Generalized scaling relations for unidirectionally coupled nonequilibrium systems

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Unidirectionally coupled systems which exhibit phase transitions into an absorbing state are investigated at the multicritical point. We find that for initial conditions with isolated particles, each hierarchy level exhibits an inhomogeneous active region, coupled and uncoupled respectively. The particle number of each level increases algebraically in time as \( N(t) \sim t^\nu \) with different exponents \( \nu \) in each domain. This inhomogeneity is a quite general feature of unidirectionally coupled systems and leads to two hyperscaling relations between dynamic and static critical exponents. Using the contact process and the branching-annihilating random walk with two offsprings, which belong to the DP and PC classes respectively, we numerically confirm the scaling relations.

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Nonequilibrium absorbing phase transitions (APT) have been studied extensively over the last decade in the hope of understanding better nonequilibrium continuous phase transitions \([1, 2]\). Also APT are one of the simplest and natural extensions of the well-established equilibrium transitions to nonequilibrium systems. In APT, a system undergoes a continuous phase transition from an active into an absorbing state where no dynamics occur furthermore. Models exhibiting APT describe wide range of phenomena such as epidemic spreading, catalytic chemical reactions, surface growth, wetting and roughening transitions, self-organized criticality, and transport in disordered media \([1, 2]\).

The basis of theoretical and numerical analysis is the understanding that the concept of universality plays a central role in classifying nonequilibrium phase transitions as it does in thermal equilibrium. However, surprisingly few universality classes for APT have been identified so far, and – as a unifying theoretical framework is not available – we are still far from a systematic classification of APT even in one spatial dimension. The best-studied directed percolation (DP) class includes the systems which have no symmetry between absorbing states and no conservation laws in the order parameter \([1, 2]\). The best-known scaling relation between these exponents is

\[
P_A(t) \sim t^{-\delta}, \quad \rho(t) \sim t^{-\alpha}, \quad N(t) \sim t^\nu, \quad R(t) \sim t^{1/\nu},
\]

with \( \alpha = \beta/\nu, \ z = \nu/\nu_\perp \) and \( d \) is the spatial dimension. The well-known scaling relation between these exponents is

\[
\eta = d/\nu - \alpha - \delta.
\]

Except in systems with infinitely many absorbing states, we have \( \alpha = \delta \).

Rather than exploring new systems exhibiting unknown critical behavior, coupled systems of known universality classes have been studied recently as an another direction of searching for new universality classes. A coupled systems is a multi-species system in which each species is coupled to the others in certain ways. Among possible ways of coupling such as bidirectional, cyclic and unidirectional coupling in linear or quadratic way \([3, 10, 11]\), a recent study on a hierarchy of linearly unidirectionally coupled DP systems shows that a new critical behavior emerges at the multicritical point where criticality of all levels coincide \([5]\). The multicritical behavior is characterized by a varying order parameter exponent \( \beta \), taking a different value in each level of the hierarchy, while the exponents \( \nu_\perp \) and \( \nu_\parallel \) are those of the DP class in all levels. Also \( \eta \) and \( \delta \) vary according to the levels of the hierarchy, but these exponents do not satisfy the ordinary scaling relation of Eq. \((3)\) in each higher level \((k > 1)\). Instead they are suggested to satisfy

\[
\eta_k = \frac{d}{z_k} - \alpha_k - \delta_1
\]

with \( \alpha_k = \delta_k \). In Eq. \((4)\), the exponent of \( P_A(t) \) of the first level, \( \delta_1 \) replaces \( \delta_k \) of \( k \)’ level. The appearance of \( \delta_1 \) in Eq. \((4)\) is unusual and there has been no clear understanding for that. It should be noted that Eq. \((4)\) comes from taking an average where one uses configurations in
which each level survives regardless of the survival of the first level. We shall refer to this usual average as slave average to distinguish it from our average method discussed below.

In this paper, we report generalized scaling relations for linearly and unidirectionally coupled systems belonging to same or different universality classes. We propose an explanation for the appearance of $\delta_1$ in Eq. (4). We also show that Eq. (4) is not satisfied in general coupling systems because it is a consequence of the usual slave average which does not take into account an important inhomogeneity in the distribution of particles in the higher levels of the hierarchy. In addition, our scaling relations provides insight in the origin of markedly large corrections to scaling which make estimations of critical exponents difficult.

We consider a hierarchy of an arbitrary number of coupled systems with coupling $A \to B \to C \cdots$. For the derivation of scaling relations, we are interested in the dynamic behavior of the $k$'th level starting with a single particle on the first level. Then on each level, a cluster is created and spreads. Due to the coupling, the size of the cluster on each level is always larger than those of the lower levels. It means that the cluster of the $k$'th level (with size $R_k(t)$) is divided into two parts, the coupled and the uncoupled region where the dynamics on level $k$ evolve autonomously (Fig. 1). In the coupled region (with size $R_C(t)$), the $(k-1)$'th level feeds particles to the $k$'th level so that $R_C(t)$ is actually the spreading distance of the $(k-1)$'th level ($R_{k-1}(t)$). However, in the uncoupled region (with size $R_U(t)$), there is only the feeding from the boundary of $R_C(t)$. So in $R_U(t)$, the cluster evolves with its own reaction dynamics. It implies that the number of particles in the $k$'th level ($N_k(t)$) increases in time with different exponents at the multicritical point, viz., $N_C \sim t^{\delta_k}$ and $N_U \sim t^{\delta_k}$ for observables.

We shall refer to this usual average as $\eta^C_k$. For $z_U^k \geq z_k$, we have a condition, $z_U^k = d/z_k - \alpha^C_k - \delta_1$. This is of the same form as Eq. (4) for $z_U^k = z_k - 1$, $\eta^C_k$ is larger than $\eta^U_k$ because $\alpha^C_k$ and $\alpha^U_k$ always satisfy $\alpha^C_k \leq \alpha^U_k$. In that case, we have $\eta^C_k = \eta^U_k$ and $\alpha^C_k = \alpha^U_k$ and we get $\eta^C_k = d/z_k + \alpha_k + \delta_1$ with $\alpha_k \neq \delta_k$. This is of the same form as Eq. (4) for $z_U^k = z_k - 1$ but with $\alpha_k \neq \delta_k$. However the values of $\eta^C_k$ is not the same in general because it is measured in different ways.

Notice that the inhomogeneity of a cluster gives corrections, $t^{-\Delta_\rho}$, $t^{-\Delta_\eta}$ and $t^{-\Delta_z}$ with some negative coefficients to the scaling with the asymptotic exponents $\alpha_k$, $\eta_k$ and $z_k$. The quantities $\Delta_\rho$, $\Delta_\eta$ and $\Delta_z$ are defined as $\Delta_\rho = |\alpha^C_k - \alpha^U_k|$, $\Delta_\eta = |\eta^C_k - \eta^U_k|$ and $\Delta_z = d/|z_U^k - 1/z_k - 1|$ respectively. These correction exponents may be small so that they can cause a significant long time drift of effective exponents measured in
The first level, a particle with probability \( p \) is spontaneously annihilated. For PC-PC coupling, the existence of a multicritical point \( \sigma \) in unidirectionally coupled systems. We denote the coupling of two systems as Source-Slave coupling, where the source system feeds particles to slave one.

There are four ways of coupling DP and PC system. For PC-PC coupling, the existence of a multicritical point depends on the way of coupling \( M \). We will discuss it in other place. For DP-DP coupling, we consider a two-level hierarchy of contact process (CP) \( \sigma \). In the first level, a particle \( A \) is spontaneously annihilated with probability \( p \) and it creates one \( A \) particle on one of the nearest neighboring \( (nn) \) sites with \( (1-\sigma)(1-p) \). In the second level, the annihilation and branching occur with \( p \) and \( \sigma(1-p) \) respectively. In branching, if the target site is already occupied by an other particle, the branching process is rejected. The probability \( \sigma \) is introduced for the coupling. The criticality of each level is \( pC_A = 0.232674(4) \) at \( \sigma = 0 \) for the first and \( \sigma = 1 \) for the second level \( \frac{1}{2} \). Using the fact that the ratio of creation and annihilation probabilities, \( R = (1-p_c)/p_c \) should be same at criticality for any value of \( \sigma \), we find that the critical line of each level in \( \sigma-p \) phase diagram is \( \sigma_c^A = 1-pR/(1-p) \) and \( \sigma_c^B = pR/(1-p) \) with \( R = 3.29785 \) (Fig. 2). The coupling dynamics \( A \) \( \rightarrow \) \( A+B \) with \( \sigma(1-p) \) linearly and unidirectionally couple two CPs without feedback from the second to the first level. The multicritical point \( (\sigma_M, p_M) \) is the intersection point of the two critical lines, \( \sigma_c^A \) and \( \sigma_c^B \). We find \( \sigma_M = 1/2 \) and \( p_M = 0.13165 \). If the multicritical point is approached along the line \( \Delta = (p_M - p) \rightarrow 0 \) with fixed \( \sigma = \sigma_M \), new critical behavior emerges.

By measuring \( \eta_C \) and \( \eta_U \) of Eq. (7), we can predict \( \eta_B \) and \( \alpha_B \) defined as \( N_B = N_C + N_U \sim t^{\eta_B} \) and \( \rho_B = \rho_C + \rho_U \sim t^{-\alpha_B} \). Using source average at \( \Delta = 0 \), we measure effective exponents \( \eta_C(t), \eta_U(t), \alpha_C(t), \eta_B(t) \) and \( 1/\nu_U(t) \) up to \( 10^5 \) Monte Carlo time steps. \( \eta_C(t) \) is defined as \( \eta_C(t) = \log(N_C(mt)/N_C(t))/\log m \) and similarly for the others. Table shows the numerical results of source average. \( \eta_B \) is fairly underestimated due to the large correction with \( \Delta_n = 0.08(1) \). From Eq. (7) we predict \( \alpha_C = 0.0725(100), \alpha_U = 0.1525(125) \). Because of \( \eta_C > \eta_U \) and \( \alpha_C < \alpha_U \), we predict \( \eta_B = \eta_C = 0.40(1) \) and \( \alpha_B = \alpha_C = 0.0725(100) \). The prediction agree very well with the numerical results. Fig. 3 shows \( \eta_C(t), \eta_U(t) \) and \( \eta_B(t) \). Our results are also equivalent to the previous result of the second level of DP-DP coupling measured in slave average, \( \eta_B = 0.39(2) \) and \( \alpha_B = 0.075(10) \).

Next, we consider a two-level hierarchy of systems belonging to DP and PC class. We only discuss PC-DP coupling here.

We consider the CP and branching annihilating random walks with two offsprings (BAW(2)) \( M \). BAW(2) belongs to PC class. In BAW(2), a particle, A hops to one of the \( nn \) sites with probability \( p \) and it creates two particles on two \( nn \) sites to the left or right direction with \( (1-\sigma)(1-p) \). If two particles happen to be on a same site by branching or hopping, they annihilate each other instantaneously. The critical point of BAW(2) at \( \sigma = 0 \) is at \( p_c^A = 0.5105(7) \). The critical line is \( \sigma_c^A = 1 - pR_A/(1-p) \), where \( R_A = (1-p_c^A)/p_c^A = 0.95886 \). In CP, annihilation and branching of a \( B \) particle occur with \( p \) and \( \sigma(1-p) \). Criticality of CP at \( \sigma = 1 \) is same as before. The critical line is now \( \sigma_c^B = pR/(1-p) \). The critical point is \( \sigma_M = 0.77474 \) and \( p_M = 0.19023 \). At the multicritical point, we estimate \( \eta_C = 0.14(2) \).
\(\eta_U = 0.1875(25)\) and \(2/z_U = 1.26(1)\). Using random initial conditions of \(\rho_A(0) = \rho_B(0) = 1/2\), we also measure the exponent \(\beta\) of \(\rho_B^\alpha\) in steady-states up to system size \(1.2 \times 10^4\) and \(\Delta = 6 \times 10^{-3}\). We estimate \(\beta_B = 0.28(1)\). The numerical results show that the CP still belongs to the DP class.

With \(z_A = 1.750(5), \delta_A = 0.285(2)\) for BAW(2)[15], Eq. (7) predicts \(\alpha_C = 0.146(20)\) and \(\eta_U = 0.1575(96)\). The value of \(\eta_C\) and \(\eta_U\) agree with the DP value \(\alpha = 0.15964(6)\)[2] within numerical error. It means that \(\rho_B\) uniformly decays but very slowly converges due to the coupling effect in the coupled region. Due to \(\eta_C < \eta_U\), Eq. (7) predicts \(\eta_B = \eta_U = 0.1875(25)\) which is not the value of DP class, \(\eta = 0.313686(8)\)[2]. We directly measure \(\eta_B = 0.18(1)\) which is somewhat underestimated but agrees well with the prediction. We conclude that even though CP is not changed by BAW(2), Eq. (7) is satisfied very well while Eq. (4) gives a wrong prediction for this coupling. On the other hand, with the DP values for \(\eta_B\) and \(z_B\), Eq. (4) predicts \(\alpha_B = 0.0338\), which is completely different from the true DP value, \(\alpha = 0.15964\). Eq. (4) also gives a wrong prediction for BAW-CP-CP coupling in which BAW(2) is on the first level. The CP of the third level is affected by the second level. But the second level is not affected by the first. In slave average, exponents of the third level are same as those of the second level in CP-CP coupling without BAW(2). These results cannot satisfy Eq. (4) because \(\delta_1\) is not a DP value.

In order to interpret these observations we stress that the dynamic exponent \(\eta\) can depend on average methods, but \(\alpha\) does not because it is the ratio of \(\beta\) and \(\nu_{||}\) which characterize off-critical behavior. Since away from the critical point the survival probability for source is unity and hence the source and the slave average coincide. Therefore the results of two average methods should give the same value of \(\alpha\). So it is natural that two values of \(\alpha\) measured in two different ways are same in CP-CP and BAW-CP coupling. The method of average is a matter of choice but one obtains distinct scaling relations which are valid for any coupling. It means that Eq. (4) is not a relation for the slave average because it is not always satisfied by the results of slave average. Also Eq. (4) is not a general relation for unidirectionally coupled systems because it is a relation for quantities of the total system. As the relation for quantities of a total system is obtained by comparing Eq. (5) and (6), it can change according to the nature of the coupling.

In summary, unidirectionally coupled systems show an inhomogeneity in the number of particles or density. A cluster in a higher hierarchy is divided into a coupled and an uncoupled region. In each region, dynamic quantities show different scaling behavior. In order to describe this property quantitatively we employ a special average method, the source average. Using scaling arguments we derive two generalized scaling relations which describes the inhomogeneity. The presence of \(\delta\) of the first level in the scaling relations arises naturally from the source average. Our scaling relations also explain the existence of large corrections to scaling of dynamic quantities of a total system which cause a long time drift of critical exponents. Our scaling relations should be tested for other couplings between different universality classes and also in higher dimensions. The DP-PC coupling will be a possible candidate for future study.

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| \(\eta_C\) | \(\eta_U\) | \(2/z_U\) | \(\alpha_C\) | \(\eta_B\) |
|----------|----------|---------|----------|----------|
| 0.40(1)  | 0.32(1)  | 1.265(5) | 0.075(5) | 0.375(25) |

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