First Integrals of Dynamical Systems And Their Numerical Preservation

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Abstract

We calculate Noether like operators and first integrals of scalar equation $y'' = -k^2 y$ using complex Lie symmetry method, by taking values of $k$ and $y$ to be real as well as complex. We numerically integrate the equations using a symplectic Runge-Kutta method and check for preservation of these first integrals. It is seen that these structure preserving numerical methods provide qualitatively correct numerical results and good preservation of first integrals is obtained.

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1 Introduction

Marius Sophus Lie proposed a symmetry based method for the analytical solution of differential equations using group of continuous transformations known as Lie groups [11,14,15,23]. Amalie Emmy Noether later presented her remarkable theorem which relates variational symmetries with conservation laws or the first integrals in [18]. In literature, different methods are available to calculate first integrals of ordinary differential equations (ODEs) including the direct method, the characteristic or multiplier method, the Noether approach and partial Noether approach [16,13,16,17]. In this paper, we use classical Noether approach to calculate first integrals of harmonic oscillator equation. We then apply complex symmetry method in the restricted domain to find first integrals of system of harmonic oscillators by considering the Lagrangian in the
Concerning the numerical solutions of ODEs with quadratic first integrals, it is well known that symplectic numerical methods are a suitable candidate [21]. These methods are a subclass of geometric integrators which preserve geometric properties of the exact flow of ODEs. One class of symplectic methods with optimal order are the Gauss-Legendre Runge–Kutta methods. They are one step numerical methods for ODEs and preserve all linear and quadratic first integrals of a dynamical system [9]. If we intend to preserve cubic or higher order first integrals, we do not have a general numerical scheme for such purpose but we can design a numerical method that has this as its specific goal, for example splitting method and discrete gradient method [9]. In this paper, we present a way of constructing symplectic Runge–Kutta methods. We then take order four Gauss-Legendre Runge–Kutta method for the numerical integration of ODEs and report good preservation of first integrals by the numerical solution.

## 2 Symmetries and First Integrals

Consider a second-order ODE,

\[ y'' = f(t, y, y'), \]  

(1)

which admits a Lagrangian \( L(t, y, y') \) satisfying the Euler-Lagrange equation,

\[ \frac{\partial L}{\partial y} - \frac{d}{dt} \left( \frac{\partial L}{\partial y'} \right) = 0. \]  

(2)

To explain the invariance criteria for variational problems under a group of transformation we consider the operator,

\[ X = \xi(t, y) \frac{\partial}{\partial t} + \eta(t, y) \frac{\partial}{\partial y}, \]  

(3)

known as the infinitesimal operator or the symmetry generator. The functions \( \xi \) and \( \eta \) are the components of tangent vector \( X \) and are defined as,

\[ \xi(t, y) = \frac{\partial \tilde{t}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \eta(t, y) = \frac{\partial \tilde{y}}{\partial \epsilon} \bigg|_{\epsilon=0}. \]  

(4)

The operator \( X \) is called Noether symmetry generator corresponding to the Lagrangian \( L(t, y, y') \), if there exist a gauge function \( B(t, y) \) such that the following condition holds,

\[ X^{(1)}(L) + D_t(\xi)L = D_t(B), \]  

(5)

where \( X^{(1)} \) is a first order prolongation of \( X \) and \( D \) is total derivative operator given by,

\[ D_t = \frac{\partial}{\partial t} + y' \frac{\partial}{\partial y}. \]  

(6)

According to Noether theorem, for each Noether symmetry of an Euler-Lagrange equation, there corresponds a function \( I \)

\[ I = \xi L + (\eta - \xi y') \frac{\partial L}{\partial y'} - B(t, y), \]  

(7)

called the first integral or conserved quantity of the equation (1), with respect to the symmetry generator \( X \).
2.1 Complex Symmetry Analysis

We first discuss some important results related to complex Noether symmetries, complex Lagrangian and Noether Theorem in the restricted complex domain. We will use them to determine first integrals of second-order restricted complex ODEs \cite{2}. We then present expressions for Euler-Lagrange like equations, conditions for Noether-like operators and expressions for first integrals corresponding to these operators. For more details, see \cite{3} and references therein.

Consider a system of two second-order ODEs of the form,

\[ f'' = w_1(t, f, g, f', g'), \]
\[ g'' = w_2(t, f, g, f', g'). \]  

Suppose we have a transformation \( y(t) = f + ig \) and \( w = w_1 + iw_2 \) which converts the system (8) to second order restricted complex ODE,

\[ y'' = w(t, y, y'). \]  

Assume that the equation admits a complex Lagrangian \( L(t, f, g, f', g') \), i.e.,

\[ L = L_1 + iL_2. \]  

Therefore, we have two Lagrangians \( L_1 \) and \( L_2 \) for the system (8) that satisfy Euler-Lagrange like equations,

\[ \frac{\partial L_1}{\partial f} + \frac{d}{dt} \left( \frac{\partial L_1}{\partial f'} \right) = 0, \]
\[ \frac{\partial L_2}{\partial f} - \frac{d}{dt} \left( \frac{\partial L_2}{\partial f'} \right) = 0. \]  

The operators,

\[ X_1 = \varsigma_1 \frac{\partial}{\partial t} + \chi_1 \frac{\partial}{\partial f} + \chi_2 \frac{\partial}{\partial g}, \]
\[ X_2 = \varsigma_2 \frac{\partial}{\partial t} + \chi_2 \frac{\partial}{\partial f} - \chi_1 \frac{\partial}{\partial g}. \]  

are called Noether-like operators for the Lagrangians \( L_1 \) and \( L_2 \), if they satisfy following conditions,

\[ X_1^{(1)}(L_1) - X_2^{(1)}(L_2) + (D_t\varsigma_1)L_1 - (D_t\varsigma_2)L_2 = D_tA_1, \]
\[ X_1^{(1)}(L_2) + X_2^{(1)}(L_1) + (D_t\varsigma_1)L_2 + (D_t\varsigma_2)L_1 = D_tA_2, \]  

where \( A_1 \) and \( A_2 \) are suitable gauge functions. The two first integral corresponding to the Noether-like operators \( X_1 \) and \( X_2 \) can be found as,

\[ I_1 = \varsigma_1 L_1 - \varsigma_2 L_2 + \partial_f L_1(\chi_1 - f'\varsigma_1 - g'\varsigma_2) - \partial_f L_2(\chi_2 - f'\varsigma_2 - g'\varsigma_1) - A_1, \]
\[ I_2 = \varsigma_1 L_2 + \varsigma_2 L_1 + \partial_f L_2(\chi_1 - f'\varsigma_1 - g'\varsigma_2) + \partial_f L_1(\chi_2 - f'\varsigma_2 - g'\varsigma_1) - A_2. \]  

(13)
Runge–Kutta methods

Runge–Kutta methods \cite{4} are one-step numerical methods for solving initial value problems (IVPs),

\[ y'(t) = f(y(t)), \quad y(t_0) = y_0, \quad y(t) \in \mathbb{R}^n. \]  

These methods provide an approximation \( y_n = y(t_n) \) of the exact solution \( y(t) \) at time \( t_n = nh \), where \( n = 0, 1, \cdots \) and \( h \) corresponds to the stepsize. The general form of an \( s \)-stage Runge–Kutta method is,

\[ Y_i = y_{n-1} + \sum_{j=1}^{s} a_{ij} hf(Y_j), \quad i = 1, 2, \cdots, s, \]  

\[ y_n = y_{n-1} + \sum_{i=1}^{s} b_i hf(Y_i), \]  

where \( b_i \) are the quadrature weights of the method and \( c_i \) are the nodes at which the stages \( Y_i \) are evaluated. A Runge–Kutta method can be represented by a Butcher tableau,

\[
\begin{array}{c|ccc}
 c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
  & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
 \hline
 b_1 & b_2 & \cdots & b_s
\end{array}
\]  

The Runge–Kutta methods are explicit if \( a_{ij} = 0 \) for \( i \leq j \), otherwise, they are implicit.

3.1 Symplectic Runge–Kutta methods

If the initial value problem \cite{14} has a quadratic first integral

\[ I(y) = \langle y, Sy \rangle = y^T S y, \]  

where \( S \) is a symmetric square matrix, then we have

\[ \langle y, f(y) \rangle = y^T S f(y) = 0. \]  

We want to determine numerical solutions \( y_n \) such that the first integral \( I(y) \) is preserved numerically, i.e.,

\[ \langle y_n, S y_n \rangle = \langle y_{n-1}, S y_{n-1} \rangle = 0. \]  

It has been shown in \cite{3,5,12} that only symplectic Runge–Kutta methods preserve the quadratic first integrals while numerically integrating \cite{13}. Moreover, in this paper we will only be considering implicit Runge–Kutta methods to check the numerical preservation of first integrals because explicit methods cannot be symplectic \cite{22}. A Runge–Kutta method is symplectic if its coefficients satisfy the following condition \cite{5,12,20},

\[ b_i a_{ij} + b_j a_{ji} - b_i b_j = 0 \quad \text{for all} \quad i, j = 1, 2, \cdots, s, \]  

(16)
which can be derived as follows. Firstly, apply the Runge-Kutta method \( (15) \) to solve the IVP \( (14) \). The stage values are

\[
Y_i = y_{n-1} + h \sum_{j=1}^{s} a_{ij} f(Y_j).
\]

Since,

\[
\langle Y_i, Sf(Y_i) \rangle = 0,
\]

\[
\Rightarrow \langle y_{n-1}, Sf(Y_i) \rangle + h \sum_{j=1}^{s} a_{ij} \langle f(Y_j), Sf(Y_i) \rangle = 0.
\] (17)

Moreover, for the output values we have,

\[
y_n = y_{n-1} + \sum_{i=1}^{s} b_i h f(Y_i).
\]

Thus

\[
\langle y_n, Sy_n \rangle = \langle y_{n-1}, Sy_{n-1} \rangle + h \sum_{i=1}^{s} b_i \langle y_{n-1}, Sf(Y_i) \rangle
\]

\[
+ h \sum_{j=1}^{s} b_j \langle f(Y_j), Sy_{n-1} \rangle + h^2 \sum_{i,j=1}^{s} b_i b_j \langle f(Y_i), Sf(Y_j) \rangle.
\] (18)

Evidently from (17) and (18) we have,

\[
\langle y_n, Sy_n \rangle = \langle y_{n-1}, Sy_{n-1} \rangle,
\]

provided

\[
b_i a_{ij} + b_j a_{ji} - b_i b_j = 0.
\] (19)

### 3.2 Construction of symplectic Runge-Kutta methods

Although there exist several techniques to construct symplectic Runge-Kutta methods in literature [9,24], here we construct symplectic Runge-Kutta methods with the help of Vandermonde transformation. This was first discussed in [8]. The idea is to pre and post multiply the Vandermonde matrix with the matrix of symplectic condition for Runge–Kutta method \( (16) \). Our strategy is to write the values of \( a \) and \( b \) in terms of \( c \) using the Vandermonde transformation. We then choose the values of \( c \) as the zeros of the shifted Legendre polynomial on the interval \([0,1]\).

Consider the Vandermonde matrix \( V \) given as,

\[
V = c_i^{j-1} = \begin{bmatrix}
1 & c_1 & c_1^2 & \cdots & c_1^{s-1} \\
1 & c_2 & c_2^2 & \cdots & c_2^{s-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & c_s & c_s^2 & \cdots & c_s^{s-1}
\end{bmatrix}.
\]

Multiply the symplectic condition \( (16) \) of Runge-Kutta methods with matrix \( V \) as follows,

\[
c_i^{k-1} (b_i a_{ij} + b_j a_{ji} - b_i b_j) c_j^{l-1} = 0, \quad \forall \ i, j, k, l = 1, 2, \ldots, s.
\] (20)
For methods with two stages \((s = 2)\), we take \(l, k = 1, 2\), and then take summation over \(i\) and \(j\) from 1 to 2.

For \(l = 1, k = 1\),

\[
\sum_{i,j} b_i a_{ij} + \sum_{i,j} b_j a_{ji} - \sum_{i,j} b_i b_j = 0. \tag{21}
\]

For \(l = 1, k = 2\),

\[
\sum_{i,j} b_i c_i a_{ij} + \sum_{i,j} b_j c_j a_{ji} - \sum_{i,j} b_i c_i b_j = 0. \tag{22}
\]

For \(l = 2, k = 1\),

\[
\sum_{i,j} b_i a_{ij} c_j + \sum_{i,j} b_j c_j a_{ji} - \sum_{i,j} b_i c_i c_j = 0. \tag{23}
\]

For \(l = 2, k = 2\),

\[
\sum_{i,j} b_i c_i a_{ij} c_j + \sum_{i,j} b_j c_j a_{ji} c_i - \sum_{i,j} b_i c_i b_j c_j = 0. \tag{24}
\]

The following order two conditions must be satisfied.

\[
\sum_{i=1}^{s} b_i = 1, \quad \sum_{i=1}^{s} b_i c_i = \frac{1}{2}. \tag{25}
\]

Using equations (25) in equations (21)-(24) we have,

\[
\sum_{i} b_i c_i = \frac{1}{2},
\]

\[
\sum_{i,j} b_i a_{ij} c_j + \sum_{i,j} b_j c_j a_{ji} = \frac{1}{2},
\]

\[
\sum_{i,j} b_i c_i a_{ij} + \sum_{i,j} b_j a_{ji} c_i = \frac{1}{2},
\]

\[
\sum_{i,j} b_i c_i a_{ij} c_j = \frac{1}{2}.
\]

Consider the relation,

\[
b_i (c_i - c_1) = b_i c_i - b_i c_1,
\]

Take summation over \(i\) from 1 to \(s\) we get,

\[
\sum_{i} b_i (c_i - c_1) = \sum_{i} b_i c_i - \sum_{i} b_i c_1,
\]

\[
b_2 (c_2 - c_1) = \frac{1}{2} - c_1,
\]

\[
b_2 = \frac{\frac{1}{2} - c_1}{c_2 - c_1}.
\]

Similarly we can get,

\[
b_1 = \frac{\frac{1}{2} - c_2}{c_1 - c_2}.
\]
Now consider the relation,
\[ b_i(c_i - c_1)a_{ij}(c_j - c_1) = b_i c_i a_{ij} c_j - b_i a_{ij} c_j c_1 + b_i a_{ij} c_1 c_1. \]

Take summation over \(i\) and \(j\), and use previous equations we get,
\[ a_{22} = \frac{\frac{1}{3} - \frac{c_1}{3} - \frac{c_2}{6} + \frac{c_1 c_2}{2}}{b_2(c_2 - c_1)(c_2 - c_1)}. \]

Similarly we get,
\[ a_{11} = \frac{\frac{1}{3} - \frac{c_1}{3} - \frac{c_2}{6} + \frac{c_1 c_2}{2}}{b_1(c_1 - c_2)(c_1 - c_2)}, \]
\[ a_{21} = \frac{\frac{1}{3} - \frac{c_1}{3} - \frac{c_2}{6} + \frac{c_1 c_2}{2}}{b_2(c_2 - c_1)(c_1 - c_2)}, \]
\[ a_{12} = \frac{\frac{1}{3} - \frac{c_1}{3} - \frac{c_2}{6} + \frac{c_1 c_2}{2}}{b_1(c_1 - c_2)(c_2 - c_1)}. \]

A class of Runge-Kutta methods can be found by suitably choosing \(c_1\) and \(c_2\).

For Gauss methods, the abscissa \(c_i\) are the zeros of the shifted Legendre polynomials \(P_s^\ast\) on the interval \([0, 1]\) where,
\[ P_s^\ast(x) = \frac{s!}{2^s} \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} \binom{s + k}{k} x^k. \]

For Radau methods, the first step is to choose the abscissa \(c_1 = 0\) or \(c_s = 1\) or both of them. The rest of the abscissa are chosen such that, for Radau I methods, the abscissa are the zeros of the polynomial \(P_s^\ast(x) + P_{s-1}^\ast(x)\) of order \(2s - 1\) or, for Radau II methods, the abscissa are the zeros of the polynomial \(P_s^\ast(x) - P_{s-1}^\ast(x)\) of order \(2s - 1\). For Lobatto III methods, the abscissa are the zeros of the polynomial \(P_s^\ast(x) - P_{s-2}^\ast(x)\) of order \(2s - 2\). Thus we have the following symplectic methods,

**Gauss, s=2:**

| \( \frac{1}{2} - \frac{\sqrt{3}}{6} \) | \( \frac{1}{4} \) | \( \frac{1}{4} - \frac{\sqrt{3}}{6} \) |
| --- | --- | --- |
| \( \frac{1}{2} + \frac{\sqrt{3}}{6} \) | \( \frac{1}{4} + \frac{\sqrt{3}}{6} \) | \( \frac{1}{4} \) |

**Radau I, s=2:**

| 0 | \( \frac{1}{8} \) | \( \frac{1}{8} \) |
| \( \frac{1}{4} \) | \( \frac{7}{24} \) | \( \frac{3}{8} \) |
| \( \frac{1}{4} \) | \( \frac{1}{4} \) |
Radau II, s=2:

| 3 | −1/24 |
|---|-------|
| 8/3 | 1/8 |
| 1/4 | 1/4 |

Similarly, we can construct methods with more stages and higher order.

4 Construction of first integrals and their numerical preservation

We construct first integrals of system of harmonic oscillators (both coupled and uncoupled) determined by the second order ODE,

$$y'' = -k^2 y.$$  \hfill (26)

We take different values of $k$ and $y$ as follows.

Case I: $(k^2 = 1$ and $y$ is real)$

When $k^2 = 1$ and $y(t)$ is real valued, (26) becomes one-dimensional harmonic oscillator equation

$$y'' = -y,$$ \hfill (27)

which possesses the standard Lagrangian

$$L = \frac{y'^2}{2} - \frac{y^2}{2}. \hfill (28)$$

Taking the Lagrangian and inserting in (5), yields the following determining system of equations

$$-\eta y + \eta y' + (\eta_y - \frac{1}{2} \xi_t) y'^2 - \frac{1}{2} \xi_y y'^3 - \frac{1}{2} \xi_t y^2 - \frac{1}{2} \xi_y y^2 y' - B_t - y' B_y = 0. \hfill (29)$$

Comparing different powers of $y'$ we have a system of four partial differential equations whose solution gives rise to

$$\xi(t, y) = c_1 + c_2 \sin 2t + c_3 \cos 2t,$$
$$\eta(t, y) = (c_2 \cos 2t - c_3 \sin 2t) y + c_4 \sin t + c_5 \cos t,$$
$$B(t, y) = -(c_2 \sin 2t + c_3 \cos 2t) y^2 + (c_4 \cos t - c_5 \sin t) y. \hfill (30)$$

We thus obtain the following 5-Noether symmetry generators

$$X_1 = \frac{\partial}{\partial t},$$
$$X_2 = \sin 2t \frac{\partial}{\partial t} + y \cos 2t \frac{\partial}{\partial y'},$$
$$X_3 = \cos 2t \frac{\partial}{\partial t} - y \sin 2t \frac{\partial}{\partial y'},$$
$$X_4 = \cos t \frac{\partial}{\partial y},$$
$$X_5 = \sin t \frac{\partial}{\partial y}. \hfill (31)$$
Using the symmetries (31) and the Lagrangian (28) in the Noether’s theorem (7), we get following first integrals,

\[ I_1 = \frac{y'^2}{2} + \frac{y^2}{2}, \]
\[ I_2 = y' \cos t + y \sin t, \]
\[ I_3 = y' \sin t - y \cos t, \]
\[ I_4 = -\frac{1}{2} y'^2 \cos 2t - yy' \sin 2t + \frac{1}{2} y^2 \cos 2t, \]
\[ I_5 = -\frac{1}{2} y'^2 \sin 2t + yy' \cos 2t + \frac{1}{2} y^2 \sin 2t. \] (32)

Amongst these five first integrals only two are independent [16]. We numerically integrate (27) using order four Gauss symplectic Runge-Kutta method which we refer from now on as Gauss2. We take step-size \( h = 0.01 \) and number of steps \( n = 10,000 \). By employing a symplectic integrator, we expect the first integrals of the system to be preserved by the numerical scheme and this is what we have achieved. We look at the deviation of numerically evaluated first integral \( I(y_n) \) from the actual value of first integral \( I(y_0) \). We calculate error by taking difference of the first integral evaluated at an initial value \( I(y_0) \) with the value of the first integral evaluated at all subsequent numerically approximated values \( I(y_n) \) given by the formula: \( \text{Error} = |I(y_n) - I(y_0)| \). Figure 1 and Figure 2 represent the absolute error in the integral \( I_2 \) and \( I_3 \) respectively. It is clear from the figures that the error is very small and bounded, depicting qualitatively correct numerical results. Similar error behavior is obtained for other first integrals.

![Figure 1: Error in integral I_2](image1)

![Figure 2: Error in integral I_3](image2)
Case II: \((k^2 = 1 \text{ and } y \text{ is complex})\)

When \(k^2 = 1\) and \(y(t)\) is a complex function \(y = f + ig\) for \(f\) and \(g\) being real functions of \(t\), yields the following system of ODEs

\[
\begin{align*}
f'' &= -f, \\
g'' &= -g,
\end{align*}
\]

which admits the Lagrangians

\[
\begin{align*}
L_1 &= \frac{1}{2}(f'^2 - g'^2 - f^2 + g^2), \\
L_2 &= fg' - fg.
\end{align*}
\]

Using the Lagrangians (34) in (12) we get 9-Noether like operators

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= \sin t \frac{\partial}{\partial f}, \\
X_3 &= \sin t \frac{\partial}{\partial g}, \\
X_4 &= \cos t \frac{\partial}{\partial f}, \\
X_5 &= \cos t \frac{\partial}{\partial g}, \\
X_6 &= \sin 2t \frac{\partial}{\partial t} + f \cos 2t \frac{\partial}{\partial f} + g \cos 2t \frac{\partial}{\partial g}, \\
X_7 &= g \cos 2t \frac{\partial}{\partial f} - f \cos 2t \frac{\partial}{\partial g}, \\
X_8 &= \cos 2t \frac{\partial}{\partial f} - \sin 2t \frac{\partial}{\partial f} - g \sin 2t \frac{\partial}{\partial g}, \\
X_9 &= -g \sin 2t \frac{\partial}{\partial f} + f \sin 2t \frac{\partial}{\partial g}.
\end{align*}
\]

Invoking Eq. (13), we obtain following invariants,

\[
\begin{align*}
I_{1,1} &= (f'^2 - g'^2 - f^2 + g^2) \sin 2t - 2(ff' - gg') \cos 2t, \\
I_{1,2} &= 2(fg') \sin 2t - 2(fg' + fg) \cos 2t, \\
I_{2,1} &= (f'^2 - g'^2 - f^2 + g^2) \cos 2t + 2(f'g' + fg) \sin 2t, \\
I_{2,2} &= 2(f'g' - fg) \cos 2t + 2(fg' + f'g) \sin 2t, \\
I_{3,1} &= -2f^2 \cos t - 2f \sin t, \\
I_{3,2} &= -2g^2 \cos t - 2g \sin t, \\
I_{4,1} &= -2f^2 \sin t + 2f \cos t, \\
I_{4,2} &= -2g^2 \sin t + 2g \cos t, \\
I_{5,1} &= f'^2 - g'^2 - f^2 + g^2, \\
I_{5,2} &= 2f'g' + fg.
\end{align*}
\]

associated with Noether-like operators (35). System of equations (33) is integrated using Gauss2 method with stepsize \(h = 0.01\) and number of steps \(n = 10,000\). The absolute error in the first integrals \(I_{2,1}, I_{2,2}, I_{4,1}\) and \(I_{4,2}\) is plotted in Figures 3, 4, 5 and 6 respectively. Similar error behaviour is obtained for \(I_{1,1}, I_{1,2}, I_{3,1}, I_{3,2}, I_{5,1}\) and \(I_{5,2}\). We observe that the error does not grow out of bound which shows that the numerical method is able to mimic the true qualitative feature of the dynamical system.
Figure 3: Error in integral $I_{2,1}$

Figure 4: Error in integral $I_{2,2}$

Figure 5: Error in integral $I_{4,1}$

Figure 6: Error in integral $I_{4,2}$
Case III: \((k\text{ and } y\text{ are complex})\)

When \(k\) and \(y(t)\) are both complex, i.e., \(k = \alpha_1 + i\alpha_2\) and \(y = f + ig\) for \(f, g, \alpha_1,\) and \(\alpha_2\) being real, following coupled system of harmonic oscillators is obtained,

\[
\begin{align*}
  f'' &= -(\alpha_1^2 - \alpha_2^2) f + 2\alpha_1\alpha_2 g, \\
  g'' &= -(\alpha_1^2 - \alpha_2^2) g - 2\alpha_1\alpha_2 f,
\end{align*}
\]

which admits a pair of Lagrangians \([17]\),

\[
\begin{align*}
  L_1 &= \frac{1}{2}(f'^2 - g'^2) - \frac{1}{2}(\alpha_1^2 - \alpha_2^2)(f^2 - g^2) + 2\alpha_1\alpha_2 fg, \\
  L_2 &= f'g' - \alpha_1\alpha_2(f^2 - g^2) - (\alpha_1^2 - \alpha_2^2)fg.
\end{align*}
\]

The system \([17]\) admits following 9 Noether-like operators,

\[
\begin{align*}
  X_1 &= \frac{\partial}{\partial t}, \\
  X_2 &= \sin(\alpha_1 t) \cosh(\alpha_2 t) \frac{\partial}{\partial f} + \cos(\alpha_1 t) \sinh(\alpha_2 t) \frac{\partial}{\partial g}, \\
  X_3 &= \cos(\alpha_1 t) \sinh(\alpha_2 t) \frac{\partial}{\partial f} - \sin(\alpha_1 t) \cosh(\alpha_2 t) \frac{\partial}{\partial g}, \\
  X_4 &= \cos(\alpha_1 t) \cosh(\alpha_2 t) \frac{\partial}{\partial f} - \sin(\alpha_1 t) \sinh(\alpha_2 t) \frac{\partial}{\partial g}, \\
  X_5 &= -\sin(\alpha_1 t) \sinh(\alpha_2 t) \frac{\partial}{\partial f} - \cos(\alpha_1 t) \cosh(\alpha_2 t) \frac{\partial}{\partial g}, \\
  X_6 &= \sin(2\alpha_1 t) \cosh(2\alpha_2 t) \frac{\partial}{\partial f} + \{(\alpha_1 f - \alpha_2 g) \cos(2\alpha_1 t) \cosh(2\alpha_2 t) + (\alpha_1 g + \alpha_2 f) \sin(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial f} \\
  &\quad + \{(\alpha_1 g + \alpha_2 f) \cos(2\alpha_1 t) \cosh(2\alpha_2 t) - (\alpha_1 f - \alpha_2 g) \sin(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial g}, \\
  X_7 &= \cos(2\alpha_1 t) \sinh(2\alpha_2 t) \frac{\partial}{\partial f} + \{(\alpha_1 g + \alpha_2 f) \cos(2\alpha_1 t) \cosh(2\alpha_2 t) - (\alpha_1 f - \alpha_2 g) \sin(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial f} \\
  &\quad - \{(\alpha_1 f - \alpha_2 g) \cos(2\alpha_1 t) \cosh(2\alpha_2 t) + (\alpha_1 g + \alpha_2 f) \sin(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial g}, \\
  X_8 &= \cos(2\alpha_1 t) \cosh(2\alpha_2 t) \frac{\partial}{\partial f} + \{(\alpha_1 f - \alpha_2 g) \sin(2\alpha_1 t) \cosh(2\alpha_2 t) - (\alpha_1 g + \alpha_2 f) \cos(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial f} \\
  &\quad + \{(\alpha_1 f - \alpha_2 g) \cos(2\alpha_1 t) \sinh(2\alpha_2 t) + (\alpha_1 g + \alpha_2 f) \sin(2\alpha_1 t) \cosh(2\alpha_2 t)\} \frac{\partial}{\partial g}, \\
  X_9 &= -\sin(2\alpha_1 t) \sinh(2\alpha_2 t) \frac{\partial}{\partial f} + \{(\alpha_1 f - \alpha_2 g) \cos(2\alpha_1 t) \sinh(2\alpha_2 t) + (\alpha_1 g + \alpha_2 f) \sin(2\alpha_1 t) \cosh(2\alpha_2 t)\} \frac{\partial}{\partial f} \\
  &\quad - \{(\alpha_1 f - \alpha_2 g) \sin(2\alpha_1 t) \cosh(2\alpha_2 t) - (\alpha_1 g + \alpha_2 f) \cos(2\alpha_1 t) \sinh(2\alpha_2 t)\} \frac{\partial}{\partial g}.
\end{align*}
\]

Using Noether-like operators \([39]\) with pair of Lagrangians \([38]\) in \([13]\), we obtain
following ten first integrals,
\[ I_{1,1} = (\alpha_1^2 - \alpha_2^2)(f^2 - g^2) - 4\alpha_1\alpha_2 fg + f'^2 - g'^2, \]
\[ I_{1,2} = 2(\alpha_1^2 - \alpha_2^2)fg + 2\alpha_1\alpha_2(f^2 - g^2) + 2f'g', \]
\[ I_{2,1} = f' \sin(\alpha_1 t) \cosh(\alpha_2 t) - g' \cos(\alpha_1 t) \sinh(\alpha_2 t) - (\alpha_1 f - \alpha_2 g) \cos(\alpha_1 t) \cosh(\alpha_2 t) \]
\[ \quad - (\alpha_1 g + \alpha_2 f) \sin(\alpha_1 t) \sinh(\alpha_2 t), \]
\[ I_{2,2} = g' \sin(\alpha_1 t) \cosh(\alpha_2 t) + f' \cos(\alpha_1 t) \sinh(\alpha_2 t) - (\alpha_1 f + \alpha_2 g) \sin(\alpha_1 t) \cosh(\alpha_2 t) \]
\[ \quad + (\alpha_1 g - \alpha_2 f) \cos(\alpha_1 t) \sinh(\alpha_2 t), \]
\[ I_{3,1} = f' \cos(\alpha_1 t) \cosh(\alpha_2 t) + g' \sin(\alpha_1 t) \sinh(\alpha_2 t) + (\alpha_1 f - \alpha_2 g) \sin(\alpha_1 t) \cosh(\alpha_2 t) \]
\[ \quad - (\alpha_1 g + \alpha_2 f) \cos(\alpha_1 t) \sinh(\alpha_2 t), \]
\[ I_{3,2} = g' \cos(\alpha_1 t) \cosh(\alpha_2 t) - f' \sin(\alpha_1 t) \sinh(\alpha_2 t) + (\alpha_1 g + \alpha_2 f) \sin(\alpha_1 t) \cosh(\alpha_2 t) \]
\[ \quad + (\alpha_1 f - \alpha_2 g) \cos(\alpha_1 t) \sinh(\alpha_2 t), \]
\[ I_{4,1} = \frac{1}{2} \left\{ \left( \alpha_1^2 - \alpha_2^2 \right)(f^2 - g^2) - 4\alpha_1\alpha_2 fg - (f'^2 - g'^2) \right\} \sin(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad + \left\{ 2\alpha_1\alpha_2(f^2 - g^2) + 2(\alpha_1^2 - \alpha_2^2)fg - 2f'g' \right\} \cos(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ \quad + \left\{ \alpha_1(f'f' - gg') - \alpha_2(fg' + gf') \right\} \cos(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad + \left\{ \alpha_1(fg' + gf') + \alpha_2(f'f' - gg') \right\} \sin(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ I_{4,2} = \frac{1}{2} \left\{ \left( \alpha_1^2 - \alpha_2^2 \right)(f^2 - g^2) - 4\alpha_1\alpha_2 fg - (f'^2 - g'^2) \right\} \cos(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ \quad + \left\{ 2\alpha_1\alpha_2(f^2 - g^2) + 2fg(\alpha_1^2 - \alpha_2^2) - 2f'g' \right\} \sin(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad + \left\{ \alpha_1(fg' + f'g) + \alpha_2(f'f' - gg') \right\} \cos(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad - \left\{ \alpha_1(f'f' - gg') - \alpha_2(fg' + gf') \right\} \sin(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ I_{5,1} = \frac{1}{2} \left\{ \left( \alpha_1^2 - \alpha_2^2 \right)(f^2 - g^2) - 4\alpha_1\alpha_2 fg - (f'^2 - g'^2) \right\} \cos(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad + \left\{ 2\alpha_1\alpha_2(f^2 - g^2) + 2(\alpha_1^2 - \alpha_2^2)fg - 2f'g' \right\} \sin(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ \quad + \left\{ \alpha_1(fg' + g'f) + \alpha_2(ff' - gg') \right\} \cos(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ \quad - \left\{ \alpha_1(ff' - gg') - \alpha_2(fg' + gf') \right\} \sin(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ I_{5,2} = \frac{1}{2} \left\{ \left( \alpha_1^2 - \alpha_2^2 \right)(f^2 - g^2) + 4\alpha_1\alpha_2 fg + (f'^2 - g'^2) \right\} \sin(2\alpha_1 t) \sinh(2\alpha_2 t) \]
\[ \quad + \left\{ 2\alpha_1\alpha_2(f^2 - g^2) + 2fg(\alpha_1^2 - \alpha_2^2) - 2f'g' \right\} \cos(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad - \left\{ \alpha_1(ff' - gg') - \alpha_2(fg' + gf') \right\} \sin(2\alpha_1 t) \cosh(2\alpha_2 t) \]
\[ \quad - \left\{ \alpha_1(ff' - gg') + \alpha_2(fg' + gf') \right\} \cos(2\alpha_1 t) \sinh(2\alpha_2 t) \]
(40)
Gauss2 method is again used to integrate (37) with stepsize $h = 0.01$ and number of steps $n = 10,000$. The absolute error in the first integrals is calculated as before. The absolute error in integrals $I_{1,1}$, $I_{1,2}$, $I_{3,1}$ and $I_{3,2}$ is plotted in Figures 7, 8, 9 and 10 respectively, which remains bounded for long time. Similar error behaviour is obtained for $I_{2,1}$, $I_{2,2}$, $I_{4,1}$, $I_{4,2}$, $I_{5,1}$ and $I_{5,2}$. Symplectic Gauss2 method is able to preserve all first integrals obtained by employing complex symmetry analysis.

Figure 7: Error in integral $I_{1,1}$

Figure 8: Error in integral $I_{1,2}$

Figure 9: Error in integral $I_{3,1}$

Figure 10: Error in integral $I_{3,2}$
5 Conclusion

First integrals of the dynamical system \(y'' = -k^2y\) are obtained via classical Noether approach and complex symmetry method. The later approach yields invariant energy as a particular example, that is stored in both oscillators. Since these first integrals are quadratic in nature, symplectic Runge–Kutta method, whose construction is also given in this paper, has successfully been applied to the system and numerical preservation of these first integrals have been obtained. Interestingly, the numerical method presented in this paper was able to preserve energy of the single oscillator as well as the energy stored in the pair of coupled oscillators that arise from complex Noether approach. The error in the first integrals remain bounded for long time which would not have been possible if we have employed non-symplectic integrators.

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