HOMOGENIZATION OF A STATIONARY PERIODIC MAXWELL SYSTEM IN A BOUNDED DOMAIN IN THE CASE OF CONSTANT MAGNETIC PERMEABILITY

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To the memory of Mikhail Zaharovich Solomyak

Abstract. In a bounded domain $\Omega \subset \mathbb{R}^3$ of class $C^{1,1}$, we consider a stationary Maxwell system with the boundary conditions of perfect conductivity. It is assumed that the magnetic permeability is given by a constant positive $(3 \times 3)$-matrix $\mu_0$, and the dielectric permittivity is of the form $\eta(x/\varepsilon)$, where $\eta(x)$ is a $(3 \times 3)$-matrix-valued function with real entries, periodic with respect to some lattice, bounded and positive definite. Here $\varepsilon > 0$ is the small parameter. Suppose that the equation involving the curl of the magnetic field intensity is homogeneous, and the right-hand side $r$ of the second equation is a divergence-free vector-valued function of class $L^2$. It is known that, as $\varepsilon \to 0$, the solutions of the Maxwell system, namely, the electric field intensity $u_\varepsilon$, the electric displacement vector $w_\varepsilon$, the magnetic field intensity $v_\varepsilon$, and the magnetic displacement vector $z_\varepsilon$ weakly converge in $L^2$ to the corresponding homogenized fields $u_0$, $w_0$, $v_0$, $z_0$ (the solutions of the homogenized Maxwell system with effective coefficients). We improve the classical results. It is shown that $v_\varepsilon$ and $z_\varepsilon$ converge to $v_0$ and $z_0$, respectively, in the $L^2$-norm, the error terms do not exceed $C\varepsilon \|r\|_{L^2}$. We also find approximations for $v_\varepsilon$ and $z_\varepsilon$ in the energy norm with error $C\sqrt{\varepsilon} \|r\|_{L^2}$. For $u_\varepsilon$ and $w_\varepsilon$ we obtain approximations in the $L^2$-norm with error $C\sqrt{\varepsilon} \|r\|_{L^2}$.

Introduction

The paper concerns homogenization of periodic differential operators. The literature on homogenization is very extensive; we mention the books [BeLPap, BaPa, Sa, ZhKO].

0.1. Operator error estimates. Let $\Gamma \subset \mathbb{R}^d$ be a lattice. For $\Gamma$-periodic functions in $\mathbb{R}^d$ we denote

$f^\varepsilon(x) := f(x/\varepsilon), \quad \varepsilon > 0.$

In a series of papers [BSu1, BSu2, BSu3] by Birman and Suslina, an operator-theoretic approach to homogenization theory was suggested and developed. This approach was applied to a wide class of matrix strongly elliptic second order operators $A_\varepsilon$ acting in $L^2(\mathbb{R}^d; C^n)$ and admitting a factorization of the form

$A_\varepsilon = b(D)^*g^\varepsilon(x)b(D). \quad (0.1)$

Here the matrix-valued function $g(x)$ is bounded, positive definite, and periodic with respect to the lattice $\Gamma$. Next, $b(D)$ is the matrix first order operator of the form $b(D) = \sum_{j=1}^d b_j D_j$ such that its symbol has maximal rank. The simplest example of the operator (0.1) is the scalar elliptic operator

$A_\varepsilon = -\text{div } g^\varepsilon(x)\nabla = D^*g^\varepsilon(x)D$ (the acoustics operator).

The elasticity operator also can be written in the form (0.1). In electrodynamics, the auxiliary operator $A_\varepsilon = \text{curl } a^\varepsilon(x)\text{curl} - \nabla \nu^\varepsilon(x)\text{div}$ arises, it can be represented in the form (0.1).

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In [BSu1], it was shown that the resolvent \((A_\varepsilon + I)^{-1}\) converges in the operator norm in \(L_2(\mathbb{R}^d; \mathbb{C}^n)\) to the resolvent of the effective operator \(A^0 = b(D)^* g^0 b(D)\), as \(\varepsilon \to 0\). Here \(g^0\) is a constant positive matrix called the effective matrix. We have
\[
\| (A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.2}
\]

In [BSu3], approximation of the resolvent \((A_\varepsilon + I)^{-1}\) in the operators acting from \(L_2(\mathbb{R}^d; \mathbb{C}^n)\) to the Sobolev space \(H^1(\mathbb{R}^d; \mathbb{C}^n)\) was found:
\[
\| (A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.3}
\]
Here \(K(\varepsilon)\) is the so-called corrector. It involves a rapidly oscillating factor, and so depends on \(\varepsilon\); herewith,
\[
\| K(\varepsilon) \|_{L_2 \to H^1} = O(\varepsilon^{-1}).
\]

Estimates (0.2) and (0.3) are order-sharp. The results of such type are called operator error estimates in homogenization theory. A different approach to operator error estimates (the modified method of the first order approximation or the shift method) was suggested by Zhikov. In [Zh, ZhPas1], estimates (0.2) and (0.3) for the acoustic and elasticity operators were obtained by this method. Further results are discussed in the survey [ZhPas2].

Operator error estimates were also studied for boundary value problems in a bounded domain \(O \subset \mathbb{R}^d\) with sufficiently smooth boundary; see [ZhPas1, ZhPas2, Gr1, Gr2, KeLiS, PSu, Su3, Su4, Su5]. Let \(A_{D,\varepsilon}\) and \(A_{N,\varepsilon}\) be the operators in \(L_2(O; \mathbb{C}^n)\) given by the expression
\[
b(D)^* g^\varepsilon(x) b(D)
\]
with the Dirichlet or Neumann conditions on the boundary. Let \(A^0_D\) and \(A^0_N\) be the corresponding effective operators. We have
\[
\| (A_{D,\varepsilon} + I)^{-1} - (A^0_D + I)^{-1} \|_{L_2(O) \to L_2(O)} \leq C\varepsilon, \tag{0.4}
\]
\[
\| (A_{N,\varepsilon} + I)^{-1} - (A^0_N + I)^{-1} - \varepsilon K_\varepsilon(\varepsilon) \|_{L_2(O) \to H^1(O)} \leq C\varepsilon^{1/2}. \tag{0.5}
\]
Here \(b = D, N\), and \(K_\varepsilon(\varepsilon)\) is the corresponding corrector. Estimate (0.4) is of sharp order \(O(\varepsilon)\) (as for the problem in \(\mathbb{R}^d\)). The order of estimate (0.5) is worse than the order of (0.3); this is caused by the boundary influence.

In [ZhPas1], by the shift method, estimate (0.5) and the analog of estimate (0.4) with error \(O(\sqrt{\varepsilon})\) were obtained for the acoustic and elasticity operators. Independently, similar results for the acoustic operator were obtained by Griso [Gr1, Gr2] with the help of the unfolding method. For the first time, sharp-order estimate (0.4) was proved in [Gr2]. The case of matrix elliptic operators was studied in [KeLiS] (where uniformly elliptic operators were considered under some regularity assumptions on the coefficients) and in [PSu, Su3, Su4, Su5] (where estimates (0.4) and (0.5) were obtained for the class of strongly elliptic operators described above).

0.2. Homogenization of the Maxwell system in \(\mathbb{R}^3\). Now, we discuss the homogenization problem for the stationary Maxwell system in \(\mathbb{R}^3\).

Suppose that the dielectric permittivity and the magnetic permeability are given by the matrix-valued functions \(\eta^\varepsilon(x)\) and \(\mu^\varepsilon(x)\), where \(\eta(x)\) and \(\mu(x)\) are bounded, positive definite, and periodic with respect to some lattice \(\Gamma\). By \(J(\mathbb{R}^3)\) we denote the subspace of vector-valued functions \(f \in L_2(\mathbb{R}^3; \mathbb{C}^d)\) such that \(\text{div } f = 0\) (in the sense of distributions). Let \(u_\varepsilon\) and \(v_\varepsilon\) be the intensities of the electric and magnetic fields: \(w_\varepsilon = \eta^\varepsilon u_\varepsilon\) and \(z_\varepsilon = \mu^\varepsilon v_\varepsilon\) are the electric and magnetic displacement vectors. We write the Maxwell operator \(M_\varepsilon\) in terms of the displacement vectors, assuming that \(w_\varepsilon\) and \(z_\varepsilon\) are divergence-free. Then the operator \(M_\varepsilon\) acts in the space \(J(\mathbb{R}^3) \oplus J(\mathbb{R}^3)\) and is given by the expression
\[
M_\varepsilon = \begin{pmatrix} 0 & i\text{curl}(\mu^\varepsilon)^{-1} \\ -i\text{curl}(\eta^\varepsilon)^{-1} & 0 \end{pmatrix}
\]
on the natural domain. The operator \(M_\varepsilon\) is selfadjoint, if \(J(\mathbb{R}^3) \oplus J(\mathbb{R}^3)\) is considered as a subspace of the weighted space \(L_2(\mathbb{R}^3; \mathbb{C}^3; (\eta^\varepsilon)^{-1}) \oplus L_2(\mathbb{R}^3; \mathbb{C}^3; (\mu^\varepsilon)^{-1})\). The point \(\lambda = i\) is a regular point for the operator \(M_\varepsilon\).
Let us discuss the question about the behavior of the resolvent \((\mathcal{M}_\varepsilon - iI)^{-1}\) for small \(\varepsilon\). In other words, we are interested in the behavior of the solutions \((w_\varepsilon, z_\varepsilon)\) of the Maxwell system

\[
(\mathcal{M}_\varepsilon - iI) \begin{pmatrix} w_\varepsilon \\ z_\varepsilon \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad q, r \in J(\mathbb{R}^3; \mathbb{C}^3),
\]

(0.6)

and also in the behavior of the fields \(u_\varepsilon = (\eta^\varepsilon)^{-1}w_\varepsilon\) and \(v_\varepsilon = (\mu^\varepsilon)^{-1}z_\varepsilon\).

The homogenized Maxwell operator \(\mathcal{M}_0\) has the coefficients \(\eta^0\) and \(\mu^0\); it is well known that the effective matrices \(\eta^0\) and \(\mu^0\) are the same as for the scalar elliptic operators \(-\text{div}\eta^0\nabla\) and \(-\text{div}\mu^0\nabla\). Let \((w_0, z_0)\) be the solution of the homogenized Maxwell system

\[
(\mathcal{M}_0 - iI) \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}.
\]

Let \(u_0 = (\eta^0)^{-1}w_0\) and \(v_0 = (\mu^0)^{-1}z_0\). From the classical results (see, e. g., [BeLPap, Sa, ZhKO]) it is known that, as \(\varepsilon \to 0\), the vector-valued functions \(u_\varepsilon, w_\varepsilon, v_\varepsilon, z_\varepsilon\) weakly converge in \(L_2(\mathbb{R}^3; \mathbb{C}^3)\) to the corresponding homogenized fields \(u_0, w_0, v_0, z_0\).

Operator error estimates for the Maxwell system (0.6) were studied in [BSu1, Chapter 7], [BSu2, §14], [BSu3, §22], [Su1, BSu4], and [Su2]. In [BSu1, BSu2, BSu3], the case where \(\mu = 1\) was considered and approximations were found not for all physical fields; in [Su1], the general case was considered, but approximations were found not for all fields; in [BSu4], the problem was solved completely in the case of constant magnetic permeability; finally, in [Su2], a complete solution was achieved in the general case. The method was to reduce the problem to homogenization of some auxiliary second order equation. The solution of system (0.6) can be written as \(w_\varepsilon = w_\varepsilon^{(1)} + w_\varepsilon^{(2)}, z_\varepsilon = z_\varepsilon^{(1)} + z_\varepsilon^{(2)}\), where \((w_\varepsilon^{(1)}, z_\varepsilon^{(1)})\) is the solution of the system with \(r = 0\), and \((w_\varepsilon^{(2)}, z_\varepsilon^{(2)})\) is the solution of the system with \(q = 0\). For instance, let us consider \((w_\varepsilon^{(2)}, z_\varepsilon^{(2)})\). We substitute the first equation \(w_\varepsilon^{(2)} = \text{curl}(\mu^\varepsilon)^{-1}z_\varepsilon^{(2)}\) in the second one and arrive at the following problem for \(z_\varepsilon^{(2)}\):

\[
\text{curl}(\eta^\varepsilon)^{-1}\text{curl}(\mu^\varepsilon)^{-1}z_\varepsilon^{(2)} + z_\varepsilon^{(2)} = i\varepsilon^\varepsilon, \quad \text{div}z_\varepsilon^{(2)} = 0.
\]

Substituting \(\varphi_\varepsilon^{(2)} = (\mu^\varepsilon)^{-1/2}z_\varepsilon^{(2)}\) and lifting the divergence-free condition, we see that \(\varphi_\varepsilon^{(2)}\) is the solution of the second order elliptic equation

\[
\mathcal{L}_\varepsilon\varphi_\varepsilon^{(2)} + \varphi_\varepsilon^{(2)} = i(\mu^\varepsilon)^{-1/2}r,
\]

(0.7)

where

\[
\mathcal{L}_\varepsilon = (\mu^\varepsilon)^{-1/2}\text{curl}(\eta^\varepsilon)^{-1}\text{curl}(\mu^\varepsilon)^{-1/2} - (\mu^\varepsilon)^{1/2}\nabla\text{div}(\mu^\varepsilon)^{1/2}.
\]

(0.8)

The field \(w_\varepsilon^{(2)}\) is expressed in terms of the derivatives of the solution:

\[
w_\varepsilon^{(2)} = \text{curl}(\mu^\varepsilon)^{-1/2}\varphi_\varepsilon^{(2)}.
\]

In the case of constant \(\mu\), the operator (0.8) belongs to the class of operators (0.1), which allows one to apply general results of the papers [BSu1, BSu2, BSu3] to equation (0.7). If \(\mu\) is variable, this is not the case, but it is possible to use the abstract scheme from [BSu1, BSu2, BSu3] to study the operator (0.8); this was done in [Su1, Su2]. The result of these considerations was approximation of the resolvent \((\mathcal{M}_\varepsilon - iI)^{-1}\). In contrast to the resolvent of the operator (0.1), this resolvent has no limit in the operator norm, but it can be approximated by the sum of the resolvent \((\mathcal{M}_0 - iI)^{-1}\) and some corrector of zero order (which weakly tends to zero); the error estimate is of sharp order \(O(\varepsilon)\). In terms of the solutions, this implies approximations for all physical fields in the \(L_2(\mathbb{R}^3; \mathbb{C}^3)\)-norm with error estimates of order \(O(\varepsilon)\). For instance, we write down the result for \(u_\varepsilon\):

\[
\|u_\varepsilon - u_0 - u_\varepsilon^{(1)}\|_{L_2(\mathbb{R}^3)} \leq C\varepsilon(\|q\|_{L_2(\mathbb{R}^3)} + \|r\|_{L_2(\mathbb{R}^3)}).
\]

Here \(u_\varepsilon^{(1)}\) is interpreted as the zero order corrector; it is expressed in terms of \(u_0\), the solution of some “correction” Maxwell system, and some rapidly oscillating factor. The weak limit of \(u_\varepsilon^{(1)}\) is equal to zero.
0.3. Statement of the problem. Main results. In the present paper, we study homogenization of the stationary Maxwell system in a bounded domain \( \Omega \subset \mathbb{R}^3 \) of class \( C^{1,1} \). We rely on the general theory of the Maxwell operator in arbitrary domains developed in the papers [BS1, BS2] by Birman and Solomyak.

Suppose that the magnetic permeability is given by the constant positive matrix \( \mu_0 \), and the dielectric permittivity is given by the oscillating matrix \( \eta^r(x) \). The boundary conditions of perfect conductivity are imposed. The notation for the physical fields is the same as above in Subsection 0.2. The Maxwell operator \( M_\varepsilon \), written in terms of the displacement vectors, acts in the space \( H(\Omega) \oplus J_0(\Omega) \). Here \( J(\Omega) \) and \( J_0(\Omega) \) are the divergence-free subspaces of \( L_2(\Omega; C^3) \) defined below in (5.1), (5.2). The operator \( M_\varepsilon \) is given by

\[
M_\varepsilon = \begin{pmatrix}
0 & i\text{curl}\mu_0^{-1} \\
-\text{i curl}(\eta^r)^{-1} & 0
\end{pmatrix}
\]

on the natural domain with the boundary conditions taken into account (see (5.4) below). The operator \( M_\varepsilon \) is selfadjoint, if \( J(\Omega) \oplus J_0(\Omega) \) is considered as a subspace of the weighted space \( L_2(\Omega; C^3; (\eta^r)^{-1}) \oplus L_2(\Omega; C^3; \mu_0^{-1}) \).

We study the resolvent \( (M_\varepsilon - iI)^{-1} \). In other words, we are interested in the behavior of the solutions \( (w_\varepsilon, z_\varepsilon) \) of the Maxwell system

\[
(M_\varepsilon - iI) \begin{pmatrix} w_\varepsilon \\ z_\varepsilon \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad q \in J(\Omega), \ r \in J_0(\Omega),
\]

and also in the behavior of the fields \( \varepsilon_\varepsilon = (\eta^r)^{-1}w_\varepsilon \) and \( \zeta_\varepsilon = \mu_0^{-1}z_\varepsilon \).

Let \( M^0 \) be the homogenized Maxwell operator with the coefficients \( \eta^0 \) and \( \mu_0 \). The homogenized Maxwell system is of the form

\[
(M^0 - iI) \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}.
\]

We put \( u_0 = (\eta^0)^{-1}w_0 \) and \( v_0 = \mu_0^{-1}z_0 \). As for the problem in \( \mathbb{R}^3 \), the classical results (see [BeLPap, Sa, ZhKO]) give weak convergence in \( L_2(\Omega; C^3) \) of the vector-valued functions \( u_\varepsilon, w_\varepsilon, v_\varepsilon, z_\varepsilon \) to the corresponding homogenized fields \( u_0, w_0, v_0, z_0 \).

We improve the classical results in the case where \( q = 0 \). Let us describe our main results. If \( q = 0 \), the fields \( v_\varepsilon \) and \( z_\varepsilon \) converge in the \( L_2(\Omega; C^3) \)-norm to \( v_0 \) and \( z_0 \). The following sharp-order estimates hold:

\[
\| v_\varepsilon - v_0 \|_{L_2(\Omega)} \leq C\varepsilon \| r \|_{L_2(\Omega)},
\]

\[
\| z_\varepsilon - z_0 \|_{L_2(\Omega)} \leq C\varepsilon \| r \|_{L_2(\Omega)}.
\]

In addition, we find approximations for \( v_\varepsilon \) and \( z_\varepsilon \) in the \( H^1(\Omega; C^3) \)-norm:

\[
\| v_\varepsilon - v_0 - \varepsilon v^{(1)}_\varepsilon \|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \| r \|_{L_2(\Omega)},
\]

\[
\| z_\varepsilon - z_0 - \varepsilon z^{(1)}_\varepsilon \|_{H^1(\Omega)} \leq C\varepsilon^{1/2} \| r \|_{L_2(\Omega)}.
\]

Here the correctors \( v^{(1)}_\varepsilon \) and \( z^{(1)}_\varepsilon \) involve rapidly oscillating factors, their norms in \( H^1(\Omega; C^3) \) are of order \( O(\varepsilon^{-1}) \). Finally, we obtain approximations for \( u_\varepsilon \) and \( w_\varepsilon \) in the \( L_2(\Omega; C^3) \)-norm:

\[
\| u_\varepsilon - u_0 - u^{(1)}_\varepsilon \|_{L_2(\Omega)} \leq C\varepsilon^{1/2} \| r \|_{L_2(\Omega)},
\]

\[
\| w_\varepsilon - w_0 - w^{(1)}_\varepsilon \|_{L_2(\Omega)} \leq C\varepsilon^{1/2} \| r \|_{L_2(\Omega)}.
\]

The correction terms \( u^{(1)}_\varepsilon \) and \( w^{(1)}_\varepsilon \) can be interpreted as correctors of zero order, they weakly tend to zero.

The case of system (0.9) with \( r = 0 \) is more difficult and is not considered in the present paper.
0.4. The method. As for the problem in \( \mathbb{R}^3 \), the method is based on reduction to the study of some second order operator \( L_\varepsilon \). First, we study this operator, and next we derive the results for the Maxwell system.

The operator \( L_\varepsilon \) acts in \( L_2(\mathcal{O}; \mathbb{C}^2) \) and is formally given by

\[
L_\varepsilon = \mu_0^{-1/2} \text{curl}(\eta^\varepsilon(x))^{-1} \text{curl} \mu_0^{-1/2} - \mu_0^{-1/2} \nabla \nu^\varepsilon(x) \text{div} \mu_0^{1/2}
\]  

(0.10)

with the boundary conditions

\[
(\mu_0^{1/2} \varphi)_n|_{\partial \mathcal{O}} = 0, \quad ((\eta^\varepsilon(x))^{-1} \text{curl} (\mu_0^{-1/2} \varphi))_n|_{\partial \mathcal{O}} = 0.
\]  

(0.11)

The precise definition of the operator \( L_\varepsilon \) is given in terms of the quadratic form. For application to the Maxwell system, we can put \( \nu(x) = 1 \), but for generality we study the operator (0.10) with variable coefficient \( \nu^\varepsilon(x) \). The operator \( L_\varepsilon \) can be written in a factorized form \( b(D)^* g^\varepsilon(x) b(D) \), but the direct reference to the results of [PSu, Su3, Su4] is impossible, since in those papers the cases of the Dirichlet or Neumann boundary conditions were studied, and in the present case the boundary conditions (0.11) are of mixed type. Therefore, we need to prove analogs of estimates (0.4) and (0.5) for the resolvent of the operator \( L_\varepsilon \).

The method of proving such estimates is based on consideration of the associated problem in \( \mathbb{R}^3 \) and using the results for this problem, introduction of the boundary layer correction term \( s_\varepsilon \), and a careful analysis of this term. A crucial role is played by using the Steklov smoothing operator (initially borrowed from [Zh, ZhPas1]), estimates in the \( \varepsilon \)-neighborhood of the boundary, and the duality arguments.

0.5. The plan of the paper. The paper consists of five sections. In Section 1, the model second order operator \( L_\varepsilon \) in \( L_2(\mathbb{R}^3; \mathbb{C}^2) \) is considered; the effective operator is constructed, and the known results about approximation of the resolvent \( (L_\varepsilon + I)^{-1} \) are formulated. In Section 2, the model operator \( L_\varepsilon \) in \( L_2(\mathcal{O}; \mathbb{C}^2) \) is introduced, the effective operator is described, and some auxiliary statements (about estimates in the \( \varepsilon \)-neighborhood of the boundary) are given. In Section 3, we formulate main results about approximation of the resolvent \( (L_\varepsilon + I)^{-1} \) and give the first two steps of the proofs: the associated problem in \( \mathbb{R}^3 \) is considered, the boundary layer correction term \( s_\varepsilon \) is introduced, and the proof of main theorems is reduced to estimation of \( s_\varepsilon \) in \( H^1(\mathcal{O}; \mathbb{C}^2) \) and in \( L_2(\mathcal{O}; \mathbb{C}^2) \). In Section 4, we obtain the required estimates for the norms of the correction term \( s_\varepsilon \) and complete the proof of theorems from Section 3. Section 5 is devoted to homogenization of the stationary Maxwell system with \( q = 0 \). We reduce the problem to the question about the behavior of the resolvent of \( L_\varepsilon \). The final result on approximation for the solutions of the Maxwell system (Theorem 5.3) is obtained.

0.6. Notation. Let \( \mathfrak{H} \) and \( \mathfrak{H}_* \) be complex separable Hilbert spaces. The symbols \( (\cdot, \cdot)_\mathfrak{H} \) and \( \| \cdot \|_\mathfrak{H} \) stand for the inner product and the norm in \( \mathfrak{H} \); the symbol \( \| \cdot \|_{\mathfrak{H}_0 \to \mathfrak{H}_*} \) denotes the norm of a linear continuous operator acting from \( \mathfrak{H}_0 \) to \( \mathfrak{H}_* \).

The symbols \( (\cdot, \cdot) \) and \( | \cdot | \) stand for the inner product and the norm in \( \mathbb{C}^n \); \( 1 = 1_n \) is the identity \((n \times n)\)-matrix. If \( a \) is an \((n \times n)\)-matrix, then \( |a| \) denotes the norm of \( a \) as a linear operator in \( \mathbb{C}^n \). We denote \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), \( iD_j = \partial_j = \partial/\partial x_j \), \( j = 1, 2, 3 \), \( \mathbf{D} = -i\nabla = (D_1, D_2, D_3) \). The classes \( L_p \) of \( \mathbb{C}^n \)-valued functions in a domain \( \mathcal{O} \subset \mathbb{R}^3 \) are denoted by \( L_p(\mathcal{O}; \mathbb{C}^n) \), \( 1 \leq p \leq \infty \). The Sobolev spaces of \( \mathbb{C}^n \)-valued functions in a domain \( \mathcal{O} \) are denoted by \( H^s(\mathcal{O}; \mathbb{C}^n) \). Next, \( H^s_0(\mathcal{O}; \mathbb{C}^n) \) is the closure of \( C_0^\infty(\mathcal{O}; \mathbb{C}^n) \) in \( H^s(\mathcal{O}; \mathbb{C}^n) \). If \( n = 1 \), we write simply \( L_p(\mathcal{O}), H^s(\mathcal{O}) \), etc., but sometimes we use such simple notation for the spaces of vector-valued or matrix-valued functions. Various constants in estimates are denoted by \( c, C, \tilde{C}, \tilde{c}, \mathfrak{C} \) (possibly, with indices and marks).

0.7. The author plans to devote a separate paper to more general problem about homogenization of the stationary Maxwell system in a bounded domain in the case where both coefficients are given by the rapidly oscillating periodic matrix-valued functions. Problem (0.9) with \( r = 0 \) (which is not considered in the present paper) will be a particular case of this more general problem.
1. The model second order operator in \( \mathbb{R}^3 \)

1.1. Lattice. Let \( \Gamma \subset \mathbb{R}^3 \) be a lattice generated by the basis \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \), i. e.,

\[
\Gamma = \{ \mathbf{a} \in \mathbb{R}^3 : \mathbf{a} = z_1 \mathbf{a}_1 + z_2 \mathbf{a}_2 + z_3 \mathbf{a}_3, \ z_j \in \mathbb{Z} \}.
\]

By \( \Omega \subset \mathbb{R}^3 \) we denote the elementary cell of the lattice \( \Gamma \):

\[
\Omega = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t_1 \mathbf{a}_1 + t_2 \mathbf{a}_2 + t_3 \mathbf{a}_3, \ -1/2 < t_j < 1/2 \}.
\]

For \( \Gamma \)-periodic functions \( f(\mathbf{x}) \) in \( \mathbb{R}^3 \), we use the notation \( f^\varepsilon(\mathbf{x}) := f(\mathbf{x}/\varepsilon) \), where \( \varepsilon > 0 \). For periodic square matrix-valued functions \( f(\mathbf{x}) \), we denote

\[
\mathbf{T} := |\Omega|^{-1} \int_{\Omega} f(\mathbf{x}) \, d\mathbf{x}, \quad \mathbf{f} := \left( |\Omega|^{-1} \int_{\Omega} f(\mathbf{x}) \, d\mathbf{x} \right)^{-1}.
\]

Here, in the definition of \( \mathbf{T} \) it is assumed that \( f \in L_{1,\text{loc}}(\mathbb{R}^3) \), and in the definition of \( \mathbf{f} \) it is assumed that the matrix \( f(\mathbf{x}) \) is non-degenerate and \( f^{-1} \in L_{1,\text{loc}}(\mathbb{R}^3) \).

By \( H^1(\Omega; \mathbb{C}^n) \) we denote the subspace of functions in \( H^1(\Omega; \mathbb{C}^n) \), whose \( \Gamma \)-periodic extension to \( \mathbb{R}^3 \) belongs to \( H^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C}^n) \).

1.2. The Steklov smoothing. The operator \( S_{\varepsilon}^{(k)} \), \( \varepsilon > 0 \), acting in \( L_2(\mathbb{R}^3; \mathbb{C}^k) \) (where \( k \in \mathbb{N} \)) and given by

\[
(S_{\varepsilon}^{(k)})_\varepsilon(\mathbf{u})(\mathbf{x}) = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x} - \varepsilon \mathbf{z}) \, d\mathbf{z}, \quad \mathbf{u} \in L_2(\mathbb{R}^3; \mathbb{C}^k),
\]

is called the Steklov smoothing operator. We omit the index \( k \) and write simply \( S_{\varepsilon} \). Obviously, \( S_{\varepsilon} D^\alpha \mathbf{u} = D^\alpha S_{\varepsilon} \mathbf{u} \) for \( \mathbf{u} \in H^\sigma(\mathbb{R}^3; \mathbb{C}^k) \) and any multiindex \( \alpha \) such that \( |\alpha| \leq \sigma \). Note that

\[
\|S_{\varepsilon}\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leq 1. \tag{1.2}
\]

We need the following properties of the operator \( S_{\varepsilon} \) (see [ZhPas1, Lemmas 1.1 and 1.2] or [PSu, Propositions 3.1 and 3.2]).

**Proposition 1.1.** For any function \( \mathbf{u} \in H^1(\mathbb{R}^3; \mathbb{C}^k) \), we have

\[
\|S_{\varepsilon} \mathbf{u} - \mathbf{u}\|_{L_2(\mathbb{R}^3)} \leq \varepsilon r_1 \|D\mathbf{u}\|_{L_2(\mathbb{R}^3)},
\]

where \( 2r_1 = \text{diam } \Omega \).

**Proposition 1.2.** Let \( f \) be a \( \Gamma \)-periodic function in \( \mathbb{R}^3 \) such that \( f \in L_2(\Omega) \). Let \( [f^\varepsilon] \) be the operator of multiplication by the function \( f^\varepsilon(\mathbf{x}) \). Then the operator \( [f^\varepsilon] S_{\varepsilon} \) is continuous in \( L_2(\mathbb{R}^3) \) and

\[
\|[f^\varepsilon] S_{\varepsilon}\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leq |\Omega|^{-1/2} \|f\|_{L_2(\Omega)}.
\]

1.3. Definition of the operator \( L_{\varepsilon} \). Suppose that \( \mu_0 \) is a symmetric positive \( (3 \times 3) \)-matrix with real entries. Suppose that a symmetric \((3 \times 3)\)-matrix-valued function \( \eta(\mathbf{x}) \) with real entries and a real-valued function \( \nu(\mathbf{x}) \) are periodic with respect to the lattice \( \Gamma \) and such that

\[
\eta, \eta^{-1} \in L_\infty, \quad \eta(\mathbf{x}) > 0; \quad \nu, \nu^{-1} \in L_\infty, \quad \nu(\mathbf{x}) > 0. \tag{1.3}
\]

In \( L_2(\mathbb{R}^3; \mathbb{C}^3) \), we consider the operator \( L_{\varepsilon} \) given formally by the differential expression

\[
L_{\varepsilon} = \mu_0^{-1/2} \text{curl} (\eta^\varepsilon(\mathbf{x}))^{-1} \text{curl} \mu_0^{-1/2} - \mu_0^{1/2} \nabla \nu^\varepsilon(\mathbf{x}) \text{div} \mu_0^{1/2}. \tag{1.4}
\]

The operator \( L_{\varepsilon} \) belongs to the class of operators admitting a factorization of the form (0.1), i. e., \( L_{\varepsilon} = b(D) g^\varepsilon(\mathbf{x}) b(D) \). This class was studied in the papers [BSu1, BSu2, BSu3]. In our case,
$g(x)$ is the $(4 \times 4)$-matrix-valued function, and $b(D)$ is the $(4 \times 3)$-matrix first order differential operator. Namely,

$$b(D) = \begin{pmatrix}
-ic\text{curl} \mu_0^{1/2} \\
-ic\text{div} \mu_0^{1/2}
\end{pmatrix}, \quad g(x) = \begin{pmatrix}
\eta(x)^{-1} & 0 \\
0 & \nu(x)
\end{pmatrix}. \quad (1.5)
$$

From (1.3) it follows that the matrix $g(x)$ is positive definite and bounded. Obviously,

$$\|g\|_{L_\infty} = \max \{ \|\eta^{-1}\|_{L_\infty} : \|\nu\|_{L_\infty} \}, \quad \|g^{-1}\|_{L_\infty} = \max \{ \|\eta\|_{L_{\infty}} : \|\nu^{-1}\|_{L_{\infty}} \}.
$$

The operator $b(D)$ can be written as $b(D) = \sum_{j=1}^{3} b_j D_j$, where $b_j$ are constant matrices. The symbol $b(\xi) = \sum_{j=1}^{3} b_j \xi_j$ of the operator $b(D)$ is given by

$$b(\xi) = \begin{pmatrix}
\xi_2 \\
\xi_3 \\
\xi_1 \\
\xi_0
\end{pmatrix}, \quad r(\xi) = \begin{pmatrix}
0 & -\xi_3 & \xi_2 \\
\xi_3 & 0 & -\xi_1 \\
-\xi_2 & \xi_1 & 0
\end{pmatrix}, \quad \xi' = (\xi_1 \xi_2 \xi_3).
$$

We have

$$\text{rank } b(\xi) = 3, \quad 0 \neq \xi \in \mathbb{R}^3. \quad (1.6)
$$

This condition is equivalent to the estimates

$$\alpha_0 1_3 \leqslant b(\xi)^* b(\xi) \leqslant \alpha_1 1_3, \quad |\xi| = 1, \quad (1.7)
$$

with positive constants $\alpha_0$ and $\alpha_1$. It is easy to check these estimates with the constants

$$\alpha_0 = \min \{ |\mu_0|^{-1} : |\mu_0^{-1}|^{-1} \}, \quad \alpha_1 = |\mu_0| + |\mu_0^{-1}|.
$$

The precise definition of the operator $L_\varepsilon$ is given in terms of the quadratic form

$$L_\varepsilon[\varphi, \varphi] := \int_{\mathbb{R}^3} \langle g^\varepsilon(x) b(D) \varphi, b(D) \varphi \rangle \, dx
$$

$$= \int_{\mathbb{R}^3} \left( \langle \tilde{\eta}^\varepsilon(x) \rangle^{-1} \text{curl}(\mu_0^{-1/2} \varphi), \text{curl}(\mu_0^{-1/2} \varphi) \rangle + \nu^\varepsilon(x) \text{div}(\mu_0^{1/2} \varphi) \rangle^2 \right) \, dx, \quad \varphi \in H^1(\mathbb{R}^3; \mathbb{C}^3).
$$

Under our assumptions, the following two-sided estimates hold:

$$c_1 \|D\varphi\|_{L_2(\mathbb{R}^3)}^2 \leqslant L_\varepsilon[\varphi, \varphi] \leqslant c_2 \|D\varphi\|_{L_2(\mathbb{R}^3)}^2, \quad \varphi \in H^1(\mathbb{R}^3; \mathbb{C}^3),
$$

$$c_1 = \alpha_0 \|g^{-1}\|_{L_\infty}^{-1}, \quad c_2 = \alpha_1 \|g\|_{L_\infty}. \quad (1.8)
$$

Thus, the form $L_\varepsilon$ is closed and nonnegative. The selfadjoint operator in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ generated by this form is denoted by $L_\varepsilon$.

1.4. The effective operator $L^0$. According to the general rules, we define the effective operator

$$L^0 = b(D)^* g^0 b(D), \quad (1.9)
$$

where $g^0$ is a constant positive matrix called the effective matrix. It is defined in terms of the solution of the auxiliary problem on the cell $\Omega$. Let $\Lambda(x)$ be a $(3 \times 4)$-matrix-valued function which is a $\Gamma$-periodic solution of the problem

$$b(D)^* g(x)(b(D) \Lambda(x) + 1) = 0, \quad \int_{\Omega} \Lambda(x) \, dx = 0. \quad (1.10)
$$

Then

$$g^0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(x) \, dx, \quad \tilde{g}(x) := g(x)(b(D) \Lambda(x) + 1). \quad (1.11)
$$

It is easy to check that

$$|\Lambda|_{H^1(\Omega)}^4 \leqslant C_L |\Omega|^{1/2}, \quad (1.12)
$$

where the constant $C_L$ depends only on $|\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_{\infty}}, \|\eta^{-1}\|_{L_{\infty}}, \|\nu\|_{L_{\infty}}, \|\nu^{-1}\|_{L_{\infty}}$, and the parameters of the lattice $\Gamma$. 
In other words, \( v \) is the solution of the equation

\[
\mu_0^{1/2} \text{curl} (\eta(x)^{-1} \left( \text{curl} (\mu_0^{-1/2} v) + i\tilde{C} \right)) = \mu_0^{1/2} \nabla_{\Omega} (\mu_0^{1/2} v) + iC_4 = 0.
\]

Integrating (1.15) and using (1.17), we find

\[
\Phi(\Omega) = \text{curl} \left( \int \nabla_{\Omega} (\mu_0^{1/2} v) + iC_4 \right) d\Omega = \nabla_{\Omega} (\mu_0^{1/2} v) + iC_4 = 0.
\]

Thus, \( \Phi(\Omega) \in \mathbb{C} \) with some \( \Phi(\Omega) \). From (1.14) it follows that

\[
\left( \eta(x) \right)^{-1} \left( \text{curl} (\mu_0^{-1/2} v(x)) + i\tilde{C} \right) = i(\nabla \Phi(x) + c)
\]

with some \( \Phi \in \tilde{H}^1(\Omega) \) and \( c \in \mathbb{C}^3 \). Hence,

\[
\text{curl} (\mu_0^{-1/2} v(x)) + i\tilde{C} = i\eta(x)(\nabla \Phi(x) + c).
\]

By (1.15),

\[
\int_\Omega \langle \eta(x)(\nabla \Phi(x) + c), \nabla F(x) \rangle d\Omega = 0, \quad F \in \tilde{H}^1(\Omega).
\]

Thus, \( \Phi \in \tilde{H}^1(\Omega) \) is the solution of the equation

\[
\text{div} \eta(x)(\nabla \Phi(x) + c) = 0.
\]

Recalling the definition of the effective matrix \( \eta^0 \) for the operator \(-\text{div} \eta(x) \nabla\), we have

\[
\eta^0 c = |\Omega|^{-1} \int_\Omega \eta(x)(\nabla \Phi(x) + c) d\Omega.
\]

Integrating (1.15) and using (1.17), we find

\[
\tilde{C} = \eta^0 c.
\]

On the other hand, (1.13) and (1.14) imply that

\[
\int_\Omega \nu(x) \left( \text{div} (\mu_0^{1/2} v) + iC_4 \right) \text{div} (\mu_0^{1/2} z) d\Omega = 0, \quad z \in \tilde{H}^1(\Omega; \mathbb{C}^3).
\]

This means that there exists a constant \( \alpha \in \mathbb{C} \) such that

\[
\nu(x) \left( \text{div} (\mu_0^{1/2} v) + iC_4 \right) = i\alpha.
\]

Multiplying (1.19) by \( \nu(x)^{-1} \) and integrating, we obtain \( C_4 |\Omega| = \alpha \int_\Omega \nu(x)^{-1} d\Omega \). Hence,

\[
\alpha = \mu C_4.
\]
Hence,

\[
\Psi(x) = \text{constant} \frac{\sum(x)}{i\mu_0} = \text{constant} \frac{\sum(x)}{\sin(\omega_x t)}
\]

By (1.22) and (1.23),

\[
g(x) = \text{constant} \frac{\sum(x)}{i\mu_0} = \text{constant} \frac{\sum(x)}{\sin(\omega_x t)}
\]

Consequently, the effective operator (1.9) is given by the differential expression

\[
\mathcal{L}^0 = \mu_0^{-1/2} \text{curl} (\eta^0)^{-1} \text{curl} \mu_0^{-1/2} - \mu_0^{-1/2} \nabla \nu \text{div} \mu_0^{1/2}
\]

on the domain \( H^2(\mathbb{R}^3; \mathbb{C}^3) \).

The following estimates for the effective coefficients are well known:

\[
|\eta^0| \leq ||\eta||_{L_\infty}, \quad (|\eta^0|^{-1}) \leq ||\eta^{-1}||_{L_\infty},
\]

\[
|\nu| \leq ||\nu||_{L_\infty}, \quad (|\nu|^{-1}) \leq ||\nu^{-1}||_{L_\infty}.
\]

The symbol of the effective operator is given by

\[
a(\xi) = \mu_0^{-1/2} r(\xi)^i (\eta^0)^{-1} r(\xi) \mu_0^{-1/2} + \mu_0^{1/2} \xi \bar{\xi} \mu_0^{1/2}.
\]
Taking (1.28) into account, we see that the symbol \( a(\xi) \) satisfies
\[
c_1|\xi|^21 \leq a(\xi) \leq c_2|\xi|^21, \quad \xi \in \mathbb{R}^3.
\] (1.29)

Here the constants \( c_1 \) and \( c_2 \) are the same as in (1.8).

1.5. The properties of the effective matrix \( \eta^0 \). The properties of the functions \( \Phi_j \).

The effective matrix \( \eta^0 \) satisfies the estimates
\[
\underline{\eta} \leq \eta^0 \leq \bar{\eta},
\] (1.30)

known as the Voigt–Reuss bracketing. See, e.g., [BSu1, Chapter 3, Theorem 1.5]. We distinguish the cases where one of the inequalities in (1.30) becomes an identity; see, e.g., [BSu1, Chapter 3, Propositions 1.6 and 1.7].

**Proposition 1.3.** 1) The identity \( \eta^0 = \bar{\eta} \) is equivalent to the relations \( \text{div} \eta_j(x) = 0, \ j = 1, 2, 3 \), for the columns \( \eta_j(x) \) of the matrix \( \eta(x) \).

2) The identity \( \eta^0 = \underline{\eta} \) is equivalent to the following representations for the columns \( \kappa_j(x) \) of the matrix \( \eta(x)^{-1} \): \( \kappa_j(x) = c_j^0 + \nabla f_j(x), \ j = 1, 2, 3 \), with some \( c_j^0 \in \mathbb{C}^3 \) and \( f_j \in \bar{H}^1(\Omega) \).

**Remark 1.4.** 1) If \( \eta^0 = \bar{\eta} \), then \( \Phi_j(x) = 0, \ j = 1, 2, 3 \), and \( \Sigma(x) = 0 \). According to (1.25), in this case we have \( \tilde{g}(x) = \theta^0 \).

2) If \( \eta^0 = \underline{\eta} \), then \( \eta(x)(\nabla \Phi_j(x) + c_j) = \tilde{c}_j, \ j = 1, 2, 3 \); see [BSu2, Remark 3.5]. In this case, we have \( \nu_j(x) = 0, \ j = 1, 2, 3 \), i.e., \( \Psi(x) = 0 \). If, in addition, \( \nu(x) = \text{Const} \), then \( \nu_4(x) = 0 \).

Hence, in this case we have \( \Lambda(x) = 0 \).

In what follows, we will need some properties of the functions \( \Phi_j, \ j = 1, 2, 3 \).

**Remark 1.5.** The columns of the matrix \( \Sigma(x) \) are vector-functions \( \nabla \Phi_j(x), \ j = 1, 2, 3 \), where \( \Phi_j \) is the periodic solution of the problem
\[
\text{div} \eta(x)(\nabla \Phi_j(x) + c_j) = 0, \quad \int_{\Omega} \Phi_j(x) \, dx = 0,
\] (1.31)

with \( c_j = (\eta^0)^{-1} \tilde{c}_j \). According to [LaUr, Chapter 3, Theorem 13.1], the solution of this problem is bounded: \( \Phi_j \in L_{\infty} \), and the norm \( \|\Phi_j\|_{L_{\infty}} \) is controlled in terms of \( \|\eta\|_{L_{\infty}}, \|\eta^{-1}\|_{L_{\infty}} \), and the parameters of the lattice \( \Gamma \).

The following statement was checked in [PSu, Corollary 2.4].

**Proposition 1.6.** For any \( u \in H^1(\mathbb{R}^3) \), we have
\[
\int_{\mathbb{R}^3} |(\nabla \Phi_j)^\varepsilon|^2|u|^2 \, dx \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^3)} + \beta_2 \varepsilon^2 \|\Phi_j\|^2_{L_{\infty}} \|D u\|^2_{L_2(\mathbb{R}^3)},
\]

where the constants \( \beta_1 \) and \( \beta_2 \) depend only on \( \|\eta\|_{L_{\infty}} \) and \( \|\eta^{-1}\|_{L_{\infty}} \).

1.6. Approximation of the resolvent of the operator \( \mathcal{L}_0 \). Applying Theorem 2.1 from [BSu1, Chapter 4] to the operator (1.4), we obtain the following result.

**Theorem 1.7.** Let \( \mathcal{L}_0 \) be the operator (1.4). Suppose that the effective operator \( \mathcal{L}^0 \) is defined by (1.27). For \( \varepsilon > 0 \) we have
\[
\|(\mathcal{L}_0 + I)^{-1} - (\mathcal{L}^0 + I)^{-1}\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leq C_1 \varepsilon.
\]
The constant \( C_1 \) depends only on \( |\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_{\infty}}, \|\eta^{-1}\|_{L_{\infty}}, \|\nu\|_{L_{\infty}}, \|\nu^{-1}\|_{L_{\infty}}, \) and the parameters of the lattice \( \Gamma \).
Approximation of the resolvent in the norm of operators acting from $L_2(\mathbb{R}^3; \mathbb{C}^3)$ to the Sobolev space $H^1(\mathbb{R}^3; \mathbb{C}^3)$ was obtained in [BSu3, Theorem 10.6]; that approximation contained a corrector with the smoothing operator of different type than $S_\varepsilon$. In [PSu, Theorem 3.3], it was shown that this smoothing operator can be replaced by the Steklov smoothing operator. Let us formulate the result of [PSu] as applied to (1.4). We introduce a corrector

$$K_\varepsilon = \Lambda^\varepsilon S_\varepsilon b(D)(L^0 + I)^{-1}. \quad (1.32)$$

Here $S_\varepsilon$ is the Steklov smoothing operator defined by (1.1), and the matrix $\Lambda$ is the periodic solution of problem (1.10). The operator

$$b(D)(L^0 + I)^{-1}$$

is continuous from $L_2(\mathbb{R}^3; \mathbb{C}^3)$ to $H^1(\mathbb{R}^3; \mathbb{C}^4)$. By Proposition 1.2 and relation $\Lambda \in \tilde{H}^1(\Omega)$, the operator $\Lambda^\varepsilon S_\varepsilon$ is a continuous mapping of $H^1(\mathbb{R}^3; \mathbb{C}^4)$ to $H^1(\mathbb{R}^3; \mathbb{C}^3)$. Hence, the corrector (1.32) is continuous from $L_2(\mathbb{R}^3; \mathbb{C}^3)$ to $H^1(\mathbb{R}^3; \mathbb{C}^3)$. Taking (1.5) and (1.24) into account, we obtain

$$K_\varepsilon = \left( \mu_0^{-1/2} \Psi^\varepsilon S_\varepsilon \text{curl} \mu_0^{-1/2} + \mu_0^{1/2} (\nabla \rho)^\varepsilon S_\varepsilon \text{div} \mu_0^{1/2} \right) (L^0 + I)^{-1}. \quad (1.33)$$

**Theorem 1.8.** Suppose that the assumptions of Theorem 1.7 are satisfied. Suppose that the corrector $K_\varepsilon$ is defined by (1.33). Then for $\varepsilon > 0$ we have

$$\| (L_\varepsilon + I)^{-1} - (L^0 + I)^{-1} - \varepsilon K_\varepsilon \|_{L_2(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3)} \leq C_2 \varepsilon. \quad (1.34)$$

The constant $C_2$ depends only on $|\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_\infty}, \|\eta^{-1}\|_{L_\infty}, \|\nu\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

It is easy to deduce approximation for the “flux” $g^\varepsilon b(D)(L_\varepsilon + I)^{-1}$ from Theorem 1.8; see [Su4, Theorem 1.8]. We have

$$\| g^\varepsilon b(D)(L_\varepsilon + I)^{-1} - g^\varepsilon S_\varepsilon b(D)(L^0 + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)} \leq \tilde{C}_3 \varepsilon, \quad (1.35)$$

where $\tilde{C}_3$ depends only on $|\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_\infty}, \|\eta^{-1}\|_{L_\infty}, \|\nu\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$. By (1.5) and (1.25),

$$g^\varepsilon b(D)(L_\varepsilon + I)^{-1} = -i \left( (\eta^\varepsilon)^{-1} \text{curl} \mu_0^{-1/2}(L_\varepsilon + I)^{-1} \right), \quad (1.35)$$

$$\tilde{g}^\varepsilon S_\varepsilon b(D)(L^0 + I)^{-1} = -i \left( (\eta^0)^{-1} + \Sigma^\varepsilon S_\varepsilon \text{curl} \mu_0^{-1/2}(L_\varepsilon + I)^{-1} \right). \quad (1.36)$$

Let us show that in estimate (1.34) the operator $S_\varepsilon$ can be replaced by the identity; only the constant in estimate will change.

**Lemma 1.9.** For $\varepsilon > 0$ we have

$$\| \tilde{g}_\varepsilon(S_\varepsilon - I) b(D)(L^0 + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)} \leq C^\prime \varepsilon. \quad (1.37)$$

The constant $C^\prime$ depends only on $|\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_\infty}, \|\eta^{-1}\|_{L_\infty}, \|\nu\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

**Proof.** By (1.36), the left-hand side of (1.37) does not exceed

$$\| g^0(S_\varepsilon - I) b(D)(L^0 + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)} + \| \Sigma^\varepsilon (S_\varepsilon - I) \text{curl} \mu_0^{-1/2}(L_\varepsilon + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)}. \quad (1.38)$$

Using Proposition 1.1, we estimate the first term in (1.38):

$$\| g^0(S_\varepsilon - I) b(D)(L^0 + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)} \leq \varepsilon \| g \|_{L_\infty r_1} \| Db(D)(L^0 + I)^{-1} \|_{L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)}. \quad (1.39)$$
Recalling that the columns of the matrix \( \Sigma^\varepsilon \) are \( (\nabla \Phi_j)^\varepsilon \), \( j = 1, 2, 3 \), we estimate the second term in (1.38). Applying Proposition 1.6, and next Proposition 1.1 and inequality (1.2), we have:

\[
\begin{align*}
\left\| (\nabla \Phi_j)^\varepsilon (S^\varepsilon - I) \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} & \\
& \leq \sqrt{\frac{1}{\varepsilon}} \left\| (S^\varepsilon - I) \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \\
& \quad + \sqrt{\frac{1}{\varepsilon}} \| \Phi_j \|_{L^\infty} \left\| (S^\varepsilon - I) \text{D curl} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \\
& \leq \varepsilon (\sqrt{\beta_1 r_1 + 2 \sqrt{\beta_2} \| \Phi_j \|_{L^\infty}}) \left\| \text{D} b(\text{D})(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)}.
\end{align*}
\]

From (1.7) and (1.29) it follows that

\[
\| \text{D} b(\text{D})(L^0 + I)^{-1} \|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq \sup_{\xi \in \mathbb{R}^3} | \xi b(\xi)(\alpha(\xi) + 1)^{-1} | \leq \alpha_1^{1/2} \epsilon_1^{-1}. \tag{1.41}
\]

As a result, inequalities (1.39)–(1.41) together with Remark 1.5 yield the required estimate (1.37).

Now, relations (1.34)–(1.37) imply the following result.

**Theorem 1.10.** For \( \varepsilon > 0 \) we have

\[
\left\| g^\varepsilon b(\text{D})(L^0 + I)^{-1} - \tilde{g}^\varepsilon b(\text{D})(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq C_3 \varepsilon.
\]

In other words,

\[
\left\| (\eta^\varepsilon)^{-1} \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} - (\eta^0)^{-1} \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq C_3 \varepsilon,
\]

\[
\left\| \| \nu^\varepsilon \text{div} \mu_0^{-1/2}(L^0 + I)^{-1} - \| \nu^0 \| \text{div} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq C_3 \varepsilon.
\]

The constant \( C_3 \) depends only on \( |\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L^\infty}, \|\eta^{-1}\|_{L^\infty}, \|\nu\|_{L^\infty}, \|\nu^{-1}\|_{L^\infty} \), and the parameters of the lattice \( \Gamma \).

Now we distinguish particular cases. Taking Proposition 1.3 and Remark 1.4 into account, we deduce the following result from Theorems 1.8 and 1.10.

**Proposition 1.11.**

1) Let \( \eta^0 = \eta \), i.e., the columns of the matrix \( \eta(x) \) are divergence free. Then for \( \varepsilon > 0 \) we have

\[
\left\| (\eta^\varepsilon)^{-1} \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} - (\eta^0)^{-1} \text{curl} \mu_0^{-1/2}(L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq C_3 \varepsilon.
\]

2) Let \( \eta^0 = \eta \), i.e., the columns of the matrix \( \eta(x)^{-1} \) are potential. Suppose, in addition, that \( \nu(x) = \text{Const} \). Then the corrector (1.33) is equal to zero, and for \( \varepsilon > 0 \) we have

\[
\left\| (L^0 + I)^{-1} - (L^0 + I)^{-1} \right\|_{L^2(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3)} \leq C_2 \varepsilon.
\]

**2. The second order model operator in a bounded domain**

**2.1. Definition of the operator** \( L_\varepsilon \). Let \( \mathcal{O} \subset \mathbb{R}^3 \) be a bounded domain of class \( C^{1,1} \). Let \( n(x) \) be the unit outer normal vector to \( \partial \mathcal{O} \) at the point \( x \in \partial \mathcal{O} \). The projection of the vector-valued function \( u(x) \) onto the normal vector on the boundary is denoted by

\[
u_n(x) := \langle u(x), n(x) \rangle,
\]

its tangential component is denoted by

\[
u_r(x) := u(x) - \nu_n(x)n(x).
\]
Suppose that the coefficients $\mu_0$, $\eta$, and $\nu$ satisfy the assumptions of Subsection 1.3. In $L_2(O; \mathbb{C}^3)$, we consider the quadratic form

$$l_\varepsilon[\varphi, \varphi] := \int_O \langle g^\varepsilon(x)b(D)\varphi, b(D)\varphi \rangle \, dx$$

$$= \int_O \left( \left(\eta^\varepsilon(x)\right)^{-1} \text{curl} \left(\mu_0^{-1/2}\varphi\right), \text{curl} \left(\mu_0^{-1/2}\varphi\right) \right) + \nu^\varepsilon(x) |\text{div} \left(\mu_0^{1/2}\varphi\right)|^2 \right) \, dx,$$

(2.1)
defined on the domain

$$\text{Dom } l_\varepsilon = \{ \varphi \in L_2(O; \mathbb{C}^3) : \text{curl} (\mu_0^{-1/2}\varphi) \in L_2(O; \mathbb{C}^3), \text{div} \left(\mu_0^{1/2}\varphi\right) \in L_2(O), (\mu_0^{1/2}\varphi)|_{\partial O} = 0 \}.$$

(2.2)

Apriori, conditions from (2.2) on a vector-valued function $\varphi \in L_2(O; \mathbb{C}^3)$ (in particular, the boundary condition) are understood in the generalized sense; see [BS1, BS2] and Definition 5.1 below. Since $\partial O \in C^{1,1}$, the set (2.2) coincides with

$$\text{Dom } l_\varepsilon = \{ \varphi \in H^1(O; \mathbb{C}^3) : (\mu_0^{1/2}\varphi)|_{\partial O} = 0 \}.$$

Then the boundary condition can be understood in the sense of the trace theorem. Under our assumptions, the form (2.1) is coercive. The following two-sided estimates hold:

$$c_1 ||\varphi||_{H^1(O)} \leq l_\varepsilon[\varphi, \varphi] + ||\varphi||_{L_2(O)}^2 \leq c_2 ||\varphi||_{H^1(O)}^2, \quad \varphi \in \text{Dom } l_\varepsilon.$$  

(2.3)

The constant $c_1$ depends on $|\mu_0|$, $|\mu_0^{-1}|$, $|\eta|$, $|\nu|$, and the domain $O$, the constant $c_2$ depends on $|\mu_0|$, $|\mu_0^{-1}|$, $|\eta|$, $|\nu|$, and the domain $O$. These properties were checked in [BS1, Theorem 2.3] under the assumption that $\partial O \in C^2$ and in [F, Theorem 2.6] under the assumption that $\partial O \in C^{3/2+\delta}$, $\delta > 0$.

Thus, the form (2.1) is closed and nonnegative. A selfadjoint operator in $L_2(O; \mathbb{C}^3)$ generated by this form is denoted by $L_\varepsilon$. Formally, $L_\varepsilon$ is given by the differential expression

$$L_\varepsilon = \mu_0^{-1/2} \text{curl} (\eta^\varepsilon(x))^{-1} \text{curl} \mu_0^{-1/2} - \mu_0^{1/2} \nabla \nu^\varepsilon(x) \text{div} \mu_0^{1/2}$$

with the boundary conditions

$$(\mu_0^{1/2}\varphi)|_{\partial O} = 0, \quad ((\eta^\varepsilon)^{-1} \text{curl} (\mu_0^{-1/2}\varphi))_{\tau}|_{\partial O} = 0.$$  

The second condition is “natural” and is not reflected in the domain of the quadratic form $l_\varepsilon$.

**Remark 2.1.** In [Su4], when studying the general operators of the form $b(D)^*g^\varepsilon(x)b(D)$ with the Neumann boundary condition, it was assumed that the rank of the symbol $b(\xi)$ is maximal for $0 \neq \xi \in \mathbb{C}^n$. This condition ensured the coercivity of the corresponding quadratic form on the class $H^1(O; \mathbb{C}^n)$. In our case, this condition is not satisfied, though for $0 \neq \xi \in \mathbb{R}^3$ the rank of the matrix $b(\xi)$ is maximal; see (1.6). We emphasize that the form $l_\varepsilon$ is coercive due to the boundary condition $((\mu_0^{-1/2}\varphi))_{\tau}|_{\partial O} = 0$.

**Our goal** is to approximate the generalized solution of the problem

$$\mu_0^{-1/2} \text{curl} (\eta^\varepsilon(x))^{-1} \text{curl} (\mu_0^{-1/2}\varphi_\varepsilon(x)) - \mu_0^{-1/2} \nabla \nu^\varepsilon(x) \text{div} (\mu_0^{1/2}\varphi_\varepsilon(x)) + \varphi_\varepsilon(x) = F(x), \quad x \in O;$$

$$(\mu_0^{1/2}\varphi_\varepsilon)|_{\partial O} = 0, \quad ((\eta^\varepsilon)^{-1} \text{curl} (\mu_0^{-1/2}\varphi_\varepsilon))_{\tau}|_{\partial O} = 0,$$

(2.4)

for small $\varepsilon$. Here $F \in L_2(O; \mathbb{C}^3)$. The solution is understood in the weak sense: $\varphi_\varepsilon \in H^1(O; \mathbb{C}^3)$, $(\mu_0^{1/2}\varphi_\varepsilon)|_{\partial O} = 0$, and

$$l_\varepsilon[\varphi_\varepsilon, \zeta] + \langle \varphi_\varepsilon, \zeta \rangle_{L_2(O)} = \langle F, \zeta \rangle_{L_2(O)}; \quad \zeta \in H^1(O; \mathbb{C}^3), \quad (\mu_0^{1/2}\zeta)|_{\partial O} = 0.$$  

(2.5)

Then $\varphi_\varepsilon = (L_\varepsilon + I)^{-1} F$. Thus, we are interested in the behavior of the resolvent $(L_\varepsilon + I)^{-1}$ for small $\varepsilon$. 
2.2. The effective operator $L^0$. Suppose that the matrix $\eta^0$ is defined by (1.16), (1.17). Recall that $\nu$ is the harmonic average of the coefficient $\nu(x)$. Let $g^0$ be the matrix (1.26). The effective operator $L^0$ is a selfadjoint operator in $L_2(\mathcal{O}; \mathbb{C}^3)$ generated by the quadratic form

$$
l^0[\varphi, \varphi] := \int_\mathcal{O} (g^0 b(\mathbf{D}) \varphi, b(\mathbf{D}) \varphi) \, dx
$$

$$
= \int_\mathcal{O} \left( \left( \left( \nu \right)^{-1} \text{curl} (\mu_0^{-1/2} \varphi), \text{curl} (\mu_0^{-1/2} \varphi) \right) + \nu \text{div} \left( \mu_0^{1/2} \varphi \right) \right) \, dx,
$$

(2.6)

$$
\varphi \in H^1(\mathcal{O}; \mathbb{C}^3), \quad (\mu_0^{1/2} \varphi)|_{\partial \mathcal{O}} = 0.
$$

By (1.28), the form (2.6) satisfies the estimates

$$
\epsilon_1 \| \varphi \|^2_{H^1(\mathcal{O})} \leq l^0[\varphi, \varphi] + \| \varphi \|^2_{L_2(\mathcal{O})} \leq \epsilon_2 \| \varphi \|^2_{H^1(\mathcal{O})},
$$

(2.7)

$$
\varphi \in H^1(\mathcal{O}; \mathbb{C}^3), \quad (\mu_0^{1/2} \varphi)|_{\partial \mathcal{O}} = 0,
$$

with the same constants as in (2.3).

Due to the smoothness of the boundary, the following regularity property holds: the operator $L^0$ is given by the differential expression

$$
L^0 = \mu_0^{-1/2} \text{curl} (\eta^0)^{-1} \text{curl} \mu_0^{-1/2} - \mu_0^{1/2} \nabla \text{div} \mu_0^{1/2}
$$

on the domain

$$
\text{Dom} L^0 = \{ \varphi \in H^2(\mathcal{O}; \mathbb{C}^3), \ (\mu_0^{1/2} \varphi)|_{\partial \mathcal{O}} = 0, \ ((\eta^0)^{-1} \text{curl} (\mu_0^{-1/2} \varphi))|_{\partial \mathcal{O}} = 0 \}.
$$

Herewith,

$$
\| (L^0 + I)^{-1} \|_{L_2(\mathcal{O}) \to H^2(\mathcal{O})} \leq \tilde{c},
$$

(2.8)

where the constant $\tilde{c}$ depends on $|\mu_0|, |\mu_0^{-1}|, \|\eta\|_{L_\infty}, \|\eta^{-1}\|_{L_\infty}, \|\nu\|_{L_\infty}, \|\nu^{-1}\|_{L_\infty}$, and the domain $\mathcal{O}$.

Remark 2.2. Under the assumption that $\partial \mathcal{O} \subset C^{1,1}$ (and for sufficiently smooth coefficients), such regularity property for the solutions of the Dirichlet or Neumann problems for the second order strongly elliptic equations can be found, e. g., in the book [McI, Chapter 4]. The proof is based on the method of difference quotients and essentially relies on the coercivity condition for the quadratic form. In our case, the coefficients of the operator $L^0$ are constant and the coercivity condition (2.7) holds, but the boundary conditions are of mixed type. It is easy to check the regularity for the operator $L^0$ by the same method as before.

Let $\varphi_0$ be the solution of the “homogenized” problem

$$
\mu_0^{-1/2} \text{curl} (\eta^0)^{-1} \text{curl} (\mu_0^{-1/2} \varphi_0(x)) - \mu_0^{1/2} \nabla \text{div} (\mu_0^{1/2} \varphi_0(x)) + \varphi_0(x) = F(x), \quad x \in \mathcal{O};
$$

$$
(\mu_0^{1/2} \varphi_0)|_{\partial \mathcal{O}} = 0, \quad ((\eta^0)^{-1} \text{curl} (\mu_0^{-1/2} \varphi_0))|_{\partial \mathcal{O}} = 0.
$$

(2.9)

In other words, the function $\varphi_0 \in H^1(\mathcal{O}; \mathbb{C}^3)$ satisfies the boundary condition $(\mu_0^{1/2} \varphi_0)|_{\partial \mathcal{O}} = 0$ and the identity

$$
l^0[\varphi_0, \zeta] + (\varphi_0, \zeta)_{L_2(\mathcal{O})} = (F, \zeta)_{L_2(\mathcal{O})}, \quad \zeta \in H^1(\mathcal{O}; \mathbb{C}^3), \quad (\mu_0^{1/2} \zeta)|_{\partial \mathcal{O}} = 0.
$$

(2.10)

Then $\varphi_0 = (L^0 + I)^{-1} F$. Estimate (2.8) means that $\varphi_0 \in H^2(\mathcal{O}; \mathbb{C}^3)$ and

$$
\| \varphi_0 \|^2_{H^2(\mathcal{O})} \leq \tilde{c} \| F \|^2_{L_2(\mathcal{O})}.
$$

(2.11)
2.3. Estimates in the neighborhood of the boundary. We put
\[(\partial \mathcal{O})_\varepsilon := \{ x \in \mathbb{R}^d : \text{dist} \{ x; \partial \mathcal{O} \} < \varepsilon \}, \quad \varepsilon > 0.\]
We choose the numbers \(\varepsilon_0, \varepsilon_1 \in (0, 1]\) satisfying the following condition.

**Condition 2.3.** The number \(\varepsilon_0 \in (0, 1]\) is such that the strip \((\partial \mathcal{O})_\varepsilon\) can be covered by a finite number of open sets admitting diffeomorphisms of class \(C^{0,1}\) rectifying the boundary \(\partial \mathcal{O}\). Let \(\varepsilon_1 := \varepsilon_0(1 + r_1)^{-1}\), where \(2r_1 = \text{diam} \Omega\).

Clearly, \(\varepsilon_1\) depends only on the domain \(\mathcal{O}\) and the parameters of the lattice \(\Gamma\). Note that Condition 2.3 is ensured only by the Lipschitz property of the boundary. We have imposed a more restrictive assumption \(\partial \mathcal{O} \in C^{1,1}\) in order to ensure estimate (2.8).

The following statements were checked in [PSu, Section 5]; Lemma 2.5 is similar to Lemma 2.6 from [ZhPas1].

**Lemma 2.4.** Suppose that Condition 2.3 is satisfied. Let \(0 < \varepsilon \leq \varepsilon_0\). Denote \(B_\varepsilon := \mathcal{O} \cap (\partial \mathcal{O})_\varepsilon\).

1) For any function \(u \in H^1(\mathcal{O})\) we have
\[
\int_{B_\varepsilon} |u|^2 \, dx \leq \beta \varepsilon \| u \|_{H^1(\mathcal{O})} \| u \|_{L^2(\mathcal{O})}.
\]

2) For any function \(u \in H^1(\mathbb{R}^3)\) we have
\[
\int_{(\partial \mathcal{O})_\varepsilon} |u|^2 \, dx \leq \beta \varepsilon \| u \|_{H^1(\mathbb{R}^3)} \| u \|_{L^2(\mathbb{R}^3)}.
\]

The constant \(\beta\) depends only on the domain \(\mathcal{O}\).

**Lemma 2.5.** Suppose that Condition 2.3 is satisfied. Let \(h(x)\) be a \(\Gamma\)-periodic function in \(\mathbb{R}^3\) such that \(h \in L^2(\Omega)\). Let \(S_\varepsilon\) be the operator (1.1). Denote \(\beta_\varepsilon := \beta(1 + r_1)\), where \(2r_1 = \text{diam} \Omega\). Then for \(0 < \varepsilon \leq \varepsilon_1\) and \(u \in H^1(\mathbb{R}^3; C^k)\) we have
\[
\int_{(\partial \mathcal{O})_\varepsilon} |h(x)|^2 |(S_\varepsilon u)(x)|^2 \, dx \leq \beta_\varepsilon \varepsilon |\Omega|^{-1} \| h \|_{L^2(\Omega)}^2 \| u \|_{H^1(\mathbb{R}^3)} \| u \|_{L^2(\mathbb{R}^3)}.
\]

3. The results for the model second order equation in a bounded domain

3.1. Approximation of the resolvent of the operator \(L_\varepsilon\). Now, we formulate our main results about approximation of the solution of problem (2.4). For convenience of further references, the following set of the parameters is called the “problem data”:

\[
|\mu_0|, \left| \mu^{-1}_0 \right|, \| \eta \|_{L^\infty}, \left\| \eta^{-1} \right\|_{L^\infty}, \| \nu \|_{L^\infty}, \left\| \nu^{-1} \right\|_{L^\infty};
\]
the parameters of the lattice \(\Gamma\); and the domain \(\mathcal{O}\).

**Theorem 3.1.** Let \(\varphi_\varepsilon\) be the solution of problem (2.4), and let \(\varphi_0\) be the solution of the homogenized problem (2.9) with \(F \in L^2(\mathcal{O}; C^3)\). Suppose that the number \(\varepsilon_1\) satisfies Condition 2.3. Then for \(0 < \varepsilon \leq \varepsilon_1\) we have
\[
\| \varphi_\varepsilon - \varphi_0 \|_{L^2(\mathcal{O})} \leq C_1 \varepsilon \| F \|_{L^2(\mathcal{O})}.
\]

In operator terms,
\[
\|(L_\varepsilon + I)^{-1} - (L^0 + I)^{-1}\|_{L^2(\mathcal{O})} \leq C_1 \varepsilon.
\]

The constant \(C_1\) depends only on the problem data (3.1).
To approximate the solution in $H^1(\Omega; \mathbb{C}^3)$, we need to introduce a corrector. We fix a linear continuous extension operator

$$P_\Omega : H^s(\Omega; \mathbb{C}^3) \rightarrow H^s(\mathbb{R}^3; \mathbb{C}^3), \quad s = 0, 1, 2.$$  

Such an operator exists for any bounded domain with Lipschitz boundary (see, e.g., [St]).

Denote

$$\|P_\Omega\|_{H^{s}(\Omega) \rightarrow H^{s}(\mathbb{R}^3)} =: C^{(s)}_\Omega, \quad s = 0, 1, 2. \quad (3.3)$$

The constants $C^{(s)}_\Omega$ depend only on the domain $\Omega$. Next, let $[\Lambda^\varepsilon]$ be the operator of multiplication by the matrix-valued function $\Lambda(\varepsilon^{-1}x)$, and let $R_\Omega$ be the restriction operator of functions in $\mathbb{R}^3$ onto the domain $\Omega$. Let $S_\varepsilon$ be the Steklov smoothing operator; see (1.1). We introduce a corrector

$$K_\varepsilon := R_\Omega [\Lambda^\varepsilon] S_\varepsilon b(D) P_\Omega (L^0 + I)^{-1}.$$  

The operator $b(D)P_\Omega (L^0 + I)^{-1}$ is continuous from $L_2(\Omega; \mathbb{C}^3)$ to $H^1(\mathbb{R}^3; \mathbb{C}^4)$. As has been already mentioned, the operator $[\Lambda^\varepsilon] S_\varepsilon$ is continuous from $H^1(\mathbb{R}^3; \mathbb{C}^4)$ to $H^1(\mathbb{R}^3; \mathbb{C}^3)$. Hence, the corrector $K_\varepsilon$ is a continuous mapping of $L_2(\Omega; \mathbb{C}^3)$ to $H^1(\Omega; \mathbb{C}^3)$. Using (1.5) and (1.24), we write the corrector as

$$K_\varepsilon = R_\Omega \left( \mu_0^{-1/2} \psi^\varepsilon \text{curl} \mu_0^{-1/2} + \mu_0^{1/2} \left( \nabla \rho \right)^\varepsilon \text{div} \mu_0^{1/2} \right) P_\Omega (L^0 + I)^{-1}. \quad (3.4)$$

Let $\varphi_0$ be the solution of problem (2.9). We put $\tilde{\varphi}_0 := P_\Omega \varphi_0$ and

$$\tilde{\psi}_\varepsilon(x) := \tilde{\varphi}_0(x) + \varepsilon \mu_0^{-1/2} \psi^\varepsilon(x) (S_\varepsilon \text{curl} \mu_0^{-1/2} \tilde{\varphi}_0)(x) + \varepsilon \mu_0^{1/2} (\nabla \rho)^\varepsilon(x) (S_\varepsilon \text{div} \mu_0^{1/2} \tilde{\varphi}_0)(x), \quad x \in \mathbb{R}^3, \quad (3.5)$$

$$\psi_\varepsilon := \tilde{\psi}_\varepsilon|_{\Omega}.$$  

Then

$$\psi_\varepsilon = \varphi_0 + \varepsilon \Lambda^\varepsilon S_\varepsilon b(D) \tilde{\varphi}_0 = (L^0 + I)^{-1} F + \varepsilon K_\varepsilon F. \quad (3.6)$$

**Theorem 3.2.** Suppose that the assumptions of Theorem 3.1 are satisfied. Let $\psi_\varepsilon$ be defined by (3.5). Then for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\| \varphi_\varepsilon - \psi_\varepsilon \|_{H^1(\Omega)} \leq C_2 \varepsilon^{1/2} \| F \|_{L_2(\Omega)}. \quad (3.7)$$

In operator terms,

$$\|(L_\varepsilon + I)^{-1} - (L^0 + I)^{-1} - \varepsilon K_\varepsilon\|_{L_2(\Omega) \rightarrow H^1(\Omega)} \leq C_2 \varepsilon^{1/2}. \quad (3.8)$$

The constant $C_2$ depends only on the problem data (3.1).

**Theorem 3.3.** Suppose that the assumptions of Theorem 3.1 are satisfied. We put

$$\mathbf{u}_\varepsilon := (\eta^\varepsilon)^{-1} \text{curl} \mu_0^{1/2} \varphi_0, \quad \mathbf{u}_0 := (\eta^0)^{-1} \text{curl} \mu_0^{1/2} \varphi_0.$$  

Then for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\| \mathbf{u}_\varepsilon - \mathbf{u}_0 - \Sigma^\varepsilon \text{curl} (\mu_0^{1/2} \varphi_0) \|_{L_2(\Omega)} \leq C_3 \varepsilon^{1/2} \| F \|_{L_2(\Omega)}, \quad (3.9)$$

$$\| \eta^\varepsilon \text{div} (\mu_0^{1/2} \varphi_0) - \eta^0 \text{div} (\mu_0^{1/2} \varphi_0) \|_{L_2(\Omega)} \leq C_3 \varepsilon^{1/2} \| F \|_{L_2(\Omega)}.$$  

The constant $C_3$ depends only on the problem data (3.1).

Now, we distinguish the special cases. By Proposition 1.3 and Remark 1.4, Theorems 3.2 and 3.3 directly imply the following statement.

**Proposition 3.4.** 1) Suppose that $\eta^0 = \eta$, i.e., the columns of the matrix $\eta(x)$ are divergence free. Then for $0 < \varepsilon \leq \varepsilon_1$ we have

$$\| \mathbf{u}_\varepsilon - \mathbf{u}_0 \|_{L_2(\Omega)} \leq C_3 \varepsilon^{1/2} \| F \|_{L_2(\Omega)}.$$


2) Suppose that \( \eta^0 = \eta \), i.e., the columns of the matrix \( \eta(x)^{-1} \) are potential. Suppose, in addition, that \( \nu(x) = \text{Const.} \) Then the corrector (3.4) is equal to zero and for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\| \varphi_\varepsilon - \varphi_0 \|_{H^1(\Omega)} \leq C_2 \varepsilon^{1/2} \| F \|_{L^2(\Omega)}.
\]

3.2. The first step of the proof. The associated problem in \( \mathbb{R}^3 \). Obviously, we have \( \|(L^0 + I)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq 1 \), whence \( \| \varphi_0 \|_{L^2(\Omega)} \leq \| F \|_{L^2(\Omega)}. \) By (2.11) and (3.3),

\[
\| \tilde{\varphi}_0 \|_{L^2(\Omega)} \leq C_0 \| F \|_{L^2(\Omega)},
\]

(3.9)

\[
\| \tilde{\varphi}_0 \|_{H^2(\Omega)} \leq C_4 \| F \|_{L^2(\Omega)},
\]

(3.10)

We put

\[\tilde{F} := L^0 \tilde{\varphi}_0 + \varphi_0.\]

(3.11)

Then \( \tilde{F} \in L^2(\mathbb{R}^3; \mathbb{C}^3) \) and \( \tilde{F}|_{\partial \Omega} = F. \) By (1.29), (3.9), and (3.10),

\[
\| \tilde{F} \|_{L^2(\Omega)} \leq C_2 \| \tilde{\varphi}_0 \|_{H^2(\Omega)} + \| \varphi_0 \|_{L^2(\Omega)} \leq C_4 \| F \|_{L^2(\Omega)},
\]

(3.12)

where \( C_4 = c_0 C_0^{(0)} \) + \( C_0^{(0)}. \) We also need the following inequality which directly follows from (3.11) and (3.12):

\[
\| \varphi_0 \|_{L^2(\Omega)} \leq \| \tilde{F} \|_{L^2(\Omega)} \leq C_0 \| F \|_{L^2(\Omega)}.
\]

(3.13)

Let \( \tilde{\varphi}_\varepsilon \in H^1(\mathbb{R}^3; \mathbb{C}^3) \) be the generalized solution of the following equation in \( \mathbb{R}^3 \):

\[
\mathcal{L}_\varepsilon \tilde{\varphi}_\varepsilon + \varphi_\varepsilon = \tilde{F},
\]

i.e., \( \tilde{\varphi}_\varepsilon = (\mathcal{L}_\varepsilon + I)^{-1} \tilde{F}. \) We apply Theorems 1.7, 1.8, and 1.10. Using also (3.12), we arrive at the estimates

\[
\| \tilde{\varphi}_\varepsilon - \varphi_0 \|_{L^2(\Omega)} \leq C_1 \varepsilon^{1/2} \| F \|_{L^2(\Omega)},
\]

(3.14)

\[
\| \tilde{\varphi}_\varepsilon - \varphi_0 \|_{H^1(\Omega)} \leq C_1 \varepsilon^{1/2} \| F \|_{L^2(\Omega)},
\]

(3.15)

\[
\| g^b(D)\tilde{\varphi}_\varepsilon - g^b(D)\varphi_0 \|_{L^2(\Omega)} \leq C_3 \varepsilon \| \tilde{F} \|_{L^2(\Omega)} \leq C_3 C_0 \varepsilon \| F \|_{L^2(\Omega)}.
\]

(3.16)

3.3. The second step of the proof. Introduction of the correction term \( s_\varepsilon. \) Now, we introduce the “boundary layer correction term” \( s_\varepsilon \in H^1(\Omega; \mathbb{C}^n) \), as the function satisfying the following identity and boundary condition:

\[
(g^b(D)s_\varepsilon b(D)\zeta)_{L^2(\Omega)} + (s_\varepsilon, \zeta)_{L^2(\Omega)} = (g^b(D)\varphi_0 b(D)\zeta)_{L^2(\Omega)} - (F, \zeta)_{L^2(\Omega)} + (\varphi_0, \zeta)_{L^2(\Omega)},
\]

\( \forall \zeta \in H^1(\Omega; \mathbb{C}^3) \), \( (\mu_{0}^{1/2} \zeta)|_{\partial \Omega} = 0 \),

\[
((\mu_{0}^{1/2} s_\varepsilon)_{n})|_{\partial \Omega} = \varepsilon (\mu_{0}^{1/2} \Lambda^\varepsilon S e b(D)\varphi_0)_{n}|_{\partial \Omega}.
\]

(3.17)

Let us show that taking \( s_\varepsilon \) into account allows us to obtain approximation of the solution \( \varphi_\varepsilon \) in the \( H^1 \)-norm with an error of sharp order \( O(\varepsilon). \)

**Theorem 3.5.** For \( \varepsilon > 0 \) we have

\[
\| \varphi_\varepsilon - \psi_\varepsilon + s_\varepsilon \|_{H^1(\Omega)} \leq C_5 \varepsilon \| F \|_{L^2(\Omega)}.
\]

(3.18)

The constant \( C_5 \) depends only on the problem data (3.1).

**Proof.** Denote \( V_\varepsilon := \varphi_\varepsilon - \psi_\varepsilon + s_\varepsilon \). Then from (2.5), (3.6), (3.17), and the boundary conditions \( (\mu_{0}^{1/2} \varphi_\varepsilon)_{n}|_{\partial \Omega} = 0 \), \( (\mu_{0}^{1/2} \varphi_0)_{n}|_{\partial \Omega} = 0 \) it follows that \( V_\varepsilon \in H^1(\Omega; \mathbb{C}^3) \), \( (\mu_{0}^{1/2} V_\varepsilon)_{n}|_{\partial \Omega} = 0 \), and

\[
L_\varepsilon [V_\varepsilon, \zeta] + (V_\varepsilon, \zeta)_{L^2(\Omega)} = (g^b(D)\varphi_0 - g^b(D)\psi_\varepsilon, b(D)\zeta)_{L^2(\Omega)} + (\varphi_0 - \psi_\varepsilon, \zeta)_{L^2(\Omega)},
\]

\( \forall \zeta \in H^1(\Omega; \mathbb{C}^3) \), \( (\mu_{0}^{1/2} \zeta)|_{\partial \Omega} = 0 \).

(3.19)

The first term on the right can be written as

\[
(g^b(D)\varphi_0 - g^b(D)\psi_\varepsilon, b(D)\zeta)_{L^2(\Omega)} + (g^b(D)(\tilde{\varphi}_\varepsilon - \psi_\varepsilon), b(D)\zeta)_{L^2(\Omega)}.
\]
By (1.8), (3.15), and (3.16), it does not exceed
\[ \|g^s b(D)\tilde{\varphi}_0 - g^s b(D)\tilde{\varphi}_\varepsilon\|_{L_2(\mathbb{R}^3)}^2 + \left( \|s_\varepsilon \|_{L_2(\mathbb{R}^3)}^2 \right)^{1/2} \leq C_5 \varepsilon \|b(D)\varphi\|_{L_2(\mathbb{O})}^2, \]
where \( C_5' = C_4 \left( \|g^{-1}\|_{L_\infty}^2 + \sqrt{\varepsilon} C_2 \right). \) The second term in the right-hand side of (3.19) can be written as \( \langle \tilde{\varphi}_0 - \tilde{\varphi}_\varepsilon, \zeta \rangle_{L_2(\mathbb{O})} + \langle \tilde{\varphi}_\varepsilon - \tilde{\psi}_\varepsilon, \zeta \rangle_{L_2(\mathbb{O})}. \) By (3.14) and (3.15), it does not exceed \( C_5'' \varepsilon \|b(D)\|_{L_2(\mathbb{O})} \|\zeta\|_{L_2(\mathbb{O})}, \) where \( C_5'' = C_5 (C_1 + C_2). \) As a result, we see that the right side of identity (3.19) is majorated by \( \tilde{C_5} \varepsilon \left( \|b(D)\|_{L_2(\mathbb{O})} \left( \|l_\varepsilon[\zeta, \zeta] + \|\zeta\|_{L_2(\mathbb{O})}^2 \right)^{1/2} \right), \) where \( \tilde{C_5} = (C_5')^2 + (C_5'')^2. \)

Substituting \( \zeta = V_\varepsilon \) in (3.19) and using the obtained estimate, we arrive at the inequality
\[ \left( \|l_\varepsilon[V_\varepsilon, V_\varepsilon] + \|V_\varepsilon\|_{L_2(\mathbb{O})}^2 \right)^{1/2} \leq \tilde{C_5} \varepsilon \|b(D)\|_{L_2(\mathbb{O})}. \]
Together with the lower estimate (2.3), this implies the required inequality (3.18) with the constant \( C_5 = \tilde{C_5} \varepsilon^{1/2}. \)

**Conclusions.**
1) From (3.18) it follows that
\[ \|\varphi - \psi_\varepsilon\|_{H^1(\mathbb{O})} \leq C_5 \varepsilon \|b(D)\|_{L_2(\mathbb{O})} + \|s_\varepsilon\|_{H^1(\mathbb{O})}. \]
So, for the proof of Theorem 3.2, we need to prove a suitable estimate for the norm \( \|s_\varepsilon\|_{H^1(\mathbb{O})}. \)
2) From (3.6) and (3.18) it follows that
\[ \|\varphi - \varphi_0\|_{L_2(\mathbb{O})} \leq C_5 \varepsilon \|b(D)\|_{L_2(\mathbb{O})} + \varepsilon \|s_\varepsilon\|_{L_2(\mathbb{O})}. \]
By Proposition 1.2 and estimates (1.12), (3.13),
\[ \|s_\varepsilon\|_{L_2(\mathbb{O})} \leq C_6 \varepsilon \|b(D)\|_{L_2(\mathbb{O})} + \|s_\varepsilon\|_{L_2(\mathbb{O})}. \]
Together with (3.21), this yields
\[ \|\varphi - \varphi_0\|_{L_2(\mathbb{O})} \leq C_6 \varepsilon \|b(D)\|_{L_2(\mathbb{O})} + \|s_\varepsilon\|_{L_2(\mathbb{O})}, \]
where \( C_6 = C_5 + C_6 A C_4 \varepsilon \|g^{-1}\|_{L_\infty}. \) Hence, to prove Theorem 3.1, we have to estimate the norm \( \|s_\varepsilon\|_{L_2(\mathbb{O})} \) in appropriate way.

4. **Estimation of the correction term. Proof of Theorems 3.1–3.3**

First, we estimate the \( H^1(\mathbb{O}) \)-norm of the correction term \( s_\varepsilon \) and prove Theorem 3.2 and also Theorem 3.3. Next, using the already proved Theorem 3.2 and the duality arguments, we estimate the \( L_2(\mathbb{O}) \)-norm of the correction term \( s_\varepsilon \) and prove Theorem 3.1.

4.1. Estimate for the correction term in \( H^1(\mathbb{O}) \). Proof of Theorem 3.2. We rewrite identity (3.17), using (2.10):
\[ (g^s b(D)s_\varepsilon, b(D)\zeta)_{L_2(\mathbb{O})} + (s_\varepsilon, \zeta)_{L_2(\mathbb{O})} = (g^s - g^0 b(D)\varphi_0, b(D)\zeta)_{L_2(\mathbb{O})} =: I_\varepsilon[\zeta], \]
\[ \forall \zeta \in H^1(\mathbb{O}; \mathbb{C}^3), \quad (\mu_0^{-1/2} \zeta)_{n|_{\partial \mathbb{O}}} = 0. \]
According to (1.5), (1.25), and (1.26),
\[ I_\varepsilon[\zeta] = (\Sigma^c \text{curl} \mu_0^{-1/2} \varphi_0, \text{curl} \mu_0^{-1/2} \zeta)_{L_2(\mathbb{O})}. \]

**Lemma 4.1.** For \( 0 < \varepsilon \leq \varepsilon_0 \) we have
\[ |I_\varepsilon[\zeta]| \leq C_7 \|b(D)\|_{L_2(\mathbb{O})} (l_\varepsilon[\zeta, \zeta])^{1/2}, \quad \zeta \in H^1(\mathbb{O}; \mathbb{C}^3), \quad (\mu_0^{-1/2} \zeta)_{n|_{\partial \mathbb{O}}} = 0. \]
The constant \( C_7 \) depends only on the problem data (3.1).
Proof. Recall that $\Sigma(x)$ is the matrix with the columns $\nabla \Phi_j(x)$, $j = 1, 2, 3$. Hence, the matrix $\Sigma^e(x)$ has the columns $(\nabla \Phi_j)^e(x) = \varepsilon \nabla \Phi_j^e(x)$, $j = 1, 2, 3$. The components of the vector-valued function $\text{curl}\mu_0^{-1/2}\varphi_0$ are denoted by $[\text{curl}\mu_0^{-1/2}\varphi_0]$, $j = 1, 2, 3$. Then

$$\Sigma^e\text{curl}\mu_0^{-1/2}\varphi_0 = \varepsilon \sum_{j=1}^{3} (\nabla \Phi_j^e)[\text{curl}\mu_0^{-1/2}\varphi_0]_j = \varepsilon \sum_{j=1}^{3} \left( \nabla \left( \Phi_j^e[\text{curl}\mu_0^{-1/2}\varphi_0]_j \right) - \Phi_j^e \nabla[\text{curl}\mu_0^{-1/2}\varphi_0]_j \right).$$

Together with (4.2), this implies

$$I_\varepsilon[\zeta] = I^{(1)}_\varepsilon[\zeta] + I^{(2)}_\varepsilon[\zeta], \quad I^{(1)}_\varepsilon[\zeta] := \varepsilon \sum_{j=1}^{3} \left( \nabla \left( \Phi_j^e[\text{curl}\mu_0^{-1/2}\varphi_0]_j \right) , \text{curl}\mu_0^{-1/2}\zeta \right)_{L_2(O)}, \quad I^{(2)}_\varepsilon[\zeta] := -\varepsilon \sum_{j=1}^{3} (\Phi_j^e \nabla[\text{curl}\mu_0^{-1/2}\varphi_0]_j, \text{curl}\mu_0^{-1/2}\zeta)_{L_2(O)}. \quad (4.4)$$

(4.5)

By (2.11) and the boundedness of $\Phi_j$ (see Remark 1.5), the term (4.6) admits the estimate

$$|I^{(2)}_\varepsilon[\zeta]| \leq C'_\varepsilon \varepsilon \|\mathbf{F}\|_{L_2(O)}(I_\varepsilon[\zeta, \zeta])^{1/2}, \quad (4.7)$$

where the constant $C'_\varepsilon$ depends only on the problem data (3.1). We have taken into account the obvious inequality

$$\|\text{curl}\mu_0^{-1/2}\zeta\|_{L_2(O)} \leq \|\eta\|_{L_\infty}(I_\varepsilon[\zeta, \zeta])^{1/2}. \quad (4.8)$$

Let $0 < \varepsilon \leq \varepsilon_0$. We fix a cut-off function $\theta_\varepsilon(x)$ in $\mathbb{R}^3$ such that

$$\theta_\varepsilon \in C_0^\infty(\mathbb{R}^3); \quad \text{supp}\ \theta_\varepsilon \subset (\partial O)_\varepsilon; \quad 0 \leq \theta_\varepsilon(x) \leq 1; \quad \theta_\varepsilon(x) = 1 \text{ for } x \in \partial O; \quad \varepsilon|\nabla \theta_\varepsilon| \leq \kappa = \text{ Const.} \quad (4.9)$$

We put

$$f_{j,\varepsilon} := \varepsilon \nabla \left( \theta_\varepsilon \Phi_j^e[\text{curl}\mu_0^{-1/2}\varphi_0]_j \right), \quad j = 1, 2, 3, \quad (4.10)$$

and represent the term (4.5) as

$$I^{(1)}_\varepsilon[\zeta] = \sum_{j=1}^{3} (f_{j,\varepsilon}, \text{curl}\mu_0^{-1/2}\zeta)_{L_2(O)}. \quad (4.11)$$

Here we have used the identity

$$\left( \nabla \left( (1 - \theta_\varepsilon)\Phi_j^e[\text{curl}\mu_0^{-1/2}\varphi_0]_j \right), \text{curl}\mu_0^{-1/2}\zeta \right)_{L_2(O)} = 0,$$

which can be checked by integration by parts and using the identity $\text{div}\text{curl} = 0$ (when checking, we can assume that $\zeta \in H^2(O; \mathbb{C}^3)$).

It remains to estimate the term (4.11). By (4.9), (4.10), and Remark 1.5, we have

$$\|f_{j,\varepsilon}\|_{L_2(O)} \leq \kappa \|\Phi_j\|_{L_\infty} \|\text{curl}\mu_0^{-1/2}\varphi_0\|_{L_2(\partial O)} \quad (4.12)$$

The first summand in (4.12) does not exceed $C\varepsilon^{1/2}\|\mathbf{F}\|_{L_2(O)}$, due to Lemma 2.4 and estimate (2.11). By (2.11), the third summand in (4.12) is majorated by $C\varepsilon\|\mathbf{F}\|_{L_2(O)}$. To estimate
the second term in (4.12), we apply Proposition 1.6 and (4.9):

\[
\|\theta_\varepsilon (\nabla \Phi_j) \varepsilon [\text{curl } \mu_0^{-1/2} \varphi_0]_j \|_{L^2(\Omega)} \leq \|\theta_\varepsilon (\nabla \Phi_j) \varepsilon [\text{curl } \mu_0^{-1/2} \varphi_0]_j \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq \sqrt{\beta_1} \|\text{curl } \mu_0^{-1/2} \varphi_0]_j \|_{L^2(\partial \Omega, \mathbb{R}^3)} + \sqrt{\beta_2 \varepsilon} \|\Phi_j \|_{L^\infty} \|\nabla (\theta_\varepsilon [\text{curl } \mu_0^{-1/2} \varphi_0]_j) \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq \left( \sqrt{\beta_1} + \sqrt{\beta_2} \|\Phi_j \|_{L^\infty} \right) \|\text{curl } \mu_0^{-1/2} \varphi_0]_j \|_{L^2(\partial \Omega, \mathbb{R}^3)} + \sqrt{\beta_2 \varepsilon} \|\Phi_j \|_{L^\infty} \|\nabla [\text{curl } \mu_0^{-1/2} \varphi_0]_j \|_{L^2(\mathbb{R}^3)}.
\]

By Lemma 2.4 and estimate (3.10), the first term on the right does not exceed \(C_\varepsilon^{1/2} \|F\|_{L^2(\Omega)}\). From (3.10) it follows that the second term is estimated by \(C \varepsilon \|F\|_{L^2(\Omega)}\). We arrive at

\[
\sum_{j=1}^{3} \|f_{j, \varepsilon}\|_{L^2(\Omega)} \leq C'' \varepsilon^{1/2} \|F\|_{L^2(\Omega)},
\]

where the constant \(C''\) depends only on the problem data (3.1). Hence, by (4.8), the term (4.11) satisfies

\[
\|\mathcal{I}^{(1)}[\zeta]\| \leq C'' \varepsilon^{1/2} \|F\|_{L^2(\Omega)} \left( \|\zeta\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\mathbb{R}^3)} \right)^{1/2}.
\]

Now, relations (4.4), (4.7), and (4.14) imply the required inequality (4.3). \(\square\)

We introduce the following function in \(\mathbb{R}^3\):

\[
\phi_\varepsilon(x) := \varepsilon \theta_\varepsilon(x) \lambda^{\varepsilon}(x) (S_\varepsilon b(D) \tilde{\varphi}_0)(x).
\]

**Lemma 4.2.** Let \(\phi_\varepsilon\) be defined by (4.15). For \(0 < \varepsilon \leq \varepsilon_0\) we have

\[
\|s_\varepsilon\|_{H^1(\Omega)} \leq C_8 \left( \varepsilon^{1/2} \|F\|_{L^2(\Omega)} + \|\phi_\varepsilon\|_{H^1(\mathbb{R}^3)} \right),
\]

where the constant \(C_8\) depends only on the problem data (3.1).

**Proof.** By (3.17), (4.1), and (4.9), the function \(s_\varepsilon - \phi_\varepsilon \in H^1(\Omega; \mathbb{C}^3)\) satisfies the boundary condition

\[
\left( \mu_0^{1/2} (s_\varepsilon - \phi_\varepsilon) \right)_n |_{\partial \Omega} = 0
\]

and the identity

\[
l_\varepsilon [s_\varepsilon - \phi_\varepsilon, \zeta] + (s_\varepsilon - \phi_\varepsilon, \zeta)_{L^2(\Omega)} = \mathcal{I}_\varepsilon[\zeta] - \mathcal{J}_\varepsilon[\zeta],
\]

\[
\zeta \in H^1(\Omega; \mathbb{C}^3), \quad \left( \mu_0^{1/2} \zeta \right)_n |_{\partial \Omega} = 0,
\]

where

\[
\mathcal{J}_\varepsilon[\zeta] := (g \cdot b(D) \phi_\varepsilon, b(D) \zeta)_{L^2(\Omega)} + (\phi_\varepsilon, \zeta)_{L^2(\Omega)}.
\]

By (1.8), we have

\[
|\mathcal{I}_\varepsilon[\zeta]| \leq \sqrt{c_0} \|D \phi_\varepsilon\|_{L^2(\mathbb{R}^3)} \left( l_\varepsilon \|\zeta\|_{L^2(\mathbb{R}^3)} \right)^{1/2} + \|\phi_\varepsilon\|_{L^2(\mathbb{R}^3)} \|\zeta\|_{L^2(\Omega)}.
\]

Substituting \(\zeta = s_\varepsilon - \phi_\varepsilon\) in (4.17) and using (4.3) and (4.19), we arrive at

\[
l_\varepsilon [s_\varepsilon - \phi_\varepsilon, s_\varepsilon - \phi_\varepsilon] + \|s_\varepsilon - \phi_\varepsilon\|_{L^2(\Omega)} \leq 2 C_7^2 \varepsilon \|F\|_{L^2(\Omega)} + 2 C_2 \|D \phi_\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \|\phi_\varepsilon\|_{L^2(\mathbb{R}^3)}^2.
\]

Combining this with the lower estimate (2.3), we obtain

\[
\|s_\varepsilon - \phi_\varepsilon\|_{H^1(\Omega)} \leq C_8 \left( \varepsilon^{1/2} \|F\|_{L^2(\Omega)} + \|\phi_\varepsilon\|_{H^1(\mathbb{R}^3)} \right),
\]

where the constant \(C_8\) depends only on the problem data (3.1). This implies (4.16). \(\square\)

**Lemma 4.3.** Suppose that the number \(\varepsilon_1\) satisfies Condition 2.3. Let \(\phi_\varepsilon\) be defined by (4.15). For \(0 < \varepsilon \leq \varepsilon_1\) we have

\[
\|\phi_\varepsilon\|_{L^2(\mathbb{R}^3)} \leq C_9 \varepsilon \|F\|_{L^2(\Omega)},
\]

\[
\|\phi_\varepsilon\|_{H^1(\mathbb{R}^3)} \leq C_{10} \varepsilon^{1/2} \|F\|_{L^2(\Omega)},
\]

where the constants \(C_9\) and \(C_{10}\) depend on the problem data (3.1).
The norm of the first summand on the right is estimated with the help of (4.9) and Lemma 2.5:

\[ \varepsilon \| (D_j \theta_\varepsilon) \Lambda^\varepsilon S_\varepsilon b(D) \tilde{\varphi}_0 \|_{L_2(\Omega)} \leq C \| (D_j \Lambda^\varepsilon S_\varepsilon b(D) \tilde{\varphi}_0) \|_{L_2(\partial \Omega)}, \]

where \( C' = C \sqrt{\beta \varepsilon^{1/2}} \). We have taken (1.7), (1.12), and (3.10) into account. Similarly, Lemma 2.5 and relations (1.7), (1.12), (3.10) imply the following estimate for the norm of the second term in (4.22):

\[ \| (D_j \Lambda^\varepsilon S_\varepsilon b(D) \tilde{\varphi}_0) \|_{L_2(\Omega)} \leq C'' \| F \|_{L_2(\Omega)}, \]

where \( C'' = C \sqrt{\beta \varepsilon^{1/2}} \). As a result, we arrive at the estimate

\[ \| D \phi_\varepsilon \|_{L_2(\Omega)} \leq C_{10} \varepsilon^{1/2} \| F \|_{L_2(\Omega)}, \]

where the constant \( C_{10} \) depends only on the problem data (3.1). Together with (4.20), this implies (4.21). \( \square \)

Lemmas 4.2 and 4.3 directly imply the following statement.

**Corollary 4.4.** For \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[ \| s_\varepsilon \|_{H^1(\Omega)} \leq C_{11} \varepsilon^{1/2} \| F \|_{L_2(\Omega)}, \]

where the constant \( C_{11} \) depends only on the problem data (3.1).

**Completion of the proof of Theorem 3.2.** Relations (3.20) and (4.23) imply the required estimate (3.7) with the constant \( C_2 = C_5 + C_{11} \). \( \square \)

### 4.2. Proof of Theorem 3.3.

From (3.7) it follows that

\[ \| g^\varepsilon b(D)(\varphi_\varepsilon - \psi_\varepsilon) \|_{L_2(\Omega)} \leq C_{12} \varepsilon^{1/2} \| F \|_{L_2(\Omega)}, \]

where the constant \( C_{12} \) depends only on the problem data (3.1). According to (3.6),

\[
\begin{align*}
g^\varepsilon b(D)\psi_\varepsilon &= g^\varepsilon b(D)\varphi_0 + \varepsilon g^\varepsilon b(D)(\Lambda^\varepsilon S_\varepsilon b(D)\tilde{\varphi}_0) \\
&= g^\varepsilon b(D)\varphi_0 + g^\varepsilon(b(D)\Lambda)^\varepsilon S_\varepsilon b(D)\tilde{\varphi}_0 + \varepsilon \sum_{j=1}^{3} g^\varepsilon b_j \Lambda^\varepsilon S_\varepsilon D_j b(D)\tilde{\varphi}_0 \\
&= \tilde{g}^\varepsilon b(D)\varphi_0 + g^\varepsilon(b(D)\Lambda)^\varepsilon(S_\varepsilon - I) b(D)\tilde{\varphi}_0 + \varepsilon \sum_{j=1}^{3} g^\varepsilon b_j \Lambda^\varepsilon S_\varepsilon D_j b(D)\tilde{\varphi}_0.
\end{align*}
\]

By Proposition 1.2 and relations (1.12) and (3.10), the norm of the third summand on the right does not exceed \( C \varepsilon \| F \|_{L_2(\Omega)} \). The second term on the right can be written as

\[
g^\varepsilon(b(D)\Lambda)^\varepsilon(S_\varepsilon - I) b(D)\tilde{\varphi}_0 = -i \left( (\Sigma^\varepsilon + (\eta^0)^{-1} - (\eta^\varepsilon)^{-1})(S_\varepsilon - I) \text{curl} \mu_0^{-1/2} \tilde{\varphi}_0, (\nu - \nu^\varepsilon)(S_\varepsilon - I) \text{div} \mu_0^{-1/2} \tilde{\varphi}_0 \right),
\]
Similarly to the proof of Lemma 1.9, using Propositions 1.1 and 1.6, it is easy to check that
\[
\|g^\epsilon(b(D)\Lambda^\epsilon)(S\epsilon - I)b(D)\tilde{\varphi}_0\|_{L_2(\mathbb{R}^3)} \leq C_1\epsilon\|F\|_{L_2(O)},
\]
where the constant $C_1$ depends only on the problem data (3.1). As a result, we obtain
\[
\|g^\epsilon(b(D)\psi_\epsilon - \tilde{g}^\epsilon(b(D)\varphi_0\|_{L_2(O)} \leq \tilde{C}_1\epsilon\|F\|_{L_2(O)},
\]
where the constant $\tilde{C}_1$ depends only on the problem data (3.1).

Relations (4.24) and (4.25) imply the required estimate
\[
\|g^\epsilon(b(D)\varphi_\epsilon - \tilde{g}^\epsilon(b(D)\varphi_0\|_{L_2(O)} \leq (C_{12} + \tilde{C}_{12})\epsilon^{1/2}\|F\|_{L_2(O)},
\]
which is equivalent to the pair of inequalities (3.8). □

4.3. Estimate for the correction term in $L_2(O)$. Completion of the proof of Theorem 3.1.

Lemma 4.5. For $0 < \epsilon \leq \epsilon_1$ we have
\[
\|s_\epsilon\|_{L_2(O)} \leq C_{13}\epsilon\|F\|_{L_2(O)},
\]
where the constant $C_{13}$ depends only on the problem data (3.1).

Proof. Let $G \in L_2(O; \mathbb{C}^2)$. We put $\zeta_\epsilon := (L_\epsilon + I)^{-1}G$. We substitute $\zeta = \zeta_\epsilon$ in the identity (4.17). Then the left-hand side of this identity takes the form
\[
s_\epsilon - \phi_\epsilon, G)_{L_2(O)} = \mathcal{I}_\epsilon[\zeta_\epsilon] - \mathcal{J}_\epsilon[\zeta_\epsilon].
\]

Combining (4.4), (4.7), (4.18), (4.20), (4.27), and the obvious estimate
\[
\mathcal{J}_\epsilon[\zeta_\epsilon] \leq \|\zeta_\epsilon\|_{L_2(O)}^2 \leq \|G\|_{L_2(O)}^2,
\]
we have
\[
|(s_\epsilon - \phi_\epsilon, G)_{L_2(O)}| \leq (C_1 + C_9)\epsilon\|F\|_{L_2(O)}\|G\|_{L_2(O)} + |\mathcal{I}_\epsilon^{(1)}[\zeta_\epsilon]| + \|g^\epsilon(b(D)\phi_\epsilon, b(D)\zeta_\epsilon)\|_{L_2(O)}|.
\]

Since the functions $f_{j\epsilon}$ and $\phi_\epsilon$ are supported in the $\epsilon$-neighborhood of the boundary $\partial O$ (see (4.9), (4.10), and (4.15)), from (4.11), (4.13), and (4.21) it follows that
\[
|\mathcal{I}_\epsilon^{(1)}[\zeta_\epsilon]| + \|g^\epsilon(b(D)\phi_\epsilon, b(D)\zeta_\epsilon)\|_{L_2(O)} \leq C_{13}^\epsilon\epsilon^{1/2}\|F\|_{L_2(O)}\|\mathcal{D}\zeta_\epsilon\|_{L_2(B_\epsilon)},
\]
where the constant $C_{13}$ depends only on the problem data (3.1).

Applying the already proved Theorem 3.2, we approximate the function $\zeta_\epsilon$ by $\zeta_0 + \epsilon\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0$, where $\zeta_0 = (\tilde{L}_0 + I)^{-1}G$ and $\tilde{\zeta}_0 = P_\partial O\zeta_0$. We have:
\[
\|\mathcal{D}\zeta_\epsilon\|_{L_2(B_\epsilon)} \leq \|\mathcal{D}(\zeta_\epsilon - \zeta_0 - \epsilon\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0)\|_{L_2(O)} + \|\mathcal{D}\zeta_0\|_{L_2(B_\epsilon)} + \epsilon\|\mathcal{D}(\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0)\|_{L_2((\partial O)_\epsilon)}.
\]

By Theorem 3.2, the first term on the right does not exceed $C_2\epsilon^{1/2}\|G\|_{L_2(O)}$. The second term is estimated by $\sqrt{\beta}\epsilon^{1/2}\|G\|_{L_2(O)}$, due to Lemma 2.4 and estimate (2.8). Let us estimate the third term:
\[
\epsilon\|\mathcal{D}(\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0)\|_{L_2((\partial O)_\epsilon)} \leq \|\mathcal{D}(\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0\|_{L_2((\partial O)_\epsilon)} + \epsilon\|\Lambda^\epsilon S_\epsilon b(D)\tilde{\zeta}_0\|_{L_2(\mathbb{R}^3)} \\
\leq \beta\epsilon\epsilon^{1/2}\|b(D)\tilde{\zeta}_0\|_{L_2^{1/2}(\mathbb{R}^3)}\|b(D)\tilde{\zeta}_0\|_{L_2(\mathbb{R}^3)} + C_\epsilon\epsilon^{1/2}\|b(D)\tilde{\zeta}_0\|_{L_2(\mathbb{R}^3)}.
\]

We have used Lemma 2.5, Proposition 1.2, and estimate (1.12). Combining this with the analog of estimate (3.10) for $\tilde{\zeta}_0$, we see that the third term in (4.30) does not exceed $C_{13}^\epsilon\epsilon^{1/2}\|G\|_{L_2(O)}$, where $C_{13}^\epsilon$ depends only on the problem data (3.1). As a result, we arrive at the inequality
\[
\|\mathcal{D}\zeta_\epsilon\|_{L_2(B_\epsilon)} \leq (C_2 + \sqrt{\beta}\epsilon + C_{13}^\epsilon)\epsilon^{1/2}\|G\|_{L_2(O)}, \quad 0 < \epsilon \leq \epsilon_1.
\]
Relations (4.28), (4.29), and (4.31) imply that
\[ \| \mathbf{s}_e - \phi_e, \mathbf{G} \|_{L^2(\Omega)} \leq \tilde{C}_{13} \varepsilon \| \mathbf{F} \|_{L^2(\Omega)} \| \mathbf{G} \|_{L^2(\Omega)}, \quad \forall \mathbf{G} \in L^2(\Omega; \mathbb{C}^3), \]
where the constant \( \tilde{C}_{13} \) depends only on the problem data (3.1). Hence,
\[ \| \mathbf{s}_e - \phi_e \|_{L^2(\Omega)} \leq \tilde{C}_{13} \varepsilon \| \mathbf{F} \|_{L^2(\Omega)}. \]
Together with estimate (4.20), this implies (4.26).

完成功命的证明定理 3.1. 关系式（3.23）和（4.26）意味着所需的估计式（3.2）与常数 \( C_1 = C_6 + C_{13} \). □

5. The stationary Maxwell system

5.1. Functional classes. As above, we assume that \( \Omega \subset \mathbb{R}^3 \) is a bounded domain of class \( C^{1,1} \). Recall the following definitions; see [BS1, BS2].

Definition 5.1. Let \( \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \) and \( \text{div} \mathbf{u} \in L^2(\Omega) \). Then, by definition, the relation \( \mathbf{u}|_{\partial \Omega} = 0 \) means that
\[ (\mathbf{u}, \nabla \omega)_{L^2(\Omega)} = -(\text{div} \mathbf{u}, \omega)_{L^2(\Omega)}, \quad \forall \omega \in H^1(\Omega). \]

Definition 5.2. Let \( \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \) and \( \text{curl} \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \). Then, by definition, the relation \( \mathbf{u}|_{\partial \Omega} = 0 \) means that
\[ (\mathbf{u}, \text{curl} \mathbf{z})_{L^2(\Omega)} = (\text{curl} \mathbf{u}, \mathbf{z})_{L^2(\Omega)}, \quad \forall \mathbf{z} \in L^2(\Omega; \mathbb{C}^3) : \text{curl} \mathbf{z} \in L^2(\Omega; \mathbb{C}^3). \]
Suppose that the matrix \( \mu_0 \) and the matrix-valued function \( \eta(x) \) satisfy the assumptions of Subsection 1.3. Along with the ordinary space \( L^2(\Omega; \mathbb{C}^3) \), we need to define the weighted \( L^2 \)-spaces of vector-valued functions: the space
\[ L^2(\Omega; (\eta^\varepsilon)^{-1}) = L^2(\Omega; \mathbb{C}^3; (\eta^\varepsilon)^{-1}) \]
with the inner product
\[ (f_1, f_2)_{L^2(\Omega; (\eta^\varepsilon)^{-1})} = \int_{\Omega} \langle (\eta^\varepsilon(x))^{-1} f_1(x), f_2(x) \rangle \, dx \]
and the similar space
\[ L^2(\Omega; \mu_0^{-1}) = L^2(\Omega; \mathbb{C}^3; \mu_0^{-1}) \]
with the inner product
\[ (f_1, f_2)_{L^2(\Omega; \mu_0^{-1})} = \int_{\Omega} \langle \mu_0^{-1} f_1(x), f_2(x) \rangle \, dx. \]
We introduce two subspaces of divergence-free vector-valued functions in \( L^2 \):
\begin{align*}
J(\Omega) & := \{ \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \int_{\Omega} \langle \mathbf{u}, \nabla \omega \rangle \, dx = 0, \quad \forall \omega \in H^1_0(\Omega) \}, \quad (5.1) \\
J_0(\Omega) & := \{ \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \int_{\Omega} \langle \mathbf{u}, \nabla \omega \rangle \, dx = 0, \quad \forall \omega \in H^1(\Omega) \}. \quad (5.2)
\end{align*}
The subspace (5.1) consists of all functions \( \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \) such that \( \text{div} \mathbf{u} = 0 \) in the sense of distributions. The subspace (5.2) consists of all functions \( \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \) such that \( \text{div} \mathbf{u} = 0 \) and \( \mathbf{u}|_{\partial \Omega} = 0 \) (in the sense of Definition 5.1).
5.2. Statement of the problem. We study an electromagnetic resonator filling the domain $\mathcal{O}$. Suppose that the magnetic permeability is given by the constant matrix $\mu_0$, and the dielectric permittivity is given by the matrix $\eta^\varepsilon(x) = \eta(x^{-1}x)$. The intensities of the electric and magnetic fields are denoted by $u_\varepsilon(x)$ and $v_\varepsilon(x)$, respectively. The electric and magnetic displacement vectors are expressed in terms of $u_\varepsilon$, $v_\varepsilon$ by $w_\varepsilon(x) = \eta^\varepsilon(x)u_\varepsilon(x)$, $z_\varepsilon(x) = \mu_0v_\varepsilon(x)$.

The operator $M_\varepsilon$ written in terms of the displacement vectors acts in the space $\mathcal{J}(\mathcal{O}) \oplus J_0(\mathcal{O})$ considered as a subspace of

$$L_2(\mathcal{O}; \mathbb{C}^3; (\eta^\varepsilon)^{-1}) \oplus L_2(\mathcal{O}; \mathbb{C}^3; \mu_0^{-1}),$$

and is given by

$$M_\varepsilon = \begin{pmatrix} 0 & i\text{curl}\mu_0^{-1} \\ -i\text{curl}\eta^\varepsilon -1 & 0 \end{pmatrix}$$

(5.3)
on the domain

$$\text{Dom } M_\varepsilon = \{(w, z) \in \mathcal{J}(\mathcal{O}) \oplus J_0(\mathcal{O}) : \text{curl } (\eta^\varepsilon)^{-1}w \in L_2(\mathcal{O}; \mathbb{C}^3),$$

$$\text{curl } \mu_0^{-1}z \in L_2(\mathcal{O}; \mathbb{C}^3), \ (((\eta^\varepsilon)^{-1}w)_\tau)|_{\partial \mathcal{O}} = 0 \}. \quad (5.4)$$

Here the boundary condition for $w$ is understood in the sense of Definition 5.2.

The operator $M_\varepsilon$ is selfadjoint; see [BS1, BS2]. The point $\lambda = i$ is a regular point for the operator $M_\varepsilon$. Our goal is to study the behavior of the resolvent $(M_\varepsilon - iI)^{-1}$. In other words, we are interested in the behavior of the solutions $(w_\varepsilon, z_\varepsilon)$ of the equation

$$(M_\varepsilon - iI) \begin{pmatrix} w_\varepsilon \\ z_\varepsilon \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad q \in \mathcal{J}(\mathcal{O}), \ r \in J_0(\mathcal{O}), \quad (5.5)$$

and also in the behavior of the fields $u_\varepsilon = (\eta^\varepsilon)^{-1}w_\varepsilon$ and $v_\varepsilon = \mu_0^{-1}z_\varepsilon$. In details, the Maxwell system (5.5) takes the form

$$\begin{aligned}
&i\text{curl}\mu_0^{-1}z_\varepsilon - iw_\varepsilon = q, \\
&-i\text{curl}(\eta^\varepsilon)^{-1}w_\varepsilon - iz_\varepsilon = r, \\
&\text{div } w_\varepsilon = 0, \ \text{div } z_\varepsilon = 0, \\
&((\eta^\varepsilon)^{-1}w_\varepsilon)_\tau|_{\partial \mathcal{O}} = 0, \ (z_\varepsilon)_n|_{\partial \mathcal{O}} = 0.
\end{aligned}$$

(5.6)

Let $\eta^0$ be the effective matrix defined by (1.16) and (1.17). Let $M^0$ be the effective Maxwell operator with the coefficients $\eta^0$ and $\mu_0$ (defined similarly to (5.3) and (5.4)). Consider the homogenized equation

$$(M^0 - iI) \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (5.7)$$

and define the functions $u_0 = (\eta^0)^{-1}w_0$ and $v_0 = \mu_0^{-1}z_0$. In details, (5.7) takes the form

$$\begin{aligned}
&i\text{curl}\mu_0^{-1}z_0 - iw_0 = q, \\
&-i\text{curl}(\eta^0)^{-1}w_0 - iz_0 = r, \\
&\text{div } w_0 = 0, \ \text{div } z_0 = 0, \\
&((\eta^0)^{-1}w_0)_\tau|_{\partial \mathcal{O}} = 0, \ (z_0)_n|_{\partial \mathcal{O}} = 0.
\end{aligned}$$

(5.8)

The classical results (see [BeLPap, Sa, ZhKO]) show that the fields $u_\varepsilon, w_\varepsilon, v_\varepsilon, z_\varepsilon$ weakly converge in $L_2(\mathcal{O}; \mathbb{C}^3)$ to the corresponding homogenized fields $u_0, w_0, v_0, z_0$, as $\varepsilon \to 0$.

5.3. The case where $q = 0$. Reduction of the problem to the model second order equation. If $q = 0$, the system (5.6) takes the form

$$\begin{aligned}
&w_\varepsilon = \text{curl } \mu_0^{-1}z_\varepsilon, \\
&\text{curl } (\eta^\varepsilon)^{-1}w_\varepsilon + z_\varepsilon = i\varepsilon r, \\
&\text{div } w_\varepsilon = 0, \ \text{div } z_\varepsilon = 0, \\
&((\eta^\varepsilon)^{-1}w_\varepsilon)_\tau|_{\partial \mathcal{O}} = 0, \ (z_\varepsilon)_n|_{\partial \mathcal{O}} = 0.
\end{aligned}$$

(5.9)
From (5.9) it follows that $z_{\varepsilon}$ is the solution of the problem
\[
\begin{align*}
\text{curl}(\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1}z_{\varepsilon} + z_{\varepsilon} &= i\varepsilon r, & \text{div}z_{\varepsilon} &= 0, \\
(z_{\varepsilon})_{n}|_{\partial\Omega} &= 0, & ((\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1}z_{\varepsilon})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.10)

Then the function $\varphi_{\varepsilon} := \mu_0^{-1/2}z_{\varepsilon}$ is the solution of the problem
\[
\begin{align*}
\mu_0^{-1/2}\text{curl}(\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{\varepsilon} + \varphi_{\varepsilon} &= i\mu_0^{-1/2}r, & \text{div}\mu_0^{1/2}\varphi_{\varepsilon} &= 0, \\
(\mu_0^{1/2}\varphi_{\varepsilon})_{n}|_{\partial\Omega} &= 0, & ((\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{\varepsilon})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.11)

Obviously, the solution of problem (5.10) is simultaneously the solution of the problem
\[
\begin{align*}
\mu_0^{-1/2}\text{curl}(\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{\varepsilon} - \mu_0^{-1/2}\nabla\text{div}\mu_0^{1/2}\varphi_{\varepsilon} + \varphi_{\varepsilon} &= i\mu_0^{-1/2}r, \\
(\mu_0^{1/2}\varphi_{\varepsilon})_{n}|_{\partial\Omega} &= 0, & ((\eta^\varepsilon)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{\varepsilon})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.11)

(Note that the condition $r \in J_0(\mathcal{C}; \mathbb{C}^3)$ automatically implies that $\text{div}\mu_0^{1/2}\varphi_{\varepsilon} = 0$.) The problem (5.11) coincides with (2.4) if $\nu = 1$ and $F = i\mu_0^{-1/2}r$.

Let $L_{\varepsilon}$ be the operator defined in Subsection 2.1 with the coefficients $\mu_0$, $\eta^\varepsilon$, and $\nu = 1$. We see that the solution $\varphi_{\varepsilon}$ of problem (5.10) can be written as $\varphi_{\varepsilon} = i(L_{\varepsilon} + I)^{-1}(\mu_0^{-1/2}r)$.

Similarly, in the case where $q = 0$, the effective system (5.8) takes the form
\[
\begin{align*}
\text{curl}(\eta^0)^{-1}\text{curl}\mu_0^{-1}z_{0} + z_{0} &= i\varepsilon r, & \text{div}z_{0} &= 0, \\
(z_{0})_{n}|_{\partial\Omega} &= 0, & ((\eta^0)^{-1}\text{curl}\mu_0^{-1}z_{0})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.12)

Then $z_{0}$ is the solution of the problem
\[
\begin{align*}
\text{curl}(\eta^0)^{-1}\text{curl}\mu_0^{-1}z_{0} + z_{0} &= i\varepsilon r, & \text{div}z_{0} &= 0, \\
(z_{0})_{n}|_{\partial\Omega} &= 0, & ((\eta^0)^{-1}\text{curl}\mu_0^{-1}z_{0})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.13)

Hence, the function $\varphi_{0} := \mu_0^{-1/2}z_{0}$ is the solution of the problem
\[
\begin{align*}
\mu_0^{-1/2}\text{curl}(\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{0} + \varphi_{0} &= i\mu_0^{-1/2}r, & \text{div}\mu_0^{1/2}\varphi_{0} &= 0, \\
(\mu_0^{1/2}\varphi_{0})_{n}|_{\partial\Omega} &= 0, & ((\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{0})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.13)

Clearly, the solution of problem (5.13) is simultaneously the solution of the problem
\[
\begin{align*}
\mu_0^{-1/2}\text{curl}(\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{0} - \mu_0^{-1/2}\nabla\text{div}\mu_0^{1/2}\varphi_{0} + \varphi_{0} &= i\mu_0^{-1/2}r, \\
(\mu_0^{1/2}\varphi_{0})_{n}|_{\partial\Omega} &= 0, & ((\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_{0})_{\tau}|_{\partial\Omega} &= 0.
\end{align*}
\] (5.14)

(The condition $r \in J_0(\mathcal{C}; \mathbb{C}^3)$ automatically implies that $\text{div}\mu_0^{1/2}\varphi_{0} = 0$.) The problem (5.14) coincides with (2.9), if $\nu = 1$ and $F = i\mu_0^{-1/2}r$.

Let $L^0$ be the effective operator defined in Subsection 2.2 with the coefficients $\mu_0$, $\eta^0$, and $\nu = 1$. Then the solution $\varphi_{0}$ of problem (5.13) can be written as $\varphi_{0} = i(L^0 + I)^{-1}(\mu_0^{-1/2}r)$.

5.4. Results for the Maxwell system. Applying Theorem 3.1 and using the relations
\[
z_{\varepsilon} = \mu_0^{1/2}\varphi_{\varepsilon}, \quad v_{\varepsilon} = \mu_0^{1/2}\varphi_{\varepsilon}, \quad z_{0} = \mu_0^{1/2}\varphi_{0}, \quad v_{0} = \mu_0^{1/2}\varphi_{0}, \quad F = i\mu_0^{-1/2}r,
\]
for $0 < \varepsilon \leq \varepsilon_1$ we obtain
\[
\|z_{\varepsilon} - z_{0}\|_{L_2(\Omega)} \leq \|\mu_0^{1/2}\|_{\varepsilon} - \varphi_{0}\|_{L_2(\Omega)} \leq C_1\|\mu_0^{1/2}\|_{\varepsilon} \|\varphi_{0}\|_{L_2(\Omega)},
\]
\[
\|v_{\varepsilon} - v_{0}\|_{L_2(\Omega)} \leq \|\mu_0^{1/2}\|_{\varepsilon} - \varphi_{0}\|_{L_2(\Omega)} \leq C_1\|\mu_0^{1/2}\|_{\varepsilon} \|\varphi_{0}\|_{L_2(\Omega)}.
\]

Now we apply Theorem 3.2. If $\nu = 1$, the solution of equation (1.23) is equal to zero: $\rho(x) = 0$, whence the function (3.6) takes the form
\[
\Psi_{\varepsilon} = \varphi_{0} + \varepsilon\mu_0^{-1/2}\Psi_{\varepsilon} S_{\varepsilon}\text{curl}\mu_0^{-1/2}\varphi_{0}.
\]
Denote \( \tilde{w}_0 = \text{curl}\mu_0^{-1/2}\varphi_0 \). Clearly, \( \tilde{w}_0 \) is an extension of the function \( w_0 = \text{curl}\mu_0^{-1/2}\varphi_0 \). From (3.7) it follows that for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\begin{align*}
\|z_e - z_0 - \varepsilon \Psi^e S_e \tilde{w}_0\|_{H^1(\Omega)} & \leq C_2 \|\mu_0\|^{1/2}\|\mu_0^{-1}\|^{1/2} \varepsilon^{1/2} \|r\|_{L^2(\Omega)}, \\
\|v_e - v_0 - \varepsilon \Psi^e S_e \tilde{w}_0\|_{H^1(\Omega)} & \leq C_2 \|\mu_0\|^{1/2}\|\mu_0^{-1}\|^{1/2} \varepsilon^{1/2} \|r\|_{L^2(\Omega)}.
\end{align*}
\]

(5.17) (5.18)

Next, the first equation in (5.9) implies that \( w_e = \text{curl}\mu_0^{-1/2}\varphi_e \), whence \( u_e = (\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_e \). Similarly, \( w_0 = \text{curl}\mu_0^{-1/2}\varphi_0 \) and \( u_0 = (\eta^0)^{-1}\text{curl}\mu_0^{-1/2}\varphi_0 \). Applying Theorem 3.3, we see that

\[
\|u_e - u_0 - \Sigma^e w_0\|_{L^2(\Omega)} \leq C_3 \|\mu_0^{-1}\|^{1/2} \varepsilon^{1/2} \|r\|_{L^2(\Omega)}.
\]

(5.19)

for \( 0 < \varepsilon \leq \varepsilon_1 \). Recalling that \( \Sigma(x) \) is the matrix with the columns \( \nabla \Phi_j(x), j = 1, 2, 3 \), where \( \Phi_j(x) \) is the \( \Gamma \)-periodic solution of problem (1.31), we represent this matrix in the form \( \Sigma(x) = \Xi(x)(\eta^0)^{-1} \), where \( \Xi(x) \) is the matrix with the columns \( \nabla Y_j(x), j = 1, 2, 3 \), and \( Y_j(x) \) is the \( \Gamma \)-periodic solution of the problem

\[
\text{div} \eta(x)(\nabla Y_j(x) + \tilde{e}_j) = 0, \quad \int_\Omega Y_j(x) \, dx = 0.
\]

(5.20)

Since \( (\eta^0)^{-1}w_0 = u_0 \), we rewrite (5.19) as

\[
\|u_e - u_0 - \Xi^e u_0\|_{L^2(\Omega)} \leq C_3 \|\mu_0^{-1}\|^{1/2} \varepsilon^{1/2} \|r\|_{L^2(\Omega)}.
\]

(5.21)

Combining the relations \( w_e = \eta^e u_e, w_0 = \eta^0 u_0 \), and (5.21), we deduce

\[
\|w_e - w_0 - \Upsilon^e w_0\|_{L^2(\Omega)} \leq C_3 \|\mu_0^{-1}\|^{1/2} \|\eta\|_{L^\infty} \varepsilon^{1/2} \|r\|_{L^2(\Omega)}.
\]

(5.22)

where \( \Upsilon(x) = \tilde{\eta}(x)(\eta^0)^{-1} - 1, \tilde{\eta}(x) := \eta(x)(\Xi(x) + 1) \).

Relations (5.15)–(5.18), (5.21), and (5.22) imply the following final result about homogenization of the solutions of the Maxwell system with \( q = 0 \).

**Theorem 5.3.** Suppose that \( \Omega \subset \mathbb{R}^3 \) is a bounded domain of class \( C^{1,1} \). Suppose that \( \mu_0 \) is a positive matrix with real entries and \( \eta(x) \) is a \( \Gamma \)-periodic matrix-valued function with real entries such that \( \eta(x) > 0 \) and \( \eta^{-1}\in L^\infty \). Let \( (w_e, z_e) \) be the solution of system (5.9) with \( r \in J_0(\Omega; C^3) \). Let \( u_e = (\eta^0)^{-1}w_e \) and \( v_e = \mu_0^{-1}z_e \). Suppose that \( (w_0, z_0) \) is the solution of the homogenized system (5.12) with the effective matrix \( \eta^0 \) defined by (1.16) and (1.17). Let \( u_0 = (\eta^0)^{-1}w_0 \) and \( v_0 = \mu_0^{-1}z_0 \). Suppose that \( \varepsilon_1 \) is subject to Condition 2.3. Then the following statements hold.

1) The fields \( v_e, z_e \) converge to \( v_0, z_0 \), respectively, in the \( L_2(\Omega; C^3) \)-norm, as \( \varepsilon \to 0 \). Moreover, for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\begin{align*}
\|v_e - v_0\|_{L^2(\Omega)} & \leq C_1 \varepsilon \|r\|_{L^2(\Omega)}, \\
\|z_e - z_0\|_{L^2(\Omega)} & \leq C_2 \varepsilon \|r\|_{L^2(\Omega)}.
\end{align*}
\]

2) Let \( S_e \) be the Steklov smoothing operator defined by (1.1). Let \( \Psi(x) \) be the matrix with the columns \( \text{curl}p_j(x), j = 1, 2, 3 \), where \( p_j \) is the \( \Gamma \)-periodic solution of problem (1.22). Let \( \tilde{w}_0(x) \) be the extension of the function \( w_0(x) \) to \( \mathbb{R}^3 \) constructed above. Then for \( 0 < \varepsilon \leq \varepsilon_1 \) the fields \( v_e, z_e \) satisfy approximations in the \( H^1(\Omega; C^3) \)-norm with the following error estimates:

\[
\begin{align*}
\|v_e - v_0 - \varepsilon \Psi^e S_e \tilde{w}_0\|_{H^1(\Omega)} & \leq C_3 \varepsilon^{1/2} \|r\|_{L^2(\Omega)}, \\
\|z_e - z_0 - \varepsilon \Psi^e S_e \tilde{w}_0\|_{L^2(\Omega)} & \leq C_4 \varepsilon^{1/2} \|r\|_{L^2(\Omega)}.
\end{align*}
\]

(5.23) (5.24)

3) Let \( \Xi(x) \) be the matrix with the columns \( \nabla Y_j(x), j = 1, 2, 3 \), where \( Y_j \) is the \( \Gamma \)-periodic solution of problem (5.20). Let \( \tilde{\eta}(x) := \eta(x)(\Xi(x) + 1) \) and \( \Upsilon(x) := \tilde{\eta}(x)(\eta^0)^{-1} - 1 \). Then for
0 < \varepsilon \leq \varepsilon_1 \), the fields \( u_\varepsilon \), \( w_\varepsilon \) satisfy approximations in the \( L_2(O; \mathbb{C}^3) \)-norm with the following error estimates:

\[
\| u_\varepsilon - u_0 - \Xi^\varepsilon u_0 \|_{L_2(O)} \leq C_5 \varepsilon^{1/2} \| r \|_{L_2(O)},
\]

(5.25)

\[
\| w_\varepsilon - w_0 - \Upsilon^\varepsilon w_0 \|_{L_2(O)} \leq C_6 \varepsilon^{1/2} \| r \|_{L_2(O)}.
\]

(5.26)

The constants \( C_1, C_2, C_3, C_4, C_5, \) and \( C_6 \) depend only on \( |\mu_0|, |\mu_0^{-1}|, \| \eta \|_{L_\infty}, \| \eta^{-1} \|_{L_\infty}, \) the parameters of the lattice \( \Gamma \), and the domain \( O \).

**Remark 5.4.** 1) We see that there is no symmetry in the results for the magnetic fields \( v_\varepsilon \), \( z_\varepsilon \) and the electric fields \( u_\varepsilon \), \( w_\varepsilon \). The magnetic fields converge in the \( L_2 \)-norm, with error estimates being of sharp order \( O(\varepsilon) \), and admit approximations in \( H^1 \) with the error terms of order \( O(\sqrt{\varepsilon}) \). The electric fields are approximated only in \( L_2 \) with the error terms of order \( O(\varepsilon) \). This is explained by the absence of symmetry in the statement of the problem: we assume that \( q = 0 \) in the right-hand side of system (5.5). For this reason, the function \( z_\varepsilon \) is a solution of the auxiliary second order equation, while \( w_\varepsilon \) is given in terms of the derivatives of this solution.

2) Note that the mean values of the periodic matrix-valued functions \( \Xi(x) \) and \( \Upsilon(x) \) are equal to zero. Therefore, by the mean value property, the correction terms \( \Xi^\varepsilon u_0 \) and \( \Upsilon^\varepsilon w_0 \) weakly tend to zero in \( L_2 \). Then relations (5.25) and (5.26) imply that the fields \( u_\varepsilon \) and \( w_\varepsilon \) weakly converge in \( L_2 \) to \( u_0 \) and \( w_0 \), respectively. This agrees with the classical results. The terms \( \Xi^\varepsilon u_0 \) and \( \Upsilon^\varepsilon w_0 \) can be interpreted as the zero order correctors.

3) Similarly, the correction terms \( \varepsilon \mu_0^{-1} \Psi^\varepsilon S_v w_0 \) and \( \varepsilon \Psi^\varepsilon S_v w_0 \) from (5.23), (5.24) weakly tend to zero in \( H^1 \). Hence, the fields \( v_\varepsilon \) and \( z_\varepsilon \) weakly converge in \( H^1 \) to \( v_0 \) and \( z_0 \), respectively.

Now, we distinguish the special cases. By Proposition 1.3 and Remark 1.4, from Theorem 5.3 we deduce the following statement.

**Proposition 5.5.** 1) Suppose that \( \eta^0 = \eta \), i.e., the columns of the matrix \( \eta(x) \) are divergence free. Then for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\| u_\varepsilon - u_0 \|_{L_2(O)} \leq C_5 \varepsilon^{1/2} \| r \|_{L_2(O)}.
\]

2) Suppose that \( \eta^0 = \eta_x \), i.e., the columns of the matrix \( \eta(x)^{-1} \) are potential. Then for \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
\| v_\varepsilon - v_0 \|_{H^1(O)} \leq C_3 \varepsilon^{1/2} \| r \|_{L_2(O)},
\]

\[
\| z_\varepsilon - z_0 \|_{H^1(O)} \leq C_4 \varepsilon^{1/2} \| r \|_{L_2(O)}.
\]
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