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Nonselfintersecting magnetic orbits on the plane. 
Proof of Principle of the Overthrowing of the Cycles. 

1. Introduction. Overthrowing of the Cycles. Unsolved problems

Beginning from 1981 one of the present authors (S.Novikov) published a series of papers [1, 2, 3, 4] (some of them in collaboration with I.Schmelzer and I.Taimanov) dedicated to the development of the analog of Morse theory for the closed 1-forms–multivalued functions and functionals –on the finite- and infinite-dimensional manifolds (Morse-Novikov Theory). This theory was developed very far for the finite dimensional manifolds (many people worked in this direction later). The notion of ”Multivalued action” was understood and ”Topological quantization of the coupling constant” for them was formulated by Novikov in 1981 as a Corollary from the requirement, that the Feinmann Amplitude should be one-valued on the space of fields–maps, by Deser-Jackiv-Templeton in 1982 for the special case of Chern-Simons functional and by Witten in 1983). This idea found very important applications in the quantum field theory. Very beautiful analog of this theory appeared also in the late 80-ies in the Symplectic Geometry and Topology, when the so-called Floer Homology Theory was discovered.

A very first topological idea of this theory, formulated in early 80-ies, was the so-called Principle of the Overthrowing of the Cycles. It led to the results which were not proved rigorously until now. Our goal is to prove some of them.

We remind here that Novikov studied in particularly an important class of classical Hamiltonian systems of the different physical origin, formally equiv-

\footnote{This paper will be published in 1995 by American Mathematical Society in proceedings of the seminar by S.P.Novikov in seria “Advances in Mathematical Sciences.”}
alent to the motion of the charged particle on the Riemannian manifolds $M^n$ in the external magnetic field $\Omega$, which is a closed 2-form on the manifold (see [3]). In terms of Symplectic Geometry, these Hamiltonian Systems on the Phase spaces like $W^{2n} = T^*(M^n)$ are generated by the standard Hamiltonian functions (the same as the so-called ”Natural Systems” in Classical Mechanics) corresponding to the nonstandard Symplectic Structure, determined by the External Magnetic Field. In the most interesting cases our symplectic form is topologically nontrivial (i.e. it may have nontrivial cohomology class in $H^2(W^{2n}, R)$).

Periodic orbits are the extremals of the (possibly multivalued) action functional $S$ on the space $L(M^n)$ of the closed loops (i.e. smooth or piecewise smooth mappings of the circle in the manifold $M^n$):

$$S\{\gamma(t)\} = \oint_\gamma 1/2 \left(\frac{d\gamma}{dt}\right)^2 + e \oint_\gamma d^{-1}(\Omega)$$

This quantity is not well-defined in general as a functional, but its variation $\delta S$ is well-defined as a closed 1-form on the space of closed loops $L(M^n)$ (this is the situation of Dirac monopole).

Even if the closed 1-form $\delta S$ is exact, its integral $S$ may be not bounded from below. In both these cases standard Morse theory does not work. For the fixed energy $E$ we replace the action functional by the ”Maupertui–Fermat” functional with the same extremals:

$$F_E(\gamma) = (2E)^{1/2}l(\gamma) + e \oint_\gamma d^{-1}\Omega$$

This functional is also multivalued in general. Here $l(\gamma)$ is an ordinary Riemannian length. Let the charge $e$ will be equal to 1 and the form $\Omega$ is exact $\Omega = dA$ and small enough. The functional above is positive. We have some very nice special case of the Finsler metric (its geometry was investigated by E.Cartan many years ago). We may apply the ordinary Morse-Lusternik-Schnirelmann theory in this case.

**Definition 1** We call the functional $F_E$ not everywhere positive if the form $p^*\Omega$ is exact on the universal covering $p : M \to M^n$ and there exist a closed curve $\gamma$ on the universal covering (or the curve homotopic to zero in $M^n$), such that $F_E(\gamma) < 0$. By definition $F_E(\gamma_0) = 0$ for any constant curve $\gamma_0$. 

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Definition 2 We call the functional $F_E$ essentially multivalued if the form $p^*\Omega$ is not exact on the universal covering $M$. It is well-defined as a functional on some regular nontrivial free abelian covering space $\hat{L} \to L(M)$ with discreet fiber $\mathbb{Z}^k$ over the loop space for the manifold $M$.

In this case there is a natural imbedding of the (trivial) covering space over the one-point curves $M \times \mathbb{Z}^k \subset \hat{L}$, such that $F_E(M \times 0) = 0$ for some selected point $0 \in \mathbb{Z}^k$ and $F_E(M \times j) \neq 0$ for $j \neq 0$.

In the last case the functional $F_E$ is obviously not everywhere positive on the covering space $\hat{L}$. There exist an index $j \in \mathbb{Z}^k$ such that

$$F_E(M \times j) < 0$$

There is a natural free action of the group $\pi_1(M^n)$ on the loop space $L(M)$, such that the factor space is isomorphic to the space $L_0(M^n)$. Here $L_0(X)$ denotes the space of loops on $X$, homotopic to zero. This action extends naturally to the space $\hat{L}$ and we are coming to the factor-space $\hat{L}_0$ of the space $\hat{L}$ by the group $\pi_1(M^n)$.

On the last space $\hat{L}_0$ our functional $F_E$ is well-defined and not everywhere positive. There is an imbedding

$$M^n \times \mathbb{Z}^k \subset \hat{L}_0$$

such that

$$F_E(M^n \times 0) = 0, F_E(M^n \times j) < 0$$

for the same indices as above. It makes sense only if our functional is essentially multivalued: there exist an index $j$ different from zero. For one valued functionals we have $k = 0$ and $\mathbb{Z}^k, k = 0$ contains only one point $0$.

The following two lemmas are trivial, but important.

Lemma 1 All imbeddings $M^n \times j \to \hat{L}_0$ are homotopic to one with index 0.

Lemma 2 Our functional has nondegenerate manifolds of local minima in all one-point families $M_j = M^n \times j \subset \hat{L}_0$.
Definition 3  By the Overthrowing of the Cycle (set) \( Z \subset M^n \) (in the negative domain) for the given multivalued or not everywhere positive functional \( F_E \) we call any continuous map

\[
f: Z \times I[0, 1] \to \hat{L}_0
\]

such that \( f(Z \times 0) = Z \subset M_0 \) and \( F_E(Z \times 1) < 0 \).

The existence of such overthrowing was pointed out in early 80-ies by Novikov as a main topological reason for the existence of periodic orbits, homotopic to zero, in the magnetic field. There are two important examples.

Example 1  For the essentially multivalued functionals we may take \( Z = M^n \). Overthrowing here is a homotopy between \( M^n \times 0 \) and \( M^n \times j \) as above.

Example 2  For the case of one-valued but not everywhere positive functionals we may take \( Z \) as one point in \( M^n \). Later Taimanov proved in [5] that there exist overthrowing with \( Z = M^n \) for any not everywhere positive functional.

As a Corollary from the overthrowing an analog of the Morse inequalities was formulated. Let all critical points are nondegenerate. For the number of them with Morse index equal to \( i \) and positive value of the functional we have inequality:

\[
m_i(F_E) \geq b_{i-1}(M^n), i \geq 1
\]

Here \( b_i \) are Betti numbers or any their improvements of the Smale type. Critical points may be degenerate or they may be multiples of one smaller extremal. Therefore we expect to prove existence of one periodic extremal from this arguments. However there exist an important difficulty (pointed out by Bolotin many years ago):

We prove the existence of the positive critical values \( c_s > 0 \) for the functional \( F_E \) by the minimax arguments, but actual critical points may not exist. Our functional violates the important Compactness Principle.

The critical value \( F_E = c_s > 0 \) may be realized by the infinitely long curve \( \gamma \), which satisfies to the Euler-Lagrange equation and may be approximated
by the locally convergent sequence of the closed curves $\gamma_i(t) \to \gamma(t)$, such that:

$$F_E(\gamma_i) \to c_* + 0, l(\gamma_i) \to \infty$$

Until now we don’t know any examples of such infinitely long extremals obtained through the overthrowing of the cycles.

In the present paper we are going to prove completely the Overthrowing Principle for the important case $M^2 = \mathbb{T}^2$ with Euclidean metric and arbitrary nonzero magnetic field.

We may think about the Euclidean plane $\mathbb{R}^2$ with everywhere nonzero double-periodic magnetic field, directed along $z$-axis. All 4 periodic extremals for any generic energy $E$ (with the Morse indices $1,2,2,3$ of the Maupertui-Fermat functional) will be found as convex nonselfintersecting curves. Therefore they are geometrically distinct. Of course we obtain other extremals from them by the discreet translations on periods. In principle, homological arguments don’t give anything else.

**Remark 1.** In the paper [4] after long story a right criteria were found (in the Theorem 1) for the existence of the nonselfintersecting extremals of multivalued and not everywhere positive functionals on the 2-sphere. The idea of the proof was incomplete for the essentially multivalued case. Later it was completed and finally proved by Taimanov (see proof and all history of this problem in the survey article [6]).

**Remark 2.** It is clear for us now that no analogs of the Morse type theory can be constructed on the space of immersions. Therefore the theorem 2 of the paper [4] is unnatural. Its most general form is probably wrong. It should be replaced by the stronger result—by the main theorem of the present paper for the nonzero magnetic field.

**Remark 3.** In the very interesting papers [7, 8] V.Ginzburg proved the existence of periodic orbits with energy small enough and large enough, using the perturbation of the limiting pictures. In particularly, he pointed out to us that the theorem 3, announced without proof in the paper [4], is wrong. In fact, it contradicts to the example of the constant negative curvature and magnetic field equal to the Gaussian 2-form with the right sign, such that the extremals are exactly the horocycles: there is no periodic horocycles on the compact surfaces$^2$. This mistake is interesting: theorem 3 was extracted from the lemma 3 which claims that our functional is bounded from below in

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$^2$As Ginzburg wrote, this example was pointed out to him by Marina Ratner
any free homotopy class of loops, if it is true for the trivial one. This lemma is wrong. It is true for the homotopy classes of mappings \((S^1, s) \to (M^2, x)\) representing any element of fundamental group \(\pi_1(M^2, x)\), but may be wrong for some free homotopy classes containing the infinite number of elements of fundamental group. It is exactly what is going on in this counterexample.

For the finite conjugacy classes our theorem can be true. However there is no proof of it: the lack of compactness presents here analogous difficulty as before.

We may find some finite critical value \(c > -\infty\), but corresponding extremal may be infinitely long as before. Which kind of extremal we may get? For the surfaces with negative curvature and horocycles we have \(c = -\infty\), so this case is out of our arguments.

Nontrivial example we get on the 2-torus \(T^2\) with the exact magnetic field \(\Omega = dA\), such that our functional \(F_E\) is positive on the space of the closed curves homotopic to zero.

This property is always true for the energy, larger than some critical energy \(E_0\). In many cases an interval of energies exists such that the Maupertui-Fermat functional is not a well-defined Finsler metric, but positive on the space of loops homotopic to zero.

Another interesting example we get for \(n = 3\). Let the manifold \(M^n\) is a fiber bundle with fiber isomorphic to the circle \(S^1 \subset M^3\). For the magnetic fields \(\Omega\) with homology classes from the base we may ask about the periodic extremals homotopic to the fiber. For the 3-torus this restriction means that our magnetic field has no more than one rationally independent flux over the integral 2-cycles.
2. Nonzero double periodic magnetic field on the plane. 
Proof of Overthrowing of the Cycles for Convex Polygons.

We consider now nonzero smooth double periodic magnetic field on the Euclidean plane, directed orthogonal to this plane $\mathbb{R}^2$:

$$B(x + 1, y) = B(x, y + 1) = B(x, y) > 0$$

For the energy level such that

$$(2E)^{1/2} = 1$$

we denote a Maupertui-Fermat functional by $F = F_E$. We shall consider only this case without any losses of generality.

Consider the space of closed convex curves, oriented in such a way that:

$$F\{\gamma\} = l(\gamma) - \int \int_K B(x, y)dx\,dy, x_1 = x, x_2 = y$$

In this formula $K$ means a positively oriented domain inside of the curve $\gamma$, magnetic field $B$ is positive. The second term we call a \textbf{magnetic area}. It comes with the negative sign.

In this section we consider the functional $F$ on the \textbf{space $P_N$ of the straight-line convex ”parameterized” polygons $\gamma \in P_N$ containing exactly $N$ equal straight-line pieces of any length $L$}. By definition, ”parameterized polygon” means ”polygon with some natural numeration of vertices”

$$AB...CDA = A_1A_2...A_{N-1}A_NA_{N+1} = A_1$$

Cyclic permutation of this numeration leads to the free action of the group $\mathbb{Z}/N$ on the space $P_N$. We denote factor-space by $\bar{P}_N$.

Let $B_{\min}, B_{\max}$ denote minimum and maximum of $B(x, y)$ on the torus $T^2$. We introduce the following parameters:

$$N_0 = \left\lceil \frac{8B_{\max}}{B_{\min}} \right\rceil + 1$$

$$\alpha_0 = \min\{\frac{1}{1000N}, \frac{2}{N} \arctan \left( \frac{9B_{\min}}{20B_{\max}(2N^3 + N/2)} \right) \}$$
\[ L_0 = \frac{4N}{\sin \left( \frac{\alpha_0}{2} \right) B_{\text{min}}} \]

This parameter depends on \( N, B_{\text{min}}, B_{\text{max}} \).

We shall consider the spaces \( P_N \) for \( N > N_0 \) only. For \( N \to \infty \) we have \( \alpha_0 \to 0 \) and \( L_0 \to \infty \). Let \( AB, BC \) are the neighboring edges of the convex polygon. In the point \( B \) we have an external angle \( \alpha \) and internal angle \( \beta \), such that \( \alpha + \beta = \pi \).

**Definition 4** We call a convex closed polygon from the space \( P_N \) admissible if all its external angles are larger than \( \frac{\alpha_0}{2} \) and \( L < 2L_0 \).

On the subspace of admissible polygons \( P^a_N \) we define a corrected functional:

\[ F_a(\gamma) = F(\gamma) + \sum_{k=1}^{N} \phi\left( \frac{\alpha_0}{\alpha_k} \right) + L_0 \psi\left( \frac{L}{L_0} \right) \]

Here \( \alpha_k \) means the k-th external angle of our admissible convex polygon \( \gamma \), \( \phi \) and \( \psi \) are such real nonnegative functions on the closed interval \([0, 2]\) that

\[
\begin{align*}
\phi(x) &= \psi(x) = 0, x \leq 1 \\
\frac{d\phi}{dx} &> 0, \frac{d\psi}{dx} > 0, x > 1 \\
\phi(x) &\to +\infty, \psi(x) \to 1, x \to 2
\end{align*}
\]

and both \( x \)-derivatives of these functions converge to the \( +\infty \) if \( x \to 2 \).

**Theorem 1** Let \( \gamma \in P_N \) is a convex polygon with \( N > N_0 \). It is an extremal for the functional \( F \) on this space such that \( F(\gamma) > 0 \) if and only if \( \gamma \) is an admissible curve, an extremal also for the functional \( F_a \) and \( F(\gamma) = F_a(\gamma) > 0 \).

The obvious geometric facts are true:

**Lemma 3** Let \( \gamma_i, i = 1, 2 \) are two convex polygons, such that \( \gamma_1 \) lies completely inside of \( \gamma_2 \). In this case we have \( l(\gamma_1) < l(\gamma_2) \).
Lemma 4 Let $\gamma \in P_N$ is a convex polygon with total length $NL$, such that there exist two internal angles in it less than $\frac{\pi}{3}$. It follows that the distance between these two vertices (say $A, B$) is at least $LN/4$.

For the proof of last lemma we observe that this vertices $A, B$ cannot be neighboring in $\gamma$. All our curve $\gamma$ belongs to the interior of the romb whose two opposite vertices are exactly $A, B$ with corresponding internal angles equal to $\frac{2\pi}{3}$ for this romb. The perimeter of the romb is less than $4|AB|$. We conclude therefore that $LN < 4|AB|$ from lemma 3. Lemma is proved.

Lemma 5 Let $D$ is any convex subset in the Euclidean space $R^2$, bounded by the polygon $\gamma \in P_N, l(\gamma) = NL$, and $C$ is any point inside it. After the rotation of this set on the small angle $\delta \alpha$ around the point $C$ we get a domain $D_1$ whose magnetic area satisfies to inequality:

$$|\int \int_{D_1} B \, dx \, dy - \int \int_{D} B \, dx \, dy| < N^2 L^2 (B_{\max} - B_{\min}) \frac{\delta \alpha}{2}$$

For the proof of this lemma we point out that after the rotation on the small angle $\delta \alpha$ total set $D_1$ minus the original one will have area no more than $N^2 L^2 \frac{\delta \alpha}{2}$. Combining this with the obvious estimates for the integral we get our final estimate. Lemma is proved.

Lemma 6 Let $\gamma \in P_N, N > N_0$ and there exist two different vertices (say, $A, B$), such that the corresponding internal angles are less than $\frac{\pi}{3}$. There exist a small deformation $\gamma_t$ of the curve $\gamma = \gamma_0$ in the space $P_N$ (with fixed length), such that magnetic area increases in the linear approximation and all external angles don’t decrease. Therefore the curve $\gamma$ cannot be an extremal for any one of the functionals $F, F_a$.

Proof of the lemma. Let the segment $AB$ be horizontal in our picture. It divides $\gamma$ on 2 pieces $AC \ldots DB$ (upper piece) and $AE \ldots FB$ (lower piece). Our deformation will be such that the lower piece $E \ldots F$ does not move $E_t = E, \ldots, F_t = F$ and the upper piece $C \ldots D$ moves up perpendicular to the segment $AB$ parallel to itself on the distance $t, (C_t \ldots D_t) || (C \ldots D)$.

Position of the vertices $A_t, B_t$ we define from this completely, because the length $L$ does not change.

This deformation has the desired properties (see elementary trigonometric calculation in the Appendix). Lemma is proved.

So we cannot have two internal angles less than $\frac{\pi}{3}$ for the extremals.
Lemma 7 No polygon $\gamma \in P_N$ can have all external angles except may be one angle (say, $\alpha_N$) less than $\alpha_0$.

Proof of this statement follows immediately from the definition of $\alpha_0$ and elementary geometric facts: total sum of all external angles is equal to $2\pi$, each of them is less than $\pi$. Therefore we have

$$\sum_{k=1}^{N-1} \alpha_k = 2\pi - \alpha_N < (N - 1)\alpha_0 \leq \frac{N - 1}{1000N}$$

At the same time we have $\alpha_N < \pi$. It leads to the contradiction, which proves our lemma.

Consider now a curve (polygon from the space $P_N$) containing at least two vertices with external angles more than $\alpha_0$. Let this vertices are $A, B$ and all vertices between them (from one side) have ”small” external angles (i.e. less than $\alpha_0$). We construct a deformation $\gamma_t$ of this curve $\gamma = \gamma_0$ in the space $P_N$ with fixed length: let $[AB]$ is a segment between this two points and $C, D$ are the vertices with orthogonal projections on the segment $[AB]$ closest to the centrum. Here $C$ belongs to the arc with small external angles and $D$ belongs to the other arc of $\gamma$. Let $A_t = A$ and $B_t$ is obtained from $B$ by the small shift $\delta x = t$ along the segment $[AB]$ in the direction of $A$. We rotate the arcs $A \ldots C$ and $A \ldots D$ around the point $A = A_t$. The arcs $B \ldots C$ and $B \ldots D$ we shift parallel to themselves on the same distance as the point $B$. After that we rotate them around the point $B_t$. Finally we find the points $C_t, D_t$ as a crossing points. Following lemma is true for this deformation.

Lemma 8 The deformation $\gamma_t$ described above does not change the length. It is such that all external angles (except the angles in the vertices $A, B$ with external angles more than $\alpha_0$) don’t decrease; $t$-derivative of the magnetic area for $t = 0$ is nonzero. Therefore the curve $\gamma$ cannot be an extremal for the functionals $F, F_a$: Any curve which is an extremal for each of them is such that all external angles are more than $\alpha_0$.

Proof of this lemma uses the lemmas above. It is based on the elementary trigonometric calculations using the values of parameters $\alpha_0, N_0, L_0$, fixed in the beginning of this section (see Appendix for the details).
Lemma 9 Let the curve $\gamma \in P_N$ is such that all external angles are bigger than $\alpha_0$ and $L > L_0$. In this case we have $F(\gamma) < 0$. If the curve is admissible $L < 2L_0$ we have $F_a(\gamma) < 0$

For the proof of this lemma it is enough to estimate the magnetic area of any triangle $ABC$ based on two edges $AB, BA$ of our polygon. The external angle is bigger than $\alpha_0$. Therefore its area $S$ is bigger than $S_0$:

$$S > S_0 = \frac{1}{2}L^2 \sin \left( \frac{\alpha_0}{2} \right)$$

And its magnetic area is bigger than $B_{\text{min}}S_0 > 2NL$. We have $l(\gamma) = NL$. As a corollary we are coming to inequality:

$$F(\gamma) < NL - 2NL < 0$$

All external angles are bigger than $\alpha_0$. So the contribution of the function $\phi$ in the value of the functional $F_a(\gamma)$ is equal to zero. By definition, we always have $\psi \leq 1$. Therefore we conclude for $L > L_0$ that

$$F_a(\gamma) = F(\gamma) + L_0 \psi \left( \frac{L}{L_0} \right) < -NL + L_0 < 0$$

Lemma is proved.

Proof of the theorem 1 follows now from the lemmas. Theorem 1 is proved.

We are going to construct now a natural analog of the Morse theory for the functional $F_a$ on the space $P_N$ of the admissible polygons— or, more exactly on the space $\tilde{P}_N$ of the admissible polygons completed naturally by the one-point curves and factorized by the discreet group $Z^2 \times (Z/N)$ generated by the basic translations of the plane $R^2$ and cyclic permutation of the order of vertices.

This space is homotopy equivalent to the torus $T^2$ (i.e. to the subspace of the one point curves). This space without one point curves is homotopy equivalent to the 3-torus $T^2 \times S^1$.

We are going to use the Morse type estimates ”modulo subspace $P^0$”, where the functional $F_a$ is less or equal to zero

$$P^0 = \{F_a \leq 0\}$$

An easy lemma is true:
Lemma 10 The space $P^0$ is not connected. It contains at least two components. One of them is exactly set of all one point curves $T^0_0 \subset P^0$. Another one $P^0_1$ contains all $N$-polygons $\gamma$ with equal angles, such that the length of edges $L$ is big enough (but less than $2L_0$).

We already know that the set of one-point curves is a local minimum for the functional $F$. By definition, the value of the functional $F_0$ on it is equal to zero. So this set is a local minimum also for $F_0$, because $F_0 \geq F$ for any admissible curve. Therefore the set of one point curves is isolated in $P^0$. The curves with equal angles have all external angles equal to $\frac{2\pi}{N} > \alpha_0$. For $L = L_0$ and large $N > N_0$ we have $F = F_0$ for them and $F < 0$.

Lemma is proved.

For any point in the plane we construct an Overthrowing of it, continuously depending from this point.

Definition 5 By definition, our Initial Overthrowing is a set of all $N$-polygons with centrum in this point and with equal angles and edges with the length $L < L_0$. It determines a map

$$f : (T^2 \times S^1) \times [01] \to \bar{P}^a_N$$

such that

$$f(T^2 \times S^1 \times 0) = T^2_0 \subset P_0, f(T^2 \times S^1 \times 1) \subset P^0_1$$

Parameter along the circle $S^1$ here is exactly an angle numerating all polygons with the same centrum and same length, parameter in the interval $[01]$ coincides with radius divided by the maximal radius, such that all image belongs to the negative values of our functionals for all central points in the plane.

An obvious lemma is true:

Lemma 11 An Overthrowing

$$f : (T^2 \times S^1 \times [01], T^2 \times S^1 \times 0 \bigcup T^2 \times S^1 \times 1) \to (\bar{P}^a_N, P^0)$$

generates monomorphisms in Homology groups:

$$H^{i-1}(T^2 \times S^1) \to H^i(\bar{P}^a_N, P^0), i \geq 1$$
Our functional $F_a$ generates a cell decomposition of the space $\bar{P}_N$ modulo $P_0$, corresponding to the critical points such that $F_a(\gamma) > 0$. This is a corollary from our lemmas, because this space is invariant under the gradient flow (all gradient lines go inside of it). So we may apply standard arguments of the Morse theory to this space modulo negative subspace $P_0$. Combining this fact with the previous lemma, we are coming to the theorem:

**Theorem 2** For any value $N > N_0$ there exist at least two different extremals of the Maupertuis-Fermat functional $F$ in the space $P_N^a$ of the admissible convex polygons. If critical points are nondegenerate, there exist at least 8 of them in the same space with Morse indices equal to 1, 2, 2, 2, 3, 3, 3, 4.

Proof of the theorem. By the minimax principle, we always have at least one extremal in this space. Let we have only one critical point. After the long gradient deformation, starting from the initial Overthrowing Process

$$f = f_0 : T^2 \times S^1 \times [01] \to P_N^a$$

we are coming to the new Overthrowing Process $f_1$ in which almost all image is below the critical level and the remaining part is concentrated in the small neighborhood of the critical point. After removing from the space $\bar{P}_N^a$ some small neighborhood of the critical point the new overthrowing will split on some pieces (at least two) such that the image of its boundaries $T^2 \times S^1 \times 0$ and $T^2 \times S^1 \times 1$ belongs to different components.

It follows from the fact that any new overthrowing of one point should pass through the same small neighborhood as the new overthrowing of all torus $T^2$. By any new overthrowing of the point we have in mind any curve $f_1 : \tau(t) \to \bar{P}_N^a$ where $\tau(t)$ is any continuous curve in $T^2 \times S^1 \times [01]$ such that $\tau(0) \in T^2 \times S^1 \times 0$, $f_1(\tau(1)) \in P^0$.

Our space $P_N^a$ is locally contractible. Using this, we deform its identity map onto itself in such a way that after deformation all small neighborhood of the critical point will collapse to this point. Finally we constructed a deformation of the set of one point curves to one (critical) point in the space $P_N$. However this set is nonhomotopic to zero in the space $P_N$. We are coming to contradiction. So we have at least two extremals.

Other part of this theorem is a standard obvious corollary from the handle decomposition generated by the critical points of the functional $F_a$. It follows
from the lemmas above that we may apply the standard arguments of Morse Theory here. Theorem 2 is proved.
3. Compactness Property for $N \to \infty$. Main results.

**Definition 6** For any convex polygon $\gamma \in P_N$ we call by the Maximal Diameter $D_{\text{max}}$ a maximal distance between two points of this polygon. By the diameter in the direction $\phi$ we call a maximal distance $D_\phi$ between two straight lines parallel to the direction $\phi$, which have nontrivial intersection with $\gamma$. We call by the maximal and minimal diameters $D_{\text{max}}, D_{\text{min}}$ exactly maximum and minimum of the function $D_\phi$, corresponding to the directions $\phi_{\text{max}}, \phi_{\text{min}}$.

**Theorem 3** Let $\gamma_N \in P_N$ is an extremal of the functional $F$, such that $F(\gamma_N) > 0$. Following estimate for its maximal diameter and for the maximal length $L_0$ are true:

$$D_{\text{max}} \leq \frac{8 (3 + B_{\text{max}} B_{\text{min}}^{-1})}{B_{\text{min}} (1 - 8N^{-1})}, \quad L_0 \leq 4D_{\text{max}}$$

The proof of this theorem follows from the lemma:

**Lemma 12** For any extremal $\gamma$ of the functional $F$ on the space $P_N$, such that $F(\gamma) > 0$, the estimate is true:

$$\frac{D_{\text{max}}}{D_{\text{min}}} \leq \frac{B_{\text{max}} B_{\text{min}}^{-1} + 3}{1 - 8N^{-1}}$$

Proof of the lemma. We describe a deformation, which preserves length of $\gamma$ and changes the magnetic area in the linear approximation if the inequality is not true. The $y$-axis is exactly direction $\phi_{\text{min}}$ in our picture.

Let $AB, HG$ are the most left edges such that their angles with $x$-axis are no more than $\frac{\pi}{4}$ and $CD, FE$ are the rightest edges with the same property. The arcs $AB \ldots CD$ and $HG \ldots FE$ belong to the upper and lower parts of $\gamma$. The points $A$ and $H$ or $D$ and $E$ may coincide, but it is not important. Our deformation $\gamma_t, \gamma_0 = \gamma$ is such that $A_t \ldots H_t = A \ldots H$, the arc $D_t \ldots E_t$ is obtained from $D \ldots E$ by the parallel shift on the distance $\delta x = t$ to the left. The arc $B_t \ldots C_t$ is obtained by the parallel shift of the arc $B \ldots C$
up (on the distance \(\delta y_1\) and left (on the distance \(\delta x_1\)). The arc \(G_t \ldots F_t\) is obtained from \(G \ldots F\) by the parallel shift down (on the distance \(\delta y_2\)) and left (on the distance \(\delta x_2\). The value of all these parameters as function from the variable \(t\) follows from the requirement that all lengths are the same and the new polygon \(\gamma_t\) is closed.

Proof of the lemma follows from the trigonometric calculations (see Appendix).

Proof of the Theorem: let \(A, B, C, D\) are the most left, most upper, rightest and most lower vertices in \(\gamma\). A polygon \(ABCD\) with four edges belongs completely to the interior of \(\gamma\). Therefore its area is at least \(\frac{D_{\text{min}}D_{\text{max}}}{2}\) and the magnetic area \(Q\) is at least \(1/2D_{\text{min}}D_{\text{max}}B_{\text{min}}\). Combining this with the trivial estimate \(l(\gamma) \leq 4D_{\text{max}}\) and \(F(\gamma) > 0\) we are coming to inequality:

\[
0 < F = l - Q < 4D_{\text{max}} - 1/2D_{\text{min}}D_{\text{max}}B_{\text{min}}
\]

or finally

\[
D_{\text{min}} < 8B_{\text{min}}^{-1}
\]

Using the lemma above, we are coming to the desired inequality for \(D_{\text{max}}\). Theorem is proved.

Consider now a sequence \(\gamma_N\) of the extremals of functional \(F\) on the spaces \(P_N\) with \(F > 0\) and \(N > N_0\). In fact we consider only a subsequence \(N_k = N_12^k\).

For the large \(N\), small external angles and bounded total length \(NL\) the polygons \(\kappa\) and \(p_N(\kappa)\) are very closed.

From the theorem above we conclude that There exist a subsequence \(k_j \to \infty\) such that \(\kappa_j = \gamma_{N_{k_j}} \to \gamma\) where \(\gamma\) is a continuous curve, because all family of our extremal \(N_{k_j}\)-polygons \(\kappa_j\) with positive value of the functional \(F\) is precompact.

**Theorem 4** The limiting curve \(\gamma\) is a periodic smooth extremal of the functional \(F\) with positive value of \(F(\gamma) > 0\).

**Lemma 13** Let \(\kappa_j \in P_N, N = N_{k_j}\) as above is a sequence of ”relative extremals” with positive value of the functional \(F\) and \(\gamma\) is a limiting continuous curve. For \(N \to \infty\) all family of external angles of the curves \(\kappa_j\) converges to zero as \(O(1/N)\).
Let A is a vertex with largest external angle $\alpha$ and B is "opposite" vertex, such that the arc $A \ldots B$ contains exactly $N/2$ edges. Consider following deformation $\gamma_t$ of the polygon $\kappa_j = \gamma_0$: $B_t = B$, all vertices except $A$ move along their own edges towards $B$ in such a way that the distance between any two vertices will be exactly $L - \delta L, \delta L = t$. A shift of the vertex $A$ will be completely determined by the requirement that new polygon has equal edges with length $L - t$.

For the variation of the functional $F$ in the point $t = 0$ we get inequality (see elementary trigonometric calculation in Appendix):

$$|\delta F| > \delta L(N - \pi L_0 B_{\max} - L_0 B_{\max} (\sin \alpha)^{-1})$$

At the same time we remember that $\delta F = 0$ for $t = 0$.

Finally we are coming to inequality:

$$\sin \alpha < \left( \frac{N}{L_0 B_{\max}} - \pi \right)^{-1}$$

Therefore we proved that for the large enough $N$ there exist such constant $c$ that $\alpha < cN^{-1}$. Lemma is proved.

**Lemma 14** The limiting curve $\gamma$ belongs to the class $C^1$.

We found already the upper estimate for the length of "relative extremals" in the lemma above. It is easy to find also the lower estimate for this length. Consider any point $x$ inside $\gamma$. We apply homotety with centrum in this point and with coefficient $1 + p$. For the variation we have

$$0 = \delta F = \delta l(\gamma) - \delta \int \int_K B dxdy > pl(\gamma) - pl(\gamma)d_{max}B_{max}$$

Here $d_{max}$ means maximal distance from the point $x$ to $\gamma$. We deduce from this an inequality:

$$d \geq B_{max}^{-1}, NL = l(\gamma) \geq l_0 = 2B_{max}^{-1}$$

Consider any arc $P \ldots Q \ldots R \ldots S$ on the extremal $\gamma$, containing $n$ edges, $n \leq N/2$. By the previous lemma, the angle $\phi$ between the lines $PQ$ and $RS$ in the point of their intersection (outside of $\gamma$) has the order $O(nN^{-1})$:

$$\phi \leq nc/N$$
For the length of the arc $P \ldots S$ we have $l(P \ldots S) = nL \geq n l_0 N_{-1}$. Combining this with previous inequality, we get:

$$\phi \geq c l(P \ldots S) l_0^{-1}$$

For the limiting curve $N \to \infty$ we have an upper estimate for the angle between two "tangent" lines in the points $P, S$

$$\phi \geq c(B_{\min}, B_{\max}) \times l(P, S)$$

(distance along the curve).

**Definition 7** By the "tangent" line for any convex curve we call any straight line which has all our curve from one side. For the vertices of convex polygons "tangent line" means that it has only one common point with our polygon.

Our curve is convex because it is a limit of convex curves. Lemma follows from this estimate.

The polygons from our sequence $\kappa_j$ belong to the spaces $P_{2^k_j, N_1}$. We fix numeration such that all the vertices $P_{j,s}$ with numbers $2^k_j$ s converge to some points $P_s$ on the limiting curve for fixed values of $s, j \to \infty, s = 1, \ldots N_1$.

For $N_1$ large enough and any two vertices $P_{0,s} = R_0, P_{0,r} = Q_0$ on this curve we have two sequences

$$P_{j,s} = R_j \to R, P_{j,t} = Q_j \to Q$$

**Lemma 15** Following estimate is true for the angle $\phi$ between two straight lines, "tangent" to the polygons $\kappa_j$ in the vertices $R_j, Q_j$:

$$\phi = \int_{R_j}^{Q_j} B(\kappa_j(t))dl(t) + O\left(\frac{1}{2^k_j N_1}\right), j \to \infty$$

Here integral is taken along the curve $\kappa_j$ using a natural parameter $l$.

Proof. Consider the arc $TR_jU \ldots A \ldots VQ_jS$ where $A$ is a "central" vertex between $R_j$ and $Q_j$ (or one of two central vertices, if the number of edges in the arc is odd). Let $B$ is a "central" vertex of the opposite arc $R_jT \ldots B \ldots SQ_j$ in the same sense, $T_1, T_2$ means two "tangent" lines in the vertices $R_j, Q_j$ and $\phi$ means external angle in their crossing point.
We construct such deformation $\gamma_t, \gamma_0 = \kappa_j$ of this extremal that:

All arc $R_jT..., B...SQ_j$ doesn’t move;

The points $R_j, Q_j$ we move at first inside of the curve $\gamma_0$ on the small distance $\delta s = t$ in the direction perpendicular to the edges $R_j U, Q_j V$. We denote new vertices by $R^1, Q^1$. We move all edges of the arc $R_j U...VQ_j \rightarrow R^1 U^1...V^1 Q^1$ inside on the distance $\delta s$ in the directions perpendicular to each edge and construct from their pieces a new arc $R^1 U^1......V^1 Q^1$ with slightly smaller edges (not necessary equal to each other).

After that we make all edges equal by the deformation, such that the point $B$ does not move, all vertices from the arc $A^1...Q^1...B^1$ move along their own edges on this arc in the direction towards $B$, all vertices on the arc $A^1...R^1...B$ move along their own edges on this arc towards $B$, the vertex $A$ moves along one of two edges (which one will be uniquely defined by the condition that we have to get finally equal edges).

On the first step of deformation we have an estimate for the length:

$$\delta l = \phi \delta s + O(\frac{1}{2^k_j N_1})$$

(from lemma 13)

and for the magnetic area:

$$\delta \int \int_K B dx dy = \int_{R_j}^{Q_j} B dl + O(\frac{1}{2^k_j N_1})$$

From the same lemma 13 we conclude that the total product over all vertices is equal to one plus something small:

$$\cos \alpha_1 \times \ldots \cos \alpha_{2^k_j N_1} = 1 + O(\frac{1}{2^k_j N_1})$$

for the polygon $\kappa_j = \gamma_0$. Changing the length of one edge on the arc $A^1 R^1 B$ on the value $\delta L$, we have a shift of the point $A^1$ on the distance

$$\delta L \prod_l \cos \alpha_l = \delta L (1 + O(\frac{1}{2^k_j N_1}))$$

Therefore the variation of length on the second step is small enough, and a shift of any vertex is no more than $c \delta s$, where $c$ is some constant independent from $j$. Variation of the magnetic area is small enough on the second step.
Total variation of the functional we get after summation of all our contributions:

\[ 0 = \delta F = (\phi - \int_{R_j}^Q Bdl + O(\frac{1}{2^{k_j}N_1})) \]

Lemma is proved.

Proof of the theorem 4. From the lemma 15 above we have for the limiting curve \( j \to \infty \) exactly the statement of the theorem. Theorem is proved.

**Theorem 5** For any smooth positive double periodic magnetic field on the Euclidean plane (directed along the third axis, orthogonal to the plane) there exist at least two different periodic convex extremals, such that the value of the Maupertui-Fermat functional is positive for them.

Proof. Let we have only one extremal for the functional \( F \) after the limit \( j \to \infty \). Consider the overthrowing process \( f_j : T^2 \times [01] \to P_{2^{k_j}N_1}^a \) after the long gradient deformation, determined by the corrected functional \( F^a \). For the large \( j \) we have several (at least two) different extremals in this space, which have the same limit for \( j \to \infty \).

Therefore our "relative extremals" \( \kappa_{j,p}, p = 1,2,... \) for all large values \( j \) belong to the same very small contractible neighborhood \( W \) of the limiting extremal \( \kappa \) in the space of convex piecewise smooth curves.

After long gradient deformation (mentioned above) the new overthrowing process belongs to the negative subspace \( P^0 \subset P_{2^{k_j}N_1}^a \), everywhere outside of the neighborhood \( W \). As above in the proof of the theorem 2 any overthrowing process of the point, determined by the map \( f_j \), should pass through this set \( W \). By this reason, the imbedding of the manifold of all one-point curves is contractible in the space of nonparameterized closed convex curves (i.e. factor by the action of the group \( SO_2 \), changing the initial point in the natural parameterization). But this is an obvious contradiction. Theorem is proved.

**Theorem 6** Let all periodic convex extremals with the positive value of Maupertui-Fermat functional are nondegenerate in sense of Morse in the space of non-parameterized curves. In this case there exist at least four periodic convex extremals for any fixed value of energy such that their Morse indices are equal to \((1,2,2,3)\).
For the proof of this theorem we are going to use theorem 3 and the comparison of Morse indices of periodic extremals with the Morse indices of ”relative extremals” \( \kappa_j \) for all values of \( j \) large enough. This comparison (which looks easy), never was proved rigorously for our spaces. So we shall finish the complete proof later. Let us present here the idea of the proof. A very first question is: **What is the Morse index for the Maupertui-Fermat functional on the space of all smooth curves?**

For the definition of this quantity we have to introduce some unique receipt of parameterization of curves, because our functional does not depend on it. A natural parameter (length) is OK for our goals.

After that we consider a Morse index on the space \( P \) of the convex curves with natural parameterization. This index is finite. There is a trivial ”nullity” of this critical point equal to 1. It corresponds to the choice of initial point on the curve. Our functional is invariant under the free action of the group \( SO_2 \) on the space \( P \), as it was mentioned above. The factor-space \( \tilde{P} \) of the space \( P \) by this action is homotopy equivalent to the torus \( T^2 \). Our functional in the generic case has only nondegenerate critical points in \( \tilde{P} \).

In process of approximation we use the spaces \( P_N \) with the natural action of the group \( Z/N \subset SO_2 \). After the approximation our functional is only \( Z/N \)-invariant. Corresponding factor-spaces \( \tilde{P}_N \) have the homotopy type \( T^2 \times S^1 \). We found more critical points in the theorem 3, than we need for the limit \( N \to \infty \). In fact pairs of them with the neighboring indices \( i, i+1 \) should have the same limit. Returning to the spaces \( P_N \) with discreet parameterization, we get free \( Z/N \)-critical orbits instead of the points. This orbits converge to the \( SO_2 \)-orbits for \( N \to \infty \). Each nondegenerate critical \( SO_2 \)-orbit with Morse index \( i \) generates at least 2 nondegenerate critical \( Z/N \)-orbits with Morse indices \( i, i + 1 \) by the obvious homological reasons if we shall be able to prove that this approximation is really equivalent to the small perturbation of our functional in the \( C^2 \) norm.

Following the classical papers of Marston Morse, we introduce a finite-dimensional approximation of the space \( P \) in the small neighborhood of the given closed extremal \( \gamma \) with Morse index equal to \( i \). It is convenient for us to use the same spaces \( P_N \) of polygons with \( N \) equal edges and total length not far from the length \( NL \) of our extremal \( \gamma \).

The approximation of the functional by Morse is the following:

For \( L \) small enough we join all pairs of the neighboring vertices by the unique small extremal and construct therefore the **Extremal Polygon** with
the same vertices for any polygon from the space $P_N$. This space is canonically isomorphic to the space $P_N$, but the value of our functional on the extremal polygon is different than its value for the straight-line polygon with the same vertices. We denote this functional on the space $P_N$ by $F^e$. Its extremals are exactly the same as smooth extremals $\gamma$ on the space of all smooth curves.

Consider the small neighborhood of the extremal $\gamma$. All external angles of polygons in this neighborhood have order $O(N^{-1})$. We are going to compare functionals $F^e$ and $F$ in this part of the space $P_N$. An easy trigonometric estimate shows that difference between these functionals has order $O(N^{-2})$.

More exactly, consider a small straight-line interval $AC$ with length equal to $L$ and a small piece of extremal with the same vertices $A,C$ (it look like an arc of the circle with radius equal to $R_0 = B^{-1}$ in our approximation, for very small values of $L$). Here $B$ is a value of the magnetic field in the central point of the interval $AC$. Calculating the terms of the order $O(N^{-3})$ for the length of the extremal arc and the magnetic area of this small domain we shall need also the first derivatives of the field $B$ in the same point. After some elementary trigonometry we are coming to the following lemma:

**Lemma 16** Let our magnetic field belongs to the class $C^2$ on the torus. The value of Maupertuis-Fermat functional for all such ”local” geometric figures bounded by the small straight-line edge and a small piece of the extremal is less or equal than the quantity $O(L^3)$ with coefficient depending from the maximal values of the magnetic field $B$ and its first derivatives on the torus.

Combining this result with elementary properties of the Euclidean Geometry, we see that any small variation of these local geometric figure leads to the variation of the functional of the order $O(L^3)$ and $O(L^2\delta L)$ in the variable $N^{-1}$. If we consider any variation of the polygon from the small neighborhood of the extremal $\gamma$ with length $l$, we know a priori that $L \sim O(N^{-1})$ and $\delta L \sim O(N^{-1})$.

We use now the additivity property of our functional: the difference between functionals $F - F^e$ is equal to the sum of $N$ ”local” terms corresponding to the individual edges, described in the lemma above. Therefore we are coming to the following result:

**Lemma 17** In the small neighborhood of our extremal $\gamma$ in the space $P_N$ all
derivatives of any order of the difference \( F - F^e \) are the quantities of the order \( O(N^{-2}) \).

Now we are going to finish the proof of the theorem. In the process of approximation of the space of curves by the extremal polygons we consider a sequence \( N = 2^{k_j} N_1 \) as above, for which we have a convergence of the "relative" extremals of the functional \( F \) on the subspaces \( P_N \) to the smooth extremal \( \gamma \). From the old results of Morse we know that for all large values of \( N \) we have the same curve \( \gamma \) as extremal of the functional \( F^e \), i.e. of the same functional on the space of the extremal polygons. We know also that we may consider the tangent spaces to \( P_N \) in the point \( \gamma \) for all values of \( N = 2^{k_j} N_1 \) as the finite dimensional subspaces \( T_j \) of the same Hilbert space \( T \). This sequence \( T_j \) converges in the sense that all \( T_j \) with larger numbers "almost " contain the previous ones: there exist a natural projector

\[
\pi_{j,j+s} : T_j \to T_{j+s}
\]

such that \( \|\pi_{j,j+s}(u) - u\| \to 0 \) for \( j \to \infty \) homogeneously in \( s \) and for all unit vectors \( u \). We shall identify the subspaces \( T_j \) and \( \pi_{j,j+s}(T_j) \) in our notations.

The second variation of the functional \( F^e \) also converges. It means that this second variation is strictly positive on the subspaces orthogonal to the image of \( T_j \) in \( T_{j+s} \) with lowest eigenvalue, which converges to the \( +\infty \) for \( j \to \infty \). On the image of \( T_j \) all lower eigenvalue and eigenvectors converge to their values on the space of normal vector fields along the curve \( \gamma \). Therefore we may use the finite spaces \( P_N \) for the calculation of the Morse index. Our theorem follows now from the lemmas above, because the Morse index is stable under the perturbations of the function \( F^e \) on the spaces \( P_N \), which are small with first, second (and third) derivatives in all points of our neighborhood under investigation. The role of 1-dimensional nullity is the following: it leads generically to the splitting of one nondegenerate critical circle and creation of some nonzero even number of nondegenerate critical points in the spaces \( P_N \) in the process of approximation: half of them with index \( i \) and another half with index \( i+1 \); all of them converge to our extremal circle which is one point \( \gamma \) in the space of the nonparameterized curves.

Theorem is proved.
4. Appendix. Trigonometric calculations. Proof of the lemmas.

Proof of the Lemma 6. It is convenient to decompose the deformation in two steps.

Fig 1.

Fig 2a.

Fig 2b.
Step 1:
We shift the arc $ACDB$ up at the distance $t$. The images of the points $A$, $B$, $C$, $D$ under the shifts are denoted by $A'$, $B_t$, $C_t$, $D'$. We also add small vertical segments connecting the points $A$, $A'$ and $B$, $B'$.

Step 2: We rotate the segments $AE$, $A'C_t$, $BF$, $B'D_t$ round the points $E$, $C_t$, $F$, $D_t$ at such angles that the images of the points $A$, $A'$ coincide and the images of $B$, $B'$ coincide too. We shall denote them $A_t$ and $B_t$ respectively.

Let us denote the variation of the magnetic area in the first step and in the second one by $\delta_1 Q$ and $\delta_2 Q$ respectively. We have

$$\delta_1 Q \geq B_{\min} |AB| t,$$

$$\delta_2 Q \geq -B_{\max} \cdot \frac{1}{2} (|AE||AA_t| + |C_tA'||A'A_t| + |BF||BB_t| + |D_tB'||B'B_t|).$$

The infinitesimal triangles $AA'A_t$ and $BB'B_t$ have the magnetic area $O(t^2)$ and may be neglected in the first-order calculations.

The angles $AA_tA'$ and $BB_tB'$ are greater then $2\pi/3$ thus

$$|AA_t| < |AA'| = t, \ |A'A_t| < t, \ |BB_t| < t, \ |B'B_t| < t,$$

$$\delta_2 Q \geq -B_{\max} 2Lt.$$

$$|AB| \geq \frac{LN}{4} \geq \frac{LN_0}{4} > 2\frac{B_{\max}}{B_{\min}} L$$

Combining all this estimates we get:

$$\delta Q = \delta_1 Q + \delta_2 Q > 2B_{\max}Lt - 2B_{\max}Lt = 0.$$

The Lemma 6 is proved.
Proof of the Lemma 8. It is convenient to decompose the deformation in two steps.

Step 1:

We rotate the whole arcs $AC$, $BC$, $AD$, $BD$ around the points $A$ and $B$ to the corresponding angles

$$
\delta \varphi_1 = \frac{1}{\cos \varphi_1} \cdot \frac{1}{\tan \varphi_1 + \tan \varphi_2} \cdot \frac{\delta x}{|AC|},
\delta \varphi_2 = \frac{1}{\cos \varphi_2} \cdot \frac{1}{\tan \varphi_1 + \tan \varphi_2} \cdot \frac{\delta x}{|BC|}
$$

$$
\delta \varphi_3 = \frac{1}{\cos \varphi_3} \cdot \frac{1}{\tan \varphi_3 + \tan \varphi_4} \cdot \frac{\delta x}{|BD|},
\delta \varphi_4 = \frac{1}{\cos \varphi_4} \cdot \frac{1}{\tan \varphi_3 + \tan \varphi_4} \cdot \frac{\delta x}{|AD|}
$$

Here $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ are the angles between the $x$-axes and the intervals $AC$, $BC$, $BD$, $AD$ respectively, $|AC|$ denotes the length of the span $AC$.

Let us denote the images of the points $C$, $D$ after the rotations around the point $A$ by $C_t$, $D_t$, the images of the points $C$, $D$ after the rotations around the point $B$ by $C'$, $D'$, the orthogonal projections of the points $C$ and $D$ to the interval $AB$ by $\tilde{C}$, $\tilde{D}$.

Finally we add small horizontal intervals connecting $C_t$ and $C'$, $D_t$ and $D'$.
Step 2:

We shift the whole arc $C'B'D'$ left at the distance $\delta x$.

Let us denote the magnetic area above the line $AB$ and below the line $AB$ by $Q_+$ and $Q_-$ respectively, $Q = Q_+ + Q_-$, $Q_{AD}$ be the magnetic area of the polygon bounded by the arc $AD$ and by the span $AD$, $Q_{BD}$ be the magnetic area of the polygon bounded by the arc $BD$ and by the span $BD$, $\delta_1$ be the variation in the first step, $\delta_2$ be the variation in the second step. Then

\[ \delta_1 Q_+ \geq \frac{1}{2} B_{\min} |AC||CC_t| + \frac{1}{2} B_{\min} |BC||CC'| \] (1)

\[ \delta_1 Q_- \geq \delta_1 Q_{AD} + \delta_1 Q_{BD} + B_{\min} \cdot \text{area of the triangle } ADD_t + + B_{\min} \cdot \text{area of the triangle } BDD' \] (2)

\[ \delta_2 Q \geq -\frac{NL}{2} B_{\max} \delta x \] (3)

We shall use the following estimates:

1) $\varphi_1 + \varphi_2 < N\alpha_0/2$, $\varphi_1 < N\alpha_0/2$, $\varphi_2 < N\alpha_0/2$, $\cos \varphi_1 > 0.9$, $\cos \varphi_2 > 0.9$, $\tan \varphi_1 < \frac{9B_{\min}}{20B_{\max}(2N^3 + N/2)}$, $\tan \varphi_2 < \frac{9B_{\min}}{20B_{\max}(2N^3 + N/2)}$. 

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2) The angles between the $x$-axis and all the segments of the arc $AB$ are less than $N\alpha_0/2$, their cosines are greater than 0.9.

3) $|AB| > 1.8L$, $|A\hat{C}| > 0.4L$, $|A\hat{D}| > 0.4L$, $|B\hat{C}| > 0.4L$, $|B\hat{D}| > 0.4L$, $|AC| > 0.4L$, $|AD| > 0.4L$, $|BC| > 0.4L$, $|BD| > 0.4L$.

4) $|D\hat{D}| > L/2N$.

The estimates 1) - 3) follow directly from the definition of $\alpha_0$. Let us prove the estimate 4).

Due to the Lemma 6 it follows that if $\gamma$ is an extremal then at least one of the arcs $AD$ or $BD$ has no internal angles less than $\pi/3$. For the sake of concreteness let us assume that the arc $AD$ has this property. Then there are two possibilities.

1) Moving from the point $A$ to the point $D$ along the arc $AD$ we always move right. Let us denote $Q$ the neighboring vertex to $A$ in the arc $AD$. The angle between the $x$-axes and $AQ$ is greater than $\pi/3 - N\alpha_0/2 > \pi/3 - 0.001$.

2) Moving from the point $A$ to the point $D$ along the arc $AD$ we move left and then right. Let $T$ be turning vertex, $P$ and $Q$ be the preceding and the succeeding ones. Then the projection of the interval $PQ$ to the $y$-axes is greater than $\sqrt{3}L/2$.

In the both cases the distance between $Q$ and the line $AB$ is greater then $(\sqrt{3}/2 - 0.001)L$. Let us connect $Q$ with $B$ and denote the crossing point of the intervals $QB$ and $D\hat{D}$ by $S$. $|QB| < NL/2$, $|SB| > 0.4$ then

$$|D\hat{D}| > |SD| > \frac{|SB|}{|QB|} \left(\frac{\sqrt{3}}{2} - 0.001\right) L > \frac{0.4L}{NL/2} \left(\frac{\sqrt{3}}{2} - 0.001\right) L > \frac{L}{2N}.$$
For the infinitesimal intervals $CC_t$, $CC''$ we have:

$$|CC_t| = |AC|\delta \varphi_1 = \frac{1}{\cos \varphi_1} \cdot \frac{1}{\tan \varphi_1 + \tan \varphi_2} \delta x > \frac{1}{\tan \varphi_1 + \tan \varphi_2} \delta x,$$

thus

$$|CC_t| > \frac{10B_{max}(2N^3 + N/2)}{9B_{min}}\delta x, \ |CC''| > \frac{10B_{max}(2N^3 + N/2)}{9B_{min}}\delta x.$$

$|AC| + |BC| > |A\tilde{C}| + |B\tilde{C}| > 1.8L$, thus

$$\delta_1 Q_+ > 0.9B_{min}L\frac{10B_{max}(2N^3 + N/2)}{9B_{min}}\delta x = (2N^3 + \frac{N}{2})LB_{max}\delta x. \quad (4)$$

$$\delta \varphi_3 = \frac{1}{\cos \varphi_3} \cdot \frac{1}{\tan \varphi_3 + \tan \varphi_4} \cdot \delta x \cdot \frac{1}{\cos \varphi_3} \cdot \frac{\delta x}{|BD|} < \frac{1}{\tan \varphi_3} \cdot \frac{\delta x}{|BD|} = \frac{1}{\sin \varphi_3} \cdot \frac{\delta x}{|BD|} \sin \varphi_3|BD| = |D\tilde{D}| \quad \text{thus}$$

$$\delta \varphi_3 < \frac{2N}{L}\delta x, \ \delta \varphi_4 < \frac{2N}{L}\delta x.$$

Applying the Lemma 5 and taking into account that the magnetic areas of the triangles $ADD_t$ and $BDD'$ are positive we get

$$\delta_1 Q_- \geq -2N^3LB_{max}\delta x. \quad (5)$$

Combining (3), (4),(5) we get:

$$\delta Q = \delta_1 Q_+ + \delta_1 Q_- + \delta_2 Q > 0.$$  

This completes the proof.
Proof of the Lemma 12. To estimate the variation of the magnetic area $Q$ under this deformation it is convenient to decompose this deformation in two steps.

Step 1:

We shift the arc $BC$ up at the distance $\delta y_1$, the arc $FG$ down at the distance $\delta y_2$ and we rotate the segments $AB$, $CD$, $EF$, $GH$ over the points $A$, $D$, $E$, $H$ at the angles

$$
\delta \varphi_1 = \frac{\delta y_1}{L \cos \varphi_1}, \quad \delta \varphi_2 = \frac{\delta y_1}{L \cos \varphi_2}, \quad \delta \varphi_3 = \frac{\delta y_2}{L \cos \varphi_3}, \quad \delta \varphi_4 = \frac{\delta y_2}{L \cos \varphi_4},
$$

where $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ are the angles between the real line and the segments $AB$, $CD$, $EF$, $GH$, $|\tan \varphi_k| \leq 1$, $k = 1, \ldots, 4$. Let us denote the images of the points $B$, $C$, $F$, $G$ under the shifts $B'$, $C'$, $F'$, $G'$, the images under the rotations $B_1$, $C''$, $F''$, $G_1$. 

Fig 6.
Step 2:

We shift the arc $B'C'$ left at the distance $\delta x_1 = L \sin \varphi_1 \delta \varphi_1 = \tan \varphi_1 \delta y_1$
the arc $F'G'$ left at the distance $\delta x_2 = L \sin \varphi_4 \delta \varphi_4 = \tan \varphi_4 \delta y_2$, the arc $C''DEF''$ left at the distance

$$\delta x = \delta y_1 (\tan \varphi_1 + \tan \varphi_2) = \delta y_2 (\tan \varphi_3 + \tan \varphi_4).$$

(6)

From (6) it follows that $\delta y_1 = t/(\tan \varphi_1 + \tan \varphi_2)$, $\delta y_2 = t/(\tan \varphi_3 + \tan \varphi_4)$.

Let us denote $BC_x$ and $FG_x$ the lengths of the projections of the arcs $BC$ and $FG$ to the $x$-axes.

For the change of magnetic area in the step 1 we have

$$\delta_1 Q \geq B_{\min} (\delta y_1 BC_x + \delta y_2 FG_x) = B_{\min} \left[ \frac{t BC_x}{\tan \varphi_1 + \tan \varphi_2} + \frac{t FG_x}{\tan \varphi_3 + \tan \varphi_4} \right]$$

thus

$$\delta_1 Q \geq B_{\min} t \min(BC_x, FG_x)$$

The angles between the $x$-axes and all segments of the arcs $AH$ and $DE$ are greater then $\pi/4$ thus the projection of these arcs to the $x$-axes are smaller then $D_{\min}$ and

$$\min(BC_x, FG_x) \geq D_x - 2D_{\min} - 2L,$$
where $D_x$ denotes the diameter of $\gamma$ in the direction $x$. It is easy to show that

$$D_x \geq D_{\text{max}} - D_{\text{min}}, \quad L \leq \frac{4}{N} D_{\text{max}}.$$

Thus

$$\delta_1 Q \geq B_{\text{min}} t \left[ \left( 1 - \frac{8}{N} \right) D_{\text{max}} - 3D_{\text{min}} \right].$$

For the variation of the magnetic area in the second step we have

$$\delta_2 Q \geq -B_{\text{max}} D_{\text{min}} t.$$

Thus

$$\delta Q = \delta_1 Q + \delta_2 Q \geq \left\{ B_{\text{min}} \left[ \left( 1 - \frac{8}{N} \right) D_{\text{max}} - 3D_{\text{min}} \right] - B_{\text{max}} D_{\text{min}} \right\} t.$$

If

$$\frac{D_{\text{max}}}{D_{\text{min}}} > \frac{B_{\text{max}} B_{\text{min}}^{-1} + 3}{1 - 8N^{-1}}$$

then

$$\delta Q \geq \left\{ B_{\text{min}} \left[ B_{\text{max}} B_{\text{min}}^{-1} D_{\text{max}} + 3D_{\text{min}} - 3D_{\text{min}} \right] - B_{\text{max}} D_{\text{min}} \right\} t > 0.$$

Lemma 12 is proved.

Proof of the Lemma 13.

The variation of the magnetic area under this deformation consists of two parts:

1) The magnetic area of the small triangles near all the vertices except $A$ and $B$ (we shall denote it by $\delta_1 Q$).
2) The magnetic area of the small quadrangle $AP_1A_tQ_t$ near the vertex $A$. Here we denote the neighbors of $A$ by $P$ and $Q$, the shifts of the points $A$, $P$, $Q$ by $A_t$, $P_t$, $Q_t$, the magnetic area of the quadrangle by $\delta_2Q$.

We use the following estimate. Let $F$ be a vertex of our polygon, the arc $BF$ contains $k$ segments. Then the shift of the vertex $F$ under the deformation $FF_t$ is less than $kt$. It is easy to prove it by induction.

![Fig 9.](image)

Let $E$ be the neighbor of $F$ on the arc $BF$, $\alpha_E$ be the external angle in the vertex $E$, $E_t$ be the shift of $E$. Then

$$|FF_t| = |EE_t| \cos \alpha_E + t < |EE_t| + t,$$

Then the shifts of all the vertices except $A$ are less than $Nt/2$. For the area of the triangle near the vertex $E$ we have

$$\text{area of the triangle } E_tEF = \frac{1}{2}|EE_t||EF_t| \sin \alpha_E < \frac{1}{2}NLt\alpha_E < \frac{L_0}{2}t\alpha_E.$$

The sum of all external angles is equal to $2\pi$ thus the sum of the areas of all triangles is less then $\pi L_0 t$ and the magnetic area of these triangles is less then $B_{\max} \pi L_0 t$.

The distances $|AA_t|$ and $|P_tQ_t|$ can be estimated by

$$|AA_t| \leq 2\frac{Nt}{2} \frac{1}{\sin \alpha}, \quad |P_tQ_t| < 2\frac{L_0}{N},$$

where $\alpha$ is the external angle in the vertex $A$. Then for the area of the quadrangle $AP_1A_tQ_t$ we have

$$\text{area of the quadrangle } AP_1A_tQ_t \leq \frac{1}{2}|P_tQ_t||AA_t| < L_0 t (\sin \alpha)^{-1}.$$

Finally we get

$$\delta F = \delta l(\gamma) - \delta Q < -Nt + B_{\max} \pi L_0 t + L_0 B_{\max} t (\sin \alpha)^{-1}.$$
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