Platonicons: the Platonic solids start rolling

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Abstract

We describe the construction of a new family of developable rollers based on the Platonic solids. In this way kinetic sculptures may be realised, with the Platonic solids quite literally in their heart. We also describe the strong way in which the Platonicons circumscribe the Platonic solids.

A new family of Platonic developable rollers

Recently we announced the construction and some properties of a new family of solids, the polycons [4], which generalise the sphericon [1, 6]. Over the course of their motion on a plane, rolling in an amusing manner, their entire surface makes contact with it. The polycons are based on regular polygons, and are named for the multiple pieces of identical cones which comprise their surface. In principle, there are infinitely many polycons, as many as there are regular polygons. The polycons are examples of what we termed developable rollers, defined fully in [4].

In this paper we introduce another family of developable rollers discovered by David Hirsch in 2017, which are based on the fundamental Platonic solids, and which we call collectively the Platonicons. Though the number of Platonic solids is only five, there are more than five members of this family, though not infinitely many. Figure 1 shows a Platonicon, based on the tetrahedron.

Figure 1: The first Platonicon discovered by Hirsch, a Tetrahedcon (A). 3D printed, 10×8.5×10 cm.

From antiquity, the Platonic solids have enchanted artists and mathematicians. We hope that seeing them give rise to kinetic, rolling objects will inspire the artists and mathematicians among our readers.

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Constructing the Platonicons

Pieces of cones

Take any of the Platonic solids, with its vertices, edges and faces. The associated Platonicon is made up of the Platonic solid enhanced with identical modules ‘glued’ onto its faces. Each module is made from pieces of two or three identical cones; the vertex angle of the cones is in each case the dihedral angle of the dual Platonic solid (see Table 1). This angle choice gives the correct symmetries to result in an object that will roll smoothly with its centre of mass maintaining constant height.

Table 1: Platonic Solid Properties.

| Solid           | Vertices | Faces | Shape of face | Dual solid   | Dihedral angle |
|-----------------|----------|-------|---------------|--------------|----------------|
| Tetrahedron     | 4        | 4     | triangle      | self-dual    | arccos \(\frac{1}{3}\) |
| Cube            | 8        | 6     | square        | octahedron   | \(\frac{\pi}{2}\) |
| Octahedron      | 6        | 8     | triangle      | cube         | \(\pi - \arccos \left(\frac{1}{3}\right)\) |
| Dodecahedron    | 20       | 12    | pentagon      | icosahedron  | \(\pi - \arctan(2)\) |
| Icosahedron     | 12       | 20    | triangle      | dodecahedron | \(\pi - \arccos \left(\frac{\sqrt{5}}{3}\right)\) |

To describe the construction of the modules, we distinguish one face of the solid and one vertex of this face. The modules are created using the following steps:

I Construct a right circular cone of the specified vertex angle such that its apex coincides with the distinguished vertex of the solid, and such that the adjacent vertices of the solid (i.e., connected to the chosen vertex by edges) lie on the cone. The edges of the solid connecting these vertices to the distinguished vertex are generators of the surface of the cone. We require only the part of this cone that lies above the distinguished face.

II In this step, a second identical cone is constructed, with its position determined by the nature of the distinguished face, as shown in Figure 2.

- If the face is a triangle, this second cone has its apex at one of the vertices adjacent to the distinguished vertex. The edge connecting the vertices lies on both cones, and their curved surfaces cut each other in a conic section that lies above the altitude of the triangle that contains the third vertex. The volume above the distinguished face and common to both cones forms the required module.
- For the square face of the cube, this second cone is placed so that its apex lies on the vertex diagonally opposite the distinguished vertex. The two cones cut each other in a conic section, lying above the other diagonal of the square. Again, the volume above the distinguished face and common to both cones forms the required module.
- Above the pentagonal face of the dodecahedron, not one but two further cones are constructed, with their apices at the two vertices which lie on the edge of the pentagon opposite to the distinguished vertex. The volume retained to form the module is common to the three cones, and lies above the pentagon; there are three conic section edges associated with each module.

III Construct \(n - 1\) more identical modules, where \(n\) is the number of faces of the platonic solid.

Orienting the modules to form the Platonicons

To create an object that can develop its entire surface as it rolls, the \(n\) modules are constructed on the faces of the Platonic solid in such a way that the Platonicon has a single sinuous surface. Although there are 3, 4
or 5 orientations for each module (depending on the underlying polygon), not all of these result in distinct solids (up to symmetry and chirality), and not all are consistent with the rolling condition. In line with the naming of the underlying Platonic solid, we call these the tetrahedcon, cubicon, octahedcon, icosahecon and dodecahedcon. Five octahedcons, three dodecahedcons and two of each of the other Platonicons have been identified (to date). Animated images of some of these are available to view [2].

**Platonicon properties**

**Vertices and edges**

A curved conic section edge of a Platonicon may end at a vertex which coincides with a vertex of the underlying Platonic solid, or it may be distinct, or in some cases it merges into the surface of the Platonicon. Three of the octahedcons discovered do not have an axis of rotational symmetry, but the properties of all other Platonicons are summarised in Table 2. Note that when two conic section edges join each other to form a single edge, this reduces the number of distinct edges.

**Table 2**: Properties of the rotationally symmetric Platonicons.

| Conic sections | Tetrahedcon | Cubicon | Octahedcon | Dodecahedcon | Icosahedcon |
|----------------|-------------|---------|------------|--------------|-------------|
| Designation    | elliptic    | parabolic | parabolic  | elliptic and hyperbolic | elliptic    |
| Edges          | 2 A 4 B     | 6 A B   | 8 A B      | 36 A B C      | 10 A B      |
| Vertices       | 4 A 4 B     | 8 A B   | 6 A 6 B    | 32 A 32 B     | 12 A 12 B   |

For generalised sphericons, Phillips [5] investigated a connection with mazes. The edges of the Platonicons create even more complex mazes which warrant further investigation in future.

**Circumscribing the Platonic solids**

Generally when one speaks of a Platonic solid being circumscribed, one thinks of the solid being inside a hollow sphere with only the vertices of the solid touching the sphere’s surface, à la Kepler’s model of the planets. But a cylinder may also be circumscribed by a sphere: the two circular edges of the cylinder are latitudes (small circles) on the surface of the sphere. The Platonicons circumscribe the Platonic solids in both ways. Not only are the vertices of the Platonic solid part of the Platonicon surface, but so too are its edges; if a circumscribing sphere is imagined to kiss the Platonic solid, the Platonicon hugs it!

**Rolling behaviour**

Like the other members of their extended family the developable rollers, the Platonicons roll smoothly but with meanderings in their motion, as first one and then another part of a cone’s surface comes in contact with the surface upon which the rolling is taking place.

However, some of the Platonicons, those described above as having a vertex that merges into their surface, exhibit a rolling behaviour not previously seen among the developable rollers. Rather than “knowing” which
overall direction to move in, these Platonicons can exhibit different modes of rolling. They may or may not develop their whole surface when they roll. They can be gently encouraged to exhibit these behaviours by choosing the slope of the surface upon which they are rolling. Compare the motion of the cubicons which follow a predefined and unchangeable overall path, with those of the tetrahedcons or octahedcons [3].

Summary and Conclusions

We have introduced the Platonicons in this short paper, but we have only been able to touch on some of their features. We intend to continue to investigate these objects and their properties, and report on them. For instance, an obvious comparison to make will be that of the surface area and volume of each of the circumscribing Platonicons to the surface area and volume of the sphere that circumscribes the same Platonic solid in the customary (vertex only) way. The connection to mazes will be fascinating.

As static objects, the Platonic solids have long inspired artists and mathematicians. We have shown in this paper that in the embrace of the Platonicons they can also dance.

Acknowledgements

The underlying polyhedra in Figure 2 are based on those of https://en.wikipedia.org/wiki/User:Cyp.

Recently, Daniel Walsh made us aware of his construction of ‘super advanced sphericons’ [7], which he also terms polyhedricons (various hexahedricons and the icosahedricon). There is potential to explore the similarities and differences between these and the Platonicons in consultation with him in future.

References

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