A Note on Koldobsky’s Lattice Slicing Inequality

Oded Regev

Abstract

We show that if $K \subseteq \mathbb{R}^d$ is an origin-symmetric convex body, then there exists a vector $y \in \mathbb{Z}^d$ such that

$$|K \cap \mathbb{Z}^d \cap y^\perp|/|K \cap \mathbb{Z}^d| \geq \min(1, c \cdot d^{-1} \cdot \text{vol}(K)^{-1/(d-1)})$$

for some absolute constant $c > 0$, where $y^\perp$ denotes the subspace orthogonal to $y$. This gives a partial answer to a question by Koldobsky.

1 Introduction

The following question was asked by Koldobsky during the 2013 AIM workshop on “Sections of convex bodies.”

Question 1.1. Is it the case that for all $d \geq 1$ there exists an $\alpha = \alpha(d) > 0$ satisfying the following: for all origin-symmetric convex bodies $K \subseteq \mathbb{R}^d$ such that $\text{span}(K \cap \mathbb{Z}^d) = \mathbb{R}^d$ there exists a nonzero $y \in \mathbb{R}^d$ such that

$$|K \cap \mathbb{Z}^d \cap y^\perp|/|K \cap \mathbb{Z}^d| \geq \alpha \cdot \text{vol}_d(K)^{-1/d}$$

In other words, the question asks to find a dimension $d - 1$ subspace that contains at least an $\alpha \cdot \text{vol}_d(K)^{-1/d}$ fraction of the lattice points in $K$. The requirement on the span is in order to avoid degenerate cases of bodies of very small volume that would force $\alpha$ to be small.

Alexander, Henk, and Zvavitch [AHZ15] gave a positive answer to this question by showing that one can take $\alpha = C^{-d}$ for some absolute constant $C$. They also showed that for the special case of unconditional bodies $K$ one can take $\alpha = c/d$ for some absolute constant $c > 0$, and observed that this is tight (as follows by taking $K$ to be the cross-polytope $\text{conv}(\pm e_1, \ldots, \pm e_d)$). It remains an open question whether one can take $\alpha = c/d$ for general bodies. In this note we show that this is the case for bodies whose volume is at most $C^{d^2}$ for any constant $C > 0$ (see Theorem 3.1 for the full statement). We refer to [AHZ15] for further background on Koldobsky’s question and its connection to the slicing problem of Bourgain.

Acknowledgements

I am grateful to Assaf Naor for suggesting that I look at Koldobsky’s question.
2 Orthogonal lattice points

We use the convention that \( c \) is an arbitrary absolute positive constant which might differ from one occurrence to the next. A lattice \( L \subset \mathbb{R}^d \) is defined as the set of all integer linear combinations of \( d \) linearly independent vectors in \( \mathbb{R}^d \). The dual lattice of \( L \) is \( L^* = \{ y \in \mathbb{R}^d : \langle y, x \rangle \in \mathbb{Z}, \forall x \in L \} \). For any \( s > 0 \), we define the function \( \rho_s : \mathbb{R}^n \rightarrow \mathbb{R} \) as \( \rho_s(x) = \exp(-\pi \|x\|^2/s^2) \). For a discrete set \( A \subset \mathbb{R}^n \) we define \( \rho_s(A) = \sum_{x \in A} \rho_s(x) \) and denote by \( D_{A,s} \) the probability distribution assigning mass \( \rho_s(x)/\rho_s(A) \) to any \( x \in A \). Recalling that the Fourier transform of \( \rho_s(\cdot) \) is \( s^d \rho_{1/s}(\cdot) \), the following is an immediate application of the Poisson summation formula.

Lemma 2.1. For any lattice \( L \subset \mathbb{R}^d \) and \( s > 0 \),

\[
\rho_s(L) = (\det L)^{-1} s^d \cdot \rho_{1/s}(L^*) .
\]

In particular,

\[
\rho_s(L) \geq (\det L)^{-1} s^d ,
\]

or equivalently,

\[
\Pr_{y \sim D_{L,s}} [ y = 0 ] \leq (\det L) \cdot s^{-d} .
\]

The following is another easy corollary of the Poisson summation formula (already used in \[Ban93\]), and holds because \( \rho_s \) is a positive definite function, i.e., a function with a non-negative Fourier transform.

Lemma 2.2. For any lattice \( L \subset \mathbb{R}^d, s > 0, \) and \( x \in \mathbb{R}^d \),

\[
\rho_s(L + x) \leq \rho_s(L) .
\]

Corollary 2.3. For any lattice \( L \subset \mathbb{R}^d, s > 0, \) and \( x \in L \),

\[
\Pr_{y \sim D_{L,s}} [ \langle x, y \rangle = 0 ] \geq \Pr_{y \sim D_{Z/\|x\|,s}} [ y = 0 ] = \rho_s(\|x\|)(Z)^{-1} \geq c \cdot \min(1, (s\|x\|)^{-1}) .
\]

Proof. We start with the first inequality. Clearly, \( \langle x, y \rangle \) takes integer values. For any \( i \in \mathbb{Z} \), the set of points \( y \) in \( L^* \) with \( \langle x, y \rangle = i \) is either empty or a translation of \( L^* \cap x^\perp \) whose affine span is at distance \( i/\|x\| \) from the origin. The \( \rho_s \) mass of this set is obviously zero in the former case and at most \( \rho_s(i/\|x\|) \rho_s(L^* \cap x^\perp) \) in the latter by Lemma 2.2 and the product property of \( \rho_s \). The inequality follows. The last inequality is an easy calculation. \( \square \)

3 Application to Koldobsky’s question

Theorem 3.1. Let \( K \subset \mathbb{R}^d \) be an origin-symmetric convex body. Then there exists a vector \( y \in \mathbb{Z}^d \) such that

\[
|K \cap \mathbb{Z}^d \cap y^\perp|/|K \cap \mathbb{Z}^d| \geq \min(1, c \cdot d^{-1} \cdot \text{vol}(K)^{-1/(d-1)}) .
\]

We note that this bound improves on that of Alexander et al. \[AHZ15\] for any body whose volume is at most \( c^d \).
Proof. By John’s theorem (see, e.g., [Bal97]), there exists a linear transformation $T$ of determinant 1 so that $TK$ has circumradius at most $d \cdot \text{vol}(K)^{1/d}$. Therefore, by considering the body $TK$ and the determinant 1 lattice $T\mathbb{Z}^d$, it suffices to prove the following: for any origin-symmetric convex body $K \subset \mathbb{R}^d$ with circumradius at most $d \cdot \text{vol}(K)^{1/d}$, and any lattice $L \subset \mathbb{R}^d$, there exists a nonzero vector $y \in L^*$ such that $$|K \cap L \cap y^\perp| / |K \cap L| \geq \min(1, c \cdot d^{-1} \cdot \text{vol}(K)^{-1/(d-1)}) .$$

Notice that if $\text{vol}(K) < d^{-d}$ then the circumradius of $K$ is less than 1, in which case $L \cap K$ is not full dimensional (by Hadamard’s inequality and $\det L = 1$). As a result, we can choose a vector $y$ so that $K \cap L \subseteq y^\perp$, making the quotient above 1. We therefore assume from now on that $\text{vol}(K) \geq d^{-d}$.

We will use a simple application of the probabilistic method. Namely, let us choose $y$ from the distribution $DL^*$, where $s = C \cdot \text{vol}(K)^{1/(d(d-1))} \geq 1$ for a large enough absolute constant $C > 0$. Then, by Corollary 2.3, for any fixed $x \in K \cap L$, the probability that $\langle x, y \rangle = 0$ is at least $$c \cdot (sd \cdot \text{vol}(K)^{1/d})^{-1} = c \cdot d^{-1} \cdot \text{vol}(K)^{-1/(d-1)} .$$

Let $p$ denote the latter quantity. It follows that the expected fraction of vectors $x$ in $K \cap L$ satisfying $\langle x, y \rangle = 0$ is at least $p$. Moreover, by Markov’s inequality, with probability at least $p/2$ over the choice of $y$, the fraction of vectors $x$ in $K \cap L$ satisfying $\langle x, y \rangle = 0$ is at least $p/2$. To complete the proof, notice by Lemma 2.1 that the probability that $y = 0$ is $s^{-d} < p/2$. Therefore, there is a positive probability over the choice of $y$ that $y \neq 0$ and moreover, that the fraction of vectors $x$ in $K \cap L$ satisfying $\langle x, y \rangle = 0$ is at least $p/2$. This completes the proof.

References

[AHZ15] Matthew Alexander, Martin Henk, and Artem Zvavitch. A discrete version of Koldobsky’s slicing inequality, 2015. Available at [http://arxiv.org/abs/1511.02702](http://arxiv.org/abs/1511.02702).

[Bal97] Keith Ball. An elementary introduction to modern convex geometry. In Flavors of geometry, volume 31 of Math. Sci. Res. Inst. Publ., pages 1–58. Cambridge Univ. Press, Cambridge, 1997.

[Ban93] Wojciech Banaszczyk. New bounds in some transference theorems in the geometry of numbers. Mathematische Annalen, 296(4):625–635, 1993.