The Nambu–Jona-Lasinio Model of QCD on the Lattice

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FSU-SCRI-93-127
October 16, 2018

Abstract

In an effort to investigate some of the low energy properties of QCD, in particular those related to chiral symmetry breaking, as well as to obtain insights on the behavior of an interacting theory of fermions on the lattice, the two flavor Nambu–Jona-Lasinio model with $SU(2) \times SU(2)$ chiral symmetry is studied on the four–dimensional hypercubic lattice using large $N$ techniques and numerical simulations. Naive and Wilson fermions are considered and transparent results are obtained regarding the following: the scalar and pseudoscalar spectrum, the approach to the continuum and chiral limits, the size of the $1/N$ corrections, and the effects of the zero momentum fermionic modes on finite lattices. Also, some interesting observations are made by viewing the model as an embedding theory of the Higgs sector.
When the high frequency modes of the gauge and fermionic fields of QCD are integrated down to the energy scale $E$ corresponding the correlation length of the gauge field ($E \approx \text{glueball mass} \approx 1550 \text{ MeV}$) the resulting effective theory will essentially be a theory of fermions with contact interactions and cutoff $\Lambda \lesssim 1550 \text{ MeV}$. The resulting effective Lagrangian will maintain the original chiral symmetry but will be more complicated. At energies much smaller than $\Lambda$ it should be enough to keep in the Lagrangian the least irrelevant operator, namely the four-Fermi dimension six operator. This is one way to motivate the Nambu–Jona-Lasinio (NJL) model.

Unfortunately, by only keeping the four-Fermi operator, valuable information was lost and the NJL model does not confine the quarks. Furthermore, if, for example, we want to study the $\sigma$ particle, which on phenomenological grounds is believed to have mass $\approx 750 \text{ MeV}$, then the separation of scales is probably not large enough to justify the neglect of operators with dimension higher than six. Nevertheless, the NJL model possesses the same chiral symmetry as QCD and it can realize this symmetry in the Goldstone mode. It is this feature that is most crucial in the understanding of the lightest hadrons. Furthermore, our interest in the model is not so much aimed at its quantitative predictability but rather on the qualitative insights that can provide as a low energy theory of QCD, as an embedding theory of the Scalar Sector of the Minimal Standard Model and as a four-dimensional interacting theory of fermions on the lattice.

The NJL model has been studied extensively for various cases with continuum type regularizations. For a comprehensive review the reader is referred to and references therein. The model has also been studied on the lattice in connection with the possible equivalence of the top quark condensate with the Higgs field. In that work, however, the separation of scales is very large (the cutoff is of the order $10^{14} \text{ GeV}$), and it is therefore quite a different problem than the one considered here.

In this paper we consider the two flavor (up and down) NJL model with $SU(2) \times SU(2)$ chiral symmetry and $SU(N)$ color symmetry, with scalar and pseudoscalar couplings on the four-dimensional hypercubic lattice. We consider both naive and Wilson fermions and we study the model using a large $N$ expansion as well as a Hybrid Monte Carlo (HMC) numerical simulation. In this paper we will only present the main results. A detailed version of this work will appear elsewhere.

The Lagrangian density in Minkowski space and in continuum notation is:

$$\mathcal{L} = \overline{\Psi}(i\partial - m_0)\Psi + \frac{G_1}{2} \left[ (\overline{\Psi}\Psi)^2 + (\overline{\Psi}i\gamma_5 \tau \Psi)^2 \right].$$

(1)

In the above expression all indices have been suppressed. The fermionic field $\Psi$ is a flavor $SU(2)$ doublet and a color $SU(N)$ $N$-column vector. The Lagrangian is diagonal in color, in contrast with the full QCD Lagrangian which is diagonal in flavor. $\tau = \{\tau_1, \tau_2, \tau_3\}$ are the three isospin Pauli matrices, $\partial = \gamma^\mu \partial_\mu$, and $m_0$ is the bare quark mass (if $m_0 \neq 0$ the chiral symmetry is explicitly broken). To obtain a Lagrangian that is quadratic in the fermionic fields we introduce the scalar auxiliary field $\sigma$ and the three pseudoscalar auxiliary fields $\pi = \{\pi_1, \pi_2, \pi_3\}$.

Because of the chiral couplings the fermionic determinant has a phase. This phase is related to the chiral anomaly and the Wess–Zumino term and we will not consider it in this work. Going to Euclidean space and appropriately discretizing the above Lagrangian
we obtain the model on the hypercubic lattice. On the lattice, as it is well known, we have species doubling. The doubling in the NJL model will be interpreted as a doubling of the color degrees of freedom. To treat this problem we add to the Lagrangian density an irrelevant operator (Wilson term) of the form \( \frac{aq^2}{2} \Psi \partial^2 \Psi \), \( a \) being the lattice spacing and \( r \) a constant. We consider the \( r = 0 \) case where no effort is made to remove the doublers (naive fermions) and also the \( r \neq 0 \) case where the doubler masses are raised to the cutoff and the chiral symmetry is explicitly broken (Wilson fermions). With these considerations and after appropriate scaling of the fields and couplings, so that only dimensionless quantities appear, we obtain:

\[
Z = \int \left[ d\Psi d\bar{\Psi} d\sigma d\pi \right] e^{-S}
\]

\[
S = \sum_{x,y} \left\{ \sum_{i=1}^{N/2} \left( \Psi_x^i M_{xy} \Psi_y^i + \bar{\Psi}_x^{i+N/2} M_{xy}^\dagger \Psi_y^{i+N/2} \right) \right\} + n_f \beta_1 (\sigma_x^2 + \pi_x^2) \delta_{xy}
\]

\[
M_{xy} = \frac{1}{2} \sum_\mu \left[ (\gamma_\mu - r) \delta_{x+\mu,y} - (\gamma_\mu + r) \delta_{x-\mu,y} \right] + (4r + m_0 + \sigma_x + i\gamma_5 \pi_x \cdot \tau) \delta_{xy}
\]  

(2)

with \( \gamma_\mu \) hermitian, \( n_f = 2 \) the number of flavors and \( \beta_1 = \frac{1}{2n_f G} \). For \( r = 0 \) and \( m_0 = 0 \) the symmetry is not explicitly broken. In that case at \( \beta_1 = \beta_1_c \) the model undergoes a second order phase transition from a symmetric phase with \( \langle \sigma \rangle = 0 \), massive \( \pi \) and \( \sigma \) fields and massless quarks, to a phase with spontaneously broken symmetry with \( \langle \sigma \rangle \neq 0 \), massless pion fields (Goldstone particles) and massive \( \sigma \) and quark fields with dynamically generated quark mass equal to \( \langle \sigma \rangle \). In this work we will always stay in the broken phase.

We study the above action using large \( N \) techniques and obtain analytical results both on finite and infinite volumes. The infinite volume results are obtained sufficiently close to the continuum limit using asymptotic expansions. We also study this action for \( N = 2 \) using an HMC numerical simulation with Conjugate Gradient and leap–frogs algorithms. The seven important results that stem from our analysis are presented below.

I

For naive fermions, at large \( N \) and with pion mass \( M_\pi = 140 \) MeV, we calculate the \( \sigma \) mass \( (M_\sigma) \), the \( \sigma \) width \( (\Gamma_\sigma) \) and the quark mass \( (M_q) \) in physical units as functions of the \( \sigma \) mass \( m_\sigma \) in lattice units. The results are given in Figure 1 in units of the pion decay constant \( F_\pi = 93 \) MeV. By setting \( M_q = 310 \) MeV we find \( M_\sigma = 726 \) MeV, \( \Gamma_\sigma = 135 \) MeV, and \( \Lambda = \pi/a = 1150 \) MeV (dotted vertical line). \( M_\sigma \) is consistent with phenomenological expectations and \( \Lambda \) is consistent with the expectation that the cutoff should be close and below the mass of the lightest glueball (1550 MeV). The width however is underestimated. The reason is traced to the fact that to leading order in large \( N \) the width receives contributions only from the quark bubble and not from the pion bubble because the quark bubble is of order \( 1/N \). Because the phase space available for the \( \sigma \) to decay to two quarks is much smaller than the phase space to decay to two pions, the pion loop contribution, although of order \( 1/N \), is probably more important than the quark loop contribution.

II

The above result is relevant not only for the low energy QCD but also for the Higgs sector. It is well known that there is an equivalence between the \( \sigma-\pi \) sector of QCD with
the scalar sector of the Minimal Standard Model. In the former the scale is set by the pion decay constant \( F_\pi = 93 \text{ MeV} \) and in the latter by the weak scale \( F_\pi = 246 \text{ GeV} \). As mentioned in I we find that in accordance with phenomenological expectations in the \( \sigma - \pi \) QCD sector \( M_\sigma / F_\pi \approx 8 \) but in the Higgs sector all previous analysis predicts a triviality bound of the Higgs mass with \( M_\sigma / F_\pi \approx 2.8 \) (see for example [10]). In the past this has been a reason for concern since it could imply that the Higgs mass bound may be underestimated. Our analysis suggests that this apparent discrepancy appears because the Higgs mass bounds are traditionally obtained for \( m_\sigma \lesssim 0.5 \) while the \( M_\sigma / F_\pi \approx 8 \) ratio is obtained for \( m_\sigma \approx 2 \) and it should therefore be accompanied by large deviations from the low energy behavior. Nevertheless, this is only a suggestion since we have not calculated deviations from the low energy behavior of a physical process that would enable us to make exact statements. However, the value of the width serves as an indication of the size of such deviations. In a way, departure from low energy behavior will be signaled by an increasing width of the \( \sigma \) to two quark decay. At \( m_\sigma \approx 2 \) the width is already fairly large.

Figure 1. \( r = 0, \frac{M_\sigma}{F_\pi} = \frac{140 \text{ MeV}}{93 \text{ MeV}} \) and the number of colors is set equal to three. From top to bottom the lines correspond to the \( \sigma \) mass, quark mass and \( \sigma \) width calculated at large \( N \) and to leading order in \( m_\sigma \). The vertical line denotes the point where \( \frac{M_\sigma}{F_\pi} = \frac{310 \text{ MeV}}{93 \text{ MeV}} \).

III

If the Higgs sector is the low energy effective field theory of a NJL model with \( N_c = 3, n_f = 2 \) and \( M_\pi = 0 \) (the pions should now be viewed as the would-be Goldstone Bosons),
we obtain for the larger values of \( m_\sigma \) a figure that is almost exactly the same as Figure 1 with \((M_\sigma)\), \((\Gamma_\sigma)\) and \((M_q)\) measured in units of \( F_\pi = 246 \text{ GeV} \). Strictly speaking such a figure does not contain any physical predictions but there is a very interesting observation that can be made.

If we set the fermion mass to \( \frac{M_q}{F_\pi} \approx \frac{310}{93} \), as is the case for the low energy sector of QCD, then the Higgs mass will be \( M_\sigma = 1915 \text{ GeV} \). This point will correspond to \( m_\sigma = 2 \) as denoted by the dotted vertical line in Figure 1. As discussed in II, at \( m_\sigma = 2 \) one would expect very large deviations from the low energy behavior of scattering cross sections. This suggests that as the CM energy is turned up first the deviations from the low energy behavior will become large, signaling the onset of new physics, and later the Higgs particle would be observed.

**Figure 2.** The \( \beta_1, \kappa = 1/(8 + 2m_0) \) plane for \( r = 1, N = 2 \). The solid lines are constant \( m_\pi \) lines. From top to bottom they correspond to \( m_\pi = 0, 0.1, 0.2, 0.3 \). The dotted lines are constant \( m_q \) lines. From right to left they correspond to \( m_q = -0.2, -0.1, 0.0, 0.1, 0.2, 0.3 \). The \( m_q = 0, m_\pi = 0 \) point is located at \( \beta_{\text{chiral}} = 0.3416, \kappa_{\text{chiral}} = 0.3994 \).

**IV**

With Wilson fermions we obtain at large \( N \) analytical expressions of the pion mass \( (m_\pi) \) and quark mass \( (m_q) \) in lattice units as functions of the bare parameters of the model. We are then able to make exact statements regarding the approach to the continuum and chiral limits. For small \( m_q \) and \( m_\pi \) the constant \( m_q \) and \( m_\pi \) lines including the \( m_q = 0 \) and \( m_\pi = 0 \)
lines are presented in Figure 2. The point $\beta_{1\text{chiral}}, \kappa_{\text{chiral}}$, where the $m_q = 0$ and $m_\pi = 0$ lines intersect, is denoted by a circle and corresponds to the true chiral limit.

From this figure we see that if for a fixed $\beta_1$ we were to change $\kappa = \frac{1}{s_{r-m_0}}$ from smaller to larger values (as is often done in QCD with dynamical Wilson fermions) then if $\beta_1 < \beta_{1\text{chiral}}$ we would reach the $m_\pi = 0$ limit before we reach the continuum limit. On the other hand if $\beta_1 > \beta_{1\text{chiral}}$ we would reach the continuum limit before we reach the $m_\pi = 0$ limit. As mentioned above there is only one point in the $\beta_1, \kappa$ plane where we can obtain a true chiral limit. This may provide an insight on how the retrieval of the chiral limit is achieved in QCD.

**Figure 3.** $r = 0, m_0 = 0, N = 2$. The diamonds are the values of $m_q$ and the crosses are the values of $m'_q$ from the numerical simulation for $L_x = 8, L_t = 16$. The solid lines are the large $N$ numbers with the zero mode included. From right to left they correspond to $(L_x = 8, L_t = 16), (L_x = 16, L_t = 16), (L_x = 32, L_t = 32), (L_x = 64, L_t = 64)$. The dotted lines are the large $N$ numbers with the zero mode excluded. From right to left they correspond to $(L_x = 16, L_t = 16), (L_x = 8, L_t = 16)$. The solid vertical line denotes the infinite volume $\beta_{1c}$ from large $N$. For $\beta_1 \leq \beta_{1c}$ the model is in the broken phase.

In Figure 2 the constant $m_\pi$ lines were drawn under the assumption that $m^2_\pi \ll 4m^2_q$ which can not be satisfied on the $m_q = 0$ line, except on the one point where it intersects the $m_\pi = 0$ line. The constant $m_\pi$ lines are therefore valid only in the regions where $m^2_\pi \ll 4m^2_q$. 
At large $N$ and for Wilson fermions the $\sigma$ particle has mass proportional to the cutoff. Our analysis traces this fact to two related reasons. First, although the Wilson term has raised the masses of the doublers to the cutoff, it has not decoupled them from the theory. Through vacuum polarization these contribute to the $\sigma$ self energy and raise its mass. Second, although the Wilson term has not altered the low frequency behavior of the propagating quark, it has however altered its high frequency behavior. Again through vacuum polarization the high frequency modes contribute to the $\sigma$ self energy and also raise its mass. Such a phenomenon may also be responsible for the difficulty in observing the $\sigma$ particle in numerical simulations of QCD with dynamical Wilson fermions.

**Figure 4.** a) The $\sigma$ propagator in momentum space for 10 small momenta with $r = 0$, $m_0 = 0$, $\langle \sigma \rangle = 0.4840$, $N = 2$, $L_x = 16$, $L_t = 16$. The crosses are the values from the numerical simulation. The diamonds are the large $N$ results. b) Same as in a but for $\langle \sigma \rangle = 0.35$. c) Same as in a but for the pion propagator. d) Same as in b but for the pion propagator.

The numerical and the leading order large $N$ results are in good agreement, indicating that the $1/N$ corrections are small for the quantities we were able to measure. The first indication that the model at $N = 2$ has same type of behavior as at $N = \infty$ appears in
Figure 3. There we plot \( m_q \) and \( m'_q = \{-N \langle \bar{\Psi} \Psi \rangle / (2\beta_1)\} + m_0 \) vs. \( \beta_1 \) as determined from the numerical simulation. At large \( N \) the gap equation gives \( m_q = m'_q \). This relation is satisfied nicely. Furthermore, the agreement with large \( N \) of dynamically determined quantities vs. bare quantities is also quite good and helps us to get oriented in the bare parameter space. Figure 3 is an example of this. Some more results that demonstrate this agreement are presented in [8].

\[ m_q \quad \text{vs.} \quad \beta_1 \]

The important comparison with large \( N \) that will help us get a feel for the size of the \( 1/N \) corrections comes from comparisons of dynamically determined quantities vs. other dynamically determined quantities. In particular we exchange one of the bare parameters for \( \langle \sigma \rangle \). In Figure 4 we present the \( \sigma \) and \( \pi \) propagators in momentum space for ten small momenta and \( r = 0 \). It is from these figures that we would have to extract the pion wave function renormalization constant \( Z_\pi \). As it can be seen the large \( N \) prediction for the same \( \langle \sigma \rangle \) as the one measured in the simulation is in good agreement with the numerical results. The large \( N \) predictions in Figures 4a, 4b, 4c, and 4d “fit” the numerical results with \( \chi^2 \) per degree of freedom 0.38, 0.67, 0.32, 0.42 respectively. This means that the pion decay constant in lattice units \( f_\pi = \langle \sigma \rangle \sqrt{\frac{Z_\pi}{\sigma}} \) [6] as a function of \( \langle \sigma \rangle \) has very small \( 1/N \) corrections. Another dynamically determined quantity that agrees very well with the large \( N \) prediction when plotted vs. \( \langle \sigma \rangle \) is the pion mass \( m_\pi \). This plot is shown in Figure 5.

\[ \langle \sigma \rangle \quad \text{vs.} \quad m_\pi \]

Figure 5. \( r = 1, N = 2, L_x = 8, L_t = 16. \) The crosses are the MC data. The solid line is the large \( N \) prediction on the same size lattice.
This figure suggests that the $1/N$ corrections to $m_\pi$ are very small.

At large $N$ we find that on a finite lattice $m_q$ can become zero only if $\beta_1$ becomes infinite (free field). On the other hand we find that on a finite lattice $m_\pi$ can be made to vanish by adjusting the bare parameter $m_0$. Indeed we can see from Figure 5 that $m_\pi$ can be made very small. In fact it turns out that the value of $m_0$ where this happens is predicted by large $N$ quite well.

As it has already been discussed in $\text{V}$, for $r \neq 0$ the $\sigma$ mass is of order cutoff and therefore very heavy to be able to measure from the decay of the $\sigma - \sigma$ correlation function. However for $r = 0$ one would expect to be able to measure $m_\sigma$. As we will discuss in $\text{VII}$ below this is not possible with the lattice sizes accessible to us. This is unfortunate since $m_\sigma$ is another very important quantity. However, we expect the size of the $1/N$ corrections of $m_\sigma$ to be similar to the ones of $m_\pi$ and therefore very small. Also, we did not measure the $\sigma$ width in the numerical simulation, but, as discussed in $\text{I}$, we expect the $1/N$ corrections to be large.

$\text{VII}$

On finite lattices the zero momentum modes of the quarks affect the inversion speed of the CG algorithm and introduce finite size effects. To leading order at large $N$ the matrix $M^\dagger M$ of eq. 2 is diagonal in momentum, spin, flavor, and color spaces. The smallest eigenvalue of this matrix is $m_q^2$ and corresponds to the $p = 0$ matrix element. This in turn corresponds to the zero momentum mode of the quarks which from now on we will simply call “zero modes”.

On a finite lattice the role of the zero modes is crucial in the inversion of $M^\dagger M$. For small $m_q$, the condition number of the matrix is $4/m_q^2$ for $r = 0$ and $64r/m_q^2$ for $r = 1$ and it is clear that it depends strongly on the presence of the zero modes. A large condition number will make the inversion of $M^\dagger M$ very slow. Furthermore the spacing of the smallest eigenvalues behaves like $1/L^2$ and for larger lattices the inversion times will rapidly get worse. An important observation can be made by noticing the dependence of the condition number on $r$. This suggests that performing the simulation with smaller $r$ will yield a quite faster inversion. It is possible that this may also be the case for QCD.

But the unwelcomed effect of the zero modes on a finite lattice is not limited to large inversion times. Because on a finite lattice their effects are not suppressed by the measure but instead by an inverse volume factor, it turns out that in certain cases they severely obscure the physics. By simply looking at the numerical results in Figure 3 we would not only be unable to estimate the critical point but also we would be unable to see any indication of a phase transition. The large $N$ result on the same size lattice also has the same problems. As we increase the lattice size in the large $N$ calculation (solid lines from top to bottom) we see that a picture of an order parameter slowly materializes. At $L_x = 64$, $L_t = 64$ a fairly good prediction of the large $N$ infinite volume critical point is achieved. If we now do the same large $N$ calculation but neglect from the momentum sums the zero modes, we obtain as a result the two dotted lines for $L_x = 8$, $L_t = 16$ and $L_x = 8$, $L_t = 16$ (from left to right). We see that neglecting the zero modes on an $L_x = 8$, $L_t = 16$ lattice gives very similar results as the ones obtained on a $L_x = 64$, $L_t = 64$ lattice with the zero modes included.

As mentioned earlier although one would expect to be able to measure the $\sigma$ mass in the
\( r = 0 \) case we were not able to do so. Large \( N \) provides an explanation of this unexpected problem. If we plot the real part of the inverse \( \sigma \) propagator for a finite lattice and set the external momentum to \( q = \{ im_\sigma, 0, 0, 0 \} \) we can obtain the sigma mass at the zero of this function. We find that because of the presence of a discontinuity we do not obtain a root at all. A root will eventually be obtained but not until \( m_\sigma \) becomes very heavy [8]. The presence of this discontinuity is again due to the zero modes.

There are some very interesting issues relevant to lattice work that have not been considered in this paper. It would be important to calculate the three and four point vertices and therefore be able to calculate scattering amplitudes and their departure from low energy behavior as well as the \( 1/N \) corrections to the width. It would also be interesting to study the NJL model at finite temperature and investigate the finite temperature transition in connection with the approach to the chiral limit. Finally it would be important to include vector meson couplings (see, for example, [6], [9]) and confirm that the vector meson masses scale appropriately and do not become of the order cutoff as the \( \sigma \) particle does.

Acknowledgments

We would like to thank U.M. Heller and R. Edwards for useful discussions concerning the subject of this paper. This research was supported in part by the DOE under grant \# DE-FG05-85ER250000 and \# DE-FG05-92ER40742.

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