Further Inequalities for Power Series with Nonnegative Coefficients Via a Reverse of Jensen Inequality

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Abstract. Some inequalities for power series with nonnegative coefficients via a new reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

1. Introduction

In 1994, Dragomir & Ionescu obtained the following reverse of Jensen’s discrete inequality:

Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior $\bar{I}$ of the interval $I$. If $x_i \in \bar{I}$ and $w_i \geq 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the inequality:

\begin{equation}
0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \\
\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i.
\end{equation}

In order to improve Grüss’ discrete inequality, Cerone & Dragomir established in 2002 the following result [1]:

\begin{equation}
\left| \sum_{i=1}^{n} w_i a_i b_i - \sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} w_i b_i \right| \\
\leq \frac{1}{2} (A - a) \sum_{i=1}^{n} w_i \left| b_i - \sum_{j=1}^{n} w_j b_j \right| \\
\leq \frac{1}{2} (A - a) \left[ \sum_{i=1}^{n} w_i b_i^2 - \left( \sum_{i=1}^{n} w_i b_i \right)^2 \right]^{1/2},
\end{equation}

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provided \( \infty < a \leq a_i \leq A < \infty \), and \( w_i \geq 0 \) (\( i = 1, \ldots, n \)) with \( W_n := \sum_{i=1}^{n} w_i = 1 \).

In addition, if \( \infty < b \leq b_i \leq B < \infty \), (\( i = 1, \ldots, n \)) then we have the string of inequalities

(3) \[
\left| \sum_{i=1}^{n} w_i a_i b_i - \sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} w_i b_i \right| \\
\leq \frac{1}{2} (A - a) \sum_{i=1}^{n} w_i \left| b_i - \sum_{j=1}^{n} w_j b_j \right| \\
\leq \frac{1}{2} (A - a) \left[ \sum_{i=1}^{n} w_i b_i^2 - \left( \sum_{i=1}^{n} w_i b_i \right)^2 \right]^{1/2} \\
\leq \frac{1}{4} (A - a) (B - b).
\]

Utilising these results, we observe that if \( \Phi \) is differentiable convex on a finite interval, say \( [m, M] \), then we have the inequalities:

(4) \[
0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \\
\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i \\
\leq \frac{1}{2} (M - m) \sum_{i=1}^{n} w_i \left| \Phi' (x_i) - \sum_{j=1}^{n} w_j \Phi' (x_j) \right| \\
\leq \frac{1}{2} (M - m) \left[ \sum_{i=1}^{n} w_i \left[ \Phi' (x_i) \right]^2 - \left( \sum_{i=1}^{n} w_i \Phi' (x_i) \right)^2 \right]^{1/2}
\]

for \( x_i \in (m, M) \) (\( i = 1, \ldots, n \)).

If the lateral derivatives \( \Phi'_+ (m) \) and \( \Phi'_- (M) \) are finite, then we also have

(5) \[
0 \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left( \sum_{i=1}^{n} w_i x_i \right) \\
\leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i \\
\leq \frac{1}{2} \left[ \Phi'_- (M) - \Phi'_+ (m) \right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|
\]
\[
\leq \frac{1}{2} \left[ \Phi'_{-} (M) - \Phi'_{+} (m) \right] \left[ \sum_{i=1}^{n} w_i x_i^2 - \left( \sum_{i=1}^{n} w_i x_i \right)^2 \right]^{1/2}
\]

\[
\leq \frac{1}{4} (M - m) \left[ \Phi'_{-} (M) - \Phi'_{+} (m) \right]
\]

for \( x_i \in [m, M] \) \((i = 1, \ldots, n)\).

The most important power series with nonnegative coefficients are:

\[\exp (z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \quad z \in \mathbb{C}, \quad \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n, \quad z \in D (0, 1),\]

\[\ln \frac{1}{1 - z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \quad z \in D (0, 1), \quad \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \quad z \in \mathbb{C},\]

\[\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)!} z^{2n+1}, \quad z \in \mathbb{C}.
\]

Other important examples of functions as power series representations with nonnegative coefficients are:

\[\frac{1}{2} \ln \left( \frac{1 + z}{1 - z} \right) = \sum_{n=1}^{\infty} \frac{1}{2n - 1} z^{2n-1}, \quad z \in D (0, 1),\]

\[\sin^{-1} (z) = \sum_{n=0}^{\infty} \frac{\Gamma (n + \frac{1}{2})}{\sqrt{\pi} (2n + 1) n!} z^{2n+1}, \quad z \in D (0, 1),\]

\[\tanh^{-1} (z) = \sum_{n=1}^{\infty} \frac{1}{2n - 1} z^{2n-1}, \quad z \in D (0, 1),\]

\[2F_1 (\alpha, \beta, \gamma, z) := \sum_{n=0}^{\infty} \frac{\Gamma (n + \alpha) \Gamma (n + \beta) \Gamma (\gamma)}{n! \Gamma (\alpha) \Gamma (\beta) \Gamma (n + \gamma)} z^n, \quad \alpha, \beta, \gamma > 0\]

\[z \in D (0, 1),\]

where \( \Gamma \) is \textit{Gamma function}.

On utilizing the above reverses of Jensen inequality we obtained in [5]:

**Theorem 1.1.** Let \( f (z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with nonnegative coefficients and convergent on the open disk \( D (0, R) \) with \( R > 0 \) or \( R = \infty \). If \( p \geq 1, 0 < \alpha < R \) and \( x > 0 \) with \( \alpha x^p, \alpha x^{p-1} < R \), then

\[0 \leq \frac{f (\alpha x^p)}{f (\alpha)} - \left( \frac{f (\alpha x)}{f (\alpha)} \right)^p \leq p \left[ \frac{f (\alpha x^p)}{f (\alpha)} - \frac{f (\alpha x^{p-1})}{f (\alpha)} \right].\]

Moreover, if \( 0 < x \leq 1 \), then

\[0 \leq \frac{f (\alpha x^p)}{f (\alpha)} - \left( \frac{f (\alpha x)}{f (\alpha)} \right)^p \leq p \left[ \frac{f (\alpha x^p)}{f (\alpha)} - \frac{f (\alpha x^{p-1})}{f (\alpha)} \right].\]
\begin{align*}
\leq \frac{1}{2} p \left( \frac{f(\alpha x^{2(p-1)})}{f(\alpha)} - \left[ \frac{f(\alpha x^{p-1})}{f(\alpha)} \right]^2 \right)^{1/2} \leq \frac{1}{4} p
\end{align*}

and

\begin{align*}
0 \leq f(\alpha x^p) - \left[ \frac{f(\alpha x^p)}{f(\alpha)} \right]^p \leq p \left[ \frac{f(\alpha x^p)}{f(\alpha)} - \frac{f(\alpha x^{p-1})}{f(\alpha)} f(\alpha) \right] \\
\leq \frac{1}{2} p \left( \frac{f(\alpha x^2)}{f(\alpha)} - \left[ \frac{f(\alpha x)}{f(\alpha)} \right]^2 \right)^{1/2} \leq \frac{1}{4} p.
\end{align*}

**Corollary 1.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with nonnegative coefficients and convergent on the open disk \( D(0, R) \) with \( R > 0 \) or \( R = \infty \). If \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( u, v > 0 \) with \( v^p \leq u^q < R \), then

\begin{align*}
\left[ \frac{f(uv)}{f(u^q)} \right]^p &\leq f(v^p) \leq \frac{1}{4} p + \left[ \frac{f(uv)}{f(u^q)} \right]^p \\
\text{and}
\end{align*}

\begin{align*}
0 &\leq [f(v^p)]^{1/p} [f(u^q)]^{1/q} - f(uv) \leq \frac{1}{4^{1/p}} p^{1/p} f(u^q).
\end{align*}

For some similar exponential and logarithmic inequalities see [5].

For other recent results for power series with nonnegative coefficients, see [2], [7], [11] and [12]. For more results on power series inequalities, see [2] and [7]-[10].

Motivated by the above results and utilizing a new reverse of Jensen inequality we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

## 2. Reverses of Jensen’s Inequality

The following reverse of the Jensen’s inequality holds:

**Theorem 2.1.** Let \( f : I \rightarrow \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R} \), \( m < M \) with \( [m, M] \subset \bar{I} \), \( \bar{I} \) is the interior of \( I \). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \((i = 1, \ldots, n)\) with \( W_n := \sum_{i=1}^{n} w_i = 1 \) and \( \sum_{i=1}^{n} w_i x_i \in (m, M) \), then

\begin{align*}
0 &\leq \sum_{i=1}^{n} w_i f(x_i) - f \left( \sum_{i=1}^{n} w_i x_i \right) \\
&\leq \frac{(M - \sum_{i=1}^{n} w_i x_i) (\sum_{i=1}^{n} w_i x_i - m)}{M - m} \Psi_f \left( \sum_{i=1}^{n} w_i x_i; m, M \right) \\
&\leq \frac{(M - \sum_{i=1}^{n} w_i x_i) (\sum_{i=1}^{n} w_i x_i - m)}{M - m} \sup_{t \in (m, M)} \Psi_f (t; m, M)
\end{align*}
\[\begin{align*}
\leq & \left( M - \sum_{i=1}^{n} w_{i}x_{i} \right) \left( \sum_{i=1}^{n} w_{i}x_{i} - m \right) \frac{f'_{-}(M) - f'_{+}(m)}{M - m} \\
\leq & \frac{1}{4} (M - m) \left[ f'_{-}(M) - f'_{+}(m) \right],
\end{align*}\]

where \( \Psi_f (:; m, M) : (m, M) \to \mathbb{R} \) is defined by

\[ \Psi_f (t; m, M) = \frac{f(M) - f(t)}{M - t} - \frac{f(t) - f(m)}{t - m}. \]

We also have the inequality

\[0 \leq \sum_{i=1}^{n} w_{i}f (x_{i}) - f \left( \sum_{i=1}^{n} w_{i}x_{i} \right) \leq \frac{1}{4} (M - m) \left[ f'_{-}(M) - f'_{+}(m) \right],\]

provided that \( \sum_{i=1}^{n} w_{i}x_{i} \in (m, M) \).

**Proof.** By the convexity of \( f \) we have that

\[\begin{align*}
\sum_{i=1}^{n} w_{i}f (x_{i}) - f \left( \sum_{i=1}^{n} w_{i}x_{i} \right) &= \sum_{i=1}^{n} w_{i} \left[ \frac{m (M - x_{i}) + M (x_{i} - m)}{M - m} \right] \\
&\quad - f \left( \sum_{i=1}^{n} w_{i} \left[ \frac{m (M - x_{i}) + M (x_{i} - m)}{M - m} \right] \right) \\
&\leq \sum_{i=1}^{n} w_{i} \left( M - x_{i} \right) f (m) + (x_{i} - m) f (M) \\
&\quad - f \left( \frac{m (M - \sum_{i=1}^{n} w_{i}x_{i}) + M (\sum_{i=1}^{n} w_{i}x_{i} - m)}{M - m} \right) \\
&\quad = \frac{(M - \sum_{i=1}^{n} w_{i}x_{i}) f (m) + (\sum_{i=1}^{n} w_{i}x_{i} - m) f (M)}{M - m} \\
&\quad - f \left( \frac{m (M - \sum_{i=1}^{n} w_{i}x_{i}) + M (\sum_{i=1}^{n} w_{i}x_{i} - m)}{M - m} \right) := B.
\]
By denoting
\[ \Delta_f(t; m, M) := \frac{(t - m) f(M) + (M - t) f(m)}{M - m} - f(t), \quad t \in [m, M] \]
we have
\[ \Delta_f(t; m, M) = \frac{(t - m) f(M) + (M - t) f(m) - (M - t + t - m) f(t)}{M - m} \]
\[ = \frac{(t - m) \left[ f(M) - f(t) \right] - (M - t) \left[ f(t) - f(m) \right]}{M - m} \]
\[ = \frac{(M - t) (t - m)}{M - m} \Psi_f(t; m, M) \]
for any \( t \in (m, M) \).

Therefore we have the equality
\[ B = \frac{(M - \sum_{i=1}^{n} w_i x_i) (\sum_{i=1}^{n} w_i x_i - m)}{M - m} \Psi_f \left( \sum_{i=1}^{n} w_i x_i; m, M \right) \]
provided that \( \sum_{i=1}^{n} w_i x_i \in (m, M) \).

If \( \sum_{i=1}^{n} w_i x_i \in (m, M) \), then
\[ \Psi_f \left( \sum_{i=1}^{n} w_i x_i; m, M \right) \leq \sup_{t \in (m, M)} \Psi_f (t; m, M) \]
\[ = \sup_{t \in (m, M)} \left[ \frac{f(M) - f(t)}{M - t} - \frac{f(t) - f(m)}{t - m} \right] \]
\[ \leq \sup_{t \in (m, M)} \left[ \frac{f(M) - f(t)}{M - t} \right] + \sup_{t \in (m, M)} \left[ - \frac{f(t) - f(m)}{t - m} \right] \]
\[ = \sup_{t \in (m, M)} \left[ \frac{f(M) - f(t)}{M - t} \right] - \inf_{t \in (m, M)} \left[ \frac{f(t) - f(m)}{t - m} \right] \]
\[ = f'_- (M) - f'_+ (m), \]
which by (15) and (17) produces the desired result (13).

Since, obviously
\[ \frac{(M - \sum_{i=1}^{n} w_i x_i) (\sum_{i=1}^{n} w_i x_i - m)}{M - m} \leq \frac{1}{4} (M - m), \]
then by (15) and (17) we deduce the second inequality (14).

The last part is clear.

For similar integral versions see [4].
Remark 2.1. a) For \( p > 1 \) and \( 0 < m < M < \infty \) consider the function \( \Psi_p (\cdot ; m, M) : (m, M) \to \mathbb{R} \) defined by

\[
\Psi_p (t; m, M) = \frac{M^p - t^p}{M - t} - \frac{t^p - m^p}{t - m} = \frac{t (M^p - m^p) - t^p (M - m) - mM (M^{p-1} - m^{p-1})}{(M - t) (t - m)}.
\]

If \( x_i \in [m, M] \) and \( w_i \geq 0 \ (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^n w_i = 1 \) and \( \sum_{i=1}^n w_i x_i \in (m, M) \), then

\[
0 \leq \sum_{i=1}^n w_i x_i^p - \left( \sum_{i=1}^n w_i x_i \right)^p \leq \frac{(M - \sum_{i=1}^n w_i x_i) (\sum_{i=1}^n w_i x_i - m)}{M - m} \Psi_p \left( \sum_{i=1}^n w_i x_i; m, M \right)
\]

\[
\leq \frac{(M - \sum_{i=1}^n w_i x_i) (\sum_{i=1}^n w_i x_i - m)}{M - m} \sup_{t \in (m, M)} \Psi_p (t; m, M)
\]

\[
\leq p \left( M - \sum_{i=1}^n w_i x_i \right) \left( \sum_{i=1}^n w_i x_i - m \right) \frac{M^{p-1} - m^{p-1}}{M - m}
\]

\[
\leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1})
\]

and

\[
0 \leq \sum_{i=1}^n w_i x_i^p - \left( \sum_{i=1}^n w_i x_i \right)^p \leq \frac{(M - \sum_{i=1}^n w_i x_i) (\sum_{i=1}^n w_i x_i - m)}{M - m} \Psi_p \left( \sum_{i=1}^n w_i x_i; m, M \right)
\]

\[
\leq \frac{1}{4} (M - m) \Psi_p \left( \sum_{i=1}^n w_i x_i; m, M \right)
\]

\[
\leq \frac{1}{4} (M - m) \sup_{t \in (m, M)} \Psi_p (t; m, M) \leq \frac{1}{4} p (M - m) (M^{p-1} - m^{p-1}) .
\]

For \( 0 < m < M < \infty \) consider the function \( \Psi_{\ln} (\cdot ; m, M) : (m, M) \to \mathbb{R} \) defined by

\[
\Psi_{\ln} (t; m, M) = \frac{- \ln M + \ln t}{M - t} - \ln t + \frac{\ln m}{t - m} = \frac{(M - m) \ln t - (M - t) \ln m - (t - m) \ln M}{(M - t) (t - m)}.
\]
\[ = \ln \left( \frac{t^{M-m}}{m^{M-t} M^{t-m}} \right) \]

b) If \( x_i \in [m, M] \) and \( w_i \geq 0 \) (\( i = 1, \ldots, n \)) with \( W_n := \sum_{i=1}^{n} w_i = 1 \) and \( \sum_{i=1}^{n} w_i x_i \in (m, M) \), then

\[
0 \leq \ln \left( \sum_{i=1}^{n} w_i x_i \right) - \sum_{i=1}^{n} w_i \ln x_i
\]

\[
\leq \ln \left( \frac{\sum_{i=1}^{n} w_i x_i}{M - \sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n} w_i x_i - m} \right)
\]

\[
\leq \frac{(M - \sum_{i=1}^{n} w_i x_i) (\sum_{i=1}^{n} w_i x_i - m)}{M - m} \sup_{t \in (m, M)} \Psi - \ln (t; m, M)
\]

\[
\leq \frac{1}{Mm} \left( M - \sum_{i=1}^{n} w_i x_i \right) \left( \sum_{i=1}^{n} w_i x_i - m \right) \leq \frac{1}{4} \frac{(M - m)^2}{Mm},
\]

and

\[
0 \leq \ln \left( \sum_{i=1}^{n} w_i x_i \right) - \sum_{i=1}^{n} w_i \ln x_i
\]

\[
\leq \ln \left( \frac{\sum_{i=1}^{n} w_i x_i}{M - \sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n} w_i x_i - m} \right)
\]

\[
\leq \frac{1}{4} (M - m)
\]

\[
\times \ln \left( \frac{(\sum_{i=1}^{n} w_i x_i)^{M-m}}{M - \sum_{i=1}^{n} w_i x_i - \sum_{i=1}^{n} w_i x_i - m} \right) \left( \frac{\sum_{i=1}^{n} w_i x_i - m}{\sum_{i=1}^{n} w_i x_i} \right)
\]

\[
\leq \frac{1}{4} (M - m) \sup_{t \in (m, M)} \Psi - \ln (t; m, M) \leq \frac{1}{4} \frac{(M - m)^2}{Mm}.
\]

3. Power Inequalities

For \( p > 1 \), \( f(t) := t^p \), \( m = 0 \) and \( M = 1 \) we have

\[
\Psi_f (t; m, M) = \frac{t^p - 1}{t - 1} - t^{p-1} = \frac{1 - t^{p-1}}{1 - t} =: B_p (t).
\]

If \( p \in (1, 2) \), the function \( \Gamma (t) = t^{p-1} \) is concave on \((0, 1)\) and then \( B_p (\cdot) \) is decreasing on \((0, 1)\). Therefore

\[
\sup_{t \in (0,1)} B_p (t) = \lim_{t \to 0^+} B_p (t) = 1.
\]
If \( p = 2 \), then \( B_p(t) = 1 \) for \( t \in (0, 1) \). If \( p \in (2, \infty) \), the function \( \Gamma(t) = t^{p-1} \) is convex on \( (0, 1) \) and then \( B_p(\cdot) \) is increasing on \( (0, 1) \). Therefore
\[
\sup_{t \in (0,1)} B_p(t) = \lim_{t \to 1^-} B_p(t) = p - 1.
\]

In conclusion
\[
M_p := \sup_{t \in (0,1)} B_p(t) = \begin{cases} 1 & \text{if } p \in (1, 2], \\ p - 1 & \text{if } p \in (2, \infty). \end{cases}
\]

If \( z_i \in [0, 1] \) and \( w_i \geq 0 \) (\( i = 1, \ldots, n \)) with \( W_n := \sum_{i=1}^{n} w_i = 1 \) and \( \sum_{i=1}^{n} w_i z_i \in (0, 1) \), then from (13) and (14) we have the inequalities:
\begin{equation}
0 \leq \sum_{i=1}^{n} w_i z_i^p - \left( \sum_{i=1}^{n} w_i z_i \right)^p \leq M_p \left( 1 - \sum_{i=1}^{n} w_i z_i \right) \sum_{i=1}^{n} w_i z_i \leq \frac{1}{4} M_p
\end{equation}
and
\begin{equation}
0 \leq \sum_{i=1}^{n} w_i z_i^p - \left( \sum_{i=1}^{n} w_i z_i \right)^p \leq \frac{1}{4} \cdot \frac{1 - \left( \sum_{i=1}^{n} w_i z_i \right)^{p-1}}{1 - \sum_{i=1}^{n} w_i z_i} \leq \frac{1}{4} M_p.
\end{equation}

**Proposition 3.1.** If \( x_i \geq 0, y_i > 0 \) for \( i \in \{1, \ldots, n\} \), \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and such that
\begin{equation}
0 \leq \frac{x_i}{y_i^{q-1}} \leq 1 \quad \text{for } i \in \{1, \ldots, n\},
\end{equation}
then we have
\begin{equation}
0 \leq \sum_{i=1}^{n} x_i - \left( \sum_{i=1}^{n} \frac{x_i y_i}{y_i^q} \right)^p \leq M_p \left( 1 - \sum_{i=1}^{n} \frac{x_i y_i}{y_i^q} \right) \left( \sum_{i=1}^{n} \frac{x_i y_i}{y_i^{q-1}} \right) \leq \frac{1}{4} M_p
\end{equation}
and
\begin{equation}
0 \leq \sum_{i=1}^{n} x_i - \left( \sum_{i=1}^{n} \frac{x_i y_i}{y_i^q} \right)^p \leq \frac{1}{4} \cdot \frac{1 - \left( \sum_{i=1}^{n} \frac{x_i y_i}{y_i^q} \right)^{p-1}}{1 - \sum_{i=1}^{n} \frac{x_i y_i}{y_i^{q-1}}} \leq \frac{1}{4} M_p,
\end{equation}
where \( M_p \) is defined above.

**Proof.** The inequalities (25) and (26) follow from (22) and (23) by choosing
\[
z_i = \frac{x_i}{y_i^{q-1}}, \quad \text{and } w_i = \frac{y_i^q}{\sum_{j=1}^{n} y_j^q}.
\]
The details are omitted. \( \square \)
Remark 3.1. Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. Assume that

$0 \leq \frac{a_i}{b_i^{q-1}} \leq 1$, for $i \in \{1, \ldots, n\}$.

If $p_i > 0$ for $i \in \{1, \ldots, n\}$, then for $x_i := p_i^{1/p} a_i$ and $y_i := p_i^{1/q} b_i$ we have

$$
\frac{x_i}{y_i^{q-1}} = \frac{p_i^{1/p} a_i}{(p_i^{1/q} b_i)^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{(q-1)/q} b_i^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{1/p} b_i^{q-1}} \leq \frac{a_i}{b_i^{q-1}} \in [0, 1]
$$

for $i \in \{1, \ldots, n\}$.

If we write the inequalities (25) and (26) for these choices, we get the weighted inequalities

$$(28) \quad 0 \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} - \left( \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} \right)^p \leq M_p \left(1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} \right) \left( \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} \right) \leq \frac{1}{4} M_p$$

and

$$(29) \quad 0 \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} - \left( \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} \right)^p \leq \frac{1}{4} \cdot \frac{1 - \left( \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} \right)^{p-1}}{1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}} \leq \frac{1}{4} M_p.$$

Theorem 3.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $p > 1, 0 < \alpha < R$ and $0 < x \leq 1$, then

$$(30) \quad 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left( \frac{f(\alpha x)}{f(\alpha)} \right)^p \leq M_p \left(1 - \frac{f(\alpha x)}{f(\alpha)} \right) \frac{f(\alpha x)}{f(\alpha)} \leq \frac{1}{4} M_p$$

and

$$(31) \quad 0 \leq 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left( \frac{f(\alpha x)}{f(\alpha)} \right)^p \leq \frac{1}{4} \cdot \frac{1 - \left( \frac{f(\alpha x)}{f(\alpha)} \right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \leq \frac{1}{4} M_p.$$

Proof. Let $m \geq 1$ and $0 < \alpha < R, 0 < x \leq 1$. If we write the inequality (22) for

$$w_j = \frac{a_j \alpha^j}{\sum_{k=0}^{m} a_k \alpha^k} \quad \text{and} \quad z_j := x^j \in [0, 1], \quad j \in \{0, \ldots, m\},$$

then we get

$$(32) \quad 0 \leq \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j - \left( \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j \right)^p.$$
\[ \leq M_p \left( 1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j \right) \]
\[ \leq \frac{1}{4} M_p. \]

Since all series whose partial sums involved in the inequality (32) are convergent, then by letting \( m \to \infty \) in (32) we deduce (30).

The inequality (31) follows from (23) in a similar way and the details are omitted. □

**Remark 3.2.** We observe that from (9) we have for \( p > 1 \)
\[ 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left[ \frac{f(\alpha x)}{f(\alpha)} \right]^p \leq \frac{1}{4} p, \]
which is not as good as the inequality
\[ 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left( \frac{f(\alpha x)}{f(\alpha)} \right)^p \leq \frac{1}{4} \times \begin{cases} 1 & \text{if } p \in (1, 2], \\ p-1 & \text{if } p \in (2, \infty). \end{cases} \]
that has been obtained in (30).

**Corollary 3.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with nonnegative coefficients and convergent on the open disk \( D(0, R) \) with \( R > 0 \) or \( R = \infty \). If \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( u, v > 0 \) with \( v^p \leq u^q < R \), then
\[ 0 \leq \frac{f(v^p)}{f(u^q)} - \left( \frac{f(uv)}{f(u^q)} \right)^p \leq M_p \left( 1 - \frac{f(uv)}{f(u^q)} \right) \frac{f(uv)}{f(u^q)} \leq \frac{1}{4} M_p \]
and
\[ 0 \leq \frac{f(v^p)}{f(u^q)} - \left( \frac{f(uv)}{f(u^q)} \right)^p \leq \frac{1}{4} \cdot \frac{1 - \frac{f(uv)}{f(u^q)}}{1 - \frac{f(uv)}{f(u^q)}} \leq \frac{1}{4} M_p. \]

**Proof.** Follows by taking into (30) and (31) \( \alpha = u^q \) and \( x = \frac{v}{u^{q/p}} \). The details are omitted. □

**Remark 3.3.** From (35) we have
\[ \left( \frac{f(uv)}{f(u^q)} \right)^p \leq \frac{f(v^p)}{f(u^q)} \leq \left( \frac{f(uv)}{f(u^q)} \right)^p + \frac{1}{4} M_p \]
and
\[ 0 \leq \left[ f \left( \frac{v^p}{1/p} \right) f \left( \frac{u^q}{1/q} \right) ^{1/q} \right] - f(uv) \leq \frac{1}{4^{1/p}} M_p \left( 1 - f(uv) \right) \]
provided that \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( u, v > 0 \) with \( v^p \leq u^q < R \).

These inequalities are better that the corresponding ones from Corollary 1.1.
If we take \( p = q = 2 \) in (37) and (38), then we get

\[
(39) \quad \left( \frac{f(uv)}{f(u^2)} \right)^2 \leq \frac{f(v^2)}{f(u^2)} \leq \left( \frac{f(uv)}{f(u^2)} \right)^2 + \frac{1}{4}
\]

and

\[
(40) \quad 0 \leq \left[ f(v^2) \right]^{1/2} \left[ f(u^2) \right]^{1/2} - f(uv) \leq \frac{1}{2} f(u^2),
\]

provided that \( u, v > 0 \) with \( v^2 \leq u^2 < R \).

**Example 3.1.** a) If we write the inequalities (30) and (31) for the function

\[
\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad z \in D(0, 1),
\]

then we have

\[
(41) \quad 0 \leq \frac{1 \cdot \alpha}{1 - \alpha x^p} - \left( \frac{1 - \alpha}{1 - \alpha x} \right)^p \leq M_p \frac{\alpha (1 - \alpha)(1 - x)}{(1 - \alpha x)^2} \leq \frac{1}{4} M_p
\]

and

\[
(42) \quad 0 \leq \frac{1 \cdot \alpha}{1 - \alpha x^p} - \left( \frac{1 - \alpha}{1 - \alpha x} \right)^p
\]

\[
\leq \frac{1}{4} \cdot \frac{1 - \alpha x}{\alpha (1 - x)} \left[ 1 - \left( \frac{1 - \alpha}{1 - \alpha x} \right)^{p-1} \right] \leq \frac{1}{4} M_p
\]

for any \( \alpha, x \in (0, 1) \) and \( p > 1 \).

b) If we write the inequalities (30) and (31) for the function \( \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \), \( z \in \mathbb{C} \), then we have

\[
(43) \quad 0 \leq \exp \left[ \alpha (x^p - 1) \right] - \exp \left[ \alpha (x - 1) \right]
\]

\[
\leq M_p \left( 1 - \exp \left[ \alpha (x - 1) \right] \right) \exp \left[ \alpha (x - 1) \right] \leq \frac{1}{4} M_p
\]

and

\[
(44) \quad 0 \leq \exp \left[ \alpha (x^p - 1) \right] - \exp \left[ \alpha (x - 1) \right]
\]

\[
\leq \frac{1}{4} \cdot \frac{1 - \exp \left[ \alpha (p - 1)(x - 1) \right]}{1 - \exp \left[ \alpha (x - 1) \right]} \leq \frac{1}{4} M_p
\]

for any \( \alpha > 0 \), \( p > 1 \) and \( x \in (0, 1) \).

**4. Logarithmic Inequalities**

If we consider the convex function \( f(t) = t \ln t, \ t > 0 \), then

\[
(45) \quad \Psi_{\ln(\cdot)} (t; m, M) = \frac{M \ln M - t \ln t}{M - t} - \frac{t \ln t - m \ln m}{t - m}
\]

for \( 0 < m < M < \infty \).

If we take \( M = 1 \) and \( m \to 0^+ \) in (45) then we have

\[
\lim_{m \to 0^+} \Psi_{\ln(\cdot)} (t; m, 1) = \lim_{m \to 0^+} \left[ \frac{-t \ln t}{1 - t} - \frac{t \ln t - m \ln m}{t - m} \right]
\]
\[
\frac{-t \ln t - t \ln t}{1 - t} = \frac{\ln t}{t - 1}
\]
for \( t \in (0, 1) \).

From (14) we have
\[
0 \leq \sum_{i=1}^{n} w_i x_i \ln x_i - \sum_{i=1}^{n} w_i x_i \ln \left( \sum_{i=1}^{n} w_i x_i \right) \leq \frac{1}{4} \left( \sum_{i=1}^{n} w_i x_i \right)^{-1}
\]
for any \( x_i \in (0, 1) \), \( w_i \geq 0 \) \((i = 1, \ldots, n)\) with \( W_n := \sum_{i=1}^{n} w_i = 1 \).

**Theorem 4.1.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with nonnegative coefficients and convergent on the open disk \( D(0, R) \) with \( R > 0 \) or \( R = \infty \). If \( 0 < \alpha < R, p > 0 \) and \( x \in (0, 1) \), then
\[
0 \leq \frac{p \alpha x^p f'(\alpha x^p)}{f(\alpha)} \ln x - \frac{f(\alpha x^p)}{f(\alpha)} \ln \left( \frac{f(\alpha x^p)}{f(\alpha)} \right) \leq \frac{1}{4} \frac{1}{1 - \frac{f(\alpha x^p)}{f(\alpha)}}.
\]

**Proof.** If \( 0 < \alpha < R \) and \( m \geq 1 \), then by (46) for \( x_j = (x^p)^j \), we have
\[
0 \leq \sum_{k=0}^{m} a_k \alpha^k \sum_{j=0}^{m} a_j \alpha^j x^p j \ln x^p j
\]
\[
- \sum_{k=0}^{m} a_k \alpha^k \sum_{j=0}^{m} a_j \alpha^j x^p j \ln \left( \sum_{k=0}^{m} a_k \alpha^k \sum_{j=0}^{m} a_j \alpha^j x^p j \right)
\]
\[
\leq \frac{1}{4} \left( \sum_{k=0}^{m} a_k \alpha^k \sum_{j=0}^{m} a_j \alpha^j x^p j \right)^{-1}
\]
where \( p > 0 \) and \( x \in (0, 1) \).

This is equivalent to
\[
0 \leq \frac{\ln x^p}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} j a_j \alpha^j (x^p)^j
\]
\[
- \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j \ln \left( \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j \right)
\]
\[
\leq \frac{1}{4} \left( \sum_{k=0}^{m} a_k \alpha^k \sum_{j=0}^{m} a_j \alpha^j (x^p)^j \right)^{-1}
\]

Since all series whose partial sums involved in the inequality (48) are convergent, then by letting \( m \to \infty \) in (48) we deduce (47). \(\square\)
Example 4.1. a) If we write the inequality (47) for the function \( \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \), \( z \in D(0, 1) \), then we have for \( \alpha, x \in (0, 1) \) and \( p > 0 \) that

\[
0 \leq \frac{p\alpha x^p (1 - \alpha)}{(1 - \alpha x^p)^2} \ln x - \frac{1 - \alpha}{(1 - \alpha x^p)} \ln \left( \frac{1 - \alpha}{1 - \alpha x^p} \right)
\]

\[
\leq \frac{1}{4} \frac{(1 - \alpha x^p) \ln \left( \frac{1 - \alpha x^p}{1 - \alpha} \right)}{\alpha (1 - x^p)}.
\]

b) If we write the inequality (47) for the function \( \exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \), \( z \in \mathbb{C} \), then we have

\[
0 \leq [p\alpha x^p \ln x - \alpha (x^p - 1)] \exp [\alpha (x^p - 1)] \leq \frac{1}{4} \frac{\alpha (1 - x^p)}{1 - \exp [\alpha (x^p - 1)]}
\]

for \( x \in (0, 1) \) and \( \alpha, p > 0 \).

5. Exponential Inequalities

If we consider the exponential function \( f : \mathbb{R} \to (0, \infty) \), \( f(t) = \exp(\beta t) \) with \( \beta > 0 \) then

\[
\Psi_{\exp(\beta)}(t; m, M) = \frac{\exp(\beta M) - \exp(\beta t)}{M - t} - \frac{\exp(\beta t) - \exp(m)}{t - m}.
\]

If we take \( M = 0 \) we have

\[
\Psi_{\exp(\beta)}(t; m, 0) = \frac{1 - \exp(\beta t)}{-t} - \frac{\exp(\beta t) - \exp(m)}{t - m}
\]

and letting \( m \to -\infty \), then we get

\[
\lim_{m \to -\infty} \Psi_{\exp(\beta)}(t; m, 0) = \frac{\exp(\beta t) - 1}{t} =: \Psi_{\exp(\beta)}(t)
\]

with \( t \in (-\infty, 0) \).

Since \( \exp(\beta \cdot) \) is convex on \( (-\infty, 0) \), then \( \Psi_{\exp(\beta)}(\cdot) \) is monotonic non-decreasing on \( (-\infty, 0) \) and then

\[
\sup_{t \in (-\infty, 0)} \Psi_{\exp(\beta)}(t) = \lim_{t \to 0^-} \frac{\exp(\beta t) - 1}{t} = \beta.
\]

From (13) we have

\[
0 \leq \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp \left( \beta \sum_{i=1}^{n} w_i x_i \right) \leq -\beta \sum_{i=1}^{n} w_i x_i
\]

for any \( x_i \leq 0, w_i \geq 0 \) \((i = 1, \ldots, n)\) with \( W_n := \sum_{i=1}^{n} w_i = 1 \).
Theorem 5.1. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series with nonnegative coefficients and convergent on the open disk \( D(0, R) \) with \( R > 0 \) or \( R = \infty \). If \( x \leq 0 \), \( \beta > 0 \) with \( \exp(\beta x) < R \) and \( 0 < \alpha < R \), then

\[
0 \leq \frac{f(\alpha \exp(\beta x))}{f(\alpha)} - \exp\left[\frac{\alpha \beta x f'(\alpha)}{f(\alpha)}\right] \leq -\frac{\alpha \beta x f'(\alpha)}{f(\alpha)}.
\]

Proof. If \( 0 < \alpha < R \) and \( m \geq 1 \), then by (51) for \( x_j = jx \), we have

\[
0 \leq \sum_{j=0}^{m-1} a_j \alpha^j \sum_{j=0}^{m} a_j \alpha^j [\exp(\beta x)]^j - \exp \left( \sum_{j=0}^{m} \frac{\beta x}{a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j \right)
\]

for \( x \in (-\infty, 0) \).

Since all series whose partial sums involved in the inequality (53) are convergent, then by letting \( m \to \infty \) in (53) we deduce (52). \( \square \)

Example 5.1. a) If we write the inequality (52) for the function \( \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \), \( z \in D(0, 1) \), then we have for \( x \leq 0 \), \( \beta > 0 \) and \( 0 < \alpha < 1 \), that

\[
0 \leq \frac{1}{1-\alpha \exp(\beta x)} - \exp \left( \frac{\alpha \beta x}{1-\alpha} \right) \leq -\frac{\alpha \beta x}{1-\alpha}.
\]

b) If we write the inequality (52) for the function \( \exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \), \( z \in \mathbb{C} \), then we have

\[
0 \leq \exp (\alpha [\exp(\beta x) - 1]) - \exp (\alpha \beta x) \leq -\alpha \beta x
\]

for any \( \alpha > 0 \) and \( x \leq 0 \), \( \beta > 0 \).

References

[1] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, Tamkang J. Math. 38 (2007), No. 1, pp. 37-49. Preprint RGMIA Res. Rep. Coll., 5 (2) (2002), Art. 14. [Online http://rgmia.org/papers/v5n2/RGIApp.pdf].

[2] P. Cerone and S. S. Dragomir, Some applications of de Bruijn’s inequality for power series. Integral Transform. Spec. Funct. 18 (6) (2007), pp. 387-396.

[3] S. S. Dragomir, Discrete Inequalities of the Cauchy-Bunyakovsky-Schwarz Type, Nova Science Publishers Inc., N.Y., 2004.

[4] S. S. Dragomir, Some reverses of the Jensen inequality with applications. Bull. Aust. Math. Soc. 87 (2013), no. 2, pp. 177–194.

[5] S. S. Dragomir, Inequalities for power series with nonnegative coefficients via a reverse of Jensen inequality, Preprint RGMIA Res. Rep. Coll., 17 (2014), Art. 47. [Online http://rgmia.org/papers/v17/v17a47.pdf].

[6] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen’s inequality and applications. Rev. Anal. Numér. Théor. Approx. 23 (1994), no. 1, pp. 71-78. MR1325895 (96c:26012).
Inequalities for Power Series with Nonnegative Coefficients

[7] A. Ibrahim and S. S. Dragomir, *Power series inequalities via Buzano’s result and applications*. Integral Transform. Spec. Funct. **22** (12) (2011), pp. 867-878.

[8] A. Ibrahim and S. S. Dragomir, *Power series inequalities via a refinement of Schwarz inequality*. Integral Transform. Spec. Funct. **23** (10) (2012), pp. 769-778.

[9] A. Ibrahim and S. S. Dragomir, *A survey on Cauchy–Bunyakovsky–Schwarz inequality for power series*, p. 247-p. 295, in G. V. Milovanović and M. Th. Rassias (eds.), Analytic Number Theory, Approximation Theory, and Special Functions, Springer, 2013. DOI 10.1007/978-1-4939-0258-3-10,

[10] A. Ibrahim, S. S. Dragomir and M. Darus, *Some inequalities for power series with applications*. Integral Transform. Spec. Funct. **24** (5) (2013), pp. 364-376.

[11] A. Ibrahim, S. S. Dragomir and M. Darus, *Power series inequalities related to Young’s inequality and applications*. Integral Transforms Spec. Funct. **24** (2013), no. 9, pp. 700-714.

[12] A. Ibrahim, S. S. Dragomir and M. Darus, *Power series inequalities via Young’s inequality with applications*. J. Inequal. Appl. **2013**, 2013:314, 13 pp.

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