JACOB'S LADDERS, CONJUGATE INTEGRALS, EXTERNAL MEAN-VALUES AND OTHER PROPERTIES OF A MULTIPLY π(T)-AUTOCORRELATION OF THE FUNCTION |ζ(1/2 + it)|^2

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Abstract. In this paper we obtain a new class of transformation formulae (without an explicit presence of a derivative) for the integrals containing products of factors |ζ(1/2 + it)|^2 with respect to two components of a disconnected set on the critical line.

1. Introduction

1.1. In the work of reference [3] (comp. also [1] and [2]) we have introduced the following disconnected set

\[ \Delta(n + 1) = \Delta(n + 1; T, U) = \bigcup_{k=0}^{n+1} [\phi_k^1(T), \phi_k^1(T + U)] \]

where

\[ y = \frac{1}{2} \varphi(t) = \varphi_1(t); \quad \varphi_0^1(t) = t, \quad \varphi_1^1(t) = \varphi_1(t), \]

\[ \phi_k^1(t) = \varphi_1[\varphi_1(t)], \ldots, \phi_k^1(t) = \varphi_1[\varphi_k^{-1}(t)], \quad t \in [T, T + U], \]

and \( \phi_k^1(t) \) stands for the \( k \)-th iteration of the Jacob's ladder

\[ \varphi_1(t), \quad t \geq T_0[\varphi_1]. \]

The set (1.1) has the following properties

\[ t \sim \phi_k^1(t), \quad \phi_k^1(T) \geq (1 - \epsilon)T, \quad k = 0, 1, \ldots, n + 1, \]

\[ \phi_k^1(T + U) - \phi_k^1(T) < \frac{T}{2n + 5 \ln T}, \quad k = 1, \ldots, n + 1, \]

\[ \phi_k^1(T) - \phi_k^{k+1}(T + U) > 0.18 \times \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n, \]

\[ U \in \left( 0, \frac{T}{\ln^2 T} \right), \]

and, in the macroscopic domain, i. e. for

\[ U \in \left[ T^{1/3 + \epsilon}, \frac{T}{\ln^2 T} \right], \]

we have a more detailed information about the set (1.1), namely

\[ ||[\phi_k^1(T), \phi_k^1(T + U)]|| = \varphi_k^1(T + U) - \varphi_k^1(T) \sim U, \quad k = 1, \ldots, n + 1, \]

\[ \varphi_k^1(T) - \varphi_k^{k+1}(T + U) \sim (1 - \epsilon) \frac{T}{\ln T}, \quad k = 0, 1, \ldots, n, \]

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where \( c \) is the Euler constant. We have that (see (1.3))
\[
\phi_{n+1}(T + U) - \phi_{1}(T + U) \prec \cdots \prec [\phi_{1}(T), \phi_{1}(T + U)] \prec [T, T + U],
\]
i. e. the segments are ordered from \([T, T + U]\) to the left.

**Remark 1.** The asymptotic behavior of the disconnected set (1.1) is as follows: if \( T \to \infty \) then the components of this set recede unboundedly each from other (see (1.3), (1.5)) and all together are receding to infinity. Hence, if \( T \to \infty \) then the set (1.1) behaves as an one-dimensional Friedman-Hubble expanding universe.

1.2. Next, we have shown (see [3]) that for the weighted mean-value of the integral
\[
\frac{1}{U} \int_{T}^{T+U} \prod_{k=0}^{n} \left| \frac{1}{2} + i \varphi_{1}^{k}(t) \right| dt, \quad U \in \left( 0, \frac{T}{\ln^{2} T} \right)
\]
the following factorization formula
\[
g_{n+1} = \frac{1}{U} \int_{T}^{T+U} \prod_{k=0}^{n} \left| \frac{1}{2} + i \varphi_{1}^{k}(t) \right| dt \sim \prod_{l=1}^{s} \frac{1}{U} \int_{T}^{T+U} a_{n-1} \prod_{k=0}^{a_{n-1}} \left| \frac{1}{2} + i \varphi_{1}^{k}(t) \right| dt, \quad T \to \infty
\]
holds true for every fixed natural number \( n \) and for every proper partition (the partition \( n + 1 = a_{1} + a_{2} + \cdots + a_{s}, \quad a_{l} \in [1, n], \quad l = 1, \ldots, s, \) and
\[
g_{l} = \frac{U}{\varphi_{1}^{a_{l}}(T + U) - \varphi_{1}^{a_{l}}(T), \quad l = 1, \ldots, s,}
\]
\[
g_{n+1} = \frac{U}{\varphi_{1}^{n+1}(T + U) - \varphi_{1}^{n+1}(T)}.
\]

1.3. Next, by [3], (6.5), \( n + 1 \to k \), we have
\[
t - \varphi_{1}^{k} \sim k(1 - c)\pi(t), \quad k = 0, 1, \ldots, n
\]
where \( \pi(t) \) is the prime-counting function. Hence
\[
\frac{1}{2} + i \varphi_{1}^{k}(t) = \frac{1}{2} + it - i[t - \varphi_{1}^{k}(t)] \sim \frac{1}{2} + it - ik(1 - c)\pi(t), \quad k = 0, 1, \ldots, n.
\]

**Remark 2.** By (1.8) the arguments in the product (1.6) performs some complicated oscillations around the sequence
\[
\frac{1}{2} + it - ik(1 - c)\pi(t), \quad k = 0, 1, \ldots, n
\]
of the lattice points. Based on this, the integral (1.6) represents the multiple (for \( k \geq 2 \)) \( \pi(t) \)-autocorrelation of the function \( \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \), i. e. we have certain type of the complicated nonlinear and nonlocal interaction of the function \( \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \) with itself.
1.4. After this we turn back to the formula (1.7). This formula binds the corresponding set of integrals over the same segment \([T, T + U]\). However, the segment \([T, T + U]\) is only one component of the disconnected set \(\Delta(n + 1)\) (see (1.1)). This is the reason for the following.

**Question.** Is there some formula that binds the integral (1.6) with the integral of the type
\[
\int_{\varphi_1^{p(n)}(T)}^{\varphi_1^{p(n)}(T + U)} \prod_{k=0}^{n} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(u) \right) \right|^2 du, \quad 1 \leq p(n) \leq n,
\]
i. e. with the integral over the component
\([\varphi_1^{p(n)}(T), \varphi_1^{p(n)}(T + U)] \neq [T, T + U]\).

2. The main formula and its structure

2.1. We obtain the following theorem in the direction of our Question

**Theorem.** For every disconnected set
\(\Delta(2l) = \Delta(2l; T, U) = \bigcup_{k=0}^{2l} [\varphi_1^k(T), \varphi_1^k(T + U)]\), \(l = 1, \ldots, L_0\)
where \(L_0 \in \mathbb{N}\) is an arbitrary fixed number, and for every
\[
U \in \left(0, \frac{T}{\ln^2 T}\right]
\]
the following asymptotic transformation formula
\[
\int_{\varphi_1(T)}^{\varphi_1(T + U)} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(u) \right) \right|^2 du_1 \sim \frac{\varphi_1^2(T + U) - \varphi_1^2(T)}{\varphi_1^2(T + U) - \varphi_1^2(T)} \int_{T}^{T + U} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt, \quad T \to \infty
\]
holds true.

**Remark 3.** We call the integrals that are bind by the formula (2.1) the conjugate integrals.

Let
\[
\frac{1}{2} + i\gamma, \quad \frac{1}{2} + i\gamma', \quad \gamma < \gamma'
\]
be consecutive zeros of the Riemann zeta-function lying on the critical line and \(l = 7, T = \gamma, U = \gamma' - \gamma\). Thus, for example, the following formula (see (2.1))
\[
\int_{\varphi_1^{14}(\gamma)}^{\varphi_1^{14}(\gamma)} \prod_{k=0}^{6} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(u) \right) \right|^2 du_7 \sim \frac{\varphi_1^{14}(T + U) - \varphi_1^{14}(T)}{\varphi_1^{14}(T + U) - \varphi_1^{14}(T)} \int_{\gamma}^{\gamma'} \prod_{k=0}^{6} \left| \zeta \left( \frac{1}{2} + i\varphi_1^k(t) \right) \right|^2 dt, \quad \gamma \to \infty
\]
holds true.

**Remark 4.** Nor the formula (2.2) for seven factors and \(U = \gamma' - \gamma\) is not accessible for the current methods in the theory of the Riemann zeta-function.
2.2. By the continuity of the function $\varphi_1(v)$ we have (see (2.1)) that if

$$u_l = \varphi_1(t), \ t \in [T, T + U]$$

then

$$\varphi_k(u_l) = \varphi_k[\varphi_1(t)] = \varphi^{k+1}(t) \in [\varphi_1^{k+1}(T), \varphi_1^{k+1}(T + U)].$$

Consequently, the product

$$\prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(u_l) \right) \right|^2$$

corresponds to the disconnected set

$$\bigcup_{k=l}^{2l-1} [\varphi^k(T), \varphi^k(T + U)] = \Delta(l, 2l - 1),$$

and similarly the product

$$\prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2$$

corresponds to the disconnected set

$$\bigcup_{k=0}^{l-1} [\varphi^k(T), \varphi^k(T + U)] = \Delta(0, l - 1),$$

where the sets (2.3), (2.4) are subsets of the set $\Delta(2l)$.

Next (comp. (1.3)), we have

(2.5) $\rho\{[\varphi^k(T), \varphi^k(T + U)]; [\varphi^{k+1}(T), \varphi^{k+1}(T + U)]\} > 0.17 \times \pi(T)$

where $\rho$ represents the distance of corresponding segments.

Remark 5. The formula (2.1) controls a quasi-chaotic behavior of the values of the function $|\zeta (1/2 + it)|^2$ with respect to the disconnected set $\Delta(2l)$ in spite of big distances separating the components of the set $\Delta(2l)$ (see (2.5)).

3. Some external mean-values

3.1. Using the mean-value theorem on the left-hand side of (2.1) we obtain

(3.1) $$\frac{1}{U} \int_T^{T+U} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(t) \right) \right|^2 \, dt \sim$$

$$\sim \frac{\{\varphi^1(T + U) - \varphi^1(T)\}^2}{\{\varphi^2(T + U) - \varphi^2(T)\}} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_1^k(\alpha_l) \right) \right|^2$$

where (see the paragraph 2.2)

$$\alpha_l \in (\varphi^1(T), \varphi^1(T + U)), \ \alpha_l = \varphi^l(t_l),$$

i. e.

$$\varphi^k(\alpha_l) = \varphi^{k+1}(t_l) \in (\varphi_1^{k+1}(T), \varphi_1^{k+1}(T + U)).$$

Hence, by (3.1) and (3.2) we have the following
Corollary 1. There are the values
\[ \tau_k = \tau_k(T, U, l) \in (\varphi_k^1(T), \varphi_k^1(T + U)), \quad k = l, \ldots, 2l - 1 \]
such that
\[
\frac{1}{U} \int_T^{T+U} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_k^1(t) \right) \right|^2 dt \sim \frac{\varphi_1^1(T + U) - \varphi_1^1(T)}{\varphi_1^1(T + U) - \varphi_1^1(T)} U \prod_{k=0}^{2l-1} \left| \zeta \left( \frac{1}{2} + i \tau_k \right) \right|^2
\]
where
\[
U \in \left( 0, \frac{T}{\ln^2 T} \right], \quad l = 1, \ldots, L_0, \quad T \to \infty.
\]

Remark 6. Since:
(a) the integral
\[
\int_T^{T+U} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_k^1(t) \right) \right|^2 dt
\]
corresponds to the disconnected set \( \Delta(0, l - 1) \), (see (2.4)),
(b) the product
\[
\prod_{k=l}^{2l-1} \left| \zeta \left( \frac{1}{2} + i \tau_k \right) \right|
\]
corresponds to the disconnected set \( \Delta(l, 2l - 1) \), (see (2.3)),
(c) the sets \( \Delta(0, l - 1) \) and \( \Delta(l, 2l - 1) \) are separated by the big distance
\[
\rho \{ \Delta(0, l - 1); \Delta(l, 2l - 1) \} > 0.17 \times \pi(T)
\]
(see (2.3), (2.4)),
it is quite natural to call the right-hand side of the equation (3.3) the external mean-value of the integral on the left-hand side.

3.2. Next, by the similar way, we obtain the following

Corollary 2. There are the values
\[ \tau_k = \tau_k(T, U, l) \in (\varphi_k^1(T), \varphi_k^1(T + U)), \quad k = 0, 1, \ldots, l - 1 \]
such that
\[
\frac{1}{U} \int_T^{T+U} \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \varphi_k^1(u) \right) \right|^2 du \sim \frac{\varphi_1^2(T + U) - \varphi_1^2(T)}{\varphi_1^2(T + U) - \varphi_1^2(T)} U \prod_{k=0}^{l-1} \left| \zeta \left( \frac{1}{2} + i \tau_k \right) \right|^2,
\]
where
\[
U \in \left( 0, \frac{T}{\ln^2 T} \right] \cup \left( 0, \frac{T}{\ln^2 T} \right), \quad l = 1, \ldots, L_0, \quad T \to \infty.
\]

Remark 7. The formula (3.4) gives us the second variant of the external mean-value theorem.
4. Other properties of the distribution of the values of \( |\zeta(\frac{1}{2} + it)| \) with respect to the disconnected set \( \Delta(2l) \)

4.1. Similarly to (3.3), (3.4), we obtain the following formula

\[
\prod_{k=0}^{l} \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \sim \frac{\varphi'_1(T + U) - \varphi'_1(T)}{\sqrt{\varphi'^2_1(T + U) - \varphi'^2_1(T)U}} \prod_{k=l}^{2l-1} \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right|,
\]

(4.1)

where \( \tau_k \in (\varphi'_k(T), \varphi'_k(T + U)), k = 0, 1, \ldots, 2l - 1 \).

Next, we obtain from (4.1) the following

**Corollary 3.**

\[
G^{(l)}_{0^{-1}} \left[ \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right] \sim \left\{ \frac{\varphi'_1(T + U) - \varphi'_1(T)}{\sqrt{\varphi'^2_1(T + U) - \varphi'^2_1(T)U}} \right\}^{1/l} G^{(2l-1)}_{l^{-1}} \left[ \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right], T \to \infty
\]

(4.2)

where the following symbols

\[
G^{(l)}_{0^{-1}} \left[ \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right] = \left\{ \prod_{k=0}^{l} \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right\}^{1/l},
\]

(4.3)

\[
G^{(2l-1)}_{l^{-1}} \left[ \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right] = \left\{ \prod_{k=1}^{2l-1} \left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right| \right\}^{1/l}
\]

stand for the geometric means.

4.2. Since (see (4.3))

\[
\frac{G^{(l)}_{0^{-1}}}{G^{(2l-1)}_{l^{-1}}} = G^{(l)}_{0^{-1}} \left[ \frac{\left| \zeta\left(\frac{1}{2} + i\tau_{k+1}\right) \right|}{\left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right|} \right],
\]

(4.4)

and we have for arithmetic and geometric means (for example)

\[
\bar{x}_A \geq \bar{x}_G; \bar{x}_A = \frac{1}{n} \sum_{i=1}^{n} x_i, \bar{x}_G = \sqrt{n} \prod_{i=1}^{n} x_i, x_i > 0.
\]

(4.5)

Then we obtain from (4.1)-(4.4) the formula

\[
\bar{G}^{(l)}_{0^{-1}} \left[ \frac{\left| \zeta\left(\frac{1}{2} + i\tau_k\right) \right|}{\left| \zeta\left(\frac{1}{2} + i\tau_{k+1}\right) \right|} \right] \sim \left\{ \frac{\varphi'_1(T + U) - \varphi'_1(T)}{\sqrt{\varphi'^2_1(T + U) - \varphi'^2_1(T)U}} \right\}^{1/l} = \Omega_l.
\]

Next, from the inequality

\[
\bar{G}^{(l)}_{0^{-1}} > (1 - c)\Omega_l, T \to \infty
\]
we obtain that (see (4.5))

\[
1 \frac{1}{l} \left\{ \sum_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right| \right\}_m > (1 - \epsilon) \Omega_l.
\]

The numbers \((\tau_0, \tau_1, \ldots, \tau_{l-1})\) may be ordered by \(l!\)-ways in the product

\[
\prod_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right|,
\]

and the same holds for the sequence of numbers \((\tau_l, \ldots, \tau_{2l-1})\). Therefore we have \((l!)^2\) inequalities of the type (4.6). In this sense we use the symbol

\[
\left\{ \sum_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right| \right\}_m, \quad m = 1, \ldots, (l!)^2.
\]

Hence, we obtain from (4.6) the following

**Corollary 4.** We have \((l!)^2\) inequalities

\[
1 \frac{1}{l} \left\{ \sum_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right| \right\}_m > (1 - \epsilon) \left\{ \frac{\varphi^1_l(T + U) - \varphi^1_l(T)}{\sqrt{\varphi^2_l(T + U) - \varphi^2_l(T)U}} \right\}^{1/l},
\]

for \(\tau_0, \tau_1, \ldots, \tau_{2l-1}\), where

\[
m = 1, \ldots, (l!)^2, \quad l = 1, \ldots, L_0, \quad U \in \left(0, \frac{T}{\ln^2 T} \right], \quad l = 1, \ldots, L_0, \quad T \to \infty.
\]

**Remark 8.** There are certain multiplicative effects also in the genetics, among the polygenic systems, and consequently the geometric means is used there, see, for example, [4], pp. 336, 337. We also note that we have used the formula for multiplication of independent variables as a motivation for our paper [3].

5. **Remarks about essential influence of the Riemann hypothesis on the sequence** \(\left\{ \varphi^k_l(T + U) - \varphi^k_l(T) \right\}_{k=1}^{L_0} \)

5.1. Let us remind that in the macroscopic case (1.4) we have the asymptotic formula (see (1.5))

\[
\varphi^k_l(T + U) - \varphi^k_l(T) \sim U, \quad k = 1, \ldots, L_0.
\]

In connection with (5.1) we ask the question: what is the influence of the Riemann hypothesis on measures of the segments

\[ [\varphi_1(T), \varphi_1(T + U)] \]

in the case (comp. (1.4))

\[
U \in (0, T^{1/3 - \epsilon_0}],
\]

for example, in the case \(\epsilon_0 = \frac{1}{12} \), i.e.

\[
U \in (0, T^{1/4}].
\]

First of all we have, on the Riemann hypothesis, that (see [3], p. 300)

\[
\zeta \left( \frac{1}{2} + it \right) = O \left( t^{1/4+\epsilon} \right), \quad t \to \infty,
\]

we obtain that (see (4.5))

\[
(4.6)
\]

\[
1 \frac{1}{l} \left\{ \sum_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right| \right\}_m > (1 - \epsilon) \Omega_l.
\]

\[
\prod_{k=0}^{l-1} \left| \frac{\zeta \left( \frac{1}{2} + i \tau_k \right)}{\zeta \left( \frac{1}{2} + i \tau_{k+1} \right)} \right|,
\]

and the same holds for the sequence of numbers \((\tau_l, \ldots, \tau_{2l-1})\). Therefore we have \((l!)^2\) inequalities of the type (4.6). In this sense we use the symbol

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Hence, we obtain from (4.6) the following

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\]

for \(\tau_0, \tau_1, \ldots, \tau_{2l-1}\), where

\[
m = 1, \ldots, (l!)^2, \quad l = 1, \ldots, L_0, \quad U \in \left(0, \frac{T}{\ln^2 T} \right], \quad l = 1, \ldots, L_0, \quad T \to \infty.
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\[
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In connection with (5.1) we ask the question: what is the influence of the Riemann hypothesis on measures of the segments

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\]

First of all we have, on the Riemann hypothesis, that (see [3], p. 300)

\[
(5.3)
\]

\[
\zeta \left( \frac{1}{2} + it \right) = O \left( t^{1/4+\epsilon} \right), \quad t \to \infty,
\]
(5.4) \[ \zeta \left( \frac{1}{2} + it \right) = O \left( T^{\frac{1}{2 + \epsilon}} \right), \quad t \in [(1 - \epsilon)T, T + U] \]

(comp. (1.3) and [3], (6.17)). Next we obtain for (5.2) from our formula (see [2], (2.5))

\[ \int_T^{T+V} \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt \sim [\varphi_1(T + V) - \varphi_1(T)] \ln T, \quad V \in \left( 0, \frac{T}{\ln T} \right), \]

by (5.4) that

\[ \varphi_1^1(T + U) - \varphi_1^1(T) = O \left( \frac{U}{\ln T} T^{\frac{1}{2 + \epsilon}} \right), \]
\[ \varphi_1^2(T + U) - \varphi_1^2(T) = O \left( \frac{U}{\ln^2 T} T^{\frac{1}{2 + 2\epsilon}} \right), \]
\[ \vdots \]
\[ \varphi_1^{L_0}(T + U) - \varphi_1^{L_0}(T) = O \left( \frac{U}{\ln^{L_0} T} T^{\frac{1}{L_0 + \epsilon}} \right). \]

Since

\[ T^{\frac{1}{L_0 + \epsilon}} = T^{\frac{1}{S \ln \ln T}} < T^{\frac{1}{\ln \ln T}}, \]

then by (5.4), (5.6) we obtain the following

Remark 9. On the Riemann hypothesis the following estimates hold true

\[ U \in (0, T^{1/3 - \epsilon}] \Rightarrow \varphi_k^1(T + U) - \varphi_k^1(T) = O \left( \frac{U}{\ln^{\frac{1}{S \ln \ln T}}} \right), \quad k = 1, \ldots, L_0. \]

For example, if \( U = 1 \) then on Riemann hypothesis we have that

\[ \varphi_k^1(T + 1) - \varphi_k^1(T) = O \left( \frac{U}{\ln^{\frac{1}{S \ln \ln T}}} \right), \quad k = 1, \ldots, L_0 \]

either for

\[ L_0 = S = 10^{10^{10^{34}} \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln \ln

Page 8 of 11
Remark 10. In the general case we are able to guarantee only that
\[ (5.8) \quad \varphi_1^1(T + 1) - \varphi_1^1(T) \in (0, T^{1/3 - \epsilon_0}], \quad \epsilon \leq \frac{\epsilon_0}{2}. \]

Hence, the comparison of (5.7), \( U = 1 \), with (5.8) shows the essential influence of the Riemann hypothesis on our subject.

6. The proof of Theorem

6.1. By using our formula (see [2], (9.1))
\[ \tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt} \]
we obtain (see [12])
\[ \int_T^{T+U} \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)]dt = \]
\[ = \int_T^{T+U} \tilde{Z}^2[\varphi_1^0(t)] \tilde{Z}^2[\varphi_1^{n-1}(t)] \cdots \tilde{Z}^2[\varphi_1^1(t)] \tilde{Z}^2[t]dt = \]
\[ = \int_T^{T+U} \tilde{Z}^2[\varphi_1^{n-1}(u_1)] \tilde{Z}^2[\varphi_1^{n-2}(u_1)] \cdots \tilde{Z}^2[\varphi_1^1(u_1)] \tilde{Z}^2[u_1] du_1 = \]
\[ = \int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)} \tilde{Z}^2[\varphi_1^{n-2}(u_1)] \cdots \tilde{Z}^2[\varphi_1^1(u_1)] \frac{d\varphi_1^1(u_1)}{du_1} du_1 = \]
\[ = \int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)} \tilde{Z}^2[\varphi_1^{n-1}(u_1)] \cdots \tilde{Z}^2[\varphi_1^0(u_1)] du_1, \quad l = 1, \ldots, n, \]
\[ \text{i.e. the following formula} \]
\[ (6.1) \quad \int_T^{T+U} \prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)]dt = \int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)} \prod_{k=0}^{n-l} \tilde{Z}^2[\varphi_1^k(u_1)] du_1, \quad l = 1, \ldots, n \]
holds true.

6.2. Let us remind that (see [3], (6.14))
\[ \tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi'[\varphi(t)]} = \frac{|\zeta \left( \frac{1}{2} + it \right)|^2}{\{1 + O\left( \frac{\ln \ln t}{\ln t} \right) \ln t \}}, \]
\[ t \in [T, T + U], \quad U \in \left( \frac{T}{\ln T} \right), \quad (\varphi_1^1(T), \varphi_1^1(T + U)) \subset (\varphi_1^{n+1}(T), T + U). \]
Putting (6.2) into (6.1) and using the mean-value theorem on both integrals in (6.1) we obtain the following formula (comp. [3], (6.17))

\[
\int_T^{T+U} \prod_{k=0}^n \left| \frac{1}{2} + i\varphi_k(t) \right|^2 dt \sim \ln^l T \int_{\varphi_1(T)}^{\varphi_1(T+U)} \prod_{k=0}^{n-1} \left| \frac{1}{2} + i\varphi_k(u) \right|^2 du, \quad l = 1, \ldots, n, \quad T \to \infty.
\]

(6.3)

Next, the formula (see [3], (3.1))

\[
\int_T^{T+U} \prod_{k=0}^n \left| \frac{1}{2} + i\varphi_k(t) \right|^2 dt \sim \{\varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)\} \ln^{n+1} T;
\]

\[
\ln^{n+1} T = \ln^{(l-1)+1} T \ln^{(n-l)+1} T
\]

together with the formula (6.3) gives the following asymptotic equality

\[
\frac{\int_T^{T+U} \varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)}{\int_T^{T+U} \varphi_1^{n+1-i}(T+U) / \varphi_1^{n+1-i}(T)} \sim \frac{\int_T^{T+U} \varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)}{\int_T^{T+U} \varphi_1^n(T) / \varphi_1^{n+1}(T)}
\]

i. e.

\[
\{\varphi_1^{n+1}(T+U) - \varphi_1^{n+1}(T)\} \int_T^{T+U} \prod_{k=0}^{n-1} \left| \frac{1}{2} + i\varphi_k(t) \right|^2 dt \sim \int_T^{T+U} \prod_{k=0}^{n-l} \left| \frac{1}{2} + i\varphi_k(u) \right|^2 du \times
\]

\[
\times \int_{\varphi_1^{n+1-i}(T)}^{\varphi_1^{n+1-i}(T+U)} \prod_{k=0}^{l-1} \left| \frac{1}{2} + i\varphi_k(v) \right|^2 dv, \quad l = 1, \ldots, n, \quad T \to \infty.
\]

(6.5)

6.3. Next, in the case

\[
n - l = l - 1 \Rightarrow n = 2l - 1,
\]

we obtain that (see (6.4), (6.5))

\[
\left\{ \int_{\varphi_1(T)}^{\varphi_1(T+U)} \prod_{k=0}^{l-1} \left| \frac{1}{2} + i\varphi_k(u) \right|^2 du \right\}^2 \sim \{\varphi_1^{2l}(T+U) - \varphi_1^{2l}(T)\} \int_T^{T+U} \prod_{k=0}^{2l-1} \left| \frac{1}{2} + i\varphi_k(t) \right|^2 dt \sim \{\varphi_1^{2l}(T+U) - \varphi_1^{2l}(T)\}^2 \ln^{2l} T,
\]

i. e. the following formula holds true

\[
\int_{\varphi_1(T)}^{\varphi_1(T+U)} \prod_{k=0}^{l-1} \left| \frac{1}{2} + i\varphi_k(u) \right|^2 du \sim \{\varphi_1^{2l}(T+U) - \varphi_1^{2l}(T)\} \ln^l T.
\]

(6.6)
Consequently, we obtain from (6.6) by (6.4), in the case \( n = l - 1 \), the formula

\[
\int_{\varphi_1(T)}^{\varphi_1(T+U)} \prod_{k=0}^{l-1} \left| \frac{1}{2} + i \varphi_k^1(u_t) \right|^2 \, du_t \sim \\
\frac{\varphi_1^2(T+U) - \varphi_1^2(T)}{\varphi_1^2(T+U) - \varphi_1^2(T)} \int_T^{T+U} \prod_{k=0}^{l-1} \left| \frac{1}{2} + i \varphi_k^1(t) \right|^2 \, dt
\]

that verifies (2.1).

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