A New Representation Theorem for Many-valued Modal Logics

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Abstract

We propose a new definition of the representation theorem for many-valued logics, with modal operators as well, and define the stronger relationship between algebraic models of a given logic and relational structures used to define the Kripke possible-world semantics for it. Such a new framework offers a new semantics for many-valued logics based on the truth-invariance entailment. Consequently, it is substantially different from current definitions based on a matrix with a designated subset of logic values, used for the satisfaction relation, often difficult to fix. In the case when the many-valued modal logics are based on the set of truth-values that are complete distributive lattices we obtain a compact autoreferential Kripke-style canonical representation. The Kripke-style semantics for this subclass of modal logics have the joint-irreducible subset of the carrier set of many-valued algebras as set of possible worlds. A significant member of this subclass is the paraconsistent fuzzy logic extended by new logic values in order to also deal with incomplete and inconsistent information. This new theory is applied for the case of autoepistemic intuitionistic many-valued logic, based on Belnap's 4-valued bilattice, as a minimal extension of classical logic used to manage incomplete and inconsistent information as well.

1 Introduction

Many-valued logic was conceived as a logic for uncertain, incomplete and possibly inconsistent information which is very close to the statements containing the words "necessary" and "possible", that is, to the statements that make an assertion about the mode of truth of some other statement. Algebraic semantics interprets modal connectives as operators, while Relational semantics uses relational structures, often called Kripke models, whose elements are thought of variously as being possible worlds; for example, moments of time, belief situations, states of a computer, etc. The two approaches are closely related: the subsets of relational structures form an algebra with modal operators, while conversely any modal algebra can be embedded into an algebra of subsets of a relational structure via extensions of Stone's Boolean representation theory. For example, the first (1934) and the most known Stone's representation theorem for Boolean algebras, is the duality between the category of Boolean algebras and the category of Stone spaces. Every Boolean algebra \((BA, +, -, \cdot, \ 0, 1)\), where \(+, -, \cdot\) are corresponding algebraic operations (addition, multiplication and complement) for classical logic connectives \(\lor, \land, \neg\) respectively, is isomorphic to an algebra of particular clopen (i.e., simultaneously closed and open) subsets of its Stone space. Stone's theorem has since been the model for many other similar representation theorems. Our representation theorem, in the case of distributive complete lattice of truth values, is a particular Stone-like autoreferential representation based on the particular subsets of these truth-values.

In order to be able to follow this paper the readers must have clear in mind the difference between a many-valued logic and its underlying algebra of truth-values (for example, the propositional logic and its Boolean algebra, the intuitionistic logic and its Heyting algebra), so that we can informally use the term lattice (of algebraic truth values) speaking about logics as well. Given two sets \(A\) and \(B\), we denote by \(A^B\), the set of all functions from \(B\) to \(A\), by \(A^n\) the \(n\)-th cartesian product \(A \times \ldots \times A\), and by \(\mathcal{P}(A)\) the powerset of \(A\). The representation theorems are based on Lindenbaum algebra of a logic \(L = (Var, O, \models)\), where \(Var\) is a set of propositional symbols of a language \(L\), \(O\) is the set of logical connectives and \(\models\) is the entailment relation of this logic. We denote by \(F(L)\) the set of all formu-
lae. Notice that the truth-values in $A \subset Var \subseteq F(L)$ are the constant propositional symbols as well, and we will use the same symbols for them as those used for elements in $A$, with the bottom and top elements 0,1 respectively. Lindenbaum algebra of $L$ is the quotient algebra $F(L)/\equiv$, where for any two formulae $\phi, \psi \in F(L)$, it holds that $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$.

The algebraic existential modal operators $o_i : A \to A$, $i = 1, 2, \ldots$, are monotonic, additive ($o_i(x \lor y) = o_i(x) \lor o_i(y)$) and $o_i(0) = 0$ (the universal modal operators are monotonic and multiplicative $\overline{\alpha}(x \land y) = \overline{\alpha}(x) \land \overline{\alpha}(y)$, $\overline{\alpha}(1) = 1$). They appear often in many-valued logics, for example, as conflation operator (knowledge negation $[2]$) and Moore’s autoepistemic work, based on algebraic

Relevant work: we will briefly present the previous work, based on algebraic matrices, and explain some weak points of such a matrix-based approach. The standard approach to representation theorems uses a subset $D \subset A$ of the set of truth values $A$, denominated designated elements; informally the designated elements represent the equivalence class of the theorems of $L$. Given an algebra $A = (A, \{o\}_{o \in \mathcal{O}})$, the $\mathcal{O}$-matrix is the pair $(A, D)$, where $D \subset A$ is a subset of designated elements. The algebraic satisfaction relation $\models a$ (‘a’ stands for ‘algebraic’) is defined as follows:

**Definition 1** Let $L = (Var, \mathcal{O}, \models)$ be a logic, $(A, D)$ a $\mathcal{O}$-matrix, and $\phi \in F(L)$. Let $I : Var \to A$ be a map that assigns logic values to propositional variables, and $\overline{I} : F(L) \to A$ be its unique extension to all formulae in a language $L$. Let $M$ be a class of $\mathcal{O}$-matrices. We define the relation $\models a$ inductively as follows:

1. $(A, D) \models a \phi$ iff $\overline{I}(\phi) \in D$,
2. $(\mathcal{D}, D) \models a \phi$ iff $\overline{I}(\phi) \in D$ for every $I : Var \to A$,
3. $M \models a \phi$ iff $(A, D) \models a \phi$ for every $(A, D) \in M$.

A logic $L$ is sound w.r.t. $M$ iff for every $\phi \in F(L)$, if $L \models \phi$ then $M \models a \phi$.

$L$ is complete w.r.t. $M$ iff for every $\phi \in F(L)$, if $M \models a \phi$ then $L \models \phi$.

Dual to algebraic semantics, based on the class $M$ of $\mathcal{O}$-matrices we also have the Kripke-style semantics based on a class $\mathcal{R}$ of relational models where the satisfiability relation $\models r$ is defined by induction on the structure of the formulae. Substantially, each relational model $K \in \mathcal{R}$ is a Kripke frame over a set of possible worlds with additional accessibility relations between possible worlds associated with logical operators. The distinctive feature of this relational semantics is that the accessibility relations are used in the definition of satisfiability, which is not just a mechanical truth-functional translation of the formula structure into the model. The definition of the algebraic/relational duality is based on the following assumption:

**Definition 2** Representation Assumption [12]

Assume that there exists a class $\mathcal{R}$ of relational structures such that there exist $\mathcal{D} : \mathcal{M} \to \mathcal{R}$, $\mathcal{E} : \mathcal{R} \to \mathcal{M}$ such that (C):

(i) for every $K \in \mathcal{R}$, $\mathcal{E}(K) = (A_K, D_K) \in \mathcal{M}$, where $A_K$ is an algebra of subsets of the support of $K$;

(ii) for every $M \in \mathcal{M}$ such that (C): $\mathcal{E}(M) = (\mathcal{D}(M)) = (A_D(M), D_D(M))$ then there is an injective homomorphism $i_n : A \to A_D(M)$ with $i_n^{-1}(D_D(M)) \subseteq D$.

Let $m : Var \to A_K$ be a meaning function (assigns logic values to propositional variables), then $(K,m)$ is the Kripke model for a frame $K$. Then, the definition of the relation $\models r$ can be given as follows:

**Definition 3** [12] Assume that $\mathcal{M}$ and $\mathcal{R}$ satisfy condition (C)(i). Let $K \in \mathcal{R}$, $m : Var \to A_K$, and $\overline{m} : F(L) \to A_K$ be the unique homomorphism of $\mathcal{O}$-algebras that extends $m$. Let $y$ be an element in the support of $K$, then:

1. $K \models_{m,y} \phi$ iff $y \in \overline{m}(\phi)$;
2. $K \models m \phi$ iff $\overline{m}(\phi) \in D_K$;
3. $K \models r \phi$ iff for every $m$, $K \models m \phi$.

A logic $L$ is sound w.r.t. $\mathcal{R}$ iff for every $\phi \in F(L)$, if $L \models \phi$ then $\mathcal{R} \models r \phi$. $L$ is complete w.r.t. $\mathcal{R}$ iff for every $\phi \in F(L)$, if $\mathcal{R} \models r \phi$ then $L \models \phi$.
• In a matrix-based many-valued logic, a formula is satisfied if its logic value is a designated value. Such an approach, based on $O$-matrices, is very effective for all kinds of 2-valued logic where the set of designated elements is a singleton set composed by only true value, $D = \{1\}$, as in the case of classical, intuitionistic and 2-value modal logics (extension of Boolean algebra). It is only a partially valid solution for the case when a set of truth-values can not be easily divided into two complementary subsets: $D \subset A$ for values for which we retain that a formula can be considered satisfied, and its complement $A \setminus D$ for those which we retain that a formula cannot be considered satisfied. For example, in the case of fuzzy logic where $A = [0, 1]$ (the closed set of reals between 0 and 1) we can assume that $D = [a, 1]$ is the closed set between some prefixed value $0 < a < 1$ and 1. But is not clear what is the correct value for $a$ for generally acceptable fuzzy logic (otherwise we will have an infinite number of different logics for each an arbitrary value $a$).

An analog difficulty we can find in the case of bisemilattices [32, 13, 11].

• The second observation is that the representation theorems define the isomorphism between a many-valued algebra and the set-based algebra that is a subalgebra of the canonical extension of the original many-valued algebra. It will be useful to define directly such an isomorphism based on the duality assumption (C).

Main contribution: The main contribution, presented in Section 3, is a general representation theorem for many-valued logics with the truth-invariance entailment for any set of truth-values $A$ (also if it is not a lattice). It is substantially different w.r.t the previous representation theorems that are all based on matrices, and is based on algebraic models of a logic. We replace the duality Algebras (Matrices) - Relational structures described in previous work, by the semantic duality Algebraic models - Kripke models. The novelty is that the set of models of a given logic can be obtained by using Gentzen-like sequent calculi [19] without using necessarily the subset of designated elements (matrices). As a guiding example instead, here in Section 2, will be presented a more specific case, when a logic is based on complete distributive lattice $A$. These sequent-based representations of many-valued logics with truth-invariance entailment allows us to define, without using the matrices, the set of models of a given many-valued logic, required by general definitions in Section 3. This particular example in Section 2, when $A$ is complete distributive lattice, is then used in Section 4 for a concrete definition of Kripke frames based on an autoreferential assumption [20] where the set of possible worlds is fixed by a subset of algebraic truth values in $A$.

This paper is based on the idea that the satisfaction relation (and the entailment) in the case of many-valued logics can be defined without using the subset of designated elements. For example, in the case of logic programs, let $I : Var \to A$ be a many-valued valuation, and $(A, \sqsubseteq)$ be the set $A$ of logic values with partial truth order $\sqsubseteq$. Then, given any rule $B \leftarrow B_1 \land \ldots \land B_n$ where $B$ is a propositional letter and $B_i$ is a ground literal (propositional letter or negation of them), we say that it is satisfied iff $I(B) \sqsubseteq I(B_1) \land \ldots \land I(B_n)$; the valuation that satisfies all rules is a model for such a logic program. As we have seen in this case, instead of the subset $D \subseteq A$ of designated elements, we simply use the truth ordering between logic values.

The simple way to extend this example to any propositional logic $L = (Var, \sqsubseteq, \vdash)$ is to consider equivalently this logic as a sequent system of (structural and logical) rules $R : \frac{\ldots}{\ldots}$ where each $s_i$ is a sequent $\phi_1, \ldots, \phi_n \vdash \psi_1, \ldots, \psi_m$ where, accordingly to Gentzen, the commas in the left are conjunctions while those on the right are disjunctions, and $\phi_i, \psi_j \in F(L)$ are logic formulae. We say that a valuation $I : Var \to A$ satisfies this sequent $I(\phi_1) \land \ldots \land I(\phi_n) \sqsubseteq I(\psi_1) \lor \ldots \lor I(\psi_m)$, and that $I$ satisfies a rule $R$ iff $I$ satisfies the conclusion sequent $s$ of this rule whenever it satisfies all sequent premises $s_1, \ldots, s_k$ of this rule. Then, a model of this logic is any valuation $I$ which satisfies all logic sequent rules of this logic (the structural sequent rules as Identity, Cut, Weakening, Permutation, Contraction and Associativity rules are satisfied by all valuations).

Notice that this sequent-based approach is always possible, independently of the algebraic properties of the set of truth-values in $A$, for example by transforming the original many-valued logic into 2-valued modal logic [21, 22], and defining the classical 2-valued sequent rules as presented in [19] with the truth-invariance entailment for many-valued logics. This truth-invariance entailment be used for a new representation theorem in this paper (in Definition 5). Notice that the sequent system can be used also as a basis for an autoreferential algebraic/relational semantics of many-valued logics [20].

In what follows we denote by $y \sqsubseteq x$ iff $(y \sqsubseteq x$ and not $x \sqsubseteq y$), and we denote by $x \triangleright y$ two unrelated elements in $A$ (so that not $(x \sqsubseteq y$ or $y \sqsubseteq x$)).

We define the following mapping $\downarrow : A \to \mathcal{P}(A)$ such that for any element $x \in A$, we obtain the closed set $\downarrow x = \{a \in A \mid a \sqsubseteq x \}$. It is well known that for any two elements of a complete lattice $x, y \in A$ holds the set...
intersection closure property \( \downarrow x \cap \downarrow y = \downarrow (x \land y) \), but does not hold the union closure property, that is, generally does not exists \( z \in A \) such that \( \downarrow x \cup \downarrow y = \downarrow z \).

But the closure property for the intersection and union holds for the more general case of hereditary subsets: a set \( B \in \mathcal{P}(A) \) is hereditary if it is closed downwards under \( \subseteq \), i.e., if we have that whenever \( x \in B \) and \( y \subseteq x \) then \( y \in B \). Notice that the bottom hereditary subset of any complete lattice \( A \) is the set \( \downarrow 0 = \{ 0 \} \) where 0 is the bottom element of \( A \). Thus while \( \mathcal{P}(A) \) is a topological space, its subset composed by only hereditary subsets of \( A \), used to define the canonical representation isomorphic to the algebra \( A \), will not be topological space (because the empty set will not be an element of the carrier set of this canonical subalgebra of the powerset canonical extension algebra). This is also seen in power-domains in the domain theory, where the empty set is often excluded.

This paper follows the following plan:

In Section 2 we show, in a particular example, how we are able to avoid the matrices used in previous Representation theorem frameworks for many-valued logics: we present an autoreferential semantics for many-valued logics, based on sequents and many-valued valuations. In Section 3 we define the main result of this paper: a new general Representation theorem framework for many-valued logics with truth-invariance entailment, where we replace the duality Algebras (Matrices) - Relational structures by the semantic duality Algebraic models - Kripke models. In Section 4 we apply this new Representation theorem framework to modal many-valued logics, in the particular case when it is based on complete distributive lattices of truth values (an autoreferential representation). In Section 5 we consider a concrete example of Belnap’s bilattice, composed by two (truth and knowledge) complete distributive lattices, used for applications in logic programming with incomplete and inconsistent information.

2 Sequents for Many-valued logics based on complete distributive lattices

The main result of this work is a new representation theorem for any many-valued logic, based on models of such a logic, and will be presented in the next section. In this section instead we will introduce an example of defining the set of models of a given many-valued logic \( L \), based on binary sequent systems for many-valued logics.

Sequent calculus, introduced by Gentzen \cite{22} and Hertz \cite{24} for classical logic, was generalized to the many-valued (m-sequents) case by Rouseau \cite{25} and others. The tableaux calculi were presented in \cite{20, 27}. The strict correspondence between the cut-free m-sequent calculus and closed tableaux has been presented in \cite{28}. The more detailed information for interested readers can be found in \cite{29, 30}. This ad-hoc m-sequent system is not standard one. Consequently, it is interesting to consider a calculus for many-valued logics based on standard binary sequents. Such a standard two-sides sequent calculi for lattice-based many-valued logics has been elaborated recently (with an autoreferential Kripke-style semantics for such logics) in two complementary ways in \cite{20, 31}.

A sequent system for the truth-invariance semantics of the entailment, used in a new representation theorem in the next Section, was recently presented in \cite{19}. Such a general system does not use the partial ordering of the truth values in \( A \).

Here we will present another example of a sequent system, for many-valued logics with a complete distributive lattice \( A \), with truth-preserving semantics of the entailment. It is another example, more specific than that in \cite{19}, of how we can define the models of many-valued logics without using the matrices. We justify this significant case because the algebras for all many-valued logics with finite set of logic values are complete lattices. And also the algebras for fuzzy logic, belief based logic, etc., \cite{32}, with infinite number of logic values, are complete and distributive lattices as well. In what follows we will use the approach in \cite{31}, with the valuation-based semantics for many-valued logic. Given a propositional logic \( L \) a sequent is a consequence pair of formulae \( s = (\phi; \psi) \in F(L) \times F(L) \), denoted also by \( \phi \vdash \psi \).

A Gentzen system, denoted by a pair \( \mathcal{G} = (\langle L, \vdash \rangle) \), where \( \vdash \) is finitary consequence relation on set of sequents in \( L = F(L) \times F(L) \), is said to be normal if it satisfies the following conditions: for any sequent \( s = \phi \vdash \psi \in L \) and a set of sequents \( \Gamma \subseteq L \),

1. (reflexivity) if \( s \in \Gamma \) then \( \Gamma \vdash s \)
2. (transitivity) if \( \Gamma \vdash s \) and for every \( s' \in \Gamma \), \( \Theta \vdash s' \), then \( \Theta \vdash s \)
3. (finiteness) if \( \Gamma \vdash s \) then there is finite \( \Theta \subseteq \Gamma \) such that \( \Theta \vdash s \).
4. for any homomorphism \( \sigma \) from \( L \) into itself (i.e., substitution), if \( \Gamma \vdash s \) then \( \sigma(\Gamma) \vdash \sigma(s) \), i.e., \( \{ \sigma(\phi_i) \vdash \sigma(\psi_i) \mid \phi_i \vdash \psi_i \in \Gamma \} \vdash (\sigma(\phi) \vdash \sigma(\psi)) \).

Notice that from (1) and (2) we obtain the monotonic property:

5. if \( \Gamma \vdash s \) and \( \Gamma \subseteq \Theta \), then \( \Theta \vdash s \).

We denote by \( C : \mathcal{P}(L) \to \mathcal{P}(L) \) the closure operator such that \( C(\Gamma) = \{ s \in L \mid \Gamma \vdash s \} \), with the properties: \( \Gamma \subseteq C(\Gamma) \) (from reflexivity (1)); it is monotonic, i.e., \( \Gamma \subseteq \Gamma_1 \) implies \( C(\Gamma) \subseteq C(\Gamma_1) \) (from
AXIOMS

Any sequent theory \( \Gamma \subseteq L \) is said to be a closed theory iff \( \Gamma = C(\Gamma) \). This closure property corresponds to the fact that \( \Gamma \vdash s \) iff \( s \in C(\Gamma) \).

Each sequent theory \( \Gamma \) can be considered as a valuation (characteristic function) \( \beta : L \rightarrow [0,1] \) such that for any sequent \( s \in L \), \( \beta(s) = 1 \) iff \( s \in \Gamma \). We will use this 2-valued valuation-based semantics in order to define the sound and complete many-valued logic.

Example 1: Let us consider the many-valued modal logic with a distributive complete lattice \( (A,\subseteq) \) of truth values (where all truth-values in \( A \) are language primitives as well), that is an extension of the Distributive modal logic (distributive lattice logic DLL) \( [9,33] \) (with \( \Box \) universal modal operator, and its left adjoint existential modal operator \( \lozenge \), with \( \dag \dag \Box \) and with negative modal additive operator \( \neg : (A,\subseteq,\vee) \rightarrow (A,\subseteq,\vee)^{OP} \), where \( \vee^{OP} = \land, \subseteq^{OP} = \geq \)). The binary consequence system \( \mathcal{G} \), in this logic \( L \), is as follows:

(AXIOMS) \( \mathcal{G} \) contains the following sequents:
1. \( \phi \vdash \phi \) (reflexive)
2. \( \phi \vdash \bot \) (top/bottom axioms)
3. \( \phi \land \psi \vdash \phi, \phi \land \psi \vdash \psi \) (projection: axioms for meet)
4. \( \phi \land \psi \vdash \psi \vdash \phi \vee \psi \) (injections: axioms for join)
5. \( \phi \land (\psi \lor \varphi) \vdash (\phi \land \psi) \lor (\phi \land \varphi) \) (distributivity axiom)
6. \( \Box(\phi \land \psi) \vdash \Box \phi \land \Box \psi, \bot \vdash \bot \) (multiplicative lattice property axioms)
7. \( \Diamond(\phi \lor \psi) \vdash \Diamond \phi \lor \Diamond \psi, \Diamond 0 \vdash \Diamond 0 \) (additive lattice property axioms)
8. \( \neg \phi \land \neg \psi \vdash \neg (\phi \lor \psi), \bot \vdash \neg \) (additive lattice negation axiom)
9. The set of sequents that define the poset of the lattice of truth values \( (A,\subseteq) \): for any two \( x, y \in A \), if \( x \subseteq y \) then \( x \vdash y \) is an axiom.

(INFERENCERULES) \( \mathcal{G} \) is closed under the following inference rules:
1. \( \phi \vdash \psi, \phi \lor \psi \) (cut/transitivity rule)
2. \( \phi, \psi \vdash \phi \lor \psi \) (lower/upper lattice bond rules)
3. \( \phi \lor \psi \vdash \phi, \phi \lor \psi \vdash \psi \) (monotonicity of modal operators rules)
4. \( \phi \vdash \psi \) (antitonicity of modal negation rule)
5. \( \sigma(\phi) \vdash \sigma(\psi) \) (substitution rule: \( \sigma \) is substitution (\( \gamma/p \))).

Notice that the rules in point 2 are the consequences of the diagonal mapping \( \Delta : A \rightarrow Y \), where \( Y = A \times A \) and \( \Delta x = (x,x) \), (which is both an additive and multiplicative modal operator), and its Galois adjunctions with the meet (multiplicative) and join (additive) operators \( \land, \lor : Y \rightarrow A \), i.e., with \( \Delta \land \land \lor \lor \Delta \); that is \( \Delta x \subseteq Y(y,z) \) (i.e., \( x \subseteq y \) and \( x \subseteq z \)) iff \( x \subseteq y \lor (y,z) = y \lor z \), and \( x \lor y \lor (x,y) \subseteq z \) iff \( (x,y) \leq_Y \Delta z \) (i.e., \( x \subseteq z \) and \( y \subseteq z \)).

The axioms from 1 to 5 and the rules 1 and 2 are taken from [9] for the DLL and it was shown that this sequent-based Gentzen-like system is sound and complete. The system \( \mathcal{G} \) in Example 1 in only a guiding example, that will be consider in the rest of this section. We are able to introduce another logical connectives for any given many-valued modal logic (where existential modal operators are monotonic, additive and normal) based on the complete distributive lattice of truth values in \( A \), in order to obtain a similar sequent system as this in Example 1. Notice that in a Gentzen-like deductive system \( \mathcal{G} \) above each sequent is a valid truth-preserving consequence pair defined by the poset of the complete distributive lattice \( (A,\subseteq) \) of truth values (which are also the constants of this modal propositional language \( L \)). Consequently, each occurrence of the symbol \( \vdash \) can be substituted by the partial order \( \subseteq \) of this complete lattice.

Definition 4 Truth-Preserving entailment: For any two formulae \( \phi, \psi \in \mathcal{F}(L) \), the truth-preserving consequence pair (sequent) denoted by \( \phi \vdash \psi \) is satisfied by a given valuation \( I \in \mathcal{V}_m \) if \( \mathcal{T}(\phi) \subseteq \mathcal{T}(\psi) \).

This sequent is a tautology if it is satisfied by all valuations, i.e., when \( \forall I \in \mathcal{V}_m (\mathcal{T}(\phi) \subseteq \mathcal{T}(\psi)) \).

For a normal Gentzen-like sequent system \( \mathcal{G} \) of the many-valued logic \( L \), with the set of sequents \( \text{Seq}_L \subseteq L \) and a set of inference rules \( \text{Rul}_L \), we say that a many-valued valuation \( I \) is its model if it satisfies all sequents in \( \mathcal{G} \). The set of all models of a given set of sequents (theory) \( \Gamma \) is denoted by \( \text{Mod}_\Gamma = \{ I \in \mathcal{V}_m | \forall (\phi \vdash \psi) \in \Gamma (\mathcal{T}(\phi) \subseteq \mathcal{T}(\psi)) \} \subseteq \mathcal{V}_m ^A \).

Proposition 1 Soundness: All axioms of the Gentzen-like sequent system \( \mathcal{G} \) of a many-valued logic \( L \) based on complete distributive lattice \( (A,\subseteq) \) of algebraic truth values are the tautologies, and all its rules are sound for the model satisfiability and preserve the tautologies.

Proof: It is straightforward to verify (see the Example 1) that all axioms are tautologies (all constant sequents specify the poset of a complete lattice \( (A,\subseteq) \), thus are tautologies). It is straightforward to verify
that all rules preserve the tautologies. Moreover, if all
premises of any rule in \( \mathcal{G} \) are satisfied by a given
many-valued valuation \( \overline{T}: F(\mathcal{L}) \to A \), then also the deduced
sequent of this rule is satisfied by the same valuation,
i.e., the rules are sound for the model satisfiability.

\[ \Box \]

It is easy to verify that for any two \( x, y \in A \) we have that
\( x \sqsubseteq y \) iff \( x \vdash y \), that is the truth-preserving en-
tailment coincides with the partial truth-ordering in a
lattice \((A, \sqsubseteq)\). Notice that it is compatible with the latt-
icate operators, that is, for any two formulae \( \phi, \psi \in L \),
\( \phi \land \psi \vdash \psi \) and \( \phi \vdash \psi \lor \phi \). This entailment imposes
the following restrictions on the logic implication: in
order to satisfy the Deduction Theorem \( \vdash \neg \phi \Rightarrow (\phi \land \psi) \)
iff \( \neg \phi \land (\phi \land \psi) \) (i.e., inference rules for elimination and
introduction of the logic connective \( \Rightarrow \), that is, for any \( \phi, \psi \in L \)
and \( \neg \phi \land \psi \vdash \phi \lor \psi \), and \( \neg \phi \lor \psi \vdash \phi \land \psi \) by
\[ \overline{T}(\phi) \land \overline{T}(\psi) = \overline{T}(\phi \lor \psi), \quad \overline{T}(\phi) \lor \overline{T}(\psi) = \overline{T}(\phi \land \psi). \]
Thus the characteristic function, with
\( \overline{T}(\phi) \land \overline{T}(\psi) \) by this entailment, the logic implication must
satisfy (the case when \( z = 1 \)) the requirement that
for any \( x, y \in X \), \( x \Rightarrow y \) iff \( y \sqsubseteq x \), while
modest satisfying \( x \land (y \Rightarrow x) \sqsubseteq y \) in order to satisfy
the Modus Ponens inference rule. The particularity
of this entailment is that any consequent pair (se-
quent) \( \phi \lor \psi \) is algebraically an equation \( \phi \land \psi = \phi \lor \psi \)
(or, \( \phi \lor \psi = \psi \lor \phi \)). It is easy to verify, that in the case
\[ \Box \]

Remark: It is easy to observe that each sequent is,
from the logic point of view, a 2-valued object so that all
inference rules are embedded into the classical 2-valued
framework, i.e., given a bivaluation \( \beta : L \to 2 \), we have
that a sequent \( s = \phi \vdash \psi \) is satisfied when \( \beta(s) = 1 \), so
that we have the relationship between sequent bival-
vations and many-valued valuations \( I \) used in Definition

In fact we have that \( \beta = \exists_{o \phi < \pi_1, \land > o(\overline{T}(\phi) \land \overline{T}(\psi)) : L \to 2} \) is the characteristic function, with \( \pi_1 \) first pro-
jection, a valuation \( \overline{T}: F(\mathcal{L}) \to A \), and \( eq : A \times A \to 2 \)
(defined by \( eq(a, b) = 1 \) iff \( a = b \)).

Consequently, \( \beta(\phi \lor \psi) = \beta(\phi; \psi) = eq(\phi < \pi_1, \land > o(\overline{T}(\phi) \land \overline{T}(\psi)) =
\overline{T}(\phi) \land \overline{T}(\psi) \land \overline{T}(\phi) \land \overline{T}(\psi) \Rightarrow
\overline{T}(\phi) \land \overline{T}(\psi) \land \overline{T}(\phi) \land \overline{T}(\psi)) \). Thus
\( \beta(\phi \lor \psi) = 1 \) iff \( \overline{T}(\phi) \land \overline{T}(\psi) \), i.e., when this
sequent is satisfied by \( \Gamma \).

From my point of view, this sequent feature, which is
only an alternative formulation for the 2-valued clas-
sical logic, is fundamental in the framework of many-
valued logics, where the semantics for the entailment,
based on algebraic matrices \((A, D)\) is often arbitrary.

\[ \Box \]

Thus, this correct definition of the 2-valued entailment
in the sequent system \( \mathcal{G} \), based only on the lattice or-
dering, can replace the current entailment based on the
algebraic matrices \((A, D)\), where \( D \subseteq A \) is a subset of
designated elements, which is upward closed. That is,
if \( x \in D \) and \( x \sqsubseteq y \) then \( y \in D \) (thus \( 1 \in D \)). Conse-
sequently, the matrix-entailment, defined by \( \phi \vdash_D \psi \), is
valid iff \( \forall I \in \mathcal{V}_m. (\overline{T}(\phi) \in D \Rightarrow \overline{T}(\psi) \in D) \). It is
easy to verify also that \( \phi \lor \psi \) implies \( \phi \lor_D \psi \). Thus,
we are now able to introduce the model-theoretic seman-
tics for the many-valued logics:

**Definition 5** A many-valued model-theoretic seman-
tics of a given many-valued logic \( L \), with a Gentzen
system \( \mathcal{G} = (\mathcal{L}, \vdash) \), is a semantic deducibility relation
\( \models_I \), defined for any \( \Gamma \subseteq \mathcal{L} \) and sequent \( s = (\phi \vdash
\psi) \in L \) by:

\[ \Gamma \models_I s \iff \text{"all many-valued models of } \Gamma \text{ are}
the models of } s. \]

**Example 2:** Let us consider a many-valued logic with
\( \{p, q, r, r_1\} \subseteq \mathcal{V} \) and many-valued clauses \( P_r = \{p \Leftarrow
a, q \Leftarrow b, r \Leftarrow p, r \Leftarrow q\} \), with \( a, b \in A \). The sequen-
t-based translation of \( P_r \) results in a sequent theory \( \Gamma =
\{p \Leftarrow a, q \Leftarrow b, r \Leftarrow q, p \Leftarrow r\} \), so
that the set of models of \( P_r \) is equal to \( \mathcal{L}_r = \{I : \forall V a \Rightarrow A \mid I(p) = a, I(q) = b, I(r) = a \lor b \}
\text{ and } I(r_1) \in A \}. \) Thus we have that \( \Gamma \models_{m_r} (a \land q \Leftarrow
a \lor b) \) and \( \Gamma \models_I (a \land b \Leftarrow q \Leftarrow r_1) \), while for every \( c \in A \),
\( \Gamma \models_{m_r} (a \land b \Leftarrow q \Leftarrow r_1) \), while for every \( c \in A \),
\( \Gamma \models_I (a \land b \Leftarrow q \Leftarrow r_1) \), while for every \( c \in A \),
\( \Gamma \models_{m_r} (a \land b \Leftarrow q \Leftarrow r_1) \), while for every \( c \in A \),
\( \Gamma \models_I (a \land b \Leftarrow q \Leftarrow r_1) \), while for every \( c \in A \),

It is easy to verify that the Gentzen-like system
\( \mathcal{G} = (\mathcal{L}, \vdash) \) of a complete-lattice based many-valued
is a normal logic.

**Theorem 1** The many-valued model theoretic semantics
is an adequate semantics for a many-valued logic
\( L \) specified by a Gentzen-like logic system \( \mathcal{G} = (\mathcal{L}, \vdash) \),
that is, it is sound and complete.

Consequently, \( \Gamma \models_I s \iff \Gamma \models s. \)

**Proof:** Let us prove that for any many valued model
\( I \in Mod_I \), the obtained sequent bivaluation \( \beta =
\overline{T}(\phi) \land \overline{T}(\psi) : L \to 2 \) is the characteristic
function of the closed theory \( \Gamma \vdash I = C(\mathcal{T}) \) with
\( T = \{\phi \vdash x, x \vdash \phi \mid \phi \in L, x = \overline{T}(\phi)\}. \) From
the definition of \( \beta \) we have that \( \beta(\phi \lor \psi) = \beta(\phi; \psi) =
\overline{T}(\phi) \land \overline{T}(\psi) \land \overline{T}(\phi) \land \overline{T}(\psi) \Rightarrow
\overline{T}(\phi) \land \overline{T}(\psi) \land \overline{T}(\phi) \land \overline{T}(\psi)) \). Thus
\( \beta(\phi \lor \psi) = 1 \) iff \( \overline{T}(\phi) \land \overline{T}(\psi) \), i.e., when this
sequent is satisfied by \( I \).
1. Let us show that for any sequent $s$, $s \in \Gamma_I$ implies $\beta(s) = 1$: First of all any sequent $s \in T$ is of the form $\varphi \vdash x$ or $x \vdash \varphi$, where $x = I(\varphi)$, so that it is satisfied by $I$ (it holds that $\overline{T(\varphi)} \subseteq \overline{T(\varphi)}$ in both cases). Consequently, all sequents in $T$ are satisfied by $I$. From Proposition 1 we have that all inference rules in $\mathcal{G}$ are sound w.r.t. the model satisfiability, thus for any deduction $T \vdash s$ (i.e., $s \in \Gamma_I$) where all sequents in premises are satisfied by the many-valued valuation (model) $I$, also the deduced sequent $s = (\phi \vdash \psi)$ must be satisfied, that is, it must hold that $\overline{T(\phi)} \subseteq \overline{T(\psi)}$, i.e., $\beta(s) = 1$.

2. Let us show that for any sequent $s$, $\beta(s) = 1$ implies $s \in \Gamma_I$: For any sequent $s = (\phi \vdash \psi)$ if $\beta(s) = 1$ then $x = \overline{T(\phi)} \subseteq \overline{T(\psi)} = y$ (i.e., $s$ is satisfied by $I$). From the definition of $T$, we have that $\phi \vdash x, y \vdash \psi \in T$, and from $x \in y$ we have $x \in y \in Ax_y$ (where $Ax_y$ are axioms (sequents) in $\mathcal{G}$, with $\{x \in y \mid x, y \in A, x \subseteq y \} \subseteq Ax_y$, thus satisfied by every valuation) by the transitivity rule we obtain that $T \vdash (\phi \vdash \psi)$, i.e., $s = (\phi \vdash \psi) \in C(T) = \Gamma_I$. So, from (1) and (2) we obtain that $\beta(s) = 1$ iff $s \in \Gamma_I$, i.e., the sequent bivaluation $\beta$ is the characteristic function of a closed set. Consequently, any many-valued model $v$ of this many-valued logic $L$ corresponds to the closed bivaluation $\beta$ which is a characteristic function of a closed theory of sequents: we define the set of all closed bivaluations obtained from the set of many-valued models $I \in \text{Mod}_L$: $\text{Biv}_L = \{\Gamma_I \mid I \in \text{Mod}_L\}$. From the fact that $\Gamma$ is satisfied by every $I \in \text{Mod}_L$ we have that for every $\Gamma_I \in \text{Biv}_L$, $\Gamma \subseteq \Gamma_I$, so that $C(\Gamma) = \bigcap \text{Biv}_L$ (the intersection of closed sets is also a closed set). Thus, for $s = (\phi \vdash \psi)$, $\Gamma \models m s$ iff $\forall I \in \text{Mod}_L (\forall (\phi \vdash \psi) \in C(\overline{T(\phi)}) \subseteq \overline{T(\psi)})$ implies $\overline{T(\phi)} \subseteq \overline{T(\psi)}$ iff $\forall I \in \text{Mod}_L (\forall (\phi \vdash \psi) \in C(\beta(\phi \vdash \psi) = 1) \subseteq \overline{T(\phi)} \subseteq \overline{T(\psi)})$ implies $\beta(\phi \vdash \psi) = 1$ iff $\forall v \in \text{Mod}_L (\forall (\phi \vdash \psi) \in C((\phi \vdash \psi) \in \Gamma_I) \subseteq \overline{T(\phi)}) \subseteq \overline{T(\psi)})$ implies $s \in \Gamma_I$ iff $\forall \Gamma_I \in \text{Biv}_L (\Gamma \subseteq \Gamma_I \text{ implies } s \in \Gamma_I)$ iff $\forall \tau \in \text{Biv}_L (s \in \Gamma_I)$, because $\Gamma \subseteq \Gamma_I$ for each $\Gamma_I \in \text{Biv}_L$ iff $s \in \bigcap \text{Biv}_L = C(\Gamma)$, that is, $\Gamma \vdash s$.

Consequently, in order to define the model-theoretic semantics for a many-valued logics, we do not need to define the "problematic" matrices: we are able to use only the many-valued valuations, and many-valued models (i.e., valuations which satisfy all sequents in $\Gamma$ of a given many-valued logic $L$).

Differently from the classical logic where a formula is a theorem if it is true in all models of the logic, here, in a many-valued logic $L$, but specified by a set of sequents in $\Gamma$, for a formula $\phi \in F(L)$ that has the same value $x \in X$ (for any algebraic truth-value $x$) for all many-valued models $I \in \text{Mod}_L$, we have that its sequent-based version $\phi \vdash x$ and $x \vdash \phi$ are theorems; that is, $\forall I \in \text{Mod}_L (\overline{T(\phi)} = x)$ iff $\Gamma \vdash (\phi \vdash x)$ and $\Gamma \vdash (x \vdash \phi)$). (For instance, in the case of classical logic, a formula $\phi$ is a theorem iff $\Gamma \vdash (\phi \vdash 1)$ and $\Gamma \vdash (1 \vdash \phi)$), while $\neg \phi$ is a theorem iff $(\Gamma \vdash (\phi \vdash 0)$ and $\Gamma \vdash (0 \vdash \phi))$. But such a value $x \in X$ does not need to be a designated element $x \in D$, as in the matrix semantics for a many-valued logic, and it explains why we do not need the rigid semantic specification by matrix designated elements.

Thus, by translating a many-valued logic $L$ into its "meta" sequent-based $2$-valued logic, we obtain an unambiguous theory of truth-invariance inference without using the matrices.

Remark: There is also another way to reduce the many-valued logics into "meta" $2$-valued logics, based on the ontological encapsulation [31], where each many-valued proposition (or many-valued ground atom $p(a_1, \ldots, a_n)$) is ontologically encapsulated into $2$-valued atom $pF(a_1, \ldots, a_n, x)$ (by enlarging original atoms with new logic variable whose domain of values is the set $A$). Roughly, "$p(a_1, \ldots, a_n)$ has a value $x"$ iff $pF(a_1, \ldots, a_n, x)$ is true). In fact such an atom is equivalent to the following formula of sequents: $(p(a_1, \ldots, a_n) \vdash x) \land (x \vdash p(a_1, \ldots, a_n))$.

Autoreferential possible world semantics:
Based on this Gentzen-like sequent deductive system $\mathcal{G}$, or more general sequent system in [19], with truth-invariance semantics for the entailment used in the rest of this paper (in Definition 6), we are able to define the equivalence relation $\approx_L$ between the formulae of any propositional logic based on a complete distributive lattice $A$ in order to define the Lindenbaum algebra for this logic, $(L/\approx_L, \subseteq)$, where for any two formulae $\phi, \psi \in L$,

(a) $\phi \approx_L \psi$ iff $\phi \vdash \psi$ and $\psi \vdash \phi$, i.e., iff $\forall I \in \text{Mod}_L (\overline{T(\phi)} = T(\psi))$.

Thus, each element of the quotient algebra $L/\approx_L$ is an equivalence classes, denoted by $[\phi]$; the partial ordering $\subseteq$ is defined by

(b) $[\phi] \subseteq [\psi]$ iff $\phi \vdash \psi$ (i.e., if $\phi \subseteq \psi$).

In particular we will consider an equivalence class (set of all equivalent formulae w.r.t. $\approx_L$) $[\phi]$ that has exactly one constant $x \in A$, which is an element of this equivalence class (we abuse a denotation here by denoting by $x$ a formula (logic language constant), such that has a constant logic value $x \in A$ for every interpretation $I$, as well), and we can use it as the representation element for this equivalence class $[x]$. Thus,
every formula in this equivalence class has the same truth-value as this constant. Consequently, we have the injection \( i_A : A \rightarrow L \) between elements in \( (A, \sqsubseteq) \) and elements in the Lindenbaum algebra, such that for any logic value \( x \in A \), we obtain the equivalence class \( [x] = i_A(x) \in L \). It is easy to extend this injection into a monomorphism between the original algebra and this Lindenbaum algebra, by extension of corresponding connectives in this Lindenbaum algebra. For example: \( [x \land y] = i_A(x \land y) = i_A(x) \land i_A(y) = [x] \land [y], \left[\neg x\right] = i_A(\neg x) = \neg_L i_A(x) = \neg_L [x], \) etc.. In an autoreferential semantics we will assume that each equivalence class of formulae \([\phi]\) in this Lindenbaum algebra corresponds to one "state - description". In particular, we are interested to the subset of "state - descriptions" that are invariant w.r.t. many-valued worlds in the Kripke-style semantics for the original many-valued modal logic. But from the injection \( i_A \) we can take for such an invariant "state -description" \( [x] \in L \), only its inverse image \( x = i_A^{-1}([x]) \in A \). Consequently, the set of possible worlds in this autoreferential semantics corresponds to a particular subset of truth values in the complete lattice \( (A, \sqsubseteq) \): in this paper we will use the set of join irreducible elements (Birkhoff’s representation), as semantics based on prime filters, and one more possible world for the bottom algebraic truth value. Thus, it is from the economical point of view analogous to the semantics based on prime filters.

3 A new representation theorem

Based on the considerations in the previous paragraph, we intend to define an algebraic/relation dual in the way that we do not need to define a subset of designated elements \( D \) of a many-valued algebra. Let \( I : Var \rightarrow A \) be a given many-valued model of the logic \( L \), then we can use the algebraic model \((A, I)\), instead of \( \omega \)-matrices \((A, D)\). Let \( \Gamma \) be a sequent theory for this logic \( L \). The intuitive idea is to use the models \( Mod_L \) of the logic \( L \) (notice that \( I \in Mod_L \) is not any valuation for the propositional variables but is a model, and that the representation theorem is interesting only for logics that have at least one model, i.e., when \( Mod_L \) is not empty).

In what follows we will consider a poset \( A \) of truth values (with partial ordering \( \sqsubseteq \) such that at least for each \( x \in A \) we have that \( x \sqsubseteq x \)) of truth values (nullary operators of the algebra) for this many-valued logic, and \( \{o_i\}_{o_i \in \Omega} \) the set of functions \( o_i : A^n \rightarrow A \) (with arity \( n \geq 1 \)) assigned to operation names in \( \Omega \) of the logic \( L = (Var, O, \sqsubseteq) \). We assume that the carrier set of every algebra for a logic \( L \) contains also a set of propositional variables in \( Var \), so that the terms of an algebra \( A \) are the terms with variables in \( Var \).

Consequently, any pair \((A, I)\) can be seen as a ground term algebra obtained by assigning to \( Var \) the values in a model \( I \) of \( L \).

Thus, the satisfaction relation \( \models^a \) will be relative to a model \( I \) of the logic \( L \) instead of the prefixed set of elements in \( D \). For example, in the case of a logic program \( L \) we can use the Fitting’s 3-valued fixed point operator to obtain its well-founded 3-valued model. Here we will apply the truth-invariance entailment principle, the idea originally used to define the inference closure in the bilattice based logics \([21]\), and used recently to develop a new sequent system for many-valued logics presented in \([19]\) as well: in these two papers has been described a kind of transformation of the original many-valued logic into the ‘meta’ 2-valued logic. The set of models \( Mod_L \) of a given set \( \Gamma \) of formulae has to satisfy this truth-invariance principle \([19]\).

\[ \text{(MV) } \forall \phi \in \Gamma)(\exists x \in A)(\forall I \in Mod_L)(\overline{I}(\phi) = x), \]

that is, the value of each formula in \( \Gamma \) is invariant in \( Mod_L \).

In any case, in the representation theorem framework we are interested in establishing what is a canonical isomorphic algebra for a logic \( L \), and its relationship with Kripke relational structures. So, we can use models \( I \in Mod_L \) of a logic \( L \) only as mean to obtain these results. The algebraic satisfaction relation \( \models^a \) is defined as follows:

**Definition 6** Let \( L = (Var, O, \sqsubseteq) \) be a logic, \((A, I)\) be an algebraic logic model of a logic \( L \), defined by a mapping \( I : Var \rightarrow A \), and \( \phi \in F(L) \), and \( \overline{I} : F(L) \rightarrow A \) be its unique standard extension to all formulae in a language \( L \). Let \( \mathcal{M} \) be a class of algebraic models.

We define the relation \( \models^a \) as follows \((x \in A)\):

1. \((A, I); x \models^a \phi \iff \overline{I}(\phi) = x\),
2. \(\mathcal{M}; x \models^a \phi \iff (A, I); x \models^a \phi \) for every \((A, I) \in \mathcal{M}\).

We define the entailment relation of a logic \( L \) by: for every \( \phi \in F(L), x \in A \), \( L; x \models^a \phi \iff M; x \models^a \phi \).

Notice that in this definition, analogous to Definition \([1]\), we do not use the set of designated values \( D \), and we are able to determine which set of formulae is deduced for each algebraic logic value \( x \in A \). It is a generalization of classical deduction, where \( L \models^a \phi \) is equivalent to this new definition \( L; 1 \models^a \phi \), and \( L \models^a \neg \phi \) is equivalent to \( L; 0 \models^a \phi \) (i.e., \( L; 1 \models^a \neg \phi \)). The inference of \( \phi \) defined by Definition \([1]\) based on set \( D \) of designated values, can be expressed from this more accurate definition above by \( \bigvee_{x \in D} L; x \models^a \phi \). Thus, this new entailment relation \( \models^a \) given by Definition \([1]\) is more powerful and
more general than the entailment relation of L given by Definition \[1\]
Notice that if \( \Gamma \) is a sequent theory for L, then \( L; x \vdash \phi \) \( \iff \) \( \forall \in ModL(\overline{\Gamma}(\phi) = x) \), that is, in the case of the sequent system presented in Section 2, \( \Gamma \vdash (\phi \vdash x) \) and \( \Gamma \vdash (x \vdash \phi) \). Consequently, \( \vdash \) satisfies the truth-invariance principle (MV). Now we can introduce a new definition of the algebraic/relational duality, as follows:

**Definition 7** Let \( M \) be a class of all algebraic models for a given logic L. Assume that there exists a class \( \mathcal{K}_M \) of Kripke-models of a logic L, \( (K, I_K) \in \mathcal{K}_M \), with a Kripke-frame \( K = (I_K, \{R_i\}_{i \leq n}) \) where \( I_K \) is the set of possible worlds, \( \{R_i\}_{j \leq n} \) a finite set of accessibility relations between them (relational structure), with a mining mapping \( I_K : \text{Var} \rightarrow \mathcal{P}(1_K) \), such that there exists a mapping \( D : M \rightarrow \mathcal{K}_M \), with \( \mathcal{K}_M = \{D(M) \mid M = (A, I) \in M\} \), and there exists a mapping \( E : \mathcal{K}_M \rightarrow M \) such that:

(i) for every Kripke model \( M_K = (K, I_K) \in \mathcal{K}_M \) of L, the \( (A_K, I_K) = E((K, I_K)) \in M \) is an algebraic model of L, where \( A_K = (\mathcal{P}(1_K), \{0\}_0 \in \phi) \) is an algebra of subsets of the support \( 1_K \) of \( K \);

(ii) for every algebraic model \( M = (A, I) \in M \) of L, the \( (K, I_K) = D(M) \) is a Kripke model over a set \( 1_K \), so that, if \( E(D(M)) = (A_K, I_K) \) then there is a monotone injection mapping \( i_n : A \rightarrow A_K \) between truth values of algebras \( A = (A, \subseteq, \{0\}_0 \in \phi) \) and \( A_K = (\mathcal{P}(1_K), \subseteq, \{0\}_0 \in \phi) \), where \( A_K = \mathcal{P}(1_K) \), such that \( I_K = i_n \circ I \) and \( D_K = \{i_n(x) \mid x \in A\} \).

A representation is autoreferential when \( 1_K \subseteq A \).

**Example:**
Let us consider the two following autoreferential representations:

Case A: Let us consider the standard propositional logic L = \( \{\text{Var}, 0, \vdash\} \), where \( 0 = \{\land, \land\} \) and its simple Boolean algebra \( A = (A, \subseteq, \{0\}_0 \in \phi) \) with logic operators ‘and’, \( \land \), and logic negation \( \sim \) respectively, with \( 0 \subseteq 1 \). Let us take \( 1_K = A = \{0, 1\} = 2 \), so that the canonical extension of the Boolean algebra A is the power set algebra \( (\mathcal{P}(\{0, 1\}), \subseteq, \{0\}_0 \in \phi) \), with inclusion homomorphism \( i_n : (\mathcal{P}(\{0, 1\}), \subseteq, \{0\}_0 \in \phi) \rightarrow (\mathcal{P}(\{0, 1\}), \subseteq, \{0\}_0 \in \phi) \), which preserves ordering, such that for its bottom and top elements hold, \( 0_K = i_n(0) = \{0\}, 1_K = i_n(1) = \{0, 1\} \).

The negation algebraic operator \( \sim \) is defined by \( X = X \Rightarrow \{0\} \), where the operator (implication) \( \Rightarrow \) is defined by \( X \Rightarrow Y = \bigcup\{Z \in \mathcal{P}(\{0, 1\}) \mid Z \cap X \subseteq Y\} \), for any \( X, Y \in \mathcal{P}(\{0, 1\}) \).

Notice that \( \sim \) is not an involution in \( \mathcal{P}(\{0, 1\}) \), because \( \sim(\{1\}) = \{0, 1\} \neq \{0\} \). But it is an involution negation operator for the subalgebra of this canonical extension, \( (D_K, \{\land, \sim\}) \), where \( D_K = \{\{0\}, \{0, 1\}\} \subset \mathcal{P}(\{0, 1\}) \), isomorphic to algebra A and defined by the image of the inclusion \( i_n \).

Case B: Let us consider the 4-valued Belnap’s distributive bilattice \( A = \{f, \bot, \top, t\} \) with \( \bot \) for unknown and \( \top \) for inconsistent logic value, \( f = 0, t = 1 \) are bottom and top values w.r.t the truth ordering \( 0 \subseteq \bot, 0 \subseteq \top, \bot \subseteq 1, \top \subseteq 1 \) and \( \bot \approx \top \). It is the smallest many-valued logic capable of dealing with incomplete (unknown) and inconsistent logics. In this case we can take \( 1_K = \{0, \bot, \top, t\} \subset A \), with monotone injection \( i_n : (A, \subseteq, \{\land, \lor\}) \rightarrow (\mathcal{P}(1_K), \subseteq, \{\wedge, \lor\}) \) such that: \( 0_K = i_n(0) = \{0\}, i_n(\bot) = \{0, \bot\}, i_n(\top) = \{0, \top\}, 1_K = i_n(1) = \{0, \bot, \top\} \), i.e., \( D_K = \{\{0\}, \{0, \bot\}, \{0, \top\}, \{0, \bot, \top\}\} \).

In this new definition we replaced the old duality Algebra - Relational structures by the semantic duality Algebraic models - Kripke models of a logic L.

Notice that in the definition above we do not require the injection \( i_n \) to be an injective homomorphism, as in the assumption \( 2 \) but we require that the following diagram commutes (here \( id_{A_K} \) is the identity mapping for \( A_K \):

\[
\begin{array}{ccc}
V & \xrightarrow{I} & A \\
\downarrow & & \downarrow \rightarrow \\
\overline{A} & \xrightarrow{id_{A_K}} & A_K
\end{array}
\]

**Definition 8** Assume that \( M \) and \( \mathcal{K}_M \) satisfy the assumptions in \( 7 \). Let \( (K, m) \in \mathcal{K}_M \), \( 1_K \) be the support of K, with \( m : \text{Var} \rightarrow \mathcal{P}(1_K) \) and \( \overline{m} : F(L) \rightarrow \mathcal{P}(1_K) \) be the unique extension of m for all formulae in \( F(L) \). Let \( y \in 1_K \) and \( \phi \in F(L) \), then:

1. \( (K, m) \models_y \phi \iff y \in \overline{m}(\phi) \);
2. \( (K, m) \models \phi \iff \overline{m}(\phi) \in D_K \);
3. \( \mathcal{K}_M \models \phi \iff \forall(K, m_i), (K, m_j) \in \mathcal{K}_M(\overline{m_i}(\phi) = \overline{m_j}(\phi) \in D_K) \).

The following theorem is the basic result for the next representation theorem, and shows that from Definition \( 7 \) the new relational inference \( \models \) is sound and complete w.r.t. the algebraic inference \( \models^0 \).

**Theorem 2** Assume that \( M \) and \( \mathcal{K}_M \) satisfy the assumptions in \( 7 \). Then, for every \( \phi \in F(L) \), if \( M; x \models^0 \phi \) then \( \mathcal{K}_M \models \phi \) with \( i_n(x) = \overline{m}(\phi) \) for any \( (K, m) \in \mathcal{K}_M \). The converse also holds.
Proof: Assume that \(\mathcal{M} \) and \(\mathcal{R} \) satisfy the assumptions in [1] and \(\phi \in F(L) \) such that \((A, I); x \models_\mathcal{A} \phi\), i.e., \(x = T(\phi)\). Let \((K, I_K) \in K_M; \overline{m}: F(L) \rightarrow A_K \) be the unique extension of \(m: Var \rightarrow \mathcal{P}(1_K)\). By (C)(i) we have that \(\mathcal{E}(K, I_K) = (A_K, I_K) \in \mathcal{M}\), with \(A_K = \mathcal{P}(1_K)\). Then, from \(T_K = i_n \circ T\) it holds that \(T_K(\phi) = i_n(T(\phi)) = i_n(x)\) and \((A_K, I_K); i_n(x) \models_\mathcal{A} \phi\). That is, for any \(g: Var \rightarrow A_K\) (thus also for \(m\), \(\overline{m}(\phi) = T_K(\phi) = i_n(x) \in D_K\), in the way that \((K, m) \models_\mathcal{A} \phi\). It is valid for any \((K, I_K) \in K_M\), thus \(K_M \models_\mathcal{A} \phi\). Let \(\phi \in F(L)\), with \(K_M \models_\mathcal{A} \phi\). Then for any \(M = (A, I) \in \mathcal{M}\), we have \((K, I_K) = \mathcal{D}(M) \in K_M\). Since for any \((K, m) \in K_M\) we know that \((K, m) \models_\mathcal{A} \phi\), that is, for any \(m: Var \rightarrow A_K\) (thus for \(I_K\) also), \(\overline{m}(\phi) = i_n(x) \in D_K\) for some \(x \in A\), so that \(T_K(\phi) = i_n(x) \in D_K\), and from the fact that \(I_K = i_n \circ I\), we can take \(x = I(\phi)\). Thus \((A, I); x \models_\mathcal{A} \phi\).

Since it holds for any \(M = (A, I) \in \mathcal{M}\), we obtain \(\mathcal{M}; x \models_\mathcal{A} \phi\).

\[\square\]

Corollary 1 Let \(L = (Var, \mathcal{O}, \models)\) be sound and complete logic w.r.t. a class \(\mathcal{M}\) of algebraic models. Assume that there exists a class \(K_M\) such that the assumption in [2] holds. Then \(L\) is sound and complete w.r.t. the class \(K_M\), which can be regarded as a class of Kripke-style models.

From this corollary we are able to define a direct duality between algebraic and Kripke-style semantics for a logic \(L\):

\[\mathcal{M} \xrightarrow{\mathcal{D}} K_M \xrightarrow{\mathcal{E}} \mathcal{M}\]

Theorem 3 Representation Theorem: Assume that \(\mathcal{M} \) and \(\mathcal{R} \) satisfy the assumptions in [2]. Injective mapping \(i_n: (A, I) \hookrightarrow (A_K, I_K)\), where \((K, I_K) = \mathcal{D}(A, I)\). Thus, the dual representation of the algebra \(A\) is the subalgebra of \(A_K\) defined by image of the homomorphism \(i_n\).

Proof: It comes from the fact that \(I\) and \(I_K\) are the homomorphisms between \(O\)-algebras. So we can show it by structural induction on the formulae in \(F(L)\). For example, for a formula composed by conjunction, \(\phi \land \psi\), with \(x = T(\phi), y = T(\psi)\), we have that, \(i_n(x \land A y) = i_n(T(\phi) \land A T(\psi)) = i_n(T(\phi \land \psi)), \) from the homomorphic property of \(T\)

\[= (i_n \circ T)(\phi \land \psi) = T_K(\phi \land \psi), \] from the commutativity of (C)(ii)

\[= T_K(\phi) \land K T_K(\psi) = i_n(T(\phi)) \land_K i_n(T(\psi)) = i_n(x) \land_K i_n(y). \] Thus, we obtained the homomorphism holds for the restriction of \(i_n\) to the image of \(I\), but it is generally valid for any \(I\).

\[\square\]

Example 4: (The continuation of Example 3)

Let us consider now the algebraic models for \(L\), based on the Boolean algebra, \((A, I) \in \mathcal{M}\), where \(I: Var \rightarrow 2\) is the interpretation for propositional variables in \(Var\), and on its canonical extension \((A_K, I_K) = \mathcal{E}(\mathcal{D}(A, I))\)

\[\mathcal{A}_K = (\mathcal{P}(\{0, 1\}), \subseteq, \{\cap, \cup\}).\]

We have that for any \(p \in Var\), \(I(p) = 1\) iff \(I_K(p) = \{0, 1\}\) and \(I(p) = 0\) iff \(I_K(p) = \{\emptyset\}\).

We do not have any modal operator in these algebras, thus the frame \((K, I_K) = \mathcal{D}(A, I)\) has the set of only two accessibility worlds equal to \(1_K = 2 = \{0, 1\}\) and an empty accessibility relation, that is \(K = \{\{0, 1\}, \{\}\}\).

4 Autoreferential representation for complete distributive lattices

In Examples 3 and 4 we have shown the cases for this new definition of representation theorem, based on models of a logic \(L\), which define only relational structures \(K \in (K, I_K) = \mathcal{D}(A, I)\), with a set of possible worlds (support) equal to the set \(1_K \subseteq A\).

In the rest of this paper we will consider the subclass of complete lattices in which each lattice of truth values \((A, \subseteq, \wedge, \vee)\) is isomorphic to the complete sublattice of the powerset lattice \((\mathcal{P}(A), \subseteq, \cap, \cup)\). Consequently, we will consider the cases when there exists the subset \(S = C_L(\mathcal{P}(A)) \subseteq \mathcal{P}(A)\), closed under intersection \(\cap\) and union \(\cup\), with the isomorphism \(i_S: (A, \subseteq, \wedge, \vee) \sim (C_L(\mathcal{P}(A)), \subseteq, \cap, \cup)\), so that we obtain the inclusion map \(i_n = \subseteq \circ i_S: A \hookrightarrow A_K = \mathcal{P}(1_K)\) as required in Definition 4.

For such a subclass of complete lattices we will obtain that the carrier set \(A\), of the many-valued logic algebra \(A\), is the set of possible worlds for the Kripke frame for the dual relational representation of the algebraic semantics: this is an autoreferential assumption [20]. The relational semantic of other modal operators of the algebra \(A\) will be obtained successively by a correct definition of the accessibility relations of the Kripke frame.

It is well known that any complete lattice \(A\) has the following property: each (also infinite) subset \(X\) of \(A\) has the least upper bound (supremum) denoted by \(\bigvee X\) (when \(X\) has only two elements, the supremum corresponds to the join operator \(\vee\)), and the greatest lower bound (infimum) denoted by \(\bigwedge X\) (when \(X\) has only two elements the infimum corresponds to the meet operator \(\wedge\)). Thus, it has the bottom element \(0 = \bigwedge A \in A\), and the top element \(1 = \bigvee A \in A\). The cardinality of the set of hereditary subsets of \(A\) is gen-
eraly greater than the cardinality of the lattice $A$. But in what follows we will consider the class of complete distributive lattices $A$, for which we are able to define an isomorphism between the original lattice and the particular collection $A^+$ of hereditary subsets of $A$. Thus, in each distributive lattice we are able to define the implication and negation logical operators based on relative pseudocomplement and pseudocomplement relatively, i.e., $a \rightarrow b = \bigvee S$, $S = \{x \in A | x \wedge a \subseteq b\}$ and $\sim a = a \rightarrow 0$.

**Example 5:** Many-valued logics for approximate truth enriched by approximation of unknown and inconsistent information: The class of poset lattices can also be used for enabling standard fuzzy logic over the closed interval $[0,1]$ of reals, with whenever $x \leq y$ then $x \subseteq y$ where $\leq$ is the standard ordering of numbers, used for approximation of the truth value, with ability to consider incomplete (unknown) and mutually inconsistent information as well. For example, let us consider an enriched fuzzy logic with the set of truth values in $[0,.5 - \Delta] \cup [.5 + \Delta,1] \cup \{\perp,\top\}$ where $.5 - \Delta \subseteq \perp \subseteq 5 + \Delta$ and $.5 - \Delta \subseteq \top \subseteq 5 + \Delta$, for an sufficiently small value $\Delta < .5$. In the simplest case we can substitute $.5$ value with two unrelated values $0.5^- = \perp$ and $0.5^+ = \top$. This more expressive fuzzy logic we will denominate PO-fuzzy logic. This enrichment of the fuzzy logic is obtained by replacement of the closed subinterval $[x - \Delta, x + \Delta]$ of reals by the discrete Belnap’s bilattice $\{x - \Delta, \perp, \top, x + \Delta\}$ for an enough small $\Delta$. We are able also to repeat such an operation for a number of such replacements for different values for $x \in [0,1]$, with the family of unknown and inconsistent values such that if $x \leq y$ then $\perp \subseteq \perp$ and $\top \subseteq \top$, and $x + \Delta < y - \Delta$, in order to have not only the fuzzy approximation of truth values, but also the approximations of unknown and inconsistent values. Each such an enrichment is a distributive lattice.

Obviously, each finite or infinite many-valued logic with total ordering can be enriched by the family of values $\perp$ and $\top$, for the approximations of the unknown and inconsistent values, in order to be able to deal with any kind of incomplete and inconsistent information.

□

From the Birkhoff’s representation theorem for distributive lattices, every finite (thus complete) distributive lattice is isomorphic to the lattice of lower sets of the poset of join-irreducible elements. An element $x \neq 0$ in $A$ is a join-irreducible element if $x = a \lor b$ implies $x = a$ or $x = b$ for any $a,b \in A$. Lower set (down closed) is any subset $Y$ of a given poset $(A, \subseteq)$ such that, for all elements $x$ and $y$, if $x \subseteq y$ and $y \in Y$ then $x \in Y$.

**Proposition 2** 0-Lifted Birkhoff isomorphism: Let $A$ be a complete distributive lattice, then we define the following mapping $\downarrow^+: A \rightarrow \mathcal{P}(A)$: for any $x \in A$, $\downarrow^+ x = \downarrow x \cap \hat{A}$, where $\hat{A} = \{y \mid y \in A \text{ and } y \text{ is join-irreducible} \} \cup \{0\}$.

We define the set $A^+ = \{\downarrow^+ a \mid a \in A \} \subseteq \mathcal{P}(A)$, so that $\downarrow \lor \mathcal{V} = \mathcal{I}_A : A^+ \rightarrow A^+$ and $\downarrow \lor \mathcal{V} = \mathcal{I}_A : A \rightarrow A$.

Thus, the operator $\downarrow^+$ is inverse of the supremum operation $\lor : A^+ \rightarrow A$. The set $(A^+, \subseteq)$ is a complete lattice, such that there is the following 0-lifted Birkhoff isomorphism $\downarrow^+: (A, \subseteq, \lor, \top) \simeq (A^+, \subseteq, \lor, \top)$.

□

The name lifted here is used to denote the difference from the original Birkhoff’s isomorphism. That is, we have that for any $x \in A$, $0 \in \downarrow^+ x$, so that $\downarrow^+ x$ is never empty set (it is lifted by bottom element 0).

Notice that $(A^+, \subseteq, \lor, \top)$ is a subalgebra of the powerset algebra $(\mathcal{P}(A), \subseteq, \lor, \top)$.

**Example 6:** Belnap’s bilattice in the Example 5, is a distributive lattice w.r.t. the $\leq_i$ ordering, with two join-irreducible elements $\perp$ and $\top$, so that $\hat{B} = \{0, \perp, \top\}$. In this case we have that $\downarrow^+ 1 = \downarrow^+ (\downarrow \lor \mathcal{V}) = \downarrow^+ \perp \lor \top = \downarrow \perp \lor \top = (0, \perp, \top) = \hat{B} \neq \hat{B} = \downarrow 1 = B$.

□

It is easy to verify that $\downarrow^+ 0 = \{0\}$ is the bottom element in $A^+$.

**Remark:** For a many-valued logic with distributive complete lattice of truth values we have that $A_K = \mathcal{P}(1_K) \subseteq \mathcal{P}(A)$, with $1_K = \hat{A}$ and $D_K = A^+$, and the injective homomorphism $\downarrow^+: A \rightarrow \mathcal{P}(A)$ corresponds to the injective homomorphism $i_n : (A, I) \hookrightarrow (A_K, I_K)$ in the representation theorem. Thus, the dual representation of this algebra (in this case a distributive complete lattice) $A$ is the subalgebra $(A^+, \subseteq, \lor, \top)$ of $A_K$, defined by the image of the homomorphism $i_n = \downarrow^+$.

Based on these results we are able to extend the complete distributive lattices with other unary algebraic operators $\{o_i\}_{i \in N} : A \rightarrow A$ and binary operators $\{\odot_i\}_{i \in N} : A \times A \rightarrow A$ in order to obtain a class of algebras $((A, \subseteq, \lor, \top), \{o_i\}_{i \in N}, \{\odot_i\}_{i \in N})$, with the following set-based canonical representation:
Proposition 3 CANONICAL REPRESENTATION: Let $A = ((A, \sqcap, \wedge, \vee), \{o_i\}_{i \in N}, \{\tilde{o_i}\}_{i \in N})$ be a complete distributive lattice-based algebra.

We define its canonical representation by the algebra $A^+ = ((A^+, \leq, \sqcap, \bigvee, \{o_i^+\}_{i \in N}, \{\tilde{o_i}^+\}_{i \in N})$, such that,

$$o_i^+(\bigvee (\downarrow A)) = (\downarrow A, o_i \bigvee \downarrow A) \in A$$

and $\tilde{o_i}^+(x, y) = \tilde{o_i}^+(\downarrow x, \downarrow y)$. Thus, $\downarrow$ is an isomorphism $\downarrow^+: A \cong A^+$.

Example 7: Let us consider the binary implication operator $\rightarrow$ equal to the relative pseudocomplement $\lor$ over a complete distributive lattice. Then, we have that $\downarrow (x, y) = (\downarrow x, \downarrow y)$ and $\downarrow (x, y) = (\downarrow x, \downarrow y) = (\downarrow x, \downarrow y) = (\downarrow x, \downarrow y) = (\downarrow x, \downarrow y)$.

Example 8: The smallest nontrivial distributive bilattice is Belnap's 4-valued bilattice $B = \{t, f, \perp, \top\}$ where $t$ is true, $f$ is false, $\top$ is inconsistent (both true and false) or possible, and $\perp$ is unknown. As Belnap observed, these values can be given two natural orders: truth order, $\leq$, and knowledge order, $\leq_k$, such that $f \leq_k \top \leq t$, $f \leq_k \perp \leq t$, $\perp \leq k \top$, $\perp \leq k t$, $f \perp k t$. That is, bottom element 0 for $\leq_k$ ordering is $f$, and for $\leq_k$ ordering is $\perp$, and top element 1 for $\leq_k$ ordering is $t$, and for $\leq_k$ ordering is $\top$. Meet and join operators under $\leq_k$ are denoted $\land$ and $\lor$; they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under $\leq_k$ are denoted $\and$ and $\lor$, such that
hold: \( f \otimes t = \bot, f \oplus t = \top, \top \land \bot = f \) and \( \top \lor \bot = t \). There is a natural notion of the bilattice truth negation, denoted \( \neg \), (reverses the \( \leq_t \) ordering, while preserving the \( \leq_k \) ordering): switching \( f \) and \( t \), leaving \( \bot \) and \( \top \), and corresponding knowledge negation (confutation), denoted \( \neg \), (reverses the \( \leq_k \) ordering, while preserving the \( \leq_t \) ordering), switching \( \bot \) and \( \top \), leaving \( f \) and \( t \). These two kinds of negation commute: \( \neg \neg x = \neg x \) for every member \( x \) of a bilattice.

In what follows we will use the relative pseudocomplements, defined by \( x \rightarrow y = \bigvee \{ z \mid z \land x \leq_t y \} \), and pseudocomplements, defined by \( \neg_t x = \sim x = x \rightarrow f \) (and, analogously, for \( \leq_k \) ordering, \( x \rightarrow y \) and \( \neg_k x = x \rightarrow \bot \)).

The confutation is a monotone function that preserves all finite meets (and joins) w.r.t. the lattice \((B, \leq_t)\), thus it is the universal (and existential, because \( \bot = \neg \bot = \neg \bot \)) modal many-valued operator: "it is believed that" for a bilattice (as in ordinary 2-valued logic, the epistemic negation is composition of strong negation \( \neg \) and this belief operator, \( \neg = \neg \neg \)), which extends the 2-valued belief of the autoepistemic logic as follows:

1. if \( A \) is true than "it is believed that \( A \)" i.e., \( \neg A \), is true;
2. if \( A \) is false than "it is believed that \( A \)" is false;
3. if \( A \) is unknown than "it is believed that \( A \)" is inconsistent: it is really inconsistent to believe in something that is unknown;
4. if \( A \) is inconsistent (that is both true and false) than "it is believed that \( A \)" is unknown: really, we can not tell nothing about believing in something that is inconsistent.

**Remark:** Notice that the knowledge negation operator \( \neg_k \) is normal additive modal operator w.r.t. the \( \leq_t \) ordering. As we will see in the next definition, its dual is truth negation \( \neg \) which is a normal modal operator w.r.t. the \( \leq_k \) ordering. Thus, in the case of the believe (confutation) modal operator \( o_i = \neg \in \) Belnap’s bilattice, \( \hat{A} = \{ f, \bot, \top \} \), such that \( \neg f = f, \neg t = t, \neg \bot = \top, \neg \top = \bot \) (see more in the next section), we obtain that \( R\neg = \{(f, f), (f, \bot), (\top, \bot), (f, f), (\top, \top), (\bot, \bot)\} \), while for the autoepistemic Moore’s operator \( [13] \), \( o_i = \mu : B \rightarrow B \), defined by \( \mu(x) = t \) if \( x \in \{ \top, t \} \); \( f \) otherwise, we have that \( R\mu = \{(f, f), (f, \bot), (f, \top), (\bot, \bot), (\top, \bot), (\bot, \top)\} \).

Both of these modal operators are additive and normal. For the modal negation additive operator \( \neg \), we have that \( R\neg = \{(f, f), (f, \bot), (f, \top), (\bot, \bot), (\top, \bot), (\bot, \top)\} \).

Now we are able to define the relational Kripke-style semantics for a propositional modal logic \( L \), based on the modal Heyting algebras in Proposition 3.

**Definition 10** For a complete distributive lattice-based logics, the mapping \( \mathbb{D} : M \rightarrow \mathbb{K}_M \) is defined as follows: Let \((A, I) \in M \) be an algebraic model of \( L \), then \( M_k = (K, I_k) = \mathbb{D}(A, I) \) is the correspondent Kripke model, such that \( K = \langle \{1_k, \bot\}, (R_j)_{j \leq n} \rangle \) is a frame, where \( 1_k = \hat{A} = \hat{R}_j \) is an accessibility relation (given by Definition 7) for a modal operator \( o_j \), and \( I_k : \text{Var} \rightarrow \mathbb{P}(A) \) is a canonical valuation, such that for any atomic formula (propositional variable) \( p \in \text{Var} \), \( I_k(p) = \uparrow_p \) \((I(p)) \) \( D_k = A^\uparrow \). Then, for any world \( x \in k \), and formulae \( \psi, \phi \in F(L) \), \( M_k \models_x \phi \Leftrightarrow x \in I_k(p) \).

Notice that in the world \( x = 0 \) (bottom element in \( A \)) each formula \( \phi \in F(L) \) is satisfied: because of that we will denominate this world by inconsistent or trivial world. The semantics for the implication is the Kripke modal semantics for the implication of the intuitionistic logic (with only inverted ordering for the accessibility relation \( \square \)).

In any modal logic the set of worlds where a formula \( \phi \) is satisfied is denoted by \( \| \phi \| = \{ x \mid M_k \models_x \phi \} \), so that we have \( M_k \models_x \phi \Leftrightarrow x \in \| \phi \| \).

**Theorem 4** Soundness and Completeness: Let \((A, I) \in M \) be an algebraic model of \( L \) and \( (K, I_k) = \mathbb{D}(A, I) \) be the correspondent Kripke model, with a frame \( K = \langle \{1_k, \bot\}, (R_j)_{j \leq n} \rangle \), where \( 1_k = A \), and the canonical valuation \( I_k : \text{Var} \rightarrow \mathbb{P}(A) \) given by Definition 7. Then, for any propositional formula \( \phi \), the set of worlds where \( \phi \) holds is equal to \( \| \phi \| = I_k(\hat{\phi}) = \{ n \mid I_n(\hat{\phi}) \} \in D_k = A^\uparrow \), where the monotone injection \( i_n : A \rightarrow A_k \), \( A_k = \mathbb{P}(1_k) \), from Definition 7 satisfies \( i_n = \downarrow^+ \).

**Proof:** By structural induction:
1. For any proposition variable \( p \in \text{Var} \), \( x \in \hat{A} \), \( M_k \models_x p \Leftrightarrow x \in I_k(p) = i_n \circ I(p) = \downarrow^+ I(p) \), thus \( \| p \| = \downarrow^+ I(p) \).
2. From \( M_k \models_x \phi \land \psi \Leftrightarrow M_k \models_x \phi \land M_k \models_x \psi \), holds that \( \| \phi \land \psi \| = \| \phi \| \cap \| \psi \| = \downarrow^+ \hat{T}(\phi) \cap \downarrow^+ \hat{T}(\psi) \).
(by structural induction), $= \downarrow^+ T(\varphi \land \psi)$ (Prop. 2).
3. Similarly, $\| \varphi \lor \psi \| = \| \varphi \| \cup \| \psi \| = \downarrow^+ T(\varphi) \cup \downarrow^+ T(\psi) = \downarrow^+ T(\varphi \lor \psi)$.
4. Suppose that $\| \varphi \| = \downarrow^+ T(\varphi)$ and $\| \psi \| = \downarrow^+ T(\psi)$. Then for any $x \in A$ we have that $x \in \| \varphi \| \Rightarrow \psi$ iff $M_K \models x \varphi \Rightarrow \psi$ if $\forall y \in A \models (y \subseteq x \land M_K \models y \varphi)$ implies $M_K \models y \psi$ if $\forall y \in A \models (y \subseteq x \land \| \varphi \|)$ implies $y \in \| \psi \|$. So that $S = \| \varphi \| \Rightarrow \psi = \{ x \subseteq \downarrow^+ x \wedge \| \varphi \| \subseteq \| \psi \| \}$. Then, $S = id_{A^+} * S = \downarrow^+ \| S = \downarrow^+ S$ (from the homomorphism $\downarrow^+$).
5. For any additive algebraic modal operator $\alpha_i$ we obtain an existential logic modal operator $\diamond_i$, so that for any $x \in A$, $M_K \models x \diamond_i \varphi/g$ iff $\exists y \in A(x, y) \in R_i$ and $M_K \models y \varphi$ (g), i.e., $\exists y \in A(x, y) \in R_i$ and $y \in \downarrow^+ i \alpha_i$.
6. For any additive algebraic negation operator $\bar{o}_i$ we obtain a logic modal negation operator $\neg_i$, so that for any $x \in A$, $M_K \models x \neg_i \varphi$ iff $\forall y (M_K \models y \varphi$ implies $(x, y) \in R_i$), i.e., $\forall y \in \downarrow^+ i \alpha_i$ (x, y) \in R_i), where $\alpha_i = \bar{T}(\varphi/g)$.
7. From the definition of $\neg_i$ in Def [9], we let us show that $\| \neg_i \varphi \| \subseteq \downarrow^+ \neg_i \varphi$. Suppose that there exists $x \in A$ (i.e., join-irreducible) such that $x \notin \| \neg_i \varphi \|$ but $x \notin \downarrow^+ \neg_i \varphi$. Then we define $\beta = \bigvee \{ x \subseteq \downarrow^+ \neg_i \varphi \}$ (from Birkhoff Th. for distributive lattice each element is uniquely defined by the specific subset of join-irreducible elements). Thus (6.1) $\beta > \neg_i \varphi$.
8. Then for every $\gamma$ such that $\alpha_i \gamma > \beta$ (it always exists, at least for $0 = 0$, i.e., $\neg_i \varphi = 1$) we have that $\neg_i \gamma \in R_i$, $(x, \gamma) \in R_i$. In order to have that $x \in \| \neg_i \varphi \|$ implies $\forall y \in \downarrow^+ \alpha_i(x, y) \in R_i$ it must hold that $\downarrow^+ \alpha \subseteq \downarrow^+ \gamma$.
9. Then from $\neg_i \varphi = (\downarrow^+ \varphi \downarrow^+$ is an identity) $= \alpha_i(\varphi) \downarrow^+$ (from the additive property of the modal negation $\alpha_i = \bigwedge \{ \alpha_i y \mid y \in \downarrow^+ \alpha \} = \bigwedge \{ \alpha_i y \mid y \in \downarrow^+ \varphi \} = \alpha_i(\varphi) \downarrow^+$).
10. In contradiction with (6.1). Thus, we have that $\| \neg_i \varphi/g \| = \downarrow^+ \neg_i \varphi = \downarrow^+ \varphi(\neg_i \varphi/g) = \downarrow^+ \bigwedge \neg_i \varphi(\varphi/g)$.

This theorem demonstrates that the satisfaction relation in Definition [10] satisfies the general property for relational semantics given by point 1 of Definition [8] that is, it holds $(K, m) \models x \varphi$ iff $x \in \neg_i \varphi$. In fact, it holds from the fact that for $m = I_K$, $(K, m) \models x \varphi$ if $x \in \| \varphi \|$ and from this theorem we have that $\| \varphi \| = \neg_i \varphi$.

Notice that in the case when a lattice $A$ is a complete ordering where for any $x \in A, \downarrow^+ x = \downarrow x$ (for example in the fuzzy logic), then the minimum requirement for an unary modal operators $\alpha_i$ is to be monotonic.

We do not require it to be surjective, by defining the accessibility relation as $R_i = \{ (\alpha_i x) \mid x \in A \} \bigcup \{ (y, x) \mid x \in A \}$. In that case we have that $M_K \models x \alpha_i \varphi$ if $\exists y \in A((x, y) \in R_i$ and $M_K \models y \varphi$ (g) iff $\exists y \in A(x = o_i(y) \in A$ and $M_K \models y \varphi$). In fact, from the monotonicity of $\alpha_i$ $\exists y \in A(x = o_i(y) \in A(\alpha_i(x) = o_i(\alpha_i(x))) \iff dom$ for any $x \in \downarrow^+ \alpha_i(\alpha_i(\varphi)) = \downarrow^+ \bigwedge \neg_i \varphi(\neg_i(\varphi))$.
5 Application to Belnap’s bilattice

In this section we will apply the results obtained in the previous section to the 4-valued Belnap’s bilattice based logic $L$. Such a logic is a significant extension of normal strong Kleene’s 3-valued logic to the paraconsistent type of logics, where we are able to obtain a non-explosive inconsistency.

That is a very important class of logics which is able to deal also with mutually-inconsistent information, in typical Web data integration of different and independent source data with mutually inconsistent information [36]. That is the main reason that we applied a new representation theorem to this case instead of more complex bilattices.

Bilattice theory is a ramification of multi-valued logic by considering both truth $\leq_t$ and knowledge $\leq_k$ partial orderings. Given two truth values $x$ and $y$, if $x \leq_t y$ then $y$ is at least as true as $x$, i.e., $x \leq y$ if $x < y$ or $x = y$. The negation operation for these two orderings, $\neg$ and $\neg k$ respectively, are defined as the invocation operators which satisfy De Morgan law between the join and meet operations.

**Definition 11** (Ginsberg [33]) A bilattice $B$ is defined as a sixtuple $(B, \leq_t, \land, \lor, \bot, \top)$, such that: The $t$-lattice $(B, \leq_t, \land, \lor)$ and the $k$-lattice $(B, \leq_k, \land, \lor)$ are both complete lattices, and $\neg : B \rightarrow B$ is an involution ($\neg \neg$ is the identity) mapping such that $\neg$ is a lattice homomorphism from $(B, \land, \lor)$ to $(B, \land, \lor)$ and $(B, \land, \lor)$ to itself.

The following definition introduces the subclass of D-bilattices [32] (the Belnap’s bilattice is the smallest non trivial D-bilattice). For more information and a more compact definition of D-bilattices and their properties, as well as a number of significant examples, the reader can use [37].

**Definition 12** [34] A $D$-bilattice $B$ is a distributive bilattice $(B, \land, \lor, \bot, \top, \neg)$ with the isomorphism of truth-knowledge lattices $\partial : (B, \leq_t) \simeq (B, \leq_k)$, which is an involution. Let us define the unary operator $\neg =_{df} \partial \neg \partial : B \rightarrow B$. Then we say that a $D$-bilattice is **perfect** if two truth negations, the intuitionistic negation $\neg t$ (pseudocomplement), such that $\neg_t x = \sqrt[3]{\{z \mid z \land x = 0\}}$, and the bilattice negation $\neg$, are correlated by $\neg = \neg t \neg$.

In each $D$-bilattice $(B, \land, \lor, \bot, \top, \neg)$, the operator $\neg$ is selfadjoint modal operator w.r.t. the $\leq_t$, and the bilattice negation operator for $k$-lattice satisfy $-1_k = 0_k$, $-0_k = 1_k$, while $-1_t = 1_t$, $-0_t = 0_t$.

**Corollary 2** [37] For any $D$-bilattice $B$ the duality operator $\partial$ can be extended to the following isomorphism of modal Heyting algebras $\partial : (B, \leq_t, \alpha_t) \simeq (B, \leq_k, \alpha_k)$, with $\alpha_t = \{\land, \neg_t, \lor\}$, $\alpha_k = \{\land, \neg k, \lor\}$, where $\neg$ and $\neg k$ are the intuitionistic implications (the relative pseudocomplements) w.r.t. the $\leq_t$ and $\leq_k$ respectively.

Informally, these dual lattices are the modal extensions of Heyting algebras. The conjunct modal operators are the belief operators. As we will see, they correspond also to default negations in dual algebras.

The approach that we will use in order to find the representation theorem for a Belnap’s bilattice (defined in Example 8), based on the fact that it is a $D$-bilattice, is different than the standard one, based on the natural duality theorems [38]. (A natural duality for a quasi-variety gives us a uniform method to represent each algebra in the quasi-variety as the algebra of all continuous homomorphisms over some structured Boolean space), but close in spirit to the higher-order Herbrand model types [49].

A many-valued interpretation of a logic $L$ in an algebraic model $(A, I) = (B, \leq_k, \alpha_k)$, $I$ is of the form $I : Var \rightarrow B$, while for its extension $(A_K, I_K) = E(\mathbb{D}(A, I))$ the interpretation is of the higher-order type $I_K : Var \rightarrow A_K \subseteq P(B) \simeq 2^n$. That is, it maps each propositional variable in $Var$ to a logical value which is a function $f$ in $2^n$. Really, it maps to some subset $S$ of $B$, but such a set can be equivalently represented by its characteristic function $f \in 2^n$, such that $S = \{a \in B \mid f(a) = 1\}$. In what follows we will use both of these equivalent set-based and functional representations.

Both lattices $(B, \leq_t)$ and $(B, \leq_k)$ are distributive lattices, thus, from the Proposition 2 we obtain that

1. For the truth-ordered lattice $(B, \leq_t)$: $B_t^+ = \{1^+, a \in B \}$
   $$\{\{f\}, \{f, \bot\}, \{f, \top\}, \{f, \bot, \top\}\} \subseteq P((f, \bot, \top)),$$
   with bottom $0_t = \bot^+ f = \bot f = \{f\}$, and top element $1_t = 1^+ t = \bigcup_{x \in S_t = \{ot, \top\}} \downarrow x = \{f, \bot, \top\}$.

   That is, we have the isomorphism $i_t = 1^+: (B, \leq_t) \simeq (B_t^+, \subseteq) \subseteq (P(1_t), \subseteq)$, such that $i_t(f) = \{f\}, i_t(\bot) = \{f, \bot\}, i_t(\top) = \{f, \top\}$ and $i_t(t) = \{f, \bot, \top\}$, which satisfies the requirement (C)(ii) for inclusion $i_n \equiv i_t$.

2. For the knowledge-ordered lattice $(B, \leq_k)$: $B_k^+ = \{1^+, a \in B \}$
   $$\{\{1\}, \{\bot, f\}, \{\bot, t\}, \{\bot, f, t\}\} \subseteq P((\bot, f, t)),$$
   with bottom $0_k = \bot^+ \bot = \downarrow \bot = \{\bot\}$, and top element $1_k = 1^+ t = \bigcup_{x \in S_k = \{f, t\}} \downarrow x = \{\bot, f, t\}$.

   That is, we have the isomorphism $i_k = 1^+: (B, \leq_k)$
The accessibility relation is \( R_\rightarrow \) (Definition 9), and the composition \( \neg\neg \rightarrow \) of the selfadjoint (existential and universal) operator \( \neg \) (conflation) and pseudocomplement \( \neg \). It is represented as selfadjoint modal operator in dual (knowledge ordering) lattice instead. Thus, for the propositional intuitionistic autoepistemic 4-valued logic \( L = (\text{Var}, \{\land, \rightarrow, \neg\}) \), where \( \rightarrow \) is the intuitionistic implication and \( \neg \) the belief modal operator, we have:

**Theorem 5 (Representation Theorem for Belnap’s D-bilattice)**

Let \( \partial : (B, \leq_\ell, \alpha_l) \cong (B, \leq_k, \alpha_k) \) be a D-bilattice isomorphism for Belnap’s bilattice \( B \), with \( \alpha_l = \{\land, \neg, \rightarrow, \neg\} \) and \( \alpha_k = \{\land, \rightarrow, \neg\} \), and \( I : \text{Var} \to B \) be a many-valued interpretation of intuitionistic autoepistemic logic \( L = (\text{Var}, \{\land, \Rightarrow, \neg\}) \). Let the isomorphism \( \partial_P \) be the extension of the isomorphism \( \partial \) to sets, that is, for any set \( X \in \mathcal{P}(1_k) \), \( \partial_P X = \{\partial x \mid x \in X \} \in \mathcal{P}(1_k) \), while \( \partial_P \) be its reduction to \( B^+_k \) and \( B^-_k \) respectively. Then the following commutative diagram, where \( I' = \partial I \), \( I_k = \downarrow^+_k \partial I \), \( B^+_k = \partial_P(B^+_k) \), \( 1_k = \partial_P(1_k) \), for algebraic models of \( L \) holds:

\[
\begin{array}{ccc}
((B, \leq_\ell, \alpha_l), I) & \xymatrix{ & ((B, \leq_k, \alpha_k), I')} \\
\downarrow^+_{\ell} & \downarrow^+_{k} \\
((B^+_\ell, \{\land, \neg, \alpha_l\}), I_{\ell}) & \xymatrix{ & ((B^+_k, \{\land, \neg, \alpha_k\}), I_k)} \\
\downarrow_{\ell} & \downarrow_{k} \\
((\mathcal{P}(1_k), \{\land, \neg, \alpha_k\}), I_k) & \xymatrix{ & ((\mathcal{P}(1_k), \{\land, \neg, \alpha_k\}), I_k)} \\
\downarrow_{\ell} & \downarrow_{k} \\
\mathcal{E}(\mathcal{O}(\mathcal{D})(B, \leq_\ell, \alpha_l), I) & \xymatrix{ & \mathcal{E}(\mathcal{O}(\mathcal{D})(B, \leq_k, \alpha_k), I')} \\
\end{array}
\]

where \( \partial_P \) is injective homomorphisms, and \( \downarrow^+_\ell \) and \( \downarrow^+_k \) are the isomorphisms of \( \downarrow^+ \) w.r.t the truth and knowledge ordering respectively.

**Proof:** it is easy to verify, based on the precedent propositions and Theorem 4 for Kripke frames of these modal Heyting algebras we have that \( K_{\ell} = (1_{\ell}, \leq_{\ell}, R_{\rightarrow}, \neg) \), where the modal operator \( \alpha_{\ell} \) for the accessibility relation is \( R_{\rightarrow} = \{(x, y) \mid y \in 1_{\ell} \land x \in y \} \). Dually, for knowledge ordering we obtain the Kripke frame \( K_k = (1_k, \leq_k, R_{\rightarrow}, \neg) \), where for a modal operator \( \alpha_k \) the accessibility relation is \( R_{\neg} = \{(x, y) \mid y \in 1_k \land x \in y \} \). It is easy to verify that these two Kripke frames are dual, i.e., \( \partial_P : K_{\ell} \simeq K_k \).

Notice that we do not represent the bilattice negation \( \neg \) as an independent modal negation operator (in the truth-ordering lattice) with an incomparability relation (in Definition 9), because in Belnap’s bilattice (see Example 7) it is derived as the composition \( \neg\neg \rightarrow \neg \rightarrow \) of the selfadjoint (existential and universal) operator \( \neg \) (conflation) and pseudocomplement \( \neg \). It is represented as selfadjoint modal operator in dual (knowledge ordering) lattice instead. Thus, for the propositional intuitionistic autoepistemic 4-valued logic \( L = (\text{Var}, \{\land, \Rightarrow, \neg\}) \), where \( \Rightarrow \) is the intuitionistic implication and \( \neg \) the belief modal operator, we have:

**Theorem 5 (Representation Theorem for Belnap’s D-bilattice)**

Let \( \partial : (B, \leq_\ell, \alpha_l) \cong (B, \leq_k, \alpha_k) \) be a D-bilattice isomorphism for Belnap’s bilattice \( B \), with \( \alpha_l = \{\land, \neg, \rightarrow, \neg\} \) and \( \alpha_k = \{\land, \rightarrow, \neg\} \), and \( I : \text{Var} \to B \) be a many-valued interpretation of intuitionistic autoepistemic logic \( L = (\text{Var}, \{\land, \Rightarrow, \neg\}) \). Let the isomorphism \( \partial_P \) be the extension of the isomorphism \( \partial \) to sets, that is, for any set \( X \in \mathcal{P}(1_k) \), \( \partial_P X = \{\partial x \mid x \in X \} \in \mathcal{P}(1_k) \), while \( \partial_P \) be its reduction to \( B^+_k \) and \( B^-_k \) respectively. Then the following commutative diagram, where \( I' = \partial I \), \( I_k = \downarrow^+_k \partial I \), \( B^+_k = \partial_P(B^+_k) \), \( 1_k = \partial_P(1_k) \), for algebraic models of \( L \) holds:

\[
\begin{array}{ccc}
((B, \leq_\ell, \alpha_l), I) & \xymatrix{ & ((B, \leq_k, \alpha_k), I')} \\
\downarrow^+_{\ell} & \downarrow^+_{k} \\
((B^+_\ell, \{\land, \neg, \alpha_l\}), I_{\ell}) & \xymatrix{ & ((B^+_k, \{\land, \neg, \alpha_k\}), I_k)} \\
\downarrow_{\ell} & \downarrow_{k} \\
((\mathcal{P}(1_k), \{\land, \neg, \alpha_k\}), I_k) & \xymatrix{ & ((\mathcal{P}(1_k), \{\land, \neg, \alpha_k\}), I_k)} \\
\downarrow_{\ell} & \downarrow_{k} \\
\mathcal{E}(\mathcal{O}(\mathcal{D})(B, \leq_\ell, \alpha_l), I) & \xymatrix{ & \mathcal{E}(\mathcal{O}(\mathcal{D})(B, \leq_k, \alpha_k), I')} \\
\end{array}
\]

where \( \partial_P \) is injective homomorphisms, and \( \downarrow^+_\ell \) and \( \downarrow^+_k \) are the isomorphisms of \( \downarrow^+ \) w.r.t the truth and knowledge ordering respectively.

**Proof:** it is easy to verify, based on the precedent propositions and Theorem 4 for Kripke frames of these modal Heyting algebras we have that \( K_{\ell} = (1_{\ell}, \leq_{\ell}, R_{\rightarrow}, \neg) \), where the modal operator \( \alpha_{\ell} \) for the accessibility relation is \( R_{\rightarrow} = \{(x, y) \mid y \in 1_{\ell} \land x \in y \} \). Dually, for knowledge ordering we obtain the Kripke frame \( K_k = (1_k, \leq_k, R_{\rightarrow}, \neg) \), where for a modal operator \( \alpha_k \) the accessibility relation is \( R_{\neg} = \{(x, y) \mid y \in 1_k \land x \in y \} \). It is easy to verify that these two Kripke frames are dual, i.e., \( \partial_P : K_{\ell} \simeq K_k \).

Notice that we do not represent the bilattice negation \( \neg \) as an independent modal negation operator (in the truth-ordering lattice) with an incomparability relation (in Definition 9), because in Belnap’s bilattice (see Example 7) it is derived as the composition \( \neg\neg \rightarrow \neg \rightarrow \) of the selfadjoint (existential and universal) operator \( \neg \) (conflation) and pseudocomplement \( \neg \). It is represented as selfadjoint modal operator in dual (knowledge ordering) lattice instead. Thus, for the propositional intuitionistic autoepistemic 4-valued logic \( L = (\text{Var}, \{\land, \Rightarrow, \neg\}) \), where \( \Rightarrow \) is the intuitionistic implication and \( \neg \) the belief modal operator, we have:
inal D-bilattice, its set-based *isomorphic Representation*, and its powerset *extension*. Notice that all arrows (homomorphism between modal Heyting algebras) of the commutative diagram on the top are *isomorphisms*. The lower part of the commutative diagram represents the fact that the modal Heyting algebras of isomorphic representations are the *subalgebras* of the powerset extensions.

6 Conclusion

In this paper we defined a new framework for representation theorem, based on models of a given many-valued modal logic L with truth-invariance entailment, which is able to establish more close link between algebraic and Kripke-style models for such non-classical logics. The truth-invariance semantics of the entailment is different from the matrix-based entailment, and, consequently, this representation theorem is substantially different from all previous representation theorems with matrix-based semantics. For the particular subclass of *distributive* complete lattices we obtain the possibility to define the canonical powerset extension algebra, based on the subsets of its carrier set of logic values, and its unique subalgebra isomorphic to the original many-valued algebra with modal operators. The resulting Kripke frame of the correspondent Kripke-style models of L has as the set of possible values the join-irreducible subset (with 0 element also) of the carrier set of logic values of the many-valued algebra, in the way that we are able to represent the concrete Kripke models for a logic L. Unlike the standard method based on the natural duality theorem \[12\], where a class \( \mathcal{R} \) of relational structures would be the family of duals of algebras, difficult to describe in a simple logic language, our approach offers a very simple and compact autoreferential description. I believe that main results (representation theorem) can also be obtained by Priestley duality in a different manner. The second contribution of this paper is dedicated to the representation theorem for Belnap’s bilattice, which has recently been used for logic programs in Semantic Web applications \[38\] in order to deal with incomplete and partially inconsistent information.

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