Periodic Points of a \( p \)-Adic Operator and their \( p \)-Adic Gibbs Measures

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Abstract—In this paper we investigate generalized Gibbs measure (GGM) for \( p \)-adic Hard-Core (HC) model with a countable set of spin values on a Cayley tree of order \( k \geq 2 \). This model is defined by \( p \)-adic parameters \( \lambda_i \), \( i \in \mathbb{N} \). We analyze \( p \)-adic functional equation which provides the consistency condition for the finite-dimensional generalized Gibbs distributions. Each solutions of the functional equation defines a GGM by \( p \)-adic version of Kolmogorov’s theorem. We define \( p \)-adic Gibbs distributions as limit of the consistent family of finite-dimensional generalized Gibbs distributions and show that, for our \( p \)-adic HC model on a Cayley tree, such a Gibbs distribution does not exist. Under some conditions on parameters \( p \), \( k \) and \( \lambda_i \) we find the number of translation-invariant and two-periodic GGMs for the \( p \)-adic HC model on the Cayley tree of order two.

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1. INTRODUCTION

Let \( \mathbb{Q} \) be the set of rational numbers. It is known that all numbers found in nature through rational numbers (by changing definition of metric on \( \mathbb{Q} \)) are of only two types:

1) The real numbers (the field of numbers that have Archimedean property\(^1\));
2) \( p \)-adic numbers, where \( p \) is a prime number. The non-Archimedean field of numbers.

So, the problems of studying nature (related to the rational numbers) are modeled only by these two types of numbers, there are no different, third types of numbers! This is a mathematically proven assertion (Ostrowski’s theorem, [12]).

The theory of \( p \)-adic numbers is one of very actively developing area in mathematics. Today, numerous applications of \( p \)-adic numbers are found in many branches of mathematics, biology, physics and other sciences (see for example [1, 9, 25–28, 31] and the references therein).

In physics, it is confirmed that it is possible to study very small objects through \( p \)-adic numbers. For example, in standard quantum mechanics, the wave functions describing the evolution of free particles were expressed as wave functions of \( p \)-adic strings. This result is explained by the fact that the energy of a simple quantum particle actually consists of the sum of the energies of its \( p \)-adic components. In this paper we consider a \( p \)-adic operator related to a physical system with a countable set of spin values and having \( p \)-adic valued potential. We study periodic (in particular, fixed) points of this operator and give corresponding to them \( p \)-adic Gibbs measures.

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In [8, 17–19] infinite-dimensional recurrence equations over \( p \)-adic/non-Archimedean spaces are investigated. In this paper, first we consider an infinite-dimensional recurrence equation over \( p \)-adic \( c_0 \) space. Then we investigate the existence of periodic solution of the recurrence equation. We also use solutions to construct the generalized Gibbs measure for the \( p \)-adic Hard-Core(HC) model with a countable set of spin values on a Cayley tree.

2. PRELIMINARIES

2.1. \( p \)-Adic Numbers

Let \( \mathbb{Q} \) be the field of rational numbers. It is clear that, for a fixed prime number \( p \), every \( x \in \mathbb{Q} \), \( x \neq 0 \) can be represented in the form \( x = \frac{p^r a}{m} \), where \( r, n \in \mathbb{Z} \), \( m \) is a positive integer, moreover \( n \) and \( m \) are relatively prime with \( p \). This number \( r \) is usually (see [6, 12]) denoted by \( \text{ord}_p x \), i.e.,

\[
\text{ord}_p x = \begin{cases} 
   \text{the highest power of } p \text{ which divides } x, & \text{if } x \in \mathbb{Z}, \\
   \text{ord}_p a - \text{ord}_p b, & \text{if } x = a/b, \ a, b \in \mathbb{Z}, \ b \neq 0.
\end{cases}
\]

The \( p \)-adic norm of \( x \in \mathbb{Q} \) is given by

\[
|x|_p = \begin{cases} 
   p^{-r}, & x \neq 0, \\
   0, & x = 0.
\end{cases}
\]

It is well-known that (see [12, 31]) this norm is non-Archimedean, i.e. it satisfies the strong triangle inequality:

\[
|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad \forall x, y \in \mathbb{Q}.
\]

From this property one get the following

1) if \( |x|_p \neq |y|_p \), then \( |x + y|_p = \max\{|x|_p, |y|_p\} \);

2) if \( |x|_p = |y|_p \), then \( |x - y|_p \leq |x|_p \).

The completion of \( \mathbb{Q} \) with respect to the \( p \)-adic norm defines the \( p \)-adic field \( \mathbb{Q}_p \). Any \( p \)-adic number \( x \neq 0 \) can be uniquely represented in the canonical form

\[
x = p^\gamma(x)(x_0 + x_1 p + x_2 p^2 + ...),
\]

where \( \gamma(x) = \text{ord}_p x \in \mathbb{Z} \) and \( x_0 \neq 0, x_j \in \{0, 1, ..., p - 1\}, j = 1, 2, ... \). In this case \( |x|_p = p^{-\gamma(x)} \).

In [15] authors have introduced new symbols “O” and “o” which allowed to simplify certain calculations. Roughly speaking, these symbols replace the notation \( \equiv \) (mod \( p^k \)) without noticing about power of \( k \). Let us recall them. For a given \( p \)-adic number \( x \) by \( O[x] \) we mean a \( p \)-adic number such that \( |x|_p = |O(x)|_p \). By \( o[x] \), we mean a \( p \)-adic number such that \( |o(x)|_p < |x|_p \). For instance, if \( x = 1 - p + p^2 \), we can write \( O[1] = x, o[1] = x - 1 \) or \( o[p] = x - 1 + p \). Therefore, the symbols \( O[] \) and \( o[] \) make our work easier when we need to calculate the \( p \)-adic norm of \( p \)-adic numbers. It is easy to see that \( y = O[x] \) if and only if \( x = O[y] \).

2.2. \( p \)-Adic Quadratic Equation

We recall that an integer \( a \in \mathbb{Z} \) is called quadratic residue modulo \( p \) if the congruent equation \( x^2 \equiv a \) (mod \( p \)) has a solution \( x \in \mathbb{Z} \).

Let \( p \) be odd prime and \( a \) an integer not divisible by \( p \). The Legendre symbol (see [11]) is defined by

\[
\left( \frac{a}{p} \right) = \begin{cases} 
   1, & \text{if } a \text{ is quadratic residue of } p; \\
   -1, & \text{if } a \text{ is quadratic nonresidue of } p.
\end{cases}
\]

Lemma 2.1. [31] The equation

\[
x^2 = a, \ a = p^{\gamma_1(a)}(a_0 + a_1 p + a_2 p^2 + ...), \ 0 \leq a_j \leq p - 1, \ a_0 > 0
\]

has a solution in \( \mathbb{Q}_p \) if and only if the following:

i) \( \gamma(a) \) is even;

ii) \( x^2 \equiv a_0 \) (mod \( p \)) is solvable for \( p \neq 2 \); the equality \( a_1 = a_2 = 0 \) hold if \( p = 2 \).
2.3. Cubic Equations on $\mathbb{Q}_p$

It should be noted that the solvability of the general cubic equation over $\mathbb{Q}_p$ is equivalent to the solvability of the depressed cubic equation over $\mathbb{Q}_p$. Namely, we know that the general cubic equation can be reduced to the following depressed cubic equation

$$x^3 + ax = b,$$  \hspace{1cm} (2.2)

here $a, b \in \mathbb{Q}_p$, and $ab \neq 0$.

In [14] and [30], criteria for a solvability of the depressed cubic equation over $\mathbb{Q}_p$ are given. We use these criteria in this paper, therefore here we give them.

**Case** $p > 3$. Let $a = \frac{a^*}{|a_p|}, b = \frac{b^*}{|b_p|}$ and $a^*, b^* \in \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}$.

Suppose that

$$a^* = a_0 + a_1p + a_2p^2 + ..., \quad b^* = b_0 + b_1p + b_2p^2 + ....$$

We denote by $D_0 = -4a_0^3 - 27b_0^2$, $u_1 = 0$, $u_2 = -a_0$, $u_3 = b_0$, $u_{n+3} = b_nu_n - a_0u_{n+1}$, for any $n = 1, 2, \ldots, p - 3$.

**Theorem 2.2.** [14] Let $p > 3$ be a prime. Equation (2.2) has a solution in $\mathbb{Q}_p$ if and only if one of the following conditions holds:

1. $|a_p|^3 < |b_p|^2, 3|\text{ord}_p b$ and $b_0^{\frac{p-1}{3}} \equiv 1 (\text{mod} p)$;
2. $|a_p|^3 = |b_p|^2$ and $D_0u_{p-2}^2 \not\equiv 9a_0^2 (\text{mod} p)$;
3. $|a_p|^3 > |b_p|^2$.

Let $D = -4(a|a_p|^3 - 27(b|b_p|^2)^2 \neq 0$, $D = \frac{D^*}{|D_p|^3}$, $D^* \in \mathbb{Z}_p^*$, $D^* = d_0 + d_1p + ....$

**Theorem 2.3.** [14] Let $p > 3$ be a prime and $N$ is number of the solutions of (2.2) in $\mathbb{Q}_p$. Then the following statements hold true:

$$N = \begin{cases}
3, & |a_p|^3 < |b_p|^2, 3|\text{ord}_p b, p \equiv 1 (\text{mod} 3), b_0^{\frac{p-1}{3}} \equiv 1 (\text{mod} p); \\
3, & |a_p|^3 = |b_p|^2, D = 0; \\
3, & |a_p|^3 = |b_p|^2, 0 < |D_p| < 1, 2|\text{ord}_p D, a_0 \not\equiv 1 (\text{mod} p); \\
3, & |a_p|^3 = |b_p|^2, |D_p| = 1 \text{ and } u_{p-2} \equiv 0 (\text{mod} p); \\
3, & |a_p|^3 > |b_p|^2, 2|\text{ord}_p a, (-a_0)^{\frac{p-1}{2}} \equiv 1 (\text{mod} p); \\
1, & |a_p|^3 < |b_p|^2, 3|\text{ord}_p b, p \equiv 2 (\text{mod} 3); \\
1, & |a_p|^3 = |b_p|^2, 0 < |D_p| < 1, 2 \nmid \text{ord}_p D; \\
1, & |a_p|^3 = |b_p|^2, D_0u_{p-2}^2 \not\equiv 0 (\text{mod} p), D_0u_{p-2}^2 \not\equiv 9a_0^2 (\text{mod} p); \\
1, & |a_p|^3 > |b_p|^2, 2|\text{ord}_p a, (-a_0)^{\frac{p-1}{2}} \not\equiv 1 (\text{mod} p); \\
1, & |a_p|^3 > |b_p|^2, 2 \nmid \text{ord}_p a; \\
0, & \text{otherwise.}
\end{cases} \hspace{1cm} (2.3)
3. PERIODIC POINTS OF THE $p$-ADIC OPERATOR

Let $\mathbb{Q}_p$ be the field of $p$-adic numbers and let

$$ c_0 = \left\{ \{x_n\}_{n=1}^\infty \subset \mathbb{Q}_p \mid \lim_{n \to \infty} x_n = 0 \right\}, \quad \mathcal{P} = \left\{ \{x_n\}_{n=1}^\infty \in c_0 \mid \sum_{n=1}^{\infty} x_n = -1 \right\}. $$

Note that $c_0$ is a linear space over $\mathbb{Q}_p$. We define an operator $F : x = \{x_n\}_{n=1}^\infty \in c_0 \setminus \mathcal{P} \rightarrow y = Fx = \{y_n\}_{n=1}^\infty \in c_0$ by the following formula

$$ y_n = \lambda_n \cdot \left( \frac{1}{1 + \sum_{n=1}^{\infty} x_n} \right)^2, \quad (3.1) $$

where $\lambda = \{\lambda_n\}_{n=1}^\infty \in c_0$ is parameter-vector.

Now we investigate fixed points of the operator (3.1).

### 3.1. Fixed Points

First, we classify the set of fixed points $\text{Fix}(F)$. To do this, we find all solutions of the following equation:

$$ x_n = \lambda_n \cdot \left( \frac{1}{1 + \sum_{n=1}^{\infty} x_n} \right)^2, \quad \lambda = \{\lambda_n\}_{n=1}^\infty, \quad x = \{x_n\}_{n=1}^\infty \in c_0 \setminus \mathcal{P}. \quad (3.2) $$

It is known from $p$-adic analysis that for a series to be convergent, it is necessary and sufficient that the limit of its $n$-term is zero. Since $\lambda = \{\lambda_n\}_{n=1}^\infty \in c_0$, $x = \{x_n\}_{n=1}^\infty \in c_0 \setminus \mathcal{P}$, we can denote $z = \sum_{n=1}^{\infty} x_n$ and $\theta = \sum_{n=1}^{\infty} \lambda_n$.

Note that $z \neq -1$. Summarizing both hand sides of the equation (3.2) we get

$$ z(1 + z)^2 = \theta, \quad z, \theta \in \mathbb{Q}_p. \quad (3.3) $$

By substitution $z = t - \frac{2}{3}$ we get

$$ t^3 - \frac{t}{3} - \left( \theta + \frac{2}{27} \right) = 0. \quad (3.4) $$

This is equation (2.2) with $a = -\frac{1}{3}, b = \theta + \frac{2}{27}$. For $p > 3$ we get $|a|_p = 1$.

We denote

$$ -\frac{1}{3} = a_0 + a_1p + a_2p^2 + ..., \quad \theta + \frac{2}{27} = \frac{1}{|\theta + \frac{2}{27}|_p} \cdot (b_0 + b_1p + b_2p^2 + ...), \quad (3.5) $$

$$ D_0 = -4a_0^3 - 27b_0^3, \quad u_1 = 0, \quad u_2 = -a_0, \quad u_3 = b_0, $$

$$ u_{n+3} = b_0u_n - a_0u_{n+1}, \text{for any} \ n = 1, 2, \ldots, p - 3. $$

Then using Theorem 2.2 we get the following:

**Lemma 3.1.** Let $p > 3$ be a prime and $\theta \neq -\frac{2}{27}$. Equation (3.4) has a solution in $\mathbb{Q}_p$ if and only if one of the following conditions holds true:

1. $|\theta + \frac{2}{27}|_p > 1.3 \text{ord}_p \left( \theta + \frac{2}{27} \right)$ and $b_0 \frac{p-1}{3(p-1)} \equiv 1 \text{ (mod p)}$;
2. $|\theta + \frac{2}{27}|_p = 1$ and $D_0u_2^2 \not\equiv 9a_0^2 \text{ (mod p)}$;
3. $|\theta + \frac{2}{27}|_p < 1$. 

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$p$-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS  Vol. 14  Suppl. 1  2022
Remark 3.2. If \( \theta = -\frac{2}{27} \) then equation (3.4) has at least one solution in \( \mathbb{Q}_p \), i.e. \( t = 0 \) is solution of equation (3.4).

Let \( D = \frac{4}{27} - 27(\theta + \frac{2}{27})^2 |\theta + \frac{2}{27}|^2 \). If \( D \neq 0 \), then we denote \( D = \frac{d_0}{D_p}, D^* \in \mathbb{Z}_p^* \). \( D^* = d_0 + d_1 p + \ldots \). Then using Theorem 2.3 we get the following theorem about the classification of the set \( \text{Fix}(F) \).

Theorem 3.3. Let \( p > 3 \) be a prime and \(|\text{Fix}(F)|\) is number of the fixed points of the operator \( F \). Then the following statements hold:

\[
|\text{Fix}(F)| = \begin{cases} 
3, & |\theta + \frac{2}{27}| > 1, 3 |\text{ord}_p(\theta + \frac{2}{27}), p \equiv 1(\text{mod } 3), b_0^{\nu-1} \equiv 1(\text{mod } p); \\
3, & |\theta + \frac{2}{27}| = 1, D = 0; \\
3, & |\theta + \frac{2}{27}| = 1, 0 < |D|_p < 1, 2 |\text{ord}_p D, d_0^{\nu-1} \equiv 1(\text{mod } p); \\
3, & |\theta + \frac{2}{27}| = 1, |D|_p = 1 \text{ and } u_{p-2} \equiv 0(\text{mod } p); \\
3, & |\theta + \frac{2}{27}| < 1, (-a_0)^{\nu-1} \equiv 1(\text{mod } p); \\
1, & |\theta + \frac{2}{27}| > 1, 3 |\text{ord}_p(\theta + \frac{2}{27}), p \equiv 2(\text{mod } 3); \\
1, & |\theta + \frac{2}{27}| = 1, 0 < |D|_p < 1, 2 |\text{ord}_p D, a_0^{\nu-1} \neq 1(\text{mod } p); \\
1, & |\theta + \frac{2}{27}| = 1, 0 < |D|_p < 1, 2 \nmid \text{ord}_p D; \\
1, & |\theta + \frac{2}{27}| = 1, D_0 u_{p-2}^2 \neq 0(\text{mod } p), D_0 u_{p-2}^2 \neq 9a_0^2(\text{mod } p); \\
1, & |\theta + \frac{2}{27}| < 1, (-a_0)^{\nu-1} \neq 1(\text{mod } p); \\
0, & \text{otherwise}. 
\end{cases}
\]

Proof. To prove this theorem, we have to check all conditions of Theorem 2.3. However, we shall content ourselves with checking the first condition, since the rest can be checked similarly. First condition of Theorem 2.3 has the following form:

\[
|a|^3 |p| < |b|^3 |p| \text{ord}_p b, p \equiv 1(\text{mod } 3), b_0^{\nu-1} \equiv 1(\text{mod } p).
\]

In our case \( a = -\frac{1}{3} \) and \( b = \theta + \frac{2}{27} \). Since \( p > 3 \), by (3.5) we have

\[
|\theta + \frac{2}{27}| > 1, 3 |\text{ord}_p(\theta + \frac{2}{27}), p \equiv 1(\text{mod } 3), b_0^{\nu-1} \equiv 1(\text{mod } p).
\]

Case \( p = 3 \). In this case we denote

\[
\mathbb{Z}_3^*[i_0, i_1, \ldots, i_k] = \left\{ x^* \in \mathbb{Z}_3^* : x^* = i_0 + i_1 3^1 + \ldots + i_k 3^k + x_{k+1} 3^{k+1} + \ldots \right\},
\]

\[
\mathbb{Z}_3^*[i_0, i_1, \ldots, i_k | j_0, j_1, \ldots, j_s] = \mathbb{Z}_3^*[i_0, i_1, \ldots, i_k] \times \mathbb{Z}_3^*[j_0, j_1, \ldots, j_s],
\]

\[
\Delta_{11} = \bigcup_{i,j=0}^2 \mathbb{Z}_3^*[2, i, j | 1, 2, i, j], \quad \Delta_{12} = \bigcup_{i,j=0}^2 \mathbb{Z}_3^*[2, 1, j | 1, 2, 1, j + 1],
\]

\[
\Delta_{13} = \bigcup_{i,j=0}^2 \mathbb{Z}_3^*[2, i + 1, j + 1 | 1, 2, i + 1, j].
\]
$\Delta_{21} = \bigcup_{i,j=0}^{2} Z_3^*[2, i + j, i|1, 0, 2 - (i + j), j]$, $\Delta_{22} = \bigcup_{i,j=0}^{2} Z_3^*[2, 0, 2 - j|2, 0, 2, j]$,

$\Delta_{23} = \bigcup_{i,j=0}^{2} Z_3^*[2, 3 + i, j|2, 0, i - 1, 1 - (i + j)]$,

$\Delta_1 = \Delta_{11} \cup \Delta_{12} \cup \Delta_{13}$, $\Delta_2 = \Delta_{21} \cup \Delta_{22} \cup \Delta_{23}$, $\Delta = \Delta_1 \cup \Delta_2$.

**Theorem 3.4.** [30] Equation (2.2) has a solution in $\mathbb{Q}_3$ if and only if one of the following conditions holds true:

1. $|a_3^3| > |b_3^2|$;
2. $|a_3^3| = |b_3^2| \text{ and } a^* \in \mathbb{Z}_3^*[1]$;
3. $|a_3^3| < |b_3^2|$, $3 \ord_3 b$ and
   
   i) $\frac{4}{3}|^3 = |b_3^2|$, $(a^*, b^*) \in \mathbb{Z}_3^*[1, 1] \cup \mathbb{Z}_3^*[1, 2, 1] \cup \Delta$;
   
   ii) $\frac{4}{3}|^3 < |b_3^2|$, $b^* \in \mathbb{Z}_3^*[1, 0] \cup \mathbb{Z}_3^*[2, 2]$.

**Theorem 3.5.** [30] Let $N_2$ denotes number of the solutions of (2.2) in $\mathbb{Q}_3$. Then the following statements hold true:

$$N_2 = \begin{cases} 3, & |a_3^3| > |b_3^2|, 2 \ord_3 a, a^* \in \mathbb{Z}_3^*[2]; \\ 1, & |a_3^3| > |b_3^2|, 2 \not\in \ord_3 a; \\ 1, & |a_3^3| = |b_3^2|, 2 \not\in \ord_3 a; \\ 1, & |a_3^3| = |b_3^2|, a^* \in \mathbb{Z}_3^*[1]; \\ 1, & |a_3^3| < |b_3^2|, 3 \ord_3 b, \frac{4}{3}|^3 = |b_3^2|, (a^*, b^*) \in \mathbb{Z}_3^*[1, 2, 1] \cup \mathbb{Z}_3^*[1, 1, 1] \cup \Delta; \\ 1, & |a_3^3| < |b_3^2|, 3 \ord_3 b, \frac{4}{3}|^3 < |b_3^2|, \quad b^* \in \mathbb{Z}_3^*[1, 0] \cup \mathbb{Z}_3^*[2, 2]. \end{cases}$$ (3.6)

In our case $a = -\frac{1}{3} = 3^{-1}(2 + 2 \cdot 3 + 2 \cdot 3^2 + \ldots)$, $b = \theta + \frac{2}{27}$ and $|a_3| = 3$. From this we have $\ord_3 a = -1$, $a^* = a|a|_p = 2 + 2 \cdot 3 + 2 \cdot 3^2 + \ldots$ and $a^* \in \mathbb{Z}_3^*[2, 2, \ldots, 2]$.

Using Theorem 3.4 we get the following

**Lemma 3.6.** Equation (3.4) has a solution in $\mathbb{Q}_3$ if and only if one of the following conditions holds true:

- $|\theta + \frac{2}{27}|_3^3 < 27$;
- $|\theta + \frac{2}{27}|_3^2 > 27$, $3 \ord_3 (\theta + \frac{2}{27})$, $b^* \in \mathbb{Z}_3^*[1, 0] \cup \mathbb{Z}_3^*[2, 2]$.
Proof. We are going to check all conditions of Theorem 3.4 for our case.
1) $|a^3_3| > |b^2_3|$ yields $|\theta + \frac{b^2_3}{a^3_3}|^2 < 27$. Hence, the first condition of Theorem 3.4 is satisfied.
2) $|a^3_3| = |b^2_3|$ and $a^* \in \mathbb{Z}_3[1]$ yield $|\theta + \frac{b^2_3}{27a^3_3}| = 27$, but in our case $a^* \in \mathbb{Z}_3[2]$. Hence, the second condition of Theorem 3.4 does not hold in our case.
3) $|a^3_3| < |b^2_3$, $3|\text{ord}_3 b$ yield $|\theta + \frac{b^2_3}{27a^3_3}| > 27$, $3|\text{ord}_3 (\theta + \frac{b^2_3}{27})$ respectively.
i) $|\frac{27}{3}| \in |b^2_3|, (a^*, b^*) \in \mathbb{Z}_3[1, 1] \cup \mathbb{Z}_3[2, 1] \cup \Delta$ yield $|\theta + \frac{b^2_3}{27}| = 27, (a^*, b^*) \in \mathbb{Z}_3[1, 1] \cup \mathbb{Z}_3[2, 1] \cup \Delta$. Hence, this condition of Theorem 3.4 does not hold in our case.
ii) $|\frac{3}{3}| \in |b^2_3, b^* \in \mathbb{Z}_3[1, 0] \cup \mathbb{Z}_3[2, 2]$ yield $|\theta + \frac{b^2_3}{27}| > 27, b^* \in \mathbb{Z}_3[1, 0] \cup \mathbb{Z}_3[2, 2]$. Hence, this condition of Theorem 3.4 is satisfied.

Theorem 3.7. Let $p = 3$. If one of the conditions of Lemma 3.6 is satisfied, then the operator $F$ has a unique fixed point.

Otherwise, there is no fixed point of the operator $F$.

Proof. Note that the number of fixed points of the operator $F$ is equal to the number of solutions of equation (3.4). We use Theorem 3.5 to find the number of solutions of equation (3.4).

First, we have all the conditions for the existence of a fixed point of the operator $F$ from Lemma 3.6. Theorem 3.5 considers six cases, for the operator $F$ there are only two of these cases, i.e., only in the cases $|a^3_3| > |b^2_3|$, $2 \not| \text{ord}_3 a$ and $|a^3_3| < |b^2_3|$, $3|\text{ord}_3 b$, $|\frac{3}{3}| \not| |b^2_3|$, $b^* \in \mathbb{Z}_3[1, 0] \cup \mathbb{Z}_3[2, 2]$ there is a fixed point of the operator $F$. According to Theorem 3.5, the solution in these conditions is unique. It follows that if the operator $F$ has a fixed point, it will be unique.

Remark 3.8. Since there is no any result about the solutions of cubic equations in the space of 2-adic numbers, the above problem remains open in the case of $p = 2$.

3.2. 2-Periodic Points

We now deal with the classification of the set of 2-periodic points of the operator $F$. Denote

$$\text{Per}_2 \{F\} = \{x \in c_\infty \setminus P \mid FFx = x \setminus \text{Fix}\{F\}.$$

By (3.1), the equation $FFx = x$ has the following form

$$x_n = \frac{\lambda_n}{\left(1 + \sum_{j \in \mathbb{N}} \lambda_j \frac{1}{\sum_{i \in \mathbb{N}} x_i^2}\right)^2}, \lambda = \{\lambda_n\}_{n=1}^\infty, x = \{x_n\}_{n=1}^\infty \in c_\infty \setminus P. \quad (3.7)$$

Let $\theta = \sum_{j \in \mathbb{N}} \lambda_j$ and $z = \sum_{j \in \mathbb{N}} x_j$. Summarizing both hand sides of (3.7) we get

$$z = \frac{\theta (1 + z)^4}{(1 + z)^2 + \theta^2}. \quad (3.8)$$

Let $f(z) = \frac{\theta}{1+z}$ then $f(f(z)) = \frac{\theta(1+z)^4}{(1+z)^2 + \theta^2}$. In order to find 2-periodic points (different from the fixed points) of $f(z)$ we consider the following equation:

$$f(f(z)) - z = 0. \quad (3.9)$$

From (3.9) we get

$$z^2 + (2 - \theta)z + 1 = 0. \quad (3.10)$$
The solutions of equation (3.10) are

\[ z_{1,2} = \frac{\theta - 2 \pm \sqrt{\theta^2 - 4\theta}}{2}. \]  

(3.11)

Substituting (3.11) into (3.3) we have

\[ \theta \sqrt{\theta^2 - 4\theta} \left( \sqrt{\theta^2 - 4\theta} \pm (\theta - 2) \right) = 0. \]  

(3.12)

It is clear that if \( \theta \neq 0 \) and \( \theta \neq 4 \) then the solutions of equation (3.10) are not fixed points of \( f(z) \).

If \( \theta = 0 \) solution (3.11) gives \( z = -1 \), however, we have \( z \neq -1 \).

If \( \theta = 4 \) solution (3.11) gives \( z = 1 \). In this case \( z = 1 \) is fixed point of \( f(z) \).

Thus, we can assume that \( \theta \neq 0 \) and \( \theta \neq 4 \). Let \( \theta = p^{\gamma(\theta)}(\theta_0 + \theta_1p + \theta_2p^2 + \ldots) \), \( \theta_0 \neq 0 \) and \( D(\theta) = \theta^2 - 4\theta \).

**Theorem 3.9.** Let \( p \geq 3 \) be a prime and \(|\text{Per}_2\{F\}|\) be the number of 2-periodic points of the operator \( F \). Then the following statements hold:

\[
|\text{Per}_2\{F\}| = \begin{cases} 
2, & \text{if } |\theta|_p > 1; \\
2, & \text{if } |\theta|_p < 1, 2 \mid \text{ord}_p \theta, \left(\frac{-\theta_0}{p}\right) = 1; \\
0, & \text{if } |\theta|_p < 1, 2 \nmid \text{ord}_p \theta \text{ or } \left(\frac{-\theta_0}{p}\right) = -1; \\
2, & \text{if } |\theta|_p = 1, \left(\frac{\theta_0^2 - 4\theta_0}{p}\right) = 1; \\
0, & \text{if } |\theta|_p = 1, 2 \nmid \text{ord}_p (\theta - 4) \text{ or } \left(\frac{|\theta - 4|p(\theta - 4)(\text{mod } p)}{p}\right) = -1; \\
0, & \text{if } |\theta|_p = 1, 2 \mid \text{ord}_p (\theta - 4) \text{ or } \left(\frac{|\theta - 4|p(\theta - 4)(\text{mod } p)}{p}\right) = 1. 
\end{cases}
\]  

(3.13)

**Proof.** Let \( |\theta|_p > 1 \), i.e. \( \gamma(\theta) < 0 \). Then we rewrite \( D(\theta) = p^{2\gamma(\theta)} (\theta_0^2 + o[1]) \). Due to the Lemma 2.1 equation (3.10) has two solutions.

Let \( |\theta|_p < 1 \), i.e. \( \gamma(\theta) > 0 \). Then we can rewrite \( D(\theta) = p^{\gamma(\theta)} (-4\theta_0 + o[1]) \). Then, again thanks to the Lemma 2.1 equation (3.10) has two solutions if \( \gamma(\theta) \) is even (it means that \( 2 \mid \text{ord}_p \theta \)) and the congruence \( x^2 \equiv -\theta_0 \text{ (mod } p) \) is solvable (It means that \( \left(\frac{-\theta_0}{p}\right) = 1 \)).

Let \( |\theta|_p = 1 \), i.e. \( \gamma(\theta) < 0 \). Then \( D(\theta) = \theta_0^2 - 4\theta_0 + o[1] \), here \( \theta_0 \neq 0 \). So, we have the following two cases:

Let \( \theta_0 \neq 4 \). It follows that equation (3.10) has two solutions if the congruence \( x^2 \equiv \theta_0^2 - 4\theta_0 \text{ (mod } p) \) is solvable.

Let \( \theta_0 = 4 \). Since \( \theta \neq 4 \), then there exists \( s \in \mathbb{N} \) such that \( \theta = 4 + \theta_s p^s + o[p^s] \). It yields that

\[
D(\theta) = p^s \left(4\theta_s + o[1]\right). 
\]

According to the Lemma 2.1, equation (3.10) has two solutions if and only if \( s \) is even and the congruence \( x^2 \equiv \theta_s \text{ (mod } p) \) is solvable. It means that \( \left(\frac{|\theta - 4|p(\theta - 4)(\text{mod } p)}{p}\right) = 1. \) \( \square \)
Theorem 3.10. Let $p = 2$ and $|\text{Per}_2(F)|$ is number of 2-periodic points of the operator $F$. Then the following statements hold:

\[
|\text{Per}_2(F)| = \begin{cases} 
2, \text{ if } |\theta|_2 > 1; \\
2, \text{ if } |\theta|_2 = 1 \text{ and } \theta_2 = 1; \\
0, \text{ if } |\theta|_2 = 1 \text{ and } \theta_2 = 0; \\
0, \text{ if } |\theta|_2 \leq \frac{1}{32}, 2 \mid \text{ord}_p \theta; \\
2, \text{ if } |\theta|_2 \leq \frac{1}{32}, 2 \mid \text{ord}_p \theta, \theta_1 = \theta_2 = 0; \\
0, \text{ if } |\theta|_2 \leq \frac{1}{32}, 2 \mid \text{ord}_p \theta, \theta_1 = 1 \text{ or } \theta_2 = 1; \\
2, \text{ if } |\theta|_2 = \frac{1}{2} \text{ and } \theta_1 = 0; \\
0, \text{ if } |\theta|_2 = \frac{1}{2} \text{ and } \theta_1 = 1; \\
0, \text{ if } |\theta|_2 = \frac{1}{8}; \\
2, \text{ if } |\theta|_2 = \frac{1}{16} \text{ and } \theta_1 = 0, \theta_2 = 1; \\
0, \text{ if } |\theta|_2 = \frac{1}{16} \text{ and } \theta_1 = 1 \text{ or } \theta_2 = 0. 
\end{cases}
\]

(3.14)

Proof. Let $|\theta|_2 > 1$. Then $D(\theta) = \theta^2 - 4\theta = \theta^2(1 - \frac{4}{\theta})$. We have $|\frac{4}{\theta}|_2 \leq \frac{1}{2}$. Thus, by Lemma 2.1 equation (3.10) has two solutions.

Let $|\theta|_2 = 1$. We have $\gamma(\theta) = 0$. Then $D(\theta) = 1 + (\theta_1^2 + \theta_1 + \theta_2 - 1)2^2 + o[4]$. $\theta_1^2 + \theta_1 \equiv 0 \text{ (mod } 2)$. Thanks to Lemma 2.1 equation (3.10) has two solutions if $\theta_2 = 1$, equation (3.10) has no solution if $\theta_2 = 0$.

Let $|\theta|_2 \leq \frac{1}{32}$. It means $\gamma(\theta) \geq 5$. Then $D(\theta) = 2^{\gamma(\theta)+2}(-1 - \theta_12 - \theta_22^2 - \theta_32^3 + o[8])$. By Lemma 2.1 equation (3.10) has two solutions if $\theta_1 = \theta_2 = 0$ and $\gamma(\theta)$ is even.

Let $|\theta|_2 = \frac{1}{32}$, i.e. $\gamma(\theta) = 1$. Then $D(\theta) = 2^2(-1 - \theta_1^22^2 + \theta_22^4 + o[2^4])$. By Lemma 2.1 equation (3.10) has two solutions if $\theta_1 = 0$.

Let $|\theta|_2 = \frac{1}{4}$, i.e. $\gamma(\theta) = 2$. Then $D(\theta) = 2^4(-\theta_12 + (\theta_1^2 + \theta_1 - \theta_2)2^2 + (\theta_2 - \theta_3)2^3 + (\theta_3 + \theta_1\theta_2 + \theta_2^2 - \theta_4)2^4 + o[2^4])$. Equation (3.10) has two solutions if $\theta_1 = 0, \theta_2 = 1, \theta_3 = 1, \theta_4 = 0$.

Let $|\theta|_2 = \frac{1}{8}$, i.e. $\gamma(\theta) = 3$. Then $D(\theta) = 2^5(1 - \theta_12 - \theta_22^2 + o[2^2])$.

Again by Lemma 2.1 equation (3.10) has no solution.

Let $|\theta|_2 = \frac{1}{16}$, i.e. $\gamma(\theta) = 4$. Then $D(\theta) = 2^6(-1 - \theta_12 + (1 - \theta_2)2^2 + o[2^2])$. Equation (3.10) has two solutions if $\theta_1 = 0, \theta_2 = 1$. \qed

4. APPLICATION TO THE THEORY OF GIBBS MEASURES

In this section we give $p$-adic Gibbs measures corresponding to fixed and periodic points of operator $F$. 

\[p\text{-ADIC NUMBERS, ULTRAMETRIC ANALYSIS AND APPLICATIONS} \quad \text{Vol. 14 Suppl. 1 2022}\]
The Cayley tree $\mathbb{S}^k = (V, L)$ of order $k \geq 1$ is an infinite tree, i.e., graph without cycles, each vertex of which has exactly $k + 1$ edges (see Chapter 1 in [22]). Here $V$ is the set of vertices of $\mathbb{S}^k$ and $L$ is the set of its edges. For $l \in L$ its endpoints $x, y \in V$ are called nearest neighbor vertices and denoted by $l = \langle x, y \rangle$.

The distance $d(x, y)$ between the vertices $x$ and $y$ on the Cayley tree, is number of edges of the shortest path connecting vertices $x$ and $y$.

For a fixed $x^0 \in V$ we put

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m.$$  

The set $S(x)$ of direct successors of the vertex $x$ is defined as follows. If $x \in W_n$ then

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \ldots, k\}.$$

We consider the hard core (HC) model of nearest neighbors with a countable number of states from $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ on the Cayley tree. The configuration $\sigma = \{\sigma(x) \mid x \in V\}$ on the Cayley tree is given as a function from $V$ to the set $\mathbb{N}_0$. For a subset $A \subset \mathbb{N}_0$ we denote by $\Omega_A$ the set of all configurations defined on $A$.

A configuration $\sigma$ defined on $V$ is called admissible if $\sigma(x)\sigma(y) = 0$ for any neighbor $\langle x, y \rangle$ in $V$, i.e., if the vertex $x$ has a spin value $\sigma(x) = 0$, then on the neighbor vertices we can put any value from $\mathbb{N}_0$, if on the vertex $x$ there is any value from $\mathbb{N}_0$, then we put only zeros on the neighbor vertices.

The activity set (see [3]) on the set of states $\mathbb{N}_0$ is a bounded function $\lambda : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

Now we give definition of $p$-Adic Gibbs measure. Let $z : x \rightarrow z_x = (z_{1,x}, z_{2,x}, \ldots) \in \mathbb{Q}_p^{\mathbb{N}_0}$ be vector-valued function on $V$, where $z_{i,x} \in \mathbb{Q}_p, i \in \mathbb{N}_0$.

Let $\nu = \{\nu(i) \in \mathbb{Q}_p, i \in \mathbb{N}_0\}$ be a fixed $p$-adic probability measure [10]. Denote $\text{supp}(\nu) = \{i \in \mathbb{N}_0 : \nu(i) \neq 0\}$.

We construct generalized probability Gibbs distribution $\mu^{(n)}$ (as a HC-model) on $\Omega_{V_n}$ for any $n \in \mathbb{N}$ and $\lambda_{\sigma(x)}$ as the following

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{x \in V_n} \lambda_{\sigma_n(x)}^{-\nu(\sigma_n(x))} \prod_{x \in W_n} z_{\sigma_n(x), x}, \quad (4.1)$$

here $\text{supp}(\lambda) \cap \text{supp}(\nu) \neq \emptyset$ and $Z_n$ is the corresponding normalizing factor or partition function given by

$$Z_n = \sum_{\sigma_n \in \Omega_{V_n}} \prod_{x \in V_n} \lambda_{\sigma_n(x)}^{-\nu(\sigma_n(x))} \prod_{x \in W_n} z_{\sigma_n(x), x}. \quad (4.2)$$

Definition 4.1. Generalized probability Gibbs distribution is called $p$-adic Gibbs distribution if (see [4], [16])

$$\lambda_i, z_{i,x} \in \mathbb{E}_p := \left\{a \in \mathbb{Q}_p : |a - 1|_p < p^{-1/(p-1)}\right\}, \text{ for any } i \in \mathbb{N}_0, x \in V.$$  

Remark 4.2. Our condition $z_{i,x} \in \mathbb{E}_p$ comes from the real-valued analogue of Gibbs measures. In real case $z_{i,x}$ is given as $\exp(h_{i,x})$, so we assumed that $z_{i,x}$ belongs to the set $\text{Im}(\exp_p(x))$. In general, this condition is not necessary (see [21]).

$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{x \in V_n} \lambda_{\sigma_n(x)}^{-\nu(\sigma_n(x))} \prod_{x \in W_n} z_{\sigma_n(x), x}. \quad (4.1)$
By \( p \)-adic version of Kolmogorov’s theorem on extension of measures (see [5]) one can show that if measures (4.1) are consistent (compatible) then there is unique \( p \)-adic Gibbs measure, which coincides with \( \mu^{(n)} \) on the cylinders with base \( \sigma_n \).

Here we consider \( p \)-adic Gibbs measures on the set of admissible configurations. By similar arguments of the real case (see proof of Theorem 2.1 in [2]) one can show that the measure \( \mu^{(n)}, n = 1, 2, \ldots \) associated with (4.1), on the set of admissible configurations, satisfy compatibility condition if and only if for any \( x \in V \setminus \{x^0\} \) the following equation holds:

\[
  z_{i,x} = \lambda_i \prod_{y \in S(x)} \frac{1}{1 + \sum_{j \in \mathbb{N}_0} z_{j,y}}, \quad i \in \mathbb{N}_0, \tag{4.3}
\]

where \( \lambda_i \) is redefined as function of previous \( \lambda_i \) and \( \nu(i) \). Assume that \( 1 + \sum_{j \in \mathbb{N}_0} z_{j,y} \neq 0 \) for any \( y \in S(x) \).

Finding the general form of solutions to equation (4.3) seems to be a very difficult task.

Let we consider following series \( \sum_{j \in \mathbb{N}_0} \lambda_j \) and \( \sum_{j \in \mathbb{N}_0} z_{j,y}, y \in V \). If these series are not convergent, then the above equation does not make sense.

Recall that a \( p \)-adic series converges if and only if its terms have zero limit. Also, if the \( p \)-adic series is convergent, its sum does not depend on the numbering of terms (see page 75 of [6]).

**Theorem 4.3.** There does not exist a \( p \)-adic Gibbs distribution on the set of admissible configurations (with finite-volume distributions defined as (4.1)).

**Proof.** It can be seen that there is no any solution \( z_{i,x} \in \mathcal{E}_p \) to equation (4.3). Indeed, assume a solution \( z_{i,x} \in \mathcal{E}_p \) exists. Then by above mentioned remark we should have

\[
  \lim_{i \to \infty} z_{i,x} = 0, \quad \text{for all } x \in V.
\]

That is for any \( \varepsilon > 0 \) there is \( i_0 \in \mathbb{N} \) such that

\[
  |z_{i,x}|_p < \varepsilon, \quad i \geq i_0.
\]

Take now \( \varepsilon < 1 \) then by the property 1) of the \( p \)-adic norm (see Section 2.1) we get

\[
  |z_{i,x} - 1|_p = 1 > p^{-1/(p-1)},
\]

that is a contradiction to the assumption that \( z_{i,x} \in \mathcal{E}_p \). \( \square \)

**Remark 4.4.** In [7] for the real-valued case of the HC-model it is proven that there is unique TIGM. But Theorem 4.3 shows that \( p \)-adic version of the model is different from the real one.

Now, we investigate generalized Gibbs measures (GGMs). Consider translation-invariant solutions, i.e., \( z_x = z \in \mathbb{Q}_p^{\mathbb{N}_0} \), for all \( x \in V \). GGM corresponding to such a solution, is called translation-invariant \( p \)-adic GGM (TIGpGM)

Similarly, one can define a periodic GGM (PGpGM) which corresponds to the periodic solution \( z_x \), in sense of periodicity with respect to shifts on vertices of the Cayley tree (see, for example, page 78 of [29] and page 931 of [23]). Moreover, as it was shown in [24], only translation-invariant and two-period Gibbs measures may exist for our HC-model.
4.2. Translation-Invariant Solutions for $p \geq 3$

To classify the TIGpGMs it is necessary to find all translation-invariant solutions of (4.3), i.e., $z_x = z \in \mathbb{Q}_p^N$.

To do this, we rename the series terms in the equation and get

$$z_i = \lambda_i \cdot \frac{1}{\left(1 + \sum_{j \in \mathbb{N}} z_j\right)^k}, \quad i \in \mathbb{N}. \quad (4.4)$$

It can be seen that the equation (4.4) is equivalent to equation (3.2) in the previous section. Hence, the number of TIGpGMs for the HC model on the Cayley tree of order two is the same as the number of fixed points of the operator $F$ considered in the previous section. To use these fixed points we introduce, for $p > 3$,

$$\theta = \sum_{n=1}^{\infty} \lambda_n, \quad b = \theta + \frac{2}{27},$$

then

$$-\frac{1}{3} = a_0 + a_1 p + a_2 p^2 + \ldots, \quad b |_p = b_0 + b_1 p + \ldots,$$

and

$$D_0 = -4a_0^3 - 27b_0^3, \quad u_1 = 0, u_2 = -a_0, u_3 = b_0,$$

$$u_{n+3} = b_0 u_n - a_0 u_{n+1}, \quad \text{for any } n = 1, 2, \ldots, p - 3.$$ Also, $D = \frac{4}{27} - 27 |_p^2 |_p$. If $D \neq 0$, then $D = \frac{D^*}{|_p^2}$, $D^* \in \mathbb{Z}_p$, $D^* = d_0 + d_1 p + \ldots$.

Now as a corollary of Theorem 3.3 we have

**Theorem 4.5.** Let $p > 3$ be a prime and $N_{\text{TIGpGM}}$ is number of the translation-invariant generalized $p$-adic Gibbs measures for HC model on the Cayley tree of order two. Then the following statements hold:

\[
N_{\text{TIGpGM}} = \begin{cases} 
3, & |b|_p^2 > 1, 3 |_{\text{ord}_p b}, p \equiv 1 \pmod{3}, b_0^{p-1} \equiv 1 \pmod{p}; \\
3, & |b|_p = 1, D = 0; \\
3, & |b|_p = 1, 0 < |D|_p < 1, 2 |_{\text{ord}_p D}, a_0^{p-1} \equiv 1 \pmod{p}; \\
3, & |b|_p = 1, |D|_p = 1 \text{ and } u_{p-2} = 0 \pmod{p}; \\
3, & |b|_p < 1, (-a_0)^{\frac{p-1}{2}} \equiv 1 \pmod{p}; \\
1, & |b|_p > 1, 3 |_{\text{ord}_p b}, p \equiv 2 \pmod{3}; \\
1, & |b|_p = 1, 0 < |D|_p < 1, 2 |_{\text{ord}_p D}, a_0^{p-1} \equiv 1 \pmod{p}; \\
1, & |b|_p = 1, 0 < |D|_p < 1, 2 \nmid \text{ord}_p D; \\
1, & |b|_p = 1, D_0 u_{p-2}^2 \not\equiv 0 \pmod{p}, D_0 u_{p-2}^2 \not\equiv 9 a_0^2 \pmod{p}; \\
1, & |b|_p < 1, (-a_0)^{\frac{p-1}{2}} \not\equiv 1 \pmod{p}; \\
0, & \text{otherwise.}
\end{cases} \quad (4.5)
\]

**Remark 4.6.** Comparing with Remark 4.4 one notes that Theorem 4.5 shows regions of parameters where $TIpGGM$ is non unique. That is another interesting difference between real and $p$-adic cases.
In the case \( p = 3 \), by Lemma 3.6 and Theorem 3.7 we have the following

**Theorem 4.7.** Let \( p = 3 \). Then we have the following:

1. The translation-invariant generalized 3-adic Gibbs measures for HC model on the Cayley tree of order two exist if and only if one of the following statements hold:

   1.a) \( |b|_3^2 < 27 \),
   1.b) \( |b|_3 > 27, 3|\text{ord}_3 b, b^* \in \mathbb{Z}_3^* \cup \mathbb{Z}_3^* \cup \mathbb{Z}_3^* \).

2. If the translation-invariant generalized 3-adic Gibbs measures for HC model on the Cayley tree of order two exist, it will be unique.

### 4.3. Two-Periodic GpGM

By results of Section 3.2 we get the following theorems related to two-periodic GpGMs.

**Theorem 4.8.** Let \( p \geq 3 \) be a prime and \( N_{\text{PGpGM}} \) is number of \( G^{(2)}_p \)-periodic generalized \( p \)-adic Gibbs measures for HC model on the Cayley tree of order two. Then the following statements hold:

\[
N_{\text{PGpGM}} = \begin{cases} 
2, & \text{if } |\theta|_p > 1; \\
2, & \text{if } |\theta|_p < 1, 2 \mid \text{ord}_p \theta, \left( \frac{-\theta}{p} \right) = 1; \\
0, & \text{if } |\theta|_p < 1, 2 \nmid \text{ord}_p \theta \text{ or } \left( \frac{-\theta}{p} \right) = -1; \\
2, & \text{if } |\theta|_p = 1, \left( \frac{\theta^2 - 4\theta}{p} \right) = 1; \\
0, & \text{if } |\theta|_p = 1, \left( \frac{\theta^2 - 4\theta}{p} \right) = -1; \\
2, & \text{if } |\theta|_p = 1, 2 \mid \text{ord}_p (\theta - 4), \left( \frac{\theta - 4}{p}(\theta - 4)(\text{mod } p) \right) = 1; \\
0, & \text{if } |\theta|_p = 1, 2 \nmid \text{ord}_p (\theta - 4) \text{ or } \left( \frac{\theta - 4}{p}(\theta - 4)(\text{mod } p) \right) = -1.
\]

(4.6)
Theorem 4.9. Let \( p = 2 \). Then the following statements hold true:

\[
N_{PGpGM} = \begin{cases} 
2, & \text{if } |\theta|_2 > 1; \\
2, & \text{if } |\theta|_2 = 1 \text{ and } \theta_2 = 1; \\
0, & \text{if } |\theta|_2 = 1 \text{ and } \theta_2 = 0; \\
2, & \text{if } |\theta|_2 = \frac{1}{2} \text{ and } \theta_1 = 0; \\
0, & \text{if } |\theta|_2 = \frac{1}{2} \text{ and } \theta_1 = 1; \\
2, & \text{if } |\theta|_2 = \frac{1}{4} \text{ and } \theta_1 = 0, \theta_2 = \theta_3 = 1, \theta_4 = 0; \\
0, & \text{if } |\theta|_2 = \frac{1}{4} \text{ and } (\theta_1 - 1)\theta_2\theta_3(\theta_4 - 1) = 0; \\
0, & \text{if } |\theta|_2 = \frac{1}{8}; \\
2, & \text{if } |\theta|_2 = \frac{1}{16} \text{ and } \theta_1 = 0, \theta_2 = 1; \\
0, & \text{if } |\theta|_2 = \frac{1}{16} \text{ and } \theta_1 = 1 \text{ or } \theta_2 = 0; \\
0, & \text{if } |\theta|_2 \leq \frac{1}{32}, 2 \nmid \text{ord}_p \theta; \\
2, & \text{if } |\theta|_2 \leq \frac{1}{32}, 2 \mid \text{ord}_p \theta, \theta_1 = \theta_2 = 0; \\
0, & \text{if } |\theta|_2 \leq \frac{1}{32}, 2 \mid \text{ord}_p \theta, \theta_1 = 1 \text{ or } \theta_2 = 1.
\end{cases}
\]

Remark 4.10. In [7] for the real-valued case of the HC-model it is proven that there are 1 or 3 two-periodic GMs. But Theorem 4.9 shows that \( p \)-adic version of the model may have 0 or 2 two-periodic GpGMs.

Remark 4.11. In \([4, 5, 13, 16, 20, 21]\) (see also references therein) the authors studied existence of a phase transition for \( p \)-adic models. But definition of the phase transition varies as “non-uniqueness of GMs”, “boundedness/unboundedness of GMs” or “existence of many generalized/quasi GMs”. The problem of existence some kind of phase transition for our model will be studied in a separate paper.

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