On Feller and Strong Feller Properties and Irreducibility of
Regime-Switching Jump Diffusion Processes with Countable
Regimes

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Abstract

This work focuses on a class of regime-switching jump diffusion processes with a countably
infinite state space for the discrete component. Such processes can be used to model complex
hybrid systems in which both structural changes, small fluctuations as well as big spikes coexist
and are intertwined. The paper provides weak sufficient conditions for Feller and strong Feller
properties and irreducibility for such processes. The conditions are presented in terms of the
coefficients of the associated stochastic differential equations.

Keywords: Regime-switching jump diffusion, Feller property, strong Feller property, irreducibil-
ity.

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1 Introduction

Motivated by the increasing need of modeling complex systems, in which both structural changes
and small fluctuations as well as big spikes coexist and are intertwined, this paper continues the
study on regime-switching jump diffusion processes with countable regimes. Our focus is on Feller
and strong Feller properties and irreducibility for such processes. We provide weak sufficient con-
ditions for Feller and strong Feller properties and irreducibility.

Roughly speaking, a regime-switching jump diffusion process can be considered as a two com-
ponent process \((X(t), A(t))\), an analog (or continuous state) component \(X(t)\) and a switching (or
discrete event) component \(A(t)\). The analog component models the state of interest while the
switching component can be used to describe the structural changes of the state or random en-
vironment or random factors that are not represented by the usual jump diffusion formulation.
For instance, a regime-switching Black-Scholes model is considered in Zhang (2001), in which the
continuous component \(X(t)\) models the price evolution of a risky asset and the switching compo-
nent \(A(t)\) delineates the overall economy state. Regime-switching jump diffusion is also used in

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mathematical biology such as the recent paper Tuong et al. (2019), in which a stochastic SIRS model subject to both white and color noises is analyzed. We refer to Mao and Yuan (2006), Shao (2015a,b), Shao and Xi (2014), Yin and Zhu (2010) and the references therein for more work on regime-switching jump diffusions and their applications.

In the theory of Markov processes and their applications, dealing with a Markov process $\xi(t)$ with $\xi(0) = x$, for a suitable function $f$, often one must consider the function $P_t f(x) := E_x[f(\xi(t))]$. Following Dynkin (1965), the process $\xi(t)$ is said to be Feller if $P_t f$ is continuous for any $t \geq 0$ and $\lim_{t\to 0} P_t f(x) = f(x)$ for any bounded and continuous function $f$ and it is said to be strong Feller if $P_t f$ is continuous for any $t > 0$ and any bounded and measurable function $f$. This is a natural condition in physical or social modeling: a slight perturbation of the initial data should result in a small perturbation in the subsequent movement. In addition, Feller and strong Feller properties are intrinsically related to the existence and uniqueness of invariant measures of the underlying process; see, for example, Meyn and Tweedie (1992, 1993a,b).

While Feller and strong Feller properties for regime-switching (jump) diffusion processes have been investigated in the literature, this paper makes substantial improvements over the aforementioned papers. It presents weak local non-Lipschitz conditions for Feller and strong Feller properties. A standing assumption in the literature (such as Nguyen et al. (2017), Shao (2015b), Xi and Zhu (2017), Yin and Zhu (2010)) is that the coefficients of the associated stochastic differential equations are (locally) Lipschitz. While it is a convenient assumption, it is rather restrictive in many applications. For example, the diffusion coefficients in the Feller branching diffusion and the Cox-Ingersoll-Ross model are only Hölder continuous. For another example, many control and optimization problems often require the handling of systems where the (local) Lipschitz condition is violated. Motivated by these considerations, this paper further improves the results in the recent paper Xi et al. (2019) by presenting weak non-Lipschitz conditions for Feller and strong Feller properties. The sufficient conditions are spelled in Theorems 2.5 and 3.3. While certain technical aspects are similar in both papers, the assumptions on the coefficients of the associated stochastic differential equations in this paper are substantially weakened; see Remarks 2.3 and 3.2 for details. It is also worth mentioning that the sufficient condition for strong Feller property in Theorem 3.3 is inspired by Priola and Wang (2006), which deals with gradient estimate for diffusion semigroups. The extension from diffusions to regime-switching jump diffusions with countable regimes is non-trivial as the interactions between the analog and switching components add much subtlety and difficulty to the analyses.

The paper next considers irreducibility of regime-switching jump diffusions. Irreducibility plays an important role in establishing the uniqueness of an invariant measure for the underlying Markov process; see, for example, Hairer (2016). Unfortunately such a property for regime-switching jump diffusions has not been systematically investigated in the literature yet. In this paper, we derive irreducibility for regime-switching jump diffusions (Theorem 4.4) by using an important identity concerning the transition probability of such processes. An intermediate step, which is interesting in its own right, is to show that the sub-systems consists of jump diffusions are irreducible under weaker conditions than those in the recent papers such as Qiao (2014), Xi and Zhu (2019). As an application, we present in Proposition 4.10 a set of sufficient conditions under which a unique invariant measure for regime-switching jump diffusions exists.

The rest of the paper is arranged as follows. We give the precise formulation of regime-switching jump diffusion processes in Section 1.1; some frequently used notations and terminologies are also presented in the Section 1.1. Feller and strong Feller properties for regime-switching jump diffusions are established in Sections 2 and 3 respectively. Section 4 derives irreducibility for regime-switching jump diffusions. Two examples are arranged in Section 5 for demonstration.
1.1 Formulation

Let \((U, \Omega)\) be a measurable space, \(\nu\) a \(\sigma\)-finite measure on \(U\), and \(S = \{1, 2, \ldots\}\). Assume further that \(d \geq 1\) is an integer, \(b : \mathbb{R}^d \times S \to \mathbb{R}^d\), \(\sigma : \mathbb{R}^d \times S \to \mathbb{R}^{d \times d}\), and \(c : \mathbb{R}^d \times S \times U \to \mathbb{R}^d\) are Borel measurable functions. Suppose \((X, \Lambda)\) is a right continuous, strong Markov process with left-hand limits on \(\mathbb{R}^d \times S\) such that the first component \(X\) satisfies the following stochastic differential equation (SDE),

\[
dX(t) = b(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dW(t) + \int_U c(X(t^-), \Lambda(t^-), u)\tilde{N}(dt, du),
\]

where \(W\) is a standard \(d\)-dimensional Brownian motion, \(N\) is a Poisson random measure on \([0, \infty) \times U\) with intensity \(dt \nu(du)\), and \(\tilde{N}\) is the associated compensated Poisson random measure. Here the second component \(\Lambda\) is supposed to be a continuous-time stochastic process taking values in the set \(S\) and satisfies

\[
\mathbb{P}\{\Lambda(t + \Delta) = l | \Lambda(t) = k, X(t) = x\} = \begin{cases}
q_{kl}(x)\Delta + o(\Delta) & \text{if } k \neq l \\
1 + q_{kk}(x)\Delta + o(\Delta) & \text{if } k = l,
\end{cases}
\]

uniformly in \(\mathbb{R}^d\), provided that \(\Delta \downarrow 0\).

To obtain the structure of the process \(\Lambda\), let us consider the family of disjoint intervals \(\{\Delta_{kl}(x) : k, l \in S\}\) defined on the positive half of the real line as follows

\[
\Delta_{12}(x) = [0, q_{12}(x)),
\]

\[
\Delta_{13}(x) = [q_{12}(x), q_{12}(x) + q_{13}(x)),
\]

\[
\vdots
\]

\[
\Delta_{21}(x) = [q_{1}(x), q_{1}(x) + q_{21}(x)),
\]

\[
\Delta_{23}(x) = [q_{1}(x) + q_{21}(x), q_{1}(x) + q_{21}(x) + q_{23}(x)),
\]

\[
\vdots
\]

\[
\Delta_{31}(x) = [q_{1}(x) + q_{2}(x), q_{1}(x) + q_{2}(x) + q_{31}(x)),
\]

\[
\vdots
\]

where \(q_k(x) := \sum_{l \in S \setminus \{k\}} q_{kl}(x)\) and we set \(\Delta_{kl}(x) = \emptyset\) in the case of \(q_{kl}(x) = 0, k \neq l\). We note that \(\{\Delta_{kl}(x) : k, l \in S\}\) are disjoint intervals and that the length of the interval \(\Delta_{kl}(x)\) is equal to \(q_{kl}(x)\). Define a function \(h : \mathbb{R}^d \times S \times \mathbb{R}_+ \to \mathbb{R}\) by

\[
h(x, k, r) = \sum_{l \in S \setminus \{k\}} 1_{\Delta_{kl}(x)}(r).
\]

In other words, we set

\[
h(x, k, r) = \begin{cases}
  l - k & \text{if } r \in \Delta_{kl}(x) \\
  0 & \text{otherwise},
\end{cases}
\]

for each \(x \in \mathbb{R}^d\) and \(k \in S\). As a result, the process \(\Lambda\) can be described as a solution to the following stochastic differential equation

\[
\Lambda(t) = \Lambda(0) + \int_0^t \int_{\mathbb{R}_+} h(X(s^-), \Lambda(s^-), r)N_1(ds, dr),
\]

\[1.4\]
where $N_1$ is a Poisson random measure on $[0, \infty) \times [0, \infty)$ with characteristic measure $m(dz)$, the Lebesgue measure.

We make the following standing assumption throughout the paper:

**Assumption 1.1.** For any $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, the system of stochastic differential equations (1.1) and (1.4) has a non-explosive weak solution $(X^{(x,k)}, \Lambda^{(x,k)})$ with initial condition $(x, k)$ and the solution is unique in the sense of probability law.

Consequently we can consider the semigroup

$$P_t f(x, k) := \mathbb{E}_{x,k}[f(X(t), \Lambda(t))] = \mathbb{E}[X^{(x,k)}(t), \Lambda^{(x,k)}(t)], \quad f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S}).$$

The main focus of this paper is to investigate the continuity properties of the semigroup $P_t$. We say that the semigroup $P_t$ or the process $(X, \Lambda)$ is *Feller continuous* if $P_t f \in C_b(\mathbb{R}^d \times \mathbb{S})$ for every $f \in C_b(\mathbb{R}^d \times \mathbb{S})$ and $t \geq 0$ and $\lim_{t \to 0} P_t f(x, k) = f(x, k)$ for all $f \in C_b(\mathbb{R}^d \times \mathbb{S})$ and $(x, k) \in \mathbb{R}^d \times \mathbb{S}$. Furthermore, we say that the semigroup $P_t$ or the process $(X, \Lambda)$ is *strong Feller continuous* if $P_t f \in C_b(\mathbb{R}^d \times \mathbb{S})$ for every $f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S})$ and $t > 0$.

For convenience, we state the infinitesimal generator of the regime-switching jump diffusion $(X, \Lambda)$:

$$\mathcal{A} f(x, k) := \mathcal{L}_k f(x, k) + Q(x) f(x, k),$$

where $f(\cdot, k) \in C^2(\mathbb{R}^d)$ for each $k \in \mathbb{S}$,

$$\mathcal{L}_k f(x, k) := \frac{1}{2} \text{tr} (a(x, k) \nabla^2 f(x, k)) + \langle b(x, k), \nabla f(x, k) \rangle + \int_U (f(x + c(x, k, u), k) - f(x, k) - \langle \nabla f(x, k), c(x, k, u) \rangle) \nu(du),$$

and

$$Q(x) f(x, k) := \sum_{l \in \mathbb{S}} q_{kl}(x) [f(x, l) - f(x, k)] = \int_{[0, \infty)} [f(x, k + h(x, k, z)) - f(x, k)] m(dz).$$

### 2 Feller Property

In this section, we demonstrate that the process $(X, \Lambda)$ possesses the Feller property under the following assumptions:

**Assumption 2.1.** Assume the following conditions hold.

(i) If $d = 1$, then there exist a positive number $\delta_0$ and a nondecreasing and concave function $\rho : [0, \infty) \to [0, \infty)$ satisfying

$$\int_{0+} \frac{dr}{\rho(r)} = \infty,$$

such that for all $k \in \mathbb{S}, R > 0$ and $x, z \in \mathbb{R}$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$,

$$\text{sgn}(x - z)(b(x, k) - b(z, k)) \leq \kappa_R \rho(|x - z|),$$

$$|\sigma(x, k) - \sigma(z, k)|^2 + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(du) \leq \kappa_R |x - z|,$$
where $\kappa_R$ is a positive constant and $\text{sgn}(a) = 1_{\{a > 0\}} - 1_{\{a \leq 0\}}$. In addition, for each $k \in S$, the function $c$ satisfies that
\[ \text{the function } x \mapsto x + c(x, k, u) \text{ is nondecreasing for all } u \in U; \tag{2.4} \]
or, there exists some $\beta > 0$ such that
\[ |x - z + \theta(c(x, k, u) - c(z, k, u))| \geq \beta|x - z|, \forall (x, z, u, \theta) \in \mathbb{R} \times \mathbb{R} \times U \times [0, 1]. \tag{2.5} \]
i (i) If $d \geq 2$, then there exist a positive number $\delta_0$ and a nondecreasing and concave function $\rho : [0, \infty) \to [0, \infty)$ satisfying
\[ 0 < \rho(r) \leq (1 + r)^2 \rho(r/(1 + r)) \text{ for all } r > 0, \text{ and } \int_{0^+} \frac{dr}{\rho(r)} = \infty, \tag{2.6} \]
so that for all $k \in S, R > 0$ and $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$,
\[ 2(x - z, b(x, k) - b(z, k)) + |\sigma(x, k) - \sigma(z, k)|^2 + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(du) \leq \kappa_R \rho(|x - z|^2), \tag{2.7} \]
where $\kappa_R$ is a positive constant.

Assumption 2.2. For each $k \in S$, there exists a concave function $\gamma_k : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(0) = 0$ such that for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$, there exists a positive constant $\kappa_R$ (which, without loss of generality, can be assumed to be the same positive constant as in (2.2) and (2.3)) such that
\[ \sum_{l \in S \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq \kappa_R \gamma_k(|x - y|). \tag{2.8} \]

Remark 2.3. We note that Assumption 2.1 is comparable to the corresponding assumption in Xi et al. (2019), except that the non-local term in (2.3) and (2.7) only requires the regularity of $\int_U |c(x, k, u) - c(z, k, u)|^2 \nu(du)$. In Xi et al. (2019), the corresponding term is $\int_U ||c(x, k, u) - c(z, k, u)||^2 \wedge |x - z| \cdot |c(x, k, u) - c(z, k, u)||\nu(du).

Assumption 2.2 is weaker than that in Xi et al. (2019). Indeed, that paper assumes that $Q(x) = (q_{kl}(x))$ satisfies
\[ \sum_{l \in S \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq \kappa_R \rho \left( \frac{|x - y|}{1 + |x - y|} \right), \text{ for each } k \in S, \]
for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$, in which $\kappa_R > 0$ and $\rho$ is an increasing and concave function satisfying (2.6). In contrast, the function $\gamma_k$ in Assumption 2.2 may depend on $k$, and is only required to be concave with $\gamma_k(0) = 0$. In particular, the non-integrability condition $\int_0^\infty \frac{dr}{\rho(r)} = \infty$ is dropped. This relaxation is significant and renders that the analyses in Xi et al. (2019) are not applicable.

We will use the coupling method to establish the Feller property. To this end, let us first construct a basic coupling operator $\mathcal{A}$ for $\mathcal{B}$: For $f(x, i, z, j) \in C^2_c(\mathbb{R}^d \times S \times \mathbb{R}^d \times S)$, we define
\[ \mathcal{A}f(x, i, z, j) := \left[ \tilde{\Omega}_d + \tilde{\Omega}_z + \tilde{\Omega}_s \right] f(x, i, z, j), \tag{2.9} \]
For any function $k$ corresponding to the operator $\delta$ in addition $\Lambda$ denotes the first time when the switching components $s$ have settled. Under Assumptions 2.4 and 2.5, the following assertion holds:

$$\tilde{\Omega}_d f(x, i, z, j) := \frac{1}{2} \text{tr}(a(x, i, z, j) D^2 f(x, i, z, j)) + \langle b(x, i, z, j), Df(x, i, z, j) \rangle,$$

$$\tilde{\Omega}_j f(x, i, z, j) := \int_{U} \left[ f(x + c(x, i, u), i, z + c(z, j, u), j) - f(x, i, z, j) \right. $$

$$- \langle D_x f(x, i, z, j), c(x, i, u) \rangle - (D_z f(x, i, z, j), c(z, j, u)) \nu(du),$$

where $Df(x, i, z, j) = (D_x f(x, i, z, j), D_z f(x, i, z, j))$ is the gradient and $D^2 f(x, i, z, j)$ is the Hessian matrix of $f$ with respect to the variables $x$ and $z$, and

$$\tilde{\Omega}_a f(x, i, z, j) := \sum_{l \in S} [q_{il}(x) - q_{il}(z)]^+(f(x, l, z, j) - f(x, i, z, j))$$

$$+ \sum_{l \in S} (q_{ij}(z) - q_{ij}(x))^+(f(x, i, z, l) - f(x, i, z, j))$$

$$+ \sum_{l \in S} [q_{il}(x) \wedge q_{il}(z)](f(x, l, z, l) - f(x, i, z, j)).$$

For any function $f : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}$, let $\tilde{f} : \mathbb{R}^d \times S \times \mathbb{R}^d \times S \mapsto \mathbb{R}$ be defined by $\tilde{f}(x, i, z, j) := f(x, z)$. Now we denote for each $k \in S$

$$\tilde{\mathcal{L}}_k f(x, z) = (\tilde{\Omega}_d^{(k)} + \tilde{\Omega}_j^{(k)}) f(x, z) := (\tilde{\Omega}_d + \tilde{\Omega}_j) \tilde{f}(x, k, z, k), \quad \forall f \in C_c^2(\mathbb{R}^d \times \mathbb{R}^d).$$

We introduce the following notations. Let $(X(\cdot), \lambda(\cdot), \tilde{X}(\cdot), \tilde{\lambda}(\cdot))$ denote the coupling process corresponding to the operator $\tilde{\mathcal{L}}$ with initial condition $(x, k, z, k)$, in which $\delta_0 > |x - z| > 0$, and $\delta_0$ is the positive constant in Assumption 2.1. For any $R > 0$, let

$$\tau_R := \inf\{t \geq 0 : |\tilde{X}(t)| \vee |X(t)| \vee |\tilde{\lambda}(t)| \vee |\lambda(t)| > R\}.$$  

In view of Assumption 1.1, $\lim_{R \to \infty} \tau_R = \infty$ a.s. Also denote $\Delta_t = \tilde{X}(t) - X(t)$ and

$$S_{\delta_0} := \inf\{t \geq 0 : |\Delta_t| > \delta_0\} = \inf\{t \geq 0 : |\tilde{X}(t) - X(t)| > \delta_0\}.$$  

In addition

$$\zeta := \inf\{t \geq 0 : \lambda(t) \neq \tilde{\lambda}(t)\}$$

denotes the first time when the switching components $\lambda$ and $\tilde{\lambda}$ differ.

We need the following lemma whose proof is arranged in Appendix A:

**Lemma 2.4.** Under Assumption 2.1, the following assertion holds:

$$\lim_{|\tilde{x} - x| \to 0} \mathbb{E}[|\Delta_{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta}|] = 0, \quad \forall t \geq 0.$$  

Now we are ready to show that the process $(X, \lambda)$ has Feller property.

**Theorem 2.5.** Under Assumptions 2.1 and 2.2, the process $(X, \lambda)$ has the Feller property.
Proof. We need to show that for each \((x, k) \in \mathbb{R}^d \times \mathbb{S}\) and each \(f \in C_b(\mathbb{R}^d \times \mathbb{S})\), the limit \((P_t f)(\tilde{x}, \tilde{k}) \to (P_t f)(x, k)\) as \((\tilde{x}, \tilde{k}) \to (x, k)\) holds for all \(t \geq 0\). Since \(\mathbb{S} = \{1, 2, \ldots\}\) has a discrete topology, it is enough to consider only \((\tilde{x}, k) \to (x, k)\). First, observe that
\[
| (P_t f)(\tilde{x}, \tilde{k}) - (P_t f)(x, k) | = | E[f(\tilde{X}(t), \tilde{A}(t))] - E[f(X(t), A(t))] |
\leq | E[f(\tilde{X}(t), \tilde{A}(t))] - E[f(\tilde{X}(t), A(t))] |
+ | E[f(\tilde{X}(t), \tilde{A}(t))] - E[f(X(t), A(t))] |
= | E[(f(\tilde{X}(t), \tilde{A}(t)) - f(\tilde{X}(t), A(t))) 1_{\{\zeta \leq t\}}] |
+ | E[(f(\tilde{X}(t), \tilde{A}(t)) - f(\tilde{X}(t), A(t))) 1_{\{\zeta > t\}}] |
\leq 2 ||f||_\infty P\{\zeta \leq t\} + | E[f(\tilde{X}(t), A(t))] - E[f(X(t), A(t))] |. \tag{2.17}
\]
We will show that both terms on the right-hand side of (2.17) converge to 0 as \(\tilde{x} \to x\).

Consider the function \(\Xi(x, k, z, l) := 1_{\{k \neq l\}}\). It follows directly from the definition that
\[
\widetilde{\omega} \Xi(x, k, z, l) = \overline{\Omega}_k \Xi(x, k, z, l) \leq 0, \text{ if } k \neq l.
\]

When \(k = l\), we have from (2.8) that
\[
\widetilde{\omega} \Xi(x, k, z, l) = \overline{\Omega}_k \Xi(x, k, z, k)
= \sum_{i \in \mathbb{S}} | q_{ki}(x) - q_{ki}(z) |^+ (1_{\{i \neq k\}} - 1_{\{k \neq k\}}) + \sum_{i \in \mathbb{S}} | q_{ki}(z) - q_{ki}(x) |^+ (1_{\{i \neq k\}} - 1_{\{k \neq k\}})
\leq \sum_{i \in \mathbb{S}, i \neq k} | q_{ki}(x) - q_{ki}(z) | \leq \kappa_R \gamma_k (|x - y|).
\]

Hence
\[
\widetilde{\omega} \Xi(x, k, z, l) \leq \kappa_R \gamma_k (|x - y|) \tag{2.18}
\]
for all \(k, l \in \mathbb{S}\) and \(x, z \in \mathbb{R}^d\) with \(|x| \vee |z| \leq R\).

Note that \(\zeta \leq t \wedge \tau_R \wedge S_{\delta_0}\) if and only if \(\tilde{A}(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta) \neq A(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta)\). Thus we can use (2.18) to compute
\[
P\{\zeta \leq t \wedge \tau_R \wedge S_{\delta_0}\}
= E[\Xi(\tilde{X}(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta), \tilde{A}(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta), X(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta), A(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta))]
= \Xi(\tilde{x}, k, x, k) + E \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta} \omega \Xi(\tilde{X}(s), \tilde{A}(s), X(s), A(s)) ds \right]
\leq \kappa_R E \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta} \gamma_k (|\tilde{X}(s) - X(s)|) ds \right]
\leq \kappa_R \int_0^t E[\gamma_k (|\tilde{X}(s \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta) - X(s \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta)|) ds
\leq \kappa_R \int_0^t \gamma_k (E[|\Delta_s \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta|]) ds,
\]
where the last inequality follows from the assumption that \(\gamma_k\) is concave. Then it follows from (2.16), the assumption that \(\gamma_k(0) = 0\), and the bounded convergence theorem that
\[
\lim_{|\tilde{x} - x| \to 0} P\{\zeta \leq t \wedge \tau_R \wedge S_{\delta_0}\} = 0. \tag{2.19}
\]
Note also that on the set \( \{ S_{\delta_0} \leq t \land \zeta \land \tau_R \} \) we have \( \delta_0 \leq |\Delta S_{\delta_0} \land \zeta \land \tau_R| \). This implies
\[
\delta_0 \mathbb{P}\{ S_{\delta_0} \leq t \land \zeta \land \tau_R \} \leq \mathbb{E}[|\Delta S_{\delta_0} \land \zeta \land \tau_R| |1_{\{ S_{\delta_0} \leq t \land \zeta \land \tau_R \}}] \leq \mathbb{E}[|\Delta S_{\delta_0} \land \zeta \land \tau_R|].
\]
Therefore, it follows from (2.16) that
\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ S_{\delta_0} \leq t \land \zeta \land \tau_R \} = 0. \tag{2.20}
\]
Fix an arbitrary positive number \( \epsilon \). We have from (2.16) that
\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ |\Delta t \land S_{\delta_0} \land \tau_R \land \zeta| > \epsilon \} \leq \lim_{|\tilde{x} - x| \to 0} \frac{\mathbb{E}[|\Delta t \land S_{\delta_0} \land \tau_R \land \zeta|]}{\epsilon} = 0. \tag{2.21}
\]
Since \( \lim_{R \to \infty} \tau_R = \infty \) a.s., we can choose \( R \) sufficiently large so that
\[
\mathbb{P}\{ \tau_R < t \} < \epsilon. \tag{2.22}
\]
Then
\[
\mathbb{P}\{ |\Delta t| > \epsilon \} = \mathbb{P}\{ |\Delta t| > \epsilon, \tau_R < t \} + \mathbb{P}\{ |\Delta t| > \epsilon, \tau_R \geq t, \zeta \leq t \land S_{\delta_0} \land \tau_R \}
+ \mathbb{P}\{ |\Delta t| > \epsilon, \tau_R \geq t, \zeta > t \land S_{\delta_0} \land \tau_R \}
+ \mathbb{P}\{ |\Delta t| < \epsilon, \tau_R \geq t, \zeta > t \land S_{\delta_0} \land \tau_R \}
\leq \epsilon + \mathbb{P}\{ \zeta \leq t \land S_{\delta_0} \land \tau_R \} + \mathbb{P}\{ |\Delta t| > \epsilon, t \leq S_{\delta_0} \land \tau_R \land \zeta \}
\leq \epsilon + \mathbb{P}\{ \zeta \leq t \land S_{\delta_0} \land \tau_R \} + \mathbb{P}\{ \zeta \leq t \land \tau_R \land \zeta \} + \mathbb{P}\{ |\Delta t \land S_{\delta_0} \land \tau_R \land \zeta| > \epsilon \}.
\]
From (2.19), (2.20) and (2.21) we have
\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ |\Delta t| > \epsilon \} \leq \epsilon.
\]
Since \( \epsilon \) is arbitrary, we conclude that \( \lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ |\Delta t| > \epsilon \} = 0 \). In other words, \( \tilde{X}(t) \to X(t) \) in probability as \( \tilde{x} \to x \). With the metric \( d \) on \( \mathbb{R}^d \times \mathbb{S} \) defined by \( d((x, i), (y, j)) := |x - y| + 1_{\{i \neq j\}} \), we see immediately that \( (\tilde{X}(t), \Lambda(t)) \to (X(t), \Lambda(t)) \) in probability as \( \tilde{x} \to x \). Because the function \( f \) is continuous, we also have \( f(\tilde{X}(t), \Lambda(t)) \to f(X(t), \Lambda(t)) \) in probability as \( \tilde{x} \to x \). Then the bounded convergence theorem implies
\[
|\mathbb{E}[f(\tilde{X}(t), \Lambda(t))] - \mathbb{E}[f(X(t), \Lambda(t))]| \to 0 \text{ as } \tilde{x} \to x. \tag{2.23}
\]
Next, we show that \( \lim_{\tilde{x} \to x} \mathbb{P}\{ \zeta \leq t \} = 0 \) holds. Recall that \( R \) is chosen so that (2.22) holds. Then we compute
\[
\mathbb{P}\{ \zeta \leq t \} = \mathbb{P}\{ \zeta \leq t, \tau_R < t \} + \mathbb{P}\{ \zeta \leq t, \tau_R \geq t \}
\leq \mathbb{P}\{ \tau_R < t \} + \mathbb{P}\{ \zeta \leq t, \tau_R \geq t, S_{\delta_0} \leq t \land \zeta \} + \mathbb{P}\{ \zeta \leq t, \tau_R \geq t, S_{\delta_0} > t \land \zeta \}
\leq \epsilon + \mathbb{P}\{ S_{\delta_0} \leq t \land \zeta \land \tau_R \} + \mathbb{P}\{ \zeta \leq t \land \tau_R \land S_{\delta_0} \}.
\]
It then follows from (2.19) and (2.20) that \( \lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ \zeta \leq t \} \leq \epsilon \). Again since \( \epsilon \) is arbitrary, we conclude that
\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{P}\{ \zeta \leq t \} = 0. \tag{2.24}
\]
Finally we plug (2.23) and (2.24) into (2.17) to complete the proof. \( \square \)
3 Strong Feller Property

In this section, we study the strong Feller property for the process \((X, \Lambda)\).

**Assumption 3.1.** For every \(k \in \mathbb{S}\) the following assertions hold:

(i) For every \(R > 0\) there exists a constant \(\lambda_R > 0\) such that
\[
\langle \xi, a(x, k) \xi \rangle \geq \lambda_R |\xi|^2, \quad \xi \in \mathbb{R}^d,
\]  
for all \(x \in \mathbb{R}^d\) with \(|x| \leq R\), where \(a(x, k) := \sigma(x, k) \sigma(x, k)^T\).

(ii) There exists a nonnegative function \(g \in C(0, \infty)\) satisfying
\[
\int_0^1 g(r) dr < \infty,
\]  
and
\[
\|\sigma_{\lambda_R}(x, k) - \sigma_{\lambda_R}(z, k)\|^2 + 2(x - z, b(x, k) - b(z, k)) + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(du) \leq 2\kappa_R|x - z|g(|x - z|),
\]  
for all \(x, z \in \mathbb{R}^d\) with \(|x| \vee |z| \leq R\) and \(|x - z| \leq \delta_0\), where \(\delta_0\) is a positive constant and \(\sigma_{\lambda_R}\) the unique symmetric nonnegative definite matrix-valued function such that \(\sigma_{\lambda_R}^2(x, k) = a(x, k) - \lambda_R I\).

**Remark 3.2.** We remark that Assumption 3.1 improves significantly over those in the literature such as Shao (2015b), Xi and Zhu (2017), which require Lipschitz condition for the coefficients of the associated stochastic differential equations. By contrast, (3.3) places very mild conditions on the coefficients; it allows to treat for example the case of Hölder continuous coefficients by taking \(g(r) = r^{-p}\) for \(0 \leq p < 1\); see Example 5.1.

The main result of this section is:

**Theorem 3.3.** Suppose Assumptions 2.2 and 3.1 hold. Then the process \((X, \Lambda)\) has strong Feller property.

As in Section 2, we will use the coupling method to prove Theorem 3.3. To this end, we first define
\[
\tilde{a}(x, i, z, j) := \begin{pmatrix} a(x, i) & \hat{g}(x, i, z, j) \\ \hat{g}(x, i, z, j)^T & a(z, j) \end{pmatrix} \quad \text{and} \quad b(x, i, z, j) := \begin{pmatrix} b(x, i) \\ b(z, j) \end{pmatrix}
\]  
where
\[
\hat{g}(x, i, z, j) := \lambda_R(I - 2u(x, z)u(x, z)^T) + \sigma_{\lambda_R}(x, i) \sigma_{\lambda_R}(z, j)^T,
\]  
and \(u(x, z) := \frac{x - z}{|x - z|}\). Then we define the coupling operator for \(\hat{\mathcal{A}}\) of (1.6) as follows:
\[
\hat{\mathcal{A}} f(x, i, z, j) := [\hat{\Omega}_d + \hat{\Omega}_j + \hat{\Omega}_s] f(x, i, z, j), \quad f \in C^2_c(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S}),
\]  
(3.4)
This implies and hence \( \hat{\Omega}_d f(x, i, z, j) = \frac{1}{2} \text{tr} (\hat{a}(x, i, z, j)D^2 f(x, i, z, j)) + (b(x, i, z, j), Df(x, i, z, j)) \). \tag{3.5}

and \( \hat{\Omega}_j \) and \( \hat{\Omega}_k \) are defined as in (2.11) and (2.12), respectively. In addition, as in Section 2, for each \( k \in \mathbb{S} \) and any \( F \in C^2_c(\mathbb{R}^d \times \mathbb{R}^d) \), we write \( f(x, k, z, k) := F(x, z) \) and denote

\[
\hat{L}_k F(x, z) = [\hat{\Omega}_d^{(k)} + \hat{\Omega}_j^{(k)}] f(x, k, z, k) := \hat{s} f(x, k, z, k). \tag{3.6}
\]

We can regard \( \hat{L}_k \) as the coupling operator for \( L_k \) defined in (1.7).

Furthermore, to facilitate future presentations, we introduce the following notations. For any \( x, z \in \mathbb{R}^d \) and \( i, j \in \mathbb{S} \), put

\[
A(x, i, z, j) := a(x, i) + a(z, j) - 2\hat{g}(x, i, z, j),
\]

\[
\bar{A}(x, i, z, j) := \frac{1}{|x - z|^2} (x - z, A(x, i, z, j)(x - z)),
\]

\[
B(x, i, z, j) := (x - z, b(x, i) - b(z, j)).
\]

**Lemma 3.4.** For all \( x, z \in \mathbb{R}^d \) and \( i, j \in \mathbb{S} \), we have

(i) \( \hat{a}(x, i, z, j) \) is symmetric and uniformly positive definite,

(ii) \( \text{tr} A(x, i, z, j) = \|\sigma_{\lambda_R}(x, i) - \sigma_{\lambda_R}(z, j)\|^2 + 4\lambda_R \), and

(iii) \( \bar{A}(x, i, z, j) \geq 4\lambda_R \).

**Proof.** The proof involves elementary and straightforward computations; similar computations can be found in Chen and Li (1989) and Priola and Wang (2006). We shall omit the details here. \( \square \)

**Remark 3.5.** Note that when \( d = 1 \), we have

\[
\bar{A}(x, i, z, j)|x - z|^2 = |(\sigma_{\lambda_R}(x, i) - \sigma_{\lambda_R}(z, j))(x - z)|^2 + 4\lambda_R|x - z|^2
\]

\[
= |(\sigma_{\lambda_R}(x, i) - \sigma_{\lambda_R}(z, j))|^2|x - z|^2 + 4\lambda_R|x - z|^2,
\]

and hence

\[
\bar{A}(x, i, z, j) = |(\sigma_{\lambda_R}(x, i) - \sigma_{\lambda_R}(z, j))|^2 + 4\lambda_R = \text{tr} A(x, i, z, j). \tag{3.7}
\]

This implies

\[
\text{tr} A(x, i, z, j) - \bar{A}(x, i, z, j) + 2B(x, i, z, j) = 2B(x, i, z, j). \tag{3.8}
\]

Now, let \( \phi \in C^2([0, \infty)) \). As in Chen and Li (1989), for each \( k \in \mathbb{S} \) and all \( x, z \in \mathbb{R}^d \) with \( x \neq z \), we can verify that

\[
\hat{\Omega}_d^{(k)} \phi(|x - z|) = \frac{\phi''(|x - z|)}{2} \bar{A}(x, k, z, k)
\]

\[
+ \frac{\phi'(|x - z|)}{2|x - z|} [\text{tr} A(x, k, z, k) - \bar{A}(x, k, z, k) + 2B(x, k, z, k)]. \tag{3.9}
\]

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Moreover, we have

\[
\hat{\Omega}_j^{(k)} \phi(|x - z|) = \int_U (\phi(|x + c(x, k, u) - z - c(z, k, u)|) - \phi(|x - z|)) - \frac{g'(|x - z|)}{|x - z|} (x - z, c(x, k, u) - c(z, k, u))\nu(du).
\] (3.10)

Motivated by Priola and Wang (2006), we consider the function \(G\) given by

\[
G(r) := \int_0^r \exp \left\{ - \int_0^s \frac{R}{2\lambda R} g(w)dw \right\} \int_s^1 \exp \left\{ \int_0^v \frac{R}{2\lambda R} g(u)du \right\} dvds, \quad r \in [0, 1]
\]
where \(g\) is the function given in Assumption 3.1 (ii). Since \(g \geq 0\), we see that

\[
G'(r) = e^{-\int_0^r \frac{R}{2\lambda R} g(w)dw} \int_r^1 e^{\int_0^v \frac{R}{2\lambda R} g(u)du} dv \geq 0, \quad \text{and} \quad G''(r) = -1 - \frac{R}{2\lambda R} g(r) G'(r) \leq 0. \] (3.11)

Note also that \(G\) is concave and \(\lim_{r \to 0} G(r) = 0\). Since \(G'(0) \geq 1\) and \(G(0) = 0\), there exists a constant \(0 < \alpha \leq 1\) so that

\[
r \leq G(r) \quad \text{for all} \quad r \in [0, \alpha]. \] (3.12)

**Lemma 3.6.** Suppose Assumptions 3.1 holds. Then for any \(R > 0\) and \(k \in \mathbb{S}\) there exits a positive constant \(\beta_R > 0\) such that

\[
\hat{\Lambda}_k G(|x - z|) \leq -\beta_R
\] (3.13)
for all \(x, z \in \mathbb{R}^d\) with \(|z| \vee |x| \leq R\) and \(0 < |x - z| \leq \alpha \wedge \delta_0\), where \(\delta > 0\) is given in (3.12).

This lemma follows directly from straightforward but involved computations. To preserve the flow of reading, we arrange it to Appendix A.

Throughout the rest of the section, we use the following notations. For any \(x, \tilde{x} \in \mathbb{R}^d\) and \(k \in \mathbb{S}\), denote by \((X(\cdot), \Lambda(\cdot), \tilde{X}(\cdot), \tilde{\Lambda}(\cdot))\) the process corresponding to the coupling operator \(\tilde{\mathscr{A}}\) with initial condition \((x, k, \tilde{x}, k)\). As in Section 2, \(\Delta_t := \tilde{X}(t) - X(t), t \geq 0\). Let \(\tau_R, S_\delta, \text{ and } \zeta\) be defined as in (2.13), (2.14), and (2.15), respectively. In addition, for each \(n \in \mathbb{N}\), we define

\[
T_n := \inf \left\{ t \geq 0 : |X(t) - \tilde{X}(t)| < \frac{1}{n} \right\}.
\] (3.14)

Then \(\lim_{n \to \infty} T_n = T\), where

\[
T := \inf \{ t \geq 0 : X(t) = \tilde{X}(t) \}.
\] (3.15)

**Lemma 3.7.** Suppose Assumption 3.1 holds. Then the following assertions hold for every \(t \geq 0\):

\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{E}[G(|\Delta_t \wedge \tau_R \wedge S_\delta \wedge \zeta|)] = 0.
\] (3.16)

\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{E}[G(|\Delta_t \wedge \tau_R \wedge S_\delta \wedge \zeta^-|)] = 0.
\] (3.17)

In particular,

\[
\lim_{|\tilde{x} - x| \to 0} \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_\delta \wedge \zeta^-|] = 0,
\] (3.18)

where \(\delta := \delta_0 \wedge \alpha\), \(\delta_0\) is the constant given in Assumption 3.1 (ii), and \(\alpha \in (0, 1)\) is the constant given in (3.12).
Proof. Assume without loss of generality that $\tilde{\delta} \geq |x - \bar{x}| > 0$. We apply Itô’s formula to the process $G(|X(\cdot) - X(\cdot)|) = G(|\Delta|)$:

$$
\mathbb{E}[G(\Delta_{t\wedge \tau_R \wedge S_\delta \wedge \zeta})] = G(\Delta_0) + \mathbb{E}\left[\int_0^{t\wedge \tau_R \wedge S_\delta \wedge \zeta} \hat{L}G(\Delta_s)ds\right]
$$

$$
\leq G(\Delta_0) - \beta_R \mathbb{E}[t \wedge \tau_R \wedge S_\delta \wedge \zeta],
$$

where the last inequality follows from (3.13). Hence

$$
\mathbb{E}[G(\Delta_{t\wedge \tau_R \wedge S_\delta \wedge \zeta})] + \beta_R \mathbb{E}[t \wedge \tau_R \wedge S_\delta \wedge \zeta] \leq G(\Delta_0) = G(|x - \bar{x}|)
$$

Since $\lim_{r \to 0} G(r) = 0$, (3.16) follows. The same argument implies (3.17).

Since $|\Delta_{t\wedge \tau_R \wedge S_\delta \wedge \zeta}| \leq \tilde{\delta} \leq \alpha$, it follows from (3.12) that

$$
|\Delta_{t\wedge \tau_R \wedge S_\delta \wedge \zeta}| \leq G(\Delta_{t\wedge \tau_R \wedge S_\delta \wedge \zeta})
$$

and therefore (3.18) follows.

\[\square\]

Lemma 3.8. Suppose Assumptions 2.2 and 3.1 hold. Then

$$
\lim_{|\bar{x} - x| \to 0} \mathbb{P}\{\zeta \leq t\} = 0.
$$

holds for every $t \geq 0$.

Proof. Given $\epsilon > 0$. Choose $R$ sufficiently large so that $\mathbb{P}\{\tau_R \leq t\} < \epsilon$. Observe that

$$
P\{\zeta \leq t\} = P\{\zeta \leq t, \tau_R < t\} + P\{\zeta \leq t, \tau_R \geq t\}
$$

$$
\leq P\{\tau_R < t\} + P\{\zeta \leq t, \tau_R \geq t, S_\delta \leq t \wedge \zeta\} + P\{\zeta \leq t, \tau_R \geq t, S_\delta > t \wedge \zeta\}
$$

$$
\leq \epsilon + P\{\zeta \leq t, \tau_R \geq t, S_\delta \leq t \wedge \zeta\} + P\{\zeta \leq t, \tau_R \geq t, S_\delta > t \wedge \zeta\}
$$

$$
\leq \epsilon + P\{S_\delta \leq t \wedge \zeta \wedge \tau_R\} + P\{\zeta \leq t \wedge \tau_R \wedge S_\delta\}. \quad (3.20)
$$

As in the proof of Theorem 2.5, condition (2.8) enables us to derive

$$
P\{\zeta \leq t \wedge \tau_R \wedge S_\delta\} \leq \kappa_R \int_0^t \gamma_k(\mathbb{E}[|\Delta_{s \wedge \tau_R \wedge S_\delta \wedge \zeta}|])ds.
$$

Furthermore, (3.18) implies that

$$
\lim_{|\bar{x} - x| \to 0} P\{\zeta \leq t \wedge \tau_R \wedge S_\delta\} = 0. \quad (3.21)
$$

Note that on the set $\{S_\delta \leq t \wedge \zeta \wedge \tau_R\}$ we have $\tilde{\delta} \leq |\Delta_{s \wedge \tau_R \wedge \zeta \wedge \tau_R}|$. Since $G$ is increasing, we have

$$
0 < G(\tilde{\delta}) \leq G(|\Delta_{s \wedge \tau_R \wedge \zeta \wedge \tau_R}|).
$$

Thus

$$
G(\tilde{\delta}) P\{S_\delta \leq t \wedge \zeta \wedge \tau_R\} \leq \mathbb{E}[G(|\Delta_{s \wedge \tau_R \wedge \zeta \wedge \tau_R}|)1_{\{S_\delta \leq t \wedge \zeta \wedge \tau_R\}}] \leq \mathbb{E}[G(|\Delta_{s \wedge \tau_R \wedge \zeta \wedge \tau_R}|)]
$$

This, together with (3.16), implies that

$$
\lim_{|\bar{x} - x| \to 0} P\{S_\delta \leq t \wedge \zeta \wedge \tau_R\} = 0. \quad (3.22)
$$

In view of (3.20), it follows from (3.21) and (3.22) that $\lim_{|\bar{x} - x| \to 0} P\{\zeta \leq t\} \leq \epsilon$. Since $\epsilon$ is arbitrary, we obtain (3.19).

\[\square\]
Lemma 3.9. Suppose Assumptions 2.2 and 3.1 hold. Then
\[
\lim_{|x-\tilde{x}| \to 0} \mathbb{P}\{t < T\} = 0. \tag{3.23}
\]
holds for every \(t \geq 0\).

Proof. We may assume without loss of generality that \(\bar{\delta} \geq |x - \tilde{x}| > \frac{1}{n_0} > 0\) for some \(n_0 \in \mathbb{N}\). Let \(\epsilon > 0\) and choose a sufficiently large \(R\) so that \(\mathbb{P}\{\tau_R \leq t\} < \epsilon\). For each \(n \geq n_0\), define \(T_n\) and \(T\) as in (3.14) and (3.15), respectively.

We first observe that
\[
\mathbb{P}\{t < T\} = \mathbb{P}\{t < T, \tau_R < t\} + \mathbb{P}\{t < T, \tau_R \geq t\}
\]
\[
\leq \mathbb{P}\{\tau_R < t\} + \mathbb{P}\{t < T, \tau_R \geq t, S_\delta < t\} + \mathbb{P}\{t < T, \tau_R \geq t, S_\delta \geq t\}
\]
\[
\leq \epsilon + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R\} + \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta\}
\]
\[
= \epsilon + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R, S_\delta \leq \xi\} + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R, S_\delta > \xi\}
\]
\[
+ \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta, t < \xi\} + \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta, t \geq \xi\}
\]
\[
\leq \epsilon + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R \wedge \xi\} + \mathbb{P}\{\xi < S_\delta \wedge t \wedge T \wedge \tau_R\}
\]
\[
+ \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta \wedge \xi\} + \mathbb{P}\{\xi \leq t\}
\]
\[
\leq \epsilon + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R \wedge \xi\} + \mathbb{P}\{\xi \leq t\} + \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta \wedge \xi\} + \mathbb{P}\{\xi \leq t\}
\]
\[
= \epsilon + \mathbb{P}\{S_\delta \leq t \wedge T \wedge \tau_R \wedge \xi\} + \mathbb{P}\{t \leq T \wedge \tau_R \wedge S_\delta \wedge \xi\} + 2\mathbb{P}\{\xi \leq t\}
\]
\[
\leq \epsilon + \mathbb{P}\{S_\delta \leq T \wedge \tau_R \wedge \xi\} + \frac{\mathbb{E}[T \wedge \tau_R \wedge S_\delta \wedge \xi]}{t} + 2\mathbb{P}\{\xi \leq t\}. \tag{3.24}
\]

Note that on the set \(\{S_\delta \leq T_n \wedge \tau_R \wedge \xi\}\) we have \(\tilde{\delta} \leq |\Delta S_\delta \wedge T_n \wedge \tau_R \wedge \xi|\). Since \(G\) is increasing, \(0 < G(\tilde{\delta}) \leq G(|\Delta S_\delta \wedge T_n \wedge \tau_R \wedge \xi|)\).

Thus
\[
G(\tilde{\delta})\mathbb{P}\{S_\delta \leq T_n \wedge \tau_R \wedge \xi\} \leq \mathbb{E}[G(|\Delta S_\delta \wedge T_n \wedge \tau_R \wedge \xi|)1\{S_\delta \leq T_n \wedge \tau_R \wedge \xi\}] \leq \mathbb{E}[G(|\Delta S_\delta \wedge T_n \wedge \tau_R \wedge \xi|)]
\]
\[
= G(|x - \tilde{x}|) + \mathbb{E} \left[ \int_0^{S_\delta \wedge T_n \wedge \tau_R \wedge \xi} \tilde{L}_k G(|\Delta_s|) ds \right]
\]
\[
\leq G(|x - \tilde{x}|) - \beta_R \mathbb{E}[T_n \wedge \tau_R \wedge S_\delta \wedge \xi],
\]
where that last inequality follows from (3.13). So
\[
G(\tilde{\delta})\mathbb{P}\{S_\delta \leq T_n \wedge \tau_R \wedge \xi\} + \beta_R \mathbb{E}[T_n \wedge \tau_R \wedge S_\delta \wedge \xi] \leq G(|x - \tilde{x}|).
\]

Passing to the limit as \(n \to \infty\), we obtain
\[
G(\tilde{\delta})\mathbb{P}\{S_\delta \leq T \wedge \tau_R \wedge \xi\} + \beta_R \mathbb{E}[T \wedge \tau_R \wedge S_\delta \wedge \xi] \leq G(|x - \tilde{x}|).
\]

Then, in view of (3.24), we have
\[
\mathbb{P}\{t < T\} \leq \epsilon + \mathbb{P}\{S_\delta \leq T \wedge \tau_R \wedge \xi\} + \frac{\mathbb{E}[T \wedge \tau_R \wedge S_\delta \wedge \xi]}{t} + 2\mathbb{P}\{\xi \leq t\}
\]
\[
\leq \epsilon + \frac{G(|x - \tilde{x}|)}{G(\tilde{\delta})} + \frac{G(|x - \tilde{x}|)}{t\beta} + 2\mathbb{P}\{\xi \leq t\}.
\]

From (3.19) and the fact that \(\lim_{|x - \tilde{x}| \to 0} G(|x - \tilde{x}|) = 0\), we obtain \(\lim_{|x - \tilde{x}| \to 0} \mathbb{P}\{t < T\} \leq \epsilon\). Since \(\epsilon\) was arbitrary, we obtain (3.23). \(\square\)
Now we are ready to present the proof of Theorem 3.3.

**Proof of Theorem 3.3.** Given $x \in \mathbb{R}^d$ and $k \in S$. We want to show that for every bounded Borel measurable function $f$ on $\mathbb{R}^d$ the limit $(P_t f)(\tilde{x}, \tilde{k}) \to (P_t f)(x, k)$ as $(\tilde{x}, \tilde{k}) \to (x, k)$ holds for all $t > 0$. Since $S = \{1, 2, \ldots\}$ has a discrete topology, we may consider only when $(\tilde{x}, k) \to (x, k)$, that is, when $\tilde{k} = k$.

For any given $\epsilon > 0$ we can choose a sufficiently large $R$ so that $\mathbb{P}\{\tau_R \leq t\} < \epsilon$. Let $\tilde{x} \in \mathbb{R}^d$ be such that $\tilde{d} \geq |x - \tilde{x}| > 0$, where $\tilde{d} := \delta_0 \wedge \alpha$. Denote the coupling process corresponding to the coupling operator $\tilde{L}$ with initial condition $(x, k, \tilde{x}, \tilde{k})$ by $(X(t), \Lambda(t), \tilde{X}(t), \tilde{\Lambda}(t))$. Denote by

$$
\tilde{T} := \inf\{t \geq 0 : (X(t), \Lambda(t)) = (\tilde{X}(t), \tilde{\Lambda}(t))\}
$$

(3.25)

the coupling time of $(X(t), \Lambda(t))$ and $(\tilde{X}(t), \tilde{\Lambda}(t))$. Recall the stopping time $T$ defined in (3.15). We make the following observations: (i) $T \leq \tilde{T}$, and (ii) $T < \zeta$ implies $T = \tilde{T}$. We then have

$$
1_{\{t < \tilde{T}\}} = 1_{\{t < T\}} + 1_{\{T \leq t < \tilde{T}\}}
$$

\begin{align*}
&= 1_{\{t < T\}} + 1_{\{T \leq t < \tilde{T}, \zeta \leq t\}} + 1_{\{T \leq t < \tilde{T}, \zeta > t\}} \\
&\leq 1_{\{t < T\}} + 1_{\{\zeta \leq t\}} + 1_{\{T \leq \tilde{T}, t < \tilde{T}\}} \\
&\leq 1_{\{t < T\}} + 1_{\{\zeta \leq t\}} + 1_{\{T < \zeta, t < \tilde{T}\}} \\
&\leq 1_{\{t < T\}} + 1_{\{\zeta \leq t\}} + 1_{\{T = \tilde{T}, t < \tilde{T}\}} \\
&\leq 1_{\{t < T\}} + 1_{\{\zeta \leq t\}} + 1_{\{t < T\}} \\
&= 2 \cdot 1_{\{t < T\}} + 1_{\{\zeta \leq t\}}.
\end{align*}

Then it follows that

$$
\|(P_t f)(\tilde{x}, \tilde{k}) - (P_t f)(x, k)| = |\mathbb{E}[f(\tilde{X}(t), \tilde{\Lambda}(t))] - \mathbb{E}[f(X(t), \Lambda(t))]| \\
\leq \mathbb{E}[|f(\tilde{X}(t), \tilde{\Lambda}(t)) - f(X(t), \Lambda(t))|1_{\{t < \tilde{T}\}}] \\
+ \mathbb{E}[|f(\tilde{X}(t), \tilde{\Lambda}(t)) - f(X(t), \Lambda(t))|1_{\{t \geq \tilde{T}\}}] \\
= \mathbb{E}[|f(\tilde{X}(t), \tilde{\Lambda}(t)) - f(X(t), \Lambda(t))|1_{\{t < \tilde{T}\}}] \\
\leq 2 ||f||_{\infty} \mathbb{E}[1_{\{t < \tilde{T}\}}] \\
\leq 2 ||f||_{\infty} \mathbb{E}[2 \cdot 1_{\{t < T\}} + 1_{\{\zeta \leq t\}}] \\
= 4 ||f||_{\infty} \mathbb{P}\{t < T\} + 2 ||f||_{\infty} \mathbb{P}\{\zeta \leq t\}.
$$

A combination of (3.19) and (3.23) then gives

$$
\lim_{|\tilde{x} - x| \to 0} \|(P_t f)(\tilde{x}, \tilde{k}) - (P_t f)(x, k)| = 0.
$$

This establishes the strong Feller property and concludes the proof. \qed

4 Irreducibility

Denote the transition probability of the process $(X, \Lambda)$ by

$$
P(t, (x, k), B \times \{l\}) := P_t 1_{B \times \{l\}}(x, k) = \mathbb{P}\{(X(t), \Lambda(t)) \in B \times \{l\}|(X(0), \Lambda(0)) = (x, k)\},
$$

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for $B \in \mathfrak{B}(\mathbb{R}^d)$ and $l \in S$. The semigroup $P_t$ of (1.5) is said to be irreducible if for any $t > 0$ and $(x, k) \in \mathbb{R}^d \times S$

$$P(t, (x, k), B \times \{l\}) > 0$$

for all $l \in S$ and all nonempty open set $B \in \mathfrak{B}(\mathbb{R}^d)$.

We make the following assumptions:

**Assumption 4.1.** For each $k \in S$ and $x \in \mathbb{R}^d$, the stochastic differential equation

$$X^{(k)}(t) = x + \int_0^t b(X^{(k)}(s), k)ds + \int_0^t \sigma(X^{(k)}(s), k)dW(s) + \int_0^t \int_{\mathcal{U}} c(X^{(k)}(s-), k, u)\tilde{N}(ds, du)$$

has a non-explosive weak solution $X^{(k)}$ with initial condition $x$ and the solution is unique in the sense of probability law.

**Assumption 4.2.** For any $x \in \mathbb{R}^d$ and $k \in S$, we have

$$2\langle x, b(x, k) \rangle + \|\sigma(x, k)\|^2 + \int_{\mathcal{U}} |c(x, k, u)|^2\nu(du) \leq \kappa(|x|^2 + 1),$$

where $a(x, k) := \sigma(x, k)\sigma(x, k)^T$, and $\lambda, \kappa$ are positive constants.

**Assumption 4.3.** (i) There exists a positive constant $\kappa_0 > 0$ such that

$$0 \leq q_{kl}(x) \leq \kappa_0 l^2 k^{-1}$$

for all $x \in \mathbb{R}^d$ and $k \neq l \in S$.

(ii) For any $k, l \in S$, there exist $k_0, k_1, ..., k_n \in S$ with $k_i \neq k_{i+1}, k_0 = k$, and $k_n = l$ such that the set $\{x \in \mathbb{R}^d : q_{k_i k_{i+1}}(x) > 0\}$ has positive Lebesgue measure for all $i = 0, 1, ..., n - 1$.

**Theorem 4.4.** Suppose Assumptions 3.1, 4.1, 4.2, and 4.3 hold. Then the semigroup $P_t$ of (1.5) is irreducible.

In order to obtain the irreducibility of the process $(X, \Lambda)$ we first show that, for any given $k \in S$, the process $X^{(k)}$ of (4.1) is strong Feller and irreducible. Then we use a result in Xi et al. (2019) to write $P(t, (x, k), B \times \{l\})$ as a convergent series in terms of sub-transition probabilities of the killed processes $\tilde{X}^{(j)}, j \in S$ and the transition rates $q_{ji}(x)$. Denote the transition probability of the process $X^{(k)}$ by

$$P^{(k)}(t, x, B) := \mathbb{P}\{X^{(k)}(t) \in B | X^{(k)}(0) = x\}, \quad B \in \mathfrak{B}(\mathbb{R}^d).$$

The corresponding semigroup $P_t^{(k)}$ is said to be irreducible if $P^{(k)}(t, x, B) > 0$ for all nonempty open set $B \subset \mathbb{R}^d$. We next kill the process $X^{(k)}$ with killing rate $q_k(\cdot)$ and denote the killed process by $\tilde{X}^{(k)}$, that is, we define

$$\tilde{X}^{(k)}(t) = \begin{cases} X^{(k)}(t) & \text{if } t < \tau, \\ \partial & \text{if } t \geq \tau, \end{cases}$$
where $\tau := \inf\{t \geq 0 : A(t) \neq A(0)\}$ and $\varnothing$ is a cemetery point added to $\mathbb{R}^d$. Then the semigroup of the killed process $\tilde{X}^{(k)}$ is given by

$$\tilde{P}^k_t f(x) := \mathbb{E}_x[f(\tilde{X}^{(k)}(t))] = \mathbb{E}\left[ f(X^{(k)}(t)) \exp\left\{ \int_0^t g_{kk}(X^{(k)}(s))ds \right\} | X^{(k)}(0) = x \right],$$

where $f \in \mathcal{B}_b(\mathbb{R}^d)$. We also denote its transition probability by

$$\tilde{P}^k(t, x, B) := \mathbb{E}_x[1_B(\tilde{X}^{(k)}(t))] = \mathbb{P}\{ \tilde{X}^{(k)}(t) \in B | \tilde{X}^{(k)}(0) = x \}, \quad B \in \mathcal{B}(\mathbb{R}^d).$$

**Lemma 4.5.** Under Assumptions 3.1 and 4.1, the semigroup $P^k_t$ is strong Feller.

**Proof.** Let $(\tilde{X}^{(k)}, X^{(k)})$ be the coupling process corresponding to $\tilde{L}_k$ of (3.6) with initial condition $(\bar{x}, x)$. Suppose without loss of generality that $0 < |\bar{x} - x| < \delta_0$, where $\delta_0$ is the positive constant in Assumption 3.1. Define $T := \inf\{t \geq 0 : \tilde{X}(t) = X(t)\}$. Using very similar calculations as those in the proof of Lemma 3.9, we can show that $\lim_{|\bar{x} - x| \to 0} \mathbb{P}\{t < T\} = 0$. Then it follows that for any $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t > 0$, we have

$$|(P^k_t f)(\bar{x}) - (P^k_t f)(x)| = |\mathbb{E}[f(\tilde{X}^{(k)}(t))] - \mathbb{E}[f(X^{(k)}(t))]| \leq 2\|f\|_{\infty} \mathbb{P}\{t < T\} \to 0,$$

as $\bar{x} - x \to 0$. This implies that $P^k_t f$ is a continuous function and hence completes the proof. $\square$

**Lemma 4.6.** Suppose Assumption 4.2 holds. Then for every $T > 0$ there exists a constant $K := K(T, X(0)) > 0$ so that

$$\mathbb{E}[|X(t)|^2] \leq K \tag{4.5}$$

for all $t \in [0, T]$.

**Proof.** This lemma follows from (4.2) and standard arguments. For brevity, we omit the details here. $\square$

To derive irreducibility for the semigroup $P^k_t$, we consider the function $F$ given by

$$F(r) := \int_0^{1+r} e^{-\int_0^{s} g(w)dw} ds, \quad r \in [0, \infty) \tag{4.6}$$

where $g$ is the function given in Assumption 3.1(ii). Since $g \geq 0$, we see that

$$0 \leq F(r) \leq \frac{r}{1+r} \leq 1 \tag{4.7}$$

$$0 \leq F'(r) = \frac{1}{(1+r)^2} e^{-\int_0^{1+r} g(w)dw} \leq \frac{1}{(1+r)^2} \leq 1 \tag{4.8}$$

$$0 \geq F''(r) = -\frac{2}{(1+r)^3} e^{-\int_0^{1+r} g(w)dw} - \frac{g(\frac{r}{1+r})}{(1+r)^4} e^{-\int_0^{1+r} g(w)dw} = -\left[ \frac{2}{1+r} + \frac{g(\frac{r}{1+r})}{(1+r)^2} \right] F'(r). \tag{4.9}$$

In addition, for any $x \in \mathbb{R}^d$, we have

$$\nabla F(|x|^2) = 2F'(|x|^2)x, \quad \nabla^2 F(|x|^2) = 4F''(|x|^2)x x^T + 2F'(|x|^2)I.$$

**Lemma 4.7.** Under Assumptions 1.1, 3.1 (ii), and 4.2, the semigroup $P^k_t$ is irreducible.
**Remark 4.8.** While irreducibility for jump diffusions has been considered in the literature such as Qiao (2014), Xi and Zhu (2019), it is worth pointing out that Assumption 3.1(ii) is much weaker than Assumptions (H′) and (H′′) of Qiao (2014) and Assumption 2.5 of Xi and Zhu (2019). In particular, as we mentioned in Remark 4.8, Assumption 3.1(ii) allows to treat SDEs with merely Hölder continuous coefficients. The relaxations make the analyses more involved and subtle than those in the literature. To preserve the flow of reading, we defer the proof of Lemma 4.10 to Appendix A.

**Proof of Theorem 4.4.** Given \( t > 0 \) and \( (x, k) \in \mathbb{R}^d \times \mathbb{S} \). We want to show that \( P(t, (x, k), B \times \{l\}) > 0 \) for all \( l \in \mathbb{S} \) and all \( B \in \mathcal{B}(\mathbb{R}^d) \) with positive Lebesgue measure. Under Assumption 4.3 and from Lemma 4.5, as in the proof of Theorem 4.8 of Xi et al. (2019), we can write

\[
P(t, (x, k), B \times \{l\}) = \delta_{kl} \tilde{P}^{(k)}(t, x, B) + \sum_{m=1}^{\infty} \int \cdots \int \sum_{l_0, l_1, l_2, \ldots, l_m \in \mathbb{S}, l_i \neq k, l_m = l} \tilde{P}^{(l_0)}(t_1, x, dy_1) q_{0} t_1(y_1) \\
\times \tilde{P}^{(l_1)}(t_2 - t_1, y_1, dy_2) \cdots q_{m-1,m}(y_m) \tilde{P}^{(l_m)}(t - t_m, y_m, B) dt_1 dt_2 \cdots dt_m,
\]

where \( \delta_{kl} \) is the Kronecker symbol. From Assumption 4.3 (ii), we know that the set \( \{ y \in \mathbb{R}^d : q_{l_i, l_{i+1}}(y) > 0 \} \) has positive Lebesgue measure. Then it suffices to show that \( \tilde{P}^{(k)}(s, y, B) > 0 \) for all \( k \in \mathbb{S}, s > 0 \) and \( B \in \mathcal{B}(\mathbb{R}^d) \). We calculate

\[
\tilde{P}^{(k)}(s, y, B) = \mathbb{P}\{X^{(k)}_y(s) \in B\} \\
= \mathbb{E}_k \left[ 1_B(X^{(k)}_y(s)) \exp \left( - \int_0^s q_k(X^{(k)}_y(r)) dr \right) \right] \\
\geq \mathbb{E}_k \left[ 1_B(X^{(k)}_y(s)) e^{-M} \right] \\
\geq e^{-M} \mathbb{P}\{X^{(k)}_y(s) \in B\} \\
= e^{-M} P^{(k)}(s, y, B).
\]

From Lemma 4.7, the semigroup associated with the process \( X^{(k)} \) is irreducible and therefore \( P^{(k)}(s, y, B) > 0 \). This completes the proof. \( \square \)

Finally we notice that the following results can be proved in the same manner as in Xi (2004), Xi and Zhu (2019):

**Proposition 4.9.** Suppose Assumptions 1.1, 2.1, and 2.2 hold. In addition, assume there exist constants \( \alpha, \beta > 0 \), a compact subset \( C \subset \mathbb{R}^d \), a compact subset \( N \subset \mathbb{S} \), a measurable function \( f : \mathbb{R}^d \times \mathbb{S} \to [1, \infty) \), and a twice continuously differentiable function \( V : \mathbb{R}^d \times \mathbb{S} \to [0, \infty) \) such that

\[
\mathcal{A} V(x, k) \leq -\alpha f(x, k) + \beta 1_{C \times N}(x, k), \quad \forall (x, k) \in \mathbb{R}^d \times \mathbb{S}.
\]

Then the semigroup \( P_t \) of (1.5) has an invariant probability measure \( \pi \).

**Proposition 4.10.** Suppose Assumptions 1.1, 2.2, 3.1, 4.1, 4.2, and 4.3 hold. If there exists a twice continuously differentiable function \( V : \mathbb{R}^d \times \mathbb{S} \to [0, \infty) \) such that (4.11) holds, then the semigroup \( P_t \) of (1.5) has a unique invariant measure.
5 Examples

Example 5.1. Consider the following SDE

\[ dX(t) = b(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dW(t) + \int_U c(X(t^-), \Lambda(t^-), u)\tilde{N}(dt, du), \]  

(5.1)

\( X(0) = x \in \mathbb{R} \) where \( W \) is a standard 1-dimensional Brownian motion, \( \tilde{N} \) is the associated compensated Poisson random measure on \([0, \infty) \times U\) with intensity \( dt \nu(du) \) in which \( U = \{ u \in \mathbb{R} : 0 < |u| < 1 \} \) and \( \nu(du) := \frac{du}{|u|^2} \). Note that \( \nu \) is a \( \sigma \)-finite measure on \( U \) with \( \nu(U) = \infty \). The component \( \Lambda \) is the continuous-time stochastic process taking values in \( S = \{1, 2, \ldots \} \) generated by \( \bar{Q}(x) = (q_{kl}(x)) \) where

\[ q_{kl}(x) = \begin{cases} \frac{k}{3^{1+k}} \frac{1}{(1+l|x|^2)^2} & \text{if } k \neq l \\ -\sum_{l \neq k} q_{kl}(x) & \text{otherwise.} \end{cases} \]

Furthermore, suppose the coefficients of (5.1) are given by

\[ \sigma(x, k) = x^2 + 1, \quad b(x, k) = -\frac{x}{2k^2}, \quad c(x, k, u) = \frac{\gamma}{k^2}ux, \quad (x, k) \in \mathbb{R} \times S, \]

where \( \gamma = \frac{1}{\sqrt{3}} \).

We make the following observations.

(i) Assumption 1.1 is satisfied. Indeed, one can verify directly that the coefficients of (5.1) satisfy the linear growth condition and Assumption 2.2 of Xi et al. (2019). Therefore thanks to Theorem 2.5 of Xi et al. (2019), (5.1) has a unique strong non-explosive solution. This of course implies Assumption 1.1. In addition, Assumption 4.2 holds.

(ii) It is clear that Assumption 2.1 (i) holds true. Next we verify Assumption 2.2. To this end, we compute

\[ \sum_{l \in S \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| = \sum_{l \in S \setminus \{k\}} \left| \frac{k}{3^{1+k}} \frac{1}{(1+l|x|^2)^2} - \frac{k}{3^{1+k}} \frac{1}{(1+l|y|^2)^2} \right| 
\]

\[ = k \frac{3^k}{3^k} \sum_{l \in S \setminus \{k\}} \frac{1}{3^l} \left| \frac{1}{1+l|x|^2} - \frac{1}{1+l|y|^2} \right| 
\]

\[ \leq \sum_{l \in S} \frac{l}{3^l} \frac{||y|^2 - |x|^2|}{(1+l|x|^2)(1+l|y|^2)} 
\]

\[ = \sum_{l \in S} \frac{l}{3^l} \frac{(|y| + |x|)|y| - |x|}{(1+l|x|^2)(1+l|y|^2)} 
\]

\[ \leq \sum_{l \in S} \frac{l}{3^l} |y - x| = \frac{3}{4} |x - y|, \]

where the last inequality follows from the triangle inequality \( ||y| - |x|| \leq |x - y| \) and the observation that

\[ \frac{|y| + |x|}{(1+l|x|^2)(1+l|y|^2)} \leq \frac{|y|}{1+|y|^2} + \frac{|x|}{1+|x|^2} \leq \frac{|y|}{1+|y|^2} + \frac{|x|}{1+|x|^2} \leq \frac{1}{2} + \frac{1}{2} = 1. \]
(iii) We can further verify that Assumption 3.1 holds. Indeed, since \( a(x, k) = \sigma^2(x, k) = x^\frac{4}{3} + 2x^\frac{2}{3} + 1 \), for each \( R > 0 \), we can take \( \lambda_R = 1 \) and \( \sigma_{\lambda_R}(x, k) = (x^\frac{4}{3} + 2x^\frac{2}{3} + 1)^\frac{3}{2} \) for all \( (x, k) \in \mathbb{R} \times \mathbb{S} \). Then it is straightforward to verify that for all \( x, z \in \mathbb{R} \) with \( |x| \vee |z| \leq R \) and \( k \in \mathbb{S} \)

\[
\|\sigma_{\lambda_R}(x, k) - \sigma_{\lambda_R}(z, k)\|^2 + 2(x - z, b(x, k) - b(z, k)) + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(du)
\leq 2(\frac{2}{3} x^\frac{2}{3} + x^\frac{2}{3}) - \frac{1}{k^2} |x - z|^2 + \frac{1}{k^2} |x - z|^2
\leq 4(R^\frac{2}{3} + 1)|x - z|^2
= 4(R^\frac{2}{3} + 1)|x - z|g(|x - z|),
\]

where \( g(r) = r^{-\frac{1}{3}}. \) Note that the function \( g \) satisfies (3.2). As a result, (5.1) is strong Feller continuous thanks to Theorem 3.3.

(iv) Next we see immediately that Assumptions 4.3 holds and hence (5.1) is irreducible by virtue of Theorem 4.4.

**Example 5.2.** Consider the following SDE

\[
dX(t) = b(X(t), \Lambda(t))dt + \sigma(X(t), \Lambda(t))dW(t) + \int_U c(X(t^-), \Lambda(t^-), u)\tilde{N}(dt, du),
X(0) = x \in \mathbb{R}^2,
\]

where \( W \) is a standard 2-dimensional Brownian motion, \( \tilde{N} \) is the associated compensated Poisson random measure on \([0, \infty) \times U\) with intensity \( dt\nu(du) \) in which \( U = \{u \in \mathbb{R}^2 : 0 < |u| < 1\} \) and \( \nu(du) := \frac{du}{|u|^3 + 3} \) for some \( \delta \in (0, 2) \). The component \( \Lambda \) is the continuous-time stochastic process taking values in \( \mathbb{S} = \{1, 2, \ldots\} \) generated by \( Q(x) = (q_{kl}(x)) \) with \( q_{kl}(x) = \frac{k}{2^{l+k}(1+|x|^2)} \) for \( x \in \mathbb{R}^2 \) and \( k \neq l \in \mathbb{S} \) and \( q_k(x) = -q_{kk}(x) = \sum_{l \neq k} q_{kl}(x) \). The coefficients of (5.2) are given by

\[
\sigma(x, k) = (|x| + 1)I, \quad b(x, k) = -2kx, \quad c(x, k, u) = \gamma \sqrt{k} |u|x
\]

where \( \gamma \) is a positive constant so that \( \gamma^2 \int_U |u|^2 \nu(du) = 2. \)

Detailed calculations as those in Example 5.1 reveal that (5.2) has a unique non-explosive weak solution, which is strong Feller continuous and irreducible. Next we verify that \( V(x, k) := 1 + k|x|^2 \) satisfies (4.11) and hence by Proposition 4.10, (5.2) has a unique invariant measure.

Observe that \( \nabla V(x, k) = 2kx \) and \( \nabla^2 V(x, k) = 2kI \). Then we compute

\[
\mathcal{A} V(x, k) := \frac{1}{2} \text{tr} \left( a(x, k) \nabla^2 V(x, k) \right) + \langle b(x, k), \nabla V(x, k) \rangle + \sum_{l \in \mathbb{S}} q_{kl}(x) [V(x, l) - V(x, k)]
\]

\[
+ \int_U (V(x + c(x, k, u), k) - V(x, k) - \nabla V(x, k), c(x, k, u)) \nu(du)
\leq \frac{1}{2} \text{tr} \left( (|x| + 1)^2 I \right) + \langle -2kx, 2kx \rangle + \sum_{l \in \mathbb{S}} q_{kl}(x)V(x, l)
\]

\[
+ \int_U \left( [1 + k|x| + \gamma \sqrt{k} |u|x^2] - [1 + k|x|^2] - \langle 2kx, \gamma \sqrt{k} |u|x \rangle \right) \nu(du)
\]

\[
= (|x|^2 + 2|x| + 1) - 4k^2|x|^2 + \sum_{l \in \mathbb{S}} \frac{k}{3^{l+k} + l|x|^2[1 + l|x|^2]}
\]
Moreover, for all $x > 0$, we have $1 + 2|x| \leq |x|^2$. Thus it follows that

$$\mathcal{A}V(x, k) \leq \left( -\frac{k|x|^2}{1 + k|x|^2} + \frac{1}{1 + k|x|^2} \right) V(x, k), \quad \forall (x, k) \in \{x \in \mathbb{R}^d : |x| \geq r_0\} \times \mathbb{S}. $$

Moreover, for all $k \in \mathbb{S}$ and $|x| \geq r_0$, we have $\frac{k|x|^2}{1 + k|x|^2} \geq \frac{|x|^2}{1 + |x|^2} \geq \frac{r_0^2}{1 + r_0^2} =: 2\alpha > 0$. Also notice that there exists some $r_1 > 0$ such that for all $|x| \geq r_1$ and $k \in \mathbb{S}$, we have $\frac{1}{1 + k|x|^2} \leq \frac{1}{1 + |x|^2} \leq \alpha$. Consequently it follows that for some sufficiently large $\beta > 1$, we have

$$\mathcal{A}V(x, k) \leq -\alpha V(x, k) + \beta C_{\times N}(x, k), \quad \forall (x, k) \in \mathbb{R}^d \times \mathbb{S},$$

where $C := \{x \in \mathbb{R}^d : |x| \leq r_0 \vee r_1\}$ and $N := \{1\}$. This implies (4.11) and hence a unique invariant probability measure $\pi$ for (5.2) exists.

## A Proofs of Several Technical Results

**Proof of Lemma 2.4.** We will prove the lemma separately for the cases $d = 1$ and $d \geq 2$.

**Case (i):** $d = 1$. Let $\{a_n\}$ be a strictly decreasing sequence of real numbers satisfying $a_0 = 1$, $\lim_{n \to \infty} a_n = 0$, and $\int_{a_n}^{a_{n-1}} \frac{dr}{r} = n$ for each $n \geq 1$. For each $n \geq 1$, let $\rho_n$ be a nonnegative continuous function with support on $(a_n, a_{n-1})$ so that

$$\int_{a_n}^{a_{n-1}} \rho_n(r) dr = 1 \text{ and } \rho_n(r) \leq 2(nr)^{-1} \text{ for all } r > 0.$$

For $x \in \mathbb{R}$, define

$$\psi_n(x) = \int_0^{x} \int_0^{y} \rho_n(z) dz dy. \quad (A.1)$$

We can immediately verify that $\psi_n$ is even and twice continuously differentiable, with

$$\psi'_n(r) = \text{sgn}(r) \int_0^{r} \rho_n(z) dz = \text{sgn}(r) |\psi'_n(r)|, \quad (A.2)$$

and

$$|\psi'_n(r)| \leq 1, \quad 0 \leq |r| \psi''_n(r) = |r| \rho_n(|r|) \leq \frac{2}{n}, \quad \text{and} \quad \lim_{n \to \infty} \psi_n(r) = \frac{2}{r} \quad (A.3)$$
for $r \in \mathbb{R}$. Furthermore, for each $r > 0$, the sequence $\{\psi_n(r)\}_{n \geq 1}$ is nondecreasing. Note also that for each $n \in \mathbb{N}$, $\psi_n$, $\psi'_n$, and $\psi''_n$ all vanish on the interval $(-a_n, a_n)$. Moreover the classical arguments using Assumption 2.1 (i), (A.2) and (A.3) reveal that

$$
\tilde{L}_k \psi_n(x-z) = \frac{1}{2} \psi''_n(x-z)|\sigma(x,k) - \sigma(z,k)|^2 + \psi'_n(x-z)(b(x,k) - b(z,k))
$$

$$
+ \int_U \{\psi_n(x-z + c(x,k,u) - c(z,k,u)) - \psi_n(x-z) - \psi'_n(x-z)(c(x,k,u) - c(z,k,u))\} \nu(du)
$$

$$
\leq K \frac{KR}{n} + \kappa_R \rho(|x-z|),
$$

(A.4)

for all $x, z$ with $|x| \vee |z| \leq R$ and $0 < |x-z| \leq \delta_0$, where $K$ is a positive constant independent of $R$ and $n$. Then it follows that

$$
\mathbb{E}[\psi_n(\Delta t \wedge S_{k_0} \wedge \tau_R \wedge \zeta)] = \mathbb{E}[\psi_n(\tilde{X}(t \wedge S_{k_0} \wedge \tau_R \wedge \zeta)) - X(t \wedge S_{k_0} \wedge \tau_R \wedge \zeta)]
$$

$$
= \psi_n(\tilde{x} - x) + \mathbb{E}\left[\int_0^{t \wedge \tau_R \wedge S_{k_0} \wedge \zeta} \tilde{L}_k \psi_n(\tilde{X}(s) - X(s))ds\right]
$$

$$
\leq \psi_n(|\Delta_0|) + \mathbb{E}\left[\int_0^{t \wedge \tau_R \wedge S_{k_0} \wedge \zeta} \left(\kappa_R \rho(|\Delta_s|) + K \frac{KR}{n} \right)ds\right]
$$

$$
\leq \psi_n(|\Delta_0|) + K \frac{KR}{n} t + \kappa R \int_0^t \rho(\mathbb{E}[|\Delta_s \wedge \tau_R \wedge S_{k_0} \wedge \zeta|])ds,
$$

where the first inequality follows from (A.4) and the second inequality follows from the concavity of $\rho$ and Jensen’s inequality. Then we use the monotone convergence theorem and (A.3) to derive

$$
\mathbb{E}[|\Delta t \wedge \tau_R \wedge S_{k_0} \wedge \zeta|] \leq |\Delta_0| + \kappa R \int_0^t \rho(\mathbb{E}[|\Delta_s \wedge \tau_R \wedge S_{k_0} \wedge \zeta|])ds.
$$

Let $u(t) := \mathbb{E}[|\Delta_t \wedge \tau_R \wedge S_{k_0} \wedge \zeta|]$. Then $u$ satisfies

$$
0 \leq u(t) \leq v(t) := |\Delta_0| + \kappa R \int_0^t \rho(u(s))ds.
$$

Define the function $\Gamma(r) := \int_1^r \frac{ds}{\rho(s)}$ for $r > 0$. Thanks to (2.1), we can verify that $\Gamma$ is nondecreasing and satisfies $\Gamma(r) > -\infty$ for all $r > 0$ and $\lim_{r \to 0} \Gamma(r) = -\infty$. Then we have

$$
\Gamma(u(t)) \leq \Gamma(v(t)) = \Gamma(|\Delta_0|) + \int_0^t \Gamma'(v(s))v'(s)ds = \Gamma(|\Delta_0|) + \kappa R \int_0^t \rho(u(s))\rho(s)ds
$$

$$
\leq \Gamma(|\Delta_0|) + \kappa R \int_0^t 1ds = \Gamma(|\Delta_0|) + \kappa R t,
$$

where we use the assumption that $\rho$ is nondecreasing to obtain the last inequality. Taking the limit $|\Delta_0| = |\tilde{x} - x| \to 0$ we have $\Gamma(u(t)) \to -\infty$ since $\lim_{r \to 0} \Gamma(r) = -\infty$. Moreover, since $\Gamma(r) > -\infty$ for all $r > 0$ we must have $\lim_{|\tilde{x} - x| \to 0} u(t) = 0$. This gives (2.16) as desired.

Case (ii) $d \geq 2$. Consider the function $f(x, z) := |x-z|^2$. Then Assumption 2.1 (ii) implies that

$$
\tilde{L}_k f(x, z) = 2(x-z, b(x,k) - b(z,k)) + |\sigma(x,k) - \sigma(z,k)|^2 + \int_U |c(x,k,u) - c(z,k,u)|^2 \nu(du)
$$
\[ \leq \kappa_R \rho(|x - z|^2), \]
for all \( x, z \in \mathbb{R}^d \) with \(|x| \vee |z| \leq R\) and \(|x - z| \leq \delta_0 \). Consequently
\[ \mathbb{E}[|\Delta_{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta}|^2] = \mathbb{E}[f(\tilde{X}(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta), X(t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta))] \]
\[ = f(\tilde{x}, x) + \mathbb{E} \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta} \mathbb{E}[\mathcal{L}_k f(\tilde{X}(s), X(s))]ds \right] \]
\[ \leq |\Delta_0| + \mathbb{E} \left[ \int_0^{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta} \kappa_R \rho(|\tilde{X}(s) - X(s)|^2)^2ds \right] \]
\[ \leq |\Delta_0| + \kappa_R \int_0^t \rho(\mathbb{E}[|\Delta_{s \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta}|^2])ds, \]
where the last inequality follows from the concavity of \( \rho \) and Jensen’s inequality. Using the same argument as that in Case (i), we can show that \( \lim_{|x - z| \to 0} \mathbb{E}[|\Delta_{t \wedge \tau_R \wedge S_{\delta_0} \wedge \zeta}|^2] = 0 \); which, together with Hölder’s inequality, leads to (2.16).

Combining the two cases completes the proof. \( \square \)

**Proof of Lemma 3.6.** In view of (3.9), it follows from (3.11) that
\[ \hat{\Omega}_d^{(k)} G(|x - z|) \]
\[ = \frac{G''(|x - z|)}{2} A(x, k, z, k) + \frac{G'(|x - z|)}{2|x - z|} [\text{tr} A(x, k, z, k) - A(x, k, z, k) + 2B(x, k, z, k)] \]
\[ \leq \frac{G''(|x - z|)}{4\lambda_R} + \frac{G'(|x - z|)}{2|x - z|} [||\sigma_{\lambda_R}(x, k) - \sigma_{\lambda_R}(z, k)||^2 + 2B(x, k, z, k)] \]
\[ = 2\lambda_R \left( -1 - \frac{\kappa_R}{2\lambda_R} g(|x - z|) F'(|x - z|) \right) \]
\[ + \frac{G'(|x - z|)}{2|x - z|} [||\sigma_{\lambda_R}(x, k) - \sigma_{\lambda_R}(z, k)||^2 + 2B(x, k, z, k)] \]
\[ = -2\lambda_R + \left( -\kappa_R g(|x - z|) + \frac{||\sigma_{\lambda_R}(x, k) - \sigma_{\lambda_R}(z, k)||^2 + 2B(x, k, z, k)}{2|x - z|} \right) G'(|x - z|). \] (A.5)

Since the function \( G \) is concave, we have \( G(r_1) - G(r_0) \leq G'(r_0)(r_1 - r_0) \) for all \( r_0, r_1 \geq 0 \). Take \( r_0 = |x - z| \) and \( r_1 = |x + c(x, k, u) - z - c(z, k, u)| \) to obtain
\[ G(|x + c(x, k, u) - z - c(z, k, u)|) - G(|x - z|) - \frac{G'(|x - z|)}{|x - z|} (x - z, c(x, k, u) - c(z, k, u)) \]
\[ \leq G'(|x - z|) \left( |x + c(x, k, u) - z - c(z, k, u)| - |x - z| - \frac{(x - z, c(x, k, u) - c(z, k, u))}{|x - z|} \right). \]

Furthermore, with \( a := x - z \) and \( b := c(x, k, u) - c(z, k, u) \), we can verify directly that
\[ |a + b| - |a| - \frac{\langle a, b \rangle}{|a|} = -\frac{(|a + b| - |a|)^2 + |b|^2}{2|a|} \leq \frac{|b|^2}{2|a|}. \]
Hence it follows that
\[ G(|x + c(x, k, u) - z - c(z, k, u)|) - G(|x - z|) - \frac{G'(|x - z|)}{|x - z|} (x - z, c(x, k, u) - c(z, k, u)) \]
\[ \leq \left( \frac{|c(x,k,u) - c(z,k,u)|^2}{2|x-z|} \right) G'(|x-z|). \]

Then we have
\[ \hat{\Omega}_j^k G(|x-z|) \leq G'(|x-z|) \int_U \frac{|c(x,k,u) - c(z,k,u)|^2}{2|x-z|} \nu(du). \quad (A.6) \]

From (A.5) and (A.6), we see that
\[
\hat{\mathcal{L}}_k G(|x-z|) = [\hat{\Omega}_d^k + \hat{\Omega}_j^k] G(|x-z|) \\
\leq -2\lambda_R + G'(|x-z|) \left( -\kappa_R g(|x-z|) + \frac{||\sigma_{\lambda,R}(x,i) - \sigma_{\lambda,R}(z,j)||^2 + 2B(x,k,z,k)}{2|x-z|} \right) \\
\quad + \int_U \frac{|c(x,k,u) - c(z,k,u)|^2}{2|x-z|} \nu(du) \\
\leq -2\lambda_R.
\]

The proof is complete. \qed

**Proof of Lemma 4.7.** Let \( T > 0, r > 0 \) and \( x,a \in \mathbb{R}^d \) be arbitrary but fixed. We will show that
\[ P^{(k)}(T,x,B(a;r)) = \mathbb{P}\{|X^{(k)}(T) - a| < r|X^{(k)}(0) = x| > 0, \]
or equivalently, \( \mathbb{P}\{|X^{(k)}(T) - a| \geq r|X^{(k)}(0) = x| < 1 \). Let us choose some \( t_0 \in (0,T) \). For any \( n \in \mathbb{N} \), we set \( X^{(k)}_n(t_0) := X^{(k)}(t_0)1_{\{|X^{(k)}(t_0)| \leq n\}} \). Since \( \lim_{r \to 0} F(r) = 0 \) and \( 0 \leq F \leq 1 \), the bounded convergence implies that
\[ \lim_{n \to \infty} \mathbb{E}[F(|X^{(k)}_n(t_0) - X^{(k)}(t_0)|^2)] = 0. \quad (A.7) \]

For \( t \in [t_0,T] \), define
\[ J^n(t) := \frac{T-t}{T-t_0} X^{(k)}_n(t_0) + \frac{t-t_0}{T-t_0} a, \quad \text{and} \quad h^n(t) := \frac{a - X^{(k)}_n(t_0)}{T-t_0} - b(J^n(t),k). \]

We see that \( J^n(t_0) = X^{(k)}_n(t_0) \) and \( J^n(T) = a \). In addition, \( J^n \) satisfies the following stochastic differential equation
\[ J^n(t) = X^{(k)}_n(t_0) + \int_{t_0}^t b(J^n(s),k)ds + \int_{t_0}^t h^n(s)ds, \quad t \in [t_0,T]. \]

Consider the stochastic differential equation
\[
Y(t) = X^{(k)}(t_0) + \int_{t_0}^t [b(Y(s),k) + h^n(s)]ds + \int_{t_0}^t \sigma(Y(s),k)dW(s) \\
+ \int_{t_0}^t \int_U c(Y(s),k,u)\tilde{N}(ds,du), \quad t \in [t_0,T]. \quad (A.8) \]

Also denote \( \Delta_t := Y(t) - J^n(t) \) for \( t \in [t_0,T] \). Note that \( \Delta_{t_0} = X^{(k)}(t_0) - X^{(k)}_n(t_0) \) and \( \Delta_T = Y(T) - a \). In addition, \( \Delta_t \) satisfies the stochastic differential equation
\[
\Delta_t = \Delta_{t_0} + \int_{t_0}^t [b(Y(s),k) - b(J^n(s),k)]ds + \int_{t_0}^t \sigma(Y(s),k)dW(s) + \int_{t_0}^t \int_U c(Y(s),k,u)\tilde{N}(ds,du). \]
Consequently the generator of the process $\Delta_t$ is given by

$$
L f(x) = L_d f(x) + L_j f(x)
$$

$$
: = \frac{1}{2} \text{tr} \left( \sigma(Y(s), k) \sigma(Y(s), k)^T \nabla^2 f(x) \right) + \langle b(Y(s), k) - b(J^n(s), k), \nabla f(x) \rangle
$$

$$
+ \int_U \left( f(x + c(Y(s), k, u)) - f(x) - \langle \nabla f(x), c(Y(s), k, u) \rangle \nu(du), \quad f \in C^2_c(\mathbb{R}^d) \right).
$$

We compute

$$
L_d F(|\Delta_s|^2) = \frac{1}{2} \text{tr} \left( \sigma(Y(s), k) \sigma(Y(s), k)^T \nabla^2 F(|\Delta_s|^2) \right) + \langle b(Y(s), k) - b(J^n(s), k), \nabla F(|\Delta_s|^2) \rangle
$$

$$
= \frac{1}{2} \text{tr} \left( \sigma(Y(s), k) \sigma(Y(s), k)^T \left[ 4F''(|\Delta_s|^2)\Delta_s \Delta_s^T + 2F'(|\Delta_s|^2)I \right] \right)
$$

$$
+ \langle b(Y(s), k) - b(J^n(s), k), 2F'(|\Delta_s|)\Delta_s \rangle
$$

$$
= 2F''(|\Delta_s|^2)|\Delta_s^T \sigma(Y(s), k)|^2 + F'(|\Delta_s|)|\sigma(Y(s), k)||^2
$$

$$
+ 2F'(|\Delta_s|^2)(b(Y(s), k) - b(J^n(s), k), \Delta_s)
$$

$$
\leq F'(|\Delta_s|) \left( \langle \sigma(Y(s), k)|^2 + 2(b(Y(s), k) - b(J^n(s), k), \Delta_s) \right)
$$

$$
\leq |\sigma(Y(s), k)|^2 + 2(b(Y(s), k) - b(J^n(s), k), \Delta_s),
$$

where the inequalities follow from (4.8) and (4.9). Likewise, the concavity of $F$ leads to

$$
L_j F(|\Delta_s|^2) = \int_U \left( F(|\Delta_s| + c(Y(s), k, u))^2 - F(|\Delta_s|^2) - \langle \nabla F(|\Delta_s|^2), c(Y(s), k, u) \rangle \right) \nu(du)
$$

$$
\leq \int_U \left[ F'(|\Delta_s|^2)||\Delta_s + c(Y(s), k, u)||^2 - |\Delta_s|^2 - 2F'(|\Delta_s|^2)\langle \Delta_s, c(Y(s), k, u) \rangle \nu(du) \right]
$$

$$
= \int_U F'(|\Delta_s|^2)|c(Y(s), k, u)|^2 \nu(du)
$$

$$
\leq \int_U |c(Y(s), k, u)|^2 \nu(du).
$$

Therefore by adding the above two inequalities, we have

$$
\mathcal{L} F(|\Delta_s|^2) \leq |\sigma(Y(s), k)|^2 + 2(b(Y(s), k) - b(J^n(s), k), \Delta_s) + \int_U |c(Y(s), k, u)|^2 \nu(du).
$$

Furthermore, when $|Y(s)| \leq R, |J^n(s)| \leq R$ and $|\Delta_s| \leq \delta_0$, we can use (4.2) and (3.3) to obtain

$$
\mathcal{L} F(|\Delta_s|^2) \leq \kappa(|Y(s)|^2 + 1) + 2\kappa R|\Delta_s|g(|\Delta_s|) \leq K_R + \kappa|Y(s)|^2,
$$

where $K_R = \kappa + 2\kappa R \max_{r \in [0, \delta_0]} \{rg(r)\} < \infty$. In view of (4.2) and (4.5), we can use the standard arguments to show that $E[\sup_{t \leq s \leq t_0} |Y(s)|^2] \leq K$, where $K$ is a positive constant independent of $t_0$. For any $R > 0$, we define $\tau_R := \inf\{t \geq t_0 : |Y(t)| \lor |J^n(t)| > R\} \wedge T$ and $S_\delta_0 := \inf\{t \geq t_0 : |Y(t) - J^n(t)| \geq \delta_0\} \wedge T$. Then we can compute

$$
E[F(|\Delta_{T \wedge \tau_R \wedge S_\delta_0}|^2)] = E[F(|\Delta_{t_0}|^2)] + E\left[ \int_{t_0}^{T \wedge \tau_R \wedge S_\delta_0} \mathcal{L} F(|\Delta_s|^2) ds \right]
$$

$$
\leq E[F(|\Delta_{t_0}|^2)] + E\left[ \int_{t_0}^{T \wedge \tau_R \wedge S_\delta_0} (K_R + \kappa|Y(s^-)|^2) ds \right].
$$
Consequently we have \( \mathbb{E}[F(|\Delta_{t_0}|^2)] \leq \frac{1}{F(\delta_0^2)} \mathbb{E}[F(|\Delta_{T \wedge S_{t_0}}|^2)] \). \hspace{1cm} (A.10)

Indeed, we observe that \(|\Delta_{T \wedge S_{t_0} \wedge \tau_R}| \geq \delta_0\) on the set \(\{S_{t_0} < T \wedge \tau_R\}\). Since \(F\) is increasing, we have \(F(\delta_0^2) \leq F(|\Delta_{T \wedge S_{t_0}}|^2)\). This together with the fact that \(0 \leq F \leq 1\) give the following

\[
\begin{align*}
\frac{\mathbb{E}[F(|\Delta_{T \wedge \tau_R \wedge S_{t_0}}|^2)]}{F(\delta_0^2)} &= \frac{\mathbb{E}[F(|\Delta_{T \wedge \tau_R} \wedge S_{t_0}|^2)1_{\{T \wedge \tau_R \leq S_{t_0}\}}] + \mathbb{E}[F(|\Delta_{T \wedge \tau_R \wedge S_{t_0}}|^2)1_{\{T \wedge \tau_R > S_{t_0}\}}]}{F(\delta_0^2)} - \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)] \\
&\geq \frac{\mathbb{E}[F(|\Delta_{T \wedge \tau_R} \wedge S_{t_0}|^2)1_{\{T \wedge \tau_R \leq S_{t_0}\}}] + \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)]}{F(\delta_0^2)} \\
&\geq \mathbb{P}\{T \wedge \tau_R > S_{t_0}\} + \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)]1_{\{T \wedge \tau_R \leq S_{t_0}\}} - \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)] \\
&\geq \mathbb{P}\{T \wedge \tau_R > S_{t_0}\} - \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)]1_{\{T \wedge \tau_R > S_{t_0}\}} \\
&\geq \mathbb{P}\{T \wedge \tau_R > S_{t_0}\} - \mathbb{E}[\mathbb{1} \cdot 1_{\{T \wedge \tau_R > S_{t_0}\}}] = 0.
\end{align*}
\]

Consequently we have \( \mathbb{E}[F(|\Delta_{T \wedge \tau_R}|^2)] \leq \frac{\mathbb{E}[F(|\Delta_{T \wedge \tau_R \wedge S_{t_0}}|^2)]}{F(\delta_0^2)} \). Since \(\lim_{R \to \infty} \tau_R = \infty\) a.s. and \(0 \leq F \leq 1\), the bounded convergence theorem gives (A.10).

Recall that \(Y\) satisfies the stochastic differential equation (A.8) for \(t \in [t_0, T]\). For \(t \in [0, t_0]\), we define \(Y(t) := X^{(k)}(t)\) and \(X^{(k)}(t)\) is the weak solution to (4.1) with initial condition \(x\). Then the process \(Y\) satisfies the following stochastic differential equation:

\[
Y(t) = x + \int_0^t [b(Y(s), k) + h^\delta(s)1_{\{s > t_0\}}]ds + \int_0^t \sigma(Y(s), k)dW(s) + \int_0^t \int_U c(Y(s), k, u)\tilde{N}(ds, du)
\]

for \(t \in [0, T]\). Next we set

\[
H(t) := 1_{\{t > t_0\}}\sigma^{-1}(Y(t), k)h^\delta(t),
\]

\[
M(t) := \exp \left\{ \int_0^t \langle H(s), dW(s) \rangle - \frac{1}{2} \int_0^t |H(s)|^2 ds \right\}.
\]

As argued in Qiao (2014), it follows from (4.3) that \(|H(t)|^2\) is bounded and hence \(M\) is a martingale under \(\mathbb{P}\) by Novikov’s criteria. Moreover, \(\mathbb{E}[M(T) = 1\}. Define

\[
\mathbb{Q}(B) := \mathbb{E}[M(T)1_{\{B\}}], \quad B \in \mathcal{F}_T
\]

\[
\tilde{W}(t) := W(t) + \int_0^t H(s)ds.
\]

It follows from Theorem 132 of Situ (2005) that \(\mathbb{Q}\) is a probability measure, \(\tilde{W}\) is a \(\mathbb{Q}\)-Brownian motion and \(\tilde{N}(dt, du)\) is a \(\mathbb{Q}\)-compensated Poisson random measure with compensator \(dt\nu(du)\). Furthermore, under the measure \(\mathbb{Q}\), \(Y\) solves the following stochastic differential equation

\[
Y(t) = x + \int_0^t b(Y(s), k)ds + \int_0^t \sigma(Y(s), k)d\tilde{W}(s) + \int_0^t \int_U c(Y(s), k, u)\tilde{N}(ds, du)
\]
for \( t \in [0,T] \). By the uniqueness in law of the solution to the SDE, we have that the law of \( \{X^{(k)}(t) : t \in [0,T]\} \) under \( \mathbb{P} \) is the same as the law of \( \{Y(t) : t \in [0,T]\} \) under \( \mathbb{Q} \). In particular, we have \( \mathbb{P}\{|X^{(k)}(T) - a| \geq r|X^{(k)}(0) = x\} = \mathbb{Q}\{|Y(T) - a| \geq r|Y(0) = x\} \). Since \( \mathbb{P} \) and \( \mathbb{Q} \) are equivalent, the desired assertion \( \mathbb{P}\{|X^{(k)}(T) - a| \geq r|X^{(k)}(0) = x\} = \mathbb{Q}\{|Y(T) - a| \geq r|Y(0) = x\} < 1 \) will follow if we can show that \( \mathbb{P}\{|Y(T) - a| \geq r|Y(0) = x\} < 1 \). To this end, for any \( \varepsilon > 0 \), we first choose an \( R > 0 \) sufficiently large so that \( \mathbb{P}\{\tau_R < T\} < \varepsilon \). Next we use the facts that the function \( F \) is bounded and increasing, (A.10) and (A.9) to compute

\[
\mathbb{P}\{|Y(T) - a| \geq r|Y(0) = x\} = \mathbb{P}\{|Y(T) - a| \geq r^2|Y(0) = x\}
\leq \frac{E[F(|Y(T) - a|^2)]}{F(r^2)} = \frac{E[F(|\Delta_T|^2)]}{F(r^2)}
\leq \frac{E[F(|\Delta_T\wedge S_{\delta_0}|^2)]}{F(r^2)F(\delta_0^2)}
\leq \frac{E[F(|\Delta_{T\wedge S_{\delta_0}}|\wedge T - t_0) + P(\tau_R < T)]}{F(r^2)F(\delta_0^2)}
\leq \frac{E[F(|\Delta_{t_0}|^2)]}{F(r^2)F(\delta_0^2)} + (K_R + K)(T - t_0) + \varepsilon.
\]

Note that by virtue of (A.7), \( E[F(|\Delta_{t_0}|^2)] \to 0 \) as \( n \to \infty \). Therefore we can choose \( n \) sufficiently large and \( t_0 \) close enough to \( T \) to make the last term less than \( 1 \) as desired. \( \square \)

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