Controlling the motion of a travelling wave

Robert A. Van Gorder*

July 19, 2021

Abstract

Travelling waves arise in several areas of science, hence modification of travelling wave properties is of great interest. While many studies have demonstrated how to control the form or shape of a solitary travelling wave by employing soliton or dispersion management, far less is known about controlling the motion of a travelling wave while keeping its form unchanged. We present a technique for control of travelling wave motion using time-varying coefficients, which we refer to as wave management. The technique allows one to alter the trajectory of a travelling wave, slowing, stopping, or reversing the direction of the wave, all while ensuring that the wave form is unchanged, and we illustrate this through multiple examples. Our results suggest that wave management is a promising tool for applications where one needs to modify the motion of a wave while preserving its form, and we highlight several potential applications.

keywords: travelling waves, nonlinear wave equations, reaction-diffusion equations, wavespeed control, non-autonomous partial differential equations

1 Introduction

The theory of travelling waves has a long and storied history. The linear one-dimensional wave equation was first derived as a model for a vibrating string by d’Alembert [17, 18], who also solved it, obtaining the d’Alembert formula which shows how the initial data for the wave equation is transported in opposite directions along the real line in the form of travelling waves. The d’Alembert formula is still used today. Another major advance to the science of travelling waves came due to Russel [67], who reported observing a solitary wave of translation within a canal. This report motivated the work of Lord Rayleigh [64], Boussinesq [11], and Korteweg and de Vries [39], all of which eventually resulted in a theory of solitary water waves and the development of the Korteweg-de Vries (KdV) equation, with its famous soliton solution. The most interesting feature of solitary waves is that they maintain their form while propagating at a constant velocity. Solitons are a special type of solitary wave which have the property that they maintain their form (except for a change in their phase shift) after a collision. Interest in solitary waves and the more specialized soliton has only increased over the last 70 years, with applications found in fluid mechanics and plasma dynamics, mathematical physics, optics, and theoretical biology [69]. The study of sigmoidal wavefronts, where one side of the space domain has a low concentration and the other a high concentration, with the wave itself connecting these two extremes, has origins in Fisher’s study of

*Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin 9054, New Zealand (rvangorder@maths.otago.ac.nz)
population genetics\[25\]. Fisher proposed a model for the spread of an advantageous gene within a population, determined the existence of travelling wavefronts, and studied their properties. In the same year, Kolmogorov, Petrovsky, and Piskunov proposed the same equation for this application\[38\], and hence the equation is sometimes referred to as the Fisher-KPP equation. The model is actually a diffusion equation with a reaction term (in the modern literature, these are referred to as reaction-diffusion equations). These studies paved the way for much of the work on reaction-diffusion systems which has taken place over the past 80 years, with applications of these systems to genetics, ecology, physiology, chemistry, and physics\[29\].

With all of this interest in travelling waves, there has been some effort directed toward understanding how the wavespeed of such waves changes as a function of system parameters, and applying this understanding in order to control some aspect of a given travelling wave. Control of the wavespeed of a travelling wave has been shown to be useful in several applications, ranging from optics\[77\] to neuroscience\[65\] \[44\]. The most common approach is to vary the model parameters, with the constant wavespeed adjusting accordingly, although there are limitations to this approach depending on relevant parameter regimes for which waves might exist. Another option discussed recently for controlling the constant speed of a travelling wave is to couple the governing equation to a type of controller employing a Stefan condition at one moving boundary. This approach was used to give a desired constant wavespeed in variations of the Fisher-KPP equation\[21\] \[24\] (even though the standard Fisher-KPP equation defined on the real line might not support such wavespeeds). Feedback controllers have been used to control the position of a travelling wave in reaction diffusion systems\[46\], and it has been known for some time that such controllers can stabilize travelling waves\[9\]. These approaches still result in a constant wavespeed and can be viewed as passive controls, with each new application of the control resulting in a new value of the constant wavespeed. Another option is an active control mechanism where the wavespeed is varied during transit of the wave. Despite the possible applications, the latter option is barely explored, and will be the focus of the present paper.

Dispersion management or soliton management is a popular technique for the modification of nonlinear waves in optics\[42\] \[2\] \[8\], atomic matter waves\[20\], and Bose-Einstein condensates\[36\], with the approach employing temporal or spatial modification of the dispersion and loss/gain terms, resulting in a non-autonomous problem. We refer to all of these techniques collectively as wave management. Wave management has been used to modify the amplitude or width of a solitary wave, resulting in a kind of self-similar transform of the original wave variables\[72\] \[73\]. Wave management not only allows for the control of wave properties, but can also be employed to stabilize waves against perturbations\[85\]. Despite the myriad uses for wave management, it was only recently shown that wave management can be used to modify the speed - rather than the structure - of a travelling wave, with the method used to slow, stop, or even reverse a solitary wave in the cubic nonlinear Schrödinger equation\[6\].

The focus of this paper is to more generally discuss the utility of wave management for controlling the motion of a travelling wave over time. Rather than modifying the wave envelope as commonly done in soliton management, we demonstrate that wave management can be used to preserve the structure of the travelling wave, while allowing it to propagate differently over different time intervals, resulting in a time-dependent wavespeed which is user controlled. We first outline the general method for employing wave management to control the speed and direction of a travelling wave. We exhibit several interesting choices of controls which change the wavespeed either gradually or rapidly, either of which may be useful depending upon the application. We
next highlight the application of wavespeed control to solitary waves and wavefronts, using the Korteweg-de Vries and Fisher-KPP equations as respective prototypical examples. We finally discuss our results, and briefly describe how our approach may find application in various areas of science and engineering.

2 Management of a travelling wave

Consider the nonlinear partial differential equation

\[ \frac{\partial u}{\partial t} + \delta_0 L[u] + \gamma_0 N[u] = 0, \]  

(1)

where \( L \) is a differential operator (the dispersion or diffusion term) on the space \( \mathbb{R}^m \), \( N \) is a nonlinear differential operator (the gain or loss term), while \( \delta_0 \) and \( \gamma_0 \) are parameters which control the relative strength of each term. We allow for \( N \) to contain spatial derivatives, although these should generally be lower order than the highest order derivatives in \( L \). Similarly, we allow \( L \) to be nonlinear, so long as it contains the highest order space derivatives. Assume that the transformation \( u(x, t) = U(z) \), \( z = k \cdot x - ct \), \( k \in \mathbb{R}^m \) a constant vector, puts (1) into the form of the ordinary differential equation

\[ -c \frac{dU}{dz} + \delta_0 \hat{L}[U] + \gamma_0 \hat{N}[U] = 0, \]  

(2)

where the hat denotes operators which are evaluated along the wave coordinate, \( z \). If this equation admits a solution \( U(z) \), then this gives a travelling wave solution of (1). The wavespeed, \( c \), will in general be a function of \( \delta_0 \) and \( \gamma_0 \), e.g., \( c = c(\delta_0, \gamma_0) \). As such, a specific choice of \( \delta_0 \) and \( \gamma_0 \) may permit (or, prohibit) a certain wavespeed. Another option is that no such \( c \) exists for this choice of parameter values, in which case we have failed to find a travelling wave solution.

Time-management of dispersion and nonlinearity require that the respective scaling of each term be allowed to vary in time, and the managed analogue of (1) reads

\[ \frac{\partial u}{\partial t} + \delta(t) L[u] + \gamma(t) N[u] = 0, \]  

(3)

The partial differential equation (3) is now non-autonomous, since the coefficients depend on time. As part of the wave management, we choose \( \delta(t) \) and \( \gamma(t) \) so that \( u(x, t) = U(Z) \) where \( Z = k \cdot x - C(t) \) for some differentiable function \( C(t) \) and \( U \) is a solution of the relevant travelling wave ODE with \( z \) replaced by \( Z \),

\[ -\frac{dC}{dt} \frac{dU}{dZ} + \delta(t) \hat{L}[U] + \gamma(t) \hat{N}[U] = 0. \]  

(4)

Plugging the solution \( u(x, t) = U(Z) \) into (3) results in a coupled system of parameter constraints

\[ S \left( \delta(t), \gamma(t), \frac{dC}{dt} \right) = 0, \]  

(5)

and a managed solution \( U(Z) \) to (3) exists provided that the system (5) admits a solution for \( \delta(t) \) and \( \gamma(t) \). We refer to (5) as the constraint system, as it determines the solvability condition for the management parameters. The constraint system (5) depends on \( k \) as well as any other parameters present in \( U(z) \), such as the amplitude of the wave.
If (5) admits a solution \(\delta(t) = \delta \left( \frac{dC}{dt} \right), \gamma(t) = \gamma \left( \frac{dC}{dt} \right)\), then the non-autonomous managed system in (3) admits the exact solution \(u(x, t) = U(k \cdot x - C(t))\). The envelope \(U\) corresponds to the solution \(u(x, t) = U(k \cdot x - ct)\) of the autonomous system (1), and hence the shape of the wave is unchanged under the control. In addition to preserving the shape, we remark that the managed solution is integrable provided the autonomous system (1) is integrable, whereas the managed solution is unchanged under the control. A management strategy which involves space will distort the shape of the wave, and temporal changes to the wavenumber vector \(k\) or the amplitude of the wave will also distort the wave. Although we allow for wave solutions on the space domain \(\mathbb{R}^m\), we consider propagation in a one-dimensional manner based on the choice of wavenumber vector \(k \in \mathbb{R}^m\). Although other types of waves are possible (such as radial waves which propagate outward from a single point), the mass or concentration measured by the wave must become diluted as the wavefront expands in size and hence the form of the wave is not preserved (even in the autonomous setting), regardless of controlling the wavespeed. Some of these points are discussed further in the SI Appendix.

### 2.1 Asymptotic wavespeed controls

Assuming one or more pairs of \(\delta(t)\) and \(\gamma(t)\) satisfying (5) exist, it is natural to explore how to choose \(C(t)\) so that the wave motion is controlled. A choice \(C(t) = c_1 \cdot t\) for \(c \neq 0\) a constant will result in a wave with constant wavespeed \(c_1\) in time, as is standard. In this case, (5) reduces to an algebraic equation, giving constant solutions \(\delta\) and \(\gamma\) (if they exist).

Suppose one wishes to change the motion of a travelling wave initially travelling with wavespeed \(c_1\) so that it later travels with wavespeed \(c_2\). The seemingly obvious choice would be to consider a function of the form

\[
C(t) = \begin{cases} 
  c_1, & \text{for } t \leq T, \\
  c_2, & \text{for } t > T.
\end{cases}
\]

However, this function is not differentiable at \(t = T\), and hence the transforms we have taken would not generate a differentiable solution to the problem. To remedy this, we must somehow mollify the transition between distinct wavespeeds.

In order to modify a wave travelling with initial wavespeed \(c_1\) so that it eventually travels with wavespeed \(c_2\), while keeping the function \(C(t)\) smooth, we remark that any of the following functions will accomplish this goal:

\[
C(t) = \frac{c_2 + c_1}{2} (t - T) + \frac{c_2 - c_1}{2} \sqrt{(t - T)^2 + X^2},
\]

\[
C(t) = \left( \frac{c_2 + c_1}{2} + \frac{c_2 - c_1}{2} \tanh(X(t - T)) \right) (t - T),
\]

\[
C(t) = \frac{c_2 + c_1}{2} (t - T) + \frac{c_2 - c_1}{2X} \log \cosh(X(t - T)).
\]

Each of these functions scales like \(C(t) \sim c_1 \cdot t\) for \(t \ll T\), \(C(t) \sim c_2 \cdot t\) for \(t \gg T\), and remains smooth for all time. The constants \(T \in \mathbb{R}\) and \(X > 0\) determine the time and sharpness of the change in
motion. Of course, the choice of $C(t)$ is not unique, with many other examples possible. For all of these functions, the agreement with a fixed wavespeed of either $c_1$ or $c_2$ is asymptotic, yet in specific applications of experiments it will likely be more desirable to transition between wavespeeds over a finite time interval.

### 2.2 Precise wavespeed controls

All of the functions \([7]\) tend exactly to the asymptotic scalings $C(t) \sim c_1 t$ for $t \ll T$, $C(t) \sim c_2 t$ for $t \gg T$. However, for practical applications, it will be useful to completely transition to a new wavespeed in a finite and fixed time, rather than just asymptotically. Taking our motivation from bump functions which are smooth yet compactly supported, consider

$$C(t) = \begin{cases} 
 c_1(t-T), & \text{for } t \leq T, \\
 c_1(t-T) + \frac{(c_2 - c_1)(t-T)}{1 + \exp \left( \frac{1}{t-T} - \frac{1}{t-X} \right)}, & \text{for } T < t < T + X, \\
 c_2(t-T), & \text{for } t \geq T + X.
\end{cases} \tag{8}$$

This choice of $C(t)$ gives a wavespeed of $c_1$ for all $t \leq T$ and a wavespeed of $c_2$ for all $t \geq T + X$. The duration of the transition, $X$, must remain positive yet can be made arbitrarily small with \([8]\) remaining a smooth function. Note that the function is smooth although not analytic, as is characteristic of bump functions.

In order to stop a wave initially travelling with wavespeed $c$, one would take $c_1 = c$ and $c_2 = 0$. On the other hand, suppose at $t = T$ we wish to stop a travelling wave with wavespeed $c_1$ for a duration of $\Delta t$ units of time, before allowing the wave to again propagate with wavespeed $c_2$. To do so, we may choose

$$C(t) = \begin{cases} 
 c_1(t-T), & \text{for } t \leq T, \\
 c_1(t-T) + \frac{-c(t-T)}{1 + \exp \left( \frac{1}{t-T} - \frac{1}{t+X-T} \right)}, & \text{for } T < t < X + T, \\
 0, & \text{for } X + T \leq t \leq X + T + \Delta t, \\
 c_2(t-T-2X-\Delta t), & \text{for } X + T + \Delta t < t < 2X + T + \Delta t, \\
 1 + \exp \left( \frac{1}{t-(T+X+\Delta t)} - \frac{1}{t+2X+\Delta t-T} \right), & \text{for } X + T + \Delta t < t < 2X + T + \Delta t, \\
 c_2(t-T-2X-\Delta t), & \text{for } t \geq 2X + T + \Delta t.
\end{cases} \tag{9}$$

One may similarly consider additional transitions, in order to obtain more complicated wavespeed management schemes. We illustrate several representative control strategies for slowing, stopping, or even reversing travelling waves, in Fig. \[4\].
Figure 1: Plots of several distinct control schemes $C(t)$ corresponding to $8$ in (a,b,c) and $9$ in (d). In particular, we choose (a) $c_1 = 1$, $c_2 = 2$, (b) $c_1 = 1$, $c_2 = 0$, (c) $c_1 = 1$, $c_2 = -1$ all with a transition at $T = 5$, and transition interval of size $X = 0.1$. These correspond to cases where the solitary wave is sped up ($c_2 > c_1$), stopped ($c_2 = 0$), and reversed ($\text{sgn}(c_2) = -\text{sgn}(c_1)$) beyond time $t = T = 5$. For (d), we take $c_1 = c_2 = 1$ with $\Delta t = 5$, $X = 0.1$ in $9$. The wave is stopped at $t = 0$ for a duration of $\Delta t = 5$ units, before moving along with original wavespeed. For each management choice (a-d) we plot the managed KdV solitary wave $13$ obtained from using $C(t)$ as given in $8$ over space interval $x \in [-10, 10]$ and indicated time interval. The color scale ranges from dark blue (value of 0) to white (value of 0.5); white regions comprise the cores of respective solitary waves.
3 Examples of wave management

3.1 The controlled Korteweg-de Vries soliton

We first demonstrate wave management using the Korteweg-de Vries (KdV) equation \[11, 39\], which admits the well-known soliton solution \[19\]

\[ u(x, t) = \frac{1}{2} \text{sech}^2 \left( \frac{x - t + x_0}{2} \right), \]

where \( x_0 \) is a constant which selects the location of the peak at \( t = 0 \). Consider the managed KdV equation

\[
\frac{\partial u}{\partial t} + \delta(t) \frac{\partial^3 u}{\partial x^3} + 6\gamma(t) u \frac{\partial u}{\partial x} = 0. \tag{10}
\]

Choosing \( u(x, t) = U(Z) \), \( Z = x - C(t) + x_0 \), we have

\[
- \frac{dC}{dt} \frac{dU}{dZ} + \delta(t) \frac{d^3U}{dZ^3} + 6\gamma(t) U \frac{dU}{dZ} = 0. \tag{11}
\]

The constraint system (5) comprises \( \frac{dC}{dt} = \delta(t) \), \( \frac{dC}{dt} = \gamma(t) \), so we choose the management parameters

\[ \delta(t) = \gamma(t) = \frac{dC}{dt}. \tag{12} \]

Under (12), we find that (10) has the managed solution

\[ u(x, t) = \frac{1}{2} \text{sech}^2 \left( \frac{x - C(t) + x_0}{2} \right). \tag{13} \]

To illustrate the approach, we plot the exact solution (13) in Fig. 1 under four different control strategies.

It is useful to see whether our managed exact solutions (13) agree with predictions from numerical simulations. We choose two different management techniques. First, we choose (8) with \( c_1 = -1, c_2 = 1, T = 10, X = 0.1 \), which corresponds to an initial wavespeed of \(-1\) and then a reverse in direction to a wavespeed of \(1\) after \( t = 10 \). The resulting exact solution is plotted in the top left panel of Fig. 2, while a comparison between this exact solution (13) and a numerical simulation is plotted in the top right panel of Fig. 2. The second managed wavespeed we choose takes the form

\[
C(t) = \int_0^t \frac{2 - \tanh(\xi - 17) + \tanh(\xi - 33)}{2} d\xi. \tag{14}
\]

This management technique takes a wavespeed of \(1\), slows it to zero for a time, and then speeds it back up again to \(1\), where it remains. The resulting exact solution (13) using (14) is plotted in the top left panel of Fig. 2, while a comparison between this exact solution (13) and a numerical simulation is plotted in the top right panel of Fig. 2. For more details on the numerical method, see the SI Appendix.

The numerical simulation shows good agreement with the exact solution, supporting the assertion that the managed KdV solitons given by (13) are indeed robust. The slight offset at late times is due to the \( O(10^{-3}) \) error in the numerical simulations which presents as a small translational error.
Figure 2: Plots of managed KdV solitons for cases where the wave is reversed (top) and slowed, stopped, and then sent on its way (bottom). The left panels show the exact solution (13) plotted over space and time. The right panels show both the exact solution (13) and the numerical simulation at indicated times. In the top row, we use the management technique (8) (with $c_1 = -1$, $c_2 = 1$, $T = 10$, $X = 0.1$). The soliton is moved left until $t = 10$, at which time it turns and then moves right for the remainder of the simulation. In the bottom row, we use the management technique (14). The soliton is moved to the right, frozen in place, and then sent on its way.
3.2 The controlled Fisher-KPP wavefront

In addition to travelling wave solutions of nonlinear wave equations, travelling waves are commonly studied in the context of reaction-diffusion systems. Unlike the linear wave equation, for which travelling waves are a natural solution, the linear diffusion equation admits no travelling waves. However, the combination of diffusion with nonlinear reaction terms allows for the existence of travelling waves in some reaction-diffusion systems. Unlike solitary waves that have one or multiple discrete peaks, the most commonly encountered travelling waves in the reaction-diffusion setting are moving sigmoidal wavefronts which comprise heteroclinic connections between two constant states. One such reaction-diffusion equation is the Fisher-KPP equation \[25\], and a managed form of this equation reads

\[
\frac{\partial u}{\partial t} - \delta(t) \frac{\partial^2 u}{\partial x^2} - \gamma(t)u(1 - u) = 0. \tag{15}
\]

The standard Fisher equation admits travelling wavefronts scaling like \(u \sim (1 + \exp(x - ct + x_0))^2\) \[3\]. Assuming a similar type of solution for the non-autonomous \eqref{15}, and placing this into \eqref{15} we find that the constraint system \eqref{5} takes the form

\[
4\delta(t) + \gamma(t) - 2\frac{dC}{dt} = 0, \quad \gamma(t) - \delta(t) - \frac{dC}{dt} = 0. \tag{16}
\]

Therefore, choosing the management parameters

\[
\delta(t) = \frac{1}{5} \frac{dC}{dt}, \quad \gamma(t) = \frac{6}{5} \frac{dC}{dt}, \tag{17}
\]

the managed Fisher equation \eqref{15} has the exact solution

\[
u(x, t) = (1 + \exp(x - C(t) + x_0))^2. \tag{18}\]

We simulate the managed Fisher-KPP equation \eqref{15}, and compare the resulting solutions with the exact solution \eqref{18}. We first choose the management technique \eqref{14}, which takes a travelling wavefront, stops it for a while, and then sends it along at the original wavespeed. The Fisher-KPP wavefront obtained using \eqref{14} is shown in the top two panels of Fig. 3. We also use the management technique

\[
C(t) = \int_0^t \frac{1 - \tanh(\xi - 27)}{2} d\xi, \tag{19}
\]

which takes a travelling wavefront with wavespeed of 1 and gradually stops it after \(t \approx 27\). (Although the value of the wavespeed is exponentially small past \(t \approx 27\), this is as good as zero when comparing between the exact solutions and the numerical simulations.) The wavefront obtained using \eqref{19} is shown in the lower two panels of Fig. 3. Again, numerical simulations of the managed Fisher-KPP equation \eqref{15} show good agreement with the corresponding exact solutions \eqref{18}, supporting the view that the managed wavefronts given by \eqref{18} are robust.

3.3 Additional examples and considerations

In addition to the two example illustrated above, we provide a number of other examples of myriad nonlinear waves which are possible to control in the SI Appendix. We also discuss how to extend the method of wave management to systems of equations.
Figure 3: Plots of managed Fisher-KPP wavefronts for cases where the wave is slowed, stopped, and then sent on its way (top) or stopped and held stationary for all time (bottom). The left panels show the exact solution (18) plotted over space and time. The right panels show both the exact solution (18) and the numerical simulation at indicated times. In the top row, the wavefront is stopped for a while, before being sent along at the original wavespeed, according to the management technique (14). In the bottom row, the wavefront is stopped and frozen in place for all remaining time, using the management technique (19).
4 Discussion

We have described a method for controlling the motion (speed and direction) of a travelling wave while keeping its form unchanged; we refer to this method as wave management, as it is motivated by the dispersion management and soliton management literature. Wave management involves treating the dispersion and loss/gain parameters as functions of time, resulting in a non-autonomous partial differential equation which must then be solved. Under the assumption of a wave envelope equivalent to that of the autonomous case yet with a time-varying wavespeed, we obtain a solvability condition that, when satisfied, results in a set of specific management parameters which ensures the existence of the desired travelling wave. When these management parameters exist, one is able to then control the motion of a travelling wave as desired, all while preserving the structure of the wave.

The preservation of the wave structure is a key point to the novelty of our method relative to other methods in the literature, such as dispersion or soliton management (as typically applied), which modify the envelope and structure of the nonlinear wave – meaning that any attempt at control using these methods tends to distort the structure of the wave. After outlining the theory, we demonstrate the utility of the method by way of specific examples, comprising both nonlinear wave equations and nonlinear reaction-diffusion equations. For all cases considered, the exact solutions show excellent agreement with the numerical simulations, suggesting that our exact solutions to the non-autonomous problem indeed robust. As a result, we anticipate it will be possible to employ wave management to control the motion of waves in real-world systems.

In light of the analysis and examples provided, wave management appears to be a useful yet relatively straightforward strategy for controlling the motion of a travelling wave. Of course, to successfully apply wave management, one or both of the dispersion and loss/gain terms should be controllable, and the controllability of these features will depend upon the specific system under consideration. We regard this possible lack of controllability as the primary limitation of implementing the method of wave management experimentally. Still, there are many applications for which these parameters may be controlled and hence wave management applied, and we now point to several applications which might stand to benefit from our approach.

A number of examples from mathematical physics allow for the requisite ability to control the dispersion and loss/gain. This controllability is certainly present in nonlinear optics, where light has been slowed and even stopped in experiments [45, 31, 23]. Dispersion management is frequently employed to modify the structure of optical waves [42, 24, 8], although in most of these works, the soliton envelope is modified in some manner, and management of the wavespeed under the cubic NLS was later discussed in [6]. Our approach would greatly improve these contributions to nonlinear optics, by allowing one to better control the motion of a solitary wave while ensuring it retains its form and various nice properties (such as integrability). The approach also works well with vector systems in nonlinear optics and in the SI Appendix we demonstrate how to control vector bright-dark solitons as just one application. Additional examples of systems which may be controlled during experiments are found in mathematical physics, such as the control of atomic matter waves in Bose-Einstein condensations [20, 36]. Experiments on these systems have shown that both dispersion and loss/gain terms may be modified over time, making wave management of the resulting travelling waves feasible. Since the manner of wave management we describe preserves integrability and stability properties of nonlinear waves, our approach can be used to preserve adiabaticity [30] in non-autonomous systems arising in mathematical physics.

Reaction-diffusion systems arising in chemistry often have temperature sensitivity, both through
reaction rates (such as those arising from scaling reaction terms with temperature activated Arrhenius parameters [5, 54]) and temperature-dependent diffusion (such as the linear Einstein, Wright-Sullivan, or Stokes-Einstein-Sutherland relations in liquid reactions [32] or more general power-law relations in gas reactions [58]). The role of temperature on Turing pattern formation from reaction-diffusion systems was recently studied under a variety of mechanisms in [74], and it was shown that localized Turing patterns can be modified through spatial or temporal changes in local temperature. Since temperature can be used to modify both diffusion and reaction rates in experiments, it may be an attractive method by which to implement wave management in chemical systems that permit travelling wave solutions. Temperature-dependent wavespeeds have already been observed in experimental work in fields as drastically different as physiology [62] and energy management [81], and a more precise control through wave management techniques outlined in this paper could be of service to any of these applications.

There are a number of potential biological applications for wave management, although these will likely be more theoretical than experimental owing to the difficulties inherent in modifying biological systems in a closed laboratory setting. Certain epidemiological systems are known to permit travelling waves, with a few mathematical examples can be found in [34, 80, 49, 78]. Travelling waves in epidemics are not only limited to theory, having been observed in the spread of influenza through different geographic regions [4, 76], as well as in spatiotemporal data for outbreaks of measles [25], rabies [27], and dengue fever [16]. Reduction of the number of contacts relative to the size of the population through travel restrictions or social distancing can modify the diffusion rates within an epidemic model [22, 14], whereas vaccination can be used to lower the reproduction rate of the epidemic within the model, with vaccination influencing the dynamics of a travelling wave [82]. Used in conjunction, these may then slow the propagation of a wave of infection over time. Conversely, ineffective policies or seasonal affects may increase the speed the wave of infection over time. Ecological migrations or invasions can manifest as travelling waves, with a number of factors influence the wavespeed [56, 20], and we provide an explicit example of the control of an invasion wave under a Lotka-Volterra population model in the SI Appendix. Changes in climate or habitat may cause long-term changes in wave structure or speed, whereas seasonal affects will also likely play a role.

A Supplementary Information

We further discuss the theory behind wave management and then provide several more concrete examples of the method.

A.1 Theoretical development of the wave management technique

Consider the nonlinear dispersive partial differential equation

$$\frac{\partial u}{\partial t} + \delta_0 L[u] + \gamma_0 N[u] = 0,$$

(20)

where $L$ is a differential operator (the dispersion term) on the space $\mathbb{R}^m$, $N$ is a nonlinear differential operator (the gain or loss term), while $\delta_0$ and $\gamma_0$ are parameters which control the relative strength of each term. We allow for $N$ to contain spatial derivatives, although these should generally be lower order than the highest order derivatives in $L$. Similarly, we allow $L$ to be nonlinear, so long
as it contains the highest order space derivatives. Assume that the transformation \( u(x,t) = U(z) \), \( z = \mathbf{k} \cdot \mathbf{x} - ct \), \( \mathbf{k} \in \mathbb{R}^m \) a constant vector, puts (1) into the form of the ordinary differential equation

\[-c \frac{dU}{dz} + \delta_0 \hat{L}[U] + \gamma_0 \hat{N}[U] = 0, \tag{21}\]

where the hat denotes operators which are evaluated along the wave coordinate, \( z \). If this equation admits a solution \( U(z) \), then this gives a travelling wave solution of (1). The wavespeed, \( c \), will in general be a function of \( \delta_0 \) and \( \gamma_0 \), e.g., \( c = c(\delta_0, \gamma_0) \). As such, a specific choice of \( \delta_0 \) and \( \gamma_0 \) may permit (or prohibit) a certain wavespeed. Another option is that no such \( c \) exists for this choice of parameter values, in which case we have failed to find a travelling wave solution.

Time-management of dispersion and nonlinearity require that the respective scaling of each term be allowed to vary in time, and the managed analogue of (1) reads

\[\frac{\partial u}{\partial t} + \delta(t) L[u] + \gamma(t) N[u] = 0, \tag{22}\]

The partial differential equation (3) is now non-autonomous, since the coefficients depend on time. As part of the wave management, we choose \( \delta(t) \) and \( \gamma(t) \) so that \( u(x,t) = U(Z) \) where \( Z = \mathbf{k} \cdot \mathbf{x} - C(t) \) for some differentiable function \( C(t) \) and \( U \) is a solution of the relevant travelling wave ODE with \( z \) replaced by \( Z \),

\[-\frac{dC}{dt} \frac{dU}{dZ} + \delta(t) \hat{L}[U] + \gamma(t) \hat{N}[U] = 0. \tag{23}\]

Using \( u(x,t) = U(Z) \) in (3) results in a coupled system of parameter constraints

\[S\left( \delta(t), \gamma(t), \frac{dC}{dt} \right) = 0, \tag{24}\]

and a solution \( U(Z) \) to (3) exists provided that the system (5) admits a solution for \( \delta(t) \) and \( \gamma(t) \). We refer to (5) as the constraint system, as it determines the solvability condition for the management parameters. The constraint system (5) depends on \( k \) as well as any other parameters present in \( U(z) \), such as the amplitude of the wave.

If (5) admits a solution \( \delta(t) = \delta \left( \frac{dC}{dt} \right), \gamma(t) = \gamma \left( \frac{dC}{dt} \right) \), then the non-autonomous managed system in (3) admits the exact solution \( u(x,t) = U(\mathbf{k} \cdot \mathbf{x} - C(t)) \). The envelope \( U \) corresponds to the solution \( u(x,t) = U(\mathbf{k} \cdot \mathbf{x} - ct) \) of the autonomous system (1), and hence the shape of the wave is unchanged under the control. In addition to preserving the shape, we remark that the managed (3) is integrable provided the autonomous system (1) is integrable, whereas the managed solution \( u(x,t) = U(\mathbf{k} \cdot \mathbf{x} - C(t)) \) to (3) is stable provided that the corresponding autonomous solution \( u(x,t) = U(\mathbf{k} \cdot \mathbf{x} - ct) \) to (1) is stable. As such, the managed solution to (3) is as robust as is the autonomous solution to (1).

A.1.1 Integrability of the managed equation

We note that wave management does not require integrability of the original autonomous equation in order to be applied. However, in the case where the original autonomous equation is integrable, we would like this integrability to be retained in the controlled equation. Let us then consider the question of whether a non-autonomous managed equation maintains integrability relative to
the original autonomous equation. To this end, assume (1) is integrable. Then, choosing the
management parameters to satisfy the constraint system (5) (from this system, the management
parameters will be linear combinations of \( \frac{dC}{dt} \)), and introducing the new timescale \( \tau = C(t) \), we
obtain the autonomous equation
\[
\frac{\partial v}{\partial \tau} + \delta_0 L[v] + \gamma_0 N[v] = 0,
\]
for the solution in terms of the new timescale, viz., \( u(x,t) = v(x,\tau) \). Note that (25) admits the
exact solution \( v(x,\tau) = V(k \cdot x - \tau) = U(k \cdot x - C(t)) \), which is exactly the managed travelling
wave solution of (3) discussed above.

Since (25) and (1) differ only by a timescale, (25) is integrable if and only if (1) is integrable.

Although this preservation of integrability appears fairly straightforward, it is worth seeing
explicit examples, and we consider the integrability of the managed KdV equation as well as the
conditional integrability of the Fisher-KPP equation. To this end, we apply the Painlevé test
to determine if a given managed equation has the Painlevé property and hence is integrable [79]. We
do not provide a detailed description of these techniques; the interested reader may consult the
references [1, 79, 63, 68]. In applying the Painlevé test, we expand a general solution in a Laurent
series about a singular manifold \( \mathcal{M} \). As we are concerned with travelling waves, we take the
singular manifold to be \( \mathcal{M} = \{ (x,t) \mid k \cdot x - C(t) = Z_0 \} \), which is just a one-dimensional manifold
parametrized by the choice of wave management resulting in \( C(t) \). For sake of example, we consider
one nonlinear wave equation (the KdV equation) and one reaction-diffusion equation (the Fisher-
KPP equation), however we remark that similar results hold for all of the other managed forms of
integrable equations we have considered later in this Appendix.

Integrability of the non-autonomous KdV equation Recall that the managed KdV equation
reads
\[
\frac{\partial u}{\partial t} + \delta(t) \frac{\partial^3 u}{\partial x^3} + 6\gamma(t) u \frac{\partial u}{\partial x} = 0,
\]
and the management parameters are
\[
\delta(t) = \gamma(t) = \frac{dC}{dt}.
\]
Choosing the parameters (12), the KdV equation (10) is put into the form
\[
\frac{dC}{dt} \left\{ \frac{d^3 U}{dZ^3} - \frac{dU}{dZ} + 6U \frac{dU}{dZ} \right\} = 0.
\]
Note that the choice of timescale \( \tau = C(t) \) taken to arrive at (25) would have cleared the factor \( \frac{dC}{d\tau} \) from this equation, however (28) still separates into a factor which involves \( Z \) alone. Although
we allow \( \frac{dC}{d\tau} = 0 \) for some values of \( t \) (for those times at which the wave is stopped), we take the
stronger condition of
\[
\frac{d^3 U}{dZ^3} - \frac{dU}{dZ} + 6U \frac{dU}{dZ} = 0,
\]
which arises from setting the \( Z \) factor in (28) to zero to balance this equation. To determine the
strength of the pole solution to (29), we assume a leading order term \( U(Z) \sim U_0(Z - Z_0)^{-\alpha} \). We
then have from the leading-order balance:

\[ \frac{d^3U}{dZ^3} \sim -U_0 \alpha (\alpha + 1) (\alpha + 2) (Z - Z_0)^{-\alpha - 3} = -6U_0 \alpha (Z - Z_0)^{-2\alpha - 1} \sim 6U \frac{dU}{dZ}, \]  

(30)

which gives a pole solution only when

\[ \alpha = 2 \quad \text{and} \quad U_0 = -2. \]  

(31)

To find the resonances, we next assume

\[ U(Z) \sim U_0(Z - Z_0)^{-\alpha} + b(Z - Z_0)^r = -2(Z - Z_0)^{-2} + b(Z - Z_0)^r. \]

From the leading-order terms, we find resonances at \( r = -3, r = 2, \) and \( r = 4. \) The \( r = -3 \) resonance is one degree lower than the degree of the pole, and corresponds to the arbitrariness of the singular manifold. On the other hand, the \( r = 2 \) and \( r = 4 \) resonances correspond to positions in the Laurent series expansion of \( U(Z) \) where we should have an arbitrary term. We then assume a solution taking the form of a Laurent series

\[ U(Z) = \frac{-2}{(Z - Z_0)^2} + \frac{a_{-1}}{Z - Z_0} + \sum_{\ell=0}^{\infty} a_{\ell}(Z + Z_0)\ell, \]  

(32)

and using this solution representation in (29) we obtain successively the first several terms

\[ a_{-1} = 0, \quad a_0 = \frac{1}{6}, \quad a_1 = a_3 = a_5 = a_7 = 0, \quad a_6 = -\frac{a_2^2}{6}, \quad a_8 = -\frac{3a_2a_4}{22}. \]  

(33)

Here \( a_2 \) and \( a_4 \) are arbitrary constant parameters (as was suggested by the resonance analysis). All subsequent even terms can be written in terms of \( a_2 \) and \( a_4, \) while all subsequent odd terms are zero. We conclude that the non-autonomous KdV equation (10) passes the Painlevé test for integrability provided that the wavespeed \( C(t) \) is controlled using (12). We remark that our analysis has recovered the same pole and resonance structure as the standard KdV equation, the analysis for which was previously outlined in [79].

### Integrability of the non-autonomous Fisher-KPP equation

The managed form of the Fisher-KPP equation reads

\[ \frac{\partial u}{\partial t} - \delta(t) \frac{\partial^2 u}{\partial x^2} - \gamma(t)u(1 - u) = 0, \]  

(34)

while the management parameters are given by

\[ \delta(t) = \frac{1}{5} \frac{dC}{dt}, \quad \gamma(t) = \frac{6}{5} \frac{dC}{dt}. \]  

(35)

Choosing the parameters (17), the Fisher-KPP equation (15) is put into the form

\[ \frac{dC}{dt} \left\{ \frac{1}{5} \frac{d^2U}{dZ^2} + \frac{dU}{dZ} + \frac{6}{5} U (1 - U) \right\} = 0. \]  

(36)

As (36) shows a separation of \( t \) and \( Z \) variables, we will take the \( Z \) factor to zero, and write

\[ \frac{1}{5} \frac{d^2U}{dZ^2} + \frac{dU}{dZ} + \frac{6}{5} U (1 - U) = 0 \]  

(37)
for all \( t \). We will again carry out the Painlevé test. To determine the strength of the pole solution (37), we again assume a leading order term \( U(Z) \sim U_0(Z - Z_0)^{-\alpha} \), resulting in the leading-order balance:

\[
\frac{1}{5} \frac{d^2 U}{d Z^2} \sim \frac{\alpha (\alpha + 1) U_0}{5} (Z - Z_0)^{-\alpha - 2} = \frac{6 U_0^2}{5} (Z - Z_0)^{-2\alpha} \sim \frac{6}{5} U^2. \tag{38}
\]

We find that (38) admits a pole solution only when \( \alpha = 2 \) and \( U_0 = 1 \).

To find the resonances, we next assume \( U(Z) \sim U_0(Z - Z_0)^{-\alpha} + b(Z - Z_0)^\ell = (Z - Z_0)^{-\alpha} + b(Z - Z_0)^\ell \). From the leading-order terms, we find resonances at \( r = -3 \) and \( r = 4 \). The \( r = -3 \) resonance is one degree lower than the order of the pole and corresponds to the arbitrariness of the singular manifold. The \( r = 4 \) resonance corresponds to the position in the Laurent series expansion of \( U(Z) \) where we should find an arbitrary term. Assuming a Laurent series solution

\[
U(Z) = \frac{1}{(Z - Z_0)^2} + \frac{a_{-1}}{Z - Z_0} + \sum_{\ell = 0}^{\infty} a_\ell (Z + Z_0)\ell, \tag{40}
\]

and using this solution representation in (37), we successively obtain

\[
a_{-1} = -1, \quad a_0 = \frac{5}{12}, \quad a_1 = -\frac{1}{12}, \quad a_2 = \frac{1}{240}, \quad a_3 = \frac{1}{720}, \quad a_5 = -4a_4 - \frac{1}{1440},
\]

\[
a_6 = \frac{49}{6} a_4 + \frac{703}{518400}, \quad a_7 = \frac{34}{3} a_4 - \frac{971}{518400}, \quad a_8 = \frac{961}{80} a_4 + \frac{151}{76032}. \tag{41}
\]

The coefficient \( a_4 \) is arbitrary, and this is again due to the resonance at \( r = 4 \). All subsequent terms \( a_\ell \), for \( \ell \geq 5 \), can be given in terms of this arbitrary \( a_4 \). We conclude that the non-autonomous Fisher-KPP equation (15) passes the Painlevé test for integrability provided that the management parameters are taken as in (17). We have again recovered a pole and resonance structure just like that in the literature for the standard Fisher-KPP equation (3). Our Laurent series is slightly different, however, since in (3) the wave speed is solved for through a comparability condition which emerges from the equation involving \( a_4 \). In contrast, our configuration assumes a wave speed is prescribed, while instead the diffusion and reaction parameters are solved for, so the managed parameters (17) automatically satisfy the condition in the equation involving \( a_4 \). The result is that the coefficients in the Laurent series differ between what we show in (41) and the work in (3), although the integrability properties (the respective orders of the pole and resonances) remain the same between our work and theirs.

**Integrability of the non-autonomous KdV equation, revisited** In our Painlevé analysis of the KdV and Fisher-KPP equations, we started out using the proper choice of managed parameters. What if these parameters are unknown? Can one obtain useful management parameters from an integrability condition? To this end, we revisit the integrability of the non-autonomous KdV equation (10) under the assumption that the management parameters are unknown, and show that requiring the managed equation to be integrable actually results in a type of constraint system which is more broad than that given by (5). One may then choose a control from within this constraint system to modify the wavespeed of the KdV soliton.
Assuming again that \( u(x, t) = U(Z) \) with \( Z = x - C(t) + x_0 \), we obtain

\[
- \frac{dC}{dt} \frac{dU}{dZ} + \delta(t) \frac{d^3U}{dZ^3} + 6\gamma(t)U \frac{dU}{dZ} = 0.
\] (42)

We again assume a leading order term \( U(Z) \sim U_0(Z - Z_0)^{-\alpha} \). We then have, from the leading-order balance,

\[
\delta(t) \frac{d^3U}{dZ^3} \sim -U_0\delta(t)(\alpha + 1)(\alpha + 2)(Z - Z_0)^{-\alpha - 3} = -6U_0\gamma(t)\alpha(Z - Z_0)^{-2\alpha - 1} \sim 6U \frac{dU}{dZ}.
\] (43)

From (43) we must again have \( \alpha = 2 \). However, the dominant balance is only consistent when either \( U_0 = 0 \) (removing the pole, which is not what we want) or \( \gamma(t)U_0 + 2\delta(t) = 0 \). Since we desire a wave with amplitude independent of time, we cannot have \( U_0 = U_0(t) \), which means that we must have that \( U_0 = -2\delta(t)/\gamma(t) = \text{constant} \neq 0 \), hence \( \delta(t)/\gamma(t) \) must be a a non-zero constant. We must have \( \gamma(t) = \gamma_0\delta(t) \) for some constant \( \gamma_0 \). In this case, we find \( U_0 = -2/\gamma_0 \).

We next assume \( U(Z) \sim U_0(Z - Z_0)^{-\alpha} + b(Z - Z_0)^r \), and again find resonances at \( r = -3, r = 2, \) and \( r = 4 \). Expressing the solution \( U(Z) \) in a Laurent series

\[
U(Z) = \frac{-2}{\gamma_0(Z - Z_0)^2} + \frac{a_{-1}}{Z - Z_0} + \sum_{\ell=0}^{\infty} a_\ell(Z + Z_0)^\ell,
\] (44)

and using this solution representation in (11), we obtain successively the first several terms

\[
a_{-1} = 0, \quad a_0 = \frac{1}{6\gamma_0\delta(t)} \frac{dC}{dt}, \quad a_1 = a_3 = a_5 = 0,
\] (45)

where \( a_2 \) and \( a_4 \) are again arbitrary. In order for \( a_0 \) to be independent of time, we must have \( \delta(t) = \delta_0 \frac{dC}{dt} \). Then, \( a_0 = (6\gamma_0\delta_0)^{-1} \). Beyond \( a_4 \), all subsequent odd-indexed coefficients are zero, while all subsequent even-indexed coefficients will involve \( a_2, a_4, \) and \( \gamma_0 \), and will hence be constant.

We conclude that the non-autonomous KdV equation (10) passes the Painlevé test for integrability provided that the equation is controlled using management parameters

\[
\delta(t) = \delta_0 \frac{dC}{dt} \quad \text{and} \quad \gamma(t) = \gamma_0 \delta(t) = \gamma_0 \delta_0 \frac{dC}{dt}.
\] (46)

The functions (46) are more general linear scalings of those in (12), and the non-autonomous KdV equation is actually integrable for these generic scalings. The choice of constants \( \delta_0 \) and \( \gamma_0 \) will obviously influence the form of the solution, with the KdV soliton

\[
u(x, t) = \frac{1}{2} \sech^2 \left( \frac{x - C(t) + x_0}{2} \right).
\] (47)

emerging for \( \delta_0 = \gamma_0 = 1 \).

Similar results can be obtained for the other systems we use as examples in this Appendix. Interestingly, this means that the constraint system (3) appears to give sufficient conditions for the integrability of the non-autonomous managed equation (3). However, one still needs to do more work to find the management parameters from this set of \( \delta(t) \) and \( \gamma(t) \) which ensure integrability, since as in the example above we need to further set \( \delta_0 = \gamma_0 = 1 \) to obtain the bright-soliton solution for the KdV equation.
A.1.2 Stability of the managed wave solution

We show that the management procedure will preserve stability properties of an autonomous traveling wave.

Consider (3) and choose the management parameters to satisfy the constraint system (5) (the management parameters will again be linear combinations of $\frac{\partial C}{\partial \tau}$), and introducing the new timescale $\tau = C(t)$, we obtain the autonomous equation

$$\frac{\partial v}{\partial \tau} + \delta_0 L[v] + \gamma_0 N[v] = 0.$$  \hfill (48)

If (1) admits the travelling wave solution $u(x, t) = U(k \cdot x - ct)$ then (25) admits the corresponding exact solution $v(x, \tau) = V(k \cdot x - \tau) = U(k \cdot x - C(t))$, which is exactly the managed travelling wave solution of (3) discussed above.

Assume that $u(x, t) = U(k \cdot x - ct)$ is a stable solution to the autonomous equation (1). Then, we claim that the corresponding managed solution with management parameters satisfying the constraint system (5) is stable. To demonstrate this we consider linear stability, although we note that similar remarks are in order for other kinds of stability.

For the autonomous equation (1), consider a linear perturbation of the travelling wave solution $u(x, t) = U(k \cdot x - ct)$, say

$$u(x, t) = U(k \cdot x - ct) + \epsilon \tilde{u}(x, t),$$

where $0 < \epsilon \ll 1$ is a small perturbation function and $\tilde{u}(x, t)$ is the perturbation function giving the shape of the perturbation in space and growth or decay of the perturbation in time. We will assume that $U$ holds any boundary conditions so that $\tilde{u}(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$. Placing (49) into (1), and expanding the operators $L$ and $N$ like

$$L[U + \epsilon \tilde{u}] = L[U] + \epsilon L'[U, \tilde{u}] + O(\epsilon^2),$$

$$N[U + \epsilon \tilde{u}] = N[U] + \epsilon N'[U, \tilde{u}] + O(\epsilon^2),$$

we arrive at the equation governing the dynamics of the perturbation $\tilde{u}$,

$$\frac{\partial \tilde{u}}{\partial t} + \delta_0 L'[U, \tilde{u}] + \gamma_0 N'[U, \tilde{u}] = 0 \quad \text{subject to} \quad |\tilde{u}(x, t)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$  \hfill (52)

If (52) admits only solutions satisfying $|\tilde{u}(x, t)| \rightarrow 0$ as $t \rightarrow \infty$ when we say that the solution $u(x, t) = U(k \cdot x - ct)$ of (1) is linearly stable. Let us assume that this is true, i.e., that $u(x, t) = U(k \cdot x - ct)$ is a linearly stable solution of (1).

For the non-autonomous system (3), we select management parameters satisfying (5), and the change of time variable $\tau = C(t)$, resulting in (48). We choose a perturbation of the managed solution $u(x, t) = U(k \cdot x - C(t))$ taking the form

$$u(x, t) = U(k \cdot x - C(t)) + \epsilon \tilde{v}(x, \tau) = V(k \cdot x - \tau) + \epsilon \tilde{v}(x, \tau),$$

and upon placing this into (48) we obtain the equation governing the dynamics of perturbations to the managed wave,

$$\frac{\partial \tilde{v}}{\partial \tau} + \delta_0 L'[V, \tilde{v}] + \gamma_0 N'[V, \tilde{v}] = 0 \quad \text{subject to} \quad |	ilde{v}(x, t)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$  \hfill (54)
However, note that (54) is exactly (52) under a change of time variable. As such, $\tilde{v}(x, \tau) \to 0$ as $\tau \to \infty$ provided $|u(x, t)| \to 0$ as $t \to \infty$. In other words, the managed solution $u(x, t) = U(k \cdot x - C(t))$ to (3) is linearly stable provided the travelling wave solution $u(x, t) = U(k \cdot x - ct)$ to the autonomous system (1) is linearly stable.

A.1.3 What if space-varying rather than time-varying coefficients are employed?

We have used time-varying coefficients to modify the motion (speed and direction) of a travelling wave. Since we desire to change the motion of the traveling wave in time, this is the logical choice. Still, one may wonder what would happen if the control parameters varied in space. Generically, this will change the structure of the wave, as highlighted in many papers on non-autonomous solitons (see, for instance, [71]). Many authors studying applications in quantum field theory have employed a space-varying potential to change the structure of a nonlinear matter wave. See [50, 51, 52, 47, 75] and references therein for applications of space-varying external potentials with regard to Bose-Einstein condensates.

Indeed, a spatial potential is often used to confine a region of mass or concentration in quantum mechanical and quantum field theoretic models. Treating the parameter $\gamma$ as a function of space has similar effect, confining more mass to within the core of the wave, or repelling mass away from the core of the wave.

On the other hand, treating the dispersion parameter as a function of time makes the diffusive lengths scale local rather than global. While time-dependent dispersion arises in applications, these seldom feature traveling wave solutions. There is also more care needed, as simply writing $\delta(x)L[u]$ for the dispersion operator may not make sense, so the $x$-dependence needs to enter into the operator $L$. As an example, consider the diffusion operator with a dependence of the diffusion parameter on space. The proper way to write such an operator is $L[u] = \nabla \cdot (\delta(x)\nabla u)$ rather than $L[u] = \delta(x)\nabla^2 u$. While the time-dependent dispersion parameter will pass through spatial operators, the space-dependent dispersion parameter does not.

While space-varying parameters can be useful if one wishes to change the shape or structure of a wave, they are not desirable candidates for controlling the motion of a wave.

A.1.4 What if time-varying wavenumbers are employed?

Let us assume that a time-varying wavenumber, $k = k(t)$, is employed. Then, assuming a solution of the form $u(x, t) = U(Z)$ where now $Z = k(t) \cdot x - C(t)$, the managed form of the equation (3) takes the form

$$- \frac{dC}{dt} \frac{dU}{dZ} + \left( \frac{dk}{dt} \cdot x \right) \frac{dU}{dZ} + \delta(t)\hat{L}[U] + \gamma(t)\hat{N}[U] = 0. \quad (55)$$

Wave management as outlined above involves selecting $\delta$ and $\gamma$ so that we obtain only an ordinary differential equation for $U(Z)$. However, the second term in (55) explicitly involves the space variable $x$, hence there is no way to clear this expression so that it only depends on $Z$. So a solution involving only $Z$ is not possible, and (55) does not reduce to (21) for any choice of $\delta(t)$ and $\gamma(t)$. The problem here is that a time-varying wavenumber vector results in an explicit dependence on the space variables, and as discussed above this type of space dependence requires that the envelope of the wave, $U(Z)$, changes in space and hence cannot depend only on the travelling wave variable $Z$.
A.1.5 What if the wavefront is curved?

Although we allow for wave solutions on the space domain \( \mathbb{R}^m \), all of our wave solutions have one-dimensional motion, in that the solitary wave or wavefront always propagates in a one-dimensional manner based on the choice of wavenumber vector \( \mathbf{k} \in \mathbb{R}^m \) used in the travelling wave variable \( Z = \mathbf{k} \cdot \mathbf{x} - C(t) \). This mirrors all of the literature on autonomous travelling waves (solitons, for instance, most commonly display one-dimensional motion, even if they exist in higher-dimensional domains), and so should not be surprising. Other types of waves are certainly possible, such as radial waves which propagate outward from a single point, resulting in a curved wavefront. However, such waves do not conserve concentration or mass while also maintaining their structure, since the mass or concentration must become diluted as the wavefront expands in size. Spherical waves in \( \mathbb{R}^3 \) are one example of this behaviour. Considering a wave which depends only on the radius \( r \) from the origin and time, the spherical form of d’Alembert’s formula (obtained by Euler) reads \( u(r,t) = r^{-1} U_+(r + ct) + r^{-1} U_-(r - ct) \). This solution tends to zero as the radius \( r \) becomes large.

While one could consider applying wave management to waves with curved or higher-dimensional wavefronts, the benefit of maintaining the wave envelope offered under wave management will no longer hold even though the wavespeed can be controlled. This is not a shortcoming of the proposed method of wave management, but rather a reality of dealing with curved wavefronts in more than one space dimensions.

A.1.6 What if the governing equation is linear?

Although the wave management technique is designed for the control of solitary waves or wavefronts in nonlinear equations, it is natural to wonder whether it can be used to control linear waves. To this end, consider the managed linear wave equation

\[
\frac{\partial^2 u}{\partial t^2} = \delta(t) \frac{\partial^2 u}{\partial x^2},
\]

subject to the initial data

\[
u = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t} = g(x) \quad \text{at} \quad t = 0.
\]

The standard linear wave equation on \( \mathbb{R} \) admits waves of the form \( U(x \pm ct) \) for generic envelope \( U \). The functional form of \( U \) is then determined by the initial data. If we assume a solution \( u = U(x - C(t)) = U(Z) \), we find

\[
\left\{ \left( \frac{dC}{dt} \right)^2 - \delta(t) \right\} \frac{d^2 U}{dZ^2} - \frac{d^2 C}{dt^2} \frac{dU}{dZ} = 0.
\]

Even if we choose \( \delta(t) = \left( \frac{dC}{dt} \right)^2 \), the second term persists and hence the only possibility is either \( U = \text{constant} \) or \( C(t) \) is a linear function (which is just the standard case of a constant wavespeed \( c \), \( C(t) = ct \)). As such, the standard linear wave equation is not controllable under wave management.

From (58) it is clear that (56) does not have a sufficient number of degrees of freedom to allow us to manage the wavespeed: Although it has a dispersion term, it lacks a loss/gain type term. Consider, then, the amended wave equation

\[
\frac{\partial^2 u}{\partial t^2} + \gamma(t) \frac{\partial u}{\partial t} = \delta(t) \frac{\partial^2 u}{\partial x^2}
\]

(59)
subject again to the initial data (57). Here $\gamma(t)$ plays the role of a friction parameter, resulting in a de-amplification of a travelling wave in the autonomous setting. Assuming solutions of the form $u(x, t) = U(x \pm C(t))$, the constraint system (5) takes the from

$$\left(\frac{dC}{dt}\right)^2 - \delta(t) = 0, \quad \frac{d^2C}{dt^2} + \gamma(t) \frac{dC}{dt} = 0.$$ (60)

Choosing management parameters

$$\delta(t) = \left(\frac{dC}{dt}\right)^2, \quad \gamma(t) = -\frac{d}{dt} \log \left(\frac{dC}{dt}\right),$$ (61)

we have that any $u(x, t) = U(x \pm C(t))$ is a solution of the PDE (59). The solution of (59) then takes the form $u(x, t) = f(x - C(t)) + f(x + C(t))$. Taking into account the initial data (57), from d’Alembert’s formula we have the exact solution to (59):

$$u(x, t) = f(x - C(t)) + f(x + C(t)) + \frac{1}{2} \left(\frac{dC}{dt}(0)\right)^{-1} \int_{x-C(t)}^{x+C(t)} g(\xi) d\xi.$$ (62)

Although the wave equation (59) is linear, the wave management works since there is both a dispersion and a loss/gain term. The fact that both of these terms is linear does not preclude application of wave management.

### A.2 Numerical simulation details

We have employed numerical simulations in our study of the KdV and Fisher-KPP equations, and we discuss the particulars of these simulations here.

We simulate the managed KdV equation (10) using the Finite Element Method (FEM) by employing the package FlexPDE [60]. The package employs adaptive timestepping and mesh refinement in order to obtain solutions to within a desired error tolerance. To approximate the problem on a finite space domain, for our simulations we choose a domain $x \in [0, L]$, where $L = 50$ for the top right panel in Fig. 1 and $L = 60$ for the bottom right panel in Fig. 1. We employ periodic boundary conditions, so that $u(x + L, t) = u(x)$ for all $t$. The KdV solution uses 100 cells which gives a root-mean-square error of less than $10^{-3}$ at all times. When simulating integrable equations, it is important to choose the initial data correctly, otherwise a 1-soliton might not be observed due to instability, with instead a collection of solitons being observed [84]. The common choice of initial data is a “stationary” soliton envelope corresponding to the exact solution with $c = 0$. Our initial data for the KdV equation is taken to be $u(x, 0) = \frac{1}{2} \sech^2 \left(\frac{x-10}{2}\right)$ for the top panels and $u(x, 0) = \frac{1}{2} \sech^2 \left(\frac{x-15}{2}\right)$ for the bottom panels.

To carry out our numerical simulations for the Fisher equation, we again use the FEM package FlexPDE [60]. To approximate the problem on a finite space domain, we choose a domain $x \in [0, L]$, where $L = 60$ for the top right panel in Fig. 2 and $L = 50$ for the bottom right panel in Fig. 2. We employ Dirichlet boundary conditions $u(0, t) = 1$ and $u(L, t) = 0$ for all $t$. The Fisher solutions use 234 and 225 cells, respectively, which ensures a root-mean-square error of less than $10^{-5}$ at all times. The initial data is taken to be $u(x, t) = (1 + \exp(x - 10))^{-2}$ for both cases.
A.3 Additional examples of controlled waves

Having outlined the theory of wave management and provided two explicit examples in terms of the KdV and Fisher-KPP equations, we now demonstrate several more examples to highlight the wide applicability of wave management.

A.3.1 Controlled solitary waves under the nonlinear Schrödinger equation

Akin to what was done in [70], we consider the nonlinear Schrödinger equation (NLS) for a homogeneous media,

\[ i \frac{\partial u}{\partial t} + \delta(t) \frac{\partial^2 u}{\partial x^2} + \gamma(t) |u|^2 u = 0. \]  

(63)

We begin by searching for a bright soliton of the form

\[ u = \exp(i[x - \Omega(t)]) \text{sech}(Z) \]  

(64)

with \( Z = x - C(t) + x_0 \), where we are assuming that there is a phase factor which evolves in time (as is typical when studying non-autonomous NLS equations). The set of conditions (5) becomes

\[-\frac{dC}{dt} + 2\delta(t) = 0, \quad \frac{d\Omega}{dt} = 0, \quad \gamma(t) - 2\delta(t) = 0. \]  

(65)

Choosing management parameters

\[ \delta(t) = \frac{1}{2} \frac{dC}{dt}, \quad \gamma(t) = \frac{dC}{dt}, \]  

(66)

we find that (63) has the exact bright soliton solution

\[ u(x,t) = \exp(i[x - \Omega_0]) \text{sech}(x - C(t) - x_0), \]  

(67)

where \( \Omega_0 \) is a constant.

We next consider a dark soliton of the form

\[ u = \exp(i[x - \Omega(t)]) \text{tanh}(Z). \]  

(68)

The condition set (5) is then

\[-\frac{dC}{dt} + 2\delta(t) = 0, \quad \frac{d\Omega}{dt} - \delta(t) + \gamma(t) = 0, \quad \gamma(t) + 2\delta(t) = 0. \]  

(69)

For management parameters

\[ \delta(t) = \frac{1}{2} \frac{dC}{dt}, \quad \gamma(t) = -\frac{dC}{dt}, \]  

(70)

(63) has the exact dark soliton solution

\[ u(x,t) = \exp \left( i \left[ x - \frac{3}{2} C(t) - \Omega_0 \right] \right) \text{tanh}(x - C(t) - x_0). \]  

(71)

Note that in this case the phase factor \( \Omega(t) \) does indeed depend on time, and will evolve as the control \( C(t) \) evolves. As this only enters into the phase, it will not influence the structure or shape of the wave in space.
A.3.2 Controlled wavefront under the Bateman-Burgers equation

The managed form of the Bateman-Burgers equation \cite{7, 13} reads

\[
\frac{\partial u}{\partial t} - \delta(t) \frac{\partial^2 u}{\partial x^2} - \gamma(t) u \frac{\partial u}{\partial x} = 0. \quad (72)
\]

The classical Burgers equation admits travelling wave solutions scaling like \( u \sim (1 + \exp(x - ct + x_0))^{-1} \) \cite{15, 33}. Assuming that \( ct \) is replaced by \( C(t) \), and placing this into (72), we find that the constraint system \cite{5} takes the form

\[
\delta(t) - \frac{dC}{dt} = 0, \quad \delta(t) + \gamma(t) + \frac{dC}{dt} = 0. \quad (73)
\]

Therefore, choosing the management parameters

\[
\delta(t) = \frac{dC}{dt}, \quad \gamma(t) = -2 \frac{dC}{dt}, \quad (74)
\]

the managed Burgers equation (72) has the exact solution

\[
u(x, t) = (1 + \exp(x - C(t) + x_0))^{-1}. \quad (75)
\]

A.3.3 Controlled wavefronts under a scalar FitzHugh-Nagumo equation

Consider the managed form of a scalar reaction-diffusion equation

\[
\frac{\partial u}{\partial t} - \delta(t) \frac{\partial^2 u}{\partial x^2} + \gamma(t) u (1 - u) (\alpha - u) = 0, \quad (76)
\]

where \( \alpha \in [0, 1] \) is a parameter. This cubic nonlinearity results in a type of scalar Nagumo equation \cite{35, 41, 43, 53}, which admits travelling wave solutions scaling like \( u \sim (1 + \exp(x - ct + x_0))^{-1} \).

We search for similar solutions, finding that the constraint system \cite{5} takes the form

\[
\delta(t) - (1 - \alpha) \gamma(t) + \frac{dC}{dt} = 0, \quad \alpha \gamma(t) - \delta(t) + \frac{dC}{dt} = 0. \quad (77)
\]

Choosing the management parameters

\[
\delta(t) = \frac{1}{1 - 2\alpha} \frac{dC}{dt}, \quad \gamma(t) = \frac{2}{1 - 2\alpha} \frac{dC}{dt}, \quad (78)
\]

for \( \alpha \neq 1/2 \), the managed (76) has the exact solution

\[
u(x, t) = (1 + \exp(x - C(t) + x_0))^{-1}. \quad (79)
\]

When \( \alpha = 1/2 \), the management approach does not work. However, even the standard travelling wave with constant wavespeed does not exist for \( \alpha = 1/2 \), so this lack of existence is not a flaw in the wave management approach.
A.3.4 Managed nonlinear wave trains

In addition to solitary waves and wavefronts, periodic wave trains are also controllable under the management technique. Cnoidal waves, involving the Jacobi elliptic function cn, were known even to Korteweg and deVries \[39\] have been studied in a number of subsequent works \[48, 10\]. Consider again the managed form of the KdV equation, and consider a solution of the form

$$u(x,t) = \text{cn}^2(Z,\kappa),$$

where the parameter $0 < \kappa < 1$ determines the period and structure of the cnoidal wave. The constraint system (5) becomes

$$4(1 + \kappa^2)\delta(t) - 6\gamma(t) + \frac{dC}{dt} = 0, \quad 2\kappa^2\delta(t) - \gamma(t) = 0.$$  

(81)

Taking the management parameters to be

$$\delta(t) = \frac{1}{4(2\kappa^2 - 1)} \frac{dC}{dt}, \quad \gamma(t) = \frac{\kappa^2}{2(2\kappa^2 - 1)} \frac{dC}{dt},$$

(82)

we find that the managed KdV equation has the exact solution

$$u(x,t) = \text{cn}^2(x - C(t) + x_0,\kappa).$$

(83)

Cnoidal waves also exist for the NLS equation (63) \[12, 50, 51\]. Choosing

$$u = \exp(i[x - \Omega(t)]) \text{cn}(Z,\kappa),$$

(84)

the constraint system becomes

$$-\frac{dC}{dt} + 2\delta(t) = 0, \quad \frac{d\Omega}{dt} - 2\delta(t) + \gamma(t) = 0, \quad \gamma(t) - 2\kappa^2\delta(t) = 0.$$  

(85)

Choosing management parameters

$$\delta(t) = \frac{1}{2} \frac{dC}{dt}, \quad \gamma(t) = \kappa^2 \frac{dC}{dt},$$

(86)

we find that (63) has the exact solution

$$u(x,t) = \exp(i[x - (1 - \kappa^2)C(t)]) \text{cn}(x - C(t) - x_0,\kappa).$$

(87)

Note again that this solution of the NLS equation involves a time-varying phase which depends on the control $C(t)$. Similar results can be found for wavetrains involving Jacobi sn or dn functions.

A.3.5 Compactons under the Rosenau-Hyman equation

In addition to waves defined over the real line, it is possible to apply wave management to travelling waves which are compactly supported, e.g., compactons \[66\]. As an example, consider the managed Rosenau-Hyman equation $K(2,2)$ \[66\],

$$\frac{\partial u}{\partial t} + \delta(t) \frac{\partial^3}{\partial x^3} (u^2) + \gamma(t) \frac{\partial}{\partial x} (u^2) = 0.$$  

(88)
For this case, observe that \( L \) is nonlinear, while \( N \) contains lower-order derivatives. Provided that the management parameters are chosen to satisfy the constraint system

\[
\delta(t) = \gamma(t) = -\frac{4}{3} \frac{dC}{dt},
\]

we find that (88) admits a compacton solution

\[
u(x, t) = \begin{cases} 
\cos^2 \left( \frac{x - C(t) + x_0}{4} \right), & \text{if } |x - C(t) + x_0| \leq 2\pi, \\
0, & \text{otherwise}.
\end{cases}
\]

(90)

### A.4 Extension of wave management to systems of equations

Wave management can easily be extended to the study of systems of partial differential equations which admit traveling wave solutions, and we demonstrate this here, while also providing two more examples.

#### A.4.1 Theoretical development for systems

Consider the system of nonlinear dispersive partial differential equations for \( n \) unknown functions \( u_1, u_2, \ldots, u_n \),

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + \delta_{01} L_1[u_1] + \gamma_{01} N_1[u_1, u_2, \ldots, u_n] &= 0, \\
\frac{\partial u_2}{\partial t} + \delta_{02} L_2[u_2] + \gamma_{02} N_2[u_1, u_2, \ldots, u_n] &= 0, \\
&\vdots \\
\frac{\partial u_n}{\partial t} + \delta_{0n} L_n[u_n] + \gamma_{0n} N_n[u_1, u_2, \ldots, u_n] &= 0,
\end{align*}
\]

(91)

where the \( L_\ell (\ell = 1, 2, \ldots, n) \) are differential operators (the dispersion or diffusion terms) on the space \( \mathbb{R}^m \), \( N_\ell (\ell = 1, 2, \ldots, n) \) are nonlinear differential operators, while \( \delta_{0\ell} \) and \( \gamma_{0\ell} \) are parameters which control the relative strength of each term. We allow for the \( N_\ell \) to contain spatial derivatives, although these should generally be lower order than the highest order derivatives in \( L_\ell \). Similarly, we allow the \( L_\ell \) to be nonlinear, so long as they contain the highest order space derivatives.

Assume that the transformation \( u_\ell(x, t) = U_\ell(z), z = k \cdot x - ct, k \in \mathbb{R}^m \) a constant vector, puts (91) into the form of the ordinary differential equation

\[
\begin{align*}
-c \frac{dU_1}{dz} + \delta_{01} \hat{L}_1[U_1] + \gamma_{01} \hat{N}_1[U_1, U_2, \ldots, U_n] &= 0, \\
-c \frac{dU_2}{dz} + \delta_{02} \hat{L}_2[U_2] + \gamma_{02} \hat{N}_2[U_1, U_2, \ldots, U_n] &= 0, \\
&\vdots \\
-c \frac{dU_n}{dz} + \delta_{0n} \hat{L}_n[U_n] + \gamma_{0n} \hat{N}_n[U_1, U_2, \ldots, U_n] &= 0,
\end{align*}
\]

(92)
where the hat denotes operators which are evaluated along the wave coordinate, z. If this equation
admits a solution \( U(z) \), then this gives a travelling wave solution of (91). The wavespeed, c, will in
general be a function of all of the parameters \( \delta_0, \gamma_0, \ldots, \delta_n, \gamma_n \).

The wave managed analogue of (91) reads

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + \delta_1(t)L_1[u_1] + \gamma_1(t)N_1[u_1, u_2, \ldots, u_n] &= 0, \\
\frac{\partial u_2}{\partial t} + \delta_2(t)L_2[u_2] + \gamma_2(t)N_2[u_1, u_2, \ldots, u_n] &= 0, \\
&\quad \vdots \\
\frac{\partial u_n}{\partial t} + \delta_n(t)L_n[u_n] + \gamma_n(t)N_n[u_1, u_2, \ldots, u_n] &= 0. 
\end{align*}
\] (93a) (93b) (93c) (93d)

Consider a solution to the managed system (91) taking the form

\[
u_1(x, t) = U_1(z), \quad u_2(x, t) = U_2(z), \quad \ldots, \quad u_n(x, t) = U_n(z), \quad \text{where } Z = k \cdot x - C(t), \]

(94)

where the wave envelopes \( U_\ell \) are the solutions to the autonomous traveling wave system (92). We
then choose the \( \delta_\ell(t) \) and \( \gamma_\ell(t) \), \( \ell = 1, 2, \ldots, n \), so that each \( u_\ell(x, t) = U_\ell(Z) \) where \( Z = k \cdot x - C(t) \)
for some differentiable function \( C(t) \), and the \( U_\ell \) are solutions of of the relevant travelling wave
system with \( z \) replaced by \( Z \),

\[
\begin{align*}
-dC \frac{dU_1}{dt} \frac{dZ}{dt} + \delta_1(t)\dot{L}_1[U_1] + \gamma_1(t)\dot{N}_1[U_1, U_2, \ldots, U_n] &= 0, \\
-dC \frac{dU_2}{dt} \frac{dZ}{dt} + \delta_2(t)\dot{L}_2[U_2] + \gamma_2(t)\dot{N}_2[U_1, U_2, \ldots, U_n] &= 0, \\
&\quad \vdots \\
-dC \frac{dU_n}{dt} \frac{dZ}{dt} + \delta_n(t)\dot{L}_n[U_n] + \gamma_n(t)\dot{N}_n[U_1, U_2, \ldots, U_n] &= 0. 
\end{align*}
\] (95a) (95b) (95c) (95d)

Using the solution envelopes found in (92) and placing these into (95) results in a system of
parameter constraints,

\[
S\left(\delta_1(t), \delta_2(t), \ldots, \delta_n(t), \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t), \frac{dC}{dt}\right) = 0. \]

(96)

Therefore, a managed solution (94) to the system (93) exists provided that the constraint system (95) admits a solution for each \( \delta_\ell(t) \) and \( \gamma_\ell(t) \) in terms of the quantity \( \frac{dC}{dt} \). In this case, we can
control the motion of a traveling wave solution to the system (93) through wave management.

As was true for the scalar case, the managed system (93) with management parameters satisfying
(96) is integrable if and only if the autonomous system (91) is integrable. Furthermore, the managed
solution (94) is stable provided that the solution to the autonomous system (92) is stable. We
now provide two explicit examples of wave management applied to nonlinear systems of partial
differential equations.
A.4.2 Controlled vector bright-dark solitons under the NLS

Consider the non-autonomous coupled NLS system

\[
\begin{align*}
\frac{i}{\partial t}u_1 + \delta_1(t) \frac{\partial^2 u_1}{\partial x^2} + \gamma_1(t) \left( |u_1|^2 + |u_2|^2 \right) u_1 &= 0, \\
\frac{i}{\partial t}u_2 + \delta_2(t) \frac{\partial^2 u_2}{\partial x^2} + \gamma_2(t) \left( \beta|u_1|^2 + |u_2|^2 \right) u_2 &= 0,
\end{align*}
\]

where \(0 < \alpha, \beta < 1\) are cross-phase modulation parameters. Systems of this form are known to permit vector solitons in the autonomous case \([55, 37, 83]\), with bright-bright, dark-dark, or bright-dark pairs possible. Considering bright-dark vector solutions of the form

\[
\begin{align*}
u_1 &= \exp(i[x - \Omega_1(t)]) \text{sech}(Z) \quad \text{and} \quad u_2 = \exp(i[x - \Omega_2(t)]) \text{tanh}(Z),
\end{align*}
\]

the constraint system \((96)\) takes the form

\[
\begin{align*}
\frac{dC}{dt} + 2\delta_1(t) &= 0, \\
\frac{dC}{dt} + 2\delta_2(t) &= 0, \\
\frac{d\Omega_1}{dt} + \alpha\gamma_1(t) &= 0, \\
\frac{d\Omega_2}{dt} + \gamma_2(t) - \delta_2(t) &= 0, \\
(1 - \alpha)\gamma_1(t) - 2\delta_1(t) &= 0, \\
(1 - \beta)\gamma_2(t) + 2\delta_2(t) &= 0.
\end{align*}
\]

Solving this system of six equations assuming \(\alpha \neq 1\) and \(\beta \neq 1\), we find management parameters

\[
\delta_1(t) = \delta_2(t) = \frac{1}{2} \frac{dC}{dt}, \quad \gamma_1(t) = \frac{1}{1 - \alpha} \frac{dC}{dt}, \quad \gamma_2(t) = -\frac{1}{1 - \beta} \frac{dC}{dt},
\]

as well as the time-varying phase parameters

\[
\begin{align*}
\Omega_1(t) &= -\frac{\alpha}{1 - \alpha} C(t), \\
\Omega_2(t) &= \frac{3 - \beta}{2(1 - \beta)} C(t).
\end{align*}
\]

The result is a controlled bright-dark soliton pair solution of the vector NLS \((97)\) taking the form

\[
\begin{align*}
u_1(x, t) &= \exp \left( i \left[ x + \frac{\alpha}{1 - \alpha} C(t) \right] \right) \text{sech}(x - C(t) + x_0), \\
u_2(x, t) &= \exp \left( i \left[ x - \frac{3 - \beta}{2(1 - \beta)} C(t) \right] \right) \text{tanh}(x - C(t) + x_0).
\end{align*}
\]

A.4.3 Controlled waves of invasion under a Lotka-Volterra population competition model

In addition to travelling wavefronts emerging from scalar systems, it is possible to apply wave management to systems of equations. To illustrate this, first consider a managed Lotka-Volterra competition model \([59]\)

\[
\begin{align*}
\frac{\partial u_1}{\partial t} - \delta_1(t) \frac{\partial^2 u_1}{\partial x^2} - \gamma_1(t) u_1(1 - u_1 - \alpha u_2) &= 0, \\
\frac{\partial u_2}{\partial t} - \delta_2(t) \frac{\partial^2 u_2}{\partial x^2} - \gamma_2(t) u_1(1 - \beta u_1 - u_2) &= 0,
\end{align*}
\]

where \(\delta_1(t) = \delta_2(t)\) and \(\gamma_1(t) = \gamma_2(t)\) are management parameters.
which arises in competition between different populations for the same resources or predator-prey interactions. Here $\alpha$ and $\beta$ are constants that are not equal to one. The autonomous form of (103) is known to admit solutions of the form $u_1 = (1 + \exp(x - c_0 t + x_0))^{-2}$ and $u_2 = 1 - u_1$ (see [57, 41, 40]) which can model the invasion of one species habitat by the other (depending on the sign of the wavespeed $c_0$). Making a similar assumption on the form of a solution pair for the non-autonomous managed case, equation (103) has a solution of the desired form when the constraint system (96) takes the form

$$
\delta_1(t) - (1 - \alpha)\gamma_1(t) + \frac{dC}{dt} = 0, \quad (1 - \alpha)\gamma_1(t) + 4\delta_1(t) - 2\frac{dC}{dt} = 0,
$$

$$
\delta_2(t) + (1 - \beta)\gamma_2(t) + \frac{dC}{dt} = 0, \quad (1 - \beta)\gamma_2(t) - 4\delta_2(t) + 2\frac{dC}{dt} = 0.
$$

(104a)

Choosing the management parameters

$$
\delta_1(t) = \delta_2(t) = \frac{1}{5} \frac{dC}{dt}, \quad \gamma_1(t) = \frac{6}{5(1 - \alpha)} \frac{dC}{dt}, \quad \gamma_2(t) = -\frac{6}{5(1 - \beta)} \frac{dC}{dt},
$$

(105)

the managed Lotka-Volterra competition model (103) has the exact solution

$$
u_1(x, t) = (1 + \exp(x - C(t) + x_0))^{-2}, \quad u_2(x, t) = 1 - (1 + \exp(x - C(t) + x_0))^{-2}.
$$

(106)

References

[1] M. Ablowitz, A. Ramani, and H. Segur. Nonlinear evolution equations and ordinary differential equations of Painlevé type. *Lettere al Nuovo Cimento (1971-1985)*, 23(9):333–338, 1978.

[2] M. J. Ablowitz and G. Biondini. Multiscale pulse dynamics in communication systems with strong dispersion management. *Optics Letters*, 23(21):1668–1670, 1998.

[3] M. J. Ablowitz and A. Zeppetella. Explicit solutions of fisher’s equation for a special wave speed. *Bulletin of Mathematical Biology*, 41(6):835–840, 1979.

[4] W. J. Alonso, C. Viboud, L. Simonsen, E. W. Hirano, L. Z. Daufenbach, and M. A. Miller. Seasonality of influenza in brazil: a traveling wave from the amazon to the subtropics. *American Journal of Epidemiology*, 165(12):1434–1442, 2007.

[5] S. Arrhenius. Über die reaktionsgeschwindigkeit bei der inversion von rohrzucker durch säuren. *Zeitschrift für physikalische Chemie*, 4(1):226–248, 1889.

[6] L. W. Baines and R. A. Van Gorder. Soliton wave-speed management: Slowing, stopping, or reversing a solitary wave. *Physical Review A*, 97(6):063814, 2018.

[7] H. Bateman. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43(4):163–170, 1915.

[8] A. Biswas. Dispersion-managed solitons in optical fibres. *Journal of Optics A: Pure and Applied Optics*, 4(1):84, 2001.
[9] M. E. Bleich and J. E. Socolar. Controlling spatiotemporal dynamics with time-delay feedback. *Physical Review E*, 54(1):R17, 1996.

[10] N. Bottman and B. Deconinck. Kdv cnoidal waves are spectrally stable. *Discrete and Continuous Dynamical Systems-Series A (DCDS-A)*, 25(4):1163, 2009.

[11] J. Boussinesq. *Essai sur la théorie des eaux courantes*. Memoires presentes par divers savants a l’Academie des Sciences, 1877.

[12] J. C. Bronski, L. D. Carr, B. Deconinck, and J. N. Kutz. Bose-Einstein condensates in standing waves: The cubic nonlinear schrödinger equation with a periodic potential. *Physical Review Letters*, 86(8):1402, 2001.

[13] J. M. Burgers. A mathematical model illustrating the theory of turbulence. In *Advances in Applied Mechanics*, volume 1, pages 171–199. Elsevier, 1948.

[14] S. L. Chang, N. Harding, C. Zachreson, O. M. Cliff, and M. Prokopenko. Modelling transmission and control of the covid-19 pandemic in australia. *Nature Communications*, 11(1):1–13, 2020.

[15] J. D. Cole. On a quasi-linear parabolic equation occurring in aerodynamics. *Quarterly of Applied Mathematics*, 9(3):225–236, 1951.

[16] D. A. Cummings, R. A. Irizarry, N. E. Huang, T. P. Endy, A. Nisalak, K. Ungchusak, and D. S. Burke. Travelling waves in the occurrence of dengue haemorrhagic fever in thailand. *Nature*, 427(6972):344–347, 2004.

[17] J. l. R. d’Alembert. Recherches sur la courbe que forme une corde tendue mise en vibration. *Histoire de l’académie royale des sciences et belles lettres de Berlin*, 3:214–219, 1747.

[18] J. l. R. d’Alembert. Suite des recherches sur la courbe que forme une corde tenduê, mise en vibration. *Histoire de l’académie royale des sciences et belles lettres de Berlin*, 3:220–249, 1747.

[19] P. G. Drazin and R. S. Johnson. *Solitons: an introduction*, volume 2. Cambridge University Press, 1989.

[20] B. Eiermann, P. Treutlein, T. Anker, M. Albiez, M. Taglieber, K.-P. Marzlin, and M. Oberthaler. Dispersion management for atomic matter waves. *Physical Review Letters*, 91(6):060402, 2003.

[21] M. El-Hachem, S. W. McCue, W. Jin, Y. Du, and M. J. Simpson. Revisiting the Fisher–Kolmogorov–Petrovsky–Piskunov equation to interpret the spreading–extinction dichotomy. *Proceedings of the Royal Society A*, 475(2229):20190378, 2019.

[22] J. M. Epstein, D. M. Goedcke, F. Yu, R. J. Morris, D. K. Wagener, and G. V. Bobashev. Controlling pandemic flu: the value of international air travel restrictions. *PloS One*, 2(5):e401, 2007.

29
[23] J. L. Everett, G. T. Campbell, Y.-W. Cho, P. Vernaz-Gris, D. B. Higginbottom, O. Pinel, N. P. Robins, P. K. Lam, and B. C. Buchler. Dynamical observations of self-stabilizing stationary light. *Nature Physics*, 13(1):68–73, 2017.

[24] N. T. Fadai. Semi-infinite travelling waves arising in a general reaction-diffusion Stefan model. *Nonlinearity*, 34(2):725, 2021.

[25] R. A. Fisher. The wave of advance of advantageous genes. *Annals of Eugenics*, 7(4):355–369, 1937.

[26] G. García-Ramos and D. Rodríguez. Evolutionary speed of species invasions. *Evolution*, 56(4):661–668, 2002.

[27] B. Grenfell. Rivers dam waves of rabies. *Proceedings of the National Academy of Sciences*, 99(6):3365–3367, 2002.

[28] B. T. Grenfell, O. N. Bjørnstad, and J. Kappey. Travelling waves and spatial hierarchies in measles epidemics. *Nature*, 414(6865):716–723, 2001.

[29] P. Grindrod. *The theory and applications of reaction-diffusion equations: patterns and waves*. Clarendon Press, 1996.

[30] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot, and J. G. Muga. Shortcuts to adiabaticity: Concepts, methods, and applications. *Reviews of Modern Physics*, 91(4):045001, 2019.

[31] G. Heinze, C. Hubrich, and T. Halfmann. Stopped light and image storage by electromagnetically induced transparency up to the regime of one minute. *Physical Review Letters*, 111(3):033601, 2013.

[32] J. O. Hirschfelder, C. F. Curtiss, R. B. Bird, and M. G. Mayer. *Molecular theory of gases and liquids*, volume 165. Wiley New York, 1964.

[33] E. Hopf. The partial differential equation $u_t + uu_x = \mu_{xx}$. *Communications on Pure and Applied Mathematics*, 3(3):201–230, 1950.

[34] Y. Hosono and B. Ilyas. Traveling waves for a simple diffusive epidemic model. *Mathematical Models and Methods in Applied Sciences*, 5(07):935–966, 1995.

[35] T. Kawahara and M. Tanaka. Interactions of traveling fronts: an exact solution of a nonlinear diffusion equation. *Physics Letters A*, 97(8):311–314, 1983.

[36] P. Kevrekidis, G. Theocharis, D. Frantzeskakis, and B. A. Malomed. Feshbach resonance management for Bose-Einstein condensates. *Physical Review Letters*, 90(23):230401, 2003.

[37] Y. S. Kivshar and S. K. Turitsyn. Vector dark solitons. *Optics Letters*, 18(5):337–339, 1993.

[38] A. Kolmogorov, I. Petrovskii, and N. Piskunov. A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem. *Bull. Moscow Univ., Math. Mech.*, 1:1–25, 1937.
[39] D. J. Korteweg and G. De Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 39(240):422–443, 1895.

[40] A. L. Krause and R. A. Van Gorder. A non-local cross-diffusion model of population dynamics II: Exact, approximate, and numerical traveling waves in single-and multi-species populations. *Bulletin of Mathematical Biology*, 82(8):113, 2020.

[41] N. A. Kudryashov and A. S. Zakharchenko. Analytical properties and exact solutions of the Lotka–Volterra competition system. *Applied Mathematics and Computation*, 254:219–228, 2015.

[42] T. Lakoba, J. Yang, D. Kaup, and B. Malomed. Conditions for stationary pulse propagation in the strong dispersion management regime. *Optics Communications*, 149(4):366–375, 1998.

[43] H. Li and Y. Guo. New exact solutions to the fitzhugh–nagumo equation. *Applied Mathematics and Computation*, 180(2):524–528, 2006.

[44] T. Litschel, M. M. Norton, V. Tserunyan, and S. Fraden. Engineering reaction–diffusion networks with properties of neural tissue. *Lab on a Chip*, 18(5):714–722, 2018.

[45] C. Liu, Z. Dutton, C. H. Behroozi, and L. V. Hau. Observation of coherent optical information storage in an atomic medium using halted light pulses. *Nature*, 409(6819):490–493, 2001.

[46] J. Löber and H. Engel. Controlling the position of traveling waves in reaction-diffusion systems. *Physical Review Letters*, 112(14):148305, 2014.

[47] E. K. Luckins and R. A. Van Gorder. Bose–Einstein condensation under the cubic–quintic Gross–Pitaevskii equation in radial domains. *Annals of Physics*, 388:206–234, 2018.

[48] A. Ludu and J. P. Draayer. Patterns on liquid surfaces: cnoidal waves, compactons and scaling. *Physica D: Nonlinear Phenomena*, 123(1-4):82–91, 1998.

[49] N. A. Maidana and H. M. Yang. Describing the geographic spread of dengue disease by traveling waves. *Mathematical Biosciences*, 215(1):64–77, 2008.

[50] K. Mallory and R. A. Van Gorder. Stationary solutions for the 1+1 nonlinear Schrödinger equation modeling repulsive Bose-Einstein condensates in small potentials. *Physical Review E*, 88(1):013205, 2013.

[51] K. Mallory and R. A. Van Gorder. Stationary solutions for the 2+1 nonlinear Schrödinger equation modeling Bose-Einstein condensates in radial potentials. *Physical Review E*, 90(2):023201, 2014.

[52] K. Mallory and R. A. Van Gorder. Stationary solutions for the nonlinear Schrödinger equation modeling three-dimensional spherical Bose-Einstein condensates in general potentials. *Physical Review E*, 92(1):013201, 2015.

[53] M. Mansour. Accurate computation of traveling wave solutions of some nonlinear diffusion equations. *Wave Motion*, 44(3):222–230, 2007.
[54] A. D. McNaught, A. Wilkinson, et al. Compendium of Chemical Terminology, volume 1669. Blackwell Science Oxford, 1997.

[55] C. Menyuk. Nonlinear pulse propagation in birefringent optical fibers. *IEEE Journal of Quantum Electronics*, 23(2):174–176, 1987.

[56] D. Mollison. Modelling biological invasions: chance, explanation, prediction. *Philosophical Transactions of the Royal Society of London. B, Biological Sciences*, 314(1167):675–693, 1986.

[57] Y. Morita and K. Tachibana. An entire solution to the Lotka–Volterra competition-diffusion equations. *SIAM Journal on Mathematical Analysis*, 40(6):2217–2240, 2009.

[58] I. Mostinsky. “Diffusion coefficient”, in: *International Encyclopedia of Heat & Mass Transfer*, Hewitt, G.F., Shires, G.L. and Polezhaev, Y.V. (eds.), volume 10. CRC Press, Florida, USA, 1996.

[59] A. Okubo, P. K. Maini, M. H. Williamson, and J. D. Murray. On the spatial spread of the grey squirrel in britain. *Proceedings of the Royal Society of London. B. Biological Sciences*, 238(1291):113–125, 1989.

[60] PDE Solutions, inc. Flexpde. *URL http://www.pdesolutions.com*, 2017.

[61] M. G. Pedersen. Wave speeds of density dependent nagumo diffusion equations–inspired by oscillating gap-junction conductance in the islets of langerhans. *Journal of Mathematical Biology*, 50(6):683–698, 2005.

[62] F. R. Pereira, J. C. Machado, and F. S. Foster. Ultrasound characterization of coronary artery wall in vitro using temperature-dependent wave speed. *IEEE transactions on ultrasonics, ferroelectrics, and frequency control*, 50(11):1474–1485, 2003.

[63] A. Ramani, B. Grammaticos, and T. Bountis. The Painlevé property and singularity analysis of integrable and non-integrable systems. *Physics Reports*, 180(3):159–245, 1989.

[64] L. Rayleigh. On waves. *Philosophical Magazine*, 1:257–259, 1876.

[65] K. A. Richardson, S. J. Schiff, and B. J. Gluckman. Control of traveling waves in the mammalian cortex. *Physical Review Letters*, 94(2):028103, 2005.

[66] P. Rosenau and J. M. Hyman. Compactons: solitons with finite wavelength. *Physical Review Letters*, 70(5):564, 1993.

[67] J. S. Russell. *Report on Waves: Made to the Meetings of the British Association in 1842-43*. Richard and John Edward Taylor, 1845.

[68] R. Schmitz. The WTC and ARS painlevé tests. *Applied Mathematics Letters*, 10(4):5–9, 1997.

[69] A. Scott. *Encyclopedia of nonlinear science*. Routledge, 2006.

[70] V. Serkin and A. Hasegawa. Soliton management in the nonlinear Schrödinger equation model with varying dispersion, nonlinearity, and gain. *Journal of Experimental and Theoretical Physics Letters*, 72(2):89–92, 2000.
[71] V. Serkin, A. Hasegawa, and T. Belyaeva. Nonautonomous solitons in external potentials. *Physical Review Letters*, 98(7):074102, 2007.

[72] V. N. Serkin and A. Hasegawa. Novel soliton solutions of the nonlinear Schrödinger equation model. *Physical Review Letters*, 85(21):4502, 2000.

[73] V. N. Serkin and A. Hasegawa. Exactly integrable nonlinear Schrödinger equation models with varying dispersion, nonlinearity and gain: application for soliton dispersion. *IEEE Journal of selected topics in Quantum Electronics*, 8(3):418–431, 2002.

[74] R. A. Van Gorder. Influence of temperature on Turing pattern formation. *Proceedings of the Royal Society A*, 476(2240):20200356, 2020.

[75] R. A. Van Gorder. Perturbation theory for Bose–Einstein condensates on bounded space domains. *Proceedings of the Royal Society A*, 476(2243):20200674, 2020.

[76] C. Viboud, O. N. Bjørnstad, D. L. Smith, L. Simonsen, M. A. Miller, and B. T. Grenfell. Synchrony, waves, and spatial hierarchies in the spread of influenza. *Science*, 312(5772):447–451, 2006.

[77] T. A. Vieira, M. R. Gesualdi, M. Zamboni-Rached, and E. Recami. Production of dynamic frozen waves: controlling shape, location (and speed) of diffraction-resistant beams. *Optics Letters*, 40(24):5834–5837, 2015.

[78] Z.-C. Wang and J. Wu. Travelling waves of a diffusive kermack–mckendrick epidemic model with non-local delayed transmission. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2113):237–261, 2010.

[79] J. Weiss, M. Tabor, and G. Carnevale. The Painlevé property for partial differential equations. *Journal of Mathematical Physics*, 24(3):522–526, 1983.

[80] P. Weng and X.-Q. Zhao. Spreading speed and traveling waves for a multi-type sis epidemic model. *Journal of Differential Equations*, 229(1):270–296, 2006.

[81] A. White, J. McTigue, and C. Markides. Wave propagation and thermodynamic losses in packed-bed thermal reservoirs for energy storage. *Applied energy*, 130:648–657, 2014.

[82] Z. Xu and C. Ai. Traveling waves in a diffusive influenza epidemic model with vaccination. *Applied Mathematical Modelling*, 40(15-16):7265–7280, 2016.

[83] J. Yang and Y. Tan. Fractal structure in the collision of vector solitons. *Physical Review Letters*, 85(17):3624, 2000.

[84] N. J. Zabusky and M. D. Kruskal. Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, 15(6):240, 1965.

[85] V. Zharnitsky, E. Grenier, C. K. Jones, and S. K. Turitsyn. Stabilizing effects of dispersion management. *Physica D: Nonlinear Phenomena*, 152:794–817, 2001.