Two-loop QCD Corrections to the Heavy Quark Form Factors: the Vector Contributions

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Abstract

We present closed analytic expressions of the electromagnetic vertex form factors for heavy quarks at the two-loop level in QCD for arbitrary momentum transfer. The calculation is carried out in dimensional regularization. The electric and magnetic form factors are expressed in terms of 1-dimensional harmonic polylogarithms of maximum weight 4.

Key words: Feynman diagrams, Multi-loop calculations, Vertex diagrams, Heavy quarks.

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1 Introduction

The forward-backward asymmetry $A^{b}_{fb}$ in the production of bottom quarks at electron-positron colliders displays presently a substantial discrepancy between experimental measurement and theoretical expectation \cite{1}. This theoretical expectation is determined using the electroweak parameters obtained from a global fit to a number of different electroweak precision observables, including this forward-backward asymmetry itself. In turn, $A^{b}_{fb}$ exercises a strong pull on the global fit, in particular towards larger masses of the Higgs boson. In particular, among the electroweak precision observables, $A^{b}_{fb}$ is more sensitive on the Higgs mass than most other quantities.

The present theoretical description of $A^{b}_{fb}$ includes the fully massive next-to-leading order (NLO) electroweak \cite{2} and fully massive NLO QCD \cite{3,4} corrections as well as the leading terms from the next-to-next-to-leading order (NNLO) QCD corrections \cite{5} (see also \cite{6,7}), which were obtained based on the massless approximation plus leading logarithmic mass terms. Given the substantial discrepancy between the experimental result and the theoretical expectation and the high impact on the Higgs mass determination, a more precise theoretical understanding of $A^{b}_{fb}$ is clearly desired.

At a future linear collider \cite{8}, precision determinations of electroweak parameters will again involve the forward-backward asymmetries. In this setting, the top quark asymmetry $A^{t}_{fb}$, is experimentally accessible, and of high interest in understanding the interplay of quark mass generation and electroweak symmetry breaking. For a precise theoretical description of this asymmetry, inclusion of mass corrections is clearly mandatory.

The NNLO QCD corrections to $A^{Q}_{fb}$ for massive quarks $Q$ involve three classes of contributions: (1) the tree level matrix elements for the decay of a vector boson into four partons, (at least) two of which being the heavy quark-antiquark pair; (2) the one-loop corrected matrix elements for the decay of a vector boson into a heavy quark-antiquark pair plus a gluon; (3) the two-loop corrections to the decay of a vector boson into a heavy quark-antiquark pair. While the former two contributions can be obtained \cite{9} along the lines of the calculations of three jet production involving heavy quarks \cite{10,11,12}, the latter remain to be calculated.

It is the aim of this paper (and of two companion papers) to contribute to the NNLO QCD corrections to $A^{Q}_{fb}$ for massive quarks by computing the virtual two-loop QCD corrections to the form factors of a massive quark. These form factors describe the full structure of the $(Z^*, \gamma^*) \rightarrow Q\bar{Q}$ vertex function, involving the vector and axial vector couplings of the vector boson. In the present paper, we derive the two-loop corrections to the vector form factors, while two following papers will discuss the parity-violating form factors at two loops. The two-loop corrections to the vector form factors were considered previously only in \cite{13}, where the contribution from closed fermion loops was calculated.

This paper is organized as follows.

In Section 2 we give the notations followed throughout all the paper, defining the form factors and their expansions in terms of the coupling constant. In Section 3 we
Figure 1: Tree-level and one-loop diagrams, involved in the calculation of the heavy-quark vertex form factors. The curly line represents a gluon; the double straight lines, quarks of mass $m$. The external fermion lines are on the mass-shell: $p_1^2 = p_2^2 = m^2$. The wavy line on the l.h.s. carries momentum $Q = p_1 + p_2$, with the metrical convention $Q^2 < 0$ when $Q$ is space-like.

give the virtual contributions to the one- and two-loop form factors before the subtraction of the UV divergences. In Section 4 we discuss in detail the renormalization and in Section 5 we give the UV-renormalized form factors at the one- and two-loop level. These results are presented for the case of choosing the renormalization scale $\mu$ equal to the mass $m$ of the heavy quark. In the Subsection 5.3 the logarithms of the ratio $m/\mu$, that are present if $\mu \neq m$, are explicitly given. Finally, Section 6 deals with the analytical continuation of our formulas above the threshold.

2 The QCD Form Factors

We call $V_{c_1c_2}^{\mu}(p_1, p_2)$ the QCD vertex amplitude, corresponding to the decay of a virtual photon of momentum $Q = p_1 + p_2$ into a quark and an anti-quark, of momenta $p_1$, $p_2$ and colors $c_1$ and $c_2$ respectively. The two fermions are on the mass-shell, $p_1^2 = p_2^2 = m^2$, where $m$ denotes the mass of the heavy quark in the on-shell scheme. Let us define the following two vectors:

\begin{equation}
Q^\mu = p_1^\mu + p_2^\mu, \quad t^\mu = p_2^\mu - p_1^\mu,
\end{equation}

such that $Q^2 = S$, where $S$ is the c.m. energy squared, and the dimensionless variable

\begin{equation}
s = \frac{S}{m^2} = \frac{Q^2}{m^2}.
\end{equation}

Within QCD the $V_{c_1c_2}^{\mu}(p_1, p_2)$ can be expressed in terms of two dimensionless scalar form factors $F_i(s), i = 1, 2$, depending only on the dimensionless variable $s$ of Eq. (2), as follows:

\begin{equation}
V_{c_1c_2}^{\mu}(p_1, p_2) = \bar{u}_{c_1}(p_1) \Gamma_{c_1c_2}^{\mu}(p_1, p_2) u_{c_2}(p_2),
\end{equation}
Figure 2: The two-loop vertex diagrams involved in the calculation of the form factors at order $\mathcal{O}(\alpha_s^2)$. The curly lines are gluons; the double straight lines, quarks of mass $m$; the single straight lines, massless quarks and the dashed lines ghosts. The external fermion lines are on the mass-shell: $p_1^2 = p_2^2 = m^2$. The wavy line on the l.h.s. carries momentum $Q = p_1 + p_2$, with the metrical convention $Q^2 < 0$ when $Q$ is space-like.
\[ \Gamma_{c_1c_2}^\mu(p_1,p_2) = -i v_Q \delta_{c_1c_2} \left[ F_1(s) \gamma^\mu + \frac{1}{2m} F_2(s) i \sigma^{\mu\nu} Q^\nu \right], \quad (4) \]

where \( \bar{u}_{c_1}(p_1), v_{c_2}(p_2) \) are the spinor wave functions of the quark and the anti-quark, \( \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \), and where:

\[ v_Q = e Q_Q, \quad (5) \]

\( Q_Q \) being the charge of the heavy quark in units of the positron charge \( e > 0 \).

The indices \( c_1, c_2 \), in Eq. (4), are the color indices: \( c_1, c_2 = 1, \cdots, N_c \), where \( N_c \) is the number of colors.

The extraction of each form factor \( F_i(s) \) from Eq. (4) can be carried out by the following general projector operators \( P_{\mu}^{(i)} \):

\[ P_{\mu}^{(i)}(m,p_1,p_2) = \frac{p_2-m}{m} \left[ i g_1^{(i)} \gamma_\mu + \frac{i}{2m} g_2^{(i)} t_\mu \right] \frac{p_1+m}{m}, \quad (6) \]

performing a trace over the spinor and the color indices. The constants \( g_j^{(i)}, j = 1, 2 \), are properly chosen such that:

\[ \text{Tr} \left( P_{\mu}^{(i)}(m,p_1,p_2) \Gamma^\mu(p_1,p_2) \right) = F_i(s). \quad (7) \]

Let us observe that since we work in a \( D \)-dimensional space (to regularize the divergences arising in the computation) the trace over the spinor indices is consistently performed in \( D \) dimensions as well. We use the convention of keeping the trace of the unit matrix equal to four also in \( D \) dimensions.

The explicit values of the constants are:

\[ g_1^{(1)} = -\frac{1}{v_Q N_c} \frac{1}{4(1-\epsilon)} \frac{1}{s-4}, \quad (8) \]
\[ g_2^{(1)} = \frac{1}{v_Q N_c} \frac{(3-2\epsilon)}{(1-\epsilon)} \frac{1}{(s-4)^2}, \quad (9) \]
\[ g_1^{(2)} = \frac{1}{v_Q N_c} \frac{1}{(1-\epsilon)} \frac{1}{s(s-4)}, \quad (10) \]
\[ g_2^{(2)} = -\frac{1}{v_Q N_c} \frac{1}{(1-\epsilon)} \frac{1}{(s-4)^2} \left[ \frac{4}{s} - 2 + 2\epsilon \right]. \quad (11) \]

The form factors are given as an expansion in powers of \( \alpha_S/(2\pi) \), where \( \alpha_S \) is defined as the standard \( \overline{\text{MS}} \) coupling in QCD (with \( N_f \) massless and one massive quark) at the renormalization scale \( \mu \):

\[ F_1(\epsilon, s, \frac{\mu^2}{m^2}) = 1 + \left( \frac{\alpha_S}{2\pi} \right) F_1^{(1)}(\epsilon, s, \frac{\mu^2}{m^2}) + \left( \frac{\alpha_S}{2\pi} \right)^2 F_1^{(2)}(\epsilon, s, \frac{\mu^2}{m^2}) + \mathcal{O}\left( \left( \frac{\alpha_S}{2\pi} \right)^3 \right), \quad (12) \]
\[ F_2(\epsilon, s, \frac{\mu^2}{m^2}) = \left( \frac{\alpha_S}{2\pi} \right) F_2^{(1)}(\epsilon, s, \frac{\mu^2}{m^2}) + \left( \frac{\alpha_S}{2\pi} \right)^2 F_2^{(2)}(\epsilon, s, \frac{\mu^2}{m^2}) + \mathcal{O}\left( \left( \frac{\alpha_S}{2\pi} \right)^3 \right). \quad (13) \]
The superscripts “1l” and “2l” stand for one- and two-loop contributions and the first term 1 in $F_1(\epsilon, s, \mu^2/m^2)$ is the tree-level approximation. The subscript “R” stands for “renormalized”, meaning that $F_{1,R}^{(1l)}(\epsilon, s, \mu^2/m^2)$ and $F_{1,R}^{(2l)}(\epsilon, s, \mu^2/m^2)$ come from the sum of the contributions of the virtual diagrams of Figs. 1 and 2 and the relative counterterms, shown in Fig. 3, for the subtraction of the UV divergences. In the following the expressions of the unsubtracted as well as the UV-renormalized form factors will be given at the one- and two-loop level.

3 Unsubtracted Contributions

In this Section we consider the contribution of the diagrams shown in Figs. 1 and 2 to the expansions in Eqs. (12,13). As explained in [14, 15, 16], the form factors coming from the computation of the trace operation introduced in the previous Section are expressed in terms of several hundreds of scalar integrals. It is possible to express all these integrals as a combination of only 17 independent scalar integrals, called the Master Integrals (MIs) of the problem, by means of the so-called Laporta algorithm [17], using integration-by-parts identities [18], Lorentz invariance [19] and general symmetry relations. This reduction algorithm is performed exactly in $D = (4 - 2\epsilon)$ dimensions [20]. Once the expression in terms of the MIs is found, we expand the result in powers of $\epsilon$ around $\epsilon = 0$ ($D = 4$), using the expansions of the MIs given in [14, 16] (the MIs are evaluated with the differential equations method [21, 22, 23]). The result will be therefore given as a Laurent expansion in $\epsilon$ where both UV- and IR-poles are regularized with the same parameter $\epsilon$.

The form factors that we are going to present in this Section are given for spacelike $Q^2 (S = Q^2 < 0)$ and they are expressed in terms of 1-dimensional harmonic polylogarithms (HPLs) [24, 25] of the variable:

$$x = \frac{\sqrt{-s + 4} - \sqrt{-s}}{\sqrt{-s + 4} + \sqrt{-s}} = \frac{\sqrt{-s + 4m^2} - \sqrt{-S}}{\sqrt{-s + 4m^2} + \sqrt{-S}}, \quad (0 \leq x \leq 1).$$

(14)

$C_F$ and $C_A$ are the Casimir operators for the fundamental and adjoint representation of the color gauge group $SU(N_c)$ respectively:

$$C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c,$$

(15)

where $N_c$ is the number of colors. $T_R$ is the normalization factor of the generators of the fundamental representation:

$$tr (t^a t^b) = T_R \delta^{ab} = \frac{1}{2} \delta^{ab}. \quad (16)$$

Because all the calculations are done in $D$ dimensions, we have to take into account for each loop a factor

$$C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon = (4\pi)^\epsilon \Gamma (1 + \epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon,$$

(17)

where $\mu$ is the mass scale of dimensional regularization, that we choose equal to the renormalization scale.
3.1 One-Loop Unsubtracted Form Factors
At the one-loop level only the diagram shown in Fig. 1(b) contributes to the form factors \( F_1(\epsilon, s, \mu^2/m^2) \) and \( F_2(\epsilon, s, \mu^2/m^2) \). We will write:

\[
F_{i, \text{Fig.1(b)}}^{(1)}(\epsilon, s, \frac{\mu^2}{m^2}) = C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^{\epsilon} \mathcal{F}_i^{(1)}(\epsilon, s), \quad \text{with } i = 1, 2, \quad (18)
\]

where:

\[
\mathcal{F}_1^{(1)}(\epsilon, s) = \frac{1}{\epsilon} \left\{ C_F \left[ \frac{1}{2} + \left( 1 - \frac{1}{1 + x} - \frac{1}{1 - x} \right) H(0; x) \right] \right\} + C_F \left[ \frac{1}{2} \left( 1 - \frac{2}{1 - x} \right) H(0; x) - \left( 1 - \frac{1}{1 + x} - \frac{1}{1 - x} \right) \zeta(2) \right.

\[
\left. - H(0; x) - H(0,0;x) + 2H(-1,0;x) \right] \right\} - \epsilon \left\{ C_F \left[ \frac{1}{2} \left( 1 - \frac{2}{1 - x} \right) \zeta(2) - H(0,0;x) + 2H(-1,0;x) \right. \right.

\[
\left. + \left( 1 - \frac{1}{1 + x} - \frac{1}{1 - x} \right) \zeta(2) + 2\zeta(3) \right.

\[
-(4-\zeta(2))H(0;x) - 2\zeta(2)H(-1;x) - H(0,0;x) + 2H(-1,0;x)

\[
+ 2H(0,0;x) - 4H(-1,0;x) \right\} + \mathcal{O}(\epsilon^2), \quad (19)
\]

\[
\mathcal{F}_2^{(1)}(\epsilon, s) = -C_F \left( \frac{1}{1 - x} - \frac{1}{1 + x} \right) H(0; x)

\[
+ \epsilon \left\{ C_F \left[ \left( \frac{1}{1 - x} - \frac{1}{1 + x} \right) \zeta(2) - 4H(0;x) - H(0,0;x) \right.ight.

\[
\left. + 2H(-1,0;x) \right] \right\} + \mathcal{O}(\epsilon^2). \quad (20)
\]

3.2 Two-Loop Unsubtracted Form Factors
At the two-loop level, all the 9 Feynman diagrams of Fig. 2 contribute to the form factors \( F_1(\epsilon, s, \mu^2/m^2) \) and \( F_2(\epsilon, s, \mu^2/m^2) \). The total contribution comes from the sum of the diagrams involving only the heavy quark, Fig. 2 (a)–(e) and (g)–(i), and the diagrams in which a light quark runs in the internal loop, Fig. 2 (f). If we consider \( N_f \) light quarks, the latter contribution is simply the contribution of diagram (f), in which the quark in the internal loop is considered as massless, multiplied by \( N_f \).

We will write:

\[
F_{i, \text{Fig.2}}^{(2)}(\epsilon, s, \frac{\mu^2}{m^2}) = C^2(\epsilon) \left( \frac{\mu^2}{m^2} \right)^{2\epsilon} \mathcal{F}_i^{(2)}(\epsilon, s), \quad \text{with } i = 1, 2, \quad (21)
\]

where:

\[
\mathcal{F}_1^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ C_F C_A \left[ \frac{11}{24} + \left( \frac{11}{12} \frac{11}{12(1-x)} - \frac{11}{12(1+x)} \right) H(0;x) \right] \right\}
\]

\[6\]
\[ + C_f T_R N_f \left[ - \frac{1}{6} - \left( \frac{1}{3} - \frac{1}{3(1-x)} - \frac{1}{3(1+x)} \right) H(0; x) \right] \]

\[ + C_f T_R \left[ - \frac{1}{3} - \left( \frac{2}{3} - \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0; x) \right] \]

\[ + C_F^2 \left[ \frac{25}{8} + \frac{6}{(1+x)^2} - \frac{6}{(1+x)} + \left( \frac{1}{2} - \frac{7}{2(1-x)} + \frac{6}{(1+x)^3} \right) \right. \]

\[ \left. - \frac{9}{(1+x)^2} + \frac{11}{2(1+x)} \right] H(0; x) + \left( 1 + \frac{1}{(1-x)^2} \right) \]

\[ - \frac{1}{(1-x)} + \frac{1}{(1+x)^2} - \frac{1}{(1+x)} \right) H(0,0; x) \]\n
\[ + \frac{1}{\epsilon} \left\{ C_F C_A \left[ \frac{185}{144} + \frac{4 \zeta(2)}{3(1-x)} + \frac{4 \zeta(2)}{3(1+x)} - \frac{\zeta(2)}{3} - \frac{\zeta(3)}{2(1-x)^2} \right. \]

\[ + \frac{\zeta(3)}{2(1-x)} - \frac{\zeta(3)}{2(1+x)^2} + \frac{\zeta(3)}{2(1+x)} - \frac{\zeta(3)}{2} \]

\[ - \left( \frac{14}{3} - \frac{14}{3(1-x)} - \frac{14}{3(1+x)} \right) H(-1,0; x) \]

\[ + \left( \frac{83}{18} - \frac{\zeta(2)}{2(1-x)^2} + \frac{3 \zeta(2)}{2(1-x)} - \frac{\zeta(2)}{2(1+x)^2} + \frac{3 \zeta(2)}{2(1+x)} \right. \]

\[ - \frac{3 \zeta(2)}{2} - \frac{199}{36(1-x)} - \frac{133}{36(1+x)} \right) H(0; x) + \left( 1 + \frac{1}{(1-x)^2} \right) \]

\[ + \left( \frac{1}{1-x} + \frac{1}{(1+x)^2} - \frac{1}{(1+x)} \right) H(0,0; x) \]

\[ + \left( \frac{23}{6} - \frac{17}{6(1-x)} - \frac{17}{6(1+x)} \right) H(0,0; x) - \left( 2 + \frac{1}{(1-x)^2} \right) \]

\[ - \frac{2}{(1-x)} + \frac{1}{(1+x)^2} - \frac{2}{(1+x)} \right) H(0,0,0; x) \]

\[ - \left( 1 + \frac{1}{(1-x)^2} - \frac{1}{(1-x)} + \frac{1}{(1+x)^2} - \frac{1}{(1+x)} \right) H(0,1; x) \]

\[ + \left. \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(1,0; x) \right] \]

\[ + C_f T_R N_f \left[ - \frac{13}{36} - \frac{2 \zeta(2)}{3(1-x)} - \frac{2 \zeta(2)}{3(1+x)} + \frac{2 \zeta(2)}{3} + \left( \frac{4}{3} - \frac{4}{3(1-x)} \right) \right. \]

\[ - \frac{4}{3(1+x)} \right) H(-1,0; x) - \left( \frac{14}{9} - \frac{17}{9(1-x)} \right) \]

\[ - \frac{11}{9(1+x)} \right) H(0; x) - \left( \frac{2}{3} - \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0,0; x) \]\n
\[ + C_f T_R \left[ - \frac{1}{4} - \frac{2 \zeta(2)}{3(1-x)} - \frac{2 \zeta(2)}{3(1+x)} + \frac{2 \zeta(2)}{3} + \left( \frac{4}{3} - \frac{4}{3(1-x)} \right) \right. \]

\[ - \frac{4}{3(1+x)} \right) H(-1,0; x) - \left( \frac{1}{4} - \frac{4}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0; x) \]
\[
\begin{align*}
&\left[-\left(\frac{2}{3} - \frac{2}{3(1-x)} - \frac{2}{3(1+x)}\right) H(0, 0; x)\right] \\
&+ C_F^2 \left[-\left(\frac{35}{16} + \frac{7\zeta(2)}{2(1-x)} - \frac{6\zeta(2)}{(1+x)^2} + \frac{9\zeta(2)}{(1+x)^2} - \frac{11\zeta(2)}{2(1+x)}\right) \\
&- \frac{\zeta(2)}{2} + \frac{8}{(1+x)^2} - \frac{8}{(1+x)} - \left(1 - \frac{7}{(1-x)} + \frac{12}{(1+x)^3}\right) \\
&- \frac{18}{(1+x)^2} + \frac{11}{(1+x)}\right) H(-1, 0; x) - \left(4 + \frac{4}{(1-x)^2}\right) \\
&- \left(4 - \frac{\zeta(2)}{(1-x)^2} + \frac{\zeta(2)}{(1-x)} - \frac{\zeta(2)}{(1+x)^2} + \frac{\zeta(2)}{(1+x)} - \frac{\zeta(2)}{(1+x)^2}\right) H(0; x) \\
&- \left(2 + \frac{2}{(1-x)^2} + \frac{2}{(1-x)} - \frac{2}{(1+x)^2} - \frac{2}{(1+x)}\right) H(0, -1; x) \\
&+ \left(\frac{7}{2} + \frac{4}{(1-x)^2} - \frac{2}{(1-x)} + \frac{2}{(1+x)^2} - \frac{2}{(1+x)}\right) \\
&+ \left(\frac{7}{2(1+x)}\right) H(0, 0; x) + \left(3 + \frac{3}{(1-x)^2} - \frac{3}{(1-x)}\right) \\
&+ \left(\frac{3}{(1+x)^2} - \frac{3}{(1+x)}\right) H(0, 0, 0; x)\right] \\
&+ C_F C_A \left[\frac{2585}{864} + \frac{36\zeta(2) \log(2)}{(1+x)^2} - \frac{36\zeta(2) \log(2)}{(1+x)} + 18\zeta(2) \log(2)\right] \\
&+ \frac{145\zeta(2)}{18(1-x)} - \frac{168\zeta(2)}{(1+x)^4} + \frac{363\zeta(2)}{(1+x)^3} - \frac{543\zeta(2)}{2(1+x)^2} \\
&+ \frac{1501\zeta(2)}{18(1+x)} - \frac{179\zeta(2)}{9} - \frac{37\zeta^2(2)}{20(1-x)^2} + \frac{679\zeta^2(2)}{160(1+x)} \\
&- \frac{879\zeta^2(2)}{10(1+x)^5} + \frac{879\zeta^2(2)}{4(1+x)^4} - \frac{7481\zeta^2(2)}{40(1+x)^3} + \frac{943\zeta^2(2)}{16(1+x)^2} \\
&- \frac{441\zeta^2(2)}{160(1+x)} - \frac{71\zeta^2(2)}{20} + \frac{53\zeta(3)}{6(1-x)} + \frac{324\zeta(3)}{(1+x)^4} \\
&- \frac{648\zeta(3)}{(1+x)^3} + \frac{374\zeta(3)}{(1+x)^2} - \frac{247\zeta(3)}{6(1+x)} - \frac{35\zeta(3)}{6} + \frac{3}{(1+x)^2} \\
&- \frac{3}{(1+x)} - \frac{19\zeta(2)}{3(1-x)^4} + \frac{90\zeta(2)}{(1+x)^4} - \frac{180\zeta(2)}{(1+x)^3} + \frac{135\zeta(2)}{(1+x)^2} \\
&- \frac{116\zeta(2)}{3(1+x)} - \frac{2\zeta(3)}{3(1-x)^2} + \frac{2\zeta(3)}{(1-x)} - \frac{2\zeta(3)}{(1+x)^2} \\
&+ \frac{2\zeta(3)}{(1+x)} - \frac{2\zeta(3)}{(1-x)}\right) H(-1; x) + \left(\frac{74}{3} - \frac{74}{3(1-x)}\right)
\end{align*}
\]
\[-\frac{74}{3(1+x)} H(-1, -1, 0; x) - \left(\frac{161}{18} - \frac{2\zeta(2)}{(1-x)^2}\right)\]
\[-\frac{2\zeta(2)}{(1+x)^2} - \frac{109}{9(1-x)} - \frac{6}{(1+x)^3} + \frac{4}{(1+x)^2}\]
\[-\frac{79}{9(1+x)} H(-1, 0; x) - \left(4 + \frac{4}{(1-x)^2} - \frac{4}{(1-x)}\right)\]
\[+ \frac{4}{(1+x)^2} - \frac{4}{(1+x)} H(-1,0,-1; x) - \left(\frac{64}{3} - \frac{46}{3(1-x)}\right)\]
\[+ \frac{282}{(1+x)^4} - \frac{564}{(1+x)^3} + \frac{339}{(1+x)^2} - \frac{217}{3(1+x)}\]
\[H(-1,0,0; x)\]
\[+ \left(2 + \frac{4}{(1-x)^2} - \frac{2}{(1-x)} + \frac{4}{(1+x)^2}\right]\]
\[-\frac{2}{(1+x)} H(-1,0,0,0; x) + \left(4 + \frac{4}{(1-x)^2} - \frac{4}{(1-x)}\right)\]
\[+ \frac{4}{(1+x)^2} - \frac{4}{(1+x)} H(-1,0,1,0; x) - \left(6 - \frac{6}{(1-x)}\right)\]
\[-\frac{6}{(1+x)} H(-1,1,0; x) + \left(\frac{4129}{216} + \frac{11\zeta(2)}{2(1-x)}\right)\]
\[-\frac{258\zeta(2)}{(1+x)^3} + \frac{744\zeta(2)}{(1+x)^4} - \frac{779\zeta(2)}{(1+x)^3} + \frac{357\zeta(2)}{(1+x)^2} - \frac{145\zeta(2)}{2(1+x)}\]
\[+ 3\zeta(2) - \frac{2\zeta(3)}{2(1-x)^2} + \frac{2\zeta(3)}{(1-x)} + \frac{324\zeta(3)}{(1+x)^5} - \frac{810\zeta(3)}{(1+x)^4}\]
\[+ \frac{662\zeta(3)}{(1+x)^3} + \frac{367\zeta(3)}{2(1+x)^2} + \frac{20\zeta(3)}{(1+x)} - \frac{15\zeta(3)}{2} - \frac{563}{27(1-x)}\]
\[+ \frac{9}{(1+x)^3} - \frac{27}{2(1+x)^2} - \frac{1391}{108(1+x)}\]
\[H(0; x)\]
\[+ \left(\frac{\zeta(2)}{(1-x)^2} - \frac{35\zeta(2)}{8(1-x)} - \frac{90\zeta(2)}{(1+x)^5} + \frac{225\zeta(2)}{(1+x)^4} - \frac{363\zeta(2)}{2(1+x)^3}\right)\]
\[+ \frac{193\zeta(2)}{4(1+x)^2} - \frac{83\zeta(2)}{8(1+x)} + \frac{7\zeta(2)}{2}\]
\[H(0,-1; x) - \left(10 + \frac{10}{(1-x)^2}\right)\]
\[-\frac{10}{(1-x)} + \frac{10}{(1+x)^2} - \frac{10}{(1+x)}\]
\[H(0,-1,-1,0; x)\]
\[-\left(\frac{79}{3} - \frac{46}{3(1-x)} - \frac{12}{(1+x)^4} + \frac{24}{(1+x)^3} - \frac{14}{(1+x)^2}\right)\]
\[-\frac{40}{3(1+x)} H(0,-1,0; x) + \left(14 + \frac{8}{(1-x)^2} - \frac{83}{8(1-x)}\right)\]
\[-\frac{282}{(1+x)^5} + \frac{705}{(1+x)^4} - \frac{1171}{2(1+x)^3} + \frac{725}{4(1+x)^2}\]
\[-\frac{227}{8(1+x)} H(0,-1,0,0; x) + \left(6 + \frac{6}{(1-x)^2}\right)\]
\[-\frac{6}{(1-x)} + \frac{6}{(1+x)^2} - \frac{6}{(1+x)} \right) H(0,-1,1,0;x) \\
\left( 59 \zeta(2) - \frac{59}{18} \zeta(2) - \frac{317}{16} \zeta(2) + \frac{59}{16} \zeta(2) - \zeta(2) + \frac{91}{18(1-x)} \right) \}
\left( \frac{24}{(1+x)^4} + \frac{39}{(1+x)^3} - \frac{3}{2(1+x)^2} - \frac{227}{18(1+x)} \right) H(0,0;x) \\
\left( 22 + \frac{12}{(1-x)^2} - \frac{75}{4(1-x)} + \frac{12}{1+x} - \frac{30}{(1+x)^4} \right) H(0,0,-1,0;x) \\
\left( \frac{50}{3} - \frac{19}{6(1-x)} - \frac{258}{(1+x)^5} + \frac{816}{(1+x)^4} - \frac{923}{(1+x)^3} \right) \\
\left( \frac{436}{(1+x)^2} - \frac{529}{6(1+x)} \right) H(0,0,0;x) - \left( 12 + \frac{6}{(1-x)^2} - \frac{12}{(1-x)} \right) \\
\left( 16(1-x) - \frac{177}{16(1+x)} \right) H(0,0,0,0;x) - \left( 14 + \frac{8}{(1-x)^2} - \frac{12}{(1-x)} \right) \\
\left( \frac{12}{(1+x)} \right) H(0,0,1,0;x) + \left( \frac{\zeta(2)}{(1-x)^2} - \frac{\zeta(2)}{(1-x)} \right) \\
\left( \frac{6}{(1-x)} + \frac{6}{(1+x)^2} - \frac{6}{(1+x)} \right) H(0,1,-1,0;x) \\
+ \left( \frac{8}{(1-x)} + \frac{4}{(1+x)^2} \right) \right) H(0,1,0;x) - \left( 2 + \frac{4}{(1-x)^2} - \frac{5}{(1-x)} \right) \\
\left( \frac{24}{(1+x)^5} + \frac{60}{(1+x)^4} - \frac{72}{(1+x)^3} + \frac{52}{(1+x)^2} \right) \\
\left( \frac{11}{(1+x)} \right) H(0,1,0,0;x) - \left( 2 + \frac{2}{(1-x)^2} - \frac{2}{(1-x)} \right) \\
\left( \frac{2}{(1+x)^2} + \frac{2}{(1+x)} \right) H(0,1,1,0;x) + \left( \frac{\zeta(2)}{(1-x)} + \frac{\zeta(2)}{(1+x)} \right) \\
- \zeta(2) - \frac{2\zeta(3)}{(1-x)^2} + \frac{2\zeta(3)}{(1-x)} - \frac{2\zeta(3)}{(1+x)^2} + \frac{2\zeta(3)}{(1+x)}
\[-2\zeta(3) H(1; x) - \left(6 - \frac{6}{1-x} - \frac{6}{1+x}\right) H(1, -1; 0; x)\]
\[-\left(3 + \frac{2\zeta(2)}{(1-x)^2} - \frac{2\zeta(2)}{1-x} + \frac{336\zeta(2)}{1+x} - \frac{840\zeta(2)}{(1+x)^4}\right) H(1, 0; x) + \left(4 + \frac{4}{(1-x)^2}\right) H(1, 0, -1; 0; x)\]
\[-\left(4 + \frac{4}{(1-x)^2} + \frac{24}{(1+x)^2} - \frac{48}{(1+x)^3} - \frac{8}{(1+x)^4}\right) H(1, 0, 0; x) + \left(2 - \frac{4}{(1-x)^2} + \frac{4}{(1-x)} - 336\right) H(1, 0, 1, 0; x)\]
\[+ \left(2 - \frac{2}{(1-x)} - \frac{2}{1+x}\right) H(1, 1, 0; x)\]
\[+ C_{FR} \left[\begin{array}{c}
- \frac{169}{216} - \frac{34\zeta(2)}{9(1-x)} - \frac{22\zeta(2)}{9(1+x)} + \frac{22\zeta(2)}{9} - \frac{8\zeta(3)}{3(1-x)} \\
- \frac{8\zeta(3)}{3(1+x)} - \frac{8\zeta(3)}{3} - \frac{8\zeta(2)}{3} \left(1 - \frac{1}{1-x} - \frac{1}{1+x}\right) H(-1; x) \\
- \left(\frac{16}{3} - \frac{16}{3(1-x)} - \frac{16}{3(1+x)}\right) H(-1, -1; 0; x) \\
+ \left(\frac{56}{9} - \frac{68}{9(1-x)} - \frac{44}{9(1+x)}\right) H(-1, 0; x) + \left(\frac{8}{3} - \frac{8}{3(1-x)}\right) \\
- \frac{8}{3(1+x)} H(-1, 0, 0; x) - \left(\frac{353}{54} - \frac{178}{27(1-x)}\right) \\
- \frac{175}{27(1+x)} H(0; x) + \left(\frac{8}{3} - \frac{8}{3(1-x)}\right) \\
- \frac{8}{3(1+x)} H(0, -1; 0; x) - \left(\frac{28}{9} - \frac{34}{9(1-x)}\right) \\
- \frac{22}{9(1+x)} H(0, 0; x) - \left(\frac{4}{3} - \frac{4}{3(1-x)}\right) \\
- \frac{4}{3(1+x)} H(0, 0, 0; x)\end{array}\right]\]
\[+ C_{FR} \left[\begin{array}{c}
\frac{223}{216} - \frac{4\zeta(2)}{3(1-x)} + \frac{392\zeta(2)}{3(1+x)^4} - \frac{784\zeta(2)}{3(1+x)^3} + \frac{458\zeta(2)}{3(1+x)^2}\end{array}\right]\]

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\[-68\zeta(2) + 8\zeta(2) - \frac{4\zeta(3)}{3(1 - x)} - \frac{4\zeta(3)}{3(1 + x)} + \frac{4\zeta(3)}{3} + 196 \frac{1}{9(1 + x)^2} - 196 \frac{4\zeta(2)}{9(1 + x)} + \frac{1}{3} \left( \frac{1}{(1 - x)} + \frac{1}{(1 + x)} \right) \]
\[-1 \right) H(-1; x) - \frac{8}{3} \left( \frac{1}{1 - x} - \frac{1}{1 + x} \right) H(-1, -1; 0; x) + \left( 2 - \frac{8}{3(1 - x)} - \frac{4}{3(1 + x)} \right) H(-1, 0; x) + \left( \frac{4}{3} - \frac{4}{3(1 + x)} \right) H(0, 0; x) \]
\[+ \left( \frac{10}{9} + \frac{4}{3(1 - x)} + \frac{248}{9(1 + x)^2} - \frac{496}{9(1 + x)^3} + \frac{326}{9(1 + x)^2} \right) H(0, 0; x) - \left( \frac{8}{3} - \frac{7}{3(1 - x)} + \frac{7}{(1 + x)^5} \right) H(0, 0; x) \]
\[+ C_F^2 \left\{ \frac{383}{32} - \frac{72\zeta(2) \log(2)}{(1 + x)^2} + \frac{72\zeta(2) \log(2)}{(1 + x)} - 36\zeta(2) \log(2) + \frac{14\zeta(2)}{(1 - x)} \right. \]
\[+ \frac{240\zeta(2)}{(1 + x)^3} + \frac{508\zeta(2)}{(1 + x)^3} - \frac{270\zeta(2)}{(1 + x)^2} - \frac{8\zeta(2)}{(1 + x)} + 29\zeta(2) \]
\[+ \frac{61\zeta^2(2)}{10(1 - x)^2} - \frac{1219\zeta^2(2)}{80(1 - x)} - \frac{171\zeta^2(2)}{(1 + x)^5} + \frac{855\zeta^2(2)}{2(1 + x)^4} - \frac{6867\zeta^2(2)}{20(1 + x)^3} \]
\[+ \frac{749\zeta^2(2)}{8(1 + x)^2} - \frac{1731\zeta^2(2)}{80(1 + x)} + \frac{181\zeta^2(2)}{10} - \frac{4\zeta(3)}{(1 - x)^2} + \frac{11\zeta(3)}{(1 - x)} \]
\[+ \frac{168\zeta(3)}{(1 + x)^4} - \frac{348\zeta(3)}{(1 + x)^3} + \frac{208\zeta(3)}{(1 + x)^2} - \frac{33\zeta(3)}{(1 + x)} - 4\zeta(3) \]
\[+ \frac{52}{(1 + x)^2} - \frac{52}{(1 + x)} - \left( \frac{7\zeta(2)}{(1 - x)} - \frac{180\zeta(2)}{(1 + x)^4} + \frac{348\zeta(2)}{(1 + x)^3} \right) \]
\[- \frac{252\zeta(2)}{(1 + x)^2} + \frac{79\zeta(2)}{(1 + x)} - \frac{13\zeta(2)}{(1 + x)^2} \right) H(-1; x) + \left( 2 - \frac{14}{(1 - x)} \right) \]
\[+ \frac{24}{(1 + x)^3} - \frac{36}{(1 + x)^2} + \frac{22}{(1 + x)} \right) H(-1, -1, 0; x) + \left( \frac{16}{(1 + x)^3} - \frac{16}{(1 + x)^2} - \frac{16}{(1 + x)} \right) H(-1, -1, 0; x) \]
\[-\left(36 - \frac{4\zeta(2)}{(1-x)^2} + \frac{4\zeta(2)}{(1-x)} - \frac{4\zeta(2)}{(1+x)^2} + \frac{4\zeta(2)}{(1+x)} - 4\zeta(2) \right) \frac{1}{1-x} - \frac{152}{(1+x)^3} + \frac{228}{(1+x)^2} - \frac{88}{(1+x)} \right) H(-1,0;x) + \left(8 + \frac{8}{(1-x)^2} - \frac{8}{(1-x)} + \frac{8}{(1+x)^2} - \frac{8}{(1+x)} \right) H(-1,0,-1,0;x) - \left(9 + \frac{16}{(1-x)^2} - \frac{23}{(1-x)} + \frac{252}{(1+x)^4} - \frac{492}{(1+x)^3} + \frac{252}{(1+x)^2} \right) \frac{7}{(1+x)} H(-1,0,0;x) - \left(12 + \frac{12}{(1-x)^2} - \frac{12}{(1-x)} \right) H(-1,0,0,0;x) + \left(12 - \frac{12}{(1+x)^2} \right) H(-1,0,0,0;x) + \left(19 - \frac{8\zeta(2)}{(1-x)^2} \right) + \frac{6\zeta(2)}{(1-x)} - \frac{60\zeta(2)}{(1+x)^5} + \frac{156\zeta(2)}{(1+x)^4} - \frac{142\zeta(2)}{(1+x)^3} + \frac{15\zeta(2)}{(1+x)^2} + \frac{52\zeta(2)}{(1+x)} + \frac{37\zeta(3)}{(1-x)^2} - \frac{7\zeta(3)}{(1-x)} + \frac{168\zeta(3)}{(1+x)^3} - \frac{420\zeta(3)}{(1+x)^2} + \frac{364\zeta(3)}{(1+x)} \right) \frac{5\zeta(3)}{(1+x)^2} + \frac{8\zeta(3)}{(1-x)} + \frac{45}{2(1-x)} + \frac{68}{(1+x)^3} - \frac{102}{(1+x)^2} + \frac{207}{4(1+x)} \right) H(0;x) + \left(\frac{2\zeta(2)}{(1-x)^2} + \frac{19\zeta(2)}{4(1-x)} + \frac{180\zeta(2)}{(1+x)^5} + \frac{363\zeta(2)}{(1+x)^3} - \frac{185\zeta(2)}{2(1+x)^2} + \frac{67\zeta(2)}{4(1+x)} \right) \frac{450\zeta(2)}{(1+x)^4} - \frac{10\zeta(2)}{4(1+x)} H(0,-1;x) + \left(4 + \frac{4}{(1-x)^2} - \frac{4}{(1-x)} \right) H(0,-1,-1,0;x) + \left(13 + \frac{7}{(1-x)} + \frac{384}{(1+x)^4} \right) H(0,-1,0;x) - \frac{780}{(1+x)^3} + \frac{460}{(1+x)^2} - \frac{69}{(1+x)} \right) H(0,-1,0,0;x) + \left(14 + \frac{10}{(1-x)^2} \right) \frac{19}{4(1+x)} + \frac{125}{(1-x)} + \frac{517}{(1+x)^4} - \frac{271}{(1+x)^3} + \frac{252}{2(1+x)^2} \right) H(0,-1,0,0,0;x) + \left(\frac{123}{2} + \frac{5\zeta(2)}{(1-x)^2} - \frac{105\zeta(2)}{8(1-x)} \right) H(0,-1,0,0,0,0;x) + \left(165\zeta(2) - \frac{281\zeta(2)}{2(1+x)^3} + \frac{203\zeta(2)}{4(1+x)^2} - \frac{137\zeta(2)}{8(1+x)} \right) \frac{66\zeta(2)}{(1+x)^5} + \frac{16\zeta(2)}{(1+x)^4} - \frac{54}{(1-x)^2} + \frac{192}{(1+x)^4} - \frac{484}{(1+x)^3} + \frac{436}{(1+x)^2} \right) H(0,0;x) - \left(22 + \frac{21}{(1-x)^2} - \frac{21}{2(1-x)} \right) H(0,0,0;x) + \left(\frac{22 + 10}{(1-x)^2} - \frac{10}{2(1-x)} + \frac{384}{(1+x)^5} \right) \frac{158}{(1+x)} \right) H(0,0,0,0;x) + \left(\frac{746}{(1+x)^3} + \frac{169}{(1+x)^2} - \frac{45}{2(1+x)} \right) H(0,0,0,0,0;x) + \left(\frac{746}{(1+x)^3} + \frac{169}{(1+x)^2} - \frac{45}{2(1+x)} \right) H(0,0,0,0,0,0;x)
\[
-\left(\frac{17}{2} - \frac{4}{(1-x)^2} + \frac{13}{(1-x)} + \frac{60}{(1+x)^5} - \frac{216}{(1+x)^2} + \frac{250}{(1+x)^3}ight)
- \frac{80}{(1+x)^2} - \frac{40}{(1+x)}\right) H(0, 0, 0; x) + \left(27 + \frac{17}{(1-x)^2}\right)
- \frac{217}{8(1-x)} - \frac{6}{(1+x)^5} + \frac{15}{(1+x)^4} - \frac{17}{2(1+x)^3} + \frac{59}{4(1+x)^2}
- \frac{201}{8(1+x)} H(0, 0, 0, 0; x) + \left(12 + \frac{4}{(1-x)^2} - \frac{8}{(1-x)}\right)
- \frac{96}{(1+x)^5} + \frac{240}{(1+x)^4} - \frac{176}{(1+x)^3} + \frac{28}{(1+x)^2}
- \frac{8}{(1+x)} H(0, 0, 1, 0; x) - \left(10 + \frac{8}{(1-x)^2} - \frac{8}{(1-x)}\right)
+ \frac{96}{(1+x)^4} - \frac{192}{(1+x)^3} + \frac{116}{(1+x)^2} - \frac{20}{(1+x)}\right) H(0, 1, 0; x)
- \left(4 + \frac{7}{(1-x)} - \frac{360}{(1+x)^5} + \frac{900}{(1+x)^4} - \frac{700}{(1+x)^3} + \frac{150}{(1+x)^2}\right)
- \frac{5}{(1+x)} H(0, 1, 1, 0; x) + \left(\frac{4\zeta(3)}{(1-x)^2} - \frac{4\zeta(3)}{(1-x)} + \frac{4\zeta(3)}{(1+x)^2}\right)
- \frac{4\zeta(3)}{(1+x)} + 4\zeta(3)\right) H(1; x) + \left(16 + \frac{4\zeta(2)}{(1-x)^2} - \frac{6\zeta(2)}{(1-x)}\right)
- \frac{144\zeta(2)}{(1+x)^5} + \frac{360\zeta(2)}{(1+x)^4} - \frac{312\zeta(2)}{(1+x)^3} + \frac{112\zeta(2)}{(1+x)^2} - \frac{14\zeta(2)}{(1+x)}
+ 4\zeta(2) - \frac{16}{(1-x)} - \frac{48}{(1+x)^3} + \frac{72}{(1+x)^2} - \frac{40}{(1+x)}\right) H(1, 0; x)
- \left(8 + \frac{8}{(1-x)^2} - \frac{8}{(1-x)} + \frac{8}{(1+x)^2} - \frac{8}{(1+x)}\right) H(1, 0, -1, 0; x)
+ \left(16 + \frac{360}{(1+x)^4} - \frac{720}{(1+x)^3} + \frac{394}{(1+x)^2} - \frac{34}{(1+x)}\right) H(1, 0, 0; x)
+ \left(8 + \frac{8}{(1-x)^2} - \frac{10}{(1-x)} - \frac{144}{(1+x)^5} + \frac{360}{(1+x)^4} - \frac{312}{(1+x)^3}\right)
+ \frac{116}{(1+x)^2} - \frac{18}{(1+x)}\right) H(1, 0, 0, 0; x) + \left(8 + \frac{8}{(1-x)^2}\right)
- \frac{8}{(1-x)} + \frac{8}{(1+x)^2} - \frac{8}{(1+x)}\right) H(1, 0, 1, 0; x)\right],
\]

\[\mathcal{F}_2^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2}\left\{ C_F^2 \left[ -\frac{6}{(1+x)^2} + \frac{6}{(1+x)} + \left(\frac{3}{2(1-x)} - \frac{6}{(1+x)^3} + \frac{9}{(1+x)^2}\right) - \frac{9}{2(1+x)}\right] H(0; x) \right\}
+ \frac{1}{\epsilon} \left\{ C_F C_A \left[ -\left(\frac{11}{6(1-x)} - \frac{11}{6(1+x)}\right) H(0; x) \right] \right\},\]
\[ + C_f T_R N_f \left[ + \left( \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0; x) \right] \\
+ C_f T_R \left[ + \left( \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0; x) \right] \\
+ C_F^2 \left\{ - \frac{3 \zeta(2)}{2(1-x)} + \frac{6 \zeta(2)}{(1+x)^3} - \frac{9 \zeta(2)}{(1+x)^2} + \frac{9 \zeta(2)}{2(1+x)} - \frac{20}{1+x} \right.
\left. - \frac{9}{(1+x)^2}\right\} H(-1,0;x) + \left( \frac{3}{2(1-x)} - \frac{35}{2(1+x)} \right) H(0;x) + \\
+ \left( \frac{7}{(1+x)^3} - \frac{5}{2(1+x)} \right) H(0;0;x) \right\} \\
+ C_F C_A \left[ + \frac{2 4 \zeta(2) \log(2)}{(1+x)^2} + \frac{2 4 \zeta(2) \log(2)}{(1+x)} + \frac{3 \zeta(2)}{(1+x)^2} - \frac{13 \zeta(2)}{12(1-x)} \right]
\left. + \frac{168 \zeta(2)}{(1+x)^4} - \frac{363 \zeta(2)}{(1+x)^3} + \frac{491 \zeta(2)}{(1+x)^2} - \frac{629 \zeta(2)}{12(1+x)} + \frac{69 \zeta(2)}{40(1-x)^3} \right.
\left. - \frac{207 \zeta(2)}{(1+x)^2} + \frac{45 \zeta(2)}{(1+x)} \right\} H(-1;x) + \\
\left. + \left( \frac{9 \zeta(2)}{(1+x)^2} - \frac{9 \zeta(2)}{(1+x)} \right) \right\} H(-1,0;x) \\
\left. - \frac{90 \zeta(2)}{(1+x)^3} + \frac{189 \zeta(2)}{(1+x)^2} - \frac{231 \zeta(2)}{2(1+x)^2} + \frac{51 \zeta(2)}{2(1+x)} \right\} H(0;x) \\
\left. - \frac{23}{6(1-x)} - \frac{1}{(1+x)^3} + \frac{9}{(1+x)^2} - \frac{41}{6(1+x)} \right\} H(-1,0;x) \\
\left. - \frac{1}{2(1-x)^2} - \frac{1}{2(1-x)} - \frac{1}{(1+x)^4} + \frac{564}{(1+x)^3} + \frac{623}{2(1+x)^2} \right\} H(-1,0,0;x) + \\
\left. + \left( \frac{3 \zeta(2)}{(1-x)^3} - \frac{11 \zeta(2)}{4(1-x)^2} + \frac{3 \zeta(2)}{8(1-x)} \right) \right\} \\
\left. + \frac{258 \zeta(2)}{(1+x)^3} + \frac{744 \zeta(2)}{(1+x)^4} + \frac{1515 \zeta(2)}{(1+x)^3} - \frac{621 \zeta(2)}{2(1+x)^2} + \frac{307 \zeta(2)}{8(1+x)} \right\} H(0;x) \\
\left. + \frac{13 \zeta(3)}{(1+x)^5} + \frac{324 \zeta(3)}{(1+x)^4} + \frac{810 \zeta(3)}{(1+x)^3} - \frac{635 \zeta(3)}{(1+x)^2} + \frac{285 \zeta(3)}{2(1+x)^2} \right\} \\
\left. + \frac{13 \zeta(3)}{(1+x)^5} - \frac{259}{4(1-x)^3} + \frac{9}{9(1+x)^3} + \frac{27}{2(1+x)^2} + \frac{89}{9(1+x)^3} \right\} H(0;x) \\
\left. - \frac{9 \zeta(2)}{2(1-x)^3} - \frac{27 \zeta(2)}{4(1-x)^2} + \frac{21 \zeta(2)}{8(1-x)} - \frac{90 \zeta(2)}{(1+x)^5} + \frac{225 \zeta(2)}{(1+x)^4} \right\} \]
\[-\frac{174\zeta(2)}{(1 + x)^3} + \frac{36\zeta(2)}{(1 + x)^2} + \frac{21\zeta(2)}{8(1 + x)} \]  
\[-\frac{1}{(1 - x)^4} + \frac{12}{1 + x^4} - \frac{24}{(1 + x)^3} + \frac{9}{1 + x^2} \]  
\[-\left( \frac{1}{2(1 - x)} - \frac{3}{4(1 - x)^2} + \frac{5}{8(1 - x)} - \frac{282}{(1 + x)^5} + \frac{705}{(1 + x)^4} \right) \]  
\[-\frac{562}{(1 + x)^5} + \frac{138}{(1 + x)^2} + \frac{5}{8(1 + x)} \right) \]  
\[+ \left( \frac{7\zeta(2)}{4(1 - x)^3} - \frac{21\zeta(2)}{8(1 - x)^2} - \frac{5\zeta(2)}{16(1 - x)} + \frac{69\zeta(2)}{(1 + x)^5} - \frac{345\zeta(2)}{2(1 + x)^4} \]  
\[+ \frac{135\zeta(2)}{(1 + x)^5} - \frac{30\zeta(2)}{(1 + x)^2} - \frac{5\zeta(2)}{16(1 + x)} + \frac{3}{(1 - x)^2} - \frac{59}{12(1 - x)} \]  
\[-\frac{24}{(1 + x)^4} + \frac{39}{(1 + x)^3} - \frac{7}{2(1 + x)^2} - \frac{115}{12(1 + x)} \right) \]  
\[-\frac{1}{(1 - x)^3} - \frac{2(1 - x)^2}{4(1 - x)} + \frac{11}{(1 + x)^5} - \frac{30}{(1 + x)^4} \]  
\[+ \left( \frac{48}{(1 + x)^3} - \frac{42}{(1 + x)^2} + \frac{25}{4(1 + x)} \right) \]  
\[+ \left( \frac{3}{(1 - x)^3} - \frac{15}{4(1 - x)^2} + \frac{11}{8(1 - x)} + \frac{258}{(1 + x)^5} - \frac{816}{(1 + x)^4} \right) \]  
\[+ \frac{1803}{2(1 + x)^3} - \frac{767}{2(1 + x)^2} + \frac{315}{8(1 + x)} \right) \]  
\[+ \left( \frac{3}{4(1 - x)^3} - \frac{9}{8(1 - x)^2} + \frac{11}{16(1 - x)} - \frac{3}{(1 + x)^5} + \frac{15}{2(1 + x)^4} \right) \]  
\[-\frac{4}{(1 + x)^3} - \frac{3}{2(1 + x)^2} + \frac{11}{16(1 + x)} \right) \]  
\[-\frac{3}{(1 - x)} - \frac{48}{(1 + x)^5} + \frac{120}{(1 + x)^4} - \frac{84}{(1 + x)^3} + \frac{6}{(1 + x)^2} \]  
\[+ \frac{3}{(1 + x)} \right) \]  
\[H(0, 0, 0; x) - \left( \frac{48}{(1 + x)^4} - \frac{96}{(1 + x)^3} \right) \]  
\[-\frac{1}{(1 - x)} + \frac{24}{(1 + x)^5} - \frac{60}{(1 + x)^4} \]  
\[+ \frac{70}{(1 + x)^3} - \frac{45}{(1 + x)^2} + \frac{11}{2(1 + x)} \right) \]  
\[-\left( \frac{3\zeta(2)}{(1 - x)} - \frac{336\zeta(2)}{(1 + x)^5} + \frac{840\zeta(2)}{(1 + x)^4} - \frac{660\zeta(2)}{(1 + x)^3} + \frac{150\zeta(2)}{(1 + x)^2} \right) \]  
\[H(1, 0; x) \]  
\[-\left( \frac{3\zeta(2)}{(1 + x)^3} - \frac{24}{(1 + x)^2} + \frac{36}{(1 + x)} \right) \]  
\[-\frac{24}{(1 + x)^4} - \frac{48}{(1 + x)^3} + \frac{40}{(1 + x)^2} - \frac{16}{(1 + x)} \right) \]  
\[H(1, 0, 0; x) \]
\[
- \left( \frac{3}{(1-x)} - \frac{336}{(1+x)^5} + \frac{840}{(1+x)^4} - \frac{660}{(1+x)^3} + \frac{150}{(1+x)^2} \right) H(1,0,0;0;x)
+ \frac{3}{(1+x)} H(1,0,0;0;x)
+ C_F T_R N_f \left[ - \frac{4 \zeta(2)}{3(1-x)} + \frac{4 \zeta(2)}{3(1+x)} - \left( \frac{8}{3(1-x)} \right) H(-1,0;x) + \left( \frac{49}{9(1-x)} - \frac{49}{9(1+x)} \right) H(0;x)
+ \left( \frac{4}{3(1-x)} - \frac{4}{3(1+x)} \right) H(0,0;x) \right]
+ C_F T_R \left[ - \frac{2 \zeta(2)}{3(1-x)} + \frac{136 \zeta(2)}{(1+x)^3} + \frac{272 \zeta(2)}{(1+x)^3} - \frac{132 \zeta(2)}{(1+x)^2} + \frac{10 \zeta(2)}{3(1+x)} - \frac{68}{3(1+x)^2} + \frac{68}{3(1+x)^3} + \left( \frac{4}{3(1-x)} + \frac{4}{3(1+x)} \right) H(-1,0;x)
- \frac{6 \zeta(2)}{2(1-x)} + \frac{24 \zeta(2)}{(1+x)^3} - \frac{60 \zeta(2)}{(1+x)^4} + \frac{42 \zeta(2)}{(1+x)^3} - \frac{3 \zeta(2)}{(1+x)^2}
- \frac{3 \zeta(2)}{2(1+x)} - \frac{44}{9(1-x)} - \frac{124}{3(1+x)^3} - \frac{62}{(1+x)^2}
- \frac{142}{9(1+x)} H(0;x) + \left( \frac{2}{3(1-x)} - \frac{88}{3(1+x)^4} + \frac{176}{3(1+x)^3} \right) H(0,0;x) - \left( \frac{3}{2(1-x)} - \frac{24}{(1+x)^5} \right) H(0,0,0;x) \right]
+ C_F^2 \left[ \frac{48 \zeta(2) \log(2)}{(1+x)^2} - \frac{48 \zeta(2) \log(2)}{(1+x)} - \frac{9 \zeta(2)}{(1-x)} + \frac{240 \zeta(2)}{(1+x)^4} - \frac{496 \zeta(2)}{(1+x)^3} + \frac{256 \zeta(2)}{(1+x)^2} + \frac{9 \zeta(2)}{(1-x)} - \frac{69 \zeta^2(2)}{20(1-x)^3} + \frac{207 \zeta^2(2)}{40(1-x)^2} - \frac{327 \zeta^2(2)}{80(1-x)}
+ \frac{171 \zeta^2(2)}{(1+x)^5} + \frac{855 \zeta^2(2)}{2(1+x)^4} + \frac{3291 \zeta^2(2)}{10(1+x)^3} - \frac{1323 \zeta^2(2)}{20(1+x)^2} - \frac{327 \zeta^2(2)}{80(1+x)}
- \frac{10 \zeta(3)}{(1-x)^2} + \frac{7 \zeta(3)}{(1+x)} - \frac{168 \zeta(3)}{(1+x)^4} + \frac{348 \zeta(3)}{(1+x)^3} - \frac{200 \zeta(3)}{(1+x)^2} + \frac{23 \zeta(3)}{(1+x)} - \frac{44}{(1+x)^2} + \frac{44}{(1+x)} + \left( \frac{9 \zeta(2)}{(1-x)^2} - \frac{6 \zeta(2)}{(1-x)} - \frac{180 \zeta(2)}{(1+x)^4} \right) H(-1;x) + \left( \frac{6}{(1-x)} \right)
+ \frac{348 \zeta(2)}{(1+x)^3} + \frac{213 \zeta(2)}{(1+x)^2} + \frac{42 \zeta(2)}{(1+x)} \right) H(-1,0;x)
- \frac{24}{(1+x)^3} + \frac{36}{(1+x)^2} - \frac{18}{(1+x)} \right) H(-1,-1,0;x)
- \left( \frac{10}{(1-x)} + \frac{128}{(1+x)^3} - \frac{192}{(1+x)^2} + \frac{54}{(1+x)} \right) H(-1,0;x) \right]
\]
\[
\begin{align*}
&+ \left( \frac{1}{(1 - x)^2} - \frac{4}{1 - x} + \frac{252}{(1 + x)^4} - \frac{492}{(1 + x)^3} + \frac{239}{(1 + x)^2} \\
&+ \frac{4}{1 + x} \right) H(-1, 0; 0) \\
&+ \left( \frac{156\zeta(2)}{(1 + x)^3} - \frac{137\zeta(2)}{(1 + x)^3} + \frac{19\zeta(2)}{(1 + x)^2} + \frac{125\zeta(2)}{4(1 + x)} + \frac{7\zeta(3)}{2(1 + x)} \\
&+ \frac{168\zeta(3)}{(1 + x)^5} - \frac{105\zeta(3)}{(1 + x)^4} - \frac{7\zeta(3)}{2(1 + x)} \\
&- \frac{15}{4(1 - x)} + \frac{84}{(1 + x)^3} - \frac{126}{(1 + x)^2} + \frac{183}{4(1 + x)} \right) H(0; x) \\
&+ \left( \frac{9\zeta(2)}{(1 - x)^3} - \frac{27\zeta(2)}{2(1 - x)^2} + \frac{21\zeta(2)}{4(1 - x)} - \frac{180\zeta(2)}{(1 + x)^5} + \frac{450\zeta(2)}{(1 + x)^4} \\
&- \frac{348\zeta(2)}{(1 + x)^3} + \frac{72\zeta(2)}{(1 + x)^2} + \frac{21\zeta(2)}{4(1 + x)} \right) H(0, -1; x) \\
&- \left( \frac{2}{(1 - x)^2} + \frac{1}{1 - x} + \frac{384}{(1 + x)^4} - \frac{780}{(1 + x)^3} + \frac{412}{4(1 + x)^2} \\
&- \frac{19}{4(1 + x)} \right) H(0, -1, 0; x) - \left( \frac{1}{(1 - x)^3} - \frac{3}{2(1 - x)^2} - \frac{7}{4(1 - x)} \\
&- \frac{252}{(1 + x)^5} - \frac{630}{(1 + x)^4} + \frac{496}{(1 + x)^3} - \frac{114}{(1 + x)^2} \\
&- \frac{7}{4(1 + x)} \right) H(0, -1, 0, 0; x) - \left( \frac{7\zeta(2)}{2(1 - x)^3} - \frac{21\zeta(2)}{4(1 - x)^2} \\
&+ \frac{\zeta(2)}{8(1 - x)} - \frac{66\zeta(2)}{(1 + x)^5} + \frac{165\zeta(2)}{(1 + x)^4} - \frac{135\zeta(2)}{(1 + x)^3} + \frac{75\zeta(2)}{2(1 + x)^2} \\
&- \frac{\zeta(2)}{8(1 + x)} - \frac{2}{(1 - x)^2} - \frac{1}{1 - x} + \frac{192}{(1 + x)^4} - \frac{472}{(1 + x)^3} \\
&- \frac{394}{14} \right) H(0, 0; x) + \left( \frac{2}{(1 - x)^3} - \frac{3}{(1 + x)^2} \\
&+ \frac{14}{1 - x} - \frac{14}{(1 + x)^5} + \frac{14}{(1 + x)^4} = \frac{14}{(1 + x)^3} + \frac{14}{(1 + x)^2} \\
&- \frac{14}{(1 + x)} \right) H(0, 0, -1; x) - \left( \frac{7}{2(1 - x)^2} - \frac{63}{4(1 - x)} \\
&- \frac{60}{1 + x} + \frac{216}{(1 + x)^4} - \frac{245}{(1 + x)^3} + \frac{79}{(1 + x)^2} + \frac{89}{4(1 + x)} \right) H(0, 0, 0; x) \\
&- \left( \frac{3}{2(1 - x)^3} - \frac{9}{4(1 - x)^2} + \frac{11}{8(1 - x)} - \frac{6}{(1 + x)^5} + \frac{15}{(1 + x)^4} \\
&- \frac{8}{(1 + x)^3} - \frac{3}{(1 + x)^2} + \frac{11}{8(1 + x)} \right) H(0, 0, 0, 0; x) \\
&- \left( \frac{6}{1 - x} - \frac{96}{(1 + x)^5} + \frac{240}{(1 + x)^4} - \frac{168}{(1 + x)^3} + \frac{12}{(1 + x)^2} \\
&- \frac{15}{4(1 + x)} \right)
\end{align*}
\]
\[ + \frac{6}{(1 + x)} H(0, 0, 1, 0; x) - \left( \frac{4}{(1 - x)^2} - \frac{4}{1 - x} \right) \]
\[ - \frac{96}{(1 + x)^4} + \frac{192}{(1 + x)^3} - \frac{100}{(1 + x)^2} + \frac{4}{(1 + x)} H(0, 1, 0; x) \]
\[ + \left( \frac{25}{2(1 - x)} - \frac{360}{(1 + x)^5} + \frac{900}{(1 + x)^4} - \frac{670}{(1 + x)^3} + \frac{105}{(1 + x)^2} \right) \]
\[ + \frac{25}{2(1 + x)} H(0, 1, 0, 0; x) + \left( \frac{3\zeta(2)}{(1 - x)} + \frac{144\zeta(2)}{(1 + x)^5} - \frac{360\zeta(2)}{(1 + x)^4} \right) \]
\[ + \frac{300\zeta(2)}{(1 + x)^3} - \frac{90\zeta(2)}{(1 + x)^2} + \frac{3\zeta(2)}{(1 + x)} - \frac{4}{(1 - x)} + \frac{48}{(1 + x)^3} \]
\[ - \frac{72}{(1 + x)^2} + \frac{28}{(1 + x)} H(1, 0; x) - \left( \frac{4}{(1 - x)^2} - \frac{4}{1 - x} \right) \]
\[ + \frac{360}{(1 + x)^4} - \frac{720}{(1 + x)^3} + \frac{356}{(1 + x)^2} + \frac{4}{(1 + x)} H(1, 0, 0; x) \]
\[ + \left( \frac{3}{1 - x} + \frac{144}{(1 + x)^5} - \frac{360}{(1 + x)^4} + \frac{300}{(1 + x)^3} - \frac{90}{(1 + x)^2} \right) \]
\[ + \frac{3}{(1 + x)} H(1, 0, 0, 0; x) \] \hspace{1cm} (23)

4 Renormalization

The subtraction of the one-loop subdivergences and the two-loop overall divergence in the two-loop graphs of Fig. 2 is performed in a hybrid scheme: we renormalize the heavy-quark wave function and mass in the \textit{on-shell} (OS) renormalization scheme, while the coupling $\alpha_S$ and the gluon wave function are renormalized in the modified minimal subtraction (\overline{MS}) scheme. The counterterm diagrams to add to the unsubtracted form factors given in Eqs. (19,20,22,23), are shown in Fig. 3.

4.1 One-Loop Counterterms

The renormalization of the one-loop form factors and the subtraction of the one-loop subdivergences from the two-loop graphs shown in Fig. 2 require the renormalization constants $Z_{g,\overline{\text{MS}}}^{(1)}(\epsilon), Z_{3,\overline{\text{MS}}}^{(1)}(\epsilon), \delta n_{\text{OS}}(\epsilon, m, \mu^2/m^2)$ and $Z_{2,\text{OS}}(\epsilon, \mu^2/m^2)$, at the one-loop level. Once $Z_{g,\overline{\text{MS}}}^{(1)}(\epsilon), Z_{3,\overline{\text{MS}}}^{(1)}(\epsilon)$ and $Z_{2,\text{OS}}(\epsilon, \mu^2/m^2)$ are defined, the expression of $Z_{1F}(\epsilon, \mu^2/m^2)$ is obtained using the Slavnov-Taylor identities.

The expressions for the coupling and gluon wave function renormalization constants (the latter one in the Feynman gauge) are in the \overline{MS}-scheme:

\[ Z_{g,\overline{\text{MS}}}^{(1)}(\epsilon) = -\frac{\alpha_S}{2\pi} C(\epsilon) \frac{1}{4\epsilon} \left( \frac{11}{3} C_A - \frac{4}{3} T_R(N_f + 1) \right) , \] \hspace{1cm} (24)
\[ Z_{3,\overline{\text{MS}}}^{(1)}(\epsilon) = \frac{\alpha_S}{2\pi} C(\epsilon) \frac{1}{2\epsilon} \left( \frac{5}{3} C_A - \frac{4}{3} T_R(N_f + 1) \right) . \] \hspace{1cm} (25)
Figure 3: Counterterm diagrams. For the renormalization of the two-loop form factors we use diagrams (a)–(i). Diagram (j) is employed in the renormalization of the one-loop form factors.
For what concerns the renormalization of the heavy quark mass and wave function, we need the following constants at the one-loop level, defined in the OS-scheme:

\[
\delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2}) = -m \frac{\alpha_s}{2\pi} C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{C_F}{2} \frac{(3-2\epsilon)}{(1-2\epsilon)}, \tag{26}
\]

\[
Z_{\text{OS}}^{(1)}(\epsilon, \frac{\mu^2}{m^2}) = -\frac{\alpha_s}{2\pi} C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{C_F}{2} \frac{(3-2\epsilon)}{(1-2\epsilon)}. \tag{27}
\]

The constant \(Z_{1F}^{(1)}(\epsilon, \frac{\mu^2}{m^2})\), needed for the renormalization of the \(Q\bar{Q}\)-gluon vertex, is obtained from a Slavnov-Taylor identity as follows:

\[
Z_{1F}^{(1)}(\epsilon, \frac{\mu^2}{m^2}) = Z_{\text{g,MS}}^{(1)}(\epsilon) + \frac{1}{2} Z_{3,\text{MS}}^{(1)}(\epsilon) \tag{28}
\]

\[
= -\frac{\alpha_s}{2\pi} C(\epsilon) \left( \frac{1}{2} C_A + \left( \frac{\mu^2}{m^2} \right)^\epsilon \frac{C_F}{(1-2\epsilon)} \right). \tag{29}
\]

The renormalization of the UV divergences of the one-loop form factors, Eqs. (19,20), is straightforward. It is sufficient to add the counterterm of Fig. 3 (j), defined as:

\[
Z_{2,\text{OS}}^{(1)}(\epsilon, \frac{\mu^2}{m^2}) \underset{\text{def}}{=} Z_{2,\text{OS}}^{(1)}(\epsilon, \frac{\mu^2}{m^2}) \times \overset{\text{def}}{\raisebox{1cm}{\includegraphics{counterterm.png}}}, \tag{30}
\]

where the constant \(Z_{2,\text{OS}}^{(1)}(\epsilon, \frac{\mu^2}{m^2})\) is given by Eq. (27), to the diagram of Fig. 1 (b). The corresponding UV-renormalized form factors are given in Section 5. Let us note that \(F_{2}^{(1)}(\epsilon, s)\) in Eq. (20) is IR and UV finite. The counterterm diagram defined in Eq. (30) is, in fact, proportional to \(\gamma^\mu\) and affects only \(F_{1}^{(1)}(\epsilon, s)\).

For the subtraction of the one-loop subdivergences from the two-loop diagrams of Fig. 2 we have to define as well the counterterm diagrams shown in Fig. 3 (a)–(h).

The first two involve the constant \(\delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2})\):

- graph (a) in Fig. 3:

\[
\delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2}) \underset{\text{def}}{=} -\frac{1}{m} \delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2}) \times \left( m \overset{\text{def}}{\raisebox{1cm}{\includegraphics{counterterm.png}}} \right); \tag{31}
\]

- graph (b) in Fig. 3:

\[
\delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2}) \underset{\text{def}}{=} -\frac{1}{m} \delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2}) \times \left( m \overset{\text{def}}{\raisebox{1cm}{\includegraphics{counterterm.png}}} \right); \tag{32}
\]

The one-loop diagram multiplying \(\delta m_{\text{OS}}^{(1)}(\epsilon, m, \frac{\mu^2}{m^2})/m\) in Eq. (31) is defined as follows:

\[
m \overset{\text{def}}{\raisebox{1cm}{\includegraphics{counterterm.png}}} = m C_F \frac{\alpha_s}{2\pi} C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon
\]
\[ \times \int \mathcal{D}^{Dk} \frac{U^\mu}{[(p_1 + k)^2 - m^2]^2[(p_2 - k)^2 - m^2]k^2} \]  

where the measure \( \mathcal{D}^{Dk} \) is such that:

\[ \int \mathcal{D}^{Dk} = \frac{1}{C(\epsilon)} \left( \frac{m^2}{\mu^2} \right)^\epsilon \int \frac{d^Dk}{(2\pi)^{2(1-\epsilon)}} \]  

and where:

\[ U^\mu = v_Q \gamma_\sigma [\not{p}_1 + \not{k} + m] [\not{p}_1 + \not{k} + m] \gamma^\nu [\not{k} - \not{p}_2 + m] \gamma_\sigma . \]  

The corresponding form factors are:

\[ F_{i,(\otimes)}^{(1)}(\epsilon, s, \frac{\mu^2}{m^2}) = C(\epsilon) \left( \frac{\mu^2}{m^2} \right)^\epsilon F_{i,(\otimes)}^{(\otimes)}(\epsilon, s) \quad \text{with} \ i = 1, 2 , \]  

where:

\[
\begin{align*}
F_{1,(\otimes)}^{(\otimes)}(\epsilon, s) &= -\frac{1}{\epsilon} \left\{ C_F \left[ 4 - \frac{12}{1 + x} + \frac{12}{(1 + x)^2} \right] \\
&\quad + \left[ \left( \frac{1}{1 - x} - \frac{2}{1 + x} + \frac{3}{(1 + x)^2} - \frac{2}{(1 + x)^3} \right) \zeta(2) + \left( \frac{4}{1 - x} + \frac{12}{1 + x} - \frac{18}{(1 + x)^2} + \frac{4}{(1 + x)^3} \right) \zeta(3) \right] H(0; x) \\
&\quad - \left( \frac{2}{1 - x} - \frac{4}{1 + x} + \frac{6}{(1 + x)^2} - \frac{4}{(1 + x)^3} \right) \zeta(3) - H(0, 0; x) + 2H(-1, 0; x) \right) \\
&\quad - \left[ \frac{1}{1 - x} - \frac{3}{1 + x} + \frac{6}{(1 + x)^2} - \frac{4}{(1 + x)^3} \right] \zeta(2) \\
&\quad - H(0, 0; x) + 2H(-1, 0; x) \right) \\
&\quad - \left[ \frac{1}{1 - x} - \frac{2}{1 + x} + \frac{3}{(1 + x)^2} - \frac{2}{(1 + x)^3} \right] \times \\
&\quad \times \left[ H(0, 0, 0; x) + 4H(-1, -1, 0; x) \\
&\quad - 2H(-1, 0, 0; x) - 2H(0, -1, 0; x) \right] \right\} + O(\epsilon^2) ,
\end{align*}
\]  

22
\( \mathcal{F}_2^{(\otimes)}(\epsilon, s) = -\frac{1}{\epsilon} \left\{ C_F \left[ \frac{2}{1+x} \left(1 - \frac{1}{1+x}\right) + \frac{1}{2} \left(\frac{1}{1-x} - \frac{3}{1+x}\right) + 6 \left(1 + x\right)^2 - 4 \left(1 + x^3\right) \right] \frac{3}{2} \right\} \)

\[ + \frac{4}{(1+x)^2} \left(1 - \frac{3}{1+x}\right) + \frac{2}{(1+x)^2} \right\} H(0; x) \]

\[ + \frac{1}{2} \left(\frac{1}{1-x} - \frac{3}{1+x} + 6 \frac{4}{(1+x)^3} \right) [\zeta(2)] \]

\[ - H(0, 0; x) + 2 H(-1, 0; x) \right\} \]

\[ - \epsilon \left\{ C_F \left[ \frac{4}{(1+x)} \left(1 - \frac{1}{1+x}\right) \right] \right. \]

\[ + \frac{2}{(1+x)} \left(1 - \frac{3}{1+x} + \frac{2}{1+x} \right) \left[2 \zeta(2) \right. \]

\[ - 3 H(0; x) - 2 H(0, 0; x) + 4 H(-1, 0; x) \right\] \]

\[ - \frac{1}{2} \left(\frac{1}{1-x} - \frac{3}{1+x} + 6 \frac{4}{(1+x)^3} \right) [\zeta(3)] \]

\[ + \zeta(2) (H(0; x) - 2 H(-1; x)) - H(0, 0; x) \] \]

\[ - 4 H(-1, -1, 0; x) + 2 H(-1, 0, 0; x) \]

\[ + 2 H(0, -1, 0; x) \right\} + \mathcal{O} (\epsilon^2) \right\}

The one-loop diagram appearing in Eq. (32) has exactly the same form factors given in Eqs. (37, 38). Therefore, in the following we will refer to both diagrams using the picture of the first one only.

The counterterm diagrams which involve \( Z_{2,OS}(\epsilon, \mu^2/m^2) \), \( Z_{1F}(\epsilon, \mu^2/m^2) \), and \( Z_{3,MS}(\epsilon) \) are defined as the product of the renormalization constants times the one-loop vertex diagram:

- graph (c) in Fig. (3):

\[ Z_2^{(ui)} \]

\[ \overset{\text{def}}{=} Z_{2,OS}^{(ui)} \left( \epsilon, \frac{\mu^2}{m^2} \right) \times \]

\[ ; \quad (39) \]

- graph (d) in Fig. (3):

\[ Z_3^{(ui)} \]

\[ \overset{\text{def}}{=} Z_{2,OS}^{(ui)} \left( \epsilon, \frac{\mu^2}{m^2} \right) \times \]

\[ ; \quad (40) \]

- graph (e) in Fig. (3):

\[ \]

\[ ; \quad (38) \]
\[ Z_3^{(1)} \overset{\text{def}}{=} - Z_{3,\text{MS}}^{(1)}(\epsilon) \times \gamma \quad ; \quad (41) \]

• graph (f) in Fig. 3:

\[ Z_1^{(1)} \overset{\text{def}}{=} Z_1^{(1)}(\epsilon, \frac{\mu^2}{m^2}) \times \gamma \quad ; \quad (42) \]

• graph (g) in Fig. 3:

\[ Z_1^{(1)} \overset{\text{def}}{=} Z_1^{(1)}(\epsilon, \frac{\mu^2}{m^2}) \times \gamma \quad ; \quad (43) \]

• graph (h) in Fig. 3:

\[ Z_2^{(1)} \overset{\text{def}}{=} Z_2^{(1)}_{2,\text{OS}}(\epsilon, \frac{\mu^2}{m^2}) \times \gamma \quad ; \quad (44) \]

The one-loop vertex diagram, appearing in Eqs. (39–44) is defined as:

\[ \mathcal{V}^\mu = vQ\gamma_\sigma[p_1 + k + m]\gamma^\mu[k - p_2 + m]\gamma_\sigma , \quad (46) \]

and the corresponding form factors are given in Eqs. (19,20).

### 4.2 Two-Loop Counterterm

In order to complete the UV-renormalization of the form factor \( F_1 \), we have also to subtract its value at \( s = 0 \), or, which is the same, to add the counterterm diagram shown in Fig. 3(i). We need, therefore, the constant \( Z_{2,\text{OS}}(\epsilon, \mu^2/m^2) \) at the two-loop level, that was computed in [26, 27]. We use the result of [27] and we express it in terms of the renormalized \( \overline{\text{MS}} \) coupling \( \alpha_S \), finding:

\[
Z_{2,\text{OS}}^{(2)}(\epsilon, \frac{\mu^2}{m^2}) = \left( \frac{\alpha_S}{2\pi} \right)^2 C^2(\epsilon) \left\{ \left( \frac{\mu^2}{m^2} \right)^{2\epsilon} \left[ C_F^2 \left( \frac{9}{8\epsilon^2} + \frac{51}{16\epsilon} + \frac{433}{32} - 6\zeta(3) + 24\zeta(2) \log 2 \right) \right. \\
- \left. \frac{39}{2}\zeta(2) \right] + C_F C_A \left( \frac{11}{8\epsilon^2} - \frac{101}{16\epsilon} - \frac{803}{32} + 3\zeta(3) - 12\zeta(2) \log 2 \right) \\
+ \frac{15}{2}\zeta(2) \right\} + C_F T_R N_f \left( \frac{1}{2\epsilon^2} + \frac{9}{4\epsilon} + \frac{59}{8} + 2\zeta(2) \right) + C_F T_R \left( \frac{1}{\epsilon^2} + \frac{19}{12\epsilon} \right) 
\]
\[ + \frac{1139}{72} - 8\zeta(2) \right) + \left( \frac{\mu^2}{m^2} \right)^\epsilon \left[ C_F C_A \left( \frac{33}{12\epsilon^2} + \frac{11}{3\epsilon} + \frac{22}{3} \right) \right. \\
\left. - C_F T_R (N_f + 1) \left( \frac{1}{\epsilon^2} + \frac{4}{3\epsilon} + \frac{8}{3} \right) \right] \right) \right] . \]

The counterterm diagram is defined, therefore, as follows:

\[ Z_2^{(2l)} \overset{\text{def}}{=} Z_2^{(2l)}(\epsilon, \frac{\mu^2}{m^2}) \times \text{counterterm} . \] (48)

5 Renormalized Form Factors

We report now the analytic expression of the UV-renormalized form factors at the one- and two-loop level, \( F_{i,R}^{(1l)}(\epsilon, s, \frac{\mu^2}{m^2}) \) and \( F_{i,R}^{(2l)}(\epsilon, s, \frac{\mu^2}{m^2}) \), in the space-like region \( s = q^2 < 0 \), in terms of HPLs of the variable \( x \) already introduced in Eq. (14), where:

\[ x = \sqrt{-s + 4} - \sqrt{-s} \]

\[ \sqrt{-s + 4} + \sqrt{-s} . \]

The one-loop form factors are given up to the first term in the expansion in \( \epsilon \), while the two-loop ones up to the finite part.

The renormalization of the UV divergences is carried out in the hybrid scheme explained in the previous Section. Note that adding the diagrams of Figs. 1 and 2 and the corresponding counterterms we will have a non-trivial dependence on the renormalization scale \( \mu \), due to the fact that the virtual contributions coming from the one- and two-loop diagrams reported in Section 3 depend on the ratio \( (\frac{\mu^2}{m^2})^2 \), while in the counterterms we have a dependence on \( (\frac{\mu^2}{m^2})^2 \) (those calculated in the OS-scheme) as well as on \( (\frac{\mu^2}{m^2})\epsilon \) (for the counterterms in the MS-scheme). The expansion of these ratios in powers of \( \epsilon \) generates terms proportional to \( \log (\frac{\mu^2}{m^2}) \) and \( \log^2 (\frac{\mu^2}{m^2}) \). In this Section we give the one- and two-loop UV-renormalized form factors for \( \mu = m \). For the terms proportional to \( \log (\frac{\mu^2}{m^2}) \) and \( \log^2 (\frac{\mu^2}{m^2}) \) we refer the reader to Section 5.3.

5.1 One-Loop UV-Renormalized Form Factors

The one-loop UV-renormalized form factors are recovered adding the diagram of Fig. 4(b) and the corresponding counterterm:

\[ CT^{(1l)}(\epsilon, s) = Z_{2,OS}^{(1l)}(\epsilon) \times \text{counterterm} \]. (49)

We put \( \mu = m \) and we define:

\[ F_{i,R}^{(1l)}(\epsilon, s) = C(\epsilon) F_{i,R}^{(1l)}(\epsilon, s) , \quad \text{with} \quad i = 1, 2, \] (50)
finding:

\[
F_{1,R}^{(1)}(\epsilon, s) = \frac{1}{\epsilon} \left\{ C_F \left[ -1 + \left( 1 - \frac{1}{1-x} - \frac{1}{1+x} \right) H(0; x) \right] \right\} \\
- C_F \left[ 2 + \left( \frac{1}{2} - \frac{1}{1+x} \right) H(0; x) + \left( 1 - \frac{1}{1-x} - \frac{1}{1+x} \right) \zeta(2) \right] \\
- 2H(0; x) - H(0, 0; x) + 2H(-1, 0; x) \right\] \\
- \epsilon \left\{ C_F \left[ 4 - \frac{1}{2} \left( 1 - \frac{2}{1+x} \right) [\zeta(2) - H(0, 0; x) + 2H(-1, 0; x)] \right] \\
+ \left( \frac{1}{1-x} - \frac{1}{1+x} \right) [2(\zeta(2) + \zeta(3)) + (\zeta(2) - 4) H(0; x) \right]
- 2\zeta(2) H(-1; x) - 2H(0, 0; x) + 4H(-1, 0; x)
- H(0, 0; x) - 4H(-1, -1; 0; x) + 2H(-1, 0, 0; x) \\
+ 2H(0, -1, 0; x) \right\} + \mathcal{O}(\epsilon^2), \quad (51)
\]

\[
F_{2,R}^{(1)}(\epsilon, s) = -C_F \left( \frac{1}{1-x} - \frac{1}{1+x} \right) H(0; x) \\
+ \epsilon \left\{ C_F \left[ \left( \frac{1}{1-x} - \frac{1}{1+x} \right) [\zeta(2) - 4H(0; x) \right]
- H(0, 0; x) + 2H(-1, 0; x) \right\} \right\} + \mathcal{O}(\epsilon^2). \quad (52)
\]

5.2 Two-Loop UV-Renormalized Form Factors

Adding the contributions of Eqs. (31, 32) and (39–44), we find, diagrammatically, the following counterterm:

\[
CT^{(2)}(\epsilon, s) = -\frac{2}{m} \frac{\delta m_{\text{OS}}^{(1)}(\epsilon, m)}{m} \times \left( m \begin{array}{c}
\end{array} \right) \\
+ \left[ 2Z_{g,\text{MS}}^{(1)}(\epsilon) + Z_{2,\text{OS}}^{(1)}(\epsilon) \right] \times \begin{array}{c}
\end{array} + Z_{2,\text{OS}}^{(2)}(\epsilon) \times \begin{array}{c}
\end{array}. \quad (53)
\]

The two-loop UV-renormalized form factors are recovered adding the diagrams in Fig. 2 and the counterterm in Eq. (53).

We put \( \mu = m \) and we define:

\[
F_{i,R}^{(2)}(\epsilon, s) = C_2(\epsilon) F_{i,R}^{(2)}(\epsilon, s), \quad \text{with } i = 1, 2, \quad (54)
\]

finding:

\[
F_{1,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ \frac{11}{12} C_F C_A \left[ 1 - \left( 1 - \frac{1}{1-x} - \frac{1}{1+x} \right) H(0; x) \right] \right\}
\]
\[-\frac{1}{3} C_F T_R N_f \left[ 1 - \left( 1 - \frac{1}{1 - x} - \frac{1}{1 + x} \right) H(0; x) \right] \\
+ C_F^2 \left[ \frac{1}{2} - \left( 1 - \frac{1}{1 - x} - \frac{1}{1 + x} \right) H(0; x) \\
+ \left( 1 - \frac{1}{1 - x} + \frac{1}{(1-x)^2} - \frac{1}{1 + x} + \frac{1}{(1+x)^2} \right) H(0, 0; x) \right] \}
\\n+ \frac{1}{\epsilon} \left\{ C_F C_A \left[ -\frac{49}{36} + \zeta(2) + H(0, 0; x) \\
+ \left( 1 - \frac{1}{1 - x} - \frac{1}{1 + x} \right) \left[ \frac{\zeta(2)}{2} + \left( \frac{67}{36} - \zeta(2) \right) H(0; x) \right] \\
+ H(0, 0; x) - H(-1, 0; x) + H(1, 0; x) - H(0, 0, 0; x) \right] \\
- \left( 1 - \frac{1}{1 - x} + \frac{1}{(1-x)^2} - \frac{1}{1 + x} + \frac{1}{(1+x)^2} \right) \left[ \frac{\zeta(3)}{2} \right] \\
+ \frac{1}{2} \zeta(2) H(0; x) - H(0, -1, 0; x) + H(0, 0, 0; x) \right[ \\
+ H(0, 1, 0; x) \right] \\
+ \frac{5}{9} C_F T_R N_f \left[ 1 - \left( 1 - \frac{1}{1 - x} - \frac{1}{1 + x} \right) H(0; x) \right] \\
+ C_F^2 \left[ 2 - \left( \frac{1}{2} - \frac{1}{1 - x} \right) H(0; x) \\
+ \left( \frac{1}{1 + x} - \frac{1}{1 - x} - \frac{2}{(1+x)^2} \right) H(0, 0; x) \\
+ \left( 1 - \frac{1}{1 - x} - \frac{1}{1 + x} \right) \left( \zeta(2) - 3H(0; x) - 2H(0, 0; x) \right) \\
+ 2H(-1, 0; x) \right] - \left( 1 - \frac{1}{1 - x} + \frac{1}{(1-x)^2} - \frac{1}{1 + x} \right) \\
+ \frac{1}{(1+x)^2} \left( \zeta(2) H(0; x) - 4H(0, 0; x) - 3H(0, 0, 0; x) \right) \\
+ 2H(0, -1, 0; x) + 4H(-1, 0, 0; x) \right] \}
\\n+ C_F C_A \left[ -\frac{1595}{108} + \frac{3}{(1+x)^2} - \frac{3}{(1+x)} + \zeta(2) \left( -\frac{347}{36} \right) \\
+ \frac{36 \log 2}{(1+x)^2} - \frac{36 \log 2}{(1+x)} + 6 \log 2 + \frac{79}{18(1-x)} - \frac{168}{(1+x)^4} \\
+ \frac{363}{(1+x)^3} - \frac{543}{2(1+x)^2} + \frac{734}{9(1+x)} \right] + \zeta(2)^2 \left( -\frac{71}{20} \right) \\
- \frac{37}{20(1-x)^2} + \frac{679}{160(1-x)} - \frac{879}{10(1+x)^5} + \frac{879}{4(1+x)^4} \]
\[-\frac{7481}{40(1+x)^3} + \frac{943}{16(1+x)^2} - \frac{441}{160(1+x)} + \zeta(3)\left(\frac{5}{6}\right) + \zeta(2)\left(-\frac{31}{3(1-x)} + \frac{324}{648} - \frac{648}{374} + \frac{374}{269} - \frac{269}{135}\right) + \frac{127}{3(1+x)}H(-1; x) + \zeta(3)\left(2 + \frac{2}{(1-x)^2} - \frac{2}{(1-x)}\right) + \frac{2}{(1+x)^2} - \frac{2}{(1+x)}H(-1; x) + \frac{\zeta(2)}{3(1-x)}H(-1, -1, 0; x)\]

\[+ \zeta(2)\left(\frac{2}{(1+x)^2} + \frac{2}{(1+x)^2}\right)H(-1, 0; x) + \left(-\frac{31}{9} + \frac{43}{9(1-x)} + \frac{6}{(1+x)^3} - \frac{9}{(1+x)^2}\right) + \frac{46}{9(1+x)}H(-1, 0; x) + \left(-4 - \frac{4}{(1-x)^2} + \frac{4}{(1-x)}\right)\]

\[+ \frac{4}{(1+x)^2} + \frac{4}{(1+x)}H(-1, 0, -1, 0; x) + \left(-\frac{53}{3}\right) + \frac{282}{3(1-x)} - \frac{564}{3(1-x)^3} - \frac{339}{(1+x)^2}\]

\[+ \frac{206}{3(1+x)}H(-1, 0, 0; x) + \left(2 + \frac{4}{(1-x)^2} - \frac{2}{(1-x)}\right) + \frac{4}{(1+x)^2} - \frac{2}{(1+x)}H(-1, 0, 0, 0; x) + \left(4 + \frac{4}{(1-x)^2}\right)\]

\[- \frac{4}{(1-x)} + \frac{4}{(1+x)^2} - \frac{4}{(1+x)}H(-1, 0, 1, 0; x) + \left(-6 + \frac{6}{(1-x)} + \frac{6}{(1+x)}\right)H(-1, 1, 0; x)\]

\[+ \zeta(2)\left(\frac{29}{6} + \frac{11}{3(1-x)} - \frac{258}{258} + \frac{744}{(1+x)^5} + \frac{744}{1(1+x)^4} - \frac{779}{(1+x)^3}\right) + \frac{357}{6}H(0; x) + \zeta(3)\left(-\frac{15}{2} - \frac{1}{2(1-x)^2}\right)\]

\[+ \frac{223}{2(1+x)^2} + \frac{810}{3(1+x)} - \frac{662}{2(1-x)} - \frac{2(1-x)^2}{3(1+x)^3} + \frac{324}{2(1+x)}H(0; x) + \left(\frac{2545}{216} - \frac{365}{27(1-x)} + \frac{9}{(1+x)^3}\right)\]

\[- \frac{27}{2(1+x)^2} - \frac{599}{108(1+x)}H(0; x) + \zeta(2)\left(7 + \frac{1}{(1-x)^2}\right)\]
\[-\frac{35}{8(1-x)} - \frac{90}{(1+x)^5} + \frac{225}{(1+x)^4} - \frac{363}{2(1+x)^3} + \frac{193}{4(1+x)^2} \]
\[-\frac{83}{8(1+x)} H(0, -1; x) + \left(-10 - \frac{10}{(1-x)^2} + \frac{10}{(1-x)} \right) \]
\[-\frac{10}{(1+x)^2} + \frac{10}{(1+x)} H(0, -1, 0; x) + \left(-\frac{68}{3} \right) \]
\[+\frac{35}{3(1-x)} + \frac{12}{(1+x)^4} - \frac{24}{(1+x)^3} + \frac{14}{(1+x)^2} \]
\[+\frac{29}{3(1+x)} H(0, -1, 0; x) + \left(14 + \frac{8}{(1-x)^2} - \frac{83}{8(1-x)} \right) \]
\[-\frac{282}{(1+x)^5} + \frac{705}{(1+x)^4} - \frac{1171}{2(1+x)^3} + \frac{725}{4(1+x)^2} \]
\[-\frac{227}{8(1+x)} H(0, -1, 0, 0; x) + \left(6 + \frac{6}{(1-x)^2} - \frac{6}{(1-x)} \right) \]
\[+\frac{6}{(1+x)^2} - \frac{6}{(1+x)} H(0, -1, 1, 0; x) \]
\[+\zeta(2) \left(1 + \frac{1}{(1-x)^2} + \frac{5}{16(1-x)} - \frac{69}{(1+x)^5} + \frac{345}{2(1+x)^4} \right) \]
\[-\frac{563}{4(1+x)^3} + \frac{317}{8(1+x)^2} - \frac{59}{16(1+x)} \right) H(0, 0; x) \]
\[-\left(\frac{217}{36} + \frac{25}{18(1-x)} - \frac{24}{(1+x)^4} + \frac{39}{(1+x)^3} - \frac{3}{2(1+x)^2} \right) \]
\[-\left(\frac{130}{9(1+x)} \right) H(0, 0; x) + \left(22 + \frac{12}{(1-x)^2} - \frac{75}{4(1-x)} \right) \]
\[+\frac{12}{(1+x)^5} - \frac{30}{(1+x)^4} + \frac{49}{(1+x)^3} - \frac{63}{2(1+x)^2} \]
\[-\frac{51}{4(1+x)} H(0, 0, -1, 0; x) + \left(89 + \frac{4}{6} - \frac{4}{3(1-x)} - \frac{258}{(1+x)^5} \right) \]
\[-\frac{816}{(1+x)^4} - \frac{923}{(1+x)^3} + \frac{436}{(1+x)^2} - \frac{259}{3(1+x)} \right) H(0, 0, 0; x) \]
\[+\left(-12 - \frac{6}{(1-x)^2} + \frac{193}{16(1-x)} + \frac{3}{(1+x)^5} - \frac{15}{2(1+x)^4} \right) \]
\[+\frac{17}{4(1+x)^3} - \frac{39}{8(1+x)^2} + \frac{177}{16(1+x)} \right) H(0, 0, 0, 0; x) \]
\[+\left(-14 - \frac{8}{(1-x)^2} + \frac{12}{(1-x)} + \frac{48}{(1+x)^5} - \frac{120}{(1+x)^4} \right) \]
\[+\frac{88}{(1+x)^3} - \frac{20}{(1+x)^2} + \frac{12}{(1+x)} \right) H(0, 0, 1, 0; x) \]
\[+\zeta(2) \left(1 + \frac{1}{(1-x)^2} + \frac{1}{(1-x)} + \frac{1}{(1+x)^2} \right) \]

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\[-\frac{1}{(1+x)} H(0,1;x) + \left(6 + \frac{6}{(1-x)^2} - \frac{6}{1-x}\right)\]
\[+ \frac{6}{(1+x)^2} - \frac{6}{(1+x)} H(0,1,-1,0;x) + \left(8 - \frac{4}{1-x}\right)\]
\[+ \frac{48}{(1+x)^4} \left(\frac{96}{(1+x)^3} + \frac{56}{(1+x)^2} - \frac{12}{1+x}\right) H(0,1,0;x)\]
\[+ \left(\frac{-2}{(1-x)^2} + \frac{5}{1-x} + \frac{24}{1+x} - \frac{60}{(1+x)^3}\right) H(0,1,0,0;x)\]
\[+ \frac{72}{(1+x)^3} - \frac{52}{(1+x)^2} + \frac{11}{1+x}\]
\[ + C_T R N_f \left[ \frac{106}{27} + \zeta(2) \left( \frac{31}{9} - \frac{22}{9(1-x)} - \frac{16}{9(1+x)} \right) \right. \\
+ \zeta(3) \left( \frac{4}{3} - \frac{4}{3(1-x)} - \frac{4}{3(1+x)} \right) \\
+ \zeta(2) \left( -\frac{8}{3} + \frac{8}{3(1-x)} - \frac{8}{3(1+x)} \right) H(-1; x) \\
+ \left( -\frac{8}{3} + \frac{8}{3(1-x)} - \frac{8}{3(1+x)} \right) H(-1, -1, 0; x) \\
+ \left( \frac{38}{9} - \frac{44}{9(1-x)} - \frac{32}{9(1+x)} \right) H(-1, 0; x) \\
+ \left( \frac{4}{3} - \frac{4}{3(1-x)} - \frac{4}{3(1+x)} \right) H(-1, 0, 0; x) \\
+ \zeta(2) \left( -\frac{2}{3} + \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0; x) \\
+ \left( -\frac{209}{54} + \frac{106}{27(1-x)} - \frac{103}{27(1+x)} \right) H(0; x) \\
+ \left( \frac{4}{3} - \frac{4}{3(1-x)} - \frac{4}{3(1+x)} \right) H(0, -1, 0; x) \\
+ \left( -\frac{19}{9} + \frac{22}{9(1-x)} - \frac{16}{9(1+x)} \right) H(0, 0; x) \\
+ \left( \frac{2}{3} + \frac{2}{3(1-x)} - \frac{2}{3(1+x)} \right) H(0, 0, 0; x) \right] \\
+ C_T R \left[ \frac{383}{27} + \frac{196}{9(1+x)^2} - \frac{196}{9(1+x)} \\
- \zeta(2) \left( 1 - \frac{392}{3(1+x)^4} + \frac{784}{3(1+x)^3} - \frac{458}{3(1+x)^2} + \frac{22}{1+x} \right) \right. \\
+ \zeta(2) \left( -\frac{2}{3} + \frac{5}{3(1-x)^4} - \frac{24}{3(1+x)^5} + \frac{60}{(1+x)^4} - \frac{44}{(1+x)^3} \right. \\
+ \left. \frac{6}{(1+x)^2} - \frac{5}{3(1+x)} \right) H(0; x) + \left( \frac{265}{54} + \frac{236}{27(1-x)} \right. \\
+ \left. \frac{356}{9(1+x)^3} - \frac{178}{3(1+x)^2} + \frac{563}{27(1+x)} \right) H(0; x) \\
+ \left( \frac{19}{9} + \frac{248}{9(1+x)^4} - \frac{496}{9(1+x)^3} + \frac{326}{9(1+x)^2} \right. \\
- \frac{26}{3(1+x)} \right) H(0, 0; x) + \left( \frac{2}{3} + \frac{5}{3(1-x)} - \frac{24}{(1+x)^5} \right. \\
+ \left. \frac{60}{(1+x)^4} - \frac{44}{(1+x)^3} + \frac{6}{3(1+x)^2} + \frac{5}{3(1+x)} \right) H(0, 0, 0; x) \right] \\
+ C_T \left[ \frac{23}{2} + \zeta(2) \left( \frac{55}{4} - \frac{72 \log 2}{(1+x)^3} + \frac{72 \log 2}{(1+x)} - 12 \log 2 + \frac{2}{1-x} \right) \right] \]
\[- \frac{240}{(1+x)^4} + \frac{528}{(1+x)^3} - \frac{300}{(1+x)^2} + \frac{11}{2(1+x)} \]
\[+ \zeta(2) \left( \frac{181}{10} - \frac{61}{10(1-x)^2} - \frac{1219}{80(1-x)} - \frac{171}{(1+x)^5} \right) \]
\[+ \frac{855}{2(1+x)^4} - \frac{6867}{20(1+x)^3} + \frac{749}{8(1+x)^2} - \frac{1731}{80(1+x)} \]
\[+ \zeta(3) \left( -7 - \frac{4}{(1-x)^2} + \frac{2}{(1-x)} + \frac{168}{(1+x)^4} - \frac{336}{(1+x)^3} \right) \]
\[+ \frac{190}{(1+x)^2} - \frac{24}{(1+x)} + \zeta(2) \left( 10 + \frac{2}{1-x} + \frac{180}{(1+x)^4} \right) \]
\[+ \frac{360}{(1+x)^3} - \frac{270}{(1+x)^2} - \frac{88}{(1+x)} \right) H(-1; x) - \left( 4 - \frac{4}{1-x} \right) \]
\[+ \frac{4}{(1+x)} \right) H(-1, -1, 0; x) + \left( 16 + \frac{16}{(1-x)^2} - \frac{16}{(1-x)} \right) \]
\[+ \frac{16}{(1+x)^2} - \frac{16}{(1+x)} \right) H(-1, -1, 0; x) + \zeta(2) \left( 4 \right) \]
\[+ \left( -\frac{55}{2} \frac{36}{(1-x)} + \frac{192}{(1+x)^3} - \frac{288}{(1+x)^2} \right) \]
\[+ \frac{115}{(1+x)} \right) H(-1, 0; x) \left( 8 + \frac{8}{(1-x)^2} - \frac{8}{1-x} \right) \]
\[+ \frac{8}{(1+x)^2} + \frac{8}{(1+x)} \right) H(-1, 0, -1; x) - \left( 6 + \frac{16}{(1-x)^2} \right) \]
\[+ \frac{14}{(1-x)} + \frac{252}{(1+x)^4} - \frac{504}{(1+x)^3} + \frac{270}{(1+x)^2} \]
\[+ \frac{16}{(1+x)} \right) H(-1, 0, 0; x) + \left( -12 - \frac{12}{(1-x)^2} + \frac{12}{1-x} \right) \]
\[+ \frac{12}{(1+x)^2} + \frac{12}{(1+x)} \right) H(-1, 0, 0; x) + \zeta(2) \left( -17 \right) \]
\[+ \frac{8}{(1-x)^2} + \frac{2(1-x)}{(1+x)^5} - \frac{60}{(1+x)^4} - \frac{156}{(1+x)^3} - \frac{136}{(1+x)^2} \]
\[+ \frac{6}{(1+x)^2} + \frac{113}{2(1+x)} \right) H(0; x) + \zeta(3) \left( 8 - \frac{7}{1-x} \right) \]
\[+ \frac{168}{(1+x)^5} - \frac{420}{(1+x)^4} + \frac{364}{(1+x)^3} - \frac{126}{(1+x)^2} \]
\[+ \frac{5}{(1+x)} \right) H(0; x) - \left( \frac{85}{8} - \frac{19}{2(1-x)} - \frac{47}{4(1+x)} \right) H(0; x) \]
\[+ \zeta(2) \left( -10 + \frac{2}{(1-x)^2} + \frac{19}{4(1-x)} + \frac{180}{(1+x)^5} \right) \]
\[
\begin{align*}
-\frac{450}{(1+x)^4} + \frac{363}{(1+x)^3} - \frac{185}{2(1+x)^2} + \frac{67}{4(1+x)} H(0, -1; x) \\
+ \left( 4 + \frac{4}{(1-x)^2} - \frac{4}{(1-x)} + \frac{4}{(1+x)^2} \right) H(0, -1, -1, 0; x) + \left( 16 - \frac{2}{(1-x)} + \frac{384}{(1+x)^4} \right) \\
- \frac{768}{(1+x)^3} + \frac{442}{(1+x)^2} - \frac{60}{(1+x)} H(0, -1, 0; x) \\
+ \left( -14 - \frac{10}{(1-x)^2} + \frac{67}{4(1-x)} - \frac{252}{(1+x)^5} + \frac{630}{(1+x)^4} \right) H(0, -1, 0, 0; x) \\
+ \zeta(2) \left( 13 + \frac{5}{(1-x)^2} - \frac{105}{8(1-x)} - \frac{66}{(1+x)^5} \right) \\
+ \frac{165}{(1+x)^4} - \frac{281}{2(1+x)^3} + \frac{203}{4(1+x)^2} - \frac{137}{8(1+x)} H(0, 0; x) \\
+ \left( 4 + \frac{229}{16} - \frac{16}{(1-x)^2} + \frac{42}{(1+x)} H(0, 0, -1, 0; x) + \left( -22 - \frac{10}{(1-x)^2} \right) \\
+ \frac{466}{(1+x)^2} - \frac{343}{2(1+x)} \right) H(0, 0, 0, 0; x) + \left( -22 - \frac{10}{(1-x)^2} \right) \\
+ \frac{45}{17} + \frac{384}{2(1+x)} - \frac{960}{8(1+x)} + \frac{746}{169} H(0, 0, 0, -1; x) + \left( -10 + \frac{10}{(1-x)^2} \right) \\
+ \frac{60}{17} + \frac{216}{2(1+x)} - \frac{256}{8(1+x)} + \frac{89}{169} H(0, 0, 0, 0; x) + \left( -10 + \frac{10}{(1-x)^2} \right) \\
- \frac{71}{2(1+x)} + \frac{17}{(1+x)^4} - \frac{217}{8(1-x)} \\
+ \frac{6}{(1+x)^3} + \frac{15}{(1+x)^2} - \frac{256}{8(1+x)^2} \right) H(0, 0, 0, 0; x) + \left( 12 + \frac{4}{(1-x)^2} - \frac{8}{(1-x)} \right) \\
+ \frac{96}{8(1+x)} H(0, 0, 0, 0; x) + \left( 12 + \frac{4}{(1-x)^2} - \frac{8}{(1-x)} \right) \\
- \frac{96}{(1+x)^3} + \frac{240}{(1+x)^4} - \frac{176}{(1+x)^3} + \frac{28}{(1+x)^2} \right) H(0, 0, 1, 0; x) + \left( -10 - \frac{8}{(1-x)^2} + \frac{8}{(1-x)} \right) \\
+ \left( -10 - \frac{8}{(1-x)^2} + \frac{8}{(1-x)} \right) H(0, 0, 1, 0; x) + \left( -10 - \frac{8}{(1-x)^2} + \frac{8}{(1-x)} \right) \\
- \frac{96}{(1+x)^4} + \frac{192}{(1+x)^3} - \frac{116}{(1+x)^2} + \frac{20}{(1+x)} \right) H(0, 1, 0; x) + \left( -4 - \frac{7}{(1-x)} + \frac{360}{(1+x)^5} - \frac{900}{(1+x)^4} + \frac{700}{(1+x)^3} \right)
\end{align*}
\]
\[ F_{2,R}(\epsilon, s) = \frac{1}{\epsilon} \left\{ C_F \left[ \frac{1}{1-x} - \frac{1}{1+x} \right] H(0; 1, 0, 0; x) - 2 \left( \frac{1}{1-x} - \frac{1}{(1-x)^2} \right) \right\} \]

\[ + C_F C_A \left[ \frac{3}{(1-x)^2} - \frac{3}{(1+x)^2} + \frac{3}{(1+x)} + \zeta(2) \left( -\frac{24 \log 2}{(1-x)^2} + \frac{24 \log 2}{1+x} \right) \right. \]

\[ - \frac{6}{(1-x)^2} + \frac{6}{(1+x)^2} + \frac{3}{12(1-x)} + \frac{16}{(1+x)^3} + \frac{16}{2(1+x)^2} \]

\[ - \frac{607}{12(1+x)} + \zeta(2) \left( \frac{69}{40(1-x)^3} - \frac{207}{80(1-x)^2} \right) \]

\[ + \frac{32(1-x)}{10(1+x)^5} - \frac{4(1+x)^4}{10(1+x)^3} \]

\[ - \frac{249}{5(1+x)^2} + \frac{32(1+x)}{45} + \zeta(3) \left( -\frac{324}{(1+x)^4} + \frac{648}{1+x} \right) \]

\[ - \frac{344}{(1+x)^2} + \frac{20}{1+x} + \zeta(2) \left( -\frac{9}{2(1-x)^2} + \frac{9}{2(1-x)} \right) \]

\[ + \frac{90}{(1+x)^4} - \frac{180}{(1+x)^3} + \frac{231}{2(1+x)^2} - \frac{51}{2(1+x)} \]

\[ H(-1; x) \]

\[ 34 \]
\[
\begin{aligned}
&+ \left( \frac{1}{6(1-x)} - \frac{6}{(1+x)^3} + \frac{9}{(1+x)^2} - \frac{19}{6(1+x)} \right) H(-1, 0; x) \\
&+ \left( -\frac{1}{2(1-x)^2} + \frac{1}{2(1-x)} + \frac{282}{(1+x)^4} - \frac{564}{(1+x)^3} \right) \frac{H(-1, 0; x) + \zeta(2)}{(1+x)^2} \frac{3}{(1-x)^3} \\
&+ \frac{632}{2(1+x)^2} - \frac{59}{2(1+x)} \right) H(-1, 0; x) + \zeta(2) \frac{3}{(1-x)^3} \\
&- \frac{4(1-x)^2 + 8(1-x) + (1+x)^5}{(1+x)^4} + 2(1+x)^3 \\
&- \frac{621}{2(1+x)^2} + \frac{307}{8(1+x)} \right) H(0; x) + \zeta(3) \frac{13}{4(1-x)} \\
&- \frac{324}{(1+x)^5} + \frac{1}{(1+x)^4} - \frac{635}{(1+x)^3} + \frac{285}{2(1+x)^2} \\
&+ \frac{13}{4(1+x)} \right) H(0; x) + \left( -\frac{127}{18(1-x)} - \frac{9}{(1+x)^3} + \frac{27}{2(1+x)^2} \\
&+ \frac{23}{9(1+x)} \right) H(0; x) + \zeta(2) \left( -\frac{9}{2(1-x)^3} + \frac{27}{4(1-x)^2} \\
&- \frac{8(1-x)^4 + (1+x)^5}{(1+x)^4} + (1+x)^3 - (1+x)^2 \\
&- \frac{21}{8(1+x)} \right) H(0, -1; x) + \left( -\frac{1}{(1-x)^2} + \frac{1}{(1-x)^2} - \frac{12}{(1+x)^4} \\
&+ \frac{24}{(1+x)^3} - \frac{9}{(1+x)^2} + \frac{9}{(1+x)} \right) H(0, -1, 0; x) \\
&+ \left( -\frac{1}{2(1-x)^2} + \frac{3}{4(1-x)^2} - \frac{5}{8(1-x)} + \frac{282}{(1+x)^5} \\
&- \frac{705}{(1+x)^4} + \frac{562}{(1+x)^3} - \frac{138}{(1+x)^2} - \frac{5}{8(1+x)} \right) \frac{H(0, -1, 0; x)}{1+x} \\
&+ \zeta(2) \left( \frac{7}{4(1-x)^3} - \frac{81}{4(1-x^2)} - \frac{5}{16(1-x)} + \frac{69}{(1+x)^5} \\
&- \frac{345}{2(1+x)^4} + \frac{135}{(1+x)^3} - \frac{30}{(1+x)^2} - \frac{30}{16(1+x)} \right) \frac{H(0, 0; x)}{(1-x)^2} \\
&+ \left( -\frac{1}{(1-x)^3} + \frac{12}{12(1-x)} + \frac{12}{(1+x)^3} + \frac{12}{12(1-x)} \right) H(0, 0; x) + \left( -\frac{1}{(1-x)^3} + \frac{3}{2(1-x)^2} \\
&- \frac{4(1-x)}{(1+x)^5} + (1+x)^4 - (1+x)^3 + (1+x)^2 \\
&- \frac{25}{4(1+x)} \right) H(0, 0, -1, 0; x) + \left( \frac{3}{(1-x)^3} - \frac{15}{4(1-x)^2} \\
&+ \frac{11}{8(1-x)} + \frac{258}{(1+x)^5} - \frac{816}{(1+x)^4} + \frac{1803}{2(1+x)^3} - \frac{767}{2(1+x)^2} \right) \\
\end{aligned}
\]
\[
\begin{align*}
&+ \frac{315}{8(1 + x)} H(0, 0, 0; x) + \left( \frac{3}{4(1 - x)^3} - \frac{9}{8(1 - x)^2} \right) \\
&+ \frac{11}{16(1 - x)} - \frac{3}{(1 + x)^5} + \frac{15}{2(1 + x)^4} - \frac{4}{(1 + x)^3} + \frac{3}{2(1 + x)^2} \\
&+ \frac{11}{16(1 + x)} H(0, 0, 0; x) + \left( \frac{3}{1 - x} - \frac{48}{(1 + x)^5} \right) \\
&+ \frac{120}{(1 + x)^4} - \frac{84}{(1 + x)^3} + \frac{6}{(1 + x)^2} + \frac{3}{(1 + x)} H(0, 0, 1; x) \\
&+ \left( -\frac{48}{(1 + x)^4} + \frac{96}{(1 + x)^3} - \frac{48}{(1 + x)^2} \right) H(0, 1; x) \\
&- \left( \frac{11}{2(1 - x)} + \frac{24}{(1 + x)^5} - \frac{60}{(1 + x)^4} + \frac{70}{(1 + x)^3} - \frac{45}{(1 + x)^2} \right) H(1, 0; x) \\
&- \left( \frac{840}{(1 + x)^4} + \frac{660}{(1 + x)^3} - \frac{150}{(1 + x)^2} - \frac{3}{(1 + x)} \right) H(1, 0; x) \\
&- \left( \frac{24}{(1 + x)^3} - \frac{36}{(1 + x)^2} + \frac{12}{1 + x} \right) H(1, 0; x) \\
&- \left( \frac{24}{(1 + x)^4} - \frac{48}{(1 + x)^3} + \frac{40}{(1 + x)^2} - \frac{16}{1 + x} \right) H(1, 0, 0; x) \\
&- \left( \frac{3}{(1 - x)} - \frac{336}{(1 + x)^5} + \frac{840}{(1 + x)^4} - \frac{660}{(1 + x)^3} + \frac{150}{1 + x} \right) \\
&+ \left( \frac{3}{(1 + x)} \right) H(1, 0, 0, 0; x) \\
&+ C_F T_R N_f \left[ \zeta(2) \left( -\frac{2}{3(1 - x)} + \frac{2}{3(1 + x)} \right) \right] + \left( -\frac{4}{3(1 - x)} \right) \\
&+ \frac{4}{3(1 + x)} H(-1, 0; x) + \left( \frac{25}{9(1 - x)} - \frac{25}{9(1 + x)} \right) H(0; x) \\
&+ \left( \frac{2}{3(1 - x)} - \frac{2}{3(1 + x)} \right) H(0, 0; x) \\
&+ C_F T_R \left[ -\frac{68}{3(1 + x)^2} + \frac{68}{3(1 + x)} + \zeta(2) \left( -\frac{136}{(1 + x)^4} + \frac{272}{1 + x} \right) \right] \\
&- \frac{132}{(1 + x)^2} - \frac{4}{(1 + x)} + \zeta(2) \left( -\frac{3}{2(1 - x)} + \frac{24}{(1 + x)^5} \right) \\
&- \frac{60}{(1 + x)^4} + \frac{42}{(1 + x)^3} - \frac{3}{(1 + x)^2} - \frac{3}{2(1 + x)} H(0; x) \\
&- \left( \frac{9}{(1 - x)} + \frac{3(1 + x)^3}{(1 + x)^2} - \frac{62}{(1 + x)^2} + \frac{118}{9(1 + x)} \right) H(0; x) \\
&- \left( \frac{88}{3(1 + x)^4} - \frac{176}{3(1 + x)^3} + \frac{92}{3(1 + x)^2} - \frac{4}{3(1 + x)} \right) H(0, 0; x) \\
\end{align*}
\]
\[-\left(\frac{3}{2(1-x)} - \frac{24}{(1+x)^3} + \frac{60}{(1+x)^4} - \frac{42}{(1+x)^5} + \frac{3}{(1+x)^2}\right) H(0, 0, 0; x)\]\[\quad + C_F^2 \left[\zeta(2) \left(\frac{48 \log 2}{(1+x)^2} - \frac{48 \log 2}{(1+x)} - \frac{17}{2(1-x)}\right) + \frac{240}{(1+x)^4} - \frac{528}{(1+x)^3} + \frac{304}{(1+x)^2} - \frac{15}{2(1+x)}\right]\]
\[-\zeta^2(2) \left(\frac{69}{20(1-x)^3} - \frac{207}{40(1-x)^2} + \frac{327}{80(1-x)} - \frac{171}{(1+x)^5}\right) + \frac{855}{2(1+x)^4} - \frac{10(1+x)^3}{1323} + \frac{80(1+x)^2}{327} + \frac{80(1+x)}{327}\]\[\quad + \zeta(3) \left(\frac{10}{(1-x)^2} + \frac{10}{(1-x)} - \frac{168}{(1+x)^4} + \frac{336}{(1+x)^3}\right) - \frac{182}{(1+x)^2} + \frac{14}{(1+x)}\right) + \zeta(2) \left(\frac{9}{(1-x)^2} - \frac{9}{(1-x)}\right)\]
\[-\frac{180}{(1+x)^4} + \frac{360}{(1+x)^3} - \frac{231}{(1+x)^2} + \frac{51}{(1+x)}\right) H(-1; x)\]
\[-\left(\frac{9}{(1-x)} + \frac{192}{(1+x)^3} - \frac{288}{(1+x)^2} + \frac{87}{(1+x)}\right) H(-1, 0; x)\]
\[-\left(\frac{1}{(1-x)^2} - \frac{1}{(1-x)} + \frac{252}{(1+x)^4} - \frac{504}{(1+x)^3} + \frac{257}{(1+x)^2}\right) \frac{5}{(1+x)}\right) H(-1, 0, 0; x) + \zeta(2) \left(-\frac{19}{2(1-x)^2} + \frac{81}{4(1-x)}\right)\]
\[\frac{60}{(1+x)^5} - \frac{156}{(1+x)^4} + \frac{131}{(1+x)^3} - \frac{10}{(1+x)^2}\]
\[-\frac{143}{4(1+x)}\right) H(0; x) + \zeta(3) \left(-\frac{7}{2(1-x)} - \frac{168}{(1+x)^5}\right) + \frac{420}{(1+x)^4} - \frac{350}{(1+x)^3} + \frac{105}{(1+x)^2} - \frac{7}{2(1+x)}\right) H(0; x)\]
\[-\left(\frac{31}{4(1-x)} - \frac{31}{4(1+x)}\right) H(0; x)\]
\[+ \zeta(2) \left(\frac{9}{(1-x)^3} - \frac{27}{2(1-x)^2} + \frac{21}{4(1-x)} - \frac{180}{(1+x)^5}\right) + \frac{450}{(1+x)^4} - \frac{348}{(1+x)^3} + \frac{72}{(1+x)^2} + \frac{21}{4(1+x)}\right) H(0, -1; x)\]
\[-\left(\frac{2}{(1-x)^2} - \frac{2}{(1-x)} + \frac{384}{(1+x)^4} - \frac{768}{(1+x)^3} + \frac{394}{(1+x)^2}\right) \frac{10}{(1+x)}\right) H(0, -1, 0; x) + \left(\frac{1}{(1-x)^3} - \frac{3}{2(1-x)^2}\right)
\]

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- 7
\frac{1}{4(1-x)} + \frac{252}{(1+x)^5} - \frac{630}{(1+x)^4} + \frac{496}{(1+x)^3} - \frac{114}{(1+x)^2} - \frac{7}{4(1+x)} H(0, -1, 0; x) + \zeta(2) \left(-\frac{7}{2(1-x)^3}\right)
+ \frac{21}{4(1-x)^2} - \frac{1}{8(1-x)} + \frac{66}{(1+x)^5} - \frac{165}{(1+x)^4} + \frac{135}{(1+x)^3} - \frac{7}{2(1+x)^2} - \frac{1}{8(1+x)} H(0, 0; x) + \left(\frac{2}{(1-x)^2} + \frac{1}{2(1-x)}\right) H(0, 0; x)
+ \left(\frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{14}{(1-x)} - \frac{3}{(1+x)^2} + \frac{1}{(1+x)}\right) H(0, 0, -1, 0; x)
+ \frac{2}{(1-x)^2} - \frac{1}{4(1-x)} - \frac{60}{(1+x)^5} + \frac{216}{(1+x)^4} - \frac{251}{(1+x)^3} - \frac{88}{(1+x)^2} + \frac{71}{4(1+x)} H(0, 0, 0; x) + \left(-\frac{3}{2(1-x)^3}\right)
+ \frac{9}{4(1-x)^2} - \frac{11}{8(1-x)} + \frac{6}{(1+x)^5} + \frac{1}{(1+x)^4} + \frac{8}{(1+x)^3} - \frac{3}{(1+x)^2} - \frac{11}{8(1+x)} H(0, 0, 0, 0; x) + \left(-\frac{6}{(1-x)}\right)
+ \frac{96}{(1+x)^5} - \frac{240}{(1+x)^4} + \frac{168}{(1+x)^3} - \frac{12}{(1+x)^2} - \frac{6}{(1+x)} H(0, 0, 1, 0; x) + \left(-\frac{4}{(1-x)^2} + \frac{4}{(1-x)}\right)
+ \frac{96}{(1+x)^4} - \frac{192}{(1+x)^3} + \frac{100}{(1+x)^2} - \frac{4}{(1+x)} H(0, 1, 0; x)
+ \left(\frac{25}{2(1-x)} - \frac{360}{(1+x)^5} + \frac{900}{(1+x)^4} - \frac{670}{(1+x)^3} + \frac{105}{(1+x)^2}\right) H(0, 1, 0, 0; x)
+ \left(\frac{25}{2(1+x)}\right) H(0, 1, 0, 0; x) + \zeta(2) \left(\frac{3}{(1-x)} + \frac{144}{(1+x)^5}\right)
+ \frac{360}{(1+x)^4} + \frac{300}{(1+x)^3} - \frac{90}{(1+x)^2} + \frac{3}{(1+x)} H(1, 0; x)
+ \left(\frac{4}{(1-x)} - \frac{1}{(1+x)^3} + \frac{48}{(1+x)^2} - \frac{72}{(1+x)}\right) H(1, 0; x)
+ \left(\frac{4}{(1-x)^2} - \frac{4}{(1-x)} + \frac{360}{(1+x)^4} - \frac{720}{(1+x)^3} + \frac{356}{(1+x)^2}\right) H(1, 0, 0; x)
+ \left(\frac{4}{(1+x)}\right) H(1, 0, 0; x) + \left(\frac{3}{(1-x)} + \frac{144}{(1+x)^5} - \frac{360}{(1+x)^4}\right)
\[ \frac{300}{(1+x)^3} - \frac{90}{(1+x)^2} + \frac{3}{(1+x)} H(1, 0, 0; x) \] . \quad (56)

### 5.3 Form Factors for \( \mu \neq m \)

In this Section we report the expressions for the renormalized form factors in the case we keep \( \mu \neq m \).

At the one-loop level we do not have an explicit dependence on the logarithm of the ratio of the renormalization scale and the mass of the heavy quark, because an overall factor \( \left( \frac{\mu^2}{m^2} \right) ^{\epsilon} \) can be taken out:

\[ F_{i,R}^{(1)} (\epsilon, s, \frac{\mu^2}{m^2}) = C(\epsilon) \left( \frac{\mu^2}{m^2} \right) ^{\epsilon} \mathcal{F}_{i,R}^{(1)} (\epsilon, s), \quad (57) \]

where the functions \( \mathcal{F}_{i,R}^{(1)} (\epsilon, s) \) are given in Eqs. (51,52).

At the two-loop level, such a dependence results from the coupling constant renormalization, first appearing at this level. Factoring an overall \( \left( \frac{\mu^2}{m^2} \right) ^{2\epsilon} \), we have:

\[ F_{i,R}^{(2)} (\epsilon, s, \frac{\mu^2}{m^2}) = C^2(\epsilon) \left( \frac{\mu^2}{m^2} \right) ^{2\epsilon} \left\{ \mathcal{F}_{i,R}^{(2)} (\epsilon, s) + \mathcal{M}_{i}^{(2)} (\epsilon, s) \log \left( \frac{\mu^2}{m^2} \right) + \mathcal{N}_{i}^{(2)} (s) \log^2 \left( \frac{\mu^2}{m^2} \right) \right\}, \quad (58) \]

where the functions \( \mathcal{F}_{i,R}^{(2)} (\epsilon, s) \) are given in Eqs. (55,56) and the functions \( \mathcal{M}_{i}^{(2)} (\epsilon, s) \) and \( \mathcal{N}_{i}^{(2)} (s) \) can be derived from the renormalization group equation.

Introducing \( \beta_0 \), the first coefficient of the QCD \( \beta \)-function:

\[ \beta_0 = \frac{11C_A - 4T_R(N_f + 1)}{6}, \quad (59) \]

we can write

\[ F_{i,R}^{(2)} (\epsilon, s, \frac{\mu^2}{m^2}) = \left( \frac{\alpha_s}{2\pi} \right) ^2 C^2(\epsilon) \left( \frac{\mu^2}{m^2} \right) ^{2\epsilon} \left\{ \mathcal{F}_{i,R}^{(2)} (\epsilon, s) - \frac{\beta_0}{\epsilon} \left[ \left( \frac{\mu^2}{m^2} \right) ^{-\epsilon} - 1 \right] \mathcal{F}_{i,R}^{(1)} (\epsilon, s) \right\} \quad (60) \]

\[ = \left( \frac{\alpha_s}{2\pi} \right) ^2 C^2(\epsilon) \left( \frac{\mu^2}{m^2} \right) ^{2\epsilon} \left\{ \mathcal{F}_{i,R}^{(2)} (\epsilon, s) + \beta_0 \mathcal{F}_{i,R}^{(1)} (\epsilon, s) \log \left( \frac{\mu^2}{m^2} \right) - \epsilon \frac{\beta_0}{2} \mathcal{F}_{i,R}^{(1)} (\epsilon, s) \log^2 \left( \frac{\mu^2}{m^2} \right) \right\}. \quad (61) \]

From these, the coefficients appearing in Eq. (58) can be readily read off:

\[ \mathcal{M}_{i}^{(2)} (\epsilon, s) = \frac{1}{\epsilon} \left\{ -C_F C_A \frac{11}{6} \left[ 1 - \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \right\}. \]
\[ + C_F T_R(N_f + 1) \frac{2}{3} \left[ 1 - \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \]

\[ + C_F T_R(N_f + 1) \frac{2}{3} \left[ 2 - \frac{(2)}{(1-x)} - \frac{\zeta(2)}{(1+x)} + \zeta(2) - \left( 2 - \frac{2}{(1-x)} \right) \right] \]

\[ \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0, 0, x) \]

\[ + C_F T_R(N_f + 1) \frac{2}{3} \left[ 2 - \frac{\zeta(2)}{(1-x)} - \frac{\zeta(2)}{(1+x)} + \zeta(2) + \left( 2 - \frac{2}{(1-x)} \right) \right] \]

\[ \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0, 0, x) \]

\[ N_1^{(2)}(s) = C_F C_F \frac{11}{12} \left[ 1 - \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \]

\[ - C_F T_R(N_f + 1) \frac{1}{3} \left[ 1 - \left( 1 - \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \]

\[ M_2^{(2)}(\epsilon, s) = -C_F C_F \frac{11}{6} \left[ \left( \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \]

\[ + C_F T_R(N_f + 1) \frac{2}{3} \left[ \left( \frac{1}{(1-x)} - \frac{1}{(1+x)} \right) H(0; x) \right] \]

\[ N_2^{(2)}(s) = 0. \]

### 6 Analytical Continuation above Threshold

Eqs. (61, 62, 63, 64) are written in terms of the “space-like” variable \( x \), defined in Eq. (13), for negative c.m. energy squared \( S < 0 \). In particular, for \( S < 0 \), \( x \) is real and positive and varies from \( x = 1 \), when \( S = 0 \), to \( x = 0 \) when \( S = -\infty \).

The physical form factors, defined in the time-like region \( S = Q^2 > 0 \) (in particular above the physical threshold \( S > 4m^2 \), where an imaginary part appears) can be recovered by analytical continuation with the usual \( i\epsilon \)-prescription, i.e. giving a small positive imaginary part to \( S \): \( S + i\epsilon \).

In so doing, if \( S > 0 \), but still \( S < 4m^2 \), the variable \( x \) becomes a phase factor:

\[ x = r = \frac{\sqrt{4m^2 - S} - \sqrt{-S - i\epsilon}}{\sqrt{4m^2 - S} + \sqrt{-S - i\epsilon}} = \frac{\sqrt{4m^2 - S} + i\sqrt{S}}{\sqrt{4m^2 - S} - i\sqrt{S}} = e^{i\phi}, \]

where:

\[ \phi = \arctan \sqrt{\frac{S}{4m^2 - S}}. \]
Above threshold, $S > 4m^2$, we define:

$$y = \frac{\sqrt{S} - \sqrt{S - 4m^2}}{\sqrt{S} + \sqrt{S - 4m^2}},$$

with $y = 1$ at $S = 4m^2$ and $y = 0$ at $S = \infty$, and the continuation in $x$ is performed by the replacement:

$$x \to -y + i\epsilon.$$

The real and imaginary parts of the form factors are defined through the relations:

$$F_{1,R}(\epsilon, s + i\epsilon) = \Re F_{1,R}(\epsilon, s) + i\pi \Im F_{1,R}(\epsilon, s),$$

$$F_{2,R}(\epsilon, s + i\epsilon) = \Re F_{2,R}(\epsilon, s) + i\pi \Im F_{2,R}(\epsilon, s),$$

where $s = S/m^2$.

In the following two Sections we will give real and imaginary parts of the one- and two-loop analytically continued form factors for $\mu = m$. The renormalization scale dependence follows from the pattern outlined in Section 5.3.

### 6.1 One-Loop Form Factors above Threshold

As in Section 5 we write:

$$F_{i,R}^{(1l)}(\epsilon, s) = C(\epsilon) F_{i,R}^{(1l)}(\epsilon, s) \quad \text{with } i = 1, 2.$$

We have:

$$\Re F_{1,R}^{(1l)}(\epsilon, s) = C_F \left\{ \frac{1}{\epsilon} \left[ -1 + \left( 1 - \frac{1}{1-y} - \frac{1}{1+y} \right) H(0; y) \right] \right.$$  

$$-2 - \left( 1 - \frac{1}{2} - \frac{1}{1-y} \right) H(0; y)$$

$$- \left( 1 - \frac{1}{1-y} - \frac{1}{1+y} \right) \left[ 4\zeta(2) - 2H(0; y) - H(0, 0; y) \right]$$

$$- 2H(1, 0; y)$$

$$- \epsilon \left[ 4 - \left( \frac{1}{2} - \frac{1}{1-y} \right) \left[ 4\zeta(2) - H(0, 0; y) - 2H(1, 0; y) \right] \right]$$

$$+ \left( 1 - \frac{1}{1-y} - \frac{1}{1+y} \right) \left[ 8\zeta(2) + 2\zeta(3) \right.$$

$$- 4(1 - \zeta(2))H(0; y) + 8\zeta(2)H(1; y) - 2H(0, 0; y)$$

$$- 4H(1, 0; y) - H(0, 0, 0; y) - 2H(0, 1, 0; y)$$

$$- 2H(1, 0, 0; y) - 4H(1, 1, 0; y) \right]\right\} + \mathcal{O}(\epsilon^2),$$

$$\Im F_{1,R}^{(1l)}(\epsilon, s) = C_F \left\{ \frac{1}{\epsilon} \left[ 1 - \frac{1}{1-y} - \frac{1}{1+y} \right] \right.$$
\[ \frac{1}{2} - \frac{1}{1+y} + \left(1 - \frac{1}{1-y} - \frac{1}{1+y}\right)[1 + H(0; y) + 2H(1; y)] + \epsilon \left[\frac{1}{2} \left(1 - \frac{2}{1+y}\right)\right] H(0; y) + 2H(1; y)] \\
+ \left(1 - \frac{1}{1-y} - \frac{1}{1+y}\right)[4 - 2\zeta(2) + H(0; y) + 2H(1; y) + H(0, 0; y) + 2H(0, 1; y) + 2H(1, 0; y) + 4H(1, 1; y)] \right\} + \mathcal{O}(\epsilon^2), \tag{74} \]

\[ \Re \mathcal{F}^{(1n)}_{2,R}(\epsilon, s) = C_F \left\{ \left(\frac{1}{1-y} - \frac{1}{1+y}\right) H(0; y) - \epsilon \left[\left(\frac{1}{1-y} - \frac{1}{1+y}\right) [4\zeta(2) - 4H(0; y) - H(0, 0; y) - 2H(1, 0; y)]\right]\right\} + \mathcal{O}(\epsilon^2), \tag{75} \]

\[ \Im \mathcal{F}^{(1n)}_{2,R}(\epsilon, s) = C_F \left\{ \frac{1}{1-y} - \frac{1}{1+y} + \epsilon \left[\left(\frac{1}{1-y} - \frac{1}{1+y}\right) [4 + H(0; y) + 2H(1; y)]\right]\right\} + \mathcal{O}(\epsilon^2). \tag{76} \]

### 6.2 Two-Loop Form Factors above Threshold

We define:

\[ F_{i,R}^{(2l)}(\epsilon, s) = C^2(\epsilon) \mathcal{F}_{i,R}^{(2l)}(\epsilon, s) \quad \text{with} \quad i = 1, 2. \tag{77} \]

We have:

\[ \Re \mathcal{F}^{(2l)}_{1,R}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ \frac{11}{12} C_F C_A \left[ 1 - \left(1 - \frac{1}{1-y} - \frac{1}{1+y}\right) H(0, y) \right] \\
- \frac{1}{3} C_F T_R N_f \left[ 1 - \left(1 - \frac{1}{1-y} - \frac{1}{1+y}\right) H(0, y) \right] \\
+ C_F^2 \left[ \frac{1}{2} - \zeta(2) \right] \left(3 + \frac{3}{(1-y)^2} - \frac{3}{(1-y)^2} + \frac{3}{(1+y)^2} - \frac{3}{(1+y)^2} - \frac{1}{(1+y)^2} \right) H(0, y) \\
+ \left(1 + \frac{1}{(1-y)^2} - \frac{1}{(1-y)^2} + \frac{1}{(1+y)^2} - \frac{1}{(1+y)^2} \right) H(0, 0, y) \right\} \right\} \\
+ \frac{1}{\epsilon} \left\{ C_F C_A \left[ - \frac{49}{36} - \zeta(2) \left(\frac{9}{2} - \frac{5}{2(1-y)} - \frac{5}{2(1+y)} \right) \right] \right\}. \]
\[-\zeta(3) \left( \frac{1}{2} + \frac{1}{2(1-y)^2} - \frac{1}{2(1+y)^2} \right) - \frac{1}{2(1+y)} \right) - \left( 1 - \frac{1}{1-y} - \frac{1}{1+y} \right) H(-1, 0, y)

+ \zeta(2) \left( \frac{9}{2} + \frac{5}{2(1-y)^2} - \frac{9}{2(1+y)^2} + \frac{5}{2} \right)

- \frac{9}{2(1+y)} H(0, y) + \left( \frac{67}{36} - \frac{67}{36(1-y)} \right)

- \frac{1}{(1+y)^2} \left( \frac{1}{1-y} \right) H(0, -1, 0, y) + \left( 2 - \frac{1}{1-y} \right)

- \frac{1}{(1+y)^2} H(0, 0, y) - \left( 2 + \frac{1}{1-y} - \frac{2}{1-y} \right)

+ \frac{1}{(1+y)^2} - \frac{2}{(1+y)} H(0, 0, 0, y) - \left( 1 + \frac{1}{1-y} \right)

- \frac{1}{1-y} + \frac{1}{1+y} \right) H(0, 1, 0, y)

+ \left( 1 - \frac{1}{1-y} - \frac{1}{1+y} \right) \left[ H(1, 0, y) \right]

+ C_F T_R N_f \left[ \frac{5}{9} - \left( \frac{5}{9} - \frac{5}{9(1-y)} - \frac{5}{9(1+y)} \right) H(0, y) \right]

+ C_F^2 \left[ 2 - \zeta(2) \left( \frac{5}{6} + \frac{6}{(1-y)^2} - \frac{2}{1-y} + \frac{12}{(1+y)^2} \right)

- \frac{8}{(1+y)} \right) - \zeta(2) \left( 10 + \frac{10}{(1-y)^2} - \frac{10}{1-y} \right)

+ \frac{10}{(1+y)^2} - \frac{10}{(1+y)} H(0, y) - \left( \frac{7}{2} - \frac{3}{1-y} \right)

- \frac{4}{(1+y)} H(0, y) + \left( 2 + \frac{2}{1-y} - \frac{1}{1-y} \right)

+ \frac{3}{(1+y)^2} - \frac{3}{(1+y)} H(0, 0, 0, y) + \left( 3 + \frac{3}{1-y} \right)

- \frac{3}{(1-y)} + \frac{3}{1+y} \right) H(0, 0, 0, y)

+ \left( 2 + \frac{2}{(1-y)^2} - \frac{2}{1-y} + \frac{2}{(1+y)^2} \right)

- \frac{2}{(1+y)} H(0, 1, 0, y) - \zeta(2) \left( 12 + \frac{12}{(1-y)^2} \right)

- \frac{12}{(1-y)} + \frac{12}{(1+y)^2} - \frac{12}{1+y} \right) H(1, y) \]
\[-\left(2 - \frac{2}{(1-y)} - \frac{2}{(1+y)} \right) H(1,0,y) \]
\[+ \left(4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} + \frac{4}{(1+y)^2} \right) \]
\[- \frac{4}{(1+y)} H(1,0,0,y) \right\} \]
\[+ C_F C_A \left[ -\frac{1595}{108} + \frac{3}{(1-y)^2} - \frac{3}{(1-y)} + \zeta(2) \left( \frac{76}{9} + \frac{36 \log 2}{(1-y)^2} \right) \right. \]
\[- \frac{36 \log 2}{(1-y)} + 6 \log 2 - \frac{240}{(1-y)^4} + \frac{480}{(1-y)^3} - \frac{276}{(1-y)^2} \]
\[+ \frac{344}{9(1-y)} + \frac{77}{9(1+y)} - \zeta(2) \left( \frac{491}{20} - \frac{618}{5(1-y)^5} \right) \]
\[- \frac{309}{1208} + \frac{277}{5(1-y)^3} + \frac{20(1+y)^2}{4(1-y)^2} - \frac{10(1-y)}{(1-y)^4} \]
\[- \frac{648}{374} + \frac{107}{6(1-y)^3} + \frac{269}{6(1+y)} \]
\[- \zeta(2) \left( 11 - \frac{72}{(1-y)^4} + \frac{144}{(1-y)^3} - \frac{114}{(1-y)^2} \right) \]
\[+ \frac{55}{(1-y)} + \frac{13}{(1+y)} \right] H(-1,y) + \zeta(3) \left( 2 + \frac{2}{(1-y)^2} \right) \]
\[- \frac{2}{(1-y)} + \frac{2}{(1+y)^2} - \frac{2}{(1+y)} \right) H(-1,0,y) \]
\[+ \left( 2 - \frac{2}{(1-y)} - \frac{2}{(1+y)} \right) H(-1,-1,0,y) \]
\[+ \zeta(2) \left( 2 - \frac{672}{(1-y)^5} + \frac{1680}{(1-y)^4} - \frac{1376}{(1-y)^3} + \frac{374}{(1-y)^2} \right) \]
\[- \frac{30}{(1-y)} - \frac{10}{(1+y)^2} + \frac{16}{(1+y)} \right) H(-1,0,y) \]
\[+ \left( 3 - \frac{24}{(1-y)^3} + \frac{36}{(1-y)^2} - \frac{16}{(1-y)} \right) \]
\[- \frac{2}{(1+y)} \right) H(-1,0,y) - \left( 4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} \right) \]
\[+ \frac{4}{(1+y)^2} - \frac{4}{(1+y)} \right) H(-1,0,-1,0,y) \]
\[+ \left( 4 - \frac{24}{(1-y)^4} + \frac{48}{(1-y)^3} - \frac{38}{(1-y)^2} + \frac{18}{(1-y)} \right) \]
\[+ \frac{4}{(1+y)} \right) H(-1,0,0,y) \left( 2 - \frac{336}{(1-y)^5} + \frac{840}{(1-y)^4} \right) \]
\[
\begin{align*}
&\frac{-688}{(1 - y)^3} + \frac{188}{(1 - y)^2} - \frac{16}{(1 - y)} - \frac{4}{(1 + y)^2} \\
&+ \frac{4}{(1 + y)} \bigg( H(-1, 0, 0, y) + \left( 4 + \frac{4}{(1 - y)^2} \right) \bigg) H(-1, 0, 1, 0, y) \\
&- \frac{4}{(1 - y)} + \frac{4}{(1 + y)^2} - \frac{4}{(1 + y)} \bigg( 6 - \frac{6}{(1 - y)} - \frac{6}{(1 + y)} \bigg) H(-1, 1, 0, y) - \zeta(2) \left( \frac{119}{3} \right) \\
&- \frac{516}{(1 - y)^5} + \frac{1704}{(1 - y)^4} - \frac{1990}{(1 - y)^3} + \frac{951}{(1 - y)^2} \\
&- \frac{554}{3(1 - y)} - \frac{23}{3(1 + y)} \bigg( H(0, y) - \zeta(3) \left( \frac{15}{2} - \frac{324}{(1 - y)^3} \right) \bigg) \\
&+ \frac{810}{1 - y^4} - \frac{2}{1 - y} + \frac{2}{2(1 - y)^2} - \frac{367}{(1 - y)} \\
&+ \frac{1}{2(1 + y)^2} - \frac{2}{(1 + y)} \bigg( H(0, y) + \left( \frac{2545}{216} + \frac{9}{(1 - y)^4} \right) \bigg) \\
&- \frac{2(1 - y)^2}{2(1 - y)^2} - \frac{108(1 - y)}{27(1 + y)} - \frac{365}{(1 - y)} \bigg) H(0, y) \\
&- \zeta(2) \left( 7 - \frac{72}{(1 - y)^5} + \frac{180}{(1 - y)^4} - \frac{216}{(1 - y)^3} \right) \\
&+ \frac{157}{(1 - y)^2} - \frac{34}{(1 - y)^2} + \frac{13}{(1 + y)^2} - \frac{16}{(1 + y)} \bigg( H(0, -1, y) \\
&- \left( 2 + \frac{2}{(1 - y)^2} - \frac{2}{(1 - y)} + \frac{2}{(1 + y)^2} \right) \\
&- \frac{2}{(1 + y)} \bigg) H(0, -1, -1, 0, y) - \left( 8 + \frac{48}{(1 - y)^4} \right) \\
&- \frac{96}{(1 - y)^3} + \frac{56}{(1 - y)^2} - \frac{12}{(1 - y)} \\
&- \frac{4}{(1 + y)} \bigg( H(0, -1, 0, y) + \left( 2 - \frac{24}{(1 - y)^5} + \frac{60}{(1 - y)^4} \right) \bigg) \\
&- \frac{72}{(1 - y)^3} + \frac{52}{(1 - y)^2} - \frac{11}{(1 - y)} + \frac{4}{(1 + y)^2} \\
&- \frac{5}{(1 + y)} \bigg( H(0, -1, 0, 0, y) + \left( 6 + \frac{6}{(1 - y)^5} \right) \bigg) \\
&- \frac{6}{(1 - y)} + \frac{6}{(1 + y)^2} - \frac{6}{(1 + y)} \bigg) H(0, -1, 1, 0, y) \\
+ \zeta(2) \left( 37 - \frac{78}{(1 - y)^5} + \frac{195}{(1 - y)^4} - \frac{307}{2(1 - y)^3} \right) \\
+ \frac{217}{4(1 - y)^2} - \frac{295}{8(1 - y)} + \frac{19}{(1 + y)^2} - \frac{287}{8(1 + y)} \bigg) H(0, 0, y)
\end{align*}
\]
\[
\begin{align*}
&-\left(\frac{217}{36} - \frac{24}{(1-y)^4} + \frac{39}{(1-y)^3} - \frac{3}{2(1-y)^2} - \frac{130}{9(1-y)} \right) H(0,0,y) + \\
&\frac{25}{18(1+y)} H(0,0,y) + \left(14 - \frac{48}{(1-y)^5} + \frac{120}{(1-y)^4} \right) \\
&\frac{12}{(1-y)^3} - \frac{20}{(1-y)^2} - \frac{12}{(1-y)} + \frac{8}{(1+y)^2} \\
&- \frac{2}{816} H(0,0,-1,0,y) + \left(\frac{89}{6} - \frac{258}{(1-y)^5} \right) \\
&\frac{923}{4(1-y)^3} - \frac{436}{(1-y)^2} - \frac{3}{3(1-y)} \\
&- \frac{4}{3(1+y)} H(0,0,0,y) - \left(12 - \frac{3}{(1-y)^5} \right) \\
&\frac{17}{2(1-y)^4} - \frac{12}{4(1-y)^3} + \frac{39}{8(1-y)^2} - \frac{177}{16(1-y)} \\
&\frac{6}{(1+y)^2} - \frac{193}{16(1+y)} H(0,0,0,0,y) \\
&- \left(\frac{22}{4(1-y)} + \frac{12}{(1+y)^2} - \frac{75}{4(1+y)} \right) H(0,0,1,0,y) \\
&- \frac{51}{10} \left(\frac{35}{(1-y)^5} + \frac{756}{(1-y)^4} - \frac{1890}{(1-y)^3} \right) \\
&\frac{15}{2(1-y)^2} - \frac{12}{4(1-y)} + \frac{23}{(1+y)^2} - \frac{107}{4(1+y)} \Big) H(0,1,y) \\
&+ \left(\frac{6}{(1-y)^2} - \frac{6}{(1-y)} + \frac{6}{(1+y)^2} \right) \\
&- \frac{6}{(1+y)} H(0,1,-1,0,y) + \left(\frac{68}{3} - \frac{12}{(1-y)^4} \right) \\
&\frac{24}{(1-y)^3} - \frac{14}{(1-y)^2} - \frac{29}{3(1-y)} - \frac{35}{3(1+y)} \Big) H(0,1,0,y) \\
&- \left(14 - \frac{282}{(1-y)^5} + \frac{705}{(1-y)^4} - \frac{1171}{2(1-y)^3} + \frac{725}{4(1-y)^2} \right) \\
&\frac{227}{8(1-y)} + \frac{8}{(1+y)^2} - \frac{83}{8(1+y)} \Big) H(0,1,0,0,y) \\
&- \left(10 + \frac{10}{(1-y)^2} - \frac{10}{(1-y)} + \frac{10}{(1+y)^2} \right) \\
&- \frac{10}{(1+y)} H(0,1,1,0,y) - \zeta(2) \left(\frac{149}{3} \right) \\
&\frac{756}{(1-y)^4} - \frac{1512}{(1-y)^3} + \frac{882}{(1-y)^2} - \frac{491}{3(1-y)} \\
\end{align*}
\]
\[-\frac{113}{3(1+y)} H(1,y) - \zeta(3)\left(2 + \frac{2}{(1-y)^2}\right)\]
\[-\frac{2}{(1-y)} + \frac{2}{(1+y)^2} - \frac{2}{(1+y)} H(1,y)\]
\[-\left(6 - \frac{6}{(1-y)} - \frac{6}{(1+y)}\right) H(1,-1,0,y)\]
\[+ \zeta(2)\left(6 + \frac{10}{(1-y)^2} - \frac{6}{(1-y)} + \frac{10}{(1+y)^2}\right)\]
\[-\frac{6}{(1+y)} H(1,0,y) + \left(\frac{31}{9} - \frac{6}{(1-y)^3} + \frac{9}{(1+y)^2}\right)\]
\[-\frac{46}{9(1-y)} - \frac{43}{9(1+y)} H(1,0,y) + \left(4 + \frac{4}{(1-y)^2}\right)\]
\[-\left(\frac{53}{3} + \frac{282}{(1-y)^4} - \frac{564}{(1-y)^3} + \frac{339}{(1-y)^2} - \frac{206}{3(1+y)}\right)\]
\[-\frac{35}{3(1+y)} H(1,0,0,y) - \left(2 + \frac{4}{(1-y)^2}\right)\]
\[-\frac{2}{(1-y)} + \frac{4}{(1+y)^2} - \frac{2}{(1+y)} H(1,0,0,0,y)\]
\[-\left(4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} + \frac{4}{(1+y)^2}\right)\]
\[-\frac{4}{(1+y)} H(1,0,1,0,y) + \left(\frac{52}{3} - \frac{52}{3(1-y)}\right)\]
\[-\frac{52}{3(1+y)} H(1,1,0,y)\]
\[+ C_F T_R N_F \left[\frac{106}{27} + \zeta(2)\left(\frac{88}{9} - \frac{64}{9(1-y)} - \frac{88}{9(1+y)}\right)\right]\]
\[+ \zeta(3)\left(\frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)}\right)\]
\[+ \zeta(2)\left(\frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)}\right) H(0,y)\]
\[-\left(\frac{209}{54} - \frac{103}{27(1-y)} - \frac{106}{27(1+y)}\right) H(0,0,y)\]
\[-\left(\frac{19}{9} - \frac{16}{9(1-y)} - \frac{22}{9(1+y)}\right) H(0,0,0,y)\]
\[-\left(\frac{2}{3} - \frac{2}{3(1-y)} - \frac{2}{3(1+y)}\right) H(0,0,0,0,y)\]
\[-\left(\frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)}\right) H(0,1,0,y)\]
\[\begin{align*}
&+ \zeta(2) \left( \frac{16}{3} - \frac{16}{3(1-y)} - \frac{16}{3(1+y)} \right) H(1, y) \\
&- \left( \frac{38}{9} - \frac{32}{9(1-y)} - \frac{44}{9(1+y)} \right) H(1, 0, y) \\
&- \left( \frac{4}{3} - \frac{4}{3(1-y)} - \frac{4}{3(1+y)} \right) H(1, 0, 0, y) \\
&- \left( \frac{8}{3} - \frac{8}{3(1-y)} - \frac{8}{3(1+y)} \right) H(1, 1, 0, y) \\
&+ C_F T_R \left[ \frac{383}{27} + \frac{196}{9(1-y)^2} - \frac{196}{9(1-y)} \\
&- \zeta(2) \left( \frac{22}{3} - \frac{48}{3(1-y)^2} + \frac{96}{9(1-y)^3} - \frac{44}{9(1-y)^2} \right) \\
&- \frac{4}{1-y} \right] + \zeta(2) \left( \frac{4}{3} + \frac{48}{(1-y)^3} - \frac{120}{(1-y)^2} \right) \\
&+ \frac{88}{(1-y)^3} - \frac{12}{(1-y)^2} - \frac{10}{3(1-y)} - \frac{10}{3(1+y)} \right) H(0, y) \\
&- \left( \frac{236}{27(1+y)} \right) H(0, 0, y) + \left( \frac{19}{9} + \frac{248}{9(1-y)^4} - \frac{496}{9(1-y)^3} \right) \\
&+ \frac{326}{9(1-y)^2} - \frac{26}{3(1-y)} \right) H(0, 0, y) - \left( \frac{2}{3} + \frac{24}{(1-y)^5} \right) \\
&- \frac{5}{(1-y)^4} + \frac{44}{(1-y)^3} - \frac{6}{(1-y)^2} - \frac{5}{3(1-y)} \right) H(0, 0, 0, y) \\
&+ C_F^2 \left[ \frac{23}{2} - \zeta(2) \left( 158 + \frac{72 \log 2}{(1-y)^2} - \frac{72 \log 2}{(1-y)} + 12 \log 2 \right) \\
&+ \frac{816}{(1-y)^4} - \frac{2040}{(1-y)^3} + \frac{1698}{(1-y)^2} - \frac{520}{(1+y)^2} + \frac{48}{(1+y)^3} \right) \\
&- \frac{128}{(1+y)} \right) + \zeta^2(2) \left( \frac{98}{5} + \frac{18}{(1-y)^5} - \frac{45}{(1-y)^4} \right) \\
&+ \frac{327}{5(1-y)^3} - \frac{20(1-y)^2}{(1-y)} + \frac{159}{5(1+y)^2} + \frac{83}{5(1+y)^3} \right) \\
&- \frac{331}{20(1+y)} \right) - \zeta(3) \left( 7 - \frac{168}{(1-y)^4} + \frac{336}{(1-y)^3} \right) \\
&- \frac{190}{(1-y)^2} + \frac{24}{(1-y)} + \frac{4}{(1+y)^2} - \frac{2}{(1+y)} \right) \right) \\
&+ \zeta(2) \left( 48 + \frac{1080}{(1-y)^4} - \frac{2160}{(1-y)^3} + \frac{1182}{(1-y)^2} \right)
\end{align*}\]
\[-\frac{102}{(1-y)} H(-1, y) - \zeta(3) \left( 4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)^5} \right) + \frac{4}{(1+y)^2} - \frac{4}{(1+y)^5} \]
\[+ \frac{720}{(1-y)^4} - \frac{236}{(1-y)^3} + \frac{40}{(1-y)^2} + \frac{20}{(1+y)^2} \]
\[-\frac{24}{(1+y)} H(-1, 0, y) - \left( 16 - \frac{48}{(1-y)^3} + \frac{72}{(1-y)^2} \right) \]
\[-\frac{40}{(1-y)} - \frac{16}{(1+y)} \right) H(-1, 0, y) + \left( 8 + \frac{8}{(1-y)^2} \right) \]
\[-\left( 16 + \frac{360}{(1-y)^4} - \frac{720}{(1-y)^3} + \frac{394}{(1-y)^2} \right) \]
\[-\frac{34}{(1-y)} H(-1, 0, 0, y) - \left( 8 - \frac{144}{(1-y)^5} \right) \]
\[+ \frac{360}{(1-y)^4} - \frac{312}{(1-y)^3} + \frac{116}{(1-y)^2} - \frac{8}{(1+y)^2} \]
\[-\frac{10}{(1+y)} H(-1, 0, 0, 0, y) - \left( 8 + \frac{8}{(1-y)^2} \right) \]
\[-\frac{8}{(1-y)} + \frac{8}{(1+y)^2} - \frac{8}{(1+y)} \right) H(-1, 0, 1, 0, y) \]
\[+ \zeta(2) \left( 13 + \frac{120}{(1-y)^5} - \frac{492}{(1-y)^4} + \frac{632}{(1-y)^3} - \frac{261}{(1-y)^2} \right) \]
\[-\frac{50}{(1+y)} - \frac{20}{(1+y)^2} + \frac{27}{(1+y)} \right) H(0, y) \]
\[+ \zeta(3) \left( 8 + \frac{168}{(1-y)^5} - \frac{420}{(1-y)^4} + \frac{364}{(1-y)^3} \right) \]
\[-\frac{126}{(1-y)^2} + \frac{5}{(1-y)} - \frac{7}{(1+y)} \right) H(0, y) \]
\[-\left( \frac{85}{8} - \frac{47}{4(1-y)} - \frac{19}{2(1+y)} \right) H(0, y) \]
\[-\zeta(2) \left( 12 - \frac{1080}{(1-y)^5} + \frac{2700}{(1-y)^4} - \frac{2100}{(1-y)^3} + \frac{450}{(1-y)^2} \right) \]
\[-\frac{15}{(1-y)} + \frac{21}{(1+y)} \right) H(0, -1, y) + \left( 10 + \frac{96}{(1-y)^4} \right) \]
\[-\frac{192}{(1-y)^3} + \frac{116}{(1-y)^2} - \frac{20}{(1-y)} + \frac{8}{(1+y)^2} \]
\[-\frac{8}{(1+y)} \right) H(0, -1, 0, y) + \left( 4 - \frac{360}{(1-y)^5} \right) \]
\[
\begin{align*}
+\frac{900}{(1-y)^4} &- \frac{700}{(1-y)^3} + \frac{150}{(1-y)^2} - \frac{5}{(1-y)} \\
+\frac{7}{(1+y)} &H(0,-1,0,0,y) - \zeta(2) \left( 68 + \frac{48}{(1-y)^5} \right) \\
-\frac{120}{(1-y)^4} &+ \frac{115}{(1-y)^3} - \frac{13}{2(1-y)^2} - \frac{4}{(1-y)} \\
+\frac{46}{(1+y)^2} &- \frac{273}{4(1+y)} H(0,0,y) + \left( \frac{229}{4} + \frac{192}{(1-y)^2} \right) \\
-\frac{504}{(1-y)^3} &+ \frac{28}{(1-y)^2} - \frac{343}{2(1-y)} + \frac{16}{(1+y)^2} \\
-\frac{42}{(1+y)} &H(0,0,y) - \left( 12 - \frac{96}{(1-y)^5} + \frac{240}{(1-y)^4} \right) \\
-\frac{8}{(1+y)} &H(0,0,-1,0,y) - \left( 10 + \frac{60}{(1-y)^5} \right) \\
-\frac{216}{(1-y)^4} &+ \frac{256}{(1-y)^3} - \frac{89}{(1-y)^2} + \frac{4}{(1+y)^2} \\
+\frac{17}{2(1+y)} &H(0,0,0,y) + \left( 27 - \frac{6}{(1-y)^5} + \frac{15}{(1-y)^4} \right) \\
+\frac{17}{2(1+y)} &H(0,0,0,0,y) + \left( 22 - \frac{384}{(1-y)^5} + \frac{960}{(1-y)^4} \right) \\
-\frac{217}{8(1+y)} &H(0,0,0,0,0,y) + \left( 22 - \frac{384}{(1-y)^5} + \frac{960}{(1-y)^4} \right) \\
-\frac{217}{746} &H(0,0,1,0,y) - \zeta(2) \left( 32 + \frac{936}{(1-y)^5} \right) \\
+\frac{1914}{(1-y)^3} &- \frac{499}{(1-y)^2} + \frac{32}{2(1-y)} + \frac{32}{(1+y)^2} \\
-\frac{91}{2(1+y)} &H(0,1,0,y) - \left( 16 + \frac{384}{(1-y)^4} - \frac{768}{(1-y)^3} \right) \\
+\frac{442}{(1-y)^2} &- \frac{60}{(1-y)} - \frac{2}{(1+y)} H(0,1,0,y) \\
+\left( 14 + \frac{252}{(1-y)^5} - \frac{630}{(1-y)^4} + \frac{517}{(1-y)^3} - \frac{271}{2(1-y)^2} \right) H(0,1,0,0,y) \\
+\left( 4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} + \frac{4}{(1+y)^2} \right)
\end{align*}
\]
\[
\Im F_{1,R}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ -\frac{11}{12} C_F C_A \left( 1 - \frac{1}{1 - y} - \frac{1}{1 + y} \right) + \frac{1}{3} C_F T_R N_f \left( 1 - \frac{1}{1 - y} - \frac{1}{1 + y} \right) - C_F^2 \left( \left( 1 - \frac{1}{1 - y} - \frac{1}{1 + y} \right) - \left( 1 - \frac{1}{1 - y} + \frac{1}{(1 - y)^2} - \frac{1}{1 + y} + \frac{1}{(1 + y)^2} \right) H(0, y) \right) \right\} + \frac{1}{\epsilon} \left\{ C_F C_A \left[ + \frac{67}{36} - \frac{67}{36(1 - y)} - \frac{67}{36(1 + y)} \right] + \zeta(2) \left( \frac{1}{2} + \frac{1}{2(1 - y)^2} - \frac{1}{2(1 - y)} - \frac{1}{2(1 + y)^2} \right) \right\}, \tag{78}
\]
\[ -\frac{1}{2(1+y)} - \left(1 - \frac{1}{(1-y)} - \frac{1}{(1+y)} \right) H(-1; y) \\
+ \left(2 - \frac{1}{(1-y)} - \frac{1}{(1+y)} \right) H(0; y) + \left(1 + \frac{1}{(1-y)} \right) H(1; y) \\
+ \left( \frac{1}{(1-y)} + \frac{1}{(1+y)^2} - \frac{1}{(1+y)} \right) H(0, -1; y) \\
- \left(2 + \frac{1}{(1-y)^2} - \frac{2}{(1-y)} + \frac{1}{(1+y)^2} \right) H(0, 0; y) - \left(1 + \frac{1}{(1-y)^2} - \frac{1}{(1-y)} \right) H(0, 1; y) + \left(1 - \frac{1}{(1-y)} \right) H(1, 0; y) \\
- \left( \frac{1}{(1-y)} \right) H(1; y) \right] \\
- \frac{5}{9} C_F T R N_f \left[ 1 - \frac{1}{(1-y)} - \frac{1}{(1+y)} \right] \\
+ C_F \left[ -\frac{7}{2} + \frac{3}{(1-y)} + \frac{4}{(1+y)} - \zeta(2) \left(4 + \frac{4}{(1-y)^2} \right) \\
- \frac{4}{(1-y)} + \frac{4}{(1+y)^2} - \frac{4}{(1+y)} \right] + \left(2 + \frac{2}{(1-y)^2} \right) \\
- \left( \frac{1}{(1-y)} + \frac{4}{(1+y)^2} - \frac{3}{(1+y)} \right) H(0; y) \\
+ \left(3 + \frac{3}{(1-y)^2} - \frac{3}{(1-y)} + \frac{3}{(1+y)^2} \right) H(0, 0; y) + \left(2 + \frac{2}{(1-y)^2} - \frac{2}{(1-y)} \right) \\
+ \left(2 + \frac{2}{(1+y)^2} - \frac{2}{(1+y)} \right) H(0, 1; y) - \left(2 - \frac{2}{(1-y)} \right) \\
- \left( \frac{2}{(1+y)} \right) H(1; y) + \left(4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} \right) \\
+ \left( \frac{4}{(1+y)^2} - \frac{4}{(1+y)} \right) H(1, 0; y) \right] \} \\
+ C_F C_A \left[ \frac{2545}{216} + \frac{9}{(1-y)^3} - \frac{27}{2(1-y)^2} - \frac{599}{108(1-y)} \\
- \frac{365}{27(1+y)} - \zeta(2) \left(10 + \frac{72}{(1-y)^4} - \frac{144}{(1-y)^3} \right) \\
+ \frac{79}{(1-y)^2} - \frac{12}{(1-y)} - \frac{5}{(1+y)} - \zeta(3) \left(\frac{15}{2} \right) \\
- \frac{324}{810} + \frac{662}{(1-y)^4} - \frac{662}{(1-y)^3} + \frac{367}{2(1-y)^2} \right] \] (79)
\[
\frac{6}{(1+y)^2} - \frac{6}{(1+y)^3} \cdot H(0, -1, 1; y) + \left(\frac{89}{6} - \frac{258}{923}\right) + \left(\frac{436}{(1-y)^2} - \frac{3}{1-y}\right) \\
- \left(\frac{4}{3(1+y)} \cdot H(0, 0; y) + \left(\frac{14}{1-y} + \frac{120}{39}\right) - \frac{20}{1-y} + \frac{12}{1+y} - \frac{8}{1+y}\right) \\
- \left(\frac{12}{(1+y)^2} + \frac{17}{16(1+y)}\right) H(0, 0, 0; y) \\
- \left(\frac{22}{1-y} - \frac{30}{(1-y)^2} + \frac{49}{(1-y)^3} - \frac{63}{2(1-y)^2}\right) \\
- \left(\frac{16}{1+y} - \frac{12}{(1+y)^2} - \frac{24}{4(1+y)}\right) H(0, 0, 1; y) \\
+ \left(\frac{68}{3} - \frac{12}{1-y} + \frac{24}{(1-y)^2} - \frac{14}{1-y} - \frac{29}{(1-y)^3}\right) \\
- \left(\frac{15}{3(1+y)}\right) H(0, 1; y) + \left(6 + \frac{6}{1-y} - \frac{6}{1-y}\right) \\
+ \left(\frac{6}{1+y} - \frac{6}{1+y}\right) H(0, 1, -1; y) - \left(14 - \frac{282}{705}\right) \\
+ \left(\frac{2}{1-y} - \frac{2}{1-y} - \frac{2}{1+y}\right) H(1; y) \\
+ \left(\frac{3}{9} - \frac{6}{(1-y)^2} + \frac{9}{(1-y)^3} - \frac{46}{9(1-y)} - \frac{43}{9(1+y)}\right) H(1; y) \\
- \left(\frac{6 - \frac{6}{1-y}}{1+y}\right) H(1, -1; y) \\
+ \left(\frac{53}{3} - \frac{282}{(1-y)^4} - \frac{564}{(1-y)^5} + \frac{339}{(1-y)^2} - \frac{306}{(1-y)^3}\right) \\
- \left(\frac{35}{3(1+y)}\right) H(1, 0; y) + \left(4 + \frac{4}{1-y} - \frac{4}{1-y}\right)
\]
\[
\begin{align*}
&+ \frac{4}{(1+y)^2} - \frac{4}{(1+y)} H(1, 0, -1; y) + \left(-2 - \frac{4}{(1 - y)^2}ight) \\
&+ \frac{2}{(1 - y)} - \frac{4}{(1+y)^2} + \frac{2}{(1+y)} H(1, 0, 0; y) \\
&- \left(4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} + \frac{4}{(1+y)^2}\right) \\
&- \frac{4}{(1+y)} H(1, 0, 1; y) + \left(\frac{52}{3} - \frac{52}{3(1-y)}\right) \\
&- \frac{52}{3(1+y)} H(1, 1; y) \right] \\
&+ C_F T_R N_f \left[ -\frac{209}{54} + \frac{103}{27(1-y)} + \frac{106}{27(1+y)} \\
&+ \left(-\frac{19}{9} + \frac{16}{9(1-y)} + \frac{22}{9(1+y)}\right) H(0; y) \\
&+ \left(-\frac{2}{3} + \frac{2}{3(1-y)} + \frac{2}{3(1+y)}\right) H(0, 0; y) \\
&+ \left(-\frac{4}{3} + \frac{4}{3(1-y)} + \frac{4}{3(1+y)}\right) H(0, 1; y) \\
&+ \left(-\frac{38}{9} + \frac{32}{9(1-y)} + \frac{44}{9(1+y)}\right) H(1; y) \\
&+ \left(-\frac{4}{3} + \frac{4}{3(1-y)} + \frac{4}{3(1+y)}\right) H(1, 0; y) \\
&+ \left(-\frac{8}{3} + \frac{8}{3(1-y)} + \frac{8}{3(1+y)}\right) H(1, 1; y) \right] \\
&+ C_F T_R \left[ -\frac{265}{54} + \frac{356}{9(1-y)^3} - \frac{178}{3(1-y)^2} + \frac{563}{27(1-y)} \\
&+ \frac{236}{27(1+y)} + \left(\frac{19}{9} + \frac{248}{9(1-y)^4} - \frac{496}{9(1-y)^3}\right) H(0; y) - \left(\frac{2}{3} + \frac{24}{(1-y)^5}\right) \\
&- \frac{326}{9(1-y)^2} - \frac{26}{3(1-y)} H(0, 0; y) - \frac{6}{(1-y)^4} + \frac{44}{(1-y)^3} - \frac{6}{(1-y)^2} - \frac{5}{3(1-y)} \\
&- \frac{5}{3(1+y)} H(0, 0; y) \right] \\
&+ C_F^2 \left[ -\frac{85}{8} + \frac{47}{4(1-y)} + \frac{19}{2(1+y)} \\
&+ \zeta(2) \left(-7 - \frac{60}{(1-y)^4} + \frac{120}{(1-y)^3} - \frac{83}{(1-y)^2}\right) \\
&+ \frac{21}{(1-y)} - \frac{12}{(1+y)^2} + \frac{10}{(1+y)} \right]
\end{align*}
\]
\[
+\zeta(3)\left(8 + \frac{168}{(1-y)^5} - \frac{420}{(1-y)^3} + \frac{364}{(1-y)}\right) - \frac{126}{(1-y)^2} + \frac{5}{(1-y)^3} - \frac{7}{(1-y)^4} + \zeta(2)\left(4 + \frac{4}{(1-y)^2}\right)
\]
\[
- \frac{4}{(1-y)^2} + \frac{4}{(1+y)^2} - \frac{4}{(1+y)^3} \right) H(-1; y)
\]
\[
- \left(16 - \frac{48}{(1-y)^3} + \frac{72}{(1-y)^2} - \frac{40}{(1-y)}\right) H(-1; y) - \left(16 + \frac{360}{(1-y)^4} - \frac{720}{(1-y)^3}\right) H(-1; 1; y)
\]
\[
+ \left(8 - \frac{144}{(1-y)^5} + \frac{360}{(1-y)^4} - \frac{312}{(1-y)^3} + \frac{116}{(1-y)^2}\right) H(-1, 0; y) - \frac{8}{(1-y)^2} + \frac{8}{(1-y)^3} - \frac{8}{(1+y)} \right) H(-1, 0; -1; y)
\]
\[
- \left(8 + \frac{8}{(1-y)^2} - \frac{8}{(1-y)^3} + \frac{8}{(1+y)^2}\right) H(-1, 0; 0; y)
\]
\[
- \left(8 + \frac{8}{(1-y)^2} - \frac{8}{(1-y)^3} + \frac{8}{(1+y)^2}\right) H(-1, 0; 1; y) + \zeta(2)\left(-14 - \frac{60}{(1-y)^5}\right)
\]
\[
+ \frac{150}{(1-y)^3} - \frac{132}{(1-y)^4} + \frac{36}{(1-y)^5} - \frac{8}{(1-y)^2} + \frac{14}{(1+y)} \right) H(0; y) + \left(229 + \frac{192}{(1-y)^4} - \frac{504}{(1-y)^3}\right) H(0; y)
\]
\[
+ \left(10 + \frac{96}{(1-y)^4} - \frac{192}{(1-y)^3} + \frac{116}{(1-y)^2} - \frac{20}{(1-y)}\right) H(0, -1; y) + \left(4 - \frac{360}{(1-y)^5}\right) H(0, -1; y)
\]
\[
+ \frac{900}{(1-y)^2} - \frac{700}{(1+1+y)} \right) H(0, 0; y) - \frac{5}{(1+y)^2} - \frac{5}{(1+y)^3} + \frac{5}{(1+y)^4}\right) H(0, 0; y) + \left(216 - \frac{12}{(1-y)^5} + \frac{240}{(1-y)^4}\right) H(0, 0; y)
\]
\[
- \frac{17}{2(1+y)} \right) H(0, 0; y) + \left(-12 + \frac{96}{(1-y)^5} - \frac{240}{(1-y)^4}\right) H(0, 0; y) + \left(-12 + \frac{96}{(1-y)^5} - \frac{240}{(1-y)^4}\right)
\]
\[
\]
\begin{align}
+ & \frac{176}{(1-y)^3} - \frac{28}{(1-y)^2} + \frac{8}{(1-y)} - \frac{4}{(1+y)^2} \\
+ & \frac{8}{(1+y)} H(0, 0, -1; y) + \left( \frac{27}{(1-y)^5} - \frac{6}{(1-y)^4} + \frac{15}{1-y^4} \right) \\
+ & \frac{17}{2(1-y)^3} + \frac{59}{4(1-y)^2} - \frac{201}{8(1-y)} + \frac{17}{(1+y)^2} \\
+ & \frac{217}{8(1+y)} H(0, 0, 0; y) + \left( \frac{22}{(1-y)^5} - \frac{384}{(1-y)^4} + \frac{960}{1-y^4} \right) \\
+ & \frac{169}{(1-y)^3} + \frac{16}{(1-y)^2} - \frac{2}{2(1-y)} + \frac{10}{(1+y)^2} \\
+ & \frac{21}{2(1+y)} H(0, 0, 1; y) + \left( \frac{16}{(1-y)^4} - \frac{384}{(1-y)^3} + \frac{768}{1-y^3} \right) \\
+ & \frac{442}{(1-y)^2} + \frac{60}{(1-y)} + \frac{2}{(1+y)} \right) H(0, 1; y) \\
+ & \left( 14 + \frac{252}{(1-y)^5} - \frac{630}{(1-y)^4} + \frac{517}{(1-y)^3} - \frac{271}{2(1-y)^2} \\
+ & \frac{19}{4(1-y)} + \frac{10}{(1-y)^2} - \frac{67}{4(1+y)} \right) H(0, 1, 0; y) \\
+ & \left( 4 + \frac{4}{(1-y)^2} - \frac{4}{(1-y)} + \frac{4}{(1+y)^2} \\
+ & \frac{4}{(1+y)} \right) H(0, 1, 1; y) + \zeta(2) \left( -16 - \frac{16}{(1-y)^2} \\
+ & \frac{16}{(1-y)} - \frac{16}{(1+y)^2} + \frac{16}{(1+y)} \right) H(1; y) \\
+ & \left( \frac{55}{2} - \frac{192}{(1-y)^3} + \frac{288}{(1-y)^2} - \frac{115}{1-y} \\
- & \frac{36}{(1+y)} H(1; y) + \left( \frac{6}{(1-y)^5} - \frac{252}{(1-y)^4} + \frac{504}{1-y^4} \right) \\
+ & \frac{270}{(1-y)^2} - \frac{16}{(1-y)} + \frac{16}{(1+y)^2} - \frac{14}{(1+y)} \right) H(1, 0; y) \\
+ & \left( 12 + \frac{12}{(1-y)^2} - \frac{12}{(1-y)} + \frac{12}{(1+y)^2} \\
- & \frac{12}{(1+y)} \right) H(1, 0, 0; y) + \left( 8 + \frac{8}{(1-y)^2} - \frac{8}{(1-y)} \\
+ & \frac{8}{(1+y)^2} - \frac{8}{(1+y)} \right) H(1, 0, 1; y) - \left( 4 - \frac{4}{(1-y)} \\
- & \frac{4}{(1+y)} \right) H(1, 1; y) + \left( 16 + \frac{16}{(1-y)^2} - \frac{16}{(1-y)} \\
+ & \frac{16}{(1+y)^2} - \frac{16}{(1+y)} \right) H(1, 1, 0; y) \right],
\end{align}
and

\[
\mathfrak{F}_{2,R}(\epsilon, s) = \frac{1}{\epsilon} \left\{ C_F^2 \left[ \zeta(2) \left( \frac{6}{(1 - y)^2} - \frac{6}{(1 - y)^2} - \frac{6}{(1 + y)^2} + \frac{6}{(1 + y)} \right) \right. \\
- \left( \frac{1}{(1 - y)^2} \right) H(0, y) - \left( \frac{2}{(1 - y)^2} - \frac{2}{(1 - y)} \right) H(0, 0, y) \right\} \\
+ C_F C_A \left[ -\frac{3}{(1 - y)^2} + \frac{3}{(1 - y)} - \zeta(2) \left( \frac{24 \log 2}{(1 - y)^2} - \frac{24 \log 2}{(1 - y)} \right) \right. \\
- \left( \frac{6}{(1 - y)^2} + \frac{6}{(1 + y)^2} + \frac{19}{3(1 + y)} \right) \zeta^2(2) \left( \frac{618}{5(1 - y)^4} - \frac{309}{5(1 - y)^4} \right) \\
+ \left( \frac{10}{(1 - y)^3} - \frac{20}{(1 - y)^2} - \frac{8}{(1 - y)^2} + \frac{5}{(1 + y)^3} \right) \zeta(3) \left( \frac{324}{(1 - y)^4} - \frac{648}{(1 - y)^3} \right) \\
- \left( \frac{5}{(1 - y)^2} + \frac{27}{8(1 + y)} \right) - \zeta(2) \left( \frac{72}{(1 - y)^4} - \frac{144}{(1 - y)^3} \right) \\
+ \left( \frac{120}{(1 - y)^2} - \frac{48}{(1 - y)} \right) H(-1, y) + \zeta(2) \left( \frac{672}{(1 - y)^5} \right. \\
- \left( \frac{5}{(1 - y)^3} - \frac{30}{(1 - y)^2} - \frac{6}{(1 - y)} \right) \\
- \left( \frac{16}{(1 + y)} \right) H(-1, 0, y) + \left( \frac{24}{(1 - y)^3} - \frac{36}{(1 - y)^2} \right) \\
+ \left( \frac{12}{(1 - y)} \right) H(-1, 0, y) + \left( \frac{24}{(1 - y)^4} - \frac{48}{(1 - y)^3} \right) \\
+ \left( \frac{40}{(1 - y)^2} - \frac{16}{(1 - y)} \right) H(-1, 0, 0, y) - \left( \frac{336}{(1 - y)^5} \right. \\
- \left( \frac{840}{(1 - y)^3} + \frac{660}{(1 - y)^2} - \frac{150}{(1 - y)} \right) \\
- \left( \frac{3}{(1 + y)} \right) H(-1, 0, 0, y) - \zeta(2) \left( \frac{516}{(1 - y)^5} \right) \\
- \left( \frac{1704}{(1 - y)^4} + \frac{1947}{(1 - y)^3} - \frac{840}{(1 - y)^2} + \frac{319}{4(1 - y)} \right) H(0, y) \\
- \zeta(3) \left( \frac{324}{(1 - y)^5} - \frac{810}{(1 - y)^4} + \frac{635}{(1 - y)^3} - \frac{285}{2(1 - y)^2} \right)
\]
\[-\frac{13}{4(1-y)} - \frac{13}{4(1+y)} H(0,y) - \left(\frac{9}{(1-y)^3} - \frac{27}{2(1-y)^2}\right)\]
\[-\frac{23}{9(1-y)} + \frac{127}{18(1+y)} H(0,y) - \zeta(2)\left(\frac{72}{(1-y)^5}\right)\]
\[-\frac{1}{(1-y)^4} + \frac{33}{2(1+y)} H(0,-1,y) + \frac{48}{(1-y)^4} - \frac{96}{(1-y)^3}\]
\[+ \frac{48}{(1-y)^2} H(0,-1,0,y) + \frac{24}{(1-y)^5} - \frac{60}{(1-y)^4}\]
\[+ \frac{70}{(1-y)^3} - \frac{45}{(1-y)^2} + \frac{11}{2(1-y)}\]
\[+ \frac{11}{2(1+y)} H(0,-1,0,0,y) + \zeta(2)\left(\frac{78}{(1-y)^5}\right)\]
\[-\frac{1}{(1-y)^3} + \frac{147}{(1-y)^3} - \frac{51}{2(1-y)^2} + \frac{3}{8(1-y)} H(0,0,y)\]
\[-\left(\frac{24}{(1-y)^4} - \frac{39}{(1-y)^3} + \frac{7}{2(1-y)^2} + \frac{137}{12(1-y)}\right) H(0,0,0,y) + \frac{48}{(1-y)^5}\]
\[-\frac{3}{(1+y)^2} + \frac{37}{12(1+y)} H(0,0,0,0,y) + \frac{258}{(1-y)^4} - \frac{816}{(1-y)^3}\]
\[+ \frac{2}{(1-y)^3} - \frac{3}{2(1-y)^2} + \frac{4}{8(1-y)} + \frac{315}{(1+y)^3}\]
\[-\frac{15}{4(1+y)^2} + \frac{11}{8(1+y)} H(0,0,0,0,y) - \left(\frac{3}{(1-y)^5}\right)\]
\[-\frac{15}{4(1+y)^2} + \frac{11}{8(1+y)} H(0,0,0,0,0,y) - \frac{3}{11}\]
\[-\frac{3}{(1+y)^3} + \frac{9}{8(1+y)^2} - \frac{11}{16(1+y)} H(0,0,0,0,0,y)\]
\[+ \frac{12}{(1-y)^5} - \frac{30}{(1-y)^4} + \frac{48}{(1-y)^3} - \frac{42}{(1-y)^2}\]
\[+ \frac{25}{4(1-y)} + \frac{1}{(1+y)^3} - \frac{2(1+y)^2}\]
\[+ \frac{25}{4(1+y)} H(0,0,1,0,y) + \zeta(2)\left(\frac{756}{(1-y)^5} - \frac{1890}{(1-y)^4}\right)\]
\[
\frac{1512}{(1 - y)^3} - \frac{378}{(1 - y)^2} + \frac{3}{4(1 - y)} + \frac{3}{(1 + y)^3} - \frac{9}{2(1 + y)^2} + \frac{3}{4(1 + y)} \right) H(0, 1, y) + \left( \frac{12}{(1 - y)^4} - \frac{24}{(1 - y)^3} + \frac{21}{(1 - y)^2} - \frac{9}{1 - y} + \frac{1}{1 + y} \right) H(0, 1, 0, y) - \left( \frac{282}{(1 - y)^5} - \frac{705}{(1 - y)^4} \right) H(0, 1, 0, y) = \\
+ \frac{1}{562} \left( \frac{3}{(1 - y)^3} - \frac{138}{(1 - y)^2} - \frac{8}{1 - y} - \frac{5}{2(1 + y)^3} \right) \left( \frac{3}{1 + y} \right)^2 - \frac{5}{8(1 + y)} \right) H(0, 1, 0, y) \\
+ \zeta(2) \left( \frac{756}{(1 - y)^4} - \frac{1512}{(1 - y)^3} + \frac{819}{(1 - y)^2} - \frac{63}{1 - y} \right) H(1, y) + \left( \frac{6}{(1 - y)^3} - \frac{9}{(1 - y)^2} + \frac{19}{6(1 - y)} - \frac{1}{6(1 + y)} \right) H(1, 0, y) \\
- \left( \frac{282}{(1 - y)^4} - \frac{564}{(1 - y)^3} + \frac{623}{2(1 - y)^2} - \frac{59}{2(1 - y)^2} + \frac{1}{2(1 + y)^2} + \frac{1}{2(1 + y)} \right) H(1, 0, 0, y) \\
+ C_F T_R N_f \left[ \zeta(2) \left( \frac{8}{3(1 - y)} - \frac{8}{3(1 + y)} \right) - \left( \frac{25}{9(1 + y)} \right) H(0, y) - \left( \frac{2}{3(1 - y)} - \frac{2}{3(1 + y)} \right) H(0, y) - \left( \frac{4}{3(1 - y)} - \frac{4}{3(1 + y)} \right) H(1, 0, y) \right] \\
+ C_F T_R \left[ \frac{68}{3(1 - y)^2} + \frac{68}{3(1 - y)} - \zeta(2) \left( \frac{48}{3(1 - y)^4} - \frac{96}{3(1 - y)^3} + \frac{40}{3(1 - y)^2} + \frac{8}{3(1 - y)} - \zeta(2) \left( \frac{48}{3(1 - y)^5} - \frac{120}{(1 - y)^4} + \frac{84}{(1 - y)^3} - \frac{6}{(1 - y)^2} - \frac{3}{(1 - y)} \right) H(0, y) - \left( \frac{124}{3(1 - y)^3} - \frac{62}{3(1 - y)^2} \right) H(0, y) - \left( \frac{88}{3(1 - y)^4} - \frac{118}{9(1 - y)} + \frac{68}{9(1 + y)} \right) H(0, y) - \left( \frac{176}{3(1 - y)^3} + \frac{92}{3(1 - y)^2} - \frac{4}{3(1 - y)} \right) H(0, 0, y) \right] 
\]
\[ + \left( \frac{24}{(1-y)^5} - \frac{60}{(1-y)^4} + \frac{42}{(1-y)^3} - \frac{3}{(1-y)^2} \right) \\
- \frac{3}{2(1-y)} H(0,0,0,y) \right] \\
+ C_F^2 \left[ \zeta(2) \left( \frac{48 \log 2}{(1-y)^2} - \frac{48 \log 2}{(1-y)} + \frac{816}{(1-y)^4} - \frac{2040}{(1-y)^5} \right) \\
+ \frac{1630}{(1-y)^2} - \frac{390}{(1-y)} - \frac{60}{(1-y)^3} - \frac{10}{(1+y)} \right] \\
- \zeta^2(2) \left( \frac{18}{(1-y)^5} - \frac{45}{(1-y)^4} + \frac{639}{(1-y)^3} - \frac{1017}{(1-y)^2} \right) \\
+ \frac{231}{(1-y)^2} - \frac{24}{(1-y)} - \frac{36}{(1+y)^2} + \frac{231}{(1+y)} \right] \\
- \zeta(3) \left( \frac{168}{(1-y)^5} - \frac{336}{(1-y)^4} + \frac{182}{(1-y)^3} - \frac{14}{(1-y)} \right) \\
+ \frac{10}{(1+y)^2} - \frac{10}{(1+y)} \right) + \zeta(2) \left( - \frac{1080}{(1-y)^4} \right) \\
+ \frac{2160}{(1-y)^3} - \frac{1068}{(1-y)} - \frac{12}{(1+y)^2} \right] \\
+ \frac{12}{(1+y)} \right) H(-1,y) + \zeta(2) \left( \frac{288}{(1-y)^5} - \frac{720}{(1-y)^4} \right) \\
+ \frac{600}{(1-y)^4} - \frac{180}{(1-y)^3} + \frac{6}{(1-y)} + \frac{6}{(1+y)} \right) H(-1,0,y) \\
- \left( \frac{48}{(1-y)^3} - \frac{72}{(1-y)^2} + \frac{28}{(1-y)} - \frac{4}{(1+y)} \right) H(-1,0,y) \\
+ \left( \frac{360}{(1-y)^4} - \frac{720}{(1-y)^3} + \frac{356}{(1-y)^2} + \frac{4}{(1-y)} + \frac{4}{(1+y)^2} \right) \\
- \frac{4}{(1+y)} \right) H(-1,0,0,y) - \left( \frac{144}{(1-y)^5} - \frac{360}{(1-y)^4} + \frac{300}{(1-y)^3} \right) \\
- \frac{90}{(1-y)^2} + \frac{3}{(1-y)} + \frac{3}{(1+y)} \right) H(-1,0,0,0,y) \\
- \zeta(2) \left( \frac{120}{(1-y)^5} - \frac{492}{(1-y)^4} + \frac{622}{(1-y)^3} - \frac{254}{(1-y)^2} \right) \\
- \frac{35}{2(1-y)} - \frac{1}{(1+y)^2} + \frac{45}{2(1+y)} \right) H(0,y) \\
- \zeta(3) \left( \frac{168}{(1-y)^5} - \frac{420}{(1-y)^4} + \frac{350}{(1-y)^3} - \frac{105}{(1-y)^2} \right) \\
+ \frac{7}{2(1-y)} + \frac{7}{2(1+y)} \right) H(0,y) - \left( \frac{31}{4(1-y)} \right) \\
- \frac{31}{4(1+y)} \right) H(0,y) - \zeta(2) \left( \frac{1080}{(1-y)^5} - \frac{2700}{(1-y)^4} \right) \]
\[ + \frac{2010}{(1-y)^3} - \frac{315}{(1-y)^2} - \frac{75}{2(1-y)} - \frac{75}{2(1+y)} \] \[ - \left( \frac{96}{(1-y)^4} - \frac{192}{(1-y)^3} + \frac{100}{(1-y)^2} - \frac{4}{(1-y)} - \frac{4}{(1+y)^2} \right) H(0,-1,y) \]
\[ + \frac{4}{(1+y)} \left( \frac{255}{(1-y)^3} + \frac{2}{(1-y)^2} - \frac{25}{2(1+y)} \right) H(0,-1,0,y) \]
\[ + \zeta(2) \left( \frac{48}{(1-y)^5} - \frac{120}{(1-y)^4} + \frac{111}{(1-y)^3} - \frac{93}{2(1-y)^2} \right) H(0,0,y) \]
\[ + \frac{4}{(1-y)} + \frac{1}{(1+y)^3} - \frac{2}{2(1+y)^2} + \frac{4}{(1+y)} \right) H(0,0,0,y) \]
\[ + \frac{4}{(1-y)^4} - \frac{504}{(1-y)^3} + \frac{442}{(1-y)^2} - \frac{2}{(1-y)^2} - \frac{2}{(1+y)^2} \] \[ - \frac{1}{2(1+y)} H(0,0,y) - \left( \frac{96}{(1-y)^5} - \frac{240}{(1-y)^4} + \frac{168}{(1-y)^3} \right) \]
\[ - \frac{12}{(1-y)^2} - \frac{6}{(1-y)} - \frac{6}{(1+y)} \right) H(0,0,-1,0,y) \]
\[ + \left( \frac{60}{(1-y)^5} - \frac{216}{(1-y)^4} + \frac{251}{(1-y)^3} - \frac{88}{(1-y)^2} \right) \]
\[ - \frac{71}{4(1-y)} - \frac{7}{2(1+y)^2} + \frac{57}{4(1+y)} \right) H(0,0,0,0,y) \]
\[ + \left( \frac{6}{(1-y)^5} - \frac{15}{(1-y)^4} + \frac{8}{(1-y)^3} + \frac{3}{(1-y)^2} - \frac{11}{8(1-y)} \right) \]
\[ - \frac{3}{2(1+y)^3} + \frac{9}{4(1+y)^2} - \frac{11}{8(1+y)} \right) H(0,0,0,0,0,y) \]
\[ + \left( \frac{384}{(1-y)^5} - \frac{960}{(1-y)^4} + \frac{714}{(1-y)^3} - \frac{111}{(1-y)^2} - \frac{14}{(1-y)} \right) \]
\[ - \frac{2}{(1+y)^3} + \frac{3}{(1+y)^2} - \frac{14}{(1+y)} \right) H(0,0,1,0,y) \]
\[ + \zeta(2) \left( \frac{936}{(1-y)^5} - \frac{2340}{(1-y)^4} + \frac{1836}{(1-y)^3} - \frac{414}{(1-y)^2} \right) \]
\[ - \frac{21}{2(1-y)} - \frac{6}{(1+y)^3} + \frac{9}{(1+y)^2} - \frac{21}{2(1+y)} \right) H(0,1,y) \]
\[ + \left( \frac{384}{(1-y)^4} - \frac{768}{(1-y)^3} + \frac{394}{(1-y)^2} - \frac{10}{(1-y)} + \frac{2}{(1+y)^2} \right) \]
\[ - \frac{2}{(1+y)} \right) H(0,1,0,y) - \left( \frac{252}{(1-y)^5} - \frac{630}{(1-y)^4} + \frac{496}{(1-y)^3} \right) \]
\[ - \frac{114}{(1-y)^2} - \frac{7}{4(1-y)} + \frac{1}{(1+y)^3} - \frac{3}{2(1+y)^2} \]
\[
3 \mathcal{F}_{2,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon} \left\{ C_F^2 \left[ \left( \frac{1}{1+y} - \frac{1}{1-y} \right) + 2 \left( \frac{1}{1-y} - \frac{1}{1-y^2} \right) \right] \right.
\]
\[
- \frac{1}{1+y} + \frac{1}{(1+y)^2} \right] \right. \left. \times \mathcal{H}(0, y) \right\} + C_F C_A \left[ - \frac{9}{(1-y)^3} + \frac{27}{2(1-y)^2} + \frac{23}{9(1-y)} - \frac{127}{18(1+y)} \right.
\]
\[
+ \zeta(2) \left( \frac{72}{(1-y)^4} - \frac{144}{(1-y)^3} - \frac{73}{(1-y)^2} - \frac{1}{1-y} \right)
\]
\[
+ \frac{1}{(1+y)^2} - \frac{1}{(1+y)} + \zeta(3) \left( - \frac{324}{(1-y)^5} + \frac{810}{(1-y)^4} \right)
\]
\[
- \frac{635}{(1-y)^3} + \frac{285}{2(1-y)^2} + \frac{13}{4(1-y)} + \frac{13}{4(1+y)} \right)
\]
\[
+ \left( \frac{24}{(1-y)^3} - \frac{36}{(1-y)^2} + \frac{12}{(1-y)} \right) \mathcal{H}(-1; y)
\]
\[
+ \left( \frac{24}{(1-y)^4} - \frac{48}{(1-y)^3} + \frac{40}{(1-y)^2} - \frac{16}{(1-y)} \right) \mathcal{H}(-1, 0; y)
\]
\[
+ \left( - \frac{336}{(1-y)^5} + \frac{840}{(1-y)^4} - \frac{660}{(1-y)^3} + \frac{150}{(1-y)^2} \right)
\]
\[
+ \frac{3}{(1-y)} + \frac{3}{(1+y)} \right) \mathcal{H}(-1, 0, 0; y) + \zeta(2) \left( \frac{72}{(1-y)^5} \right)
\]
\[
- \frac{180}{(1-y)^4} + \frac{139}{(1-y)^3} - \frac{57}{(1-y)^2} - \frac{1}{(1-y)} + \frac{1}{(1+y)^3} \right)
\]
\[
- \frac{3}{2(1+y)^2} - \frac{1}{(1+y)} \right) \mathcal{H}(0; y) + \left( - \frac{24}{(1-y)^4} + \frac{39}{(1-y)^3} \right)
\]
\[
+ \left( \frac{7}{12(1-y)} + \frac{3}{(1+y)^2} - \frac{37}{12(1+y)} \right) \mathcal{H}(0; y)
\]
\[
+ \left( \frac{48}{(1-y)^4} - \frac{96}{(1-y)^3} + \frac{48}{(1-y)^2} \right) \mathcal{H}(0, -1; y)
\]
\[
+ \left( \frac{24}{(1-y)^5} - \frac{60}{(1-y)^4} + \frac{70}{(1-y)^3} - \frac{45}{(1-y)^2} + \frac{11}{2(1-y)} \right)
+ \frac{11}{2(1+y)} \left( H(0, -1, 0; y) + \left( \frac{258}{(1-y)^5} - \frac{816}{(1-y)^4} \right) - \frac{3}{1-y} \right)
+ \frac{1803}{2(1-y)^3} - \frac{767}{2(1-y)^2} + \frac{315}{8(1-y)} + \frac{3}{1+y} - \frac{15}{4(1+y)^2}
+ \frac{11}{8(1+y)} \left( H(0, 0; y) + \left( \frac{48}{(1-y)^5} - \frac{120}{(1-y)^4} \right) - \frac{3}{1-y} \right)
+ \frac{84}{(1-y)^3} - \frac{6}{(1-y)^2} - \frac{3}{1-y} - \frac{3}{1+y} \right) H(0, 0, -1; y)
+ \left( - \frac{3}{(1-y)^5} + \frac{15}{2(1-y)^4} - \frac{4}{(1-y)^3} - \frac{3}{1-y} \right) H(0, 0, 0; y)
+ \left( \frac{12}{(1-y)^5} - \frac{30}{(1-y)^4} + \frac{48}{(1-y)^3} - \frac{42}{1-y} \right) H(0, 0, 1; y)
+ \left( \frac{25}{4(1-y)} + \frac{1}{(1+y)^3} - \frac{3}{2(1+y)^2} + \frac{25}{4(1+y)} \right) H(0, 0, 1; y)
+ \left( \frac{1}{(1+y)^2} - \frac{1}{(1+y)} \right) H(0, 1; y) + \left( - \frac{282}{(1-y)^5} + \frac{705}{(1-y)^4} \right)
- \frac{562}{(1-y)^3} + \frac{138}{(1-y)^2} + \frac{5}{8(1-y)} + \frac{1}{2(1+y)^3}
- \frac{3}{4(1+y)^2} + \frac{5}{8(1+y)} \right) H(0, 1, 0; y) + \left( \frac{6}{(1-y)^3} \right)
- \frac{9}{(1-y)^2} + \frac{19}{6(1-y)} - \frac{1}{6(1+y)} \right) H(1; y)
+ \left( - \frac{282}{(1-y)^4} + \frac{564}{(1-y)^3} - \frac{623}{2(1-y)^2} + \frac{59}{2(1-y)} \right)
+ \frac{1}{2(1+y)^2} - \frac{1}{2(1+y)} \right) H(1, 0; y)
\]

\[
+ C_F T_R N_f \left[ - \frac{25}{9(1-y)} + \frac{25}{9(1+y)} + \left( - \frac{2}{3(1-y)} \right) H(0; y) + \left( - \frac{4}{3(1-y)} + \frac{4}{3(1+y)} \right) H(1; y) \right]
\]

\[
+ C_F T_R \left[ - \frac{124}{3(1-y)^3} + \frac{62}{(1-y)^2} - \frac{118}{9(1-y)} - \frac{68}{9(1+y)} \right.
- \left( \frac{88}{3(1-y)^4} - \frac{176}{3(1-y)^3} + \frac{92}{3(1-y)^2} - \frac{4}{3(1-y)} \right) H(0; y)
\]

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\[
\begin{align*}
&\quad + \left( \frac{24}{(1-y)^5} - \frac{60}{(1-y)^4} + \frac{42}{(1-y)^3} - \frac{3}{(1-y)^2} - \frac{3}{2(1-y)} \right) H(0,0;y) \\
&\quad + C_F^2 \left[ -\frac{31}{4(1-y)} + \frac{31}{4(1+y)} + \zeta(2) \left( \frac{60}{(1-y)^4} - \frac{120}{(1-y)^5} \right) \right] \\
&\quad + \zeta(3) \left( -\frac{168}{(1-y)^5} + \frac{420}{(1-y)^4} - \frac{350}{(1-y)^3} + \frac{105}{(1-y)^2} \right) \\
&\quad + \zeta(3) \left( -\frac{7}{2(1-y)} - \frac{7}{2(1+y)} \right) + \left( \frac{3}{4(1+y)} \right) H(-1; y) + \left( \frac{360}{(1-y)^4} - \frac{720}{(1-y)^3} \right) H(-1, 0; y) \\
&\quad - \left( \frac{144}{(1-y)^5} - \frac{360}{(1-y)^4} + \frac{300}{(1-y)^3} - \frac{90}{(1-y)^2} + \frac{3}{(1-y)} \right) H(-1, 0, 0; y) + \zeta(2) \left( \frac{60}{(1-y)^5} - \frac{150}{(1-y)^4} \right) \\
&\quad + \frac{127}{(1-y)^3} - \frac{81}{2(1-y)^2} + \frac{5}{4(1-y)} - \frac{2}{(1-y)^3} + \frac{3}{(1+y)^2} \\
&\quad + \frac{5}{4(1+y)} \left( \frac{255}{192} - \frac{2}{1(1+y)^2} + \frac{1}{2(1+y)} \right) H(0; y) + \left( \frac{360}{(1-y)^5} - \frac{900}{(1-y)^4} + \frac{670}{(1-y)^3} \right) \\
&\quad + \frac{105}{(1-y)^2} - \frac{2}{(1-y)^2} - \frac{25}{2(1+y)} \right) H(0, -1; y) + \left( \frac{360}{(1-y)^5} - \frac{900}{(1-y)^4} + \frac{670}{(1-y)^3} \right) \\
&\quad + \left( \frac{60}{(1-y)^5} - \frac{216}{(1-y)^4} + \frac{251}{(1-y)^3} - \frac{88}{(1-y)^2} - \frac{71}{(1-y)} \right) H(0, 0; y) + \left( \frac{96}{(1-y)^5} \right) \\
&\quad + \left( \frac{7}{2(1+y)^2} + \frac{57}{4(1+y)} \right) H(0, 0, -1; y) + \left( \frac{6}{(1-y)^5} - \frac{15}{(1-y)^4} \right)
\end{align*}
\]
\[
\begin{align*}
+ \frac{8}{(1-y)^3} + \frac{3}{(1-y)^2} - \frac{11}{8(1-y)} - \frac{3}{2(1+y)^3} + \frac{9}{4(1+y)^2} \\
- \frac{11}{8(1+y)} H(0,0,0;y) + \left( \frac{384}{(1-y)^5} - \frac{960}{(1-y)^4} \right) \\
+ \frac{3}{(1-y)^3} - \frac{11}{(1-y)^2} - \frac{14}{(1-y)} + \frac{3}{2(1+y)^3} + \frac{9}{3(1+y)^2} \\
- \frac{14}{(1+y)} H(0,0,1;y) + \left( \frac{384}{(1-y)^4} - \frac{768}{(1-y)^3} \right) \\
+ \frac{394}{(1-y)^2} - \frac{10}{(1-y)} + \frac{2}{2(1+y)^2} - \frac{2}{(1+y)} H(0,1;y) \\
+ \left( - \frac{252}{(1-y)^5} + \frac{630}{(1-y)^4} - \frac{496}{(1-y)^3} + \frac{114}{(1-y)^2} \right) H(0,1,0;y) \\
+ \left( - \frac{7}{4(1-y)} - \frac{1}{(1+y)^3} + \frac{3}{2(1+y)^2} + \frac{7}{4(1+y)} \right) H(0,1,0;y) \\
+ \left( \frac{192}{(1-y)^3} - \frac{288}{(1-y)^2} + \frac{87}{(1-y)} + \frac{9}{(1+y)} \right) H(1;y) \\
- \left( \frac{252}{(1-y)^4} - \frac{504}{(1-y)^3} + \frac{257}{(1-y)^2} - \frac{5}{(1-y)} + \frac{1}{(1+y)^2} \right) \\
- \frac{1}{(1+y)} H(1,0;y) \right] . \tag{82}
\end{align*}
\]

The two-loop contribution to the heavy quark vector form factors arising from closed massive and massless fermion loops were computed previously in [13]. In this work, the result for the massless fermion loops was obtained from a more general (unintegrated) result with different masses for the external and virtual quarks, by expanding in the mass of the virtual quark.

Comparing to [13], we fully agree on the form factors for the massive fermion loop, and on the leading logarithmic terms for the massless fermion loop. Owing to the different regularization procedures used in the latter case (small quark mass versus dimensional regularization), the finite terms differ.

### 6.3 Threshold Expansions

In this Section we provide the expansions of our results in the threshold limit \( S \sim 4m^2 \) (\( y \to 1 \) in the transformed variable). We define:

\[
\beta = \sqrt{1 - \frac{4m^2}{S}} , \tag{83}
\]

as the small parameter in which we expand. Keeping terms up to the zeroth order in \( \beta \), we have:

\[
\Re \mathcal{F}_{1,R}^{(1)}(\epsilon, s) = C_F \left[ \frac{3\zeta(2)}{\beta} - 3 \right] , \tag{84}
\]
\[ \Re \mathcal{F}_{1,R}^{(1)}(\epsilon, s) = C_F \left\{ \frac{1}{\epsilon} \left[ -\frac{1}{2\beta} \right] + \frac{1}{\beta} \left[ -\frac{1}{2} \ln 2 + \ln \beta \right] \right\}, \quad (85) \]

\[ \Re \mathcal{F}_{2,R}^{(1)}(\epsilon, s) = -C_F, \quad (86) \]

\[ \Im \mathcal{F}_{1,R}^{(1)}(\epsilon, s) = \frac{C_F}{2\beta^3}, \quad (87) \]

\[ \Re \mathcal{F}_{1,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ \frac{1}{\beta^2} C_F^2 \left( -\frac{3}{4} \zeta(2) \right) + C_F^2 \left( -\frac{3}{2} \zeta(2) \right) \right\} \]

\[ + \frac{1}{\epsilon} \left\{ \frac{1}{\beta^2} \left[ C_F^2 \left( -\frac{3}{2} \zeta(2) + 3 \zeta(2) \ln \beta + 3 \zeta(2) \ln 2 \right) \right. \]

\[ + C_F^2 \left( -\frac{3}{2} \zeta(2) + 6 \zeta(2) \ln \beta + 6 \zeta(2) \ln 2 \right) \right\} \]

\[ \frac{1}{\beta^2} \left[ C_F^2 \left( -7 \zeta(2) + \frac{9}{2} \zeta^2(2) + 6 \zeta(2) \ln \beta - 6 \zeta(2) \ln^2 \beta + 6 \zeta(2) \ln 2 \right) \right. \]

\[ - 12 \zeta(2) \ln 2 \ln \beta - 6 \zeta(2) \ln^2 2 \right] \]

\[ \frac{1}{\beta} \left[ C_F^2 \left( -9 \zeta(2) \right) \right. \]

\[ + C_F C_A \left( \frac{73}{6} \zeta(2) - 11 \zeta(2) \ln \beta - 11 \zeta(2) \ln 2 \right) \]

\[ - C_F T_R N_f \left( \frac{16}{3} \zeta(2) - 4 \zeta(2) \ln \beta - 4 \zeta(2) \ln 2 \right) \]

\[ + C_F^2 \left( \frac{421}{60} + 5 \zeta(2) \ln 2 - \frac{963}{50} \zeta(2) + 9 \zeta^2(2) - \frac{81}{20} \zeta(3) + \frac{14}{5} \zeta(2) \ln \beta \right. \]

\[ - 12 \zeta(2) \ln^2 \beta + \frac{14}{5} \zeta(2) \ln 2 - 24 \zeta(2) \ln 2 \ln \beta - 12 \zeta(2) \ln^2 2 \right) \]

\[ + C_F C_A \left( -\frac{379}{60} - \frac{42}{5} \zeta(2) \ln 2 + \frac{2741}{150} \zeta(2) - \frac{83}{10} \zeta(3) - \frac{36}{5} \zeta(2) \ln \beta \right. \]

\[ - \frac{36}{5} \zeta(2) \ln 2 \right) \]

\[ + C_F T_R N_f + C_F T_R \left( \frac{37}{3} - \frac{104}{15} \zeta(2) \right), \quad (88) \]

\[ \Im \mathcal{F}_{1,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ \frac{1}{\beta} \left[ C_F C_A \left( \frac{11}{24} \right) + C_F T_R N_f \left( -\frac{1}{6} \right) \right] \right\} \]

\[ + \frac{1}{\epsilon} \left\{ \frac{1}{\beta^2} \left[ C_F^2 \left( \frac{3}{2} \zeta(2) \right) \right] \right\} \]

\[ + \frac{1}{\beta} \left[ C_F^2 \left( \frac{3}{2} \right) - C_F C_A \left( \frac{31}{72} \right) + C_F T_R N_f \left( \frac{5}{18} \right) \right] + C_F^2 \left( -3 \zeta(2) \right) \}

\[ + \frac{1}{\beta^2} \left[ C_F^2 \left( -3 \zeta(2) + 6 \zeta(2) \ln \beta + 6 \zeta(2) \ln 2 \right) \right] \]

\[ + \frac{1}{\beta} \left[ C_F^2 \left( -3 \ln \beta - 3 \ln 2 \right) \right] \]
The above results are in perfect agreement with the results already known in the literature. In particular, the threshold limits of the 1-loop form factors are in agreement with Eqs. (9,10) of [28] (if we put $C_F = 1$ in our results), once the
following replacement is carried out:

\[
\ln \left( \frac{\lambda}{M} \right) = \frac{1}{2\epsilon}.
\]  

(92)

Consequently, we found also agreement with the 1-loop correction to the cross-section of \(e^+e^- \rightarrow f\bar{f}\) given in Eqs. (6,11,12) of [28].

At 2-loop level, Eqs. (26,28) of [28], concerning the abelian contributions matching our \(C_F^2\) terms, are written regularizing the IR divergences with a small mass for the photon. If we replace \(\ln \left( \frac{\lambda}{M} \right)\) using Eq. (92), we are able to match the poles in \(1/\epsilon\) with our expressions calculated directly in dimensional regularization. Nevertheless, we can not find agreement for the finite parts because of the differences between the two regularization schemes. Eqs. (27,29) of [28], instead, are in complete agreement with our \(C_F^2\) terms. Let us note, however, that the contributions at two loops to the cross section \(e^+e^- \rightarrow f\bar{f}\) are IR finite and they can not depend on the regularization scheme. This is actually the case and we found complete agreement with Eqs. (31,32) of [28].

The 2-loop corrections in the threshold limit to the cross-section \(e^+e^- \rightarrow Q\bar{Q}\) were calculated in [31, 32]:

\[
\sigma_{e^+e^- \rightarrow Q\bar{Q}} = \sigma^{(0)} \left[ 1 + \left( \frac{\alpha_S}{2\pi} \right) \tilde{\Delta}^{(1)} + \left( \frac{\alpha_S}{2\pi} \right)^2 \tilde{\Delta}^{(2)} \right],
\]  

(93)

where \(\sigma^{(0)}\) is the tree-level cross-section and \(\tilde{\Delta}^{(1)}\) and \(\tilde{\Delta}^{(2)}\) can be expressed, up to \(\mathcal{O}(\beta^2)\), in terms of the form factors as:

\[
\tilde{\Delta}^{(1)} = 2 \left( \Re F^{(1l)}_{1,R}(\epsilon, s) + \Re F^{(1l)}_{2,R}(\epsilon, s) \right),
\]

(94)

\[
\tilde{\Delta}^{(2)} = \left( \Re F^{(1l)}_{1,R}(\epsilon, s) \right)^2 + 2 \left( \Re F^{(1l)}_{1,R}(\epsilon, s) \Re F^{(1l)}_{2,R}(\epsilon, s) \right) + 2 \Re F^{(2l)}_{1,R}(\epsilon, s)
\]

\[+ \pi^2 \left( \Im F^{(1l)}_{1,R}(\epsilon, s) \right)^2 + 2 \pi^2 \left( \Im F^{(1l)}_{1,R}(\epsilon, s) \Im F^{(1l)}_{2,R}(\epsilon, s) \right) + \left( \Re F^{(1l)}_{2,R}(\epsilon, s) \right)^2
\]

\[+ 2 \Re F^{(2l)}_{2,R}(\epsilon, s) + \pi^2 \left( \Im F^{(1l)}_{2,R}(\epsilon, s) \right)^2.
\]  

(95)

We found complete agreement with the results presented in these papers.

### 6.4 Asymptotic Expansions

In this Section we provide the expansions of our results in the limit \(S \gg m^2 \ (y \rightarrow 0\) in the transformed variable). Putting \(L = \ln (S/m^2)\) and keeping terms up to the second order in \((m^2/S)\), we have:

\[
\Re F^{(1l)}_{1,R}(\epsilon, s) = C_F \left\{ \frac{1}{\epsilon} \left[ \left( \frac{m^2}{S} \right)^2 \left( -3 + 2L \right) - 2 \left( \frac{m^2}{S} \right) - 1 + L \right] \right.
\]

\[+ \left( \frac{m^2}{S} \right)^2 \left( -4 + 8\zeta(2) + 9L - L^2 \right) - \left( \frac{m^2}{S} \right) \left( 1 - 3L \right) \]

\[+ \left( \frac{m^2}{S} \right)^2 \left( -12 + 18\zeta(2) - 18L + 2L^2 \right) \]
\[ -2 + 4\zeta(2) + \frac{3}{2}L - \frac{1}{2}L^2 \],

(96)
\[ + \frac{727}{5} \zeta^2(2) + 246 \zeta(3) \]
\[ + C_F C_A \left( \frac{3749}{36} + \frac{119}{3} \zeta(2)L - 238 \zeta(3)L + \frac{12607}{108} L \right) \]
\[ + 2 \zeta(2)L^2 + \frac{419}{18} L^2 + \frac{77}{18} L^3 - \frac{2}{3} L^4 + 144 \zeta(2) \ln 2 \]
\[ - \frac{7879}{18} \zeta(2) + \frac{118}{5} \zeta^2(2) + \frac{3527}{6} \zeta(3) \]
\[ + C_F T_R N_f \left( \frac{199}{18} + \frac{8}{3} \zeta(2)L - \frac{731}{27} L + \frac{37}{9} L^2 - \frac{2}{9} L^3 \right) \]
\[ - \frac{260}{9} \zeta(2) - \frac{8}{3} \zeta(3) \]
\[ + C_F T_R \left( \frac{2345}{18} - \frac{16}{3} \zeta(2)L - \frac{3730}{27} L + \frac{289}{9} L^2 + \frac{4}{9} L^3 \right) \]
\[ + 28 \zeta(2) \right] \]
\[ - \left( \frac{m^2}{s} \right) \left[ C_F \left( \frac{107}{4} - 99 \zeta(2)L + 12 \zeta(3)L + \frac{97}{4} L + 5 \zeta(2)L^2 \right) \right. \]
\[ - \frac{77}{4} L^2 + \frac{14}{3} L^3 - \frac{1}{12} L^4 + 72 \zeta(2) \ln 2 + 110 \zeta(2) \]
\[ - \frac{43}{5} \zeta^2(2) - 6 \zeta(3) \]
\[ + C_F C_A \left( \frac{167}{108} + 3 \zeta(2)L + 18 \zeta(3)L - \frac{269}{12} L + \frac{1}{2} \zeta(2)L^2 \right) \]
\[ - \frac{1}{4} L^2 - \frac{2}{3} L^3 + \frac{1}{24} L^4 - 36 \zeta(2) \ln 2 + \frac{139}{3} \zeta(2) \]
\[ - \frac{7}{2} \zeta^2(2) - 63 \zeta(3) \]
\[ + C_F T_R N_f \left( \frac{59}{27} + \frac{25}{3} L - L^2 + \frac{16}{3} \zeta(2) \right) \]
\[ + C_F T_R \left( \frac{853}{27} + \frac{49}{3} L - \frac{1}{5} L^2 + \frac{20}{3} \zeta(2) \right) \]
\[3 \mathcal{F}_{1,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon^2} \left\{ \left( \frac{m^2}{S} \right)^2 \right\} \left[ C_F^2 \left( 5 - 4L \right) + C_F C_A \left( \frac{11}{6} \right) - C_F T_R N_f \left( \frac{2}{3} \right) \right] + \left( \frac{m^2}{S} \right) \left[ C_F^2 \left( 2 \right) \right] + C_F^2 \left( 1 - L \right) + C_F C_A \left( \frac{11}{12} \right) - C_F T_R N_f \left( \frac{1}{3} \right) \right\} + \frac{1}{\epsilon} \left\{ \left( \frac{m^2}{S} \right)^2 \right\} \left[ C_F^2 \left( \frac{55}{2} - 29L + 6L^2 - 16\zeta(2) \right) + C_F C_A \left( -\frac{47}{9} + 2L - L^2 + 2\zeta(2) \right) + C_F T_R N_f \left( \frac{10}{9} \right) \right] \left( \frac{m^2}{S} \right) \left[ C_F^2 \left( 7 - 8L \right) \right] + C_F^2 \left( \frac{7}{2} - 4L + \frac{3}{2} L^2 - 4\zeta(2) \right) + C_F C_A \left( -\frac{67}{36} + \frac{1}{2} \zeta(2) \right) + C_F T_R N_f \left( \frac{5}{9} \right) \right\} \left( \frac{m^2}{S} \right)^2 \left[ C_F^2 \left( \frac{1}{2} + 110\zeta(2)L - 216L + \frac{101}{2} L^2 - \frac{28}{3} L^3 - 232\zeta(2) \right) + 148\zeta(3) \right] + C_F C_A \left( -\frac{12607}{108} + 28\zeta(2)L - \frac{419}{9} L - \frac{77}{6} L^2 + \frac{8}{3} L^3 \right) - 91\zeta(2) + 238\zeta(3) \right) + C_F T_R N_f \left( \frac{731}{27} - \frac{74}{9} L + \frac{2}{3} L^2 \right) + C_F T_R \left( \frac{3730}{27} - \frac{578}{9} L - \frac{4}{3} L^2 \right) \right\} \left( \frac{m^2}{S} \right) \left[ C_F^2 \left( -\frac{97}{4} - 6\zeta(2)L + \frac{77}{2} L - 14L^2 + \frac{1}{3} L^3 + 43\zeta(2) - 12\zeta(3) \right) + C_F C_A \left( \frac{269}{12} - 3\zeta(2)L + \frac{1}{2} L + 2L^2 - \frac{1}{6} L^3 + 5\zeta(2) - 18\zeta(3) \right) + C_F T_R N_f \left( -\frac{25}{3} + 2L \right) + C_F T_R \left( -\frac{49}{3} + 10L \right) \right\} + C_F^2 \left( \frac{85}{8} + 10\zeta(2)L - \frac{55}{4} L + 5L^2 - \frac{7}{6} L^3 - 11\zeta(2) - 8\zeta(3) \right) + C_F C_A \left( -\frac{2545}{216} - \zeta(2)L + \frac{233}{36} L - \frac{11}{12} L^2 + \frac{13}{2} \zeta(3) \right) + C_F T_R N_f \left( \frac{209}{54} - \frac{19}{9} L + \frac{1}{3} L^2 \right) \right\} \]

\[72\]
\[ \Re F_{2,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon} \left\{ \left( \frac{m^2}{S} \right)^2 \left[ \mathcal{C}_F^2 \left( -4 + 12L - 4L^2 + 24\zeta(2) \right) \right] \right. \\
\left. \left. - \left( \frac{m^2}{S} \right) \left[ \mathcal{C}_F^2 \left( -2L + 2L^2 - 12\zeta(2) \right) \right] \right\} \\
+ \left( \frac{m^2}{S} \right)^2 \left[ \mathcal{C}_F^2 \left( -89 - 244\zeta(2)L + 112\zeta(3)L + 7L + 28\zeta(2)L^2 \\
- 79L^2 + \frac{28}{3}L^3 - \frac{1}{3}L^4 + 192\zeta(2) \ln 2 + 656\zeta(2) \\
- \frac{276}{5}\zeta^2(2) - 224\zeta(3) \right) \right. \\
+ \mathcal{C}_F \mathcal{C}_A \left( -\frac{343}{9} - 44\zeta(2)L + 136\zeta(3)L - \frac{341}{9}L \\
+ 10\zeta(2)L^2 - \frac{55}{3}L^2 - \frac{8}{3}L^3 + \frac{1}{6}L^4 - 96\zeta(2) \ln 2 \\
+ \frac{944}{3}\zeta(2) - \frac{174}{5}\zeta^2(2) - 404\zeta(3) \right) \right. \\
+ \mathcal{C}_F \mathcal{T}_R N_f \left( \frac{76}{9} + \frac{148}{9}L - \frac{4}{3}L^2 + \frac{32}{3}\zeta(2) \right) \\
+ \mathcal{C}_F \mathcal{T}_R \left( \frac{916}{9} + \frac{868}{9}L - \frac{52}{3}L^2 - 16\zeta(2) \right) \\
- \left. \left( \frac{m^2}{S} \right) \left[ \mathcal{C}_F^2 \left( 48\zeta(2)L - \frac{31}{2}L + \frac{17}{2}L^2 - 2L^3 - 48\zeta(2) \ln 2 \\
- 36\zeta(2) + 4\zeta(3) \right) \right. \\
+ \mathcal{C}_F \mathcal{C}_A \left( 3 + \frac{173}{18}L + \frac{1}{6}L^2 + 24\zeta(2) \ln 2 - \frac{64}{3}\zeta(2) \\
+ 20\zeta(3) \right) \right. \\
+ \mathcal{C}_F \mathcal{T}_R N_f \left( -\frac{50}{9}L + \frac{2}{3}L^2 - \frac{16}{3}\zeta(2) \right) \\
+ \mathcal{C}_F \mathcal{T}_R \left( \frac{68}{3} - \frac{50}{9}L + \frac{2}{3}L^2 - 8\zeta(2) \right) \right\}, \tag{102} \]

\[ \Im F_{2,R}^{(2)}(\epsilon, s) = \frac{1}{\epsilon} \left\{ \left( \frac{m^2}{S} \right)^2 \left[ \mathcal{C}_F^2 \left( -12 + 8L \right) \right] \right. \\
- \left( \frac{m^2}{S} \right) \left[ \mathcal{C}_F^2 \left( 2 - 4L \right) \right] \right\} \\
+ \left( \frac{m^2}{S} \right)^2 \left[ \mathcal{C}_F^2 \left( -7 - 40\zeta(2)L + 158L - 28L^2 + \frac{4}{3}L^3 + 132\zeta(2) \\
- 112\zeta(3) \right) \right. \\
+ \mathcal{C}_F \mathcal{C}_A \left( \frac{341}{9} - 28\zeta(2)L + \frac{110}{3}L + 8L^2 - \frac{2}{3}L^3 \right) \right\}. \]
\[ +76\zeta(2) - 136\zeta(3) \] 
\[ -C_F T_R N_f \left( \frac{148}{9} - \frac{8}{3} L \right) - C_F T_R \left( \frac{868}{9} - \frac{104}{3} L \right) \] 
\[ - \left( \frac{m^2}{S} \right) \left[ C_F^2 \left( \frac{31}{2} - 17L + 6L^2 - 24 \zeta(2) \right) \right. \]
\[ - C_F C_A \left( \frac{173}{18} + \frac{1}{3} L \right) + C_F T_R N_f \left( \frac{50}{9} - \frac{4}{3} L \right) \]
\[ + C_F T_R \left( \frac{50}{9} - \frac{4}{3} L \right) \].

All the results of this Section can be obtained in an electronic form by downloading the source of this manuscript from [http://www.arxiv.org](http://www.arxiv.org).

## 7 Summary and Outlook

In this paper, we calculated the two-loop QCD corrections to the vector vertex form factors for heavy quarks. The result for the electric and magnetic form factors was obtained keeping the full dependence of the form factors on the mass of the heavy quarks, as well as on the momentum transfer, which is maintained arbitrary.

The extraction of the form factors from the Feynman diagrams involved in the calculation was carried out by means of standard projector operators. Each form factor is expressed, in this way, as a combination of several hundreds of scalar integrals, whose UV and IR divergences are regularized, within the Dimensional Regularization procedure, by the same parameter \( D \), dimension of the space-time. Using the Laporta algorithm, it was possible to reduce the problem of the calculation of all these integrals to the calculation of 17 master integrals, already present in the literature.

The renormalization of the UV divergences was carried out in a hybrid scheme in which the coupling constant and the gluon wave function are renormalized in the \( \overline{\text{MS}} \) scheme, while the mass and wave function of the heavy quark are renormalized in the on-shell scheme.

The expressions of the unsubtracted as well as the UV-renormalized form factors are given in a closed analytic form as a Laurent expansion in \( \epsilon = (4 - D)/2 \). The coefficients of this expansion have a suitable representation in terms of 1-dimensional harmonic polylogarithms. The presence of poles in \( \epsilon \) in our results is related to the fact that IR divergences are still present. These divergences have to be canceled against the divergences arising from the real radiation, which in this paper was not taken into account.

Besides being part of the full NNLO corrections to the forward-backward asymmetry of heavy quarks, the results presented in this paper can, on their own, already be used in a number of applications.

An immediate point of interest is the behaviour of the inclusive heavy quark production cross section above the threshold. In the continuum the \( 4\pi \)-integrated cross section was computed to order \( \alpha_s^3 \) [29] and, in view of top quark pair production
at a future linear collider, very detailed NNLO studies have been carried out in the threshold region (see [30] for a review), where the cross section is most sensitive on the top quark mass. The form factors derived here can be extrapolated at the two-loop level to their threshold values. We find complete agreement with all threshold results available in the literature [28, 31, 32].

All available calculations of NNLO QCD corrections to inclusive heavy quark production at asymptotically large energies were made using the optical theorem (see [33] for a review), which avoids the explicit calculation of the two-loop form factors. Computing more differential quantities such as rapidity distributions requires the explicit knowledge of the two-loop form factors as well as of the massive (or leading-mass) single and double real radiation corrections. In the massless case, such calculations could be performed only very recently [34], and a first step towards a fully massive calculation would certainly only consider the leading mass terms. In view of this application, we also provided the leading mass expansions of our full two-loop corrections to the form factors.

The singularity structure of the two-loop heavy quark form factors could also provide insight into the generic singularity structure of two-loop integrals involving massive quarks. The corresponding structure of massless two-loop QCD amplitudes has been predicted from non-abelian exponentiation in [35, 36], and proven very valuable in the calculation of massless two-loop four-point amplitudes [37]. For QCD amplitudes involving massive quarks, the singularity structure is only understood at the one-loop level at present [38]. At the two-loop level, we observe that the infrared divergent contributions to the form factors which are proportional to the colour factor $\mathcal{C}_F$ exponentiate naively [39], as already seen in the QED calculation [15]. The other colour factors contain explicit $1/\epsilon$ singularities which can not be explained from naive abelian exponentiation and deserve further investigation.

Finally, this calculation demonstrates that it is possible to analytically compute two-loop vertex functions with at least one internal mass scale. These functions appear in a variety of applications, ranging from electroweak corrections to heavy quark physics, where first genuine two-loop results were obtained only very recently [40, 41, 42, 43], using the same methods (reduction to master integrals, differential equations, and basis of harmonic polylogarithms) as employed here. Many more results in this domain are yet outstanding, and the methods here can be clearly instrumental for their derivation.

To compute the NNLO QCD corrections to the forward-backward asymmetry of heavy quarks, one also needs the two-loop corrections to the axial vector form factors. These corrections subdivide into two independent classes: anomalous and non-anomalous diagrams. The calculation of these is currently in progress, and results will be reported in two future publications.

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