A tensor interpretation of the 2D Dirac equation

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Abstract

We consider the Dirac equation in flat Minkowski 3–space and rewrite it as the Maxwell equation in Minkowski 4–space with torsion. The torsion tensor is defined as the dual of the electromagnetic vector potential. Our model clearly distinguishes the electron and the positron without resorting to “negative frequencies”: we produce a real scalar invariant (charge) which indicates whether we are looking at an electron or a positron. Another interesting feature of our model is that the free electron and positron are identified with gradient type solutions of the standard (torsion free) Maxwell equation; such solutions have traditionally been disregarded on the grounds of gauge invariance.

1 Introduction

The Dirac equation is the following system of 4 partial differential equations in Minkowski 4–space:

\[
\begin{pmatrix}
  i\nabla_0 - 1 & 0 & i\nabla_3 & i\nabla_1 + \nabla_2 \\
  0 & i\nabla_0 - 1 & i\nabla_1 - \nabla_2 & -i\nabla_3 \\
 -i\nabla_3 & -i\nabla_1 - \nabla_2 & i\nabla_0 - 1 & 0 \\
 -i\nabla_1 + \nabla_2 & i\nabla_3 & 0 & -i\nabla_0 - 1 \\
\end{pmatrix}
\begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \chi_1 \\
  \chi_2 \\
\end{pmatrix} = 0 \quad (1)
\]

where

\[
\nabla = \partial + iA, \quad (2)
\]

\(\partial\) being the operator of partial differentiation and \(A\) the vector potential of the external electromagnetic field (given real valued vector function). Equation (1) is often referred to as the 3D (3–dimensional) Dirac equation, with 3 indicating the number of spatial variables.

The set of complex quantities \(\psi = (\phi_1 \; \phi_2 \; \chi_1 \; \chi_2)^T\) is a bispinor, and the way it behaves under Lorentz transformations of coordinates is quite extraordinary.

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For example, a spatial rotation of the coordinate system by an angle $2\pi$ changes the sign of $\psi$. More important, time reversal leads to complex conjugations resulting in the well known difficulty of distinguishing the electron and the positron (problem of “negative frequencies”). See Sections 18 and 19 in [2] for details.

Our aim is to provide a satisfactory tensor interpretation of (1). We fail to achieve this goal in full, but succeed in handling the simpler case of the 2D Dirac equation. The 2D Dirac equation is a special case of (1) arising when $A$ and $\psi$ do not depend on $x^3$ and $A_3 \equiv 0$.

The essential new elements our mathematical model are as follows.

- The Dirac equation is interpreted as a perturbation of the polarised Maxwell equation

\[ *du = \pm i du \]  

rather than an independent equation in its own right.

- The external field (perturbation) is introduced into the model as a classical real differential geometric connection with torsion

\[ T = *A \]  

rather than by means of the complex formula (2).

- In our model the electron (solution of the perturbed Maxwell equation) is considered simultaneously with the photon (solution of the unperturbed Maxwell equation). Algebraically this seems to be the only way of avoiding the appearance of multiple copies of the Dirac equation as in [1].

- The electron mass is introduced into the model as a prescribed oscillation along the $x^3$ coordinate. Note that the idea of viewing mass in terms of oscillation along an additional space–like coordinate is a classical one, going back to Oskar Klein [3]. The peculiarity of the 2D Dirac equation is that one can use the third spatial coordinate for this purpose, thus avoiding the necessity of introducing a fifth dimension.

The paper has the following structure.

In Section 2 we specify our notation. In Section 3 we define the electron mass. In Section 4 we describe our mathematical model for the photon. Section 5 gives basic geometric facts concerning Minkowski 4–space with torsion. In Section 6 we describe our mathematical model for the electron/positron. In Section 7 we identify a basic symmetry of our model with respect to complex conjugation; in particular, we explain why our tensor model is free of the problem of “negative frequencies”. In Section 8 we state and prove the main result of this paper, Theorem 1 this theorem establishes the equivalence of our tensor model and the 2D Dirac equation. In Section 9 we introduce the notion of a free ($A \equiv 0$) electron/positron. Finally, in Section 10 we define a real scalar invariant (charge) which allows us to distinguish the electron and positron solutions.
2 Principal notation

We work in Minkowski 4–space equipped with coordinates \((x^0, x^1, x^2, x^3)\) and metric \(g_{\mu \nu} = \text{diag}(+1, -1, -1, -1)\). We use Greek letters for tensor indices, with the exception of the letters \(\alpha\) and \(\beta\) which have a special meaning. Tensor indices take the values 0, 1, 2, 3. We denote \(\partial_\mu = \partial / \partial x^\mu\). Our system of units is such that the speed of light \(c\), Planck’s constant \(\hbar\), and the electron mass \(m\) have the value 1. The Dirac equation (1), (2) is written as in [2], Section 21 (standard representation). The only difference is that we have incorporated the electron charge into the vector potential: our \(A\) corresponds to the \(eA\) of [2].

We work with complex valued antisymmetric tensor functions of various ranks. All functions are assumed to be infinitely smooth. We denote complex conjugation with an “overline”.

Given a pair of antisymmetric tensors \(Q\) and \(R\) of the same rank \(q\) we denote \(Q \cdot R := \frac{1}{q!} Q_{\mu_1 \ldots \mu_q} R^{\mu_1 \ldots \mu_q}.\) We write the condition \(Q \cdot R = 0\) as \(Q \perp R\).

Tensor functions of the form

\[
\text{constant tensor} \times e^{-ik \cdot x},
\]

\(k\) real, are called plane waves. The vector \(k\) is called the wave vector. In defining a plane wave as \(\sim e^{-ik \cdot x}\) rather than \(\sim e^{ik \cdot x}\) we follow the convention of [4], [2].

By \(\varepsilon_{\kappa \lambda \mu \nu}\) we denote the totally antisymmetric tensor. We specify an orientation of our Minkowski 4–space, and put

\[
\varepsilon_{0123} := +1
\]

for all coordinate systems with positive orientation.

We define the action of the Hodge star (duality transformation) on an antisymmetric tensor \(Q\) of rank \(q\) as

\[
(*Q)_{\mu_1 \ldots \mu_q} := \frac{1}{q!} Q^{\mu_1 \ldots \mu_q} \varepsilon_{\mu_1 \ldots \mu_q},
\]

(7)

Let \(Q\) and \(R\) be antisymmetric tensors of rank \(q\) and \(r\), respectively. We define their exterior product as

\[
(Q \wedge R)_{\lambda_1 \ldots \lambda_{q+r}} := \frac{1}{q! r!} \sum \text{sgn}(P) Q_{\mu_1 \ldots \mu_q} R_{\nu_1 \ldots \nu_r}
\]

where summation is carried out over all permutations \(P = (\lambda_1 \ldots \lambda_{q+r})\).

Let \(Q\) and \(R\) be antisymmetric tensors functions of the same rank \(q\). We define their inner product as

\[
(Q, R) := \int Q \cdot R \, dx^0 \, dx^1 \, dx^2 \, dx^3.
\]

(9)

We denote by

\[
dQ := \partial \wedge Q
\]

(10)
the exterior derivative and by $\delta$ its adjoint with respect to the inner product $(9)$.

Our definitions (6)–(10) agree with those in [5], modulo the fact that we use the language of antisymmetric tensors rather than that of differential forms.

Lorentz transformations are assumed to be “passive” in the sense that we transform the coordinate system and not the tensors themselves.

We assume that our Minkowski 4-space has a specified coordinate axis $x^3$. This means that we only allow Lorentz transformations which preserve the equation of the hyperplane $\{x^3 = 0\}$. Such Lorentz transformations are not necessarily proper: an example of an improper one is the reversal of the $x^3$ coordinate.

Given an antisymmetric tensor $Q$ we define another antisymmetric tensor $(RQ)_{\mu_1 \ldots \mu_q} := \begin{cases} Q_{\mu_1 \ldots \mu_q} & \text{if } 3 \not\in \{\mu_1, \ldots, \mu_q\}, \\ -Q_{\mu_1 \ldots \mu_q} & \text{if } 3 \in \{\mu_1, \ldots, \mu_q\}. \end{cases}$ (11)

The tensor $RQ$ is the reflection of $Q$ about the hyperplane $\{x^3 = 0\}$. The “active” reflection operator $R$ should not be confused with the “passive” reversal of the $x^3$ coordinate.

3 Mass

Throughout this paper we will be dealing with tensor functions of the form

$$Q(x^0, x^1, x^2, x^3) = \tilde{Q}(x^0, x^1, x^2)e^{\pm ix^3}. \quad (12)$$

Condition (12) introduces a length scale into our model, which we interpret as the Compton wave length of the electron. In view of our choice of the system of units $c = \hbar = m = 1$ we can use (12) as the definition of the electron mass.

In the next section we shall acquire a second set of $\pm$ signs, independent of the one in (12). In order to avoid a clash of notation we shall write (12) as

$$Q(x^0, x^1, x^2, x^3) = \tilde{Q}(x^0, x^1, x^2)e^{-i\alpha x^3} \quad (13)$$

where the index $\alpha$ takes the values $\pm 1$. We put an extra minus in the right hand side of (13) because it is convenient in view of our definition of a plane wave (5).

4 Mathematical model for the photon

In the absence of sources the Maxwell equation in vector form is

$$\delta du = 0 \quad (14)$$

where $u$ is the unknown vector function. A solution $u$ is said to be polarised if the corresponding electromagnetic tensor $du$ is an eigenvector of the linear operator $\ast$. This polarisation condition is precisely formula (3).
Let us now view the polarisation condition (3) as a differential equation and compare it with (14). Using the fact that $\delta * d = 0$, it is easy to see that $u$ is a polarised solution of (14) if and only if it is a solution of (3). Therefore we shall call (3) the polarised Maxwell equation.

In order to avoid a clash of notation we shall write (3) as

$$\star du = i\beta du$$

(15)

where the index $\beta$ takes the values $\pm 1$.

Equation (15) is under-determined because it is actually a system of 3 equations with 4 unknowns. This under-determinacy does not cause problems because (13) admits an obvious gauge transformation: if $u$ is a solution of (15) then so is $u + ds$, where $s$ is an arbitrary scalar function. One may find it convenient to complement (15) by a gauge condition which would exclude the possibility of adding an arbitrary gradient and bring the total number of equations up to 4. In our setting the natural gauge is

$$u_3 = 0.$$  

(16)

The gauge condition (16) is perfectly suited for our purposes: it totally excludes the possibility of adding a gradient because in view of (13) $(ds)_3 = -i\alpha s$, and the only way $(ds)_3$ can be zero is if $s$ is zero. Nevertheless, in the following definition we do not insist on a particular gauge. The reason for not doing this will become clear later, when it will emerge (see (23)) that our equation for the electron/positron does not depend on the choice of the gauge for the photon.

We call a solution of (15) trivial if it is the gradient of a scalar function.

**Definition 1** A nontrivial plane wave solution $u$ of the under-determined equation (15) is called a photon.

5 Minkowski space perturbed by torsion

5.1 Connection generated by the external field

Let us now equip our Minkowski 4–space with a non-trivial connection. This means that we will have to start distinguishing the usual partial derivative $\partial$ and the covariant derivative $\nabla$. When acting on a vector function $v$ the general formulae relating the two are

$$\nabla_\mu v^\lambda = \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\nu, \quad \nabla_\mu v^\lambda = \partial_\mu v^\lambda - \Gamma^\lambda_{\mu\nu} v_\nu.$$  

(17)

Here the notation is from $[3]$. We take the connection coefficients to be

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} A_\mu \varepsilon^{\lambda\nu\lambda}_{\mu\nu}$$  

(18)

where $A$ is the vector potential of the external electromagnetic field.
Remark 1 In a general coordinate system the right hand side of (18) would have
the Christoffel symbol \( \{_{\mu}^{\lambda}^{\nu}\} = \frac{1}{2} g^{\lambda\kappa} (\partial_{\mu} g_{\nu\kappa} + \partial_{\kappa} g_{\mu\nu} - \partial_{\nu} g_{\mu\kappa}) \) as an additional term. We dropped it because following the traditions of special relativity we restrict ourselves to coordinate systems in which the metric tensor is constant.

The connection coefficients (18) satisfy \( \Gamma_{\kappa\lambda\mu} g_{\kappa\nu} + \Gamma_{\kappa\lambda\nu} g_{\kappa\mu} = 0 \) so our connection is metric compatible.

Defining the torsion tensor by the standard formula \( T_{\lambda\mu\nu} = \Gamma_{\lambda\mu\nu} - \Gamma_{\lambda\nu\mu} \) we arrive at (4). Conversely, given torsion (4) we uniquely recover (see formula (7.34) in [5]) the coefficients (18) of our metric compatible connection.

5.2 Generalisation of the notion of exterior derivative

The natural generalisation of (10) is the operator
\[ d_A Q := \nabla \wedge Q. \] (19)

For an antisymmetric tensor function \( Q \) of rank \( q \) formula (19) is understood in the following way: we write \( \nabla \wedge Q \) in accordance with (8), getting \( (q + 1)! \) terms of the type \( \nabla_{\mu} Q_{\nu_1...\nu_q} \), and expand each of these terms in accordance with the standard rules of covariant differentiation of a rank \( q \) tensor (see formula (7.26) in [5]). In particular, when \( Q = v \) is a vector function we get, by applying (17),
\[ (d_A v)_{\mu\nu} = \nabla_{\mu} v_{\nu} - \nabla_{\nu} v_{\mu} = \partial_{\mu} v_{\nu} - \Gamma_{\mu\nu\lambda} v_{\lambda} - \partial_{\nu} v_{\mu} + \Gamma_{\nu\mu\lambda} v_{\lambda} = (dv)_{\mu\nu} - T_{\lambda\mu\nu} v_{\lambda}. \] (20)

We see that the operator \( d_A \) differs from \( d \) by terms with torsion.

Remark 2 The author’s impression is that in mathematics literature it is not customary to work with the operator (19) and to view it as a natural generalisation of (10). On the other hand, it appears that in physics literature (19) is accepted as the natural way of forming a higher rank antisymmetric tensor; see, for example, [4], Section 90. For the vast majority of applications the matter of distinguishing (10) and (19) is, however, irrelevant, because these applications normally concern Levi–Civita connections, in which case (10) and (19) define the same operator.

Using (4) we can rewrite (20) in shorter form as
\[ d_A v = dv - * (A \wedge v). \] (21)

Formula (21) will play a central role in the proof of Theorem 1.
6 Mathematical model for the electron/positron

The perturbed analogue of the polarised Maxwell equation (15) is

\[ *d_A v = i \beta d_A v. \] (22)

The crucial difference between (15) and (22) is that the latter does not admit the usual gauge transformation because for a scalar function \( s \) we have \( d_A d_As = -*(A \wedge ds) \neq 0 \). Therefore the choice of a condition complementing (22) becomes a matter of principle rather than a matter of convenience.

We fix an arbitrary photon \( u \) and impose not one, but two conditions

\[ v \perp u, \quad v \perp k \] (23)

where \( k \) is the wave vector of the photon. As any two photons corresponding to the same \( k \) differ by a gradient, an equivalent way of imposing the conditions (23) is to require \( v \) to be orthogonal to all photons \( u \) with given wave vector.

We call a solution of (22), (23) trivial if it is identically zero.

Definition 2 A nontrivial solution \( v \) of the over-determined system (22), (23) is called an electron/positron.

7 Basic symmetry

Before proceeding to the actual analysis of our tensor model let us point out its basic symmetry: if \( u, v \) are solutions of (15), (22), (23) with indices \( \alpha = \alpha_0, \beta = \beta_0 \), then \( u, v \) are solutions of (15), (22), (23) with indices \( \alpha = -\alpha_0, \beta = -\beta_0 \). This is obvious because \( i \) comes into our model multiplied by \( \alpha \) (see (15)) or \( \beta \) (see (15), (22)). The argument relies on the fact that we introduced the external field as a real connection as opposed to the traditional complex formula (2).

As a consequence, our model is free of the problem of “negative frequencies”. Without loss of generality we shall assume further on that the wave vector \( k \) of the chosen photon \( u \) lies on the forward light cone, i.e., \( k_0 > 0 \).

8 Main result

Let us write down explicitly the 2D Dirac equation. In doing this we should avoid using the notation (2) because now \( \nabla \) has a different meaning, see subsection 5.1.

Put \( \nabla^{\pm} = \partial \pm iA \). Then the 2D Dirac equation is

\[
\begin{pmatrix}
  i\nabla_0^+ - 1 & 0 & 0 & i\nabla_1^+ + \nabla_2^+ \\
  0 & i\nabla_0^+ - 1 & i\nabla_1^+ - \nabla_2^+ & 0 \\
  0 & -i\nabla_1^- - \nabla_2^- & -i\nabla_0^- - 1 & 0 \\
-i\nabla_1^+ + \nabla_2^+ & 0 & 0 & -i\nabla_0^- - 1
\end{pmatrix}
\begin{pmatrix}
  \phi^- \\
  \phi^+ \\
  \chi^+ \\
  \chi^-
\end{pmatrix} = 0.
\] (24)
Here we chose to use new notation for the components of the bispinor; the relation with the traditional notation is \( \psi = (\phi_1 \phi_2 \chi_1 \chi_2)^T = (\phi^{-+} \phi^{++} \chi^{++} \chi^{-+})^T \).

We shall also need the equation
\[
\begin{pmatrix}
  i\nabla_0^- - 1 & 0 & 0 & i\nabla_1^- + \nabla_2^-
\
  0 & i\nabla_0^- - 1 & i\nabla_1^- - \nabla_2^- & 0
\
  0 & -i\nabla_1^- - \nabla_2^- & -i\nabla_0^- - 1 & 0
\
  -i\nabla_1^- + \nabla_2^- & 0 & 0 & -i\nabla_0^- - 1
\end{pmatrix}
\begin{pmatrix}
  \phi^{+-}
\
  \phi^{--}
\
  \chi^{--}
\
  \chi^{-+}
\end{pmatrix} = 0.
\]

which is the 2D Dirac equation for the antiparticle; see formula (32.5) in [2].

We shall write \( \nabla_\beta, \phi_\alpha^\beta, \chi_\alpha^\beta \) for \( \nabla^\pm, \phi^{\pm\pm}, \chi^{\pm\pm} \), and later \( d^\beta \) for \( d^\pm \). Here we admit abusing notation because \( \alpha \) and \( \beta \) were actually introduced (see Sections 3 and 4) as numbers and not signs.

The combined system (25), (24) can be written as
\[
\begin{pmatrix}
  i\nabla_0^\beta - 1 & i\nabla_1^\beta & i\nabla_2^\beta
\
  0 & \nabla_0^\beta & \alpha\nabla_2^\beta
\
  0 & -\alpha\nabla_1^\beta & \nabla_0^\beta - 1
\end{pmatrix}
\begin{pmatrix}
  \phi_\alpha^\beta
\
  \chi_\alpha^\beta
\end{pmatrix} = 0.
\]

**Theorem 1** The system (22), (23) is equivalent to (26).

**Proof** Using (21) and (17) we rewrite (22) as
\[
*(\partial \wedge v - *(A \wedge v)) = i\beta*(\partial \wedge v - *(A \wedge v)).
\]

An elementary rearrangement of terms transforms the latter into
\[
*(\nabla^\beta \wedge v) = i\beta(\nabla^\beta \wedge v).
\]

Denoting \( d^\beta := \nabla^\beta \wedge Q \), we see that (22) takes the form
\[
*d^\beta v = i\beta d^\beta v.
\]

Let us now write down explicitly our chosen photon \( u \). It is convenient to work in the coordinate system in which the wave vector \( k \) of our photon has components \( k_\mu = (1, 0, 0, \alpha) \); this can always be achieved by a proper Lorentz transformation. Straightforward calculations give
\[
u_\mu = \begin{pmatrix}
  C \\
  1 \\
  -i\alpha\beta \\
  C\alpha
\end{pmatrix} e^{-i(x^0 + \alpha x^3)}
\]

where \( C \) is a constant (depending on the gauge).

Using (28) it is easy to see that \( v \) satisfies (23) if and only if
\[
v_\mu = \begin{pmatrix}
  \phi_\alpha^\beta \\
  0 \\
  \chi^{\alpha^\beta} \\
  \alpha
\end{pmatrix}
\begin{pmatrix}
  1 & 0 & 0 & \frac{i}{\alpha}\beta e^{-i\alpha x^3}
\end{pmatrix} e^{-i\alpha x^3}
\]

(29)
where $\phi^{\alpha\beta}$, $\chi^{\alpha\beta}$ are some functions of $(x^0, x^1, x^2)$.

It remains to substitute (29) into (27) and obtain the equations for $\phi^{\alpha\beta}$, $\chi^{\alpha\beta}$. We have

$$
\begin{pmatrix}
(*d^3 v - i\beta d^3 v)^{03} \\
(*d^3 v - i\beta d^3 v)^{13} \\
(*d^3 v - i\beta d^3 v)^{23}
\end{pmatrix}
= \begin{pmatrix}
-(d^3 v)_{12} + i\beta(d^3 v)_{03} \\
(d^3 v)_{02} - i\beta(d^3 v)_{13} \\
-(d^3 v)_{01} - i\beta(d^3 v)_{23}
\end{pmatrix} = 0.
$$

The last two lines of the matrix in the right hand side of (30) are linearly dependent, so (27) reduces to

$$
\begin{pmatrix}
-i\beta \nabla_2^\beta \\
-\nabla_2^\beta \\
\nabla_1^\beta - i\alpha \beta \nabla_2^\beta
\end{pmatrix}
\begin{pmatrix}
\chi^{\alpha\beta} \\
\phi^{\alpha\beta}
\end{pmatrix}
= \begin{pmatrix}
\alpha \beta (i\nabla_0^\beta - 1) \\
-i\alpha \beta \nabla_1^\beta - \nabla_2^\beta \\
\nabla_1^\beta - i\alpha \beta \nabla_2^\beta
\end{pmatrix}
\begin{pmatrix}
\nabla_0^\beta \\
\nabla_0^\beta \\
\nabla_0^\beta - i\nabla_1^\beta - i\nabla_2^\beta
\end{pmatrix}
\begin{pmatrix}
\nabla_0^\beta \\
\nabla_0^\beta \\
\nabla_0^\beta - i\nabla_1^\beta - i\nabla_2^\beta
\end{pmatrix}
\begin{pmatrix}
\phi^{\alpha\beta} \\
\chi^{\alpha\beta}
\end{pmatrix}
= 0.
$$

Multiplying the latter by $\alpha \beta$ we arrive at (27).

9 Free particles

Let us consider the situation when there is no external electromagnetic field, i.e., $A \equiv 0$. In this case our system (22), (23) has a variety of plane wave solutions, out of which we single out one particular in accordance with the following

**Physical Assumption 1** The only physically meaningful plane wave solution $v$ is the one whose wave vector is the same as for $u$.

This physical assumption is made in the spirit of Feynman diagrams. One would expect that on the basis of (13), (22), (23) it would be possible to develop a full perturbation theory describing the interaction of electrons, positrons, and photons (tensor analogue of Feynman diagrams), and the above physical assumption would emerge as a natural consequence of this theory. In its absence we have to content ourselves with introducing Physical Assumption 1 as an axiom.

Up to a proper Lorentz transformation and complex conjugation (see Section 1) all our physically meaningful plane wave solutions can be written as

$$
u_\mu = \begin{pmatrix}
1 \\
-i\alpha \beta \\
C \alpha
\end{pmatrix} e^{-i(x^0 + \alpha x^3)}, \\
\nu_\mu = \begin{pmatrix}
1 \\
0 \\
0 \\
\alpha
\end{pmatrix} e^{-i(x^0 + \alpha x^3)}.
$$

(31)
Here, as in (28), \( C \) is an arbitrary constant.

We shall call the vector function \( v \) in (31) the free electron/positron. We see that the free electron/positron is a gradient type solution of the polarised Maxwell equation.

**10 Distinguishing the electron and the positron**

Let us now separate the plane wave solutions (31) corresponding to the electron and the positron. As we already have Theorem 1 and formula (29), the separation procedure reduces to the analysis of the case of a weak constant purely electric vector potential \( A \). Namely, we look for plane wave solutions \( v \) of the form

\[
\text{constant vector } \times e^{-i(\varepsilon x^0 + \alpha x^3)}
\]

which are perturbations of (31), i.e., \( \varepsilon \approx 1 \). We say that we are dealing with an electron if \( \varepsilon = 1 + A_0 \), and with a positron if \( \varepsilon = 1 - A_0 \). As a result we arrive at the following classification of plane wave solutions (31): solutions

\[
\begin{align*}
\begin{pmatrix} C \\ 1 \\ -i\alpha \\ C\alpha \end{pmatrix} e^{-i(x^0 + \alpha x^3)}, \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ \alpha \end{pmatrix} e^{-i(x^0 + \alpha x^3)}
\end{align*}
\]

(32)

correspond to the free electron, whereas solutions

\[
\begin{align*}
\begin{pmatrix} C \\ 1 \\ i\alpha \\ C\alpha \end{pmatrix} e^{-i(x^0 + \alpha x^3)}, \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ \alpha \end{pmatrix} e^{-i(x^0 + \alpha x^3)}
\end{align*}
\]

(33)

correspond to the free positron. As usual, formulae (32), (33) are written up to a proper Lorentz transformation and complex conjugation.

Comparing (32) with (33) we see that looking only at the vector function \( v \) it is impossible to distinguish the free electron from the free positron: the difference occurs in the formulae for the associated photon \( u \). Physically this means that it is impossible to tell whether we are dealing with an electron or a positron until we examine how the particle interacts with the electromagnetic field.

A convenient way of distinguishing the two cases is to define the notion of charge in accordance with

\[
c := -\text{sgn} (i \ast (du \wedge \mathcal{R}du))
\]

(34)

where \( \mathcal{R} \) is the reflection operator (11). Substituting (32) and (33) into (34) and performing straightforward calculations we conclude that for the electron \( c = -1 \), whereas for the positron \( c = +1 \).
It is easy to see that \( c \) is a true scalar in that it is invariant under Lorentz transformations (proper and improper) and does not depend on the choice of gauge for \( u \). Moreover, at a formal mathematical level our definition of charge \( (4) \) works in the case of an external field \( A \), and even irrespective of the strength of this field.

On the other hand \( c \) is not invariant under the transformation \( u \rightarrow \Pi \). This means that the notion of charge can only be used if we distinguish the forward and backward light cones, i.e., specify the positive direction of time.

References

[1] Benn, I.M. and Tucker, R.W.: Fermions without Spinors. Commun. Math. Phys. 89, 341–362 (1983)

[2] Berestetskii, V.B., Lifshitz, E.M., and Pitaevskii, L.P.: Quantum Electrodynamics, Course of Theoretical Physics Vol. 4, 2nd Edition (Pergamon Press, Oxford 1982)

[3] Klein, O.: Quantentheorie und fünfdimensionale Relativitätstheorie. Z. f. Phys. 37, 895–906 (1926)

[4] Landau, L.D. and Lifshitz, E.M.: The Classical Theory of Fields, Course of Theoretical Physics Vol. 2, 4th Edition (Pergamon Press, Oxford 1975)

[5] Nakahara, M.: Geometry, Topology and Physics (Institute of Physics Publishing, Bristol and Philadelphia 1998)