On Alternating and Symmetric Groups Which Are Quasi OD-Characterizable

Ali Reza Moghaddamfar

Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box 16315-1618, Tehran, Iran.

E-mails: moghadam@ipm.ir and moghadam@kntu.ac.ir

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Abstract

Let \( \Gamma(G) \) be the prime graph associated with a finite group \( G \) and \( D(G) \) be the degree pattern of \( G \). A finite group \( G \) is said to be \( k \)-fold OD-characterizable if there exist exactly \( k \) non-isomorphic groups \( H \) such that \(|H| = |G|\) and \( D(H) = D(G) \). The purpose of this article is twofold. First, it shows that the symmetric group \( S_{27} \) is 38-fold OD-characterizable. Second, it shows that there exist many infinite families of alternating and symmetric groups, \( \{A_n\} \) and \( \{S_n\} \), which are \( k \)-fold OD-characterizable with \( k > 3 \).

Keywords: OD-characterization, alternating group, symmetric group, prime graph, spectrum, degree pattern, split extension, subdirect product.

1 Introduction

Throughout the article, all the groups under consideration are finite and simple groups are nonabelian. For a natural number \( n \), we denote by \( \pi(n) \) the set of all prime divisors of \( n \) and put \( \pi(G) = \pi(|G|) \). The spectrum \( \omega(G) \) of a group \( G \) is the set of orders of elements in \( G \). The set \( \omega(G) \) determines the prime graph \( \Gamma(G) \) whose vertex set is \( \pi(G) \), and two vertices \( p \) and \( q \) are adjacent if and only if \( pq \in \omega(G) \). For two vertices \( p \) and \( q \) we will write \( (p \sim q) \) to indicate that \( p \) is adjacent to \( q \) in \( \Gamma(G) \). Denote by \( s(G) \) the number of connected components of \( \Gamma(G) \) and by \( \pi_i(G) \) \( (i = 1, 2, \ldots, s(G)) \), the set of vertices of its \( i \)th connected component. If \( 2 \in \pi(G) \) then we assume that \( 2 \in \pi_1(G) \). We recall that the set of vertices of connected components of all finite simple groups are obtained in [5] and [13].

As usual, the degree \( \deg(p) \) of a vertex \( p \in \pi(G) \) in \( \Gamma(G) \) is the number of edges incident on \( p \). We denote the set of all vertices of the prime graph \( \Gamma(G) \)
which are joined to all other vertices by $\Lambda(G)$. If the prime divisors of $|G|$ are $p_1 < p_2 < \cdots < p_k$, then we define $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, which is called the degree pattern of $G$.

Given a group $H$, denote by $h_{OD}(H)$ the number of isomorphism classes of groups $G$ such that $|G| = |H|$ and $D(G) = D(H)$. Clearly, there are only finitely many isomorphism types of groups of order $n$, because there are just finitely many ways that an $n \times n$ multiplication table can be filled in. Hence $1 \leq h_{OD}(H) < \infty$ for any group $H$. In terms of function $h_{OD}$, groups $H$ are classified as follows:

**Definition 1.1** Any group $H$ satisfying $h_{OD}(H) = k$ is said to be $k$-fold OD-characterizable. Usually, a 1-fold OD-characterizable group is simply called an OD-characterizable group, and it is called quasi OD-characterizable if it is $k$-fold OD-characterizable for some $k > 1$.

This article is a continuation of my investigations of the OD-characterizability of alternating and symmetric groups initiated in [4]. We keep the notation created and the conventions made therein.

In a series of articles (see [4], [8], [9], [11] and [17]), it has been shown that many of the alternating and symmetric groups are $n$-fold OD-characterizable for $n \in \{1, 2, 3, 8\}$. These results are summarized in the following proposition.

**Proposition 1.2** The following statements hold:

(a) The alternating groups $A_p, A_{p+1}, A_{p+2}$ and the symmetric groups $S_p$ and $S_{p+1}$, where $p$ is a prime number, are OD-characterizable.

(b) The alternating group $A_{10}$ is 2-fold OD-characterizable, while the symmetric group $S_{10}$ is 8-fold OD-characterizable.

(c) The alternating groups $A_{p+3}$, where $p \neq 7$ is a prime number less than 100, are OD-characterizable.

(d) The symmetric groups $S_{p+3}$, where $p \neq 7$ is a prime number less than 100, are OD-characterizable or 3-fold OD-characterizable.

In addition, it was shown in [4, Corollary 1.5] that all alternating groups $A_m$ for which $m \leq 100$, except $A_{10}$, are OD-characterizable.

**Proposition 1.3** All alternating groups $A_m$, where $m$ is a natural number less than 100, except $A_{10}$, are OD-characterizable.

These observations convinced us to propose the following conjecture in [4]:

**Conjecture 1.4** All alternating groups $A_m$, with $m \neq 10$, are OD-characterizable.

On the other hand, in recent years we have not found any simple group $S$ with $h_{OD}(S) \geq 3$. Therefore, we asked in [4] the following question:

**Problem 1.5** Is there a simple group $S$ with $h_{OD}(S) \geq 3$?
Our recent investigations show that Conjecture 1.4 does not hold in general. Recently, the authors showed in [6] that the alternating group $A_{125}$ satisfying $h_{OD}(A_{125}) \geq 3$ (see also [7]). Here, we will show that there exist infinite families of alternating groups $A_m$ which are $k$-fold OD-characterizable with $k \geq 3$. We notice that Problem 1.5 is also answered positively through these examples.

**Theorem 1.6** There are infinitely many alternating groups $A_m$ which satisfy $h_{OD}(A_m) > 1$. In particular, there is no upper bound for $h_{OD}(A_m)$.

It is also worth mentioning that a similar description as Proposition 1.3 is exhibited about OD-characterizability of symmetric groups $S_m$, where $m$ is a natural number less than 100 (see [4, Theorem 1.7]). Nevertheless, in checking the list of such groups, we found out that it contains a mistake (in fact, Proposition 4.1 in [4] asserts erroneously that the symmetric group $S_{27}$ is 3-fold OD-characterizable). Therefore, another result of the present article can be stated as follows:

**Theorem 1.7** The symmetric group $S_{27}$ is 38-fold OD-characterizable.

Now, we give a revised list of symmetric groups in question.

**Corollary 1.8** All symmetric groups $S_m$, where $m$ is a natural number less than 100, except $m = 10, 27$, are OD-characterizable or 3-fold OD-characterizable. Moreover, the symmetric group $S_{19}$ is 8-fold OD-characterizable, while the symmetric group $S_{27}$ is 38-fold OD-characterizable.

The following conjecture involving symmetric groups is posed in [4]:

**Conjecture 1.9** All symmetric groups $S_m$, with $m \neq 10$, are OD-characterizable or 3-fold OD-characterizable.

It turns out that a negative answer to this conjecture is provided by symmetric group $S_{27}$ (see also [6, 7]). In addition, we will get many other examples of symmetric groups which are $k$-fold OD-characterizable with $k > 3$.

**Theorem 1.10** There are infinitely many symmetric groups $S_m$ which satisfy $h_{OD}(S_m) > 3$. In particular, there is no upper bound for $h_{OD}(S_m)$.

We conclude the introduction with some further notation and definitions. Given a natural number $m$, we denote by $l_m$ the largest prime less than or equal to $m$ and we let $\Delta(m) = m - l_m$. It is clear that $l_m = m$ (or equivalently $\Delta(m) = 0$) if and only if $m$ is a prime number. Note that if $m > 2$, then $l_m$ is always an odd prime, and so $\Delta(m)$ is even if $m$ is odd. In addition, from the definition, it is easy to see that

$$l_m = l_{m-1} = l_{m-2} = \ldots = l_{m-\Delta(m)+1}.$$   

We shall use the notation $\nu(m)$ (resp. $\nu_a(n)$) to denote the number of types of groups (resp. abelian groups) of order $m$. Clearly, $\nu_a(m) \leq \nu(m)$. We also denote the set of partitions of $m$ by Par$(m)$. It is known that for any prime $p$, $\nu_a(p^m) = |\text{Par}(m)|$. Finally, we use $A_m$ and $S_m$ to denote an alternating and, respectively, a symmetric group of degree $m$.  

3
2 Auxiliary results

In this section we give several definitions and auxiliary results to be used later. The first of them is the following definition of subdirect products.

**Definition 2.1** Let \( n \geq 2 \). A subdirect product of the groups \( G_1, G_2, \ldots, G_n \) is a subgroup \( G \leq G_1 \times G_2 \times \cdots \times G_n \) of the direct product such that the canonical projections \( G \to G_i \) are surjective for all \( i \).

One way of obtaining a subdirect product of two groups is via the fibre product construction. This is illustrated here for two groups. Given some groups \( G_1 \) and \( G_2 \) with normal subgroups \( N_1 \) and \( N_2 \) such that \( G_1/N_1 \) and \( G_2/N_2 \) are isomorphic, we want to construct a group \( G \) having a normal subgroup \( N \) isomorphic to \( N_1 \times N_2 \) such that \( G/N_2 \) is isomorphic to \( G_1 \) and \( G/N_1 \) is isomorphic to \( G_2 \). Notice that we will usually identify \( N_1 \times 1 \) with \( N_1 \) and \( 1 \times N_2 \) with \( N_2 \). To carry out the construction, let \( \pi_1 \) and \( \pi_2 \) be homomorphisms from \( G_1 \) and \( G_2 \) onto some group \( Q \). Now let

\[
G := \{ (g_1, g_2) \in G_1 \times G_2 \mid \pi_1(g_1) = \pi_2(g_2) \}.
\]

It is easy to check that \( G \) constitutes a subgroup of \( G_1 \times G_2 \), and the projection maps onto the coordinates map \( G \) onto \( G_1 \) and \( G_2 \), respectively. We call \( G \) the fibre product associated with \( \pi_1 \) and \( \pi_2 \) (Remak [12] called it the meromorphic product of \( G_1 \) and \( G_2 \) with normal subgroups \( N_1 \) and \( N_2 \)). Also, \( N_1 \times N_2 \) is a normal subgroup of \( G \) and the map \( \pi \) on \( G \) defined by \( \pi((g_1, g_2)) = \pi_1(g_1) \) maps \( G \) onto \( Q \) with kernel \( N_1 \times N_2 \), so \( G/(N_1 \times N_2) \) is isomorphic to \( Q \). In fact, we have

\[
G_1/N_1 \cong G/(N_1 \times N_2) \cong G_2/N_2.
\]

It is a basic observation that every subdirect product of \( G_1 \) and \( G_2 \) is a fibre product (or a meromorphic product) of these groups.

Information on the adjacency of vertices in the prime graphs associated with alternating and symmetric groups can be found in [13, 16]. Consider the function \( S : \mathbb{N} \to \mathbb{N} \), defined as follows: \( S(1) = 1 \) and for \( m > 1 \) with prime factorization \( m = \prod p_i^{\alpha_i} \), \( S(m) = \sum p_i^{\alpha_i} \). Then one has [16 Lemma 4]:

**Lemma 2.2** Let \( m \) and \( n \) be natural numbers. Then there hold:

1. \( A_n \) has an element of order \( m \) if and only if \( S(m) \leq n \) for odd \( m \) and \( S(m) \leq n - 2 \) for even \( m \).

2. \( S_n \) has an element of order \( m \) if and only if \( S(m) \leq n \).

The next two Corollaries are the adjacency criteria for two vertices of \( \Gamma(A_n) \) and \( \Gamma(S_n) \), respectively.

**Corollary 2.3** Let \( p, q \in \pi(A_n) \setminus \{ 2 \} \). Then \( (p \sim q)_{A_n} \) if and only if \( p + q \leq n \), while \( (p \sim q)_{A_n} \) if and only if \( p + 2 \leq n - 2 \).

**Corollary 2.4** Let \( p, q \in \pi(S_n) \). Then \( (p \sim q)_{S_n} \) if and only if \( p + q \leq n \).
The Goldbach conjecture says that every even natural number \( n \) greater than 4 can be written as the sum of two odd primes. In what follows, we will need a stronger conjecture:

**Strong Goldbach Conjecture.** Every even natural number \( n \) greater than six can be written as the sum of two distinct odd primes.

We can now state the connection between the strong Goldbach conjecture and the adjacency of vertices in the prime graph of a symmetric group:

**Theorem 2.5** [2, Theorem 3] The following statements are equivalent:

1. the strong Goldbach conjecture is true;
2. for each even \( n > 6 \), \( \Gamma(S_{n-1}) \subseteq \Gamma(S_n) \).

**Proof.** It follows immediately from Lemma 2.2 (2). \( \square \)

The coincidence criterion for pairwise nonisomorphic symmetric groups (statement (1) of the Lemma 2.6) is obtained modulo strong Goldbach conjecture.

**Lemma 2.6** Let \( m \) and \( n \) be natural numbers with \( 2 \leq m < n \). The prime graphs of symmetric groups \( S_m \) and \( S_n \) are equal if and only if \( m = n - 1 \) and one of the following holds:

1. both \( n \) and \( n - 2 \) are non-prime odd numbers.
2. \( n = 4 \) or 6.

**Proof.** \((\Rightarrow)\) In the case when \( n \leq 6 \), it is a straightforward verification. In fact, the equality of the prime graphs \( \Gamma(S_n) \) and \( \Gamma(S_{n-1}) \), for \( n \in \{ 4, 6 \} \), can be obviously verified using Corollary 2.4. Assume now that \( n > 6 \). We first claim that \( m = n - 1 \). If not, then \( m < n - 1 \) and hence one of the numbers \( n \) and \( n - 1 \) is even. Since \( n \geq 7 \), it follows from strong Goldbach conjecture that there exist two distinct odd primes \( p \) and \( q \) with \( m \leq n - 2 < p + q \leq n \). Hence \( p \) is adjacent to \( q \) in \( \Gamma(S_n) \), while \( p \) is nonadjacent to \( q \) in \( \Gamma(S_m) \), so these graphs are not equal. This contradiction shows \( m = n - 1 \) as claimed.

By Theorem 2.5, we may assume that \( n \) is an odd number. Assume that \( \Gamma(S_n) \neq \Gamma(S_{n-1}) \). The sets of vertices are distinct if and only if \( n \) is a prime. If \( n \) is a composite number, then the sets of vertices are equal, and so the sets of edges should be distinct. Hence there exist primes \( p, q \in \pi(S_n) \) with \( n - 1 < p + q \leq n \). Then obviously \( p + q = n \), but \( n \) is odd. The only possible case is \( \{ p, q \} = \{ 2, n - 2 \} \), and so \( n - 2 \in \pi(S_n) \) and \( n - 2 \) is a prime.

\((\Leftarrow)\) The conclusion follows immediately from Corollary 2.4 \( \square \)

We need the following lemma to find some infinite families of alternating and symmetric groups which are not OD-characterizable.

**Lemma 2.7** Let \( p \) be an odd prime number. There are infinitely many natural numbers \( n \) such that \( \Delta(p^n) > 4 \).

\(^*\)The idea of proof was borrowed from \[13\]
Proof. Take \( n \) to be an even natural number. Clearly, \( p^n - 4 \) is always composite in this case, and so we need only consider \( p^n - 2 \). Let \( a = p^2 \) and \( b = 2 \). We will now show that given any positive integers \( a, b \), we can find an infinite number of values for \( N \) such that \( a^N - b \) is composite. If \( a^N - b \) is always composite, we are done. Otherwise, there exists a positive integer \( k \) such that \( a^k - b = q > a \) is prime. Then for all positive integers \( m \), we have

\[ a^{k+(q-1)m} \equiv b \pmod{q}, \]

and certainly \( a^{k+(q-1)m} - b > q \) for all \( m > 0 \), so \( a^{k+(q-1)m} - b \) must be composite for all positive integers \( m \).

A possible generalization of Lemma 2.7 is the following statement, which seems intuitive to be true. Let \( p \) be an odd prime. Then, there are infinitely many positive integers \( n \) such that

\[ p^n - 2, \quad p^n - 4, \quad p^n - 6, \quad \ldots, \quad p^n - (p - 1), \]

are composite. Alternatively, this problem can be formulated as follows.

Problem 2.8 Let \( p \) be an odd prime. Do there exist infinitely many positive integers \( n \) such that \( \Delta(p^n) > p \).

3 The symmetric group \( S_{27} \)

The aim of this section is to find the number of non-isomorphic groups with the same order and degree pattern as the symmetric group \( S_{27} \). Indeed, we will show that there are 38 such groups.

Proof of Theorem 1.7 Let \( G \) be a group satisfying the following conditions:

1. \( |G| = |S_{27}| = 2^{23} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \) and
2. \( D(G) = D(S_{27}) = (8, 8, 7, 7, 5, 5, 4, 4, 2) \).

Under these conditions, we conclude immediately that \( \Gamma(G) = \Gamma(S_{27}) \) (see also [4, Lemma 2.15]). Letting \( R \) be the solvable radical of \( G \), we break the proof into a number of separate lemmas.

Lemma 3.1 \( R \) is a \( \{2, 3, 5, 7, 11\} \)-group. In particular, \( G \) is nonsolvable.

Proof. First, we show that \( R \) is a \( 23' \)-group. Assume the contrary. Let \( 23 \in \pi(R) \) and let \( x \) be an element of \( R \) of order 23. Put \( C = C_G(x) \) and \( N = N_G(x) \). By the structure of \( \Gamma(G) \), it follows that \( C \) is a \( \{2, 3, 23\} \)-group. Since \( N/C \) is embedded in \( \text{Aut}(\langle x \rangle) \cong \mathbb{Z}_{22} \), \( N \) is a \( \{2, 3, 11, 23\} \)-group. Using Frattini argument we get \( G = RN \), and so \( 19 \in \pi(R) \). Thus \( R \) contains a Hall \( \{19, 23\} \)-subgroup \( L \) of order \( 19 \cdot 23 \). Clearly, \( L \) is cyclic, and so \( (19 \sim 23)_G \), which is a contradiction.

Next, we show that \( R \) is a \( q' \)-group for \( q \in \{13, 17, 19\} \). Let \( q \in \pi(R) \), \( R_q \in \text{Syl}_q(R) \) and \( N = N_G(R_q) \). Again, by Frattini argument \( G = RN \) and
since $R$ is a $23'$-group we deduce that $23$ divides the order of $N$. Let $L$ be a subgroup of $N$ of order $23$. Since $L$ normalizes $R_q$, $LR_q$ is a subgroup of order $23 \cdot |R_q|$, which is abelian. But then $(q \sim 23)G$, which is a contradiction.

Finally, $R$ is a $\{2, 3, 5, 7, 11\}$-group, and since $R \neq G$, it follows that $G$ is nonsolvable. This completes the proof of lemma. □

In what follows, we put $G = G/R$ and $S = \text{Soc}(G)$. Clearly, $S$ is a direct product

$$S = P_1 \times P_2 \times \cdots \times P_m,$$

where the $P_i$ are nonabelian simple groups, and we have

$$S \trianglelefteq G \trianglelefteq \text{Aut}(S).$$

We show now that $S$ is a simple group, or equivalently $m = 1$.

**Lemma 3.2** $m = 1$. In particular, $\overline{G}$ is an almost simple group.

**Proof.** By way of contradiction, let $m \geq 2$. In this case $23$ does not divide $|S|$, for otherwise $(2 \sim 23)G$, which is a contradiction. Hence, for every $i$ we have $P_i \in S_{19}$. Therefore $23 \in \pi(G) \subseteq \pi(\text{Aut}(S))$, and so $23$ divides the order of $\text{Out}(S)$. But

$$\text{Out}(S) = \text{Out}(Q_1) \times \text{Out}(Q_2) \times \cdots \times \text{Out}(Q_r),$$

where $Q_i$ is a direct product of $n_i$ isomorphic copies of a simple group $P_i$ such that

$$S \cong Q_1 \times Q_2 \times \cdots \times Q_r.$$

Therefore, for some $j$, $23$ divides the order of the outer automorphism group of $Q_j$ of $n_j$ isomorphic simple groups $P_j$. Since $P_j \in S_{19}$, it follows that $|\text{Out}(P_j)|$ is not divisible by $23$ (see [3, 15]). Moreover, since $\text{Out}(Q_j) = \text{Out}(P_j) \wr S_{n_j}$, it follows that

$$|\text{Out}(Q_j)| = |\text{Out}(P_j)|^{n_j} \cdot (n_j)!,$$

it follows that $n_j \geq 23$, and so $2^{46}$ must divide the order of $G$, which is a contradiction. Therefore $m = 1$ and $S = P_1$. □

**Lemma 3.3** There are exactly $38$ possibilities for the group $G$.

**Proof.** By Lemma 3.2, we have $S \trianglelefteq G \trianglelefteq \text{Aut}(S)$, where $S$ is a nonabelian simple group in $S_{23}$, and so $\{13, 17, 19, 23\} \cap \pi(\text{Out}(S)) = \emptyset$, (see [3, 15]). Now, it follows from Lemma 3.1 and condition (1) that

$$|S| = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e \cdot 13^2 \cdot 17 \cdot 19 \cdot 23,$$

where $2 \leq a \leq 23$, $0 \leq b \leq 13$, $0 \leq c \leq 6$, $0 \leq d \leq 3$ and $0 \leq e \leq 2$. Comparing this with the nonabelian simple groups in $S_{23}$, we obtain $S \cong A_{26}$ or $A_{27}$. We refer to [15] for the list of nonabelian simple groups in $S_{23}$. In the sequel, we deal with two cases separately.
(1) \( S \cong A_{27} \). In this case, we have \( A_{27} \leq G/R \leq S_{27} \). Thus \( G/R \cong A_{27} \) or \( G/R \cong S_{27} \). If \( G/R \cong A_{27} \), then \(|R| = 2\). Clearly, \( R \leq \mathbb{Z}(G) \) and \( G \) is a central extension of \( R \cong \mathbb{Z}_2 \). If \( G \) splits over \( R \), then \( G \cong \mathbb{Z}_2 \times A_{27} \), otherwise we have \( G \cong \mathbb{Z}_2 \cdot A_{27} \) (non-split extension). Next, we assume \( G/R \cong S_{27} \). In this case \( R = 1 \) and so \( G \cong S_{27} \). Finally, in the case when \( S \cong A_{27} \) there are three possibilities for \( G \).

(2) \( S \cong A_{26} \). In this case, we have \( A_{26} \leq G/R \leq S_{26} \), and so \( G/R \cong A_{26} \) or \( G/R \cong S_{26} \). If \( G/R \cong A_{26} \), then \(|R| = 54\). We claim that the only possibilities for \( G \) are \( G = R \times A_{26} \), where \( R \) is an arbitrary group of order 54, and \( G = Q \times (\mathbb{Z}_2 \cdot A_{26}) \) where \( Q \) is an arbitrary group of order 27. In the first case, since there are exactly 15 groups of order 54, there are 15 possibilities for \( G \). In the second case, since there are just 5 groups of order 27, there are 5 possibilities for \( G \). To prove our claim, we first observe that the automorphism group of \( R \) has order smaller than \(|A_{26}|\).

Note that \( M/C \) splits over \( A \) and \( M/R \) is a normal subgroup of \( G \). Since \( G/R \) is simple, it follows that \( CR \) is simple, and thus \( |CR| = 26 \) and \( CR \) is an abelian group of order 26, with \( C \) a normal subgroup of \( CR \). We call such a 

If \( G/R \cong S_{26} \), then \(|R| = 27\). Actually, we want to find (up to isomorphism) all groups \( G \) having a normal subgroup \( R \) of order 27 such that 

\[
|S_{26} \times S_6| = (a!)(b!) < (26!/2).
\]

This proves that the action is transitive, and it follows that \( R \) is elementary abelian. Now

\[
|\text{Aut}(R)| = |\text{GL}(3, 3)| < |A_{26}| \leq |M/C|.
\]

This is a contradiction, so \( RC = M \).

Let \( Z = Z(R) = R \cap C \). Then \( C/Z \) is isomorphic to \( M/R \), which is \( A_{26} \). The Schur multiplier of \( A_{26} \) has order 2, and it follows that \( C' \cap Z \)
is trivial, and thus $C'$ is a normal subgroup of $M$ isomorphic to $A_{26}$. It follows that $M$ is the direct product of $R$ and $C'$. Let be write $A = C'$, so $A$ is isomorphic to $A_{26}$. Note that $A$ is characteristic in $M$, so $A$ is normal in $G$. Now $G/R \cong S_{26}$ and $G/A$ has order 54, so $G$ is a subdirect product of $G_1 = S_{26}$ and $G_2 = G/A$. (Note that, every group of order 54 has a normal subgroup of index 2.) Actually, since each of these groups has a unique homomorphism onto the group of order 2, it follows that $G$ can be constructed as follows: let $\pi_1$ and $\pi_2$ be the homomorphisms from $G_1$ and $G_2$ onto the group $\mathbb{Z}_2$. Now, we consider the fibre product associated with $\pi_1$ and $\pi_2$, that is

$$G = \{ (a_1, a_2) \mid \pi_1(a_1) = \pi_2(a_2) \}.$$ 

Clearly, $A_{26} \times P$ is a normal subgroup of $G$ and the map $\pi$ on $G$ defined by

$$\pi((a_1, a_2)) = \pi_1(a_1),$$

maps $G$ onto $\mathbb{Z}_2$ with kernel $A_{26} \times P$, so $G/(A_{26} \times P)$ is isomorphic to $\mathbb{Z}_2$. This gives 15 groups, including all direct products of $S_{26}$ with groups of order 27.

This completes the proof of lemma and Theorem 1.7. □

4 Non OD-characterizable alternating groups

We start this section with a result of M. A. Zvezdina [18, Theorem] which is concerning simple groups whose prime graphs coincide with the prime graph of an alternating simple group. More precisely, she proved:

**Lemma 4.1** [18, Theorem] Let $G$ be an alternating group $A_n$, $n \geq 5$, and let $S$ be a finite simple group. Then the prime graphs of $G$ and $S$ coincide if and only if one of the following holds:

(a) $G = A_5$, $S = A_6$;
(b) $G = A_6$, $S = A_5$;
(c) $G = A_7$, $S \in \{L_2(49), U_4(3)\}$;
(d) $G = A_9$, $S \in \{J_2, S_6(2), O_8^+(2)\}$;
(e) $G = A_n$, $S = A_{n-1}$, $n$ is odd, and the numbers $n$ and $n - 4$ are composite.

Although the groups in the statement (e) of this lemma have the same prime graph and so the same degree pattern, but they have different orders. In fact, we have

$$|A_n| = |A_{n-1}| \times n.$$
Now, if we can choose the number \( n \) such that \( \pi(n) \) is contained in the set of vertices of the prime graph \( \Gamma(A_n) \) which are joined to all other vertices, then the groups \( A_n \) and \( A_{n-1} \times H \), where \( H \) is an arbitrary group of order \( n \), have the same order and degree pattern. This will enable us to give a positive answer to problem 1.5.

Let \( G \) be a finite group satisfying \( |G| = |A_m| = m!/2 \) and \( D(G) = D(A_m) \). By [4, Lemma 2.15], the prime graph \( \Gamma(G) \) coincides with \( \Gamma(A_m) \). Simply, \( \Gamma(G) \) is a graph with vertex set \( \pi(G) = \{2, 3, 5, \ldots, l_m\} \) in which two distinct vertices \( r \) and \( s \) are joined by an edge iff \( r + s \leq m \) if \( r \) and \( s \) are odd primes and \( r + 2 \leq m - 2 \) if \( s = 2 \).

If \( \triangle(m) \leq 2 \) and \( p = l_m \), then we will deal with the alternating groups \( A_p \), \( A_{p+1} \) and \( A_{p+2} \), which are OD-characterizable (see [10, Theorem 1.5]). Therefore, we may consider the alternating groups \( A_m \) for which \( \triangle(m) \geq 3 \).

Referring to the already mentioned fact (Proposition 1.3) that all alternating groups \( A_m \), \( 10 \neq m \leq 100 \), are OD-characterizable, we restrict our attention to the alternating groups \( A_m \) where \( m > 100 \). In what follows, we prove the following result, which shows that there is an infinite family \( \{A_m\} \) of alternating groups such that \( h_{OD}(A_m) \geq 3 \).

**Proposition 4.2** Let \( m \) be an odd number satisfying \( \triangle(m) > 4 \) and \( \pi(m) \subseteq \pi(\triangle(m)! \)). Then \( h_{OD}(A_m) \geq 2 \). In particular, if \( \triangle(3^n) > 4 \) (resp. \( \triangle(5^n) > 4 \)), then \( h_{OD}(A_{3^n}) \geq 16 \) (resp. \( h_{OD}(A_{5^n}) \geq 4 \)).

**Proof.** First of all, it follows by Lemma 2.17 in [4] that \( \pi(\triangle(m)! \)) \subseteq \Lambda(A_m) \), this would mean that every vertex in \( \pi(\triangle(m)! \)) \) and so in \( \pi(m) \) is adjacent to all other vertices in \( \pi(A_m) \). Furthermore, since \( m \) is an odd number with \( \triangle(m) > 4 \), the prime graphs \( \Gamma(A_m) \) and \( \Gamma(A_{m-1}) \) coincide by Lemma 1.1. Now, if \( H \) is an arbitrary group of order \( m \), then the groups \( A_m \) and \( A_{m-1} \times H \) have the same order and degree pattern, and hence

\[
h_{OD}(A_m) \geq 1 + \nu(m) \geq 2.
\]

In the case when \( \triangle(3^n) > 4 \), it is routine to check that \( n \geq 7 \), and so

\[
h_{OD}(A_{3^n}) \geq 1 + \nu(3^n) \geq 1 + \nu_3(3^n) = 1 + |\text{Par}(n)| \geq 1 + |\text{Par}(7)| = 16.
\]

Similarly, if \( \triangle(5^n) > 4 \), then \( n \geq 3 \), and we obtain

\[
h_{OD}(A_{5^n}) \geq 1 + \nu(5^n) \geq 1 + \nu_5(5^n) = 1 + |\text{Par}(n)| \geq 1 + |\text{Par}(3)| = 4.
\]

The proof is complete. \( \square \)

**Proof of Theorem 1.6** The result follows immediately from Lemma 2.7 and Proposition 1.2. Note that, the proof of Proposition 1.2 shows that there is no upper bound to \( h_{OD}(A_m) \). \( \square \)

By Proposition 4.2, we can find many examples of alternating groups \( A_m \) with \( h_{OD}(A_m) \geq 3 \). We point out here some of them.
Some alternating groups $A_m$, $m \leq 1000$, with $h_{OD}(A_m) \geq 3$.

In this case, we obtain the following simple groups amongst $A_m$ with $h_{OD}(A_m) \geq 3$ (see Table 1):

$$A_{125}, A_{147}, A_{189}, A_{539}, A_{625}, A_{875},$$

and for each of these groups, we have (see Table 1):

1. $h_{OD}(A_{125}) \geq 1 + \nu(125) = 6$ (see also [6]),
2. $h_{OD}(A_{147}) \geq 1 + \nu(147) = 7$,
3. $h_{OD}(A_{189}) \geq 1 + \nu(189) = 14$,
4. $h_{OD}(A_{539}) \geq 1 + \nu(539) = 3$,
5. $h_{OD}(A_{625}) \geq 1 + \nu(625) = 16$,
6. $h_{OD}(A_{875}) \geq 1 + \nu(875) = 6$.

Table 1.

| $m$ | $m-4$ | $l_m$ | $\triangle(m)$ | $\pi(\triangle(m)!)$ | $\nu(m)$ |
|-----|-------|-------|-----------------|-----------------------|----------|
| 125 = $5^3$ | 121 = $11^2$ | 113 | 12 | 2, 3, 5, 7, 11 | 5 |
| 147 = $3 \cdot 7^2$ | 143 = $11 \cdot 13$ | 139 | 8 | 2, 3, 5, 7 | 6 |
| 189 = $3^3 \cdot 7$ | 185 = $5 \cdot 37$ | 181 | 8 | 2, 3, 5, 7 | 13 |
| 539 = $7^2 \cdot 11$ | 535 = $5 \cdot 107$ | 523 | 16 | 2, 3, 5, 7, 11, 13 | 2 |
| 625 = $5^4$ | 621 = $3^3 \cdot 23$ | 619 | 6 | 2, 3, 5 | 15 |
| 875 = $5^3 \cdot 7$ | 871 = $13 \cdot 67$ | 863 | 12 | 2, 3, 5, 7, 11 | 5 |

We shall use Lemma 2.7 to show that there are infinite families of alternating groups $A_n$, with $h_{OD}(A_n) \geq 3$.

An infinite family of alternating groups $A_{3^n}$, with $h_{OD}(A_{3^n}) \geq 136$.

The existence of infinite number of values $n$ for which $\triangle(3^n) > 4$ is immediate from Lemma 2.7. Indeed, if we take $n$ to be a natural number such that $n \equiv 14 \pmod{144}$, then it follows directly that $\{7, 17\} \subseteq \pi(3^n - 2)$ and similarly $\{5, 19\} \subseteq \pi(3^n - 4)$, so the numbers $3^n - 2$ and $3^n - 4$ are composite. This shows that $\triangle(3^n) \geq 8$, for all $n \equiv 14 \pmod{144}$ (note that, this provides an alternate proof of Lemma 2.7 for $p = 3$). Reasoning as in the proof of preceding Proposition 4.2, we have

$$h_{OD}(A_{3^n}) \geq 1 + \nu(3^n) \geq 1 + \nu_a(3^n) = 1 + |\Par(n)| \geq 1 + |\Par(14)| = 136,$$

where $\Par(n)$ denotes the set of all partitions of $n$. In particular, we have $h_{OD}(A_{3^{14}}) \geq 136$.

An infinite family of alternating groups $A_{5^a}$, with $h_{OD}(A_{5^a}) \geq 4$.

By Lemma 2.7 again, there exist infinitely many values of $n$ for which $\triangle(5^n) > 4$. Now for every such $n$, we have $h_{OD}(A_{5^n}) \geq 4$ by Proposition 4.2 (see also [7]).
5 On the symmetric groups $S_m$ with $h_{OD}(S_m) \geq 4$

In this section we are looking for finite non-isomorphic groups having the same order and degree pattern as a symmetric group. Suppose that $G$ is a finite group satisfying $|G| = |S_m| = m!$ and $D(G) = D(S_m)$, for some natural number $m$. First of all, we conclude from [4] Lemma 2.15 that the prime graph $\Gamma(G)$ coincides with $\Gamma(S_m)$. Actually, $\Gamma(G)$ is a graph with vertex set $\pi(G) = \{2, 3, 5, \ldots, l_m\}$ in which two distinct vertices $r$ and $s$ are joined by an edge iff $r + s \leq m$. In the case when $\Delta(m) \leq 1$, we deal with the symmetric groups $S_p$ and $S_{p+1}$, which are OD-characterizable by [10] Theorem 1.5. We now consider the symmetric groups $S_m$ for which $\Delta(m) \geq 2$, that is

$$S_{p+2}, S_{p+3}, S_{p+4}, \ldots, S_{p+\Delta(m)},$$

where $p = l_m$. On the other hand, in view of [4] Theorem 1.7 and Theorem 4.1 (see also [4, Table 7]). Now, if $H$ and $K$ are two arbitrary groups of order $m$ and $2m$, respectively, then the groups $S_m$, $\mathbb{Z}_2 \times A_m$, $\mathbb{Z}_2 \cdot A_m$, $S_{m-1} \times H$, $(\mathbb{Z}_2 \cdot A_{m-1}) \times H$, $(\mathbb{Z}_2 \cdot A_{m-1}) \times H$ and $A_{m-1} \times K$, have the same order and degree pattern, and hence $h_{OD}(S_m) \geq 4$. The proof is complete. $\square$

Proof of Theorem 1.10 The result follows immediately from Lemma 2.7 and Proposition 5.1. Note that, the proof of Proposition 5.1 shows that there is no upper bound to $h_{OD}(S_m)$. $\square$

Considering Proposition 5.1 we can now find many examples of symmetric groups $S_m$ satisfying $h_{OD}(S_m) \geq 4$. We point out here some of them.

(5.a) Some symmetric groups $S_m$, $m \leq 1000$, with $h_{OD}(S_m) \geq 4$.

As before in (4.a), we can obtain the following symmetric groups amongst $S_m$, which are $k$-fold OD-characterizable with $k \geq 4$ (see Table 1):

$$S_{125}, S_{147}, S_{189}, S_{539}, S_{625}, S_{875}.$$ 

The case $S_{125}$ had already been studied in [6].

(5.b) There is an infinite family of symmetric groups $S_{p^n}$, with $p \in \{3, 5\}$, such that $h_{OD}(S_{p^n}) \geq 4$. 

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Actually, reasoning as before in (4.b) and (4.c), there are an infinite number of values \( n \) for which \( \triangle(p^n) > 4 \), and the result is now immediate from Proposition 5.1 (see also [7]).

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