From Clock Synchronization to Dark Matter as a Relativistic Inertial Effect

Luca Lusanna

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Sezione INFN di Firenze, Polo Scientifico, Via Sansone 1, 50019 Sesto Fiorentino (FI), Italy lusanna@fi.infn.it
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1 Introduction

One of the main open problems in astrophysics is the dominance of dark entities, the dark matter and the dark energy, in the existing description of the universe given by the standard $\Lambda$CDM cosmological model [1] based on the cosmological principle (homogeneity and isotropy of the space-time), which selects the class of Friedmann-Robertson-Walker (FWR) space-times. After the transition from quantum cosmology to classical astrophysics, with the Heisenberg cut roughly located at a suitable cosmic time ($\approx 10^5$ years after the big bang) and at the recombination surface identified by the cosmic microwave background (CMB), one has a description of the universe in which the known forms of baryonic matter and radiation contribute only with a few percents of the global budget. One has a great variety of models trying to explain the composition of the universe in accelerated expansion (based on data on high red-shift supernovae, galaxy clusters and CMB): WIMPS (mainly super-symmetric particles), $f(R)$ modifications of Einstein gravity (with a modified Newton potential), MOND (with a modification of Newton law), for dark matter; cosmological constant, string theory, back-reaction (spatial averages, non-linearity of Einstein equations), inhomogeneous space-times (Lemaître-Tolman-Bondi, Szekeres), scalar fields (quintessence, k-essence, phantom), fluids (Chaplygin fluid), ... for dark energy.

Most of these developments rest on a description based on a family of FRW space-times with nearly flat 3-spaces (as required by CMB data) as the reference space-times where to interpret the astronomical data (luminosity, light spectrum, angles) on the 2-dimensional sky vault. Therefore, the starting point is the extension of the standards of relativistic metrology near the Earth and in the Solar System to astronomy: to reconstruct a 4-dimensional space-time one needs new standards of time and length like the cosmic time and the luminosity distance (or any other astrometric definition, see Ref.[2]) allowing to define an International Celestial Reference System (ICRS) [3], namely a 4-coordinate system describing a 3-universe evolving in time, where the astronomical data have to be dynamically interpreted according to Einstein gravity or some of its extensions.

The aim of this Lecture is to suggest a new viewpoint on the origin of dark matter, and maybe also of dark energy, starting from a re-reading of the general covariance of Einstein general relativity (GR), which could be also applied to every generally covariant extension of this theory if needed. It is an extended version of the review paper [4]. In this Introduction one will delineate the framework of our approach and then in the subsequent Sections one will give more details of the various topics.

The gauge group of the Lagrangian formulation of Einstein GR, the diffeomorphism group, implies that the 4-coordinates of the space-time are gauge
variables. As a consequence, the search of GR observables is restricted to 4-scalars and at the theoretical level one tries to describe gravitational dynamical properties in term of them. However, inside the Solar System the experimental localization of macroscopic classical objects is unavoidably done by choosing some convention for the local 4-coordinates of space-time. Atomic physicists, NASA engineers and astronomers have chosen a series of reference frames and standards of time and length suitable for the existing technology [5]. These conventions determine certain Post-Minkowskian (PM) 4-coordinate systems of an asymptotically Minkowskian space-time, in which the instantaneous 3-spaces are not strictly Euclidean. Then these reference frames are seen as a local approximation of a reference frame in ICRS, where however the space-time has become a cosmological FWR one, which is only conformally asymptotically Minkowskian at spatial infinity. A search of a consistent patching of the 4-coordinates from inside the Solar System to the rest of the universe will start when the data from the future GAIA mission [6] for the cartography of the Milky Way will be available. This will allow a PM definition of a Galactic Reference System containing at least our galaxy. Let us remark that notwithstanding the FRW instantaneous 3-spaces are not strictly Euclidean, all the books on galaxy dynamics describe the galaxies by means of Kepler theory in Galilei space-time.

This state of affairs requires to revisit Einstein GR to see whether it is possible to identify which components of the 4-metric tensor are connected with the gauge freedom in the choice of the 4-coordinates and which ones describe the dynamical degrees of freedom of the gravitational field. Since this cannot be done at the Lagrangian level, one must restrict himself to the class of globally hyperbolic, asymptotically flat space-times allowing a Hamiltonian description starting from the description of Einstein GR in terms of the ADM action [7] instead than in terms of the Einstein-Hilbert one. In canonical ADM gravity one can use Dirac theory of constraints [8] to describe the Hamiltonian gauge group, whose generators are the first-class constraints of the model. The basic tool of this approach is the possibility to find so-called Shanmugadhasan canonical transformations [9], which identify special canonical bases adapted to the first-class constraints (and also to the second-class ones when present). In these special canonical bases the vanishing of certain momenta (or of certain configurational coordinates) corresponds to the vanishing of well defined Abelianized combinations of the first-class constraints (Abelianized because the new constraints have exactly zero Poisson brackets even if the original constraints were not in strong involution). As a consequence, the variables conjugate to these Abelianized constraints are inertial Hamiltonian gauge variables describing the Hamiltonian gauge freedom. The remaining 2+2 conjugate variables describe the dynamical tidal degrees of freedom of the gravitational field (the two polarizations of gravitational waves in the linearized theory). If one would be able to include all the constraints in the Shanmugadhasan canonical basis, these 2+2 variables
would be the Dirac observables of the gravitational field, invariant under the Hamiltonian gauge transformations. However such Dirac observables are not known; one only has statements about their existence [10]. Moreover, in general they are not 4-scalar observables. The problem of the connection between the 4-diffeomorphism group and the Hamiltonian gauge group was studied in Ref.[11] by means of the inverse Legendre transformation and of the notion of dynamical symmetry. The conclusion is that on the space of solutions of Einstein equations there is an overlap of the two types of observables: there should exist special Shanmugadhassan canonical bases in which the 2+2 Dirac observables become 4-scalars when restricted to the space of solutions of the Einstein equations. In any case the identification of the inertial gauge components of the 4-metric is what is needed to make a fixation of 4-coordinates as required by relativistic metrology.

Another problem is that asymptotically flat space-times have the SPI group of asymptotic symmetries (direction-dependent asymptotic Killing symmetries) [12] and this is an obstruction to the existence of asymptotic Lorentz generators for the gravitational field [13]. However if one restricts the class of space-times to those not containing super-translations [14], then the SPI group reduces to the asymptotic ADM Poincaré group [15]: these space-times are asymptotically Minkowskian, they contain an asymptotic Minkowski 4-metric (to be used as an asymptotic background at spatial infinity in the linearization of the theory) and they have asymptotic inertial observers at spatial infinity whose spatial axes may be identified by means of the fixed stars of star catalogues 1. Moreover, in the limit of vanishing Newton constant ($G = 0$) the asymptotic ADM Poincaré generators become the generators of the special relativistic Poincaré group describing the matter present in the space-time. This is an important condition for the inclusion into GR of the classical version of the standard model of particle physics, whose properties are all connected with the representations of this group in the inertial frames of Minkowski space-time. In absence of matter a sub-class of these space-times is the (singularity-free) family of Christodoulou-Klainermann solutions of Einstein equations [16] (they are near to Minkowski space-time in a norm sense and contain gravitational waves).

Moreover, in this restricted class of space-times the canonical Hamiltonian is the ADM energy [17], so that there is no frozen picture like in the "spatially compact space-times without boundaries" used in loop quantum gravity 2.

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1 The fixed stars can be considered as an empirical definition of spatial infinity of the observable universe.

2 In these space-times the canonical Hamiltonian vanishes and the Dirac Hamiltonian is a combination of first-class constraints, so that it only generates Hamiltonian gauge transformations. In the reduced phase space, quotient with respect the Hamiltonian gauge group, the reduced Hamiltonian is zero and one has a frozen picture of dynamics. This class of space-times fits well with Machian ideas (no boundary conditions) and with interpretations...
To take into account the fermion fields present in the standard particle model one must extend ADM gravity to ADM tetrad gravity. Since our class of space-times admits orthonormal tetrads and a spinor structure [19], the extension can be done by simply replacing the 4-metric in the ADM action with its expression in terms of tetrad fields, considered as the basic 16 configurational variables substituting the 10 metric fields.

To study ADM tetrad gravity the preliminary problem is to choose a coordinatization of the space-time compatible with relativistic metrology. This requires a definition of global non-inertial frames, because the equivalence principle forbids the existence of global inertial frames in GR. Due to the Lorentz signature of the space-time this is a non-trivial task already in special relativity (SR): there is no notion of instantaneous 3-space, because the only intrinsic structure is the conformal one, i.e. the light-cone as the locus of incoming and outgoing radiation. The existing coordinatizations, like either Fermi or Riemann-normal coordinates, hold only locally. They are based on the 1+3 point of view, in which only the world-line of a time-like observer is given. In each point of the world-line the observer 4-velocity determines an orthogonal 3-dimensional space-like tangent hyper-plane, which is identified with an instantaneous 3-space. However, these tangent planes intersect at a certain distance from the world-line (the so-called acceleration length depending upon the 4-acceleration of the observer [20]), where 4-coordinates of the Fermi type develop a coordinate singularity. Another type of coordinate singularity is developed in rigidly rotating coordinate systems at a distance \( r \) from the rotation axis where \( \omega r = c \) (\( \omega \) is the angular velocity and \( c \) the two-way velocity of light). This is the so-called "horizon problem of the rotating disk": a time-like 4-velocity becomes a null vector at \( \omega r = c \), like it happens on the horizon of a black-hole. See Ref.[21] for a classification of the possible pathologies of non-inertial frames and on how to avoid them.

In this Lecture one will review the way out from these problems based on the 3+1 point of view in which, besides the world-line of a time-like observer, one gives a global nice foliation of the space-time with instantaneous 3-spaces. Then a metrology-oriented notion of 4-coordinates, the so-called radar 4-coordinates first introduced by Bondi [22], is introduced in these global non-inertial frames. One will give the conditions for a foliation to be nice, i.e. for the absence of pathologies like the ones of the rotating disk and of the Fermi coordinates.

Let us remark that the theory of global non-inertial frames is also needed to speak of predictability in a (either classical or quantum) theory in which the basic equations of motion are partial differential equations (PDE). To be able to use the existence and unicity theorem for the solutions of PDE’s, one in which there is no physical time like the one in Ref.[18]. However, it is not clear how to include in this framework the standard model of particle physics.
needs a well-posed Cauchy problem, whose prerequisite is a sound definition of an instantaneous 3-space (i.e. of a clock synchronization convention) where the Cauchy data are given. To give the data on a space-like surface is not factual, but with the data on the backward light-cone of an observer it is not yet possible to demonstrate the theorem. However, also the 1+3 point of view is non factual, because it requires the knowledge of a world-line from the whole past to all the future.

A Section of this Lecture will be devoted to the developments in relativistic particle mechanics made possible by the 3+1 point of view in SR [23], [21], [24]. By means of parametrized Minkowski theories [23], [21], one can get the description of arbitrary isolated systems (particles, strings, fluids, fields) admitting a Lagrangian formulation in arbitrary non-inertial frames with the transition among non-inertial frames described as a "gauge transformation" (general covariance under the frame-preserving diffeomorphisms of Ref.[25]). Moreover this framework allows us to define the inertial and non-inertial rest frames of the isolated systems, where to develop the rest-frame instant form of the dynamics and to build the explicit form of the Lorentz boosts for interacting systems. This makes possible to study the problem of the relativistic center of mass [26], relativistic bound states [27, 28, 29], relativistic kinetic theory and relativistic micro-canonical ensemble [30] and various other systems [31, 32]. Moreover a Wigner-covariant relativistic quantum mechanics [33], with a solution of all the known problems introduced by SR, has been developed after some preliminary work done in Ref.[34]. This will allow us to study relativistic entanglement.

After this digression in SR one defines global non-inertial frames with radar 4-coordinates in the asymptotically Minkowskian space-times of GR and one gives the parametrization of the tetrads and of the 4-metric in them. The absence of super-translations implies that these non-inertial frames are non-inertial rest frames of the 3-universe. Starting from the ADM action for tetrad gravity one defines the Hamiltonian formalism in a phase space containing 16 configurational field variables and 16 conjugate moments. One identifies the 14 first-class constraints of the system and one finds that the canonical Hamiltonian is the weak ADM energy (it is given as a volume integral over 3-space). The existence of these 14 first-class constraints implies that 14 components of the tetrads (or of the conjugate momenta) are Hamiltonian gauge variables describing the inertial aspects of the gravitational field (6 of these inertial variables describe the extra gauge freedom in the choice of the tetrads and in their transport along world-lines). Therefore there are only 2+2 degrees of freedom for the description of the tidal dynamical aspects of the gravitational field. The asymptotic ADM Poincaré generators

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3 While in SR Minkowski space-time is an absolute notion, unifying the absolute notions of time and 3-space of the non-relativistic Galilei space-time, in GR there is no absolute notion: space-time becomes dynamical [35] with its metric structure satisfying Einstein equations.
can be evaluated explicitly. Till now the type of matter studied in this framework consists of the electro-magnetic field and of $N$ charged scalar particles, whose signs of the energy and electric charges are Grassmann-valued to regularize both the gravitational and electro-magnetic self-energies (it is both an ultraviolet and an infrared regularization),

Then it will be shown that there is a Shanmugadhasan canonical transformation [36] (implementing the so-called York map [37] and diagonalizing the York-Lichnerowics approach [38]) to a so-called York canonical basis adapted to 10 of the 14 first-class constraints. Only the super-Hamiltonian and super-momentum constraints, whose general solution is not known, are not included in the basis, but it is clarified which variables are to be determined by their solution. Among the inertial gauge variables there is the York time [39] $^3K$, i.e. the trace of the extrinsic curvature of the 3-spaces as 3-manifolds embedded into the space-time. It is the only gauge variable which is a momentum in the York canonical basis $^4$: this is due to the Lorentz signature of space-time, because the York time and three other inertial gauge variables can be used as 4-coordinates of the space-time (see Ref.[35] for this topic and for its relevance in the solution of the hole argument). Therefore an identification of the inertial gauge variables to be fixed to get a 4-coordinate system in relativistic metrology was found. In the first paper of Ref.[40] there is the expression of the Hamilton equations for all the variables of the York canonical basis.

An important remark is that in the framework of the York canonical basis the natural family of gauges is not the harmonic one, but the family of 3-orthogonal Schwinger time gauges in which the 3-metric in the 3-spaces is diagonal.

Both in SR and GR an admissible 3+1 splitting of space-time has two associated congruences of time-like observers [21], geometrically defined and not to be confused with the congruence of the world-lines of fluid elements, when relativistic fluids are added as matter in GR [41, 42, 43]. One of the two congruences, with zero vorticity, is the congruence of the Eulerian observers, whose 4-velocity field is the field of unit normals to the 3-spaces. This congruence allows us to re-express the non-vanishing momenta of the York canonical basis in terms of the expansion ($\theta = -^3K$) and of the shear of the Eulerian observers. This allows us to compare the Hamilton equations of ADM canonical gravity with the usual first-order non-Hamiltonian ADM equations deducible from Einstein equations given a 3+1 splitting of space-time but without using the Hamiltonian formalism. As a consequence, one can extend our Hamiltonian identification of the inertial and tidal variables of the gravitational field to the Lagrangian framework and use it in the cosmological (conformally asymptotically flat) space-times: in them it is not possible to formulate the Hamiltonian formalism but the standard ADM

$^4$ Instead in Yang-Mills theory all the gauge variables are configurational.
equations are well defined. The time inertial gauge variable needed for relativistic metrology is now the expansion of the Eulerian observers of the given 3+1 splitting of the globally hyperbolic cosmological space-time.

The next step (see the second paper of Ref.[40]) is the definition of a PM linearization of ADM tetrad gravity in the family of 3-orthogonal Schwinger time gauges in which one chooses 3-coordinates diagonalizing the 3-metric in the 3-spaces and an arbitrarily given numerical function for the York time $^3K$. The cosmic time $\tau_{\text{cosm}}$ has to be chosen so that the 3-spaces $\tau_{\text{cosm}} = \text{const.}$ have an extrinsic curvature with the given value of $^3K$. This PM linearization uses the asymptotic Minkowski 4-metric as an asymptotic background, so that one never splits the 4-metric with respect to a fixed Minkowski metric in the bulk like in the standard approach to gravitational waves. A ultraviolet cutoff on the matter is needed.

This leads to a PM formulation of gravitational waves in non-harmonic 3-orthogonal gauges. All the constraints can be solved, an explicit expression of the PM 4-metric can be given and the explicit form of the Hamilton equations for the tidal degrees of freedom of the gravitational field and the matter can be obtained. It is non-trivial to show that all the standard results about gravitational waves in harmonic gauges [44] can be reproduced in the 3-orthogonal gauges with the help of the formalism of Ref.[45]. As shown in the third paper of Ref.[40] (where the matter is restricted only to scalar particles), all the 4- and 3-curvature tensors of the space-time can be explicitly evaluated and the time-like and null geodesics can be studied. It is also possible to evaluate the red-shift of light rays and the luminosity distance finding their dependence on the York time and verify the old Hubble red-shift-distance law (see ref.[46]), which becomes the usual Hubble law (a velocity-distance relation) when one uses the standard cosmological model. In the Solar System the results in the 3-orthogonal gauges are compatible with the ones in the harmonic gauges used in relativistic metrology [5].

The main important result or this lecture are the PM Hamilton equations and the implied PM second-order equations of motion for the particles. Their PN limit identifies the Newton forces acting on the particles at the lowest order augmented with 1PN forces compatible with the known results on binaries in harmonic gauges [47]. However, there are extra 0.5PN forces, depending linearly on the non-local York time $\triangle^3K = \frac{1}{\triangle}^3K$ ($\triangle$ is the asymptotic Laplacian of the 3-space), representing either a friction or an anti-friction force according to the sign of $^3K$. These 0.5PN forces are our main result, because their effect can be re-interpreted as an extra effective (time-, position- and velocity-dependent) contribution to the inertial mass of the particles in the equations used in the three main signatures for the existence of dark matter: the rotation curves of spiral galaxies [48] and the masses of clusters of galaxies from the virial theorem [49] and from weak gravitational lensing [50], [49].
While gravitational and inertial masses are equal in Einstein GR, the PM limit, followed by the PN one, shows that the non-Euclidean nature of the 3-spaces implies a breaking of the Newtonian equality of the two types of masses, which holds only in the absolute Euclidean 3-space of Galilei space-time.

As a consequence the data on dark matter can be re-read as a partial fixation of the non-local York time $^{\Delta}^3K$. However to fix the York time $^3K = \Delta^3K$ one needs a global information on $^3K$ on the whole 3-space, in particular in the voids among galaxy clusters.

Therefore one has an indication that (at least part of) dark matter could be re-absorbed in a PM extension of the conventions in the existing ICRS, such that the 3-spaces $\tau_{ICRS} = const.$ determined by a suitable ICRS time have a York time $^3K$ such that the derived non-local York time reproduces the data for the signatures of dark matter.

In the Conclusions it will be suggested that also the open problem of dark energy could be rephrased as the determination of a suitable York time in inhomogeneous cosmological space-times. Therefore there is the possibility of an understanding of the ”dark” aspects of the universe in terms of relativistic metrology.
2 Relativistic Metrology

As shown in Ref. [51] modern relativistic metrology is not only deeply rooted in Maxwell theory and its quantization but is also beginning to take into account GR.

The basic metrological conventions on the Earth surface are:

a) An atomic clock as a standard of time. The fundamental conceptual time scale is the \textit{SI atomic second}: it is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom. This definition refers to a cesium atom at rest at a temperature of 0°. However the practical time standard is the International Atomic Time (TAI), which is defined as a suitable weighted average of the SI time kept by over 200 atomic clocks in about 70 national laboratories worldwide. Time scales connected with TAI are the GPS Station Time and the Universal Time (UC) based on Earth’s rotation \(^5\). All the other existing time scales inside the Solar System are connected to this standard by fixed conventions.

b) The 2-way velocity of light (only one clock is involved in its definition), fixed to the value \(c = 299 792 458 \text{ m s}^{-1}\), in place of the standard of length \(^6\). To measure the 3-distance between two objects in an inertial frame one sends a ray of light from the first object, to which is associated an atomic clock, to the second one, where it is reflected and then reabsorbed by the first object. The measure of the flight time and the 2-way velocity of light determine the 3-distance between the objects. This convention is compatible with the Euclidean 3-space of inertial frames in Minkowski space-time. When the technology will allow one to measure the deviations from Euclidean 3-space implied by PN gravity one will need a modified convention taking into account a general relativistic notion of length.

Given these standards one can think to the Global Positioning System (GPS) as a local standard of space-time. To define GPS one needs a conventional reference frame centered on a given time-like observer. Inside the Solar System one has well defined conventions for the following reference frames:

A) The description of satellites around the Earth is done by means of NASA coordinates either in the International Terrestrial Reference System (ITRS; it is a frame fixed on the Earth surface) or in the Geocentric Celestial Reference System (GCRS) centered on the world-line of the Earth center (see

\(^5\) It is based on Very Long Baseline Interferometry (VLBI) observations of distant quasars, on Lunar Laser Ranging (LLR) and on determination of GPS satellite orbits.

\(^6\) The meter is the length of the path traveled by light in vacuum during a time interval of \(1/c\) of a second.
Both of them use a *geocentric coordinate time* $t_G$ connected to TAI.

B) The description of planets and other objects in the Solar System uses the Barycentric Celestial Reference System (BCRS), centered in the barycenter of the Solar System (see Ref.[5]). It uses a *barycentric coordinate time* $t_B$ connected to $t_G$ and TAI.

While ITRS is essentially realized as a non-relativistic non-inertial frame in Galilei space-time, BCRS is defined as a *quasi-inertial frame, non-rotating* with respect to some selected fixed stars, in Minkowski space-time with nearly-Euclidean 3-spaces (one ignores the perturbations induced from the Milky Way). It can also be considered as a PM space-time with 3-spaces having a very small extrinsic curvature of order $c^{-2}$. GCRS is obtained from BCRS by taking into account Earth’s rotation around the Sun with a suitable Lorentz boost with corrections from PN gravity\(^7\). By taking into account the extension of the geoid and Earth revolution around its axis one goes from the quasi-Minkowskian GCRS to the quasi-Galilean ITRS.

New problems emerge by going outside the Solar System. In astronomy the positions of stars and galaxies are determined from the data (luminosity, light spectrum, angles) on the sky, i.e. on a 2-dimensional spherical surface around the Earth with the relations between it and the observatory on the Earth done with GPS. To get a description of stars and galaxies as living in a 4-dimensional space-time one introduces the International Celestial Reference System ICRS (see Refs. [3]). Its time scale is a ”second” connected to GPS, TAI and SI and therefore to $t_G$ and $t_B$. ICRS has the origin in the solar system barycenter, which is considered as quasi-inertial observer carrying a quasi-inertial (essentially non-relativistic) reference frame with rectangular 3-coordinates in a *nearly Galilei space-time* whose 3-spaces are nearly Euclidean. The directions of the spatial axes are effectively defined by the adopted coordinates of 212 extragalactic radio sources observed by VLBI. These radio sources (quasars and AGN, active galactic nuclei) are assumed to have no observable intrinsic angular momentum. Thus, the ICRS is a *space-fixed* system, more precisely a *kinematically non-rotating* system, which provides the orientation of BCRS.

In astronomy the unit of length is the *astronomical unit* AU, approximately equal to the mean Earth-Sun distance. Measurements of the relative positions of planets in the Solar System are done by radar: one measures the time taken for light to be reflected from an object using the conventional value of the velocity of light $c$. Both for objects inside the Solar System and for the nearest stars one measure the distance with the *trigonometric parallax* by using the propagation of light and its velocity $c$ in inertial frames. One measures the difference (the inclination angle) in the apparent position of an object viewed

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\(^7\) See Ref.[52] for possible gravitational anomalies inside the Solar System.
along two different lines of sight at two different times and then uses Euclidean geometry to evaluate the distance. The used unit in astronomy is the parsec, which is 3.26 light-years or $3.26 \times 10^{16}$ meters.

This convention cannot be used for more distant either galactic or extra-galactic objects. New notions like standard candles, dynamical parallax, spectroscopic parallax, luminosity distance,..... are needed [2]. These notions involve both aspects of light propagation in curved space-times and cosmological assumptions like the Hubble law (velocity-red-shift linear relation).

However if one takes into account the description of the universe given by cosmology, the actual cosmological space-time cannot be a nearly Galilei space-time but it must be a cosmological space-time with some theoretical cosmic time. In the standard cosmological model [1] it is a homogeneous and isotropic FRW space-time whose instantaneous 3-spaces have nearly vanishing internal 3-curvature, so that they may locally be replaced with Euclidean 3-spaces as it is done in galactic dynamics. However they have a time-dependent conformal factor (it is one in Galilei space-time) responsible for the Hubble constant regulating the expansion of the universe. As a consequence the transition from the astronomical ICRS to an astrophysical description taking into account cosmology is far from being understood.

What is still lacking is a PM extension of the celestial frame such that the PM BCRS frame is its restriction to the solar system inside our galaxy. In particular one needs the definition of a coordinate time $t_{ICRS}$ connected to $t_B$ such that the 3-spaces $t_{ICRS} = const.$ have a very small internal 3-curvature and a suitable extrinsic curvature as sub-manifolds of the space-time connected with the Hubble constant. In this way this astronomical PM ICRS would be consistent with the FRW cosmological space-times used in astrophysics except for the conformal factor determining the accelerated expansion of the universe and creating problems in the metrological use of fixed stars.

Hopefully at least an PM extension of ICRS including our galaxy (with the definition of a galactic coordinate system) will be achieved with the ESA GAIA mission devoted to the cartography of the Milky Way [6].
3 Clock Synchronization and Global Non-Inertial Frames in Minkowski Space-Time

Since in the Minkowski space-time of SR time is not absolute, there is no intrinsic notion of 3-space and of synchronization of clocks: both of them have to be defined with some convention. As a consequence the 1-way velocity of light from one observer A to an observer B has a meaning only after a choice of a convention for synchronizing the clock in A with the one in B. Therefore the crucial quantity in special relativity is the 2-way (or round trip) velocity of light $c$ involving only one clock. It is this velocity (a kind of mean velocity) which is isotropic and constant in SR and replaces the standard of length in relativistic metrology.

Einstein convention for the synchronization of clocks in Minkowski space-time uses the 2-way velocity of light to identify the Euclidean 3-spaces of the inertial frames centered on an inertial observer A by means of only its clock. The inertial observer A sends a ray of light at $x_i^0$ towards the (in general accelerated) observer B; the ray is reflected towards A at $x_j^0$ by convention P is synchronous with the mid-point between emission and absorption on A’s world-line, i.e.

$$x_p^0 = x_i^0 + \frac{1}{2} (x_j^0 - x_i^0) = \frac{1}{2} (x_i^0 + x_j^0).$$

This convention selects the Euclidean instantaneous 3-spaces $x^\mu = ct = \text{const.}$ of the inertial frames centered on A. Only in this case the one-way velocity of light between A and B coincides with the two-way one, $c$. However, as said in the Introduction, if the observer A is accelerated, the convention can breaks down due the possible appearance of coordinate singularities.

As a consequence, a theory of global non-inertial frames in Minkowski space-time has to be developed in a metrology-oriented way to overcame the pathologies of the 1+3 point of view. This has been done in the papers of Ref.[21] based on the 3+1 point of view and on the use of observer-dependent Lorentz scalar radar 4-coordinates. This theory and its implications for the description of isolated systems in SR will be reviewed in this Section.

3.1 3+1 Splittings of Minkowski Spacetime and Radar 4-Coordinates

Assume that the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock is given: $\tau$ is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible 3+1 splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces $\Sigma_\tau$. It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of $\Sigma_\tau$ sign the same
time of the atomic clock of the observer. On each 3-space $\Sigma_\tau$ one chooses curvilinear 3-coordinates $\sigma^r$ having the observer as origin. These are the Lorentz-scalar and observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$.

If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from the Cartesian 4-coordinates $x^\mu$ of a reference inertial observer to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the embedding functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces $\Sigma_\tau$ as embedded 3-manifold into Minkowski space-time. The induced 4-metric on $\Sigma_\tau$ is the following functional of the embedding $g_{AB}(\tau, \sigma^r) = [z^\mu_A \eta_{\mu\nu} z^\nu_B](\tau, \sigma^r)$, where $z^\mu_A = \partial z^\mu / \partial \sigma^A$ and $4 \eta_{\mu\nu} = \epsilon (+ - - -)$ is the flat metric. While the 4-vectors $z^\mu_A(\tau, \sigma^u)$ are tangent to $\Sigma_\tau$, so that the unit normal $l^\mu(\tau, \sigma^u)$ is proportional to $\epsilon^{\mu\alpha\beta\gamma} [z^\alpha_1 z^\beta_2 z^\gamma_3](\tau, \sigma^u)$, one has $z^\mu_a(\tau, \sigma^r) = [N l^\mu + N^r z^\mu_r](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon [z^\mu_A l^\mu](\tau, \sigma^r) = 1 + n(\tau, \sigma^r)$ and $N_r(\tau, \sigma^r) = -\epsilon g_{rr}(\tau, \sigma^r)$ being the lapse and shift functions.

As a consequence, the components of the 4-metric $g_{AB}(\tau, \sigma^r)$ have the following expression

$$\epsilon^4 g_{\tau\tau} = N^2 - N_r N^r,$$

$$g_{rs} = -\epsilon^4 g_{rs} = \sum_{a=1}^{3} c(\alpha) r^3 e(\alpha) s =$$

$$= \tilde{\phi}^{2/3} \sum_{a=1}^{3} e^2 \sum_{b=1}^{3} \gamma_{ab} R_b V_{ra}(\theta^i) V_{sa}(\theta^i), \quad (1)$$

where $c(\alpha) = c(\alpha)(\tau, \sigma^u)$ are cotriads on $\Sigma_\tau$, $\tilde{\phi}^2 = \text{det}^3 g_{rs}(\tau, \sigma^r)$ is the 3-volume element on $\Sigma_\tau$, $\lambda_{\alpha}(\tau, \sigma^r) = \tilde{\phi}^{1/3} e \sum_{b=1}^{3} \gamma_{ab} R_b(\tau, \sigma^r)$ are the positive eigenvalues of the 3-metric ($\gamma_{ab}$ are suitable numerical constants) and $V(\theta^i(\tau, \sigma^r))$ are diagonalizing rotation matrices depending on three Euler angles.

Therefore starting from the four independent embedding functions $z^\mu(\tau, \sigma^r)$ one obtains the ten components $g_{AB}$ of the 4-metric (or the quantities $N$, $N_r$, $\tilde{\phi}$, $R_3$, $\theta^i$), which play the role of the inertial potentials generating the relativistic apparent forces in the non-inertial frame. It can be shown [21] that the usual non-relativistic Newtonian inertial potentials are hidden in the functions $N$, $N_r$ and $\theta^i$. The extrinsic curvature tensor $K_{rs}(\tau, \sigma^u) = \left[\sum_{a=1}^{3} (N_{ra} + N_{ra} - \partial_r g_{rs})\right](\tau, \sigma^u)$, describing the shape of the instantaneous 3-spaces of the non-inertial frame as embedded 3-sub-manifolds of Minkowski

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8 It is the non-factual idealization required by the Cauchy problem generalizing the existing protocols for building coordinate system inside the future light-cone of a time-like observer.

9 $\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention.
space-time, is a secondary inertial potential, functional of the ten inertial potentials $^4g_{AB}$.

The foliation is nice and admissible if it satisfies the conditions:

1) $N(\tau, \sigma^r) > 0$ in every point of $\Sigma_\tau$ so that the 3-spaces never intersect, avoiding the coordinate singularity of Fermi coordinates;

2) $\epsilon^4g_{\tau\tau}(\tau, \sigma^r) > 0$, so to avoid the coordinate singularity of the rotating disk, and with the positive-definite 3-metric $^3g_{rs}(\tau, \sigma^u) = -\epsilon^4g_{rs}(\tau, \sigma^u)$ having three positive eigenvalues (these are the Møller conditions [53]);

3) all the 3-spaces $\Sigma_\tau$ must tend to the same space-like hyperplane at spatial infinity with a unit normal $\epsilon^\mu_\tau$, which is the time-like 4-vector of a set of asymptotic ortho-normal tetrads $\epsilon^A_\mu$. These tetrads are carried by asymptotic inertial observers and the spatial axes $\epsilon^\mu_r$ are identified by the fixed stars of star catalogues. At spatial infinity the lapse function tends to 1 and the shift functions vanish.

### 3.2 Global Non-Inertial Frames in Minkowski Spacetime

By using the asymptotic tetrads $\epsilon^\mu_\mu$ one can give the following parametrization of the embedding functions

$$z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon^\mu_\mu F^A(\tau, \sigma^r), \quad F^A(\tau, 0) = 0,$$

$$x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\mu f^A(\tau),$$

where $x^\mu(\tau)$ is the world-line of the observer. The functions $f^A(\tau)$ determine the 4-velocity $u^\mu(\tau) = \dot{x}^\mu(\tau)/\sqrt{\epsilon \dot{x}^2(\tau)}$ ($\dot{x}^\mu(\tau) = \frac{dx^\mu(z)}{d\tau}$) and the 4-acceleration $a^\mu(\tau) = \frac{du^\mu(\tau)}{d\tau}$ of the observer.

The Møller conditions are non-linear differential conditions on the functions $f^A(\tau)$ and $F^A(\tau, \sigma^r)$, so that it is very difficult to construct explicit examples of admissible 3+1 splittings. When these conditions are satisfied Eqs.(2) describe a global non-inertial frame in Minkowski space-time.

Till now the solution of Møller conditions is known in the following two cases in which the instantaneous 3-spaces are parallel Euclidean space-like hyper-planes not equally spaced due to a linear acceleration.

A) **Rigid non-inertial reference frames with translational acceleration.** An example are the following embeddings
\[ z^\mu(\tau, \sigma^u) = x^\mu_0 + \epsilon^\mu_i f(\tau) + \epsilon^\mu_i \, \sigma^i, \]

\[ g_{\tau\tau}(\tau, \sigma^u) = \epsilon \left( \frac{df(\tau)}{d\tau} \right)^2, \quad g_{\tau\sigma}(\tau, \sigma^u) = 0, \quad g_{\sigma\sigma}(\tau, \sigma^u) = -\epsilon \delta_{rs}. \]

(3)

This is a foliation with parallel hyper-planes with normal \( l^\mu = \epsilon^\mu_i = \text{const.} \) and with the time-like observer \( x^\mu(\tau) = x^\mu_0 + \epsilon^\mu_i f(\tau) \) as origin of the 3-coordinates. The hyper-planes have translational acceleration \( \ddot{x}^\mu = \epsilon^\mu_i \ddot{f}(\tau) \), so that they are not uniformly distributed like in the inertial case \( f(\tau) = \tau \).

B) **Differentially rotating non-inertial frames** without the coordinate singularity of the rotating disk. The embedding defining this frames is

\[ z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_i R^r_s(\tau, \sigma) \sigma^s \rightarrow_{\sigma \rightarrow \infty} x^\mu(\tau) + \epsilon^\mu_i \sigma^i, \]

\[ R^r_s(\tau, \sigma) = R^r_s(\alpha_i(\tau, \sigma)) = R^r_s(F(\sigma) \delta_i(\tau)), \]

\[ 0 < F(\sigma) < \frac{1}{A \sigma}, \quad \frac{d F(\sigma)}{d\sigma} \neq 0 \text{ (Moller conditions)}, \]

\[ z^\mu(\tau, \sigma^u) = \dot{x}^\mu(\tau) - \epsilon^\mu_i R^r_s(\tau, \sigma) \delta^u_{swv} \epsilon_{wuv} \sigma^u \left( \frac{\Omega^r(\tau, \sigma)}{c} \right), \]

\[ z^\mu(\tau, \sigma^u) = \epsilon^\mu_i R^r_s(\tau, \sigma) \left( \delta^u_r + \Omega^r_{(\tau)u}(\tau, \sigma) \sigma^u \right), \]

(4)

where \( \sigma = |\sigma| \) and \( R^r_s(\alpha_i(\tau, \sigma)) \) is a rotation matrix satisfying the asymptotic conditions \( R^r_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^r_s, \quad \partial \rightarrow_{\sigma \rightarrow \infty} 0 \), whose Euler angles have the expression \( \alpha_i(\tau, \sigma) = F(\sigma) \delta_i(\tau), \quad i = 1, 2, 3 \). The unit normal is \( l^\mu = \epsilon^\mu_i = \text{const.} \) and the lapse function is \( 1 + n(\tau, \sigma^u) = \epsilon (z^\mu_i l^i_\mu)(\tau, \sigma^u) = 0 \). In Eq.(4) one uses the notations \( \Omega(\tau, \sigma) = R^{-1}(\tau, \sigma) \partial r R(\tau, \sigma) \) and \( \left( R^{-1}(\tau, \sigma) \partial_r R(\tau, \sigma) \right)^u_v = \delta^u_m \epsilon_{muv} \Omega^r_{(\tau, \sigma)} \), with \( \Omega^r(\tau, \sigma) = F(\sigma) \delta_i(\tau, \sigma) \)

\[ \hat{n}^r(\tau, \sigma) \]

10 \( \hat{n}^r(\tau, \sigma) \) defines the instantaneous rotation axis and \( 0 < \frac{\hat{n}(\tau, \sigma)}{2} < \max(\hat{\alpha}(\tau), \hat{\beta}(\tau), \hat{\gamma}(\tau)) \).
the rotation axis is fixed and $\Omega = \omega = \text{const.}$, a simple choice for the function $F(\sigma)$ is $F(\sigma) = \frac{1}{1 + \frac{\omega^2 \sigma^2}{c^2}}$. 11

To evaluate the non-relativistic limit for $c \to \infty$, where $\tau = ct$ with $t$ the absolute Newtonian time, one chooses the gauge function $F(\sigma) = \frac{1}{1 + \frac{\omega^2 \sigma^2}{c^2}} \to \infty 1 - \frac{\omega^2 \sigma^2}{c^2} + O(c^{-4})$. This implies that the corrections to rigidly-rotating non-inertial frames coming from Møller conditions are of or-der $O(\varepsilon^{-2})$ and become important at the distance from the rotation axis where the horizon problem for rigid rotations appears.

As shown in the first paper in Refs.[21], global rigid rotations are forbidden in relativistic theories, because, if one uses the embedding $z^\mu(\tau, \sigma^\alpha) = x^\mu(\tau) + \epsilon^\mu R_s(\tau) \sigma^s$ describing a global rigid rotation with angular velocity $\Omega^r = \Omega^\alpha(\tau)$, then the resulting $g_{\tau\tau}(\tau, \sigma^\alpha)$ violates Møller conditions, because it vanishes at $\sigma = \sigma_R = \frac{1}{\Omega(\tau)} \left[ \sqrt{\dot{x}^2(\tau) + [\dot{x}_\mu(\tau) \epsilon^\mu R_s(\tau) (\dot{\sigma} \times \hat{\Omega}(\tau))]^2} - \dot{x}_\mu(\tau) \epsilon^\mu R_s(\tau) (\dot{\sigma} \times \hat{\Omega}(\tau)) \right]$ ( $\sigma^u = \sigma \delta^u, \Omega^r = \Omega \hat{\Omega}, \dot{\sigma}^2 = \Omega^2 = 1$). At this distance from the rotation axis the tangential rotational velocity becomes equal to the velocity of light. This is the horizon problem of the rotating disk (the horizon is often named the light cylinder). Let us remark that even if in the existing theory of rotating relativistic stars [54] one uses differential rotations, notwithstanding that in the study of the magnetosphere of pulsars often the notion of light cylinder is still used.

The search of admissible 3+1 splittings with non-Euclidean 3-spaces is much more difficult. The simplest case is the following parametrization of the embeddings (1) in terms of Lorentz matrices $\Lambda^A_B(\tau, \sigma) \to \sigma \to \infty \delta^A_B$, with $\Lambda^A_B(\tau, 0)$ finite. The Lorentz matrix is written in the form $\Lambda = \mathcal{B} \mathcal{R}$ as the product of a boost $\mathcal{B}(\tau, \sigma)$ and a rotation $\mathcal{R}(\tau, \sigma)$ like the one in Eq.(4) ($\mathcal{R}_{\tau}^\tau = 1$, $\mathcal{R}_{\tau}^\nu = 0$, $\mathcal{R}_{\nu}^\nu = \mathcal{R}_{\nu}^\nu$). The components of the boost are $\mathcal{B}_{\tau}(\tau, \sigma) = \gamma(\tau, \sigma) = 1/\sqrt{1 - \beta^2(\tau, \sigma)}$, $\mathcal{B}_\nu(\tau, \sigma) = \gamma(\tau, \sigma) \beta_\nu(\tau, \sigma)$, $\mathcal{R}_{\nu}(\tau, \sigma) = \delta^\nu_x + \frac{2 \beta^\nu \beta^x}{1 + \gamma(\tau, \sigma)}$, with $\beta^\nu(\tau, \sigma) = G(\sigma) \beta^\nu(\tau)$, where $\beta^\nu(\tau)$ is defined by the 4-velocity of the observer $u^\mu(\tau) = \epsilon^\mu A(\tau)/\sqrt{1 - \beta^2(\tau)}$, $\beta^A(\tau) = (1; \beta^\tau(\tau))$. The Møller conditions are restrictions on $G(\sigma) \to \sigma \to \infty 0$ with $\Theta(0)$ finite, whose explicit form is still under investigation.

See the second paper of Ref.[21] for the description of the electro-magnetic field and of phenomena like the Sagnac effect and the Faraday rotation in this

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11 Nearly rigid rotating systems, like a rotating disk of radius $\sigma_o$, can be described by using a function $F(\sigma)$ approximating the step function $\theta(\sigma - \sigma_o)$.

12 It corresponds to the locality hypothesis of Ref.[20], according to which at each instant of time the detectors of an accelerated observer give the same indications as the detectors of the instantaneously comoving inertial observer.
framework for non-inertial frames. Moreover the embedding (4) has been used in the first paper of Ref.[34] on quantum mechanics in non-inertial frames.

### 3.3 Congruences of Timelike Observers Associated with a 3+1 Splitting

Each admissible 3+1 splitting of space-time allows one to define two associated congruences of time-like observers.

i) The congruence of the Eulerian observers with the unit normal $l^\mu(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) l^A(\tau, \sigma^r)$ to the 3-spaces embedded in Minkowski space-time as unit 4-velocity. The world-lines of these observers are the integral curves of the unit normal and in general are not geodesics. In adapted radar 4-coordinates the contro-variant orthonormal tetrads carried by the Eulerian observers are $l^A(\tau, \sigma^r)$, $4\bar{E}_{(a)}(\tau, \sigma^r) = (0; 3\bar{e}_{(a)}^u(\tau, \sigma^r))$, where $3\bar{e}_{(a)}^u(\tau, \sigma^r)$ ($a = 1, 2, 3$) are triads on the 3-space.

If $4\nabla$ is the covariant derivative associated with the 4-metric $4g_{AB}(\tau, \sigma^r)$ induced by a 3+1 splitting, the equation

$$4\nabla_A l_B = \epsilon l_A a_B + \sigma_{AB} + \frac{1}{3} \theta h_{AB} - \omega_{AB}, \quad (3h_{AB} = 4g_{AB} - \epsilon l_A l_B), \quad (5)$$

defines the acceleration $3a^A$ ($3a^A l_A = 0$), the expansion $\theta$, the shear $\sigma_{AB} = \sigma_{BA}$ ($\sigma_{AB} l^B = 0$) and the vorticity or twist $\omega_{AB} = -\omega_{BA}$ ($\omega_{AB} l^B = 0$) of the Eulerian observers with $\omega_{AB} = 0$ since they are surface-forming by construction. They will be useful in GR as shown in Section 7.

ii) The skew congruence with unit 4-velocity $v^\mu(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) v^A(\tau, \sigma^r)$ (in general it is not surface-forming, i.e. it has a non-vanishing vorticity, like the one of a rotating disk). The observers of the skew congruence have the world-lines (integral curves of the 4-velocity) defined by $\sigma^r = \text{const.}$ for every $\tau$, because the unit 4-velocity tangent to the flux lines $x^\mu_{\sigma^r}(\tau) = z^h(\tau, \sigma^r)$ is $v^\mu_{\sigma^r}(\tau) = z^h(\tau, \sigma^r)/\sqrt{4g_{\tau \tau}(\tau, \sigma^r)}$ (there is no horizon problem because it is everywhere time-like in admissible 3+1 splittings). They carry contro-variant orthonormal tetrads, given in Ref.[41], not adapted to the foliation, connected in each point by a Lorentz transformation to the ones of the Eulerian observer present in this point.
3.4 Parametrized Minkowski Theories

In the global non-inertial frames of Minkowski space-time it is possible to describe isolated systems (particles, strings, fields, fluids) admitting a Lagrangian formulation by means of parametrized Minkowski theories [23, 24], [21].

The existence of a Lagrangian, which can be coupled to an external gravitational field, makes possible the determination of the matter energy-momentum tensor and of the ten conserved Poincaré generators $P^\mu$ and $J^{\mu\nu}$ (assumed finite) of every configuration of the isolated system.

First of all one must replace the matter variables of the isolated system with new ones knowing the clock synchronization convention defining the 3-spaces $\Sigma_\tau$. For instance a Klein-Gordon field $\phi(x)$ will be replaced with $\phi(\tau, \sigma') = \phi(z(\tau, \sigma'))$; the same for every other field. Instead for a relativistic particle with world-line $x^\mu(\tau)$ one must make a choice of its energy sign: then the positive- (or negative-) energy particle will be described by 3-coordinates $\eta^\mu(\tau)$ defined by the intersection of its world-line with $\Sigma_\tau$: $x^\mu(\tau) = z^\mu(\tau, \eta^\mu(\tau))$. Differently from all the previous approaches to relativistic mechanics, the dynamical configuration variables are the 3-coordinates $\eta^\mu(\tau)$ and not the world-lines $x^\mu(\tau)$ (to rebuild them in an arbitrary frame one needs the embedding defining that frame). This fact eliminates the possibility to have time-like excitations in the spectrum of relativistic bound states: inside each 3-space only space-like correlations among the particles are possible.

Then one replaces the external gravitational 4-metric in the coupled Lagrangian with the 4-metric $\eta_{AB}(\tau, \sigma')$, which is a functional of the embedding defining an admissible 3+1 splitting of Minkowski space-time, and the matter fields with the new ones knowing the instantaneous 3-spaces $\Sigma_\tau$.

Parametrized Minkowski theories are defined by the resulting Lagrangian depending on the given matter and on the embedding $z^\mu(\tau, \sigma')$. The resulting action is invariant under to take into account of fermion fields the frame-preserving diffeomorphisms $\tau \mapsto \tau'(\tau, \sigma')$, $\sigma' \mapsto \sigma'(\sigma')$ firstly introduced in Ref.[25]. As a consequence, there are four first-class constraints with exactly vanishing Poisson brackets (an Abelianized analogue of the super-Hamiltonian and super-momentum constraints of canonical gravity) determining the momenta conjugated to the embeddings in terms of the matter energy-momentum tensor. This implies that the embeddings $z^\mu(\tau, \sigma')$ are gauge variables, so that all the admissible non-inertial or inertial frames are gauge equivalent, namely physics does not depend on the clock synchronization convention and on the choice of the 3-coordinates $\sigma'$: only the appear-
ances of phenomena change by changing the notion of instantaneous 3-space.

Even if the gauge group is formed by the frame-preserving diffeomorphisms, the matter energy-momentum tensor allows the determination of the ten conserved Poincaré generators $P^\mu$ and $J^{\mu\nu}$ (assumed finite) of every configuration of the system (in non-inertial frames they are asymptotic generators at spatial infinity like the ADM ones in GR).

As an example one may consider $N$ free scalar particles with masses $m_i$ and sign of the energy $\eta_i = \pm$, whose world-lines are identified by the configurational variables $\eta^i_\tau (\tau): x^i_\tau (\tau) = z^i_\tau (\tau, \eta^i_\tau (\tau)), \ i = 1, \ldots, N$. In parametrized Minkowski theories they are described by the following action depending on the configurational variables $\eta^i_\tau (\tau)$ and $z^i_\tau (\tau, \sigma^r)$

$$S = \int d\tau d^3\sigma L(\tau, \sigma^u) = \int d\tau L(\tau),$$

$$L(\tau, \sigma^u) = -\sum_{i=1}^{N} \delta^3(\sigma^u - \eta^u_i(\tau))$$

$$m_i c \eta_i \sqrt{\epsilon [4g_{rr}(\tau, \sigma^u) + 24g_{rr}(\tau, \sigma^u) \dot{\eta}^r_\tau(\tau) + 4g_{rs}(\tau, \sigma^u) \dot{\eta}_r^s(\tau) \eta^i_s(\tau)\eta^i_r(\tau)]}.$$ (6)

The resulting canonical momenta $\kappa^r_\tau(\tau) = \frac{\partial L(\tau)}{\partial \dot{\eta}^r_\tau(\tau)}$, $\rho^\mu(\tau, \sigma^u) = -\epsilon \frac{\partial L(\tau, \sigma^u)}{\partial \dot{\sigma}^\mu(\tau, \sigma^u)}$ satisfy the Poisson brackets $\{\eta^r_\tau (\tau), \kappa^s_\tau(\tau)\} = -\delta^r_s \delta_{ij}$, $\{\eta^r_\tau (\tau), \rho^\mu(\tau, \sigma^u)\} = -\delta^r_\mu \delta^3(\sigma^u - \sigma^u')$. The Poincaré generators and the energy-momentum tensor of this system are $(h^{rs} = -\epsilon \gamma^{rs}$ with $\gamma^{ru} 4g_{us} = \delta^r_s; \gamma = -\epsilon \det 4g_{rs})$

$$P^\mu = \int d^3\sigma \rho^\mu(\tau, \sigma^u), \quad J^{\mu\nu} = \int d^3\sigma (z^\mu_\tau \rho^\nu - z^\nu_\tau \rho^\mu)(\tau, \sigma^u),$$

$$T^{AB}(\tau, \sigma^u) = -\frac{2}{\sqrt{-\det 4g_{CD}(\tau, \sigma^u)}} \delta S \delta^4 g_{AB}(\tau, \sigma^u), \quad T^{\mu\nu} = z_\mu^A z_\nu^B T^{AB},$$

$$T_{\perp \perp}(\tau, \sigma^u) = \left(l_\mu l_\nu T^{\mu\nu}\right)(\tau, \sigma^u) = \sum_{i=1}^{N} \frac{\delta^3(\sigma^u - \eta^u_i(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}}$$

$$\eta_i \sqrt{m_i^2 c^2 + h^{rs}(\tau, \sigma^u) \kappa^r_\tau(\tau) \kappa^s_\tau(\tau)},$$

13 In the first paper of Ref. [34] there is the definition of parametrized Galilei theories, non-relativistic limit of the parametrized Minkowski theories. Also the inertial and non-inertial frames in Galilei space-time are gauge equivalent in this formulation.
\[ T_{\perp}(\tau, \sigma^u) = \left( l^\mu \, z_{r\nu} \, T^{\mu\nu}\right)(\tau, \sigma^u) = \sum_{i=1}^{N} \frac{\delta^3(\sigma^u - \eta^u_{i}(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}} \kappa_{ir}(\tau), \]

\[ T_{rs}(\tau, \sigma^u) = \left( z_{r\mu} \, z_{s\nu} \, T^{\mu\nu}\right)(\tau, \sigma^u) = \sum_{i=1}^{N} \frac{\delta^3(\sigma^u - \eta^u_{i}(\tau))}{\sqrt{\gamma(\tau, \sigma^u)}} \eta_{i} \frac{\kappa_{ir}(\tau) \kappa_{is}(\tau)}{\sqrt{m^2 c^2 + h_{vw}(\tau, \sigma^u) \kappa_{ir}(\tau) \kappa_{is}(\tau)}}. \] (7)

The four first-class constraints implying the gauge nature of the embedding and the gauge equivalence of the description in different non-inertial frames are

\[ \rho_{\mu}(\tau, \sigma^u) - \sqrt{\gamma(\tau, \sigma^u)} \left[ l_{\mu} \, T_{\perp} - z_{r\mu} \, h^{rs} \, T_{\perp s}\right] \approx 0. \] (8)

The same description can be given for the Klein-Gordon [55] and Dirac [56] fields and for the electro-magnetic field [21].

To describe the physics in a given admissible non-inertial frame described by an embedding \( z^\mu_r(\tau, \sigma^u) \) one must add the gauge-fixings \( z^\mu_r(\tau, \sigma^u) \approx 0 \).

### 3.5 The Instant Form of Dynamics in the Inertial Rest Frames and the Problem of the Relativistic Center of Mass

If one restricts himself to inertial frames, one can define the inertial rest-frame instant form of dynamics for isolated systems by choosing the 3+1 splitting corresponding to the intrinsic inertial rest frame of the isolated system centered on an inertial observer: the instantaneous 3-spaces, named Wigner 3-spaces due to the fact that the 3-vectors inside them are Wigner spin-1 3-vectors [23, 24], are orthogonal to the conserved 4-momentum \( P^\mu \) (assumed time-like, \( \epsilon P^2 > 0 \)) of the configuration.

In this framework one can give the final solution to the old problem of the relativistic extension of the Newtonian center of mass of an isolated system. In its rest frame there are only three notions of collective variables, which can be built by using only the Poincaré generators:

- the canonical non-covariant Newton-Wigner center of mass (or center of spin) \( \tilde{x}^\mu(\tau) \),
- the non-canonical covariant Fokker-Pryce center of inertia \( Y^\mu(\tau) \)
- the non-canonical non-covariant Møller center of energy \( R^\mu(\tau) \).

While \( Y^\mu(\tau) \) is a 4-vector, \( \tilde{x}^\mu(\tau) \) and \( R^\mu(\tau) \) are not 4-vectors. All of them tend to the Newtonian center of mass in the non-relativistic limit. Since the
Poincaré generators know the whole $\Sigma$, they and therefore also these three collective variables are non-local quantities: as a consequence they are non-measurable with local means [21, 26, 28, 33].

If one centers the inertial rest frame on the world-line of the Fokker-Planck center of inertia thought as an inertial observer, then the corresponding embedding has the expression [23, 24, 28]

$$z_{\mu W}^\tau(\tau, \sigma) = Y^\mu(\tau) + \epsilon^\mu_h(\hbar)\sigma^\tau,$$

where $Y^\mu(\tau)$ is the Fokker-Planck center-of-inertia 4-vector, $\hbar = P/\sqrt{\epsilon P^2}$ and $\epsilon^\mu_{A=\nu}(h) = L^\mu_{A=\nu}(P, P)$ are the columns of the standard Wigner boost for time-like orbits sending $P^\mu = \sqrt{\epsilon P^2}(\sqrt{1 + h^2}; \hbar)$ to $P = \sqrt{\epsilon P^2}(1; 0)$. Their expression is $\epsilon^\mu_h(\hbar) = h^\mu = (\sqrt{1 + h^2}; h)$ and $\epsilon^\rho_h(\hbar) = (\hbar; \delta^\rho_i + \frac{h^i h^j}{1 + \sqrt{1 + h^2}})$ as shown in Appendix B of Ref.[57].

As shown in Ref.[26, 28, 29, 33], the three collective variables can be expressed as known functions of the Lorentz-scalar rest time $\tau = c T_s = \hbar \cdot \tilde{x} = \hbar \cdot Y = \hbar \cdot R$, of canonically conjugate Jacobi data (frozen Cauchy data) $z = M c x_{NW}(0)$ and $\hbar = P/M c$ 14, of the invariant mass $M c = \sqrt{\epsilon P^2}$ of the system and of its rest spin $S$.

While the world-line of the non-canonical covariant external Fokker-Planck 4-center of inertia is

$$Y^\mu(\tau) = z^\mu_{\mu W}(\tau, 0) = \left(\tilde{x}^\sigma(\tau); Y(\tau)\right)$$

$$= \left(\sqrt{1 + h^2}(\tau + \frac{h \cdot z}{Mc}); \frac{z}{Mc} + (\tau + \frac{h \cdot z}{Mc}) h + \frac{S \times h}{M c (1 + \sqrt{1 + h^2})}\right),$$

the pseudo-world-line of the canonical non-covariant external 4-center of mass is ($\sigma = -\frac{S \times h}{M c (1 + \sqrt{1 + h^2})}$ from Ref.[26])

$$\tilde{x}^\rho(\tau) = \left(\tilde{x}^\sigma(\tau); \tilde{x}(\tau)\right) = z^\mu_{\mu W}(\tau, \sigma) = Y^\mu(\tau) + \left(0; \frac{-S \times h}{M c (1 + \sqrt{1 + h^2})}\right) =$$

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14 Their Poisson brackets are $\{z^i, h^j\} = \delta^{ij}$. $x_{NW}(\tau)$ is the standard Newton-Wigner non-covariant 3-position, classical counterpart of the corresponding position operator; the use of $z$ avoids to take into account the mass spectrum of the isolated system in the description of the center of mass. The non-covariance of $z$ under Poincaré transformations $(a, A)$ has the following form [33, 57] $z^i \rightarrow z'^i = \left( A^i_j - \frac{A^i_{\mu} h^\mu}{A^\mu \cdot h^\mu} \lambda^\nu_j \right) z^j + \left( A^i_{\mu} - \frac{A^i_{\mu} h^\mu}{A^\mu \cdot h^\mu} A^\nu_{\mu} \right) (A^{-1} a)^\nu.$
\[
\sqrt{1 + \frac{\hbar^2}{M c}} \left( \frac{\mathbf{z}}{M c} + \frac{\tau + \mathbf{h} \cdot \mathbf{z}}{M c} \right).
\]

(11)

The world-lines of the positive-energy particles are parametrized by the Wigner 3-vectors \( \eta_i(\tau), i = 1, 2, ..., N \), and are given by

\[
x^\mu_i(\tau) = Y^\mu(\tau) + \epsilon^\mu_r(\tau) \eta^r_i(\tau).
\]

(12)

The world-lines \( x^\mu_i(\tau) \) of the particles are derived (interaction-dependent) quantities. Also the standard particle 4-momenta are derived quantities, whose expression is

\[
p^\mu_i(\tau) = \epsilon^\mu(\hbar) \kappa^i(\tau) = \hbar^\mu \sqrt{m^2 c^2 + \kappa^2_i(\tau)} - \epsilon^\mu_r(\hbar) \kappa^r_i(\tau)
\]

with \( \epsilon^2 = m^2 c^2 \) in the free case.

In the case of interacting particles the reconstruction of the world-lines requires a complex interaction-dependent procedure delineated in Ref.[29], where there is also a comparison of the present approach with the other formulations of relativistic mechanics developed for the study of the problem of relativistic bound states. See Ref.[21] for the extension to non-inertial frames.

In general the world-lines \( x^\mu_i(\tau) \) do not satisfy vanishing Poisson brackets (they are relativistic predictive coordinates, see Ref.[29]): already at the classical level a non-commutative structure emerges due to the Lorentz signature [33].

In each Lorentz frame one has different pseudo-world-lines describing \( R^\mu \) and \( \tilde{x}^\mu \): the canonical 4-center of mass \( \tilde{x}^\mu \) lies in between \( Y^\mu \) and \( R^\mu \) in every (non rest)-frame. As discussed in Subsection IIF of Ref.[28], this leads to the existence of the Møller non-covariance world-tube, around the world-line \( Y^\mu \) of the covariant non-canonical Fokker-Pryce 4-center of inertia \( Y^\mu \).

The invariant radius of the tube is \( \rho = \sqrt{-W^2/p^2} = |S|/\sqrt{\epsilon P^2} \) where \( W^2 = -P^2 S^2 \) is the Pauli-Lubanski invariant when \( \epsilon P^2 > 0 \). This classical intrinsic radius is a non-local effect of Lorentz signature absent in Euclidean spaces and delimits the non-covariance effects (the pseudo-world-lines) of the canonical 4-center of mass \( \tilde{x}^\mu \). They are not detectable because the Møller radius is of the order of the Compton wave-length: an attempt to test its interior would mean to enter in the quantum regime of pair production. The Møller radius \( \rho \) is also a remnant of the energy conditions of general relativity in flat Minkowski space-time [23].

Finally Eqs.(7) can be used to extend the multipolar expansions of Ref.[58] to this framework for relativistic isolated systems as it is shown in the third paper of Refs.[26].

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15 In the rest-frame the world-tube is a cylinder: in each instantaneous 3-space there is a disk of possible positions of the canonical 3-center of mass orthogonal to the spin. In the non-relativistic limit the radius \( \rho \) of the disk tends to zero and one recovers the non-relativistic center of mass.
3.6 The Description of Isolated Systems in the Rest Frame and their Poincaré Generators

In the inertial rest frame of an isolated system Eqs.(7) are the starting point to get the explicit form of its Poincaré generators, in particular of the Lorentz boosts, which, differently from the Galilei ones, are interaction dependent.

As shown in Ref.[28], every isolated system (i.e. a closed universe) can be visualized as a decoupled non-covariant collective (non-local) pseudo-particle (the external center of mass), described by the frozen Jacobi data \( z, h \), carrying a pole-dipole structure, namely the invariant mass \( M_c \) (the monopole) and the rest spin \( \bar{S} \) (the dipole) of the system, and with an associated external realization of the Poincaré group \(^{16}\):

\[
P^\mu = M c h^\mu = M c \left( \sqrt{1 + h^2}; h \right),
\]

\[
J^{ij} = z^i h^j - z^j h^i + \epsilon^{ijk} S^k, \quad K^i = J^{oi} = -\sqrt{1 + h^2} z^i + \frac{(S \times h)^i}{1 + \sqrt{1 + h^2}}.
\]

The universal breaking of Lorentz covariance is connected to this decoupled non-local collective variable and is irrelevant because all the dynamics of the isolated system leaves inside the Wigner 3-spaces and is Wigner-covariant. The invariant mass and the rest spin are built in terms of the Wigner-covariant variables of the given isolated system \((\eta_i(\tau) \text{ and } \kappa_i(\tau))\) inside the Wigner 3-spaces [21, 26, 28, 33].

In each Wigner 3-space \( \Sigma_\tau \) there is a unfaithful internal realization of the Poincaré algebra, whose generators are built by using the energy-momentum tensor (7) of the isolated system. While the internal energy and angular momentum are \(M c\) and \(\bar{S}\) respectively, the internal 3-momentum vanishes: it is the rest frame condition. Also the internal (interaction dependent) Lorentz boost vanishes: this condition identifies the covariant non-canonical Fokker-Pryce center of inertia as the natural inertial observer origin of the 3-coordinates \(\sigma^\tau\) in each Wigner 3-space.

For \(N\) free particles the internal Poincaré generators have the following expression

\(^{16}\)The last term in the Lorentz boosts induces the Wigner rotation of the 3-vectors inside the Wigner 3-spaces.
\[ M c = \frac{1}{c} E_{(\text{int})} = \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \kappa_i^2}, \]

\[ P_{(\text{int})} = \sum_{i=1}^{N} \kappa_i \approx 0, \]

\[ S = J_{(\text{int})} = \sum_{i=1}^{N} \eta_i \times \kappa_i, \]

\[ K_{(\text{int})} = - \sum_{i=1}^{N} \eta_i \sqrt{m_i^2 c^2 + \kappa_i^2} \approx 0. \] (14)

Since one is in an instant form of the dynamics, in the interacting case only \(Mc\) and \(K_{(\text{int})}\) become interaction dependent.

The three pairs of second-class (interaction dependent) constraints \(P_{(\text{int})} \approx 0, K_{(\text{int})} \approx 0\), eliminate the internal 3-center of mass and its conjugate momentum inside the Wigner 3-spaces\(^\text{17}\); this avoids a double counting of the collective variables (external and internal center of mass). As a consequence the dynamics inside the Wigner 3-spaces is described in terms of internal Wigner-covariant relative variables. In the case of \(N\) relativistic particles one defines the following canonical transformation [33] (see Ref.[26] for other variants) \((m = \sum_{i=1}^{N} m_i)\)

\[ \eta_+ = \sum_{i=1}^{N} \frac{m_i}{m} \eta_i, \quad \kappa_+ = P_{(\text{int})} = \sum_{i=1}^{N} \kappa_i, \]

\[ \rho_a = \sqrt{N} \sum_{i=1}^{N} \gamma_{ai} \eta_i, \quad \pi_a = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Gamma_{ai} \kappa_i, \quad a = 1, \ldots, N - 1, \]

\[ \eta_i = \eta_+ + \frac{1}{\sqrt{N}} \sum_{a-1}^{N-1} \Gamma_{ai} \rho_a, \quad \kappa_i = \frac{m_i}{m} \kappa_+ + \sqrt{N} \sum_{a=1}^{N-1} \gamma_{ai} \pi_a, \] (15)

with the following canonicity conditions\(^\text{18}\)

---

\(^{17}\) One can show [26, 28] that one has \(K_{(\text{int})} = -M R_+\), where \(R_+\) is the internal Møller 3-center of energy inside the Wigner 3-spaces. The rest frame condition \(P_{(\text{int})} \approx 0\) implies \(R_+ \approx q_+ \approx y_+\), where \(q_+\) is the internal 3-center of mass and \(y_+\) the internal Fokker-Pryce 3-center of inertia.

\(^{18}\) Eqs.(15) describe a family of canonical transformations, because the \(\gamma_{ai}\)'s depend on \(\frac{1}{2}(N - 1)(N - 2)\) free independent parameters.
\[
\sum_{i=1}^{N} \gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \gamma_{aj} = \delta_{ij} - \frac{1}{N},
\]
\[
\Gamma_{ai} = \gamma_{ai} - \sum_{k=1}^{N} \frac{m_{k}}{m} \gamma_{ak}, \quad \gamma_{ai} = \Gamma_{ai} - \frac{1}{N} \sum_{k=1}^{N} \Gamma_{ak},
\]
\[
\sum_{i=1}^{N} \frac{m_{i}}{m} \Gamma_{ai} = 0, \quad \sum_{i=1}^{N} \gamma_{ai} \Gamma_{bi} = \delta_{ab}, \quad \sum_{a=1}^{N-1} \gamma_{ai} \Gamma_{aj} = \delta_{ij} - \frac{m_{i}}{m}.
\]

(16)

Since Eqs.(14) imply \(\kappa_+(\tau) = \mathcal{P}_{(\text{int})} \approx 0\) and \(\eta_+(\tau) \approx f_+\left(\rho_a(\tau), \pi_a(\tau)\right)\) due to \(\mathcal{K}_{(\text{int})} \approx 0\), the invariant mass \(Mc\) and the rest spin \(\bar{S}\) become functions only of the N-1 pairs of relative canonical variables.

As a consequence, Eqs.(10), (12) and (15) imply that the world-lines \(x^\mu_i(\tau)\) can be expressed in terms of the Jacobi data \(z, h\), and of the relative variables \(\rho_a(\tau), \pi_a(\tau), a = 1, \ldots, N - 1\). See Ref.[55] for the collective and relative variables of the Klein-Gordon field and the second paper in Ref.[28] for such variables for the electro-magnetic field in the radiation gauge. For these systems one can give for the first time the explicit closed form of the interaction-dependent Lorentz boosts.

One finds that disregarding the unobservable external center of mass all the dynamics is described only by relative variables: this is a form of \textit{weak relationism} without the heavy foundational problem of approaches like the one in Ref.[59].

The non-relativistic limit of this description [33] is Newton mechanics with the Newton center of mass decoupled from the relative variables and moreover after a canonical transformation to the frozen Hamilton-Jacobi description of the center of mass.

An important remark is that the internal space of relative variables is independent from the reference inertial frame used for the description of the isolated system. As shown in Ref.[33], the formalism is built in such a way that the Wigner rotation induced on the relative variables by a Lorentz transformation connecting two reference inertial frames is the identity, i.e. the space of the relative variables in an \textit{abstract internal space} insensitive to Lorentz transformations carried by the external center of mass (or in a more covariant description carried by the Fokker-Pryce center-of-inertia 4-vector origin of the embedding (9)).

Finally in Ref.[21] there is the extension of the formalism to admissible \textit{non-inertial rest frames}, where \(P^\mu\) is orthogonal to the asymptotic space-like hyper-planes to which the instantaneous 3-spaces tend at spatial infinity. In
these non-inertial rest frames the internal Poincaré generators are asymptotic (constant of the motion) symmetry generators like the asymptotic ADM ones in the asymptotically Minkowskian space-times.
4 Implications for Relativistic Mechanics and Classical Field Theory in Special Relativity and the Multi-Temporal Quantization Approach

In the rest-frame instant form of the dynamics it has been possible to find the explicit form of the internal Poincaré generators (in particular of the interaction-dependent invariant mass and Lorentz boosts) not only for the Klein-Gordon [55] and Dirac [31, 56] fields, but also for the electro-magnetic field in the radiation gauge (the only one suitable for the Shanmugadhasan canonical transformations of constraint theory [60]) [21, 28], for relativistic fluids [41, 42], spinning particles [27, 31] and for massless particles, the Nambu string and the two-level atom [32].

In this Section some other developments in SR will be reviewed.

4.1 Relativistic Atomic Physics

Standard atomic physics [61] is a semi-relativistic treatment of quantum electro-dynamics (QED) in which the matter fields are approximated by scalar (or spinning) particles, the relevant energies are below the threshold of pair production and the electro-magnetic field is described in the Coulomb gauge at the order $1/c$.

In Refs. [27, 28] a fully relativistic formulation of classical atomic physics in the rest-frame instant form of dynamics was given with the electro-magnetic field in the radiation gauge and with the electric charges $Q_i$ of the positive-energy particles being Grassmann-valued ($Q_i^2 = 0$, $Q_i Q_j = Q_j Q_i$ for $i \neq j$) to regularize the electro-magnetic self-energies on the world-lines of particles. In the language of QED this is both a ultraviolet regularization (no loop contributions) and an infrared one (no brehmstrahlung), so that only the one-photon exchange diagram contributes and its static and non-static effects are replaced by potentials in a formulation based on the Cauchy problem. Therefore the starting point is a parametrized Minkowski theory with $N$ charged positive-energy particles mutually interacting with a Coulomb potential and coupled to a dynamical transverse electro-magnetic field described by the canonical variables $A(\tau, \sigma)$ and $\pi(\tau, \sigma) = E(\tau, \sigma)$.

In the first paper of Ref. [27] (the second paper is devoted to spinning particles) it is shown that the use of the Lienard-Wiechert solution (see the third paper in Ref. [27]) with "no incoming radiation field" allows one to arrive at a description of $N$ charged particles dressed with a Coulomb cloud and mutually interacting through the Coulomb potential augmented with the full relativistic Darwin potential. This happens independently from the choice of the Green function (retarded, advanced, symmetric,...) due to the Grassmann regularization. The quantization allows one to recover the standard instan-
taneous approximation for relativistic bound states, which till now had only
been obtained starting from QED (either in the instantaneous approxima-
tions of the Bethe-Salpeter equation or in the quasi-potential approach). In
the case of spinning particle the relativistic Salpeter potential was iden-
tified.
Moreover in Ref.[28] it is shown that by using the previous results one
can find a canonical transformation from the canonical basis \( \eta_i(\tau), \kappa_i(\tau), \)
\( A_\perp(\tau, \sigma'), \pi_\perp(\tau, \sigma') \), in which the internal Poincaré generators have the ex-
pression in the case \( N=2 \) \( (B = \partial \times A_\perp, c(\sigma) = -1/4\pi|\sigma|) \)
\[
\mathcal{E}_{(int)} = M c^2 = c \sum_{i=1}^{N} \sqrt{m_i^2 c^2 + \left( \kappa_i(\tau) - \frac{Q_i}{c} A_\perp(\tau, \eta_i(\tau)) \right)^2} + \\
+ \sum_{i \neq j} \frac{Q_i Q_j}{4\pi |\eta_i(\tau) - \eta_j(\tau)|} + \frac{1}{2} \int d^3 \sigma \left[ \pi^2_\perp + B^2 \right](\tau, \sigma),
\]
\[
P_{(int)} = \sum_{i=1}^{N} \kappa_i(\tau) + \frac{1}{c} \int d^3 \sigma \left[ \pi_\perp \times B \right](\tau, \sigma) \approx 0,
\]
\[
\vec{S}^r = \sum_{i=1}^{N} \left( \eta_i(\tau) \times \kappa_i(\tau) \right)^r + \frac{1}{c} \int d^3 \sigma \times \left[ \left( \pi_\perp \times B \right)^r \right](\tau, \sigma),
\]
\[
\mathcal{K}_{\tau}^{(int)} = - \sum_{i=1}^{N} \eta_i^r(\tau) \sqrt{m_i^2 c^2 + \left( \kappa_i(\tau) - \frac{Q_i}{c} A_\perp(\tau, \eta_i(\tau)) \right)^2} + \\
+ \frac{1}{c} \sum_{i=1}^{N} \left[ \sum_{j \neq i}^{1..N} Q_i Q_j \left[ \frac{1}{\Delta \eta_i} \frac{\partial}{\partial \eta_j} c(\eta_i(\tau) - \eta_j(\tau)) \right] \right] - \\
- \eta_j^r(\tau) c(\eta_i(\tau) - \eta_j(\tau)) + \\
+ Q_i \int d^3 \sigma \pi_\perp^r(\tau, \sigma) c(\sigma - \eta_i(\tau)) - \frac{1}{2c} \int d^3 \sigma \sigma^r(\pi^2_\perp + B^2)(\tau, \sigma)^r
\]
to a new canonical basis \( \hat{\eta}_i(\tau), \hat{\kappa}_i(\tau), A_{\perp rad}(\tau, \sigma'), \pi_{\perp rad}(\tau, \sigma') \) so that
in the rest frame there is a decoupled free radiation transverse field and a
system of charged particles mutually interacting with Coulomb plus Darwin
potential. See the first paper in Ref.[27] for the explicit form of the relativistic
Darwin potential. The new internal Poincaré generators in the \( N=2 \) case are
\[
\mathcal{E}_{(int)} = M c^2 = c \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \hat{\kappa}_i^2(\tau)} + \frac{Q_1 Q_2}{4\pi |\eta_1(\tau) - \eta_2(\tau)|} + \\
+ V_{\text{Darwin}}(\hat{\kappa}_1(\tau), \hat{\kappa}_2(\tau), \hat{\eta}_1(\tau) - \hat{\eta}_2(\tau))
\]
\[ + \frac{1}{2} \int d^3 \sigma \left( \pi_{\perp \text{rad}}^2 + B_{\text{rad}}^2 \right)(\tau, \sigma) = \mathcal{E}_{\text{matter}} + \mathcal{E}_{\text{rad}}, \]

\[ \mathcal{P}_{(\text{int})} = \sum_{i=1}^{2} \kappa_i(\tau) + \frac{1}{c} \int d^3 \sigma \left( \mathbf{\pi}_{\perp \text{rad}} \times \mathbf{B}_{\text{rad}} \right)(\tau, \sigma) = \mathcal{P}_{\text{matter}} + \mathcal{P}_{\text{rad}} \approx 0, \]

\[ \bar{S} = \sum_{i=1}^{2} \bar{\eta}_i \times \kappa_i + \frac{1}{c} \int d^3 \sigma \times \left( \mathbf{\pi}_{\perp \text{rad}} \times \mathbf{B}_{\text{rad}} \right)(\tau, \sigma) = \bar{S}_{\text{matter}} + \bar{S}_{\text{rad}}, \]

\[ \mathcal{K}_{(\text{int})} = -\sum_{i=1}^{2} \bar{\eta}_i \sqrt{m_i^2 c^2 + \kappa_i^2} - \]

\[ -\frac{1}{2} \frac{Q_1 Q_2}{c} \left[ \bar{\eta}_1 \frac{1}{\pi} \frac{1}{m_1^2 c^2 + \kappa_1^2} - 2 A_{\perp S2}(\hat{\kappa}_2, \hat{\rho}_{12}) \right] + \]

\[ + \bar{\eta}_2 \frac{1}{\pi} \frac{1}{m_2^2 c^2 + \kappa_2^2} - 2 A_{\perp S1}(\hat{\kappa}_1, \hat{\rho}_{12}) \]

\[ - \frac{1}{2} \frac{Q_1 Q_2}{c} \left( \sqrt{m_1^2 c^2 + \kappa_1^2} \frac{\partial}{\partial \kappa_1} + \sqrt{m_2^2 c^2 + \kappa_2^2} \frac{\partial}{\partial \kappa_2} \right) \hat{K}_{12}(\hat{\kappa}_1, \hat{\kappa}_2, \hat{\rho}_{12}) - \]

\[ - \frac{Q_1 Q_2}{4 \pi c} \int d^3 \sigma \left( \hat{\pi}_{\perp S1}(\sigma - \bar{\eta}_1, \bar{\kappa}_1) + \hat{\pi}_{\perp S2}(\sigma - \bar{\eta}_2, \bar{\kappa}_2) \right) - \]

\[ - \frac{Q_1 Q_2}{c} \int d^3 \sigma \left( \hat{\pi}_{\perp S1}(\sigma - \bar{\eta}_1, \bar{\kappa}_1) \cdot \hat{\pi}_{\perp S2}(\sigma - \bar{\eta}_2, \bar{\kappa}_2) + \right. \]

\[ \left. + B_{S1}(\sigma - \bar{\eta}_1, \bar{\kappa}_1) \cdot B_{S2}(\sigma - \bar{\eta}_2, \bar{\kappa}_2) \right) - \]

\[ - \frac{1}{2} \frac{Q_1 Q_2}{c} \left( \frac{\pi_{\perp \text{rad}}^2 + B_{\text{rad}}^2}{\pi_{\perp \text{rad}}^2} \right)(\tau, \sigma) = \mathcal{K}_{\text{matter}} + \mathcal{K}_{\text{rad}} \approx 0. \quad (18) \]

The only restriction on the two decoupled systems is the elimination of their overall internal 3-center of mass inside the Wigner 3-spaces. Therefore, at the classical level there is a way out from the Haag theorem forbidding the existence of the interaction picture in QED, so that there is no unitary evolution based on interpolating fields from the "in" states to the "out" ones in scattering processes. While the extension of these results to the non-inertial rest frame is done in Ref.[21], the quantization of this framework is under investigation.

In the first paper of Ref.[32] there is the formulation in the rest-frame instant form of the relativistic quark model in the radiation gauge for the SU(3) Yang-Mills fields with scalar quarks having Grassmann-valued color charges. While in Eq.(101) of that paper there is the rest-frame condition,
in Eqs.(97) there is the invariant mass $M c^2$ for a quark-antiquark system. In it the electro-magnetic Coulomb potential of Eq.(17) is replaced with a potential, given in Eq.(95), depending on the color transverse vector potential through the Green function of the SU(3) covariant divergence. The non-linearity of the problem does not allow to evaluate a Lienard-Wiechert solution and to find the analogue of Eqs.(18).

\section*{4.2 Relativistic Kinetic Theory and Relativistic Micro-Canonical Ensemble}

In the rest-frame instant form of dynamics it is also possible to give a finally consistent treatment of relativistic kinetic theory and relativistic statistical mechanics [30]. In particular one can give a definition of the relativistic micro-canonical ensemble for an isolated system of $N$ interacting particles with fixed internal energy $E$ and rest spin $S$ only in terms of the internal Poincaré generators in the Wigner 3-spaces by means of the partition function (V is the volume defined by the function $\chi(V)$ vanishing outside it)

$$\tilde{Z}(E, S, V, N) = \frac{1}{N!} \int \prod_i^1 \int \prod_j^N d^3 \eta_i \chi(V) \int \prod_j^N d^3 \kappa_j \delta(M c^2 - E) \delta^3(\vec{S} - \vec{S}) \delta^3(P_{(int)}) \delta^3(\frac{K_{(int)}}{M c}).$$

Also it extension to non-inertial rest frames can be given by using the results of Ref.[21] with the result that notwithstanding the presence of long-range inertial forces one has still an equilibrium distribution.

\section*{4.3 Relativistic Quantum Mechanics and Relativistic Entanglement}

A new formulation of relativistic quantum mechanics in the Wigner 3-spaces of the inertial rest frame is developed in Ref.[33] in absence of the electro-magnetic field. It englobes all the known results about relativistic bound states (absence of relative times) and avoids the causality problems of the Hegerfeldt theorem [62] (the instantaneous spreading of wave packets).

In it one quantizes the frozen Jacobi data $z$ and $h$ of the canonical non-covariant decoupled external center of mass and the relative variables in the Wigner 3-spaces. Since the center of mass is decoupled, its non-covariance is irrelevant: like for the wave function of the universe, who will observe it?
The resulting Hilbert space has the following tensor product structure: 
\[ H = H_{\text{com},H,J} \otimes H_{\text{rel}}, \]
where \( H_{\text{com},H,J} \) is the Hilbert space of the external center of mass (in the Hamilton-Jacobi formulation due to the use of frozen Jacobi data) while \( H_{\text{rel}} \) is the Hilbert space of the relative variables in the abstract internal space living in the Wigner 3-spaces. While at the non-relativistic level this presentation of the Hilbert space is unitarily equivalent to the tensor product of the Hilbert spaces \( H_i \) of the individual particles \( H = H_1 \otimes H_2 \otimes \ldots \), this is not true at the relativistic level.

If one considers two scalar quantum particles with Klein-Gordon wave functions belonging to Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), in the tensor-product Hilbert space \( (\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \ldots \) there is no correlation among the times of the particles (their clocks are not synchronized) so that in most of the states there are some particles in the absolute future of the others. As a consequence the two types of Hilbert spaces lead to unitarily inequivalent descriptions and have different scalar products (compare Refs. [33] and [57]).

As a consequence, at the relativistic level the zeroth postulate of non-relativistic quantum mechanics does not hold: the Hilbert space of composite systems is not the tensor product of the Hilbert spaces of the sub-systems. Contrary to Einstein’s notion of separability (separate objects have their independent real states) [63] one gets a kinematical spatial non-separability induced by the need of clock synchronization for eliminating the relative times and to be able to formulate a well-posed relativistic Cauchy problem.

Moreover one has the non-locality of the non-covariant external center of mass which implies its non-measurability with local instruments. While its conjugate momentum operator must be well defined and self-adjoint, because its eigenvalues describe the possible values for the total momentum of the isolated system (the momentum basis is therefore a preferred basis in the Hilbert space), it is not clear whether it is meaningful to define center-of-mass wave packets.

These non-locality and kinematical spatial non-separability are due to the Lorentz signature of Minkowski space-time and this fact reduce the relevance of quantum non-locality in the study of the foundational problems of quantum mechanics [63] which have to be rephrased in terms of relative variables.

The quantization defined in Ref.[33] leads to a first formulation of a theory for relativistic entanglement, which is deeply different from the non-relativistic entanglement due to these kinematical non-locality and spatial non-separability. To have control on the Poincaré group one needs an isolated systems containing all the relevant entities (for instance both Alice and Bob) of the experiment under investigation and also the environment when needed. One has to learn to reason in terms of relative variables adapted to

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19 In Ref. [64] it was shown that the quantum Newton-Wigner position should not be a self-adjoint operator, but only a symmetric one, with an implication of bad localization.
the experiment like molecular physicists do when they look to the best system of Jacobi coordinates adapted to the main chemical bonds in the given molecule. This theory has still to be developed together with its extension to non-inertial rest frames.

4.4 Multitemporal Quantization in Non-Inertial Frames

This quantization of relativistic mechanics can be extended to the class of global non-inertial frames with space-like hyper-planes as 3-spaces and differentially rotating 3-coordinates defined in Ref.[21] by using the multi-temporal quantization approach developed in Ref.[65].

As shown in Ref.[34], in this type of quantization one quantizes only the 3-coordinates $\eta^i(\tau)$ of the particles and not the inertial effects (like the Coriolis and centrifugal ones): they remain c-numbers describing the appearances of phenomena. The known results in atomic and nuclear physics are reproduced.

4.5 Open Problem

The main open problem in SR is the quantization of fields in non-inertial frames due to the no-go theorem of Ref.[66] showing the existence of obstructions to the unitary evolution of a massive quantum Klein-Gordon field between two space-like surfaces of Minkowski space-time. It turns out that the Bogoljubov transformation connecting the creation and destruction operators on the two surfaces is not of the Hilbert-Schmidt type, i.e. that the Tomonaga-Schwinger approach in general is not unitary. One must reformulate the problem using the nice foliations of the admissible 3+1 splittings of Minkowski space-time and to try to identify all the 3+1 splittings allowing unitary evolution. This will be a prerequisite to any attempt to quantize canonical gravity taking into account the equivalence principle (global inertial frames do not exist) with the further problem that in general the Fourier transform does not exist in Einstein space-times.
5 Non-Inertial Frames in Asymptotically Minkowskian Einstein Space-Times and ADM Tetrads Gravity

After this description of SR induced by the metrology-oriented problem of clock synchronization, one has to face the same problems in the globally hyperbolic, topologically trivial, asymptotically Minkowskian space-times without super-translations of GR. As shown in the first paper of Ref.[17], in the chosen class of space-times the ten strong asymptotic ADM Poincaré generators $P_{\text{ADM}}^A$, $J_{\text{ADM}}^{AB}$ (they are fluxes through a 2-surface at spatial infinity) are well defined functionals of the 4-metric fixed by the boundary conditions at spatial infinity.

While in SR Minkowski space-time is an absolute notion, in Einstein GR also the space-time is a dynamical object [35] and the gravitational field is described by the metric structure of the space-time, namely by the ten dynamical fields $^4g_{\mu\nu}(x)$ ($x^\mu$ are world 4-coordinates). The 4-metric $^4g_{\mu\nu}(x)$ tends in a suitable way to the flat Minkowski 4-metric $^4\eta_{\mu\nu}$ at spatial infinity [17]: having an asymptotic Minkowskian background the usual splitting of the 4-metric in the bulk in a background plus perturbations in the weak field limit can be avoided as shown in Section VII.

The ten dynamical fields $^4g_{\mu\nu}(x)$ are not only a (pre)potential for the gravitational field (like the electro-magnetic and Yang-Mills fields are the potentials for electro-magnetic and non-Abelian forces) but also determines the chrono-geometrical structure of space-time through the line element $ds^2 = ^4g_{\mu\nu}(x) dx^\mu dx^\nu$. Therefore the 4-metric teaches relativistic causality to the other fields: it says to massless particles like photons and gluons which are the allowed world-lines in each point of space-time. This basic property is lost in every quantum field theory approach to gravity with a fixed background 4-metric.

In these space-times one can define global non-inertial frames by using the same admissible 3+1 splittings, centered on a time-like observer, and the observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ employed in SR. This will allow to separate the inertial (gauge) degrees of freedom of the gravitational field (playing the role of inertial potentials) from the dynamical tidal ones at the Hamiltonian level.

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20 These space-times must also be without Killing symmetries, because, otherwise, at the Hamiltonian level one should introduce complicated sets of extra Dirac constraints for each existing Killing vector.

21 The ACES mission of ESA [67] will give the first precision measurement of the gravitational red-shift of the geoid, namely of the $1/c^2$ deformation of Minkowski light-cone caused by the geo-potential. In every quantum field theory approach to gravity, where the definition of the Fock space requires the use of the Fourier transform on a fixed background space-time with a fixed light-cone, this is a non-perturbative effect requiring the re-summation of the perturbative expansion.
In GR the dynamical fields are the components $^4g_{\mu\nu}(x)$ of the 4-metric and not the embeddings $x^\mu = z^\mu(\tau, \sigma^r)$ defining the admissible 3+1 splittings of space-time like in the parametrized Minkowski theories of SR. Now the gradients $z_\mu^A(\tau, \sigma^r)$ of the embeddings give the transition coefficients from radar to world 4-coordinates, so that the components $^4g_{AB}(\tau, \sigma^r) = z_\mu^A(\tau, \sigma^r) z_\nu^B(\tau, \sigma^r)$ of the 4-metric will be the dynamical fields in the ADM action. Like in SR the 4-vectors $z_\mu^A(\tau, \sigma^r)$, tangent to the 3-spaces $\Sigma_\tau$, are used to define the unit normal $l^\mu(\tau, \sigma^r) = z_\mu^A(\tau, \sigma^r) l^A(\tau, \sigma^r)$ to $\Sigma_\tau$, while the 4-vector $z^\mu_\tau(\tau, \sigma^r)$ has the lapse function as component along the unit normal and the shift functions as components along the tangent vectors. Since the world-line of the time-like observer can be chosen as the origin of a set of the spatial world coordinates, i.e. $x^\mu(\tau) = (x^\alpha(\tau); 0)$, it turns out that with this choice the space-like surfaces of constant coordinate time $x^\alpha(\tau) = \text{const.}$ coincide with the dynamical instantaneous 3-spaces $\Sigma_\tau$ with $\tau = \text{const.}$. By using asymptotic flat tetrads $\epsilon_\mu^A = \delta_\mu^0 \delta_\tau^A + \delta_\mu^i \delta_i^A$ (with $\epsilon_\mu^A$ denoting the inverse flat cotetrads) and by choosing a coordinate world time $x^\alpha(\tau) = x_\alpha^o + \epsilon_\alpha^0 \tau = x_\alpha^o + \tau$, one gets the following preferred embedding corresponding to these given world 4-coordinates

$$x^\mu = z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \epsilon_\mu^\tau \sigma^r = \delta_\mu^0 x_\alpha^0 + \epsilon_\mu^A \sigma^A. \tag{20}$$

This choice implies $z_\mu^A(\tau, \sigma^r) = \epsilon_\mu^A$ and $^4g_{\mu\nu}(x = z(\tau, \sigma^r)) = \epsilon_\mu^A \epsilon_\nu^B ^4g_{AB}(\tau, \sigma^r)$.

As shown in Ref.[35], the dynamical nature of space-time implies that each solution (i.e. an Einstein 4-geometry) of Einstein’s equations (or of the associated ADM Hamilton equations) dynamically selects a preferred 3+1 splitting of the space-time, namely in GR the instantaneous 3-spaces are dynamically determined in the chosen world coordinate system. Eq.(20) can be used to describe this 3+1 splitting and then by means of 4-diffeomorphisms the solution can be written in an arbitrary world 4-coordinate system in general not adapted to the dynamical 3+1 splitting. This gives rise to the 4-geometry corresponding to the given solution.

To define the canonical formalism the Einstein-Hilbert action for metric gravity (depending on the second derivative of the metric) must be replaced with the ADM action (the two actions differ for a surface term at spatial infinity). As shown in the first paper of Refs.[17], the Legendre transform and the definition of a consistent canonical Hamiltonian require the introduction of the DeWitt surface term at spatial infinity: the final canonical Hamiltonian turns out to be the strong ADM energy (a flux through a 2-surface at spatial infinity), which is equal to the weak ADM energy (expressed as a volume integral over the 3-space) plus constraints. Therefore there is not a frozen picture but an evolution generated by a Dirac Hamiltonian equal to the weak ADM energy plus a linear combination of the first class constraints. Also the other strong ADM Poincaré generators are replaced by their weakly equivalent weak form $P^A_{ADM}, J^{AB}_{ADM}$.
In the first paper of Ref. [17] it is also shown that the boundary conditions on the 4-metric required by the absence of super-translations imply that the only admissible 3+1 splittings of space-time (i.e. the allowed global non-inertial frames) are the *non-inertial rest frames*: their 3-spaces are asymptotically orthogonal to the weak ADM 4-momentum. Therefore one gets \( P_{\text{ADM}}^r \approx 0 \) as the rest-frame condition of the 3-universe with a mass and a rest spin fixed by the boundary conditions. Like in SR the 3-universe can be visualized as a decoupled non-covariant (non-measurable) external relativistic center of mass plus an internal non-inertial rest-frame 3-space containing only relative variables (see the first paper in Ref. [40]).

### 5.1 The Parametrization of Tetrads for ADM Tetrad Gravity

To take into account the coupling of fermions to the gravitational field metric gravity has to be replaced with tetrad gravity. This can be achieved by decomposing the 4-metric on cotetrad fields (by convention a sum on repeated indices is assumed)

\[
4 g_{AB}(\tau, \sigma^r) = E^A_{(\alpha)}(\tau, \sigma^r) 4 \eta_{(\alpha)(\beta)} E^B_{(\beta)}(\tau, \sigma^r),
\]

(21)

by putting this expression into the ADM action and by considering the resulting action, a functional of the 16 fields \( E^A_{(\alpha)}(\tau, \sigma^r) \), as the action for ADM tetrad gravity. In Eq.(21) \( (\alpha) \) are flat indices and the cotetrad fields \( E^A_{(\alpha)} \) are the inverse of the tetrad fields \( E^A_{(\alpha)} \), which are connected to the world tetrad fields by \( E^\mu_{(\alpha)}(x) = \varepsilon^\mu_A(\tau, \sigma^r) E^A_{(\alpha)}(z(\tau, \sigma^r)) \) by the embedding of Eq.(20).

This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer’s gyroscopes. This framework was developed in the second and third paper of Refs. [17].

Even if the action of ADM tetrad gravity depends upon 16 fields, the counting of the physical degrees of freedom of the gravitational field does not change, because this action is invariant not only under the group of 4-diffeomorphisms but also under the O(3,1) gauge group of the Newman-Penrose approach [68] (the extra gauge freedom acting on the tetrads in the tangent space of each point of space-time).

The cotetrads \( E^A_{(\alpha)}(\tau, \sigma^r) \) are the new configuration variables. They are connected to cotetrads \( E^A_{(\alpha)}(\tau, \sigma^r) \) adapted to the 3+1 splitting of space-time, namely such that the inverse adapted time-like tetrad \( 4 E^A_{(\alpha)}(\tau, \sigma^r) \) is
the unit normal to the 3-space \( \Sigma_\tau \), by a standard Wigner boosts for time-like Poincaré orbits with parameters \( \varphi_{(a)}(\tau, \sigma^r) \), \( a = 1, 2, 3 \)

\[
E_{A}^{(\alpha)} = L^{(\alpha)}_{(\beta)}(\varphi_{(a)}) E_{A}^{(\beta)} \quad 4 g_{AB} = 4 E^{(\alpha)}_{A} 4 \eta_{(\alpha)(\beta)} 4 E^{(\beta)}_{B} ,
\]

\[
L^{(\alpha)}_{(\beta)}(\varphi_{(a)}) \overset{\text{def}}{=} L^{(\alpha)}_{(\beta)}(V(z(\sigma)); \bar{\phi}) = \delta_{(\beta)}^{(\alpha)} + 2 \epsilon V^{(\alpha)}(z(\sigma)) \bar{V}^{(\beta)} - \epsilon \frac{(V^{(\alpha)}(z(\sigma)) + \bar{V}^{(\alpha)})(V^{(\beta)}(z(\sigma)) + \bar{V}^{(\beta)})}{1 + V^{(\alpha)}(z(\sigma))} .
\]

In each tangent plane to a point of \( \Sigma_\tau \) this point-dependent standard Wigner boost sends the unit future-pointing time-like vector \( \bar{V}^{(\alpha)} = (1; 0) \) into the unit time-like vector \( V^{(\alpha)} = 4 E^{(\alpha)}_{A} l^{A} = \left( \sqrt{1 + \sum_{a} \bar{\varphi}_{(a)}^{2}} \right)^{-1} \bar{\varphi}^{(a)} = -\epsilon \bar{\varphi}^{(a)} \). As a consequence, the flat indices \( (a) \) of the adapted tetrads and cotetrads and of the triads and cotriads on \( \Sigma_\tau \) transform as Wigner spin-1 indices under point-dependent SO(3) Wigner rotations \( R_{(a)(b)}(V(z(\sigma))); \Lambda(z(\sigma)) \) associated with Lorentz transformations \( A^{(\alpha)}_{(\beta)}(z) \) in the tangent plane to the space-time in the given point of \( \Sigma_\tau \). Instead the index \( (o) \) of the adapted tetrads and cotetrads is a local Lorentz scalar index.

The adapted tetrads and cotetrads have the expression

\[
4 \bar{E}^{A}_{(o)} = \frac{1}{1 + n} (1; - \sum_{a} n_{(a)} 3 e^{r}_{(a)}) = l^{A} , \quad 4 \bar{E}^{A}_{(a)} = (0; 3 e^{r}_{(a)}),
\]

\[
4 E^{(a)}_{A} = (1 + n)(1; 0) = \epsilon l_{A} , \quad 4 E^{(a)}_{A} = (n_{(a)}; 3 e^{r}_{(a)}),
\]

where \( 3 e_{(a)}^{r} \) and \( 3 e_{(a)r} \) are triads and cotriads on \( \Sigma_\tau \) and \( n_{(a)} = n^{r} 3 e^{r}_{(a)} = n_{(a)} 3 e^{r}_{(a)} \) are adapted shift functions. In Eqs.(23) \( N(\tau, \sigma) = 1 + n(\tau, \sigma) > 0 \), with \( n(\tau, \sigma) \) vanishing at spatial infinity (absence of super-translations), so that \( N(\tau, \sigma) d\tau \) is positive from \( \Sigma_\tau \) to \( \Sigma_{\tau + d\tau} \), is the lapse function; \( N^{r}(\tau, \sigma) = n^{r}(\tau, \sigma) \) vanishing at spatial infinity (absence of super-translations), are the shift functions.

The adapted tetrads \( 4 \bar{E}^{A}_{(a)} \) are defined modulo SO(3) rotations \( 4 E^{A}_{(a)} = \sum_{b} R_{(a)(b)}(\alpha_{(a)}) 4 \bar{E}^{A}_{(b)} \), \( 3 e^{r}_{(a)} = \sum_{b} R_{(a)(b)}(\alpha_{(a)}) 3 e^{r}_{(b)} \), where \( \alpha_{(a)}(\tau, \sigma) \) are \( \delta_{(a)(b)} = 3 e^{r}_{(a)} 3 e_{(b)r} \).
three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin \( \alpha(\tau, \sigma) = 0 \), one arrives at the following adapted tetrads and cotetrads \( \bar{n}_{(a)} = \sum_b n_{(b)} R_{(b)(a)}(\alpha(\epsilon_c)) \), \( \sum_a n_{(a)} \eta^r_{(a)} = \sum_a \bar{n}_{(a)} \eta^r_{(a)} \)

\[
4 \bar{E}^A_{(a)} = 4 \bar{E}^A_{(o)} = \frac{1}{1 + n} (1; - \sum_a \bar{n}_{(a)} \eta^r_{(a)}) = l^A, \quad 4 \bar{E}^A_{(a)} = (0; \eta^r_{(a)}),
\]

\[
4 \bar{E}^A_{(o)} = 4 \bar{E}^A_{(a)} = (1 + n)(1; 0) = \epsilon l_A, \quad 4 \bar{E}^A_{(o)} = (\bar{n}_{(a)}; 3 \bar{e}_{(a)r}), \quad (24)
\]

which one will use as a reference standard.

The expression for the general tetrad

\[
4 E^A_{(o)} = 4 \bar{E}^A_{(\beta)} L_{(\beta)(\alpha)}(\varphi(\alpha)) = 4 \bar{E}^A_{(o)} L^{(\alpha)}(\varphi(\alpha)) + \sum_{ab} 4 \bar{E}^A_{(b)} R^T_{(b)(a)}(\alpha(\epsilon_c)) L_{(\alpha)}(\varphi(\alpha)), \quad (25)
\]

shows that every point-dependent Lorentz transformation \( A \) in the tangent planes may be parametrized with the (Wigner) boost parameters \( \varphi(\alpha) \) and the Euler angles \( \alpha(\alpha) \), being the product \( \varphi = RL \) of a rotation and a boost.

The future-oriented unit normal to \( \Sigma_\tau \) and the projector on \( \Sigma_\tau \) are \( l_A = \epsilon (1 + n)(1; 0) \), \( 4 g^{AB} l_A l_B = \epsilon, \quad l^A = \epsilon (1 + n) 4 g^{Ar} = \frac{1}{1 + n} (1; -n^r) = \frac{1}{1 + n} (1; - \sum_a \bar{n}_{(a)} \eta^r_{(a)}), \quad 3 h^B_A = \delta^B_A - \epsilon l_A l_B.

The 4-metric has the following expression

\[
4 g_{rr} = \epsilon [(1 + n)^2 - 3 g^{rs} n_r n_s] = \epsilon [(1 + n)^2 - \sum_a \bar{n}_{(a)}^2],
\]

\[
4 g_{rr} = - \epsilon n_r = - \epsilon \sum_a \bar{n}_{(a)} \eta^r_{(a)r},
\]

\[
4 g_{rs} = - \epsilon^3 g_{rs} = - \epsilon \sum_a n^r_{(a)r} n^s_{(a)s} = - \epsilon \sum_a \eta^r_{(a)r} \eta^s_{(a)s},
\]

\[
4 g^{rr} = \frac{\epsilon}{(1 + n)^2}, \quad 4 g^{rr} = - \epsilon \frac{n^r_{(a)}}{(1 + n)^2} = - \epsilon \sum_a \eta^r_{(a)} \bar{n}_{(a)},
\]

\[
4 g^{rs} = - \epsilon^3 g^{rs} - \epsilon \frac{n^r n^s_{(a)}}{(1 + n)^2} = - \epsilon \sum_{ab} \eta^r_{(a)} \eta^s_{(b)} (\delta_{(a)(b)} - \bar{n}_{(a)} \bar{n}_{(b)}) - (1 + n)^2).
\]
\[
\sqrt{g} = \sqrt{|g|} = \sqrt[3]{g} = \sqrt[3]{\epsilon^3 g^\tau\tau} = \sqrt{\gamma} (1 + n) = \gamma^3 (1 + n),
\]
\[
^3 g = \gamma = (\gamma^3 e)^2, \quad ^3 e = \det^3 e_{(a)\tau}.
\]

The 3-metric \(^3 g_{\tau\sigma}\) has signature \((+++)\), so that one may put all the flat 3-indices down. One has \(^3 g^{\tau\mu}^3 g_{\mu\sigma} = \delta^\tau_\sigma\).

### 5.2 The ADM Phase Space and the ADM Hamilton Equations

The given parametrization of the cotetrad fields leads to rewrite the action of ADM tetrad gravity in terms of the following 16 fields as configuration variables: three boost parameters \(\varphi_{(a)}(\tau, \sigma^u)\); the lapse \(N(\tau, \sigma^u) = 1 + n(\tau, \sigma^u)\) and shift \(n_{(a)}(\tau, \sigma^u)\) functions; the nine components of cotriad fields \(^3 e_{(a)\tau}(\tau, \sigma^u)\) on the 3-spaces \(\Sigma_\tau\). As shown in the second and third paper of Ref.[17], in Ref.[36] and in the first paper of Ref.[40], the ADM action for the gravitational field has the expression

\[
S_{\text{grav}} = \frac{c^3}{16 \pi G} \int d\tau d^3 \sigma \left[(1 + n) \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \right. \\
+ \frac{3 \epsilon}{2(1 + n)} (G^{c}_{a})_{(a)\tau}(b) e_{(c)\tau}(d) \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \right. \\
\left. \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) - \partial^\tau \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \right) \left(\tau, \sigma^u\right).
\]

In it \(^3 \Omega_{\tau\sigma}(a) = \partial^\tau \psi_{(a)\tau}(b) - \partial^\sigma \psi_{(a)\tau}(b) - \psi_{(a)\tau}(b) \psi_{(c)\tau}(d) \psi_{(a)\tau}(b) \psi_{(c)\tau}(d)\) is the field strength associated with the 3-spin connection \(^3 \omega_{(a)\tau}(b) = \frac{1}{2} \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \left(\partial^\tau \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) - \partial^\tau \gamma^3 e_{(a)\tau}(b) e_{(c)\tau}(d) \right)\) and \((G^{c}_{a})_{(a)\tau}(b) e_{(c)\tau}(d)\) is the flat (with lower indices) inverse of the flat Wheeler-DeWitt super-metric \(G_{(a)\tau}(b) e_{(c)\tau}(d) = \delta_{(a)\tau}(b) e_{(c)\tau}(d)\).

The canonical momenta \(\pi_{\varphi_{(a)}(\tau, \sigma^u)}, \pi_{n(\tau, \sigma^u)}, \pi_{n_{(a)}(\tau, \sigma^u)}, \gamma^3 e_{(a)\tau}(\tau, \sigma^u)\), conjugate to the configuration variables satisfy 14 first-class constraints: the ten primary constraints (the last three constraints generate rotations on quantities with flat indices \(a\) like the cotriads)

\[\pi_{\varphi_{(a)}(\tau, \sigma^u)} \approx 0, \quad \pi_{n(\tau, \sigma^u)} \approx 0, \quad \pi_{n_{(a)}(\tau, \sigma^u)} \approx 0,\]
\[ 3M_{(a)}(\tau, \sigma^u) = \epsilon_{(a)(b)(c)} 3\epsilon_{(b)r}(\tau, \sigma^u)^3\pi^r_{(c)}(\tau, \sigma^u) \approx 0, \quad (28) \]

and the secondary super-Hamiltonian and super-momentum constraints

\[
\mathcal{H}(\tau, \sigma^u) = \left[ \frac{c^3}{16\pi G} 3\epsilon \epsilon_{(a)(b)(c)} 3\epsilon^r_{(a)} 3\epsilon_{(b)} 3\Omega_{rs(c)} - \right. \\
- \left. \frac{2\pi G}{c^3} 3G_{\sigma(a)(b)(c)} 3\epsilon_{(a)r} 3\pi^r_{(b)} 3\pi_{(c)s} 3\pi^s_{(d)} \right] (\tau, \sigma^u) + \mathcal{M}(\tau, \sigma^u) \approx 0, \\
\mathcal{H}_{(a)}(\tau, \sigma^u) = \left[ \partial_r 3\pi^r_{(a)} - \epsilon_{(a)(b)(c)} 3\omega_{r(b)} 3\pi^r_{(c)} + 3\epsilon^r_{(a)} \mathcal{M}_r \right] (\tau, \sigma^u) \approx 0. \quad (29) \]

The functions \( \mathcal{M}(\tau, \sigma^u) \) and \( \mathcal{M}_r(\tau, \sigma^u) \) describe the matter present in the space-time: \( \mathcal{M}(\tau, \sigma^u) \) is the (matter- and metric-dependent) internal mass density, while \( \mathcal{M}_r(\tau, \sigma^u) \) is the universal (metric-independent) internal momentum density. If the action of matter is added to Eq.(27), one can evaluate the energy-momentum tensor \( T^{AB}(\tau, \sigma^u) = -\left[ \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{AB}} \right] (\tau, \sigma) \) of the matter \(^{23}\) and determine these functions from the following parametrization

\[
T^{rr}(\tau, \sigma^u) = \frac{\mathcal{M}(\tau, \sigma^u)}{[\epsilon(1+n)^2(\tau, \sigma^u)],} \\
T^{rr}(\tau, \sigma^u) = \frac{3\epsilon^r_{(a)} [1+n] 3\epsilon_{(a)} M_s - n_{(a)} M_s}{3\epsilon (1+n)^2} (\tau, \sigma^u). \quad (30) \]

The extrinsic curvature tensor of the 3-spaces \( \Sigma_r \) as 3-manifolds embedded into the space-time has the following expression in terms of the barred cotriads of Eq.(24) and their conjugate barred momenta

\[
3K_{rs} = -\frac{4\pi G}{c^3} \sum_{abu} \left[ \left( 3\bar{\epsilon}^{(a)r}_{(b)s} + 3\bar{\epsilon}^{(a)s}_{(b)r} \right)^3\bar{\epsilon}^{(a)u}_{(b)} \pi^{u}_{(b)} - \right. \\
- \left. 3\bar{\epsilon}^{(a)r}_{(b)s} 3\bar{\epsilon}^{(a)s}_{(b)r} 3\bar{\epsilon}^{(a)u}_{(b)} \pi^{u}_{(b)} \right]. \quad (31) \]

Therefore the basis of canonical variables for this formulation of tetrad gravity, naturally adapted to 7 of the 14 first-class constraints, is

\[
\begin{array}{ccc}
\varphi_{(a)} & n & \pi_{(a)} \\
\pi_{\varphi_{(a)}} \approx 0 & \pi_n \approx 0 & \pi_{\pi_{(a)}} \approx 0 \end{array} \quad (32) \]

\(^{23}\) The Hamilton equations imply \( 4\nabla_A T^{AB} \equiv 0 \) in accord with Einstein’s equations and the Bianchi identity.
The behavior of these fields at spatial infinity (compatible with the absence of super-translations) is given in Eqs. (5.5) of the third paper in Refs. [17]; in particular for the cotriads one has
\[ \lim_{r \to \infty} (1 + \text{const.} \cdot \frac{1}{r}) \delta_{ur} + O(r^{-3/2}) \quad (r = \sqrt{\sum_r (\sigma^r)^2}). \]

From the action (29), after having added the matter action, one can obtain the standard non-Hamiltonian ADM equations \(|r|\) denotes the 3-covariant derivative in the 3-space \(\Sigma_\tau\) with 3-metric \(3g_{rs}\); \(3R_{rs}\) is the 3-Ricci tensor of \(\Sigma_\tau\)

\[ \partial_\tau g_{rs} \overset{\circ}{=} n_{r|s} + n_{s|r} - 2(1 + n)K_{rs}, \]
\[ \partial_\tau K_{rs} \overset{\circ}{=} (1 + n)\left(3R_{rs} + 3K_{rs} - 2K_{ru}K_{us} + n_{u|r}K_{us} + n_{u|u}K_{rs} - 2K_{ru}\right), \]

with the quantities appearing in these equations re-expressed in terms of the configurational variables of Eq. (32).

Instead at the Hamiltonian level one can get the Hamilton equations for all the variables of the canonical basis (32), as shown in the first paper of Ref. [40], by using the Dirac Hamiltonian. As shown in Refs. [17], the Dirac Hamiltonian has the form (if the matter contains the electro-magnetic field there are extra terms with the electro-magnetic first-class constraints)

\[ H_D = \frac{1}{c} \hat{E}_{ADM} + \int d^3\sigma \left[ nH - n(\alpha)H(\alpha) \right](\tau, \sigma^u) + \]
\[ + \int d^3\sigma \left[ \lambda_n \pi_n + \lambda_{\alpha(\alpha)} \pi_{\alpha(\alpha)} + \lambda_{\varphi(\alpha)} \pi_{\varphi(\alpha)} + \mu(\alpha)3M(\alpha) \right](\tau, \sigma^u), \]

where \(\hat{E}_{ADM}\) is the weak ADM energy and the \(\lambda\)'s are arbitrary Dirac multipliers.

See Eqs. (2.22), (3.43) and (3.47) of the first paper of Ref. [40] for the expression of the ten weak asymptotic ADM Poincaré generators \(\hat{E}_{ADM}, \hat{P}_{ADM}^r, \hat{J}_{adm}^r, \hat{K}_{ADM}^r\). Since one is in a non-inertial rest frame (due to the absence of super-translations), one has the rest-frame conditions \(\hat{P}_{ADM}^r \approx 0\) like in SR. Then one has to add the conditions \(\hat{K}_{ADM}^r \approx 0\) to eliminate the internal 3-center of mass of the 3-universe like in SR [21]. Therefore the 3-universe can be seen as a decoupled external canonical non-covariant center of mass carrying a pole-dipole structure: the invariant mass \(M_c = \frac{1}{c} \hat{E}_{ADM}\) and the rest spin \(\hat{J}_{ADM}^r\). This view is in accord with an old suggestions of Dirac [8].
In Ref.[40] the study of ADM canonical tetrad gravity was done with the following type of matter: N charged scalar particles (described by the canonical variables $\eta^r_i(\tau), \kappa_{ir}(\tau)$) and the electro-magnetic field in the non-covariant radiation gauge (described by the canonical variables $A^r_\perp(\tau,\sigma^u), \pi^r_\perp(\tau,\sigma^u)$) as shown in Ref.[21]). The particles (described by an action like the one in Eq.(6)) have not only Grassmann-valued electric charges $Q_i$ ($Q_i^2 = 0, Q_i Q_j = Q_j Q_i$ for $i \neq j$) to regularize the electro-magnetic self-energies, but also Grassmann-valued signs of the energy ($\eta^2_i = 0, \eta_i \eta_j = \eta_j \eta_i$ for $i \neq j$) to regularize the gravitational self-energies $^{24}$.

Instead in Ref.[41] the matter is a perfect fluid described by the action of Ref.[43] re-expressed in the 3+1 point of view in Refs.[42].

In the case of N particles the functions $M$ and $M_r$ have the expression (see Ref.[40] for their form in presence of the electro-magnetic field)

$$M(\tau, \sigma^u) = \sum_{i=1}^{N} \delta(\sigma^u, \eta^u_i(\tau)) \eta_i \sqrt{m_i^2 c^2 + 3e_{(a)}^r(\tau, \sigma^u) k_{ir}(\tau, \sigma^u) k_{is}(\tau)},$$

$$M_r(\tau, \sigma^u) = \sum_{i=1}^{N} \eta_i k_{ir}(\tau), \quad (35)$$

while in the case of dust [41], described by canonical coordinates $\alpha^i(\tau, \sigma^u)$, $\Pi_i(\tau, \sigma^u), i = 1, 2, 3$, they have the expression

$$M(\tau, \sigma^u) = \sqrt{\mu^2 [\det(\partial \alpha^j)]^2 + \tilde{\phi}^{-2/3} \sum_{a r s i j} Q_a^- V_{ra} V_{sa} \partial_r \alpha^i \partial_s \alpha^j \Pi_i \Pi_j(\tau, \sigma^u)},$$

$$M_r(\tau, \sigma^u) = \sum_i \partial_r \alpha^i(\tau, \sigma^u) \Pi_i(\tau, \sigma^u). \quad (36)$$

---

$^{24}$ Both quantities are two-valued. The elementary electric charges are $Q = \pm e$, with $e$ the electron charge. Analogously the sign of the energy of a particle is a topological two-valued number (the two branches of the mass-shell hyperboloid). The formal quantization of these Grassmann variables gives two-level fermionic oscillators. At the classical level the self-energies make the classical equations of motion ill-defined on the world-lines of the particles. The ultraviolet and infrared Grassmann regularization allows to cure this problem and to get consistent solution of regularized equations of motion. See Refs.[28] for the electro-magnetic case.
6 The York Canonical Basis and the Inertial and Tidal Degrees of Freedom of the Gravitational Field

The presence of 14 first-class constraints in the phase space having the 32 fields of Eq.(32) as a canonical basis implies that there are 14 gauge variables describing inertial effects and 2 canonical pairs of physical degrees of freedom describing the tidal effects of the gravitational field (namely gravitational waves in the weak field limit). To disentangle the inertial effects from the tidal ones one needs a canonical transformation to a new canonical basis adapted to all the ten primary constraints (28) and containing the barred variables defined in Eq.(24). This is the topic of this Section.

6.1 The York Canonical Basis

A canonical transformation adapted to the ten primary constraints (28) was found in Ref.[36]. It implements the York map of Ref.[37] in the cases in which the 3-metric $g_{rs}$ has three distinct eigenvalues and diagonalizes the York-Lichnerowicz approach (see Ref.[38] for a review).

As said before Eq.(24), one can decompose the cotriads on $\Sigma_\tau$ in the product of a rotation matrix, belonging to the subgroup SO(3) of the tetrad gauge group and depending on three Euler angles $\alpha(a)$, and of barred cotriads depending only on six independent fields. The canonical transformation Abelianizes the constraints $3M(a) (\tau, \sigma^u) \approx 0$ of Eqs.(28), satisfying $\{3M(a)(\tau, \sigma^u), 3M(b)(\tau, \sigma^w)\} = \epsilon(a)(b)(c) 3M(c)(\tau, \sigma^w) \delta^3(\sigma^u, \sigma^w)$, and replaces them with the vanishing of the three momenta $\pi_{(a)}(\tau, \sigma^r) \approx 0$ conjugate to the Euler angles.

The new canonical basis, named York canonical basis, is ($a = 1, 2, 3; \bar{a} = 1, 2$)

\[
\begin{array}{cccccc}
\phi(a) & \alpha(a) & n & \bar{n}(a) & \theta^r & \tilde{\phi} \\
\pi_{\phi(a)} \approx 0 & \pi_{(a)} \approx 0 & \pi_n \approx 0 & \pi_{\bar{n}(a)} \approx 0 & \pi_{\theta^r} & \pi_{\tilde{\phi}} = \frac{\pi^3}{12\pi\alpha_0} 3K \Pi_{\bar{a}}
\end{array}
\]

(37)

In it the cotriads and the components of the 4-metric have the following expression

\[
3e_{(a)r} = \sum_b R_{(a)(b)}(\alpha(c)) 3\tilde{e}_{(b)r} = \sum_b R_{(a)(b)}(\alpha(c)) V_{rb}(\theta^l) \tilde{\phi}^{1/3} \epsilon \sum_k^2 \gamma_{ak} R_a,
\]
\[ 4 g_{\tau\tau} = \epsilon \left[ (1 + n)^2 - \sum_a \bar{n}_a^2 \right], \]
\[ 4 g_{\tau r} = -\epsilon \sum_a n_{(a)} e_{(a)r} = -\epsilon \sum_a \bar{n}_a e_{(a)r}, \quad \tilde{\phi} = \phi^6 = \sqrt{\det 3 g_{rs}}, \]
\[ 4 g_{rs} = -\epsilon^3 g_{rs} = -\epsilon \tilde{\theta}^{2/3} \sum_a V_r a (\theta^i) V_{sa} (\theta^s) Q_a^2, \quad Q_a = \epsilon \sum_{i=1}^3 \gamma_{ai} R_i, \quad (38) \]

The set of numerical parameters \( \gamma_{ai} \) appearing in \( Q_a \) satisfies [17] \( \sum_a \gamma_{ai} = 0, \sum_u \gamma_{au} \gamma_{bu} = \delta_{ab}, \sum_u \gamma_{au} \gamma_{bu} = \delta_{uv} - \frac{4}{3} \). Each solution of these equations defines a different York canonical basis.

This canonical basis has been found due to the fact that the 3-metric \( g_{rs} \) is a real symmetric \( 3 \times 3 \) matrix, which may be diagonalized with an orthogonal matrix \( V(\theta^i) \), \( V^{-1} = V^T \left( \sum_u V_{ua} V_{ub} = \delta_{ab}, \sum_u V_{ua} V_{va} = \delta_{uv}, \sum_{uv} \epsilon_{uus} V_{ua} V_{vb} = \sum_{c} \epsilon_{abc} V_{cw} \right) \), \( \det V = 1 \), depending on three parameters \( \theta^i (i = 1, 2, 3) \) \(^{25}\), whose conjugate momenta \( H_i (\theta^i) \) are to be determined as solutions of the super-momentum constraints. If one chooses these three gauge parameters to be Euler angles \( \hat{\theta} (\tau, \sigma) \), one gets a description of the 3-coordinate systems on \( \Sigma_\tau \) from a local point of view, because they give the orientation of the tangents to the three 3-coordinate lines through each point. However, for the calculations (see Refs. [40]) it is more convenient to choose the three gauge parameters as first kind coordinates \( \theta^i (\tau, \sigma) \) \( (-\infty < \theta^i < +\infty) \) on the \( \text{O}(3) \) group manifold, so that by definition one has \( V_{ru} (\theta^i) = \left( e^{-\sum_i \hat{T}_i \theta^i} \right)_{ru} \), where \( \left( \hat{T}_i \right)_{ru} = \epsilon_{ru} \) are the generators of the \( \text{o}(3) \) Lie algebra in the adjoint representation, and the Euler angles may be expressed as \( \hat{\theta} = f^i (\theta^i) \). The Cartan matrix has the form \( A(\theta^n) = \frac{1 - e^{-\sum_i \hat{T}_i \theta^i}}{\sum_i \hat{T}_i \theta^i} \) and satisfies \( A_{ri} (\theta^r) \theta^i = \delta_{ri} \theta^i; \ B (\theta^i) = A^{-1} (\theta^i) \).

From now on for the sake of notational simplicity the symbol \( V \) will mean \( V (\theta^i) \).

The extrinsic curvature tensor of the 3-space \( \Sigma_\tau \) has the expression

\[ 3 K_{ra} (\tau, \sigma^u) = -\frac{4\pi G}{c^3} \left[ \tilde{\phi}^{-1/3} \left( \sum_a Q_a^2 V_{ra} V_{sa} \left[ 2 \sum_b \gamma_{ba} H_b - \tilde{\phi} \tau_{ga} \right] + \right. \right. \]
\[ + \left. \left. \sum_{ab} Q_a Q_b \left( V_{ra} V_{sb} + V_{rb} V_{sa} \right) \sum_{tuv} \epsilon_{abu} V_{wt} B_{uw} \tau_{vt} (\theta) \right] Q_b Q^a Q^b_1 - Q_a Q^a_1 \right) \right] (\tau, \sigma^u). \]

\(^{25}\) Due to the positive signature of the 3-metric, one defines the matrix \( V \) with the following indices: \( V_{ru} \). Since the choice of Shankugadhasan canonical bases breaks manifest covariance, one will use the notation \( V_{ua} = \sum_v V_{uv} \delta_v (a) \) instead of \( V_{u(a)} \).
This canonical transformation realizes a York map because the gauge variable $\pi_{\tilde{\phi}}$ (describing the freedom in the choice of the trace of the extrinsic curvature of the instantaneous 3-spaces $\Sigma_\tau$) is proportional to York internal extrinsic time $^3K$. It is the only gauge variable among the momenta: this is a reflex of the Lorentz signature of space-time, because $\pi_{\tilde{\phi}}$ and $\theta^n$ can be used as a set of 4-coordinates for the space-time [35]. The York time describes the effect of gauge transformations producing a deformation of the shape of the 3-space along the 4-normal to the 3-space as a 3-sub-manifold of space-time.

Its conjugate variable, to be determined by the super-Hamiltonian constraint (interpreted as the Lichnerowicz equation), is $\tilde{\phi} = \phi^6 = ^3\tilde{\varepsilon} = \sqrt{\det^3g_{rs}}$, which is proportional to Misner’s internal intrinsic time; moreover $\tilde{\phi}$ is the 3-volume density on $\Sigma_\tau$: $V_R = \int_R d^3\sigma \tilde{\phi}$, $R \subset \Sigma_\tau$. Since one has $^3g_{rs} = ^2\tilde{\varepsilon}^3^3g_{rs}$ with $\det^3g_{rs} = 1$, $\tilde{\phi}$ is also called the conformal factor of the 3-metric.

The two pairs of canonical variables $R_\bar{a}, \Pi_\bar{a}$, $\bar{a} = 1, 2$, describe the generalized tidal effects, namely the independent physical degrees of freedom of the gravitational field. They are 3-scalars on $\Sigma_\tau$ and the configuration tidal variables $R_\bar{a}$ parametrize the two eigenvalues of the 3-metric $^3g_{rs}$ with unit determinant. They are Dirac observables only with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if one fixes completely the gauge and one goes to Dirac brackets, then the only surviving dynamical variables $R_\bar{a}$ and $\Pi_\bar{a}$ become two pairs of non canonical Dirac observables for that gauge: the two pairs of canonical Dirac observables have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the $R_\bar{a}, \Pi_\bar{a}$ variables.

Therefore, the 14 arbitrary gauge variables are $\varphi_{(a)}(\tau, \sigma^u)$, $\alpha_{(a)}(\tau, \sigma^u)$, $n(\tau, \sigma^u)$, $\bar{n}_{(a)}(\tau, \sigma^u)$, $\theta^i(\tau, \sigma^u)$, $\pi_{\tilde{\phi}}(\tau, \sigma^u)$: they describe the following generalized inertial effects [36]:

a) $\alpha_{(a)}(\tau, \sigma^u)$ and $\varphi_{(a)}(\tau, \sigma^u)$ are the 6 configuration variables parametrizing the O(3,1) gauge freedom in the choice of the tetrads in the tangent plane to each point of $\Sigma_\tau$ and describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \sigma)$. They fix the unit 4-velocity of the observer and the conventions for the orientation of three gyroscopes and their transport along the world-line of the observer. The Schwinger time gauges are defined by the gauge fixings $\alpha_{(a)}(\tau, \sigma^u) \approx 0$, $\varphi_{(a)}(\tau, \sigma^u) \approx 0$.

b) $\theta^i(\tau, \sigma^u)$ (depending only on the 3-metric) describe the arbitrariness in the choice of the 3-coordinates in the instantaneous 3-spaces $\Sigma_\tau$ of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice will induce a pattern of relativistic inertial forces for the gravitational field,
whose potentials are the functions $V_{ra}(\theta^i)$ present in the weak ADM energy $\hat{E}_{ADM}$.

(c) $\bar{n}_{(a)}(\tau, \sigma^u)$, the shift functions, describe which points on different instantaneous 3-spaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components $\hat{g}_{\tau \tau}(\tau, \sigma^u)$ of the 4-metric, namely they are the gravito-magnetic potentials 26 responsible of effects like the dragging of inertial frames (Lense-Thirring effect) in the post-Newtonian approximation. The shift functions are determined by the $\tau$-preservation of the gauge fixings determining the gauge variables $\theta^i(\tau, \sigma^u)$.

d) $\tilde{\pi}(\tau, \sigma^u)$, i.e. the York time $^3K(\tau, \sigma^u)$, describes the non-dynamical arbitrariness in the choice of the convention for the synchronization of distant clocks which remains in the transition from SR to GR. Since the York time is present in the Dirac Hamiltonian, it is a new inertial potential connected to the problem of the relativistic freedom in the choice of the shape of the instantaneous 3-space, which has no Newtonian analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored. Instead the other components of the extrinsic curvature of $\Sigma_\tau$ are dynamically determined once a 3-coordinate system has been chosen in the 3-space.

e) $1 + n(\tau, \sigma^u)$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces $\Sigma_\tau$, namely how these surfaces are packed in the 3+1 splitting. The lapse function is determined by the $\tau$-preservation of the gauge fixing for the gauge variable $^3K(\tau, \sigma^u)$.

As shown in Ref.[35], the dynamical nature of space-time implies that each solution (i.e. an Einstein 4-geometry) of Einstein’s equations (or of the associated ADM Hamilton equations) dynamically selects a preferred 3+1 splitting of the space-time, namely in GR the instantaneous 3-spaces are dynamically determined modulo only one inertial gauge function (the gauge freedom in clock synchronization in GR). In the York canonical basis this function is the York time, namely the trace of the extrinsic curvature of the 3-space. Instead in SR the gauge freedom in clock synchronization depends on four basic gauge functions, the embeddings $z^\mu(\tau, \sigma^u)$, and both the 4-metric and the whole extrinsic curvature tensor were derived inertial potentials. Instead in GR the extrinsic curvature tensor of the 3-spaces is a mixture

---

26 In the post-Newtonian approximation in harmonic gauges they are the counterpart of the electro-magnetic vector potentials describing magnetic fields [38]: A) $N = 1 + n$, $n \equiv -\frac{1}{2c^2} \Phi_G$ with $\Phi_G$ the gravito-electric potential; B) $n_r \equiv \frac{1}{c^2} A_G r$ with $A_G r$ the gravito-magnetic potential; C) $E_G r = \partial_r \Phi_G - \partial_\tau (\frac{1}{2} A_G r)$ (the gravito-electric field) and $B_G r = \tau_r \partial_r \Phi_G - c A_G r$ (the gravito-magnetic field). Let us remark that in arbitrary gauges the analogy with electro-magnetism breaks down.
of dynamical (tidal) pieces and inertial gauge variables playing the role of inertial potentials.

6.2 3-Orthogonal Schwinger Time Gauges and Hamilton Equations

As shown in the first paper in Refs.[40], in the York canonical basis the Dirac Hamiltonian (34) becomes (the \( \lambda \)'s are arbitrary Dirac multipliers; the Dirac multiplier \( \lambda_r(\tau) \) implements the rest frame condition \( \hat{P}^r_{\text{ADM}} \approx 0 \))

\[
H^D = \frac{1}{c} \hat{E}_{\text{ADM}} + \int d^3\sigma \left[ n \mathcal{H} - n_{(a)} \mathcal{H}_{(a)} \right] (\tau, \sigma^u) + \lambda_r(\tau) \hat{P}^r_{\text{ADM}} + \int d^3\sigma \left[ \lambda_n \pi_n + \lambda_{\tilde{n}}(\tau) \pi_{\tilde{n}}(\tau) + \lambda_{\varphi_{(a)}} \pi_{\varphi_{(a)}} + \lambda_{\alpha_{(a)}} \pi_{\alpha_{(a)}} \right] (\tau, \sigma^u),
\]

(40)

with the following expression for the weak ADM energy

\[
\hat{E}_{\text{ADM}} = c \int d^3\sigma \left[ \hat{M} - \frac{c^3}{16\pi G} S + \frac{2\pi G}{c^3} \phi^{-1} \left( -3 (\phi \pi_\phi)^2 + 2 \sum_b \Pi^2_b \right) + 2 \sum_{abtwiuuvj} \epsilon_{abt} \epsilon_{aba} V_{tw} B_{iuu} V_{uv} B_{jv} \pi^{(\theta)}_{i} \pi^{(\theta)}_{j} \right] (\tau, \sigma^u).
\]

(41)

In it \( S(\tau, \sigma^u) \) is a function of \( \hat{\phi}, \theta^i \) and \( R_a \) (given in Eq.(B8) of the first paper in Ref.[40]), which play the role of an inertial potential depending on the choice of the 3-coordinates in the 3-space (it is the \( \Gamma - \Gamma \) term in the scalar 3-curvature of the 3-space).

Eq.(41) shows that the kinetic term, quadratic in the momenta, is not positive definite. While the kinetic energy of the tidal variables and the last term \( 27 \) are positive definite, there is the negative kinetic terms (vanishing only in the gauges \( 3K(\tau, \sigma^u) = 0 \)) \(- \frac{\phi^2}{24\pi G} \int D^3\sigma \hat{\phi}(\tau, \sigma^u)^4 K^2(\tau, \sigma^u) \). It is an inertial potential associated with the inertial gauge variable York time, which is a momentum due to the Lorentz signature of space-time. It was known that this quadratic form is not definite positive, but only in the York canonical basis this can be made explicit.

In the York canonical basis it is possible to follow the procedure for the fixation of a gauge natural from the point of view of constraint theory when there are chains of first-class constraints [9]. This procedure implies that

\( 27 \) It describes gravito-magnetic effects.
one has to add six gauge fixings to the primary constraints without secondaries \( \pi\varphi(a)(\tau,\sigma^u) \approx 0,\pi\alpha(a)(\tau,\sigma^u) \approx 0 \) and four gauge fixings to the secondary super-Hamiltonian and super-momentum constraints. These ten gauge fixings must be preserved in time, namely their Poisson brackets with the Dirac Hamiltonian must vanish. The \( \tau \)-preservation of the six gauge fixings determining the gauge variables \( \alpha(a)(\tau,\sigma^u) \) and \( \varphi(a)(\tau,\sigma^u) \) produces the equations determining the six Dirac multipliers \( \lambda\varphi(a)(\tau,\sigma^u),\lambda\alpha(a)(\tau,\sigma^u) \).

The \( \tau \)-preservation of the other four gauge fixings, determining the gauge variables \( \theta^i(\tau,\sigma^u) \) and the York time \( 3K(\tau,\sigma^u) \), produces four secondary gauge fixing constraints for the determination of the lapse and shift functions. Then the \( \tau \)-preservation of these secondary gauge fixings determines the four Dirac multipliers \( \lambda_n(\tau,\sigma^u),\lambda_{\bar{n}}(\tau,\sigma^u) \). Instead in numerical gravity one gives independent gauge fixings for both the primary and secondary gauge variables in such a way to minimize the computer time.

In Section V of the first paper in Refs.[40] there is a review of the gauges usually used in canonical gravity. It is shown that the commonly used family of the harmonic gauges is not natural according to the above procedure. The harmonic gauge fixings imply hyperbolic PDE for the lapse and shift functions, to be added to the hyperbolic PDE for the tidal variables. Therefore in harmonic gauges both the tidal variables and the lapse and shift functions depend (in a retarded way) from the \textit{no-incoming radiation} condition on the Cauchy surface in the past (so that the knowledge of \( 3K \) from the initial time till today is needed).

Instead the natural gauge fixings in the York canonical basis of ADM tetrad gravity are the family of Schwinger time gauges, where the O(3,1) gauge freedom of the tetrads is eliminated with the gauge fixings (implying \( \lambda\varphi(a)(\tau,\sigma^u) = \lambda\alpha(a)(\tau,\sigma^u) = 0 \))

\[
\alpha(a)(\tau,\sigma^u) \approx 0, \quad \varphi(a)(\tau,\sigma^u) \approx 0,
\]

and the subfamily of the \textit{3-orthogonal gauges}

\[
\theta^i(\tau,\sigma^u) \approx 0, \quad 3K(\tau,\sigma^u) \approx F(\tau,\sigma^u) = \text{numerical function},
\]

in which the 3-coordinates are chosen in such a way the the 3-metric in the 3-spaces \( \Sigma_\tau \) is diagonal. The \( \tau \)-preservation of Eqs.(43) gives four coupled elliptic PDE for the lapse and shift functions. Therefore in these gauges only the tidal variables (the gravitational waves after linearization), and therefore only the eigenvalues of the 3-metric with unit determinant inside \( \Sigma_\tau \), depend (in a retarded way) on the no-incoming radiation condition. The solutions \( \phi \) and \( \pi^i(\theta) \) of the constraints and the lapse \( 1+n \) and shift \( \bar{n}(a) \) functions depend only on the 3-space \( \Sigma_\tau \) with fixed \( \tau \). If the matter consists of positive energy particles (with a Grassmann regularization of the gravitational self-energies)
these solutions will contain action-at-a-distance gravitational potentials (replacing the Newton ones) and gravito-magnetic potentials.

In the family of 3-orthogonal gauges the weak ADM energy and the super-Hamiltonian and super-momentum constraints (they are coupled elliptic PDE for their unknowns) have the expression (see Eq.(3.47) of the first paper in Ref.[40] for the other weak ADM Poincaré generators)

\[
\hat{E}_{ADM}|_{\theta^i=0} = c \int d^3 \sigma \left[ M|_{\theta^i=0} - \frac{c^3}{16 \pi G} S|_{\theta^i=0} + \frac{2 \pi G}{c^3} \phi^{-1} \left( -3 \left( \phi \pi^0_\phi \right)^2 + 2 \sum_b \Pi^2_b + \frac{2}{\sum_{abij}} \frac{\epsilon_{abij} \epsilon_{abj} \pi^{(\theta)}_i \pi^{(\theta)}_j}{\left[ Q_a Q_b^{-1} - Q_b Q_a^{-1} \right]^2} \right) (\tau, \sigma^u) \right],
\]

\[
\mathcal{H}(\tau, \sigma^u)|_{\theta^i=0} = \frac{c^3}{16 \pi G} \hat{\Delta}^{1/6}(\tau, \sigma^u) \left[ 8 \hat{\Delta}^{1/6} - 3 \hat{R}|_{\theta^i=0} \hat{\Delta}^{1/6} \right] (\tau, \sigma^u) + M|_{\theta^i=0}(\tau, \sigma^u) +
\]

\[
\hat{\mathcal{H}}_{(a)}|_{\theta^i=0}(\tau, \sigma^u) = \phi^{-2}(\tau, \sigma) \left[ \sum_{b \neq a} \sum_i \frac{\epsilon_{abij} \epsilon_{abj} Q^{-1}_a Q^{-1}_b}{Q_b Q_a^{-1} - Q_a Q_b^{-1}} \partial_b \pi^{(\theta)}_i \right] +
\]

\[
\hat{\Delta} = \sum_r \sum_a \frac{Q^{-1}_r}{Q^{-1}_a} \left[ \partial_r^2 + \sum_{a} \gamma_{ar} \partial_r R_a(\tau, \sigma^u) \partial_r \right],
\]

\[
S_{\theta^i=0}(\tau, \sigma^u) = \left( \phi^{1/3} \sum_a Q^{-2}_a \left[ \frac{2}{9} \left( \phi^{-1} \partial_\phi \phi \right)^2 + \right] \sum_b \left( 2 \gamma_{ba} \gamma_{bc} - \delta_{bc} \right) \partial_a R_b - \frac{2}{3} \gamma_{ba} \phi^{-1} \partial_a \phi \right) \partial_a R_b \right)(\tau, \sigma^u).
In the first paper in Refs. [40] there is the explicit form of the Hamilton equations for all the canonical variables of the gravitational field and of the matter replacing the standard 12 ADM equations and the matter equations \(4\nabla_A T^{AB} = 0\) in the Schwinger time gauges and their restriction to the 3-orthogonal gauges. They could also be obtained from the effective Dirac Hamiltonian of the 3-orthogonal gauges, which is evaluated by means of a \(\tau\)-dependent canonical transformation sending the gauge momentum \(\pi_\tilde{\phi}(\tau, \sigma^u)\) in the gauge-fixing conditions \(\pi_\tilde{\phi}'(\tau, \sigma^u) = \frac{\gamma^3}{12\pi G} \left(3K(\tau, \sigma^u) - F(\tau, \sigma^u)\right) \approx 0\) and which is given in Eq. (4.39) of the second paper of Ref. [40].

These equations are divided in five groups:

A) The contracted Bianchi identities, namely the evolution equations for the solutions \(\tilde{\phi}(\tau, \sigma^u)\) and \(\pi_\theta^i(\tau, \sigma^u)\) of the super-Hamiltonian and super-momentum constraints: they are identities saying that, given a solution of the constraints on a Cauchy surface, it remains a solution also at later times.

B) The evolution equation for the four basic gauge variables \(\theta^i(\tau, \sigma^u)\) and \(3K(\tau, \sigma^u)\) (the equation for the York time is the Raychaudhuri equation \(^{28}\)) : these equations determine the lapse and the shift functions once four gauge-fixings for the basic gauge variables are given.

C) The equations \(\partial_\tau n(\tau, \sigma^u) = \lambda_n(\tau, \sigma^u)\) and \(\partial_\tau \bar{n}(\alpha)(\tau, \sigma^u) = \lambda_{\bar{n}}(\alpha)(\tau, \sigma^u)\). Once the lapse and shift functions of the chosen gauge have been found, they determine the associated Dirac multipliers.

D) The hyperbolic evolution PDE for the tidal variables \(R_\alpha(\tau, \sigma^u), \Pi_\alpha(\tau, \sigma^u)\). When the equations for \(\partial_\tau R_\alpha(\tau, \sigma^u)\) is inverted to get \(\Pi_\alpha(\tau, \sigma^u)\) in terms of \(R_\alpha(\tau, \sigma^u)\) and its derivatives, then the Hamilton equations for \(\Pi_\alpha(\tau, \sigma^u)\) become hyperbolic PDE for the evolution of the physical tidal variable \(R_\alpha(\tau, \sigma^u)\).

E) The Hamilton equations for matter, when present.

Given a solution of the super-momentum and super-Hamiltonian constraints and the Cauchy data for the tidal variables on an initial 3-space, one can find a solution of Einstein’s equations in radar 4-coordinates adapted to a time-like observer in the chosen gauge.

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\(^{28}\) This equation is relevant for studying the developments of caustics in a congruence of time-like geodesics for converging values of the expansion \(\theta\) and of singularities in Einstein space-times \([69]\). However the boundary conditions of asymptotically Minkowskian space-times without super-translations should avoid the singularity theorems as it happens with their subfamily without matter of Ref. [16].
6.3 The Congruence of Eulerian Observers and the non-Hamiltonian First-Order ADM Equations of Cosmological Spacetimes

Like in SR one can consider the congruence of the Eulerian observers with zero vorticity associated with the 3+1 splitting of space-time, whose properties are described by Eq.(5). In the first paper of Ref.[40] it is shown that in ADM tetrad gravity the congruence has the following properties in each point (\(\tau; \sigma^i\))

\[a)\] The acceleration \(\dot{a}^A = l^B \dot{A}^B = 4g^{AB} \dot{a}_B\) has the components \(\dot{a}^r = \dot{\phi} - \frac{2}{3} \frac{\dot{\phi}}{\varphi^2} V_r a^2 V_a \partial_a \ln(1 + n)\), \(\dot{a}_r = -\frac{2}{3} \frac{\dot{\phi}}{\varphi^2} Q^{-1}_a \partial_a \ln(1 + n)\).

\[b)\] The expansion \(\theta = 4 \nabla_A l^A = -\epsilon \frac{12\pi G}{c^3} \frac{\dot{\pi}}{\dot{\phi}}\). In cosmology the expansion is proportional to the Hubble constant and the dimensionless cosmological deceleration parameter is \(q = \frac{3 H^4 \nabla A}{A^4} - 1 = -3 \theta^2 - l^A \partial_A \theta - 1\).

\[c)\] By using Eqs.(24) it can be shown that the shear \(\dot{\sigma}^{AB} = \sigma^{BA} = -\frac{1}{2} (3 a_A l_B + a_B l_A) + \frac{1}{2} (4 \nabla_A l_B + 4 \nabla_B l_A) - \frac{1}{3} \theta 3 h_{AB} = \dot{\sigma}^{(a)(\beta)} - \dot{\sigma}^{(\beta)(a)}\) has the following components \(\sigma^{(a)(a)} = \sigma^{(a)(r)} = 0\), \(\sigma^{(a)(b)} = \sigma^{(b)(a)} = (3 Krs - \frac{1}{3} g_{rs} 3 K) \dot{\phi}\). The diagonal elements \(\sigma^{(a)(a)}\) are connected with the canonical variables \(\phi, R_r, \pi^{(i)}_r\) and \(\Pi^a\).

By using Eqs.(39) for the extrinsic curvature tensor one finds that the diagonal elements \(\sigma^{(a)(a)}\) of the shear are also connected with the tidal momenta \(\Pi^a\), while the non-diagonal elements \(\sigma^{(a)(b)}\) are connected with the momenta \(\pi^{(b)}_r\) (the unknowns in the super-momentum constraints)

\[\Pi^a = \frac{-c^3}{8\pi G} \dot{\phi} \sum_a \gamma_{aa} \sigma^{(a)(a)}\].

\[29\] See the "1+3 point of view" of Ref.[70] for a discussion of gravity in terms of the second non-surface-forming congruence of time-like observers associated with a 3+1 splitting of space-time.

\[30\] It measures the average expansion of the infinitesimally nearby world-lines surrounding a given world-line in the congruence.

\[31\] It measures how an initial sphere in the tangent space to the given world-line, which is Lie-transported along the world-line tangent \(l^a\) (i.e. it has zero Lie derivative with respect to \(l^a \partial_a\)), is distorted towards an ellipsoid with principal axes given by the eigenvectors of \(\sigma^{(a)(b)}\), with rate given by the eigenvalues of \(\sigma^{(a)(b)}\).
\[ \pi_i^{(\theta)} = \frac{c^3}{8\pi G} \phi \sum_{u,t,a,b} A_{ui} V_{at} Q_a Q_{b}^{-1} \epsilon_{tab} \sigma_{(a)(b) | a \neq b}, \]

\[ 3K_{rs} = \frac{\dot{\phi}^2}{2} \sum_{ab} \left( -\frac{\epsilon}{3} \theta \delta_{ab} + \sigma_{(a)(b)} \right) Q_a Q_b V_{ra} V_{sb} \rightarrow \]

\[ \rightarrow_{\theta \rightarrow 0} \frac{\dot{\phi}^2}{2} Q_r Q_s \left( -\frac{\epsilon}{3} \theta + \sigma_{(a)(b)} \right). \] (46)

Therefore the Eulerian observers associated to the 3+1 splitting of space-time induce a geometrical interpretation of some of the momenta of the York canonical basis:

1) their expansion \( \theta \) is the gauge variable York time \( 3K = \frac{12\pi G}{c^3} \pi_3 \) determining the non-dynamical gauge part of the shape of the instantaneous 3-spaces \( \Sigma_\tau \) as a sub-manifold of space-time;

2) the diagonal elements of their shear describe the tidal momenta \( \Pi_{\bar{a}}, \) while the non-diagonal elements are connected to the variables \( \pi_i^{(\theta)} \), determined by the super-momentum constraints.

In Eq.(44), valid in the 3-orthogonal gauges, the term quadratic in the momenta \( \pi_{i}^{(\theta)} \) in the weak ADM energy and in the super-Hamiltonian constraint can be written as \( \frac{c^3}{16\pi G} \phi \sum_{ab,a \neq b} a_{(a)(b)}^2 \), while the super-momentum constraints can be written in the form of PDE for the non-diagonal elements of the shear

\[ \mathcal{H}_{(a)}|_{\theta=0}(\tau, \sigma^u) = -\frac{c^3}{8\pi G} \frac{\dot{\phi}^2}{2} (\tau, \sigma^u) \left( \sum_{b \neq a} Q_b^{-1} \left[ \partial_b \sigma_{(a)(b)} + \left( \phi^{-1} \partial_b \phi + \sum_b (\gamma_{ba} - \gamma_{bb}) \partial_b R_b \right) \frac{\sigma_{(a)(b)}}{\sigma_{(b)(b)}} \right] - \right) \]

\[ - \frac{8\pi G}{c^3} \frac{\dot{\phi}^{-1}}{Q_a^{-1}} \left[ \phi \partial_a \pi_{\phi} + \sum_b (\gamma_{ba} \partial_a \partial_b R_b) + \right. \]

\[ + \left. M_a \right]\right)(\tau, \sigma^u) \approx 0. \] (47)

As a consequence, by using \( 3g_{rs} \) of Eq.(38) and \( 3K_{rs} \) of Eq.(46), the first-order non-Hamiltonian ADM equations (33) can be re-expressed in terms of the configurational variables \( n, \tilde{n}_{(a)}, \phi, \theta^i, R_{a}, \) and of the expansion \( \theta \) and shear \( \sigma_{(a)(b)} \) of the Eulerian observers. Then the 12 equations can be put in the form of equations determining \( \partial_\tau \phi, \partial_\tau R_{a}, \partial_\tau \theta^i, \partial_\tau \theta \) and \( \partial_\tau \sigma_{(a)(b)} \). In
Eqs. (2.17) of the first paper in Ref. [40] this manipulation is explicitly done for the first six equations (33).

These results are important for extending the identification of the inertial and tidal variables of the gravitational field, achieved with the York canonical basis, to cosmological space-times. Since these space-times are only conformally asymptotically flat, the Hamiltonian formalism is not defined. However, they are globally hyperbolic and admit 3+1 splittings with the associated congruence of Eulerian observers. As a consequence, in them Einstein’s equations are usually replaced with the non-Hamiltonian first-order ADM equations plus the super-Hamiltonian and super-momentum constraints. Our analysis implies that, since the 4-metric can always be put in the form of Eqs. (38), the inertial gauge variables of the cosmological space-times are \( n, \hat{n}_{(a)}, \theta^i \) and the expansion \( \theta = -\epsilon^3 K \), while the physical tidal variables are \( R_{\bar{a}} \) and the diagonal components of the shear \( \sigma_{(a)(a)} (\sum_a \sigma_{(a)(a)} = 0) \). The unknown in the super-Hamiltonian constraint is the conformal factor \( \bar{\phi} \) of the 3-metric in \( \Sigma_\tau \), while the unknowns in the super-Hamiltonian constraints are the non-diagonal components of the shear \( \sigma_{(a)(b)} |_{a \neq b} \).
7 Post-Minkowskian Linearization in Non-Harmonic 3-Orthogonal Gauges and Post-Minkowskian Gravitational Waves

In the second paper of Ref.[40] it was shown that in the family of non-harmonic 3-orthogonal Schwinger gauges it is possible to define a consistent linearization of ADM canonical tetrad gravity plus matter (N charged scalar particles of masses $m_i$, Grassmann-valued signs of energy $\eta_i$, Grassmann-valued electric charges $Q_i$, plus the electro-magnetic field in the radiation gauge) in the weak field approximation, to obtain a formulation of Hamiltonian Post-Minkowskian (HPM) gravity with non-flat Riemannian 3-spaces and asymptotic Minkowski background.

In the standard linearization one introduces a fixed Minkowski background space-time, introduces the decomposition $^4g_{\mu\nu}(x) = ^4\eta_{\mu\nu} + ^4h_{\mu\nu}(x)$ in an inertial frame and studies the linearized equations of motion for the small Minkowskian fields $^4h_{\mu\nu}(x)$. The approximation is assumed valid over a big enough characteristic length $L$ interpretable as the reduced wavelength $\lambda/2\pi$ of the resulting gravitational waves (GW) (only for distances higher of $L$ the linearization breaks down and curved space-time effects become relevant). For the Solar System there is a PN approximation in harmonic gauges, which is adopted in the BCRS [5] and whose 3-spaces $t_B = const.$ have deviations of order $c^{-2}$ from Euclidean 3-spaces.

See Refs.[44, 71] and Appendix A of the second paper in Refs.[40] for a review of all the results of the standard approach and of the existing points of view on the subject [72, 73, 74, 75, 76].

In the class of asymptotically Minkowskian space-times without super-translations the 4-metric tends to an asymptotic Minkowski metric at spatial infinity, $^4g_{AB} \rightarrow ^4\eta_{AB}$, which can be used as an asymptotic background. The decomposition $^4g_{AB} = ^4\eta_{AB} + ^4h_{(1)AB}$, with a first order perturbation $^4h_{(1)AB}$ vanishing at spatial infinity, is defined in a global non-inertial rest frame of an asymptotically Minkowskian space-time deviating for first order effects from a global inertial rest frame of an abstract Minkowski space-time $M(\infty)$. The non-Euclidean 3-spaces $\Sigma_\tau$ will deviate by first order effects from the Euclidean 3-spaces $\Sigma_{\tau(\infty)}$ of the inertial rest frame of $M(\infty)$ coinciding with the limit of $\Sigma_\tau$ at spatial infinity. When needed differential operators like the Laplacian in $\Sigma_\tau$ will be approximated with the flat Laplacian in $\Sigma_{\tau(\infty)}$.

If $\zeta << 1$ is a small a-dimensional parameter, a consistent Hamiltonian linearization implies the following restrictions on the variables of the York canonical basis in the family of 3-orthogonal gauges with $^3K(\tau, \sigma) = F(\tau, \sigma) = \text{numerical function}$ (in this Section one uses the notation $\sigma$ for the curvilinear 3-coordinates $\sigma^r$)
\[ R_{a}(τ, σ) = R_{(1)a}(τ, σ) = O(ζ) < < 1, \]
\[ Π_{a}(τ, σ) = Π_{(1)a}(τ, σ) = \frac{1}{L} G O(ζ), \]
\[ ϕ(τ, σ) = \sqrt{\text{det}^{3} g_{rs}(τ, σ)} = 1 + 6 \phi(1)(τ, σ) + O(ζ^2), \]
\[ N(τ, σ) = 1 + n(τ, σ) = 1 + n_{(1)}(τ, σ) + O(ζ^2), \]
\[ \epsilon^4 g_{ττ}(τ, σ) = 1 + \epsilon^4 h_{(1)ττ}(τ, σ) = 1 + 2 n_{(1)}(τ, σ) + O(ζ^2), \]
\[ \bar{η}_{(a)}(τ, σ) = -\epsilon^4 g_{ra}(τ, σ) = -\epsilon^4 h_{(1)r}(τ, σ) = \bar{η}_{(1)(a)}(τ, σ) + O(ζ^2), \]

\[ 3 K(τ, σ) = \frac{12 π G}{c^4} \pi_{δ}(τ, σ) = 3 K_{(1)}(τ, σ) = \frac{12 π G}{c^3} ζ_{(1)δ}(τ, σ) = \frac{1}{L} O(ζ), \]
\[ \sigma_{(a)(b)}|_{a≠b}(τ, σ) = \sigma_{(1)(a)(b)}|_{a≠b}(τ, σ) = \frac{1}{L} O(ζ), \]
\[ 3 g_{rs}(τ, σ) = -\epsilon^4 g_{rs}(τ, σ) = δ_{rs} - \epsilon^4 h_{(1)rs}(τ, σ) = [1 + 2 (Γ_{r}^{(1)}(τ, σ) + 2 \phi_{(1)}(τ, σ))] δ_{rs} + O(ζ^2), \]
\[ Γ_{a}(τ, σ) = \sum_{a=1}^{3} γ_{āa} R_{ā}(τ, σ), \quad R_{ā}(τ, σ) = \sum_{a=1}^{3} γ_{āa} Γ_{a}^{(1)}(τ, σ). \quad (48) \]

The tidal variables \( R_{a}(τ, σ) \) are slowly varying over the length \( L \) and times \( L/c \); one has \( (\frac{L}{2c})^2 = O(ζ) \), where \( 4\mathcal{R} \) is the mean radius of curvature of space-time.

The consistency of the Hamiltonian linearization requires the introduction of an ultra-violet cutoff \( M \) for matter. For the particles, described by the canonical variables \( η_{i}(τ) \) and \( κ_{i}(τ) \), this implies the conditions \( \frac{aL}{M}, \frac{aM}{M} = O(ζ) \). With similar restrictions on the electro-magnetic field one gets that the energy-momentum tensor of matter is \( T^{AB}(τ, σ) = T^{AB}_{(1)}(τ, σ) + O(ζ^2) \). Therefore also the mass and momentum densities have the behavior \( M(τ, σ) = M_{(1)}(τ, σ) + O(ζ^2), \quad M_{r}(τ, σ) = M_{(1)r}(τ, σ) + O(ζ^2) \). This approximation is not reliable at distances from the point particles less than the gravitational radius \( R_{M} = \frac{M}{G} ≈ 10^{-20} M \) determined by the cutoff mass. The weak ADM Poincaré generators become equal to the Poincaré generators of this matter in the inertial rest frame of the Minkowski space-time \( M_{(∞)} \) plus terms of order \( O(ζ^2) \) containing GW and matter. Finally the GW described by this linearization must have wavelengths satisfying \( λ/2π ≈ L >> R_{M} \). If all the particles are contained in a compact set of radius \( l_{c} \) (the source), one must have \( l_{c} >> R_{M} \) for particles with relativistic velocities and \( l_{c} \geq R_{M} \) for slow particles (like in binaries). See Ref.[44] for more details.
With this Hamiltonian linearization one can avoid to make PN expansions: one gets fully relativistic expressions, i.e. a HPM formulation of gravity.

The effective Hamiltonian adapted to the 3-orthogonal gauges and replacing the weak ADM energy is

$$\frac{1}{c^2} \left( E_{ADM(1)} + E_{ADM(2)} \right) + \frac{c^2}{12\pi G} \int d^3\epsilon \left( \partial_r \frac{3}{c^3} K_{(1)}^r \right) \left( \tau, \sigma^\nu \right) + O(\zeta^3)$$

In the second paper of Refs.[40] one has found the solutions of the super-momentum and super-Hamiltonian constraints and of the equations for the lapse and shift functions with the Bianchi identities satisfied. Therefore one knows the first order quantities $\pi^{(\theta)}(\tau, \sigma), \tilde{\phi}(\tau, \sigma) = 1 + 6 \phi^{(1)}(\tau, \sigma), 1 + n^{(1)}(\tau, \sigma), \tilde{n}^{(1)(a)}(\tau, \sigma)$ (the quantities containing the action-at-a-distance part of the gravitational interaction in the 3-orthogonal gauges) with an explicit expression for the PM Newton and gravito-magnetic potentials. In absence of the electro-magnetic field they are (the terms in $\Gamma_{(1)}^{(1)}(\tau, \sigma)$ describe the contribution of GW)

$$\tilde{\phi}(\tau, \sigma) = 1 + 6 \phi^{(1)}(\tau, \sigma) = 1 + \frac{3G}{c^3} \sum_i \eta_i \frac{\sqrt{m_i^2 c^2 + \kappa_i^2(\tau)}}{|\sigma - \eta_i(\tau)|} -$$

$$- \frac{3}{8\pi} \int d^3\sigma \sum_a \frac{\partial_1^2 \Gamma_a^{(1)}(\tau, \sigma_1)}{|\sigma - \sigma_1|},$$

$$\epsilon^4 g_{rr}(\tau, \sigma) = 1 + 2 n^{(1)}(\tau, \sigma) = 1 - 2 \partial_r \frac{3}{c^3} K_{(1)}^r(\tau, \sigma) -$$

$$- \frac{2G}{c^3} \sum_i \eta_i \frac{\sqrt{m_i^2 c^2 + \kappa_i^2(\tau)}}{|\sigma - \eta_i(\tau)|} \left( 1 + \frac{\kappa_i^2}{m_i^2 c^2 + \kappa_i^2} \right),$$

$$- \epsilon^4 g_{ra}(\tau, \sigma) = \tilde{n}^{(1)(a)}(\tau, \sigma) = \partial_a \frac{3}{c^3} K_{(1)}^r(\tau, \sigma) -$$

$$- \frac{G}{c^3} \sum_i \frac{\eta_i}{|\sigma - \eta_i(\tau)|} \frac{7}{2} \kappa_{ia}(\tau) -$$

$$- \frac{1}{2} \frac{(\sigma^a - \eta_i^a(\tau)) \kappa_i(\tau) \cdot (\sigma - \eta_i(\tau))}{|\sigma - \eta_i(\tau)|^2} -$$

$$- \int \frac{d^3\sigma_1}{4\pi |\sigma - \sigma_1|} \partial_1 \partial_r \left[ 2 \Gamma_a^{(1)}(\tau, \sigma_1) -$$

32 Quantities like $|\eta_i(\tau) - \eta_j(\tau)|$ are the Euclidean 3-distance between the two particles in the asymptotic 3-space $\Sigma_{r(\sigma)}$, which differs by quantities of order $O(\zeta)$ from the real non-Euclidean 3-distance in $\Sigma_r$ as shown in Eq.(3.3) of the third paper in Ref.[40].
\[- \int d^3 \sigma_2 \frac{\sum_{a} \partial_a^2 \Gamma^{(1)}_c(\tau, \sigma_2)}{8\pi |\sigma_1 - \sigma_2|}, \]

\[\sigma_{(1)(a)(b)}|_{a \neq b}(\tau, \sigma) = \frac{1}{2} \left( \partial_a n_{(1)(b)} + \partial_b n_{(1)(a)} \right)|_{a \neq b}(\tau, \sigma). \quad (49)\]

Instead the linearization of the Hamilton equations for the tidal variables \(R_{\vec{a}}(\tau, \sigma)\) implies that they satisfy the following wave equation 33 (\(\triangle\) and \(\Box\) are the flat Laplacian and the flat D’Alambertian on \(\Sigma_{\tau(\infty)}\))

\[\partial^2_{\tau} R_{\vec{a}}(\tau, \sigma) = \triangle R_{\vec{a}}(\tau, \sigma) + \sum_{a} \gamma_{\vec{a}a} \left[ \partial_\tau \partial_a \bar{n}_{(1)(a)} + \partial^2_a n_{(1)} - 2 \partial^2_a \phi_{(1)} + 2 \partial^2_a \Gamma^{(1)}_{\vec{a}} + \frac{8\pi G}{c^3} T_{aa}^{(1)}(\tau, \sigma) \right]. \quad (50)\]

By using Eqs.(49) this wave equation becomes

\[\Box \sum_b \tilde{M}_{\vec{a}b} R_{\vec{b}}(\tau, \sigma) = E_{\vec{a}}(\tau, \sigma),\]

\[M_{\vec{a}b} = \delta_{\vec{a}b} - \sum_{a} \gamma_{\vec{a}a} \frac{\partial^2_a}{\triangle} \left( 2 \gamma_{ba} - \frac{1}{2} \sum_{b} \gamma_{bb} \frac{\partial^2_a}{\triangle} \right),\]

\[E_{\vec{a}}(\tau, \sigma) = \frac{4\pi G}{c^3} \sum_{a} \gamma_{\vec{a}a} \left[ \partial_\tau \frac{\partial_a}{\triangle} \left( 4 M_{(1)a} - \frac{\partial_a}{\triangle} \sum_{c} \partial_c M_{(1)c} \right) + \frac{1}{2} \partial^2_a \frac{\partial^2_a}{\triangle} \sum_{b} T_{bb}^{(1)}(\tau, \sigma) \right],\]

\[\downarrow\]

\[\Box \sum_b \tilde{M}_{ab} R_{\vec{b}}^{(1)}(\tau, \sigma) = \sum_{a} \gamma_{\vec{a}a} E_{\vec{a}}(\tau, \sigma),\]

\[\tilde{M}_{ab} = \sum_{\vec{a} \vec{b}} \gamma_{\vec{a}a} \gamma_{\vec{b}b} M_{\vec{a} \vec{b}} = \delta_{ab} \left( 1 - 2 \frac{\partial^2_a}{\triangle} \right) + \frac{1}{2} \left( 1 + \frac{\partial^2_a}{\triangle} \right) \frac{\partial^2_b}{\triangle},\]

\[\sum_{a} \tilde{M}_{ab} = 0, \quad M_{ab} = \sum_{\vec{a} \vec{b}} \gamma_{\vec{a}a} \gamma_{\vec{b}b} \tilde{M}_{ab}. \quad (51)\]

33 For the tidal momenta one gets \(\frac{8\pi G}{c^3} \Pi_{\vec{a}}(\tau, \sigma) = [\partial_\tau \tilde{R}_{\vec{a}} - \sum_{a} \gamma_{\vec{a}a} \partial_\tau \bar{\tilde{n}}_{(1)(a)}](\tau, \sigma) + O(\zeta^2),\) so that the diagonal elements of the shear are \(\sigma_{(1)(a)(a)}(\tau, \sigma) = [- \sum_{a} \gamma_{\vec{a}a} \partial_\tau \tilde{R}_{\vec{a}} + \tilde{\bar{n}}_{(1)(a)} - \frac{1}{3} \sum_{b} \bar{n}_{(1)(b)}](\tau, \sigma) + O(\zeta^2).\)
To understand the meaning of the spatial operators $M_{ab}$ and $\tilde{M}_{ab}$, one must consider the perturbation $4h_{(1)rs}(\tau, \sigma)$ to the spatial part of the York canonical basis to the TT components of the 3-metric. By applying the wave equation $\Box$ one verifies that like in the harmonic gauges [44] the TT part of the 3-metric condition gives the following expression for the tidal variables (the H PM-GW) of Eq.(48) and apply to it the following decomposition, given in Ref[45],

$$4h_{(1)rs}(\tau, \sigma) = \left(4h^{TT}_{(1)rs} + \frac{1}{3} \delta_{rs} H_{(1)} + \frac{1}{2} \left(\partial_{r} \epsilon_{(1)s} + \partial_{s} \epsilon_{(1)r}\right) + \left(\partial_{r} \partial_{s} - \frac{1}{3} \delta_{rs} \Delta\right) \lambda_{(1)} \right)(\tau, \sigma), \quad (52)$$

with $\sum_{r} \partial_{r} \epsilon_{(1)r} = 0$ and $4h^{TT}_{(1)rs}$ traceless and transverse (TT), i.e. $\sum_{r} 4h^{TT}_{(1)rs} = 0$, $\sum_{r} \partial_{r} 4h^{TT}_{(1)rs} = 0$. Since one finds $H_{(1)}(\tau, \sigma) = -12 \epsilon \phi_{(1)}(\tau, \sigma)$, $\lambda_{(1)}(\tau, \sigma) = -3 \epsilon \sum_{u} \frac{\partial^{2}}{\Delta} \Gamma_{u}^{(1)}(\tau, \sigma)$ and $\epsilon_{(1)r}(\tau, \sigma) = -4 \epsilon \frac{\partial_{\tau}}{\Delta} \left(\Gamma_{r}^{(1)} - \sum_{u} \frac{\partial^{2}}{\Delta} \Gamma_{u}^{(1)}\right)(\tau, \sigma)$, it turns out that the TT part of the spatial metric is independent from $\phi_{(1)}$ and has the expression

$$4h^{TT}_{(1)rs}(\tau, \sigma) = -\epsilon \left[\left(2 \Gamma_{r}^{(1)} + \sum_{u} \frac{\partial^{2}}{\Delta} \Gamma_{u}^{(1)}\right) \delta_{rs} - 2 \left(\partial_{r} \partial_{s} \Gamma_{r}^{(1)} + \partial_{r} \partial_{s} \Gamma_{s}^{(1)}\right) + \partial_{\tau} \sum_{u} \frac{\partial^{2}}{\Delta} \Gamma_{u}^{(1)}\right](\tau, \sigma),$$

$$\Rightarrow \quad 4h^{TT}_{(1)aa}(\tau, \sigma) = -2 \epsilon \sum_{b} \tilde{M}_{ab} \Gamma_{b}^{(1)}(\tau, \sigma). \quad (53)$$

Therefore the spatial operator $\tilde{M}_{ab}$ connects the tidal variables $R_{a}(\tau, \sigma)$ of the York canonical basis to the TT components of the 3-metric. By applying the decomposition (52) to the spatial part $T_{rs}^{TT}(\tau, \sigma)$ of the energy-momentum one verifies that like in the harmonic gauges [44] the TT part of the 3-metric satisfies the wave equation $\Box 4h^{TT}_{rs}(\tau, \sigma) = -\epsilon \frac{16\pi G}{c^{3}} T_{rs}^{TT}(\tau, \sigma)$.

The retarded solution of the wave equation with a no-incoming radiation condition gives the following expression for the tidal variables (the HPM-GW)

$$R_{a}(\tau, \sigma) = -\sum_{a} \gamma_{aa} \Gamma_{a}^{(1)}(\tau, \sigma) \frac{8\pi G}{c^{3}} \sum_{ab} \gamma_{aa} \tilde{M}^{-1}_{ab}(\tau, \sigma) \frac{2G}{c^{3}} \int d^3\sigma_{1} T_{(1)bb}^{TT}(\tau - |\sigma - \sigma_{1}|; \sigma_{1}),$$

$$8\pi G \frac{\partial}{c^{3}} \Pi_{aa}(\tau, \sigma) = \left(\sum_{b} M_{ab} \partial_{r} R_{b} - \sum_{a} \gamma_{aa} \left[4\pi G \frac{1}{c^{3}} (4 \partial_{a} M_{(1)a} - \partial_{a}^{2} \frac{1}{\Delta}) + \frac{\partial^{2}}{\Delta} \sum_{c} \partial_{c} M_{(1)c} + \partial_{a}^{2} K_{(1)}\right]\right)(\tau, \sigma). \quad (54)$$
The explicit form of the inverse operator is given in the second paper of Ref.\[40\]. By using the multipolar expansion of the energy-momentum $T^{AB}_{(1)}$ of Ref.\[58\] in the HPM version adapted to the rest-frame instant form of dynamics of Ref.\[26\], one gets

$$R_{a}(\tau, \sigma) = -\frac{G}{c^3} \sum_{ab} \gamma_{aa} \tilde{M}_{ab}^{-1} \frac{\partial^2 q^{(TT)aa|\tau\tau}(\tau - |\sigma|)}{|\sigma|} + \text{(higher multipoles)},$$

where $q^{(TT)aa|\tau\tau}(\tau)$ is the $TT$ mass quadrupole with respect to the center of energy (put in the origin of the radar 4-coordinates). An analogous result holds for $^{4}h^{TT}_{rs}(\tau, \sigma)$ and this implies a HPM relativistic version of the standard mass quadrupole emission formula.

Moreover, notwithstanding there is no gravitational self-energy due to the Grassmann regularization, the energy, 3-momentum and angular momentum balance equations in HPM-GW emission are verified by using the conservation of the asymptotic ADM Poincaré generators (the same happens with the asymptotic Larmor formula of the electro-magnetic case with Grassmann regularization as shown in the last paper of Ref.\[27\]). See Refs.\[44, 76, 77\] for the use of the self-energy in the standard derivation of this result by means of PN expansions.

Eqs.\(49\) and \(54\) show that the HPM linearization with no-incoming radiation condition and Grassmann regularization is a theory with only dynamical matter interacting through suitable action-at-a-distance and retarded effective potentials. Instead in relativistic atomic physics in SR the no-incoming radiation condition and the Grassmann regularization kill also the retardation leaving only the action-at-a-distance inter-particle Coulomb plus Darwin potentials. See Eq.\(7.22\) of the second paper of Ref.\[40\] for the expression of the weak ADM energy till order $O(\zeta^3)$.

Moreover it can be shown that the coordinate transformation $\bar{r} = r, \bar{\sigma}^r = \sigma^r + \frac{1}{2} \frac{\partial}{\partial \bar{r}} \left( 4 \Gamma_{1} - \sum_c \frac{\partial^2}{\partial \bar{r}} \Gamma_{c}^{(1)} \right)(r, \sigma)$, introducing new $\tau$-dependent radar 3-coordinates on the 3-space $\Sigma_\tau$, allows one to make a transition from the 3-orthogonal gauge with the 4-metric given by Eqs.\(48\) and \(49\) to a generalized non-3-orthogonal $TT$ gauge containing the $TT$ 3-metric \(53\)

$$^{4}g_{(1)AB} = ^{4}\eta_{AB} +$$

$$+ \epsilon \left( -2 \frac{\partial}{\partial \bar{r}} 3K_{(1)} + \alpha(\text{matter}) \right) - \frac{\partial}{\partial \bar{r}} 3K_{(1)} + A_{r}(\Gamma_{a}^{(1)}) + \beta_{r}(\text{matter})$$

$$- \frac{\partial}{\partial \bar{r}} 3K_{(1)} + A_{s}(\Gamma_{a}^{(1)}) + \beta_{s}(\text{matter}) \right)$$

$$\left[ B_{r}(\Gamma_{a}^{(1)}) + \gamma(\text{matter}) \right] \delta_{rs}$$
\[ + O(\zeta^2), \]

\[ \downarrow \]

\[ 4g_{AB} = 4\eta_{AB} + \epsilon \left( -\frac{2}{3} \delta \Delta^3K_{(1)} + \alpha(\text{matter}) - \frac{2}{3} \Delta^3K_{(1)} + \beta_r(\text{matter}) \right) + \]

\[ + O(\zeta^2). \] (56)

The functions appearing in Eqs. (56) are:

\[ A_r(\Gamma^a_{(1)}) = -\frac{1}{2} \partial_r \frac{\delta}{\Delta} \left( 4 \Gamma^a_r - \sum_c \frac{\delta^2}{\Delta} \Gamma^c_{(1)} \right), \]

\[ B_r(\Gamma^a_{(1)}) = -2 \left( \Gamma^r_r(1) + \frac{1}{2} \sum_c \frac{\delta^2}{\Delta} \Gamma^c_{(1)} \right), \]

\[ \alpha(\text{matter}) = \frac{8\pi G}{c^3} \left( 4M_{(1)r} - \frac{\partial}{\partial r} \sum_c \partial_c M_{(1)c} \right), \]

\[ \beta_r(\text{matter}) = -\frac{4\pi G}{c^3} \frac{1}{\Delta} \left( 4M_{(1)r} - \frac{\partial}{\partial r} \sum_c \partial_c M_{(1)c} \right), \]

\[ \gamma(\text{matter}) = \frac{8\pi G}{c^3} \frac{1}{\Delta} M_{(1)}. \]

Also in absence of matter this TT gauge differs from the usual harmonic ones for the non-spatial terms depending upon the inertial gauge variable non-local York time

\[ \frac{3}{1} K_{(1)}(\tau, \sigma) = 1 \Delta K_{(1)}(\tau, \sigma), \] (57)

describing the HPM form of the gauge freedom in clock synchronization.

If one uses the coordinate system of the generalized TT gauge, one can introduce the standard polarization pattern of GW for \( 4h_{TT}^{\tau\tau} \) (see Refs. [44, 45, 78]) and then the inverse transformation gives the polarization pattern of HPM-GW in the family of 3-orthogonal gauges.

If the matter sources have a compact support and if the matter terms \( \frac{1}{\Delta} M_{(1)}(\tau, \sigma) \) and \( \frac{1}{\Delta} M_{(1)r}(\tau, \sigma) \) are negligible in the radiation zone far away from the sources, then Eq. (56) gives a spatial TT-gauge with still the explicit dependence on the inertial gauge variable \( 3K_{(1)}(\tau, \sigma) \) (non existing in Newtonian gravity), which determines the non-Euclidean nature of the instantaneous 3-spaces. Then one can study the far field of compact matter sources: the restriction to the Solar System of the resulting HPM 4-metric \( ^4g_{\tau\tau}(\tau, \sigma) \) and \( ^4g_{\tau r}(\tau, \sigma) \) are explicitly depending on the non-local York time is of order \( c^{-2} \). The resulting shift function should be used for the HPM description of gravito-magnetism (see Refs. [38, 79, 80] for the Lense-Thirring and other associated effects).

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34 See Eq. (7.20) of the second paper in Ref. [40], where \( ^4g_{\tau\tau}(\tau, \sigma) \) and \( ^4g_{\tau r}(\tau, \sigma) \) are explicitly depending on the non-local York time.
The TT gauge allows one to reproduce the various descriptions of the GW detectors and of the reference frames used in GW detection in terms of HPM-GW: this is done in Subsection VIID of the second paper of Ref.[40], where the effect of a HPM-GW on a test mass is given in terms of the proper distance between two nearby geodesics.

The HPM-GW propagate in non-Euclidean instantaneous 3-spaces $\Sigma_\tau$ differing from the inertial asymptotic Euclidean 3-spaces $\Sigma_{\tau(\infty)}$ at the first order. In the family of 3-orthogonal gauges with York time $3 K_{(1)}(\tau, \sigma) \approx F_{(1)}(\tau, \sigma) = \text{numerical function}$, the dynamically determined 3-spaces $\Sigma_\tau$ have an intrinsic 3-curvature $3 \tilde{R}|_{\theta=0} = 2 \sum_a \partial_a^2 \tilde{R}_a^{(1)}$ determined only by the HPM-GW (and therefore by the matter energy-momentum tensor in the past as shown by Eq.(54)). Their extrinsic curvature tensor as sub-manifolds of space-time is

$$3 K_{(1)rs} \approx \sigma_{(1)(r)(s)}|_{r \neq s} + \delta_{rs} \left( \frac{1}{3} F_{(1)} - \partial_\tau \Gamma_r^{(1)} + \partial_r \tilde{n}_{(1)(r)} - \sum_a \partial_a \tilde{n}_{(1)(a)} \right),$$

(58)

with $\tilde{n}_{(1)(r)}$, $\sigma_{(1)(r)(s)}|_{r \neq s}$ and $\Gamma_r^{(1)}$ given in Eqs. (49) and (54). The York time appears only in Eq.(58): all the other PM quantities depend on the non-local York time $3 K_{(1)}(\tau, \sigma) \approx \frac{1}{3} F_{(1)}(\tau, \sigma)$.

In the third paper of Refs.[40], where the matter is restricted only to the particles $35$, one evaluates all the properties of these HPM space-times:

a) the 3-volume element, the 3-distance and the intrinsic and extrinsic 3-curvature tensors of the 3-spaces $\Sigma_\tau$;

b) the proper time of a time-like observer;

c) the time-like and null 4-geodesics (they are relevant for the definition of the radial velocity of stars as shown in the IAU conventions of Ref.[81] and in study of time delays [82], [80]);

d) the red-shift and luminosity distance. In particular one finds that the recessional velocity of a star is proportional to its luminosity distance from the Earth at least for small distances. This is in accord with the Hubble old red-shift-distance relation which is formalized in the Hubble law (velocity-distance relation) when the standard cosmological model is used (see for instance Ref.[46] on these topics). These results have a dependence on the non-local York time, which could play a role in giving a different interpretation of the data from super-novae, which are used as a support for dark energy [1].

$35$ The properties of HPM transverse electro-magnetic fields have still to be explored.
Finally, in Subsection IIIB of the second paper in Refs.[40] it is shown that this HPM linearization can be interpreted as the first term of a HPM expansion in powers of the Newton constant $G$ in the family of 3-orthogonal gauges. This expansion has still to be studied. In particular it will be useful to check whether in the HPM formulation there are phenomena (appearing at high orders in the standard PN expansions) like the hereditary tails starting from 1.5PN [$O((\frac{v}{c})^3)$] and the non-linear (Christodoulou) memory starting from 3PN (see Ref.[83] for a review)\(^{36}\). This would allow one to make a comparison with all the results of the PN expansions, in which today there is control on the GW solution and on the matter equations of motion till order 3.5PN [$O((\frac{v}{c})^7)$] (for binaries see the review in chapter 4 of Ref.[44]) and well established connections with numerical relativity (see the review in Ref.[84]) especially for the binary black hole problem (see the review in Ref.[85]).

\(^{36}\)They imply that GW propagate not only on the flat light-cone but also inside it (i.e. with all possible speeds $0 \leq v \leq c$): there is an instantaneous wavefront followed by a tail traveling at lower speed (it arrives later and then fades away) and a persistent zero-frequency non-linear memory.
8 Post-Minkowskian Hamilton Equations for Particles, their Post-Newtonian Limit and Dark Matter as a Relativistic Inertial Effect

The PM Hamilton equations and their PN limit in 3-orthogonal gauges for a system of $N$ scalar particles of mass $m_i$ and Grassmann-valued signs of energy $\eta_i$ is discussed in this Section by using the results of the third paper in Ref.[40]. See Refs.[71, 78, 86, 87] for classical texts on the motion of particles in gravitational fields and Refs.[74, 88, 89] for more recent developments \(^{37}\).

The treatment in the 3-orthogonal gauges of the PM Hamilton equations for the electro-magnetic field in the radiation gauge is given in the second paper of Ref.[40], while the PM Hamilton equations for perfect fluids are given in Ref.[41].

With only particles the PM approximation with the ultraviolet cutoff $M$ implies $\kappa_i(\tau) = \frac{m_i c \dot{\eta}_i(\tau)}{\sqrt{1-\dot{\eta}_i^2(\tau)}} + O(\zeta^2)$, $M_{(1)}(\tau, \sigma) = \sum_i \delta^3(\sigma, \eta_i(\tau)) \eta_i \frac{m_i c \dot{\eta}_i(\tau)}{\sqrt{1-\dot{\eta}_i^2(\tau)}} + O(\zeta^2)$. Moreover one has $\ddot{\eta}_i(\tau) = O(\zeta)$. The notation $\dot{a}(\tau) = \frac{da(\tau)}{d\tau}$ is used.

One can make a equal time development of the retarded kernel in Eq.(54) like in Ref.[27] for the extraction of the Darwin potential from the Lienard-Wiechert solution (see Eqs. (5.1)-(5.21) of Ref.[27] with $\sum_s P^r_{rs}(\sigma) \dot{\eta}_s(\tau) \rightarrow \sum_{uv} \mathcal{P}_b^{lu}(\sigma) \dot{\eta}_u(\tau) \dot{\eta}_v(\tau) \frac{c}{\sqrt{1-\dot{\eta}_i^2(\tau)}}$). In this way one gets the following expression of the HPM GW from point masses

$$
\Gamma_a^{(1)}(\tau, \sigma) \equiv -\frac{2G}{c^2} \sum_b \tilde{M}_{ab}^{-1} \left( \sum_i \eta_i m_i \sum_{uv} \mathcal{P}_{buv}(\sigma) \frac{\hat{\eta}_u(\tau) \hat{\eta}_v(\tau)}{\sqrt{1-\dot{\eta}_i^2(\tau)}} \right) \\
\left[ |\sigma - \eta_i(\tau)|^{-1} + \sum_{\sigma=1}^{\infty} \frac{1}{(2m)!} \left( \dot{\eta}_i(\tau) \cdot \frac{\partial}{\partial \sigma} \right)^m |\sigma - \eta_i(\tau)|^{2m-1} \right] +

O(\zeta^2),

\mathcal{P}_{rsv} = \frac{1}{2} (\dot{\eta}_r \dot{\eta}_s + \dot{\eta}_s \dot{\eta}_r) - \frac{1}{2} \left( \delta_{rs} - \frac{\partial_r \partial_s}{\Delta} \right) \delta_{uv} + \frac{1}{2} \left( \delta_{rs} + \frac{\partial_r \partial_s}{\Delta} \right) \frac{\partial_u \partial_v}{\Delta}
$$

\(^{37}\) In this approach point particles are considered as independent matter degrees of freedom with a Grassmann regularization of the self-energies to get well defined world-lines (see also Ref.[89]): they are not considered as point-like singularities of solutions of Einstein’s equations (the point of view of Ref.[86]). Solutions of this type have to be described with distributions and, as shown in Ref.[90], the most general class of such solutions under mathematical control includes singularities simulating matter shells, but not either strings or particles. See also Ref.[91].
\[
-\frac{1}{2} \left[ \frac{\partial_u}{\Delta} (\delta_{ru} \partial_s + \delta_{se} \partial_r) + \frac{\partial_v}{\Delta} (\delta_{rv} \partial_s + \delta_{su} \partial_r) \right],
\]
(59)

to order \(O(\zeta^2)\).

If the lapse and shift functions are rewritten in the form \(n_{(1)} = \bar{n}_{(1)} - \partial_c \mathcal{K}_{(1)}\), \(\bar{n}_{(1)(c)} = \hat{n}_{(1)(c)} + \partial_r \mathcal{K}_{(1)}\), to display their dependence on the inertial gauge variable non-local York time, it can be shown that the PM Hamilton equations for the particles imply the following form of the PM Grassmann regularized second-order equations of motion showing explicitly the equality of the inertial and gravitational masses of the particles.

\[
m_i \eta_i \ddot{\eta}_i (\tau) \triangleq \eta_i \sqrt{1 - \eta_i^2 (\tau)} \left( F_i^r - \dot{\eta}_i (\tau) \dot{\eta}_i (\tau) : \mathcal{F}_i \right) (\tau | \eta_i (\tau) | \eta_i \neq i (\tau)) = \\
\quad \text{def} \eta_i^r (\tau) (\eta_i (\tau) | \eta_i \neq i (\tau)),
\]

\[
\eta_i F_i^r (\tau | \eta_i (\tau) | \eta_i \neq i (\tau)) = \frac{m_i \eta_i}{\sqrt{1 - \eta_i^2 (\tau)}} \left( - \frac{\partial \bar{n}_{(1)} (\tau), \eta_i (\tau)}{\partial \eta_i^2} + \\
+ \frac{\dot{\eta}_i^r (\tau)}{1 - \eta_i^2 (\tau)} \sum_u \left[ \ddot{\eta}_i^u (\tau) \frac{\partial \bar{n}_{(1)}}{\partial \eta_i^u} + \sum_{j \neq i} \ddot{\eta}_i^j (\tau) \frac{\partial \hat{n}_{(1)}}{\partial \eta_j^u} \right] (\tau, \eta_i (\tau)) + \\
+ \left( \sum_u \ddot{\eta}_i^u (\tau) \left[ \frac{\partial \hat{n}_{(1)(u)}}{\partial \eta_i^u} - \frac{\partial \hat{n}_{(1)(\tau)}}{\partial \eta_i^u} \right] - \sum_{j \neq i} \sum_u \ddot{\eta}_i^j (\tau) \frac{\partial \hat{n}_{(1)(\tau)}}{\partial \eta_j^u} - \\
- \frac{\ddot{\eta}_i^i (\tau)}{1 - \eta_i^2 (\tau)} \sum_u \ddot{\eta}_i^u (\tau) \sum_s \left[ \ddot{\eta}_i^s (\tau) \frac{\partial \hat{n}_{(1)(u)}}{\partial \eta_i^s} + \\
+ \sum_{j \neq i} \ddot{\eta}_j^s (\tau) \frac{\partial \hat{n}_{(1)(u)}}{\partial \eta_j^s} \right] \right) (\tau, \eta_i (\tau)) + \left( \sum_u \ddot{\eta}_i^u (\tau)^2 \frac{\partial (\Gamma_u^{(1)} + 2 \phi_{(1)})}{\partial \eta_i^u} \right) - \\
\right. \\
\left. - \ddot{\eta}_i^i (\tau) \left( 2 \frac{\partial (\Gamma_i^{(1)} + 2 \phi_{(1)})}{\partial \eta_i^2} + \sum_c \frac{\ddot{\eta}_i^c (\tau)^2}{1 - \eta_i^2 (\tau)} \frac{\partial (\Gamma_c^{(1)} + 2 \phi_{(1)})}{\partial \eta_i^c} \right) + \\
+ \sum_{j \neq i} \ddot{\eta}_j^s (\tau) \left( 2 \frac{\partial (\Gamma_j^{(1)} + 2 \phi_{(1)})}{\partial \eta_j^2} \right) + \\
+ \sum_c \frac{\ddot{\eta}_i^c (\tau)^2}{1 - \eta_i^2 (\tau)} \frac{\partial (\Gamma_i^{(1)} + 2 \phi_{(1)})}{\partial \eta_i^c} \right) \right) (\tau, \eta_i (\tau)) - \\
- \frac{\ddot{\eta}_i^i (\tau)}{1 - \eta_i^2 (\tau)} \left[ \frac{\partial}{\partial r} | \eta_i \mathcal{K}_{(1)} + 2 \sum_s \ddot{\eta}_i^s (\tau) \frac{\partial \partial_r \eta_i \mathcal{K}_{(1)}}{\partial \eta_i^s} \right] + 
\]

From Clock Synchronization 65
\[ + \sum_{s} \dot{\eta}_{r}^s(\tau) \dot{\eta}_{i}^s(\tau) \frac{\partial^2 3K^{(1)}}{\partial \eta_{r}^s \partial \eta_{i}^s}(\tau, \eta_{i}(\tau)) \bigg] + O(\zeta^2). \] (60)

The effective action-at-a-distance force \( F_i(\tau) \) contains

a) the contribution of the lapse function \( \dot{n}_{(1)} \), which generalizes the Newton force;

b) the contribution of the shift functions \( \dot{n}_{(1)}(r) \), which gives the gravito-magnetic effects;

c) the retarded contribution of HPM GW, described by the functions \( \Gamma^{(1)}_r \) of Eq.(59);

d) the contribution of the volume element \( \phi_{(1)} \) \( \dot{\phi} = 1 + 6 \phi_{(1)} + O(\zeta^2) \), always summed to the HPM GW, giving forces of Newton type;

e) the contribution of the inertial gauge variable (the non-local York time)

\[ 3K^{(1)} = \frac{1}{\Delta} 3K^{(1)}. \]

In the electro-magnetic case in SR [28] the regularized coupled second-order equations of motion of the particles obtained by using the Lienard-Wiechert solutions for the electro-magnetic field are independent by the type of Green function (retarded or advanced or symmetric) used. The electromagnetic retardation effects, killed by the Grassmann regularization, are connected with QED radiative corrections to the one-photon exchange diagram. This is not strictly true in the gravitational case. The effect of retardation is not killed by the Grasmann regularization but only pushed to \( O(\zeta^2) \): at this order it should give extra contributions to the second-order equations of motion. This shows that our semi-classical approximation, obtained with our Grassmann regularization, of an unspecified "quantum gravity" theory does not take into account only a "one-graviton exchange diagram": in the spin 2 case there is an extra retardation effect showing up only at higher HPM orders \( \text{38} \).

8.1 The Center-of-Mass Problem in General Relativity

and in the HPM Linearization.

As said in Section 5, the 3-universe is described in a non inertial rest frame with non-Euclidean 3-spaces \( \Sigma_r \) tending to Euclidean inertial ones \( \Sigma_{r(\infty)} \).

\[ 38 \text{ In the electro-magnetic case the Grassmann regularization implies } Q_i \eta_{r}^i(\tau - |\sigma|) = Q_i \eta_{i}(\tau) \text{ and equations of motion of the type } \dot{\eta}_{r}^i(\tau) = Q_i \ldots \text{ with } Q_i^2 = 0. \text{ In the gravitational case the equations of motion are of the type } \eta_{r}^i(\tau) = \eta_{i} \ldots \text{ with } \eta_{r}^i = 0, \text{ but the Grassmann regularized retardation in Eq.(54) gives Eq.(59) only at the lowest order in } \zeta \text{ and has contributions of every order } O(\zeta^k). \]
at spatial infinity. Both matter and gravitational degrees of freedom live inside $\Sigma_\tau$ and their internal 3-center of mass is eliminated by the rest-frame condition $\hat{P}_{ADM} \approx 0$ (implied by the absence of super-translations) if also the condition $\hat{K}_{ADM}^r = \hat{J}_{ADM}^r \approx 0$ is added like in SR. The 3-universe may be described as an external decoupled center of mass carrying a pole-dipole structure: $\hat{E}_{ADM}$ is the invariant mass and $\hat{J}_{ADM}^r$ the rest spin. As in SR the condition $\hat{K}_{ADM}^r \approx 0$ selects the Fokker-Pryce center of inertia as the natural time-like observer origin of the radar coordinates: it follows a non-geodetic straight world-line like the asymptotic inertial observers existing in these space-times.

This is a way out from the the problem of the center of mass in general relativity and of its world-line, a still open problem in generic space-times as can be seen from Refs. [58, 92] (and Ref. [74] for the PN approach). Usually, by means of some supplementary condition, the center of mass is associated to the monopole of a multipolar expansion of the energy-momentum of a small body (see Ref. [26] for the special relativistic case).

In SR the elimination of the internal 3-center of mass leads to describe the dynamics inside $\Sigma_\tau$ only in terms of relative variables (see Eqs.(15) in the case of particles). However relative variables do not exist in the non-Euclidean 3-spaces of curved space-times, where flat objects like $r_{ij}(\tau) = \eta_{i}(\tau) - \eta_{j}(\tau)$ have to be replaced with a quantity proportional to the tangent vector to the space-like geodesics joining the two particles in the non-Euclidean 3-space $\Sigma_\tau$ (see Ref. [93] for an implementation of this idea). Quantities like $r_{ij}^2(\tau)$ have to be replaced with the Synge world function [82, 87, 89] \(^{39}\). This problem is another reason why extended objects tend to be replaced with point-like multipoles, which, however, do not span a canonical basis of phase space (see Refs. [26] for SR).

However, at the level of the HPM approximation one can introduce relative variables for the particles, like the SR ones of Eq.(15), defined as 3-vectors in the asymptotic inertial rest frame $\Sigma_\tau(\infty)$ by putting $\eta_{i}(\tau) = \eta_{(o)i}(\tau) + \eta_{(1)i}(\tau)$ and $\kappa_{i}(\tau) = \kappa_{(o)i}(\tau) + \kappa_{(1)i}(\tau)$ with $\eta_{(o)i}(\tau)$, $\kappa_{(o)i}(\tau) = O(\zeta)$. This allows one to define HPM collective and relative canonical variables for the particles, with the collective variables eliminated by the conditions $\hat{P}_{ADM}^r \approx 0$ and $\hat{K}_{ADM}^r \approx 0$ (at the lowest order they become the SR conditions).

In the case of two particles (with total and reduced masses $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{M}$) one puts $\eta_{1}(\tau) = \eta_{12}(\tau) + \frac{m_1}{M} \rho_{12}(\tau)$, $\eta_{2}(\tau) = \eta_{12}(\tau) - \frac{m_2}{M} \rho_{12}(\tau)$, $\kappa_{1}(\tau) = \frac{m_1}{M} \kappa_{12}(\tau) + \pi_{12}(\tau)$, $\kappa_{2}(\tau) = \frac{m_2}{M} \kappa_{12}(\tau) - \pi_{12}(\tau)$ and goes to the new canonical basis $\eta_{12}(\tau) = \frac{m_1 \eta_{1}(\tau) + m_2 \eta_{2}(\tau)}{M}$, $\rho_{12}(\tau) = \frac{m_1 \kappa_{1}(\tau) + m_2 \kappa_{2}(\tau)}{M}$, $\kappa_{12}(\tau) = \kappa_{1}(\tau) + \kappa_{2}(\tau)$, $\pi_{12}(\tau) = \frac{m_1 \kappa_{1}(\tau) - m_2 \kappa_{2}(\tau)}{M}$.

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\(^ {39}\) It is a bi-tensor, i.e. a scalar in both the points $\eta_{i}(\tau)$ and $\eta_{j}(\tau)$, defined in terms of the space-like geodesic connecting them in $\Sigma_\tau$. See Eq.(3.13) of the third paper in Ref.[40].
It can be shown that the conditions \( \hat{P}_{ADM}^r \approx 0 \) and \( \hat{K}_{ADM}^r \approx 0 \) imply

\[
\eta_1(\tau) \approx \left( \frac{m_2}{M} - A_{(o)}(\tau) \right) \rho_{(o)12}(\tau) + \frac{m_2}{M} \rho_{(1)12}(\tau) + f_1(\tau) [\text{rel. var., GW}], \\
\eta_2(\tau) \approx -\left( \frac{m_1}{M} + A_{(o)}(\tau) \right) \rho_{(o)12}(\tau) - \frac{m_1}{M} \rho_{(1)12}(\tau) + f_1(\tau) [\text{rel. var., GW}], \\
A_{(o)}(\tau) = \frac{m_2}{M} \sqrt{\frac{m_1 c^2 + \pi_1^2 (1)_{12}(\tau)}{m_1 c^2 + \pi_1^2 (1)_{12}(\tau)}} - \frac{m_1}{M} \sqrt{\frac{m_2 c^2 + \pi_2^2 (1)_{12}(\tau)}{m_1 c^2 + \pi_1^2 (1)_{12}(\tau)}},
\]

(61)

for some function \( f_1[\text{rel. var., GW}](\tau) \approx \eta_{(1)12}(\tau) \) depending on the relative variables and the HPM GW of Eq.(59) in absence of incoming radiation. Then the equations of motion (60) imply

\[
\mu \ddot{\rho}_{(o)12}(\tau) \approx \frac{m_2}{M} F_1^r(\tau | \eta_{(o)1}(\tau) | \eta_{(o)2}(\tau)) - \frac{m_1}{M} F_2^r(\tau | \eta_{(o)2}(\tau) | \eta_{(o)1}(\tau)),
\]

(62)

for the relative configurational variable. The collective configurational variable has \( \eta_{(o)12}(\tau) \approx -A_{(o)}(\tau) \rho_{(o)12}(\tau) \) at the lowest order, while at the first order there is an equation of motion equivalent to \( \ddot{\eta}_{(1)12}(\tau) \approx \frac{d^2}{c^2} \eta_{(1)12}(\tau) \) \( (\text{rel. var., GW}) \).

### 8.2 The Post-Newtonian Expansion at all Orders in the Slow Motion Limit.

If all the particles are contained in a compact set of radius \( l_c \), one can add a slow motion condition in the form \( \sqrt{\varepsilon} = \frac{v}{c} \approx \sqrt{\frac{R_{mi}}{l_c}}, \ i = 1, \ldots N \) \( (R_{mi} = 2Gm_i/c^2) \) is the gravitational radius of particle \( i \) with \( l_c \geq R_M \) and \( \lambda >> l_c \) (see the Introduction). In this case one can do the PN expansion of Eqs.(60).

After having put \( \tau = ct \), one makes the following change of notation

\[
\eta_i(\tau) = \tilde{\eta}_i(t), \quad v_i(t) = \frac{d\tilde{\eta}_i(t)}{dt}, \quad a_i(t) = \frac{d^2 \tilde{\eta}_i(t)}{dt^2},
\]

\[
\dot{\eta}_i(t) = \frac{v_i(t)}{c}, \quad \ddot{\eta}_i(t) = \frac{a_i(t)}{c^2}.
\]

(63)
For the non-local York time one uses the notation $^3\tilde{K}_{(1)}(t, \sigma) = ^3K_{(1)}(\tau, \sigma)$.

Then one studies the PN expansion of the equations of motion (60) with the result (kPN means of order $O(c^{-2k})$)

$$m_i \frac{d^2 \tilde{\eta}^i(t)}{dt^2} = m_i \left[ -G \frac{\partial}{\partial \tilde{\eta}^i} \sum_{j \neq i} \frac{m_j}{|\tilde{\eta}_i(t) - \tilde{\eta}_j(t)|} - \frac{1}{c} \frac{d\tilde{\eta}^i(t)}{dt} \left( \frac{\partial^2}{\partial \tilde{\eta}^i \partial \tilde{\eta}^j} ^3\tilde{K}_{(1)}(t, \tilde{\eta}_i(t)) \right) + 
+ 2 \sum_u v^u_i(t) \frac{\partial}{\partial \tilde{\eta}^i} ^3\tilde{K}_{(1)}(t, \tilde{\eta}_i(t)) + \sum_{uv} v^u_i(t) v^v_i(t) \frac{\partial^2}{\partial \tilde{\eta}^i \partial \tilde{\eta}^j} ^3\tilde{K}_{(1)}(t, \tilde{\eta}_i(t)) \right] + 
+ F_{i(1PN)}(t) + (\text{higher PN orders}).$$

(64)

At the lowest order one finds the standard Newton gravitational force

$$F_{i(\text{Newton})}(t) = -m_i G \frac{\partial}{\partial \tilde{\eta}^i} \sum_{j \neq i} \frac{m_j}{|\tilde{\eta}_i(t) - \tilde{\eta}_j(t)|}.$$ 

The unexpected result is a 0.5PN force term containing all the dependence upon the non-local York time. The (arbitrary in these gauges) double rate of change in time of the trace of the extrinsic curvature creates a 0.5 PN damping (or anti-damping since the sign of the inertial gauge variable $^3K_{(1)}$ of Eq.(57) is not fixed) effect on the motion of particles. This is an inertial effect (hidden in the lapse function) not existing in Newton theory where the Euclidean 3-space is absolute.

Then there are all the other kPN terms with $k = 1, 1.5, 2,...$. Since these results have been obtained without introducing ad hoc Lagrangians for the particles, are not in the harmonic gauge and do not contain terms of order $O(\zeta^2)$ and higher, it is not possible to make a comparison with the standard PN expansion (whose terms are known till the order 3.5PN [44]). Therefore only the 1PN and 0.5PN terms will be considered in the next two Subsections.

### 8.3 The HPM Binaries at the 1PN Order

Since in the next Subsection the 0.5PN term depending on the non-local York time will be connected with dark matter at the level of galaxies and clusters of galaxies and since there is no convincing evidence of dark matter in the Solar System and near the galactic plane of the Milky Way [96], it is reasonable to assume $^3K_{(1)}(\tau, \sigma) = ^3F_{(1)}(\tau, \sigma) \approx 0$ near a star with planets and near a binary.

In the description of Subsection 8.1 of the 1PN two-body problem, which is relevant for the treatment of binary systems as shown in Chapter VI

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40 For binaries one assumes $\frac{\tilde{r}}{l_c} \approx \sqrt{\frac{m_2}{m_1}} << 1$, where $l_c \approx |\tilde{r}|$ with $\tilde{r}(t)$ being the relative separation after the decoupling of the center of mass. Often one considers the case $m_1 \approx$
of Refs. [44] based on Ref. [47, 94, 95], it can be shown that the relative momentum in the rest frame has the 1PN expression \( \pi_{12}(\tau) = \pi_{(1)12}(\tau) \approx \mu \mathbf{v}_{(\text{rel})(o)12}(t) \left[ 1 + \frac{m_1^2 + m_2^2}{2M^3} \left( \frac{v_{(\text{rel})(o)12}(t)}{c} \right)^2 \right] \), where \( \mathbf{v}_{(\text{rel})(o)12}(t) = \frac{d \rho_{(o)12}(t)}{dt} \) is the velocity of the lowest order \( \rho_{(o)12}(\tau) \) of the relative variable.

If one ignores the York time and considers only positive energy particles \((\eta_1, \eta_2 \rightarrow +1)\), the 1PN equations of motion for the relative variable of the binary implied by Eqs. (62) and 1PN expression of the weak ADM energy \( \mathcal{E}_{ADM} \) and of the rest spin \( \mathcal{J}_{ADM}^{\text{rs}} \) (being determined by the boundary conditions they are constants of the motion implying planar motion in the plane orthogonal to the rest spin) can be shown to be

\[
\frac{d \mathbf{v}_{(\text{rel})(o)12}(t)}{dt} = -GM \frac{\rho_{(o)12}(t)}{[\rho_{(o)12}(t)]^3} \left[ 1 + \left( 1 + \frac{3\mu}{M} \right) \frac{\mathbf{v}_{(\text{rel})(o)12}(t)}{c} \right] - \frac{3\mu}{2M} \left( \frac{\mathbf{v}_{(\text{rel})(o)12}(t) \cdot \mathbf{\rho}_{(o)12}(t)}{[\rho_{(o)12}(t)]} \right)^2 + \frac{GM}{[\rho_{(o)12}(t)]^3} \left( 4 - \frac{2\mu}{M} \right) \mathbf{v}_{(\text{rel})(o)12}(t) \frac{\mathbf{v}_{(\text{rel})(o)12}(t) \cdot \mathbf{\rho}_{(o)12}(t)}{[\rho_{(o)12}(t)]}.
\]

\[
\mathcal{E}_{ADM(1PN)} = \sum_i m_i c^2 + \mu \left( \frac{1}{2} v_{(\text{rel})(o)12}^2(t) \right) \left[ 1 + \frac{m_1^2 + m_2^2}{M^3} \left( \frac{v_{(\text{rel})(o)12}(t)}{c} \right)^2 \right] - \frac{GM}{[\rho_{(o)12}(t)]^3} \left[ 1 + \frac{1}{2} \left( 3 + \frac{\mu}{M} \right) \frac{v_{(\text{rel})(o)12}(t)}{c} \right] + \frac{\mu}{M} \left( \frac{\mathbf{v}_{(\text{rel})(o)12}(t)}{c} \cdot \frac{\mathbf{\rho}_{(o)12}(t)}{[\rho_{(o)12}(t)]^2} \right),
\]

\[
\mathcal{J}_{ADM(1PN)}^{\text{rs}} = \left( \rho_{(o)12}(t) v_{(\text{rel})(o)12}(t) - \rho_{(o)12}(t) v_{(\text{rel})(o)12}(t) \right) \left[ 1 + \frac{m_1^2 + m_2^2}{2M^3} \left( \frac{v_{(\text{rel})(o)12}(t)}{c} \right)^2 \right].
\]

(65)

Our 1PN equations (65) in the 3-orthogonal gauges coincide with Eqs. (2.5), (2.13) and (2.14) of the first paper in Ref. [47] (without \( G^2 \) terms since they are \( O(\zeta^2) \)), which are obtained in the family of harmonic gauges starting from an ad hoc 1PN Lagrangian for the relative motion of two test particles (first derived by Infeld and Plebanski [86]) \(^{41}\). These equations are the starting point for studying the post-Keplerian parameters of the binaries, which,

\(^{41}\) See chapter 4 of Ref. [44] for a review of the emission of GW’s from circular and elliptic Keplerian orbits and of the induced inspiral phase.
together with the Roemer, Einstein and Shapiro time delays (both near Earth and near the binary) in light propagation, allow one to fit the experimental data from the binaries (see the second paper in Ref.[47] and Chapter VI of Ref.[44]). Therefore these results are reproduced also in our 3-orthogonal gauge with $\tilde{\mathcal{K}}^{(1)}(\tau, \sigma) = 0$.

8.4 From the Three Signatures for Dark Matter
Reinterpreted as Relativistic Inertial Effects
Induced by the York Time to the Need of a PM ICRS

To study the effects induced by the 0.5PN velocity-dependent (friction or anti-friction) force term in Eqs.(64), depending on the inertial gauge variable non-local York time $\tilde{\mathcal{K}}^{(1)}(t, \tilde{\eta}_i(t)) = \frac{1}{c^2} \tilde{\mathcal{K}}^{(1)}(\tau, \sigma) \approx \frac{1}{c} F^{(1)}(\tau, \sigma)$ with $F^{(1)}(\tau, \sigma)$ arbitrary numerical function, it is convenient to rewrite such equations in the form

$$\frac{d}{dt} \left[ m_i \left( 1 + \frac{1}{c} \frac{d}{dt} \tilde{\mathcal{K}}^{(1)}(t, \tilde{\eta}_i(t)) \right) \frac{d\tilde{\eta}^r_i(t)}{dt} \right] = -G \frac{\partial}{\partial \eta^r_i} \sum_{j \neq i} \eta_j \frac{m_i m_j}{|\tilde{\eta}_i(t) - \tilde{\eta}_j(t)|} + O(\zeta^2),$$

(66)

because the damping or anti-damping factors in Eq.(64) are $\gamma_{il}(t, \tilde{\eta}_i(t)) = \tilde{\mathcal{K}}^{(1)}(t, \tilde{\eta}_i(t))$ and $\tilde{\eta}_i(t) = O(\zeta)$.

As a consequence the velocity-dependent force can be reinterpreted as the introduction of an effective (time-, velocity- and position-dependent) inertial mass term for the kinetic energy of each particle:

$$m_i \rightarrow m_i \left( 1 + \frac{1}{c} \frac{d}{dt} \tilde{\mathcal{K}}^{(1)}(t, \tilde{\eta}_i(t)) \right) = m_i + (\Delta m)_i(t, \tilde{\eta}_i(t)),$$

(67)

in each instantaneous 3-space. Instead in the Newton potential there are the gravitational masses of the particles, equal to the inertial ones in the 4-dimensional space-time due to the equivalence principle. Therefore the effect is due to a modification of the effective inertial mass in each non-Euclidean 3-space depending on its shape as a 3-sub-manifold of space-time: it is the equality of the inertial and gravitational masses of Newtonian gravity to be violated! In Galilei space-time the Euclidean 3-space is an absolute time-independent notion like Newtonian time: the non-relativistic non-inertial frames live in
this absolute 3-space differently from what happens in SR and GR, where they are (in general non-Euclidean) 3-sub-manifolds of the space-time.

Eqs. (64), (66) and (67) can be applied to the three main signatures of the existence of dark matter in the observed masses of galaxies and clusters of galaxies, where the 1PN forces are not important, namely the virial theorem [49], the weak gravitational lensing [50], [49] and the rotation curves of spiral galaxies (see Ref.[48] for a review), to give a reinterpretation of dark matter as a relativistic inertial effect.

A) Masses of clusters of galaxies from the virial theorem. For a bound system of N particles of mass m (N equal mass galaxies) at equilibrium, the virial theorem relates the average kinetic energy \( \langle E_{\text{kin}} \rangle \) in the system to the average potential energy \( \langle U_{\text{pot}} \rangle \) in the system: \( \langle E_{\text{kin}} \rangle = -\frac{1}{2} \langle U_{\text{pot}} \rangle \) assuming Newton gravity. For the average kinetic energy of a galaxy in the cluster one takes \( \langle E_{\text{kin}} \rangle \approx \frac{1}{2} m < v^2 > \), where \( < v^2 > \) is the average of the square of the radial velocity of single galaxies with respect to the center of the cluster (measured with Doppler shift methods; the velocity distribution is assumed isotropic). The average potential energy of the galaxy is assumed of the form \( \langle U_{\text{pot}} \rangle = -\frac{G m M}{R} \), where \( M = N m \) is the total mass of the cluster and \( R = \alpha R \) is a "effective radius" depending on the cluster size \( R \) (the angular diameter of the cluster and its distance from Earth are needed to find \( R \)) and on the mass distribution on the cluster (usually \( \alpha \approx 1/2 \)). Then the virial theorem implies \( M \approx \frac{R^2}{G} < v^2 > \). It turns out that this mass \( M \) of the cluster is usually at least an order of magnitude bigger that the baryonic matter of the cluster \( M_{\text{bar}} = N m \) (spectroscopically determined).

By applying Eqs.(64) to the equilibrium condition for a self-gravitating system, i.e. \( \frac{d^2}{dt^2} \sum_i m_i \left| \vec{\eta}_i(t) \right|^2 = 0 \) with \( m_i = m \), one gets \( \sum_{i>j} G \frac{m_i m_j}{|\vec{\eta}_i(t) - \vec{\eta}_j(t)|} - \frac{1}{c^2} \sum_i m_i \left( \vec{\eta}_i(t) \cdot \vec{v}_i(t) \right) \gamma_i(t, \vec{\eta}_i(t)) = 0 \) with \( m_i = m_j = m \). Therefore one can write \( \langle U_{\text{pot}} \rangle = -\frac{1}{R^2} \sum_{i>j} \frac{G m^2}{|\vec{\eta}_i(t) - \vec{\eta}_j(t)|} \approx G \frac{m M_{\text{bar}}}{R} \) (with \( R = R/2 \)) and \( \frac{1}{2} m < v^2 > = -\frac{1}{2} \langle U_{\text{pot}} \rangle + \frac{m^2}{2c^2} \langle (\vec{\eta} \cdot \vec{v}) \gamma(t, \vec{\eta}) \rangle \) with the notation \( \langle (\vec{\eta} \cdot \vec{v}) \gamma(t, \vec{\eta}) \rangle = \frac{1}{R} \sum_i \left( \vec{\eta}_i(t) \cdot \vec{v}_i(t) \right) \gamma_i(t, \vec{\eta}_i(t)) \) (it contains the non-local York time). Therefore for the measured mass \( M \) (the effective inertial mass in 3-space) one has

\[
M = \frac{R}{G} < v^2 > = M_{\text{bar}} + \frac{R}{G c} \langle (\vec{\eta} \cdot \vec{v}) \gamma(t, \vec{\eta}) \rangle \overset{\text{def}}{=} M_{\text{bar}} + M_{\text{DM}},
\]

(68)

and one sees that the non-local York time can give rise to a dark matter contribution \( M_{\text{DM}} = M - M_{\text{bar}} \).

B) Masses of galaxies or clusters of galaxies from weak gravitational lensing. Usually one considers a galaxy (or a cluster of galaxies) of big mass \( M \)
behind which a distant, bright object (often a galaxy) is located. The light from the distant object is bent by the massive one (the lens) and arrives on the Earth deflected from the original propagation direction. As shown in Ref. [50] one has to evaluate Einstein deflection of light, emitted by a source $S$ at distance $d_S$ from the observer $O$ on the Earth, generated by the big mass at a distance $d_D$ from the observer $O$. The mass $M$, at distance $d_{DS}$ from the source $S$, is considered as a point-like mass generating a 4-metric of the Schwarzschild type (Schwarzschild lens). The ray of light is assumed to propagate in Minkowski space-time till near $M$, to be deflected by an angle $\alpha$ by the local gravitational field of $M$ and then to propagate in Minkowski space-time till the observer $O$. The distances $d_S$, $d_D$, $d_{DS}$, are evaluated by the observer $O$ at some reference time in some nearly-inertial Minkowski frame with nearly Euclidean 3-spaces (in the Euclidean case $d_{DS} = d_S - d_D$). If $\xi = \theta d_D$ is the impact parameter of the ray of light at $M$ and if $\xi \gg R_s = \frac{2GM}{c^2}$ (the gravitational radius), Einstein’s deflection angle is

$$\alpha = \frac{2 R_s}{\xi} = \frac{4GM}{c^2\xi}$$

and the so-called Einstein radius (or characteristic angle) is

$$\alpha_o = \sqrt{2 R_s} \frac{d_{DS}}{d_D} = \sqrt{\frac{4GM}{c^2} \frac{d_{DS}}{d_D}}.$$ A measurement of the deflection angle and of the three distances allows to get a value for the mass $M$ of the lens, which usually turns out to be much larger of its mass inferred from the luminosity of the lens. For the calculation of the deflection angle one considers the propagation of ray of light in a stationary 4-metric of the BCRS type and uses a version of the Fermat principle containing an effective index of refraction $n$. One has $n = \gamma_{\tau\tau} = \epsilon [1 - \frac{2\omega}{c^2} - 2 \partial_{\tau}^3\mathcal{K}]$ in the PM approximation. Since one has $\frac{d^2\psi}{ds^2} = -\frac{GM_{\text{bar}}}{c^2\epsilon}$, the definition $2 \partial_{\tau}^3\mathcal{K}_{(1)}^{\text{def}} = -\frac{GM_{\text{bar}}}{c^2\epsilon}$ leads to an Einstein deflection angle

$$\alpha = \frac{4GM}{c^2\xi} \quad \text{with} \quad M^{\text{def}} = M_{\text{bar}} + M_{DM}.$$ (69)

Therefore also in this case the measured mass $M$ is the sum of a baryonic mass $M_{\text{bar}}$ and of a dark matter mass $M_{DM}$ induced by the non-local York time at the location of the lens.

C) Masses of spiral galaxy masses from their rotation curves. In this case one considers a two-body problem (a point-like galaxy and a body circulating around it) described in terms of an internal center of mass $\tilde{\eta}_{12}(t) \approx \tilde{\eta}_{(1)12}(t)$ ($\tilde{\eta}_{(G)12}(t) = 0$ is the origin of the 3-coordinates) and a relative variable $\tilde{\rho}_{12}(t)$. Then the sum and difference of Eqs. (64) imply the equations of motion for $\tilde{\eta}_{(1)12}(t)$ and $\tilde{\rho}_{12}(t)$. While the first equation implies a small motion of the overall system, the second one has the form

$$\frac{d^2\tilde{\rho}_{(1)12}(t)}{dt^2} = -GM\frac{\tilde{\rho}_{12}(t)}{|\tilde{\rho}_{12}(t)|^3} \frac{1}{c} \frac{d\tilde{\rho}_{12}(t)}{dt} \gamma_{\tau}(t, \tilde{\rho}_{12}(t), \psi(t)), $$
\[
\gamma_+(t, \tilde{\rho}_{12}(t), v(t)) = \frac{m_1}{M} \gamma_1(t, \frac{m_2}{M} \tilde{\rho}_{12}(t), v(t)) + \frac{m_2}{M} \gamma_2(t, -\frac{m_1}{M} \tilde{\rho}_{12}(t), v(t)),
\]

(70)

where \(\gamma_i\) are the damping or anti-damping factors defined after Eq.(66). Eq.(70) gives the two-body Kepler problem with an extra perturbative force. Without it a Keplerian solution with circular trajectory such that \(|\tilde{\rho}_{12}(t)| = R = \text{const.}\) implies that the Keplerian velocity \(v_o(t) = v_o \hat{n}(t)\) has the modulus vanishing at large distances, \(v_o = \sqrt{\frac{GM}{R}} \to R \to \infty 0\). Instead the rotation curves of spiral galaxies imply that the relative 3-velocity goes to constant for large \(R\), i.e. \(v = \sqrt{\frac{GM}{R}} + \Delta M(r) \to R \to \infty \text{const.} (M_{\text{bar}}\) is the spectroscopically determined baryon mass), so that the extra required term \(\Delta M(r)\) is interpreted as the mass \(M_{DM}\) of a dark matter halo.

The presence of the extra force term implies that the velocity must be written as \(v(t) = v_o(t) + v_1(t)\) with \(v_1(t)\) a first order perturbative correction satisfying \(\frac{dv_1(t)}{dt} = -\frac{\gamma_+}{c} \hat{n}(t) \gamma_+(t, \tilde{\rho}_{12}(t), v_o(t))\). Therefore at the first order in the perturbation one gets \(v^2(t) = v_o^2 \left(1 - \frac{2}{c} \hat{n}(t) \cdot \int_0^t dt_1 \hat{n}(t_1) \gamma_+(t_1, \tilde{\rho}_{12}(t_1), v_o(t_1))\right)\). Therefore, after having taken a mean value over a period \(T\) (the time dependence of the mass of a galaxy is not known) the effective mass of the two-body system is

\[
M_{\text{eff}} = \frac{(v^2) R}{G} = M \left(1 - \left\langle \frac{2}{c} \hat{n}(t) \cdot \int_0^t dt_1 \hat{n}(t_1) \gamma_+(t_1, \tilde{\rho}_{12}(t_1), v_o(t_1))\right\rangle\right) = M_{\text{bar}} + M_{DM}.
\]

(71)

with a \(\Delta M(r) = M_{DM}\) function only of the mean value of the total time derivative of the non-local \(3K_{(1)}\) to be fitted to the experimental data.

Therefore, the existence of the inertial gauge variable York time, a property of the non-Euclidean 3-spaces as 3-sub-manifolds of Einstein space-times (connected only to the general relativistic remnant of the gauge freedom in clock synchronization, independently from cosmological assumptions) implies the possibility of describing part (or maybe all) dark matter as a relativistic inertial effect in Einstein gravity without alternative explanations using:

1) the non-relativistic MOND approach [97] (where one modifies Newton equations);

2) modified gravity theories like the \(f(R)\) ones (see for instance Refs.[98]; here one gets a modification of the Newton potential);

3) the assumption of the existence of WIMP particles [99].
Let us also remark that the 0.5PN effect has origin in the lapse function and not in the shift one, as in the gravito-magnetic elimination of dark matter proposed in Ref.[100].

The open problem with this explanation of dark matter is the determination of the non-local York time from the data on dark matter. From what is known about dark matter in the Solar System and inside the Milky Way near the galactic plane, it seems that \( \mathcal{K}_{(1)}(\tau, \sigma) \) is negligible near the stars inside a galaxy. Instead the non-local York time (or better a mean value in time of its total time derivative) should be relevant around the galaxies and the clusters of galaxies, where there are big concentrations of mass and well defined signatures of dark matter. Instead there is no indication on its value in the voids existing among the clusters of galaxies.

Therefore the known data on dark matter do not allow one to get an experimental determination of the York time \( \mathcal{K}_{(1)}(\tau, \sigma) = \Delta \mathcal{K}_{(1)}(\tau, \sigma) \), because to do it one needs to know the non-local York time on all the 3-universe at a given \( \tau \).

Since, as said in the Introduction, at the experimental level the description of matter is intrinsically coordinate-dependent, namely is connected with the conventions used by physicists, engineers and astronomers for the modeling of space-time, one has to choose a gauge (i.e. a 4-coordinate system) in non-modified Einstein gravity which is in agreement with the observational conventions in astronomy. This way out from the gauge problem in GR requires a choice of 3-coordinates on the instantaneous 3-spaces identified by a choice of time and by a clock synchronization convention, i.e. a fixation of the York time \( \mathcal{K}_{(1)}(\tau, \sigma) \). The convention resulting by one set of such choices would give a PM extension of ICRS, with BCRS being its quasi-Minkowskian approximation for the Solar System. Since the existing ICRS [3, 5] has diagonal 3-metric, 3-orthogonal gauges are a convenient choice.

The real problem is the extraction of an indication of which kind of function of time and 3-coordinates to use for the York time \( \mathcal{K}_{(1)}(\tau, \sigma) \) from astrophysical data different from the ones giving information about dark matter. Once one would have a phenomenological parametrization of the York time, then the data on dark matter would put restrictions on the induced phenomenological parametrization of the non-local York time \( \mathcal{K}_{(1)}(\tau, \sigma) = \frac{1}{2} \mathcal{K}_{(1)}(\tau, \sigma) \). As it will be delineated in the final Section, to implement this program one has to look at the astrophysical data on dark energy after having succeeded to interpret also it as a relativistic inertial effect in suitable cosmological space-times in which one can induce the distinction between inertial and tidal degrees of freedom of the gravitational field from the previously discussed Hamiltonian framework.
9 Dark Energy and Other Open Problems

This Lecture contains a full review of an approach to SR and to asymptotically Minkowskian classical canonical Einstein GR based on a description of global non-inertial frames centered on a time-like observer which is suggested by relativistic metrology. The gauge freedom in clock synchronization, which does not exist in Galilei space-time (Newton time and Euclidean 3-spaces are absolute) and is not restricted in Minkowski space-time (it spans the class of the admissible 3+1 splittings of this absolute space-time), is restricted in GR to the gauge freedom connected with the inertial gauge variable $^3K$, the York time, which determines the shape of the instantaneous non-Euclidean 3-spaces as 3-sub-manifolds of the space-time.

The study of canonical ADM tetrad gravity in asymptotically Minkowskian space-times without super-translations (so that they admit an asymptotic ADM Poincaré algebra at spatial infinity) in the York canonical basis allowed one to disentangle the tidal degrees of freedom of the gravitational field from the inertial gauge ones (they include the York time), to find the family of non-harmonic 3-orthogonal Schwinger time gauges and to define a HPM linearization in them. The main properties of these non-harmonic gauges are that only the HPM-GW (but not the lapse and shift functions) are retarded quantities with a no-incoming radiation condition and that one can naturally find which quantities depend upon the York time.

Relativistic particle mechanics, coupled to the electro-magnetic field in the radiation gauge, has been studied both in SR and GR with a suitable Grassmann regularization of the self-energies so to get well defined equations of motion.

In SR, after a clarification of the problem of the relativistic center of mass and the definition of inertial and non-inertial rest frames of isolated systems, it was possible to develop a formulation, the parametrized Minkowski theories, in which the transitions among global non-inertial frames are gauge transformations. Then isolated systems were described in the rest-frame instant form of dynamics and the structure of their Poincaré generators and of their relative variables in the instantaneous Wigner 3-spaces was clarified. With this approach it was possible to give a new formulation of the microcanonical ensemble in relativistic kinetic theory and to develop a formulation of relativistic quantum mechanics and relativistic entanglement taking into account the known results about relativistic bound states and the spatial non-separability and non-locality induced by the Lorentz signature of Minkowski space-time.

In GR it was possible to derive regularized equations of motion of the particles in the non-inertial rest frame and to study their PM limit in the HPM linearization in the 3-orthogonal gauges and the emission of HPM GW (with the energy balance under control even in absence of self-forces). Then
the PN limit of these PM equations allows one to recover the known 1PN results of harmonic gauges. The more surprising result is that in the PN expansion of the PM equations of motion there is a 0.5PN term in the forces depending upon the York time. This opens the possibility to describe dark matter as a relativistic inertial effect implying that the effective inertial mass of particles in the 3-spaces is bigger of the gravitational mass because it depends on the York time (i.e. on the shape of the 3-space as a 3-sub-manifold of the space-time: this is impossible in Newton gravity in Galilei space-time and leads to a violation of the Newtonian equivalence principle).

The proposed solution to the gauge problem in GR based on the conventions of relativistic metrology for ICRS and the results of the last Section on the re-interpretation of dark matter as a relativistic inertial effect arising as a consequence of a convention on the York time in an extended PM ICRS push toward the necessity of similar re-interpretation also of dark energy in cosmology [1, 101, 102, 103]. As it has been shown, the identification of the tidal and inertial degrees of freedom of the gravitational field can be reformulated in the framework of the non-Hamiltonian first-order ADM equations by means of the replacement of the Hamiltonian momenta with the expansion and the shear of the Eulerian observers associated with the 3+1 splitting of the space-time. Therefore this identification can also be applied to the cosmological space-times which do not admit a Hamiltonian formulation: also in them the identification of the instantaneous 3-spaces \( \Sigma_\tau \), now labeled by a cosmic time, requires a conventional choice of clock synchronization, i.e. a convention on the York time \( ^3K \) defining the shape of the 3-spaces as 3-sub-manifolds of the space-time, and of 3-coordinates (the 3-orthogonal ones are acceptable also in cosmology).

In the standard \( \Lambda \)CDM cosmological model the class of cosmological solutions of Einstein equations is restricted to Friedmann-Robertson-Walker (FRW) space-times with nearly Euclidean 3-spaces (i.e. with a small internal 3-curvature). In them the Killing symmetries connected with homogeneity and isotropy imply (\( \tau \) is the cosmic time, \( a(\tau) \) the scale factor) that the York time is no more a gauge variable but coincides with the Hubble constant: \( ^3K(\tau) = -\frac{\dot{a}(\tau)}{a(\tau)} = -H(\tau) \). However at the first order in cosmological perturbations (see Ref.[104] for a review) one has \( ^3K = -H + ^3K(1) \) with \( ^3K(1) \) being again an inertial gauge variable to be fixed with a metrological convention. Therefore the York time has a central role also in cosmology and one needs to know the dependence on it of the main quantities, like the red-shift and the luminosity distance from supernovae, which require the introduction of the notion of dark energy to explain the 3-universe and its accelerated expansion in the framework of the standard \( \Lambda \)CDM cosmological model.

Instead in inhomogeneous space-times without Killing symmetries like the Szekeres ones [105, 106] the York time remains an arbitrary inertial gauge variable. Therefore the main open problem of the present approach is to see whether it is possible to find a 3-orthogonal gauge in a inhomogeneous
Einstein space-time (at least in a PM approximation) in which the convention on the inertial gauge variable York time allows one to accomplish the following two tasks simultaneously: a) to eliminate both dark matter and dark energy through the choice of a 4-coordinate system (suggested by astrophysical data) to be used in a consistent PM reformulation of ICRS and b) to save the main good properties of the standard $\Lambda$CDM cosmological model due to the inertial and dynamical properties of the space-time. As matter one will take the dust, whose description in the York canonical basis is given in Ref.[41].

Also in the back-reaction approach [107, 108] to cosmology, according to which dark energy is a byproduct of the non-linearities of GR when one considers spatial averages of 3-scalar quantities in the 3-spaces on large scales to get a cosmological description of the universe taking into account its observed inhomogeneity, one gets that the spatial average of the product of the lapse function and of the York time (a 3-scalar gauge variable) gives the effective Hubble constant. Since this approach starts from the Hamiltonian description of an asymptotically flat space-time and since all the canonical variables in the York canonical basis, except the angles $\theta^i$, are 3-scalars, the formalism presented in this Lecture will allow to study the spatial average of nearly all the Hamilton equations and not only of the super-Hamiltonian constraint and of the Hamilton equation for the York time as in the existing formulation of the approach. This will be done by using the perfect fluids of Ref.[41] as matter.

Also the recent point of view of Ref.[109], taking into account the relevance of the voids among the clusters of galaxies, has to be reformulated in terms of the York time.

Finally one should find the dependence upon he York time of the Landau-Lifschitz energy-momentum pseudo-tensor and re-express it as the effective energy-momentum tensor of a viscous pseudo-fluid. One will have to check whether for certain choices of the York time the resulting effective equation of state of the fluid has negative pressure, realizing also in this way a simulation of dark energy.

Other open problems in GR under investigation are:

A) Find the second order of the HPM expansion to see whether in PM space-times there is the emergence of hereditary terms [44, 83] like the ones present in harmonic gauges.

B) Study the PM equations of motion of the transverse electro-magnetic field trying to find Lienard-Wiechert-type solutions in GR. Study astrophysical problems where the electro-magnetic field is relevant.

C) Find the expression in the York canonical basis of the Weyl scalars of the Newman-Penrose approach [68] and then of the four Weyl eigenvalues, which are tetrad-independent 4-scalar invariants of the gravitational field.
Is it possible to find a canonical transformation replacing the 3-scalar tidal variables with four 4-scalar functions of the Weyl eigenvalues? Are Weyl eigenvalues Dirac observables?

D) Try to make a multi-temporal quantization (see Refs.[34, 65]) of the linearized HPM theory over the asymptotic Minkowski space-time, in which, after a Shanmugadhasan canonical transformation to a new York canonical basis adapted to all the constraints, only the tidal variables are quantized but not the inertial gauge ones. After this type of quantization, in which the lapse and shift functions remain c-numbers, the space-time would still be a classical 4-manifold: only the two eigenvalues of the 3-metric describing GW are quantized and therefore only 3-metric properties like 3-distances, 3-areas, 3-volumes become quantum properties. After having re-expressed the Ashtekar variables [110] for asymptotically Minkowskian space-times (see Appendix B of Ref.[4]) in this final York canonical basis it will be possible to compare the outcomes of this new type of quantization with loop quantum gravity.
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