Dual Lie Bialgebras of Witt and Virasoro Types\textsuperscript{1}

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Abstract. Structures of dual Lie bialgebras on the one sided Witt algebra, the Witt algebra and the Virasoro algebra are investigated. As a result, we obtain some infinite dimensional Lie algebras.

Key words: the Virasoro algebra, Lie algebras of Witt type, Lie bialgebras, dual Lie bialgebras

1. Introduction

Lie bialgebras, subjects of intensive study in recent literature (e.g., [1–3,5–14,17,19,20]), are important ingredients in quantum groups [4], which have close relations with the Yang-Baxter equation. In general, a Lie bialgebra is a vector space endowed with both the structure of Lie algebra and the structure of Lie coalgebra simultaneously, satisfying some compatibility condition, which was suggested by a study of Hamiltonian mechanics and Poisson Lie group [4,6]. Michaelis [7] studied a class of Witt type Lie bialgebras, and introduced techniques on how to construct coboundary or triangular Lie bialgebras from Lie algebras which contains two linear independent elements $a$ and $b$ satisfying condition $[a, b] = kb$ for some non-zero scalar $k$.

Ng and Taft [13] proved that all structures of Lie bialgebras on the one sided Witt algebra, the Witt algebra and the Virasoro algebra are coboundary triangular (see also [12]). Furthermore, they gave a complete classification of all structures of Lie bialgebras on the one sided Witt algebra. For the cases of generalized Witt type Lie algebras, the authors [14] obtained that all structures of Lie bialgebras on them are coboundary triangular. Similar results also hold for some other kinds of Lie algebras (see, e.g., [19,20]).

It may sound that coboundary triangular Lie bialgebras have relatively simple structures. However, even for the (two-sided) Witt algebra and the Virasoro algebra, a classification of coboundary triangular Lie bialgebra structures on them is still an open problem. Thus it seems to us that it is worth paying more attention on them. This is one of our motivations in the present paper, here we investigate Lie bialgebra structures from the point of view of dual Lie bialgebras. Investigating dual structures on some algebraic structures has also drawn some authors’ much attention, e.g., [2,3,5,8–10]. One may have noticed that the dual of a finite dimensional Lie bialgebra is naturally a Lie bialgebra, and so one would not predict anything new in this case. Thus we start our investigation by considering the “dual” of the one sided Witt algebra, the Witt algebra and the Virasoro algebra. Notice that the (full) dual of an

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infinite dimensional space is always an infinite dimensional space with uncountable basis, thus here “dual” means some restricted dual (cf. (2.3)). A surprising thing is that dualizing Lie bialgebras may produce new Lie algebras, as can be seen from Remark 4.10. This is another motivation in the present paper.

The paper is organized as follows. Some definitions and preliminary results are briefly recalled in Section 2. Then in Section 3, structures of dual coalgebras of $\mathbb{F}[x]$ and $\mathbb{F}[x^{\pm 1}]$ are addressed. Finally in Section 4, structures of dual Lie bialgebras of the one side Witt Lie bialgebras, the Witt Lie bialgebras and the Virasoro Lie bialgebras are investigated. As a result, we obtain some infinite dimensional Lie algebras. The main results of the present paper are summarized in Theorems 3.5, 4.6 and 4.9.

2. Definitions and preliminary results

Throughout the paper, $\mathbb{F}$ denoted an algebraically closed field of characteristic zero unless otherwise stated. All vector spaces are assumed to be over $\mathbb{F}$. As usual, we use $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ to denote the sets of integers, nonnegative integers, positive integers, respectively.

We briefly recall some notions on Lie bialgebras, for details, we refer readers to, e.g., [4,14].

**Definition 2.1.** (1) A Lie bialgebra is a triple $(L, [\cdot, \cdot], \delta)$ such that

(i) $(L, [\cdot, \cdot])$ is a Lie algebra;

(ii) $(L, \delta)$ is a coalgebra;

(iii) $\delta(x, y) = x \cdot \delta(y) - y \cdot \delta(x)$ for $x, y \in L$,

where $x \cdot (y \otimes z) = [x, y] \otimes z + y \otimes [x, z]$ for $x, y, z \in L$.

(2) A Lie bialgebra $(L, [\cdot, \cdot], \delta)$ is coboundary if $\delta$ is coboundary in the sense that there exists $r \in L \otimes L$ written as $r = \sum r^{[1]} \otimes r^{[2]}$, such that $\delta(x) = x \cdot r$ for $x \in L$.

(3) A coboundary Lie bialgebra $(L, [\cdot, \cdot], \delta)$ is triangular if $r$ satisfies the following classical Yang-Baxter Equation (CYBE),

$$C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

where $r_{12} = \sum r^{[1]} \otimes r^{[2]} \otimes 1$, $r_{13} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}$, $r_{23} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}$ are elements in $U(L) \otimes U(L) \otimes U(L)$, and $U(L)$ is the universal enveloping algebra of $L$.

Two Lie bialgebras $(g, [\cdot, \cdot], \delta)$ and $(g', [\cdot, \cdot]', \delta')$ are said to be dually paired if there bialgebra structures are related via

$$\langle [f, h]' , \xi \rangle = \langle f \otimes h, \delta \xi \rangle, \quad \langle \delta' f, \xi \otimes \eta \rangle = \langle f, [\xi, \eta] \rangle \quad \text{for } f, h \in g', \xi, \eta \in g,$$

where $\langle \cdot, \cdot \rangle$ is a nondegenerate bilinear form on $g' \times g$, which is naturally extended to a nondegenerate bilinear form on $(g' \otimes g') \times (g \otimes g)$. In particular, if $g' = g$ as a vector space, then $g$ is called a self-dual Lie bialgebra.

The following easily obtained result can be found in [6].
Proposition 2.2. Let $(\mathfrak{g}, [\cdot, \cdot], \delta)$ be a finite dimensional Lie bialgebra, then so is the linear dual space $\mathfrak{g}^* := \text{Hom}_F (\mathfrak{g}, F)$ by dualisation, namely $(\mathfrak{g}^*, [\cdot, \cdot]', \delta')$ is the Lie bialgebra defined by (2.2) with $\mathfrak{g}' = \mathfrak{g}^*$. In particular, $\mathfrak{g}$ and $\mathfrak{g}^*$ are dually paired.

Thus a finite dimensional $(\mathfrak{g}, [\cdot, \cdot], \delta)$ is always self-dual as there exists a vector space isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ which pulls back the bialgebra structure on $\mathfrak{g}^*$ to $\mathfrak{g}$ to obtain another bialgebra structure on $\mathfrak{g}$ to make it to be self-dual. However, in sharp contrast to finite dimensional case, infinite dimensional Lie bialgebras are not self-dual in general.

For convenience, we denote by $\varphi$ the Lie bracket of Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, which can be regarded as a linear map $\varphi : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$. Let $\varphi^* : \mathfrak{g}^* \to (\mathfrak{g} \otimes \mathfrak{g})^*$ be the dual of $\varphi$.

Definition 2.3. [8] Let $(\mathfrak{g}, \varphi)$ be a Lie algebra over $F$. A subspace $V$ of $\mathfrak{g}^*$ is called a good subspace if $\varphi^*(V) \subset V \otimes V$. Denote $\mathcal{R} = \{ V \mid V$ is a good subspace of $\mathfrak{g}^* \}$. Then

$$\mathfrak{g}^o = \sum_{V \in \mathcal{R}} V,$$

is also a good subspace of $\mathfrak{g}^*$, which is obviously the maximal good subspace of $\mathfrak{g}^*$.

Proposition 2.4. [8] For any good subspace $V$ of $\mathfrak{g}^*$, the pair $(V, \varphi^*)$ is a Lie coalgebra. In particular, $(\mathfrak{g}^o, \varphi^*)$ is a Lie coalgebra.

It is clear that if $\mathfrak{g}$ is a finite dimensional Lie algebra, then $\mathfrak{g}^o = \mathfrak{g}^*$. However if $\mathfrak{g}$ is infinite dimensional, then $\mathfrak{g}^o \subset \mathfrak{g}^*$ in general.

For any Lie algebra $\mathfrak{g}$, the dual space $\mathfrak{g}^*$ has a natural right $\mathfrak{g}$-module structure defined for $f \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$ by

$$(f \cdot x)(y) = f([x, y]) \text{ for } y \in \mathfrak{g}.$$ 

We denote $f \cdot \mathfrak{g} = \text{span}\{ f \cdot x \mid x \in \mathfrak{g} \}$, the space of translates of $f$ by elements of $\mathfrak{g}$.

We summarize some results of [1–3] as follows.

Proposition 2.5. Let $\mathfrak{g}$ be a Lie algebra. Then

1. $\mathfrak{g}^o = \{ f \in \mathfrak{g}^* \mid f \cdot \mathfrak{g} \text{ is finite dimensional} \}$.
2. $\mathfrak{g}^o = (\varphi^*)^{-1}(\mathfrak{g}^* \otimes \mathfrak{g}^*)$, the preimage of $\mathfrak{g}^* \otimes \mathfrak{g}^*$ in $\mathfrak{g}^*$.

The notion of good subspaces of an associative algebra can be defined analogously. In the next two sections, we shall investigate $\mathfrak{g}^o$ for some associative or Lie algebras $\mathfrak{g}$.

3. THE STRUCTURE OF $F[x, x^{-1}]^o$ 

Let $(\mathcal{A}, \mu, \eta)$ be an associative $F$-algebra with unit, where $\mu$ and $\eta$ are respectively the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit $\eta : F \to \mathcal{A}$, satisfying

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

$$(\eta \otimes id)(k \otimes a) = (id \otimes \eta)(a \otimes k) : F \otimes \mathcal{A} \cong \mathcal{A} \otimes F \cong \mathcal{A},$$
for $k \in \mathbb{F}$, $a,b,c \in \mathcal{A}$. Then a coassociative coalgebra is a triple $(\mathcal{C}, \Delta, \varepsilon)$, which is obtained by conversing arrows in the definition of an associative algebra. Namely, $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and $\varepsilon : \mathcal{C} \to \mathbb{F}$ are respectively comultiplication and counit of $\mathcal{C}$, satisfying

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C},$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta : \mathcal{C} \otimes \mathcal{C} \to \mathbb{F} \otimes \mathcal{C} \cong \mathcal{C} \otimes \mathcal{F} \cong \mathcal{C}.$$ 

For any vector space $\mathcal{A}$, there exists a natural injection $\rho : \mathcal{A}^* \otimes \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$ defined by $\rho(f,g)(a,b) = \langle f(a), g(b) \rangle$ for $f, g \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$. In case $\mathcal{A}$ is finite dimensional, $\rho$ is an isomorphism. For any algebra $(\mathcal{A}, \mu)$, the multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ naturally induces the map $\mu^* : \mathcal{A}^* \to (\mathcal{A} \otimes \mathcal{A})^*$. If $\mathcal{A}$ is finite dimensional, then the isomorphism $\rho$ insures that $(\mathcal{A}^*, \mu^*, \varepsilon^*)$ is a coalgebra, where for simplicity, $\mu^*$ denotes the composition of the maps: $\mathcal{A}^* \xrightarrow{\mu^*} (\mathcal{A} \otimes \mathcal{A})^* \xrightarrow{\rho^{-1}} \mathcal{A}^* \otimes \mathcal{A}^*$. However, if $\mathcal{A}$ is infinite dimensional, the situation is different.

Let $\mathcal{A} = \mathbb{F}[x]$ (resp., $\mathbb{F}[x, x^{-1}]$) be the algebra of polynomials (resp., Laurent polynomials) on variable $x$. As a vector space, we have $\mathcal{A}^* = \mathbb{F}[[\varepsilon]]$ (resp., $\mathbb{F}[[\varepsilon, \varepsilon^{-1}]]$), the space of formal power series on $\varepsilon$ (resp., $\varepsilon, \varepsilon^{-1}$), where $\varepsilon^i$ is the dual element of $x^i$, namely, $\langle \varepsilon^i, x^j \rangle := \varepsilon^i(x^j) = \delta_{ij}$ for $i, j \in \mathbb{Z}_+$ (resp., $i, j \in \mathbb{Z}$). In this way, for any $f = \sum_i f_i \varepsilon^i \in \mathcal{A}^*$ (which can be an infinite sum) and any $g = \sum_j g_j x^j \in \mathcal{A}$ (a finite sum), we have

$$\langle f, g \rangle = f(g) = \sum_{i,j} f_i \langle \varepsilon^i, g \rangle = \sum_{i,j} f_i g_j \langle \varepsilon^i, x^j \rangle = \sum_{i,j} f_i g_j \varepsilon^i(x^j) = \sum_j f_j g_j,$$  \quad (3.1)$$

where the right-hand side is a finite sum.

We summarize some results of [11] as follows.

**Proposition 3.1.** [11] For any $f = \sum_{i=0}^\infty f_i \varepsilon^i \in \mathbb{F}[[\varepsilon]]$ with $f_i \in \mathbb{F}$,

$$f \in \mathbb{F}[x]^\circ \iff \exists r \in \mathbb{N} \text{ such that } f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r},$$  \quad (3.2)$$

for some $h_i \in \mathbb{F}$ and all $n > r$.

One can easily generalize this result to the Laurent polynomial algebra $\mathbb{F}[x, x^{-1}]$ as follows.

**Theorem 3.2.** For any $f = \sum_{j=-\infty}^\infty f_j \varepsilon^j \in \mathbb{F}[[\varepsilon, \varepsilon^{-1}]]$ with $f_i \in \mathbb{F}$,

$$f \in \mathbb{F}[x, x^{-1}]^\circ \iff \exists r \in \mathbb{N} \text{ such that } f_n = h_1 f_{n-1} + h_2 f_{n-2} + \cdots + h_r f_{n-r},$$  \quad (3.3)$$

for some $h_i \in \mathbb{F}$ and all $n \in \mathbb{Z}$.

**Remark 3.3.** Note that there is a significant difference between $\mathbb{F}[x]^\circ$ and $\mathbb{F}[x, x^{-1}]^\circ$. From (3.2), we can easily see that every polynomial of $\varepsilon$ is in $\mathbb{F}[x]^\circ$, namely, $\mathbb{F}[[\varepsilon]] \subset \mathbb{F}[x]^\circ$. In fact one can prove that $\mathbb{F}[x]^\circ$ is the subspace of $\mathbb{F}[[\varepsilon]]$ consisting of rational functions $r(\varepsilon)$ on $\varepsilon$ such that zero is not a singular point of $r(\varepsilon)$, i.e., $\mathbb{F}[x]^\circ = \{ \frac{f(\varepsilon)}{g(\varepsilon)} \mid f(\varepsilon), g(\varepsilon) \in \mathbb{F}[\varepsilon], g(0) \neq 0 \}$. However, from (3.3), one can obtain that if $0 \neq f \in \mathbb{F}[x, x^{-1}]^\circ$, then for any $N \in \mathbb{N}$, there exist $i > N$ and $j < -N$ such that $f_i \neq 0 \neq f_j$. In particular, $\mathbb{F}[\varepsilon, \varepsilon^{-1}] \cap \mathbb{F}[x, x^{-1}]^\circ = \emptyset$. 


Proof of Theorem 3.2. Note that $F[x, x^{-1}]$ is a principle ideal domain. Let $f \in F[x, x^{-1}]$. By [17], $\text{Ker}(f) \supset I$, where $I$ is a finite codimensional ideal of $F[x, x^{-1}]$. Thus there exists a polynomial $h(x) = x^r - h_1x^{r-1} - \cdots - h_r \in F[x]$ with $r \in \mathbb{N}$ and $h_i \in F$ such that $I = (h(x))$. Then for $n \in \mathbb{Z}$, we have $x^{n-r}h(x) \in I \subseteq \text{Ker}(f)$, which means $f(x^{n-r}h(x)) = 0$, i.e., $f_n = h_1f_{n-1} + h_2f_{n-2} + \cdots + h_rf_{n-r}$.

Now assume $f_n = h_1f_{n-1} + h_2f_{n-2} + \cdots + h_rf_{n-r}$ for $n \in \mathbb{Z}$. Let $h(x) = x^r - h_1x^{r-1} - \cdots - h_r$. Then $f(x^{n-r}h(x)) = 0$, i.e., the ideal $I = (h(x))$ is a finite codimensional ideal such that $I \subseteq \text{Ker}(f)$. Thus, $f \in F[x, x^{-1}]^\circ$ by Proposition 2.5.

A sequence $\{f_n\}_{n=-\infty}^{\infty}$ satisfying the condition $f_n = h_1f_{n-1} + h_2f_{n-2} + \cdots + h_rf_{n-r}$ for $n \in \mathbb{Z}$ is called a recursive sequence, the minimal integer $r$ of the recursive relation is called the degree of the sequence. For convenience, we call a formal power series $f = \sum_{n=-\infty}^{\infty} f_nx^n$ a recursive power series whenever $\{f_n\}_{n=-\infty}^{\infty}$ is a recursive sequence.

A coalgebra $C$ over a field $F$ is called irreducible if any two nonzero subcoalgebras of $C$ have nonzero intersection. A subcoalgebra of $C$ is called an irreducible component of $C$ if it is a maximal irreducible subcoalgebra.

The following result can be found in [17].

Proposition 3.4. If $C$ be a cocommutative coalgebra, then $C$ is direct sum of its irreducible component.

Let $\mathcal{A} = \mathbb{F}[x, x^{-1}]$ (or $\mathbb{F}[x]$). For $a \in \mathbb{F}$, let $I_a$ be the ideal generated by $(x-a)$ in $\mathcal{A}$. Then $U = \{I_a \mid 0 \neq a \in \mathbb{F}\}$ (or $U = \{I_a \mid a \in \mathbb{F}\}$) is the set of maximal ideals of $\mathcal{A}$. By [17], $I_a^\perp := \{f \in \mathcal{A} \mid f(I_a) = 0\}$ is a simple subcoalgebra of $\mathcal{A}^\circ$. For $n \in \mathbb{N}$, denote $I_a^\circ = ((x-a)^n)$. Then we have the chain of ideals of $\mathcal{A}$: $I_a \supset I_{a^2} \supset \cdots \supset I_{a^n} \supset \cdots$. Since each ideal $I_a^\circ$ is a finite codimensional ideal of $\mathcal{A}$, we have $I_a^\perp \subset I_{a^2}^\perp \subset \cdots \subset I_{a^n}^\perp \subset \cdots$, which is a chain of subcoalgebras of $\mathcal{A}^\circ$. We denote $S_a = \bigcup_{n=0}^{\infty} I_a^\perp = \lim_{n \to \infty} I_a^\perp$. Then $S_a$ is a maximal irreducible subcoalgebra of $\mathcal{A}^\circ$.

Theorem 3.5. Let $S_a$ be defined as above. Then $\mathcal{A}^\circ = \bigoplus_{a \in \mathbb{F}} S_a$ if $\mathcal{A} = \mathbb{F}[x, x^{-1}]$, or $\mathcal{A}^\circ = \bigoplus_{a \in \mathbb{F}} S_a$ if $\mathcal{A} = \mathbb{F}[x]$.

Proof. We prove the case $\mathcal{A} = \mathbb{F}[x, x^{-1}]$ as the other case is analogous. For $0 \neq a \in \mathbb{F}$, $S_a$ is obviously a maximal irreducible subcoalgebra of $\mathcal{A}^\circ$. If $S_a \cap S_b \neq 0$ for some $a, b \in \mathbb{F}$ with $a \neq b$, it must contain a simple subcoalgebra of $\mathcal{A}^\circ$. Since $I_a^\perp$ is a unique simple subcoalgebra of $S_a$, we obtain $I_a^\perp \subset S_a \cap S_b$. Analogously, $I_b^\perp \subset S_a \cap S_b$, which forces $I_a^\perp = I_b^\perp$, contradiction with $a \neq b$. Thus the sum is directed. Since the irreducible components of $\mathcal{A}^\circ$ are in 1-1 correspondence with the finite codimensional maximal ideals of $\mathcal{A}$, we see that $S_a$’s with $0 \neq a \in \mathbb{F}$ run over all the irreducible components of $\mathcal{A}^\circ$. □
4. Dual Lie bialgebras of Witt type

Let $(\mathfrak{g}, \varphi, \delta)$ be a Lie bialgebra, and $\varphi^* : \mathfrak{g}^* \to (\mathfrak{g} \otimes \mathfrak{g})^*$ the dual of $\varphi$. By Proposition 2.4, $\varphi^*$ induces a map $\varphi^\circ := \varphi^*|_{\mathfrak{g}^\circ} : \mathfrak{g}^\circ \to \mathfrak{g} \otimes \mathfrak{g}^\circ$, which makes $(\mathfrak{g}^\circ, \varphi^\circ)$ to be a Lie coalgebra. By [12, Proposition 3], the map $\delta^* : \mathfrak{g}^\circ \otimes \mathfrak{g}^* \hookrightarrow (\mathfrak{g} \otimes \mathfrak{g})^* \delta^* \mathfrak{g}^*$ induces a map $\delta^\circ := \delta^*|_{\mathfrak{g}^\circ \otimes \mathfrak{g}^\circ} : \mathfrak{g}^\circ \otimes \mathfrak{g}^\circ \to \mathfrak{g}^\circ$, which makes $(\mathfrak{g}^\circ, \delta^\circ)$ to be a Lie algebra. Thus we obtain a Lie bialgebra $(\mathfrak{g}^\circ, \delta^\circ, \varphi^\circ)$, called the dual Lie bialgebra of $(\mathfrak{g}, \varphi, \delta)$.

Convention 4.1. When there is no confusion, we will use $[\cdot, \cdot]$ to denote the bracket in $\mathfrak{g}$ or $\mathfrak{g}^\circ$, i.e., $[\cdot, \cdot] = \varphi$ or $\delta^\circ$, and we also use $\Delta$ to denote the cobracket in $\mathfrak{g}$ or $\mathfrak{g}^\circ$, i.e., $\Delta = \delta$ or $\varphi^\circ$.

If $\mathfrak{g}$ is an infinite dimensional Lie algebra, there do not exist effective approaches to describe the structure of $\mathfrak{g}^\circ$ explicitly. However, if $\mathfrak{g}$ is a Lie algebra “induced” from some associative algebra (in sense of (4.1) below), it is sometimes possible to use the theory of dual associative algebras to describe $\mathfrak{g}^\circ$. So, let $(\mathcal{A}, \mu)$ be a commutative associative $F$-algebra with multiplication $\mu$. Let $\partial \in \text{Der}_F(\mathcal{A})$ be a derivation of $\mathcal{A}$. Then we obtain a Lie algebra, denoted $\mathcal{A}_\partial$, on the space $\mathcal{A}$, with bracket defined by

$$id \otimes \partial - \partial \otimes id, \text{ namely, } [a, b] = a\partial(b) - \partial(a)b \text{ for } a, b \in \mathcal{A}. \quad (4.1)$$

Such a Lie algebra $\mathcal{A}_\partial$ is usually referred to as a Witt type Lie algebra (e.g., [15,18]).

For the Lie algebra $\mathcal{A}_\partial$, there are two natural ways to produce Lie coalgebras over some subspaces of $\mathcal{A}^*$, denoted as $(\mathcal{A}_\partial)^\circ, (\mathcal{A}^\circ)_{\varphi^\circ}$ respectively:

1. The Lie coalgebra $(\mathcal{A}_\partial)^\circ$ is defined by (2.3) (cf. Proposition 2.4), with cobracket, denoted $\Delta_\partial$, defined by

$$\Delta_\partial(f) = (\mu(id \otimes \partial - \partial \otimes id))^*(f) = (id \otimes \partial - \partial \otimes id)^*\mu^*(f) \text{ for } f \in (\mathcal{A}_\partial)^\circ. \quad (4.2)$$

2. The Lie coalgebra $(\mathcal{A}^\circ)_{\varphi^\circ}$ is defined on the space $\mathcal{A}^\circ$, with cobracket, denoted $\Delta_{\varphi^\circ}$, being induced from the coalgebra, i.e.

$$\Delta_{\varphi^\circ}(f) = (id \otimes \varphi^\circ - \varphi^\circ \otimes id)\mu^\circ(f) \text{ for } f \in \mathcal{A}^\circ, \quad (4.2)$$

where $\varphi^\circ = \varphi^*|_{\mathcal{A}^\circ}$ is defined by $\varphi^\circ(f)(a) = f(\varphi(a))$ for $a \in \mathcal{A}$, and $\mu^\circ = \mu^*|_{\mathcal{A}^\circ}$.

Note the difference that $(\mathcal{A}_\partial)^\circ$ is the maximal good subspace of the dual space $(\mathcal{A}_\partial)^*$ of the Lie algebra $\mathcal{A}_\partial$, while $(\mathcal{A}^\circ)_{\varphi^\circ}$ as a space is the maximal good subspace of the dual space $\mathcal{A}^*$ of the associative algebra $\mathcal{A}$.

The following result (which holds for any field $F$ of characteristic not 2) can be found in [10].

Proposition 4.2. Let $\mathcal{A}$ be a commutative $F$-algebra. Let $\partial \in \text{Der}_F(\mathcal{A})$. Then $(\mathcal{A}^\circ)_{\varphi^\circ}$ is a Lie subcoalgebra of $(\mathcal{A}_\partial)^\circ$. Furthermore, if there exist $c_1, c_2, \cdots, c_n \in \mathcal{A}$ such that the ideal $I = (2\partial(c_1), 2\partial(c_2), \cdots, 2\partial(c_n))$ has finite codimension, then $(\mathcal{A}^\circ)_{\varphi^\circ} = (\mathcal{A}_\partial)^\circ$. 


Now let $\mathcal{A} = \mathbb{F}[x]$ (or $\mathbb{F}[x^\pm]$) and $\partial = \frac{d}{dx}$. Then we obtain the one-side Witt algebra $\mathcal{W}^+ := \mathbb{F}[x]_{\partial}$ (or the Witt algebra $\mathcal{W} := \mathbb{F}[x,x^{-1}]_{\partial}$). By Proposition 4.2, $(\mathcal{A}^o)_{\partial^o} = (\mathcal{A}_\partial)^o$ in both cases. The well-known Virasoro algebra $\mathcal{V}$ is the universal central extension of the Witt algebra, namely, it has a basis $\{x^{m+1}, c \mid m \in \mathbb{Z}\}$ with $c$ being a central element and

$$[x^{m+1}, x^{n+1}] = (n-m)x^{m+n+1} + \frac{n^2-m^2}{12}\delta_{m+n,0} c \quad \text{for} \quad m, n \in \mathbb{Z}. \quad (4.3)$$

It is proven in [10] that the Virasoro coalgebra $\mathcal{V}^o$ is isomorphic to $\mathcal{W}^o$, the dual of Witt algebra. Thus from the discussions above and Proposition 4.2, we obtain

**Theorem 4.3.** Let $\mathcal{L}$ be the one side Witt algebra $\mathcal{W}^+$ (resp., the Witt algebra $\mathcal{W}$ or the Virasoro algebra $\mathcal{V}$). Then the Lie coalgebra $\mathcal{L}^o$ is equal to $(\mathbb{F}[x]^o)_{\partial^o}$ (resp., $(\mathbb{F}[x,x^{-1}]^o)_{\partial^o}$), with the underlining space described by (3.2) (resp., (3.3)) and the cobracket uniquely determined by

$$\Delta(\varepsilon^n) = \sum_{i+j=n+1} (j-i)\varepsilon^i \otimes \varepsilon^j \quad \text{for} \quad n \in \mathbb{N} \ (\text{resp.}, \ n \in \mathbb{Z}), \quad (4.4)$$

where the sum is over $i, j \in \mathbb{N} \ (\text{resp.}, \ i, j \in \mathbb{Z})$.

**Remark 4.4.**

(1) Note that in case $\mathcal{L} = \mathcal{W}$ or $\mathcal{V}$, the right-hand side of (4.4) is always an infinite sum, which is an element in $(\mathcal{L} \otimes \mathcal{L})^*$, and its action on any element of $\mathbb{F}[x^\pm] \otimes \mathbb{F}[x^\pm]$ can only produce finite many nonzero terms (cf. (3.1)). Also note from Remark 3.3 that (4.4) is not necessarily in $\mathcal{L}^o \otimes \mathcal{L}^o$ (which is the case when $\mathcal{L} = \mathcal{W}$ or $\mathcal{V}$ because $\varepsilon^n \notin \mathbb{F}[x,x^{-1}]^o$), and that an element $f$ in $\mathbb{F}[x]^o$ or in $\mathbb{F}[x,x^{-1}]^o$ may be an infinite combination of $\varepsilon^i$s, thus $\Delta(f)$ is in general an infinite combination of elements of the form (4.4) such that the resulting combination is in $\mathcal{L}^o \otimes \mathcal{L}^o$, which is guaranteed by Proposition 2.4.

(2) We also remark that in general, if elements $f = \sum_i f_i, g = \sum_j g_j$ are infinite sums in a Lie bialgebra, it does not mean that

$$\Delta(f) = \sum_i \Delta(f_i) \quad \text{or} \quad [f, g] = \sum_{i,j} [f_i, g_j] \quad \text{if} \quad f = \sum_i f_i, \quad g = \sum_j g_j \quad \text{are infinite sums.} \quad (4.5)$$

However in our case, since we consider dual structures, elements $f, g$ are in certain dual spaces, because of the pairing like in (3.1), we see that (4.5) holds as long as it makes sense. Thus in Theorem 4.3, the cobracket is uniquely determined by (4.4).

**Proof of Theorem 4.3.** We only need to prove the cobracket relations. We give a proof only for the case $\mathcal{L} = \mathcal{W}^+$ as the proof for other cases is analogous. Thus let $\varepsilon^n \in (\mathcal{W}^+)^o$ with $n \in \mathbb{N}$, and suppose $\mu^o(\varepsilon^n) = \sum_{s,t \in \mathbb{N}} c_{st}\varepsilon^s \otimes \varepsilon^t$ for some $c_{st} \in \mathbb{F}$. Then for $i, j \in \mathbb{N}$,

$$c_{ij} = \sum_{s,t \in \mathbb{N}} c_{st}\varepsilon^s(x^i)\varepsilon^t(x^j) = \mu^o(\varepsilon^n)(x^i \otimes x^j) = \varepsilon^n(\mu(x^i \otimes x^j)) = \varepsilon^n(x^{i+j}) = \delta_{n,i+j}.$$
Thus \( \mu(\varepsilon^n) = \sum_{i,j \in \mathbb{N}; i+j=n} \varepsilon^i \otimes \varepsilon^j \). Similarly, if we assume \( \partial^\circ(\varepsilon^n) = \sum_{s \in \mathbb{N}} c_s \varepsilon^s \), then \( c_i = \partial^\circ(\varepsilon^n)(x^i) = \varepsilon^n(\partial(x^i)) = i \delta_{n,i-1} \), i.e., \( \partial^\circ(\varepsilon^n) = (n+1)\varepsilon^{n+1} \). By Proposition 4.2 and (4.2), we have

\[
\Delta(\varepsilon^n) = (id \otimes \partial^\circ - \partial^\circ \otimes id) \mu(\varepsilon^n) = \sum_{i+j=n} (id \otimes \partial^\circ - \partial^\circ \otimes id)(\varepsilon^i \otimes \varepsilon^j) = \sum_{i+j=n+1} (j-i)\varepsilon^i \otimes \varepsilon^j,
\]

where the sums are over \( i,j \in \mathbb{N} \), proving (4.4).

From now on, \( \mathcal{L} \) always denotes one of the one sided Witt algebra \( \mathcal{W}^+ \), the Witt algebra \( \mathcal{W} \) and the Virasoro algebra \( \mathcal{V} \). We summarize some results of [13] as follows.

**Proposition 4.5.**

1. Every Lie bialgebra structure on \( \mathcal{L} \) is coboundary triangular associated to a solution \( r \) of CYBE (2.1) of the form \( r = a \otimes b - b \otimes a \) for some nonzero \( a,b \in \mathcal{L} \) satisfying \([a,b] = k b \) for some \( 0 \neq k \in \mathbb{F} \).

2. Let \( \mathfrak{g} \) be an infinite dimensional Lie subalgebra of \( \mathcal{W} \) such that \( x \in \mathfrak{g} \) and \( \mathfrak{g} \not\cong \mathcal{W} \) as Lie algebras. Let \( \mathfrak{g}^{(n)} \) be the Lie bialgebra structure on \( \mathfrak{g} \) associated to the solution \( r_n = x \otimes x^n - x^n \otimes x \) of the CYBE for any \( x^n \in \mathfrak{g} \). Then every Lie bialgebra structure on \( \mathfrak{g} \) is isomorphic to \( \mathfrak{g}^{(n)} \) for some \( n \) with \( x^n \in \mathfrak{g} \).

Using Theorem 4.3 and Proposition 4.5, we can obtain the dual Lie bialgebra structures on \( \mathcal{L} \) as follows.

**Theorem 4.6.** Let \((\mathcal{L},[\cdot,\cdot],\Delta)\) be the coboundary triangular Lie bialgebra with cobracket associated to the solution \( r = x \otimes x^n - x^n \otimes x \) of CYBE for some \( 1 \neq n \in \mathbb{Z} \). Then the dual Lie bialgebra is \((\mathcal{L}^\circ,[\cdot,\cdot],\Delta)\) with the underlining space \( \mathcal{L}^\circ \) and the cobracket \( \Delta \) determined by Theorem 4.3, and the bracket uniquely determined by the following (cf. (4.5)).

1. If \( \mathcal{L} = \mathcal{W}^+ \), then \( n \in \mathbb{Z}_+ \) and for \( i,j \in \mathbb{Z}_+ \),

\[
[\varepsilon^i,\varepsilon^j] = \begin{cases} 
(2n-j-1)\varepsilon^{j+1-n} & \text{if } i=1, \ 1 \neq j \geq n-1, \\
(j-1)\varepsilon^j & \text{if } i=n, \ j \neq 1, i, n, \\
0 & \text{if } i,j \notin \{1,n\} \text{ or } i=1, \ j<n-1.
\end{cases}
\]  

(4.6)

2. If \( \mathcal{L} = \mathcal{W} \) or \( \mathcal{V} \), then for \( i,j \in \mathbb{Z} \),

\[
[\varepsilon^i,\varepsilon^j] = \begin{cases} 
(2n-j-1)\varepsilon^{j+1-n} & \text{if } i=1, \ j \neq 1, \\
(j-1)\varepsilon^j & \text{if } i=n, \ j \neq 1, i, n, \\
0 & \text{if } i,j \notin \{1,n\}.
\end{cases}
\]  

(4.7)

**Proof.** By Theorem 4.3, it remains to prove (4.6) and (4.7). We only prove (4.6) as the proof of (4.7) is similar. Using notation as in (3.1), by Proposition 2.4, we obtain

\[
\langle [\varepsilon^i,\varepsilon^j],x^m \rangle = \langle \varepsilon^i \otimes \varepsilon^j, \Delta(x^m) \rangle = \langle \varepsilon^i \otimes \varepsilon^j, x \cdot r \rangle \\
= \langle \varepsilon^i \otimes \varepsilon^j, [x^m,x] \otimes x^n + x \otimes [x^m,x^n] - [x^m,x^n] \otimes x - x^n \otimes [x^m,x] \rangle \\
= (1-m)(\delta_{i,m}\delta_{j,n} - \delta_{i,n}\delta_{j,m}) + (n-m)(\delta_{i,1}\delta_{j,m+n-1} - \delta_{i,m+n-1}\delta_{j,1}) \\
= \langle (1-i)\delta_{j,n}\varepsilon^i - (1-j)\delta_{i,n}\varepsilon^j + (2n-j-1)\delta_{i,1}\varepsilon^{j+1-n} - (2n-i-1)\delta_{j,1}\varepsilon^{i+1-n},x^m \rangle. 
\]
Thus
\[
[\varepsilon^i, \varepsilon^j] = (1-i)\delta_{j,n}\varepsilon^i \cdot (1-j)\delta_{i,n}\varepsilon^j + (2n-j-1)\delta_{i,1}\varepsilon^{j+1-n} - (2n-i-1)\delta_{j,1}\varepsilon^{i+1-n}.
\]

(4.8)

From this one immediately obtains \([\varepsilon^i, \varepsilon^j] = 0\) if \(i, j \in \{1, n\}\). Now suppose \(i = 1 \neq j\). Noting that \(n \neq 1\), we obtain from (4.8) that \([\varepsilon, \varepsilon^j] = (2n-j-1)\varepsilon^{j+1-n}\). In particular \([\varepsilon, \varepsilon^j] = 0\) if \(j < n-1\) as in this case \(\varepsilon^{j+1-n}\) is not defined in \(\mathbb{F}[x]^o\) (cf. Proposition 3.1). Thus we have the first and third cases of (4.6).

Finally suppose \(i = n\) and \(j \neq 1, i, n\). Then (4.8) gives \([\varepsilon^n, \varepsilon^j] = (j-1)\delta_{i,n}\varepsilon^j\), which is the second case of (4.6). This completes the proof. \(\square\)

**Remark 4.7.** Note from Proposition 4.5(2) that every nonzero solution \(r\) of CYBE of \(\mathcal{W}^+\) has the form \(r = x \otimes x^n - x^n \otimes x\) for some \(1 \neq n \in \mathbb{Z}_+\). Thus Theorem 4.6 in particular gives a complete classification of dual Lie bialgebras of \(\mathcal{W}^+\). However for the case \(\mathcal{L} = \mathcal{W}\) or \(\mathcal{V}\), a classification of noncommutative 2-dimensional Lie subalgebras of \(\mathcal{L}\) (namely a classification of pairs \((a, b)\) of nonzero elements \(a, b \in \mathcal{L}\) satisfying \([a, b] = kb\) for some \(0 \neq k \in \mathbb{F}\)) is still an open problem, thus a classification of dual Lie bialgebras of \(\mathcal{L}\) remains open.

A series of 2-dimensional Lie subalgebras of \(\mathcal{L}\) for \(\mathcal{L} = \mathcal{W}\) or \(\mathcal{V}\) were presented in [16]. We present another series of 2-dimensional Lie subalgebras of \(\mathcal{L}\) as follows.

**Lemma 4.8.** Let \(\mathcal{L} = \mathcal{W}\) or \(\mathcal{V}\), and fix \(n \in \mathbb{Z}, \ell, k \in \mathbb{F}\) with \(n \neq 1\) and \(\ell, k \neq 0\). Denote
\[
X = -\frac{1}{n-1} x + \ell x^n, \quad Y = -\frac{k}{2(n-1)} x^{-n+2} + kx - \frac{(n-1)\ell}{2(n-1)} x^n \in \mathcal{L}.
\]
Then \(\text{span}_\mathbb{F}\{X, Y\}\) is a 2-dimensional Lie subalgebra of \(\mathcal{L}\), and \([X, Y] = Y\).

**Proof.** The proof is straightforward. \(\square\)

Now we are able to obtain the following main result of this section.

**Theorem 4.9.** Let \(\mathcal{L}, X, Y\) be as in Lemma 4.8. Let \((\mathcal{L}, [\cdot, \cdot], \Delta)\) be the coboundary triangular Lie bialgebra associated to the solution \(r = X \otimes Y - Y \otimes X\) of CYBE. Then the dual Lie bialgebra is \((\mathcal{L}^o, [\cdot, \cdot], \Delta)\) with the underlining space \(\mathcal{L}^o\) and the cobracket \(\Delta\) determined by Theorem 4.3, and the bracket uniquely determined by (cf. (4.5))

\[
[\varepsilon^i, \varepsilon^j] = \begin{cases} 
-k(\delta_{j,n} + \delta_{j,2-n})\varepsilon^j - \frac{k(j+2n-3)}{2(n-1)} \varepsilon^{i+n-1} - \frac{k\ell}{2} (2n-j-1)\varepsilon^{j-n+1} & \text{if } i = 1 \neq j, \\
-\frac{k}{2} \delta_{j,n}\varepsilon^j - \frac{k\ell}{2} (n-1)\delta_{j,n}\varepsilon^{2-n} + \frac{k(j-1)}{2(n-1)} \varepsilon^{j-2n} + \frac{k(2n-j-1)}{2(n-1)} \varepsilon^{j-2n+1} & \text{if } i = -n, j \neq 1, 2-n, \\
\frac{k(j+2n-3)}{2(n-1)} \varepsilon^{j+n-1} + \frac{k\ell}{2} (1-j)\varepsilon^j & \text{if } i = n, j \neq 1, 2-n, n, \\
0 & \text{if } i, j \notin \{1, 2-n, n\}.
\end{cases}
\]

(4.9)
Proof. For convenience, we denote $\ell_0 = -\frac{1}{n-1}$, $k_0 = -\frac{k}{2(n-1)\ell}$, $k_1 = -\frac{(n-1)k}{2}$. Thus $X = \ell_0 x + \ell x^n$ and $Y = k_0 x^{-n+2} + kx + k_1 x^n$. Then

$$\langle [\varepsilon^i, \varepsilon^j], x^m \rangle = \langle \varepsilon^i \otimes \varepsilon^j, \Delta(x^m) \rangle = \langle \varepsilon^i, [x^m, X\varepsilon^j(Y) - Y\varepsilon^j(X)] \rangle + \langle \varepsilon^j, [x^m, Y\varepsilon^i(X) - X\varepsilon^i(Y)] \rangle = A_{ij} - A_{ji},$$

where,

$$A_{ij} = \left\langle \varepsilon^i, [x^m, (\ell_0 x + \ell x^n)\varepsilon^j(k_0 x^{2-n} + kx + k_1 x^n) - (k_0 x^{2-n} + kx + k_1 x^n)\varepsilon^j(\ell_0 x + \ell x^n)] \right\rangle$$

$$= \left\langle (k_0 \delta_{j,2-n} + k_1 \delta_{j,n})(\ell_0 (1-i)\varepsilon^i + \ell(2n-i-1)\varepsilon^{i-n+1})
- (\ell_0 \delta_{j,1} + \ell \delta_{j,n})(k_0 (3-i-2n)\varepsilon^{i+n-1} + k(1-i)\varepsilon^i + k_1 (2n-i-1)\varepsilon^{i-n+1}), x^m \right\rangle. \tag{4.10}$$

From this, we see that $A_{ij} = 0$ if $j \neq 1, 2, n$. Thus $[\varepsilon^i, \varepsilon^j] = 0$ if $i, j \notin \{1, 2, n\}$, which proves the last case of (4.9).

Now assume $i = 1 \neq j$. By (4.10), we obtain

$$[\varepsilon, \varepsilon^j] = (k_0 \delta_{j,2-n} + k_1 \delta_{j,n}) \ell(2n-2)\varepsilon^{2-n} - \ell \delta_{j,n}(k_0 (2-2n)\varepsilon^n + k_1 (2n-2)\varepsilon^{2-n})
-k(\ell_0 (1-j)\varepsilon^i + \ell(2n-j-1)\varepsilon^{i-n+1})
+ \ell_0 (k_0 (3-j-2n)\varepsilon^{i+n-1} + k(1-j)\varepsilon^i + k_1 (2n-j-1)\varepsilon^{i-n+1})
= -k_0 \delta_{j,2-n} \varepsilon^{2-n} - k_1 \delta_{j,n} \varepsilon^n - \frac{k}{2} (2n-j-1)\varepsilon^{i-n+1} + \frac{k}{2(n-1)\ell} (3-j-2n)\varepsilon^{i+n-1},$$

which proves the first case of (4.9).

Next assume $i = 2-n$ and $j \neq 1, 2-n$. By (4.10), we obtain

$$[\varepsilon^{2-n}, \varepsilon^j] = k_1 \delta_{j,n} \ell_0 (n-1)\varepsilon^{2-n} + \ell(3n-3)\varepsilon^{3-2n}
- \ell \delta_{j,n}(k_0 (1-n)\varepsilon + k(n-1)\varepsilon^{2-n} + k_1 (3n-3)\varepsilon^{3-2n})
-k_0 (\ell_0 (1-j)\varepsilon^i + \ell(2n-j-1)\varepsilon^{i-n+1})
= -\frac{k}{2} (n-1)\delta_{j,n} \varepsilon^{2-n} - \frac{k}{2} \delta_{j,n} \varepsilon + \frac{k(1-j)}{2(n-1)\ell} \varepsilon^i + \frac{k(2n-j-1)}{2(n-1)\ell} \varepsilon^{i-n+1},$$

which proves the second case of (4.9).

Finally assume $i = n$ and $j \neq 1, 2-n, n$. By (4.10), we obtain

$$[\varepsilon^n, \varepsilon^j] = -k_1 (\ell_0 (1-j)\varepsilon^i + \ell(2n-j-1)\varepsilon^{i-n+1})
+ (k_0 (3-j-2n)\varepsilon^{i+n-1} + k(1-j)\varepsilon^i + k_1 (2n-j-1)\varepsilon^{i-n+1})
= \frac{k(2n-j-3)}{2(n-1)\ell} \varepsilon^{i+n-1} + \frac{k(1-j)}{2} \varepsilon^i,$$

which proves the third case of (4.9). This completes the proof of the theorem. \qed

**Remark 4.10.** Theorem 4.9 in particular gives a series of infinite dimensional Lie algebras (which seem to be new to us) on the space $S := \mathbb{F}[\varepsilon, \varepsilon^{-1}]$ with bracket defined by (4.9), for
various \( n \in \mathbb{Z}, \ell, k \in \mathbb{F} \) with \( n \neq 1 \) and \( \ell, k \neq 0 \) (note from Remark 3.3 that in fact \( S \cap L^0 = 0 \), however the bracket in (4.9) is well-defined on \( S \)).

We close the paper by proposing the following question: In which case will the dual Lie bialgebra of a coboundary triangular Lie bialgebra be coboundary triangular?

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