Openness of uniformly valuative stability on the Kähler cone of projective manifolds

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Abstract
Assume that a projective variety is uniformly valuatively stable with respect to a polarization. We show that the projective variety is uniformly valuatively stable with respect to any polarization sufficiently close to the original polarization. The definition of uniformly valuatively stability in this paper is stronger than that given by Dervan and Legendre (Valuative stability of polarised varieties, arXiv:2010.04023, 2020). We also define the valuative stability for the transcendental Kähler classes. Our openness result can be extended to the Kähler cone of projective manifolds.

Keywords Uniformly valuatively stability · $\beta$-invariant · Positive intersection product · Volume function

Mathematics Subject Classification 14C17 · 32J25 · 32J27

1 Introduction
Finding a canonical metric on Kähler manifolds is the central problem in Kähler geometry, especially finding a constant scalar curvature Kähler (cscK) metric. The Yau–Tian–Donaldson (YTD) conjecture predicts that the existence of a cscK metric is equivalent to an algebro-geometric stability, the so-called $K$-polystability due to Tian [38] and Donaldson [21], on a polarised manifold. When the automorphism group of a variety is discrete, K-polystability reduces to K-stability since the classical Futaki invariant [27] vanishes automatically. The notion of K-(poly)stability of a polarised variety has played an important role in algebraic geometry, especially Fano varieties, in recent years.

The YTD conjecture is widely open in general. There have been considerable strides on these ideas for the Fano case in recent years. Chen–Donaldson–Sun [13] and Tian [39] independently proved that K-polystability implies the existence of Kähler–Einstein metrics on Fano manifolds, solving this conjecture in the Fano case (also see [3, 14, 17, 47] for other...
different methods). In the algebraic side, the theory has achieved substantial progress. The main breakthrough is due to Fujita [23] (also Fujita and Odaka [25]) and Li [32], which re-interprets K-stability in terms of valuations by the algebraic invariant, the so-called δ-invariant [25] or β-invariant [23, 32]. One can test K-stability of a Fano variety by computing its δ-invariant or β-invariant. This is the so-called Fujita–Li criterion.

People study K-stability of Fano varieties from the viewpoint of birational geometry. An almost complete theory of K-stability of Fano varieties is established. From this powerful theory, one can construct a desirable moduli space of K-stable Fano varieties, the so-called K-moduli space. There are many important works along these lines, due to Xu, Liu, Zhuang, Blum, etc. (see [1, 4, 6, 7, 15, 43, 44], etc.). We refer the reader to an excellent survey [42] for the algebraic theory of K-stability of Fano varieties. Very recently, Liu–Xu–Zhuang [34] proved that the K-moduli space is proper by solving two profound and challenging conjectures, the so-called Higher Rank Finite Generation conjecture and Optimal Destabilization conjecture. As an application of those conjectures, they also show that K-stability is equivalent to uniform K-stability for a log Fano pair with the discrete automorphism group. Moreover, their argument also holds for the non-discrete automorphism group. The Fujita–Li criterion for K-stability of Fano varieties has played an essential role in all of these developments.

To study K-stability of polarised varieties, the next step is to develop the Fujita–Li criterion in the polarised case. The original definition of K-stability involves C∗-degenerations of a polarised variety, the so-called test configurations. Donaldson [21] associates a numerical invariant to each test configuration, the so-called Donaldson–Futaki invariant. K-stability means that this invariant is always positive. By works of Boucksom, Jonsson, etc. (see [10, 12]), we can identify a test configuration with a finitely generated Z-filtration on the section ring of the polarization.

For any valuation, one can associate a filtration to this valuation. When this filtration is finitely generated, such a valuation is called a dreamy valuation. A valuation is called a divisorial valuation if it is induced by a prime divisor. A divisor is called a dreamy divisor if the corresponding divisorial valuation is dreamy. Dervan and Legendre [19] define a new β-invariant for polarised varieties, which generalizes Fujita’s original β-invariant, by computing the Donaldson–Futaki invariant of the test configuration associated with a dreamy divisor. They showed that K-stability over integral test configurations is equivalent to valuative stability over dreamy divisors. Here an integral test configuration means that its central fiber is integral. It gives an expectation to establish the Fujita–Li criterion in the polarised case.

Unfortunately, examples in [2] show that positivity of the Donaldson–Futaki invariant for algebraic test-configurations may not be enough to ensure the existence of a cscK metric. A stronger notion, the so-called uniform K-stability, is introduced by the thesis [37] and deeply developed in [12] and [18], which becomes a new candidate for the stability criterion of the existence of a cscK metric. When the automorphism group of a manifold is discrete, the uniform YTD conjecture states that uniform K-stability is equivalent to the existence of a cscK metric. Very recently, Li [33] proved the existence of cscK metrics under the condition of uniform K-stability for model filtration, which is stronger than the original uniform K-stability. Moreover, his approach also holds when the automorphism group is non-discrete.

A basic question about uniform stability is whether it is preserved under small perturbations of the polarization or not. This question is motivated by a classical result of LeBrun–Simanca [31], in which they establish openness results for perturbations of cscK metrics. Fujita [24] proved the openness of uniform K-stability for the log canonical and log anti-canonical polarization. Note that Fujita’s result requires that the base variety can have bad singularities, the so-called demi-normal pair (see [28] or [24]). Zhang [46] proved that
the valuative stability threshold (δ-invariant) is continuous on the big cone of Fano manifolds. Thus, the openness of uniformly valuative stability holds for Fano manifolds.

In this paper, we establish openness of uniformly valuative stability for general projective varieties. This gives an affirmative answer to the above question for uniformly valuative stability. Note that our definition of uniformly valuative stability is stronger than that given by Dervan and Legendre in [19], see Definition 7 and Remark 3. Our main theorem is

**Theorem 1** (see Theorem 8) For a normal projective variety X, the uniformly valuative stability locus

\[ \text{UVs} := \{ [L] \in \text{Amp}(X) : (X, L) \text{ is uniformly valuatively stable} \} \]

is an open subcone of the ample cone \( \text{Amp}(X) \).

Together with LeBrun–Simanca’s openness, our result fits the expectation of YTD conjecture.

A main difficulty of Theorem 1 is to control the difference of the derivative part in the expression of \( \beta \)-invariant for two nearby ample divisors. It is hard to control the difference for all prime divisors in general. In addition, the log discrepancy has no control generally. By considering the derivative part of \( \beta \)-invariant together with the log discrepancy, we obtain a partial control of \( \beta \)-invariant (see Theorem 10), which is enough to show our main theorem.

As an immediate application of the main theorem, we obtain

**Corollary 2** (see Theorem 14) For a normal projective variety X, the uniformly valuative stability threshold

\[ \text{Amp}(X) \ni L \mapsto \zeta(L) \in \mathbb{R} \]

is continuous on the ample cone \( \text{Amp}(X) \) (see Definition 9 for \( \zeta(L) \)).

The invariant \( \zeta \) is motivated by δ-invariant since \( \delta - 1 \) can be viewed as the stability threshold (in the sense of Definition 9) of the original \( \beta \)-invariant. Corollary 2 gives a similar result with Zhang [46] for projective varieties. According to the expression of \( \beta \)-invariant, we do not have a canonical formulation to define its corresponding δ-invariant for polarised varieties. Studying the invariant \( \zeta \) is a good candidate to test valuative stability.

The definition of cscK metrics does not need a polarization. In [20] and [36], they independently define K-stability for the transcendental Kähler classes. It is natural to extend the valuative stability to any Kähler class of compact Kähler manifolds (see Definition 10).

Due to some well-known results about analytic geometry, it is straightforward to see that our argument for the algebraic class can also work for the Kähler class on projective manifolds. We state it as follows,

**Theorem 3** (see Theorem 17) For a projective manifold X, the uniformly valuative stability locus

\[ \tilde{\text{UVs}} := \{ \alpha \in \mathcal{K} : (X, \alpha) \text{ is uniformly valuatively stable} \} \]

is an open subcone of the Kähler cone \( \mathcal{K} \).

To facilitate access to the individual topics, the sections are rendered as self-contained as possible.

This article is organized as follows. In Sect. 2, we review basic theories of the volume and the positive intersection product. Moreover, we collect some well-known properties and useful theorems. In Sect. 3, we recall these definitions of the \( \beta \)-invariant and the valuative stability of polarised varieties. Moreover, we formulate our main theorem. In Sect. 4, we finish the proof of the main theorem. In Sect. 5, as an application of the main theorem, we
prove continuity of uniformly valuative stability threshold. In Sect. 6, we compile some basic theories of the analytic volume and positive intersection product. Moreover, we state some related facts and extend our result to the Kähler cone of projective manifolds. In Sect. 7, we propose some interesting further questions.

Notation

We work throughout over the complex number \( \mathbb{C} \). A variety is always assumed to be a connected, reduced, separated and of finite type scheme over \( \text{Spec} \ \mathbb{C} \). Unless we say specifically, in this article, we fix \( X \) as an \( n \)-dimensional normal projective \( \mathbb{Q} \)-Gorenstein varieties, and fix all divisors as Cartier divisors. For the convenience of writing, we do not distinguish between divisors and line bundles.

2 Preliminaries

2.1 Volume function

In this subsection, we review some of the standard facts on the volume function.

The Néron–Severi space \( N^1(X) \) is the real vector space of numerical equivalence classes of \( \mathbb{R} \)-Cartier divisors on \( X \). In general, the Néron–Severi space is denoted by \( N^1(X)_{\mathbb{R}} \). But for simplicity, we denote it by \( N^1(X) \). For any Cartier divisor \( D \), the volume of \( D \) is defined to be

\[
\text{Vol}(D) = \limsup_{m \to \infty} \frac{h^0(X, mD)}{m^n/n!}.
\]

For any natural number \( a > 0 \), we have

\[
\text{Vol}(aD) = a^n \text{Vol}(D).
\]

It follows that the volume of a \( \mathbb{Q} \)-Cartier divisor \( D \) can be defined as

\[
\text{Vol}(D) = \frac{1}{a^n} \text{Vol}(aD),
\]

for some \( a \in \mathbb{N} \), such that \( aD \) is Cartier divisor. This is independent of the choice of \( a \). The volume of a \( \mathbb{Q} \)-Cartier divisor depends only on its numerical equivalence class. Thus, the volume function can be descended to \( N^1(X)_{\mathbb{Q}} \). Then the volume function extends to \( N^1(X) \) continuously. The volume satisfies the homogeneous property, i.e.

\[
\text{Vol}(aD) = a^n \text{Vol}(D).
\]

for any \( a > 0 \) and \( D \) in \( N^1(X) \).

We recall some definitions of positivity of \( \mathbb{R} \)-divisors. An \( \mathbb{R} \)-divisor \( D \) in \( N^1(X) \) is called nef if the intersection number \( L \cdot C \) is nonnegative for any curve \( C \) on \( X \). The volume of a nef \( \mathbb{R} \)-divisor \( D \) is equal to the top self-intersection number \( D^n \). All nef classes in \( N^1(X) \) form a convex cone, called the nef cone and denoted by Nef \( (X) \), whose interior is called the ample cone, denoted by Amp \( (X) \). An \( \mathbb{R} \)-divisor \( D \) in \( N^1(X) \) is called big if

\[
\text{Vol}(D) > 0.
\]

All big classes in \( N^1(X) \) form a convex open cone, called the big cone and denoted by Big \( (X) \), whose closure is called the pseudo-effective (psef for short) cone. For any two big
$\mathbb{R}$-divisors $D$ and $B$, one obtains
\[ \text{Vol}(D + B) \geq \text{Vol}(D). \]

For more details of the volume function, we refer to the standard reference [30].

### 2.2 Positive intersection product

In this subsection, we present some preliminaries about the positive intersection product. We follow the notion of [11]. References to this subsection are [8, 11, 35, Section 7], [16].

In general, the volume of a big divisor is not equal to its top self-intersection number. But it can be computed as the movable intersection number (see [30, Chapter 11]) by Fujita’s approximation theorem. In other words, for any big divisor $D$, let $\pi_m : X_m \to X$ be the resolution of base locus $b(|mD|)$ with the exceptional divisor $E_m$ and set $D_m = \pi_m^* D - \frac{1}{m} E_m$,
\[ \text{Vol}(D) = \lim_m D_n^m. \]

To compute the volume of a big divisor, in [11] the authors introduce a valid notion, the so-called positive intersection product. Next we recall the notion (also see [8, 11] for details).

Recall that the Riemann–Zariski space of $X$ is the locally ringed space defined by
\[ X := \lim_{\to} X_\pi, \]
where $X_\pi$ runs over all birational models of $X$ with the birational morphism $\pi : X_\pi \to X$. Here the projective limit is taken in the category of locally ringed spaces. We do not use the theory of Riemann–Zariski spaces in an essential way in this paper. We refer to [40, 45] for more discussions on the structure of this space.

**Definition 1** ([11, Definition 1.1]) For any integer $0 \leq p \leq n$,

(i) the space of $p$-codimensional Weil classes on the Riemann–Zariski space $X$ is defined as
\[ N^p(X) := \lim_{\to} N^p(X_\pi), \]
with arrows defined by push-forward, where $N^p(X_\pi)$ is the real vector space of numerical equivalence classes of codimension $p$-cycles (see [26, Chapter 19]).

(ii) the space of $p$-codimensional Cartier classes on $X$ is defined as
\[ CN^p(X) := \lim_{\to} N^p(X_\pi), \]
with arrows defined by pullback.

By definition of projective limit, a Weil class $\alpha$ in $N^p(X)$ is given by its incarnations $\alpha_\pi$ in $N^p(X_\pi)$ on each smooth birational model of $X$, satisfying
\[ \nu_*(\alpha_\pi') = \alpha_\pi'' \]
for any birational morphism $\nu : X_{\pi'} \to X_{\pi''}$ with $\pi' = \pi'' \circ \nu$.

Further for each $\pi$, given a class $\alpha$ in $N^p(X_\pi)$, one can extend it to a Cartier class by its pullback. Thus, we have the natural injection
\[ N^p(X_\pi) \hookrightarrow CN^p(X). \]
When $p = 1$, we refer to the space $CN^1(\mathfrak{X})$ as the Néron–Severi space of $\mathfrak{X}$. Its elements are the so-called Shokurov’s b-divisors.

In the sequel, we use the notation $\alpha \geq 0$ for a psef class $\alpha$ in $N^p(X)$ (see [26]). We consider positive Cartier classes in $\mathfrak{X}$. For a birational morphism $\nu : V' \to V$, a class $\alpha$ in $N^1(V)$ is nef (resp. psef, big) if and only if $\nu^* \alpha$ is nef (resp. psef, big). Therefore, one can extend these definitions to the Riemann–Zariski space.

**Definition 2** ([11, Definition 1.6]) A Cartier class $\alpha$ in $CN^1(\mathfrak{X})$ is called nef (resp. psef, big) if its incarnation $\alpha_\pi$ is nef (resp. psef, big) for some $\pi$.

On a smooth projective variety $V$, for any $p$-classes $\alpha_1, \ldots, \alpha_p$ in $N^1(V)$, their intersection product $\alpha_1 \cdots \alpha_p \in N^p(V)$ (see [26]). Further for any birational morphism $\nu : V' \to V$, one has $v^*\alpha_1 \cdots v^*\alpha_p = v^*(\alpha_1 \cdots \alpha_p)$, see [26, Chapter 19]. One can define the intersection product of $p$-Cartier classes $\alpha_1, \ldots, \alpha_p$ in $CN^1(\mathfrak{X})$, which have a common determination $X_\pi$, as the Cartier class in $CN^p(\mathfrak{X})$ determined by $\alpha_1, \pi \cdots \alpha_p, \pi$.

**Definition 3** ([11, Definition 2.5]). For any big classes $\alpha_1, \ldots, \alpha_p$ in $CN^1(\mathfrak{X})$, their positive intersection product

$$\langle \alpha_1 \cdots \alpha_p \rangle \in N^p(\mathfrak{X})$$

is defined as the least upper bound of the set of classes

$$(\alpha_1 - D_1) \cdots (\alpha_p - D_p) \in N^p(\mathfrak{X})$$

where $D_i$ is an effective Cartier $\mathbb{Q}$-divisor on $\mathfrak{X}$ such that $\alpha_i - D_i$ is nef.

**Remark 1** In [8, Theorem 3.5], the authors give an analytic definition of the positive intersection product (they call it as the movable intersection product) for Kähler manifolds. For any big classes $\alpha_1, \ldots, \alpha_p$ on the Kähler manifold $V$, in which big class means that each $\alpha_j$ can be represented by a Kähler current $T$, i.e. a closed positive $(1, 1)$-current $T$ such that $T \geq \theta \omega$ for some smooth Hermitian metric $\omega$ and a small positive constant $\theta$, one defines

$$\langle \alpha_1 \cdots \alpha_p \rangle := \sup_{T_j, \nu} v_\nu(\gamma_1 \wedge \ldots \wedge \gamma_p)$$

where $T_j \in \alpha_j$ is a Kähler current with logarithmic poles, i.e. there is a modification $\nu_j : V_j \to V$ such that $v_j^*T_j = [E_j] + \gamma_j$ for some effective $\mathbb{Q}$-divisor $E_j$ and closed semi-positive form $\gamma_j$. Here we take a common modification $\nu : V' \to V$, and write

$$v^*T_j = [E_j] + \gamma_j.$$

**Definition 4** ([11, Definition 2.10]) For any psef classes $\alpha_1, \ldots, \alpha_p$ in $CN^1(\mathfrak{X})$, their positive intersection product

$$\langle \alpha_1 \cdots \alpha_p \rangle \in N^p(\mathfrak{X})$$

is defined as the limit

$$\lim_{\varepsilon \to 0^+} \langle (\alpha_1 + \varepsilon \gamma) \cdots (\alpha_p + \varepsilon \gamma) \rangle,$$

where $\gamma$ in $CN^1(\mathfrak{X})$ is any big Cartier class.

This definition is independent of the choice of the big class $\gamma$ (see [11, Definition 2.10]).

For any big $\mathbb{R}$-divisor $D$ in $N^1(X)$, we have

$$\text{Vol}(D) = \langle D^n \rangle,$$
also see [8, Definition 3.2] or [9, Definition 1.17] for an analytic definition.

An interesting fact about the volume function on the big cone, due to Boucksom–Favre–Jonsson [11], is stated as follows,

**Theorem 4** ([11, Theorem A]). The volume function is $C^1$-differentiable on the big cone of $N^1(X)$. If $\alpha$ in $N^1(X)$ is big and $\gamma$ in $N^1(X)$ is arbitrary, then

$$\frac{d}{dt} \bigg|_{t=0} \text{Vol}(\alpha + t\gamma) = n(\alpha^{n-1}) \cdot \gamma.$$  

We collect some facts about the positive intersection product as follows, for using later,

**Proposition 5** ([11, Proposition 2.9, Corollary 3.6])

(i) The positive intersection product is symmetric, homogeneous of degree 1, and super-additive in each variable. Moreover, it is continuous on the $p$-fold product of the big cone of $CN^1(X)$.

(ii) For any psef class $\alpha$ in $CN^1(X)$, one obtains

$$\langle \alpha^n \rangle = \langle \alpha^{n-1} \rangle \cdot \alpha.$$  

**Remark 2** In general, the positive intersection product is not multilinear, see [9, Definition 1.17] for an analytic explanation.

### 3 Valuative stability

In this section, we review the $\beta$-invariant given by [19] and state the definition of valuative stability.

In [19], Dervan and Legendre compute the Donaldson–Futaki invariant of the test configuration associated to a dreamy divisor for a polarised variety and obtain a new numerical invariant, which generalizes Fujita’s original $\beta$-invariant. Then they show that valuative stability over dreamy divisors is equivalent to $K$-stability over integral test configurations. Here an integral test configuration means that its central fiber is integral. In this paper, we do not involve the explicit definition of $K$-stability, refer to [12, 19, 21].

Let $(X, L)$ be a polarised variety, let $\pi: Y \to X$ be a surjective birational morphism.

**Definition 5** A prime divisor $F \subset Y$ for some birational model $Y$ over $X$ is called a prime divisor over $X$. Denote by $\text{PDiv}/X$ the set of all prime divisors over $X$.

One can view $F$ as a divisorial valuation $\text{ord}_F$ on $X$, defined on the function field of $X$. In particular, we can always assume that $Y$ is smooth by taking a resolution of singularities. Since the information of the valuation associated to $F$, which we are interested in, does not change under the resolution of singularities, see [29, Remark 2.23].

**Definition 6** For any $F$ in $\text{PDiv}/X$, the log discrepancy $A_X(F)$ is defined to be

$$A_X(F) := 1 + \text{ord}_F(K_Y - \pi^*K_X).$$

Note that the log discrepancy is well-defined, since we always assume that the canonical divisor $K_X$ is $\mathbb{Q}$-Cartier.

For any prime divisor $F$ over $X$, one can define a subspace $H^0(X, mL -xF) \subset H^0(X, mL)$ by the identifications

$$H^0(X, mL -xF) := H^0(Y, m\pi^*L -xF) \subset H^0(Y, m\pi^*L) \cong H^0(X, mL).$$
For any ample divisor $L$, one defines the slope of $(X, L)$ to be

$$\mu(L) := \frac{-K_X \cdot L^{n-1}}{L^n}.$$  

For any $F$ in $\text{PDiv}/X$, Dervan and Legendre define

$$\beta_L(F) := A_X(F) \text{Vol}(L) + n\mu(L) \int_0^{+\infty} \text{Vol}(L - xF) dx + \int_0^{+\infty} \text{Vol}'(L - xF) K_X dx,$$  

(3)

where

$$\text{Vol}(L - xF) := \text{Vol}(\pi^* L - x F),$$

and

$$\text{Vol}'(L - xF) K_X := \frac{d}{dt} \Big|_{t=0} \text{Vol}(\pi^* L - x F + t \pi^* K_X).$$

For simplicity, we always omit $\pi^*$ in the above notations. It follows from Theorem 4 that the notation $\text{Vol}'(L - xF) K_X$ is well-defined for any $L$ in $\text{Big}(X)$ and $F$ in $\text{PDiv}/X$. It is straightforward that $\beta_L(\cdot)$ depends only on the numerical equivalence class of $L$.

There are three numerical invariants on the space of prime divisors over $X$. Roughly speaking, these can be viewed as norms. For any $F$ in $\text{PDiv}/X$, we set

$$S_L(F) := \int_0^{+\infty} \text{Vol}(L - xF) dx,$$

and

$$j_L(F) := \text{Vol}(L) \tau_L(F) - S_L(F),$$

where $\tau_L(F)$ is the pseudo-effective threshold of $F$ with respect to $L$, defined by

$$\tau_L(F) := \sup \{ x \in \mathbb{R} : \text{Vol}(L - x F) > 0 \}.$$

Note that our notation $S_L(F)$ is different from the usual one, which is equal to $S_L(F)/\text{Vol}(L)$. But just for convenience, we use this notation.

**Lemma 6** When $L$ is ample, for any prime divisor $F$, $\tau_L(F)$, $S_L(F)$ and $j_L(F)$ have following relations

$$\frac{1}{n+1} \text{Vol}(L) \tau_L(F) \leq j_L(F) \leq \frac{n}{n+1} \text{Vol}(L) \tau_L(F),$$  

(4)

and

$$\frac{1}{n+1} \text{Vol}(L) \tau_L(F) \leq S_L(F) \leq \frac{n}{n+1} \text{Vol}(L) \tau_L(F).$$  

(5)

The invariant $j_L(\cdot)$ can be viewed as a norm corresponding to non-Archimedean functional $J^\text{NA}$ and $S_L(\cdot)$ corresponds to $J^\text{NA} - J^\text{NA}$, see [19, Section 2], [10] and [12, Section 7.2]. The proof of this lemma is essentially same as that of Fujita [22] in Fano case, also see [5, Proposition 3.11].

**Proof of Lemma 6** We only need to show (5). The first inequality of (5) is given by the concavity of the volume function, which gives

$$\text{Vol}(L - xF) \geq \text{Vol}(L) \left( \frac{x}{\tau_L(F)} \right)^n.$$  

It follows that

$$S_L(F) \geq \frac{1}{n+1} \text{Vol}(L) \tau_L(F).$$
The second inequality is proved in [22, Proposition 2.1] (In [22], $L = -K_X$, but this condition is not used in the proof).

For any $L$ in $\text{Amp}(X)$, we define two numerical invariants:

$$s(L) := \sup \{ s \in \mathbb{R} : -K_X - sL \text{ is ample} \},$$

and

$$\tilde{s}(L) := \inf \{ s \in \mathbb{R} : K_X + sL \text{ is ample} \}.$$

By definitions of $s(L)$ and $\tilde{s}(L)$, we have $\mu(L) \geq s(L)$ and $\mu(L) \leq \tilde{s}(L)$. Indeed, if one assumes that $-K_X - \mu(L)L$ is ample, then

$$0 < (-K_X - \mu(L)L) \cdot L^{n-1} = (-K_X \cdot L^{n-1} - \mu(L)L^n)$$

$$= \left( -\frac{K_X \cdot L^{n-1}}{L^n} - \mu(L) \right) L^n.$$

This leads to a contradiction. It follows that $\mu(L) \geq s(L)$. Another one is similar.

**Lemma 7** For any big divisor $L$ in $N_1^1(X)$ and any prime divisor $F$ over $X$, we have

$$\int_0^{\tau_L(F)} n((L - xF)^{n-1}) \cdot Ldx = (n+1) \int_0^{\tau_L(F)} \text{Vol}(L - xF)dx. \quad (6)$$

This Lemma is due to [19, Corollary 3.11]. Its proof is a standard computation by using integration by part, left to the reader.

By this lemma, we can re-write $\beta$ as

$$\beta_L(F) = A_X(F) \text{Vol}(L) + (n\mu(L) - (n+1)s(L)) S_L(F)$$

$$- \int_0^{\tau_L(F)} \text{Vol}'(L - xF).(-s(L)L - K_X)dx,$$

or

$$\beta_L(F) = A_X(F) \text{Vol}(L) + (n\mu(L) - (n+1)\tilde{s}(L)) S_L(F)$$

$$+ \int_0^{\tau_L(F)} \text{Vol}'(L - xF).(\tilde{s}(L)L + K_X)dx. \quad (7, 8)$$

**Definition 7** For any $L$ in $\text{Amp}(X)$, a polarised variety $(X, L)$ is called

(i) *valuatively semistable* if $\beta_L(F) \geq 0$ for any prime divisor $F$ over $X$;

(ii) *valuatively stable* if $\beta_L(F) > 0$ for any non-trivial prime divisor $F$ over $X$, in which non-trivial prime divisor $F$ means that the divisorial valuation associated to $F$ is non-trivial;

(iii) *uniformly valuatively stable* if there exists an $\varepsilon_L > 0$ such that

$$\beta_L(F) \geq \varepsilon_L S_L(F) \quad (9)$$

for any prime divisor $F$ over $X$. 
Remark 3  
(i) Note that in [19], valuative stability means that $\beta_L$ satisfies the demanded inequality for all dreamy divisors (see [19, Definition 2.6]). If $\beta_L$ is nonnegative for all prime divisors over $X$, it is called strongly valuatively semistable in [19].

(ii) In [19] the authors use the norm $j_L(\cdot)$ to define uniformly valuative stability. By Lemma 6, $j_L$ and $S_L$ are equivalent.

In this paper, we wish to investigate the openness of uniformly valuative stability. Our main theorem is stated as follows,

**Theorem 8** The uniformly valuative stability locus

\[ \text{UVs} := \{ [L] \in \text{Amp}(X) : (X, L) \text{ is uniformly valuatively stable} \} \]

is an open subcone of the ample cone $\text{Amp}(X)$.

### 4 Proof of openness of valuative stability

In this section, we give a proof of Theorem 8.

We first give a rough idea of setup: Fix an ample $\mathbb{R}$-divisor $L$, which is uniformly valuably stable, and choose a constant $\varepsilon_L > 0$ such that

$$\beta_L(F) \geq \varepsilon_L S_L(F)$$

for any prime divisor $F$ over $X$. Our goal is to show that there exists a small open neighborhood $U$ of $L$ in $\text{Amp}(X)$ such that, for any $L'$ in $U$ there is a constant $\varepsilon_{L'} > 0$ satisfying

$$\beta_{L'}(F) \geq \varepsilon_{L'} S_{L'}(F)$$

for all prime divisor $F$ over $X$.

To define such an open neighborhood of $L$, we fix any norm $\| \cdot \|$ on $N^1(X)$ and define an open subset

$$U_\varepsilon := \{ L' \in \text{Amp}(X) : \| L' - L \| < \varepsilon \}.$$ 

If necessary, we shrink this neighborhood, i.e. shrink $\varepsilon$.

It suffices to prove following these two estimates

$$\beta_{L'}(F) - \beta_L(F) \geq -f(\varepsilon) S_{L'}(F) \quad (10)$$

and

$$S_L(F) \geq s^-(\varepsilon) S_{L'}(F) \quad (11)$$

for any prime divisor $F$ over $X$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $s^- : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with $f(\varepsilon) \to 0$ and $s^-(\varepsilon) \to 1$ as $\varepsilon \to 0$. Indeed,

$$\beta_{L'}(F) = \beta_L(F) + \beta_{L'}(F) - \beta_L(F) \geq \varepsilon_L S_L(F) - f(\varepsilon) S_{L'}(F) \geq (\varepsilon_L s^- - f(\varepsilon)) S_L(F). \quad (12)$$

**Lemma 9** For any $L$ in $\text{Amp}(X)$, there exists a small constant $\varepsilon > 0$, such that for any $L'$ in $U_\varepsilon$ satisfying the following inequality

$$s^-(\varepsilon) S_L(F) \leq S_L(F) \leq s^+(\varepsilon) S_{L'}(F)$$

for all $F$ in $\text{PDiv}_X$, where $s^- : \mathbb{R}^+ \to \mathbb{R}^+$ and $s^+ : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions with $s^-(\varepsilon) \to 1$ and $s^+(\varepsilon) \to 1$ as $\varepsilon \to 0$. Moreover, $s^-(\varepsilon) < 1$ and $s^+(\varepsilon) > 1$.
Proof For any $L'$ in $U_\varepsilon$, we write it as $L' = L + \varepsilon H$ for some $\mathbb{R}$-divisor $H$ in $N^1(X)$. For any $s > 0$, we can write

$$L + \varepsilon H = \frac{1}{1 + s} \left( L + s(L + \frac{(1 + s)\varepsilon}{s} H) \right),$$

and set

$$L_s := L + \frac{(1 + s)\varepsilon}{s} H.$$  

Then by choosing $s$ small enough (determined later), which depends on $\varepsilon$, we can assume that both $(1 + s)L - L_s$ and $L_s - (1 - s)L$ are big. Indeed,

$$(1 + s)L - L_s = s \left( L - \frac{(1 + s)\varepsilon}{s^2} H \right),$$

for instance, take $s = \varepsilon^{1/4}$, then $(1 + s)L - L_s$ is big when $\varepsilon$ is small enough.

Thus we have

$$S_{L'}(F) = \int_{0}^{+\infty} \Vol(L' - x F) dx$$

$$= (1 + s)^{-n} \int_{0}^{+\infty} \Vol(L + s L_s - (1 + s) x F) dx$$

$$\geq (1 + s)^{-n} \int_{0}^{+\infty} \Vol(L + (s - s^2)L - (1 + s) x F) dx$$

$$= \left( \frac{1 + s - s^2}{1 + s} \right)^n \int_{0}^{+\infty} \Vol(L - \frac{1 + s}{1 + s - s^2} x F) dx$$

$$= \left( \frac{1 + s - s^2}{1 + s} \right)^{n+1} S_L(F). \quad (13)$$

On the other hand, similarly, we have

$$S_{L'}(F) \leq \left( \frac{1 + s + s^2}{1 + s} \right)^{n+1} S_L(F). \quad (14)$$

By taking

$$s^- (\varepsilon) = \left( 1 - \frac{\varepsilon^{1/2}}{1 + \varepsilon^{1/4}} \right)^{n+1} \text{ and } s^+ (\varepsilon) = \left( 1 + \frac{\varepsilon^{1/2}}{1 + \varepsilon^{1/4}} \right)^{n+1},$$

we finish the proof of Lemma 9. \(\square\)

In what follows, we aim to establish inequality (10). In fact, we do not need to show the inequality (10) for any prime divisor $F$ over $X$. By definition of uniformly valuative stability, we introduce a subset of prime divisors over $X$ as in the next definition, on which it is clearly sufficient to test uniformly valuative stability.

Definition 8 For any $L$ in $\Amp(X)$, set

$$\mathcal{D}_L^{\mathrm{ind}} := \{ F \in \mathrm{PDiv}_X : \beta_L(F) \leq C_L S_L(F) \},$$

for some constant $C_L > 0$ (determined later).
It follows that we only need to prove the inequality (10) for any $F$ in $\mathcal{D}^{\text{ud}}_{U_{L}'}$. Since it automatically satisfies the condition of uniformly valuative stability when $F$ is not contained in $\mathcal{D}^{\text{ud}}_{U_{L}'}$.

Then for any $F$ in $\mathcal{D}^{\text{ud}}_{U_{L}'}$, we have

$$AX(F)\text{Vol}(L') + \int_{0}^{+\infty} n((L' - xF)^{n-1}) \cdot (K_X + \tilde{s}(L')L')dx \leq (C_{L'} - n\mu(L') + (n+1)\tilde{s}(L'))S_{L'}(F),$$

where we have used Lemma 7 or (8). Now we choose $C_{L'} > 0$ such that $C_{L'} - n\mu(L') + (n+1)\tilde{s}(L') \geq 0$.

**Theorem 10** Given a divisor $L$ in $\text{Amp}(X)$, there exists a constant $\varepsilon_0 > 0$ and a continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$, such that for any $0 < \varepsilon \leq \varepsilon_0$ and any $L' \in U_{\varepsilon}$, the inequality

$$\beta_{L'}(F) - \beta_L(F) \geq -f(\varepsilon)S_{L'}(F)$$

is satisfied for all $F \in \mathcal{D}^{\text{ud}}_{U_{L}'}$. Moreover, the choice of $f$ only depends on $X$ and $L$.

We first show the estimate of the second term of $\beta$-invariant, i.e. $\mu S$.

**Lemma 11** For any $L$ in $\text{Amp}(X)$, there exists a constant $\varepsilon_0 > 0$ and a continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{\varepsilon \to 0} h(\varepsilon) = 0$, such that for any $0 < \varepsilon \leq \varepsilon_0$ and any $L' \in U_{\varepsilon}$, the inequality

$$n\mu(L')S_{L'}(F) - n\mu(L)S_{L}(F) \geq -h(\varepsilon)nS_{L'}(F)$$

is satisfied for all $F \in \mathcal{D}^{\text{ud}}_{U_{L}'}$. Moreover, the choice of $h$ only depends on $X$ and $L$.

**Proof** For simplicity, we denote

$$s_-(n) := \left(1 - \frac{s^2}{1+s}\right)^n < 1 \text{ and } s_+(n) := \left(1 + \frac{s^2}{1+s}\right)^n > 1.$$  

For any $L' \in U_{\varepsilon}$, we can write

$$L' = L + \varepsilon H = \frac{1}{1+s} (L + sL_s)$$

in the same way as the proof of Lemma 9, for some $\mathbb{R}$-divisor $H$ and $L_s$ in $N^1(X)$.

Thus we have

$$\text{Vol}(L') = (1+s)^{-n}\text{Vol}(L+sL_s) \geq (1+s)^{-n}\text{Vol}(L+(s-s^2)L) = s_-(n)\text{Vol}(L).$$

Similarly, one obtains

$$\text{Vol}(L') \leq s_+(n)\text{Vol}(L).$$

The proof falls naturally into two cases.

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(1) When $\mu(L') \geq 0$, then we compute
\[
\begin{align*}
n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) & \geq nS_L(F)(s^-\varepsilon\mu(L') - \mu(L)) \\
& = nS_L(F)\left((s^-\varepsilon - 1)\mu(L') + \frac{-K_X \cdot (L')^{n-1}}{\text{Vol}(L')} - \frac{-K_X \cdot L^{n-1}}{\text{Vol}(L)}\right) \\
& \geq nS_L(F)\left((s^-\varepsilon - 1)\mu(L') + \frac{-K_X \cdot (L')^{n-1}}{s_+(n)\text{Vol}(L)} - \frac{-K_X \cdot L^{n-1}}{\text{Vol}(L)}\right) \\
& \geq nS_L(F)\left((s^-\varepsilon - 1)\mu(L') + (s_+(n) - 1)\frac{-K_X \cdot (L')^{n-1}}{\text{Vol}(L)} + \frac{1}{\text{Vol}(L)}(-K_X \cdot (L')^{n-1} - (-K_X) \cdot L^{n-1})\right) \\
& \geq nS_L(F)\left((s^-\varepsilon - 1)\mu(L') + \frac{1}{s_+(n)} - 1)s_+(n)\mu(L') + \frac{1}{\text{Vol}(L)}((-K_X) \cdot H((L')^{n-2} + (L')^{n-3} \cdot L + \cdots + L^{n-2})\right) \\
& \geq nS_L(F)\left((s^-\varepsilon - s_+(n))\mu(L') + \varepsilon\frac{1}{\text{Vol}(L)}((-K_X) \cdot H((L')^{n-2} + (L')^{n-3} \cdot L + \cdots + L^{n-2})\right).
\end{align*}
\]
In general, we do not know the sign of
\[
\frac{1}{\text{Vol}(L)}((-K_X) \cdot H((L')^{n-2} + (L')^{n-3} \cdot L + \cdots + L^{n-2}).
\]
But we can cancel it directly if it is nonnegative. Therefore, without loss of generality, we may assume that it is negative. Then
\[
\begin{align*}
n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) & \geq h_1(\varepsilon) - g(\varepsilon)nS_L(F) \\
& \geq (-h_1(\varepsilon) - g(\varepsilon))s^-\varepsilon^{-1}nS_{L'}(F),
\end{align*}
\]
where
\[
g(\varepsilon) = -\varepsilon\frac{1}{\text{Vol}(L)}((-K_X) \cdot H((L')^{n-2} + (L')^{n-3} \cdot L + \cdots + L^{n-2}) \tag{17}
\]
which is a polynomial in $\varepsilon$ with degree $n - 1$ and $g(0) = 0$, whose coefficients depend on $-K_X$, $L$, $H$, and the leading term is $\text{Vol}(L)^{-1}(-K_X) \cdot H^{n-1}$, and $h_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $h_1(\varepsilon) \to 0$ as $\varepsilon \to 0$, which depends on $\mu(L')$.
In fact, $h_1$ is independent of the choice of $L'$. Since $g$ is a polynomial with degree $n - 1$ and $L'$ can be represented by a basis of Nef cone (see the following Lemma 12), then the choice of $g$ only depends on $X$ and $L$.

(2) When $\mu(L') \leq 0$, the computation is similar. We omit it.

This completes the proof of Lemma 11 by taking $h = (h_1 + g)s^-\varepsilon^{-1}$. $\square$

**Lemma 12** There exists a constant $a > 0$, which depends on $\varepsilon$ and $L$, such that
\[
(1 - a)L \leq L' \leq (1 + a)L
\]
for any $L'$ in $U_\varepsilon$. 

\begin{affirmation}
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\end{affirmation}
Moreover, such a can be chosen as small as we wish by choosing \( \varepsilon \) small.

**Proof** For any \( L' \) in \( U_\varepsilon \), we write it as \( L' = L + H \) for some \( \mathbb{R} \)-divisor \( H \) in \( N^1(X) \) with \( \| H \| < \varepsilon \). Set \( \rho := \dim_{\mathbb{R}} N^1(X) \). Since \( L \) is ample, there exists a basis \( (A_1, \ldots, A_\rho) \) of \( N^1(X) \) with each \( A_i \) in \( \text{Nef}(X) \), and there exists some \( t_1, \ldots, t_\rho \in \mathbb{R}_{>0} \) such that \( L = \sum_{i=1}^\rho t_i A_i \) with \( \sum_{i=1}^\rho t_i = 1 \). Set \( t_0 = \min_i t_i \in \mathbb{R}_{>0} \). We may assume that the norm \( \| \cdot \| \) is given by

\[
\left\| \sum_{i=1}^\rho s_i A_i \right\| := \sum_{i=1}^\rho |s_i|.
\]

Set \( H = \sum_{i=1}^\rho r_i A_i \) with \( \| H \| < \varepsilon \) (i.e. \( \sum_{i=1}^\rho |r_i| < \varepsilon \)). Then we have

\[
L' = L + H = \sum_{i=1}^\rho (t_i + r_i) A_i < \sum_{i=1}^\rho (t_i + \varepsilon) A_i = \sum_{i=1}^\rho (t_i + \frac{\varepsilon}{t_0}) A_i \leq (1 + \frac{\varepsilon}{t_0}) \sum_{i=1}^\rho t_i A_i,
\]

also

\[
L' = L + H = \sum_{i=1}^\rho (t_i + r_i) A_i > \sum_{i=1}^\rho (t_i - \varepsilon) A_i = \sum_{i=1}^\rho (t_i - \frac{\varepsilon}{t_0}) A_i \geq (1 - \frac{\varepsilon}{t_0}) \sum_{i=1}^\rho t_i A_i.
\]

The proof is completed by taking \( a = \varepsilon/t_0 \), where \( t_0 = \min_i t_i \).

**Remark 4** Consistent with the notation in Sect. 2.2, " \( \leq \) " means that their difference is a psef class. In fact, \( (1 + a)L - L' \) and \( L' - (1 - a)L \) are nef according to the proof of Lemma 12.

We now turn to the proof of Theorem 10.

**Proof of Theorem 10** For any \( L' \) in \( U_\varepsilon \), we write

\[
L' = L + \varepsilon H = \frac{1}{1+s} (L + s L_s)
\]

in the same way as the proof of Lemma 9, for some \( \mathbb{R} \)-divisor \( H \) and \( L_s \) in \( N^1(X) \) such that both \( (1 + s)L - L_s \) and \( L_s - (1 - s)L \) are big when \( \varepsilon \) is small enough, where \( s = \varepsilon^{1/4} \).

We divide into following these two cases. 

\( \text{Springer} \)
One assumes that \( \mu(L') \geq 0 \), then \( \tilde{s}(L') \geq 0. \)

\[
\beta_{L'}(F) - \beta_L(F) = A_X(F) (\text{Vol}(L') - \text{Vol}(L)) + n \mu(L') S_{L'}(F) - n \mu(L) S_L(F) \\
+ \int_0^{+\infty} n \langle (L' - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx \\
- \tilde{s}(L') \int_0^{+\infty} n \langle (L' - x F)^{n-1} \rangle \cdot L' dx \\
+ \int_0^{+\infty} n \langle (L - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx \\
- \int_0^{+\infty} n \langle (L - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx \\
\geq n \mu(L') S_{L'}(F) - n \mu(L) S_L(F) + A_X(F) \text{Vol}(L') (1 - s_-(n)^{-1}) \\
- \tilde{s}(L') \int_0^{+\infty} n \langle (L' - x F)^{n-1} \rangle \cdot L' dx + \tilde{s}(L') \int_0^{+\infty} n \langle (L - x F)^{n-1} \rangle \cdot L' dx \\
+ \int_0^{+\infty} n \left( \langle (L' - x F)^{n-1} \rangle - \langle (L - x F)^{n-1} \rangle \right) \cdot (K_X + \tilde{s}(L')L') dx. 
\] 

(19)

By Lemma 12, we can take a small positive constant \( a \) (recall \( a = \varepsilon / t_0 \)) such that

\[
(1 - a)L \leq L' \leq (1 + a)L,
\]

for any \( L' \) in \( U_\varepsilon \). Then one obtains

\[
(1 - a)L - x F \leq L' - x F \leq (1 + a)L - x F.
\]

Therefore, by continuity and homogeneity of the positive intersection product (see Proposition 5 or [11, Proposition 2.9]), we have

\[
(1-a)^{n-1} \langle (L - \frac{x}{1-a} F)^{n-1} \rangle \leq \langle (L' - x F)^{n-1} \rangle \leq (1+a)^{n-1} \langle (L - \frac{x}{1+a} F)^{n-1} \rangle. 
\] 

(20)

Since \( K_X + \tilde{s}(L')L' \) is nef, we have

\[
(1 - a)^{n-1} \langle (L - \frac{x}{1-a} F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') \leq \langle (L' - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') \\
\leq (1 + a)^{n-1} \langle (L - \frac{x}{1+a} F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L').
\]

It follows that

\[
\int_0^{+\infty} n \langle (L' - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx \\
\geq (1 - a)^{n-1} \int_0^{+\infty} n \langle (L - \frac{x}{1-a} F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx \\
= (1 - a)^n \int_0^{+\infty} n \langle (L - x F)^{n-1} \rangle \cdot (K_X + \tilde{s}(L')L') dx.
\]
Thus, we obtain
\[
\int_{0}^{+\infty} n \left( (\langle L' - x F \rangle^{n-1}) - (\langle L - x F \rangle^{n-1}) \right) \cdot (K_X + \tilde{s}(L')L') \, dx
\geq (1 - (1 - a)^{-n}) \int_{0}^{+\infty} n(\langle L' - x F \rangle^{n-1}) \cdot (K_X + \tilde{s}(L')L') \, dx.
\]

Recall \( s = \varepsilon^{1/4} \), when we choose \( \varepsilon \) small enough, then \( a (= \varepsilon/t_0, \text{see Lemma 12}) \) can be chosen small enough, such that
\[
1 - s_-(n)^{-1} \leq 1 - (1 - a)^{-n}.
\]

Then, we obtain
\[
\begin{align*}
A_X(F) \text{Vol}(L') (1 - s_-(n)^{-1}) \\
+ \int_{0}^{+\infty} n \left( (\langle L' - x F \rangle^{n-1}) - (\langle L - x F \rangle^{n-1}) \right) \cdot (K_X + \tilde{s}(L')L') \, dx \\
\geq (1 - s_-(n)^{-1}) \left( A_X(F) \text{Vol}(L') + \int_{0}^{+\infty} n(\langle L' - x F \rangle^{n-1}) \cdot (K_X + \tilde{s}(L')L') \, dx \right) \\
\geq (1 - s_-(n)^{-1}) \left( C_{L'} - n\mu(L') + (n + 1)\tilde{s}(L') \right) S_{L'}(F) \tag{21}
\end{align*}
\]
where we used (15) and Lemma 7 for the second inequality.

Since \( L' \) is ample, by (20), we have
\[
(1 - a)^{n-1} \langle (L - \frac{x}{1 - a} F)^{n-1} \rangle \cdot L' \leq \langle (L' - x F)^{n-1} \rangle \cdot L' \\
\leq (1 + a)^{n-1} \langle (L - \frac{x}{1 + a} F)^{n-1} \rangle \cdot L'.
\]

Then one obtains
\[
\int_{0}^{+\infty} n(\langle L' - x F \rangle^{n-1}) \cdot L' \, dx \leq \int_{0}^{+\infty} (1 + a)^{n-1} n(\langle L - \frac{x}{1 + a} F \rangle^{n-1}) \cdot L' \, dx \\
= (1 + a)^{n} \int_{0}^{+\infty} n(\langle L - x F \rangle^{n-1}) \cdot L' \, dx.
\]

It follows that
\[
\tilde{s}(L') \int_{0}^{+\infty} n(\langle L - x F \rangle^{n-1}) \cdot L' \, dx - \tilde{s}(L') \int_{0}^{+\infty} n(\langle L' - x F \rangle^{n-1}) \cdot L' \, dx \\
\geq \tilde{s}(L')((1 + a)^{-n} - 1) \int_{0}^{+\infty} n(\langle L' - x F \rangle^{n-1}) \cdot L' \, dx \\
= \tilde{s}(L')((1 + a)^{-n} - 1) (n+1) S_{L'}(F). \tag{22}
\]

Note that here we have used \( \tilde{s}(L') \geq 0 \) and Lemma 7.

Combining (19), (21), (22), and (16), we have
\[
\begin{align*}
\beta_{L'}(F) - \beta_L(F) \\
\geq - h(\varepsilon) n S_{L'}(F) + \left( (1 - s_-(n)^{-1}) (C_{L'} - n\mu(L') + (n + 1)\tilde{s}(L')) \right) \\
+ (n + 1)\tilde{s}(L')((1 + a)^{-n} - 1) S_{L'}(F) \\
\geq - f(\varepsilon) S_{L'}(F).
\end{align*}
\]
where \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function with \( f(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), which depends on \( \mu(L') \), \( s(L') \) and \( C_{L'} \) and intersection numbers \( (\mathcal{K}_X) \cdot (L')^k \cdot L^{n-1-k} \) for \( k = 0, \ldots, n-1 \).

By definition, we know that \( \mu(L') \) and \( s(L') \) are continuous with respect to \( L' \). Thus, we can choose \( C_{L'} \) continuously depending on \( L' \). Therefore, the choice of \( f \) only depends on \( X \) and \( L \).

(2) One assumes that \( \mu(L') \leq 0 \), then \( s(L') \leq 0 \). We use the same idea of case (1).

\[
\beta_{L'}(F) - \beta_L(F) = A_X(F)(\text{Vol}(L') - \text{Vol}(L)) + n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) \\
+ \int_{0}^{+\infty} n((L' - xF)^{n-1}) \cdot K_X dx \\
+ \int_{0}^{+\infty} n((L' - xF)^{n-1})(-K_X - s(L')L') dx \\
- \int_{0}^{+\infty} n((L' - xF)^{n-1})(-K_X - s(L')L') dx \\
+ \int_{0}^{+\infty} n((L - xF)^{n-1})(-K_X - s(L')L') dx \\
+ s(L') \int_{0}^{+\infty} n((L - xF)^{n-1}) \cdot L' dx \\
\geq n\mu(L')S_{L'}(F) - n\mu(L)S_L(F) + A_X(F)\text{Vol}(L')(1 - s_-(n)^{-1}) \\
- s(L') \int_{0}^{+\infty} n((L' - xF)^{n-1}) - ((L - xF)^{n-1}) \cdot L' dx \\
+ \int_{0}^{+\infty} n(((L - xF)^{n-1}) - ((L' - xF)^{n-1})) \cdot (-K_X - s(L')L') dx. \tag{23}
\]

By (20) and Lemma 7, we have

\[
\int_{0}^{+\infty} n(((L - xF)^{n-1}) - ((L' - xF)^{n-1})) \cdot (-K_X - s(L')L') dx \\
\geq ((1 + a)^{-n} - 1) \int_{0}^{+\infty} n((L' - xF)^{n-1}) \cdot (-K_X - s(L')L') dx \\
= (1 - (1 + a)^{-n}) \int_{0}^{+\infty} n((L' - xF)^{n-1}) \cdot K_X \\
+ ((1 + a)^{-n} - 1)(-s(L'))(n + 1)S_{L'}(F). \tag{24}
\]

Since \( F \) belongs to \( D_{L'}^{nd} \), one obtains

\[
A_X(F)\text{Vol}(L')(1 - s_-(n)^{-1}) \\
\geq (1 - s_-(n)^{-1}) \left( (C_{L'} - n\mu(L'))S_{L'}(F) - \int_{0}^{+\infty} n((L' - xF)^{n-1}) \cdot K_X dx \right). \tag{25}
\]
Since $L'$ is ample, by (20) and Lemma 7, we obtain

$$
(-s(L')) \int_0^{+\infty} n \left( ((L' - x F)^{n-1}) - (L - x F)^{n-1} \right) \cdot L'dx
$$

$$
\geq (-s(L'))(1 - (1 - a)^{-n}) \int_0^{+\infty} n \left( (L' - x F)^{n-1} \right) \cdot L'dx
$$

$$
= (-s(L'))(1 - (1 - a)^{-n})(n + 1)S_{L'}(F). \tag{26}
$$

In addition, we have the following natural lower bound,

$$
\int_0^{+\infty} n \left( (L' - x F)^{n-1} \right) \cdot K_X dx = \int_0^{+\infty} n \left( (L' - x F)^{n-1} \right) \cdot (K_X + \tilde{s}(L') L')dx
$$

$$
- \tilde{s}(L') \int_0^{+\infty} n \left( (L' - x F)^{n-1} \right) \cdot L'dx
$$

$$
\geq - \tilde{s}(L')(n + 1)S_{L'}(F). \tag{27}
$$

Combining (16) and (23–27), we have

$$
\beta_{L'}(F) - \beta_L(F)
$$

$$
\geq -h(\varepsilon)nS_{L'}(F) + \left( (1 - s_-(n)^{-1})(C_{L'} - n\mu(L')) + ((1 + a)^{-n} - 1)(-s(L'))(n + 1) + (-s(L'))(1 - (1 - a)^{-n})(n + 1) \right)S_{L'}(F)
$$

$$
+ \left( 1 - (1 + a)^{-n} + s_-(n)^{-1} - 1 \right) \int_0^{+\infty} n \left( (L' - x F)^{n-1} \right) \cdot K_X dx
$$

$$
\geq -h(\varepsilon)nS_{L'}(F) + \left( (1 - s_-(n)^{-1})(C_{L'} - n\mu(L')) + ((1 + a)^{-n} - (1 - a)^{-n})(-s(L'))(n + 1) \right)S_{L'}(F)
$$

$$
+ \left( s_-(n)^{-1} - (1 + a)^{-n} \right)(-\tilde{s}(L'))(n + 1)S_{L'}(F)
$$

$$
\geq -f(\varepsilon)S_{L'}(F).
$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, which depends on $\mu(L')$, $\tilde{s}(L')$, $s(L')$ and $C_{L'}$ and intersection numbers $(-K_X) \cdot (L')^k \cdot L'^{-k}$ for $k = 0, \ldots, n - 1$.

Similar to case (1), we can choose a continuous function $f$ which only depends on $X$ and $L$. By combining these two cases above, we complete the proof of Theorem 10. \hfill \Box

Finally, we finish the proof of the main theorem.

**Proof of Theorem 8** For any $L$ in UVs, by Theorem 10, there exists a constant $\varepsilon_0 > 0$ and a continuous function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$, which only depends on $X$ and $L$, such that for any $0 < \varepsilon \leq \varepsilon_0$ and any $L' \in U_\varepsilon$, the inequality

$$
\beta_{L'}(F) - \beta_L(F) \geq -f(\varepsilon)S_{L'}(F)
$$

is satisfied for all $F$ in $D_{L'}^{\text{nd}}$.

Since $L$ is uniformly valuatively stable, combining with (12), we have

$$
\beta_{L'}(F) \geq \left( \frac{\varepsilon L}{2} - f(\varepsilon) \right)S_{L'}(F),
$$
for all $F$ in $\mathcal{D}^\text{ud}_{L'}$.

It follows that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\varepsilon L}{2} - f(\varepsilon) > 0$$

for all $L$ in $U_\varepsilon$ and any $0 < \varepsilon \leq \varepsilon_0$. Then for any $L$ in $U_\varepsilon$, we have

$$\beta_{L'}(F) \geq \varepsilon L' S_{L'}(F)$$

for some constant $\varepsilon L' > 0$ and all $F$ in $\mathcal{D}^\text{ud}_{L'}$. Thus, $L'$ belongs to UVs for any $L'$ in $U_\varepsilon$.

Finally, by definitions of $\beta$ and $S$-invariant, we have

$$\beta_{kL'}(F) = k^n \beta_{L'}(F), \quad S_{kL'}(F) = k^{n+1} S_{L'}(F),$$

for $k > 0$. Then $\mathbb{R}_+ U_\varepsilon \subset \text{UVs}$. Therefore, the uniformly valuative stability locus UVs is an open subcone of $\text{Amp}(X)$. $\Box$

5 Continuity of the uniformly valuative stability threshold

As an immediate application of Theorem 10 and 8, in this section, we show continuity of the uniformly valuative stability threshold.

Definition 9 For any $L$ in $\text{Amp}(X)$, the uniformly valuative stability threshold of $L$ is defined to be

$$\zeta(L) := \sup\{x \in \mathbb{R} : \beta_L(F) \geq x S_L(F) \text{ for any } F \in \text{PDiv}_X\}.$$

In fact, when $(X, L) = (X, -K_X)$ is Fano, we have $\zeta(L) = \delta(X) - 1$. This is the main motivation to study the $\zeta$-invariant.

Recall the definition of $\delta$-invariant, due to Blum and Jonsson [5],

$$\delta(L) = \inf_{F \in \text{PDiv}_X} \frac{A_X(F) \text{Vol}(L)}{S_L(F)},$$

see [25] for the original definition of $\delta$-invariant. Thus, one obtains

$$A_X(F) \text{Vol}(L) \geq \delta(L) S_L(F)$$

for any $F$ in $\text{PDiv}_X$. By (27), we have a natural lower bound

$$\beta_L(F) \geq (\delta(L) + n \mu(L) - (n + 1) \tilde{s}(L)) S_L(F),$$

i.e.

$$\zeta(L) \geq \delta(L) + n \mu(L) - (n + 1) \tilde{s}(L).$$

One can take a $c_L > 0$ such that $\delta(L) + n \mu(L) - (n + 1) \tilde{s}(L) + c_L > 0$. Thus now we set $C_L := \delta(L) + n \mu(L) - (n + 1) \tilde{s}(L) + c_L > 0$ in the definition of $\mathcal{D}^\text{ud}_L$. We also define

$$\zeta^\text{ud}(L) := \sup\{x \in \mathbb{R} : \beta_L(F) \geq x S_L(F) \text{ for any } F \in \mathcal{D}^\text{ud}_L\}.$$

By definition, one obtains

$$C_L \geq \zeta^\text{ud}(L).$$

Lemma 13 For any $L$ in $\text{Amp}(X)$, we have

$$\zeta(L) = \zeta^\text{ud}(L).$$
Proof By definitions of $\zeta^{ud}(L)$ and $\zeta(L)$, we have

$$\zeta(L) \leq \zeta^{ud}(L).$$

For any $F \notin D^\text{ud}_L$, then

$$\beta_L(F) \geq C_L S_L(F) \geq \zeta^{ud}(L) S_L(F).$$

Thus, for any $F$ in PDiv$_X$, we have

$$\beta_L(F) \geq \zeta^{ud}(L) S_L(F),$$

i.e.

$$\zeta(L) \geq \zeta^{ud}(L).$$

$\square$

Theorem 14 The uniformly valuative stability threshold

$$\text{Amp}(X) \ni L \mapsto \zeta(L) \in \mathbb{R}$$

is continuous on the ample cone.

Proof For any $L$ in Amp$(X)$ and any $\varepsilon > 0$, we aim to show that there exists a small open neighborhood $U_0$ of $L$ in Amp$(X)$ such that for any $L'$ in $U_0$ satisfying

$$|\zeta(L') - \zeta(L)| < \varepsilon.$$ 

By Theorem 10, for any $L'$ in $U_0$, it satisfies the following inequality

$$\beta_{L'}(F) - \beta_L(F) \geq - f(\theta) S_{L'}(F)$$

for any $F$ in $D^\text{ud}_{L'}$, where $f$ is a continuous function with $f(\theta) \to 0$ as $\theta \to 0$. Moreover, $f$ only depends on $X$ and $L$.

Thus, we have

$$\beta_{L'}(F) \geq \zeta(L) S_L(F) - f(\theta) S_{L'}(F)$$

$$= (\zeta(L) + c_L S_L(F)) - f(\theta) S_{L'}(F)$$

$$\geq (\zeta(L) + c_L s^-(\theta) - c_L s^+ (\theta) - f(\theta)) S_{L'}(F)$$

$$= \left( \zeta(L) - (1 - s^- (\theta)) \zeta(L) + c_L (s^+ (\theta) - s^- (\theta)) + f(\theta) \right) S_{L'}(F)$$

for any $F$ in $D^\text{ud}_{L'}$. Thus one obtains

$$\zeta(L') = \zeta^{ud}(L') \geq \zeta(L) - \left( (1 - s^- (\theta)) \zeta(L) + c_L (s^+ (\theta) - s^- (\theta)) + f(\theta) \right).$$

We can take a small enough constant $\theta > 0$ such that

$$(1 - s^- (\theta)) \zeta(L) + c_L (s^+ (\theta) - s^- (\theta)) + f(\theta) < \varepsilon.$$ 

Thus, we have

$$\zeta(L') - \zeta(L) > -\varepsilon. \quad (28)$$

On the other hand, by replacing $L$ by $L'$ and writing $L = L' - \theta H$ in Theorem 10, we have

$$\beta_L(F) - \beta_{L'}(F) \geq - f(\theta) S_L(F)$$

for any $F$ in $D^\text{ud}_L$, where $f$ is a continuous function with $f(\theta) \to 0$ as $\theta \to 0$. Moreover, $f$ only depends on $X$ and $L$. 

$\square$
Similarly, we can compute
\[ \beta_L(F) \geq (\zeta(L')s^+(\theta)^{-1} - c_L'(s^-)(\theta)^{-1} - s^+(\theta)^{-1}) - f(\theta)) S_L(F), \]
for any \( F \) in \( \mathcal{D}_L^{ud} \). One obtains
\[ \zeta(L) = \zeta^{ud}(L) \geq \zeta(L')s^+(\theta)^{-1} - c_L'(s^-)(\theta)^{-1} - s^+(\theta)^{-1}) - f(\theta). \]
Then, we have
\[ \zeta(L') \leq \zeta(L) + (s^+(\theta) - 1)\zeta(L) + c_L'(s^+(\theta)s^-(\theta)^{-1} - 1) + s^+(\theta)f(\theta). \]
One can choose a \( c_L' \) depending on \( L' \) continuously since \( \delta(\cdot), \mu(\cdot) \) and \( \tilde{s}(\cdot) \) are continuous on \( \text{Amp}(X) \). Then we take \( \theta > 0 \) small enough such that
\[ (s^+(\theta) - 1)\zeta(L) + c_L'(s^+(\theta)s^-(\theta)^{-1} - 1) + s^+(\theta)f(\theta) < \varepsilon. \]
Thus, we have
\[ \zeta(L') - \zeta(L) < \varepsilon. \]
Together with (28), we finish the proof of Theorem 14.

6 Valuative stability for transcendental classes

In this section, let \( X \) be a projective manifold. We extend the valuative stability to the Kähler cone of projective manifolds.

Denote by \( \mathcal{K} \) the Kähler cone of \( X \) and \( \mathcal{E} \) the pseudo-effective cone in \( H^{1,1}(X, \mathbb{R}) \). The interior \( \mathcal{E}^\circ \) of the psef cone is an open subcone, whose element is called big class.

Recall the definition of the volume of a big class \( \alpha \) in \( \mathcal{E}^\circ \) ([8, Definition 3.2]),
\[ \text{Vol}(\alpha) := \sup_{T \in \alpha} \int_{\tilde{X}} \gamma^n > 0, \]
where the supremum is taken over all Kähler currents \( T \in \alpha \) with logarithmic poles, and \( \pi^* T = [E] + \gamma \) with respect to some modification \( \pi : \tilde{X} \to X \) for an effective \( \mathbb{Q} \)-divisor \( E \) and a closed semi-positive form \( \gamma \) (or see [9, Definition 1.17] for a definition in the sense of pluripotential theory).

Let \( \alpha \in \mathcal{K} \) be a Kähler class of \( X \), for any prime divisor \( F \) over \( X \), then \( \text{Vol}(\alpha - x[F]) \) is well-defined for some \( x > 0 \). Since \( \pi^* \alpha \) may not be Kähler on \( Y \), but it is still big. Therefore, by openness of the big cone \( \mathcal{E}^\circ \), we can define
\[ \tau_\alpha(F) := \{ x \in \mathbb{R} : \text{Vol}(\alpha - x[F]) > 0 \}. \tag{29} \]
It follows that the \( S \)-invariant is well-defined, denoted by \( S_\alpha(\cdot) \). Similarly, for any Kähler class \( \alpha \), we also denote by
\[ \mu(\alpha) := \frac{c_1(X) \cdot \alpha^{n-1}}{\alpha^n}, \]
\[ s(\alpha) := \sup \{ s \in \mathbb{R} : c_1(X) - s\alpha \text{ is Kähler} \}, \]
and
\[ \tilde{s}(\alpha) := \inf \{ s \in \mathbb{R} : -c_1(X) + s\alpha \text{ is Kähler} \}. \]
We have \( s(\alpha) \leq \mu(\alpha) \leq \tilde{s}(\alpha) \).
In [8] the authors established the perfect theory of the positive intersection product of big classes on compact Kähler manifolds.

**Theorem 15** ([8, Theorem 3.5]) Let $X$ be a compact Kähler manifold. We denote here by $H^{k,k}_{>0}(X)$ the cone of cohomology classes of type $(k,k)$ which have non-negative intersection with all closed semi-positive smooth forms of bidegree $(n-k,n-k)$.

(i) For each integer $k = 1, 2, \ldots, n$, there exists a canonical “movable intersection product”

$E \times \cdots \times E \rightarrow H^{k,k}_{>0}(X)$, \hspace{1em} $(\alpha_1, \ldots, \alpha_k) \mapsto \langle \alpha_1 \cdot \alpha_2 \cdots \alpha_k \rangle$

such that $\text{Vol}(\alpha) = \langle \alpha^n \rangle$ whenever $\alpha$ is a big class (see Remark 1).

(ii) The product is increasing, homogeneous of degree 1 and super-additive in each argument, i.e.

$\langle \alpha_1 \cdots (\alpha_j' + \alpha_j'') \cdots \alpha_k \rangle \geq \langle \alpha_1 \cdots \alpha_j' \cdots \alpha_k \rangle + \langle \alpha_1 \cdots \alpha_j'' \cdots \alpha_k \rangle$.

It coincides with the ordinary intersection product when the $\alpha_j$ in $K$ are nef classes.

(iii) The movable intersection product satisfies the Teissier-Hovanskii inequality

$\langle \alpha_1 \cdot \alpha_2 \cdots \alpha_n \rangle \geq (\langle \alpha_1^n \rangle)^{1/n} \cdots (\langle \alpha_n^n \rangle)^{1/n}$.

It follows that $\beta$-invariant is well-defined for any Kähler class. For any $\alpha$ in $K$, we define

$\beta_\alpha(F) := A_X(F) \text{Vol}(\alpha) + n \mu(\alpha) \int_0^{+\infty} \text{Vol}(\alpha - x[F])dx$ 

$- \int_0^{+\infty} n \langle (\alpha - x[F])^{n-1} \rangle \cdot c_1(X)dx$.

Therefore, we can extend the valuative stability to any Kähler class.

**Definition 10** For any $\alpha$ in $K$, we say that $(X, \alpha)$ is

(i) valuatively semistable if

$\beta_\alpha(F) \geq 0$

for any prime divisor $F$ over $X$;

(ii) valuatively stable if

$\beta_\alpha(F) > 0$

for any non-trivial prime divisor $F$ over $X$;

(iii) uniformly valuatively stable if there exists an $\varepsilon_\alpha > 0$ such that

$\beta_\alpha(F) \geq \varepsilon_\alpha S_\alpha(F)$ (30)

for any prime divisor $F$ over $X$.

The positive intersection product $\langle \alpha_1 \cdots \alpha_p \rangle$ depends continuously on the $p$-tuple $(\alpha_1, \ldots, \alpha_p)$ for any big classes $\alpha_1, \ldots, \alpha_p$ (see [9, Definition 1.17]).

If $\gamma$ is psef and $\alpha$ is big, by (ii) and (iii) of Theorem 15, then we have

$\text{Vol}(\alpha + \gamma) \geq \text{Vol}(\alpha)$.

A well-known result about the differentiability of the volume function on $E^\circ$, due to D. Witt Nyström [41], is stated as follows,
**Theorem 16** ([41, Theorem C]) On a projective manifold $X$, the volume function is continuously differentiable on the big cone $\mathcal{E}^\circ$ with

$$
\frac{d}{dt} \bigg|_{t=0} \text{Vol}(\alpha + t \gamma) = n(\alpha^{n-1}) \cdot \gamma.
$$

(31)

for any $\alpha$ in $\mathcal{E}^\circ$ and any $\gamma$ in $H^{1,1}(X, \mathbb{R})$.

Therefore, we have a similar integration by part type formula:

$$
\int_0^{+\infty} n(\alpha - x[F])^{n-1} \cdot x dx = (n + 1) \int_0^{\infty} \text{Vol}(\alpha - x[F]) dx,
$$

(32)

for any $\alpha$ in $\mathcal{K}$ and any prime divisor $F$ over $X$.

It follows that these proofs of Theorem 10 and 8 can also remain valid for the Kähler cone. In other words, the openness of uniformly valuative stability also holds on the Kähler cone.

We summary as follows,

**Theorem 17** For a projective manifold $X$, the uniformly valuative stability locus

$$
\hat{\text{UVs}} := \{ \alpha \in \mathcal{K} : (X, \alpha) \text{ is uniformly valuatively stable} \}
$$

is an open subcone of the Kähler cone $\mathcal{K}$.

### 7 Further questions

In this section, we propose some further questions.

#### 7.1 Relation between K-stability and valuative stability

In [19], Dervan and Legendre prove that K-stability over integral test configurations is equivalent to valuative stability over dreamy divisors. But for non-integral test configurations and non-dreamy divisors, this relation between K-stability and valuative stability is still open. According to Fujita’s idea in [23], we have the following setup.

Take any prime divisor $F$ over $X$, it induces a filtration $\mathcal{F}$ on

$$
R := \bigoplus_{r=0}^{\infty} R_k := \bigoplus_{r=0}^{\infty} H^0(X, kL),
$$

which is defined by

$$
\mathcal{F}^x R_k := \begin{cases} 
H^0(X, kL - \lfloor xF \rfloor) & \text{if } x \in \mathbb{R}_{\geq 0}, \\
R_k & \text{otherwise},
\end{cases}
$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$. Then $\mathcal{F}$ is a decreasing, left continuous, multiplicative and linearly bounded $\mathbb{R}$-filtration. We define

$$
I_{(r,x)} := \text{Image}(\mathcal{F}^x R_r \otimes L^{-r} \to \mathcal{O}_X),
$$

where the homomorphism is the evaluation.
Take any \( e^+, e^- \in \mathbb{Z} \) with \( e^+ > \tau L(F) \) and \( e^- < 0 \), set \( e := e^+ - e^- \). Let \( r_1 \in \mathbb{Z}_{>0} \) be a sufficiently large positive integer. For any \( r \geq r_1 \), we can define a family of flag ideals \( \mathcal{I}_r \subset \mathcal{O}_{X \times \mathbb{A}^1} \) by
\[
\mathcal{I}_r := I_{(r, r e^+)} + I_{(r, r e^- - 1)} t^1 + \cdots + I_{(r, r e^- + 1)} t^{r e^- - 1} + (t^r).
\] (33)

Let \( \pi_r : X_r \rightarrow X \times \mathbb{A}^1 \) be the blow up of \( X \times \mathbb{A}^1 \) along flag ideal \( \mathcal{I}_r \) with exceptional divisor \( E_r \subset X_{r, 0} \). Let \( p_1 : X \times \mathbb{A}^1 \rightarrow X \) be the projection of the first factor of \( X \times \mathbb{A}^1 \). We obtain a family of semiample test configurations \((X_r, \mathcal{L}_r)\) (see [12, Definition 2.2]), where \( \mathcal{L}_r := \rho^*_r L - 1/r E_r \) is a relative semiample \( \mathbb{Q} \)-line bundle and \( \rho_r = p_1 \circ \pi_r \).

It is interesting to understand the limiting behavior of the Donaldson-Futaki invariant of this sequence \((X_r, \mathcal{L}_r)\). We expect to obtain one direction of the relation between K-stability and valuative stability by studying the limiting behavior above.

We make the following question:

**Question 18** Does K-stability imply valuative stability?

### 7.2 Openness of uniformly valuative stability on compact Kähler manifolds

We have seen from Sect. 6 that valuative stability is well-defined on compact Kähler manifolds. But, in general, we do not know the differentiability of the volume function on the big cone \( \mathcal{E}^0 \) for compact Kähler manifolds. Then the useful formula (32) may not hold for compact Kähler manifolds. In fact, in our argument, we only need the following inequality,
\[
C^{-1} \int_0^{+\infty} \Vol(\alpha - x[F]) dx \leq \int_0^{+\infty} \langle (\alpha - x[F])^{n-1} \rangle \cdot d\alpha \leq C \int_0^{+\infty} \Vol(\alpha - x[F]) dx,
\]
for some constant \( C > 0 \), which only depends on \( \alpha \) continuously. We expect that this inequality holds.

We make the following question:

**Question 19** Does the openness of uniformly valuative stability hold on the Kähler cone of compact Kähler manifolds?

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