Central charges, black-hole entropy and geometrical structure of $N$–extended supergravities in $D = 4$

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Abstract: The derivation of absolute (moduli-independent) U-invariants for all $N > 2$ extended supergravities at $D = 4$ in terms of (moduli-dependent) central and matter charges is reported. These invariants give a general definition of the “topological” Bekenstein–Hawking entropy formula for extremal black-holes and reduce to the square of the black-hole ADM mass for “fixed scalars” which extremize the black-hole “potential” energy.

1 Introduction

Recently, considerable progress has been made in the study of general properties of black holes arising in supersymmetric theories of gravity such as extended supergravities, string theory and M-theory [1]. Of particular interest are extremal black holes in four dimensions which correspond to BPS saturated states [2] and whose ADM mass depends, beyond the quantized values of electric and magnetic charges, on the asymptotic value of scalars at infinity. The latter describe the moduli space of the theory.

Another physical relevant quantity, which depends only on quantized electric and magnetic charges, is the black hole entropy, which can be defined macroscopically, through the Bekenstein-Hawking area-entropy relation or microscopically, through D-branes techniques [3] by counting of microstates [4]. It has been further realized that the scalar fields, independently of their values at infinity, flow towards the black hole horizon to a fixed value of pure topological nature given by a certain ratio of electric and magnetic charges [5].

These “fixed scalars” correspond to the extrema of the ADM mass in moduli space while the black-hole entropy is the actual value of the squared ADM mass at this point [5].

In theories with $N > 2$, extremal black-holes preserving one supersymmetry have the further property that all central charge eigenvalues other than the one equal to the BPS mass flow to zero for “fixed scalars”. The black-hole entropy is still given by the square of the ADM mass for “fixed scalars” [5].

Recently [5], the nature of these extrema has been further studied and shown that they generically correspond to non degenerate minima for $N = 2$ theories whose relevant moduli space is the special geometry of $N = 2$ vector multiplets.

The entropy formula turns out to be in all cases a U-duality invariant expression (homogeneous of degree two) built out of electric and magnetic charges and as such can be in fact also computed through certain (moduli-independent) topological quantities which only depend on the nature of the U-duality groups and the appropriate representations of electric and magnetic charges. For example, in the $N = 8$ theory the entropy was shown to correspond to the unique quartic $E_7$ invariant built with its 56 dimensional representation [5].
In this talk we intend to report on further progress made on this subject in collaboration with Riccardo D’Auria and Sergio Ferrara [10] by deriving, for all $N > 2$ theories, topological (moduli-independent) U-invariants constructed in terms of (moduli-dependent) central charges and matter charges, and show that, as expected, they coincide with the squared ADM mass at “fixed scalars”.

2 Central charges, U-invariants and entropy

Extremal black-holes preserving one supersymmetry correspond to $N$-extended multiplets with

$$M_{ADM} = |Z_1| > |Z_2| \cdots > |Z_{[N/2]}|$$

where $Z_\alpha$, $\alpha = 1, \cdots, [N/2]$, are the proper values of the central charge antisymmetric matrix written in normal form [11]. The central charges $Z_{AB} = -Z_{BA}$, $A, B = 1, \cdots, N$, and matter charges $Z_I$, $I = 1, \cdots, n$ are those (moduli-dependent) symplectic invariant combinations of field strenghts and their duals (integrated over a large two-sphere) which appear in the gravitino and gaugino supersymmetry variations respectively [12], [13], [14]. Note that the total number of vector fields is $n_v = N(N - 1)/2 + n$ (with the exception of $N = 6$ in which case there is an extra singlet graviphoton) [15].

It was shown in ref. [7] that at the attractor point, where $M_{ADM}$ is extremized, supersymmetry requires that $Z_\alpha$, $\alpha > 1$, vanish together with the matter charges $Z_I$, $I = 1, \cdots, n$ ($n$ is the number of matter multiplets, which can exist only for $N = 3, 4$).

Moreover, S. Ferrara, G. Gibbons, R. Kallosh and B. Kol showed that the properties of extreme non rotating black holes are completely encoded in the metric of the manifold $G_{ij}$ spanned by scalar fields and on an effective potential, function of scalars and electric and magnetic charges, $V(\Phi, e, g)$, known as geodesic potential [16], defined by:

$$V = -\frac{1}{2} P^t \mathcal{M}(N) P.$$  \hspace{1cm} (2)

Here $P$ is the symplectic vector $P = (p^\Lambda, q_\Lambda)$ of quantized electric and magnetic charges and $\mathcal{M}(N)$ is a symplectic $2n_v \times 2n_v$ matrix whose $n_v \times n_v$ blocks are given in terms of the $n_v \times n_v$ vector kinetic matrix $\mathcal{N}_{\Lambda \Sigma}$ ($-Im \mathcal{N}, Re \mathcal{N}$ are the normalizations of the kinetic $F^2$ and the topological $F^*F$ terms respectively appearing in the black-hole supergravity lagrangian) and

$$\mathcal{M}(N) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$  \hspace{1cm} (3)

with:

$$A = Im \mathcal{N} + Re \mathcal{N} Im \mathcal{N}^{-1} Re \mathcal{N}$$

$$B = -Re \mathcal{N} Im \mathcal{N}^{-1}$$

$$C = -Im \mathcal{N}^{-1} Re \mathcal{N}$$

$$D = Im \mathcal{N}^{-1}$$  \hspace{1cm} (4)

In particular, they showed [16] that the area of the event horizon is proportional to the value of $V$ at the horizon:

$$\frac{A}{4\pi} = V(\Phi_h, e, g)$$  \hspace{1cm} (5)

where $\Phi_h$ is the value taken by scalar fields at the horizon. It was also shown that in a general (even not supersymmetric) setting, Einstein equations can be reduced to a system of scalar equations that give, near the horizon, the solution:

$$\Phi^i = \left(\frac{2\pi}{A}\right) \frac{\partial V}{\partial \Phi^i} \log \tau + \Phi^i_h.$$  \hspace{1cm} (6)
in terms of the evolution parameter \( \tau = \frac{1}{r - r_h} \). From eq. (6) we see that the request that the horizon is a fixed point \( \left( \frac{d\Phi^i}{d\tau} = 0 \right) \) implies that the geodesic potential is extremized in moduli space:

\[
\Phi_h : \frac{d\Phi^i}{d\tau} = 0 \iff \frac{\partial V}{\partial \Phi^i}|_{\Phi_h} = 0 \tag{7}
\]

To summarize, the area \( A_h \) of the event horizon, and then, from the area–entropy formula \( S_{B-H} = \frac{A_h}{4} \), the Bekenstein–Hawking entropy \( S_{B-H} \) of the black hole, are given by the geodesic potential evaluated at the horizon, and we have a tool for finding this value: the geodesic potential gets an extremum at the horizon.

However, the geodesic potential \( V(\Phi, e, g) \) defined in eq.s (2) and (3) has a particular meaning in supergravity theories, that allows to find its extremum in a very easy way. Indeed, an expression exactly coinciding with (2) has been found in [14] as the result of a sum rule among central and matter charges in supergravity theories. So, in every supergravity theory, the geodesic potential has the general form:

\[
V \equiv -\frac{1}{2} P_{ABCD} Z^{AB} + \frac{1}{2} Z_I P_{IJ}^I \tag{8}
\]

Central and matter charges in extended supergravities were found to satisfy some differential relations [14], that are a direct consequence of the geometrical properties of the manifolds spanned by scalar fields, in particular the fact of being coset manifolds of the form \( U/H \), embedded in the symplectic group \( Sp(2n_v) \) (\( n_v \) being the total number of vectors) [14].

Then, to find the extremum of \( V \) we can apply the differential relations among central and matter charges found in [14], that in general read:

\[
\nabla Z_{AB} = Z_I P_{AB}^I + \frac{1}{2} Z^{CD} P_{ABCD} \\
\nabla Z_I = \frac{1}{2} Z^{AB} P_{ABI} + Z^I P_{IJ} 
\]

where the matrices \( P_{ABCD}, P_{ABI}, P_{IJ} \) are the subblocks of the vielbein of \( U/H \) [14]:

\[
\mathcal{P} \equiv L^{-1} \nabla L = \begin{pmatrix} P_{ABCD} & P_{ABI} \\ P_{IAB} & P_{IJ} \end{pmatrix} \tag{9}
\]

written in terms of the indices of \( H = H_{Aut} \times H_{matter} \).

Applying eq.s (8) to the geodesic potential, we find that the extremum is given by:

\[
dV = \frac{1}{2} \nabla Z_{AB} Z^{AB} + \nabla Z_I Z^I + c.c. = \\
= \frac{1}{2} \left(\frac{1}{2} Z^{CD} P_{ABCD} + Z_I P_{AB}^I \right) Z^{AB} \\
+ \left(\frac{1}{2} Z^{AB} P_{ABI} + Z^I P_{IJ} \right) Z^I = 0 \tag{11}
\]

that is \( dV = 0 \) for:

\[
Z_I = 0 ; Z^{AB} Z^{CD} P_{ABCD} = 0 \tag{12}
\]

However, in the following we will show one more technique for finding the entropy, exploiting the fact that it is a ‘topological quantity’ not depending on scalars. This last procedure is particularly interesting because it refers only to group theoretical properties of the coset manifolds spanned by scalars, and do not need the knowledge of any details of the black-hole horizon. This allows to give the entropy formula as a moduli–independent quantity in the entire moduli space and not just at the critical points. Namely, we are looking for quantities \( S \left( Z_{AB}(\phi), Z^{AB}(\phi), Z_I(\phi), Z^I(\phi) \right) \) such that \( \frac{\partial}{\partial \phi^i} S = 0, \phi^i \) being the moduli
coordinates. These formulae generalize the quartic \( E_{7(-7)} \) invariant of \( N = 8 \) supergravity to all other cases.

Let us first consider theories where matter can be present (\( N = 3, 4 \)) [9], [18], [19]. We focus in particular on the \( N = 4 \) case.

The U–duality group is, in this case, \( SU(1, 1) \times SO(6, n) \). The central and matter charges \( Z_{AB}, Z_I \) transform under the isotropy group:

\[
H = SU(4) \times O(n) \times U(1)
\]

(13)

Under the action of the elements of \( U/H \) the charges get mixed with their complex conjugate. The transformation of the charges under infinitesimal transformations \( \xi \) on the coset \( K = SU(1, 1) U(1) \times O(6, n) O(6) \times O(n) \) can be read from the differential relations (9) satisfied by the charges [14]:

\[
\delta Z_{AB} = \frac{1}{2} \xi \epsilon_{ABCD} Z^{CD} + \xi_{ABI} Z^B
\]

(14)

\[
\delta Z_I = \xi \eta_{IJ} Z^J + \frac{1}{2} \xi_{ABI} Z^{AB}
\]

(15)

where we used the fact that for the \( N = 4 \) theory the vielbein components have the form:

\[
P_{ABCD} = \epsilon_{ABCD} P, P_{IJ} = \eta_{IJ} P, P_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} P^{CDJ},
\]

(16)

and that, once the covariant derivatives (11) are known, the infinitesimal variations are simply obtained by the substitution \( \nabla \rightarrow \delta \), \( P \rightarrow \xi \) (with \( \xi_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} \xi_{CDJ} \)).

There are three \( O(6, n) \) invariants given by

\[
I_1 = \frac{1}{2} Z_{AB} \overline{Z}_{AB} - Z_I \overline{Z}^I
\]

(17)

\[
I_2 = \frac{1}{4} \epsilon_{ABCD} Z_{AB} Z_{CD} - \overline{Z}_I \overline{Z}^I
\]

(18)

and the unique \( SU(1, 1) \times O(6, n) \) invariant \( S, \nabla S = 0 \), is given by:

\[
S = \sqrt{(I_1)^2 - |I_2|^2}
\]

(19)

At the attractor point \( Z_I = 0 \) and \( \epsilon_{ABCD} Z_{AB} Z_{CD} = 0 \) so that \( S \) reduces to the square of the BPS mass.

For \( N = 5, 6, 8 \) the U-duality invariant expression \( S \) is the square root of a unique invariant under the corresponding U-duality groups \( SU(5, 1), O^*(12) \) and \( E_{7(-7)} \). The strategy is to find a quartic expression \( S^2 \) in terms of \( Z_{AB} \) such that \( \nabla S = 0 \), i.e. \( S \) is moduli-independent.

As before, this quantity is a particular combination of the \( H \) quartic invariants.

Let us analyse in particular the \( N = 8 \) case. For \( N = 8 \) the \( SU(8) \) invariants are [4]:

\[
I_1 = (Z_{AB} \overline{Z}^{BA})^2 = (TrA)^2
\]

(20)

\[
I_2 = Z_{AB} \overline{Z}^{BC} Z_{CD} \overline{Z}^{DA} = Tr(A^2)
\]

(21)

\[
I_3 = PfZ = \frac{1}{24} \epsilon_{ABCDEFGH} Z_{AB} \overline{Z}_{CD} \overline{Z}_{EF} \overline{Z}_{GH}
\]

(22)

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1. The Bekenstein-Hawking entropy \( S_{BH} = \frac{A}{4} \) is actually \( \pi S \) in our notation.
2. Here we denote by U-duality group the isometry group \( G \) acting on the scalars, although only a restriction of it to integers is the proper U-duality group.
3. The Pfaffian of an \((n \times n)\) (n even) antisymmetric matrix is defined as \( PfZ = \frac{1}{2^n n!} \epsilon^{A_1 \cdots A_n} Z_{A_1} A_2 \cdots Z_{A_{n-1}} A_n \), with the property: \( |PfZ| = |detZ|^{1/2} \).
where $A^B \equiv Z_{AC} Z^C B$.

The $E_{7(-7)}$ transformations are:

$$\delta Z_{AB} = \frac{1}{2} \xi_{ABCD} Z^{CD}$$

(23)

where $\xi_{ABCD}$ satisfies the reality constraint:

$$\xi_{ABCD} = \frac{1}{24} \xi_{ABCDEFGH} \xi^{EFGH}$$

(24)

One finds the following $E_{7(-7)}$ invariant [9]:

$$S = \frac{1}{2} \sqrt{4Tr(A^2) - (TrA)^2 + 32Re(PfZ)}$$

(25)

Note that at the attractor point $PfZ = 0$, $(TrA)^2 = 2Tr(A^2)$ and $S$ becomes proportional to the square of the BPS mass.

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**References**

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