS sector we verify immediately that the two states, which are also the absolute cohomology representatives, are $|Q^M, iQ^L; -1\rangle$ and $c_0|Q^M, iQ^L; -1\rangle$. In the R sector, such states are linear combinations of the following basis states

$$(\gamma_0)^{m_+}|+\rangle, \ (\gamma_0)^{m_-}|-\rangle, \ (\gamma_0)^{n_+}c_0|+\rangle, \ (\gamma_0)^{n_-}c_0|-\rangle,$$

where $m_\pm, n_\pm \geq 0$, and $|+\rangle$ and $|-\rangle$ denote the two degenerate vacua $|Q^M, iQ^L; -\frac{1}{2}\rangle$ and $\psi_0^+|Q^M, iQ^L; -\frac{1}{2}\rangle$, respectively. Since for $p^+ = p^- = 0$ the fermion zero mode terms are absent in $G_0$, these two vacua span the relative cohomology. The full BRST operator acting on the states in (2.13) is simply $d = -\gamma_0\gamma_0b_0$, so one easily sees that the absolute cohomology is doubly degenerate at ghost number 0 and 1, and is generated by $(\gamma_0)^{m_\pm}|\pm\rangle$ with $m_\pm = 0, 1$.

For all other cases, the reason that there is no infinite degeneracy due to the bosonic zero mode $\gamma_0$ goes back to the simple fact that in $F^{L_0}(p^M, p^L)$, any state in Ker $G_0$ is in fact in Im $G_0$. [Note that on this space $G_0$ and $F$ coincide.] The complete calculation of this result is deferred to Appendix B, here we simply outline the procedure. The two absolute cohomology states corresponding to a given relative cohomology state $\psi_0$ are: $\psi^{(1)} = \psi_0$ and $\psi^{(2)} = \phi_0 + \gamma_0\phi_1 + c_0\psi_0$, where $\phi_0, \phi_1 \in F^{L_0}(p^M, p^L)$ are solutions to the following system of equations

$$G_0\phi_1 - \psi_0 = 0,$$
$$G_0\phi_0 + \hat{d}\phi_1 = 0,$$
$$\hat{d}\phi_0 - N\phi_1 - M\psi_0 = 0,$$

equivalent to $d\psi^{(2)} = 0$. Let us verify that this system has a solution for any relative cohomology state $\psi_0$. Since $G_0\psi_0 = 0$, we can always solve the first one and determine
Then $G_0 \hat{\phi}_1 = 0$, and we may determine $\phi_0$ using the second equation. In fact both $\phi_0$ and $\phi_1$ are determined only up to $G_0$ exact terms. Using this freedom, we may rewrite the third equation as the statement that there exists $\rho \in \mathcal{F}_{rel}(p^M, p^L)$ such that
\[ \hat{d}\phi_0 - N\phi_1 - M\psi_0 = \hat{d}\rho. \quad (2.15) \]

Now, using the first two equations in (2.14) and identities in (2.3) we verify that the l.h.s. is annihilated both by $G_0$ and $\hat{d}$, and thus is an element of relative cohomology. However, since there can be no relative cohomology at ghost number $(\text{gh})(\psi_0) + 2$, we conclude that a $\rho$ satisfying (2.15) must exist.

Of course, a more detailed analysis is still required to show that the states $\psi^{(1)}$ and $\psi^{(2)}$ are indeed nontrivial and that linear combinations of such states exhaust all of the absolute cohomology. This is discussed in detail in Appendix B.

Some comments are in order.

1. The above results give the cohomology in a particular picture for the $\beta\gamma$ system; namely $Q_{3/2} = -\frac{1}{2} - \kappa$. However, as was realized in [24] and clearly explained in [25], there exists isomorphic copies in any picture differing by integer charge. The isomorphism is given explicitly by repeated action of the zero mode $X_0$ of the picture changing operator, and the zero mode $Y_0$ which is the inverse to $X_0$ in cohomology. This is not generally an isomorphism in relative cohomology; although it is true that $[b_0, X_0] = 0$, this is not so for $Y_0$. The picture changing isomorphism will be essential in Section 3.

2. The projection to the $\hat{c} < 1$ superconformal minimal models is obtained via an appropriate free field resolution of the irreducible representations, as detailed in [6,8,26,16].

3. It is well known that the gh = 0 representatives for the 2D bosonic string can be written in the form of “$c = 1$ singular vector $\times$ Liouville vacuum.” The singular vectors, in turn, have known expressions in terms of Schur polynomials. The analogous result holds for the 2D NSR string, but now in terms of “super” Schur polynomials. We detail the construction in Appendix D.

### 3. The chiral structure of the $\hat{c} = 1$ model

For the remainder of the paper we will restrict our attention to the case $\hat{c} = 1$, where $iQ^L = 1$ and thus $\alpha_{\pm} = \pm 1/\sqrt{2}$, which we may think of as the 2D NSR string. We will exploit the analysis of Section 2, together with standard results of conformal field theory, to obtain some insight into the structure of this model.
It was recently established for the 2D bosonic string that the operator cohomology contains a natural ring structure, and that relative cohomology states at $gh = 0$ give rise to a symmetry algebra which acts as derivations of the ring [9]. Specifically, to a given $\psi \in H^{(-1)}_{\text{abs}}(\mathcal{F}(p^M, p^L), d) \simeq H^{(-1)}_{\text{rel}}(\mathcal{F}(p^M, p^L), d)$ we may associate an $L_0$-weight zero operator $\Psi(z)$ such that

$$\psi = \lim_{z \to 0} \Psi(z)|0\rangle,$$

(3.1)

where $|0\rangle$ denotes the $sl(2, \mathbb{R})$ vacuum of the model. Given two such, $\Psi_1(z)$ and $\Psi_2(z)$, the product

$$(\Psi_1 \Psi_2)(z) = \frac{1}{2\pi i} \oint_z \frac{dx}{x-z} \Psi_1(x) \Psi_2(z)$$

(3.2)

is commutative and associative for the states of the bosonic string – indeed all singular terms in the OPE are $d$-trivial by standard arguments [9]. Further, from a representative $\psi \in H^{(0)}_{\text{rel}}(\mathcal{F}(p^M, p^L), d)$ we may derive a spin one current; it is just defined as that operator $J(z)$ associated to $b_{-1}\psi$. We write

$$J(z) = (b_{-1}\Psi)(z).$$

(3.3)

The corresponding charge is BRST invariant. Amongst other things, it was shown in [9] that the ring can be identified with the polynomial ring generated by two elements, $x$ and $y$ say, on which the charges are represented as vector fields. The problem was greatly simplified by the fact that two of the symmetry charges act as $\partial_x$ and $\partial_y$.

This product, and the definition of symmetry current, may be immediately taken over to the NSR string, and we will soon consider just how much of the other structure does also. First, however, we may expect to gain some understanding by a simple examination of the “kinematics” involved. From the results of Section 2 we see that the representatives of $H^{(-1)}_{\text{abs}}(\mathcal{F}(p^M, p^L), d)$ are parametrized by two negative integers. The element $\psi_{(r,s)}$, at level $\frac{r^2}{2}$, has momenta

$$[p^M, ip^L] = \left[\frac{1}{2}(r-s), \frac{1}{2}(r+s+2)\right].$$

(3.4)

Half the states, those with $r - s \in 2\mathbb{Z}$, appear in the NS sector, while those with $r - s \in 2\mathbb{Z} + 1$ come from the R sector. By adding the momenta, we clearly find

$$(\psi_{(r,s)} \psi_{(r',s')})(z) \sim \psi_{(r+r'+2, s+s'+2)}(z),$$

(3.5)

where $\sim$ emphasizes that the identification on the r.h.s. is forced, but only once it is established that the r.h.s. is nonzero. It is now clear that, with this proviso, the entire
chiral ring is generated by the operators \( x \equiv \Psi_{(-1,-2)} \) and \( y \equiv \Psi_{(-2,-1)} \), with momenta \( [\frac{1}{2}, -\frac{1}{2}] \) and \( [-\frac{1}{2}, -\frac{1}{2}] \) respectively, which arise in the R sector. For

\[
\Psi_{(r,s)} \sim x^{-s-1}y^{-r-1}, \quad r, s < 0, \tag{3.6}
\]

and thus “kinematically” one obtains all possibilities.

Representatives of \( H^{(0)}_{rel}(\mathcal{F}(p^M,p^L),d) \) are parametrized by two numbers \( r, s \in \mathcal{C} \), with either \( r = 0, s = 0 \) or both \( r, s \in \mathbb{Z} \) with \( rs > 0 \) (see section 2). We will restrict our attention to the subset parametrized by integers \( r, s \leq 0 \), since the remainder of the associated currents \( (3.3) \) will act trivially on the ring, as follows (a posteriori) by kinematical reasoning as in [9]. From these states \( \psi_{(r,s)} \), we construct the spin 1 current \( J_{(r,s)} \) as in \( (3.3) \). The kinematics of these operators is exactly as in equation \( (3.5) \) above, so we may, for example, immediately deduce that the only candidate for \( \partial_x \) is \( \partial_x \sim J_{(-1,0)} \). For, it is the only current which acts on \( x \) to produce a state with the momenta \([0,0]\) of the identity operator. Similarly we have the candidate \( \partial_y \sim J_{(0,-1)} \). So far, the combined NS and R structure does not look much different from the one that occurred in the bosonic case [9].

To make more concrete statements requires some calculation, but first we should clear up several points which may be troubling the reader at this stage. The cohomology calculation in Section 2 was done in a particular picture, and thus a more precise notation for the ring elements and currents above is \( \Psi_{(r,s)}^{(q)} \) and \( J_{(r,s)}^{(q)} \), where \( q = -\frac{1}{2} - \kappa, \kappa = 0, \frac{1}{2} \) depending on which sector the operators arise from. However, now the result of the product of operators from pictures \(-\frac{1}{2} - \kappa_1 \) and \(-\frac{1}{2} - \kappa_2 \) will be an operator in the \((-1 - \kappa_1 - \kappa_2)\) picture. For \( \kappa_1 = \kappa_2 = 0 \) we straightforwardly have the product \( R \times R \rightarrow NS \), but otherwise we go outside the set \( q = -\frac{1}{2}, -1 \). Fortunately this is not a problem, since – as emphasized in Section 2 – picture changing provides an isomorphism in cohomology. By standard contour deformation arguments one can verify that picture changing is, in fact, a ring isomorphism. Thus we may simply identify all states which are related by picture changing, and the operator cohomology still defines a ring modulo this identification. The remaining products are then \( NS \times NS \rightarrow NS \), \( NS \times R \rightarrow R \), and \( R \times NS \rightarrow R \). In the above notation we easily find for example
\[
\psi^{(-\frac{1}{2})}_{(-1, -2)} = \left( b_{-1} - \sqrt{\frac{1}{2}} \beta_{-1} \psi^+_0 \right) | \frac{1}{2}, -\frac{1}{2}; -\frac{1}{2} \rangle,
\]
\[
\psi^{(-\frac{1}{2})}_{(-2, -1)} = \left( \beta_{-1} + \sqrt{2} b_{-1} \psi^+_0 \right) | -\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2} \rangle,
\]
\[
\psi^{(-1)}_{(-1, -3)} = \left( \alpha_{-1} \beta_{-\frac{1}{2}} + b_{-1} \psi_{-\frac{1}{2}} - \sqrt{2} \beta_{-\frac{3}{2}} \right) | 1, -1; 1 \rangle,
\]
\[
\psi^{(-1)}_{(-3, -1)} = \left( \alpha_{-1} \beta_{-\frac{1}{2}} + b_{-1} \psi_{-\frac{1}{2}} + \sqrt{2} \beta_{-\frac{3}{2}} \right) | -1, -1; 1 \rangle,
\]

and upon bosonizing we have
\[
\begin{align*}
x &= [S^(-\frac{1}{2}) - \frac{1}{\sqrt{2}} S \partial \xi e^{-\frac{3}{2}i\phi} c] e^{\frac{1}{2}(\phi M + i\phi L)}, \\
y &= [S \partial \xi e^{-\frac{3}{2}i\phi} c - \sqrt{2} S^1 e^{-\frac{1}{2}i\phi}] e^{\frac{1}{2}(\phi M - i\phi L)}, \\
\Psi^{(-1)}_{(-1, -3)} &= \left( -c \sqrt{2} \partial^2 \xi e^{-2i\phi} - c i \partial \phi \partial \xi e^{-2i\phi} + \psi e^{-i\phi} + i \partial \phi \partial \xi e^{-2i\phi} c \right) e^{i(\phi M + i\phi L)}, \\
\Psi^{(-1)}_{(-3, -1)} &= \left( c \sqrt{2} \partial^2 \xi e^{-2i\phi} + c i \partial \phi \partial \xi e^{-2i\phi} + \psi e^{-i\phi} + i \partial \phi \partial \xi e^{-2i\phi} c \right) e^{-i(\phi M - i\phi L)}.
\end{align*}
\]

This still leaves a question for the currents, however, since these have been defined via the action of \( b_{-1} \) in a given picture. We may define the currents in the \( q \) picture by
\[
J^{(q)}_{(r, s)}(\Psi^{(q')}) = J^{(q-1)}_{(r, s)}(\Psi^{(q'+1)}),
\]
where \( \Psi^{(q)}_{(r, s)} \) corresponds to a state in \( H^{(0)}(\mathcal{F}(p^M, p^L), d) \) picture changed to the \( q \) picture as discussed above. To see that this is sensible, it is sufficient to note the identity
\[
\oint dz (b_{-1}(X_0 \Psi^{(q-1)}_{(u, v)})) (z) (\Psi^{(q')}) (0) = \oint dz (b_{-1} \Psi^{(q-1)}_{(r, s)}) (z) (X_0 \Psi^{(q')}) (0).
\]
So, in fact, we have
\[
J^{(q)}_{(r, s)} (z) = (b_{-1} \Psi^{(q)}_{(r, s)}) (z).
\]

A last comment along these lines is that, since it is always possible to choose a representative \( \psi^{(-1)} \) of \( H^{(0)}_{\text{rel}}(\mathcal{F}(p^M, p^L), d) \) which is a superconformal highest weight (and ghost vacuum) as we show in Appendix D, one might try to introduce spin 1 “superpartners” to the currents as, for example, \( \tilde{J} \equiv \beta^{-1/2} G_{-1/2} \psi \). This obeys all the required properties in the same way as the usual current. However, explicit computation shows that there are no new currents here.

With this in hand, we may construct some examples of the currents in this notation,
\[
\begin{align*}
J^{(-\frac{1}{2})}_{(-1, 0)} &= S^{-\frac{1}{2}} e^{-\frac{1}{2} \phi} e^{-\frac{1}{2}(\phi M + i\phi L)}, \\
J^{(-\frac{1}{2})}_{(0, -1)} &= S^{-\frac{1}{2}} e^{\frac{1}{2} \phi} e^{\frac{1}{2}(\phi M - i\phi L)},
\end{align*}
\]
\[
J_+(z) \equiv J_{(0,-2)}^0(z) = (\psi^M) \ e^{i\phi^M}, \\
J_0(z) \equiv J_{(-1,-1)}^0(z) = i\partial \phi^M, \\
J_-(z) \equiv J_{(-2,0)}^0(z) = (\psi^M) \ e^{-i\phi^M},
\]

(3.13)

Let us now consider how well the kinematical predictions are borne out by explicit computation. Most important is the identification of \(\partial_x\) and \(\partial_y\), which we can check from (3.12) and (3.8), e.g.

\[
J_{(-1,0)}^{(-1)}(z)x(w) \sim \frac{1}{(z-w)} \frac{1}{\sqrt{2}} \partial_w \xi e^{-2i\tilde{\phi}} c(w).
\]

(3.14)

The residue of the pole is easily identified as the picture changed representative of the identity in the \(q=-1\) picture. Similarly we may check that \(\partial_y\) is correctly identified. By this calculation we have verified (in exactly the same way as the analogous statement was proven in [9]) that indeed the cohomology ring is precisely the polynomial ring generated by the two elements \(x\) and \(y\)!

Thus we immediately determine that for \(r, s > 0\), \(J_{(r,s)}\) act trivially on the ring – since they map the generators to states which vanish in cohomology.

In fact, as perhaps one could also anticipate from the kinematic analysis, the chiral structure for the NSR string is almost exactly parallel to that of the bosonic string. In particular, the currents \(J_+, J_-\) and \(J_0\) defined in (3.13), with \(r+s = -2\), generate an \(sl(2)\) algebra under which \(x\) and \(y\) transform as a doublet. Indeed the complete spectrum may be decomposed under it (see also the discussion in Appendix D). States with the same Liouville momentum are in the same multiplet (from the kinematics (3.7), \((r' + s') \rightarrow (r' + s')\) when acted on by a current with \(r + s = -2\)). Thus the symmetry currents occur in \(sl(2)\) multiplets with spin \(j = |r + s|/2\), the highest weight in the multiplet always being one of the “discrete tachyon” vertices, \(J_{(0,-2)}\). The ring decomposes into \(sl(2)\) multiplets with spin \(j = (|r + s| - 2)/2\). One may now check that the arguments in [9] go through for the nontriviality of the action of \(J_{(r,s)}\), \(r, s \leq 0\), on the ring, and the identification of the symmetry algebra with \(W_\infty\). Indeed this action is just proportional to that of the area preserving polynomial vector fields on the \(x - y\) plane; i.e., given the “hamiltonian” \(h(x, y) = x^{-s}y^{-r}\), \(r, s \leq 0\), \(J_{(r,s)}\) acts on the ring as the vector field

\[
\partial_y h(x, y) \frac{\partial}{\partial x} - \partial_x h(x, y) \frac{\partial}{\partial y}.
\]

(3.15)

There are, however, two last remarks which should be made regarding this analysis of one chiral sector. The first is that in the above analysis we have made a particular choice
of cocycles for bosonizing the fields such that the ring product is commutative. In general it is “phase commutative,” in the sense that the opposite orders of product agree when a well-defined relative phase is included. We give a brief outline of this choice in Appendix C. One easily shows that associativity is still guaranteed, so we do not believe there is any physical distinction between different choices. In particular, one may show by direct calculation that \( x^2 \sim \Psi_{(-1,-3)} \) and \( y^2 \sim \Psi_{(-3,-1)} \), completely independent of the choice of cocycles.

The second remark is that there is a well-defined “GSO projection” from which an analogue of the superstring may be defined [20]. Motivated by the corresponding analysis of compactified critical NSR strings, this is imposed by restricting to the set of operators which are local with respect to the spacetime supersymmetry charge. For the 2D NSR string, because of the background charge required in the Liouville sector, spacetime translation invariance is ruined along with the usual supersymmetry algebra, and the analogue operator is the spin 1 current

\[
Q_S(z) = e^{-\frac{4i}{\beta}}S e^{i\phi^M}, \tag{3.16}
\]

which defines a nilpotent charge. [Note that this is obtained by acting with \( b_{-1} \) as before, but on the cohomology state in the Ramond sector with \( r = 0, s = -2 \).] However, it may still be used to make the projection [27], and the resulting projected ring is just the restriction to even powers of \( y \), with no restriction on the power of \( x \). The induced symmetry currents are those preserving this restriction. From (3.15) we are clearly left with those currents corresponding to hamiltonians which are odd powers of \( y \) – with again no restriction on the \( x \) power. All this may be summarized by introducing the variable \( z = y^2 \). The GSO projected chiral ring is just the polynomial ring generated by \( x \) and \( z \). The symmetry algebra of this ring is the subset of the area preserving vector fields on the \( x - y \) plane which map the GSO projected ring to itself. This constraint implies the hamiltonians of the subalgebra are just those of the form \( x^{-s}y^{-2r+1}, r, s \leq 0 \), and the corresponding vector fields on the \( x - z \) plane are obtained as in (3.15). One sees immediately that these are not area preserving maps of the \( x - z \) plane – or, more precisely, they preserve the area in the metric induced by the transformation \( x \to x, y \to z = y^2 \). It should also be noted that \( Q_S \) acts trivially on the projected ring, simply by kinematic reasoning, so there are no spacetime fermionic elements in this ring.
4. Conclusions: the ground rings and symmetries

For the 2D NSR closed string we must now join left and right sectors to obtain the physical theory. From the discussion of Section 3 we see that the analysis follows precisely that of the bosonic string given in [9]. For completeness we sketch those results in the present context. From the NSR string one can make another interesting string theory – the 2D superstring – by imposing the GSO projection. We will also discuss the structure of this theory.

The ground ring is obtained by putting together the left moving and right moving (distinguished by a ′) rings with the constraint that \( p^L = p^{L′} \). For the 2D NSR string one immediately finds that the ground ring – at the “critical” point where the cosmological constant vanishes – is just the ring of polynomial functions on the quadric cone \( Q \) defined by

\[
a_1 a_2 - a_3 a_4 = 0, \tag{4.1}
\]

where the generators are

\[
\begin{align*}
a_1 &= xx′, \\
a_2 &= yy′, \\
a_3 &= xy′, \\
a_4 &= yx′.
\end{align*} \tag{4.2}
\]

Note that we have allowed the matter scalar \( \phi^M \) to be compactified. For the uncompactified string we have the further constraint that \( p^M = p^{M′} \), and the ground ring reduces to that generated by just \( a_1 \) and \( a_2 \).

The quantum symmetry currents are the spin (1,0) and (0,1) operators with \( SL(2) \) spin

\[
\begin{align*}
J_{j,m,m′} &= J(-j-m,-j+m)Ψ(-j-m′,-j+m′), \\
J’_{j,m,m′} &= Ψ(-j-m,-j+m)J’(-j-m′,-j+m′),
\end{align*} \tag{4.3}
\]

where \(-j \leq m \leq j, -j + 1 \leq m′ \leq j - 1\). Again note that the constraint \( p^L = p^{L′} \) has been imposed. The resulting symmetry algebra is just the diffeomorphisms generated by the volume preserving polynomial vector fields on the quadric \( Q \). When the matter sector is uncompactified, we must further restrict to those currents which preserve the matching \( p^M = p^{M′} \). One is left with those area preserving vector fields on the \( a_1 - a_2 \) plane which preserve the locus \( a_1 a_2 = 0 \).
All this has simply been a copy of the bosonic string [9]. We may now project to
the 2D superstring by restricting to those operators which are local with respect to the
“spacetime supersymmetry” current, applied to each chiral sector. As we saw in Section
3, the chiral ring is then generated by \( x \) and \( z = y^2 \), and on putting them together we find
a polynomial ground ring generated by
\[
\begin{align*}
    b_1 &= xx', \\
    b_2 &= zz', \\
    b_3 &= x^2 z', \\
    b_4 &= zz'^2,
\end{align*}
\]
which satisfy the constraint
\[
    b_1^2 b_2 - b_3 b_4 = 0. \tag{4.5}
\]
One may check that the quantum symmetry algebra, obtained analogously to (4.3) from
the chiral algebra of the GSO projected chiral rings, acts on this new subspace as vector
fields.

There are some simple observations which can be made in conclusion. The “energy
operator” \( i \partial \phi^M \) is exactly the current \( J_0 \) of (3.13), so it appears that the arguments in [9]
do not characterize any difference between the matrix model description of the 2D bosonic
string and an analogous description in the NSR case – if it exists. In fact the real distinction
in BRST structure seems to only be in the doubling of relative cohomology when \( r \) or \( s \)
vanishes (in the notation of Section 2), where the constraint on the difference \( r - s \) is lifted.
It is worth noting that the energy operator clearly survives the GSO projection. This is
true even if the GSO projection is defined by one of the other Ramond operators with
one of \( r \) or \( s \) vanishing, and \( r - s \in 2\mathbb{Z} \). [The choice used in the present paper is most
natural within the class of extended models discussed in [20,27], where our analysis could
also be applied.] Finally, for the GSO projected ring, the symmetry charges act as vector
fields which must preserve the volume of the metric induced from that on the quadric,
\[
    \Theta = db_1 \frac{db_2}{\sqrt{b_2}} \frac{db_4}{b_4}. \tag{4.4}
\]
The significance of these remarks for understanding 2D superstrings is
yet to be determined.

Note added: While preparing this manuscript we received a paper by Kutasov, Martinec
and Seiberg [28], in which they discuss consequences of the existence of ring structure in
2D string models. They also argue that the ground ring of the uncompactified 2D NSR
string is as we determined above.
Appendix A. Notations and conventions

By scalar supermultiplet with background charge we mean the pair $\phi(z)$, a free scalar field with background charge, and $\psi(z)$, a free spin $\frac{1}{2}$ real fermion, with fundamental two-point functions

$$
\langle \phi(z)\phi(w) \rangle = -\ln(z-w), \quad \langle \psi(z)\psi(w) \rangle = \frac{1}{(z-w)}.
$$

(A.1)

We take the following conventions for the $N = 1$ superconformal currents

$$
T(z) = -\frac{1}{2} :\partial\phi\partial\phi : +iQ\partial^2\phi - \frac{1}{2}\psi\partial\psi,
$$

$$
G(z) = i\partial\phi\psi + 2\partial\psi,
$$

(A.2)

so that the central charge is given by

$$
\hat{c} = 2 \frac{c}{3} = 1 - 8Q^2.
$$

(A.3)

The half-integer-spin fields can be consistently subjected to different boundary conditions, leading to two distinct sectors which we parametrize by $\kappa = \frac{1}{2}$ (Neveu-Schwarz) and $\kappa = 0$ (Ramond). In terms of modes, $i\partial\phi(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$, $p = \alpha_0$, and $\psi(z) = \sum_{m\in\mathbb{Z}+\kappa} \psi_m z^{-m-1/2}$, and we have

$$
[\alpha_m, \alpha_n] = m\delta_{m+n,0}, \quad \{\psi_m, \psi_n\} = \delta_{m+n,0}.
$$

(A.4)

Throughout the paper operators are always normal ordered with respect to the $SL(2,\mathbb{R})$ vacuum (which is contained in the NS sector). We will denote the Fock space built on the vacuum state $|p\rangle$ with momentum $p$ by $\mathcal{F}(p)$. The conformal dimension of the corresponding Virasoro representation is

$$
\Delta(p) = \frac{1}{2}p(p-2Q) + \frac{1}{16}(1-2\kappa).
$$

(A.5)

In the text we distinguish between the Liouville and matter fields by writing superscripts $L$ and $M$ respectively. The total fermion number charge will be denoted $Q_{1/2}$.

The $N = 1$ superconformal algebra of the combined ghost system is generated by

$$
T^G(z) = :c(z)\partial b(z) + 2\partial c(z)b(z) - \frac{1}{2}\gamma(z)\partial\beta(z) - \frac{3}{2}\partial\gamma(z)\beta(z) :,
$$

$$
G^G(z) = -2b(z)\gamma(z) + c(z)\partial\beta(z) + \frac{3}{2}\partial c(z)\beta(z).
$$

(A.6)
The choice of NS or R sector for the bosonic ghosts must be the same as that for $\psi^M$ and $\psi^L$.

The ghost number current of the spin $\lambda$ pair ("fermion number" if $\lambda = \frac{1}{2}$) is denoted $j_\lambda$, and its charge by $Q_\lambda$. We will use the notation $|p^M, ip^L; q\rangle$ for vacuum states, where $q$ denotes the "picture" for the $\beta\gamma$-system. The vacuum of the $q$ picture has $Q_{3/2} = q$. For $\kappa = \frac{1}{2}$, $q$ takes integer values, and for $\kappa = 0$, half-integer. We make a fixed choice for the vacuum of the spin $\frac{1}{2}$ fields; namely, for NS we use the $SL(2, \mathbb{R})$ vacuum, and for R we demand $\psi^\pm_n (n > 0)$ and $\psi^-_0$ annihilate the vacuum. Similarly, we choose once and for all the $bc$ ghost system's vacuum to be $c_1|0\rangle$, where $|0\rangle$ denotes its $SL(2, \mathbb{R})$ vacuum. Neither of these fixed choices is displayed in the notation. In this way, the "physical vacuum" $|p^M, ip^L; -\kappa - \frac{1}{2}\rangle$ (\(\kappa = 0, \frac{1}{2}\)) used in the cohomology calculation of Section 2, is exactly that for which all negatively moded oscillators are creation operators. The "total" ghost number $gh \equiv Q_{3/2} + Q_2$ is normalized so that $d$ has ghost number one, and the physical vacuum has ghost number zero (independent of the sector).

After bosonization of the half-integer-spin fields, which is discussed in Appendix C, the vacuum states are labeled by the charges $Q_\lambda$. The spin 1 fermionic ghost system which arises is by definition always in the $SL(2, \mathbb{R})$ vacuum sector.

For the computation of cohomology it is convenient to introduce a set of "lightcone" combinations of modes

$$q^\pm = \sqrt{\frac{1}{2}}(q^M \pm iq^L), \quad p^\pm = \sqrt{\frac{1}{2}}((p^M - Q^M) \pm i(p^L - Q^L)),$$

$$\alpha_n^\pm = \sqrt{\frac{1}{2}}(\alpha_n^M \pm i\alpha_n^L), \quad n \neq 0, \quad \psi_m^\pm = \sqrt{\frac{1}{2}}(\psi_m^M \pm i\psi_m^L).$$

with nonvanishing commutation relations

$$[q^\pm, p^\mp] = i, \quad [\alpha_n^\pm, \alpha_n^\mp] = m\delta_{m+n,0}, \quad \{\psi_m^\pm, \psi_n^\mp\} = \delta_{m+n,0},$$

as well as shifted momenta

$$P^\pm(n) = \sqrt{\frac{1}{2}}((p^M - (n + 1)Q^M) \pm i(p^L - (n + 1)Q^L)),$$

In particular $p^\pm = P^\pm(0)$.

\[\text{1}\] Note that, a priori, choosing the NS vacuum to have ghost number zero would fix the R vacuum to have ghost number $-\frac{1}{2}$. To have a more symmetric result for the cohomology we have decided to normalize the ghost number operator differently in each sector.
In terms of these the BRST operator $\hat{d}$ is given by

$$\hat{d} = \sum_{n \neq 0} c_{-n} \left( P^+(n) \alpha_n^- + P^-(n) \alpha_n^+ \right) + \sum_{m \in \mathbb{Z} + \kappa} \gamma_m \left( P^+(2m) \psi_m^- + P^-(2m) \psi_m^+ \right)$$

$$+ \sum_{m, n \in \mathbb{Z}, m, n, m+n \neq 0} c_{-n} \left( \alpha_{-m}^- \alpha_{-m+n}^- + \frac{1}{2} (m-n) c_{-m} b_{m+n} \right)$$

$$+ \sum_{n \in \mathbb{Z}, m \in \mathbb{Z} + \kappa, m, n, m+n \neq 0} \gamma_{-m} \left( \gamma_{-m} \psi_{-m}^- \psi_{m+n}^- + \left( \frac{1}{2} n - m \right) \gamma_{-m} \beta_{m+n} \right)$$

$$+ \sum_{n \in \mathbb{Z}, m \in \mathbb{Z} + \kappa, m, n, m+n \neq 0} \gamma_{-m} \left( \alpha_{-n}^+ \psi_{n+m}^- + \alpha_{-n}^- \psi_{n+m}^+ - b_{-n} \gamma_{m+n} \right).$$

(A.10)

### Appendix B. Cohomology computations

This appendix consists of three parts. First we present a summary of mathematical results on the computation of cohomology using a spectral sequence technique. Their proofs can be found in standard textbooks on homological algebra (e.g. [29,30]), but are usually presented at a more abstract level than the pedestrian approach adopted here. In the simplest cases, in particular those pertinent to the 2D (super) string, elementary proofs of some of the results below have been discussed recently in [8,16] and [15]. The second part reviews these computations of the relative cohomology on $\mathcal{F}(p^M, p^L)$, as a concrete application of these general techniques. Finally, in the third part, we discuss the calculation of the absolute cohomology.

#### B.1. Cohomology of a filtered (graded) complex

Consider a complex $(\mathcal{C}, d)$ of vector spaces, where $\mathcal{C} = \bigoplus_n \mathcal{C}^{(n)}$ and the differential $d : \mathcal{C}^{(n)} \to \mathcal{C}^{(n+1)}$. We will consider a spectral sequence which arises when there is an additional gradation, such that for each order $n$,\footnote{This is stronger than the usual assumption that $\mathcal{C}$ must be a filtered complex. A standard filtration in our case is given by subspaces $\mathcal{C}_{(k)} = \bigcup_{k' \geq k} \mathcal{C}_{k'}$.}

$$\mathcal{C}^{(n)} = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}_k^{(n)}.$$  

(B.1)

We will refer to the integer $k$ as the degree, and denote the projection onto the subspace of degree $k$ by $\pi_k$. This gradation by the degree must satisfy the following properties:

1. The differential $d$ has only terms of nonnegative degree, i.e.

$$d = d_0 + d_1 + \ldots = d_0 + d_+,$$

(B.2)
where
\[ d_i : C_k^{(n)} \rightarrow C_{k+i}^{(n+1)}. \] (B.3)

2. In each order only a finite number of nontrivial degrees are present, i.e. for each \( n \),
spaces \( C_k^{(n)} \) are nontrivial for a finite number of \( k \)'s.

The problem is to set up a systematic method of computing cohomology classes of \( d \),
which we denote by \( H_d(C) \). The first observation is that \( d^2 = 0 \) implies
\[
\sum_{i,j \text{ } i+j=k} d_id_j = 0, \quad k = 0, 1, \ldots \tag{B.4}
\]
and, in particular,
\[
d_0^2 = 0. \tag{B.5}
\]

Thus we can consider another complex \((C, d_0)\), with the same underlying space \( C \) and \( d_0 \)
as the differential. Note that \((C, d_0)\) is in fact a direct sum of complexes labeled by the
degree, and therefore its cohomology is much easier to investigate. Moreover, there is a
simple necessary condition for the cohomology of \( d \) being nontrivial, namely \( H_d^{(n)}(C) = 0 \)
whenever \( H_{d_0}^{(n)}(C) = 0 \) (see [8]).

Any \( \psi \in C^{(n)} \) can be expanded as finite sum of terms with definite degree,
\[ \psi = \psi_k + \psi_{k+1} + \ldots + \psi_p. \]
By examining the condition \( d\psi = 0 \) in each degree it is also easy to prove [8]
that one can always choose representative \( \psi = \psi_k + \ldots + \psi_p \) of a nontrivial cohomology class
in \( H_d^{(n)}(C) \) such that the lowest degree term \( \psi_k \) in \( \psi \) represents a nontrivial cohomology
class in \( H_{d_0}^{(n)}(C) \).

A spectral sequence is a “gadget” that allows a systematic investigation of which
\( \psi_k \in H_{d_0}^{(n)}(C) \) extend to a cohomology class \( \psi \in H_d^{(n)}(C) \), where \( \psi = \psi_k + \psi_> \)
and \( \psi_> \) stands for “corrections” of degree higher than \( k \). The \( r \)th term in the spectral sequence is
simply the space of those \( \psi_k + \ldots + \psi_{k+r} \) which survive this extension “nontrivially” through
\( r \) degrees. The successive terms then give finer and finer approximations to \( H_d^{(n)}(C) \). We
will first establish the construction of successive terms of the spectral sequence, but the
reader will recognize where the definitions are heading by keeping in mind the obvious
necessary conditions at each degree imposed by \( d\psi = 0 \).

The first term of the spectral sequence associated with our gradation is \( E_1 = H_{d_0}(C) \).
Next observe that (B.4) for \( k = 1, 2 \) gives
\[
d_0d_1 + d_1d_0 = 0,
\]
\[
d_0d_2 + d_1d_1 + d_2d_0 = 0. \tag{B.6}
\]
The first equation tells us that $d_1$ induces a well defined transformation $d_1 : E_1 \rightarrow E_1$, while the second says that this induced map is nilpotent, $d_1^2 = 0$, on $E_1$. Call $\delta_1 = d_1$ and consider the complex $(E_1, \delta_1)$.

The second term, $E_2$, in the spectral sequence is equal to the cohomology of the previous term, namely $E_2 = H_{\delta_1}(E_1)$. This one also has a canonical differential obtained from $d$, which is explicitly constructed as follows:

Let $\psi_k \in C$ represent an element of $E_2$. Then we must have

$$d_0 \psi_k = 0,$$

$$d_1 \psi_k + d_0 \psi_{k+1} = 0,$$  \hspace{1cm} (B.7)

for some $\psi_{k+1} \in C_{k+1}$. The first condition is needed so that $\psi_k$ defines an element in $E_1$, while the second is simply $\delta_1 \psi_k = 0$ in $E_1$. Define $\delta_2 : E_2 \rightarrow E_2$ by

$$\delta_2 \psi_k = d_2 \psi_k + d_1 \psi_{k+1}$$

$$= \pi_{k+2} d(\psi_k + \psi_{k+1}).$$  \hspace{1cm} (B.8)

With a little of work one verifies that $\delta_2$ does not depend on any of the choices made, and that $\delta_2 \delta_2 = 0$. Thus we can define $E_3 = H_{\delta_2}(E_2)$, and so on.

In general, $\psi_k \in C$ represents an element in $E_r, r > 1$, if there exist $\psi_{k+1}, \ldots, \psi_{k+r-1}$ such that

$$\pi_i d(\psi_k + \psi_{k+1} + \ldots + \psi_{k+r-1}) = 0,$$  \hspace{1cm} for $i = k, \ldots, k + r - 1.$  \hspace{1cm} (B.9)

Then $\delta_r : E_r \rightarrow E_r$,

$$\delta_r \psi_k = \pi_{k+r} d(\psi_k + \psi_{k+1} + \ldots + \psi_{k+r-1}),$$  \hspace{1cm} (B.10)

is well defined and nilpotent. We now see precisely what it means that the elements of $E_r$ are those cohomology classes of $d_0$ which can be extended to approximate cohomology classes of $d$ through $r$ degrees.

The spectral sequence in this context is simply the sequence of complexes $(E_r, \delta_r)_{r=1}^{\infty}$ constructed as above. In interesting cases this sequence converges, which means that the spaces $E_r$ stabilize, i.e.

$$E_r = E_{r+1} = \ldots = E_{\infty},$$  \hspace{1cm} (B.11)

for some $r \geq 1$. Obviously, this requires

$$\delta_r = \delta_{r+1} = \ldots = \delta_{\infty} = 0.$$  \hspace{1cm} (B.12)
In such a case one also says that the sequence collapses at $E_r$.

Thus, as stated, computing subsequent terms in the spectral sequence is nothing other than systematically correcting $\psi_k \in H_{d_0}(C)$ such that, if it lives till $E_\infty$, $\psi_k + \psi_\succ$ should represent a class in $H_d(C)$. Indeed, the main theorem in the subject is that under the assumptions above [29,30]

$$E_\infty \simeq H_d(C).$$

For example, (B.7) is precisely the statement that the nontrivial $d_0$-cohomology state $\psi_k$ may be corrected by a degree $k + 1$ term so that the result is annihilated by $d$ through degree one terms. But moreover, the second term of the spectral sequence also throws away states which are $d_1$ trivial in $d_0$ cohomology, which is equivalent to demanding the extension is nontrivial to this order. Similar arguments may be given at higher order.

It is useful to keep the following two observations in mind.

1. One might get the wrong impression that the magic of the spectral sequence absolves one from doing any work. This is not true. At each stage one must compute the cohomology of $\delta_r$ on $E_r$, and this may become quite complicated. Rather, the spectral sequence provides an algorithm for a systematic computation, which can be formulated quite abstractly and then applied in different settings.

2. Most spectacular applications of these techniques are in cases when one needs to do very little calculations to get an answer. In particular, a rather trivial observation is that $\delta_r$ increases the degree by $r$, so often one can deduce that it is zero by simply examining the set of degrees in which $E_r$ is nontrivial. For example, if all of the cohomology $H_{d_0}(C)$ is concentrated in a single degree $k$ (note that this allows for various orders $n$ with nontrivial $H_{d_0}^{(n)}(C)$) then the sequence must collapse at the first term, i.e. $E_1 \simeq E_\infty$, or, equivalently, $H_{d_0}^{(n)}(C) \simeq H_d^{(n)}(C)$.

**B.2. The relative cohomology of $\mathcal{F}(p^M, p^L)$**

A simple application of the general formalism described above is the computation of the relative BRST cohomology in Section 2. We will consider NS and R sectors separately.

In the NS sector the calculation initially follows that in [8] for the bosonic case. The lightcone combinations (A.7) allow us to assign a degree to the oscillators,

$$\text{deg}(\alpha^+_n) = \text{deg}(c_n) = \text{deg}(\psi^+_r) = \text{deg}(\gamma_r) = +1,$$

$$\text{deg}(\alpha^-_n) = \text{deg}(b_n) = \text{deg}(\psi^-_r) = \text{deg}(\beta_r) = -1,$$

(B.14)
under which $\hat{d}$ decomposes as
\[ \hat{d} = \hat{d}_0 + \hat{d}_1 + \hat{d}_2. \] (B.15)

Here $\hat{d}_k$ denotes terms with degree $k$, and, in particular,
\[ \hat{d}_0 = \sum_{n \in \mathbb{Z}} P^+(n) c_{-n} \alpha^-_n + \sum_{m \in \mathbb{Z} + \frac{1}{2}} P^+(2m) \gamma^-_m \psi^-_m. \] (B.16)

The coefficients $P^\pm(n)$ are given by (A.9). Note that the parametrization (2.10) follows from the condition $P^+(r) = P^-(s) = 0$ for integers $r$ and $s$.

Consider now the proof of the results of Section 2. The exceptional case (i) has been discussed in Section 2. When $P^+(n) \neq 0 \ \forall n \neq 0$, a “contracting homotopy” can be constructed, and case (ii) follows immediately. In fact, nontrivial exceptions can only arise when there are a pair of integers $r$ and $s$, $rs > 0$, such that both $P^+(r) = P^-(s) = 0$. [For more details see [8] for the directly analogous discussion in the bosonic case.]

The calculation now splits into several cases, whose proofs all run quite parallel to each other. Thus we consider first the case $r, s \in \mathbb{Z}_+$, i.e. case (iii), and then we must further distinguish between even and odd values.

For $r \in 2\mathbb{Z}_+$, the $\hat{d}_0$ cohomology is built from the oscillators $\alpha^+_r$ and $c_{-r}$, and at the level $\frac{r}{2}$ ($L_0 = 0$) there are only two such states (both of the same degree) for $s \in 2\mathbb{Z}_+$, but none otherwise. Clearly in this case the cohomology of $\hat{d}_0$ coincides with that of $\hat{d}$ (see remark 2 below (B.13)).

For $r \in 2\mathbb{Z}_+ - 1$, the cohomology of $\hat{d}_0$ is spanned by the states of the form
\[ (\alpha^+_r)^{a_1} (\psi^+_r)^{a_2} (c_{-r})^{b_1} (\gamma^-_r)^{b_2} |p^M, ip^L; -\frac{1}{2} \rangle, \] (B.17)
where the integers $a_1, a_2, b_1$ and $b_2$ satisfy
\[ (a_1 + b_1) + \frac{1}{2}(a_2 + b_2) = \frac{s}{2}, \quad a_1, b_2 \geq 0, \quad a_2, b_1 = 0, 1. \] (B.18)

Thus the cohomology of $\hat{d}_0$ clearly is not generally restricted to a single degree, and further analysis is required. We identify the space spanned by the states in (B.17) with the first term $E_1$ of the spectral sequence. The induced differential $\delta_1$ is explicitly given by
\[ \delta_1 = \psi^+_{-\frac{r}{2}} \gamma^-_{-\frac{r}{2}} \alpha^-_r - \gamma^-_{-\frac{r}{2}} \gamma^-_{-\frac{r}{2}} b_r. \] (B.19)

By examining the action of $\delta_1$ on the states (B.17), one finds that if $s \in 2\mathbb{Z}_+ - 1$ the cohomology of $\delta_1$ on $E_1$ is concentrated in only one degree, and is two dimensional with
one state at ghost number 0 and 1, respectively. Otherwise the cohomology of \( \delta_1 \) on \( E_1 \) vanishes, and thus no nontrivial state is obtained. [An elementary derivation of these results, which in fact is equivalent to the above spectral sequence argument, has recently been given in [15].] This completes the analysis for case (iii), and case (iv) is analogous.

For the \( R \) sector there is one essential difficulty, beyond the \( \text{NS} \) calculation above, due to the fact that \( G_0 \) does not act reducibly on \( \mathcal{F}(p^M, p^L) \). Thus the restriction to the \( \mathcal{F}_{rel}(p^M, p^L) \) subspace is not trivial to carry out. Of course in the case that either \( r \) or \( s \) vanishes, the \( \hat{d} \) cohomology on the whole Fock spaces is anyway at most one vacuum state (by the same arguments as in [8]) and case (ii) again follows easily. The other cases may be dealt with by introducing “rotated” oscillators in terms of which \( \mathcal{F}_{rel} \) can be constructed explicitly.

The operator \( \vartheta = \psi_0^+/p^+ \) is well defined for \( r \neq 0 \), and satisfies

\[
\{G_0, \vartheta\} = 1. \tag{B.20}
\]

Of course \( G_0^2 = 0 \) on \( \mathcal{F}^{L_0}(p^M, p^L) \), so the existence of such an operator implies that the cohomology of \( G_0 \) on \( \mathcal{F}^{L_0}(p^M, p^L) \) is trivial, and indeed \( \vartheta \) is the corresponding contracting homotopy. In particular we have

\[
\mathcal{F}_{rel} = G_0\mathcal{F}^{L_0}, \quad \mathcal{F}^{L_0} = \mathcal{F}_{rel} \oplus \vartheta\mathcal{F}_{rel}, \quad \dim(\mathcal{F}_{rel}) = \dim(\vartheta\mathcal{F}_{rel}) = \frac{1}{2}\dim(\mathcal{F}^{L_0}). \tag{B.21}
\]

Further, if \( o_m \) denotes any fundamental oscillator, we may define rotated oscillators [17]

\[
\tilde{o}_m \equiv [G_0, \vartheta o_m], \tag{B.22}
\]

which again satisfy the same algebra as the original oscillators \( o_m \) (except that \( \tilde{\psi}_0^+ = 0 \) by this definition).

If \( \mathcal{O} \) is an operator built from the fundamental oscillators, normal ordered with respect to the physical vacuum with zero modes of \( \psi^\pm(z) \) to the right, we denote by \( \tilde{\mathcal{O}} \) the operator obtained by rotating all the oscillators in \( \mathcal{O} \). This is achieved by \( \tilde{\mathcal{O}} = [G_0, \vartheta\mathcal{O}] \), as can easily be seen by pulling the commutator through all the oscillators; for example, if we have two even oscillators \( o_1 \) and \( o_2 \), \( \{G_0, \vartheta o_1 o_2\} = o_1 o_2 - \vartheta[G_0, o_1]o_2 - \vartheta o_1 [G_0, o_2] = \tilde{o}_1 \tilde{o}_2 \) upon using \( \vartheta^2 = 0 \). Since \( \tilde{\mathcal{O}} = \mathcal{O} - \vartheta[G_0, \mathcal{O}] \), we find that any operator commuting with \( G_0 \) can be rewritten in terms of the rotated oscillators. In particular this applies to \( \tilde{d} \).

Moreover, we may also show that \( \mathcal{F}_{rel} \) is contained in the subspace freely generated by the rotated creation oscillators acting on the vacuum \( |p^M, ip^L, -\frac{1}{2}\rangle \). Indeed, let \( \psi =
\( \mathcal{O}|p^M, ip^L; -\frac{1}{2} \rangle \) denote a state in this subspace, where \( \mathcal{O} \) is some operator built with creation operators with respect to this vacuum. Then

\[
\psi = G_0 \partial \mathcal{O}|p^M, ip^L; -\frac{1}{2} \rangle = \bar{\mathcal{O}}|p^M, ip^L; -\frac{1}{2} \rangle \pm \partial \mathcal{O}G_0|p^M, ip^L; -\frac{1}{2} \rangle , \tag{B.23}
\]

and by examining the zero modes \( \psi_0^\pm \) in \( G_0 \) one easily verifies that the last term vanishes.

This analysis applies to all states built on the Fock space with \( p^+ \neq 0 \), i.e. \( r \neq 0 \). When \( r = 0 \), but \( s \neq 0 \) we may go through precisely the same analysis, but using \( \bar{\vartheta} = \frac{\psi^-}{p} \), with the obvious changes. Thus one effectively needs to use two “patches” in this oscillator space. The exceptional case \( p^+ = p^- = 0 \) has been dealt with separately in Section 2.

With these results in hand, the computation of cohomology is straightforward. We introduce a filtration on \( \mathcal{F}_{rel} \) by assigning degrees as in (B.14) but to the rotated oscillators. Furthermore, since \( \tilde{\psi}_0^+ = 0 \), only non-zero modes appear in \( \hat{d}_0 \),

\[
\hat{d}_0 = \sum_{n \neq 0} \left[ P^+ n \tilde{c}_- n \tilde{c}_- + P^+ (2n) \tilde{\gamma}_- n \tilde{\psi}_- n \right] . \tag{B.24}
\]

The rest of the computation proceeds as in NS sector, in particular one must consider at most the second term of the spectral sequence.

### B.3. The absolute cohomology of \( \mathcal{F}(p^M, p^L) \)

In this subsection we give the remaining details for the proof of the relation between the relative and absolute cohomologies in R sector discussed in Section 2. Our approach has been inspired by that of [21] for the case of ten dimensional superstring.

First note that any state \( \varphi \) in the product of Fock spaces \( \mathcal{F}(p^M, p^L) \) corresponding to the \( q = -\frac{1}{2} \) picture may be decomposed with respect to ghost zero modes \( c_0 \) and \( \gamma_0 \) as

\[
\varphi = \phi + c_0 \psi , \quad b_0 \phi = b_0 \psi = 0 , \tag{B.25}
\]

and

\[
\phi = \sum_{n=0}^{n(\phi)} \gamma_0^n \phi_n , \quad \beta_0 \phi_n = 0 , \tag{B.26}
\]

\[
\psi = \sum_{n=0}^{n(\psi)} \gamma_0^n \psi_n , \quad \beta_0 \psi_n = 0 ,
\]

where \( n(\phi) \) and \( n(\psi) \) are finite. [Strictly speaking the finiteness of the expansion into series in \( \gamma_0 \) should be viewed as a definition of the Fock space topology.] We will denote \( (\psi)_> = \psi - \psi_0 \).
Let us consider the case in which one of $p^+$ or $p^-$ is nonzero. As we saw in the previous section, the cohomology of $F = G_0$ on $F^{L_0}(p^M, p^L)$ is then trivial. This allows us to prove the following technical result about the states in $K(p^M, p^L)$:

For $\psi \in K(p^M, p^L)$, i.e. $\psi = \sum_{m=0}^{n(\psi)} \gamma_0^m \psi_m$, $\psi_m \in F^{L_0}(p^M, p^L)$, a general solution to

$$(d\psi)_> = 0,$$

is of the form

$$\psi = \overrightarrow{d} \rho + F \rho'_0,$$

where $\rho, \rho'_0 \in K(p^M, p^L)$ and in addition $\beta_0 \rho'_0 = 0$ and $n(\rho) = n(\psi) - 1$.

In fact, recall that $\overrightarrow{d} = \gamma_0 F + \beta_0 N + \tilde{d}$, so that $\overrightarrow{d} \psi = 0$ written in components reads ($n = n(\psi)$)

$$\begin{align*}
\tilde{d}\psi_0 - N \psi_1 &= 0 \\
\tilde{d}\psi_1 + F \psi_0 - 2N \psi_2 &= 0 \\
&\vdots \\
\tilde{d}\psi_{n-1} + F \psi_{n-2} - nN \psi_n &= 0 \\
\tilde{d}\psi_n + F \psi_{n-1} &= 0 \\
F \psi_n &= 0.
\end{align*}$$

(B.29)

We must show that solving all, but the first, of the above equations yields $\psi$ of the form (B.28). Clearly the solution to the last equation is $\psi_n = F \chi_n$. Since $\{F, \overrightarrow{d}\} = 0$, the next equation gives $F(\tilde{d}\chi_n - \psi_{n-1}) = 0$, which is solved by $\psi_{n-1} = F \chi_{n-1} + \tilde{d}\chi_n$. Continuing, we obtain

$$\psi_k = F \chi_k - (k + 1)N \chi_{k+2} + \tilde{d}\chi_{k+1}, \quad k \geq 0, \quad \chi_i = 0, \ i > n.$$

(B.30)

Introducing

$$\rho = \sum_{m=0}^{n-1} (\gamma_0)^m \chi_{m+1}, \quad \rho_0 = \chi_0,$$

we obtain (B.28).

Using this result it is easy to verify that any absolute cohomology state $\varphi = \phi + c_0 \psi$ is equivalent to a linear combination of states (2.12) given by the relative cohomology. Indeed, in components, $d\varphi = 0$ reads

$$\overrightarrow{d}\phi - (M + \gamma_0 \gamma_0) \psi = 0, \quad \overrightarrow{d} \psi = 0.$$

(B.32)
while the “gauge transformation” $\varphi \rightarrow \varphi + d\omega$, where $\omega = \xi + c_0 \chi$, is

$$\phi \rightarrow \phi + \overline{d}\xi - (M + \gamma_0 \gamma_0) \chi, \quad \psi \rightarrow \psi - \overline{d}\chi. \quad (B.33)$$

Since $\overline{d}\psi = 0$, by setting $\chi = \rho$ one can immediately gauge away all terms in $\psi$ except $\psi_0$. Thus we may assume $\psi = \psi_0$, where $\overline{d}\psi_0 = F\psi_0 = 0$. The remaining gauge freedom is

$$\psi_0 \rightarrow \psi_0 + \overline{d}\chi, \quad (\overline{d}\chi)_> = 0. \quad (B.34)$$

Thus $\chi$ must be of the form $\chi = \overline{d}\sigma + F\sigma'$, and (B.34) simply becomes $\psi_0 \rightarrow \psi_0 + \overline{d}F\sigma'$; i.e. only the relative cohomology component in $\psi_0$ cannot be gauged away. From the discussion in Section 2, we know that if $\psi_0$ is a relative cohomology state, there exists $\lambda = \lambda_0 + \gamma_0 \lambda_1$ such that $d(\lambda + c_0 \psi_0) = 0$. Writing $\phi = (\phi - \lambda) + (\lambda + c_0 \psi)$, there is an $\omega$ as above such that $\varphi = (\phi' - \lambda) + (\lambda + c_0 \psi_0) + d\omega$, and we deduce from (B.32) that $\overline{d}(\phi - \lambda) = 0$. Therefore all terms in $\phi - \lambda$, with the possible exception of the lowest order one, can be gauged away using a suitable $\xi$ in (B.33). This concludes the proof that any absolute cohomology class can be written as a sum of two $(d$ closed states) of the form (2.12) , canonically constructed from relative cohomology classes.

We must still verify that for a given relative cohomology representative $\psi_0$, the corresponding states, $\psi^{(1)}$ and $\psi^{(2)}$ defined in (2.12) are nontrivial representatives of the absolute cohomology. For $\psi^{(2)}$ this clearly follows from the discussion above. In the case of $\psi^{(1)}$, we certainly cannot gauge it away using $\xi$ alone. Thus we must consider the most general gauge transformation. Remarkably, using the general solution for $\chi$ (which clearly must satisfy $\overline{d}\chi = 0$ so that no “$c_0$” term is induced by the gauging), and the identities (2.5), it is straightforward to verify that $(M + \gamma_0 \gamma_0) \chi = \overline{d}\chi'$, for some $\chi'$. [In particular, using (B.28) and further counting of available ghost numbers for relative cohomology, $\overline{d}\chi = 0$ implies $\chi = \overline{d}\sigma$ here.] This shows that $\psi^{(1)}$ is also nontrivial.

Appendix C. Bosonization

Our cocycles for the bosonized operators, and thus the spin fields, are constructed as follows. The basic rule is that all “fermionic” first order fields anticommute (in this regard we follow the conventional choice of [31]). A “minimal” set may be obtained explicitly as

$$\psi^\pm(z) = e^{\pm iH} I^\pm, \quad \gamma(z) = e^{i\phi} \tilde{I} \eta, \quad \beta(z) = \partial \xi e^{-i\phi} \tilde{I}^{-1}, \quad (C.1)$$
where

\[ I = e^{i\pi(Q_1 + Q_2)}, \quad \tilde{I} = e^{i\pi(Q_1 + Q_2 + Q_{1/2})}. \quad \text{(C.2)} \]

Here \( Q_n \) just denotes the fermion number corresponding to a spin \( n \) pair. The spin fields are given the “obvious” square root cocycle, *e.g.* \( S^\pm = e^{\pm iH/2}I \pm \frac{i}{2} \). Note that writing out the cocycle operators is not necessary in calculations, but useful for setting consistent definitions. With the above we find, for example

\[ \tilde{I} - m e^{i n H(z)} \tilde{I} = e^{- i m n \pi} e^{i n H(z)}. \quad \text{(C.3)} \]

For the calculations in Section 3, however, we found it useful to enlarge the set of mutually anticommuting operators to include \( e^{\pm i\phi^M} \) and \( e^{\pm \phi^L} \) – in particular this allows a choice so that the \( sl(2) \) symmetry currents (3.13) are “manifestly bosonic”. Moreover, for Section 4 it is necessary to include further cocycles so that the first order fields mutually anticommute between left and right moving sectors. The complete set of cocycles used for these calculations are defined by (C.4) together with

\[ e^{\pm i\phi^M} \to e^{\pm i\phi^M} I_M^{\pm 1}, \quad e^{\pm \phi^L} \to e^{\pm \phi^L} I_L^{\pm 1}, \quad \text{(C.4)} \]

and similar definitions for the right moving sector distinguished by ‘, where

\[
\begin{align*}
I &= \exp\{i\pi(Q_1 + Q_2 - Q_1' - Q_2' - Q_3' - Q_\frac{3}{2}' - p^{M'} - i p^{L'})\} \\
\tilde{I} &= \exp\{i\pi(Q_1 + Q_2 + Q_3' - Q_1' - Q_2' - Q_3' - Q_\frac{3}{2}' - p^{M'} - i p^{L'})\} \\
I' &= \exp\{-i\pi(Q_1' + Q_2' - Q_1 - Q_2)\} \\
\tilde{I}' &= \exp\{-i\pi(Q_1' + Q_2' + Q_3' - Q_1 - Q_2)\} \\
I'_M &= \exp\{-i\pi(Q_1' + Q_2' + Q_3' - Q_1' - Q_2' - Q_3' - p^{M'} - i p^{L'})\} \\
I'_M &= \exp\{-i\pi(Q_1' + Q_2' + Q_3' + p^M - Q_1' - Q_2' - Q_3' - p^{M'} - i p^{L'})\} \\
I'_L &= \exp\{-i\pi(Q_1' + Q_2' + Q_3' + p^M - Q_1 - Q_2)\}. \quad \text{(C.5)}
\end{align*}
\]

As an aside, note that if suitable cocycles were included in the minimal set (C.2) to ensure left and right mutual anticommutation of the appropriate fields, then the operators \( a_1 \) and \( a_2 \) of Section 4 – which generate the ground ring of the uncompactified matter model – would still be mutually commuting. However, the remaining generators – as required for
the compactified matter model—commute up to relative phases, making the determination of the symmetries induced for the uncompactified model more difficult.

Recall that since $\gamma$ and $\beta$ commute with the operator $Q_P \equiv Q_{3/2} + Q_1$, $Q_P$ has eigenvalue $q$ for every $\beta \gamma$ Fock space state in a given $q$ picture. Thus, after bosonization, the $q$ picture is just found as that “slice” of the complete bosonized space with a definite eigenvalue $q$ for $Q_P$. The picture changing operator is

$$X(z) = \{d, \xi(z)\}, \quad (C.6)$$

which has $Q_{3/2} + Q_2 = 1$, and $Q_1 - Q_2 = 0$. It clearly increases the picture by one unit, commutes with $d$, and has $L_0 = 0$. The bosonized expression for $X(z)$ is

$$X(z) = (G^M + G^L)e^{i\tilde{\phi}} + c\partial \xi - be^{2i\tilde{\phi}} \partial \eta - \partial(be^{2i\tilde{\phi}} \eta). \quad (C.7)$$

The $d$-invariant operator with $L_0 = 0$ which decreases the picture by one unit is denoted $Y(z)$

$$Y(z) = c\partial \xi e^{-2i\tilde{\phi}}. \quad (C.8)$$

It has $Q_{3/2} + Q_2 = -1$ and $Q_1 - Q_2 = 0$.

The explicit action on the states was clearly discussed in [25]. The zero modes $X_0$ and $Y_0$ (e.g. $X_0 = \oint \frac{dz}{2\pi i z}X(z)$) commute with each other. More importantly

$$:X_0 :: Y_0 := 1 + [d, \epsilon], \quad (C.9)$$

for some operator $\epsilon$, and thus they are inverse in cohomology. Similarly

$$:X_0 :^n: Y_0 :^n := 1 + [d, \epsilon'], \quad (C.10)$$

and we can go from one picture to another with these operators which clearly provide an isomorphism in cohomology.

**Appendix D. Explicit representatives for $\hat{c} = 1$**

In this appendix we will present some results on the structure of the $\hat{c} = 1$ Fock space modules of the $N = 1$ superconformal algebra. We will show how to obtain expressions for the singular vectors in terms of certain “super-Schur polynomials,” and how these can
be used to obtain explicit representatives of the BRST-cohomology. The line of reasoning closely follows the discussion in the bosonic case [16].

From the Kac determinant [32] and the existence of a hermitian form, it follows that the \( \hat{c} = 1 \) Fock space \( \mathcal{F}(p) \) is reducible if and only if \( p = \frac{1}{2}(r - s) \) for some \( r, s \in \mathbb{Z}, rs > 0, (r - s) \in 2\mathbb{Z} + (1 - 2\kappa) \). In this case the Fock space contains a singular vector at level \( \frac{rs}{2} \). Moreover, in this case, the Fock space module is completely reducible, i.e.

\[
\mathcal{F}(p = \frac{1}{2}(r - s)) = \bigoplus_{\ell \geq 0} \mathcal{L}(h = \frac{1}{8}(|r - s| + 2\ell)^2, \hat{c} = 1).
\]

The submodules obtained by restricting this sum to \( \ell \geq \ell_0 \geq 0 \) are isomorphic to \( \mathcal{F}(\frac{1}{2}(r_k - s_k)) \), where \( k = \max(0, s - r) + \ell_0, r_k = r + k, s_k = s - k \). The isomorphism is given by the operator \((J_-)^k\), where

\[
J_- = \frac{1}{2\pi i} \oint dz \, \psi(z)e^{-i\phi(z)}, \quad (D.1)
\]

is the “screening charge” for the \( \hat{c} = 1 \) free field realization. As we have seen, \( J_- \) also equals the negative root operator of an \( sl(2) \) algebra. Clearly, this construction of singular vectors is intimately related to the fact that the \( N = 1 \) superconformal algebra is the commutant of the horizontal \( sl(2) \) algebra on \( c = \frac{3}{2} \) highest weight modules of affine \( sl(2) \). The isomorphism \((J_-)^k\) can be used to obtain explicit formulas for the singular vectors. Let us first introduce some special functions.

Elementary Schur polynomials \( S_k(x), x = (x_1, x_2, \ldots), k \geq 0 \) are defined through a generating function

\[
\sum_{k \geq 0} S_k(x)z^k = \exp \left( \sum_{k \geq 1} x_k z^k \right). \quad (D.2)
\]

For convenience we put \( S_k(x) = 0 \) for \( k < 0 \). They can be generalized to the supercase by introducing Grassmannian variables \( \theta_m \), where \( m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \) in the NS-case and \( m = 1, 2, \ldots \) in the R-case as well as an additional Clifford variable \( \theta_0, \theta_0^2 = 1 \) in the R-case.

We now define NSR-Schur polynomials by

\[
S^{NS}_k(x, \theta) = \sum_{\ell \geq 0} S_{k-\ell}(x) \theta_\ell, \\
S^R_k(x, \theta) = \sum_{\ell \geq 0} S_{k-\ell}(x) \theta_\ell, \quad (D.3)
\]

in terms of which we can expand

\[
\psi^{NS/R}_<(z)e^{-i\phi_<(z)} = -\sqrt{\frac{1}{2}} \sum_{m \geq \kappa} S^{NS/R}_m(x, \theta) z^{m-\frac{1}{2}}, \quad (D.4)
\]

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where
\[ \psi_{<}^{NS/R}(z) = \sum_{m \leq -\kappa} \psi_m^{NS/R} z^{-m-\frac{1}{2}}, \quad \phi_<(z) = \sum_{n < 0} \frac{1}{n} \alpha_n z^{-n}, \quad (D.5) \]
and \( x_j = -\alpha_{-j}/j, \theta_m = -\sqrt{2}\psi_-m \).

Now suppose \( r,s \in \mathbb{Z}_+ \). As we have argued above, the singular vector \( \Psi_{(r,s)} \) at level \( \frac{r+s}{2} \) in \( \mathcal{F}(p = \frac{1}{2}(r-s)) \) is given by \[ (D.6) \]
\[ \left( \frac{1}{2\pi i} \oint dz \psi(z) e^{-i\phi(z)} \right)^s |\frac{1}{2}(r+s)\rangle. \]

After normal ordering and expanding the resulting expression by making use of \((D.4)\), the contour integrals can be evaluated. This results in explicit expressions for the singular vectors in terms of the NSR-Schur polynomials \((D.3)\). For example
\[ \psi_{(r,1)}^{NS} = S_{\frac{r}{2}}^{NS}(x,\theta)|\frac{1}{2}(r-1)\rangle, \quad r \in 2\mathbb{Z} + 1, \]
\[ \psi_{(r,2)}^{NS} = \left( S_{\frac{r}{2}}^{NS}(x,\theta) S_{\frac{r+1}{2}}^{NS}(x,\theta) + S_{\frac{r+1}{2}}^{NS}(x,\theta) S_{\frac{r}{2}}^{NS}(x,\theta) \right) |\frac{1}{2}(r-2)\rangle, \quad r \in 2\mathbb{Z}, \]
\[ \psi_{(r,1)}^{R} = S_{\frac{r}{2}}^{R}(x,\theta)|\frac{1}{2}(r-1)\rangle, \quad r \in 2\mathbb{Z}, \]
\[ \psi_{(r,2)}^{R} = S_{\frac{r+1}{2}}^{R}(x,\theta) S_{\frac{r+1}{2}}^{R}(x,\theta) |\frac{1}{2}(r-2)\rangle, \quad r \in 2\mathbb{Z} + 1. \]

In the NS case it is relatively straightforward to write expressions for generic \((r,s)\) in terms of determinants of the elementary NS-Schur polynomials, as in the bosonic case. In the R-sector there does not seem to be a comparably simple expression, due to the fact that the R-Schur polynomials are not mutually anticommuting because of the presence of the Clifford variable \( \theta_0 \). We refrain from giving more general expressions since the above will suffice for the purpose of this paper. The case \( r,s \in \mathbb{Z}_- \) can be treated analogously, or simply by observing that \( \psi_{(-r,-s)} \sim \psi_{(s,r)} \). We also remark that, because of a symmetry of \( G_r \), the expression for \( \psi_{(s,r)} \) can be obtained from \( \psi_{(r,s)} \) by letting \( (x,\theta) \to (-x,-\theta) \).

We will now discuss – in complete analogy with the bosonic case – how the above singular vectors immediately lead to representatives of the (discrete) BRST cohomology at ghost number zero.

First of all, observe that
\[ [d, \psi^M e^{\pm i\phi^M}] = \partial \left( c\psi^M e^{\pm i\phi^M} \pm \gamma e^{\pm i\phi^M} \right), \]
\[ [d, i\partial\phi^M] = \partial \left( c\partial\phi^M + \psi^M \gamma \right), \quad (D.8) \]

\[ ^3 \text{For R, one can take either of the two possible vacua } |\pm\rangle \text{ where } |+\rangle = \psi_0|-. \]
implying that the $sl(2)$ algebra generated by $\{J_\pm, J_0\}$ (see (3.13)) acts on the cohomology. In particular, the BRST operator commutes with the operator $(J_-)^k$ above, which provided the embedding $\mathcal{F}^M(\frac{1}{2}(r_k - s_k)) \to \mathcal{F}^M(\frac{1}{2}(r - s))$. Moreover, since the BRST operator $d$ acts on each irreducible constituent of $\mathcal{F}^M(p)$ separately, we might as well restrict the computation of physical states to each $\mathcal{F}^M(\frac{1}{2}(r_k - s_k)) \otimes \mathcal{F}^L(\frac{1}{2}(r + s + 2)) \otimes \mathcal{F}^G$ (notice we defined $r_k, s_k$ such that $r_k + s_k = r + s$). Choosing $k = s$ for $r, s > 0$ (or $k = -r$ for $r, s < 0$) we find that the vacuum $|p^M, ip^L; q\rangle = |\frac{1}{2}(r_k - s_k), \frac{1}{2}(r_k + s + 2); -\frac{1}{2} - \kappa\rangle$ provides a nontrivial cohomology state. Since this state has $p^- = 0$, it is also in Ker $G_0$ (for R), i.e. it represents a relative cohomology state. Mapping with the operator $(J_-)^k$ provides the nontrivial (relative) cohomology state

$$\psi(r, s)(x^M, \theta^M)|\frac{1}{2}(r - s), \frac{1}{2}(r + s + 2); -\frac{1}{2} - \kappa\rangle \quad \text{(D.9)}$$

By comparison to the results of Section 2, we conclude that these exhaust the nontrivial (relative) cohomology states at ghost number zero. So we see that, as in the bosonic case, it is possible to choose a “material gauge,” i.e. a representative only containing matter excitations, for the ghost number zero sector. In particular it also follows that the associated currents (3.3) can be chosen to be superconformal primaries.

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