Collective directional emission from distant emitters in waveguide QED

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We study a system of two distant quantum emitters coupled via a one-dimensional waveguide where the electromagnetic field has a direction-dependent velocity. As a consequence, the onset of collective emission is non-simultaneous and, for appropriate parameters, while one of the emitters exhibits superradiance the other can be subradiant. Interference effects enable the system to radiate in a preferential direction depending on the atomic state and the field propagation phases. We characterize such directional emission as a function of various parameters, delineating the conditions for optimal directionality.

Introduction.— Engineering atom-photon interactions by manipulating electromagnetic (EM) fields is a significant aspect of design and implementation of quantum technologies [1, 2]. For instance, reducing the mode volume of the EM field enhances the light-matter coupling [3] and controlling the field polarization allows for chiral interactions between quantum emitters with polarization-dependent transitions [4]. Current platforms allow one to change yet another property of the EM field: its propagation velocity [5–7]. In particular, one can envision the possibility of having an EM field with unequal velocities when propagating to the left or to the right, here referred to as anisotachy [8]. Such feature is, as yet, an unexplored aspect of quantum optical systems, which could be implemented with state-of-the art non-reciprocal components [9, 10]. Since the propagation velocity is an essential ingredient in connecting the distant parts of a larger system, the effects of anisotachy are expected to appear when measuring properties that depend on the interaction between delocalized subsystems, such as quantum correlations.

Quantum correlations among emitters can collectively enhance or inhibit light absorption and emission [11]. For example, a collection of emitters can radiate faster or slower than individual ones depending on their correlations, phenomena known as super- and sub-radiance respectively [12]. These effects have been extensively studied both theoretically [11, 13–17] and experimentally across various platforms [18–30]. Recent works have proposed collective effects for controlling the direction of emission using the non-local correlations between two emitters, with potential applications in quantum information processing and quantum error correction [31–33].

In this Letter we propose a system comprising of two distant quantum emitters or atoms coupled to a one-dimensional waveguide with an effective direction-dependent field velocity, or anisotachy. A direction-dependent time delay can allow the two emitters to exhibit disparate collective effects such that while one atom decays superradiantly, the other exhibits subradiance. In such a system, interference effects in the radiated field lead to a directional emission. We characterize such directional emission as a function of initial states of the emitters, field propagation phases, and the waveguide coupling efficiency. Our results demonstrate that collective directional emission is a rather prevalent quantum optical phenomenon that needs further exploration, to understand its advantages, limitations and dependence on a broader set of parameters.

We first present a theoretical model of the system, describing the disparate cooperative decay dynamics of the emitters and the radiated field intensity in the presence of anisotachy. We then characterize the optimum conditions for directional emission. Finally we discuss the experimental feasibility and give a brief outlook of the phenomena.

Model.— Let us consider two two-level quantum emitters coupled to the EM field modes of a waveguide. Using field circulators, the field modes propagate through different waveguides with unequal index of refraction, as

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shown in Fig. 1, leading to an effective anisotachy. With this possible implementation in mind, we describe the system as a single waveguide with direction-dependent field velocities.

The Hamiltonian for the system is given as \( H = H_E + H_F + H_{EF} \), where \( H_E = \sum_{m=1,2} \hbar \omega_0 \hat{\sigma}_m^+ \hat{\sigma}_m^- \) is the Hamiltonian for the emitters, with \( \{ \hat{\sigma}^+_m, \hat{\sigma}^-_m \} \) as the raising and lowering operators for the \( m \)th atom. \( H_F = \int_0^\infty d\omega \hbar \omega \left[ \hat{\alpha}_L^\dagger(\omega) \hat{\alpha}_R(\omega) + \hat{\alpha}_L^\dagger(\omega) \hat{\alpha}_L(\omega) \right] \), corresponds to the Hamiltonian for the guided modes of the waveguide, with \( \hat{\alpha}_L^\dagger \) and \( \hat{\alpha}_R^\dagger \) as the bosonic operators for the right and left propagating modes, respectively. The interaction Hamiltonian in the interaction picture with respect to the free Hamiltonians \( H_E + H_F \) is [34, 35]

\[
\hat{H}_{EF} = \sum_{m=1,2} \int_0^\infty d\omega \hbar g(\omega) \left[ \hat{\sigma}_m^+(\omega) e^{i k_m x_m} + \hat{\sigma}_m^-(\omega) e^{-i k_m x_m} \right] e^{-i(\omega-\omega_0)t} + \text{H.C.},
\]

where \( x_1 = -x_2 = -d/2 \) denotes the position of the emitters, \( g(\omega) \) represents the atom-field coupling strength and \( k_{LR} = \omega/v_{LR} \) corresponds to the asymmetric left and right wavenumbers. Considering the initial state of the system with the emitters being in the single excitation sub-space and the field in vacuum,

\[
|\Psi(t)\rangle = \left[ \sum_{m=1,2} c_m(t) \hat{\sigma}_m^+ + \int_0^\infty d\omega \left\{ c_R(\omega,t) \hat{\alpha}_R^\dagger(\omega) + c_L(\omega,t) \hat{\alpha}_L^\dagger(\omega) \right\} |gg\rangle \otimes |\{0\}\rangle \right],
\]

one can derive the equations of motion for the atomic coefficients, \( c_1 \) and \( c_2 \), as (see Supplemental Material (SM) for details)

\[
\dot{c}_1(2)(t) = -\frac{\gamma}{2} \left[ c_1(2)(t) \right] - \beta c_2(1)(t - T_{LR}(R)) \Theta(t - T_{LR}(R)) e^{i \omega_0 T_{LR}(R)}.
\]

Here \( \gamma \) is the total spontaneous emission rate, \( \beta = 4\pi |g(\omega_0)|^2 \) is the decay rate into the guided modes, and \( T_{LR}(L) = d/v_{LR}(L) \) is the propagation time of the field traveling from one emitter to the one on the right (left).

**Dynamics.**—Let us consider the initial state of the emitters to be \( |\Psi_0\rangle = \cos \theta e^{i \phi_{A1}} |eg\rangle + \sin \theta e^{i \phi_{A2}} |ge\rangle \). The equations of motion (Eq. (3)) can be solved using a Wigner-Weisskopf approach to obtain (see SM):

\[
c_1(2)(t) = c_1(2)(0) \sum_{n=0}^{\infty} \frac{\left( \beta \gamma e^{i \phi / 2} / 2 \right)^{2n}}{(2n)!} (t - 2nT)^{2n} e^{-\gamma/2(t-2nT)} \Theta(t-2nT) - c_2(1)(0) \sum_{n=0}^{\infty} \frac{\left( \beta \gamma e^{i \phi / 2} / 2 \right)^{2n+1}}{(2n+1)!} e^{i \phi_{LR}(R)-i \phi (t - 2nT - T_{LR}(R))} (t - 2nT - T_{LR}(R))^{2n+1} e^{-\gamma/2(t-2nT-T_{LR}(R))} \Theta(t-2nT-T_{LR}(R)),
\]
where $\phi_{R(L)} = \omega_0 T_{R(L)}$ is the phase acquired by the resonant field upon propagation between the emitters, and $T = (T_R + T_L)/2$ and $\phi = (\phi_R + \phi_L)/2$ are the average propagation time and phase respectively [36]. The first term in the equation above represents the modification of atomic decay after $n$ round trips of the field between the emitters. The second term represents an odd-number of trips $(2n + 1)$ from one emitter to the other, where the directional propagation phase $\langle \phi_{R(L)} \rangle$ determines the interference properties.

Fig. 2 (a) shows the decay of the atomic excitation coefficients as a function of time. Destructive (constructive) interference in the left (right) propagating modes leads to a subradiant atom $A1$ and a superradiant atom $A2$, after the field from one atom reaches the other. One can thus interpret collective decay as a mutually stimulated emission process, as is evident from the series expansion in Eq. (4). For a negligible separation between emitters $T \to 0$, the series converges to yield the standard superradiant exponential decay. In the presence of delay, the resulting dynamics is more precisely described as a cascade of stimulated emission processes [37]. For instance, the field from one emitter can stimulate emission of the other, leading to a non-exponential decay that is faster than superradiance [38–41], or completely suppress its emission, leading to bound states in the continuum (BIC) [42, 43]. More generally, this effect can accelerate the decay of one atom while slowing the decay of the other, as shown in Fig. 2 (a). This demonstrates that the phenomena of super- and subradiance are not a characteristic of the system as a whole, rather an effective description of the local atom-photon interference effects.

The EM field intensity emitted by the system, as a function of position $x$ and time $t$, can be evaluated as

$$I(x,t) = \frac{\beta \sin \phi}{P_{tot}} \left( \frac{\sin \Delta \phi \sin 2\theta + \beta \cos 2\theta \sin \phi}{1 + (\beta \sin \phi)^2} \right),$$

(6)

where $P_{tot} = P_R + P_L$ is the total probability of emitting into the waveguide:

$$P_{tot}(\Psi_0) = \beta \left[ \frac{1 - \beta \cos^2 \phi - (\beta - 1) \cos \Delta \phi \cos \phi \sin 2\theta}{1 - (\beta \cos \phi)^2} \right].$$

(7)

We note from the above that $P_{tot} = \beta$ only for $\phi = (n + \frac{1}{2})\pi$, more generally, the interference in the field enhances or inhibits the effective coupling efficiency between the emitters and the waveguide.

Fig. 3 shows the directionality of photon emission $\chi$ as a function of the parameters of the initial atomic state for the optimum directionality condition $\phi = (n + \frac{1}{2})\pi$, for two particular waveguide coupling efficiencies $(\beta = 1$ and $0.01)$. Considering two orthogonal entangled atomic states $|\Psi_{a(b)}\rangle = \frac{1}{\sqrt{2}} (|eg\rangle + e^{i\phi_{a(b)}} |ge\rangle)$ with $\varphi_a - \varphi_b = \pi$, we obtain a direction parameter value $\chi_{a(b)} = \sin (\phi_R - \phi_L - \varphi_{a(b)}) / (1 + \beta^2)$. It can be thus seen that appropriately manipulating the relative field propagation phase $(\phi_R - \phi_L)$ can allow one to distinguish any two orthogonal entangled states based on the direction of emission, as illustrated by the points $\chi_+$ and $\chi_-$ on Fig. 3.
directionality parameter $\chi$ for waveguide coupling efficiencies. We note that the directional emission from an entangled state benefits from $C$, where $C$ is the concurrence that characterizes the entanglement of the emitters. Such directional emission and field propagation phase (Fig. 2(b)). Such directional emission is a rather general feature of collective delocalized systems (Eq. (6)). We analyze the directionality of emission as a function of various parameters, characterizing the optimal conditions for directional emission (Fig. 3).

The proposed system can be implemented with field circulators, that are readily available for fiber optics and an active element of research in superconducting circuits [48, 49] and integrated photonics [50, 51]. These can be integrated into state-of-the-art waveguide QED platforms [52]. Our results suggest that such directional emission can also be observed for waveguides with low coupling efficiencies. We further remark that an analog of the phenomena described here can also be observed in classical systems [53]. Nonetheless, in the proposed model, the directionality of the coupling aids the detection of entanglement (Eq. (8)) and helps distinguish between the symmetric and asymmetric entangled states of two emitters (Fig. 3).

While on the one hand our results show that directional emission could be used for state tomography and measuring entanglement, on the other hand one can use directional driving fields to prepare particular entangled states of the emitters. Such directional emission and state preparation protocols can allow for efficient and controllable routing of quantum information in quantum networks [31, 32, 54, 55].

When preparing the emitters in an initial state with
directional parameter zero, any perturbation away from their initial positions or internal state can break the symmetry towards an emission in a specific direction, suggesting directionality as a resource for sensing applications. We quantify the optimum estimation of an atomic or field phase using the Fisher information and show that directional emission offers a metrological advantage compared to ignoring directional effects.

The phenomena described in this work can be extended to study directional emission from collective many-body quantum states. Such states can be particularly advantageous in sensing and metrological applications. Additionally, for strongly driven systems, the effects of atomic nonlinearity become relevant. It has been shown, for example, that nonlinearity can assist in directional emission. It would therefore be pertinent to analyze and optimize the directionality over a broader set of parameters including general atomic states, field propagation phases in nonlinear systems and anisotachy.

Anisotachy in waveguide QED platforms could offer new ways to manipulate light-matter interactions. In particular, we show here that it can be used to couple delocalized correlated state of two emitters to a specific direction of collective radiation. This effect expands the toolbox for quantum optics applications while enriching our understanding of waveguide QED systems.

Acknowledgments. — We are grateful to Pierre Meystre and Alejandro González-Tudela for insightful comments on the manuscript. This work was supported in part by CONICYT-PAI grant 77190033, FONDECYT grant N° 11200192 from Chile, and grant No. UNAM-DGAPA-PAPIIT IG101421 from Mexico.

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Here $\hat{E}(x,t) = \int_0^\infty dk \mathcal{E}_k [\hat{a}_L(k) e^{-ikLx} + \hat{a}_R(k) e^{ikRx}] e^{-i\omega t}$ is the electric field operator, and we have assumed $\mathcal{E}_k \approx \mathcal{E}_{k_0}$ to be constant near the atomic resonance frequency.

Here $\varphi$ could refer to either the atomic phases $\{\phi_{A1}, \phi_{A2}\}$, the field propagation phases $\{\phi_R, \phi_L\}$, or the relative phases $\{\Delta \phi, \phi\}$.

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Supplemental Material

I. DERIVATION OF THE EQUATIONS OF MOTION

We can write the coupled emitter-field equations of motion from Eqs.(2) and (3) in the main text as:

\[ \dot{c}_R (\omega, t) = -i \sum_{m=1,2} c_m (t) \left(g (\omega)\right)^* e^{-i\omega x_m / v_R} e^{i(\omega - \omega_0) t}, \]  
\[ \dot{c}_L (\omega, t) = -i \sum_{m=1,2} c_m (t) \left(g (\omega)\right)^* e^{i\omega x_m / v_L} e^{i(\omega - \omega_0) t}, \]  
\[ \dot{c}_m (t) = -i \int_0^\infty d\omega \ g (\omega) e^{-i(\omega - \omega_0) t} \left[ c_R (\omega, t) e^{i\omega x_m / v_R} + c_L (\omega, t) e^{-i\omega x_m / v_L} \right]. \]  

Formal integration of (S1) and (S2) yields

\[ c_R (\omega, t) = -i \int_0^t d\tau \sum_{m=1,2} g^* (\omega) c_m (\tau) e^{-i\omega x_m / v_R} e^{i(\omega - \omega_0) \tau}, \]  
\[ c_L (\omega, t) = -i \int_0^t d\tau \sum_{m=1,2} g^* (\omega) c_m (\tau) e^{i\omega x_m / v_L} e^{i(\omega - \omega_0) \tau}. \]

Substituting the above in Eq.(S3), we can rewrite the atomic equation as follows

\[ \dot{c}_m (t) = - \int_0^\infty d\omega \ |g (\omega)|^2 \int_0^t d\tau \sum_{n=1,2} c_n (\tau) e^{-i(\omega - \omega_0) (t-\tau)} \left[ e^{i\omega (x_m - x_n) / v_R} + e^{-i\omega (x_m - x_n) / v_L} \right]. \]

We now define the field correlation function \( F (s) = \int_0^\infty d\omega \ |g (\omega)|^2 e^{-i(\omega - \omega_0) s} \), to obtain

\[ \dot{c}_m (t) = - \int_0^t d\tau \left[ 2c_m (\tau) F (t-\tau) + c_n (\tau) \left\{ e^{i\omega_0 T_R^{mn}} F (t-\tau - T_R^{mn}) + e^{-i\omega_0 T_L^{mn}} F (t-\tau + T_L^{mn}) \right\} \right], \]

where \( T_R^{mn} = (x_m - x_n) / v_R, T_L^{mn} \) is the direction dependent delay time for the light propagating between the emitters.

In the standard Markov approximation \( F (\tau) = 2\pi |g (\omega_0)|^2 \delta (\tau) \), though more generally \( F (\tau) \) is a narrow distribution symmetric around \( s = 0 \). We assume that the temporal width \( \sigma \) of such distribution is narrower than the delay time between the emitters (\( \sigma < |T_L^{mn} - T_R^{mn}|) \),

\[ \dot{c}_m (t) = -2 \int_0^t d\tau \ c_m (t-\tau) F (\tau) - e^{i\omega_0 T_R^{mn}} \int_{-T_R^{mn}}^{T_R^{mn}} d\tau \ c_n (t-\tau - T_R^{mn}) F (\tau), \]

where \( T^{12} = T_L^{12} = T_L \) or \( T^{21} = T_R^{21} = T_R \). If \( \sigma \) is small enough we can assume that the amplitude of the coefficients does not vary significantly over the region where \( F (\tau) \) is non-zero, such that \( c_m (t-\tau) \approx c_m (t) \). Thus given that \( F (\tau) \) is symmetric, centered around \( \tau = 0 \) and narrower than \( T_R^{mn} \) we have

\[ \dot{c}_m (t) \approx -2c_m (t) \int_0^\infty ds \ F (s) - c_n (t - T_R^{mn}) \Theta (t - T_R^{mn}) e^{i\omega_0 T_R^{mn}} \int_{-\infty}^\infty d\tau \ F (\tau). \]

The term \( F (\tau) \) is a complex function with the real and imaginary part being even and odd functions respectively. We define

\[ \frac{\gamma}{2} = 2 \text{Re} \left[ \int_0^\infty d\tau \ F (\tau) \right] = \text{Re} \left[ \int_{-\infty}^\infty d\tau \ F (\tau) \right], \]
\[ \Delta_L = \text{Im} \left[ \int_0^\infty d\tau \ F (\tau) \right], \]

where \( \Delta_L \) is the Lamb shift, which we include as a part of the emitters renormalized resonance frequency \( \omega_0 \). Introducing a phenomenological cross-coupling efficiency \( \beta \) between the emitters (\( 0 \leq \beta \leq 1 \)), one can simplify Eq. (S9) to obtain the atomic equations of motion (Eq. (4)) in the main text.
II. ATOMIC DYNAMICS

A. Lambert W-function solution

Taking the Laplace transform of Eq.(4) in the main text, one gets

\[ s \tilde{c}_1 (s) - c_1 (0) = -\frac{\gamma}{2} \left[ \tilde{c}_1 (s) + \beta \tilde{c}_2 (s) e^{-sT_L e^{i\phi_L}} \right], \quad (S12) \]
\[ s \tilde{c}_2 (s) - c_2 (0) = -\frac{\gamma}{2} \left[ \tilde{c}_2 (s) + \beta \tilde{c}_1 (s) e^{-sT_R e^{i\phi_R}} \right], \quad (S13) \]

which can be solved to obtain the Laplace coefficients pertaining to the two emitters as follows

\[ \tilde{c}_1 (s) = \frac{c_1 (0) (s + \frac{\gamma}{2}) - c_2 (0) \beta \frac{T_L}{2} e^{sT_L e^{i\phi_L}}}{(s + \frac{\gamma}{2})^2 - \beta \frac{T_L}{2} e^{sT_L e^{i\phi_L}}^2}, \quad (S14) \]
\[ \tilde{c}_2 (s) = \frac{c_2 (0) (s + \frac{\gamma}{2}) - c_1 (0) \beta \frac{T_R}{2} e^{sT_R e^{i\phi_R}}}{(s + \frac{\gamma}{2})^2 - \beta \frac{T_R}{2} e^{sT_R e^{i\phi_R}}^2}. \quad (S15) \]

The poles of the above Laplace coefficients are given by

\[ s_n^\pm = -\frac{\gamma}{2} + \frac{1}{T} W_n \left[ \mp \beta \frac{T}{2} e^{\gamma T/2} e^{i\phi} \right], \quad (S16) \]

where \( W_n \) is the \( n \)th branch of the Lambert W-function [S61].

We can thus rewrite the Eq. (S14) and (S15) as

\[ \tilde{c}_m (s) = \sum_{\pm} \sum_{n=-\infty}^{\infty} \alpha_{n,m}^\pm \frac{s - s_n^\pm}{s}, \quad (S17) \]

where the coefficients \( \alpha_{n,m}^\pm \) are obtained as

\[ \alpha_{n,m}^\pm = \lim_{s \to s_n^\pm} \tilde{c}_m (s) \left( s - s_n^\pm \right). \quad (S18) \]

Thus taking the inverse Laplace transform of Eq. (S17), we get

\[ c_m (t) = \sum_{\sigma = \pm} \sum_{n=-\infty}^{\infty} \alpha_{n,m}^\sigma e^{-\gamma_n^\sigma t}, \quad (S19) \]

where

\[ \gamma_n^\pm = \frac{\gamma}{2} - \frac{1}{T} W_n \left[ \mp \frac{\beta T}{2} e^{\gamma T/2} e^{i\phi} \right], \quad (S20) \]
\[ \alpha_{n,1}^\pm = \frac{1}{2} \frac{c_1 (0) \pm c_2 (0) e^{i(\phi_L - \phi_R)/2} e^{(T_L - T_R) \gamma_n^\pm/2}}{1 + W_n \left( \mp \frac{\beta T}{2} e^{\gamma T/2} e^{i\phi} \right)}, \quad (S21) \]
\[ \alpha_{n,2}^\pm = \frac{1}{2} \frac{c_2 (0) \pm c_1 (0) e^{-i(\phi_L - \phi_R)/2} e^{-(T_L - T_R) \gamma_n^\pm/2}}{1 + W_n \left( \mp \frac{\beta T}{2} e^{\gamma T/2} e^{i\phi} \right)}. \quad (S22) \]

B. Series expansion solution

An alternative way of expressing the atomic excitation amplitudes as the inverse Laplace transform of Eq. (S14) and (S15) in terms of a series solution is as follows [S37]:

\[ \tilde{c}_m (s) = \sum_{\pm} \sum_{n=-\infty}^{\infty} \alpha_{n,m}^\pm \frac{s - s_n^\pm}{s}, \quad (S17) \]
\[ c_1 (t) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-i\infty + \epsilon}^{+i\infty - \epsilon} ds \left[ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right] \frac{(s + \gamma/2)^2}{(s + \gamma/2)^2 + \frac{\gamma}{\pi}^2 \left( \sum_{n=0}^{\infty} \frac{(s + \gamma/2)^2}{(s + \gamma/2)^2 + \frac{\gamma}{\pi}^2} \right)^{2n}} \right], \quad (S23) \]

\[ = c_1 (0) \left( \frac{1}{2\pi i} \int ds \left[ \frac{1}{s + \gamma/2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right] \right), \quad (Ia) \]

\[ - c_2 (0) \left( \frac{1}{2\pi i} \int ds \left[ \frac{1}{s + \gamma/2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right] \right), \quad (S24) \]

\[ c_2 (t) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-i\infty + \epsilon}^{+i\infty - \epsilon} ds \left[ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right] \frac{(s + \gamma/2)^2}{(s + \gamma/2)^2 + \frac{\gamma}{\pi}^2 \left( \sum_{n=0}^{\infty} \frac{(s + \gamma/2)^2}{(s + \gamma/2)^2 + \frac{\gamma}{\pi}^2} \right)^{2n}} \right], \quad (S25) \]

\[ = c_2 (0) \left( \frac{1}{2\pi i} \int ds \left[ \frac{1}{s + \gamma/2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right] \right), \quad (Ib) \]

\[ - c_1 (0) \left( \frac{1}{2\pi i} \int ds \left[ \frac{1}{s + \gamma/2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right] \right). \quad (S26) \]

We identify the terms (Ia) = (Ib) \equiv (I) as corresponding to the round trip times (even number) of the field between the atoms and the terms (IIa) and (IIb) (not necessarily equal to each other) as the terms coming from odd number of trips between the atoms. Simplifying each of the above terms:

\[ (I) = \frac{1}{2\pi i} \int ds \left[ \frac{1}{s + \gamma/2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right], \quad (S27) \]

\[ = \sum_{n} \left( \frac{\beta \gamma e^{i\phi/2}}{(2n)!} \right)^{2n} (t - 2nT)^{2n} e^{-\gamma/2(2nT)} \Theta (t - 2nT), \quad (S28) \]

\[ (IIa) = \frac{1}{2\pi i} \int ds \left[ \frac{\beta \gamma e^{-sT e^{i\phi}L}}{2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right], \quad (S29) \]

\[ = \sum_{n} \left( \frac{\beta \gamma/2}{(2n + 1)!} \right)^{2n+1} e^{2n\phi + i\phi L} (t - 2nT - T_L)^{2n+1} e^{-\gamma/2(2nT + T_L)} \Theta (t - 2nT - T_L), \quad (S30) \]

\[ (IIb) = \frac{1}{2\pi i} \int ds \left[ \frac{\beta \gamma e^{-sT e^{i\phi}R}}{2} \left\{ \sum_{n=0}^{\infty} \left( \frac{\beta \gamma}{2e^{-sT e^{i\phi}}} \right)^{2n} \right\} \right], \quad (S31) \]

\[ = \sum_{n} \left( \frac{\beta \gamma/2}{(2n + 1)!} \right)^{2n+1} e^{2n\phi + i\phi R} (t - 2nT - T_R)^{2n+1} e^{-\gamma/2(2nT + T_R)} \Theta (t - 2nT - T_R). \quad (S32) \]

We substitute the above in Eqs. (S24) and (S24) to obtain the dynamics of general initial states given by Eq. (5) in the main text.

### III. INTENSITY DYNAMICS

The intensity of the field emitted by the atoms as a function of position and time can be evaluated as \[ I (x, t) = \frac{e^{\frac{\omega}{2}}}{\pi} \left| \left\langle \Psi (t) | \hat{E}^\dagger (x, t) \hat{E} (x, t) | \Psi (t) \right\rangle \right|^2, \] where \[ \hat{E} (x, t) = \int_0^\infty d\omega \xi_\omega [\hat{a}_L (\omega) e^{-iklx} + \hat{a}_R (\omega) e^{ikrx}] e^{-i\omega t} \] is the electric field
operator at position $x$ and time $t$. More explicitly, we obtain

$$I(x,t) = \frac{\epsilon_0 c |E_0|^2}{2} \langle \Psi(t) \rangle \left[ \int d\omega_1 \left\{ \hat{a}_L^\dagger(\omega_1) e^{i k_1 L x} + \hat{a}_R(\omega_1) e^{-i k_1 R x} \right\} e^{i \omega_1 t} \right. $$

$$\left. \int d\omega_2 \left\{ \hat{a}_L(\omega_2) e^{-i k_2 L x} + \hat{a}_R(\omega_2) e^{i k_2 R x} \right\} e^{-i \omega_2 t} \right] |\Psi(t)\rangle, \tag{S33}$$

$$= \frac{\epsilon_0 c |E_0|^2}{2} \langle \Psi(t) \rangle \int d\omega_1 \int d\omega_2 \left[ e^{i(k_1 L - k_2 L) x} \epsilon^*_L(\omega_1, t) \epsilon_L(\omega_2, t) + e^{-i(k_1 R - k_2 R) x} \epsilon_L(\omega_1, t) \epsilon_R(\omega_2, t) \right. $$

$$\left. + e^{-i(k_1 L + k_2 L) x} \epsilon^*_R(\omega_1, t) \epsilon_L(\omega_2, t) + e^{i(k_1 L + k_2 L) x} \epsilon^*_L(\omega_1, t) \epsilon_R(\omega_2, t) \right] e^{i(\omega_1 - \omega_2) t}, \tag{S34}$$

$$= \frac{\epsilon_0 c |E_0|^2}{2} \langle \Psi(t) \rangle \int d\omega_1 \left[ c_L(\omega, t) e^{-i \omega x/v_L} + c_R(\omega, t) e^{i \omega x/v_R} \right] e^{-i \omega t} \tag{S35}$$

$$= \frac{\epsilon_0 c |E_0|^2}{4\pi} \gamma \beta \int d\omega e^{-i \omega t} \left[ \int_0^t \frac{d\tau}{v_L} \left\{ c_1(\tau) e^{i \omega(x-x_1)/v_L} + c_2(\tau) e^{i \omega(x-x_2)/v_L} \right\} e^{i(\omega-\omega_0) \tau} \right]^2, \tag{S36}$$

where we have used Eqs. (S1) and (S2) to substitute the field amplitudes in terms of the atomic excitation amplitudes. Using the W-function solution for the atomic coefficients (Eq. (S19)) and performing the integrals over time and frequency, we obtain

$$I/I_0 = \sum_{\sigma = \pm, n = \pm \infty} \left[ \sum_{\sigma = \pm, n = \pm \infty} c_{n,1}^\sigma e^{-i(\omega_n - i \gamma_n)(t+(x-x_1)/v_L)} \left\{ \Theta [t + (x - x_1)/v_L] - \Theta [(x - x_1)/v_L] \right\} $$

$$+ c_{n,2}^\sigma e^{-i(\omega_n - i \gamma_n)(t+(x-x_2)/v_L)} \left\{ \Theta [t + (x - x_2)/v_L] - \Theta [(x - x_2)/v_L] \right\} $$

$$+ c_{n,1,1} e^{-i(\omega_n - i \gamma_n)(t-(x-x_1)/v_R)} \left\{ \Theta [t - (x - x_1)/v_R] - \Theta [- (x - x_1)/v_R] \right\} $$

$$+ c_{n,2,1} e^{-i(\omega_n - i \gamma_n)(t-(x-x_2)/v_R)} \left\{ \Theta [t - (x - x_2)/v_R] - \Theta [- (x - x_2)/v_R] \right\} \right] \tag{S37}.$$}

We can rewrite the above expression of the atomic excitation amplitudes using Eq. (S19) as

$$I/I_0 = \left[ c_1( t + (x + d/2)/v_L) e^{-i \omega_0(x+d/2)/v_L} \left\{ \Theta [t + (x + d/2)/v_L] - \Theta [(x + d/2)/v_L] \right\} $$

$$+ c_2( t + (x - d/2)/v_L) e^{-i \omega_0(x-d/2)/v_L} \left\{ \Theta [t + (x - d/2)/v_L] - \Theta [(x - d/2)/v_L] \right\} $$

$$+ c_1( t - (x + d/2)/v_R) e^{i \omega_0(x+d/2)/v_R} \left\{ \Theta [t - (x + d/2)/v_R] - \Theta [(x + d/2)/v_R] \right\} $$

$$+ c_2( t - (x - d/2)/v_R) e^{i \omega_0(x-d/2)/v_R} \left\{ \Theta [t - (x - d/2)/v_R] - \Theta [(x - d/2)/v_R] \right\} \right]^2, \tag{S38}$$

which corresponds to Eq.(6) in the main text.

**IV. DIRECTIONAL EMISSION**

Let us consider the dynamics for the atomic coefficients given by Eq. (5) in the main text. In the limit $\gamma T \ll 1$, neglecting the delay but keeping the propagation phases, we obtain:

$$c_1(t) = \left[ c_1(0) \sum_{n=0}^{\infty} \frac{\beta^2 t e^{i \phi}}{2n!} \sum_{n=0}^{\infty} \frac{\beta^2 t e^{i \phi}}{(2n+1)!} \right] e^{-\frac{1}{2} \Theta(t)} \tag{S39}$$

$$c_2(t) = \left[ c_2(0) \sum_{n=0}^{\infty} \frac{\beta^2 t e^{i \phi}}{2n!} \right] - c_1(0) \left[ e^{i (\phi_L - \phi)} \sum_{n=0}^{\infty} \frac{\beta^2 t e^{i \phi}}{(2n+1)!} \right] e^{-\frac{1}{2} \Theta(t)} \tag{S40}.$$}

These series converges to

$$c_1(t) = \left[ c_1(0) \cosh \left\{ \frac{\beta^2 t e^{i \phi}}{2} \right\} - c_2(0) \left[ e^{i (\phi_L - \phi)} \sinh \left\{ \frac{\beta^2 t e^{i \phi}}{2} \right\} \right] \right] e^{-\frac{1}{2} \Theta(t)} \tag{S41}$$

$$c_2(t) = \left[ c_2(0) \cosh \left\{ \frac{\beta^2 t e^{i \phi}}{2} \right\} - c_1(0) \left[ e^{i (\phi_R - \phi)} \sinh \left\{ \frac{\beta^2 t e^{i \phi}}{2} \right\} \right] \right] e^{-\frac{1}{2} \Theta(t)} \tag{S42}.$$
We now consider the field coefficients in the steady state \( (t \to \infty) \), which can be simplified to:

\[
c_R(\omega, t \to \infty) = -ig^*(\omega)e^{-i\frac{2P}{\hbar}} \left[ (c_1(0) e^{i\phi_R} + c_2(0)) \frac{(\frac{\gamma}{2} - i\Delta)}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} - (c_2(0) e^{2i\phi} + c_1(0) e^{i\phi_R}) \frac{\beta \frac{\gamma}{2}}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right],
\]

\[\text{(S43)}\]

\[
c_L(\omega, t \to \infty) = -ig^*(\omega)e^{-i\frac{2P}{\hbar}} \left[ (c_1(0) + c_2(0) e^{i\phi_L}) \frac{(\frac{\gamma}{2} - i\Delta)}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} - (c_2(0) e^{2i\phi} + c_1(0) e^{i\phi_L}) \frac{\beta \frac{\gamma}{2}}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right].
\]

\[\text{(S44)}\]

The probability of emitting the photon to the right (left) is thus given by

\[
P_{R(L)} = \int_0^\infty d\omega |c_{R(L)}(\omega, t \to \infty)|^2.
\]

\[\text{(S45)}\]

We parametrize the initial atomic coefficients as \( c_1(0) = e^{i\phi_1} \cos \theta \) and \( c_2(0) = e^{i\phi_2} \sin \theta \), to obtain the right and left emission probabilities as follows:

\[
P_R = |g(\omega_0)|^2 \int_0^\infty d\omega \left| (\sin \theta + \cos \theta e^{i\Delta \phi} e^{i\phi}) \frac{(\frac{\gamma}{2} - i\Delta)}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} - (\sin \theta + \cos \theta e^{i\Delta \phi} e^{-i\phi}) e^{2i\phi} \frac{\beta \frac{\gamma}{2}}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2,
\]

\[\text{(S46)}\]

\[
P_L = |g(\omega_0)|^2 \int_0^\infty d\omega \left| (\sin \theta + \cos \theta e^{i\Delta \phi} e^{-i\phi}) \frac{(\frac{\gamma}{2} - i\Delta)}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} - (\sin \theta + \cos \theta e^{i\Delta \phi} e^{i\phi}) \frac{\beta \frac{\gamma}{2}}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2.
\]

\[\text{(S47)}\]

We consider the integral:

\[
I_0 = \int_0^\infty d\omega \left| \frac{T_1(\frac{\gamma}{2} - i\Delta) - T_2 \beta \frac{\gamma}{2}}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2,
\]

\[\text{(S48)}\]

\[
= |T_1|^2 \int_0^\infty d\omega \left| \frac{\Delta^2}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2 - 2\text{Im}[T_1 T_2^*] \beta \frac{\gamma}{2} \int_0^\infty d\omega \left| \frac{\Delta}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2
\]

\[+ |T_1 - T_2 \beta|^2 \left( \frac{\gamma}{2} \right)^2 \int_0^\infty d\omega \left| \frac{1}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2,
\]

\[\equiv |T_1|^2 I_3 - 2\text{Im}[T_1 T_2^*] \beta \left( \frac{\gamma}{2} \right) I_2 + |T_1 - T_2 \beta|^2 \left( \frac{\gamma}{2} \right)^2 I_1.
\]

\[\text{(S49)}\]

where we can simplify the integrals \( I_1, I_2 \) and \( I_3 \) as follows:

\[
I_1 = \int_0^\infty d\omega \left| \frac{1}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2 = \frac{\pi}{2} \frac{1}{(\frac{\gamma}{2})^3} \left( 1 - (\beta \cos \phi)^2 \right) \left( 1 + (\beta \sin \phi)^2 \right)
\]

\[\text{(S50)}\]

\[
I_2 = \int_0^\infty d\omega \left| \frac{\Delta}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2 = -\frac{\pi}{4} \frac{1}{(\frac{\gamma}{2})^3} \left( 1 - (\beta \cos \phi)^2 \right) \left( 1 + (\beta \sin \phi)^2 \right)
\]

\[\text{(S51)}\]

\[
I_3 = \int_0^\infty d\omega \left| \frac{\Delta^2}{(\frac{\gamma}{2} - i\Delta)^2 - (\beta \frac{\gamma}{2} e^{i\phi})^2} \right|^2 = \frac{\pi}{2} \frac{1}{(\frac{\gamma}{2})^3} \left( 1 - (\beta \cos \phi)^2 \right) \left( 1 + (\beta \sin \phi)^2 \right)
\]

\[\text{(S52)}\]
Substituting the above in Eq. (S49), we get

$$I_0 = \frac{\pi |T_1|^2 [1 - \beta^2 \cos 2\phi] + \text{Im} [T_1 T_2^* \beta^3 \sin 2\phi + |T_1 - T_2\beta|^2]}{(1 - (\beta \cos \phi)^2) (1 + (\beta \sin \phi)^2)}. \quad (S53)$$

Plugging this result back into Eq. (S47) and (S46) and considering $4\pi |g(\omega_0)|^2 = \beta \gamma$ yields

$$P_R = \frac{1}{2} \frac{\beta}{1 + (\beta \sin \phi)^2} \left[ \frac{(1 - \beta + (\beta \sin \phi)^2) (1 - \beta \cos \Delta \phi \cos \phi \sin \theta)}{1 - (\beta \cos \phi)^2} + \cos (\Delta \phi + \phi) \sin 2\theta + 2\beta \sin \theta \sin^2 \phi \right], \quad (S54)$$

$$P_L = \frac{1}{2} \frac{\beta}{1 + (\beta \sin \phi)^2} \left[ \frac{(1 - \beta + (\beta \sin \phi)^2) (1 - \beta \cos \Delta \phi \cos \phi \sin \theta)}{1 - (\beta \cos \phi)^2} + \cos (\Delta \phi - \phi) \sin 2\theta + 2\beta \cos^2 \theta \sin^2 \phi \right]. \quad (S55)$$

which depends on the four parameters: $\beta, \theta, \phi, \Delta \phi$, as discussed in the main text. We use the above equations to obtain the total probability of emitting into the waveguide $P_{\text{tot}}$ (Eq.(8)) and the directionality parameter $\chi$ (Eq.(9)) in the main text.