Skyrmions and Quantum Hall Ferromagnets in Improved Composite-Boson Theory

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Abstract

An improved composite-boson theory of quantum Hall ferromagnets is proposed. It is tightly related with the microscopic wave-function theory. The characteristic feature is that the field operator describes solely the physical degrees of freedom representing the deviation from the ground state. It presents a powerful tool to analyze excited states within the lowest Landau level. Excitations include a Goldstone mode and nonlocal topological solitons. Solitons are vortices and Skyrmions carrying the U(1) and SU(2) topological charges, respectively. Their classical configurations are derived from their microscopic wave functions. The activation energy of one Skyrmion is estimated, which explains experimental data remarkably well.

I. INTRODUCTION

The quantum Hall (QH) effect [1] is a remarkable macroscopic quantum phenomenon observed in the two-dimensional electron system at low temperature $T$ and in strong magnetic field $B$. The Hall conductivity is quantized with extreme accuracy and develops a series of plateaux at magic values of the filling factor $\nu = 2\pi h \rho_0 / e B$. Here, $\rho_0$ is the average electron density. The QH effect comes from the realization of an incompressible quantum fluid. The composite-boson (CB) picture [2–4] and the composite-fermion picture [5,6] have proved to be quite useful to understand all essential aspects of QH effects. Electrons may condense into an incompressible quantum Hall liquid as composite bosons. The QH state is such a condensate of composite bosons, where quasiparticles are vortices [7].

When the spin degree of freedom is incorporated, a quantum coherence develops spontaneously provided the Zeeman effect is sufficiently small, turning the QH system into a QH ferromagnet. New excitations are Skyrmions. Skyrmions were initially considered as solutions of the effective nonlinear sigma model [8], and later studied also in a Hartree-Fock approximation [9]. Their existence has been confirmed experimentally [10,11].

In this paper we present a field-theoretical formulation of QH ferromagnets and Skyrmions based on an improved composite-boson (CB) theory [12], proposed based on a suggestion due to Girvin [1], Read [3] and Rajaraman et al. [4]. In this scheme, the field operator describes solely the physical degrees of freedom representing the deviation from the ground state, and the semiclassical property of excitations is determined directly...
by their microscopic wave functions. This clarifies the relation between the microscopic wave function and the classical Skyrmion, which is missing in the effective theory \[8\]. It is explicitly shown that the Skyrmion excitation is reduced to the vortex excitation in the small-size limit. We estimate its activation energy, which accounts for the observed data due to Schmeller et al. \[11\] remarkably well provided certain physical assumptions are made. Thus, our field-theoretical approach presents a new insight into the Skyrmion physics.

This paper is composed as follows. In Section \[1\] we give an overview of the improved CB theory. In Section \[2\] CB fields are defined. First, as in the standard CB theory \[3\], we attach an odd number of flux quanta to an electron by way of a singular phase transformation. We call the resulting electron-flux composite the *bare composite boson*. In order to soften the singularity brought in, we dress it with a cloud of an effective magnetic field that bare composite bosons feel. The resulting object turns out to be the *dressed composite boson*. In Section \[4\] the relation is established between the electron wave function and the CB wave function. We also verify that the ground state is given by the Laughlin state within the semiclassical approximation. In Section \[5\] we make a semiclassical analysis of vortex excitations. In Section \[6\] we include the spin degree of freedom and show that a quantum coherence develops spontaneously when the Zeeman effect is sufficiently small. In Section \[7\] we discuss vortex and Skyrmion excitations in the QH ferromagnet. The Skyrmion classical configuration is also derived directly from its microscopic wave function. We evaluate the excitation energy of one Skyrmion and compare it with experimental data. Throughout the paper we use the natural units \(\hbar = c = 1\).

II. OVERVIEW

We start with a review of the improved CB theory applied to the spin-frozen QH system \[12\]. We denote the electron field by \(\psi(x)\) and its position by the complex coordinate normalized as \(z = (x + iy)/2\ell_B\). Any state \(|\Phi\rangle\) in the lowest Landau level at the filling factor \(\nu = 1/m\) \((m\ \text{odd})\) is represented by the wave function,

\[
\Phi[x] \equiv \langle 0|\psi(x_1)\cdots\psi(x_N)|\Phi\rangle = \omega[z]\Phi_{\text{LN}}[x], \tag{2.1}
\]

where

\[
\Phi_{\text{LN}}[x] = \prod_{r<s}(z_r - z_s)^m \exp[-\sum_{r=1}^{N}|z_r|^2] \tag{2.2}
\]

is the Laughlin function, and \(\omega[z] \equiv \omega(z_1, z_2, \ldots, z_N)\) is an analytic function symmetric in all \(N\) variables. The mapping from the fermionic wave function \(\Phi[x]\) to the bosonic function \(\omega[z]\) defines a bosonization. We call the underlying boson the *dressed composite boson* and denote its field operator by \(\varphi(x)\). The field operator turns out to be the one considered first by Read \[3\] and revived recently by Rajaraman et al. \[4\]. We derive that

\[
\Phi_{\varphi}[x] \equiv \langle 0|\varphi(x_1)\cdots\varphi(x_N)|\Phi\rangle = \omega[z]. \tag{2.3}
\]

The Laughlin state is represented by \(\Phi_{\varphi}[x] = 1\).
A. Vortices

A typical vortex state is represented by $\Phi_\phi[x] = \prod_r z_r$. When the wave function is factorizable, $\omega[z] = \prod_r \omega(z_r)$, as in this example, we obtain a semiclassical equation,

$$\langle \varphi(x) \rangle = \omega(z). \tag{2.4}$$

This is a highly nontrivial constraint since it connects directly the classical field $\langle \varphi(x) \rangle$ to the microscopic wave function $\omega(z)$. It dictates how the electron density $\rho(x)$ is modulated around the topological excitation.

The topological charge density $Q(x)$ is determined in terms of the classical field. It is the vorticity, $Q(x) \equiv Q^V(x)$ given by

$$Q^V(x) = \frac{1}{2\pi i} \epsilon_{ij} \partial_i \partial_j \ln \langle \varphi(x) \rangle = \delta(x), \tag{2.5}$$

for a vortex sitting at the origin. The topological soliton induces the density modulation according to the soliton equation,

$$\frac{\nu}{4\pi} \nabla^2 \ln \rho(x) - \rho(x) + \rho_0 = \nu Q(x), \tag{2.6}$$

as follows from the semiclassical constraint (2.4). The topological soliton carries the electric charge $\nu q e$, where $q = \int d^2 x Q(x)$ is the topological charge.

B. Skyrmions

We next summarize the idea of an improved CB theory applied to QH ferromagnets. We denote the spin component by the index $\alpha(=\uparrow, \downarrow)$. Any state at $\nu = 1/m$ is represented by the wave function similar to (2.1). Let us explicitly consider the case when the spinor component is factorizable,

$$\Phi_{\text{spin}}^\alpha[x] = \prod_r \left( \begin{array}{c} \omega^\uparrow(z_r) \\ \omega^\downarrow(z_r) \end{array} \right) \Phi_{\text{LN}}[x]. \tag{2.7}$$

The ground state is given by

$$\Phi_{g\text{spin}}^\alpha[x] = \prod_r \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \Phi_{\text{LN}}[x]. \tag{2.8}$$

From a set of two analytic functions $\omega^\alpha(z)$ we construct the complex-projective field $n(x)$ whose two components $n^\alpha(x)$ are

$$n^\alpha(x) = \frac{\omega^\alpha(z)}{\sqrt{(|\omega^\uparrow(z)|^2 + |\omega^\downarrow(z)|^2)}}. \tag{2.9}$$

This represents the most general Skyrmion configuration \cite{13} together with the asymptotic boundary condition $n^\uparrow = 1$ and $n^\downarrow = 0$. The normalized spin field, or the nonlinear sigma field, is defined by
\[ s^a(x) = n(x)^\dagger \tau^a n(x), \quad (2.10) \]

with \( \tau^a \) the Pauli matrices. The Skyrmion configuration is indexed by the Pontryagin number \([13]\), whose density is

\[ Q^P(x) = \frac{1}{8\pi} \varepsilon_{abc} \varepsilon_{ij} s^a(x) \partial^i s^b(x) \partial^j s^c(x). \quad (2.11) \]

The Skyrmion excitation modulates not only the spin density but also the electron density. The electron density modulation is determined by the same soliton equation as (2.6), where \( Q(x) \equiv Q^P(x) \) is the Pontryagin number density (2.11).

The simplest excitation is given by one Skyrmion with scale \( \kappa \) sitting at \( x = 0 \), whose wave function is \([14]\)

\[ \Phi^{\text{spin}}_{\text{sky}}[x] = \prod_r \left( \frac{z_r}{\kappa/2} \right)_r \Phi_{\text{LN}}[x]. \quad (2.12) \]

The scale \( \kappa \) is to be fixed dynamically to minimize the excitation energy. The Pontryagin number density (2.11) is calculated for the simplest Skyrmion (2.12) as

\[ Q^P(x) = \frac{1}{\pi} \frac{(\kappa \ell_B)^2}{p^2 + (\kappa \ell_B)^2}, \quad (2.13) \]

We study the limit \( \kappa \to 0 \), where the Skyrmion wave function (2.12) is reduced to the vortex wave function in the spin-polarized ground state, and the Pontryagin number density is reduced to the vorticity (2.7), \( Q^P(x) \to \delta(x) \), as \( \kappa \to 0 \). Hence, the soliton equation (2.6) yields a vortex configuration for the zero-size Skyrmion. This makes clear the relation between the Skyrmion and the vortex in QH ferromagnets.

III. BOSONIZATION

The microscopic Hamiltonian for spinless planar electrons in external magnetic field \((0,0,-B)\) is given by

\[ H = \frac{1}{2M} \int d^2x \psi^\dagger(x)(P_x^2 + P_y^2)\psi(x) + H_C \]

\[ = \frac{1}{2M} \int d^2x \psi^\dagger(x)(P_x - iP_y)(P_x + iP_y)\psi(x) + \frac{N}{2} \omega_c + H_C, \quad (3.1a) \]

\[ \text{where } \psi(x) \text{ is the electron field; } P_j = -i \partial_j + eA_j^{\text{ext}} \text{ is the covariant momentum with } A_j^{\text{ext}} = \frac{1}{2} \varepsilon_{jk} x_k B; \text{ } N \text{ is the electron number and } \omega_c \text{ is the cyclotron frequency. The Coulomb interaction term is} \]

\[ H_C = \frac{e^2}{2\varepsilon} \int d^2x d^2y \frac{\phi(x) \phi(y)}{|x - y|}, \quad (3.2) \]

where \( \phi(x) \equiv \rho(x) - \rho_0 \) stands for the deviation of the electron density \( \rho(x) \equiv \psi^\dagger(x)\psi(x) \) from its average value \( \rho_0 \).
The state $|\Phi\rangle$ in the lowest Landau level obeys

$$(P_x + iP_y)\psi(x)|\Phi\rangle = -\frac{i}{\ell_B} \left(z + \frac{\partial}{\partial z}\right)\psi(x)|\Phi\rangle = 0,$$  

(3.3)
on which the kinetic Hamiltonian is trivial. We call it the LLL condition. The generic solution of this equation yields the $N$-body wave function \((2.1)\) for the state $|\Phi\rangle$. We are concerned about the state $|\Phi\rangle$ in the lowest Landau level.

The bosonization scheme was pioneered by Girvin and MacDonald \([2]\). We define the field $\phi(x)$ by an operator phase transformation,

$$\phi(x) = e^{-i\Theta(x)}\psi(x),$$  

(3.4)

where $\Theta(x)$ is the phase field,

$$\Theta(x) = m \int d^2y \theta(x - y)\rho(y),$$  

(3.5)

with the azimuthal angle $\theta(x - y)$. When $m$ is an odd integer, we can prove that $\phi(x)$ is a bosonic operator. Let us call the underlying boson the bare composite boson. It is a hardcore boson satisfying the exclusion principle, $\phi(x)^2 = 0$. The LLL condition \((3.3)\) reads

$$(\tilde{P}_x + i\tilde{P}_y)\phi(x)|\Phi\rangle = 0,$$  

(3.6)

where $\tilde{P}_j \equiv P_j + \partial_j\Theta(x)$ is the covariant momentum for the bare CB field. By solving this condition, the $N$-body wave function is found to be $\Phi_\phi[x] \equiv \omega[z]|\Phi_{LN}[x]\rangle$. Namely, the standard bosonization is a mapping from the fermionic wave function $\Phi[x]$ to a bosonic wave function $\Phi_\phi[x]$,

$$\Phi[x] \mapsto \Phi_\phi[x] \equiv \omega[z]|\Phi_{LN}[x]\rangle.$$  

(3.7)

The mapping is to attach $m$ Dirac flux quanta $m\phi_D$ to each electron by way of the phase transformation \((3.4)\), where $\phi_D = 2\pi/e$. Thus, the ground state of bare composite bosons is described by the modulus of the Laughlin function, $|\Phi_{LN}[x]\rangle$.

We wish to introduce another CB field $\varphi(x)$ which makes the LLL condition as simple as possible. We set

$$\varphi(x) = e^{-A(x) - i\Theta(x)}\psi(x),$$  

(3.8)

together with an operator $A(x)$ to be determined later. By substituting this into the LLL condition \((3.3)\), provided $A(x)$ satisfies

$$\partial_jA(x) = \varepsilon_{jk}\{\partial_k\Theta(x) + eA^\text{ext}_k(x)\} = \varepsilon_{jk}\partial_k\Theta(x) - \frac{1}{2\ell_B^2}x_j, \quad (3.9)$$

it is reduced to a simple formula,

$$(P_x + iP_y)\varphi(x)|\Phi\rangle = -\frac{i}{\ell_B}\frac{\partial}{\partial z}\varphi(x)|\Phi\rangle = 0,$$  

(3.10)
where \( P_j \) is the covariant momentum for the new CB field \( \varphi(x) \): See (4.3). We solve (3.9) as

\[
\mathcal{A}(x) = m \int d^2y \ln \left( \frac{|x-y|}{2\ell_B} \right) \rho(y) - |z|^2.
\]  

(3.11)

The \( N \)-body wave function \( \Phi_\varphi[x] \) of dressed composite bosons is obtained by solving the LLL condition (3.10), and it is given simply by an analytic function \( \omega[z] \) as in (2.3). We shall verify in the next section that one analytic function \( \omega[z] \) characterizes one state \( |\Phi\rangle \) as in (2.1) in terms of electrons, or as in (3.7) in terms of bare composite bosons, or as in (2.3) in terms of dressed composite bosons.

The effective magnetic potential for the bare composite boson is \( A_{\text{ext}}^k(1/e) \partial_k \Theta \), which is rewritten as \( (1/e)\varepsilon_{kj} \partial_j A(x) \). Bare composite bosons feel the effective magnetic field \( B_{\text{eff}}(x) \),

\[
B_{\text{eff}}(x) = e^{-1} \nabla^2 A(x) = m\phi_D \rho(x) - B.
\]  

(3.12)

The effective field vanishes, \( \langle B_{\text{eff}} \rangle = 0 \), on the homogeneous state \( \langle \rho(x) \rangle = \rho_0 \) if \( m\phi_D \rho_0 = B \). It occurs at the filling factor \( \nu \equiv \rho_0 \phi_D / B = 1/m \). At \( \nu = 1/m \) we rewrite the effective magnetic field as

\[
e B_{\text{eff}}(x) = \nabla^2 A(x) = 2\pi m \varrho(x),
\]  

(3.13)

with \( \varrho(y) \equiv \rho(y) - \rho_0 \). It is solved as

\[
A(x) = m \int d^2y \ln \left( \frac{|x-y|}{2\ell_B} \right) \varrho(y).
\]  

(3.14)

This formula is equivalent to (3.11) at \( \nu = 1/m \) up to an irrelevant integration constant. Since the field \( \varphi(x) \) is constructed by dressing a cloud of the effective magnetic field, we have termed it the dressed CB field.

**IV. QUANTUM HALL STATES**

We derive the Hamiltonian in terms of dressed composite bosons by substituting (3.8) into (3.1B),

\[
H = \frac{1}{2M} \int d^2x \varphi^\dagger(x)(\mathcal{P}_x - i\mathcal{P}_y)(\mathcal{P}_x + i\mathcal{P}_y)\varphi(x) + \frac{N}{2} \omega_c + H_C,
\]  

(4.1)

where we have defined

\[
\varphi^\dagger(x) \equiv \varphi^\dagger(x)e^{2A(x)},
\]  

(4.2)

with which \( \rho(x) = \psi^\dagger(x)\psi(x) = \varphi^\dagger(x)\varphi(x) \). The covariant momentum \( \mathcal{P}_j = P_j + \partial_j \Theta - i \partial_j \mathcal{A} \) reads,

\[
\mathcal{P}_j = -i \partial_j + e(\delta_{jk} - i\varepsilon_{jk})\mathcal{A}_k, \quad \mathcal{A}_k(x) = -\frac{1}{e} \varepsilon_{kj} \partial_j \mathcal{A}(x).
\]  

(4.3)
As we noticed in (3.10), it yields

\[ \mathcal{P}_x + i\mathcal{P}_y = -\frac{i}{\ell_B} \frac{\partial}{\partial z^*}, \]  

(4.4)

with which the Hamiltonian (4.1) is rewritten as

\[ H = \frac{\omega_c}{2} \int d^2x \left( \frac{\partial}{\partial z^*} \varphi(x) \right) \mathcal{A}(x) \frac{\partial}{\partial z^*} \varphi(x) + \frac{N}{2} \omega_c + H_C. \]  

(4.5)

It is manifestly hermitian.

The Lagrangian density is

\[ L = \psi^\dagger (i\partial_t - eA^{\text{ext}}_t) \psi - \mathcal{H} = \varphi^\dagger (i\partial_t - eA^{\text{ext}}_t - \partial_t \Theta + i\partial_t A) \varphi - \mathcal{H}, \]  

(4.6)

where \( \mathcal{H} \) is the Hamiltonian density and \( A^{\text{ext}}_\mu = (A^{\text{ext}}_t, A^{\text{ext}}_k) \) is the potential of the external electromagnetic field. Because the canonical conjugate of \( \varphi(x) \) is not \( i\varphi(x) \) but \( i\varphi(x) \), the equal-time canonical commutation relations are

\[ [\varphi(x), \varphi^\dagger(y)] = \delta(x - y), \quad [\varphi(x), \varphi(y)] = [\varphi^\dagger(x), \varphi^\dagger(y)] = 0. \]  

(4.7)

They are also derived [4] by an explicit calculation from those of the electron fields \( \psi(x) \) and \( \psi^\dagger(x) \) based on the definition (3.8).

The CB wave function is defined by

\[ \Phi_\varphi[x] = \langle 0 | \varphi(x_1) \varphi(x_2) \cdots \varphi(x_N) | \Phi \rangle. \]  

(4.8)

The LLL condition (3.10) implies that the wave function \( \Phi_\varphi[x] \) is an analytic function, \( \Phi_\varphi[x] = \omega[z] \). With the use of the formula (3.11) it is an easy exercise to derive the following relation [4],

\[ \varphi^\dagger(x_1) \varphi^\dagger(x_2) \cdots \varphi^\dagger(x_N) | 0 \rangle = \Phi_{LN}[x] \psi^\dagger(x_1) \psi^\dagger(x_2) \cdots \psi^\dagger(x_N) | 0 \rangle, \]  

(4.9)

where \( \Phi_{LN}[x] \) is the Laughlin function (2.2).

Because of the commutation relations (4.7) the state \( | \Phi \rangle \) associated with the CB wave function (4.8) is given by

\[ | \Phi \rangle = \int [dx] \Phi_\varphi[x] \varphi^\dagger(x_1) \varphi^\dagger(x_2) \cdots \varphi^\dagger(x_N) | 0 \rangle \]  

(4.10)

\[ = \int [dx] \omega[z] \Phi_{LN}[x] \psi^\dagger(x_1) \psi^\dagger(x_2) \cdots \psi^\dagger(x_N) | 0 \rangle, \]  

(4.11)

where use was made of (4.9), and \([dx] = d^2x_1 d^2x_2 \cdots d^2x_N \). It follows from (4.11) that the electron wave function is \( \Phi[x] = \omega[z] \Phi_{LN}[x] \), as verifies the basic formulas (2.1) \( \sim \) (2.3).

The semiclassical ground state is the one that minimizes the total energy \( \langle H \rangle \). The Coulomb energy (3.2) is minimized by the state where \( \langle \rho(x) \rangle = 0 \). It is realized when the electron density is homogeneous,

\[ \langle \rho(x) \rangle = \langle \varphi^\dagger(x) \varphi(x) \rangle = e^{2\mathcal{A}(x)} \langle \varphi^\dagger(x) \varphi(x) \rangle = \rho_0. \]  

(4.12)
In the semiclassical approximation we obtain
\[ N \prod_{r=1}^{N} \langle \rho(x_r) \rangle = \prod_{r=1}^{N} e^{2\langle A(x_r) \rangle} \langle \Phi | \varphi^\dagger(x_N) \cdots \varphi^\dagger(x_2) \varphi^\dagger(x_1) \varphi(x_1) \varphi(x_2) \cdots \varphi(x_N) | \Phi \rangle. \] (4.13)

We insert a complete set \( \sum |n\rangle \langle n| = 1 \) between two operators \( \varphi^\dagger(x_1) \) and \( \varphi(x_1) \). When the state \( |\Phi\rangle \) contains \( N \) electrons, only the vacuum term \( |0\rangle \langle 0| \) survives in the complete set because \( \varphi(x_r) \) decreases the electron number by one. Hence, \( N \)-body wave function (4.8) is given by
\[ \Phi_{\varphi}[x] = \rho_0^{N/2} \prod_{r=1}^{N} e^{-\langle A(x_r) \rangle}, \] (4.14)
up to an irrelevant phase factor. On the other hand, to suppress the kinetic energy we impose the LLL condition (3.10), as requires \( \Phi_{\varphi}[x] \) to be analytic. Consequently, it follows that \( \langle A(x) \rangle = \text{constant} \) or \( B_{\text{eff}} = 0 \), which is possible only at \( \nu = 1/m \) from (3.12). Namely, the ground state is realized only at the magic filling factor \( \nu = 1/m \). At \( \nu = 1/m \) the ground-state wave function is given by \( \Phi_{\varphi}[x] = \text{constant} \) in terms of composite bosons, and therefore by the Laughlin wave function (2.2) in terms of electrons. In this way the Laughlin state is proved to be the ground state in the improved CB theory.

For the sake of completeness, we briefly recall the results \([2]\) of the corresponding analysis with use of bare composite bosons. We can derive the equations similar to (4.10) and (4.11), which verifies the mapping (3.7). The semiclassical ground state is similarly given by \( \Phi_{\varphi}[x] = \text{constant} \) in terms of bare composite bosons. The wave function in terms of electrons is singular,
\[ \Phi[x] = e^{im \sum_{r<s} \theta(z_r - z_s)}. \] (4.15)
The state does not belong to the lowest Landau level. We can remove this singular short-distance behavior by including a higher order perturbation correction \([2,15]\). Nevertheless, it makes the naive CB theory less attractive.

V. SEMICLASSICAL ANALYSIS

We analyze excitations on the QH state. A priori two types of excitations are possible, that is, perturbative and nonperturbative ones in terms of the density fluctuation \( \rho(x) \) and its conjugate phase \( \chi(x) \). To carry out a perturbative analysis, we parametrize the bare field as \( \phi(x) = e^{ix(x)} \sqrt{\rho_0 + \varrho(x)} \) in terms of the density deviation \( \varrho(x) \) and its canonical phase \( \chi(x) \). The dressed field \( \varphi(x) \) is a nonlocal operator due to the factor \( e^{-A(x)} \) with (3.14),
\[ \varphi(x) = e^{-A(x)} e^{ix(x)} \sqrt{\rho_0 + \varrho(x)}. \] (5.1)
Substituting (5.1) into the Hamiltonian (4.1) and expanding various quantities in term of \( \varrho(x) \) and \( \chi(x) \), we obtain the perturbative Hamiltonian, which is found to be identical to the one analyzed already in the bare CB theory \([2]\), as should be the case. The result is that there exist no perturbative fluctuations confined to the lowest Landau level. We
conclude that all excitations in the lowest Landau level are nonperturbative objects. The improved theory confirms this assertion by showing explicitly how they are created on the ground state. Indeed, any excited state is represented as in (4.10), which is a nonlocal object because the creation operator $\varphi^\dagger(x)$ is a nonlocal operator as in (5.1).

A. Vortices

We first examine excited states when the wave function (4.8) is factorizable, $\Phi[x] = \omega[z] = \prod_r \omega(z_r)$. In this case we can easily make a semiclassical analysis of the one-point function $\langle \varphi(x) \rangle = \omega(z)$ by setting

$$e^{-A(x)}e^{i\chi(x)}\sqrt{\rho_0 + \varrho(x)} = \omega(z),$$

(5.2)

based on the parametrization (5.1). Here and hereafter, we use the same symbols $A(x)$, $\varrho(x)$ and $\chi(x)$ also for the classical fields. When an analytic function $\omega(z)$ is given, (5.2) is an integral equation determining the density deviation $\varrho(x)$ confined to the lowest Landau level.

We transform (5.2) into a differential equation. The Cauchy-Riemann equation for the analytic function (5.2) yields,

$$\partial_j(A(x) - \ln \sqrt{\rho_0 + \varrho(x)}) = -\varepsilon_{jk}\partial_k\chi(x).$$

(5.3)

Using (3.13) we obtain that

$$\frac{\nu}{4\pi}\nabla^2 \ln \left(1 + \frac{\varrho(x)}{\rho_0}\right) - \varrho(x) = \nu Q^V(x),$$

(5.4)

which we call the soliton equation, where

$$Q^V(x) = \frac{1}{2\pi}\varepsilon_{jk}\partial_j\partial_k\chi(x) = \frac{1}{2\pi}\varepsilon_{jk}\partial_j\partial_k\ln \omega(z)$$

(5.5)

is the topological charge density associated with the excitation. This is nonvanishing since $\ln \omega(z)$ is a multivalued function unless $\omega(z) = \text{constant}$. The topological current is

$$Q^V(\mu)(x) = \frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\chi(x),$$

(5.6)

with $Q^V(x) = Q^V_0(x)$. It is a conserved quantity, $\partial^\nu Q^V(\mu)(x) = 0$. Topological solitons are generated around the zeros of $\omega(z)$ according to the soliton equation (5.4). The density modulation is induced in order to confine the excitation within the lowest Landau level.

The topological charge is evaluated as

$$Q^V = \int d^2x Q^V(x) = \frac{1}{2\pi}\oint d\mathbf{x}_k\partial_k\ln \omega(z),$$

(5.7)

where the loop integration $\oint$ is made to encircle the excitation at infinity ($|x| \to \infty$) provided $\omega(z)$ is regular everywhere. The topological charge is uniquely determined by the asymptotic behavior of the classical field $\varphi(x)$,
\[
\varphi(x) \to \sqrt{\rho_0} z^q, \quad \text{as} \quad |z| \to \infty,
\] (5.8)

for which the electron number is
\[
\Delta N = \int d^2 x \vartheta(x) = -\nu Q_V = -\nu q,
\] (5.9)

as follows from the soliton equation (5.4). The electric charge carried by this soliton is \(-e\Delta N = \nu q e\). It is a hole made in the condensate of composite bosons. We may as well derive the electric number (5.9) more directly from the parametrization (5.2). We take the asymptotic behavior \(|z| \to \infty\) in (5.2) and equate it with (5.8),
\[
\vartheta(x) \to 0, \quad \chi(x) \to q \theta, \quad \mathcal{A}(x) \to -q \ln |z|, \quad \text{as} \quad |z| \to \infty.
\] (5.10)

Taking the limit \(|x| \to \infty\) in (3.14) we recover the quantization of the electron number (5.9). Actually we have determined the coefficient \(\sqrt{\rho_0}\) in the asymptotic behavior (5.8) in this way.

A topological excitation carries a quantized topological charge, which has to be created all at once. It cannot be created by an accumulation of perturbative fluctuations, as agrees with the perturbative result mentioned at the beginning of this section. In the absence of the Coulomb term all these excitations are degenerate with the ground state, which explains the degeneracy in the lowest Landau level at \(\nu < 1\). The degeneracy is removed since any density modulation acquires a Coulomb energy,
\[
\langle H_C \rangle = \frac{e^2}{2\varepsilon} \int d^2 x d^2 y \frac{\varrho(x) \varrho(y)}{|x-y|} = \gamma \nu^2 q^2 \frac{e^2}{\varepsilon \ell_B},
\] (5.11)

where \(\gamma\) is a constant of order one.

An explicit example is given by a vortex (quasihole) sitting at \(x = 0\), whose wave function is \(\Phi_\varphi[x] = \prod_r z_r\) up to a normalization factor, or \(\omega(z) = \sqrt{\rho_0} z\). In this example the topological charge (5.5) is concentrated at the vortex center, \(Q^V_0(x) = \delta(x)\). A crude approximation of the soliton equation (5.4) reads
\[
\frac{\ell_B^2}{2} \nabla^2 \varrho(x) - \varrho(x) = \nu \delta(x),
\] (5.12)

since \(\nu = 2\pi \rho_0 \ell_B^2\). Its exact solution is \(\varrho(x) = -(\nu/\pi \ell_B^2) K_0(s)\) with \(s = \sqrt{2r}/\ell_B\) and \(K_0(s)\) the modified Bessel function. This is a rather poor approximation because of its singular behavior, \(\varrho(x) \to -\infty\), at the vortex center \((x = 0)\). A better approximation is given by
\[
\varrho(x) = -\rho_0 \left(1 + s - \frac{s^2}{6}\right) e^{-s},
\] (5.13)

which has the correct behavior both at \(x = 0\) and \(|x| \gg \ell_B\). Furthermore, it has the correct topological charge. For a numerical analysis, it is convenient to set \(\varrho(x) = \rho_0 e^{u(s)}\) in the soliton equation (5.4), as yields \[10\],
\[
\frac{d^2 u}{ds^2} + \frac{1}{s} \frac{du}{ds} + 1 = e^{u(s)},
\] (5.14)

for \(s > 0\). The result of a numerical analysis shows that the density modulation is well approximated by (5.13), as in Fig.1. The Coulomb energy is given by (5.11) with \(\gamma \simeq 0.39\).
VI. QUANTUM HALL FERROMAGNET

We proceed to analyze the QH system with the SU(2) symmetry. The electron field $\psi^\alpha(x)$ has the index $\alpha = \uparrow, \downarrow$. It denotes the electron spin in the monolayer QH system with the spin SU(2) symmetry, or the layer index in a certain bilayer QH system with the pseudospin SU(2) symmetry. For definiteness we analyze the monolayer spin system in this paper.

A. Hamiltonian

The Hamiltonian depends on the electron spin through the Zeeman energy term,

$$H_Z = -g^* \mu_B B \int d^2 x S_z(x),$$

with $S_z = \frac{1}{2}(\psi_{\uparrow}^{\dagger}\psi_{\uparrow} - \psi_{\downarrow}^{\dagger}\psi_{\downarrow})$, where $g^*$ is the gyromagnetic factor and $\mu_B$ the Bohr magneton. Each Landau level contains two energy levels with the one-particle gap energy $g^* \mu_B B$. The lowest Landau level is filled at $\nu = 2$. We consider the case where the Zeeman energy is much smaller than the Coulomb energy. Though one Landau level contains two degenerate energy levels in the vanishing limit of the Zeeman coupling ($g^* = 0$), the system becomes incompressible at $\nu = 1/m$. The physical reason is the Coulomb exchange energy, as we now see.

We define the bare CB field $\phi^\alpha(x)$ and the dressed CB field $\varphi^\alpha(x)$ by

$$\phi^\alpha(x) = e^{-i\Theta(x)} \psi^\alpha(x), \quad \varphi^\alpha(x) = e^{-A(x)} \phi^\alpha(x),$$

where the phase field $\Theta(x)$ and the auxiliary field $A(x)$ are given by (3.5) and (3.14), respectively, with the total electron density $\rho(x) = \sum_{\alpha} \psi_{\alpha}^{\dagger}(x)\psi_{\alpha}(x) = \sum_{\alpha} \varphi_{\alpha}^{\dagger}(x)e^{2A(x)}\varphi_{\alpha}(x)$. The Hamiltonian is

$$H = \frac{1}{2M} \sum_{\alpha} \int d^2 x \psi_{\alpha}^{\dagger}(x)(P_x^2 + P_y^2)\psi_{\alpha}(x) + H_C + H_Z$$

$$= \omega_e \sum_{\alpha} \int d^2 x \left( \frac{\partial}{\partial z^*} \varphi_{\alpha}(x) \right)^{\dagger} e^{2A(x)} \frac{\partial}{\partial z^*} \varphi_{\alpha}(x) + \frac{N}{2} \omega_e + H_C + H_Z,$$

with the Coulomb term $H_C$ and the Zeeman term $H_Z$. The Coulomb term depends on the deviation $\varrho(x)$ of the total electron density from the average density, $\varrho(x) = \rho(x) - \rho_0$, as in (3.2).

B. Wave Function

We may decompose the bare CB field into the U(1) field $\phi(x)$ and the SU(2) field $n^\alpha(x)$,

$$\phi^\alpha(x) = \phi(x)n^\alpha(x), \quad \sum_{\alpha} n_{\alpha}^{\dagger}(x)n^\alpha(x) = 1.$$
The field \( n^\alpha(x) \) is the complex-projective (CP) field \([13]\), whose overall phase has been removed and given to the U(1) field \( \phi(x) \). The spin operator is expressed as

\[
S^a(x) = \frac{1}{2} \rho(x) \Sigma^a(x),
\]

(6.6)

where

\[
\Sigma^a(x) = n^\dagger(x) \tau^a n(x), \quad n(x) = \begin{pmatrix} n^\dagger(x) \\ n^\dagger(x) \end{pmatrix}.
\]

(6.7)

In terms of the dressed CB field the decomposition reads

\[
\varphi^\alpha(x) = \varphi(x) n^\alpha(x), \quad \varphi(x) = e^{-A(x)} \phi(x).
\]

(6.8)

The SU(2) component \( n^\alpha(x) \) is common between the bare and dressed fields (6.5) and (6.8): It is a local field. On the other hand, \( \varphi(x) \) is a nonlocal field due to the factor \( e^{-A(x)} \) as in the spinless theory.

The ground state minimizes both the Coulomb and Zeeman energies. The Coulomb energy is minimized by the homogeneous electron density, \( \langle \rho(x) \rangle = \rho_0 \). The Zeeman energy is minimized when all electrons are polarized into the positive \( z \) axis, \( \langle n^\uparrow(x) \rangle = 1 \) and \( \langle n^\downarrow(x) \rangle = 0 \). The ground state is unique, which we denote by \( |g_0\rangle \).

The two-component CB field is \( \Phi(x) = \varphi(x)n(x) \). With the Hamiltonian (6.4), the LLL condition for the state \( |\Phi\rangle \) is

\[
\frac{\partial}{\partial z^*} \Phi(x)|\Phi\rangle = 0.
\]

(6.9)

Because of this condition the \( N \)-body wave function is analytic,

\[
\Phi_\varphi[x] = \langle 0| \Phi(x_1) \Phi(x_2) \cdots \Phi(x_N) |\Phi\rangle = \Omega[z],
\]

(6.10)

where \( \Omega[z] \) is totally symmetric in \( N \) variables. When it is factorizable, \( \Omega[z] = \prod_r \Omega(z_r) \), the electron wave function reads

\[
\Phi[x] = \prod_r \begin{pmatrix} \omega^\dagger(z_r) \\ \omega^\dagger(z_r) \end{pmatrix} \Phi_{LN}[x].
\]

(6.11)

The ground-state wave function is

\[
\Phi[x] = \prod_r \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Phi_{LN}[x],
\]

(6.12)

where all spins are polarized.

C. Spin Texture

We consider a spin texture given by performing an SU(2) transformation on the ground state \( |g_0\rangle \),
\[ |\Phi| = e^{i\Omega}|g_0\rangle, \tag{6.13} \]

where \( \Omega \) is its generator,
\[ \Omega = \sum_a \int \! d^2 x f^a(x) S^a(x) = \sum_a \int \! d^2 q f^a_{-q} \hat{S}^a_q. \tag{6.14} \]

The spin texture is described by the classical sigma field \( s^a(x) = \langle \Phi| \Sigma^a(x)|\Phi \rangle \), which we parametrize as
\[ s^x(x) = \sigma(x), \quad s^y(x) = \sqrt{1 - \sigma^2(x)} \sin \vartheta(x), \quad s^z(x) = \sqrt{1 - \sigma^2(x)} \cos \vartheta(x). \tag{6.15} \]

It is classified by the Pontryagin number \[13\],
\[ Q^P_\mu = \int \! d^2 x Q^P_\mu(x), \]
with the topological current
\[ Q^P_\mu(x) = \frac{1}{8\pi} \varepsilon_{abc} \varepsilon_{\mu\nu\lambda} s^a \partial^\nu s^b \partial^\lambda s^c. \tag{6.16} \]

It is absolutely conserved, \( \partial^\mu Q^P_\mu = 0 \).

The spin texture \[6.13\] does not belong to the lowest Landau level. The excitation energy is naively given by \( \langle \Psi| H |\Psi \rangle \) with the Hamiltonian \[6.4\], to which the kinetic term and the Zeeman term contribute but the Coulomb term does not. The kinetic energy is of the order of \( \omega_c \), as implies the spin stiffness of this order. Such an excitation is impossible at sufficiently low temperature. It is necessary to excite merely the component \( |\hat{\Phi}\rangle \) of \( |\Phi\rangle \) belonging to the lowest Landau level. Furthermore, it is also necessary to extract the LLL component of the potential term since it kicks out the state out of the lowest Landau level.

### D. LLL Projection

The LLL component \( |\hat{\Phi}\rangle \) is extracted from the spin texture \( |\Phi\rangle \) by extracting the LLL component \( \langle f^a(x) \rangle \) from the “wave packet” \( f^a(x) \) in the generator \( \hat{\Omega} \) of the SU(2) transformation \[6.14\]. This turns out to replace the plane wave \( e^{ixq} \) with \[17,18\]
\[ \langle e^{ixq} \rangle \equiv e^{-\frac{1}{2} q^2 \ell^2_B} e^{iXq} \tag{6.17} \]

in the Fourier representation of \( f^a(x) \), where \( X = (X,Y) \) is the guiding center. We call \( \langle e^{ixq} \rangle \) the LLL projection of \( e^{ixq} \). The generator \[6.14\] is projected as
\[ \hat{\Omega} = \sum_a \int \! d^2 x \langle f^a(x) \rangle S^a(x) = \sum_a \int \! d^2 q f^a_{-q} \hat{S}^a_q, \tag{6.18} \]
where
\[ \hat{S}_q^a = (2\pi)^{-1} \int \! d^2 x \langle e^{-iqx} \rangle S^a(x). \tag{6.19} \]

Similarly we define
\[ \hat{\rho}_q = (2\pi)^{-1} \int \! d^2 x \langle e^{-iqx} \rangle \rho(x) \tag{6.20} \]
for the electron density operator.

From the commutation relation \([X, Y] = -i \ell_B^2\) between the \(X\) and \(Y\) components of the guiding center, we obtain

\[
[e^{iqX}, e^{ipX}] = 2ie^{i(q+p)X} \sin[\ell_B^2 \frac{q \wedge p}{2}],
\]

(6.21)

with \(q \wedge p \equiv qxpy - qypx\). The translation \(e^{iqx}\) is Abelian, but the magnetic translation \(e^{iqX}\) is non-Abelian. It governs the symmetric structure of the two-dimensional space after the LLL projection. It is straightforward to derive the following \(W_\infty \times SU(2)\) algebra [18],

\[
[\hat{\rho}_p, \hat{\rho}_q] = \frac{i}{\pi} \hat{\rho}_{p+q} \sin[\ell_B^2 \frac{pq}{2}] \exp[\ell_B^2 \frac{pq}{2}],
\]

(6.22a)

\[
[\hat{S}_p, \hat{\rho}_q] = \frac{i}{\pi} \hat{S}_{p+q} \sin[\ell_B^2 \frac{pq}{2}] \exp[\ell_B^2 \frac{pq}{2}],
\]

(6.22b)

\[
[\hat{S}_p, \hat{S}_q] = \frac{i}{2\pi} \varepsilon^{abc} \hat{S}_c \hat{S}_{p+q} \cos[\ell_B^2 \frac{pq}{2}] \exp[\ell_B^2 \frac{pq}{2}]
\]

\[+ \frac{i}{4\pi} \delta^{ab} \hat{\rho}_{p+q} \sin[\ell_B^2 \frac{pq}{2}] \exp[\ell_B^2 \frac{pq}{2}],
\]

(6.22c)

based on the algebra (6.21) of the magnetic translation.

We make the LLL projection of the Hamiltonian (6.4), whose result we denote by \(\hat{H}\).

The Coulomb term reads

\[
\hat{H}_C = \frac{1}{2} \int d^2x d^2y \rho(x) \langle \langle V(x-y) \rangle \rangle \rho(y) = \pi \int d^2q V(q) \hat{\rho}_{-q} \hat{\rho}_q,
\]

(6.23)

where \(V(q)\) is the Fourier transformation of the potential \(V(x) = e^2/\varepsilon|x|\).

E. Exchange Energy

We evaluate the energy \(\langle \hat{\Phi}|\hat{H}|\hat{\Phi} \rangle\) by making a perturbative expansion of the spin texture around the ground state,

\[
H_{\text{eff}} \equiv \langle \hat{\Phi}|\hat{H}|\hat{\Phi} \rangle = \langle g_0|\hat{H}|g_0 \rangle - \langle g_0|\hat{O}, \hat{H}|g_0 \rangle + \cdots.
\]

(6.24)

Making a straightforward algebraic calculation, making a gradient expansion and taking the lowest order term in \(f_{-q}^a\), we obtain the exchange energy [18,14,19] from the Coulomb term (5.23),

\[
H_{\text{stiff}} = \frac{1}{2} \rho_s \sum_a \int d^2x \partial_k s^a(x))^2,
\]

(6.25)

where \(s^a(x)\) is the nonlinear sigma field with \(\rho_s = \nu e^2/(16\sqrt{2\pi\varepsilon\ell_B})\). It describes the spin stiffness. Combining the exchange energy and the Zeeman energy we obtain the effective Hamiltonian,

\[
H_{\text{eff}} = \frac{1}{2} \rho_s \sum_a \int d^2x \partial_k s^a(x))^2 - \frac{\rho_0}{2} g^* \mu_B B \int d^2x \tilde{s}^a(x),
\]

(6.26)
The exchange energy arises because of the following reason: The local spin rotation has components in higher Landau level since it is not a symmetry of the Hamiltonian. Only its LLL component is excited at sufficiently low temperature. Since the LLL components of the spin operators and the density operator do not commute as in (6.22b), the local spin rotation induces a local density modulation and affects the Coulomb energy.

The spin-stiffness term is precisely the nonlinear-sigma-model Hamiltonian [8]. Though the term is derived perturbatively in the present framework, there is much numerical evidence [8,20,21] that it captures correctly the long-distance physics associated with Skyrmion excitations.

### F. Goldstone Mode

The Zeeman effect is quite small in actual samples. We consider the vanishing limit of the Zeeman term ($g^* = 0$). According to the effective Hamiltonian (6.26), the energy is minimized for any constant value of the sigma field, $s(x) = s_0 =$constant. Hence, there exists a degeneracy in the ground states as indexed by $s_0$. The choice of a ground state implies a spontaneous magnetization, or a quantum Hall ferromagnetism. When a continuous symmetry is spontaneously broken, there should arise a gapless mode known as the Goldstone mode and quantum coherence develops spontaneously.

When we include a small Zeeman effect, the ground state $|g_0\rangle$ is chosen where $s_0 = (0, 0, 1)$. We can treat the Zeeman interaction as a perturbation because it is much less important than the Coulomb interaction. The key property of the QH ferromagnet is that it is a macroscopic coherent state though the coherent length is finite, where all spin components are simultaneously measurable, $s^a(x) = 2\rho_0^{-1}\langle S^a(x)\rangle$, with extremely good accuracy. An evidence is the existence of coherent excitations such as Skyrmions.

The Goldstone mode describes small fluctuations of the CP field around the ground state (6.12). Up to the lowest order of the perturbation in the CP field $n(x)$, it is parametrized as [18]

$$n^\dagger(x) = 1, \quad n^\dagger(x) = \frac{\zeta(x)}{\sqrt{\rho_0}} \quad (6.27)$$

with $[\zeta(x), \zeta^\dagger(y)] = \delta(x - y)$. The LLL condition (6.3) yields two conditions,

$$\frac{\partial}{\partial z^*} \varphi(x)|\Phi\rangle = 0, \quad \frac{\partial}{\partial z^*} \zeta(x)|\Phi\rangle = 0, \quad (6.28)$$

up to this order. Although they look similar, they describe very different excitation modes. As in the spinless QH system, $\varphi(x)$ is a nonlocal field and generates extended objects. On the other hand, $\zeta(x)$ is a local field, and it describes the Goldstone mode.

We may relate $\zeta(x)$ to the classical sigma field (6.13),

$$\langle \zeta(x) \rangle = \frac{\sqrt{\rho_0}}{2} \{\sigma(x) + i\vartheta(x)\}. \quad (6.29)$$

The effective Hamiltonian (6.26) is recognized as a classical counterpart of the quantum version,
\[ H_{\text{eff}} = \frac{2\rho_s}{\rho_0} \int d^2 x [\partial_k \zeta^\dagger (x) \partial_k \zeta(x) + \xi_L^{-2} \zeta^\dagger (x) \zeta(x)], \]  

on the coherent state. Here, \( \xi_L \) is the coherent length,

\[ \xi_L = \sqrt{\frac{2\rho_s}{g^* \mu_B B \rho_0}} = \frac{(2\pi)^{1/4} \ell_B}{2\sqrt{2\tilde{g}}}, \]

where

\[ \tilde{g} = \frac{g^* \mu_B B}{e^2 / \varepsilon \ell_B} \]

is the ratio of the Zeeman energy to the Coulomb energy. We call it the normalized g-factor. We have \( \xi_L \sim 4\ell_B \) in typical samples at \( B \simeq 10 \text{ Tesla} \), where \( \tilde{g} \simeq 0.02 \). In the momentum space the effective Hamiltonian reads

\[ H_{\text{eff}}(k) = E_k \zeta^\dagger_k \zeta_k, \]

with \( [\zeta_k, \zeta_l^\dagger] = \delta(k - l) \), and the dispersion relation is

\[ E_k = \frac{2\rho_s}{\rho_0} k^2 + g^* \mu_B B. \]

The Goldstone mode has acquired a gap \( E_0 = g^* \mu_B B \).

**VII. TOPOLOGICAL EXCITATIONS**

We analyze topological excitations on the QH ferromagnet. We use the semiclassical approximation. When the \( N \)-body wave function is factorizable, the one-point function is analytic, \( \langle \phi^a(x) \rangle = \omega^a(z) \). From (5.8), the one-point function is parametrized as

\[ e^{-A(x)} \omega(x) \sqrt{\rho_0 + \rho(x)} n^a(x) = \omega^a(z), \]

since \( |\phi(x)|^2 = \rho_0 + \rho(x) \). Here and hereafter, all fields are classical fields. When the wave function \( \omega^a(z) \) is given, the electron density \( \rho(x) \) and the spin field \( S^a(x) \) are determined by this equation. There are two types of excitations associated with the U(1) part and the SU(2) part of the CB field. The U(1) excitation has a characteristic length \( \ell_B \), while the SU(2) excitation has no scale provided the Zeeman term is neglected.

**A. Vortex Excitations**

The U(1) excitation is generated on the spin-polarized ground state (6.12) when \( \partial_k \chi(x) \neq 0 \) and \( \partial_k n^a(x) = 0 \) in (7.1). We may set \( \langle \phi^\dagger \rangle = 0 \). The one-point function \( \langle \phi^\dagger (x) \rangle \) is essentially Abelian, and the Cauchy-Riemann equation for (7.1) yields precisely the same soliton equation (5.4). The topological charge density is given by (5.3) with an analytic function \( \omega(z) = \omega^\dagger(z) \). The U(1) excitation is the vortex. The Coulomb energy of the vortex excitation is given by (5.11) with \( \gamma \simeq 0.39 \). There is no antivortex excitation. Instead of it an electron is placed into the spin-down state, as would increase the Coulomb energy of the same order as the vortex excitation and the Zeeman energy.
B. Skyrmion Excitations

The SU(2) excitation is generated on the spin-polarized ground state (6.12) when \( \partial_k \chi(x) = 0 \) and \( \partial_k n^\alpha(x) \neq 0 \) in (7.1). The CP field is

\[
n^\alpha(x) = \frac{\omega^\alpha(z)}{\sqrt{\omega^\dagger(z)\omega(z)}}.
\]  

yielding the wave function

\[
\Psi_{\text{Skyrmion}}[x] = \prod_r \left( \frac{\omega^\dagger(z_r)}{\omega^\dagger(z_r)} \right)_r \Phi_{\text{LN}}[x].
\]  

A simplest choice is given by

\[
\Psi_{\text{Skyrmion}}[x] = \left( \frac{z^q}{(\kappa/2)^q} \right) \Phi_{\text{LN}}[x],
\]  

with a positive integer \( q \), which describes a classical Skyrmion with scale \( \kappa \) sitting at the origin of the system. It is clear in (7.4) that the Skyrmion is reduced to the vortex in the limit \( \kappa \to 0 \), where there is no distinction between the U(1) and SU(2) excitations. Because the vortex is regarded as the small limit of the skyrmion, we do not make a clear distinction between them in the QH ferromagnet.

For the Skyrmion (7.4) the classical sigma field (6.15) is calculated as

\[
s^x = \sqrt{1 - (s^z)^2} \cos(q\theta), \quad s^y = -\sqrt{1 - (s^z)^2} \sin(q\theta), \quad s^z = \frac{r^{2q} - (\ell_B\kappa)^{2q}}{r^{2q} + (\ell_B\kappa)^{2q}},
\]  

and the Pontryagin number density (2.11) as

\[
Q^P(x) = \frac{q^2}{\pi} \frac{r^{2q-2}(\ell_B\kappa)^{2q}}{[r^{2q} + (\ell_B\kappa)^{2q}]^2}.
\]  

The spin flips at the Skyrmion center, \( s = (0,0,-1) \) at \( r = 0 \), while the spin-polarized ground state is approached away from it, \( s = (0,0,1) \) for \( r \gg \kappa\ell_B \).

A Skyrmion excitation modulates not only the SU(2) part but also the U(1) part via the relation (7.1). The Cauchy-Riemann equation reads

\[
\partial_j(A(x) - \ln \sqrt{\rho_0 + \varrho(x)}) = -\varepsilon_{jk} K_k,
\]  

where

\[
K_k = -i \sum_\alpha n^\alpha \partial_k n^\alpha.
\]  

From (7.1) the same soliton equation as (5.4) is derived,

\[
\frac{\nu}{4\pi} \nabla^2 \ln \left( 1 + \frac{\varrho(x)}{\rho_0} \right) - \varrho(x) = \nu Q^P(x),
\]  

where \( \nu = \frac{\ell_B}{\kappa} \).
but the topological charge density now reads

\[ Q^P(x) = \frac{1}{2\pi} \varepsilon_{j} \partial_j K_k(x). \quad (7.10) \]

It is a straightforward calculation [13] to show that the charge (7.10) is identical to the Pontryagin number density (2.11).

The topological charge is evaluated as

\[ Q^P = \int d^2x Q^P(x) = \frac{1}{2\pi i} \oint d^x j_k K_j(x), \quad (7.11) \]

where the loop integration \( \oint \) is made to encircle the excitation at infinity (\( |x| \to \infty \)). The electron number associated with the topological soliton is

\[ \Delta N = \int d^2x \varrho(x) = -\nu Q^P. \quad (7.12) \]

The topological charge is determined by the asymptotic value of the CP field (7.2). We find \( Q^P = q \) for the Skyrmion (7.5). The electron number of this soliton is \( \Delta N = -\nu q \): It represents the number of electrons removed by the Skyrmion excitation.

The Skyrmion spin is given by

\[ \Delta N_s = -\frac{1}{2} \int d^2x \{2S^z(x) - \rho_0\} = \frac{1}{2} \int d^2x \{\rho_0 - \rho(x)s^z(x)\}. \quad (7.13) \]

It seems to diverge logarithmically for the Skyrmion (7.5) with \( q = 1 \). This is a fake since the Zeeman term breaks the spin SU(2) symmetry explicitly and introduces a coherent length \( \xi_L \) into the SU(2) component. The Skyrmion configuration (7.5) is valid only within the coherent domain because the coherent behavior of the spin texture is lost outside it. By cutting the upper limit of the integration at \( r = \kappa \xi_L/2 \) in (7.13), we obtain

\[ \Delta N_s = \frac{\kappa^2}{2} \ln\left(\frac{\xi_L^2}{4\ell_B} + 1\right) = \frac{\kappa^2}{2} \ln\left(\frac{\sqrt{2\pi}}{32\tilde{g}} + 1\right), \quad (7.14) \]

where \( \xi_L \) is the coherent length given by (6.31) and \( \tilde{g} \) is the normalized g-factor (6.32).

The density modulation around the Skyrmion is governed by the soliton equation (7.9). This equation has formally the same expression as the soliton equation (5.4) for the vortex excitation. We may obtain an approximate solution in the two limits, the large Skyrmion limit (\( \kappa \gg 1 \)) and the small Skyrmion limit (\( \kappa \ll 1 \)). First, in the large limit we may solve (7.9) iteratively as

\[ \varrho(x) = -\nu Q^P(x) - \frac{\nu^2}{8\pi \rho_0} \nabla^2 Q^P(x) + \cdots. \quad (7.15) \]

We may approximate it as

\[ \varrho(x) \simeq -\nu Q^P(x) = \frac{\nu}{\pi} \frac{(\ell_B \kappa)^2}{r^2 + (\ell_B \kappa)^2}, \quad \text{for} \quad \kappa \gg 1, \quad (7.16) \]
for the Skyrmion with \( q = 1 \), where we have used (7.6). It agrees with the formula due to Sondhi et al. [8]. On the other hand, the topological charge \( Q^P(x) \) is localized in the small limit, \( Q^P(x) \to g \delta(x) \) as \( \kappa \to 0 \) in (7.6). Hence, the solution is given by the vortex configuration,

\[
g(x) \simeq -\rho_0 \left( 1 + \frac{\sqrt{2}r}{\ell_B} - \frac{r^2}{3\ell_B^2} \right) e^{-\sqrt{\frac{2}{\pi}}r/\ell_B}, \quad \text{for } \kappa \ll 1,
\]

which has been derived in (5.13).

We evaluate the excitation energy of a Skyrmion with \( q = 1 \) at \( \nu = 1 \). It consists of the exchange energy, the electrostatic term and the Zeeman term, where the exchange energy (6.25) is exactly calculable for one Skyrmion,

\[
E_{\text{Skyrmion}} = 4\pi \rho_s + \frac{e^2}{2\varepsilon} \int d^2x d^2y \frac{g(x) g(y)}{|x-y|} + g^* \mu_B B \Delta N_s,
\]

where the Skyrmion spin \( \Delta N_s \) is given by (7.13). It is calculated by using (7.16) and (7.14) for a large Skyrmion,

\[
E_{\text{Skyrmion}} = \frac{e^2}{\varepsilon \ell_B} \left[ \sqrt{\frac{\pi}{32}} + \frac{\beta}{\kappa} + \frac{\tilde{g}}{2} \ln \left( \frac{\sqrt{2\pi}}{32\tilde{g}} + 1 \right) \right],
\]

with \( \beta = 3\pi^2/64 \). For a sufficiently small Skyrmion, the Coulomb energy is calculated with the vortex configuration and is given by (5.11).

The Coulomb energy increases for a smaller Skyrmion while the Zeeman energy increases for a larger Skyrmion. The optimized scale \( \kappa \) is obtained by minimizing the total energy (7.19),

\[
\kappa = \beta^{1/3} \left\{ \frac{\tilde{g}}{2} \ln \left( \frac{\sqrt{2\pi}}{32\tilde{g}} + 1 \right) \right\}^{-1/3}.
\]

The Skyrmion excitation energy is given by (7.19) together with (7.20).

The parameter \( \beta \) describes a strength of the Coulomb energy. In general it is a function of the size \( \kappa \), obeying \( \beta(\kappa) \to 0.39\kappa \) as \( \kappa \to 0 \) and \( \beta(\kappa) \to 3\pi^2/64 \) as \( \kappa \to \infty \). We expand it around an arbitrary point \( \kappa_0 \) as \( \beta(\kappa) = \beta_0 + \beta_1 (\kappa - \kappa_0) \). The \( \kappa \)-dependence of the Skyrmion energy is the same as in (7.13) with \( \beta = \beta_0 - \beta_1 \kappa_0 \). The optimized scale is given by (7.20) with the same replacement. The parameter \( \beta \) is calculable only if the soliton equation (7.9) is solved for a Skyrmion with arbitrary scale \( \kappa \). Furthermore, there will be a correction to it from a finite thickness of the layer. In this paper we treat \( \beta \) as a phenomenological parameter.

The creation energy of a Skyrmion-antiSkyrmion pair will be given by \( 2E_{\text{Skyrmion}} \) if they are sufficiently far apart one another. However, it is not this quantity that is observed experimentally. The activation energy \( \Delta \) is usually determined by the Arrhenius formula,

\[
R_{xx} \propto \exp\left[ -\frac{\Delta}{2kT} \right],
\]

where \( R_{xx} \) is the magnetoresistivity. Various factors affect the activation energy [1]. For instance, impurities make Coulomb potentials around them and lead to a Landau-level
broadening. Phenomenologically, their effects result in a subtraction of a certain amount of offset $\Gamma_{\text{offset}}$ from the Skyrmion creation energy,

$$\Delta = 2E_{\text{Skyrmion}} - \Gamma_{\text{offset}},$$  \hspace{1cm} (7.22)

where $\Gamma_{\text{offset}}$ increases with disorder. In FIG. 2 we have fitted the experimental data due to Schmeller et al. \[11\] by the formula (7.19), where we have used $\beta = 0.24$ and an appropriate offset $\Gamma_{\text{offset}}$ phenomenologically for each curve. The theoretical curve reproduces all the data remarkably well. The Skyrmion spin is estimated that $\Delta N_s \simeq 3.7$ at $\tilde{g} = 0.015$ ($B = 3.05$ Tesla). This estimation is consistent with the Hartree-Fock result \[4\] and the experimental result based on Knight-shift measurements \[10\].

**VIII. DISCUSSIONS**

We have studied QH ferromagnets based on the improved CB theory. We have investigated excitations confined to the lowest Landau level. Charged excitations are Skyrmions, carrying the quantized charge $\nu e$ at the filling factor $\nu = 1/m$. Small Skyrmions are identified with vortices. We have successfully explained the experimental data due to Schmeller et al. \[11\].

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REFERENCES

[1] The Quantum Hall Effect, edited by S. Girvin and R. Prange (Springer-Verlag, New York, 1990), 2nd ed.
[2] S.M. Girvin and A.H. MacDonald: Phys. Rev. Lett. 58 (1987) 1252; S.C. Zhang, T.H. Hansen and S. Kivelson: Phys. Rev. Lett. 62 (1989) 82; Z.F. Ezawa and A. Iwazaki: Phys. Rev. B 43 (1991) 2637; Z.F. Ezawa, M. Hotta and A. Iwazaki: Phys. Rev. B 46 (1992) 7765; S.C. Zhang: Int. J. Mod. Phys. B 6 (1992) 25.
[3] N. Read: Phys. Rev. Lett. 62 (1989) 86.
[4] R. Rajaraman and S.L. Sondhi: Int. J. Mod. Phys. B 10 (1996) 793.
[5] J.K. Jain: Phys. Rev. Lett. 63 (1989) 199.
[6] A. Lopez and E. Fradkin: Phys. Rev. B 44 (1991) 5246; V. Kalmeyer and S.C. Zhang: Phys. Rev. B 46 (1992) 9889; B.I. Halperin, P.A. Lee and N. Read: Phys. Rev. B 47 (1993) 7312.
[7] R.B. Laughlin: Phys. Rev. Lett. 50 (1983) 1395.
[8] S.L. Sondhi, A. Karlhede, S. Kivelson and E.H. Rezayi: Phys. Rev. B 47 (1993) 16419; D.H. Lee and C.L. Kane: Phys. Rev. Lett. 64 (1990) 1313.
[9] H.A. Fertig, L. Brey, R. Côté and A.H. MacDonald: Phys. Rev. B 50 (1994) 11018; M. Abolfath, J.J. Palacios, H.A. Fertig, S.M. Girvin and A.H. MacDonald: Phys. Rev. B 56 (1997) 6795.
[10] S.E. Barrett, G. Dabbagh, L.N. Pfeiffer, K.W. West and R. Tycko: Phys. Rev. Lett. 74 (1995) 5112; E.H. Aifer, B.B. Goldberg and D.A. Broido: Phys. Rev. Lett. 76 (1996) 680.
[11] A. Schmeller, J.P. Eisenstein, L.N. Pfeiffer and K.W. West: Phys. Rev. Lett. 75 (1995) 4290; D.K. Maude, M. Potemski, J.C. Portal, M. Henini, L. Eaves, G. Hill and M.A. Pate: Phys. Rev. Lett. 76 (1996) 4604.
[12] Z.F. Ezawa: Physica B 249–251 (1998) 841; Z.F. Ezawa: Phys. Lett. A 249 (1998) 223.
[13] A.A. Belavin and A.M. Polyakov: JETP Letters 22 (1975) 245; A. D’Adda, A. Luscher and P. DiVecchia: Nucl. Phys. B146 (1978) 63; D.J. Gross: Nucl. Phys. B132 (1978) 439.
[14] K. Moon, H. Mori, K. Yang, S.M. Girvin, A.H. MacDonald, L. Zheng, D. Yoshioka and S.C. Zhang: Phys. Rev. B 51 (1995) 5138.
[15] Z.F. Ezawa and A. Iwazaki: Int. J. Mod. Phys. B 6 (1992) 3205; Z.F. Ezawa and A. Iwazaki: Phys. Rev. B 47 (1993) 7295; Z.F. Ezawa and A. Iwazaki: Phys. Rev. B 48 (1993) 15189.
[16] Z.F. Ezawa, M. Hotta and A. Iwazaki: Phys. Rev. D 44 (1991) 452.
[17] S.M. Girvin, A.H. MacDonald and P.M. Platzman: Phys. Rev. B 33 (1986) 2481; S.M. Girvin and T. Jach: Phys. Rev. B 29 (1984) 5617.
[18] Z.F. Ezawa: Phys. Lett. A 229 (1997) 392; Z.F. Ezawa: Phys. Rev. B 55 (1997) 7771.
[19] C. Kallin and B.I. Halperin: Phys. Rev. B 30 (1984) 5655.
[20] D. Lilliehöök, K. Leijnell, A. Karlhede and S.L. Sondhi: Phys. Rev. B 56 (1997) 6805.
[21] E.H. Rezayi: Phys. Rev. B 56 (1997) R7104; D. Yoshioka: J. Phys. Soc. Jpn. 67 (1998) 3356.
FIGURES

FIG. 1. The density modulation around a vortex with \( q = 1 \) is plotted. The solid curve is obtained by solving the differential equation (5.14) numerically. The dashed curve is drawn by using the approximate formula (5.13).

FIG. 2. Theoretical curves versus experimental data due to Schmeller et al. [11] for the activation energy in QH ferromagnets. All the data are fitted excellently by the theoretical formula (7.22) with (7.19), where an appropriate offset \( \Gamma_{\text{offset}} \) is assumed for each sample.