Ricci-flat Graphs with Girth Four

Wei Hua HE  
School of Mathematics and Statistics, Guangdong University of Technology,  
Guangzhou 510520, P. R. China  
E-mail: hwh12@gdut.edu.cn

Jun LUO  
School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P. R. China  
E-mail: luojun3@mail.sysu.edu.cn

Chao YANG  
School of Mathematics and Statistics, Guangdong University of Foreign Studies,  
Guangzhou 510006, P. R. China  
E-mail: sokoban2007@163.com

Wei YUAN  Hui Chun ZHANG  
School of Mathematics, Sun Yat-sen University, Guangzhou 510275, P. R. China  
E-mail: yuanw9@mail.sysu.edu.cn zhanghc3@mail.sysu.edu.cn

Abstract  Lin–Lu–Yau introduced a notion of Ricci curvature for graphs and obtained a complete classification for all Ricci-flat graphs with girth at least five. In this paper, we characterize all Ricci-flat graphs of girth four with vertex-disjoint 4-cycles.

Keywords  Ricci curvature, Ricci-flat graph, vertex-disjoint

MR(2010) Subject Classification  05C75

1 Introduction  
Ricci curvature is one of the fundamental notions in Riemannian geometry. Researches about Ricci curvature on manifolds inspires many attempts of transferring this concept to non-smooth spaces, including graphs.

The first notion of Ricci curvature on graphs was given by Chung and Yau in 1996 [6]. As developed for the last two decades, there are mainly two ways to define Ricci curvature on singular spaces: one is utilising the optimal transportation on Wasserstein spaces and the other one is from the viewpoint of Bakry–Emery condition. Adopting the first method, Ollivier found a concept of coarse Ricci curvature for Markov chains on metric spaces and particularly...
on graphs [15], via the $L^1$-Wasserstein spaces. Lott–Villani [13] and Sturm [17] introduced the lower Ricci bounds for metric measure spaces via $L^2$-Wasserstein spaces. Based on these works, Ollivier and Villani together compared these two notions of Ricci curvature for hypercubes [16]. On the other hand, Lin and Yau defined a type of Ricci curvature on graphs via Barky–Emery methods in [12]. By modifying Ollivier’s approach, Lin–Lu–Yau also proposed a slightly different Ricci curvature for graphs [10].

These new concepts have been discussed considerably. Please see, for example [2, 3]. Ollivier’s Ricci curvature being viewed as a function, as well as its relation with Lin–Lu–Yau Ricci curvature are also studied in [4]. Similarly, there are also many interesting works about other curvatures on planar graphs, for example [8, 9] et al.

As a notion from geometry, Ricci curvature on graphs also finds its applications in combinatorics and even in computer science. For instance, we refer to [5] for the applications of Ollivier’s Ricci curvature to the study of coloring of graphs, and also [14] for the employment of Ricci curvature in understanding the internet topology.

A manifold is Ricci-flat if its Ricci curvature vanishes everywhere. In particular, the famous Calabi–Yau manifolds are Ricci-flat and they provide a potential model to describe the physical world [18]. Similarly, a graph is said to be Ricci-flat, if its Ricci curvature (in the referred sense) vanishes identically on each edges. In this article, we only consider Ricci-flat graphs in the sense of Lin–Lu–Yau [11].

A very recent pioneering work by Lin–Lu–Yau [11] and Cushing et al. [7] completely classifies all Ricci-flat graphs with girth at least five.

**Theorem 1.1** ([7, 11]) A Ricci-flat graph with girth at least five is isomorphic to: (1) the infinite path, (2) a cycle of length at least six, (3) the dodecahedral graph, (4) the half-dodecahedral graph, (5) the Petersen graph, or (6) the Triplex graph.

As for graphs with girth four, the authors of [11] also gave infinitely many examples of Ricci-flat graphs. We observe that there exist 4-cycles sharing common edges in all of their examples. Thus, a natural problem is to classify all Ricci-flat graphs with girth four and edge-disjoint, i.e., any two 4-cycles share no common edges.

We first construct a family of examples to show there exist infinitely many Ricci-flat graphs with edge-disjoint 4-cycles as follows.

**Example 1.2** The graphs in Figure 1.

![Figure 1](image_url) A family of Ricci-flat graphs with edge-disjoint 4-cycles

In this example, some 4-cycles in the graph share common vertices. This shows that there are still a great number of instances even if we only focus on graphs with 4-cycles sharing
no edge. Thus it is wise to consider Ricci-flat graphs with girth four but all two 4-cycles are vertex-disjoint, i.e. sharing no common vertices.

For these graphs, we can obtain the following simple characterization, which is the main result of this article:

**Theorem 1.3** A Ricci-flat graph with girth four such that no vertex is shared by two 4-cycles is isomorphic to one of the following two graphs (Figure 2).

![Figure 2](image)

**Remark 1.4** A very recent preprint of Bai et al. [1] also extends Theorem 1.1 in a different direction, by classifying all Ricci-flat graphs with maximum degree at most 4.

The article is organized as follows: Section 2 is for a brief review of the notion of Ricci curvature on graphs and a proof for the fact that graphs in Example 1.2 are Ricci-flat; Section 3 is about the local structures of Ricci-flat graphs; Section 4 is for the proof of Theorem 1.3.

## 2 Preliminaries

We follow Lin–Lu–Yau for the definition of Ricci curvature [10, 11]. Let $G$ be a simple (no loops and no multi-edges) undirected graph with vertex set $V$ and edge set $E$. For $x, y \in V$, let $N(x)$ be the set of neighbors of $x$, $d_x = |N(x)|$ be the degree of vertex $x$, and $d(x, y)$ be the distance between $x$ and $y$ in $G$.

A probability distribution is a function $m : V \to [0, 1]$ with $\sum_{x \in V} m(x) = 1$. To define Ricci curvature for each edge of the graph, we only consider distributions $m^\alpha_x$ in the following form,

$$m^\alpha_x(v) = \begin{cases} 
\alpha, & v = x; \\
1 - \alpha d_x^{-1}, & v \in N(x); \\
0, & \text{otherwise,}
\end{cases}$$

where $\alpha \in [0, 1]$ and $x \in V$.

Let $xy \in E$, and let $m^\alpha_x$ and $m^\alpha_y$ be two distributions. A transportation problem between the two distributions can be stated as a linear programming problem. That is, to find the minimum transportation distance

$$\min \sum_{u, v \in V} d(u, v) X_{uv},$$

where $X_{uv}$ is the transportation matrix.
subject to the constraints

\[
\begin{align*}
\sum_{v \in V} X_{uv} &= m_x^\alpha(u), \quad u \in V; \\
\sum_{v \in V} X_{uv} &= m_y^\alpha(v), \quad v \in V; \\
X_{uv} &\geq 0,
\end{align*}
\]

where the variable \( X_{uv} \) denotes the mass transferred from vertex \( u \) to \( v \).

We define the transportation distance between \( m_x^\alpha \) and \( m_y^\alpha \) to be optimal solution to the above linear programming problem, namely

\[
W(m_x^\alpha, m_y^\alpha) = \min \sum_{u,v \in V} d(u,v)X_{uv}.
\]

It is called the Wasserstein distance.

Ricci curvature can be defined for any unordered pair of vertices \( x \) and \( y \), but for our purpose in this article, we only need the case that \( x \) and \( y \) are adjacent. For any edge \( xy \in E \), it was showed in [10] that the function

\[
1 - \frac{W(m_x^\alpha, m_y^\alpha)}{1 - \alpha}
\]

is increasing on \( \alpha \) and is bounded. The Ricci curvature \( \kappa(x,y) \) is defined to be

\[
\kappa(x,y) = \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1 - \alpha}.
\]

Recall that a graph \( G \) is Ricci-flat if \( \kappa(x,y) = 0 \) for all edges \( xy \in E \).

A function \( f \) over the vertex set \( V \) of \( G \) is said to be \( c \)-Lipschitz if \( |f(u) - f(v)| \leq c \cdot d(u,v) \) for all \( u,v \in V \). By the theory of linear programming, the dual problem of the above defined transportation problem between \( m_x^\alpha \) and \( m_y^\alpha \) is to find the maximum value

\[
\max \sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u))
\]

subject to

\[
|f(u) - f(v)| \leq d(u,v), \quad u,v \in V.
\]

In other words, the maximum is taken over all 1-Lipschitz function \( f \). Because the optimal solution of a linear programming problem is equal to that of its dual problem, we have

\[
W(m_x^\alpha, m_y^\alpha) = \max \sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u)). \tag{2.1}
\]

Immediately, we derive

**Lemma 2.1** ([11]) Let \( f \) be any 1-Lipschitz function. Then

\[
W(m_x^\alpha, m_y^\alpha) \geq \sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u)).
\]

With the aid of the dual formula (2.1), we can easily show graphs in Example 1.2 are Ricci-flat:

**Proof (Graphs in Example 1.2 are Ricci-flat)** Let \( G \) be a graph in Figure 1. From the construction of graphs in Figure 1, it is clear that these graphs have constant Ricci curvature.
Namely, there exists a constant $k \in \mathbb{R}$ such that $\kappa(x, y) = k$ for all edge $xy$. Moreover, this $k$ is independent of the number of vertices. It suffices to show $k = 0$.

We first show $k \leq 0$. Assuming otherwise, due to [10, Theorem 4.1], we have $\text{diam}(G) \leq 2/k$. It contradicts to the construction of graphs in Figure 1. It yields $k \leq 0$.

Now we prove $\kappa(x, y) \geq 0$ for any edge $xy \in E$. Given any $\alpha \in [0, 1]$, it suffices to show

$$W(m_x^\alpha, m_y^\alpha) \leq 1.$$ 

According to the construction of graphs in Figure 1, we know that one end of edge has degree 4 and another one has degree 2. Without loss of generality, we can assume $d_x = 4$ and $d_y = 2$. Set vertices $x_1, x_2, x_3, y_1$ such that $N(y) = \{y_1, x\}$, $N(x) = \{x_1, x_2, x_3, y\}$ and $x_1 \sim y_1$.

From the definition of the distributions, we get

$$m_x^\alpha(u) - m_y^\alpha(u) = \begin{cases} 
\alpha - \frac{1 - \alpha}{2}, & u = x; \\
\frac{1 - \alpha}{4} - \alpha, & u = y; \\
\frac{1 - \alpha}{4}, & u \in \{x_1, x_2, x_3\}; \\
\frac{1 - \alpha}{2}, & u = y_1; \\
0, & \text{otherwise}.
\end{cases}$$

Set $\beta = \frac{1 - \alpha}{4}$. Giving any 1-Lipschitz function $f$, we have

$$\sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u)) = f(x)(\alpha - 2\beta) + f(y)(\beta - \alpha) + \beta \sum_{i=1}^3 f(x_i) - 2\beta f(y_1)$$

$$= (\alpha - \beta)(f(x) - f(y)) + \beta(f(x_2) + f(x_3) - 2f(x)) + 2\beta(f(x_1) - f(y_1)) + \beta(f(x) - f(x_1)). \quad (2.2)$$

Since $f$ is 1-Lipschitz, by (2.2), we conclude that

$$\sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u)) \leq (\alpha - \beta) + 2\beta + 2\beta + \beta$$

$$= \alpha + 4\beta = 1.$$

By (2.1), we obtain

$$W(m_x^\alpha, m_y^\alpha) = \sup_{f \text{ is } 1-\text{Lip}} \sum_{u \in V} f(u)(m_x^\alpha(u) - m_y^\alpha(u)) \leq 1.$$ 

This implies $\kappa(x, y) \geq 0$, and hence concludes the proof. \hfill \Box

### 3 Local Structures

Before our discussion on the local structure of Ricci-flat graphs of girth 4, we recall the following lemma from [11].

**Lemma 3.1 ([11])** Suppose that an edge $xy$ in a graph $G$ is not in any 3-cycles or 4-cycles, and assume $d_x \leq d_y$. Then one of the following statements holds.
1. $d_x = d_y = 2$, and $xy$ is not in any 5-cycle.
2. $d_x = d_y = 3$, and $xy$ is shared by two 5-cycles.
3. $d_x = 2, d_y = 3$. Let $x_1$ be the other neighbor of $x$ besides $y$, and let $y_1$ and $y_2$ be the two neighbors of $y$ besides $x$, then $\{d(x_1, y_1), d(x_1, y_2)\} = \{2, 3\}$.
4. $d_x = 2, d_y = 4$. Let $x_1$ be the other neighbor of $x$ besides $y$, and let $y_1$, $y_2$, and $y_3$ be the three neighbors of $y$ besides $x$, then at least two of $y_1, y_2, y_3$ have distance 2 from $x_1$.

This lemma provides us important inspirations in analyzing the local structure of Ricci-flat graphs of girth 4. Analogous to Lemma 3.1, we obtain the following very useful observation:

**Lemma 3.2** Let $xy$ be an edge of a graph $G$, and $xy$ is in exactly one 4-cycle but is not in any 3-cycle. Then $\kappa(x, y) \leq \frac{2}{d_x} + \frac{2}{d_y} - 1$.

**Proof** Since the edge $xy$ is in a unique 4-cycle, let $z$ be the other neighbor of $x$ in this cycle. Let

$$f(u) = \begin{cases} 0 & \text{if } u \in N[x] \setminus \{y, z\}, \\ 2 & \text{if } u \in N(y) \setminus \{x\}, \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $f$ is a 1-Lipschitz function over graph $G$. By Lemma 2.1,

$$W(m_x^\alpha, m_y^\alpha) \geq \sum_{u \in V} f(u)(m_y^\alpha(u) - m_x^\alpha(u))$$

$$= \left(\alpha - \frac{0}{d_x}\right) + \left(0 - \frac{0}{d_x}\right) + 2(d_y - 1)\left(\frac{1}{d_y} - 0\right)$$

$$= (2 - \alpha) - (1 - \alpha)\left(\frac{2}{d_x} + \frac{2}{d_y}\right).$$

Hence

$$\kappa(x, y) = \lim_{\alpha \to 1} \frac{1 - W(m_x^\alpha, m_y^\alpha)}{1 - \alpha} \leq \frac{2}{d_x} + \frac{2}{d_y} - 1. \quad \square$$

The next lemma characterizes the local structures for edges in a 4-cycle of a Ricci-flat graph $G$ with girth 4 and disjoint 4-cycles.

**Lemma 3.3** Suppose that $G$ is a graph with girth 4, and the 4-cycles of $G$ are mutually vertex-disjoint. Let $xy$ be an edge of $G$ in a 4-cycle with Ricci curvature $\kappa(x, y) = 0$. Without loss of generality, we assume $d_x \leq d_y$. Then one of the following statements holds.
1. \(d_x = 2, d_y = 4\), and \(xy\) is not in any 5-cycle.

2. \(d_x = d_y = 3\), and \(xy\) is not in any 5-cycle.

3. \(d_x = 3, d_y = 4\). Let \(x_1\) and \(x_2\) be the two neighbors of \(x\) besides \(y\) with \(x_1\) in the 4-cycle, and let \(y_1\) and \(y_2\) be the two neighbors of \(y\) not in the 4-cycle. Then either \(d(x_1, y_1) = d(x_2, y_2) = 2\) (Type A), or \(d(x_2, y_1) = d(x_2, y_2) = 2\) (Type B).

4. \(d_x = d_y = 4\). Let \(x_1\) and \(x_2\) be the two neighbors of \(x\) not in the 4-cycle, and let \(y_1\) and \(y_2\) be the two neighbors of \(y\) not in the 4-cycle, then \(d(x_1, y_1) = d(x_2, y_2) = 2\).

**Figure 4** Local structures in 4-cycle

**Remark 3.4** Lemma 3.3 claims that under certain conditions, for each edge \(xy\) with Ricci curvature 0, there are four possible degree combinations \(\{d_x, d_y\}\). But there are two possible local structures in the \(\{3, 4\}\) combination, which are denoted by type A and type B, resulting in five local structures in all.

**Proof** By Lemma 3.2 and the hypothesis that \(\kappa(x, y) = 0\), we have \(0 \leq \frac{d_x}{d_x} + \frac{d_y}{d_y} - 1\). Solving this inequality, we have the following solutions.

1. \(d_x = 2, d_y \geq 2\).

2. \(d_x = 3, d_y = 3, 4, 5, 6\).

3. \(d_x = d_y = 4\).

Because the 4-cycles of \(G\) are disjoint, the vertex \(y\) must be incident to another edge which is not in any 3-cycles or 4-cycles, if \(d_y \geq 3\). By Lemma 3.1, \(d_y \leq 4\). So the possible values of \(d_x\) and \(d_y\) have to be the following.

1. \(d_x = 2, d_y = 2, 3, 4\).

2. \(d_x = 3, d_y = 3, 4\).

3. \(d_x = d_y = 4\).

Simple calculation shows that if \(d_x = d_y = 2\), then \(\kappa(x, y) = 1\). So no local structure is possible for this degree combination.
If \( d_x = 2 \) and \( d_y = 3 \), let \( x_1 \) be the other neighbor of \( x \) besides \( y \), and let \( y_1 \) be the neighbor of \( y \) that is not in the 4-cycle. Because the 4-cycles of \( G \) are disjoint, \( d(x_1, y_1) \geq 2 \). If \( d(x_1, y_1) = 2 \), we have \( \kappa(x, y) = \frac{1}{2} \). If \( d(x_1, y_1) \geq 3 \), we have \( \kappa(x, y) = \frac{1}{3} \).

If \( d_x = 2 \) and \( d_y = 4 \), let \( x_1 \) be the other neighbor of \( x \) besides \( y \), and let \( y_1 \) and \( y_2 \) be the two neighbors of \( y \) that is not in the 4-cycle. If \( d(x_1, y_1) = 2 \) or \( d(x_1, y_2) = 2 \), then \( \kappa(x, y) = \frac{1}{4} \). If \( d(x_1, y_1) \geq 3 \) and \( d(x_1, y_2) \geq 3 \), then \( \kappa(x, y) = 0 \). Therefore, the edge \( xy \) is not in any 5-cycle.

The above calculations for the case \( d_x = 2 \) and \( d_y = 2, 3, 4 \) can be summarized in Table 1.

| \( d_x \) | \( d_y \) | \( d(x_1, y_1) \) | \( d(x_1, y_1), d(x_1, y_2) \) | \( \kappa \) |
| --- | --- | --- | --- | --- |
| 2 | 2 | - | - | 1 |
| 2 | 3 | 2 | - | \( \frac{1}{2} \) |
| 2 | 3 | \( \geq 3 \) | - | \( \frac{1}{4} \) |
| 2 | 4 | - | \( 2, \geq 2 \) or \( \geq 2, 2 \) | \( \frac{1}{4} \) |
| 2 | 4 | - | \( \geq 3, \geq 3 \) | 0 |

Table 1  \( d_x = 2 \)

If \( d_x = 3 \) and \( d_y = 3 \), let \( x_1 \) and \( x_2 \) be the other neighbors of \( x \) besides \( y \), with \( x_1 \) in the 4-cycle. And let \( y_1 \) and \( y_2 \) be the two neighbors of \( y \) besides \( x \), with \( y_1 \) in the 4-cycle. If \( d(x_2, y_2) = 2 \), then \( \kappa(x, y) = \frac{1}{3} \). If \( d(x_2, y_2) \geq 3 \), then \( \kappa(x, y) = 0 \).

If \( d_x = 3 \) and \( d_y = 4 \), let \( x_1 \) and \( x_2 \) be the other neighbors of \( x \) besides \( y \), with \( x_1 \) in the 4-cycle. And let \( y_1 \) and \( y_2 \) be the two neighbors of \( y \) not in the 4-cycle. Note that either \( d(x_i, y_j) = 2 \) or \( d(x_i, y_j) = 3 \) for all \( i, j = 1, 2 \), so the complete calculations are divided into 8 subcases according to the distances between \( x_1, x_2 \) and \( y_1, y_2 \). The result are listed in Table 2. In three of the subcases (Lines 4, 6, 8 of Table 2), the Ricci curvature of edge \( xy \) vanishes. Line 4 is Type B. Since the vertices \( y_1 \) and \( y_2 \) are interchangeable, Line 6 and Line 8 of the table can be combined to obtain the local structure of Type A.

| \( d(x_1, y_1), d(x_1, y_2) \) | \( d(x_2, y_1), d(x_2, y_2) \) | \( \kappa \) |
| --- | --- | --- |
| 3, 3 | 3, 3 | \( - \frac{1}{6} \) |
| 3, 3 | 2, 3 | \( - \frac{1}{12} \) |
| 3, 3 | 2, 2 | 0 |
| 2, \( \geq 2 \) | 3, 3 | \( - \frac{1}{6} \) |
| 2, 2 | 2, 3 | 0 |
| 2, 3 | 2, 3 | \( - \frac{1}{12} \) |
| 2, \( \geq 2 \) | 3, 2 | 0 |
| 2, \( \geq 2 \) | 2, 2 | \( \frac{1}{12} \) |

Table 2  \( d_x = 3, d_y = 4 \)

If \( d_x = 4 \) and \( d_y = 4 \), let \( x_1 \) and \( x_2 \) be the two neighbors of \( x \) not in the 4-cycle. And let \( y_1 \) and \( y_2 \) be the two neighbors of \( y \) not in the 4-cycle. Table 3 shows the Ricci curvature of edge \( xy \) for different subcases. The unique subcase that the Ricci curvature vanishes is illustrated in bottom right of Figure 4.
Ricci-flat Graphs with Girth Four

1687

\[d(x_1, y_1), d(x_1, y_2)\]

\[d(x_2, y_1), d(x_2, y_2)\]

\[\kappa\]

\begin{array}{|c|c|c|}
\hline
3, 3 & 3, 3 & -\frac{1}{2} \\
\hline
2, 3 & \geq 2, 3 & -\frac{1}{4} \\
\hline
2, \geq 2 & \geq 2, 2 & 0 \\
\hline
\end{array}

Table 3 \( d_x = 4, d_y = 4 \)

Lemma 3.1 and Lemma 3.3 are fundamental tools in this article and will be applied repeatedly in proving the main result in the next section.

4 The Main Result

In this section, we prove Theorem 1.3 by exhausting all possible cases.

Proof of Theorem 1.3 \( \) We start by investigating a 4-cycle of \( G \). By Lemma 3.3, the degree sequence of a 4-cycle of \( G \) in cyclic order can be only one of the following cases.

1. \((2, 4, 2, 4)\)
2. \((2, 4, 4, 4)\)
3. \((3, 3, 3, 3)\)
4. \((3, 3, 3, 4)\)
5. \((3, 3, 4, 4)\)
6. \((3, 4, 4, 4)\)
7. \((3, 4, 3, 4)\)
8. \((4, 4, 4, 4)\)

We will show that in the first six cases, the graph \( G \) could not exist and there is exactly one graph associated to each of the last two cases.

Case 1 \((2, 4, 2, 4)\) \( \) Let \( a, b, c, d \) be the four vertices of the 4-cycle, in the order of the degree sequence. That is \( d(a) = d(c) = 2 \) and \( d(b) = d(d) = 4 \). Let \( b_1 \) and \( b_2 \) be the other two neighbors of \( b \), and let \( d_1 \) and \( d_2 \) be the other two neighbors of \( d \). Obviously, \( b_i \) and \( d_j \) \((1 \leq i, j \leq 2)\) are distinct vertices, otherwise there would be 4-cycles with common edges. In the remaining cases, we will denote and refer to the vertices in the 4-cycle and their neighbors in a similar manner.

Because the edge \( bb_1 \) does not lie in any 4-cycle by the hypothesis of the theorem, so it must satisfy the local structure of Lemma 3.1. Since \( d(b) = 4 \), so \( d(b_1) = 2 \). By the same reason \( d(b_2) = d(d_1) = d(d_2) = 2 \). Let \( z \) be the other neighbor of \( b_1 \) besides \( b \). See Figure 5. Note that \( z \) must be distinct from \( d_1 \) or \( d_2 \). Suppose to the contrast that the other neighbor of \( b_1 \) is \( d_1 \), then the edge \( b_1d_1 \) does not satisfy Lemma 3.1.

Now we apply Lemma 3.1 to edge \( bb_1 \), at least two vertices of \( a, c, b_2 \) have distance 2 from \( z \). But this is impossible (because there is no way to form a 2-path from \( z \) to either \( a \) or \( c \)), so no graph exists for this case.
Case 2 (2, 4, 4, 4)  See Figure 6. The same as Case 1, because the degree of vertices b, c, d are 4, the degree of vertices $b_i, c_i, d_i$ ($i = 1, 2$) are all 2. By applying Lemma 3.3 to edge bc, without loss of generality, let $z_i$ be the common neighbor of $b_i$ and $c_i$ ($i = 1, 2$). By applying Lemma 3.3 again to edge cd, we know that the vertices $c_i$ and $d_i$ have a common neighbor, for $i = 1, 2$. But since all the vertices $c_1, c_2, d_1, d_2$ have degree 2, the common neighbor of $c_i$ and $d_i$ has to be $z_i$, for $i = 1, 2$. Now, all vertices in Figure 6 cannot be extended except $z_1$ and $z_2$. Therefore, the edge $bb_1$ does not satisfy Lemma 3.1 (the edge $bb_1$ does not lie in two 5-cycles), no graph exists for this case, either.

Case 3 (3, 3, 3, 3)  In this case, each vertex of the 4-cycle, a, b, c and d, has exactly one neighbor outside the 4-cycle, denoted by $a_1, b_1, c_1$ and $d_1$, respectively. By Lemma 3.1, the degree of $a_1, b_1, c_1$ and $d_1$ can be either 2 or 3.

If $d(b_1) = 3$, by Lemma 3.1, the edge $bb_1$ is shared by two 5-cycles. This contradicts with the fact the the edge $ab$ cannot lie in any 5-cycles. If $d(b_1) = 2$, by Lemma 3.1, the edge $bb_1$ needs to form a 5-cycle with either $ab$ or $bc$, which is also a contradiction. So no graph exists for this case.
Case 4 (3, 3, 3, 4) The same as Case 3, by applying Lemma 3.1 to edge $bb_1$, there will be a contradiction. So no graphs exist for this case.

Case 5 (3, 3, 4, 4) See Figure 7. Both $a_1$ and $b_1$ must have degree 2, otherwise by the same argument in Case 3 the edge $ab$ lies in a 5-cycle, a contradiction. Also, the degree of $c_1, c_2, d_1$ and $d_2$ are all 2. By applying Lemma 3.3 to edge $cd$, let $z_i$ be the common neighbor of $c_i$ and $d_i$, for $i = 1, 2$. Thus the edges $bc$ and $da$ have no way to satisfy the local condition. Again, no graph exists for this case.

Case 6 (3, 4, 4, 4) The structure of the graph is similar to that of Case 2, except that $a$ will have a neighbor $a_1$. To satisfy the local condition for edges $ab$ and $da$, the vertex $a_1$ must be adjacent to both $z_1$ and $z_2$, see Figure 8. But then the edge $a_1z_1$ does not satisfy the local condition. So no graph exists for this case.

Case 7 (3, 4, 3, 4) The degree combination for the two vertices of each edge in the 4-cycle is \{3, 4\}. There are two types of local structures for the \{3, 4\} combination, namely type A and type B. It is easy to show that the four edges in the 4-cycle must satisfy the same type of local condition. If all of them are type A, no graph is possible. If all of them are type B, we obtain the graph $R_2$, see Figure 2.
Case 8 (4, 4, 4, 4) By applying Lemma 3.3 to edge $ab$, let $z_i$ be the common vertex of $a_i$ and $b_i$ ($i = 1, 2$), respectively (see Figure 9). Then by applying Lemma 3.1 to edge $bb_1$, the degree of $b_1$ must be two. In other words, $b_1$ has no other neighbors besides $b$ and $z_1$. Now by applying Lemma 3.3 to edges $bc$ and $cd$, we obtain the graph $R_1$, see Figure 2.

Finally, it is easy to check that the graphs $R_1$ and $R_2$ are indeed Ricci-flat. □

Remark 4.1 (Further Study) Our method in proving Theorem 1.3 might be extended further to study the Ricci-flat graphs of girth 4 with edge-disjoint 4-cycles. In such an extension, more discussions involved are expected, especially when one tries to generalize Lemma 3.3 in which even more local degree combinations for an edge $xy$ in a 4-cycle would be involved. From examples given in Figure 1, we see that any characterization of Ricci-flat graphs of girth 4 with edge-disjoint 4-cycles have to contain infinitely many types.

References
[1] Bai, S., Lu, L., Yau, S. T.: Ricci-flat graphs with maximum degree at most 4. arXiv:2103.00941v2
[2] Bauer, F., Chung, F., Lin, Y., et al.: Curvature aspects of graphs. Proc. Am. Math. Soc., 145(5), 2033–2042 (2017)
[3] Bhattacharya, B., Mukherjee, S.: Exact and asymptotic results on coarse ricci curvature of graphs. Discrete Math., 338(1), 23–42 (2015)
[4] Bourne, D., Cushing, D., Liu, S., et al.: Ollivier–Ricci idleness functions of graphs. SIAM J. Discrete Math., 32(2), 1408–1424 (2018)
[5] Cho, H., Paeng, S. H.: Ollivier’s Ricci curvature and the coloring of graphs. Euro. J. Comb., 34(5), 916–922 (2013)
[6] Chung, F., Yau, S. T.: Logarithmic Harnack inequalities. Math. Res. Lett., 3, 793–812 (1996)
[7] Cushing, D., Kangaslampi, R., Lin, Y., et al.: Erratum for Ricci-flat graphs with girth at least five. Commun. Anal. Geom., to appear in 29(8) (2021)
[8] Hua, B., Lin, Y.: Curvature notions on graphs. Front. Math. China, 11(5), 1275–1290 (2016)
[9] Hua, B., Su, Y.: The set of vertices with positive curvature in a planar graph with nonnegative curvature. Adv. Math., 343, 789–820 (2019)
[10] Lin, Y., Lu, L., Yau, S. T.: Ricci curvature of graphs. Tohoku Math. J. (2), 63(4), 605–627 (2011)
[11] Lin, Y., Lu, L., Yau, S. T.: Ricci-flat graphs with girth at least five. Commun. Anal. Geom., 22(4), 671–687 (2014)
[12] Lin, Y., Yau, S. T.: Ricci curvature and eigenvalue estimate on locally finite graphs. Math. Res. Lett., 17(2), 343–356 (2010)
[13] Lott, J., Villani, C.: Ricci curvature for metric measure spaces via optimal transport, *Ann. of Math.* (2), 169, 903–991 (2009)
[14] Ni, C., Lin, Y., Gao, J., et al.: Ricci curvature of the internet topology. In: 2015 IEEE Conference on Computer Communications (INFOCOM), IEEE, Piscataway, 2015, 2758–2766
[15] Ollivier, Y.: Ricci curvature of markov chains on metric spaces. *J. Funct. Anal.*, 256(3), 810–864 (2009)
[16] Ollivier, Y., Villani, C.: A curved Brunn–Minkowski inequality on the discrete hypercube, or: What is the ricci curvature of the discrete hypercube? *SIAM J. Discrete Math.*, 26(3), 983–996 (2012)
[17] Sturm, K. T.: On the geometry of metric measure spaces I & II. *Acta Math.*, 196(1), 65–177 (2006)
[18] Yau, S. T., Nadis, S.: The Shape of Inner Space: String Theory and the Geometry of the Universe’s Hidden Dimensions, Basic Books, New York, 2010