A Note on John Simplex with Positive Dilation

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Abstract
We prove a John’s theorem for simplices in $\mathbb{R}^d$ with positive dilation factor $d + 2$, which improves the previously known $d^2$ upper bound. This bound is tight in view of the $d$ lower bound. Furthermore, we give an example that $d$ isn’t the optimal lower bound when $d = 2$. Our results answered both questions regarding John’s theorem for simplices with positive dilation raised by [Leme and Schneider 2020].

1 Introduction
In the problem of linear function reconstruction ([Khachiyan 1995, Summa et al. 2014, Nikolov 2015]), one needs to efficiently reconstruct a linear function $f$ defined on a set $X$ by using a zeroth order oracle. When the cost of oracle querying grows along with accuracy, a John like simplex serves as a good basis as in [Leme and Schneider 2020].

A crucial step of their method is to find a simplex $T$ with vertices in $X$, such that $X$ can be contained in some translate of $T$ with dilation, where a smaller (absolute value) dilation factor indicates higher efficiency. When considering negative dilation, a translate $-dT$ is able contain $X$ so the upper bound is $O(d)$ which matches its lower bound.

However, if we look at positive dilation, it seems to be less efficient than negative dilation because only a worse $d^2$ upper bound was given in [Leme and Schneider 2020]. Though a better bound for positive dilation does not immediately help design better algorithms for the reconstruction problem, it is natural to ask if we can improve it.

We prove a John’s theorem for simplices with positive dilation factor $d + 2$ which answers the question affirmatively. Furthermore, we give a counterexample that the $d$ lower bound given in [Leme and Schneider 2020] isn’t tight when $d = 2$.

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2 Background

In the seminal work [John 2014], it’s proven that any convex \( X \subset R^d \) can be sandwiched between two concentric ellipsoids of ratio \( d \). Specifically, the following theorem was given in [John 2014]:

**Theorem 2.1.** Given any convex body \( X \subset R^d \), let \( x + TB_d \) denote the minimal volume ellipsoid containing \( X \), then it holds

\[
x + \frac{1}{d}TB_d \subset X \subset x + TB_d
\]  

(1)

The ellipsoid \( x + TB_d \) is called ‘John ellipsoid’. For a more comprehensive review we refer the readers to [Henk 2012]. In this paper we consider general \( X \) which is potentially non-convex, and we aim to find a John like simplex with positive dilation factor \( O(d) \) so that it can be seen as an analogue to John ellipsoid.

**Definition 2.2.** Given \( d + 1 \) points \( v_1, ..., v_{d+1} \) in \( R^d \), we call their convex hull a simplex with vertices \( v_1, ..., v_{d+1} \).

We consider the following: given a compact set \( X \subset R^d \) which is not necessarily convex or connected, we want to find \( d + 1 \) points in \( X \) forming a simplex \( T \) such that \( X \) is contained in some translate of \( T \) with positive dilation.

A simple method to do so is by considering the maximum volume simplex (MVS) of \( X \) as in [Leme and Schneider 2020].

**Definition 2.3.** A maximum volume simplex \( T \) with vertices in a bounded set \( X \subset R^d \) is one of the simplices whose euclidean measure is no less than any other simplex of \( X \).

**Proposition 2.4.** There always exists a MVS as long as \( X \) is compact.

**Proof.** Because \( X \) is bounded, the volume of its simplex has a least upper bound \( c \geq 0 \), therefore we can find a sequence of simplices \( T_j \) with vertices \( v_1^{(j)}, ..., v_{d+1}^{(j)} \), whose volumes converge to \( c \). Due to the compactness of \( X \), any sequence in \( X \) has a converging sub-sequence which converges to a point in \( X \), thus we can find indexes \( a_j \) such that \( v_1^{(a_j)} \) converges to a point \( \tilde{v}_1 \in X \).

We consider a new sequence of simplices \( T_{a_j} \) whose first vertex is replaced by \( \tilde{v}_1 \) while others keep unchanged. It’s easy to see that their volumes still converge to \( c \). Again we find a converging sub-sequence of \( v_2^{(a_j)} \) which converges to a point \( \tilde{v}_2 \in X \), and repeat this procedure until we have all \( \tilde{v}_i \). Then the simplex with vertices \( \tilde{v}_i \) is the desired MVS.

[Leme and Schneider 2020] prove the following lemma by using the ‘maximum volume’ property:

**Lemma 2.5.** Let \( T \) be the MVS with vertices in \( X \), then \( X \) is contained in a translate of \( -dT \).
Using lemma 2.5 twice we get a $d^2$ upper bound for positive dilation. We are interested in proving tighter bounds for the positive dilation factor. Existing upper and lower bounds are $d^2$ and $d$ respectively as in Leme and Schneider [2020], in their paper two questions regarding John’s theorem for simplices with positive dilation were raised.

**Question 1:** Can we get a John’s theorem for simplices with positive dilation factor $O(d)$?

**Question 2:** Given compact $X \subset \mathbb{R}$, can we always find a triangle $T$ with vertices in $X$ so that $X$ is contained in some translate of $2T$?

We give an affirmative answer to question 1 by proving a $d + 2$ upper bound, and provide a counter-example to question 2.

### 3 $d + 2$ upper bound

In this section we prove a $d + 2$ upper bound based on a simple observation: by fully exploiting the 'maximum volume' property, we can squeeze $-dT$ to a much smaller set which still contains $X$.

We already know that a simplex $T \in \mathbb{R}^d$ can be covered by a translate of $-dT$, therefore a naive method (choose $T$ to be MVS and use $-d$ dilation twice) directly leads to a $d^2$ upper bound for positive dilation.

However, the first step of this method is extravagant in that we can find a smaller set covering $X$ when $d > 2$. Think about the following example in $\mathbb{R}^3$: assume $T$ (the MVS of $X$) is the regular simplex with vertices $v_1, ..., v_4$ and 0 as its center, then $-3T$ has $-3v_1, ..., -3v_4$ as its vertices. We denote $S_{v,i}$ to be the hyperplane parallel to hyperplane $v_1, ..., v_{i-1}, v_{i+1}, ..., v_4$, with $v$ lying on it. By the definition of $T$, $X$ lies between $S_{v,i}, S_{\hat{v},i}$ for any $i = 1, ..., 4$, therefore much space is wasted in the 'cones' of $-3T$.

The $d^2$ bound can be decomposed as $1 + (d+1)(d-1)$ where for each $v_i$, $T$ extends $d-1$ times along its direction. However, we only need to extend once which yields a $1 + (d+1) = d + 2$ upper bound.

**Theorem 3.1.** For any compact set $X \in \mathbb{R}^d$, there exists $d + 1$ points in $X$ forming a simplex $T$ as its vertices, such that $X$ can be covered in a translate of $(d+2)T$.

**Proof.** We choose the MVS of $X$ to be $T$ with vertices $v_1, ..., v_{d+1}$ and we assume 0 is its center (of gravity) without loss of generality. We denote $\hat{v}_i$ to be the symmetric point of $v_i$ with respect to hyperplane $v_1, ..., v_{i-1}, v_{i+1}, ..., v_{d+1}$ along direction $v_i$, and $S_{\hat{v},i}$ to be the hyperplane parallel to hyperplane $v_1, ..., v_{i-1}, v_{i+1}, ..., v_{d+1}$, with $v$ lying on it. By the definition of $T$, $X$ lies between $S_{v,i}, S_{\hat{v},i}$ for any $i = 1, ..., d+1$, otherwise the 'maximum volume' property will be contradicted.

We take a close look at the simplex $T'$ made up of $S_{\hat{v},i}$. Obviously $T'$ is a translate of $T$ with positive dilation and center (of gravity) 0 unchanged. Define the intersection point between line $v_i\hat{v}_i$ (we overload the notation of vectors to denote endpoints of a segment when the context is clear) and hyperplane $v_1, ..., v_{i-1}, v_{i+1}, ..., v_{d+1}$ as $w_i$, we have that $v_i = \frac{d}{d+1} w_i v_i$ by the definition of
Figure 1: Example in $R^3$. The red pyramid is $T$, the blue one is $-3T$ and the yellow ones are the wasted space.

center (of gravity). Combing with the fact that $w_i v_i = \hat{v}_i w_i$, the dilation factor is $(1 + \frac{1}{d+1})/\frac{1}{d+1} = d + 2$.

Denote the simplex made up of $S_{v_i,i}$ as $\tilde{T}$ and the region enclosed by $S_{v_i,i}, S_{\hat{v}_i,i}$ as $\hat{T}$, we have the following inclusion:

$$X \subset \hat{T} = T' \cap \tilde{T} \subset T'$$

which finishes our proof.

4 Lower Bound $> d$ when $d = 2$

We give a counter-example to question 2 in this section. The idea behind is intuitive: we construct several discrete points as a hard case, so that these discrete points won’t help much when we construct the covering triangle, but hurts a lot when we consider covering them since the whole convex hull needs to be covered implicitly.

**Theorem 4.1.** There exists a compact set $X \in R^2$, such that for any triangle $T$ with vertices in $X$, $X$ can’t be contained in any translate of $2T$.

**Proof.** We pick 5 points:

$$A = (-1, 0), B = (1, 0), C = (-\epsilon - \delta, 1), D = (\epsilon + \delta, 1), E = (0, \epsilon - 1)$$
to construct \( X = \{ A, B, C, D, E \} \), where constants (TBD) satisfy \( \epsilon, \delta \in (0, 1) \). Due to symmetry, we discuss different choices of \( T \) by 6 cases and how they lead to contradiction.

**Case 1:** \( T = \triangle CDE \)

The intercept along \( y = 0 \) of any translate of \( 2T \) has length at most \( 4(\epsilon + \delta) < 2 \), thus \( AB \) can't be covered by any translate of \( 2T \).

**Case 2:** \( T = \triangle ABE \)

The intercept along \( x = 0 \) of any translate of \( 2T \) has length at most \( 2 - 2\epsilon < 2 - \epsilon \), thus \( C, E \) can't be simultaneously covered by any translate of \( 2T \).

**Case 3:** \( T = \triangle ACD \)

The intercept along \( y = 0 \) of any translate of \( 2T \) has length at most \( 4(\epsilon + \delta) < 2 \), thus \( AB \) can't be covered by any translate of \( 2T \).

**Case 4:** \( T = \triangle ABC \)

The intercept along \( y = 1 \) of any translate of \( 2T \) has length at most \( 2\epsilon < 2\epsilon + 2\delta \) when its 'bottom' is below \( y = \epsilon - 1 \), thus \( CD \) and \( E \) can't be simultaneously covered by any translate of \( 2T \).

**Case 5:** \( T = \triangle ACE \)

The intercept along \( y = 0 \) of any translate of \( 2T \) has length at most \( \frac{2 - 2\epsilon}{\epsilon + \delta}(1 - \epsilon) < 2 \), thus \( AB \) can't be covered by any translate of \( 2T \).

**Case 6:** \( T = \triangle ADE \)

We would like to prove that any translate of \( 2T \) can’t cover \( \triangle ABC \). We extend \( \overrightarrow{AD} \) by twice to \( D' = (1 + 2\epsilon + 2\delta, 1) \) and \( \overrightarrow{AE} \) by twice to \( E' = (1, 2\epsilon - 2) \), then try to move \( \triangle ABC \) to fit in \( \triangle A'D'E' \) where \( A' = A \).

Because \( A' + \overrightarrow{CB} = (\epsilon + \delta, -1) \), in order for \( C \) to be contained in \( \triangle A'D'E' \), \( B \) must lie below line

\[
y = \frac{1}{1 + \epsilon + \delta}(x - \epsilon - \delta) - 1 \tag{3}
\]

Therefore the largest possible \( y \)-coordinate of \( B \) is that of the intersection point between line 3 and

\[
y = \frac{2 - \epsilon}{\epsilon + \delta}(x - 1 - \frac{2(\epsilon + \delta)(1 - \epsilon)}{2 - \epsilon}) \tag{4}
\]

By straightforward computation, we have that

\[
y = -\frac{2}{x + \epsilon + 1 - \epsilon} \tag{5}
\]

is the largest possible \( y \)-coordinate of \( B \). However, the intercept along line 5 of \( \triangle A'D'E' \) equals

\[
(2 - 2\epsilon - \frac{2}{\epsilon + \delta + 1 - \epsilon}) \times \frac{1 + \frac{(\epsilon + \delta)(1 - \epsilon)}{2 - \epsilon}}{2 - \epsilon} = 2 + \frac{(\epsilon + \delta)(1 - \epsilon)}{2 - \epsilon} - \frac{2(\epsilon + \delta)}{(1 - \epsilon)(2 - \epsilon)} = 2 - \frac{(\epsilon + \delta)(1 + 2\epsilon - \epsilon^2)}{(1 - \epsilon)(2 - \epsilon)} < 2
\]
thus $C, A, B$ can’t be simultaneously covered by any translate of $2T$.

For the choice of constants, any constant pair additionally satisfying $\epsilon+\delta < \frac{1}{2}$ is feasible.

\[ \square \]

5 Conclusion

In this note, we analyze John’s theorem for simplices with positive dilation and answer related open questions raised by Leme and Schneider [2020]. We prove a tight $d+2$ upper bound which matches the $d$ lower bound, improving the previously known $d^2$ bound. We also give a simple counter-example showing that the $d$ lower bound isn’t optimal.

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