From Brans-Dicke theory to Newtonian gravity

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Abstract

We present the new interpretation of scalar field for the Brans-Dicke theory. This interpretation is obtained by considering a fixed spacetime structure of manifold.

Keywords: scalar-tensor theory, Newtonian physics

1 Comparisons with Newtonian gravity

The scalar-tensor theory first time was invented by P. Jordan [1] in the 1950’s, and then taken over by C. Brans and R.H. Dicke [2] some years later. In this paper we restrict our discussion to the Brans-Dicke theory [1, 2] that, among of all the alternative theories of classical Einstein’s gravity, is the most studied and hence the best known theory.

Scalar-tensor theories of gravity describe the universe as grounded on differentiable arbitrary manifold $M^4$ enveloped by a principal bundle formed with isometric representations of a finite continuous Poincaré group. Einstein’s principle of general relativity asserts the invariance under general coordinate transformations of the actions integral grounded on a $M^4$ manifold parameterized by variables $x^\mu, \mu = 0, 1, 2, 3$.

As it is well known field equation and conservation law of the relativity theory can be obtained from principle of least action. The same principle is

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the basis of the Brans - Dicke theory

\[ S = S_G + S_m \]  

\[ S_G = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ \Phi R - \omega \frac{\Phi^{,\mu} \Phi^{,\mu}}{\Phi} \right\} + S_m, \]

where \( R \) is the scalar curvature, \( \phi \) is a scalar field, \( \omega \) is a dimensionless coupling parameter, and \( S_m \) is an action of ordinary matter (not including the scalar field).

An unsatisfactory feature of relativistic theories is that the components of Lagrangian do not have any direct physical interpretation. Note that in abstract manifold, the Ricci scalar and tensor \( g_{\mu\nu} \) lose their geometrical meaning that they had in a spacetime and now can be viewed only as a source for the metric.

The scalar - tensor theories appears as a theory in which the gravity is described simultaneously by two fields the metric tensor and the scalar fields, the latter being an essential part of the geometrical property of spacetime manifesting its presence in all geometrical phenomena, such as curvature, geodesic motion and so on.

On another hand, the Newtonian description of gravitating systems was developed in the 17th century using a scalar potential field and is nowadays a part of most classical mechanics textbooks. It seems reasonable to interpret Newtons absolute space as an absolute Euclidean embedding-space that acts as a container for non-Euclidean geometry. But there may be as well other reasons to contemplate Minkowski space from considerations of scalar gravity.

In a gravity theories, a model of spacetime is usually a pair \((M, T)\) where \( M \) is \( N \)-dimensional manifold with suitable topological and differentiable properties and \( T \) represent a collection of matter fields on \( M \). In approach with fixed spacetime structure, like Newtonian mechanics and special relativity, this model suggest an interpretation of manifold \( M \) as independently existing "container" for the histories of fields and particles. Obviously, the action (2) can be endowed with a structure of a manifold. To obtain the physical interpretation of the scalar field, one may write the action (2) for Minkowski metric in the following manner: \( g_{\mu\nu} = \eta_{\mu\nu}, \ R = 0 \). We need field equations for \( \phi \), so the action (2) for this field must be supplied,

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1Units \( 8\pi G = c = 1 \) are used throughout the paper. Greek indices range over the coordinates of the 4-manifold and Roman indices over the coordinates of the 3-surfaces.
$$S_G = \int dx^4 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi,$$

where \( \phi = e^\Phi \). Without losing generality, suppose that \( \omega = 1 \).

The overall action for the aggregate of \( N \) point particles is

$$S_m = - \sum_{i=1}^{n} \int_{-\infty}^{\infty} dt \frac{1}{2} m_i \dot{q}_i^\mu \dot{q}_i^\nu \eta_{\mu\nu} - \sum_{i=1}^{n} \int_{R^4} dx m_i \delta(x - \mathbf{q}_i) \phi. \quad (4)$$

where \( q_I(t) = \{q_I^\mu(t)\}, I = 1, ..., N \) are the trajectory of point particles with mass \( m_I \).

As in Newtonian mechanics, we can consider arbitrary spacelike section \( t = \text{const} \) given by Euclidean metric. The action (3) provides the following field equations [3]:

$$\frac{\delta S_G}{\delta \phi} = \Delta \phi = 0, \quad (5)$$

where \( \Delta := \partial_1^2 + \partial_2^2 + \partial_3^2 \) is a three dimensional Laplace, the gradient of scalar field \( \partial \phi \) is taken at the point \( q_I = \{t, q_I^\mu\} \), where at the time the particle is located.

The field equations read

$$\frac{\delta S}{\delta \phi} = \frac{1}{4\pi G} \Delta \phi - \sum_i m_i (x - \mathbf{q}_i) = 0, \quad (6)$$

$$\frac{\delta S}{\delta q_i} = m_i (\ddot{q}_i^\mu - \partial_\mu \phi) = 0, \quad (7)$$

Then, with this definition, the equation of gravity field have the form

$$\Delta \phi = \sum_I m_I \delta(x - \mathbf{q}_I). \quad (8)$$

Thus, we see that the scalar field variables of Brans-Dicke theory play the role of a standard Newtonian potential.

The field equation (8) yields the following solution [4]

$$\phi(t, x) = - \sum_I \frac{m_I}{|x - \mathbf{q}_I(t)|}, \quad (9)$$
where
\[ |x - q_I| := \sqrt{-\eta_{\mu\nu}(x^\mu - q_I^\mu)(x^\nu - q_I^\nu)} \] (10)

Now we consider the model with two particle with masses \( m_I \) and \( m_J \). The force \( F = F^\mu \), acting on a particle \( m_I \) by particle \( m_J \), is
\[ F^\mu = m_I m_J \frac{q_I^\mu - q_J^\mu}{|q_I - q_J|^3}. \] (11)

It is Newton’s law of gravitation.

2 Discussion

In order to make a physical prediction from theory a manifold must be endowed with a structure (metric, connection, curvature ...). In contrast with usual formalism of general relativity one can perform a model of spacetime with fixed background geometry, like Newtonian mechanics and special theory of relativity. Then the set up of action implies the imposing the Ricci scalar not as a scalar curvature of spacetime but as a matter source. The standard derivation of the Newtonian gravity as a weak field limit of relativistic theories do not expose the specific feature of Brans-Dicke theory which include Newtons law of gravitation as an exact solution.

References

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