AN ANALOGUE OF THE FIELD-OF-NORMS FUNCTOR AND THE GROTHENDIECK CONJECTURE

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ABSTRACT. The paper contains a construction of an analogue of the Fontaine-Wintenberger field-of-norms functor for higher dimensional local fields. This construction is done completely in terms of the ramification theory of such fields. It is applied to deduce the mixed characteristic case of a local analogue of the Grothendieck Conjecture for these fields from its characteristic $p$ case, which was proved earlier by the author.

0. Introduction.

The field-of-norms functor [FW1,2] allows to identify the Galois groups of some infinite extensions of $\mathbb{Q}_p$ with those of complete discrete valuation fields of characteristic $p$. This functor is an essential component of Fontaine’s theory of $\varphi\Gamma$-modules — one of most powerful tools in the modern study of $p$-adic representations cf. e.g. [Ber]. Other areas of very impressive applications are the Galois cohomology of local fields [He], arithmetic aspects of dynamical systems [LMS], explicit reciprocity formulae [Ab2,3], [Ben], a description of the structure of ramification filtration [Ab7], the proof of an analogue of the Grothendieck Conjecture for 1-dimensional local fields [Ab4].

A local analogue of the Grothendieck Conjecture establishes an opportunity to recover the structure of a local field from the structure of its absolute Galois group together with its filtration by ramification subgroups. The study of this conjecture in the context of higher dimensional local fields became actual due to a recent development of ramification theory for such fields [Zh2], [Ab5]. The characteristic $p$ case of the Conjecture has been already considered in [Ab6]. (Notice that the restriction to 2-dimensional fields is not essential in [Ab6] — the method works for any dimension $N \geq 2$.) This result could lead to the proof of the mixed characteristic case of that conjecture if there were a suitable analogue of the field-of-norms functor for higher dimensional local fields.

The construction of such a functor is suggested in the present paper. In our setting we replace the appropriate category of infinite extensions of $\mathbb{Q}_p$ by the category $\mathcal{B}^a(N)$ of infinite increasing field towers $K_0 \subset K_1 \subset \cdots \subset K_n \subset \ldots$ with restrictions on the upper ramification numbers of the intermediate extensions $K_{n+1}/K_n$. In order to introduce the set of elements of the corresponding field-of-norms one can’t use in such towers the sequences of norm compatible elements but it is still possible to work with the sequences of elements $a_n \in \mathcal{O}_{K_n}$ such that $a_n \equiv a_{n+1}^p \mod p^c$, where $0 < c \leq 1$ is independent on $n$.

The main difficulty in the realization of this idea comes from the fact that the construction of ramification theory for an $N$-dimensional local field $L$ depends on
the choice of its $F$-structure, i.e. on the choice of the subfields $L(i)$ of $i$-dimensional constants, where $1 \leq i \leq N$. On the other hand, in order to work with elements of $L$ one should use one or another choice of its local parameters. This choice can be made compatible with a given $F$-structure only after passing to some finite “semistable” extension of $L$. This explains why we have a precise analogue of the Fontaine-Wintenberger functor only for a subcategory of “special” towers $B^{fa}(N)$ in $B^a(N)$. Nevertheless, the construction of the functor can be extended to the whole category $B^a(N)$ and can be applied to deduce the mixed characteristic case of the Grothendieck Conjecture from the characteristic $p$ case.

We now briefly explain the content of the paper.

Section 1 contains preliminaries: definitions and simplest properties of $N$-dimensional local fields $L$. We pay the special attention to the concept of $P$-topology — this is the topology on $L$, which accumulates properties of $N$ valuation topologies which can be attached to $L$. Then the Witt-Artin-Schreier duality and the Kummer theory allow us to transfer the $P$-topological structure to the group $\Gamma^{ab}_L(p)$, where $\Gamma_L(p)$ is the Galois group of the maximal $p$-extension of $L$. This structure gives an opportunity to work with $\Gamma_L(p)$ in terms of generators, cf. [Ab6].

Section 2 contains a “co-analogue” of Epp’s elimination wild ramification. This statement deals with a subfield of $(N-1)$-dimensional constants in an $N$-dimensional local field. (Most widely known interpretation of Epp’s procedure deals with a subfield of 1-dimensional constants.) Our proof establishes an elimination procedure which is similar to the procedure developed in [ZhK], where it was shown that an essential part of such elimination can be done inside a given deeply ramified extension in the sense of [CG]. This elimination procedure is required to justify the main starting point in the construction of the ramification theory for higher dimensional local fields from [Ab5]. (The original arguments from [Ab5] were not complete, cf. remark in n.2.1.)

Section 3 contains a brief introduction into the ramification theory and contains a version of Krasner’s Lemma in the context of higher dimensional local fields. In Section 4 we introduce and study the categories of special towers $B^a(N)$ and $B^{fa}(N)$. These towers play a role of strict arithmetic profinite extensions from the Fontaine-Wintenberger construction of the field-of-norms functor.

In section 5 we explain the construction of a family of characteristic $p$ local fields $X(K)$, where $K \in B^{fa}(N)$, and prove that all such fields can be identified after (roughly speaking) taking inseparable extensions of constant subfields of lower dimension. These fields will play a role of the field-of-norms attached to a given tower $K \in B^{fa}(N)$. In section 6 we apply Krasner’s Lemma from section 3 to establish all expected properties of the correspondence $K \mapsto \mathcal{K} \in X(K)$, where $K \in B^{fa}(N)$. In section 7 we use these properties to define the analogue $\mathcal{X}_K$, where $K \in B^{fa}(N)$, of the field-of-norms functor. In addition, we use the operation of radical closure to extend this construction to the whole category $B^a(N)$. In section 8 we prove that the identification of the Galois groups $\Gamma_\mathcal{K}$ (where $\mathcal{K}$ is the $p$-adic closure of the composite of all fields from the tower $K$) and $\Gamma_K$ becomes $P$-continuous when being restricted to their maximal abelian $p$-quotients. The proof is based on a higher dimensional version of the relation between the Witt-Artin-Schreier theory for $\mathcal{K}$ and the Kummer theory for $\mathcal{K}$ from [Ab2]. This relation together with the proof of the compatibility of the proposed field-of-norms functor with the class field theories for $\mathcal{K}$ and $\mathcal{K}$, leads to another proof of the explicit reciprocity formula from [Vo]
(cf. also [Ka]) — the details will appear later elsewhere.

Finally, the $P$-continuity result from n.8 allows us to prove in section 9 the mixed characteristic case of the Grothendieck Conjecture. Notice that the construction of the higher dimensional version of the field-of-norms functor from this paper is especially adjusted to the proof of this conjecture and was motivated by Deligne’s paper [De]. It should be also mentioned that there are definite ideological links with methods of the paper [Fu], where the construction of Coleman power series was developed in the context of 2-dimensional local fields with further applications to the construction of $p$-adic $L$-functions.

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1. Preliminaries.

1.1. The concept of higher dimensional local field.

Let $K$ be an $N$-dimensional local field, where $N \in \mathbb{Z}_{\geq 0}$. In other words, if $N = 0$ then $K$ is a finite field and for $N \geq 1$, $K$ is a complete discreet valuation field with residue field $K^{(1)}$ which is an $(N - 1)$-dimensional local field. We use the notation $K^{(N)}$ for the last residue field of $K$.

Let $\mathcal{O}_K^{(1)}$ be the valuation ring of $K$ with respect to first valuation and let $\alpha : \mathcal{O}_K^{(1)} \rightarrow K^{(1)}$ be a natural projection. Define the valuation ring $\mathcal{O}_K$ of $K$ by setting for $N = 0$, $\mathcal{O}_K = K$ and for $N \geq 1$, $\mathcal{O}_K = \alpha^{-1}(\mathcal{O}_{K^{(1)}})$. Recall that a system $t_1, \ldots, t_N \in \mathcal{O}_K$ is a system of local parameters in $K$ if $t_1$ is a uniformiser in $\mathcal{O}_K^{(1)}$ and $\alpha(t_2), \ldots, \alpha(t_N)$ is a system of local parameters in $K^{(1)}$.

In terms of such system of local parameters any element $\xi \in K$ can be uniquely presented as a power series of the following form

$$\xi = \sum_{\bar{a} = (a_1, \ldots, a_N)} \alpha_{\bar{a}} t_1^{a_1} \cdots t_N^{a_N}.$$  

Here all coefficients $\alpha_{\bar{a}}$ are either elements of $K^{(N)}$ if $\text{char } K = p > 0$ or the Teichmuller representatives of those if $\text{char } K = 0$. All indices $a_i \in \mathbb{Z}$ and there are integers (which depend on $\xi$) $A_1, A_2(a_1), \ldots, A_N(a_1, \ldots, a_{N-1})$ such that $\alpha_{\bar{a}} = 0$ if either $a_1 < A_1$, or $a_2 < A_2(a_1), \ldots$, or $a_N < A_N(a_1, \ldots, a_{N-1})$.

There is an important concept of $P$-topology on $K$ which brings into correlation all $N$ valuation topologies related to $K$. The $P$-topological structure provides us with a reasonable treatment of morphisms of higher dimensional local fields. We discuss this structure briefly in n.1.2 below. Notice that if $f : K \rightarrow L$ is a $P$-continuous morphism of higher dimensional local fields then $E = f(K)$ is a closed subfield in $L$ (i.e. $\mathcal{O}_E^{(1)}$ is closed in $\mathcal{O}_L^{(1)}$ with respect to first valuation and $E^{(1)}$ is closed in $L^{(1)}$), for any system $t_1, \ldots, t_N$ of local parameters in $K$ their images $f(t_1), \ldots, f(t_N)$ are local parameters in $E$ and their knowledge determines the morphism $f$ uniquely.

Our considerations will be limited with local fields $K$ such that $\text{char } K^{(1)} = p$ where $p$ is a fixed prime number (such fields possess most interesting arithmetic structure). Under this assumption there is the following classification of $N$-dimensional local fields:

— if $\text{char } K = p$, then $K = k((t_N)) \cdots ((t_1))$ where $k = K^{(N)}$ is the last residue field of $K$. As a matter of fact, this result is equivalent to the existence of a system of local parameters $t_1, \ldots, t_N$ in $K$.
— if char $K = 0$, then $K \supset \mathbb{Q}_p$ and we can introduce a canonical subfield $K(1)$ of 1-dimensional constants in $K$: this is the algebraic closure of $\mathbb{Q}_p$ in $K$. Suppose a uniformising element $t_1$ of $K(1)$ can be included in a system of local parameters $t_1, t_2, \ldots, t_N$ of $K$. Then $K = K(1)\{\{t_N\}\ldots\{t_2\}$ and such $K$ is called standard. Otherwise, there is a finite extension $E$ of $K(1)$ such that the composite $KE$ is standard.

The above result about the characteristic 0 fields is implied by the following version of Epp’s theorem [Epp], which holds for all (not necessarily characteristic 0) higher dimensional local fields $K$:

— suppose $K$ is an $N$-dimensional field and $K(1)$ is its subfield of 1-dimensional constants; then there is a finite extension $E$ of $K(1)$ such that the fields $KE$ and $E$ have a common uniformising element (with respect to the first valuation in $K$).

1.2. The concept of $P$-topology.

Let $K$ be an $N$-dimensional local field. Its $P$-topology can be described explicitly by induction on $N$ in terms of any chosen system $t_1, \ldots, t_N$ of local parameters of $K$ by constructing a basis of open 0-neighborhoods $\mathcal{U}_b(K)$, cf. [Zh1]. We shall consider the following three cases:

a) char $K = p$;

b) char $K = 0$, char $K^{(1)} = p$ and $t_1$ is a local parameter in $K(1)$;

c) $K$ is a finite extension of $E$, which satisfies the above assumptions from b).

The case a).

Here $K = k((t_N))\ldots((t_1))$, where $k$ is a finite field of characteristic $p$. If $N = 0$ then $\mathcal{U}_b(K)$ contains by definition only one set $\{0\}$. Then the family $\mathcal{U}(K)$ of all open sets in $K$ consists of all subsets of $K$. Suppose $N \geq 1$. Let $\tilde{t}_N, \ldots, \tilde{t}_2$ be the images of $t_N, \ldots, t_2$ in $K^{(1)}$. Then $K^{(1)} = k((\tilde{t}_N))\ldots((\tilde{t}_2))$ and we can use the correspondences $\tilde{t}_N \mapsto t_N, \ldots, \tilde{t}_2 \mapsto t_2$ and $\alpha \mapsto \alpha$ for $\alpha \in k$, to define the embedding $h : K^{(1)} \hookrightarrow K$. Then $\mathcal{U}_b(K)$ consists of the sets $\sum_{a \in \mathbb{Z}} t_1^a h(U_a)$, where all $U_a \in \mathcal{U}(K^{(1)})$ and $U_a = K^{(1)}$ for $a \gg 0$.

The case b).

Here again the images $\tilde{t}_2, \ldots, \tilde{t}_N$ give a system of local parameters of $K^{(1)}$ and the family of all open subsets of $K^{(1)}$ is already defined by induction. So, we use again the map $h : K^{(1)} \hookrightarrow K$, which is determined by the correspondences $\tilde{t}_i \mapsto t_i$, $i = 2, \ldots, N$, and $\alpha \mapsto [\alpha]$ for $\alpha \in k$, and proceed along lines in the case a).

The case c).

If $[K : E] = n$, then the $P$-topological structure on $K$ comes from any isomorphism of $E$-vector spaces $K \simeq E^n$ and the $P$-topological structure on $E$.

It is well-known that $K$ is an additive $P$-topological group but the multiplication in $K$ has very bad $P$-topological properties. Later we need to study the $P$-continuity of maps between objects obtained from $K$-spaces by duality. For this reason we shall use the following description of compact subsets in $K$.

Introduce a basis $\mathcal{C}_b(K)$ of compact subsets in $K$. In other words, if $\mathcal{C}_b(K)$ is such a family then any compact subset $D$ in $K$ will appear as a closed subset of some $C \in \mathcal{C}_b(K)$. Proceed again by induction on the dimension $N$ of $K$ according to above assumptions a)-c) about $K$.

In the case a) $\mathcal{C}_b(K)$ will consist of the only one set $\{K\}$ if $N = 0$. If $N \geq 1$ then in the cases a) and b) we can use the map $h : K^{(1)} \hookrightarrow K$ to define $\mathcal{C}_b(K)$ as
In the case c), we just set $C = \mathcal{C}_b(K)$ and $C = \{0\}$ for $a \ll 0$. In the case $c)$, we just set $\mathcal{C}_b(K) = \{C^n \mid C \in \mathcal{C}_b(E)\}$.

**Proposition 1.1.** The above defined family $\mathcal{C}_b(K)$ is a basis of $P$-compact subsets in $K$.

**Proof.** Proceed by induction on $N$ when $K$ satisfies the assumptions from the cases a) and b). The case $N = 0$ is clear.

Let $N \geq 1$. Prove first that $\mathcal{C}_b(K)$ consists of compact subsets in $K$. Suppose $C = \sum a_i h(C_a) \in \mathcal{C}_b(K)$. Notice first, that each $h(C_a)$ is $P$-compact in $K$. For any $b \in \mathbb{Z}$, set $C_{\leq b} = \sum_{a \leq b} t_i h(C_a)$. Then $C_{\leq b}$ is $P$-homeomorphic to the product of finitely many compact sets $h(C_a)$, $a \leq b$. Therefore, $C_{\leq b}$ is $P$-compact. Finally, $C = \lim_b C_{\leq b}$ as $P$-topological sets. So, $C$ is compact.

Suppose $D$ is a $P$-compact subset in $K$. Take $a_0 \in \mathbb{Z}$ such that $D \subset \sum_{a \geq a_0} t_i h(K^{(1)})$ (such $a_0$ exists because $D$ is compact). From the definition of the $P$-topology it follows that all projections $pr_a : D \rightarrow K^{(1)}$ (where for any $d \in D$, $d = \sum t_i h(pr_a(d))$) are open maps. Therefore, all $pr_a(D)$ are compact subsets in $K^{(1)}$. By induction there are $C_a \in \mathcal{C}_b(K^{(1)})$ such that $pr_a(D)$ are closed subsets in $C_a$. So, $D$ is a subset in the $P$-compact set $\sum t_i h(C_a) \in \mathcal{C}_b(K)$.

Finally, the case c) follows from the definition of the $P$-topology as the product topology associated with the $P$-topology on $E$. The proposition is proved.

The following proposition can be proved easily by induction on $N$.

**Proposition 1.2.** For any $C_1, C_2 \in \mathcal{C}_b(N)$, $C_1 + C_2 \in \mathcal{C}_b(K)$ and $C_1 C_2 \in \mathcal{C}_b(K)$.

**Remark.** A small modification of the above arguments proves the existence of a base of compact subsets $\mathcal{C}_b(K)$, which consists of additive subgroups of $K$.

2. Higher dimensional elimination of wild ramification.

2.1. Introduce the category $\mathcal{LC}$ of higher dimensional local fields with a given subfield of constants of codimension 1. The objects in $\mathcal{LC}$ are couples $(K, E)$ where $K$ is a local field of dimension $N \geq 1$ and $E$ is a topologically closed subfield of dimension $N - 1$ which is algebraically closed in $K$. If $N = 1$ and char $K = 0$ we shall agree by definition to take as $E$ the maximal unramified extension of $\mathbb{Q}_p$ in $K$, i.e. in this case a 1-dimensional field will play a role of a subfield of 0-dimensional constants. Morphisms $(K, E) \rightarrow (K', E')$ in the category $\mathcal{LC}$ are given by $P$-continuous morphisms of local fields $f : K \rightarrow K'$ such that $f(E) \subset E'$.

We shall use the notation $\mathcal{LC}(N)$ for the full subcategory in $\mathcal{LC}$ consisting of $(K, E)$, where $K$ is an $N$-dimensional field. Notice that $\mathcal{LC}(1)$ is equivalent to the usual category of complete discrete valuation fields with finite residue field of characteristic $p$.

**Remark.** Suppose $(K, E) \in \mathcal{LC}$. Then there is a natural embedding of first residue fields $E^{(1)} \subset K^{(1)}$ but $(K^{(1)}, E^{(1)})$ is not generally an object of the category $\mathcal{LC}(N - 1)$, because $E^{(1)}$ is not generally algebraically closed in $K^{(1)}$. Notice that it is separably closed in $K^{(1)}$: otherwise, $E$ will possess a non-trivial unramified extension in $K$.

**Definition.** $(K, E) \in \mathcal{LC}(N)$ is standard if there is a system of local parameters $t_1, \ldots, t_N$ in $K$ such that $t_1, \ldots, t_{N-1}$ is a system of local parameters in $E$. In other words, if $(K, E)$ is standard then there is a $t_N \in K$ which extends any system of
local parameters in $E$ to a system of local parameters in $K$. Such an element $t_N$ of $K$ will be called an $N$-th local parameter in $K$ (with respect to a given subfield of $(N - 1)$-dimensional constants $E$).

We mention the following simple properties:

a) for any $(K, E) \in \text{LC}$, there is always a closed subfield $K_0$ in $K$ containing $E$ such that $(K_0, E') \in \text{LC}$ is standard; this field $K_0$ appears in the form $E\{\{t\}\}$ with a suitably chosen element $t$ of $\mathcal{O}_K$;

b) if $(\tilde{K}, E) \in \text{LC}(N)$ and $K$ is a closed subfield in $\tilde{K}$ such that $\tilde{K} \supset E$ and $(K, E) \in \text{LC}(N)$, then $(K, E)$ is standard; (One can see easily, that $[\tilde{K} : K] < \infty$ and if $\tilde{t}_N$ is an $N$-th local parameter for $\tilde{K}$ then $N_{\tilde{K}/K}\tilde{t}_N$ is an $N$-th local parameter for $K$.)

c) if $(K, E) \in \text{LC}$ is standard then for any finite extension $E'$ of $E$, $(KE', E') \in \text{LC}$ is standard; (Any $N$-th local parameter in $K$ is still an $N$-th local parameter in $KE'$.)

d) any $(K, E) \in \text{LC}(1)$ is standard;

e) for any $(K, E) \in \text{LC}(2)$, there is a finite extension $E'$ of $E$ such that $(KE', E') \in \text{LC}(2)$ is standard. (This follows from Epp’s Theorem.)

The following property plays a very important role in the construction of ramiﬁcation theory for higher dimensional fields.

**Proposition 2.1.** Suppose $(K, E), (L, E) \in \text{LC}(N), L \supset K$ and $(L, E)$ is standard. Then $\mathcal{O}_L = \mathcal{O}_K[t_N]$, where $t_N$ is an $N$-th local parameter in $L$.

**Proof.** Clearly, $\mathcal{O}_K[t_N] \subset \mathcal{O}_L$.

Let $t_1, \ldots, t_{N-1}$ be local parameters in $E$. It will be sufﬁcient to prove that

$$t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} t_N^{a_N} \in \mathcal{O}_K[t_N]$$

if $(a_1, \ldots, a_{N-1}, a_N) \geq \tilde{0}_N$.

We can assume that $a_N < 0$ (otherwise, there is nothing to prove).

Notice that $\tilde{t}_N = N_{L/K}t_N$ is an $N$-th local parameter for $K$ and $\tilde{t}_N t_N^{-1} \in \mathcal{O}_K[t_N]$.

Therefore,

$$t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} t_N^{a_N} = t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} \tilde{t}_N^{-1} (\tilde{t}_N t_N^{-1})^{-a_N} \in \mathcal{O}_K[t_N]$$

because $t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} \tilde{t}_N^{a_N} \in \mathcal{O}_K$. The proposition is proved.

2.2. The following theorem plays in our setting a role of a higher dimensional version of Epp’s Theorem.

**Theorem 1.** If $(K, E) \in \text{LC}(N)$, then there is a finite separable extension $E'$ of $E$ such that $(KE', E') \in \text{LC}(N)$ is standard.

**Proof.** Use induction on $N$.

If $N = 1$ there is nothing to prove. Notice that the case $N = 2$ follows from Epp’s Theorem.

Suppose $N > 1$ and the theorem holds for local fields of dimension $< N$. 


Proposition 2.2. Suppose \((K, E) \in \text{LC}(N)\), then there is a finite separable extension \(\tilde{E}\) of \(E\) such that if \(\tilde{K} = KE\) then 
1) \(\tilde{K}\) and \(\tilde{E}\) have a common first uniformiser;
2) \(\tilde{E}^{(1)}\) is algebraically closed in \(\tilde{K}^{(1)}\).

This proposition will be proved in nn.2.3-2.9 below. It implies the statement of Theorem 1 as follows.

By the above property 2), \((\tilde{K}^{(1)}, \tilde{E}^{(1)}) \in \text{LC}(N - 1)\). Therefore, by the inductive assumption there is a finite separable extension \(E_1\) of \(\tilde{E}^{(1)}\) such that \((K_1, E_1)\) is standard (where \(K_1 = \tilde{K}^{(1)}E_1\)). Denote by \(t_2, \ldots, t_N\) a system of local parameters in \(K_1\) such that \(t_2, \ldots, t_{N-1}\) is a system of local parameters of \(E_1\). Let \(E'\) be an unramified extension \(\tilde{E}\) such that \(E'^{(1)} = E_1\). Notice that if \(K' = KE'\) then \(K''^{(1)} = K_1\). Let \(t_2, \ldots, t_{N-1}\) be liftings of \(\tilde{t}_2, \ldots, \tilde{t}_{N-1}\) to \(O^{(1)}_{E_1}\) and let \(t_N\) be a lifting of \(\tilde{t}_N\) to \(O^{(1)}_{K_1}\). Then \(t_1, \ldots, t_N\) is a system of local parameters in \(K'\) and \(t_1, \ldots, t_{N-1}\) is a system of local parameters in \(E'\), i.e. \((K', E') \in \text{LC}(N)\) is standard.

Proof of Proposition 2.2.

2.3. Choose a standard \((K_0, E) \in \text{LC}(N)\) such that \(K_0 \subset K\), and denote by \(t_1, \ldots, t_N\) a system of local parameters in \(K_0\) such that the first \(N - 1\) of them give a system of local parameters in \(E\).

It will be sufficient to prove our theorem for extensions \(K/K_0\) satisfying one of the following conditions (because any finite extension of \(K_0\) can be embedded into a bigger extension obtained as a sequence of such subextensions):

a) there is a finite extension \(\tilde{E}\) of \(E\) such that \(\tilde{K} := KE\) is unramified over \(\tilde{K}_0 := K_0E\), i.e. such that both fields \(K\) and \(\tilde{K}_0\) have the same first uniformiser and \(\tilde{K}^{(1)}\) is separable over \(\tilde{K}_0^{(1)}\);

b) \(K/K_0\) is a cyclic extension of a prime to \(p\) degree \(m\);

c) \(K/K_0\) is a purely non-separable extension of degree \(p\).

Following the terminology from [Zh2] we can call \((K, E)\) an almost constant extension of \((K_0, E)\) in the case a) and an infernal elementary extension in the case b).

2.4 The case a).

This case easily follows from the following observation. Consider the natural field embedding \(\tilde{E}^{(1)} \subset \tilde{K}_0^{(1)}\). Then \((\tilde{K}_0^{(1)}, \tilde{E}^{(1)}) \in \text{LC}(N - 1)\). Indeed, \((\tilde{K}_0, \tilde{E}) \in \text{LC}(N)\) is standard, then \(\tilde{E}^{(1)}\) is a field of \((N - 2)\)-dimensional constants in \(\tilde{K}_0^{(1)}\), which is algebraically closed in \(\tilde{K}_0^{(1)}\). On the other hand, \(\tilde{E}^{(1)}\) is separably closed in \(\tilde{K}^{(1)}\) (otherwise, \(\tilde{E}\) will have a non-trivial unramified extension in \(\tilde{K}\)). This implies that any finite extension \(E'\) of \(\tilde{E}^{(1)}\) in \(\tilde{K}^{(1)}\) is either purely inseparable or trivial.
Therefore, \( E' \subset \tilde{K}_0^{(1)} \) (because \( \tilde{K}^{(1)}/\tilde{K}_0^{(1)} \) is separable) and \( E' = \tilde{E}^{(1)} \) (because \( \tilde{E}^{(1)} \) is algebraically closed in \( \tilde{K}_0^{(1)} \)).

2.5. The case \( a_1 \).

We can assume that \( E \) contains a primitive \( m \)-th root of unity. Then \( K = K_0(\sqrt{t_1^{a_1}, \ldots, t_N^{a_N}}) \), where \( a_1, \ldots, a_N \in \mathbb{Z}_{\geq 0} \), and we can assume that \( \gcd(a_N, m) = 1 \). Let \( \tilde{E} = E(\sqrt{t_1}, \ldots, \sqrt{t_{N-1}}) \), then \( \tilde{E} \) has local parameters \( \sqrt{t_1}, \ldots, \sqrt{t_{N-1}} \) and this system can be extended to a system of local parameters in \( \tilde{K} = K \tilde{E} \) by adding \( \sqrt{t_N} \). So, \((\tilde{K}, \tilde{E})\) is standard and \( \tilde{E}^{(1)} \) is algebraically closed in \( \tilde{K}^{(1)} \).

2.6. Special extensions.

For our future targets we need to keep control on the choice of the extension \( \tilde{E} \) of \( E \) in the proposition 2.2. This idea goes back to the paper [ZhK] where it was proved that Epp’s elimination of wild ramification for an infernal extension can be done by the use of subextensions of a given deeply ramified extension.

Consider an increasing sequence of finite extensions

\[ E \subset \tilde{E}_0 \subset E_0 \subset \tilde{E}_1 \subset E_1 \subset \cdots \subset \tilde{E}_n \subset E_n \subset \cdots \]

such that each \( \tilde{E}_n \) and \( E_n \) have a system of local parameters \( \tilde{t}_{1n}, \ldots, \tilde{t}_{N-1,n} \) and, respectively, \( t_{1n}, \ldots, t_{N-1,n} \), satisfying the following condition:

C. There is a \( c > 0 \) such that for all \( 1 \leq i \leq N-1 \) and \( n \geq 1 \),

\[ v^1 \left( \frac{t_{im}}{t_{i,n-1}} - 1 \right) \geq c \]

where \( v^1 \) is a \( t_1 \)-adic (1-dimensional) valuation on \( \tilde{K} \) normalized by the condition \( v^1(t_1) = 1 \).

Proposition 2.2 will be implied in the cases b) and c) by the following statement.

**Proposition 2.3.** Suppose that \( K, K_0 \) and \( E \) satisfy the assumptions from the cases b) or c). Then there is an \( n^* \in \mathbb{Z}_{\geq 0} \) (depending only on the extension \( K/K_0 \) and the \( c \) from the above condition C) such that proposition 2.1 holds with \( E' = E_{n^*} \).

2.7. The case \( b_2 \).

In the case \( b_2 \) we have \( K = K_0(\theta), \theta^p - \theta = \xi \), where \( \xi \in K_0 \) is a power series

\[ \xi = \sum_{\bar{a}} [\alpha_{\bar{a}}] t_1^{a_1} \cdots t_N^{a_N} \]

with restrictions on its coefficients described in the beginning of section 1. Applying the Artin-Schreier equivalence we can assume also that it contains only non-zero terms with \( \bar{a} \leq \bar{0}_N \) and \( \bar{a} \not\equiv 0 \mod p \) if \( \bar{a} \neq 0 \).

Set \( \xi = \xi' + \xi'' \), where

\[ \xi' = \sum_{a_N = 0} [\alpha_{\bar{a}}] t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} \quad \xi'' = \sum_{a_N \neq 0} [\alpha_{\bar{a}}] t_1^{a_1} \cdots t_{N-1}^{a_{N-1}} t_N^{a_N} \]

Let

\[ A = \min \{ a_1 \mid \alpha_{\bar{a}} \neq 0, a_N = 0 \} = v^1(\xi') \]
\[ B = \min\{a_1 \mid \alpha_\alpha \neq 0, a_N \neq 0\} = v^1(\xi^\gamma), \]

where \( v^1 \) is a \( t_1 \)-adic valuation from the above condition \( C \).

Notice that the first set can be empty. In this case we set by definition \( A = 0 \).

The second set is never empty: otherwise, \( K \) is a composite of an algebraic extension of \( E \) and \( K_0 \), i.e. \( E \) is not algebraically closed in \( K \). For any \( s \in \mathbb{Z}_{\geq 0} \), let

\[ B(s) = \min\{a_1 \mid \alpha_\alpha \neq 0, v_p(a_N) = s\} \]

(we set \( B(s) = 0 \) if the corresponding subset of indices is empty). Then \( B = \min\{B(s) \mid s \geq 0\} \).

Notice that if we pass from \( E \) to its finite extension \( \tilde{E}_0 \), cf. condition \( C \), then \( \tilde{t}_1, \ldots, \tilde{t}_N \) is a system of local parameters for \( K_0 \tilde{E}_0 \). Rewrite \( \xi \) in terms of these local parameters and apply to this expression the Artin-Schreier equivalence to get rid of all \( p \)-th powers and terms from the maximal ideal of \( O_{K_0 \tilde{E}_0} \). This procedure gives an analogue \( \tilde{\xi}_0 \) of \( \xi \) for the extension \( K\tilde{E}_0/K_0 \tilde{E}_0 \). As earlier, use the \( t_1 \)-adic valuation \( v^1 \) to define the analogues \( \tilde{A}_0, \tilde{B}_0, \tilde{B}_0^{(s)} \) of, respectively, \( A, B \) and \( B(s) \), \( s \geq 0 \).

**Lemma 2.4.** a) \( \tilde{A}_0 \geq A \);

b) for all \( s \geq 0 \), \( \tilde{B}_0^{(s)} \geq \min\left\{ \frac{1}{p^u} B^{(s+u)} \mid u \geq 0 \right\} \).

Apply the similar procedure to the extensions \( E_0, \tilde{E}_1, E_1, \ldots \) to get the invariants \( A_0, B_0, B_0^{(s)}, \tilde{A}_1, \tilde{B}_1, \tilde{B}_1^{(s)}, A_1, B_1, B_1^{(s)}, \ldots \).

Similarly, we have the following property.

**Lemma 2.5.** For all \( i, s \geq 0 \),
a) \( \tilde{A}_{i+1} \geq A_i \);

b) \( \tilde{B}_i^{(s)} \geq \min\left\{ \frac{1}{p^u} B_i^{(s+u)} \mid u \geq 0 \right\} \).

When passing through the special extensions \( E_i/\tilde{E}_i \), \( i \geq 0 \), we have the better estimates:

**Lemma 2.6.** For all \( i \geq 0 \) and \( s \geq 1 \),
a) \( A_i \geq \min\left\{ \frac{1}{p^u} \tilde{A}_i, \tilde{A}_i + c \right\} \);

b) \( B_i^{(0)} \geq min\left\{ \frac{1}{p^u} B_i^{(0)}; \frac{1}{p^u} B_i^{(1)}; \frac{1}{p^u} (\tilde{B}_i^{(u)} + c), u \geq 2 \right\} \);

c) \( B_i^{(s)} \geq \min\left\{ \frac{1}{p^u} B_i^{(s+1)}; \frac{1}{p^u} (\tilde{B}_i^{(s+u)} + c), u \geq 0 \right\} \).

**Corollary 2.7.** a) \( \lim_{i \to \infty} A_i = 0 \);

b) if \( \gamma_i = \min\{B_i^{(s)} \mid s \geq 1\} \), then \( \lim_{i \to \infty} \gamma_i = 0 \).

**Lemma 2.8.** If \( i \geq 0 \) is such that \( B_i^{(0)} < B_i^{(s)} \) for all \( s \geq 1 \), then for all \( u \geq i \), \( B_u = B_i^{(0)} \).

**Corollary 2.9.** There is an index \( n^* \) such that \( A_n^* > B_n^* \).

So, if \( n \geq n^* \), then \( K^{(1)}_n = K^{(1)}_{0n}(\tilde{\theta}) \) with

\[ \tilde{\theta}^p = \tilde{\eta} := (t_1^{-B_{n^*}} \xi_n) \mod m^{(1)}_{K_{0n}}. \]
where \( B = B_n = B_{n^*} \) and \( e_n^{-1} = v^1(t_{1n}) \). Clearly, \( \bar{\eta} \notin K_{0n}^{(1)p} + E_n^{(1)} \) and, therefore, \( E_n^{(1)} \) remains to be algebraically closed in \( K_n^{(1)} \).

Besides, if \( n \geq \min\{n^*, 1\} \) then the first uniformiser \( t_{1n} \) appears in the leading term of the \( \xi_n'' \) with an exponent divisible by \( p \) and, therefore, it is also a uniformiser for \( K_n \). So, proposition 1.3 is proved in the case \( b_2 \).

2.8 The case \( c \).

In this case we have \( K = K_0(\theta) \), \( \theta^p = \xi \), where \( \xi \in K_0 \) is the power series from n.2.7, containing non-zero terms only with \( \bar{a} \neq 0 \) mod \( p \).

Set \( \xi = \xi' + \xi'' \), where

\[
\xi' = \sum_{a_N \equiv 0 \mod p} [\alpha]\tilde{t}_1^{a_1}\ldots t_{N-1}^{a_{N-1}}t_N^{a_N}, \quad \xi'' = \sum_{a_N \neq 0 \mod p} [\alpha]\tilde{t}_1^{a_1}\ldots t_{N-1}^{a_{N-1}}t_N^{a_N}
\]

Let

\[
A = \min\{a_1 \mid \alpha_{\bar{a}} \neq 0, a_N \equiv 0 \mod p\} = v^1(\xi')
\]
\[
B = \min\{a_1 \mid \alpha_{\bar{a}} \neq 0, a_N \neq 0 \mod p\} = v^1(\xi''),
\]

where \( v^1 \) is a \( t_1 \)-adic valuation from the above condition \( C \).

Notice that the first set can be empty. In this case we set by definition \( A = +\infty \). The second set is never empty: otherwise, \( \theta \) is algebraic over \( E \), i.e. \( E \) is not algebraically closed in \( K \).

If we pass from \( E \) to its finite extension \( \tilde{E}_i \), where \( i = 0, 1, \ldots \), cf. condition \( C \), then \( \tilde{t}_{i_1}, \ldots, \tilde{t}_{i_{N-1}}, t_N \) is a system of local parameters for \( K_0\tilde{E}_i \). Rewrite \( \xi \) in terms of these local parameters and take away all \( p \)-th power terms. This procedure gives an analogue \( \tilde{\xi}_i \) of \( \xi \) for the extension \( K\tilde{E}_i/K_0\tilde{E}_i \). As earlier, use \( t_1 \)-adic valuation \( v^1 \) to define the analogues \( \tilde{A}_i \) and \( \tilde{B}_i \).

Similarly, introduce the invariants \( A_i \) and \( B_i \), where \( i = 0, 1, \ldots \), when passing in the above procedure from \( E \) to \( E_i \).

We have the following estimates.

Lemma 2.10. a) \( \tilde{A}_0 \geq A \) and \( \tilde{B}_0 = B \);

b) for all \( i \geq 0 \), \( \tilde{A}_{i+1} \geq A_i \) and \( \tilde{B}_{i+1} = B_i \);

c) for all \( i \geq 0 \), \( A_i \geq A_i + c \) and \( B_i = B_i \).

This implies immediately that there is an index \( n^* \) such that \( A_{n^*} > B_{n^*} = B \). Therefore, for all \( n \geq n^* \), \( K_n^{(1)} = K_{0n}^{(1)}(\bar{\theta}) \), where \( \bar{\theta} \) is the image of \( t_{1n}^{-B_{en}} \xi_n \) in \( K_{0n}^{(1)} \), where \( e_n^{-1} = v^1(t_{1n}) \). Clearly, \( E_{0n}^{(1)} \) is still algebraically closed in \( K_{0n}^{(1)} \). Even more, if \( n \geq \min\{n^*, 1\} \) then \( t_{1n}, \ldots, t_{N-1,n} \) appear in the leading term of \( \xi_n'' \) with divisible by \( p \) exponents. In particular, \( t_{1n} \) is still a (first) uniformiser for \( K_n \).

The case \( c \) is also considered.

2.9. Characteristic 0 analogue of the Artin-Schreier theory.

The characteristic 0 case \( b_1 \) can be treated similarly to the characteristic \( p \) case \( b_2 \) due to the characteristic 0 analogue of the Artin-Schreier theory from [Ab1].

This construction can be briefly reminded as follows.

Suppose \( L_0 \) is a complete discrete valuation field of characteristic 0 with the maximal ideal \( m_{L_0} \) and the residue field \( k \) of characteristic \( p \). Assume that \( \zeta_p \in L_0 \) (where \( \zeta_p \) is a primitive \( p \)-th root of unity) and let \( \pi_1 \in L_0 \) be such that \( \pi_1^{p-1} = -p \).
Proposition 2.11.

a) \( L = L_0(\sqrt[p]{v}) \) with \( v \in 1 + \pi_1 m_{L_0} \) if and only if \( L = L_0(\theta), \) where \( \theta^p - \theta = w \) with \( w \in p^{-1} m_{L_0}; \)

b) With the above notation and assumptions \( L \) admits another presentation \( L = L_0(\theta_1), \) where \( \theta^p - \theta_1 = w_1 \in p^{-1} m_{L_0}, \) if \( w_1 = w + \eta^p - \eta \) with \( \eta \in L_0 \) such that \( \eta^p \in p^{-1} m_{L_0}. \)

Proof. We only sketch the idea of the proof.

Let \( E(X) = \exp \left( X + X^p/p + \cdots + X^{p^n}/p^n + \cdots \right) \in \mathbb{Z}_p[[X]] \)
be the Artin-Hasse exponential. Then \( v = E(\pi_1 V) \) with \( V \in m_{L_0} \) and if \( u^p = v, \)
\( u \in L, \) then \( u = E(U) \) with \( U \in m_L. \) Then the equivalence
\[
E(X^p) = E(X^p) \exp(pX) \equiv E(X^p + pX) \mod(p^2 X, pXP)
\]
implies that
\[
U^p + pU \equiv \pi_1 V \mod \pi_1 p m_L
\]
(notice that \( U^p \in \pi_1 m_L). \)

Divide both sides of the above equivalence by \( \pi_1^p \) and deduce that \( L = L_0(\theta), \)
where \( \theta^p - \theta = w \in p^{-1} m_{L_0} \) with \( \theta \equiv \pi_1^{-1} U \mod m_L \) and \( w \equiv p^{-1} V \mod m_{L_0}. \)

2.10 The case \( b_1). \)

Proposition 2.12. Suppose \( K, K_0 \) and \( E \) satisfy the condition \( b_1) \) from n.2.3. Then there is an \( n^* \in \mathbb{Z}_{\geq 0} \) such that proposition 2.3 holds with \( E' = E_{n^*}. \)

Proof. Assume first that \( \zeta_p \in E. \)

Then \( K E_0 = (K_0 E_0)(\sqrt[p]{v_0}), \) where \( v_0 = \tilde{v}_0 \equiv \tilde{t}_0^{c_1} \cdots \tilde{t}_N^{c_N}(1 + \tilde{a}), \) \( \tilde{a} \in m_{K E_0} \) and \( c_1, \ldots, c_N \in \mathbb{Z}_{\geq 0}. \)

Then the condition \( C \) from n.2.6 implies that
\[
\tilde{v}_0 = \tilde{t}_0^{p c_1} \cdots \tilde{t}_N^{p c_N} (1 + \tilde{a}^p + \tilde{b}),
\]
where \( \tilde{a}, \tilde{b} \in m_{K E_0} \) and \( v^1(\tilde{b}) \geq c. \) This implies that \( K E_0 = K_0 E_0(\sqrt[p]{v_0}), \)
where \( v_0 = 1 + pa + b \) with \( a, b \in m_{K_0 E_0} \) such that \( v^1(b) \geq c. \)

By continuing the above procedure we obtain that \( K E_n = (K_0 E_n)(\sqrt[p]{v_n}) \) where \( v_n = 1 + pb_n \) with \( b_n \in m_{K_0 E_n}. \) Since \( v_n = 1 + \pi_1 m_{K_0 E_n}, \) the extension \( K E_n/K_0 E_n \)
can be given via the analogue of the Artin-Schreier theory from n.2.9 and we can proceed further as in n.2.7 to finish the proof of our proposition.

Suppose now that \( \zeta_p \notin E. \)

Let \( K' = K(\zeta_p), K'_0 = K_0(\zeta_p), E' = E(\zeta_p) \) and \( E'_n = E_n(\zeta_p), E'_0 = E_0(\zeta_p) \) for all \( n \geq 0. \) Then the tower
\[
E'/K' \subset E'_0 \subset E'_1 \subset \cdots \subset E'_n \subset \cdots
\]
satisfies the condition \( C \) from n.2.6. Therefore, there is an \( n^* \) such that if \( K' = K E'_n \) and \( K'_n = K_0 E'_n, \) then \( E'_n(1) \) is algebraically closed in \( K'_n(1). \)

Let \( F \) be a non-trivial purely inseparable extension of \( E_n(1) \) in \( K_n(1). \) Then \( F E'_n(1) \)
is a non-trivial purely inseparable extension of \( E'_n(1) \) in \( K'_n(1) \)
(use that \( [E'_n(1) : E_n(1)] < p). \) But this contradicts to the fact that \( E'_n(1) \)
is algebraically closed in \( K'_n(1). \) Therefore, \( E_n(1) \) is algebraically closed in \( K_n(1), \) because it is its separably closed subfield.

The proposition is completely proved.
3. Ramification theory and Krasner’s Lemma.

3.1 The category of local fields with $F$-structures.

This category $\text{LF}(N)$ will appear as the disjoint union of its two full subcategories $\text{LF}_0(N)$ and $\text{LF}_p(N)$.

The category $\text{LF}_0(N)$.

Choose a simplest $N$-dimensional local field $L_0 = \mathbb{Q}_p\{\{t_N\}\} \ldots \{\{t_2\}\}$. Define its $F$-structure as an increasing sequence of closed subfields $\{L_0(i) \mid 1 \leq i \leq N\}$ with the system of local parameters $p = t_1, t_2, \ldots, t_N$. Choose an algebraic closure $\bar{L}_0$ of $L_0$. Denote by $\mathbb{C}(N)_p$ the completion of $\bar{L}_0$ with respect to its first $(p$-adic) valuation. For $1 \leq i \leq N$, denote by $\mathbb{C}(i)_p$ the completion of the algebraic closure of $L_0(i)$ in $\mathbb{C}(N)_p$. It will be convenient to have a special agreement for $i = 0$.

By definition, $\mathbb{C}(0)_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}(N)_p$ and $L_0(0) = L_0 \cap \mathbb{C}(0) = \mathbb{Q}_p$. Notice that $\mathbb{C}(1)_p = \mathbb{C}_p$ is the $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$.

Clearly, the $P$-topological structure of finite extensions of $L_0$ induces the $P$-topological structures on the fields $\mathbb{C}(0)_p \subset \mathbb{C}(1)_p \subset \cdots \subset \mathbb{C}(N)_p$.

The objects of the category $\text{LF}_0(N)$ are finite extensions $K$ of $L_0$ in $\mathbb{C}(N)_p$ with the induced $F$-structure. This structure is given by the sequence of algebraically closed and $P$-closed subfields $\{K(i) \mid 0 \leq i \leq N\}$, where $K(i) = K \cap \mathbb{C}(i)_p$. Notice that $K(0)$ is the maximal unramified extension of $\mathbb{Q}_p$ in $K$. We agree to use the notation $\mathcal{K}$ for the algebraic closure of $K$ in $\mathbb{C}(N)_p$. Notice that $\Gamma_K = \text{Aut}(\bar{K}/K)$ consists of $P$-continuous field automorphisms $\tau$ of $\mathbb{C}(N)_p$ such that $\tau|_K = \text{id}$ and for all $0 \leq i \leq N$, $\tau(\mathbb{C}(i)_p) = \mathbb{C}(i)_p$. It is well-known [Hy], that $\mathcal{C}(N)_p^{\Gamma_K} = K$ and, therefore, for all $0 \leq i \leq N$, $\mathbb{C}(i)_p^{\Gamma_K} = K(i)$.

Suppose $K, L \in \text{LF}_0(N)$. Then the corresponding set of morphisms $\text{Hom}_{\text{LF}(N)}(K, L)$ consists of all $P$-continuous field morphisms $\phi : \mathbb{C}(N)_p \rightarrow \mathbb{C}(N)_p$ such that for $0 \leq i \leq N$,

a) $\phi(\mathbb{C}(i)_p) = \mathbb{C}(i)_p$;

b) $\phi(K) \subset L$.

Notice that any $\phi \in \text{Hom}_{\text{LF}(N)}(K, L)$ transforms the $F$-structure of $K$ to the $F$-structure of $L$.

The category $\text{LF}_p(N)$.

We proceed similarly to the above characteristic 0 case. Choose a basic $N$-dimensional local field $L_p = \mathbb{F}_p((t_N)) \ldots ((t_1))$ and define its $F$-structure by a sequence of subfields $\{L_p(i) \mid 0 \leq i \leq N\}$ such that $L_p(i)$ has local parameters $t_1, \ldots, t_i$. Choose an algebraic closure $\bar{L}_p$ of $L_p$. Denote by $\mathcal{C}(N)_p$ the completion of $\bar{L}_p$ with respect to its first valuation. For $0 \leq i \leq N$, denote by $\mathcal{C}(i)_p$ the completion of the algebraic closure of $L_p(i)$ in $\mathbb{C}(N)_p$. As earlier, the $P$-topological structure of finite extensions of $L_p$ induces the $P$-topological structures on the fields $\mathbb{F}_p = \mathcal{C}(0)_p \subset \mathcal{C}(1)_p \subset \cdots \subset \mathcal{C}(N)_p$.

The objects of the category $\text{LF}_p(N)$ are finite extensions $K$ of $L_p$ in $\mathbb{C}(N)_p$ with the induced $F$-structure $\{K(i) \mid 0 \leq i \leq N\}$, where $K(i) = K \cap \mathcal{C}(i)_p$. Notice that $\mathcal{C}(N)_p^{\Gamma_K} = \mathcal{R}(K)$ — the radical closure (=the completion of the maximal purely non-separable extension) of $K$ in $\mathbb{C}(N)_p$. Similarly for $0 \leq i \leq N$, it holds that $\mathcal{C}(i)_p^{\Gamma_K} = \mathcal{R}(K(i))$. The morphisms in $\text{LF}_p(N)$ are defined also along lines in the above characteristic 0 case.
3.2 Standard $F$-structure.

We say that the $F$-structure on $L \in LF(N)$ is standard if there is a system of local parameters $t_1, \ldots, t_N$ in $L$ such that for all $1 \leq r \leq N$, $t_1, \ldots, t_r$ is a system of local parameters for $L(r)$. Applying the above Theorem 1 we obtain easily the following

**Proposition 3.1.** For any $E \in LF(N)$, there is a finite separable extension $E'$ of $E(N-1)$ such that $EE'$ has a standard $F$-structure.

**Remark.** The above proposition played a fundamental role in the construction of the higher dimensional ramification theory in [Ab5], but its proof in [Ab5] was not complete, due to reasons mentioned in the Remark from 2.1. Notice that the construction of ramification theory, cf. n.3.3 below, can be based only on the result of Theorem 1.

Note that the $F$-structure allows to treat higher dimensional local fields in a very similar way to classical complete discrete valuation fields with finite residue fields. For example, for any finite extension of local fields with $F$-structure we can introduce:

a) a vector ramification index $\bar{e}(L/K) = (e_1, \ldots, e_N)$.

Any finite extension of $K$ in $\bar{K}$ appears with a natural $F$-structure and a natural $P$-toplogy. In particular, if $L \subset M$ are such subfields in $\bar{K}$ then its vector ramification index equals $e(M/K) = (e_1, \ldots, e_N)$, where for $1 \leq r \leq N$, $e_r = [M(r) : L(r)]/[M(r-1) : L(r-1)]$. This index plays a role of the usual ramification index in the theory of 1-dimensional local fields.

b) a canonical $N$-valuation $v_L : L \to \mathbb{Q}^N \cup \{\infty\}$.

If $L$ has a standard $F$-structure and $t_1, \ldots, t_N$ is a corresponding system of local parameters, then $v_L$ is uniquely defined by the conditions $v_L(t_1) = (1, 0, \ldots, 0)$, $v_L(t_2) = (0, 1, 0, \ldots, 0)$, $\ldots$, $v_L(t_N) = (0, 0, \ldots, 0, 1)$. Otherwise, one should use a finite extension $L_1$ of $L$ with standard $F$-structure and set $v_L = \bar{e}(L_1/L)^{-1}v_{L_1}$.

3.3 Review of ramification theory.

Suppose $K \in LF(N)$. Then $\Gamma_K = \text{Aut}(\bar{K}/K)$ has a canonical decreasing filtration by ramification subgroups $\{\Gamma_K^{(j)} \mid j \in J(N)\}$ with the set of indices $J(N) = \bigsqcup_{1 \leq r \leq N} J_r$. Here $J_r = \{j \in \mathbb{Q}^r \mid j \geq 0_r\}$ with respect to the lexicographic ordering on $\mathbb{Q}^r$, where $0_r = (0, \ldots, 0) \in \mathbb{Q}^r$. By definition, if $r_1 > r_2$ then any element from $J_{r_1}$ is bigger than any element from $J_{r_2}$.

The definition of this filtration can be described as follows.

Let $E/K$ be a finite extension in $\bar{K}$ (this is a subfield in $\mathbb{C}(N)_p$ or $\mathbb{C}(N)_p$). Consider the finite set $I_{E/K}$ of all $P$-continuous embeddings of $E$ into $\bar{K}$ which are the identity on $K$.

There is a natural filtration of this set

$I_{E/K} \supset I_{E/K,0} \supset I_{E/K,(0,0)} \supset \cdots \supset I_{E/K,0_N}$

where for $1 \leq r \leq N$, $I_{E/K,0_r}$ are embeddings which are the identity on the subfield of $(r-1)$-dimensional constants $E(r-1)$.

For $1 \leq r \leq N$ and $j \in J_r$, define the set $I_{E/K,j} \subset I_{E/K,0_r}$ as follows.

Take a suitable finite extension $E'$ of $E(r-1)$ in $\bar{K}$ such that if $\bar{E}(r) = E'E(r)$ and $\bar{K}(r) = K(r)E'$ then $\mathcal{O}_{\bar{E}(r)} = \mathcal{O}_{\bar{K}(r)}[\theta]$. (Recall, if $L \in LF(r)$ then
\( \mathcal{O}_L = \{ l \in L \mid v_L(l) \geq 0 \} \). Then use the natural identification \( I_{E/K, \bar{\alpha}} = I_{E(r)/\bar{K}(r)} \) to define the ramification filtration of \( I_{E/K} \) in lower numbering

\[
I_{E/K, j} = \{ \tau \in I_{E(r)/\bar{K}(r)} \mid v_{E(r)}(\tau(\theta) - \theta) \geq v_{E(r)}(\theta) + j \}.
\]

Introduce an analogue of the Herbrand function \( \varphi_{E/K} : J(N) \longrightarrow J(N) \) by setting for \( 1 \leq r \leq N \) and \( j \in J_r \),

\[
\varphi_{E/K}(j) = \bar{e}_{E(r)/K(r)}^{-1} \int_{\bar{0}_r}^{j} |I_{E/K, j}| dj \in J_r.
\]

This gives the upper numbering such that for any \( j \in J(N) \), \( I_{E/K}^{(j)} = I_{E/K, \varphi_{E/K}(j)} \). As in the classical situation, if \( E_2 \supset E_1 \supset K \), then the natural projection \( I_{E_2/K} \longrightarrow I_{E_1/K} \) induces for any \( j \in J(N) \), an epimorphic map from \( I_{E_2/K}^{(j)} \) onto \( I_{E_1/K}^{(j)} \) and \( \lim_j I_{E/K}^{(j)} = \Gamma_K^{(j)} \) is the ramification subgroup of \( \Gamma_K \) with the upper number \( j \).

As an example, consider the case of an extension \( E/K \) in \( LF(N) \) such that \( [E : K] = p^N \) and \( e(E/K) = (p, \ldots, p) \in \mathbb{Q}^N \). Then for \( 1 \leq r \leq N \), there are \( \alpha_r > 0 \) such that for all \( j \in J_r \),

\[
\varphi_{E/K}(j) = \begin{cases} 
    j, & \text{if } j < \alpha_r; \\
    \alpha_r + \frac{j - \alpha_r}{p}, & \text{if } j \geq \alpha_r
  \end{cases}
\]

As in the classical case for any finite extension \( E/K \), the Herbrand function \( \varphi_{E/K} : J(N) \longrightarrow J(N) \) is a piece-wise linear function with finitely many edge points. Define \( i(E/K) \in J(N) \) and \( j(E/K) \in J(N) \) as the first and the second coordinates of the last edge point of the graph of \( \varphi_{E/K} \). Notice that if \( 1 \leq r \leq N \) and \( j \in J_r \), then \( j \in J_r \) is an edge point iff \( \varphi'_-(j) \neq \varphi'_+(j) \), where \( \varphi'_-(j) \) and \( \varphi'_+(j) \) are slopes of \( \varphi_{E/K} \) in the left and right neighbourhoods of \( j \), respectively. (By definition, \( \varphi'_-(0_r) = g_{r0} \bar{e}_{E(r)/K(r)}^{-1} \), where \( g_{r0} = [E(r) : K(r)E(r-1)] \).

If \( 1 \leq r \leq N \) and \( j \in J_r \), then \( \varphi'_-(j) = g_-(j) \bar{e}_{E(r)/K(r)}^{-1} \) and \( \varphi'_+(j) = g_+(j) \bar{e}_{E(r)/K(r)}^{-1} \), where \( g_-(j) \) and \( g_+(j) \in \mathbb{N} \). We shall call \( g_-(j)/g_+(j) := \text{mult}_{E/K}(j) \) — the multiplicity of \( \varphi_{E/K} \) in \( j \in J_r \). We have:

- \( \text{mult}_{E/K}(j) = 1 \) if and only if \( j \) is not an edge point;
- \( \prod_{j \in J(N)} \text{mult}_{E/K}(j) = [E : K] \).

3.4. Krasner’s lemma.

Suppose \( L, K \in LF(N), L \supset K, L(N - 1) = K(N - 1) \) and \( E \) is a finite extension of \( L(N - 1) \) such that \( (LE, E) \in LC(N) \) is standard. Then \( \mathcal{O}_L = \mathcal{O}_K[\theta] \) where \( \bar{L} = LE, \bar{K} = KE \) and \( \theta \) is an \( N \)-th local parameter in \( \bar{K} \).

Let \( F(T) = T^d + a_1 T^{d-1} + \cdots + a_d \in \mathcal{O}_K[T] \) be the minimal unitary polynomial for \( \theta \) over \( \bar{K} \). Denote by \( \bar{\theta}_1 = \theta, \bar{\theta}_2, \ldots, \bar{\theta}_d \in \bar{K} \) all roots of \( F(T) \). Notice that \( v_{\bar{L}}(\bar{\theta}_1) = \cdots = v_{\bar{L}}(\bar{\theta}_d) = (0, 0, 1, 1) \).

In this situation the Krasner Lemma can be given by the following proposition.
Proposition 3.2. If $\alpha \in \bar{K}$ is such that $v_K(F(\alpha)) = A+(0,\ldots,0,1)$ with $A > 0_N$, then

1) there is an index $1 \leq l_0 \leq d$ such that $v_L(\alpha - \theta_{l_0}) = a + (0,\ldots,0,1)$, where $\varphi_{L/K}(a) = A$;

b) if $A > j(L/K)$ then the above index $l_0$ is unique.

Proof. Choose an index $l_0$ such that

$$v_L(\alpha - \theta_{l_0}) = \max\{v_L(\alpha - \theta_l) \mid 1 \leq l \leq d\}.$$ 

Let $a \in J_N$ be such that $v_L(\alpha - \theta_{l_0}) = a + (0,\ldots,0,1)$.

Lemma 3.3. $v_K(F(\alpha)) = \varphi_{L/K}(a) + (0,\ldots,0,1)$.

Proof of lemma. Let $i_1 < i_2 < \cdots < i_s$ be the lower indices which correspond to all jumps of the ramification filtration on $I_{L/K}$. Then for some integers $d = g_0 > g_1 > \cdots > g_{s-1} > g_s = 1$ and all $1 \leq i \leq n$, $v_L(\theta - \theta_i)$ takes $g_0 - g_1$ times the value $i_1 + v_L(\theta), \ldots, g_{s-1} - g_s$ times the value $i_s + v_L(\theta)$. Notice that $i_s = i(L/K)$, $\bar{e}_{L/K} = (1,\ldots,1,d)$ and if $i_t \leq a < i_{t+1}$ for some $0 \leq t \leq s$ (with the agreements $i_0 = 0_N$ and $i_{s+1} = \infty$) then

$$\varphi_{L/K}(a) = \bar{e}_{L/K}^{-1}(g_0 i_1 + \cdots + g_t(i_t - i_{t-1}) + g_t(a - i_t)).$$

Clearly, for all $1 \leq l \leq s$, $v_L(\alpha - \theta_l) = \min\{v_L(\alpha - \theta_{l_0}), v_L(\theta_{l_0} - \theta_l)\}$. This implies

$$v_L(F(\alpha)) = \sum_{1 \leq l \leq n} v_L(\alpha - \theta_l)$$

$$= (g_0 - g_1)(i_1 + v_L(\theta)) + \cdots + (g_t - g_{t-1})(i_t + v_L(\theta_{l_0})) + g_t(a + v_L(\theta_{l_0}))$$

$$= g_0 v_L(\theta_{l_0}) + g_0 i_1 + g_1(i_2 - i_1) + \cdots + g_{t-1}(i_t - i_{t-1}) + g_t(a - i_t)$$

$$= \bar{e}_{L/K}(v_L(\theta_{l_0}) + \varphi_{L/K}(a)).$$

The lemma is proved, because $v_K = \bar{e}_{L/K}^{-1} v_L$.

It remains to prove the part 2) of our proposition.

Suppose $\theta_{l_1}$ is a root of $F$ with the same property $v_L(\alpha - \theta_{l_1}) = a + (0,\ldots,0,1)$. Then $v_L(\theta_{l_1} - \theta_{l_0}) > a + (0,\ldots,0,1)$. But if $A > j(L/K)$ then $a > i(L/K)$ and $\theta_{l_1} = \theta_{l_2}$.

The proposition is proved.

Corollary 3.4. With the above assumption and notation

$$v_K(D(F)) = (1,\ldots,1,d)j(L/K) - i(L/K) + (0,\ldots,0,d-1)$$

where $D(F)$ is the discriminant of $F$. 
Proof. Let $\delta(F) = (\theta - \theta_2) \ldots (\theta - \theta_d)$ be the different of $F$. Then

$$v_K(D(F)) = v_L(\delta(F)) = \sum_{2 \leq i \leq d} v_L(\theta - \theta_i)$$

$$= (g_0 - g_1)(i_1 + v_L(\theta)) + \cdots + (g_{s-1} - g_s)(i_s + v_L(\theta))$$

$$= \bar{e}(L/K) \varphi_{L/K}(i_s) - i_s + (d - 1)v_L(\theta)$$

It remains to notice that $\bar{e}(L/K) = (1, \ldots, 1, d)$, $i_s = i(L/K)$ and $\varphi_{L/K}(i_s) = j(L/K)$.

**Corollary 3.5.** $j(L/K) \leq 2v_K(D(F))$.

**Proof.** This follows from the last corollary because $i(L/K) \leq v_L(\delta(F)) = v_K(D(F))$.

4. Families of increasing towers.

In this section we work with local fields of characteristic 0 from $LF_0(N)$.

4.1. The category $\mathcal{B}(N)$.

The objects of $\mathcal{B}(N)$ are increasing sequences $K_\ast = \{K_n \mid n \geq 0\}$ of $K_n \in$ $LF_0(N)$. If $K_\ast, L_\ast \in \mathcal{B}(N)$, then $\text{Hom}_{\mathcal{B}(N)}(K_\ast, L_\ast)$ consists of field automorphisms $f : \mathcal{C}(N)_p \to \mathcal{C}(N)_p$ such that

- $f$ is $P$-continuous;
- $f$ is compatible with $F$-structure;
- $f(K_n) \subset L_n$ for all $n \geq 0$.

Clearly, if $K_\ast = \{K_n \mid n \geq 0\} \in \mathcal{B}(N)$ then for any $1 \leq r \leq N$, the subfields of constants of dimension $r$, $\{K_n(r) \mid n \in \mathbb{Z}_{\geq 0}\}$, give an object of the category $\mathcal{B}(r)$. This object will be usually denoted by $K_\ast(r)$.

Notice that two towers $K_\ast$ and $L_\ast$ are naturally isomorphic if $K_n = L_n$ for all $n \geq 0$ (all sufficiently large $n$). Such towers will be called almost equal.

Let $K_\ast, L_\ast \in \mathcal{B}(N)$. Then by definition $K_\ast \subset L_\ast$ or $L_\ast$ is an extension of $K_\ast$ if for all $m \geq 0$, $K_m \subset L_m$. $L_\ast$ is a finite extension of $K_\ast$ of degree $d = d(L_\ast/K_\ast)$ if for all $m \geq 0$, $[L_m : K_m] = d$. Clearly, if $L_\ast/K_\ast$ and $M_\ast/L_\ast$ are finite extensions then $M_\ast/K_\ast$ is also finite and $d(M_\ast/K_\ast) = d(L_\ast/K_\ast)d(M_\ast/L_\ast)$.

An extension $L_\ast/K_\ast$ will be called separable if there is an index $m_0$ and an algebraic extension $E$ of $K_{m_0}$ such that $L_\ast$ is almost equal to $EK_{\ast} := \{EK_m \mid m \geq 0\}$. Clearly, if $L_\ast/K_\ast$ and $M_\ast/L_\ast$ are separable then $M_\ast/K_\ast$ is also separable. Notice also, that the composite of finitely many separable extensions of $K_\ast$ is again separable over $K_\ast$. Therefore, any finite extension $L_\ast/K_\ast$, contains a “unique” maximal separable over $K_\ast$ subextension $L_\ast^{(s)}$ (i.e. any another maximal separable subextension is almost equal to $L_\ast^{(s)}$).

An extension $L_\ast/K_\ast$ will be called purely inseparable if for any $n \geq 0$, there is an $m = m(n) \geq 0$ such that $L_n \subset K_m$. The simplest example of a purely inseparable extension of $K_\ast$ is $K_\ast'$ such that for all $m$, $K_m' = K_{m+1}$.

Suppose $L_\ast \subset K_\ast$ is a finite extension in $\mathcal{B}(N)$ of degree $d = d(L_\ast/K_\ast)$. Let $\tilde{L}$ and $\tilde{K}$ be the $p$-adic completions of the $\bigcup_{m \geq 0} L_m$ and, resp., $\bigcup_{m \geq 0} K_m$. Suppose that $[\tilde{L} : \tilde{K}] = \tilde{d}$. Then there are the following simple properties:

- $\tilde{d} \leq d$;
Proof. Suppose \( K, L, \in \mathcal{B}(N) \) and \( L \) is a finite separable extension of \( K \). If \( K, \in \mathcal{B}^a(N) \) then \( L, \in \mathcal{B}^a(N) \).

\( \varphi \) is determined by its values on \( L \) and \( K \). Therefore, we can work with \( L \) and \( K \) instead of \( K_{m+1} \) and \( L_{m+1} \). We may assume that \( n \) is a prime power.

\( \varphi \) is determined by its values on \( K \) and \( L \). Therefore, we can work with \( K \) and \( L \) instead of \( L_{m+1} \) and \( L_{m+1} \).

We may assume that \( m \) is a prime power. Therefore, we can work with \( m \) and \( L \) instead of \( L_{m+1} \) and \( K_{m+1} \).

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Remark. a) The category \( \mathcal{B}^a(N) \) is a full subcategory in \( \mathcal{B}(N) \) consisting of \( K, \in \mathcal{B}(N) \) such that there is an index \( n^* = n^*(K) \) and \( c^* = c^*(n^*, K) > 0 \) such that for all \( n \geq n^* \),

\( a) \quad [K_{n+1} : K_n] = p^n \) and \( \bar{e}(K_{n+1}/K_n) = (p, \ldots, p) \in \mathbb{Z}^N; \)
\( b) \quad \text{if } 1 \leq r \leq N, \text{ then } \text{pr}_1(j(K_{n+1}(r)/K_n(r))) \geq p^n c^*. \) (\( \text{pr}_1(j) \) denotes the first coordinate of \( j \in J_r \subset \mathbb{Q}^r \).)

Lemma 4.2. If \( m \geq m_0 \) then \( \alpha_{m+1} \leq \max\{p\alpha_m - (p - 1)j_m, \alpha_m\} \).

Proof. By the composition property of Herbrand’s function we have

\[
\varphi_{L_{m+1}/K_m}(j) = \varphi_{K_{m+1}/K_m}(\varphi_{L_{m+1}/K_{m+1}}(j))
\]

for any \( j \in J(N) \). Looking at the last edge points we obtain

\[
j(L_{m+1}/K_m) = \max\{\varphi_{K_{m+1}/K_m}(\alpha_m), j_m\}.
\]

On the other hand, \( L_{m+1} = L_m K_{m+1} \) implies that \( j(L_{m+1}/K_m) = \max\{\alpha_m, j_m\} \).

Therefore,

\( \alpha_m \geq j_m \) then \( \varphi_{K_{m+1}/K_m}(\alpha_m) \leq \alpha_m; \)
\( \alpha_m < j_m \) then \( \alpha_m \) and \( \varphi_{K_{m+1}/K_m}(\alpha_m) \) coincide because the both appear as 2nd coordinates of the prelast edge point of the \( \varphi_{L_{m+1}/K_m} \).

It remains only to notice that for \( j \in J_N \),
\[ \varphi_{K_{m+1}/K_m}(j) = \begin{cases} j, & \text{if } j \leq j_m \\ j_m + \frac{1}{p}(j - j_m), & \text{if } j \geq j_m. \end{cases} \]

The lemma is proved.

**Lemma 4.3.** If \( m \geq m_0 \) and \( \alpha_m < j_m \) then \( \varphi_{L_m/K_m} = \varphi_{L_{m+1}/K_{m+1}} \).

**Proof.** Notice first that \( j(L_{m+1}/K_m) = \max\{\alpha_m, j_m\} = j_m \).

Let \( j'_m = j(L_{m+1}/L_m) \). Then for all \( j \in \mathcal{J}(N) \),

\[ \varphi_{L_{m+1}/K_m}(j) = \varphi_{L_m/K_m}(\varphi_{L_{m+1}/L_m}(j)) \]

implies that \( j(L_{m+1}/K_m) = \max\{\alpha_m, \varphi_{L_m/K_m}(j'_m)\} \). Therefore, \( j_m = \varphi_{L_m/K_m}(j'_m) \).

Notice that \( \tilde{e}(L_{m+1}/L_m) = \tilde{e}(K_{m+1}/K_m) = (p, \ldots, p) \) implies that \( \varphi_{L_{m+1}/L_m}(j) = j \) and \( \varphi_{K_{m+1}/K_m}(j) = j \) for all \( \bar{0}_N \leq j \leq j'_m \) and, resp., \( \bar{0}_N \leq j \leq j_m \). Therefore, the above relations (1) and (2) imply that for all \( \bar{0}_N \leq j \leq j'_m \),

\[ \varphi_{L_{m+1}/K_{m+1}}(j) = \varphi_{L_{m+1}/K_m}(j) = \varphi_{L_m/K_m}(j). \]

In addition, the point \( (j'_m, j_m) \) is the last edge point of the graph of \( \varphi_{L_{m+1}/K_m} \) with the minimal possible multiplicity \( p \). But all edge points of \( \varphi_{L_{m+1}/K_{m+1}} \) and \( \varphi_{L_m/K_m} \) are situated in the area \( j < j'_m \). Therefore, these functions coincide for all \( j \).

The lemma is proved.

We continue the proof of our proposition.

If \( m \geq m_0 \), then \( \text{pr}_1(j_m/p^m) \geq c^* \). Then Lemma 4.2 implies that

\[ \frac{\alpha_{m+1}}{p^{m+1}} \leq \max \left\{ \frac{\alpha_m}{p^m} - \left( 1 - \frac{1}{p} \right)(c^*, 0, \ldots, 0), \frac{\alpha_m}{p^m} \right\}. \]

Therefore, \( \alpha_m/p^m \) tends to 0 and taking (if necessary) a bigger \( m_0 \) we can assume that for all \( m \geq m_0 \), \( \text{pr}_1(\alpha_m/p^m) < c^* \) and, therefore, \( \alpha_m < j_m \). Then by Lemma 4.3 the Herbrand functions of the extensions \( L_m/K_m \) with \( m \geq m_0 \) coincide. Denote this function by \( \varphi_{L_m/K} \), and use the relation \( \varphi_{L_m/K}(j'_m) = j_m \), where \( j'_m = j(L_{m+1}/L_m) \), from the proof of Lemma 4.3.

Because \( \varphi_{L_m/K} \) is a piece-wise linear function the condition \( \text{pr}_1 j_m \geq p^m c^* \) implies the existence of \( 0 < c_1^* = c^*(m_0, L) < c^* \) such that \( \text{pr}_1(j'_m) \geq p^m c_1^* \) for all \( m \geq m_0 \).

The proposition is proved.

4.3. The category \( \mathcal{B}^{f_{\alpha}}(N) \).

4.3.1. Suppose \( K, \in \mathcal{B}^{\alpha}(N) \) with parameters \( n^* = n^*(K) \) and \( c^*(n^*, K) \).

**Definition.** If indices \( u_1, \ldots, u_N \) are such that \( n^* \leq u_N \leq u_{N-1} \leq \cdots \leq u_1 \) then \( K_{u_1 \ldots u_N} = K_{u_1}(1) \cdots K_{u_{N-1}}(N-1)K_{u_N} \). We usually denote this field (with its natural \( F \)-structure) as \( K_{\bar{u}} \), where \( \bar{u} = (u_1, \ldots, u_N) \).

**Definition.** Denote by \( \mathcal{B}^{f_{\alpha}}(N) \) the full subcategory of all \( K, \in \mathcal{B}^{\alpha}(N) \) such that for some index parameter \( \bar{u}^0 = \bar{u}^0(K) \), \( K_{\bar{u}^0} \) has a standard \( F \)-structure.

**Remark.** If \( \bar{u}^0 = \bar{u}^0(K) \) is the above index parameter then we always assume that \( n^*(K) = u_0^N \).
Proposition 4.4. Suppose $L, \supset K$, is a finite extension in $\mathcal{B}^a(N)$. Then there is a finite Galois extension $\bar{L}$ of $K$, such that $\bar{L}, \supset L$, and $\bar{L} \in \mathcal{B}^f_a(N)$.

Proof. Let $n^* = n^*(L) = n^*(K)$. Choose a finite Galois extension $E$ of $K_{n^*}$ such that $E, = EK, \supset L$. Then $E, \in \mathcal{B}^a(N)$, cf. n.4.2, and we can assume that $n^* = n^*(E)$. Take a finite extension $F$ of $E_{n^*}(N - 1)$ such that $(E_{n^*}, F, F)$ is standard in the category $LC(N)$.

Let $F, = FK, (N - 1)$. We can assume that $m^* := n^*(E) = n^*(K, (N - 1)) \geq n^*$. By induction there is a finite Galois extension $H$ of $K(N - 1)m^*$ such that $H, = HK, (N - 1) \supset F$, and $H, \in \mathcal{B}^f_a(N - 1)$. Then $(E_{n^*}, H, H) \in LC(N)$ is still standard and, therefore, $HE, \in \mathcal{B}^f_a(N)$. At the same time, $HE$, is Galois over $K$, as a composite of Galois extensions.

The proposition is proved.

4.3.2. The following proposition (or more precisely, its applications below) plays a crucial role in the construction of an analogue of the field-of-norms functor.

Proposition 4.5. Suppose $E, \in \mathcal{B}^f_a(N)$. Then for any $u \geq u^0_{N}(E,)$, there is a $v = v(u) \geq u$ such that $(E_uE_v(N - 1), E_v(N - 1)) \in LC(N)$ is standard.

In nn.4.3.3-4.3.6 below we assume that this proposition is proved and consider its applications. We need these applications later in our construction of the field-of-norms functor. We also need them in dimension $< N$, when proving the above Proposition 4.5 by induction on $N$ in n.4.4.

4.3.3. Functions $m_r$, $1 \leq r < N$.

Proposition 4.6. Suppose $K, \in \mathcal{B}^f_a(N)$ with the index parameter $\bar{u}^0(K,)$ = $(u_1^0, \ldots, u_N^0)$. Then for $1 \leq r < N$, there are non-decreasing functions $m_r : \mathbb{Z}_{\geq u^0_{r+1}} \rightarrow \mathbb{Z}_{\geq u^0_r}$ such that for any $u_1, \ldots, u_N$ such that $u_{N-1} \geq m_{N-1}(u_N), \ldots$, $u_1 \geq m_1(u_2)$, $K_{u_1u_2...u_N}$ has a standard $F$-structure.

Proof. Use induction on $N$.

Then for $K, (N - 1) \in \mathcal{B}^f_a(N - 1)$, there are functions $m_r : \mathbb{Z}_{\geq u^0_{r+1}} \rightarrow \mathbb{Z}_{\geq u^0_r}$, where $1 \leq r \leq N - 2$, such that if $u_{N-1} \geq u^0_{N-1}$, $u_{N-2} \geq m_{N-2}(u_{N-1}), \ldots$, $u_1 \geq m_1(u_2)$ then $K(N - 1)_{u_2...u_N}$ has a standard $F$-structure.

If $u \geq u^0_N$, take $v = v(u) \geq u^0_{N-1}$ from Proposition 4.5. Then define $m_{N-1} : \mathbb{Z}_{\geq u^0_N} \rightarrow \mathbb{Z}_{\geq u^0_{N-1}}$ by the relation

$$m_{N-1}(u) = \max\{v(u') \mid u^0_N < u' < u\}.$$ 

Then this collection of functions $m_r$, $1 \leq r < N$, satisfies the requirements of our proposition.

Remark. With the above notation, suppose the indices $(v^0_1, \ldots, v^0_N)$ are such that $v^0_1 \geq \cdots \geq v^0_N$ and the functions $n_r : \mathbb{Z}_{\geq v^0_{r+1}} \rightarrow \mathbb{Z}_{\geq v^0_r}$, $1 \leq r < N$, are such that $v^0_{r+1} \geq u^0_{r+1}$ and $n_r(u) \geq m_r(u)$ for all $u \geq v^0_{r+1}$. Then the proposition holds also with the indices $v^0_1, \ldots, v^0_N$ and the functions $n_{N-1}, \ldots, n_1$. In particular, we can assume (if necessary) that the functions $m_r$ from our proposition are strictly increasing.
4.3.4. Local parameters.

Suppose $K_r \in \mathcal{B}^f_a(N)$ and for $1 \leq r < N$, $m_r : \mathbb{Z}_{\geq u_r} \rightarrow \mathbb{Z}_{\geq u^0_r}$ are corresponding functions from the above proposition. We always agree to assume in this situation that $n^*(K_r) = u^0_r$ and $m_r(u^0_{r+1}) = u^0_r$ for all $1 \leq r < N$.

Let $1 \leq r \leq N$ and let indices $u_1, \ldots, u_r$ be such that $u_r \geq u_r^0$, $u_{r-1} = m_r(u_r)$, \ldots, $u_1 = m_1(u_2)$. Let $t_r^{(r)}$ be an $r$-th local parameter in the field $K_{u_1(1)}K_{u_2}(2) \ldots K_{u_r}(r)$.

**Proposition 4.7.** For any indices $u_1, \ldots, u_N$ such that $u_N \geq u_N^0$, $u_{N-1} \geq m_{N-1}(u_N)$, \ldots, $u_1 \geq m_1(u_2)$, the above introduced elements $t^{(1)}_{u_1}, \ldots, t^{(N)}_{u_N}$ give a system of local parameters in the field $K_{u_1} = K_{u_1(1)} \ldots K_{u_{N-1}}(N-1)K_{u_N}$.

**Proof.** If $N = 1$ there is nothing to prove.

If $N > 1$ we can assume by induction that $t^{(1)}_{u_1}, \ldots, t^{(N-1)}_{u_{N-1}}$ is a system of local parameters in $E = K_{u_1(1)} \ldots K_{u_{N-1}}(N-1)$.

Let $u_{N-1}' = m_{N-1}(u_N)$, $u_{N-1}' = m_{N-2}(u_{N-1}')$, \ldots, $u_1' = m_1(u_2')$. Let $E' = K_{u_1}(1) \ldots K_{u_{N-1}}(N-1)$ and set $\bar{K} = E/K_{u_N}$. Then $E' \subseteq E$ and $(K_{u'}, E') \in \text{LC}(N)$ is standard. Therefore, $(K_{u'}, E')$ is also standard, i.e. $t^{(N)}_{u_N}$ extends the system of local parameters $t^{(1)}_{u_1}, \ldots, t^{(N-1)}_{u_{N-1}}$ of $E$ to a system of local parameters of $K_{u_1} = K_{u'}E$.

The proposition is proved.

4.3.5. Construction of special extensions.

Assume that $K_r \in \mathcal{B}^f_a(N)$ is given via the above notation. Assume in addition that the functions $m_r$, $1 \leq r < N$, are strictly increasing.

For any $n \in \mathbb{Z}_{\geq 0}$, set $v_N^n = u_N^0 + n$ and define the vector $v^n = (v_1^n, \ldots, v_N^n)$ by the relations $v_{N-1} = m_{N-1}(v_N^n + 1)$, \ldots, $v_1^n = m_1(v_2^n + 1)$. Notice that for any indices $u_1, \ldots, u_N$ such that $v_1^n \leq w_r \leq v_1^n + 1$ with $1 \leq r \leq N$, the field $K_{w_1, \ldots, w_N}$ has a standard $F$-structure. Indeed, for any $1 \leq r < N$, $m_r(w_{r+1}) \leq m_r(v_{r+1}^{n+1}) = v_{r+1}^{n}$.

Set for all $n \geq 0$, $w_{r+1} = (v_1^n + 1, \ldots, v_N^n + 1)$, $O_{w^n} = O_{K_{w^n}}$ and $O_w = O_{K_w}$.

Notice that we have a natural embedding $O_{w^n} \subset O_{w^n}$. In addition, $O_{w^n} \subset O_{w^n+1}$. Indeed, $w_{N+1}^n = v_{N+1}^n + 1 = v_{N+1}^{n+1}$ and if for some $1 \leq r < N$, $u_{r+1}^n \leq u_{r+1}^{n+1}$, then

$$w_r^n = v_{r+1}^n + 1 = m_r(v_{r+1}^n + 1) + 1 = m_r(u_{r+1}^n) + 1$$

For any $u \geq u_0^0$, let $v_{K_u}$ be the canonical $N$-valuation associated with $K_u$. Then $v_{K_u} := v_{K_u}/p^n$ does not depend on the choice of $u$. We have also $1$-valuations $v_{K_u}^1 := pr_1 v_{K_u}$ and $v_{K_u}^1 = pr_1 v_{K_u}$. For any $c > 0$, set

$$m_{K_u}(c) = \{ o \in O_{C(N)} \mid v_{K_u}(o) \geq c \}.$$
**Proposition 4.8.** Let $c_1^* = c^*(u_0^0, K_\cdot)/p$. Then for all $n \geq 0$, the $p$-th power map induces a ring epimorphism

$$\mathcal{O}_{\bar{u}^{n+1}} \mod m_{K_\cdot}^1(c_1^*) \rightarrow \mathcal{O}_{u^n} \mod m_{K_\cdot}^1(c_1^*).$$

**Proof.** Remind that

$$\bar{u}^{n+1} = (u_1^{n+1}, \ldots, u_N^{n+1}) = (v_1^n + 1, \ldots, v_N^n + 1).$$

Let $1 \leq r \leq N$ and let $t_{u_r}^{(r)}$ be the $r$-th local parameter for $K_{\bar{u}^{n+1}}(r)$ from n.4.3.4. It will be sufficient to prove that its $p$-th power is congruent modulo $m_{K_\cdot}^1(c_1^*)$ to some $r$-th local parameter of the field $K_{\bar{u}^n}(r)$. By induction we can assume that $r = N$.

Let $E = K_{\bar{u}^n}$, $E' = K_{\bar{u}^n}(t_{u_r}^{(N)}) \subset K_{\bar{u}^{n+1}}$. Then $[E' : E] = p$ and both these fields have a standard $F$-structure. If $\tau \in I_{E'/E}$ and $\tau \neq \text{id}$, then

$$v_{K_{\bar{u}^{n+1}}}^1 \left( \tau t_{u_r}^{(N)} - t_{u_r}^{(N)} \right) \geq p v_{u_r}^n c^*$$

by the definition of the parameter $c^* = c^*(n^*, K_\cdot)$. This implies that all conjugates to $t_{u_r}^{(N)}$ over $E$ are congruent modulo $m_{K_\cdot}^1(c^*/p) = m_{K_\cdot}^1(c_1^*)$. Therefore, $p$-th power of $t_{u_r}^{(N)}$ is congruent modulo $m_{K_\cdot}^1(c_1^*)$ to the norm $N_{E'/E} \left( t_{u_r}^{(N)} \right)$, which is an $N$-th local parameter in $K_{\bar{u}^n}$.

The proposition is proved.

**Corollary 4.9.** With the above notation and assumptions there is a field tower

$$K_{\bar{u}^0} \subset K_{\bar{u}^1} \subset K_{\bar{u}^1} \subset \cdots \subset K_{\bar{u}^n} \subset K_{\bar{u}^{n+1}} \subset \cdots$$

such that all extensions $K_{\bar{u}^{n+1}}/K_{\bar{u}^n}$ satisfy the condition $C$ from n.2.6.

4.3.6. **Modified system of local parameters.**

As earlier, $K_\cdot \in B^f(N)$ together with the corresponding strictly increasing functions $m_r : \mathbb{Z}_{\geq u_r^0} \rightarrow \mathbb{Z}_{\geq u_r^0}$ for $1 \leq r < N$.

Define $U(m_1, \ldots, m_{N-1}) \subset \mathbb{Z}^N$ as the set of $\bar{u} = (u_1, \ldots, u_N)$ such that $u_N \geq u_0^0$, $u_N-1 \geq m_{N-1}(u_N + 1)$, $u_1 \geq m_1(u_2 + 1)$.

**Proposition 4.10.** For all $1 \leq r \leq N$ and $u \geq u_r^0$, there are $\tau_{u_r}^{(r)} \in K_u(r)$ such that

a) $\tau_{u_r+1}^{(r)} \equiv \tau_{u_r}^{(r)} \mod m_{K_\cdot}^1(c_1^*)$;

b) if $\bar{u} = (u_1, \ldots, u_N) \in U(m_1, \ldots, m_{N-1})$ then $\tau_{u_1}^{(1)}, \ldots, \tau_{u_r}^{(r)}$ is a system of local parameters in $K_u(r)$.

**Proof.** Use induction on $r$. Then it will be sufficient to define $\tau_{u}^{(N)}$ with $u \geq u_N^0$.

Set $\tau_{u_N}^{(N)} = t_{u_N}^{(0)}$, cf. n.4.3.4.

Then use induction on $n \geq 1$. Take $\tau_{u_N^{n}+n}^{(N)} \in \mathcal{O}_{\bar{u}^n}$ such that

$$\tau_{u_N^{n}+n}^{(N)} \equiv \tau_{u_N^{n}+n-1}^{(N)} \mod m_{K_\cdot}^1(c_1^*)$$

Clearly, this is an $N$-th local parameter in $K_{\bar{u}}$ and, therefore, it completes the system $\tau_{u_{N-1}}^{(N-1)}, \ldots, \tau_{u_1}^{(1)}$ to a system of local parameters in $K_{\bar{u}}$.  

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4.4. **Proof of proposition 4.5.**

Notice that there is nothing to prove if $N = 1$ and use induction on $N$ by assuming that the proposition holds in dimensions $< N$.

Therefore, we can use the result of Corollary 4.9 in dimensions $< N$. It remains to notice that if $K_{\nu_n}$ is $F$-standard then $K_{\nu_{n+1}}$ is infersal over $K_{\nu_n}$. So, Proposition 4.5. follows from the case b) of the procedure of elimination of wild ramification from n.2.3.

5. **Family of fields $X(K_r)$, $K_r \in \mathcal{B}^f_o(N)$.**

5.1. **Fontaine’s field $R_0(N)$.**

Recall that objects $K \in \text{LF}_0(N)$ are realised as subfields in $\mathbb{C}(N)_p$. They are closed subfields with induced $F$-structure and $P$-topology. Any $K \in \text{LF}_0(N)$ has a canonical valuation $v_K$ of rank $N$.

Notice that if $K' \in \text{LF}_0(N)$ then $v_{K'} = \bar{\alpha}v_K$ with some $\bar{\alpha} \in \mathbb{Q}^N$, $\bar{\alpha} > \bar{0}$, and therefore, all such valuations belong to the same class of equivalent valuations. If $K \in \text{LF}_0(N)$ and $v_K$ is the extension of its canonical valuation of rank $N$ to $\mathbb{C}(N)_p$ then

$$\mathcal{O}_{\mathbb{C}(N)_p} = \{o \in \mathbb{C}(N)_p \mid v_K(o) \geq \bar{0}_N\}.$$ 

Set $R(N) = \lim_n(\mathcal{O}_{\mathbb{C}(N)_p} \pmod{p})_n$, where connecting morphisms are induced by the $p$-th power map. Then $R(N)$ is an integral domain and its fraction field $R_0(N)$ is a perfect field of characteristic $p$. The $F$-structure on $\mathbb{C}(N)_p$ induces an $F$-structure on $R_0(N)$ given by the decreasing sequence of subfields

$$R_0(N) \supset R_0(N-1) \supset \cdots \supset R_0(1) \supset R_0(0).$$

In addition, the field $R_0(0)$ consists of the sequences $\{\alpha^{p^{-n}}\}_{n \geq 0}$, where $\alpha \in \bar{\mathbb{F}}_p$. The map $\{\alpha^{p^{-n}}\}_{n \geq 0} \mapsto \alpha$ identifies $R_0(0)$ with $\bar{\mathbb{F}}_p$, in particular, any finite field of characteristic $p$ can be embedded naturally into $R_0(N)$.

Notice that $R = R(1)$ and $\text{Frac } R = R_0(1)$ are original notations introduced for the corresponding 1-dimensional objects by J.-M. Fontaine.

Let $K_r \in \mathcal{B}^a_o(N)$. It determines a valuation of rank $N$ on $\mathbb{C}(N)_p$ given by the formula $v_{K_r} = \lim_{n \to \infty} (v_{K_n}/p^n)$. This determines the valuation $v_{R,K_r}$ of rank $N$ on $R_0(N)$ such that if $\bar{r} = (r_n)_{n \geq 0} \in R(N)$ then

$$v_{R,K_r}(\bar{r}) = \lim_{n \to \infty} p^n v_{K_r}(\hat{r}_n) = \lim_{n \to \infty} v_{K_n}(\hat{r}_n)$$

where $\hat{r}_n \in \mathcal{O}_{\mathbb{C}(N)_p}$ is such that $\hat{r}_n \pmod{p} = r_n$.

Notice that if $L_r \in \mathcal{B}^a_o(N)$ then $v_{R,L_r} = \bar{\alpha}v_{R,K_r}$, with $\bar{\alpha} \in \mathbb{Q}^N$, $\bar{\alpha} > \bar{0}$. Therefore, the equivalence class of such valuations does not depend on the choice of $K_r$.

If $c > 0$, then (as earlier)

$$\mathcal{M}_{R,K_r}(c) = \{o \in R(N) \mid v^1_{R,K_r}(o) \geq c\}$$

where $v^1_{R,K_r} = \text{pr}_1 v_{R,K_r}$.

The following proposition is just an easy consequence of the above definitions.
Proposition 5.1. For any $c > 0$ such that $p \in m^1_{K_u}(c)$,
a) $R(N) = \lim_{\rightarrow}(O_{\mathbb{C}(N)} \mod m^1_{K_u}(c))$, where connecting morphisms are induced by 
the $p$-th power map;
b) for any $u \geq 0$, the $u$-th projection $\text{pr}_u : R(N) \rightarrow O_{\mathbb{C}(N)} \mod m^1_{K_u}(c)$ induces a 
ring identification of $R(N) \mod m^1_{R,K}(p^uc)$ and $O_{\mathbb{C}(N)} \mod m^1_{K_u}(c)$.

Remark. $O_{\mathbb{C}(N)}$ is equipped with the $P$-topology given by the inductive limit of $P$-topologies on all $K \in LF_0(N)$. This topology induces the $P$-topology on $R(N)$ and $R_0(N)$.

5.2. The family of fields $X(K_u)$.
Suppose $K_u \in B^f_a(N)$ with the parameters $u^0(K_u) = (u^0_1, \ldots, u^0_N)$ and $c^* = c^*(u^0_N, K_u)$. As earlier in n.4.3.3, choose for all $1 \leq r < N$ the corresponding strictly 
increasing functions $m_r : \mathbb{Z}_{\geq u^0_{r+1}} \leftarrow \mathbb{Z}_{\geq u^0_r}$ and $r$-th local parameters $\tau_u^r \in K_u(r)$, 
where $u \geq u^0_r$.

Set (as earlier, $c^*_r = c^*/p$)

$$\tau^r = (\tau_u^r \mod m^1_{K_u}(c^*_r))_{u \geq u^0_r} \subset \lim_{\rightarrow}(O_{\mathbb{C}(N)} \mod m^1_{K_u}(c^*_r)) = R(N).$$

Let $k = k(K_u)$ be the last residue field of $K_{u^0_N}$ (this is also the residue field for 
all $K_u$ with $u \geq u^0_N$). As it was mentioned in n.5.1, $k$ can be naturally indentified 
with a subfield in $R_0(0) \subset R_0(N)$.

Proposition 5.3. The correspondences $T_1 \mapsto \tau^{(1)}$, \ldots, $T_N \mapsto \tau^{(N)}$ determine a 
unique continuous embedding of the $N$-dimensional local field $k((T_1)) \ldots ((T_N))$ into 
$R_0(N)$. Its image is an $N$-dimensional local subfield $K$ in $R_0(N)$ with the system 
of local parameters $\tau^{(1)}, \ldots, \tau^{(N)}$.

Proof. We need the following obvious lemma.

Lemma 5.4. Suppose $L \in LF_0(N)$ has a standard $F$-structure, which is 
compatible with given local parameters $t_1, \ldots, t_N$. Let $c > 0$ and $m^1_L(c) = \{o \in O_{\mathbb{C}(N)} \mid \text{pr}_1 v_L(o) \geq c\}$. Then any $o \in O_L$ can be uniquely presented modulo 
m^1_L(c) in the form

$$\sum_{a_1 < c} [\alpha_a] t_1^{a_1} \ldots t_N^{a_N}.$$

Remark. The coefficients $[\alpha_a]$ are the Teichmuller representatives of the elements of 
the last residue field of $L$ and satisfy the standard restrictions from the beginning 
of n.1.1.

Continue the proof of proposition 5.3.
We prove first that the power series

$$\sum_{\bar{a} \geq 0_N} \alpha_{\bar{a}} \tau^{(1)}_{a_1} \ldots \tau^{(N)}_{a_N}$$

converges in $R(N)$ if its coefficients $\alpha_{\bar{a}}$ satisfy the restrictions described in n.1.1. 
This is equivalent to the fact that for all $u \geq u^0_N$, the series

$$\sum_{\bar{a} \geq 0_N} [\alpha_{\bar{a}}] p^{-u} \tau^{(1)}_{a_1} \ldots \tau^{(N)}_{a_N}$$

converges in $R(N)$.
converge to elements \( f_u \in O_{\mathcal{C}(N)} \) such that \( f_u^p \equiv f_{u+1} \mod m_{K_i}^1(c_i^*) \).

Let \( \bar{u}^n = (u_1^n, \ldots, u_{N-1}^n, u_N^n) \) with \( u_N^n = u \). Then for \( 1 \leq r \leq N \), it holds \( u \leq u_r^n \) and

\[
\tau_u^{(r)} \equiv \tau_u^{(r)p^n-u} \mod m_{K_i}^1(c_i^*)
\]

This means that the above series (4) can be expressed in terms of local parameters of the field \( K_{\bar{u}^n} \), its coefficients \( \alpha_1^{p^n-u} \) satisfy the restrictions from n.1.1 and, therefore, these series converge in \( O_{\mathcal{G}^N} \subset O_{\mathcal{C}(N)} \).

Then the uniqueness property from lemma 5.4 implies that

\[
f_{u+1}^p \equiv f_u \mod m_{K_i}^1(c_i^*)
\]

and the series (3) converges in \( R(N) \).

Even more, Lemma 5.4 implies that any element from \( R(N) \) can be presented in at most one way as a sum of the series (3). So, the image \( \mathcal{K} \) of \( k((T_N)) \) is an \( N \)-dimensional local field with the set of local parameters \( \tau^{(1)}, \ldots, \tau^{(N)} \).

The proposition is proved.

Notice that the above fields \( \mathcal{K} \subset R_0(N) \) are not uniquely determined by a given \( K_i \in B^{f,a}(N) \). They depend also on the choice of functions \( m_1, \ldots, m_{N-1} \) and the choice of compatible systems of local parameters \( \{\tau_u^{(r)}\} \) for \( u \geq 0 \), \( 1 \leq r \leq N \). Denote by \( X(K_i; m_1, \ldots, m_{N-1}) \) the family of all subfields \( \mathcal{K} \) which can be constructed for a given tower by the use of given invariants \( \bar{u}^0(K_i) \) together with an appropriate choice of strictly increasing functions \( m_1, \ldots, m_{N-1} \). Notice that taking a bigger invariant \( \bar{u}^0(K_i) \) together with the contraction of the domain of definition of functions \( m_1, \ldots, m_{N-1} \) doesn’t affect this family. Clearly, for a given tower \( K_i \), the sets \( X(K_i; m_1, \ldots, m_{N-1}) \) form an inductive system. Its inductive limit will be denoted by \( X(K_i) \).

5.3. The categories \( LF_R(N) \) and \( \bar{LF}_R(N) \).

Consider the category \( LF_R(N) \) of all \( N \)-dimensional closed subfields \( \mathcal{K} \) in \( R_0(N) \) together with the induced \( F \)-structure given by the subfields of \( r \)-dimensional constants \( \mathcal{K}(r) = R_0(r) \cap \mathcal{K} \), \( 0 \leq r \leq N \). If \( \mathcal{K}, \mathcal{L} \in LF_R(N) \) then \( \text{Hom}_{LF_R(N)}(\mathcal{K}, \mathcal{L}) \) consists of compatible with \( F \)-structure and \( P \)-continuous morphisms \( f : R_0(N) \rightarrow R_0(N) \) such that \( f(\mathcal{K}) \subset \mathcal{L} \).

Suppose that \( v^1 \) is a 1-dimensional valuation coinciding with one of (equivalent valuations) \( pr_{1, v^1 R, K_i} \), where \( K_i \in B^{f,a}(N) \).

For any \( v^1 \)-adic closed subfield \( \mathcal{L} \) in \( R_0(N) \) denote by \( \mathcal{R}(\mathcal{L}) \) the \( v^1 \)-adic closure of the maximal inseparable extension of \( \mathcal{L} \) in \( R_0(N) \).

**Definition.** If \( \mathcal{K}, \mathcal{L} \in LF_R(N) \) then \( \mathcal{K} \sim \mathcal{L} \) if for \( 1 \leq r \leq N \), \( \mathcal{K}(r)\mathcal{R}(\mathcal{K}(r-1)) = \mathcal{L}(r)\mathcal{R}(\mathcal{L}(r-1)) \), where the composite is taken in the category of \( v^1 \)-adic closed subfields of \( R_0(N) \).

Clearly, the above defined relation \( \sim \) is an equivalence relation. Denote by \( \bar{LF}_R(N) \) the category such that its objects are equivalence classes \( \text{cl}(\mathcal{K}) \) of all \( \mathcal{K} \in LF_R(N) \) and for any \( \text{cl}(\mathcal{K}), \text{cl}(\mathcal{L}) \in \bar{LF}_R(N) \), \( \text{Hom}_{\bar{LF}_R(N)}(\text{cl}(\mathcal{K}), \text{cl}(\mathcal{L})) \) consists of compatible with \( F \)-structure and \( P \)-continuous field morphisms \( f : R_0(N) \rightarrow R_0(N) \) such that for any \( 1 \leq r \leq N \), \( f(\mathcal{K}(r)) \subset \mathcal{L}(r)\mathcal{R}(\mathcal{L}(r-1)) \).

**Remark.** The usual “1-dimensional” Krasner’s Lemma implies that:

- if \( \mathcal{L}_1, \mathcal{L} \in LF_R(N) \), \([\mathcal{L}_1 : \mathcal{L}] = m \) and \( \mathcal{L}' \sim \mathcal{L} \) then there is a unique \( \mathcal{L}'_1 \in LF_R(N) \).
such that $\mathcal{L}_1' \sim \mathcal{L}_1$ and $\mathcal{L}_1'$ is an extension of $\mathcal{L}'$ of degree $m$.

In particular, we can use the concepts of finite algebraic, separable, Galois and purely inseparable extensions in $\widetilde{LF}_R(N)$.

5.4. Identification of elements from $X(K_r)$, $K_r \in \mathcal{B}^{fa}(N)$.

**Proposition 5.5.** Suppose $K_r \in \mathcal{B}^{fa}(N)$. Then any two elements from $X(K_r)$ represent the same object in $\widetilde{LF}_R(N)$.

**Proof.** Let $\bar{u}^0(K_r) = (u^0_1, \ldots, u^0_N)$ and $c^*_1 = c^*(u^0_N, K_r)/p$.

Suppose $\mathcal{K} \in X(K_r)$ is obtained via a choice of strictly increasing functions $m_r: \mathbb{Z}_{\geq u^0_{r+1}} \rightarrow \mathbb{Z}_{\geq u^0_r}$, and a special system of local parameters $\tau^{(r)}_u$, $1 \leq r < N$, $u \geq u^0_1$.

Take some $u \geq u^0_N$ and choose $\bar{u} = (u_1, \ldots, u_{N-1}, u) \in U(m_1, \ldots, m_{N-1})$.

Set $\mathcal{K}_{\bar{u}} = \mathcal{K}(\sigma^{u-u_{N-1}}K(N-1)) \cdots (\sigma^{u-u_1}K(1))$. (Here $\sigma$ is as usually the $p$-th power map.) Then

$$\sigma^{u-u_1} \tau^{(1)}, \ldots, \sigma^{u-u_{N-1}} \tau^{(N-1)}, \tau^{(N)}$$

is a system of local parameters in $\mathcal{K}_{\bar{u}}$ which is compatible with a given (standard) $F$-structure of $\mathcal{K}_{\bar{u}}$.

It is easy to see that for $1 \leq r \leq N$, the correspondences $\sigma^{u-u_r} \tau^{(r)} \mapsto \tau^{(r)}$ give the identification

$$\psi_{\bar{u}} : \mathcal{O}_{\mathcal{K}_{\bar{u}}} \mod m^1_{R, \mathcal{K}_r}(p^u c^*_1) \simeq \mathcal{O}_{\bar{u}} \mod m^1_{K_r}(c^*_1).$$

If $\bar{u}' = (u'_1, \ldots, u'_{N-1}, u) \in U(m_1, \ldots, m_{N-1})$ is such that $u'_r \geq u_r$ for all $1 \leq r \leq N$, then $\psi_{\bar{u}}$ and $\psi_{\bar{u}'}$ are compatible via natural inclusions $\mathcal{K}_{\bar{u}} \subset \mathcal{K}_{\bar{u}'}$ and $\mathcal{O}_{\bar{u}} \subset \mathcal{O}_{\bar{u}'}$. Therefore, the $u$-th projection $\text{pr}_u : R(N) \rightarrow \mathcal{O}_{\mathcal{C}(N)_p} \mod m^1_{K_r}(c^*_1)$ induces the identification

$$\psi_u : \mathcal{O}_{\mathcal{K}R(K(N-1))} \rightarrow \mathcal{O}^{(u)} \mod m^1_{K_r}(c^*_1)$$

where $\mathcal{O}^{(u)}$ is the valuation ring of the composite of all $\mathcal{K}_{\bar{u}}$ with $\bar{u}$ running over the set of all $\bar{u} = (u_1, \ldots, u_{N-1}, u_N)$ such that $u_N = u$.

In order to understand the relation between different $\psi_u$, notice that if $\bar{u}' = (u'_1, \ldots, u'_{N-1}, u + 1) \in U(m_1, \ldots, m_{N-1})$, then $\bar{u} = (u'_1 - 1, \ldots, u'_{N-1} - 1, u) \in U(m_1, \ldots, m_{N-1})$ (because the functions $m_r$ are strictly increasing) and $\mathcal{K}_{\bar{u}} = \mathcal{K}_{\bar{u}'}$. This implies that $\psi_{\bar{u}}$ and $\psi_{\bar{u}'}$ fit into a commutative diagram via the natural projection

$$\mathcal{O}_{\mathcal{K}_{\bar{u}}} \mod m^1_{R, \mathcal{K}_r}(p^{u+1} c^*_1) \rightarrow \mathcal{O}_{\mathcal{K}_{\bar{u}}} \mod m^1_{R, \mathcal{K}_r}(p^{u} c^*_1)$$

and the restriction of the transition morphism of the projective system $\mathcal{O}_{\mathcal{C}(N)_p} \mod m^1_{K_r}(c^*_1)$ from the definition of $R(N)$. Therefore, $\lim \psi_u$ identifies $\mathcal{O}_{\mathcal{K}R(K(N-1))}$ with $\lim \mathcal{O}^{(u)} \mod m^1_{K_r}(c^*_1) \subset R(N)$. In particular, $\mathcal{K}R(K(N-1))$ does not depend on the choice of $\mathcal{K}$ in $X(K_r)$.

The proposition is proved.

5.5. Let $K_r, L_r \in \mathcal{B}^{fa}(N), \mathcal{K} \in X(K_r)$ and $\mathcal{L} = X(L_r)$. Let $\widetilde{K}$ and $\widetilde{L}$ be the $p$-adic completions of $\bigcup_{m \geq 0} K_m$ and, resp., $\bigcup_{m \geq 0} L_m$. Notice that if $\mathcal{K}$ is purely inseparable over $\mathcal{L}$ then $\tilde{K} = \tilde{L}$. Inversely, we have the following property.
Proposition 5.6. With the above notation, if \( \overline{K} = \overline{L} \) then \( R(K) = R(L) \).

The proof is straightforward.

6. Separable extensions in \( B^{fa}(N) \) and \( \overline{LF}_R(N) \).

6.1. In this subsection we prove that the correspondence \( K, \mapsto cl(K) \), where \( K \in X(K_n) \), transforms finite separable extensions in \( B^{fa}(N) \) to finite separable extensions of the same degree in \( \overline{LF}_R(N) \).

Proposition 6.1. Suppose \( L, K, \in B^{fa}(N) \) and \( L, \supset K, \) is separable and finite of degree \( d(L/K) = d \). Then for any \( \overline{K} \in X(K_n) \), there is an \( \overline{L} \in X(L_n) \) such that \( \overline{L} \) is a separable extension of \( K \) of degree \( d \).

Proof. By induction on \( N \geq 0 \), we can assume that \( K, (N - 1) = L, (N - 1) \).

We can assume also that:

- \( \bar{u}^0(K, r) = \bar{u}^0(L, r) = (u_0^0, \ldots, u_N^0) \);
- \( c^*(u_0^0, K), = c^*(u_0^N, L, r) = pc^*_r \);
- there are strictly increasing functions \( m_r : \mathbb{Z}_{\geq u_0^r} \to \mathbb{Z}_{\geq u_0^r} \), where \( 1 \leq r < N \), such that \( m_r(u_0^0 + 1) = u_0^r \) and if \( \bar{u} \in U(m_1, \ldots, m_{N-1}) \) then both \( L_{\bar{u}} \) and \( K_{\bar{u}} \) have a standard \( F \)-structure;
- \( K \in X(K_n; m_1, \ldots, m_{N-1}) \);
- for all \( u \geq u_0^N \), the Herbrand functions of extensions \( L_u/K_u \) coincide and are equal to \( \varphi_{L_u/K_u} \);
- the initial choice of \( u_0^N \) provides us with the inequality \( \text{pr}_1(j(L_u/K_u)) + \delta_1 N < p^{u_0^N}c^*_1/2 \) (here and everywhere below \( \delta_1 N \) is the Kronecker symbol).

As usually, we denote by \( (i(L_u/K_u), j(L_u/K_u)) \) the last edge point of the graph of \( \varphi_{L_u/K_u} \), and use the notation \( \text{pr}_1(i(L_u/K_u)) = i^1, \text{pr}_1(j(L_u/K_u)) = j^1, \text{pr}_1(e(L_u/K_u)) = e^1 \).

Consider the corresponding sequence of multi-indices \( \bar{u}^0, \bar{u}^0, \ldots, \bar{u}^n, \bar{v}^n, \ldots \) from n.4.3.5 and the corresponding field towers

\[
L_{\bar{u}^0} \subset L_{\bar{u}^1} \subset L_{\bar{u}^2} \subset \cdots \subset L_{\bar{u}^n} \subset L_{\bar{v}^n} \subset \ldots
\]

\[
K_{\bar{v}^0} \subset K_{\bar{v}^1} \subset K_{\bar{v}^2} \subset \cdots \subset K_{\bar{v}^n} \subset K_{\bar{v}^n} \subset \ldots
\]

For any \( u \geq u_0^N \), set \( n = n(u) = u - u_0^N \). So, \( \bar{u}^n = (u_0^n, \ldots, u_{N-1}^n, u) \).

Consider \( a_{iu} \in \mathcal{O}_{K_{\bar{v}^n}} \), where \( 1 \leq i \leq d \) and \( u \geq u_0^N \), such that

- there is \( N \)-th local parameter \( \eta_{u_0^N} u^N \) in \( L_{\bar{u}^0} \) such that

\[
\eta_{u_0^N} u^N a_{1u} \eta_{u_0^N} u_{u_0^N} + \cdots + a_{du} u_{u_0^N} = 0;
\]

- for all \( 1 \leq i \leq d \) and \( u \geq u_0^N \), \( a_{iu} = a_i^{u_{u_0^N}} \mod m_{K_i}^1(c_i^1) \) or, equivalently, there are \( \alpha_i \in \mathcal{O}_K \) such that \( \text{pr}_u(\alpha_i) = a_{iu} \mod m_{K_i}^1(c_i^1) \), where \( \text{pr}_u \) is the projection from \( R(N) = \lim_\mu(\mathcal{O}_{C(N)}\mu \mod m_{K_i}^1(c_i^1))_u \) to its \( u \)-th component.
Let \( F_u(T) = T^d + a_1 u T^{d-1} + \cdots + a_d u \), where \( u \geq u_N^0 \). Then all \( F_u(T) \) are \( N \)-th Eisenstein polynomials in \( \mathcal{O}_{K_u^n} [T] \) (i.e. their images in \( \mathcal{O}_{K^{(N-1)}_u} [T] \) are Eisenstein polynomials, where \( K_u^{(N-1)} \) is the pre-last residue field of \( K_u^n \) ) and \( F_u^0 (\eta_u^0) = 0 \). We want to prove that for \( c_2^* = e^1 c_1^*/2 \), there are \( N \)-th local parameters \( \eta_u \in L_u^n \) with \( u > u_N^0 \), such that for all \( u \geq u_N^0 \),

\[
\eta_u^p + \eta_u \in m_{L_u}^1 (p^u c_2^*)
\]

(notice that \( \eta_u^0 \) has been chosen earlier).

Suppose \( u \geq u_N^0 \) and we have already constructed such elements \( \eta_v \) for all \( v \) such that \( u_N^0 \leq v \leq u \).

**Lemma 6.2.** If \( \theta_u + 1 \in \mathcal{O}_{C(N)_p} \) is a root of \( F_u+1(T) \) then there is a unique root \( \theta_u \in \mathcal{O}_{C(N)_p} \) of \( F_u(T) \) such that \( \theta_u - \theta_u^p + 1 \in m_{L_u}^1 (p^u c_2^*) \).

**Proof of lemma.** Clearly, \( F_u(\theta_u^p) \in m_{K_u}^1 (p^u c_1^*) \). Let \( v_{K_u} (F_u(\theta^p)) = j_u + (0, \ldots, 0, 1) \). Then by assumptions from n.6.1

\[
pr_1 (j_u) \geq p^u c_1^* - \delta_1 N > 2j^1 + \delta_1 N \geq j^1.
\]

Therefore, \( j_u > j(L_u/K_u) = j(L/K) \) and we can apply Krasner’s lemma, cf. n.3.4. This lemma gives the existence of a unique root \( \theta_u \in \mathcal{O}_{C(N)_p} \) of \( F_u(T) \) such that \( v_{L_u} (\theta_u^p - \theta_u) = i_u + (0, \ldots, 0, 1) \) with \( \varphi_{L/K}(i_u) = j_u \). Because \( j_u \geq j(L_u/K_u) \), we have

\[
\frac{j_u - j(L_u/K_u)}{i_u - i(L_u/K_u)} = e^{-1}(L_u/K_u)
\]

and this implies

\[
pr_1 (i_u) \geq e^1 (pr_1 (j_u - j^1) \geq e^1 (p^u c_1^* - \delta_1 N - j^1) > e^1 (p^u c_1^* - \frac{1}{2} p^u N c_2^*) \geq \frac{1}{2} e^1 p^u c_1^* = p^u c_2^*.
\]

The lemma is proved.

Notice that \( L_u^n + 1 = L_u^n K_u^n + 1 = K_u^n + 1(\theta_u) \) is of degree \( d \) over \( K_u^n + 1 \) and, therefore, \( F_u(T) \) is still irreducible over \( K_u^n + 1 \). Therefore, there is a \( \tau \in \Gamma_{K_u^n + 1} \) such that \( \tau(\theta_u) = \eta_u \). Take \( \eta_{u+1} = \tau(\theta_{u+1}) \). Then the uniqueness of \( \theta_u \) in the above lemma implies that the field \( K_u^n + 1(\eta_{u+1}) \) contains the field \( K_u^n + 1(\eta_u) = L_u^n + 1 \). Therefore, these fields are equal, because they are both of the same degree \( d \) over \( K_u^n + 1 \).

Finally, \( \eta_{u+1} \) is an \( N \)-th local parameter in \( L_u^n + 1 \) because \( F_{u+1} \) is an \( N \)-th Eisenstein polynomial in \( \mathcal{O}_{K^n + 1} [X] \) and the existence of the sequence \( \eta_u \), \( u \geq u_N^0 \), is proved.

Let

\[
\eta = \lim_u \eta_u \in \lim_u (\mathcal{O}_{C(N)_p} \mod m_{L_u}^1 (c_2^*))_u = R(N)
\]

Then \( \eta \) is a root of \( N \)-th Eisenstein polynomial

\[
F(T) = T^d + \alpha_1 T^{d-1} + \cdots + \alpha_d \in \mathcal{O}_{K} [T].
\]

Therefore, \( \mathcal{L} = K(\eta) \) is of degree \( d \) over \( K \), \( \mathcal{L} \) has standard \( F \)-structure and \( \eta \) is its \( N \)-th local parameter. Clearly, \( \mathcal{L} \in X(L; m_1, \ldots, m_{N-1}) \).
We now prove that $L$ is separable over $K$. Indeed, notice first that

a) any other root of $F_{u_0}$ equals $\tau \eta_u$ for a suitable automorphism $\tau$ of $\mathbb{C}(N)$ such that $\tau|_{K_1} = \text{id}$ for all $n \geq 0$;

b) with the above notation, for a sufficiently large $u$, $i^1 + \delta_1N < p^u c_2^*$ and, therefore, $\tau \eta_u \not\equiv \eta_u \mod m_{L_1}(c_2^*)$.

Therefore, $\tau \eta := \lim \tau \eta_u \in R(N)$ is again a root of $F(X)$ which is different from $\eta$. Therefore, $F(X)$ has $d$ distinct roots in $R_0(N)$.

The proposition is proved.

**Corollary 6.3.** Under assumptions from the above proposition:

a) there is a natural identification of the set of all isomorphic embeddings $\iota$ of $L$, into $\mathbb{C}(N)$ such that $\iota|_{K_1} = \text{id}$ and the set of all isomorphic embeddings $\iota : L \to R_0(N)$ such that $\iota|_K = \text{id}$;

b) $\varphi_{L_1}/K_1 = \varphi_{L}/K$.

6.2. With the above notation we are going to prove now that for a sufficiently large separable extension $E_1$ of $K_1$, the appropriate $E \in \mathcal{X}(E_1)$ contains any given separable extension of $K$ in $R_0(N)$.

**Proposition 6.4.** Suppose $K_1 \in B^{f_0}(N)$, $K \in X(K_1)$ and $L$ is a finite separable extension of $K$ with standard $F$-structure such that $K(N-1) = L(N-1)$. Then there is an $L_1 \in B^{f_0}(N)$ and a field embedding $\iota : L \to R_0(N)$ such that

a) $L$ is a separable extension of $K$, of degree $d = [L : K]$;

b) $\iota(L) \in X(L_1)$.

**Proof.** We can assume that:

— there are parameters $\bar{u}^0(K_1) = (u_1^0, \ldots, u_N^0)$, $c^*(u_0^0, K_1) = pc_1^*$ and strictly increasing functions $m_r : \mathbb{Z}_{\geq u_0^0} \to \mathbb{Z}_{\geq u_0^0}$, where $1 \leq r < N$, such that $K \in \mathcal{X}(K_1, m_1, \ldots, m_{N-1})$;

— $\mathcal{O}_L = \mathcal{O}_K[\theta]$ where $\theta$ is a root of $N$-th Eisenstein polynomial $\mathcal{F}(T) = T^d + \alpha_1 T^{d-1} + \cdots + \alpha_d \in \mathcal{O}_K[T]$;

— $2v_1^1(D(\mathcal{F})) < p^{u_0^0} c_1^* - 1$, where $D(\mathcal{F})$ is the discriminant of $\mathcal{F}$ over $\mathcal{O}_K$.

As earlier consider the sequence $\bar{u}^0 = u^0(K_1), \bar{u}^1, \ldots, \bar{u}^n, \ldots$ and set $u = n + u_0^0$. For $u \geq u_0^0$, introduce the polynomials

$$F_u(T) = T^d + a_1 u T^{d-1} + \cdots + a_d u \in \mathcal{O}_{K_u}[T]$$

where $a_i u \mod m_{K_u}(c_1^*) = \text{pr}_{u^0} (\alpha_i u^n)$ for $1 \leq i \leq d$. Notice that for $u \geq u_0^0$,

$$D(F_u) \mod m_{K_u}(c_1^*) = \text{pr}_{u^0} D(\mathcal{F}) \neq 0.$$

For $u \geq u_0^0$, we will prove the existence of roots $\eta_u \in \mathcal{O}_{\mathbb{C}(N)}$ of $F_u(T)$ such that if $M_u = K_{u^0}(\eta_u)$, then

1) $\eta_u^0$ is $N$-th local parameter in $M_u^0$;

2) $\eta_u - \eta_{u+1}^0 \in m_{M_u}(p^u c_1^*/2)$. 

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Suppose such roots $\eta_u, \ldots, \eta_i$ have been already constructed.

Let $\text{pr}_1 \overline{e}(K_{\tilde{a}^n}/K_u) = e_{1u}$.

Let $\theta_u+1 \in \mathcal{O}_C(N)_{\mathfrak{p}}$ be a root of $F_{u+1}(T)$. Then $F_u(\theta_{u+1}^p) \in \mathcal{m}_1(K_{\tilde{a}^n})(e_{1u}p^u c_1^*)$ and, therefore, $v_{K_{\tilde{a}^n}}(F_u(\theta_{u+1}^p)) = j_u + (0, \ldots, 0, 1)$ with
\[
\text{pr}_1(j_u) + \delta_{1N} \geq e_{1u}p^u c_1^*.
\]

Lemma 6.5. $j_u > j(M_u/K_{\tilde{a}^n})$.

Proof of lemma. From Corollary 3.5 we have
\[
j(M_u/K_{\tilde{a}^n}) \leq 2v_{K_{\tilde{a}^n}}(D(F_u)) = 2\overline{e}(K_{\tilde{a}^n}/K_u)v_K(D(F))
\]
This implies
\[
\text{pr}_1(j(M_u/K_{\tilde{a}^n})) \leq 2e_{1u}v_K^1(D(F)) < e_{1u}(p^u c_1^* - 1) \leq e_{1u}p^u c_1^* - 1 \leq \text{pr}_1(j_u)
\]
The lemma is proved.

Continue the proof of our proposition.

The above lemma implies the existence of a unique root $\theta_u$ of $F_u$ such that
\[
v_{M_u}(\theta_{u+1}^p - \theta_u) = i_u + (0, \ldots, 0, 1)
\]
where $\varphi_{M_{\tilde{a}^n}/K_{\tilde{a}^n}}(i_u) = j_u$. Similarly to the proof of Lemma 6.2
\[
\frac{j_u - j(M_u/K_{\tilde{a}^n})}{i_u - i(M_u/K_{\tilde{a}^n})} = \overline{e}(M_u/K_{\tilde{a}^n})
\]
where $\overline{e}(M_u/K_{\tilde{a}^n}) = (1, \ldots, 1, d)$. Applying Corollary 3.4 we obtain
\[
i_u = (1, \ldots, 1, d)j_u - v_{K_{\tilde{a}^n}}(D(F_u)) + (0, \ldots, 0, d - 1).
\]
Therefore,
\[
\text{pr}_1(i_u) + \delta_{1N} \geq \text{pr}_1 j_u + \delta_{1N} - v_{K_{\tilde{a}^n}}^1(D(F_u)) \geq e_{1u}p^u c_1^* - e_{1u}v_K^1(D(F)) \geq e_{1u}p^u c_1^*/2
\]
i.e. $\eta_u - \eta_{u+1}^p \in \mathcal{m}_1(M_u)(p^u c_1^*/2)$.

As earlier, the uniqueness of $\theta_u$ implies the existence of $\theta_{u+1} := \eta_{u+1}$ such that $\theta_u = \eta_u$ and the required sequence $\{\eta_u \mid u \geq u_0^N\}$ is constructed.

The uniqueness property of $\theta_u = \eta_u$ implies also that $K_{\tilde{a}^{u+1}}(\eta_{u+1}) = K_{\tilde{a}^{u+1}}(\eta_u)$.

Consider the tower $M_u = M_{u^N}(K)$. Then
\begin{itemize}
  \item $M_u \in \mathcal{B}^{\mathfrak{a}}(N)$ and has parameters $\bar{\eta}^*(K)$ and $c^*(u_0^N, K)/2$;
  \item $K_{\bar{\eta}}(\eta) \in X(M_u; m_1, \ldots, m_{N-1})$, where $\eta = \lim u_0$ is a root of $\mathcal{F}(T)$ in $R_0(N)$. The choice of this root $\eta$ of $\mathcal{F}(T)$ determines an embedding of $\mathcal{L}$ into $R_0(N)$ which induces the identity on $\mathcal{K}$;
  \item by taking $L_u = K_{u^N}(\eta_{u^N})K_u$, we obtain a separable extension of $K_u$ with the parameters $u_0^N$ and $c^*(u_0^N, K)/2$ such that $\mathcal{L} \in X(L_u)$.
\end{itemize}
The proposition is proved.
**Corollary 6.6.** Suppose $K, \in \mathcal{B}^{f,a}(N)$ with parameters $\tilde{w}^{0}(K, \iota) = (u_{1}, \ldots, u_{N})$ and $c^{*}(u_{N}^{0}, K_{i})$. Suppose that $\mathcal{K} \in X(K)$ and $L/K$ is a finite separable extension in $R_{0}(N)$ with standard $F$-structure. Then there is an $L_{\iota} \in \mathcal{B}^{f,a}(N)$ such that

a) $L_{\iota}$ is a finite separable extension of $K_{i}$;

b) $L \in X(L_{\iota})$;

c) $L_{\iota}$ has parameters of the form $v^{0} = v^{0}(L_{\iota})$ and $c^{*}(v^{0}, L_{\iota}) = c^{*}(u_{N}^{0}, K_{i})/2^{N}$.

**Proof.** Apply the construction from the proof of the above proposition to the sequence of extensions

$$\mathcal{K} \subset \mathcal{K}(L_{1}) \subset \cdots \subset \mathcal{K}(L_{N}) = L$$

and notice that for sufficiently large first parameters $n_{1}^{*} = n^{*}(L_{1}(i))$, where $1 \leq i \leq N$, the second parameter can be taken in the form $c^{*}(n_{1}^{*}, L_{1}(i)) = c^{*}(u_{N}^{0}, K_{i})/2^{N}$.

**Corollary 6.7.** The correspondence $K_{i} \mapsto cl(K) \in \tilde{\mathcal{L}}(N)$, where $K_{i} \in \mathcal{B}^{f,a}(N)$, induces the identification of absolute Galois groups $\psi: \Gamma_{K} \longrightarrow \Gamma_{K}$ (here $\tilde{K}$ is the $p$-adic closure of the $\cup_{m \geq 0}K_{m}$). This identification is compatible with ramification filtrations, i.e. for any $j \in J(N)$,

$$\Gamma_{K_{0}}^{(j)} \cap \Gamma_{\tilde{K}} = \Gamma_{\tilde{K}}^{(\varphi_{K_{0}}/K_{0})^{(j)}}$$

(where $\varphi_{K_{0}}/K_{0} = \lim_{m \rightarrow \infty} \varphi_{K_{m}/K_{0}}$).

6.3. The above results give that if $\mathcal{K} \in X(K)$ with $K_{i} \in \mathcal{B}^{f,a}(N)$, then $R_{0}(N)$ contains a separable closure of $\mathcal{K}$. Because $R_{0}(N)$ is perfect the algebraic closure of $\mathcal{K}$ in $R_{0}(N)$ is algebraically closed. Even more, $R_{0}(N)$ is $v_{K}^{1}$-complete, therefore, $R_{0}(N)$ contains the $v_{K}^{1}$-completion $\mathcal{R}(\tilde{\mathcal{K}})$ of $\tilde{\mathcal{K}}$.

**Proposition 6.8.** $R_{0}(N) = \mathcal{R}(\tilde{\mathcal{K}})$.

**Proof.** Suppose $K_{i}$ has parameters $n^{*} = n^{*}(K_{i})$ and $c^{*} = c^{*}(n^{*}, K_{i})$. Let $c^{**} = c^{*}(n^{*}, K_{i})/2^{N}$. The proposition easily follows from the following Lemma.

**Lemma 6.9.** For any $\alpha \in R(N)$, there is a finite separable extension $\mathcal{L}$ of $\mathcal{K}$ and $\beta \in \mathcal{O}_{\mathcal{R}(\mathcal{L})}$ such that $\alpha \equiv \beta \mod \mathcal{m}_{\mathcal{K}}^{1}(c^{**})$.

**Proof of Lemma.** Suppose $\alpha = (a_{u} \mod p)_{u \geq 0}$, where $a_{u} \in \mathcal{O}_{\mathcal{C}(N)}p$ and $a_{u+1} \equiv a_{u} \mod p$ for all $u \geq 0$. We can assume that $a_{0} \in L_{0}$, where $L_{0}$ is a finite extension of $K_{0}$ such that $L_{0} = L_{0}K_{i} \in \mathcal{B}^{f,a}(N)$.

By the above Corollary 6.6, $L_{\iota}$ has parameters $m^{*} = m^{*}(L_{\iota}) \geq n^{*}$ and $c^{**}$.

Suppose $\tilde{v}^{0} = (v_{1}^{0}, \ldots, v_{N}^{0})$ with $v_{N}^{0} \geq m^{*}$ is an index parameter from the construction of some $L' \in X(L_{\iota})$.

Then $\mathcal{O}_{L_{0}} \subset \mathcal{O}_{L_{\tilde{v}^{0}}}$ and $pr_{v_{N}^{0}}(\mathcal{O}_{L'}) = \mathcal{O}_{L_{\tilde{v}^{0}}} \mod \mathcal{m}_{\mathcal{K}}^{1}(c^{**})$.

In particular, there is an $\alpha' \in \mathcal{O}_{L'}$ such that $pr_{v_{N}^{0}} \alpha' = a_{0} \mod \mathcal{m}_{\mathcal{K}}^{1}(c^{**})$, or equivalently,

$$\sigma^{v_{N}^{0}} \alpha' \equiv \alpha' \mod \mathcal{m}_{\mathcal{K}}^{1}(p^{v_{N}^{0}}c^{**})$$

Therefore, $\alpha \equiv \sigma^{-v_{N}^{0}} \alpha' \mod \mathcal{m}_{\mathcal{K}}^{1}(c^{**})$, and the lemma is proved because $\sigma^{-v_{N}^{0}} \alpha' \in \mathcal{O}_{\mathcal{R}(\mathcal{L})}$, where $\mathcal{L}$ is a separable extension of $\mathcal{K}$ such that $cl(L') = cl(\mathcal{L})$. 

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7. The functors $\mathcal{X}$ and $\mathcal{X}_K$.

7.1. The functor $\mathcal{X}_K$, $K \in \mathcal{B}^{fa}(N)$.

Let $K \in \mathcal{B}^{fa}(N)$ and let $\mathcal{B}^{a}_{K}(N)$ be the category of finite separable extensions $L$, of $K$, in $\mathcal{B}^{a}(N)$. Morphisms in $\mathcal{B}^{a}_{K}(N)$ are morphisms $f$ in the category $\mathcal{B}^{a}(N)$ such that $f|_{K} = \text{id}$. Let $\tilde{\mathcal{B}}^{a}_{K}(N)$ be the category of finite separable extensions of $\text{cl}(K) \in \tilde{\mathcal{B}}^{a}_{R}(N)$, where $K \in X(K)$. In this section we use results of n.6 about the correspondence $K \mapsto \text{cl}(K)$, where $K \in \mathcal{B}^{fa}(N)$, to construct an equivalence between the categories $\mathcal{B}^{a}_{K}(N)$ and $\tilde{\mathcal{B}}^{a}_{R}(N)_{K}$.

Let $L$, be a finite separable extension of $K \in \mathcal{B}^{fa}(N) \in \mathcal{B}(N)$. Then $L \in \mathcal{B}^{a}(N)$, cf. n.4.2. Choose a finite Galois extension $E$, of $K$, such that $E \in \mathcal{B}^{fa}(N)$ and $E \supset L$, cf. Prop.4.4. If $K \in X(K)$ then there is a unique separable extension $E$ of $K$ in $R_{0}(N)$ such that $E \in X(E')$ and $[E : K] = [E' : K]$, cf. n.6. Therefore, $G = \text{Gal}(E/L)$ acts on $E$ and we can set $L = E^{H}$, where $H \subset G$ is such that $E^{H} = L$.

**Proposition 7.1.** With the above notation, $\text{cl}(L) \in \tilde{\mathcal{B}}^{a}_{R}(N)$ does not depend on the choice of $K \in X(K)$ and $E \in \mathcal{B}^{fa}(N)$.

The proof is straightforward.

Suppose $L, L' \in \mathcal{B}^{a}_{K}(N)$ and $f : L, \longrightarrow L'$ is a morphism in $\mathcal{B}^{a}_{K}(N)$. In other words, $f$ is a $P$-continuous and compatible with the corresponding $F$-structures automorphism of $\mathcal{C}(N)_{\mathfrak{p}}$ such that $f(L_{m}) = L'_{m}$ for $m \gg 0$ and $f|_{K} = \text{id}$. Choose $E \in \mathcal{B}^{fa}(N)$ such that $E \supset L$, and $E$, is finite Galois over $K$. Let $E' = f(E)$, $G = \text{Gal}(E/K)$, $G' = \text{Gal}(E'/K)$, $H = \text{Gal}(E/L)$ and $H' = \text{Gal}(E'/L')$.

Let $K \in X(K)$ and let $E$ be its field Galois extension in $X(E)$ of degree $[E : K]$. Let $f_{R}$ be an automorphism of $R_{0}(N)$ induced by $f$. Then $f_{R}$ is $P$-continuous and compatible with $F$-structures, $f_{R}(E) \in X(E')$ and $f_{R}(E)^{H'} = f(E^{H}) \in X(L')$.

So, $f_{R} \in \text{Hom}_{\tilde{\mathcal{B}}^{a}_{R}(N)_{K}}(\mathcal{X}_{K}(L), \mathcal{X}_{K}(L'))$. Clearly, if we set $f_{R} = \mathcal{X}_{K}(f)$ then we get a functor $\mathcal{X}_{K}$ from $\mathcal{B}^{a}_{K}(N)$ to $\tilde{\mathcal{B}}^{a}_{R}(N)_{K}$.

Summarizing the results of n.6 we obtain the following principal result of this paper.

**Theorem 2.** a) The above defined functor $\mathcal{X}_{K}$, where $K \in \mathcal{B}^{fa}(N)$, is an equivalence of the categories $\mathcal{B}^{a}_{K}(N)$ and $\tilde{\mathcal{B}}^{a}_{R}(N)_{K}$;

b) $\mathcal{X}_{K}$ induces an identification $\psi_{K}$, of groups $\Gamma_{K} = \text{Gal}(\bar{K}/K)$ and $\Gamma_{K} = \text{Gal}(\mathcal{K}_{\text{sep}}/K)$, where $K \in X(K)$ and $\mathcal{K}_{\text{sep}}$ is the separable closure of $K$ in $R_{0}(N)$;

c) the identification $\psi_{K}$, is compatible with ramification filtrations on $\Gamma_{\bar{K}}$ and $\Gamma_{K}$, i.e. for any $j \in J(N)$, $\psi_{K}$ identifies the groups $\Gamma_{\bar{K}} \cap \Gamma_{K_{0}}^{\phi_{\bar{K}}/K_{0}(j)}$ and $\Gamma_{K}^{(j)}$.

7.2 The functor $\mathcal{X} : \mathcal{B}^{a}(N) \longrightarrow \text{RLF}_{R}(N)$.

Let $\text{RLF}_{R}(N)$ be the category of $P$-closed perfect subfields in $R_{0}(N)$. These subfields are considered with their natural $F$-structure and $P$-topology. Morphisms are $P$-continuous isomorphisms of such fields, which are compatible with corresponding $F$-structures.
If \( K, \epsilon \in B^a(N) \) choose \( L, \in B^{f_a}(N) \) such that \( L_K \), is a finite Galois extension. If \( L \in X(L) \), then \( G = Gal(L/K) \) acts on \( R(L) \). Indeed, for any \( g \in G \) and any \( L \in X(L) \), the action of \( g \) on \( L \), induces a field isomorphism \( g : L \longrightarrow L' \), where \( L' \in X(L) \), and we have a natural identification \( R(L) = R(L') \). With the above notation set \( \mathcal{X}(K) = R(L)^G \in RLF_{R}(N) \).

**Proposition 7.2.** \( \mathcal{X}(K) \) does not depend on a choice of \( L, \in B^{f_a}(N) \).

**Proof.** Suppose \( L', \in B^{f_a}(N) \) is such that \( L'/K \), is a finite Galois extension with the Galois group \( G' \). Choose \( M, \in B^{f_a}(N) \) such that \( M, \supset L', M, \supset L' \), and \( M \) is a finite Galois extension of \( K \), with the Galois group \( S \).

Let \( H = Gal(M/L), H' = Gal(M/L') \). If \( L \in X(L) \), \( L' \in X(L') \), then there are \( M \in X(M) \) and \( M' \in X(M) \) such that \( M/L \) and \( M'/L' \) are Galois extensions with Galois groups \( H \) and \( H' \), respectively. Then \( \mathcal{R}(M) = \mathcal{R}(M') \) and \( \mathcal{R}(L') = \mathcal{R}(M') = \mathcal{R}(M) = \mathcal{R}(L)^G \).

The proposition is proved.

Suppose \( K, K' \in B^a(N) \) and \( f \in Hom_{B^a(N)}(K, K') \), i.e. \( f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p \) is a \( P \)-continuous and compatible with \( F \)-structure field automorphism such that \( f(K) = K' \). As earlier, denote by \( f_R \) the automorphism of \( R_0(N) \) which is induced by \( f \).

Choose \( L, \in B^{f_a}(N) \) such that \( L_K \), is a finite Galois extension with the group \( G \). Then \( L' = f(L) \) is a Galois extension of \( K' \), with the same group \( G \). If \( L \in X(L) \), then \( f_R(L') = L' \in X(L) \) and \( f_R(\mathcal{X}(K)) = f_R(\mathcal{R}(L)^G) = f_R(L)^G = \mathcal{X}(K') \).

So, \( f \in Hom_{RLF_{R}(N)}(\mathcal{X}(K), \mathcal{X}(K')) \) and \( \mathcal{X} : B^a(N) \longrightarrow RLF_{R}(N) \) is a functor. The following property follows directly from the above definitions.

**Proposition 7.3.** a) \( \mathcal{X} \) is a strict functor;

b) if \( L, K, \in B^{f_a}(N) \) and \( L, \) is a finite separable extension of \( K \), then

\[
\mathcal{R}(\mathcal{X}_K(L)) = \mathcal{X}(L).
\]

7.3. Let \( \epsilon = (\epsilon^{(n)} \mod p)_{n \geq 0} \in R(1) \subset R(N) \), where \( \epsilon^{(0)} = 1 \), \( \epsilon^{(1)} \neq 1 \), and \( \epsilon^{(n+1)p} = \epsilon^{(n)} \) for all \( n \geq 0 \), be Fontaine’s element. Let \( < \epsilon > = \epsilon^{Z_p} \subset R(1)^* \) be the multiplicative subgroup of all Fontaine’s elements. Notice, if \( f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p \) is a field automorphism then \( f_R(< \epsilon >) = < \epsilon > \), where \( f_R \) is induced by \( f \).

**Lemma 7.4.** The correspondence \( f \mapsto f_R \) identifies \( Aut \mathbb{C}(N)_p \) and the subgroup \( Aut' R_0(N) \) of \( g \in Aut R_0(N) \) such that \( g(< \epsilon >) = < \epsilon > \).

**Proof.** We have noticed already that for any \( f \in Aut \mathbb{C}(N)_p \), \( f_R(< \epsilon >) = < \epsilon > \).

Suppose \( g \in Aut R_0(N) \) and \( g(< \epsilon >) = < \epsilon > \), i.e. \( g(\epsilon) = \epsilon^a \) with \( a \in Z_p^* \).

Notice that \( g : R(N) \longrightarrow R(N) \) induces the automorphism \( W(g) : W(R(N)) \longrightarrow W(R(N)) \), where \( W \) is the functor of Witt vectors. Consider the Fontaine map

\[
\gamma : W(R(N)) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p}
\]

given by the correspondence \((r_0, r_1, \ldots, r_n, \ldots) \mapsto r^{(0)} + pr^{(1)} + \ldots + p^n r^{(n)} + \ldots \), where for any \( r = (r_m \mod p)_{m \geq 0} \in R(N) \) and \( n \geq 0 \), \( r^{(n)} = \lim_{m \to \infty} r^m_{m+n} \). This map is a surjective morphism of \( p \)-adic algebras and its kernel \( J \) is a principal ideal generated by \( 1 + \epsilon^{1/p} + \ldots + \epsilon^{(p-1)/p} \). Therefore, \( W(g)(J) = J \) and \( W(g) \)
induces an automorphism \( f = W(g) \mod J \) of \( \mathbb{C}(N)_p \). Clearly, \( f_R = g \). The lemma is proved.

**Remark.** From the above description of the correspondence \( f \mapsto f_R \) it easily follows that \( f \) is \( P \)-continuous (resp., compatible with \( F \)-structure) if and only if \( f_R \) possess the same property.

7.4. Introduce the following definition.

**Definition.** A subfield \( \widetilde{K} \) of \( \mathbb{C}(N)_p \) is an SAPF-field if there is \( K, j \in \mathcal{B}^a(N) \) such that \( \widetilde{K} \) is the \( p \)-adic closure of \( \bigcup_{n \geq 0} K_n \).

**Remark.** The above defined SAPF-fields are higher dimensional analogues of strict arithmetic profinite extensions introduced in [FW1-2].

Denote by \( \text{SAPF}(N) \) the category of SAPF-fields in \( \mathbb{C}(N)_p \), such that if \( \widetilde{K}, \widetilde{K}' \in \text{SAPF}(N) \), then \( \text{Hom}_{\text{SAPF}(N)}(\widetilde{K}, \widetilde{K}') \) consists of \( P \)-continuous and compatible with \( F \)-structures \( f \in \text{Aut} \mathbb{C}(N)_p \) such that \( f(\widetilde{K}) = \widetilde{K}' \).

Let \( \widetilde{K} \in \text{SAPF}(N) \). Set \( \tilde{\mathcal{X}}(\widetilde{K}) = \mathcal{X}(K), \) where \( K, j \in \mathcal{B}^a(N) \) is such that \( \widetilde{K} \) is a \( p \)-adic closure of \( \bigcup_{n \geq 0} K_n \).

**Lemma 7.5.** The above defined \( \tilde{\mathcal{X}}(\widetilde{K}) \) does not depend on the choice of \( K, j \in \mathcal{B}^a(N) \).

**Proof.** The proof follows directly from the construction of the functor \( \mathcal{X} \) and proposition 5.6.

The correspondence \( \widetilde{K} \mapsto \tilde{\mathcal{X}}(\widetilde{K}) \) is naturally extended to the functor \( \tilde{\mathcal{X}} : \text{SAPF}(N) \longrightarrow \text{RLF}_R(N) \).

Taking together the above results about the functor \( \mathcal{X} \) we obtain the following theorem.

**Theorem 3.** Suppose \( K, j \in \mathcal{B}^a(N) \) and \( \widetilde{K} \) is a \( p \)-adic closure of \( \bigcup_{n \geq 0} K_n \). Then the functor \( \tilde{\mathcal{X}} \) induces the identification \( i : \Gamma_{\widetilde{K}} \longrightarrow \Gamma_{\widetilde{K}} \) where \( \widetilde{K} = \mathcal{X}(K) \). If \( K, j \in \mathcal{B}^{f,a}(N) \) and \( \mathcal{K} \in \mathcal{X}(K) \) then \( \mathcal{R}(\mathcal{K}) = \widetilde{K} \) and under a natural identification \( \Gamma_{\mathcal{K}} = \Gamma_{\widetilde{K}} \), the identification \( i \) is compatible with ramification filtrations, i.e. for any \( j \in J(N) \),

\[
\Gamma_{\widetilde{K}} \cap \Gamma_{K_0}^{(\varphi_{\widetilde{K}/K_0})}(j) = \Gamma_{\mathcal{K}}^{(j)}.
\]

8. A property of the \( P \)-continuity for the functor \( \mathcal{X} \).

8.1. Suppose \( \mathcal{K} \in \text{LF}_p(N) \).

Let \( \Gamma_{\mathcal{K}}^{ab}(p) \) be the Galois group of the maximal abelian \( p \)-extension of \( \mathcal{K} \).

For any \( M \geq 1 \), consider the Witt-Artin-Schreier duality

\[
\Gamma_{\mathcal{K}}^{ab}(p)/p^M \times W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \longrightarrow W_M(\mathbb{F}_p)
\]

where \( \sigma \) is the Frobenius endomorphism of the additive group \( W_M(\mathcal{K}) \) of Witt vectors of length \( M \) with coefficients in \( \mathcal{K} \). This allows us to provide \( \Gamma_{\mathcal{K}}^{ab}(p)/p^M \) with the \( P \)-topological structure. Its basis of open 0-neighborhoods consists of the annihilators of the compact subsets of \( W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \). By results of n.1.2
the basis of such compact subsets consists of the images in $W_M(K)/(\sigma - \text{id})W_M(K)$ of all subsets of the form

$$W_M(D) = \{(a_0, \ldots, a_{M-1}) \in W_M(K) \mid a_0, \ldots, a_{M-1} \in D\}$$

where $D \in C_b(K)$ is the basis of compact subsets in $K$.

Finally, the $P$-topology on $\Gamma^\text{ab}_K(p)$ appears as the projective limit topology of the projective system of $P$-topological groups $\Gamma^\text{ab}_K(p)/p^M$.

8.2. Suppose $K \in LF_0(N)$ and $K$ contains a primitive $p^M$-th root of unity $\zeta_{p^M}$. Then the $P$-topological structure on $K^*$ induces the $P$-topological structure on $\Gamma^\text{ab}_K(p)/p^M$, where $\Gamma_K(p)$ is the Galois group of the maximal abelian $p$-extension of $K$. This structure is defined similarly to the characteristic $p$ case by the use of the Kummer duality

$$\Gamma^\text{ab}_K(p)/p^M \times K^*/K^{*p^M} \rightarrow <\zeta_{p^M}>$$

We don’t need this structure in a full generality. Let $\tilde{\Gamma}_K(p)/p^M$ be the quotient of $\Gamma^\text{ab}_K(p)/p^M$ by the annihilator of the subgroup in $K^*$ generated by the elements of $(1 + p\mathcal{O}_K)^*$. Then we have the induced pairing

$$\tilde{\Gamma}^\text{ab}_K(p)/p^M \times (1 + p\mathcal{O}_K)^* \rightarrow <\zeta_{p^M}>$$

and a basis of open subgroups in $\tilde{\Gamma}^\text{ab}_K(p)/p^M$ consists of the annihilators of the subsets $1 + pD$, where $D \in C_b(K)$ and $C_b(K)$ is a basis of compact subsets in $K$ from n.1.2.

8.3. Suppose $K_n \in B^{f_a}(N)$ and for a sufficiently large $n$, $K_n$ contains a primitive $p^M$-th root of unity.

Let $\tilde{K}$ be the $p$-adic closure of $\cup_{n \geq 0}K_n$. Then for any $M \in \mathbb{N}$, we have a natural identification

$$\Gamma^\text{ab}_K(p)/p^M = \lim_n \Gamma^\text{ab}_{K_n}/p^M.$$ 

Applying the arguments from n.2.9 we can also write

$$\Gamma^\text{ab}_K(p)/p^M = \lim_n \tilde{\Gamma}^\text{ab}_{K_n}/p^M.$$ 

Therefore, the basis of $P$-open neighborhoods in $\Gamma^\text{ab}_K(p)/p^M$ consists of annihilators of all compact subsets $1 + pD \subset (1 + p\mathcal{O}_{\tilde{K}})^*$, where $D \in C_b(K_n)$ for some $n \geq 0$.

8.4. Suppose $K_n \in B^{f_a}(N), K \in X(K_n)$ and $\iota : \Gamma_{\tilde{K}} \rightarrow \Gamma_K$ is the identification of Galois groups (where $\tilde{K}$ is the $p$-adic closure of the $\cup_{n \geq 0}K_n$) from Theorem 3. Suppose for each $M \in \mathbb{N}$, $\zeta_{p^M} \in K_n$ if $n \gg 0$ and consider the groups $\Gamma^\text{ab}_{\tilde{K}}/p^M = \lim_n \tilde{\Gamma}^\text{ab}_{K_n}/p^M$ and $\Gamma^\text{ab}_K/p^M$ with the above $P$-topological structures.

**Theorem 4.** With the above notation, the identification

$$\iota \mod p^M : \Gamma^\text{ab}_K/p^M \rightarrow \Gamma^\text{ab}_{\tilde{K}}/p^M$$

is $P$-continuous.
Proof.

8.4.1. Consider the dual morphism

\[ \tilde{\iota}_M : W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \to \tilde{K}^*/\tilde{K}^{*p^M}. \]

Then \( \iota \mod p^M \) is \( P \)-continuous if and only if \( \tilde{\iota}_M \) transforms each \( P \)-compact subset in \( W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \) onto a \( P \)-compact subset in \( \tilde{K}^*/\tilde{K}^{p^M} \).

Notice that the map \( \tilde{\iota}_M \) can be characterised as follows.

Choose a primitive \( p^M \)-th root of unity \( \zeta_M \). Let \( \bar{w} \in W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \) and let \( w \in W_M(\mathcal{K}) \) be a lifting of \( \bar{w} \). Consider \( T \in W_M(R(N)) \) such that \( \sigma T - T = w \) then for any \( \tau \in \Gamma_{\mathcal{K}}, \tau T - T = a_\tau \in W_M(\mathbb{F}_p) \). Let \( \bar{v} \in \tilde{K}^*/\tilde{K}^{*p^M} \) and \( v \in \tilde{K}^* \) be a lifting of \( \bar{v} \). Consider \( Z \in C(N) \) such that \( Z^{p^M} = v \). Then for any \( \tau \in \Gamma_{\tilde{K}} \), \( \tau Z/Z = \zeta_M^{b_\tau} \), where \( b_\tau \in W_M(\mathbb{F}_p) \). With the above notation, with respect to the identification \( \Gamma_{\tilde{K}} = \Gamma_{\mathcal{K}} \), we have the following criterion:

\[ \tilde{\iota}_M(\bar{w}) = \bar{v} \quad \iff \quad a_\tau = b_\tau \quad \forall \tau \in \Gamma_{\tilde{K}} = \Gamma_{\mathcal{K}} \]

8.4.2. As earlier, let \( \mathcal{R}(\mathcal{K}) \) be the completion of the radical closure of \( \mathcal{K} \) (with respect to 1st valuation). Denote by \( \mathcal{R}(\mathcal{O}_{\mathcal{K}}) \) its valuation ring.

Notice first that the natural embedding \( \mathcal{K} \subset \mathcal{R}(\mathcal{K}) \) induces a natural identification of \( P \)-topological groups

\[ W_M(\mathcal{R}(\mathcal{K}))/\langle \sigma - \text{id} \rangle W_M(\mathcal{R}(\mathcal{K})) = W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \]

Let \( \varepsilon \) be Fontaine's elements. Recall, \( \varepsilon = (\varepsilon^{(n)})_{n \geq 0} \in R = R(1) \subset R(N) \) is such that \( \varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1 \) and we can assume that \( \varepsilon^{(M)} = \zeta_M \) — this is the primitive \( p^M \)-th root of unity chosen in 8.4.1. From the construction of \( \mathcal{K} \in X(\tilde{K},) \) it follows that \( \varepsilon \in \mathcal{R}(\mathcal{O}_{\mathcal{K}}) \). Consider the map

\[ \text{pr} : \frac{1}{|\varepsilon| - 1} W_M(\mathcal{R}(\mathcal{O}_{\mathcal{K}})) \to W_M(\mathcal{R}(\mathcal{K}))/\langle \sigma - \text{id} \rangle W_M(\mathcal{R}(\mathcal{K})) \]

induced by the projection \( W_M(\mathcal{R}(\mathcal{K})) \to W_M(\mathcal{R}(\mathcal{K}))/\langle \sigma - \text{id} \rangle W_M(\mathcal{R}(\mathcal{K})) \).

**Lemma 8.1.** \( \text{pr} \) is surjective.

**Proof.** This follows from the formula

\[ W_M(\mathcal{K}) = \bigcup_{s \geq 0} \frac{1}{\sigma^s(|\varepsilon| - 1)} W_M(\mathcal{O}_{\mathcal{K}}). \]

**Remark.** It can be easily seen that the family of sets

\[ \left\{ \text{pr} \left( \frac{1}{|\varepsilon| - 1} W_M(\sigma^{-s} D) \right) \mid s \in \mathbb{Z}_{\geq 0}, D \subset \mathcal{O}_{\mathcal{K}}, D \in \mathcal{C}_b(\mathcal{K}) \right\} \]

is a basis of compact subsets in \( W_M(\mathcal{K})/(\sigma - \text{id})W_M(\mathcal{K}) \).

Let \( w \in W_M(\mathcal{K}) \) be the element from n.8.4.1. By the above lemma, there is an \( f \in W(\mathcal{R}(\mathcal{O}_{\mathcal{K}})) \) such that \( w = f/((|\varepsilon| - 1) \mod p^M) \). Therefore, if \( U \in W(R_0(N)) \) is
such that $\sigma U - U = f/([\epsilon] - 1)$ then for any $\tau \in \Gamma_K$, $\tau U - U = \tilde{a}_\tau \in W(\bar{F}_p)$, where $\tilde{a}_\tau \mod p^M = a_\tau$.

8.4.3. Let $\varepsilon_1 = \sigma^{-1}\varepsilon$, then

$$s = ([\varepsilon] - 1)/([\varepsilon_1] - 1) \in W^1(R(1)) \subset W(R(1)) \subset W(R(N))$$

where $W^1(R(1)) = \text{Ker} \gamma : W(R(1)) \rightarrow \mathcal{O}_{C_p}$ is Fontaine's map. It is known [Ab2] that $s$ generates the ideal $W^1(R(1))$. Notice that similar arguments show that $s$ generates also the kernel $W^1(R(N))$ of the analogue of Fontaine's map from $W(R(N))$ to $\mathcal{O}_{C(N)_{\bar{p}}}$.

Let $T_1 = U([\varepsilon_1] - 1)$. Then $T_1 \in W(R(N))$ and $\sigma T_1 - s T_1 = f$. Let $X = U([\varepsilon] - 1) = s T_1 \in W^1(R)$, then

$$\frac{\sigma X}{\sigma s} - X = f$$

and for any $\tau \in \Gamma_K$, $\tau X - X = \tilde{a}_\tau ([\varepsilon] - 1)$.

8.4.4. Let $A(N)_{\text{cris}}$ be an analogue of Fontaine’s $A_{\text{cris}}$ constructed by the use of $R(N)$ instead of $R$. This is the divided power envelope of the $W(R(N))$ with respect to the ideal $W^1(R(N))$, which is generated by $s$. Proceeding as in [Ab2] we obtain that if

$$\sigma m - m = f$$

where $m \in \text{Fil}^1 A(N)_{\text{cris}}$, then for any $\gamma \in \Gamma_{\bar{K}}$, $\tau m - m = \tilde{a}_\tau \log[\varepsilon]$.

Multiplying both parts of the equality (5) by $p$ and taking exponentials we obtain the equality

$$\sigma Y = Y^p \exp(p f)$$

where $Y \in 1 + \text{Fil}^1 A(N)_{\text{cris}}$ and for any $\gamma \in \Gamma_{\bar{K}}$, $\tau Y / Y = [\varepsilon] \tilde{a}_\tau$. Proceeding again as in [Ab2] we can prove that $Y \in 1 + W^1(R(\bar{N}))$ (and therefore can forget about the cristalline ring $A(N)_{\text{cris}}$).

8.4.5. The equation (6) implies that

$$\sigma^M Y = Y^p \exp(p \sigma^M - 1 f + \cdots + p^M f)$$

and, because $\sigma$ is injective on $W(R(N))$, this gives

$$Y = (\sigma^{-M} Y)^p \exp(p \sigma^{-1} f + \cdots + p^M \sigma^{-M} f)$$

Notice that for any $\gamma \in \Gamma_{\bar{K}}$, $\tau (\sigma^{-M} Y) = (\sigma^{-M} Y) [\sigma^{-M} \varepsilon] \tilde{a}_\tau$.

Apply Fontaine’s map $\gamma : W(R(N)) \rightarrow \mathcal{O}_{C(N)_{\bar{p}}}$ to the both parts of (7). Notice that $\gamma(Y) = 1$, $\gamma(\sigma^{-M} Y) = Z = 1 + p \mathcal{O}_{C(N)_{\bar{p}}}$, $\gamma([\sigma^{-M} \varepsilon]) = \zeta_M$ and $\gamma(\sigma^{-s} f) \in \mathcal{O}_{\bar{K}}$ for any $s \in \mathbb{Z}$. This gives

$$Z^p \gamma = \exp(-p \gamma(\sigma^{-1} f) \cdots - p^M \gamma(\sigma^{-M} f)) \in 1 + p \mathcal{O}_{\bar{K}}$$
and for any $\tau \in \Gamma_K$, $\tau Z/Z = \zeta_M^a$.

8.4.6. The above computations imply (with the notation from n.8.4.1) that if $\bar{w} = f/([e] - 1) \mod (\sigma - \text{id}) W_M(\mathcal{R}(\mathcal{K}))$ with $f \in W(\mathcal{R}(\mathcal{O}_K))$ then $\bar{i}_M(\bar{w}) = \bar{v}$, where

$$\bar{v} = \exp(-p\gamma(\sigma^{-1}f) - \cdots - p^M \gamma(\sigma^{-M}f)) \mod \widetilde{K^*p^M}.$$

It remains to notice that by Proposition 1.2 the correspondence

$$f \mapsto \exp(-p\gamma(\sigma^{-1}f) - \cdots - p^M \gamma(\sigma^{-M}f)) \mod \widetilde{K^*p^M}$$

maps all $P$-compact subsets in $W_M(\mathcal{R}(\mathcal{O}_K))$ to $P$-compact subsets in $1 + p\mathcal{O}_K$.

The theorem is proved.

Remark. The above computations in nn.8.4.3-8.4.6 can be used to deduce (in the similar way as in [Ab2]) the explicit formula for Hilbert symbol for higher dimensional fields from [Vo].

9. The Grothendieck Conjecture for higher dimensional local fields.

9.1. Suppose $K, K'$ are 1-dimensional local fields from the category $LF(1) = LF_0(1) \coprod LF_{01}(1)$. Then any isomorphism $f \in \text{Hom}_{LF(1)}(K, K')$ is given by an automorphism of $\mathbb{C}(1)_p$ or $\mathbb{C}(1)_p$ such that $f(K) = K'$. Therefore, $f$ induces the isomorphism of profinite groups

$$f^*: \Gamma_{K'} \longrightarrow \Gamma_K$$

such that for any $v \geq 0$, $f^*(\Gamma_{K'}^{(v)}) = \Gamma_K^{(v)}$.

The inverse statement was proved in [Mo] in the mixed characteristic case and in [Ab4] if the characteristic of the residue fields of $K$ and $K'$ is $\geq 3$. It is known as a local (1-dimensional) analogue of the Grothendieck Conjecture and can be stated in the following form:

If $\iota: \Gamma_{K'} \longrightarrow \Gamma_K$ is an isomorphism of profinite groups such that for any $v \geq 0$, $\iota(\Gamma_{K'}^{(v)}) = \Gamma_K^{(v)}$, then there is an $f \in \text{Hom}_{LF(1)}(K, K')$ such that $\iota = f^*$.

9.2. Suppose $N \geq 1$ and $\mathcal{K}, \mathcal{K}' \in LF_{R_0}(N)$. Suppose $f \in \text{Hom}_{LF_{R_0}(N)}(\mathcal{K}, \mathcal{K}')$ is isomorphism. In other words, $f: R_0(N) \longrightarrow R_0(N)$ is a $P$-continuous and compatible with F-structures field automorphism such that for all $1 \leq i \leq N$, $f(\mathcal{K}(i) R(\mathcal{K}(i - 1))) = \mathcal{K}'(i) R(\mathcal{K}'(i - 1))$. Then $f^*: \Gamma_{\mathcal{K}'} \longrightarrow \Gamma_{\mathcal{K}}$ is an isomorphism of profinite groups such that for any $j \in J(N)$, $f^*(\Gamma_{\mathcal{K}'}^{(j)}) = \Gamma_{\mathcal{K}}^{(j)}$.

In addition, suppose $\mathcal{E}$ is a finite extension of $\mathcal{K}$ in $R_0(N)$ and $f(\mathcal{E}) = \mathcal{E}'$. Then $\mathcal{E}'$ is a finite extension of $\mathcal{K}'$ such that $f^*(\Gamma_{\mathcal{E}'}) = \Gamma_{\mathcal{E}}$. Let $M \in \mathbb{N}$. Consider the induced isomorphism of the maximal abelian quotients modulo $p^M$-th powers

$$f^*_M: \Gamma_{\mathcal{E}'}^{ab}/p^M \longrightarrow \Gamma_{\mathcal{E}}^{ab}/p^M.$$

It is dual to the isomorphism of additive groups

$$f_M: W_M(\mathcal{E})/(\sigma - \text{id}) W_M(\mathcal{E}) \longrightarrow W_M(\mathcal{E}')/(\sigma - \text{id}) W_M(\mathcal{E}').$$

Clearly, $f_M$ is $P$-continuous and, therefore, maps $P$-compact subsets to $P$-compact subsets. This implies that $f^*_M$ is $P$-continuous for all $M \in \mathbb{N}$.

The inverse statement appears as an analogue of the Grothendieck Conjecture for higher dimensional local fields of characteristic $p$. 

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Theorem 5. With the above notation suppose that \( p \geq 3 \) and
\[
\iota : \Gamma_{K'} \longrightarrow \Gamma_K
\]
is an isomorphism of profinite groups such that
\begin{enumerate}[
  \text{a)}]
  \item for any \( j \in J(N) \), \( \iota(\Gamma^{(j)}_{K'}) = \Gamma^{(j)}_K \);
  \item if \( E \) and \( E' \) are finite extensions of \( K \) and, resp., \( K' \) in \( R_0(N) \) such that the both \( E \) and \( E' \) have a standard \( F \)-structure, then for all \( M \geq 1 \), the induced isomorphism
  \[
  \iota_M : \Gamma^{ab}_{E'}/p^M \longrightarrow \Gamma^{ab}_E/p^M
  \]
is \( P \)-continuous.
\end{enumerate}
Then there is an \( f \in \text{Hom}_{\text{LF}}(K, K') \) such that \( f^* = \iota \).

This statement was proved in [Ab6] in the case \( N = 2 \). The case of general \( N \) can be done along the same lines.

Remark. Actually, in the statement of the main theorem in [Ab6] there was no requirement that \( E \) and \( E' \) have standard \( F \)-structure. But in the proof we applied this condition only to fields with standard \( F \)-structure. Also, in [Ab6] there was a requirement about the \( P \)-continuity of the induced group isomorphism \( \iota^{ab} : \Gamma^{ab}_{E'} \longrightarrow \Gamma^{ab}_E \) but again in the proof we applied this property only to the induced isomorphism of the Galois groups \( \Gamma^{ab}_{E'}(p) \) and \( \Gamma^{ab}_E(p) \) of the maximal \( p \)-extensions of \( E' \) and \( E \).

9.3. Suppose \( N \geq 1 \) and \( K, K' \in \text{LF}_0(N) \). Any \( P \)-continuous and compatible with \( F \)-structures field automorphism \( f : \mathbb{C}_p(N)_p \longrightarrow \mathbb{C}(N)_p \) such that \( f(K) = K' \) induces an isomorphism of profinite groups \( f^* : \Gamma_{K'} \longrightarrow \Gamma_K \) such that \( f^*(\Gamma^{(j)}_{K'}) = \Gamma^{(j)}_K \) for any \( j \in J(N) \).

Suppose \( E \) is a finite extension of \( K \), then \( E' = f(E) \) is a finite extension of \( K' \). If both \( E \) and \( E' \) contain a primitive \( p^M \)-th root of unity then the groups \( \Gamma^{ab}_{K'}/p^M \) and \( \Gamma^{ab}_{K'}/p^M \) are provided with the \( P \)-topological structure, cf. n.8.2, and the induced isomorphism
\[
f^*_M : \Gamma^{ab}_{K'}/p^M \longrightarrow \Gamma^{ab}_K/p^M
\]
is \( P \)-continuous.

Consider the inverse statement.

Theorem 6. With the above notation suppose that \( p \geq 3 \) and \( \iota : \Gamma_{K'} \longrightarrow \Gamma_K \) is an isomorphism of profinite groups such that
\begin{enumerate}[
  \text{a)}]
  \item for all \( j \in J(N) \), \( \iota(\Gamma^{(j)}_{K'}) = \Gamma^{(j)}_K \);
  \item if \( E, E' \) are finite extensions of \( K \) and, resp., \( K' \) such that the both contain \( \zeta_{p^M} \), then the induced isomorphism
  \[
  \iota_M : \Gamma^{ab}_{E'}/p^M \longrightarrow \Gamma^{ab}_{E}/p^M
  \]
is \( P \)-continuous.
\end{enumerate}
Then there is a (unique) field isomorphism \( f : \mathbb{C}_p(N) \longrightarrow \mathbb{C}_p(N) \) such that \( f(K) = K' \) and \( f = \iota^* \).

Remark. Modulo some technical details and notation this statement has been announced in [Ab5].
Proof.

9.3.1. Notice first, that \( \iota \) induces for \( 1 \leq r \leq N \), the group isomorphisms \( \iota(r) : \Gamma_{K^r} \rightarrow \Gamma_{K^r} \). All these isomorphisms are also compatible with the corresponding ramification filtrations.

In particular, \( \iota(1) \) is a compatible with ramification filtration isomorphism of the absolute Galois groups of 1-dimensional local fields \( K(1) \) and \( K'(1) \). Therefore, by the 1-dimensional case of a local analogue of the Grothendieck conjecture, cf. n.8.1, \( \iota(1) \) is induced by a field isomorphism \( f(1) : C_p \rightarrow C_p \) such that \( f(1)(K(1)) = K'(1) \).

9.3.2. Prove the existence of \( F, F' \in \mathcal{B}^{fa}(N) \) such that for all \( n \geq 0 \),

a) \( F_0 \supset K, F'_0 \supset K' \);

b) \( \iota(\Gamma_{F_n}) = \Gamma_{F'_n} \);

c) \( \zeta_n \in F_n \) and \( \zeta_n \in F'_n \), where \( \zeta_n \) is a primitive \( p^n \)-th root of unity.

Let \( E_0 = \mathbb{Q}_p \{ \{ \tau_N \} \} \ldots \{ \{ \tau_2 \} \} \) be a basic \( N \)-dimensional local field. Then \( K \) and \( K' \) are its finite extensions with induced \( F \)-structures. Consider \( E_\iota \in \mathcal{B}(N) \) such that for all \( n \geq 1 \), \( E_n = E_0(\zeta_n, \sqrt[n]{\tau_2}, \ldots, \sqrt[n]{\tau_n}) \). Clearly, \( E_\iota \in \mathcal{B}^a(N) \) (even more, \( E_\iota \in \mathcal{B}^{fa}(N) \)).

Let \( L_\iota = KE_\iota \). Then \( L_\iota \in \mathcal{B}^a(N) \) by Prop.4.1. Introduce \( L'_\iota = \{ L'_n \mid n \geq 0 \} \in \mathcal{B}(N) \) such that \( \iota(\Gamma_{L'_n}) = \Gamma_{L_n} \). Then \( L'_\iota \in \mathcal{B}^a(N) \) because \( \iota \) is compatible with ramification filtrations.

Suppose \( n^* = n^*(L) \) is the parameter for \( L_\iota \) introduced in n.4.2. Clearly, \( n^* \) can be taken also as a parameter for \( L'_\iota \). Choose a finite extension \( M(N - 1) \) of \( L_{n^*}(N - 1) \) such that if \( M = L_{n^*}M(N - 1) \) then \( (M, M(N - 1)) \in \text{LC}(N) \) is standard, cf. Theorem 1. If necessary, we can enlarge \( M(N - 1) \) to satisfy the following property: if \( M(N - 1)' \) is such that \( \iota(N - 1)(\Gamma_{M(N - 1)')} = \Gamma_{M(N - 1)} \) and \( M' = L'_{n^*}M(N - 1)' \) then \( (M', M(N - 1)') \in \text{LC}(N) \) is standard. Therefore, the towers \( M'_n = ML_n, M'_n = L'_{n^*}M(N - 1)' \) are such that for all \( n \geq 0 \), \( \iota(\Gamma_{M'_n}) = \Gamma_{M_n} \) and \( (M'_n, M'_n, M(N - 1)), (M'_n, M'_n, M(N - 1)) \in \text{LC}(N) \) are standard.

Apply the above procedure to \((N - 1)\)-dimensional towers \( M'_n, M'_n, (N - 1) \in \mathcal{B}^a(N - 1) \) with a parameter \( m^* \geq n^* \) and so on. Finally, we obtain finite separable extensions \( F_\iota \) and \( F'_\iota \) of \( L_\iota \) and, resp., \( L'_\iota \), which still satisfy the above requirements a)-c) but are already objects of the category \( \mathcal{B}^{fa}(N) \).

9.3.3. Let \( \mathcal{F} \in X(F) \) and \( \mathcal{F}' \in X(F'_\iota) \), cf. section 5. By Theorem 2 the group isomorphism \( \iota \) induces the identification

\[
\iota_\mathcal{F} : \Gamma_{\mathcal{F}'_\iota} \rightarrow \Gamma_{\mathcal{F}}
\]

which is compatible with ramification filtrations on these groups.

Suppose finite extensions \( \mathcal{E}/\mathcal{F} \) and \( \mathcal{E}'/\mathcal{F}' \) are such that \( \iota_\mathcal{F}(\Gamma_{\mathcal{E}'_\iota}) = \Gamma_{\mathcal{E}} \). If \( \mathcal{E} \) and \( \mathcal{E}' \) have standard \( F \)-structures then \( \mathcal{E} \in X(E) \) and \( \mathcal{E}' \in X(E'_\iota) \), where \( E, E'_\iota \in \mathcal{B}^{fa}(N) \) are finite separable extensions of \( F \) and \( F'_\iota \), respectively. Therefore, we can apply Theorem 4 to deduce from the condition b) of the statement of our theorem that for any \( M \in \mathbb{N} \), the induced identification

\[
\iota_{\mathcal{F},M} : \Gamma_{\mathcal{E}/p^M} \rightarrow \Gamma_{\mathcal{E}/p^M}
\]

is \( p \)-continuous.
Therefore, by the characteristic $p$ case of the Grothendieck Conjecture, cf. Theorem 5 in n.9.2, the isomorphism $\iota_F$ is induced by a field isomorphism $f_R : R_0(N) \longrightarrow R_0(N)$ such that $f_R(\mathcal{R}(\mathcal{F})) = \mathcal{R}(\mathcal{F}')$.

9.3.4. Clearly, $f_R|_{R_0(1)}$ is induced by the $f(1) : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$ from n.9.3.1. Therefore, $f_R$ leaves invariant the subgroup of Fontaine’s elements $< \varepsilon >$ and by Lemma 7.4, $f_R$ is induced by a field automorphism $f : \mathbb{C}(N)_p \longrightarrow \mathbb{C}(N)_p$.

The characteristic property of the field automorphism $f_R$ is that it transforms the action of any $\tau \in \Gamma_{K'}$ on $R_0(N)$ into the action of $\iota(\tau) \in \Gamma_K$ on $R_0(N)$. Therefore, $f$ satisfies the same property and we have

$$f(K) = f(\mathbb{C}(N)_p^{\Gamma_K}) = \mathbb{C}(N)_p^{\Gamma_{K'}} = K'.$$

So, $f \in \text{Hom}_{\text{LF}_0(N)}(K, K')$ and Theorem 6 is proved.

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