su(2) and su(1,1) displaced number states and their nonclassical properties

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Abstract

We study su(2) and su(1,1) displaced number states. Those states are eigenstates of density-dependent interaction systems of quantized radiation field with classical current. Those states are intermediate states interpolating between number and displaced number states. Their photon number distribution, statistical and squeezing properties are studied in detail. It is show that these states exhibit strong nonclassical properties.

1 Introduction

Coherent states and their various generalizations play very important roles in many fields of physics [1]. They were first discovered in 1926 by Schrödinger, as an example of non-spreading wave packet [2]. In 1950’s, Senitzky, Plebanski, Husimi and Epstein found other wave packets [3], which keep their shape and follow the classical motion, before the modern work of coherent states by Glauber and Sudarshan in 1963 [4]. These wave packets are essentially the displaced number states, \( D(\alpha)|n\rangle \), where \( D(\alpha) \equiv \exp(\alpha a^\dagger - \alpha^* a) \) is the displacement operator and \( a^\dagger, a \) are the creation and annihilation operators satisfying \([a, a^\dagger] = 1\).

Since Stoler et al. introduced the binomial states (BS) in 1985 [5], the so-called intermediate states which interpolate between two fundamental quantum optical states have attracted much attention [6]. In a previous paper [7] Sasaki and one of authors of this paper proposed the the intermediate number-squeezed states. One of these intermediate states is the so-called su(2) displaced number states (DNS)

\[
|M, \xi, n\rangle \equiv D_2(M, \xi)|n\rangle; \quad D_2(M, \xi) \equiv \exp(\xi J_+ - \xi^* J_-),
\]  

(1.1)

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in which \(|n\rangle\) \((0 \leq n \leq M)\) is the number state and
\[
J^+_M = a^\dagger \sqrt{M-N}, \quad J^-_M = \sqrt{M-N}a, \quad J^0_M = N - M/2
\] (1.2)
are the Holstein-Primakoff (HP) realization of su(2) algebra. It is obvious that \(|M, \xi, n\rangle\) is a
finite superposition of number states \(|n\rangle\) for \(n = 0, 1, \cdots, M\).

One can also define the non-compact counterpart of the su(2) DNS, the su(1,1) displaced number state
\[
\|M, \zeta, n\rangle \equiv D_{11}(M, \zeta)|n\rangle, \quad D_{11}(M, \zeta) \equiv \exp(\zeta K^+_M - \zeta^* K^-_M),
\] (1.3)
where \(K^+_M, K^-_M\) and \(K^0_M\) are generators of su(1,1) Lie algebra via its HP realization
\[
K^+_M = a^\dagger \sqrt{M+N}, \quad K^-_M = \sqrt{M+N}a, \quad K^0_M = N + M/2,
\] (1.4)
and \(M/2\) (\(M\) is non-negative integer) is the Bargmann index. This is a natural generalization
of the negative binomial states \(\ref{eq:binomial}\) which correspond to \(n = 0\).

In this paper we shall prove that both su(2) and su(1,1) DNS are eigenstates of density-dependent interaction systems of quantized radiation field with classical current, and that both are intermediate states between the number and the displaced number states and reduce to them in two different limits. We also study their expansion in Fock space and their photon number distributions, statistical properties and their squeezing effect. It is of interest that these states which interpolate between number and displaced number states, neither of which exhibit squeezing, are nevertheless squeezing states.

2 Physical interpretation

2.1 Physical systems

In this section we shall see that both su(2) and su(1,1) DNS are eigenstates of some interaction systems of quantum radiation field with a classical current. To see this, we first see that the su(2) and su(1,1) DNS satisfy the following eigenvalue equations (writing \(\xi \equiv re^{i\theta}, \zeta = Re^{i\vartheta}\))
\[
H_2|M, \xi, n\rangle = E_n(2)|M, \xi, n\rangle, \quad H_{11}||M, \zeta, n\rangle = E_n(1, 1)||M, \zeta, n\rangle,
\] (2.1)
where \(H_2\) and \(H_{11}\)
\[
H_2 = \omega N - \frac{\omega}{2} \tan(2r) \left( e^{i\theta} a^\dagger \sqrt{M-N} + e^{-i\theta} \sqrt{M-N}a \right),
\] (2.2)
\[
H_{11} = \omega N - \frac{\omega}{2} \tanh(2R) \left( e^{i\vartheta} a^\dagger \sqrt{M+N} + e^{-i\vartheta} \sqrt{M+N}a \right),
\] (2.3)
are Hermitian operators and can be cased as the Hamiltonians describing interaction between a single mode radiation field with frequency $\omega$ and a classical current and this interaction is density-dependent, and $E_n(2)$ and $E_n(11)$ are eigenvalues (energy)

$$E_n(2) = \frac{M}{2} + \frac{2n - M}{2\cos(2r)}, \quad E_n(11) = \frac{M + 2n}{2\cosh(2R)} - \frac{M}{2}. \quad (2.4)$$

Requirement of non-negative energy gives

For su(2) case: $\cos(2r) > \max(0, 1 - 2n/M)$ or $\cos(2r) < \min(0, 1 - 2n/M)$;

For su(1,1) case: $\cosh(2R) \leq 1 + \frac{2n}{M}$.

### 2.2 Limiting states

We now prove that both su(2) and su(1,1) DNS are intermediate states interpolating between number and displaced number states of radiation field. For su(2) case, it is easy to see that, in the limit $M \to \infty$, $|\xi| \to 0$ with $|\xi|^2 M = \alpha^2$ ($\alpha$ real) fixed and $n$ finite

$$\xi J^+_M \to \alpha e^{i\theta} a^\dagger, \quad \xi^* J^-_M \to \alpha e^{-i\theta} a, \quad D_2(M, \xi) \to D(\alpha e^{i\theta}) \equiv e^{\alpha(e^{i\theta}a^\dagger - e^{-i\theta}a)}, \quad (2.5)$$

and thus the su(2) DNS degenerate to the displaced number state of radiation field

$$|M, \xi, n\rangle \longrightarrow D(\alpha e^{i\theta})|n\rangle = e^{\alpha(e^{i\theta}a^\dagger - e^{-i\theta}a)}|n\rangle, \quad (2.6)$$

while in a different limit $\xi \to m\pi e^{i\theta}$ with $m$ integer, we have $D_2(m\pi e^{i\theta}) \to 1$ and the su(2) DNS degenerates to the number state $|n\rangle$ of radiation field.

Similarly, the su(1,1) DNS tend to the displaced number states of radiation field in the limit $M \to \infty$, $|\zeta| \to 0$ keeping $M|\zeta|^2 = \alpha^2$, and to the number state $|n\rangle$ in the limit $\zeta \to m\pi e^{i\theta}$.

### 3 Photon number distributions

#### 3.1 su(2) case

The su(2) DNS can be expanded as a finite linear combination of Fock states

$$D_2(M, \xi)|n\rangle = \sum_{m=0}^{M} \langle m|D_2(M, \xi)|n\rangle|m\rangle \quad (3.1)$$

and what we need to do is to determine the matrix element $\langle m|D_2(M, \xi)|n\rangle$. Using the disentangling theorem of su(2)

$$D_2(M, \xi) = \exp(\xi J^+_M - \xi^* J^-_M) = \exp(\zeta J^+_M)(1 + |\zeta|^2)^J_M \exp(-\zeta^* J^-_M), \quad (3.2)$$
where $\varsigma = e^{i\theta} \tan r$, the matrix elements can be written as

$$
\langle m | D_2(M, \xi) | n \rangle = \langle m | \exp(\varsigma J_+^M)(1 + |\varsigma|^2)^p \exp(-\varsigma^* J^-_M) | n \rangle. \tag{3.3}
$$

From

$$
(J^-_M)^k | n \rangle = \begin{cases} 
\sqrt{\frac{(M - n + k)! n!}{(M - n)!(n - k)!}} | n - k \rangle, & k \leq n, \\
0, & k > n,
\end{cases} \tag{3.4}
$$

we have

$$
\exp(-\varsigma^* J^-_M) | n \rangle = \sqrt{\frac{n!}{(M - n)!}} \sum_{k=0}^{n} \frac{(-\varsigma^*)^{n-k}}{(n - k)!} \sqrt{\frac{(M - k)!}{k!}} | k \rangle, \tag{3.5}
$$

$$
\langle m | \exp(\varsigma J_+^M) = \sqrt{\frac{m!}{(M - m)!}} \sum_{k=0}^{m} \frac{(\varsigma)^{m-k}}{(m - k)!} \sqrt{\frac{(M - k)!}{k!}} | k \rangle. \tag{3.6}
$$

Inserting Eqs. (3.5, 3.6) into Eq. (3.3) we finally obtain the desired matrix elements

$$
\langle m | D_2(M, \xi) | n \rangle = e^{i\theta(m-n)} D_m(M, n, |\varsigma|), \tag{3.7}
$$

where

$$
D_m(M, n, |\varsigma|) \equiv (-1)^n |\varsigma|^{m+n} \left[ \frac{m! n!}{(M - m)!(M - n)!} \right]^{1/2} F(M, n, m, |\varsigma|^2), \tag{3.8}
$$

and

$$
F(M, n, m, |\varsigma|^2) = \sum_{k=0}^{\min(m,n)} \frac{(M - k)!(-|\varsigma|^2)^k(1 + |\varsigma|^2)^{k-M/2}}{k!(n - k)!(m - k)!} \tag{3.9}
$$

is a Hypergeometric function.

The photon distribution is easily obtained as

$$
P_m(M, n, |\varsigma|^2) = \frac{m! n!}{(M - m)!(M - n)!} |\varsigma|^{2(M+n)} \left[ F(M, n, m, |\varsigma|^2) \right]^2. \tag{3.10}
$$

In the special case $n = 0$, $P_m(M, n, |\varsigma|^2)$ reduces to the binomial distribution

$$
P_m(M, 0, |\varsigma|^2) = \binom{M}{n} p^n (1 - p)^{M-n}, \quad p = \frac{|\varsigma|^2}{1 + |\varsigma|^2}, \tag{3.11}
$$

as we expected.

### 3.2 su(1,1) case

Using the disentangling theorem of su(1,1) (writing $\lambda = e^{i\theta} \tanh R$)

$$
D_{11}(M, \zeta) = \exp(\zeta K_+ - \zeta^* K_-) = \exp(\lambda K_+)(1 - |\lambda|^2)^{K_0} \exp(-\lambda^* K_-), \tag{3.12}
$$
\[
\exp(-\lambda^* K_- |n\rangle = \sqrt{n!(M + n - 1)!} \sum_{k=0}^{n} \frac{(-\lambda^*)^{n-k}}{(n-k)!} \sqrt{k!(M + k - 1)!} |k\rangle,
\]
\[
\langle m | \exp(\lambda K_+ ) = \sqrt{m!(M + m - 1)!} \sum_{k=0}^{m} \frac{\lambda^{m-k}}{(m-k)!} \sqrt{k!(M + k - 1)!} \langle k|,
\]
we obtain
\[
\langle m | D_{11}(M, \xi)|n\rangle = e^{i\theta(m-n)} G_m(M, n, |\lambda|),
\]
where
\[
G_m(M, n, |\lambda|) \equiv (-1)^n |\lambda|^{m+n} \sqrt{m!n!(M + m - 1)!(M + n - 1)!} G(M, n, m, |\lambda|^2),
\]
and
\[
G(M, n, m, |\lambda|^2) = \sum_{k=0}^{\min(m,n)} \frac{(-|\lambda|^2)^{-k} (1 - |\lambda|^2)^{k+M/2}}{k!(M + k - 1)!(n-k)!(m-k)!},
\]
is also a hypergeometric function.

The photon distribution is then obtained as
\[
P_{11}(m) = m!n!(M + m - 1)!(M + n - 1)! |\lambda|^{2(m+n)} G^2(M, n, m, |\lambda|^2).
\]
When \(n = 0\), \(P_{11}(m)\) reduces to the negative binomial distribution as we expected
\[
P_{11}(m) = \binom{M + m - 1}{m} |\lambda|^{2m} (1 - |\lambda|^2)^m.
\]

**4 Photon statistics**

**4.1 su(2) case**

Writing \(\langle M, \xi, n | N | M, \xi, n \rangle = \langle n | D_{2}^{-1}(M, \xi) J^0_M D_2(M, \xi) | n \rangle + M/2\), where \(N \equiv a^\dagger a\), and using
\[
D_{2}^{-1}(M, \xi) J^0_M D_2(M, \xi) = \frac{1}{2}(e^{i\theta} J^+_M + e^{-i\theta} J^-_M) \sin(2r) + J^0_M \cos(2r),
\]
we obtain
\[
\langle M, \xi, n | N | M, \xi, n \rangle = M \sin^2 r + n \cos(2r).
\]
Similarly we have
\[
\langle M, \xi, n | N^2 | M, \xi, n \rangle = M^2 \sin^4 r + 2Mn \sin^2 r \cos(2r) + n^2 \cos^2(2r)
+ \frac{1}{4} \sin^2(2r)[(M - n + 1)n + (M - n)(n + 1)].
\]
So Mandel’s $Q$-index \[Q = \frac{\langle \Delta N^2 \rangle - \langle N \rangle}{\langle N \rangle} = \frac{-A \sin^4 r + (A - M + 2n) \sin^2 r - n}{(M - n) \sin^2 r + n \cos^2 r}, \tag{4.4}\]
where
\[A = 2Mn - 2n^2 + M = 2n(M - n) + M > 0. \tag{4.5}\]
Since the denominator $(M - n) \sin^2 r + n \cos^2 r > 0$, we only need to consider the numerator
\[Q' \equiv -A \sin^4 r + (A - M + 2n) \sin^2 r - n, \tag{4.6}\]
which is clearly a parabola with respect to $\sin^2 r$. The $Q'$ has a maximum at the point
\[\sin^2 r_{\text{max}} = \frac{A - M + 2n}{2A}, \tag{4.7}\]
and the maximum is
\[Q'_{\text{max}} = \frac{n^2(M - n)^2 - n(M - n)}{A}. \tag{4.8}\]

It is obvious that, in the cases $n = 0$, $M$ or in the case $M = 2$, $n = 1$, $Q'_{\text{max}} = 0$. In those cases, the field is sub-Poissonian except at the points that $Q'$ takes its maximum, namely at the points $\sin^2 r = 0, \sin^2 r = 1$ and $\sin^2 r = 1/2$ respectively.

In any other cases, $Q'_{\text{max}} > 0$ and there exist two $r$ values, say $r_-$ and $r_+$,
\[\sin^2 r_{\pm} = \frac{n(M - n + 1) \pm \sqrt{n^2(M - n)^2 - n(M - n)}}{2Mn - 2n^2 + M}, \tag{4.9}\]
such that $Q' = 0$. It is easy to see that $0 < \sin^2 r_- < \sin^2 r_+ < 1$. We illustrate $Q'$ as a function of $\sin^2 r$ in Fig. 1(su(2) case), from which we find that

1. When $0 \leq \sin^2 r < \sin^2 r_-$ or $\sin^2 r_+ < \sin^2 r \leq 1$, the field is sub-Poissonian.
2. When $\sin^2 r_- < \sin^2 r < \sin^2 r_+$, the field is super-Poissonian.
3. When $\sin^2 r = \sin^2 r_-$ or $\sin^2 r = \sin^2 r_+$, the field is Poissonian.

**4.2 su(1,1) case**

From
\[D_{11}^{-1}(M, \zeta)K^{0}_{M}D_{11}(M, \zeta) = \cosh(2R)K^{0}_{M} + \frac{1}{2} \sinh(2R) \left( e^{i\vartheta}K^{+}_{M} + e^{-i\vartheta}K^{-}_{M} \right), \tag{4.10}\]
we can easily obtain the Mandel’s $Q$-index
\[Q = \frac{(4kn + 2n^2 + 2k) \sinh^4 R - 2n(2k + n - 1) \sinh^2 R - n}{2(k + n) \sinh^2 R + n}. \tag{4.11}\]
Since the denominator $2(k + n) \sinh^2 R + n > 0$, we only need to consider the numerator

$$Q' = A \sinh^4 R - (A - 2k - 2n) \sinh^2 R - n,$$

which is also a parabola with respect to $\sinh^2 R$.

The parabola $Q'$ has a minimum $Q'_{\min}$ at the point $\sinh^2 R_{\min}$, where

$$\sinh^2 R_{\min} = -\frac{4kn + 2n(n - 1)}{2(4kn + 2n^2 + 2k)} \geq 0,$$

$$Q'_{\min} = -\left(\frac{(2kn + n^2 - n)^2}{4kn + 2n^2 + 2k} + n\right) \leq 0. \quad (4.13)$$

Since $Q' = -n \leq 0$ for $\sinh^2 R = 0$, so the parabola has only one point intersecting with the $\sinh^2 R$-axis. We denote this point by $\sinh^2 R_+$ and it is given by

$$\sinh^2 R_+ = -\frac{(4kn + 2n^2 - 2n) + \sqrt{(4kn + 2n^2 - 2n)^2 + 4(4kn + 2n^2 + 2n)n}}{2(4kn + 2n^2 + 2k)} \geq 0. \quad (4.14)$$

We illustrate this parabola in Fig.1(su(1,1) case). We first note that in the case $n = 0$ (negative binomial states), $\sinh^2 R_{\min} = \sinh^2 R_+ = Q'_{\min} = 0$, and $Q' > 0$ which means the field is super-Poissonian except the point $\sinh^2 R = 0$ on which $Q' = 0$ and the field is Poissonian.

When $n \geq 1$, from Fig.1(su(1,1) case), we find that, (1) when $0 \leq \sinh^2 R < \sinh^2 R_+$, the field is sub-Poissonian; (2) when $\sinh^2 R = \sinh^2 R_+$, the field is Poissonian; (3) when $\sinh^2 R_+ < \sinh^2 R$, the field is super-Poissonian.

We find that both su(2) and su(1,1) displaced number states exhibit sub-Poissonian statistics in some ranges of parameters involved.

## 5 Squeezing effect

Define two quadratures $x$ (coordinate) and $p$ (momentum)

$$x = (a^\dagger + a)/\sqrt{2}, \quad p = (a^\dagger - a)/\sqrt{2}. \quad (5.1)$$

Then their variances are

$$(\Delta x^2) \equiv \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2} + \langle N \rangle + \text{Re}\langle a^2 \rangle - 2(\text{Re}\langle a \rangle)^2,$$

$$(\Delta p^2) \equiv \langle p^2 \rangle - \langle p \rangle^2 = \frac{1}{2} + \langle N \rangle - \text{Re}\langle a^2 \rangle - 2(\text{Im}\langle a \rangle)^2. \quad (5.2)$$

If $(\Delta x)^2 < 1/2$ (or $(\Delta p)^2 < 1/2$), we say that the quadrature $x$ (or $p$) is squeezed.
For \( \text{su}(2) \), we have

\[
\langle a \rangle = e^{i\theta} \sum_{m=0}^{M-1} \sqrt{m+1}D_m(M, n, |\varsigma|)D_{m+1}(M, n, |\varsigma|),
\]

\[
\text{Re}\langle a^2 \rangle = \cos 2\theta \sum_{m=0}^{M-2} \sqrt{(m+1)(m+2)}D_m(M, n, |\varsigma|)D_{m+2}(M, n, |\varsigma|),
\]

\[
\langle N \rangle = \sum_{m=0}^{M} mD_m(M, n, |\varsigma|)^2,
\]  \hspace{1cm} (5.3)

and for \( \text{su}(1,1) \),

\[
\langle a \rangle = e^{i\vartheta} \sum_{m=0}^{\infty} \sqrt{m+1}G_m(M, n, |\lambda|)G_{m+1}(M, n, |\lambda|),
\]

\[
\text{Re}\langle a^2 \rangle = \cos 2\vartheta \sum_{m=0}^{\infty} \sqrt{(m+1)(m+2)}G_m(M, n, |\lambda|)G_{m+2}(M, n, |\lambda|),
\]

\[
\langle N \rangle = \sum_{m=0}^{\infty} mG_m(M, n, |\lambda|)^2.
\]  \hspace{1cm} (5.4)

We first note that \((\Delta x)^2\) and \((\Delta p)^2\) are related with each other by the following relation

\[
(\Delta x)^2_\theta = (\Delta p)^2_{\theta \pm \pi/2}, \quad (\Delta x)^2_\vartheta = (\Delta p)^2_{\vartheta \pm \pi/2}.
\]  \hspace{1cm} (5.5)

So hereafter we only consider the quadrature \( x \). Then, it is easy to see that \((\Delta x)^2\) is a \( \pi \)-periodic function of \( \theta \) (or \( \vartheta \) for \( \text{su}(1,1) \) case) and it is symmetric with respect to \( \theta = \pi/2 \) (or \( \vartheta = \pi/2 \)).

Fig. 2 shows how \((\Delta x)^2\) of the state \( D_2(M, \xi)|n\rangle \) depends on parameters \( |\varsigma| \) and \( \theta \), respectively. From these plots we find that

1. When \( \theta = 0 \). In the starting point \( |\varsigma| = 0 \) (corresponding to number state \(|n\rangle\)) \((\Delta x)^2 = \frac{1}{2} + n \) and the quadrature \( x \) is not squeezed. Then, with the increase of \( |\varsigma| \), it becomes squeezed drastically until the maximum of squeezing (minimum of \((\Delta x)^2\)) is reached. By further increasing \( |\varsigma| \), the squeezing becomes weaker and weaker until it disappears for a large enough \( |\varsigma| \). The squeezing range depends on \( n \): the larger \( n \), the narrower the squeezing range. For large enough \( n \), there is no squeezing.

2. Dependence on \( \theta \). Since \((\Delta x)^2\) is symmetric with respect to \( \theta_m = \pi/2 \), so we only plot \( 0 \leq \theta \leq \pi/2 \) part. We see that, with the decrease of \( \theta \) form 0, the squeezing becomes weaker and weaker until it disappears for large enough \( \theta \).

Fig. 3 shows how \((\Delta x)^2\) of the state \( D_{11}(M, \zeta)|n\rangle \) depends on parameters \( |\lambda| \) and \( \vartheta \), respectively. From these plots we find that

1. When \( \vartheta = \pi/2 \). In the starting point \( |\lambda| = 0 \) (corresponding to number state \(|n\rangle\)) \((\Delta x)^2 = \frac{1}{2} + n \) and the quadrature \( x \) is not squeezed, as in the \( \text{su}(2) \) case. Then, with the
increase of $|\lambda|$, it becomes squeezed. The larger $|\lambda|$, the stronger the squeezing. The squeezing range depends on $n$: the larger $n$, the narrower the squeezing range.

2. Dependence on $\vartheta$. We only plot $0 \leq \vartheta \leq \pi/2$ part. We see that, with the decrease of $\vartheta$ form $\pi/2$, the squeezing becomes weaker and weaker drastically and finally disappears.

So we find that both su(2) and su(1,1) displaced number states exhibit squeezing effect in some ranges of parameters involved.

6 Conclusion

In this paper we have systematically investigated the su(2) and su(1,1) DNS and their various properties. These states are eigenstates of some Hamiltonians describing density-dependent interaction between single mode radiation field and classical current. As intermediate states both su(2) and su(1,1) DNS degenerate to the number and displaced number states in two different limits. We obtain their explicit expansion in the Fock space and photon distributions. We analytically studied their statistical properties and find that these states are sub-Poissonian states in some ranges of parameters involved. It is of interest that these states exhibit strong squeezing effects, although their limiting states, number and displaced number states, do not.

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FIG. 1. Illustration of the $Q'$. 

FIG. 2. Squeezing effect of the state $D_0(M, \zeta)|n\rangle$: variance $(\Delta x)^2$ as a function of $|\zeta|$ for $M = 50$ and $\theta = 0$; and as a function of $\theta$ for $M = 50$ and $|\zeta| = 2.1$. 

FIG. 3. Squeezing effect of the state $D_{11}(M, \zeta)|n\rangle$: variance $(\Delta x)^2$ as a function of $|\lambda|$ for $M = 2$ and $\theta = \pi/2$, and as a function of $\theta$ for $M = 50$ and $|\lambda| = 0.9$. 

1