Vacuum Energy Densities of a Field in a Cavity with a Mobile Boundary

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We consider the zero-point field fluctuations, and the related field energy densities, inside a one-dimensional and a three-dimensional cavity with a mobile wall. The mechanical degrees of freedom of the mobile wall are described quantum-mechanically and they are fully included in the overall system dynamics. In this optomechanical system, the field and the wall can interact with each other through the radiation pressure on the wall, given by the photons inside the cavity or even by vacuum fluctuations. We consider two cases: the 1D electromagnetic field and the 3D scalar field, and use the Green’s functions formalism, that allows extension of the results obtained for the scalar field to the electromagnetic field. We show that the quantum fluctuations of the position of the cavity’s mobile wall significantly affect the field energy density inside the cavity, in particular at the very proximity of the mobile wall. The dependence of this effect from the ultraviolet cut-off frequency, related to the plasma frequency of the cavity walls, is discussed. We also compare our new results for the 1D electromagnetic field and the 3D massless scalar field to results recently obtained for the 1D massless scalar field. We show that the presence of a mobile wall also changes the Casimir-Polder force on a polarizable body placed inside the cavity, giving possibility to detect experimentally the new effects we have considered.

I. INTRODUCTION

The existence of vacuum field fluctuations, and the related vacuum energy density of the field, is a striking consequence of quantum electrodynamics and quantum field theory in general [1]. Vacuum fluctuations have observable effects, for example the Casimir force, that is a usually attractive force of quantum origin between neutral macroscopic metallic or dielectric bodies placed in the vacuum space [2,3]. Casimir forces originate from the change of the field energy associated to the vacuum fluctuations, when one or more boundary conditions such as dielectric or conducting objects, are changed.

A more thoroughly description of quantum vacuum effects can be obtained by considering local field quantities such as the field energy density. Obtaining the energy densities of the field in the vacuum state is relevant also in view of the fact that they are strictly related to atom-surface Casimir-Polder dispersion interactions [4,5]. Moreover, it has been shown that the vacuum electric and magnetic energy density, as well as vacuum field fluctuations, can become singular in the proximity of sharp metallic boundaries [6,8]. The presence of such surface divergences of the energy density could be relevant also in view of the fact that they should yield significant gravitational effects because the energy density acts as a source term for gravity [6,10].

New effects arise when one or more boundaries are allowed to move. A known effect is the dynamical Casimir effect, that is the emission of real quanta from the vacuum when a boundary is forced to move with nonuniform acceleration [11,13]. For a very small mass of a mobile wall, quantum effects relative to its mechanical degrees of freedom such as position fluctuations, may be relevant. It is thus worth to consider the effect of a mobile boundary, whose mechanical degrees of freedom are treated quantum mechanically, on a quantum field; in this case an effective coupling between the wall and the field arises due to the radiation pressure, as well as an effective coupling between the field modes [14]. These effects are also related to the growing field of quantum optomechanics, which studies the coupling of optical cavity modes with mechanical degrees of freedom [15,17]; this subject is also relevant for building more sensitive force detectors, to be used for example for the detection of gravitational waves [18]. The effect of vacuum fields on the position fluctuations of a single mirror in the vacuum space, and the role of vacuum friction, has been recently considered [19].

We consider in this paper how a moving conducting boundary such as a cavity wall, being treated quantum mechanically, can affect the field fluctuations and the related field energy density inside the cavity. This effect can be in principle observed because the field energy density can be probed through the Casimir-Polder interaction with a polarizable body placed inside the cavity. In a previous paper [20], we have investigated this aspect in the simple case of a massless scalar field in a one-dimensional cavity with one fixed and one mobile wall, and we have found a change of the field fluctuations in the cavity, particularly relevant in the proximity of the mobile wall, and of the Casimir force between the cavity walls. Also, this new effect, in the case considered in [20], has a size such that it should allow its experimental observation for a sufficiently small mass of the mobile wall, and masses down to $10^{-21}$ Kg can be nowadays reached in modern optomechanics experiments [17].
In this paper we extend the results previously obtained for the 1D massless scalar field to the more realistic cases of a 1D electromagnetic field and the 3D massless scalar field. We give a local description of vacuum field fluctuations in terms of the zero-point field energy density and local field fluctuations, inside a cavity with a mobile wall, both for the 1D electromagnetic field and the 3D scalar field. In both cases the motion of the mobile wall, which is assumed bound to an equilibrium position by a harmonic potential, is described quantum-mechanically and the effects of radiation pressure and of the wall’s quantum position fluctuations are included in the formalism. Our description is based on an appropriate generalization, that we introduce in this paper, of the effective Hamiltonian obtained in [14] for the 1D massless scalar case. For the 1D electromagnetic case, using a formalism based on the Green’s functions, we obtain the change of the renormalized electric and magnetic energy densities inside the 1D cavity due to the motion of the cavity wall; we show that, similarly to the 1D scalar results, it is particularly relevant close to the equilibrium position of the mobile wall. We also discuss their dependence on the ultraviolet cutoff frequency, that is related to the plasma frequency of the mobile wall, and show that, when the cutoff frequency is increased, the energy-density change becomes more and more concentrated near the mobile wall. In the 3D scalar case, we first obtain the renormalized Green’s function of the scalar field on the interacting ground state, and we then use it to obtain the renormalized field energy density change in the cavity. We show that, contrarily to both scalar and electromagnetic 1D cases, the peak of the energy density change is not located at the equilibrium position of the mobile wall. This peak, however, moves towards the mobile wall when the cutoff frequency is increased. We also discuss the dependence of the change of the field energy density on the mass and oscillation frequency of the mobile wall, and how the new effects we have obtained can be observed through the Casimir-Polder force on a polarizable body placed inside the cavity in the proximity of the mobile wall.

Although our system has some analogy with the Dynamical Casimir Effect, we wish to stress that it is quite different, because in the present case the mobile wall is not moving according to a prescribed law as in the dynamical Casimir effect, but it is described quantum-mechanically according to the quantum dynamics induced by the Hamiltonian of the interacting wall-field system. Finally, we point out that the results we obtain for the 3D scalar field can be also useful for an extension to the 3D electromagnetic field case.

This paper is organized as follows. In Sec. II we introduce part of the local formalism that will be used in the subsequent Sections of the paper. In Sec. III we consider the problem of the interaction between a one-dimensional electromagnetic field and a movable wall, using the local formalism and exploiting the results obtained in Refs. [14, 20] for the simpler case of a massless scalar field in a one-dimensional cavity. We obtain the change of the renormalized zero-point energy densities of the electric and magnetic field components on the interacting ground state of the system. In Sec. IV we consider the case of a massless scalar field in a 3D cavity with one mobile wall. We first generalize the effective Law Hamiltonian to the 3D case, and then obtain the correction to the field energy density consequent to the wall’s motion due to quantum position fluctuations and radiation pressure. We discuss the main physical features of the change of the field energy density and compare our new results with previous ones obtained for the 1D scalar case and discuss observability of the new effects found. Section V is devoted to our conclusive remarks.

II. THE LOCAL FORMALISM

In this Section we introduce the local field formalism we will use in the following and the relative notations. A field theoretical approach to the study of the properties of the vacuum starts from the analysis of the behavior of local field quantities. For our purposes, the energy-momentum tensor $T^{\mu\nu}$ represents the appropriate quantity, because $T^{00}$ is the energy density of the field, the components $T^{\mu\nu}$ are related to the energy and momentum flow, and the stress components $T^{ik}$ are related to general mechanical properties of the vacuum.

In the presence of boundaries, a local formulation requires the introduction of the (renormalized) energy-momentum tensor of the vacuum $\Theta^{\mu\nu}_{vac}$, in the form [21–23]

$$\Theta^{\mu\nu}_{vac} = \langle 0|T^{\mu\nu}|0\rangle - \langle 0|T^{\mu\nu}|0\rangle_0 . \quad (1)$$

In this equation, the measurable vacuum energy density is defined as the difference between that in the confined field configuration $\langle 0|T^{\mu\nu}|0\rangle$ and that corresponding to the unbounded configuration $\langle 0|T^{\mu\nu}|0\rangle_0$. An advantage of a local description is that it permits a different and thoughtful point of view yielding a deeper understanding of the nature of vacuum energy and vacuum stresses. It is known that $\Theta^{\mu\nu}_{vac}$ can be expressed in terms of the field propagators. The presence of quantum field fluctuations in a specific configuration, and the consequent observable quantum vacuum effects, can be understood from the modifications of the emission/reabsorption of virtual field quanta under external constraints. When boundaries are introduced, the propagation is modified due to surface interactions, and consequently this perturbs the homogeneity of the corresponding propagator. For example, the propagator of the scalar field $G(x, x')$ must satisfy the appropriate boundary conditions, for instance Dirichlet or von Neumann boundary conditions in the case of perfect reflectors. The energy-momentum tensor of the electromagnetic vacuum for the
confined field configuration, can be written as \[\Theta_{\text{vac}}^{\mu\nu}(x) = -i \left\{ \tau_{x,x}^{\mu\nu} (G(x, x') - G_0(x - x')) \right\} \bigg|_{x'=x} \] (2)
where \(G\) is the scalar propagator in the presence of the boundaries, \(G_0\) is the scalar propagator in the unbounded space, and \(G_R = G - G_0\) is the renormalized propagator. We have also introduced the differential operator \[
\tau_{x,x}^{\mu\nu} = 2 \left( \partial^\mu \partial^\nu + \frac{1}{4} \eta^{\mu\nu} \partial^\alpha \partial^\alpha \right),
\] (3)
with \(\eta^{\mu\nu} = \text{diag}(-1,+1,+1,+1)\). The vacuum subtraction in Eq. (1) is now obtained from the difference between the confined and the free propagators, respectively \(G(x, x')\) and \(G_0(x - x')\), that is the renormalized Green function of the system \(G_R(x, x')\). For explicit evaluations all one has to do is to construct \(G_R(x, x')\) for the considered configuration (see Refs. [21, 26], for example) that, except for special cases with a simple geometry, can be a challenging task. The local method has however the advantage that it allows to obtain local quantities of the electromagnetic field such as its energy density from the Green’s functions of the scalar field with Dirichlet and von Neumann boundary conditions relative to the problem under investigation, using Eq. (2) [21, 27, 28]. We will use this formalism for tackling our problem in the next Section. In addition, electric and magnetic field two-point correlation functions can be also obtained from the scalar propagator, using [21, 26]
\[
D^{\mu\nu;\lambda\kappa}(x-x') = i\langle 0| F^{\mu\nu} (x) F^{\lambda\kappa} (x') |0\rangle = d^{\mu\nu;\lambda\kappa} G_0(x - x'),
\] (4)
where \(F^{\mu\nu}\) is the electromagnetic strength tensor and we have defined the differential operator
\[
d^{\mu\nu;\lambda\kappa} = \partial^\mu \partial^\lambda g^{\nu\kappa} - \partial^\nu \partial^\lambda g^{\mu\kappa} + \partial^\nu \partial^\kappa g^{\mu\lambda} - \partial^\mu \partial^\kappa g^{\nu\lambda}.
\] (5)
In the next Sections we will use Eq. (4), by applying the operator (5) to the renormalized Green’s function in the presence of the boundaries.

III. THE 1D ELECTROMAGNETIC CASE

In order to consider the electromagnetic field inside a 1D cavity using the approach outlined in the previous Section, we first consider a one-dimensional cavity formed by two perfectly reflecting mirrors and a massless scalar field \(\phi(x, t)\) at zero temperature. One of the mirrors is fixed at the position \(x = 0\) while the other is bounded by a harmonic potential \(V(q)\) to its equilibrium position \(L_0\), and has mass \(M\) and oscillation frequency \(\omega_{osc}\). We label the position of the movable mirror by \(q(t)\), which is an operator because we are treating the mirror’s motion quantum-mechanically. The effective nonrelativistic Hamiltonian describing our 1D coupled mirror-field system is \(H = H_0 + H_{\text{int}}\), where
\[
H_0 = \hbar \omega_{osc} b^\dagger b + \hbar \sum_j \omega_j a_j^\dagger a_j
\] (6)
is the unperturbed Hamiltonian. The first and second term of (6) are respectively the mirror and the field Hamiltonian, with: \(b\) and \(b^\dagger\) annihilation and creation operators of the mechanical degrees of freedom of the movable mirror; \(a_j\) and \(a_j^\dagger\) annihilation and creation operators for the mode \(j\) of the scalar field. We impose Dirichlet boundary conditions on the field operator; the field modes are relative to the equilibrium position \(L_0\) of the moving mirror, and thus the possible wave numbers are \(k_j = j\pi/L_0\), with \(j\) an integer number. The effective interaction Hamiltonian, describing the mobile mirror-field interaction and an effective interaction between different field modes (due to the motion of the wall), is \(H_{\text{int}}\),
\[
H_{\text{int}} = -\sum_{j\ell} C_{j\ell} (b + b^\dagger) \mathcal{N} [(a_j + a_j^\dagger)(a_\ell + a_\ell^\dagger)],
\] (7)
where
\[
C_{j\ell} = (-1)^{j+\ell} \left( \frac{\hbar}{2} \right)^{3/2} \frac{1}{L_0 \sqrt{M \omega}} \left[ \sqrt{\frac{\omega_j \omega_\ell}{\omega_{osc}}} \right] \] (8)
is the coupling constant and \(\mathcal{N}\) is the normal ordering operator, while \(j\) and \(\ell\) are integer numbers specifying the field modes (evaluated for the equilibrium position of the wall).

From the Hamiltonian (7), using perturbation theory at the lowest significant order, it is possible to obtain the dressed ground state \(|g\rangle\) of the field-mirror system as done in [25]
\[
|g\rangle = |\{0_p\}, 0\rangle + \sum_{j\ell} D_{j\ell} |\{1_j, 1_\ell\}, 1\rangle,
\] (9)
where the elements of the states in curly brackets indicate field excitations, while the other element indicates excitations of the wall’s mechanical degrees of freedom. We have also defined
\[
D_{j\ell} = (-1)^{j+\ell} \frac{1}{L_0} \left[ \sqrt{\frac{\hbar \omega_j \omega_\ell}{8M \omega_{osc}} \left( \omega_{osc} + \omega_j + \omega_\ell \right)} \right]
\] (10)
In order to obtain local quantities of the field, as outlined in Sec. II, it is useful to calculate first the renormalized scalar field propagator on the dressed vacuum state (9), that is the difference of the Green’s function for the confined field with Dirichlet boundary conditions and for the free field (form now on we explicitly write space and time components)
\[
G_R(x, t; x', t') = \langle g| \phi_{BC}(x, t) \phi_{BC}(x', t') |g\rangle - \langle \{0_r\} | \phi_{un}(x, t) \phi_{un}(x', t') |\{0_r\}\rangle,
\] (11)
where $\phi_{BC}(x,t)$ satisfies the Dirichlet boundary condition at the wall’s position and $\phi_{un}(x,t)$ is the free-field operator in the unbounded space. We obtain

$$G_R(x,t;x',t') = G_{R0}(x,t;x',t') + \Delta G_R(x,t;x',t') ,$$

with

$$G_{R0}(x,t;x',t') = \langle \{0_r\}|\phi_{BC}(x,t)\phi_{BC}(x',t')|\{0_r\} \rangle - \langle \{0_r\}|\phi_{un}(x,t)\phi_{un}(x',t')|\{0_r\} \rangle = \left( \sum_p \frac{\hbar c^2}{L_0\omega_p} e^{-i\omega_p(t-t')} \sin(k_p x) \sin(k_p x') - \int dp \frac{\hbar c^2}{2\pi^2}\omega_p e^{-i\omega_p(t-t')} e^{ik_p(x-x')} \right) ,$$

$$\Delta G_R(x,t;x',t') = 8 \sum_{j\ell} \sum_r \frac{\hbar c^2}{L_0(\omega_j\omega_r)^{1/2}} D_{j\ell}D_{lr} \left[ \cos(\omega_j t - \omega_r t') \right] [\sin(k_j x) \sin(k_r x') ] .$$

Equation (13) shows that the renormalized field propagator is given by two terms. The first term $G_{R0}$, at the zeroth order in the atom-mirror coupling, takes into account that the field is confined in the cavity and it is the difference between the fixed-wall propagator and the free propagator. The second term $\Delta G_R$ is a correction term taking into account the effective interaction between the field and the mobile mirror, and it is related to the quantum fluctuations of the position of the mobile wall.

We can now face the 1D electromagnetic case. Starting from the scalar Green’s function (12) and (13), using Eq. (14) by applying the appropriate differential operators, we can obtain the field fluctuations and energy densities (they just differ by a multiplicative factor) associated to the electric and magnetic field components along the $z$ and $y$ directions respectively. They are given by a zeroth-order term (the same obtained for fixed walls) and a first-order term, coming from the zeroth- and first order Green’s functions (13), respectively. The zeroth-order terms are given by

$$\langle E_z^2(x) \rangle_0 = \lim_{(x',t') \to (x,t)} c^{-2} \partial_t \partial_{x'} \langle G_{R0}(x,t;x',t') \rangle = - \frac{\hbar c \pi}{24L_0^2} \frac{e^{\frac{2\pi x}{L_0}}}{\left( e^{\frac{2\pi x}{L_0}} - 1 \right)^2} , (14)$$

$$\langle B_y^2(x) \rangle_0 = \lim_{(x',t') \to (x,t)} \partial_x \partial_{x'} \langle G_{R0}(x,t;x',t') \rangle = \frac{\hbar c}{8\pi(x-L)^2} \frac{e^{\frac{2\pi x}{L_0}}}{\left( e^{\frac{2\pi x}{L_0}} - 1 \right)^2} , (15)$$

$$\langle E_z^2(x) \rangle_1 = \lim_{(x',t') \to (x,t)} c^{-2} \partial_t \partial_{x'} \langle \Delta G_R(x,t;x',t') \rangle = \sum_{j\ell r} (-1)^{\ell+r} \frac{\hbar^2}{L_0^2 M\omega_{osc}} \frac{\omega_{j}\omega_{r} - \omega_{r} + \omega_{j}}{\left( \omega_{osc} + \omega_{j} + \omega_{r} \right)} \sin(k_j x) \sin(k_r x) , (18)$$
\begin{align}
\langle B^2_y(x) \rangle_1 &= \lim_{(x',t') \to (x,t)} \partial_{x'} \partial_{t'} \langle \Delta G_R(x,t;x',t') \rangle \\
&= \sum_{j \ell r} (-1)^{j+r} \frac{\hbar^2}{2L_0^3 \omega_{osc}} \frac{\omega_j \omega_\ell}{(\omega_{osc} + \omega_j + \omega_\ell)(\omega_{osc} + \omega_j + \omega_\ell)} \cos(k_\ell x) \cos(k_\ell x).
\end{align}
(19)

The corrections [18] and [19] to the electric and magnetic energy densities take into account of the effective field-mirror interaction and of the effects of radiation pressure, related to the wall’s quantum fluctuations of its position. Summing up these two corrections, we obtain the correction to the field energy density in the cavity

\begin{align}
\frac{1}{2} \left[ \langle E^2_x(x) \rangle_1 + \langle B^2_y(x) \rangle_1 \right] = \sum_{j \ell r} (-1)^{j+r} \frac{\hbar^2}{2L_0^3 \omega_{osc}} \frac{\omega_j \omega_\ell}{(\omega_{osc} + \omega_j + \omega_\ell)(\omega_{osc} + \omega_j + \omega_\ell)} \cos((k_\ell - k_\tau)x).
\end{align}
(20)

This quantity can be evaluated numerically. In Fig. 1 we show the corrections to the electric and magnetic components of the electromagnetic energy density, as well as the correction to the total energy density, in the proximity of the mobile wall, where the effects we are investigating are more relevant. We have used the typical value of the mass of a commercial MEMS (MicroElectroMechanical System), that is $M \approx 10^{-11}$ kg. However, much smaller masses in the range $10^{-15} - 10^{-21}$ kg can be obtained nowadays in optomechanical devices [15, 28, 29], and this should allow to make even more significant the effect we are considering, because it scales as $1/M$. The energy densities are plotted for typical values $\omega_{osc} = 10^9$ kg, $L_0 = 10 \mu$m of the mirror’s oscillation frequency and cavity length respectively, and the cutoff frequency of a typical plasma frequency of a metal. The figures show that the motion of the wall significantly affects the field energy density inside the cavity and that this effect is particularly important near the moving wall. In addition, this effect become more and more relevant when the mass of the wall and its oscillation frequency are decreased, consistently with our previous results for the 1D scalar field [21]. In Fig. 2 we have plotted the correction to the vacuum field energy density caused by the wall’s motion for different cut-off frequencies. We observe that when the cut-off frequency increases, the effect becomes more and more relevant and sharply localized in the vicinity of the wall’s equilibrium position.

The changes to the energy density can be probed exploiting the Casimir-Polder dispersion interactions with an electrically or magnetically polarizable body placed inside the cavity. If this polarizable body has static electric polarizability $\alpha_E$ and static magnetic polarizability $\alpha_M$, its interaction energy with the electric and magnetic field fluctuations, under appropriate conditions, can be written as [4, 20]

\begin{equation}
\delta E = -\frac{1}{2} \alpha_E \langle E^2(x_{pb}) \rangle - \frac{1}{2} \alpha_M \langle B^2(x_{pb}) \rangle,
\end{equation}
(21)

where $x_{pb}$ is the position of the polarizable body. Thus the interaction with the polarizable body permits to measure both electric and magnetic energy densities.

**IV. THE 3D SCALAR CASE**

We now discuss the case of a scalar field in a 3D cavity. We consider a three-dimensional massless scalar field that satisfies Dirichlet boundary conditions $\phi(0, y, z) = \phi(L_x(t), y, z) = 0$, inside a three-dimensional cavity with walls along the axis $x$, $y$, $z$ of length $L_x$, $L_y$, $L_z$ respectively. One of the two walls perpendicular to the $x$ axis is free to move and its position is $x = L_x(t)$, with $L_x(t)$ the time-dependent cavity length along the $x$ direction. All other boundaries are fixed in space. The movable wall has mass $M$ and, similarly to the case discussed in the previous Section, it is bounded by a (harmonic) potential $V(q)$ at its equilibrium position.

In order to deal with our 3D problem, we need to generalize the effective Hamiltonian [10, 11] used in the previous Section to the three-dimensional case. We start with the situation in which all walls have fixed positions. In this case, the scalar field is given by

\begin{equation}
\phi(r, t) = \sum_n \sqrt{\frac{\hbar c}{\omega_n SL_x}} \sin \left( q^n_x x \right) e^{i q^n_x r t} e^{-i \omega_n t} + H.c.,
\end{equation}
(22)

where we have used periodic boundary conditions in the $y$ and $z$ directions, and

\begin{align}
\omega_n &= c \sqrt{\left( \frac{n_x \pi}{L_x} \right)^2 + \frac{(2\pi)^2}{S} \left( n_y^2 + n_z^2 \right)}, \\
q^n_x &= \frac{2\pi}{S} (n_y y + n_z z), \quad q^n_x = \frac{\pi}{L_x} n_x,
\end{align}
(23)

are, respectively, the frequency and the wavevector components along the $yz$ plane and the $x$ direction. They
depend on the three integer numbers \( n = (n_x, n_y, n_z) \). We have assumed \( L_y = L_z \) and defined \( S = L_y L_z \); also, we have defined the component \( r_\parallel = (y\hat{y} + z\hat{z}) \) of the position in the plane \( yz \).

When the wall perpendicular to the \( x \) axis at \( L_x(t) \) is movable, we can write the field in terms of a set of an

\[
\Theta^{xx} = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial y} \right)^2 - \left( \frac{\partial \phi}{\partial z} \right)^2 \right],
\]

which involves both space and time derivatives of the field operator evaluated at the wall’s position. After calculating this quantity in the reference frame comoving with the movable wall, and afterwards making a Lorentz transformation to the laboratory frame in the non-relativistic limit, we obtain that only the term \( \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 \) contributes to the force due to the radiation pressure, while the term involving the time derivative is negligible being proportional to \( L_x^2(t) \).
Therefore, the equations of motion for the field operator and the wall’s position, in the nonrelativistic limit, are
\[ \square \phi = 0 \]
\[ m \ddot{q} = -\frac{\partial V(q)}{\partial q} + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2, \quad (27) \]
where \(-\frac{\partial V(q)}{\partial q}\) is the force due to the potential binding

\[ \hat{G}^{(n)}_k + \omega^2_k(t) \hat{Q}^{(n)}_k = 2\lambda(t) \sum_j g_{kj} \hat{Q}^{(n)}_j + \lambda(t) \sum_j g_{kj} \hat{Q}^{(n)}_j + \lambda^2(t) \sum_{j,l} g_{kj} g_{jl} \hat{Q}^{(n)}_j, \]
\[ m \ddot{q} = -\frac{\partial V(q)}{\partial q} + \frac{1}{q(t)} \sum_{kj} (-1)^{k_x+j_x} \hat{Q}^{(n)}_k \omega_{k_x} \omega_{j_x} \delta_{k_y,-j_y} \delta_{k_z,-j_z}, \quad (28) \]

where \(j = (j_x, j_y, j_z)\) indicates a set of three integer numbers, \(\omega_{k_x} = \pi k_x/L_x, \omega_{j_x} = \pi j_x/L_x, \lambda = \lambda_x(t)/L_x(t)\) and
\[ g_{kj} = \begin{cases} (-1)^{k_x+j_x} \frac{2k_x}{j_x-k_x} \delta_{k_y,-j_y} \delta_{k_z,-j_z} & (k_x \neq j_x) \\ 0 & (k_x = j_x) \end{cases}. \quad (29) \]

Following a procedure analogous to that used in Ref. [14], we can show that the equations (28) can be obtained from a set of Euler-Lagrange equations relative to an appropriate Lagrangian. The corresponding Hamiltonian, associated with equations of motion (28), after a canonical quantization procedure and renormalization, is
\[ H = \frac{(p + \Gamma)^2}{2m} + V(q) + \hbar \sum_k \omega_k a_k a_k^\dagger - E_{Cas}, \quad (30) \]
where \(\Gamma\) is the following operator
\[ \Gamma = \frac{i\hbar}{2q} \sum_{kj} g_{kj}\left( \frac{\omega_k}{\omega_j} \right)^{1/2} \left[ a_k^\dagger a_j - a_k a_j - a_k^\dagger a_j^\dagger + a_j a_j^\dagger \right], \quad (31) \]
and the quantity \(E_{Cas}\) is the Casimir energy for the two fixed-walls configuration.

Our Hamiltonian (30) is the generalization of the Hamiltonian obtained by Law in [14] to the three-dimensional case. It shows a non-linear character of the coupling between the field and the movable wall; however, in many situations the wall is bounded to an equilibrium position by an external potential, and the effects of the radiation pressure can be treated as a small perturbation. We assume this is the case in our system, and thus we now derive a linearized form of the Hamiltonian (30). We suppose that the mobile wall is confined near the equilibrium position \(L_0\) by the potential \(V(q)\). When the displacement of the wall \(x_m = q - L_0\) is small compared to \(L_0\), we can write
\[ \Gamma \approx \Gamma_{q=L_0} = \Gamma_0, \]
\[ a_k(q) \approx a_k^0 + \frac{x_m}{2\omega_k} \left( \frac{\partial \omega_k}{\partial q} \right)_0 a_k^\dagger, \quad (32) \]
\[ \omega_k(q) \approx \omega_k^0 + x_m \left( \frac{\partial \omega_k}{\partial q} \right)_0, \quad (33) \]
where \(a_k^0\) and \(\omega_k^0\) are, respectively, the annihilation operator and the frequency associated with the equilibrium position \(L_0\). We now substitute equations (32) into the Hamiltonian (30) and make a unitary transformation \(H' = T^\dagger H T\), where the transformation operator is given by
\[ T = \exp\{ixm\Gamma_0/\hbar\}. \quad (34) \]
We also assume that \(V(q)\) is a harmonic potential with frequency \(\omega_{osc}\). After lengthy algebraic calculations, we can write our Hamiltonian in the form \(H = H_0 + H_{int}\), where
\[ H_0 = \hbar \omega_{osc} b^\dagger b + \hbar \sum_k \omega_k a_k^\dagger a_k, \quad (35) \]
is an unperturbed Hamiltonian where \(b\) and \(b^\dagger\) are the annihilation and creation operators of the movable wall, respectively; the effective Hamiltonian \(H_{int}\) describing the interaction between the three-dimensional massless scalar field and the movable wall is
\[ H_{int} = \sum_{k,j} C_{kj} \left\{ (b + b^\dagger) N[(a_k^0 + a_{k0}^\dagger)(a_{j0} + a_{j0}^\dagger)] \right\}, \quad (36) \]
where
\[ C_{kj} = \frac{\hbar}{2} \sqrt{\frac{\hbar}{2M\omega_{osc}}} \left[ \frac{\partial \omega_k}{\partial q} \bigg|_{L_0} \delta_{kj} - \frac{g_{kj}}{L_0} \left( \frac{\omega_k}{\omega_j} \right)^{1/2} \omega_{k0} \right]. \quad (37) \]
The quantity $C_{kj}$ given in (39) is the new coupling constant for the wall-field interaction in the 3D case, which can be compared with the analogous 1D coupling constant of Sec. III, given by Eq. (8).

From our effective Hamiltonian (35), using first-order perturbation theory we can obtain the interacting (dressed) ground state of the field-mirror system, having virtual excitations of both field and mirror (we are using, for the states, the same notations as in Sec. III)

$$|g⟩ = |{0_p}, 0⟩ + \sum_{kj} D_{kj} |{1_k, 1_l}, 1⟩,$$

where we have defined

$$D_{kj} = \frac{C_{kj}}{\hbar(\omega_{osc} + \omega_k + \omega_l)}.$$  \hspace{1cm} (38)

We can now evaluate the average number of excitations of the 3D cavity in the field mode specified by the integers $m = (m_x, m_y, m_z)$, in the dressed ground state (37), due to the quantum fluctuations of the movable wall’s position. It results

$$\langle \langle x | N_m | x⟩ = \langle \langle x | N_{(m_x, m_y, m_z)} | x⟩ = \frac{\pi^4 \hbar^4}{2ML^2_0^2} \frac{m_x^4}{\omega_{osc}(\omega_m)^2(\omega_{osc} + 2\omega_m)^2}$$

$$+ \frac{\hbar}{2ML^2_0^2} \sum_{m_x, m_y \neq m_x} \frac{(m_x m_y')^2}{(m_x'^2 - m_x^2)^2} \frac{\omega_{osc} \omega_m \omega_{(m_x', m_y, m_z)}(\omega_{osc} + \omega_m + \omega_{(m_x', m_y, m_z)})^2}{},$$

where we used Eq. (24).

This is the photon spectrum due to the motion of the mirror. The field excitations originate from the vacuum state as a consequence of the mirror-field interaction.

The one-dimensional case can be recovered from our result (39) by setting $m_y = m_z = 0$, obtaining

$$\langle \langle x | N_{(m, 0, 0)} | x⟩ = \frac{\pi^4 \hbar^4}{2ML^2_0^2} \frac{m_x^4}{\omega_{osc}(\omega_m)^2(\omega_{osc} + 2\omega_m)^2}$$

$$+ \frac{\hbar}{2ML^2_0^2} \sum_{m_x} \frac{\omega_{(m_x, 0, 0)} \omega_{(m_x', 0, 0)}}{(\omega_{osc} + \omega_{(m_x, 0, 0)} + \omega_{(m_x', 0, 0)})^2},$$

which coincides with the result already obtained in [20] for the 1D scalar case.

Eq. (39) shows that, similarly to the 1D case [20], the number of virtual quanta inside the cavity decreases with increasing oscillation frequency $\omega_{osc}$ and mass $M$ of the mobile wall. From a physical point of view, this is due to the fact that when the mirror oscillation frequency increases, its action in mediating the effective wall-field interaction is weaker. An analogous consideration holds for the dependence from the mirror’s mass.

In order to obtain local field quantities for the 3D scalar field, that, as mentioned in the Introduction, are also useful to evaluate analogous quantities for the 3D electromagnetic field, we first calculate the renormalized field propagator. The renormalized propagator is the difference between two different Green’s functions. The first function is the propagator of the field with Dirichlet boundary conditions on the dressed vacuum state $|g⟩$ given by (37), while the second one is the free-field propagator

$$G_R(x, t; x', t') = \langle \langle x | \phi_{BC}(x, t) \phi_{BC}(x', t') | x⟩ - \langle \langle x | \phi_{un}(x, t) \phi_{un}(x', t') | x⟩ = G_{R0}(x, t; x', t') + \Delta G_R(x, t; x', t'),$$

where $G_{R0}(x, t; x', t')$ is a zeroth-order term in the effective field-wall interaction, giving the renormalized (meaning that the unbounded-field propagator has been subtracted) for a fixed-walls cavity, and $\Delta G_R(x, t; x', t')$ is a first-order correction due to the motion of the mobile wall. An explicit calculation of these two terms yields

$$G_R(x, t; x', t') = G_{R0}(x, t; x', t') + \Delta G_R(x, t; x', t'),$$

where $(x, t)$ is the space-time coordinate, and subscripts $BC$ and $un$ indicate quantities in the presence of boundary conditions and in the unbounded space, respectively.

The renormalized Green’s function of the system results to be

$$G_R(x, t; x', t') = G_{R0}(x, t; x', t') + \Delta G_R(x, t; x', t'),$$

(42)
where we have we defined

\[
\Delta G(x, t; x', t') = \sum_{mj} \sum_r 4D_{mj} D_{jr} [u_m(x, t) u_r(x', t') + u'^*_m(x, t) u_r(x', t')] ,
\]

where we have we defined

\[
u_m(x, t) \equiv u_m(x, r||, t) = \sqrt{\frac{\hbar c^2}{\omega_m S L_0}} \sin \left( \frac{m_x \pi}{L_0} x \right) e^{ik^m_r r||} e^{-i\omega_m t} .
\]

The zero-order term takes into account the fact that the field is confined, while the correction term takes also into account the field-wall interaction and the wall’s quantum position fluctuations.

From the expressions obtained above for the scalar propagator, we can now obtain the energy density of the scalar field in the cavity by applying the appropriate differential operators according to Eq. (2)

\[
\langle \rho \rangle = \frac{1}{2} \lim_{(x', t') \to (x, t)} \left( c^{-2} \partial_t \partial_t' + \nabla_x \cdot \nabla_{x'} \right) G_R(x, t; x', t')
\]

\[
= \frac{1}{2} \lim_{(x', t') \to (x, t)} \left( c^{-2} \partial_t \partial_t' + \nabla_x \cdot \nabla_{x'} \right) [G_{R0} + \Delta G_R]
\]

\[
= \langle \rho_0 \rangle + \langle \Delta \rho \rangle .
\]

Therefore, also the field energy density can be written as a zero-order term \( \langle \rho_0 \rangle \) plus a first-order correction \( \langle \Delta \rho \rangle \) related to the quantum fluctuations of the wall’s position and radiation pressure, respectively given by

\[
\langle \rho_0(x) \rangle = \frac{1}{2} \lim_{(x', t') \to (x, t)} \left( c^{-2} \partial_t \partial_t' + \nabla_x \cdot \nabla_{x'} \right) G_{R0}(x, t; x', t') = -\pi^2 \hbar c \left( \frac{2p_x \pi}{L_0} \right)^2 \left( \frac{2p_x^2 + 2p^2_{\parallel}}{2\omega_p} \right) e^{-\eta \omega_p} ,
\]

\[
\langle \Delta \rho(x) \rangle = \frac{1}{2} \lim_{(x', t') \to (x, t)} \left( c^{-2} \partial_t \partial_t' + \nabla_x \cdot \nabla_{x'} \right) \Delta G_R(x, t; x', t')
\]

\[
= \frac{1}{2} \sum_{mj} \sum_{r||} 8D_{mj} D_{jr} \frac{\hbar c^2}{S L_0} \left[ \frac{\omega_p \omega_m}{c^2} + \frac{4\pi^2 m_x r_z + m_y r_y}{\sqrt{\omega_p \omega_m}} \right] \sin \left( \frac{m_x \pi}{L_0} x \right) \sin \left( \frac{r_x \pi}{L_0} x \right)
\]

\[
+ \frac{m_x r_x}{\sqrt{\omega_p \omega_m}} \left( \frac{r_x \pi}{L_0} \right)^2 \cos \left( \frac{m_x \pi}{L_0} x \right) \cos \left( \frac{r_x \pi}{L_0} x \right) \cos \left( k_m^r \cdot r|| \right) ,
\]

where \( \eta^{-1} \) is an ultraviolet cut-off parameter related to the plasma frequency of the cavity walls. We note that in Eq. (10), because of the relations (8) and (58), we obtain \( k_m^r = k_r^\perp \) and thus the correction to the energy density does not depend on \( r|| \).

The zero-order term contains the well-known Casimir energy density \( -\frac{\pi^2 \hbar c}{1440 L_0^4} \) for a three-dimensional scalar field with Dirichlet boundary conditions (see [3, 26, 34], for example), and a term dependent on the position inside the cavity that, in the limit of perfectly conducting walls (infinite cut-off frequency, that is \( \eta \to 0 \), or \( \omega_c \to \infty \)) diverges in the proximity of the wall’s positions. This divergence is related to the presence of a sharp ideal boundary condition, in agreement with the results in Refs. [2] [26, 34], and in Ref. [3] for a single boundary. By integration over \( x \), the two contributions in \( \langle \rho_0 \rangle \) yield the 3D scalar field Casimir energy \( E_0 = -\frac{\pi^2 \hbar c}{1440 L_0^4} \) between two fixed walls.
with parameters such that $0.00001$ clearly shows. The distance of the peak wall, as Fig. shifted with respect to the equilibrium position of the equilibrium position, in the present 3D case the peak is maximum change of the energy density is at the wall’s difference compared to the scalar 1D case discussed measured with high precision [5, 35].

The movable wall. These effects should be currently measured since nowadays Casimir interactions can be experimentally probed using the Casimir-Polder interaction with a polarizable body placed inside the cavity, analogously to the electromagnetic 1D case discussed in the previous Section. This effect is more relevant for decreasing mass and oscillation frequency of the mobile wall. Because in actual optomechanical experiments it is possible to obtain extremely small masses, of the order of $10^{-21}$ Kg, corrections to Casimir-Polder potentials of some few percent seem realistic if the polarizable body is placed very close the movable wall. These effects should be currently measurable since nowadays Casimir interactions can be measured with high precision [2, 32].

Our case of a 3D scalar field has some qualitative significant difference compared to the scalar 1D case discussed in [20]. In fact, contrarily to the 1D case, where the maximum change of the energy density is at the wall’s equilibrium position, in the present 3D case the peak is shifted with respect to the equilibrium position of the wall, as Fig. clearly shows. The distance of the peak from the equilibrium position is strongly related to the cut-off frequency of the cavity walls. A numerical evaluation carried out for different values of the cut-off frequency shows that, increasing the cut-off frequency, the peak more and more approaches the wall’s equilibrium position. Finally, we wish to mention that our results for the 3D massless scalar field can be the basis to obtain the field energy densities for the 3D electromagnetic field too, both for transverse electric and magnetic modes, using the relations outlined in Section III.

V. CONCLUSIONS

We have considered the interaction between a moving conducting wall, whose mechanical degrees of freedom are treated quantum-mechanically, and a field. We have considered the cases of the electromagnetic field in a 1D cavity and a massless scalar field in a 3D cavity, generalizing previous results obtained for the simpler case of a 1D scalar field. The movement of the wall, which we have assumed bound to its equilibrium position by a harmonic potential, yields an effective wall-field interaction and an effective interaction between the field modes, mediated by the mobile wall. For the 1D electromagnetic case, using the Green’s function formalism, we have been able to obtain the electric and magnetic energy densities exploiting previous results obtained from the scalar 1D case using the Law effective Hamiltonian. For the 3D scalar case, we have first generalized the Law Hamiltonian, originally obtained only for the 1D case, to our 3D case with Dirichlet boundary conditions, and then obtained the renormalized Green’s function in the interacting ground state. For both cases, we have evaluated the corrections to the field energy densities inside the cavity in the dressed ground state, and found that they are particularly significant in the vicinity of the movable wall. We have also found that these effects become more relevant with decreasing mass and oscillation frequency of the movable wall around its equilibrium position. We have also discussed measurability of these effects, exploiting Casimir-Polder interactions with a polarizable body placed inside the cavity, and shown that they should be observable by the optomechanical techniques available nowadays. Finally, we point out that our results for the 3D scalar case could be the basis for obtaining also the renormalized electromagnetic energy densities for the transverse electric modes in a 3D cavity with a mobile wall; obtaining the transverse magnetic modes would require solving an analogous scalar problem with von Neumann boundary conditions. We shall discuss these points in a subsequent paper.

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