Logical, conditional, and classical probability

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Abstract

The propositional logic is generalized on the real numbers field. The logical function with all properties of the classical probability function is obtained. The logical analog of the Bernoulli independent tests scheme is constructed. The logical analog of the Large Number Law is deduced from properties of these functions. The logical analog of the conditional probability is defined. Consistency ensured by a model on a suitable variant of the nonstandard analysis.

1 Introduction

There is the evident near affinity between the classical probability function and the Boolean function of the classical propositional logic [1]. These functions are differed by the range of value, only. That is if the range of values of the Boolean function shall be expanded from the two-elements set \{0; 1\} to the segment [0; 1] of the real numeric axis then the logical analog of the Bernoulli Large Number Law [2] can be deduced from the logical axioms. These topics is considered in this article.

2 The classical logic

Definition 2.1 Sentence \(\ll \Theta \gg\) is a true sentence if and only if \(\Theta\).
For example: sentence \(\ll \text{it rains} \gg\) is the true sentence if and only if it rains.

Definition 2.2 Sentence \(\ll \Theta \gg\) is a false sentence if and only if it is not \(\Theta\).

Definition 2.3 Sentences \(A\) and \(B\) are equal \((A = B)\) if \(A\) is true if and only if \(B\) is true.
Hereinafter we use the usual notions of the classical propositional logic [4].

Definition 2.4 Sentence \(C\) is a conjunction of the sentences \(A\) and \(B\) \((C = (A \land B))\) if \(C\) is true if and only if \(A\) is true and \(B\) is true.

Definition 2.5 Sentence \(C\) is a negation of the sentence \(A\) \((C = \overline{A})\), if \(C\) is true if and only if \(A\) is false.

Theorem 2.1
1) \((A \land A) = A\);
2) \((A \land B) = (B \land A)\);
3) \((A \land (B \land C)) = ((A \land B) \land C)\);
4) if \(T\) is the true sentence then for every sentence \(A\): \((A \land T) = A\);
5) if \(F\) is false sentence then \(F\) is true sentence.

**Proof of the Theorem 2.1:** From Definitions 2.1, 2.2, 2.3, 2.4.

**Definition 2.6** Each function \(g\) with domain in the set of the sentences and with the range of values on the two-elements set \(\{0; 1\}\) is a Boolean function if:
1) \(g(\overline{A}) = 1 - g(A)\) for every sentence \(A\);
2) \(g(A \land B) = g(A) \cdot g(B)\) for all sentences \(A\) and \(B\).

**Definition 2.7** Set \(\Im\) of the sentences is a basic set if for every element \(A\) of this set there exist Boolean functions \(g_1\) and \(g_2\) such that the following conditions fulfill:
1) \(g_1(A) \neq g_2(A)\);
2) \(g_1(B) = g_2(B)\) for each element \(B\) of \(\Im\) such that \(B \neq A\).

**Definition 2.8** Set \([\Im]\) of the sentences is a propositional closure of the set \(\Im\) if the following conditions fulfill:
1) if \(A \in \Im\) then \(A \in [\Im]\);
2) if \(A \in [\Im]\) then \(\overline{A} \in [\Im]\);
3) if \(A \in [\Im]\) and \(B \in [\Im]\) then \((A \land B) \in [\Im]\);
4) there do not exist other elements of \([\Im]\) except the listed by 1), 2), 3) points of this definition.

In the following text the elements of \([\Im]\) are called as the \(\Im\)-sentences.

**Definition 2.9** \(\Im\)-sentence \(A\) is a tautology if for all Boolean functions \(g:\)

\[ g(A) = 1. \]

**Definition 2.10** A disjunction and an implication are defined by the usual way:

\[ (A \lor B) = (\overline{A} \land \overline{B}), \]
\[ (A \Rightarrow B) = (A \land \overline{B}). \]

By this definition and the Definitions 2.4 and 2.5:
\((A \lor B)\) is the false sentence if and only if \(A\) is the false sentence and \(B\) is the false sentence.
\((A \Rightarrow B)\) is the false sentence if and only if \(A\) is the true sentence and \(B\) is the false sentence.

**Definition 2.11** A \(\Im\)-sentence is a propositional axiom \([4]\) if this sentence has got one some amongst the following forms:

A1. \( (A \Rightarrow (B \Rightarrow A)) \);
A2. \(( (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \));
A3. \(( (B \Rightarrow A) \Rightarrow ((B \Rightarrow A) \Rightarrow B)) \).

Let \(\Im\) be some basic set. In the following text I consider \(\Im\)-sentences, only.

**Definition 2.12** Sentence \(B\) is obtained from the sentences \((A \Rightarrow B)\) and \(A\) by the logic rule ”modus ponens”.

**Definition 2.13** \([4]\) Array \(A_1, A_2, \ldots, A_n\) of the sentences is a propositional deduction of the sentence \(A\) from the hypothesis list \(\Gamma\) (denote: \(\Gamma \vdash A\)) if \(A_n = A\)
and for all numbers \( l \) (\( 1 \leq l \leq n \)): \( A_l \) is either the propositional axiom or \( A_l \) is obtained from some sentences \( A_{l-k} \) and \( A_{l-s} \) by the modus ponens or \( A_l \in \Gamma \).

**Definition 2.14** A sentence is a propositional proved sentence if this sentence is the propositional axiom or this sentence is obtained from the propositional proved sentences by the modus ponens.

Hence, if \( A \) is the propositional proved sentence then the propositional deduction

\[ \vdash A \]

exists.

**Theorem: 2.2** [4] If sentence \( A \) is the propositional proved sentence then for all Boolean function \( g: g (A) = 1 \).

**Proof of the Theorem 2.2:** [4].

**Theorem: 2.3 (The completeness Theorem).** [4] All tautologies are the propositional proved sentences.

**Proof of the Theorem 2.3:** [4].

### 3 B-functions

**Definition 3.1** Each function \( b(x) \) with domain in the sentences set and with the range of values on the numeric axis segment \([0; 1]\) is called as a B-function if

\[ b (C) = 1 \]

for some sentence \( C \) and

\[ b (A \land B) + b (A \land \overline{B}) = b (A) \]

for every sentences \( A \) and \( B \).

**Theorem: 3.1** For each B-function \( b \):
1) for every sentences \( A \) and \( B \): \( b (A \land B) \leq b (A) \);  
2) for every sentence \( A \): if \( T \) is the true sentence, then \( \overline{A} \)
\[ b (A) + b (\overline{A}) = b (T) \]
3) for every sentence \( A \): if \( T \) is the true sentence, then \( b (A) \leq b (T) \);

**Proof of the Theorem 3.1:**
1)From Definitions 3.1.  
2) From the points 4 and 2 of the Theorem 2.1:
\[ b (T \land A) + b (T \land \overline{A}) = b (A) + b (\overline{A}) \].
3) From previous point of that Theorem.  
Therefore, if \( T \) is the true sentence, then

\[ b (T) = 1. \] (1)

Hence, for every sentence \( A \):
Theorem: 3.2 If sentence $D$ is the propositional proved sentence then for all B-functions $b$: $b(D) = 1$.

Proof of the Theorem 3.2:
If $D$ is $A1$ then by Definition 2.10:

$$b(D) = b\left(A \land \overline{B \land A}\right).$$

By (2):

$$b(D) = 1 - b\left(A \land \overline{B \land A}\right).$$

By the Definition 3.1 and the Theorem 2.1:

$$b(D) = 1 - b(A) + b\left(A \land \overline{B \land A}\right),$$

$$b(D) = 1 - b\left((A \land B) \land \overline{A}\right),$$

$$b(D) = 1 - b(A \land B) + b\left((A \land A) \land B\right),$$

$$b(D) = 1 - b(A \land B) + b(A \land B).$$

The proof is similar for the rest propositional axioms.

Let for all B-function $b$: $b(A) = 1$ and $b(A \Rightarrow D) = 1$.

By Definition 2.10:

$$b(A \Rightarrow D) = b\left(\overline{A \land \overline{D}}\right).$$

By (2):

$$b(A \Rightarrow D) = 1 - b\left(A \land \overline{D}\right).$$

Hence,

$$b\left(A \land \overline{D}\right) = 0.$$
\[ b(A \land D) = b(D) - b(D \land \overline{A}) = 1. \]

Therefore, for all B-function \( b \):

\[ b(D) = 1. \]

**Theorem: 3.3**

1) If for all Boolean functions \( g \):

\[ g(A) = 1 \]

then for all B-functions \( b \):

\[ b(A) = 1. \]

2) If for all Boolean functions \( g \):

\[ g(A) = 0 \]

then for all B-functions \( b \):

\[ b(A) = 0. \]

**Proof of the Theorem 3.3:**

1) This just follows from the preceding Theorem and from the Theorem 2.3.
2) If for all Boolean functions \( g \): \( g(A) = 0 \), then by the Definition 2.6: \( g(\overline{A}) = 1 \). Hence, by the point 1 of this Theorem: for all B-function \( b \): \( b(\overline{A}) = 1 \). By (2): \( b(A) = 0 \).

**Theorem: 3.4** All Boolean functions are the B-functions.

Therefore, the B-function is the generalization of the logic Boolean function.

**Proof of the Theorem 3.4:** If \( C \) is \( A1 \) then \( g(C) = 1 \).

By Definition 2.6: for all Boolean functions \( g \):

\[ g(A \land B) + g(A \land \overline{B}) = g(A) \cdot g(B) + g(A) \cdot (1 - g(B)) = g(A). \]

**Theorem: 3.5**

\[ b(A \lor B) = b(A) + b(B) - b(A \land B). \]

**Definition 3.2** Sentences \( A \) and \( B \) are inconsistent sentences for the B-function \( b \) if

\[ b(A \land B) = 0. \]

**Proof of the Theorem 3.5:** By the Definition 2.10 and (2):

\[ b(A \lor B) = 1 - b(\overline{A} \land \overline{B}). \]
By Definition 3.1:
\[ b(A \lor B) = 1 - b(A) + b(A \land B) = b(A) + b(B) - b(A \land B). \]

**Theorem: 3.6** If sentences \( A \) and \( B \) are the inconsistent sentences for the \( B \)-function \( b \) then
\[ b(A \lor B) = b(A) + b(B). \]

**Proof of the Theorem 3.6:** This just follows from the preceding Theorem and Definition 3.2.

**Theorem: 3.7** If \( b(A \land B) = b(A) \cdot b(B) \) then \( b(A \land B) = b(A) \cdot b(B) \).

**Proof of the Theorem 3.7:** By the Definition 3.1:
\[ b(A \land B) = b(A) - b(A \land B). \]

Hence,
\[ b(A \land B) = b(A) - b(A) \cdot b(B) = b(A) \cdot (1 - b(B)). \]

Hence, by (2):
\[ b(A \land B) = b(A) \cdot b(B). \]

**Theorem: 3.8** \( b(A \land \overline{A} \land B) = 0. \)

**Proof of the Theorem 3.8:** By the Definition 3.1 and by the points 2 and 3 of the Theorem 2.1:
\[ b(A \land \overline{A} \land B) = b(A \land B) - b(A \land A \land B), \]

hence, by the point 1 of the Theorem 2.1:
\[ b(A \land \overline{A} \land B) = b(A \land B) - b(A \land B). \]

**Theorem: 3.9**
\[ P(A \land (B \lor C)) = P(A \land B) + P(A \land C) - P(A \land B \land C). \]

**Proof of the Theorem 3.9:**
By Definition 3.1:
\[ P(A \land (B \lor C)) = P((A \land (B \land \overline{C})) = P(A) - P(A \land B \land \overline{C}) = P(A) - P(A \land B) + P(A \land C) - P(A \land B \land C). \]
4 The independent tests

Definition 4.1 Let \( st(n) \) be a function such that \( st(n) \) has got the domain on the set of natural numbers and has got the range of values in the set of the \( \exists \)-sentences.

In this case \( \exists \)-sentence \( A \) is a \([st]\)-series of range \( r \) with \( V \)-number \( k \) if \( A, r \) and \( k \) fulfill to some one amongst the following conditions:

1) \( r = 1 \) and \( k = 1 \), \( A = st(1) \) or \( k = 0 \), \( A = st(1) \);
2) \( B \) is \([st]\)-series of range \( r - 1 \) with \( V \)-number \( k \) and

\[
A = (B \land st(r)).
\]

or \( B \) is \([st]\)-series of range \( r - 1 \) with \( V \)-number \( k \) and

\[
A = (B \land st(r)).
\]

Let us denote a set of \([st]\)-series of range \( r \) with \( V \)-number \( k \) as \([st](r, k)\).

For example, if \( st(n) \) is a sentence \( B_n \) then the sentences:

\[
(B_1 \land B_2 \land B_3), (B_1 \land B_2 \land B_3), (B_1 \land B_2 \land B_3)
\]

are the elements of \([st](3, 2)\), and

\[
(B_1 \land B_2 \land B_3 \land B_4 \land B_5) \in [st](5, 3).
\]

Definition 4.2 Function \( st(n) \) is independent for \( B \)-function \( b \) if for \( A \) such that:

\[
b(A) = \prod_{n=1}^{r} b(st(n)).
\]

Definition 4.3 Let \( st(n) \) be a function such that \( st(n) \) has got the domain on the set of natural numbers and has got the range of values in the set of the \( \exists \)-sentences.

In this case \( \exists \)-sentence \( A \) is a \([st]\)-disjunction of range \( r \) with \( V \)-number \( k \) (denote: \( t[st](r, k) \)) if \( A \) is the disjunction of all elements of \([st](r, k)\).

For example, if \( st(n) \) is a sentence \( C_n \) then:

\[
(C_1 \land C_2 \land C_3) = t[st](3, 0),
\]

\[
t[st](3, 1) = ((C_1 \land \overline{C}_2 \land C_3) \lor (\overline{C}_1 \land C_2 \land C_3) \lor (C_1 \land \overline{C}_2 \land \overline{C}_3)),
\]

\[
t[st](3, 2) = ((C_1 \land C_2 \land C_3) \lor (C_1 \land \overline{C}_2 \land C_3) \lor (C_1 \land \overline{C}_2 \land \overline{C}_3)),
\]

\[
(C_1 \land C_2 \land C_3) = t[st](3, 3).
\]

Definition 4.4 A rational number \( \omega \) is called as a frequency of sentence \( A \) in the \([st]\)-series of \( r \) independent for \( B \)-function \( b \) tests (designate: \( \omega = \nu_r[st](A) \)) if

1) \( st(n) \) is independent for \( B \)-function \( b \),
2) for all \( n \): \( b(st(n)) = b(A) \),
3) \( t[st](r, k) \) is true and \( \omega = k/r \).

Theorem: 4.1 (the J.Bernoulli formula [2]) If \( st(n) \) is independent for \( B \)-function \( b \) and there exists a real number \( p \) such that for all \( n \): \( b(st(n)) = p \) then
\[ b(t[st](r,k)) = \frac{r!}{k!(r-k)!} \cdot p^k \cdot (1-p)^{r-k}. \]

**Proof of the Theorem 4.1:** By the Definition 4.2 and the Theorem 3.7: if \( B \in [st](r,k) \) then:

\[ b(B) = p^k \cdot (1-p)^{r-k}. \]

Since \([st](r,k)\) contains \( \frac{r!}{(k! \cdot (r-k)!)} \) elements then by the Theorems 3.7, 3.8 and 3.6 this Theorem is fulfilled.

**Definition 4.5** Let function \( st(n) \) has got the domain on the set of the natural numbers and has got the range of values in the set of the \( \mathcal{S} \)-sentences.

Let function \( f(r,k,l) \) has got the domain in the set of threes of the natural numbers and has got the range of values in the set of the \( \mathcal{S} \)-sentences.

In this case \( f(r,k,l) \) has got the domain in the set of threes of the natural numbers and has got the range of values in the set of the \( \mathcal{S} \)-sentences.

By the Definition 4.5:

\[ T[st](r,k,k+1) = (T[st](r,k,k) \lor t[st](r,k+1)). \]

**Definition 4.6** If \( a \) and \( b \) are real numbers and \( k-1 \leq a \leq k \) and \( l \leq b < l+1 \) then \( T[st](r,a,b) = T[st](r,k,l) \).

**Theorem 4.2**

\[ T[st](r,a,b) = \langle \frac{a}{r} \leq \nu_r [st] (A) \leq \frac{b}{r} \rangle. \]

**Proof of the Theorem 4.2:** By the Definition 4.6: there exist natural numbers \( r \) and \( k \) such that \( k-1 < a \leq k \) and \( l \leq b < l+1 \).

The recursion on \( l \):

1. Let \( l = k \).

In this case by the Definition 4.4:

\[ T[st](r,k,k) = t[st](r,k) = \langle \nu_r [st] (A) \leq \frac{k}{r} \rangle. \]

2. Let \( n \) be any natural number.

The recursive assumption: Let

\[ T[st](r,k,k+n) = \langle \frac{k}{r} \leq \nu_r [st] (A) \leq \frac{k+n}{r} \rangle. \]

By the Definition 4.5:

\[ T[st](r,k,k+n+1) = (T[st](r,k,k+n) \lor t[st](r,k+n+1)). \]

By the recursive assumption and by the Definition 4.4:

\[ T[st](r,k,k+n+1) = \]

\[ = (\langle \frac{k}{r} \leq \nu_r [st] (A) \leq \frac{k+n}{r} \rangle \lor \langle \nu_r [st] (A) = \frac{k+n+1}{r} \rangle). \]
Hence, by the Definition 2.10:

\[ T_{\text{st}}(r, k, k + n + 1) = \ll \frac{k}{r} \leq \nu_r [\text{st}] (A) \leq \frac{k + n + 1}{r} \gg. \]

**Theorem: 4.3** If \( st(n) \) is independent for \( B \)-function \( b \) and there exists a real number \( p \) such that \( b(st(n)) = p \) for all \( n \) then

\[ b(T_{\text{st}}(r, a, b)) = \sum_{a \leq k \leq b} \frac{r!}{k! \cdot (r - k)!} \cdot p^{k} \cdot (1 - p)^{r - k}. \]

**Proof of the Theorem 4.3:** This is the consequence from the Theorem 4.1 by the Theorem 3.6.

**Theorem: 4.4** If \( st(n) \) is independent for the \( B \)-function \( b \) and there exists a real number \( p \) such that \( b(st(n)) = p \) for all \( n \) then

\[ b(T_{\text{st}}(r, r \cdot (p - \varepsilon), r \cdot (p + \varepsilon))) \geq 1 - \frac{p \cdot (1 - p)}{r \cdot \varepsilon^2} \]

for every positive real number \( \varepsilon \).

**Proof of the Theorem 4.4:** Because

\[ \sum_{k=0}^{r} (k - r \cdot p)^2 \cdot \frac{r!}{k! \cdot (r - k)!} \cdot p^{k} \cdot (1 - p)^{r - k} = r \cdot p \cdot (1 - p) \]

then if

\[ J = \{ k \in \mathbb{N} | 0 \leq k \leq r \cdot (p - \varepsilon) \} \cap \{ k \in \mathbb{N} | r \cdot (p + \varepsilon) \leq k \leq r \} \]

then

\[ \sum_{k \in J} \frac{r!}{k! \cdot (r - k)!} \cdot p^{k} \cdot (1 - p)^{r - k} \leq \frac{p \cdot (1 - p)}{r \cdot \varepsilon^2}. \]

Hence, by (2) this Theorem is fulfilled.

Hence

\[ \lim_{r \to \infty} b(T_{\text{st}}(r, r \cdot (p - \varepsilon), r \cdot (p + \varepsilon))) = 1 \quad (3) \]

for all tiny positive numbers \( \varepsilon \).

## 5 The logic probability function

**Definition 5.1** \( B \)-function \( P \) is \( P \)-function if for every \( \exists \)-sentence \( \ll \Theta \gg \):

If \( P(\ll \Theta \gg) = 1 \) then \( \ll \Theta \gg \) is true sentence.

Hence from Theorem 4.2 and (3) if \( b \) is a \( P \)-function then the sentence

\[ \ll (p - \varepsilon) \leq \nu_r [st] (A) \leq (p + \varepsilon) \gg \]
is almost true sentence for large \( r \) and for all tiny \( \varepsilon \). Therefore, it is almost truely that

\[
\nu_r[st](A) = p
\]

for large \( r \).
Therefore, it is almost true that

\[
b(A) = \nu_r[st](A)
\]

for large \( r \).
Therefore, the function, defined by the Definition 5.1 has got the statistical meaning. That is why I'm call such function as the logic probability function.

6 Conditional probability

**Definition 6.1:** Conditional probability \( B \) for \( C \) is the following function:

\[
b(B/C) \overset{\text{def}}{=} \frac{b(C \land B)}{b(C)}.
\] (4)

**Theorem 6.1** The conditional probability function is a B-function.

**Proof of Theorem 6.1** From Definition 6.1:

\[
b(C/C) = \frac{b(C \land C)}{b(C)}. \tag{1}
\]

Hence by point 1 of Theorem 2.1:

\[
b(C/C) = \frac{b(C)}{b(C)} = 1.
\]

Form Definition 6.1:

\[
b((A \land B)/C) + b((A \land (\neg B))/C) = \frac{b(C \land (A \land B))}{b(C)} + \frac{b(C \land (A \land (\neg B)))}{b(C)}.
\]

Hence:

\[
b((A \land B)/C) + b((A \land (\neg B))/C) = \frac{b(C \land (A \land B)) + b(C \land (A \land (\neg B)))}{b(C)}.
\]

By point 3 of Theorem 2.1:

\[
b((A \land B)/C) + b((A \land (\neg B))/C) = \frac{b((C \land A) \land B) + b((C \land A) \land (\neg B))}{b(C)}.
\]

Hence by Definition 3.1:

\[
b((A \land B)/C) + b((A \land (\neg B))/C) = \frac{b(C \land A)}{b(C)}.
\]

Hence by Definition 6.1:

\[
b((A \land B)/C) + b((A \land (\neg B))/C) = b(A/C) \quad \Box
\]
7 Classical probability

Let $P$ be $P$-function.

Definition 7.1 \{$B_1, B_2, \ldots, B_n$\} is called as complete set if the following conditions are fulfilled:

1. if $k \neq s$ then $(B_k \land B_s)$ is a false sentence;
2. $(B_1 \lor B_2 \lor \ldots \lor B_n)$ is a true sentence.

Definition 7.2 $B$ is favorable for $A$ if $(B \land A)$ is a false sentence, and $B$ is unfavorable for $A$ if $(B \land A)$ is a false sentence.

Let

1. \{$B_1, B_2, \ldots, B_n$\} be complete set;
2. for $k \in \{1, 2, \ldots, n\}$ and $s \in \{1, 2, \ldots, n\}$: $P(B_k) = P(B_s)$;
3. if $1 \leq k \leq m$ then $B_k$ is favorable for $A$, and if $m + 1 \leq s \leq n$ then $B_s$ is unfavorable for $A$.

In that case from point 5 of Theorem 2.1 and from (1) and (2):

$$P(A \land B_k) = 0$$
for $k \in \{1, 2, \ldots, m\}$ and

$$P(A \land B_s) = 0$$
for $s \in \{m + 1, m + 2, \ldots, n\}$.

Hence from Definition 3.1:

$$P(A \land B_k) = P(B_k)$$
for $k \in \{1, 2, \ldots, n\}$.

By point 4 of Theorem 2.1:

$$A = (A \land (B_1 \lor B_2 \lor \ldots \lor B_m \lor B_{m+1} \ldots \lor B_n)).$$

Hence by Theorem 3.9:

$$P(A) = P(A \land B_1) + P(A \land B_2) + \ldots + P(A \land B_m) + P(A \land B_{m+1}) + \ldots + P(A \land B_n) = P(B_1) + P(B_2) + \ldots + P(B_m).$$

Therefore

$$P(A) = \frac{m}{n}.$$  

8 Conclusion

The logic probability function is the extension of the logic B-function. Therefore, the probability is some generalization of the classic propositional logic. That is the probability is the logic of events such that these events do not happen, yet.
9 Appendix. Consistency

9.1 THE NONSTANDARD NUMBERS

Let us consider the set $\mathbb{N}$ of natural numbers.

**Definition A.1:** The $n$-part-set $S$ of $\mathbb{N}$ is defined recursively as follows:
1) $S_1 = \{1\};$
2) $S_{n+1} = S_n \cup \{n + 1\}.$

**Definition A.2:** If $S_n$ is the $n$-part-set of $\mathbb{N}$ and $A \subseteq \mathbb{N}$ then $\|A \cap S_n\|$ is the quantity elements of the set $A \cap S_n,$ and if

$$\varpi_n(A) = \frac{\|A \cap S_n\|}{n},$$

then $\varpi_n(A)$ is the frequency of the set $A$ on the $n$-part-set $S_n.$

**Theorem A.1:**
1) $\varpi_n(\mathbb{N}) = 1;$
2) $\varpi_n(\emptyset) = 0;$
3) $\varpi_n(A) + \varpi_n(\mathbb{N} - A) = 1;$
4) $\varpi_n(A \cap B) + \varpi_n(A \cap (\mathbb{N} - B)) = \varpi_n(A).$

**Proof of the Theorem A.1:** From Definitions A.1 and A.2.

**Definition A.3:** If ”lim” is the Cauchy-Weierstrass ”limit” then let us denote:

$$\Phi_{ix} = \{ A \subseteq \mathbb{N} \mid \lim_{n \to \infty} \varpi_n(A) = 1 \}.$$

**Theorem A.2:** $\Phi_{ix}$ is the filter [5], i.e.:
1) $\mathbb{N} \in \Phi_{ix},$
2) $\emptyset \notin \Phi_{ix},$
3) if $A \in \Phi_{ix}$ and $B \in \Phi_{ix}$ then $(A \cap B) \in \Phi_{ix};$
4) if $A \in \Phi_{ix}$ and $A \subseteq B$ then $B \in \Phi_{ix}.$

**Proof of the Theorem A.2:** From the point 3 of Theorem A.1:

$$\lim_{n \to \infty} \varpi_n(\mathbb{N} - B) = 0.$$

From the point 4 of Theorem A.1:

$$\varpi_n(A \cap (\mathbb{N} - B)) \leq \varpi_n(\mathbb{N} - B).$$

Hence,

$$\lim_{n \to \infty} \varpi_n(A \cap (\mathbb{N} - B)) = 0.$$

Hence,

$$\lim_{n \to \infty} \varpi_n(A \cap B) = \lim_{n \to \infty} \varpi_n(A).$$
In the following text we shall adopt to our topics the definitions and the proofs of the Robinson Nonstandard Analysis [6]:

**Definition A.4:** The sequences of the real numbers \( \langle r_n \rangle \) and \( \langle s_n \rangle \) are \( Q \)-equivalent (denote: \( \langle r_n \rangle \sim \langle s_n \rangle \)) if
\[
\{ n \in \mathbb{N} | r_n = s_n \} \in \Phiix.
\]

**Theorem A.3:** If \( r, s, u \) are the sequences of the real numbers then
1) \( r \sim r \),
2) if \( r \sim s \) then \( s \sim r \);
3) if \( r \sim s \) and \( s \sim u \) then \( r \sim u \).

**Proof of the Theorem A.3:** By Definition A.4 from the Theorem A.2 is obvious.

**Definition A.5:** The \( Q \)-number is the set of the \( Q \)-equivalent sequences of the real numbers, i.e. if \( \tilde{a} \) is the \( Q \)-number and \( r \in \tilde{a} \) and \( s \in \tilde{a} \), then \( r \sim s \); and if \( r \in \tilde{a} \) and \( r \sim s \) then \( s \in \tilde{a} \).

**Definition A.6:** The \( Q \)-number \( \tilde{a} \) is the standard \( Q \)-number \( a \) if \( a \) is some real number and the sequence \( \langle r_n \rangle \) exists, for which: \( \langle r_n \rangle \in \tilde{a} \) and \( \{ n \in \mathbb{N} | r_n = a \} \in \Phiix \).

**Definition A.7:** The \( Q \)-numbers \( \tilde{a} \) and \( \tilde{b} \) are the equal \( Q \)-numbers (denote: \( \tilde{a} = \tilde{b} \)) if \( a \subseteq \tilde{b} \) and \( \tilde{b} \subseteq a \).

**Theorem A.4:** Let \( f(x, y, z) \) be a function, which has got the domain in \( \mathbb{R} \times \mathbb{R} \times \mathbb{R} \), has got the range of values in \( \mathbb{R} \) (\( \mathbb{R} \) is the real numbers set).

Let \( \langle y_1, n \rangle, \langle y_2, n \rangle, \langle y_3, n \rangle, \langle z_1, n \rangle, \langle z_2, n \rangle, \langle z_3, n \rangle \) be any sequences of real numbers.

In this case if \( \langle z_i, n \rangle \sim \langle y_i, n \rangle \) then \( \langle f(y_1, n, y_2, n, y_3, n) \rangle \sim \langle f(z_1, n, z_2, n, z_3, n) \rangle \).

**Proof of the Theorem A.4:** Let us denote:
if \( k = 1 \) or \( k = 2 \) or \( k = 3 \) then
\[
A_k = \{ n \in \mathbb{N} | y_{k, n} = z_{k, n} \}.
\]

In this case by Definition A.4 for all \( k \):
\[
A_k \in \Phiix.
\]

Because
\[
(A_1 \cap A_2 \cap A_3) \subseteq \{ n \in \mathbb{N} | f(y_{1, n}, y_{2, n}, y_{3, n}) = f(z_{1, n}, z_{2, n}, z_{3, n}) \}
\]
then by Theorem A.2:
\[
\{ n \in \mathbb{N} | f(y_{1, n}, y_{2, n}, y_{3, n}) = f(z_{1, n}, z_{2, n}, z_{3, n}) \} \in \Phiix.
\]

**Definition A.8:** Let us denote: \( QR \) is the set of the \( Q \)-numbers.

**Definition A.9:** The function \( \tilde{f} \), which has got the domain in \( QR \times QR \times QR \), has got the range of values in \( QR \), is the \( Q \)-extension of the function \( f \),
which has got the domain in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, has got the range of values in $\mathbb{R}$, if the following condition is accomplished:

Let $\langle x_n \rangle$, $\langle y_n \rangle$, $\langle z_n \rangle$ be any sequences of real numbers. In this case: if

$\langle x_n \rangle \in \tilde{x}$, $\langle y_n \rangle \in \tilde{y}$, $\langle z_n \rangle \in \tilde{z}$, $\tilde{u} = \tilde{f}(\tilde{x}, \tilde{y}, \tilde{z})$, then

$\langle f(x_n, y_n, z_n) \rangle \in \tilde{u}$.

**Theorem A.5:** For all functions $f$, which have the domain in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, have the range of values in $\mathbb{R}$, and for all real numbers $a$, $b$, $c$, $d$: if $\tilde{f}$ is the Q-extension of $f$; $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ are standard Q-numbers $a$, $b$, $c$, $d$, then:

if $d = f(a, b, c)$ then $\tilde{d} = \tilde{f}(\tilde{a}, \tilde{b}, \tilde{c})$ and vice versa.

**Proof of the Theorem A.5:** If $\langle r_n \rangle \in \tilde{a}$, $\langle s_n \rangle \in \tilde{b}$, $\langle u_n \rangle \in \tilde{c}$, $\langle t_n \rangle \in \tilde{d}$ then by Definition A.6:

$$\begin{align*}
\{ n \in \mathbb{N} | r_n = a \} &\in \Phi_{ix}, \\
\{ n \in \mathbb{N} | s_n = b \} &\in \Phi_{ix}, \\
\{ n \in \mathbb{N} | u_n = c \} &\in \Phi_{ix}, \\
\{ n \in \mathbb{N} | t_n = d \} &\in \Phi_{ix}.
\end{align*}$$

1) Let $d = f(a, b, c)$.
In this case by Theorem A.2:

$$\{ n \in \mathbb{N} | t_n = f(r_n, s_n, u_n) \} \in \Phi_{ix}.$$ 

Hence, by Definition A.4:

$$\langle t_n \rangle \sim \langle f(r_n, s_n, u_n) \rangle.$$ 

Therefore by Definition A.5:

$$\langle f(r_n, s_n, u_n) \rangle \in \tilde{d}.$$ 

Hence, by Definition A.9:

$$\tilde{d} = \tilde{f}(\tilde{a}, \tilde{b}, \tilde{c}).$$

2) Let $\tilde{d} = \tilde{f}(\tilde{a}, \tilde{b}, \tilde{c})$.
In this case by Definition A.9:

$$\langle f(r_n, s_n, u_n) \rangle \in \tilde{d}.$$ 

Hence, by Definition A.5:

$$\langle t_n \rangle \sim \langle f(r_n, s_n, u_n) \rangle.$$ 

Therefore, by Definition A.4:

$$\{ n \in \mathbb{N} | t_n = f(r_n, s_n, u_n) \} \in \Phi_{ix}.$$
Hence, by the Theorem A.2:

\[ \{ n \in \mathbb{N} | t_n = f(r_n, s_n, u_n), r_n = a, s_n = b, u_n = c, t_n = d \} \in \Phi \text{ix}. \]

Hence, since this set does not empty, then

\[ d = f(a, b, c). \]

By this Theorem: if \( \tilde{f} \) is the Q-extension of the function \( f \) then the expression "\( f(\tilde{x}, \tilde{y}, \tilde{z}) \)" will be denoted as "\( f(\tilde{x}, \tilde{y}, \tilde{z}) \)" and if \( \tilde{u} \) is the standard Q-number then the expression "\( \tilde{u} \)" will be denoted as "\( u \)".

Theorem A.6: If for all real numbers \( a, b, c \):

\[ \varphi(a, b, c) = \psi(a, b, c) \]

then for all Q-numbers \( \tilde{x}, \tilde{y}, \tilde{z} \):

\[ \varphi(\tilde{x}, \tilde{y}, \tilde{z}) = \psi(\tilde{x}, \tilde{y}, \tilde{z}). \]

Proof of the Theorem A.6: If \( \langle x_n \rangle \in \tilde{x}, \langle y_n \rangle \in \tilde{y}, \langle z_n \rangle \in \tilde{z}, \tilde{u} = \varphi(\tilde{x}, \tilde{y}, \tilde{z}) \), then by Definition A.9: \( \langle \varphi(x_n, y_n, z_n) \rangle \in \tilde{u} \).

Because \( \varphi(x_n, y_n, z_n) = \psi(x_n, y_n, z_n) \) then \( \langle \psi(x_n, y_n, z_n) \rangle \in \tilde{u} \).

If \( \tilde{u} = \psi(\tilde{x}, \tilde{y}, \tilde{z}) \) then by Definition A.9: \( \langle \psi(x_n, y_n, z_n) \rangle \in \tilde{v} \), too.

Therefore, for all sequences \( \langle t_n \rangle \) of real numbers: if \( \langle t_n \rangle \in \tilde{u} \) then by Definition A.5: \( \langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle \).

Hence, \( \langle t_n \rangle \in \tilde{v} \); and if \( \langle t_n \rangle \in \tilde{v} \) then \( \langle t_n \rangle \sim \langle \varphi(x_n, y_n, z_n) \rangle \); hence, \( \langle t_n \rangle \in \tilde{u} \).

Therefore, \( \tilde{u} = \tilde{v} \).

Theorem A.7: If for all real numbers \( a, b, c \):

\[ f(a, \varphi(b, c)) = \psi(a, b, c) \]

then for all Q-numbers \( \tilde{x}, \tilde{y}, \tilde{z} \):

\[ f(\tilde{x}, \varphi(\tilde{y}, \tilde{z})) = \psi(\tilde{x}, \tilde{y}, \tilde{z}). \]

Consequences from Theorems A.6 and A.7: \( \Phi \): For all Q-numbers \( \tilde{x}, \tilde{y}, \tilde{z} \):

\[ \Phi_1: (\tilde{x} + \tilde{y}) = (\tilde{y} + \tilde{x}), \]
\[ \Phi_2: (\tilde{x} + (\tilde{y} + \tilde{z})) = ((\tilde{x} + \tilde{y}) + \tilde{z}), \]
\[ \Phi_3: (\tilde{x} + 0) = \tilde{x}, \]
\[ \Phi_5: (\tilde{x} \cdot \tilde{y}) = (\tilde{y} \cdot \tilde{x}), \]
\[ \Phi_6: (\tilde{x} \cdot (\tilde{y} \cdot \tilde{z})) = ((\tilde{x} \cdot \tilde{y}) \cdot \tilde{z}), \]
\[ \Phi_7: (\tilde{x} \cdot 1) = \tilde{x}, \]
\[ \Phi_10: (\tilde{x} \cdot (\tilde{y} + \tilde{z})) = ((\tilde{x} \cdot \tilde{y}) + (\tilde{x} \cdot \tilde{z})). \]

Proof of the Theorem A.7: Let \( \langle w_n \rangle \in \tilde{w}, f(\tilde{x}, \tilde{w}) = \tilde{u}, \langle x_n \rangle \in \tilde{x}, \langle y_n \rangle \in \tilde{y}, \langle z_n \rangle \in \tilde{z}, \varphi(\tilde{y}, \tilde{z}) = \tilde{w}, \psi(\tilde{x}, \tilde{y}, \tilde{z}) = \tilde{v} \).

By the condition of this Theorem: \( f(x_n, \varphi(y_n, z_n)) = \psi(x_n, y_n, z_n) \).
By Definition A.9: \( \langle \psi(x_n, y_n, z_n) \rangle \in \tilde{v}, \langle \varphi(x_n, y_n) \rangle \in \tilde{w}, \langle f(x_n, w_n) \rangle \in \tilde{u} \).

For all sequences \( \langle t_n \rangle \) of real numbers:

1) If \( \langle t_n \rangle \in \tilde{v} \) then by Definition A.5: \( \langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle \).

Hence \( \langle t_n \rangle \sim \langle f(x_n, \varphi(y_n, z_n)) \rangle \).

Therefore, by Definition A.4:

\[ \{ n \in \mathbb{N} | t_n = f(x_n, \varphi(y_n, z_n)) \} \in \Phi_{ix} \]

and

\[ \{ n \in \mathbb{N} | w_n = \varphi(y_n, z_n) \} \in \Phi_{ix} \]

Hence, by Theorem A.2:

\[ \{ n \in \mathbb{N} | t_n = f(x_n, w_n) \} \in \Phi_{ix} \]

Hence, by Definition A.4:

\[ \langle t_n \rangle \sim \langle f(x_n, w_n) \rangle \]

Therefore, by Definition A.5: \( \langle t_n \rangle \in \tilde{u} \).

2) If \( \langle t_n \rangle \in \tilde{u} \) then by Definition A.5: \( \langle t_n \rangle \sim \langle f(x_n, w_n) \rangle \).

Because \( \langle w_n \rangle \sim \langle \varphi(y_n, z_n) \rangle \) then by Definition A.4:

\[ \{ n \in \mathbb{N} | t_n = f(x_n, w_n) \} \in \Phi_{ix} \]

and

\[ \{ n \in \mathbb{N} | w_n = \varphi(y_n, z_n) \} \in \Phi_{ix} \]

Therefore, by Theorem A.2:

\[ \{ n \in \mathbb{N} | t_n = f(x_n, \varphi(y_n, z_n)) \} \in \Phi_{ix} \]

Hence, by Definition A.4:

\[ \langle t_n \rangle \sim \langle f(x_n, \varphi(y_n, z_n)) \rangle \]

Therefore,

\[ \langle t_n \rangle \sim \langle \psi(x_n, y_n, z_n) \rangle \]

Hence, by Definition A.5: \( \langle t_n \rangle \in \tilde{v} \).

From above and from 1) by Definition A.7: \( \tilde{u} = \tilde{v} \).

**Theorem A.8: \Phi_{ix}**: For every Q-number \( \tilde{x} \) the Q-number \( \tilde{y} \) exists, for which:

\( \langle \tilde{x} + \tilde{y} \rangle = 0 \).

**Proof of the Theorem A.8**: If \( \langle x_n \rangle \in \tilde{x} \) then \( \tilde{y} \) is the Q-number, which contains \( \langle -x_n \rangle \).

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Theorem A.9: Φ9: There is not that 0 = 1.
Proof of the Theorem A.9: is obvious from Definition A.6 and Definition A.7.

Definition A.10: The Q-number $\tilde{x}$ is $Q$-less than the Q-number $\tilde{y}$ (denote: $\tilde{x} < \tilde{y}$) if the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of real numbers exist, for which: $\langle x_n \rangle \in \tilde{x}, \langle y_n \rangle \in \tilde{y}$ and

$$\{ n \in \mathbb{N} | x_n < y_n \} \in \Phi^{ix}.$$ 

Theorem A.10: For all Q-numbers $\tilde{x}, \tilde{y}, \tilde{z}$: [8]
\begin{align*}
\Omega_1: & \text{ there is not that } \tilde{x} < \tilde{x}; \\
\Omega_2: & \text{ if } \tilde{x} < \tilde{y} \text{ and } \tilde{y} < \tilde{z} \text{ then } \tilde{x} < \tilde{z}; \\
\Omega_4: & \text{ if } \tilde{x} < \tilde{y} \text{ then } (\tilde{x} + \tilde{z}) < (\tilde{y} + \tilde{z}); \\
\Omega_5: & \text{ if } 0 < \tilde{z} \text{ and } \tilde{x} < \tilde{y}, \text{ then } (\tilde{x} \cdot \tilde{z}) < (\tilde{y} \cdot \tilde{z}); \\
\Omega_3': & \text{ if } \tilde{x} < \tilde{y} \text{ then there is not, that } \tilde{y} < \tilde{x} \text{ or } \tilde{x} = \tilde{y} \text{ and vice versa}; \\
\Omega_3'': & \text{ for all standard Q-numbers } x, y, z: x < y \text{ or } y < x \text{ or } x = y.
\end{align*}

Proof of the Theorem A.10: is obvious from Definition A.10 by the Theorem A.2.

Theorem A.11: Φ8: If $0 < |\tilde{x}|$ then the Q-number $\tilde{y}$ exists, for which $(\tilde{x} \cdot \tilde{y}) = 1$.

Proof of the Theorem A.11: If $\langle x_n \rangle \in \tilde{x}$ then by Definition A.10: if

$$A = \{ n \in \mathbb{N} | 0 < |x_n| \}$$

then $A \in \Phi^{ix}$.

In this case: if for the sequence $\langle y_n \rangle$: if $n \in A$ then $y_n = 1/x_n$

- then

$$\{ n \in \mathbb{N} | x_n \cdot y_n = 1 \} \in \Phi^{ix}.$$ 

Thus, Q-numbers are fulfilled to all properties of real numbers, except $\Omega_3$ [9]. The property $\Omega_3$ is accomplished by some weak meaning ($\Omega_3'$ and $\Omega_3''$).

Definition A.11: The Q-number $\tilde{x}$ is the infinitesimal Q-number if the sequence of real numbers $\langle x_n \rangle$ exists, for which: $\langle x_n \rangle \in \tilde{x}$ and for all positive real numbers $\varepsilon$:

$$\{ n \in \mathbb{N} | |x_n| < \varepsilon \} \in \Phi^{ix}.$$ 

Let the set of all infinitesimal Q-numbers be denoted as $I$.

Definition A.12: The Q-numbers $\tilde{x}$ and $\tilde{y}$ are the infinite closed Q-numbers (denote: $\tilde{x} \approx \tilde{y}$) if $|\tilde{x} - \tilde{y}| = 0$ or $|\tilde{x} - \tilde{y}|$ is infinitesimal.

Definition A.13: The Q-number $\tilde{x}$ is the infinite Q-number if the sequence $\langle r_n \rangle$ of real numbers exists, for which $\langle r_n \rangle \in \tilde{x}$ and for every natural number $m$:

$$\{ n \in \mathbb{N} | m < r_n \} \in \Phi^{ix}.$$ 

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9.2 Model

Let us define the propositional calculus like to (4), but the propositional forms shall be marked by the script greek letters.

Definition C1: A set \( \mathcal{R} \) of the propositional forms is a \textit{U-world} if:
1) if \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{R} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \vdash \beta \) then \( \beta \in \mathcal{R} \),
2) for all propositional forms \( \alpha \): it is not that \( \alpha \& (\neg \alpha) \in \mathcal{R} \),
3) for every propositional form \( \alpha \): \( \alpha \in \mathcal{R} \) or \( \neg \alpha \in \mathcal{R} \).

Definition C2: The sequences of the propositional forms \( \langle \alpha_n \rangle \) and \( \langle \beta_n \rangle \) are \textit{Q-equivalent} (denote: \( \langle \alpha_n \rangle \sim \langle \beta_n \rangle \)) if
\[
\{ n \in \mathbb{N} | \alpha_n = \beta_n \} \in \Phi_{\text{fix}}.
\]

Let us define the notions of the \textit{Q-extension of the functions} for like as in the Definitions A.5, A.2, A.9, A.5, A.6.

Definition C3: The Q-form \( \bar{\alpha} \) is \textit{Q-real} in the U-world \( \mathcal{R} \) if the sequence \( \langle \alpha_n \rangle \) of the propositional forms exists, for which: \( \langle \alpha_n \rangle \in \bar{\alpha} \) and
\[
\{ n \in \mathbb{N} | \alpha_n \in \mathcal{R} \} \in \Phi_{\text{fix}}.
\]

Definition C4: The set \( \bar{\mathcal{R}} \) of the Q-forms is the \textit{Q-extension of the U-world} \( \mathcal{R} \) if \( \bar{\mathcal{R}} \) is the set of Q-forms \( \bar{\alpha} \), which are Q-real in \( \mathcal{R} \).

Definition C5: The sequence \( \bar{\mathcal{R}}_k \) of the Q-extensions is the \textit{S-world}.

Definition C6: The Q-form \( \bar{\alpha} \) is \textit{S-real} in the S-world \( \bar{\mathcal{R}}_k \) if
\[
\{ k \in \mathbb{N} | \bar{\alpha} \in \bar{\mathcal{R}}_k \} \in \Phi_{\text{fix}}.
\]

Definition C7: The set \( A \) (\( A \subseteq \mathbb{N} \)) is the \textit{regular set} if for every real positive number \( \varepsilon \) the natural number \( n_0 \) exists, for which: for all natural numbers \( n \) and \( m \), which are more or equal to \( n_0 \):
\[
|w_n(A) - w_m(A)| < \varepsilon.
\]

Theorem C1: If \( A \) is the regular set and for all real positive \( \varepsilon \):
\[
\{ k \in \mathbb{N} | w_k(A) < \varepsilon \} \in \Phi_{\text{fix}}.
\]
then
\[
\lim_{k \to \infty} w_k(A) = 0.
\]

Proof of the Theorem C1: Let be
\[
\lim_{k \to \infty} w_k(A) \neq 0.
\]
That is the real number \( \varepsilon_0 \) exists, for which: for every natural number \( n' \) the natural number \( n \) exists, for which:
Let $\delta_0$ be some positive real number, for which: $\varepsilon_0 - \delta_0 > 0$. Because $A$ is the regular set then for $\delta_0$ the natural number $n_0$ exists, for which: for all natural numbers $n$ and $m$, which are more or equal to $n_0$:

$$|w_m(A) - w_n(A)| < \delta_0.$$ 

That is

$$w_m(A) > w_n(A) - \delta_0.$$ 

Since $w_n(A) \geq \varepsilon_0$ then $w_m(A) \geq \varepsilon_0 - \delta_0$.

Hence, the natural number $n_0$ exists, for which: for all natural numbers $m$:

if $m \geq n_0$ then $w_m(A) \geq \varepsilon_0 - \delta_0$.

Therefore,

$$\{m \in \mathbb{N} | w_m(A) \geq \varepsilon_0 - \delta_0\} \in \Phi_{\text{ix}}.$$ 

and by this Theorem condition:

$$\{k \in \mathbb{N} | w_k(A) < \varepsilon_0 - \delta_0\} \in \Phi_{\text{ix}}.$$ 

Hence,

$$\{k \in \mathbb{N} | \varepsilon_0 - \delta_0 < \varepsilon_0 - \delta_0\} \in \Phi_{\text{ix}}.$$ 

That is $0 \notin \Phi_{\text{ix}}$. It is the contradiction for the Theorem 2.2.

**Definition C8:** Let $\langle \overline{R}_k \rangle$ be a S-world.

In this case the function $\mathfrak{M}(\overline{\beta})$, which has got the domain in the set of the Q-forms, has got the range of values in $\mathbb{Q}$, is defined as the following:

If $\mathfrak{M}(\overline{\beta}) = \overline{p}$ then the sequence $\langle p_n \rangle$ of the real numbers exists, for which:

$\langle p_n \rangle \in \overline{p}$ and

$$p_n = w_n\left(\left\{k \in \mathbb{N} | \overline{\beta} \in \overline{R}_k\right\}\right).$$

**Theorem C2:** If $\left\{k \in \mathbb{N} | \overline{\beta} \in \overline{R}_k\right\}$ is the regular set and $\mathfrak{M}(\overline{\beta}) \approx 1$ then $\overline{\beta}$ is S-resl in $\langle \overline{R}_k \rangle$.

**Proof of the Theorem C2:** Since $\mathfrak{M}(\overline{\beta}) \approx 1$ then by Definitions 2.12 and 2.11: for all positive real $\varepsilon$:

$$\left\{n \in \mathbb{N} | w_n\left(\left\{k \in \mathbb{N} | \overline{\beta} \in \overline{R}_k\right\}\right) > 1 - \varepsilon\right\} \in \Phi_{\text{ix}}.$$ 

Hence, by the point 3 of Theorem 2.1: for all positive real $\varepsilon$:
\( \{ n \in \mathbb{N} \mid (N - w_n \left( \{ k \in \mathbb{N} \mid \tilde{\beta} \in \tilde{\mathbb{R}}_k \} \right) ) < \varepsilon \} \in \Phi_{\text{ix}}. \)

Therefore, by the Theorem C1:
\[
\lim_{n \to \infty} (N - w_n \left( \{ k \in \mathbb{N} \mid \tilde{\beta} \in \tilde{\mathbb{R}}_k \} \right) ) = 0.
\]

That is:
\[
\lim_{n \to \infty} w_n \left( \{ k \in \mathbb{N} \mid \tilde{\beta} \in \tilde{\mathbb{R}}_k \} \right) = 1.
\]

Hence, by Definition 2.3:
\[ \{ k \in \mathbb{N} \mid \tilde{\beta} \in \tilde{\mathbb{R}}_k \} \in \Phi_{\text{ix}}. \]

And by Definition C6: \( \tilde{\beta} \) is S-real in \( \langle \tilde{\mathbb{R}}_k \rangle \).

Theorem C3: The P-function exists.

Proof of the Theorem C3: By the Theorems C2 and 2.1: \( \mathfrak{M}(\tilde{\beta}) \) is the P-function in \( \langle \tilde{\mathbb{R}}_k \rangle \).

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