SMOOTH FREE INVOLUTION OF $HCP^3$ AND SMITH CONJECTURE FOR IMBEDDINGS OF $S^3$ IN $S^6$

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Abstract. This paper establishes an equivalence between existence of free involutions on $HCP^3$ and existence of involutions on $S^6$ with fixed point set an imbedded $S^3$, then a family of counterexamples of the Smith conjecture for imbeddings of $S^3$ in $S^6$ are given by known result on $HCP^3$. In addition, this paper also shows that every smooth homotopy complex projective 3-space admits no orientation preserving smooth free involution, which answers an open problem [Pe]. Moreover, the study of existence problem for smooth orientation preserving involutions on $HCP^3$ is completed.

1. Introduction

The original Smith conjecture, which states that no periodic transformation of $S^3$ can have a tame knotted $S^1$ as its fixed point set, has been solved in the DIFF category, but it is generally false in the TOP and PL categories except for some special cases (see [MB]). However, the generalized Smith conjecture of codimension two is not true in any category (see [Gi], [Go], [Su], and [Lü1]). The generalized Smith conjecture of codimension greater than two is directly associated with the knot theory of the imbedded $S^m$ in $S^n$ for $n - m > 2$. It is well known that any imbedded $S^m$ in $S^n$ is unknotted in the TOP and PL categories if $n - m > 2$, and in the DIFF category if $2n > 3(m+1)$ (see [Ha1], [Le], [St], [Ze]). Haefliger [Ha2], [Ha3] and Levine [Le] showed that there exists infinite imbeddings of $S^m$ into $S^n$ which are knotted in the DIFF category if $2n \leq 3(m+1)$ and $m+1 \equiv 0 \mod 4$, and that there exists a knotted $S^{4k+1}$ in $S^{6k+3}$ in the DIFF category. Using Brieskorn manifolds, some explicit counterexamples for the generalized conjecture of codimension greater than two were given in [Lü2] if $2n \leq 3(m+1)$ and $m+1 \equiv 0 \mod 4$ with $n - m$ being even more than two and $n$ being odd.

The motivation of this paper is to consider the generalized Smith conjecture in the DIFF category for the extreme case $2n = 3(m+1)$ with $m+1 \equiv 0 \mod 4$; especially for $n = 6, m = 3$. Montgomery and Yang [MY] established the one-to-one correspondence $\eta$ between $\Pi$ and $C_3^3$, where $\Pi$ is the group of diffeomorphism classes of all homotopy complex projective 3-spaces, denoted by $HCP^3$ (for the sum operation, see [MY]), and $C_3^3$ is the group of isotopy classes of all imbeddings of $S^3$ into $S^6$. Note that $C_3^3$ is infinite cyclic (see [Ha3]), so is $\Pi$. This provides a way of dealing with the Smith conjecture for imbeddings of $S^3$ in $S^6$ by associating to $HCP^3$. We will show that there exist infinite distinct imbeddings $i: S^3 \to S^6$ in $C_3^3$.

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such that each knot \((S^6, i(S^3))\) admits a smooth involution, i.e., there is a smooth involution on \(S^6\) with \(i(S^3)\) as its fixed point set. This implies that the Smith conjecture for imbeddings of \(S^3\) in \(S^6\) is false (see Corollary 1.3 below). Notice that for \(k > 2\), there is no \(\mathbb{Z}_k\)-action on \(S^6\) with an imbedded \(S^3\) as the set of fixed points. So the only case concerned with Smith conjecture is the \(\mathbb{Z}_2\)-actions.

Our strategy to study the \(\mathbb{Z}_2\)-actions on \(S^6\) with an imbedded \(S^3\) as fixed point set is to establish its relation with free involutions on \(HCP^3\), which turns out to be equivalent as precisely stated by the following theorem.

**Theorem 1.1.** Let \([M]\) be an element in \(\Pi\). Then \(M\) admits a smooth orientation reversing free involution if and only if for any \(i : S^3 \to S^6\) in \(\eta[M]\), there is a smooth involution on \(S^6\) with \(i(S^3)\) as the set of fixed points.

With respect to \(HCP^3\), Petrie [Pe], Dovermann, Masuda and Schultz [DMS] have already the following

**Theorem 1.2** ([Pe], [DMS]). There are infinitely many homotopy complex projective 3-spaces which admit a smooth orientation reversing free involution.

Notice that although Petrie’s original assertion that every \(HCP^3\) has an orientation reversing free involution is not really proved as pointed out in [DMS, p.4], but his proof still yields Theorem 1.2.

As a consequence of Theorems 1.1 and 1.2, we have

**Corollary 1.3.** There exist infinitely many knotted imbeddings of \(S^3\) into \(S^6\) which offer counterexamples for the Smith conjecture.

In addition, we are also concerned with an open problem. In [Pe], Petrie said that the question of existence of an orientation preserving (free) involution on every \(HCP^3\) is still open. The following answers this question negatively.

**Theorem 1.4.** On every smooth \(HCP^3\), there is no smooth orientation preserving free involution.

Theorem 1.4 means that if an \(HCP^3\) admits a smooth free involution, then the involution must be orientation reversing.

**Remark.** In his paper [Ma], Masuda studied smooth (nonfree but orientation preserving) involutions on \(HCP^3\), and proved using Montgomery and Yang correspondence that every \(HCP^3\) admits a smooth involution with two copies of \(\mathbb{Z}_2\)-cohomology \(\mathbb{C}P^1\) as fixed point set. In contrast to this, he also proved that only the standard \(\mathbb{C}P^3\) admits a smooth involution with \(\mathbb{Z}_2\)-cohomology \(\mathbb{C}P^2\) and \(\mathbb{C}P^0\) as fixed point set. Thus, Theorem 1.4 with Masuda’s results together completes the study of existence problem for smooth orientation preserving involutions on \(HCP^3\).

The paper is organized as follows. In Section 2, we first give a proof of Theorem 1.4, and then prove a basic lemma concerning the 2-dimensional homology of the orbit space of a smooth free involution on an \(HCP^3\). In Section 3, we show that the surgery processes for Montgomery and Yang correspondence can still be carried on for \(\mathbb{Z}_2\)-actions. This establishes Theorem 1.1.
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2. Smooth free involutions of $HCP^3$

First, we prove Theorem 1.4 and then give a lemma which is fundamental for further results.

Proof of Theorem 1.4. Let $M = HCP^n$. Then there is a class $x \in H^2(M; \mathbb{Z})$ such that $H^*(M; \mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1})$ and $x^n \in H^{2n}(M; \mathbb{Z})$ is dual to the fundamental homology class of $M$ ([Sp], Theorem 5, p. 265). If $M$ admits a free involution $\tau$, then $\tau^*(x) = \pm x$, but $\tau^*(x) = x$ implies that $\tau_*$ is the identity on $H_*(M; \mathbb{Z})$ and so $\tau$ has a fixed point by the Lefschetz theorem ([Sp], Theorem 7, p. 195) which is impossible since $\tau$ is assumed to be free. So $\tau^*(x) = -x$ and $\tau_*$ sends a generator of $H_2(M; \mathbb{Z})$ into its negative and since $\pi_2(M) = H_2(M; \mathbb{Z})$, the action of $\pi_1(N)$ on $\pi_2(N)$ is nontrivial ([Sp], Corollary 7, p. 383), where $N$ is the orbit space of the action $\tau$ on $M$. Since $\tau^*(x^n) = (-1)^n x^n$, this means that $\tau$ preserves the orientation if and only if $n$ is even. Thus, when $n = 3$, there exists no orientation preserving free involution on $HCP^3$. The proof is completed. \hfill $\square$

The proof of Theorem 1.4 also gives the following result in the general case.

Corollary 2.1. Suppose that $M$ is an $HCP^n$ admitting a free involution $\tau$. Then $\tau$ preserves the orientation if and only if $n$ is even.

Lemma 2.1. Let $M$ be an $HCP^3$ with a smooth free involution $\tau$, and $p : M \to N = M/\tau$ be the orbit space projection. Then $H_2(N; \mathbb{Z}) = 0$.

Proof. Let $B$ be a closed Möbius band with $S^1$ as center, and $f : S^1 \to N$ be a smooth imbedding representing the generator of $\pi_1(N) \cong \mathbb{Z}_2$. It is easy to see that $f$ may extend to a map from $B$ to $N$, still denoted by $f$. Since $f : \partial B \to N$ represents twice of the generator, it is homotopic to zero. Let $\mathbb{R}P^2 = B \cup_\lambda D$, where $D$ is a 2-disk, and $\lambda$ is a diffeomorphism of $\partial D$ and $\partial B$. Then $f$ extends to a map from $\mathbb{R}P^2$ to $N$, also denoted by $f$. Change $f$ on the interior of $D$ by a map $\alpha : S^2 \to N$, we get a map $f_\alpha : \mathbb{R}P^2 \to N$.

Let $\mathbb{Z}_\xi$ be the local integer coefficient on $N$ twisted by the line bundle $\xi$ determined by $\tau$. As shown in [Ol], there is a canonical isomorphism between $H_2(N; \mathbb{Z}_\xi)$ and $\Omega_2(N; \xi)$, so $f$ and $f_\alpha$ may represent two elements in $H_2(N; \mathbb{Z}_\xi)$, denoted by $[f]$ and $[f_\alpha]$ respectively. Notice that $\Omega_n(X; \phi)$ was defined in [Ko] where $\phi$ is a stable bundle over a space $X$, and Olk in his dissertation [Ol] proved that

$$\tilde{\Omega}_n(X; \phi) \cong H_n(X; \mathbb{Z}_\phi) \text{ for } n = 0, 1, 2, 3$$
where $Z_{\phi}$ is the local integer coefficient associated to $\phi$ (see also [Li]). To see the picture more clearly, we assume $f : \mathbb{R}P^2 \rightarrow N$ is an imbedding (this is guaranteed by reason of dimension), and $f_\alpha = f \circ \alpha$ is the connected sum of $f$ and an imbedding $\alpha : S^2 \rightarrow N$ with $\alpha(S^2) \cap f(\mathbb{R}P^2) = \emptyset$. Then $p^{-1}f(\mathbb{R}P^2)$ and $p^{-1}f_\alpha(\mathbb{R}P^2)$ are embedded 2-dimensional spheres in $M$. Let $\alpha' : S^2 \rightarrow M$ be such that $p \circ \alpha' = \alpha$.

From the proof of Theorem 1.4, we know that the action of $\pi_1(N)$ on $\pi_2(N)$ is nontrivial, i.e., the action of the generator of $\pi_1(N)$ sends $x \in \mathbb{Z}$ to $-x \in \mathbb{Z}$. Then the nontriviality of the action of $\pi_1(N)$ on $\pi_2(N)$ means that $\alpha'$ and $\tau \circ \alpha'$ represent the elements in $H_2(M; \mathbb{Z})$ (via Hurewicz homomorphism) with opposite signs. Geometrically, $p^{-1}f_\alpha(\mathbb{R}P^2)$ is the connected sum of $p^{-1}f(\mathbb{R}P^2)$ with $\alpha'$ and $\tau \circ \alpha'$. When making connected sum with $\alpha'$, the 2-disk $D$ in $p^{-1}f(\mathbb{R}P^2)$ is removed, and when making connected sum with $\tau \circ \alpha'$, the 2-disk $\tau(D)$ is removed. Since $\tau : p^{-1}f(\mathbb{R}P^2) \rightarrow p^{-1}f(\mathbb{R}P^2)$ is orientation reversing, we have

$$[p^{-1}f_\alpha(\mathbb{R}P^2)] = [p^{-1}f(\mathbb{R}P^2)] + 2[\alpha']$$

in $H_2(M; \mathbb{Z})$. By the Gysin homology sequence

$$0 = H_3(M; \mathbb{Z}) \xrightarrow{p^*} H_3(N; \mathbb{Z}) \xrightarrow{t} H_2(N; \mathbb{Z}_2) \xrightarrow{\alpha} H_2(M; \mathbb{Z}_2) \xrightarrow{\beta} H_1(N; \mathbb{Z}_2)$$

(2.1) where $t$ is the transfer and the fact that $H_1(N; \mathbb{Z}_2) = 0$ (which can be seen by $\Omega_1(N; \xi) \cong H_1(N; \mathbb{Z}_2)$, and that a map $g : S^1 \rightarrow N$ with $g^*\xi \cong T(S^1)$ must be null-homotopic), we see that $H_2(N; \mathbb{Z}_2)$ contains at least one factor of $\mathbb{Z}$, and $H_2(N; \mathbb{Z}) = 0$ or $\mathbb{Z}_2$. By another Gysin homology sequence ([Sp], Problem J, pp. 282-283)

$$H_2(N; \mathbb{Z}) \xrightarrow{t} H_2(M; \mathbb{Z}) \xrightarrow{p^*} H_2(N; \mathbb{Z}_2) \xrightarrow{\beta}$$

$$H_1(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \xrightarrow{\beta} H_1(M; \mathbb{Z}) = 0$$

(2.2)

we have that $H_2(N; \mathbb{Z}_2)$ contains only one factor of $\mathbb{Z}$ and that

$$H_2(N; \mathbb{Z}_2) \cong \mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}_2.$$

Next we shall prove that

$$H_2(N; \mathbb{Z}_2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

is impossible.

Suppose that $H_2(N; \mathbb{Z}_2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Let $l$ be a generator of $H_2(M; \mathbb{Z}) \cong \mathbb{Z}$. Then there is a pair $(m_0, n_0)$ for $m_0 \in \mathbb{Z}, n_0 \in \mathbb{Z}_2$ such that $p_*(l) = (m_0, n_0)$. If $m_0 = 0$, then all $(m, 0) \in \mathbb{Z} \oplus \mathbb{Z}_2$ with $m \neq 0$ are not in the image of $p_*$, which is impossible by (2.2). If $m_0 \neq 0$, then $(0, 1)$ is not in the image of $p_*$. We claim that

there exists $(m_1, n_1)$ with $m_1 \neq 0$ in $\mathbb{Z} \oplus \mathbb{Z}_2$ such that $(m_1, n_1)$ does not belong to the image of $p_*$ in (2.2).

Choose an $f_\alpha$ defined above having the property $[p^{-1}f_\alpha(\mathbb{R}P^2)] \neq 0$, then $[f_\alpha] = (m_1, n_1)$ with $m_1 \neq 0$. For the homomorphism $\beta$, it can be seen by the relation of
normal bordism groups and homology groups, or by the Thom isomorphism with local coefficients ([Sp], Problem J, pp. 282-283) that \( \beta(x) = w_1(\xi) \cap x \) for \( x \in H_2(N; \mathbb{Z}_2) \) and \( w_1(\xi) \in H^1(N; \mathbb{Z}_2) \) the unreduced first Stiefel-Whitney class of \( \xi \) as explained in [Ste]. By the first point of view, \( \beta[f_\alpha] \) can be taken as follows.

For any section \( s \) of \( f_\alpha^* \xi \) over \( \mathbb{R}P^2 \) transversal to the zero section, let

\[ S = \{ x \in \mathbb{R}P^2 | s(x) = 0 \}, \]

then \( f_\alpha(S) \) represents \( \beta[f_\alpha] \). Obviously we may take \( S \) as being the circle in \( \mathbb{R}P^2 \) representing the generator of \( \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \), thus \( f_\alpha(S) \) represents the generator of \( H_1(N; \mathbb{Z}) \cong \mathbb{Z}_2 \), and the claim holds. Then by (2.2) the image of \( \beta \) will have \( \mathbb{Z}_2 \) as a proper subgroup. This leads to a contradiction.

Thus \( H_2(N; \mathbb{Z}_2) \cong \mathbb{Z}_2 \), and all \([f_\alpha]\) are odd elements.

Now by (2.1) we have that

\[ H_3(N; \mathbb{Z}) = 0. \tag{2.3} \]

If \( H_2(N; \mathbb{Z}) \cong \mathbb{Z}_2 \), then by (2.3) and \( H_1(N; \mathbb{Z}) \cong \mathbb{Z}_2 \) and the universal coefficient theorem, we have that

\[ H_0(N; \mathbb{Z}_2) = H_1(N; \mathbb{Z}_2) = H_3(N; \mathbb{Z}_2) = \mathbb{Z}_2, \quad H_2(N; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \]

By Poincaré duality,

\[ H_4(N; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_5(N; \mathbb{Z}_2) = H_6(N; \mathbb{Z}_2) = \mathbb{Z}_2. \]

Thus the Euler characteristic number of \( N \) would be 3, but it should be half of that of \( M \) which is 2. This contradiction shows that \( H_2(N; \mathbb{Z}) \cong \mathbb{Z}_2 \) is impossible.

Thus we have proved that \( H_2(N; \mathbb{Z}) = 0 \). This completes the proof. \( \square \)

### 3. Montgomery and Yang Correspondence in \( \mathbb{Z}_2 \)-actions

The proof of Theorem 1.1 is equivalent to establishing the surgery processes for Montgomery and Yang correspondence under \( \mathbb{Z}_2 \)-actions. For this, we first prove some lemmas.

**Lemma 3.1.** Let \( M \) be a smooth \( HCP^3 \) with a smooth free involution \( \tau \), and \( p : M \to N = M/\tau \) be the orbit space projection. Then there is an imbedding \( j : \mathbb{R}P^2 \to N \) such that \( p^{-1}j(\mathbb{R}P^2) \) is represented by an imbedded sphere which represents a generator of \( \pi_2(M) \).

**Proof.** We look at all \( p^{-1}f_\alpha(\mathbb{R}P^2) \) stated in the proof of Lemma 2.1. Notice that we have seen in the proof of Lemma 2.1 that \([f_\alpha(\mathbb{R}P^2)]\) are odd elements. Since \( H_2(N; \mathbb{Z}) = 0 \) and \( H_3(N; \mathbb{Z}) = 0 \) by Lemma 2.1 and (2.3), \( t \) is an isomorphism in (2.2), so all \([p^{-1}f_\alpha(\mathbb{R}P^2)]\) are odd elements in \( H_2(M; \mathbb{Z}) = \pi_2(M) = \mathbb{Z} \). Thus there is some \( \alpha_1 \) such that \( p^{-1}f_{\alpha_1}(\mathbb{R}P^2) \) represents a generator of \( \pi_2(M) \). This completes the proof. \( \square \)

Recall from [MY] that the standard \( CP^3 \) can be obtained by gluing two \( S^2 \times D^4 \) on their boundaries by a map

\[ f : S^2 \times S^3 \to S^2 \times S^3 \tag{3.1} \]
where \( f(Gu, v) = (Gu, v^{-1}) \), \( u \mapsto Gu \) is the Hopf map \( S^3 \to S^2 \), and \( S^3 \) is regarded as the space of unit quaternions, and \( G \subset S^3 \) consists of unit complex numbers.

Regard \( S^2 \) as the unit sphere in \( \mathbb{R}^3 = \mathbb{R} \times \mathbb{C} \), and \( \mathbb{R}^4 \) as the quaternion field which is identified with \( \mathbb{C}^2 \) by

\[
x_0 + x_1 i + x_2 j + x_3 k = x_0 + x_1 i + (x_2 + x_3 i)j \mapsto (x_0 + x_1 i, x_2 + x_3 i).
\]

Then the Hopf map is given by \( \mathbb{C}^2 \to \mathbb{R} \times \mathbb{C} \) sending

\[
(\psi_1, \psi_2) \mapsto (|\psi_1|^2 - |\psi_2|^2, 2\psi_1 \psi_2).
\]

It is easy to check that multiplying \( j \) on the left side of the above mapping induces the antipodal map on \( S^2 \), i.e., there is a commutative diagram

\[
\begin{array}{ccc}
S^3 & \xrightarrow{w} & S^3 \\
\downarrow & & \downarrow \\
S^2 & \xrightarrow{Gu} & S^2
\end{array}
\]

Therefore

\[
f(Gju, v) = (Gju, v^{-1}) = (-Gu, v^{-1}) = f(-Gu, v)
\]

on \( S^2 \times D^4 \) defines a smooth involution by mapping \((Gu, \ast)\) to \((-Gu, \ast)\). This involution commutes with \( f \), so there is a smooth free involution \( \tau_0 \) on \( \mathbb{C}P^3 \) such that \( \mathbb{C}P^3/\tau_0 \) is glued by two copies of \( \mathbb{R}P^2 \times D^4 \) along their boundaries. Thus, \( \mathbb{C}P^3/\tau_0 \) is actually a \( \mathbb{R}P^2 \)-bundle over \( S^4 \). This leads to the following

**Lemma 3.2.** Let \( M \) be an \( HCP^3 \) with a smooth free involution \( \tau \) and \( N = M/\tau \). Then \( N_0 = \mathbb{C}P^3/\tau_0 \) and \( N \) are homotopy equivalent.

**Proof.** First we claim that every \( \mathbb{R}P^2 \)-bundle over \( S^4 \) has a CW-complex structure such that it contains a cell in each dimension \( i = 0, 1, 2, 4, 5, 6 \). Every \( \mathbb{R}P^2 \)-bundle over \( S^4 \) is the union of two copies of \( \mathbb{R}P^2 \times D^4 \) by gluing boundaries of two \( \mathbb{R}P^2 \times D^4 \)'s. It is well-known that \( \mathbb{R}P^2 \) has a CW-decomposition such that it contains a cell in each dimension \( i = 0, 1, 2 \), and it is the union of those three cells. Thus, one of two copies of \( \mathbb{R}P^2 \times D^4 \) offers one cell in each dimension \( i = 4, 5, 6 \). On the other hand, another of two copies of \( \mathbb{R}P^2 \times D^4 \) has a natural deformation to \( \mathbb{R}P^2 \), so the \( i \)-cell for \( i = 0, 1, 2 \) is given by this \( \mathbb{R}P^2 \), and the \( i \)-cell for \( i = 4, 5, 6 \) from the first copy extends to an \( i \)-cell by the deformation. Then we obtain the required CW-decomposition.

As a special \( \mathbb{R}P^2 \)-bundle over \( S^4 \), \( N_0 \) has such one CW-decomposition as above. Then we see that the inclusion of \( \mathbb{R}P^2 \) in \( N_0 \) induces isomorphisms of homotopy groups up to dimension 2. Now, for any \( HCP^3 \) space \( M \) with free involution \( \tau \), by Lemma 3.1 there is an imbedding \( j : \mathbb{R}P^2 \to N \) such that the inclusion \( j(\mathbb{R}P^2) \hookrightarrow N \) also induces isomorphisms of homotopy groups up to dimension 2. Let \( f \) be a homeomorphism from the \( \mathbb{R}P^2 \) in \( N_0 \) to \( j(\mathbb{R}P^2) \) in \( N \). Since the CW-decomposition of \( N_0 \) contains no 3-dimensional cells, the 2-skeleton and 3-skeleton of the CW-decomposition of \( N_0 \) are equal, and both are just \( \mathbb{R}P^2 \). Thus, \( f \) has been defined on the boundary of the 4-cell in the CW-decomposition of \( N_0 \). Since \( \pi_3(N) = 0 \), the
obstruction theory tells us that \( f \) can extend to the 4-cell, denoted still by \( f \). Now \( f \) is defined on the 4-skeleton of \( N_0 \), i.e., on the boundary of the 5-cell of \( N_0 \). Since \( \pi_4(N) = \pi_5(N) = 0 \), the same argument as above shows that finally \( f \) can extend to the 6-skeleton of \( N_0 \), i.e., \( f \) is exactly defined on \( N_0 \). By the definition of \( f \), \( f \) induces isomorphisms of homotopy groups up to dimension 2. Since \( \pi_i(N_0) = \pi_i(N) = 0 \) for \( i = 3, 4, 5, 6 \), by Theorem 3.1 in page 107 of [Hi], we conclude that \( f \) is a homotopy equivalence.

\[ \square \]

**Lemma 3.3.** Let \( M \) be an \( H\mathbb{C}P^3 \) with a smooth free involution \( \tau \), and let \( j: \mathbb{R}P^2 \to N = M/\tau \) be the smooth imbedding described in Lemma 3.1. Then the normal bundle of \( j(\mathbb{R}P^2) \) is trivial.

**Proof.** Since
\[ \mathbb{C}P^3/\tau_0 = (\mathbb{R}P^2 \times D^4) \cup_{\lambda} (\mathbb{R}P^2 \times D^4) \]
where \( \lambda: \mathbb{R}P^2 \times S^3 \to \mathbb{R}P^2 \times S^3 \) is such that \( \lambda: \mathbb{R}P^2 \times v \to \mathbb{R}P^2 \times v^{-1} \) is a homeomorphism, an easy argument by using Mayer-Vietoris sequence shows that \( H_2(\mathbb{C}P^3/\tau_0; \mathbb{Z}_2) = \mathbb{Z}_2 \) and its generator is represented by \( \mathbb{R}P^2 \times * \). The proof of Lemma 3.1 contains the fact that a generic inclusion \( j: j(\mathbb{R}P^2) \to N \) induces an isomorphism of homotopy groups at levels 1 and 2 and so it induces an integral homology isomorphism at these levels by the Whitehead theorem ([Sp], Theorem 9, p. 399). It follows from the universal coefficient theorems that \( j \) induces homology and cohomology isomorphism at levels 1 and 2 with arbitrary coefficients. In particular, \( j(\mathbb{R}P^2) \) represents the generator of \( H_2(N; \mathbb{Z}_2) = \mathbb{Z}_2 \). Since the tangent bundle \( T(\mathbb{C}P^3/\tau_0) \) restricted to \( \mathbb{R}P^2 \times * \) is
\[ T(\mathbb{C}P^3/\tau_0)|_{\mathbb{R}P^2 \times *} = T(\mathbb{R}P^2 \times *) \oplus \text{trivial bundle} \]
we see that the Stiefel-Whitney classes \( w_1(\mathbb{C}P^3/\tau_0) = w_1(\mathbb{R}P^2 \times *) \neq 0 \) and \( w_2(\mathbb{C}P^3/\tau_0) = w_2(\mathbb{R}P^2 \times *) \neq 0 \). By the proof of Lemma 3.2, a homotopy equivalence \( f: \mathbb{C}P^3/\tau_0 \to M/\tau = N \) can be chosen so that \( f: \mathbb{R}P^2 \times * \to j(\mathbb{R}P^2) \) is a homeomorphism.

It is well known that Stiefel-Whitney classes are homotopy invariant, so \( w_1(N) \neq 0 \) in \( H^1(N; \mathbb{Z}_2) \) and \( w_2(N) \neq 0 \) in \( H^2(N; \mathbb{Z}_2) = \mathbb{Z}_2 \). Thus \( w(T(N)|_{j(\mathbb{R}P^2)}) = w(j(\mathbb{R}P^2)) \), and the Stiefel-Whitney classes of the normal bundle are trivial. This means that the normal bundle of \( j(\mathbb{R}P^2) \) is stably trivial since its Stiefel-Whitney classes vanish at levels 1 and 2 and \( j(\mathbb{R}P^2) \) is a 2-complex. Thus, the normal bundle of \( j(\mathbb{R}P^2) \) is trivial since its fiber has dimension 4 (which is strictly larger than 2). This completes the proof. \( \square \)

With the above understood, we are going to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that \( M \) admits a smooth free involution \( \tau \). Then by Lemma 3.3 there is an imbedding \( h: \mathbb{R}P^2 \times D^4 \to M/\tau \). Hence there is an imbedding \( h: S^2 \times D^4 \to M \) such that
\[ h(S^2 \times D^4) = p^{-1}(\mathbb{R}P^2 \times D^4) \]
where \( p: M \to M/\tau \) is the projection. By Lemma 3.1, \( h: S^2 \times 0 \to M \) is a primary imbedding defined in [MY]. Now, by Lemma 5 in [MY], there is another primary
imbeeding $h': S^2 \times D^4 \to M$ such that
\[ h(S^2 \times D^4) \cup h'(S^2 \times D^4) = M \]
and
\[ h(S^2 \times D^4) \cap h'(S^2 \times D^4) = h(S^2 \times S^3) = h'(S^2 \times S^3). \]
Clearly, on $h(S^2 \times D^4)$,
\[ \tau h(x, y) = h(-x, y). \]
According to [MY, Appendix and Lemma 11], an imbedding $i: S^3 \to S^6$ is given by
\[ S^6 = (M - \text{int}(h(S^2 \times D^4))) \cup \alpha(D^3 \times S^3) \]
where $\alpha: h(S^2 \times S^3) \to \partial(D^3 \times S^3) = S^2 \times S^3$ is defined by $\alpha = fh^{-1}$ and $f(Gu, v) = (Guv, v^{-1})$.

Now on $(M - \text{int}(h(S^2 \times D^4))) = h'(S^2 \times D^4) \subset S^6$, there is the involution $\tau_1(= \tau)$. We define an involution $\tau_2$ on $D^3 \times S^3$ by $\tau_2(x, y) = (-x, y)$. Since on $h(S^2 \times S^3)$, we have
\[ \alpha \tau_1 h(x, y) = fh^{-1} \tau h(x, y) = fh^{-1}h(-x, y) = f(-x, y) \]
and
\[ \tau_2 \alpha h(x, y) = \tau_2 fh^{-1}h(x, y) = \tau_2 f(x, y). \]
Represent $(x, y) \in S^2 \times S^3$ by $(Gu, v)$, then
\[ f(Gu, v) = (Guv, v^{-1}) \]
and
\[ \tau_2 f(Gu, v) = (-Guv, v^{-1}) = f(-Gu, v). \]
Hence
\[ \alpha \tau_1 = \tau_2 \alpha \]
on $h(S^2 \times S^3)$. Combining $\tau_1$ with $\tau_2$ together, we obtain then an involution on $S^6$ with $i(S^3)$ as the set of fixed points.

Conversely, let $i: S^3 \to S^6$ be an imbedding such that there is an involution $\tau$ on $S^6$ with $i(S^3)$ as the set of fixed points. Take a $\tau$-invariant Riemannian metric on $S^6$, then $\tau$ induces a bundle map of the normal bundle of $i(S^3)$, i.e., the orthogonal bundle of $T(i(S^3))$ in $T(S^6)$ according to the Riemannian metric, covering the identity map of $i(S^3)$, and on every fiber of the normal bundle of $i(S^3)$, the bundle map must set $v$ to $-v$. Then by using the exponential map, it follows that there is an $Z_2$-equivariant imbedding $h: D^3 \times S^3 \to S^6$ such that
\[ h(0, x) = i(x) \text{ and } \tau h(v, x) = h(-v, x). \]
By [MY], there is an imbedding $k: S^2 \times D^4 \to S^6$ such that
\[ S^6 = k(S^2 \times D^4) \cup h(D^3 \times S^3) \]
and
\[ k(S^2 \times D^4) \cap h(D^3 \times S^3) = k(S^2 \times S^3) = h(S^2 \times S^3). \]
By the proof of Lemma 12 in [MY], we may change $h$ by a map $\mu : S^3 \to SO(3)$ to get an imbedding $k' : D^3 \times S^3 \to S^6$, so that $k$ and $k'$ satisfy the condition of [MY, Lemma 12], and

$$\tau k'(v, x) = k'(-v, x).$$

Now let $\lambda : k(S^2 \times S^3) \to S^2 \times S^3$ be the map defined by $\lambda = f k'^{−1}$, where $f$ is the map stated in (3.1), then

$$M = k(S^2 \times D^4) \cup_\lambda (S^2 \times D^4)$$

is an $HCP^3$ by [MY, Lemma 12], and $\eta[M] = [i]$. To see that $M$ has a free involution, let $\tau_1 = \tau$ on $k(S^2 \times D^4)$, and $\tau_2$ on $S^2 \times D^4$ is given by

$$\tau_2(v, x) = (-v, x).$$

Then on $k(S^2 \times S^3) = k'(S^2 \times S^3)$, $\lambda \tau_1 k'(v, x) = \lambda k'(-v, x) = f(-v, x)$ and

$$\tau_2 \lambda k'(v, x) = \tau_2 f(v, x).$$

As we have seen before,

$$\tau_2 f(v, x) = f(-v, x)$$

so

$$\lambda \tau_1 = \tau_2 \lambda$$

on $k(S^2 \times S^3)$. Therefore, combining $\tau_1$ and $\tau_2$ gives a free involution on $M$. This completes the proof. \(\square\)

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