Localized Superluminal solutions to the wave equation in (vacuum or) dispersive media, for arbitrary frequencies and with adjustable bandwidth(†)

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Abstract — In this paper we set forth new exact analytical Superluminal localized solutions to the wave equation for arbitrary frequencies and adjustable bandwidth. The formulation presented here is rather simple, and its results can be expressed in terms of the ordinary, so-called “X-shaped waves”. Moreover, by the present formalism we obtain the first analytical localized Superluminal approximate solutions which represent beams propagating in dispersive media. Our solutions may find application in different fields, like optics, microwaves, radio waves, and so on.

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1. – Introduction

For many years it has been known that localized (non-dispersive) solutions exist to the (homogeneous) wave equation\cite{1,2,3}, endowed with subluminal or Superluminal\cite{4,5,6,7,8,9} velocities. These solutions propagate without distortion for long distances in vacuum.

Particular attention has been paid to the localized Superluminal solutions like the so-called X-waves\cite{5,6,8} and their finite energy generalizations\cite{7,8}. It is well known that such Superluminal Localized Solutions (SLS) have been experimentally produced in acoustics\cite{10}, optics\cite{11} and more recently microwave physics\cite{12}.

As is well known, the standard X-wave has a broad band frequency spectrum, starting from zero\cite{8,9} (it being therefore appropriate for low frequency applications). This fact can be viewed as a problem, because it is difficult or even impossible to define a carrier frequency for that solution, as well as to use it in high frequency applications.

Therefore, it would be very interesting to obtain exact SLSs to the wave equations with spectra localized at higher (arbitrary) frequencies and with an adjustable bandwidth (in other words, with a well defined carrier frequency).

To the best of our knowledge, only two attempts were made in this direction: one by Zamboni-Rached \textit{et al.}\cite{8}, and the other by Saari\cite{13}. The former showed how to shift the spectrum to higher frequencies without dealing with its bandwidth, while the latter worked out an analytical approximation to optical pulses only.

In this work we are presenting analytical and exact Superluminal localized solutions in vacuum, whose spectra can be localized inside any range of frequency with adjustable bandwidths, and therefore with the possibility of choosing a well defined carrier frequency. In this way, we can get (without any approximation) radio, microwave, optical, etc., localized Superluminal waves.

Taking advantage of our methodology, we obtain the first analytical approximations to the SLS’s in dispersive media (i.e., in media with a frequency dependent refractive index).

One of the interesting points of this work, let us stress, is that all results are obtained from simple mathematical operations on the standard “X-wave”.

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\[\text{Equation}\]

\[\text{Figure}\]

\[\text{Table}\]
2. – Superluminal localized waves in dispersionless media

Let us start by dealing with SLSs in dispersionless media. From the axially symmetric solution to the wave equation in vacuum \((n = 1)\), in cylindrical coordinates, one can easily find that

\[
\psi(\rho, z, t) = J_0(k_\rho \rho) e^{+i k_z z} e^{-i \omega t} \tag{1}
\]

with the conditions

\[
k^2_\rho = \frac{\omega^2}{c^2} - k^2_z ; \quad k^2_\rho \geq 0 , \tag{2}
\]

where \(J_0\) is the zeroth-order ordinary Bessel function; \(k_z\) and \(k_\rho\) are the axial and the transverse wavenumber respectively, \(\omega\) is the angular frequency and \(c\) is the light velocity.

It is essential to call attention right now to the dispersion relation (2). Positive (but not constant, a priori) values of \(k^2_\rho\), with real \(k_z\), do allow both subluminal and Superluminal solutions, while implying truly propagating waves only (with exclusion of the evanescent ones). We shall pay attention in this paper to the Superluminal solutions. Conditions (2) correspond in the \((\omega, k_z)\) plane to confining ourselves to the sector shown in Fig.1; that is, to the region delimited by the straight lines \(\omega = \pm ck_z\).

Important consequences can be inferred from (2), when performing the coordinate transformation

\[
\begin{aligned}
\omega &= \omega \\
k_z &= (\omega/c) \cos \theta,
\end{aligned} \tag{3}
\]

which yields

\[
k_\rho = (\omega/c) \sin \theta . \tag{4}
\]

With this transformation, solution (1) can be rewritten, in the new coordinates \((\omega, \theta)\), as

\[
\psi(\rho, \zeta) = J_0\left(\frac{\omega}{c} \rho \sin \theta\right) e^{+i \omega \zeta \cos \theta} , \tag{5}
\]
where $\zeta \equiv z - Vt$, and where the propagation speed (group velocity) is obviously $V = c/\cos \theta$. Equation (5) states that the beam is transversally localized in energy, and propagates without suffering any dispersion. It should be noticed also the relationship between $V$ and $\theta$: namely, each value of $\theta$ yields a different wave velocity. This fact will be used in the next Section.

Equation (5) represents the well known “Bessel beam”. As can be seen, such an equation has two free parameters, $\omega$ and $\theta$. Considering $\theta$ constant, and making a superposition of waves for different frequencies, one can obtain localized (non-dispersive), Superluminal solutions; namely

$$\Psi(\rho, \zeta) = \int_0^\infty S(\omega)J_0\left(\frac{\omega}{c}\rho \sin \theta\right) e^{+i\omega\zeta \cos \theta} d\omega.$$  

In eq.(6), if an exponential spectrum $\Psi(\omega) = e^{-a\omega}$ is considered, one obtains, by use of identity (6.611.1) of ref.[14], the ordinary X-shaped wave:

$$\Psi(\rho, \zeta) = \frac{1}{\sqrt{(aV - i\zeta)^2 + (\frac{V^2}{c^2} - 1)\rho^2}},$$  

where $a$ is a positive constant.

This solution is a wave that propagates in free space without distortion and with the Superluminal velocity $V = 1/\cos \theta$. Because of its non-dispersive properties, and its low frequency spectrum, the X-wave is being particularly applied in fields like acoustics[5]. The illustration of an X-wave, with parameters $a = 10^{-7}$ s and $V = 5 c$, is shown in Fig.2.

### 3. – Superluminal localized waves for arbitrary frequencies and adjustable bandwidths

In the last Section, it has been shown that a superposition of Bessel beams can be used to obtain a localized and Superluminal solution to the wave equation in a dispersionless medium. It is known that it may be a difficult task, it being possible, or not, finding analytical expressions for eq.(4). Its numerical solutions usually brings in some

*It is easy to see that this spectrum starts from zero, it being suitable for low frequency applications, and has the bandwidth $\Delta \omega = 1/a$
inconveniences for further analysis, uncertainties concerning the fast oscillating field components, etc.; besides implying a loss in the physical interpretation of the results. Thus, it is always worth looking for analytical expressions.

Actually, the kind of solution found by us for eq.(6) is strictly related to the chosen spectrum $S(\omega)$. Following previous work of ours[8], we are going to present our spectrum together with its main characteristics.

3.1 – The $S(\omega)$ spectrum

One of our main objectives is finding out a spectrum which can preserve the integrability of eq.(6) for any frequency range. In order to be able to shift our spectrum towards the desired frequency, let us locate it around a central frequency, $\omega_c$, with an arbitrary bandwidth $\Delta \omega$.

Then, let us choose the spectrum

$$S(\omega) = \left(\frac{\omega}{V}\right)^m e^{-a\omega}$$

(8)

where $V$ is the wave velocity, while $m$ and $a$ are free parameters. For $m = 0$, it is $S(\omega) = \exp[-a\omega]$, and one gets the (standard) X-wave spectrum.

After some mathematical manipulations, one can easily find the following relations, valid for $m \neq 0$:

$$m = \frac{1}{(\Delta \omega/\omega_c) - \ln (1 + (\Delta \omega/\omega_c))}$$

(8.1)

$$\omega_c = \frac{m}{a}.$$  

(8.2)

Here, because of the non-symmetric character of spectrum (8), let us call $\Delta \omega_+ (>0)$ the bandwidth to the right, and $\Delta \omega_- (<0)$ the bandwidth to the left of $\omega_c$; so that $\Delta \omega = \Delta \omega_+ - \Delta \omega_-$. It should be noted however that, already for small values of $m$ (typically, for $m \geq 10$), one has $\Delta \omega_+ \approx -\Delta \omega_-$. Once defined $\omega_c$ and $\Delta \omega$, one can determine $m$ from the first equation. Then, using the second one, $a$ is found.

Figure 3 illustrates the behavior of relation (8.1). From this figure, one can observe that the smaller $\Delta \omega/\omega_c$ is, the higher $m$ must be. Thus, one can notice that $m$ plays the
fundamental role of controlling the spectrum bandwidth.

From the X-wave spectrum, it is known that $a$ is related to the (negative) slope of the spectrum. Contrarily to $a$, quantity $m$ has the effect of rising the spectrum. In this way, one parameter compensates for the other, producing the localization of the spectrum inside a certain frequency range. At the same time, this fact also explains (because of relation (8.2)) why an increase of both $m$ and $a$ is necessary to keep the same $\omega_c$. This can be seen from Fig.4.

In Fig.4, both spectra have the same $\omega_c$. Taking the narrow spectrum as a reference, one can observe that, to get such a result, both quantities $m$ and $a$ have to increase. Moreover, this figure shows the important role of $m$ for generating a wider, or narrower, spectrum.

### 3.2 – X-type waves in a dispersionless medium

To illustrate the use of the proposed solutions, let us define the ordinary X-wave, by rewriting eq. (6) with $S(\omega) = \exp[-a\omega]$:

$$
\Psi(\rho, \zeta) = \int_0^\infty J_0(\frac{\omega}{V}\rho \sqrt{n_0^2 \frac{V^2}{c^2} - 1}) e^{-(aV - i\zeta)\omega/V} d\omega 
$$

$$
= V \int_0^\infty J_0(\frac{\omega}{V}\rho \sqrt{n_0^2 \frac{V^2}{c^2} - 1}) e^{-(aV - i\zeta)\omega/V} d(\frac{\omega}{V}) ,
$$

which is the same as

$$
X \equiv \Psi(\rho, \zeta) = \frac{V}{\sqrt{(aV - i\zeta)^2 + \rho^2(n_0^2 \frac{V^2}{c^2} - 1)}},
$$

where $n_0$ is the refractive index of the medium underlying these considerations. Applying our spectrum expressed by eq.(8), equation (6) can be rewritten as

$$
\Psi(\rho, \zeta) = V \int_0^\infty (\frac{\omega}{V})^n J_0 \left(\frac{\omega}{V}\rho \sqrt{n_0^2 \frac{V^2}{c^2} - 1}\right) e^{-(aV - i\zeta)\omega/V} d(\frac{\omega}{V}) .
$$
We have therefore seen that the use of a spectrum like (8) allows shifting it towards any frequency and confining it within the desired frequency range. In fact, this is one of its most important characteristics.

It can be seen that eqs.(9) and (10) are equivalent. Deriving eq.(9) with respect to \((aV - i\zeta)\), a multiplicative factor \((\frac{\omega}{V})\) is each time produced (an obvious property of Laplace transforms). In this way, it is possible to write eq.(11) as:

\[
\Psi(\rho, \zeta) = (-1)^m \frac{\partial^m X}{\partial(aV - i\zeta)^m}.
\]  

(12)

A different expression for eq.(11), without any need of calculating the \(m\)-th derivative of the X-wave, can be found by using identity (6.621) of ref.[14]:

\[
\Psi(\rho, \zeta) = \frac{\Gamma(m + 1)}{V^m} F\left(\frac{m + 1}{2}, -\frac{m}{2}; 1; \left(\frac{\rho^2 X^2}{V^2} - 1\right) \left(\frac{n_0 V^2}{c^2} - 1\right) \right),
\]

(12′)

where \(X\) is the ordinary X-wave given in eq.(10), and \(F\) is a Gauss’ hypergeometric function. Equation (12′) can be useful in the cases of large values of \(m\).

Let us call attention to equations (12) and (12′): to our knowledge, no analytical expression had been previously met for X-type waves, which can be localized in the neighbourhood of any chosen frequency with an adjustable bandwidth. Equations (12) allow getting one or more of them in a simple way: All that has to be done is calculating the \(m\)-th derivative of \(X\) with respect to \((aV - i\zeta)\). Alternatively, one can have recourse to eq.(12′).

Fig.5 shows an example of an X-shaped wave for microwave frequencies. To that aim, it was chosen \(\omega_c = 6 \times 10^9\) GHz and \(\Delta\omega = 0.9 \omega_c\), and the values of \(n\) and \(a\) were calculated by using eqs.(8.1) and (8.2): thus obtaining \(m = 10\) and \(a = 1.6667 \times 10^{-9}\). As one can see, the resulting wave has really the same shape and the same properties as the classical X-waves: namely, both a longitudinal and a transverse localization.

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†It can be noticed that \(\partial X/\partial(aV - i\zeta) = (iV)^{-1}\partial X/\partial t\). Time derivatives of the X-wave have been actually considered by J.Fagerholm et al.[15]: however the properties of the spectrum generating those solutions (like its shifting in frequency and its bandwidth) did not find room in that previous work.
4. – Superluminal localized waves in dispersive media

We shall now pass to dealing with dispersive media.

In Section (2), equations (3) and (4) were written for a dispersionless medium ($n_0 = $ constant, independent of the frequency). However, for a typical medium, when the refractive index depends on the wave frequency, $n(\omega)$, those equations become\[13\]

\[
\begin{align*}
k_{\rho}(\omega) &= \frac{\omega}{c} n(\omega) \sin(\theta) \\
k_{z}(\omega) &= \frac{\omega}{c} n(\omega) \cos(\theta)
\end{align*}
\]

The above equations describe one of the basic points of this work. In Section (2) it was mentioned that $\theta$ determines the wave velocity: a fact that can be exploited when one looks for a localized wave that does not suffer dispersion. In other words, one can choose a particular frequency dependence of $\theta$ to compensate for the (geometrical) dispersion due to the variation with the frequency of the refractive index\[13\].

If the frequency dependence of the refractive index in a medium is known, within a certain frequency range, let us see how the consequent dispersion can be compensated for. When a dispersionless pulse is desired, the constraint $k_z = a + \omega b$ must be satisfied. And, by using the last term in eq.(13), one infers that such a constraint is forwarded by the following relationship between $\theta$ and $\omega$:

\[
\cos(\theta(\omega)) = \frac{a + b \omega}{\omega n(\omega)} ,
\]

where $a$ and $b$ are arbitrary constants (and $b$ is related to the wave velocity: $b = 1/V$). For convenience, we shall consider $a = 0$. Then, eq.(13) can be rewritten as

\[
\Psi(\rho, \zeta) = \int_0^\infty S(\omega) J_0(\rho \omega \sqrt{\frac{n^2(\omega)}{c^2} - b^2}) e^{+i\omega\zeta} d\omega .
\]

Let us stress that this equation is a priori suited for many kinds of applications. In fact, whatever its frequency be (in the optical, acoustic, microwave,... range), it constitutes the integral formula representing a wave which propagates without dispersion in a dispersive medium.
Now, let us mention how it is possible to realize relation (14) for optical frequencies. Although limited to the case of the air, or of low-dispersion media, the axicon[5,6,11,16] is one of the simplest means to realize it. Another possibility is using “spectral hole burning filters”, or holograms[17]. More in general, one can follow a procedure similar to the one illustrated in Figure 6.

The process illustrated in Fig.6 is actually simple. In fact, there is a different deviation of the wave vector for each spectral component in passing through the chosen device (axicon, hologram, and so on): and such a deviation, associated with the dispersion due to the medium, makes the phase velocity equal for each frequency. This corresponds to no dispersion for the group-velocity. More details about the physics under consideration can be found in Ref.[13].

Now, let us consider a nearly gaussian spectrum as that given by eq.(8), and assume the presence of a dispersive medium whose refractive index (for the frequency range of interest) can be written in the form

\[ n(\omega) = n_0 + \omega \delta, \]  

where \( n_0 \) is a constant, while \( \delta \) is a free parameter that makes it possible a linear behavior of \( n(\omega) \): something that is actually realizable for frequencies far from the resonances associated with the used material. Notice that the linear relationship between the refractive index and the wave frequency assumed in eq.(16) is not necessary: but its existence gets our calculations simplified.

In this way, substituting eq.(16) into eq.(15), and considering the spectrum, shifted towards optical frequencies, given by eq.(8), a relation similar to eq.(17) is found:

\[ \Psi(\rho, \zeta, \delta) = V \int_0^{\infty} \left( \frac{\omega}{V} \right)^n J_0 \left( \frac{\sqrt{V^2 c^2 (n_0 + \delta \omega)^2 - 1}}{\omega V} \right) e^{-(aV-i\zeta)\omega/V} d\left( \frac{\omega}{V} \right). \]  

To the purpose of evaluating eq.(17), let us make a Taylor expansion and rewrite it as

\[ \Psi(\rho, \zeta, \delta) = \Psi(\rho, \zeta, 0) + \delta \frac{\partial \Psi}{\partial \delta} \bigg|_{\delta=0} + \frac{\delta^2}{2!} \frac{\partial^2 \Psi}{\partial \delta^2} \bigg|_{\delta=0} + \frac{\delta^3}{3!} \frac{\partial^3 \Psi}{\partial \delta^3} \bigg|_{\delta=0} + \ldots \]  

For the above equation it is known that, if \( \delta \) is small enough, it is possible to truncate the series at its first derivative. For the time being, let us assume this is the case and
that there is no problem on truncating eq.(18). One can check Fig.7, which shows typical values of $\delta$ for SiO$_2$, a typical raw-material in fiber optics.

Looking at eq.(18), one can notice that its first term $\Psi(\rho, \zeta, 0)$ is already known to us, because it coincides with the solution given by our eq.(11). To complete the expansion (18), one must find $\frac{\partial \Psi}{\partial \delta}|_{\delta=0}$. After some simple mathematical manipulations, one gets that

$$\frac{\partial \Psi}{\partial \delta}|_{\delta=0} = -\frac{V^4 \rho n_0}{c^2 \sqrt{n_0^2 \frac{V^2}{c^2} - 1}} \int_0^\infty (\frac{\omega}{V})^{m+2} e^{-(aV-i\zeta)\omega/V} J_1\left(\frac{\omega}{V} \sqrt{n_0^2 \frac{V^2}{c^2} - 1}\right) d(\frac{\omega}{V}) .$$

This integral can be easily evaluated by using identity 6.621-4 of ref.[14], so to obtain

$$\frac{\partial \Psi}{\partial \delta}|_{\delta=0} = (-1)^{m+4} \frac{V^3 n_0}{c^2 \left(n_0^2 \frac{V^2}{c^2} - 1\right)} \frac{\partial^{m+2}}{\partial (aV-i\zeta)^{m+2}} [(aV - i\zeta) X] ,$$

(20)

As in the case of eqs.(12), (12‘), another form for expressing eq.(19) can be found by having recourse once more to the identity (6.621) of ref.[14]:

$$\frac{\partial \Psi}{\partial \delta}|_{\delta=0} = -\frac{n_0 \rho^2 \Gamma(m+4) X^{m+4}}{2 c^2 V^m} F\left(m+4, -m-1; 2; \frac{n_0^2 V^2}{c^2} - 1\right) \left(\frac{X^2}{V^2}\right) .$$

(20’)

where, as before, $X$ is the ordinary X-wave given by eq.(10) and $F$ is again a Gauss’ hypergeometric function. Once more, equation (20’) can be useful in the cases of large values of $m$.

Finally, from our basic solution (12) and its first derivative (20), one can write the desired solution of eq.(15) as

$$\Psi(\rho, \zeta) = (-1)^n \frac{\partial^n X}{\partial (aV - i\zeta)^n} + (-1)^{n+4} \frac{V^3 n_0}{c^2 \left(n_0^2 \frac{V^2}{c^2} - 1\right)} \frac{\partial^{n+2}}{\partial (aV - i\zeta)^{n+2}} [(aV - i\zeta) X] \delta .$$

(21)
However, if one wants to use equations (12') and (20'), instead of eqs. (12) and (20), the solution (21) can be written in the form

\[
\Psi(\rho, \zeta) = \frac{\Gamma(m+1)}{V_m} X^{m+1} F\left(\frac{m+1}{2}; \frac{-m}{2}; 1; (n_0^2 V^2 - 1) \rho^2 X^2 \right) \\
- n_0 \rho^2 \frac{\Gamma(m+4)}{2 c^2 V_m} X^{m+4} F\left(\frac{m+4}{2}; \frac{-m-1}{2}; 2; (n_0^2 V^2 - 1) \rho^2 X^2 \right),
\]

(21')

It is also interesting to notice that, e.g., the approximated Superluminal localized solution (21) for a dispersive medium has been obtained from simple mathematical operations (derivatives) applied to the standard “X-wave”.

5. – Optical applications

To illustrate what was said before, two practical examples will be considered, both in optical frequencies. When mentioning optics, it is natural to refer ourselves to optical fibers. Then, let us suppose the bulk of the dispersive medium under consideration to be fused Silica (SiO$_2$).

Far from the medium resonances (which is our case), the refractive index can be approximated by the well-known Sellmeier equation[18]

\[
n^2(\omega) = 1 + \sum_{j=1}^{N} \frac{B_j \omega_j^2}{\omega_j^2 - \omega^2},
\]

(22)

where \( \omega_j \) is the resonance frequency, \( B_j \) is the strength of the \( j \)th resonance, and \( N \) is the total number of the material resonances that appear in the frequency range of interest. For typical frequencies of “long-haul transmission” in optics, it is necessary to choose \( N = 3 \), which leads us to the values[18] \( B_1 = 0.6961663 \), \( B_2 = 0.4079426 \), \( B_3 = 0.8974794 \), \( \lambda_1 = 0.0684043 \) μm, \( \lambda_2 = 0.1162414 \) μm and \( \lambda_3 = 9.896161 \) μm.

Figure 7 illustrates the relation between \( N \) and \( \omega \), and specifies the range that will be adopted here. In the two examples, the spectra are localized around the angular frequency \( \omega_c = 23.56 \times 10^{14} \) Hz (which corresponds to the wavelength \( \lambda_c = 0.8 \) μm), with two different
bandwidth $\Delta \omega_1 = 0.55\omega_c$ and $\Delta \omega_2 = 0.4\omega_c$. The values of $a$ and $N$ corresponding to these two situations are $a = 1.14592 \times 10^{-14}$, $N = 27$, and $a = 1.90986 \times 10^{-14}$, $N = 41$, respectively.

Looking at these “windows”, one can notice that Silica does not suffer strong variations of its refractive index. As a matter of fact, a linear approximation to $n = n(\omega)$ is quite satisfactory in these cases. Moreover, for both situations, and for their respective $n_0$ values, the value of parameter $\delta$ results to be very small, verifying condition (15); which means that it is quite acceptable our truncation of the Taylor expansion. The beam intensity profiles for both bandwidths are shown in Figs. 8 and 9.

In the first figure, one can see a pattern similar to that of Fig.2; but here, of course, the pulse is much more localized spatially and temporally (typically, it is a femtosecond pulse).

In the second figure, one can observe some little differences with respect to the first one, mainly in the spatial oscillations inside the wave envelope[19]. This may be explained by taking into account that, for certain values of the bandwidth, the carrier wavelength become shorter than the width of the spatial envelope; so that one meets a well defined carrier frequency.

Let us point out that both these waves are transversally and longitudinally localized, and that, since the dependence of $\Psi$ on $z$ and $t$ is given by $\zeta = z - Vt$, they are free from dispersion, just like a classical X-shaped wave.

6. – Conclusions

In this paper we have first worked out analytical Superluminal localized solutions to the wave equation for arbitrary frequencies and with adjustable bandwidth in vacuum. The same methodology has been then used to obtain new, analytical expressions representing X-shaped waves (with arbitrary frequencies and adjustable bandwidth) which propagate in dispersive media. Such expressions have been obtained, on one hand, by adopting the appropriate spectrum (which made possible to us both choosing the carrier frequency rather freely, and controlling the spectral bandwidth), and, on the other hand, by having recourse to simple mathematics. Finally, we have illustrated some examples of our
approach with applications in optics, considering fused Silica as the dispersive medium.

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Figure Captions

Figure 1 – Geometrical representation, in the plane $(\omega, k_z)$, of the condition (2): see the text.

Figure 2 – Illustration of the real part of an X-wave with bandwidth, $\Delta \omega$, of 10 MHz starting from zero.

Figure 3 – Behavior of the derivative number, $m$, as a function of the normalized bandwidth frequency, $\Delta \omega_\pm / \omega_c$. Given a central frequency, $\omega_c$, and a bandwidth, $\Delta \omega_\pm$, one finds the exact value of $m$ by substituting these values into eq.(8.1).

Figure 4 – Normalized spectra for $\omega_c = 23.56 \times 10^{14}$ Hz and different bandwidths. The first with $N = 27$ (solid line), and the second spectrum with $N = 41$ (dotted line). See the text.

Figure 5 – The real part of an X-shaped beam for microwave frequencies in a dispersion-less medium.

Figure 6 – Sketch of a generic device (axicon, hologram, etc.) suited to properly deviating the wave vector of each spectral component.

Figure 7 – Variation of the refractive index $n(\omega)$ with frequency for fused Silica. The solid line is its behaviour, according to Sellmeir’s formulae. The open circles and squares are the linear approximations for $N = 41$ and $N = 27$, respectively.

Figure 8 – The real part of an X-shaped beam for optical frequencies in a dispersive medium, with $N = 27$. It refers to the larger window in Fig.7.

Figure 9 – The real part of an X-shaped beam for optical frequencies in a dispersive medium, with $N = 41$. It refers to the inner window in Fig.7.
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