ON CLASSIFICATION OF CONTINUOUS FIRST ORDER THEORIES

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Abstract

We give several new characterizations of IP (the independence property) and SOP (the strict order property) for continuous first order logic and study their relations to the function theory and the Banach space theory. We suggest new dividing lines of unstable theories by the study of subclasses of Baire-1 functions and argue why one should not expect a perfect analog of Shelah’s theorem, namely a theory is unstable iff it has IP or SOP, for real-valued logics, especially for continuous logic.

1 Introduction

In 1971 in [She71], Saharon Shelah proved the following crucial theorem:

Shelah’s Theorem[

A complete first order theory has the order property (OP) if and only if it has the independence property (IP) or the strict order property (SOP).

This theorem is of the great importance of Shelah’s classification of (classical) first order theories and is also the focal point of [Kha19b] and the present

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1In this article, when we refer to Shelah’s theorem, we mean this theorem (for classical logic).
paper. In [Ben13], Ben Yaacov claims that a straightforward translation of
the proof of Shelah’s theorem for classical first order logic would provide a
proof of this theorem for continuous logic of [BBHU08]. We conjecture that
this claim is not true and in Section 5 we explain the strategy for finding
a counterexample. We argue that finding a counterexample is related to a
specific class of Banach spaces, namely the nonreflexive Banach spaces not
containing $c_0$ or $\ell_1$. However, what is more important to us than rejecting
this claim is the classification of continuous first order theories and character-
ization of model theoretic properties in the terms of the function theory and
the Banach space theory. This classification even gives a better understand-
ing of classical logic and provides fundamental insights for it. We believe that
the fact that “there is no a perfect analog of Shelah’s theorem in continuous
logic” is not only bad, but also has “positive” aspects and achievements.

In [Kha19b], we studied the relationship between OP/IP/SOP in classi-
cal logic and subclasses of Baire-1 functions, and showed the correspondence
between Shelah’s theorem and the Eberlein–Smulian Theorem (cf. Fact 2.3
below). The present paper generalizes the results of [Kha19b] on a new
approach to Shelah classification theory to the continuous first order logic
[BBHU08]. This approach is based on the fact that the study of the model-
theoretic properties of formulas in ‘models’ instead of only these properties in
‘theories’ develops a sharper stability theory and establishes important links
between model theory and other areas of mathematics. We search to find an
answer to the question of whether there is a Shelah’s theorem for continuous
logic by exploring the exact position of ‘continuous’ model theoretic proper-
ties in the world of function spaces. However, the latter is the goal itself and
the former will be the result.

Continuous logic [BBHU08, BU10] is a generalization of the usual first or-
der order logic to study analysis structures such as metric spaces, measure algebra
and Banach spaces, including Banach lattice and $C^*$-algebra, etc. There are
several other good alternatives of usual first order logic, such as Henson’s logic
[Hen76, HI97] and the setting of compact abstract theories [Ben03], to study
metric structures, but continuous logic has many advantages over them. Al-
though this logic is appropriate for structures from functional analysis, it is
almost strikingly parallel to the usual first order logic and preserves many of
desirable characteristics of first order model theory, e.g., compactness theo-

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2See Section 5 for a discussion.
3See Section 5 for some non-parallelism.
rem, L"owenheim–Skolem theorems, omitting type theorem, and fundamental results of stability theory. As we will see in the present paper, it provides a natural setting for classification theory, in the sense of [She90].

Surprisingly, the study of desirable properties within a model and for the extensions/theory of a model in the Banach space theory has already been done by Bessaga–Pełczyński [BP58], Odell–Rosenthal [OR75], and Haydon–Odell–Rosenthal [HOR91]. Theorem A and Theorem B of [HOR91, pages 1, 2] are witnesses to the fact, as we shall see in Section 5 below. The first is about the properties within a model and the second is about the properties for a theory. In fact, Theorem B of [HOR91] is very similar to Shelah’s theorem above, namely a theory is unstable if and only if it has IP or SOP. In the following, we provide a brief explanation of it. Let $X$ be a compact metric space. We will shortly define proper subclasses $C(X) \subseteq B_{1/4}(X) \nsubseteq B_{1/2}(X)$ of (real-valued) Baire-1 functions on $X$, denoted by $B_1(X)$. (Cf. Definition 2.1 below.) Suppose that $(f_n)$ is a sequence of real-valued continuous function on $X$ which converges pointwise to a function $f$. Theorem B(a) of [HOR91] asserts that: If $f \in B_1(X) \setminus B_{1/2}(X)$, then $(f_n)$ has a subsequence whose spreading model is equivalent to the unit vector basis of $\ell_1$. Theorem B(b) of [HOR91] asserts that: If $f \in B_{1/4}(X) \setminus C(X)$, there exists $(g_n)$ a convex block subsequence of $(f_n)$, whose spreading model is equivalent to the summing basis for $c_0$. We can assume that each spreading model of a sequence $(f_n)$ is a sequence of formulas of the form $(\phi(a_n, y) : n < \omega)$ such that $a_n$’s are in the monster model of a theory. Then we will see that Theorem B(a) should be compared with Theorem 4.3(v) below; it corresponds to the independence property for a theory, and Theorem B(b) should be compared with Proposition 3.10 below; it is related to the strict order property for a theory. However, we will argue that there is no a perfect analog of Shelah’s theorem in continuous logic as well as the Banach space theory. In fact, the lack of a perfect analog arises from the existence of functions $B_{1/2} \setminus B_{1/4}$ so the two alternatives (a), (b) in Theorem B don’t cover all cases.

There are some considerations. The notions and results for the continuous

4Loosely (and possibly incorrect) speaking, a spreading model can be consider as a sequence $(\phi(a_n, y) : n < \omega)$ where $\phi$ is a formula and $(a_n)$ a sequence in an elementary extension of the original model in the sense of model theory.

5Roughly, a convex block subsequence of $(f_n)$ can be consider as a sequence of Boolean combinations of the instances of $\phi(a_n, y)$ where $a_n$ is in the monster model and $\phi$ a formula.

6See Section 5 for a discussion.
framework are not the literal translations of their classical logic analogs, and the proofs are often more complex than the classical ones. On the other hand, not all results in classification theory of classical (\(\{0,1\}\)-valued) logic are established here and in fact, classification in continuous case is finer.

The following is a summary of the main results of this paper:

- Theorems \(3.6\) and \(3.9\) on a generalization and refinement of Shelah’s theorem and a characterization of SOP in the terms of function spaces.
- Theorem \(4.3\) on a characterization of NIP (for a continuous theory) in the terms of function spaces and the Banach spaces theory, and its consequence (Corollary \(4.5\)) on a characterization of NIP formulas in classical logic.
- Theorem \(4.10\) on Baire-1/2 definability of NIP formulas in continuous logic and its consequence (Corollary \(4.11\)) on DBSC definability of NIP formulas in classical logic.
- Lemma \(3.8\) on a generalization of alternation number in model theory, Proposition \(3.10\) on embedding of \(c_0\) in the space of formulas in SOP theories, and Proposition \(5.2\) on many Shelah-like theorems. Also, the topics and theses presented in Chapter 5 are (in our view) illuminating and important.

We think that the benefits, results and importance of the present paper are as follows:

(i): This article is a step towards a classification theory for continuous first order theories similar to the classical case, which is important in itself.

(ii): The results of this study will lead to a better understanding of the classical model theory, and even provide new result for it. On the other hand, it explains why some theorems are not established in continuous logic and why such theorems have already existed in classical case.

\(^7\)As an example, NIP in continuous logic corresponds to Baire-1/2 functions which is strictly larger than DBSC functions. Recall from [Kha19b] that NIP corresponds to DBSC in classical logic.

\(^8\)In continuous logic, based on the fact that function spaces \(DBSC \subsetneq B_{1/2}\), there is a classification that corresponds to subclasses between \(DBSC\) and \(B_{1/2}\), but in classical case \(DBSC = B_{1/2}\). (cf. Section \(5\))
(iii): This study provides classifications for the Banach spaces that have not been previously seen or considered.

(iv): This study provides in-depth links to basic questions and topics in model theory, Banach space theory, and function theory, as well as creating new tools, new questions, and new insights into all of these areas.

The reader may add something else to the above list.

This paper is organized as follows: In Section 2 we give notation and preliminaries. In Section 3 we study SOP for continuous logic and give characterizations of SOP which are new (even for classical logic). In Section 4 we study NIP for continuous logic and give characterization of NIP in the terms of function spaces and definability of coheirs. In Section 5 we argue why there is no Shelah’s theorem for continuous logic, why there are pathological Banach spaces (i.e. not containing infinite-dimensional reflexive subspace or ℓ₁ or c₀) and what the relationship between these Banach spaces and the failure of Shelah’s theorem for continuous logic is. Finally, we give new classes of (continuous) theories using function spaces.

2 Notation and preliminaries

We work in continuous first order logic [BBHU08, BU10], and assume that the reader is familiar with this logic. As continuous logic is an extension of the classical (∋0, 1-valued) logic; thus our results hold in the latter case. Our model theory notation is standard, and a text such as [BBHU08] will be sufficient background for the model theory part of the paper. For the function theory part, read this paper with [HOR91] and [Kha19b] in your hand. Although, we provide some necessary functional analysis background.

2.1 Subclasses of Baire 1 functions

In this subsection we give definitions of the functions spaces with which we shall concerned, with some of elementary relations between them.

Let $X$ be a set and $A$ a subset of $\mathbb{R}^X$. The topology of pointwise convergence on $X$ is that inherited from the usual product topology of $\mathbb{R}^X$. A typical neighborhood of a function $f$ is

$$U_f(x_1, \ldots, x_n; \epsilon) = \{g \in \mathbb{R}^X : |f(x_i) - g(x_i)| < \epsilon \text{ for } i \leq n\},$$
where $\epsilon > 0$ and $\{x_1, \ldots, x_n\}$ is a finite subset of $X$. In this paper, the topology on subsets of $\mathbb{R}^X$ is pointwise convergence. Otherwise, we explicitly state what is our desired topology.

**Definition 2.1.** Let $X$ be a compact metric space. The following is a list of the spaces of real-valued functions defined on $X$. The important ones to note immediately are (iii) and (iv).

(i) $C(X)$ is the space of continuous real-valued functions on $X$.
(ii) $B_1(X)$, the first Baire class, is the space of functions $f : X \to \mathbb{R}$ such that $f$ is the pointwise limit of a sequence of continuous functions.
(iii) $DBSC(X)$, the difference of bounded semi-continuous functions, is the space of real-valued functions $f$ on $X$ such that there are semi-continuous functions $F_1, F_2$ with $f = F_1 - F_2$.
(iv) $B_{1/2}(X)$ or Baire-$1/2$, is the space of real-valued functions $f$ on $X$ such that $f$ is the uniform limit of a sequence $(F_n) \in DBSC$.
(v) $B_{1/4}(X)$ or Baire-$1/4$, is the space of real-valued functions $f$ on $X$ such that there are $C > 0$ and sequence $(F_n) \in DBSC$ such that (1) $(F_n)$ uniformly converges to $f$ and (2) for all $n$ there is a sequence $(f^n_i) \in C(X)$ with $f^n_i \to F_n$ pointwise and $\sum_{i=1}^\infty |f^n_i - f^{n+1}_i(x)| \leq C$.

We give some elementary relation between these function spaces. (However, we will use more content of [HOR91].)

**Fact 2.2.** (i) [HOR91, page 3] $f \in DBSC(X)$ if and only if there are a uniformly bounded sequence $(f_n) \in C(X)$ and $C > 0$ such that $f_n \to f$ pointwise and $\sum_{i=1}^\infty |f_n(x) - f_{n+1}(x)| \leq C$ for all $x \in X$.
(ii) [HOR91, page 3] The classes $DBSC$, $B_{1/4}$ and $B_{1/2}$ are Banach algebras with respect to suitable norms.
(iii) [HOR91, Proposition 5.1] For uncountable compact metric space $X$,

$$C(X) \subsetneq DBSC(X) \subsetneq B_{1/4}(X) \subsetneq B_{1/2}(X) \subsetneq B_1(X).$$

(iv) [CMR96, Proposition 2.2] Let $f$ be a simple real-valued function on $X$. Then $f \in B_{1/2}(X)$ if and only if $f \in DBSC(X)$.

Let $Y$ be a topological space. A subset $A \subseteq Y$ is **relatively compact** if it has compact closure in $Y$. We recall from [Kha19b, Fact 3.1] the well-known compactness theorem of Eberlein and Šmulian.

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9 A function is called simple if its range is a finite set.
**Fact 2.3** (Eberlein–Šmulian Theorem). Let $X$ be a compact Hausdorff space and $A$ a norm-bounded subset of $C(X)$. Then for the topology of pointwise convergence the following are equivalent:

1. $A$ is relatively compact in $C(X)$.
2. The following two properties hold:
   - $(RSC)$ Every sequence of $A$ has a convergent subsequence in $\mathbb{R}^X$, and
   - $(SCP)$ the limit of every convergent sequence of $A$ is continuous.

**Explanation.** See [Whi67] for a short proof of the Eberlein–Šmulian Theorem. Recall from [Fre06, 462F] that for each compact metric space $X$, the weak topology and the pointwise convergence topology on $C(X)$ are the same. Notice that, each sequence contains a subsequence converging to an element of $C(X)$ if and only if (i) each sequence has a convergent subsequence in $\mathbb{R}^X$, and (ii) the limit of every convergent sequence is continuous. So, (2) is precisely the condition $B$ in the main theorem of [Whi67]. Also, (1) is the condition $A$ in [Whi67]. Although, the Eberlein–Šmulian theorem is proved for arbitrary Banach spaces (even normed spaces), but it follows easily from the case $C(X)$ (see [Fre06, Theorem 462D]).

We want to recall a crucial result of Bourgain, Fremlin and Talagrand [BFT78]. First recall that a subset $A$ of a topological space $Y$ is called *relatively countably compact* if each sequence of elements of $A$ has a cluster point in $Y$.

**Definition 2.4.** A regular Hausdorff space $Y$ is *angelic* if (i) every relatively countably compact set $A$ is relatively compact, and (ii) every point in the closure of a relatively compact set $A$ is the limit of a sequence in $A$.

**Fact 2.5** (Bourgain–Fremlin–Talagrand). Let $X$ be a compact Hausdorff space and $A$ a countable norm-bounded subset of $C(X)$. Then for the topology of pointwise convergence the following are equivalent:

1. $A$ is relatively compact in $B_1(X)$.
2. The closure of $A$ in $\mathbb{R}^X$ is angelic.

\(^{10}\text{Relative sequential compactness in } \mathbb{R}^X\)
\(^{11}\text{Sequential completeness}\)
(3) If \( r < s \) and \((f_n)\) is a sequence in \( A \), then there are disjoint finite sets \( E, F \) such that

\[
\bigcap_{n \in E} \left[ f_n \leq r \right] \cap \bigcap_{n \in F} \left[ f_n \geq s \right] = \emptyset.
\]

(Here \( \left[ f \leq r \right] = \{ x : f(x) \leq r \} \) and \( \left[ f \geq s \right] = \{ x : f(x) \geq s \} \).)

**Proof.** The equivalence (2) \( \iff \) (3) follows from the equivalence (vi) \( \iff \) (viii) of [BFT78, Theorem 4D]. (Indeed, notice that since \( X \) is compact, we can work with *infinite* subsets \( E, F \subseteq \mathbb{N} \), so the condition (3) above and (viii) of Theorem 4D of [BFT78] are equivalent. See also the equivalence (iii) \( \iff \) (iv) of [Kha20, Lemma 3.12].)

(2) \( \Rightarrow \) (1) is evident, since the limits of sequences of continuous functions are Baire-1.

(1) \( \Rightarrow \) (2): By Theorem 3F of [BFT78], \( B_1(X) \) is angelic. As any subspace of an angelic space is angelic [Fre06, 462C(a)], the closure of \( A \) in \( R^X \) is angelic. (Note that since the closure of \( A \) in \( B_1(X) \) is compact, this closure is closed (and compact) in the space \( R^X \). In particular it implies that the closure of \( A \) in \( R^X \) is contained in \( B_1(X) \).)

\[
\begin{proof}
\end{proof}
\]

### 2.2 Continuous model theory

As mentioned model theory notation is standard [BBHU08]. We fix an \( L \)-formula \( \phi(x,y) \), a complete \( L \)-theory \( T \) and a subset \( A \) of the monster model of \( T \). We let \( \bar{\phi}(y,x) = \phi(x,y) \). We define \( p = tp_\phi(a/A) \) as the function \( \phi(p,y) : A \to [0,1] \) by \( b \mapsto \phi(a,b) \). This function is called a complete \( \phi \)-types over \( A \). The set of all complete \( \phi \)-types over \( A \) is denoted by \( S_\phi(A) \). We equip \( S_\phi(A) \) with the least topology in which all functions \( p \mapsto \phi(p,b) \) (for \( b \in A \)) are continuous. It is compact and Hausdorff, and is not necessarily totally disconnected. Let \( X = S_\phi(A) \) be the space of complete \( \phi \)-types on \( A \). Note that the functions \( q \mapsto \phi(a,q) \) (for \( a \in A \)) are continuous, and as \( \phi \) is fixed we can identify this set of functions with \( A \). So, \( A \) is a subset of all bounded continuous functions on \( X \), denoted by \( A \subseteq C(X) \). Similar to the above, one can define \( B_1(X) \), \( DBSC(X) \), \( B_{1/2}(X) \) and \( B_{1/4}(X) \).

The only additional thing we need to remark on is the following result (see [Kha20, Corollary 2.10] and [Iov99, Proposition 2.6] for real-valued logics and [Pil16, Proposition 2.2] for classical logic):

**Fact 2.6** (Grothendieck Criterion). Let \( (a_i) \) be a sequence in some model of \( T \) and \( \phi(x,y) \) a formula. Then the following are equivalent:
(i) There is no any sequence \((b_j)\) and \(r < s\) such that either \(\phi(a_i, b_j) \leq r \land \phi(a_j, b_i) \geq s\) holds iff \(i < j\), or \(\phi(a_j, b_i) \leq r \land \phi(a_i, b_j) \geq s\) holds iff \(i < j\).

(ii) For any sequence \((b_j)\), \(\lim_i \lim_j \phi(a_i, b_j) = \lim_j \lim_i \phi(a_i, b_j)\) when the limits on both sides exist.

(iii) Every function in the pointwise closure of \(A = \{\phi(a_i, y) : S_\bar{a} (\{a_i\}) \to [0, 1]: i < \omega\}\) is continuous. Equivalently, \(A\) is relatively pointwise compact in \(C(S_\bar{a} (\{a_i\}))\).

### 3 Strict order property

In this section and the next section we introduce and study the “correct” generalizations of SOP/IP for continuous logic. We argue that they are the appropriate generalizations for our purpose.

First, we need some notion and fact on the indiscernible sequences. Let \(\phi(x, y)\) be formula, \(n\) a natural number and \(A \subseteq [0, 1]\). We say that a condition \(\psi(x_1, \ldots, x_n)\) is a \(\phi\)-\(n\)-\(A\)-condition if it is of the forms \(\exists y (\land \phi(x_i, y) \leq r_i \land \land \phi(x_j, y) \geq r_j)\) or \(\forall y (\lor \phi(x_i, y) \leq r_i \lor \lor \phi(x_j, y) \geq r_j)\) where \(r_i, r_j \in A\). In this case, \(\psi(x_1, \ldots, x_n)\) has \(n\) free variables \(x_1, \ldots, x_n\) and a bounded variable \(y\). If \(M\) is a model of a theory and \(\bar{a} = (a_1, \ldots, a_n) \in M^n\), the \(\phi\)-\(n\)-\(A\)-type of \(\bar{a}\), denoted by \(tp_{\phi,n,A}(\bar{a})\), is the set of all \(\phi\)-\(n\)-\(A\)-conditions \(\psi(x)\) such that \(\models \psi(\bar{a})\). (Although these notions seems very restrictive and unnatural, they are very useful for proving the main theorem of this section, i.e., Theorem \[3.6\] below. Note that the notion \(\phi\)-\(n\)-\(A\)-type is completely different from the notion \(\phi\)-type we defined earlier.)

**Definition 3.1.** Let \(T\) be a complete \(L\)-theory, \(\phi(x, y)\) an \(L\)-formula, \(n\) a natural number, \(A \subseteq [0, 1]\), and \((a_i)\) a sequence in some model. (i) We say that the sequence \((a_i)\) is a \(\phi\)-\(n\)-\(A\)-indiscernible sequence (over the empty set) if for each \(i_1 < \cdots < i_n < \omega, j_1 < \cdots < j_n < \omega,\)

\[
    tp_{\phi,n,A}(a_{i_1}, \ldots, a_{i_n}) = tp_{\phi,n,A}(a_{j_1}, \ldots, a_{j_n}).
\]

(ii) We say \((a_i)\) is a \(\phi\)-\(n\)-indiscernible sequence, if it is \(\phi\)-\(n\)-\(A\)-indiscernible for \(A = [0, 1]\).

(iii) We say \((a_i)\) is a \(\phi\)-indiscernible sequence, if it is \(\phi\)-\(n\)-indiscernible for all \(n \in \mathbb{N}\).

In classical logic, the following result is well-known (cf. Theorem I.2.4 of [She90]). Although there is a technical consideration in continuous case,
Fact 3.2. Let $T$ be a complete $L$-theory, $\phi(x, y)$ an $L$-formula, $M$ a model of $T$, and $n$ a natural number.

(i) If $(a_i)_{i \in I}$ is an infinite sequence in $M$, there is an infinite $\phi$-indiscernible sequence $(c_i)$ (possibly in an elementary extension of $M$) such that for every finite set $A \subseteq [0, 1] \cap \mathbb{Q}$ and every $k \in \mathbb{N}$ there is a $\phi$-$k$-$A$-indiscernible subsequence $(b_i) \subseteq (a_i)$ such that $tp_{\phi,k,A}(c_1, \ldots, c_k) = tp_{\phi,k,A}(b_1, \ldots, b_k)$.

(ii) If $I \subset J$ are two (infinite) linear ordered sets and $(a_i)_{i \in I}$ is an infinite $\phi$-indiscernible sequence in $M$, there is a sequence $(b_j)_{j \in J}$ (possibly in an elementary extension of $M$) which is a $\phi$-indiscernible sequence.

Proof. (i): As Ramsey’s theorem (Theorem 2.1, appendix 2 of [She90]) works for “finitely” many types (colors) and continuous logic is $[0, 1]$-valued, we need to work with $A$-valued logics at the beginning, where $A$ is a finite subset of $[0, 1]$, and then generalize it to $[0, 1]$-valued case. Indeed, let $(A_k)$ be an increasing sequence of finite subsets of rational numbers in $[0, 1]$ such that its union is $[0, 1] \cap \mathbb{Q}$; that is, $A_k$’s are finite, $A_k \subseteq A_{k+1}$ and $\bigcup_k A_k = [0, 1] \cap \mathbb{Q}$. By induction on $k$, we obtain a sequence $((a_k^i) : k < \omega)$ of subsequences of $(a_i)$ as follows. Let $(a_0^i) = (a_i)$ and suppose that $(a_{k-1}^i)$ is given. By Ramsey’s theorem, let $(a_k^i)$ be a $\phi$-$k$-$A$-indiscernible subsequence of $(a_{k-1}^i)$. (Note that Ramsey’s theorem now works because $A_k$ and $k$ are finite and so $\phi$-$k$-$A$-typs are finite.) Now, by the compactness theorem, there is a sequence $(c_i)$ (possibly in a saturated model) such that for all $k \in \mathbb{N}$, $(c_i)$ is $\phi$-$k$-$A_k$-indiscernible and $tp_{\phi,k,A_k}(c_1, \ldots, c_k) = tp_{\phi,k,A_k}(a_1^k, \ldots, a_k^k)$. As $\bigcup_k A_k$ is dense in $[0, 1] \cap \mathbb{Q}$ the proof is completed.

The above result can be presented for a more general state, but that is enough for our purpose.

In the following, we introduce the notion $SOP$ for a continuous theory. We remind that the interpretation of $\phi(x, y)$ is a function into the interval $[0, 1]$, and identifying True with the value zero and False with one, the following notion is a generalization of the $SOP$ in classical ($\{0, 1\}$-valued) model theory.

Definition 3.3 ($SOP$ for a continuous theory). Let $T$ be a continuous theory and $\mathcal{U}$ the monster model of $T$.

(i) Let $\phi(x, y)$ be a formula and $\epsilon > 0$. We say the formula $\phi(x, y)$ (in continuous logic) has the $\epsilon$-strict order property ($\epsilon$-$SOP$) if there exists a
sequence \((a_i b_i : i < \omega)\) in \(U\) such that for all \(i < j\),
\[
\phi(U, a_i) \leq \phi(U, a_{i+1}) \quad \text{and} \quad \phi(b_j, a_i) + \epsilon \leq \phi(b_{i+1}, a_j).
\]

(ii) We say the formula \(\phi(x, y)\) (in continuous logic) has the strict order property (SOP) if it has \(\epsilon\)-SOP for some \(\epsilon > 0\).

(iii) We say that \(T\) has SOP if there is a formula \(\phi(x, y)\) which has SOP.

**Remark 3.4.** (i) It is easy to verify that, in classical logic, a theory has SOP (in the usual sense) if and only if it has SOP (as in Definition 3.3). Indeed, suppose that Definition 3.3(i) holds for \(\phi(x, y)\), \((a_i b_i : i < \omega)\), and \(\epsilon > 0\). Then, \(\phi(U, a_i) \supseteq \phi(U, a_{i+1})\) for all \(i\). For \(j = i + 1\), as the formula is \(\{0, 1\}\)-valued, we have \(\epsilon \leq \phi(b_{i+1}, a_{i+1})\), and so \(\models \neg \phi(b_{i+1}, a_{i+1})\). Therefore, \(\phi(b_{i+1}, a_i) \leq 1 - \epsilon\), and so \(\models \phi(b_{i+1}, a_i)\). To summarize, \(b_{i+1} \in \phi(U, a_i) \setminus \phi(U, a_{i+1})\) and \(\phi(U, a_i) \supseteq \phi(U, a_{i+1})\) for all \(i\). The converse is similar.

(ii) In [Ben13, Question 4.14], Ben Yaacov introduced a notion of SOP for continuous logic and claimed that Shelah’s theorem holds using this notion. As mentioned before, we will argue that this claim does not hold. On the other hand, almost two years after a version of this article was submitted to publish, we learned that James Hanson studied the notion of SOP in the sense of [Ben13] in his thesis [Han21, pages 485-490]. In fact, he showed that Ben Yaacov’s notion is equivalent to instability.

12Hanson informed us of this equivalence. Notice that he uses continuous connectives to defining of a quasi-metric with an infinite chain, even in the classical case.

13The terminology was borrow from [Bal19].

**Definition 3.5.** (i) We say that the independence property is uniformly blocked for \(\phi(x, y)\) on \((a_i)\), if there are a natural number \(N\) and a set \(E \subseteq\)
\( \{1, \ldots, N\} \) such that for each \( i_1 < \cdots < i_N < \omega \), and each real numbers \( r < s \) the following does not hold

\[
\exists x \left( \bigwedge_{j \in E} \phi(x, a_{i_j}) \leq r \land \bigwedge_{j \in N \setminus E} \phi(x, a_{i_j}) \geq s \right).
\]

(ii) We say \((b_i), (a_i)\) witness the order property with \( r < s \) if \( \phi(b_j, a_i) \leq r \) and \( \phi(b_i, a_j) \geq s \) for all \( i < j < \omega \).

The following result generalizes and refines a crucial theorem of Shelah that every unstable NIP theory has SOP.

**Theorem 3.6.** Let \( T \) be a complete continuous theory. The following are equivalent:

(i) \( T \) has SOP.

(ii) There are a formula \( \phi(x,y) \), sequences \((a_i), (b_i)\), and real numbers \( r < s \) such that

(1) the independence property is uniformly blocked for \( \phi(x,y) \) on \((a_i)\), and

(2) \((b_i), (a_i)\) witness the order property with \( r < s \).

Before giving the proof let us remark:

**Remark 3.7.** The condition (ii)(1) is related to a subclass of Baire-1 functions. (See Lemma 3.8 below.) This property is local and in one sense weaker than NIP (it can be hold for IP formulas) and in another sense stronger than NIP (not every NIP formula satisfies it). In contrast with continuous logic, NIP implies (ii)(1) in classical case. We will see shortly, the latter is the reason why one should not expect a result similar to Shelah’s theorem for continuous logic.

As Theorem 3.6 is a generalization of Theorem 2.6 of [Kha19b], for easier reading, see the proof of the latter. The proofs are basically the same, although the continuous case is more complex and has one more key point.

**Proof of Theorem 3.6.** (i) \( \Rightarrow \) (ii): Immediate by definition. Indeed, suppose that \( \phi(x,y) \) has \( \epsilon \)-SOP with \((b_i), (a_i)\) as witnesses. Since the sequence \((\phi(x,a_i))\) is increasing, let \( N = \{1,2\} \) and \( E = \{1\} \). It is easy to see that the conditions (1),(2) hold.
(ii) ⇒ (i): By Fact 3.2(i) we can assume that the sequence \((a_i : i < \omega)\) is \(\bar{\phi}\)-indiscernible. As the independence property is uniformly blocked for \(\phi(x,y)\) on \((a_i)\), there is a natural number \(N\) such that the conditions of Definition 3.5 hold for \((a_i)\). Therefore, for any \(s' < s\), there are some integer \(n\) and \(\eta : N \rightarrow \{0, 1\}\) such that \(\bigwedge_{i<n}\phi(x,a_i)^{\eta(i)}\) is inconsistent, where for a formula \(\varphi\), we use the notation \(\varphi^r\) to mean \(\varphi \leq r\) and \(\varphi^0\) to mean \(\varphi \geq s'\). (Recall that unlike classical model theory, in continuous logic Trus is 0 and False is 1.) Starting with that formula, we change one by one instances of \(\phi(x,a_i)\geq s' \land \phi(x,a_{i+1}) \leq r\) to \(\phi(x,a_i) \leq r \land \phi(x,a_{i+1}) \geq s'\). Finally, we arrive at a formula of the form \(\bigwedge_{i<n}\phi(x,a_i) \leq r \land \bigwedge_{n \leq i < N}\phi(x,a_i) \geq s'\). The tuple \(b_n\) satisfies that formula. Therefore, for such \(r < s'\), there is some \(i_0 < N, \eta_0 : N \rightarrow \{0, 1\}\) such that

\[
(*) \quad \bigwedge_{i \neq i_0, i_0+1} \phi(x,a_i)^{\eta_0(i)} \land \phi(x,a_{i_0}) \geq s' \land \phi(x,a_{i_0+1}) \leq r
\]

is inconsistent, but

\[
(**) \quad \bigwedge_{i \neq i_0, i_0+1} \phi(x,a_i)^{\eta_0(i)} \land \phi(x,a_{i_0}) \leq r \land \phi(x,a_{i_0+1}) \geq s'
\]

is consistent. Let us define \(\varphi'_{s'}(x, \bar{a}) = \bigwedge_{i \neq i_0, i_0+1} \phi(x,a_i)^{\eta_0(i)}\).

The new point is that by (ii)(1), since the number of Boolean combinations of the length \(N\) of formulas is finite, so there are \(\text{infinitely many}\) \(r + \frac{1}{n} = s' < s\) such that there exist a fixed pattern similar to \((*)\), \((**)\) where \((*)\) is inconsistent and \((**)\) is consistent for all such \(s'\)’s. For simplicity, we can assume from now on that \((*)\), \((**)\) is this pattern. Also, we fix a \(s_0\) with this pattern and consider \(\varphi_{s_0}(x, \bar{a})\) defined as above. Note that for these \(s'\)’s if \(s' \nless r\) then the statement

\[
(I) \quad \varphi_{s_0}(x, \bar{a}) \land \phi(x,a_{i_0}) \geq s' \land \phi(x,a_{i_0+1}) \leq r
\]

is still inconsistent. (For inconsistency of \((I)\), note that for \(s' < s_0\) since the pattern is fixed, so \(\varphi'(x, \bar{a}) \land \phi(x,a_{i_0}) \geq s' \land \phi(x,a_{i_0+1}) \leq r\) is inconsistent and therefore it is easy to check that \(\varphi_{s_0}(x, \bar{a}) \land \phi(x,a_{i_0}) \geq s' \land \phi(x,a_{i_0+1}) \leq r\) is inconsistent. Of course, in continuous logic, we can not conclude that \(\varphi_{s_0}(x, \bar{a}) \land \phi(x,a_{i_0}) \geq r \land \phi(x,a_{i_0+1}) \leq r\) is inconsistent.)

By Fact 3.2(ii), increase the sequence \((a_i : i < \omega)\) to an \(\tilde{\phi}\)-indiscernible sequence \((a_i : i \in \mathbb{Q})\). Then for \(i_0 \leq i < i' \leq i_0 + 1\), the formula \(\varphi_{s_0}(x, \bar{a}) \land\)
\( \phi(x, a_i) \leq r \land \phi(x, a_{i'}) \geq s_0 \) is consistent, but \( \varphi_{s_0}(x, \bar{a}) \land \phi(x, a_i) \geq s' \land \phi(x, a_{i'}) \leq r \) is inconsistent, for all \( s' > r \) where \( s_0 > s' \). Thus the formula \( \psi(x, y) = \varphi_{s_0}(x, \bar{a}) \land \phi(x, y) \) has the strict order property. \( \square \)

We want to give a characterization of SOP in the terms of function spaces. First we give a lemma which is a generalization and abstraction of the alternation number in model theory for continuous logic. Compare with Lemma 2.8 of [Kha19b].

**Lemma 3.8.** Let \((f_i)\) be a sequence of [0,1]-valued functions on a set \(X\). Suppose that the independence property is uniformly blocked on \((f_n)\). That is, there are a natural number \(N\) and a set \(E \subseteq \{1, \ldots, N\}\) such that for each \(i_1 < \cdots < i_N < \omega\), and real numbers \(r < s\),

\[
\bigcap_{j \in E} [f_{i_j} \leq r] \cap \bigcap_{j \in N \setminus E} [f_{i_j} \geq s] = \emptyset.
\]

(Here \([f \leq r] = \{x : f(x) \leq r\}\) and \([f \geq s] = \{x : f(x) \geq s\}\).) Then the following property hold:

(i) there is a real number \(C\) such that for all \(x \in X\),

\[
\sum_{i=1}^{\infty} |f_i(x) - f_{i+1}(x)| \leq C.
\]

Suppose moreover\(^1\) that \(X\) is a compact metric space and \(f_n\)'s are continuous, then the following property hold:

(ii) \((f_i)\) converges pointwise to a function \(f\) which is DBSC.

**Proof.** (i): We first consider discrete valued functions. For \(n \in \mathbb{N}\), set \(f_i^n(x) = \lfloor n f_i(x) \rfloor / n\) for all \(i < \omega\). (Here \(\lfloor x \rfloor\) is the integer floor function. That is, the greatest integer less than or equal to \(x\).) By the assumption, it is easy to check that for all \(x \in X\) and \(n \in \mathbb{N}\), \(\sum_{i=1}^{\infty} |f_i^n(x) - f_{i+1}^n(x)| \leq 2N - 2\). (This was shown for \(n = 1\) in [Kha19b].) Therefore, for all \(k \in \mathbb{N}\),

\[
\sum_{i=1}^{k} |f_i(x) - f_{i+1}(x)| = \lim_{n} \sum_{i=1}^{k} |f_i^n(x) - f_{i+1}^n(x)| \leq 2N - 2,
\]

\(^1\)As we study types over separable models, or equivalently countable models in the classical case, the spaces of types are Polish. This means that additional assumption is fulfilled throughout the paper.
and so \( \sum_{i=1}^{\infty} |f_i(x) - f_{i+1}(x)| \leq 2N - 2. \)

(ii): By Fact 2.2(i), \( f \in DBSC \) if and only if there are \( C > 0 \) and (uniformly bounded) sequence \( (g_i) \) of continuous functions such that \( f_n \to f \) pointwise and \( \sum_{i=1}^{\infty} |g_i(x) - g_{i+1}(x)| \leq C \) for all \( x \in X \). By (i), the latter condition holds and so the limit is in \( DBSC \).

Note that, in contrast with [Kha19b, Lemma 2.8], one cannot expect a converse to the direction (i) \( \Rightarrow \) (ii) of Lemma 3.8.

We can now give a characterization of \( SOP \) in the term of function spaces.

**Theorem 3.9.** Let \( T \) be a complete continuous theory and \( U \) its monster model. The following are equivalent:

(i) \( T \) has \( SOP \).

(ii) There are a formula \( \phi(x,y) \) and sequence \( (a_i) \) such that the independence property is uniformly blocked for \( \phi(x,y) \) on \( (a_i) \), but the sequence \( (\phi(x,a_i) : S_{\phi}(U) \to [0,1]) \) converges to a \( DBSC \) function which is not continuous.

**Proof.** (i) \( \Rightarrow \) (ii): Suppose that \( T \) has \( SOP \) witnessed by the formula \( \phi(x,y) \) and sequences \( (b_i), (a_i) \). Then the independence property is uniformly blocked for \( \phi(x,y) \) on \( (a_i) \). In this case, \( N = \{1,2\} \) and \( E = \{1\} \). By Lemma 3.8, the sequence \( (\phi(x,a_i) : S_{\phi}(U) \to [0,1]) \) converges to a function \( f \) which is \( DBSC \). (Note that the sequence is also increasing.) Since \( (b_i), (a_i) \) witness the order property and by Grothendieck’s criterion (Fact 2.6) the non order property and continuity are equivalent, so \( f \) is not continuous.

(ii) \( \Rightarrow \) (i): This is a result of Theorem 3.6, Lemma 3.8 and Grothendieck’s criterion.

Recall that the unit vector basis \( (e_n) \) of \( c_0 \) is defined by \( e_n = (0, \ldots, 0, 1, 0, \ldots) \), the 1 occurring in the \( n^{th} \) coordinate. The summing basis \( (s_n) \) is defined by \( s_n = e_1 + \cdots + e_n \).

**Proposition 3.10.** Let \( T \) be a complete theory and \( U \) the monster model. If \( T \) has \( SOP \), then there are a formula \( \phi(x,y) \) and sequence \( (a_i) \) such that the sequence \( (\phi(x,a_i) : S_{\phi}(U) \to [0,1]) \) is equivalent to the summing basis for \( c_0 \).

**Proof.** By Theorem 3.6, there are \( \epsilon > 0 \) and sequence \( (a_i b_i : i < \omega) \) in the monster model \( U \) such that for all \( i < j \), \( \phi(U,a_i) \leq \phi(U,a_{i+1}) \) and \( \phi(b_j, a_i) + \epsilon < \phi(b_i, a_j) \). Since the sequence \( (\phi(x, a_i) : S_{\phi}(U) \to [0,1]) \) is increasing, its limit is semi-continuous, and since the sequence has the order property, the limit is non-continuous. Now, one can either use Lemma 3.4
in [HOR91] or directly prove the desired result. Indeed, it is easy to check that for every $k$ and scalers $c_1, \ldots, c_k$, $\| \sum_{i=1}^k c_i \phi(x, a_i) \|_\infty \leq \| \sum_{i=1}^k c_i s_i \|_\infty$, where $(s_i)$ denotes the summing basis of $c_0$.

**Remark 3.11.** For some reasons one can not expect a converse to Proposition 3.10. Indeed, by Theorem 1.1 of [Far94], if a sequence $(f_i)$ is equivalent to the summing basis of $c_0$, then its limit is a Baire-$1/4$ function. Recall that, by Lemma 3.8 above, in the SOP case, the limit is DBSC, and this class is a proper subclass of Baire-$1/4$. Also, it does not seem that a weaker property than the property (ii) in Theorem 3.6 implies SOP. In contrast with SOP (for a theory), we will shortly see that NIP is equivalent to having no copy of $\ell_1$. So maybe one of the reasons why Rosenthal’s $\ell_1$-theorem [Ros74] is more important than his $c_0$-theorem [Ros94] is that the former is a first order property but the latter is not.

In [Kha19b] the notion “NSOP in a model” was introduced for classical logic. It is natural to generalize it to continuous logic.

**Definition 3.12 (NSOP in a model).** Let $T$ be a complete $L$-theory, $\phi(x, y)$ an $L$-formula, and $M$ a model of $T$.

(i) A set $\{a_i : i < \kappa\}$ of $(y)$-tuples from $M$ is said to be a **SOP-guarantee** for $\phi(x, y)$ if the following conditions (1),(2) hold, simultaneously.

1. the independence property is uniformly blocked for $\phi(x, y)$ on $(a_i : i < \kappa)$, and
2. there are $(b_i : i < \kappa)$ (in the monster model) and $r < s$ such that $(b_i), (a_i)$ witness the order property with $r < s$.

(ii) Let $A$ be a set of $(x)$-tuples from $M$. Then $\phi(x, y)$ has **SOP-guarantee in $A$** if there is a countably infinite sequence $(a_i : i < \omega)$ of elements of $A$ which is a SOP-guarantee for $\phi(x, y)$.

(iii) Let $A$ be a set of $(x)$-tuples in $M$. We say that $\phi(x, y)$ has resistance to SOP-guarantee in $A$ if it does not have SOP-guarantee in $A$.

(iv) $\phi(x, y)$ has resistance to SOP-guarantee in $M$ if it has resistance to SOP-guarantee in the set of $(x)$-tuples from $M$.

The following is Remark 2.13 of [Kha19b] for the classical first order logic, which is also true for continuous logic. For the sake of completeness we present it.
Remark 3.13. Let $T$ be a complete (continuous) $L$-theory, $\phi(x,y)$ an $L$-formula, $M$ a model of $T$, and $A$ a subset of $M$.

(i) If $\phi$ has SOP-guarantee in $A$, then a Boolean combination of instances of $\phi$ has SOP for the theory $T$. Of course, if $\phi$ has SOP for $T$, then it has SOP-guarantee in some models of $T$.

(ii) $\phi$ has NSOP for the theory $T$ iff it has resistance to SOP-guarantee in every model $M$ of $T$ iff it has resistance to SOP-guarantee in some model $M$ of $T$ in which all types over the empty set in countably many variables are realised.

(iii) If $\phi(x,y)$ has SOP-guarantee in some model $M$ of $T$, then there are arbitrarily long SOP-guarantees for $\phi$ (of course in different models).

4 Independence property

In this section we give some new characterizations of $NIP$ which are different from the classical case.

Definition 4.1. Let $T$ be a complete (continuous) theory and $\phi(x,y)$ a formula.

(i) We say $\phi(x,y)$ has the independence property (IP) if there are real numbers $r < s$ and a sequence $(a_i : i < \omega)$ in the monster model $U$ such that for all disjoint finite sets $E, F$ the following holds

$$\exists y \left( \bigwedge_{i \in E} \phi(a_i, y) \leq r \land \bigwedge_{i \in F} \phi(a_i, y) \geq s \right)$$

(ii) We say $\phi(x,y)$ has the non independence property (NIP) if it has not IP.

(iii) We say $T$ has NIP (IP) if every (some) formula has NIP (IP), respectively.

Definition 4.2. Let $T$ be a complete (continuous) theory, $\phi(x,y)$ a formula. We say the independence property is semi-uniformly blocked for $\phi(x,y)$ if for each $r < s$ there is a natural number $N_{r,s}$ and a set $E \subset \{1, \ldots, N_{r,s}\}$ such that for every sequence $(a_i)$ and each $i_1 < \cdots < i_{N_{r,s}} < \omega$, the following does not hold

$$\exists y \left( \bigwedge_{j \in E} \phi(a_{i_j}, y) \leq r \land \bigwedge_{j \in N \setminus E} \phi(a_{i_j}, y) \geq s \right).$$
It is easy to see that $\phi(x, y)$ has $NIP$ iff the independence property is semi-uniformly blocked for $\phi(x, y)$ (see also Theorem 4.3 below). Note that, as before mentioned in Remark 3.7, the condition (ii)(1) of Theorem 3.6 is stronger than $NIP$ in the sense that there is a natural number $N$ such that for all $r < s$, $N_{r,s} = N$.

### 4.1 Characterization of $NIP$

We first give a characterization of $NIP$ in the terms of function spaces. It seems that the conditions (iv), (v) are new to model theorists.

**Theorem 4.3** (Characterization of $NIP$). Let $T$ be a complete $L$-theory, $\phi(x, y)$ an $L$-formula and $U$ the monster model of $T$. Then the following are equivalent:

(i) $\phi$ has $NIP$ for $T$.

(ii) The independence property is semi-uniformly blocked for $\phi(x, y)$.

(iii) For any sequence (not necessarily indiscernible) $(a_i : i < \omega)$, there is a subsequence $(a_{j_i} : i < \omega)$ such that for each $b \in U$ the sequence $\phi(a_{j_i}, b)$ converges.

(iv) For any sequence (not necessarily indiscernible) $(a_i : i < \omega)$, there is a subsequence $(a_{j_i} : i < \omega)$ such that the sequence $\phi(a_{j_i}, y)$ converges to a function $f$ which is Baire-$1/2$.

(v) There is no sequence $(a_i)$ such that the sequence $(\phi(a_i, y) : S\phi(U) \to [0, 1])$ is equivalent in the supremum norm to the usual basis of $\ell_1$.

**Proof.** (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are evident.

(i) $\Rightarrow$ (ii) follows from the compactness theorem.

The equivalence (i) $\Leftrightarrow$ (iii) is folklore. (See Lemma 3.12 in [Kha20].)

The equivalence (iii) $\Leftrightarrow$ (v) follows from one of the prettiest theorems in Banach space theory, namely Rosenthal’s $\ell_1$-theorem [Ros74, Theorem 1].

(i) $\Rightarrow$ (iv): As (i) and (iii) are equivalent, there is a subsequence $(a_{j_i} : i < \omega)$ such that the sequence $\phi(a_{j_i}, y)$ converges to a function $f$. Suppose, for a contradiction, that $f$ is not Baire-$1/2$. So, by the equivalence (1) $\Leftrightarrow$ (5) of Proposition 2.3 of [HOR91], there are $r < s$ such that the ordinal index $\alpha(f; r, s)$ is infinite; equivalently, for all natural numbers $m$, $K_m(f; r, s) \neq \emptyset$.

(See [HOR91] page 7 for the definitions of $K_m$ and $\alpha$.) By Lemma 3.1 in [HOR91], for all $m$ there are $r < r' < s' < s$ and a subsequence $(\phi(a_{j_i}, y) :
\( i < \omega \) so that

\[
\exists y \left( \bigwedge_{i \in E} \phi(a_i, y) \leq r' \land \bigwedge_{i \in F} \phi(a_i, y) \geq s' \right)
\]

holds, for all disjoint subsets \( E, F \) of \( \{1, \ldots, m\} \). So, by the compactness theorem, \( \phi \) has \( IP \), a contradiction.

**Remark 4.4.** In general, it seems that one cannot expect the limits in Theorem 4.3(iv) belong to a proper subclass of Baire-1/2. Indeed, using Theorem 6.1 of [HOR91] it is easy to show that there are theory \( T \), NIP formula \( \phi(x, y) \) and sequence \( (a_i) \) as above such that the limit is not Baire-1/4. (Recall that DBSC \( \subsetneq \) Baire-1/4 \( \subsetneq \) Baire-1/2.)

For the sake of completeness we state the result separately for the classical (\( \{0, 1\} \)-valued) logic. The following was proved in [Kha19b, Proposition 2.14], although the condition (v) is new.

**Corollary 4.5 (Classical case).** Let \( T \) be a complete classical \( L \)-theory, \( \phi(x, y) \) an \( L \)-formula and \( U \) the monster model of \( T \). Then the following are equivalent:

(i) \( \phi \) has NIP for \( T \).
(ii) For any sequence (not necessarily indiscernible) \( (a_i : i < \omega) \), the independence property is uniformly blocked for \( \phi(x, y) \) on \( (a_i : i < \omega) \). (Cf. Definition 3.9.)
(iii) For any sequence (not necessarily indiscernible) \( (a_i : i < \omega) \), there is a subsequence \( (a_{j_i} : i < \omega) \) such that for each \( b \in U \) the sequence \( \phi(a_{j_i}, b) \) converges.
(iv) For any sequence (not necessarily indiscernible) \( (a_i : i < \omega) \), there is a subsequence \( (a_{j_i} : i < \omega) \) such that the sequence \( \phi(a_{j_i}, y) \) converges to a function \( f \) which is DBSC.
(v) There is no sequence \( (a_i) \) such that the sequence \( (\phi(a_i, y) : S_\phi(U) \to [0, 1]) \) is equivalent in the supremum norm to the usual basis of \( \ell_1 \).

**Proof.** Immediate by Theorem 4.3, since for simple functions we have DBSC = Baire-1/2 (Fact 2.2(iv)).

### 4.2 Baire-1/2 definability of coheirs

In [Kha20], the notion “NIP in a model” (in the framework of continuous logic), Definition 4.9 below, was introduced and applications of it, such as
Baire-1 definability of coheirs over NIP models, were presented. In [KP18], more applications of this notion were presented for classical logic. In this section we provide a stronger result using a stronger assumption; that is, we replace the “NIP in a model” with the “NIP for a theory.” We want to show that coheirs in NIP theories are Baire-1/2 definable (cf. Definition 4.7 below).

Let $M$ be a separable model, $M^*$ a saturated elementary extension of $M$, $X = \mathcal{S}_a(M)$ and $A = \{ \phi(a,y) : X \to [0,1] : a \in M \}$. A type $p(x) \in \mathcal{S}_a(M^*)$ is called a coheir (of a type) over $M$ if it is approximately finitely satisfiable in $M$; that is, for every condition $\varphi = 0$ in $p(x)$ (i.e. $\varphi(p) = 0$) and every $\epsilon > 0$, the condition $|\varphi| \leq \epsilon$ is satisfiable in $M$. (Clearly in the classical case (i.e. when $\epsilon = 0$ and $\varphi = 0$) this notion is exactly the same as the usual notion of coheir.) Alternatively, with the notation mentioned in Section 2.2, $p(x) \in \mathcal{S}_a(M^*)$ is a function from $M^*$ to $[0,1]$, defined by $b \mapsto \phi(p,b)$. Then, $p$ is a coheir of a type over $M$ if for each $\epsilon > 0$ and each $b_1, \ldots, b_n \in M^*$ there is some $a \in M$ such that $|\phi(p,b_i) - \phi(a,b_i)| < \epsilon$, $i \leq n$.

Let $p(x)$ be a coheir of a type over $M$, and $b_1, b_2 \in M^*$. If $b_1, b_2$ have the same $\phi$-type over $M$, then $\phi(p,b_1) = \phi(p,b_2)$. Indeed, as $p(x)$ is approximately finitely satisfiable in $M$, for an arbitrary $\epsilon > 0$, there is some $a \in M$ such that $|\phi(p,b_i) - \phi(a,b_i)| < \epsilon$, $i \leq 2$. Therefore, $|\phi(p,b_1) - \phi(p,b_2)| \leq |\phi(p,b_1) - \phi(a,b_1)| + |\phi(a,b_1) - \phi(a,b_2)| + |\phi(p,b_2) - \phi(a,b_2)| < 2\epsilon$. As $\epsilon$ is arbitrary, $\phi(p,b_1) = \phi(p,b_2)$. For simplicity, for all $b \in M^*$, we will write $\phi(p,b) = \phi(p,q)$ where $q = tp_{\phi}(b/M) \in X$. To summarize, $p(x)$ is a function in $[0,1]^X$, defined by $q \mapsto \phi(p,q)$ as above.

By Remark 3.19 of [Kha20], similar to the classical logic, there is a correspondence between the coheirs $p(x)$ of types over $M$ and the functions in the pointwise closure of $A$. (See also the explanation before Definition 3.18 of [Kha20].) For the seek of completeness we present it and give a proof.

**Fact 4.6.** Let $X$ be as above and $f \in [0,1]^X$. Then $f$ belongs to the pointwise closure of $A$ iff there is $p(x) \in \mathcal{S}_a(M^*)$, such that $p(x)$ is approximately finitely satisfiable in $M$ and $f(q) = \phi(p,q)$ for all $q \in X$.

**Proof.** Recall that a typical neighborhood of $f$ is of the form $U_f(q_1, \ldots, q_n; \epsilon) = \{ g \in [0,1]^X : |f(q_i) - g(q_i)| < \epsilon \text{ for } i \leq n \}$, where $\epsilon > 0$ and $\{q_1, \ldots, q_n\}$ is a finite subset of $X$. Therefore, $f$ belongs to the pointwise closure of $A$ iff for each $U_f(q_1, \ldots, q_n; \epsilon)$ there is a $\phi(a,y) \in A$ (i.e. $a \in M$) such that $|f(q_i) - \phi(a,q_i)| < \epsilon$ for $i \leq n$ iff $p(x) \in \mathcal{S}_a(M^*)$ is approximately finitely satisfiable in $M$, where $\phi(p,q) = f(q)$ for all $q \in X$. (In this case, recall that
\( \phi(x, q) = r \) belongs to \( p(x) \) (or \( \phi(p, q) = r \)) iff \( \phi(p, b) = r \) for all \( b \in M^* \) such that \( q = tp_\phi(b/M) \in X. \)

**Definition 4.7** (Baire-1/2 definability). Let \( M \) be a separable model, \( M^* \) a saturated elementary extension of it, and \( \phi(x, y) \) a formula. Let \( p(x) \in S_\phi(M^*) \) be approximately finitely satisfiable in \( M \). We say that \( p \) is Baire-1/2 (resp. DBSC) definable over \( M \) if there is a sequence \( \phi(a_n, y), a_n \in M \), such that the sequence \( \phi(a_n, y) \) of functions on \( S_\phi(M) \) converges pointwise to a function \( f \) which is Baire-1/2 (resp. DBSC) and \( \phi(p, q) = f(q) \) for all \( q \in S_\phi(M) \).

**Remark 4.8.** (i) Recall that, in the above definition, as \( p(x) \) is approximately finitely satisfiable in \( M \), if \( b_1, b_2 \in M^* \) have the same \( \hat{\phi} \)-type over \( M \), then \( \phi(p, b_1) = f(tp(b/M)) = \phi(p, b_2) \).

(ii) Note that, as \( f \) is a Baire-1 function, for every open set \( O \subseteq \mathbb{R} \), the set \( f^{-1}(O) \) is \( F_\sigma \), i.e. it is a countable union of closed sets. Of course, Baire-1/2 definability says more because Baire-1/2 \( \subseteq \) Baire-1. Indeed, in classical logic, \( f^{-1}(O) \) is a disjoint differences of closed sets \( W_1, \ldots, W_m \).

(iii) Note that Baire-1/2 (or DBSC) definability is a generalization of the usual definability. That is, a type is called definable if the function \( f \) above is continuous. Pillay [Pil16] showed that “stability in a model” is equivalent to the definability of “coheirs” [Kha19b]. In [Kha19b], this result was generalized to “\( NIP \) in a model”. In [Kha19b], the connection between \( NIP \) for classical theories and DBSC functions was studied, although the equivalence of \( NIP \) (for theories) and DBSC definability of coheirs was not studied. We do this work in the present paper; moreover for continuous logic.

Because we will refer to a proof for the notion ‘\( NIP \) in a model’ ([Kha20, Definition 3.11]) we recall this notion:

**Definition 4.9** (\( NIP \) in a model). Let \( M \) be a model, and \( \phi(x, y) \) a formula. We say that \( \phi(x, y) \) has \( NIP \) in \( M \) if for each sequence \( (a_n) \subseteq M \), and \( r > s \), there are some finite disjoint subsets \( E, F \) of \( \mathbb{N} \) such that the following does not hold

\[
\exists y \left( \bigwedge_{i \in E} \phi(a_i, y) \leq r \land \bigwedge_{i \in F} \phi(a_i, y) \geq s \right).
\]

\[\text{Note that such functions are Baire-1 but not converse. See [CMR96] Proposition 2.2 and [Kha19b] Remark 2.15.}\]

\[\text{Unfortunately, some authors have mistakenly claimed that “stability in a model” is equivalent to the definability of “types”, and thus made other false claims. The reason for this is probably that they were not aware of Fact 4.6 above.}\]
In Lemma 3.12 of [Kha20], there is a long list on equivalences of this notion. The only additional thing we need to remember is that NIP (for a theory) is stronger than NIP in a model. We are now ready to give definability result. The proof uses Theorem 4.3 and some crucial results of [BFT78].

**Theorem 4.10.** Let \( T \) be a theory and \( \phi(x,y) \) a formula. The following are equivalent.

(i) \( \phi(x,y) \) has NIP.

(ii) For any separable model \( M \) and saturated elementary extension \( M^* \) of it, whenever \( p(x) \in S_{\phi}(M^*) \) is approximately finitely satisfiable in \( M \) then it is Baire-1/2 definable over \( M \).

(iii) For any separable model \( M \) and saturated elementary extension \( M^* \) of it, the number of \( p(x) \in S_{\phi}(M^*) \) which is approximately finitely satisfiable in \( M \) is \(< 2^{2\omega} \).

**Proof.** As \( \phi \) has NIP (for the theory) iff for every separable model \( M \) it has NIP in \( M \), the equivalence (i) \( \iff \) (ii) is a straightforward adaptation of the argument of Proposition 2.6 of [Kha17] to continuous logic. Although there are some considerations. Indeed, recall that for separable model \( M \), the space \( X = S_\phi(M) \) is compact and Polish. Let \( M_0 \) be a countable dense subset of \( M \) and \( A_0 = \{ \phi(a,y) : X \rightarrow [0,1] : a \in M_0 \} \). As \( \phi(x,y) \) has a modulus of uniform continuity (cf. [BBHU08, Theorem 3.5]) and the set \( X_0 = \{ q \in X : q = tp_{\phi}(b/M) \text{ for some } b \in M \} \) is dense in \( X \), it is easy to check that \( A_0 \) and \( A = \{ \phi(a,y) : X \rightarrow [0,1] : a \in M \} \) have the same pointwise closure. So we can work with the countable set \( A_0 \), and the countability assumption of Fact 2.5 is fulfilled throughout the proof. By Fact 4.6, any type \( p(x) \in S_{\phi}(M^*) \) is a coheir iff there is a function \( f \) in the closure of \( A \) such that \( f(q) = \phi(p,q) \) for all \( q \in X \). By the equivalences (1) \( \iff \) (2) \( \iff \) (3) of Fact 2.5, \( \phi \) has NIP in \( M \) iff the closure of \( A \) in \([0,1]^X\) is angelic iff every \( f \) in the closure of \( A \) is Baire-1. The only additional point is that by Theorem 4.3(iv) the function \( f \), in the pointwise closure of \( A \), that defines \( p(x) \) is Baire-1/2.

(ii) \( \Rightarrow \) (iii) is evident, since every Baire-1/2 function is the limit of a sequence of continuous functions.

(iii) \( \Rightarrow \) (i) is the usual proof of many coheirs for formulas with the independence property. Namely, Suppose rather that (iii) fails; that is, there are
$r < s$ and sequence $(a_n)$ in a separable model $M$ such that
\[
D(I) = \{ b \in S_\phi(M) : \bigwedge_{n \in I} \phi(a_n, b) \leq s \land \bigwedge_{n \in \mathbb{N}\setminus I} \phi(a_n, b) \geq r \} \neq \emptyset
\]
for every $I \subseteq \mathbb{N}$. If $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters on $\mathbb{N}$, there is an $I \subseteq \mathbb{N}$ such that $I \in \mathcal{F}$ and $\mathbb{N}\setminus I \in \mathcal{G}$; So that $\lim_{\mathcal{F}} \phi(a_n, b) \leq s < r \leq \lim_{\mathcal{G}} \phi(a_n, b)$ for every $b \in A(I)$. As there are $2^{2^{\aleph_0}}$ distinct ultrafilters on $\mathbb{N}$, $(\phi(a_n, y) : S_\phi(M) \to [0, 1])$ has $2^{2^{\aleph_0}}$ distinct cluster point and (ii) fails. 

We state the result separately for the classical ($\{0, 1\}$-valued) logic.

**Corollary 4.11 (Classical case).** Let $T$ be a classical theory and $\phi(x, y)$ a formula. The following are equivalent.

(i) $\phi(x, y)$ has NIP.

(ii) For any countable model $M$ and saturated elementary extension $M^*$ of it, whenever $p(x) \in S_\phi(M^*)$ is finitely satisfiable in $M$ then it is DBSC definable over $M$.

(iii) For any countable model $M$ and saturated elementary extension $M^*$ of it, the number of $p(x) \in S_\phi(M^*)$ which is finitely satisfiable in $M$ is $< 2^{2^{\aleph_0}}$.

**Proof.** Immediate, by Theorem 4.10 and the fact that for simple functions DBSC=Baire-1/2 (cf. Fact 2.2(iv)).

Of course, the direction (i) $\Rightarrow$ (ii) can be consider as a consequence of Proposition 2.6 of [HP11]. Indeed, since assuming NIP every coheir of a type over $M$ is strongly Borel definable in the sense of [HP11], the function that defines the coheir is the indicator function of a finite Boolean combination of closed sets over $M$. It is easy to check that such function is DBSC, by Proposition 2.2 of [CMR96]. (See also Remark 2.15 of [Kha19b].)

**Remark 4.12.** Let $T$ be a classical theory, $M$ a countable model of $T$, and $\phi(x, y)$ a NIP formula. It can be proved that every local Keisler measure which is finitely satisfiable in $M$ is Baire-1/2 definable. Indeed, Ben Yaacov and Keisler [BK09, Corollary 2.10] proved that every measure in the theory $T$ corresponds to a type in the randomization $T^R$ of $T$. On the other hand the randomization of every NIP theory is a NIP continuous theory (see [Ben09]). (In fact, the argument of this result is local (formula-by-formula), and one can easily check that the corresponding formula $\phi^R$ of $\phi$ in the randomization is NIP if $\phi$ is NIP.) By Theorem 4.10 above, every $\phi^R$-type in $T^R$ is Baire-1/2 definable, and so its corresponding measure in $T$ is Baire-1/2 definable. This will be discussed in detail in a future work.
History. In [HPP08] and [HP11], the Borel definability of coheirs and the strongly Borel definability of invariant types in NIP “theories” were proved, respectively. The notion “NIP in a model” was introduced in [Kha20] and some variants of definability of types and coheirs were proved. The correspondence between Shelah’s theorem (in classical logic) and the Eberlein–Šmulian theorem was also pointed out in [Kha20]. This approach was later followed in [Kha17] and [KP18], and more applications for the notion “NIP in a model” were presented. In [Kha19b], the usefulness of this approach for the study of model theoretic properties of “theories” (in classical logic) was demonstrated. In the present paper, this utility is extended to continuous logic, and new results are presented for both classical logic and continuous logic. In the next section, we examine the effectiveness of this approach in classification of “continuous theories.”

5 Discussion and thesis

In this section we argue why there is no a result similar to Shelah’s theorem for classical logic in the Banach space theory; equivalently, there is no such theorem in real-valued function theory as well as continuous logic. We explain the strategy for finding a counterexample, and give a new classification of continuous theories in the term of function spaces. Although the following is mainly expository but is (in our view) very illuminating. (This section can’t be read without firm grasp of [Kha19b, Section 3].)

5.1 Discussion

In [Kha19a], a ‘weak’ form of Shelah’s theorem for continuous logic was proved. Although, as mentioned above, we argue that the exact form of Shelah theorem ($IP \iff IP$ or $SOP$) does not hold in continuous logic. The reader can compare the proofs of [Kha19a, Proposition 1.9] and Theorem 3.6 above to see where the key difference is. In fact, the notion $SCP$ (or $NwSOP$) in [Kha19a] is strictly stronger than the notion $SOP$ (cf. Definition 3.3 above), and the usual argument of Shelah’s theorem does work with the latter notion. On the other hand, given the above, the following table is available:
Stability | $C(X)$
--- | ---
NIP | $B_1(X) \setminus B_{1/2}(X) = \emptyset$
NSOP | $DBSC(X) \setminus C(X) = \emptyset$

(Recall that the correspondence between stability and $C(X)$ follows from the Eberlein–Šmulian Theorem, the second and third correspondences follows from Theorems 4.3 and 3.9 respectively.)

In the classical case, from [Kha19b], we have the following table:

| Classical Logic |
| --- | --- |
| Stability | $C(X)$ |
| NIP | $B_1(X) \setminus DBSC(X) = \emptyset$ |
| NSOP | $DBSC(X) \setminus C(X) = \emptyset$ |

(The correspondences in classical case follows from the Eberlein–Šmulian Theorem, Propositions 2.13 and Remark 2.10 of [Kha19b], respectively. See also Section 3 of [Kha19b], especially Remark 3.4.)

Recall from Fact 2.2(iv) that for simple functions $B_{1/2}(X) = DBSC(X)$. Of course, as mentioned in Fact 2.2(iii), $B_{1/2}(X) \neq DBSC(X)$ in real-valued functions, i.e. the continuous case. We argue that this is the reason why Shelah’s theorem holds in classical logic and no one should expect something similar to be true in continuous logic. To complete the discussion, we recall some results of the function theory and the Banach space theory.

The main result of [HOR91], i.e. Theorem B, is very similar to Shelah’s theorem but for the same reason it is not a perfect analog of the latter. Here we explain this similarity. First we recall two notions from the Banach space theory. Let $(x_n)$ be a seminormalized basic sequence in a Banach space $Y$. A basic sequence $(e_n)$ is said to be spreading model of $(x_n)$ (or $Y$) if for all natural number $k$ and $\epsilon > 0$ there exist $N$ such that if $N < n_1 < \cdots < n_k$ and $(r_i)^k \subseteq \mathbb{R}$ with $\sup_i |r_i| \leq 1$, then

$$
\left| \left\| \sum_{i=1}^{k} r_i x_{n_i} \right\| - \left\| \sum_{i=1}^{k} r_i e_i \right\| \right| < \epsilon.
$$

Roughly speaking, $(e_n)$ is a Morley sequence in an elementary extension of $Y$ in the sense of model theory. Finally recall that a sequence $(g_n)$ in a Banach space is a convex block subsequence of $(f_n)$ if $g_n = \sum_{i=p_n+1}^{p_{n+1}} r_i f_i$ where $(p_n)$ is
an increasing sequence of integers, \((r_i) \in \mathbb{R}^+\) and for each \(n, \sum_{i=p_n+1}^{p_{n+1}} r_i = 1\).

Again roughly, a convex block subsequence of \((f_n)\) can be considered as a sequence of Boolean combinations of the instances of \(f_n\). In the following theorem, we can assume that \(X\) is the type space \(S\tilde{\phi}(M)\) where \(\phi(x, y)\) is a formula, \(M\) a separable model of a continuous theory, and \((f_n)\) is a sequence of the form \((\phi(a_n, y) : n < \omega)\) where \(a_n \in M\) and \(\phi(a_n, y)\) is a continuous function on \(X\) as previously defined.

**Theorem B:** Let \(X\) be compact metric space, \(f \in B_1(X) \setminus C(X)\), and \((f_n)\) a uniformly bounded sequence in \(C(X)\) which converges to \(f\).

(a) (i) If \(f \notin B_{1/2}(X)\), then \((f_n)\) has a subsequence whose spreading model is equivalent to the unit vector basis of \(\ell_1\).

(ii) If every convex block basis of \((f_n)\) has a spreading model equivalent to the unit vector basis of \(\ell_1\), then \(f \notin B_{1/2}(X)\).

(b) (i) If \(f \in B_{1/4}(X)\), then some convex block basis of \((f_n)\) has a spreading model equivalent to the summing basis of \(c_0\).

(ii) If \((f_n)\) has a spreading model equivalent to the summing basis of \(c_0\), then \(f \in B_{1/4}(X)\).

**Explanation.** Theorem B(a) and Theorem B(b) are Theorems II.3.5 and II.3.6 of [AGR03, Chapter 23]. Theorem B(a)(i), Theorem B(a)(ii) and Theorem B(b)(i) follows from [HOR91, Theorem B(a)], [HOR91 Theorems 3.7] and [HOR91 Theorem B(b)], respectively. Theorem B(b)(ii) is due to V. Farmaki [Far94, Theorem 1.1] answering an open question raised in [HOR91].

Notice that Theorem B(a)(ii) is a converse to Theorem B(a)(i) and Theorem B(b)(ii) is a converse to Theorem B(b)(i). Theorem B(a) should be compared with Theorem 3.10 above. In other words, it corresponds to the independence property for a theory. On the other hand, Theorem B(b) should be compared with Proposition 3.10 above. Therefore, it is related to the strict order property for a theory. The reason that Theorem B(a) and Theorem B(b) cannot interact and join together to form a single theorem similar to Shelah’s theorem (or even the Eberlein-Šmulian Theorem) is that \(B_{1/4}(X) \neq B_{1/2}(X)\) in general (cf. Fact 2.2(iii)). (Moreover, recall from [OS95] that there exists a separable Banach space \(X\) so that no spreading model of \(X\) contains \(c_0\) or \(\ell_p (1 \leq p < \infty)\).) This is precisely the reason why one should not expect the exact form of Shelah’s theorem to be established.
for continuous logic. This is not the case in classical logic because for \{0, 1\}-valued (even simple) functions $DBSC(X) = B_{1/4}(X) = B_{1/2}(X)$. It seems that the exotic (or unnatural) examples in Banach space theory, those that do not contain an infinite dimensional reflexive subspace or an isomorph of $\ell_1$ or $c_0$, come from this point.

In the following we argue that finding a counterexample to a perfect analog of Shelah’s theorem for continuous logic is related to a specific type of Banach spaces. In [Kha18, Section 4.2], we assigned a Banach space $V_{\phi,M}$ to each formula $\phi$ and model $M$ as follows. Let $T$ be a (continuous) theory, $M$ a model of it, $\phi(x, y)$ a formula, and $X = S_{\hat{\phi}}(M)$ the space of complete $\hat{\phi}$-types on $M$. The space of linear $\hat{\phi}$-definable relations on $M$, denoted by $V_{\phi,M}$, is the closed subspace of $C(X)$ generated by the set \{ $\phi(a, y) : X \to [0, 1] \mid a \in M$, $\phi(a, y)$ is continuous and $V_{\phi,M}$ is a Banach space with sup-norm\}. It is easy to show that (i) a (continuous) theory is stable if and only if for each separable model $M$ and each formula $\phi$ the Banach space $V_{\phi,M}$ is reflexive (cf. [Kha18, Corollary 4.4]), and (ii) a (continuous) theory has NIP if and only if for each separable model $M$ and each formula $\phi$ the Banach space $V_{\phi,M}$ does not contain an isomorphic copy of $\ell_1$ (cf. [Kha18, Corollary 4.7]). Our observation in the present paper (Proposition 3.10) shows that SOP implies the existence of an isomorph of $c_0$ in some Banach space $V_{\phi,M}$, where $\phi$ is a formula and $M$ is a separable model. To summaries:

**Proposition 5.1.** Let $T$ be a continuous theory. Then (i) implies (ii).

(i) (a) for some formula $\phi$ and some model $M$, the Banach space $V_{\phi,M}$ is nonreflexive, and (b) for each formula $\phi$ and each separable model $M$ the Banach space $V_{\phi,M}$ does not contain $\ell_1$ or $c_0$.

(ii) $T$ is unstable but NIP and NSOP.

In the case (i), for some formula $\phi$ and model $M$, $V_{\phi,M}$ is nonreflexive and not containing an isomorph of $\ell_1$ or $c_0$. Recall that the first example of such a space was given in [Jam50] by Robert C. James. We conjecture that this type of spaces, and especially exotic Banach spaces, are the spaces that their theories are unstable but NIP and NSOP. This conjecture is related to a question due to Odell and Gowers on the existence of $c_0$ and $\ell_p$ ($1 \leq p < \infty$) in ‘explicitly defined’ Banach spaces. (See [Kha18] (Q1) and a discussion on explicit definability of norms therein.) We believe that a
complete and enlightening answer to the Odell–Gowers question is necessarily possible through the model-theoretic classification of continuous first order theories.

Let us make more comparisons on the above theorems, namely Shelah’s theorem for classical logic, the Eberlein–Šmulian Theorem, Theorem B above, and Theorems A of [HOR91]. Roughly speaking, Theorem A(a) of [HOR91, page 1] asserts that a separable Banach space $X$ has an isomorphic copy of $\ell_1$ iff its bidual has a non Baire-1 point. Theorem A(b) asserts that $X$ has an isomorphic copy of $c_0$ iff its bidual has a non-continuous but DBSC point. Notice that Theorem A and the Eberlein–Šmulian Theorem study some properties inside a model, but Shelah’s theorem and Theorem B study some properties of the theory of a model. Roughly (and possibly incorrect) speaking, assuming compactness theorem of logic, Shelah’s theorem and the Eberlein–Šmulian Theorem are equivalent (cf. [Kha19b, Section 3]). Theorems A and Theorem B are not as perfect as the other two theorems, because of the gaps DBSC $\neq B_{1/4} \neq B_{1/2} \neq B_1$. Again, we point out that Shelah’s theorem (for classical logic) works because DBSC $= B_{1/2}$ for simple functions, and the Eberlein–Šmulian Theorem works because relative sequential compactness and sequential completeness implies relative compactness, equivalently $B_1 = B_1$ in our notation!

5.2 Thesis

The above results and the observations in [Kha19b, Section 3] suggest a new classification of continuous first order theories. Indeed, let $X$ be compact metric space and $C(X)$ be some subclass of $B_1(X)$, containing $B_{1/2}(X)$. We say that a theory $T$ has the $C$-property if

for any formula $\phi(x, y)$ and any infinite sequence $(a_i : i < \omega)$, if the sequence $(\phi(a_i, y) : S_\infty(\{a_i : i < \omega\}) \rightarrow [0, 1])$ converges to a function in $C(S_\infty(\{a_i : i < \omega\}))$, then there is no infinite sequence $(b_i : i < \omega)$ such that $(a_i), (b_i)$ witness the order property for $\phi(x, y)$.

Now we can provide so many Shelah-like theorems.

**Proposition 5.2.** Let $T$ be a complete (continuous) theory and $C$ as above. Then $T$ is stable if and only if it has both NIP and $C$-property.

**Proof.** Clearly, stability implies NIP and $C$-property. For the converse, by NIP, for any $\phi(x, y)$ and any $(a_i)$, there is a subsequence $(a_{i_j})$ such that
φ(a_{ij}, y) converges to a function f which is B_{1/2}. As B_{1/2} ⊆ C, by C-property, there is no infinite sequence (b_i) such that (a_{ij}), (b_i) witness the order property for φ(x, y). By Grothendieck’s criterion, f is continuous. As φ and (a_{ij}) are arbitrary, by the Eberlein–Smulian Theorem and Grothendieck’s criterion, T is stable.

Kechris and Louveau [KL90] studied various ordinal ranks of bounded Baire-1 functions. They introduced the classes of bounded Baire-1 functions B^ξ_1 (for ξ a countable ordinal). It is shown that (i) DBSC = B^1_1, (ii) B^ξ_1 is a Banach subalgebra of Baire-1 functions (with the sup-norm), and (iii) B^ξ_1 ⊆ B^ζ_1 for countable ordinals ξ ≤ ζ < ω_1. Therefore, by Proposition 5.2 we have the following:

**Corollary 5.3.** Let T be a complete theory and ξ a countable ordinal. Then T is stable if and only if it has both NIP and B^ξ_1-property.

In Section 3 of [KL90], there is a classification of functions between DBSC and Baire-1/2. Recall from Fact 2.2(iv) that every \{0, 1\}-valued Baire-1/2 function is DBSC. This explains why one can provide a better classification for continuous first order theories that is not possible for classical case.

At the end paper let us discuss what means a dividing line in continuous logic and the Banach space theory. In [Bal18, Chapter 2.3] Baldwin distinguish between a virtuous property of a theory and a dividing line in classical logic. A property is *virtuous* if it has significant mathematical consequences for the theory or its models. A property is a *dividing line* if it and its negation are both virtuous. According to these definitions, by Theorems 4.3 and 4.10 as NIP and its negation give us a lot of information about the theory/models, NIP is a dividing line in continuous logic as well as the Banach space theory. Indeed, NIP is equivalent to Baire-1/2 definability of coheirs and its negation (IP) is equivalent to existence of \ell_1 in the function space on types. In fact, such dividing line already exists in Banach space theory, Rosenthal’s \ell_1-theorem. These theorems give the sort of control that was needed; that is, control on both sides of the dividing line. Another example of such a theorem in the Banach space theory is Gowers’ theorem [Gow96]: Every Banach space X has a subspace Y that either has an unconditional basis or is hereditarily indecomposable. As Gowers argues in [Gow94b], *the questions that were traditionally asked about nice subspaces [i.e. existence of an infinite dimensional reflexive subspace or an isomorph of \ell_1 or c_0] were of the wrong kind (although more examples of Banach spaces were needed*
As NIP corresponds to Rosenthal’s $\ell_1$ theorem, several questions arise. Is there a model theoretic property corresponding to Gowers’ theorem? Are there model theoretic dividing lines that separate natural and exotic Banach spaces? What are the connections between the known classification (in classical logic) and the classification presented in this article? These and many other questions are worth looking.

Finally, the above points strongly inspire us to believe that Shelah’s dividing lines and classification theory is the correct approach to many questions in various areas of mathematics.

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