Hecke actions on certain strongly modular genera of lattices

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Abstract

We calculate the action of some Hecke operators on spaces of modular forms spanned by the Siegel theta-series of certain genera of strongly modular lattices closely related to the Leech lattice. Their eigenforms provide explicit examples of Siegel cusp forms.

1 Introduction

One of the most remarkable lattices in Euclidean space is the Leech lattice, the unique even unimodular lattice $\Gamma_1 \subset (\mathbb{R}^{24}, (,))$ of dimension 24 that does not contain vectors of square length 2. Here a lattice $\Lambda \subset (\mathbb{R}^n, (,))$ is called unimodular, if $\Lambda$ equals its dual lattice $\Lambda^\# := \{ x \in \mathbb{R}^n \mid (x, \lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \}$ and even, if the quadratic form $x \mapsto (x, x)$ takes only even values on $\Lambda$. [13] studies spaces of Siegel modular forms generated by the Siegel theta-series of the 24 isometry classes of lattices in the genus of $\Gamma_1$. The present paper extends this investigation to further genera of lattices, closely related to $\Gamma_1$. A unified construction is given in [16]: Consider the Mathieu group $M_{23} \leq \text{Aut}(\Gamma_1)$, where the automorphism group of a lattice $\Lambda \subset (\mathbb{R}^n, (,))$ is $\text{Aut}(\Lambda) := \{ g \in O(n) \mid \Lambda g = \Lambda \}$. Let $g \in M_{23}$ be an element of square-free order $l := |\langle g \rangle|$. Then

$$l \in \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\} =: \mathcal{N} = \{ n \in \mathbb{N} \mid \sigma_1(n) : = \sum_{d \mid n} d \text{ divides } 24 \}$$

and for each $l \in \mathcal{N}$, there is an up to conjugacy unique cyclic subgroup $\langle g \rangle \leq M_{23}$ of order $l$. Let $\Gamma_l := \{ \lambda \in \Gamma_1 \mid \lambda g = \lambda \}$ denote the fixed lattice of $g$. Then $\Gamma_l$ is an extremal strongly modular lattice of level $l$ and of dimension $2k_l$, where

$$k_l := 12\sigma_0(l)/\sigma_1(l)$$

and $\sigma_0(l)$ denotes the number of divisors of $l$. In particular $\Gamma_1$ is the Leech lattice, $\Gamma_2$ the 16-dimensional Barnes-Wall lattice and $\Gamma_3$ the Coxeter-Todd lattice of dimension 12.

Let $\Lambda$ be an even lattice. The minimal $l \in \mathbb{N}$ for which $\sqrt{l}\Lambda^\#$ is even, is called the level of $\Lambda$. Then $l\Lambda^\# \subset \Lambda$. For an exact divisor $d$ of $l$ let

$$\Lambda^{\#,d} := \Lambda^\# \cap \frac{1}{d}\Lambda$$

denote the $d$-partial dual of $\Lambda$. A lattice $\Lambda$ is called strongly $l$-modular, if $\Lambda$ is isometric to $\sqrt{d}\Lambda^{\#,d}$ for all exact divisors $d$ of the level $l$ of $\Lambda$. If $l$ is a prime, this coincides with the notion of modular lattices, which just means that the lattice is similar to its dual lattice. The Siegel theta-series

$$\Theta^{(m)}_\Lambda(Z) := \sum_{(\lambda_1, \ldots, \lambda_m) \in \Lambda^m} \exp(i\pi \text{trace}((\lambda_i, \lambda_j)Z))$$

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(which is a holomorphic function on the Siegel halfspace $\mathcal{H}^{(m)} = \{ Z \in \text{Sym}_m(\mathbb{C}) \mid \exists (Z) \text{ positive definite } \}$ of a strongly $l$-modular lattice is a modular form for the $l$-th congruence subgroup $\Gamma_0^{(m)}(l)$ of $\text{Sp}_{2m}(\mathbb{Z})$ (to a certain character) invariant under all Atkin-Lehner-involutions (cf. [11]). In particular for $m = 1$ and $l \in \mathcal{N}$ the relevant ring of modular forms is a polynomial ring in 2 generators as shown in [14], [15]. Explicit generators of this ring allow to bound the minimum of an $n$-dimensional strongly $l$-modular lattice $\Lambda$ with $l \in \mathcal{N}$,

$$\min(\Lambda) := \min_{0 \neq \lambda \in \Lambda} (\lambda, \lambda) \leq 2 + 2\lfloor \frac{n}{2k_l} \rfloor.$$ 

Lattices $\Lambda$ achieving this bound are called extremal. For all $l \in \mathcal{N}$ there is a unique extremal strongly $l$-modular lattice of dimension $2k_l$ and this is the lattice $\Gamma_l$ described above. All the genera are presented in the nice survey article [17].

In this paper we investigate the spaces of Siegel modular forms generated by the Siegel theta-series of the lattices in the genus $G(\Gamma_l)$ for $l \in \mathcal{N}$ using similar methods as for the case $l = 1$ which is treated in [13]. The vector space $V := V(G)$ of all complex formal linear combinations of the isometry classes of lattices in any genus $G$ forms a finite dimensional commutative $\mathbb{C}$-algebra with positive definite Hermitian scalar product. Taking theta-series defines linear operators $\Theta^{(m)}$ from $V$ into a certain space of modular forms and hence a filtration of $V$ by the kernels of these operators. This filtration behaves nicely under the multiplication and is invariant under all Hecke-operators. With the Kneser neighbouring process we construct a family of commuting self-adjoint linear operators on $V$. Their common eigenvectors provide explicit examples of Siegel cusp forms.

The genera $G(\Gamma_l)$ ($l \in \mathcal{N}$) share the following properties:

**Corollary 1.1** Let $l \in \mathcal{N}$ and let $p$ be the smallest prime not dividing $l$. The mapping $\Theta^{(k_l)}$ is injective on $V(G(\Gamma_l))$. For $l \neq 7$, the construction described in [2] (see Paragraph 2.3) gives a non-zero cusp form $BFW(\Gamma_l, p) = \Theta^{(k_l)}(\text{Per}(\Gamma_l, p))$. The eigenvalue of the Kneser operator $K_2$ at the eigenvector $\text{Per}(\Gamma_l, p)$ is the negative of the number of pairs of minimal vectors in $\Gamma_l$ which is also the minimal eigenvalue of $K_2$.

**Remark 1.2** In Section 4 we also list the eigenvalues of some of the operators $T(q)$ defined in Subsection 2.4. These eigenvalues suggest that for even values of $k_l$, the cusp form $BFW(\Gamma_l, p)$ is a generalized Duke-Imamoglu-Ikeda lift (see [8]) of the elliptic cusp form of minimal weight $k_l$.

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## 2 Methods

The general method has already been explained in [13] (see also [19], [20], [21] and [2] for similar strategies).

### 2.1 The algebra $V = V(G)$

Let $G$ be a genus of lattices in the Euclidean space $(\mathbb{R}^{2k}, (,))$. Then $G$ is the disjoint union of finitely many isometry classes

$$G = [\Lambda_1] \cup \ldots \cup [\Lambda_k].$$
Let \( \mathcal{V} := \mathcal{V}(\mathcal{G}) \cong \mathbb{C}^h \) be the complex vector space with basis \((\Lambda_1), \ldots, (\Lambda_h)\). Let \( \mathcal{V}_\mathbb{Q} = \{ (\Lambda_1), \ldots, (\Lambda_h) \}_\mathbb{Q} \cong \mathbb{Q}^h \) be the rational span of the basis.

The space \( \mathcal{V} \) can be identified with the algebra \( \mathcal{A} \) of complex functions on the double cosets \( G(\mathbb{Q}) \backslash G(\mathbb{A}) / \text{Stab}_{G(\mathbb{A})}(\Lambda_\mathbb{A}) = \bigcup_{i=1}^h G(\mathbb{Q}) x_i \text{Stab}_{G(\mathbb{A})}(\Lambda_\mathbb{A}) \) where \( G \) is the integral form of the real orthogonal group \( G(\mathbb{R}) = O_{2k} \) defined by \( \Lambda_1 \), \( \mathbb{A} \) denotes the ring of rational adèles and \( \Lambda_\mathbb{A} \) the adèlic completion of \( \Lambda_1 \). If \( \chi_i \) denotes the characteristic function mapping \( G(\mathbb{Q}) x_j \text{Stab}_{G(\mathbb{A})}(\Lambda_\mathbb{A}) \) to \( \delta_{ij} \) and \( \Lambda_i = x_i \Lambda_1 \) \((i = 1, \ldots, h)\) then the isomorphism maps \([\Lambda_i] \) to \([\text{Aut}(\Lambda_i)]\chi_i \). The usual Petersson scalar product then translates into the Hermitian scalar product on \( \mathcal{V} \) defined by

\[
\langle [\Lambda_i], [\Lambda_j] \rangle := \delta_{ij} |\text{Aut}(\Lambda_i)|
\]

and the multiplication of \( \mathcal{A} \) defines a commutative and associative multiplication \( \circ \) on \( \mathcal{V} \) with

\[
[\Lambda_i] \circ [\Lambda_j] := \#(\text{Aut}(\Lambda_i)) \delta_{ij} [\Lambda_i]
\]

(see for instance [\ref{3} Section 1.1]). Note that the Hermitian form \( \langle \cdot, \cdot \rangle \) is associative, i.e.

\[
\langle v_1 \circ v_2, v_3 \rangle = \langle v_1, v_2 \circ v_3 \rangle \quad \text{for all } v_1, v_2, v_3 \in \mathcal{V}.
\]

### 2.2 The two basic filtrations of \( \mathcal{V} \)

For simplicity we now assume that \( \mathcal{G} \) consists of even lattices. Let \( l \) be the level of the lattices in \( \mathcal{G} \). Taking the degree-\( n \) Siegel theta-series \( \Theta^{(n)}_{\Lambda_i}(n = 0, 1, 2, \ldots) \) of the lattices \( \Lambda_i \) \((i = 1, \ldots, h)\) then defines a linear map

\[
\Theta^{(n)} : \mathcal{V} \rightarrow \mathcal{M}_{n,k}(l) \text{ by } \Theta^{(n)} \left( \sum_{i=1}^h c_i [\Lambda_i] \right) := \sum_{i=1}^h c_i \Theta^{(n)}_{\Lambda_i}
\]

with values in a space of modular forms of degree \( n \) and weight \( k \) for the group \( \Gamma^{(n)}_0(l) \) (see [\ref{1}]).

For \( n = 0, \ldots, 2k \) let \( \mathcal{V}_n := \ker(\Theta^{(n)}) \) be the kernel of this linear map. Then we get the filtration

\[
\mathcal{V} := \mathcal{V}_{-1} \supseteq \mathcal{V}_0 \supseteq \mathcal{V}_1 \supseteq \ldots \supseteq \mathcal{V}_{2k} = \{0\}
\]

where \( \mathcal{V}_0 = \{ v = \sum_{i=1}^h c_i [\Lambda_i] \mid \sum_{i=1}^h c_i = 0 \} \) is of codimension 1 in \( \mathcal{V} \).

Clearly \( \Theta^{(n)}(\mathcal{V}_{n-1}) \) is the kernel of the Siegel \( \Phi \)-operator mapping \( \Theta^{(n)}(\mathcal{V}) \) onto \( \Theta^{(n-1)}(\mathcal{V}) \). For square-free level one even has

**Theorem 2.1** (see [\ref{4} Theorem 8.1]) If \( l \) is square-free, then \( \Theta^{(n)}(\mathcal{V}_{n-1}) \) is the space of cusp forms in \( \Theta^{(n)}(\mathcal{V}) \).

Let \( \mathcal{W}_n := \mathcal{V}_n^\perp \) be the orthogonal complement of \( \mathcal{V}_n \). We then have the ascending filtration

\[
0 = \mathcal{W}_{-1} \subseteq \mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \ldots \subseteq \mathcal{W}_{2k} = \mathcal{V}.
\]

By [\ref{12} Proposition 2.3, Corollary 2.4] one has the following lemma:

**Lemma 2.2**

\[
\mathcal{W}_n \circ \mathcal{W}_m \subset \mathcal{W}_{n+m} \text{ for all } m, n \in \{-1, \ldots, 2k\}
\]

and

\[
\mathcal{W}_n \circ \mathcal{V}_m \subset \mathcal{V}_{m-n} \text{ for all } m > n \in \{-1, \ldots, 2k\}.
\]
Since theta-series have rational coefficients, both filtrations are rational, i.e. \( V_n = \mathbb{C} \otimes (V_n \cap V_{\mathbb{Q}}) \) and \( W_n = \mathbb{C} \otimes (W_n \cap V_{\mathbb{Q}}) \), hence the same statements hold when \( V \) is replaced by \( V_{\mathbb{Q}} \).

### 2.3 The Borcherds-Freitag-Weissauer cusp form

The article [5] gives a quite general construction of a cusp form of degree \( k \). Let \( \Lambda \) be a \( 2k \)-dimensional even lattice and choose some prime \( p \) such that the quadratic space \((\Lambda/p\Lambda, Q_p)\) is isometric to the sum of \( k \) hyperbolic planes. Fix a totally isotropic subspace \( F \) of \( \Lambda/p\Lambda \) of dimension \( k \). For \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda^k \) we put \( E(\lambda) := \langle \lambda_1, \ldots, \lambda_k \rangle + p\Lambda \) and \( S(\lambda) := \frac{1}{p^k}((\lambda_i, \lambda_j))_{i,j} \in \text{Sym}_k(\mathbb{R}) \). Define \( \epsilon(E(\lambda)) = \epsilon(\lambda) := (-1)^{\dim(F \cap E(\lambda))} \) if \( E(\lambda) \) is a \( k \)-dimensional totally isotropic subspace of \( \Lambda/p\Lambda \) and \( \epsilon(E(\lambda)) = \epsilon(\lambda) := 0 \) otherwise.

**Definition 2.3** \( \text{BFW}(\Lambda, p)(Z) := \sum_{\lambda \in \Lambda^k} \epsilon(\lambda) \exp(i\pi \text{trace}(S(\lambda)Z)) \).

By [5] the form \( \text{BFW}(\Lambda, p) \) is a linear combination of Siegel theta-series of lattices in the genus of \( \Lambda \): For any \( k \)-dimensional totally isotropic subspace \( E \) of \( \Lambda/p\Lambda \) let \( \Gamma(E) := (E, p\Lambda) \) be the full preimage of \( E \). Dividing the scalar product by \( p \), one obtains a lattice \( \frac{1}{p}\Gamma(E) := (\Gamma(E), \frac{1}{p}(,)) \in \mathcal{G} \). Then we define

\[
\text{Per}(\Lambda, p) := \sum_{E} \epsilon(E)\left[ \frac{1}{p}\Gamma(E) \right] \in \mathcal{V}
\]

where the sum runs over all \( k \)-dimensional totally isotropic subspaces of \( \Lambda/p\Lambda \). As \( \epsilon \) is only defined up to a sign, also \( \text{Per}(\Lambda, p) \) is only well defined up to a factor \( \pm 1 \). It is shown in [5, Theorem 2] that

\[
\Theta^{(k)}(\text{Per}(\Lambda, p)) = \text{BFW}(\Lambda, p).
\]

In analogy to the notation in [10] we call \( \text{Per}(\Lambda, p) \) the **perestroika** of \( \Lambda \). Clearly \( \text{BFW}(\Lambda, p) \) is in the kernel of the \( \Phi \)-operator and hence a cusp form, if the level of \( \Lambda \) is square-free by Theorem 2.1.

### 2.4 Hecke-actions

Strongly related to the Borcherds-Freitag-Weissauer construction are the Hecke operators \( T(p) \) which define self-adjoint linear operators on \( \mathcal{V} \) and whose action on theta series coincides with the one of \( T(p) \) in [7, Theorem IV.5.10] and [22, Proposition 1.9] up to a scalar factor (depending on the degree of the theta series). Assume that the genus \( \mathcal{G} \) consists of even \( 2k \)-dimensional lattices of level \( l \). For primes \( p \) not dividing \( l \) we define \( T(p) : \mathcal{V} \to \mathcal{V} \) by

\[
T(p)([\Lambda]) := \sum_{E} \left[ \frac{1}{p}\Gamma(E) \right]
\]

where the sum runs over all \( k \)-dimensional totally isotropic subspaces of \( \Lambda/p\Lambda, Q_p \). Note that \( T(p) \) is 0 if \( \Lambda/p\Lambda, Q_p \) is not isomorphic to the sum of \( k \) hyperbolic planes.

The following operators commute with the \( T(p) \) and are usually easier to calculate using the Kneser neighbouring-method (see [9]): For a prime \( p \) define the linear operator \( K_p \) by

\[
K_p([\Lambda]) := \sum_{\Gamma} [\Gamma], \text{ for all } \Lambda \in \mathcal{G}
\]
where the sum runs over all lattices $\Gamma$ in $G$ such that the intersection $\Lambda \cap \Gamma$ has index $p$ in $\Lambda$ and in $\Gamma$. If $p$ does not divide the level $l$ \cite[Proposition 1.10]{22} shows that the operators $K_p$ are essentially the Hecke operators $T_i^{(m-1)(p^2)}$ (up to a summand, which is a multiple of the identity and a scalar factor). Also if $p$ divides $l$, the operators $K_p$ are self-adjoint: For $\Lambda$ and $\Gamma$ in $G$, the number $n(\Gamma, [\Lambda])$ of neighbours of $\Gamma$ that are isometric to $\Lambda$ equals the number of rational matrices $X \in \text{GL}_{2k}(\mathbb{Z}) \text{diag}(p^{-1}, 1^{2k-1}, p) \text{GL}_{2k}(\mathbb{Z})$ solving

$$I(\Gamma, \Lambda) : \quad X F_{\Gamma} X^{tr} = F_{\Lambda}$$

(where $F_{\Gamma}$ and $F_{\Lambda}$ denote fixed Gram matrices of $\Gamma$ respectively $\Lambda$) divided by the order of the automorphism group of $\Lambda$ (since one only counts lattices, $X$ and $gX$ have to be identified for all $g \in \text{GL}_{2k}(\mathbb{Z})$ with $gF_{\Lambda}g^{tr} = F_{\Lambda}$). Mapping $X$ to $X^{-1}$ gives a bijection between the set of solutions of $I(\Gamma, \Lambda)$ and $I(\Lambda, \Gamma)$. Therefore

$$n(\Gamma, [\Lambda]) | \text{Aut}(\Lambda)| = n(\Lambda, [\Gamma]) | \text{Aut}(\Gamma)|.$$

Hence the linear operators $K_p$ and $T(p)$ generate a commutative subalgebra

$$\mathcal{H} := \langle T(q), K_p \mid q, p \text{ primes} , q/l \rangle \leq \text{End}^s(\mathcal{V})$$

of the space of self-adjoint endomorphisms of $\mathcal{V}$ and $\mathcal{V}$ has an orthogonal basis $(d_1, \ldots, d_k)$, consisting of common eigenvectors of $\mathcal{H}$.

For each $1 \leq i \leq h$ we define $v(i) \in \{-1, \ldots, 2k - 1\}$ by $d_i \in \mathcal{V}_{v(i)}$, $d_i \not\in \mathcal{V}_{v(i)+1}$. Analogously let $w(i) \in \{0, \ldots, 2k\}$ be defined by $d_i \in \mathcal{W}_{w(i)}$, $d_i \not\in \mathcal{W}_{w(i)-1}$.

**Lemma 2.4 (\cite[Lemma 2.5]{23})** Let $1 \leq i \leq h$ and assume that $d_i$ generates a full eigenspace of $\mathcal{H}$. Then $w(i) = v(i) + 1$.

If the genus $G$ is strongly modular of level $l$, by which we mean that $\sqrt{d}\Lambda^{#,d} \in G$ for all $\Lambda \in G$ and all exact divisors $d$ of $l$, then the Atkin-Lehner involutions

$$W_d : [\Lambda] \mapsto [\sqrt{d}\Lambda^{#,d}]$$

for exact divisors $d$ of $l$ define further self-adjoint linear operators on $\mathcal{V}$. In this case let

$$\hat{\mathcal{H}} := \langle \mathcal{H}, W_d \mid d \text{ exact divisor of } l \rangle.$$
3 Results

The explicit calculations are performed in MAGMA (12). Fix \( l \in \mathcal{N} \), let \( \mathcal{G} := \mathcal{G}(\Gamma_1) \), \( \mathcal{V} = \mathcal{V}(\mathcal{G}) \) and denote by \( \Lambda_1 := \Gamma_1, \Lambda_2, \ldots, \Lambda_l \) representatives of the isometry classes of lattices in \( \mathcal{G} \). We find that in all cases \( \mathcal{H} = \langle K_2, K_3 \rangle \cong \mathbb{C}^h \) is a maximal commutative subalgebra of End(\( \mathcal{V} \)). Therefore the common eigenspaces are of dimension one and it is straightforward to calculate an explicit orthogonal basis \((d_1, \ldots, d_h)\) of \( \mathcal{V} \) consisting of eigenvectors of \( \mathcal{H} \). In particular \( v(i) = w(i) - 1 \) for all \( i = 1, \ldots, h \) by Lemma 2.3. Here we choose \( d_1 := \sum_i^{h} |\mathrm{Aut}(\Lambda_i)|^{-1} [\Lambda_i] \in \mathcal{V}_0 - \mathcal{V}_1 \) to be the unit element of \( \mathcal{V} \) and \( (l \neq 7) \, d_h = \mathrm{Per}(\Gamma_l, p) \in \mathcal{V}_{k_l - 1} \) where \( p \) is the smallest prime not dividing \( l \). We then determine some Fourier-coefficients of the series \( \Theta^{(n)}(d_i) (n = 0, 1, \ldots, k_l) \) to get upper bounds on \( v(i) \). In all cases the degree-\( k_l \) Siegel theta-series of the lattices are linearly independent hence \( \mathcal{V}_{k_l} = \{0\} \).

Moreover \( \mathcal{V}_{k_l - 1} = \langle d_h \rangle \) if \( l \neq 7 \). We also know that \( w(1) = 0 \) and we may choose \( d_2 \) such that \( w(2) = 1 \). By Lemma 2.2 and 2.4 the product \( d_j \circ d_l \) lies in \( \mathcal{V}_{w(i) + w(j)} \). If the coefficient of \( d_h \) in the product is non-zero, this yields lower bounds on the sum \( w(i) + w(j) \) which often yield sharp lower bound for \( w(i) \) and \( w(j) \). The method is illustrated in [13, Section 3.2] and an example is given in Paragraph 3.1.

3.1 The genus of the Barnes-Wall lattice in dimension 16.

The lattices in this genus are given in [18]. The class number is \( h = 24 \) and we find

\[ \langle K_2, K_3 \rangle = \mathcal{H}_Q \cong \mathbb{Q}^{13} \oplus F_1 \oplus F_2 \oplus F_3 \]

where the totally real number fields \( F_i \cong \mathbb{Q}[x]/(f_i(x)) \) are given by

\[
\begin{align*}
 f_1 &= x^3 - 11496x^2 + 41722560x - 47249837568, \\
 f_2 &= x^3 - 1704x^2 + 400320x + 173836800, \\
 f_3 &= x^5 - 11544x^4 + 42868800x^3 - 53956108800x^2 + 1813238784000x + 20094119608320000
\end{align*}
\]

and \( \langle K_2, K_3, W_2 \rangle = \mathcal{H}_Q \cong \mathbb{Q}^{13} \oplus \text{Mat}_3(\mathbb{Q}) \oplus \text{Mat}_3(\mathbb{Q}) \oplus \text{Mat}_5(\mathbb{Q}) \).

Let \( \alpha_i, \beta_i \) and \( \gamma_j \) \((i = 1, \ldots, 3, j = 1, \ldots, 5)\) denote the complex roots of the polynomials \( f_1, f_2 \) respectively \( f_3 \). Let \( \epsilon_i \) \((i = 1, \ldots, 3)\) denote the primitive idempotents of \( \mathcal{H}_Q \) with \( \mathcal{H}_Q \epsilon_i \cong F_i \).

Since the image of \( \theta_Q \) under \( \Theta^{(n)} \) has rational Fourier-coefficients, the functions \( v \) and \( w \) are constant on the eigenspaces \( E_i = \mathcal{V} \epsilon_i \) \((i = 1, 2, 3)\). We therefore give their values in one line in the following tabular:

**Theorem 3.1** The functions \( v \) and the eigenvalues of \( ev_2 \) and \( ev_3 \) of \( K_2 \) respectively \( K_3 \) on \( (d_1, \ldots, d_{24}) \) are as follows:

| \( i \) | \( v(i) \) | \( ev_2 \) | \( ev_3 \) | \( i \) | \( v(i) \) | \( ev_2 \) | \( ev_3 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | -1  | 34560 | 7176640 | 15 | 3, 4 | 1320 | 8640 |
| 2   | 0   | 16200 | 2389440 | \( E_2 \) | 4 | \( \beta_j \) | 31680 |
| 3   | 1   | 8760  | 792000  | 19 | 3, 4, 5 | 1080 | -45120 |
| 4   | 1   | 7128  | 804288  | 20 | 3, 4, 5 | 312 | 4032 |
| \( E_1 \) | 2 | \( \alpha_j \) | 266688 | 21 | 5 | -216 | 8640 |
| 8   | 3   | 2664  | 90048   | 22 | 5 | -216 | 20928 |
| 9   | 3   | 1320  | 77760   | 23 | 6 | -936 | 13248 |
| \( E_3 \) | 3 | \( \gamma_j \) | 100800 | 24 | 7 | -2160 | 39360 |
For the dimensions of $D_v$ one finds

\[
\begin{array}{ccccccccc}
 v & -1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 \dim(D_v) & 1 & 1 & 2 & 3 & 7-10 & 3-5 & 2-4 & 1 \\
\end{array}
\]

**Proof.** By explicit calculations of the Fourier-coefficients the values given in the table are upper bounds for the $v(i)$. By Lemma 2.4 they also provide upper bounds on the $w(i) = v(i) + 1$.

We see that

\[d_i \circ d_j = A_{ij}d_{24} + \sum_{m=1}^{23} b_{ij}^m d_m\]

with a nonzero coefficient $A_{ij}$ for the following pairs $(i, j)$:

$(23, 2), (22, 3), (21, 4), (E_1, E_2), (E_3, E_3), (8, 8), (9, 9)$

(where $(E_1, E_2)$ means that there is some vector in $E_1$ and some in $E_2$ such that this coefficient is non-zero, similarly $(E_3, E_3)$). Since $d_{m} \in W_7$ for all $m \leq 23$ and $d_j \circ d_i \in W_{w(i)+w(j)}$ the inequality $w(i) + w(j) \leq 7$ together with $A_{ij} \neq 0$ implies that $d_{24} \in W_7$ which is a contradiction. Hence $w(i) + w(j) \geq 8$ for all pairs $(i, j)$ above. This yields equality for all values $v(i)$ and $v(j)$ for these pairs. Similarly we get $3 \leq v(i)$ for $i = 15, 19, 20$ since $A_{i,i} \neq 0$ for these $i$.

**Conjecture 3.2** $v(19) = 5$ and $v(20) = 5$.

Since $d_{15} \circ d_2 = \sum_{m=1}^{18} c_m d_m + A_1 d_{19} + A_2 d_{20}$ with $A_1 \neq 0 \neq A_2$, we get $w(15) + 1 \geq \max(w(19), w(20))$.

**Remark 3.3** If the conjecture is true, then $v(15) = 4$ and $\dim(D_3) = 7$, $\dim(D_4) = 4$, and $\dim(D_5) = 4$.

Using the formula in [11] Korollar 3 (resp. 22 Proposition 1.9]) we may calculate the eigenvalues of $T^{(n-1)}(3^2)$ from the one of $K_3$ and compare them with the ones given in [6] formula (7). The result suggests that $\Theta^{(2)}(d_4)$, $\Theta^{(3)}(v)$ (for some $v \in E_3$), $\Theta^{(6)}(d_{19})$ and $\Theta^{(8)}(d_{24})$ are generalized Duke-Imamoglu-Ikeda-lifts (cf. 8) of the elliptic cusp forms $\delta_8 \theta_{D_4}^i$ ($i=3,2,1,0$) where $\delta_8 = \frac{1}{10} (\theta_{D_4}^4 - \theta_{D_4})$ is the cusp form of $\Gamma_0(2)$ of weight 8 and $\theta_{D_4}$ the theta series of the 4-dimensional 2-modular root lattice $D_4$. This would imply that $v(19) = 5$ and, with Lemma 2.2, $v(15) = 4$.

### 3.2 The genus of the Coxeter-Todd lattice in dimension 12.

For $l = 3$ one has $h = 10$, all lattices in this genus are modular, and $H_Q = \langle K_2 \rangle \cong \mathbb{Q}^{10} = \hat{H}_Q$.

**Theorem 3.4** There is some $a \in \{0, 1\}$ such that the function $v$ and the eigenvalues $ev_2$ of $K_2$ and $e_2$ of $T(2)$ are as follows:

| $i$ | $v(i)$ | $ev_2$ | $e_2$ | $i$ | $v(i)$ | $ev_2$ | $e_2$ |
|-----|--------|--------|-------|-----|--------|--------|-------|
| 1   | -1     | 2079   | 151470| 6   | 3 - a  | 234    | 7560  |
| 2   | 0      | 1026   | -27540| 7   | 3      | 126    | 2376  |
| 3   | 1      | 594    | 17820 | 8   | 3      | -36    | 432   |
| 4   | 1      | 432    | 3240  | 9   | 4      | -144   | -864  |
| 5   | 2      | 288    | -5400 | 10  | 5      | -378   | 1944  |
For the dimensions of $D_v$ one finds

| $v$ | $\dim(D_v)$ |
|-----|-------------|
| $-1$ | 1           |
| $0$  | 1           |
| $1$  | 2           |
| $2$  | $1+a$       |
| $3$  | $3-a$       |
| $4$  | 1           |
| $5$  | 1           |

We conjecture that $a = 0$ but cannot prove this using Lemma 2.2.

The eigenvalues of $T(2)$ suggest that $\Theta(2)(d_3)$, $\Theta(4)(d_6)$ and $\Theta(6)(d_{10})$ are generalized Duke-Imamoglu-Ikeda-lifts (cf. [3]) of the elliptic cusp forms $\delta_6$, $\delta_6\theta_A^2$, respectively $\delta_0$, where $\delta_0 = \frac{1}{36}(\theta_A^6 - \theta_{15}^2)$ is the cusp form of $\Gamma_0(3)$ of weight 6 and $\theta_A^2$ the theta series of the hexagonal lattice $A_2$. This would imply $v(3) = 1$, $v(6) = 3$ and $v(10) = 5$ and hence $a = 0$.

3.3 The genus of the 5-modular lattices in dimension 8.

The class number of this genus is $h = 5$, all lattices in this genus are modular, and $H = \langle K_2 \rangle \cong \mathbb{Q}^5 = \hat{\mathcal{H}}_Q$.

**Theorem 3.5** For $l = 5$ one has $\dim(D_v) = 1$ for $v = -1, 0, 1, 2, 3$. The function $v$ and the eigenvalues $ev_2$ of $K_2$ and $ev_p$ of $T(p)$ ($p = 2, 3$) are given in the following table:

| $i$ | $v(i)$ | $ev_2$ | $ev_3$ | $i$ | $v(i)$ | $ev_2$ | $ev_3$ |
|-----|--------|--------|--------|-----|--------|--------|--------|
| 1   | $-1$   | 135    | 2240   | 2   | 0      | 70     | -120   |
| 2   | 0      | 70     | 160    | 3   | 1      | 42     | 84     |

3.4 The genus of the strongly 6-modular lattices in dimension 8.

The class number of $G(\Gamma_6)$ is $h = 8$, the Hecke-algebras are $\hat{\mathcal{H}}_Q = \langle K_2, W_2 \rangle \cong \mathbb{Q}^5 + \text{Mat}_3(\mathbb{Q})$ and $H = \langle K_2 \rangle \cong \mathbb{Q}^5 + \mathbb{Q}[x]/(f(x))$ where

$$f(x) = x^3 - 66x^2 - 216x + 31104.$$

Let $\delta_i \in \mathbb{R}$ ($i = 1, 2, 3$) denote the roots of $f$.

**Theorem 3.6** Then the function $v$ and the eigenvalues $ev_2$ of $K_2$ and $ev_5$ of $T(5)$ are given in the following table:

| $i$ | $v(i)$ | $ev_2$ | $ev_5$ | $i$ | $v(i)$ | $\delta_j$ |
|-----|--------|--------|--------|-----|--------|-----------|
| 1   | $-1$   | 144    | 39312  | $E$ | 1      | 1872      |
| 2   | 0      | 54     | 1872   | 7   | 2      | -6        |
| 3   | 1      | 18     | 1008   | 8   | 3      | -36       |

Hence $\dim(D_v) = 1$ for $v = -1, 0, 2, 3$ and $\dim(D_1) = 4$.

3.5 The genus of the 7-modular lattices in dimension 6.

The class number is $h = 3$, all lattices are modular, and $\hat{\mathcal{H}}_Q = \mathcal{H}_Q = \langle K_2 \rangle \cong \mathbb{Q}^3$. In contrast to the other genera, the perestroika $\text{Per}(\Gamma_7, 2)$ and hence also $\text{BFW}(\Gamma_7, 2)$ vanishes due to the fact that the image of $\text{Aut}(\Gamma_7)$ in $GO_7^+(2)$ is not contained in the derived subgroup $O_7^+(2)$. In fact, $\Theta(2)$ is already injective. Since the discriminant of the space is not a square modulo 3 and 5, the Hecke operators $T(3)$ and $T(5)$ vanish.

**Theorem 3.7** We have $v(i) = i - 2$ for $i = 1, 2, 3$ and hence $\dim(D_v) = 1$ for $v = -1, 0, 1$. The eigenvalues of $K_2$ are 35, 19, and 5, the ones of $T(2)$ are 30, -18, and 10, and $T(11)$ has eigenvalues 2928, -144, and 248.
3.6 The genus of the strongly $l$-modular lattices in dimension 4 for $l = 11, 14, 15$.

For $l = 11, 14, 15$ the genus $G(\Gamma_l)$ consists of 3 isometry classes and $\mathcal{H}_Q = \langle K_2 \rangle \cong \mathbb{Q}^3 = \hat{\mathcal{H}}_Q$ since all lattices in the genus are strongly modular.

**Theorem 3.8** For $l = 11, 14, 15$ one has $\dim(D_v) = 1$ for $v = -1, 0, 1$. The eigenvalues $ev_2$ of $K_2$ and $e_p$ of $T(p)$ for primes $p \leq 7$ not dividing $l$ are given in the following table:

| $i$ | $v(i)$ | $l = 11$ | $l = 14$ | $l = 15$ |
|-----|--------|---------|---------|---------|
|     |        | $e_2$   | $e_3$   | $e_5$   | $e_7$   |
| 1   | -1     | 9       | 6       | 12      | 16      |
| 2   | 0      | 4       | -4      | -2      | 2       |
| 3   | 1      | -6      | 1       | 3       | 7       |
|     |        | $ev_2$  | $e_3$   | $e_5$   | $e_7$   |
| 1   | -1     | 8       | 8       | 12      | 9       |
| 2   | 0      | 2       | -4      | 0       | 1       |
| 3   | 1      | -4      | 2       | 6       | -3      |

3.7 The genus of the 23-modular lattices in dimension 2.

In the smallest possible dimension 2 the genus $G(\Gamma_{23})$ consists of only 2 isometry classes and $\mathcal{H}_Q = \langle K_2 \rangle \cong \mathbb{Q}^2 = \hat{\mathcal{H}}_Q$ for the same argument that all lattices in the genus are modular.

**Theorem 3.9** For $l = 23$ one has $\dim(D_v) = 1$ for $v = -1, 0$. One has $v(1) = -1$, $v(2) = 0$, $d_1K_2 = 2d_1$ and $d_2K_2 = -d_2$. For the $T(p)$ for primes $p < 23$ we find $T(2) = T(3) = T(13) = K_2$ and $T(5) = T(7) = T(11) = T(17) = T(19) = 0$.

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