Geometric Justification of the Fundamental Interaction Fields for the Classical Long-Range Forces

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Based on the principle of Reparametrization Invariance the general structure of physically relevant classical matter systems is illuminated within the Lagrangian framework. The canonical structure of a reparametrization invariant Lagrangian for an extended object (p-brane) embedded in a bulk space M is shown to lead to some familiar Lagrangians, such as the relativistic point particle in an electromagnetic field, the string theory Lagrangian, and the Dirac-Nambu-Goto Lagrangian. The outlined systems are based on first-order homogeneous Lagrangians in the velocity/(generalized velocity) to achieve reparametrization invariance along with the usual general covariance. In a straightforward way, the matter Lagrangian contains background interaction fields, such as a 1-form field analogous to the electromagnetic vector potential and symmetric tensor for gravity. The geometric justification of the interaction field Lagrangians for the electromagnetic and gravitational interactions are emphasized. The framework naturally suggests new classical interaction fields beyond electromagnetism and gravity. The simplest model with such fields is analyzed and its relevance to dark matter and dark energy phenomena on large/cosmological scales is inferred. Unusual pathological behavior in the Newtonian limit is suggested to be a precursor of quantum effects and of inflation-like processes at microscopic scales.

**Keywords:** diffeomorphism invariant systems, reparametrization-invariant matter systems, matter Lagrangian, homogeneous singular Lagrangians, relativistic particle, string theory, extended objects, p-branes, interaction fields, classical forces beyond electromagnetism and gravity, generally covariant theory, gauge symmetries, background free theories.

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I. INTRODUCTION

Probing and understanding physical reality goes through a classical interface that shapes our thoughts as classical causality chains. Therefore, understanding the essential mathematical constructions in classical mechanics and classical field theory is important, even though quantum mechanics and quantum field theory are regarded as more fundamental than their classical counterparts. Two approaches, the Hamiltonian and the Lagrangian, are very useful in theoretical physics. In general, there is a transformation that relates these two
approaches - the Legendre transform\(^1\). For reparametrization-invariant models, however, there are problems in changing from Lagrangian to the Hamiltonian approach\(^2,5\).\(^7,8\).

Fiber bundles provide the mathematical framework for classical mechanics, field theory, and even quantum mechanics when viewed as a classical field theory. Parallel transport, covariant differentiation, and gauge symmetry are very important structures\(^9\) associated with fiber bundles. When asking: “What structures are important to physics?”, one should also ask: “Why one fiber bundle should be more ‘physical’ than another?”, “Why does the ‘physical’ base manifold seem to be a four-dimensional Lorentzian manifold?”\(^11,14,15\), and “How should one construct an action integral for a given fiber bundle?”\(^1,6,16\)-\(^18\). Starting with the tangent or cotangent bundle seems natural because these bundles are related to the notion of a classical point-like matter. Since knowledge is accrued and tested via experiments that involve classical apparatus, the physically accessible fields should be generated by matter and should couple to matter as well. Therefore, the matter Lagrangian should contain the interaction fields, not their derivatives, with which classical matter interacts\(^19\).

In what follows, the principle of reparametrization invariance is illustrated as a guiding principle in formulating physically relevant models. The symmetry of reparametrization invariance is a common feature of many important physics models but it has been often treated as an issue that needs to be resolved to make reasonable predictions within each specific model. Since any model based on a Lagrangian can be reformulated into an equivalent reparametrization invariant model\(^2,21\), then this symmetry may be signaling an important fundamental principle. Thus, by focusing the discussion on models with Lagrangians that possess such reparametrization invariance does not restrict the generality of the models considered but instead provides an important classification of the possible physical systems and their interactions. In a nutshell, the principle of reparametrization invariance is like the covariance principle but about the internal coordinates of the physical process under study.

The covariance principle is effectively related to the diffeomorphism symmetry of a manifold \(M\). In the theory of manifolds, switching from one chart on \(M\) to another is physically equivalent to switching from one observer to another in the spacetime of the observer \(M\). Thus, coordinate independence of the physical laws and their mathematical forms when formulated in the manifold framework of \(M\). However, the framework does not say anything about a specific physical process \(E\) until one makes the relevant manifold model.
A physical process $E$ can be viewed as a manifold that consists of the points involved in the process and their relationships. Thus, processes and their studies can be viewed as the embedding of manifolds. *That is, how $E$ should be embedded in $M$?* For example, the motion of a point particle is just about the trajectory of a particle as viewed as the 1-dimensional curve in the 4D space-time of the physical observers. Since a process $E$ is also viewed as a manifold then there are mathematical charts that describe $E$ locally. The description of the process (embedding $E \hookrightarrow M$) should not depend on the choice made for the charts on $E$. *This is an additional symmetry to the covariance principle that things should not depend on the choice of the observer’s coordinates for $M$.*

Thus, the mathematical framework should also possess diffeomorphisms symmetry of the manifold $E$. That is, in the case of point particle, a 1D curve (the trajectory) mapped into another topologically equivalent 1D curve (but same trajectory) should not change our understanding and the description of the motion of a point particle. Physically, one talks about covariance principle when considering the diffeomorphisms symmetry of a manifold $M$ and about reparametrization invariance when considering the diffeomorphisms symmetry of the manifold $E$ that is embedded in $M$.

The current research suggests that reparametrization invariance can be achieved by using the Lagrangian formulation with Lagrangians that are homogeneous functions of order one with respect to the velocity. This leads to all the subsequent results that justify only electromagnetic and gravitational classical forces at the macroscopic scales which is consistent with experimental observations.

This paper is aiming at illustrating the possibility that physical reality and observed physical laws are related to a mathematical construction guided by the principle of reparametrization invariance for the embedding of manifolds. This principle suggests geometric justification of the fundamental interaction fields for the classical long-range forces - electromagnetism and gravity, as well as possible new classical fields. Are there observable consequences of such fields on microscopic and/or cosmological scales? Are such fields present in nature? Under what conditions the relevant Reparametrization Invariant Lagrangians with such fields could be reduced or not to the known Lagrangians that contain only gravitational and electromagnetic fields? These are only a few of the far-reaching questions related to the idea of reparametrization invariance and its correspondence to the observed physical laws. One day, hopefully, some of the readers of this paper will be
able to address these questions fully and answer them completely.

In brief, the paper starts with the relativistic particle\textsuperscript{7,9,16,20} aiming to illustrate the main ideas and their generalization to extended objects (\textit{p}-branes). In answering the question: “What is the Lagrangian for the classical matter?” the proposed\textbf{ canonical matter Lagrangian} naturally contains background interaction fields, such as a 1-form field analogous to the electromagnetic vector potential and symmetric tensor that is usually associated with gravity. The guiding principles needed for the construction of the Lagrangians for the interaction fields are also discussed as an illustration of the uniqueness of the Lagrangians for the electromagnetic and gravitational fields. The author considers this mathematical framework to be a geometric justification of the electromagnetism and gravity. The framework presented here seems to be able to go beyond the Feynman’s proof of the Maxwell and Lorentz equations that justify electromagnetism from a few simple fundamental principles\textsuperscript{22}.

In Section \textbf{II}, the Lagrangian for a relativistic particle is given as an example of a reparametrization-invariant action. Section \textbf{III} contains arguments in favor of first-order homogeneous Lagrangians; in sub-section \textbf{IIIA} are listed some of the good and not that good properties of such models; in sub-section \textbf{IIIB} the canonical form of the first-order homogeneous Lagrangians is justified; sub-section \textbf{IIIC} considers the generalization to \textit{E}-dimensional extended objects (\textit{p}-branes). Section \textbf{IV} discusses the physical implication of such Lagrangians, in particular, in sub-section \textbf{IVA} the possibility of classical forces beyond electromagnetism and gravity is studied for the simplest possible Lagrangian System with symmetric fields $S_n$ with $n > 2$, while in sub-section \textbf{IVB} the notion of proper time is shown to be mostly related to the gravitational term ($n = 2$) of the matter Lagrangian. Section \textbf{V} justifies the field Lagrangians relevant for the interaction fields, in particular, the uniqueness of the Lagrangian for electromagnetism, as well as the uniqueness of the Hilbert-Einstein action integral for gravity. The conclusions and discussions are given in Section \textbf{VI} followed by short Section \textbf{VII} containing relevant theorems framed as problems and exercises.

\section*{II. THE RELATIVISTIC PARTICLE LAGRANGIAN}

It is well known that localized particles move with a finite 3D speed. In an extended configuration space (4D space-time), when the time is added as a coordinate ($x^0 = ct$), particles move with a constant 4-velocity ($v \cdot v = constant$). The 4-velocity is constant due to
the definition \( v^\mu = dx^\mu /d\tau \) that uses the invariance of the proper-time \((\tau)\) mathematically defined via a symmetric tensor \( g_{\mu\nu} \) \((d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu)\). Physically the proper-time \((\tau)\) is associated with the passing of time measured by a co-moving clock that is at rest with respect to the particle during its motion. In this case, the action integral for a massive relativistic particle has a nice geometrical meaning: it is the time-elapsed along the particle trajectory:

\[
S_1 = \int d\tau L_1(x,v) = \int d\tau \sqrt{g_{\mu\nu} v^\mu v^\nu},
\]

\[
\sqrt{g_{\mu\nu} v^\mu v^\nu} \to 1 \Rightarrow S_1 = \int d\tau.
\]

However, for massless particles, such as photons, the length of the 4-velocity is zero \((g_{\mu\nu} v^\mu v^\nu = 0)\). Thus, one has to use different Lagrangian to avoid problems due to division by zero when evaluating the Euler-Lagrange equations. In this case, the appropriate “good” action is:

\[
S_2 = \int L_2(x,v) d\tau = \int g_{\mu\nu} v^\mu v^\nu d\tau.
\]

Notice that the Euler-Lagrange equations obtained from \(S_1\) and \(S_2\) are equivalent, and both are equivalent to the geodesic equation as well:

\[
\frac{d}{d\tau} \vec{v} = D_\beta \vec{v} = v^\beta \nabla_\beta \vec{v} = 0,
\]

\[
v^\beta \left( \frac{\partial v^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} v^\gamma \right) = 0.
\]

In General Relativity (GR) the Levi-Civita connection \( \nabla_\beta \), with Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} = g^{\alpha\rho} \left( g_{\rho\beta,\gamma} + g_{\rho\gamma,\beta} - g_{\beta\gamma,\rho} \right) /2 \), preserves the length of the vectors \((\nabla g(\vec{v},\vec{v}) = 0)\). Therefore, these equivalences are not surprising because the Lagrangians in (1) and (2) are functions of the preserved arc length \(g(\vec{v},\vec{v}) = \vec{v}^2\). In principle, however, the parallel transport for an arbitrary connection \( \nabla_\beta \) does not have to preserve the length of a general vector.

Remarkably, however, going beyond length preserving parallel transport may still hold such equivalence. For example, the Weyl\'s integrable geometry does have such equivalence between what one expects to be the generalized geodesic equation and the equation derived from an appropriate Lagrangian. Weyl\’s Integrable geometry provides a framework that is likely to be relevant to physics. It is based on the original Weyl\’s gauge symmetry idea where the length of a vector may depend on the gauge choice as well as upon infinitesimal
local displacements. In the Weyl’s integrable geometry, however, this freedom is constrained
to constructions where the length of a vector does not change upon a transport along a
closed loop. In such geometry, one finds that only an action that is build upon a co-scalar
of order (-1) results in trajectory restricting equations of motion that do correspond to the
generalized geodesic equation\textsuperscript{12} while any other choices built upon co-scalar length \( l \) of order
\( n \neq -1 \) results in the statement that \( dl \) is a closed one-form, that is, a perfect deferential
\( (d(dl) = 0) \). In the Weyl’s geometry terminology a scaler, vector, and a general tensor object
\( Y_{\mu\ldots\nu} \) is a co-tensor of order \( n \) when \( \tilde{Y}_{\mu\ldots\nu} = \beta^n Y_{\mu\ldots\nu} \) upon the gauge change of the metric
tensor \( \tilde{g}_{\mu\nu} = \beta^2 g_{\mu\nu} \). Thus, the line element \( d\tau \) defined as usual to be \( d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \) is a
co-scalar of order (+1), then the co-tangent vector with components \( v^\mu = dx^\mu / d\tau \) is seen as
a co-vector of order (-1). The mathematical framework developed in ref.\textsuperscript{12} practically shows
that the only reasonable choice of action for a massive particle, within the Weyl’s integrable
geometry framework, is given by the action integral (I).

The equivalence between \( S_1 \) and \( S_2 \) is very robust. Since \( L_2 \) is a homogeneous function
of order 2 with respect to \( \vec{v} \), the corresponding Hamiltonian function \( (h = v \partial L / \partial v - L) \) is
exactly equal to its Lagrangian \( (h(x, v) = L_2(x, v)) \). As long as there is no explicit proper-
time dependence then \( L_2 \) is conserved, and so is the length of \( \vec{v} \). Any parameter independent
homogeneous Lagrangian in \( \vec{v} \) \( (L_n(x, \beta v) = \beta^n L_n(x, v)) \) of order \( n \neq 1 \) is conserved because
\( h = (n - 1) L_n \). When \( dL / d\tau = 0 \), then one can show that the Euler-Lagrange equations for
\( L \) and \( \tilde{L} = f (L) \) are equivalent under certain minor restrictions on \( f \). This is an interesting
type of equivalence that applies to homogeneous Lagrangians. It is different from the usual
equivalence \( L \rightarrow \tilde{L} = L + d\Lambda / d\tau \) or the more general equivalence discussed in ref.\textsuperscript{24}. Any
solution of the Euler-Lagrange equation for \( \tilde{L} = L^\alpha, \alpha \neq 1 \) would conserve \( L = L_1 \) since
\( \tilde{h} = (\alpha - 1) L^\alpha \) is conserved. All these solutions are solutions of the Euler-Lagrange equation
for \( L \) as well; thus \( L^\alpha \subset L \) in the sense of their set of solutions. In general, conservation of
\( L_1 \) is not guaranteed since \( L_1 \rightarrow L_1 + d\Lambda / d\tau \) is also a first-order homogeneous Lagrangian
in the velocities that is equivalent to \( L_1 \). This suggests that there could be a choice of \( \Lambda \), a
“gauge fixing”, such that \( L_1 + d\Lambda / d\tau \) is conserved even if \( L_1 \) was not.
III. HOMOGENEOUS LAGRANGIANS

Suppose one doesn’t know classical physics, which is mainly concerned with trajectories of point particles in some space \( M \) but is told that can derive it from a variational principle if the right action integral \( S = \int L d\tau \) is used. By following the above example, one would wonder: “should the smallest ‘time distance’ be the guiding principle?” when constructing \( L \). If yes, “How should it be defined for other field theory models?” It seems that a reparametrization-invariant theory can provide us with a metric-like structure\(^7\), and thus, a possible link between field models and geometric models\(^25\).

In the example of the relativistic particle (Sec. III above), the Lagrangian and the trajectory parameterization have a geometrical meaning. In general, however, parameterization of a trajectory is quite arbitrary for any observer. If there is the smallest time interval that sets space-time scale, then this would imply a discrete space-time structure since there may not be any events in the smallest time interval. The Planck scale is often considered to be such a special scale\(^26\). Leaving aside recent hints for quantum space-time from loop quantum gravity and other theories, one should ask: “Should there be any preferred trajectory parameterization in a smooth 4D space-time?” and “Aren’t we free to choose the standard of distance (time, using natural units \( c = 1 \))?" If so, then one should have a smooth continuous manifold and our theory should not depend on the choice of parameterization.

If one examines the Euler-Lagrange equations carefully:

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial v^\alpha} \right) = \frac{\partial L}{\partial x^\alpha}, \tag{4}
\]

one would notice that any homogeneous Lagrangian of order \( n \) \( (L(x, \alpha \vec{v}) = \alpha^n L(x, \vec{v}) \) provides a reparametrization invariance of the equations under the transformations \( \tau \to \tau/\alpha, \vec{v} \to \alpha \vec{v} \). As a side remark, notice that for homogeneous Lagrangian in \( x \), the Euler-Lagrange equations possess scale invariance \( x \to \alpha x \). In general, such symmetry is related to the freedom of choosing a system of units by the laboratory observer. However, this symmetry is often broken due to the natural scales relevant to the specific process under study. Next, note that the action integral \( S \) involves an integration that is a natural structure for orientable manifolds \( (M) \) with an \( n \)-form of the volume. Since a trajectory is a one-dimensional object, then what one is looking at is an embedding

\[
\phi : \mathbb{R}^1 \to M \tag{5}
\]
This means that the map $\phi$ pushes forward the tangential space $\phi_* : T(\mathbb{R}^1) = \mathbb{R}^1 \to T(M)$, and pulls back the cotangent space $\phi^* : T^*(\mathbb{R}^1) = \mathbb{R}^1 \leftarrow T^*(M)$. Thus, a 1-form $\omega$ on $M$ that is in $T^*(M)$ ($\omega = A_{\mu}(x) \, dx^\mu$) will be pulled back on $\mathbb{R}^1$ ($\phi^*(\omega)$) and there it should be proportional to the volume form on $\mathbb{R}^1$ ($\phi^*(\omega) = A_{\mu}(x) (dx^\mu/d\tau) d\tau \sim d\tau$), allowing one to integrate $\int \phi^*(\omega)$:

$$\int \phi^*(\omega) = \int L d\tau = \int A_{\mu}(x) \, v^\mu d\tau.$$

Therefore, by selecting a 1-form $\omega = A_{\mu}(x) \, dx^\mu$ on $M$ and using $L = A_{\mu}(x) \, v^\mu$ one is actually solving for the embedding $\phi : \mathbb{R}^1 \to M$ using a chart on $M$ with coordinates $x : M \to \mathbb{R}^n$.

The Lagrangian obtained this way is first-order homogeneous in the velocity $v$ with very simple dynamics. The corresponding Euler-Lagrange equation is $F_{\nu\mu} v^\mu = 0$ where $F$ is a 2-form ($F = dA$); in electrodynamics, this is the Faraday’s tensor. If one relaxes the assumption that $L$ is a pulled back 1-form and assume that it is just a homogeneous Lagrangian of order one, then one finds a reparametrization-invariant theory that has an important physics-related dynamics.

### A. Pros and Cons of Homogeneous Lagrangians of First Order

Although most of the features listed below are more or less self-evident, it is important to compile a list of properties of the first-order homogeneous Lagrangians in the velocity $\vec{v}$.

Some of the good properties of a theory with a first-order homogeneous Lagrangian are:

1. First of all, the action $S = \int L(x, \frac{dx^i}{d\tau}) d\tau$ is a reparametrization invariant.

2. For any Lagrangian $L(t, x^i, \frac{dx^i}{dt})$ one can construct a reparametrization-invariant Lagrangian by enlarging the space from $x^i : i = 1, ..., n$ to an extended space-time $x^\mu : \mu = 0, 1, ..., n, x^0 = t^2$. $L(t, x^i, \frac{dx^i}{dt}) \to L(x^\mu, \frac{dx^\mu}{dt}, \frac{dx^0}{d\tau})$. The Euler-Lagrange equations for these two Lagrangians are equivalent as long as $v^0 = dt/d\tau$ is well behaved and it is also a reasonable “time”-parametrization choice.

3. Parameterization-independent path-integral quantization is possible since the action $S$ is reparametrization invariant.

4. The reparametrization invariance may help in dealing with singularities$^{27}$. 

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(5) It is easily generalized to extended objects ($p$-branes) that is the subject of Section III C.

The list of trouble-making properties in a theory with a first-order homogeneous Lagrangian includes:

1. There are constraints among the Euler-Lagrange equations since $\det \left( \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta} \right) = 0$.

2. It follows that the Legendre transformation ($T(M) \leftrightarrow T^*(M)$), which exchanges velocity and momentum coordinates $(x, v) \leftrightarrow (x, p)$, is problematic.

3. There is a problem with the canonical quantization approach since the Hamiltonian function is identically ZERO ($h \equiv 0$).

Constraints among the equations of motion are not an insurmountable problem since there are procedures for quantizing such theories. For example, instead of using $h \equiv 0$ one can use some of the constraint equations available, or a conserved quantity, as Hamiltonian for the quantization procedure. Changing coordinates $(x, v) \leftrightarrow (x, p)$ seems to be difficult but it may be resolved in some special cases by using the assumption that a gauge $\Lambda$ has been chosen so that $L \rightarrow L + \frac{d\Lambda}{dt} = \tilde{L} = \text{const}$. The above-mentioned quantization difficulties would not be discussed since they are outside of the scope of this paper. A new approach that turns the problem $h \equiv 0$ into a virtue and naturally leads to a Dirac-like equation is under investigation and subject of a forthcoming paper, for some preliminary details see ref. 32.

B. Canonical Form of the First-Order Homogeneous Lagrangians

Hopefully, by now the reader is puzzled, and is wondering along the following line of thinking: “What is the general mathematical expression for first-order homogeneous functions?” In this section, the notion of the canonical form of the first-order homogeneous Lagrangian and why such form may be a useful mathematical expression from a physics point of view is justified.

First, note that any symmetric tensor of rank $n$ ($S_{\alpha_1\alpha_2...\alpha_n} = S[\alpha_1\alpha_2...\alpha_n]$, where $[\alpha_1\alpha_2...\alpha_n]$ is an arbitrary permutation of the indexes) defines a homogeneous function of order $n$.
\( (S_n(\vec{v}, \ldots, \vec{v}) = S_{\alpha_1\alpha_2 \ldots \alpha_n} v^{\alpha_1} \ldots v^{\alpha_n} ) \) in the velocity \( v \). The symmetric tensor of rank two is denoted by \( g_{\alpha\beta} \). Using this notation, the **canonical form of the first-order homogeneous Lagrangian** is defined as:

\[
L(\vec{x}, \vec{v}) = \sum_{n=1}^{\infty} \sqrt{S_n(\vec{v}, \ldots, \vec{v})} = A_\alpha v^\alpha + \sqrt{g_{\alpha\beta} v^\alpha v^\beta} + \ldots + \sqrt{S_m(\vec{v}, \ldots, \vec{v})}.
\]

Whatever is the Lagrangian for the matter, it should involve interaction fields that couple with the velocity \( \vec{v} \) to a scalar. Thus, the matter Lagrangian \( L_{\text{matter}}(\vec{x}, \vec{v}; \text{Fields } \Psi) \) would depend also on the interaction fields. When the matter action is combined with the action \( (\int L[\Psi]dV) \) for the interaction fields \( \Psi \), then one obtains a full **background independent theory**. Then, the corresponding Euler-Lagrange equations contain “dynamical derivatives” on the left-hand side and sources on the right-hand side:

\[
\partial_\gamma \left( \frac{\delta L}{\delta (\partial_\gamma \Psi)} \right) = \frac{\delta L}{\delta \Psi} + \frac{\partial L_{\text{matter}}}{\partial \Psi}.
\]

The advantage of the canonical form of the first-order homogeneous Lagrangian (6) is that each interaction field, which is associated with a symmetric tensor, has a **unique matter source** that is a monomial in the velocities:

\[
\frac{\partial L}{\partial S_{\alpha_1\alpha_2 \ldots \alpha_n}} = \frac{1}{n} (S_n(\vec{v}, \ldots, \vec{v})^{-1}) v^{\alpha_1} \ldots v^{\alpha_n}.
\]

There are many other ways to write first-order homogeneous functions. For example, one can consider the following expression \( L(\vec{x}, \vec{v}) = (h_{\alpha\beta} v^\alpha v^\beta) (g_{\alpha\beta} v^\alpha v^\beta)^{-1/2} \) where \( h \) and \( g \) are seemingly different symmetric tensors. However, each one of these fields \( (h \) and \( g) \) has the same source type \( (\sim v^\alpha v^\beta) \):

\[
\frac{\partial L}{\partial h_{\alpha\beta}} = \frac{L(\vec{x}, \vec{v})}{h_{\gamma\rho} v^\gamma v^\rho} v^\alpha v^\beta, \quad \frac{\partial L}{\partial g_{\alpha\beta}} = \frac{L(\vec{x}, \vec{v})}{g_{\gamma\rho} v^\gamma v^\rho} v^\alpha v^\beta.
\]

Theories with two metrics have been studied before\(^{33,34}\). However, at this stage of our discussion, it seems unclear why the same source type should produce different interaction fields. Therefore, the canonical form (6) is a natural choice for further discussion of the first-order homogeneous Lagrangians. Moreover, if one embraces the **principle of one-to-one correspondence between an interaction field and its source**; then the canonical form of the first-order homogeneous Lagrangian (6) seems to justify, from the mathematical point of
view, the presence of the electromagnetic and gravitational fields in nature. *If one could devise a unique procedure to express any first-order homogeneous function in the canonical form above by using only the first two terms, then this could be viewed as a mathematical explanation of the unique physical reality of only two fundamental classical interactions - the electromagnetic and gravitational interactions. Thus, it is important to investigate the additional higher-order terms and their relevance to our physical reality. However, before entering into such a discussion, which is the focus of Section IV, it will be interesting to touch upon the geometric description behind the principle of reparametrization invariance and the relevant Lagrangians that provide an illustration of the power of that principle in the justification of important models in theoretical and mathematical physics.*

**C. E-dimensional Extended Objects**

At the beginning of the current section, the classical mechanics of a point-like particle has been discussed as a mathematical problem concerned with the embedding \( \phi : \mathbb{R}^1 \to M \). The map \( \phi \) provides the description of the trajectory (the word line) of the particle in the target space \( M \). The actual coordinate realization of the map \( \phi \) depends on the choice of the Lagrangian \( L \) and the interaction fields in \( M \) due to the other objects that are already in \( M \). According to the canonical form of the first-order homogeneous Lagrangians, a point particle would interact with electromagnetic-like vector field \( A_\mu(x) \) and gravitation-like (symmetric rank 2 tensor) field \( g_{\mu\nu}(x) \), as well as with other possible classical long-range fields that are described via rank \( n > 2 \) symmetric tensors \( S_{\alpha_1\alpha_2...\alpha_n}(x) \).

These interaction fields can be viewed as an embedding of higher-dimensional objects. For example, \( A_\mu(x) \) may be viewed as an embedding of \( M \) into space with the same dimension \( m \), for electromagnetism, it is 4D space into another 4D space. For gravity, it is about 4D space into a 10D space since there are 10 independent entries in a symmetric \( 4 \times 4 \) matrix \( g_{\mu\nu}(x) \). However, one does not have to consider only the interaction fields for a point particle. One can consider a more general extended object called \( p \)-brane. In this sense, the classical mechanics of a point-like particle that has been discussed as a problem concerned with the embedding \( \phi : \mathbb{R}^1 \to M \) is actually a 0-brane that is a one-dimensional object. Although time is kept in mind as an extra dimension, one should not insist on any special structure associated with a time flow.
Let’s think of an extended object as a manifold \( E \) with dimension, denoted also by \( E, \dim E = E = p + 1 \) where \( p = 0, 1, 2, \ldots \). In this respect, one has to solve for \( \phi : E \to M \) such that some action integral is minimized. From this point of view, one is dealing with the mechanics of a \( p \)-brane. In other words, how is this \( E \)-dimensional extended object submerged in \( M \), and what are the relevant interaction fields? By using the coordinate charts on \( M (x : M \to \mathbb{R}^m) \), one can think of this as a field theory over the \( E \)-manifold with a local fiber \( \mathbb{R}^m \). Thus the field \( \tilde{\phi} \) is such that:

\[
\tilde{\phi} : \phi^a = x \circ \phi : E \to M \to \mathbb{R}^m.
\] (8)

Following the relativistic point particle discussion after equation (5), one considers the space of the \( E \)-forms over the manifold \( M \), denoted by \( \Lambda^E(M) \) and dimension \( D = (m! \ldots (m - E)! = m! \ldots (m - E)! \). In a specific coordinate basis an element \( \Omega \) in \( \Lambda^E(M) \) has the form \( \Omega = \Omega^a_{\alpha_1 \ldots \alpha_E} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_E} \). Let’s use an arbitrary label \( \Gamma = 1, 2, \ldots, D \) to index different \( E \)-forms over \( M \); thus, \( \Omega \to \Omega^\Gamma = \Omega^\Gamma_{\alpha_1 \ldots \alpha_E} dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots dx^{\alpha_E} \). Next let’s introduce “generalized velocity vectors” with components \( \omega^\Gamma \):

\[
\omega^\Gamma = \frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_E})}{\partial (z^1 z^2 \ldots z^E)}, \quad dz = dz^1 \wedge dz^2 \wedge \ldots \wedge dz^E.
\]

In the above expression, \( \frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_E})}{\partial (z^1 z^2 \ldots z^E)} \) represents the Jacobian of the transformation from coordinates \( \{x^\alpha\} \) over the manifold \( M \) to coordinates \( \{z^a\} \) over the embedded \( p \)-brane. The pull-back of an \( E \)-form \( \Omega^\Gamma \) must be proportional to the volume form over the \( p \)-brane:

\[
\phi^* (\Omega^\Gamma) = \omega^\Gamma dz^1 \wedge dz^2 \wedge \ldots \wedge dz^E = \Omega^\Gamma_{\alpha_1 \ldots \alpha_E} \frac{\partial (x^{\alpha_1} x^{\alpha_2} \ldots x^{\alpha_E})}{\partial (z^1 z^2 \ldots z^E)} dz^1 \wedge dz^2 \wedge \ldots \wedge dz^E.
\]

Therefore, it is suitable for integration over the \( E \)-manifold. Thus, the action for the embedding \( \phi \) is:

\[
S[\phi] = \int_E L(\tilde{\phi}, \omega) dz = \int_E \phi^* (\Omega) = \int_E A_\Gamma(\tilde{\phi}) \omega^\Gamma dz.
\]

This is a homogeneous function in \( \omega \) and is reparametrization (diffeomorphism) invariant with respect to the diffeomorphisms of the \( E \)-manifold. If one relaxes the linearity \( L(\tilde{\phi}, \omega) = \phi^* (\Omega) = A_\Gamma(\tilde{\phi}) \omega^\Gamma \) in \( \omega \), then the canonical expression for the first-order homogeneous
Lagrangian gives:
\[
L\left(\vec{\phi}, \omega \right) = \sum_{n=1}^{\infty} \sqrt{S_n (\omega, \ldots, \omega)} = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega^{\Gamma_1} \omega^{\Gamma_2}} + \ldots \sqrt{S_m (\omega, \ldots, \omega)}.
\] (9)

At this point, there is a strong analogy between the relativistic point particle and the \(p\)-brane. However, there is a difference in the number of components; \(\vec{x}, \vec{v}\), and \(\vec{\phi} = \vec{x} \circ \phi\) have the same number of components \((m = \text{dim}(M))\) but the “generalized velocity” \(\omega\) has a bigger number of components \(D = \binom{m}{E} \geq m\) that are related to Jacobians\(^{35}\).

Some specific examples of \(p\)-brane theories correspond to the following familiar Lagrangians in theoretical and mathematical physics:

- The Lagrangian for a 0-brane (relativistic point particle in an electromagnetic field, \(\text{dim} \, E = 1\) and \(\omega^\Gamma \rightarrow v^\alpha = \frac{dx^\alpha}{dt}\) is:

  \[
  L \left(\vec{\phi}, \omega \right) = A_\Gamma \omega^\Gamma + \sqrt{g_{\Gamma_1 \Gamma_2} \omega^{\Gamma_1} \omega^{\Gamma_2}} \rightarrow L \left(\vec{x}, \vec{v} \right)
  \]

  \[
  L \left(\vec{x}, \vec{v} \right) = qA_\alpha v^\alpha + m \sqrt{g_{\alpha \beta} v^\alpha v^\beta}.
  \]

- The Lagrangian for a 1-brane (strings, \(\text{dim} \, E = 2\)) is:

  \[
  L \left(x^\alpha, \partial_i x^\beta \right) = \sqrt{Y_{\alpha \beta} Y_{\alpha \beta}},
  \]

  using the notation:

  \[
  \omega^\Gamma \rightarrow Y_{\alpha \beta} = \frac{\partial(x^\alpha, x^\beta)}{\partial(\tau, \sigma)} = \det \begin{pmatrix}
  \partial_\tau x^\alpha & \partial_\sigma x^\alpha \\
  \partial_\tau x^\beta & \partial_\sigma x^\beta
\end{pmatrix} = \partial_\tau x^\alpha \partial_\sigma x^\beta - \partial_\sigma x^\alpha \partial_\tau x^\beta.
  \]

  In this case, the index \(\Gamma\) for labeling the components of the \textit{generalized velocity vector} \(\omega^\Gamma\) corresponds to the set of pairs \(\{\alpha, \beta\}\) out of \(m\) elements. For example, for \(m = 4\) this will be \(4!/2!^2 = 6\) not 4 like for the standard velocity vector in \(M\).

- The Lagrangian for a general \(p\)-brane has the Dirac-Nambu-Goto term (DNG)\(^{36}\):

  \[
  L \left(x^\alpha, \partial_E x^\beta \right) = \sqrt{Y^\Gamma Y_\Gamma}.
  \]
Notice that most of the Lagrangians above, except for the relativistic particle, are restricted only to gravity-like interactions. In the case of the charged relativistic particle, the electromagnetic interaction is very important. The corresponding interaction term for p-branes is known as Wess-Zumino term\textsuperscript{37}.

The above discussion can be viewed as a justification of important class of model Lagrangian systems via the principle of reparametrization invariance when applied to the mechanics of point particle as well as to the mechanics of extended objects. The principle leads naturally to important, well-known, and studied electromagnetic-like ($n = 1$) and gravity-like ($n = 2$) interaction terms. However, the framework also suggests new possible fields ($n > 2$). Thus, it is important to investigate the additional higher-order terms and their relevance to our physical reality.

IV. CLASSICAL FORCES BEYOND ELECTROMAGNETISM AND GRAVITY

So far, the focus of the paper has been to justify and encourage the study of models based on first-order homogeneous Lagrangians by emphasizing their general properties, their potential to provide a mathematical justification of the observed macroscopic physical reality, and along the way, to set the stage for the study of diffeomorphism invariant mechanics of extended objects by following the close analogy with the relativistic point particle. Consequently, it is important to understand these new terms in the canonical expression of the first-order homogeneous Lagrangians\textsuperscript{[3]}\textsuperscript{[3]}. In this respect, this section discusses the implications of such interaction terms beyond electromagnetism and gravity as given by the canonical expression of the first-order homogeneous Lagrangians\textsuperscript{[3]}\textsuperscript{[3]}.

First, one should recognize that one can circumvent the linear dependence, ($\det(\frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}) = 0$) due to the reparametrization symmetry, of the equations of motion derived from $L = \sqrt{S_n (\vec{v}, ..., \vec{v})}$ by adding an extra set of equations ($\frac{dL}{d\tau} = 0$). This way the equations of motion derived from $L = \sqrt{S_n (\vec{v}, ..., \vec{v})}$ and $\frac{dL}{d\tau} = 0$ are equivalent to the equations of motion derived from $L = S_n (\vec{v}, ..., \vec{v})$. This is similar to the discussion at the end of Section II. As noticed before, this is a specific choice of parametrization such that $g_{\alpha\beta} (x) v^\alpha v^\beta$ is constant. Indeed, if one starts with the re-parametrization invariant Lagrangian $L = q A_\alpha v^\alpha + m \sqrt{g_{\alpha\beta}(x) v^\alpha v^\beta}$ and defines proper time $\tau$ such that: $d\tau = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} \Rightarrow \sqrt{g_{\alpha\beta} v^\alpha v^\beta} = 1$, then one can
effectively consider \( L = qA_\alpha v^\alpha + (m + \chi)\sqrt{g_{\alpha\beta}v^\alpha v^\beta} - \chi \) as our model Lagrangian. Here \( \chi \) is a Lagrange multiplier to enforce \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \) that breaks the reparametrization invariance explicitly. Then one can write it as \( L = qA_\alpha v^\alpha + (m + \chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \) and using \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \) one arrives at \( L = qA_\alpha v^\alpha + (m + \chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \). One can deduce a specific value for \( \chi \) \( (\chi = -m/2) \) by requiring that \( L = qA_\alpha v^\alpha + m\sqrt{g_{\alpha\beta}(x)v^\alpha v^\beta} \) and \( L = qA_\alpha v^\alpha + (m + \chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \) produce the same Euler-Lagrange equations under the constraint \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \). Then, by dropping the overall constant term, this finally results in the familiar equivalent Lagrangian: \( L = qA_\alpha v^\alpha + m\sqrt{g_{\alpha\beta}(x)v^\alpha v^\beta} \). One can deduce a specific value for \( \chi \) \( (\chi = -m/2) \) by requiring that \( L = qA_\alpha v^\alpha + m\sqrt{g_{\alpha\beta}(x)v^\alpha v^\beta} \) and \( L = qA_\alpha v^\alpha + (m + \chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \) produce the same Euler-Lagrange equations under the constraint \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \). Then, by dropping the overall constant term, this finally results in the familiar equivalent Lagrangian: \( L = qA_\alpha v^\alpha + m\sqrt{g_{\alpha\beta}(x)v^\alpha v^\beta} \) where \( \tau \) has the usual meaning of proper-time parameterization such that \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \).

If one focuses on a specific \( n^{th} \)-term of re-parametrization invariant Lagrangian \( (L) \), that is, \( L = (S_n(v))^1/n \) in the parametrization gauge \( S_n(v) = \text{const} \) then the equations of motion are:

\[
S_{n/\alpha/\beta} \frac{dv^\beta}{d\tau} = S_{n,\alpha} - S_{n/\alpha,\beta}v^\beta.
\]

Here \( S_{n,\alpha} \) denotes partial derivative with respect to \( x^\alpha \) when \( S_{n/\alpha} \) indicates partial derivative with respect to \( v^\alpha \). From this expression, it is clear that \( n = 2 \) is a model that results in velocity independent symmetric tensor \( S_{n/\alpha/\beta}(v) \) that can be associated with the metric tensor. Usually, such metric tensor is assumed invertible and therefore the differential equations can be written in the form acceleration as a function of velocity and position. However, in general, the left-hand side \( S_{n/\alpha/\beta}(v) \) goes as \( v^{n-2} \) while the right-hand side as \( v^n \) which will result in the general behavior that the acceleration grows as \( v^2 \) at most. This is consistent with the known velocity dependence of the equation of the geodesics as well as the equation of the geodesic deviations. The growth is not usually an issue since there is a limitation on the magnitude \( v < c \) due to the finite propagation speed. Thus, for a suitably chosen units, \( c = 1 \) one should have \( |v^\alpha| \leq 1 \). However, if \( n > 2 \) and if the maximum speed limit 1 is reached along one coordinate, then there could be issues for keeping the system at rest with respect to another coordinate direction since a term like \( 1/v^{n-2} \) will grow towards infinity when \( v \rightarrow 0 \).

A. Simplest Pure \( S_n(v) \) Lagrangian Systems

To further illustrate our point above and to gain a better understanding of the \( S_n(v) \) terms, let us consider the simplest possible pure \( S_n(v) \) Lagrangian systems by assuming:
• Curvilinear coordinate system such that: $S_n(v) = f(t, r, w, u)$ where $w = dx^0/d\tau$ and $u = dr/d\tau$;

• Static fields, that is: $S_n(v) = f(r, w, u)$;

• Inertial coordinate system in the sense of Newtonian like space and time separation, that is: $S_{t...r...r} = 0$ except for $S_{t...t}$ and $S_{r...r}$ components.

Thus, the expression for $S_n(v)$ takes upon the following form:

$$S_n(r, w, u) = \psi(r)w^n + \phi(r)u^n. \quad (10)$$

Notice that here the symbols $u$ and $w$ are used to denote the spatial $r$ and time-like velocity coordinates instead of the previously used symbol $v$. Later in the discussion, this symbol $v$ will be used to denote the spatial speed in the Newtonian limit using coordinate-time parametrization $v = dr/dt$. This way the corresponding equations of motion for $L = S_n(r, w, u)$ are:

$$\frac{du}{d\tau} = -\frac{u^2\phi'(r)}{(n-1)\phi(r)} + \frac{1}{w^{n-2}n(n-1)\phi(r)}w^n\psi'(r), \quad (11)$$

$$\frac{dw}{d\tau} = -\frac{ww\psi'(r)}{(n-1)\psi(r)}. \quad (12)$$

One can recognize the connection of the fields $\psi(r)$ and $\phi(r)$ to the energy and linear momentum of a particle by looking at the generalized momentum: $p_\alpha = \frac{\partial L}{\partial v^\alpha}$. In particular, $\psi(r)$ is related to the energy of the particle $E = p_0 = \frac{\partial L}{\partial w}$, especially when considering $\tau = x^0 = ct$ in the co-moving frame $u/w = v/c \approx 0$ using coordinate-time parametrization where $w = 1$. In this respect, if the energy of the particle is conserved then $\psi(r) = constant$ and therefore $\psi'(r) = 0$. The “radial” acceleration at macroscopic scales is then:

$$a_r = \frac{dv}{dt} = -\frac{v^2\phi'(r)}{(n-1)\phi(r)}. \quad (13)$$

Here, the speed of light $c$, the maximum speed of propagation cancels out and $u = dr/d(ct) = v/c$ is related to the spatial speed $v$. If $n = 2$ and $\phi(r) = br$ then one recovers the usual kinematical expression for the normal acceleration of a particle moving in a circular orbit $(a_n = v^2/r)$.

In general, however, depending on the specifics of the model and details of $\phi(r)$ one may obtain deviations from the flat space or the metric model for gravity. Such new models
and forces could be relevant at large/cosmological scales where the dark-matter problem manifests itself in the deviation of the kinematical acceleration from the anticipated gravitational acceleration in galaxies and clusters of galaxies. Depending on the sign of $\phi'(r)/\phi(r)$ this term could be “dissipative” in the sense that the system will settle at $v = 0$ after a sufficiently long time if the sign of $\phi'(r)/\phi(r)$ is positive. If the sign is negative then one has “repulsive gravity” that could be relevant to the dark-energy problem since the system will have an unstable $v = 0$ configuration. Any further speculations about this equation are poorly justified without any underlining theory that predicts $\phi(r)$ and compares it to experimental observations.

At the microscopic scale, however, one may have $\psi'(r) \neq 0$. This could suggest that the energy $p_0$ may not be conserved due to the interaction of the particle with the environment; thus, it may be subject to energy exchange. However, $p_0$ should nevertheless be conserved since the model under consideration has no explicit coordinate-time dependence. This can be illustrated using the equation for $w$ (12). The equation can be rewritten in a form that makes it easy to be integrated and to see the conservation of the energy $p_0$:

$$d \ln(w) = -\frac{1}{(n-1)}d \ln(\psi) \Rightarrow$$

$$\psi(r)w^{n-1} = \text{constant} = \frac{p_0}{n}$$  \hspace{1cm} (14)

Looking back at the first equation (11) for $u$, when $n = 2$ the spatial force has two parts, one is velocity independent force proportional to $\propto \frac{v^2\psi'(r)}{2\psi(r)\phi(r)}$, and the other part could be “dissipative” if the sign of $\phi'(r)/\phi(r)$ is positive or “repulsive gravity” if the sign of $\phi'(r)/\phi(r)$ is negative as discussed earlier.

For $n > 2$ the physics interpretation of the equation of motion (11) leads to unusual behavior:

$$\frac{(n-1)}{w^2} \frac{du}{d\tau} = -\frac{v^2\phi'(r)}{c^2\phi(r)} + \frac{c^{n-2} \psi'(r)}{v^n n\phi(r)}.$$  \hspace{1cm} (15)

It seems that an observer cannot detect a particle to be in complete rest ($u/w = v/c = 0$) for a finite time interval $\Delta t$. If the speed of a particle was zero ($v = 0$) at some moment then the particle should have an infinite acceleration $du/d\tau$ at that moment since $du/d\tau \propto w^2 v^{2-n} \rightarrow \infty$. Thus, the particle will instantaneously leave the state $v = 0$ for a non-zero velocity state rather than staying in the zero velocity state. Depending on the details of the fields $\psi'(r)$ and $\phi'(r)$ there may not be a zero external force configuration for such particle
in general. Nevertheless, specific fields $\psi'(r)$ and $\phi'(r)$ may allow for zero acceleration state $du/d\tau = 0$ and non-zero spatial velocity state $u/w = v/c \neq 0$:

$$v^n = \frac{c^n \psi'(r)}{n \phi'(r)}.$$ 

However, such state would imply that the observer cannot be in the co-moving frame of the particle anymore since the coordinate time $t$ will not be synchronized with the “proper-time” $\tau$ of the particle $dw/d\tau \neq 0$ and thus $w = dx^0/d\tau \neq constant$.

The above-discussed pathology is strikingly similar to the manner that quantum mechanical particles behave: particles cannot be localized with speed as close to zero as one wishes to; even more, the conservation of energy needs to be amended due to external fields (14). Therefore, such terms with $n > 2$ may play important role in the understanding of the mechanism behind the inflation driven early stage of the universe, as well as in the derivation of the Dirac equation containing fundamental sub-atomic interactions beyond electromagnetism and gravity (for preliminary discussion see32 and 41).

Not being able to observe a particle at rest seems somewhat in contradiction to our classical physics reality. However, the more appropriate Lagrangian should take into account that “empty space” has Minkowski geometry:

$$L = m \sqrt{\eta_\alpha^\beta v^\alpha v^\beta} + \kappa n \sqrt{S_n (\vec{v}, \ldots, \vec{v})}.$$ 

Here $\eta_{\alpha\beta} = (1,-1,\ldots,-1)$ is the Lorentz invariant metric tensor. For Lagrangians that contain gravity ($S_2(v)$ term) the problem for spatial velocity limit $v \to 0$ does not exist as discussed above for the case $n = 2$. In the non-relativistic limit ($v/c \to 0$), the present model of pure $S_n$ interaction in Minkowski spacetime results in an acceleration $\frac{dv}{d\tau}$ that is the same up to $O(v^2)$ terms for $L = \text{const}$ parametrization as well as for $\sqrt{\eta_{\alpha\beta} v^\alpha v^\beta} = \text{const}$ parametrization. Thus the non-relativistic limit cannot distinguish these two choices of parametrization.

B. Choice of Proper Time Parametrization

It was mentioned earlier that for parameter independent homogeneous Lagrangians of order $\alpha$ one has $h = (\alpha - 1)L$ and thus $dL/d\lambda = 0$ except for $\alpha = 1$ that singles out first-order homogeneous Lagrangians. When working with re-parametrization invariant Lagrangian
model, one can choose parametrization so that $L d\lambda = d\tau$ or effectively thinking of $L(x, v) = \text{const}$. This brings the homogeneous Lagrangians of first order back in the family $dL/d\lambda = 0$.

This appears to be the choice of parametrization to be made $\lambda \rightarrow \tau$ if the structure of $L$ is not known. However, it seems that $\sqrt{g_{\alpha\beta} v^\alpha v^\beta} = \text{const}$ is preferred as physically more relevant due to its connection to the lifetime of unstable elementary particles. Especially, due to the lack of experimental evidence that the lifetime of charged elementary particles is affected by the presence of electromagnetic fields. This can be related to the observation that for any Lagrangian of the form $L = v^\mu A_\mu(x, v)$, where $x$ is space-time coordinate and $v$ is a world-line velocity vector (4-vector for 3+1 space-time), one can define a velocity dependent symmetric tensor $g_{\alpha\beta}(x, v) = \frac{1}{2} (A_{\alpha/\beta}(x, v) + A_{\beta/\alpha}(x, v))$ where $A_{\beta/\alpha}(x, v)$ denotes partial derivative with respect to $v^\alpha$ of $A_\beta(x, v)$. Then one can show that $\frac{d}{dx} (v^\alpha g_{\alpha\beta}(x, v) v^\beta) = 0$ along the trajectory determined by the Euler-Lagrange equation for $L = v^\mu A_\mu(x, v)$ - just like the usual geodesic equation of motion as in the discussion presented in Section III. This symmetric tensor $g_{\alpha\beta}(x, v)$ does not depend on the velocity independent electromagnetic vector potential $A_\mu(x)$ and thus the length of the vector as calculated with $g_{\alpha\beta}(x, v)$ is not affected by the presence of electromagnetic interaction. Therefore, a proper time parametrization that coincides with the traditional definition: $d\tau = \sqrt{g_{\alpha\beta}(x,v) dx^\alpha dx^\beta}$ can be introduced.

The name of this special choice of $\tau$ parametrization derives from the fact that it is generally covariant and thus independent of the observer’s coordinate system and can be interpreted as the passing of time measured in the rest frame of the system under study. Therefore, it is often of the form $d\tau = \sqrt{g_{00}(t)} dt$ and thus can be integrated along the laboratory coordinate time $t$. The laboratory coordinate time $t$ is up to the observer at rest as part of the laboratory measuring tools for various processes. Unfortunately, for first-order homogeneous Lagrangians, one has $v^\alpha g_{\alpha\beta}(x, v) v^\beta = 0$ because $A_\mu(x, v)$ is a homogeneous function of zero degree and thus $v^\beta A_{\mu/\beta}(x, v) = 0$. This seems to make it difficult to define the proper time parametrization in the usual way: $d\tau = \sqrt{g_{\alpha\beta}(x,v) dx^\alpha dx^\beta}$ for such first-order homogeneous Lagrangians $L = v^\mu A_\mu(x, v)$.

In this respect, for first-order homogeneous Lagrangians in the velocity, it is not clear if one has to chose “proper time” parametrization so that $L = \text{const}$, or $\sqrt{g_{\alpha\beta} v^\alpha v^\beta} = \text{const}$, or $L - A_\mu(x) v^\mu = \text{const}$. The choice $L - A_\mu(x) v^\mu = \text{const}$ may very well be the appropriate choice since the weak and the strong forces do have an effect on the lifetime of
elementary particles; for example, neutrons are unstable in free space but stable within nuclei. In connection to this, note that the other terms beyond gravity ($S_n$ with $n > 2$) are seemingly related to the internal degrees of freedom of the elementary particles. This should become more clear once a non-commutative quantization ($v \rightarrow \gamma$) is applied to the re-parametrisation invariant Lagrangian, which will be discussed elsewhere (for some preliminary results see\textsuperscript{32} and \textsuperscript{41}). Unfortunately, it is not clear how to extract the $A_\mu(x)v^\mu$ component of any first-order homogeneous Lagrangian $L$ mathematically, which is applicable to a physically relevant process, that is not assuming electromagnetic interaction a priory. Mathematically, one can extract $A_\mu(x)$ from first-order homogeneous Lagrangian $L$ by considering $A_\mu(x) = L(x,v)/\mu = p_\mu$ at $v^\alpha \rightarrow 0$; however, physically $v^0$ should never be zero. Nevertheless, in the case of the Simplest Pure $S_n(v)$ Lagrangian Systems discussed in the previous subsection, one can define \textit{“proper time”\,} parametrisation under certain conditions.

The condition for reasonable parametrisation such as \textit{“proper time”\,} for $S_n(v)$ is surprisingly restrictive! It demands $n = 2$ so that the laboratory clock could be at rest with respect to the particle studied. If $n > 2$ there is this pathological unsuitability that moves the particle \textquotedblright{instantaneously}\textquotedblright{} away from the rest frame of the clock. Thus, only $n = 2$ allows for a rest frame within the model Lagrangians discussed. Then by using the conservation of $p_0$ \textsuperscript{141} one has: $2\psi(r)cdt = p_0d\tau$ where almost everything is a constant ($c$ and $p_0$) and $\psi(r)$ seems to be related to the gravitational potential at the location $r$ where the particle is. In conclusion, it seems that \textit{“proper time”\,} parametrisation is only possible for $n = 2$ systems based on the analyses of the Simplest Pure $S_n(v)$ Lagrangian Systems and the discussion above. Thus, gravity is essential for the notion of the \textit{“proper time”\,} parametrisation and no other Simple $S_n(v)$ Lagrangian System provides an alternative parametrisation that makes sense as the passing of time in the rest frame of a particle.

To conclude this section, one may naively extrapolate the scale at which such new forces may be dominant. Considering that electromagnetic forces are relevant at atomic and molecular scale when gravity is dominating the solar system and at galactic and cosmological scales, then one may deduce that terms beyond gravity may be relevant at galactic and intergalactic scales. Along this line of reasoning, a possible determination of the structure of such forces from the velocity distribution of stars in galaxies is an interesting possibility. In this respect, such forces can be of relevance to the dark matter and dark energy cosmology.
problems. The pathological $dv/d\tau \to \infty$ when $v \to 0$ behavior of pure $S_n$ for $n > 2$ interactions could also be of relevance to inflation models. Finally, as already mentioned, such terms are essential for bringing in fields beyond the electromagnetic fields into the Dirac equation when considering the quantization of the first-order homogeneous Lagrangians in the velocity.

V. THE BACKGROUND FIELDS AND THEIR LAGRANGIANS

The uniqueness of the interaction fields and their source types has been essential for the selection of the matter Lagrangian (9). The first two terms in the Lagrangian are easily identified as electromagnetic and gravitational interaction. The other terms describe new classical forces. It is not yet clear if these new terms are actually present in nature or not, so one shall not engage them actively in the following discussion but our aim is to start preparing the stage for such research and discussions. At this point, one has a theory with background fields since the equations for the interaction fields are not known. To complete the theory, one needs to introduce actions for these interaction fields.

One way to write the action integrals for the interaction fields $S_n$ in (9) follows the case of the $p$-brane discussion. There, one has been solving for $\phi : E \to M$ by selecting a Lagrangian that is more than a pull-back of an $E$-form over the manifold $M$. In a similar way, one may view $S_n$ as an $M$-brane field theory, where $S_n : M \to S_nM$ and $S_nM$ is the fiber of symmetric tensors of rank $n$ over $M$. This approach, however, cannot terminate itself since new interaction fields would be generated as in the case of $\phi : E \to M$.

Another way assumes that $A_\Gamma$ is an $n$-form. Thus, one may use the structure of the external algebra $\Lambda (T^*M)$ over $M$ to construct objects proportional to the volume form over $M$. For any $n$-form ($A$) objects proportional to the volume form $\Omega_{\text{Vol}}$ can be constructed by using operations in $\Lambda (T^*M)$, such as the external derivative $d$, external multiplication $\wedge$, and the Hodge dual $\ast$. For example, $A \wedge \ast A$ and $dA \wedge \ast dA$ are forms proportional to the volume form $\Omega_{\text{Vol}}$.

The next important ingredient comes from the symmetry in the matter equation. That is, if there is a transformation $A \to A'$ that leaves the matter equations unchanged, then there is no way to distinguish $A$ and $A'$ by experiments and measurements via the matter that is obeying these equations. Thus the action for the field $A$ should obey the same symmetry
(gauge symmetry) as those found in the equations of motion for the matter.

A. Justifying the Electromagnetic Action

Let us consider now the matter equation for 4D electromagnetic interaction which is \( d\vec{v}/d\tau = F \cdot \vec{v} \) where \( F \) is the 2-form obtained by differentiation of the 1-form \( A \) (\( F = dA \)), and the gauge symmetry for \( A \) is \( A \rightarrow A' = A + df \) since the external differential operator \( d \) obeys \( d^2 = 0 \). The reasonable terms, which can result in the volume form \( \Omega_{\text{Vol}} \) for the field Lagrangian \( \mathcal{L}(A) \) of a 1-form field \( A \), are then: \( A \wedge *A, dA \wedge dA, \) and \( dA \wedge *dA \) and of course \( A \wedge A \wedge A \wedge A \). The first and last terms do not conform with the gauge symmetry \( A \rightarrow A' = A + df \) and the second term \( (dA \wedge dA) \) is a boundary term since \( dA \wedge dA = d(A \wedge dA) \) that gives \( \int_M d(A \wedge dA) = A \wedge dA \) at the boundary of \( M \); this term is interesting in the quantum Hall effect. Therefore, one is left with a unique action for fields based on a one-form \( A = A_\mu(x) dx^\mu \) that respects the gauge symmetry of the corresponding Euler-Lagrange equations of motion for matter: \( A \rightarrow A' = A + df \) - this is exactly the electromagnetic field generated by moving charges \( j^\mu = \rho v^\mu \) and described by the standard action:

\[
S[A] = \int_M dA \wedge *dA + A_\mu j^\mu = \int_M F \wedge *F + A_\mu j^\mu.
\]

Note that if \( F \) was considered as a fundamental field rather than \( A \) then in 4D one can also consider the term \( F \wedge F \). However, as soon as one recognizes that \( F = dA \) then this becomes the boundary term \((dA \wedge dA)\) discussed above. Furthermore, once \( F = dA \) is recognized as a two-form and expressed in the coordinate basis \( F_{\mu\nu} dx^\mu \wedge dx^\nu \) then one can also consider a gauge invariant term of the form: \( F_{\mu\nu} dx^\mu \wedge dx^\nu \wedge *(dx^\mu \wedge dx^\nu) \) as part of the action. However, such term is zero due to permutation symmetry since \( W^{\nu\mu} = W^{\mu\nu} = dx^\mu \wedge dx^\nu \wedge *(dx^\mu \wedge dx^\nu) \propto \eta^{\mu\nu} \eta^{\rho\sigma} dx^\rho \wedge dx^1 \wedge dx^2 \wedge dx^3 \); thus, the anti-symmetric \( F \) and the symmetric \( W \) contract to zero \( (F_{\mu\nu} W^{\mu
u} = 0) \).

B. Justifying the Einstein-Hilbert-Cartan Action

For our next example, let’s look at the terms related to the matter equations that involve gravity. There are two possible choices of matter equation. The first one is the geodesic equation \( d\vec{v}/d\tau = \vec{v} \cdot \Gamma \cdot \vec{v} \) where \( \Gamma \) is considered as a connection 1-form that transforms
in the usual way $\Gamma \rightarrow \Gamma + \partial g$ under coordinate transformations by the group element $g$. This type of transformation, however, is not a "good" symmetry since restricting the gauge transformation $\Gamma \rightarrow \Gamma + \Sigma$ to transformations $\Sigma = \partial g$ such that $\vec{v} \cdot \Sigma \cdot \vec{v} = 0$, would mean to select a subset of coordinate systems, inertial systems, for which the action $S$ is well defined and satisfies $S[\Gamma] = S[\Gamma + \Sigma]$. Selecting a specific class of coordinate systems for the description of a physical phenomenon is not desirable, so this option shall not be explored any further.

In general, the Euler-Lagrange equations assume a background observer who defines the coordinate system. For electromagnetism, this is acceptable since neutral particles are such privileged observers. In gravity, however, there is no such observer, and the equation for matter should be relational. Such an equation then is the equation of the geodesic deviation:

$$d^2\xi/d\tau^2 = R(v, v) \cdot \xi,$$

where $R$ is a Lie algebra $(TM)$ valued curvature 2-form $R = d\Gamma + [\Gamma, \Gamma]$. A general curvature 2-form is denoted by $F \rightarrow (F_{\alpha\beta})^i_j$. Here, $\alpha$ and $\beta$ are related to the tangential space $(TM)$ of the base manifold $M$. The $i$ and $j$ are related to the fiber structure of the bundle over $M$ where the connection $(\Gamma_a)^i_j$ that defines $(F_{\alpha\beta})^i_j$ is given. Clearly, the Riemann curvature tensor $R$ is very special curvature because all of its indices are of $TM$ type. For that reason, it is possible to contract the fiber degree of freedom with the base manifold degree of freedom (indices). Thus, an action linear in $R$ is possible. In general, one needs to consider a quadratic action $(F^i_{\alpha\beta} \wedge \ast F^j_{\alpha\beta})$, i.e. trace of $F \wedge \ast F$.

Using the symmetries of the Riemann curvature tensor $R$ ($R_{\alpha\beta,\gamma\rho} = -R_{\beta\alpha,\gamma\rho} = -R_{\alpha\beta,\rho\gamma} = R_{\gamma\rho,\alpha\beta}$) one has two possible expressions that can be proportional to the volume form $\Omega$. The first expression is possible in all dimensions and can be denoted by $R^*$, which means that a Hodge dual operation has been applied to the second pair of indices $(R_{\alpha\beta,*}^{\gamma\rho})$. The $R^*$ action seems to be related to the Cartan-Einstein action for gravitation $S[R] = \int R_{\alpha\beta} \wedge * (dx^\alpha \wedge dx^\beta)^{\wedge 3}$. The other expression is only possible in a four-dimensional space-time and involves full anti-symmetrization of $R$ ($R_{\alpha[\beta,\gamma]\rho}$) denoted by $R^{\wedge}$. However, the fully anti-symmetric tensor $R^{\wedge}$ is identically zero due to symmetry considerations related to the permutation group$^{32}$. Since the symmetries of the equation of the geodesic deviation are encoded in the Riemann curvature tensor $R$, then once again one arrives at the unique Einstein-Hilbert-Cartan action for gravity based on $R$.  

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VI. CONCLUSIONS AND DISCUSSIONS

In conclusion, the discussion in this paper has shown the potency of the principle of reparametrization invariance when realized via the canonical-form of the first-order homogeneous Lagrangians in the velocity or the generalized velocity by using the principle of one-to-one correspondence between an interaction field and its source to justify the fundamental interaction fields for the classical long-range forces via the geometrical concepts of embedding of manifolds as well as the natural differential structures over manifolds. In summary, the structure of the matter Lagrangian \( L \) for extended objects, and in particular the point particle, have been discussed. Imposing reparametrization invariance of the Lagrangian based action \( S = \int_E L(x, \omega) \) naturally leads to a first-order homogeneous Lagrangian. In its canonical form, the Lagrangian \( L \) contains electromagnetic and gravitational interactions, as well as interactions that are not yet experimentally discovered.

The fields \( A_\mu(x) \) and \( g_{\mu\nu}(x) \) associated with \( n = 1 \) and \( n = 2 \) homogeneous Lagrangians build form monomials in the velocities \( S_n(v, \ldots, v) \) are clearly related to electromagnetic and gravitational interactions. Especially, if one recognizes that the gauge symmetry of these interaction fields are encoded in the 2-forms \( F \) and \( R \) that naturally appear in the corresponding equations of motion - the Euler-Lagrange equation that corresponds to the Lorentz force \( d\vec{v}/d\tau = qF\vec{v} \) for charged particles and the equation of the geodesic deviations for massive particles \( d^2\vec{\xi}/d\tau^2 = R(v, v)\vec{\xi} \).

If one extrapolates from the strengths of the two known classical long-range interactions, then it is natural to expect that the new terms in \( L \) should be important, if present in nature at all, at big cosmological scales, such as those relevant to the dynamics of galactic and galactic clusters. Thus, perhaps relevant to the dark matter and dark energy phenomena. Furthermore, the pathological behavior \( (15) \) discussed for the simplest model of \( S_n(v, \ldots v) \) fields when \( n > 2 \) may be relevant to the inflation processes in the early universe. At microscopic scales such \( n > 2 \) fields may be useful in justifying the interactions in the standard model of elementary particles upon suitable quantization that recovers the Dirac equation but with additional interactions beyond electromagnetism and gravitation.

If one is going to study the new interaction fields \( S_n(v, \ldots v), n > 2 \), then the guiding
principles for writing field Lagrangians, as discussed in the examples of electromagnetism and gravity (Section V), may be a useful starting point. Furthermore, it may be useful to apply the outlined constructions to gravity by considering it as a 3-brane in a 10-dimensional target space \((g_{\alpha\beta} : M \to S_2M)\) and to compare it with the 10D supergravity.

*If such \(S_n(v, \ldots v)\) related forces are not present in nature then one needs to understand why nature is not taking advantage of such possibilities.* The choice of the canonical Lagrangian is based on the assumption of one-to-one correspondence between interaction fields and the type of their sources. *If one can show that any first-order homogeneous function can be written in the canonical form proposed, then this would be a significant step towards our understanding of the fundamental interactions in nature - especially if one can show that only \(n = 1\) and \(n = 2\) effective terms are needed.* Note that an equivalent expression can be considered as well: \(L = A_\alpha(\vec{x}, \vec{v})v^\alpha\). This expression is simpler and is concerned with the structure of the homogeneous functions of order zero \(A_\alpha(\vec{x}, \vec{v})\). In any case, understanding the structure of the homogeneous functions of any order seems to be an important mathematical problem with significant implications for physics.

**VII. EXERCISE PROBLEMS**

1. Show that \(g_{\mu\nu}v^\mu v^\nu = \text{constant}\) along the trajectory of a particle is necessary and sufficient condition for the Euler-Lagrange equations corresponding to \(S_1\) \(^{(1)}\) and \(S_2\) \(^{(2)}\) to be equivalent to each other and to the geodesic equation \(^{(3)}\).

2. Show that for any Lagrangian \(L(x, v)\) that is a homogeneous function in the velocity \(\vec{v}\) of order \(n \neq 1\) the corresponding Hamiltonian function \(h = v^\alpha \left( \frac{\partial L}{\partial v^\alpha} \right) - L\) is proportional to the Lagrangian, that is, \(h = (n - 1)L\).

3. Show that any time independent Lagrangian \(L(\vec{x}, \vec{v})\), which is a homogeneous function in velocity \(\vec{v}\) of order \(n \neq 1\), is an integral of the motion with respect to the corresponding Euler-Lagrange equations for \(L\).

4. Consider a Lagrangian that is a constant of the motion; that is, \(dL/d\tau = 0\). Show that any solution of the Euler-Lagrange equations for \(L\) is also a solution for \(\tilde{L} = f(L)\) under certain minor and reasonable requirements on \(f\), such as \(\tilde{L} = f(L) \neq 0\) and \(\tilde{L}' = f' \neq 0\).
5. Show that if \( v^0 = \frac{dt}{d\tau} \) is well behaved (\( v^0 \neq 0 \) over the duration of the process studied) then the Euler-Lagrange equations for the reparametrization-invariant Lagrangian
\[
L(x^\mu, v^\mu) = L(x^\mu, v^i/v^0)v^0,
\]
where \( i = 1, \ldots, n \), \( \mu = 0, 1, \ldots, n \) and \( x^0 = t, v^i = dx^i/d\tau, v^0 = dt/d\tau \), are equivalent to the Euler-Lagrange equations for coordinate-time parametrization (\( \tau = t \)) choice for \( L(t, x^i, dx^i/dt) \). Hint: use that \( L(x^\mu, v^i/v^0) \) is zero-order homogeneous function with respect to \( v^\mu \) and notice the relationship between the Hamiltonian function \( h \) for the initial Lagrangian \( L(t, x^i, dx^i/dt) \) and the generalized momentum \( p_0 = \partial L/\partial v^0 \) for the reparametrization-invariant Lagrangian \( L(x^\mu, v^\mu) = L(x^\mu, v^i/v^0)v^0 \).

6. Show that \( \sum_\beta v^\beta \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta} = 0 \) if \( L \) is first-order homogeneous Lagrangian. Thus, \( \det \left( \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta} \right) = 0 \), since in an extended space-time one usually expects \( v^0 \neq 0 \).

7. Consider the constraint \( \sqrt{g_{\alpha\beta}v^\alpha v^\beta} = 1 \) implemented via a Lagrangian multiplier \( \chi \) in the Lagrangian \( L = qA_\alpha v^\alpha + (m+\chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \). Show that the value of \( \chi \) is required to be \( \chi = -m/2 \) if \( L = qA_\alpha v^\alpha + m\sqrt{g_{\alpha\beta}}v^\alpha v^\beta \) and \( L = qA_\alpha v^\alpha + (m+\chi)g_{\alpha\beta}v^\alpha v^\beta - \chi \) are to result in the same Euler-Lagrange equations.

8. Show that the function \( S_n(r, w, u) \) defined in equation (10) is an integral of motion for the equations given by (11) and (12).

9. Consider the Lagrangian \( L = m\sqrt{\eta_{\alpha\beta}v^\alpha v^\beta + \kappa \sqrt{S_n(v, \ldots, v)}} \), where \( \eta_{\alpha\beta} = (1, -1, \ldots, -1) \) is the Lorentz invariant metric tensor. Show that in the non-relativistic limit (\( v \to 0 \)), the Euler-Lagrange equations for the acceleration \( \frac{dv}{d\tau} \) are the same up to \( O(v^2) \) terms whether \( L = \text{const} \) or \( \sqrt{\eta_{\alpha\beta}v^\alpha v^\beta} = \text{const} \) parametrization is imposed. Thus the non-relativistic limit cannot distinguish these two choices of trajectory parametrization.

10. Show that solutions of the Euler-Lagrange equations for \( L = v^\mu A_\mu(x, v) \), where \( x \) is space-time coordinate and \( v^\mu \) is a world-line velocity vector (4-vector for 3+1 space-time), satisfy \( \frac{d}{dx} \left( v^\alpha g_{\alpha\beta}(x, v) v^\beta \right) = 0 \) for the velocity dependent metric \( g_{\alpha\beta}(x, v) = \frac{1}{2}(A_{\alpha/\beta}(x, v) + A_{\beta/\alpha}(x, v)) \) with \( A_{\beta/\alpha}(x, v) \) being partial derivative with respect to \( v^\alpha \) of \( A_\beta(x, v) \).
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