FREE-FIELD REPRESENTATION OF PERMUTATION BRANES IN GEPNER MODELS.

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Abstract

We consider free-field realization of Gepner models basing on free-field realization of N=2 superconformal minimal models. Using this realization we analyse A/B-type boundary conditions starting from the ansatz when left-moving and right-moving free-fields degrees of freedom are glued at the boundary by an arbitrary constant matrix. It is shown that the only boundary conditions consistent with the singular vectors structure of unitary minimal models representations are given by permutation matrices and give thereby explicit free-field construction of permutation branes of Recknagel.

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0. Introduction

The investigation of D-branes on Calabi-Yau manifolds on string scales is interesting and important problem. There is a significant progress in this direction achieved mainly due to the intensive study of D-branes at Gepner points of Calabi-Yau moduli space initiated by Recknagel and Shomerus [1].

Because of Gepner models are defined by purely algebraic construction [2], [3] it is natural that the symmetry preserving boundary states (D-branes) in these models can be described by algebraic objects also [4]- [6]. Thus the question of their geometric interpretation appears to be nontrivial and interesting. The considerable progress in the understanding of the geometry of D-branes in Gepner models has been achieved recently in [7]- [16]. The main idea developed in these papers is to relate the intersection index of boundary states [17] with the bilinear form of the K-theory classes of bundles on the large volume CY manifold and use this relation to associate the K-theory classes to the boundary states.

The natural question which appears in this concern but hard to answer is if one can find direct CFT description of geometry of D-branes in Gepner models instead of interpolation of large-volume topological dates of bundles into the Gepner point? Trying to find the direct description (as well as to develop integral representation for the boundary correlation functions) the free-field construction of D-branes in Gepner models has been developed in [18]. It was shown there that the free-field representations of the open string spectrum between the Recknagel-Shomerus boundary states can be described in terms of representations of Malikov, Schechtman and Vaintrob chiral de Rham complex [19] on the Landau-Ginzburg orbifold. The chiral de
Rham complex is string generalization of the usual de Rham complex and is a sheaf of vertex algebras \[19\]-\[21\]. Hence it is a geometric object and this property has been used in \[18\] to interpret geometrically the boundary states in Gepner models (constructed in purely algebraic terms) as a fractional branes on Landau-Ginzburg orbifolds. This suggests that chiral de Rham complex might be natural and efficient object for the description of D-brane geometry at string scales.

Having this in mind we try in this note to extend the free-field representation of \[18\], \[22\] for the case of permutation branes \[6\]. Our aim here is to analyse and represent the free-field construction of permutation branes leaving the important question of study and comparing of the free-field geometry of D-branes to the results of \[23\], \[24\] for the future.

In section 1 we briefly review free-field construction of irreducible representations in \(N = 2\) minimal models developed by Feigin and Semikhatov. In section 2 we schematically consider the free-field realization of Gepner models. In section 3 we investigate A and B-type gluing conditions in terms of the free-fields. We start from the ansatz where the left-moving and right-moving free-field degrees of freedom are glued at the boundary by constant arbitrary matrix and analyse A and B-type boundary conditions in terms of free-fields. The section 4 is a main part of the paper. We analyse the consistency of the boundary conditions with the singular vectors structure of minimal models (butterfly resolution) and show that only permutation matrices survive giving thereby the free-field representation of permutation Ishibashi states. In section 5 we use the Recknagel solution \[6\] of Cardy’s constraints as well as orbifold construction to obtain free-field realization of permutations branes in Gepner models.

1. Free-field realization of \(N = 2\) minimal models irreducible representations.

In this section we briefly discuss free-field construction of Feigin and Semikhatov \[25\] of the irreducible modules in \(N = 2\) superconformal minimal models. Free-field approach to \(N = 2\) minimal models considered also in \[26\]-\[27\].

1.1. Free-field representations of \(N = 2\) super-Virasoro algebra.

We introduce (in the left-moving sector) the free bosonic fields \(X(z), X^*(z)\) and free fermionic fields \(\psi(z), \psi^*(z)\), so that its OPE’s are given by

\[
X^*(z_1)X(z_2) = \ln(z_{12}) + \text{reg.},
\]

\[
\psi^*(z_1)\psi(z_2) = z_{12}^{-1} + \text{reg.},
\]

where \(z_{12} = z_1 - z_2\). Then for an arbitrary number \(\mu\) the currents of \(N = 2\) super-Virasoro algebra are given by

\[
G^+(z) = \psi^*(z)\partial X(z) - \frac{1}{\mu}\partial\psi^*(z), \quad G^-(z) = \psi(z)\partial X^*(z) - \partial\psi(z),
\]

\[
J(z) = \psi^*(z)\psi(z) + \frac{1}{\mu}\partial X^*(z) - \partial X(z),
\]

\[
T(z) = \partial X(z)\partial X^*(z) + \frac{1}{2}(\partial\psi^*(z)\psi(z) - \psi^*(z)\partial\psi(z)) - \frac{1}{2}\partial^2 X(z) + \frac{1}{\mu}\partial^2 X^*(z),
\]

and the central charge is

\[
c = 3(1 - \frac{2}{\mu}).
\]
As usual, the fermions $\psi(z), \psi^*z, G^\pm(z)$ in NS (R) sector are expanded into half-integer (integer) modes. The bosons $X(z), X^*(z), J(z), T(z)$ are expanded into integer modes in both sectors.

In NS sector $N = 2$ Virasoro superalgebra is acting naturally in Fock module $F[p,p^*]$ generated by the fermionic operators $\psi^*[r], \psi[r], r < \frac{1}{2}$, and bosonic operators $X^*[n], X[n], n < 0$ from the vacuum state $|p,p^*\rangle$ such that

$$\psi[r]|p,p^*\rangle = \psi^*[r]|p,p^*\rangle = 0, r \geq \frac{1}{2},$$
$$X[n]|p,p^*\rangle = X^*[n]|p,p^*\rangle = 0, n \geq 1,$$
$$X[0]|p,p^*\rangle = p|p,p^*\rangle, \quad X^*[0]|p,p^*\rangle = p^*|p,p^*\rangle.$$ (4)

It is a primary state with respect to the $N = 2$ Virasoro algebra

$$G^\pm[r]|p,p^*\rangle = 0, r > 0,$$
$$J[n]|p,p^*\rangle = L[n]|p,p^*\rangle = 0, n > 0,$$
$$J[0]|p,p^*\rangle = \frac{j}{\mu}|p,p^*\rangle = 0,$$
$$L[0]|p,p^*\rangle = \frac{h(h+2)-j^2}{4\mu}|p,p^*\rangle = 0,$$ (5)

where $j = p^* - \mu p, \; h = p^* + \mu p$.

The character $f[0,p^*](q,u)$ of the Fock module $F[p,p^*]$ is given by

$$f[0,p^*](q,u) \equiv Tr_{F[p,p^*]}(q^{L[0]}-\frac{c}{24}uJ[0]) = q \frac{h(h+2)-j^2}{4\mu} \frac{\Theta(q,u)}{\eta(q)^3},$$ (6)

where the Jacobi theta-function

$$\Theta(q,u) = q^{\frac{1}{8}} \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}m^2} u^{-m}$$ (7)

and the Dedekind eta-function

$$\eta(q) = q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$$ (8)

have been used.

The $N = 2$ Virasoro algebra has the following set of automorphisms which is known as spectral flow [28]

$$G^\pm[r] \rightarrow G^\pm_t[r] \equiv G^\pm[r \pm t], \quad L[n] \rightarrow L_t[n] \equiv L[n] + tJ[n] + t^2 \frac{c}{6} \delta_{n,0}, \quad J[n] \rightarrow J_t[n] \equiv J[n] + t \frac{c}{3} \delta_{n,0},$$ (9)

where $t \in \mathbb{Z}$.

The spectral flow action on the free fields can be easily described if we bosonize fermions $\psi^*, \psi$

$$\psi(z) = \exp(-y(z)), \quad \psi^*(z) = \exp(+y(z)).$$ (10)
and introduce spectral flow vertex operator

$$U^t(z) = \exp(-t(y + \frac{1}{\mu}X^* - X)(z)).$$  \hspace{1cm} (11)$$

It gives the action of spectral flow on the modes of the free-fields

$$\psi[r] \rightarrow \psi[r - t], \ \psi^*[r] \rightarrow \psi^*[r + t],$$

$$X^*[n] \rightarrow X^*[n] + t\delta_{n,0}, \ X[n] \rightarrow X[n] - \frac{t}{\mu}\delta_{n,0}. \hspace{1cm} (12)$$

The action of the spectral flow on the vertex operator $V_{(p,p^*)}(z)$ is given by the normal ordered product of the vertex $U^t(z)$ and $V_{p,p^*}(z)$. It follows from (12) that spectral flow generates twisted sectors.

### 1.2. Irreducible $N = 2$ super-Virasoro representations and butterfly resolution.

The $N = 2$ minimal models are characterized by the condition that $\mu$ is integer and $\mu \geq 2$. In NS sector the irreducible highest-weight modules, constituting the (left-moving) space of states of the minimal model, are unitary and labeled by two integers $h,j$, where $h = 0, ..., \mu - 2$ and $j = -h, -h + 2, ..., h$. The highest-weight vector $|h,j\rangle$ of the module satisfies the conditions

$$G^\pm[r]|h,j\rangle = 0, r > 0,$$

$$J[n]|h,j\rangle = L[n]|h,j\rangle = 0, n > 0,$$

$$J[0]|h,j\rangle = \frac{j}{\mu}|h,j\rangle,$$

$$L[0]|h,j\rangle = \frac{h(h+2) - j^2}{4\mu}|h,j\rangle.$$  \hspace{1cm} (13)

The Fock modules are highly reducible representations of $N = 2$ Virasoro algebra and hence contain infinite number of singular vectors. To describe the singular vectors structure we introduce following to [25] the pair of fermionic screening currents $S^\pm(z)$ and the screening charges $Q^\pm$

$$S^+(z) = \psi^* \exp(X^*)(z), \ S^-(z) = \psi \exp(\mu X)(z),$$

$$Q^\pm = \oint dz S^\pm(z).$$  \hspace{1cm} (14)

The screening charges commute with the generators of $N = 2$ super-Virasoro algebra [4]. But they do not act within each Fock module. Instead they relate to each other the different Fock modules. The space where the screening charges are acting naturally is the direct sum of Fock modules

$$F_\pi = \oplus_{(p,p^*)\in \pi} F_{p,p^*}, \hspace{1cm} (15)$$

where $\pi$ is the lattice of momentums:

$$\pi = \{(p,p^*)|p = \frac{n}{\mu}, p^* = m, n, m \in \mathbb{Z}\}. \hspace{1cm} (16)$$

Application of the screening charge to an arbitrary vector $|p,p^*\rangle \in F_\pi$ gives the singular vector from another Fock module.
The screening charges are nilpotent and mutually anti-commute

\[(Q^+)^2 = (Q^-)^2 = \{Q^+, Q^-\} = 0.\] \hspace{1cm} (17)

Due to this important properties (17) one can combine the charges \(Q^\pm\) into BRST operator acting in \(F_\pi\) and build a BRST complex of Fock modules \(F_{p,p'} \in F_\pi\). This complex which has been constructed in [25] describes the structure of N=2 Virasoro superalgebra singular vectors and corresponding submodules such that the cohomology of the complex gives the irreducible module \(M_{h,j}\).

Let us consider first free-field construction for the chiral module \(M_{h,j} = h\). In this case the complex (which is known due to Feigin and Semikhatov as butterfly resolution) can be represented by the following diagram

\[\begin{array}{cccccccc}
  & & & & & \cdots & \leftarrow F_{1,h+\mu} & \leftarrow F_{0,h+\mu} \\
  & & & & \uparrow & & \uparrow & \\
  & & \cdots & \leftarrow F_{1,h} & \leftarrow F_{0,h} & \cdots & \leftarrow F_{-1,h-\mu} & \leftarrow F_{-2,h-\mu} & \leftarrow \cdots \\
  & & \uparrow & & \uparrow & & \uparrow & \\
  & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\end{array}\] \hspace{1cm} (18)

We shall denote this resolution by \(C_h\) and denote by \(\Gamma\) the set where the momentums of the Fock spaces of the resolution take values. The horizontal arrows in this diagram are given by the action of \(Q^+\) and vertical arrows are given by the action of \(Q^-\). The diagonal arrow at the middle of butterfly resolution is given by the action of \(Q^+ Q^-\) (which equals \(-Q^- Q^+\) due to (17)).

Ghost number operator \(g\) of the complex is defined for an arbitrary vector \(|v_{n,m}\rangle \in F_{n,m\mu+h}\) by

\[g|v_{n,m}\rangle = (n + m)|v_{n,m}\rangle, \text{ if } n, m \geq 0,
\]
\[g|v_{n,m}\rangle = (n + m + 1)|v_{n,m}\rangle, \text{ if } n, m < 0.\] \hspace{1cm} (19)

The main statement of [25] is that the complex (18) is exact except at the \(F_{0,h}\) module, where the cohomology is given by the chiral module \(M_{h,j=\h}\).

The butterfly resolution allows to write the character \(\chi_h(q,u) \equiv Tr_{M_{h,h}}(q^{L[0]} - \frac{c}{24} u^{J[0]})\) of the module \(M_{h,h}\) as the Euler characteristic of the complex:

\[\chi_h(q,u) = \chi_h^{(l)}(q,u) - \chi_h^{(r)}(q,u),\]

\[\chi_h^{(l)}(q,u) = \sum_{n,m \geq 0} (-1)^{n+m} f_{n,h+\mu}(q,u),\]

\[\chi_h^{(r)}(q,u) = \sum_{n,m > 0} (-1)^{n+m} f_{-n,h-\mu}(q,u),\] \hspace{1cm} (20)

where \(\chi_h^{(l)}(q,u)\) and \(\chi_h^{(r)}(q,u)\) are the characters of the left and right wings of the resolution.
To get the resolutions for other (anti-chiral and non-chiral) modules one can use the observation [25] that all irreducible modules can be obtained from the chiral modules $M_{h,j} = h$, $h = 0, ..., \mu - 2$ by the spectral flow action $U^{-t}$, $t = h, h - 1, ..., 1$. Equivalently one can restrict the set of chiral modules by the range $h = 0, ..., \lceil \frac{\mu}{2} \rceil - 1$ and extend the spectral flow action by $t = \mu - 1, ..., 1$ (when $\mu$ is even and $h = \lceil \frac{\mu}{2} \rceil - 1$ the spectral flow orbit becomes shorter: $t = \lceil \frac{\mu}{2} \rceil - 1, ..., 1$) [29]. Thus the set of irreducible modules can be labeled also by the set $(h, t) | h = 0, ..., \lceil \frac{\mu}{2} \rceil - 1, t = \mu - 1, ..., 0$, except the case when $\mu$ is even and the spectral flow orbit becomes shorter. It turns out that one can get all the resolutions by the spectral flow action also.

Due to this discussion it is more convenient to change the notation for irreducible modules. In what follows we shall denote the irreducible modules as $M_{h,t}$, indicating by $t$ spectral flow parameter.

As well as the modules and resolutions one can get the characters by the spectral flow action [25]:

$$\chi_{h,t}(q, u) = q^{\frac{\mu t^2}{2}} u^{\frac{\mu t}{2}} \chi_h(q, u^t).$$

(21)

There are the following important automorphism properties of irreducible modules and characters [25, 29].

$$M_{h,t} \equiv M_{h, -2, t - h - 1}, \chi_{h, t}(q, u) = \chi_{h, t - h - 1}(q, u), \quad (22)$$

$$M_{h,t} \equiv M_{h, t + \mu}, \chi_{h, t + \mu}(q, u) = \chi_{h, t}(q, u),$$

(23)

where $\mu$ is odd and

$$M_{h,t} \equiv M_{h, t + \mu}, \chi_{h, t + \mu}(q, u) = \chi_{h, t}(q, u), \quad h \neq \lceil \frac{\mu}{2} \rceil - 1,$$

$$M_{h,t} \equiv M_{h, t + \lceil \frac{\mu}{2} \rceil}, \chi_{h, t + \lceil \frac{\mu}{2} \rceil}(q, u) = \chi_{h, t}(q, u), \quad h = \lceil \frac{\mu}{2} \rceil - 1,$$

(24)

where $\mu$ is even.

Note that the butterfly resolution is not periodic under the spectral flow as opposed to the characters. It is also not invariant with respect to the automorphism [22]. Instead, the periodicity and invariance are recovered on the level of cohomology. Thus, $U^{\pm \mu}$ spectral flow and automorphism [22] are the quasi-isomorphisms of complexes.

The modules, resolutions and characters in R sector are generated from the modules, resolutions and characters in NS sector by the spectral flow operator $U^{-\frac{\mu}{2}}$.

2. Free-field realization of Gepner model.

2.1. Free-field realization of the product of minimal models.

It is easy to generalize the free-field representation of the Sect.1. to the case of tensor product of $r N = 2$ minimal models which can be characterized by $r$ dimensional vector $\mu = (\mu_1, ..., \mu_r)$, where $\mu_i \geq 2$ and integer.

Let $E$ be a real $r$ dimensional vector space and let $E^*$ be the dual space to $E$. Let us denote by $<,>$ the natural scalar product in the direct sum $E \oplus E^*$. In the subspaces $E$ and $E^*$ we fix the sets of basic vectors $R$ and $R^*$

$$R = \{ s_i, i = 1, ..., r \},$$

$$R^* = \{ \mu_i s_i^*, i = 1, ..., r \},$$

$$< s_i, s_j^* > = \delta_{i,j}. \quad (25)$$
According to the $\mathcal{R}$ and $\mathcal{R}^*$ we introduce (in the left-moving sector) the free bosonic fields $X_i(z), X_i^*(z)$ and free fermionic fields $\psi_i(z), \psi_i^*(z), i = 1, \ldots, r$ so that its singular OPE’s are given by (2) as well as the following fermionic screening currents and their charges

$$S_i^+(z) = s_i \psi^* \exp(s_i X^*)(z),$$
$$S_i^-(z) = s_i^* \psi \exp(\mu_i s_i^* X)(z),$$
$$Q_i^\pm = \oint dz s_i^\pm(z).$$

(26)

For each $i = 1, \ldots, r$ one can define by the formulas (2) N=2 $c_i = 3(1-\frac{2}{\mu_i})$ Virasoro superalgebra

$$G_i^+ = s_i \psi^* s_i^* \partial X - \frac{1}{\mu_i} s_i \partial \psi^*, \quad G_i^- = s_i^* \psi s_i \partial X^* - s_i^* \partial \psi,$$
$$J_i = (s_i \psi^* s_i^* \psi + \frac{1}{\mu_i} s_i \partial X^* - s_i^* \partial X),$$
$$T_i(z) = \frac{1}{2}(s_i \partial \psi^* s_i^* \psi - s_i \psi^* s_i^* \partial \psi) + s_i \partial X^* s_i^* \partial X - \frac{1}{2}(s_i^* \partial^2 X + \frac{1}{\mu_i} s_i \partial^2 X^*)$$

(27)

as well as the vertex operators

$$V(p_i, p_i^*) = \exp(p_i^* s_i^* X + p_i s_i X^*),$$

(28)

which are the conformal fields whose conformal dimensions and charges are labeled by integers $h_i = p_i^* + \mu_i p_i, j_i = p_i + \mu_i p_i$.

The vertex operators are naturally associated to the lattice $\Pi = P \oplus P^* \in E \oplus E^*$, where $P \in E, P^* \in E^*$ such that $P$ is generated by $\frac{1}{\mu_i} s_i$ and $P^*$ is generated by the basis $s_i^*, i = 1, \ldots, r$.

For an arbitrary vector $(p, p^*) \in \Pi$, we introduce in NS sector Fock vacuum state $\lvert \mathbf{p}, p^* \rangle$ and denote by $F_{\mathbf{p}, p^*}$ the Fock module generated from $\lvert \mathbf{p}, p^* \rangle$ by the creation operators of the fields $X_i(z), X_i^*(z), \psi_i(z), \psi_i^*(z)$.

Let $F_{\Pi}$ be the direct sum of Fock modules associated to the lattice $\Pi$. As an obvious generalization of the results from Sec.1, we form for each vector $\mathbf{h} = \sum_i h_i s_i^* \in P^*$, where $h_i = 0, 1, \ldots, \mu_i - 2$ butterfly resolution $C^*_\mathbf{h}$ as the product $\otimes_{i=1}^r C^*_{h_i}$ of butterfly resolutions of minimal models. The corresponding ghost number operator $g$ is given by the sum of ghost number operators of each of the resolutions. The differential $\partial$ acting on ghost number $N$ subspace of the resolution is given by the sum of differentials of each of the complexes $C^*_{h_i}$. It is obvious that the complex $C^*_\mathbf{h}$ is exact except at the $F_{0, \mathbf{h}}$ module, where the cohomology is given by the product $M_{\mathbf{h}, 0} = \otimes_{i=1}^r M_{h_i, 0}$ of the chiral modules of each minimal model. Hence one can represent the character

$$\chi_{\mathbf{h}, 0}(q, u) \equiv Tr_{M_{\mathbf{h}, 0}}(q^{L[0]} - \frac{c}{24})u^{J[0]}$$

(29)

of $M_{\mathbf{h}, 0}$ as the product of characters $\chi_{\mathbf{h}, 0}(q, u) = \prod_i \chi_{h_i, 0}(q, u)$.

According to the discussion at the end of Sec.1, we obtain the resolution and character for the product of arbitrary irreducible modules of minimal models acting on $C^*_\mathbf{h}$ by the spectral flow operators $U^{-t} = \prod_i U_i^{-t_i}$ of the minimal models. Hence one can label the resolutions, modules and characters by the pairs of vectors $(\mathbf{h}, \mathbf{t})$, from the set $\Delta = \{(\mathbf{h}, \mathbf{t}) | h_i = 0, \ldots, \lfloor \frac{d_i}{2} \rfloor - 1\}$. 
1, \ t_i = 0, \ldots, \mu_i - 1, \ i = 1, \ldots, r\}. \text{ On the equal footing one can use the set } \tilde{\Delta}' = \{(h,t)|h_i = 0, \ldots, \mu_i - 2, \ t_i = 0, \ldots, h_i, \ i = 1, \ldots, r\}.

It is also clear that R sector resolutions, modules and characters are generated from NS sector by the spectral flow \( U^{-v/2} = \prod_{i=1}^{r} U_i^{-1/2} \), where \( v = (1, \ldots, 1) \).

The same free-field realization can be used in the right-moving sector. Thus the sets of screening vectors \( \bar{\mathcal{R}} \) and \( \bar{\mathcal{R}}^* \) have to be fixed in the right-moving sector. It can be done in many ways, the only restriction is that the corresponding cohomology group has to be isomorphic to the space of states of the product of minimal models in the right-moving sector. Therefore \( \bar{\mathcal{R}} \) and \( \bar{\mathcal{R}}^* \) are determined modulo \( O(r,r) \) transformations which left unchanged the matrix of scalar products \( <s_i,s_j^*> \). In what follows we put \( \bar{\mathcal{R}} = \mathcal{R}, \bar{\mathcal{R}}^* = \mathcal{R}^* \).

Hence, one can use the same complex to describe the irreducible modules in the right-moving sector.

2.2. Free-field realization and Calabi-Yau extension.

It is well known that product of minimal models can not be applied straightforward to describe the string theory on 2D-dimensional CY manifold. First, one has to demand that \( \sum c_i = 3D \). Second, the so called simple current orbifold \( CY_{\mu} \) of the product of minimal models has to be constructed. The currents of \( N = 2 \) Virasoro superalgebra of this model are given by the sum of currents of each minimal model

\[
G^\pm(z) = \sum_i G^\pm_i, \\
J(z) = \sum_i J_i, \ T(z) = \sum_i T_i.
\]

The left-moving (as well as the right-moving) sector of the \( CY_{\mu} \) is given by projection of the space of states on the subspace of integer \( J[0] \)-charges and organizing the projected space into the orbits \( (h,t) \) under the spectral flow operator \( U^{-v} = \prod_{i=1}^{r} U_i \).

The partition function in NS sector of \( CY_{\mu} \) sigma model is diagonal modular invariant of the spectral flow orbits characters restricted to the subset of integer \( J[0] \) charges. From \( N = 2 \) Virasoro superalgebra representations there is no difference what of the sets \( \tilde{\Delta} \) or \( \tilde{\Delta}' \) we use to parameterize the orbit characters (though their free-field realizations are different). In what follows we combine these to sets into the extended set \( \Delta = \{(h,t)|h_i = 0, \ldots, \mu_i - 2, \ t_i = 0, \ldots, h_i, \ i = 1, \ldots, r\} \) and take into account this extension by corresponding multiplier ("field identification") \[2\].

The orbit characters (with the restriction on integer charges subspace) can be written in explicit form so that the structure of simple current extension becomes clear \[3\], \[31\]:

\[
ch_{h,t}(q,u) = \frac{1}{\kappa^2} \sum_{n,m=0}^{\kappa-1} T_{rM_{h,t}}(U^{nv}q^{(L[0] - \scriptstyle{\hat{\chi}^r})}u^{J[0]}\exp(\imath2\pi mJ[0])U^{-nv}) = \\
\frac{1}{\kappa^2} \sum_{n,m=0}^{\kappa-1} \chi_{h,t+nv}^\epsilon(\tau,\upsilon+m),
\]

where \( q = \exp(\imath2\pi\tau), \ u = \exp(\imath2\pi\upsilon) \) and \( \kappa = \text{lcm}\{\mu_i\} \). The partition function of \( CY_{\mu} \) model is
given by

\[
Z_{CY}(q, \bar{q}) = \frac{1}{2^r} \sum_{[h, t] \in \Delta_{CY}} \kappa |ch_{[h, t]}(q)|^2, \tag{33}
\]

where \(\Delta_{CY}\) denotes the subset of \(\Delta\) restricted to the space of integer \(J[0]\) charges. \([h, t]\) denotes the spectral flow orbit of the point \((h, t)\). Factor \(\frac{1}{2^r}\) corresponds to the extended set \(\Delta\) of irreducible modules and \(\kappa\) is the length of the orbit \([h, t]\). In general case the orbits with different lengths could appear but we will not consider these cases to escape the problem of fixed point resolution \([30], [31], [32]\).

2.3. Free-field realization of Gepner models.

The Gepner models \([2]\) of CY superstring compactification are given by (generalized) GSO projection \([2], [3]\) which is carrying out on the product of the space of states of \(CY\) model and space of states of external fermions and bosons describing space-time degrees of freedom of the string. In the framework of simple current extension formalism the Gepner’s construction has been farther developed in \([31], [32], [30]\).

Let us introduce so called supersymmetrized (Green-Schwarz) characters \([2], [3]\)

\[
Ch_{[h, t]}(q, \bar{q}) = \frac{1}{4\kappa^2} \sum_{n,m=0}^{2\kappa-1} Tr_{(M_{h,t} \otimes \Phi)}(U_{tot}^{-m} \exp (i\pi n J_{tot}[0]) q^{(L_{tot} - \frac{c_{tot}}{2})} u^{J_{tot}[0]} U_{tot}^{-m}), \tag{34}
\]

where the trace is calculated in the product of \(M_{h,t}\) and Fock module \(\Phi\) generated by the external (space-time) fermions and bosons in NS sector, \(J_{tot}[0]\) and \(L_{tot}[0]\) are zero modes of the total \(U(1)\) current and stress-energy tensor which includes the contributions from space-time degrees of freedom, \(c_{tot} = c + \frac{3}{2}(8 - 2D) = 12\) is a total central charge and \(U_{tot}\) is a total spectral flow operator acting in the product \(M_{h,t} \otimes \Phi\).

The modular invariant Gepner model partition function is given by \([2], [3], [31]\)

\[
Z_{Gep}(q, \bar{q}) = \frac{1}{2^r} (Im \tau)^{(1-D)/2} \sum_{[h, t] \in \Delta_{CY}} \kappa |Ch_{[h, t]}(q)|^2. \tag{35}
\]

3. The Ishibashi states in Fock modules.

The boundary states we are going to construct can be considered as a bilinear forms on the space of states of the model. Thus, it will be implied in what follows that the right-moving sector of the model is realized by the free-fields \(\bar{X}_i(\bar{z}), \bar{X}_i^*(\bar{z}), \bar{\psi}_i(\bar{z}), \bar{\psi}_i^*(\bar{z}), i = 1, ... , r\) and the right-moving \(N = 2\) super-Virasoro algebra is given by the formulas similar to \([2]\).

There are two types of boundary states preserving \(N = 2\) super-Virasoro algebra \([33]\), usually called \(B\)-type

\[
(L[n] - \bar{L}[-n])|B >> = (J[n] + \bar{J}[-n])|B >> = 0, \tag{36}
\]

\[
(G^+[r] + \eta \bar{G}^+[-r])|B >> = (G^-[r] + \eta \bar{G}^-[-r])|B >> = 0
\]

and \(A\)-type states

\[
(L[n] - \bar{L}[-n])|A >> = (J[n] - \bar{J}[-n])|A >> = 0, \tag{37}
\]

\[
(G^+[r] + \eta \bar{G}^-[-r])|A >> = (G^-[r] + \eta \bar{G}^+[-r])|A >> = 0
\]
where $\eta = \pm 1$.

In the tensor product of the left-moving Fock module $F_{p, p^*}$ and right-moving Fock module $F_{\bar{p}, \bar{p}^*}$ we construct the most simple states fulfilling the solutions (36) and (37). We shall call these states as Fock space Ishibashi [34] states.

3.1. B-type Ishibashi states in Fock module.

Let us consider in NS sector the following ansatz for fermions

$$\left( \psi^*_i[r] - \eta \Omega^*_{ij} \bar{\psi}^*_j[-r] \right)|p, p^*, \eta, B > > 0, 
\left( \psi_i[r] - \eta \Omega^*_{ij} \bar{\psi}_j[-r] \right)|p, p^*, \eta, B > > 0$$

where $\Omega_{ij}, \Omega^*_{ij}$ are the arbitrary nondegenerate matrices. Substituting these relations into (36) and using (27), (31) we find

$$\Omega_{ik} \Omega^*_{in} = \delta_{kn},$$
$$\Omega_{ij} d_i = d_j, \quad \Omega^*_{ij} d^*_i = d^*_j,$$
$$\bar{p}_k = -\Omega_{jk} p_j - d_k, \quad \bar{p}^*_k = -\Omega^*_{jk} p^*_j - d^*_k,$$

$$\left( \Omega_{jk} X_j[n] + \bar{X}_k[-n] + d_k \delta_{n,0} \right)p, p^*, \eta, B > > 0,$$
$$\left( \Omega^*_{jk} X^*_j[n] + \bar{X}^*_k[-n] + d^*_k \delta_{n,0} \right)p, p^*, \eta, B > > 0,$$

where $d_k = \frac{1}{\mu_k}, \quad d^*_k = 1$ and we combine these coefficients into the $r$-dimensional vectors $d = (d_1, ..., d_r), \quad d^* = (d^*_1, ..., d^*_r)$.

It is helpful to rewrite the boundary conditions in toric coordinates on the target space:

$$\theta_i[n] = \frac{i}{\sqrt{2\mu_i}}(X^*_i[n] - \mu_i X_i[n]), \quad R_i[n] = \frac{1}{\sqrt{2\mu_i}}(X^*_i[n] + \mu_i X_i[n]),$$
$$\gamma_i[s] = \frac{i}{\sqrt{2\mu_i}}(\psi^*_i[s] - \mu_i \psi_i[s]), \quad \sigma_i[s] = \frac{1}{\sqrt{2\mu_i}}(\psi^*_i[s] + \mu_i \psi_i[s]).$$

Then (38) and (39) take the form

$$\left( \sigma_i[s] - \frac{\eta}{2} \sqrt{\frac{\mu_j}{\mu_i}} \Omega_{ij} + \sqrt{\frac{\mu_j}{\mu_i}} \Omega^*_{ij} \bar{\sigma}_j[-s] - \frac{\eta}{2} \sqrt{\frac{\mu_j}{\mu_i}} \Omega_{ij} - \sqrt{\frac{\mu_j}{\mu_i}} \Omega^*_{ij} \bar{\sigma}_j[-s] \right)B > > 0,$$
$$\left( \gamma_i[s] + \frac{\eta}{2} \sqrt{\frac{\mu_j}{\mu_i}} \Omega_{ij} - \sqrt{\frac{\mu_j}{\mu_i}} \Omega^*_{ij} \bar{\sigma}_j[-s] - \frac{\eta}{2} \sqrt{\frac{\mu_j}{\mu_i}} \Omega_{ij} + \sqrt{\frac{\mu_j}{\mu_i}} \Omega^*_{ij} \bar{\sigma}_j[-s] \right)B > > 0,$$

$$\left( \bar{R}_j[-n] + \frac{1}{2} \left( \sqrt{\frac{\mu_i}{\mu_j}} \Omega^*_{ij} - \sqrt{\frac{\mu_i}{\mu_j}} \Omega_{ij} \right)R_i[n] - \frac{1}{2} \left( \sqrt{\frac{\mu_i}{\mu_j}} \Omega^*_{ij} - \sqrt{\frac{\mu_i}{\mu_j}} \Omega_{ij} \right) \theta_i[n] + \sqrt{2 \mu_i} \delta_{n,0} \right)B > > 0,$$
$$\left( \bar{\theta}_j[-n] + \frac{1}{2} \left( \sqrt{\frac{\mu_i}{\mu_j}} \Omega^*_{ij} - \sqrt{\mu_i} \Omega_{ij} \right)R_i[n] + \frac{1}{2} \left( \sqrt{\mu_i} \Omega^*_{ij} - \mu_i \Omega_{ij} \right) \theta_i[n] \right)B > > 0.$$
3.2. A-type Ishibashi states in Fock module.

The A-type Ishibashi states in Fock module can be found analogously. The linear ansatz for fermions has the form

\[(\psi_i^*[r] - m^r\Upsilon_{ij}\tilde{\psi}_j[-r])|p, p^*, \tilde{p}, \tilde{p}^*, \eta, A >= 0,\]
\[(\tilde{\psi}_i[r] - m^r\Upsilon_{ij}^*\tilde{\psi}_j^*[r])|\bar{p}, \bar{p}^*, \bar{\tilde{p}}, \bar{\tilde{p}}^*, \eta, A >= 0\]  

(44)

where \(\Upsilon_{ij}, \Upsilon_{ij}^*\) are the arbitrary nondegenerate matrices. Substituting these relations into (37) and using (2) we find

\[\tilde{p}_k = - \Upsilon_{jk}^*p_j^* - d_k, \quad \tilde{p}_k^* = - \Upsilon_{jk}p_j\]
\[(\Upsilon_{jk}X_j[n] + \tilde{X}_k[-n] + d_k^*\delta_{n,0})|p, p^*, \tilde{p}, \tilde{p}^*, \eta, A >= 0,\]
\[(\Upsilon_{jk}^*X_j^*[n] + \tilde{X}_k^*[-n] + d_k\delta_{n,0})|\bar{p}, \bar{p}^*, \bar{\tilde{p}}, \bar{\tilde{p}}^*, \eta, A >= 0.\]  

(45)

In the toric coordinates (40) the conditions take the form

\[(\sigma_i[s] - \frac{m^s\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\tilde{\sigma}_j[-s]) + \frac{m^s\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\tilde{\sigma}_j[-s])|A >= 0,\]
\[(\gamma_i[s] + \frac{m^s\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} - \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\tilde{\sigma}_j[-s]) + \frac{m^s\Upsilon_{ij}}{\sqrt{\mu_i\mu_j}} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*\tilde{\sigma}_j[-s])|A >= 0,\]
\[(\tilde{R}_j[-n] + \frac{1}{2}(\Upsilon_{ij} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*)R_i[n] + \frac{1}{2}(\Upsilon_{ij} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*)\tilde{R}_i[n] + \frac{1}{2}(\Upsilon_{ij} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*)\tilde{R}_i[n])|A >= 0.\]
\[(\tilde{\theta}_j[-n] + \frac{1}{2}(\Upsilon_{ij} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*)R_i[n] - \frac{1}{2}(\Upsilon_{ij} + \sqrt{\mu_i\mu_j}\Upsilon_{ij}^*)\tilde{R}_i[n])|A >= 0.\]  

(46)

The reality constraint takes the form

\[\Upsilon_{ij}^* = \frac{1}{\mu_i\mu_j}\tilde{\Upsilon}_{ij}.\]  

(47)

The linear A-type Ishibashi state in NS sector is given similar to B-type

\[|p, p^*, \Upsilon, \eta, A >= \prod_{n=1}\exp\left(-\frac{1}{n}(X_i[-n]\Upsilon_{ik}\tilde{X}_k[-n] + X_i^*[n]\Upsilon_{ik}^*\tilde{X}_k^*[n])\right)\]
\[\prod_{r=1/2}\exp(m(\psi_i[-r]\Upsilon_{ik}\tilde{\psi}_k[-r] + \psi_i^*[r]\Upsilon_{ik}^*\tilde{\psi}_k^*[r]))|p, p^*, -(\Upsilon^*)^T p^* - d, -\Upsilon^T p - d^* >.\]  

(48)

4. Permutation Ishibashi states in the product of minimal models.

4.1. B-type permutation Ishibashi states.

Free-field realizations of the irreducible modules described in Sect. 1.2 and the constructions (43, 48) allows to suggest that Ishibashi states in the product of minimal models can also be represented by the free-fields. Let us consider the following superposition of B-type Fock modules Ishibashi states (43)

\[|I_h, \Omega, \eta, B >= \delta(\Omega h - h) \sum_{(p, p^*) \in \Gamma_h} c_{p, p^*} |p, p^*, \Omega, \eta, B >,\]  

(49)
where the coefficients $c_{\mathbf{p}, \mathbf{p}^*}$ are arbitrary and the summation is performed over the momentums of the butterfly resolution $C_h^\star$. Since the partition function is diagonal the delta-function $\delta(\Omega - h)$ has been inserted. It is clear that this state satisfies the relations (56).

Before GSO projection, the closed string states of the model which can interact with the Ishibashi state come from the product of left-moving and right-moving Fock modules $F_{\mathbf{p}, \mathbf{p}^*} \otimes \bar{F}_{-(\Omega^\star)^T \mathbf{p} - d^*}$, where $(\mathbf{p}, \mathbf{p}^*) \in \Gamma_h$. The left-moving modules of the superposition constitute the butterfly resolution $C_h$ whose cohomology is given by the module $M_h$. What about the Fock modules from the right-moving sector? To have nontrivial interaction with the states from the model the right-moving Fock modules have to from the product of resolutions of minimal models also. But this contradicts to the relations between left-moving and right-moving momentums from (39). This contradiction may be resolved if we allow that right-moving Fock modules form the product of resolutions each of which is dual to $\Gamma_h$. The dual resolution $\bar{C}_h$ to the minimal model resolution is given by the following diagram

\[
\begin{array}{cccc}
\vdots & \vdots \\
\downarrow & \downarrow \\
\cdots & \rightarrow F_{\frac{1}{\mu}, -1 - h - \mu} & \rightarrow F_{\frac{1}{\mu}, -1 - h - \mu} & \\
\downarrow & \downarrow & \\
\cdots & \rightarrow F_{\frac{1}{\mu}, -1 - h} & \rightarrow F_{\frac{1}{\mu}, -1 - h} & \\
\downarrow & \downarrow & \downarrow & \\
F_{\frac{1}{\mu}, -1 - h + \mu} & \rightarrow F_{\frac{1}{\mu}, -1 - h + \mu} & \rightarrow \cdots \\
\downarrow & \downarrow & \\
F_{\frac{1}{\mu}, -1 - h + 2\mu} & \rightarrow F_{\frac{1}{\mu}, -1 - h + 2\mu} & \rightarrow \cdots \\
\downarrow & \downarrow & \\
\vdots & \vdots & \\
\end{array}
\]

(here, $h$ is an integer number taking values from 0 to $\mu - 2$) The arrows on this diagram are given by the same operators as on the diagram (18).

Hence the right-moving Fock modules have to form dual resolution $\bar{C}_h = \otimes_{i=1}^r \bar{C}_{h_i}$ and matrices $\Omega^T$, $(\Omega^\star)^T$ have to map the set of left-moving momentums $\Gamma_h$ on the set of momentums $\Gamma_h$ which has to be isomorphic to $\Gamma_h$. Therefore we conclude that $\Omega^T$ has to be an element of the direct product of permutation groups $\mathbb{R}_{r_i}$ of $r_i$-elements

\[
\Omega \in \mathbb{R}_{r_1} \times \cdots \times \mathbb{R}_{r_N} = \mathbb{R}_{r_1} \otimes \mathbb{R}_{r_2} \cdots \otimes \mathbb{R}_{r_N},
\]

which are determined by the sets $r_1, ..., r_N$ of coinciding elements in the vector $\mu$. In other words, it is implied here that we have $\mu_1 = ... = \mu_{r_1}$, $\mu_{r_1 + 1} = ... = \mu_{r_1 + r_2}$, ... In view of (52) we have also

\[
\Omega_{ij} = \Omega_{ij}.
\]
Thus the relations (41) take the form
\[
(σ_i[s] - ηΩ_{ij}σ_j[-s])|B| = 0,
\]
\[
(γ_i[s] - ηΩ_{ij}γ_j[-s])|B| = 0,
\]
\[
(\bar{R}_j[-n] + Ω_{ij}R_i[n] + \sqrt{2\mu_j}δ_{n,0})|B| = 0,
\]
\[
(\bar{θ}_j[-n] + Ω_{ij}θ_i[n])|B| = 0.
\]

Hence, \(i\)-th minimal model in the right-moving sector interacts to \(Ω^{-1}(i)\)-th minimal model from the left-moving sector.

Having the matrix \(Ω\) fixed by (51) one can define the coefficients \(c_{p,p^*}\) from the \(BRST\) invariance condition. It is a straightforward generalization of the condition found in [22] for \(N = 2\) minimal models. To formulate this condition one has to describe by the free fields the total space of states of the model.

To do that we form first the product of complexes \(C_h \otimes \bar{C}_h\) to build the complex
\[
\ldots \to C_h^{-2} \to C_h^{-1} \to C_h^0 \to C_h^{+1} \to \ldots,
\]
which is graded by the sum of the ghost numbers \(g\) and \(\bar{g}\). For an arbitrary ghost number \(I\) the space \(C_h^I\) is given by the sum of products of the Fock modules from the resolutions \(C_h\) and \(\bar{C}_h\) such that \(g + \bar{g} = I\). The differential \(δ\) of the complex is defined by the differentials \(\partial\) and \(\bar{\partial}\) of the complexes \(C_h\) and \(\bar{C}_h\)
\[
δ|v_g \otimes \bar{v}_{\bar{g}}| = |\partial v_g \otimes \bar{v}_{\bar{g}}| + (-1)^g|v_g \otimes \partial \bar{v}_{\bar{g}}|,
\]
where \(|v_g|\) is an arbitrary vector from the complex \(C_h\) with ghost number \(g\), while \(|\bar{v}_{\bar{g}}|\) is an arbitrary vector from the complex \(\bar{C}_h\) with the ghost number \(\bar{g}\) and \(g + \bar{g} = I\). The cohomology of the complex (54) is nonzero only at grading 0 and is given by the product of irreducible modules \(M_h \otimes \bar{M}_{h,t=2h}\), where \(M_{h,t=2h}\) is the product of anti-chiral modules of minimal models.

The Ishibashi state we are looking for can be considered as a linear functional on the Hilbert space of the product of models, then it has to be an element of the homology group. Therefore, the \(BRST\) invariance condition for the state can be formulated as follows.

Let us define the action of the differential \(δ\) on the state \(|I_h, Ω, \eta, B|\) by the formula
\[
<< δ^*|I_h, Ω, \eta, B|v_g \otimes \bar{v}_{\bar{g}}| >> \equiv << I_h, Ω, \eta, B|δ_{g+\bar{g}}v_g \otimes \bar{v}_{\bar{g}}| >>,
\]
where \(v_g \otimes \bar{v}_{\bar{g}}\) is an arbitrary element from \(C_h^{g+\bar{g}}\). Then, \(BRST\) invariance condition means that
\[
δ^*|I_h, Ω, \eta, B| = 0.
\]

As a straightforward generalization of Theorem 2 from [22] we find that superposition (49) satisfies \(BRST\) invariance condition (57) if the coefficients \(c_{p,p^*}\) take values \(±1\) according to the expression
\[
c_{p,p^*} = \sqrt{2} \cos((2g_{p,p^*} + 1)\frac{π}{4})c_{0,h},
\]
where \(g_{p,p^*}\) is the ghost number.
Thus the superposition (59) respects the singular vector structure of the product of minimal N=2 Virasoro algebra representations and gives explicit construction of permutation Ishibashi states. Note also that BRST condition doesn’t fix the phase of the overall coefficient c_{0,h}.

Now we consider the closed string transition amplitude between a pair of permutation Ishibashi states with the permutations Ω' and Ω. It is given by the following expression

\[<< I_{h'}, \Omega', \eta, B | (-1)^{g(\Omega', \Omega)} q^{L[0]} - \frac{c}{\pi^2} u^{J[0]} | I_h, \Omega, \eta, B >> = \delta(h - h') \delta(\Omega' h' - h') \delta(\Omega h - h) \]

\[\sum_{(p, p') \in \Gamma_h} (-1)^{g(\Omega', \Omega)} |c_{p, p'}|^2 \delta(\Omega' \Omega^{-1} p - p') \delta(\Omega' \Omega^{-1} p^* - p^*) \]

\[<< p, p^*, \Omega', \eta | (-1)^{g(\Omega', \Omega)} q^{L[0]} - \frac{c}{\pi^2} u^{J[0]} | p, p^*, \Omega, \eta, B >> . \quad (59)\]

Due to the insertion \((-1)^{g(\Omega', \Omega)}\) the amplitude is calculating according to the ghost number of the intermediate closed string states and the ghost number operator \(g(\Omega', \Omega)\) depends on the permutation matrices. To simplify the calculation we put here the number \(N\) of permutation groups to be equal 1 (and hence \(\mu_1 = ... \mu_r = \mu\)). Due to the factor \(\delta(\Omega' \Omega^{-1} p - p') \delta(\Omega' \Omega^{-1} p^* - p^*)\) the summation is restricted to the subspace of \(\Gamma_h\) which is invariant with respect to the permutation \(\Omega' \Omega^{-1}\). It allows us to write

\[<< p, p^*, \Omega', \eta | (-1)^{g(\Omega', \Omega)} q^{L[0]} - \frac{c}{\pi^2} u^{J[0]} | p, p^*, \Omega, \eta, B >> = q^{\frac{1}{2}(\sum_{i \in \Omega(\Omega^c)} 2p_i^2 p_i + p_i^2 \rho_i^2 + \rho_i^2 \rho_i + 2\rho_i \rho_i) - \frac{c}{\pi^2} u^{\sum_{i \in \Omega(\Omega^c)} 2p_i^2 p_i + p_i^2 \rho_i^2 + \rho_i^2 \rho_i + 2\rho_i \rho_i}} (oscillator contribution), \quad (60)\]

where \(|\Xi|\) is the length of \(i_{th}\) cycle of the permutation \(\Xi \equiv \Omega' \Omega^{-1}\) and \(\nu(\Xi)\) is the number of cycles of the permutation.

The oscillator contribution calculation is useful to carry out for the bosons and fermions separately. The bosonic contribution can be calculated as follows. First of all we have from (59)

\[\prod_{a,c} < p, p^*, -(\Omega')^T p - d, -(\Omega')^T p^* - d^* | \prod_{n=1} \exp(-\frac{1}{n} X_a[n] \tilde{X}_{\Omega(a)}[n]) \prod_{m=1} \exp(-\frac{q}{m} X_c[-m] \tilde{X}_{\Omega(c)}[-m]) | p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >= \]

\[\prod_{n=1}^{k_1+\cdots+k_r} \sum_{l_1+\cdots+l_r=0}^{1} \frac{1}{n^{k_1+\cdots+k_r}} q^{\sum_{n=1}^{k_1+\cdots+k_r}} \]

\[< p, p^*, -(\Omega')^T p - d, -(\Omega')^T p^* - d^*[X_1[n]]^{k_1} (\tilde{X}_{\Omega(1)}[n])^{k_1} \cdots [X_r[n]]^{k_r} (\tilde{X}_{\Omega(r)}[n])^{k_r} (X_1[-n])^{l_1} (\tilde{X}_{\Omega(1)}[-n])^{l_1} \cdots (X_r[-n])^{l_r} (\tilde{X}_{\Omega(r)}[-n])^{l_r} | p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >= \]

\[\prod_{n=1} (1 - q^n)^{|\Xi|-1} (1 - q^{n|\nu(\Xi)|})^{-1}. \quad (61)\]
By the similar reasons

\[
\prod_{a,c} \langle p, p^*, -\Omega'^T p - d, -\Omega'^T p^* - d^* \mid \prod_{n=1}^{\infty} \exp(-\frac{1}{n} X_n \mid \tilde{X}_{\tilde{\Omega}(a)} \mid n) \rangle \\
\prod_{m=1} \exp(-\frac{m}{m} (X_c \mid -m \mid \tilde{X}_{\tilde{\Omega}(c)} \mid -m))) \mid p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >
\]

\[
(1 - q^{n \mid \Xi \mid 1})^{-1} \cdots (1 - q^{n \mid \Xi \mid \nu(\Xi)})^{-1}. \quad (62)
\]

The first part of the fermionic contribution is given by

\[
\prod_{b,d} \langle p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* \mid \prod_{r=1/2} (1 - m\eta \psi_b^* [r] \tilde{\psi}_{\tilde{\Omega}(b)} [r]) \rangle \\
\prod_{s=1/2} (1 + m\eta u^{-1} q^s \psi_d [-s] \tilde{\psi}_{\tilde{\Omega}(d)} [-s]) \mid p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >
\]

\[
\prod_{s=1/2} (1 - (-1)^{\Xi \mid 1} u^{-1} \mid \Xi \mid q^s \mid \Xi \mid 1) \cdots (1 - (-1)^{\Xi \mid \nu(\Xi)} u^{-1} \mid \Xi \mid \nu(\Xi) q^s \mid \Xi \mid \nu(\Xi)). \quad (63)
\]

where \( Tr(\Xi) \mid _{\wedge k} V \) means the trace of matrix \( \Xi = \Omega' \Omega^{-1} \) acting by permutation components in the space \( \wedge k V \) of the \( r \)-dimensional real vector space \( V \). One can see that the last expression can be rewritten similar to (61)

\[
\prod_{b,d} \langle p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* \mid \prod_{r=1/2} (1 - m\eta \psi_b^* [r] \tilde{\psi}_{\tilde{\Omega}(b)} [r]) \rangle \\
\prod_{s=1/2} (1 + m\eta u^{-1} q^s \psi_d [-s] \tilde{\psi}_{\tilde{\Omega}(d)} [-s]) \mid p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >
\]

\[
\prod_{s=1/2} (1 - (-1)^{\Xi \mid 1} u^{-1} \mid \Xi \mid q^s \mid \Xi \mid 1) \cdots (1 - (-1)^{\Xi \mid \nu(\Xi)} u^{-1} \mid \Xi \mid \nu(\Xi) q^s \mid \Xi \mid \nu(\Xi)). \quad (64)
\]

Analogously

\[
\prod_{a,c} \langle p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* \mid \prod_{r=1/2} (1 - m\eta \psi_a^* [r] \tilde{\psi}_{\tilde{\Omega}(a)} [r]) \rangle \\
\prod_{s=1/2} (1 + m\eta u^{-1} q^s \psi_c [-s] \tilde{\psi}_{\tilde{\Omega}(c)} [-s]) \mid p, p^*, -\Omega^T p - d, -\Omega^T p^* - d^* >
\]

\[
\prod_{s=1/2} (1 - (-1)^{\Xi \mid 1} u^{-1} \mid \Xi \mid q^s \mid \Xi \mid 1) \cdots (1 - (-1)^{\Xi \mid \nu(\Xi)} u^{-1} \mid \Xi \mid \nu(\Xi) q^s \mid \Xi \mid \nu(\Xi)). \quad (65)
\]

Collecting the results we obtain

\[
<< H_{\Omega'}, \Omega', \eta, B \mid (1 - g(\Omega', \Omega)) q^{L[0] - \frac{1}{2}} u^{J[0]} \mid H_{\Omega'}, \Omega, \eta, B >> = \\
\delta(\mathbf{h} - \mathbf{h}') \delta(\Omega' \mathbf{h} - \mathbf{h}) \delta(\Omega \mathbf{h} - \mathbf{h}) |_{c_0, h}^2 \sum_{(p, p^*) \in \Gamma_h} (-1)^{g(\Omega', \Omega)} \delta(\Xi \mathbf{p} - \mathbf{p}) \delta(\Xi \mathbf{p}^* - \mathbf{p}^*) \]

\[
\sum_{\nu(\Xi)} \frac{1}{2} \left( \sum_{i=1}^{\nu(\Xi)} \mid \Xi \mid (2p_i^* p_i + p_i + \frac{m}{n}) - \frac{1}{2} u \sum_{i=1}^{\nu(\Xi)} \mid \Xi \mid (\frac{m}{n} - p_i) \right)
\]

\[
\prod_{i=1}^{\nu(\Xi)} \prod_{n=1}^{(1 - (-1)^{\Xi \mid 1} u^{-1} \mid \Xi \mid q^{(n-1)/2} \mid \Xi \mid 1)} (1 - (-1)^{\Xi \mid 1} u^{-1} \mid \Xi \mid q^{(n-1)/2} \mid \Xi \mid 1) (1 - \bar{q}^{n \mid \Xi \mid 1})^{-2}. \quad (66)
\]
The transition amplitude between $|I_{h'}, \Omega', -\eta, B >>$ and $|I_h, \Omega, \eta, B >>$ is given by the similar expression. Indeed, the change $\eta \rightarrow -\eta$ affects only on the fermionic contribution (63) so that
\[
<< I_{h'}, \Omega', -\eta, B | (-1)^g(\Omega', \Omega) q^{L[0]} - \frac{e^{i\pi}}{2} u^{J[0]} | I_h, \Omega, \eta, B >> =
\]
\[
\delta(h - h') \delta(\Omega' h - h) \delta(\Omega h - h) |c_{0, h}|^2 \sum_{(p, p') \in \Gamma_h} \nu(\Xi) q^{\Xi} (\sum_{i=1}^{\nu(\Xi)} \Xi_i (2p_i^2 + p_i^2 + \frac{\mu^2}{\mu^2}) - \frac{e^{i\pi}}{2} \sum_{i=1}^{\nu(\Xi)} |\Xi_i|^2 - p_i)
\]
\[
\prod_{i=1}^{\nu(\Xi)} \prod_{n=1}^{\nu(\Xi)} (1 - u^{-|\Xi_i|} q^{(n-1/2)|\Xi_i|}) (1 - u^{-|\Xi_i|} q^{(n-1/2)|\Xi_i|}) (1 - q^n|\Xi_i|)^{-2}. 
\]

Now one can fix the dependence of the ghost number operator on the permutation matrices. Taking into account the representation (20) we find that the amplitude is given by the product of minimal model characters if
\[
g(\Omega', \Omega) = \sum_{i=1}^{\nu(\Xi)} g_i. 
\]

Thus the ghost number receives the contribution $g_i$ from $i_{th}$ invariant subspace of $\Gamma_h$. In other words we consider the space of intermediate closed string states as a product of minimal model butterfly resolutions (18) in amount of the number $\nu(\Xi)$. Hence the amplitude is given by the following product of minimal model characters
\[
<< I_{h'}, \Omega', \eta, B | (-1)^g(\Omega', \Omega) q^{L[0]} - \frac{e^{i\pi}}{2} u^{J[0]} | I_h, \Omega, \eta, B >> =
\]
\[
\delta(h - h') \delta(\Omega' h - h) \delta(\Omega h - h) |c_{0, h}|^2 \prod_{i=1}^{\nu(\Xi)} \exp(-i\pi(1 - |\Xi_i|) \frac{h_i}{\mu}) \chi_{h_i}(\tau, \nu) \exp(-i\pi(1 - |\Xi_i|) \frac{h_i}{\mu}) \chi_{h_i}(\tau, \nu + \frac{1}{2}) 
\]

where $f$ is fermion number operator and we have used the relation
\[
Tr_{M_{h_i}}((-1)^{|\Xi_i|} f q^{L[0]} - \frac{e^{i\pi}}{2} u^{J[0]}) = \exp(-i\pi(1 - |\Xi_i|) \frac{h_i}{\mu}) \chi_{h_i}(\tau, \nu + \frac{1}{2}) 
\]

Analogously
\[
<< I_{h'}, \Omega', -\eta, B | (-1)^g(\Omega', \Omega) q^{L[0]} - \frac{e^{i\pi}}{2} u^{J[0]} | I_h, \Omega, \eta, B >> =
\]
\[
\delta(h - h') \delta(\Omega' h - h) \delta(\Omega h - h) |c_{0, h}|^2 \prod_{i=1}^{\nu(\Xi)} \exp(-i\pi \frac{h_i}{\mu}) \chi_{h_i}(\tau, \nu) \exp(-i\pi \frac{h_i}{\mu}) \chi_{h_i}(\tau, \nu + \frac{1}{2}) 
\]

It was mentioned in Section 2 that the irreducible representations are generated by the spectral flow action. Hence for an arbitrary module $M_{h, t}, (h, t) \in \Delta$, the Ishibashi state is generated by the action of spectral flow operators on the Ishibashi state (49). It is easy to check that the state
\[
|I_{h, t}, \Omega, \eta, B >> = \prod_i U_i^{t_i} U_i^{-t_i} |I_h, \Omega, \eta, B >>, 
\]
satisfy B-type boundary conditions if
\[
\Omega t - t = 0. 
\]
One has to take into account however that the right-moving space of states of the model is governed by dual butterfly resolutions (twisted by right-moving spectral flow operators). The representative of chiral primary field from the dual resolution is

$$U^h G^+ \left[ \frac{1}{2} - h \right] \ldots G^+ \left[ - \frac{1}{2} \right] - \frac{1}{\mu}, -1 - h \succ \left| - \frac{1}{\mu} + h \right|, -1 \succ . \quad (74)$$

Thus the highest-weight vectors of the model are given by the products of the following minimal model states

$$|p_i = \frac{t_i}{\mu_i} \rangle, p_i^\ast = h_i - t_i, \bar{p}_i = -1 + \frac{1 + h_i - t_i}{\mu_i} - l, \bar{p}_i^\ast = - 1 - t_i - l \succ . \quad (75)$$

Therefore the Ishibashi states (72) have nontrivial overlap with the states (75) if in addition to (73) we have

$$h_{\Omega^{-1}(i)} = h_i, \quad h_i - 2t_i - l = 0 \ mod \ \mu_i. \quad (76)$$

It is easy to see from (12) that this state satisfy the boundary conditions (38), (39). Hence (36) is fulfilled. It is also BRST closed because the spectral flow commutes with screening charges.

The transition amplitude between such states is spectral flow twist of the amplitudes (69), (71).

$$\langle \langle I_{h',t',\Omega',\eta',B}|(-1)^{g(\Omega',\Omega)}q^{L[0]} - \sum_{i,t} J[0]|I_{h,t,\Omega,\eta,B} \rangle \rangle = \delta(h - h')\delta(\Omega' - h)\delta(\Omega - h)\delta(\mu)\delta(t - t')$$

$$|c_{0,h}^\nu(\Xi)\prod_{i=1}^{\nu(\Xi)} \exp(-i\pi(1 - |\Xi|_i) \frac{h_i - 2t_i}{\mu_i})\chi_{h_i,t_i}(\tau, v + \frac{1 - |\Xi|_i}{2}) \rangle. \quad (77)$$

$$\langle \langle I_{h',t',\Omega',-\eta,B}|(-1)^{g(\Omega',\Omega)}q^{L[0]} - \sum_{i,t} J[0]|I_{h,t,\Omega,\eta,B} \rangle \rangle = \delta(h - h')\delta(\Omega' - h)\delta(\Omega - h)\delta(\mu)\delta(t - t')$$

$$|c_{0,h}^\nu(\Xi)\prod_{i=1}^{\nu(\Xi)} \exp(-i\pi(1 - |\Xi|_i) \frac{h_i - 2t_i}{\mu_i})\chi_{h_i,t_i}(\tau, v + \frac{1}{2}) \rangle. \quad (78)$$

4.2. A-type permutation Ishibashi states.

Let us consider free-field representation for A-type Ishibashi states. It is obvious that A-type Ishibashi states are given by superpositions like (49). Similar to the B-type case one can conclude that matrix $\Upsilon^T$ is proportional to the element of the permutation group $\mathbb{N}_{r_1 \ldots r_N}$. More precisely

$$\Upsilon = \mu_1 \Omega_1 \otimes \ldots \otimes \mu_N \Omega_N, \quad \Upsilon^* = \frac{1}{\mu_1} \Omega_1 \otimes \ldots \otimes \frac{1}{\mu_N} \Omega_N \quad (79)$$

where $\Omega_i \in \mathbb{N}_{r_i}, i = 1, \ldots, N$.
The boundary conditions (46) take the form which is mirror to (53)

\[
\begin{align*}
(\sigma_i[s] - m\Omega_{ij}\bar{\sigma}_j[-s])|A>>&= 0, \\
(\gamma_i[s] + m\Omega_{ij}\bar{\gamma}_j[-s])|A>>&= 0, \\
(\bar{R}_j[-n] + \Omega_{ij}R_i[n] + \sqrt{2}\mu_j\delta_{n,0})|A>>&= 0, \\
(\bar{\theta}_j[-n] - \Omega_{ij}\theta_i[n])|A>>&= 0.
\end{align*}
\]

(80)

The BRST condition for A-type states is slightly different from B-type case. The reason is that the application of one of the left-moving BRST charges, say \(Q^+_i\) to A-type state gives according to (44) and (45) the right-moving BRST charge \(\bar{Q}^-_i\) multiplied by \(\mu_i\) as opposed to the B-type case. In fact we are free to rescale arbitrary the right-moving BRST charges because it does not change the cohomology of the complex in the right-moving sector and the cohomology of the total complex (54). Hence we define the right-moving BRST charges in such a way to cancel this effect

\[
\begin{align*}
\bar{S}^+_i(z) &= \frac{m}{\mu_i} s_i \bar{\psi}^* \exp(s_i\bar{X}^*)(z), \\
\bar{S}^-_i(z) &= m\mu_i s_i^* \bar{\psi} \exp(\mu_i s_i^* \bar{X})(z), \\
Q^+_i &= \oint d\bar{z} \bar{S}^+_i(z),
\end{align*}
\]

(81)

As a result BRST invariant A-type Ishibashi state |\(I_h, \Omega, \eta, A\rangle\rangle\) is given similar to (49), (58) with the restriction \(\delta(\Omega h - h)\) and similar to B-type case the phase of coefficient \(c_{0,h}\) is arbitrary also.

A-type version of the transition amplitude (69) can be calculated similar to B-type case so the result is given by

\[
\langle\langle I_{h'}, \Omega', \eta, A|(-1)^{\varphi(\Omega',\Omega)} q^{L[0]} - \varpi u^{J[0]}|I_h, \Omega, \eta, A\rangle\rangle = \\
\delta(h - h')\delta(\Omega' h - h)\delta(\Omega h - \Omega')|c_{0,h}|^2 \\
\prod_{i=1}^{\nu(\Xi)} \exp(-i\pi(1 - |\Xi|\bar{h}_i)v\frac{1 - |\Xi|\bar{h}_i}{2}) \chi(h, \tau, v + \frac{1 - |\Xi|\bar{h}_i}{2}),
\]

(82)

where \(\Xi = \Omega'\Omega^{-1}\) and we have putted for simplicity the number \(N\) of permutation groups to be equal 1.

For an arbitrary module \(M_{h,t}\), \((h, t) \in \Delta\), the A-type Ishibashi state is generated by the action of spectral flow operators. It is easy to check that the state

\[
|I_{h,t}, \Omega, \eta, A\rangle\rangle = \prod_i U_{t_i}^{h_i} |I_h, \Omega, \eta, A\rangle\rangle,
\]

(83)

satisfy A-type boundary condition if the spectral flow parameter \(t\) satisfy (73). Though the right-moving space of states of the model is governed by dual butterfly resolutions (twisted by right-moving spectral flow operators) the only restrictions on \(h, t\) are

\[
\Omega h = h, \ \Omega t = t.
\]

(84)
Corresponding transition amplitude is given similar to B-type case.

\[
<< I_{h',t', \Omega', \eta, A} | (-1)^{g(\Omega', \Omega)} q^{L[0]} u^{J[0]} | I_{h,t}, \Omega, \eta, A >> = \delta(h - h') \delta(\Omega' h - h) \delta(\Omega h - h) \delta(\mu(t - t')) |c_0,h|^2 \\
\prod_{i=1}^{\nu(\Xi)} \exp(-i\pi(1 - |\Xi|_i) h_i - 2t_i) \chi_{h_i,t_i}(\tau, v + \frac{1 - |\Xi|_i}{2}).
\]

(85)

Thus the expressions (85), (86) reproduce correctly (with the correct fermionic contribution) the corresponding results from [6]. It allows to use the Cardy’s constraint solution found for permutation branes in [6] to construct the free-field representation of permutation branes.

5. Free-field representation of permutation branes in Gepner model.

5.1. A-type boundary states in Calabi-Yau extension.

It has already been noticed that the product of minimal models can not be applied straightforward to describe in the bulk the string theory on CY manifold. Instead, the so called simple current orbifold whose partition function is diagonal modular invariant partition function with respect to orbit characters describes. The extension of this technique to the conformal field theory with a boundary has been developed in [1], [6], [31], [5], [8].

As we have seen \( BRST \) invariance fixes the free-field permutation Ishibashi states up to the arbitrary constant \( c_{h,t} \). Hence our problem is to apply the (simple current) orbifold construction and Cardy’s constraint to the superposition of free-field permutation Ishibashi states with arbitrary coefficients \( c_{h,t} \). Fortunately the Cardy’s constraint for the perturbation branes has been found by Recknagel [6]. Hence it is sufficient only to quote his solution.

Thus the free-field realization of permutation A-type branes can be given as follows. We start first from the spectral flow invariant permutation boundary states

\[
|[[\Lambda, \lambda], \Omega, \eta, A >> = \frac{\alpha}{\kappa^2} \sum_{(h,t) \in \Delta} \delta(\Omega h - h) \delta(\Omega t - t) \\
W_{h,t}^{\Lambda, \lambda, \Omega} \sum_{m,n=0}^{\kappa-1} \exp(i2\pi n J[0]) U^{mv} U^{mv} |I_{h,t}, \Omega, \eta, A >> .
\]

(87)

They are labeled by the spectral flow orbit classes \( [\Lambda, \lambda] \) of the vectors \( (\Lambda, \lambda) \in \Delta \). The
coefficients $W_{\lambda,\tilde{\lambda},\mu}$ which solve the Cardy’s constraint are given by

$$W_{\lambda,\tilde{\lambda},\Omega}^{h,t} = \sum_{a=1}^{\nu(\Omega)} S_{(\lambda_a,\tilde{\lambda}_a)(h_a, t_a)}(S_{(0,0), (h_a, t_a)})^{-\frac{1}{2} |\Omega|_a},$$

$$S_{(\lambda_a,\tilde{\lambda}_a)(h_a, t_a)} = S_{\lambda_a, h_a} \exp(i\pi \frac{(h_a - 2 t_a)(\lambda_a - 2 \tilde{\lambda}_a)}{\mu}),$$

$$S_{\lambda_a, h_a} = \sqrt{\frac{2}{\mu}} \sin(\pi \frac{(h_a + 1)(\lambda_a + 1)}{\mu}).$$ (88)

The summation over $n$ makes $J[0]$-projection, while summation over $m$ introduce spectral flow twisted sectors. This state depends only on the spectral flow orbit class. Moreover, the $J[0]$ integer charge restriction of the orbits $[\lambda, \tilde{\lambda}]$ is necessary for the self-consistency of the expression (87). $\alpha$ is the normalization constant.

Now we apply the internal automorphism group of Gepner model to construct additional boundary states. Namely one can use the operator $\exp(-i2\pi \sum_i \phi_i J_i[0]) \in U(1)^r$ to generate new boundary states. Let us consider the properties of the state

$$[[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> \exp(-i2\pi \sum_i \phi_i J_i[0]) [[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> .$$ (89)

It satisfies the conditions similar to (77) except the relations for fermionic fields

$$(G^\pm[r] + i\eta \sum_i \exp(\pm i2\pi \phi_i \Omega(i)) \tilde{G}^\pm_{\Omega(i)}[-r]) [[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> \phi = 0,$$

$$(\psi^*_i[r] - i\eta \mu_i \exp(i2\pi \phi_i \Omega(i)) \tilde{\psi}^*_{\Omega(i)}[-r]) [[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> \phi = 0,$$

$$(\psi_i[r] - i\eta \mu_i \exp(-i2\pi \phi_i \Omega(i)) \tilde{\psi}_{\Omega(i)}[-r]) [[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> \phi = 0.$$ (90)

This state does not invariant with respect to diagonal $N=2$ Virasoro algebra unless

$$\phi_i \in Z, \ i = 1, ..., r.$$ (91)

Hence the group $U(1)^r$ reduces to $Z^r$. It is worth to note that one can ignore the case when all $\phi_i$ are half-integer because it can be canceled by the $\eta \rightarrow -\eta$ redefinition. It is easy to see directly what kind of states we obtain by this way.

$$[[\lambda, \tilde{\lambda}], \Omega, \eta, A] >> =$$

$$\frac{\alpha}{\kappa^2} \sum_{(h,t) \in \Delta_0} W^{h,t}_{[\lambda,\tilde{\lambda}]} \sum_{m,n=0}^{\nu(\Omega)} \exp(i2\pi n J[0]) \exp(-i2\pi m \sum_i \phi_i c_i) U^{mv} \tilde{U}^{mv}$$

$$\exp(-i2\pi \sum_i \phi_i \frac{h_i - 2 t_i}{\mu_i}) |I_{h,t}, \Omega, \eta, A >> =$$

$$\frac{\alpha}{\kappa^2} \sum_{(h,t) \in \Delta_0} \prod_{e=1}^{\nu(\Omega)} S_{\lambda_e, h_e}(S_{0, h_e})^{-\frac{|\Omega|_e}{2}} \exp(i\pi \frac{(\Lambda_e - 2 \lambda_e)(h_e - 2 t_e)}{\mu})$$

$$\exp(-i\pi \frac{(h_e - 2 t_e)}{\mu} \sum_{a=1}^{2 \phi_{e+a}} \sum_{m,n=0}^{\nu(\Omega)} \exp(i2\pi m \sum_{a=1}^{\phi_{e+a} = \alpha} \phi_{e+a})$$

$$\sum_{m,n=0}^{\nu(\Omega)} \exp(i2\pi n J[0]) U^{mv} \tilde{U}^{mv} |I_{h,t}, \Omega, \eta, A >> .$$ (92)
It allows to parameterize the boundary states by
\[ ||[\Lambda, \lambda], \Omega, \eta, A|| = \exp(-i2\pi \sum_i \lambda_i J_i[0])||[\Lambda, 0], \Omega, \eta, A|| \] \tag{93}
so that the different boundary states are labeled by different values of \(|\lambda|_e = \sum_{a=1}^{[\Omega]_e} \lambda_{e+a}, \ e = 1, ..., \nu(\Omega)\) and spectral flow invariant boundary states are recovered when
\[ \frac{2}{\mu} \sum_{e=1}^{\nu(\Omega)} \lambda|_e \in \mathbb{Z}. \] \tag{94}

5.2. B-type boundary states in Calabi-Yau extension.

Let us denote by \(\Delta_\Omega\) the subset of \(\Delta\) satisfying (73), (76). Then for an arbitrary pair of vectors \((\Lambda, \lambda) \in \Delta_{\text{CY}}\) the free-filed realization of spectral flow invariant \(B\)-type boundary state is given by
\[ ||[\Lambda, \lambda], \Omega, \eta, B|| = \frac{\alpha}{\kappa^2} \sum_{(h, t) \in \Delta_\Omega} W^{h,t}_{\Lambda, \lambda, \Omega} \sum_{m,n=0}^{\kappa-1} \exp(i2\pi n J[0])U^{m\nu}U^{\nu m}||I_{h,t}, \Omega, \eta, B||, \] \tag{95}
where the coefficients \(W^{h,t}_{\Lambda, \lambda, \Omega}\) are given by (88). One can check that this state depends only on the spectral flow orbit class \([\Lambda, \lambda]\) of vectors \((\Lambda, \lambda)\). It is also obvious that \([\Lambda, \lambda]\) has to be restricted to the set of \(J[0]\) integer charges by the reasons similar to the \(A\)-type case.

The other boundary states are generated by internal automorphism group of Gepner model similar to the \(A\)-type case. Namely the state
\[ ||[\Lambda, \lambda], \Omega, \eta, B||_\phi = \exp(-i2\pi \sum_i \phi_i J_i[0])||[\Lambda, \lambda], \Omega, \eta, B|| \] \tag{96}
satisfies the conditions similar to (90) and does not invariant with respect to diagonal \(N=2\) Virasoro algebra unless
\[ \phi_i \in \mathbb{Z}, \ i = 1, ..., r. \] \tag{97}
Hence the group \(U(1)^r\) reduces to \(Z^r\) and one can parameterize the boundary states by
\[ ||[\Lambda, \lambda], \Omega, \eta, B|| = \exp(-i2\pi \sum_i \lambda_i J_i[0])||[\Lambda, 0], \Omega, \eta, B|| \] \tag{98}
so that the different boundary states are labeled by different values of \(|\lambda|_e = \sum_{a=1}^{[\Omega]_e} \lambda_{e+a}, \ e = 1, ..., \nu(\Omega)\).

In conclusion of this section we would like to make the following remarks. First of all we note that our free-field construction allows to interpretate of \(A/B\)-type gluing conditions (37), (36) geometrically. Indeed, in terms of the free-fields \(B\)-type gluing conditions for example are given by (53). Thus \pm 1 eigne-values of the permutation matrix \(\Omega\) can be interpreted as labeling Newman and Dirichlet boundary conditions. While the complex eigne-values realize mixed boundary conditions [33]. This result seems to contradict to the calculation of D-brane charges performed in [23] and [24]. It has been found there that D0-branes correspond to transposition matrices permuting only one pair of minimal models. It follows from [53] that in
this case we have only one Dirichlet condition and the corresponding free-field boundary state gives codimension one D-brane. We do not know at the moment how to resolve or explain the contradiction. Perhaps more profound geometric investigation of the open string spectrum in terms of chiral de Rham complex has to be performed. But we postpone it for the next publication.

As the next remark we note that the free-field representations of permutation boundary states are determined modulo BRST-exact states satisfying \( A \) or \( B \)-type boundary conditions. We interpret this ambiguity in the free-field representation as a result of adding brane-antibrane pairs annihilating under the tachyon condensation process \[37\]. In this context the free-field representations of boundary states can be considered as the superpositions of branes flowing under the tachyon condensation to nontrivial boundary states in Gepner models. It is also important to note that the automorphisms \( [22] \) give different free-field representations of boundary states because the corresponding butterfly resolutions are not invariant with respect to these automorphisms. However their cohomology are invariant. Hence these different representations have to be identified. Thus the free-field boundary states construction have to be considered in derived category sense \[38\].

5.3. Free-field representation of permutation boundary states in Gepner models.

It is completely clear from \[21\] and \[30\] how to incorporate in our construction the space-time degrees of freedom to obtain the free-field construction of permutation branes in Calabi-Yau extension to the case of Gepner models. It is straightforward (see for example \[1\], \[2\], \[3\], \[18\]) and we shall not represent the details here.

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