Fermion masses in $E_6$ grand unification with family permutation symmetries

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Abstract

Neutrino masses have been extensively studied in the context of discrete symmetries. A family permutation symmetry easily explains all neutrino oscillations and their mixing angles. However it is not obvious the embedding of such symmetry in a grand unified theory where all fermions live in the same representation. We show that it is possible to embed such a horizontal discrete symmetry in a grand unified theory. We discuss an $E_6$ GUT model as in [1], but differently from [1], we take the SM Higgs doublet in the $351'$ to have $(+3,+5)$ charges with respect the additional $U(1)_r \times U(1)_t$. This choice makes possible two different horizontal symmetry breaking patterns and a distinction between neutrinos and the rest of matter fermions.

A grand unified model for fermion masses

In this paper we will study a $E_6$ unified gauge group [1, 2, 3] that could explain at the same time, neutrino, lepton and quark masses. In addition to the gauge symmetry group we will require a permutation symmetry [4, 5] of the three fermion families. An horizontal permutation symmetry means that the three families can be considered identical in the sense of third quantization[6]. The breaking of this $S_3$ symmetry into a $S_2$ symmetry exchanging the $\nu_\mu, \nu_\tau$ neutrinos can easily explain large solar and atmospheric mixing angles[7]. However this permutation symmetry does not directly apply to the quark and lepton masses, in which case both $S_3$ and $S_2$ are strongly broken. The mass ratio of the muon and the electron is roughly 200. The issue is how to embed a neutrino mass matrix that is $S_2$ symmetric and with just a small breaking mass term of the full $S_3$ group, in the same unified group with lepton and quarks where the breaking of $S_3$ and $S_2$ is instead strong. The idea is the following. Yukawa couplings for quark and leptons strongly break the permutation symmetry, thus we expect that they are proportional to the v.e.v. of some scalar fields responsible of the $S_3$ spontaneous symmetry breaking. Probably yukawa couplings in the neutrino sectors are not proportional to the same fields in order to explain this difference. This possibility is not obvious when we are dealing with gauge groups with all fermions living in the
same irreducible representation. In SU(5) the Yukawa couplings can be written in the following way

\[ L = g_u T^{\alpha \beta} T^{\gamma \delta} H^\sigma \varepsilon_{\alpha \beta \gamma \delta \sigma} + g_d T^{\alpha \beta} F_\alpha \bar{H}_\beta + g_v F_\alpha \nu_R H^\alpha + M \nu_R \nu_R \]

(1)

where \( T^{\alpha \beta} \), \( F_\alpha \) are weyl fermions belonging to the 10 and \( \bar{5} \) of SU(5). \( \nu_R \) is the SU(5) singlet right-handed neutrino and \( M \) is a majorana mass term. In the lagrangian above, Yukawa interactions are distinct, but we want to understand why these are proportional to the v.e.v. of different scalars. If we embed SU(5) into SO(10) we have an additional U(1) gauge group that commutes with the full SU(5). The U(1) charges for the representation above are \( H(+q), \bar{H}(-q), T(-1) \), \( F(+3) \) and \( \nu_R(-5) \). From these charges we derive that each mass operators have \( U_r(1) \) charges

| SU(5) mass operator | \( U_r(1) \) |
|---------------------|-------------|
| \( T^{\alpha \beta} F_\alpha \) | +2          |
| \( F_\alpha \nu_R \) | -2          |
| \( \nu_R \nu_R \) | -10         |
| \( T^{\alpha \beta} T^{\gamma \delta} \) | -2          |

We observe that the mass operators \( T^{\alpha \beta} T^{\gamma \delta} \) and \( F_\alpha \nu_R \) have the same charges, thus we expect that the same SU(5) singlet is at the origin of their Yukawa interaction. While neutrinos have an approximate \( S_3 \) symmetry [5], the same symmetry is not observed in the up sector. If we embed SU(5) into \( E_6 \) we have an additional \( U_t(1) \) and the table above becomes

| SU(5) mass operator | \( U_r(1) \) | \( U_t(1) \) |
|---------------------|-------------|-------------|
| \( T^{\alpha \beta} F_\alpha \) | +2          | +2          |
| \( F_\alpha \nu_R \) | -2          | +2          |
| \( \nu_R \nu_R \) | -10         | +2          |
| \( T^{\alpha \beta} T^{\gamma \delta} \) | -2          | +2          |
| \( F_\alpha \nu_L \) | +3          | +5          |
| \( \nu_R \nu_L \) | -5          | +5          |
| \( x_L^t x_L \) | 0            | +8          |

The advantage here is that we have two right-handed neutrinos \( \nu_R \) and \( x_L \), and the Dirac mass operator \( F_\alpha x_L \) has different quantum numbers from all the others and in particular is different from \( T^{\alpha \beta} T^{\gamma \delta} \) giving mass to the up sector. Thus we explore the possibility that the fundamental lagrangian has an \( E_6 \) unifying gauge symmetry times a \( S_3 \) permutation symmetry of the three fermion families that belong to the 27 of \( E_6 \). As already said the 27 contains two standard model singlets that will play the role of right-handed neutrinos. Now we have to choose the representation for the Higgs SU(2)_W doublet. We prefer to keep just one Higgs doublet, that will give mass both for the up and the down sector. This is because we want to have the Standard Model with just one higgs at the weak scale where the FCNC are strongly suppressed due to the GIM mechanism. So we choose the Higgs to be also a \( S_3 \) singlet. Now we have to decide to which \( E_6 \) representation we have to assign the Higgs doublet. The Yukawa interaction for fermions at the tree level can be

\[ 27^i_i \ 27^i_i \ 351'_\alpha \beta \]  

\[ i = 1,2,3 \text{ family index} \]

(2)

where the 351’ is symmetric under the exchange of \( \alpha \) and \( \beta \) the gauge symmetry indices. The reason why we put the Higgs doublet in the 351’ is that it contains a SU(2)_W doublet with the (-3,-5) charges with respect the \( U_r(1) \times U_t(1) \). Also the 351 contains a Higgs doublet with same charges,
\[
T_{(\alpha\beta)} = \begin{pmatrix}
0 & D_1 & D_2 & D_3 & -N_L & E_L \\
-D_1 & 0 & u_{R3}^c & -u_{R2}^c & -d_{L1} & u_{L1}^c \\
-D_2 & -u_{R3}^c & 0 & u_{R1}^c & -d_{L2} & u_{L2}^c \\
-D_3 & u_{R2}^c & -u_{R1}^c & 0 & -d_{L3} & u_{L3}^c \\
-N_L & d_{L1} & d_{L2} & d_{L3} & 0 & e_R^c \\
-E_L & -u_{L1} & -u_{L2} & -u_{L3} & -e_R^c & 0
\end{pmatrix}
\]

\[
R_\alpha^\alpha = \begin{pmatrix}
\nu_{eR1} & D_1 & D_2 & D_3 & N_L & E_L \\
x_L & d_{R1}^c & d_{R2}^c & d_{R3}^c & \nu_L & e_L
\end{pmatrix}_{\alpha\beta}
\]

Table 1: Branching of 27 in SU(2) × SU(6): 27 = (2, 6) + (1, 15) = T_{(\alpha\beta)} + R_\alpha^\alpha

but this would give zero in [2]. At the tree level of the fundamental high energy lagrangian, we have just one Yukawa interaction that comes from [2], since this is the unique U_r(1) × U_t(1) gauge invariant operator

\[g \times i_{L} \nu_{L} h_{0}\]

As already mentioned the higgs is a S_3 singlet and does not carry any family index, while fermions are assigned to the 27_i and i runs over the three fermion families. In this way we have been able to accommodate a Dirac mass proportional to the identity matrix as proposed in [5]. So, before the E_6 symmetry breaking, quark and charged lepton yukawa couplings are zero, since they do not form a gauge invariant operator with the Standard Model Higgs. The up quark yukawa operator (in a SU(5) notation) is \(T^{\alpha\beta} T^{\gamma\delta} H^\sigma \varepsilon_{\alpha\beta\gamma\delta\sigma}\) and its charges are (+5, +3). We need a SU(5) singlet with opposite U_r(1) × U_t(1) to make an invariant operator. At first sight such a singlet is contained both in the 78 and in the 650, it has the correct U(1) charges. But we will see after that in order to give a Yukawa coupling to the up quarks we have to write an interaction

\[27^\alpha 27^\beta 351^\gamma 351^\sigma \Sigma^{\gamma\sigma}_{\alpha\beta}\]

where \(\Sigma^{\gamma\sigma}_{\alpha\beta}\) is the irrep 2430.

The breaking \(E_6 \supset SU(2) \times SU(6) \supset U(1) \times SU(5)\)

The breaking \(E_6 \supset SU(2) \times SU(6)\) can be achieved after the irrep 650 takes a vev. If \(Q_r\) and \(Q_t\) are the U(1) charges corresponding to the embedding \(E_6 \supset Q_t \times SO(10) \supset Q_t \times Q_t \times SU(5)\) while \(Q_{T_3}\) and \(Q'\) correspond to the embedding \(E_6 \supset SU(2) \times SU(6) \supset Q_{T_3} \times SU(6) \supset Q_{T_3} \times Q' \times SU(5)\), then these charges are related as follows

\[Q_r = -\frac{1}{3} Q' - \frac{5}{3} Q_{T_3} \quad Q' = -\frac{2}{3} Q_r + \frac{4}{3} Q_t \quad Q_{T_3} = -\frac{1}{3} Q_r - \frac{1}{3} Q_t\]

All the quantum numbers are reported in table 2. The branching rule for the 27 is 27 = (2, 6) + (1, 15) where the branching rule for 6 in SU(5) × U(1)' is 6 = 1(5) + 5(-1) and for 15 is 15 = 5(-4) + 10(2). That means that we can take two tensors that contain all fermions \(T_{(\alpha\beta)}\) and \(R_\alpha^\alpha\) (see table 1). We use the following convention for the tensors, upper indices and lower indices differs for a complex conjugation, i.e. \(R_\alpha^\alpha R_\alpha^\beta\) is a SU(2) × SU(6) invariant. Latin indices \(a, b, c, \ldots = 1, 2\) are the SU(2) indices, while greek indices \(\alpha, \beta, \gamma, \ldots = 1, 6\) are the SU(6) indices. \(i, j, k = 1, 2, 3\) are family indices. Sometimes, whenever confusion is possible, underlined indices \(\underline{\alpha}\) mean \(E_6\) indices and run from
The Yukawa interaction \( i K \) the indices within the parenthesis () while it is symmetric if indices are within \([\]\) parenthesis as in \(6\)

\[
\begin{bmatrix}
\bar{D}_1 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & -2 & -4 & 0 \\
\bar{D}_2 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & -2 & -4 & 0 \\
u^c_{R3} & -\frac{1}{3} & -\frac{1}{3} & 0 & -1 & 1 & 2 & 0 \\
\bar{D}_3 & -\frac{1}{3} & -\frac{1}{3} & 0 & 2 & -2 & -4 & 0 \\
u^c_{R2} & -\frac{1}{3} & -\frac{1}{3} & 0 & -1 & 1 & 2 & 0 \\
u^c_{R1} & -\frac{1}{3} & -\frac{1}{3} & 0 & -1 & 1 & 2 & 0 \\
N_L & 0 & -\frac{1}{3} & -\frac{1}{3} & 2 & -2 & -4 & 0 \\
d_{L1} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & 2 \\
d_{L2} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & 2 \\
d_{L3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & 2 \\
E_l & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 & -2 & -4 \\
u^c_{l1} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & 1 & 2 \\
u^c_{l2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & 1 & 2 \\
u^c_{l3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -1 & 1 & 2 \\
\nu^c_R & 0 & 0 & 0 & 0 & -5 & 1 & 5 \\
D^c_1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & -2 & -2 & -1 \\
D^c_2 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & -2 & -2 & -1 \\
D^c_3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & -2 & -2 & -1 \\
\bar{N}_L & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 & -2 & -1 \\
E^c_L & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -2 & -2 & -1 \\
x_L & 0 & 0 & 0 & 0 & 4 & 5 & -1 \\
n^c_{R1} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 3 & 1 & 1 \\
n^c_{R2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 3 & 1 & 1 \\
n^c_{R3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 3 & 1 & 1 \\
x_L & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 3 & 1 & -1 \\
\nu^c_L & 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 3 & 1 & -1 \\
e_L & -1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 3 & 1 & -1 \\
\end{bmatrix}
\]

Table 2: Quantum numbers of the irrep \(27\)

1 to 27. We also use the convention that a tensor \( T_{(\alpha\beta\gamma)} \) is antisymmetric under permutations of the indices within the parenthesis () while it is symmetric if indices are within \([\]\) parenthesis as in \( T_{[\alpha\beta\gamma]} \). The higgs doublet is in the 351' = (1, 15) + (3, 21) + (2, 34) + (1, 105'), and more precisely it lives in the (3, 21), i.e. we can consider a tensor \( K^{[ab]}_{[\alpha\beta]} \) of SU(2) × SU(6) that includes the standard Higgs doublet. The higgs doublet has charges \( Q_r = -3 \) and \( Q_t = -5 \) which corresponds to the U(1) charges \( Q_T = 2 \) and \( Q' = 4 \) with respect \( U_T(1) \times U(1)' \).

The only representation with correct charges is (3,21). The usual two components vector of the Higgs doublet are written in terms of the K components as follows \( (h_0, h^-) = (K_{[22]}^{[15]}, K_{[16]}^{[22]}) \).

The Yukawa interaction \((i = 1, 2, 3 \text{ is the family index})\)

\[
L = 277^{\alpha} \bar{27}^{\beta}_{\gamma} \ 351'^{\alpha\beta}_{\alpha\beta} = R^{\alpha}_{\alpha\alpha} R^{\beta}_{\beta\beta} K^{[ab]}_{[\alpha\beta]} = R^{\alpha}_{\alpha R} R^{\beta}_{\beta R} K^{[22]}_{[15]} = x_L \nu_L h_0
\]

Thus there is only one Yukawa interaction in the fundamental \( E_6 \) symmetric and renormalizable lagrangian this is the Dirac neutrino mass \( \bar{\nu}_L \) does not introduce any mass neither for quarks nor for charged leptons. This result is important since a Yukawa interaction \( u^c_R \) \( u_L \) \( h_0 \) (symmetric
under $S_3$ family permutations) would give $m_{top} = m_{charm} = m_{up}$ that is clearly unacceptable.

Quark and leptons can take a yukawa interaction only after the $S_3$ symmetry is broken. We expect they arise through higher order operators in effective lagrangian approach after the spontaneous symmetry breaking of $E_6 \times S_3$. This result is automatic because we have chosen the Higgs doublet to be in the $351'$, with $U(1)_r \times U(1)_t$ charges $(-3,-5)$.

| SU(5) mass operator | $U_r(1)$ | $U_t(1)$ |
|---------------------|----------|----------|
| $\tilde{D}^\alpha x_L H_*^\alpha$ | 5        | 7        |
| $x_L^T x_L$          | 0        | 8        |
| $\nu_R^T x_L$        | -5       | 5        |
| $F^\alpha x_L H^\alpha$ | 0        | 0        |
| $T^{\alpha\beta} T^{\gamma\delta} H^\sigma_{\epsilon\alpha\beta\gamma\delta}$ | -5 | -3 |
| $T^{\alpha\beta} F^\alpha H^\beta_\gamma$ | 5 | 7 |
| $\nu_R^T \nu_R$     | -10      | 2        |
| $F^\alpha \nu_R H^\alpha$ | -5 | -3 |
| $D^\alpha x_L H^\alpha$ | -5 | -3 |
| $T^{\alpha\beta} D^\alpha H^\beta_\gamma$ | 0 | 4 |
| $D^\alpha \nu_R H^\alpha$ | -10 | -6 |
| $D^\alpha D^\alpha$ | 0 | -4 |
| $\nu_R \tilde{D}^\alpha$ | 5 | -1 |
| $\tilde{D}^\alpha \nu_R H^\alpha_\beta$ | 0 | 4 |

Table 3: Mass terms in SU(5).

Now suppose that we want to give mass to the $DD$, then we have to break $U(1)_r \times U(1)_t$ in the direction $(0,4)$, but in this case also the $x_L, x_L$ gets a mass and would be heavier than the $\nu_R$. One could break the $U(1)$ and give a mass slightly smaller than the $x_L$. This is because we want the mixing between the $\nu_L$ and $x_L$, and a too heavy $x_L$ would not allow this possibility. The situation looks better if we first break $U(1)_r \times U(1)_t$ in the direction $(5,-5)$ giving a Dirac mass $\nu_R x_L$ (see table 3). This can be achieved through the $351'$ that could come from the product $78 \times 351$ (since the Higgs doublet is already in the $351'$). Thus the breaking could proceed as follows. First the 650 takes a v.e.v. and breaks the $E_6 \supset SU(2) \times SU(6)$, after that the 78 could break $SU(2) \times SU(6) \supset U(1) \times SU(6)$ and the 351 breaks $U(1) \times SU(6) \supset SU(5) \times U(1)$, then the tensor $351' \in 78 \times 351$ can give dirac mass $\nu_R x_L$ through the operator $27^a 27^b 351_{a \gamma} 78_{\beta}$. So we are left with $SU(5) \times U(1)_x$ and the charges with respect this new $U(1)_x$ are in the table 4.

Thus we see that the breaking of the remaining $U(1)$ will give mass to all fermions and the necessary Yukawa couplings. Note that if the breaking occurs through a field with charges $(0,4)$, it will appear as a second power for the up-sector and at the third power in the down sector, since they have respectively charges 8 and 12. This is compatible with the fact that the up-quark sector is heavier than the down sector. Concerning the $S_3$ symmetry breaking, we can imagine in first approximation that the 351 is responsible for the $S_3$ breaking. Let us discuss first the neutrino sector. We can have the following lagrangian

$$L = 27^a_1 27^b_j 351_k \alpha_\gamma 78_{\beta}^\gamma + 27^a_1 27^b_i 351_i \alpha_\gamma 78_{\beta}^\gamma$$

Light left handed neutrinos will mix with the heavy dirac neutrino through the mass matrix
SU(5) mass operator \( U_x(1) \)

\[
\begin{align*}
\bar{D}^\alpha x_L H_\alpha^* & \quad 12 \\
x_L^i x_L & \quad 8 \\
\nu_R^i x_L & \quad 0 \\
F_\alpha x_L H^\alpha & \quad 0 \\
T^{\alpha\beta} T^{\gamma\delta} H^\sigma & \quad -8 \\
T^{\alpha\beta} F_\alpha H^\beta_\delta & \quad 12 \\
\nu_R^i \nu_R & \quad -8 \\
F_\alpha \nu_R H^\alpha & \quad -8 \\
D_\alpha x_L H^\alpha & \quad -8 \\
T^{\alpha\beta} D_\alpha H^\beta_\delta & \quad 4 \\
D_\alpha \nu_R H^\alpha & \quad -16 \\
D_\alpha D^\alpha & \quad -4 \\
F_\alpha D^\alpha & \quad 4 \\
\bar{D}^\alpha \nu_R H^\alpha_\alpha & \quad 4 \\
\end{align*}
\]

Table 4: \( U(1)_x \) charges

(i and j family index)

\[
M = \left( \begin{array}{ccc} \nu_L & \nu_R^* & x_L \end{array} \right)^i \left( \begin{array}{ccc} 0 & 0 & m \delta_{ij} \\ 0 & m_v \delta_{ij} & M_{ij} \\ m \delta_{ij} & M_{ij} & m_x \delta_{ij} \end{array} \right) \left( \begin{array}{c} \nu_L \\ \nu_R^* \\ x_L \end{array} \right)^j
\]

that will give the following mass matrix elements for the left-handed neutrinos \( (M >> m, m_v, m_x) \)

\[
m_{ij}^{\text{light}} = m_v \left( \frac{m^2}{M^2} \right)_{ij}
\]

If \( m \) and \( m_v \) are proportional to the identity matrix and \( m_v \sim m_x \) (since from table 3 we see that both have \( U(1)_x \) (8)) only \( M \) will be responsible for \( S_3 \) breaking in the neutrino sector. Then we have to deal with the up and down quark \( S_3 \) breaking. From tensor analysis we see that to introduce the yukawa for up quarks we need the 2430 acquiring a v.e.v. In fact even if the 2430 can be obtained as the product of the 351\( \times 27 \times 27 \), and even if the SU(5) singlet with \( U(1)_x \) charge +8 in the 2430 can be obtained as the tensor product 351\( \times 27 \times 27 \) where both the 351 and the 27 are SU(5) singlets, in order to turn on the up quark yukawa coupling we need to add the 2430 to the list of representations of the scalar fields. The reason is the following.

The top Yukawa interaction is \( T^{\alpha\beta} T^{\gamma\delta} H^\sigma \varepsilon_{\alpha\beta\gamma\delta\sigma} \). If the field \( H \) is in the 351', and it has charges (-3,-5), then indices with these charges correspond to the 351'_{22,26} (see table 3 to see the \( U(1) \) charges relative to 22nd and 26th rows of the 27).

The up quark yukawa coupling, written in \( E_6 \) notation becomes

\[
(u_L^1 u_R^{1C} + u_L^2 u_R^{2C} + u_L^3 u_R^{3C}) h = (27^{12}27^6 + 27^{13}27^5 + 27^{14}27^3) 351'_{22,26}.
\]

In order to make an \( E_6 \) invariant, we need to multiply the operator above with few scalar representations that acquire some v.e.v. At the same time, these v.e.v.'s must be neutral under
the four U(1) charges of the cartan subalgebra of SU(3)×SU(2)×U(1) (i.e. they must be standard model singlets). A tensor, that is a standard model singlet, either must simultaneously contain all indices 12,6,26 (see table 2) or none of them. The minimal representation containing $T_{12,6}^{26}$ is the 1728 of $E_6$. We prefer to add the 2430 and make directly the invariant

$$(27^{12} 27^6 2430^{22,26}_{12,6} + 27^{13} 27^5 2430^{22,26}_{13,5} + 27^{14} 27^3 2430^{22,26}_{14,3}) 351^{1}_{22,26}$$

since it could be helpful to explain the top-bottom hierarchy.

This is interesting since the 2430 could explain while the $S_3$ breaking in the quark sector is not the same as in the neutrino sector. The minimization of an effective potential could give a vev for the 2430 not proportional to the 351. Namely it could give a reversed hierarchy. In addition we expect the vev in the 2430 to be proportional to the 351×27×27 since its charge is (5,3) that is the sum of (5,-5)+(0,4)+(0,4). Similar arguments lead us to introduce the 1728 to turn on the yukawa coupling for the down quarks. We expect it could take a vev proportional to 351×27×27×27. The additional 27 can be used to explain the mass ratio between the top and the bottom quarks. Note that the up quark yukawa interaction arises from just one yukawa interaction

$$27^a 27^b 351^{a}_{a} 2430^{b}_{b}$$

while the down quark yukawa we have two distinct operators available

$$k_1 27^a 27^b 351^{a, b}_{a, b} 1728^{a, b}_{a, b} + k_2 27^a 27^b 351^{a, b}_{a, b} 1728^{a, b}_{a, b}$$

Their distinction arises because there are two distinct way to contract the indices of the $E_6$ invariant tensor $\varepsilon_{a, b, c}$. While the up matrix must be left-right symmetric the down quark matrix could be not symmetric due to these two distinct contraction available. Namely

$$g_0 27^i 27^j 351^{i, j}_{i, j} 1728^{i, j}_{i, j} + g_1 27^i 27^j 351^{i, j}_{i, j} 1728^{i, j}_{i, j} + \sum_{i, j, k} g_3 27^i 27^j 351^{i, j}_{i, j} 1728^{i, j}_{i, j}$$

after the 1728 takes a v.e.v. we arrive at the following yukawa for the down quark

$$\sum_{ij} (g_0 d_{Li} d_{Rj} h \langle 1728_i \rangle + g_1 d_{Li} d_{Rj} h \langle 1728_j \rangle + g_2 d_{Li} d_{Rj} h \langle 1728_k \rangle + \sum_{i, j, k} g_3 d_{Li} d_{Rj} h \langle 1728_k \rangle$$

from which we get the following mass matrix

$$M = \begin{pmatrix}
    x_1 & g_1 x_1 + g_2 x_2 + g_3 x_3 & g_1 x_1 + g_3 x_2 + g_2 x_3 \\
g_2 x_1 + g_1 x_2 + g_3 x_3 & x_2 & g_3 x_1 + g_1 x_2 + g_2 x_3 \\
g_2 x_1 + g_3 x_2 + g_1 x_3 & g_3 x_1 + g_2 x_2 + g_1 x_3 & x_3
\end{pmatrix}$$

where $x_i = h \langle 1728_i \rangle$ with $i = 1, 2, 3$. The latter fits the experimental data (the six quark masses and the three angles, plus one phase in the CKM mixing matrix) with the following parameters

\footnote{This result is similar to SU(5), where the up quark matrix must be left-right symmetric, while the down matrix can be not symmetric.}
We observe that $x_1, x_2 \ll x_3$ and in the down sector $g_1 \gg g_2$ ( $g_0$ has been reabsorbed in the definition of $x_i$). The up quark mass matrix is very close to the identity matrix and the left-right symmetry is not strongly broken.

We are now able to write the full lagrangian in $E_6$ notation responsible for all fermion masses

$$L = 27^i_\alpha 27^\beta_\gamma 351^i_\alpha \delta + 27^i_\alpha 27^\beta_\gamma (351^k_\alpha 78^\gamma_\beta + 351^k_\gamma 78^\gamma_\alpha) + 27^i_\beta 27^\beta_\gamma (351^i_\gamma 78^\gamma_\beta + 351^i_\gamma 78^\gamma_\alpha)$$

and finally (after all standard model singlets take a vev) we obtain the following lagrangian in $SU(2) \times SU(6)$ notation

$$L = \nu_R^i x_L^j ((1, 21)_i^{[1]}(3, 1)^{[12]} + \delta^j(1, 21)_i^{[1]}(3, 1)^{[12]}) + u_R^i u_R^j \ h_0 ((3, 35)_i^{[2]} - (3, 35)_i^{[5]} + 5) + (x_L^i N_{Li} + d_R^i d_{Li} + e_L^i e_{Li}) \ h_0 (4, 6)_i^{[22]} + (x_L^i N_{Li} + d_R^i d_{Li} + e_L^i e_{Li}) \ h_0 (4, 6)_i^{[22]} + (\bar{D}D + \bar{N}_L N_L + \bar{E}_L E_L) (2, 6)_i^{[12]}(2, 6)_j^{[12]} + x_L^i x_L^j (2, 6)_i^{[2]}(2, 6)_j^{[2]} + x_L^i \nu_{Li} \ h_0$$

As already said the vev of 2430 and 1728 could show a reversed hierarchy with respect the 351.

We can imagine the following gauge symmetry breaking pattern $E_6 \rightarrow SU(2) \times SU(6) \rightarrow U(1) \times SU(5) \rightarrow SU(5)$, and scalars take a vev following the hierarchy pattern $\langle 650 \rangle \sim \langle 351 \rangle \sim \langle 27_{16} \rangle \sim \langle 27_{22} \rangle$ and the $\langle 2430 \rangle \sim \langle 351 \rangle \langle 27 \rangle^2$ and $\langle 1728 \rangle \sim \langle 351 \rangle \langle 27 \rangle^3$.

A list of all relevant masses and their origin in this $E_6$ breaking pattern.

$v^i_R x_L$}

This Dirac mass term appears when both the 351 and the 78 of $E_6$ take a vev. $E_6$ has been broken into $SU(2) \times SU(6)$ by the 650, and after both the 351 and the 78 break $SU(2) \times SU(6)$ into $SU(5) \times U(1)$. The $351_{\alpha \beta}$ has charge $(5, -5)$, but it is antisymmetric under the exchange of the indices $\alpha \beta$. So we need to add the 78 to build a non zero invariant operator

$$27^i_\alpha 27^\beta_\gamma (351^k_\alpha 78^\gamma_\beta + 351^k_\gamma 78^\gamma_\alpha) + 27^i_\beta 27^\beta_\gamma (351^i_\gamma 78^\gamma_\beta + 351^i_\gamma 78^\gamma_\alpha)$$

$i, j, k$ are family indices, and the $351^k$ breaks the $S_3$ symmetry, but this operator gives a mass only to the $S_3$ singlet component of the dirac heavy neutrinos. In $SU(2) \times SU(6)$ notation the operator above can be written ( $\lambda$ is a dimensionless constant)

$$R^i_{\alpha \beta} R^j_{\beta \gamma}(1, 21)_i^{[12]}(3, 1)^{[12]} = \lambda(1, 21)_i^{[12]}(3, 1)^{[12]} + \delta^j(1, 21)_i^{[12]}(3, 1)^{[12]}$$
where the (1,21) and (3,1) under SU(2) × SU(6) are contained respectively in the 351 and 78. We remind that latin letters run from 1 to 2 and are SU(2) indices while greek letters are SU(6) indices. This mass term can be written as follows

\[ M = c \left( x_{L1} \ x_{L2} \ x_{L3} \right) \begin{pmatrix} (1, 21)_{[11]}/c + 1 & 1 & 1 \\ 1 & (1, 21)_{[11]}/c + 1 & 1 \\ 1 & 1 & (1, 21)_{[11]}/c + 1 \end{pmatrix} \begin{pmatrix} \nu_{R1}^c \\ \nu_{R2}^c \\ \nu_{R3}^c \end{pmatrix} \]

and \( c = \lambda \sum_k (1, 21)_k^k (3, 1)_{[12]} \). In the limit \( \lambda \to \infty \) the above matrix is diagonalized by the tri-bimaximal matrix

\[ \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \]

So this matrix could explain neutrino oscillations. To do that, we need to mix these heavy neutrinos with the light left-handed neutrino and to give a majorana mass to the \( \nu_{cR} \nu_{cR} \).

\( g v_i^L x_i^L h \)

This mass arises directly from the tree level Yukawa interaction in the fundamental \( E_6 \) invariant lagrangian.

\[ 27_i^\alpha 27_i^\beta 351_i^{\alpha \beta} = R_{ai}^\alpha R_{bi}^\beta (3, 21)_{[ab]}^{\alpha \beta} = R_{2i}^1 R_{2i}^5 (3, 21)_{[15]}^{22} = x_{Li} \nu_{Li} h_0 \]

The Standard model higgs doublet \( h \) is in the 351' and more precisely in the component \((3, 21)_{[15]}^{22}\) of SU(2) × SU(6). This dirac mass does not break \( S_3 \), in fact this mass matrix is proportional to the identity matrix.

\( \nu_R^c \nu_R^c \) and \( x_i^L x_i^L \)

These majorana masses are necessary to break the lepton number and give mass to the light neutrinos. They could arise from the the operator

\[ 27_i^\alpha 27_i^\beta 27_{k \alpha} 27_{k \beta} = R_{ai}^\alpha R_{bi}^\beta (\bar{2}, 6)_{\alpha}^{\alpha k} (\bar{2}, 6)_{\beta}^{\beta k} = R_{2i}^1 R_{2i}^1 (\bar{2}, 6)_{1}^{2k} (\bar{2}, 6)_{1}^{2k} \]

or

\[ 27_i^\alpha 27_i^\beta 27_i^{\gamma} 27_i^{\delta} 351_i^{\gamma \alpha} 351_i^{\delta \beta} \]

These masses should not break the \( S_3 \) symmetry and should be proportional to the identity matrix. That is the reason why family indices \( i \) and \( k \) are not the same for fermions and scalars in the operator above.

\( u_i^L u_i^R h_0 \)

The Yukawa coupling for up quark fermions comes after the 2430 of \( E_6 \) take a vev. This scalar contains a SU(5) singlet with U(1) charges (5,3) with respect U_r(1)×U_l(1) and charge (-2,0) with respect U_T(1)×U(1)'. This belongs to the \((3, 35)_{[ab;j]}^{\gamma} \) of SU(2) × SU(6). The SU(5) singlet is in the direction \((3, 35)_{[12]}^{11} \), where we have also added the family index \( i \), in order to break the permutation symmetry and to induce a hierarchy among the top, the charm and the up quark.
\( T_{(\alpha \beta)} \) stands for the fermions contained in the (1,15) of SU(2) × SU(6). The SU(6) invariant and completely antisymmetric tensor is \( \varepsilon^{\alpha \beta \delta \sigma \rho} \). The operator responsible for these masses is

\[
\begin{align*}
27_i^\alpha & \ 27_i^\beta & 351'_{\gamma \delta} & 2430_i^{\alpha \beta} & \gamma^\delta = & \ T_{(\alpha \beta)} T_{(\delta \sigma)} (3, 21)_{[ab]}^{\gamma \delta} (3, 35)_{[ab]}^{\gamma \delta} \\
& & & = & \ T_{(\alpha \beta)} T_{(\delta \sigma)} (3, 21)_{[15]}^{22} (3, 35)_{[22]}^{11} \gamma^\delta = & \ u_L^i u_R^i \ h_0 ((3, 35)_{[22]}^{11} - (3, 35)_{[22]}^{5i}) \\
& & & = & \ DD + \bar{N}_L N_L + \bar{E}_L E_L \\
\end{align*}
\]

These masses are necessary and are expected to be very heavy, close to the unification scale. Note that the fermions \( D \) and \( F \) belong to the same SU(5) representation, so we have to be sure that the mass \( T_{(1 \beta)} R_{2i}^\beta = (\bar{D} + \bar{N}_L N_L + \bar{E}_L E_L) \) is much larger than \( T_{(1 \beta)} R_{2i}^\beta = (\bar{D} + \bar{N}_L N_L + \bar{E}_L E_L) \) otherwise the lightest neutrinos and down quarks would have not the desired U(1) charges. To to that we need that the 27_{22} , (i.e. the scalar component with charges (0,4)) takes a vev much larger than the 22_{16}.

\[
\begin{align*}
27_i^\alpha & \ 27_i^\beta & 27_k^\gamma \varepsilon_{\alpha \beta \gamma} & = & \ T_{(\alpha \beta)} R_{\lambda a}^\beta (2, \bar{6}) A_{\alpha}^a \varepsilon^{\alpha \beta} = & \ T_{(1 \beta)} R_{1i}^\beta (2, \bar{6})_{2}^1 \varepsilon^{12} = \\
& & & = & \ DD + \bar{N}_L N_L + \bar{E}_L E_L ) (2, \bar{6})_{2}^1 \\
& & & = & \ (d_R^c d_L + e_L e_R) h_0 \\
\end{align*}
\]

Finally we have the yukawa coupling for the down quarks and the charged leptons. They appear after the 1728_{i \beta} takes a vev. This representation contains a SU(5) singlet with charges (5,7) that is within the (4,\bar{6})_{[a b c]} of SU(2) × SU(6). Differently from the up quark Yukawa coupling, the 1728 can accommodate not left-right symmetric mass matrices. This is because we have the choice between 1728_{i \beta} and 1728_{j \beta} in the operator below. In other words the family index \( i \) in the (4,\bar{6})_{[22]} can be attached either to the \( d_L \) or the \( d_R^c \) giving rise to distinct matrices.

\[
\begin{align*}
27_i^\alpha & \ 27_j^\beta & 351'_{\alpha \gamma} \gamma^\delta & 1728 \varepsilon_{i \beta \delta} \varepsilon_{\alpha \gamma \rho} = & \ T_{(\alpha \beta)} R_{ij}^\delta (3, 21)_{[abc]}^\gamma (4, \bar{6})_{i}^[\gamma [abc]} = \\
& & & = & \ T_{(5 \beta)} R_{ij}^\beta (3, 21)_{[22]}^{[5]} (4, \bar{6})_{i}^{22} = \\
& & & = & \ (x L_j N_L + d_R^c d_L + e_L e_R) h_0 (4, \bar{6})_{i}^{22} \\
\end{align*}
\]

**Conclusion**

We have studied a grand unified model based on the E_6 unification group. The three families are identical in the context of third quantization, that means that the fundamental lagrangian is invariant under permutations of the three fermion families. This permutation symmetry is manifest in the neutrino sector, where large mixing angles together with \( s_{13} \approx 0 \) result, clearly point toward a soft S_3 symmetry breaking pattern. In the quark sector, mixing angles are small and this clearly restrict the class of grand unified theories that could be considered as good candidate to describe the full spectrum of fermion masses. In fact the first compelling issue is how to make a distinction between the up quarks and the neutrino sector. In SO(10) we have an additional U(1) belonging to the cartan algebra, and an additional right handed neutrino that
is a standard model singlet. From a simple analysis of the U(1) charges is not obvious how to make a distinction between the up quark and the dirac neutrino mass. The 16 16 10 invariant yukawa interaction obtained from the product of two fermion families belonging to the 16 and the scalar representation belonging to the 10 of SO(10), give the same yukawa both for up quarks and neutrinos. Since we want Dirac neutrino mass matrix proportional to the identity matrix we would obtain the unacceptable $m_{\text{top}} = m_{\text{charm}} = m_{\text{up}}$ up quark masses. If we enlarge the group to $E_6$, we have two additional right-handed neutrino instead of one. This new neutrino can mix with the light left-handed neutrino through a Yukawa operator that is distinct from up quark yukawa. This can be achieved if we put the standard Higgs doublet in the 351' with $U(1)_r \times U(1)_t$ charges equal to (+3,+5). We have studied the feasibility of this scenario. At the tree level there is only only one yukawa interaction, mixing the the light left-handed neutrino and standard model singlet $\nu_L x_L h_0$. The remaining Yukawa coupling, necessary to turn on masses for all matter fermions, arise only after loop correction are included, and appear as higher order operators. We have found that is necessary to turn on the vev of the 2430 or 1728, otherwise the top mass would be zero. From the tensor analysis, we conclude this scenario is feasible but further work, including radiative corrections and an explicit analysis of the effective potential, is needed to have a complete and exhaustive description of the model.

References

[1] B. Stech and Z. Tavartkiladze, Phys. Rev. D 70, 035002 (2004) [arXiv:hep-ph/0311161].

[2] N. Maekawa and T. Yamashita, Prog. Theor. Phys. 110, 93 (2003) [arXiv:hep-ph/0303207].
   M. Bando and T. Kugo, Prog. Theor. Phys. 109, 87 (2003) [arXiv:hep-ph/0209088].
   N. Maekawa and T. Yamashita, Prog. Theor. Phys. 107, 1201 (2002) [arXiv:hep-ph/0202050].
   M. Bando and N. Maekawa, Prog. Theor. Phys. 106, 1255 (2001) [arXiv:hep-ph/0109018].

[3] R. Slansky, “Group Theory For Unified Model Building,” Phys. Rept. 79, 1 (1981).
   I. G. Koh, J. Patera and C. Rousseau, J. Math. Phys. 25, 2863 (1984).
   G. W. Anderson and T. Blazek, arXiv:hep-ph/0101349.

[4] C. S. Lam, arXiv:hep-ph/0503159. K. Siyeon, Phys. Rev. D 71, 036005 (2005) [arXiv:hep-ph/0411343].
   F. Plentinger and W. Rodejohann, Phys. Lett. B 625, 264 (2005) [arXiv:hep-ph/0507143].
   T. Araki, J. Kubo and E. A. Paschos, arXiv:hep-ph/0502164.
   G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005) [arXiv:hep-ph/0504165].
   S. Morisi and M. Picariello, arXiv:hep-ph/0505113.
   E. Ma, New J. Phys. 6, 104 (2004).
   S. L. Chen, M. Frigerio and E. Ma, Phys. Rev. D 70, 073008 (2004) [Erratum-ibid. D 70, 079905 (2004)] [arXiv:hep-ph/0404084].
   E. Ma, arXiv:hep-ph/0409075.
   P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002) arXiv:hep-ph/0202074.
K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. 64, 2747 (1990).
R. N. Mohapatra, Nucl. Phys. Proc. Suppl. 138, 257 (2005) [arXiv:hep-ph/0402035].
W. Grimus and L. Lavoura, Phys. Lett. B 572, 189 (2003) [arXiv:hep-ph/0305046].
J. Kubo, A. Mondragon, M. Mondragon and E. Rodriguez-Jauregui, Prog. Theor. Phys. 109, 795 (2003) [arXiv:hep-ph/0302196].
Y. Koide, Phys. Rev. D 60, 077301 (1999) [arXiv:hep-ph/9905416].

[5] F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234

[6] F. Caravaglios, arXiv:hep-ph/0211183; F. Caravaglios, arXiv:hep-ph/0211129; V. A. Rubakov, Phys. Lett. B 214, 503 (1988).
M. McGuigan, Phys. Rev. D 38, 3031 (1988).
S. B. Giddings and A. Strominger, Nucl. Phys. B 321, 481 (1989).

[7] Q.R. Ahmad et al. Phys. Rev. Lett. 89 (2002) 011301;
S. Fukuda et al. Phys. Lett. B 539(2002) 179;
J. W. Hampel et al. Phys. Lett. B 447(1999) 127.
K. Eguchi et al. Phys. Rev. Lett. 90(2003) 021802.
Y. Fukuda et al. Phys. Rev. Lett. 85 (2000) 3999;
Y. Fukuda et al. Phys. Rev. Lett. 81 (1998) 1562.
M. Apollonio et al. Eur. Phys. J. C27 (2003) 331;
M. Apollonio et al. Phys. Lett. B 466(1999) 415.
M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, New J. Phys. 6, 122 (2004).