The method proposed by T. I. Zelenjak is applied to the mean curvature flow in the plane. A new type of monotonicity formula for star-shaped curves is obtained. Bibliography: 6 titles.

1. Introduction

One of the classical problems combining geometry and PDE’s is the mean curvature flow. Gerhard Huisken proved that convex surfaces converge in finite time to points in asymptotically spheric fashion (see [3]). In dimension two, this result was proved by M. Gage and R. Hamilton in [1]. In [2], Grayson showed that in the plane any closed embedded curve shrinks to a convex one in finite time and thus also shrinks to a point. This result is not true in higher dimensions where other types of singularities may occur if the initial curve is not convex (see [6]).

A powerful tool in proving many properties of solutions is the monotonicity formula of Gerhard Huisken (see [4]). In the present paper, we apply a general method developed by T. Zelenjak in [5] to mean curvature flow in the plane and derive the monotonicity formula of Huisken. We also derive a new monotonicity formula for star-shaped curves. The presented approach is general and systematic, and we believe that it can be very useful in generalizations of the mean curvature flow, where no monotonicity formula is known. Our main motivation was the derivation of such a monotonicity formula for the anisotropic mean curvature flow, which still remains a challenge.

2. The formulation of the problem

We consider a closed curve in $\mathbb{R}^2$ moving by its curvature with an anisotropy given by a function $g$:

$$\partial_t \gamma = g(\nu)\kappa \nu,$$

where $\gamma : \mathbb{R}_+ \times S^1 \to \mathbb{R}^2$ is the curve parametrization, $\kappa$ is the curvature, and $\nu$ is the normal vector.

Note that in this form we fix a certain parametrization which has no tangential component. For a general parametrization, we will get

$$\partial_t \gamma \cdot \nu = g(\nu)\kappa.$$ (1)

Now if $\gamma(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}$, then

$$\begin{pmatrix} \partial_t u_1 \\ \partial_t u_2 \end{pmatrix} = g(u_1', u_2') \frac{-u_1'' u_2' + u_1' u_2''}{(u_1'^2 + u_2'^2)^2} \begin{pmatrix} -u_2' \\ u_1' \end{pmatrix},$$

where $'$ means the $x$-derivative.

Assume that the first singularity appears at the point 0 after finite time $T$. We rescale the parametrization in the following way

$$\tau = -\log(T - t), \tilde{\gamma}(\tau, x) = (T - t)^{-\frac{1}{2}} \gamma(t, x)$$
and arrive at
\[
\begin{pmatrix}
\frac{\partial \tau}{\partial v_1} \\
\frac{\partial \tau}{\partial v_2}
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} + g(v'_1, v'_2) - \frac{v''_1 v'_2 + v'_1 v''_2}{(v'^2_1 + v'^2_2)^2}
\begin{pmatrix}
v'_2 \\
v'_1
\end{pmatrix},
\] (2)
where \( \tilde{\gamma}(\tau, x) = \begin{pmatrix} v_1(\tau, x) \\ v_2(\tau, x) \end{pmatrix} \).

In the present paper, a significant part of the computations is carried out for anisotropic flow, but at the final part of the computations, we are able to deal with the isotropic case only.

**Remark 1.** Note that in the isotropic case \( g \equiv 1 \), the stationary solution of (2) is the circle with the radius \( \sqrt{2} \).

**Main result.** For the solutions of (2) with \( g \equiv 1 \), which are star-shaped with respect to the origin, we prove the monotonicity formula
\[
\frac{d}{d\tau} \int _{S^1} \sqrt{v'^2_1 + v'^2_2} \left( f(\psi) + \left( \frac{\log(v'^2_1 + v'^2_2)}{2} + \frac{v'^2_1 + v'^2_2}{4} \right) \cos \psi \right) dx
= -\int _{S^1} |{\partial \tau, \gamma} \cdot {\nu}|^2 \sqrt{v'^2_1 + v'^2_2} \frac{1}{\cos \psi} dx,
\] (3)
where \( \psi \) is the angle between the outer normal direction \( (v'_2, -v'_1) \) and the position vector \( (v_1, v_2) \), and
\[
f(\psi) = \psi \sin \psi + \cos \psi \log(\cos \psi)
\] (4)
is a positive, even, and convex function defined in the interval \((-\pi/2, \pi/2)\) (see Fig. 1).

![Fig. 1](image)

**Remark 2.** One can check numerically that the function
\[
f(\psi) + \alpha \cos \psi,
\]
is positive and strictly convex in the interval \((-\pi/2, \pi/2)\) for \( \alpha < 1 \). Since asymptotically the curve converges to a circle of radius \( \sqrt{2} \), the coefficient of the \( \cos \psi \)-term on the left-hand side of formula (3) is
\[
\frac{\log(v'^2_1 + v'^2_2)}{2} + \frac{v'^2_1 + v'^2_2}{4} \approx \frac{\log 2}{2} + \frac{1}{2} = 0.8465 \ldots
\]
This means that the quantity depending on $\psi$ on the left-hand side of (3), at least for large times, is a convex function with minimum in the origin, and thus measures the $W^{1,2}$-deviation of the curve from its final limit, which is the circle.

3. Monotonicity formula by Zelenjak’s approach

In this section, we adapt the method proposed by T. I. Zelenjak in [5] to the mean curvature flow in the plane.

For system (2), we want to obtain a monotonicity formula of the form

$$\frac{d}{dt} \int F(v_1, v_2, v_1', v_2') \, dx = - \int |\partial_\gamma \cdot \nu|^2 \rho(v_1, v_2, v_1', v_2') \, dx,$$

(5)

where $\rho$ is positive.

Note that in the isotropic case $g \equiv 1$, the well-known Huisken’s monotonicity formula (see [4,6]) corresponds in this notations to

$$F(\xi_1, \xi_2, \eta_1, \eta_2) = \rho(\xi_1, \xi_2, \eta_1, \eta_2) = e^{-\frac{|\xi|^2}{4}} |\eta|.$$  

Differentiating the left-hand side of (5) and integrating by parts, we get

$$\partial_\gamma v_1 \left[ \frac{\partial F}{\partial \xi_1} - \frac{\partial^2 F}{\partial \xi_1 \partial \eta_1} v_1' - \frac{\partial^2 F}{\partial \xi_2 \partial \eta_1} v_2' - \frac{\partial^2 F}{\partial \eta_1 \partial \eta_2} v_1'' \right] + \partial_\gamma v_2 \left[ \frac{\partial F}{\partial \xi_2} - \frac{\partial^2 F}{\partial \xi_2 \partial \eta_2} v_1' - \frac{\partial^2 F}{\partial \eta_1 \partial \eta_2} v_2' - \frac{\partial^2 F}{\partial \eta_2 \partial \eta_2} v_2'' \right].$$

(6)

(7)

On the right-hand side of (5), we use (2) to obtain

$$-\rho(\xi, \eta) \frac{\partial_\gamma v_1 v_2' + \partial_\gamma v_2 v_1'}{(v_1^2 + v_2^2)^{\frac{3}{2}}} \left( \frac{-v_1 v_2' + v_2 v_1'}{2(v_1^2 + v_2^2)^{\frac{3}{2}}} + g(v_1', v_2') \frac{-v_1'' v_2' + v_2'' v_1'}{(v_1^2 + v_2^2)^{\frac{3}{2}}} \right)$$

(8)

$$= -\rho \partial_\gamma v_1 \left[ \frac{v_1 v_2^2 - v_2 v_1 v_2'}{2(v_1^2 + v_2^2)^2} + g(v_1', v_2') \frac{v_2^2}{(v_1^2 + v_2^2)^2} v_1'' - g(v_1', v_2') \frac{v_1 v_2'}{(v_1^2 + v_2^2)^2} v_2'' \right]$$

(9)

$$-\rho \partial_\gamma v_2 \left[ \frac{v_2 v_1^2 - v_1 v_1 v_2'}{2(v_1^2 + v_2^2)^2} - g(v_1', v_2') \frac{v_1^2}{(v_1^2 + v_2^2)^2} v_1'' + g(v_1', v_2') \frac{v_2^2}{(v_1^2 + v_2^2)^2} v_1'' \right].$$

(10)

Remark 3. Note that the calculation use only a weak form of (2) similar to (1), which means that the obtained result is still valid for any parametrization of the curve.

Now we require the terms inside the square brackets in (6) and (9) as well as in (7) and (10) to be equal. Moreover, we require that

$$D^2_\eta F(\xi, \eta) = \rho(\xi, \eta) g(\eta) \left( \begin{array}{cc} \frac{\eta_1^2}{(\eta_1 + \eta_2)^2} & \frac{\eta_2^2}{(\eta_1 + \eta_2)^2} \\ \frac{\eta_1 \eta_2}{(\eta_1 + \eta_2)^2} & \frac{\eta_1 \eta_2}{(\eta_1 + \eta_2)^2} \end{array} \right) = \rho(\xi, \eta) g(\eta) |\eta|^{-1} D^2 |\eta|,$$

(11)

where $g$ is homogeneous of order 0.

Introducing radial coordinates $(|\eta|, \phi)$ for $\eta$, it is easy to check that given $\rho$ one can find $F$ satisfying (11) if and only if

$$\rho(\xi, \eta) = c(\xi, |\eta|),$$

where $c$ is homogeneous of order 0 with respect to $\eta$, and

$$\int_0^{2\pi} c(\xi, \phi) g(\phi) \cos \phi \, d\phi = \int_0^{2\pi} c(\xi, \phi) g(\phi) \sin \phi \, d\phi = 0 \quad \text{for all } \xi.$$
Moreover, $F$ is homogeneous of order 1 in $\eta$ and we may write $F(\xi, \eta) = f(\xi, \phi)|\eta|$. Formula (11) becomes now

$$D^2 F = (\partial_{\phi\phi} f + f)D^2|\eta| = c(\xi, \phi)g(\phi)D^2|\eta|.$$  

(12)

The solution of $f'' + f = h$ can be calculated by the following formula:

$$f(\phi) = c_1 \cos \phi + c_2 \sin \phi + \int_0^\phi h(\tau) \sin(\phi - \tau) d\tau.$$  

(13)

To complete the proof of equalities (6)=(9) and (7)=(10), it suffices to verify that

$$\frac{\partial^2 F}{\partial \xi_1 \partial \eta_1} - \frac{\partial^2 F}{\partial \xi_2 \partial \eta_1} \eta_1 - \frac{\partial^2 F}{\partial \xi_2 \partial \eta_2} \eta_2 = \rho \frac{-\xi_1\eta_2^2 + \xi_2\eta_1\eta_2}{2(\eta_1^2 + \eta_2^2)}$$  

(14)

and

$$\frac{\partial^2 F}{\partial \xi_2} - \frac{\partial^2 F}{\partial \xi_1 \partial \eta_2} \eta_1 - \frac{\partial^2 F}{\partial \xi_2 \partial \eta_2} \eta_2 = \rho \frac{-\xi_2\eta_1^2 + \xi_1\eta_1\eta_2}{2(\eta_1^2 + \eta_2^2)}.$$  

(15)

After differentiating these equations in $\eta_1$ and $\eta_2$, respectively, we obtain

$$-\frac{\partial^2 F}{\partial \xi_1 \partial \eta_1} \eta_1 - \frac{\partial^3 F}{\partial \xi_2 \partial \eta_1^2} \eta_1 = \partial_{\eta_1} \left( \rho \frac{-\xi_1\eta_2^2 + \xi_2\eta_1\eta_2}{2(\eta_1^2 + \eta_2^2)} \right)$$

and

$$-\frac{\partial^3 F}{\partial \xi_1 \partial \eta_1} \eta_1 - \frac{\partial^3 F}{\partial \xi_2 \partial \eta_1^2} \eta_1 = \partial_{\eta_2} \left( \rho \frac{-\xi_2\eta_1^2 + \xi_1\eta_1\eta_2}{2(\eta_1^2 + \eta_2^2)} \right),$$

where we can substitute the value of $D^2 F$ from (11). The two equations turn out to be the same and can be written in terms of $c$ as follows:

$$2g(\eta)\langle \eta, D\xi \rangle - |\eta|^2 \langle \xi, D\eta \rangle = -c \cdot (\xi, \eta).$$  

(16)

**Remark 4.** If we differentiate (14) with respect to $\eta_2$ and add (15) differentiated with respect to $\eta_1$, then the result is the same as (16).

Taking $c = e^b$ and rewriting (16) in polar coordinates in $\eta$ variable, we arrive at

$$\nabla_{\xi_1, \xi_2, \phi} b \cdot \begin{pmatrix} 2g(\phi) \cos \phi \\ 2g(\phi) \sin \phi \\ \xi_1 \sin \phi - \xi_2 \cos \phi \end{pmatrix} = -\xi_1 \cos \phi - \xi_2 \sin \phi.$$  

The coordinate transformation

$$\tilde{\xi}_1 = \xi_1 \cos \phi + \xi_2 \sin \phi,$$

$$\tilde{\xi}_2 = \xi_1 \sin \phi - \xi_2 \cos \phi,$$

$$\tilde{\phi} = \phi,$$

brings us to the following first order linear PDE:

$$\nabla_{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\phi}} b \cdot \begin{pmatrix} 2g(\phi) \tilde{\xi}_2 \\ \tilde{\xi}_1 \tilde{\xi}_2 \\ \tilde{\xi}_2 \end{pmatrix} = -\tilde{\xi}_1.$$  

(17)

In the isotropic case $g \equiv 1$, the solution $b(\tilde{\xi}, \tilde{\phi}) = -\frac{\tilde{\xi}^2}{4}$ gives us Huisken’s famous monotonicity formula.

It remains a challenge to find a solution to (17) in the general anisotropic case, which would correspond to Huisken’s one.
4. A NEW MONOTONICITY FORMULA

We observe that another obvious solution to (17) different from Huisken’s one is

\[ b(\tilde{\xi}, \phi) = -\log|\tilde{\xi}_2|. \]

This gives the function

\[ \rho(\xi, \eta) = |\eta|c(\xi, \eta) = \frac{|\eta|}{|\xi_1\sin\phi - \xi_2\cos\phi|} = \frac{|\eta|^2}{|\langle\xi, \eta\rangle|}, \]

where \( \eta \) is the vector \( \eta \) rotated by 90 degrees clockwise and thus showing in outer normal direction.

From now on, we consider the isotropic case \( g \equiv 1 \). Obviously, we cannot solve (12) globally, because \( \rho \) is not integrable. However if the domain is always star-shaped with respect to the origin and the angle \( \psi \) between \( \xi \) and \( \eta \) remains between \(-\pi/2\) and \( \pi/2\), then we can solve (12) locally. Let us solve the equation

\[ \partial_\psi f + f = \frac{1}{\cos \psi} \]

in the interval \((-\pi/2, \pi/2)\). The general solution is

\[ f(\psi) + a(\xi)\cos \psi + b(\xi)\sin \psi, \]

where

\[ f(\psi) = \psi\sin \psi + \cos \psi \log(\cos \psi). \]

First we take \( a = b = 0 \).

As mentioned before, \( f(\psi) \) is a positive, bounded, convex, and even function in the interval \((-\pi/2, \pi/2)\) (see Fig. 1). The corresponding function \( F \) is

\[ F(\xi, \eta) = \frac{|\eta|}{|\xi|}f(\psi) = \frac{|\eta|}{|\xi|}(\psi\sin \psi + \cos \psi \log(\cos \psi)), \]

where \( \psi \) is the angle between the position vector \( \xi = (v_1, v_2) \) and the outer normal \( \nu \) directed to \((\eta_2, -\eta_1) = (v_2', -v_1')\).

Now we need to check whether the function \( F \) satisfies equations (14) and (15). The answer is no. Relations (14) and (15) yield

\[ \frac{\eta_2}{|\xi|^2} \neq -\frac{\eta_2}{2} \]

and

\[ -\frac{\eta_1}{|\xi|^2} \neq -\frac{\eta_1}{2}, \]

respectively (see Remark 4). This means that there is an additional term in formula (5),

\[ \frac{d}{d\tau} \int_{S1} \sqrt{\frac{v_1'^2 + v_2'^2}{v_1^2 + v_2^2}} f(\psi) \, dx + \int_{S1} |\partial_\nu \gamma \cdot \nu|^2 \sqrt{\frac{v_1'^2 + v_2'^2}{v_1^2 + v_2^2}} \frac{1}{\cos \psi} \, dx \]

\[ = -\int_{S1} (v_2'\partial_\nu v_1 - v_1'\partial_\nu v_2)(\frac{1}{2} + \frac{1}{v_1^2 + v_2^2}) \, dx. \]
5. The “repaired” formula

In order to obtain a monotonicity formula without additional terms, we need to go back to the general solution of (19). The idea is that adding to $F$ a term linear in $\eta$ makes no problems in (11). Let us find a function $a(r)$ such that the function

$$F(\xi, \eta) = \frac{|\eta|}{|\xi|}f(\psi) + a(|\xi|)|\eta| \cos \psi$$

solves (14) and (15); then there are no additional terms. Substituting $F$, we obtain

$$\frac{\eta_2}{|\xi|^2} - \eta_2\left(\frac{a'(|\xi|)}{|\xi|} + \frac{a(|\xi|)}{|\xi|}\right) = -\frac{\eta_2}{2}$$

and

$$-\frac{\eta_1}{|\xi|^2} + \eta_1\left(\frac{a'(|\xi|)}{|\xi|} + \frac{a(|\xi|)}{|\xi|}\right) = \frac{\eta_1}{2},$$

respectively. Now we need to solve the equation

$$ra'(r) + a(r) = \frac{r^2}{2} + \frac{1}{r}. \quad (23)$$

The solution is $a(r) = \frac{r}{4} + \frac{\log r}{r}$ and

$$F(\xi, \eta) = \frac{|\eta|}{|\xi|}f(\psi) + |\eta|\left(\frac{|\xi|}{4} + \frac{\log |\xi|}{|\xi|}\right) \cos \psi.$$ 

Thus we obtain the following monotonicity formula

$$\frac{d}{dt}\int_{s^1} \sqrt{v_1'^2 + v_2'^2} \left(f(\psi) + \left(\frac{\log(v_1^2 + v_2^2)}{2} + \frac{v_1^2 + v_2^2}{4}\right) \cos \psi\right) dx \quad (24)$$

$$= -\int_{s^1} |\partial_r \gamma \cdot \nu|^2 \sqrt{\frac{v_1'^2 + v_2'^2}{v_1^2 + v_2^2}} \frac{1}{\cos \psi} dx.$$ 

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