LOCAL TO GLOBAL PROPERTY IN FREE GROUPS

OFIR DAVID

Abstract. The local to global property for an equation ψ over a group G asks to show that ψ is solvable in G if and only if it is solvable in every finite quotient of G. In this paper we focus show that in order to prove this local to global property for free groups $G = F_k$, it is enough to prove for $k \leq$ the number of parameters in ψ. In particular we use it to show that the local to global property holds for $m$-powers in free groups.

1. Introduction

An interesting concept that appears in many parts of mathematics is the local to global principle. One of the most well known examples is the problem of finding integer solutions to $x^2 + y^2 = p$ where $p \in \mathbb{Z}$ is some prime. If there was an integer solution, then there is a solution mod $n$ for every $n$. In particular we have a solution mod 4, and since the squares mod 4 are 0 and 1, this implies that $p \equiv \not 0, 1, 2 \pmod{4}$, so there is no solution if $p \equiv 3 \pmod{4}$. Similar results hold for other equations over $\mathbb{Z}$.

A natural question is if the converse holds as well - if there is a family of solutions mod $n$ for all $n$, does it implies a solution over $\mathbb{Z}$? In our example above, if $p \not\equiv 3 \pmod{4}$ is a prime, then it is well known that there is an integer solution to $x^2 + y^2 = p$ (so in this case, it is enough to find a solution mod 4).

In this paper we study this local to global principle for groups with respect to their finite quotients. Namely, if $G$ is a group and ψ is an equation over $G$ (defined below), is it true that ψ has a solution in $G$ if and only if it has a solution in every finite quotient of $G$? For example, in the additive group $\mathbb{Z}$, given $m, n \in \mathbb{N}$, we may ask whether $n = \sum_1^m x = mx$ has an integer solution $x \in \mathbb{Z}$, and this can be easily seen to be equivalent to whether $n = mx$ has a solution in the finite quotient $\mathbb{Z}/m\mathbb{Z}$.

Definition 1.1. Denote by $F_n = \left\langle x_1, \ldots, x_n \right\rangle$ the free group on $n$ letters, and for a group $G$ denote by $F_n \ast G$ the free product. If $\varphi : F_n \to G$ is any homomorphism, we will also denote by $\varphi : F_n \ast G \to G$ its natural extension which is the identity on $G$.

An equation $\psi$ over $G$ is simply an element $\psi \in F_n \ast G$. We say that $\psi$ has a solution over $G$ if there is some homomorphism $\varphi \in \text{Hom}(F_n, G)$ such that $\varphi(\psi) = e_G$.

If $\pi : G \to H$ is a surjective homomorphism, then we say that $\psi$ has a solution in $H$ (with respect to $\pi$) if there is some $\varphi \in \text{Hom}(F_n, G)$ such that $\pi(\varphi(\psi)) = e_H$. 

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Example 1.2.  (1) In the example before the definition we had the equation \( \sum_{i=1}^{m} x - n \) where \( F_1 = \langle x \rangle \) as an additive group. We can write the equation as an additive equation because \( \mathbb{Z} \) is abelian. More generally, for \( g \in G \) we can consider the equation \( x^m g^{-1} \), which is solvable in \( G \) exactly when \( g \) is an \( m \)-power.

(2) If \( G \) is any group and \( g \in G \), we consider the equation \( \psi(x,y) = [x,y] g^{-1} = x y x^{-1} y^{-1} g^{-1} \).

This equation has a solution over \( G \) exactly if \( g \) is a commutator. If \( G \) is any abelian group and \( e \neq g \in G \), then \( \psi \) doesn't have a solution in \( G \) - this is because every commutator in an abelian group is trivial. However, if \( \pi : G \to H \) is a projection such that \( \pi(g) = e_H \), then \( \psi \) has a solution over \( H \) because, for example, \( \pi(\psi(e,e)) = \pi(g) = e_H \).

(3) If \( g,h \in G \), then the equation \( x g x^{-1} h^{-1} \in \langle x \rangle * G \) is solvable if and only if \( g \) and \( h \) are conjugate to one another.

Remark 1.3. We will mostly be interested in equations as in (1) and (2) above with the form \( w = g \) with \( w \in F_n \), \( g \in G \) (i.e. \( x^m = g \) and \( [x,y] = g \)). However, some of the result here are true in the more general definition and we prove it for them.

Definition 1.4. Let \( \psi \) be an equation over \( G \). We say that \( \psi \) has the local to global property, if \( \psi \) is solvable over \( G \) (global solution) if and only if it is solvable over \( G/K \) for any \( K \unlhd G \) (local solutions).

For a general group, there is no reason that local solutions will imply a global solution. Indeed, there might be some \( e \neq g \in G \) which is trivial in every finite quotient, so we can’t even distinguish it from \( e \). But even if this is not the case and \( G \) is residually finite, then it might still not be enough, and we will need to work with its profinite completion \( \hat{G} \) (see Definitions 2.4, 2.6). This completion is a topological completion, and this topological approach let us use notations and results from topology which contribute in both understanding better these local to global problems and solve them.

In this paper we are mainly interested in the local to global principle for free groups. In the two examples mentioned above this principle was shown to hold, first in [4] by Thompson (though the proof is by Lubotzky) for powers (namely \( x^m = g \)) and then in [2] by Khelif for commutators (namely \( [x,y] = g \)). In this paper we give another proof to a more generalized form of a reduction step appearing in [2] for equations over free groups.

Theorem 1.5. Let \( w \in F_n \) be a word on \( n \) variables. Then \( w g^{-1} \) has the local to global property for any \( g \in G = F_k \) and any \( k \geq 0 \), if this claim is true for \( 0 \leq k \leq n \).

Actually, the theorem above will be slightly stronger - it is enough to prove the local to global property if we further assume that for any finite quotient \( G/K \), the solution \( h_1, \ldots, h_k \) to the equation \( w = g \) also generate \( G/K \).

This reduction together with the discussion above about the equation \( n = mx \) over \( \mathbb{Z} \) produce another, very simple, proof for the local to global principle for the equation \( x^m = g \) over free groups.

Theorem 1.6. Let \( G = F_k \) be the free group on \( k \) variables. Then for any \( g \in G \), we can write \( g = h^m \) for some \( h \in G \), if and only if in any finite quotient \( G/K \) of \( G \) we can find \( h_K \in G \) such that \( gK = h_K^m K \).
In section §2 we will begin by recalling the main definitions and results for profinite groups and their completions, and in particular how to interpret solution of equations in a topological language.

After giving a sketch of the idea of the reduction step in section 3.1, we recall the definition of Stallings graphs in section 3.2 which is a very useful topological tool to study free groups, and finally in section 3.3 we give the proof of the reduction step, and its application to $m$-powers.

For the reader convenience, we added section §A where we give proofs for some of the well known results about profinite groups that we used in the reduction step.

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2. **Profinite topology and completion - motivation**

Let us fix some group $G$ and an equation $\psi \in F_n * G$. As we mentioned before, if $\psi$ has a solution in $G$, then it has a solution in every quotient (and in particular finite quotients) of $G$. We now try to understand what we need for the inverse direction to hold as well.

**Definition 2.1.** Let $F_n$ be the free group generated by $x_1, ..., x_n$. Given $\bar{g} = (g_1, ..., g_n) \in G^n$ we let $\varphi_\bar{g} : F_n \to G$ be the homomorphism defined by $\varphi_\bar{g}(x_i) = g_i$. Note that $\varphi_\bar{g}$ depends on our choice of the basis $x_1, ..., x_n$ for $F_n$, but once this basis is chosen, every homomorphism can be written uniquely as $\varphi_\bar{g}$, so we can identify $\text{Hom}(F_n, G)$ with $G^n$. We will also use the same notation for the extension $\varphi_\bar{g} : F_n * G \to G$ which is the identity on $G$. Finally, for $\psi \in F_n * G$ we will write $\psi(\bar{g}) := \varphi_\bar{g}(\psi)$.

**Example 2.2.** Fix some group $G$, and element $g \in G$ and let $\psi = [x, y]g^{-1} \in F_2 * G$. If $g_1, g_2 \in G$, then our two notations will produce

$$\varphi_{(g_1, g_2)}(\psi) = \psi(g_1, g_2) = [g_1, g_2]g^{-1}.$$  

We can now talk about the set of solutions in $G^n$ for this equation.

**Definition 2.3.** Given any normal subgroup $K \trianglelefteq G$ we write

$$\Omega_{\psi, K} = \{ \bar{g} \in G^n \mid \psi(\bar{g}) \in K \}.$$  

In other words, these are all the tuples which solve the equation $\psi$, after projecting to $G/K$. We will also write $\Omega_\psi := \Omega_{\psi, \{e\}}$ which are the actual solutions for $\psi$ in $G$.

We should think of $\Omega_{\psi, K}$ as approximate solutions - they might not solve the original equation, but up to an “error” in $K$ they do. Clearly, if $K_1 \trianglelefteq K_2$ are normal in $G$, then $\Omega_{\psi, K_1} \subseteq \Omega_{\psi, K_2}$, so we have better and better solutions as we decrease $K$.

If we run over the finite index normal subgroups $K$, then we get that

$$\Omega_\psi \subseteq \bigcap_{K \trianglelefteq G} \Omega_{\psi, K}.$$
This is equivalent to our statement from before that a global solution in \( G \) implies a local solution in every finite quotient. Assume now that we have a solution in every finite quotient, namely the \( \Omega_{\psi,K} \) are not empty, and we want to find a global solution, namely \( \Omega_{\psi} \neq \emptyset \).

This will follow if we can prove the next two properties:

1. There is an equality \( \Omega_{\psi} = \bigcap_{K \leq G} \Omega_{\psi,K} \), and
2. The intersection \( \bigcap_{K \leq G} \Omega_{\psi,K} \) is nonempty.

For the first part, suppose that \( \bar{g} \in \bigcap_{K \leq G} \Omega_{\psi,K} \), or equivalently the value \( \psi(\bar{g}) \in \bigcap K \). If we knew that the intersection of all finite index normal subgroups is trivial, then it means that \( \psi(\bar{g}) = e \), and therefore \( \bar{g} \) is a solution in \( G \) for \( \psi \). This leads to the following definition:

**Definition 2.4.** A group \( G \) is called residually finite, if for any \( e \neq g \in G \) there is a projection \( \pi: G \to H \) to some finite group \( H \) such that \( \pi(g) \neq e \). Equivalently the intersection of all finite index subgroups (resp. f.i. normal subgroups) is trivial.

**Example 2.5.**

1. The group \( \mathbb{Z} \) is residually finite. If \( n \neq 0 \), then its reduction modulo \( 2n \) is non trivial.
2. Similarly the group \( \text{SL}_k(\mathbb{Z}) \) for \( k \geq 2 \) is residually finite by considering the maps \( \text{SL}_k(\mathbb{Z}) \to \text{SL}_k(\mathbb{Z}/p\mathbb{Z}) \) for primes \( p \).
3. Subgroups of residually finite groups are also residually finite. Since the free group \( F_2 \) is residually finite, then it is residually finite. Moreover, every free group \( F_k \) can be embedded in \( F_2 \), so all the free groups are residually finite.

If \( G \) is residually finite, then the first condition above holds. For the second condition, note that if \( K_j \subseteq G \), \( j = 1, \ldots, k \), then so is \( \bigcap_{j=1}^k K_j \subseteq G \). It follows that \( \Omega_{\psi,\bigcap_{j=1}^k K_j} \subseteq \bigcap_{j=1}^k \Omega_{\psi,K_j} \), and under our assumption that there are always local solutions for every \( K \subseteq G \), we conclude that any finite intersection \( \bigcap_{j=1}^k \Omega_{\psi,K_j} \) is nonempty. If we can give a topology to \( G \) where the \( \Omega_{\psi,K} \) are compact, then the finite intersection property for compact sets implies that \( \bigcap_{K \leq G} \Omega_{\psi,K} \neq \emptyset \) which is the second condition above.

In general, this might not be the case, however there is a natural topology on \( G \), called the profinite topology, and then completing \( G \) with respect to this topology will have this property.

**Definition 2.6.** Let \( G \) be a residually finite group, and let \( \{(\pi_i,H_i) \mid i \in I\} \) be all the pairs such that \( \pi_i: G \to H_i \) is a projection to a finite group \( H_i \). Let \( H_i \) have the discrete topology and \( \prod_{i \in I} H_i \) have the product topology and identify \( G \) as a subgroup via \( g \mapsto (\pi_i(g))_{i \in I} \) (this is injective because \( G \) is residually finite).

Then the induced topology on \( G \) is called the profinite topology, and the closure of \( G \) in \( \prod_{i \in I} H_i \), denoted by \( \hat{G} \), is called the profinite completion of \( G \). For each \( i \in I \), we will denote by \( \hat{\pi}_i: \hat{G} \to H_i \) the projection of \( \hat{G} \) to \( H_i \) which is an extension of \( \pi_i \).
In the profinite topology, the cosets of finite index subgroups form a basis for the topology. Thus, we should think of two elements as "close" if they are mapped to the same element under the projection $G \rightarrow G/K$ where $|G/K|$ is large. Alternatively, the profinite topology is the weakest topology such that all the projections to the (discrete) finite groups are continuous. For more details about residually finite groups and their completion, the reader is referred to [1, 5, 3].

For our discussion, one of the main results that we need is the following:

**Theorem 2.7.** Let $G$ be a residually finite group and $\hat{G}$ its profinite completion. Then $\hat{G}$ is a totally disconnected, Hausdorff and compact topological group which is also residually finite.

**Proof.** The finite groups with discrete topologies are totally disconnected, Hausdorff and compact groups. It is easily seen that the product of such sets is totally disconnected and Hausdorff, and by Tychonoff’s theorem it is also compact. Since by definition $\hat{G}$ is closed in a product of finite groups, we conclude that it is also totally disconnected, Hausdorff and compact. Finally, product of finite group is always residually finite, and therefore $\hat{G}$ is residually finite as a subgroup of a residually finite group. $\square$

We already see the usefulness of this topological approach, and in particular Tychonoff’s theorem above shows that $\hat{G}$ is compact.

The profinite topology is defined in such a way that if $K \trianglelefteq_{f.i.} G$, then the natural map $G \rightarrow G/K$ is continuous, so the induced map $\psi_K : G^n \rightarrow G \rightarrow G/K$ is also continuous, and therefore $\Omega_{\psi,K} = \psi_K^{-1}(eK)$ is closed in $G^n$. Furthermore, their closures $\hat{\Omega}_{\psi,K}$ in $\hat{G}$ are compact so we can use their finite intersection property in $\hat{G}$.

**Corollary 2.8.** Let $G$ be a residually finite group, and let $\psi$ be an equation over $G$. Then $\psi$ has a solution in $\hat{G}$, if and only if it has a solution in every finite quotient of $G$.

**Proof.** The first part, namely local solutions imply a solution in $\hat{G}$ was presented above, and it is left to the reader to fill in the details. For the other direction, if $K \trianglelefteq_{f.i.} G$, then the map $\pi : G \rightarrow G/K$ can be extended to $\hat{\pi} : \hat{G} \rightarrow G/K$, so a solution in $\hat{G}$ implies a local solution for every such $K$. $\square$

Given an equation $\psi$ over $G$, it is also an equation over $\hat{G}$, so by the corollary above, the set of solutions there $\hat{\Omega}_\psi$ is not empty exactly if there is a solution in every finite quotient of $G$. This let us talk about the local to global principal in a more “compact” way. Namely, the local to global principal holds exactly if a solution in $\hat{G}$ implies a solution in $G$.
3. The reduction

3.1. The reduction idea. Now that we have our new topological notation, let us sketch the main idea of the reduction step, where as example we consider the equation $x^m = g$.

(1) Fix some $g \in G = F_k$ and assume that $g$ is an $m$-power in every finite quotient of $F_k$ or equivalently there is some $h \in \hat{G}$ such that $h^m = g$. Letting $H = \langle \hat{h} \rangle \leq \hat{G}$, assume first that there is some $g \in H_0 \leq F_k$ finitely generated such that $H_0 \cong \hat{H}_0 = H$, namely

$$
\begin{array}{cccc}
\hat{H}_0 = \langle \hat{h} \rangle & \longrightarrow & \hat{F}_k \\
\uparrow & & \uparrow \\
\langle g \rangle & \longrightarrow & H_0 & \longrightarrow & F_k.
\end{array}
$$

The group $H_0$ is free as a subgroup of a free group, and therefore $H_0 \cong F_{k'}$ for some $k'$ (it is also finitely generated). On the other hand, its completion $\hat{H}_0 = \langle \hat{h} \rangle$ is generated topologically by one element $h$, so we must have that $k' = 1$. In other words, this reduces the problem to $H_0 \cong \mathbb{Z}$ and $H_0 = \mathbb{Z}$ where we already know the local to global principle for $m$-powers (or additively - multiples of $m$).

(2) However, the conditions above are not true in general. In order to fix this problem, let $H_0$ be a finitely generated subgroup such that $g \in H_0$ and $\overline{H}_0 \leq H$. Suppose that we can find a surjective continuous map $\pi : H \to \overline{H}_0$ which fixes $\overline{H}_0$, so we have the diagram

$$
\begin{array}{cccc}
\overline{H}_0 & \xrightarrow{\pi} & H & \xrightarrow{} & \hat{F}_k \\
\uparrow & & \uparrow & & \uparrow \\
\langle g \rangle & \xrightarrow{} & H_0 & \xrightarrow{} & F_k,
\end{array}
$$

where $\pi \circ \iota = Id_{\overline{H}_0}$. In this case we get that (1) $\overline{H}_0$ will be generated topologically by $\pi(h)$ and (2) we have that $\pi(h)^m = \pi(h^m) = \pi(g) = g$ because $g \in \overline{H}_0$ is fixed by $\pi$. We can now use the trick from above for $\langle g \rangle \leq H_0 \leq \overline{H}_0 \leq \hat{G}$.

As we shall see, this result will be more general, and not only for the $m$-power equations. The interpretation of this result will be that if $w \in F_n$ is any word (e.g. $w = x^m, [x,y]$ etc.) and we want to prove the local to global property for it, namely that a solution to $w(\hat{g}_1, ..., \hat{g}_n) = g$ in $\hat{G}$ implies a solution in $G$, it is enough to prove this claim under the further assumption that $\hat{g}_1, ..., \hat{g}_n$ generate $\hat{F}_k$, so in particular $k \leq n$. In the finite quotients language, it means that the solution $\hat{g}_1K, ..., \hat{g}_nK$ generate the group $G/K$ for every $K \leq \hat{G}$.

(3) The two main problems in (2) above are to somehow define a homomorphism from $\hat{H}$ to $\overline{H}_0$ and moreover it needs to fix $\overline{H}_0$. However, there is one such situation where this is very easy. Suppose that $H_0$ is a free factor $H_0 \leq N$ of some group $N \leq F_k$, namely we can write $N = H_0 * H_1$ for some subgroup $H_1$. Then there is a natural projection $\pi : N \to H_0$ which fixes $H_0$. We can then extend it to their completion $\pi : \overline{N} \to \overline{H}_0$, and if $H \leq \overline{N}$ then the
restriction of \( \pi \) to \( H \) will do the job

\[
\begin{array}{c}
\pi \\
\downarrow H_0 \\
H \\
N \\
\downarrow \hat{F}_k \\
\end{array}
\]

The main result of the next section will be to show that we can actually choose such \( H_0 \) and \( N \) “wisely” so that \( H_0 \leq_* N \) and \( H \leq N \).

3.2. Stallings graphs. The Stallings graphs will be our main tool to understand subgroups of the free group, and to say when one subgroup is a free factor of another. Before we define them, consider the following example of a labeled graph (defined below).

\[
\begin{array}{c}
\langle x \rangle \\
\downarrow H_0 \\
N \\
\downarrow F_k \\
\end{array}
\]

**Figure 3.1.** A labeled graph.

This graph has two “main” cycles corresponding to \( x^2y \) on the left and \( x^{-1}y^2 \) on the right, and every other cycle can be constructed using these two cycles (up to homotopy, i.e. modulo backtracking). Furthermore, the labeling allows us to think of these cycles as elements in \( F_2 = \langle x, y \rangle \), so that the fundamental group of the graph could be considered as the subgroup generated by \( x^2y \) and \( x^{-1}y^2 \). With this example in mind, we now give the proper definitions to make this argument more precise.

One of the most basic results in algebraic topology is that the fundamental group of a graph is always a free group. Let us recall some of the details.

**Definition 3.1 (Cycle basis).** Let \( \Gamma \) be a connected undirected graph with a special vertex \( v \in V(\Gamma) \), and let \( T \subseteq E(\Gamma) \) be a spanning tree. For each edge \( e : u \to w \) let \( C_e \) be the simple cycle going from \( v \) to \( u \) on the unique path in the tree \( T \), then from \( u \) to \( w \) via \( e \) and finally from \( w \) to \( v \) via \( T \). We denote by \( C(T) = \{ e \notin T \mid C_e \} \) this collection of cycles.

It is not hard to show that any cycle in a connected graph can be written as a concatenation of cycles in \( C(T) \) and their inverses as elements in the fundamental group \( \pi_1(\Gamma) \) (namely, we are allowed to remove backtracking). More over, it has a unique such presentation which leads to the following:
Corollary 3.2. Let $\Gamma$ be a graph and $T$ a spanning tree. Then $C(T)$ is a basis for $\pi_1(\Gamma)$ which is a free group on $|E(\Gamma)| - |V(\Gamma)| + 1$ elements.

Example 3.3. In figure 3.1 the edge touching the 0 vertex form a spanning tree, and then $C_{(1,2)} = 0 \xrightarrow{\gamma_1} 1 \xrightarrow{\gamma_2} 2 \xrightarrow{\gamma_3} 0$ and $C_{(3,2)} = 0 \xleftarrow{\gamma_4} 3 \xleftarrow{\gamma_5} 2 \xleftarrow{\gamma_6} 0$, so that eventually we will think of the fundamental group as generated by $x^2y$ and $x^{-1}y^2$ per our intuition from the start of this section.

In particular, the corollary above implies that the fundamental group of the bouquet graph with a single vertex and $n$ self loops is the free group $F_n$. We can label the edges by their corresponding basis elements $x_1,\ldots,x_n$ of $F_n$. Since it is important in which direction we travel across the edge, we will think of each edge as two directed edges labeled by $x_i$ and $x_i^{-1}$ depending on the image in the fundamental group. For simplicity, we will keep only the edges with the $x_i$ labeling, understanding that we can also travel in the opposite direction via an $x_i^{-1}$ labeled edge.

It is well known that a fundamental group of a covering space correspond to a subgroup of the original space. Using the generalization of the labeling above we can produce covering using the combinatorics of labeled graphs.

For the rest of this section we fix a basis $x_1,\ldots,x_n$ of the free group $F_n$.

Definition 3.4. A labeled graph $(\Gamma, v)$ is a directed graph $\Gamma$ with a special vertex $v$, where the edges are labeled by $x_1,\ldots,x_n$ (see figure 3.2). A labeled graph morphism $(\Gamma_1, v_1) \rightarrow (\Gamma_2, v_2)$ between labeled graphs is a morphism of graphs $\Gamma_1 \rightarrow \Gamma_2$ which sends $v_1$ to $v_2$ and preserves the labels.

We denote by $\Gamma_{F_n}$ the bouquet graph with the $x_1,\ldots,x_n$ labeling. Note that another way to define a labeling on a graph $\Gamma$ is a morphism of directed graphs $\varphi : \Gamma \rightarrow \Gamma_{F_n}$, where the labeling of an edge $e \in E(\Gamma)$ is defined to be the labeling of $\varphi(e)$. In this way a labeled graph morphism is just a map which defines a commuting diagram

$$
\begin{array}{ccc}
(\Gamma_1, v_1) & \rightarrow & (\Gamma_2, v_2) \\
\downarrow & & \downarrow \\
\Gamma_{F_n} & & \\
\end{array}
$$

This labeling map $\varphi : \Gamma \rightarrow \Gamma_{F_n}$ induces a homomorphism $\hat{\varphi} : \pi_1(\Gamma, v) \rightarrow \pi_1(\Gamma_{F_n}) = F_n$. Since every path in $\Gamma$ is sent to a cycle in $\Gamma_{F_n}$, we can extend this map to general paths in $\Gamma$.

Definition 3.5. Let $(\Gamma, v)$ be a graph with a labeling $\varphi : \Gamma \rightarrow \Gamma_{F_n}$. Given a path $P$ in $\Gamma$ starting at $v$, define the labeling $L(P)$ of the path to be the (cycle) element $\varphi(P)$ in the fundamental group $\pi_1(F_n)$. In other words, this is just the element in $F_n$ created by the labels on the path.

In general, for a labeled graph $\varphi : \Gamma \rightarrow \Gamma_{F_n}$ the function $\hat{\varphi}$ is not injective. However, in the Stallings graphs case, defined below, it is.

Definition 3.6. A Stallings graph is a labeled graph $\varphi : (\Gamma, v) \rightarrow \Gamma_{F_n}$ where $\varphi$ is locally injective, namely for every vertex $u \in V(\Gamma)$ and every $i = 1,\ldots,n$ there is at most one outgoing edge from $u$ and at most one ingoing edge into $u$ labeled by $x_i$. We call the graph a covering graph if $\varphi$ is a local homeomorphism, or equivalently every vertex has exactly one ingoing and one outgoing labeled by $x_i$ for every $i$.

Remark 3.7. Given a covering graph, we can remove every edge and vertex which are not part of a simple cycle so as to not change the fundamental group. The resulting graph will be a Stallings graph, and conversely, every Stallings graph can be extended to a covering graph of $\Gamma_{F_n}$ without changing the fundamental domain.
In the Stallings graph case, it is an exercise to show that $\hat{\varphi}$ is injective, and we may consider $\pi_1(\Gamma, v)$ as a subgroup of $F_n$. Moreover, we can use 3.1 to find a basis for $\pi_1(\Gamma, v)$ as a subgroup of $F_n$.

**Example 3.8.** In figure 3.2 below, in the left most graph, the path

$$P := v_0 \xrightarrow{x} v_1 \xrightarrow{y} v_2 \xleftarrow{y} v_0$$

is labeled by $L(P) = xyyy^{-1} = xy$. Similarly, in the second graph from the right the path

$$P := v_0 \xrightarrow{x} v_1 \xrightarrow{y} v_2 \xleftarrow{x} v_3 \xleftarrow{y} v_0$$

is labeled by $\pi_1(P) = xyx^{-1}y^{-1} = [x, y]$.

The images $\hat{\varphi}(\Gamma, v)$ for the graphs in this figure from left to right are $\langle xy, xy^2, y \rangle = \langle x, y \rangle$, $\langle y, xyx^{-1} \rangle$, $\langle xyx^{-1}y^{-1} \rangle$ and $\langle x, y \rangle$. Note that the fundamental group of the left most graph is free of rank 3 (there are 3 loops in the graph) while the image $\hat{\varphi}(\Gamma, v) = \langle x, y \rangle$ is generated by only two element, which in particular indicates that it is not a Stallings graph.

![Figure 3.2. These are graphs labeled by $x, y$ where $F_2 = \langle x, y \rangle$, and the special vertices are the yellow ones. The left most graph is not Stallings because it has two $y$ labeled edges coming out of the same vertex. The rest are Stallings graphs where the right most graph is $\Gamma F_2$.](image)

One way to construct Stallings graphs is by starting with a standard labeled graph, and then taking a suitable quotient. We will use this construction to build such graphs for subgroups of $F_n$.

**Definition 3.9.** Let $\varphi : (\Gamma, v) \to \Gamma F_n$ be a labeled graph. Define an equivalence relation on vertices $v_1 \sim v_2$ if there exist paths $P_1, P_2$ leading from $v_0$ to $v_1, v_2$ respectively such that $\pi_1(P_1) = \pi_1(P_2)$. Define an equivalence relation on the edges $(v_1 \xrightarrow{x} v_1') \sim (v_2 \xrightarrow{x} v_2')$ if $v_1 \sim v_2$ and $x_1 = x_2$ (which implies that $v_1' \sim v_2'$ also).
It is easy to check that if $(\Gamma, v)$ is a labeled graph, then the quotient graph $(\Gamma/\sim, [v])$, where $[v]$ is the image of $v$, is a Stallings graph. We will usually also remove any edges and vertices which are not part of a simple cycle, since these do not change the fundamental group.

**Example 3.10.** In the left most graph in figure 3.2, the path $v_0 \xrightarrow{x} v_1 \xrightarrow{y} v_2$ and the path $v_0 \xrightarrow{x} v_1 \xrightarrow{y} v_0$ define the same element in $F_2$, so we need to identify the vertices $v_0$ and $v_2$. Similarly the paths $v_0 \xleftarrow{y} v_0$ and $v_0 \xleftarrow{y} v_1$ define the same element so we need to identify $v_0$ and $v_1$, so in the end we are left with the bouquet graph $\Gamma_{F_2}$.

**Definition 3.11.** Let $S \subseteq F_n$. For each $s \in S$, let $(P_s, v_s, 0)$ be a cycle graph on a single path $P_s$ such that $\pi_1(P_s) = s$. Let $\Gamma$ be the graph $\bigcup_{s \in S} (P_s, v_s, 0)$ where we identify all the $v_{s,0}$ into a single vertex. Denote by $\Gamma_S = \Gamma/\sim$ its quotient.

**Claim 3.12.** For any $S \subseteq F_n$ we have that $\pi_1(\Gamma_S) = \langle S \rangle$.

**Proof.** Left as an exercise to the reader.

With our new language of Stallings graphs, we can now prove how a simple condition of injectivity implies that one subgroup is a free factor of another subgroup.

**Claim 3.13.** Let $(\Gamma, v)$ be a Stallings graph and $(\Gamma', v)$ a labeled subgraph (which must be Stallings as well). Then $\pi_1(\Gamma', v)$ is a free factor of $\pi_1(\Gamma, v)$.

**Proof.** Let $T'$ be a spanning tree for $\Gamma'$ and extend it to a tree $T$ of $\Gamma$. Our construction of cycle basis $C(T)$ will contain the cycle basis $C(T')$, so that we can choose generators for $\pi_1(\Gamma', v)$ which is a subset of a set of generators for $\pi_1(\Gamma, v)$, which implies that the first is a free factor of the second.

**Example 3.14.** Consider the following Stallings graph:

![Figure 3.3. A Stallings graph where the green edges form a spanning tree.](image)

This graph contains a Stallings subgraph on the path corresponding to $xyx^{-1}y^{-1}$. This subgraph has as spanning tree the 3 green edges on it, and we add another green edge to create a spanning tree for the full graph. As generators for the fundamental group we first take $xyx^{-1}y^{-1}$ for the $x$ edge which is not in green, and the second generator is $xy^2x$ for the $y$-labeled edge which is not in the spanning tree. Using 3.13 we conclude that $\langle [x, y] \rangle$ is a free factor of $\langle [x, y], xy^2x \rangle$. Note that a priori, the group $\langle [x, y], xy^2x \rangle$ might be generated by 1 element, and then $\langle [x, y] \rangle$ is a free factor exactly if it equals the full group. The graph visualization tells us that the second group is actually bigger and needs at least two generators.
If $H \leq N \leq F_n$ then we can construct the two Stallings graphs $\Gamma_H, \Gamma_N$ and then there is a natural labeled morphism $\Gamma_H \to \Gamma_N$. If this morphism is injective, then by the claim above we know that $H \leq_s N$ is a free factor of $N$. However, the injectiveness of this map depends also on our initial choice of basis for $F_n$, and in general $H$ can be a free factor of $N$ even when the corresponding graph morphism is not injective. The next result uses the Stallings graphs to show how to naturally find two subgroups where the Stallings graphs are injective, and therefore one is a free factor of the other.

As we shall see later, this type of result is exactly what we need in our reduction in section 3.1.

**Theorem 3.15.** Let $N_j$, $j \in J$ be a directed system of subgroups of $F_n$, and let $S \subseteq N = \bigcap_{j \in J} N_j$. Then there is some $S \subseteq H \leq N$ and $j_0$ such that $H$ is a free factor of any subgroup $N'$ such that $H \leq N' \leq N_{j_0}$. If $S$ is finite, then we may take $H$ which is finitely generated.

**Proof.** Consider the map $\Gamma_S \to \Gamma_N$. The image of this map is a sub Stallings graph of $\Gamma_N$ corresponding to some $\Gamma_H$ for some $H \leq N$. Note that if $S$ is finite, then so is $\Gamma_S$ and $\Gamma_H$ and therefore $H$ is finitely generated. We claim that there is some $j_0$ such that $\Gamma_H \to \Gamma_{N_{j_0}}$ is injective. Once we know this, if $H \leq N' \leq N_{j_0}$ is any intermediate group, then $\Gamma_H \to \Gamma_{N_{j_0}}$ is the composition of $\Gamma_H \to \Gamma_{N_{j_0}} \to \Gamma_{N_{j_0}}$ making $\Gamma_H \to \Gamma_{N_{j_0}}$ injective as well. By 3.13 it follows that $H \leq_s N'$ for any such $N'$, thus completing the proof.

To find such a $j_0$, for each $\nu \in \Gamma_H$ let $P_\nu$ be a path from $v_0$ to $\nu$ in $\Gamma_H$ where $v_0$ is the special vertex. If $\nu_1 \neq \nu_2$ in $\Gamma_H$, then $g_{\nu_1, \nu_2} := \pi_1 (P_{\nu_1}) \pi_1 (P_{\nu_2})^{-1} \notin H$ and because $\Gamma_H \subseteq \Gamma_N$, this element is not in $N$ as well. Hence, we can find some $j_{\nu_1, \nu_2}$ such that $g_{\nu_1, \nu_2} \notin N_{j_{\nu_1, \nu_2}}$ and using the directedness of $N_j$, we can find $j_0$ such that $g_{\nu, \nu} \notin N_{j_0}$ for any two distinct vertices $\nu, \mu \in \Gamma_H$. This implies in turn that the map $\Gamma_H \to \Gamma_{N_{j_0}}$ is injective on the vertices, and since this is a morphism of Stallings graphs it is injective on the edges as well, which is exactly what we needed.

**Example 3.16.** Consider the set $S = \{x^3, y^2\}$ where $N = \langle y, xyx^{-1}, x^3 \rangle$. In this case the image of $S$ in $\Gamma_N$, as can be seen in the figure below, is the group $H = \langle x^3, y \rangle$. The element $x^3$ is mapped to $x^3$ while $y^2$ circles twice around $y$.

![Figure 3.4](image)

**Figure 3.4.** The set $S = \{x^3, y^2\}$ is contained in the (finitely generated) $H = \langle x^3, y \rangle$ which is a free factor of $N = \langle y, x^3, xyx^{-1} \rangle$.

### 3.3. Proof of the main theorem.

Let $g \in F_k$ and $m \in \mathbb{Z}$. Our profinite notation shows that $x^m = g$ has the local to global property if a solution over $\tilde{F}_k$ implies a solution over $F_k$. This is a very special type of equation which can be written as $w = g$, where the left side contains only
parameters and the right side contain only $g$. In this section we provide the details for the ideas in section 3.1 for this type of equations and then the specialization for $x^m = g$ will lead to the proof of theorem 1.6.

For the rest of this section, we will have two free groups. The first will be denoted by $F_n$ and the free variables will be from it, while the second $G = F_k$ will be the group in which we try to prove the local global principle.

**Definition 3.17.** Let $w \in F_n$. We say that $w$ satisfies the local to global property in $F_k$, if for any $g \in G = F_k$, the equation $w = g$ is solvable in $G$ if and only if it is solvable in $G$. We say that $w$ satisfies the local global property in free groups, if it satisfies it for every $k \geq 1$.

**Theorem 3.18.** Let $w \in F_n$. Then $w$ satisfies the local to global property in free groups, if and only if it satisfies it in $F_k$ for any $k \leq n$.

**Proof.** The $\Rightarrow$ direction is clear. Let us assume that $w$ satisfies the local to global property for $k \leq n$ and show that it holds for general $k$.

Let $k \in \mathbb{N}$, $g \in F_k$, and assume that $w = g$ is solvable in $F_k$. Let $\hat{g}_1, ..., \hat{g}_n \in \hat{F}_k$ such that $w(\hat{g}_1, ..., \hat{g}_n) = g$ and set $H = \langle \hat{g}_1, ..., \hat{g}_n \rangle$. Since $H$ is closed in $\hat{F}_k$, it is the intersection of all the closed finite index subgroups of $\hat{F}_k$ which contain it. However, by lemma A.6 these subgroups are in bijection with the finite index subgroups of $F_k$ via the maps $N \mapsto \hat{N}$ and $\hat{N} \mapsto F_k \cap \hat{N}$. Thus if we denote this set by $J = \{N \; | \; N \leq F_k, H \leq \hat{N}\}$, which is a directed set, then $H = \bigcap_{N \in J} \hat{N}$.

More over, since $g \in H \leq \hat{N}$ for every $N \in J$ we conclude that $g \in \hat{N} \cap F_k = N$, and therefore $g \in \bigcap_{N \in J} \hat{N}$. Hence we have the following subgroups:

$$\langle g \rangle \leq \bigcap_{N \in J} N \leq H = \bigcap_{N \in J} \hat{N}$$

Applying theorem 3.15 for $S = \{g\}$, we can find $\langle g \rangle \leq H_0 \leq \bigcap_{N \in J} N$ with $H_0$ finitely generated and $H_0 \leq_n N$ for some $N \in J$, so we may define a projection $\pi : \hat{N} \rightarrow H_0$ which is the identity on $H_0$. The group $N$ has finite index in $F_k$ and $H_0$ is a free factor in $N$, so by lemma A.4 their subspace topology is the profinite topology. Moreover, the group $N$ is finally generated as a finite index subgroup of the finitely generated group $F_k$, and of course $H_0$ is finitely generated by assumption. We can now use A.3 to find a continuous extension $\hat{\pi} : \hat{N} \rightarrow \hat{H}_0$ which is the identity on $\hat{H}_0$.

Since $g \in \overline{H}_0$, we get that

$$g = \pi(g) = \pi(w(\hat{g}_1, ..., \hat{g}_n)) = w(\pi(\hat{g}_1), ..., \pi(\hat{g}_n)),$$

so that $w = g$ is solvable in $\overline{H}_0$. Moreover, because $\overline{H}_0 \leq H \leq \hat{N}$, then the restriction of $\pi$ to $H$ is also surjective on $\overline{H}_0$, implying that $\overline{H}_0$ is generated topologically by $\hat{\pi}(\hat{g}_1), ..., \hat{\pi}(\hat{g}_n)$.

The group $H_0$ is finitely generated subgroup of a free group, and therefore $H_0 \cong F_{k'}$ for some $k'$. Also, since by lemma A.4 its subspace topology is the profinite topology we conclude that $\overline{H}_0 \cong H_0 \cong F_{k'}$. But $F_{k'}$ is generated topologically by $n$ elements, so that $k' \leq n$. Indeed, if $x_1, ..., x_{k'}$ is a basis for $F_{k'}$ then there is the projection $\pi : F_{k'} \rightarrow \mathbb{F}_2^{k'}$ sending $x_i$ to the standard basis element $e_i$. Using lemma A.2, we can extend $\pi$ to the projection $\hat{\pi} : \hat{F}_{k'} \rightarrow \hat{\mathbb{F}}_2^{k'}$. If $\hat{F}_{k'}$ is generated topologically by $n$ elements, then so is any of its quotients, and since $\hat{\mathbb{F}}_2$ cannot be generated by less than $k'$ elements, we conclude that $k' \leq n$.

To summarize, the equation $w = g$ has a solution in $\hat{H}_0 \cong \hat{F}_{k'}$ where $g \in H_0 \cong F_k$. Under our assumption, the word $w$ has the local to global property for $k' \leq n$, so it has a solution in $H_0 \leq G$, thus completing the proof. \[\square\]
Finally, we can use this reduction to prove that \( w = x^m \) has the local to global property for free groups.

**Proof of theorem 1.6.** Since \( w \in F_1 \), by the reduction step in theorem 3.18, it is enough to prove that \( w \) has the local to global property in \( F_1 \cong \mathbb{Z} \). But we already saw that this is true, hence \( w \) has the local to global property for free groups. \( \square \)

## Appendix A. Some profinite results

In this section we collect all sort of results about profinite groups which are well known, but we add them here for the convenience of the reader.

We start by understanding the continuous homomorphisms between profinite groups and their profinite completions.

**Lemma A.1.** Any homomorphism \( \varphi : G_1 \to G_2 \) between groups with the profinite topologies is continuous.

**Proof.** The profinite topology is defined as the weakest topology where all the projections to finite groups are continuous. Hence, we need to show that if \( H \) is finite and \( \pi : G_2 \to H \) is a homomorphism, then \( \varphi \circ \pi : G_1 \to G_2 \to H \) is continuous. But this is true by definition of the profinite topology on \( G_1 \), which completes the proof. \( \square \)

**Lemma A.2.** Let \( G \) be a dense subgroup of the metric group \( \hat{G} \) and \( H \) a compact metric group. Then any continuous homomorphism \( \varphi : G \to H \) has a unique continuous extensions to a homomorphism \( \hat{\varphi} : \hat{G} \to H \).

**Proof.** Given \( g \in \hat{G} \) we can find a sequence \( g_i \in G \) such that \( g_i \to g \). Because \( H \) is compact, by restricting to a subsequence we may assume that \( \varphi(g_i) \to h \) for some \( h \in H \). If \( g_i' \to g \) is any other such sequence with \( \varphi(g_i') \to h' \), then \( g_i'g_i^{-1} \to e \) so that

\[
    h'h^{-1} = \lim_{i \to \infty} \varphi(g_i') \varphi(g_i^{-1}) = \varphi(e) = e,
\]

so we see that \( \hat{\varphi} : g \mapsto h \) is well defined. In particular, for \( g \in G \), we may take \( g_i = g \), so that \( \hat{\varphi}(g) = \varphi(g) \), namely it is an extension of \( \varphi \).

It is now a standard exercise to show that \( \hat{\varphi} \) is a continuous homomorphism. \( \square \)

**Corollary A.3.** Let \( G, H \) be finitely generated, residually finite groups with the profinite topology. Then any homomorphism \( \varphi : G \to H \) can be extended uniquely to a continuous homomorphism \( \hat{\varphi} : \hat{G} \to \hat{H} \).

**Proof.** By lemma A.1 we know that \( \varphi \) is continuous, and therefore its composition with the embedding \( H \hookrightarrow \hat{H} \) is continuous. To apply lemma A.2 we first note that by definition \( G \) is dense in \( \hat{G} \), and every profinite completion, and in particular \( \hat{H} \) is compact. Finally, if a group \( G \) is finitely generated, then it has only countably many finite quotients. It follows that \( \prod G/K \) is a countable product of (discrete) metric spaces and therefore it is a metric space in itself. In our case, we get that \( G, H, \hat{G} \) and \( \hat{H} \) are metric space. We can now apply lemma A.2 to prove this lemma. \( \square \)

In our proofs we work with subgroups of a group with the profinite topology, so we want a simple condition when such a subgroup inherits the profinite topology as the subspace topology. Once a subgroup \( H \leq G \) has the profinite topology, we also want to show that its closure \( \overline{H} \) in \( \hat{G} \) is isomorphic to \( \hat{H} \).
Lemma A.4. Let $G$ be a group with the profinite topology and $H \leq G$ a subgroup. Then the induced topology on $H$ is the profinite topology if one of the following is true.

1. $H$ is a free factor of $G$.
2. $H$ is a finite index subgroup of $G$.
3. $H$ can be reached by a finite sequence of taking free factors and finite index subgroups.

Proof. In general, the induced topology on $H$ is the weakest topology such that any composition $H \to G \to K$, $K$ finite, is continuous. It follows that this topology is weaker than the profinite topology on $H$. To show equality we need to show that any $\varphi : H \to K$, $K$ finite, is continuous, or equivalently the kernel is open in $H$.

1. Write $G = H \ast \hat{H}$ and let $\pi : G \to \hat{H}$ be the projection which fixes $H$ and sends $\hat{H}$ to the identity. If $K$ is finite, and $\varphi : H \to K$, then $G \xrightarrow{\pi} H \xrightarrow{\varphi} K$ is continuous by the definition of the profinite topology on $G$, and since $H \hookrightarrow G$ is continuous, then so is $\varphi : H \to G \xrightarrow{\pi} H \xrightarrow{\varphi} K$, which completes this case.

2. Let $\varphi : H \to K$ where $K$ is finite and set $N = \ker(\varphi)$. Under the assumption that $[G : H] < \infty$, we get that $[G : N] < \infty$. It follows that $N$ is closed and open in $G$ and therefore in $H$, implying that $\varphi$ is continuous.

3. Follows by induction.

Lemma A.5. Let $G$ be a group with the profinite topology and $H \leq G$ a subgroup, both of which are finitely generated. If the induced topology on $H$ is the profinite topology, then $\overline{H}$ in $\hat{G}$ is naturally isomorphic to $\hat{H}$.

Proof. Consider the continuous map $\varphi : H \to \overline{H} \leq \hat{G}$. Since $\hat{G}$ is compact, then so is $\overline{H}$, so we can then use lemma A.2 to extend $\varphi$ to a continuous map $\hat{\varphi} : \hat{H} \to \overline{H}$. Since $\hat{\varphi}$ is injective on the dense subgroup $H$ inside $\hat{H}$ (both of which are metric spaces) it must also be injective on $H$. On the other hand, since $H$ is dense in $\overline{H}$ and $H \leq \text{Im}(\hat{\varphi})$, we conclude that $\hat{\varphi}$ is surjective as well. Finally, since $\hat{\varphi}$ is a continuous bijection between compact and Hausdorff spaces, its inverse is continuous as well, so we conclude that $\hat{\varphi} : \hat{H} \to \overline{H}$ is a homeomorphism as well, thus completing the proof.

Finally, the next lemma shows how to understand the topology on $\hat{G}$. This topology is generated by open (and closed) finite index subgroups of $\hat{G}$ which correspond to finite index subgroups of $G$.

Lemma A.6. Let $G$ be a finitely generated group with the profinite topology. The map $N \to \overline{N} \leq \hat{G}$ is a bijection between finite index subgroups of $G$ and finite index open and closed subgroups of $\hat{G}$ with the inverse map $\overline{N} \mapsto \overline{N} \cap G = N$.

Proof. The trick here if $\varphi : \hat{G} \to H$ is continuous for some finite group $H$ with the discrete topology, then the inverse of any subset from $H$ is closed an open. But if $U \subseteq \hat{G}$ is such a set, then since $G$ is dense in $\hat{G}$, for any $\hat{g} \in U$ we can find $g_i \in U \cap G$ which converge to $\hat{g}$. It follows that $U \subseteq \overline{U \cap G} \subseteq \overline{U} = U$ so we get an equality $U = \overline{U \cap G}$. In particular this is true for finite index closed and open subgroups of $\hat{G}$.

For the other direction, if $K \leq \hat{G}$, then the map $\pi : G \to G/K$ is continuous in the profinite topology (by definition), so by lemma A.2 we have the continuous extension $\hat{\pi} : \hat{G} \to G/K$. In particular $\hat{\pi}^{-1}(e) \leq \hat{G}$ is a closed and open subgroup such that $\hat{\pi}^{-1}(e) \cap G = \pi^{-1}(e) = K$, so by our argument above $\hat{\pi}^{-1}(e) = \overline{K}$.
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E-mail address: eofirdavid@gmail.com