BLACK HOLE AREA QUANTIZATION*

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It has been argued by several authors that the quantum mechanical spectrum of black hole horizon area must be discrete. This has been confirmed in different formalisms, using different approaches. Here we concentrate on two approaches, the one involving quantization on a reduced phase space of collective coordinates of a Black Hole and the algebraic approach of Bekenstein. We show that for non-rotating, neutral black holes in any spacetime dimension, the approaches are equivalent. We introduce a primary set of operators sufficient for expressing the dynamical variables of both, thus mapping the observables in the two formalisms onto each other. The mapping predicts a Planck size remnant for the black hole.

Keywords: black holes, quantum gravity, quantum algebra

1. What to Quantize?

The central question of obtaining the theory of Quantum Gravity seems to be: what to “quantize”. A perturbative approach has been successfully attempted especially in the elegant work of B. DeWitt and is adequate if one learns to contend with the limitations of a nonrenormalizable theory. However the nonperturbative aspects of the theory would remain inaccessible. The formulation in terms of New Canonical Variables of Ashtekar may be taken to be the minimal consistent nonperturbative approach to Gravity. It is a formulation amenable to solution on loop spaces, originally pioneered by Mandelstam for QCD. Obtaining phenomenologically interesting solutions however remains an unsolved problem.

This has spurred a number of other approaches wherein one assumes certain ground states suggested by classical General Relativity as possible vacua. By focusing on a few collective coordinates one attempts a quantization of these. Several approaches to Quantum Cosmology may be viewed in this light and seem to enjoy success, again when interpreted with caution. In the following we shall discuss such

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a procedure, formulated in [14], applicable to any of the several Black Hole solutions assumed to be a given ground state of the theory. This will be referred to as reduced phase space quantization, or Approach I.

A parallel approach to quantizing the collective coordinates of a Black Hole is due to Bekenstein et al [15]. Unlike the canonical approach wherein the canonical variables generate all the dynamical variables relevant to a particular energy or length scale, in the algebraic approach one introduces each of the operators by hand, guided by phenomenological observables. Quantization then amounts to ascertaining all the commutation relations between the complete set of dynamical variables [8]. It is necessary in this approach that all the possible dynamical variables that are relevant at a particular energy scale are consistently identified. Cautious truncation is necessitated within the complete list of dynamical variables of a supposed complete theory. Finally, in lieu of knowledge of the dynamics governing the system, one relies on symmetries to propose a set of spectrum generating operators connecting the eigenstates of the observables. We shall refer to this as the algebraic approach or Approach II.

Black Holes seem to present to us the happy situation where one may be reasonably confident that all the collective coordinates of the system are known. Equivalently, being highly symmetric solutions facilitate the task of the algebraic approach. This has to do with the well known “no hair” property of the Black Holes. The only spoiler to this seems to be the possibility that as one approaches the quantum domain there may be phenomena occurring near the horizon which, although inaccessible to the asymptotic observer, require additional dynamical variables. We shall see that our ignorance of this can however be encoded in appropriate operators \( \hat{g}_{n} \).

A classical Black Hole is characterised by a short list of observables, viz., electric and magnetic charge, angular momentum and the mass or equivalently the surface area of the horizon. It has been argued by various authors, using widely different approaches, that the spectra of above observables are discrete. In particular, the horizon area of a black hole has been shown to have a uniformly spaced spectrum. Although the spectrum found in [13] is not strictly uniformly spaced, in the context of black hole entropy, as well as in a different regularisation scheme [14], the dominant contribution is equally spaced. In this talk we limit the discussion to show the equivalence of Approach I and Approach II for the case of Schwarzschild black hole. We propose a pair of primary operators \( P \) and \( P^\dagger \) together with a set of operators \( \hat{g}_{n} \) (see eqn. (16)) which can generate quantum algebras of both approaches consistently, thus implicitly mapping one model onto the other. While Approach II leaves open the value of the spectrum spacing, Approach I predicts the unit of spacing, as also a zero point value for the same. We argue that such a zero point remnant is to be expected in Approach II as well, and can be consistently included.

2. The Reduced Phase Space

It follows from the analysis of [13,14] that the dynamics of static spherically symmetric configurations in any classical theory of gravity in \( d \)-spacetime dimensions
is governed by an effective action of the form

\[ I = \int dt \left( P_M \dot{M} - H(M) \right) \]  

(1)

where \( M \) is the mass and \( P_M \) its conjugate momentum. The above action can be rigorously derived by assuming a generic gravity action (e.g., Einstein action, low energy string effective action or with or without cosmological constant), imposing spherical symmetry on it and doing a careful constraint analysis. However, intuitively it can be understood as follows: Birkhoff’s theorem for uncharged spherically symmetric solutions of gravity states that the mass \( M \) is the only time-independent and coordinate invariant parameter of the solution. That is \( \dot{M} = 0 \). The above effective action (with \( H \) being independent of \( P_M \)) is necessary and sufficient to guarantee this time independence. The boundary conditions imposed are those of \[17, 18\]. \( P_M \) has the interpretation of difference between the Schwarzschild times between left and right infinities \[19, 20, 21\].

Now to restrict ourselves to black holes (and simultaneously exclude all other spherically symmetric configurations). Also, as mentioned earlier, since the conjugate momentum \( P_M \) plays the role of ‘time’, motivated by Euclidean quantum gravity \[22\], we assume that it is periodic with period which is inverse the Hawking temperature \( (T_H(M)) \). That is,

\[ P_M \sim P_M + \frac{\hbar}{T_H(M)} \]  

(2)

This ensures that there is no conical singularity in the two dimensional euclidean section near the black hole horizon. However, note that the above identification implies that the physical phase space is a wedge cut out from the full \((M, P_M)\) plane, bounded by the \( M \) axis and the line \( P_M = \frac{\hbar}{T_H(M)} \). Thus, we make the following canonical transformation \((M, P_M) \rightarrow (X, \Pi_X)\), which on the one hand ‘opens up’ the phase space, and on the other hand, naturally incorporates the periodicity \[1, 2\]:

\[ X = \sqrt{\frac{A}{4\pi G_d}} \cos \left( 2\pi P_M T_H / \hbar \right) \]  

(3)

\[ \Pi_X = \sqrt{\frac{A}{4\pi G_d}} \sin \left( 2\pi P_M T_H / \hbar \right) \]  

(4)

where \( A \) is the black hole horizon area and \( G_d \) the \( d \)-dimensional Newton’s constant. Note that both \( A \) and \( T_H \) are functions of \( M \). It can be shown that the validity of the first law of black hole thermodynamics ensures that the above set of transformations is indeed canonical \[23\]. Also note that fixing the periodicity of \( P_M \) to be \( \hbar / T_H(M) \) uniquely fixes the prefactors in the right hand sides of \[3\] and \[4\]. Squaring and adding \[3\] and \[4\], we get:

\[ A = 4\pi G_d \left( X^2 + \Pi_X^2 \right) \]  

(5)

The r.h.s. is nothing but the Hamiltonian of a simple harmonic oscillator defined on the \((X, \Pi_X)\) phase space with mass \( \mu \) and angular frequency \( \omega \) given by \( \mu = \)
Black hole area quantization...

Upon quantization, the ‘position’ and ‘momentum’ variables are replaced by the operators:

\[ X \rightarrow \hat{X}, \quad \Pi_X \rightarrow \hat{\Pi}_X = -i\hbar \frac{\partial}{\partial X}, \quad (6) \]

and the spectrum of the black hole area operator follows immediately. With \( \ell_{pl} \) denoting the \( d \)-dimensional Planck length, and \( \bar{a} = 8\pi \ell_{pl}^{d-2} \),

\[ A_n = \bar{a}(n + \frac{1}{2}) \equiv n\bar{a} + a_{pl} \quad n = 0, 1, 2, \ldots \quad (7) \]

Thus \( \bar{a} \) signifies the basic quantum of area, and \( a_{pl} = \bar{a}/2 \) is its ‘zero-point’ value. Hawking radiation takes place when the black hole jumps from a higher to a lower area level, the difference in quanta being radiated away. The above spectrum shows that the black hole does not evaporate completely, but a Planck size remnant is left over at the end of the evaporation process. It may be noted that the periodic orbits in the phase space under consideration admit of an adiabatic invariant. As mentioned earlier, in the present example, the latter is in fact the horizon area of the black hole, just as it had been conjectured previously. This follows from the integral

\[ \text{Adiabatic Invariant} = \oint \Pi_X dX = \frac{A}{4G}. \]

3. Area as an Adiabatic Invariant

Now let us consider Approach II. It has been argued that for a non-extremal black hole the area is an adiabatic invariant, and the spectrum emerges from a proposed algebra of black hole observables. In the present work we take the case of neutral black hole in zero angular momentum state. With slight modification of the notation of \[ \text{it is assumed that there exists an operator } \hat{R}_{ns_n} \text{ which creates a single black hole state from vacuum } |0\rangle \text{ with area } a_n \text{ in an internal quantum state } s_n: \]

\[ \hat{R}_{ns_n} |0\rangle = |n, s_n\rangle, \quad (8) \]

\[ \hat{A} |n, s_n\rangle = a_n |n, s_n\rangle \quad (9) \]

We make the caveat that \( s_n \in \{0, 1, \ldots, m_n - 1\} \) as in \[ \text{such that the degeneracy of states with same area eigenvalue } a_n, \text{ obeys in } m_n \propto a_n. \]

Bekenstein introduced a minimal set of linear operators satisfying the following requirements: (i) The commutator bracket between the operators must result in a linear combination of the operators in the set. In other words, the algebra of black hole operators must be linear and closed. (ii) The area operator must commute with generators of gauge transformations and rotations. This imposes the physical requirement of invariance of the horizon area under these transformations.

The set of linear operators for the neutral black holes will be area operator \( \hat{A} \), black hole creation operator \( \hat{R}_{ns_n} \) and its adjoint operator \( \hat{R}^\dagger_{ns_n} \) and identity operator \( \hat{I} \). Bekenstein assumes that the vacuum state |0\rangle has zero area in the construction of the algebra. We shall denote Bekenstein’s area operator as \( \hat{A} \) with
eigenvalues $a'_n$ such that the vacuum area is $a'_0 = 0$. We will shortly see the relation between the operators $\hat{A}$ and $\hat{A}'$ and their respective eigenvalues $a_n$ and $a'_n$.

With these requirements, Bekenstein’s algebra for neutral black hole will be [3]:

\begin{align}
[\hat{A}', \hat{R}_{ns_n}] &= a'_n \hat{R}_{ns_n}, \\
[\hat{R}_{ns_n}, \hat{R}_{ms_m}] &= \epsilon_{nm} \hat{R}_{ks_k} \quad (\epsilon_{nm} \neq 0 \text{ iff } a'_n + a'_m = a'_k), \\
[\hat{A}', [\hat{R}_{ms_m}, \hat{R}_{ns_n}]] &= (a'_n - a'_m)[\hat{R}_{ms_m}, \hat{R}_{ns_n}] \quad \text{if } a'_n > a'_m, \\
&= 0 \quad \text{otherwise}
\end{align}

Eqn. (11) implies that the black hole state created by a commutator of two black hole creation operators $(\hat{R}_{ns_n}, \hat{R}_{ms_m})$ will be another single black hole state $|k, s_k\rangle$ provided its area satisfies $a'_k = a'_m + a'_n$. Though the relation $[\hat{A}', \hat{R}_{ms_m}] = -a'_n \hat{R}_{ns_n}$ is used to obtain eqn. (12), the positive definite nature of area operator $\hat{A}'$ requires the inequality condition $a'_n > a'_m$. Clearly, the spectrum of the above algebra \{ $a'_n$ \} involves both addition and subtraction of area levels which is possible if and only if the area levels are equally spaced, i.e.,

$$a'_n = n\bar{b}, \quad n = 0, 1, 2, \ldots$$

where $\bar{b}$ is some positive constant with dimensions of area.

It is obvious that the neutral black hole algebra (10 - 12) is unchanged under the shift of the area operator:

$$\hat{A}' \rightarrow \hat{A}' + \bar{c} \hat{I} \equiv \hat{\tilde{A}}$$

where $\bar{c}$ is an arbitrary constant. This relation between $\hat{A}$ and $\hat{A}'$ implies their respective eigenvalues to satisfy

$$a_n = a'_n + \bar{c} .$$

Equivalently, the vacuum state will have non-zero area $a_0 = c$. The situation is similar to the problems of single particle Quantum Mechanics where nontrivial zero-point energy always exists except for a free particle. For the case of the Hydrogen atom this is due to quantizing only the relative coordinates but not the coordinates of the centre of mass. In the case of the black hole, the same is to be expected because we are not quantizing the trivial collective coordinates corresponding to its location. The $\bar{c}$ must therefore be nonzero, presumably equal to $\bar{b}$ up to a dimensionless constant of order unity. If we identify $\bar{b}$ with the unit $\bar{a}$ obtained systematically in Approach I, it is reasonable to also identify $\bar{c}$ with $a_{Pl} = \bar{a}/2 = 4\pi\hbar^2/\bar{m}$.  

4. Conciliation

Our next step is to find a realisation of the operators in Approach II in terms of the fundamental degrees of freedom $(M, \Pi_M)$ in Approach I. We propose a representation of the algebra (10), (12) with the following form for the black hole creation operator $\hat{R}_{ns_n}$ and area operator $\hat{A}$:

$$\hat{R}_{ns_n} = (P^{1/n}) \hat{g}_{sn} \; ; \; \hat{A} = (\hat{P}^2 + 1/2)\bar{a} ,$$

where $\hat{g}_{sn}$ and $\hat{P}$ are operators that satisfy the algebra (10), (12).
Black hole area quantization ...

where \( \hat{P}^\dagger \) (\( \hat{P} \)) raises (lowers) the area level \( n \) to \( n+1 \) (respectively, \( n-1 \)). The operators \( \hat{g}_{sn} \) are similar to the secret operators in Approach I. We postulate that these two sets of operators satisfy the following commutation relations:

\[
[\hat{P}, \hat{P}^\dagger] = 1, \\
[\hat{P}, \hat{g}_{sm}] = [\hat{P}^\dagger, \hat{g}_{sm}] = 0, \\
[\hat{g}_{sm}, \hat{g}_{sn}] = \epsilon_{mn}^k \hat{g}_{sk}
\]

where \( \epsilon_{mn}^k \neq 0 \) iff \( k = m + n \). (19)

Equation (19) ensures eqn. (11); however it should be remembered that the operators \( \hat{g}_{sn} \) have a meaning only within the product form \( (\hat{P}^\dagger)^n \hat{g}_{sn} \). Comparison with the reduced phase space approach (3-7) immediately gives us the form of \( \hat{P}^\dagger \) as

\[
\hat{P}^\dagger = \frac{1}{\sqrt{2\hbar}} \left[ \hat{X} - i\hat{\Pi}_X \right].
\]

(20)

Note that the area operator (16) becomes identical to that in Approach I, namely Eq.(5). The identification (20) shows that the black hole creation operator \( \hat{R}_{nsn} \) can be expressed in terms of fundamental gravitational degrees of freedom \((M, P_M)\) via (3),(4) and (16) in the following way:

\[
\hat{R}_{nsn} = (\hat{P}^\dagger)^n \hat{g}_{sn} ; \hat{P}^\dagger = \sqrt{\frac{\hat{A}(M)}{8\pi G_d\hbar}} \exp \left(-i2\pi \hat{P}_M \hat{T}_H(M)\right).
\]

(21)

We see that the secret operator \( \hat{g}_{sn} \) in algebraic approach does not have a representation in terms of the fundamental gravitational degrees of freedom. This is consistent with the no hair theorem where asymptotic observer cannot detect the internal quantum state of the black hole.

5. Conclusion

We have shown that approaches I and II are equivalent in the zero angular momentum sector from the asymptotic observer viewpoint, and hence give rise to qualitatively similar spectra for the black hole area. In Approach II, in ref. 2 the remnant (or zero-point) area was chosen to be zero. Relying on single particle Quantum Mechanics experience we advocate taking this to be non-zero; the presence of the same in no way alters any of the commutators (14) - (16). However note that the precise value of the remnant remains undetermined in this approach. In the reduced phase space approach on the other hand, the remnant is explicitly determined to be a multiple of the Planck area in the relevant dimension. Since the latter is the only natural length scale in quantum gravity, this seems satisfactory. But a fundamental conclusion it suggests is that the lowest energy state of the neutral black hole system is unique, like the Hydrogen atom ground state.

Also note that the discrete spectrum (5) means that Hawking radiation would consist of discrete spectrum lines, enveloped by the semi-classical Planckian distribution. As argued in 1, for Schwarzschild black holes of mass \( M \), the gap is order \( 1/M \), which is comparable to the frequency at which the peak of the Planckian distribution takes place. Hence the spectrum would be far from being a continuum, and
Das, Ramadevi and Yajnik

can potentially be tested if and when Hawking radiation becomes experimentally measurable. Note that this is quite distinct from the predictions of loop quantum gravity, where it was shown that the resulting Hawking spectrum is practically continuous. It would also be interesting to explore the implications of the Planck size remnant to the problem of information loss, since the presence of the former can considerably influence Hawking Radiation near the end stage of the black hole.

A further test of the correspondence elucidated in this article would be to apply it to non-spherically symmetric as well as charged black holes. Since both the approaches have dealt with electric charge, analyzing the area and charged spectrum of a charged black hole should be straightforward. However, it is to be borne in mind that at least for semi-classical configurations (those with large quantum numbers), the extremality bound has to be obeyed, at least approximately. Incorporating angular momentum might be somewhat tricky as the reduced phase space approach has not been explored beyond the realm of spherical symmetry. We hope to report on these and other related issues in the near future.

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