Non–regular McKean–Vlasov equations and calibration problem in local stochastic volatility models

Mao Fabrice DJETE†
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Abstract

In order to deal with the question of the existence of a calibrated local stochastic volatility model in finance, we investigate a class of McKean–Vlasov equations where a minimal continuity assumption is imposed on the coefficients. Namely, the drift coefficient and, in particular, the volatility coefficient are not necessarily continuous in the measure variable for the Wasserstein topology. In this paper, we provide an existence result and show an approximation by N–particle system or propagation of chaos for this type of McKean–Vlasov equations. As a direct result, we are able to deduce the existence of a calibrated local stochastic volatility model for an appropriate choice of stochastic volatility parameters. The associated propagation of chaos result is also proved.

1 Introduction

Motivated by the local stochastic volatility (LSV) model in finance that we will detailed in Section 2.3, we are interested in this paper in the question of the existence of a process $S$ satisfying

$$dS_t = \frac{S_t \sigma_D(t, S_t) \sigma_t}{\sqrt{\sigma_t^2 |S_t|}} dW_t \quad \text{or equivalently} \quad dX_t = -\frac{1}{2} \frac{\sigma_t^2 \sigma_D(t, e^{X_t})^2}{\mathbb{E}[\sigma_t^2 |X_t|]} dt + \frac{\sigma_t \sigma_D(t, e^{X_t})}{\sqrt{\mathbb{E}[\sigma_t^2 |X_t|]}} dW_t \quad \text{for} \quad X_t = \log(S_t) \quad (1.1)$$

where $(W_t)_{t \geq 0}$ is an $\mathbb{R}$–valued $\mathbb{F}$–Brownian motion and $(\sigma_t)_{t \geq 0}$ is an $\mathbb{R}$–valued $\mathbb{F}$–predictable process on a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. This question is commonly accepted as a very difficult problem. For making this question more tractable, the process $(\sigma_t)_{t \geq 0}$ is usually taken as $\sigma_t^2 = \mathbb{P}(t, X_t, Y_t)$ where $Y$ is an Itô process driven by another Brownian motion $B$ potentially correlated to $W$. Even with this simplification, the literature on this subject has remained limited so far. Notice that, if the couple $(X_t, Y_t)$ has a density $p(t, x, y)$ w.r.t. the Lebesgue measure, the expression $\mathbb{E}[\mathbb{P}(t, X_t, Y_t) |X_t]$ can be rewritten $\mathbb{E}[\mathbb{P}(t, X_t, Y_t) |X_t] = \int_{\mathbb{R}} \mathbb{P}(t, x, y) p(t, x, y) \, dy$. This observation led us to treat the variables $\int_{\mathbb{R}} \mathbb{P}(t, x, y) p(t, x, y) \, dy$ and $\int_{\mathbb{R}} p(t, x, y) \, dy$ separately. This allowed us to prove some general results, one of which is the following. Let $\theta \in [-1, 1]$ s.t. $d\langle W, B \rangle_t = \theta dt$. Let us assume that $v$ and $\sigma_D$ are Borel maps bounded above and below by positive constants, and $\mathbb{R}_+ \times \mathbb{R}^2 \ni (t, x, y) \mapsto v(t, x, y) \sigma_D(t, e^{x})^2 \in \mathbb{R}$ is Lipschitz in $(x, y)$ uniformly in $t$. Also, there are bounded Borel maps $\mathbb{R}_+ \times \mathbb{R} \ni (t, y) \mapsto \lambda(t, y) \in \mathbb{R}$ and $\mathbb{R}_+ \times \mathbb{R} \ni (t, y) \mapsto \beta(t, y) \in \mathbb{R}$ s.t. $\beta$ is Lipschitz in $y$ uniformly in $t$, $\beta^2$ bounded below by a positive constant and $\mathbb{R}_+ \times \mathbb{R} \ni (t, y) \mapsto -\theta \beta(t, y) \in \mathbb{R}$ is a non–negative function.

**Theorem 1.1.** Under some conditions on the initial density $p_0 : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ (see Equation (2.1)), for any $T > 0$, there exists an $\mathbb{R}^2$–valued $\mathbb{F}$–adapted continuous process $(X, Y)$ verifying: $\mathcal{L}(X_0, Y_0) = p_0(x)dx$, for each $t \leq T$;

$$dX_t = -\frac{1}{2} \frac{\sigma_D(t, e^{X_t})^2}{\mathbb{E}[\sigma_t^2 |X_t|]} \frac{c + p_X(t, X_t) v(t, X_t, Y_t)}{c + p_X(t, X_t) \mathbb{E}[v(t, X_t, Y_t) |X_t]} dt + \sigma_D(t, e^{X_t}) \sqrt{\frac{c + p_X(t, X_t) v(t, X_t, Y_t)}{c + p_X(t, X_t) \mathbb{E}[v(t, X_t, Y_t) |X_t]}} dW_t \quad (1.2)$$

and

$$dY_t = \lambda(t, Y_t) dt + \beta(t, Y_t) dB_t \quad (1.3)$$

where $c$ is a positive constant and $p_X(t, \cdot)$ is the density of $\mathcal{L}(X_t)$.

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†École Polytechnique Paris, Centre de Mathématiques Appliquées, mao-fabrice.djete@polytechnique.edu. This work benefits from the financial support of the Chairs Financial Risk and Finance and Sustainable Development.
This result guarantees the existence of a calibrated local stochastic volatility model for the choice of stochastic volatility parameters $\sigma^2 = \mathfrak{v}(t, X_t, Y_t) = c + p_X(t, X_t) \nu(t, X_t, Y_t)$. Although the existence of a calibrated LSV model is commonly accepted in the mathematical finance community, there are not many papers that have rigorously investigated this issue. We can evoke the paper of Abergel and Tachet [1]. In [1], by considering a discretized version (in time and space) of the Fokker–Planck equation associated to (1.1), the authors use a fixed point argument for establishing an existence result in short–time. Let us cite also Jourdain and Zhou [13] who investigate this question in the case where $Y$ only take a finite number of values. These discretizations are important in the proof of [1] and [13]. In a setting allowing continuous space and time, when all the coefficients involved are homogeneous in time $t$ (in particular $\mathfrak{v}(t, s, y) = \mathfrak{v}(y)$), Lacker, Shkolnikov, and Zhang [19] prove the existence of a stationary solution of (1.4). The homogeneous assumption is crucial for establishing their result. Moreover, they need the two Brownian motions $W$ and $B$ to be independent. Our theorem seems to be among the most general results regarding the existence of a LSV model calibrating European prices. We are able to consider continuous time and space, to allow the Brownian motions $W$ and $B$ to be correlated, we can have non–homogeneous coefficients and, it is possible to consider any maturity $T$ and to start for any initial data (as long as it is smooth enough). In addition to this new existence result, in this paper (see Section 2.2), we provide the approximation of $(X, Y)$ solution of (1.2) + (1.3) by a particle system. To the best of our knowledge, this is the first result of this kind for a system of type (1.2) + (1.3). This approximation by particle system leads naturally to a numerical scheme allowing to compute a solution $(X, Y)$.

The approach used to deal with the LSV model is sufficiently general that it allows us to consider more general framework (see Theorem 2.3). A typical example that can be considered is the question of the existence of a couple $X := (X^1, X^2)$ verifying:

$$
\begin{align*}
\text{d}X^1_t &= b(t, X_t, p(t, X_t), (ph_1(t, X^1_t)) \text{d}t + \sigma(t, X_t, p(t, X_t), (ph_1(t, X^1_t)) \text{d}W_t \\
\text{d}X^2_t &= \lambda(t, X^2_t) \text{d}t + \beta(t, X^2_t) \text{d}B_t \quad \text{with} \quad (ph_1(t, X^1_t)) := \int_{\mathbb{R}} h(t, X^1_t, x_2) p(t, X^1_t, x_2) \text{d}x_2
\end{align*}
$$

(1.4)

with $p(t, x_2, x_2)$ the density of $\mathcal{L}(X_t)$ w.r.t. the Lebesgue measure. These types of equations fall into the category of McKean–Vlasov equations. The applied mathematics researchers have been studying equations of this type due to its involvement in fields like Physics, Biology, finance, · · ·.

For the study of moderately interacting particle systems, in the absence of $(ph_1(t, X^1_t))$, (1.4) has been investigated. Oelschläger [21] shows existence and uniqueness of this kind of equation when the diffusion coefficient $\sigma$ is constant. Jourdain and Méléard [12] extend the work of [21] in the case where the drift and diffusion coefficients depend on the densities of the marginal law of the solution $X_t$. In the framework considered by [12] but without drift coefficients, Bossy and Jabin [6] weakened the assumptions necessary for the result of [12]. In the presence of $(ph_1(t, X^1_t))$, a common situation is the consideration of the conditional distribution of $X^2_t$ given $X^1_t$ as it is the case in the LSV model previously mentioned. Our existence result of a calibrated LSV model is actually a direct application of the existence of a couple $(X^1, X^2)$ satisfying (1.4).

In this article, we will establish the existence of solution of (1.4) in a more general setting under suitable assumptions over the coefficients. We will show then how this result can be used in establishing an existence result for the calibration of local stochastic volatility model. Our approach relies on the use of a fixed point theorem after proving some Sobolev estimates for the density $p$ (see Section 3.1 for the idea leading the proof). We would like to mention that most of the techniques used in this article are inspired by Bogachev, Krylov, Röckner, and Shaposhnikov [5, Chapter 6], where the authors present interesting techniques for establishing estimates of the gradient of a solution of the Fokker–Planck equation under mild assumptions.

After introducing some notations just below, in Section 2, we present the framework considered in this paper and state the main results. Namely, an existence result, an approximation result by particle system and the application to the local stochastic volatility model. Most of the technical proofs are completed in Section 3.

Notations. (i) Given a metric space $(E, \Delta)$, $p \geq 1$, we denote by $\mathcal{P}(E)$ the collection of all Borel probability measures on $E$, and by $\mathcal{P}_p(E)$ the subset of Borel probability measures $\mu$ such that $\int_E \Delta(e, e_0)^p \mu(de) < \infty$ for some $e_0 \in E$. We equip $\mathcal{P}_p(E)$ with the Wasserstein metric $\mathcal{W}_p$ defined by

$$
\mathcal{W}_p(\mu, \mu') := \left( \inf_{\lambda \in \mathcal{L}(\mu, \mu')} \int_{E \times E} \Delta(e, e')^p \lambda(de, de') \right)^{1/p},
$$

2
where $\Lambda(\mu, \mu')$ denote the collection of all probability measures $\lambda$ on $E \times E$ such that $\lambda(de, E) = \mu$ and $\lambda(E, de') = \mu'(de')$. Equipped with $\mathcal{W}_p$, $\mathcal{P}_p(E)$ is a Polish space (see [26, Theorem 6.18]). For any $\mu \in \mathcal{P}(E)$ and $\mu$-integrable function $\varphi$, we define $\langle \varphi, \mu \rangle := \int_E \varphi(e) \mu(de)$.

(ii) Let $\mathbb{N}^*$ denote the set of positive integers. Given non-negative integers $m$ and $n$, we denote by $\mathbb{S}^{m \times n}$ the collection of all $m \times n$-dimensional matrices with real entries, equipped with the standard Euclidean norm, which we denote by $\cdot$. We also denote $\mathbb{S}^n := \mathbb{S}^{n \times n}$, and denote by $0_{m \times n}$ the element in $\mathbb{S}^{m \times n}$ whose entries are all 0, and by $I_n$ the identity matrix in $\mathbb{S}^n$. Let $d \in \mathbb{N}^*$, for any open set $E \subset \mathbb{R}^d$, $\text{diam}(E)$ denotes the diameter of $E$, i.e., $\sup_{x,y \in E} |x-y|$. Given an open set $\Omega_T = (0, T) \times \Omega$ in $\mathbb{R} \times \mathbb{R}^d$, where $\Omega \subset \mathbb{R}^d$ is an open set and $T > 0$, $C^\infty_\text{c}(\Omega_T)$ denotes the class of infinitely differentiable functions with compact support in $\Omega_T$ and $C^{1,2}_\text{c}(\Omega_T)$ denotes the class of functions on $\Omega_T$ with continuous derivatives up to the second order in $x$ and a continuous derivative in $t$. When the derivatives have continuous extensions to the closure of $\Omega_T$ i.e. $\overline{\Omega_T}$, we write $C^\infty_\text{c}(\overline{\Omega_T})$ and $C^{1,2}_\text{c}(\overline{\Omega_T})$. The notation $\partial_t$ indicates the continuous derivative in $t$, and the symbols $\nabla^2$ and $\nabla$ are the second and first order derivatives in $x$.

Given an open set $U \subset \mathbb{R}^\ell$ for $\ell \geq 1$, for any Banach space $(E, \| \cdot \|)$, the space $C^{0,\delta}(U, E)$ consists of Hölder continuous functions of order $\delta \in (0, 1)$ functions $f$ on $U$ with value in $E$ with finite norm

$$
\|f\|_{C^{0,\delta}(U, E)} := \sup_{x \in U} |f(x)|_E + \sup_{x,y \in U, x \neq y} \frac{|f(x) - f(y)|_E}{|x-y|^\delta}.
$$

The support of a function $f$, i.e., the closure of the set $\{f \neq 0\}$, is denoted by $\text{supp } f$.

## 2 Setup and Main results

The general assumptions used throughout this paper are now formulated. We set $\theta \in [-1, 1]$ and the probability space $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ supporting $W$ and $B$ two $\mathbb{R}$-valued $\mathbb{P}$-Brownian motions satisfying $d(W, B)_t = \theta dt$ . We are given the numbers $q := 3 + 1/4$ and $1/2 > \beta > 1/q$ with $(1 - 2\beta)q > 2$. We refer to Section 3.2 for the definition of $L^{1,1}_\text{loc}$, $L^q$ and the Sobolev spaces $H^{q,1}$ and $\mathbb{H}^{q,1}$. We consider the following Borel measurable functions

$$
[b, \sigma] : L^{1,1}_\text{loc}(\mathbb{R}_+; L^1(\mathbb{R}^2; \mathbb{R})) \times \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ and } [\lambda, \beta] : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}.
$$

For any $T \in \mathbb{R}_+$ and Borel map $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, if we define $f^T(s, x) := f(s \wedge T, x)$ for all $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, when $f \in L^1([0, T] \times \mathbb{R}^2)$, we will consider its extension $f^T$ over $L^{1,1}_\text{loc}(\mathbb{R}_+; L^1(\mathbb{R}^2; \mathbb{R}))$ and write $[b, \sigma](f)$ instead of $[b, \sigma](f^T)$.

**Assumption 2.1.** Let $T > 0$. There exist $m$ and $M$ positive numbers satisfying: for any $f \in L^1([0, T] \times \mathbb{R}^2)$ s.t. $f \geq 0$, for each $t \in [0, T]$ and $x = (x_1, x_2) \in \mathbb{R}^2$, if we introduce the map $b$ and the symmetric matrix $a := (a_{i,j})_{1 \leq i,j \leq 2}$ by: $b^1(f)(t, x) := b(f)(t, x)$, $b^2(f)(t, x) := \lambda(t, x_2)$,

$$
a^{1,1}(f)(t, x) := \frac{1}{2} \sigma(\lambda)^2, \quad a^{2,1}(f)(t, x) := a^{1,2}(f)(t, x) := \sigma(\lambda(\lambda^2)(t, x_2) \theta \text{ and } a^{2,2}(f)(t, x) := \frac{1}{2} \beta(t, x_2)^2,
$$

one has that:

- **Growth assumption** For any $(t, x) \in [0, T] \times \mathbb{R}^2$, $|b(f)(t, x)| \leq M$ and $m I_2 \leq a(f)(t, x) \leq M I_2$. In addition, for any open set $E \subset \mathbb{R}^2$ verifying $\text{diam}(E) \leq 1

$$
\|\nabla a(f)\|_{L^q([0, T] \times \mathbb{R}^2)} \leq M \left[1 + \|f\|_{H^{1,1}([0, T] \times \mathbb{R}^2)} + \|\partial_x f\|_{L^q([0, T] \times \mathbb{R}^2)} + \|f_1\|_{L^q([0, T] \times \mathbb{R}^2)} \right]
$$

and

$$
\sup_{t \in [0, T]} \|a(f)(t, \cdot)\|_{C^{0,1-2\beta-2/q}(E)} \leq M \left[1 + \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^{0,1-2\beta-2/q}(\mathbb{R}^2)} + \sup_{t \in [0, T]} \|f_1(t, \cdot)\|_{C^{0,1-2\beta-1/q}(\mathbb{R})} \right]
$$

where

$$
f_1(t, x_1) = f_1(t, x_1) := \int_\mathbb{R} f(t, x_1, x_2) \, dx_2 \text{ and } |\partial_x f_1(t, x_1)| := \int_\mathbb{R} |\partial_x f(t, x_1, x_2)| \, dx_2;
$$
• Continuity assumption The map \([0, T] \times \mathbb{R} \ni (t, x_2) \mapsto \beta(t, x_2) \in \mathbb{R}\) is Lipschitz in \(x_2\) uniformly in \(t\). Besides, whenever \(\lim_{n \to \infty} \|f^n - f\|_{L^1([0, T] \times \mathbb{R}^2)} = 0\), for any \(\varphi \in C_c^{\infty}([0, T] \times \mathbb{R}^2)\)

\[
\lim_{n \to \infty} \int_{[0, T] \times \mathbb{R}^2} b^i(f^n)(t, x) \varphi(t, x) \, dx \, dt = \int_{[0, T] \times \mathbb{R}^2} b^i(f)(t, x) \varphi(t, x) \, dx \, dt \quad \text{for any} \ 1 \leq i \leq 2
\]

and

\[
\lim_{n \to \infty} \int_{[0, T] \times \mathbb{R}^2} a^{i,j}(f^n)(t, x) \varphi(t, x) \, dx \, dt = \int_{[0, T] \times \mathbb{R}^2} a^{i,j}(f)(t, x) \varphi(t, x) \, dx \, dt \quad \text{for any} \ 1 \leq i, j \leq 2.
\]

**Remark 2.2.** It is worth mentioning that, the constants \(m\) and \(M\) may depend on \(T\) but are independent of the choice of the map \(f\). Also notice that when \(f\) does not belong to the Sobolev space or to the Hölder space, the preceding inequalities are trivially true since the upper bound is equal to infinity. The value of \(q\) i.e. \(3 + 1/4\) follows from the use of some Sobolev embedding theorems in the proofs.

Let \(p_0 : \mathbb{R}^2 \to \mathbb{R}_+\) be a density i.e., \(\int_{\mathbb{R}^2} p_0(x_1, x_2) \, dx_1 \, dx_2 = 1\) s.t. (again see Section 3.2 for the definition of the Sobolev spaces \(H^{n,1}\) and \(\mathbb{H}^{q,1}\))

\[
p_0 \in H^{q,1}(\mathbb{R}^2), \quad (p_0)_1 \in H^{n,1}(\mathbb{R}) \quad \text{and}, \quad \text{there is} \ \alpha > 0, \quad \int_{\mathbb{R}} e^{\alpha |x_2|^2} \left| \int_{\mathbb{R}} p_0(x_1, x_2) \, dx_1 \right|^q \, dx_2 < \infty. \tag{2.1}
\]

**Theorem 2.3.** Let \(\kappa\) s.t. \(\frac{1}{2} > \beta > \kappa > \frac{1}{4}\) and \(T > 0\). Under Assumption 2.1, there exists an \(\mathbb{R}^2\)-valued \(\mathbb{F}\)-adapted continuous process \(X := (X^1, X^2)\) satisfying: \(\mathcal{L}(X_0)(dx) = p_0(x) \, dx\),

\[
dX^1_t = b(p)(t, X^1_t) \, dt + \sigma(p)(t, X^1_t) \, dW_t \quad \text{and} \quad dX^2_t = \lambda(t, X^2_t) \, dt + \beta(t, X^2_t) \, dB_t, \quad t \leq T \tag{2.2}
\]

where \(\mathcal{L}(X_t)(dx) = p(t, x) \, dx\). In addition, the density \(p\) satisfies

\[
\|p_1\|_{L^q([0, T] \times \mathbb{R})} + \|\partial_{x_1} p_1\|_{L^q([0, T] \times \mathbb{R})} + \|p\|_{H^{q,1}([0, T] \times \mathbb{R}^2)} < \infty
\]

and

\[
\|p\|_{C^{0,\kappa-1/4}([0, T], C^{0,1-2\beta-1/4}(\mathbb{R}^2))} + \|p_1\|_{C^{0,\kappa-1/4}([0, T], C^{0,1-2\beta-1/4}(\mathbb{R}))} < \infty.
\]

**Remark 2.4.** (i) We would like to point out that finding a process \(X\) satisfying the system mentioned in Theorem 2.3 is not an easy task. Indeed, in this system of equations, the coefficients (especially the diffusion coefficient \(\sigma\)) depend on the marginal distribution of the process i.e. \(\mathcal{L}(X_t)\). Given the assumption considered, this dependence can be particularly discontinuous for the Wasserstein distance, see the examples of \(b\) and \(\sigma\) given in the section just below. Consequently, the classical results of existence for McKean–Vlasov processes fail in this situation. Besides this discontinuity, the involvement of the distribution generates another technical difficulty. As we will see in the examples, \(b\) and \(\sigma\) can depend on \((pv)_1(t, x_1)\) where \(v : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^k\) is a bounded Borel map. This type of dependence through \(p\) creates a non–local+ local aspect which makes things even more difficult. It is non–local through the second variable as there is an integration in the second variable \(x_2\) in \((pv)_1(t, x_1)\), but it is local in the sense that the first variable \(x_1\) is fixed in the density \(p\) and in the map \(v\) (in the definition of \((ph)_1(t, x_1)\) ). It is worth emphasizing that the role of \(x_1\) and \(x_2\) has its importance in the sense that if the non–local part was \(x_1\) (and not \(x_2\)) and the local part was \(x_2\) (and not \(x_1\)), the difficulty would have been quite different. This case turns out to be less complex and has been investigated in the literature (see for instance [6]).

(ii) Let us mention that the shape of the coefficients makes it possible to take into consideration the conditional distribution of \(X^2\) given \(X^1\) in a non–linear and specific way as we will see in the application to the local stochastic volatility model. Also, the two Brownian motions \(W\) and \(B\) can be correlated i.e. \(d(W, B)_t = \theta dt\).

(iii) Our result contains some known results in the literature. But, the considered framework and assumptions seem to be more general compared to the literature. However, we are unable to have a more general dependency for the coefficients \(\lambda\) and \(\beta\). Also, we cannot provide a uniqueness result for this system.
2.1 Example of functions satisfying Assumption 2.1

Here, we give some examples of map \((b, \lambda, \sigma, \beta)\) for the application of Theorem 2.3. The maps \(\lambda\) and \(\beta\) need to be bounded, and \(\beta\) must be Lipschitz in \(x_2\) uniformly in \(t\). Let \((h, v) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^k \times \mathbb{R}^k\). The maps \(h\) and \(v\) satisfy: for each \(T > 0\), for all \((t, x_1, x_2) \in [0, T] \times \mathbb{R}^2\),

\[
\kappa_i^i m \leq h(t, x_1, x_2) \leq \kappa_i^M m \quad \text{and} \quad c_i^m \leq v(t, x_1, x_2) \leq c_i^M, \text{ for each } 1 \leq i \leq k,
\]

where \(\{\kappa_i^i m, \kappa_i^M m, c_i^m, c_i^M, 1 \leq i \leq k\}\) are some real numbers. Let us introduce

\[
\mathcal{E}_h^T := \{(t, x, e_0, e_1, (c^2_1, \ldots, c^2_k)) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^k : e_1 \kappa^i_m \leq c^i_2 \leq e_1 \kappa^i_M \text{ for each } 1 \leq i \leq k\} \quad (2.3)
\]
and

\[
\mathcal{E}_v^T := \{(t, x, e_0, e_1, (c^2_1, \ldots, c^2_k)) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^k : e_1 c^i_m \leq c^i_2 \leq e_1 c^i_M \text{ for each } 1 \leq i \leq k\}. \quad (2.4)
\]

We take the maps \([b^\circ, \sigma^\circ] : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^k \to \mathbb{R}^2\) satisfying: \(\mathcal{E}_h^T \supset (t, x, e_0, e_1, e_2) \to b^\circ(t, x, e_0, e_1, e_2) \in \mathbb{R}\) and \(\mathcal{E}_v^T \supset (t, x, e_0, e_1, e_2) \to \sigma^\circ(t, x, e_0, e_1, e_2) \in \mathbb{R}\) are bounded and continuous in \((e_0, e_1, e_2)\) uniformly in \((t, x)\). In addition, the matrix \(a\) defines as \(a^{11} := \frac{\kappa^2}{\sigma^2}, a^{22} := \frac{\kappa^2}{\sigma^2}, a^{12} = a^{21} := \theta \beta \sigma^2\) verifies \(0 < \inf_{\mathcal{E}_v^T} \inf_{x \neq 0} \frac{\kappa^2}{\sigma^2} \).

- A typical example is the following

\[
b(f)(t, x) := b^\circ(t, x, f(t, x), f_1(t, x_1), (f h)_1(t, x_1)) \text{ and } \sigma(f)(t, x) := \sigma^\circ(t, x, f(t, x), f_1(t, x_1), (f v)_1(t, x_1)),
\]

for each \(1 \leq i \leq k\), and for each \(T > 0\), \(\mathcal{E}_h^T \supset (t, x, e_0, e_1, e_2) \to \sigma^\circ(t, x, e_0, e_1, e_2) \in \mathbb{R}\) is Lipschitz in \((x, e_0, e_1, e_2)\) uniformly in \((t, x)\) i.e. \(\text{ess sup}_{\mathcal{E}_v^T} |\nabla \sigma^\circ| < \infty\). We refer to Section 3.5.2 for the checking of the Assumption 2.1.

- Let us give another example. We take

\[
b(f)(t, x) := b^\circ(t, x, f(t, x), \int_{\mathbb{R}^2} f(t, x') \, dx', \int_{\mathbb{R}^2} h(t, x') f(t, x') \, dx')
\]
and

\[
\sigma(f)(t, x) := \sigma^\circ(t, x, f(t, x), \int_{\mathbb{R}^2} f(t, x') \, dx', \int_{\mathbb{R}^2} v(t, x') f(t, x') \, dx'),
\]

and for each \(T > 0\), \(\mathcal{E}_h^T \supset (t, x, e_0, e_1, e_2) \to \sigma^\circ(t, x, e_0, e_1, e_2) \in \mathbb{R}\) is Lipschitz in \((x, e_0)\) uniformly in \((t, e_1, e_2)\) i.e. \(\text{ess sup}_{\mathcal{E}_v^T} |\partial_x \sigma^\circ| + |\partial_{e_1} \sigma^\circ| + |\partial_{e_2} \sigma^\circ| < \infty\).

2.2 Approximation by particle system

In this section, we provide a way to approximate a solution of the system (2.2) by using interacting processes.

We say \(f \in L^1_{\text{prob}}(\mathbb{R}^+ \times \mathbb{R}^2)\) if \(f \geq 0\), and for each \(t \in \mathbb{R}^+, \int_{\mathbb{R}^2} f(t, x) \, dx = 1\). Let \(e \geq 1\). In addition, we assume that

Assumption 2.5. For any \(T > 0, f, f' \in L^1_{\text{prob}}(\mathbb{R}^+ \times \mathbb{R}^2) \) and \((t, x, x') \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2\), we have

\[
|\langle b, \sigma \rangle(f)(t, x) - [b, \sigma](f')(t, x')| \leq M \left[ |x - x'| + |f(t, x) - f'(t, x')| + |f_1(t, x_1) - f'_1(t, x_1)| + ||f - f'||_{L^1([0, T] \times \mathbb{R}^2)} \right].
\]

Let \(\delta > 0\) and \(G_\delta : \mathbb{R} \to \mathbb{R}^+\) be a kernel satisfying: \(\int_{\mathbb{R}} G_\delta(x) \, dx = 1\), for any \(\alpha \geq 1\),

\[
\lim_{\delta \to 0} G_\delta * \varphi = \varphi, \text{ a.e.,} \quad \|G_\delta * \varphi\|_{L^\alpha(\mathbb{R})} \leq C \|\varphi\|_{L^\alpha(\mathbb{R})} \text{ and } \|\nabla (G_\delta * \varphi)\|_{L^\alpha(\mathbb{R})} \leq C \|\nabla \varphi\|_{L^\alpha(\mathbb{R})}, \text{ for all } \varphi \in C_c^\infty(\mathbb{R})
\]

where \(C\) is independent of \(\delta\) and \(*\) denotes the convolution product. We set \(G_\delta(x) := G_\delta(x_1)G_\delta(x_2)\). For any process \(\nu := (\nu_t \in \mathbb{R}_+^*) \subset \mathcal{P}(\mathbb{R}^2)\), we introduce the quantity

\[
G_\delta * \nu(t, x) := \int_{\mathbb{R}^2} G_\delta(x_1 - x'_1, x_2 - x'_2) \nu_t(dx'_1, dx'_2).
\]
Since Assumption 2.5 holds, it is easy to check that (see Section 3.6), for each \( T > 0, C([0, T]; \mathcal{P}_e(\mathbb{R}^2)) \times [0, T] \times \mathbb{R}^2 \ni (\nu, t, x) \rightarrow [b, \sigma](G_0 + \nu)(t, x) \in \mathbb{R}^3 \) is Lipschitz in \((\nu, t)\) uniformly in \( t \) where \( C([0, T]; \mathcal{P}_e(\mu)) \) is the set of continuous functions over \([0, T]\) with value in \( \mathcal{P}_e(\mathbb{R}^2) \). Let \( \mathbf{X} := (X^1, X^2) \) be an \( \mathbb{R}^2 \)-valued \( \mathbb{F} \)-adapted continuous process satisfying \( p(0, \cdot) = \varrho(\cdot, \cdot) \), for \( t \in [0, T] \),

\[
dX^1_t = b(\mathbf{G}_0 + \mu)(t, \mathbf{X}^1_t)dt + \sigma(\mathbf{G}_0 + \mu)(t, \mathbf{X}^1_t)W_t \quad \text{and} \quad dX^2_t = \lambda(t, X^2_t)dt + \beta(t, X^2_t)dB_t,
\]

where \( \mu = \mathcal{L}(\mathbf{X}^1_t) = p^\delta(t, \mathbf{x})dx \). We consider \((W_i, B^i)_{i \geq 1}\) a sequence of independent random variables s.t. for each \( i, W_i \) and \( B^i \) are two \( \mathbb{R} \)-valued Brownian motions s.t. d\((W_i, B^i)\), \( i \geq 1 \). Besides the Assumptions of Theorem 2.3, the initial density \( p_0 \) is s.t. \( \int_{\mathbb{R}^2} |x|^r p_0(dx) < \infty \) for \( r > 1 \). For each \( \delta > 0 \), let \((\mathbf{X}^i_0, \ldots, \mathbf{X}^N)\) be the unique solution of: \((\mathbf{X}^i_0, \ldots, \mathbf{X}^N)\) is i.i.d. with \( \mathcal{L}(\mathbf{X}^0_0)(dx) = p_0(x)dx \),

\[
d\mathbf{X}^N,i = b(\mathbf{G}_0 + \mu^N)(t, \mathbf{X}^N,i)dt + \sigma(\mathbf{G}_0 + \mu^N)(t, \mathbf{X}^N,i)dW_t \quad \text{and} \quad d\mathbf{X}^N,i = \lambda(t, \mathbf{X}^N,i)dt + \beta(t, \mathbf{X}^N,i)dB_t,
\]

where \( \mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{\mathbf{X}^N,i} \). Let us fixed \( T > 0 \).

Proposition 2.6. 1. The sequence \((p^\delta)_{\delta > 0}\) is relatively compact in \( C([0, T] \times \mathbb{R}^2) \) for the uniform topology and each limit point \( p \) is s.t. \( p(t, x)dx = \mathcal{L}(\mathbf{X}^t) \) where \( \mathbf{X}^t \) is a solution of the system (2.2).

2. For each \( \delta > 0 \), the sequence \((\mu^N)_{N \geq 1}\) converges towards \((\mu^\delta)_{t \in [0, T]} \) in \( W_6 \). In addition, for any \( k \geq 1 \) and any bounded measurable map \( \phi : [0, T] \times \mathbb{R}^2k \rightarrow \mathbb{R} \)

\[
\lim_{N \rightarrow \infty} \mathbb{E} \left[ \int_0^T \phi(t, \mathbf{X}^N, x^1, \ldots, \mathbf{X}^N,k) dt \right] = \int_{[0, T] \times \mathbb{R}^2k} \phi(t, x^1, \ldots, x^k) p^\delta(t, x^1) \cdots p^\delta(t, x^k) dx^1 \cdots dx^k dt.
\]

Remark 2.7. Due to its non–regular aspect, providing a numerical scheme to simulate a solution of the system (2.2) can be challenging. Proposition 2.6 shows a propagation of chaos result which naturally leads to a numerically scheme to solve the system (2.2).

2.3 Local stochastic volatility model

As mentioned in the introduction, one of the famous application of the SDE like Equation (1.4) or Equation (2.2) is in the calibration of local stochastic volatility models (see Lipton [20], Piterbarg [22], Guyon and Henry- Labordere [11], Tian, Zhu, Lee, Klebaner, and Hamza [25], Saporito, Yang, and Zubelli [23], Bayer, Belomestny, Butkovsky, and Schoenmakers [4]). Let us here describe this model more precisely. Let \((\sigma_t)_{t \in [0, T]}\) be an \( \mathbb{R} \)-valued \( \mathbb{F} \)-predictable process. The process \((\sigma_t)_{t \in [0, T]}\) plays the role of stochastic volatility process. We consider \((S_t)_{t \in [0, T]}\) an \( \mathbb{R} \)-valued \( \mathbb{F} \)-adapted which is the spot price following the dynamics (under risk neutral measure with no interest rate and no dividends for simplification)

\[
dS_t = S_t \Sigma(t, S_t) \sigma_t dW_t.
\]

If we define the map \( \Sigma \) and the process \( \overline{\Sigma} \) by

\[
\overline{\Sigma}(t, x) := (\Sigma(t, x) \sqrt{\mathbb{E}[\sigma^2_t | S_t = x]}) \quad \text{and} \quad d\overline{\Sigma}_t = \overline{\Sigma}(t, \overline{S}_t)dW_t \quad \text{with} \quad \overline{\Sigma}_0 = \overline{\Sigma}_0,
\]

by Markovian projection, we know that \( \mathcal{L}(\sigma_t) = \mathcal{L}(\overline{\Sigma}_t) \) for each \( t \in [0, T] \). With this manipulation, the marginal distribution of \( S \) matches the marginal distribution of \( \overline{S} \). The dynamics of \( \overline{S} \) has the particularity to follow the dynamics of a local volatility model. These two spot prices leads to, for instance, the same European option prices. Let \( C(t, K) := C(t, K; S_0) \) be the observed call option price of maturity \( t \), strike \( K \) with \( S_0 \) the initial price of the underlying asset. If we assume that we have access to the observed prices \( \{C(t, K), K > 0, t > 0\} \), by using Dupire’s formula (see Dupire [9]), we know that the local volatility model \( \overline{\Sigma} \) calibrates the observed call option prices if

\[
\overline{\Sigma}(t, x) = \frac{\sigma_D(t, x)}{\sqrt{\mathbb{E}[\sigma^2_t | S_t = x]}},
\]

Consequently, in order to make \( S \) fits the observed call option prices the map \( \Sigma(t, x) \) must verify

\[
\Sigma(t, x) = \frac{\sigma_D(t, x)}{\sqrt{\mathbb{E}[\sigma^2_t | S_t = x]}},
\]
This means $S$ has to satisfy
\[ dS_t = \frac{S_t \sigma_D(t, S_t) \sigma_t}{\sqrt{\mathbb{E} [\sigma_t^2 | S_t]}} dW_t \quad \text{or equivalently} \quad dX_t = -\frac{1}{2} \frac{\sigma_t^2 \sigma_D(t, e^{X_t})^2}{\mathbb{E} [\sigma_t^2 | X_t]} dt + \frac{\sigma_t \sigma_D(t, e^{X_t})}{\sqrt{\mathbb{E} [\sigma_t^2 | X_t]}} dW_t \quad \text{for} \; X_t = \log(S_t). \quad (2.5) \]

The existence of a local stochastic volatility model calibrating the European option prices is then equivalent to find $S$ or $X$ solution of $(2.5)$. This problem is considered by the mathematical finance community as a very difficult problem. With an appropriate choice of the stochastic volatility process $(\sigma_t)_{t \in [0,T]}$, when $\sigma_D$ satisfies some conditions, Theorem 2.3 allows to prove the existence of $X$ (or $S$).

We set $\sigma_t$ by
\[ \sigma_t^2 := c + p_X(t, X_t) v(t, X_t, Y_t) \]
where $c > 0$, $p_X$ is the density of $\mathcal{L}(X_0)$, and $dY_t = \lambda(t, Y_t) dt + \beta(t, Y_t) dB_t$ with $B$ is a Brownian motion verifying $d(W, B)_t = \theta dt$. We recall the assumptions that we mentioned in the introduction. The maps $v$ and $\sigma_D$ are bounded above and below by positive constants, and $\mathbb{R}^+ \times \mathbb{R} \ni (t, x, y) \mapsto v(t, x, y) \sigma_D(t, e^x)^2 \in \mathbb{R}$ is Lipschitz in $(x, y)$ uniformly in $t$. Besides, $\beta$ is non–degenerate and for simplicity, the map $\mathbb{R}^+ \times \mathbb{R} \ni (t, y) \mapsto -\theta \beta(t, y) \in \mathbb{R}$ is non–negative. We have the next result

**Proposition 2.8.** Let $T > 0$. There exists $(X, Y)$ an $\mathbb{R}^2$–valued $\mathbb{F}$–adapted continuous process verifying: $\mathcal{L}(X_0, Y_0) = p_0(x) dx$, for each $t \leq T$,
\[ dX_t = -\frac{1}{2} \frac{\sigma_D^2(t, e^{X_t})}{c + p_X(t, X_t) \mathbb{E}[v(t, X_t, Y_t) | X_t]} v(t, X_t, Y_t) dt + \frac{\sigma_D(t, e^{X_t})}{c + p_X(t, X_t) \mathbb{E}[v(t, X_t, Y_t) | X_t]} dW_t \quad (2.6) \]
and
\[ dY_t = \lambda(t, Y_t) dt + \beta(t, Y_t) dB_t \quad (2.7) \]
where $p_X(t, \cdot)$ is the density of $\mathcal{L}(X_t)$.

**Remark 2.9.** As we will see in proof (just below), this proposition is essentially an application of Theorem 2.3. Due to the assumptions we must verify for the application of Theorem 2.3 (see Assumption 2.1), we are unable to take $c = 0$. Notice that, the condition: the map $\mathbb{R}^+ \times \mathbb{R} \ni (t, y) \mapsto -\theta \beta(t, y) \in \mathbb{R}$ is non–negative, can be replaced by a general non–degenerative condition (see Assumption 2.1).

**Proof.** Let us introduce the maps
\[ \sigma^o(t, x_1, x_2, e_0, e_1, e_2)^2 := \sigma_D^2(t, e^{x_1}) \frac{c + e_1 v(t, x_1, x_2)}{c + e_2} \quad \text{and} \quad b^o(t, x_1, x_2, e_0, e_1, e_2) := -\frac{1}{2} \sigma^o(t, x_1, x_2, e_0, e_1, e_2)^2. \]

Notice that
\[ \mathcal{E}^T_v := \{(t, x, e_0, e_1, e_2) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} : e_1 m_v \leq e_2 \leq e_1 M_v\} \]
where $0 < m_v \leq v \leq M_v$ and $|\nabla v| \leq M_v$. To prove the proposition, it is enough to show that the map $(b^o, \lambda, \sigma^o, \beta)$ satisfies assumptions of the first example of Section 2.1. For any $e_1 \geq 0$,
\[ m_D \frac{c + e_1 M_v}{c + e_1 m_v} \leq \sigma^o(t, x_1, x_2, e_0, e_1, e_2)^2 \leq M_D \frac{c + e_1 M_v}{c + e_1 m_v}, \quad \text{for all} \; (t, x, e_0, e_1, e_2) \in \mathcal{E}^T_h, \]
where $0 < m_D \leq \sigma_D^2 \leq M_D$. We can verify that the map $\mathbb{R}^+ \ni z \mapsto \frac{c + e_1 M_v}{c + e_1 m_v} \in \mathbb{R}$ is decreasing and the map $\mathbb{R}^+ \ni z \mapsto \frac{c + e_1 M_v}{c + e_1 m_v} \in \mathbb{R}$ is increasing. Therefore, for $(t, x, e_0, e_1, e_2) \in \mathcal{E}^T_h$,
\[ m_D \frac{m_v}{M_v} \leq \sigma^o(t, x_1, x_2, e_0, e_1, e_2)^2 \leq M_D \frac{M_v}{m_v}. \]

Since $\beta$ is non–degenerate and, the map $\mathbb{R}^+ \times \mathbb{R} \ni (t, y) \mapsto -\theta \beta(t, y) \in \mathbb{R}$ is non–negative, we verify that $0 < \inf_{\mathcal{E}^T_v} \inf_{z \neq 0} \frac{a^{1.1}}{z} \leq a^{2.1} := \frac{1}{2} \sigma^o$, $a^{2.2} := \frac{1}{2} \beta^2$, $a^{1.2} = a^{2.1} := \theta \beta \sigma^o$. As $c > 0$ and the map $\mathbb{R}^2 \ni (x_1, x_2) \mapsto \lambda(t, y) \sigma^o(t, e^{x_1}) \frac{c + e_1 v(t, x_1, x_2)}{c + e_2}$
as we have done in the previous proof, we get

\[
\partial_x \sigma_x^2 = \partial_x \sigma_D^2 \frac{c + e_1 v}{c + e_2}, \quad \partial_x \sigma_x^2 = \frac{e_1 \sigma_D^2}{c + e_2}, \quad \partial_{t, x} \sigma_x^2 = \frac{\nu \sigma_D^2}{c + e_2} \quad \text{and} \quad \partial_{t, x} \sigma_x^2 = -\sigma_D^2 \frac{c + e_1 v}{c + e_2^2}.
\]

Therefore, as \( c > 0 \) and \( m_D \frac{\nu}{\sigma_D} \leq \inf_{(t, x, e_0, e_1, e_2)} \sigma_D^2 (t, x, e_0, e_1, e_2)^2 \), we verify that

\[
\sup_{(t, x, e_0, e_1, e_2)} \left| \nabla \sigma_D (t, x, e_0, e_1, e_2) \right| < \infty.
\]

We can conclude that the map \((b^o, \lambda, \sigma^o, \beta)\) satisfies the assumptions of the first example of Section 2.1. By applying Theorem 2.3, there exists \((X, Y)\) satisfying \(G(X, Y) = p_0(x)dx\),

\[
dX_t = -\frac{1}{2} \sigma_D^2 (t, e^{X_t}) \frac{c + p_1(t, X_t) v(t, X_t, Y_t)}{c + (pv)_1(t, Y_t)} \, dt + \sigma_D (t, e^{X_t}) \sqrt{\frac{c + p_1(t, X_t) v(t, X_t, Y_t)}{c + (pv)_1(t, Y_t)}} \, dW_t
\]

and

\[
dY_t = \lambda(t, Y_t) dt + \beta(t, Y_t) dB_t
\]

where \(p\) is the density of \(G(X, Y)\). Notice that \((pv)_1(t, X_t) = p_1(t, X_t) \mathbb{E}[v(t, X_t, Y_t)|X_t]\) with \(p_X(t, X_t) = p_1(t, X_t)\).

This is enough to deduce the proposition.

**Remark 2.10.** By putting (2.6) + (2.7) into the framework of Theorem 2.3 as we have done in the previous proof, we can check that the couple \((X, Y)\) falls into the context of Proposition 2.6. We can therefore provide an approximation by particle system of a solution \((X, Y)\) of (2.6) + (2.7).

### 3 Proof of main results

#### 3.1 Main idea leading the proofs

For ease of reading, we provide in this part the idea leading the proof. The proof of Theorem 2.3 is essentially an application of a fixed point Theorem namely Schauder fixed point Theorem. This fixed point Theorem is applied on a map defined over an appropriate set of probability density \( p \). In order to obtain the appropriate set of probability density, the key step is to provide an explicit estimate of the gradients of \( p \) and \( p_1 \) where \( p \) is the density of \( G(X) \) with \( X \) an SDE process. These estimates must be given explicitly according to the regularity of the coefficients of the SDE \( X \). For sake of simplification, we present the main idea in less general framework and only for the gradient of \( p \). We use Einstein notation and refer to Section 3.2 for the definition of some functional spaces. Let \( T_0 = 1 \) and \( a := (a^{i, j})_{1 \leq i, j \leq 2} : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R} \times S^2 \) be a Borel map s.t. \( 0 < m_I \leq a \leq M_I \). Let \( X = (X^1, X^2) \) be a solution of

\[
dX_t = a(t, X_t)^{1/2} \left( \frac{dW^1_t}{dW^2_t} \right)
\]

where \( W^1 \) and \( W^2 \) are independent Brownian motions.

The measure \(G(X)|dx|dt\) has a density \( p(t, x) \) w.r.t. the Lebesgue measure on \([0, T] \times \mathbb{R}^2\) (see for instance Proposition 3.4). Before giving the detailed proofs in the next sections, let us give some heuristic arguments on the estimate of gradient of \( p \) i.e. \( \nabla p \). First, by Itô’s formula (see also Lemma 3.5)

\[
\int_{[0, T_0] \times \mathbb{R}^2} w(t) |\partial_t \phi + a^{i, j} \partial_{x, i} \partial_{x, j} \phi |(t, x) p(t, x) \, dx \, dt \leq \| w' \phi \|_{L^1([0, T_0] \times \mathbb{R}^2)},
\]

for any \( w \in C^\infty([0, T_0]) \) with \( w(T_0) = 0 \), \( \phi \in C^{1,2}([0, T_0] \times U_R) \) with \( \phi(0, \cdot) = 0 \) and \( U_R \) is a ball of diameter \( R \).

For any \( R > 0 \), let \( f := (f^i)_{1 \leq i \leq 2} \subset C^\infty_c([0, T_0] \times U_R) \). Let \( \phi : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R} \) be a map verifying: \( \phi(0, \cdot) = 0 \), \( \phi|_{[0, T_0] \times \partial U_R} = 0 \) and

\[
\partial_t \phi + a^{i, j} \partial_{x, i} \partial_{x, j} \phi = \partial_{x, i} f^i \quad \text{on} \quad (0, T_0) \times U_R.
\]
As we will show in Proposition 3.1, there exists $N$ a constant and $\ell$ a map depending only on $m$, $M$, $T_0$, the dimension $d = 2$ and $\alpha \in (1, \infty) \setminus \{2\}$ (we write $\alpha' := \alpha/(\alpha - 1)$ see Section 3.2 just below) s.t. for any $\pi \in (0, 1)$,

$$\|\phi\|_{H^{\alpha_1}(U_{R,T_0})} \leq N\|\partial_x f^i\|_{H^{\alpha_1}(U_{R,T_0})} + \ell(\pi, \alpha, R)\{\sup_{t \in (0,T_0)} \|a(t, \cdot)\|_{C^{0,\tau}(U_R)} + \|\nabla a\|_{L^{\alpha_1}(U_{R,T_0})}\}\|\phi\|_{H^{\alpha_1}(U_{R,T_0})}$$

where $\mathbb{R} \ni r \to \ell(\pi, \alpha, r) \in \mathbb{R}_+$ is continuous and increasing with $\ell(\pi, \alpha, 0) = 0$. Let us take $0 < \overline{R} \leq 1$ s.t.

$$1 - \ell(\pi, \alpha, \overline{R}) \sup_{E \in \mathcal{S}} \left\{\sup_{t \in (0,T_0)} \|a(t, \cdot)\|_{C^{0,\tau}(E)} + \|\nabla a\|_{L^{\alpha_1}(E)}\right\} > 0 \text{ where } \mathcal{S} := \{E \subset \mathbb{R}^2 \text{ open set s.t. diam}(E) \leq 1\}.$$

In Proposition 3.6, we then show that: for any $f := (f^i)_{1 \leq i \leq 2} \in C_c^\infty((0,T_0) \times U_R)$ and $R \leq \overline{R}$,

$$\left|\int_{U_{R,T_0}} \partial_x f^i(t,x)w(t)dx\right| \leq \frac{N\|w\|_{L^{\alpha_1}(U_{R,T_0})}}{1 - \ell(\pi, \alpha, \overline{R})} \sup_{E \in \mathcal{S}} \left\{\sup_{t \in (0,T_0)} \|a(t, \cdot)\|_{C^{0,\tau}(E)} + \|\nabla a\|_{L^{\alpha_1}(E)}\right\}\|\partial_x f^i\|_{H^{\alpha_1}(U_{R,T_0})}.$$

Since $\|\partial_x f^i\|_{H^{\alpha_1}(U_{R,T_0})} \leq \|f\|_{L^{\alpha_1}(U_{R,T_0})}$, this estimate allows us to deduce by a duality argument in Proposition 3.8,

$$\|w\nabla p\|_{L^{\alpha_1}((0,T_0) \times \mathbb{R}^2)} \leq \frac{N2^{2/\alpha'}}{1 - \ell(\pi, \alpha, \overline{R})} \sup_{E \in \mathcal{S}} \left\{\sup_{t \in (0,T_0)} \|a(t, \cdot)\|_{C^{0,\tau}(E)} + \|\nabla a\|_{L^{\alpha_1}(E)}\right\}\|w\|_{L^{\alpha_1}((0,T_0) \times \mathbb{R}^2)} \leq A(\pi, \alpha, \alpha')\|w\|_{L^{\alpha_1}((0,T_0) \times \mathbb{R}^2)}.$$

When $\alpha' = q = 3 + 1/4$, by using Sobolev embedding Theorem, the result of Krylov recall in Proposition 3.4 and some recursive properties, we find in Proposition 3.16 a constant $H := H(A(\pi, q, q'), m, M, |w|_\infty, \cdots, |w|_\infty^{(2)})$ depending only on $A(\pi, q, q')$, $m$, $M$, and the supremum of $w$ and its first 2-derivatives s.t. the constant $H$ is a locally bounded function of the indicated quantities and

$$\|w\|_{L^{\alpha_1}((0,T_0) \times \mathbb{R}^2)} \leq H(A(\pi, q, q'), m, M, |w|_\infty, \cdots, |w|_\infty^{(2)}).$$

As $p$ satisfies a parabolic equation, if the initial density $p(0, \cdot)$ is regular enough, we can deduce Hölder estimate of $p$ using the $L^q$ estimates of $\nabla p$ (see Proposition 3.3 or Proposition 3.12). Therefore, using the detail of the estimates we have obtained, we can choose $L$ and $\overline{R}$ (see Section 3.4) s.t.

$$\sup_{t \in (0,T_0)} \left\{\sup_{E \in \mathcal{S}} \|a(t, \cdot)\|_{C^{0,\tau}(E)} + \|\nabla a\|_{L^q((0,T_0) \times E)}\right\} \leq L \text{ and } 1 - \ell(\pi, q', \overline{R})L > 0$$

leading to

$$\sup_{t \in (0,T_0)} \|w(t, \cdot)\|_{C^{0,\tau}(\mathbb{R}^2)} + \|w\nabla p\|_{L^q((0,T_0) \times \mathbb{R}^2)} \leq L.$$

Consequently, if $a$ depends on $p$, we see that we can realize a fixed point. By using similar techniques, we obtain in Proposition 3.13 and Proposition 3.11, similar estimates for $p_1$ using the regularity of the second component of the SDE i.e. $X^2$. In the next sections, we will provide the details of the arguments mentioned in a more general framework. In particular, we will see how to handle the map $w$ appearing in the estimates.

### 3.2 Background materials

**Reminder on functional spaces** We start by recalling some functional spaces and their properties. Let $d \in \mathbb{N}^*$. The open ball of $\mathbb{R}^d$ with radius $r/2$ (thus of diameter $r$) centered at $a$ is denoted by $U_a(r)$ or $U_{r,a}$. When the center is not important, we will simply write $U_r$. As already mentioned in the previous section, we use Einstein notation i.e. in expressions like $a^{ij}x_iy_j$ and $b_{x_i}$ the standard summation rule with respect to repeated indices will be meant.
Given an open set $\Omega \subset \mathbb{R}^d$ and a Banach space $(E, \|\cdot\|_E)$, for a nonnegative measure $\mu$ and $q \in [1, \infty)$, the symbols $L^q(\mu; E)$ or $L^q(\Omega, \mu; E)$ denote the space of equivalence classes of $\mu$–measurable functions $f$ such that the function $\|f\|_E^q$ is $\mu$–integrable. This space is equipped with the standard norm
\[
\|f\|_q := \|f\|_{L^q(\Omega, \mu; E)} := \left( \int_\Omega |f(x)|^q \mu(dx) \right)^{1/q}.
\]

We say $f \in L^q_{\text{loc}}(\Omega, \mu; E)$ if for any compact $Q \subset \Omega$, we have $\|f\|_E^q 1_Q$ is $\mu$–integrable. When $E = \mathbb{R}^\ell$ for $\ell \geq 1$ or when $E$ is obvious, we will simply write $L^q(\mu)$ or $L^q(\Omega, \mu)$. Also, the notation $L^q(\Omega)$ or $L^q_{\text{loc}}(\Omega)$ always refers to the situation where $\mu$ is the classical Lebesgue measure. As usual, for $q \in [1, \infty]$ we set $q'$ the conjugate of $q$ by
\[
q' := \frac{q}{q-1}.
\]

We denote by $W^{q,1}(\Omega)$ or $H^{q,1}(\Omega)$ the Sobolev class of all functions $f \in L^q(\Omega)$ whose generalized partial derivatives $\partial_x f$ are in $L^q(\Omega)$. A generalized (or Sobolev) derivative is defined by the equality (the integration by parts formula)
\[
\int_\Omega f(x)\partial_x \varphi(x)dx = -\int_\Omega \partial_x f(x)\varphi(x)dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\]

This space is equipped with the Sobolev norm
\[
\|f\|_{q,1} := \|f\|_{H^{q,1}(\Omega)} := \left( \|f\|_q^q + \sum_{i=1}^d \|\partial_x f\|_q^q \right)^{1/q}.
\]

The class $W^{q,1}_0(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{q,1}(\Omega)$. Let $U_R$ be an open ball of diameter $R$. The dual of the space $W^{q,1}_0(U_R)$ is denoted $W^{q,-1}(U_R)$ or $H^{q,-1}(U_R)$. When $u : U_R \to \mathbb{R}$ is a map, we will say $u \in W^{q,1}_0(U_R)$ whenever the linear map $W^{q,1}_0(U_R) \ni v \to \langle u, v \rangle \in \mathbb{R}$ is well defined and continuous. We refer to [2, Chapter 3] for an overview on the dual of Sobolev spaces.

Let $J \subset [0, T]$ be an interval (open or closed) and $U$ be an open set in $\mathbb{R}^d$. Let $H^{q,1}(J \times U)$ denote the space of all measurable functions $u$ on the set $J \times U$ such that $u(t, \cdot) \in W^{q,1}_0(U)$ for almost all $t$ and the norm
\[
\|u\|_{H^{q,1}(J \times U)} := \left( \int_J \|u(t, \cdot)\|_{H^{q,1}_0(U)}^q dt \right)^{1/q}
\]
is finite. The space $H^{q,1}_0(J \times U)$ is defined similarly, but with $H^{q,1}_0(\Omega)$ in place of $H^{q,1}(\Omega)$, and $H^{q,-1}_0(J \times U)$ denotes its dual.

**Estimates for parabolic equation** Now, we provide local estimates for a parabolic equation. This estimate is important to give estimates of solution of Fokker–Planck equation as we will see in Section 3.3. Let $T_0 > 0$, $d = 2$, $U_R \subset \mathbb{R}^d$ be an open ball of diameter $R > 0$, and we denote $U_{R,T_0} := (0, T_0) \times U_R$. Also, let us take Borel maps $b := (b^i)_{1 \leq i \leq 2} : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R}^2$, $a := (a^{ij})_{1 \leq i,j \leq 2} : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$, $g : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R}$ and a differentiable map in space $f := (f^i)_{1 \leq i \leq 2} : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R}^2$. The maps $g$ and $f$ have compact supports on $[0, T_0] \times \mathbb{R}^2$ included in $U_{R,T_0}$. We consider $\phi \in C^{1,2}(U_{R,T_0})$ satisfying: $\phi(0, \cdot) = 0$, $\phi|_{[0,T_0] \times \partial U_R} = 0$, and
\[
\partial_t \phi(t, x) + \sum_{i=1}^d b^i(t, x)\partial_x \phi(t, x) + \sum_{i,j=1}^d a^{ij}(t, x)\partial_{x_i} \phi(t, x) = \sum_{i=1}^d \partial_{x_i} f^i(t, x) + g(t, x) \quad \text{for all } (t, x) \in U_{R,T_0}.
\]

To simplify the notation, we will use the standard summation rule with respect to repeated indices and write
\[
\partial_t \phi(t, x) + b^i(t, x)\partial_x \phi(t, x) + a^{ij}(t, x)\partial_{x_i} \phi(t, x) = \partial_{x_i} f^i(t, x) + g(t, x) \quad \text{for all } (t, x) \in U_{R,T_0}.
\]

For any $r \geq 1$, recall that $r'$ is the conjugate of $r$ i.e. $r' = r/(r-1)$. We assume that $a$ satisfies $0 < m I_2 \leq a \leq M I_2$. Although presented very differently, the next proposition is inspired by [5, Lemma 6.2.5].
Proposition 3.1. Let $\alpha > 1$ and $s > (2 - s)\vee 1$, there exists $N > 0$ depending only on $T_0$, $\alpha$, $m$, $M$ and the dimension $d$ s.t. for any $\pi \in (0, 1)$

$$
\|\phi\|_{H^{\alpha - 1}(U_R, T_0)} \leq N \|\partial_x F^i + g\|_{H^{\alpha - 1}(U_R, T_0)} + \|\pi, \alpha, R\left(\text{ess sup}_{t \in (0, T_0)} \|a(t, \cdot, \cdot)\|_{C^{0, \pi}(U_R)} + \Gamma(\alpha, a, b, U_R, T_0)\right)\|\phi\|_{H^{\alpha - 1}(U_R, T_0)}
$$

where $\Gamma : (0, 1) \times (1, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function in the third variable depending only on $(T_0, \alpha, m, M, d, \pi)$ with $\Gamma(\cdot, \cdot, 0) = 0$ and

$$
\Gamma(\alpha, a, b, U_R, T_0) := \|(\nabla a, b)\|_{L^{\infty}(U_R, T_0)} I_{1 \alpha > 2} + \|(\nabla a, b)\|_{L^{2, (2 - s)}(U_R, T_0)} I_{\alpha = 2} + \|(\nabla a, b)\|_{L^{\alpha'}(U_R, T_0)} I_{\alpha < 2}.
$$

Proof. Let $x_0 \in U_R$. We set $a_0(t) := a(t, x_0)$. We rewrite $\partial_t \phi + b^j \partial_x \phi + a^j \partial_x \phi = \partial_x \phi^i(t, x) + g(t, x) - b^i \partial_x \phi + \partial_x \phi^i(t, x) + \partial_x a^j(t, x) \partial_x \phi(t, x)$.

Since $\phi |_{[0, T_0] \times \partial U_R} = 0$, for $t \in (0, T_0)$, we can extend $\phi(t, \cdot)$ by 0 over $\mathbb{R}^2 \setminus U_R$ while keeping the same regularity. By Proposition A.1, there exists $N = N(T_0, \alpha, d, m, M) > 0$ satisfying

$$
\|\phi\|_{H^{\alpha - 1}(U_R, T_0)} = \|\phi\|_{H^{\alpha - 1}([0, T_0] \times \mathbb{R}^2)} \leq N \|\partial_x f^i + g - b^i \partial_x \phi - \partial_x \phi^i\|_{H^{\alpha - 1}([0, T_0] \times \mathbb{R}^2)} = N \|\partial_x f^i + g - b^i \partial_x \phi - \partial_x \phi^i\|_{H^{\alpha - 1}(U_R, T_0)}.
$$

First, one has

$$
\|\partial_x ((a^j - a^j) \partial_x \phi)\|_{H^{\alpha - 1}(U_R, T_0)} = \|\partial_x ((a^j - a^j) \partial_x \phi)\|_{H^{\alpha - 1}(U_R, T_0)} \leq \|\partial_x ((a^j - a^j) \partial_x \phi)\|_{L^\infty(U_R, T_0)} \leq \text{ess sup}_{(t, x) \in (0, T_0) \times U_R} \|a^j(t, x) - a^j(t_0)\|_{L^\infty(U_R, T_0)}.
$$

For $\pi \in (0, 1)$, $t \in [0, T_0]$ and $x \in U_R$ s.t. $x \neq x_0$, one has

$$
|a^j(t, x) - a^j(t_0)| \leq |x - x_0|^\pi |a^j(t, x) - a^j(t_0)| \leq |x - x_0|^\pi |a(t, \cdot)|_{C^{0, \pi}(U_R)} \leq \text{diam}(U_R)^\pi \|a(t, \cdot)|_{C^{0, \pi}(U_R)}\|_{H^{\alpha - 1}(U_R, T_0)}.
$$

Secondly, by [5, Lemma 1.1.7],

- For $\alpha > 2$, there exists a universal constant $C > 0$ depending only on the dimension s.t.

$$
\|\partial_x a^j(t, \cdot) \partial_x \phi(t, \cdot)\|_{H^{\alpha - 1}(U_R, T_0)} \leq C \|\partial_x a^j(t, \cdot)\|_{L^2(U_R)} \|\nabla \phi(t, \cdot)\|_{L^\infty(U_R, T_0)}.
$$

By using Hölder inequality successively, we find a constant $C$ depending on the dimension and $T_0$ s.t.

$$
\|\partial_x a^j \partial_x \phi\|_{H^{\alpha - 1}(U_R, T_0)} \leq C \text{diam}(U_R)^{(a - 2)/\alpha} \|\nabla a\|_{L^\infty(U_R, T_0)} \|\phi\|_{H^{\alpha - 1}(U_R, T_0)}.
$$

Using similar approach, we get

$$
\|b^j \partial_x \phi\|_{H^{\alpha - 1}(U_R, T_0)} \leq C \text{diam}(U_R)^{(a - 2)/\alpha} \|b\|_{L^\infty(U_R, T_0)} \|\phi\|_{H^{\alpha - 1}(U_R, T_0)}.
$$

- For $\alpha < 2$, there exists a universal constant $C > 0$ only depending on the dimension s.t.

$$
\|\partial_x a^j \partial_x \phi\|_{H^{\alpha - 1}(U_R, T_0)} \leq C \text{diam}(U_R)^{1 - d/\alpha} \|\partial_x a^j\|_{L^\infty(U_R, T_0)} \|\nabla \phi\|_{L^\infty(U_R, T_0)} \leq C \text{diam}(U_R)^{1 - d/\alpha} \|\nabla a\|_{L^\infty(U_R, T_0)} \|\phi\|_{H^{\alpha - 1}(U_R, T_0)}.
$$

Again, we can find that

$$
\|b^j \partial_x \phi\|_{H^{\alpha - 1}(U_R, T_0)} \leq C \text{diam}(U_R)^{1 - d/\alpha} \|b\|_{L^\infty(U_R, T_0)} \|\phi\|_{H^{\alpha - 1}(U_R, T_0)}.
$$
• For $\alpha = 2$, and $s > (2-s) \vee 1$, there exists a universal constant $C > 0$ only depending on the dimension $s$ t.s.

$$\|\partial_{x_r} a^{ij}(t, \cdot) \partial_{x_r} \phi(t, \cdot)\|_{H^{2-1/(U_R)}} \leq C \text{diam}(U_R)^{2+2/s} \|\partial_{x_r} a^{ij}(t, \cdot) \nabla \phi(t, \cdot)\|_{L^4(U_R)}.$$ 

Here again, by using Hölder inequality successively, we find a constant $C$ depending on the dimension and $T_0$ s.t.

$$\|\partial_{x_r} a^{ij} \partial_{x_r} \phi\|_{H^{2-1/(U_R,T_0)}} \leq C \text{diam}(U_R)^{2+2/s} \|\nabla \alpha\|_{L^2/(2-\sigma)(U_R,T_0)} \|\phi\|_{H^{2-1}(U_R,T_0)}.$$ 

and

$$\|b \partial_{x_r} \phi\|_{H^{2-1}(U_R,T_0)} \leq C \text{diam}(U_R)^{2+2/s} \|b\|_{L^2/(2-\sigma)(U_R,T_0)} \|\phi\|_{H^{2-1}(U_R,T_0)}.$$ 

By combining the previous results, we get the proof of the proposition by setting

$$\ell(\pi, \alpha, R) := N \left\{ \text{diam}(U_R)^{\pi} + C \text{diam}(U_R)^{1-d/\alpha'} 1_{\alpha < 2} + C \text{diam}(U_R)^{2+2/s} 1_{\alpha = 2} + C \text{diam}(U_R)^{(\alpha-2)/\alpha} 1_{\alpha > 2} \right\}.$$ 

The proof shows that $\ell$ is given by

$$\ell(\pi, \alpha, R) := N \left\{ \text{diam}(U_R)^{\pi} + C \text{diam}(U_R)^{1-d/\alpha'} 1_{\alpha < 2} + C \text{diam}(U_R)^{2+2/s} 1_{\alpha = 2} + C \text{diam}(U_R)^{(\alpha-2)/\alpha} 1_{\alpha > 2} \right\}. \quad (3.1)$$

Let $\tilde{C}(\pi, \alpha, a, b, U_R, T_0)$ be defined by

$$\tilde{C}(\pi, \alpha, a, b, U_R, T_0) := \text{ess sup}_{t \in (0,T_0)} \|a(t, \cdot)\|_{C^{\alpha, \pi}(U_R)} + \Gamma(\alpha, a, b, U_R, T_0). \quad (3.2)$$

From Proposition 3.1, we easily deduce the next corollary which gives an estimate of the solution $\phi$.

**Corollary 3.2.** If $1 - \ell(\pi, \alpha, R) \tilde{C}(\alpha, a, b, U_R, T_0) > 0$, one has

$$\|\phi\|_{H^{\alpha-1}(U_R,T_0)} \leq \frac{N}{1 - \ell(\pi, \alpha, R) \tilde{C}(\pi, \alpha, a, b, U_R, T_0)} \|\partial_{x_r} f^i + g\|_{H^{\alpha-1}(U_R,T_0)}.$$ 

In the next part, we study the regularity, in terms of the Hölder norm, of functions verifying a certain type of equation. Let $d \in \mathbb{N}^*$ and $T_0 > 0$. Let $\psi := (\psi_i)_{0 \leq i \leq d} : [0, T_0] \times \mathbb{R}^d \to \mathbb{R}^{d+1}$ and $u : [0, T_0] \times \mathbb{R}^d \to \mathbb{R}$ be Borel maps satisfying:

$$d(u(t, \cdot), \phi) = (\psi(t, \cdot, \phi) + \sum_{i=1}^d \psi_i(t, \cdot) \partial_{x_i} \phi) \text{d}t \text{ for any } \phi \in C_c^\infty(\mathbb{R}^d).$$

Let $\alpha > 2$ and $\frac{1}{\alpha} > \beta > \kappa > \frac{1}{\alpha}$ with $(1-2\beta)\alpha > d$.

**Proposition 3.3.** There exists $N > 0$ depending only on $(\kappa, \alpha, d, \beta, T_0)$ s.t.

$$\|u\|_{C^{0, \kappa-1/\alpha}(0,T_0), C^{0, 1-2\beta-\alpha/\alpha}(\mathbb{R}^d)} \leq N \left[ \|u\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|\psi\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|u(0, \cdot)\|_{H^{\alpha-1}(\mathbb{R}^d)} \right].$$

**Proof.** Let $\alpha > 2$ and $\frac{1}{\alpha} > \beta > \kappa > \frac{1}{\alpha}$. By [16, Theorem 7.2] (see also [5, Theorem 6.2.2]), there exists $N > 0$ depending only on $(\kappa, \alpha, d, \beta, T_0)$ s.t.

$$\|u\|_{C^{0, \kappa-1/\alpha}(0,T_0), W^{\alpha-1,2\beta}(\mathbb{R}^d)} \leq N \left[ \|\nabla^2 u\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|\psi\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|u(0, \cdot)\|_{H^{\alpha-1}(\mathbb{R}^d)} \right].$$

where $W^{\alpha,1-2\beta}(\mathbb{R}^d)$ denotes a fractional Sobolev space (see [7, Chapter 4] for details). We have $\|\nabla^2 u\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} \leq \|u\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d}$. Let us mention that the statement in [16, Theorem 7.2] assumes $\|u\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|\psi\|_{H^{\alpha-1}(0,T_0) \times \mathbb{R}^d} + \|u(0, \cdot)\|_{H^{\alpha-1}(\mathbb{R}^d)} < \infty$. However, in the case where this quantity is not finite, the inequality is trivially true. By [7, Theorem 4.47] for $(1-2\beta)\alpha > d$, there exists a constant $C > 0$ independent of $u$ s.t.

$$\|u\|_{C^{0, \kappa-1/\alpha}(0,T_0), C^{0, 1-2\beta-\alpha/\alpha}(\mathbb{R}^d)} \leq C \|u\|_{C^{0, \kappa-1/\alpha}(0,T_0), W^{\alpha-1,2\beta}(\mathbb{R}^d)}.$$

We can conclude the result. \qed
3.3 Estimates for Fokker–Planck equation

This section is dedicated to the analysis of the regularity of solution of Fokker–Planck in terms of the regularity of the coefficients. We will provide here some explicit estimates.

Let \( \theta \in [-1, 1] \) and \( \tilde{b}, \tilde{\lambda}, \tilde{\sigma}, \tilde{\beta} : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^4 \) be a Borel map. For \( x = (x_1, x_2) \), let us define \( b^1(t, x) := \tilde{b}(t, x), \)
\( b^2(t, x) := \tilde{\lambda}(t, x), \)
\( a^{1,1}(t, x) := \frac{1}{2} \tilde{\sigma}(t, x)^2, \)
\( a^{1,2}(t, x) := a^{2,1}(t, x) := \tilde{\sigma}(t, x) \tilde{\beta}(t, x) \theta \)
and \( a^{2,2}(t, x) := \frac{1}{2} \tilde{\beta}(t, x)^2. \)

These maps satisfy
\[
m I_2 \leq a \quad \text{and} \quad |(\tilde{b}, \tilde{\lambda}, \tilde{\sigma}, \tilde{\beta})| = M.
\]

Let \( X := (X^1, X^2) \) be a weak solution of
\[
dX^1_t = \tilde{b}(t, X_t) dt + \tilde{\sigma}(t, X_t) dW_t \quad \text{and} \quad dX^2_t = \tilde{\lambda}(t, X_t) dt + \tilde{\beta}(t, X_t) dB_t \quad \text{where} \quad d(W, B)_t = \theta dt.
\]

Under these assumptions, the map \( p(t, x_1, x_2) \) which denotes the density of \( \mu_t := \mathcal{L}(X_t) \) is well defined almost every \( t \).

3.3.1 Estimates with Borel measurable coefficients

Without assuming any additional assumptions over \((a, b)\), we give an estimate of the density \( p \). This proposition is essentially an application of [14, Chapter 2 Section 3 Theorem 4]. Its proof mainly uses geometric arguments.

**Proposition 3.4.** Let \( T_0 > 0 \) and \( \gamma \in [1, (d + 1)^{'}] \). There exists a constant \( G > 0 \) depending only on \( T_0, \gamma \), the dimension \( d = 2 \), and \( M \) i.e. \( G = G(T_0, \gamma, 2, m, M) \) s.t.
\[
\int_{[0, T_0] \times \mathbb{R}^2} |p(t, x_1, x_2)|^\gamma d x_1 d x_2 d t \leq G.
\]

**Proof.** By [14, Chapter 2 Section 3 Theorem 4], there exists a constant \( G > 0 \) depending only on \( T_0, \gamma \), the dimension \( d = 2 \), and \( M \) s.t. for any Borel \( f : [0, T_0] \times \mathbb{R}^d \to \mathbb{R} \)
\[
\int_{[0, T_0] \times \mathbb{R}^2} |f(t, x)| p(t, x) d x d t \leq G\|f\|_{L^{d+1}([0, T_0] \times \mathbb{R}^d)}.
\]

By using a duality argument, we deduce the result. \( \square \)

By using Itô’s formula and taking the expectation, we find that for any \( \varphi \in C^\infty_c(\mathbb{R}_+ \times \mathbb{R}^2) \),
\[
d\langle \varphi, \mu_t \rangle = \int_{\mathbb{R}^2} [\partial_t \varphi(t, x) + b^i(t, x) \partial_x^i \varphi(t, x) + a^{i,j}(t, x) \partial_x^i \partial_x^j \varphi(t, x)] \mu_t(dx) dt = \int_{\mathbb{R}^2} [\partial_t \varphi(t, x) + \mathcal{L} \varphi(t, x)] \mu_t(dx) dt.
\]

For any \( T_0 > 0 \), let us introduce
\[
C^\infty_0(T_0) := \{ w \in C^\infty([0, \infty)) \text{ s.t. supp}(w) = [0, T_0) \}.
\]

Notice that if \( w \in C^\infty_0(T_0) \), \( w^{(j)}(T_0) = 0 \) and \( w^{(j)} \in C^\infty_0(T_0) \) for all \( j \geq 0 \) where \( w^{(j)} \) denote the \( j \)th derivative of \( w \). We provide, in the next lemma, estimates involving the generator of the process \( X \) i.e. \( \mathcal{L} \). Let \( T_0 > 0, R > 0 \) and \( U_R = U_R^1 \times U_R^2 \).

**Lemma 3.5.** Let \( w \in C^\infty_0(T_0) \) and \( \alpha \in (1, \infty) \). For any \( \phi \in C^\infty_c([0, \infty) \times U_R) \) verifying \( \phi(0, \cdot) = 0 \), one has
\[
\int_{[0, T_0] \times \mathbb{R}^2} w(t)[\partial_t \phi(t, x) + \mathcal{L} \phi(t, x)] \mu_t(dx) dt \leq \int_{U_R \times T_0} |w(t)\phi(t, x)p(t, x)| dx dt = \|w \phi p\|_{L^1(U_R \subset T_0)}
\]
and
\[
\|w \phi p\|_{L^1(U_R \subset T_0)} \leq 2\|\phi\|_{H^{\alpha,1}(U_R \subset T_0)} \int_{U_R^2} \|w p(\cdot, x_2)\|_{L^{\alpha}(U_R \subset T_0)} dx_2.
\]
Proof. Since \( w \in C_0^\infty(T_0) \), we have in particular that \( w(T_0) = 0 \). By applying Itô’s formula and taking the expectation, we find
\[
- \int_{[0,T_0] \times \mathbb{R}^2} w'(t) \phi(t, x) \mu_4(dx) dt = \int_{[0,T_0] \times \mathbb{R}^2} w(t) \left[ \partial_t \phi(t, x) + b'(t, x) \partial_x \varphi(t, x) + a^{i,j}(t, x) \partial_{x_i} \partial_{x_j} \phi(t, x) \right] \mu_4(dx) dt.
\]
Consequently, we have
\[
\left| \int_{[0,T_0] \times \mathbb{R}^2} w(t) \left[ \partial_t \phi(t, x) + b'(t, x) \partial_x \varphi(t, x) + a^{i,j}(t, x) \partial_{x_i} \partial_{x_j} \phi(t, x) \right] \mu_4(dx) dt \right| \leq \int_{U_{R,T_0}} |w'(t) \phi(t, x)p(t, x)| dx dt.
\]
Next, for any \( g \in C_c^\infty(U_R^2 \setminus \{0\}) \), by Morrey’s inequality, for any \( \alpha > 1 \), there exists a constant \( C \) depending only on \( \alpha \) s.t.
\[
\sup_{x_2 \in U_R^2} |g(x_2)| \leq C\|g\|_{H^{\alpha,1}(U_R^2)}.
\]
The constant \( C \) may be taken \( C = C(\alpha) = 2^{1-\frac{1}{\alpha}} \) (see for instance the proof of [10, Section 5.6 Theorem 4]). Then, \( C(r) \leq 2 \) for any \( r \geq 1 \). Let \( \theta \in C^\infty((0,T_0]) \) s.t. \( \theta(0) = 0 \) and \( f \in C_c^\infty(U_R^2) \). We set \( \psi(t, x_1, x_2) := \theta(t)f(x_1)g(x_2) \).

One has
\[
\|\psi\|_{L^{\alpha,1}(U_{R,T_0})} \leq \|\theta\|_{L^{\alpha}(0,T_0)} \|f\|_{L^{\alpha}(U_R^2)} \|g\|_{L^{\alpha}(U_R^2)} + \|f\|_{L^{\alpha}(U_R^2)} \|g\|_{L^{\alpha}(U_R^2)}.
\]
We observe that
\[
\|f\|_{L^{\alpha}(U_R^2)} \|\theta\|_{L^{\alpha}(0,T_0)} \sup_{y \in U_R^2} |g(y)| \leq C\|\theta\|_{L^{\alpha,1}(U_R^2)} \|f\|_{L^{\alpha}(U_R^2)} \|\theta\|_{L^{\alpha}(0,T_0)} \leq C\|\psi\|_{H^{\alpha,1}(U_{R,T_0})}.
\]
Then, we find that
\[
\int_{U_{R,T_0}} |\psi(t, x)w'(t)p(t, x)| dx dt = \int_{U_{R,T_0}} |\theta(t)f(x_1)g(x_2)w'(t)p(t, x_1, x_2)| dx_1 dx_2 dt
\]
\[
\leq \sup_{y \in U_R^2} |g(y)| \int_{U_{R,T_0}} \left( \int_{U_{R,T_0}} |\theta(t)f(x_1)| |w'(t)p(t, x_1, x_2)| dx_1 dx_2 dt \right)^{1/\alpha'} dx_2
\]
\[
\leq \|f\|_{L^{\alpha}(U_R^2)} \|\theta\|_{L^{\alpha}(0,T_0)} \sup_{y \in U_R^2} |g(y)| \int_{U_{R,T_0}} \left( \int_{U_{R,T_0}} |w'(t)p(t, x_1, x_2)|^{\alpha'} dx_1 dx_2 \right)^{1/\alpha'} dx_2.
\]
This is enough to deduce the result. \( \square \)

3.3.2 Estimates with additional assumptions over the coefficients

Now, by considering additional assumptions over the coefficients, we provide some inequalities. Let \( T_0 > 0 \), \( \alpha \in (1, \infty) \setminus \{2\} \) and \( \pi \in (0, 1) \). Recall that \( \tilde{C}(\pi, \alpha, a, b, U_R, T_0) \) is defined in (3.2) and \( \ell(\pi, \alpha, R) \) is defined in (3.1). We introduce the collection of sets \( \mathcal{S} \) and the constant \( \tilde{C}_\infty(\pi, \alpha, a, b, T_0) \) by
\[
\mathcal{S} := \{ E \subset \mathbb{R}^2 \text{ open set s.t. diam}(E) \leq 1 \} \quad \text{and} \quad \tilde{C}_\infty(\pi, \alpha, a, b, T_0) := \sup_{E \in \mathcal{S}} \tilde{C}(\pi, \alpha, a, b, E, T_0).
\]

For any \( U_R \subset \mathbb{R}^2 \) with \( R \leq 1 \), we have \( \tilde{C}(\pi, \alpha, a, b, U_R, T_0) \leq \tilde{C}_\infty(\pi, \alpha, a, b, T_0) \). Notice that \( \tilde{C}(\pi, \alpha, a, b, U_R, T_0) = \tilde{C}(\pi, \alpha', a, b, U_R, T_0) \), recall that \( \alpha' \) is the conjugate of \( \alpha \) i.e. \( \alpha' = \alpha/(\alpha - 1) \). We define \( \mathcal{R} \) by
\[
\mathcal{R}(\pi, \alpha, a, b, T_0) := \left\{ 0 < R \leq 1 \text{ s.t. } 1 - \left[ \ell(\pi, \alpha, R) + \ell(\pi, \alpha', R) \right] \tilde{C}_\infty(\pi, \alpha, a, b, T_0) > 0 \right\}.
\]
We assume that
\[
\mathcal{R}(\pi, \alpha, a, b, T_0) \text{ is non-empty.}
\]
The next proposition gives estimates that will help us find estimations for the gradients of \( p \) and \( p_1 \). The first estimate is used for estimating the gradient of \( p \) and the second for estimating the gradient of \( p_1 \).
Proposition 3.6. Let $w \in C_0^\infty(T_0)$. There exists $N > 0$ depending only on $T_0$, $\alpha$, $m$, $M$, and the dimension $d = 2$ s.t. for $\overline{R} \in \mathcal{R}(\pi, \alpha, a, b, T_0)$, any $R \leq \overline{R}$ and $g$, $(f)_1 \leq \ell d$, and $v$ belonging to $C_0^\infty(U_{R, T_0})$,\[
\left| \int_{U_{R, T_0}} [\partial_x f^i + g] v(t, x) w(t) \mu_t(dx) dt \right| \leq \frac{N}{1 - \ell(\pi, \alpha, R) \mathcal{C}_\infty(\pi, \alpha, a, b, T_0)} \| (\partial_x f^i + g) v \|_{H^{n-1}(U_{R, T_0})} \| w' p \|_{L^0'_{\alpha}(U_{R, T_0})}\]
and
\[
\left| \int_{U_{R, T_0}} [\partial_x f^i + g] v(t, x) w(t) \mu_t(dx) dt \right| \leq \frac{2N}{1 - \ell(\pi, \alpha, R) \mathcal{C}_\infty(\pi, \alpha, a, b, T_0)} \| (\partial_x f^i + g) v \|_{H^{n-1}(U_{R, T_0})} \int_{U_{R, T_0}^2} \| w' \phi^R(x_1, x_2) \|_{L^0'_{\alpha}(U_{R, T_0})} dx_2.
\]

Proof. The numbers $\pi$ and $\alpha$ are given. Since we have $\overline{R} \in \mathcal{R}(\pi, \alpha, a, b, T_0)$, this means that $\overline{R} > 0$ and $1 - [\ell(\pi, \alpha, \overline{R}) + \ell(\pi, \alpha', \overline{R})] \mathcal{C}_\infty(\pi, \alpha, a, b, T_0) > 0$, therefore
\[
\sup_{E \in \mathcal{S}} \mathcal{C}(\pi, \alpha, a, b, E, T_0) = \sup_{E \in \mathcal{S}} \left[ \text{ess sup}_{t \in (0, T_0)} \| a(t, \cdot) \|_{C_0^\infty(E)} + \Gamma(\alpha, a, b, E, T_0) \right] < \infty.
\]
Consequently, there exists a sequence of smooth functions $(a^n, b^n)_{n \geq 1} \subset C_0^\infty([0, T_0] \times \mathbb{R}^2)$ s.t. $m_2 \leq a^n \leq M_2$, $\lim_{n \to \infty} (a^n, b^n) = (a, b) \text{ a.e.}$, and for any $E \in \mathcal{S}$,
\[
\lim_{n \to \infty} \text{ess sup}_{t \in (0, T_0)} \| a^n(t, \cdot) \|_{C_0^\infty(E)} = \text{ess sup}_{t \in (0, T_0)} \| a(t, \cdot) \|_{C_0^\infty(E)} \quad \text{and} \quad \lim_{n \to \infty} \Gamma(\alpha, a - a^n, b - b^n, E, T_0) = 0.
\] (3.5)

Notice that in Equation (3.5), it is not possible to approximate $a$ in $C([0, T_0]; C_0^\infty(E))$ as $a$ is not necessary continue in $t$. For each $n \geq 1$, by [15, Theorem 10.4.1], there exists $\phi^n \in C^{1, 2}((0, T_0) \times U_R)$ with derivatives belonging to Hölder spaces satisfying: $\phi^n(0, \cdot) = 0$, $\phi^n(0, \cdot) = 0$, and for any $(t, x) \in (0, T_0) \times U_R$,
\[
\partial_t \phi^n(t, x) + b^n(t, x) \partial_x \phi^n(t, x) + a^n(t, x) \partial_x \phi^n(t, x) = \partial_t \phi^n(t, x) + L^n \phi^n(t, x) = [\partial_x f^i + g] v(t, x).
\]
Let us observe that as $(0, 1) \ni r \to [\ell(\pi, \alpha, r) + \ell(\pi, \alpha', r)] \mathcal{C}(\pi, \alpha, a, b, U_{R, T_0}) \geq 1 - [\ell(\pi, \alpha, \overline{R}) + \ell(\pi, \alpha', \overline{R})] \mathcal{C}_\infty(\pi, \alpha, a, b, T_0) > 0$.

Thanks to the previous observation, for $n$ large enough, $1 - [\ell(\pi, \alpha, \overline{R}) + \ell(\pi, \alpha', \overline{R})] \mathcal{C}(\pi, \alpha, a^n, b^n, U_{R, T_0}) = \mathcal{C}(\pi, \alpha', a^n, b^n, U_{R, T_0})$, by Corollary 3.2, there exists $N$ depending only on $(d, T_0, m, \alpha, \alpha')$ s.t. for any $z \in \{\alpha, \alpha'\}$
\[
\| \phi^n \|_{H^{n-1}(U_{R, T_0})} \leq \frac{N}{1 - \ell(\pi, z, \mathcal{C}(\pi, \alpha, a^n, b^n, U_{R, T_0}))} \| (\partial_x f^i + g) v \|_{H^{n-1}(U_{R, T_0})}.
\] (3.6)

Let $p^n(t, x_1, x_2)$ be the density of $\mu^n_t := \mathcal{L}(X^n_t) := \mathcal{L}(X^{n, 1}_t, X^{n, 2}_t)$ where: $\mathcal{L}(X^n_t) = \mathcal{L}(X^n_t)$,
\[
dX^n_t = b^n(t, X^n_t) dt + a^n(t, X^n_t)^{1/2} \left( dW^n_1 \right) + a^n(t, X^n_t)^{1/2} \left( dW^n_2 \right),
\]
and $(W^1, W^2)$ an $\mathbb{R}^2$–valued Brownian motion. By Lemma 3.5, for any $w \in C_0^\infty(T_0)$,
\[
\left| \int_{[0, T_0] \times \mathbb{R}^2} w(t) [\partial_t \phi^n(t, x) + L^n \phi^n(t, x)] \mu^n_t(dx) dt \right| \leq \int_{U_{R, T_0}} |w'(t) \phi^n(t, x)| p^n(t, x) dx dt = \| w' \phi^n p^n \|_{L^1(U_{R, T_0})}.
\]
Therefore,
\[
\lim_{n \to \infty} \int_{U_{R, T_0}} w(t) [\partial_t f^i + g] v(t, x) \mu^n_t(dx) dt
\] (3.7)

\[= \lim_{n \to \infty} \int_0^{T_0} \int_{\mathbb{R}^2} w(t) [\partial_t \phi^n(t, x) + L^n \phi^n(t, x)] \mu^n_t(dx) dt \leq \lim_{n \to \infty} \| w' \phi^n p^n \|_{L^1(U_{R, T_0})}.
\]
We know that \( \| (\partial_x, f^i + g)v\|_{H^{n'-1}(U_R, \mathcal{T}_0)} + \| (\partial_x, f^i + g)v\|_{H^{n-1}(U_R, \mathcal{T}_0)} < \infty \), then by (3.6), \( \sup_{n \geq 1} \| \phi^n \|_{H^{n', 1}(U_R, \mathcal{T}_0)} + \| \phi^n \|_{H^{1, 1}(U_R, \mathcal{T}_0)} < \infty \). We can deduce that \( (\phi^n)_{n \geq 1} \) is relatively compact in \( L^1(U_R, \mathcal{T}_0) \) where \( \hat{\alpha} := \min(\alpha, \alpha') \). Up to a sub-sequence, we can assume that there is \( \phi \) s.t. \( \lim_{n \to \infty} \phi^n = \phi \) a.e. Since \( \phi^n(t, \cdot) \) is s.t. \( \phi^n_{[0, T_0]} \times \partial U_R = 0 \), and \( \hat{\alpha} := \max(\alpha, \alpha') > d = 2 \), by Morrey’s inequality,

\[
\sup_{n \geq 1} \int_0^T \| \phi^n(t, \cdot) \|^2_{C^0(1-2/4\mathbb{R}^2)} \, dt \leq \sup_{n \geq 1} C(2, \hat{\alpha}) \| \phi^n \|^2_{H^{1, 1}(U_R, \mathcal{T}_0, \mathbb{R}^2)} = \sup_{n \geq 1} C(2, \hat{\alpha}) \| \phi^n \|^2_{H^{1, 1}(U_R, \mathcal{T}_0)} < \infty.
\]

For any \( \beta \in C^\infty([0, T_0]) \), by applying Arzelà–Ascoli theorem together with the previous result, the sequence of functions \( (\phi^{\beta, n})_{n \geq 1} \subset C(\mathbb{R}^2) \) is relatively compact for the uniform topology where

\[
\phi^{\beta, n}(x) := \int_0^T \beta(t) \phi^n(t, x) \, dt.
\]

We know that \( \lim_{n \to \infty} \phi^n = \phi \) a.e., then \( \lim_{n \to \infty} \phi^{\beta, n}(\cdot) = \int_0^T \beta(t) \phi(t, \cdot) \, dt \) for the uniform topology. Consequently,

\[
\lim_{n \to \infty} \| w' (\phi^n - \phi) \|_{L^1(U_R, \mathcal{T}_0)} = 0.
\]

Since \( a \) and \( b \) are bounded, \( a \) is non–degenerate and \( a(t, x) \) is uniformly continuous in \( x \) uniformly in \( t \), then the density \( p \) is uniquely defined. We know that \( \lim_{n \to \infty} (a^n, b^n) = (a, b) \) a.e., consequently \( \lim_{n \to \infty} p^n = p \) in the weak sense. Therefore

\[
\lim_{n \to \infty} \| w' \phi^n p^n \|_{L^1(U_R, \mathcal{T}_0)} = \lim_{n \to \infty} \| w' \phi p^n \|_{L^1(U_R, \mathcal{T}_0)} = \| w' \phi p \|_{L^1(U_R, \mathcal{T}_0)}.
\]

By (3.7) and (3.6),

\[
\int_{U_R, \mathcal{T}_0} \left[ (\partial_x, f^i + g) v(t, x) w(t) \mu_t (dx) \right] \, dt \leq \| w' \phi p \|_{L^1(U_R, \mathcal{T}_0)} \leq \| \phi \|_{L^\infty(U_R, \mathcal{T}_0)} \| w' \|_{L^{\infty}(U_R, \mathcal{T}_0)}
\]

\[
\leq \frac{N}{1 - \ell(\pi, \alpha, R)C(\pi, \alpha, a, b, U_R, T_0)} \| (\partial_x, f^i + g) v \|_{H^{n-1}(U_R, \mathcal{T}_0)} \| w' \|_{L^{\infty}(U_R, \mathcal{T}_0)}.
\]

We obtain the second estimates by using the second inequality in Lemma 3.5 and the fact that \( \| \phi \|_{H^{n', 1}(U_R, \mathcal{T}_0)} \leq \lim_{n \to \infty} \| \phi^n \|_{H^{n', 1}(U_R, \mathcal{T}_0)} \).

We set \( v \in L^{\infty}_{\ell_{\infty}}([0, T_0] \times \mathbb{R}^2) \) s.t. \( \nabla v \in L_{\ell_{\infty}}^{\alpha', \alpha}([0, T_0] \times \mathbb{R}^2) \) and \( \overline{R} \in \mathcal{R}(\pi, \alpha, a, b, T_0) \), and we define \( K(\pi, \alpha, a, b, v, T_0, \overline{R}) := K(\pi, \alpha, a, b, v, T_0) \) by

\[
K(\pi, \alpha, a, b, v, T_0) := \frac{2N \sup_{E \in \mathcal{S}} \left[ \text{ess sup}_{(t, x) \in [0, T_0] \times E} |v(t, x)| + \ell(\pi, \alpha, \overline{R}) \left[ \| \nabla v \|_{L^{\infty}([0, T_0] \times E)} 1_{a > 2} + \| \nabla v \|_{L^{\infty}([0, T_0] \times E)} 1_{a < 2} \right] \right]}{1 - \left[ \ell(\pi, \alpha, \overline{R}) + \ell(\pi, \alpha', \overline{R}) \right] C(\pi, \alpha, a, b, T_0)}.
\]

Let \( R \leq \overline{R}, f := (f^i)_{1 \leq i \leq 2} \subset C^\infty(\overline{U}_R, \mathcal{T}_0) \) and \( g \in C^\infty(\overline{U}_R, \mathcal{T}_0) \).

**Corollary 3.7.** Under the consideration of Proposition 3.6, one has

\[
\left| \int_{U_R, \mathcal{T}_0} [\partial_x, f^i + g] v(t, x) w(t) \mu_t (dx) \right| \leq K(\pi, \alpha, a, b, v, T_0) \| (f, g) \|_{L^\infty(U_R, \mathcal{T}_0)} \| w' \|_{L^{\infty}(U_R, \mathcal{T}_0)}
\]

and

\[
\left| \int_{U_R, \mathcal{T}_0} [\partial_x, f^i + g] v(t, x) w(t) \mu_t (dx) \right| \leq K(\pi, \alpha, a, b, v, T_0) \| (f, g) \|_{L^\infty(U_R, \mathcal{T}_0)} \left( \int_{U_R, \mathcal{T}_0} \| w' \|^2_{L^{\infty}(U_R, \mathcal{T}_0)} \, dx \right)^{1/2}.
\]
Proof. Similarly to the proof of Proposition 3.6, by definition, since $(0,1) \ni r \to \ell(\pi, \alpha, r) \in \mathbb{R}_+$ is increasing, for any $R \leq \overline{R}$, we have
\[
\frac{1}{1 - \ell(\pi, \alpha, R)C_{\infty}(\pi, \alpha, a, b, T_0)} \leq \frac{1}{1 - \ell(\pi, \alpha, \overline{R})C_{\infty}(\pi, \alpha, a, b, T_0)}.
\]
Then, by Proposition 3.6, we find
\[
\left| \int_{U_{R,T_0}} [\partial_x f^i + g] v(t, x) w(t) \mu_t(dx) dt \right| \leq \frac{N}{1 - \ell(\pi, \alpha, R)C_{\infty}(\pi, \alpha, a, b, T_0)} \|v_h f^i + g\|_{H^{0, -1}(U_{R,T_0})} \|w' p\|_{L^{\alpha'}(U_{R,T_0})}
\]
and
\[
\left| \int_{U_{R,T_0}} [\partial_x f^i + g] v(t, x) w(t) \mu_t(dx) dt \right| \leq \frac{2N}{1 - \ell(\pi, \alpha, R)C_{\infty}(\pi, \alpha, a, b, T_0)} \|v_h f^i + g\|_{H^{0, -1}(U_{R,T_0})} \int_{U_{R}} \|w' p(\cdot, x_2)\|_{L^{\alpha'}(U_{R,T_0})} dx_2.
\]
Notice that
\[
\|v \partial_x f^i\|_{H^{0, -1}(U_{R,T_0})} = \|\partial_x (f^i v) - f^i \partial_x v\|_{H^{0, -1}(U_{R,T_0})} \leq \sup_{(t,x) \in [0,T] \times U_R} |v(t, x)| + \ell(\pi, \alpha, R) \left[ \|v\|_{L^\infty(U_{R,T_0})} I_{\alpha > 2} + \|\nabla v\|_{L^{\alpha'}(U_{R,T_0})} I_{\alpha < 2} \right] \|f^i\|_{L^{\infty}(U_{R,T_0})}.
\]
This is enough to conclude the proof. \(\square\)

For any Borel map $(h, f) : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R}^2$, whenever it is well–defined, we recall that the notation $(h f)_1(t, x_1)$ means $(h f)(t, x_1) := \int_R h(t, x_1, x_2) f(t, x_1, x_2) dx_2$. In particular, the notations $|\partial_x (v p)_1|$ and $|(v p)_1|$ mean
\[
|(v p)_1| := \int_R |(v p)(t, x_1, x_2)| dx_2 \quad \text{and} \quad |\partial_x (v p)_1| := \int_R |\partial_x (v p)(t, x_1, x_2)| dx_2.
\]
We are now able to provide in the next proposition a relation between $(p, p_1)$ and its gradient.

Proposition 3.8. Let $\pi \in (0, 1), \alpha \in (1, \infty) \setminus \{2\}$ and $w \in C_0^\infty(T_0)$. If $R \in R(\pi, \alpha, a, b, T_0)$ is non–empty, we have
\[
\|w(p)\|_{H^{0, 1}((0,T_0) \times \mathbb{R}^2)} \leq K(\pi, \alpha, a, b, v, T_0)^N 2^d \|w' p\|_{L^{\alpha'}((0,T_0) \times \mathbb{R})}
\]
and
\[
\|w(p)\|_{H^{0, 1}((0,T_0) \times \mathbb{R}^2)} \leq K(\pi, \alpha, a, b, v, T_0)^N 2^d \|w' p\|_{L^{\alpha'}((0,T_0) \times \mathbb{R})}^{1/\alpha'} \leq K(\pi, \alpha, a, b, v, T_0)^N 2^d \|w' p(\cdot, x_2)\|_{L^{\alpha'}((0,T_0) \times \mathbb{R})} dx_2.
\]

Remark 3.9. Among other things, the previous result shows the passage of local estimates to global estimates. The existence of a small diameter $\overline{R} \in \mathcal{R}(\pi, \alpha, a, b, T_0)$ allows to propagate the estimates over all the space. It is possible in particular because $K(\pi, \alpha, a, b, v, T_0)$ is independent of the choice of the ball $U_{R}$. It only depends on the diameter $\overline{R}$. The techniques used in the proof below are similar to those used in the proof of [5, Theorem 3.2.2].

Proof. Let $0 < r < 1$ s.t. $2r \leq \overline{R}$. Let $(\tilde{E}_k)_{k \geq 1}$ be a sequence of disjoint sets of diameter $r$ s.t. $\cup k \geq 1 \tilde{E}_k = \mathbb{R}$. Also, we consider a sequence of open sets of diameter $2r$, $(\tilde{O}_k)_{k \geq 1}$, s.t. $\tilde{E}_k \subset \tilde{O}_k$. We take $(\varphi_k)_{k \geq 1}$ satisfying $\varphi_k \in C_c(\tilde{O}_k), 0 \leq \varphi_k \leq 1$ and $\varphi_k(e) = 1$ for $e \in \tilde{E}_k$. Let $E_{k,q} := \tilde{E}_k \times \tilde{E}_q, O_{k,q} := \tilde{O}_k \times \tilde{O}_q$ and $\varphi_{k,q}(x_1, x_2) := \varphi_k(x_1) \varphi_q(x_2)$. By Corollary 3.7, for $f^i := \tilde{f}^i \varphi_{k,q}$ and $g := \tilde{g} \varphi_{k,q}$,
\[
\int_{[0,T_0] \times O_{k,q}} [\partial_x (\tilde{f}^i \varphi_{k,q}) + \tilde{g} \varphi_{k,q}] v(t, x) w(t) \mu_t(dx) dt \leq K(\pi, \alpha, a, b, v, T_0) \|\tilde{f}^i \tilde{g}\|_{L^\alpha((0,T_0) \times O_{k,q})} \|w' p\|_{L^{\alpha'}((0,T_0) \times O_{k,q})} dx_2.
\]
and
\[
\left| \int_{[0,T_0] \times O_{k,q}} [\partial_x, (\bar{f} \varphi_{k,q}) + \bar{g} \varphi_{k,q}] v(t,x)w(t)\mu(x)dx \right| \\
\leq K(\pi, \alpha, a, b, v, T_0) \left( \int \|\bar{f}, \bar{g}\|_{L^{\infty}(0,T_0) \times O_{k,q}} \int_{O_k} \|w' p(\cdot, x_2)\|_{L^{\infty}(0,T_0) \times \bar{O}_k} dx_2 \right).
\]

If \( \|w' p\|_{L^{\infty}(0,T_0) \times O_{k,q}} \leq 0 \) or \( \int_{O_k} \|w' p(\cdot, x_2)\|_{L^{\infty}(0,T_0) \times \bar{O}_k} dx_2 < \infty \), the previous inequality shows that \( w \partial_x (vp) \) is well defined on \( [0,T_0] \times \bar{O}_{k,q} \) (see for instance point (i) of [10, Section 5.8 Theorem 3]). Then, by integration by part, for any \( \bar{f}^i \) and \( \bar{g} \)
\[
\left| \int_{[0,T_0] \times O_{k,q}} [\bar{f}^i \partial_x (vp) + \bar{g} (vp)] \varphi_{k,q}(t,x)w(t) \right| dt \leq K(\pi, \alpha, a, b, v, T_0) \left( \int \|\bar{f}, \bar{g}\|_{L^{\infty}(0,T_0) \times O_{k,q}} \|w' p\|_{L^{\infty}(0,T_0) \times O_{k,q}} \right)
\]
and
\[
\left| \int_{[0,T_0] \times O_{k,q}} [\bar{f}^i \partial_x (vp) + \bar{g} (vp)] \varphi_{k,q}(t,x)w(t) \right| dt \leq K(\pi, \alpha, a, b, v, T_0) \left( \int \|\bar{f}, \bar{g}\|_{L^{\infty}(0,T_0) \times O_{k,q}} \|w' p\|_{L^{\infty}(0,T_0) \times O_{k,q}} \right).
\]
Notice that, even when the quantities \( \|w' p\|_{L^{\infty}(0,T_0) \times O_{k,q}} \) or \( \int_{O_k} \|w' p(\cdot, x_2)\|_{L^{\infty}(0,T_0) \times \bar{O}_k} dx_2 \) are not finite, the previous inequalities are still true. Indeed, in this case all upper bounds are infinite. Besides, the inequalities are true for \( \bar{f}^i \) belonging to \( L^{\infty}(0,T_0) \times O_{k,q} \).
We start by dealing with the first inequality. In the first inequality, by taking the supremum over \( \|\bar{f}, \bar{g}\|_{L^{\infty}(0,T_0) \times O_{k,q}} \leq 1 \), we find
\[
\|w \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k} = \left\| w \nabla (v p) \right\|_{L^{\infty}(0,T_0) \times \bar{O}_k} + \left\| \varphi_{k,q} w (v p) \right\|_{L^{\infty}(0,T_0) \times \bar{O}_k} \leq K(\pi, \alpha, a, b, v, T_0) \left\| w' p \right\|_{L^{\infty}(0,T_0) \times \bar{O}_k}.
\]
Therefore, as the constant \( K(\pi, \alpha, a, b, v, T_0) \) is independent of \( (k,q) \), \( \bar{\varphi}_k (e) = 1 \) for \( e \in E_k \) and \( 0 \leq \bar{\varphi}_k \leq 1 \), by taking the summation, we get
\[
\|w \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k}^2 = \left\| w \nabla (v p) \right\|_{L^{\infty}(0,T_0) \times \bar{O}_k} + \left\| \varphi_{k,q} w (v p) \right\|_{L^{\infty}(0,T_0) \times \bar{O}_k} \leq K(\pi, \alpha, a, b, v, T_0) \sum_{k,q \geq 1} \left( \int \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k} \right)^2 \leq K(\pi, \alpha, a, b, v, T_0) \sum_{k,q \geq 1} \left( \int \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k} \right)^2.
\]
Now, we consider the second inequality. We take \( \bar{f}^2 = 0 \),
\[
\bar{f}^1(t,x_1,x_2) := \left| \beta^1(t,x_1) \right| \left| \right| w \partial_x (v p)(t,x_1,x_2) \right| \leq 0 \)
and
\[
\bar{g}^1(t,x_1,x_2) := \left| \beta^0(t,x_1) \right| \left| \right| w \partial_v (v p)(t,x_1,x_2) \right| \leq 0 \)
We find
\[
\left| \int_{[0,T_0] \times \bar{O}_k} \left[ w(t) \left| \beta^1(t,x_1) \right| \partial_x (v p)(t,x_1,x_2) \right] \right| dt \leq K(\pi, \alpha, T_0) \sum_{q \geq 1} \int_{O_k} \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k} dx_2 \leq K(\pi, \alpha, T_0) \sum_{q \geq 1} \left( \int \|w' p\|_{L^{\infty}(0,T_0) \times \bar{O}_k} \right)^2.
\]
We take the supremum over \( \| (\beta^1, \beta^0) \|_{L^\infty(0, T_0) \times \mathbb{R}} \leq 1 \) and find the result. It is worth mentioning that we implicitly suppose that \( \partial_{x_1} (vp) \) is well defined. Indeed, as mentioning earlier, when it is not the case, all the upper bonds are infinite.

In the proposition just below, we provide a recursive estimate making a relation between the \( L^2 \) norm of \( p \) and the \( L^{s+1} \) norm of \( p \). Its usefulness will be clear in Section 3.4.

**Proposition 3.10.** For any \( z > 2 \) and \( w \in C_0^\infty (T_0) \), if \( \mathcal{R} (\pi, z, a, b, T_0) \) is non–empty, we have

\[
\| w p \|_{L^{s+1}([0, T_0] \times \mathbb{R}^2)} \leq \| w \|_{\infty} C(2, z)^{\frac{z}{2}} K(\pi, z', a, b, 1, T_0)^{\frac{2}{d}} \| w' \|_{L^{s}([0, T_0] \times \mathbb{R}^2)}
\]

and

\[
\int_0^{T_0} \| w(t) p(t, \cdot) \|_{C^{0,1-2/z}([0, T_0] \times \mathbb{R}^2)} \, dt \leq C(2, z)^{\frac{z}{2}} K(\pi, z', a, b, 1, T_0)^{\frac{2}{d}} \| w' \|_{L^{s}([0, T_0] \times \mathbb{R}^2)}
\]

where \( C(2, z) \) is a constant appearing in the Sobolev embedding Theorem depending on \( d = 2 \) and \( z \).

**Proof.** Since \( z > d = 2 \), we use Proposition A.3 and find for any \( z < r \leq z + 1 \)

\[
\int_0^{T_0} \| w(t) p(t, \cdot) \|_{C^{0,1-2/z}([0, T_0] \times \mathbb{R}^2)} \, dt \leq C(2, z)^{\frac{z}{2}} \| w \|_{L^{s+1}([0, T_0] \times \mathbb{R}^2)}
\]

and

\[
\| w \|_{L^{s}([0, T_0] \times \mathbb{R}^2)} \leq \| w \|_{\infty} C(d, z)^{\frac{z}{s}} \| w \|_{L^{s+1}([0, T_0] \times \mathbb{R}^2)} \| w \|_{L^{s+1}([0, T_0] \times \mathbb{R}^2)}
\]

where \( s := \frac{r}{z+1} \). We can conclude by taking \( r = z + 1 \) and using Proposition 3.8.

The next result gives a more explicit estimate for the gradient of \( p_1 \). Also, in the same spirit as the previous proposition, it provides a relation between the \( L^{s+1} \) norm of \( (vp)_1 \) and the \( H^{s+1} \) norm of \( (vp)_1 \).

**Proposition 3.11.** Let \( r > 1 \) and \( 1 < s < r + 1 \) s.t. \( s \neq 2 \). For any \( w \in C_0^\infty (T_0) \), if \( \mathcal{R} (\pi, s, a, b, T_0) \) is non–empty, we have the estimate

\[
\left[ \| w \|_{\partial_{x_1} (vp)} \right]_{L^s([0, T_0] \times \mathbb{R})} + \| w \|_{(vp)} \left[ \| \partial_{x_1} (vp) \|_{L^s([0, T_0] \times \mathbb{R})} \right]^{1/s} \leq K(\pi, s, a, b, 1, T_0)^{\frac{2}{s}} \int_0^{T_0} \| w(t) p(t, \cdot) \|_{L^\infty([0, T_0] \times \mathbb{R}^2)} \, dt \leq \| w \|_{\infty} C(1, s)^{\frac{s}{s}} \| w \|_{L^{s+1}([0, T_0] \times \mathbb{R})}
\]

where \( C(1, s) \) is a constant appearing in the Sobolev embedding Theorem depending on \( 1 \) and \( s \).

**Proof.** We observe that, for \( s < r + 1 \) and \( u > 1 \),

\[
\int_0^{T_0} \int_0^R \int_0^R \left| w(t)p(t, x_1, x_2) \right|^s \, dx_1 \, dt \leq \int_0^{T_0} \int_0^R \int_0^R \left| w(t)p(t, x_1, x_2) \right|^{s-1} \left| w(t)p(t, x_1, x_2) \right| \, dx_1 \, dt \leq \int_0^{T_0} \sup_{x_1 \in \mathbb{R}^2} \int_0^R \int_0^R \left| w(t)p(t, x_1, x_2) \right| \, dx_1 \, dt \leq \int_0^{T_0} \left[ \int_0^R \left( \int_0^R \left| w(t)p(t, x_1, x_2) \right| \, dx_1 \right)^{u} \, dt \right] \frac{1}{u} \, dx_2.
\]
We take \((s-1)u = r\). Then, we have 
\[
\int_{\mathbb{R}} \left[ \int_0^{T_0} \int_{\mathbb{R}} |w'(t)p(t, x_1, x_2)|^s \, dx_1 \, dt \right]^{1/s} \, dx_2
\]
and
\[
\leq \left[ \int_0^{T_0} \sup_{x' \in \mathbb{R}^2} |w'(t)p(t, x', x)| \, dt \right]^{s-1/s} \int_{\mathbb{R}} \left[ \int_0^{T_0} \int_{\mathbb{R}} |w'(t)p(t, x_1, x_2)|^s \, dx_1 \, dt \right]^{1/s} \, dx_2.
\]
Consequently, as \(s \neq 2\) and \(\mathcal{R}(\pi, s, a, b, T_0)\) is non-empty,
\[
\left[ \int_{[0,T_0] \times \mathbb{R}} w(t) \left| \int_{\mathbb{R}} |\partial_{x_1}(vp)(t, x_1, x_2)| \, dx_2 \right|^s \, dx_1 \, dt + \int_{[0,T_0] \times \mathbb{R}} w(t) \left| \int_{\mathbb{R}} |vp(t, x_1, x_2)| \, dx_2 \right|^s \, dx_1 \, dt \right]^{1/s}
\]
\[
\leq K(\pi, s, a, b, v, T_0) 2^d \int_{[0,T_0] \times \mathbb{R}} |w'(t)p(t, x_1, x_2)|^s \, dx_1 \, dt + \int_{[0,T_0] \times \mathbb{R}} w(t) \left| \int_{\mathbb{R}} |w'(t)p(t, x_1, x_2)|^{1/s} \, dx_1 \right|^s \, dx_2
\]
\[
\leq K(\pi, s, a, b, v, T_0) 2^d \int_{[0,T_0] \times \mathbb{R}} \sup_{x' \in \mathbb{R}^2} |w'(t)p(t, x', x)| \, dt \right]^{s-1/s} \int_{\mathbb{R}} \left[ \int_0^{T_0} \int_{\mathbb{R}} |w'(t)p(t, x_1, x_2)|^s \, dx_1 \, dt \right]^{1/s} \, dx_2.
\]
The second inequality follows from Proposition A.3.

Now, we are going to establish some estimates in H"older norm. Let \(r > 2\) and \(\frac{1}{2} > \beta > \kappa > \frac{1}{r}\) with \((1-2\beta)r > 2\).

**Proposition 3.12.** We assume that \(\mathcal{R}(\pi, r, a, b, T_0)\) is non-empty then
\[
\| w \|_{C^{0, \kappa - 1/r}([0,T_0]; C^{0, 1-2\beta - 2/r}([\mathbb{R}^2]))} \leq D \left[ \left| K(\pi, r', a, b, 1, T_0) + K(\pi, r', a, b, a, T_0) \right| 2^{d/r} + 1 \right] \| w' \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| b \|_{\infty} \| w \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| w(0)(0, \cdot) \|_{H^{r-1}([\mathbb{R}^2])},
\]
where \(D\) is the constant used in Proposition 3.3 depending only on \((\beta, \kappa, r, d = 2, T_0)\).

**Proof.** In a weak sense, we can write, \(\partial_t (w p)(t, x) = w'(t) p(t, x) - \partial_{x_1} (b^i w p)(t, x) + \partial_{x_1} \partial_{x_j} (a^{i,j} w p)(t, x)\). Then by applying Proposition 3.3, we find
\[
\| w \|_{C^{\kappa - 1/r}([0,T_0]; C^{0, 1-2\beta - 2/r}([\mathbb{R}^2]))} \leq D \left[ \| \nabla w \p \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| w' p - \partial_{x_1} (b^i w) p + \partial_{x_1} \partial_{x_j} (a^{i,j} w p) \|_{H^{r-1}([0,T_0] \times \mathbb{R}^2)} + \| w(0)(0, \cdot) \|_{H^{r-1}([\mathbb{R}^2])} \right].
\]
By Proposition 3.8,
\[
\| \partial_x \partial_{x_j} (a^{i,j} w p) \|_{H^{r-1}([0,T_0] \times \mathbb{R}^2)} \leq \| \nabla (a \ p \ w) \|_{L^r([0,T_0] \times \mathbb{R}^2)} \leq K(\pi, r', a, b, a, T_0) 2^{d/r} \| w' \|_{L^r([0,T_0] \times \mathbb{R}^2)}.
\]
Also,
\[
\| \nabla w \p \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| w' \|_{H^{r-1}([0,T_0] \times \mathbb{R}^2)} + \| \partial_{x_1} (b^i w) p \|_{H^{r-1}([0,T_0] \times \mathbb{R}^2)}
\]
\[
\leq \| \nabla w \p \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| w' \|_{L^r([0,T_0] \times \mathbb{R}^2)} + \| b \|_{\infty} \| w \|_{L^r([0,T_0] \times \mathbb{R}^2)}.
\]
Let us observe that, for any \(\phi \in C_c^\infty(\mathbb{R})\), by Itô’s formula,
\[
d \int_{\mathbb{R}^2} \phi(x_1) w(t)p(t, x_1, x_2) dx_2 dx_1 = \int_{\mathbb{R}^2} \phi(x_1) w(t)p(t, x_1, x_2) dx_2 dx_1 + \int_{\mathbb{R}^2} \phi'(x_1) w(t) b^i(t, x_1, x_2)p(t, x_1, x_2) dx_2 dx_1
\]
\[
+ \int_{\mathbb{R}^2} \phi''(x_1) w(t) a^{1,1}(t, x_1, x_2)p(t, x_1, x_2) dx_2 dx_1 \, dt.
\]
For any bounded measurable map \( f : [0, T_0] \times \mathbb{R}^2 \to \mathbb{R} \), we recall that \((pf)_1(t, x_1) := \int_{\mathbb{R}} f(t, x_1, x_2)p(t, x_1, x_2)dx_2\). We rewrite
\[
d \int_{\mathbb{R}} \phi(x)(pw)_1(t, x_1)dx_1 = \left[ \int_{\mathbb{R}} \phi(x)(pw'_1)(t, x_1)dx_1 + \int_{\mathbb{R}} \phi'(x)(pw b^1)_1(t, x_1)dx_1 + \int_{\mathbb{R}} \phi''(x)(pw a^{1,1})_1(t, x_1)dx_1 \right] dt.
\]
Let \( \tau > 2 \) and \( \frac{1}{2} > \beta > \alpha > \frac{1}{d} \) with \((1 - 2\beta)\tau > 1\).

**Proposition 3.13.** We have the estimate
\[
\|wp_1\|_{C^{0, \tau-1, \tau}(0, T_0), C^{0, -\tau-1, -\tau}(R)} \leq D\left[ \|b\|_{\infty} \left\| \left[ wp_1, w, p_1, w|\partial x_1 p|, w|\partial x_1 (p a^{1,1})| \right]_{L^{\tau}(0, T_0) \times R} + \|wp_1(0, \cdot)\|_{H^{(1+\frac{1}{d})}} \right] \]
where \( D \) is the constant used in Proposition 3.3 depending only on \( (\beta, \alpha, \tau, d = 1, T_0) \).

**Remark 3.14.** Notice that, by Proposition 3.11, if \( \mathcal{R}(\pi, \tau, a, b, T_0) \) is non-empty,
\[
\|w|\partial x_1 (p a^{1,1})|_{L^{\tau}(0, T_0) \times R} \leq K(\pi, \tau, a, b, a^{1,1}, T_0)2^d \left[ \int_{0}^{T_0} \|w'(t)p(t, \cdot)\|_{\infty} dt \right]^\frac{1}{\tau} \int_{\mathbb{R}} \left[ \int_{0}^{T_0} \|w(t)p(t, x_1, x_2)\|_{x_1} \frac{d}{dt} \right] dx_2.
\]

**Proof.** By Proposition 3.3,
\[
\|wp_1\|_{C^{0, \tau-1, \tau}(0, T_0), C^{0, -\tau-1, -\tau}(R)} \leq D\left[ \|wp_1\|_{L^{\tau}(0, T_0) \times R} + \|wp_1 + w(p b^1)'_1 + w(p a^{1,1})''_1\|_{H^{(1+\frac{1}{d})}(0, T_0) \times R} + \|wp_1(0, \cdot)\|_{H^{(1+\frac{1}{d})}} \right].
\]
Since \( \|w(p a^{1,1})''_1\|_{H^{(1+\frac{1}{d})}(0, T_0) \times R} \leq \|w(p a^{1,1})'_1\|_{L^{\tau}(0, T_0) \times R} \leq \|w|\partial x_1 (p a^{1,1})|_{L^{\tau}(0, T_0) \times R} \), we obtain
\[
\|wp_1\|_{C^{0, \tau-1, \tau}(0, T_0), C^{0, -\tau-1, -\tau}(R)} \leq D\left[ \|b\|_{\infty} \left\| \left[ wp_1, w, p_1, w|\partial x_1 p|, w|\partial x_1 (p a^{1,1})| \right]_{L^{\tau}(0, T_0) \times R} + \|wp_1(0, \cdot)\|_{H^{(1+\frac{1}{d})}} \right].
\]

\[\square\]

Let us define \( \gamma := (1 + 1/2)^2 + 1 = 2 + 1/4 \). Contrary to the previous last results where we showed estimates putting in relation \( \rho \) and its gradient, in the next proposition we give estimate of the \( L^\gamma \) norm of \( p \) involving only the coefficients of the SDE.

**Proposition 3.15.** Let \( \pi \in (0, 1) \) and \( w \in C^0(T_0) \). If \( \mathcal{R}(\pi, 3, a, b, T_0) \) is non-empty, we have
\[
\|wp\|_{L^{\gamma}(0, T_0 \times R^2)} \leq C(2, 1 + 1/2)^{\gamma} \|w, w'|_\infty^{1+1/2} K(\pi, 3, a, b, 1, T_0)^{1+1/2} d G(T_0, 1 + 1/2, 2, m, M)
\]
where \( G(T_0, 1 + 1/2, 2, m, M) \) is the constant used in Proposition 3.4 and \( C(2, 1 + 1/2) \) is the constant appearing in Sobolev embedding theorem.

**Proof.** Since \( 1 + 1/2 < d = 2 \), by Proposition A.2,
\[
\|wp\|_{L^{\gamma}(0, T_0 \times R^2)} \leq C(2, 1 + 1/2)^{1+1/2} G(w)^{1/2+1/4} \|wp\|_{L^{1+1/2}(0, T_0 \times R^2)}^{1+1/2}
\]
where \( G(w) := \|w\|_\infty, \gamma = (1 + 1/2)^2 + 1 = 2 + 1/4 \) and \( C(2, 1 + 1/2) \) is a positive constant depending only on \( 1 + 1/2 \) and \( 2 \). As \( \mathcal{R}(\pi, 3, a, b, T_0) \) is non-empty, by applying proposition 3.8 and then Proposition 3.4, we obtain
\[
\|wp\|_{L^{\gamma}(0, T_0 \times R^2)} \leq \|w\|_\infty^{1+1/2} (C(2, 1 + 1/2) K(\pi, 3, a, b, 1, T_0))^{1+1/2} d G(w)^{1+1/2} \|wp\|_{L^{1+1/2}(0, T_0 \times R^2)}^{1+1/2}
\]
\[\leq \|w, w'|_\infty^{1+1/2} (C(2, 1 + 1/2) K(\pi, 3, a, b, 1, T_0))^{1+1/2} d G(T_0, 1 + 1/2, 2, m, M).
\]
\[\square\]
Summary of the estimates  In order to get a general idea of all the estimates we have proven so far, we summarize them here before using them in the next section.

- For $\gamma := 2 + 1/4$,
  \[
  \|w p\|_{L^{2}(\mathbb{R}^{2})}^{2} \leq C(2, 1 + 1/2)(w, w')^{1/2} d G(T_0, 1 + 1/2, 2, m, M);
  \]

- For any $z > 2$, \[
  \|w p\|_{L_{z}^{z+1}([0, T_0] \times \mathbb{R}^{2})}^{z+1} \leq \|w\|_{\infty} C(2, z) K(\pi, z', a, b, 1, T) K(\pi, z', a, b, 1, T_0) \leq 2d \|w' p\|_{L^{z}((0, T_0] \times \mathbb{R}^{2})}^{z+1} \]
  and \[
  \int_{0}^{T_0} \|w(t) p(\cdot, \cdot)\|_{L_{z}^{z+1}([0, T_0] \times \mathbb{R}^{2})} dt \leq C(2, z) K(\pi, z', a, b, 1, T) K(\pi, z', a, b, 1, T_0) \leq 2d \|w' p\|_{L^{z}((0, T_0] \times \mathbb{R}^{2})}^{z+1} ;
  \]

- For $\alpha > 1$, \[
  \|w v p\|_{H_{\alpha}^{\alpha'}([0, T_0] \times \mathbb{R}^{2})} \leq K(\pi, \alpha, a, b, v, T_0) \alpha' \|w' p\|_{L^{\alpha'}((0, T_0] \times \mathbb{R}^{2})}^{\alpha'};
  \]

- For any $r > 1$ and $1 < s < r + 1$ s.t. $r \neq 2$,
  \[
  \left[\|w \varphi(\cdot, \cdot)\|_{L^{s}([0, T_0] \times \mathbb{R}^{2})} + \|w v p\|_{L^{r}([0, T_0] \times \mathbb{R}^{2})}\right]^{1/s} \leq K(\pi, s, a, b, v, T_0) \leq 2d \int_{0}^{T_0} \|w' p(\cdot, \cdot)\|_{L^{s}([0, T_0] \times \mathbb{R}^{2})} \int_{\mathbb{R}} \int_{\mathbb{R}} \|w(t)p(t, x_1, x_2)\|_{L^{1}(\mathbb{R})}^{1} dt \int_{\mathbb{R}} dx_2 \]
  where $u' = \frac{r}{r-1} u$. In addition, \[
  \|w (v p)\|_{L_{z}^{z+1}([0, T_0] \times \mathbb{R}^{2})} \leq \|w\|_{\infty} C(1, s) \|w (v p)\|_{R_{z}^{r}([0, T_0] \times \mathbb{R}^{2})};
  \]

- For any $r > 2$ and $1 < \beta > \kappa > \frac{1}{2}$ s.t. $(1 - 2\beta) r > 2$,
  \[
  \|w p\|_{C_{0, s-1/r}([0, T_0]; C_{0, 2s-2/r}([0, T_0]))} \leq D \left[\|K(\pi, r', a, b, 1, T_0) + K(\pi, r', a, b, a, T_0)\|_{L_{r}^{r}([0, T_0] \times \mathbb{R}^{2})} \right] \leq 2d \|w' p\|_{L^{r}((0, T_0] \times \mathbb{R}^{2})}^{r} + \|w\|_{L^{r}((0, T_0] \times \mathbb{R}^{2})}^{r} + \|w(0) p(0, \cdot)\|_{L^{r}((0, T_0] \times \mathbb{R}^{2})};
  \]

- For any $\tau > 2$ and $\frac{1}{2} < \beta > \kappa > \frac{1}{2}$ with $(1 - 2\beta)\tau > 1$,
  \[
  \|wp_{1}\|_{C_{0, \tau-1/\tau}([0, T_0]; C_{0, 2\tau-2/\tau}([0, T_0]))} \leq D \left[\|\beta\|_{\infty} \|w p_{1}\|_{L_{\tau}^{\tau}([0, T_0] \times \mathbb{R}^{2})} + \|wp_{1}(0, \cdot)\|_{H^{\tau}((0, T_0] \times \mathbb{R})}^{\tau} \right].
  \]

3.4 Construction of the appropriate set for the density $p$

This section is devoted to find the appropriate upper bounds for the Lebesgue norm of $p$ and its gradient in terms of the coefficients of the SDE. These upper bounds will be used to defined the right space for $p$.

Let $j \geq 1$ and $\pi \in (0, 1)$. Recall that $C_{\infty}$ is defined in Equation (3.3) and $\gamma = 2 + 1/4$. For any $2 < \tau \leq \gamma + j$, we can check that \[
  \hat{C}_{\infty}(\pi, \tau', a, b, T_0) = \hat{C}_{\infty}(\pi, \tau, a, b, T_0) \leq |T_0|^{\frac{1}{4} + \frac{1}{\tau'} - \frac{1}{\pi}} \hat{C}_{\infty}(\pi, \gamma + j, a, b, T_0).
  \]
We assume that there is $0 < \mathcal{R} \leq 1$ s.t.

$$1 - \left[ \ell(\pi, 3, \mathcal{R}) + \ell(\pi, 3', \mathcal{R}) \right][T_0]^{\frac{1}{3}} - \frac{2}{\gamma + j} + j \sum_{i=0}^{j} \left[ \ell(\pi, \gamma + i, \mathcal{R}) + \ell(\pi, (\gamma + i)', \mathcal{R}) \right][T_0]^{\frac{1}{3}} - \frac{2}{\gamma + j} \right] \mathcal{C}_\infty(\pi, \gamma + j, a, b, T_0) > 0.$$

Consequently, we easily verify that, as $3 \leq \gamma + j$,

$$\mathcal{R} \in \mathcal{R}(\pi, 3, a, b, T_0) \text{ and } \mathcal{R} \in \mathcal{R}(\pi, \gamma + i, a, b, T_0) \text{ for each } 1 \leq i \leq j.$$

We set $A_j(\pi)$ by

$$A_j(\pi) := \frac{2N}{1 - \left[ \ell(\pi, 3, \mathcal{R}) + \ell(\pi, 3', \mathcal{R}) \right][T_0]^{\frac{1}{3}} - \frac{2}{\gamma + j} + j \sum_{i=0}^{j} \left[ \ell(\pi, \gamma + i, \mathcal{R}) + \ell(\pi, (\gamma + i)', \mathcal{R}) \right][T_0]^{\frac{1}{3}} - \frac{2}{\gamma + j} \right] \mathcal{C}_\infty(\pi, \gamma + j, a, b, T_0)}.$$

It is straightforward that $K(\pi, 3, a, b, \mathcal{R}) \leq A_j(\pi)$ and for any $1 \leq i \leq j$,

$$K(\pi, (\gamma + i)', a, b, \mathcal{R}) = K(\pi, \gamma + i, a, b, \mathcal{R}) \leq A_j(\pi)$$

where we recall that $K$ is defined in (3.8).

**Proposition 3.16.** Let $j \geq 1$ and $\pi \in (0, 1)$. For any $i \leq j$, there exists a positive constant

$$H_{ij} := H_i(A_j(\pi), T_0, m, M, [(w, \cdots, w^{(i+1)})]|_{(\mathcal{R}_i)}|)$$

depending only on $A_j(\pi), i, T_0, m, M$, and the supremum of $w$ and its first $(i + 1)$-derivatives s.t. the constant $H_{ij}$ is a locally bounded function of the indicated quantities and

$$\|w\|_{L^\gamma([0, T_0] \times \mathbb{R}^2)}^{\gamma + i} \leq H_{ij}.$$  \hfill (3.9)

**Proof.** We proceed in a recursive way. For $i = 0$. By the estimate obtained in Proposition 3.15 and recall in **Summary of the estimates**, we have

$$\|w\|_{L^\gamma([0, T_0] \times \mathbb{R}^2)} \leq C(2, 1 + 1/2)\|w, w'\|_{\infty}^{(1 + 1/2)} K(\pi, 3, a, b, \mathcal{R})^{1 + 1/2} 2^d G(T_0, 1 + 1/2, 2, m, M).$$

Since $3 \leq \gamma + j$, $K(\pi, 3, a, b, \mathcal{R}) \leq A_j(\pi)$, if we set

$$H_0(A_j(\pi), T_0, m, M, [(w, \cdots, w^{(1)})]|_{(\mathcal{R}_i)}|) := C(2, 1 + 1/2)\|w, w'\|_{\infty}^{(1 + 1/2)} (A_j(\pi))^{1 + 1/2} 2^d G(T_0, 1 + 1/2, 2, m, M),$$

the inequality (3.9) is true for $i = 0$. Let us assume that it is true for $i \in \{0, \cdots, j - 1\}$. We check for $i = j$. By Proposition 3.10, we know that

$$\|w\|_{L^\gamma([0, T_0] \times \mathbb{R}^2)}^{\gamma + j} \leq \|w\|_{\infty} C(2, \gamma + j - 1) K(\pi, \gamma + j - 1, a, b, \mathcal{R})^{\gamma + j - 1} 2^d \|w'\|_{L^\gamma([0, T_0] \times \mathbb{R}^2)}^{\gamma + j - 1}.$$

Again we have $K(\pi, \gamma + j - 1, a, b, \mathcal{R}) \leq A_j(\pi)$. Also, by recursive assumption

$$\|w'\|_{L^\gamma([0, T_0] \times \mathbb{R}^2)}^{\gamma + j - 1} \leq H_{j-1}(A_j(\pi), T_0, m, M, [(w', \cdots, w^{(j)})]|_{(\mathcal{R}_i)}|).$$

Let $z := \gamma + j - 1$, it is enough to define

$$H_j(A_j(\pi), T_0, m, M, [(w, \cdots, w^{(j+1)})]|_{(\mathcal{R}_i)}|) := \|w\|_{\infty} C(2, z) (A_j(\pi))^2 2^d H_{j-1}(A_j(\pi), T_0, m, M, [(w', \cdots, w^{(j)})]|_{(\mathcal{R}_i)}|)$$

to conclude the proof. \qed
Reminder of the main estimates  Combining the previous results of Summary of the estimates and Proposition 3.16, we get

- For any \( j \geq 1 \),
  \[
  \|w p\|_{L^{\gamma+j}([0,T_0] \times \mathbb{R}^2)}^\gamma \leq (A_j(\pi))^{\gamma+j} 2^d H_j(A_j(\pi), T_0, m, M, (w', \ldots, (w')^{(j+1)}))_\infty; \]

- For all \( j \geq 1 \),
  \[
  \int_0^{T_0} \|w(t) p(t, \cdot)\|_{C^{0,1-\frac{2}{\gamma+j}}(\mathbb{R}^2)}^{\gamma+j} \, dt \leq C(2, \gamma + j) (A_j(\pi))^{\gamma+j} 2^d H_j(A_j(\pi), T_0, m, M, (w', \ldots, (w')^{(j+1)}))_\infty; \]

- For any \( j \geq 1 \),
  \[
  \left[ \|w|\partial_z(v p)\|_{L^{\gamma+j}([0,T_0] \times \mathbb{R})} + \|w(v p)\|_{L^{\gamma+j}([0,T_0] \times \mathbb{R})} \right]^{\gamma+j} \leq K(\pi, \gamma + j, a, b, v, \mathbb{R}_0^2, T_0)^{2d} \left[ \int_0^{T_0} \|w(t) p(t, \cdot)\|_{C^{0,1-\frac{2}{\gamma+j}}(\mathbb{R}^2)}^{\gamma+j} \, dt \right] \left[ \int_0^{T_0} \|w'(t) p_X(t, x)\|_{C^{0,1-\frac{2}{\gamma+j}}(\mathbb{R})}^{\gamma+j} \, dt \right] \, dx_2Q_{x_2} \]

where \( p_{X_2}(t, x_2) := \int_{\mathbb{R}} p(t, x_1, x_2) \, dx_1 \);  

- For any \( j \geq 1, 2 < r = \gamma + j \) and \( \frac{1}{2} > \beta > \kappa > \frac{1}{\delta} \) s.t. \( (1 - 2\beta)r > 2 \),
  \[
  \|w p\|_{C^0,\gamma+j\pi(0, T_0), C^{0,1-2\beta/\pi(r)}(\mathbb{R}^2)} \leq D \left[ \left[ 1 + \|\pi\|_\infty + \sup_{E \in S} \|\nabla a\|_{L^{\gamma+j}(0, T_0) \times E} \right] A_j(\pi) 2^d/\pi + 1 \right] \|w p\|_{L^{\gamma+j}(0, T_0) \times \mathbb{R}^2}^\gamma \]  
  \[
  + \|\pi\|_\infty \|w p\|_{L^{\gamma+j}(0, T_0) \times \mathbb{R}^2}^\gamma + \|w(0)p(0, \cdot)\|_{H^{r,1}(\mathbb{R}^2)}; \]

- For any \( j \geq 1, \pi = \gamma + j > 2 \) and \( \frac{1}{2} > \beta > \pi > \frac{1}{\delta} \) with \( (1 - 2\beta)\pi > 1 \),
  \[
  \|w p\|_{C^0,\pi,1-\pi(0, T_0), C^{0,1-2\beta/\pi(r)}(\mathbb{R})} \leq T \left[ \|\pi\|_\infty \left[ \|wp_1, w_p 1, w|\partial_z p, 1, w|\partial_z (p a^{1,1})|_1 \right] \right]_{L^{\gamma+j}(0, T_0) \times \mathbb{R}} + \|wp_1(0, \cdot)\|_{H^{r,1}(\mathbb{R})}. \]

Choice of the appropriate constant for the upper bound  We recall that \( \gamma = 2 + 1/4 \). Let \( j \geq 1, 2 \) \( \leq \gamma + j \) with \( \frac{1}{2} > \beta > \kappa > \frac{1}{r} \) s.t. \( (1 - 2\beta)r > 2 \) and \( \pi = \gamma + j > 2 \) with \( \frac{1}{2} > \beta > \pi > \frac{1}{\delta} \) s.t. \( (1 - 2\beta)\pi > 1 \). We set \( \pi := 1 - 2\beta - \frac{1}{\gamma+j} \). It is straightforward to check that

\[
\hat{C}_\infty(\pi, \gamma + j, a, b, \mathbb{R}_0^2, T_0) \leq \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \left[ \operatorname{ess sup}_{t \in [0, T_0]} \|a(t, \cdot)\|_{C^{0,\pi}(E)} \right] \sup_{(t, x) \in [0, T_0] \times E} |b(t, x)| + \|\nabla a\|_{L^{\gamma+j}(0, T_0) \times E}; \]

\[
\leq \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \left[ \operatorname{ess sup}_{t \in [0, T_0]} \|a(t, \cdot)\|_{C^{0,\pi}(E)} \right] \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \|\nabla a\|_{L^{\gamma+j}(0, T_0) \times E}. \]

Let \( L_j^{\infty} \) and \( L_j \) be two constants s.t.

\[
\sup_{E \in \mathcal{S}(\mathbb{R}^2)} \sup_{t \in [0, T_0]} \|a(t, \cdot)\|_{C^{0,\pi}(E)} \leq L_j^{\infty} \]

and

\[
\|b\|_\infty + \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \|\nabla a\|_{L^{\gamma+j}(0, T_0) \times E} \leq \|b\|_\infty + 1 + \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \|\nabla a\|_{L^{\gamma+j}(0, T_0) \times E} \leq L_j. \]
Let \( \varepsilon > 0 \), we choose \( 0 < R_\varepsilon (L_j^\infty + L_j) := \overline{R} \leq 1 \) s.t.

\[
1 - \left[ \ell (\pi, 3, \overline{R}) + \ell (\pi, 3', \overline{R}) \right] T_0 |T_0|^4 \sum_{i=0}^j \left[ \ell (\pi, \gamma + i, \overline{R}) + \ell (\pi, (\gamma + i)', \overline{R}) \right] |T_0|^\frac{4}{\gamma + i} \right] \left( L_j^\infty + L_j \right) = \varepsilon > 0.
\]

Notice that, it is possible because \([0, 1] \ni r \rightarrow \ell (\pi, \nu, r) \in \mathbb{R}_+ \) is continuous with \( \ell (\pi, \nu, 0) = 0 \) for any \( \nu > 1 \). This leads to

\[
A_j (\pi) \leq \frac{2N}{\varepsilon}.
\]

By using the previous inequality and Reminder of the main estimates, we have

\[
\| \nabla p \|_{C^{3+\frac{1}{2}}(0, T_0)] \times \mathbb{R}^2} \leq \left( \frac{2N}{\varepsilon} \right)^{\gamma + j} 2^{dH} \left( \frac{2N}{\varepsilon}, T, m, M, \| (w', \ldots, (w'^{(j+1)})_\infty \right) =: \mathbf{L}_j^0 (|w(1), \ldots, w^{(j+2)}|_\infty),
\]

\[
\int_0^{T_0} \| w(t) p(t, \cdot) \|_{\gamma + j, C^{\alpha, 1}((0, T_0] \times \mathbb{R})} \leq C(2, \gamma + j) \mathbf{L}_j^0 (|w(1), \ldots, w^{(j+2)}|_\infty) =: \mathbf{L}_j^1 (|w, \ldots, w^{(j+2)}|_\infty)
\]

and

\[
\left[ \| \nabla_p \mathbf{p} \|_{C^{3+\frac{1}{2}}(0, T_0)] \times \mathbb{R}^2} + \| \mathbf{w} \mathbf{p} \|_{C^{3+\frac{1}{2}}(0, T_0)] \times \mathbb{R}^2} \right] \gamma + j \leq \frac{2N}{\varepsilon} 2^{dH} \left( \mathbf{L}_j^1 (|w, \ldots, w^{(j+2)}|_\infty) \right) \gamma + j \leq \frac{2N}{\varepsilon} 2^{dH} \left( \mathbf{L}_j^2 (|w, \ldots, w^{(j+2)}|_\infty) \right) \gamma + j.
\]

We choose \( L_j \) s.t.

\[
1 + \mathbf{L}_j^0 (|w(1), \ldots, w^{(j+2)}|_\infty) + \mathbf{L}_j^1 (|w, \ldots, w^{(j+2)}|_\infty) + \mathbf{L}_j^2 (|w, \ldots, w^{(j+2)}|_\infty) \leq \frac{L_j}{M}.
\]

(3.10)

Notice that \( L_j \) depends on \( \varepsilon, j, N, T_0, m, M, \int_{\mathbb{R}} \left[ \int_0^{T_0} \| w(t) p X_2 (t, x_2) \|_{\gamma + j, \mathbb{R}} \right] \frac{dt}{dx_2} \) and \( (w, \ldots, w^{(j+2)}) \).

In addition,

\[
\| \nabla_p \|_{C^{3+\frac{1}{2}}(0, T_0)] \times \mathbb{R}^2} \leq \frac{2N}{\varepsilon} 2^{dH} \left( |w(1), \ldots, w^{(j+2)}|_\infty + \| \mathbf{w} p (0, \cdot) \|_{H^{\gamma + j, 1} (\mathbb{R})} \right) =: \mathbf{L}_j^3 (|w, \ldots, w^{(j+2)}|_\infty)
\]

and

\[
\| \mathbf{w} \mathbf{p} \|_{C^{3+\frac{1}{2}}(0, T_0)] \times \mathbb{R}^2} \leq \frac{2N}{\varepsilon} 2^{dH} \left( \mathbf{L}_j^3 (|w, \ldots, w^{(j+2)}|_\infty) \right) \gamma + j.
\]

We choose \( L_j^\infty \) s.t.

\[
1 + \mathbf{L}_j^3 (|w, \ldots, w^{(j+2)}|_\infty) + \mathbf{L}_j^4 (|w, \ldots, w^{(j+3)}|_\infty) \leq \frac{L_j^\infty}{M}.
\]

(3.11)

the constant \( L_j^\infty \) depends on \( \varepsilon, j, N, D, \overline{R}, T_0, m, M, \int_{\mathbb{R}} \left[ \int_0^{T_0} \| w(t) p X_2 (t, x_2) \|_{\gamma + j, \mathbb{R}} \right] \frac{dt}{dx_2} \), \( \| \mathbf{w} p (0, \cdot) \|_{H^{\gamma + j, 1} (\mathbb{R})} \), \( \| \mathbf{w} \mathbf{p} (0, \cdot) \|_{H^{\gamma + j, 1} (\mathbb{R})} \) and \( (w, \ldots, w^{(j+3)}) \). We can easily check the next proposition
Proposition 3.17. For any \( j \geq 1 \), with the previous choice of \( L_j \) and \( L_j^\infty \), one has if
\[
\sup_{E \in \mathcal{S}(\mathbb{R}^2)} \sup_{t \in [0,T_0]} \|a(t,\cdot)\|_{C^{0,1-\frac{\gamma}{\beta}}(\mathbb{R}^2)} \leq L_j^\infty
\]
and
\[
\|b\|_\infty + 1 + \sup_{E \in \mathcal{S}(\mathbb{R}^2)} \|\nabla a\|_{L^{\gamma+j}(0,T_0) \times E} \leq L_j
\]
then
\[
1 + \|w p\|_{C^{0,1-1/(\gamma+j)}([0,T_0] \times \mathbb{R}^2)} + \int_0^{T_0} \|w(t)p(t,\cdot)\|_{C^{0,1-1/(\gamma+j)}(\mathbb{R}^2)} \, dt + \left[ \|w\|_{L^{\gamma+j}(0,T_0)} + \|w p_1\|_{L^{\gamma+j}(0,T_0) \times \mathbb{R}} \right] \leq \frac{L_j}{M}
\]
and
\[
1 + \|w p\|_{C^{0,1-1/(\gamma+j)}([0,T_0] \times \mathbb{R}^2)} + \|wp_1\|_{C^{0,1-1/(\gamma+j)}([0,T_0] \times \mathbb{R})} \leq \frac{L_j^\infty}{M}
\]
where \( r = \gamma + j < \beta > \kappa > \frac{1}{r} \) s.t. \((1-2\beta)r > 2\) and \( \frac{1}{r} > \beta > \kappa > \frac{1}{r} \) with \((1-2\beta)^r > 1\).

3.5 Existence of solution
We set \( j \geq 1, r = \gamma + j, \beta = \overline{\beta}, \kappa = \overline{\kappa} \) satisfying \( \frac{1}{r} > \beta > \kappa > \frac{1}{r} \) and \((1-2\beta)r > 2\). In particular \( \frac{1}{r} > \beta > \kappa > \frac{1}{r} \) and \((1-2\beta)^r > 1\).

Let \( T_0 > 0 \) and \( \theta \in [-1,1] \). We give ourselves the following maps
\[
\overline{\mathbf{B}}, \overline{\boldsymbol{\pi}} : L^1([0,T_0] \times \mathbb{R}^2) \rightarrow \mathbb{R} \times \mathbb{R} \quad \text{and} \quad [\overline{\mathbf{B}}, \overline{\boldsymbol{\pi}}] : [0,T_0] \times \mathbb{R} \rightarrow \mathbb{R}^2.
\]

Assumption 3.18. The positive numbers \( m \) and \( M \) are s.t. for any map \( f \in L^1([0,T_0] \times \mathbb{R}^2) \) s.t. \( f \geq 0 \), for each \( t \in [0,T_0] \) and \( x = (x_1,x_2) \in \mathbb{R}^2 \), if we introduce the vector map \( \overline{\mathbf{B}} \) and the symmetric matrix \( \overline{\boldsymbol{\pi}} \) by \( \overline{\mathbf{B}}(t,x) := \overline{\mathbf{B}}(f)(t,x), \overline{\boldsymbol{\pi}}(t,x) := \overline{\pi}(t,x) \)
\[
\overline{\pi}^{1,1}(t,x) := \frac{1}{2} \overline{\pi}^1(f)(t,x)^2, \quad \overline{\pi}^{1,2}(t,x) := \frac{1}{2} \overline{\pi}^1(f)(t,x)\overline{\pi}^2(t,x)\theta \quad \text{and} \quad \overline{\pi}^{2,2}(t,x) := \frac{1}{2} \overline{\pi}^2(t,x)^2
\]
one has that \( \|\overline{\mathbf{B}}\| \leq M, m 1_{\mathbb{R}^2} \leq \overline{\boldsymbol{\pi}} \leq M 1_{\mathbb{R}^2} \) for \( E \subset \mathbb{R}^2 \) open set with diam\((E) \leq 1\)
\[
\|\nabla \overline{\pi}\|_{L^{\gamma+j}(0,T_0)} \leq M \left[ 1 + \|f\|_{L^{\gamma+j}(0,T_0) \times \mathbb{R}^2} + \|f_1\|_{L^{\gamma+j}(0,T_0)} \right]
\]
and
\[
\sup_{t \in [0,T_0]} \|\overline{\pi}(t,\cdot)\|_{C^{0,1-2\beta-2/(\gamma+j)}(\mathbb{R}^2)} \leq M \left[ 1 + \sup_{t \in [0,T_0]} \|f(t,\cdot)\|_{C^{0,1-2\beta-2/(\gamma+j)}(\mathbb{R}^2)} + \sup_{t \in [0,T_0]} \|f_1(t,\cdot)\|_{C^{0,1-2\beta-2/(\gamma+j)}(\mathbb{R}^2)} \right].
\]
The map \( [0,T_0] \times \mathbb{R} \ni (t,x_2) \rightarrow \overline{\beta}(t,x_2) \in \mathbb{R} \) is Lipschitz in \( x_2 \) uniformly in \( t \).

Let \( w \in C_0^\infty(T_0) \) s.t. \( w \geq 0 \), and \( \varrho : [0,T_0] \times \mathbb{R}^2 \rightarrow \mathbb{R}^+ \) be a map s.t. for each \( t \in [0,T_0] \), \( \varrho(t,\cdot) \) is a density of probability i.e. \( \int_{\mathbb{R}^2} \varrho(t,x) \, dx = 1 \) and \( \varrho \) can be extended to \( [0,T_0] \times \mathbb{R}^2 \). We introduce the density \( p(t,\cdot) \) of \( \mu_t := \mathcal{L}(X_t) \) where \( X := (X_1,X_2) \) is solution of: \( p(0,\cdot) = \varrho(0,\cdot) = \varrho_0(\cdot) \),
\[
dX_1^t = \overline{\mathcal{B}}(w_0)(t,X_t) \, dt + \overline{\mathcal{F}}(w_0)(t,X_t) \, dW_t \quad \text{and} \quad dX_2^t = \overline{\mathcal{X}}(t,X_t) \, dt + \overline{\mathcal{B}}(t,X^t_2) \, dB_t
\]
where \( d(W,B)_t = \theta dt \). Let \( j \geq 1 \), we take the initial density \( \varrho_0 \) s.t.
\[
\|\varrho_0\|_{H^{\gamma+j}(\mathbb{R}^2)} + \|\varrho_0\|_{H^{\gamma+j}(\mathbb{R}^2)} < \infty \quad \text{and there is } \alpha > 0 \text{ s.t. } \int_{\mathbb{R}} e^{\alpha z^2} \left| \int_{\mathbb{R}} \varrho_0(z_1,z_2) \, dz_1 \right|^{\gamma+j} \, dz_2 < \infty.
\]
Since Assumption 3.18 holds, \( \overline{\mathcal{X}} \) is bounded and \( \overline{\mathcal{B}} \) is Lipschitz. Therefore (see Appendix Proposition A.4)
\[
\int_{\mathbb{R}} \left[ \int_{\mathbb{R}} |p(X_2(t,x)|^{\gamma+j} \, dt \right]^{(\gamma+j)^2} \, dz_2 < \infty, \quad \text{for each } \overline{\mathcal{T}} > 0,
\]
where the upper bound depends only on \( \overline{\lambda}, \overline{\beta} \) and \( \varrho_0 \). Using the properties verified by \( \overline{\mathbf{B}}, \overline{\boldsymbol{\pi}}, \overline{\lambda}, \overline{\beta} \) in Assumption 3.18, by an obvious application of Proposition 3.17, we obtain the upper bounds of \( p \) given the upper bounds of \( \varrho_0 \).
Proof. Let $\mathcal{C}$ be the locally uniform topology on $C([0, T_0], \mathbb{R}^2)$. By Proposition 3.19, $\mathcal{C}$ is compact for the locally uniform topology.

Let $j \geq 1$ and $\epsilon > 0$ be fixed. We take $w \in C_0^\infty(T_0)$ and, $L_j$ and $L_j^\infty$ chosen as previously. We say that $f \in C([0, T_0] \times \mathbb{R}^2; \mathbb{R}_+)$ belongs to $\Theta(w)$ if: $\int_{\mathbb{R}^2} f(t, x) dx = 1$ for each $t \in [0, T_0]$, $f(0, \cdot) = \varrho_0(\cdot)$.

Let $j \geq 1$ and $\epsilon > 0$ be fixed. We take $w \in C_0^\infty(T_0)$ and, $L_j$ and $L_j^\infty$ chosen as previously. We say that $f \in C([0, T_0] \times \mathbb{R}^2; \mathbb{R}_+)$ belongs to $\Theta(w)$ if: $\int_{\mathbb{R}^2} f(t, x) dx = 1$ for each $t \in [0, T_0]$, $f(0, \cdot) = \varrho_0(\cdot)$.

Proposition 3.20. For any $w \in C_0^\infty(T_0)$ the set $\Theta(w)$ is compact. For each $s < T_0$, $\inf_{t \in [0, s]} w(t) > 0$ then $\Theta(w)$ is compact for the locally uniform topology.

Proof. The convexity of $\Theta(w)$ for any $w \in C_0^\infty(T_0)$ is obvious. Let $w \in C_0^\infty(T_0)$ be s.t. for any $s < T_0$, $\inf_{t \in [0, s]} w(t) > 0$. Let $(f^n)_{n \geq 1} \subset \Theta(w)$. As $\|w f^n\|_{C^{0, 0, 1/(1+j)}([0, T_0], \mathbb{R}^2)} \leq \frac{L_j^\infty}{M} - 1$, the sequence $(f^n)_{n \geq 1}$ is relatively compact in $C([0, T_0] \times \mathbb{R}^2)$ for the uniform topology. Let $f^n$ be the limit of a sub-sequence. We use the same notation for the sequence and its sub-sequence. We set $f(t, x) := \frac{\int_{\mathbb{R}^2} f^n(t, x, w) dx}{w(t)}$ for any $(t, x) \in [0, T_0] \times \mathbb{R}^2$. It is easy to see that for $s < T_0$,

$$f(t, x) = \lim_{n \to \infty} f^n(t, x) = \lim_{n \to \infty} \frac{w(t) f^n(t, x)}{w(t)}$$

for any $(t, x) \in [0, s] \times \mathbb{R}^2$.

Notice that, for all $s < T_0$ and any $(t', x') \in [0, s] \times [0, s]$,

$$|f(t, x) - f(t', x')| \leq \frac{1}{\delta^2} \left[ \|w\|_{\infty} |w(t) - w(t')| \right],$$

where $\delta := \inf_{t \in [0, s]} w(t)$. Then $\sup_{n \geq 1} \|f^n\|_{C^{0, 0, 1/(1+j)}([0, s], \mathbb{R}^2)} < \infty$. This is true for any $s > T_0$, consequently, the (sub)sequence $(f^n)_{n \geq 1}$ converges towards $f$ for the locally uniform topology on $C([0, T_0] \times \mathbb{R}^2)$. Using $\|w f^n\|_{C^{0, 0, 1/(1+j)}([0, T_0], \mathbb{R}^2)} \leq \frac{L_j^\infty}{M} - 1$, by similar way, up to a sub-sequence, we also conclude that the sequence $(f^n)_{n \geq 1}$ converges to $f_1$ for the locally uniform topology on $C([0, T_0] \times \mathbb{R})$. We easily verify that

$$\|w f\|_{C^{0, 0, 1/(1+j)}([0, T_0], \mathbb{R}^2)} \leq \frac{L_j^\infty}{M} - 1.$$
By using \( \| \vartheta_{x_1} f \|^2_{L^\gamma_{\gamma+j}([0,t_0] \times \mathbb{R})} + \| w \nabla f \|^2_{L^\gamma_{\gamma+j}([0,t_0] \times \mathbb{R}^2)} \leq \frac{L_\gamma}{M} - 1 \), we deduce that \( w \nabla f \) and \( \vartheta_{x_1} f \) are well defined. By using the compactness results from the Sobolev embedding Theorem and Fatou lemma, we check the different inequalities for \( f \). All of these results allow us to conclude that \( f \in \Theta(w) \). Consequently, \( \Theta(w) \) is compact. \( \square \)

Let \( w \in C_0^\infty(T_0) \) be s.t. for each \( s < T_0 \), \( \inf_{t \in [0,s]} w(t) > 0 \). We introduce the application

\[
\Psi_w : \Theta(w) \ni \varrho \mapsto \Psi_w(\varrho) \in \Theta(w)
\]

where \( \Psi_w(\varrho)(t,x) = p(t,x) \) for \( (t,x) \in [0,T_0) \times \mathbb{R}^2 \), and \( p \) is the density of \( \mu_t \) := \( \mathcal{L}(X_t) \) where \( X := (X^1,X^2) \) is solution of:

\[
p(0,\cdot) = g_0(\cdot),
\]

\[
dX_t^1 = \overline{b}(w,\varrho)(t,X_t)dt + \overline{\sigma}(w,\varrho)(t,X_t)dW_t \quad \text{and} \quad dX_t^2 = \overline{\sigma}(t,X_t^2)dt + \overline{\beta}(t,X_t^2)dB_t
\]

with \( d(W,B)_t = \theta dt \). Besides Assumption 3.18, we assume that \( (\overline{b},\overline{\sigma}) \) satisfies:

**Assumption 3.21.** Whenever \( \lim_{n \to \infty} \| f^n - f \|_{L^1([0,s] \times \mathbb{R}^2)} = 0 \) for any \( s \in [0,T_0) \) and any \( \varphi \in C_c([0,T_0) \times \mathbb{R}^2) \),

\[
\lim_{n \to \infty} \int_{[0,T_0] \times \mathbb{R}^2} \overline{b}(f^n)(t,x)\varphi(t,x) \, dx \, dt = \int_{[0,T_0] \times \mathbb{R}^2} \overline{b}(f)(t,x)\varphi(t,x) \, dx \, dt
\]

and

\[
\lim_{n \to \infty} \int_{[0,T_0] \times \mathbb{R}^2} a^{i,j}(f^n)(t,x)\varphi(t,x) \, dx \, dt = \int_{[0,T_0] \times \mathbb{R}^2} a^{i,j}(f)(t,x)\varphi(t,x) \, dx \, dt
\]

where \( a^{1,1}(\rho)(t,x) := \frac{1}{2}\overline{\sigma}(\rho)(t,x)^2 \), \( a^{2,1}(\rho)(t,x) := a^{1,2}(\rho)(t,x) := \frac{1}{2}\overline{\sigma}(\rho)(t,x)\overline{\beta}(t,x) \theta \) and \( a^{2,2}(\rho)(t,x) := \frac{1}{2}\overline{\beta}(t,x)^2 \).

We say that the map \( \Psi_w : \Theta(w) \to \Theta(w) \) is well defined if for \( \varrho \in \Theta(w) \) there is a unique density \( \Psi_w(\varrho) \) and we have \( \Psi_w(\varrho) \in \Theta(w) \).

**Proposition 3.22.** The map \( \Psi_w : \Theta(w) \to \Theta(w) \) is well defined and continuous.

**Proof.** By [24, 6.4.4 Corollary], we know that for \( \varrho \in \Theta(w) \), the density \( \Psi_w(\varrho) \) is uniquely defined. By Proposition 3.19, we see that \( \Psi_w(\varrho) \in \Theta(w) \). Let us verified that \( \Psi_w \) is continuous. Let \( (\varrho^n)_{n \geq 1} \subset \Theta(w) \) be a sequence s.t. \( \lim_{n \to \infty} \varrho^n = \varrho^\infty \) for the locally uniform topology with \( \varrho^\infty \in \Theta(w) \). As \( \Theta(w) \) is a compact set, the sequence \( (\Psi_w(\varrho^n))_{n \geq 1} \subset \Theta(w) \) is relatively compact. Notice that, as \( \lim_{n \to \infty} \varrho^n = \varrho^\infty \) for the locally uniform topology, using Assumption 3.21, and passing in the limit in the Fokker–Planck equation, we can verify that the limit of any sub-sequence of \( (\Psi_w(\varrho^n))_{n \geq 1} \) is \( \Psi_w(\varrho^\infty) \). Indeed, let \( \overline{\rho}^\infty \) be the limit of a sub-sequence. For simplification, the sub–sequence and the sequence \( (\overline{\rho}^n := \Psi_w(\varrho^n))_{n \geq 1} \) share the same notation. For each \( s < T_0 \) and \( K > 0 \), we have

\[
\| \varrho^n - \varrho^\infty \|_{L^1(\mathbb{R}^2)} \leq s(2K)^d \sup_{(t,x_1,x_2) \in [0,s] \times (-K,K)^2} \| (\varrho^n - \varrho^\infty)(t,x_1,x_2) \| + \| \varrho^n - \varrho^\infty \|_{L^1([0,s] \times (\mathbb{R}^2 \setminus (-K,K)^2))} \leq s(2K)^d \sup_{(t,x_1,x_2) \in [0,s] \times (-K,K)^2} \| (\varrho^n - \varrho^\infty)(t,x_1,x_2) \| + \int_{\mathbb{R}^2} 1_{|x| \geq K}(\varrho^n + \varrho^\infty)(t, dx) \, dt.
\]

It is easy to check that for each \( t \in [0,s] \), the sequence of distribution \( (\varrho^n(t,x)dx)_{n \geq 1} \) converges weakly to \( \varrho^\infty(t,x)dx \).

As \( \int_{\mathbb{R}^2} 1_{|x| = K} \varrho^\infty(t,x)dx = 0 \) for each \( K > 0 \), by Portemanteau Theorem, we have that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} 1_{|x| \geq K} \varrho^n(t,x)dx = \int_{\mathbb{R}^2} 1_{|x| \geq K} \varrho^\infty(t,x)dx \, dt.
\]

Therefore, by taking first \( n \to \infty \) and then \( K \to \infty \), we find that \( \lim_{n \to \infty} \| \varrho^n - \varrho^\infty \|_{L^1(\mathbb{R}^2)} = 0 \). Consequently, we have \( \lim_{n \to \infty} \| w\varrho^n - w\varrho^\infty \|_{L^1(\mathbb{R}^2)} = 0 \) and \( \lim_{n \to \infty} \| \overline{\rho}^n - \overline{\rho}^\infty \|_{C^0([0,s] \times \mathbb{R}^2)} = 0 \), \( \| \overline{\varphi}^{i,j} \|_{L^1([0,s] \times \mathbb{R}^2)} = 0 \) for any \( s \in [0,T_0) \). By Assumption 3.21, we obtain that for any \( \varphi \in C_c([0,T_0) \times \mathbb{R}^2) \),

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} (\overline{b}, \overline{\lambda})(w \varrho^n)(t,x)\varphi(t,x)\overline{\rho}^n(t,x)dx = \int_{\mathbb{R}^2} (\overline{b}, \overline{\lambda})(w \varrho^\infty)(t,x)\varphi(t,x)\overline{\rho}^\infty(t,x)dx \quad (3.12)
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2_0} a^{i,j}(w \varphi^n)(t, x) \varphi^{i,j}(t, x) \overline{p^n}(t, x) \, dx \, dt = \int_{\mathbb{R}^2_0} a^{i,j}(w \varphi^\infty)(t, x) \varphi^{i,j}(t, x) \overline{p}^\infty(t, x) \, dx \, dt. \tag{3.13}
\]

For each \(1 \leq n \leq \infty\), if we introduce the vector map \(b^n\) by \(b^{n,1}(t, x) = \overline{\Phi}(w \varphi^n)(t, x)\), \(b^{n,2}(t, x) = \overline{\Theta}(t, x)\), using (3.12) and (3.13), we find that, for any \(r \in [0, T]\) and \(\varphi \in C_c^\infty(\mathbb{R}^2\beta)\)
\[
\int_{\mathbb{R}^2} \varphi(r, x) \overline{p}^\infty(r, x) \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^2} \varphi(r, x) \overline{p}^n(r, x) \, dx
\]
\[
= \int_{\mathbb{R}^2} \varphi(0, x) g_0(x) \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^2} [\partial_t \varphi(t, x) + b^{n,1}(t, x) \partial_x \varphi(t, x) + a^{i,j}(w \varphi^n)(t, x) \partial_{x_i} \varphi(t, x)] \, dx \, dt
\]
\[
= \int_{\mathbb{R}^2} \varphi(0, x) g_0(x) \, dx + \int_{\mathbb{R}^2} [\partial_t \varphi(t, x) + b^{\infty,1}(t, x) \partial_x \varphi(t, x) + a^{i,j}(w \varphi^\infty)(t, x) \partial_{x_i} \varphi(t, x)] \, dx \, dt.
\]

Let \(X := (X^1, X^2)\) be the \(\mathbb{R}^2\)-valued \(\mathcal{F}\)-adapted continuous process satisfying: \(\mathcal{L}(X_0)(dx) = g_0(x) \, dx\) and for each \(t \in [0, T]\),
\[
dX^1_t = \overline{\Phi}(w \varphi^\infty)(t, X^1_t) \, dt + \overline{\Theta}(w \varphi^\infty)(t, X^1_t) \, dW_t \quad \text{and} \quad dX^2_t = \overline{\Theta}(t, X^2_t) \, dt + \overline{\beta}(t, X^2_t) \, dB_t.
\]

We denote by \(p\) the density of \(\mathcal{L}(X_t)\) i.e. \(\mathcal{L}(X_t)(dx) = p(t, x) \, dx\). By uniqueness and equivalence between Fokker–Planck equation and McKean–Vlasov process, we deduce that for each \(t \in [0, T]\), \(p(t, x) = \overline{p}^\infty(x)\). We deduce that \(\overline{p}^\infty = \Phi^\infty(w)\). This is true for any sub-sequence of \((\Psi_n(w^n))_{n \geq 1}\). We deduce that the entire sequence \((\Psi_n(w^n))_{n \geq 1}\) converges to \(\Phi(w)\). We can conclude the proof.

Since \(\Psi_n\) is continuous and \(\Theta(w)\) is a compact convex set, by a direct application of the Schauder fixed–point theorem, we have the following results

**Theorem 3.23.** The map \(\Psi_n : \Theta(w) \to \Theta(w)\) has at least one fixed point i.e. there exists \(p \in \Theta(w)\) s.t. \(p = \Psi_n(p)\).

**Corollary 3.24.** Let \(j \geq 1\). For any \(T < T_0\), there exists an \(\mathbb{R}^2\)-valued \(\mathcal{F}\)-adapted continuous process \(X := (X^1, X^2)\) satisfying: \(p(0, \cdot) = g_0(\cdot)\)
\[
dX^1_t = \overline{\Phi}(p)(t, X^1_t) \, dt + \overline{\Theta}(p)(t, X^1_t) \, dW_t \quad \text{and} \quad dX^2_t = \overline{\Theta}(t, X^2_t) \, dt + \overline{\beta}(t, X^2_t) \, dB_t \quad \text{for any} \ t \leq T
\]
where \(\mathcal{L}(X_t)(dx) = p(t, x) \, dx\) and \(p\) satisfies
\[
1 + \|p\|_{H^{\gamma+1,1}([0, T] \times \mathbb{R}^2)} + \|\partial_x p\|_{L^{\gamma+1}([0, T] \times \mathbb{R})} + \|p\|_{L^{\gamma+1}([0, T] \times \mathbb{R})} \leq \frac{L_j}{M}
\]
and
\[
1 + \|p\|_{C_0,1/(\gamma+1)}([0, T]; C_0,1-2\beta-2/(\gamma+2)(\mathbb{R}^2)) + \|p\|_{C_0,1-1/\gamma ([0, T]; C_0,1-2\beta-1/\gamma (\mathbb{R}))} \leq \frac{L_j^\infty}{M}
\]
with \(L_j\) and \(L_j^\infty\) depend only on \(j, m, M, g_0(\cdot) := g(0, \cdot), T, T_0\).

**Proof of Corollary 3.24.** As \(T < T_0\), it is enough to choose \(w \in C_c^\infty(T_0)\) such that \(w(t) = 1\) for each \(t \in [0, T]\).
3.5.2 Checking the assumptions for the examples given in Section 2.1

In this part, we prove that the examples given in Section 2.1 satisfy Assumption 2.1. We only check the first example, the proof for the second example is similar. Recall that the coefficients \( b^\circ, \sigma^\circ, \lambda \) and \( \beta \) are given in Section 2.1. We have the definition

\[
b(f)(t,x_1,x_2) = b^\circ(t,x_1,x_2, f(t,x), f_1(t,x_1),(fh)_1(t,x_1)) \quad \text{and} \quad \sigma(f)(t,x_1,x_2) = \sigma^\circ(t,x_1,x_2, f(t,x), f_1(t,x_1),(fh)_1(t,x_1)).
\]

As in Assumption 2.1, we introduce the vector map \( \mathbf{B} \) and the symmetric matrix \( \mathbf{\Sigma} := (\mathbf{\Sigma}^{ij})_{1 \leq i,j \leq 2} \) by \( \mathbf{B}^i(f)(t,x) = b(f)(t,x), \mathbf{B}^j(f)(t,x) = \lambda(t,x_2) \),

\[
\mathbf{\Sigma}^{1,1}(f)(t,x) := \frac{1}{2}\sigma(f)(t,x)^2, \quad \mathbf{\Sigma}^{2,2}(f)(t,x) := \frac{1}{2}\sigma(f)(t,x)\beta(t,x_2)\theta, \quad \text{and} \quad \mathbf{\Sigma}^{2,1}(f)(t,x) := \frac{1}{2}\beta(t,x_2)^2
\]

**Verification of Growth assumption** Let \( T > 0 \). Notice that, for any \( (t,x_1,x_2) \in [0,T] \times \mathbb{R}^2 \) and \( f \geq 0 \), we have

\[
(t,x_1,x_2,f(t,x),f_1(t,x_1),(fh)_1(t,x_1)) \in \mathcal{E}^T_k \quad \text{and} \quad (t,x_1,x_2,f(t,x),f_1(t,x_1),(fu)_1(t,x_1)) \in \mathcal{E}^T_v,
\]

recall that \( \mathcal{E}^T_k \) and \( \mathcal{E}^T_v \) are defined in Equation (2.3) and Equation (2.4). Knowing the assumptions satisfy by \( (b^\circ, \sigma^\circ) \) in the example, we set

\[
0 < m := \inf_{\mathcal{E}^T_k} \inf_{f \geq 0} \frac{\int (t,x) z^\top a z}{\|z\|^2}, \quad M := \sup_{\mathcal{E}^T_k} |b| + |\lambda| + \sup_{\mathcal{E}^T_v} |a| + 1 + \sup_{\mathcal{E}^T_v} |\nabla a|^q + \sum_{i=1}^k |c_M^i|^q.
\]

We have for all \( (t,x) \in [0,T] \times \mathbb{R}^2 \), \( |\mathbf{B}(f)(t,x)| \leq M, m_2 \leq |\mathbf{\Sigma}(f)(t,x)| \leq M_1 \). For any \( E \subset \mathbb{R}^2 \) open set s.t. \( \text{diam}(E) \leq 1 \)

\[
\|\nabla \Sigma(f)\|_{L^g([0,T] \times E)}^g \leq M \left[ 1 + \|f\|_{L^{g+1}([0,T] \times \mathbb{R}^2)} + \|\partial_{x_1} f_1\|_{L^\infty([0,T] \times \mathbb{R}^2)} + \|f_1\|_{L^\infty([0,T] \times \mathbb{R}^2)} \right]
\]

and

\[
\sup_{t \in [0,T]} \|\Sigma(f)(t,\cdot)\|_{C^{0,1-\gamma}(E)} \leq M \left[ 1 + \sup_{t \in [0,T]} \|f(t,\cdot)\|_{C^{0,1-2/3}(\mathbb{R}^2)} + \sup_{t \in [0,T]} \|f_1(t,\cdot)\|_{C^{0,1-\gamma}(\mathbb{R})} \right].
\]

**Verification of Continuity assumption** Let \( T > 0 \) and \( (f^n)_{1 \leq n \leq \infty} \) be a sequence s.t. \( \lim_{n \to \infty} \|f^n - f^\infty\|_{L^1([0,T] \times \mathbb{R}^2)} = 0 \). We can verify that

\[
\lim_{n \to \infty} \int_{[0,T] \times \mathbb{R}} |f^n(t,x_1) - f^\infty(t,x_1)| \, dx_1 \, dt \leq \lim_{n \to \infty} \|f^n - f^\infty\|_{L^1([0,T] \times \mathbb{R}^2)} = 0.
\] \hspace{1cm} (3.14)

Let \( \varphi \in C_c([0,T] \times \mathbb{R}^2) \) and \( \varphi^{i,j} \in C_c([0,T] \times \mathbb{R}^2) \). The sequences \((z^n)_{n \geq 1} \subset \mathbb{R}^2\) and \((c^n)_{n \geq 1}\) are bounded where

\[
z^n := \int_{[0,T] \times \mathbb{R}^2} b(f^n)(t,x)\varphi(t,x) \, dx \, dt \quad \text{and} \quad c^n := \int_{[0,T] \times \mathbb{R}^2} \Sigma^{i,j}(f^n)(t,x)\varphi^{i,j}(t,x) \, dx \, dt.
\]

We only deal with \( (z^n)_{n \geq 1} \), the analysis of the sequence \( (c^n)_{n \geq 1} \) is similar. The sequence \((z^n)_{n \geq 1}\) is compact. Let \( z^\infty \) be the limit of a convergent sub-sequence \((z^{n_k})_{k \geq 1}\). By using Equation (3.14), we can find a sub-sequence \((n_k^\prime) := U(n_k)_{k \geq 1}\) where \( U \) is a non-decreasing map, s.t. \( \lim_{k \to \infty} (f^{n_k^\prime}(t,x), (f^{n_k^\prime}(t,x))_{1 \times (x_1,x_2)}) = (f^\infty(t,x),(f^\infty(t,x))_{1 \times (x_1,x_2)}) \) a.e. \((t,x)\). Consequently, since \((b^\circ, \lambda, \sigma^\circ, \lambda)(t,x_1,x_2, \cdot)\) is continuous uniformly in \((t,x_1,x_2)\), by dominated convergence theorem,

\[
z^\infty = \lim_{k \to \infty} z^{n_k} = \lim_{k \to \infty} z^{n_k} = \lim_{k \to \infty} \int_{[0,T] \times \mathbb{R}^2} b(f^{n_k})(t,x)\varphi(t,x) \, dx \, dt = \int_{[0,T] \times \mathbb{R}^2} b(f^\infty)(t,x)\varphi(t,x) \, dx \, dt.
\]

We proved that any convergent sub-sequence \((z^{n_k})_{k \geq 1}\) converges towards \( \int_{[0,T] \times \mathbb{R}^2} b(f^\infty)(t,x)\varphi(t,x) \, dx \, dt \). We can deduce that the entire sequence \((z^n)_{n \geq 1}\) converges to \( \int_{[0,T] \times \mathbb{R}^2} b(f^\infty)(t,x)\varphi(t,x) \, dx \, dt \). Therefore, Assumption 2.1 is satisfied for \((b, \lambda, \sigma, \beta)\).
3.6 Approximation by particle system

Let $\delta > 0$ and $G_\delta : \mathbb{R} \to \mathbb{R}_+$ be a kernel satisfying: $\int_\mathbb{R} G_\delta(x)dx = 1$, and for any $\alpha \geq 1$,

$$\lim_{\delta \to 0} G_\delta * \varphi = \varphi, \text{ a.e.}, \quad \|G_\delta * \varphi\|_{L^\alpha(\mathbb{R})} \leq C\|\varphi\|_{L^\alpha(\mathbb{R})} \quad \text{and} \quad \|\nabla(G_\delta * \varphi)\|_{L^\alpha(\mathbb{R})} \leq C\|\nabla \varphi\|_{L^\alpha(\mathbb{R})}, \text{ for all } \varphi \in C_c^\infty(\mathbb{R})$$

where $C$ is independent of $\delta$ and $*$ denotes the convolution product. We set $G_\delta(x) = G_\delta(x_1)G_\delta(x_2)$. We always consider $(b, \lambda, \sigma, \beta)$ as in Assumption 2.1. We define

$$(\overline{V}_\delta, \overline{\sigma}_\delta)(f)(t, x_1, x_2) := (b, \sigma)(f_\delta)(t, x_1, x_2)$$

where for any $f \in L^1_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^2))$, $f_\delta$ is defined by $f_\delta(t, x) := G_\delta * (f(t, \cdot))(x)$. For proving the approximation by particle system, we start by giving some results involving the regularization of the dependency w.r.t. the density.

Proposition 3.25. Let $j = 1, \delta > 0$ and $T > 0$. There exists an $\mathbb{R}^2$-valued $\mathbb{F}$-adapted continuous process $X^\delta := X := (X^1, X^2)$ satisfying $p(0, \cdot) = q_0(\cdot)$, for $t \in [0, T]$,

$$dX^1_t = \overline{V}_\delta(p^\delta)(t, X_t)dt + \overline{\sigma}_\delta(p^\delta)(t, X_t)W_t \quad \text{and} \quad dX^2_t = \lambda(t, X^2_t))dt + \beta(t, X^2_t)dB_t$$

where $p^\delta(t, x)dx = L(X^\delta_t)(dx)$. In addition

$$1 + \|p^\delta\|_{L^\alpha(\mathbb{R}^2)}^{\gamma_{j, j}}(0, T] \times \mathbb{R}^2) + \|\nabla_1 p^\delta(1, \cdot)\|_{L^\alpha(\mathbb{R}^2)}^{\gamma_{j, j}}(0, T] \times \mathbb{R}^2) + \|p^\delta(1, \cdot)\|_{L^\alpha(\mathbb{R}^2)}^{\gamma_{j, j}}(0, T] \times \mathbb{R}^2) \leq \frac{L_j}{M} \quad \text{and} \quad 1 + \|p^\delta\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) + \|p^\delta(1, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) + \|p^\delta(1, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) \leq \frac{L_j^\infty}{M} \quad \text{with } L_j \text{ and } L_j^\infty \text{ are independent of } \delta.$$

Proof. We want now to apply theorem 2.3. For any $f$ where we recall that $f_\delta(t, x) := G_\delta * (f(t, \cdot))(x)$. Using the property of $G_\delta$, we notice that

$$\|\nabla f_\delta(t, \cdot)\|_{L^\alpha(\mathbb{R}^2)} \leq C\|\nabla f(t, \cdot)\|_{L^\alpha(\mathbb{R}^2)}, \quad \|\nabla_1 f_\delta(1, \cdot)\|_{L^\alpha(\mathbb{R}^2)} \leq C\|\nabla_1 f(1, t, \cdot)\|_{L^\alpha(\mathbb{R}^2)}$$

and

$$\|f_\delta(t, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) + \|f_\delta(1, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) \leq C\left[\|f(t, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2) + \|f(1, t, \cdot)\|_{C^{0, \alpha-1}(0, T] \times \mathbb{R}^2)}\right].$$

Therefore the map $(\overline{V}_\delta, \lambda, \overline{\sigma}_\delta, \beta)$ satisfies Growth assumption of Assumption 2.1 with $(m, M)$ independent of $\delta$. Notice that $m$ and $M$ can depend of $C$. For applying Theorem 2.3, we need to verify Continuity assumption of Assumption 2.1. To do so, it is enough to notice that if $(f^n)_{1 \leq n \leq \infty}$ is a sequence s.t. $\lim_{n \to \infty} \|f^n - f\|_{L^1(0, T] \times \mathbb{R}^2) = 0$, one has

$$\lim_{n \to \infty} \int_0^T \|G_\delta * (f^n(t, \cdot)) - G_\delta * (f^n(t, \cdot))\|_{L^1(\mathbb{R}^2)} dt = 0. \quad (3.15)$$

We can conclude that Continuity assumption of Assumption 2.1 is verified. We can apply Theorem 2.3.

As $L_j$ and $L_j^\infty$ are independent of $\delta$, the sequence $(p^\delta)_{\delta > 0} \subset C([0, T] \times \mathbb{R}^2; \mathbb{R}_+)$ is relatively compact in $C([0, T] \times \mathbb{R}^2; \mathbb{R}_+)$. By using the property of $G_\delta$ and similar techniques as in the proof of Proposition 3.22, we can show that:

Proposition 3.26. The sequence $(p^\delta)_{\delta > 0} \subset C([0, T] \times \mathbb{R}^2; \mathbb{R}_+)$ is relatively compact and the limit $p$ of any convergent sub-sequence satisfied $p(t, x) \ dx = L(X_t)(dx)$ where $X := (X^1, X^2)$ verifies $p(0, \cdot) = q_0(\cdot)$,

$$dX^1_t = b(p)(t, X_t)dt + \sigma(p)(t, X_t)dW_t \quad \text{and} \quad dX^2_t = \lambda(t, X^2_t))dt + \beta(t, X^2_t)dB_t.$$
Proof. Let $p$ be the limit of a convergent sub-sequence of $(p^\delta)_{\delta > 0}$. We use the same notation for the sequence and the sub-sequence. By using the same technique as in the proof of Proposition 3.22, we can show that $\lim_{\delta \to 0} \| p^\delta - p \|_{L^1(\mathbb{R}^T)} = 0$. Notice that

$$
\sup_{t \in [0,T]} \sup_{x \in [-K,K]^2} |G_\delta \ast (p^\delta(t,\cdot))(x) - G_\delta \ast (p(t,\cdot))(x)| \leq \sup_{(t,x) \in [0,T] \times [-K,K]^2} |p^\delta(t,x) - p(t,x)|
$$

and by techniques from the proof of Proposition 3.22

$$
\lim_{K \to \infty} \lim_{\delta \to 0} \int_{\mathbb{R}^2 \setminus [-K,K]^2} |G_\delta \ast (p^\delta(t,\cdot))(x)| + |G_\delta \ast (p(t,\cdot))(x)| \, dx = 0.
$$

Therefore, $\lim_{\delta \to 0} \| G_\delta \ast (p^\delta(\cdot, \cdot)) - p \|_{L^1([0,T] \times \mathbb{R}^2)} = 0$. By using similar techniques as in proof of Proposition 3.25 and proof of Proposition 3.22, we can let $\delta \to 0$ in the Fokker–Planck equation (in weak sense) satisfied by $p^\delta$, we find that $p$ verified the desired equation. \qed

We now provide the approximation by particle system. Let us mention that

$$
G_\delta \ast (p^\delta(t,\cdot))(x) = \int_{\mathbb{R}^2} G_\delta(x - x') p^\delta(t,x') \, dx' = \int_{\mathbb{R}^2} G_\delta(x - x') \mu^\delta_t(dx')
$$

where $\mu^\delta_t = \mathcal{L}(X^\delta_0)$. For any $\nu := (\nu_t)_{t \in [0,T]} \in \mathcal{P}(\mathbb{R}^2)$, we recall that $G_\delta \ast \nu(t,x) := \int_{\mathbb{R}^2} G_\delta(x - x') \nu(t,x') \, dx'$. For each $\delta > 0$, we know that $G_\delta$ is smooth then for any $\nu := (\nu_t)_{t \in [0,T]}$ and $\nu' := (\nu'_t)_{t \in [0,T]} \in C([0,T]; \mathcal{P}(\mathbb{R}^2))$

$$
|G_\delta \ast \nu_1(t,x_1) - (G_\delta \ast \nu')(t,x'_1)| + |G_\delta \ast \nu(t,x) - G_\delta \ast \nu'(t,x)| \leq K(\delta) \left[ |x - x'| + \sup_{t \in [0,T]} W_1(\nu_t, \nu'_t) \right],
$$

and since $\| G_\delta \ast \nu - G_\delta \ast \nu' \|_{L^1([0,T] \times \mathbb{R}^2)} = \sup_{|\delta| \leq 1} \int_{\mathbb{R}^2} \phi(x)(G_\delta \ast \nu_1(t,x_1) - G_\delta \ast \nu'_1(t,x'_1)) \, dx$, we check that

$$
\| G_\delta \ast \nu - G_\delta \ast \nu' \|_{L^1([0,T] \times \mathbb{R}^2)} \leq K(\delta) \sup_{t \in [0,T]} W_1(\nu_t, \nu'_t)
$$

where $K(\delta)$ is a constant depending on $\delta$ and $T$. Consequently, under Assumption 2.5, for each $T > 0$ and $\delta > 0$, $C([0,T]; \mathcal{P}(\mathbb{R}^2)) \times [0,T] \times \mathbb{R}^2 \ni (\nu, t, x) \mapsto [b, \sigma](G_\delta \ast \nu)(t,x) \in \mathbb{R}^2$ is Lipschitz in $(\nu, x)$ uniformly in $t$. $(W^i, B^i)_{i \geq 1}$ is a sequence of independent random variables s.t. for each $i$, $W^i$ and $B^i$ are two $\mathbb{R}$–valued Brownian motions s.t. $d\langle W^i, B^i \rangle_t = \theta \, dt$. The initial density $\theta_0 \in H^{-1}(\mathbb{R}^2)$ is s.t. $\int_{\mathbb{R}^2} |x|^r \theta_0(dx) < \infty$ for $r > e \geq 1$.

**Proposition 3.27.** For each $\delta > 0$, if we let $(X_0^{N,1}, \ldots, X_0^{N,N})$ be the solution of: $(X_0^{N,1}, \ldots, X_0^{N,N})$ is i.i.d., $\mathcal{L}(X_0^{N,1}(\cdot)) = g_0(\cdot) \, dx$,

$$
dX_0^{N,i} = b(G_\delta \ast \mu^N(t,X_i^{N,i})) \, dt + \sigma(G_\delta \ast \mu^N(t,X_i^{N,i})) \, dW^i_t \quad \text{and} \quad dX_t^{N,i} = \lambda(t,X_t^{N,i}) \, dt + \beta(t,X_t^{N,i}) \, dB^i_t
$$

where $\mu^N := \sum_{i=1}^N \delta_{X_i^{N,i}}$, one has that

$$
\lim_{N \to \infty} \mathcal{L}(\mu^N) = \delta_{\mu^N} \in \mathcal{W}_e.
$$

In addition, for any $k \geq 1$ and any bounded measurable map $\phi : [0,T] \times \mathbb{R}^{2k} \to \mathbb{R}$

$$
\lim_{N \to \infty} \mathbb{E} \left[ \int_0^T \phi(t,X_t^{N,1}, \ldots, X_t^{N,k}) \, dt \right] = \int_0^T \phi(t,x^1, \ldots, x^k) \, p^\delta(t,x^1) \cdots p^\delta(t,x^k) \, dx^1 \cdots dx^k \, dt.
$$

**Proof.** Since, for each $\delta > 0$, the map $[0,T] \times \mathbb{R}^2 \ni (t, (x_1, x_2), \nu) \mapsto (b, \lambda, \sigma, \beta)(G_\delta \ast \nu)(t,x) \in \mathbb{R}^4$ is Lipschitz in $(x, \nu)$ uniformly in $t$. Therefore, for each $N \geq 1$, $(X^{N,1}, \ldots, X^{N,N})$ is uniquely defined. Also, the process $X^\delta$ is uniquely defined in distribution where $X^\delta := X := (X^1, X^2)$ satisfies: $p(0, \cdot) = g_0(\cdot)$, for $t \in [0,T]$

$$
dX_1^t = b(p^\delta(t,X_1^t)) \, dt + \sigma(p^\delta(t,X_1^t)) \, dW_t^1 \quad \text{and} \quad dX_t^2 = \lambda(t,X_t^2) \, dt + \beta(t,X_t^2) \, dB_t^2
$$

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where \( p^\delta(t, x)dx = \mathcal{L}(X^\delta_t)(dx) \). Let \( (Y^{N,1}, \cdots, Y^{N,N}) \) be the solution of: 

\[
Y^N_0 = X^i_0,
\]

\[
dY^{N,i}_t = \mathfrak{b}_i(p^\delta)(t, Y^{N,i}_t)dt + \sigma_i(p^\delta)(t, Y^{N,i}_t)W^i_t \quad \text{and} \quad dY^{N,i,2}_t = \lambda(t, Y^{N,i,2}_t)dt + \beta(t, Y^{N,i,2}_t)dB^i_t.
\]

Notice that the sequence \( (Y^{N,i})_{1 \leq i \leq N} \) is i.i.d. with \( \mathcal{L}(Y^{N,i}) = p^\delta(t, x)dx \). Since \( \int_{\mathbb{R}^2} |x|^r \varrho_0(dx) < \infty \) for \( r > e \), we can apply [8, Proposition 4.15] and find: for each \( i \geq 1 \),

\[
\lim_{N \to \infty} \mathbb{P}\left[ \sup_{t \in [0,T]} |Y^{N,i}_t - X^{N,i}_t| \right] = 0.
\]

We can conclude that \( \lim_{N \to \infty} \mathcal{L}(\mu^N) = \delta_{\mu^e} \) in \( \mathcal{W}_e \) and for each \( k \geq 1 \), \( \lim_{N \to \infty} \mathcal{L}(X^{N,1}, \cdots, X^{N,k}) = \mathcal{L}(X^\delta) \otimes \cdots \otimes \mathcal{L}(X^\delta) \) in \( \mathcal{W}_e \). Let us show the last result. Let \( f^N(t, x^1, \cdots, x^k) \) be the density of \( \mathcal{L}(X^{N,1}_t, \cdots, X^{N,k}_t) \) i.e. \( \mathcal{L}(X^{N,1}_t, \cdots, X^{N,k}_t) = f^N(t, x^1, \cdots, x^k)dx^1 \cdots dx^k \). As \( m_2 \leq a \) and \( |(a, b)| \leq M \), by Proposition 3.4 (see also ??), we have

\[
\sup_{N \geq 1} \int_{[0,T] \times \mathbb{R}^{2k}} |f^N(t, x^1, \cdots, x^k)|^{2k+1} dx^1 \cdots dx^k dt < \infty.
\]

The sequence \( \{f^N\}_{N \geq 1} \) is relatively compact for the weak * topology i.e. there is \( f \) and a sub-sequence \( \{N_i\}_{i \geq 1} \) s.t. for any \( \varphi : [0,T] \times \mathbb{R}^{2k} \to \mathbb{R} \) verifying \( f^{T_0}_0 |\varphi(t, x^1, \cdots, x^k)|^{2k+1} dx^1 \cdots dx^k dt < \infty \), we have

\[
\lim_{i \to \infty} \int_{[0,T] \times \mathbb{R}^{2k}} \varphi(t, x^1, \cdots, x^k) f^{N_i}(t, x^1, \cdots, x^k) dx^1 \cdots dx^k dt = \int_{[0,T] \times \mathbb{R}^{2k}} \varphi(t, x^1, \cdots, x^k) f(t, x^1, \cdots, x^k) dx^1 \cdots dx^k dt.
\]

It is easy to check that any limit point \( f \) satisfies \( \mathcal{L}(X^\delta) \otimes \cdots \otimes \mathcal{L}(X^\delta) = f(t, x^1, \cdots, x^k)dx^1 \cdots dx^k \). Therefore the entire sequence \( \{f^N\}_{N \geq 1} \) converges for the weak * topology towards \( f \). Notice that

\[
\lim_{K \to \infty} \sup_{N \geq 1} \int_{[0,T] \times \mathbb{R}^{2k}} 1_{\{|x^1| + \cdots + |x^k| \geq K\}} f^N(t, x^1, \cdots, x^k) dx^1 \cdots dx^k dt \leq \lim_{K \to \infty} \frac{T \sup_{N \geq 1} \sum_{i=1}^k \mathbb{E}[\sup_{t \in [0,T]} |X^{N,i}_t|]}{K} = 0.
\]

This is enough to take \( \varphi \) as a measurable bounded map in the previous convergence. \( \square \)

**Proof of Proposition 2.6** The proof of Proposition 2.6 is just a combination of Proposition 3.26 and Proposition 3.27.

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A Technical results

We set $r \in (1, \infty)$, $T > 0$ and $d \in \mathbb{N}^*$. Let $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a map, we introduce the linear operator

$$Lu : H^{r,1}([0, T] \times \mathbb{R}^d) \to \mathbb{R}$$

in the following way: for any $\varphi \in H^{r,1}([0, T] \times \mathbb{R}^d)$,

$$Lu(\varphi) := \int_{[0, T] \times \mathbb{R}^d} \partial_t u(t, x)\varphi(t, x) - a^{ij}_0(t) \partial_{x_i} u(t, x) \partial_{x_j} \varphi(t, x) \, dx \, dt,$$

where $a_0 := (a_0^{ij})_{1 \leq i, j \leq d} : [0, T] \to \mathbb{S}^d$ is a Borel map satisfying $0 < mI_d \leq a \leq M I_d$. The next result is essentially a reformulation of [17, Theorem 2.4] (see also [5, Theorem 1.2.1] for the dual norm indicated in the proposition below).

**Proposition A.1.** Let $f := (f^i)_{1 \leq i \leq d} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be Borel maps. The map $u$ is s.t. $u(0, \cdot) = 0$ and for any $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$,

$$Lu(\varphi) = -\int_{[0, T] \times \mathbb{R}^d} f^i(t, x) \partial_{x_i} \varphi(t, x) + g(t, x) \varphi(t, x) \, dx \, dt.$$ 

Then, there exists a positive constant $N$ depending only on $T$, $d$, $r$ and $m$ such that

$$\|u\|_{H^{r,1}([0, T] \times \mathbb{R}^d)} \leq N \left( \|f\|_{L^r([0, T] \times \mathbb{R}^d)} + \|g\|_{L^r([0, T] \times \mathbb{R}^d)} \right)^{1/r} = N \|Lu\|_{H^{-1}([0, T] \times \mathbb{R}^d)}.$$

Now, let $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a map s.t. there exists $G > 0$ satisfying

$$\int_{\mathbb{R}^d} |u(t, x)| \, dx \leq G, \quad \text{for all a.e. } t \in [0, T].$$

**Proposition A.2.** If $1 \leq r < d$, one has

$$\|u\|_{L^\gamma([0, T] \times \mathbb{R}^d)} \leq C(d, r) G^{r/d} \|\nabla u\|_{L^r([0, T] \times \mathbb{R}^d)}$$

where $\gamma := \frac{r d + 1}{d - r}$ and $C(d, r)$ is a positive constant appearing in the Sobolev embedding Theorem depending only on $r$ and $d$.

**Proof.** As $1 \leq r < d$, by Gagliardo–Nirenberg–Sobolev inequality (Sobolev embedding Theorem), there exists $C(d, r) > 0$ s.t. for each $t \in [0, T]$,

$$\|u(t, \cdot)\|_{L^{r^*}(\mathbb{R}^d)} \leq C(d, r) \|\nabla u(t, \cdot)\|_{L^r(\mathbb{R}^d)} \quad \text{where } r^* := \frac{dr}{d - r}. \]

Let $a > 1$,

$$\int_{\mathbb{R}^d} |u(t, x)|^a \, dx = \int_{\mathbb{R}^d} |u(t, x)|^{(\ell - 1/a)a} |u(t, x)|^{1/a} \, dx \leq G^{1/a} \left[ \int_{\mathbb{R}^d} |u(t, x)|^{(\ell - 1/a)a'} \, dx \right]^{1/a'} = G^{1/a} \|u(t, \cdot)\|_{L^{(\ell - 1/a)a'}(\mathbb{R}^d)}.$$

We want $(\ell - 1/a)a' = \frac{dr}{d - r}$ and $r = \gamma - 1/a$. This leads to $a = d/r$ and $\ell = r d + 1/a$. Therefore

$$\|u\|_{L^\ell([0, T] \times \mathbb{R}^d)} \leq \int_0^T \|u(t, \cdot)\|_{L^{r^*}(\mathbb{R}^d)} \, dt \leq C(d, r) G^{r/d} \|\nabla u\|_{L^r([0, T] \times \mathbb{R}^d)}.$$

**Proposition A.3.** If $d \leq r$, for any $r < \ell \leq r + 1$, one has

$$\int_0^T \|u(t, \cdot)\|_{C^{0,1-d/r}(\mathbb{R}^d)} \, dt \leq C(d, r) \|u\|_{H^{r,1}([0, T] \times \mathbb{R}^d)}$$

and

$$\|u\|_{L^r([0, T] \times \mathbb{R}^d)} \leq \|u\|_{L^{r^s}(\mathbb{R}^d)}^{r^s} \|u\|_{L^{s}(\mathbb{R}^d)}^{r/s^s} \|u\|_{H^{r,1}([0, T] \times \mathbb{R}^d)}^{r^s},$$

where $s := \frac{1}{r + 1 - r}$. 

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Proof. Since \( d < r \), by Morrey’s inequality, there exists \( C(d, r) \) s.t. for each \( t \in [0, T] \)
\[
\|u(t, \cdot)\|_{C^{0,1-d/r}(\mathbb{R}^d)} \leq C(d, r)\|u(t, \cdot)\|_{H^{r,1}(\mathbb{R}^d)}.
\]
Let \( 1 < a < \ell \), for any \( s > 1 \)
\[
\|u\|^r_{L^r([0,T] \times \mathbb{R}^d)} = \int_{[0,T] \times \mathbb{R}^d} |u(t,x)|^r |u(t,x)|^a dxdt \leq \int_{[0,T] \times \mathbb{R}^d} |u(t,x)|^r |u(t,\cdot)|^a dx dt
\]
\[
\leq \left[ \int_0^T \int_{\mathbb{R}^d} |u(t,x)|^r dx \right]^{1/s} \left[ \int_0^T |u(t,\cdot)|^{a\ell^s} dt \right]^{1/s'}
\]
\[
\leq G^{1/s'} \left[ \int_0^T \int_{\mathbb{R}^d} |u(t,x)|^{r(\ell - a - 1)} |u(t,x)| dx dt \right]^{1/s} \left[ \int_0^T \|u(t,\cdot)\|^{a\ell^s} dt \right]^{1/s'},
\]
where we used Jensen inequality for the last inequality. We want \( s(\ell - a - 1) + 1 = r \) and \( \beta \frac{s}{s-1} = r \). Then \( s = \frac{r}{r+1-\ell} \).
Since we must have \( s > 1 \), this means \( r < \ell \leq r + 1 \). We obtain
\[
\|u\|^r_{L^r([0,T] \times \mathbb{R}^d)} \leq G^{1/s'} \|u\|^r_{L^r([0,T] \times \mathbb{R}^d)} C(d, r)^{r/s'} \|u\|^r_{H^{r,1}(\mathbb{R}^d)}.
\]

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space supporting an \( \mathbb{R} \)-valued \( \mathbb{F} \)-Brownian motion \( B \). We consider bounded maps \([\lambda, \beta] : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \) s.t. \( \beta \) is Lipschitz in \( \mathbb{R} \) and \( \beta \beta^T \geq m \) with \( m > 0 \). Let \((Z_t)_{t \geq 0}\) be the \( \mathbb{F} \)-adapted continuous process satisfying
\[
dZ_t = \lambda(t, Z_t)dt + \beta(t, Z_t)dB_t.
\]
We denote by \( q \) the density of \( \mathcal{L}(Z_t) \) i.e. \( \mathcal{L}(Z_t)(dz) = q(t, z)dz \).

**Proposition A.4.** For any \((r, \delta) \in (1, \infty) \times (0, 1)\), if there is \( \alpha > 0 \) s.t.
\[
\int_\mathbb{R} \exp^{a|z|^2} q(0, z_0)^r dz_0 < \infty
\]
then
\[
\int_\mathbb{R} \left[ \int_0^T |q(t, z)|^r dt \right]^{\delta} dz < \infty, \text{ for each } T > 0.
\]

**Proof.** Let us first assume that \( \lambda = 0 \) and \( \beta = 1 \). In that case, \( q \) is the density of the normal distribution, more specifically
\[
q(t, z) = \frac{1}{2\pi t} \int_\mathbb{R} \exp^{-\frac{1}{2t}|z-z_0|^2} q(0, z_0)dz_0.
\]
We can verify that
\[
\exp^{-\theta|z-z_0|^2} \leq \exp^{-\theta|z-z_0|^2} \text{ and } \exp^{-\theta|z-z_0|^2} \leq \exp^{-\frac{2\theta}{1-\theta}|z|^2} \exp^{-\frac{\theta}{1-\theta}|z-z_0|^2} \text{ for any } \theta \in (0, 1).
\]
Therefore for any \( \theta \in (0, 1) \),
\[
\int_0^T \left| q(t, z) \right|^r dt \leq \int_0^T \left[ \frac{1}{\sqrt{2\pi t}} \int_\mathbb{R} \exp^{-\frac{1}{2t}|z-z_0|^2} q(0, z_0)dz_0 \right]^r dt \leq \int_0^T \frac{1}{\sqrt{2\pi t}} \int_\mathbb{R} \exp^{-\frac{1}{2t}|z-z_0|^2} q(0, z_0)dz_0 dt \int_\mathbb{R} \exp^{-\frac{1}{2t}|z|^2} \exp^{-\frac{1}{2t}|z-z_0|^2} dz_0 dt
\]
\[
\leq \int_0^T \frac{1}{\sqrt{2\pi t}} \int_\mathbb{R} \exp^{-\frac{1}{2t}|z|^2} \exp^{-\frac{1}{2t}|z|^2} q(0, z_0)dz_0 dt = \int_0^T \frac{1}{\sqrt{2\pi t}} dt \int_\mathbb{R} \exp^{-\frac{1}{2t}|z|^2} q(0, z_0)dz_0 \exp^{-\frac{1}{2t}|z|^2}
\]
Then
\[
\int_\mathbb{R} \left[ \int_0^T |q(t, z)|^r dt \right]^{\delta} dz \leq \left[ \int_0^T \frac{1}{\sqrt{2\pi t}} dt \right]^{\delta} \left[ \int_\mathbb{R} \exp^{-\frac{1}{2t}|z|^2} q(0, z_0)dz_0 \right]^{\delta} \int_\mathbb{R} \exp^{-\frac{1}{2t}|z|^2} dz.
\]
By taking \( \theta \) s.t. \( \frac{\theta}{2t} < \alpha \) with \((1 - \theta) > 0\), we can deduce the result. For the general case, by [18] (see [3] for the absence of Dini continuity over the drift), the density \( q \) has Gaussian upper bound. By using the same argument, we can conclude.