On the Noncommutative Geometry of Twisted Spheres

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Abstract

We describe noncommutative geometric aspects of twisted deformations, in particular of the spheres in Connes and Landi \cite{8} and in Connes and Dubois Violette \cite{7}, by using the differential and integral calculus on these spaces that is covariant under the action of their corresponding quantum symmetry groups. We start from multiparametric deformations of the orthogonal groups and related planes and spheres. We show that only in the twisted limit of these multiparametric deformations the covariant calculus on the plane gives by a quotient procedure a meaningful calculus on the sphere. In this calculus the external algebra has the same dimension of the classical one. We develop the Haar functional on spheres and use it to define an integral on forms. In the twisted limit (differently from the general multiparametric case) the Haar functional is a trace and we thus obtain a cycle on the algebra. Moreover we explicitely construct the $\ast$–Hodge operator on the space of forms on the plane and then by quotient on the sphere. We apply our results to even spheres and we compute the Chern–Connes pairing between the character of this cycle, \textit{i.e.} a cyclic $2n$–cocycle, and the instanton projector defined in \cite{8}.

1 Introduction

Noncommutative geometry is an active research field both in Physics and Mathematics. There are many examples of noncommutative spaces and many different philosophies behind them. It is therefore instructive to study the geometry of specific examples in order to gain a better understanding of the relations between different approaches.
Quantum spaces related to standard quantum groups, i.e. FRT ([10]) matrix quantum groups or Drinfeld–Jimbo quantum enveloping algebras, have a rich non-commutative structure and display remarkable properties for many respects (their connection with integrable models and invariants of knots, semiclassical structures, etc...), but, on the other hand, present some triviality when considered under the light of the noncommutative geometry à la Connes. For instance, the $C^*$–algebra of the standard Podleś quantum 2–spheres and of the quantum 4-sphere of [4] are isomorphic, i.e. the topology cannot distinguish inbetween the two different classical dimensions. Indeed the study of [13] of Podleś 2-spheres shows that the Hochschild dimension, that corresponds to the commutative notion of dimension, is zero and periodic cyclic cohomology is generated by 0–traces. The drop of dimension appearing in this example seems to be a generic property of these kind of quantum spaces.

In [8] a deformation $S^4_q$ of the 4–sphere was introduced with the property that the Hochschild dimension equals the commutative dimension. In [6] this program was generalized to any dimension: a complete classification of 3–dimensional spherical manifolds was given, while for generic dimension a family of deformed spheres, called $\theta$–deformed spheres, was introduced. In general [7] a $\theta$–deformation $M_\theta$ of a manifold $M$ equipped with a smooth action of the $n$–torus $T^n$ is determined by defining $C^\infty(M_\theta)$ as the invariant subalgebra (under the action of $T^n$) of the algebra $C^\infty(M \times T^n_\theta) = C^\infty(M) \hat{\otimes} C^\infty(T^n_\theta)$ of smooth functions on $M \times T^n_\theta$; where $T^n_\theta$ is the noncommutative $n$-torus. This construction allows for an easy definition of smooth differential forms and of the $*$–Hodge operator. Although this calculus is not connected with a quantum group action, it coincides with the covariant calculus.

In [23] (see also [20]) it was shown that $S^4_q$ is an homogeneous space of a twisted deformation of $SO(5)$. Twisted quantum groups quantize Poisson–Lie groups defined by classical $r$-matrices living in the Cartan subalgebra of the semisimple Lie algebra. A useful point of view in order to understand the different geometry of the standard and the twisted case is given by multiparametric deformations. Multiparametric quantum groups were defined by introducing in the FRT construction [10] the multiparametric $R$–matrices [19]. These quantum groups depend on two sets of parameters and contain as limit structures both the standard deformation and the twisted deformation. Homogeneous spaces with respect to multiparametric quantum groups are constructed in the same way as for standard deformations. It is easy to verify that the 4–sphere of [8] appears as the limit, to the twisted structure case, of the multiparametric orthogonal 4–sphere (the 4–sphere covariant under the action of the multiparametric quantum $SO(5)$ group); similarly the $\theta$-deformed spaces and groups of [7] can be obtained as special limits to the twisted case of the corresponding multiparametric quantum spaces and groups.

A basic tool in order to analyze the geometry of (multiparametric) quantum spaces is the covariant differential calculus. This calculus is constructed from the requirement of covariance under the action of the corresponding orthogonal quantum groups [13, 3]. It is closely related to the bicovariant noncommutative calculus on quantum groups [24] (see also [13] and [4]). In the past years the bicovariant calculus
has been studied and classification results are known. Nevertheless a relation with
the noncommutative geometry à la Connes has lacked; in particular with the notion
of cycle over an algebra (8). Cycles are differential algebras provided with an
integral, which is a closed graded trace; they are in one–to–one correspondence
with cyclic cocycles and allow the computation of topological invariants like Chern
characters.

In this letter we show that quantum groups techniques, and in particular the
covariant differential calculus on quantum planes and spheres, are useful tools in
order to describe the noncommutative geometry of twisted deformations. In Section
2 we introduce multiparametric deformations of the orthogonal groups and of the
related planes and spheres. The multiparametric covariant differential calculus on
the plane is then studied, we see that only in the twisted limit the calculus ad-
mits a euclidean reality structure, the space of exterior forms is obtained (quantum)
antisymmetrizing the space of tensors and the quotient calculus on the sphere is
nontrivial (i.e. it is not 0–dimensional). The twisted limit of the multiparametric
calculus on the plane and of the multiparametric bicovariant calculus on the cor-
responding orthogonal quantum group was first studied in [2]. In Section 3 we give
a self contained exposition of the Haar functional on twisted spheres and use it in
Section 4 to define an integral on forms. Since (differently from the case of standard
deformations) the Haar functional is a trace, the integral on forms turns out to be
a closed graded trace: we thus obtain a cycle on the algebra. A cycle defines a
cyclic cocycle, a basic tool in noncommutative geometry à la Connes. In Section 5
we define the pairing (metric) between tensorfields and exploit the construction of
forms as antisymmetrized tensors in order to define the $*$–Hodge operator. Explicit
expressions for this operator and its properties are given. The $*$–Hodge operator
is shown to coincide with the one defined in [7]. Exterior forms, integration theory and
$*$–Hodge operator are fundamental ingredients for the construction of field theories
and gauge theories.

In the last Section we apply these tools to even spheres. We make use of the
projector introduced in [8] and [7] which defines the instanton bundle and we com-
pute its charge as the Chern–Connes pairing with the character of the cycle defined
in Section 4.

\section{Orthogonal Multiparametric Quantum Groups,
Planes and Spheres}

The orthogonal multiparametric quantum groups are freely generated by the $N^2$
matrix elements $T^a_b$ (fundamental representation) and the identity 1, modulo the
quadratic $RTT$ relations and the orthogonality relations discussed below. The non-
commutativity is controlled by the multiparametric $R$-matrix $R_{q,r}$

$$
(R_{q,r})^{ab}_{ef} T^e_c T^f_d = T^b_d T^a_e (R_{q,r})^{ef}_{cd} 
$$

(1)
which satisfies the quantum Yang-Baxter equation. The multiparametric $R_{q,r}$ matrix is obtained from the uniparametric one $R_r$, via the transformation [14] (we follow the notations of [2]): $R_{q,r} = F^{-1} R_r F^{-1}$ where $(F^{-1})^{ab}_{cd}$ is a diagonal matrix in the index couples $ab$, $cd$

$$F^{-1} \equiv \text{diag}\left(\sqrt{\frac{r}{q_{11}}}, \sqrt{\frac{r}{q_{12}}}, \ldots \sqrt{\frac{r}{q_{NN}}}\right), \quad (2)$$

and where the complex parameters $q_{ab}$, $a, b = 1, \ldots, N$ satisfy the following relations

$$q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}, \quad q_{ab} = \frac{r^2}{q_{ab}} = \frac{r^2}{q_{a'b}} = q_{a'b'}, \quad (3)$$

in the last equality we defined primed indices as $a' \equiv N + 1 - a$. Relations (3) also imply $q_{aa'} = r$, therefore the $q_{ab}$ with $a < b \leq N$ give all the $q$'s. One can also easily show that the non diagonal elements of $R_{q,r}$ coincide with those of $R_r$. The matrix $F$ satisfies $F_{12} F_{21} = 1$ i.e. $F^{ab}_{ef} F^{fe}_{cd} = \delta^a_d \delta^b_c$, the quantum Yang-Baxter equation $F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12}$ and the relations $(R_{q,r})_{12} F_{13} F_{23} = F_{23} F_{13} (R_{q,r})_{12}$. Notice that for $r = 1$ the multiparametric $R$ matrix reduces to $R = F^{-2}$. Let $\hat{R}$ be the matrix defined by $\hat{R}^{ab}_{cd} \equiv (R_{q,r})^{ba}_{cd}$, then the multiparametric $\hat{R}_{q,r}$ is obtained from $\hat{R}$, via the similarity transformation

$$\hat{R}_{q,r} = F \hat{R} F^{-1}; \quad (4)$$

the characteristic equation and the projector decomposition of $\hat{R}_{q,r}$ are therefore the same as in the uniparametric case:

$$(\hat{R} - r I)(\hat{R} + r^{-1} I)(\hat{R} - r^{-1} N I) = 0 \quad (5)$$

$$\hat{R} = r P_S - r^{-1} P_A + r^{-1} N P_0 \quad (6)$$

with

$$P_S = \frac{1}{r + r^{-1}}[\hat{R} + r^{-1} I - (r^{-1} + r^{-1} N) P_0]$$

$$P_A = \frac{1}{r + r^{-1}}[-\hat{R} + r I - (r - r^{-1} N) P_0]$$

$$P_0 = Q_N(r) K$$

$$Q_N(r) \equiv (g_{ab} g^{ab})^{-1} = \frac{1 - r^{-2}}{(1 - r^{-N})(1 + r^{-N})}, \quad K^{ab}_{cd} \equiv g_{ab} g_{cd}$$

Orthogonality of $T$ reads

$$g^{bc}_{\alpha b} T^d_{\alpha c} = g^{ad} 1, \quad g_{ac} T^a_{\beta b} T^c_{\beta d} = g_{bd} 1 \quad (8)$$

where $g^{ab} = g_{ab} = \delta_{ab'}$. The consistency of (8) with the $RTT$ relations is due to

$$g_{ab} \hat{R}^{bc}_{de} = (\hat{R}^{-1})^{ef}_{ad} g_{fe'}, \quad \hat{R}^{bc}_{de} g^{ca}_{ef} = g^{bf}_{de} (\hat{R}^{-1})^{ca}_{ad} \quad (9)$$

These identities hold also for $\hat{R} \to \hat{R}^{-1}$. A multiparametric determinant is defined by $\det_{q,r} T = \epsilon_{q,r}^{1 \ldots N} T^1_{i_1} \ldots T^N_{i_N}$ cf. [21], it satisfies $(\det_{q,r} T)^2 = 1$. Imposing also the relation $\det_{q,r} T = 1$ we obtain the special orthogonal quantum group $SO_{q,r}(N)$. 4
The multiparametric orthogonal quantum plane is the algebra freely generated by the elements $x^a$ with commutation relations

$$P_{cd}^{ab}x^c x^d = 0 .$$

(10)

The element $c = x^a g_{ab} x^b$ is central and imposing the extra relation $c = 1$ we obtain the multiparametric orthogonal quantum sphere.

The costructures of the orthogonal multiparametric quantum groups have the same form as in the uniparametric case: the coproduct $\Delta$, the counit $\varepsilon$ and the antipode $S$ are given by $\Delta(T^a_b) = T^a_b \otimes T^b_c$, $\varepsilon(T^a_b) = \delta^a_b$, $S(T^a_b) = g^{ac}T^d_c g_{db}$ the coaction on the quantum plane and sphere reads $\delta(x^a) = T^a_b \otimes x^b$.

Differential Calculus

There are only two $SO_{q,r}(N)$-covariant first order differential calculi on the quantum plane such that any 1-form can be uniquely written as sum of functions on the quantum plane times the basic differentials $dx^i$: $f_i dx^i$ i.e. such that the bimodule of 1-forms is generated as a left (or right) module by the differentials $dx^i$. The deformed commutation relations are [5] (the multiparametric case appeared in [3])

$$x^a dx^b = r \hat{R}^{ab}_{cd} dx^c x^d ,$$

(11)

the other calculus is obtained by replacing $r \hat{R}^{ab}_{cd}$ with $r^{-1} \hat{R}^{-1ab}_{cd}$ (so that for $r = 1$ the calculus is unique). The exterior differential $d$ by definition satisfies the Leibniz rule and therefore, from (11) it follows that the algebra of exterior forms is generated by $x^a$ and $dx^a$ modulo the ideal generated by the relations (10), (11) and

$$dx^a dx^b = -r \hat{R}^{ab}_{cd} dx^c dx^d ;$$

(12)

recalling (3) and (7) this relation is equivalent to $P_S^{ab}_{cd} dx^c dx^d = P_0^{ab}_{cd} dx^c dx^d = 0$. Partial derivatives can be defined so that $da = dx^c \partial_c(a)$. They satisfy the deformed Leibniz rule and the commutation relations

$$\partial_c x^b = \delta^b_c 1 + r \hat{R}^{be}_{cd} x^d \partial_e , \quad P_A^{ab}_{cd} \partial_b \partial_a = 0 .$$

(13)

Remark 1 The space of 2-forms defined by (12) is equivalent to the space of 2-forms defined using the wedge product (here $\mu$ is an arbitrary coefficient)

$$dx^a \wedge dx^b \equiv \mu P_A^{ab}_{cd} dx^c \otimes dx^d$$

(14)

$$= dx^a \otimes dx^b - (I - \mu P_A)_{cd} dx^c \otimes dx^d$$

(15)

Notice however that for all $\mu$,

$$\Lambda \equiv (I - \mu P_A) = \frac{\mu}{r + r^{-1}}[-\hat{R} + \mu^{-1}(\mu r - r - r^{-1})I - (r - r^{-1} - N)P_0]$$

(16)
does not satisfy the braid equation. This situation differs from that of the \(GL_{q,r}(N)\)-covariant plane \([13]\) and from that of exterior forms on quantum groups as described in \([24]\). There the corresponding \(\Lambda\) matrix satisfies the braid equation so that the space of \(k\)-forms (not only that of 2-forms) can be defined as the space of quantum antisymmetric tensorfields. We have

\[
dx_{i_1} \land \ldots \land dx_{i_k} = W(dx^{i_1} \otimes \ldots \otimes dx^{i_k}) = W^{j_1 \ldots j_k}_{i_1 \ldots i_k} dx^{j_1} \otimes dx^{j_k},
\]

where the numerical coefficients \(W_{i_1 \ldots i_k}^{j_1 \ldots j_k}\) give the alternating sum of \(k!\) addends, these addends corresponding to the \(k!\) permutations of \(k\) elements. Since \(\Lambda\) is not in general a representation of the permutation group each permutation must be expressed via a minimal set of nearest neighbour transpositions, each transposition is then represented via \(\Lambda\). A recursion relation for \(W\) is

\[
W_{i_1 \ldots i_k} = T_{i_1 \ldots i_k} W_{i_1 \ldots i_{k-1}},
\]

where \(T_{i_1 \ldots i_k} = I - \Lambda_{k-1,k} + \Lambda_{k-2,k-1} \Lambda_{k-1,k} \ldots - (-1)^k \Lambda_{12} \Lambda_{23} \cdots \Lambda_{k-1,k}\) and \(W_{i j} = T_{j} = \delta_{ij}\). A recursion relation for \(T\) is

\[
T_{i_1 \ldots i_k} = I - T_{i_1 \ldots i_{k-1}} \Lambda_{k-1,k}.
\]

A different space of 2-forms on the quantum orthogonal plane is defined imposing only the relation \(P_{ab}^{cd}dx^c dx^d = 0\), cf. \([10]\); this relation (and only this one for generic \(r\)), is implied by the wedge product \(dx^a \land dx^b \equiv dx^a \otimes dx^b - r^{-1} \hat{R}_{ab}^{cd} dx^c \otimes dx^d\). In this case \(\Lambda = r^{-1} \hat{R}\) satisfies the braid equation and the space of exterior forms can be constructed as in \((17)\)–\((19)\). A differential calculus on the quantum orthogonal plane with this exterior algebra is studied in \([3]\); it is obtained as a bicovariant calculus \((24)\) on the inhomogeneous orthogonal quantum group \(ISO_{q,r}(N)\).

**Remark 2** In the next sections we consider the \(r \to 1\) limit of the exterior algebra defined by \((12)\), equivalently \((14)\). In this limit the \(\Lambda\) matrix \((16)\) with \(\mu = 2\) equals the \(\hat{R}_{q,r=1}\) matrix (cf. last paragraph in Remark 1), it satisfies the braid equation and it also squares to the identity matrix, therefore defining a representation of the permutation group. We can thus construct the space of exterior forms as in \((17)\)–\((19)\). Here we show that the exterior algebra \(\text{Im}(W)\) obtained as the image of the antisymmetrizer map \(W\) is isomorphic to the exterior algebra \(\Omega\) freely generated by the elements \(dx^i\) modulo the ideal generated by the relations \((12)\). Indeed consider the surjection \(\phi : \Omega \to \text{Im}W\) defined by \(\phi([T]) = \frac{1}{k!} W(T)\) where \(T\) is a polynomial of degree \(k\) in the \(dx^i\), and \([T]\) denotes the equivalence class under \((12)\). \(\phi\) is well defined, it is also injective because \([T] = [\frac{1}{k!}W(T)]\); this last relation is proven observing that \(\Lambda^{ab}_{cd}[dx^c dx^d] = -[dx^a dx^b]\). A similar argument shows that \(\Omega \cong \text{Im}(W)\) also for the \(GL_{q,r \neq 1}(N)\)-covariant plane.

Generalizing \([11]\) to the multiparametric case, because of \((12)\) and of the specific expression of \(\hat{R}_{q,r}\), we have that any monomial \(dx^{i_1} dx^{i_2} \ldots dx^{i_p}\) can be rewritten
as sum of monomials $dx^{j_1}dx^{j_2}\ldots dx^{j_p}$ with $j_1 < j_2 < \ldots < j_p$ so that the graded differential algebra of exterior forms on the quantum plane has dimension $2^N$ as in the classical case. In particular every $N$-form is proportional to the volume form

$$V_N = i^{\left\lfloor \frac{N}{2} \right\rfloor} dx^1 dx^2 \ldots dx^N, \quad \left\lfloor \frac{N}{2} \right\rfloor \equiv \text{integer part of } \frac{N}{2}.$$  \hfill (20)

The epsilon tensor is defined by

$$\epsilon^{i_1\ldots i_N}_{q,r} dx^{i_1} dx^{i_2} \ldots dx^N = dx^{i_1} dx^{i_2} \ldots dx^{i_N}. \hfill (21)$$

One can show that $x^a V_N = r^N V_N x^a$ If we extend the quantum plane algebra including the generator $c^{-1}$, the exterior differential is then given by the 1-form $\omega = \frac{r^2}{r^2 - 1} c^{-1} dc$ as follows $d\theta = \frac{1}{r} [\omega, \theta]_\pm$ where we use the commutator if $\theta$ is an even form, the anticommutator if $\theta$ is odd. Notice that $d$ is an inner differential only if $r \neq 1$. The drop of dimension discussed in the introduction is related to this property of the exterior differential. While this aspect may seem a trivialization (from an outer $d$ with $r = 1$ to an inner $d$ with $r \neq 1$) it also hints that the geometry is highly noncommutative, indeed $d$ and the partial derivatives are finite difference operators for $r \neq 1$.

It is natural to study how the calculus on the $N + 1$ dimensional quantum plane induces a calculus on the $N$ dimensional sphere. As in the commutative case we define the exterior algebra on the sphere as the quotient of the exterior algebra on the plane modulo the differential ideal generated by the relation $c = 1$. Since $c$ is not central in the differential algebra, i.e. $c dx^a = r^2 dx^a c$ and $x^a dc = dc x^a + (1 - r^{-2})c dx^a$, we immediately have that $dx^a = 0$ if $r \neq 1$. We conclude that in the $r \neq 1$ case the quotient calculus on the sphere is trivial.

\section*{Real Forms}

All real forms of (uniparametric) orthogonal quantum groups and their quantum spaces are studied in \cite{23}, (see also \cite{3}). Here we focus on the compact form $SO_{q,r}(N, \mathbb{R})$ and on the multiparametric Euclidean quantum plane $\mathbb{R}^N_{q,r}$ and sphere $S^N_{q,r}$. These are given by the conjugation

$$(T^a_b)^* = g_{ea} T^e_f g^{bf}, \quad (x^a)^* = g_{ea} x^e,$$  \hfill (22)

that is compatible with the quantum group, plane and sphere defining relations and with the coaction $\delta(a^a) = T^a_b \otimes x^b$ if $\hat{R}_{cd}^{ab} = R_{cd}^{ba}$, i.e. $q_{ab} q^{ab} = r^2, r \in \mathbb{R}$. Conjugation \cite{22} however, for $r \neq 1$, is not compatible with the differential calculus on the quantum plane in the sense that (11) implies $(dx^a)^* \neq d(x^a^*)$. Also, the conjugated partial derivatives $\partial_a^*$ are not linear combinations of the $\partial_a$’s. Rather $(dx^a)^*$ and $\partial_a^*$ generate the other calculus on the quantum orthogonal plane \cite{cf line

\footnote{We show in Subsection 5.2 that $V_N$ is real.}

\footnote{Proof. In order to show $\epsilon^{i_1\ldots i_N}_{q,r} \hat{R}_{1a_1} \hat{R}_{i_2a_2} \ldots \hat{R}_{i_Na_N} = \pm \delta_{a_N}$ apply the $SO_{q,r}(N)$-coaction to $V_N$, recall that $(\det T)^2 = 1$ and consider the $N \times N$ representation of $T_{cd}^e$ given by $\hat{R}_{ae}^{dc}$. Finally the plus sign is singled out going to the commutative limit $r = q_{ab} = 1.$}
after (11). In [16], $(dx^a)^*$ and $\partial_a^*$ are expressed nonlinearly in terms of the $x, dx, \partial$ algebra. We just mention that on the other hand the conjugations that give the signatures $n, m$ with $n + m = N$, and $n - m = 0, 1, 2$ give a real differential calculus $(dx^a)^* = d(x^a^*)$.

**Integration**

Generalizing [21] to the multiparametric case, we obtain that there exists a unique (normalized) integral of functions on the multiparametric sphere $S^N_{q,r}$ such that it is invariant under the $SO_{q,r}(N, \mathbb{R})$ coaction and it is analytic in $r - 1$ and $q_{ab} - r$. We use the notation $h(f)$ for the integral (Haar functional) of $f \in S^N_{q,r}$. On the elements $[x^{i_1} x^{i_2} \ldots x^{i_p}] \in S^N_{q,r}$ (the square brackets denote the equivalence class w.r.t. the relation $c = 1$) we have $h([x^{i_1} x^{i_2} \ldots x^{i_{2n+1}}]) = 0$ and $h([x^{i_1} x^{i_2} \ldots x^{i_{2n}}]) = \lambda_n \Delta^n(x^{i_1} x^{i_2} \ldots x^{i_{2n}})$ with $\Delta = g^{ij} \partial_i \partial_j$ and $\lambda_n$ a proportionality factor depending only on $n$ and $r$. The Haar functional on $S^N_{q,r}$ has the following reality, positivity and quantum cyclic properties:

$$h(f) = h(f^*) \quad h(f^* f) \geq 0 \quad h(f g) = h(g Df)$$

where $f, g, Df \in S^N_q$. The map $D$ is defined on the basic monomials $[x^{i_1} x^{i_2} \ldots x^{i_p}]$ as $D[x^{i_1} x^{i_2} \ldots x^{i_p}] \equiv [D^{i_1}_{j_1} x^{j_1} D^{i_2}_{j_2} x^{j_2} \ldots D^{i_p}_{j_p} x^{j_p}]$, where $D^a_e \equiv g^{ae} g_{es}$. This map $D$ is then extended by linearity to all of $S^N_{q,r}$. (It is easy to see that $D$ is well defined, in fact $D[c - 1] = 0$). In the twisted limit $r \to 1$, we have $D^a_e \to \delta^a_e$ and we obtain the cyclic property $h(f g) = h(g f)$. In the next section we give a self-contained exposition of the Haar functional on twisted spheres, we also give the explicit expression of the $\lambda_n$ coefficients.

### 3 Twisted Spheres

The twisted quantum Euclidean planes and spheres $\mathbb{R}^N_q$, $S^N_q$ are obtained considering the limit $r \to 1$ of the corresponding multiparametric structures. We have $(a' = N + 1 - a)$

$$\hat{R}^a_{\hat{b} c d} = q_{ab} \delta^a_d \delta^b_c \quad g_{ab} = g^{ab} = \delta_{ab} \quad (23)$$

$$|q_{ab}| = 1 \quad q_{aa} = q_{a'a'} = 1 \quad q_{ab} = q_{a'b'} \quad q_{ab} = q^{-1}_{ba} = q^{-1}_{ab} \quad (24)$$

For each $p$ we have that

$$\prod_{i=1}^N q_{ip} = 1 \quad (25)$$

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$^3$Hint: use induction on the number of deformation parameters. The positivity property $h([x^{i_1} x^{i_2} \ldots x^{i_{2n+1}}]) = h([x^{i_{2n+1}} x^{i_2} \ldots x^{i_{2n+1}}])$ holds because of $g_{ij} \in \mathbb{R}$, $P_\alpha^\gamma_{\beta\delta} = P_\alpha^\gamma_{\beta\delta}$ and $P_\alpha^\gamma_{\beta\delta} g_{ji} g_{hh} = P_\alpha^\gamma_{ki} g_{me} g_{ef}$, cf. [8]. Contrary to [21] reality of $\hat{R}$ is not needed.
Explicitly, the twisted quantum Euclidean plane $q$-commutation relations are
\[ x^a x^b = q_{ab} x^b x^a \] (26)

The twisted quantum Euclidean sphere is the quotient algebra $S^N_q = \mathbb{R}^{N+1}_q/I$, where $I$ is the ideal generated by the relation $c = 1$.

**Remark 3** If we specialize to $S^4_q$ the only independent deformation parameter is $q_{12} = q$. The explicit relations are $2(x^1 x^5 + x^2 x^4) + (x^3)^2 = 1$, $[x^3, \ldots] = 0$ and
\[ x^1 x^2 = q x^2 x^1, \quad x^1 x^4 = q x^4 x^1, \quad x^1 x^5 = x^5 x^1, \]
\[ x^2 x^5 = q x^5 x^2, \quad x^4 x^5 = q x^5 x^4, \quad x^2 x^4 = x^4 x^2. \]

The explicit isomorphism with Connes-Landi sphere is given by $\lambda: q \mapsto \lambda q$. Integration of functions on the sphere

**Integration of functions on the sphere**

Let $\Delta = g^{ij} \partial_i \partial_j$ be the Laplacian in $\mathbb{R}^N_q$, where the partial derivatives now satisfy, for all $f \in \mathbb{R}^N_q$,
\[ \partial_s(x^a f) = \delta^a_s f + q_{as} x^a \partial_s f, \quad \partial_a \partial_b = q_{ab} \partial_b \partial_a. \] (27)

A straightforward computation shows that (no sum on $k$)
\[ \Delta x^k x^{k'} = x^k x^{k'} \Delta + 2 + 2 x^k \partial_k + 2 x^{k'} \partial_{k'} + \] (28)

so that
\[ \Delta c = c \Delta + 2N + 4 x^j \partial_j. \] (29)

Before introducing the Haar functional we have to show the following lemma.

**Lemma 4** For each $n > 0$ we have that
\[ \Delta^{n+1}(c x^{i_1} \ldots x^{i_{2n}}) = 2(n + 1)(N + 2n)\Delta^n(x^{i_1} \ldots x^{i_{2n}}). \] (30)

**Proof.** We first show that for all $n$ and for each $0 \leq k \leq n$ we have
\[ a_k \Delta^n(x^{i_1} \ldots x^{i_{2n}}) + \Delta^{n-k+1} c \Delta^k(x^{i_1} \ldots x^{i_{2n}}) = \]
\[ a_{k+1} \Delta^n(x^{i_1} \ldots x^{i_{2n}}) + \Delta^{n-k} c \Delta^{k+1}(x^{i_1} \ldots x^{i_{2n}}) \] (31)

with $a_k = 2k(N + 4n) - 4k(k - 1)$. Indeed recalling (29) and observing that $x^j \partial_j(x^{i_1} \ldots x^{i_{2(n-k)}}) = 2(n - k)x^{i_1} \ldots x^{i_{2(n-k)}}$ the l.h.s. of (31) equals
\[ (a_k + 2N + 8(n - k))\Delta^n(c x^{i_1} \ldots x^{i_{2n}}) + \Delta^{n-k} c \Delta^{k+1}(x^{i_1} \ldots x^{i_{2n}}). \]
Since the calculus is covariant, the partial derivatives satisfy the following property:

\[ (x, a)_n = x(x + a) \ldots (x + (n - 1)a) \text{ and } (x, a)_0 = 1, \]

let us define \( h : \mathbb{R}^{N+1}_q \rightarrow \mathbb{C} \) as the linear map that on monomials is given by

\[
  h(x^{i_1} \ldots x^{i_{2n}}) = \lambda_n \Delta^n (x^{i_1} \ldots x^{i_{2n}}), \quad \lambda_n = \frac{1}{2^n n! (N, 2)_n}. \tag{32}
\]

**Proposition 5** Let \( f \in \mathbb{R}^{N+1}_q \) and let \([f] \in \mathbb{S}^N_q\) be its equivalence class. The linear functional \( h([f]) \equiv h_q(f) \) is well defined on \( \mathbb{S}^N_q \) and satisfies the following properties (we omit to denote the equivalence class):

a) \( h(1) = 1 \).

b) \( h(fg) = h(gf) \), for each \( f, g \in \mathbb{S}^N_q \).

c) \( 1 \otimes h = (\text{id} \otimes h) \circ \delta \) (where here, 1 is the identity in \( SO_{q,r}(N) \)).

d) \( h(f) = h(f^*) \) and \( h(f^*f) > 0 \), for each \( f \in \mathbb{S}^N_q \) (reality and positivity of \( h \)).

**Proof.** From the definition of \( \lambda_n \) and from Lemma (30) we directly check that

\[
  h_q(x^{i_1} \ldots x^{i_{2n}}(c - 1)) = \lambda_{n+1}\Delta^{n+1}(x^{i_1} \ldots x^{i_{2n}}) - \lambda_n\Delta^n(x^{i_1} \ldots x^{i_{2n}})
  = (2(n + 1)(N + 2n)\lambda_{n+1} - \lambda_n)\Delta^n(x^{i_1} \ldots x^{i_{2n}}) = 0,
\]

i.e. \( h_q((c - 1)\mathbb{R}^{N+1}_q) = 0 \) and \( h \) is well defined. Point a) is trivial. To prove point b) let us remark that if \( h(x^{i_1} \ldots x^{i_{2n}}) \neq 0 \) then for every index \( \ell \in \{i_1, \ldots, i_{2n}\} \) there is a companion index \( \ell' \in \{i_1, \ldots, i_{2n}\} \), i.e. \( \{i_1, \ldots, i_{2n}\} = \{j_1, j'_1 \ldots j_n, j'_n\} \). It follows

\[
  h(x^{i_1} \ldots x^{i_{2n}}) = \prod_{k=1}^{2n} q_{i_k} h(x^{i_1} \ldots x^{i_{2n-1}}) = \prod_{k=1}^{n} q_{j_k} h(x^{i_1} \ldots x^{i_{2n-1}}) = h(x^{i_1} \ldots x^{i_{2n-1}}).
\]

Since the calculus is covariant, the partial derivatives satisfy the following property:

\[
  \delta \circ \partial_{\ell} = (S^{-1}(T^k_{\ell}) \otimes \partial_{\ell}) \circ \delta.
\]

The Laplacian \( \Delta \) is then \( SO_q(N + 1) \)-invariant, i.e. \( (\text{id} \otimes \Delta) \circ \delta = \delta \circ \Delta \), and point c) is then proved.

In order to prove reality of \( h \) we first observe that \( \mathbb{R}^{N+1}_q \) can be linearly generated by the ordered monomials \( x^{e_1} \ldots x^{e_k}x^{a_1}x^{a_1'} \ldots x^{a_s}x^{a_s'} \), with \( e_1 \leq e_2 \ldots \leq e_k \) and where the \( e_i \) indices do not have a companion \( e'_i \) index. The action of \( x^k \partial_k \) (no sum on \( k \)) on these monomials doesn’t depend on the \( g_{ab} \), so that, thanks to (24), the value of \( h \) on these monomials equals the classical value. Reality then easily follows from
the reality of the classical integral. Analogously, to prove positivity it is enough to verify that if $f$ is an ordered polynomial the terms of $f^*f$ which contribute to $h$ are automatically ordered monomials.

We call $h$ the Haar functional on $S^N_q$. We have seen that $h$ on the ordered monomials $x^{e_1} \ldots x^{e_k} x^{a_1} x^{d_1^*} \ldots x^{a_s} x^{a_s^*}$ equals the commutative integral.

4 Calculus on $\mathbb{R}^N_q$ and $S^N_q$

The graded differential algebra $\Omega(\mathbb{R}^N_q) = \bigoplus_{k=0}^{N} \Omega_k(\mathbb{R}^N_q)$, with $\Omega_0(\mathbb{R}^N_q) = \mathbb{R}^N_q$, is the $r \to 1$ limit of the multiparametric one. As shown in Remark 2, for $r = 1$ we can consider $\Omega(\mathbb{R}^N_q)$ as the space of completely $q$-antisymmetrized tensors. The explicit relations are (26) and

$$dx^a x^b = q_{ab} x^b dx^a, \quad dx^a \wedge dx^b = -q_{ab} dx^b \wedge dx^a. \quad (33)$$

Conjugation (22) is now compatible with the differential calculus, we have $x^{a^*} = x^{a'}$ and $(dx^a)^* = d(x^{a^*})$. The volume form (20) is now central [this result follows also directly from (25)].

We now define a differential graded algebra on $S^N_q$. Let $J = J_1 + J_2 \subset \Omega(\mathbb{R}^N_q+1)$, where

$$J_1 = \{(c-1)\omega, \ \omega \in \Omega(\mathbb{R}^N_q+1)\}, \quad J_2 = \{\omega \in \Omega(\mathbb{R}^N_q+1), \ \omega \wedge dc = 0\}.$$

Because $c$ and $dc$ are central, both $J_1$ and $J_2$ are graded ideals. Moreover it is easy to verify that $d(J) \subset J$ and $\delta(J) \subset SO_q(N+1) \otimes J$, i.e. $J$ is a differential ideal and left coideal. We have that

$$\Omega(S^N_q) \equiv \Omega(\mathbb{R}^N_q+1)/J \equiv \bigoplus_{k=0}^{N} \Omega_k(S^N_q)$$

is a left covariant differential graded algebra with $\Omega_0(S^N_q) = S^N_q$. In the following we denote with $[\omega] \in \Omega_k(S^N_q)$ the equivalence class of $\omega \in \Omega_k(\mathbb{R}^N_q+1)$; we have that

$$d[\omega] = [d\omega].$$

Let $\omega_k \in \Omega_N(\mathbb{R}^N_q+1)$ be defined by

$$\omega_k = \frac{1}{N!} \epsilon_{(q-1)} \epsilon_{s_1 \ldots s_N k} dx^{s_1} \wedge \ldots \wedge dx^{s_N}, \quad (34)$$

where $\epsilon_{(q-1)}$ is the epsilon tensor $\epsilon_{(q-1),r=1}$ [c.f. (21) and (39)]. Thanks to (43) we have that $\omega_k \wedge dx^t = \delta_k^{V_{N+1}}$ and therefore

$$x^k \omega_k \wedge dc = x^k \omega_k \wedge (x^a g_{ab} dx^b + dx^a g_{ab} x^b) = 2x^k \omega_k \wedge dx^a g_{ab} x^b = 2x^k V_{N+1} g_{kb} x^b = 2c V_{N+1}.$$
From this formula, using that on a commutative sphere of unit radius we have \( dc/2 = d (g_{ab} x^a x^b)^{1/2} \), we read off the volume form on \( \Omega(S_q^N) \):

\[
\mathcal{V}_N = [x^k \omega_k] .
\]

Any \( N \) forms on the sphere can be expressed in terms of \( \mathcal{V}_N \) as follows. Let \( \omega \in \Omega_N(\mathbb{R}^{N+1}_q) \) so that \( \omega \wedge dc/2 \) is proportional to \( V_{N+1} \), we set \( \omega \wedge dc/2 = f_\omega V_{N+1} \). We then have

\[
c\omega \wedge dc = 2f_\omega cV_{N+1} = f_\omega x^k \omega_k \wedge dc ,
\]

and therefore \( c\omega - f_\omega x^k \omega_k \in J_2 \). Since \( \omega - c\omega \in J_1 \), we obtain that \( [\omega] = [f_\omega x^k \omega_k] = [f_\omega] \mathcal{V}_N \).

We are now ready to define an integral on \( N \)-forms:

\[
\int [\omega] = \int [f_\omega] \mathcal{V}_N = h([f_\omega]) ,
\]

where \( \omega \wedge dc/2 = f_\omega V_{N+1} \). This integral verifies the following Stokes‘ Theorem:

**Proposition 6**

\[
\int d[\theta] = 0 \quad \forall \ [\theta] \in \Omega_{N-1}(S_q^N) .
\]

**Proof.** Let \( \theta = dx^{i_1} \wedge \ldots dx^{i_{N-1}} f_{i_1 \ldots i_{N-1}} \in \Omega_{N-1}(\mathbb{R}^{N+1}_q) \). By direct computation we obtain that

\[
d\theta \wedge dc = (-1)^{N-1} 2dx^{i_1} \wedge \ldots dx^{i_{N-1}+1} x^\ell \partial_{i_N} f_{i_1 \ldots i_{N-1}} g_{i_{N+1}\ell} \\
= (-1)^{N-1} 2\epsilon_q^{i_1 \ldots i_{N+1}} x^\ell \partial_{i_N} f_{i_1 \ldots i_{N-1}} g_{i_{N+1}\ell} V_{N+1} .
\]

By using the definition (34) of the integral on \( N \)-forms we have that if \( f_{i_1 \ldots i_{N-1}} \) is odd in the coordinates \( x^\ell \) then \( \int d[\theta] = 0 \), while if \( f_{i_1 \ldots i_{N-1}} \) is even and of degree \( 2n \) in the coordinates \( x^\ell \) we have

\[
\int d[\theta] = (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} h(x^\ell \partial_{i_N} f_{i_1 \ldots i_{N-1}}) g_{i_{N+1}\ell} \\
= (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} \lambda_n \Delta^{n-1}(x^\ell \partial_{i_N} f_{i_1 \ldots i_{N-1}}) g_{i_{N+1}\ell} \\
= (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} \lambda_n \Delta^{n-1} (2\partial_{\ell} g^\ell + x^\ell \Delta) \partial_{i_N} f_{i_1 \ldots i_{N-1}} g_{i_{N+1}\ell} \\
= (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} \lambda_n \Delta^{n-1} (2\partial_{i_{N+1}} \partial_{i_N} f_{i_1 \ldots i_{N-1}} + g_{i_{N+1}\ell} x^\ell \Delta \partial_{i_N} f_{i_1 \ldots i_{N-1}}) \\
= (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} \lambda_n \Delta^{n-1} g_{i_{N+1}\ell} x^\ell \Delta \partial_{i_N} f_{i_1 \ldots i_{N-1}} ,
\]

where we used the relation \( [\Delta, x^\ell] = 2g^\ell \partial_k \) and the \( q \)-antisymmetry of \( \epsilon_q \)-tensor,

i.e. \( \epsilon_q^{i_1 \ldots i_{N+1} i_{N+1} i_N} = -\epsilon_q^{i_1 \ldots i_{N+1} i_N i_{N+1}} \)

together with \( \partial_{i_{N+1}} \partial_{i_N} = q_{i_{N+1} i_N} \partial_{i_N} \partial_{i_{N+1}} \). Since \( \Delta \partial_k = \partial_k \Delta \) we can repeat the above argument and obtain

\[
\int d[\theta] = (-1)^{N-1} \epsilon_q^{i_1 \ldots i_{N+1}} \lambda_n g_{i_{N+1}\ell} x^\ell \Delta^n \partial_{i_N} f_{i_1 \ldots i_{N-1}} = 0 ,
\]

because \( \deg(\partial_{i_N} f_{i_1 \ldots i_N}) = 2n - 1 \).
The integral (35) has also the property, for each \(a \in S_q^N\) and \([\omega] \in \Omega_N(S_q^N)\),

\[
\int [a\omega] = \int [af_\omega]V_N = h([af_\omega]) = h([f_\omega a]) = \int [f_\omega a] V_N = \int [f_\omega]V_N[a] = \int [\omega a] .
\]

Following the proof of Proposition III.4 (1 \(\Rightarrow\) 3) of [6] we can conclude that the integral \(\int\) is a closed graded trace on \(\Omega(S_q^N)\). We summarize the results of this section in the following proposition (for a definition of cycle see Section 6):

**Theorem 7** \((\Omega(S_q^N), d, \int)\) is a cycle. 

For future reference, we denote with \(\tau\) the character of the cycle \((\Omega(S_q^N), d, \int)\),

\[
\tau(a_0, a_1 \ldots a_N) = \frac{2^{[N/2]+1}[N/2]!}{i^{[N/2]}N!} \int a_0 da_1 \ldots da_N , \quad a_i \in S_q^N .
\] (36)

The normalization in this formula is chosen in order to fix the charge of the Bott projector on the classical even sphere equal to 1. Indeed if \(p_B\) is the Bott projector for \(S^{2n}\) it can be shown that its Chern character is \(ch(p_B) = 1 + \frac{\nu(2n)}{2^{2n}n!} V_{2n}\) (see [12]).

Since the definition of the Haar measure in Proposition 5 and in (32) doesn’t contain \(q\)-factors, it is natural not to \(q\)-deform the normalization of the character \(\tau\).

## 5 Hodge Theory

### 5.1 Hodge Theory on \(\mathbb{R}_q^N\)

We already observed that in the \(r \to 1\) limit the space of exterior forms is the image of the \(q\)-antisymmetrizer \(W\) introduced in (17). Moreover since in this case \(\Lambda^{ab}_{cd} \propto \Lambda_{(q=1)}^{ab}_{cd}\), every permutation \(W^{i_1 \ldots i_k}_{j_1 \ldots j_k}\) differs from the \(q_{ab} = 1\) permutation \(W_{(q=1)}^{i_1 \ldots i_k}_{j_1 \ldots j_k}\) by at most a proportionality factor given by a monomial in the \(q_{ab}\)’s. In particular (no sum on \(i\)’s)

\[
W^{i_1 \ldots i_k}_{i_1 \ldots i_k} = W_{(q=1)}^{i_1 \ldots i_k}_{i_1 \ldots i_k} ,
\]

since \(\Lambda_{(q=1)}^{ab}_{cd}\) never enters \(W_{(q=1)}^{i_1 \ldots i_k}_{j_1 \ldots j_k}\). We also have \(\forall i_1, \ldots i_N, j_1, \ldots j_N\)

\[
\epsilon_q^{i_1 \ldots i_N} W^{1 \ldots N}_{j_1 \ldots j_N} = W^{i_1 \ldots i_N}_{j_1 \ldots j_N} ,
\] (37)

indeed, applying both sides to \(dx^{j_1} \otimes \ldots dx^{j_N}\), we obtain the identity \(\epsilon_q^{i_1 \ldots i_N} dx^{1} \wedge \ldots dx^{N} = dx^{i_1} \wedge \ldots dx^{i_N}\). If all \(j\)’s are different we have \(W^{j_1 \ldots j_N}_{j_1 \ldots j_N} = 1\) and, using (37), we also have

\[
W^{i_1 \ldots i_N}_{1 \ldots N} = \epsilon_q^{i_1 \ldots i_N} ,
\]

\[
W^{1 \ldots N}_{j_1 \ldots j_N} = \epsilon_{(q^{-1})}^{j_1 \ldots j_N} \equiv \epsilon_{(q^{1})}^{j_1 \ldots j_N} .
\] (38) (39)
where in the last line we have used that \( \varepsilon_{q}^{i_{1} \ldots i_{N}} \) is just a monomial in the \( q_{ab} \)'s and that the inverse is therefore the same monomial with \( q_{ab} \rightarrow q_{ab}^{-1} \). The definition of \( \varepsilon_{(q-1)} \) with lower indices is just to preserve the index structure of \( W_{j_{1} \ldots j_{N}}^{1 \ldots N} \). Relation (39) [as well as (38)] holds also for arbitrary \( j \)'s; indeed both the l.h.s. and the r.h.s. are zero unless all \( j \)'s are different. From (38) and (39) we see that in \( W_{1 \ldots N} \) the upper indices are \( q \)-antisymmetric, while the lower indices are \( q^{-1} \)-antisymmetric; this is actually true [cf. (10)] for any \( W_{1 \ldots k} \). In other words, in the expression \( \alpha = \frac{1}{k!} \alpha_{i_{1} \ldots i_{k}} dx^{i_{1}} \wedge \ldots dx^{i_{k}} \) we can consider \( \alpha_{i_{1} \ldots i_{k}} \) with \( q^{-1} \)-antisymmetrized indices.

### Proposition 8

\[
\varepsilon_{q}^{i_{1} \ldots i_{k}l_{k+1} \ldots l_{N}} \varepsilon_{(q-1)}^{j_{1} \ldots j_{k}l_{k+1} \ldots l_{N}} = (N-k)! W_{j_{1} \ldots j_{k}}^{i_{1} \ldots i_{k}} \quad (40)
\]

**Proof.** We use an induction procedure. Indeed relation (10) holds for \( k = N \); we consider it holds for \( k \) and we show it holds for \( k - 1 \). We have to prove that

\[
W_{j_{1} \ldots j_{k-1}i_{k}}^{i_{1} \ldots i_{k} - 1} = (N-k+1) W_{j_{1} \ldots j_{k-1}}^{i_{1} \ldots i_{k} - 1} \quad (41)
\]

or, by applying both sides of (41) to \( dx^{j_{1}} \otimes \ldots dx^{j_{k-1}} \) and using (18),

\[
\mathcal{I}_{b_{1} \ldots b_{j_{1}} \ldots i_{k} - 1}^{i_{1} \ldots i_{k - 1}} dx^{b_{1}} \wedge \ldots dx^{b_{k-1}} = (N-k+1) dx^{i_{1}} \wedge \ldots dx^{i_{k-1}} \quad (42)
\]

Now from \( A^{\rho}_{q_{k}} = \delta_{q}^{\rho} \) and (19) we have \( \text{Tr}_{k} \mathcal{I}_{1 \ldots k} = N I - \mathcal{I}_{1 \ldots k-1} \), where \( \text{Tr}_{k} \) means trace on the \( k \)-factor of \( \Omega^{\otimes k}(\mathbb{R}^{N}) \). Relation (12) is then proven by observing that \( A^{i_{w} \ldots i_{v}}_{b_{w} \ldots b_{v}} dx^{b_{w} \ldots b_{v}} \wedge dx^{i_{w} \ldots i_{v}} = -dx^{i_{w} \ldots i_{v}} \wedge dx^{i_{w} \ldots i_{v}} \). \( \blacksquare \)

Notice that since the epsilont tensor up to a sign is invariant under cyclic permutations [recall (23)] we also have

\[
\varepsilon_{q}^{i_{k+1} \ldots i_{N} i_{1} \ldots i_{k} } \varepsilon_{(q-1)}^{j_{k+1} \ldots j_{N} j_{1} \ldots j_{k} } = (N-k)! W_{j_{1} \ldots j_{k}}^{i_{1} \ldots i_{k}} \quad (43)
\]

and therefore

\[
W_{i_{1} j_{2} \ldots j_{k}}^{i_{1} j_{2} \ldots j_{k}} = (N-k+1) W_{i_{1} j_{2} \ldots j_{k}}^{i_{1} j_{2} \ldots j_{k}} \quad (44)
\]

The metric on \( \mathbb{R}^{N} \) induces the following pairing\( \langle \ , \rangle : \Omega^{\otimes k}_{1}(\mathbb{R}^{N}) \otimes \Omega^{\otimes k}_{1}(\mathbb{R}^{N}) \rightarrow \mathbb{R}^{N} \) (the tensor product \( \otimes \) is over \( \mathbb{R}^{N} \)):

\[
\langle dx^{i_{1}} \otimes \ldots dx^{i_{k}}, dx^{j_{1}} \otimes \ldots dx^{j_{k}} \rangle = \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} g^{i_{1} j_{1}} \ldots g^{i_{k} j_{k}} \quad (45)
\]

\( ^{4} \) The shell structure of this pairing is uniquely determined by requiring compatibility with the wedge product, see [51]. The sign \( (-1)^{\frac{k(k-1)}{2}} = (-1)^{\frac{k(k-1)}{2}} \) is introduced in order to obtain in the commutative limit the standard metric on \( k \)-forms.

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Lemma 9

In order to study the pairing between forms we need the following properties among metric and epsilon tensors

$$\langle \theta f, \theta' \rangle = \langle \theta, f \theta' \rangle.$$ \hspace{1cm} (46)

In order to study the pairing between forms we need the following properties among metric and epsilon tensors

**Lemma 9**

$$\det_q g = \epsilon^{i_1 \cdots i_N}_q g_{i_11} \cdots g_{i_NN} = \det g = (-1)^{\frac{N^2}{2}}$$ \hspace{1cm} (47)

$$\epsilon^{i_1 \cdots i_N}_q g_{j_1i_1} \cdots g_{j_Ni_N} = \epsilon^{i_1' \cdots i_N'}_q = \epsilon^{i_1 \cdots i_N}_q \det g$$ \hspace{1cm} (48)

$$\epsilon^{i_1 \cdots j_N}_q g_{j_1i_1} \cdots g_{j Ni_N} = \epsilon^{i_1 \cdots j_N}_q \det g$$ \hspace{1cm} (49)

**Proof.** Relation (47) follows from $g_{ij} = \delta_{ij'}$ and $\epsilon^{N,N-1 \cdots 1}_q = (-1)^{\frac{N^2}{2}} \epsilon^{1 \cdots N}_q$, a consequence of (21) and (33).

In order to prove (48) we observe that if for a given $N$-tuple $(i_1, \ldots, i_N)$ we have $\epsilon^{i_1 \cdots k \cdots i_N}_q = \epsilon^{i_1' \cdots k' \cdots i_N}_q \det g$ then also $\epsilon^{i_1 \cdots k \cdots i_N}_q = \epsilon^{i_1' \cdots k' \cdots i_N}_q \det g$. Since $\epsilon^{N,N-1 \cdots 1}_q = \epsilon^{1 \cdots N}_q \det g$, relation (48) can be proven by iterating this procedure.

Relation (49) follows from $\epsilon^{1 \cdots N}_q = \epsilon^{(q-1) \cdots 1 \cdots N}_q = \epsilon^{(q-1) \cdots N,N-1 \cdots 1}_q \det g$ and an iteration argument similar to the previous one. \hfill \blacksquare

In the following proposition we describe the coupling between forms.

**Proposition 10** The pairing $\langle \ , \rangle$ satisfies the following property

$$\langle dx^{a_1} \otimes \cdots \otimes dx^{a_k}, dx^{i_k} \wedge \cdots \wedge dx^{i_1} \rangle = \langle dx^{a_1} \wedge \cdots \wedge dx^{a_k}, dx^{i_k} \otimes \cdots \otimes dx^{i_1} \rangle,$$ \hspace{1cm} (50)

and when restricted to forms reads

$$\langle dx^{a_1} \wedge \cdots \wedge dx^{a_k}, dx^{i_k} \wedge \cdots \wedge dx^{i_1} \rangle = (-1)^{\frac{k(k+1)}{2}} g^{a_1 b_1} \cdots g^{a_k b_k} W^{i_k \cdots i_1}_{b_k \cdots b_1}.$$ \hspace{1cm} (51)

**Proof.** Relation (51) is equivalent to

$$g^{a_k b_k} \cdots g^{a_1 b_1} W^{i_k \cdots i_1}_{b_k \cdots b_1} = W^{a_1 \cdots a_k}_{b_1 \cdots b_k} g^{b_1 i_1} \cdots g^{b_k i_k},$$ \hspace{1cm} (52)

which holds for $k = N$ because of (40), and the previous lemma. Now, by induction, if (52) holds for $k$ then it holds for $k - 1$: indeed just multiply by $g_{a_k i_k}$ (summing over both $a_k$ and $i_k$) and recall (41) and (44). \hfill \blacksquare

The $\ast$–Hodge operator is defined as the unique map $\ast: \Omega_k(\mathbb{R}^N_q) \to \Omega_{N-k}(\mathbb{R}^N_q)$ such that

$$\alpha \wedge \ast \beta = \langle \alpha, \beta \rangle V_N \quad \alpha, \beta \in \Omega_k(\mathbb{R}^N_q).$$ \hspace{1cm} (53)

We collect in the following proposition the main properties of the $\ast$–Hodge operator.
Proposition 11 An explicit expression for $\ast$ is given by

$$\ast(dx^{i_1} \wedge \ldots dx^{i_k}) = C_{N,k} \epsilon_q^{i_1 \ldots i_k} g_{i_{k+1} \ldots i_{N+1}} dx^{i_{N+1}} \wedge \ldots dx^{i_k},$$

where $C_{N,k} = (-1)^{\left\lfloor \frac{N}{2} \right\rfloor} (-1)^{\frac{N-k}{2}}/(N-k)!$.

For all $\alpha, \beta \in \Omega_k(\mathbb{R}^N_q)$, $f, h \in \mathbb{R}^N_q$ and $\gamma \in \Omega_{N-k}(\mathbb{R}^N_q)$ we have

$$\ast(f \alpha h) = f(\ast \alpha) h \quad \text{R}_q^N \text{ left and right linearity},$$

$$(\ast \alpha)_i = (\ast \alpha)_k = 1,$$  

$$\ast \alpha = (\ast \alpha) = -1^{k(N-k)} \alpha, \quad \text{even } \alpha \wedge \ast \alpha = 0,$$ 

$$\langle \alpha, \beta \rangle = \langle \ast \alpha, \ast \beta \rangle, \quad \langle \alpha, \gamma \rangle = \langle \ast \alpha, \gamma, V_N \rangle.$$  

Proof. In order to prove (54) we have to show that $dx^{a_1} \wedge \ldots dx^{a_k} \wedge \ast(dx^{i_1} \wedge \ldots dx^{i_k}) = \langle dx^{a_1} \wedge \ldots dx^{a_k}, dx^{i_1} \wedge \ldots dx^{i_k} \rangle V_N$ where in the l.h.s. we use (54). Equivalently we have to show

$$(-1)^{\left\lfloor \frac{N}{2} \right\rfloor} (-1)^{\frac{N-k}{2}} \epsilon_q^{i_1 \ldots i_k} g_{i_{k+1} \ldots i_{N+1}} dx^{i_{N+1}} \wedge \ldots dx^{i_k} =$$

$$(\ast \alpha)_{a_1 a_2 \ldots a_k} g^{a_{b_1} \ldots a_{b_{k-1}}} W_{b_1 \ldots b_{k-1}}.$$  

This last equality can be proven multiplying by $g_{a_1 a_2 \ldots g_{a_1 a_k}}$, using (18) and (19), recalling that the $q$-epsilons tensor is invariant (up to a sign) under cyclic permutations and finally using (44). (53) follows from (46). The second relation in (56) follows from $\langle V_N, V_N \rangle = 1$. (57) can be proven as in the commutative case, for example using twice (54) and then (48), (49) and $W_{1 \ldots k} W_{1 \ldots k} = k! W_{1 \ldots k}$. Also (58) can be proved using (54) as in the commutative case. (59) and (60) are then easily shown using (53) and (58).

Remark 12 It is easy to verify that the $\ast$-Hodge operator defined in (53) coincides with the one defined in [7]. Indeed let’s denote the relations on the $n$-torus $T^n_0$ by $U_i^i U^{i'} = g_{ij} U_j^i U^{i'} = U^{i'} = U^{-1}$ and $U^i = 1$ if $i \neq i'$, where $i = 1, \ldots, N$, $N = 2n$ or $N = 2n+1$ and $g_{ij} = e^{i \theta_0 \beta} \alpha, \beta = 1, \ldots, n$. The exterior algebra $\Omega(\mathbb{R}^N_q)$ is then homomorphic to $\Omega(\mathbb{R}^N) \otimes C_{q\alpha}(T^N_0)$ via the identification $dx^{i_1} \wedge \ldots dx^{i_k} = dx_0^{i_1} \otimes \ldots dx_0^{i_k} \otimes U^{i_1} \ldots U^{i_k}$ where the $x_0$’s are the coordinates on the commutative plane. In particular [cf. (20)] we have $\epsilon_q^{i_1 \ldots i_N} = \epsilon_0^{i_1 \ldots i_N} U^{i_1} \ldots U^{i_N}$. By applying $* \otimes \text{id}$ to $dx_0^{i_1} \ldots dx_0^{i_k} \otimes U^{i_1} \ldots U^{i_k}$ (where $*_0$ is the commutative $\ast$-Hodge operator) we obtain expression (54) thus showing that $\ast = *_0 \otimes \text{id}$.

Conjugation

We now study the star structure that $\Omega(\mathbb{R}^N_q)$ inherits from $\mathbb{R}^N_q$. We recall that $(x^a)^* = x^a$ and that on 1-forms $(fdh)^* = d(h^*) f^*$. On $\Omega^{1,0}(\mathbb{R}^N_q)$ we define

$$(dx^{i_1} \otimes \ldots dx^{i_k})^* \equiv (-1)^{\left\lfloor \frac{k(k-1)}{2} \right\rfloor} dx^{i_k} \otimes \ldots dx^{i_1}.$$
Notice that the complex conjugate of $\epsilon^{i_1...i_N}_q$ is $\epsilon^{i'_1...i'_N}_{q'}$, so that using (49) and (48) $\bar{\epsilon}^{i_1...i_N}_q = \epsilon^{i'_1...i'_N}_{q'}$, then from (10) and (43) we have $W^{i_1...i_k}_{j_1...j_k} = W^{i'_1...i'_{k-1}}_{j_1...j_{k-1}}$, i.e.

$$(dx^i \wedge ... dx^i)^* = (-1)^{\frac{k(k-1)}{2}} dx^{i_k} \wedge ... dx^{i_1},$$

thus the space of exterior forms $\Omega(\mathbb{R}^N_q)$ naturally inherits the conjugation on $\Omega_1^{\otimes k}(\mathbb{R}^N_q)$.

In particular we have that the volume element $V_N$ is real: $V_N = V_N^*$.

**Proposition 13** Reality of the $*$-Hodge operator. For any form $\alpha$

$$* \alpha^* = (*\alpha)^*$$

**Proof.** (62) can be proven by explicit computation, using (54), recalling (48),(49) and again the invariance (up to a sign) of the $q$-epsilon tensor under cyclic permutations.

5.2 Hodge Theory on $S^N_q$

Recalling that the versor normal to the commutative unit sphere $S^N$ is $dc/2$, it is easy to see that forall $\theta, \theta' \in \Omega(S^N)$, the metric on $\Omega(S^N)$ and on $\Omega(\mathbb{R}^{N+1})$ are related by $\langle \theta, \theta' \rangle|_{S^N} = \langle \theta \wedge \frac{dc}{2}, \theta' \wedge \frac{dc}{2} \rangle|_{S^N}$. It is therefore natural to define in the noncommutative case, forall $[\alpha], [\beta] \in \Omega_k(S^N_q)$,

$$\langle [\alpha], [\beta] \rangle = \frac{1}{4}[\langle \alpha \wedge dc, \beta \wedge dc \rangle].$$

Independence from the representatives $\alpha$ and $\beta$ is easily proven. The $*$-Hodge map is then the unique map $*: \Omega_k(S^N_q) \to \Omega_{N-k}(S^N_q)$ such that

$$[\alpha] * [\beta] = \langle [\alpha], [\beta] \rangle V_N \quad [\alpha], [\beta] \in \Omega_k(S^N_q).$$

It is also easy to check that $J$ is a $*$-ideal: $J^* \subset J$. Then

$$[\alpha]^* = [\alpha^*]$$

is a well defined $*$-structure on $\Omega(S^N_q)$. Since $J$ is a differential ideal and a left coideal we have $d [\alpha]^* = (d[\alpha])^*$ and we have a $SO_q(N,\mathbb{R})$-coaction on $\Omega(S^N_q)$. Reality of the volume form $V_N^*$ follows from reality of $V_{N+1}$ and of $c$; we have

$$x^k \omega_k \wedge dc = 2cV_{N+1} = (2cV_{N+1})^* = (-1)^N dc^* \wedge \omega_k^* x^k = \omega_k^* x^k \wedge dc$$

and therefore $V_N^* = [x^k \omega_k]^* = [\omega_k^* x^k]^* = [x^k \omega_k] = V_N$.

We collect in the following proposition the main properties of the $*$-Hodge operator on $\Omega(S^N_q)$. 17
Proposition 14 An explicit expression for $*$ is given by

$$\ast [dx^i \wedge \ldots dx^k] = C'_{N,k} c_{g_{ab}g_{k+1} \ldots g_{N} g_{i_{N} t_{N}}} [dx^{i_{N}} \wedge \ldots dx^{i_{k+1}} x^b],$$  

(66)

where $C'_{N,k} = (-i)^{\frac{N+k}{2}} (N-k)!$. For all $\beta \in \Omega_k(\mathbb{R}^{N+1})$, $\theta, \eta \in \Omega_k(S^N)$, $f, h \in S^N_q$ and $\nu \in \Omega_{N-k}(S^N_q)$ we have

$$\ast[\beta] = (-1)^{N-k}[\ast(\beta \wedge \frac{dc}{2})] ,$$  

(67)

$$\ast(f \theta h) = f(\ast \theta) h \quad S^N_q \text{ left and right linearity},$$  

(68)

$$\ast 1 = V_N , \quad \ast V_N = 1 ,$$  

(69)

$$\ast \ast \theta = (-1)^{k(N-k)} \theta ,$$  

(70)

$$\theta \wedge \ast \eta = (-1)^{k(N-k)} \ast \theta \wedge \eta ,$$  

(71)

$$\langle \theta, \eta \rangle = \langle \ast \theta, \ast \eta \rangle ,$$  

(72)

$$\langle \ast \theta, \nu \rangle = \langle \theta \wedge \nu, V_N \rangle ,$$  

(73)

$$\ast \theta^* = (\ast \theta)^* \quad \text{reality of the $*$-Hodge}.$$  

(74)

Proof. Relation (67) is equivalent to $(-1)^{N-k}[\alpha \wedge \ast(\beta \wedge \frac{dc}{2})] = \langle [\alpha], [\beta] \rangle V_N$ i.e. (use $[c] = 1$)

$$\frac{1}{2} (-1)^{N-k}[c \alpha \wedge \ast(\beta \wedge dc)] = \frac{1}{4} \langle [\alpha \wedge dc, \beta \wedge dc] x^k \omega_k \rangle ;$$

this last relation holds because

$$\{ \frac{1}{2} (-1)^{N-k} c \alpha \wedge \ast(\beta \wedge dc) - \frac{1}{4} (\alpha \wedge dc, \beta \wedge dc) x^k \omega_k \} \wedge dc = \frac{1}{2} c \alpha \wedge dc \ast(\beta \wedge dc) - \frac{1}{2} (\alpha \wedge dc, \beta \wedge dc) c V_{N+1} = 0 .$$

Relation (66) follows from (67) and (54). Relation (68) follows from (64) and (46). Also (69) easily follows from (64). Relation (71) is equivalent to

$$\frac{1}{4} (-1)^{N-k} (-1)^k [\ast(\ast(dx^{i_1} \wedge \ldots dx^{i_k} \wedge dc) \wedge dc)] = (-1)^{k(N-k)} [dx^{i_1} \wedge \ldots dx^{i_k}] .$$  

(75)

Applying twice (54) and then (18) and (19) the l.h.s. equals

$$(-1)^{k(N-k)} \frac{1}{k!} W^{i_1 \ldots i_k}_{s_1 \ldots s_k} [dx_s^1 \wedge \ldots dx_s^k \ast g_{ab} x^f x^b] =$$

$$(-1)^{k(N-k)} [dx^{i_1} \wedge \ldots dx^{i_k} c - \mathcal{J}^{i_{i_1} \ldots i_k}_{u_1 \ldots u_{k-1} f} dx^{u_1} \wedge \ldots dx^{u_{k-1}} \wedge dc x^f]$$

where in the second line we used (13), (11) and $\Lambda^{v_{a}}_{u_{k} f} = q_{v_{a}} \delta^{u_{f}} \delta^{a}_{uk}$. Finally notice that this last expression is the same equivalence class as the one in the r.h.s. of (73). Relation (71) is equivalent to $[\alpha \ast(\beta \wedge dc)] = (-1)^{k(N-k)} [\ast(\alpha \wedge dc) \wedge \beta]$ (with $[\alpha] = \theta, [\beta] = \eta$). This equality holds if

$$\alpha \wedge \ast(\beta \wedge dc) \wedge dc = (-1)^{k(N-k)} \ast(\alpha \wedge dc) \wedge \beta \wedge dc$$

18
(\alpha \wedge dc) \wedge *(\beta \wedge dc) = (-1)^{(k+1)(N-k)} *(\alpha \wedge dc) \wedge (\beta \wedge dc)

and this last relation is true because of (38) (with \(N+1\) instead of \(N\)). (72) and (73) are then easily shown using (54) and (71). The reality property of the \(*\)–Hodge operator on \(\Omega(S^N_q)\) follows from that on \(\Omega(\mathbb{R}^{N+1}_q)\) and from \((\alpha \wedge dc)^* = \alpha^* \wedge dc\).

6 Geometry of the instanton bundle on \(S^{2n}_q\)

The purpose of this section is to apply the calculus developed in the previous sections to the twisted sphere \(S^{2n}_q\), namely to study the geometric properties of the instanton projector introduced in [3] for the 4–dimensional case and in [7] for the general case. As shown in [7] its curvature is self dual – for a different and more explicit proof in the 4–dimensional case see the first version of this paper, math.QA/0108136v1.

We here compute the charge, \(i.e.\) the Chern–Connes pairing between the projector \(e\) and the character \(\tau\) defined in (33).

We briefly recall some basic notions about the coupling between cyclic homology and \(K\)–theory, (general references for this section are [3] and [14]). Let \(A\) be an associative \(\mathbb{C}\)–algebra. A projector \(e \in M_k(A)\), \(i.e.\) \(e^2 = e\), defines a finitely generated projective module \(E = eA^k\), which we consider as the space of sections of a quantum vector bundle. In the Grothendieck group \(K_0(A)\) the module \(E\) defines a class, that we still denote with \(e\). Topological informations of this quantum bundle are extracted by using cyclic homology and cohomology, which are defined as follows. Let \(d_i : A^\otimes(n+1) \to A^\otimes n\) be defined as \(d_i(a_0 \otimes a_1 \cdots \otimes a_n) = a_0 \otimes \cdots a_i a_{i+1} \cdots \otimes a_n\), for \(i = 0, \ldots n-1\) and \(d_n(a_0 \otimes a_1 \cdots \otimes a_n) = a_n a_0 \otimes a_1 \cdots \otimes a_{n-1}\); the Hochschild boundary is defined as \(b = \sum_{i=0}^n (-1)^i d_i\) and the Hochschild complex is \((C_*(A), b)\), with \(C_n(A) = A^\otimes n+1\). Let \(t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_1 \otimes \cdots a_n \otimes a_0\) be the cyclic operator and \(C^\lambda_n(A) = A^\otimes n+1/(1-t)A^\otimes n+1\). The Connes complex is then \((C^\lambda_*(A), b)\); its homology is denoted as \(H^\lambda_*(A)\). Analogously we define \(H^\lambda_*(A)\) as the cohomology of the complex \((C^\lambda_*(A), b)\), where \(C^\lambda_n = \{\tau : A^\otimes(n+1) \to \mathbb{C} | \tau \circ t_n = (-1)^n \tau\}\) and \(b(\tau) = \tau \circ b\).

Cycles give an alternative way of introducing cyclic cocycles. A cycle \((\Omega, d, \int)\) over \(A\) of dimension \(n\) is given by a differential graded algebra \((\Omega = \bigoplus_{k=0}^n \Omega_k, d)\), a closed graded trace \(\int : \Omega_n \to \mathbb{C}\) and an algebra morphism \(\rho : A \to \Omega_0\). More explicitly these data must satisfy:

\[
d(\omega \nu) = (d\omega) \nu + (-1)^{|\omega|} \omega d\nu \, , \quad d^2 = 0 \quad \int \omega \nu = (-1)^{|\omega||\nu|} \int \nu \omega \, , \quad \int d\omega = 0 \, .
\]

The character of the cycle \((\Omega, d, \int)\) over \(A\), defined as \(\tau : A^\otimes(n+1) \to \mathbb{C}\),

\[
\tau(a_0 \otimes \cdots a_n) = \int \rho(a_0) \rho(a_1) \cdots \rho(a_n) \, ,
\]
is a cyclic $n$–cocycle; we still denote its class in $H^n_\chi$ with $\tau$. It can be shown that all cyclic cocycles are characters of some cycle.

For each projector $e \in M_k(A)$, i.e. $e^2 = e$, the Chern character is a class in cyclic homology and is defined as $cch^\lambda_m(e) = 1/n! \text{Tr}(e^{\otimes 2n+1}) \in H^{2n}_\chi(A)$, where $\text{Tr} : M_k(A)^{\otimes n} \to A^{\otimes n}$ is the generalized trace, i.e. $\text{Tr}(N^{(1)} \otimes \ldots \otimes N^{(n)}) = \sum_{\alpha} N^{(1)}_{\alpha_1 \alpha_2} \otimes N^{(2)}_{\alpha_2 \alpha_3} \ldots \otimes N^{(n)}_{\alpha_n \alpha_1}$. For each $2n$–cocyelce $\tau$ the Chern–Connes pairing given by $\langle e, \tau \rangle = \tau(cch^\lambda_m(e))$ depends only on the class of $e$ in $K_0(A)$ and of $\tau$ in $H^{2n}_\chi$ and can be computed in the following way. Let $(\Omega, d, J)$ be the cycle over $A$ of dimension $2n$ associated to $\tau$. We canonically define a connection on $\mathcal{E} = eA^k$ in the following way. Let $f_\alpha = ev_\alpha$, where $(v_\alpha)_\beta = \delta_{\alpha \beta}$; the Levi–Civita connection $\nabla^e : \mathcal{E} \to \mathcal{E} \otimes_A \Omega_1$ is defined as

$$\nabla^e(f_\alpha) = \sum_{\beta} f_\beta \otimes de_{\beta \alpha}.$$  

The curvature $\mathcal{F}^e = \nabla^2 : \mathcal{E} \to \mathcal{E} \otimes \Omega_2$ is $A$–linear and defines a $k \times k$–matrix of two forms given by $\mathcal{F}^e = edede$. The Chern–Connes pairing between the character of the cycle and $\mathcal{E}$ is computed by the following formula:

$$\langle e, \tau \rangle = \frac{1}{n!} \tau(\text{Tr}(e^{\otimes 2n+1})) = \frac{1}{n!} \int \text{Tr}((\mathcal{F}^e)^n).$$  

Let us come back to $S^\otimes_{\mathbb{Q}}$ and let us introduce the Clifford algebra defined in $[\mathbb{I}]$. We have that $\text{Cliff}(\mathbb{R}^{2n+1})$ is generated by $2n + 1$ generators $\gamma^i$ and the following relations:

$$\gamma^i \gamma^j + q_{ij} \gamma^j \gamma^i = 2g^{ij};$$  

let us remark that the chiral $\gamma$ of $[\mathbb{I}]$ is included among the fundamental $\gamma^i$’s and corresponds to $\gamma^{n+1}$. The unique irreducible representation is given on $\otimes^n \mathbb{C}^2$ by

$$\gamma^i = \sqrt{2} \left( \begin{array}{c} -q_{i1} & 0 \\ 0 & 1 \end{array} \right) \otimes \ldots \otimes \left( \begin{array}{c} -q_{i_{n-1}} & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \otimes 1 \ldots 1, \quad i < n$$  

$$\gamma^i = \gamma^{i \dagger}$$  

$$\gamma^{n+1} = \otimes^n \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).$$  

As a consequence of (78) $e = \frac{1}{2}(1 + \gamma^i [x^j] g_{ij})$ is a projector.

Let $(\Omega(S^\otimes_{\mathbb{Q}}), d, J)$ be the cycle over $S^\otimes_{\mathbb{Q}}$ defined in Proposition $[\mathbb{I}]$ and let $\tau$ be its character defined in $[\mathbb{I}]$. Let $\mathcal{V}^e$ be the canonical connection with values in $\Omega(S^\otimes_{\mathbb{Q}})$ and let $\mathcal{F}^e = edede$ be its curvature. As a consequence of $e = e^*$ and of (78) $\mathcal{F}^e$ is antihermitian, i.e. $\mathcal{F}^e_{\alpha \beta} = -\mathcal{F}^e_{\beta \alpha}$, $\alpha, \beta = 1, \ldots 2n$. Moreover in $[\mathbb{I}]$ and in $[\mathbb{II}]$ it is shown that $cch^\lambda_m(e) = 0$ for $m < n$. In the following proposition we compute the maximal Chern character and verify that the normalization of $\tau$ discussed in $[\mathbb{II}]$ still guarantees the integrality of the pairing.
Proposition 15 \textit{The charge of the instanton projector on }$S^2_q$\textit{, i.e. the Chern–Connes pairing between }$e$\textit{ and }$\tau$\textit{, reads}

$$\langle e, \tau \rangle = \frac{1}{n!} \tau(\text{Tr}[e^{\otimes 2n+1}]) = 1.$$ 

\textit{Proof.} By the use of the faithful representation (78) it can be shown that

$$\text{Tr}(\gamma_{i_0} \cdots \gamma_{i_{2n}}) = 2^n \epsilon_{(q^{-1})_{i_0} \cdots i_{2n}}.$$ 

Since thanks to Stokes’ Theorem we can ignore $\text{Tr}([de]^{2n})$ (it can be shown that it is zero anyway), we have that

$$\int \text{Tr}[e(de)^{2n}] = \int \frac{1}{2^{2n+1}} \text{Tr}(\gamma_{i_0} \cdots \gamma_{i_{2n}})[x^{i_0} dx^{i_1} \cdots dx^{i_{2n}}]$$

$$= \int \frac{1}{2^{n+1}} \epsilon_{(q^{-1})_{i_0} \cdots i_{2n}}[x^{i_0} dx^{i_1} \cdots dx^{i_{2n}}]$$

$$= \int \frac{(-1)^n}{2^{n+1}} \epsilon_{(q^{-1})_{i_0} \cdots i_{2n}}[x^{i_0} dx^{i_1} \cdots dx^{i_{2n}}]$$

$$= \int \frac{(2n)!}{2^{n+1}} \epsilon^i \omega_i = \int \frac{(2n)!}{2^{n+1}} \epsilon^i \nu_{2n},$$

where we used (18) in the third line and the definition (34) in the last line. The result then follows by applying the definition (56) of $\tau$.

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