Massive planar and non–planar double box integrals for light $N_f$ contributions to $gg \rightarrow t\bar{t}$

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Abstract: We present the master integrals needed for the light fermionic two–loop corrections to top quark pair production in the gluon fusion channel. Via the method of differential equations we compute the results in terms of multiple polylogarithms in a Laurent series about $d = 4$, where $d$ is the space–time dimension. The most involved topology is a non–planar double box with one internal mass. We employ the coproduct–augmented symbol calculus and show that significant simplifications are possible for selected results using an optimised set of multiple polylogarithms.
1 Introduction

Analytical calculations of next-to-next-to-leading order (NNLO) corrections to top quark pair production at hadron colliders require, among other ingredients, results for various two-loop Feynman integrals. While first complete numerical NNLO predictions [12–15] for the total pair production cross section have appeared recently, in the analytical approach only a subset of the required building blocks are available [1–11] at the present time. Here, we focus on double box master integrals which contribute to the light fermionic corrections in the gluon channel, i.e. all Feynman diagrams containing at least one massless fermion in a closed loop.

The most involved integrals considered here are the three master integrals of a particular non-planar topology with one massive propagator. Our results for these integrals were sketched in [16, 17]. Numerical results in the physical region of phase space have been presented in the analysis [18] using the sector decomposition program SecDec [19, 20]. Here, we present the full analytical result and describe in more detail how we obtained it.

The outline of this paper is as follows. In section 1, we describe our calculational setup, which is based on the method of differential equations [21–28] for a Laurent expansion in $\epsilon = (4 - d)/2$. In sections 3 and 4 we present results for the planar and non-planar master integrals, respectively, given in terms of multiple polylogarithms [29–42]. The symbol calculus [30, 43] and its coproduct based extension [44, 51] are powerful tools to exploit functional identities between multiple polylogarithms and have been successfully applied to both conformal theories [45–50] and QCD [16, 17, 51–54]. For selected Laurent coefficients of our non-planar master integrals, we employ symbol and coproduct based techniques and find remarkable simplifications.

The full result up to and including weight four, as needed for the NNLO corrections to top quark pair production, is attached in form of a computer readable file to the arXiv submission of this paper. In the main text, we give the first few Laurent coefficients of the results to illustrate their structure.

2 Calculational method

Our setup for the calculation is as follows. We identify the dimensionally regularised master integrals required for the light fermionic two-loop corrections to $gg \to t\bar{t}$ by generating diagrams with QGRAF [55], matching them to sectors of integral families and reducing the loop integrals with integration-by-parts (IBP) identities through a variant of the Laporta algorithm [56–59]. The last two steps are performed with Reduze 2 [60–63] (for other public reduction programs see [64–66]). Ambiguities in the representation of Feynman integrals which are due to shifts of the loop momenta, crossings of external momenta or a combination thereof are eliminated by the program in an automated way. For completeness, we append the definition of the integral families we used as a file to the arXiv submission of this paper. These integral families are also the basis for our sector naming conventions. While many of the required master integrals are already known in the literature [1, 2, 67–76], we find several sectors for which the master integrals have not been computed in analytical form before. These are the following.
For sector tt2pC: 5: 214:
\[
\begin{align*}
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_m(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_m(k_2 - p_1) D_0(k_1 - p_3), \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_m(k_2) D_0(k_1 - k_2) D_2^m(k_2 - p_1) D_m(k_2 - p_1) D_0(k_1 - p_3).
\end{align*}
\] (2.1), (2.2)

For sector tt2pD: 5: 174:
\[
\begin{align*}
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_m(k_1 + p_23) D_m(k_1 - p_3), \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_2^m(k_1 + p_23) D_m(k_1 - p_3).
\end{align*}
\] (2.3), (2.4)

For sector tt2pD: 5: 182:
\[
\begin{align*}
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_m(k_1 + p_23) D_m(k_1 - p_3), \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_2^m(k_1 + p_23) D_m(k_1 - p_3).
\end{align*}
\] (2.5), (2.6)

For sector tt2pE: 5: 333:
\[
\begin{align*}
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_1) D_0(k_2) D_0(k_1 - p_1) D_0(k_1 - p_2) D_0(k_1 - p_3), \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int D_0(k_1) D_2^m(k_1 - k_2) D_0(k_1 - p_1) D_0(k_1 - p_2) D_0(k_1 - p_3).
\end{align*}
\] (2.7), (2.8)

For sector tt2nA: 7: 463:
\[
\begin{align*}
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int \frac{D_0(k_2) D_0(k_1 - k_2) D_0(k_1 - p_1) D_0(k_1 - p_3)}{D_{t2nA: 7: 463}(k_1, k_2, p_1, p_2, p_3)}, \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int \frac{D_0(k_1) D_0(k_2) D_0(k_1 - p_1) D_0(k_1 - p_3)}{D_{t2nA: 7: 463}(k_1, k_2, p_1, p_2, p_3)}, \\
\mathcal{D}^d k_1 \mathcal{D}^d k_2 &= \int \frac{D_0(k_1) D_0(k_2) (k_1 \cdot k_2)^2}{D_{t2nA: 7: 463}(k_1, k_2, p_1, p_2, p_3)}.
\end{align*}
\] (2.9), (2.10), (2.11)
with the denominator

\[
D_{12A:7.463}(k_1, k_2, p_1, p_2, p_3) = D_0(k_1)D_0(k_2)D_0(k_1 + p_1)D_0(k_2 + p_2)D_0(k_1 - k_2 + p_1)D_0(k_1 - k_2 - p_2)D_m(k_1 - k_2 + p_{13}).
\] (2.12)

The integrals involve propagator denominators with mass zero or mass \( m \),

\[
D_0(k) = k^2 + i\delta, \quad D_m(k) = k^2 - m^2 + i\delta, \quad (2.13, 2.14)
\]

and employ the integration measure

\[
\mathcal{D}^d k \equiv \frac{(2\pi)^2 m^{2\epsilon}}{C(\epsilon)} \frac{d^d k}{(2\pi)^d}, \quad C(\epsilon) \equiv (4\pi)^\epsilon \Gamma(1 + \epsilon), \quad (2.15)
\]

where \( d \) is the space–time dimension and \( \epsilon \equiv (4 - d)/2 \). The incoming momenta \( p_1 \) and \( p_2 \) fulfil \( p_1^2 = p_2^2 = 0 \), while the outgoing momenta \( p_3 \) and \( p_4 = p_1 + p_2 - p_3 \) fulfill \( p_3^2 = p_4^2 = m^2 \). Finally, the definitions above employ the abbreviations \( p_{12} \equiv p_1 + p_2 \), \( p_{13} \equiv p_1 - p_3 \) and \( p_{23} \equiv p_2 - p_3 \). Different choices of master integrals are possible. Our selection above leads to a (partial) decoupling of the differential equations order by order in the Laurent expansion about \( \epsilon = 0 \). This effectively allows us to solve the integrals by integrating ordinary differential equations as we discuss in more detail below.

The four-point functions we want to compute have two massless legs and two legs with the same non-vanishing top quark mass \( m \). Consequently, the generically 6 independent scalar products of the 3 linearly independent external momenta reduce to 3 independent quantities in this case. Propagators are restricted to have mass zero or \( m \) and thus do not introduce additional scales. Therefore, all of our master integrals depend on 3 independent variables, for which we choose \( m \) and 2 dimensionless quantities out of the set \( \{ x, y, z \} \), where

\[
x = \frac{\sqrt{1 - 4m^2/s} - 1}{\sqrt{1 - 4m^2/s} + 1}, \quad y = -\frac{t}{m^2}, \quad z = -\frac{u}{m^2}. \quad (2.16)
\]

The Mandelstam variables are \( s \equiv p_{12}^2, t \equiv p_{13}^2 \) and \( u \equiv p_{23}^2 \). The variable \( x \) absorbs roots in the differential equations associated with a massive two particle threshold, see e.g. [77] for more details. Momentum conservation implies \( s + t + u = 2m^2 \), which translates into the non-linear relation

\[
y + z = -\frac{1 + x^2}{x} \quad (2.17)
\]

for our dimensionless variables.

In the physical region of phase space for top quark pair production the variables fulfil

\[
m^2 > 0, \quad -1 \leq x < 0, \quad -x \leq y \leq -1/x, \quad -x \leq z \leq -1/x, \quad yz \geq 1, \quad y + z \geq 2. \quad (2.18)
\]

Branch cut ambiguities are resolved by causality, implemented via the \( i\delta \) prescription in the Feynman propagator denominators, (2.13) and (2.14). Depending on the topology,
such a branching occurs for our master integrals due to thresholds located at $s = 0$, $s = 4m^2$, $t = m^2$ or $u = m^2$, see figure 1. In the physical region with $t$ and $u$ negative and $s > 4m^2$ it is sufficient to absorb these imaginary parts into an infinitesimal positive imaginary part of $s$ for the results to be well defined. This translates to an infinitesimal positive imaginary part for $x$. In contrast to the planar cases, solving the non-planar master integrals requires us to take care of these prescriptions and the associated explicit imaginary parts of transcendental functions right from the start, see section 4. To give a well defined meaning also to all intermediate expressions we pick some reference point in phase space, where we choose a value for $x$ with a small (but finite) positive imaginary part and a value for $y$ with a small (positive or negative) imaginary part. The value of $z$, including its imaginary part, is completely determined by the mass-shell relation (2.17), which we treat in an algebraically exact manner throughout our calculation. Of course, our final results should not depend on arbitrary details of our intermediate regularisation, which we also explicitly checked.

We employ the method of differential equations to calculate the 11 unknown master integrals in analytical form. We use Reduze 2 to automatically calculate the differential equations, insert the reductions and change to an alternative basis, if required. By differentiating with respect to the overall squared scale $m^2$ we verify the correct scaling behaviour, a feature which becomes explicit only after insertion of the reductions. We integrate the differential equations in the two independent dimensionless variables and equate the solutions. In that way we fully decouple the problem of integration from the determination of the integration constants, which are pure numbers. These can in principle be determined by an independent numerical evaluation method for a couple of phase space points. However, we prefer to give exact solutions for them. We employ evaluations of independent Mellin-Barnes representations [78, 79] in kinematical limits to determine analytical expressions for the integration constants and to check the results. For the planar topologies we used Ambre [80] to generate Mellin-Barnes representations, while for the non-planar topology we prepared this representation manually, see appendix A. For expansions in kinematical

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**Figure 1.** Physical region of phase space bounded by $ut = 1$ and possible threshold singularities at $s = 0$, $s = 4m^2$, $t = m^2$ and $u = m^2$. 

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limits we used \texttt{MB.m} [81]. In order to determine the integration constants, we also exploit regularity and symmetry conditions, which serves as a more convenient alternative in some cases and as a redundant cross-check in others. We check our results by comparing them to numerical Mellin-Barnes evaluations and find good agreement for a choice of typically four to seven significant digits. For the non-planar master integral (2.9) we also compare our results at different points in phase space with the numerical results of [18] and find agreement.

Our results are expressed in terms of multiple polylogarithms. This class of iterated integrals is defined recursively,

\begin{equation}
G(w_1, \ldots, w_n; x) = \int_0^x \frac{dt}{t-w_1} G(w_2, \ldots, w_n; t) \quad \text{if at least one } w_i \neq 0, \tag{2.19}
\end{equation}

\begin{equation}
G(0, \ldots, 0; x) = \frac{1}{n!} \ln^n(x), \tag{2.20}
\end{equation}

\begin{equation}
G(; x) = 1. \tag{2.21}
\end{equation}

Here, the weights \( w_i \in \mathbb{C}, i = 1, 2, \ldots, n \) and the argument \( x \in \mathbb{C} \) are considered as functions of the indeterminates. We employ symbol and coproduct techniques based on algorithms given in [43, 51], as well as more traditional methods for their automated treatment, in particular for argument changes and projections onto alternative basis functions. To implement these ideas, we have written an in-house Mathematica [82] package, which utilises the numerical evaluation implementation [34] in \texttt{GiNaC} [62]. We emphasize that we exploit functional identities for truly complex variables and make sure all pole prescriptions and corresponding imaginary parts are consistently taken into account at all stages.

For univariate polylogarithms, we generalise from linear to polynomial denominators by defining generalised weights \([f(o)]\) with

\begin{equation}
G([f(o)], w_2, \ldots, w_n; x) = \int_0^x \frac{dt}{f(t)} G(w_2, \ldots, w_n; t) \tag{2.22}
\end{equation}

where \( f(o) \) is an irreducible rational polynomial and \( o \) is a dummy variable. Without loss of generality we normalise the leading coefficient of \( f \) to one. It is curious to note that all of our integration measures with non-linear irreducible denominators are indeed of this form. A generalised weight \([f(o)]\) with the complex factorisation

\begin{equation}
f(o) = (o - r_1) \cdots (o - r_n), \tag{2.23}
\end{equation}

where \( r_i \in \mathbb{C}, i = 1, \ldots, n \), can be expanded in terms of standard weights according to

\begin{equation}
G(\ldots, [f(o)], \ldots; x) = G(\ldots, r_1, \ldots; x) + \ldots + G(\ldots, r_n, \ldots; x). \tag{2.24}
\end{equation}

Working directly with the left hand side of this equation has the advantage that these functions give rise to a rational symbol and do not introduce spurious imaginary parts. While the irreducible denominators needed here are cyclotomic polynomials and the associated cyclotomic polylogarithms defined in [41] could be used to express them, we prefer to work with the above definitions in order to emphasize the \( \text{d} \ln f(t) \) structure (cf. [83] for linear but multivariate \( f \)). More details for these generalised weights will be given in another work [84], where also non-cyclotomic polynomials \( f(o) \) are considered.
3 Results for planar master integrals

For sector $\text{tt2pC:5:214}$ we choose variables $y$ and $x$ for the differential equations and find

\[
p_1 \rightarrow \begin{array}{c} \hline p_3 \\ \hline p_2 \rightarrow \end{array} = -\frac{x}{m^2(1-x)^2} \sum_{i=-1}^{1} a_i \epsilon^i + O(\epsilon^2), \tag{3.1}
\]

\[
a_{-1} = -\frac{1}{32} G(0; x)^2
\]

\[
a_0 = \frac{1}{16} G(-1/x, 0, -1; y) + \frac{1}{16} G(-x, 0, -1; y) - \frac{1}{8} G(-1, 0, -1; y)
+ \frac{1}{16} G(-1/x, -1; y) G(0; x) - \frac{1}{16} G(-x, -1; y) G(0; x) - \frac{1}{16} G(-1/x; y) G(1, 0; x)
+ \frac{1}{32} G(-1/x; y) G(0; x)^2 + \frac{1}{16} G(-x; y) G(1, 0; x) + \frac{3}{32} G(0, 1, 0; x) + \frac{1}{4} G(0, -1, 0; x)
- \frac{1}{32} G(0; x) G(1, 0; x) - \frac{1}{48} G(0; x)^3 + \frac{1}{48} \pi^2 G(-1/x; y) - \frac{1}{48} \pi^2 G(-1; y)
+ \frac{1}{96} \pi^2 G(0; x) + \frac{3}{16} \zeta(3) - \frac{1}{16} G(0; x)^2, \tag{3.2a}
\]

\[
p_1 \rightarrow \begin{array}{c} \hline p_3 \\ \hline p_2 \rightarrow \end{array} = -\frac{x}{m^2(1-x)^2} \sum_{i=-1}^{2} b_i \epsilon^i + O(\epsilon^3), \tag{3.3}
\]

\[
b_{-1} = \frac{1}{16} G(0; x) - \frac{1}{8} - \frac{1}{8(1-x)} G(0; x)
\]

\[
b_0 = -\frac{1}{16} G(-1; y) G(0; x) + \frac{1}{16} G(1, 0; x) - \frac{1}{4} G(-1, 0; x) + \frac{3}{32} G(0; x)^2 - \frac{1}{32} \pi^2
+ \frac{1}{8} G(-1; y) + \frac{1}{8} G(0; x) - \frac{3}{8} + \frac{1}{1-x} \left( \frac{1}{8} G(-1; y) G(0; x) - \frac{1}{8} G(1, 0; x) \right)
+ \frac{1}{2} G(-1, 0; x) - \frac{3}{32} G(0; x)^2 + \frac{1}{16} \pi^2 + \frac{1}{4} G(0; x) \right), \tag{3.4a}
\]

Since the coefficients $a_1$, $b_1$ and $b_2$ are rather lengthy we provide them only via a file on arXiv. The solution contains multiple polylogarithms with either argument $y$ and weights drawn from the set $\{-1, 0, -x, -1/x\}$ or with argument $x$ and weights drawn from $\{-1, 0, 1\}$.

For sector $\text{tt2pD:5:174}$ we choose variables $y$ and $z$ and find

\[
p_1 \rightarrow \begin{array}{c} \hline p_4 \\ \hline p_3 \rightarrow \end{array} = \frac{1}{m^2(y+1)} \sum_{i=-1}^{1} c_i \epsilon^i + O(\epsilon^2), \tag{3.5}
\]
\[ c_{-1} = \frac{1}{16} G(0, -1; y) + \frac{1}{96} \pi^2 \]  
(3.6a)

\[ c_0 = \frac{1}{16} G(1/z, -1; y) - \frac{1}{16} G(1/z, 0, -1; y) - \frac{1}{16} G(-z - 2, -1, -1; y) \]
\[- \frac{1}{16} G(1/z, -1; y) G(-1; z) + \frac{1}{16} G(-z - 2, -1; y) G(-1; z) - \frac{1}{16} G(0, -1; z) G(1/z; y) \]
\[- \frac{1}{32} G(-1; z)^2 G(-z - 2; y) + \frac{1}{32} G(-1; z)^2 G(1/z; y) + \frac{3}{16} G(-1, 0, -1; y) \]
\[+ \frac{1}{8} G(0, 0, -1; y) - \frac{1}{16} G(-2, -1, -1; z) + \frac{1}{16} G(-1, 0, -1; z) - \frac{1}{8} G(0, -1; y) G(-1; y) \]
\[- \frac{1}{32} \pi^2 G(-z - 2; y) + \frac{1}{96} \pi^2 G(1/z; y) + \frac{1}{96} \pi^2 G(-1; y) - \frac{1}{32} \pi^2 G(-2; z) \]
\[+ \frac{1}{96} \pi^2 G(-1; z) - \frac{1}{32} \pi^2 \ln 2 + \frac{29}{64} \zeta(3) + \frac{1}{8} G(0, -1; y) + \frac{\pi^2}{48}, \]  
(3.6b)

\[ p_1 p_4 = \frac{1}{m^4 (y + 1)} \sum_{i=-2}^{2} d_i \epsilon^i + \mathcal{O}(\epsilon^3), \]  
(3.7)

\[ d_{-2} = \frac{1}{48} \]  
(3.8a)

\[ d_{-1} = -\frac{1}{16} G(-1; y) - \frac{1}{48} G(-1; z) + \frac{5}{48} \]  
(3.8b)

\[ d_0 = \frac{3}{16} G(0, -1; y) + \frac{1}{16} G(-1; y) G(-1; z) + \frac{3}{32} G(-1; y)^2 + \frac{1}{96} G(-1; z)^2 - \frac{7}{288} \pi^2 \]
\[- \frac{1}{8} G(-1; y) - \frac{5}{48} G(-1; z) + \frac{13}{48} + \frac{1}{y + 1} \left( \frac{3}{16} G(0, -1; y) + \frac{1}{32} \pi^2 \right). \]  
(3.8c)

We provide the coefficients \(c_1, d_1\) and \(d_2\) via a file on arXiv. The solution contains multiple polylogarithms with either argument \(y\) and weights drawn from the set \([-1, 0, -2, z, 1/z]\) or with argument \(z\) and weights drawn from \([-2, -1, 0]\).

For sector \(tt2pD:5:182\) we choose variables \(z\) and \(y\) and find

\[ p_1 \quad \triangle \quad p_4 \]
\[ p_3 \quad \square \quad p_2 \]
\[ = \frac{1}{m^2 (y + z + 2)} \sum_{i=-1}^{0} e_i \epsilon^i + \mathcal{O}(\epsilon), \]  
(3.9)

\[ e_{-1} = -\frac{1}{16} G(1/y, -1, -1; z) + \frac{1}{16} G(1/y, 0, -1; z) + \frac{1}{16} G(-y - 2, -1, -1; z) \]
\[- \frac{1}{8} G(-1, 0, -1; y) + \frac{1}{16} G(-2, -1, -1; y) - \frac{1}{8} G(-1, 0, -1; y) \]
\[+ \frac{1}{16} G(1/y, -1; z) G(-1; y) - \frac{1}{16} G(-y - 2, -1; z) G(-1; y) + \frac{1}{16} G(0, -1; y) G(1/y; z) \]
\[+ \frac{1}{32} G(-y - 2; z) G(-1; y)^2 - \frac{1}{32} G(1/y; z) G(-1; y)^2 + \frac{1}{32} \pi^2 G(-y - 2; z) \]
\[- \frac{1}{96} \pi^2 G(1/y; z) + \frac{1}{32} \pi^2 G(-2; y) - \frac{1}{48} \pi^2 G(-1; y) - \frac{1}{48} \pi^2 G(-1; z) + \frac{1}{32} \pi^2 \ln 2 \]
\[- \frac{21}{64} \zeta(3), \]  
(3.10a)
We provide the coefficients $\epsilon_0$ and $f_1$ via a file on arXiv. The solution contains multiple polylogarithms with either argument $z$ and weights drawn from the set $\{-1, 0, -2 - y, 1/y\}$ or with argument $y$ and weights drawn from $\{-2, -1, 0\}

For sector $\texttt{tt2pE:5:333}$ we choose variables $y$ and $z$ and find

\begin{align}
g_{-2} &= -\frac{1}{16}G(-1; y)G(-1; z) + \frac{1}{32}G(-1; y)^2 + \frac{1}{32}G(-1; z)^2 + \frac{\pi^2}{32} \\
g_{-1} &= \frac{1}{4}G(-1, 0, -1; y) + \frac{1}{8}G(-2, -1, -1; z) + \frac{1}{4}G(-1, 0, -1; z) + \frac{1}{8}G(-1; y)^2G(-1; z) \\
&\quad + \frac{1}{16}G(-1; z)^2G(-z - 2; y) + \frac{1}{16}G(-1; z)^2G(1/z; y) - \frac{1}{8}G(0, -1; z)G(1/z; y) \\
&\quad - \frac{1}{8}G(-1; z)G(-z - 2, -1; y) - \frac{1}{8}G(-1; z)G(1/z, -1; y) + \frac{1}{8}G(-z - 2, -1, -1; y) \\
&\quad + \frac{1}{8}G(1/z, -1, -1; y) - \frac{1}{8}G(1/z, 0, -1; y) - \frac{1}{12}G(-1; y)^3 - \frac{1}{12}G(-1; z)^3 \\
&\quad + \frac{1}{16}\pi^2G(-z - 2; y) + \frac{1}{48}\pi^2G(1/z; y) - \frac{1}{12}\pi^2G(-1; y) + \frac{1}{16}\pi^2G(-2; z)
\end{align}
\[-\frac{1}{12}\pi^2 G(-1; z) + \frac{1}{16}\pi^2 \ln 2 + \frac{35}{32}\zeta(3), \quad (3.14b)\]

\[p_1 \quad \begin{array}{c} \ldots \end{array} \quad p_2 \quad p_3 \quad p_4 \]

\[\frac{1}{m^4(y+1)(z+1)} \sum_{i=-3}^{1} h_i \epsilon^i + O(\epsilon^2), \quad (3.15)\]

\[h_{-3} = \frac{1}{32}\]

\[h_{-2} = -\frac{1}{16} G(-1; y) - \frac{1}{16} G(-1; z) \quad (3.16b)\]

\[h_{-1} = \frac{1}{4} G(-1; y) G(-1; z) - \frac{1}{12} \pi^2 \quad (3.16c)\]

\[h_0 = -\frac{3}{8} G(1/z, -1, -1; y) + \frac{3}{8} G(1/z, 0, -1; y) - \frac{5}{8} G(-z - 2, -1, -1; y) - \frac{3}{8} G(-1, -1, 0; y) - \frac{3}{8} G(-2, -1, -1; z) - \frac{3}{8} G(-1, 0, -1; z) + \frac{5}{8} G(z - 2, -1; y) G(-1, -1; z) + \frac{3}{8} G(1/z; y) G(0, -1; z) - \frac{1}{2} G(-1; y) G(-1; z) - \frac{3}{16} G(1/z; y) G(-1; z)^2 - \frac{5}{16} G(-z - 2; y) G(-1; z)^2 - \frac{1}{16} \pi^2 G(1/z; y) - \frac{5}{16} \pi^2 G(-z - 2; y) + \frac{17}{48} \pi^2 G(-1; y) - \frac{5}{16} \pi^2 G(-2; z) + \frac{5}{16} \pi^2 \ln 2 + \frac{1}{6} G(-1; y)^3 + \frac{1}{6} G(-1; z)^3 - \frac{91}{32}\zeta(3). \quad (3.16d)\]

We provide the coefficients \(g_0\) and \(h_1\) via a file on arXiv. The solution contains multiple polylogarithms with either argument \(y\) and weights drawn from the set \(\{-1, 0, -2 - z, 1/z\}\) or with argument \(z\) and weights drawn from \(\{-2, -1, 0\}\).

4 Non-planar master integrals

The non-planar sector \(tt2nA:7:463\) is more involved than the previous planar cases and contains thresholds in all three channels, \(s, t\) and \(u\). For the integration of the differential equations, we choose the master integrals (2.9-2.11). In order to eliminate roots in \(s\) from the differential equations we choose the variable \(x\) and supplement it with \(y\). Since several subsectors occur both, in their uncrossed and their crossed version with \(y \leftrightarrow z\), integrating the differential equations with \((y, x)\) requires non-trivial argument change identities for multiple polylogarithms with explicit imaginary parts. We consider both kinematical invariants and master integrals to be complex valued and keep algebraic relations between the invariants exact, as discussed in section 2.

For the integration of the differential equations, we choose the master integrals (2.9-2.11) and the variables \(y\) and \(x\). As described before, we fix integration constants and check our results using regularity constraints, symmetry conditions and a Mellin-Barnes representation. For the Mellin-Barnes representation we choose another basis, where the
integrands contain the massive propagator to the power 1, 2 and 3, respectively, see appendix A. This Mellin-Barnes representation is described in appendix A. For the solutions in the basis (2.9-2.11) we find

$$p_1 \rightarrow \begin{array}{c} p_3 \end{array} \begin{array}{c} p_2 \rightarrow \begin{array}{c} p_4 \end{array} \end{array} = \frac{x^2}{m^6(1-x)^2(y+1)(1-x+x^2+xy)} \sum_{i=-4}^{0} k_i e^i + O(\epsilon), \quad (4.1)$$

$$k_{-4} = \frac{1}{32} \quad (4.2)$$

$$k_{-3} = \frac{1}{32} G(-(1-x+x^2)/x; y) - \frac{1}{32} G(-1; y) + \frac{1}{32} G([1-o+o^2]; x) - \frac{1}{8} G(1; x) + \frac{1}{32} G(0; x) + \frac{1}{32} i\pi + \frac{7}{96} + \frac{x(y+1)}{1-x} \left( \frac{1}{16} G(-(1-x+x^2)/x; y) - \frac{1}{16} G(-1; y) + \frac{1}{16} G([1-o+o^2]; x) - \frac{1}{16} G(1; x) \right) \quad (4.3)$$

$$k_{-2} = -\frac{1}{32} G(-(1-x+x^2)/x; y)^2 - \frac{1}{16} G(-(1-x+x^2)/x; y) G(-1; y) - \frac{1}{16} G(-1; y)^2 - \frac{1}{16} G(-(1-o+o^2); x) + \frac{1}{4} G(-1; y) G(1; x) - \frac{1}{16} G(-1; y) G(0; x) + \frac{1}{32} G([1-o+o^2]; x) G(0; x) - \frac{1}{32} G([-o+o^2]; x)^2 + \frac{1}{8} \left( G(1; x)^2 - \frac{1}{8} G(1; x) G(0; x) - \frac{1}{8} i\pi G(-(1-x+x^2)/x; y) - \frac{1}{16} i\pi G(-1; y) - \frac{1}{16} i\pi G([-1-o+o^2]; x) + \frac{1}{16} i\pi G(0; x) - \frac{7}{192} \pi^2 + \frac{1}{8} G(-(1-x+x^2)/x; y) - \frac{1}{6} G(-1; y) + \frac{1}{8} G([-1-o+o^2]; x) - \frac{1}{4} G(1; x) + \frac{1}{8} i\pi - \frac{7}{24} + \frac{x(y+1)}{1-x} \right) \left( G(1; x) - \frac{1}{8} G(0; x) - \frac{7}{24} \right) \right) \quad (4.4)$$

$$p_1 \rightarrow \begin{array}{c} p_3 \end{array} \begin{array}{c} p_2 \rightarrow \begin{array}{c} p_4 \end{array} \end{array} = \frac{x^2}{m^4(1-x)^2(1-x+x^2+xy)} \sum_{i=-4}^{0} l_i e^i + O(\epsilon), \quad (4.5)$$

$$l_{-4} = \frac{7}{384} \quad (4.6)$$

$$l_{-3} = -\frac{5}{192} G(-(1-x+x^2)/x; y) + \frac{1}{64} G(-1; y) - \frac{5}{192} G([-1-o+o^2]; x) - \frac{1}{16} G(1; x) + \frac{11}{192} G(0; x) - \frac{5}{192} i\pi \quad (4.7)$$

$$l_{-2} = +\frac{1}{192} G(-(1-x+x^2)/x; y)^2 - \frac{1}{32} G(-(1-x+x^2)/x; y) G(-1; y)$$
Using the coproduct extended symbol calculus we obtain for the first master integral the
\[ \frac{-1}{\epsilon} \] and weights drawn from arithms which we can express most naturally with the variables omitted in the following. Guided by the symbol we construct a new set of multiple polylogarithms with the third master integral and the finite parts are considerably more involved and therefore the light \( N_f \) contributions to \( gg \rightarrow t \bar{t} \) at NNLO are actually independent of this constant. The full solution contains multiple polylogarithms with either argument \( x \) and weights drawn from the set \( \{-1,0,-\frac{1}{x},-(1+x^2)/x,-(1-x+x^2)/x\} \) or with argument \( y \) and weights drawn from \( \{-1,0,1,1+o^2,1-o+o^2\} \).

Significant simplifications are possible for all poles in \( \epsilon \) of the first two master integrals, including the \( 1/\epsilon \) terms not displayed above because of their length. The \( 1/\epsilon \) pole of the third master integral and the finite parts are considerably more involved and therefore omitted in the following. Guided by the symbol we construct a new set of multiple polylogarithms which we can express most naturally with the variables \( y_1 = y + 1 \) and \( z_1 = z + 1 \).
simplified expressions

\[ k_{-4} = \frac{1}{32} \]

\[ k_{-3} = -\frac{1}{16} \ln(y_1 + z_1) + \frac{1}{16} i\pi + \frac{7}{96} + \frac{y_1 - z_1}{32(y_1 + z_1)} \ln(y_1/z_1) \]  

\[ k_{-2} = -\frac{1}{32} \ln^2(y_1 z_1) + \frac{1}{32} \ln^2(y_1 + z_1) + \frac{1}{16} \ln(y_1 + z_1) \ln(y_1 z_1) - \frac{1}{16} i\pi \ln(y_1 z_1) \]

\[ - \frac{1}{16} i\pi \ln(y_1 + z_1) - \frac{19}{192} \pi^2 - \frac{1}{48} \ln(y_1 z_1) - \frac{1}{8} \ln(y_1 + z_1) + \frac{1}{8} i\pi - \frac{7}{24} \]

\[ + \frac{y_1 - z_1}{y_1 + z_1} \ln(y_1/z_1) \left( -\frac{1}{16} i\pi + \frac{1}{8} i\pi + \frac{7}{48} \right) \]  

\[ k_{-1} = -\frac{1}{16} G\left(1, 0; 0; \frac{y_1 z_1}{y_1 + z_1}\right) - \frac{1}{16} i\pi G\left(1, 0; \frac{y_1 z_1}{y_1 + z_1}\right) + \frac{1}{48} \ln^3(y_1 z_1) \]

\[ - \frac{1}{16} \ln^2(y_1 + z_1) \ln(y_1 z_1) + \frac{1}{8} i\pi \ln(y_1 + z_1) \ln(y_1 z_1) + \frac{3}{32} \pi^2 \ln(y_1 z_1) \]

\[ + \frac{5}{48} \pi^2 \ln(y_1 + z_1) - \frac{29}{32} \zeta(3) - \frac{5}{48} i\pi^3 - \frac{5}{48} \ln^2(y_1 z_1) + \frac{7}{48} \ln^2(y_1/z_1) \]

\[ + \frac{1}{4} \ln(y_1 + z_1) \ln(y_1 z_1) - \frac{1}{4} i\pi \ln(y_1 z_1) + \frac{1}{12} \ln(y_1 + z_1) + \frac{1}{2} \ln(y_1 + z_1) \]

\[ - \frac{1}{144} \pi^2 - \frac{1}{2} i\pi + \frac{7}{6} + \frac{y_1 - z_1}{y_1 + z_1} \ln(y_1/z_1) \left( -\frac{1}{48} \ln^2(y_1 z_1) + \frac{1}{16} \ln^2(y_1 + z_1) \right) \]

\[ - \frac{1}{8} i\pi \ln(y_1 + z_1) - \frac{17}{96} \pi^2 - \frac{1}{24} \ln(y_1 z_1) - \frac{1}{4} \ln(y_1 + z_1) + \frac{1}{8} i\pi - \frac{7}{12} \]  

Here we choose a representation which makes the forward-backward symmetry \(y_1 \leftrightarrow z_1\) of the corner integral explicit. For the second master integral we find

\[ l_{-4} = \frac{7}{384} \]  

\[ l_{-3} = -\frac{1}{32} \ln(y_1 + z_1) + \frac{1}{64} \ln y_1 - \frac{5}{192} \ln z_1 + \frac{1}{32} i\pi \]  

\[ l_{-2} = \frac{1}{64} \ln^2(y_1 + z_1) - \frac{1}{64} \ln^2 y_1 + \frac{1}{192} \ln^2 z_1 - \frac{1}{32} \ln y_1 \ln z_1 \]

\[ + \frac{1}{16} \ln z_1 \ln(y_1 + z_1) - \frac{1}{32} i\pi \ln(y_1 + z_1) - \frac{1}{16} i\pi \ln y_1 \ln z_1 - \frac{47}{1452} \pi^2 \]  

\[ l_{-1} = -\frac{1}{32} G\left(1, 0; 0; \frac{y_1 z_1}{y_1 + z_1}\right) - \frac{1}{32} i\pi G\left(1, 0; \frac{y_1 z_1}{y_1 + z_1}\right) + \frac{1}{16} \ln^2 y_1 \ln z_1 \]

\[ - \frac{1}{16} \ln z_1 \ln^2(y_1 + z_1) + \frac{1}{8} i\pi \ln z_1 \ln(y_1 + z_1) + \frac{5}{96} \pi^2 \ln(y_1 + z_1) \]

\[ - \frac{1}{24} \pi^2 \ln y_1 - \frac{1}{144} \ln^3 z_1 + \frac{29}{288} \pi^2 \ln z_1 - \frac{55}{192} \zeta(3) - \frac{5}{96} i\pi^3 \]  

The original expressions for these poles in terms of \(G\) functions with argument \(y\) or \(x\) contained 65 multiple polylogarithms (22 two-dimensional and weight > 1) when all products are expanded with the shuffle relations. Systematically exploiting relations between them by a coproduct based reduction procedure reduces this number to 28 multiple polylogarithms (12 two-dimensional and weight > 1). This is reduced by an optimised choice of basis functions to just \(\text{Li}_3(y_1 z_1/(y_1 + z_1)), \text{Li}_2(y_1 z_1/(y_1 + z_1)), \ln(y_1 + z_1), \log y_1\) and
\ln z_1. Note that in the above expressions we used a more compact $G$ function based notation, which can easily be converted to classical polylogarithms via $\text{Li}_3(x) = -G(0, 0, 1; x)$, $\text{Li}_2(x) = -G(0, 1; x)$ and shuffle relations. Finally, we remark that the original expressions contained roots in $s$ through $x$, both in the rational prefactors and in the multiple polylogarithms, which could all be eliminated in the above expressions.

5 Conclusions

In this work, we presented analytical solutions for double box master integrals, which have not been available before. For the first time, we gave explicit solutions for non–planar double box integrals with a massive propagator in terms of multiple polylogarithms. Our results complete the set of master integrals required for the analytical calculation \cite{85} of the light $N_f$ corrections to $gg \to t\bar{t}$ at the two-loop level.

By carrying out a coproduct–augmented symbol analysis of the poles of two non–planar master integrals we demonstrated that remarkable simplifications are possible using an optimised set of multiple polylogarithms. It has been shown in \cite{86, 87} for the case of massless, planar two–loop and three–loop four–point topologies that it is possible to choose a basis in which the differential equations for the master integrals take a special and particularly simple form. In this basis, the master integrals have uniform transcendentality and no algebraic prefactors. Applying this method to the integrals discussed in this paper and choosing an appropriate set of multiple polylogarithms should allow to rewrite the full set of solutions in a very compact form \cite{88}.

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A A Mellin-Barnes representation for sector tt2nA:7:463

In this appendix, we give a Mellin-Barnes representation for integrals of the non–planar sector tt2nA:7:463 where the integrand contains the massive propagator taken to the power $n$. Similar to the calculation \cite{79} in the massless case, we start from a Feynman parameter representation, integrate the Feynman parameters at the expense of introducing
Mellin-Barnes contour integrals and obtain

\[ p_1 \rightarrow \prod_{n=1}^{p_3} = \int \frac{\mathcal{D}^d k_1 \mathcal{D}^d k_2}{D_{\text{tut2A}}:463(k_1, k_2, p_1, p_2, p_3) \cdot D_{\text{m}}^{n-1}(k_1-k_2+p_{13})} \]  

(A.1) 

\[ = \frac{m^{8-2d}u^{-4-2n}-1}{16 \Gamma^2(3-d/2)\Gamma(n)\Gamma(-4+d)\Gamma(-6-n+3d/2)} \int_{C_1} \frac{dz_1}{2\pi i} \int_{C_2} \frac{dz_2}{2\pi i} \int_{C_3} \frac{dz_3}{2\pi i} \int_{C_4} \frac{dz_4}{2\pi i} \int_{C_5} \frac{dz_5}{2\pi i} \]  

\[ \left( \frac{-8}{\mu^2} \right)^{-n+d-z_1-z_2-z_5} \left( \frac{-t_1}{\mu^2} \right)^{z_1} \left( \frac{-u_1}{\mu^2} \right)^{z_2} \left( \frac{m^2}{\mu^2} \right)^{z_5} \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_4)\Gamma(-z_5)}{\Gamma^2(2+z_1+z_2+z_3+z_4)} \]  

\[ \Gamma(1+z_1+z_3)\Gamma(1+z_2+z_3)\Gamma(1+z_1+z_4)\Gamma(1+z_2+z_4)\Gamma(n+z_1+z_2+2z_5) \]  

\[ \Gamma(4-d/2+z_1+z_2+z_3+z_4)\Gamma(-5-n+d-z_1-z_2-z_3-z_5) \]  

\[ \Gamma(-5-n+d-z_1-z_2-z_4-z_5)\Gamma(6+n-d+z_1+z_2+z_3+z_4+z_5) \]  

(A.2) 

where \( t_1 = t - m^2, u_1 = u - m^2 \) and \( \mu \) is an auxiliary normalisation scale. The contours \( C_1, \ldots, C_5 \) of complex integration are for imaginary parts from \(-\infty\) to \(+\infty\) and, for simplicity, fixed real parts chosen to separate the towers of increasing and decreasing poles of the different \( \Gamma \) functions. Despite the fact that this representation requires only one contour integration more than in the massless case, its evaluation is significantly more involved.

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