Abstract

We consider transport properties of the chaotic (strange) attractor along unfolded trajectories of the dissipative standard map. It is shown that the diffusion process is normal except of the cases when a control parameter is close to some special values that correspond to the ballistic mode dynamics. Diffusion near the related crises is anomalous and non-uniform in time: there are large time intervals during which the transport is normal or ballistic, or even superballistic. The anomalous superdiffusion seems to be caused by stickiness of trajectories to a non-chaotic and nowhere dense invariant Cantor set that plays a similar role as cantori in Hamiltonian chaos. We provide a numerical example of such a sticky set. Distribution function on the sticky set almost coincides with the distribution function (SRB measure) of the chaotic attractor.
The occurrence of anomalous transport properties in Hamiltonian systems with chaotic dynamics is now widely discussed. Deviations of the statistical properties of such systems from the normal (Gaussian) ones are often linked to a specific non-uniform structure of phase space. Such an abnormal behavior can also emerge in dissipative systems depending on their properties for different values of control parameters. We demonstrate this feature using the dissipative generalization of the standard map [1].

Properties of low-dimensional systems with the chaotic attractors are fairly well understood and used in numerous applications for systems with dissipation (see in reviews [2, 3, 4]). The presence of dissipation bounds the diffusion along the momentum or energy, while the diffusion along the coordinate can be unbounded. In this paper we consider dissipative generalization of the standard map and show that there exists a ballistic mode near some values of the control parameters, and that, at least, near these values of the parameters, the dynamics of the system along the coordinate is superdiffusive. The map that will be studied was introduced in [1]. Different properties of this map were discussed in [3, 4, 5], and a rigorous proof of the existence of chaotic attractor in such type of systems was obtained in [6, 7]. It was also proved in [6, 7] that the system of this type exhibits quasi-periodic attractors, periodic sinks, transient chaos with the dynamics attracting to the sink, and the existence of the SRB measure for the purely chaotic dynamics. Extended rigorous approach based on the use of a return map is developed in [8, 9]. In this paper we provide a numerical demonstration of the existence of windows in the parameter space, in which there are attracting non-chaotic trajectories, and show that the observed windows are linked to the so-called ballistic mode dynamics. The main result is that for the parameter value in the vicinity of the window edge transport along the cyclic coordinate (phase) is anomalous and superdiffusive.

The original system [1] has a stable limit cycle and the system is forced by a periodic sequence of δ-function type kicks:

\[ \dot{I} = -\Gamma(I - 1) + \epsilon \sin x \sum_{n=-\infty}^{\infty} \delta(t - n), \quad \dot{x} = \Omega + \alpha(I - 1) \]  

(1)

where \( \Gamma, \epsilon, \Omega, \) and \( \alpha \) are constants with evident physical meaning, and the limit cycle corresponds to the dimensionless action \( I = 1 \). The main construction of the model is the same as in the standard map, i.e. a nonlinear rotation periodically perturbed by fairly short pulses.
of a force to change generalized momentum (action) of the system. Since the unperturbed
dynamics ($\epsilon = 0$) has a stable limit cycle, all interesting behaviors for ($\epsilon \neq 0$) occur in its
vicinity. The system (1) can be considered in a twofold way: in the cylinder phase space
($0 < x < 2\pi$) and in the unbounded phase space. The second case is convenient to study
transport. The trajectories, being folded, are related to the first case.

Equation (1) can be replaced by a map

$$ p_{n+1} = e^{-\Gamma} p_n + K \sin x_n $$
$$ x_{n+1} = x_n + \Omega + p_{n+1}, \quad (\text{mod } 2\pi) $$

(2)

where

$$ K = \epsilon \alpha \mu e^{-\Gamma}, \quad p = \alpha \mu (I - 1); \quad \mu = (e^\Gamma - 1)/\Gamma. $$

(3)

The map (2) is also known as the dissipative standard map (DSM). The phase volume
shrinks each time step by a factor $\exp(-\Gamma)$. It is defined by two important parameters,
dissipation constant $\Gamma$ and force amplitude $K$, and by a shift $\Omega$ which does not play an
important role for our consideration. It will be put $\Omega = 0$. For large $\Gamma \gg 1$ the equations
(2) shrinks to a one-dimensional sine-map

$$ x_{n+1} = x_n + \Omega + K \sin x_n, \quad (\text{mod } 2\pi) $$

(4)

proposed in [10] and studied in many papers. Eq. (1) doesn’t reflect all properties of (2)
but it can be used to get auxiliary results. The parameter $K = K(\Gamma)$ can be presented as

$$ K(\Gamma) = K(0)/\Gamma $$

(5)

for $\Gamma \gg 1$. A typical structure of the chaotic attractor is shown in Fig. 1(a). As it was
derived in [1], chaotic attractor occurs for fairly large $K$ and fairly large dissipation $\Gamma$.

Nevertheless, as it will be shown below, these conditions are necessary but not sufficient
since there exist of windows of values $(K, \Gamma)$ where the chaotic attractor collapses to an
attracting point or cycle or strange sets that will be discussed below.

It is easy to see that there exists a solution, called “ballistic mode” in analogy to similar
solutions for the standard map [11], for which $x_n$ grows with $n$ if trajectories, defined on the
cylinder in (2), will be unfolded.

Let for example

$$ x_n = x_0 + 2\pi mn. $$

(6)
with $m \in \mathbb{N}$. Then the ballistic mode is defined by the initial conditions
\[
\sin x_0 = (p_0/K)(1 - \exp(-\Gamma)), \quad p_0 = 2\pi m
\]  
and the solution exists for
\[
K^* + \Delta K > K \geq K^* \equiv 2\pi m(1 - \exp(-\Gamma))
\]  
The domain of $K$ where the ballistic mode exists will be called window of width $\Delta K$. 

FIG. 1: Structure of the chaotic attractor for different values of $K(\Gamma = 5, \Omega = 0)$. 

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There is an infinite countable number of windows. Consecutive increasing of $K$ leads to the transformations of the solutions of Eq. (2) presented in Fig. 1 for $m = 2$. After increasing of $K > K^*$ the chaotic attractor disappears and all trajectories are attracted to a point. Continuing increasing of $K$ provides a set of bifurcations of the Feigenbaum type (Fig. 2) until the “dying” chaotic attractor appears (see Fig. 1(c)). The dynamics on the dying attractor $x > \pi$, $p < 0$ is chaotic and a symmetric dying attractor exists for $x < \pi$, $p > 0$. The basins of the two dying attractors are disjoint. After a crisis around $K = 12.835395$ the chaotic attractor has a common basin and the typical structure is restored (Figs. 1(d) and 1(e)). Fig. 1(f) shows a zoom of the dying attractor from Fig. 1(d).

The scenario of the return to the chaotic attractor from the window when $K > K^*$ is standard.

When $K$ passes the value $K^* + \Delta K$, the dying attractor appears and transport becomes pure ballistic since the symmetric part of the dying attractor has disjoint basin. The transport is calculated in the open (unfolded) space $x \in (-\infty, \infty)$, $p \in (-|p|_{\text{max}}, p_{\text{max}})$ with $p_{\text{max}} > 2\pi$. The transport properties are calculated as time dependence of moments of the coordinate $x$ displacements over unfolded trajectories

$$\langle (x_n - x_0)^{2m} \rangle \sim t^{\mu(m)}$$

where averaging is performed over large number of initial conditions $(x_0, p_0)$ and $\mu(m)$ is the transport exponent. Typically, we considered $1/4 \leq m \leq 4$. Pure ballistic transport corresponds to $\mu(m) = 2m$, for the normal transport $\mu(m) = m$. A detailed study shows that for the values of $K$ near the crisis the transport can be even superballistic, at least for
FIG. 3: Supediffusive transport near the attractor's crisis ($\Gamma = 5$, $\Omega = 0$). Averaging is performed over $10^4$ trajectories. The values of $\mu(m)$ are given for the intervals $\log_{10} n \in (8.4, 9)$ – left, and $\log_{10} n \in (7.5, 9)$ – right: $\mu(m) = 0.82; 1.54; 2.74; 4.72; 6.51; 8.23$ – left and $\mu(m) = 0.29; 0.66; 1.80; 4.10; 6.11; 8.12$ – right.

FIG. 4: Comparison of the time behavior of the moments $m = 1$ (2nd moment) and $m = 4$ (8th moment) for the anomalous case in Fig. 3 (right) - solid line, and "almost normal" case (dashed line) for $K = 12.48$ (see Fig. 1).
FIG. 5: Dispersion of two initially close trajectories; $K = 12.835395$, $\Gamma = 5$, $\Omega = 0$.

A fairly long time of $10^9$ iterations (see Fig. 3). The slight superballistic behavior is due to small accelerations along the momentum direction. The slope and the transport exponent $\mu(m)$ depend on the time interval reflecting the multi-scale behavior of trajectories.

The behavior of curves in Fig. 3 is not uniform and for the considered time $10^9$ we don’t have a clear asymptotic behavior. The situation becomes evident from Fig. 4, where the case of $K = 12.8354$ in Fig. 3 and the case of $K = 12.48$ in Fig. 1 are compared for two different moments: 2nd moment ($m = 1$) and 8th moment ($m = 4$). The latter case is close to the normal one, i.e. the slopes in Fig. 4 are close to 1 ($m = 1$) and to 4 ($m = 4$). The strong superdiffusion for the anomalous value of $K$ persists during all computational time of $10^9$ (the local slopes are between 1 and 2 for $m = 1$). The co-existence of at least two scales is clear from Fig. 5 where we plot a distance between two trajectories defined as $\Delta r = [(p_1 - p_2)^2 + (x_1 - x_2)^2]^{1/2}$. Particles spend a fairly long time before a switch of the upper-lower parts of the dying attractor to the lower one (see in Fig. 1(d)). The existence of two very different scales are also clearly seen from the distribution of the escape times from a small domain of the dying attractor (Fig. 6) and from the distribution of the first arrival time to a small domain of the same attractor (Fig. 7).

A very small, almost zero, slope for the large time behavior of both distributions indicates
FIG. 6: Distribution of the escape time from the areas $p > 11.8$ and $p < -11.8$ ($K = 12.835395$, $\Gamma = 5$, $\Omega = 0$) with 28 trajectories and $10^{10}$ iterations on each.

a “sticky dynamics” responsible for the anomalous properties of transport. This is a delicate issue that needs more discussions. There were different publications on the transient chaos [12, 13, 14, 15, 16, 17] and the so-called strange non-chaotic attracting set [12, 13, 14, 15, 16, 17] that can influence the transport (see also references in [2]). Here we demonstrate a "sticky set" in Fig. 8 that is quasi-periodic, invariant, and not strange. All these statements are contingent upon the finite time and accuracy of simulations. This set is embedded into a dying attractor. In Fig. 8 (right) we demonstrate that during the last $10^5$ iterations the points of the set are staying almost at the same positions as the points of the sticky set after $10^7$ iterations. In Fig. 9, by a sequence of zooms, we show the absence of strangeness. The sticky sets can also be compared to cantori that occur in Hamiltonian dynamics. General discussion of the sticky sets can be found in [18]. Due to a finite accuracy and computational time we can not exclude a possibility that the observed sticky set is a sink [6, 7] that has
FIG. 7: First arrivals to the area \((1.6 < x < 1.62; \ 12 < p < 12.6)\) from the area \((1.7 < x < 1.72;\ 12.7 < p < 13)\). \(K = 12.835395, \ \Gamma = 5, \ \Omega = 0; \ 10^8\) trajectories.

a very narrow basin. Our simulations show that even a small perturbation of \(K\) for the set (see Fig. 8) leads to escape of the trajectory to the main chaotic attractor. The distribution function of the points of a periodic trajectory of the sticky set and for a chaotic trajectory of the dying attractor is given in Fig. 10. Their similarity and especially similarity in location of the peaks of the distributions suggests a specific role of the stickiness in generation of the anomalous transport. The vicinity to cantori is sticky making the trajectories to stay longer in nearby area. The same happens for the chaotic trajectories near the sticky islands boarders of Hamiltonian dynamics. Such a topology of the phase space can be considered as a cause of the anomalous transport.

The last comment is related to the peaks of the distribution function \(\rho(p, x)\) shown in Fig. 10. Since \(\Gamma = 5\) is fairly large, the structure of the chaotic attractor is close to the one-dimensional case [4]. It is known since a long [19] (see also [20]) that in this case \(\rho = \rho(x)\) has a singularity \(\rho \sim |x - x^*|^{-1/2}\) near an equilibrium point \(x^*\) where \(|dx_n/dx_{n-1}| = 0\). The
FIG. 8: Incommensurate periodic sticky set for $K = 12.835395$, $\Gamma = 5$, $\Omega = 0$: left - after $10^7$ iterations; right - the last $10^5$ iterations from the trajectory on the left.

FIG. 9: Consequent zooms of the sticky set in Fig. 8.

peaks are located along the trajectory $\{x^*\}$ generated by $x^*$ as the initial condition. This trajectory may correspond to a quasi-periodic sink that may or may not coincide with the sticky set demonstrated in Figs. 8,9. We consider this issue as an open problem, i.e. is the sticky set a sink or an isolated set of points near which the trajectories are staying fairly long
FIG. 10: Distribution function \((K = 12.835395, \quad \Gamma = 5, \quad \Omega = 0)\): left - on a chaotic trajectory after \(10^{11}\) iterations; right - after \(10^7\) iterations on the sticky set trajectory with \(x_0 = 1.41861,\) \(p_0 = 12.42994.\)

time due to the singularity of the distribution function. Our simulation, within an accuracy of computations, confirms the latter one.

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