SOME PROPERTIES FOR LIPSCHITZ STRONGLY $p$-SUMMING OPERATORS

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Abstract. We consider the space of molecules endowed with the transpose version of the Chevet-Saphar norm and we identify its dual space with the space of Lipschitz strongly $p$-summing operators. We also extend some old results to the category of Lipschitz mappings and we give a factorization result of Lipschitz $(p,r,s)$-summing operators.

0. Introduction

An area of research that is currently very active is the study of non-linear geometry, of Banach spaces or even of general metric spaces, by borrowing ideas and insights from the linear theory of Banach spaces. A very powerful tool in the latter is given by the class of $p$-summing operators, so that naturally has led several authors to investigate Lipschitz versions of them starting with the seminal paper [7]. J.A. Chavez Dominguez [2] has explored more properties of this class and has defined a norm on the space of molecules of which dual space coincides with the space of Lipschitz $p$-summing. The aim of this paper is to continue to study the same ideas developed in [2]. We try to give a similar treatment to the class of Cohen strongly $p$-summing operators. We consider the transpose version of the norm of Chevet-Saphar and we show that the dual of the space of molecules endowed with this norm coincides with the space of Lipschitz Cohen strongly $p$-summing. Some old results have been established, namely a version of Grothendieck’s theorem and the relationship between the Lipschitz mapping $T : X \to Y$ and its linearization $\hat{T} : \mathcal{F}(X) \to Y$ for the concepts of $p$-summing and Cohen strongly $p$-summing.

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The paper is organized as follows.

First, we recall some standard notations which will be used throughout. In section 1, we define a norm on the space of molecules that is inspired by the Chevet-Saphar norms. We give and prove an integral characterization of a linear form on this new space. In section 2, we give the definition of Lipschitz Cohen strongly $p$-summing for maps from a metric space to a Banach space. We show that this space is precisely the dual of the space of molecules described in section 1. Finally, in section 3 we study some basic properties of these Lipschitz Cohen strongly $p$-summing operators, drawing parallels to the linear theory. Some interesting results have been obtained namely the Grothendieck theorem and the relationship between the Lipschitz mapping and its linearization for certain concept of summability. The last part of this section is devoted to study a factorization result of Lipschitz ($p^*, r, s$)-summing operators like the one given in linear case. We show that the map $T$ is Lipschitz ($p^*, r, s$)-summing if and only if, $T$ can be written as $T = T_2 \circ T_1$ where $T_1$ is Lipschitz $r$-summing and $T_2$ is Lipschitz Cohen strongly $s^*$-summing.

We recall briefly some basic notations and terminology. In this paper we will always consider metric spaces with a distinguished point (pointed metric spaces) which we denote by 0. We denote by $X^#$ the Banach space of all Lipschitz functions $f : X \rightarrow \mathbb{R}$ which vanish at 0 under the Lipschitz norm given by

$$Lip(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.$$ 

Consider $1 \leq p \leq \infty$, for sequences of the same length $(\lambda_i)_{i=1}^n$ of real numbers and $(x_i)_{i=1}^n, (x'_i)_{i=1}^n$ of points in $X$, we denote their weak Lipschitz $p$-norm by

$$u_p^{Lip}((\lambda_i, x_i, x'_i)) = \sup_{f \in B_X^#} \left( \sum_{i=1}^n |\lambda_i (f(x_i) - f(x'_i))|^p \right)^{\frac{1}{p}}$$

We denote by $\mathcal{F}(X)$ the free Banach space over $X$, i.e., $\mathcal{F}(X)$ is the completion of the space

$$AE = \left\{ \sum_{i=1}^n \lambda_i m_{x_i, x'_i}, (\lambda_i)_{i=1}^n \subset \mathbb{R}, (x_i)_{i=1}^n, (x'_i)_{i=1}^n \subset X \right\}$$

with the norm

$$\|u\|_{\mathcal{F}(X)} = \inf \left\{ \sum_{i=1}^n |a_i| d(x_i, x'_i) : m = \sum_{i=1}^n a_i m_{x_i, x'_i} \right\}$$
where the function \( m_{x,x'} : X^\# \to \mathbb{R} \) is defined as follows
\[
m_{x,x'} (f) = f(x) - f(x').
\]

We have
\[
\mathcal{F}(X)^* = X^\#.
\]

For the general theory of free Banach spaces, see [1, 8, 9, 11, 13]. Let \( X \) be a pointed metric space and \( Y \) be a Banach space, we denote by \( \text{Lip}_0(X;Y) \) the Banach space of all Lipschitz functions \( T : X \to Y \) such that \( T(0) = 0 \) with pointwise addition and Lipschitz norm. We also denote by \( \mathcal{F}(X;Y) \) the vector space of all \( Y \)-valued molecules on \( X \), i.e.,
\[
\mathcal{F}(X;Y) = \left\{ \sum_{i=1}^n \lambda_i y_i m_{x_i,x'_i}, (\lambda_i)_i \subset \mathbb{R}, (y_i)_i \subset Y, (x_i)_i, (x'_i)_i \subset X \right\}.
\]

For any \( T \in \text{Lip}_0(X;Y^*) \), we denote by \( \varphi_T \) its correspondent linear function on \( \mathcal{F}(X;Y) \) defined by
\[
\langle \varphi_T, m \rangle = \langle T, m \rangle,
\]
where \( \langle ., . \rangle \) is a pairing of \( \text{Lip}_0(X;Y^*) \) and \( \mathcal{F}(X;Y) \) defined by
\[
\langle T, m \rangle = \sum_{x \in X} \langle T(x), m(x) \rangle.
\]

Therefore, for a general molecule \( m = \sum_{i=1}^n \lambda_i y_i m_{x_i,x'_i} \),
\[
\langle T, m \rangle = \sum_{i=1}^n \lambda_i \langle T(x_i) - T(x'_i), y_i \rangle. \quad (0.1)
\]

Let \( X \) be a pointed metric space and \( Y \) be a Banach space, note that for any \( T \in \text{Lip}_0(X;Y) \), then there exists a unique linear map (linearization of \( T \)) \( \hat{T} : \mathcal{F}(X) \to Y \) such that \( \hat{T}\delta_X = T \) and \( \| \hat{T} \| = \text{Lip}(T) \), i.e., the following diagram commutes
\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\delta_X \downarrow & \nearrow \hat{T} & \\
\mathcal{F}(X) & \\
\end{array}
\] (0.2)

where \( \delta_X \) is the canonical embedding so that \( \langle \delta_X(x), f \rangle = \langle m_{x,0}, f \rangle = f(x) \) for \( f \in X^\# \). Let \( Y \) be a Banach space, then \( B_Y \) denotes its closed unit ball and \( Y^\* \) its (topological) dual. Consider \( 1 \leq p \leq \infty \) and \( n \in \mathbb{N}^* \). We denote by \( l_p^n(Y) \) the Banach space of all sequences \( (y_i)_{i=1}^n \) in \( Y \) with the norm
\[
\|(y_i)_{i=1}^n\|_{l_p^n(Y)} = \left( \sum_{i=1}^n \|y_i\|^p \right)^{\frac{1}{p}},
\]
and by $l_p^{n,\omega}(Y)$ the Banach space of all sequences $(y_i)_{i=1}^n$ in $Y$ with the norm
\[ \|(y_i)_i\|_{l_p^{n,\omega}(Y)} = \sup_{y^* \in B_{Y^*}} (\sum_{i=1}^n |\langle y_i, y^*_i \rangle|^p)^{1/p}. \]

We also have
\[ \|(y_i)_i\|_{l_p^{n,\omega}(Y)} = \sup_{y^* \in B_{Y^*}} (\sum_{i=1}^n |\langle y_i, y^*_i \rangle|^p)^{1/p} = \|(y_i)_i\|_{l_p^{n,\omega}(Y^{**})}. \quad (0.3) \]

If $Y = \mathbb{R}$, we simply write $l_p^n$ and $l_p^{n,\omega}$.

1. The Chevet Saphar norms on the space of molecules

Let $E$, $F$ be Banach spaces, in [3, 12], the Chevet-Saphar norms $d_p$ and $g_p$ are defined on tensor product $E \otimes F$ for $1 \leq p \leq \infty$ as follow
\[ d_p (u) = \inf \left\{ \|(x_i)_i\|_{l_p^{n,\omega}(E)} \|(y_i)_i\|_{l_p^{n,\omega}(F)} \right\}, \]
where the infimum is taking aver all representations of $u$ of the form $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$. If we interchange the roles of the weak and strong norms in $d_p$, we obtain the transpose norm
\[ g_p (u) = \inf \left\{ \|(x_i)_i\|_{l_p^{n}(E)} \|(y_i)_i\|_{l_p^{n,\omega}(F)} \right\}. \]

For every $p$, we have
\[ g_p = d_p^t. \]

Inspired by the tensor norm $g_p$, we give a new norm on $\mathcal{F}(X; Y)$ like the one given by J.A. Chavez Dominguez in [2] for the norm $d_p$. Note that the space $\mathcal{F}(X; Y)$ plays the role of the tensor product in the linear theory. Let $p \in [1, \infty]$ and $m \in \mathcal{F}(X; Y)$. We consider for $m \in \mathcal{F}(X; Y)$
\[ \mu_p (m) = \inf \left\{ \|(\lambda_i d(x_i, x'_i))_i\|_p \|(y_i)_i\|_{l_p^{n,\omega}(Y)} \right\}, \]
where the infimum is taken over all representations of $m$ of the form
\[ m = \sum_{i=1}^n \lambda_i y_i m_{x_i x'_i} \]
with $x_i, x'_i \in X$, $y_i \in Y$, $\lambda_i \in \mathbb{R}$; $(1 \leq i \leq n)$ and $n \in \mathbb{N}^*$. 

**Proposition 1.1.** Let $X$ be a pointed metric space and $Y$ be a Banach space. Let $p \in [1, \infty]$, then $\mu_p$ is a norm on $\mathcal{F}(X; Y)$. 
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**Proof.** It is clear that for any molecule \( m \in \mathcal{F}(X; Y) \) and any scalar \( \alpha \) we have
\[
\mu_p(m) \geq 0 \quad \text{and} \quad \mu_p(\alpha m) = |\alpha| \mu_p(m).
\]
Let \( y^* \in Y^* \), \( f \in X^# \) and \( m \in \mathcal{F}(X; Y) \). Using the pairing formula (0.1)
\[
|\langle y^* f, m \rangle| = \left| \sum_{i=1}^{n} \lambda_i y^*(y_i)(f(x_i) - f(x'_i)) \right|
\]
\[
\leq \left( \sum_{i=1}^{n} |\lambda_i (f(x_i) - f(x'_i))|^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} |y^*(y_i)|^p \right)^{\frac{1}{p}}.
\]
\[
\leq \|y^*\| \text{Lip} (f) \|\text{d} (x_i, x'_i)\|_{L_p} \|y_i\|_{\ell^p_w(Y)}.
\]
By taking the infimum over all representations of \( m \), we obtain
\[
|\langle y^* f, m \rangle| \leq \|y^*\| \text{Lip} (f) \mu_p(m).
\]
Now, suppose that \( \mu_p(m) = 0 \), then for every \( y^* \in Y^* \) and \( f \in X^# \)
\[
0 = \langle y^* f, m \rangle = \sum_{i=1}^{n} \langle f, \lambda_i y^*(y_i) m_{x_i, x'_i} \rangle,
\]
by the duality between \( \mathcal{F}(X) \) and \( X^# \), the real-valued molecule \( y^* \circ m \)
is equal to 0 for all \( y^* \in Y^* \) and consequently \( m = 0 \). Let now \( m_1, m_2 \in \mathcal{F}(X; Y) \). By the definition of \( \mu_p \) we can find a representation
\[
m_1 = \sum_{i=1}^{l} \lambda_{1i} y_{1i} m_{x_{1i}, x'_{1i}},
\]
such that
\[
\|\text{d} (x_{1i}, x'_{1i})\|_{L_p} \|y_{1i}\|_{\ell^p_w(Y)} \leq \mu_p(m_1) + \varepsilon.
\]
Replacing \( \lambda_{1i} \) and \( (y_{1i}) \) by an appropriate multiple of them,
\[
\lambda_{1i} = \frac{\|y_{1i}\|_{\ell^p_w(Y)}^{\frac{1}{p}}}{\|\text{d} (x_{1i}, x'_{1i})\|_{L_p}^{\frac{1}{p}}}, \quad y_{1i} = \frac{\|\text{d} (x_{1i}, x'_{1i})\|_{L_p}^{\frac{1}{p}}}{\|y_{1i}\|_{\ell^p_w(Y)}^{\frac{1}{p}}},
\]
we can find
\[
\|\text{d} (x_{1i}, x'_{1i})\|_{L_p} \leq (\mu_p(m_1) + \varepsilon)^{\frac{1}{p}}, \quad \|y_{1i}\|_{\ell^p_w(Y)} \leq (\mu_p(m_1) + \varepsilon)^{\frac{1}{p}}.
\]
Similarly for \( m_2 \), we choose a representation
\[
m_2 = \sum_{i=1}^{s} \lambda_{2i} y_{2i} m_{x_{2i}, x'_{2i}},
\]
such that
\[ \|\lambda_2 d(x_{2i}, x'_{2i})\|_{l_p^m} \| (y_{2i})_i \|_{p^*(Y)} \leq \mu_p(m_2) + \varepsilon. \]

Again, replacing \((\lambda_{2i})\) and \((y_{2i})\) by an appropriate multiple of them as in the above, we find
\[ \|\lambda_2 d(x_{2i}, x'_{2i})\|_{l_p^m} \leq \left( \mu_p(m_2) + \varepsilon \right)^\frac{1}{p}, \quad \| (y_{2i})_i \|_{p^*(Y)} \leq \left( \mu_p(m_2) + \varepsilon \right)^\frac{1}{p^*}. \]

Now, we have
\[
\begin{align*}
& w_p(m_1 + m_2) \\
& \leq \left( \| \lambda_1 d(x_{1i}, x'_{1i})\|_{l_p^m} + \|\lambda_2 d(x_{2i}, x'_{2i})\|_{l_p^m} \right)^\frac{1}{p} \left( \| (y_{1i})_i \|_{p^*(Y)} + \| (y_{2i})_i \|_{p^*(Y)} \right)^\frac{1}{p^*} \\
& \leq \left( \mu_p(m_1) + \mu_p(m_2) + 2\varepsilon \right)^\frac{1}{p} \left( \mu_p(m_1) + \mu_p(m_2) + 2\varepsilon \right)^\frac{1}{p^*} \\
& \leq \mu_p(m_1) + \mu_p(m_2) + 2\varepsilon.
\end{align*}
\]

By letting \(\varepsilon\) tend to zero we obtain the triangle inequality for \(\mu_p\).

We denote by \(\mathcal{F}_{\mu_p}(X; Y)\) the completion of \(\mathcal{F}(X; Y)\) for the norm \(\mu_p\).

**Proposition 1.2.** Let \(X\) be a pointed metric space and \(Y\) be a Banach space. We have
\[ \mathcal{F}_{\mu_p}(X; Y) = \mathcal{F}(X) \overset{\mu_p}{\otimes} Y, \]
where \(g_p\) is the Chevet-Saphar norm defined as above.

**Proof.** We can establish the identification via the next linear map
\[ \varphi(m) = \sum_{i=1}^n \lambda_i y_i m_{x_i x'_i} = u = \sum_{i=1}^n \left( \lambda_i m_{x_i x'_i} \right) \otimes y_i. \]

Indeed, we have
\[
\begin{align*}
\mu_p(m) &= \inf \left\{ \| (\lambda_i d(x_i, x'_i))_i \|_{l_p^m} \| (y_i)_i \|_{p^*(Y)} \right\} \\
&= \inf \left\{ \left( \sum_{i=1}^n \| \lambda_i m_{x_i x'_i} \|_{l_p^m(\mathcal{F}(X))} \right)^\frac{1}{p} \| (y_i)_i \|_{p^*(Y)} \right\} \\
&= \inf \left\{ \left\| \left( \lambda_i m_{x_i x'_i} \right)_i \right\|_{l_p^m(\mathcal{F}(X))} \| (y_i)_i \|_{p^*(Y)} \right\} = g_p(u).
\end{align*}
\]

Now, it suffices to show that \(\varphi\) is onto. Let \(u = \sum_{i=1}^n v_i \otimes y_i \in \mathcal{F}(X) \overset{\mu_p}{\otimes} Y\) where \(v_i = \sum_{s_i=1}^{k_i} \lambda_{s_i} m_{x_{s_i} x'_{s_i}}\). We put for \(1 \leq i \leq n, m_i = \frac{\lambda_i}{\lambda_{s_i}} m_{x_{s_i} x'_{s_i}}\).
\[ \sum_{s_i=1}^{k_i} \lambda_{s_i} y_i m_{x_{s_i} x'_{s_i}} \text{ and } m = \sum_{i=1}^{n} m_i. \] We will verify that \( \varphi(m) = u. \) Indeed,

\[ \varphi(m) = \sum_{i=1}^{n} \varphi(m_i) = \sum_{i=1}^{n} \varphi \left( \sum_{s_i=1}^{k_i} \lambda_{s_i} y_i m_{x_{s_i} x'_{s_i}} \right) = \sum_{s_i=1}^{k_i} \sum_{i=1}^{n} \lambda_{s_i} m_{x_{s_i} x'_{s_i}} \otimes y_i = \sum_{i=1}^{n} v_i \otimes y_i = u. \]

In the next result we give a characterization of an element of the dual of the space \( \mathcal{F}_{\mu_p}(X;Y). \) For the proof, we need the following lemma.

**Lemma 1.3 (Ky Fan).** Let \( E \) be a Hausdorff topological vector space and \( C \) be a compact convex subset of \( E \). Let \( M \) be a set of functions on \( C \) with values in \( (-\infty, \infty] \) having the following:

(a) each \( f \in M \) is convex and lower semicontinuous;
(b) if \( g \in \text{conv}(M) \), there is an \( f \in M \) such that \( g(x) \leq f(x) \), for every \( x \in C \);
(c) there is an \( r \in \mathbb{R} \) such that each \( f \in M \) has a value not greater than \( r \).

Then, there is an \( x_0 \in C \) such that \( f(x_0) \leq r \) for all \( f \in M \).

**Theorem 1.4.** Let \( X \) be a pointed metric space, \( Y \) be a Banach space and \( C > 0 \). The following properties are equivalent.

1. The function \( \varphi \) is \( \mu_p \)-continuous on \( \mathcal{F}(X;Y) \), i.e.,

\[ |\varphi(m)| \leq C \mu_p(m) \text{ for all } m \in \mathcal{F}(X;Y). \tag{1.1} \]

2. For any representation of \( m \) of the form \( m = \sum_{i=1}^{n} \lambda_i y_i m_{x_{i} x'_{i}} \), we have

\[ \sum_{i=1}^{n} |\varphi(\lambda_i y_i m_{x_i x'_i})| \leq C \mu_p(m). \tag{1.2} \]

3. There exists a Radon probability \( \mu \) on \( B_Y^* \) such that for every atom of the form \( y m_{x x'} \),

\[ |\varphi(y m_{x x'})| \leq C d(x, x') \| y \|_{L_{\mu_p}(\mu)}. \tag{1.3} \]
Proof. (1) $\Rightarrow$ (2): Let $(\alpha_i)_{1 \leq i \leq n}$ be a scalar sequence. By (1.1), we have

$$\left| \varphi \left( \sum_{i=1}^{n} \alpha_i \lambda_i y_i m_{x_i} \right) \right| = \left| \sum_{i=1}^{n} \alpha_i \varphi \left( \lambda_i y_i m_{x_i} \right) \right|$$

$$\leq C \| (\alpha_i \lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} (Y)$$

$$\leq C \| (\alpha_i)_{i \in I} \|_{\infty} \| (\lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} (Y)$$

Taking the supremum over all sequences $(\alpha_i)_{1 \leq i \leq n}$ with $\| (\alpha_i)_{i} \|_{\infty} \leq 1$, we obtain (1.2).

(2) $\Rightarrow$ (3): Let $\varphi$ be a $\mu$-continuous function on $F(X; Y)$. Let $K = B_{Y^*}$. We consider the set $C$ of probability measures on $K$. It is a convex and compact subset of $C(K)^*$ endowed with its weak *-topology. Let $M$ be the set of all functions on $C$ with values in $\mathbb{R}$ of the form

$$\Psi \left( (\lambda_i), (x_i), (y_i) \right) (\mu)$$

$$= \sum_{i=1}^{n} \varphi \left( \lambda_i y_i m_{x_i} \right) - \sum_{i=1}^{n} \left( \frac{C}{p} \| (\lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} + \frac{C}{p} \| y_i \|_{L_p^\ast (\mu)} \right)$$

where $(x_i)_{1 \leq i \leq n}, (x'_i)_{1 \leq i \leq n} \subset X, (y_i)_{1 \leq i \leq n} \subset Y$ and $(\lambda_i)_{1 \leq i \leq n} \subset \mathbb{R}$. We will verify the assumptions of Ky Fan’s lemma:

(a) It is easy to see that each element of $M$ is convex and continuous on $C$.

(b) It suffices to show that $M$ is convex. Let $\Psi_1, \Psi_2$ in $M$ such that

$$\Psi_1 \left( (\lambda_i), (x_i), (y_i) \right) (\mu)$$

$$= \sum_{i=1}^{l} \varphi \left( \lambda_i y_i m_{x_i} \right) - \sum_{i=1}^{l} \left( \frac{C}{p} \| (\lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} + \frac{C}{p} \| y_i \|_{L_p^\ast (\mu)} \right),$$

and

$$\Psi_2 \left( (\lambda_i), (x_i), (y_i) \right) (\mu)$$

$$= \sum_{i=1}^{s} \varphi \left( \lambda_i y_i m_{x_i} \right) - \sum_{i=1}^{s} \left( \frac{C}{p} \| (\lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} + \frac{C}{p} \| y_i \|_{L_p^\ast (\mu)} \right),$$

It follows that

$$\alpha \Psi_1 + (1 - \alpha) \Psi_2$$

$$= \sum_{i=1}^{n} \varphi \left( \lambda_i y_i m_{x_i} \right) - \sum_{i=1}^{n} \left( \frac{C}{p} \| (\lambda_i d (x_i, x'_i))_{i \in I} y_i \|_{p, w} + \frac{C}{p} \| y_i \|_{L_p^\ast (\mu)} \right),$$
Let \( y \in \mathbb{R}^n \), and

\[
x_i = \begin{cases} x_{1i} & \text{if } 1 \leq i \leq l, \\
x_{2(i-l)} & \text{if } l + 1 \leq i \leq n,
\end{cases}
\]

\[
x'_i = \begin{cases} x'_{1i} & \text{if } 1 \leq i \leq l, \\
x'_{2(i-l)} & \text{if } l + 1 \leq i \leq n,
\end{cases}
\]

\[
y_i = \begin{cases} \frac{\alpha}{p} y_{1i} & \text{if } 1 \leq i \leq l, \\
(1 - \alpha)^{\frac{1}{p'}} y_{2(i-l)} & \text{if } l + 1 \leq i \leq n.
\end{cases}
\]

\[
\lambda_i = \begin{cases} \frac{1}{p} \lambda_{1i} & \text{if } 1 \leq i \leq l, \\
(1 - \alpha)^{\frac{1}{p'}} \lambda_{2(i-l)} & \text{if } l + 1 \leq i \leq n.
\end{cases}
\]

\[\Psi(y_\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\(\Psi(\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \)

\[\Psi(y^\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\[\Psi(\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\[\Psi(y^\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\[\Psi(\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\[\Psi(y^\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]

\[\Psi(y^\mu) = \frac{1}{p} \left( \frac{\alpha}{p} \right)^{\frac{1}{p'}} + \frac{1}{p} \left( \frac{\epsilon}{p'} \right)^{\frac{1}{p'}} \]
Then

\[ |\varphi(ym_{xx})| \leq C \left( \frac{1}{p^p} d(x, x')^p + \frac{\epsilon^{p^*}}{p^*} \|y\|_{L^p_p(K, \mu)}^{p^*} \right) \]

\[ \leq C \left( \frac{1}{p^p} \left( \frac{d(x, x')}{\epsilon} \right)^p + \frac{1}{p^*} (\epsilon \|y\|_{L^p_p(\mu)})^{p^*} \right) \]

We take the infimum over all \( \epsilon > 0 \), we find

\[ |\varphi(ym_{xx})| \leq C d(x, x') \|y\|_{L^p_p(\mu)}. \]

(3) \( \Rightarrow \) (1): Let \( m \in F(X; Y) \) such that

\[ m = \sum_{i=1}^{n} \lambda_i y_i m_{x_i x'_i} \]

By (1.3)

\[ |\varphi(m)| \leq \sum_{i=1}^{n} |\varphi(\lambda_i y_i m_{x_i x'_i})| \]

\[ \leq C \left( \sum_{i=1}^{n} \lambda_i d(x_i, x'_i) \|y_i\|_{L^p_p(\mu)} \right) (\text{by Hölder inequality}) \]

\[ \leq C \left( \sum_{i=1}^{n} \lambda_i d(x_i, x'_i) \||y_i||_{L^p_p(\mu)} \right) \left( \sum_{i=1}^{n} \|y_i\|_{L^p_p(\mu)} \right) \]

\[ \leq C \left( \sum_{i=1}^{n} \lambda_i d(x_i, x'_i) \|y_i\|_{L^p_p(\mu)} \right) \left( \sum_{i=1}^{n} \|y_i\|_{L^p_p(\mu)} \right) \]

as \( m \) is arbitrary, we find

\[ |\varphi(m)| \leq C \mu_p(m), \]

thus, \( \varphi \) is \( \mu_p \)-continuous function on \( F(X; Y) \).

2. The space of Lipschitz Cohen strongly \( p \)-summing

Let \( u : E \to F \) be a linear operator between Banach spaces. Cohen [4] has introduced the following concept: a linear operator \( u \) between Banach spaces \( E, F \) is strongly \( p \)-summing (or Cohen strongly \( p \)-summing) \((1 < p \leq \infty)\) if there is a positive constant \( C \) such that for all \( n \in \mathbb{N}^*, x_1, \ldots, x_n \in E \) and \( y_1^*, \ldots, y_n^* \in F^* \), we have

\[ \sum_{i=1}^{n} |\langle u(x_i), y_i^* \rangle| \leq C \left( \sum_{i=1}^{n} \|x_i\|_p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} \|y_i^*\|_{p^*_w(Y^*)} \right). \quad (2.1) \]

The smallest constant \( C \), which is noted by \( d_p(u) \), such that the inequality (2.1) holds, is called the strongly \( p \)-summing norm on the space \( \mathcal{D}_p(E; F) \) of all Cohen strongly \( p \)-summing linear operators from \( E \) into \( F \), which is a Banach space. If \( p = 1 \), we have \( \mathcal{D}_1(E; F) = \mathcal{B}(E; F) \), the space of all bounded linear operators from \( E \) to \( F \).
We give the same definition to the category of Lipschitz mappings.

**Definition 2.1.** Let $X$ be a pointed metric space and $Y$ be a Banach space. Let $T : X \to Y$ be a Lipschitz map. $T$ is Lipschitz Cohen strongly $p$-summing ($1 < p \leq \infty$) if there is a constant $C > 0$ such that for any $n \in \mathbb{N}^*$, $(x_i)_i, (x'_i)_i$ in $X$; $(y_i^*)_i$ in $Y^*$ and $(\lambda_i)_i$ in $\mathbb{R}_+^*$ ($1 \leq i \leq n$), we have

$$
\sum_{i=1}^{n} \lambda_i \left| \langle T(x_i) - T(x'_i), y_i^* \rangle \right| \leq C \left\| (\lambda_i d(x_i, x'_i))_i \right\|_{l_p^\infty} \left\| (y_i^*)_i \right\|_{l_{n,w}^p(Y^*)}.
$$

(2.2)

We denote by $\mathcal{D}_p^L(X; Y)$ the Banach space of all Lipschitz Cohen strongly $p$-summing and $d_p^L(T)$ its norm

$$
d_p^L(T) = \inf \{ C > 0, C \text{ verifying (2.2)} \}.
$$

As in the linear case, if $p = 1$ we have $\mathcal{D}_1^L(X; Y) = Lip_0(X; Y)$.

It is easy to show the next Proposition.

**Proposition 2.2.** Let $u$ be a bounded linear operator from $E$ into $F$ and $1 \leq p \leq \infty$. Then

$$
d_p(u) = d_p^L(u).
$$

**Theorem 2.3 (Pietsch’s domination).** Let $X$ be a pointed metric space and $Y$ be a Banach space. The following properties are equivalent.

1. The mapping $T$ belongs to $\mathcal{D}_p^L(X; Y)$.
2. For $(x_i)_i, (x'_i)_i$ in $X$; $(y_i^*)_i$ in $Y^*$ and $(\lambda_i)_i$ in $\mathbb{R}_+^*$ ($1 \leq i \leq n$), we have

$$
\sum_{i=1}^{n} \lambda_i \langle T(x_i) - T(x'_i), y_i^* \rangle \leq d_p^L(T) \left\| (\lambda_i d(x_i, x'_i))_i \right\|_{l_p^\infty} \left\| (y_i^*)_i \right\|_{l_{n,w}^p(Y^*)}.
$$

(2.3)

3. There exist a constant $C > 0$ and a Radon probability $\mu$ on $B_{Y^{**}}$ such that for all $x, x' \in X$ and $y^* \in Y^*$, we have

$$
\left| \langle T(x) - T(x'), y^* \rangle \right| \leq Cd(x, x') \left\| y^* \right\|_{L_p^\infty(\mu)}.
$$

(2.4)

In this case,

$$
d_p^L(T) = \inf \{ C > 0, C \text{ verifying (2.4)} \}.
$$

**Proof.**
(1) \implies (2) : Immediate.

(2) \implies (3) : Let \( T \in \mathcal{D}_p^L(X;Y) \). First, we can see \( T \) as a mapping defined from \( X \) into \( Y^{**} \). Let \( \varphi_T \) its correspondent linear function on \( \mathcal{F}(X;Y^*) \). Let \( m \in \mathcal{F}(X;Y^*) \), by (2.3) we have

\[
|\varphi_T(m)| = \left| \sum_{i=1}^{n} \lambda_i \langle T(x_i) - T(x'_i), y^*_i \rangle \right| \\
\leq C \|(\lambda_i d(x_i, x'_i))_i\|_{L_p} \|(y^*_i)_i\|_{L_{p,w}(Y^*)},
\]

hence, as \( m \) is arbitrary,

\[
|\varphi_T(m)| \leq C \mu_p(m),
\]

then \( \varphi_T \) is \( \mu_p \)-continuous on \( \mathcal{F}(X;Y^*) \), by Theorem 1.4 (3), we have for all \( x, x' \in X \) and \( y^* \in Y^* \)

\[
|\langle T(x) - T(x'), y^* \rangle| = |\varphi_T(y^* m_{x,x'})| \\
\leq C d(x, x') \|y^*\|_{L_{p,w}(\mu)}.
\]

(3) \implies (1) : Let \( T \) be a Lipschitz mapping verifies (2.4). For \( x, x' \in X \) and \( y^* \in Y^* \),

\[
|\varphi_T(y^* m_{x,x'})| = |\langle T(x) - T(x'), y^* \rangle| \\
\leq C d(x, x') \|y^*\|_{L_{p,w}(\mu)}.
\]

so, by Theorem 1.4 (1), \( \varphi_T \) is \( \mu_p \)-continuous on \( \mathcal{F}(X;Y^*) \) and by (1.2)

\[
\sum_{i=1}^{n} \lambda_i |\langle T(x_i) - T(x'_i), y^*_i \rangle| = \sum_{i=1}^{n} |\varphi_T(\lambda_i y^*_i m_{x,x_i})| \\
\leq C \mu_p(m) \\
\leq C \|(\lambda_i d(x_i, x'_i))_i\|_{L_p} \|(y^*_i)_i\|_{L_{p,w}(Y^*)},
\]

therefore \( T \) is in \( \mathcal{D}_p^L(X;Y) \) and

\[
d_p^L(T) \leq C. \quad \blacksquare
\]

The main result of this section is the following identification.

**Theorem 2.4.** Let \( X \) be a pointed metric space and \( Y \) be a Banach space. Let \( p \in [1, \infty] \). We have the isometric identification

\[
\mathcal{D}_p^L(X;Y^*) = \mathcal{F}_{\mu_p}(X;Y)^*.
\]
**Proof.** Let $T \in \mathcal{D}_p^L (X; Y^*)$ and $\varphi_T$ its correspondent linear function on $\mathcal{F} (X; Y)$. We will show that $\varphi_T$ is $\mu_p$-continuous. Let $m = \sum_{i=1}^{n} \lambda_i y_i m_{x,i} x'_i \in \mathcal{F} (X; Y)$. As $y_i$ is an element in $Y^{**}$, we obtain

$$|\varphi_T (m)| = \sum_{i=1}^{n} |\lambda_i (T (x_i) - T (x'_i), y_i)| \leq d_p^L (T) \sum_{i=1}^{n} |\lambda_i| d (x_i, x'_i) \|y_i\|_{L_p^\mu(B_{Y^{**}})}$$

by Hölder inequality

$$\leq d_p^L (T) \|(\lambda_i d (x_i, x'_i))_i\|_p \left(\sum_{i=1}^{n} |y_i|_{L_p^\mu(B_{Y^{**}})}\right)^{\frac{1}{p^*}}$$

$$\leq d_p^L (T) \|(\lambda_i d (x_i, x'_i))_i\|_p \sup_{y^{**} \in B_Y^{**}} (\sum_{i=1}^{n} |y^{**} (y_i)|^{p^*})^{\frac{1}{p^*}}$$

$$\leq d_p^L (T) \|(\lambda_i d (x_i, x'_i))_i\|_p \|y_i\|_{L_p^\mu(B_{Y^{**}})}.$$ 

Hence, as $m$ is arbitrary,

$$|\varphi_T (m)| \leq d_p^L (T) \mu_p (m),$$

then $\varphi_T$ is $\mu_p$-continuous on $\mathcal{F} (X; Y)$ and $\|\varphi_T\|_{\mu_p} \leq d_p^L (T).$ Conversely, let $\varphi \in \mathcal{F}_{\mu_p} (X; Y)^*$. Note that $\varphi$ can be identified with a mapping $T_\varphi : X \rightarrow Y^*$ via the formula

$$\langle T_\varphi (x), y \rangle = \varphi (ym_{x0}).$$

It is clear that $T_\varphi$ is Lipschitz. Indeed,

$$\|T_\varphi (x) - T_\varphi (x')\| \leq \sup_{y \in B_Y} |\langle T_\varphi (x) - T_\varphi (x'), y \rangle|$$

$$= \sup_{y \in B_Y} |\langle \varphi (ym_{x0}) - \varphi (ym_{x'0}) \rangle|$$

$$= \sup_{y \in B_Y} |\varphi (ym_{x'0})|$$

$$\leq \sup_{y \in B_Y} \|\varphi\| d (x, x') \|y\|$$

$$\leq \|\varphi\| d (x, x')$$

Now, let $(x_i)_i, (x'_i)_i \subset X, (y^{**}_i)_i \subset Y^{**}$ and $(\lambda_i)_i \subset \mathbb{R}^*_+ (1 \leq i \leq n).$ Consider the finite-dimensional subspaces

$$V = \text{span} (y^{**}_i)_{i=1}^{n} \subset Y^{**},$$

and $U = \text{span} (T_\varphi (x_i) - T_\varphi (x'_i))_{i=1}^{n} \subset Y^*$. Let $\varepsilon > 0$. By the principle of local reflexivity [5, page 178], there is an injective linear map $\phi : V \rightarrow Y$ such that

$$\max \{\|\phi\|, \|\phi^{-1}\|\} \leq 1 + \varepsilon,$$

and $\langle \phi (y^{**}), u^* \rangle = \langle y^{**}, u^* \rangle$ for any $y^{**} \in V$ and $u^* \in U$. Letting $y_i = \phi (y^{**}_i)$, the latter condition together with the continuity of $\varphi$ imply

$$|\sum_{i=1}^{n} \lambda_i (T_\varphi (x_i) - T_\varphi (x'_i), y^{**}_i)| = |\sum_{i=1}^{n} \lambda_i \langle T_\varphi (x_i) - T_\varphi (x'_i), y_i \rangle|$$

$$= |\varphi (\sum_{i=1}^{n} \lambda_i y_i m_{x,i} x'_i)| \leq \|\varphi\| \mu_p (\sum_{i=1}^{n} \lambda_i y_i m_{x,i} x'_i)$$

for any $y^{**} \in V$ and $u^* \in U$.
\[
\leq \| \varphi \|_{\mu_p} \| (\lambda_i d (x_i, x'_i))_i \|_{l_p^\psi} \| (y_i)_i \|_{l_{p^*, w}^* (Y)}.
\]

Noting that
\[
\| (y_i)_i \|_{l_{p^*, w}^* (Y)} = \sup_{\| y^* \| \leq 1} \left( \sum_{i=1}^n |y^*(y_i)|^{p^*} \right)^{\frac{1}{p^*}}
\]
\[
= \sup_{\| y^* \| \leq 1} \left( \sum_{i=1}^n |y^*(\phi (y^*_i))|^{p^*} \right)^{\frac{1}{p^*}}
\]
\[
\leq \| \phi \| \| (y^*_i)_i \|_{l_{p^*, w}^* (Y^{**})}
\]
\[
\leq (1 + \varepsilon) \| (y^*_i)_i \|_{l_{p^*, w}^* (Y^{**})}
\]

Since \( \varepsilon > 0 \) was arbitrary, letting it go to zero proves that \( T_\varphi \) is Lipschitz Cohen strongly \( p \)-summing and \( d_p^L (T_\varphi) \leq \| \varphi \|_{\mu_p} \).

3. SOME INCLUSION AND COINCIDENCE PROPERTIES

The aim of this section is to explore more properties of the class of Cohen Lipschitz \( p \)-summing operators. We start by showing the relationship between the Lipschitz mapping and its linearization for the concept of strongly \( p \)-summing. A similar characterization holds for Lipschitz compact operators (see [10]).

**Proposition 3.1.** The following properties are equivalent.
1. The mapping \( T \) belongs to \( D_p^L (X; Y) \).
2. The linear operator \( \hat{T} \) belongs to \( D_p (\mathcal{F} (X); Y) \).

In other words, \( D_p^L (X; Y) = D_p (\mathcal{F} (X); Y) \) holds isometrically.

**Proof.** First, suppose that \( T \in D_p^L (X; Y) \). Let \( m \in \mathcal{F} (X) \) and \( y^* \in Y^* \). Then
\[
\left| \left\langle \hat{T} (m) , y^* \right\rangle \right| \leq \sum_{i=1}^n |\lambda_i| \left| \langle T (x_i) - T (x'_i) , y^* \rangle \right|
\]
\[
\leq d_p^L (T) \sum_{i=1}^n |\lambda_i| d (x_i, x'_i) \| y^* \|_{L_{p^*, \mu}}
\]
as \( m \) is arbitrary, we obtain the Pietsch’s domination for \( \hat{T} \)
\[
\left| \left\langle \hat{T} (m) , y^* \right\rangle \right| \leq d_p^L (T) \| m \|_{\mathcal{F} (X)} \| y^* \|_{L_{p^*, \mu}}.
\]
By [4, Theorem 2.3.1], \( \hat{T} \in D_p(\mathcal{F}(X); Y) \) and
\[
d_p\left(\hat{T}\right) \leq d_p^L(T).
\]
Conversely, suppose that \( \hat{T} \in D_p(\mathcal{F}(X); Y) \). Let \( x, x' \in X \) and \( y^* \in Y^* \)
\[
|\langle T(x) - T(x'), y^* \rangle| = \left| \left\langle \hat{T}(m_{xx'}), y^* \right\rangle \right|
\leq d_p\left(\hat{T}\right) \| m_{xx'} \| \| y^* \|_{L_p^*(\mu)}
\leq d_p\left(\hat{T}\right) d(x, x') \| y^* \|_{L_p^*(\mu)}
\]
by Theorem 2.3, \( T \) is in \( D_p^L(X; Y) \) and
\[
d_p^L(T) \leq d_p\left(\hat{T}\right). \quad \blacksquare
\]

One of the nice results of Cohen is that a linear map \( u : E \to F \) between Banach spaces is strongly \( p \)-summing if and only if the adjoint map \( u^* : F^* \to E^* \) is \( p^* \)-summing. It would be interesting to point out that an analogous situation holds in the nonlinear case: if \( X \) is a metric space and \( Y \) is a Banach space, \( T : X \to Y \) is Lipschitz Cohen strongly \( p \)-summing if and only if the "adjoint" map \( T^\# |_{Y^*} : Y^* \to X^\# \) is \( p^* \)-summing (this map is actually just the linear adjoint of the linearization \( \hat{T} : \mathcal{F}(X) \to Y \)).

For Lipschitz \( p \)-summing operators we have the following result.

**Proposition 3.2.** Let \( 1 \leq p < \infty \). Let \( T : X \to Y \) be a Lipschitz map and \( \hat{T} \) its linearization. Suppose that \( \hat{T} \) is \( p \)-summing, then \( T \) is Lipschitz \( p \)-summing.

**Proof.** If \( \hat{T} \) is \( p \)-summing then it is Lipschitz \( p \)-summing, and by (0.2) \( T \) Lipschitz factors through \( \hat{T} \), so \( T \) is Lipschitz \( p \)-summing by the ideal property of Lipschitz \( p \)-summing operators. \( \blacksquare \)

**Remark 3.3.** The converse of the precedent Proposition is not true. Indeed, the canonical inclusion
\[
\delta_\mathbb{R} : \mathbb{R} \to \delta_\mathbb{R}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})
\]
is Lipschitz \( p \)-summing since it is Lipschitz equivalent to the identity \( id_\mathbb{R} \); hence \( \delta_\mathbb{R} : \mathbb{R} \to \mathcal{F}(\mathbb{R}) \) is Lipschitz \( p \)-summing. But the linearization of this map is the identity on \( \mathcal{F}(\mathbb{R}) \), which cannot be \( p \)-summing.
because $\mathcal{F}(\mathbb{R})$ is infinite-dimensional (isometric to $L_1(\mathbb{R})$, in fact, see [6, Corollary 7]).

**Corollary 3.4.** Let $X$ be a pointed metric space and $Y$ be an $L_p$-space ($1 \leq p < \infty$). Then

$$\mathcal{D}^L_p(X;Y) \subset \Pi^L_p(X;Y).$$

**Proof.** If $T$ is in $\mathcal{D}^L_p(X;Y)$, the Proposition 3.1 implies that $\hat{T} : \mathcal{F}(X) \to Y$ is Cohen strongly $p^*$-summing. By a result of Cohen [4, Theorem 3.2.3], $\hat{T}$ is $p$-summing and by Proposition 3.2, $T$ is Lipschitz $p$-summing with

$$\pi^L_p(T) \leq d^L_p(T). \quad \blacksquare$$

We recall that (see [2]) $\text{cs}_p(X;Y)$ is the space of molecules $\mathcal{F}(X;Y)$ endowed with the next norm

$$\text{cs}_p(m) = \inf \left\{ \| (\lambda_i \|y_i\|, \|w_{p^*}^{\text{Lip}}(\lambda_i^{-1}, x_i, x'_i) \|_{L_p} \right\},$$

where the infimum is taken over all representations of $m$ of the form

$$m = \sum_{i=1}^{n} y_i m_{x_i x'_i}$$

with $x_i, x'_i \in X$, $y_i \in Y$, $\lambda_i \in \mathbb{R}_+^*$; $(1 \leq i \leq n)$ and $n \in \mathbb{N}^*$.

**Corollary 3.5.** Let $X$ be a pointed metric space, $1 < p < \infty$ and $Y$ be an $L_p$-space. The identity mapping

$$\text{id} : \text{cs}_p(X;Y) \to \mathcal{F}_{\mu_p}(X;Y),$$

is continuous with $\|\text{id}\| \leq 1$.

**Proof.** Let $m \in \text{cs}_p(X;Y)$ and $\varphi \in \mathcal{F}_{\mu_p}(X;Y)^*$ such that

$$\|\varphi\|_{\mu_p} \leq 1.$$ 

By Theorem 2.4, we can identify $\varphi$ with a function $T_\varphi \in \mathcal{D}^L_p(X;Y^*)$ with $\|\varphi\|_{\mu_p} = d^L_p(T_\varphi)$. By the above result, $T_\varphi \in \Pi^L_p(X;Y^*)$, and Theorem 4.3 in [2] asserts that $\varphi \in \text{cs}_p(X;Y)^*$ with

$$\|\varphi\|_{\text{cs}_p(X;Y)^*} = \pi^L_p(T_\varphi) \leq d^L_p(T_\varphi) = \|\varphi\|_{\mu_p},$$
consequently,

\[
\mu_p(m) = \sup_{\|\varphi\| \leq 1} \|\varphi(m)\| \\
\leq \sup_{\|\varphi\|_{cs_p(X;Y)^*} \leq 1} \|\varphi(m)\| = \|m\|_{cs_p(X;Y)}
\]

In the next result, we give a version of Grothendieck’s Theorem, we mention that other nonlinear versions have already appeared in the literature (for example in [7]).

**Corollary 3.6.** (Grothendieck’s Theorem) Let \( X = l_1 \) (or any finite dimensional Banach space) and \( H \) be a Hilbert space. Then

\[
\Pi^L_1(X;H) = Lip_0(X;H).
\]

**Proof.** In this case, the free Banach space \( F(X) \) is isomorphic to \( L_1(\mathbb{R}) \) (see [6, Corollary 7 and 8]), then \( \hat{T}: F(X) \to H \) is 1-summing, consequently \( T \) is 1-summing. ■

In the last result, we consider Lipschitz \((p, r, s)\)-summing linear operators and we combine with Theorem 5.2 and 5.4 in [2] for giving a factorization result using the language of Lipschitz Cohen strongly \(p\)-summing operators.

We recall the following definition as stated in [2].

**Definition 3.7.** Let \( X \) be a pointed metric space and \( Y \) be a Banach space. Let \( T: X \to Y \) be a Lipschitz map. \( T \) is Lipschitz \((p, r, s)\)-summing if there is a constant \( C > 0 \) such that for any \( n \in \mathbb{N}^* \), \((x_i), (x'_i)\) in \( X \); \((y^*_i)\) in \( Y^* \) and \((\lambda_i), (k_i) \) in \( \mathbb{R}^*_+ \) \((1 \leq i \leq n)\), we have

\[
\|\lambda_i(T(x_i) - T(x'_i); y^*_i)\|_{p} \leq C \omega_{r}^{Lip} \left( \left( \lambda_i k_i^{-1}, x_i, x'_i \right) \right) \|k(y^*_i)\|_{l^{p,w}(Y^*)}. \tag{3.1}
\]

We denote by \( \Pi^L_{p,r,s}(X;Y) \) the Banach space of all Lipschitz \((p, r, s)\)-summing.

**Theorem 3.8.** Let \( p, r, s \in [1, \infty] \) such that \( \frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1 \). Let \( T \in Lip_0(X;Y) \), the following are equivalent.

1. The mapping \( T \) belongs to \( \Pi^L_{p^*,r^*,s}(X;Y) \).
2. There exist a constant \( C > 0 \) and regular Borel probability measures \( \mu \) and \( \nu \) on the weak\(^*\) compact unit balls \( B_{X^*}, B_{Y^{**}} \) such that for all
\( x, x' \in X \) and \( y^* \in Y^* \)

\[
|\langle T(x) - T(x'), y^* \rangle| \leq C \left( \int_{B_X^*} |f(x) - f(x')|^r \, d\mu(f) \right)^{\frac{1}{r}} \left( \int_{B_{Y^*}} |y^{**}(y^*)|^s \, d\nu(y^{**}) \right)^{\frac{1}{s}}.
\]

(3) There exist a metric space \( \tilde{X} \) and two Lipschitz mappings \( T_1, T_2 \) such that \( T_1 \in \Pi_{L^p}^L(X; \tilde{X}) \), \( T_2 \in D_{L^s}^{L^s}(\tilde{X}; Y) \) and \( T = T_2 \circ T_1 \).

Proof. (1) \( \iff \) (2): First, by (0.3) the norm of \( T \) as an element of \( \Pi_{L^p, r, s}^L(X; Y^*) \) is the same as its norm in \( \Pi_{L^p, r, s}^L(X; Y^{**}) \). Then, the equivalence follows from [2, Theorem 5.2 and 5.4] (specialized to the case \( E = Y^* \)).

(2) \( \implies \) (3): Suppose that \( T \) verifies (3.2). Then, we have the following diagram which is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow j_X & & \nearrow \overline{T} \\
\tilde{X} & \subset & L_r(\mu)
\end{array}
\]

where \( j_X : X \rightarrow L_r(\mu) \) is the isometric injection (which is Lipschitz \( r \)-summing), \( \tilde{X} = j_X(X) \) is a pointed metric space of which the metric is defined by

For \( \tilde{x}, \tilde{x}' \in \tilde{X} : d(\tilde{x}, \tilde{x}') = \|\tilde{x} - \tilde{x}'\|_{L_r(\mu)} \),

and its origin is \( j_X(0) \). We have \( T = \overline{T} \circ j_X \). The mapping \( \overline{T} \) is well defined and is Lipschitz Cohen strongly \( s^* \)-summing. Indeed,

\[
|\langle \overline{T}(\tilde{x}) - \overline{T}(\tilde{x}'), y^* \rangle| = |\langle T(x) - T(x'), y^* \rangle| \\
\leq C \left( \int_{B_X^*} |f(x) - f(x')|^r \, d\mu(f) \right)^{\frac{1}{r}} \|y^*\|_{L_s(\nu)} \\
\leq C \left( \int_{B_X^*} |(\tilde{x} - \tilde{x}') (f)|^r \, d\mu(f) \right)^{\frac{1}{r}} \|y^*\|_{L_s(\nu)} \\
\leq C \|\tilde{x} - \tilde{x}'\|_{L_r(\mu)} \|y^*\|_{L_s(\nu)}
\]

therefore by Theorem 2.3, \( \overline{T} \) is Lipschitz Cohen strongly \( s^* \)-summing.

(3) \( \implies \) (2): Given a factorization (3.3), consider the map \( \delta_{\tilde{X}} \circ T_1 : X \rightarrow \mathcal{F}(\tilde{X}) \) and \( \widehat{T}_2 : \mathcal{F}(\tilde{X}) \rightarrow Y \subset Y^{**} \), then \( T = \widehat{T}_2 \circ \delta_{\tilde{X}} \circ T_1 \), so, we obtain what we needed for [2, Theorem 5.4 (c)]. ■
SOME PROPERTIES FOR LIPSCHITZ STRONGLY $p$-SUMMING OPERATORS

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