RESTRICTION OF $p$-ADIC REPRESENTATIONS OF $GL_2(Q_p)$ TO PARAHORIC SUBGROUPS

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ABSTRACT. Without using the $p$-adic Langlands correspondence, we prove that for many finite length smooth representations of $GL_2(Q_p)$ on $p$-torsion modules the $GL_2(Q_p)$-linear morphisms coincide with the morphisms that are linear for the normalizer of a parahoric subgroup. We identify this subgroup to be the Iwahori subgroup in the supersingular case, and $GL_2(Z_p)$ in the principal series case. As an application, we relate the action of parahoric subgroups to the action of the inertia group of $Gal(\overline{Q_p}/Q_p)$, and we prove that if an irreducible Banach space representation $\Pi$ of $GL_2(Q_p)$ has infinite $GL_2(Z_p)$-length then a twist of $\Pi$ has locally algebraic vectors. This answers a question of Dospinescu. We make the simplifying assumption that $p > 3$ and that all our representations are generic.

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1. INTRODUCTION.

Let $p \geq 5$ be a prime number. Fix a finite extension $E/Q_p$, with ring of integers $O$ and residue field $k$. The purpose of this paper is to study the behaviour of representations of $G = GL_2(Q_p)$ on $p$-torsion $O$-modules and on $E$-Banach spaces upon restriction to a parahoric subgroup. Our motivation for doing so arises from two applications of our results. The first one is the following theorem, which answers a question of Dospinescu by providing a classification of those irreducible unitary Banach space representations of $GL_2(Q_p)$ which have infinite length when restricted to the maximal compact subgroup $K = GL_2(Z_p)$. Let $Z \cong Q_p^\times$ be the centre of $G$.

**Theorem 1.0.1.** Let $\Pi$ be an absolutely irreducible, admissible, very generic, unitary $E$-Banach space representation of $GL_2(Q_p)$ with central character $\zeta : Z \to O^\times$. (See Section 2.1.3 for the precise genericity conditions we need.) Then exactly one of the following holds:

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1. $\Pi|_{KZ}$ is irreducible.
(2) \( \Pi \) has irreducible supersingular reduction, \( \Pi \cong \Pi \otimes (\nr_{-1} \circ \det) \), and \( \Pi \cong \Ind_{\mathbb{Q}_p}^{\mathbb{G}}(\Pi_0) \) for some irreducible \( G^+ \)-representation \( \Pi_0 \) such that \( \Pi_0|_{\mathbb{Q}_p} \) is irreducible. (Here \( G^+ = \ker(\nr_{-1} \circ \det) \) is the subgroup of \( \text{GL}_2(\mathbb{Q}_p) \) of elements whose determinant has even valuation.)

(3) \( \Pi \) has a closed \( KZ \)-stable \( E \)-subspace of finite dimension over \( E \).

**Corollary 1.0.2.** Let \( \Pi \) be as in the statement of Theorem 1.0.1. If \( \Pi|_K \) has infinite length then a twist of \( \Pi \) has locally algebraic vectors.

The statements above refer to topological irreducibility, and all cases of Theorem 1.0.1 can occur (see Remark 3.1.8). We remark that our proofs are independent of the \( p \)-adic Langlands correspondence for \( \text{GL}_2(\mathbb{Q}_p) \), and so they have a chance of being applicable to other groups. For example, Theorem 1.0.1 provides a different perspective on questions recently considered by Dospinescu, Paškūnas and Schraen [Paš18, DPS] about the length of certain Banach space representations of another compact-mod-centre \( p \)-adic group, namely the units \( D^\times \) of the non-split quaternion algebra \( D \) over \( \mathbb{Q}_p \). We hope that our methods may remain applicable in that context, especially to the cases left untreated in their work (namely when the corresponding Galois representation is Hodge–Tate at \( p \), see [DPS, Section 1.6]).

On the other hand, we emphasize that to get Theorem 1.0.1 we make use of one of the main results of the paper [Paš13], namely that absolutely irreducible admissible unitary Banach space representations of \( \text{GL}_2(\mathbb{Q}_p) \) are residually of finite length, and that a list of the possible reductions can be given in generic cases. This uses Colmez’s functor in a significant way. We could avoid this appeal to [Paš13] by making explicit assumptions in Theorem 1.0.1 about the reduction of a unit ball in \( \Pi \): this would yield a statement similar to Corollary 1.0.2. Similarly, using the \( p \)-adic local Langlands correspondence one can easily prove a converse to Corollary 1.0.2.

The second of our applications is concerned with a \( p \)-adic analogue of the so-called “inertial Langlands correspondence”, which is a refinement of the classical local Langlands correspondence obtained by considering compact subgroups of the groups appearing at the two sides of the correspondence. In more detail, in the setting of smooth representations of \( \text{GL}_2(\mathbb{Q}_p) \) with complex coefficients we have that two irreducible smooth representations are in the same Bernstein component if and only if their Langlands parameters have isomorphic restriction to the inertia group \( I_{\mathbb{Q}_p} \). On the other hand, as a consequence of the theory of types, one knows that two irreducible cuspidal \( G \)-representations \( \pi_1, \pi_2 \) are in the same Bernstein component if and only if \( \pi_1|_K \cong \pi_2|_K \). Since the cuspidal representations \( \pi_1 \) and \( \pi_2 \) are in the same Bernstein component if and only if they are unramified twists of each other, it follows that \( \pi_1|_{\mathbb{Q}_p} \cong \pi_2|_{\mathbb{Q}_p} \) if and only if \( \pi_1 \cong \pi_2 \otimes (\nr_{\pm 1} \circ \det) \). We prove the following analogous result in the setting of \( E \)-Banach space representations.

**Theorem 1.0.3.** Let \( \Pi_1, \Pi_2 \) be absolutely irreducible, admissible, very generic, non-ordinary, unitary \( E \)-Banach representations of \( \text{GL}_2(\mathbb{Q}_p) \) with central character \( \zeta \). Write \( Iw \) for the Iwahori subgroup of \( G \) and \( N \) for its normalizer in \( G \).

1. If \( \Pi_1, \Pi_2 \) have reducible reduction, then \( \Pi_1|_{Iw} \cong \Pi_2|_{Iw} \) if and only if \( \Pi_1 \cong \Pi_2 \otimes (\nr_{\pm 1} \circ \det) \), and \( \Pi_1|_N \cong \Pi_2|_N \) if and only if \( \Pi_1 \cong \Pi_2 \).
2. If \( \Pi_1, \Pi_2 \) have reducible reduction, then \( \Pi_1|_{KZ} \cong \Pi_2|_{KZ} \) if and only if \( \Pi_1 \cong \Pi_2 \otimes (\nr_{\pm 1} \circ \det) \).
3. Otherwise, there are no \( Iw \)-linear isomorphisms \( \Pi_1 \cong \Pi_2 \).

**Corollary 1.0.4.** Let \( \rho_1, \rho_2 : \text{Gal}_{\mathbb{Q}_p} \to \text{GL}_2(\mathbb{E}) \) be absolutely irreducible continuous Galois representations with det \( \rho_1 = \det \rho_2 \), and write \( \overline{\rho}_i \) for the semisimplified mod \( p \) reduction of \( \rho_i \). Assume \( \overline{\rho}_i|_{\mathbb{Q}_p} \) is not isomorphic to \( 1 \oplus \omega \) or \( 1 \oplus 1 \), where \( \omega \) is the cyclotomic character. Let \( \Pi_1, \Pi_2 \) be the \( E \)-Banach space representations of \( \text{GL}_2(\mathbb{Q}_p) \) corresponding to \( \rho_1 \) under Colmez’s functor.

1. If \( \overline{\rho}_1, \overline{\rho}_2 \) are irreducible, then \( \rho_1|_{\mathbb{Q}_p} \cong \rho_2|_{\mathbb{Q}_p} \) if and only if \( \Pi_1|_{Iw} \cong \Pi_2|_{Iw} \).
2. If \( \overline{\rho}_1, \overline{\rho}_2 \) are reducible, then \( \rho_1|_{\mathbb{Q}_p} \cong \rho_2|_{\mathbb{Q}_p} \) if and only if \( \Pi_1|_{KZ} \cong \Pi_2|_{KZ} \).
3. Otherwise, \( \rho_1|_{\mathbb{Q}_p} \) is not isomorphic to \( \rho_2|_{\mathbb{Q}_p} \).

The corollary follows from the theorem since by Clifford theory and the condition on the determinant we have \( \rho_1|_{\mathbb{Q}_p} \cong \rho_2|_{\mathbb{Q}_p} \) if and only if \( \rho_1 \cong \rho_2 \otimes \nr_{\pm 1} \) (see Proposition 3.4.3). We see that Corollary 1.0.4 relates the inertia group \( I_{\mathbb{Q}_p} \) to a parahoric subgroup of \( \text{GL}_2(\mathbb{Q}_p) \), as in the case of smooth representations, but the fact that the subgroup changes according to the reduction type of the Galois representation seems to be a new feature of the \( p \)-adic case.
Although Theorems 1.0.1 and 1.0.3 are about \( E \)-Banach space representations, the main input in their proof is Theorem 2.5.1, a stronger version of Theorem 1.0.3 valid for \( p \)-torsion representations, which may be of independent interest. It is proved as a combination of the results in Section 2, building on work of Morra and Paškūnas, without using the \( p \)-adic Langlands correspondence for \( GL_2(Q_p) \). Theorems 1.0.1 and 1.0.3 then follow directly in the case of supersingular reduction. In the case of reducible reduction we need a more involved argument, making use of a Banach space version of Ribet’s lemma on lattices in irreducible two-dimensional representations with reducible reduction, which we develop in Appendix A.

1.1. Acknowledgments. The problem of relating the actions of \( I_{Q_p} \) and parahoric subgroups has been considered by Caraiani–Emerton–Gee–Geraghty–Paškūnas–Shin and (independently) Gabriel Dospinescu. Their methods were different, making use of Colmez’s functor, and were not brought to completion (for instance, the role of the Iwahori subgroup seems to be unexpected). I learned about this problem from Toby Gee, and I am grateful to him as well as Matthew Emerton for helpful conversations on these and related subjects. I thank Vytautas Paškūnas for explaining me the exact sequence (2.4.8), and Stefano Morra for sharing some of his unpublished notes on [Mor17].

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1.2. Notation and conventions. We fix throughout the article a finite extension \( E/Q_p \) with ring of integers \( \mathcal{O} \) and residue field \( k \), as well as an algebraic closure \( \overline{k} \) of \( k \), to act as coefficients. Fix an algebraic closure \( \overline{Q}_p/Q_p \) and write \( G_{\overline{Q}_p} = \text{Gal}(\overline{Q}_p/Q_p) \), \( I_{Q_p} \subset G_{\overline{Q}_p} \), \( \text{for the inertia group, and } \omega: \text{Gal}(Q_p) \rightarrow \mathbb{Z}_p^\times \) for the cyclotomic character. We normalize local class field theory so that \( p \omega : \mathbb{O}^x \rightarrow \mathbb{O}^x \) for the unramified character sending \( p \) to \( \lambda \), and similarly if \( \lambda \in k^\times \). We will usually work with a fixed continuous character \( \zeta : \mathbb{Q}_p^\times \rightarrow \mathbb{O}^\times \) to act as the central character of our \( GL_2(Q_p) \)-representations.

Write \( G = GL_2(Q_p) \), \( K = GL_2(Z_p) \), \( B \) for the upper-triangular Borel subgroup, \( Z \cong Q_p^\times \) for the centre, and \( T \) for the diagonal torus. The index-two subgroup \( G^+ \subset G \) is defined to be the kernel of \( \text{nr}_1 \circ \text{det} \).

We will write \( U(p^n)Z_p = \begin{pmatrix} 1 & p^nZ_p \\ 0 & 1 \end{pmatrix} \), and similarly for the lower-triangular unipotent \( U \). Define \( K_0(p^n) = \begin{pmatrix} Z_p^\times & Z_p \\ p^nZ_p & Z_p^\times \end{pmatrix} \), so that \( K_0(p) = \text{Iw} \) is the Iwahori subgroup. We will sometimes write \( K_0(p^\infty) \) for \( B(\mathbb{Z}_p) \).

The principal congruence subgroups of \( K \) will be denoted \( K_n = \begin{pmatrix} 1 & p^nZ_p \\ p^nZ_p & 1 \end{pmatrix} \), \( 1+p^nZ_p \), and \( \text{Iw} = H \rtimes \text{Iw}_1 \) for the subgroup \( H = \begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix} \), where \([\cdot]\) denotes the Teichmüller lift to \( Z_p \) of an element of \( \mathbb{F}_p \). The character \( ad^{-1} \) of \( T(\mathbb{F}_p) \) (or its inflation to other groups such as \( H \) and \( \text{Iw} \)) is denoted \( \alpha \).

Except in the section on Banach space representations, we will write

\[
\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The group \( N = \Pi^2 \rtimes \text{Iw} \) is the normalizer of \( \text{Iw} \) in \( G \). If \( \pi \) is a smooth representation of \( \text{Iw} \), we will sometimes denote the twist \( \text{ad}(\Pi)^{\times}(\pi) \) by \( \pi^\dagger \): this is the representation of \( \text{Iw} \) with the same representation space as \( \pi \) but the action

\[
\pi^\dagger \begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \pi(\Pi \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \Pi^{-1}).
\]

If \( G \) is a locally profinite group with a closed subgroup \( K \) and an open subgroup \( H \), we write \( \text{Ind}^G_K \) for induction and \( c-\text{Ind}^G_H \) for compact induction, so that \( \text{Ind}^G_K \) is right adjoint to \( \text{Res}^G_K \) and \( c-\text{Ind}^G_H \) is left adjoint to \( \text{Res}^G_H \). If \( \pi \) is a representation of \( G \) and \( x \in \pi \), we will write \( \langle G \cdot x \rangle \) for the smallest \( G \)-subrepresentation
of $\pi$ that contains $x$. Unless stated otherwise, the notation $\mathrm{Ext}^i_G$ will denote Ext-groups computed in the category of smooth $k[G]$-representations.

2. MOD $p$ REPRESENTATIONS.

2.1. Preliminaries. We begin with some generalities about uniserial representations of profinite groups. Then we list the irreducible representations of $k[GL_2(F_p)]$ and $k[GL_2(Q_p)]$ we will work with, and we state our genericity conditions. We will fix a continuous character $\zeta : O^\times \rightarrow \mathbb{Z}$, and we will usually assume that representations of $G$ or $G^+$ have central character $\zeta$.

2.1.1. Uniserial representations. Let $H$ be a profinite group with an open pro-$p$ subgroup, and let $\pi$ be a smooth representation of $k[H]$. Even if $\pi$ has infinite $H$-length, we can define its socle filtration by letting $\text{soc}^0(\pi)$ be the maximal semisimple $H$-subrepresentation of $\pi$, and then defining $\text{soc}^i(\pi)$ via the short exact sequence

$$0 \rightarrow \text{soc}^{i-1}(\pi) \rightarrow \text{soc}^i(\pi) \rightarrow \text{soc}^0(\pi/\text{soc}^{i-1}(\pi)) \rightarrow 0.$$ 

Since every vector $x \in \pi$ generates a finite-dimensional $H$-representation we have $\pi = \bigcup_i \text{soc}^i(\pi)$. We will usually write $\text{soc}$ for $\text{soc}^0$.

Lemma 2.1.2. Let $\pi = \pi_1 \oplus \pi_2$ be a smooth representation of $k[H]$. Then $\text{soc}^i(\pi) = \text{soc}^i(\pi_1) \oplus \text{soc}^i(\pi_2)$.

Proof. By the exact sequence

$$0 \rightarrow \text{soc}(\pi) \rightarrow \text{soc}^{i+1}(\pi) \rightarrow \text{soc}^i(\pi/\text{soc}(\pi)) \rightarrow 0$$

and induction on $i$ it suffices to prove that $\text{soc}(\pi) = \text{soc}(\pi_1) \oplus \text{soc}(\pi_2)$. For this, observe that if $X$ is a semisimple smooth representation of $k[H]$, and $i : X \rightarrow \pi$ is an $H$-linear map, then $\ker(i) \subseteq \text{soc}(\pi_i)$ for $i = 1, 2$, where $\pi_i$ the projection $\pi \rightarrow \pi_i$, and so $\text{soc}(\pi) \subseteq \text{soc}(\pi_1) \oplus \text{soc}(\pi_2)$. The other direction is true by definition.

Definition 2.1.3. We say that a $k[H]$-module $\pi$ is uniserial if the set of $k[H]$-submodules of $\pi$ is totally ordered by inclusion.

Lemma 2.1.4. The representation $\pi$ is uniserial if and only if $\text{soc}^{i+1}(\pi)/\text{soc}^i(\pi)$ is irreducible for every $i \in \mathbb{Z}_{\geq 0}$.

Proof. Every semisimple reducible $H$-representation contains two submodules incomparable by inclusion. Conversely, assume that all the graded factors of the socle filtration of $\pi$ are irreducible, and let $X \subset \pi$ be a proper submodule. Let $i$ be such that $\text{soc}^i(\pi) \subseteq X$ but $\text{soc}^{i+1}(\pi) \not\subseteq X$. Consider the submodule $X/\text{soc}^i(\pi) \subset \pi/\text{soc}^i(\pi)$. Since $\pi$ is smooth, if $X/\text{soc}^i(\pi) \neq 0$ then it contains an irreducible $k[H]$-subspace. But since $\text{soc}^i(\pi/\text{soc}^i(\pi))$ is irreducible, this implies that $\text{soc}^{i+1}(\pi) \subseteq X$, which we are assuming not to be the case. So $X = \text{soc}^i(\pi)$, hence the only submodules of $\pi$ are the $\text{soc}^i(\pi)$, and the lattice of submodules of $\pi$ is totally ordered by inclusion.

Lemma 2.1.5. If $\pi_1, \pi_2$ are uniserial smooth representations of $k[H]$, then every proper $H$-submodule of $\pi_i$ is finite-dimensional. If $\pi_1$ is infinite-dimensional, then every nonzero $k[H]$-linear morphism $\lambda : \pi_1 \rightarrow \pi_2$ is surjective.

Proof. Let $\pi'_i \subset \pi_i$ be a proper $H$-submodule. If $x \in \pi_i \setminus \pi'_i$, then the definition of a uniserial representation implies that $\pi'_i \subseteq \langle H \cdot x \rangle$, which is finite-dimensional by $H$-smoothness. The second claim follows from the first, counting dimensions on the exact sequence $0 \rightarrow \ker \lambda \rightarrow \pi_1 \rightarrow \im \lambda \rightarrow 0$.

2.1.6. Serre weights. Every irreducible $k$-representation of $GL_2(F_p)$ (also known as Serre weight) has the form $\sigma_{r,s} = \text{Sym}^r k^2 \otimes \text{det}^s$ for uniquely determined integers $0 \leq r \leq p-1$ and $0 \leq s \leq p-2$. We will usually realize $\text{Sym}^r$ on the space of degree $r$ homogeneous polynomials in two variables $x$ and $y$, and coefficients in $k$, via the action

$$(a \ b \ c \ d) x^iy^{-i} = (ax + cy)^i(bx + dy)^{-i}.$$ 

The group $U(F_p)$ fixes a unique line in $\sigma_{r,s}$, spanned by $x^r$, whose eigencharacter for the Borel subgroup is $\chi : (a \ b \ c \ d) \mapsto a^r(ad)^s$. This character determines $\sigma$ whenever $r \neq 0, p-1$, or equivalently whenever $\chi$ is not
equal to its conjugate $\chi^*$. We will call such a character a generic character, and we will say that a weight $\sigma$ is generic if $\sigma^{1w}$ is a generic character. (So the nongeneric weights are the characters and the twists of the Steinberg representation.)

2.1.7. $GL_2(\mathbb{Q}_p)$-representations. Recall that we fix throughout the paper a continuous character $\zeta : \mathbb{Q}_p^\times \to O^\times$. If $\sigma$ is a Serre weight we will often implicitly regard it as a $K$-representation, and we will extend the $K$-action to an action of $KZ$ by letting $p \in Z$ act by $\zeta(p)$. (We do not require that $Z$ act by $\zeta$ on $\sigma$, because it will be notationally convenient to work with all Serre weights rather than those with central character $O_{\mathbb{Q}_p}$.)

We follow the normalizations of [Bre03a] unless stated otherwise. We will work over the finite field $k$, and we will restrict attention to the following representations. We follow the normalizations of [Bre03a] unless stated otherwise.

1. Irreducible principal series. These are of the form

$$\pi(r, \lambda, \chi) = \left(c\text{-Ind}_{KZ}^G \text{Sym}^r k^2 / (T - \lambda)\right) \otimes (\chi \circ \det)$$

for a smooth character $\chi : \mathbb{Q}_p^\times \to k^\times$ with $\chi(p) = \pm 1$, and $(r, \lambda) \not\in \{(0, \pm 1), (p - 1, \pm 1)\}$. The only intertwinings are

$$\pi(r, \lambda, \chi) \cong \pi(r, -\lambda, nr_{-1}\chi) \text{ and } \pi(0, \lambda, \chi) \cong \pi(p - 1, \lambda, \chi).$$

These representations can be written as parabolic inductions by considering the characters $\psi = nr_{-1} \otimes nr_{1}\omega^r : T(\mathbb{Q}_p) \to k^\times$. If the pair $(r, \lambda) \not\in \{(0, \pm 1), (p - 1, \pm 1)\}$, then the representation $\text{Ind}^G_B(\psi)$ is irreducible and isomorphic to

$$\pi(r, \lambda, 1) = c\text{-Ind}_{KZ}^G (\text{Sym}^r k^2) / (T - \lambda).$$

See for example [Bre03a, Remarque 4.2.5]. Here and in what follows, $T$ is the usual Hecke operator.

2. Supersingular representations. These have the form

$$\pi(r, 0, \chi) = \left(c\text{-Ind}_{KZ}^G \text{Sym}^r k^2 / T\right) \otimes (\chi \circ \det) = \left(c\text{-Ind}_{KZ}^G (\text{Sym}^r k^2 \otimes \det^s) / T\right) \otimes (nr_{\pm 1} \circ \det)$$

for $0 \leq r \leq p - 1$, $1 \leq s \leq p - 1$, where we have written $\chi = nr_{\pm 1}\omega^s$. The only intertwining isomorphisms are

$$\pi(r, 0, \chi) \cong \pi(r, 0, \chi nr_{-1}) \cong \pi(p - 1 - r, 0, \chi^* \omega^r) \cong \pi(p - 1 - r, 0, \chi^* \omega^r nr_{-1}).$$

3. Characters and Steinberg twists. These arise as the Jordan–Hölder factors of reducible principal series, and will not be studied in this article.

2.1.8. Genericity conditions. The following are the genericity conditions we use in this paper:

1. A Serre weight is generic if it is not a character or a twist of the Steinberg representation of $GL_2(\mathbb{F}_p)$.

2. An irreducible $\mathcal{O}[GL_2(\mathbb{Q}_p)]$-representation is generic if it is isomorphic to $\pi(r, \lambda, \chi)$ for $r \not\in \{0, p - 1\}$.

It is very generic if it is generic and not isomorphic to $\pi(r, \lambda, \chi)$ for $r = p - 2$ and $\lambda \not= 0$. (See Remark 2.1.12 for why we will need to exclude twists of $\text{Sym}^{p-2}$ at certain stages of our arguments.)

3. A finite length $\mathcal{O}[GL_2(\mathbb{Q}_p)]$-representation is generic, resp. very generic if all its Jordan–Hölder factors are generic, resp. very generic.

4. A unitary admissible $E$-Banach space representation $\Pi$ of $GL_2(\mathbb{Q}_p)$ is generic, resp. very generic if $\overline{\Pi}$ is generic, resp. very generic for all open bounded $GL_2(\mathbb{Q}_p)$-stable lattices $\Theta \subset \Pi$.

2.2. Restriction of principal series to parahoric subgroups. The Iwasawa decomposition $G = BK$ implies that there is a $K$-linear isomorphism

$$\text{Ind}_{B}^{G}(nr_{\lambda-1} \otimes nr_{\lambda}\omega^{r+1}) \to \text{Ind}_{B}^{K}(1 \otimes \omega^{r+1}), \quad f \mapsto f|_{K}.$$ 

In this section we study certain representations related to the one appearing in the right-hand side of this isomorphism, and their restriction to the Iwahori subgroup.

Let $\chi : T(\mathbb{Z}_p) \to k^\times$ be a smooth character. We begin with some properties of the finite induction $\pi_{n+1}(\chi) = \text{Ind}_{K_0(p)(nr_{n+1})}^{K_0(p)}(\chi)$, which is a uniserial $K_0(p)$-representation of dimension $p^n$ (see for example [Mor11 Proposition 1.6]). We always realize an induction to $K_0(p)$ as a space of smooth functions on $K_0(p)$, and we write $\varphi_{n+1}$ for the unique function in $\pi_{n+1}(\chi)$ that is supported in $K_0(p^{n+1})$ and takes value 1 at the identity. It generates the $K_0(p)$-coscle of $\pi_{n+1}(\chi)$. 
For any $n$ there is an injection $\pi_n(\chi) \to \pi_{n+1}(\chi)$ that is an inclusion of spaces of smooth functions on $K_0(p)$. There is also a surjection $\pi_{n-1}(\chi) \to \pi_n(\chi)$ that sends $\varphi_{n-1}$ to $\varphi_n$. Hence the (unique) submodule of $\pi_{n+1}(\chi)$ of dimension $p^{n-1}$ and the (unique) quotient of $\pi_{n+1}(\chi)$ of dimension $p^{n-1}$ are both isomorphic to $\pi_n(\chi)$. In fact something stronger is true, by the following lemma.

**Lemma 2.2.1.** Let $1 \leq i \leq n+1$. Then $\pi_{n+1}(\chi)$ has an Iw-stable filtration with $p^{n-1}$ graded factors of dimension $p^{i-1}$, of the form

$$\pi_i(\chi) - \pi_i(\chi\alpha) - \pi_i(\chi\alpha^2) - \cdots.$$ 

**Proof.** By [Mor11 Proposition 1.6], the socle filtration of $\text{Ind}_{K_0(p^n)}^{K_0(p^n+1)}(\chi)$ is

$$\chi - \chi\alpha - \chi\alpha^2 - \cdots - \chi\alpha^{p-2} - \chi$$

with $p$ graded factors. By transitivity of induction and exactness of the functor $\text{Ind}_{K_0(p^n)}^{K_0(p^n)}(-)$, there is a filtration on $\pi_{n+1}(\chi) \cong \text{Ind}_{K_0(p^n)}^{K_0(p^n+1)}(\chi)$ of the form

$$\pi_n(\chi) - \pi_n(\chi\alpha) - \cdots \pi_n(\chi\alpha^{p-2}) - \pi_n(\chi).$$

The lemma follows by induction on $n$. □

The following two lemmas about eigenvectors in $\pi_{n+1}(\chi)$ will be fundamental to our argument. Recall that $H$ is the Teichmüller lift of $\mathbf{F}_p^\times \times \mathbf{F}_p^\times$ in the diagonal torus $T$.

**Lemma 2.2.2.** Let $\psi : H \to k^\times$ be a character. Then

$$\text{Hom}_{K_0(p^{n+1})}(\psi, \text{Ind}_{K_0(p^n)}^{K_0(p^n+1)}(\chi)) = \text{Hom}_{B(\mathbf{Z}_p)}(\psi, \text{Ind}_{K_0(p^n)}^{K_0(p^n+1)}(\chi)),$$

and this space vanishes unless $\psi = \chi$, in which case it has dimension $n+1$ over $k$.

**Proof.** Compare [Cas73 Lemma 1]. When $\psi = \chi$, both spaces have dimension at least $n+1$. To see this, let $\varphi_i \in \pi_i(\chi)$ be the function supported in $K_0(p^n)$ taking value 1 at the identity. It is a $K_0(p^n)$-eigenvector with eigencharacter $\chi$. Since $\varphi_i$ generates the Iw-cosocle of $\pi_i(\chi)$, it is not contained in any proper Iw-subrepresentation of $\pi_i(\chi)$. By this fact and the uniseriality of $\pi_{n+1}(\chi)$, it follows that $\{\varphi_i : 1 \leq i \leq n+1\}$ is a linearly independent set of $K_0(p^{n+1})$-eigenvectors in $\pi_{n+1}(\chi)$.

Now, by the formula

$$\begin{pmatrix} a & b \\ up^i & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & u^{-1}(ad - bp^i) \end{pmatrix}$$

valid whenever $a,d,u \in \mathbf{Z}_p^\times$ and $b \in \mathbf{Z}_p$, we deduce that there is a disjoint double coset decomposition

$$K_0(p) = \bigoplus_{i=1}^{n+1} K_0(p^{n+1}) \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} K_0(p^{n+1}) = \bigoplus_{i=1}^{n+1} K_0(p^{n+1}) \begin{pmatrix} 1 & 0 \\ p^i & 1 \end{pmatrix} B(\mathbf{Z}_p).$$

The summand indexed by $n+1$ is $K_0(p^{n+1})$ itself. The lemma follows because each of these double cosets supports at most one-dimensional space of $K_0(p^{n+1})$-eigenvectors, respectively $B(\mathbf{Z}_p)$-eigenvectors. □

**Lemma 2.2.3.** Let $\chi_1, \chi_2 : H \to k^\times$ be smooth characters and let $\alpha : \pi_{n+1}(\chi_1) \to \pi_{n+1}(\chi_2)$ be an Iw-linear morphism. Then $\alpha = 0$ if $\chi_1 \neq \chi_2$. If $\chi_1 = \chi_2$ and $\alpha$ is not an isomorphism, then $\dim \ker(\alpha) = p^n - p^{n-1}$.

**Proof.** The first statement is an immediate consequence of Lemma 2.2.2 and Frobenius reciprocity. For the second, recall from the proof of Lemma 2.2.2 that the set $\{\varphi_i : 1 \leq i \leq n+1\}$ is a basis of the $\chi$-eigenspace of $B(\mathbf{Z}_p)$ in $\pi_{n+1}(\chi)$. Now, the image of $\alpha : \pi_{n+1}(\chi) \to \pi_{n+1}(\chi)$ is generated by $\alpha(\varphi_{n+1})$, which can be written as $\sum_{i=1}^{n+1} \lambda_i \varphi_i$, for some $\lambda_i \in k$. If $\lambda_{n+1} \neq 0$, then $\alpha(\varphi_{n+1})$ generates the Iw-cosocle of $\pi_{n+1}(\chi)$, hence $\alpha$ is surjective, and comparing dimensions it is an isomorphism. Otherwise, the image of $\alpha$ is contained in $\pi_n(\chi)$, which has dimension $p^{n-1}$. □

Next we consider the inductions from $B(\mathbf{Z}_p)$. Recall from (1.2.1) that $\pi^+$ denotes the twist of an Iw-representation $\pi$ by $\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. 

Theorem 2.2.4 (Morra). Let $\chi : T(\mathbb{Z}_p) \to k^\times$ be a smooth character. Write $\pi_\infty(\chi) = \text{Ind}_{K_0(p)}^{K_0(p)}(\chi)$. Then restriction of functions to $Iw = K_0(p)$ defines a split short exact sequence of $Iw$-representations

\begin{equation}
0 \to \pi_\infty^+(\chi) \to \text{Ind}_{H(\mathbb{Z}_p)}^{K_0(p)}(\chi) \xrightarrow{\text{res}} \pi_\infty(\chi) \to 0
\end{equation}

The $Iw$-representation $\pi_\infty(\chi)$ is uniserial, with socle $\chi$, and its socle filtration satisfies

$soc^{k+1}(\pi_\infty(\chi))/soc^{k}(\pi_\infty(\chi)) \cong \alpha \otimes (soc^{k}(\pi_\infty(\chi))/soc^{k-1}(\pi_\infty(\chi)))$.

Proof. The isomorphism of $\pi_\infty^+(\chi)$ with the kernel of restriction follows from the decomposition $G = BIw \coprod BsIw$. The rest of the theorem follows from [Mor11, Proposition 1.6] and the isomorphism $\pi_\infty(\chi) \cong \lim_{\to n} \pi_n(\chi)$. □

Proposition 2.2.5. Let $\chi_1, \chi_2 : H \to k^\times$ be characters. Then

$\text{Hom}_{Iw}(\pi_\infty(\chi_1), \pi_\infty^+(\chi_2)) = 0$ and $\text{Hom}_{Iw}(\pi_\infty^+(\chi_1), \pi_\infty(\chi_2)) = 0$.

Proof. Let $\lambda : \pi_\infty(\chi_1) \to \pi_\infty^+(\chi_2)$ be a nonzero $Iw$-linear homomorphism. By Lemma 2.1.5, it is surjective. But then the socle filtration of $\pi_\infty^+(\chi_2)$ has the same property

$soc^{k+1}(\pi_\infty^+(\chi_2))/soc^{k}(\pi_\infty^+(\chi_2)) \cong \alpha \otimes (soc^{k}(\pi_\infty^+(\chi_2))/soc^{k-1}(\pi_\infty^+(\chi_2)))$ as that of $\pi_\infty^+(\chi_2)$, contradicting the fact that $\pi_\infty^+(\chi_2) = \text{ad}(\Pi)^* \pi_\infty(\chi_2)$ and $\text{ad}(\Pi)^* \alpha = \alpha^{-1} \neq \alpha$ (since $p > 3$).

The claim that $\text{Hom}_{Iw}(\pi_\infty^+(\chi_1), \pi_\infty(\chi_2)) = 0$ follows upon twisting by $\Pi$. □

Our first main result in this section is the following theorem. The extra generality in considering quotients by proper subspaces will be useful in Section 5.2.

Theorem 2.2.6. Let $\chi_1, \chi_2 : H \to k^\times$ be characters and fix $Iw$-stable proper subspaces $X_1 \subset \pi_\infty(\chi_1)$. Let $\alpha : \pi_\infty(\chi_1)/X_1 \to \pi_\infty(\chi_2)/X_2$ be an $Iw$-linear morphism. If $\alpha \neq 0$, then $\chi_1 = \chi_2$, $X_1 \subset X_2$ and $\alpha$ is a scalar multiple of the canonical surjection $\pi_\infty(\chi)/X_1 \to \pi_\infty(\chi)/X_2$.

Proof. Assume first $\alpha \neq 0$ and $\dim(X_1) \geq \dim(X_2)$. For any $n$, we have $\alpha(\pi_{n+1}(\chi_1)/X_1) \subset \pi_{n+1}(\chi_2)/X_2$ (because the length is smaller and the representations are uniserial). If $n$ is large enough, we have $\ker(\alpha) \subset \pi_{n+1}(\chi_1)/X_1$. Hence

$$\dim \text{coker}(\alpha : \pi_{n+1}(\chi_1)/X_1 \to \pi_{n+1}(\chi_2)/X_2) = \dim \ker(\alpha) + \dim(X_1) - \dim(X_2).$$

For any $n$ large enough, the surjection of $\pi_{n+1}(\chi_2)$ onto $\pi_n(\chi_2)$ induces a surjection

$$\pi_{n+1}(\chi_2)/X_2 \to \pi_n(\chi_2)$$

since the dimension of $X_2$ is fixed, whereas the dimension of $\ker(\pi_{n+1}(\chi_2) \to \pi_n(\chi_2))$ tends to infinity with $n$ (it is equal to $p^n - p^{n-1} = p^n(p - 1)$). Consider the composition

$$\pi_{n+1}(\chi_1) \to \pi_{n+1}(\chi_1)/X_1 \xrightarrow{\alpha} \pi_{n+1}(\chi_2)/X_2 \to \pi_n(\chi_2).$$

For $n$ large enough, its cokernel has still dimension $\dim \ker(\alpha) + \dim(X_1) - \dim(X_2)$, since all representations here are uniserial. Furthermore, it factors through the $p^{n-1}$-dimensional quotient of $\pi_{n+1}(\chi_1)$, which is isomorphic to $\pi_n(\chi_1)$.

We conclude that for every $n$ large enough there is a morphism $\alpha_n : \pi_n(\chi_1) \to \pi_n(\chi_2)$ whose cokernel has dimension $\dim \ker(\alpha) + \dim(X_1) - \dim(X_2)$, which is independent of $n$. Taking $n$ large enough, this implies by Lemma 2.2.3 that $\chi_1 = \chi_2$, $\dim(X_1) = \dim(X_2)$ and $\alpha$ is injective. By Lemma 2.1.5, this implies that $\alpha$ is an isomorphism. Hence without loss of generality we can assume $\chi_1 = \chi_2$; $X_1 = X_2 = X$, and there remains to prove that $\alpha$ is a scalar. Looking at the induced map on the $Iw$-cosocle, we see that there exists a scalar $\lambda \in k$ such that

$$\alpha - \lambda : \pi_\infty(\chi)/X \to \pi_\infty(\chi)/X$$

has nontrivial kernel. But then $\alpha = \lambda$, since $\alpha - \lambda$ is either zero or an isomorphism, by what we have just proved.

Now assume that $\dim(X_1) < \dim(X_2)$ and $\alpha \neq 0$. Consider the injection $\overline{\alpha} : \pi_\infty(\chi_1)/\ker(\alpha) \to \pi_\infty(\chi_2)/X_2$. Since $\overline{\alpha}$ is injective, it is an isomorphism by Lemma 2.1.5. If $\dim(\ker(\alpha)) \geq \dim(X_2)$, the case we have just treated implies that $\chi_1 = \chi_2$ and $\ker(\alpha) = X_2$, and furthermore $\overline{\alpha}$ is a scalar, which implies that $\alpha$ is a multiple of the canonical surjection. On the other hand, if $\dim(\ker(\alpha)) < \dim(X_2)$ then
the inverse $\pi_\infty(\chi_2)/X_2 \rightarrow \pi_\infty(\chi_1)/\ker(\alpha)$ is of the type we have just treated. This implies that $\chi_1 = \chi_2$ and $X_2 = \ker(\alpha)$, which is a contradiction. \hfill \square

**Corollary 2.2.7.** The space $\text{Hom}_{Iw}(\pi_\infty(\chi_1), \pi_\infty(\chi_2))$ is one-dimensional if $\chi_1 = \chi_2$, and vanishes otherwise.

*Proof.* Follows immediately from Theorem 2.2.6 for $X_1 = 0$, $X_2 = 0$. \hfill \square

**Corollary 2.2.8.** Let $\pi_1, \pi_2$ be generic principal series representations of $k[GL_2(\mathbb{Q}_p)]$. If $\pi_1$ and $\pi_2$ have nonisomorphic $K$-socle, then $\text{Hom}_{Iw}(\pi_1, \pi_2) = 0$. Otherwise, $\text{Hom}_{KZ}(\pi_1, \pi_2) = \text{Hom}_N(\pi_1, \pi_2)$ and it has dimension one over $k$.

*Proof.* Both representations decompose as $\pi_i|_{Iw} \cong \pi_\infty(\chi_i) \oplus \pi_\infty^{+}(\chi_i)$, where $\chi_i \cong \left((\text{soc}_K \pi_i)^{Iw}\right)^\perp$. (To explain the twist, notice that $\pi|_K \cong \text{Ind}_{B(\mathbb{Z}_p)}^{K}(\chi_i)$ implies that $\pi^K_i \cong \text{Ind}_{Iw}^{K}(\chi_i)$, which implies the given formula for $\chi_i$.)

Let $\alpha : \pi_1 \rightarrow \pi_2$ be $Iw$-linear. By Proposition 2.2.5 and Corollary 2.2.6, it preserves the summands in the decomposition and it is given by a scalar on each summand. Furthermore, if $\chi_i \neq \chi_2$ then the scalars are zero, and since the $\chi_i$ determine the generic weights $\text{soc}_K(\pi_i)$, this implies that if $\text{soc}_K(\pi_1) \neq \text{soc}_K(\pi_2)$ then $\alpha = 0$.

Otherwise, assume $\chi_1 = \chi_2 = \chi$ and let $x, x^+ \in k$ be such that $\alpha = x \cdot \text{id}_{\pi_\infty(\chi)} + x^+ \cdot \text{id}_{\pi_\infty^{+}(\chi)}$. If $\alpha$ is $N$-linear, then since $\Pi$ switches the summands (see the proof of Theorem 2.2.4) we deduce $x = x^+$. If $\alpha$ is $KZ$-linear, then let $\varphi \in \pi_\infty(\chi)^{K_1}$ be the function supported on $Iw$ and sending 1 to 1. Then $s\varphi$ is supported on the complement of $Iw$, because $\varphi\left(\begin{array}{cc} a & b \\ pc & d \end{array}\right) = \varphi\left(\begin{array}{cc} b & a \\ d & pc \end{array}\right)$ and $d \in \mathbb{Z}_p^\times$. So we have

$$\alpha(s\varphi) = x^+ s\varphi \text{ and } \alpha(s\varphi) = s(\alpha(\varphi)) = s(x\varphi) = xs\varphi$$

which implies $x^+ = x$, because $s\varphi \neq 0$. \hfill \square

**Corollary 2.2.8** implies that there exist nonzero $K$-linear morphisms between nonisomorphic principal series of the same weight: this is also a direct consequence of the fact that the Hecke eigenvalue $\lambda$ does not appear at the right-hand side of (2.2.1). The following proposition indicates how $\lambda$ interacts with these $K$-morphisms. It will be applied in Section 2.3.

**Proposition 2.2.9.** Let $\tau_1 = c\text{-Ind}_{KZ}^G(\sigma)/(T - \lambda_i)$ for $\lambda_1, \lambda_2 \in k^\times$ be generic principal series representations of the same Serre weight and central character. Let $\alpha : \tau_1 \rightarrow \tau_2$ be a $K$-linear homomorphism and let $x_i \in \text{soc}_K(\tau_i)^{Iw_1}$ be a generator of the $Iw_1$-invariants of the $K$-socle. Assume $\alpha(x_1) = x_2$. Then

$$\alpha\left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_1 = \lambda_2^{-(n+1)} \lambda_1^{n+1} \left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_2$$

for all $n \geq 0$.

*Proof.* It suffices to prove that there exists $\xi \in k^\times$ such that

$$\alpha\left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_1 = \xi \left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_2.$$  

Indeed, by (BL94 (19)), the Hecke operator $T$ acts on $(\text{soc}_K \tau_i)^{Iw_1}$ by the formula

$$z \mapsto \sum_{\lambda \in \mathbb{F}_p} \left(\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} p & 0 \\ 0 & 1 \end{array}\right) z.$$  

Iterating this, we deduce that there exists an element $k^+_{n+1}$ of the group algebra $k[K]$ such that

$$k^+_{n+1} \left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_i = \lambda_i^{n+1} x_i.$$  

It follows that

$$k^+_{n+1} \xi \left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_2 = \lambda_1^{n+1} x_2 \text{ and } k^+_{n+1} \left(\begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array}\right) x_2 = \lambda_2^{n+1} x_2$$

which implies that $\xi = \lambda_1^{n+1} \lambda_2^{-(n+1)}$. 

\hfill \square
In order to prove (2.2.3), we use the fact that $\tau_1$ and $\tau_2$ contain a canonical $K_0(p^{n+1})$-stable line, namely the line mapping to the $k$-span of $\varphi_{n+1}$ under restriction of functions

$$\text{res}_K: \tau_i \overset{\sim}{\longrightarrow} \text{Ind}_{B}^{K}(\mathbf{Z}_p)(\chi).$$

(Recall that $\varphi_{n+1}$ is the $K_0(p^{n+1})$-eigenvector supported on $K_0(p^{n+1})$ and sending the identity to $1$. We write $\chi$ for the conjugate of $\text{soc}_{K}(\tau_i)^{Iw_1}$, which is the same for $\tau_1$ and $\tau_2$.)

Since $\alpha$ corresponds to a scalar under $\text{res}_K$, this line is preserved by $\alpha$. Hence it suffices to prove that \[
\begin{pmatrix}
0 \\
p^n \\
1 \\
0 \\
0
\end{pmatrix}
\cdot x_i \text{ is contained in this line. Since this is a } K_0(p^{n+1})-\text{eigenvector, it suffices to prove that its restriction to } K \text{ is supported in } K_0(p^{n+1}). \] To do so, notice that $\Pi x_i$ is supported in $B_{Iw}$, and so

$$\begin{pmatrix}
0 \\
p^n \\
1 \\
0 \\
0
\end{pmatrix} x_i = \begin{pmatrix}
1 \\
0 \\
p^n \\
0 \\
0
\end{pmatrix} \Pi x_i \text{ is supported in } B_{Iw} \begin{pmatrix}
1 \\
0 \\
0 \\
p^{-n}
\end{pmatrix}.$$ 

Taking inverses, it suffices to prove that

$$K \cap \begin{pmatrix}
1 \\
0 \\
0 \\
p^n
\end{pmatrix} \text{Iw}B = K_0(p^{n+1})$$ 

If $a, d \in \mathbf{Z}_p$ and $b, c \in \mathbf{Z}_p$ we have

$$\begin{pmatrix}
a \\
p^n+1c \\
b \\
p^n+1d
\end{pmatrix} = \begin{pmatrix} 1 \\
0 \\
p^n \\
0
\end{pmatrix} \begin{pmatrix} a \\
0 \\
p^n b \\
0
\end{pmatrix} \begin{pmatrix} 1 \\
0 \\
p^{-n}
\end{pmatrix},$$

proving the $\supseteq$-direction. For the other direction, fixing $b \in B$ we can write

$$\begin{pmatrix}
a \\
p^n+1c \\
b \\
p^n+1d
\end{pmatrix} b = \begin{pmatrix} 1 \\
0 \\
ap^n-1 \\
0
\end{pmatrix} \begin{pmatrix} 1 \\
0 \\
p^n+1c \\
1
\end{pmatrix} \begin{pmatrix} a \\
0 \\
p^n+1d - bp^n+1c
\end{pmatrix}.$$ 

The first two matrices are in $K_0(p^{n+1})$ and the last two are in $B$, and if the product is in $K$ we deduce that

$$K \cap \begin{pmatrix}
1 \\
0 \\
0 \\
p^n
\end{pmatrix} \text{Iw}B \subseteq K_0(p^{n+1}) \cdot (B \cap K) \subseteq K_0(p^{n+1}). \qedhere$$

\section*{2.3. Restriction to Iw and $N$ of supersingular representations.}

We begin by recalling the $K$-structure and Iw-structure of irreducible generic supersingular representations of $G$, following the viewpoint of [Pas10] (see [Mor11] for a different take on this). Let $\sigma$ be a generic Serre weight and let $\pi = c \cdot \text{Ind}_{K/K}^{G}(\sigma) / T$. The $K$-socle of $\pi$ has $K$-length two and contains two nonisomorphic weights $\{\sigma, \sigma^{[s]}\}$. If $\chi$ is the character $\sigma^{Iw_1}$, then $(\sigma^{[s]})^{Iw_1}$ is conjugate to $\chi$, hence it is the character usually denoted $\chi^*$: this explains the notation $\sigma^{[s]}$ (although we will sometimes write $\chi^*$ for $\chi^s$, for compatibility with the notation for $\text{ad}(\Pi)$). We define

$$\pi_{\sigma} = \langle G^+ \cdot \sigma \rangle.$$ 

It follows from the results of [Pas10] that $\pi_{\sigma}$ is an absolutely irreducible $G$-representation, that $\pi \cong \text{Ind}_{B}^{G}(\pi_{\sigma})$, and that $\pi |_{G^+} = \pi_{\sigma} \oplus \pi_{\sigma^{[s]}}$. In addition, $\pi_{\sigma} \otimes_{k} k$ is the representation denoted $\pi_{\sigma}$ in [Pas10], which works with coefficients in $\mathbf{F}_p$. For this reason, in this section we will sometimes work over $k$.

\subsection*{2.3.1. The Iw-representation $M_\sigma$.} Let $\nu_{\sigma} \in \pi$ be a generator of the line $\sigma^{Iw_1}$. Following [Pas10] Definition 4.5, introduce

$$M_\sigma = \left\langle \left( \begin{pmatrix}
p^{2Z_{\geq 0}} \\
0 \\
Z_{p} \\
1
\end{pmatrix} \cdot \nu_{\sigma} \right) \right\rangle.$$ 

By [Pas10, Lemma 4.6], this is Iw-stable in $\pi$. By [Pas10] Definition 4.11, Corollary 6.4, we have a short exact sequence

$$0 \rightarrow \pi^{Iw_1} \rightarrow M_\sigma \oplus \text{IM}_{\sigma^{[s]}} \rightarrow \pi_{\sigma} \rightarrow 0$$

of Iw-representations. Notice that $\Pi M_{\sigma^{[s]}}$ is an Iw-representation isomorphic to the twist $M_{\sigma^{[s]}}^{+} = \text{ad}(\Pi_{\sigma}^{+}) M_{\sigma^{[s]}}$, The space of invariants $\pi_{\sigma}^{Iw_1} \cong \chi$ is one-dimensional, and the inclusion is the diagonal embedding, hence we
have exact sequences
\[
0 \to M_\sigma \to \pi_\sigma \to \frac{M^+_{\sigma|\chi}}{\chi} \to 0
\]
\[
0 \to M^+_{\sigma|\chi} \to \pi_\sigma \to M_\sigma / \chi \to 0
\]
where the projection restricts to the canonical surjection on \(M^+_{\sigma|\chi}\) and \(M_\sigma\) respectively.

**Proposition 2.3.2.** Let \(\sigma_1, \sigma_2\) be generic weights. Then \(\text{Hom}_{Iw}(M_{\sigma_1}, M_{\sigma_2}^+/\chi_2)\) and \(\text{Hom}_{Iw}(M^+_{\sigma_1}, M_{\sigma_2}/\chi_2)\) are both zero.

**Proof.** This is similar to Proposition 2.2.5. It suffices to prove the result over \(\overline{k}\). By [Pas10, Proposition 4.7], the injection \(\chi \to M_\sigma\) is an injective envelope in the category of representations of \(HU(\mathbb{Z}_p)\). By [Pas10, Proposition 5.9], the representation \(M_{\sigma|HU(\mathbb{Z}_p)}\) is uniserial and the layers of its socle filtration satisfy
\[
\text{soc}\,^{k+1}(M_\sigma) / \text{soc}\,^k(M_\sigma) \cong \alpha^{-1} \otimes \text{soc}\,^k(M_\sigma) / \text{soc}\,^{k-1}(M_\sigma).
\]
By the following lemma, we see that (2.3.2) is also true for the Iw-socle filtration of \(M_{\sigma}\).

**Lemma 2.3.3.** The representation \(M_{\sigma}|Iw\) is uniserial. The socle filtration of \(M_{\sigma}|HU(\mathbb{Z}_p)\) is a filtration by Iw-stable subspaces, and it coincides with the socle filtration of \(M_{\sigma}|Iw\).

**Proof.** Since every Iw-subspace of \(M_{\sigma}\) is \(HU(\mathbb{Z}_p)\)-stable, the Iw-subspaces of \(M_{\sigma}\) are totally ordered by inclusion, and so \(M_{\sigma}|Iw\) is uniserial. The irreducible \(\overline{\mathbb{F}}\)-representations of \(HU(\mathbb{Z}_p)\) and Iw are both inflated from characters of \(H\), hence the rest of the lemma follows from Lemma 2.1.4. 

Twisting by \(\Pi\), we find that
\[
\text{soc}\,^1_{Iw}(M^+) / \text{soc}\,^1_{Iw}(M^+) \cong \sigma \otimes \text{soc}\,^1_{Iw}(M^+) / \text{soc}\,^0_{Iw}(M^+).
\]
because \(\sigma^+ = \sigma^{-1}\). Now the proof proceeds as for Proposition 2.2.5. 

**Definition 2.3.4.** The submodule \(M_{\sigma, n} \subset M_{\sigma}\) is defined to be \(\langle B(\mathbb{Z}_p) \cdot t^{2n}v_\sigma \rangle\). Since \(t^{2n}v_\sigma\) is a \(T(\mathbb{Z}_p)\)-eigenvector, this is the same as \(\langle U(\mathbb{Z}_p) \cdot t^{2n}v_\sigma \rangle\), which is the submodule defined in the proof of [Pas10, Proposition 4.7].

The following is the analogue of Theorem 2.2.6 for supersingular representations.

**Theorem 2.3.5.** Let \(\sigma_1, \sigma_2\) be generic weights, with \(\sigma_{1|Iw}^+ = \chi_1\), and let \(X_1 \subset M_{\sigma_1}\) be proper (hence finite-dimensional) Iw-stable subspaces. Then \(\text{Hom}_{Iw}(M_{\sigma_1}, M_{\sigma_2}/X_2)\) is one-dimensional if \(\sigma_1 = \sigma_2\) and \(X_1 \subset X_2\), and vanishes otherwise.

**Proof.** It suffices to prove the same result over \(\overline{k}\). Let
\[
\lambda : M_{\sigma_1}/X_1 \to M_{\sigma_2}/X_2
\]
be a nonzero Iw-linear morphism. Since both \(M_{\sigma}|Iw\) and \(M_{\sigma}|HU(\mathbb{Z}_p)\) are uniserial, the module \(M_{\sigma, n}\) is Iw-stable and \(M_{\sigma, n} = \langle Iw \cdot t^{2n}v_\sigma \rangle\). By the same argument as Proposition 2.2.6, we can assume that \(\dim(X_1) \geq \dim(X_2)\).

By construction, the dimension of \(M_{\sigma, n}\) does not depend on \(\sigma\) (we will give an explicit formula in what follows). Comparing dimensions, we see that for \(n\) large enough \(\lambda(M_{\sigma_1, n}/X_1) \subset M_{\sigma_2, n}/X_2\). We claim that \(\lambda\) is an isomorphism, \(\sigma_1 = \sigma_2 = \sigma\), and \(X_1 = X_2 = X\). The claim implies the theorem, because given \(\lambda : M_{\sigma}/X \to M_{\sigma}/X\) there exists \(\alpha \in \overline{k}\) such that \(\lambda - \alpha|\text{soc}(M_{\sigma}/X) = 0\), and so \(\lambda = \alpha\) since \(\lambda - \alpha\) is not an isomorphism.

To prove the claim it is enough to prove that \(\lambda\) is injective, \(\chi_1 = \chi_2\), and \(X_1 = X_2\). Assume this is false. We are going to prove that the dimension of the cokernel of the restriction \(\lambda_n : M_{\sigma_1, n}/X_1 \to M_{\sigma_2, n}/X_2\) tends to infinity with \(n\), which would contradict the fact that the dimension of \(\ker(\lambda_n)\) is independent of \(n\) for \(n\) large enough (since it coincides with \(\ker(\lambda)\) for \(n\) large enough, by uniseriality).

Since \(v_\sigma\) is an Iw-eigenvector, the vector \(t^{2n}v_\sigma\) is an eigenvector for
\[
t^{2n} \cdot Iw = \begin{pmatrix} Z_{p^n}^\times & Z_{p^n} \\ p^{-2n+1}Z_p & Z_p \end{pmatrix}
\]
hence it is an eigenvector for

\[ K_0^+(p^{2n+1}) = \Pi \cdot K_0(p^{2n+1}) \cdot \Pi^{-1} = \begin{pmatrix} Z_{p^z}^\perp & p^{2n}Z_p \\ Z_p \\ p^b \end{pmatrix} \]

with the same \( H \)-eigencharacter as \( v_{\sigma_i} \), namely \( \chi_i \equiv \sigma_{Iw}^i \). Hence \( M_{\sigma_i,n} \) is a quotient of \( \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i) \).

The dimension of \( M_{\sigma_i,n} \) is computed in the proof of Proposition 4.7, and it is equal to a quantity \( e_n + 1 \) defined by the recursion \( e_0 = 0 \) and \( e_n = r + p(p - 1 - r) + p^2e_{n-1} \). Hence it tends to infinity with \( n \).

Now fix \( n > 0 \). We are going to prove that \( \text{coker}(\lambda_m) \geq p^{n-1} \) if \( m \) is large enough: by the above, this suffices to prove the theorem. By Lemma 2.3.8 it follows, the representation \( \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i) \) is isomorphic to the twist \( \left( \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i^+) \right)^+ \), since \( K_0^+(p^{2n+1}) = \Pi K_0(p^{2n+1}) \Pi^{-1} \). Since these are uniserial \( K_0(p) \)-representations, it follows that the \( p^n \)-dimensional quotient of \( \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i) \) is isomorphic to \( \left( \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i^+) \right)^+ \cong \text{Ind}_{K_0(p)}^{K_0(p^{2n+1})}(\chi_i) \).

If \( m \) is large enough, the map \( \lambda_m : M_{\sigma_i,m} / X_1 \to M_{\sigma_2,m} / X_2 \) passes to the quotient to \( \lambda_{Y,m} : Y_{\sigma_i,m} \to Y_{\sigma_2,m} \), still \( Iw \)-linear, because the kernels of the two maps \( M_{\sigma_i,m} \to Y_{\sigma_i,m} \) have the same dimension, which tends to infinity with \( m \) (hence they will contain \( X_i \) if \( m \) is large enough—recall that \( \dim Y_{\sigma_i,m} = p^n \) is independent of \( m \)). Furthermore, \( \text{coker}(\lambda_m) \) surjects onto \( \text{coker}(\lambda_{Y,m}) \).

If \( \text{dim}(X_1) > \text{dim}(X_2) \), or \( \lambda \) is not injective, then comparing dimensions shows that \( \lambda_m \) cannot be surjective. Hence \( \lambda_{Y,m} \) is not surjective, since if it is then \( \lambda_m \) is surjective on cosocles, hence is surjective. Similarly, if \( \chi_1 \neq \chi_2 \) then Lemma 2.2.3 and Lemma 2.3.8 imply that \( \lambda_{Y,m} = 0 \). In each of these cases we deduce by Lemma 2.2.3 and Lemma 2.3.8 that \( \dim(\text{coker}(\lambda_m)) \geq \dim(\text{coker}(\lambda_{Y,m})) \geq p^n - p^{n-1} = p^{n-1}(p - 1) \). Hence \( \dim \text{coker}(\lambda_m) \geq p^{n-1} \), and the claim follows.

Corollary 2.3.6. The space \( \text{Hom}_{k[IwZ]}(\pi_{\sigma_1}, \pi_{\sigma_2}) \) is one-dimensional if \( \sigma_1 = \sigma_2 \) and vanishes otherwise. Hence \( \text{Hom}_{k[GL_2]}(\pi_{\sigma_1}, \pi_{\sigma_2}) = \text{Hom}_{k[IwZ]}(\pi_{\sigma_1}, \pi_{\sigma_2}) \).

Proof. By Proposition 2.3.2 an \( Iw \)-linear map \( \pi_{\sigma_1} \to \pi_{\sigma_2} \) has to send \( M_{\sigma_1} \) to \( M_{\sigma_2} \) and \( \Pi M_{\sigma_1} \) to \( \Pi M_{\sigma_2} \). By Theorem 2.3.3 (and after twisting by \( \Pi \)), these restrictions are zero if \( \sigma_1 \neq \sigma_2 \), and otherwise they are scalar endomorphisms.

The corollary follows as these have to agree when restricted to \( \text{soc}(\pi_{\sigma}) \).

Corollary 2.3.7. Let \( \pi_1, \pi_2 \) be generic irreducible supersingular \( k[GL_2(Q_p)] \)-representations of weight \( \sigma_1, \sigma_2 \) respectively. Then \( \text{Hom}_{k[GL_2]}(\pi_1, \pi_2) = \text{Hom}_{k[N]}(\pi_1, \pi_2) \).

Proof. Since \( \pi_1|\cdot N \cong c\text{-Ind}_{IwZ}^N(\pi_{\sigma_1}) \), by Frobenius reciprocity we have

\[ \text{Hom}_N(\pi_1, \pi_2) = \text{Hom}_{IwZ}(\pi_{\sigma_1}, \pi_{\sigma_2}) \oplus \text{Hom}_{IwZ}(\pi_{\sigma_1}, \pi_{\sigma_2^v}). \]

Corollary 2.3.6 together with the fact that \( \sigma_2 \) and \( \sigma_2^v \) are not isomorphic implies that the right-hand side is at most one-dimensional, and does not vanish if and only if \( \pi_1 \) and \( \pi_2 \) are \( G \)-isomorphic.

The following lemma was used in the proof of Theorem 2.3.3.

Lemma 2.3.8. Let \( G \) be a locally profinite group, \( H \) a closed subgroup of \( G \), and \( \alpha : G \to G \) a continuous group automorphism. If \( \theta \) is a \( k \)-representation of \( H \), we have a representation \( \theta_\alpha = (\alpha^{-1})^*(\theta) \) of \( \alpha(H) \).

Then there is a \( G \)-linear isomorphism

\[ \text{Ind}_{H}^{G}(\theta_\alpha) \to \text{Ind}_{\alpha H}^{G}(\theta_\alpha). \]

Proof. We can assume that \( \theta_\alpha \) has the same representation space of \( \theta \) with the action \( \theta_\alpha(\alpha(h)) = \theta(h)t \).

Given a function \( f : G \to \theta \) with \( f(hg) = \theta(h)f(g) \), let \( f_\alpha(g) = f(\alpha^{-1}(g)) \). Then \( f_\alpha(\alpha(h)g) = f(h\alpha^{-1}(g)) = \theta(h)f_\alpha(g) = \theta_\alpha(\alpha(h))f_\alpha(g) \), hence \( f_\alpha \in \text{Ind}_{\alpha H}^{G}(\theta_\alpha) \). The map we are looking for is \( f \to f_\alpha \).
2.3.9. Extensions. Now we study the restriction to the Iwahori subgroup of extensions of $G$-representations and $G^+$-representations.

**Theorem 2.3.10.** Let $\sigma_1, \sigma_2$ be generic weights, and let $\pi_i = \left( c\text{-Ind}_{KZ}^G \sigma_i \right) / T$. Then the restriction map $\text{Ext}^1_{k[G]}(\pi_1, \pi_2) \to \text{Ext}^1_{k[IwZ]}(\pi_1, \pi_2)$ is injective.

**Proof.** Notice that any element in the kernel of this map is contained in $\text{Ext}^1_{k[G],\zeta}(\pi_1, \pi_2)$, i.e. it has central character $\zeta$. Hence, by [Pas13 Lemma 5.7], it suffices to prove the theorem over $k$. By [Pas10 Theorem 1.1], the theorem is true if $\sigma_1$ and $\sigma_2$ are not conjugate, since then the representations $\pi_i$ are not $G$-isomorphic and the space of $G$-extensions vanishes. So it suffices to prove that if $\sigma$ is a generic weight, $\pi = \left( c\text{-Ind}_{KZ}^G \sigma \right) / T$, and

$$0 \to \pi \to X \to \pi \to 0 \quad (2.3.3)$$

is an exact sequence of $G$-representations with central character $\zeta$ that splits over $Iw$, then the sequence is split over $G$.

For this, choose $v_\sigma$ in the $K$-socle of the quotient, invariant under $Iw$, and such that $\langle K \cdot v_\sigma \rangle \cong \sigma$. Then $\langle K \cdot v_\sigma \rangle \cong \sigma^{[s]}$, because $\pi^{Iw_1}$ is two-dimensional. Let $w_\sigma$ be an $Iw_1$-invariant lift of $v_\sigma$ to $X$ with $Iw$-eigencharacter $\chi$, which exists since we assume that the extension is $Iw$-split.

We are going to study the representations $\langle K \cdot w_\sigma \rangle$ and $\langle K \cdot \Pi w_\sigma \rangle$, which are quotients of finite principal series $\text{Ind}_{KZ}^K(\chi)$ and $\text{Ind}_{KZ}^K(\chi^{[s]})$, respectively. Recall (see for instance [BP12 Section 2]) that the representation $\text{Ind}_{Iw}^K(\chi)$ has a two-dimensional space of $Iw_1$-invariants, spanned by the function $\varphi \in \text{Ind}_{Iw}^K(\chi)$ supported in $Iw$ and satisfying $\varphi(1) = 1$, and the function $f_0 = S_0 \varphi$, where we have written $S_0 = \sum_{\lambda \in \mathbb{F}_p} \left( \frac{[\lambda]}{1} \right)^{1 \ 0}$. The functions $\varphi$ and $f_0$ are $H$-eigenvectors, with eigencharacter $\chi$ and $\chi^{[s]}$ respectively. We have an exact sequence

$$0 \to \text{Ind}_{Iw}^K(\chi) \to \text{Ind}_{Iw}^K(\chi)^{res} \to \chi \to 0$$

defined by restricting functions to $Iw$, which is $Iw$-linearly split. Looking at the $H$-eigencharacter, it follows that $f_0 \in \text{Ind}_{Iw}^K(\chi)^{res}$ and generates its $Iw$-socle.

**Proposition 2.3.11.** The representations $\langle K \cdot w_\sigma \rangle$ and $\langle K \cdot \Pi w_\sigma \rangle$ are irreducible.

**Proof.** Assume that

$$\alpha : \text{Ind}_{Iw}^K(\chi) \to \langle K \cdot w_\sigma \rangle, \varphi \mapsto w_\sigma$$

is injective. Since $\langle K \cdot v_\sigma \rangle$ is irreducible, we have $S_0 \varphi \in \pi \subset X$. Since $S_0 \varphi$ is an $Iw$-eigenvector with eigencharacter $\chi^{[s]}$, we deduce furthermore that $S_0 \varphi \in \pi^{[s]}$.

By assumption, there is an $Iw$-linear retraction $r : X \to \pi$ of the inclusion of $\pi$ in $X$. Consider the composition

$$\text{Ind}_{Iw}^K(\chi)^{res} \xrightarrow{\alpha} X \xrightarrow{r} \pi \to \pi^{[s]}.$$ 

Since $S_0 \varphi$ generates the $Iw$-socle of $\text{Ind}_{Iw}^K(\chi)^{res}$, this composition is injective. But the dimension of $\text{Ind}_{Iw}^K(\chi)^{res}$ is equal to $p$. Since the first congruence subgroup $K_1$ acts trivially on $\text{Ind}_{Iw}^K(\chi)^{res}$, this contradicts the fact that $\dim(\pi^{[s]}) = p - 1$ (see [BP12 Section 20] or [Mor13 Theorem 1.4]).

The same proof works for $\text{Ind}_{Iw}^K(\chi^{[s]}) \to \langle K \cdot \Pi w_\sigma \rangle$, or argue by symmetry, using that

$$\left( c\text{-Ind}_{KZ}^G \sigma \right) / T \cong \left( c\text{-Ind}_{KZ}^G \sigma^{[s]} \right) / T$$

to conclude. \qed

It follows from Proposition 2.3.11 that $\langle K \cdot w_\sigma \rangle$ is $K$-isomorphic to $\sigma$. To complete the proof of the theorem it suffices to prove that $Tw_\sigma = 0$, because then $\langle G \cdot w_\sigma \rangle \cong \pi$ maps isomorphically to the quotient $\pi$ and defines a $G$-splitting of the exact sequence (2.3.3). Recall that we have the equality

$$Tw_\sigma = \sum_{\lambda \in \mathbb{F}_p} \left( \frac{[\lambda]}{1} \right)^{1 \ 0} \Pi w_\sigma = S_0 \Pi w_\sigma.$$
By Proposition 2.3.11 the surjection \( \text{Ind}^K_{Iw}(\chi^*) \to (K \cdot \Pi_w) \) sending \( \varphi \) to \( \Pi_w \) is equal to zero on the \( K \)-socle. Hence the image of \( S_0 \varphi \) under this map is zero, because \( S_0 \varphi \) generates the \( K \)-socle of \( \text{Ind}^K_{Iw}(\chi^*) \). But this image is \( S_0 \Pi_w = Tw_\sigma \), hence \( Tw_\sigma = 0 \).

\[ \square \]

**Corollary 2.3.12.** Let \( \sigma_1, \sigma_2 \) be generic weights. Then the restriction map

\[ \text{Ext}^1_{k[G^+]}(\pi_{\sigma_2}, \pi_{\sigma_1}) \to \text{Ext}^1_{k[Iw]}(\pi_{\sigma_2}, \pi_{\sigma_1}) \]

is injective.

**Proof.** Let \( 0 \to \pi_{\sigma_1} \to X \to \pi_{\sigma_2} \to 0 \) be a short exact sequence of \( G^+ \)-representations that is split over \( Iw \).

The exact sequence

\[ 0 \to \pi_{\sigma_1} \oplus \pi_{\sigma_2}^+ \to X \oplus X^+ \to \pi_{\sigma_2} \oplus \pi_{\sigma_2}^+ \to 0 \]

is \( Iw \)-split, because \((-)^+ = \text{ad}(\Pi)^*(\cdot) \) and \( \Pi \) normalizes \( Iw \). It is isomorphic to the restriction to \( G^+ \) of

\[ 0 \to \text{Ind}_{G^+}^G(\pi_{\sigma_1}) \to \text{Ind}_{G^+}^G(X) \to \text{Ind}_{G^+}^G(\pi_{\sigma_2}) \to 0. \]

Since \( \text{Ind}_{G^+}^G(\pi_{\sigma_1}) \cong \left( \text{c-Ind}_{KZ}^G(\pi_{\sigma_1}) \right) / T \), this sequence is \( G \)-split by Theorem 2.3.11. But then the inclusion \( \pi_{\sigma_1} \to X \oplus X^+ \) has a \( G \)-linear retraction, which we can restrict to \( X \) to prove that the original exact sequence was already \( G^+ \)-split. \( \square \)

**2.4. Restriction to \( KZ \) of atomes automorphes.** Since we will appeal to various results of Morra on the structure of atomes automorphes of length two, we will follow some of the notation of \( \text{[Mor17]} \). For instance, the element \( [1, x] \in \text{c-Ind}_{KZ}^G(\sigma) \) is the function supported on \( KZ \) and sending the identity to \( x \in \sigma \). Let \( r \in \{1, \ldots, p-4\} \) and \( \lambda \in k^\times \).

**Definition 2.4.1.** The representation \( A_{r, \lambda} \) of \( \text{GL}_2(Q_p) \) is defined to represent the only isomorphism class of nonsplit extensions from

\[ \text{Ext}^1_{k[\text{GL}_2(Q_p)]} \left( \text{Ind}_{B}^G(nr \omega^{r+1} \otimes \text{nr}_{\lambda-1} \omega), \text{Ind}_{B}^G(nr \omega^r \otimes \text{nr}_{\lambda-1} \omega^r) \right). \]

**Remark 2.4.2.** The paper \( \text{[Mor17]} \) also studies the representations \( A_{r, \lambda} \) when \( r = p - 3 \), in which case they are not generic according to our conventions. Since our results in Section 2.2 are for generic principal series, we exclude these cases from considerations. When \( r = p - 2 \) there are many atomes automorphes of length 2, whose factors are generic principal series, but they are not covered by the results in \( \text{[Mor17]} \). This is the reason why we introduced “very generic” representations in Section 2.1.8.

We write \( \pi_1 = \text{Ind}_{B}^G(nr \omega^{r+1} \otimes \text{nr}_{\lambda-1} \omega) \) and \( \pi_2 = \text{Ind}_{B}^G(nr \omega^r \otimes \text{nr}_{\lambda-1} \omega^r) \), so that \( \pi_1 \) has weight \( \text{Sym}^r \), \( \pi_2 \) has weight \( \text{Sym}^{p-3-r} \otimes \text{det}^{r+1} \), and we have an exact sequence of \( G \)-representations

\[ 0 \to \pi_1 \to A_{r, \lambda} \to \pi_2 \to 0. \tag{2.4.1} \]

We write \( \chi_i \) for the \( Iw \)-character conjugate to \( \text{soc}_K(\pi_i)^{Iw} \), so that \( \pi_i|_{Iw} \cong \pi_{\infty}(\chi_i) \oplus \pi_{\infty}^{+}(\chi_i) \) (see the proof of Corollary 2.2.8). Furthermore, we will write \( A_{r, s, \lambda} \) for \( A_{r, \lambda} \otimes (\omega^s \circ \text{det}) \). In this section we will relate the \( G \)-action and the \( K \)-action on \( G \)-representations on \( A_{r, s, \lambda} \). The following example shows that one cannot expect a direct analogue of Theorem 2.3.6.

**Example 2.4.3.** Let \( \mu \neq 0, \lambda \). Since

\[ \text{c-Ind}_{KZ}^G(\text{Sym}^r)/(T - \lambda)|_K \cong \text{c-Ind}_{KZ}^G(\text{Sym}^r)/(T - \mu)|_K, \]

there exists a nonzero \( KZ \)-linear morphism

\[ A_{p-3-r, \mu} \otimes (\omega^{r+1} \circ \text{det}) \to A_{r, \lambda} \]

but there are no nonzero \( G \)-morphisms between these representations.

However, will be able to establish an analogue of Theorem 2.3.6 once we restrict to isomorphisms (Theorem 2.3.2), or to endomorphisms of a single \( A_{r, s, \lambda} \). We begin with the case of endomorphisms.

**Proposition 2.4.4.** The spaces \( \text{Hom}_{KZ}(A_{r, s, \lambda}, A_{r, s, \lambda}) \) and \( \text{Hom}_N(A_{r, s, \lambda}, A_{r, s, \lambda}) \) are both one-dimensional, hence coincide with \( \text{Hom}_G(A_{r, s, \lambda}, A_{r, s, \lambda}) \).
Proof. It suffices to consider the case \( s = 0 \). By Corollary \[2.2.8\] we have that \( \dim \text{Hom}_{KZ}(\pi_i, \pi_j) = \dim \text{Hom}_N(\pi_i, \pi_j) \) is equal to one if \( i = j \) and zero otherwise (here \( i, j \in \{1, 2\} \)). There is an exact sequence

\[
0 \to \text{Hom}_{KZ}(\pi_2, A_{r, \lambda}) \to \text{Hom}_{KZ}(A_{r, \lambda}, A_{r, \lambda}) \to \text{Hom}_{KZ}(\pi_1, A_{r, \lambda}).
\]

Given an element \( \sigma_2 \to A_{r, \lambda} \) of the first term, composing it with the projection to \( \pi_2 \) yields a scalar. If this scalar is not zero, then we have constructed a \( KZ \)-splitting of the exact sequence \[2.4.4\], hence an Iw-splitting, contradicting (for example) the computation of the Iw-invariants of \( A_{r, \lambda} \) in \[BP12\] Section 20. Hence the first term vanishes. For the last term, given \( \alpha : \pi_1 \to A_{r, \lambda} \), the composition \( \pi_1 \to A_{r, \lambda} \to \pi_2 \) is zero, hence \( \alpha \) factors through the subspace \( \pi_1 \) in \[2.4.4\]. Hence the term \( \text{Hom}_{KZ}(\pi_1, A_{r, \lambda}) \) is one-dimensional, and so \( \text{Hom}_{KZ}(A_{r, \lambda}, A_{r, \lambda}) \) is also one-dimensional.

The proof goes through unchanged for the group \( N \). \( \square \)

2.4.5. A presentation of \( A_{r, \lambda} \). This follows [Mor17] Section 4, which in turn is based on [Bre03b]. So we introduce the notation \( \sigma_{p+1+r} \) for \( \text{Sym}^{p+1+r}(k^2) \), a reducible representation of \( KZ \). There is a \( KZ \)-linear inclusion

\[
\sigma_r \otimes \det \to \sigma_{p+1+r}, \quad x^{r-i}y^j \mapsto X^{r+1-i}Y^{i+1} - X^{r+1-i}Y^{p+i}.
\]

It gives rise to a short exact sequence

\[
0 \to c\text{-Ind}_{KZ}^G(\sigma_r \otimes \det) \to \left( c\text{-Ind}_{KZ}^G(\sigma_{p+1+r}) \right)/T \xrightarrow{pr} c\text{-Ind}_{KZ}^G(\sigma_{p-3-r} \otimes \det^{r+2}) \to 0
\]

where \( \iota(1, x^{r-i}y^j) = \frac{1}{p^2}[1, X^{p+1+r-j}Y^{i+1}] \) and \( \text{pr}[1, X^{p+1+r-j}Y^{i+1}] = [1, x^{p-3-r}] \). In turn, this fits into a diagram

\[
\begin{array}{ccc}
0 & \to & c\text{-Ind}_{KZ}^G(\sigma_r \otimes \det) \\
\downarrow & & \downarrow \\
0 & \to & \pi_1 \otimes (\omega \circ \det) \\
\downarrow & & \downarrow \\
0 & \to & A_{r, \lambda} \otimes (\omega \circ \det) \\
\downarrow & & \downarrow \\
0 & \to & \pi_2 \otimes (\omega \circ \det)
\end{array}
\]

with all vertical arrows surjective (compare [Mor17, (17)] or [Bre03b] Section 5.3).

Lemma 2.4.6. The space \( \text{Hom}_{KZ}(\sigma_{p+1+r} \otimes \det^{-1}, A_{r, \lambda}) \) is one-dimensional.

Proof. By [Bre03b] Lemma 5.1.3(ii) with \( k = p + 3 + r \), the representation \( \sigma_{p+1+r} \) has three Jordan–Hölder factors, with \( \text{soc}_{KZ}(\sigma_{p+1+r}) \cong (\sigma_r \otimes \det) \otimes \sigma_{r+2} \) and \( \text{cosoc}_{KZ}(\sigma_{p+1+r}) \cong \sigma_{p-3-r} \otimes \det^{r+2} \). The condition \( p + 3 \leq k \leq 2p \) in the reference translates to \( 0 \leq r \leq p - 3 \), hence the lemma applies to our choice of \( r \).

By [BP12] Section 20 the \( K \)-socle of \( A_{r, \lambda} \) is isomorphic to \( \sigma_r \). Hence every nonzero \( K \)-morphism \( \sigma_{p+1+r} \otimes \det^{-1} \to A_{r, \lambda} \) has to be trivial on the factor \( \sigma_{r+2} \otimes \det^{-2} \) and cannot be trivial on \( \sigma_r \), since it cannot factor through the cosocle. This implies the claim since any two such morphisms are linearly dependent when restricted to \( \sigma_r \) (since \( \text{soc}_{KZ}(A_{r, \lambda}) \) is irreducible). \( \square \)

Definition 2.4.7. A presentation of \( A_{r, \lambda} \) is a nonzero \( KZ \)-linear morphism \( \sigma_{p+1+r} \otimes \det^{-1} \to A_{r, \lambda} \). (As usual, \( p \in Z \) acts by \( \zeta(p) \) on \( \sigma_{p+1+r} \).)

2.4.8. Iwasawa modules. The choice of a presentation of \( A_{r, \lambda} \) provides us with special elements \( x_1 = [1, x^r] \in \pi_1 \) and \( x_2 = [1, x^{p-3-r}] \in \pi_2 \), which generate \( \text{soc}_{K}(\pi_i)^{Iw} \). By [Mor17] Lemma 2.8, we have an injection

\[
\text{Ind}_{K_{p}(p^{n+1})}^{K}(\chi_i) \to \pi_i, [1, 1] \mapsto \begin{pmatrix} 0 & 1 \\ p^{n+1} & 0 \end{pmatrix} x_i.
\]

We need to apply some of the results of [Mor17] concerning this inclusion. Since they are formulated in terms of Iwasawa algebras, we begin by recalling this context.

In [Mor17] Section 3, the structure of \( \pi_{\infty}(\chi_i) \) (there denoted \( \text{M}^- \)) as a module for the Iwasawa algebra \( A = k[[\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}]] \) is described explicitly. This algebra is a commutative discrete valuation ring with uniformizer

\[
X = \sum_{\lambda \in \mathbb{F}_p} \lambda^{-1} \begin{pmatrix} 1 & 0 \\ p[\lambda] & 1 \end{pmatrix}.
\]
Since the Pontrjagin dual $\pi_\infty(\chi_i)^\vee$ is free of rank one over $A$ (compare [Mor17 (13)]) we find that

\[ \pi_\infty(\chi_i)^{\mathfrak{T}(p^{n+1}Z_p)} = \pi_\infty(\chi_i)K_{n+1} = \pi_{n+1}(\chi_i) \]

and this is a free module of rank one over $A/X^{p^n}$.

This isomorphism of $A/X^{p^n}$ onto $\pi_{n+1}(\chi_i)$ endows $A/X^{p^n}$ with an action of the Iwahori subgroup $K_0(p)$. We will need the following property of this action.

**Proposition 2.4.9** (Morra). Let $b \in K_1(p) \cap B(\mathbb{Q}_p) = \left( \begin{array}{cc} 1 + pZ_p & Z_p \\ 0 & 1 + pZ_p \end{array} \right)$. Then

\[ (b - 1)X^N \in X^{p^n+2} : (A/(X^{p^n})) \]

for all $N \in \mathbb{Z}_{\geq 0}$.

**Proof.** This is [Mor17, Corollaire 3.5].

Restricting the inclusion (2.4.4) to $\pi_{n+1}(\chi_i)$ provides us with a $K_0(p)$-linear inclusion

\[ A/X^{p^n} \to \pi_i, 1 \mapsto \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) x_i, \]

and we will often identity $A/(X^{p^n})$ with its image under this map. Then Proposition 2.4.9 continues to hold, provided that we interpret $X^N$ as a shorthand for

\[ X^N \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) x_i \in \pi_i. \]

2.4.10. *Special elements of atomes automorphes.* Choose a presentation of $\mathcal{A}_{r,\lambda}$ and consider the diagram (2.4.3). As in [Mor17 Section 4.2] we write

\[ e_{n+1} = \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, X^{p-1}Y^{r+2}] \in \mathcal{A}_{r,\lambda}, \]

which is a lift of $\left( \begin{array}{cc} 0 & 1 \\ p^n & 0 \end{array} \right) [1, x^{p-3-r}]$. From $e_{n+1}$, there is constructed in [Mor17 Section 4.3] another lift of the same element, denoted $\tilde{e}_{n+1}$, with the following additional properties. (See also [Mor17 Proposition 1.3] for a summary.)

1. The cocycle of $\left( \begin{array}{cc} 1 \\ p^{n+1}Z_p & 0 \\ & 1 \end{array} \right)$ with values in $\pi_1$ defined by $u \mapsto (u-1)\tilde{e}_{n+1}$ has values in $\text{soc}_{\text{tw}} \pi_\infty(\chi_1)$, by [Mor17, Lemme 5.1].

2. Assume $n \geq 1$. Then the cocycle of $\left( \begin{array}{cc} 1 + pZ_p & Z_p \\ 0 & 1 + pZ_p \end{array} \right)$ with values in $\pi_1$ defined by $g \mapsto (g-1)\tilde{e}_{n+1}$ for $g = \left( \begin{array}{cc} 1 + pa & b \\ 0 & 1 + pd \end{array} \right)$ satisfies the congruence

\[ (g - 1)\tilde{e}_{n+1} \equiv (-1)^{n+r+1}X^{-2n}(b_{kp^{n-1}-(p-3-r)}X^{p-3-r} + (a - d)\lambda_{kp^{n-1}-(p-2-r)}X^{p-2-r}) \]

modulo $\left( \text{soc}_{\text{tw}} \pi_\infty(\chi_1) \right) \oplus X^{(p-3-r)+(p-2)}\pi_{n+1}(\chi_1)$.

3. $\tilde{e}_{n+1}$ is fixed by $B(Z_p) \cap K_{n+1} = \left( \begin{array}{cc} 1 + p^{n+1}Z_p & p^{n+1}Z_p \\ 0 & 1 + p^{n+1}Z_p \end{array} \right)$, by [Mor17, Corollaire 6.1].

**Remark 2.4.11.** We explain in more detail what part (2) means. The notation $\mathfrak{T}$ stands for reduction modulo $p$. We have decomposed $\pi_1|_{\mathfrak{T}} = \pi_\infty(\chi_1) \oplus \pi_\infty(\chi_1)$, and embedded $A/(X^{p^n})$ in $\pi_\infty(\chi_1)$ via

\[ A/(X^{p^n}) \to \pi_\infty(\chi_1), 1 \mapsto \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^r], \]
where $[1, x^r] \in \pi_1$ is defined through our fixed presentation (2.4.3). This embedding is $A$-linear with image $\pi_{n+1}(\chi_1)$, and so in the congruence we have written $X^{p^{3-r}}$ as a shorthand for

$$X^{p^{3-r}} \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^r] \in \pi_{n+1}(\chi_1).$$

Finally, the constants denoted by $\kappa$ are elements of $F'_p \subseteq k^\times$ defined in [Mor17, (14)] as the mod $p$ reduction of certain integers.

As in the following proposition, versions of these properties of $e_{n+1}$ hold for a wider class of lifts of $[1, x^{p^{3-r}}]$.

**Proposition 2.4.12** (Morra). Assume that $e_{n+1} \in A_{r,\lambda}$ is a lift of $\left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^{p^{1-3}}]$ satisfying property (1) in the above. Then it satisfies property (2) modulo

$$\pi_+^+(\chi_1) \oplus (X^{(p^{3-r}) + (p-2)} \pi_{n+1}(\chi_1)).$$

**Proof.** This is [Mor17, Corollaire 5.5], which is justified in the remark that precedes it. □

We now have all the ingredients for the proof of the following theorem.

**Theorem 2.4.13.** Let $\lambda, \mu \in k^\times$ and assume $\text{Hom}_K(A_{r,\lambda}, A_{r,\mu}) \neq 0$. Then $\lambda = \pm \mu$, hence $A_{r,\lambda} \cong A_{r,\mu} \otimes (nr_{\pm 1} \circ \text{det})$.

**Proof.** Since $A_{r,\lambda}$ is a twist of $A_{r,\lambda}$, it suffices to prove the theorem when $s = 0$. Let $\alpha : A_{r,\lambda} \to A_{r,\mu}$ be a nonzero $K$-linear morphism. We know that $\alpha$ induces a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \pi(r, \lambda, 1) & \longrightarrow & A_{r,\lambda} & \longrightarrow & \pi(p - 3 - r, \lambda^{-1}, \omega^{r+1}) & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\
0 & \longrightarrow & \pi(r, \mu, 1) & \longrightarrow & A_{r,\mu} & \longrightarrow & \pi(p - 3 - r, \mu^{-1}, \omega^{r+1}) & \longrightarrow & 0
\end{array}
$$

because there are no nonzero $Iw$-linear maps between generic principal series with nonisomorphic $K$-socle, by Corollary 2.2.8 and $\text{Sym}^r \neq \text{Sym}^{p^{3-r}} \otimes \text{det}^{r+1}$ (since $r \neq p - 2$, the det factor is not trivial). We claim that $\alpha$ is an isomorphism. For this it suffices to prove that $\alpha_1$ and $\alpha_2$ are not zero, since by Corollary 2.2.8 they are then isomorphisms. If they are both zero then $\alpha$ is factors through $\pi_2$ to give a map $\alpha_2 \to \pi_1$, which is zero by Corollary 2.2.8. If $\alpha_1 = 0$ but $\alpha_2$ is not, then Corollary 2.2.8 implies that $\alpha_2$ is a nonzero scalar, and then $\alpha_1^{-1} \alpha$ gives a $K$-section of the projection $A_{r,\mu} \to \pi(p - 3 - r, \mu^{-1}, \omega^{r+1})$. Similarly, if $\alpha_2 = 0$ but $\alpha_1$ is not, then $\alpha_1^{-1} \alpha$ gives a $K$-retraction of the inclusion $\pi(r, \lambda, 1) \to A_{r,\lambda}$. These would contradict the fact that $\text{soc}_K(A_{r,\lambda})$ is irreducible.

Now choose a presentation of $A_{r,\lambda}$, in the sense of definition (2.4.3). It induces a presentation

$$\sigma_{p^{n+1}} \otimes \text{det}^{-1} \to A_{r,\lambda} \xrightarrow{\alpha} A_{r,\mu}.$$ 

Working with these choices of presentation we can talk about special elements in $A_{r,\lambda}$ and $A_{r,\mu}$, and by definition we find that $\alpha_2[1, x^{p^{3-r}}] = [1, x^{p^{3-r}}]$. By Proposition 2.2.9 we have

$$\alpha_2 \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^{p^{3-r}}] = \mu^{n+1} \lambda^{-(n+1)} \left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^{p^{3-r}}].$$

It follows that $\mu^{-(n+1)} \lambda^{n+1} \alpha(e_{n+1})$ is a lift of $\left( \begin{array}{cc} 0 & 1 \\ p^{n+1} & 0 \end{array} \right) [1, x^{p^{3-r}}]$ satisfying the assumption of Proposition 2.4.12. So we deduce that

$$(g - 1) \mu^{-(n+1)} \lambda^{n+1} \alpha(e_{n+1}) \equiv (-1)^{n+r+1} \mu^{-2n} (b \kappa_{p^n - 1 - (p-3-r)} X^{p^{3-r}} + (a - d) \kappa_{p^n - 1 - (p-2-r)} X^{p^{2-r}})$$

for all $g \in \left( \begin{array}{cc} 1 + p & 0 \\ 0 & 1 + p \end{array} \right)$. However, by property (2) of $e_{n+1}$ and the fact that $\alpha$ commutes with $g$ we know that

$$(g - 1) \alpha(e_{n+1}) \equiv (-1)^{n+r+1} \lambda^{-2n} \alpha_1 \left( b \kappa_{p^n - 1 - (p-3-r)} X^{p^{3-r}} + (a - d) \kappa_{p^n - 1 - (p-2-r)} X^{p^{2-r}} \right).$$
Recall that $X^{p-3-r}$ is shorthand for $X^{p-3-r}(\frac{0}{p^{n+1}} \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix})$ (this is the action of an element of the Iwasawa algebra on an element of $\pi(r, \lambda, \omega)$). Because of our choice of presentations, we know that $\alpha_1[1, x'] = [1, x']$. By Proposition 2.2.9 and the fact that $\delta$ is an isomorphism, we deduce equality of

$$
\alpha_1 \left( b_{p^n - 1 - (p-3-r)} X^{p-3-r} + \left( \frac{a-d}{2} \right) \kappa_{p^n - 1 - (p-2-r)} X^{p-2-r} \right)

$$

and

$$
\mu^{-(n+1)} \lambda^{n+1} \left( b_{p^n - 1 - (p-3-r)} X^{p-3-r} + \left( \frac{a-d}{2} \right) \kappa_{p^n - 1 - (p-2-r)} X^{p-2-r} \right).
$$

Substitute this in (2.4.7) and compare (2.4.7) with (2.4.6) for an appropriate value of $g$, for instance $g = \frac{1}{0 \ 1}$. We find

$$
\mu^{-(n+1)} \lambda^{-(n+1)} = \mu^{-(n+1)} \lambda^{-(n+1)}
$$

that is, $\lambda^2 = \mu^2$. \hfill \qed

2.4.14. An exact sequence. In preparation for Section 2.4.17, we present some general material concerning resolutions of smooth $G\mathbb{L}_2(\mathbb{Q}_p)$-representations. We will apply these results to study self-extensions of $\mathcal{A}_r, \lambda$.

Write $\delta : N \to k^\times$ for the orientation character, which is trivial on $\text{Iw}Z$ and takes value $-1$ at $\begin{pmatrix} 0 \ 1 & 0 \end{pmatrix}$. We identify the representation space of $\delta$ with $k$. Then we have a complex of $k[G]$-representations

$$
0 \to \text{c-Ind}^G_N(\delta) \xrightarrow{\partial} \text{c-Ind}^G_KZ(\text{triv}) \xrightarrow{\sum} \text{triv} \to 0
$$

declared as follows. In either induction, we write $[g, 1]$ for the function supported on $KZg^{-1}$, respectively $Ng^{-1}$, and taking value 1 at $g^{-1}$. These functions span $\text{c-Ind}^G_KZ(\text{triv})$, respectively $\text{c-Ind}^G_N(\delta)$, over $k$. We define the maps sum and $\partial$ by

$$
\sum[x, 1] = 1, \ \partial[g, 1] = [g, 1] - [gP, 1].
$$

Since $[gn, 1] = \delta(n)[g, 1]$ if $n \in N$, these are well-defined. Since (in either compact induction) we have $h[g, 1] = [hg, 1]$ for all $g, h \in G$, these are $G$-linear maps.

Proposition 2.4.15. The complex (2.4.8) is exact.

Proof. This is a standard result that holds in much greater generality, but we provide a proof for completeness. We write $X$ for the Bruhat–Tits tree of $\text{PGL}_2(\mathbb{Q}_p)$, whose vertices are in bijection with $G/KZ$, and whose edges are in bijection with $G/N$. We write $[x] = xKZ$ for $x \in G$. A 0-chain on $X$ is a function $g : G/KZ \to k$ with finite support. An oriented 1-chain on $X$ is a function $f$ with finite support on the set of oriented edges of $X$, such that $f([x], [y]) = -f([y], [x])$, whenever $([x], [y])$ is an edge of $X$.

We identify $\text{c-Ind}^G_KZ(\text{triv})$ with the space of 0-chains on $X$ by letting $[x, 1]$ be the chain supported in $[x]$ with value 1 at $[x]$. We identify $\text{c-Ind}^G_N(\delta)$ with the space of oriented 1-chains on $X$ by letting $[g, 1]$ be the chain $([g], [gP]) \mapsto 1, ([gP], [g]) \mapsto 1$, and zero elsewhere.

This is well-defined because $[gn, 1] = \delta(n)[g, 1]$. (These are $k$-linear isomorphisms. We will not make use of the $G$-action on chains.)

With these identifications, the complex (2.4.8) corresponds to

$$
0 \to C^1_X(\text{triv}, k) \xrightarrow{\partial} C^0(X, k) \xrightarrow{\sum} k \to 0
$$

with $\partial(f)[x] = \sum_{\text{edges} ([x],[y])} f([x],[y])$. This complex computes the simplicial homology of $X$ with coefficients in $k$, and $X$ is contractible, hence this is an exact sequence. \hfill \qed

Lemma 2.4.16. Let $G$ be a locally profinite group and $H$ an open subgroup of $G$. Let $V, W$ be smooth $k$-representations of $G$ and $H$, respectively. Then there is a $G$-linear isomorphism

$$
(\text{c-Ind}^G_HW) \otimes_k V \to \text{c-Ind}^G_HW|_H
$$

functorial in $V$ and $W$. \hfill \qed
Proposition 2.4.4, the theorem is equivalent to the vanishing of the group $\text{Hom}_{G}$

By twisting, it suffices to prove the theorem when

Proof.
The restriction map $\text{Ext}^{1}_{G}(A, r, s, \lambda) \to \text{Ext}^{1}_{KZ}(A, r, s, \lambda)$ is injective.

Proof. By twisting, it suffices to prove the theorem when $s = 0$. Combining the exact sequence (2.4.9) with Proposition 2.4.4, the theorem is equivalent to the vanishing of the group $\text{Hom}_{N}(A, r, \delta, A, \lambda)$. We proceed by contradiction, identifying a nonzero element of this group with an $Iw$-linear morphism $\alpha : A, r, s, \lambda \to A, r, s, \lambda$ such that $\alpha \Pi = -\Pi$. Recalling Corollary 2.2.8, we see that since $\text{Hom}_{Iw}(\pi_{1}, \pi_{2}) = 0$ every $Iw$-linear morphism $A, r, s, \lambda \to A, r, s, \lambda$ such as $\alpha$, preserves $\pi_{1}$ and passes to the quotient to $\pi_{2}$.

By Proposition 2.4.4 again, we see that there exists $y' \in k$ such that $\alpha^{2} = y'$, and up to a quadratic extension of $k$ we can assume that $y'$ is a square and $\alpha^{2} = y^{2}$ for some $y \in k$. If $y = 0$ then $\alpha$ must induce the zero map on $\pi_{1}$ and $\pi_{2}$, hence it passes to the quotient to a map $\pi_{2} \to \pi_{1}$, which is necessarily zero. Hence $y \neq 0$. But then we can write every $x \in A, r, \lambda$ as

$$x = \frac{1}{2y} ((\alpha + y)(x) - (\alpha - y)(x))$$

since $p \neq 2$, which yields an $Iw$-linear decomposition $A, r, \lambda = A, r, \lambda^{+} \oplus A, r, \lambda^{-}$. Both summands are nonzero since $\alpha$ is not a scalar endomorphism of $A, r, \lambda$. We will scale $\alpha$ so that $y = 1$.

Let $\iota_{1}, \iota_{2}$ be the corresponding orthogonal idempotents in $\text{End}_{Iw}(A, r, \lambda)$. Since $\iota_{i}$ is $Iw$-linear, it preserves $\pi_{1}$ and it passes to the quotient to $\pi_{2}$. If $\iota_{i}\pi_{1} = \pi_{1}$ for some $i$, it follows that $\alpha$ is a scalar on $\pi_{1}$, but this is impossible, since the equality $\alpha \Pi = -\Pi$ still holds on $\pi_{1}$. Since $\text{End}_{Iw}(\pi_{1}) = k \times k$, it follows that (up to replacing $\alpha$ with $-\alpha$) we have $\pi_{\infty}(\chi_{1}) \subseteq A, r, \lambda^{+}$ and $\pi_{\infty}(\chi_{1}) \subseteq A, r, \lambda^{-}$. Similarly, the image of $A, r, \lambda^{+}$ in $\pi_{2}$ is one of $\pi_{\infty}(\chi_{2}), \pi_{\infty}(\chi_{2})$, but now we have to determine which one.

Proposition 2.4.19. The image of $A, r, \lambda^{+}$ in $\pi_{2}$ is $\pi_{\infty}(\chi_{2})$.

Proof. Recall that $\iota_{1}$ is the idempotent of $A, r, \lambda$ corresponding to the direct summand $A, r, \lambda^{+}$, and write $\iota_{1}$ for the idempotent induced on $\pi_{2}$. To prove the proposition it suffices to show that $\dim \iota_{1}(\text{soc}_{K} \pi_{2}) > 1$. Indeed, if $\iota_{1}$ were the idempotent corresponding to $\pi_{\infty}(\chi_{2})$, then it would be the operator that restricts functions to $Iw$. So we would have

$$\dim \iota_{1}(\text{soc}_{K} \pi_{2}) \leq \dim \iota_{1}(\text{Ind}_{Iw}^{K}(\chi_{2})) = 1$$

since

$$\text{Ind}_{Iw}^{K}(\chi_{2})|_{Iw} = k \cdot \varphi \oplus \text{Ind}_{Iw}^{K}(\chi_{2})_{0}$$

where the $p$-dimensional subspace $\text{Ind}_{Iw}^{K}(\chi_{2})|_{Iw}$ consists of functions supported on $B(Z_{p})\text{Iw}$, and $\varphi$ is the function supported on $Iw$ and sending the identity to 1.

In order to bound $\dim \iota_{1}(\text{soc}_{K} \pi_{2})$ from below we will work with a certain $K$-subspace $\sigma_{\text{cusp}}$ of $A, r, \lambda^{+}$ with two Jordan–Hölder factors. (See e.g. [BP12 Section 20].) It is isomorphic to the mod $p$ reduction of a lattice in a tame cuspidal type, its socle is $\text{soc}_{K}(\pi_{1})$ and its cosocle projects isomorphically to $\text{soc}_{K}(\pi_{2})$. Furthermore, the $Iw$-socle of $\sigma_{\text{cusp}}$ is irreducible and coincides with $\text{soc}_{K}(\pi_{1})^{Iw} \subseteq \pi_{\infty}(\chi_{1})$. We will need the following lemmas.

Lemma 2.4.20. The $Iw$-representation $\sigma_{\text{cusp}}$ is uniserial.
Proof. We know that \( \sigma_{\text{cusp}} \) is inflated from \( \text{GL}_2(\mathbb{F}_p) \), hence \( I_w \) is acting through \( HU(\mathbb{Z}_p) \). Then \( \sigma_{\text{cusp}} \) has irreducible \( HU(\mathbb{Z}_p) \)-socle, isomorphic to \( \chi_1^* \), hence it embeds \( HU(\mathbb{Z}_p) \)-linearly into \( \text{inj}(\chi_1^* \cap \pi_1) \) (an injective envelope in the category of smooth \( HU(\mathbb{Z}_p) \)-representations). By [Past03] Proposition 5.9, \( \text{inj}(\chi_1^*) \) is \( HU(\mathbb{Z}_p) \)-uniserial.

\[ \square \]

Lemma 2.4.21. We have \( t_1 \sigma_{\text{cusp}} \geq (\dim \text{soc}_K \chi_2) + 1 \).

Proof. Although it need not be true that \( \sigma_{\text{cusp}} \) is \( \alpha \)-stable, there still is an exact sequence

\[ 0 \to \sigma_{\text{cusp}}^{\alpha=1} \to \sigma_{\text{cusp}} \to t_1 \sigma_{\text{cusp}} \to 0. \]

Comparing dimensions it suffices to check that \( \sigma_{\text{cusp}}^{\alpha=1} \) is properly contained in \( \text{soc}_K(\pi_1) \). If this is not so, then we deduce from Lemma 2.4.20 that

\[ \text{soc}_K(\pi_1) \subseteq \sigma_{\text{cusp}}^{\alpha=1} \subset A_{r,\lambda}^{\alpha=1}. \]

To see this is impossible, we use the fact that \( \pi_1 \) contains \( \text{Ind}_{\text{Iw}}^K(\chi_1) \), which decomposes as

\[ \text{Ind}_{\text{Iw}}^K(\chi_1)|_{\text{Iw}} = k \cdot \varphi \oplus \text{Ind}_{\text{Iw}}^K(\chi_1)_{0}. \]

Recall from [BPT12] Lemma 2.7(i) that \( \text{Ind}_{\text{Iw}}^K(\chi_1)_{0} \) is spanned by certain functions \( f_i = S_i \varphi \) for \( 0 \leq i \leq p-1 \), and we have \( f_i + (-1)^i \varphi \in \text{soc}_K \text{Ind}_{\text{Iw}}^K(\chi_1) = \text{soc}_K(\pi_1) \). Since all the \( f_i \) are supported in \( B(\mathbb{Z}_p)_{\text{Iw}} \), they are contained in \( \pi_{\infty}(\chi_1) = \pi_{\infty}^{\alpha=1} \). But then if \( \text{soc}_K(\pi_1) \subseteq A_{r,\lambda}^{\alpha=1} \) we deduce that also \( \varphi \in \pi_{\infty}(\chi_1) \), which is not true.

Now there is an exact sequence

\[ 0 \to (t_1 \sigma_{\text{cusp}}) \cap \pi_1 \to t_1 \sigma_{\text{cusp}} \to \mathcal{T}_1(\text{soc}_K \chi_2) \to 0 \]

and the intersection \((t_1 \sigma_{\text{cusp}}) \cap \pi_1 \) coincides with \((t_1 \sigma_{\text{cusp}}) \cap \pi_{\infty}(\chi_1) \). To conclude the proof of Proposition 2.4.19 it suffices to notice that \((t_1 \sigma_{\text{cusp}}) \cap \pi_{\infty}(\chi_1) \) is at most one-dimensional, because it is contained in \( \pi_{\infty}(\chi_1)^{K_1} \).

By Proposition 2.4.19 there are short exact sequences

\[ 0 \to \pi_{\infty}^+(\chi_1) \to t_2 A_{r,\lambda} \to \pi_{\infty}(\chi_2) \to 0, \]

\[ 0 \to \pi_{\infty}(\chi_1) \to t_1 A_{r,\lambda} \to \pi_{\infty}^+(\chi_2) \to 0. \]

We have previously described an element \( \bar{e}_{n+1} \in A_{r,\lambda} \), mapping into \( \pi_{\infty}(\chi_2) \), and satisfying

\[ (u-1)\bar{e}_{n+1} \in \text{soc}_{\text{Iw}} \pi_{\infty}^+(\chi_1) \text{ for all } u \in \begin{pmatrix} 1/p^{n+1} & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ (g-1)\bar{e}_{n+1} \equiv (-1)^{n+r+1} \lambda^{-2} \left( B_{k,p^n-1-(p-3-r)} X^{p-3-r} + (a-d) p^{n-1-(p-2-r)} X^{p-2-r} \right) \in \pi_{n+1}(\chi_1) \]

modulo \( \pi_{\infty}^+(\chi_1) \oplus X^{(p-3-r)+(p-2)} \pi_{n+1}(\chi_1) \) whenever \( g = \begin{pmatrix} 1+pa & b \\ 0 & 1+pd \end{pmatrix} \). Fix such elements \( u \) and \( g \).

Since \( t_1 \) is \( \text{Iw} \)-linear, equal to 1 on \( \pi_{\infty}(\chi_1) \), and equal to 0 on \( \pi_{\infty}^+(\chi_1) \), the element \( t_1 \bar{e}_{n+1} \) is fixed by \( u \) and satisfies the same congruence as \( \bar{e}_{n+1} \). However, \( \mathcal{T}_1 \) is zero on \( \pi_{\infty}(\chi_2) \), and so \( t_1 \bar{e}_{n+1} \in \pi_{\infty}(\chi_1) \), and since it is fixed by \( \begin{pmatrix} 1/p^{n+1} & 0 \\ 0 & 1 \end{pmatrix} \) it is actually contained in \( \pi_{n+1}(\chi_1) \), by (2.4.3). But then the congruence contradicts Proposition 2.4.19 which says that \( (g-1)\pi_{n+1}(\chi_1) \subseteq X^{p-2}\pi_{n+1}(\chi_1) \), whenever \( b \notin p\mathbb{Z}_p \), because \( p-3-r < p-2 \).

\[ \square \]

2.5. Summary. We summarize some consequences of the results above. For simplicity, we are going to write \( \text{Ext}_{\mathcal{O}[G]}^1 \) for the Ext-functor computed in the category of smooth representations on \( p \)-torsion \( \mathcal{O} \)-modules.

Theorem 2.5.1. Let \( \chi_1, \chi_2 \) be finite length smooth \( \mathcal{O}[\text{GL}_2(\mathbb{Q}_p)] \)-representations of central character \( \zeta \).

(1) If the Jordan–Hölder factors of the \( \pi_i \) are all generic and supersingular, then \( \text{Hom}_{\mathcal{O}[G]}(\pi_1, \pi_2) = \text{Hom}_{\mathcal{O}[\text{Iw}Z]}(\pi_1, \pi_2) \) and the restriction map \( \text{Ext}_{\mathcal{O}[G]}^1(\pi_1, \pi_2) \to \text{Ext}_{\mathcal{O}[\text{Iw}Z]}^1(\pi_1, \pi_2) \) is injective.

Similarly, \( \text{Hom}_{\mathcal{O}[G]}(\pi_1, \pi_2) = \text{Hom}_{\mathcal{O}[N]}(\pi_1, \pi_2) \) and the restriction map \( \text{Ext}_{\mathcal{O}[G]}^1(\pi_1, \pi_2) \to \text{Ext}_{\mathcal{O}[N]}^1(\pi_1, \pi_2) \) is injective.
(2) Fix $r \in \{1,\ldots,p-4\}, s \in \{0,\ldots,p-2\}$, and $\lambda \in k^\times$. If the $\pi_i$ both admit an exhaustive $G$-stable filtration with all graded factors isomorphic to $A_{r,s,\lambda}$, then $\text{Hom}_G(\pi_1, \pi_2) = \text{Hom}_{KZ}(\pi_1, \pi_2)$ and the restriction map $\text{Ext}^1_{O[G]}(\pi_1, \pi_2) \to \text{Ext}^1_{O[KZ]}(\pi_1, \pi_2)$ is injective.

**Proof.** For part (1), the Jordan–Hölder factors of $\pi_1|_{G^+}$ are all of the form $\pi_{\sigma}$ for some generic weight $\sigma$ (depending on the factor), since we are fixing the central character. Then the first statement follows from corollaries 2.3.6 and 2.3.12 by a standard dévissage argument. (See for instance [Paš10, Lemma A.1]. Notice that this argument requires injectivity of the restriction map $\text{Ext}^1_{O[G^+]}(\pi_{\sigma_1}, \pi_{\sigma_2}) \to \text{Ext}^1_{O[IwZ]}(\pi_{\sigma_1}, \pi_{\sigma_2})$ for all generic Serre weights $\sigma_1, \sigma_2$, but this is immediate from Corollary 2.3.12 since if an extension of $\pi_{\sigma_2}$ by $\pi_{\sigma_1}$ splits over $IwZ$ then the maximal ideal of $O$ acts trivially on the extension.)

Similarly, the second statement of part (1) follows from Corollary 2.3.7 and Theorem 2.3.10 and part (2) follows from Proposition 2.4.4 and Theorem 2.4.18.

**Corollary 2.5.2.** Let $A$ be an Artin local $O$-algebra with residue field $k$ and maximal ideal $m_A$. Let $\pi_1, \pi_2$ be smooth $A[\GL_2(Q_p)]$-representations on flat $A$-modules.

1. Assume that $\pi_1 \otimes_A k \cong \pi_2 \otimes_A k$ are supersingular generic irreducible representations. Then $\pi_1|_{A[IwZ]} \cong \pi_2|_{A[IwZ]}$ if and only if $\pi_1|_{A[G^+]} \cong \pi_2|_{A[G^+]}$ and $\pi_1|_{A[N]} \cong \pi_2|_{A[N]}$ if and only if $\pi_1 \cong \pi_2$.

2. Assume that there exist $r_1 \in \{1,\ldots,p-4\}, s_i \in \{0,\ldots,p-2\}$ and $\lambda_i \in k^\times$ such that $\pi_i \otimes_A k \cong A_{r_i,s_i,\lambda_i}$. If $f: \pi_1 \to \pi_2$ is an $A[KZ]$-linear isomorphism, then either $(r_1, \lambda_1) = (r_2, \lambda_2)$ and $f$ is $A[G^+]$-linear, or $(r_1, \lambda_1) \neq (r_2, \lambda_2)$ and $f$ induces an $A[G]$-linear isomorphism $\pi_1 \to (nr_{-1} \otimes \det) \otimes_A \pi_2$.

**Proof.** For part (1) it suffices to notice that by the flatness assumption the representations $\pi_i$ satisfy the assumptions of part (1) of Theorem 2.5.1 (to see this, tensor the $m$-adic filtration of $A$ with $\pi_i$). For part (2), notice the map $f \otimes_A k$ is a $KZ$-linear isomorphism $A_{r_1,s_1,\lambda_1} \to A_{r_2,s_2,\lambda_2}$. Comparing the $K$-socle, it follows that $r_1 = r_2$ and $s_1 = s_2$. But now Theorem 2.4.13 implies that $\lambda_1^2 = \lambda_2^2$. If $\lambda_1 = -\lambda_2$, then the twist $\pi_2 \otimes_A (nr_{-1} \otimes \det)$ is a deformation to $A$ of $A_{r_1,s_1,\lambda_1}$ (here $nr_{-1}$ is valued in $A^\times$). In either case, we find that the representations $\pi_1$ and $\pi_2 \otimes_A (nr_{-1} \otimes \det)$ satisfy the assumptions of part (2) of Theorem 2.5.1 and the claim follows.

3. **Banach space representations.**

In this section we prove the two theorems in the introduction, starting from Theorem 1.0.1. Let $\Pi$ be an absolutely irreducible admissible unitary $E$-Banach space representation of $\GL_2(Q_p)$, with fixed central character $\zeta: \Q_p^\times \to O^\times$. Assume that $\Pi$ is very generic, as defined in Section 2.1.8. In this section we will often deal with open bounded $G$-invariant lattices in $\Pi$. To abbreviate, we will refer to them simply as lattices.

3.1. **Proof of Theorem 1.0.1: supersingular reduction.** Assume that $\Pi$ has a lattice $\Theta$ whose reduction $\Theta \otimes_O k \cong \pi$ is an absolutely irreducible supersingular representation of $G$. Let $\Pi^1 \subset \Pi$ be a proper $K$-stable closed $E$-subspace. Then $\Theta^1 = \Theta \cap \Pi^1$ is a lattice in $\Pi^1$, and

$$\Theta^1 \otimes_O k \to \Theta \otimes_O k$$

is injective. Write $\Theta \otimes_O k = \pi_{\sigma} \oplus \pi_{\sigma'}$, as in Section 2.3.

**Proposition 3.1.1.** Let $\sigma_1$ and $\sigma_2$ be distinct generic weights. Let $X_j \subset \pi_{\sigma_j}$ be a finite-dimensional $Iw$-stable subspace. Then there are no nonzero $Iw$-linear morphisms $\lambda: \pi_{\sigma_1}/X_1 \to \pi_{\sigma_2}/X_2$.

**Proof.** Without loss of generality, $X_1 = 0$. Since $X_2$ is finite-dimensional, it is contained in $\soc^i_{Iw}(\pi_{\sigma_2})$ for $i$ large enough. Consider the induced morphism

$$\lambda: \pi_{\sigma_1} \to \pi_{\sigma_2}/X_2 \to \pi_{\sigma_2}/\soc^i_{Iw}(\pi_{\sigma_2}),$$

and recall from 2.3.1 that

$$\pi_{\sigma_2}/\soc^i_{\pi_{\sigma_2}} \cong M_{\sigma_2}/\soc^i(M_{\sigma_2}) \oplus M^+_{\sigma_2}/\soc^i(M^+_{\sigma_2}).$$

Let us compose $\lambda_i$ with the surjection

$$M_{\sigma_1} \oplus M^+_{\sigma_2} \to \pi_{\sigma_1},$$
and apply Proposition 2.3.2 and Theorem 2.3.3. Since $\sigma_1$ and $\sigma_2$ are not isomorphic, neither are $\sigma_1^{[s]}$ and $\sigma_2^{[s]}$, and we deduce that $\lambda_i = 0$.

It follows that $\im \lambda \subseteq \soc^{t}(\sigma_\sigma)/X_2$ is finite-dimensional. Now the proposition follows from the fact that $\pi_\sigma$ has no nonzero finite-dimensional $Iw$-quotients. To see this, observe that Lemma 2.1.5 implies that $\pi_\sigma$ has no nonzero finite-dimensional $Iw$-quotients. Then the same is true for its twist $\pi_\sigma^\dagger$, so the image of $\pi_\sigma$ and $\pi_\sigma^{\dagger}$ in a finite-dimensional $Iw$-representation vanishes. But $\pi_\sigma$ is a quotient of $\pi_\sigma \oplus \pi_\sigma^{\dagger}$.

**Proposition 3.1.2.** Let $\sigma$ be a generic weight. Assume that $X$ is an infinite-dimensional $K$-stable subspace of $\pi_\sigma$. Then $X = \pi_\sigma$.

**Proof.** It suffices to prove the proposition after extending scalars to $\overline{k}$. Write $\chi = \soc^{t}(\pi_\sigma)$. Then

$$\pi_\sigma/\chi = M_\sigma/\chi \oplus M_\sigma^{\dagger}/\chi$$

is the direct sum of two uniserial $Iw$-representations, and its socle filtration is given by Lemma 2.1.2. Assume $\soc^{t}(\pi_\sigma)$ is contained in $X$ but $\soc^{t+1}(\pi_\sigma)$ is not. Then $X/\soc^{t}(\pi_\sigma)$ intersects trivially one of the summands of $\pi_\sigma/\soc^{t+1}(\pi_\sigma)$, since otherwise it would contain the socle of both summands, hence it would contain $\soc^{t}(\pi_\sigma/\soc^{t+1}(\pi_\sigma))$ and $X$ would contain $\soc^{t+1}(\pi_\sigma)$.

If $(X/\soc^{t}(\pi_\sigma)) \cap (M_\sigma^{\dagger}/\soc^{t}(M_\sigma)) = 0$, then we obtain an injection of $X/\soc^{t}(\pi_\sigma)$ in $M_\sigma/\soc^{t}(M_\sigma)$, through the canonical projection. By Lemma 2.1.2 every proper $Iw$-submodule of $M_\sigma/\soc^{t}(M_\sigma)$ is finite-dimensional, hence this injection is an isomorphism onto $M_\sigma/\soc^{t}(M_\sigma)$. By the same argument as Lemma 2.3.2 there are no nonzero $Iw$-linear maps $M_\sigma/\soc^{t}(M_\sigma) \to M_\sigma^{\dagger}/\soc^{t}(M_\sigma)$. Hence $X/\soc^{t}(\pi_\sigma)$ is contained in the first summand $M_\sigma/\soc^{t}(M_\sigma)$, hence it coincides with it. But then $X$ contains $M_\sigma$.

The other case similarly implies that $X$ contains $M_\sigma^{\dagger}$. So it suffices to prove that if $X$ is a $K$-stable subspace of $\pi_\sigma$ containing $M_\sigma$ or $M_\sigma^{\dagger}$, then $X = \pi_\sigma$. To do this, we will identify $\pi_\sigma$ with a direct summand of a supersingular $GL_2(Q_{\sigma})$-representation, and we will use the action of $G = GL_2(Q_{\sigma})$.

By the proof of [Pas10 Proposition 4.12], we have $\Pi M_\sigma^{\dagger} \subseteq M_\sigma$. Hence if $X$ contains $M_\sigma$, then it contains $\Pi M_\sigma^{\dagger}$. Applying $s$ we see that it contains $\Pi M_\sigma^{\dagger}$. Since $\pi_\sigma = M_\sigma + \Pi M_\sigma^{\dagger}$, we deduce that $X = \pi_\sigma$.

For the other case, assume $X$ contains $\Pi M_\sigma^{\dagger}$ and is $K$-stable. Let $n \geq 0$ be an even integer and let $v_\sigma^{[n]} \in \soc^{t}(M_\sigma^{[n]})$ be a generator. Then $t^n v_\sigma^{[n]} \in M_\sigma^{[n]}$ by definition, hence $\Pi t^n v_\sigma^{[n]} \in X$. Applying $s$, we see that $t^n s v_\sigma^{[n]} \in X$. Since $n + 1$ is odd, by [Pas10 Lemma 4.8] we have $t^{n+1} v_\sigma^{[n]} \in M_\sigma^{\dagger}$, and so it suffices to prove that the vectors $t^{n+1} v_\sigma^{[n]}$ for even $n$ generate the representation $M_\sigma^{\dagger}$ over $Iw$.

If this were not true, they would be contained in an $Iw$-stable subspace $Y \subseteq M_\sigma$ of finite dimension (because $M_\sigma$ is $Iw$-uniserial). But then $Y$ is a $K_1$-stable subspace of $\pi_\sigma$ of finite dimension, containing all the $t^n v_\sigma^{[n]}$ for even integers $n \geq 2$. Inducing, we find a finite-dimensional $Iw$-stable subspace of $\pi_\sigma$ containing all the $t^n v_\sigma^{[n]}$. This is a contradiction since by definition of $M_\sigma^{\dagger}$ they generate $M_\sigma^{\dagger}$ over $Iw$.

**Proposition 3.1.3.** Let $\pi$ be an absolutely irreducible, supersingular $k[G]$-representation. Let $X$ be an infinite-dimensional $K$-stable subspace of $\pi = \pi_\sigma \oplus \pi_\sigma^{\dagger}$. Then $X$ contains one of the summands $\pi_\sigma, \pi_\sigma^{\dagger}$, hence it is a direct sum of subspaces of $\pi_\sigma, \pi_\sigma^{\dagger}$.

**Proof.** Assume it does not. Then $X$ intersects the summands in finite-dimensional subspaces $X_\sigma, X_\sigma^{\dagger}$ by Proposition 3.1.2. Then the image of $X$ in $\pi/ (X_\sigma \oplus X_\sigma^{\dagger})$ does not intersect any of the summands. It follows that the projections of $X/ (X_\sigma \oplus X_\sigma^{\dagger})$ to $\pi_\sigma/X_\sigma$ and $\pi_\sigma^{\dagger}/X_\sigma^{\dagger}$ are both injective. They are surjective by Proposition 3.1.2, since otherwise $X$ is finite-dimensional. Hence they are isomorphisms, and this contradicts the fact that there are no nonzero $Iw$-linear maps between $\pi_\sigma/X_\sigma$ and $\pi_\sigma^{\dagger}/X_\sigma^{\dagger}$ by Proposition 3.1.1.

Let us now consider the image of $\Theta^1 \otimes_{\mathcal{O}} k$ in $\Theta \otimes_{\mathcal{O}} k$. If it is finite-dimensional, then $\Theta^1$ is finitely generated over $\mathcal{O}$, by the following version of Nakayama’s lemma. Hence $\Pi^1$ is finite-dimensional and we are done.

**Lemma 3.1.4.** Let $f : M_1 \to M_2$ be an $O$-linear map between $\pi_E$-adically separated and complete $O$-modules. If $f \otimes_{\mathcal{O}} k$ is surjective, then $f$ is surjective.

**Proof.** The map $f$ is $\pi_E$-adically continuous since it is $O_E$-linear. Let $x \in M_2$. By the assumption on $f \otimes_{\mathcal{O}} k$ there exists $x_0 \in M_1$ such that $x - f(x_0) \in \pi_E M_2$. Repeating, we find that there exist $x_i \in M_1$ such that $x = \sum_{n=0}^{+\infty} \pi_E f(x_i)$. The continuity of $f$ implies that if $y = \sum_{n=1}^{+\infty} \pi_E x_i$ then $x = f(y)$. □
There remains to consider the case that $\Theta^1 \otimes_{\mathcal{O}} k$ is infinite-dimensional, in which case we can assume by Proposition 3.1.3 that its image in $\Theta \otimes_{\mathcal{O}} k$ is equal to $\pi_\sigma + X_{\sigma^{|\mathcal{O}}}$ for some finite-dimensional $X_{\sigma^{|\mathcal{O}}}$ (since otherwise $\Pi^1 = \Pi$ by Lemma 3.1.4). We know that the matrix $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ swaps the two direct summands of $\pi$.

Let us introduce $\Pi^2 = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Pi^1$, an Iw-stable closed subspace of $\Pi$, and define $\Theta^2 = \Theta \cap \Pi^2 = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \Theta^1$.

**Lemma 3.1.5.** The space $\Pi^1 \cap \Pi^2$ is finite-dimensional over $E$ and Iw-stable.

**Proof.** The space $\Pi^1 \cap \Pi^2$ is Iw-stable since so are the $\Pi^i$. Notice that $\Theta^1 \cap \Theta^2 = \Theta \cap (\Pi^1 \cap \Pi^2)$ is a lattice in $\Pi^1 \cap \Pi^2$. By construction, $\Theta^2 \otimes_{\mathcal{O}} k \to \Theta \otimes_{\mathcal{O}} k$ has image $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} X_{\sigma^{|\mathcal{O}}} \oplus \pi_\sigma$. Since $X_{\sigma^{|\mathcal{O}}}$ is finite-dimensional and the injection $(\Theta^1 \cap \Theta^2) \otimes_{\mathcal{O}} k \to \Theta \otimes_{\mathcal{O}} k$ has image contained in $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} X_{\sigma^{|\mathcal{O}}} \oplus X_{\sigma^{|\mathcal{O}}}$, the claim follows again from Lemma 3.1.4.

Since $\Pi^1 \cap \Pi^2$ is Iwahori-stable, we have completed the proof of Theorem 1.0.1 in the supersingular case if $\Pi^1 \cap \Pi^2 \neq 0$ (because we can induce $\Pi^1 \cap \Pi^2$ to a finite-dimensional $K$-stable closed $E$-subspace of $\Pi$).

**Lemma 3.1.6.** Assume $\Pi^1 \cap \Pi^2 = 0$. Then $\Pi|_{Iw} = \Pi^1 \oplus \Pi^2$.

**Proof.** By Lemma 3.1.4 and our assumptions on $\Theta^1$ and $\Theta^2$ we know that the map $\Theta^1 \oplus \Theta^2 \to \Theta$ induced by the inclusion of $\Theta^2$ in $\Theta$ is surjective. Upon inverting $p$ it follows that $\Pi = \Pi^1 + \Pi^2$ (algebraic sum), and by assumption the sum is direct. Finally, by definition of the $\Theta_i$ we find that $\Pi^1 \oplus \Pi^2 \to \Pi$ is a bijective continuous morphism, hence it is a topological isomorphism.

The following lemma applied to the orthogonal idempotents defining the decomposition in Lemma 3.1.6 implies that $\Pi^1, \Pi^2$ are actually $G^+$-stable.

**Lemma 3.1.7.** Let $\Pi_1$ and $\Pi_2$ be absolutely irreducible, unitary, admissible $E$-Banach space representations admitting lattices $\Theta_1$ whose reductions are isomorphic to the same supersingular irreducible representation. Then $\text{Hom}_{G^+}(\Pi_1, \Pi_2) = \text{Hom}_{Iw}(\Pi_1, \Pi_2)$ and $\text{Hom}_G(\Pi_1, \Pi_2) = \text{Hom}_N(\Pi_1, \Pi_2)$.

**Proof.** Let $\lambda : \Pi_1 \to \Pi_2$ be continuous and Iw-linear. Multiplying $\lambda$ by a power of $\pi_E$, we can assume that $\lambda(\Theta_1) \subseteq \Theta_2$. Let $g \in G^+$. It suffices to prove that $\lambda g - g\lambda|_{\Theta_1} = 0$. To do so, since $\Theta_2$ is separated it suffices to prove that $\lambda g - g\lambda$ induces the zero map $\Theta_1/\pi_E^n \to \Theta_2/\pi_E^n$ for all $n > 0$. But this is true by part (1) of Theorem 2.5.1 since $\Theta/s_E\pi_E^n$ is flat over $O_E/\pi_E^n$.

The same proof works for $\Pi$.

Finally, stability of $\Pi^1$ under $G^+$ implies that $X_{\sigma^{|\mathcal{O}}}$ is $0$. But then Proposition 3.1.2 implies that any proper $K$-stable closed $E$-subspace of $\Pi^1$ or $\Pi^2$ is finite-dimensional. So either the $\Pi^i$ are topologically irreducible as $K$-representations or $\Pi$ has a finite-dimensional $K$-stable closed $E$-subspace. This concludes the proof of Theorem 1.0.1 in the supersingular case.

**Remark 3.1.8.** All cases of Theorem 1.0.1 occur already for representations with supersingular reduction. More precisely, case (3) holds if and only if a twist of $\Pi$ is associated to a potentially semistable irreducible Galois representation $\rho : \text{Gal}_{Q_p} \to \text{GL}_2(E)$ with distinct Hodge–Tate weights under the $p$-adic Langlands correspondence, and case (2) holds if and only if $\Pi$ is associated to the induction of a character of $\text{Gal}_{Q_p}$ that does not extend to $\text{Gal}_{Q_p}$.

### 3.2. Proof of Theorem 1.0.1: reducible reduction

Now assume that $\Pi$ is not ordinary (as defined in [Pas13]) but has no lattice with supersingular reduction. Then by the main results of [Pas13] the Jordan–Hölder factors of the reduction of any lattice are principal series representations $\{\pi_1, \pi_2\}$ with distinct $K$-socle. Assume that $\Pi^1 \subset \Pi$ is a proper $K$-stable closed $E$-subspace. We are going to prove that $\Pi^1$ is finite-dimensional over $E$. We introduce the following piece of notation: if $\Theta \subset \Pi$ is a lattice, we write $\Theta^{\text{sub}}$ for the image of the injection $$(\Theta \cap \Pi^1) \otimes_{\mathcal{O}} k \to \Theta \otimes_{\mathcal{O}} k.$$ It is a $K$-stable subspace of $\Theta \otimes_{\mathcal{O}} k$. 

3.2.1. Outline of the argument. If \( \Pi \) admits a lattice \( \Theta \) such that \( \Theta^{ab} \) is finite-dimensional, then Lemma 3.1.3 implies that \( \Pi^1 \) is finite-dimensional. Otherwise, \( \Theta^{ab} \) is infinite-dimensional for all choices of \( \Theta \), and it is a proper subspace of \( \Theta_i \). We will prove that the latter case leads to a contradiction, eventually arising from Proposition 3.2.4 about Iw-linear morphisms between quotients of \( \pi_1 \) and \( \pi_2 \). Since our argument is quite involved, we begin by giving a brief outline.

Applying the results in Appendix A we find for \( i = 1, 2 \) a lattice \( \Theta_i \) in \( \Pi \) whose reduction is the atome automorphe surjecting onto \( \pi_i \). Then we consider \( \Theta_i^{ab} \), and we prove that it contains the other factor \( \pi_{3-i} \). To do so, we notice that if \( \Theta_i^{ab} \) does not contain \( \pi_{3-i} \) then for dimension reasons \( \Theta_i^{ab} \) surjects onto \( \pi_i \). Then we prove that this implies \( \Theta_i^{ab} = \Theta_i \) and so \( \Pi^1 = \Pi \), a contradiction. (This is Theorem 3.2.8.)

On the other hand, by a result analogous to Proposition 3.1.3, we prove in Proposition 3.2.5 that if \( \Theta \) is a lattice with semisimple reduction, then \( \Theta^{ab} \) contains one the factors \( \pi_1, \pi_2 \). In Proposition 3.2.7 we go further and we prove that this factor does not depend on the choice of \( \Theta \): up to renumbering, we can therefore assume it is \( \pi_2 \).

At this point we know that \( \Theta_2 \) has a neighbour \( \Theta_i \), which may be \( \Theta_1 \) or a lattice with semisimple reduction, such that \( \Theta_i^{ab} \) contains \( \pi_2 \). Since \( \Theta_2 \) and \( \Theta_i \) are neighbours we have inclusions

\[
\pi_E \Theta_2 \subseteq \Theta_i \subseteq \Theta_2 \subseteq \pi^{-1}_E \Theta
\]

and since \( \pi_E \Theta \cap \Pi^1 = \pi_E (\Theta \cap \Pi^1) \) we see that the lattices \( \Theta_i \cap \Pi^1, \Theta_2 \cap \Pi^1 \) are also neighbours. It follows that there are one-step filtrations on \( \Theta_i^{ab}, \Theta_2^{ab} \) with the same graded factors (up to reordering). We conclude the argument by proving that this produces a morphism between certain quotients of \( \pi_1 \) and \( \pi_2 \), contradicting Proposition 3.2.8.

3.2.2. Subspaces of \( \Theta \otimes \sigma k \). We prove some analogues of the results in Section 3.1 concerning the \( K \)-stable subspaces of \( \Theta \otimes \sigma k \).

**Proposition 3.2.3.** Let \( \pi_1, \pi_2 \) be generic principal series representations of \( \text{GL}_2(\mathbb{Q}_p) \) of distinct weight, with \( \text{soc}_K(\pi_i) = \chi_i \). Let \( X_i \) be a finite-dimensional Iw-stable subspace of \( \pi_i \). Then there are no nonzero Iw-linear morphisms \( \pi_1/X_1 \to \pi_2/X_2 \).

**Proof.** The same argument as Proposition 3.1.1 goes through, substituting \( \pi_1(\chi_i) \) for \( M_{\sigma_1}, \pi_1^+(\chi_i) \) for \( \Pi M_{\sigma_i} \), and appealing to Theorem 2.2.6 and the proof of Proposition 2.2.5. \( \square \)

**Proposition 3.2.4.** Let \( \pi \) be a generic principal series representation of \( \text{GL}_2(\mathbb{Q}_p) \). Assume \( X \) is an infinite-dimensional \( K \)-stable subspace of \( \pi \). Then \( X = \pi \).

**Proof.** The same argument as Proposition 3.1.2 proves that \( X \) has to contain one of the summands in the decomposition \( \pi|_{Iw} = \pi_\infty(\chi) \oplus \pi_\infty^+(\chi) \). (Notice that the \( G \)-action was not used to establish this, and there is no need here to extend scalars to \( \mathbb{k} \).) If it contains \( \pi_\infty(\chi) \), then we are done since \( \pi_\infty(\chi) \) generates \( \pi \cong \text{Ind}_{K_0(p)}^K \pi_\infty(\chi) \). Otherwise, let \( \varphi_{n+1} \in \pi_{n+1}(\chi) \) be the \( K_0(p^{n+1}) \)-eigenvector supported in \( K \setminus \text{Iw} \) with \( \varphi_{n+1}(1) = 1 \). Then

\[
\begin{pmatrix} a & b \\ pc & d \end{pmatrix} = \varphi_{n+1} \begin{pmatrix} b & a \\ d & pc \end{pmatrix}
\]

implies that \( \varphi_{n+1} \) is supported in \( K \setminus \text{Iw} \), and so \( s_\varphi n+1 \in \pi_\infty^+(\chi) \). But this implies that if \( X \) is \( K \)-stable and contains \( \pi_\infty^+(\chi) \), then \( X = \pi \), since \( \pi_\infty^+(\chi) \) is \( K \)-generated by the \( \varphi_{n+1} \) as \( n \) varies. \( \square \)

**Proposition 3.2.5.** Write the reduction of \( \Theta \) as a direct sum of principal series representations

\[
\pi = \Theta \otimes \sigma k = \pi_1 \oplus \pi_2.
\]

If \( X \subset \Theta \otimes \sigma k \) is a proper \( K \)-stable infinite-dimensional subspace, then \( X \) contains one of the two summands, hence it is a direct sum.

**Proof.** Given Propositions 3.2.3 and 3.2.4, the same argument as Proposition 3.1.3 goes through. \( \square \)

The following lemma will be often employed together with the previous results.
Lemma 3.2.6. Let $\Theta \subset \Pi$ be a lattice with semisimple reduction, so that $\Theta^{\text{ab}} \subset \Theta \otimes_{\mathcal{O}} k$ is a proper subspace that contains $\pi_i$ for some $i \in \{1, 2\}$. Let $\Theta^{\text{ab}}$ be a $K$-stable subspace of $\Theta^{\text{ab}}$. Then there exists a unique $\Theta^{\text{ab},\infty} \in \{\text{Fil} \Theta^{\text{ab}}, \Theta^{\text{ab}} / \text{Fil} \Theta^{\text{ab}}\}$ that is infinite-dimensional. Furthermore, $\Theta^{\text{ab},\infty}$ has a finite-dimensional $K$-subspace $X$ such that $\Theta^{\text{ab},\infty} / X$ is $K$-isomorphic to an infinite-dimensional quotient of $\pi_i$.

Proof. By Proposition 3.2.5 we can write $\Theta^{\text{ab}} = \pi_i \oplus T$ for a finite-dimensional subspace $T$ of the other summand of $\Theta \otimes_{\mathcal{O}} k$. Existence of $\Theta^{\text{ab},\infty}$ follows because $\Theta^{\text{ab}}$ is infinite-dimensional, and uniqueness because an infinite-dimensional subspace of $\Theta^{\text{ab}}$ has the form $\pi_i \oplus T'$ for some finite-dimensional subspace of $T$, again by Proposition 3.2.5. This also implies the last assertion of the lemma in the case that $\Theta^{\text{ab},\infty} = \text{Fil} \Theta^{\text{ab}}$ (the subspace $X$ we are looking for is $T'$). Otherwise, $X$ can be taken to be the finite-dimensional subspace $\left(\text{Fil} \Theta^{\text{ab}} + T\right) / \text{Fil} \Theta^{\text{ab}}$ of $\Theta^{\text{ab},\infty} = \Theta^{\text{ab}} / \text{Fil} \Theta^{\text{ab}}$.

Now let $\Theta$ be a lattice in $\Pi$ with semisimple reduction. By Proposition 3.2.5 if $\Theta^{\text{ab}}$ is infinite-dimensional then there exists an index $i(\Theta) \in \{1, 2\}$ such that $\Theta^{\text{ab}}$ contains $\pi_i(\Theta)$.

Proposition 3.2.7. Let $\Theta, \Psi$ be lattices in $\Pi$ with semisimple reduction. Then $i(\Theta) = i(\Psi)$.

Proof. Without loss of generality, assume for a contradiction that $i(\Theta) = 1$ and $i(\Psi) = 2$. Applying Theorem A.0.7 it suffices to prove the theorem when $\Theta$ and $\Psi$ are neighbours. Then $\Theta^{\text{ab}}$ and $\Psi^{\text{ab}}$ have one-step $K$-stable filtrations with the same graded factors up to reordering.

We apply Lemma 3.2.6. Assume that $\Theta^{\text{ab},\infty}$ is a subspace of $\Psi^{\text{ab}}$: then $\Psi^{\text{ab}} / \Theta^{\text{ab},\infty}$ is finite-dimensional, by Lemma 3.2.6. Since $\pi_2$ has no finite-dimensional $K$-quotients (by Proposition 3.2.4) we see that our assumption that $\pi_2 \subset \Psi^{\text{ab}}$ actually implies $\pi_2 \subset \Theta^{\text{ab},\infty}$. But then Lemma 3.2.6 implies a contradiction to Proposition 3.2.8 because it allows us to construct a nonzero $K$-linear morphism from $\pi_2$ to an infinite-dimensional quotient of $\pi_1$.

Similarly, assume that there is a surjection $\Psi^{\text{ab}} \twoheadrightarrow \Theta^{\text{ab},\infty}$. By Lemma 3.2.6 its kernel is finite-dimensional, hence the restriction of this map to $\pi_2$ still has infinite-dimensional image. Then Lemma 3.2.6 again provides a contradiction to Proposition 3.2.8 since $\Theta^{\text{ab},\infty}$ surjects onto an infinite-dimensional quotient of $\pi_1$, and the kernel of this surjection is finite-dimensional.

Up to renumbering, we can therefore assume that $\pi_2 \subset \Theta^{\text{ab}}$ for all lattices $\Theta \subset \Pi$ with semisimple reduction.

3.2.8. Splitting $A_{r,s,\lambda}$. Recall that Theorem A.0.7 provides us with two lattices $\Theta_1, \Theta_2 \subset \Pi$ such that $\Theta_1$ is indecomposable and surjects onto $\pi_1$. The following theorem implies that $\pi_1 \subset \Theta_2^{\text{ab}}$, and similarly $\pi_2 \subset \Theta_1^{\text{ab}}$. Indeed, if $\Theta_2^{\text{ab}}$ is not equal to $\Theta_1$ then it does not surject onto $\pi_2$, but then it contains $\pi_1$ since otherwise it would be finite-dimensional.

Theorem 3.2.9. Let $0 \to \pi_1 \to A_{r,s,\lambda} \to \pi_2 \to 0$ be a very generic atome automorphe, and assume that $X \subseteq A_{r,s,\lambda}$ is a $K$-stable subspace that surjects onto $\pi_2$ via the given projection. Then $X = A_{r,s,\lambda}$.

Proof. By twisting, it suffices to prove the theorem when $s = 0$. Assume for a contradiction that $X \subset A_{r,\lambda}$ is a proper subspace. By Proposition 3.2.4 it intersects $\pi_1$ in a finite-dimensional subspace. Define an $Iw$-stable subspace $B_{r,\lambda} \subset A_{r,\lambda}$ as the preimage of $\pi_{\infty}(\chi_2)$, and define $Y = X \cap B_{r,\lambda}$. Introduce $C_{r,\lambda} = B_{r,\lambda} / \pi_{\infty}^\infty(\chi_1)$, so that there is an exact sequence

\[(3.2.1) \quad 0 \to \pi_{\infty}(\chi_1) \to C_{r,\lambda} \xrightarrow{\pi_2} \pi_{\infty}(\chi_2) \to 0.\]

and let $Z$ be the image of $Y$ in $C_{r,\lambda}$. This is an $Iw$-stable subspace of $C_{r,\lambda}$ surjecting onto $\pi_{\infty}(\chi_2)$. Denote by $\kappa$ the kernel of the surjection $Z \to \pi_{\infty}(\chi_2)$.

Lemma 3.2.10. The kernel $\kappa$ is finite-dimensional.

Proof. This is because $\kappa$ is the image of $Y \cap \pi_1 \subset X \cap \pi_1$, which we are assuming to be finite-dimensional. \qed
Observe that the existence of $Z$ implies that there is a short exact sequence
\[(3.2.2) \quad 0 \to \kappa \to C_{r,\lambda} \xrightarrow{\text{pr}_1 \oplus \text{pr}_2} (\pi_\infty(\chi_1)/\kappa) \oplus \pi_\infty(\chi_2) \to 0\]
such that the projection $\text{pr}_1$ restricts to $\pi_\infty(\chi_1)$ to the canonical surjection $\pi_\infty(\chi_1) \to \pi_\infty(\chi_1)/\kappa$, and the projection $\text{pr}_2$ to zero (the sequence defines $\text{pr}_1$). Indeed, this follows because the short exact sequence
\[(3.2.3) \quad 0 \to \pi_\infty(\chi_1)/\kappa \to C_{r,\lambda}/\kappa \xrightarrow{\text{pr}_2} \pi_\infty(\chi_2) \to 0\]
is Iw-split.

Let $n > 0$ be large enough that $K_{n+1}$ acts trivially on $\kappa$, and recall the special element $\tilde{e}_{n+1} \in A_{r,\lambda}$ from Section 2.4. By construction, $\tilde{e}_{n+1} \in B_{r,\lambda}$, and $\text{pr}_2(\tilde{e}_{n+1})$ is a generator of $\pi_{n+1}(\chi_2)$ under Iw (see the proof of Proposition 2.2.4). Furthermore, by property (1) of $\tilde{e}_{n+1}$, we see that the image of $\tilde{e}_{n+1} \in C_{r,\lambda}$ is invariant under $\overline{U}(p^{n+1}\mathbb{Z}_p)$.

**Lemma 3.2.11.** The element $\tilde{e}_{n+1} \in C_{r,\lambda}$ is fixed by $K_{n+1}$.

**Proof.** This follows from the $\overline{U}(p^{n+1}\mathbb{Z}_p)$-invariance together with property (3) of $\tilde{e}_{n+1}$ and the matrix identity
\[
\begin{pmatrix}
a & b \\ c & d
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}.
\]

We claim that $\text{pr}_1(\tilde{e}_{n+1})$ is contained in $\pi_{n+1}(\chi_1)/\kappa$. This implies the theorem, because then there exists a polynomial $p$ in the truncated Iwasawa algebra $A/X^{\kappa} \cong \pi_{n+1}(\chi_1)$ such that $p = \text{pr}_1(\tilde{e}_{n+1})$ in $\pi_{n+1}(\chi_1)/\kappa$.

But now property (2) of $\tilde{e}_{n+1}$ contradicts Proposition 2.24, which implies that $(1 1 1)$ has degree at least $p - 2$, whereas (2) implies that $(1 0 1)$ has degree $p - 3 < p - 2$.

To prove our claim, let us study the representation $(\text{Iw} \cdot \tilde{e}_{n+1}) \subseteq C_{r,\lambda}$. Since $\tilde{e}_{n+1}$ is fixed by $K_{n+1}$, we know that $\text{pr}(\text{Iw} \cdot \tilde{e}_{n+1})$ intersects $\pi_\infty(\chi_1)/\kappa$ in a subspace $S \subseteq \pi_{n+1}(\chi_1)/\kappa$. Indeed, if $x \in (\text{Iw} \cdot \tilde{e}_{n+1})$ then $x$ is $K_{n+1}$-invariant, and if $\text{pr}(x) \in \pi_\infty(\chi_1)/\kappa$ then $\text{pr}_2(x) = 0$, hence $x \in \pi_\infty(\chi_1)^K_{n+1} = \pi_{n+1}(\chi_1)$.

On the other hand, we know that
\[(3.2.4) \quad \text{pr}(\text{Iw} \cdot \tilde{e}_{n+1})/S \cong \text{pr}_2(\text{Iw} \cdot \tilde{e}_{n+1}) = \pi_{n+1}(\chi_2),\]
since $\text{pr}_2(\tilde{e}_{n+1})$ is a generator of $\pi_{n+1}(\chi_2)$ under Iw. Now consider the composition
\[(3.2.5) \quad \text{pr}(\text{Iw} \cdot \tilde{e}_{n+1}) \xrightarrow{\text{pr}_2} \pi_\infty(\chi_1)/\kappa \to \pi_\infty(\chi_1)/S \to \pi_\infty(\chi_1)/\pi_{n+1}(\chi_1).\]

We need to prove that it is equal to zero. The map $\text{pr}_1$ is the identity on $S$, hence by (3.2.4) the image of (3.2.5) is an Iw-linear quotient of $\pi_{n+1}(\chi_2)$. Now by Lemma 2.2.1 we know that the $p^r$-dimensional Iw-stable subspace of $\pi_\infty(\chi_1)/\pi_{n+1}(\chi_1)$ is isomorphic to $\pi_{n+1}(\chi_1)$. By Lemma 2.2.2 we will be done if we prove that $\chi_2 \neq \chi_1$.

To see this, recall that $\pi_1$ has $K$-socle $\text{Sym}^r$ whereas $\pi_2$ has $K$-socle $\text{Sym}^{p^r-3-r} \otimes \det^{r+1}$, hence $\chi_1 = d^r$ and $\chi_2 = d^{p^r-3-r}(ad)^{r+1} = a^{r+1}d^{p^r-2}$ (because the $\chi_i$ are the conjugates of the eigencharacters of the socle). It follows that $\chi_1 \neq \chi_2$ (because $r \notin \{0, p-1\}$).

3.2.12. End of proof. Now we can conclude the proof of Theorem 1.0.1 in the non-ordinary case. Let $\Theta$ be a neighbour of $\Theta_2$. We know from the discussion above that $\overline{\Theta}_{\text{sub}}$ contains $\pi_2$ and $\overline{\Theta}_{\text{sub}}^\text{ab}$ contains $\pi_1$. Furthermore, there exist one-step filtrations on $\overline{\Theta}_{\text{sub}}$ and $\overline{\Theta}_{\text{sub}}^\text{ab}$ with the same graded pieces up to reordering. By Lemma 3.2.6, precisely one between $\text{Fil} \overline{\Theta}_{\text{sub}}$ and $\overline{\Theta}_{\text{sub}}^\text{ab}/\text{Fil} \overline{\Theta}_{\text{sub}}^\text{ab}$ has infinite dimension, and we denote it $\overline{\Theta}_{\text{sub}}^\text{ab,\infty}$.

Assume that $\overline{\Theta}_{\text{sub}}^\text{ab,\infty} \cong \text{Fil} \overline{\Theta}_{\text{sub}}^\text{ab}$. By Proposition 3.2.4 $\pi_1$ has no nonzero finite-dimensional $K$-linear quotients, hence $\pi_1 \subset \text{Fil} \overline{\Theta}_{\text{sub}}^\text{ab}$. By Lemma 3.2.6 we deduce a contradiction to Proposition 3.2.3 since there exists a surjection of $\overline{\Theta}_{\text{sub}}^\text{ab,\infty}$, with finite-dimensional kernel, onto an infinite-dimensional quotient of $\pi_2$.

So there exists a surjection $\overline{\Theta}_{\text{sub}} \to \overline{\Theta}_{\text{sub}}^\text{ab}$ with finite-dimensional kernel, and so the restriction
\[
\pi_1 \subset \overline{\Theta}_{\text{sub}} \to \overline{\Theta}_{\text{sub}}^\text{ab,\infty}\]
has infinite-dimensional image. Again, Lemma 3.2.3 provides a contradiction to Proposition 3.2.3.

This completes the proof of Theorem 1.0.1 in the case that the reduction of $\Pi$ has the same semisimplification as a reducible very generic atome automorphe.

3.3. Proof of Theorem 1.0.1 ordinary representations. This is simpler than the previous two cases. By [Pas13 Theorem 1.1], if $\Pi$ as in the statement of Theorem 1.0.1 has not yet been treated, then the genericity assumption implies that the reduction of any lattice in $\Pi$ is an irreducible $G$-representation isomorphic to a generic principal series representation. But then Proposition 3.2.3 implies that if $\Pi^1 \subset \Pi$ is a $K$-stable closed $E$-subspace, then either $\Pi^1 = \Pi$ or $\Pi^1$ is finite-dimensional, and we are done by Lemma 3.1.4.

3.4. Proof of Theorem 1.0.3 The second theorem of the introduction is as follows.

Theorem 3.4.1. Let $\Pi_1, \Pi_2$ be absolutely irreducible, very generic, non-ordinary unitary admissible $E$-Banach space representations of $GL_2(\mathbb{Q}_p)$, with central character $\zeta$.

(1) If $\Pi_1$ and $\Pi_2$ have supersingular reduction, then $\Pi_1|_N \cong \Pi_2|_N$ if and only if $\Pi_1 \cong \Pi_2$, and $\Pi_1|_{iwZ} \cong \Pi_2|_{iwZ}$ if and only if $\Pi_1 \cong \Pi_2 \otimes (nr_{+} \circ \det)$.

(2) If $\Pi_1$ and $\Pi_2$ have reducible reduction, then $\Pi_1|_{KZ} \cong \Pi_2|_{KZ}$ if and only if $\Pi_1 \cong \Pi_2 \otimes (nr_{+} \circ \det)$.

(3) If $\Pi_1$ and $\Pi_2$ have different reduction type, then there are no $iwZ$-linear isomorphisms $\alpha : \Pi_1 \to \Pi_2$.

Proof. We begin with part (3). Let $\alpha : \Pi_1 \to \Pi_2$ be an $iw$-linear topological isomorphism, and let $\Theta_1 \subset \Pi_1$ be open bounded $G$-stable lattices. Assume $\Theta_1$ has supersingular reduction. Since all open bounded lattices in $\Pi_1$ are commensurable, it is possible to multiply $\alpha$ by a power of $\pi_E$ so that it induces a saturated morphism $\alpha : \Theta_1 \to \Theta_2$ (by definition, this means that $\alpha \otimes \Theta k$ is nonzero). Then it suffices to prove that there are no nonzero $iw$-linear morphisms $\pi_\sigma \to \pi(r, \lambda, \chi)$, and by looking at the socle filtration as in Propositions 2.2.5 and 2.3.4 it suffices to prove the following lemma.

Lemma 3.4.2. There are no nonzero $iw$-linear morphisms $\alpha : M_{+}^+ \to \pi_\infty(\chi)$, for any given generic $\sigma$ and $\chi$.

Proof. By Lemma 2.3.3 since $\alpha$ is nonzero the kernel $\ker(\alpha)$ is finite-dimensional. Recall from the proof of Theorem 2.3.3 that we have an exhaustive filtration of $M_{+}^+$ by $iw$-subspaces $M_{+}^+,n$, which are proper quotients of representations of the form $\pi_{2n+1}(\chi')$ (for some $\chi'$), and have the property that $(\dim M_{+}^+,n) - p^{2n-1}$ tends to infinity with $n$. Indeed, the formulas in the proof of Theorem 2.3.3 imply that all coefficients up to $p^{2n-1}$ in the $p$-adic expansion of $\dim M_{+}^+,n - 1$ are nonzero, and so $\dim M_{+}^+,n - p^{2n-1} \geq p^{2n-2}$. Let $n$ be large enough that $\dim M_{+}^+,n - p^{2n-1} > \dim \ker(\alpha)$. We find that $\alpha$ induces morphisms

$$\pi_{2n+1}(\chi') \to M_{+}^+,n \overset{\alpha}{\to} \pi_{2n+1}(\chi)$$

which are not isomorphisms, but whose image has dimension $\dim M_{+}^+,n - \dim \ker(\alpha) > p^{2n-1}$. This contradicts Lemma 2.2.3. □

For part (1), let $\alpha : \Pi_1|_N \simto \Pi_2|_N$ be a topological isomorphism inducing a saturated map $\alpha : \Theta_1 \to \Theta_2$ between two open bounded $G$-stable lattices $\Theta_i \subset \Pi_i$. By Corollary 2.3.6 the induced map $\pi_\Theta \otimes \Theta k \to \Theta_2 \otimes \Theta k$ is a bijection, and so the reductions of the $\Theta_i$ are isomorphic supersingular irreducible representations. Hence the hypotheses of Lemma 3.1.1 hold, and so $\alpha$ is $G$-linear.

If we only assume that $\alpha$ is $iwZ$-linear, the induced map $\pi_\Theta$ need not be a bijection, but since it is not zero we deduce that the representations $\Theta_i \otimes \Theta k$ have the same Serre weights, by Corollary 2.3.6. Hence we can still deduce that they are isomorphic. Then Lemma 3.1.1 implies that $\alpha$ is $G^+$-linear, and the claim follows from Clifford theory. Indeed, diagonalizing the action of $G/G^+$ on $\Hom_{cont}^{G^+}(\Pi_1, \Pi_2)$ shows that every continuous $G^+$-linear map $\alpha : \Pi_1 \to \Pi_2$ can be written as the sum of a $G$-linear continuous map $\Pi_1 \to \Pi_2$ and a $G$-linear continuous map $\Pi_1 \to \Pi_2 \otimes (nr_{+} \circ \det)$. Now one uses the fact that every nonzero $G$-linear continuous map between topologically irreducible admissible $E$-Banach space representations of $G$ is an isomorphism (which follows from the fact that the category of admissible $E$-Banach space representations is abelian, hence a morphism is an isomorphism if and only if it has trivial kernel and cokernel).

For part (2), we will apply the results of Appendix A. Let $\alpha : \Pi_1|_{KZ} \simto \Pi_2|_{KZ}$ be an isomorphism and choose a lattice $\Theta_1 \subset \Pi_1$ with nonsplit reduction isomorphic to $A_{r,s,\lambda}$. Let $\Theta_2 \subset \Pi_2$ be a lattice, and assume that $\alpha$ induces a saturated map $\alpha : \Theta_1 \to \Theta_2$.

Lemma 3.4.3. The representation $A_{r,s,\lambda}$ has no nonzero finite-dimensional $K$-stable quotients.
Proof. Let $X \subseteq \mathcal{A}_{r,s,\lambda}$ be a $K$-stable subspace such that $\mathcal{A}_{r,s,\lambda}/X$ is finite-dimensional. By Proposition 3.2.5 and the snake lemma, the image of $X$ in the principal series quotient $\pi_2$ of $\mathcal{A}_{r,s,\lambda}$ equals $\pi_2$. But then Theorem 3.2.4 implies that $X = \mathcal{A}_{r,s,\lambda}$. □

Corollary 3.4.4. There exists a surjection from $\Theta_1 \otimes \mathcal{O} k$ to one of the Jordan–Hölder factors of $\Theta_2 \otimes \mathcal{O} k$.

Proof. There exists an exact sequence

$$0 \to \pi'_1 \to \Theta_2 \otimes \mathcal{O} k \to \pi'_2 \to 0$$

where $\pi'_1, \pi'_2$ are irreducible generic principal series representations. Compose the map $\alpha \otimes \mathcal{O} k$ with the projection to $\pi'_2$. By Lemma 3.4.3 and Proposition 3.2.4 either this map is surjective or it is the zero map. If it is zero, then the image of $\alpha \otimes \mathcal{O} k$ is contained in $\pi'_1$, and since $\alpha \otimes \mathcal{O} k$ is not zero it must be a surjection onto $\pi'_1$.

By Corollary 2.2.8 and the exact sequence defining $\mathcal{A}_{r,s,\lambda}$, the existence of the surjection in Corollary 3.4.3 implies that the irreducible $G$-constituents of $\Theta_1 \otimes \mathcal{O} k$ and $\Theta_2 \otimes \mathcal{O} k$ have the same $K$-socle. We deduce from Ribet’s lemma that $I_k$ contains an open bounded $G$-stable lattice $\Theta$ such that $\Theta \otimes \mathcal{O} k \cong \mathcal{A}_{r,s,\mu}$ as $G$-representations, for some $\mu \in k^\times$.

Now we can scale $\alpha$ so that it induces a saturated morphism $\Theta_1 \to \Theta$. By Theorem 2.4.13 and Proposition 2.4.4 we deduce that $\mu = \pm \lambda$ and that $\alpha$ induces a $K$-linear isomorphism $\Theta_1 \to \Theta$ on the mod $\pi_E$ reductions (after possibly a twist by $nr_{-1}$). By the same argument as Lemma 3.1.7 the claim follows from part (2) of Corollary 2.5.2 □

Finally, the following proposition together with the previous theorem implies Corollary 1.0.4.

Proposition 3.4.5. Let $\rho_1, \rho_2 : \mathrm{Gal}_{Q_p} \to \mathrm{GL}_2(E)$ be absolutely irreducible continuous representations. Assume that $\rho_1|_{I_{Q_p}} = \rho_2|_{I_{Q_p}}$ and det $\rho_1 = \det \rho_2$. Then $\rho_1 \cong \rho_2 \otimes nr_{\pm 1}$.

Proof. Assume first that $\rho_1|_{I_{Q_p}}$ is absolutely irreducible. Then the space $\mathrm{Hom}_{I_{Q_p}}(\rho_1, \rho_2)$ is one-dimensional and the group $\mathrm{Gal}_{Q_p}/I_{Q_p}$ acts on it by a scalar $\lambda$. It follows that every $I_{Q_p}$-linear isomorphism $\rho_1 \sim \rho_2$ is a $\mathrm{Gal}_{Q_p}$-linear isomorphism $\rho_1 \sim \rho_2 \otimes nr_\lambda$ for some $\lambda \in E^\times$. Since det $\rho_1 = \det \rho_2$ we see that $\lambda^2 = 1$, and the claim follows.

Otherwise, by [Pas13, Lemma 5.1] we can pass to a finite extension of $E$ and assume that $\rho_1|_{I_{Q_p}}$ is reducible. Let $\chi : I_{Q_p} \to E^\times$ be a character occurring in $\rho_1|_{I_{Q_p}}$. If $\chi$ is normalized by $\mathrm{Gal}_{Q_p}$ then it extends to $\mathrm{Gal}_{Q_p}$, so the space $\mathrm{Hom}_{I_{Q_p}}(\chi, \rho_1)$ has an action of $\mathrm{Gal}_{Q_p}/I_{Q_p}$, and it is not zero. Since any Frobenius eigenvector in this space yields a $\mathrm{Gal}_{Q_p}$-stable proper subspace of $\rho_1$, this contradicts the assumption that $\rho_1$ is absolutely irreducible. So $\chi$ is not normalized by $\mathrm{Gal}_{Q_p}$. However, $\chi$ is normalized by $\mathrm{Gal}_{Q_p}$ since $\rho_1$ is two-dimensional. Hence there exist characters $\chi_i : \mathrm{Gal}_{Q_p} \to E^\times$ such that $\chi_1|_{I_{Q_p}} = \chi_2|_{I_{Q_p}} = \chi$ and $\rho_1 \cong \mathrm{Ind}_{I_{Q_p}}(\chi_i)$. After a quadratic extension of $E$, this implies that $\rho_1 \cong \rho_2 \otimes nr_\lambda$ for some $\lambda \in E^\times$, and again $\lambda^2 = 1$ since det $\rho_1 = \det \rho_2$. □

Appendix A. Ribet’s Lemma for Banach spaces.

Ribet’s lemma for $\mathrm{Gal}_{Q_p}$ is the statement that if $\rho : \mathrm{Gal}_{Q_p} \to \mathrm{GL}_2(E)$ is an irreducible continuous representation whose reduction has two distinct Jordan–Hölder factors then the $p$-stable homothety classes of lattices form a bounded segment of length at least two in the Bruhat–Tits tree of $E^{\Delta_2}$. Furthermore, the lattices $\rho_1, \rho_2$ corresponding to the extremal points in the segment have indecomposable reductions with nonisomorphic socle, and all the other lattices have semisimple reduction.

Now let $\Pi$ be an irreducible, admissible, unitary $E$-Banach space representation of $G = \mathrm{GL}_2(Q_p)$ with a $G$-stable open and bounded lattice $\Theta$ such that $\overline{\Theta} = \Theta/p\Theta$ is a reducible representation with two nonisomorphic Jordan–Hölder factors $\{\pi_1, \pi_2\}$. In this appendix we prove an analogue of Ribet’s lemma for $\Pi$. Using Colmez’s functor, this is straightforward to do if $\Pi$ is generic in the sense of Section 2.1.3: it suffices to invoke [CD14, Remarque III.10(iii), Proposition III.54]. We will provide a different proof in order to make this result independent of the $p$-adic Langlands correspondence for $\mathrm{GL}_2(Q_p)$. In fact, we will deal with the more general case that $\Pi$ is an $E$-Banach space with a topologically irreducible $E$-linear action of a group $G$ that stabilizes an open and bounded lattice $\Theta$, and such that $\overline{\Theta}$ is a $k[G]$-representation of length two with distinct Jordan–Hölder factors.
Our approach follows Serre’s proof of Ribet’s lemma as closely as possible. An obstruction to do this is the absence of a Bruhat–Tits building for the infinite-dimensional $E$-vector space $\Pi$, and so we begin by providing a substitute: we do this by adapting the arguments in \cite{serre}. Unless otherwise stated, in this appendix we abbreviate “open and bounded lattice” to “lattice”. Since any two open and bounded lattices in $\Pi$ are commensurable, every $G$-stable lattice in $\Pi$ has reduction of length two over $k[G]$, with Jordan–Hölder factors $\{\pi_1, \pi_2\}$. We will say that an inclusion $\Theta_1 \subset \Theta_2$ of lattices is saturated if $\Theta_1 \nsubseteq \pi E \Theta_2$.

**Definition A.0.1.** Define a graph $\Gamma$ with set of vertices given by homothety classes of $G$-stable lattices in $\Pi$, such that $\{[\Theta_0], [\Theta_1]\}$ is an edge if and only if there are representatives of these homothety classes such that $\Theta_1 \subset \Theta_0$ and $\Theta_0/\Theta_1$ is an irreducible $k[G]$-representation.

**Remark A.0.2.** By the word “graph” we mean a one-dimensional simplicial complex. These correspond to the *graphes combinatoires* in \cite{serre}.

**Definition A.0.3.** If $\Theta_1 \subset \Theta_0$ is a saturated inclusion between $G$-stable lattices, the $G$-representation $\Theta_0/\Theta_1$ has finite length. We define the distance $$d(\Theta_0, \Theta_1) = \text{length}_G(\Theta_0/\Theta_1).$$ This only depends on the homothety class of $\Theta_0, \Theta_1$.

In order to prove that the distance is a symmetric function one can use the following lemma on saturated inclusions.

**Lemma A.0.4.** Assume that $\Theta_n \subset \Theta_0$ is a saturated inclusion of $G$-stable lattices in $\Pi$ and that $\text{length}_G(\Theta_0/\Theta_n) = n$. Then $\text{Ann}_{G}(\Theta_0/\Theta_n) = \pi^n E O$.

**Proof.** Since $\pi E$ annihilates all Jordan–Hölder factors of $\Theta_0/\Theta_n$ we deduce immediately that $\pi^n E (\Theta_0/\Theta_n) = 0$. For the other direction, we need to prove that $\pi^{n-1} E \Theta_0 \nsubseteq \Theta_n$. We use induction on $n$ and we start by pulling back a Jordan–Hölder series for $\Theta_0/\Theta_1$ to a sequence of open and bounded lattices

$$\Theta_n \subset \Theta_{n-1} \subset \cdots \subset \Theta_0.$$

Since $\Theta_n \subset \Theta_0$ is saturated, the homothety classes of the $\Theta_i$ are pairwise distinct. In addition, we know that $\Theta_{n-1}$ contains both $\Theta_n$ and $\pi E \Theta_{n-2}$, and

$$\text{length}_G(\Theta_{n-1}/\Theta_n) = \text{length}_G(\Theta_{n-1}/\pi E \Theta_{n-2}) = 1.$$

Putting these together, and using the fact that $\Theta_{n-1}/\pi E \Theta_{n-2}$ has at most two proper nonzero $G$-stable subspaces, we find that $\Theta_n \cap \pi E \Theta_{n-2} = \pi E \Theta_{n-1}$. Our inductive assumption says that $\pi^{n-2} E \Theta_0 \nsubseteq \Theta_{n-1}$, and we know that $\pi^{n-2} E \Theta_0 \subset \Theta_{n-2}$. Multiplying by $\pi E$, we deduce that $\pi^{n-1} E \Theta_0 \nsubseteq \Theta_n$, which concludes the proof.

**Corollary A.0.5.** Let $[\Theta], [\Theta']$ be vertices of $\Gamma$. Then $d([\Theta], [\Theta']) = d([\Theta'], [\Theta])$.

**Proof.** Choose a saturated inclusion $\Theta' \subset \Theta$ between representatives of these lattices and let $m = d([\Theta], [\Theta'])$. Then we have a chain of lattices

$$\pi^m E \Theta \subset \Theta' \subset \Theta$$

and by Lemma A.0.4 the inclusion $\pi^m E \Theta \subset \Theta'$ is saturated. By additivity of length we find that

$$d([\Theta'], [\Theta]) + m = \text{length}_G(\Theta/\pi^m E \Theta) = 2m$$

which yields the claim.

**Corollary A.0.6.** If $\Theta' \subset \Theta$ is a saturated inclusion of $G$-stable lattices in $\Pi$, then the $G$-representation $\Theta/\Theta'$ is uniserial.

**Proof.** Let $n = \text{length}_G(\Theta/\Theta')$. If $\Theta/\Theta'$ admits a $G$-stable filtration with $m$ graded pieces, all of which are semisimple, then $\pi^m E (\Theta/\Theta') = 0$. Hence Lemma A.0.4 implies that both the socle and the cosocle filtration have length equal to $n$, so they have simple graded pieces, and so by Lemma 2.1.4 the representation is uniserial.

The following is our version of Ribet’s lemma for $\Pi$.
Theorem A.0.7. The graph $\Gamma$ is a finite line segment of length at least two, corresponding to a chain of pairwise non-homothetic $G$-stable lattices

$$\Theta_n \subset \cdots \subset \Theta_0.$$ 

The $G$-representations $\Theta_0/\pi_E \Theta_0$ and $\Theta_n/\pi_E \Theta_n$ are indecomposable and not isomorphic. The $G$-representations $\Theta_i/\pi_E \Theta_i$ for $0 < i < n$ are semisimple.

Proof. We give the proof by means of several lemmas.

Lemma A.0.8. The graph $\Gamma$ is a tree, i.e. a connected and simply connected graph.

Proof. This follows by the same argument as [Ser77, §1, Chapitre II]. Namely, given two vertices of $\Gamma$ we can find representatives $\Theta_0, \Theta_1$ and a saturated inclusion $\Theta_1 \to \Theta_0$. Pulling back a Jordan–Hölder sequence of the finite length $G$-representation $\Theta_0/\Theta_1$ then yields a path in $\Gamma$ between $[\Theta_0]$ and $[\Theta_1]$, proving that $\Gamma$ is connected.

To see that it is simply connected, it suffices to check that for all $n \geq 1$ and every path $[\Theta_0], [\Theta_1], \ldots, [\Theta_n]$ without backtracking (i.e. such that $[\Theta_i] \neq [\Theta_{i+2}]$ for all $i$) we have $[\Theta_0] \neq [\Theta_n]$. Such a path can be represented by a sequence of lattices

$$\Theta_n \subset \Theta_{n-1} \cdots \subset \Theta_0$$

where $\text{length}_G(\Theta_i/\Theta_{i+1}) = 1$ for all $i$. It suffices to prove by induction on $n$ that the inclusion $\Theta_n \subset \Theta_0$ is saturated, i.e. that $\Theta_n \not\subset \pi_E \Theta_0$. When $n = 2$, the fact that $\text{length}_G(\Theta_0/\Theta_2) = 2$ implies that if the inclusion is not saturated then $\Theta_2 = \pi_E \Theta_0$, which we are assuming not be the case.

Assuming the statement true for $n - 1$, notice that $\Theta_n$ and $\pi_E \Theta_{n-2}$ are distinct by the assumption of no backtracking, and contain $\pi_E \Theta_{n-1}$, and so they identify with the only two $G$-stable subspaces of $\Theta_{n-1}/\pi_E \Theta_{n-1}$: this implies that

$$\Theta_{n-1} = \Theta_n + \pi_E \Theta_{n-2}.$$ 

Now we see that $\Theta_n \subset \pi_E \Theta_0$ implies $\Theta_{n-1} \subset \pi_E \Theta_0$, contradicting the inductive assumption. This concludes the proof that $\Gamma$ is a tree.

Given a vertex $\Theta_0$ of $\Gamma$, we know that $\Theta_0/\pi_E \Theta_0$ is a $G$-representation of length two, and so $\Theta_0$ has at most two adjacent vertices in $\Gamma$: this implies that $\Gamma$ is a line segment (possibly infinite or half-infinite). To prove that $\Gamma$ is finite, it suffices therefore to prove that there is no infinite sequence

$$\cdots \subset \Theta_n \subset \cdots \subset \Theta_1 \subset \Theta_0$$

of pairwise non-homothetic $G$-stable lattices in $\Theta_0$ such that $\text{length}_G(\Theta_i/\Theta_{i+1}) = 1$ for all $i$. To do so, define

$$\hat{\Theta} = \lim_{\rightarrow \infty} \Theta_0/\Theta_n.$$ 

Lemma A.0.9. The natural map $\Theta_0 \to \hat{\Theta}$ is surjective, and $\hat{\Theta}$ is $\pi_E$-adically separated and complete and $\pi_E$-torsion free.

Proof. By the proof of Lemma A.0.8 each of the inclusions $\Theta_n \subset \Theta_0$ is saturated, and so by Corollary A.0.6 the quotients $\Theta_0/\Theta_n$ are uniserial $G$-representations. Let $m, j$ be positive integers and consider the map

$$\pi^m_E : \Theta_0/\Theta_{m+j} \to \Theta_0/\Theta_{m+j}.$$ 

We claim that its image is $\Theta_m/\Theta_{m+j}$, or equivalently that

$$\pi^m_E \Theta_0 + \Theta_{m+j} = \Theta_m.$$

We know that the $G$-representation $\Theta_m/\Theta_{m+j}$ is uniserial, and its radical is $\Theta_{m+1}/\Theta_{m+j}$. By Lemma A.0.4 we know that $\pi^m_E \Theta_0 \not\subset \Theta_{m+1}$. Hence the natural map $\pi^m_E \Theta_0 \to \Theta_m/\Theta_{m+j}$ is surjective, and the claim follows. Now, since $\Theta_0/\Theta_n$ is a finite length $G$-representation, we have exact sequences

$$0 \to \lim_{\rightarrow \infty} \Theta_j/\Theta_{m+j} \to \hat{\Theta} \to \lim_{\rightarrow \infty} \Theta_0/\Theta_m \to 0.$$ 

This proves that $\hat{\Theta}$ is $\pi_E$-adically separated and complete and $\pi_E$-torsion free, and that $\hat{\Theta} \otimes_O k \cong \Theta_0/\Theta_1$. By Lemma A.0.4 it follows that the natural map $\Theta_0 \to \hat{\Theta}$ is surjective. \qed
Let $\Theta^1 = \bigcap_{i \geq 0} \Theta_i$. By Lemma A.0.9, we have an exact sequence

$$0 \to \Theta^1 \to \Theta_0 \to \hat{\Theta} \to 0.$$ 

Since $\Theta_0 \to \hat{\Theta}$ is not an isomorphism (as can be seen by applying $- \otimes \mathcal{O} k$) we see that $\Theta^1$ is a nonzero closed $G$-stable $\mathcal{O}$-submodule of $\Pi$. The following lemma then provides a contradiction to the assumed topological irreducibility of $\Pi$, and it follows that $\Gamma$ is a finite line segment.

**Lemma A.0.10.** The $E$-vector space $\Pi^1 = \Theta^1[1/p]$ is a nonzero proper closed $G$-stable subspace of $\Pi$.

**Proof.** We know that $\Theta^1$ is nowhere dense in the sense that its set-theoretical complement $\Pi \setminus \Theta^1$ is open and dense: indeed, if $\Theta^1$ contained a set open in $\Pi$, then after a translation it would contain $\pi_{E_m}^n \Theta_0$ for some positive integer value of $m$, but then $\pi_{E_m}^n \Theta_0 \subset \Theta_j$ for all $j \geq 0$, contradicting Lemma A.0.4 as soon as $j > m$. Hence the Baire category theorem implies that $\Pi^1 = \Theta^1[1/p]$ is a proper subspace of $\Pi$. To check that it is closed, let $x_n \in \Pi^1$ be a sequence in $\Pi^1$ converging to $x \in \Pi$. To prove that $x \in \Pi^1$ we can assume without loss of generality that $x \in \Theta_0$, multiplying by a suitable power of $\pi_E$. Passing to a subsequence, we can find elements $\theta_n \in \Theta_0$ such that

$$x - x_n = \pi_{E}^n \theta_n.$$ 

Fix $n > 0$. If $m$ is large enough that $\pi_{E_m}^n x_n \in \Theta^1$, we find that $\pi_{E_m}^n x$ and $\pi_{E_m}^n \pi_{E_m}^n \theta_n$ have the same image in $\hat{\Theta}$. By Lemma A.0.9 this implies that the image of $x$ is contained in $\pi_{E_m}^n \hat{\Theta}$. Since this holds for all $n > 0$ and $\Theta$ is $\pi_E$-adically separated, we see that $x \in \Theta^1$, which concludes the proof.

Finally, it is part of our assumptions that $\Gamma$ is not empty, and that if $[\Theta]$ is a vertex then $\Theta/\pi_E \Theta$ is a reducible $G$-representation. Pulling back a $G$-stable proper nonzero subspace we obtain another vertex of $\Gamma$, which has therefore length at least two. Now the following lemma concludes the proof of Theorem A.0.7.

**Lemma A.0.11.** With the notation in the statement of Theorem A.0.7, the lattices $\Theta_0, \Theta_n$ have nonisomorphic reductions with different cosocle, and the lattices $\Theta_i$ for $1 < i < n$ have semisimple reduction.

**Proof.** The statement about the reduction type is an immediate consequence of the number of neighbors of $[\Theta_n]$ in $\Gamma$. There remains to prove the statement about cosocles. To do so, we can assume without loss of generality that $\text{cosoc}_G(\Theta_0/\pi_E \Theta_0) \cong \pi_1$. Then it suffices to prove that $\Theta_i/\pi_E \Theta_i \cong \pi_1$ for all $i$. Indeed, this implies that $\text{cosoc}_G(\Theta_n/\pi_E \Theta_n) = \Theta_n/\pi_E \Theta_{n-1} \cong \pi_2$, which was to be proved.

We use induction on $i$, the base case being contained in our assumption. By the inductive assumption we know that $\pi_1 \Theta_{i-1}/\pi_E \Theta_i \cong \pi_1$. Since $\Theta_{i+1}$ and $\Theta_i$ are not homothetic, the image of $\Theta_{i+1}$ in $\Theta_i/\pi_E \Theta_i$ is not $\pi_E \Theta_{i-1}/\pi_E \Theta_i$ but the other $G$-stable subspace, and so $\Theta_{i+1}/\pi_E \Theta_i \cong \pi_2$. This implies that $\Theta_i/\pi_E \Theta_{i+1} \cong \pi_1$, which was to be proved.

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