Systems with Higher-Order Shape Invariance: Spectral and Algebraic Properties

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Abstract

We study a complex intertwining relation of second order for Schrödinger operators and construct third order symmetry operators for them. A modification of this approach leads to a higher order shape invariance. We analyze with particular attention irreducible second order Darboux transformations which together with the first order act as building blocks. For the third order shape-invariance irreducible Darboux transformations entail only one sequence of equidistant levels while for the reducible case the structure consists of up to three infinite sequences of equidistant levels and, in some cases, singlets or doublets of isolated levels.

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1. Introduction

During the last two decades Supersymmetric Quantum Mechanics became an important and popular tool in establishing connections between a variety of isospectral quantum systems [1]. It gives new insight into the problem of spectral equivalence of Hamiltonians, which historically was developed as Factorization Method of Schrödinger in Quantum Mechanics [2] and as Darboux-Crum transformations in Mathematical Physics [3].

In standard 1-dim SUSY QM [4] two almost isospectral Hamiltonians $H_1$ and $H_2$ are combined into a single (matrix) Schrödinger operator in a Hilbert superspace. The two SUSY partners of a Superhamiltonian are intertwined by supercharges $Q^\pm$

$$H_1 Q^+ = Q^+ H_2; \quad Q^- H_1 = H_2 Q^-; \quad Q^- \equiv (Q^+)^\dagger$$

which indeed generate Darboux transformations. The superhamiltonian and the two supercharges form the so called SUSY algebra. Such approach appeared to be very useful in studying a variety of different kinds of quantum systems. A very fruitful approach is based on generalizations of one specific part of the SUSY algebra, namely the invariance of the Superhamiltonian in respect to supertransformations. For the SUSY partner Hamiltonians this invariance is expressed by intertwining relations.

Below we list some specific classes of problems which were investigated in this way: 1) the class of shape-invariant potentials [5] with the algebraic construction of the spectrum; 2) supercharges $Q^\pm$ of higher orders (HSUSY) in derivatives [6]–[9] including irreducible 2-nd order transformations [7] and polynomial deformation of SUSY algebra; 3) intertwining of Hamiltonians $H_{1,2}$ with complex potentials by complex supercharges up to second order[10]; 4) construction of integrable 2-dim quantum and classical systems and corresponding symmetry operators [11]; 5) investigation of systems with matrix potentials [12].

In this paper we investigate some further generalizations of 1-dim SUSY intertwining relations starting from two hermitian Hamiltonians intertwined by complex supercharges of second order in derivatives (Section 2). A system of intertwining relations arises so that the same Hamiltonians are related at the same time by first and second order Darboux transformations. The compatibility of these constraints require the potentials to satisfy a
stationary KdV equation \[13\]. This formulation automatically allows (by elimination of one of the partner Hamiltonians) for a discussion of symmetry operators \(R_i\) of third order for each partner separately \([H_i, R_i] = 0\). These symmetry operators which commute with the Hamiltonians were studied before in another approach \[14\], \[15\], and the corresponding potentials were found.

In Section 3 a further generalization of the previous relations for the symmetry operators is obtained by the ladder equation \([H, a] = 2\lambda a\). This approach reflects in part results obtained in \[16\] in their construction of dressing chains in connection with integrable bi-hamiltonian systems and in \[17\] in the context of numerical illustrations of (cyclic) shape-invariance preserving standard shape invariance in each step. However our approach also leads to new findings concerning explicitly with the role of irreducible second order Darboux transformations \[7\]. We will focus attention on the non-trivial dependence of the spectrum on the relations between particular parameters of the system, illustrating in detail the particular cases of second and third orders. We find for the third order different type of spectra involving up to three sequences of equispaced levels while the potentials are not strictly harmonic. Conditions for which one of these sequences is shrinked to one or two levels are investigated via the study of zero modes of the generalized creation operator. We stress that these zero modes are allowed only because the creation operators are here of higher order.

In the Conclusion (Section 4) we emphasize the original aspects of our investigation in contradistinction with results in the literature.

2. Complex supercharges, symmetry operators of third order and KdV equation

In this Section we introduce supercharges with complex superpotentials intertwining two hermitian partner Hamiltonians \([10]\), where:

\[
H_i = -\partial^2 + V_i(x); \quad \partial \equiv \frac{d}{dx}. \tag{2}
\]
Since complex supercharges of first order do not lead to a genuine extension we start from:

\[ Q^+ = M^+ + i\tilde{q}^+; \quad (3) \]
\[ M^+ \equiv \partial^2 - 2f(x)\partial + b(x); \quad (4) \]
\[ \tilde{q}^+ \equiv -2\tilde{f}(x)\partial + \tilde{b}(x). \quad (5) \]

As a consequence of (1) we obtain that \( \tilde{f}(x) = \text{Const} \) and one can rescale the first order \( \tilde{q}^+ \equiv -2\tilde{f}q^+ \) with \( q^+ \) of standard structure:

\[ q^+ = \partial + W(x); \quad W(x) \equiv -\frac{\tilde{b}(x)}{2f}. \quad (6) \]

In terms of the second order \( M^+ \) and the first order \( q^+ \) (1) amounts to a system of two intertwining relations for the same partner Hamiltonians:

\[ H_1 M^+ = M^+ H_2; \quad (7) \]
\[ H_1 q^+ = q^+ H_2. \quad (8) \]

Imposing (7) leads (8) to:

\[ V_{1,2}(x) = \mp 2f'(x) + f^2(x) + \frac{f''(x)}{2f(x)} - \frac{f'^2(x)}{4f^2(x)} - \frac{d}{4f^2(x)} + \alpha; \quad (9) \]
\[ b(x) = -f'(x) + f^2(x) - \frac{f''(x)}{2f(x)} + \frac{f'^2(x)}{4f^2(x)} + \frac{d}{4f^2(x)} \quad (10) \]

with arbitrary real constants \( d \) and \( \alpha \). Let us remind that the sign of \( d \) in (9), (10) is crucial to characterize the so called reducible \((d \leq 0)\) and irreducible \((d > 0)\) second order transformation \( \text{[7]} \).

Similarly, (8) gives the standard

\[ V_{1,2}(x) = \pm W'(x) + W^2(x), \quad (11) \]

where an additive constant is set equal to zero.

Consistency requires that

\[ W(x) = -2f(x) + \gamma; \quad \gamma = \text{const} \quad (12) \]

\footnote{We remind that reducible second order operator \( M^+ \) can be written as a product \((\partial + W_1(x))(\partial + W_2(x))\) where the real superpotentials \( W_1(x), W_2(x) \) satisfy a gluing condition \(-W'_1(x) + W^2_1(x) = +W'_2(x) + W^2_2(x) + \sqrt{-d}\) (see details in \text{[7]}).}
and that \( f(x) \) satisfies a nonlinear second order differential equation which can be integrated to:

\[
f'^2(x) = 4f^4(x) - 8\gamma f^3(x) - 4(\alpha - \gamma^2)f^2(x) + \beta f(x) - d; \quad \beta = \text{const.} \quad (13)
\]

Though (13) can be solved for \( x = x(f) \), we will not make use of this solution.

It is known [11], [12], that for arbitrary intertwining relation for hermitian Hamiltonians one can construct symmetry operators of even order in derivatives which can be more or less trivial in 1-dim scalar case but not for 2-dim or matrix case. Since now we deal with two intertwining relations (7), (8), we can provide non-hermitian symmetry operators \( R_1 = M^+q^- \), \( R_2 = M^-q^+ \) of third order, to which of course one can add an arbitrary constant:

\[
H_1 M^+q^- = M^+q^- H_1; \quad (14)
\]

\[
H_2 M^-q^+ = M^-q^+ H_2. \quad (15)
\]

One finds that the hermitian part of \( R_i \) is trivially proportional to \( H_i \) but its antihermitian part \( A_i \) is more involved:

\[
A_1 \simeq -\partial^3 + (6f^3 - 3f' - 6\gamma f - \alpha + \gamma^2) \cdot \partial + (-12f^3 + 18\gamma f^2 + 6(\alpha - \gamma^2)f + 6ff' - 3\gamma f') \quad (16)
\]

with similar expression for \( A_2 \).

A systematic investigation of symmetry operators of higher order in derivatives was performed for 1-dim Schrödinger operator in [14], following the basic strategy of writing an operator polynomial in derivatives and imposing that it commutes with the Hamiltonian. By this procedure one obtains that third order symmetry operators exist if the potential satisfies the stationary version of KdV equation [13]:

\[
V''(x) = 3V^2(x) - c_1 V(x) + c_0 \quad (17)
\]

where \( c_0, c_1 \) are real constants. One can check that symmetry operator (16) with \( f(x) \) satisfying (13) can be rewritten as

\[
A_1 \simeq \partial^3 + \left( -\frac{3}{2}V_1(x) + \frac{c_1}{4} \right) \cdot \partial - \frac{3}{4}V_1'(x) \quad (18)
\]
which indeed coincides with the expression obtained in \([14]\) for potentials \(V_1(x)\) for which \((17)\) holds.

In general, the solution of \((17)\) can be expressed in terms of periodic Jacobi elliptic function:

\[
V(x) = \kappa_1 \text{sn}^2(\kappa_2 x; k) + \kappa_3, \tag{19}
\]

where the constants \(\kappa_i\) are related to the constants \(c_i\) of \((17)\). We focus attention now to the interplay between existence of a symmetry operator and the "non-degeneracy" of normalizable solutions in 1-dim Quantum Mechanics. For this purpose we restrict ourselves to the period \(k = 1\), which corresponds to \(\text{sn}(\kappa_2 x; k = 1) = \text{th}(\kappa_2 x)\). It is straightforward to check explicitly that the only bound state of the associated potential \(V_1(x)\) is an eigenstate of \(A_1\). More generally, as suggested by a fundamental theorem in Mechanics, one can convince oneself that the operator \(A_1^2\) is a third order polynomial with constant coefficients in \(H_1\) for \(V_1(x)\) satisfying \((17)\).

As a final comment let us mention how one can extend the third order results to higher orders, for example to the fifth order symmetry operators. Imposing directly the commutation between the fifth order symmetry operator and the Hamiltonian, one obtains as a consistency relation that \(V(x)\) has to satisfy a higher KdV equation and that this symmetry operator can be written as a product \(HA_1\) with \(A_1\) given in \((18)\). One can notice that if \(V(x)\) satisfies \((17)\) it will also automatically satisfy the higher KdV (but not vice versa).

3. **Intertwining with shift and higher order shape-invariance**

For illustration of the construction let us start by modifying equations \((7), (8)\):

\[
H_1 M^+ = M^+ H_2; \tag{20}
\]

\[
H_1 q^+ = q^+ (H_2 + 2\lambda), \tag{21}
\]

introducing thereby a shift by the positive constant \(2\lambda\). Eliminating one partner Hamiltonian (for example, \(H_2\)) and denoting product operators \(a^+ \equiv q^+ M^-\) and \(a^- \equiv M^+ q^-\) we obtain:

\[
H_1 a^+ = a^+ (H_1 + 2\lambda). \tag{22}
\]
We will call such Hamiltonian as "third order shape-invariant". One can also work the other way around and start from (22) with an operator $a^+$ of third order represented as a product $q^+ M^-$ and $H_1$ represented by $H_1 = q^+ q^-$ as is familiar in SUSY Quantum Mechanics. After one can introduce an auxiliary Hamiltonian $H_2 + 2\lambda = q^- q^+$ to obtain (20).

Eq. (22) is a ladder equation where $a^+$ plays the role of generalized creation operator which provides an excitation energy of $2\lambda$. In order to study the spectrum it is crucial to study zero modes of $a^-$ and $a^+$. The former describe the lowest lying levels of the system, and one has to apply recursively the operator $a^+$ to them in order to generate the excitation spectrum. The energies of the zero modes can be obtained by imposing the vanishing of their norm, which involves the average of the operator product $a^+ a^-$. This product can be easily evaluated algebraically because

\[ a^+ a^- = q^+ M^- M^+ q^- = q^+ ((H_2 - \alpha)^2 + d) q^- = H_1 ((H_1 - \alpha - 2\lambda)^2 + d). \]  

Contrary to the standard harmonic oscillator, one has the possibility to have also zero modes of the operator $a^+$ which correspond to a possible truncation of the tower of excited levels. Arguing as before, the relevant operator product is in this case

\[ a^- a^+ = (H_1 + 2\lambda) ((H_1 - \alpha)^2 + d) \]  

to be averaged over the excited states.

This construction can be generalized to higher orders taking into account that on general grounds $[7]$ $n$–th order operators $a^\pm$ can always be constructed in terms of products of $q$ and $M$. In the general higher order case, (22) gives a connection between $H_1$ and $H_1$ plus shift, which is the simplest realization of the notion of shape-invariance $[5]$. Though some properties of the spectrum, like zero modes of $a^\pm$, will depend on the explicit product structure of $a^\pm$, the excitation spectrum can be mainly obtained algebraically. So it is possible to study the consequences of (22) without taking into account the specific definition of the operators $a^\pm$.

In order to provide additional arguments to the interpretation of Eq. (22) as a generalization of shape invariance, let us start to notice that if $a^\pm$ would be of first order, $H_1$
would be identified with harmonic oscillator. If $a^\pm$ is of second order, we reobtain the singular harmonic oscillator potential which is also shape-invariant (see next Subsection). We stress that if $a^\pm$ would be an operator of second order, it is known \cite{7} that it is not always possible to write it as a product of two first operators with real superpotentials: such case is referred as irreducible.

### 3.1. Intertwining with shift and second order shape-invariance

In this Subsection to keep notations the same we consider (20), (21) with $M^+$ temporarily of first order. We can thus explore in this Subsection if even a first order intertwining operator $M^+$ can now lead to nontrivial consequences. It is easy to convince oneself that this is indeed so. The superpotentials which solve (20), (21) are superpositions of a term growing as $x$ and a singular term like $1/x$ leading to two singular potentials \cite{18} $V_1(x)$ and $V_2(x)$ as follows:

$$
V_1(x) = \frac{\rho(\rho - 1)}{x^2} + \frac{\lambda^2 x^2}{4}; \quad V_2(x) = \frac{\rho(\rho + 1)}{x^2} + \frac{\lambda^2 x^2}{4} - \lambda.
$$

These potentials are shape-invariant in the standard sense \cite{3} and belong to the class of algebraically solvable models, because $V_2(x; \rho, \lambda) = V_1(x; \rho + 1, \lambda) - \lambda$.

It is instructive to display the algebraic properties of these systems. In terms of the product operators $a^+ \equiv q^+ M^-$ and $a^- \equiv M^+ q^-$ (we repeat that in this Subsection $a^\pm$ are of second order) one obtains an algebra suggestive of a generalization of the standard harmonic oscillator algebra:

$$
[H_1, a^+] = 2\lambda a^+; \quad [a^+, a^-] = -4\lambda H_1 + \text{const}.
$$

Similar results hold for $H_2$.

It is now tempting to consider, as explained before, Eqs.(20) by themselves including the case where the operators $a^\pm$ are irreducible. We know from \cite{6} with $H_2 \equiv H_1 + 2\lambda$

\footnote{On the half line for $\rho = l + 1$ and suitable choice of boundary conditions for wave functions \cite{6} potential $V_1$ can be interpreted as a radial harmonic oscillator in partial wave $l$.}
and setting $\alpha \equiv 0$ in \((3)\) that
\[ a^+ a^- = H_1^2 + d; \quad a^- a^+ = (H_1 + 2\lambda)^2 + d. \tag{27} \]

Solving \((22)\) we obtain from \((9)\) that $f(x) = \frac{1}{2}\lambda x$ and
\[ V_1(x) = \frac{\lambda^2 x^2}{4} - \left(1 + \frac{d}{\lambda^2}\right) \frac{1}{x^2} - \lambda. \tag{28} \]

Evaluating the matrix element of the first of \((27)\), we get the equation for the 0-modes of $a^-$: $E_{1,2}^{(0)} = \pm \sqrt{-d}$. For a positive value of $d$ (irreducible case) there are no solutions and physically this is clear because the potential \((28)\) describes a collapse for $d > 0$. For $d \leq 0$ the singular part of the potential is less pathological and one may have two zero modes for suitable choices of parameters. The eigenfunctions of the zero modes of $a^-$ which are also eigenfunctions of $H_1$ can be found explicitly by replacing $\partial^2$ in $a^-$ by $(V_1(x) - E^{(0)})$. The asymptotic behavior of the corresponding eigenfunctions is compatible with the normalizability.

We expect that for self-adjoint extensions of Hamiltonian $H_1$ both operators $a^+ a^-$ and $a^- a^+$ are nonnegative on physical states. There exist two zero modes of $a^-$ if $\sqrt{-d} < \lambda$. Acting with $a^+$ on these two zero modes, one can construct two sequences of levels with internal spacing given by $2\lambda$. In the limiting case $d = 0$ the two sequences coincide. When $\sqrt{-d} > \lambda$, one obtains only one zero mode with the energy $E^{(0)} = \sqrt{-d}$ and, therefore, only one sequence of equispaced levels. The existence of the second zero mode $E^{(0)} = -\sqrt{-d}$ in this case is incompatible with positivity of the second operator in Eq.\((27)\). For $\sqrt{-d} = \lambda$ this second zero mode remains non-normalizable.

### 3.2. Intertwining with shift and third order shape-invariance.

From now on we will consider the intertwining operator $M^\pm$ to be a reducible or irreducible operator of second order \((3)\) but $q^\pm$ still of first order \((3)\). The solutions of \((20)\) are unchanged in respect to Eqs.\((3), (10)\) of Section 2 but Eq.\((21)\) implies a shift for the potential $V_2(x)$ in \((11)\). Thus the consistency equations for $W(x)$ and $f(x)$ are

\[ \text{This choice of the energy scale means} \quad (3) \quad \text{that the eigenvalues of} \quad H_1 \quad \text{are bounded from below by} \quad -\sqrt{-d}. \]
modified. Eq. (12) becomes:

\[ W(x) \equiv W_3(x) = -2f(x) - \lambda x, \]  

(29)

where an additional integration constant can be ignored because of a shift of \(x\), which fixes the origin of coordinate\(^4\). The potential \(V_1(x)\) can be written from (11) as:

\[ V_1(x) = -2f'(x) + 4f^2(x) + 4\lambda xf(x) + \lambda^2 x^2 - \lambda, \]  

(30)

with \(f(x)\) satisfying:

\[ f'' = \frac{f'^2(x)}{2f(x)} + 6f^3(x) + 8\lambda xf^2(x) + 2(\lambda^2 x^2 - (\lambda + \alpha))f(x) + \frac{d}{2f(x)}. \]  

(31)

The equation (31) can be transformed by the substitution \(f(x) \equiv \frac{1}{2}\sqrt{\lambda}g(y); \, y \equiv \sqrt{\lambda}x\) to the Painleve-IV equation \([19]\):

\[ g'' = \frac{g'^2(y)}{2g(y)} + \frac{3}{2}g^3(y) + 4yg^2(y) + 2(y^2 - a)g(y) - \frac{b}{2g(y)}. \]  

(32)

where

\[ a \equiv 1 + \frac{\alpha}{\lambda}; \quad b \equiv -\frac{4d}{\lambda^2}. \]  

(33)

This equation has been studied intensively in the last years \([20]\) and in the following we will be mainly interested in asymptotic properties of its solutions which will determine the asymptotics of potentials \([30]\) and the normalizability of eigenfunctions.

Concerning the algebra, the only modification in respect to (26) in the previous Subsection is given by:

\[ [a^+, a^-] = -2\lambda(3H_1^2 - (4\alpha + 2\lambda)H_1 + \alpha^2 + d) \]  

(34)

and similarly for \(H_2\). When the index will not appear explicitly we intend to refer to the index 1.

We show how one can derive the spectrum from (22) and (34) if normalizable zero modes of the annihilation operator \(a^-\) exist. We stress that this algebraic method is very

\(^4\)In Eq. (12) there was no such fixing, so the limit \(\lambda \to 0\) cannot be taken to obtain the results of Section 2.
powerful now since the explicit form of the potential is known only in terms of Painlevé transcendents.

The equation for zero modes of $a^-$ reads:

$$a^- \Psi_k^{(0)} = M^+ q^- \Psi_k^{(0)} = 0$$

(35)

where $k$ labels the normalizable solutions.

As explained at the beginning of Section 3, Eq.(23), the algebraic equation for eigenvalues $E_k^{(0)}$ is:

$$E_k^{(0)} \cdot [(E_k^{(0)} - \alpha - 2\lambda)^2 + d] = 0,$$

(36)

and has at most three real solutions.

For the irreducible case ($d > 0$) only one zero mode $E_0^{(0)} = 0$ exists. But it is necessary to check indeed the normalizability of corresponding zero mode which can be written (35), (29), (8) in terms of superpotentials:

$$\Psi_0^{(0)}(x) = \exp(\int x W(z)dz) = \exp(\frac{-1}{2}\lambda x^2 - 2 \int x f(z)dz)$$

(37)

where $f(z)$ satisfies (31) and must have a suitable asymptotic behavior in order to make (37) normalizable. It has been found [21] that for $b < 0$ there are two possible asymptotic behavior of $g(y)$ at $\pm \infty : -2y/3$ and $-2y$. It has been demonstrated [21] that one can realize the matching between the same asymptotic value at $+\infty$ and $-\infty$. The asymptotic values which are compatible with normalizability of (37) is $g(y) \sim -2y/3$, or equivalently, $f(x) \sim -\lambda x/3$. We notice that the subleading term in $f(x)$ have oscillations $\sim cos(\lambda x^2/\sqrt{3} + o(x))$ which show up in the potential (30) as a modulation of the harmonic term.

From $[H, a^+] = 2\lambda a^+$ one can deduce that $E^{(n)} = 2\lambda n$. The related potential (30) for the increasing asymptotic behavior of $f(x)$ has asymptotics $\lambda^2 x^2/9$, which differs from conventional $\lambda^2 x^2$. We remark that the latter system does not belong to the families of potentials isospectral to harmonic oscillator [8], [22] which have the same asymptotics but differ from harmonic oscillator for finite $x$.

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5 In this case we notice that since $M^- \Psi_0^{(0)} \neq 0$ also $H_2$ has the ground state with zero eigenvalue.
Evaluating the norm of the excited state \( \Psi^{(n+1)}(x) \equiv a^+ \Psi^{(n)}(x) \), one finds after algebraic manipulations explained in the beginning of Section 3 that for the irreducible case it cannot vanish. So, the spectrum is not bounded from above. Analytic calculation support clearly this result because of the asymptotic oscillator-like growth.

For the reducible case \( M^+ \) with \( d \leq 0 \) is:

\[
M^+ \equiv (\partial + W_1(x))(\partial + W_2(x))
\]  

(38)

with real superpotentials \[ W_{1,2}(x) = -f(x) \pm \frac{f'(x) - \sqrt{-d}}{2f(x)}. \]  

(39)

Eq. (36) has the solution \( E^{(0)}_0 = 0 \) as before and two additional solutions:

\[
E^{(0)}_+ = \alpha + 2\lambda + \sqrt{-d}
\]

(40)

\[
E^{(0)}_- = \alpha + 2\lambda - \sqrt{-d}
\]

(41)

which correspond to eigenvalues of \( H = q^+ q^- \) provided \[ (10), (11) \] are non-negative and that associated eigenfunctions \( \Psi^{(0)}_k(x) \) are normalizable. Acting on the three zero modes \( \Psi^{(0)}_k(x) \) by the creation operator \( a^+ \), one creates the excited states of the system organized in three sequences. According to Eq. (22), within each sequence the levels are \( 2\lambda \)–equidistant. The non-negativity of the norm \( ||a^\pm \Psi(x)||^2 \) for all physical states \( \Psi(x) \) leads to additional necessary conditions for the spectrum\[ 6 \].

The eigenfunctions \( \Psi^{(0)}_k(x) \), annihilated by \( a^- = M^+ q^- = (\partial + W_1(x)) \cdot (\partial + W_2(x)) \cdot (-\partial + W_3(x)) \) with \( W_3(x) \) defined in (23), can be calculated explicitly:

\[
\Psi^{(0)}_0(x) = exp(\int^x W_3(x')dx');
\]

\[
\Psi^{(0)}_+(x) = (W_2(x) - W_3(x))exp(-\int^x W_2(x')dx');
\]

\[
\Psi^{(0)}_-(x) = \left(2\sqrt{-d} + (W_2(x) - W_3(x))(W_1(x) + W_2(x))\right)exp(-\int^x W_1(x')dx').
\]

(42)

Their normalizability, similarly to \( \Psi^{(0)}_0(x) \) for the irreducible case, depends on the choice for the asymptotics of \( f(x) \) and particular values of parameters \( \alpha, \lambda \) and \( d \). In reducible

\[ 6 \]

From the algebra (22) - (24) one can prove that these conditions are sufficient to ensure the non-negativity of the norm of any state created by a polynomial of \( a^\pm \): \( ||P(a^+, a^-)\Psi(x)||^2 \geq 0 \).
case $b > 0$ four possible asymptotics of solutions of Painleve-IV equation can appear: 
\[ g(y) \sim -2y/3, -2y, \text{ and } \pm \sqrt{b}/2y. \]

On the other hand, the operator $a^+$ can also have zero modes which can be found explicitly:
\[
\begin{align*}
\Psi_1(x) &= \exp(\int^x W_1(x')dx'); \\
\Psi_2(x) &= (W_1(x) + W_2(x))\exp(\int^x W_2(x')dx'); \\
\Psi_3(x) &= \left( E_3^{(0)} + (W_1(x) + W_2(x))(W_2(x) - W_3(x)) \right) \exp(-\int^x W_3(x')dx').
\end{align*}
\]

These eigenfunctions are solutions of the Schrödinger equation with eigenvalues:
\[
E_1 = \alpha - \sqrt{-d}; \quad E_2 = \alpha + \sqrt{-d}; \quad E_3 = -2\lambda.
\]

The consistency of the energy spectrum, i.e. the non-negativity of operators $a^+a^-$ and $a^-a^+$, rules out the possibility to have a normalizable zero mode with negative energy $E_3$ for nonsingular superpotentials. Concerning the other zero modes, one can conclude only that the total number of zero modes of both operators $a^-$ and $a^+$ cannot exceed three, which follows straightforwardly from the conflicting asymptotics (42) and (43).

As the spectrum of Hamiltonian is bounded from below and the operator $a^+$ raises its levels, zero modes of $a^+$ represent an obstruction to build an infinite sequence of levels. Only one zero mode of $a^+$ may exist together with two zero modes of $a^-$. In this case the spectrum consists of one infinite sequence of levels and a finite band of levels, both generated by operators $(a^+)^n$.

Let us proceed to a more detailed analysis of different spectrum patterns.

a) Three normalizable zero modes $[12]$ of $a^-$ and, respectively, three equidistant sequences of levels may arise only for $\lambda > \sqrt{-d}$ if $\sqrt{-d} < \alpha < 2\lambda - \sqrt{-d}$ or $\sqrt{-d} - 2\lambda \leq \alpha < -\sqrt{-d}$. The corresponding solutions of Painleve-IV equation must have the leading asymptotics $f(x) \sim -\lambda x/3$ with subleading oscillations $\cos(\lambda x^2/\sqrt{3} + o(x))$.

b) Two normalizable zero modes of $a^-$ (and two sequences of levels) may exist for different solutions of Painleve-IV equation $[1]$. Namely, for $\lambda > \sqrt{-d}$ the solution with

\footnote{We discuss here only the case of equal asymptotics of $f(x)$ at $\pm \infty$. However we are aware of existence of solutions with the different asymptotics, which may lead to an additional variety of potentials with one}
asymptotics \( f(x) \sim -\lambda x/3 \) provides the fall off of \( \Psi_0^{(0)}(x), \Psi_+^{(0)}(x) \) if \( \alpha > 2\lambda - \sqrt{-d} \). For \( \lambda > \sqrt{-d} \) the solution with asymptotics \( f(x) \sim -\sqrt{-d}/2\lambda x \) generates two sequences of levels starting from \( \Psi_0^{(0)}(x), \Psi_-^{(0)}(x) \) if \( 2\lambda n + \sqrt{-d} < \alpha < 2\lambda(n + 1) - \sqrt{-d}; n = 0, 1, 2... \). For \( \lambda < \sqrt{-d} \) two sequences (from \( \Psi_0^{(0)}(x) \) and \( \Psi_+^{(0)}(x) \)) are generated for two possible asymptotics \( f(x) \sim \sqrt{-d}/2\lambda x \) and \( f(x) \sim -\lambda x/3 \) if \( -2\lambda - \sqrt{-d} < \alpha < -\sqrt{-d} \).

c) For \( \alpha \leq -2\lambda - \sqrt{-d} \) and arbitrary positive \( \lambda \) one sequence of equidistant levels will be realized with ground state \( \Psi_0^{(0)}(x) \) and one of three asymptotics: \( f(x) \sim -\lambda x/3, \sim \pm \sqrt{-d}/2\lambda x \). For \( f(x) \sim -\lambda x/3 \) only one sequence of levels will be realized also for \( \lambda = \sqrt{-d}, \alpha = \pm \sqrt{-d} \) (starting from \( \Psi_0^{(0)}(x) \)), for \( \lambda = \sqrt{-d}, \alpha > -\sqrt{-d} \) (starting from \( \Psi_-^{(0)}(x) \)) and for \( \lambda < \sqrt{-d}, \alpha \geq -2\lambda - \sqrt{-d} \) (starting from \( \Psi_+^{(0)}(x) \)). For asymptotic behavior \( f(x) \sim -\sqrt{-d}/2\lambda x \) and \( \lambda > \sqrt{-d} \) the spectrum consists of one sequence of levels if \( |\alpha - 2\lambda n| < \sqrt{-d}, n = 0, 1, 2..., \) (starts from \( \Psi_-^{(0)}(x) \)) or if \( \alpha = \sqrt{-d} \) (starts from \( \Psi_0^{(0)}(x) \)). Last, for \( f(x) \sim \sqrt{-d}/2\lambda x \) one obtain one sequence with ground state \( \Psi_+^{(0)}(x) \) if \( \lambda \leq \sqrt{-d} \) and \( \alpha > -\sqrt{-d} \) or if \( \lambda > \sqrt{-d} \) and \( |\alpha - 2\lambda n| < \sqrt{-d}, n = 0, 1, 2..., \)

For specific values of parameter \( \alpha \) the existence of normalizable zero mode of \( a^+ \) can truncate one of two sequences (with ground state \( \Psi_0^{(0)}(x) \)) in item b).

d) One additional singlet state satisfies the equation

\[
\alpha^+ \Psi_0^{(0)}(x) = a^- \Psi_0^{(0)}(x) = 0.
\]

For \( \lambda > \sqrt{-d} \) (\( \lambda < \sqrt{-d} \)) it occurs when \( \alpha = \mp \sqrt{-d} \) and \( \Psi_0^{(0)}(x) = \Psi_{2,1}(x) \) which entails the equation:

\[
f'(x) = -2f^2(x) - 2\lambda xf(x) \mp \sqrt{-d}
\]

with asymptotic behavior \( f(x) \sim \mp \sqrt{-d}/2\lambda x \). The spacing between the two ground states \( \Psi_0^{(0)}(x) \) and \( \Psi_-^{(0)}(x) \) is: \( \Delta E = 2\lambda + 2\sqrt{-d} \).

or two sequences of energy levels. The admissible range of parameters and zero mode space is derived then by intersection of those ones for the particular asymptotics described below.

8Let us remark that particular value \( \alpha = -2\lambda - \sqrt{-d} \), i.e. \( E_+^{(0)} = 0, E_-^{(0)} < 0 \), just corresponds to the case when Painleve-IV equation \( (32) \) has a class of particular solutions which coincide with solutions of the Riccati equation \( g'(y) = g^2(y) + 2gg(y) + \sqrt{b} \). Substituting this Riccati equation into Eq. (30) one finds that potential is the pure harmonic oscillator: \( V_1(x) = \lambda^2x^2 - \lambda \).

9 Its existence has been mentioned in \( (23) \)
The doublet representation \((\Psi_0^{(0)}(x), a^+\Psi_0^{(0)}(x))\) of the spectrum generating algebra \((22), (34)\) is built on solutions of the equation:

\[
(a^+)^2\Psi_0^{(0)}(x) = a^-\Psi_0^{(0)}(x) = 0.
\]

It may hold when \(\alpha = 2\lambda + \sqrt{-d}\) for arbitrary positive value of \(\lambda\) and when \(\alpha = 2\lambda - \sqrt{-d}\) for \(\lambda > \sqrt{-d}\). It is equivalent to \(a^+\Psi_0^{(0)}(x) = \Psi_{1,2}(x)\), which is satisfied when \(f(x)\) obeys the following equation:

\[
8\lambda f^2(x)(f'(x) + 2f^2(x) + 2\lambda x f(x) - 2\lambda \mp \sqrt{-d}) = (f'(x) + 2f^2(x) + 2\lambda x f(x) - 2\lambda \mp \sqrt{-d}) \cdot \left[(f'(x) + 2f^2(x) + 2\lambda x f(x) - 2\lambda \mp \sqrt{-d}) - 4\lambda(\lambda \pm \sqrt{-d})\right].
\]

One can show that all solutions of this equation fulfill the Painleve-IV equation \((31)\). These solutions have the asymptotics \(f(x) \sim \pm \sqrt{-d}/2\lambda x\) and cannot have any (pole) singularity. The spectrum consists of a doublet \((0, 2\lambda)\) and infinite sequence \(E_n = \pm 2\sqrt{-d} + 2(n + 2)\lambda, \ n = 0, 1, 2,\ldots\)

It may seem to be possible to obtain a higher representation of spectrum generating algebra with three or more levels in the band. However for higher order Darboux transformations it is known \([7]\) that the zero mode of the third order operator \(a^-\) cannot have more than two nodes, \(i.e.\) it cannot be higher in energy than the second excited level of system. From our analysis it follows that states forming the finite representation lie below the first state of the infinite sequence of levels, which is obviously a zero mode of \(a^-\). Therefore this finite representation may include only the ground state and, possibly, the first excitation. For shape-invariance of \(N\)-th order the maximal dimension of finite representation may be \((N - 1)\).

4. Conclusions.

As conclusive remarks let us now emphasize the novel features of our approach. We would like to point out what in our opinion is new in respect to \([16]\). First, we introduced in the dressing chain (in their language) irreducible transformations which correspond
to quantum mechanical systems with only one sequence of levels. Second, we discovered that in suitable circumstances one sequence can be truncated because the higher order creation operators $a^+$ can have zero modes. Third, we have given attention to the asymptotic properties of the Painleve-IV equation (31) in connection with discussion of the normalizability of eigenfunctions. In particular, for the construction of potentials which do not have subleading oscillatory behavior, we revealed the importance of the special asymptotics $g(y) \sim \sqrt{b/2y}$ which can be implemented \[21\] at both $\pm \infty$ or only alternatively at $+\infty$ or at $-\infty$. We have found that a particular class of solutions of a Painleve-IV equation can be obtained as solutions of another, first order, equation (46), which does not admit singularities.

Concerning \[17\], we differ because implementation of higher order shape-invariance in our scheme does not require to preserve standard shape-invariance of first order step by step. The spectra that we can generate appear to be more rich: possibility of having three, two, one sequences; possible truncation of one sequence; non-oscillatory asymptotic behavior of the potential.

In respect to \[9\] our approach gives more importance to the distinction between reducible and irreducible second order transformation and to potentials which generate sequences of levels spaced by $2\lambda$ but do not in general have the asymptotics behavior $\lambda^2 x^2$.

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