A new criterion for macroscopic quantum states

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There have recently been a number of proposals for measures to describe the extent to which the quantum behaviour of a system extends to the macroscopic scale. We argue that measures for systems of qubits should be extended to classify a larger set of states (including two-dimensional cluster states and certain topological states) as macroscopically quantum. This is motivated by the ability to use local measurements to distil a Schrödinger’s cat state from states which are not macroscopically quantum according to current measures. We also investigate the role played by imperfect measurements.

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Despite the overwhelming successes of quantum mechanics, one of the greatest problems it still faces is to explain why it appears to break down at the everyday macroscopic scale. Of the variety of quantum effects that are not observed in the classical world, the most striking is that macroscopic objects are never seen in superpositions of different states. The well-known thought experiment of Schrödinger’s cat highlights the absurdity of allowing a cat to exist in a superposition of alive and dead states, and yet in principle this should be possible according to the laws of quantum theory. Interpretational issues aside, it is therefore of great importance to attempt to create macroscopic quantum states in experiments, in order to probe the boundary between quantum and classical mechanics – to decide if a fundamental size limit exists, or if the challenge is purely a matter of isolating a system from its noisy environment.

Some recent experiments along these lines have attempted to create ‘cat states’ (for example in photonic systems$^1$), or similar macroscopic superpositions in systems such as superconducting circuits or molecular interferometers$^2$. While the size of a true cat state is easy to define (simply the number of particles in the state), one needs a more general measure of ‘quantum macroscopicity’ in order to compare experiments in which qualitatively different states are produced. Apart from helping to determine the extent to which a given state is macroscopically quantum, having such a measure would likely enable us to identify the relevant properties of a quantum state that are involved in the transition to classical behaviour in the real world.

In this work, we make no reference to experimental considerations, choosing instead to focus on an abstract theoretical treatment of the relatively simple case of systems of qubits. In a real system, one would have to decide, for example, on the correct way to partition the degrees of freedom in the system – perhaps spatial locations would be the relevant choice, or momentum modes, and so on. Our aim is to demonstrate, by use of concrete examples, that the most comprehensive measures that have been proposed so far are too restrictive in the set of states that they classify as macroscopically quantum. We also wish to emphasise, as has been pointed out by others$^3$, the need to view quantum macroscopicity as a resource if one wants to organise these states into a hierarchy. This view also demands an understanding of the effects that different types of operations can have on a state in terms of revealing its quantum properties at the macroscopic scale.

There is no single measure generally agreed to quantify macroscopicity; typically, a proposed measure is motivated along the lines of ‘working definition 1’ given by Fröwis and Dür in$^3$. A quantum state is macroscopic if it is able to display nonclassical behaviour at a large scale that is not simply an accumulated effect. The need to exclude accumulated quantum phenomena was originally appreciated by Leggett in 1980$^4$. These would include, for example, bulk properties of condensed matter systems that are explained only by quantum physics, yet which are built up from effects extending over only the atomic scale. In other words, one expects that many-body or long-range correlations are necessary for a state to be macroscopically quantum.

It is natural to view a state of the same form as Schrödinger’s cat as the canonical example of a macroscopic quantum state, in analogy with a Bell pair being regarded as a maximally entangled two-qubit state. A cat state of $N$ qubits is usually modelled as a Greenberger-Horne-Zeilinger (GHZ) state:

$$|\text{GHZ}_N\rangle := \frac{1}{\sqrt{2}}(|0\rangle^\otimes N + |1\rangle^\otimes N).$$

We may now ask what special properties this state has that determine its macroscopic character.

Macroscopic superpositions — One answer is that it is written (at least in this basis) as a superposition of two states that are macroscopically distinct. Various precise meanings have been given to this term and applied to
states written in the form $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$. For example, Korsbakken et al. ask how many groups of particles have to be measured in order to distinguish between $|\psi_0\rangle$ and $|\psi_1\rangle$ with a high probability of success. (For a GHZ state, a single $\sigma^z$ measurement suffices to distinguish between the states with certainty.) Thus macroscopic distinguishability can also be phrased in terms of local distinguishability. A similar idea based on local operations has been proposed and subsequently shown to equivalently categorise macroscopic pure states. Alternatively, Sekatski et al. view macroscopic distinctness as the statement that noisy or coarse-grained measurements of some macroscopic variable can distinguish between the states. For instance, Korsbakken and et al. ask how many groups of particles in a macroscopic according to the other measures.

Variance and Fisher information—Another highly nonclassical property of the GHZ state is that it features an extensive variable $Z$ with a large variance: $\sigma(Z)^2 := \langle Z^2 \rangle - \langle Z \rangle^2 = N^2$. Classical states in statistical mechanics typically have variances of order $N$ for such observables; we may say that larger fluctuations are the signature of a macroscopic quantum state. This was the motivation for the $p$-index introduced by Shimizu and Miyadera. This measure makes reference to the set $A$ of local observables – those that can be written as $A = \sum_{i=1}^N A_i$, such that $A_i$ acts nontrivially only on qubit $i$. One also needs to set an $N$-independent bound on the spectrum of each term – so we impose $|A_i| \leq 1$ (implying that $|A| \leq N$ via the triangle inequality). For a pure state $|\psi\rangle$, $p$ is then defined by

$$\max_{A \in A} \sigma(A)^2_{\psi} = O(N^p).$$

(2)

To describe scaling with $N$ we use the convention that $f(N) = O(N^k)$ means $\lim_{N \to \infty} f(N)/N^k$ exists and is non-zero. The value of $p$ lies in the range [1, 2]; all fully-separable states have $p = 1$, while GHZ states have $p = 2$.

This measure really applies only to families of states (such as the set of GHZ states), characterising the behaviour of each family as a function of the number of particles $N$. Nevertheless, a way to define an ‘effective size’ for a single family has been suggested:

$$N^*(|\psi\rangle) := \max_{A \in A} \frac{\sigma(A)^2_{\psi}}{N}.$$  

(3)

This coincides exactly with the number of particles in a GHZ state and gives sensible results for other examples (one tends not to be concerned about a possible overall multiplicative factor).

The definition of $A$ given above is actually too restrictive; there are examples where long-range correlations are present not at the level of individual qubits, but rather between groups of qubits. To account for this type of behaviour, we have to allow each $A_i$ to act on groups of bounded size; in the effective size $N^*$ we then have division by the number of groups $n_i$ instead of division by $N$.

Clearly an advantage of this type of measure is that it can be applied to any pure state of qubits, not requiring the specific form of a superposition of two states. Fröwis and Dür have provided a comprehensive comparison of the measures discussed so far, showing that the $p$-index captures the macroscopicity of the largest set of pure states – encompassing all those which are macroscopic according to the other measures.

The same authors also suggest an alternative interpretation of large fluctuations in terms of usefulness for quantum metrology. Given an observable $A$ and a state described by a density operator $\rho$, a parameter $\theta \in \mathbb{R}$ may be encoded on the state via $\rho \rightarrow e^{-i\theta A}/p e^{i\theta A}$. The quantum Fisher information $\mathcal{F}(\rho, A)$ then sets a lower bound on the uncertainty with which $\theta$ can be estimated from this state. It can be calculated using

$$\mathcal{F}(\rho, A) = 2 \sum_{a,b} \frac{(p_a - p_b)^2}{p_a + p_b} \langle \psi_a | A | \psi_b \rangle^2,$$

(4)

where $p_a$ and $|\psi_a\rangle$ are the eigenvalues and eigenstates of $\rho$.

As for the variance measures, we constrain $A$ to be a local operator and define an effective size

$$N^*(\rho) := \max_{A \in A} \frac{\mathcal{F}(\rho, A)}{4n(A)},$$

(5)

where $n(A)$ is the number of separate groups acted upon by $A$. The factor of 4 appears because the Fisher information for pure states reduces to the variance in the following way: $\mathcal{F}(|\psi\rangle\langle\psi|, A) = 4\sigma(A)^2_{\psi}$ [12]. Thus the Fisher information over local observables has been suggested to be a natural extension of variance-based measures of macroscopicity to mixed states. (The variance itself is insufficient for mixed states, as it would count even a classical mixture of $|0\rangle^\otimes N$ and $|1\rangle^\otimes N$ as macroscopically quantum). Note, however, that mixed states are so far much less studied than pure states in this context.

Distillation of GHZ states—One class of states that has been studied in depth contains those of the form $|\psi\rangle = |0\rangle^\otimes N + |\epsilon\rangle^\otimes N$, where $|\epsilon\rangle = \cos \epsilon |0\rangle + \sin \epsilon |1\rangle$ – these are often referred to as ‘generalised GHZ states’. For $N \gg 1$ and $\epsilon \ll 1$, one finds $N^* \approx \epsilon^2 N$, so the macroscopicity is suppressed by a factor of $\epsilon^2$. The authors of [17] found that these states also have the property that
it is possible to construct a sequence of local operations such that a certain number of qubits are projected out to leave the remaining ones in an exact GHZ state of size \( N' \). The procedure is stochastic, so that the value of \( N' \) has a probability distribution; its average is \( e^2N/2 \). This clearly lends itself to being interpreted as an effective size, yet this idea has not been applied more generally.

We now provide an example where we believe the macroscopicity of a state may not be revealed by variance measures, but is instead captured by the notion of GHZ distillation. The essence of the problem is that the variance quantifies the total amount of two-point correlations – to see this, one expands \( \sigma(A)^2 = \sum_i \sigma(A_i)^2 + 2 \sum_{i < j} \langle A_iA_j \rangle - \langle A_i \rangle \langle A_j \rangle \). The first part contains \( N \) terms, so that any \( O(N^2) \) contribution can only come from the second part, which is a sum of all possible two-point correlators. If we think of the qubits as existing at different spatial locations, then the existence of a large-variance local operator requires long-range correlations between macroscopically many pairs of regions. More strongly, it has been shown that \( p = 2 \) implies in one sense a large amount of many-body entanglement \[15, 19\].

Where the question of macroscopicity appears to differ from that of entanglement is that the reverse implication is not true – that is, there are highly entangled states that the variance quantifies the total amount of two-point correlations – to see this, one expands \( \sigma(A)^2 = \sum_i \sigma(A_i)^2 + 2 \sum_{i < j} \langle A_iA_j \rangle - \langle A_i \rangle \langle A_j \rangle \). The first part contains \( N \) terms, so that any \( O(N^2) \) contribution can only come from the second part, which is a sum of all possible two-point correlators. If we think of the qubits as existing at different spatial locations, then the existence of a large-variance local operator requires long-range correlations between macroscopically many pairs of regions. More strongly, it has been shown that \( p = 2 \) implies in one sense a large amount of many-body entanglement \[15, 19\].

One way of summarizing this is that the Fisher information – allowing LOCC manipulation enlarges the set of macroscopic states \[3\]. We believe that operations of this type should be allowed. Clearly one should not admit much more general control over the system, otherwise any state could be prepared. One might object that local measurements require a degree of microscopic control, in contrast to the ‘coarse-grained’ macroscopic observables we want to display quantum behaviour – however, we have seen that macroscopic distinctness can be phrased entirely in terms of local measurements.

**Imperfect distillation**— Following Sekatski et al. according to whom macroscopic distinguishability must be robust against noise in the measurements, one might demand the same of GHZ distillation. Hence we propose the following criterion for a measure of macroscopicity:

A state is to be regarded as macroscopically quantum if one can distil from it a macroscopic state, according to the measure \( N^* \), using imperfect local measurements.

We adopt a particular meaning of ‘imperfect measurements’ below; however, this clearly needs to be made more precise.

Extracting a GHZ state from a cluster state requires the measurement of a macroscopic number of qubits, so it is conceivable that some small noise in each measurement could build up to destroy the macroscopicity of the final state. We investigate a particular class of cluster states \( |C_N \rangle \) described by graphs of the type shown in fig. 1.

First note that the states \( |C_N \rangle \) share the property with full 2D cluster states that they lack correlations between non-neighbouring regions, and hence have \( N^* = O(1) \) according to the variance measure. Next, we choose to write the state with all \( A \) sites in the \( x \)-basis and all \( B \) sites in the \( z \)-basis. Before applying the controlled-\( \sigma_z^2 \) gates, the state \( |+\rangle^{\otimes N} \) can be written as

\[
\bigotimes_{i \in A} |0^i_z\rangle \otimes \bigotimes_{j \in B} (|0^j_z\rangle + |1^j_z\rangle) = \sum_b |0^b_A\rangle \otimes |b_B^z\rangle,
\]

where \( b = (b_1, b_2, \ldots), b_i \in \{0, 1\} \). The action of a controlled-\( \sigma_z^2 \) gate on two qubits \( i, j \) is determined by \( |a^i_z\rangle |b_j^z\rangle \rightarrow (|a \oplus b\rangle)^i |b_j^z\rangle \), where \( \oplus \) denotes addition modulo 2. Therefore we have that

\[
|C_N \rangle \propto \sum_b |a(b)^z_A\rangle |b_B^z\rangle,
\]

where the value of \( a \) in each term is a function of \( b \) according to the following rule: \( a_i = 0 \) when the neighbouring \( b_i \) are equal, and \( a_i = 1 \) when they are different. One can see that \( a(b^{(1)}) = a(b^{(2)}) \) if and only if \( b^{(2)} = b^{(1)} \) or \( b^{(1)} \), where \( b^{-1} := 1 - b \). Hence the state may be written as

\[
|C_N \rangle = \frac{1}{2^{(N_B-1)/2}} \sum_{b'=(b_2, b_3, \ldots)} |a(0, b')^z_A\rangle |\text{GHZ}(b')^z_B\rangle,
\]

where

\[
\text{GHZ}(b')^z_B = |0^1_z\rangle |1^2_z\rangle |\ldots|1^b_z\rangle |0^{b_z+1}_z\rangle |\ldots|0^N_z\rangle,
\]

FIG. 1. The graph used to define the cluster state \( |C_N \rangle \). Measurements are performed on the \( A \) sites in order to project the \( B \) sites into a macroscopic quantum state, as described by the Fisher information.
where \( |\text{GHZ}(b')\rangle := ([0, b'] + [1, b]) / \sqrt{2} \) and \( N_B = O(N) \) is the number of \( B \) sites. (We can choose to separate out any one of the \( B \) sites without loss of generality.)

This demonstrates how a measurement in the \( x \)-basis on every \( A \) qubit will project \( B \) onto one of \( 2^{NA-1} \) orthogonal GHZ states.

To model an imperfect measurement, we replace the projectors \([+][+]\) and \([-][-]\) with the operators

\[
A = \sqrt{1-\epsilon} [+][+] + \sqrt{\epsilon} [-][-],
\]

\[
\overline{A} = \sqrt{\epsilon} [+][+] + \sqrt{1-\epsilon} [-][-],
\]

satisfying \( AA' + A'A = I \), and where \( \epsilon \) is a small error parameter. After a measurement on the \( i \)th site, the state changes according to \(|\psi\rangle \rightarrow A_i |\psi\rangle \) or \( \overline{A}_i |\psi\rangle \). Due to the symmetry of the chosen operators, without loss of generality we can examine the case where the outcome \( A \) is obtained for each measurement. For \( \epsilon = 0 \), this would result in distillation of the state \(|\text{GHZ}(0)\rangle\). For any \( \epsilon > 0 \), after tracing out \( A \), we find \( B \) in a GHZ-diagonal state

\[
\rho_B = \sum_{b'} p_{b'} |\text{GHZ}(b')\rangle \langle \text{GHZ}(b')|, \tag{10}
\]

\[
p_{b'} = \frac{\epsilon^{k(b')}(1-\epsilon)^{-k(b')}}{Z}, \tag{11}
\]

where \( k(b') = \sum_{i=1}^{N_A} a_i (0, b') \) and \( Z = \sum_{b'} \epsilon^{k(b')}(1-\epsilon)^{-k(b')} \).

We now quantify the macroscopicity of \( \rho_B \) using the Fisher information measure \([\Box]\). For states of the form \([10]\), the Fisher information with respect to the observable \( Z \) reduces to its variance: \( \mathcal{F}(\rho_B, Z) = 4 \langle (N_B - 2|b'|)^2 \rangle \), where \( |b'| = \sum_{i=2}^{N_B} b_i \). Rather than calculating this directly, we make the following statement: for sufficiently small \( \epsilon \), the variance remains \( O(N_B^2) \) as \( N_B \to \infty \). This follows from the observation that the probabilities \( p_{b'} \) are exactly those appearing in the state of a square-lattice ferromagnetic classical Ising model at finite temperature. Recall that in this model, there is an energy cost of \( 2J \) associated with every oppositely-oriented pair of neighbouring spins. To map onto our case, we replace the Boltzmann factor \( e^{-2bJ} \) by \( \frac{1}{1-\epsilon} \).

Using this analogy, the quantity \( N_B - 2|b'| \) for each configuration maps onto the total magnetisation \( M \) of the lattice, given that each site has an associated magnetic moment of \( \pm 1 \) (and we fix \( b_1 = +1 \)). Peierls gave an argument for the existence of a phase transition in this model, and this was later altered and made more rigorous by Griffiths \([22, 23]\). The latter establishes a bound for the total magnetisation of the form \( |M| \geq N_B (1-f(T)) \), where \( f \) is independent of \( N_B \) and \( \lim_{T \to 0} f(T) = 0 \).

We require a similar bound on \( (N_B - |b'|)^2 - \langle M^2 \rangle \). Using \( 0 \leq \langle (N_B - |M|)^2 \rangle = N_B^2 - 2N_B \langle |M| \rangle + \langle M^2 \rangle \), it follows that \( \langle M^2 \rangle \geq 2N_B \langle |M| \rangle - N_B^2 \geq N_B^2 (1-2f(T)) \). To leading order, \( f(T) \approx e^{-8bJ} \), which gives the bound \( \mathcal{F}(\rho_B, Z) \geq 4N_B^2 (1-O(\epsilon^4)) \). Hence our main result is

\[
N^\ast(\rho_B) \geq N_B (1 - O(\epsilon^4)) = O(N) (1 - O(\epsilon^4)). \tag{12}
\]

This means that for sufficiently small \( \epsilon \) (independent of the size of the system), imperfect measurements of this type will still result in a macroscopic state.

It is also interesting to note that this argument fails for 1D cluster states, corresponding to the fact that the Ising model has no \( T > 0 \) phase transition in one dimension \([24]\). This strongly suggests that 1D cluster states are not macroscopic according to our criterion – although a rigorous proof would have to show that there is no other protocol that can do better. The apparent dependence on the dimension of the cluster state might be compared with the fact that 2D cluster states are universal for measurement-based quantum computation, while 1D cluster states are not \([23]\). Furthermore, we have results that suggest that ground states of the Kitaev model, which is a simple model of topological order \([24]\), behave similarly to 2D cluster states in the present context.

We argue that these results imply that current variance-based measures do not capture all the macroscopic quantum properties which can exist in systems of qubits. A measure based on GHZ distillation, as originally suggested by Dür et al. could be well-motivated, but presently appears to be intractable except for some specific cases.

Alternatively, one could argue that the concept of collective quantum behaviour ought to incorporate correlations between arbitrarily many particles, while fluctuations of macroscopic observables are determined only by two-point correlations. In some cases (such as GHZ states), \( N \)-point correlations exist alongside two-point correlations, but this is not always true – and cluster states are such examples where there are very few two-point correlations. The GHZ distillation procedure could be viewed as exacttracting two-point correlations from the existing higher-order correlations. Therefore it may be possible to enlarge the set of quantum states classified as macroscopic by proposing a measure that takes into account all possible orders of correlations.

The results obtained in this work are also useful for beginning to understand the behaviour of macroscopicity under different types of operations. If one could provide a physical motivation for the class of operations under which macroscopicity should not increase, then this may be a useful starting point for proposing a new measure.

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