VARIABLE MARTINGALE HARDY SPACES AND THEIR APPLICATIONS IN FOURIER ANALYSIS

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Abstract. Let $p(\cdot)$ be a measurable function defined on a probability space satisfying $0 < p_- := \text{ess inf}_{x \in \Omega} p(x) \leq \text{ess sup}_{x \in \Omega} p(x) =: p_+ < \infty$. We investigate five types of martingale Hardy spaces $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ and prove their atomic decompositions when each $\sigma$-algebra $\mathcal{F}_n$ is generated by countably many atoms. Martingale inequalities and the relation of the different martingale Hardy spaces are proved as application of the atomic decomposition. In order to get these results, we introduce the following condition to replace (generalize) the so-called log-Hölder continuity condition in harmonic analysis:

$$P(A)^{p_-(A)-p_+(A)} \leq C_{p(\cdot)} \quad \text{for all atom } A.$$  

Some applications in Fourier analysis are given by use of the previous results. We generalize the classical results and show that the partial sums of the Walsh-Fourier series converge to the function in norm if $f \in L_{p(\cdot)}$ or $f \in L_{p(\cdot),q}$ and $p_- > 1$. The boundedness of the maximal Fejér operator on $H_{p(\cdot)}$ and $H_{p(\cdot),q}$ is proved whenever $p_- > 1/2$ and the condition $\frac{1}{p_-} - \frac{1}{p_+} < 1$ holds. It is surprising that this last condition does not appear for trigonometric Fourier series. One of the key points of the proof is that we introduce two new dyadic maximal operators and prove their boundedness on $L_{p(\cdot)}$ with $p_- > 1$. The method we use to prove these results is new even in the classical case. As a consequence, we obtain theorems about almost everywhere and norm convergence of the Fejér means.

1. Introduction

Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function (a variable exponent). The variable Lebesgue space $L_{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions $f$ such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx < \infty$. It generalizes the classical Lebesgue space: when $p(\cdot) \equiv p$ is a constant, then $L_{p(\cdot)}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$. Interest in the variable Lebesgue spaces has increased since the 1990s because of their use in a variety of applications. As we all know, the variable function spaces can be well applied in fluid dynamics [11, 22, 60], image processing [14, 36, 69], partial differential equations and variational calculus [7, 21, 68], and harmonic analysis [5, 16, 20, 41, 80].

In order to extend the techniques and results of constant exponent case to the setting of variable Lebesgue spaces, a central problem is to determine conditions on an exponent $p(\cdot)$ such that the Hardy-Littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$. The first major result is due to Diening [19], who showed that it is sufficient to assume that $p(\cdot)$ satisfies the
so-called local log-Hölder condition:

\[
|p(x) - p(y)| \leq \frac{C}{-\log |x - y|}, \quad \forall x, y \in \mathbb{R}^n, \quad |x - y| < 1/2,
\]

and is constant outside of a large ball. This result was generalized independently by Cruz-Uribe et al. [17] and Nekvinda [57], who in addition assumed that \( p(\cdot) \) is log-Hölder continuous at infinity: there is \( p_\infty > 1 \) such that

\[
|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.
\]

The conditions (1.1) and (1.2) are called log-Hölder continuity condition. We point out that the log-Hölder continuity condition is not necessary for the boundedness of the Hardy-Littlewood maximal operator on \( L^{p(\cdot)}(\mathbb{R}^n) \) (see [58]). Heavily basing on the log-Hölder continuity condition, harmonic analysis with variable exponent has gotten a rapid development. Nakai and Sawano [56] first introduced the Hardy space \( H^{p(\cdot)}(\mathbb{R}^n) \) with a variable exponent \( p(\cdot) \) and established the atomic decompositions. As applications, they proved the duality and the boundedness of singular integral operators. Independently, Cruz-Uribe and Wang [18] also investigated the variable Hardy space \( H^{p(\cdot)}(\mathbb{R}^n) \) with \( p(\cdot) \) satisfying some conditions slightly weaker than those used in [56]. Sawano [61] improved the results in [56]. Ho [39] studied weighted Hardy spaces with variable exponents. Zhuo et al. [83] investigated Hardy spaces with variable exponents on RD-spaces and applications. Very recently, Yan et al. [79] introduced the variable weak Hardy space \( H^{p(\cdot),\infty}(\mathbb{R}^n) \) and characterized these spaces via the radial maximal functions, atoms and Littlewood-Paley functions. The Hardy-Lorentz spaces \( H^{p(\cdot),q}(\mathbb{R}^n) \) were investigated by Jiao et al. in a very recent paper [45]. Similar results for the anisotropic Hardy spaces \( H^{p(\cdot)}(\mathbb{R}^n) \) and \( H^{p(\cdot),q}(\mathbb{R}^n) \) can be found in Liu et al. [48, 49].Martingale Musielak–Orlicz Hardy spaces were investigated in Xie et al. [76, 77, 78]. We also refer to [3, 4, 81] for Besov spaces with variable smoothness and integrability and their applications.

In the early 70’s of the last century, with the development of the theory of Hardy spaces on \( \mathbb{R}^n \) in harmonic analysis, martingale Hardy spaces theory was born. Until now, most of the important facts in harmonic analysis have been found to have their satisfactory counterparts in the martingale setting. For example, in martingale setting, the duality between \( H_1 \) and \( BMO \), and the Doob maximal inequality can be found in Garsia [26]; the Burkholder martingale transforms [8] can be considered as an analogue to the classical singular integral operators. On the other hand, the theory of martingale Hardy spaces has influenced the development of harmonic analysis. For example, the atomic decomposition of \( H_p \), which is one of the most powerful tool in harmonic analysis nowadays, was first shown in martingale setting by Herz [37]. Later, the theory of atomic decomposition of martingale spaces was developed in Weisz [70]. The good-\( \lambda \) inequality, which is a useful tool to compare the integrability of two related measurable functions, was discovered by Burkholder and Gundy [10, 9] in martingale setting. A much more simplified proof of \( T(b) \) theorem was given by Coifman et al. [15] via martingale approach. The theory of martingale Hardy spaces was considered in the books [26, 51, 70, 71]. The applications of martingale theory to Fourier
analysis were developed by many people, see for example monographs [64, 70, 71, 74] and the references therein.

Although the theory of variable Hardy spaces on $\mathbb{R}^n$ has rapidly been developed in recent years, the variable exponent framework has not yet been applied to the martingale setting. The first main difficulty we need to overcome is to find a suitable replacement for the log-Hölder continuity conditions (1.1) and (1.2) when the variable exponent $p(\cdot)$ is defined on a probability space. Unlike the Euclidean space $\mathbb{R}^n$, there is no natural metric in a probability space. In order to better explain it, we first introduce some basic notation. Let $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n \geq 0})$ be a complete probability space such that $\mathcal{F} = \sigma(\cup_{n} \mathcal{F}_n)$. A measurable function $p(\cdot) : \Omega \to (0, \infty)$ is called a variable exponent. For a measurable set $A \subset \Omega$, we denote $p_-(A) := \text{ess inf}_{x \in A} p(x)$, $p_+(A) := \text{ess sup}_{x \in A} p(x)$, and for convenience

$$p_- := p_-(\Omega), \quad p_+ := p_+(\Omega).$$

Denote by $\mathcal{P}(\Omega)$ the collection of all variable exponents $p(\cdot)$ such that $0 < p_- \leq p_+ < \infty$. If $p(\cdot) \in \mathcal{P}(\Omega)$, the variable Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all measurable functions $f$ for which

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} d\mathbb{P} \leq 1 \right\} < \infty.$$

For a martingale $f = (f_n)_{n \geq 0}$ with respect to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n \geq 0})$, the Doob maximal operator is defined by

$$M_m f = \sup_{0 \leq n \leq m} |f_n|, \quad M f = \sup_{n \geq 0} |f_n|.$$

The classical Doob maximal inequality implies that $M$ is bounded on $L_p(\Omega)$ for $1 < p \leq \infty$. However, according to the facts in [16, Example 3.21], the Doob maximal operator is not bounded on $L_{p(\cdot)}(\Omega)$ for general variable exponent $p(\cdot)$ with $p_- > 1$. It is natural to find sufficient condition imposed on $p(\cdot)$ such that the Doob maximal operator $M$ is bounded on $L_{p(\cdot)}(\Omega)$. Aoyama [9] proved the Doob maximal inequality under the condition that $p(\cdot)$ is $\mathcal{F}_n$-measurable for all $n \geq 0$. Obviously, this kind of condition is quite strong. Moreover, Nakai and Sadasue [52] showed that $\mathcal{F}_0$-measurability of $p(\cdot)$ is not necessary for this maximal inequality. Note that a weak type inequality was proved in [44, Theorem 3.2]. Namely, given $p(\cdot) \in \mathcal{P}(\Omega)$ with $1 \leq p_- \leq p_+ < \infty$,

$$P(M f > \lambda) \leq C_{p(\cdot)} \int_{\Omega} \left( \frac{|f_\infty(x)|}{\lambda} \right)^{p(x)} d\mathbb{P}, \quad \forall \lambda > 0. \quad (1.3)$$

Unfortunately, we can not obtain the Doob maximal inequality by means of the weak type inequality (1.3) as in the classical case. The essential reason is that the space $L_{p(\cdot)}(\Omega)$ is no longer a rearrangement invariant space and the important formula

$$\int_{\Omega} |f(x)|^p d\mathbb{P} = p \int_0^\infty t^{p-1} \mathbb{P}(x \in \Omega : |f(t)| > t) dt$$

has no variable exponent analogue.
In this paper, we introduce a condition without metric characterization of \( p(\cdot) \) to replace the log-Hölder continuity condition mentioned above. We suppose that there exists a constant \( K_{p(\cdot)} \geq 1 \) depending only on \( p(\cdot) \) such that

\[
\mathbb{P}(A)^{p_-(A)-p_+(A)} \leq K_{p(\cdot)}, \quad \forall A \in \bigcup_n A(\mathcal{F}_n),
\]

where \( \mathcal{F}_n \) is generated by countably many atoms and \( A(\mathcal{F}_n) \) denotes the family of all atoms in \( \mathcal{F}_n \) for each \( n \in \mathbb{N} \). It is known that the log-Hölder continuity (1.4) and (1.2) imply our condition (1.4) on \([0, 1)\) or even on \( \mathbb{R}^n \) (see Remark 2.16). We should mention that Nakai et al. \([53, 54, 55]\) and Ho \([38]\) studied the martingale Morrey-Hardy and Campanato-Hardy spaces associated with \( \mathcal{F}_n \) generated by countably many atoms.

We give a systematic study of martingale Hardy spaces \( H_{p(\cdot)} \) and \( H_{p(\cdot),q} \) associated with a variable exponent \( p(\cdot) \). A powerful tool used in the paper is the atomic decomposition of variable martingale Hardy and Hardy-Lorentz spaces. Our first main result, without any restriction on \( p(\cdot) \), is the \((1, p(\cdot), \infty)\)-atomic characterization of the Hardy spaces \( H_{p(\cdot)}^s \) and \( H_{p(\cdot),q}^s \) associated with the conditional square operator \( s \), that is, \( H_{p(\cdot)}^s = H_{p(\cdot),q}^{at,1,\infty} \) and \( H_{p(\cdot),q}^s = H_{p(\cdot),q}^{at,1,\infty} \) with equivalent quasi-norms. As one of the applications of atomic decompositions, we get martingale inequalities between different Hardy spaces \( H_{p(\cdot)} \) and \( H_{p(\cdot),q} \) in Section 4.

Finally, we consider the applications of the theory of variable martingale Hardy spaces in Fourier analysis. In the constant exponent case, the theory of martingales has an extensive application in dyadic harmonic analysis; see for example the monographs \([64, 70, 71, 74]\), the papers Gát and Goginava \([27, 28, 29, 31, 32, 33]\) and Schipp and Simon \([62, 63, 65, 66, 67]\). Particularly, in \([71, \text{Theorem 3.10}]\), by using dyadic martingale theory, Weisz proved that the maximal Fejér operator \( \sigma_\star \) is bounded from \( H_{p,q} \) to \( L_{p,q} \) with \( p > 1/2 \). Then, it can be deduced from Weisz’s result that the Fejér means of \( f \) converge almost everywhere to \( f \). Inspired by this result, we generalize these theorems and prove that \( \sigma_\star \) is bounded from \( H_{p(\cdot)} \) to \( L_{p(\cdot)} \) and from \( H_{p(\cdot),q} \) to \( L_{p(\cdot),q} \) under the conditions \( 1/2 < p_- < \infty, 0 < q \leq \infty \) and \( \frac{1}{p_-} - \frac{1}{p_+} < 1 \). This last condition is very surprising because the corresponding results for Fourier transforms hold without this condition (see Liu et al. \([48, 49]\) and Weisz \([75]\)).

This gives a serious difference between the trigonometric Fourier analysis and Walsh-Fourier analysis. Unlike the classical case, our proof does not depend on interpolation method. One of the key points of the proof is that we introduce two new dyadic maximal operators and prove their boundedness on \( L_{p(\cdot)} \) with \( p(\cdot) \) satisfying (1.4) and \( p_- > 1 \). This method is new even in the classical case \([72, 71]\). Finally, we show that the boundedness of \( \sigma_\star \) implies almost everywhere and norm convergence of the Fejér means as well.

The structure of the paper is as follows. In Section 2, we present preliminaries, definitions and lemmas used later in the paper. We also introduce the definition of variable Lebesgue space \( L_{p(\cdot)}(\Omega) \) and Lorentz space \( L_{p(\cdot),q}(\Omega) \). Some basic properties of these spaces are given, including duality of \( L_{p(\cdot)}(\Omega) \) and dominated convergence theorem in variable Lorentz space \( L_{p(\cdot),q}(\Omega) \). We also introduce five types of variable Hardy spaces \( H_{p(\cdot)} \) and \( H_{p(\cdot),q} \) in this section. Moreover, we show some basic inequalities for the Doob maximal operator, including the boundedness of the operator on \( L_{p(\cdot)}(\Omega) \) and \( L_{p(\cdot),q}(\Omega) \) (see Theorem 2.21 below) and the
variable version of the dual Doob inequality. We also prove that the Doob maximal operator is bounded from \( L_{p(\cdot)}(\Omega) \) to \( L_{p(\cdot),\infty}(\Omega) \) with \( p_- \geq 1 \).

The objective of Section 3 is the atomic decomposition for variable Hardy spaces \( H_{p(\cdot)} \) and \( H_{p(\cdot),q} \). We prove the desired atomic decompositions for all kinds of Hardy spaces.

Section 4 is devoted to the applications of atomic decompositions established in Section 3. In Section 4, we obtain some continuous embedding relationships among different variable martingale Hardy spaces and martingale Hardy-Lorentz spaces. Moreover, if \( (\mathcal{F}_n)_{n\geq 0} \) is regular, then different kinds of \( H_{p(\cdot)} \) (resp. \( H_{p(\cdot),q} \)) are all equivalent.

In the last section, we deal with some applications in Walsh-Fourier analysis. We prove the above mentioned results about the Walsh-Fourier series.

Throughout this paper, \( \mathbb{Z} \) and \( \mathbb{N} \) denote the integer set and nonnegative integer set, respectively. We denote by \( C \) a positive constant, which can vary from line to line, and denote by \( C_{p(\cdot)} \) a constant depending only on \( p(\cdot) \). The symbol \( A \lesssim B \) stands for the inequality \( A \leq CB \) or \( A \leq C_{p(\cdot)}B \). If we write \( A \approx B \), then it stands for \( A \lesssim B \lesssim A \).

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2. Preliminaries

2.1. Variable Lebesgue spaces \( L_{p(\cdot)} \). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space. A measurable function \( p(\cdot) : \Omega \to (0, \infty) \) is called a variable exponent. For a measurable set \( A \subset \Omega \), we denote
\[
p_- (A) := \text{ess inf}_{x \in A} p(x), \quad p_+ (A) := \text{ess sup}_{x \in A} p(x)
\]
and for convenience
\[
p_- := p_-(\Omega), \quad p_+ := p_+(\Omega).
\]
Denote by \( \mathcal{P}(\Omega) \) the collection of all variable exponents \( p(\cdot) \) such that \( 0 < p_- \leq p_+ < \infty \). The variable Lebesgue space \( L_{p(\cdot)} = L_{p(\cdot)}(\Omega) \) is the collection of all measurable functions \( f \) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) such that for some \( \lambda > 0 \),
\[
\rho(f / \lambda) = \int_{\Omega} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} d\mathbb{P} < \infty.
\]
This becomes a quasi-Banach function space when it is equipped with the quasi-norm
\[
\|f\|_{p(\cdot)} := \inf \{ \lambda > 0 : \rho(f / \lambda) \leq 1 \}.
\]
For any \( f \in L_{p(\cdot)} \), we have \( \rho(f) \leq 1 \) if and only if \( \|f\|_{p(\cdot)} \leq 1 \); see [22] Theorem 1.3. In the sequel, we always use the symbol
\[
p = \min \{ p_-, 1 \}.
\]
Throughout the paper, the variable exponent \( p'(\cdot) \) is defined pointwise by
\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in \Omega.
\]
We present some basic properties here (see [56]):

(1) $\|f\|_{p(\cdot)} \geq 0; \|f\|_{p(\cdot)} = 0 \iff f \equiv 0$.

(2) $\|cf\|_{p(\cdot)} = |c| \cdot \|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.

(3) For $0 < b \leq p$ we have

\[
\|f + g\|_{p(\cdot)}^b \leq \|f\|_{p(\cdot)}^b + \|g\|_{p(\cdot)}^b.
\]

**Lemma 2.1** ([16, Proposition 2.21]). Let $p(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L_{p(\cdot)}(\Omega)$ and $\|f\|_{p(\cdot)} \neq 0$, then

\[
\int_{\Omega} \left| f(x) \right|^{p(x)} d\mathbb{P} = 1.
\]

**Lemma 2.2** ([16, Corollary 2.28]). Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$ satisfy

\[
\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}, \quad x \in \Omega.
\]

Then there exists a constant $C$ such that for all $f \in L_{q(\cdot)}$ and $g \in L_{r(\cdot)}$, we have $fg \in L_{p(\cdot)}$ and

\[
\|fg\|_{p(\cdot)} \leq C\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.
\]

**Lemma 2.3** ([17, Theorem 2.8]). Let $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$. If $p(\cdot) \leq q(\cdot)$, then for every $f \in L_{q(\cdot)}$ we have

\[
\|f\|_{p(\cdot)} \leq 2\|f\|_{q(\cdot)}.
\]

**Lemma 2.4** ([16, Theorem 2.34]). Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $1 \leq p(\cdot)$. Then

\[
\|f\|_{p(\cdot)} \approx \sup \int_{\Omega} fgd\mathbb{P},
\]

where the supremum is taken over all $g \in L_{p'(\cdot)}$ with $\|g\|_{p'(\cdot)} \leq 1$.

### 2.2. Variable Lorentz spaces $L_{p(\cdot),q(\cdot)}$

In this subsection, we introduce the definition of Lorentz spaces $L_{p(\cdot),q(\cdot)}(\Omega)$ with variable exponents $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \leq \infty$ is a constant. For more information about general cases $L_{p(\cdot),q(\cdot)}(\Omega)$, we refer the reader to [46]. Following [46], we introduce the definition below.

**Definition 2.5.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \leq \infty$. Then $L_{p(\cdot),q(\cdot)}(\Omega)$ is the collection of all measurable functions $f$ such that

\[
\|f\|_{L_{p(\cdot),q(\cdot)}} := \left\{ \left( \int_{0}^{\infty} \lambda^{q} \left\| X_{\{|f|>\lambda\}} \right\|_{p(\cdot)}^{q} \frac{d\lambda}{\lambda} \right)^{1/q}, \quad q < \infty, \left( \sup_{\lambda} \lambda \left\| X_{\{|f|>\lambda\}} \right\|_{p(\cdot)} \right), \quad q = \infty \right\}
\]

is finite.

**Remark 2.6.** According to [46, Theorem 3.1], the spaces $L_{p(\cdot),q(\cdot)}$ are quasi-Banach spaces. Moreover,
(1) It is similar to the classical case that the equations above can be discretized:

$$\|f\|_{L_{p(\cdot), q}} \approx \left( \sum_{k = -\infty}^{\infty} 2^{kq} \|\chi_{\{|f| > 2^k\}}\|_{p(\cdot)}^q \right)^{1/q},$$

and

$$\|f\|_{L_{p(\cdot), \infty}} \leq 2 \sup_{k \in \mathbb{Z}} 2^k \|\chi_{\{|f| > 2^k\}}\|_{p(\cdot)} \leq 2 \|f\|_{L_{p(\cdot), \infty}}.$$  (2)

By [16, Lemma 2.4], we have

$$\sup_{k \in \mathbb{Z}} 2^k \|\chi_{\{|f| > 2^k\}}\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \sup_{k \in \mathbb{Z}} \int_{\Omega} \left( \frac{2^k \chi_{\{|f| > 2^k\}}}{\lambda} \right)^{p(x)} \, d\mathbb{P} \leq 1 \right\}.$$  (3)

Let $A \in \mathcal{F}$. A simple calculation based on (1) above shows that

$$\|\chi_A\|_{L_{p(\cdot), q}} \approx \|\chi_A\|_{p(\cdot)}.$$  We now show the dominated convergence theorem in $L_{p(\cdot), q}$. We begin with the following definition.

**Definition 2.7.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \leq \infty$. A function $f \in L_{p(\cdot), q}$ is said to have absolutely continuous quasi-norm in $L_{p(\cdot), q}$ if

$$\lim_{n \to \infty} \|f\chi_{A_n}\|_{L_{p(\cdot), q}} = 0$$

for every sequence $(A_n)_{n \geq 0}$ satisfying $\mathbb{P}(A_n) \to 0$ as $n \to \infty$.

The next result shows that if $0 < p_- \leq p_+ < \infty$ and $0 < q < \infty$, then all the elements in $L_{p(\cdot), q}(\Omega)$ have absolutely continuous quasi-norm.

**Lemma 2.8.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q < \infty$. Then for every $f \in L_{p(\cdot), q}$, $f$ has absolutely continuous quasi-norm.

**Proof.** Since $f \in L_{p(\cdot), q}$, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$\left( \sum_{k = N_1}^{\infty} 2^{kq} \|\chi_{\{|f| > 2^k\}}\|_{p(\cdot)}^q \right)^{1/q} < \varepsilon.$$  By the definition of $(A_n)_{n}$, there exists $N_2 \in \mathbb{N}$ such that $\mathbb{P}(A_n) < \left( \frac{1}{2^{N_1}} \right)^{p(x)}$ for $n \geq N_2$. Now let $n \geq N_2$. By Lemma 2.3, we have

$$\|f\chi_{A_n}\|_{L_{p(\cdot), q}} \leq \left( \sum_{k = N_1}^{\infty} 2^{kq} \|\chi_{\{|f| > 2^k\}}\|_{p(\cdot)}^q \right)^{1/q} + \left( \sum_{k = -\infty}^{N_1 - 1} 2^{kq} \|\chi_{A_n}\|_{p(\cdot)}^q \right)^{1/q}$$

$$\leq \varepsilon + \left( \sum_{k = -\infty}^{N_1 - 1} 2^{kq} \cdot (2\|\chi_{A_n}\|_{p_+})^q \right)^{1/q}$$

$$< \varepsilon + 2 \left( \sum_{k = -\infty}^{N_1 - 1} 2^{(k-N_1)q} \right)^{1/q} \varepsilon = 3\varepsilon.$$
This finishes the proof. \hfill \Box

The following well-known example (see [30, Example 2.5]) shows that not all functions in $L_{p(\cdot),\infty}$ have absolutely continuous quasi-norm.

**Example 2.9.** Consider the function $f(x) = x^{-1/p}$ on $\Omega = (0, 1)$ associated with Lebesgue measure $\mathbb{P}$. Then, by a simple calculation, $f \in L_{p,\infty}$ ($0 < p < \infty$) and $f$ does not have absolutely continuous quasi-norm.

Next we introduce a closed subspace of $L_{p(\cdot),\infty}$, in which simple functions are dense.

**Definition 2.10.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. We define $\mathcal{L}_{p(\cdot),\infty}(\Omega)$ as the set of measurable functions $f$ such that

$$\lim_{n \to \infty} \|f \chi_{A_n}\|_{L_{p(\cdot),\infty}} = 0$$

for every sequence $(A_n)_{n \geq 0}$ satisfying $\mathbb{P}(A_n) \to 0$ as $n \to \infty$.

**Lemma 2.11.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then $\mathcal{L}_{p(\cdot),\infty}$ is a closed subspace of $L_{p(\cdot),\infty}$.

**Proof.** Let $(f_n)_{n \geq 1} \subset \mathcal{L}_{p(\cdot),\infty}$ be a Cauchy sequence in $L_{p(\cdot),\infty}$. Then there exists $f \in L_{p(\cdot),\infty}$ such that

$$\lim_{n \to \infty} \|f_n - f\|_{L_{p(\cdot),\infty}} = 0.$$

Choose $A_k \subset \Omega$ such that $\mathbb{P}(A_k) \to 0$ as $k \to \infty$. Hence, for any $k$ we have

$$\|f \chi_{A_k}\|_{L_{p(\cdot),\infty}} \leq 2^k \left(\|f_n \chi_{A_k}\|_{L_{p(\cdot),\infty}} + \|(f_n - f) \chi_{A_k}\|_{L_{p(\cdot),\infty}}\right) \leq 2^k \left(\|f_n \chi_{A_k}\|_{L_{p(\cdot),\infty}} + \|f_n - f\|_{L_{p(\cdot),\infty}}\right).$$

Since $f_n \in \mathcal{L}_{p(\cdot),\infty}$, by taking $n \to \infty$, we get

$$\lim_{k \to \infty} \|f \chi_{A_k}\|_{L_{p(\cdot),\infty}} = 0,$$

which implies that $f \in \mathcal{L}_{p(\cdot),\infty}$.

**Lemma 2.12.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then we have $L_{p(\cdot)} \subset \mathcal{L}_{p(\cdot),\infty} \subset L_{p(\cdot),\infty}$.

**Proof.** By Lemma 2.11 it suffices to show $L_{p(\cdot)} \subset \mathcal{L}_{p(\cdot),\infty}$. Let $f \in L_{p(\cdot)}$. Applying Lemma 2.1 we get

$$\int_{\Omega} \left(\frac{2^k \chi_{\{|f|>2^k\}}(x)}{\|f\|_{p(\cdot)}} \right)^{p(x)} d\mathbb{P} \leq \int_{\Omega} \left(\frac{|f(x)|^{p(x)}}{\|f\|_{p(\cdot)}^{p(x)}} \right) d\mathbb{P} = 1$$

for any $k \in \mathbb{Z}$. We conclude by Remark 2.6 that

$$\|f\|_{L_{p(\cdot),\infty}} \leq \|f\|_{p(\cdot)}.$$

Let $A_k \subset \Omega$ such that $\mathbb{P}(A_k) \to 0$ as $k \to \infty$. We get $\rho(f \chi_{A_k}) \to 0$ as $k \to \infty$ since $f \in L_{p(\cdot)}$. Note that $0 < p_- \leq p_+ < \infty$. Hence, by [16, Theorem 2.68], we deduce that $\|f \chi_{A_k}\|_{L_{p(\cdot)}} \to 0$ as $k \to \infty$. It follows from inequality (2.2) that

$$\lim_{k \to \infty} \|f \chi_{A_k}\|_{L_{p(\cdot),\infty}} = 0.$$

Now, we obtain that $f \in \mathcal{L}_{p(\cdot),\infty}$. The proof is complete. \hfill \Box
Lemma 2.13. (Dominated convergence theorem) Let \( p(\cdot) \in \mathcal{P}(\Omega) \) and \( 0 < q \leq \infty \). Assume that \( f_n, f, g \in L_{p(\cdot)} \) satisfies \( f_n \to f \) a.e. and \( |f_n| \leq g \) for every \( n \geq 1 \). If \( g \) has absolutely continuous quasi-norm, then

\[
\lim_{n \to \infty} \|f_n - f\|_{L_{p(\cdot)}^q} = 0.
\]

Proof. Since \( g \) has absolutely continuous quasi-norm, for any \( \varepsilon > 0 \) there exists \( N_1 \) such that \( \|g\chi_{\{g > N_1\}}\|_{L_{p(\cdot)}^q} < \varepsilon \). Clearly, \( |f_n - f| \leq 2N_1 \) on the set \( \{g \leq N_1\} \). Note that, by Remark 2.6(3),

\[
\|(f_n - f)\chi_{\{g \leq N_1\}}\|_{L_{p(\cdot)}^q} = \|(f_n - f)\chi_{\{g \leq N_1\}}\chi_{\{f_n \neq f\}}\|_{L_{p(\cdot)}^q} \leq 2N_1\|\chi_{\{f_n \neq f\}}\|_{L_{p(\cdot)}^q} = 2N_1\|\chi_{\{f_n \neq f\}}\|_{L_{q(\cdot)}}.
\]

Then, by the facts \( \chi_{\{f_n \neq f\}} \to 0 \) as \( n \to \infty \) and the dominated convergence theorem in \( L_{p(\cdot)} \) (see [10]), there exists \( N_2 \) such that \( \|(f_n - f)\chi_{\{g \leq N_1\}}\|_{L_{p(\cdot)}^q} < \varepsilon \) for \( n \geq N_2 \). Finally, for \( n \geq N_2 \),

\[
\|f_n - f\|_{L_{p(\cdot)}^q} \leq \|(f_n - f)\chi_{\{g \leq N_1\}}\|_{L_{p(\cdot)}^q} + \|(f_n - f)\chi_{\{g > N_1\}}\|_{L_{p(\cdot)}^q} \leq \varepsilon + 2\|g\chi_{\{g > N_1\}}\|_{L_{p(\cdot)}^q} < 3\varepsilon,
\]

which completes the proof. \( \square \)

2.3. **Variable martingale Hardy spaces.** In this subsection, we introduce some standard notations from martingale theory. We refer to the books [20, 51, 70] for the theory of classical martingale space. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space. Let the subalgebras \( \{\mathcal{F}_n\}_{n \geq 0} \) be increasing such that \( \mathcal{F} = \sigma(\cup_{n \geq 0} \mathcal{F}_n) \), and let \( \mathbb{E}_n \) denote the conditional expectation operator relative to \( \mathcal{F}_n \). A sequence of measurable functions \( f = (f_n)_{n \geq 0} \subset L_1(\Omega) \) is called a martingale with respect to \( \{\mathcal{F}_n\}_{n \geq 0} \) if \( \mathbb{E}_n(f_{n+1}) = f_n \) for every \( n \geq 0 \). For a martingale \( f = (f_n)_{n \geq 0} \) let \( f_{-1} := 0 \) and

\[
d_n f = f_n - f_{n-1}, \quad n \geq 0,
\]

denote the martingale difference. If in addition \( f_n \in L_{p(\cdot)} \), then \( f \) is called an \( L_{p(\cdot)} \)-martingale with respect to \( \{\mathcal{F}_n\}_{n \geq 0} \). In this case, we set

\[
\|f\|_{p(\cdot)} = \sup_{n \geq 0} \|f_n\|_{p(\cdot)}.
\]

If \( \|f\|_{p(\cdot)} < \infty \), \( f \) is called a bounded \( L_{p(\cdot)} \)-martingale and it is denoted by \( f \in L_{p(\cdot)} \). For a martingale relative to \( (\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_n\}_{n \geq 0}) \), we define the maximal function, the square function and the conditional square function of \( f \), respectively, as follows \((f_{-1} = 0)\):

\[
M_m(f) = \sup_{0 \leq n \leq m} |f_n|, \quad M(f) = \sup_{n \geq 0} |f_n|;
\]

\[
S_m(f) = \left( \sum_{n=0}^{m} |d_n f|^2 \right)^{1/2}, \quad S(f) = \left( \sum_{n=0}^{\infty} |d_n f|^2 \right)^{1/2};
\]

\[
s_m(f) = \left( \sum_{n=0}^{m} \mathbb{E}_{n-1} |d_n f|^2 \right)^{1/2}, \quad s(f) = \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f|^2 \right)^{1/2}.
\]
Denote by $\Lambda$ the collection of all sequences $(\lambda_n)_{n \geq 0}$ of non-decreasing, non-negative and adapted functions with $\lambda_\infty = \lim_{n \to \infty} \lambda_n$. Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $0 < q \leq \infty$. The variable martingale Hardy spaces associated with variable Lebesgue spaces $L_{p(\cdot)}$ are defined as follows:

$$H^M_{p(\cdot)} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^M_{p(\cdot)}} = \| M(f) \|_{p(\cdot)} < \infty \};$$

$$H^S_{p(\cdot)} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^S_{p(\cdot)}} = \| S(f) \|_{p(\cdot)} < \infty \};$$

$$H^s_{p(\cdot)} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^s_{p(\cdot)}} = \| \sigma(f) \|_{p(\cdot)} < \infty \};$$

$$Q_{p(\cdot)} = \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \text{ s.t. } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot)} \},$$

$$\| f \|_{Q_{p(\cdot)}} = \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{p(\cdot)};$$

$$P_{p(\cdot)} = \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \text{ s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot)} \},$$

$$\| f \|_{P_{p(\cdot)}} = \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{p(\cdot)}.$$

Similarly, the variable martingale Lorentz-Hardy spaces associated with variable Lorentz spaces $L_{p(\cdot),q}$ are defined as follows:

$$H^M_{p(\cdot),q} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^M_{p(\cdot),q}} = \| M(f) \|_{L_{p(\cdot),q}} < \infty \};$$

$$H^S_{p(\cdot),q} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^S_{p(\cdot),q}} = \| S(f) \|_{L_{p(\cdot),q}} < \infty \};$$

$$H^s_{p(\cdot),q} = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^s_{p(\cdot),q}} = \| \sigma(f) \|_{L_{p(\cdot),q}} < \infty \};$$

$$Q_{p(\cdot),q} = \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \text{ s.t. } S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot),q} \},$$

$$\| f \|_{Q_{p(\cdot),q}} = \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{L_{p(\cdot),q}};$$

$$P_{p(\cdot),q} = \{ f = (f_n)_{n \geq 0} : \exists (\lambda_n)_{n \geq 0} \in \Lambda, \text{ s.t. } |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot),q} \},$$

$$\| f \|_{P_{p(\cdot),q}} = \inf_{(\lambda_n) \in \Lambda} \| \lambda_\infty \|_{L_{p(\cdot),q}}.$$

We define $\mathcal{H}^M_{p(\cdot),\infty}$ as the space of all martingales such that $M(f) \in L_{p(\cdot),\infty}$. Analogously, we can define $\mathcal{H}^S_{p(\cdot),\infty}$ and $\mathcal{H}^s_{p(\cdot),\infty}$, respectively.

**Remark 2.14.** If $p(\cdot) = p$ is a constant, then the above definitions of variable Hardy spaces go back to the classical definitions stated in [26] and [70].
2.4. The Doob maximal operator. In the sequel of the paper, we will often suppose that every $\sigma$-algebra $F_n$ is generated by countably many atoms. Recall that $B \in F_n$ is called an atom, if for any $A \subset B$ with $A \in F_n$ satisfying $P(A) < P(B)$, we have $P(A) = 0$. We denote by $A(F_n)$ the set of all atoms in $F_n$. It is clear that for $f \in L^1(\Omega)$

$$\mathbb{E}_n(f) = \sum_{A \in A(F_n)} \left( \frac{1}{P(A)} \int_A f(x) dP \right) \chi_A, \quad n \in \mathbb{N}.$$ 

We now recall the definition of regularity. The stochastic basis $(F_n)_{n \geq 0}$ is said to be regular, if for $n \geq 0$ and $A \in F_n$, there exists $B \in F_{n-1}$ such that $A \subset B$ and $P(B) \leq R P(A)$, where $R$ is a positive constant independent of $n$. A martingale is said to be regular if it is adapted to a regular $\sigma$-algebra sequence. This implies that there exists a constant $R > 0$ such that

$$f_n \leq R f_{n-1}$$

for all non-negative martingales $(f_n)_{n \geq 0}$ adapted to the stochastic basis $(F_n)_{n \geq 0}$. We refer the reader to [51, Chapter 7] for more details.

In the following example, the so-called dyadic stochastic basis $(F_n)_{n \geq 0}$ is regular and every $F_n$ is generated by finitely many atoms.

**Example 2.15.** Let $([0, 1), F, \mu)$ be a probability space such that $\mu$ is the Lebesgue measure and the subalgebras $\{F_n\}_{n \geq 0}$ are defined by

$$F_n = \sigma \left\{ [j2^{-n}, (j+1)2^{-n}), j = 0, \ldots, 2^n - 1 \right\}.$$ 

Then $\{F_n\}_{n \geq 0}$ is regular. A martingale with respect to $\{F_n\}_{n \geq 0}$ is called a dyadic martingale. There are a lot of other examples for (regular) $\sigma$-algebras generated by finitely many atoms, see the Vilenkin $\sigma$-algebras in Weisz [70].

In the sequel of the paper, instead of the log-Hölder continuity (1.1) and (1.2), we will suppose that every $\sigma$-algebra $F_n$ is generated by countably many atoms and there exists an absolute constant $K_p(\cdot) \geq 1$ depending only on $p(\cdot)$ such that

$$P(A)^{p(A)} \leq K_p(\cdot), \quad \forall A \in \bigcup_n A(F_n).$$

(2.4)

Note that in this paper, under condition (2.4), we also mean that every $\sigma$-algebra $F_n$ is generated by countably many atoms.

**Remark 2.16.** There exist a lot of functions $p(\cdot)$ satisfying (2.4). In fact, if the measurable function $p(\cdot)$, which is defined on $[0, 1)$, satisfies the log-Hölder continuity (1.1), then, by [16, Lemma 3.24], we find that $p(\cdot)$ satisfies (2.4). For concrete examples we mention the function $a + cx$ for parameters $a$ and $c$ such that the function is positive ($x \in [0, 1)$). All positive Lipschitz functions with order $0 < \alpha \leq 1$ also satisfy (2.4). Note that the condition (1.2) disappears on $[0, 1)$.

In this subsection, we prove a variable version of the dual Doob inequality and the weak type inequality for Doob’s maximal operator. We provide two lemmas from [35].
Lemma 2.17 ([35, Lemma 4.1]). Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4). Then, for any atom $B \in \bigcup_n A(\mathcal{F}_n)$,
\[ \mathbb{P}(B)^{1/p_-(B)} \approx \mathbb{P}(B)^{1/p(x)} \approx \mathbb{P}(B)^{1/p_+(B)} \approx \|\chi_B\|_{p(\cdot)}, \quad \forall x \in B. \]

Lemma 2.18 ([35, Lemma 4.1]). Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4) with $p_- > 1$.

(1) Then for any atom $B \in \bigcup_n A(\mathcal{F}_n)$,
\[ \|\chi_B\|_1 \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)}. \]

(2) Let $q(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4) with $q_- > 1$. Then, for any atom $B \in \bigcup_n A(\mathcal{F}_n)$,
\[ \|\chi_B\|_{r(\cdot)} \approx \|\chi_B\|_{p(\cdot)} \|\chi_B\|_{q(\cdot)}, \] where
\[ \frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \quad x \in \Omega. \]

The following lemma can be proved in the same way as Lemma 3.4 in Jiao et al. [44].

Lemma 2.19. Let $p(\cdot) \in \mathcal{P}(\Omega)$, $1 \leq p_- \leq p_+ < \infty$, satisfy (2.4). Suppose that $f \in L_{p(\cdot)}$ with $\|f\|_{p(\cdot)} \leq 1/2$ and $f = f\chi_{\{|f| \geq 1\}}$. Then, for any atom $A \in \bigcup_n A(\mathcal{F}_n)$, $x \in A$,
\[ \left( \frac{1}{\mathbb{P}(A)} \int_A |f(t)| \, dt \right)^{p(x)} \leq \left( \frac{K_{p(\cdot)}}{\mathbb{P}(A)} \int_A |f(t)|^{p(t)} \, dt \right). \]

The next result is taken from [46, Theorem 4.1].

Lemma 2.20. Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $p_+ < \infty$, $0 < q \leq \infty$, $0 < \theta < 1$ and
\[ \frac{1}{\tilde{p}(\cdot)} = \frac{1 - \theta}{p(\cdot)}. \]
Then
\[ (L_{p(\cdot)}, L_{\infty})_{\theta,q} = L_{\tilde{p}(\cdot),q}. \]

With the help of Lemma 2.19, we can prove the following Doob maximal inequality similarly to [44].

Theorem 2.21. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$. Then, for any $f \in L_{p(\cdot)}$,
\[ \|M(f)\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \]

Corollary 2.22. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. Then
\[ \|M(f)\|_{L_{p(\cdot),q}} \lesssim \|f\|_{L_{p(\cdot),q}}. \]

Proof. It follows from Theorem 2.21 that $M$ is bounded from $L_{p(\cdot)}$ to $L_{p(\cdot)}$. Hence, by combining the fact that $M$ is bounded from $L_\infty$ to $L_\infty$ and Lemma 2.20 we find that $M$ is bounded from $L_{\tilde{p}(\cdot),q}$ to $L_{\tilde{p}(\cdot),q}$. \(\square\)

The previous two results imply immediately the next corollary.
Corollary 2.23. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy (2.4) with \( 1 < p_- \leq p_+ < \infty \) and \( 0 < q \leq \infty \). Then \( H^{M}_{p(\cdot)} \) is equivalent to \( L_{p(\cdot)} \) and \( H^{M}_{p(\cdot), q} \) to \( L_{p(\cdot), q} \) with the inequalities

\[
\|f\|_{L_{p(\cdot)}} \leq \|f\|_{H^{M}_{p(\cdot), q}} \lesssim \|f\|_{L_{p(\cdot), q}}, \quad \|f\|_{L_{p(\cdot), q}} \leq \|f\|_{H^{M}_{p(\cdot), q}} \lesssim \|f\|_{L_{p(\cdot), q}}.
\]

The following result is a variable version of the dual Doob’s inequality. For its classical case, we refer the reader to [59] or [40].

Proposition 2.24. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy (2.4) with \( 1 < p_- \leq p_+ < \infty \). Let \( (\theta_n)_{n \geq 0} \) be a sequence of arbitrary random variables. Then

\[
\left\| \sum |\mathbb{E}_n(\theta_n)| \right\|_{p(\cdot)} \lesssim \left\| \sum |\theta_n| \right\|_{p(\cdot)}.
\]

Proof. Since \( |\mathbb{E}_n(\theta_n)| \leq \mathbb{E}_n(|\theta_n|) \), it suffices to prove this result assuming \( \theta_n \geq 0 \). Consider \( 0 \leq f \in L_{p(\cdot)} \) with \( \|f\|_{p(\cdot)} = 1 \) (see Lemma 2.23) such that

\[
\left\| \sum |\mathbb{E}_n(\theta_n)| \right\|_{p(\cdot)} = \mathbb{E}\left( \sum \mathbb{E}_n(\theta_n) f \right).
\]

Then, by Doob’s inequality (Theorem 2.21),

\[
\left\| \sum |\mathbb{E}_n(\theta_n)| \right\|_{p(\cdot)} = \sum \mathbb{E}|\mathbb{E}_n(f)| \lesssim \left\| \sum |\theta_n| \right\|_{p(\cdot)} \| \sup_n \mathbb{E}_n(f) \|_{p(\cdot)} \lesssim \left\| \sum |\theta_n| \right\|_{p(\cdot)}.
\]

By a similar argument to the proof of the above proposition, we may get the following Stein’s inequality: Let \( p(\cdot) \) satisfy (2.4) with \( 1 \leq r < p_- \leq p_+ < \infty \) for some \( r \). Let \( (\theta_n)_{n \geq 0} \) be a sequence of arbitrary random variables. Then

\[
\left\| \left( \sum |\mathbb{E}_n(\theta_n)|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)} \lesssim \left\| \left( \sum |\theta_n|^r \right)^{\frac{1}{r}} \right\|_{p(\cdot)}.
\]

Taking \( (\theta_n)_{n \geq 0} = \{(d_{n+1}f^2)_{n \geq 0}\}, \) the following result can be deduced from above proposition.

Corollary 2.25. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy (2.4) with \( 2 < p_- \leq p_+ < \infty \). Then

\[
\|f\|_{H^S_{p(\cdot)}} \lesssim \|f\|_{L^{p(\cdot)}}.
\]

Theorem 2.21 says that \( M \) is bounded from \( L_{p(\cdot)} \) to \( L_{p(\cdot)} \) and hence from \( L_{p(\cdot)} \) to \( L_{p(\cdot), \infty} \) if \( p_- > 1 \). In the next theorem, we extend the last statement to \( p_- \geq 1 \). This result covers the classical weak \((1, 1)\) inequality for the Doob maximal operator.

Theorem 2.26. Let \( p(\cdot) \) satisfy (2.4) with \( 1 \leq p_- \leq p_+ < \infty \). Then, for any \( f \in L_{p(\cdot)} \),

\[
\sup_{\lambda > 0} \lambda \|\chi_{(M(f) > \lambda)}\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.
\]

Proof. For \( \lambda > 0 \) and \( n \in \mathbb{N} \), we define the stopping time

\[
\tau_n = \inf\{i \leq n : |f_i| > \lambda\},
\]

with the convention that \( \inf\emptyset = \infty \). Then

\[
\{M_n f > \lambda\} = \{\tau_n < \infty\}.
\]
It is easy to see that the $\sigma$-algebra $F_{\tau_n}$ is generated by countable atoms as well. Without loss of generality, we assume that $\|f\|_{p(\cdot)} \leq 1/2$. Then, by Lemma 2.19, we have the following estimate,

$$\int (\lambda \chi_{\{M_n > \lambda\}})^{p(x)} d\mathbb{P} \leq \int_{\{\tau_n < \infty\}} |f|_{\tau_n}^{p(x)} d\mathbb{P}$$

$$= \int \sum_{A \in A(F_{\tau_n})} \left( \frac{1}{\mathbb{P}(A)} \int_A |f(x)| d\mathbb{P} \right)^{p(x)} \chi_A d\mathbb{P}$$

$$\leq K \int \sum_{A \in A(F_{\tau_n})} \left( \frac{1}{\mathbb{P}(A)} \int_A |f(x)|^{p(x)} + 1 d\mathbb{P} \right) \chi_A d\mathbb{P}$$

$$= K \int \mathbb{E}_{\tau_n}(|f(x)|^{p(x)} + 1) d\mathbb{P} \leq C.$$

Set

$$g_n = (\lambda \chi_{\{M_n > \lambda\}})^{p(x)}, \quad g = (\lambda \chi_{\{M(f) > \lambda\}})^{p(x)}.$$

Then $g_n \leq g_{n+1}$ and $g_n$ converges to $g$ a.e. as $n \to \infty$. Using the monotone convergence theorem, we have

$$\int (\lambda \chi_{\{M(f) > \lambda\}})^{p(x)} d\mathbb{P} = \lim_{n \to \infty} \int (\lambda \chi_{\{M_n > \lambda\}})^{p(x)} d\mathbb{P} \leq C,$$

which completes the proof. \qed

3. Atomic decompositions

In this section, we investigate atomic decompositions for variable martingale Hardy spaces. Let $T$ be the set of all stopping times with respect to $(F_n)_{n \geq 0}$. For a martingale $f = (f_n)_{n \geq 0}$ and $\tau \in T$, we denote the stopped martingale by $f^\tau = (f^\tau_n)_{n \geq 0} = (f_n \wedge \tau)_{n \geq 0}$, where $a \wedge b = \min(a, b)$. We recall the definition of an atom.

**Definition 3.1.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. A measurable function $a$ is called a $(1, p(\cdot), \infty)$-atom (or $(2, p(\cdot), \infty)$-atom or $(3, p(\cdot), \infty)$-atom, respectively) if there exists a stopping time $\tau \in T$ such that

1. $\mathbb{E}(a|F_n) = 0$, $\forall n \leq \tau$,
2. $\|s(a)\|_{\infty}$ (or $\|S(a)\|_{\infty}$, $\|M(a)\|_{\infty}$, respectively) $\leq \frac{1}{\|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}}$.

$a$ is called a simple $(i, p(\cdot), \infty)$-atom if $\tau$ is the special stopping time $\tau = n\chi_A$ for some $A \in A(F_n)$ and $n \in \mathbb{N}$ ($i = 1, 2, 3$).

3.1. Atomic decompositions of $H_{p(\cdot)}$. The atomic characterization of $H_{p(\cdot)}$ has been shown in [14]. In this section, we generalize this atomic decomposition.
**Definition 3.2.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $1 < r \leq \infty$. The atomic Hardy space $H_{p(\cdot)}^{\text{at},d,r}$ is defined as the space of all martingales $f = (f_n)_{n \geq 0}$ such that

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_k^n,$$

where $(a_k)_{k \in \mathbb{Z}}$ is a sequence of $(d, p(\cdot), r)$-atoms $(d = 1, 2, 3)$ associated with stopping times $(\tau_k)_{k \in \mathbb{Z}}$ and $(\mu_k)_{k \in \mathbb{Z}}$ is a sequence of positive numbers. For a fixed $0 < t < p$ and $f \in H_{p(\cdot)}^{\text{at},d,r}$, define

$$\|f\|_{H_{p(\cdot)}^{\text{at},d,r}} = \inf \left\| \left[ \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^t \right]^{\frac{1}{t}} \right\|_{p(\cdot)},$$

where the infimum is taken over all the decompositions of the form (3.1).

The following atomic decompositions can be proved in the same way as in [44].

**Theorem 3.3.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then

$$H_{p(\cdot)}^s = H_{p(\cdot)}^{\text{at},1,\infty},$$

with equivalent quasi-norms.

Similarly, we have the following result (the proof is omitted, and in fact, the proof is just similar to the one of Theorem 3.10).

**Theorem 3.4.** Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then

$$Q_{p(\cdot)} = H_{p(\cdot)}^{\text{at},2,\infty}, \quad P_{p(\cdot)} = H_{p(\cdot)}^{\text{at},3,\infty}$$

with equivalent quasi-norms.

**Theorem 3.5.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4). If $\{F_n\}_{n \geq 0}$ is regular, then

$$H_{p(\cdot)}^S = H_{p(\cdot)}^{\text{at},2,\infty}, \quad H_{p(\cdot)}^M = H_{p(\cdot)}^{\text{at},3,\infty}$$

with equivalent quasi-norms.

**Proof.** We only give the proof for the second equality since the other one is similar. Take $f \in H^{M}_{p(\cdot)}$. Consider the following stopping times with respect to $(\mathcal{F}_n)$,

$$\rho_k := \inf \{n \in \mathbb{N} : |f_n| > 2^k \}, \quad k \in \mathbb{Z}.$$  

For fixed $k \in \mathbb{Z}$, let $F^k_j \in \mathcal{F}_{j-1}$ be the smallest set which contains $\{\rho_k = j\}$. In other words, if $\{\rho_k = j\} \in \mathcal{F}_j$ is decomposed into the disjoint union of atoms $I_{k,j,i} \in A(\mathcal{F}_j)$ and $I_{k,j,i} \in A(\mathcal{F}_{j-1})$ denotes the atom which contains $\mathcal{F}_{k,j,i}$ and $\mathbb{P}(I_{k,j,i}) \leq R\mathbb{P}(I_{k,j,i})$ (this is due to regularity, and $R$ is the constant as in [2,3]), then $F^k_j = \bigcup_i I_{k,j,i}$. Define a new family of stopping times by

$$\tau_k(x) := \inf \{n \in \mathbb{N} : x \in F^k_{n+1} \}.$$  

It is obvious that $\tau_k$ is non-decreasing. By Lemma 3.6, we have

$$\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)} \lesssim \|\chi_{\{\rho_k < \infty\}}\|_{p(\cdot)} = \|\chi_{\{M(f) > 2^k\}}\|_{p(\cdot)} \leq 2^{-k} \|M(f)\|_{p(\cdot)} \to 0.$$
as $k \to \infty$, which deduces that
\[
\lim_{k \to \infty} \mathbb{P}(\tau_k = \infty) = 1.
\]
Thus $\lim_{k \to \infty} \tau_k = \infty$ a.e. and
\[
\lim_{k \to \infty} f_n^{\tau_k} = f_n \quad \text{a.e.} \quad (n \in \mathbb{N}).
\]

We define
\[
\mu_k = 3 \cdot 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)} \quad \text{and} \quad a_n^k = \frac{f_n^{\tau_k+1} - f_n^{\tau_k}}{\mu_k}.
\]
It is not hard to check that $a^k = (a_n^k)_n$ is a $(3, p(\cdot), \infty)$-atom, and $f = \sum_{k \in \mathbb{Z}} \mu_k a_k$. Note that for every $k \in \mathbb{Z}$,
\[
\{\tau_k < \infty\} = \bigcup_{j=0}^{\infty} \{\tau_k = j\} = \bigcup_{j=0}^{\infty} \sum_{i} I_{k,j,i},
\]
where $I_{k,j,i}$'s are atoms in $A(F_j)$.

Obviously,
\[
Z := \left\| \left[ \sum_{k \in \mathbb{Z}} \left( \frac{\mu_k \chi_{\{\tau_k < \infty\}}}{\| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}} \right) \right]^{1/2} \right\|_{p(\cdot)} = \left\| \left[ \sum_{k \in \mathbb{Z}} \sum_{i} \sum_{j=0}^{\infty} (3 \cdot 2^k)^i \chi_{I_{k,j,i}} \right]^{1/2} \right\|_{p(\cdot)}.
\]

Using Lemma 2.4 we may choose a positive function $g \in L_{(\mathfrak{we})}$, with $\|g\|_{(\mathfrak{we})} \leq 1$ such that
\[
Z^t = \int_{\Omega} \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \chi_{I_{k,j,i}} g d\mathbb{P}.
\]

Applying Hölder’s inequality for some $r$ satisfying $\max(1, p_+/t) < r < \infty$, we find that
\[
Z^t \leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \mathbb{P}(I_{k,j,i})^{1/2} \left( \int_{\Omega} \chi_{I_{k,j,i}} g^r d\mathbb{P} \right)^{1/r}
\]
\[
= \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \mathbb{P}(I_{k,j,i}) \left( \frac{1}{\mathbb{P}(I_{k,j,i})} \int_{\Omega} \chi_{I_{k,j,i}} g^r d\mathbb{P} \right)^{1/r}
\]
\[
\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \mathbb{P}(\overline{I_{k,j,i}}) \left( \frac{1}{\mathbb{P}(I_{k,j,i})} \int_{\Omega} \chi_{I_{k,j,i}} g^r d\mathbb{P} \right)^{1/r}
\]
\[
\leq \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \int_{\Omega} \chi_{I_{k,j,i}} (M(g^r))^{1/r} d\mathbb{P}
\]
\[
\leq \left\| \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_{i} (3 \cdot 2^k)^i \chi_{I_{k,j,i}} \right\|_{(p(\cdot)/t)^{1/r}} \mathbb{E}_{(p(\cdot)/t)^{1/r}}[M(g^r)]^{1/r},
\]
where the first “$\lesssim$” is due to the regularity. Since $p_+/t < r < \infty$, we have $((p(\cdot)/t)')_+ < \infty$ and $r' < (p(\cdot)/t)'$. Then, using Theorem 2.21, we obtain

$$\| [M(g^r)]_{(p(\cdot)/t)'}^{1/2} \|_{(p(\cdot)/t)'} = \| M(g^r) \|_{(p(\cdot)/t)'}^{1/2} \lesssim \| g^r \|_{(p(\cdot)/t)'}^{1/2} = \| g \|_{(p(\cdot)/t)} \leq 1.$$  

Observe that

$$\left\| \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} (3 \cdot 2^k)^t \chi_{I_{k,j,t}} \right\|_{p(\cdot)/t}^{1/2} = \left\| \sum_{k \in \mathbb{Z}} (3 \cdot 2^k \chi_{M(f)>2^k})^t \right\|_{p(\cdot)}^{1/2}.$$  

Since

$$\sum_{k \in \mathbb{Z}} 2^{kt} \chi_{M(f)>2^k} = \sum_{k \in \mathbb{Z}} 2^{kt} \sum_{j=k}^{\infty} \chi_{2^j < M(f) < 2^{j+1}} \approx \sum_{j \in \mathbb{Z}} 2^{jt} \chi_{2^j < M(f) < 2^{j+1}} \lesssim M(f)^t,$$

we have $\| f \|_{H^{at,3,\infty}} \leq Z \lesssim \| f \|_{M^{\prime}(p(\cdot))}$. The converse inequality $\| f \|_{M^{\prime}(p(\cdot))} \lesssim \| f \|_{H^{at,3,\infty}}$ can be easily proved. The proof is complete. \(\square\)

The next lemma is used in the proof of the previous theorem.

**Lemma 3.6.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4) and $\{F_n\}_{n \geq 0}$ be regular. Take the same stopping times $\tau_k$ and $\rho_k$ as in the proof of Theorem 3.5. Then

$$\| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)} \lesssim \| \chi_{\{\rho_k < \infty\}} \|_{p(\cdot)}.$$  

**Proof.** It is easy too see that the lemma is equivalent to the inequality

$$\| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)} \lesssim \| \chi_{\{\rho_k < \infty\}} \|_{p(\cdot)}$$

for some $0 < \varepsilon < p$.

Observe that

$$\| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)} \leq \sum_{j=1}^{\infty} \| \chi_{F_j} \|_{p(\cdot)} = \sum_{j=1}^{\infty} \sum_{t} \sum_{i} \| \chi_{I_{k,j,t}} \|_{p(\cdot)} =: Y$$  

and

$$\left\| \sum_{j=1}^{\infty} \sum_{i} \chi_{I_{k,j,t}} \right\|_{p(\cdot)} \leq \| \chi_{\{\rho_k < \infty\}} \|_{p(\cdot)}.$$  

Choose a positive function $g \in L(\frac{p(\cdot)}{t})$, with $\| g \|_{L(\frac{p(\cdot)}{t})} \leq 1$ such that

$$Y = \int_{\Omega} \sum_{j=1}^{\infty} \sum_{t} \chi_{I_{k,j,t}} g \, d\mathbb{P}.$$
Applying Hölder’s inequality for some $\frac{p}{\varepsilon} < r < \infty$ and regular property (one just follow the proof for $Z$ in Theorem 3.3), we find that

$$Y \lesssim \| \sum_{j=1}^{\infty} \sum_{i} \chi_{I_{k,j,i}} \|_{(p(\cdot)/\varepsilon)'} \| M(g^{r'}) \|^\frac{1}{r'} \|_{(p(\cdot)/\varepsilon)'},$$

where the first “$\lesssim$” is due to the regularity. Since $\frac{p}{\varepsilon} < r < \infty$, we have $r' < (p(\cdot)/\varepsilon)'$. Using Theorem 2.21, we obtain

$$\| M(g^{r'}) \|^\frac{1}{r'} \|_{(p(\cdot)/\varepsilon)'} \lesssim \| g \|_{(p(\cdot)/\varepsilon)'} \leq 1,$$

which completes the proof. \hfill \square

Combining Theorem 3.4 and Theorem 3.5, we have the following corollary.

**Corollary 3.7.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4). If $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, then

$$H^S_{p(\cdot)} = Q_{p(\cdot)}, \quad H^M_{p(\cdot)} = P_{p(\cdot)}$$

with equivalent quasi-norms.

### 3.2. Atomic decompositions of $H_{p(\cdot),q}$

In this subsection, we give the atomic decomposition for variable Lorentz-Hardy spaces $H_{p(\cdot),q}$. If $p(\cdot) = p$ is a constant, the corresponding results are studied in [12, 13].

**Definition 3.8.** Let $p(\cdot) \in \mathcal{P}(\Omega)$, $0 < q \leq \infty$. The atomic Hardy space $H^\text{at,d,\infty}_{p(\cdot),q}$ is defined as the space of all martingales $f = (f_n)_{n \geq 0}$ such that, for all $n \in \mathbb{N},$

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k, \quad \text{a.e.}$$

(3.2)

where $(a_k)_{k \in \mathbb{Z}}$ is a sequence of $(d, p(\cdot), \infty)$-atoms $(d = 1, 2, 3)$ associated with stopping times $(\tau_k)_{k \in \mathbb{Z}}$ and $\mu_k = 3 \cdot 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}$ for each $k$. For $f \in H^\text{at,d,\infty}_{p(\cdot),q}$, define

$$\| f \|_{H^\text{at,d,\infty}_{p(\cdot),q}} = \inf \left( \sum_{k \in \mathbb{Z}} 2^{kq} \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \right)^{\frac{1}{q}} \approx \inf \| (\mu_k)_{k \in \mathbb{Z}} \|_{\ell_q},$$

where the infimum is taken over all the decompositions of $f$ of the form (3.2).

**Theorem 3.9.** Let $p(\cdot) \in \mathcal{P}(\Omega)$, $0 < q \leq \infty$. Then

$$H^s_{p(\cdot),q} = H^\text{at,1,\infty}_{p(\cdot),q}$$

with equivalent quasi-norms.

**Proof.** Assume that $f \in H^s_{p(\cdot),q}$. Let us consider the following stopping times for all $k \in \mathbb{Z},$

$$\tau_k = \inf \{ n \in \mathbb{N} : s_n(f) > 2^k \}. $$

The sequence of these stopping times is obviously non-decreasing. It is easy to see that for each $n \in \mathbb{N},$

$$f_n = \sum_{k \in \mathbb{Z}} (f^{\tau_{k+1}}_n - f^{\tau_k}_n).$$
For every $k \in \mathbb{Z}$, $n \in \mathbb{N}$, let
\[
\mu_k = 3 \cdot 2^k \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)} \quad \text{and} \quad a^k_n = \frac{f^{\tau_k}_{n+1} - f^{\tau_k}_n}{\mu_k}.
\]
If $\mu_k = 0$, then set $a^k_n = 0$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Then $(a^k_n)_{n \geq 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. Since $s(f^{\tau_k}) = s_{\tau_k}(f) \leq 2^k$, and by the sublinearity of the operator $s$, we get
\[
s \left( (a^k_n)_{n \geq 0} \right) \leq \frac{s(f^{\tau_k+1}) + s(f^{\tau_k})}{\mu_k} \leq \left\| \chi_{\{\tau_k < \infty\}} \right\|^{-1}_{p(\cdot)}.
\]
Hence $(a^k_n)_{n \geq 0}$ is a bounded $L_2$-martingale. Consequently, there exists an element $a^k \in L_2$ such that $E_n a^k = a^k_n$. If $n \leq \tau_k$, then $a^k_n = 0$. Thus we conclude that $a^k$ is a $(1, p(\cdot), \infty)$-atom. For $q = \infty$, we have
\[
\sup_k \mu_k = 3 \sup_k 2^k \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}
= 3 \sup_k 2^k \left\| \chi_{\{s(f) > 2^k\}} \right\|_{p(\cdot)}
\leq C \|s(f)\|_{L_p(\cdot), \infty} = C \|f\|_{H^*_p(\cdot), \infty}.
\]
For $q < \infty$, we have
\[
\left( \sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} = 3 \left( \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \chi_{\{\tau_k < \infty\}} \right\|^q_{p(\cdot)} \right)^{\frac{1}{q}}
= 3 \left( \sum_{k \in \mathbb{Z}} 2^{kq} \left\| \chi_{\{s(f) > 2^k\}} \right\|^q_{p(\cdot)} \right)^{\frac{1}{q}}
\leq C \left( \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \lambda^{q-1} d\lambda \left\| \chi_{\{s(f) > 2^k\}} \right\|^q_{p(\cdot)} \right)^{\frac{1}{q}}
\leq C \left( \int_{0}^{\infty} \lambda^{q-1} \left\| \chi_{\{s(f) > \lambda\}} \right\|^q_{p(\cdot)} d\lambda \right)^{\frac{1}{q}}
\leq C \|s(f)\|_{L_p(\cdot), q} = C \|f\|_{H^*_p(\cdot), q}.
\]
Conversely, assume that a martingale $f$ has the decomposition \((3.2)\). For an arbitrary integer $k_0$, set
\[
f = \sum_k \mu_k a^k := F_1 + F_2,
\]
where
\[
F_1 = \sum_{k = -\infty}^{k_0-1} \mu_k a^k \quad \text{and} \quad F_2 = \sum_{k = k_0}^{\infty} \mu_k a^k.
\]
By Remark 2.6
\[
\|f\|_{H^*_p(\cdot), q} \lesssim \|s(F_1)\|_{L_p(\cdot), q} + \|s(F_2)\|_{L_p(\cdot), q}.
\]
Note that
\[
s(F_1) \leq \sum_{k=-\infty}^{k_0-1} \mu_k s(a^k), \quad s(F_2) \leq \sum_{k=k_0}^{\infty} \mu_k s(a^k).
\]

We deal with \(q = \infty\) firstly. Since \(a^k\) is a \((1, p(\cdot), \infty)\)-atom for every \(k \in \mathbb{Z}\), we find that
\[
\|s(F_1)\|_\infty \leq \sum_{k=-\infty}^{k_0-1} \mu_k \|s(a^k)\| \leq \sum_{k=-\infty}^{k_0-1} \mu_k \|\chi_{\{\tau_k < \infty\}}\|^{-1} \leq 3 \cdot 2^{k_0}.
\]

Thus we can deduce that
\[
(3.3) \quad 2^{k_0} \|\chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)}^{p(x)} \leq 2^{k_0} \|\chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}} + \chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)}^{p(x)} = 2^{k_0} \|\chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)}^{p(x)}.
\]

Next we estimate this expression. Since \(s(a^k) = 0\) on the set \(\{\tau_k = \infty\}\), we have \(\{s(a^k) > 0\} \subset \{\tau_k < \infty\}\). Then,
\[
(3.4) \quad \{s(F_2) > 3 \cdot 2^{k_0}\} \subset \{s(F_2) > 0\} \subset \bigcup_{k=k_0}^{\infty} \{s(a^k) > 0\} \subset \bigcup_{k=k_0}^{\infty} \{\tau_k < \infty\}.
\]

Hence, by Lemma 2.1 we obtain
\[
\int_\Omega \left( \frac{3 \cdot 2^{k_0} \chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}}{\sup_{k \in \mathbb{Z}} \mu_k} \right)^{p(x)} \, d\mathbb{P} \leq \sum_{k=k_0}^{\infty} \int_{\{\tau_k < \infty\}} \left( \frac{3 \cdot 2^{k_0}}{\sup_{k \in \mathbb{Z}} \mu_k} \right)^{p(x)} \, d\mathbb{P} \leq \sum_{k=k_0}^{\infty} \int_{\{\tau_k < \infty\}} \left( \frac{3 \cdot 2^{k_0}}{3 \cdot 2^{k_0} \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^{p(x)} \, d\mathbb{P} \leq \sum_{k=k_0}^{\infty} 2^{-(k-k_0)p_-} \int_{\Omega} \left( \frac{\chi_{\{\tau_k < \infty\}}}{\|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}} \right)^{p(x)} \, d\mathbb{P} \leq C_{p_-},
\]

which implies that
\[
3 \cdot 2^{k_0} \|\chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)} \leq C_{p_-} \sup_{k \in \mathbb{Z}} \mu_k.
\]

Combining (3.3) and the above inequality, we have
\[
\|f\|_{H^s_{p(\cdot), \infty}} = \|s(f)\|_{L_{p(\cdot), \infty}} \leq 2C_{p_-} \inf_{k \in \mathbb{Z}} \mu_k \lesssim \|f\|_{H^s_{p(\cdot), \infty}},
\]

where the infimum is taken over all the decompositions of \(f\) of the form (3.2).

Now we consider the case \(q < \infty\). According to (3.3), it suffices to estimate \(\|\chi_{\{s(F_2) > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)}\).

Let
\[
0 < \varepsilon < \min(p, q) \quad \text{and} \quad 0 < \delta < 1.
\]
Applying (3.4) and (2.1), we have
\[ \| \chi_{\{s(F_2) > 2^{-2k_0}\}} \|_{p(\cdot)} \leq \left( \sum_{k=k_0}^{\infty} \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q} \]
\[ = \left( \sum_{k=k_0}^{\infty} 2^{-k\delta q} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q}. \]

Using Hölder’s inequality for \( \frac{q-\varepsilon}{q} + \frac{\varepsilon}{q} = 1 \), we get
\[ \| \chi_{\{s(F_2) > 2^{-2k_0}\}} \|_{p(\cdot)} \leq \left( \sum_{k=k_0}^{\infty} 2^{-k\delta q} \left( \sum_{k=k_0}^{\infty} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q} \right)^{1/q} \]
\[ \lesssim 2^{-k_0\delta} \left( \sum_{k=k_0}^{\infty} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q}. \]

Consequently,
\[ \sum_{k_0=-\infty}^{\infty} 2^{k_0q} \| \chi_{\{s(F_2) > 2^{-2k_0}\}} \|_{p(\cdot)}^q \lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \sum_{k=k_0}^{\infty} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \]
\[ = \sum_{k=-\infty}^{\infty} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \]
\[ \lesssim \sum_{k=-\infty}^{\infty} 2^k \| \chi_{\{\tau_k < \infty\}} \|_{p(\cdot)}^q, \]
where the last “\( \lesssim \)” is due to \( 1 - \delta > 0 \). Then we obtain
\[ \| F_2 \|_{H^\ast_p(\cdot), q} = \| s(F_2) \|_{L^p(\cdot), q} \lesssim \left( \sum_{k=-\infty}^{\infty} \mu_k^q \right)^{1/q} \lesssim \| f \|_{H^\ast_{p(\cdot), q}}, \]
where the infimum is taken over all decompositions of the form (3.2).

**Theorem 3.10.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) and \( 0 < q \leq \infty \). Then
\[ Q_{p(\cdot), q} = H^\ast_{p(\cdot), q}, \quad P_{p(\cdot), q} = H^\ast_{p(\cdot), q} \]
with equivalent quasi-norms.

**Proof.** The proof is similar to those of Theorems 3.9 so we only sketch the outline. Let \( f = (f_n)_{n \geq 0} \in Q_{p(\cdot), q} \) (or \( P_{p(\cdot), q} \)). The stopping times \( \tau_k \) are defined by
\[ \tau_k = \inf\{ n \in \mathbb{N} : \lambda_n > 2^k \}, \quad (\inf \emptyset = \infty), \]
where the infimum is taken over all decompositions of the form (3.2).\( \square \)
where \((\lambda_n)_{n \geq 0}\) is the sequence in the definition of \(Q_{p(\cdot),q}\). Let \(a^k_n\) and \(\mu_k\) \((k \in \mathbb{Z})\) be the same as in the proof of Theorem 3.9. Then we get (3.2), where \((a^k)_{k \in \mathbb{Z}}\) is a sequence of \((2,p(\cdot),\infty)\)-atoms (or \((3,p(\cdot),\infty)\)-atoms). Moreover,

\[
\|(\mu_k)_{k \in \mathbb{Z}}\|_{\ell_q} \lesssim \|f\|_{Q_{p(\cdot),q}} \quad \text{or} \quad \|(\mu_k)_{k \in \mathbb{Z}}\|_{\ell_q} \lesssim \|f\|_{P_{p(\cdot),q}}
\]

still holds.

To prove the converse part, let

\[
\lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\tau_k \leq n\}} \|S(a^k)\|_{\infty} \quad \text{or} \quad \lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\tau_k \leq n\}} \|M(a^k)\|_{\infty}.
\]

Then \((\lambda_n)_{n \geq 0}\) is a nondecreasing, nonnegative and adapted sequence. Indeed, according to Remark 3.12. Theorems 3.3, 3.4, 3.9 and 3.10 do not need any restriction on \(p(\cdot)\).

**Remark 3.11.** Theorems 3.3, 3.4, 3.9 and 3.10 do not need any restriction on \(p(\cdot)\) and they do not need \(\mathcal{F}_n\) to be generated by countably many atoms. If \(p(\cdot) = p\) is a constant, then the above atomic decompositions go back to [42], [43] and [70].

**Remark 3.12.** (1) In (3.2), if \(q < \infty\), then the sum \(\sum_{k=m}^{n} \mu_k a^k\) converges to \(f\) in \(H^s_{p(\cdot),q}\) as \(m \to -\infty, n \to \infty\). Indeed,

\[
\sum_{k=m}^{n} \mu_k a^k = \sum_{k=m}^{n} (f^{\nu_{k+1}} - f^{\nu_k}) = f^{\nu_{n+1}} - f^{\nu_m}.
\]

By the sublinearity of \(s\), we have

\[
\left\| f - \sum_{k=m}^{n} \mu_k a^k \right\|_{H^s_{p(\cdot),q}} = \|s(f - f^{\nu_{n+1}} + f^{\nu_m})\|_{L_{p(\cdot),q}} \leq \|s(f - f^{\nu_{n+1}}) + s(f^{\nu_m})\|_{L_{p(\cdot),q}} \lesssim \|s(f - f^{\nu_{n+1}})\|_{L_{p(\cdot),q}} + \|s(f^{\nu_m})\|_{L_{p(\cdot),q}}.
\]
Observe that
\[ s(f - f^{\nu_{n+1}})^2 = s(f)^2 - s(f^{\nu_{n+1}})^2, \quad s(f - f^{\nu_{n+1}}) \leq s(f), \quad s(f^{\nu_{m}}) \leq s(f) \]
and
\[ s(f - f^{\nu_{n+1}}), \quad s(f^{\nu_{m}}) \to 0 \quad \text{a.e. as } m \to -\infty, \ n \to \infty. \]
Thus, by Lemma 2.8 and 2.13, we have
\[ \|s(f - f^{\nu_{n+1}})\|_{L^p(\cdot, q)}, \|s(f^{\nu_{m}})\|_{L^p(\cdot, q)} \to 0 \quad \text{as } m \to -\infty, \ n \to \infty, \]
which implies
\[ \left\| f - \sum_{k=m}^{n} \mu_k a^k \right\|_{H^s_{p(\cdot), q}} \to 0 \quad \text{as } m \to -\infty, \ n \to \infty. \]
Further, for \( k \in \mathbb{Z}, \ a^k = (a_n^k)_{n \geq 0} \) (here \( a^k \) is a \((1, p(\cdot), \infty)-\)atom) is \( L_2 \) bounded, hence \( H^s_2 = L_2 \) is dense in \( H^s_{p(\cdot), q} \). Similarly, \( L_\infty \) is dense in \( P_{p(\cdot), q} \).

(2) If \( q = \infty \) and \( s(f) \in \mathcal{L}_{p(\cdot), \infty}(\Omega) \), then by Lemma 2.13, the sum \( \sum_{k=m}^{n} \mu_k a^k \) converges to \( f \) in \( H^s_{p(\cdot), \infty} \) as \( m \to -\infty, \ n \to \infty. \)

We can show the next atomic decomposition corresponding to Theorem 3.5. The proof is omitted.

**Theorem 3.13.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4) and \( 0 < q \leq \infty \). If \( \{F_n\}_{n \geq 0} \) is regular, then
\[ H^S_{p(\cdot), q} = H^{at, 2, \infty}_{p(\cdot), q}, \quad H^M_{p(\cdot), q} = H^{at, 3, \infty}_{p(\cdot), q} \]
with equivalent quasi-norms.

Similarly to Corollary 3.7, we have a corresponding result for variable Lorentz-Hardy spaces.

**Corollary 3.14.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4) and \( 0 < q \leq \infty \). If \( \{F_n\}_{n \geq 0} \) is regular, then
\[ H^S_{p(\cdot), q} = Q_{p(\cdot), q}, \quad H^M_{p(\cdot), q} = P_{p(\cdot), q} \]
with equivalent quasi-norms.

### 4. Boundedness of the martingale operators

This section is devoted to the applications of the atomic characterizations established in Section 3 while we are dealing with martingale inequalities between different Hardy spaces. Furthermore, if we suppose that \( \{F_n\}_{n \geq 0} \) is regular, then the equivalence of different Hardy spaces will be proved.
4.1. Martingale inequalities between $H_{p(\cdot)}$. As an application of the atomic decompositions, we shall obtain a sufficient condition for a $\sigma$-sublinear operator to be bounded from the martingale Hardy spaces to $L_{p(\cdot)}$.

An operator $T : X \to Y$ is called a $\sigma$-sublinear operator if for any $\alpha \in \mathbb{C}$ it satisfies

$$
T \left( \sum_{k=1}^{\infty} f_k \right) \leq \sum_{k=1}^{\infty} |T(f_k)| \quad \text{and} \quad |T(\alpha f)| = |\alpha||T(f)|,
$$

where $X$ is a martingale space and $Y$ is a measurable function space.

Suppose that $\tau$ is a stopping time. Denote

$$
\mathcal{F}_\tau = \{ F \in \mathcal{F} : F \cap \{ \tau \leq n \} \in \mathcal{F}_n, \ n \geq 1 \}.
$$

$\mathcal{F}_\tau$ is a sub-$\sigma$-algebra of $\mathcal{F}$. Then the conditional expectation with respect to $\mathcal{F}_\tau$ is denoted by $\mathbb{E}_{\tau}$.

We need the following property taken from [78].

**Lemma 4.1** ([78, Lemma 5.1]). Let $a$ be a $(1, p(\cdot), \infty)$-atom associated with stopping time $\tau$. If $T \in \{ s, S, M \}$, then

$$
T(a\chi_F) = T(a)\chi_F, \quad \forall F \in \mathcal{F}_\tau.
$$

**Lemma 4.2.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ and let $a$ be a $(1, p(\cdot), \infty)$-atom associated with stopping time $\tau$. If for $p_+ < r < \infty$ and $T : H^s_r \to L_r$ is a bounded $\sigma$-sublinear operator and

$$
(4.1) \quad T(a)\chi_F = T(a\chi_F), \quad \forall F \in \mathcal{F}_\tau,
$$

then $T(a) = T(a)\chi_{\{ \tau < \infty \}}$ and

$$
\mathbb{E}_\tau(|T(a)|^r) \lesssim \|\chi_{\{ \tau < \infty \}}\|_{p(\cdot)}^{-r}.
$$

**Proof.** Take $F \in \mathcal{F}_\tau$. Then, by (4.1) and the boundedness of $T$, we have

$$
\int_F |T(a)|^r d\mathbb{P} = \int_\Omega |T(a\chi_F)|^r d\mathbb{P} \lesssim \int_\Omega s(a\chi_F)^r d\mathbb{P}.
$$

Since $F$ is arbitrary and $s(a)$ is supported by $\{ \tau < \infty \}$, we obtain the first assertion. By Lemma 4.1 and the fact that $a$ is a $(1, p(\cdot), \infty)$-atom, we obtain

$$
\int_F |T(a)|^r d\mathbb{P} \lesssim \int_\Omega s(a)^r \chi_F d\mathbb{P} \leq \int_\Omega \|\chi_{\{ \tau < \infty \}}\|_{p(\cdot)}^{-r} \chi_F d\mathbb{P}.
$$

Since $F$ is arbitrary, the second assertion follows. \qed

**Theorem 4.3.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4) and $1 < r < \infty$ with $p_+ < r$. If $T : H^s_r \to L_r$ is a bounded $\sigma$-sublinear operator and

$$
(4.2) \quad T(a)\chi_F = T(a\chi_F), \quad \forall F \in \mathcal{F}_\tau
$$

for all $(1, p(\cdot), \infty)$-atoms $a$, where $\tau$ is the stopping time associated with $a$, then

$$
\|Tf\|_{L_{p(\cdot)}} \lesssim \|f\|_{H^s_r}, \quad f \in H^s_{p(\cdot)}.
$$
Proof. Let a martingale \( f \in H_{p^\mathbf{\mu}}^s \). By Theorem 3.3 we know that \( f \) has a decomposition as (3.1) such that \( a^k \) is a \((1, p^\cdot, \infty)\)-atom and \( \mu_k = 3 \cdot 2^k \| \chi_{\{ \tau < \infty \}} \|_{p^\cdot} \). According to the boundedness of \( T \),

\[
\| T(a^k) \|_r \lesssim \| s(a^k) \|_r \leq \frac{\| \chi_{\{ \tau < \infty \}} \|_r}{\| \chi_{\{ \tau < \infty \}} \|_{p^\cdot}}.
\]

By the \( \sigma \)-sublinearity of the operator \( T \), we have

\[
| T(f) | \leq \sum_{k \in \mathbb{Z}} \mu_k | T(a^k) |
\]

Then, for \( 0 < t < p \leq 1 \), we have

\[
\| T(f) \|_{p^\cdot} \leq \left\| \left[ \sum_{k \in \mathbb{Z}} (\mu_k T(a^k))^t \right]^{\frac{1}{t}} \right\|_{p^\cdot} =: Z.
\]

By Lemma 2.4, we may choose a positive function \( g \in L_{p^\cdot}(\mathbb{P}) \), with \( \| g \|_{p^\cdot} \leq 1 \) such that

\[
Z^t = \int_{\Omega} \sum_{k \in \mathbb{Z}} \left[ 3 \cdot 2^k \| \chi_{\{ \tau < \infty \}} \|_{p^\cdot} | T(a^k) | \right]^t g \, d\mathbb{P}
= \int_{\Omega} \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \| \chi_{\{ \tau < \infty \}} \|_{p^\cdot} \chi_{\{ \tau < \infty \}} \mathbb{E}_{\tau_k} (| T(a^k) |^t g) \, d\mathbb{P}.
\]

The Hölder inequality for conditional expectation implies that

\[
\mathbb{E}_{\tau_k} (| T(a^k) |^t g) \leq \mathbb{E}_{\tau_k} (| T(a^k) |^r)^{t/r} \mathbb{E}_{\tau_k} (g^{(r/t)'})^{1/(r/t)'}.
\]

Applying Lemma 4.2, we can see that

\[
Z^t \leq \int_{\Omega} \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\{ \tau_k < \infty \}} \mathbb{E}_{\tau_k} (g^{(r/t)'})^{1/(r/t)'} \, d\mathbb{P}
\leq \int_{\Omega} \sum_{k \in \mathbb{Z}} 3 \cdot 2^k \chi_{\{ \tau_k < \infty \}} \left[ M(g^{(r/t)'} \right]^{1/(r/t)'} \, d\mathbb{P}
\leq \left\| \sum_{k \in \mathbb{Z}} (3 \cdot 2^k)^{t} \chi_{\{ \tau_k < \infty \}} \right\|_{p^\cdot/t} \left\| [M(g^{(r/t)'} \right]^{1/(r/t)'} \|_{(p^\cdot)/t}'.
\]

Since \( p_+ < r \), we deduce that

\[
\frac{r}{t} > \frac{p_+}{t} \quad \text{and} \quad \left( \frac{r}{t} \right)' < \left( \frac{p^\cdot}{t} \right)'.
\]

Note that \( t < p_- \). Hence, \( ((p^\cdot)/t)')_+ < \infty \). Using the maximal inequality (Theorem 2.21), we have

\[
\left\| [M(g^{(r/t)'} \right]^{1/(r/t)'} \|_{(p^\cdot)/t} \| \leq 1.
\]
Thus, by Theorem 3.3 we obtain
\[
Z \lesssim \left\| \sum_{k \in \mathbb{Z}} (3 \cdot 2^k)^t \chi_{(r_k < \infty)} \right\|_{p(t)/t}^{1/t} \lesssim \|f\|_{H^s_{p(\cdot)}},
\]
which completes the proof. □

Similarly to Theorem 4.3 we obtain the following theorem by applying Theorem 3.3:

**Theorem 4.4.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4) and \( 1 < r < \infty \) with \( p_+ < r \). If \( T : H^S_r \rightarrow L_r \) (or \( H^M_r \rightarrow L_r \)) is a bounded \( \sigma \)-sublinear operator and (4.2) holds for all \((2, p(\cdot), \infty)\)-atoms (or \((3, p(\cdot), \infty)\)-atoms), then
\[
\|Tf\|_{L_{p(\cdot)}} \lesssim \|f\|_{Q_{p(\cdot)}}, \quad f \in Q_{p(\cdot)},
\]
(\( \text{or} \) \( \|Tf\|_{L_{p(\cdot)}} \lesssim \|f\|_{P_{p(\cdot)}}, \quad f \in P_{p(\cdot)} \)).

Now we prove our main result of this section.

**Theorem 4.5.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4). Then the following inequalities hold:
\[
\|f\|_{H^M_{p(\cdot)}} \lesssim \|f\|_{H^s_{p(\cdot)}}, \quad \|f\|_{H^S_{p(\cdot)}} \lesssim \|f\|_{H^s_{p(\cdot)}}, \quad \text{if} \quad 0 < p_- \leq p_+ < 2;
\]
(4.4) \[
\|f\|_{H^M_{p(\cdot)}} \leq \|f\|_{P_{p(\cdot)}}, \quad \|f\|_{H^S_{p(\cdot)}} \leq \|f\|_{Q_{p(\cdot)}},
\]
(4.5) \[
\|f\|_{H^S_{p(\cdot)}} \lesssim \|f\|_{P_{p(\cdot)}}, \quad \|f\|_{H^M_{p(\cdot)}} \lesssim \|f\|_{Q_{p(\cdot)}},
\]
(4.6) \[
\|f\|_{H^s_{p(\cdot)}} \lesssim \|f\|_{P_{p(\cdot)}}, \quad \|f\|_{H^s_{p(\cdot)}} \lesssim \|f\|_{Q_{p(\cdot)}},
\]
(4.7) \[
\|f\|_{P_{p(\cdot)}} \lesssim \|f\|_{Q_{p(\cdot)}}, \quad \|f\|_{P_{p(\cdot)}} \lesssim \|f\|_{Q_{p(\cdot)}}.
\]
Moreover, if \( \{\mathcal{F}_n\}_{n \geq 0} \) is regular, then
\[
H^S_{p(\cdot)} = Q_{p(\cdot)} = P_{p(\cdot)} = H^M_{p(\cdot)} = H^s_{p(\cdot)}
\]
with equivalent quasi-norms.

**Proof.** According to Lemma 4.1, we know that the operators \( M, S \) and \( s \) all satisfy (4.2). First we show (4.3). Let \( f \in H^S_{p(\cdot)} \). The maximal operator \( T(f) = M(f) \) is \( \sigma \)-sublinear and \( \|M(f)\|_2 \lesssim \|s(f)\|_2 \) (see [70, Theorem 2.11(i)]). Thus it follows from Theorem 4.3 that
\[
\|f\|_{H^M_{p(\cdot)}} = \|M(f)\|_{p(\cdot)} \lesssim \|f\|_{H^s_{p(\cdot)}}.
\]
Similarly, considering the operator \( T(f) = S(f) \), we get the second inequality of (4.3) by Theorem 4.3.

(4.4) comes easily from the definition of these martingale spaces.

Next we show (4.5). Consider the operator \( T(f) = M(f) \) or \( S(f) \). Then (4.5) follows from the combination of the Burkholder-Gundy and Doob maximal inequalities
\[
\|S(f)\|_r \approx \|M(f)\|_r \approx \|f\|_r \quad (1 < r < \infty)
\]
(see [70, Theorem 2.12]) and Theorem 4.3.
can be deduced by applying the inequalities (see \[70\] Theorem 2.11(ii))
\[
\|s(f)\|_r \lesssim \|M(f)\|_r, \quad \|s(f)\|_r \lesssim \|M(f)\|_r \approx \|S(f)\|_r, \quad 2 < r < \infty,
\]
and Theorem 4.4.

To prove (4.6), we use (4.5). Assume that \( f = (f_n)_{n \geq 0} \in Q_{p(\cdot)} \), then there exists an optimal control \((\lambda_{n}^{(1)})_{n \geq 0}\) such that \( S_n(f) \leq \lambda_{n-1}^{(1)} \) with \( \lambda_{\infty}^{(1)} \in L_{p(\cdot)} \). Since
\[
|f_n| \leq M_{n-1}(f) + \lambda_{n-1}^{(1)},
\]
by the second inequality of (4.5) we have
\[
\|f\|_{P_{p(\cdot)}} \leq C(\|f\|_{H_{p(\cdot)}^M} + \|\lambda_{n-1}^{(1)}\|_{p(\cdot)}) \lesssim \|f\|_{Q_{p(\cdot)}}.
\]
On the other hand, if \( f = (f_n)_{n \geq 0} \in P_{p(\cdot)} \), then there exists an optimal control \((\lambda_{n}^{(2)})_{n \geq 0}\) such that \( |f_n| \leq \lambda_{n-1}^{(2)} \) with \( \lambda_{\infty}^{(2)} \in L_{p(\cdot)} \). Notice that
\[
S_n(f) \leq S_n(1) + 2\lambda_{n-1}^{(2)}.
\]
Using the first inequality of (4.5), we get the rest of (4.7).

Further, assume that \( \{F_n\}_{n \geq 0} \) is regular. Then according to \[70\] p. 33, we have
\[
S_n(f) \leq R^{1/2} S_n(1) \quad \text{and} \quad \|f\|_{H_{p(\cdot)}^S} \lesssim \|f\|_{H_{p(\cdot)}^*}.
\]
Since \( s_n(f) \in F_{n-1} \), by the definition of \( Q_{p(\cdot)} \) we have
\[
\|f\|_{Q_{p(\cdot)}} \lesssim \|s(f)\|_{p(\cdot)} = \|f\|_{H_{p(\cdot)}^*}.
\]
Hence, by (4.6) we obtain
\[
Q_{p(\cdot)} = H_{p(\cdot)}^S.
\]
Combining this and Corollary 3.7, we get
\[
H_{p(\cdot)}^S = Q_{p(\cdot)} = H_{p(\cdot)}^S = P_{p(\cdot)} = H_{p(\cdot)}^M.
\]

□

The next result follows from Theorem 3.3 and Theorem 4.5.

**Corollary 4.6.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4). If \( \{F_n\}_{n \geq 0} \) is regular, then
\[
H_{p(\cdot)} = H_{p(\cdot)}^{at,d,\infty}, \quad d = 1, 2, 3
\]
with equivalent quasi-norms. Here, \( H_{p(\cdot)} \) denotes any one of the five Hardy spaces in Theorem 4.5.

Next, we consider a special case of martingale transforms.

**Definition 4.7.** Let the martingale transform \( T_b \) be defined by
\[
(T_b f)_n = \sum_{k=1}^{n} b_{k-1} d_k f, \quad n \in \mathbb{N},
\]
where \( b_k \) is \( F_k \) measurable and \( |b_k| \leq 1 \).
Theorem 4.8. Let \( p(\cdot) \) satisfy (2.4) with \( 1 < p_- \leq p_+ < \infty \). If \( \{F_n\}_{n \geq 0} \) is regular, then
\[
\|T_b f\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.
\]

Proof. Since \( |b_k| \leq 1 \) for each \( k \), we have \( S(T_b f) \leq S(f) \). By Theorem 4.3 and Burkholder-Gundy inequality, we have
\[
\|T_b f\|_{p(\cdot)} \leq \|M(T_b f)\|_{p(\cdot)} \lesssim \|S(T_b f)\|_{p(\cdot)} \lesssim \|S(f)\|_{p(\cdot)} \lesssim \|M(f)\|_{p(\cdot)}.
\]
Now the result follows from Theorem 2.21. \( \Box \)

4.2. Martingale inequalities between \( H^s\). In this subsection, we extend Theorem 4.5 to the variable Lorentz-Hardy setting; see Theorem 4.11. First, we prove a result that is corresponding to Theorem 4.3.

Theorem 4.9. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4), \( 0 < q \leq \infty \) and \( 1 < r \leq \infty \) with \( p_+ < r \). If \( T : H^s \to L_r \) is a bounded \( \sigma \)-sublinear operator and satisfies (4.2), then
\[
\|T f\|_{p(\cdot),q} \lesssim \|f\|_{H^s,\cdot,q}, \quad f \in H^s_{p(\cdot),q}.
\]

Proof. Let a martingale \( f \in H^s_{p(\cdot),q} \). By Theorem 3.9, we know that \( f \) has a decomposition as (3.2) such that \( a^k \) is a \( (1, p(\cdot), \infty) \)-atom and \( \mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)} \). For an arbitrary integer \( k_0 \), we set again
\[
f = \sum_{k \in \mathbb{Z}} \mu_k a^k = F_1 + F_2,
\]
where
\[
F_1 = \sum_{k = -\infty}^{k_0-1} \mu_k a^k, \quad F_2 = f - F_1.
\]
By the \( \sigma \)-sublinearity of the operator \( T \), we have
\[
|T(F_1)| \leq \sum_{k = -\infty}^{k_0-1} \mu_k T(a^k), \quad |T(F_2)| \leq \sum_{k = k_0}^{\infty} \mu_k T(a^k).
\]
We need to estimate \( \|T(F_1)\|_{p(\cdot),q} \) and \( \|T(F_2)\|_{p(\cdot),q} \), separately.

To estimate \( \|T(F_1)\|_{p(\cdot),q} \), we let \( 0 < \varepsilon < p \). Fix \( L \in (1, \varepsilon) \) such that \( L < r/p_+ \) and choose \( \ell \) such that \( 0 < \ell < 1 - 1/L \). By Hölder’s inequality for \( \frac{1}{L} + \frac{1}{L'} = 1 \), we have
\[
T(F_1) \leq \left( \sum_{k = -\infty}^{k_0-1} 2^{k\ell L'} \right)^{1/L'} \left( \sum_{k = -\infty}^{k_0-1} [2^{-k\ell} \mu_k T(a^k)]^L \right)^{1/L} \lesssim 2^{k_0 \ell} \left( \sum_{k = -\infty}^{k_0-1} [2^{-k\ell} \mu_k T(a^k)]^L \right)^{1/L}.
\]
Then, we have

$$\| \chi_{\{T(F_1) > 2^{k_0}\}} \|_{p(\cdot)} \leq \left\| \frac{T(F_1)}{2^{k_0}L} \right\|_{p(\cdot)} \leq 2^{k_0L(\ell - 1)} \left( \sum_{k = -\infty}^{k_0 - 1} \left| 2^{-k^\ell L} \mu_k T(a^k) \right| \right)_{p(\cdot)}$$

$$\leq 2^{k_0L(\ell - 1)} \left\{ \sum_{k = -\infty}^{k_0 - 1} 2^{(1-\ell)kL} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)} \| T(a^k) \|_{p(\cdot)/\varepsilon} \right\}^{1/\varepsilon},$$

where the last “$$\leq$$” is due to Theorem 4.3 (note that $$p, L < r$$). Since $$\|s(a^k)\|_{\infty} \leq \|\chi_{\{T_k < \infty\}}\|_{p(\cdot)}^{-1}$$ for every $$k \in \mathbb{Z}$$, it follows

$$\| \chi_{\{T(F_1) > 2^{k_0}\}} \|_{p(\cdot)} \leq 2^{k_0L(\ell - 1)} \left\{ \sum_{k = -\infty}^{k_0 - 1} 2^{(1-\ell)kL} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)} \right\}^{1/\varepsilon}.$$ 

For the case $$q = \infty$$, we have

$$\| \chi_{\{T(F_1) > 2^{k_0}\}} \|_{p(\cdot)} \lesssim 2^{k_0L(\ell - 1)} \left( \sum_{k = -\infty}^{k_0 - 1} 2^{k((1-\ell)L - 1)\varepsilon} \right) \sup_{k \in \mathbb{Z}} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)} \lesssim 2^{-k_0} \sup_{k \in \mathbb{Z}} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)},$$

(note that $$(1 - \ell)L - 1 > 0$$) which implies that

$$\| T(F_1) \|_{L(\cdot), \infty} \lesssim \| f \|_{H_*^{p(\cdot), \infty}}.$$ 

Now we turn to deal with the case $$q < \infty$$. Set

$$\delta = \frac{(1 - \ell)L + 1}{2} > 1.$$ 

By the Hölder inequality for \( \frac{\varepsilon}{q} + \frac{\varepsilon}{q} = 1 \), we have

$$\| \chi_{\{T(F_1) > 2^{k_0}\}} \|_{p(\cdot)} \lesssim 2^{k_0L(\ell - 1)} \left( \sum_{k = -\infty}^{k_0 - 1} 2^{k((1-\ell)L - \delta)\varepsilon} \right)^{(q-\varepsilon)/q} \left( \sum_{k = -\infty}^{k_0 - 1} 2^{k\delta q} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q} \lesssim 2^{-k_0\delta} \left( \sum_{k = -\infty}^{k_0 - 1} 2^{k\delta q} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)}^q \right)^{1/q}.$$ 

From this, by basic calculation, we get

$$\sum_{k = -\infty}^{\infty} 2^{kq} \| \chi_{\{T(F_1) > 2^{k_0}\}} \|_{p(\cdot)}^q \lesssim \sum_{k = -\infty}^{\infty} 2^{kq} \| \chi_{\{T_k < \infty\}} \|_{p(\cdot)}^q.$$
We deduce that

\[ \|T(F_1)\|_{L^p(q)} \lesssim \|f\|_{H^s_p(q)}. \]

Now we start to estimate \( \|T(F_2)\|_{L^p(q)}. \) According to condition (4.2) and Lemma 4.2,

\[ \{T(F_2) > 2^{k_0}\} \subset \{T(F_2) > 0\} \subset \bigcup_{k = k_0}^\infty \{T(a^k) > 0\} \subset \bigcup_{k \geq k_0} \{\tau_k < \infty\}. \]

Then

\[ \|\chi_{\{T(F_2) > 2^{k_0}\}}\|_{p(\cdot)} \leq \left\| \sum_{k = k_0}^\infty \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}. \]

So, repeating the calculation of the proof of Theorem 3.9, we easily obtain

\[ \|T(F_2)\|_{L^p(q)} \lesssim \|f\|_{H^s_p(q)}. \]

The proof is complete now. \( \square \)

The following result can be similarly prove. We omit the proof.

**Theorem 4.10.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4), \( 0 < q \leq \infty \) and \( 1 < r \leq \infty \) with \( p_+ < r. \) If \( T : H^S_r \rightarrow L_r \) (or \( H^S_{r*} \rightarrow L_r \)) is a bounded \( \sigma \)-sublinear operator and (4.2) holds for all \( (2, p(\cdot), \infty) \)-atoms (or \( (3, p(\cdot), \infty) \)-atoms), then we have

\[ \|Tf\|_{L^p(q)} \lesssim \|f\|_{Q^p(q)}, \quad f \in Q^p(q), \]

(or \( \|Tf\|_{L^p(q)} \lesssim \|f\|_{P^p(q)}, \quad f \in P^p(q). \))

Similarly to Theorem 4.5, applying the two theorems above, we can prove the result below. We omit the details of the proof.

**Theorem 4.11.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4) and \( 0 < q \leq \infty. \) Then the following inequalities hold:

\[ \|f\|_{H^M_{p(\cdot),q}} \lesssim \|f\|_{H^s_{p(\cdot),q}}, \quad \|f\|_{H^S_{p(\cdot),q}} \lesssim \|f\|_{H^s_{p(\cdot),q}}; \quad \text{if} \quad 0 < p_+ \leq p_+ < 2; \]

\[ \|f\|_{H^M_{p(\cdot),q}} \leq \|f\|_{P_{p(\cdot),q}}; \quad \|f\|_{H^S_{p(\cdot),q}} \leq \|f\|_{Q_{p(\cdot),q}}; \]

\[ \|f\|_{H^M_{p(\cdot),q}} \leq \|f\|_{P_{p(\cdot),q}}; \quad \|f\|_{H^S_{p(\cdot),q}} \leq \|f\|_{Q_{p(\cdot),q}}; \]

\[ \|f\|_{H^M_{p(\cdot),q}} \leq \|f\|_{P_{p(\cdot),q}}; \quad \|f\|_{H^S_{p(\cdot),q}} \leq \|f\|_{Q_{p(\cdot),q}}; \]

Moreover, if \( \{F_n\}_{n \geq 0} \) is regular, then

\[ H^S_{p(\cdot),q} = P_{p(\cdot),q} = H^M_{p(\cdot),q} = H^s_{p(\cdot),q} \]

with equivalent quasi-norms.

Combining Theorem 3.9 and Theorem 4.11 we obtain the following result.
Corollary 4.12. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $0 < q \leq \infty$ and $1 < r \leq \infty$ with $p_+ < r$. If $\{F_n\}_{n \geq 0}$ is regular, then

$$H_{p(\cdot),q} = H_{p(\cdot),q}^{at,d,\infty}, \quad d = 1, 2, 3$$

with equivalent quasi-norms. Here, $H_{p(\cdot),q}$ denotes any one of the five Hardy spaces in Theorem 4.11.

The following result is corresponding to Theorem 4.8. The proof is omitted.

Theorem 4.13. Let $p(\cdot)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. If $\{F_n\}_{n \geq 0}$ is regular, then

$$\|T_b f\|_{L^{p(\cdot),q}} \lesssim \|f\|_{L^{p(\cdot),q}}.$$ 

5. Applications in Fourier analysis

This section is devoted to applications of the previous results in Fourier Analysis. We mainly investigate the boundedness of the maximal Fejér operator on variable Hardy space $H_{p(\cdot)}$ and variable Lorentz-Hardy space $H_{p(\cdot),q}$ (see Corollary 4.6 and Corollary 4.12). To this end, in Section 5.1, we first introduce two new dyadic maximal operators $U$ and $V$ which play a crucial role in this section. We also prove that they are bounded on $L^{p(\cdot)}$ with $p(\cdot)$ satisfying (2.4) and $1 < p_- \leq p_+ < \infty$.

5.1. Walsh system and Fejér means. Let us investigate the dyadic martingales. Namely, let $\Omega = [0, 1)$, $\mathbb{P}$ be the Lebesgue measure and $\mathcal{F}$ be the Lebesgue measurable sets. By a dyadic interval, we mean one of the form $[k2^{-n}, (k+1)2^{-n})$ for some $k, n \in \mathbb{N}$, $0 \leq k < 2^n$. Given $n \in \mathbb{N}$ and $x \in [0, 1)$, let $I_n(x)$ denote the dyadic interval of length $2^{-n}$ which contains $x$. The $\sigma$-algebras generated by the dyadic intervals $\{I_n(x) : x \in [0, 1)\}$ will be denoted by $\mathcal{F}_n (n \in \mathbb{N})$. Such $(\mathcal{F}_n)_{n \geq 0}$ is regular, see Example 2.15 or [51].

The Rademacher functions are defined by

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1), \end{cases}$$

and

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbb{N}).$$

The product system generated by the Rademacher functions is the Walsh system:

$$w_n := \prod_{k=0}^\infty r_k^{n_k} \quad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^\infty n_k 2^k, \quad (0 \leq n_k < 2).$$
Recall (see Fine [23]) that the Walsh-Dirichlet kernels

\[ D_n := \sum_{k=0}^{n-1} w_k \]

satisfy

\[ D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 2^{-n}); \\ 0, & \text{if } x \in [2^{-n}, 1) \end{cases} \quad (n \in \mathbb{N}). \]

If \( f \in L_1 \), then the number

\[ \hat{f}(n) := \mathbb{E}(f w_n) \quad (n \in \mathbb{N}) \]

is said to be the \( n \)th Walsh-Fourier coefficient of \( f \). We can extend this definition to martingales as follows. If \( f = (f_k)_{k \geq 0} \) is a martingale, then let

\[ \hat{f}(n) := \lim_{k \to \infty} \mathbb{E}(f_k w_n) \quad (n \in \mathbb{N}). \]

Since \( w_n \) is \( \mathcal{F}_k \) measurable for \( n < 2^k \), it can immediately be seen that this limit does exist. We remember that if \( f \in L_1 \), then \( \mathbb{E}_k f \to f \) in the \( L_1 \)-norm as \( k \to \infty \), hence

\[ \hat{f}(n) = \lim_{k \to \infty} \mathbb{E}((\mathbb{E}_k f) w_n) \quad (n \in \mathbb{N}). \]

Thus the Walsh-Fourier coefficients of \( f \in L_1 \) are the same as the ones of the martingale \((\mathbb{E}_k f)_{k \geq 0}\) obtained from \( f \).

Denote by \( s_n f \) the \( n \)th partial sum of the Walsh-Fourier series of a martingale \( f \), namely,

\[ s_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k. \]

If \( f \in L_1 \), then

\[ s_n f(x) = \int_0^1 f(t) D_n(x + t) \, dt \quad (n \in \mathbb{N}), \]

where \( \dot{+} \) denotes the dyadic addition (see e.g. Schipp, Wade, Simon and Pál [64] or Golubov, Efimov and Skvortsov [34]). It is easy to see that

\[ s_{2^n} f = f_n \quad (n \in \mathbb{N}) \]

and so, by martingale results,

\[ \lim_{n \to \infty} s_{2^n} f = f \quad \text{in the } L_p \text{-norm} \]

when \( f \in L_p \) and \( 1 \leq p < \infty \). This theorem was extended in Schipp, Wade, Simon and Pál [64] (see also Golubov, Efimov and Skvortsov [34]) for the partial sums \( s_n f \) and for \( 1 < p < \infty \). More exactly,

\[ \lim_{n \to \infty} s_n f = f \quad \text{in the } L_p \text{-norm} \]

when \( f \in L_p \) and \( 1 < p < \infty \). We generalize this theorem as follows.
Theorem 5.1. Let $p(\cdot)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$. If $f \in L_{p(\cdot)}$, then
\[ \sup_{n \in \mathbb{N}} \| s_n f \|_{p(\cdot)} \lesssim \| f \|_{p(\cdot)}. \]

Proof. It was proved by Schipp, Wade, Simon and Pál [64, p. 95] that
\[ s_n f = w_n T_0 (f w_n), \]
where
\[ T_0 f := \sum_{k=1}^{\infty} n_{k-1} d_k f \]
and the binary coefficients $n_k$ are defined in (5.1). Obviously $T_0$ is a martingale transform and Theorem 4.8 implies that
\[ \| s_n f \|_{p(\cdot)} = \| T_0 (f w_n) \|_{p(\cdot)} \lesssim \| f \|_{p(\cdot)}, \]
which shows the result.

Corollary 5.2. Let $p(\cdot)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$. If $f \in L_{p(\cdot)}$, then
\[ \lim_{n \to \infty} s_n f = f \quad \text{in the } L_{p(\cdot)}-\text{norm.} \]

Proof. Note that (5.3) implies that the Walsh polynomials are dense in $L_p$ for $1 \leq p < \infty$. Then it is easy to see that the Walsh polynomials are dense in $L_{p(\cdot)}$ as well. Notice for any Walsh polynomial $T$ and any integer $n$ which exceeds the degree of this polynomial, that
\[ s_n (T) = T. \]
Using Theorem 5.1 for such $n$, we obtain
\[ \| T - s_n (f) \|_{p(\cdot)} = \| s_n (T - f) \|_{p(\cdot)} \leq C \| T - f \|_{p(\cdot)}. \]
We choose $T$ such that $\| T - f \|_{p(\cdot)} < \varepsilon/(C + 1)$ for any $\varepsilon > 0$. Consequently,
\[ \| f - s_n (f) \|_{p(\cdot)} \leq \| f - T \|_{p(\cdot)} + \| T - s_n (f) \|_{p(\cdot)} \leq (C + 1) \| T - f \|_{p(\cdot)} < \varepsilon. \]
This finishes the proof.

Similarly, for variable Lorentz spaces, we can prove the following two results.

Theorem 5.3. Let $p(\cdot)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. If $f \in L_{p(\cdot),q}$, then
\[ \sup_{n \in \mathbb{N}} \| s_n f \|_{L_{p(\cdot),q}} \lesssim \| f \|_{L_{p(\cdot),q}}. \]

Corollary 5.4. Let $p(\cdot)$ satisfy (2.4) with $1 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. If $f \in L_{p(\cdot),q}$, then
\[ \lim_{n \to \infty} s_n f = f \quad \text{in the } L_{p(\cdot),q}-\text{norm.} \]

The above results are not true if $p_- \leq 1$, see e.g. [13] and Example 5.4.2 in [34]. However, in this case we can consider a summability method. For $n \in \mathbb{N}$ and a martingale $f$, the Fejér mean of order $n$ of the Walsh-Fourier series of $f$ is given by
\[ \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} s_k f. \]
Of course, \( \sigma_n f \) has better convergence properties than \( s_k f \). It is simple to show that
\[
\sigma_n f(x) = \int_0^1 f(t)K_n(x+t) \, dt \quad (n \in \mathbb{N})
\]
if \( f \in L_1 \), where the **Walsh-Fejér kernels** are defined by
\[
K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \quad (n \in \mathbb{N}).
\]
The maximal operator \( \sigma^* \) is defined by
\[
\sigma^* f = \sup_{n \in \mathbb{N}} |\sigma_n f|.
\]

It is known (see Fine [23] or Schipp, Wade, Simon and Pál [64]) that
\[
\begin{align*}
|K_n(x)| &\leq \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} (D_{2^i}(x) + D_{2^i}(x+2^{-j-1})) \\
K_{2^n}(x) &= \frac{1}{2} \left( 2^{-n} D_{2^n}(x) + \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x+2^{-j-1}) \right),
\end{align*}
\]
where \( x \in [0, 1), n, N \in \mathbb{N} \) and \( 2^{N-1} \leq n < 2^N \).

**Remark 5.5.** In this section, if there is no special statement, we always assume that \((F_n)_{n \geq 0}\) is the sequence of the dyadic \( \sigma \)-algebras. It follows from Theorem 4.5 that the five variable Hardy spaces in the theorem are equivalent if \((F_n)_{n \geq 0}\) is regular. We use \( H_{p(\cdot)} \) to denote one of them. Similarly, according to Theorem 4.11, we use \( H_{p(\cdot),q} \) to denote any one of the variable Lorentz Hardy spaces.

5.2. **The maximal operator** \( U \). Let us define \( I_{k,n} := [k2^{-n}, (k+1)2^{-n}] \) with \( 0 \leq k < 2^n, n \in \mathbb{N} \). Motivating by the kernel functions (5.4) and (5.5), we introduce two versions of dyadic maximal functions. For a martingale \( f = (f_n) \), the first one is given by
\[
U_s f(x) := \sup_{x \in I} \left\{ \sum_{j=0}^{n-1} 2^{(j-n)s} \frac{1}{\mathbb{P}(I+2^{-j-1})} \left| \int_{I+2^{-j-1}} f_n d\mathbb{P} \right| \right\},
\]
where \( I \) is a dyadic interval with length \( 2^{-n} \) and \( s \) is a positive constant. Of course, if \( f \in L_1 \), then we can write in the definition \( f \) instead of \( f_n \). The definition can be rewritten to
\[
U_s f(x) = \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{(j-n)s} \frac{1}{\mathbb{P}(I_{k,n})} \left| \int_{I_{k,n}} f_n d\mathbb{P} \right|,
\]
where, for brevity, we use the notation \( I_{k,n}^j := I_{k,n} + 2^{-j-1} \).
In order to show that $\sigma_s$ is bounded from $H_p(\mathbb{L})$ to $L_p(\mathbb{L})$, first we have to prove that $U$ is bounded from $L_p(\mathbb{L})$ to $L_p(\mathbb{L})$ ($p_+ > 1$). We need to apply the following well-known theorem in martingale theory (see e.g. Weisz [71]).

**Theorem 5.6.** Let $p$ be a constant and $0 < p \leq 1 < r \leq \infty$. Suppose that $T : L_r \to L_r$ is a bounded $\sigma$-sublinear operator and

\[(5.6)\quad \|T a I^c\|_p \leq C_p \]

for all simple $(3, p, \infty)$-atoms $a$, where $I$ is the support of $a$. Then we have

\[
\|Tf\|_p \lesssim \|f\|_{H_p}, \quad f \in H_p.
\]

**Theorem 5.7.** For all $0 < p \leq \infty$ and all $0 < s < \infty$, we have

\[(5.7)\quad \|U_s f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p).\]

**Proof.** The theorem will be proved by applying Theorem 5.6 with $r = \infty$. Observe that (5.7) holds for $p = \infty$. Indeed,

\[
\|U_s f\|_{\infty} \leq \sup_{n \in \mathbb{N}} \sum_{j=0}^{n-1} 2^{(j-n)s} \|f\|_{\infty} \leq C \|f\|_{\infty}.
\]

By interpolation, the proof will be complete if we show that the operator $U_s$ satisfies (5.6) for each $0 < p \leq 1$. Choose a simple $(3, p, \infty)$-atom $a$ with support $I$, where $I$ is a dyadic interval with length $|I| = 2^{-K}$ ($K \in \mathbb{N}$). We can assume that $I = [0, 2^{-K})$. It is easy to see that

\[
\sum_{j=0}^{n-1} 2^{(j-n)s} \frac{1}{\mathbb{P}(J+2^{-j-1})} \left| \int_{J+2^{-j-1}} a d\mathbb{P} \right| = 0
\]

if $n \leq K$, where $J$ is a dyadic interval with length $2^{-n}$. Therefore we can suppose that $n > K$. Observe that $x \not\in [0, 2^{-K})$ and $x \in J$ imply that $J + 2^{-j-1} \cap [0, 2^{-K}) = \emptyset$ if $j \geq K$. Thus $\int_{J+2^{-j-1}} a = 0$ for $j \geq K$. Hence, we may assume that $j < K$. The same holds if $x \in [2^{-j-1} + 2^{-K}, 2^{-j})$, because $x + 2^{-j-1} \not\in [0, 2^{-K})$. Hence

\[
|U_s a(x)| \leq \sup_{n \in \mathbb{N}} \chi_J(x) \sum_{j=0}^{K-1} 2^{(j-n)s} \chi_{[2^{-j-1, 2^{-j-1}+2^{-K})]}(x) \frac{1}{\mathbb{P}(J+2^{-j-1})} \left| \int_{J+2^{-j-1}} a d\mathbb{P} \right|
\]

\[
\leq 2^{K/p} \sum_{j=0}^{K-1} 2^{(j-K)s} \chi_{[2^{-j-1, 2^{-j-1}+2^{-K})]}(x)
\]

and

\[
\int_{I^c} |U_s a(x)|^p \leq 2^K \sum_{j=0}^{K-1} 2^{(j-K)s} 2^{-K} \leq C_p,
\]

which completes the proof of the theorem. \qed
Since $H_p$ is equivalent to $L_p$ when $1 < p \leq \infty$ (see also Corollary 2.23), the preceding result implies that

$$\|U_s f\|_p \leq C_p \|f\|_p \quad (1 < p \leq \infty, 0 < s < \infty, f \in L_p).$$

This inequality remains true for Lebesgue spaces with variable exponents.

**Theorem 5.8.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $1 < p_- \leq p_+ < \infty$ and $0 < s < \infty$. If

$$\frac{1}{p_-} - \frac{1}{p_+} < s,$$

then

$$\|U_s f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).$$

**Proof.** We assume that $\|f\|_{p(\cdot)} \leq 1/2$. Then

$$\int_{\Omega} |U_s f(x)|^{p(x)} \, dx \lesssim \int_{\Omega} |U_s (f \chi_{|f| \geq 1})(x)|^{p(x)} \, dx + \int_{\Omega} |U_s (f \chi_{|f| < 1})(x)|^{p(x)} \, dx$$

$$\lesssim \int_{\Omega} |U_s (f \chi_{|f| \geq 1})(x)|^{p(x)} \, dx + C.$$

So it is enough to prove that

$$\int_{\Omega} |U_s (f \chi_{|f| \geq 1})(x)|^{p(x)} \, dx \lesssim C.$$

Let us denote by $\sum_{j=0}^{n-1}$ the sum over all $j = 1, \ldots, n - 1$ for which

$$\frac{1}{\mathbb{P}(I_{k,n}^j)} \int_{I_{k,n}^j} |f(t)| \, dt \leq 1.$$

In this case

$$\int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{k,n}(x) \sum_{j=0}^{n-1} t_2^{(j-n)s} \frac{1}{\mathbb{P}(I_{k,n}^j)} \int_{I_{k,n}^j} |f(t)| \, dt \right)^{p(x)} \, dx$$

$$\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{k,n}(x) \sum_{j=0}^{n-1} t_2^{(j-n)s} \right)^{p(x)} \, dx \leq C.$$

Hence, we may suppose that $\|f\|_{p(\cdot)} \leq 1/2$, $|f| \geq 1$ or $f = 0$ and

$$\frac{1}{\mathbb{P}(I_{k,n}^j)} \int_{I_{k,n}^j} |f(t)| \, dt > 1$$

for all $j = 1, \ldots, n - 1, k = 0, \ldots, 2^n - 1, n \in \mathbb{N}$.
Let us denote by $I_{k,n,j,1}$ (resp. $I_{k,n,j,2}$) those points $x \in I_{k,n}$ for which $p(x) \leq p_+(I_{k,n}^j)$ (resp. $p(x) > p_+(I_{k,n}^j)$). Then

$$
\int_{\Omega} |U_s f(x)|^{p(x)} \, dx \lesssim \sum_{l=1}^{2^n} \int_{I_{k,n}^l} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \sum_{j=0}^{n-1} 2^{(j-n)s} \chi_{I_{k,n,j,1}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)} \, dx
=: (A) + (B).
$$

Let $q(x) := p(x)/p_0 > 1$ for some $1 < p_0 < p_-$. Using the fact that the sets $I_{k,n}$ are disjoint for a fixed $n$ and the convexity of the function $t \mapsto t^{q(x)}$ ($x$ is fixed), we conclude

$$(A) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \left( \sum_{j=0}^{n-1} 2^{(j-n)s} \chi_{I_{k,n,j,1}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)} \right)^{p_0} \, dx
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \sum_{j=0}^{n-1} 2^{(j-n)s} \chi_{I_{k,n,j,1}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)} \, dx
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \sum_{j=0}^{n-1} 2^{(j-n)s} \chi_{I_{k,n,j,1}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)+p_0} \, dx
\lesssim \|U_s(|f|^{q(x)})\|_{p_0} \lesssim \|f\|^{q(x)} \|p_0 \leq C.
$$

To investigate $(B)$, let us observe that

$$(B) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \sum_{j=0}^{n-1} 2^{(j-n)(s-r)} 2^{(j-n)s} \chi_{I_{k,n,j,2}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)} \, dx
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}^l}(x) \sum_{j=0}^{n-1} 2^{(j-n)(s-r)} \chi_{I_{k,n,j,2}}(x) \int_{I_{k,n}^l} |f(t)| \, dt \right)^{q(x)} \, dx.$$


for some $0 < r < s$. By Hölder’s inequality,

\[
(B) \lesssim \int_\Omega \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{j-n}(s-r)2^{j-n}r q(x) \left( \frac{\chi_{I_{k,n,j},2}(x)}{P(I_{k,n}^j)} \int_{I_{k,n}^j} |f(t)|^{q-(I_{k,n}^j)} \, dt \right)^{p_0} \right)^{\frac{1}{p_0}} dx
\]

\[
\lesssim \int_\Omega \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{j-n}(s-r)2^{j-n}r q(x) 2^{nq(x)/q-(I_{k,n}^j)} \chi_{I_{k,n,j},2}(x) \left( \int_{I_{k,n}^j} |f(t)|^{q-(I_{k,n}^j)} \, dt \right)^{p_0} \right)^{\frac{1}{p_0}} dx.
\]

Since $|f| \geq 1$ or $f = 0$, $q(x) > q-(I_{k,n}^j)$ on $I_{k,n,j,2}$, $q-(I_{k,n}^j) \leq q(t) < p(t)$ for all $t \in I_{k,n}^j$ and

\[
\int_{I_{k,n}^j} |f(t)|^{q-(I_{k,n}^j)} \, dt \leq \int_{I_{k,n}^j} |f(t)|^{p(t)} \, dt \leq \frac{1}{2},
\]

we conclude

\[
(B) \lesssim \int_\Omega \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{j-n}(s-r)2^{j-n}r q(x) 2^{nq(x)/q-(I_{k,n}^j)} \chi_{I_{k,n,j},2}(x) \left( \int_{I_{k,n}^j} |f(t)|^{q-(I_{k,n}^j)} \, dt \right)^{p_0} \right)^{\frac{1}{p_0}} dx
\]

\[
\lesssim \int_\Omega \left( \sum_{n \in \mathbb{N}} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{j-n}(s-r)2^{j-n}r q(x) 2^{nq(x)/q-(I_{k,n}^j)} \chi_{I_{k,n,j},2}(x) \left( \int_{I_{k,n}^j} |f(t)|^{q-(I_{k,n}^j)} \, dt \right)^{p_0} \right)^{\frac{1}{p_0}} dx.
\]

For fixed $k, n$ let $J_j$ denote the dyadic interval with length $2^{-j}$ and $I_{k,n} \subset J_j$. Then $I_{k,n}^j \subset J_j + 2^{-j-1} = J_j$. Inequality (2.4) implies that $2^{-jp(x)} \sim 2^{-jp-(I_{k,n}^j)}$ for $x \in I_{k,n}$. It is easy to check that for $x \in I_{k,n,j,2}$,

\[
2^{jq(x)} = 2^{jq(x)}2^{jq(x)}2^{-jq(x)} \lesssim 2^{j(r+1)q-(I_{k,n}^j)} \quad 2^{-jq(x)} < 2^{j\left(\frac{q(x)-q-(I_{k,n}^j)}{q-(I_{k,n}^j)}\right)} = 2^{j\left(\frac{q(x)-q-(I_{k,n}^j)}{q-(I_{k,n}^j)}+1\right)}
\]

which is equivalent to the obvious inequality $q-(I_{k,n}^j) > 1/(r+1)$. Furthermore,

\[
rq(x) - \frac{q(x)}{q-(I_{k,n}^j)} + 1 \geq q(x) \left( r - \frac{1}{q_-} \right) + 1 \geq \begin{cases} 1, & \text{if } r - \frac{1}{q_-} \geq 0; \\ q_+ \left( r - \frac{1}{q_-} \right) + 1, & \text{if } r - \frac{1}{q_-} < 0. \end{cases}
\]
Let \( r_0 := \min \left( 1, q_+ \left( r - \frac{1}{q_-} \right) + 1 \right) \). Then \( r_0 > 0 \) if and only if
\[
\frac{1}{q_-} - \frac{1}{q_+} < r.
\]

We estimate \((B)\) further by
\[
(B) \lesssim \int_\Omega \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{(j-n)(s-r)} \left( \left( \frac{rq(x) - \frac{q(x)}{q_- I_{k,n}}}{q_-} \right) + 1 \right) \right)
\frac{1}{\mathbb{P}(P_{k,n})} \int_{I_{k,n}} |f(t)|^{q(t)} \, dt \right)^{p_0} \, dx
\lesssim \int_\Omega \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} 2^{(j-n)(s-r)} \frac{1}{\mathbb{P}(P_{k,n})} \int_{I_{k,n}} |f(t)|^{q(t)} \, dt \right)^{p_0} \, dx
\lesssim \|U_{s-r+r_0}([f]^{q(\cdot)})\|_{p_0}^{p_0} \lesssim \|f\|_{q(\cdot)}^{p_0} \leq C.
\]
Since \( p_0 \) can be arbitrarily near to 1 and \( r \) to \( s \), inequality \((5.10)\) proves the theorem with the range \((5.8)\).

Remark 5.9. Inequality \((5.8)\) and Theorem 5.8 hold if \( s \geq 1 \), or more generally if \( p_- > \max(1/s, 1) \).

The operator \( U_s \) is not bounded on \( L_{p(\cdot)} \) outside the range of \((5.8)\). More exactly, the following theorem holds.

Theorem 5.10. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition \((2.4)\), \( 1 < p_- \leq p_+ < \infty \) and \( 0 < s < \infty \). If
\[
\frac{1}{p_-(I_{0,n} + 2^{-1})} - \frac{1}{p_+(I_{0,n})} > s
\]
for all \( n \in \mathbb{N} \), then \( U_s \) is not bounded on \( L_{p(\cdot)} \).

Proof. It is easy to see that
\[
\int_\Omega |U_s f(x)|^{p(x)} \, dx \geq \int_\Omega \chi_{I_{0,n}}(x) \left( \frac{1}{\mathbb{P}(I_{0,n} + 2^{-1})} \int_{I_{0,n} + 2^{-1}} |f(t)| \, dt \right)^{p(x)} \, dx.
\]
If
\[
f(t) := \chi_{I_{0,n} + 2^{-1}}(t) 2^{n/p_-(I_{0,n} + 2^{-1})},
\]
by Lemma 2.17,
\[
\|f\|_{p(\cdot)} = 2^{n/p_-(I_{0,n} + 2^{-1})} \|\chi_{I_{0,n} + 2^{-1}}\|_{p(\cdot)} \leq C.
\]
This implies that
\[
\int_{\Omega} |U_s f(x)|^{p(x)} \, dx \geq \int_{I_0,n} 2^{-np(x)} 2^{np(x)/p_-(I_0,n+2^{-1})} \, dx
\]
\[
\geq C \int_{I_0,n} 2^{np_+(I_0,n)(1/p_-(I_0,n+2^{-1})-s)} \, dx = C 2^{np_+(I_0,n)(1/p_-(I_0,n+2^{-1})-s)} 2^{-n}
\]
which tends to infinity as \( n \to \infty \) if (5.11) holds.

\[\square\]

**Remark 5.11.** Combining the fact that \( U \) is bounded on \( L_\infty \), the above theorem and Lemma 2.20, we know that \( U \) is bounded on \( L_{p, q} \) for \( p(s) \in P(\Omega) \) satisfying (2.4) and (5.8), \( 1 < p_- \leq p_+ < \infty \) and \( 0 < q \leq \infty \).

These results, including the above theorem and the remark, should be compared with Theorem 2.21 and Corollary 2.22.

### 5.3. The maximal operator \( V \)

We define the second version of dyadic maximal function by

\[
V_{\alpha, s} f(x) := \sup_{x \in I} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha_2(j-i)s} \frac{1}{P(I+[2^{-j-1}, 2^{-j+1}+2^{-1}]})} \int_{I+[2^{-j-1}, 2^{-j+1}+2^{-1}]} f_n dP,
\]

where \( I \) is a dyadic interval with length \( 2^{-n} \) and \( f = (f_n) \) is a martingale and \( s, \alpha \) are positive constants. Obviously,

\[
V_{\alpha, s} f(x) := \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \sum_{j=0}^{n-1} 2^{(j-n)\alpha_2(j-i)s} \frac{1}{P(I^{j,i}_{k,n})} \int_{I^{j,i}_{k,n}} f_n dP,
\]

where, for brevity, we use the notation

\[
I^{j,i}_{k,n} := I_{k,n}+[2^{-j-1}, 2^{-j+1}+2^{-i}].
\]

**Theorem 5.12.** Suppose that \( 0 < p \leq \infty \) and \( 0 < \alpha, s < \infty \). Then

\[
\|V_{\alpha, s} f\|_p \leq C_p \|f\|_{H_p} \quad (f \in H_p).
\]

**Proof.** The inequality holds for \( p = \infty \) because

\[
\|V_{\alpha, s} f\|_\infty \leq \sup_{n \in \mathbb{N}} \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha_2(j-i)s} \|f\|_\infty \leq C \|f\|_\infty.
\]

Again, we are going to show that the operator \( V_{\alpha, s} \) satisfies (5.6) for each \( 0 < p \leq 1 \). We choose again a simple \((3,p,\infty)\)-atom \( a \) with support \( I = [0,2^{-K}) \). If \( i \leq K \), then \( \int_{I+[2^{-j-1}, 2^{-j+1}+2^{-i}]} a = 0 \). Thus \( i > K \) and so \( n > K \). Similarly to the proof of Theorem 5.7
$j < K$ and $x \in [2^{-j-1}, 2^{-j-1} + 2^{-K})$. Hence, in case $x \not\in [0, 2^{-K})$, 

$$
|V_{\alpha, s}a(x)| \leq \sup_{n > K} \chi_J(x) \sum_{j=0}^{K-1} \sum_{i=K}^{n-1} 2^{(j-i)\alpha} \int_{J_{j,i}} a \chi_{[2^{-j-1}, 2^{-j-1} + 2^{-K})}(x) 
$$

$$
\leq 2^{K/p} \sup_{n > K} \chi_J(x) 2^{-\alpha n} \sum_{j=0}^{K-1} 2^{j\alpha} \sum_{i=K}^{n-1} 2^{(j-i)\alpha} \chi_{[2^{-j-1}, 2^{-j-1} + 2^{-K})}(x),
$$

where $J$ is a dyadic interval with length $2^{-n}$. Since 

$$
\sum_{i=K}^{n-1} 2^{(j-i)\alpha} \leq 1,
$$

we have 

$$
|V_{\alpha, s}a(x)| \leq 2^{K/p} 2^{-K\alpha} \sum_{j=0}^{K-1} 2^{j\alpha} \chi_{[2^{-j-1}, 2^{-j-1} + 2^{-K})}(x).
$$

Consequently, 

$$
\int_{I^c} |V_{\alpha, s}a(x)|^p \leq 2^K 2^{-Kp\alpha} \sum_{j=0}^{K-1} 2^{j\alpha p} 2^{-K} \leq C_p,
$$

which finishes the proof. \hfill \Box

Under the same conditions, the inequality 

$$
\|V_{\alpha, s}f\|_p \leq C_p \|f\|_p \quad (1 < p \leq \infty, 0 < \alpha, s < \infty, f \in L_p)
$$

follows from Theorem 5.12.

**Theorem 5.13.** Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $1 < p_- \leq p_+ < \infty$ and $0 < \alpha, s < \infty$. If 

$$
(5.12) \quad \frac{1}{p_-} - \frac{1}{p_+} < \alpha + s,
$$

then 

$$
\|V_{\alpha, s}f\|_{p(\cdot)} \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \quad (f \in L_{p(\cdot)}).
$$

**Proof.** Similarly to the proof of Theorem 5.8, we may suppose again that $\|f\|_{p(\cdot)} \leq 1/2$, $|f| \geq 1$ or $f = 0$ and 

$$
\frac{1}{\mathbb{P}(I_{k,n}^i)} \int_{I_{k,n}^i} |f(t)| dt > 1.
$$
We denote by \( I_{k,n,j,i,l} \) (resp. \( I_{k,n,j,i,2} \)) those points \( x \in I_{k,n} \) for which \( p(x) \leq p_+(I_{k,n}^{j,i}) \) (resp. \( p(x) > p_+(I_{k,n}^{j,i}) \)). Then

\[
\int_{\Omega} |V_{\alpha,s}f(x)|^{p(x)} \, dx \\
y \lesssim \sum_{l=1}^{2} \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha} 2^{(j-i)s} \frac{\chi_{I_{k,n,j,i,l}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| \, dt \right)^{p(x)} \, dx \\
=: (C) + (D).
\]

Again, let \( q(x) := p(x)/p_0 > 1 \) for some \( 1 < p_0 < p_- \). By convexity and Lemma 2.19

\[(C) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha} 2^{(j-i)s} \left( \frac{\chi_{I_{k,n,j,i,l}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| \, dt \right)^{q(x)} \right)^{p_0} \, dx
\]

\[
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha} 2^{(j-i)s} \left( \frac{\chi_{I_{k,n,j,i,l}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| \, dt \right)^{q_+(I_{k,n}^{j,i})} \right)^{p_0} \, dx
\]

\[
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha} 2^{(j-i)s} \frac{\chi_{I_{k,n,j,i,l}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)|^{q(t)} \, dt \right)^{p_0} \, dx
\]

\[
\lesssim \|V_{\alpha,s}(|f|^{q(\cdot)})\|_{p_0} \lesssim \| |f|^{q(\cdot)}\|_{p_0} \leq C.
\]

Again by convexity and Hölder’s inequality, we obtain for some \( 0 < \alpha_0 < \alpha \) and \( 0 < r < s + \alpha_0 \) that

\[(D) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \left( \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)\alpha} 2^{(j-i)s} \frac{\chi_{I_{k,n,j,i,2}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| \, dt \right)^{q(x)} \right)^{p_0} \, dx
\]

\[
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-\alpha_0)} 2^{(j-i)(\alpha_0+s-r)} \left( \frac{\chi_{I_{k,n,j,i,2}}(x)}{P(I_{k,n}^{j,i})} \int_{I_{k,n}^{j,i}} |f(t)| \, dt \right)^{q(x)} \right)^{p_0} \, dx
\]
and so
\[
(D) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-a_0)}2^{(j-i)(\alpha_0+s-r)}2^{(j-i)rq(x)} \right) \left( \frac{\chi_{\mathcal{I}_{k,n,i,2}(x)}}{\mathbb{P}(\mathcal{I}_{k,n,i}^+)} \int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{q-(I_{k,n,i}^+)} dt \right)^{p_0} dx
\]
\[
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-a_0)}2^{(j-i)(\alpha_0+s-r)}2^{(j-i)rq(x)}2^{iq(x)/q-(I_{k,n,i}^+)} \right) \chi_{\mathcal{I}_{k,n,i,2}}(x) \left( \int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{q-(I_{k,n,i}^+)} dt \right)^{p_0} dx.
\]
Since
\[
\int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{q-(I_{k,n,i}^+)} dt \leq \int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{p(t)} dt \leq \frac{1}{2},
\]
we can see that
\[
(D) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-a_0)}2^{(j-i)(\alpha_0+s-r)}2^{(j-i)rq(x)}2^{iq(x)/q-(I_{k,n,i}^+)} \right) \chi_{\mathcal{I}_{k,n,i,2}}(x) \left( \int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{q-(I_{k,n,i}^+)} dt \right)^{p_0} dx.
\]
Similarly to the proof of Theorem 5.8, we get
\[
(D) \lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-a_0)}2^{(j-i)(\alpha_0+s-r)}2^{(j-i)\left(\frac{rq(x) - \frac{q(x)}{q-(I_{k,n,i}^+)} + 1}{(\alpha_0+s-r)} \right)} \right) \left( \int_{\mathcal{I}_{k,n,i}^+} \frac{1}{\mathbb{P}(\mathcal{I}_{k,n,i}^+)} |f(t)|^{q(t)} dt \right)^{p_0} dx
\]
\[
\lesssim \int_{\Omega} \left( \sup_{n \in \mathbb{N}} \sum_{k=0}^{2^n-1} \chi_{I_{k,n}}(x) \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} 2^{(j-n)(\alpha-a_0)}2^{(j-i)(\alpha_0+s-r+r_0)} \frac{1}{\mathbb{P}(\mathcal{I}_{k,n,i}^+)} \int_{\mathcal{I}_{k,n,i}^+} |f(t)|^{q(t)} dt \right)^{p_0} dx
\]
\[
\lesssim \|V_{\alpha-a_0,\alpha_0+s-r+r_0}\|^{p_0}_{p_0} \lesssim \|f|^{q(t)}\|^{p_0}_{p_0} \leq C,
\]
whenever (5.10) holds. Note that $r_0$ was defined just before (5.10). Since $r$ can be arbitrarily near to $s + \alpha_0$ and $\alpha_0$ to $\alpha$, this completes the proof. \hfill \Box

**Remark 5.14.** Inequality (5.12) and Theorem 5.13 hold if $p_0 > \max(1/(\alpha + s), 1)$.

The operator $V_{\alpha,s}$ is not bounded on $L_{p_0}$ if (5.12) is not true.
Theorem 5.15. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $1 < p_+ \leq p_+ < \infty$ and $0 < \alpha, s < \infty$. If

$$
\frac{1}{p_-(I_{0,n+2}-1)} - \frac{1}{p_+(I_{0,n})} > \alpha + s
$$

for all $n \in \mathbb{N}$, then $V_{\alpha,s}$ is not bounded on $L^p(\cdot)$.

Proof. Choosing $j = 0$ and $i = n-1$, the theorem can be shown in the same way as Theorem 5.10. □

Remark 5.16. Similarly to Remark 5.11, we know that $V_{\alpha,s}$ is also bounded on $L^p(\cdot, q)$ for $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying (2.4) and (5.12), $1 < p_+ \leq p_+ < \infty$ and $0 < q \leq \infty$.

5.4. The maximal Fejér operator on $H^p(\cdot)$. In this subsection, we apply the atomic characterization via (3, $p(\cdot)$, $\infty$)-atoms to prove the boundedness of $\sigma_s$ from $H^p(\cdot)$ to $L^p(\cdot)$. We first generalize Theorem 5.6 to the result below.

Theorem 5.17. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy (2.4) and $0 < t < \frac{1}{2}$. Suppose that the $\sigma$-sublinear operator $T : L^\infty \to L^\infty$ is bounded and

$$
(5.13) \quad \left\| \sum_k \mu_k T(a^k)^j \chi_{\{\tau_k = \infty\}} \right\|_{p(\cdot)} \lesssim \left\| \sum_k 2^k \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)},
$$

where $\tau_k$ is the stopping time associated with $(3, p(\cdot), \infty)$-atom $a^k$. Then we have

$$
\|Tf\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}},
$$

Proof. According to Corollary 4.6, $f$ can be written as

$$
f = \sum_k \mu_k a^k, \quad \text{where} \quad \mu_k = 3 \cdot 2^k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}
$$

and $a^k$'s are $(3, p(\cdot), \infty)$-atoms associated with stopping times $(\tau_k)_{k \in \mathbb{Z}}$. Then

$$
\|Tf\|_{p(\cdot)} \lesssim \left\| \sum_k \mu_k T(a^k) \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)} + \left\| \sum_k \mu_k T(a^k) \chi_{\{\tau_k = \infty\}} \right\|_{p(\cdot)}
$$

$$
= Z_1 + Z_2.
$$

We first estimate $Z_1$. The sets $\{\tau_k = j\}$ are disjoint and there exist disjoint atoms $I_{k,j,i} \in \mathcal{F}_j$ such that $\{\tau_k = j\} = \bigcup_i I_{k,j,i}$, Thus

$$
\{\tau_k < \infty\} = \bigcup_{j \in \mathbb{N}} \bigcup_i I_{k,j,i},
$$

where $I_{k,j,i}$ are disjoint for fixed $k$. For convenience, we will write

$$
\{\tau_k < \infty\} = \bigcup_i I_{k,i}.
$$
Since $0 < t < p \leq 1$, we have

$$Z_1 \leq \left\| \sum_k \mu_k \sum_l T(a_k)^t \chi_{I_{kl}} \right\|_{\frac{1}{t}}.$$

By Lemma 2.4, choose $g \in L_{(p(\cdot), t)}$, with norm less than 1 such that

$$\left\| \sum_k \mu_k \sum_l T(a_k)^t \chi_{I_{kl}} \right\|_{\frac{1}{t}} = \int_{\Omega} \sum_k \mu_k \sum_l T(a_k)^t \chi_{I_{kl}} g d\mathbb{P}.$$

Note that $T : L_{\infty} \to L_{\infty}$. Then, by Hölder’s inequality for $p_+ / t < r < \infty$ and the definition of $(3, p(\cdot), \infty)$-atoms, we obtain

$$Z_1' \leq \int \sum_k \mu_k \sum_l T(a_k)^t \chi_{I_{kl}} g d\mathbb{P}$$

$$\leq \sum_k \mu_k \sum_l \|T(a_k)^t \chi_{I_{kl}}\|_r \|\chi_{I_{kl}} g\|_{r'}$$

$$\lesssim \sum_k \sum_l (3 \cdot 2^k)^t \|\chi_{\tau_k < \infty}\|_{p(\cdot)} \|T(a_k)^t \chi_{I_{kl}}\|_r \|\chi_{I_{kl}} g\|_{r'}$$

$$\leq \sum_k \sum_l (3 \cdot 2^k)^t \mathbb{P}(I_{kl}) \left( \frac{1}{\mathbb{P}(I_{kl})} \int_{I_{kl}} g^{r'} \right)^{\frac{1}{r'}}$$

$$\leq \sum_k \sum_l (3 \cdot 2^k)^t \int_{I_{kl}} \chi_{I_{kl}} [M(g^{r'})]^{\frac{1}{r'}} d\mathbb{P}$$

$$\leq \left\| \sum_k \sum_l (3 \cdot 2^k)^t \chi_{I_{kl}} \right\|_{\frac{1}{t}} \| [M(g^{r'})]^{\frac{1}{r'}} \|_{\frac{1}{t}(p(\cdot))^{r'}}.$$

Note that $t < p_-$ and $p_+ / t < r$ imply that

$$\left( \left( \frac{p(\cdot)}{t} \right)^{r'} \right)^t < \infty \quad \text{and} \quad \left( \frac{p(\cdot)}{t} \right)^{r'} > r'.$$

By Theorems 2.21 and Theorem 3.3, we get

$$Z_1 \lesssim \left\| \sum_k \sum_l (3 \cdot 2^k)^t \chi_{\tau_k < \infty} \right\|_{\frac{1}{t}} \| g \|_{\frac{1}{t}(p(\cdot))^{r'}}$$

$$\lesssim \left\| \sum_k (3 \cdot 2^k)^t \chi_{\tau_k < \infty} \right\|_{\frac{1}{t}} \lesssim \| f \|_{H_{p(\cdot)}}.$$
Again, by the condition of the theorem, Corollary 4.6 and Theorem 3.3, we have

\[ Z_2 \leq \left\| \sum_k \mu_k t^k T(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{\p} \lesssim \left\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{\p} \lesssim \|f\|_{H_t}^p. \]

Combing the estimates of \( Z_1 \) and \( Z_2 \), we complete the proof. \( \square \)

**Theorem 5.18.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy (2.4) and \( 1/2 < t < p \). If

\[ \frac{1}{p_-} - \frac{1}{p_+} < 1, \]

then

\[ \left\| \sum_k \mu_k^t \sigma^*(a^k)^t \chi_{\{\tau_k = \infty\}} \right\|_{\p} \lesssim \left\| \sum_k 2^{kt} \chi_{\{\tau_k < \infty\}} \right\|_{\p}, \]

where \( \tau_k \) is the stopping time associated with \( a^k \).

**Proof.** We divide this proof into three steps.

**Step 1: estimate for \( \sigma^*(a) \).** The sets \( \{\tau = j\} \) are disjoint and there exist disjoint dyadic intervals \( I_{j,i} \in \mathcal{F}_j \) such that

\[ \{\tau = j\} = \bigcup_i I_{j,i}. \]

Thus

\[ \{\tau < \infty\} = \bigcup_{j \in \mathbb{N}} \bigcup_i I_{j,i}, \]

where the dyadic intervals \( I_{j,i} \) are disjoint. It follows from the definition of the atom that \( \int_{I_{j,i}} a \, d\mathbb{P} = 0 \). For simplicity, instead of \( a \chi_{I_{j,i}} \), we will write \( b^l \), and so

\[ a = \sum_{j \in \mathbb{N}} \sum_i a \chi_{I_{j,i}} = \sum_l b^l. \]

Then the support of \( b^l \) is the dyadic interval \( I_l \) with length \( 2^{-K_l} \) \( (K_l \in \mathbb{N}) \), the sets \( I_l \) are disjoint and \( \int_{I_l} b^l \, d\lambda = 0 \).

It is easy to see that \( \hat{b}^l(n) = 0 \) if \( n < 2^{K_l} \) and in this case \( \sigma_n a = 0 \). Therefore we can suppose that \( n \geq 2^{K_l} \). If \( j \geq K_l \) and \( x \notin I_l \), then \( x + 2^{-j-1} \notin I_l \). Thus for \( x \notin I_l, t \in I_l \) and \( i \geq j \geq K_l \), we have

\[ b^l(t) D_2(x+t) = b^l(t) D_2(x+t+2^{-j-1}) = 0. \]
Since \( n \geq 2^{K_i} \) and \( 2^N > n \geq 2^{N-1} \), one has \( N - 1 \geq K_i \). By (5.4) we obtain for \( x \notin I_l \) that

\[
|\sigma_n b^l(x)| \leq \sum_{j=0}^{N-1} 2^{i-N} \sum_{i=j}^{N-1} \int_0^1 |a(t)| \left( (D_1(x+t) + D_2(x+t+2^{-j-1})) \right) dt
\]

\[
\lesssim \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} 2^{-K_i} \sum_{j=0}^{K_i-1} 2^j \sum_{i=K_i}^{K_i-1} \int_{I_l} \left( D_2(x+t) + D_2(x+t+2^{-j-1}) \right) dt
\]

\[
+ \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \sum_{j=0}^{K_i-1} 2^j \sum_{i=K_l}^{K_i-1} 2^{-i} \int_{I_l} \left( D_2(x+t) + D_2(x+t+2^{-j-1}) \right) dt.
\]

Observe that the right hand side is independent of \( n \). Using (5.2), we can verify that for \( x \notin I_l \),

\[
\int_{I_l} D_2(x+t+2^{-j-1}) dt = 2^{i-K_1} 1_{I_l+[2^{-j-1},2^{-j-1}+2^{-j-1}]}(x) = 2^{i-K_1} 1_{I_{l,i}}(x)
\]

if \( j \leq i \leq K_i - 1 \),

\[
\int_{I_l} D_2(x+t) dt = 2^{i-K_1} 1_{I_l+[2^{-K_1},1]}(x)
\]

if \( i \in \mathbb{N} \) and

\[
\int_{I_l} D_2(x+t+2^{-j-1}) dt = 1_{I_l+[2^{-j-1},2^{-j-1}+2^{-K_1}]}(x) = 1_{I_{l,i}}(x)
\]

if \( i \geq K_i \). Therefore, for \( x \notin I_l \),

\[
\sigma_n b^l(x) \lesssim \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \sum_{j=0}^{K_i-1} 2^j \sum_{i=K_l}^{K_i-1} 2^{-i} 1_{I_{l,i}}(x)
\]

\[
+ \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \sum_{j=0}^{K_i-1} 2^j \sum_{i=K_l}^{K_i-1} \left( 2^{i-K_1} 1_{I_{l,i}}(x) + 1_{I_{l,i}}(x) \right)
\]

\[
\lesssim \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \sum_{j=0}^{K_i-1} 2^{-K_1} 1_{I_{l,i}}(x) + \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \sum_{j=0}^{K_i-1} 2^{j-K_1} \sum_{i=j}^{K_i-1} 2^{i-K_1} 1_{I_{l,i}}(x).
\]

Consequently, for \( x \in \{ \tau = \infty \} \),

\[
\sigma_n a(x) \lesssim \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} \left( \sum_{l=0}^{K_i-1} 2^{-K_1} 1_{I_{l,i}}(x) + \sum_{l=0}^{K_i-1} 2^{j-K_1} 1_{I_{l,i}}(x) \right)
\]

(5.16) =: \| \chi_{\{\tau<\infty\}} \|_{L^p}^{-1} (A(x) + B(x)).

For the atom \( a^k \), we denote \( l, K_l, A \) and \( B \) above by \( k_l, K_{k_l}, A_k \) and \( B_k \). Then

\[
\left\| \sum_k \mu_k \sigma_n (a^k)^t \chi_{\{\tau_k=\infty\}} \right\|_{L^p(t)} \lesssim \left\| \sum_k 2^{kt} A_k^t \right\|_{L^p(t)} + \left\| \sum_k 2^{kt} B_k^t \right\|_{L^p(t)} =: Z_1 + Z_2.
\]
Step 2: estimate for $Z_1$. By Lemma 2.41 there is $g \in L_{(p/t)'}$, with norm less than 1 such that

\[
Z_1 \lesssim \int \Omega \sum_k 2^k \sum_l \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \chi_{I_{kl}^j} |g| d\mathbb{P}
\]

\[
\leq \sum_k 2^k \sum_l \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \left| \chi_{I_{kl}^j} \right| \left| \chi_{I_{kl}^j} g \right|_{(\frac{r}{r'})'}
\]

\[
\lesssim \sum_k 2^k \sum_l \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \int \chi_{I_{kl}^j} \left( \frac{1}{\mathbb{P}(I_{kl}^j)} \int_{I_{kl}^j} |g|^{(\frac{r}{r'})'} \right)^{1/(\frac{r}{r'})'} d\mathbb{P}
\]

because $\mathbb{P}(I_{kl}^j) = \mathbb{P}(I_{kl}^{j-2}) = 2^{-K_{kl}}$. Choosing $\max(1, p) < r < \infty$ and applying Hölder’s inequality again, we conclude

\[
Z_1 \lesssim \int \sum_k 2^k \sum_l \chi_{I_{kl}} \left( \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \right)^{1/(\frac{r}{r'})'} \left( \frac{1}{\mathbb{P}(I_{kl}^j)} \int_{I_{kl}^j} |g|^{(\frac{r}{r'})'} \right)^{1/(\frac{r}{r'})'} d\mathbb{P}
\]

\[
\lesssim \int \sum_k 2^k \sum_l \chi_{I_{kl}} \left( \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \right)^{1/(\frac{r}{r'})'} \left( \frac{1}{\mathbb{P}(I_{kl}^j)} \int_{I_{kl}^j} |g|^{(\frac{r}{r'})'} \right)^{1/(\frac{r}{r'})'} d\mathbb{P}
\]

Furthermore,

\[
Z_1 \lesssim \int \sum_k 2^k \sum_l \chi_{I_{kl}} \left( \sum_{j=0}^{K_{kl}-1} 2^{(j-K_{kl})t} \right)^{1/(\frac{r}{r'})'} \left| \mathcal{U}_t(|g|^{(\frac{r}{r'})'}) \right| \left| \mathcal{U}_t(|g|^{(\frac{r}{r'})'}) \right|_{(p'/t)'.}
\]

Using Theorem 5.8 and Corollary 4.6 we get

\[
Z_1 \lesssim \left\| \sum_k 2^k \chi_{I_{kl}} \right\|_{p(\cdot)/t} \left\| g \right\|_{L_{(p(\cdot)/t)'}} \lesssim \left\| \sum_k 2^k \chi_{\tau_k < \infty} \right\|_{p(\cdot)/t}.
\]
whenever

\[(5.17) \quad \frac{1}{((p(\cdot)/t)^+/t)}_+ - \frac{1}{((p(\cdot)/t)^+/t)}_- = \frac{r/(r-t)}{p_+/p_-} - \frac{r/(r-t)}{p_-/(p_- - t)} < t.\]

Since \( r \) can be arbitrarily large, this means that

\[
\frac{p_+ - t}{p_+} - \frac{p_- - t}{p_-} < t,
\]

which is exactly \((5.14)\).

**Step 3: estimate for \( Z_2 \).** In this estimate, we have to use \( p_- > t > 1/2 \). We choose again a function \( g \in L_{(\frac{p(\cdot)}{t})_+} \) with \( \|g\|_{L_{(\frac{p(\cdot)}{t})_+}} \leq 1 \) such that

\[
\left\| \sum_k \mu_k B_k \right\|_{p(\cdot)/t} = \int \Omega \sum_k \mu_k B_k(x)^t \chi_{\{\tau = \infty\}} g d\mathbb{P}.
\]

Take \( \max(1, p_+) < r < \infty \) large enough such that \( 2t > r/(r-t) \). Let us apply Hölder’s inequality to obtain

\[
Z_2 \lesssim \int \Omega \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(1-(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} g \, dx
\]

\[
\lesssim \int \Omega \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(2(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} g \, dx
\]

Moreover,

\[
Z_2 \lesssim \int \Omega \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(2(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} \left( \frac{1}{\mathbb{P}(I_{k,l}^{t,i})} \int_{I_{k,l}^{t,i}} |g|^{(\frac{1}{r})'} \right)^{1/(\frac{1}{r})'} \, d\mathbb{P}
\]

\[
\lesssim \int \Omega \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(2(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} \left( \frac{1}{\mathbb{P}(I_{k,l}^{t,i})} \int_{I_{k,l}^{t,i}} |g|^{(\frac{1}{r})'} \right)^{1/(\frac{1}{r})'} \, d\mathbb{P}
\]

\[
\lesssim \int \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(2(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} \left( \frac{1}{\mathbb{P}(I_{k,l}^{t,i})} \int_{I_{k,l}^{t,i}} |g|^{(\frac{1}{r})'} \right)^{1/(\frac{1}{r})'} \, d\mathbb{P}
\]

\[
\lesssim \int \sum_k 2^{kt} \sum_{l=0}^{K_l-1} \sum_{i=0}^{K_l-1} 2^{(j-K_l)l_2(2(i-K_l))l_2} \chi_{I_{k,l}^{t,i}} \left( \frac{1}{\mathbb{P}(I_{k,l}^{t,i})} \int_{I_{k,l}^{t,i}} |g|^{(\frac{1}{r})'} \right)^{1/(\frac{1}{r})'} \, d\mathbb{P}
\]
Note that \(((p\cdot)/t)\)'_+ < \infty and \((\frac{2}{t})\)' < \((p\cdot)/t)\)' . Taking into account the definition of the maximal operator \(V\), Theorem [5.13] and Corollary [4.6], we obtain

\[
Z_2 \leq \int \sum_k \sum_l 2^{kt} \chi_{I_k} \left(V_{2t^{-r/(r-t)},r/(r-t)-t}\left(|g|\left(\frac{2}{t}\right)\right)\right)^{1/(\frac{2}{t})'} d\mathbb{P}
\]

\[
\leq \left\| \sum_k \sum_l 2^{kt} \chi_{I_k} \right\|_{p(t)/t} \left\| \left(V_{2t^{-r/(r-t)},r/(r-t)-t}\left(|g|\left(\frac{2}{t}\right)\right)\right)^{1/(\frac{2}{t})'} \right\|_{(p\cdot)/t'}
\]

\[
\lesssim \left\| \sum_k 2^{kt} \chi_{(r<\infty)} \right\|_{\frac{p(t)}{t}},
\]

whenever (5.17) and (5.14) hold. Combining the estimates of \(Z_1\) and \(Z_2\), we finish the proof. \(\Box\)

**Remark 5.19.** If \(1 \leq p_- < \infty\), then (5.14) holds for all \(p_+\). If \(1/2 < p_- < 1\), then \(p_+\) can be chosen such that \(p_+ > 1\) and (5.14) holds.

We immediately get the boundedness of \(\sigma_*\) from \(H_{p(\cdot)}\) to \(L_{p(\cdot)}\) by the above theorems. For the constant \(p = 1\) it is due to Fujiwara [25] (see also Schipp and Simon [63]). For other constant \(p\)'s with \(1/2 < p \leq \infty\), the theorem was proved by the third author in [72].

**Theorem 5.20.** Let \(p(\cdot) \in \mathcal{P}(\Omega)\) satisfy conditions (2.4) and (5.14). If \(1/2 < p_- < \infty\), then

\[
\|\sigma_* f\|_{p(\cdot)} \lesssim \|f\|_{H_{p(\cdot)}}, \quad f \in H_{p(\cdot)}.
\]

If \(p(\cdot) = p\) and \(p \leq 1/2\), then the theorem is not true anymore (see Simon and Weisz [67], Simon [65] and Gáét and Goginava [30]). If (5.14) does not hold, then a counterexample can be found in Theorem [5.29] below. This theorem implies the next consequences about the convergence of \(\sigma_n f\). First we consider the almost everywhere convergence.

**Corollary 5.21.** Let \(p(\cdot) \in \mathcal{P}(\Omega)\) satisfy conditions (2.4) and (5.14). If \(1/2 < p_- < \infty\) and \(f \in H_{p(\cdot)}\), then \(\sigma_n f\) converges almost everywhere on \([0,1]\).

**Proof.** Fix \(f \in H_{p(\cdot)}\) and set

\[
g_N(x) := \sup_{n,k \geq N} |\sigma_n f(x) - \sigma_k f(x)|, \quad g(x) := \lim_{N \to \infty} g_N(x) \quad (x \in [0,1]).
\]

It is sufficient to show that \(g = 0\) almost everywhere.

Observe that \(f_m\) is a Walsh polynomial,

\[
\|f - f_n\|_{H_{p(\cdot)}} \to 0 \quad \text{and} \quad \sigma_n f_m \to f_m
\]
as \(n \to \infty\). Since

\[
|\sigma_n f(x) - \sigma_k f(x)| \leq 2\sigma_* f(x)
\]

and

\[
g_N(x) \leq \sup_{n \geq N} |\sigma_n (f - f_m)(x)| + \sup_{n,k \geq N} |\sigma_n f_m(x) - \sigma_k f_m(x)| + \sup_{k \geq N} |\sigma_k (f_m - f)(x)|,
\]

the theorem follows.
we conclude that
\[ g(x) \leq 4\sigma_*(f - f_m)(x) \]
for all \( m \in \mathbb{N} \) and \( x \in I \). Henceforth, by Theorem 5.20,
\[ \|g\|_{p(\cdot)} \leq 4\|\sigma_*(f - f_m)\|_{p(\cdot)} \lesssim \|f - f_m\|_{H_p(\cdot)} \to 0 \]
as \( m \to \infty \). Hence \( g = 0 \) almost everywhere. \( \square \)

For an integrable function \( f \) belonging to the Hardy spaces, the limit of \( \sigma_nf \) is exactly the function \( f \). Let \( I \in \mathcal{F}_k \) be an atom of \( \mathcal{F}_k \). The restriction of a martingale \( f \) to the atom \( I \) is defined by
\[ f\chi_I := (\mathbb{E}_n f\chi_I, n \geq k). \]

**Corollary 5.22.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty \) and \( f \in H_{p(\cdot)} \). If there exists a dyadic interval \( I \) such that the restriction \( f\chi_I \in L_1(I) \), then
\[ \lim_{n \to \infty} \sigma_nf(x) = f(x) \quad \text{for a.e. } x \in I. \]

**Proof.** For \( f \in H_{p(\cdot)} \), let
\[ g(x) := \limsup_{n \to \infty} |\sigma_nf(x) - f(x)| \quad (x \in I). \]
There exists \( k \in \mathbb{N} \) such that \( I \) is an atom of \( \mathcal{F}_k \). Observe that
\[ g(x) \leq \limsup_{n \to \infty} |\sigma_n(f - f_m)(x)| + \limsup_{n \to \infty} |\sigma_nf_m(x) - f_m(x)| + |f_m(x) - f(x)| \]
\[ \leq \sigma_*(f - f_m)(x) + \|f(x) - f_m(x)\| \]
for all \( m \in \mathbb{N} \) and \( x \in I \). Theorem 5.40 implies
\[ \|g\|_{p(\cdot)} \leq \|\sigma_*(f - f_m)\chi_I\|_{p(\cdot)} + \|(f - f_m)\chi_I\|_{p(\cdot)} \]
\[ \leq \|\sigma_*(f - f_m)\|_{p(\cdot)} + \sup_{n \geq k} \|\mathbb{E}_n(f - f_m)\chi_I\| \]
\[ \leq 2 \|f - f_m\|_{H_{p(\cdot)}} \to 0 \]
as \( m \to \infty \). Hence \( g = 0 \) almost everywhere. \( \square \)

Since \( f \in H_{p(\cdot)} \) with \( 1 \leq p_- < \infty \) implies that \( f \) is integrable, we obtain the next corollary.

**Corollary 5.23.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1 \leq p_- < \infty \) and \( f \in H_{p(\cdot)} \). Then
\[ \lim_{n \to \infty} \sigma_nf(x) = f(x) \quad \text{for a.e. } x \in [0, 1). \]

In the next subsection in Corollary 5.44, we will show the almost everywhere convergence for all integrable functions.

For the norm convergence, we can prove the following consequences similarly.

**Corollary 5.24.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14). If \( 1/2 < p_- < \infty \) and \( f \in H_{p(\cdot)} \), then \( \sigma_nf \) converges in the \( L_{p(\cdot)} \)-norm.
Corollary 5.25. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty \) and \( f \in H_p(\cdot) \). If there exists a dyadic interval \( I \) such that the restriction \( f \chi_I \in L_1(I) \), then

\[
\lim_{n \to \infty} \sigma_n f = f \quad \text{in the } L_p(I)\text{-norm.}
\]

Corollary 5.26. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1 \leq p_- < \infty \) and \( f \in H_p(\cdot) \). Then

\[
\lim_{n \to \infty} \sigma_n f = f \quad \text{in the } L_p\text{-norm.}
\]

Note that \( H_p(\cdot) \) is equivalent to \( L_p(\cdot) \) if \( 1 < p_- < \infty \). Considering only \( \sigma_2^n f \), we do not need the restriction \( 1/2 < p_- \) about \( p(\cdot) \in \mathcal{P}(\Omega) \).

Theorem 5.27. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4) and (5.14) and \( 0 < t < p \). Then

\[
\left(5.18\right) \quad \left\| \sum_k \mu_k^{(1)} \sup_{n \in \mathbb{N}} |\sigma_2^n (a^k)| |x|_{\tau_k=\infty} \right\|_{p(\cdot)} \lesssim \left\| \sum_k 2^{kt} |x|_{\tau_k<\infty} \right\|_{p(\cdot)},
\]

where \( \tau_k \) is the stopping time associated with \( a^k \).

Proof. Taking into account (5.5) and the proof of Theorem 5.18 we can suppose that \( 2^n \geq 2^{K_t} \). For \( x \not\in I_t \), we obtain that

\[
|\sigma_2^n b^l(x)| \lesssim \sum_{j=0}^n 2^{j-n} \int_0^1 |a(t)| D_2^n (x + t + 2^{-j-1}) \, dt
\]

\[
\lesssim \|x|_{\tau<\infty}\|_{p(\cdot)}^{K_t-1} \sum_{j=0}^{K_t-1} 2^{j-K_t} \int_{I_t} D_2^n (x + t + 2^{-j-1}) \, dt
\]

\[
\lesssim \|x|_{\tau<\infty}\|_{p(\cdot)}^{-1} \sum_{j=0}^{K_t-1} 2^{j-K_t} |x|_{I_t^l}(x).
\]

If \( x \in \{\tau = \infty\} \), then

\[
\sup_{n \in \mathbb{N}} |\sigma_2^n a| \lesssim \|x|_{\tau<\infty}\|_{p(\cdot)}^{-1} \sum_{j=0}^{K_t-1} \sum_{l=0}^{K_t-1} 2^{j-K_t} |x|_{I_t^l}(x) = \|x|_{\tau<\infty}\|_{p(\cdot)}^{-1} A(x)
\]

and the proof can be finished as in Theorem 5.18.

We deduce the next result from this and Theorem 5.14.

Theorem 5.28. If \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfies conditions (2.4) and (5.14), then

\[
\left\| \sup_{n \in \mathbb{N}} |\sigma_2^n f| \right\|_{p(\cdot)} \lesssim \|f\|_{H_p(\cdot)}, \quad f \in H_p(\cdot).
\]

Neither Theorem 5.28 nor 5.20 hold if (5.14) is not satisfied. More exactly, we show
Theorem 5.29. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4). If
\begin{equation}
\frac{1}{p_-(I_{0,n-1})} - \frac{1}{p_+(I_{0,n}^0)} > 1
\end{equation}
for all $n \in \mathbb{N}$, then $\sigma_*$ as well as $\sup_{n \in \mathbb{N}} |\sigma_2^n|$ are not bounded from $H_{p(\cdot)}$ to $L_{p(\cdot)}$.

Proof. Let $a_{n-1}(t) = 2^{(n-1)/p_-(I_{0,n-1})}(L_{n_0,n} - L_{1,n})$
and $x \notin I_{0,n-1}$. It is easy to see that $a_{n-1}$ is an atom for all $n \geq 1$, and so $\|a_{n-1}\|_{H_{p(\cdot)}} \leq 1$.

As in Theorem 5.27
\[
|\sigma_2^n a_{n-1}(x)| = \left| \sum_{j=0}^{n} 2^{j-n} \int_{0}^{1} a_{n-1}(t) D_2(t+x+2^{-j-1}) dt \right|
\]
\[
= \left| \sum_{j=0}^{n-2} 2^{j-n} \chi_{I_{0,n-1}}(x) \int_{I_{0,n-1}} a_{n-1}(t) D_2(t+x+2^{-j-1}) dt \right|
\]
\[
= \left| \sum_{j=0}^{n-2} 2^{j-n} \chi_{I_{0,n-1}}(x) \int_{I_{0,n-1}} a_{n-1}(t) D_2(t+x+2^{-j-1}) dt \right|
\]

We choose $j = 0$ and the left half of $I_{0,n-1}$:
\[
|\sigma_2^n a_{n-1}(x)| \geq \chi_{I_{0,n}}(x) 2^{-n} 2^{(n-1)/p_-(I_{0,n-1})}.
\]

Then
\[
\int_{\Omega} \sup_{k \in \mathbb{N}} |\sigma_2^k a_{n-1}(x)|^{p(x)} dx \geq \int_{\Omega} |\sigma_2^n a_{n-1}(x)|^{p(x)} dx
\]
\[
\geq \int_{I_{0,n}} 2^{-n p(x)} 2^{(n-1)p(x)/p_-(I_{0,n-1})} dx
\]
\[
\geq C \int_{I_{0,n}} 2^{n p_+(I_{0,n}) (1/p_-(I_{0,n-1})-1)} dx
\]
\[
= C 2^{n p_+(I_{0,n}) (1/p_-(I_{0,n-1})-1)} 2^{-n}
\]

which tends to infinity as $n \to \infty$ if (5.19) holds.

The following corollaries can be shown as above.

Corollary 5.30. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy conditions (2.4) and (5.14). If $f \in H_{p(\cdot)}$, then $\sigma_2^n f$ converges almost everywhere on $[0,1]$.

Corollary 5.31. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy conditions (2.4) and (5.14). If $f \in H_{p(\cdot)}$ and there exists a dyadic interval $I$ such that the restriction $f \chi_I \in L_1(I)$, then
\[
\lim_{n \to \infty} \sigma_2^n f(x) = f(x) \quad \text{for a.e.} \; x \in I.
\]
Corollary 5.32. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14). If \( f \in H_{p(\cdot)} \), then \( \sigma_{2^n} f \) converges in the \( L_{p(\cdot)} \)-norm.

Corollary 5.33. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14). If \( f \in H_{p(\cdot)} \) and there exists a dyadic interval \( I \) such that the restriction \( f \chi_I \in L_1(I) \), then

\[
\lim_{n \to \infty} \sigma_{2^n} f = f \quad \text{in the } L_{p(\cdot)}(I)\text{-norm.}
\]

5.5. The maximal Fejér operator on \( H_{p(\cdot),q}^s \). In this subsection, we extend the main results in Subsection 5.4 to the variable Hardy-Lorentz space setting. Our method is new even in the classical case ([72]).

Theorem 5.34. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4), \( 0 < q \leq \infty \) and \( 1 < r \leq \infty \) with \( p_+ < r \). Suppose that \( T : H_r^s \to L_r \) is a bounded \( \sigma \)-sublinear operator and

\[
\|T a^k \chi_{\{\tau = \infty\}}\|_{p(\cdot)} \leq C \|\chi_{\{\tau < \infty\}}\|_{p(\cdot)}^{1-\beta}
\]

for some \( 0 < \beta < 1 \) and all \((1, p(\cdot), \infty)\)-atoms \( a \), where \( \tau \) is the stopping time associated with \( a \). Then we have

\[
\|Tf\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}^s}, \quad f \in H_{p(\cdot),q}^s.
\]

Proof. Let \( r = \infty \). We decompose again the martingale \( f \in H_{p(\cdot),q}^s \) into the sum of \( F_1 \) and \( F_2 \), \( f = F_1 + F_2 \) as in the proof of Theorem 4.9. Then (5.20) holds and

\[
\|TF_1\|_\infty \leq \sum_{k=-\infty}^{k_0-1} \mu_k \|T a^k\|_\infty \leq \sum_{k=-\infty}^{k_0-1} \mu_k \|s(a^k)\|_\infty
\]

\[
\leq \sum_{k=-\infty}^{k_0-1} \mu_k \|\chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}^{-1} \leq 3 \cdot 2^{k_0}.
\]

Thus

\[
2^{k_0} \|\chi_{\{TF > 6 \cdot 2^{k_0}\}}\|_{p(\cdot)} \leq 2^{k_0} \|\chi_{\{TF_2 > 3 \cdot 2^{k_0}\}}\|_{p(\cdot)},
\]

so we have to consider

\[
|TF_2| \leq \sum_{k=k_0}^{\infty} \mu_k |T a^k \chi_{\{\tau_k < \infty\}} + \sum_{k=k_0}^{\infty} \mu_k |T a^k \chi_{\{\tau_k = \infty\}}|.
\]

For the first term, we obtain similarly to Step 2 that

\[
\|\chi_{\{\sum_{k=k_0}^{\infty} \mu_k |T a^k \chi_{\{\tau_k < \infty\}} > 3 \cdot 2^{k_0} - 1\}}\|_{p(\cdot)} \leq \sum_{k=k_0}^{\infty} \chi_{\{\tau_k < \infty\}}\|_{p(\cdot)}
\]

and

\[
\sum_{k=k_0}^{\infty} \mu_k |T a^k \chi_{\{\tau_k < \infty\}}|_{L_{p(\cdot),q}} \lesssim \left( \sum_{k=-\infty}^{\infty} \mu_k^q \right)^{1/q} \lesssim \|f\|_{H_{p(\cdot),q}^s}, \quad f \in H_{p(\cdot),q}^s.
\]
if \( q < \infty \). In the last inequality we have used Theorem 3.9. If \( q = \infty \), then similarly to \((3.4)\),

\[
\| \sum_{k=0}^{\infty} \mu_k a_k |T a_k| \chi \{ \tau_k < \infty \} \|_{L^p(\cdot)} \leq \sup_{k} \mu_k \lesssim f \|_{H^s_{p(\cdot), \infty}} \quad (f \in H^s_{p(\cdot), \infty}).
\]

To investigate the second term of \((5.21)\), let \( \beta < \delta < 1 \) and \( \epsilon < \min\{ p, q \} \). Observe that inequality \((5.20)\) is equivalent to

\[
\| |\sigma a|^{\beta} \chi \{ \tau = \infty \} \|_{p(\cdot)/\epsilon} \leq C \| \chi \{ \tau < \infty \} \|_{p(\cdot)/\epsilon} \| \chi \{ \tau < \infty \} \|_{p(\cdot)}^{-\beta \epsilon}.
\]

From this it follows that

\[
\| \chi \{ \sum_{k=0}^{\infty} \mu_k a_k |T a_k| \chi \{ \tau_k < \infty \} > 3 \cdot 2^{k_0 - 1} \} \|_{p(\cdot)} \leq \frac{\sum_{k=0}^{\infty} \mu_k^{\beta} |T a_k|^{\beta} \chi \{ \tau_k = \infty \}}{3^{\beta} 2^{k_0 - 1}} \|_{p(\cdot)}^{1/\epsilon} \leq 2^{-\beta k_0} \sum_{k=0}^{\infty} \mu_k^{\beta \epsilon} \| |T a_k|^{\beta \epsilon} \chi \{ \tau_k = \infty \} \|_{p(\cdot)/\epsilon}^{1/\epsilon}
\]

\[
\lesssim 2^{-\beta k_0} \left( \sum_{k=0}^{\infty} \mu_k^{\beta \epsilon} \| |T a_k|^{\beta \epsilon} \chi \{ \tau_k = \infty \} \|_{p(\cdot)/\epsilon} \right)^{1/\epsilon}
\]

\[
\lesssim 2^{-\beta k_0} \left( \sum_{k=0}^{\infty} 2^{k \beta \epsilon} \| \chi \{ \tau_k < \infty \} \|_{p(\cdot)/\epsilon} \right)^{1/\epsilon}
\]

\[
(5.22)
\]

\[
\leq 2^{-\beta k_0} \left( \sum_{k=0}^{\infty} 2^{k(\beta - \delta) \epsilon} 2^{k \epsilon} \| \chi \{ \tau_k < \infty \} \|_{p(\cdot)}^{\epsilon} \right)^{1/\epsilon}.
\]
If \( q < \infty \), let us again use Hölder’s inequality with \( \frac{\frac{1}{q} + \frac{\epsilon}{q}}{q} = 1 \):

\[
\left\| X \{ \sum_{k=k_0}^{\infty} \mu_k |Ta^k| \chi_{\{\tau_k=\infty\} > 3 \cdot 2^k - 1} \right\|_{p(\cdot)} \leq 2^{-\beta k_0} \left( \sum_{k=k_0}^{\infty} 2^{k(\beta-\delta) \frac{\epsilon}{q-\epsilon}} \right)^{q-\epsilon} \left( \sum_{k=k_0}^{\infty} 2^{k\delta q} \|X(\tau_k < \infty)\|_{p(\cdot)}^q \right)^{1/q}.
\]

By changing the order of the sums, we obtain

\[
\sum_{k_0=-\infty}^{\infty} 2^{kq} \left\| X \{ \sum_{k=k_0}^{\infty} \mu_k |Ta^k| \chi_{\{\tau_k=\infty\} > 3 \cdot 2^k - 1} \right\|_{p(\cdot)}^q \lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \sum_{k=k_0}^{\infty} 2^{k\delta q} \|X(\tau_k < \infty)\|_{p(\cdot)}^q
\]

\[
= \sum_{k=-\infty}^{\infty} 2^{k\delta q} \|X(\tau_k < \infty)\|_{p(\cdot)}^q \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q}
\]

\[
\lesssim \sum_{k=-\infty}^{\infty} 2^{kq} \|X(\tau_k < \infty)\|_{p(\cdot)}^q.
\]

This implies that

\[
\left\| \sum_{k=k_0}^{\infty} \mu_k |Ta^k| \chi_{\{\tau_k=\infty\}} \right\|_{L_{p(\cdot)},q} \lesssim \left( \sum_{k=-\infty}^{\infty} \mu_k^q \right)^{1/q} \lesssim \|f\|_{H_{p(\cdot),q}^s}.
\]

If \( q = \infty \), we use \( 5.22 \) with \( \delta = 1 \) to obtain

\[
\left\| X \{ \sum_{k=k_0}^{\infty} \mu_k |Ta^k| \chi_{\{\tau_k=\infty\} > 3 \cdot 2^k - 1} \right\|_{p(\cdot)} \leq 2^{-\beta k_0} \left( \sum_{k=k_0}^{\infty} 2^{k(\beta-1) \epsilon} 2^k \|X(\tau_k < \infty)\|_{p(\cdot)}^\epsilon \right)^{1/\epsilon}
\]

\[
\leq 2^{-\beta k_0} \left( \sum_{k=k_0}^{\infty} 2^{k(\beta-1) \epsilon} \right)^{1/\epsilon} \sup_k \mu_k
\]

\[
\lesssim 2^{-k_0} \sup_k \mu_k
\]

and so

\[
\left\| \sum_{k=k_0}^{\infty} \mu_k |Ta^k| \chi_{\{\tau_k=\infty\}} \right\|_{L_{p(\cdot),\infty}} \lesssim \sup_k \mu_k \lesssim \|f\|_{H_{p(\cdot),\infty}^s} \quad (f \in H_{p(\cdot),\infty}^s).
\]

Now let \( r < \infty \). Similarly to the proof of Theorem 4.9, we obtain

\[
\|T(F_1)\|_{L_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}^s}.
\]
On the other hand, the inequality
\[ \| T(F_2) \|_{L_{p(\cdot), q}} \lesssim \| f \|_{H^s_{p(\cdot), q}} \]
holds in the same way as above. This completes the proof. \( \square \)

The \( \sigma \)-sublinearity cannot be omitted in general (see Bownik, Li, Yang and Zhou [11, 12, 82]). However, if \( T \) is a linear operator and \( q < \infty \), then \( T \) can be uniquely extended.

**Theorem 5.35.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4), \( 0 < q < \infty \) and \( 1 < r < \infty \) with \( p_+ < r \). Suppose that \( T : H^s_r \to L_r \) is a bounded linear operator and
\[ \| Ta^\beta \chi_{\{\tau = \infty\}} \|_{p(\cdot)} \leq C \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)}^{1-\beta} \]
for some \( 0 < \beta < 1 \) and all \( (1, p(\cdot), \infty) \)-atoms \( a \), where \( \tau \) is the stopping time associated with \( a \). Then
\[ \| Tf \|_{L_{p(\cdot), q}} \lesssim \| f \|_{H^s_{p(\cdot), q}} \cdot \ f \in H^s_{p(\cdot), q}, \quad f \in H^s_{p(\cdot), q} \]
and \( T \) can be uniquely extended to a bounded linear operator from \( H^s_{p(\cdot), q} \) to \( L^s_{p(\cdot), q} \). The theorem holds for \( q = \infty \) as well if we change \( H^s_{p(\cdot), \infty} \) by \( \mathcal{H}^s_{p(\cdot), \infty} \).

**Proof.** By Remark 3.12, the atomic decomposition converges in the \( H^s_{p(\cdot), q} \)-norm. Similarly, writing \( p(\cdot) = q = r \), the atomic decomposition converges to \( f \) in the \( H^s_r \)-norm if \( f \in H^s_r \).

Moreover, \( H^s_r \cap H^s_{p(\cdot), q} \) is dense in \( H^s_{p(\cdot), q} \). Let us define \( F_1 \) and \( F_2 \) again as in the proof of Theorem 4.9. Then for \( f \in H^s_r \cap H^s_{p(\cdot), q} \),
\[ F_1 = \sum_{k=0}^{k_0-1} \mu_k a^k \quad \text{in the } H^s_r \text{-norm}. \]

Since \( T : H^s_r \to L_r \) is a bounded linear operator, we have
\[ TF_1 = \sum_{k=0}^{k_0-1} \mu_k T a^k \quad \text{in the } L_r \text{-norm} \]
and
\[ |TF_1| = \sum_{k=0}^{k_0-1} \mu_k |Ta^k|. \]

The analogous inequality holds for \( TF_2 \). The proof can be finished as in Theorem 5.34. \( \square \)

The next two theorems can be shown similarly.

**Theorem 5.36.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy condition (2.4), \( 0 < q \leq \infty \) and \( 1 < r \leq \infty \) with \( p_+ < r \). Suppose that \( T : L_r \to L_r \) is a bounded \( \sigma \)-sublinear operator and
\[ \| Ta^\beta \chi_{\{\tau = \infty\}} \|_{p(\cdot)} \leq C \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)}^{1-\beta} \]
for some \( 0 < \beta < 1 \) and all \( (3, p(\cdot), \infty) \)-atoms \( a \), where \( \tau \) is the stopping time associated with \( a \). Then
\[ \| Tf \|_{L_{p(\cdot), q}} \lesssim \| f \|_{P_{p(\cdot), q}}, \quad f \in P_{p(\cdot), q}. \]
Theorem 5.37. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $0 < q < \infty$ and $1 < r < \infty$ with $p_+ < r$. Suppose that $T : L_r \to L_r$ is a bounded linear operator and
\[
\| |T a|^\beta \chi_{\{\tau=\infty\}} \|_{p(\cdot)} \leq C \| \chi_{\{\tau<\infty\}} \|_{p(\cdot)}^{1-\beta}
\]
for some $0 < \beta < 1$ and all $(3, p(\cdot), \infty)$-atoms $a$, where $\tau$ is the stopping time associated with $a$. Then
\[
\| T f \|_{L_{p(\cdot)}, q} \lesssim \| f \|_{P_{p(\cdot), q}}, \quad f \in L_r \cap P_{p(\cdot), q}
\]
and $T$ can be uniquely extended to a bounded linear operator from $P_{p(\cdot), q}$ to $L_{p(\cdot), q}$. The theorem holds for $q = \infty$ as well if we change $P_{p(\cdot), \infty}$ by $\mathcal{P}_{p(\cdot), \infty}$.

For linear operators $T_n$ let the maximal operators be defined by
\[
T_n f := \sup_{n \in \mathbb{N}} |T_n f|, \quad T_{N_n} f := \sup_{n \leq N} |T_n f|.
\]

Theorem 5.38. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy condition (2.4), $0 < q < \infty$ and $1 < r \leq \infty$ with $p_+ < r$. Suppose that $T_n : L_1 \to L_1$ is a bounded linear operator for each $n \in \mathbb{N}$ and
\[
(5.23) \quad T_k f_n = T_k f \quad \text{for} \quad 0 \leq k \leq 2^n.
\]
Suppose that $T_* : L_r \to L_r$ is bounded and
\[
\| |T_* a|^\beta \chi_{\{\tau=\infty\}} \|_{p(\cdot)} \leq C \| \chi_{\{\tau<\infty\}} \|_{p(\cdot)}^{1-\beta}
\]
for some $0 < \beta < 1$ and all $(3, p(\cdot), \infty)$-atoms $a$, where $\tau$ is the stopping time associated with $a$. Then
\[
\| T_* f \|_{L_{p(\cdot), q}} \lesssim \| f \|_{H_{p(\cdot), q}^S}, \quad f \in H_{p(\cdot), q}^S.
\]
The theorem holds for $q = \infty$ as well if we change $H_{p(\cdot), \infty}^S$ by $\mathcal{H}_{p(\cdot), \infty}^S$.

Proof. It is easy to see that the atomic decomposition of Theorem 3.9 converges in the $L_1$-norm,
\[
\sum_{k \in \mathbb{Z}} \mu_k a^k = f \quad \text{in the } L_1\text{-norm}
\]
if $f \in H_1^S$. Thus, in this case,
\[
T_n f = \sum_{k \in \mathbb{Z}} \mu_k T_n a^k
\]
and
\[
T_* f \leq \sum_{k \in \mathbb{Z}} |\mu_k| T_* a^k.
\]
Observe that for $f \in H_{p(\cdot), q}^S$, $f_n \in H_1^S$ because $f_n$ is integrable ($n \in \mathbb{N}$). Theorem 5.36 implies that
\[
\| T_* f \|_{L_{p(\cdot), q}} \lesssim \| f \|_{H_{p(\cdot), q}^S} \quad \text{(f \in H_1^S)}.
\]
Hence
\[
\| T_{2^n} f_n \|_{L_{p(\cdot), q}} \lesssim \| T_* f_n \|_{L_{p(\cdot), q}} \lesssim \| f_n \|_{H_{p(\cdot), q}^S} \quad \text{(f \in H_{p(\cdot), q}^S)}.
\]
Since
\[
\lim_{n \to \infty} f_n = f \quad \text{in the } H^{S}_{p(\cdot),q}\text{-norm}
\]
because of the dominated convergence theorem Lemma 2.13, \(T_2^{n,*} f_n\) converges in the \(L_{p(\cdot),q}\)-norm, say
\[
\lim_{n \to \infty} T_2^{n,*} f_n = V f \quad \text{in the } L_{p(\cdot),q}\text{-norm}.
\]
However,
\[
T_2^{n,*} f_n = T_2^{n,*} f \quad (f \in H^S_{p(\cdot),q})
\]
by the condition of the theorem. Obviously,
\[
\lim_{n \to \infty} T_2^{n,*} f = T_\ast f \quad \text{a.e.} \quad (f \in H^S_{p(\cdot),q})
\]
increasingly and so in the \(L_{p(\cdot),q}\)-norm, too. Hence \(T_\ast f = V f\) for all \(f \in H^S_{p(\cdot),q}\), which proves the theorem.

Now we are able to prove the boundedness of \(\sigma_{*}\) from \(H^S_{p(\cdot),q}\) to \(L_{p(\cdot),q}\).

**Theorem 5.39.** Let \(p(\cdot) \in \mathcal{P}(\Omega)\) satisfy conditions (5.13) and (5.14). If \(1/2 < p_- < \infty\), then
\[
(5.24) \quad \| \sigma_{*} a^{|\beta\epsilon} \chi_{\{\tau = \infty\}} \|_{p(\cdot)} \leq C \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)}^{1-\beta}\epsilon
\]
for some \(0 < \beta < 1\) and for all \((3, p(\cdot), \infty)\)-atoms \(a\), where \(\tau\) is the stopping time associated with \(a\).

**Proof.** We can choose \(0 < \beta < 1\) and \(1/2 < \epsilon < p\) such that \(\beta \epsilon > 1/2\). Instead of (5.24), we will show that
\[
\| \sigma_{*} a^{|\beta\epsilon\chi_{\{\tau = \infty\}}} \|_{p(\cdot)/\epsilon} \leq C \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)/\epsilon} \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)}^{-\beta\epsilon}.
\]
We use same symbols as in the proof of Theorem 5.18. Then, by the estimate of \(\sigma_{*}(a)\) in (5.16), we have
\[
(5.25) \quad \| \sigma_{*} a^{|\beta\epsilon\chi_{\{\tau = \infty\}}} \|_{p(\cdot)/\epsilon} \leq \| \chi_{\{\tau < \infty\}} \|_{p(\cdot)/\epsilon}^{-\beta\epsilon} \left( \| A^{\beta \epsilon \chi_{\{\tau = \infty\}}} \|_{p(\cdot)/\epsilon} + \| B^{\beta \epsilon \chi_{\{\tau = \infty\}}} \|_{p(\cdot)/\epsilon} \right).
\]
By Lemma 2.14, we can choose a function \(g \in L_{p(\cdot)/\epsilon}(\Omega)\) with \(\|g\|_{p(\cdot)/\epsilon} \leq 1\) such that
\[
\| A^{\beta \epsilon \chi_{\{\tau = \infty\}}} \|_{p(\cdot)/\epsilon} = \int_{\Omega} A(x)^{\beta \epsilon \chi_{\{\tau = \infty\}}} gd\mathbb{P}.
\]
Choosing \( \max(1, \beta p_+) < r < \infty \) and applying Hölder’s inequality, we obtain

\[
\| A^{\beta \epsilon} \chi_{(\tau=\infty)} \|_{p(\cdot)/\epsilon} \leq \int \sum_{l} \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \chi_{l}^j |g| d\mathbb{P}
\]

\[
\leq \sum_{l} \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \| \chi_{l}^j \|_{\| \|} \| \chi_{l}^j g \|_{(\frac{r}{p})^\gamma} \leq \sum_{l} \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \int \chi_{l} \left( \frac{1}{\mathbb{P}(I_l^j)} \int |g|^{(\frac{r}{p})^\gamma} \right) d\mathbb{P}
\]

\[
\because \mathbb{P}(I_l) = \mathbb{P}(I_l^j) = 2^{-K_l}. \text{ Again by Hölder’s inequality,}
\]

\[
\| A^{\beta \epsilon} \chi_{(\tau=\infty)} \|_{p(\cdot)/\epsilon} \leq \int \sum_{l} \chi_{l} \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \left( \frac{1}{\mathbb{P}(I_l^j)} \int |g|^{(\frac{r}{p})^\gamma} \right) d\mathbb{P}
\]

\[
\leq \int \sum_{l} \chi_{l} \left( \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \left( \frac{1}{\mathbb{P}(I_l^j)} \int |g|^{(\frac{r}{p})^\gamma} \right) \right)^{1/(\frac{r}{p})^\gamma} d\mathbb{P}.
\]

From this it follows that

\[
\| A^{\beta \epsilon} \chi_{(\tau=\infty)} \|_{p(\cdot)/\epsilon} \leq \int \sum_{l} \chi_{l} \left( \sum_{j=0}^{K_l-1} 2^{(j-K_l)\beta \epsilon} \left( \frac{1}{\mathbb{P}(I_l^j)} \int |g|^{(\frac{r}{p})^\gamma} \right) \right)^{1/(\frac{r}{p})^\gamma} d\mathbb{P}
\]

\[
\leq \int \sum_{l} \chi_{l} \left( U_{\beta \epsilon}(|g|^{(\frac{r}{p})^\gamma}) \right)^{1/(\frac{r}{p})^\gamma} d\mathbb{P}
\]

\[
\leq \left\| \sum_{l} \chi_{l} \right\|_{p(\cdot)/\epsilon} \left\| U_{\beta \epsilon}(|g|^{(\frac{r}{p})^\gamma}) \right\|_{(p(\cdot)/\epsilon)^\gamma} \leq \| g \|_{(p(\cdot)/\epsilon)^\gamma} \leq 1,
\]

Since \( r > \beta p_+ \) and \( \epsilon < p_- \), we get

\[
\left( \frac{r}{\beta \epsilon} \right)^\gamma < (p(\cdot)/\epsilon)^\gamma \text{ and } ((p(\cdot)/\epsilon)^\gamma)_+ < \infty.
\]

Then Theorem 5.6 implies that

\[
\left\| U_{\beta \epsilon}(|g|^{(\frac{r}{p})^\gamma}) \right\|_{(p(\cdot)/\epsilon)^\gamma} = \left\| U_{\beta \epsilon}(|g|^{(\frac{r}{p})^\gamma}) \right\|_{(p(\cdot)/\epsilon)^\gamma}^{1/(\frac{r}{p})^\gamma} \leq \| g \|_{(p(\cdot)/\epsilon)^\gamma} \leq 1,
\]

which shows that

\[
(5.26) \quad \| A^{\beta \epsilon} \chi_{(\tau=\infty)} \|_{p(\cdot)/\epsilon} \leq \| \chi_{(\tau<\infty)} \|_{p(\cdot)/\epsilon},
\]
whenever

\[\frac{1}{((p(\cdot)/\epsilon)'/(r/\beta \epsilon)')} - \frac{1}{((p(\cdot)/\epsilon)'/(r/\beta \epsilon)')} = \frac{r/(r - \beta \epsilon)}{p_+/(p_+ - \epsilon)} - \frac{r/(r - \beta \epsilon)}{p_-/(p_- - \epsilon)} < \beta \epsilon.\]

This means that

\[\frac{p_+ - \epsilon}{p_+} - \frac{p_- - \epsilon}{p_-} < \beta \epsilon,\]

in other words,

\[\frac{1}{p_-} - \frac{1}{p_+} < \beta.\]

Since \(\beta\) can arbitrarily near to 1, we obtain (5.14).

Now let us investigate the second term of (5.25). We choose again a function \(g \in L_{(p(\cdot)/\epsilon)'}\) with \(\|g\|_{(p(\cdot)/\epsilon)'} \leq 1\) such that

\[\left\| B^{\beta \epsilon} \chi_{\{\tau = \infty\}} \right\|_{p(\cdot)/\epsilon} = \int_{\Omega} B(x)^{\beta \epsilon} \chi_{\{\tau = \infty\}} g d\mathbb{P}.\]

Let us apply Hölder’s inequality to obtain

\[\left\| B^{\beta \epsilon} \chi_{\{\tau = \infty\}} \right\|_{p(\cdot)/\epsilon} \lesssim \int_{\Omega} \sum_{l} \sum_{j=0}^{K_l-1} \sum_{i=j}^{K_l-1} 2^{(j-K_l)\beta \epsilon} 2^{(i-K_l)\beta \epsilon} \chi_{I_l^j} |g| \cdot d\mathbb{P}.\]

Furthermore,

\[\sum_{l} \sum_{j=0}^{K_l-1} \sum_{i=j}^{K_l-1} 2^{(j-K_l)\beta \epsilon} 2^{(i-K_l)\beta \epsilon} \cdot \chi_{I_l^j} \cdot \inf_{x'} \|X^{(i,j)}_{l,t} g\|_{(p_{x'})'}\]

\[\lesssim \sum_{l} \sum_{j=0}^{K_l-1} \sum_{i=j}^{K_l-1} 2^{(j-K_l)\beta \epsilon} 2^{(i-K_l)\beta \epsilon} 2^{K_l-i} \int \chi_{I_l^j} \left( \frac{1}{\mathbb{P}(I_l^j t)} \int_{I_l^j} |g|_{(p_{x'})'}^{1/(p_{x'})'} d\mathbb{P} \right)^{1/(p_{x'})'} d\mathbb{P},\]
whenever \( \max(1, \beta p_+) < r < \infty \) is large enough such that \( 2\beta \epsilon > r/(r - \beta \epsilon) \). Moreover,
\[
\| B^{\beta \epsilon} \chi_{\{\tau = \infty\}} \|_{p(\cdot)/\epsilon} 
\leq \int \sum_l \chi_l \left( \sum_{j=0}^{K_i-1} \sum_{i=j}^{K_i-1} 2^{(i+j-2K_i)\beta \epsilon(1/(\pi r) + 1/(\pi r)')} 2^{K_i-i} \left( \frac{1}{|I_{l^i}|} \int_{I_{l^i}} |g|^{(\frac{1}{\pi r})'} \right)^{1/(\frac{1}{\pi r'})} \right) \ d\mathbb{P}
\]
\[
\leq \int \sum_l \chi_l \left( \sum_{j=0}^{K_i-1} \sum_{i=j}^{K_i-1} 2^{(j-K_i)(2\beta \epsilon - r/(r - \beta \epsilon))} 2^{(i-j)(r/(r - \beta \epsilon) - \beta \epsilon)} \frac{1}{|I_{l^i}|} \int_{I_{l^i}} |g|^{(\frac{1}{\pi r})'} \right)^{1/(\frac{1}{\pi r'})} \ d\mathbb{P}.
\]
Taking into account the definition of the maximal operator \( V_{\alpha,t} \) and Theorem 5.13, we obtain
\[
\| B^{\beta \epsilon} \chi_{\{\tau = \infty\}} \|_{p(\cdot)/\epsilon} \leq \int \sum_l \chi_l \left( V_{2\beta \epsilon - r/(r - \beta \epsilon), r/(r - \beta \epsilon) - \beta \epsilon} \left( |g|^{(\frac{1}{\pi r})'} \right)^{1/(\frac{1}{\pi r'})} \right) \ d\mathbb{P}
\]
\[
\leq \left\| \sum_l \chi_l \right\|_{p(\cdot)/\epsilon} \left\| \left( V_{2\beta \epsilon - r/(r - \beta \epsilon), r/(r - \beta \epsilon) - \beta \epsilon} \left( |g|^{(\frac{1}{\pi r})'} \right)^{1/(\frac{1}{\pi r'})} \right) \right\|_{p(\cdot)/\epsilon},
\]
\[
\leq \left\| \chi_{\{\tau < \infty\}} \right\|_{p(\cdot)/\epsilon}
\]
as in (5.20), whenever (5.27) and (5.14) hold. The proof of the theorem is complete. \( \square \)

Now we are able to prove the next theorem. It was proved by the third author in [72] for \( H_{p,q} \) with constant \( p \).

**Theorem 5.40.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14). If \( 1/2 < p_- < \infty \) and \( 0 < q \leq \infty \), then
\[
\| \sigma_s f \|_{L_{p(\cdot),q}} \lesssim \| f \|_{H_{p(\cdot),q}}, \quad f \in H_{p(\cdot),q}.
\]

**Proof.** It is easy to see that (5.23) holds in this case, i.e.,
\[
\sigma_k f_n = \sigma_k f \quad \text{for} \quad 0 \leq k \leq 2^n.
\]
Applying Theorems 5.38 and 5.39 we can complete the proof. \( \square \)

For a constant \( p \) with \( p \leq 1/2 \), the theorem does not hold (see Simon and Weisz [67], Simon [65] and Gát and Goginava [30]). Since the Walsh polynomials are dense in \( H_{p(\cdot),q} \) as well, the next three consequences can be proved as Corollaries 5.21 and 5.22 and 5.23.

**Corollary 5.41.** Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty \) and \( 0 < q \leq \infty \). If \( f \in H_{p(\cdot),q} \), then \( \sigma_n f \) converges almost everywhere on \([0,1] \).
Corollary 5.42. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty, \) \( 0 < q \leq \infty \) and \( f \in H_{p(\cdot),q} \). If there exists a dyadic interval \( I \) such that the restriction \( f \chi_I \in L_1(I) \), then
\[
\lim_{n \to \infty} \sigma_n f(x) = f(x) \quad \text{for a.e. } x \in I.
\]

Corollary 5.43. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1 \leq p_- < \infty, \) \( 0 < q \leq \infty \) and \( f \in H_{p(\cdot),q} \). Then
\[
\lim_{n \to \infty} \sigma_n f(x) = f(x) \quad \text{for a.e. } x \in [0,1).
\]

Now we prove that for integrable functions, the limit of \( \sigma_n f \) is exactly the function. Since \( L_1 \subset H_{1,\infty} \), more exactly,
\[
\|f\|_{H_{1,\infty}} = \sup_{\rho > 0} \rho \mathbb{P}(M(f) > \rho) \leq C\|f\|_1 \quad (f \in L_1),
\]
(see e.g. Weisz [70]), we obtain the next corollary, which was shown by Fine [24], Schipp [62] and Weisz [73].

Corollary 5.44. If \( f \in L_1 \), then
\[
\lim_{n \to \infty} \sigma_n f(x) = f(x) \quad \text{for a.e. } x \in [0,1).
\]

The results about the norm convergence can be shown in the same way.

Corollary 5.45. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty \) and \( 0 < q \leq \infty \). If \( f \in H_{p(\cdot),q} \), then \( \sigma_n f \) converges in the \( L_{p(\cdot),q} \) norm.

Corollary 5.46. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1/2 < p_- < \infty, \) \( 0 < q \leq \infty \) and \( f \in H_{p(\cdot),q} \). If there exists a dyadic interval \( I \) such that the restriction \( f \chi_I \in L_1(I) \), then
\[
\lim_{n \to \infty} \sigma_n f = f \quad \text{in the } L_{p(\cdot),q}(I) \text{-norm}.
\]

Corollary 5.47. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14), \( 1 \leq p_- < \infty, \) \( 0 < q \leq \infty \) and \( f \in H_{p(\cdot),q} \). Then
\[
\lim_{n \to \infty} \sigma_n f = f \quad \text{in the } L_{p(\cdot),q} \text{-norm}.
\]

Note that \( H_{p(\cdot),q} \) is equivalent to \( L_{p(\cdot),q} \) if \( 1 < p_- < \infty \).

Similarly to Theorem 5.27, we do not need the restriction \( 1/2 < p_- \) in the next results.

Theorem 5.48. If \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfies conditions (2.4) and (5.14). If \( 0 < q \leq \infty \), then
\[
\left\| \sup_{n \in \mathbb{N}} \| \sigma_{2^n} f \|_{L_{p(\cdot),q}} \right\|_{H_{p(\cdot),q}} \lesssim \|f\|_{H_{p(\cdot),q}}, \quad f \in H_{p(\cdot),q}.
\]

This implies the following corollaries.

Corollary 5.49. Let \( p(\cdot) \in \mathcal{P}(\Omega) \) satisfy conditions (2.4) and (5.14). If \( 0 < q \leq \infty \) and \( f \in H_{p(\cdot),q} \), then \( \sigma_{2^n} f \) converges almost everywhere on \([0,1)\).
Corollary 5.50. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy conditions (2.4) and (5.14). If there exists a dyadic interval $I$ such that the restriction $f \chi_I \in L_1(I)$, then
\[
\lim_{n \to \infty} \sigma_n^2 f(x) = f(x) \quad \text{for a.e. } x \in I.
\]

Corollary 5.51. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy conditions (2.4) and (5.14). If $0 < q \leq \infty$ and $f \in \mathcal{H}^{p(\cdot),q}$, then $\sigma_n^2 f$ converges in the $L_{p(\cdot),q}^{\infty}$-norm.

Corollary 5.52. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy conditions (2.4) and (5.14). If there exists a dyadic interval $I$ such that the restriction $f \chi_I \in L_1(I)$, then
\[
\lim_{n \to \infty} \sigma_n^2 f = f \quad \text{in the } L_{p(\cdot),q}(I)\text{-norm}.
\]

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