THE NUMBER OF LINES IN A MATROID WITH NO $U_{2,n}$-MINOR

JIM GEELEN AND PETER NELSON

This paper is dedicated to the memory of Michel Las Vergnas.

Abstract. We show that, if $q$ is a prime power at most 5, then every rank-$r$ matroid with no $U_{2,q+2}$-minor has no more lines than a rank-$r$ projective geometry over $\text{GF}(q)$. We also give examples showing that for every other prime power this bound does not hold.

1. Introduction

This paper is motivated by the following special case of a conjecture due to Bonin; see Oxley [3, p. 582].

Conjecture 1.1. For each prime power $q$ and positive integer $r$, every rank-$r$ matroid with no $U_{2,q+2}$-minor has at most $\left[\frac{r}{2}\right]_q$ lines.

Here $\left[\frac{r}{2}\right]_q = \frac{(q^r-1)(q^{r-1}-1)}{(q-1)(q^2-1)}$ is a $q$-binomial coefficient. The projective geometry $\text{PG}(r-1,q)$ has $\left[\frac{r}{2}\right]_q$ lines, so the conjectured bound is attained. Blokhuis gave examples refuting Conjecture 1.1 for all $q \geq 13$; see Nelson [2]. Our main result is the following.

Theorem 1.2. Conjecture 1.1 holds if and only if $q \leq 5$.

We construct counterexamples to Conjecture 1.1 for $q \geq 7$ in Section 3; they are refinements of Blokhuis’s construction. These examples all have rank three, and it is quite plausible that the conjecture holds whenever $r \geq 4$; this is supported by a result of Nelson [2] that the conjecture holds when $r$ is sufficiently large relative to $q$.

For $q \in \{2,3,4\}$, our proof of Conjecture 1.1 is fairly simple, and is found in Section 4. For $q = 5$, our argument is more elaborate and is found in Section 5. This argument for $q = 5$ also appeals to a computer search, discussed in Lemma 2.3. In all four cases we devote most of our attention to the rank-three case, to which the general case is easily reduced by an easy inductive argument in Section 6.

Key words and phrases. matroids, Whitney numbers.

This research was partially supported by a grant from the Office of Naval Research [N00014-12-1-0031].

© 2015. This manuscript version is made available under the Elsevier user license
http://www.elsevier.com/open-access/userlicense/1.0/
2. Preliminaries

We follow the notation of Oxley [3]. We write \( U(\ell) \) for the class of matroids with no \( U_{2,\ell+2} \)-minor. If \( e \in E(M) \) then we write \( W_1(M) \) for the number of points of \( M \), \( W_2(M) \) for the number of lines of \( M \), \( W_2^*(M) \) for the number of lines of \( M \) not containing \( e \), and \( \delta_M(e) \) for the number of lines of \( M \) containing \( e \). For a simple rank-3 matroid \( M \), we have \( M \in U(\ell) \) iff \( \delta_M(e) \leq \ell + 1 \) for all \( e \in E(M) \). \( W_1 \) and \( W_2 \) are the first two Whitney numbers of the second kind.

The following theorem was proved by Kung [1].

**Theorem 2.1.** If \( \ell \geq 2 \) is an integer and \( M \in U(\ell) \) has rank \( r \), then \( W_1(M) \leq \frac{\ell r - 1}{\ell - 1} \).

Surprisingly, we require a small graph theory result. A 1-factorisation of a graph is a partition of its edge set into perfect matchings.

**Lemma 2.2.** Any two 1-factorisations of the graph \( K_6 \) have an element in common.

**Proof.** A 1-factorisation of \( K_6 \) is a 5-edge-colouring. The union of any two colour classes is a 2-regular bipartite graph on six vertices and edges, so is a 6-cycle, and it is easy to check that for any 6-cycle \( C \) there is a unique 5-edge-colouring having \( C \) as the union of two of its colour classes. Each 5-edge-colouring has ten pairs of colour classes and \( K_6 \) has sixty 6-cycles, so \( K_6 \) has six 1-factorisations.

Suppose that there exist disjoint 1-factorisations \( F_1 \) and \( F_2 \). Each edge is in exactly three perfect matchings, so the set \( F_3 \) of perfect matchings not in \( F_1 \) or \( F_2 \) is also a 1-factorisation. Let \( F \) be a 1-factorisation that is not \( F_1, F_2 \) or \( F_3 \). Since \( |F| = 5 \) there is some \( i \) such that \( |F \cap F_i| \geq 2 \), but now \( F \) and \( F_i \) share two colour classes and are thus equal by our above observation. This is a contradiction. \( \square \)

Our next lemma, invoked twice in Section 5, was proved by a computer search whose structure we briefly sketch.

**Lemma 2.3.** Let \( A \) be a twelve-element set. There do not exist partitions \( \mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_5 \) of \( A \) satisfying the following conditions:

1. \( \mathcal{L}_0 \) has exactly six blocks, each of size 2,
2. for each \( i \in \{1, \ldots, 5\} \), the partition \( \mathcal{L}_i \) has at most five blocks and each has size at most four,
3. for every distinct \( x, y \in A \), there is exactly one \( i \in \{0, \ldots, 5\} \) such that \( \mathcal{L}_i \) has a block containing \( x \) and \( y \),
4. for each \( i \in \{1, \ldots, 5\} \), if \( \mathcal{L}_i \) has exactly five blocks then it has a block of size one.
Sketch of computational proof: Fix $L_0$ arbitrarily and suppose that partitions $L_1, \ldots, L_5$ exist. For convenience we assume they each have exactly five parts and allow parts to be empty. The block sizes of each $L_i : i \in \{1, \ldots, 5\}$ gives an integer partition $(n_{i,1}, \ldots, n_{i,5})$ of 12 so that $4 \geq n_{i,1} \geq n_2 \geq \ldots \geq n_{i,5} \geq 0$ and $n_{i,5} \leq 1$. Moreover, there are 66 unordered pairs of distinct elements of $A$ and six of these pairs are contained in blocks of $L_0$, so $\sum_{j=1}^{5} \sum_{i=1}^{5} \binom{n_{i,j}}{2} = 60$.

We say two set partitions $P, P'$ are compatible if each block of $P$ intersects each block of $P'$ in at most one element. For each integer partition $p$ of 12 into nonnegative parts, let $C(p)$ denote the set of partitions of $A$ that are compatible with $L_0$ and whose block sizes are the integers in $p$. Let $C'(p)$ denote the set of orbits of $C(p)$ under the action of the group of the $6! \cdot 2^6$ permutations of $A$ that fix $L_0$. The following table shows the nine possible $p$ that satisfy our constraints and their associated parameters.

| $p$             | $|C(p)|$ | $|C'(p)|$ | $\sum_{j=1}^{5} \binom{p_j}{2}$ |
|-----------------|---------|----------|---------------------------------|
| $(3, 3, 3, 2, 1)$ | 71040   | 5        | 10                              |
| $(3, 3, 3, 3, 0)$ | 4960    | 3        | 12                              |
| $(4, 3, 2, 2, 1)$ | 136320  | 9        | 11                              |
| $(4, 3, 3, 1, 1)$ | 41280   | 5        | 12                              |
| $(4, 3, 3, 2, 0)$ | 38400   | 4        | 13                              |
| $(4, 4, 2, 1, 1)$ | 27360   | 5        | 13                              |
| $(4, 4, 2, 2, 0)$ | 12720   | 4        | 14                              |
| $(4, 4, 3, 1, 0)$ | 15360   | 2        | 15                              |
| $(4, 4, 4, 0, 0)$ | 960     | 1        | 18                              |

The tuple $(L_1, \ldots, L_5)$ must belong to $\mathcal{C} = C(p_1) \times C(p_1) \times \ldots \times C(p_5)$, where $p_1, \ldots, p_5$ are drawn from rows of the table above whose last column sums to 60; there are 68 such (unordered) 5-tuples $p_1, \ldots, p_5$. Moreover, the partitions $L_0, \ldots, L_5$ must be pairwise compatible. For each of the 68 possible $\mathcal{C}$, a backtracking search shows this cannot occur; by considering our choice for $L_1$ up to a permutation of $A$ that preserves $L_0$, we need only consider one choice of $L_1$ from each orbit in $C'(p_1)$. Our search was performed with a Python program that runs in under two hours on a single CPU. \hfill \square

3. Counterexamples

In this section we construct counterexamples to Conjecture 1.1. They are more elaborate versions of the aforementioned construction of Blokhuis.
Lemma 3.1. Let $q$ be a prime power and $t$ be an integer with $3 \leq t \leq q$. There is a rank-3 matroid $M(q,t)$ with no $U_{2,q+t}$-minor such that $W_2(M(q,t)) = q^2 + (q+1)\binom{q}{2}$.

Proof. Let $N \cong \text{PG}(2,q)$. Let $e \in E(N)$ and let $L_1, L_2, L_3$ be distinct lines of $N$ not containing $e$ and so that $L_1 \cap L_2 \cap L_3$ is empty. Note that every line of $M$ other than $L_1, L_2$ and $L_3$ intersects $L_1 \cup L_2 \cup L_3$ in at least two and at most three elements.

Let $\mathcal{L}$ be the set of lines of $N$ and $\mathcal{L}_e$ be the set of lines of $N$ containing $e$. For each $L \in \mathcal{L}_e$, let $T(L)$ be a $t$-element subset of $L - \{e\}$ containing $L \cap (L_1 \cup L_2 \cup L_3)$. Observe that the $T(L)$ are pairwise disjoint. Let $X = \cup_{L \in \mathcal{L}_e} T(L)$, noting that $L_1 \cup L_2 \cup L_3 \subseteq X$ and so each line in $\mathcal{L}$ intersects $X$ in at least two elements. Let $M(q,t)$ be the simple rank-3 matroid with ground set $X$ whose set of lines is $\mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1 = \{L \cap X : L \in \mathcal{L} - \mathcal{L}_e\}$, and $\mathcal{L}_2$ is the collection of two-element subsets of the sets $T(L)$ for $L \in \mathcal{L}_e$. Note that $\mathcal{L}_1$ and $\mathcal{L}_2$ are disjoint. Every $f \in X$ lies in $q$ lines in $\mathcal{L}_1$ and in $(t-1)$ lines in $\mathcal{L}_2$, so $M(q,t)$ has no $U_{2,q+t}$-minor. Moreover, we have $\mathcal{L}_1 = |\mathcal{L} - \mathcal{L}_e| = q^2$ and $\mathcal{L}_2 = |\mathcal{L}_e|\binom{q}{2} = (q+1)\binom{q}{2}$. This gives the lemma. \hfill $\Box$

This next theorem refutes Conjecture 1.1 for all $q \geq 7$.

Theorem 3.2. If $\ell$ is an integer with $\ell \geq 7$, then there exists $M \in \mathcal{U}(\ell)$ such that $r(M) = 3$ and $W_2(M) > \ell^2 + \ell + 1$.

Proof. If $\ell \geq 127$, let $q$ be a power of 2 such that $\frac{1}{2}(\ell+2) < q \leq \frac{1}{2}(\ell+2)$. We have $W_2(M(q,q)) = q^2 + q\binom{q}{2}(q+1) > \frac{1}{2}q^3 \geq \frac{1}{128}(\ell+1)^3 \geq (\ell+1)^2 > \ell^2 + \ell + 1$.

If $7 \leq \ell < 127$, then it is easy to check that there is some prime power $q \in \{5, 7, 9, 13, 19, 32, 59, 113\}$ such that $\frac{1}{2}(\ell + 2) \leq q \leq \ell - 2$. Note that $3 < \ell + 2 - q \leq q$. Let $f_q(x) = q^2 + (q+1)(\frac{x^2+q}{2}) - (x^2+x+1)$. This function $f_q(x)$ is quadratic in $x$ with positive leading coefficient and $f_q(q) = f_q(q+1) = 0$; it follows that $f(x) > 0$ for every integer $x \notin \{q, q+1\}$. Now the matroid $M = M(q,\ell+2-q)$ satisfies $M \in \mathcal{U}(\ell)$ and $W_2(M) - (\ell^2 + \ell + 1) = f_q(\ell) > 0$. \hfill $\Box$

The construction in Lemma 3.1 can also be applied to projective planes of order not equal to a prime power, which may or may not exist. We conjecture that, for large $\ell$, the construction above for projective planes and $t = q$ gives the correct upper bound for the number of lines in a rank-three matroid in $\mathcal{U}(\ell)$.

Conjecture 3.3. If $\ell$ is a sufficiently large integer and $M \in \mathcal{U}(\ell)$ has rank-3, then $W_2(M) \leq q^2 + \binom{q}{2}(q+1)$, where $q$ is the largest integer such that $2q \leq \ell + 2$ and there exists a projective plane of order $q$. 

Lemma 4.1. Let $q \geq 2$ be an integer. If $M \in \mathcal{U}(q)$ has rank 3 and
has a $U_{2,q+1}$-restriction, then $W_2(M) \leq q^2 + q + 1$ and $W_2^e(M) \leq q^2$
for each nonloop $e$ of $M$.

Proof. We may assume that $M$ is simple; let $M|L$ be a $U_{2,q+1}$-restriction
of $M$. If some line $L'$ of $M$ does not intersect $L$ then contracting
a point of $L'$ yields a $U_{2,q+2}$-minor, so every line of $M$ intersects $L$.
Therefore $W_2(M) = \sum_{x \in L}(\delta_M(x) - 1) + 1 \leq (q + 1)((q + 1) - 1) + 1 = q^2 + q + 1$. For each $e \in E(M) - L$ we clearly have $\delta_M(e) = q + 1$
so $W_2^e(M) \leq (q^2 + q + 1) - (q + 1) = q^2$. For each $e \in L$ we have $W_2^e(M) = \sum_{x \in L-e}(\delta_M(x) - 1) \leq q(q+1-1) = q^2$. □

Lemma 4.2. If $q \in \{2, 3, 4\}$ and $M \in \mathcal{U}(q)$ is a rank-3 matroid with a
$U_{2,q}$-restriction $L$ and no $U_{2,q+1}$-restriction, then at most $q$ lines of $M$
are disjoint from $L$.

Proof. We may assume that $M$ is simple. Suppose that there is a set $L$
of lines disjoint from $L$ such that $|L| = q + 1$. Since each $x \in E(M) - L$
lies on $q$ lines intersecting $L$ it lies on at most one line in $L$, so the
lines in $L$ are pairwise disjoint. Let $X$ be a set formed by choosing two
points from each line in $L$; note that $|X| = 2(q + 1) = X \cap L = \emptyset$.

Since each element of $X$ lies on at most one line disjoint from $L$, at most
$(q + 1)$ pairs of elements of $X$ span lines disjoint from $L$, so
at least $\left(\frac{2(q+1)}{2}\right) - (q + 1) = 2q(q + 1)$ pairs of elements of $X$ span a
line intersecting $L$. Since $|L| = q$, there is some $y \in L$ such that at
least $2(q + 1)$ pairs of elements of $X$ span $y$. Let $L_y$ be the set of
lines of $M|\{\{y\} \cup X\}$ that contain $y$. Every line in $L_y$ spans a line
of $M$ containing $y$ and none spans $L$ itself, so $|L_y| \leq q$. We also have $\sum_{L \in L_y}|L - 1| = |X| = 2(q + 1)$ and $\sum_{L \in L_y}\frac{|L|}{2} - 1 \geq 2(q + 1)$ by choice of $y$. Since $M$ has no $U_{2,q+1}$-restriction, we also have $|L| - 1 \leq q - 1$
for each $L \in L_y$. It remains to check that, for $q \in \{2, 3, 4\}$ there are no
solutions to the system $n_1 + n_2 + \ldots + n_q = 2(q + 1)$, $\binom{n_1}{2} + \ldots + \binom{n_q}{2} \geq 2(q + 1)$ subject to $n_i \in \{0, \ldots, q - 1\}$ for each $i$. This is easy. □

Lemma 4.3. Let $q \in \{2, 3, 4\}$. If $M \in \mathcal{U}(q)$ has rank 3 and has a
$U_{2,q}$-restriction, then $W_2(M) \leq q^2 + q + 1$ and $W_2^e(M) \leq q^2$
for each nonloop $e$ of $M$.

Proof. We may assume that $M$ is simple and, by Lemma 4.1, that $M$
has no $U_{2,q+1}$-restriction; let $M|L$ be a $U_{2,q}$-restriction of $M$ and
let $f \in L$. If $W_2^f(M) \geq q^2 + 1$ then, since each $x \in L - \{f\}$ is
on at most $q$ lines not containing $f$, there are at most $(|L| - 1)q =$
Proof. We may assume that \( q \) is simple; let \( n = |M| \). If \( q = 2 \) then the result is vacuous and if \( q = 3 \) then \( M \) has no \( U_{2,3} \)-restriction so \( M \cong U_{3,n} \) and \( n \leq 5 \) so both conclusions are clear. It remains to resolve the \( q = 4 \) case.

Suppose that \( W_2(M) \geq 4^2 + 4 + 2 = 22 \). Every line of \( M \) contains either two or three points; for each \( f \in E(M) \) let \( \ell_f \) be the number of 3-point lines of \( M \) containing \( f \). Let \( \ell \) be the total number of 3-point lines of \( M \). Each 3-point line of \( M \) contains three pairs of points of \( M \), so \( 22 \leq W_2(M) = \binom{n}{2} - 2\ell \). Moreover, every \( e \in E(M) \) is in at most five lines so \( n \leq 1 + 2\ell_f + (5 - \ell_f) = 6 + \ell_f \). Summing this expression over all \( f \in E(M) \) gives \( n^2 \leq 6n + 3\ell \). Therefore \( 2(6n + 3\ell) + 3(\binom{n}{2} - 2\ell) \geq 2n^2 + 66 \), giving \( 0 \geq n^2 - 21n + 132 = (n - \frac{21}{2})^2 + \frac{87}{4} \), a contradiction; therefore \( W_2(M) \leq 4^2 + 4 + 1 \). From here, it is easy to show that \( W_2(M) \leq 4^2 \) in a manner similar to the last paragraph of the proof of Lemma 4.3. \( \square \)

5. Five

We now consider the number of lines in rank-3 matroids in \( U(5) \), first dealing with those that have no \( U_{2,5} \)-restriction.

Lemma 5.1. If \( M \in U(5) \) has rank 3 and has no \( U_{2,5} \)-restriction, then \( W_2(M) \leq 5^2 + 5 + 1 \).

Proof. We may assume that \( M \) is simple. Let \( n = |M| \) and for each \( i \in \{2, 3, 4\} \), let \( \ell_i \) be the number of lines of length \( i \) in \( M \), noting
that every line of \( M \) has length 2, 3 or 4. Suppose for a contradiction that \( \ell_2 + \ell_3 + \ell_4 \geq 32 \). Let \( P \) be the set of pairs \((e, L)\) where \( e \in L \). We have \( 2\ell_2 + 3\ell_3 + 4\ell_4 = |P| = \sum_{e \in E(M)} \delta_M(e) \leq 6n \). There are \( \binom{n}{2} \) pairs of elements of \( M \), each of which is contained in exactly one line of \( M \), and an \( i \)-element line contains \( \binom{i}{2} \) such pairs. We therefore have
\[
\ell_4 = (\ell_2 + 3\ell_3 + 6\ell_4) + 3(\ell_2 + \ell_3 + \ell_4) - 2(2\ell_2 + 3\ell_3 + 4\ell_4)
\geq \binom{n}{2} + 3 \cdot 32 - 2 \cdot 6n.
\]
Now we have
\[
0 \leq \ell_1 + 3\ell_3 = \binom{n}{2} - 6\ell_4 \\
\leq 72n - 18 \cdot 32 - 5\binom{n}{2}
= -\frac{5}{2} \left(n - \frac{149}{16}\right)^2 - \frac{839}{40},
\]
a contradiction. \( \square \)

Lemma 5.2. If \( M \in U(5) \) is a rank-3 matroid with no \( U_{2,5} \)-restriction and \( e \) is a nonloop of \( M \), then \( W^e_2(M) \leq 5^2 \).

**Proof.** We may assume that \( M \) is simple. If \( \delta_M(e) = 6 \) then \( W^e_2(M) \leq 31 - 6 = 25 \) by the previous lemma, so we may assume that \( \delta_M(e) \leq 5 \). Let \( n = |M| \) and let \( \ell_2^e, \ell_3^e \) and \( \ell_4^e \) be the number of lines of length 2, 3 and 4 respectively that do not contain \( e \). Suppose for a contradiction that \( \ell_2^e + \ell_3^e + \ell_4^e \geq 26 \). Let \( P \) be the set of pairs \((f, L)\), where \( L \) is a line not containing \( e \) and \( f \in L \). Clearly \( |P| = 2\ell_2^e + 3\ell_3^e + 4\ell_4^e \), but also, since every \( f \neq e \) is on at most five lines not containing \( e \), we have \( |P| \leq 5(n - 1) \), so \( 2\ell_2^e + 3\ell_3^e + 4\ell_4^e \leq 5(n - 1) \). Finally, let \( Q \) be the set of two-element sets \( \{f_1, f_2\} \subset E(M) \) that span a line not containing \( e \). As before, we have \( |Q| = \ell_2^e + 3\ell_3^e + 6\ell_4^e \). On the other hand, there are at most five lines of \( M \) through \( e \) and each contains at most three other points, so there are at most \( 5 \binom{5}{2} = 15 \) two-element subsets of \( E(M) - \{e\} \) that are not in \( Q \). Therefore \( |Q| = \binom{n - 1}{2} - s \) for some \( s \in \{0, \ldots, 15\} \), and \( \ell_2^e + 3\ell_3^e + 6\ell_4^e = \binom{n - 1}{2} - s \). Now
\[
\ell_4^e = (\ell_2^e + 3\ell_3^e + 6\ell_4^e) + 3(\ell_2^e + \ell_3^e + \ell_4^e) - 2(2\ell_2^e + 3\ell_3^e + 4\ell_4^e)
\geq \binom{n - 1}{2} - s + 3 \cdot 26 - 2(5(n - 1))
= \binom{n - 1}{2} - 10n + 88 - s.
Therefore, using \( s \leq 15 \) we have
\[
0 \leq \ell_2^5 + 3\ell_4^5 \\
= |Q| - 6\ell_4 \\
\leq (n-1) - s - 6((n-1) - 10n + 88 - s) \\
= 60n - 528 - 5(n-1) + 5s \\
\leq 60n - 453 - 5(n-1) \\
= -\frac{5}{2} \left( n - \frac{27}{2} \right)^2 - \frac{19}{8},
\]
again a contradiction. \( \square \)

**Lemma 5.3.** If \( M \in \mathcal{U}(5) \) has rank 3, then \( W_2(M) \leq 5^2 + 5 + 1 \).

**Proof.** Let \( M \) be a counterexample for which \( |M| \) is minimized. Note that \( M \) is simple, that \( W_2(M) \geq 32 \), and that, by Lemma 4.1, \( M \) has no \( U_{2,6} \)-restriction. Furthermore, Lemma 4.4 implies that \( M \) has a \( U_{2,5} \)-restriction \( M|L = M|\{x_1, x_2, x_3, x_4, x_5\} \).

Each element of \( L \) lies on at most five other lines, so there are at least \( 32 - 5 \cdot 5 - 1 = 6 \) lines \( L_{0,1}, L_{0,2}, \ldots, L_{0,6} \) of \( M \) that do not intersect \( L \). For each \( i \in \{1, \ldots, 6\} \) let \( a_{2i-1} \) and \( a_{2i} \) be distinct elements of \( L_{0,i} \). Note that each \( e \in E(M) - L \) lies on five lines meeting \( L \) so lies on at most one other line; it follows that the set \( A = \{a_1, a_2, \ldots, a_{12}\} \) has twelve elements and that \( L_0 = \{L_{0,1}, \ldots, L_{0,6}\} \) is a partition of \( A \) into pairs.

For each \( i \in \{1, \ldots, 5\} \) let \( L'_i \) be the set of lines of \( M \) containing \( x_i \) other than \( L \) and let \( L_i = \{L' - \{x_i\} : L' \in L'_i\} \). We have \( |L_i| \leq 5 \) and clearly \( L_i \) is a partition of \( A \). If every line through \( x_i \) contains at least two other points, then \( W_2(M \setminus x_i) = W_2(M) \), contradicting minimality of \( |M| \). Therefore \( |L'| \leq 1 \) for some \( L' \in L_i \) if \( |L_i| = 5 \). Since \( M \) has no \( U_{2,6} \)-restriction we also have \( |L'| \leq 4 \) for each \( L' \in L_i \). Finally, since each two-element subset of \( A \) either spans a line in \( L_0 \) or a line in \( L'_i \) for a unique \( i \), each such pair is contained in a block of exactly one of the partitions \( L_0, \ldots, L_5 \). By Lemma 2.3 this is impossible. \( \square \)

**Lemma 5.4.** If \( M \in \mathcal{U}(5) \) has rank 3 then \( W_2^e(M) \leq 5^2 \) for each nonloop \( e \) of \( M \).

**Proof.** Let \( (M, e) \) be a counterexample for which \( |M| \) is minimized. Note that \( M \) is simple and, by Lemma 4.1, has no \( U_{2,6} \)-restriction. If \( \delta_M(e) \geq 6 \) then \( W_2^e(M) \leq 5^2 + 5 + 1 - 6 = 25 \) by Lemma 5.3, so \( \delta_M(e) \leq 5 \). If there is some \( f \in E(M) - \{e\} \) such that every line through \( f \) contains two other points, then \( W_2^e(M \setminus f) = W_2^e(M) \), contradicting
minimality. Therefore every $x \in E(M)$ is on at most five lines that contain two other points (note that $e$ also has this property).

If $e$ is contained in a $U_{2,5}$-restriction $L'$ of $M$, then observe that each $f \in L' - \{e\}$ is on at most five lines not containing $e$, so there are at least $26 - 20 = 6$ lines of $M$ disjoint from $L'$. Let $B$ be a set formed by choosing a pair of elements from each of these lines. In a similar manner to the previous lemma, we obtain six partitions of $B$ that contradict Lemma 2.3. Therefore $e$ is contained in no such restriction. On the other hand, Lemma 5.2 implies $M$ has a $U_{2,5}$-restriction $M|L = M|\{x_1, \ldots, x_5\}$; we know that $e \notin L$.

Each $x \in L$ lies on at most four lines other than $L$ not containing $e$, so there exist $26 - 1 - 20 = 5$ lines $L_{0,1}, \ldots, L_{0,5}$ of $M$ disjoint from $L \cup \{e\}$. If there are six such disjoint lines, then we again obtain a contradiction with Lemma 2.3; we therefore assume that every $x_i$ in $L$ lies on exactly four other lines of $M$ disjoint from $L \cup \{e\}$, so $\delta_M(x_i) = 6$ for each $i \in \{1, \ldots, 5\}$.

For each $j \in \{1, \ldots, 5\}$ let $a_{2j-1}, a_{2j}$ be distinct elements of $L_{0,j}$. Let $A = \{a_1, \ldots, a_{10}\}$ and let $N = M\langle L \cup A \cup \{e\}\rangle$. As in the proof of the previous lemma, the lines $L_{0,j}$ partition $A$ into pairs, and so $|N| = 16$. Since $e$ lies on at most five lines of $N$ and each contains at most three other points, there is a partition $\{L_{1,e}, \ldots, L_{5,e}\}$ of $E(N) - \{e\}$ into three-element blocks such that $L_{j,e} \cup \{e\}$ is a four-point line of $N$ for each $j \in \{1, \ldots, 5\}$. We may also assume that $x_j \in L_{j,e}$ for each $j$.

As before, we consider the lines through each element of $L$, and for each $x_i \in L$ we obtain a partition $L_i = \{L_{i,1}, \ldots, L_{i,5}\}$ of $A \cup \{e\}$ into five blocks corresponding to the lines of $N$ through $x_i$, other than $L$. Again we have $4 \geq |L_{i,1}| \geq |L_{i,2}| \geq \ldots \geq |L_{i,5}| = 1$. (We have $|L_{i,1}| \leq 4$ here because $M$ has no $U_{2,6}$-restriction, and $|L_{i,5}| = 1$ because $\delta_M(x_i) = 6$ and not every line through $x_i$ has at least two other points.) and $\sum_{j=1}^5 |L_{i,j}| = 11$ for each $i$. Moreover, for each $i$ the point $x_i$ is on the four-element line $L_{i,e}$, so for some $j$ we have $|L_{i,j}| = 3$. Finally, there are $\binom{11}{2} - 5 = 50$ pairs of elements in $A \cup \{e\}$ that do not span one of the lines $L_{0,j}$, so $\sum_{i=1}^5 \sum_{j=1}^5 \binom{|L_{i,j}|}{2} = 50$.

If $4 \geq n_1 \geq \ldots \geq n_5 = 1$ are integers summing to 11 such that some $n_i$ is 3, then $\binom{n_1}{2} + \ldots + \binom{n_5}{2} \leq 10$ with equality only if $(n_1, n_2, \ldots, n_5) = (4, 3, 2, 1, 1)$. Therefore $(|L_{i,1}|, |L_{i,2}|, \ldots, |L_{i,5}|) = (4, 3, 2, 1, 1)$ for each $i$; note that $L_{i,e} \cup \{e\} = L_{i,2} \cup \{x_i\}$. Therefore, in the fifteen-element matroid $N \setminus e$, each $x_i \in L$ lies on two five-element lines; two three-element lines and two two-element lines. For each integer $k$, let $J_k$ be the set of $k$-element lines of $N \setminus e$. 

Let \( Y \) be the union of the lines in \( J_5 \). By the above reasoning each \( y \in Y \) lies on exactly two lines in \( J_5 \), so it follows that \( 5|\mathcal{J}_5| = 2|Y| \) and so \( |Y| \equiv 0 \pmod{5} \). Since three 5-point lines account for at least 13 points, it is clear that \( |Y| > 10 \) and so we must have \( |Y| = 15 \) and \( |Y| = E(N \setminus e) \). Therefore every element of \( N \setminus e \) lies on exactly two lines in \( J_5 \), \( |J_5| = 2 \cdot \frac{\mathcal{J}_5}{|Y|} = 6 \), and the elements of \( N \setminus e \) are exactly the intersections of the \( \binom{6}{2} \) pairs of lines in \( J_5 \). There is now a natural mapping of \( E(N \setminus e) \) to the edge set of the complete graph \( K_6 \) with vertex set \( J_5 \), where the elements of each \( J \in \mathcal{J}_5 \) are the edges incident with the vertex \( J \). The lines in \( \mathcal{L}_3 \) map to three-edge matchings. We know the lines in \( L_i \), where \( i \in \mathcal{J}_3 \) and partition \( E(N \setminus e) \), and each \( f \in E(N \setminus e) \) is contained in exactly two lines in \( \mathcal{L}_3 \), so \( \mathcal{L}_3 \) is the union of two disjoint partitions of \( E(N \setminus e) \). This gives two disjoint 1-factorisations of \( K_6 \), a contradiction by Lemma 2.2.

\[ \square \]

6. Higher Rank

Combining Lemmas 4.3, 4.4, 5.3, and 5.4 gives the following:

**Theorem 6.1.** If \( q \in \{2, 3, 4, 5\} \) and \( M \) is a rank-3 matroid in \( U(q) \), then \( W_2(M) \leq q^2 + q + 1 \) and \( W_2(M) \leq q^2 \) for each nonloop \( e \) of \( M \).

We now generalise the upper bound on \( W_2(M) \) to arbitrary rank. For a matroid \( M \) and a nonloop \( e \in E(M) \), let \( \mathcal{P}_M(e) \) denote the set of planes of \( M \) containing \( e \). Note that \( |\mathcal{P}_M(e)| = W_2(M/e) \). When we contract a nonloop \( e \) in a matroid \( M \), every line through \( e \) becomes a point and for each plane \( P \) in \( \mathcal{P}_M(e) \), the set of lines of \( M \) contained in \( P \) but not containing \( e \) are identified into a single line. This gives the following lemma:

**Lemma 6.2.** If \( M \) is a matroid and \( e \in E(M) \) is a nonloop, then \( W_2(M) = W_1(M/e) + \sum_{P \in \mathcal{P}_M(e)} W_2(M|P) \).

From here we can easily verify Conjecture 1.2 for all \( q \leq 5 \). Combined with Theorem 3.2, this gives our main result.

**Theorem 6.3.** If \( q \in \{2, 3, 4, 5\} \) and \( M \in U(q) \) then \( W_2(M) \leq \left[ \frac{r(M)}{2} \right]_q \).

**Proof.** If \( r \leq 2 \) then the result is obvious. Suppose inductively that \( r \geq 3 \) and that the result holds for smaller \( r \), and let \( e \) be a nonloop of \( M \). By Theorem 2.1 we have \( W_1(M/e) \leq \frac{q^{r-1}-1}{q-1} \) and by Theorem 6.1 we have \( W_2(M|P) \leq q^2 \) for each \( P \in \mathcal{P}_M(e) \). Therefore, by Lemma 6.2
and the inductive hypothesis,

\[ W_2(M) = W_1(M/e) + \sum_{P \in \mathcal{P}_M(e)} W_2^e(M|P) \]

\[ \leq \frac{q^{r-1}-1}{q-1} + q^2 |\mathcal{P}_M(e)| \]

\[ = \frac{q^{r-1}-1}{q-1} + q^2 W_2(M/e) \]

\[ \leq \left[ \frac{r-1}{q} \right] + q^2 \left[ \frac{r-1}{2} \right] \]

\[ = \left[ \frac{g}{q} \right]. \]

as required. \qed

References

[1] J.P.S. Kung, Extremal matroid theory, in: Graph Structure Theory (Seattle WA, 1991), Contemporary Mathematics 147 (1993), American Mathematical Society, Providence RI, 21–61.

[2] P. Nelson, The number of rank-$k$ flats in a matroid with no $U_{2,n}$-minor, J. Combin. Theory, Ser. B, in press.

[3] J. G. Oxley, Matroid Theory, Oxford University Press, New York, 2011.

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada