Probabilistic Soft Type Assignment

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July 6, 2020

Abstract
We model randomized complexity classes in the style of Implicit Computational Complexity. We introduce PSTA, a probabilistic version of STA, the type-theoretical counterpart of Soft Linear Logic. PSTA is a type assignment for an extension of Simpson’s Linear Lambda Calculus and its surface reduction, where Linear additives express random choice. Linear additives are weaker than the usual ones; they allow for duplications harmlessly affecting the computational cost of normalization. PSTA is sound and complete w.r.t. probabilistic polynomial time functions and characterizes the probabilistic complexity classes PP and BPP, the latter slightly less implicitly than PP.

1 Introduction
Probabilistic complexity is a central topic in randomized computation. Many interesting decision problems have efficient and highly trustworthy randomized algorithms for which no good deterministic counterpart is known. Examples of them are in BPP, which collects all those problems that can be solved in polynomial time with error probability bounded by a constant strictly smaller than \(\frac{1}{2}\). The nice point with this class is that the error probability can be exponentially lowered at will while incurring only a polynomial slowdown, so increasing the reliability of the answer without affecting the efficiency.

We here focus on the problem of characterizing probabilistic polynomial time complexity classes in the style of Implicit Computational Complexity (ICC), which merges arguments from computational complexity, mathematical logic and formal systems, yielding machine independent characterizations of complexity classes that do not directly rely on explicit bounds on the computation length.

Starting from Mitchell et al. [22], several type systems were proposed to capture implicitly the probabilistic polynomial time functions by means of higher-order languages. Examples are Zhang [28], or Dal Lago and Toldin [4], all

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based on Hofmann’s system SLR (Safe Linear Recursion) [14]. In particular, the latter work also discusses the inherent difficulties of characterizing the class BPP implicitly, due to the presence of external error bounds. Recently, Seiller has proposed a promising semantic approach to ICC based on the notion of Interaction Graphs [24], showing how to capture the classes PL (Probabilistic Logarithmic space) and PP (Probabilistic Polynomial time), the latter being the class of those problems that a probabilistic polynomial time Turing Machine solves with error probability at most \( \frac{1}{2} \).

Our starting observation is that all type systems introduced in [22, 28, 5] to characterize probabilistic polytime functions and problems share the same principles:

(i) they are probabilistic higher-order generalizations of the recursion-theoretic characterization of FPTIME based on Bellantoni and Cook’s safe recursion [2], which limits the expressive power of the recursion scheme;

(ii) they extend Hofmann’s SLR [14], which models deterministic computations, by means of a primitive for randomness that has the typical oracular nature.

The goal of this paper is then twofold. First, we model randomized computational complexity classes in the style of ICC by exploiting those proof-theoretical techniques derived from Girard’s Linear Logic (LL) that lead to the characterizations of PTIME, FPTIME, NPTIME and PSPACE [11, 21, 16, 9]. A clear advantage with respect to (i) is that we deal with fully-fledged higher-order languages (and polymorphism), while in all type systems developed in [22, 28, 5] functional arguments have to be used linearly, i.e. at most once.

Secondly, we introduce randomness according to the principle that any computational step should correspond to some step of normalization in a proof system or typed calculus. Probabilistic choice is then considered as the result of an interaction between a constructor and its corresponding destructor, and so it does not depend on the answer of a “black-box”, like random primitives in (ii). Matsuoka explores this idea in the non-deterministic setting [20], introducing a self-dual additive connective into restrictions of LL to characterize non-deterministic complexity classes. Applying the same approach in a probabilistic setting is less obvious, because random choice cannot be self-dual, as recently observed by Horne in [15]. Due to this reason, Horne proposes a Deep-inference logical system that introduces sub-additives [15] which enjoy De Morgan dualities and lie “half-way” in between LL additive conjunction, that models an external choice, and the additive disjunction, that models the internal one.

We achieve the above goals by means of PSTA, a new type system that merges ideas and techniques from Lafont’s Soft Linear Logic (SLL) [16], Gaboardi and Ronchi Della Rocca’s Soft Type Assignment (STA) [10], Simpson’s Linear Lambda Calculus (LLC) [25], and Ronchi Della Rocca and Roversi’s calculus \( \Lambda! \) [23]. Probabilistic features in PSTA are expressed by means of the interaction between a pair \((M, N)\) (constructor) and a new projection operator \(\text{proj}(\text{destructor})\), which randomly selects a component of \((M, N)\). Constructor and destructor are the subject of type-assignment rules that operate on Linear additives, which are weaker than standard additives.

Linear additives trigger a restricted form of duplication that causes no exponential blow up in normalization. This way, PSTA inherits the polynomial time
computational complexity bounds from STA. Moreover, Linear additives turn out to be expressive enough to encode the transition function of a probabilistic Turing Machine running in polynomial time, which is the key to establish PSTA completeness with respect to the probabilistic polytime functions. The resulting characterization is fully implicit and does not depend on the choice of the reduction strategy: this is where Linear additives play a crucial role, since the standard additive rules require a lazy strategy to avoid exponentially costing normalizations [12].

Last, by slightly modifying the encoding of the probabilistic Turing Machine in PSTA, we can show that this system is both sound and complete w.r.t. the complexity classes PP and BPP; the latter is not entirely captured implicitly due to explicit error-bounds in the statement of the characterization theorem.

Perhaps a better result for BPP is at hand by exploiting the stochastic denotational models for deductive systems based on LL (e.g. probabilistic coherence spaces [6] or weighted relational semantics [17]), once adapting them to PSTA. The idea is to find a semantic characterization of BPP in the style of [18] able to suggest some insights about the nature of this class.

Having discussed motivations about PSTA, we illustrate the key ideas behind it. We start from the inference rules for the additive connective & of LL, seen as a type-assignment:

$$
\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2 \quad \Gamma \vdash \pi_i(M) : A_i \quad i \in \{1, 2\}
$$

(1)

The rule &I affects the complexity of normalization. Indeed, it gives a type to the terms $\text{add}^n_x$ defined, for all $x$ and $n \in \mathbb{N}$, as follows:

$$
\text{add}^0_x \triangleq x \quad \text{add}^n_x \triangleq (\lambda y. \text{add}^{n-1}_y)(x, x) \quad (n > 0).
$$

The application of $\lambda x. \text{add}^n_x$ to some $M$ reduces to $M[n]$, defined as:

$$
M[0] \triangleq M \quad M[n] \triangleq (M[n-1], M[n-1]) \quad (n > 0).
$$

The size of $M[n]$ and the number of its redexes (if any) are exponential with respect to those of $M$. This example shows that linear normalization fails in presence of additive rules.

For this reason, in [3] the first author develops Linear additives, weaker than standard additives, which imply a strong linear normalization property. Linear additives come from replacing the above rule &I in (1) by the following one:

$$
\Gamma \vdash N : A \quad x_1 : A \vdash M_1 : A_1 \quad x_2 : A \vdash M_2 : A_2 \quad \vdash U : A \\
\Gamma \vdash \text{copy}^U N \text{ as } x_1, x_2 \text{ in } \langle M_1, M_2 \rangle : A_1 \& A_2
$$

(2)

with the proviso that $U$ is a closed and normal inhabitant of $A$, and the types $A, A_1, A_2$ are free from negative occurrences of the second-order quantifier; this last proviso applies to the above &E in (1) too. Intuitively, the operator copy “freezes” the substitutions of $N$ in the pair $\langle M_1, M_2 \rangle$ until $N$ has been fully evaluated to a closed normal form $V$. The corresponding reduction rule is then the following one:

$$
\text{copy}^U V \text{ as } x_1, x_2 \text{ in } \langle M_1, M_2 \rangle \rightarrow \langle M_1[V/x_1], M_2[V/x_2] \rangle.
$$

(3)
Since the above rule duplicates normal terms only, redexes cannot be copied during reduction and linear time normalization can be recovered. Moreover, since the type $A$ in (2) has only finitely many closed normal inhabitants, due to the absence of $\forall$ in negative position, by always taking $U$ in (2) as the largest term among such inhabitants, the size of the construct $\text{copy}^U$ bounds the size of the new copy of $V$; so, normalization strictly decreases the size of terms.

To let the reduction rule in (3) preserve types, in [3] we introduced a further inference rule, which is $\text{&I}^1$ in (1) with $\Gamma = \emptyset$. This rule allows to give a type to pairs $\langle M, N \rangle$ of closed terms. Here, for the sake of simplicity, we shall consider this rule as a special case of (2).

Linear additives in PSTA justify a projection $\text{proj}$, new, as compared to the standard $\pi_1$ in (1), which non-deterministically selects a component in a pair:

$$M_1 \leftarrow \text{proj}(M_1, M_2) \rightarrow M_2.$$  

Probabilistic computation can then be expressed in PSTA by turning the one step non-deterministic reduction $\rightarrow$ into a multi-step reduction $\Rightarrow$ between terms and probability distributions. As expected, probabilistic choices in a higher-order calculus may lead to the failure of confluence, as distinct evaluation strategies may produce distinct distributions.

Example 1. Let $M \triangleq (\lambda x.\langle x, x \rangle)\text{coin}$, where $\text{coin} \triangleq \text{proj}(T, F)$, $T \triangleq \lambda xy.x$ and $F \triangleq \lambda xy.y$. A call-by-name reduction strategy first passes $\text{coin}$ to $\lambda x.\langle x, x \rangle$. Then, it evaluates the two copies of $\text{coin}$ produced, obtaining the terms $\langle T, T \rangle$, $\langle T, F \rangle$, $\langle F, T \rangle$ and $\langle F, F \rangle$, as a result, each one with probability $\frac{1}{4}$. By contrast, call-by-value evaluates $M$ by first reducing $\text{coin}$, then passing the result to $\lambda x.\langle x, x \rangle$. The results are $\langle T, T \rangle$ and $\langle F, F \rangle$, both with probability $\frac{1}{2}$. Thus, the two parameter-passing policies give different distributions. 

The solution we adopt in PSTA, also studied in [7, 8], is to move from standard $\lambda$-calculus to Simpson’s Linear Lambda Calculus (LLC) and its surface reduction. This is an untyped term calculus $\Lambda'$ closely related to LL [13]. It has two $\lambda$-abstractions. One is the linear abstraction $\lambda x.M$; the other is the non-linear $\lambda x.M$. The latter can duplicate arguments with form $!A$, whose evaluation is suspended, according to the following rule:

$$\lambda x.M)!A \Rightarrow M[N/x]$$  

Then, uniqueness of distributions in our probabilistic extension of $\Lambda'$ can be recovered. For example, $M$ in Example 1 turns into $M' \triangleq (\lambda x.\langle x, x \rangle)\text{coin}$. Since reduction is forbidden in the scope of a ! operator, $\text{coin}$ is passed to the function before being evaluated.

Unfortunately, typed variants of (extensions of) $\Lambda'$ may lead to the failure of Subject reduction, as the following example shows on STA [10].

Example 2. Pretending that STA is a type-assignment for $\Lambda'$, we would have the derivation:

$$
\begin{array}{ll}
\frac{y_1 : A \vdash y_1 : A}{x : A \vdash x : A} & \text{ax} \\
\frac{y_1 : A, y_2 : A \vdash (y_1, y_2) : A \otimes A}{y_1 : A, y_2 : A \vdash y_2 : A} & \otimes R \\
\frac{s : !!A \vdash (z, z) : A \otimes A}{!!A \vdash s : A \otimes A} & \text{m} \\
\frac{\lambda z.(z, z) : !!A \rightarrow A \otimes A}{\lambda x.(z, z) : !!A \Rightarrow A \otimes A} & \Rightarrow I \\
\frac{!!A \vdash (\lambda x.(z, z))!!x : A \otimes A}{!!A \vdash (\lambda x.(z, z))!!x : A \otimes A} & \Rightarrow E
\end{array}
$$
$A \vdash x : A$

$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B}$  \text{-} \text{Il}$

$\frac{\Gamma \vdash \lambda x. M : !\rightarrow B}{\Gamma \vdash \lambda x. M : !\rightarrow B}$  \text{-} \text{I}_e$

$\frac{\Gamma \vdash M : \sigma \rightarrow A}{\Gamma, \Gamma' \vdash M N : A}$  \text{-} \text{E}$

$\frac{\Delta \vdash N : C \quad x_1 : C \vdash M_1 : C_1 \quad x_2 : C \vdash M_2 : C_2 \quad \vdash V : C}{\Delta \vdash \text{copy}^V N \text{ as } x_1, x_2 \text{ in } (M_1, M_2) : C_1 \& C_2}$  \&\text{I}$

$\frac{\Delta \vdash \text{proj}(M) : C}{\Delta \vdash \text{proj}(M) : C}$  \&\text{E}$

$\frac{y_1 : !\sigma_1, \ldots, y_n : !\sigma_n \vdash M[d(y_1)/x_1, \ldots, d(y_n)/x_n] : !\tau}{y_1 : !\sigma_1, \ldots, y_n : !\sigma_n \vdash M[d(y_1)/x_1, \ldots, d(y_n)/x_n] : !\tau}$  \text{-} \text{sp}$

$\frac{\Gamma, x_1 : \sigma, \ldots, x_n : \sigma \vdash M : \tau \quad (n \geq 0)}{\Gamma, x : !\sigma \vdash M[d(x)/x_1, \ldots, d(x)/x_n] : \tau}$  \text{-} \text{sp}$

$\frac{\Gamma \vdash M : A(\gamma/\alpha) \quad \gamma \notin \text{FV}(\Gamma)}{\Gamma \vdash M : \forall \alpha.A}$  \forall\text{I}$

$\frac{\Gamma \vdash M : \forall \alpha.A}{\Gamma \vdash M : A(B/\alpha)}$  \forall\text{E}$

\begin{align*}
\Delta &\vdash N : C \\
x_1 &: C_1 \vdash M_1 : C_1 \\
x_2 &: C_2 \vdash M_2 : C_2 \\
\vdash V : C
\end{align*}

$\Delta \vdash \text{copy}^V N \text{ as } x_1, x_2 \text{ in } (M_1, M_2) : C_1 \& C_2$  \&\text{I}$

$\Delta \vdash M : C \& C$  \&\text{E}$

$\Delta \vdash \text{proj}(M) : C$  \&\text{E}$

Figure 1: The system PSTA: $C, C_1, C_2$ are $\forall!$-lazy types, $\Delta$ is a $\forall!$-lazy context, and $V \in \mathcal{V}$.

where double line means multiple applications of a rule. Let us apply the surface reduction step in (4) to $(\lambda !z. \langle z, z \rangle)!!x$. We obtain a judgment $x : !!A \vdash \langle x, x \rangle : A \& A$ without derivations in STA. Subject reduction fails as both occurrences of $!$ in $(\lambda !z. \langle z, z \rangle)!!x$ should be erased during surface reduction, while only one is.

The last steps toward PSTA, in order to avoid the above issue, both introduce explicit dereliction $d$, and generalize the surface reduction rule in (4). For example, in PSTA, the conclusion of the derivation in Example 2 turns into $x : !!A \vdash (\lambda !z. \langle d(d(z)), d(d(z)) \rangle)!!(d(d(x)))) : A \& A$. Intuitively, according to the “general” surface reduction rule, the normalization of the term that this judgment gives a type to, first performs a beta-reduction, yielding $(d(d!!(d(d(x))))), d(d!!(d(d(x))))$; then it rewrites each $d!!(d!!(M))$ into $M$. The resulting term is $(d(d(x)), d(d(x)))$, with type in PSTA.

Many proofs are postponed in the Appendix.
2 The type assignment system 𝑃𝑆𝑇𝐴

The type assignment system 𝑃𝑆𝑇𝐴 is in Figure 1. It extends 𝑆𝑇𝐴 [10] with a non-deterministic version of Linear additives from [3] (rules &I and &E), 𝑃𝑆𝑇𝐴 derives judgments 𝜋 ⊢ 𝑀 : 𝜎, where 𝜎 is generated by a grammar of essential types, like [10], 𝜋 is the context that gives types to the free variables of 𝑀, and 𝑀 belongs to the term calculus 𝐿𝐿𝐶 [25], which is Simpson’s Linear Lambda Calculus (LLC) [25] endowed with explicit dereliction 𝑑 (as in [23]), a copy operator (as in [3]), pairs (𝑀, 𝑁) and a non-deterministic projection operator proj.

2.1 The types of 𝑃𝑆𝑇𝐴

The following grammar generates the language of types in 𝑃𝑆𝑇𝐴:

\[
\sigma, \tau ::= A \mid !\sigma \\
A, B ::= \alpha \mid \sigma \Rightarrow A \mid A \& A \mid \forall \alpha. A .
\]

The start symbol 𝜎 yields exponential types, and 𝐴 the linear types. A type !\tau is strictly exponential. The set of free variables of 𝜎 is FV(\sigma). A type 𝜎 is closed if FV(𝐴) = ∅. The ∀!-lazy types, crucial to prove the relevant properties of 𝑃𝑆𝑇𝐴, are types free of negative occurrences of ∀ and of any occurrence of !.

Example 3. Typical examples of ∀!-lazy types are the unit 1 ≜ ∀α. α → α and the boolean data type B ≜ ∀α. α → α → α, where tensor σ ⊗ τ is introduced by means of the second-order definition ∀α.(α → τ → α) → α. Moreover, if 𝐴 and 𝐵 are ∀!-lazy types then both 𝐴 ⊗ 𝐵 and 𝐴 & 𝐵 are. However, neither the type N ≜ ∀α.!(α → α) → (α → α) for natural numbers, nor the type 𝐵 → 𝐵 are ∀!-lazy types, the former because of the occurrence of !, the latter because it has negative occurrences of ∀.

The types in 𝑃𝑆𝑇𝐴 merge the structure of types from both Soft Type Assignment (STA) [10] and Linearly Additive Multiplicative Type Assignment (LAM) [3]. We recall that 𝑆𝑇𝐴 is a type-assignment that characterizes polynomial time functions (FPTIME) and problems (PTIME) under the formulas-as-types paradigm. The types of 𝑆𝑇𝐴, called essential, restrict the formulas of Soft Linear Logic (SLL) [16] in order to assure Subject reduction while preserving the polynomial time bound on term normalization. The key point about essential types is to forbid topmost occurrences of the “of course” modality “!” in the right-hand side of an implication. i.e., 𝐴 → !𝐵 is neither a type of 𝑆𝑇𝐴 nor of 𝑃𝑆𝑇𝐴. Let us also recall that 𝐿𝐴𝑀 [3] is obtained from Intuitionistic Second-Order Multiplicative Additive Linear Logic (IMALL2) by replacing the standard additives with weaker versions, called Linear additives, which avoid exponentially costing normalizations, typical of known additive rules.
2.2 Terms and one-step surface reduction of \( \text{PSTA} \)

The following grammar generates the language of raw terms in \( \text{PSTA} \):

\[
M, N ::= \Lambda | A | d(M) \\
\Lambda ::= x | \lambda x . M | \lambda ! x . M | MM | ! M \\
A ::= (M, M) | \text{proj}(M) | \text{copy}^V M \text{ as } x, y \text{ in } (M, M) \\
V, U ::= x | \lambda x . V | V V | (V, V)
\]  

(7) (8) (9) (10)

where \( M \) is the start symbol and \( \Lambda \) highlights the structure of terms that we take from LLC. We observe that \( \Lambda \) generates both a linear abstraction \( \lambda x . M \) and a non-linear one \( \lambda x . M \), the latter duplicating arguments with shape \( ! N \). Moreover, \( A \) generates additive terms and \( V \) gives the language in which we shall identify the so-called values, as we shall see.

The set of free variables of \( M \) is \( FV(M) \), where both \( \lambda x . M \) and \( \lambda ! x . M \) bind \( x \) in \( M \), and \( \text{copy}^V M \text{ as } x, y \text{ in } (P, Q) \) binds both \( x \) in \( P \) and \( y \) in \( Q \). If \( FV(M) = \emptyset \), then \( M \) is closed. The meta-level capture-avoiding substitution of \( N \) for the free variables of \( M \) is \( M[N/x] \). The inductive definition of the size \| M \| of \( M \) is standard, with \( \text{copy} \) requiring:

\[
\text{copy}^V M \text{ as } x, y \text{ in } (P, Q) \triangleq |V| + |M| + |P| + |Q| + 2 .
\]  

(11)

A variable \( x \) in \( M \) is surface-linear (s-linear) if \( x \) occurs free exactly once in \( M \), but not in the sub-terms \( ! N \) and \( d(N) \) of \( M \). A term \( M \) is surface-linear (s-linear) if both:

- \( x \) is s-linear in \( N \), for every \( \lambda x . N \) in \( M \), and
- \( x \) is s-linear in \( P \) and \( y \) is s-linear in \( Q \), for every \( \text{copy}^V N \text{ as } x, y \text{ in } (P, Q) \text{ in } M \).

We let \( !^n M \) and \( d^n(M) \) denote \( ! \cdots ! M \) and \( d(\cdots d(M) \cdots) \), respectively.

**Definition 1.** \( \Lambda_{\oplus} \) is the language of all s-linear raw terms generated by the grammar (7).

Since \( \Lambda_{\oplus} \) is endowed with a dereliction operator \( d \), that is missing in LLC, we need to generalize the reduction step \( (\lambda x . M)!N \rightarrow M[N/x] \) of LLC in order to take \( d \) into account.

**Definition 2** (Surface-preserving substitution). Let \( M, N \in \Lambda_{\oplus} \). The surface-preserving substitution \( M[N/x] \) of \( N \) for the free occurrences of \( x \) in \( M \) is:

\[
M[N/x] \triangleq \begin{cases} 
P(Q/y) & \text{if } N = ! Q \text{ and } M = P[d(x)/y], \text{ with } x \not\in FV(P), \\
M[N/x] & \text{otherwise}.
\end{cases}
\]

Moreover, \( M[N/x_1, \ldots, N/x_n] \) denotes \( (M[N/x_1]) \ldots (N/x_n) \).

**Example 4.** Let us take \( z d^3(x) d^2(x) \) in \( \Lambda_{\oplus} \). The surface-preserving substitution of \( !^2 y \) for the free occurrences of \( x \) in \( z d^3(x) d^2(x) \) is:

\[
(z d^3(x) d^2(x)) \{(!^2 y)/x\} \\
= (z d^3(x') d(x')) \{(!^2 y)/x'\} \quad \text{because } z d^3(x) d^2(x) \triangleq (z d^3(x') d(x'))[d(x)/x'] \\
= (z d(x’')) \{y/x''\} \quad \text{because } z d^2(x') d(x') \triangleq (z d(x’’) x’’) [d(x’)/x’’] \\
= z d(y) .
\]  

\( \square \)
Definition 3. The set $V$ of values in $\Lambda^!_\oplus$ contains any closed term generated by the grammar (10) that is normal with respect to the reduction step $(\lambda x.U)V \rightarrow U[V/x]$.

Definition 4 (One-step surface reduction for $\Lambda^!_\oplus$). A surface context is a term in $\Lambda^!_\oplus$ with a unique hole $[\cdot]$ in it. The following grammar generates surface contexts:

$$
C ::= [\cdot] \mid \lambda x.C \mid \lambda x,y.C \mid CM \mid MC \mid d(C) \mid \langle C, M \rangle \mid \langle M, C \rangle \mid \text{proj}(C) \\
\text{copy}^V C \text{ as } x, y \text{ in } \langle M, N \rangle \mid \text{copy}^V M \text{ as } x, y \text{ in } \langle C, N \rangle
$$

where $C[M]$ is the term obtained by filling the hole in $C$ with $M$, possibly capturing free variables.

The one-step surface reduction $\rightarrow \subseteq \Lambda^!_\oplus \times (\Lambda^!_\oplus)^2$ is:

$$
(\lambda x.M)N \rightarrow M[N/x] \\
(\lambda x.M)!N \rightarrow M\{N/x\} \\
\text{proj}\langle M, N \rangle \rightarrow M, N \\
\text{copy}^U V \text{ as } x, y \text{ in } \langle M, N \rangle \rightarrow \langle M[V/x], N[V/y] \rangle
$$

where $M \rightarrow N$, in fact, means $M \rightarrow N, N$, for any $M, N$. We can apply $\rightarrow$ in surface contexts only. A term of $\Lambda^!_\oplus$ is in (or is a) surface normal form if no reduction applies to it. Surface normal forms are ranged over by $S$, and the set of all surface normal forms is $\text{SNF}$.

2.3 Judgments, inference rules and derivations of PSTA

Once given the types in Section 2.1, terms, values and reduction steps in Section 2.2, comments and notations relative to the rules of PSTA in Figure 1 become simpler.

Let us recall that a context is a finite multi-set of assumptions $x : A$. If $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, then $FV(\Gamma) \triangleq \bigcup_{i=1}^n FV(A_i)$ and $|\Gamma| \triangleq \sum_{i=1}^n |A_i|$. A context $\Gamma$ is strictly exponential if it contains strictly exponential types only. A context $\Gamma$ is $\forall!$-lazy if it contains $\forall!$-lazy types only. If $\Gamma$ is $x_1 : A_1, \ldots, x_n : A_n$, then $\forall!\Gamma$ is $x_1 : !A_1, \ldots, x_n : !A_n$. By $D \triangleleft \Gamma \vdash M : A$ we denote a derivation $D$ with conclusion $\Gamma \vdash M : A$. The size $|D|$ of a derivation $D$ counts the number of rule instances it contains.

We conclude by commenting the inference rules of PSTA:

- Two introduction rules of the linear implication $\rightarrow$ exist. The subject in the conclusion of $\rightarrow I_l$ is $\lambda x.M$ and the antecedent of $\rightarrow$ is a linear type. The subject in the conclusion of $\rightarrow I_e$ is $\lambda x,y.M$ and the antecedent of $\rightarrow$ is strictly exponential.

- The linear additive rule $&I$ replaces the standard one in (1). The types $C, C_1, C_2$ in $&I$ and $&E$ must be $\forall!$-lazy. Likewise, $\Delta$ is $\forall!$-lazy in $&E$ and $&I$. Finally, the term $V$ in the last premise of $&I$ is a value.


• We shall consider the instance of (1) with $\Gamma = \emptyset$ as a special case of &I in PSTA. This allows us to give a type to some pairs $(M, N)$ of $\Lambda_{!0}$ and to let the reduction rule for copy in (13) preserve types in PSTA.

• The rule &E introduces non-determinism in PSTA by means of a projection that non-deterministically selects one of the two components in a pair.

• Finally, $sp$ and $m$ come from STA. They are the type-theoretical formulations of the logical rules soft promotion and multiplexor of SLL to introduce controlled duplications.

The key property of $\forall!$-lazy types, analogous to the one in [3], is that their size gives a bound on the size of any value that inhabits them:

**Proposition 1.** Let $V \in \mathcal{V}$. If $A$ is a $\forall!$-lazy type and $\mathcal{D}\vDash V : A$, then $|V| \leq |A|$.

**Proof.** The statement follows by proving by induction on the last rule of $\mathcal{D}$ the following stronger statement: “Let $V$ be generated by (10) and normal. If $x_1 : A_1, \ldots, x_n : A_n \vdash M : A$, and $A_1 \circ \ldots \circ A_n \rightarrow A$ is $\forall!$-lazy, then $|M| \leq \sum_{i=1}^{n} |A_i| + |A|$”. By assumption, the last rule of $\mathcal{D}$ cannot be $m$, $sp$, &E or &I.

**Remark 1.** **Proposition 1** implies that, for any $\forall!$-lazy type $A$, a value $V$ of type $A$ exists such that $|U| \leq |V|$, for all values $U$ in the type $A$. W.l.o.g., we shall assume that the value $V$ in the last premise of &I in Figure 1 has largest size among all the values of the same type. Therefore, as long as we consider typable terms in PSTA, the reduction rule in (13) is such that $|\text{copy}^V U|$ as $x, y$ in $(M, N) > |(M[U/x], N[U/y])|$, because $V$ is a bound on the size of the new copy of the value $U$ that the reduction generates. So, Linear additives do not problematically affect the complexity of normalization, even though they allow duplications.

### 3 A probabilistic multi-step surface reduction for PSTA

We here turn the non-deterministic reduction in **Definition 4** into a probabilistic multi-step reduction relation $\Rightarrow$ between terms of $\Lambda_{!0}$ and distributions of Surface normal forms.

We recall that a probability distribution over a countable set $X$ is a function $f : X \rightarrow [0, 1]$ such that $\sum_{x \in X} f(x) = 1$. The support $\text{supp}(\mathcal{D})$ of a distribution $\mathcal{D}$ is the subset of all the elements in $X$ such that $\mathcal{D}(x) > 0$. Given $x_1, \ldots, x_n \in X$, then $p_1 \cdot x_1 + \ldots + p_n \cdot x_n$ denotes the distribution $\mathcal{D}$ with finite $\text{supp}(\mathcal{D}) = \{x_1, \ldots, x_n\}$, such that $\mathcal{D}(x_i) = p_i$, for every $i \leq n$. Moreover, $x \in X$ denotes both an element in $X$ and the distribution having all its mass on $x$, i.e., $1 \cdot x$.

Finally, let $I$ be a finite set of indexes, let $\{p_i\}_{i \in I}$ be a family of $\forall!$-lazy numbers such that $\sum_{i \in I} p_i = 1$, and let $\{\mathcal{D}_i\}_{i \in I}$ be a family of distributions. Then, for all $x \in X$, we define $(\sum_{i \in I} p_i \cdot \mathcal{D}_i)(x) \triangleq \sum_{i \in I} p_i \cdot \mathcal{D}_i(x)$.

**Definition 5** (Multi-step surface reduction for $\Lambda_{!0}$),
Consider the term Example 5.

\[ \Omega(\lambda \cdot \langle T, d(x) \rangle) \Rightarrow (\lambda \cdot \langle F, d(x) \rangle) \]

Figure 2: Multi-step surface reduction \( \Rightarrow \) for \( \Lambda^1_{\mathbb{B}} \).

\[ (\lambda \cdot \langle \text{coin}, d(x) \rangle) \]

Figure 3: Different surface reduction strategies for \( (\lambda \cdot \langle \text{coin}, d(x) \rangle)I \), where \( \text{coin} \to T, F \).

- A surface distribution is a probability distribution over SNF (see Definition 4), i.e. a function \( \mathcal{D} : \text{SNF} \to [0, 1] \) such that \( \sum_{S \in \text{SNF}} \mathcal{D}(S) = 1 \).

- The multi-step surface reduction \( \Rightarrow \) is the relation between terms of \( \Lambda^1_{\mathbb{B}} \) and surface distributions defined in Figure 2. Both \( \pi \) and \( \rho \) range over derivations of \( M \Rightarrow \mathcal{D} \).

- The size \( |\pi| \) of a derivation \( \pi : M \Rightarrow \mathcal{D} \) is 0 if \( \pi \) is \( s1 \), and max(|\pi_1|, |\pi_2|) + 1 if \( \pi \) is \( s2 \) with premises \( M \Rightarrow M_1, M_2, \pi_1 : M_1 \Rightarrow \mathcal{D}_1 \) and \( \pi_2 : M_2 \Rightarrow \mathcal{D}_2 \).

Example 5. Consider the term \( (\lambda \cdot \langle \text{coin}, d(x) \rangle)I \), where \( \text{coin} \) is as in Example 1 and \( I \triangleq \lambda \cdot x \cdot x \). We can apply surface reduction to this term in two different ways as in Figure 3. In particular, the one with dashed lines corresponds to the derivation of the multi-step reduction \( (\lambda \cdot \langle \text{coin}, d(x) \rangle)I \Rightarrow \frac{1}{2} \cdot \langle T, I \rangle + \frac{1}{2} \cdot \langle F, I \rangle \) in Figure 4.

Example 6. Let \( \Omega_1 \triangleq \Delta_1(\langle \Delta_2 \rangle) \), where \( \Delta_1 \triangleq \lambda \cdot x \cdot d(x)!d(x) \). Since \( \Omega_1 \Rightarrow (d(x)\!d(x))!\langle \Delta_2/x \rangle = ((y\!y)\!d(x)/y)!\langle \Delta_2/y \rangle = \Omega_1 \), no surface distribution \( \mathcal{D} \) exists such that \( \Omega_1 \Rightarrow \mathcal{D} \).

The calculus \( \Lambda^1_{\mathbb{B}} \) enjoys the following confluence property:

**Theorem 1 (Confluence for \( \Lambda^1_{\mathbb{B}} \)).** Let \( M \in \Lambda^1_{\mathbb{B}} \). If \( M \Rightarrow \mathcal{D} \) and \( M \Rightarrow \mathcal{E} \) then \( \mathcal{D} = \mathcal{E} \).

**Sketch.** Following [4], we define a relation \( \Rightarrow \) between terms and distributions over \( \Lambda^1_{\mathbb{B}} \), where rule \( s1 \) is relaxed to allow \( M \Rightarrow M \), for all \( M \in \Lambda^1_{\mathbb{B}} \), and such that \( \Rightarrow \subseteq \Rightarrow \). So, if \( \Rightarrow \) is confluent, then \( \Rightarrow \) is. To show this, we first establish confluence for \( \Rightarrow \), which requires to prove “If \( M \Rightarrow M_1', M_2' \) and \( M \Rightarrow M_1'', M_2'' \), then there exist \( N_1, N_2, N_3, N_4 \) distinct such that \( M_1' \Rightarrow N_1, N_2, M_2' \Rightarrow N_3, N_4, \) and \( \exists i \in \{1, 2\} \) such that \( M_1'' \Rightarrow N_1, N_3, \) and \( M_2'' \Rightarrow N_2, N_4 \)” among other lemmas. Then, we lift this confluence property from \( \Rightarrow \) to \( \Rightarrow \).
4 Probabilistic Polytyme Soundness of PSTA

We show that the evaluation of any term of \( \Lambda_0 \) with type in PSTA (according to the multi-step reduction \( \Rightarrow \)) can be simulated by a polynomial time Probabilistic Turing Machine (pPTM), i.e. by a Probabilistic Turing Machine (PTM) whose running time is bounded by some polynomial in the input size. We adapt the proof developed for STA [10], known since [16], to the probabilistic setting. We show that surface reduction preserves types and shrinks the weight of derivations; so, in fact, we prove a version of Subject reduction (Theorem 2) a bit stronger than usual. From this we derive that the number of surface reduction steps rewriting a typable term into its surface normal forms is polynomially bounded (Lemma 3.) This, eventually, implies Probabilistic Polytyme Soundness (Theorem 3.)

We start recalling the notions of rank (here \( m \)-rank) and depth from [16, 10]. We introduce the \( sp \)-rank; the treatment of both \(!\) and \( d \), which affect the size of a term, requires it.

**Definition 6** \( (m\text{-rank}, sp\text{-rank}, \text{depth}). \)

- The \( m \)-rank of a rule \( m \) of the form:

\[
\frac{\Gamma, x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau \quad (n \geq 0)}{\Gamma, x : \sigma \vdash M[d(x)/x_1, \ldots, d(x)/x_n] : \tau} m
\]

is the number \( k \leq n \) of variables \( x_i \) such that \( x_i \in FV(M) \). The \( m \)-rank \( \text{rk}(D) \) of a derivation \( D \) is \( \max(1, k) \), with \( k \) the maximum \( m \)-rank among the instances of \( m \) in \( D \).

- The \( sp \)-rank of a rule \( sp \) of the form:

\[
\frac{x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau}{y_1 : \sigma_1, \ldots, y_n : \sigma_n \vdash M[d(y_1)/x_1, \ldots, d(y_n)/y_n] : \tau} sp
\]

is the number \( k \leq n \) of variables \( x_i \) such that \( x_i \in FV(M) \).

- The depth \( d(D) \) of a derivation \( D \) is the maximum number of occurrences of \( sp \) in a path from the conclusion of \( D \) to one axiom in \( D \).

**Definition 7** \( (\text{Weight}). \) Let \( r \geq 1 \). The weight \( w(D, r) \) \( (\text{relative to} \ r) \) of a derivation \( D \) is defined by structural induction on \( D \):

- if the last rule of \( D \) is \( ax \), then \( w(D, r) = 1 \);
- if \( D \) is obtained from \( D' \) by applying \( \neg \neg l \), \( \neg l \lor \text{or} \) \( \& \text{E} \), then \( w(D, r) = w(D', r) + 1 \);
Lemma 1 (Propositional Substitution). Let $\Gamma \vdash M : \sigma$. Then:

1. $\text{rk}(D) \leq |M|$;
2. $w(D, r) \leq r^{|D|} \cdot w(D, 1)$;
3. $w(D, 1) = |M|$. Moreover, if $D$ has no occurrences of $sp$ and $m$, then $w(D, r) = |M|$.

Theorem 2 (Weighted Subject reduction). Let $D \triangleleft \Gamma \vdash M : \sigma$ and $r \geq \text{rk}(D)$. If $M \Rightarrow M_1, M_2$, then there exist $D_1$ and $D_2$ such that:

1. $D_i \triangleleft \Gamma \vdash M_i : \sigma$.
2. $w(D_i, r) < w(D, r)$, for $i \in \{1, 2\}$.

Sketch. The proof is by induction on the definition of the one-step reduction relation $\Rightarrow$. It requires to prove a Weighted Substitution property: “For all $r \geq \text{rk}(D_1)$, if $D_1 \triangleleft \Gamma, x : \sigma \vdash \tau$ and $D_2 \triangleleft \Delta \vdash N : \sigma$ then $\Gamma \vdash M [N/x] : \tau$ and $w(D^*, r) \leq w(D_1, r) + w(D_2, r)$.” The proof of the Weighted Substitution property relies on the lemma: “If $\Gamma \vdash M : \sigma$ is derivable in PSTA, then $\Gamma$ is a strictly exponential context.”

The above theorem implies that terms typable in PSTA are strong normalizing with respect to Surface reduction $\Rightarrow$, and hence that, for any $M$ with type in PSTA, a surface distribution $\mathcal{D}$ exists such that $M \Rightarrow \mathcal{D}$. By Theorem 1, this surface distribution is unique.

Every derivation $M \Rightarrow D$, with $M$ having a type in PSTA, enjoys the following:

Lemma 2 (Uniformity). Let $\Gamma \vdash M : \sigma$. If $\pi' : M \Rightarrow \mathcal{D}$ and $\pi'' : M \Rightarrow \mathcal{D}$, then $|\pi'| = |\pi''|$.

Sketch. Reductions take place at a “surface level”, i.e. never in the scope of any $!, \text{ so that redexes are never duplicated or erased}$.

The above lemma says that an upper bound on $M \Rightarrow \mathcal{D}$ exists on the length of each non-deterministic branching of all possible reduction strategies applied to $M$. That bound is limited by a polynomial in the size of $M$:

Lemma 3 (Strong polystep soundness). Let $D \triangleleft \Gamma \vdash M : \sigma$ and $\pi : M \Rightarrow \mathcal{D}$. Then:
Proof. Let $\mathcal{D} \triangleq \Gamma \vdash M : \sigma$. Lemma 1.1-3 implies:

$$w(\mathcal{D}, \text{rk}(\mathcal{D})) \leq w(\mathcal{D}, |M|) \leq |M|^{d(\mathcal{D})} \cdot w(\mathcal{D}, 1) = |M|^{d(\mathcal{D})} \cdot |M| = |M|^{d(\mathcal{D})+1}.$$  

By induction on the size of $\pi : M \Rightarrow \mathcal{S}$, for all $r \geq \text{rk}(\mathcal{D})$, we can prove:

i. $|\sigma| \leq w(\mathcal{D}, r)$;  

ii. $|N| \leq w(\mathcal{D}, r)$, for every $N \Rightarrow N'$, $N''$ premise of $s2$ in $\pi$.

If the last rule of $\pi$ is $s1$, then both i and ii here above hold trivially. Otherwise, the last rule of $\pi$ is $s2$ with premises $M \Rightarrow M_1, M_2$, $\pi_1 : M_1 \Rightarrow \mathcal{S}_1$, and $\pi_2 : M_2 \Rightarrow \mathcal{S}_2$. By Theorem 2, there exist $\mathcal{D}_1$ and $\mathcal{D}_2$ such that both $\mathcal{D}_i \triangleq \Gamma \vdash M_i : \sigma$ and $w(\mathcal{D}_i, r) < w(\mathcal{D}, r)$. Concerning point i, by induction, $|\pi_i| \leq w(\mathcal{D}_i, r)$, with $i \in \{1, 2\}$. Hence, $|\pi| = \max(|\pi_1|, |\pi_2|) + 1 \leq \max(w(\mathcal{D}_1, r), w(\mathcal{D}_2, r)) + 1 \leq w(\mathcal{D}, r)$. Concerning point ii, $|N| \leq w(\mathcal{D}, r) < w(\mathcal{D}, r)$ holds by induction, for all $i \in \{1, 2\}$ and for all $N \Rightarrow N', N''$, premise of some $s2$ in $\pi_i$. Finally, by Lemma 1.3, we have $|M| = w(\mathcal{D}, 1) \leq w(\mathcal{D}, r)$. \]

Remark 2. From [27], we know that a Turing Machine simulates a $\beta$-reduction $M \rightarrow_M M'$ in a time bounded by $O(|M|^2)$. Similarly, for every step $M \rightarrow M_1, M_2$ in Definition 4, a PTM exists which, receiving an encoding of $M$ as input, produces an encoding of $M$, as output with probability a half, in a time bounded by $O(|M|^2)$. \]

Theorem 3 (Probabilistic Polytime Soundness of PSTA). Let $\mathcal{D} \triangleq \Gamma \vdash M : \sigma$ be such that $M \Rightarrow \mathcal{S}$. A PTM $\mathcal{P}$ exists such that, for all $S \in \text{supp}(\mathcal{S})$:

- $\mathcal{P}$ takes an encoding of $M$ as input and produces an encoding of the surface normal form $S$ as output, with probability $\mathcal{P}(S)$, and

- $\mathcal{P}$ runs in a time bounded by $O(|M|^{3d(\mathcal{D})+1})$, i.e. $\mathcal{P}$ is a pPTM.

Proof. By Lemma 3.2 and Remark 2, each reduction step $P \rightarrow P_1, P_2$, premise of $s2$ in $\pi : M \Rightarrow \mathcal{S}$, can be simulated by a PTM that runs in a time bounded by $O(|M|^{2d(\mathcal{D})+1})$. By Lemma 3.1 there can be at most $O(|M|^{d(\mathcal{D})+1})$ instances of $s2$ in $\pi$. So, a PTM exists that simulates the evaluation of $M$, running in time bounded by $O(|M|^{3d(\mathcal{D})+1})$. \]

5 Probabilistic Polytime Completeness of PSTA

We prove that the terms of $\lambda'$, with a type in PSTA are expressive enough to encode any polyomial time Probabilistic Turing Machine (pPTM), i.e. a Probabilistic Turing Machine (PTM) whose running time is bounded by some polynomial in the input size. This allows us to show that PSTA is complete with respect to the functions computed by the pPTM. Typically, encoding a Turing Machine by means of ($\lambda$-)terms requires to represent configurations, transitions between configurations, a phase of initialization, and one of output extraction. Here we focus on the main details of the key step to get completeness, i.e. the definition of the transition function of any pPTM in PSTA.
To that purpose, we recall that tensors (⊗) and unit (1) exist in PSTA as second-order types (see [19] for example.) So, inference rules for ⊗ and 1 are derivable and we can fairly assume that the reduction rules let I be I in N \rightarrow_{β} N and let M_{1} \otimes M_{2} be x_{1} \otimes x_{2} in N \rightarrow_{β} N[M_{1}/x_{1}, M_{2}/x_{2}] are available. Given tensors and unit, the types and terms of PSTA:

B ≡ ∀α.(α → α → α → α)

0 ≡ λxy.x ⊗ y

1 ≡ λxy.y ⊗ x  (14)

can represent booleans [19]. As a notation, B^{n} stands for B \otimes \ldots \otimes B and 0^{n} (resp. 1^{n}) for 0 \otimes \ldots \otimes 0 (resp. 1 \otimes \ldots \otimes 1.)

We recall that the transition function \(\delta_{P}\) of a PTM \(P\) can be seen as superposing the transition functions \(\delta_{0}\) and \(\delta_{1}\) of two deterministic Turing Machines; every computation step of \(P\) selects one between \(\delta_{0}\) and \(\delta_{1}\) with probability \(\frac{1}{2}\). So, let \(\delta_{0}, \delta_{1} : Q \times \{0, 1\} \rightarrow Q \times \{0, 1\} \times \{\text{left, right}\}\) be the transition functions of two deterministic Turing Machines \(M_{1}\) and \(M_{2}\) with \(Q\) containing at most 2\(n\) states. Following [10], these transition functions can be encoded by suitable terms \(\delta_{0}\) and \(\delta_{1}\) of type \(B^{n+1} \rightarrow B^{n+2}\). We can define:

\[
\delta_{P} \triangleq \Delta \! \alpha. \! \text{proj} \! (\text{copy}^{n+1} \! x \! \text{as} \! x_{0}, \! x_{1} \! \text{in} \! (\delta_{0} \! x_{0}, \! \delta_{1} \! x_{1})) \, ,
\]

the transition function of \(P\), whose derivation in PSTA is in Figure 5. Let \(q \otimes p\) be a pair that encodes the configuration \((q, b) \in Q \times \{0, 1\}\) of a PTM. Let \(\delta_{i} q \otimes p \Rightarrow q_{i} \otimes b_{i} \otimes m_{i}\) for \(i \in \{1, 2\}\). Then \(\delta_{P} q \otimes p \Rightarrow \frac{1}{2} q_{0} \otimes b_{0} \otimes m_{0} + \frac{1}{2} q_{1} \otimes b_{1} \otimes m_{1}\) can be easily derived.

Having an encoding for booleans, we can now show how to represent boolean strings in PSTA. For every \(i \geq 1\), the indexed type \(S_{i}\) and the indexed \(n\)-ary boolean strings \(\mathbf{s}_{i}\), whose type is \(S_{i}\), exist in PSTA:

\[
S_{1} \triangleq \forall \alpha. !'(B \rightarrow \alpha \rightarrow \alpha \rightarrow \alpha) \rightarrow (\alpha \rightarrow \alpha)

\mathbf{s}_{i} \triangleq \lambda \epsilon. \lambda z. d^{i}(c) b_{1}(\ldots (d^{i}(c) b_{n} z)\ldots) \quad \text{where} \quad s = b_{1} \ldots b_{n} \in \{0, 1\}^{n} \quad \text{and} \quad n \in \mathbb{N} \, .
\]

If \(n = 1\), we write \(S\) (resp. \(\mathbf{s}\)) in place of \(S_{1}\) (resp. \(\mathbf{s}_{1}\)). The need to introduce families of terms and families of types in (16) is due to the inference rule \(m\), as already noticed in [10].

The following result states that PSTA characterizes the functions computed in polynomial time by a PTM.

**Theorem 4** (Probabilistic Polyt ime Completeness of PSTA). Let \(P\) be a PTM. If:
1. \( \mathcal{P} \) runs in \( p(n) \)-time, for some polynomial \( p : \mathbb{N} \to \mathbb{N} \) with \( \text{deg}(p) = d_1 \), and

2. \( \mathcal{P} \) runs in \( q(n) \)-space, for some polynomial \( q : \mathbb{N} \to \mathbb{N} \) with \( \text{deg}(q) = d_2 \), and

3. for every \( s \in \{0,1\}^* \), \( \mathcal{S} : \{0,1\}^* \to [0,1] \) is the probabilistic distribution of the strings that \( \mathcal{P} \) outputs when applied to input \( s \), then, a term \( \mathcal{P} \) with type \( \text{max}(d_1, d_2, 1)+1 \) \( \mathcal{S} \) exists in \( \text{PSTA} \) such that, for every \( s \in \{0,1\}^* \), there exists a surface distribution \( \mathcal{D}_s \) satisfying the following conditions:

   i. \( \mathcal{P}(\text{max}(d_1, d_2, 1)+1) \Rightarrow \mathcal{D}_s \);
   
   ii. \( \mathcal{D}_s(s') = \mathcal{S}_s(s') \), for every \( s' \in \{0,1\}^* \).

**sketch.** The basic scheme of the proof comes from [10]. We first encode natural numbers (with indexed types \( \mathbb{N}_i \)), all polynomials \( p : \mathbb{N} \to \mathbb{N} \), and we define a term \( \text{len}_i : \mathbb{S}_i \to \mathbb{N}_i \) which, when applied to the encoding \( s \) of a boolean string, returns \( |s|_i \) (where \( |s| \) is the size of \( s \)). Then, we firstly represent configurations with indexed types \( \text{PTM}_i \). Secondly, we encode the transition \( \text{tr} : \text{PTM}_i \to \text{PTM}_i \) between configurations; \( \text{tr} \) relies on the transition functions in (15). We also introduce the terms \( \text{init}_i : \mathbb{N}_i \to \text{PTM}_i \) and \( \text{in}_i : \mathbb{S} \to \text{PTM}_i \). The former returns a configuration \( C_0 \) having an empty tape with \( n \) cells, when applied to the numeral \( n_i \). The latter fills the empty tape of \( C_0 \) with the encodings of the booleans in \( s \), whenever applied to the encoding \( s \) of a boolean string and to \( C_0 \). Finally, we require the term \( \text{ext}_i^S : \text{PTM}_i \to \mathbb{S}_i \). It extracts the boolean string on the tape when applied to the encoding of a configuration. To sum up, we construct \( \mathcal{P} \) in such a way that, when applied to the encoding of a boolean string \( s \):

   - it produces the numerals \( p(|s|) \) and \( q(|s|) \), where \( p : \mathbb{N} \to \mathbb{N} \) is the polynomial bounding the running time of \( \mathcal{P} \), and \( p : \mathbb{N} \to \mathbb{N} \) is the polynomial bounding the working tape of \( \mathcal{P} \);
   - by applying the terms \( \text{init}_i \) and \( \text{in}_i \), it constructs the encoding of the initial configuration having \( q(|s|) \) cells and the input string \( s \) written on the tape;
   - it iterates \( p(|s|) \) times the transition \( \text{tr} \) to the encoding of the initial configuration, in order to obtain the encoding of the final configuration;
   - by applying the term \( \text{ext}_i^S \) to the encoding of the final configuration, it extracts the encoding of the output string.

\( \square \)

### 6 PSTA characterizes both \( \text{PP} \) and \( \text{BPP} \)

Previous sections show that \( \text{PSTA} \) is sound and complete with respect to the functions that a \( \text{PTM} \) computes in polynomial time. What about probabilistic polytime complexity classes?

Let us recall a first basic definition from [1].
Definition 8 (Recognizing a language with error probability $\epsilon$ by a PTM). Let $\epsilon \in [0,1]$. Let $T : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Let $L \subseteq \{0,1\}^\ast$ be a language. We say that a PTM $\mathcal{P}$ recognizes $L$ with error probability $\epsilon$ in $T(n)$-time if:

1. $x \in L$ implies $\Pr[\mathcal{P} \text{ accepts } x] \geq 1 - \epsilon$;
2. $x \notin L$ implies $\Pr[\mathcal{P} \text{ rejects } x] \geq 1 - \epsilon$;
3. $\mathcal{P}$ answers “Accept” or “Reject”, regardless of its random choices, in at most $T(|x|)$ steps, on every input $x$,

where $\Pr[\mathcal{P} \text{ accepts } x]$ (resp. $\Pr[\mathcal{P} \text{ rejects } x]$) denotes the probability that $\mathcal{P}$ terminates in an accepting (resp. rejecting) state on input $x$.

Being our goal the characterization of probabilistic complexity classes by means of PSTA, we have to set how a term $M$, with type in PSTA, accepts a language. The natural counterpart of Definition 8 is:

Definition 9 (Recognizing a language with error probability $\epsilon$ by PSTA). Let $\epsilon \in [0,1]$. Let $L \subseteq \{0,1\}^\ast$ be a language. By definition, $M : \mathcal{I}^\ast \mathcal{S} \rightarrow \mathcal{B}$ in PSTA, for some $n \in \mathbb{N}$, recognizes $L$ with error probability $\epsilon$ whenever, for every $x \in \{0,1\}^\ast$, the (unique) surface distribution $\mathcal{D}_x$ such that $M \mathbb{I}^x \Rightarrow \mathcal{D}_x$ satisfies the following conditions:

1. if $x \in L$ then $\mathcal{D}_x(\emptyset) \geq 1 - \epsilon$;
2. if $x \notin L$ then $\mathcal{D}_x(\{1\}) \geq 1 - \epsilon$.

Definition 10 (The class PP (from [1])). PP contains all the languages $L$ for which a pPTM $\mathcal{P}$ exists that recognizes $L$ in $p(n)$-time with error probability $0 \leq \epsilon \leq \frac{1}{2}$, where $p$ is a polynomial that depends on $\mathcal{P}$ only.

Theorem 5 (PSTA characterizes PP). PSTA is sound and complete w.r.t. PP.

Proof. Concerning the soundness of PSTA w.r.t. PP, let us fix $M$ with type in PSTA such that $\pi : M \Rightarrow \mathcal{D}$. Theorem 3 assures that a pPTM $\mathcal{P}_M$ exists which simulates $\pi$ with a polynomial overhead and with the same probability distribution as $\mathcal{D}$. So, if $M$ recognizes a language $L$ with error probability $0 \leq \epsilon \leq \frac{1}{2}$, then $\mathcal{P}_M$ does, hence $\mathcal{P}_M$ is in PP.

Concerning completeness of PSTA w.r.t. PP, let $\mathcal{P}$ be a pPTM in PP. The proof is the one for Theorem 4, but we have to represent a pPTM that decides a problem instead of one that computes a function. W.l.o.g., we assume that a final state is either accepting or rejecting. Then, we simply replace the term $\text{ext}^B_\mathcal{I} : \mathcal{P}_\mathcal{I} \rightarrow \mathcal{B}$, which extracts the final state from the final configuration (see [10]), for $\text{ext}^S_\mathcal{I} : \mathcal{P}_\mathcal{I} \rightarrow \mathcal{S}_\mathcal{I}$, which extracts the output string from the final configuration. (We recall that $\mathcal{P}_\mathcal{I}$ is the indexed type for configurations.) So, $\mathcal{P}$ accepts a language $L$ with the same probability error $0 \leq \epsilon \leq \frac{1}{2}$ as $\mathcal{P}$.

Here above, PP is instance of a general notion, formalized in Definition 8. However, the interval that the error probability identifying PP belongs to allows for a further definition of this class, equivalent to Definition 10.
Definition 11 (PP recognizes by majority). PP contains all the languages $L$ for which a pPTM $P$ exists such that, for every $x \in \{0,1\}^*$, both the following points (1) and (2) hold:

1. if $x \in L$, then $\Pr[P \text{ accepts } x] \geq \Pr[P \text{ rejects } x]$;
2. if $x \notin L$, then $\Pr[P \text{ rejects } x] \geq \Pr[P \text{ accepts } x]$.

Definition 12 (PSTA recognizes by majority). Let $L \subseteq \{0,1\}^*$ be a language. Let $M$ be a term with type $!^{n}S \rightarrow B$ in PSTA, for some $n \in \mathbb{N}$. We say that $M$ accepts $L$ by majority whenever, for every $x \in \{0,1\}^*$, the (unique) surface distribution $D_x$ such that $M !^n x \Rightarrow D_x$ satisfies the following conditions:

1. if $x \in L$ then $D_x(0) \geq D_x(1)$;
2. if $x \notin L$ then $D_x(1) \geq D_x(0)$.

A proof analogous to the one for Theorem 5 exists for the following theorem which, however, refers to Definition 11 and Definition 12:

Theorem 6 (PSTA characterizes PP by majority). PSTA is sound and complete w.r.t. PP.

Let us now turn our attention to the relation between PSTA and BPP.

Definition 13 (The class BPP (from [1])). BPP is the class of all languages $L$ for which a pPTM $P$ exists that recognizes $L$ in $p(n)$-time with error probability $0 \leq \epsilon < \frac{1}{2}$, and $p$ is a polynomial that depends on $P$ only.

Remark 3. The value $\epsilon$ cannot be equal to $\frac{1}{2}$ in BPP. Due to this restriction the error probability can be made exponentially small at the cost of a polynomial slowdown [26]. This is why BPP is widely considered as the class capturing efficient (probabilistic) computations.

Theorem 7 (PSTA characterizes BPP). PSTA is sound and complete w.r.t. BPP.

Proof. It is like the proof of Theorem 5.

As far as we know, no alternative definition of BPP, analogous to Definition 11 and referring to an error probability implicitly, exists. Our feeling is that one can achieve a better insight on this class by moving to a semantic framework. This is where PSTA can play a role. One can indeed exploit denotational semantics, available for deductive systems based on LL, to semantically characterize probabilistic computational complexity classes which, currently, PSTA characterizes operationally. Conclusions elaborate slightly on this.

7 Conclusions

We illustrate how the relevant features of PSTA, i.e. both its polynomially costing non-deterministic normalization, with a natural probabilistic interpretation, and its connections with LL structural proof-theory, can be the base for generalizing known results or shading some light on open issues.

We think that PSTA can be used to improve known characterizations of the class NPTIME, as given in STA$_+$ by Marion et al. [9]. We recall that STA$_+$ is
STA extended with a sum-rule. That sum-rule gives a type to a choice operator $M + N$, i.e. to an oracle that autonomously “decides” when reducing to either $M$ or $N$. The normalization steps associated with the sum-rule suffer the typical drawback of additives in deductive systems based on LL: the cost of normalizing terms with a type in $\text{STA}_+$ may result in an exponential blow up. To recover NPTIME soundness, the normalization of terms with a type in $\text{STA}_+$ must be a variant of the leftmost outermost strategy, delaying substitutions as long as possible. By contrast, thanks to the inherently linear nature of non-determinism in PSTA, arising from a careful managing of context-sharing in Linear additives, PSTA enjoys a strong polynomial time normalization. Therefore, non-deterministic Linear additives can be employed to make the characterization of NPTIME free of any explicit reference to reduction strategies.

We also think that PSTA, which stems from proof-theoretical principles, will be useful to address the problem of characterizing implicitly the class BPP. As pointed out also in [4], characterizing BPP by purely syntactical means is far from obvious, for it boils down to identify some structural invariant that allows to recognize a language with an error probability strictly smaller than $\frac{1}{2}$. Given that invariant, possibly captured inside an inductively defined formal system, one could be, in principle, to enumerate all the algorithms of BPP.

Denotational semantics can be a way to suggest such a structural invariant, and PSTA can play a crucial role. PSTA is a probabilistic type-theoretical formulation of SLL, a subsystem of LL capturing the complexity class PTIME. Probabilistic denotational models for LL exist, e.g. Probabilistic Coherence Spaces PCoh [6] or Weighted Relational Semantics [17], so they can be easily adapted to PSTA. What we are looking for in these models is a probabilistic version of the notion of obsessionality [18], an invariant found in relational models for SLL, and used to characterize PTIME denotationally.

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A Confluence for $\Lambda^1_{\oplus}$

In this section we prove that the probabilistic multi-step reduction $\Rightarrow$ defined in Figure 2 is confluent, that is, each term of $\Lambda^1_{\oplus}$ can be associated with at most one surface distribution. This property is shown by adapting the techniques in Dal Lago and Toldin [4].

The first step is to prove that $\to$ enjoys a strong confluence property for $\Lambda^1_{\oplus}$:

**Lemma 4.** Let $M, N \in \Lambda^1_{\oplus}$:

1. If $M \to M', M''$ then $M\{N/x\} \to M'\{N/x\}, M''\{N/x\}$
2. If $N \to N', N''$ and $x$ is linear in $M$ then $M[N/x] \to M[N'/x], M[N''/x]$.

**Proof.** Easy induction on the structure of $M$. \qed
Lemma 5. Let $M \in \Lambda^\circ_0$. If $M \rightarrow M'$ and $M \rightarrow M''$, with $M'$ and $M''$ distinct, then there exists a term $N$ such that $M' \rightarrow N$ and $M'' \rightarrow N$.

Proof. By induction on the structure of $M$. We just consider the most interesting cases. If $M = (\lambda x.P)Q \rightarrow P[Q/x] = M'$, then either $M'' = (\lambda x.P')Q$ with $P \rightarrow P'$ or $M'' = (\lambda x.P)Q'$ with $Q \rightarrow Q'$. Since $M$ is s-linear, $x$ is s-linear in $P$ and hence $x$ does not lie within the scope of a $d$-operator. This means that $P[Q/x] = P[Q/x]$ by definition. In the first case, we have $M' \rightarrow P''[Q/x]$ by Lemma 4.1 and also $M'' \rightarrow P''[Q/x]$. In the second case, we have $M' \rightarrow P'[Q'/x]$ by Lemma 4.2, and also $M'' \rightarrow P'[Q'/x]$. Similarly, if $M = (\lambda x.P)Q \rightarrow P[Q/x] = M'$ then the only case is $M'' = (\lambda x.P')Q$ where $P \rightarrow P'$, since reduction is forbidden in $Q$. By Lemma 4.1, $M' \rightarrow P'[Q'/x]$, and also $M'' \rightarrow P''[Q'/x]$. Last, we consider the case where $M = \text{copy}^\circ V$ as $x_1, x_2$ in $\langle N_1, N_2 \rangle$, $M' = \langle N_1[V/x_1], N_2[V/x_2] \rangle$, and $M'' = \text{copy}^\circ V$ as $x_1, x_2$ in $\langle N_1', N_2 \rangle$. Since $M$ is s-linear, $x_1$ is s-linear in $N_1$ and hence $x$ does not lie within the scope of a $d$-operator. This means that $N_1[V/x] = N_1[V/x]$ by definition. Then $M' \rightarrow \langle N_1'[V/x_1], N_2[V/x_2] \rangle$ by Lemma 4.1 and also $M'' \rightarrow \langle N_2[V/x_1], N_2[V/x_2] \rangle$.

Lemma 6. Let $M \in \Lambda^\circ_0$. If $M \rightarrow M_1', M_2'$ and $M \rightarrow M''$, with $M_1'$ and $M_2'$ distinct, then there exist terms $N_1$ and $N_2$ such that $M_1' \rightarrow N_1$, $M_2' \rightarrow N_2$ and $M'' \rightarrow N_1, N_2$.

Proof. The proof is by induction on the structure of $M$. The only possible situation is when both the surface reductions $M \rightarrow M_1', M_2'$ and $M \rightarrow M''$ are applied in surface contexts $C \neq \text{[]}$, and we proceed by case analysis. We just consider a possible case. Suppose $M = PQ \rightarrow P_1'Q, P_2'Q$, where $P_1'Q = M_1'$ and $P_2'Q = M_2'$. Then either $M'' = P''Q$, where $P \rightarrow P''$, or $M'' = PQ''$, where $Q \rightarrow Q''$. In the first case we apply the induction hypothesis on $P \rightarrow P_1', P_2'$ and $P \rightarrow P''$ and we get that there exist $R_1$ and $R_2$ such that $P_1' \rightarrow R_1$, $P_2' \rightarrow R_2$ and $P'' \rightarrow R_1, R_2$, so that $P_1'Q \rightarrow R_1Q, P_2'Q \rightarrow R_2Q$ and $P''Q \rightarrow R_1Q, R_2Q$. In the second case, we have $P_1'Q \rightarrow P_1''Q, P_2'Q \rightarrow P_2''Q$ and $P''Q \rightarrow P_1''Q, P_2''Q$.

Lemma 7. Let $M \in \Lambda^\circ_0$. If $M \rightarrow M_1', M_2'$ and $M \rightarrow M_1'', M_2''$, with $M_1', M_2', M_1'', M_2''$ all distinct, then there exist $N_1, N_2, N_3, N_4$ such that $M_1' \rightarrow N_1, N_2$, $M_2' \rightarrow N_3, N_4$ and $\exists i \in \{1, 2\}$ such that $M_1'' \rightarrow N_1, N_3$ and $M_2'' \rightarrow N_2, N_4$.

Proof. The proof is by induction on the structure of $M$. The only possible situation is when both the surface reductions $M \rightarrow M_1', M_2'$ and $M \rightarrow M_1'', M_2''$ are applied in surface contexts $C \neq \text{[]}$, and we proceed by case analysis. We just consider a possible case. Suppose $M = PQ \rightarrow P_1'Q, P_2'Q$, where $P_1'Q = M_1'$ and $P_2'Q = M_2'$. Then either $M_1'' = P_1''Q$, $M_2'' = P_2''Q$ or $M_1'' = PQ_1$, $M_2'' = PQ_2$. In the first case we apply the induction hypothesis on $P \rightarrow P_1', P_2'$ and $P \rightarrow P_1'', P_2''$ and we have that there exist $R_1, R_2, R_3, R_4$ such that $P_1' \rightarrow R_1, R_2, P_2' \rightarrow R_3, R_4$ and $\exists i \in \{1, 2\}$ such that $P_1'' \rightarrow R_1, R_3$ and $P_2'' \rightarrow R_2, R_4$. Then, we have $P_1'Q \rightarrow R_1Q, R_2Q, P_2'Q \rightarrow R_3Q, R_4Q, P_1''Q \rightarrow R_1Q, R_3Q$, and $P_2''Q \rightarrow R_2Q, R_4Q$. In the second case we have $P_1'Q \rightarrow P_1''Q, PQ_1'$, $P_2'Q \rightarrow P_2''Q, PQ_2'$, $PQ_1' \rightarrow P_1''Q, P_1'Q_2'$, and $PQ_2' \rightarrow P_1'Q_2', P_2'Q_2'$.
\[
\frac{M \in \Lambda_{\Theta}^1}{M \Rightarrow M} \quad (t_1) \quad \frac{M \Rightarrow M_1, M_2 \quad M_1 \Rightarrow \mathcal{D}_1 \quad M_2 \Rightarrow \mathcal{D}_2}{M \Rightarrow \frac{1}{t} \cdot \mathcal{D}_1 + \frac{1}{t} \cdot \mathcal{D}_2} \quad (t_2)
\]

Figure 6: Multi-step reduction \(\Rightarrow\) for \(\Lambda_{\Theta}^1\).

The next step is to introduce a probabilistic multi-step reduction relation \(\Rightarrow\), which is “laxer” than \(\Rightarrow\), i.e. such that \(\Rightarrow \subseteq \Rightarrow\).

**Definition 14** (Multi-step reduction \(\Rightarrow\)).

- A term distribution is a probability distribution over \(\Lambda_{\Theta}^1\), i.e. a function \(\mathcal{D} : \Lambda_{\Theta}^1 \rightarrow [0, 1]\) such that \(\sum_{M \in \Lambda_{\Theta}^1} \mathcal{D}(M) = 1\).
- The multi-step reduction \(\Rightarrow\) is the relation between terms of \(\Lambda_{\Theta}^1\) and term distributions, defined by the rules in Figure 6. Derivations of \(M \Rightarrow \mathcal{D}\) are ranged over by \(\pi, \rho\).
- The size \(|\pi|\) of a derivation \(\pi : M \Rightarrow \mathcal{D}\) is 0 if \(\pi\) is \(t_1\), and \(|\pi| \triangleq \max(|\pi_1|, |\pi_2|) + 1\) if \(\pi\) is \(t_2\) with premises \(M \Rightarrow M_1, M_2, \pi_1 : M_1 \Rightarrow \mathcal{D}_1\) and \(\pi_2 : M_2 \Rightarrow \mathcal{D}_2\). Henceforth, with a little abuse of notation, we shall write \(|M \Rightarrow \mathcal{D}|\) in place of \(|\pi|\), whenever \(\pi : M \Rightarrow \mathcal{D}\).

Notice that the only difference between the relations \(\Rightarrow\) and \(\Rightarrow\) is that \(t_1\) applies to surface normal forms only, while \(t_2\) applies to all terms. The following states that \(\Rightarrow \subseteq \Rightarrow\):

**Lemma 8.** If \(\pi : M \Rightarrow \mathcal{D}\) then there exists a derivation \(\pi'\) such that \(\pi' : M \Rightarrow \mathcal{D}\) and \(|\pi| = |\pi'|\).

Confluence for \(\Rightarrow\) follows directly from two technical results about \(\Rightarrow\):

**Lemma 9.** Let \(M \in \Lambda_{\Theta}^1\). Let \(M \Rightarrow \mathcal{D}\) be such that \(\mathcal{D} = p_1 \cdot N_1 + \ldots + p_n \cdot N_n\), and let \(N_i \Rightarrow \mathcal{D}_i\) for all \(i \leq n\). Then:

1. \(M \Rightarrow \sum_{i=1}^n p_i \cdot \mathcal{D}_i\)
2. \(|M \Rightarrow \sum_{i=1}^n p_i \cdot \mathcal{D}_i| \leq |M \Rightarrow \mathcal{D}| + \max_{i=1}^n |N_i \Rightarrow \mathcal{D}_i|\).

**Proof.** The proof is by induction on the structure of the derivation \(M \Rightarrow \mathcal{D}\), and follows exactly [4].

**Lemma 10.** Let \(M \in \Lambda_{\Theta}^1\). If \(M \Rightarrow \mathcal{D}\) and \(M \Rightarrow \mathcal{E}\), where \(\mathcal{D} = p_1 \cdot P_1 + \ldots + p_n \cdot P_n\) and \(\mathcal{E} = q_1 \cdot Q_1 + \ldots + q_m \cdot Q_m\), then there exist \(\mathcal{L}_1, \ldots, \mathcal{L}_n\) and \(\mathcal{F}_1, \ldots, \mathcal{F}_m\) such that:

- \(P_i \Rightarrow \mathcal{L}_i\) and \(Q_j \Rightarrow \mathcal{F}_j\), for all \(i \leq n, j \leq m\);
- \(\max_{i=1}^n |P_i \Rightarrow \mathcal{L}_i| \leq |M \Rightarrow \mathcal{D}|\) and \(\max_{j=1}^m |Q_j \Rightarrow \mathcal{F}_j| \leq |M \Rightarrow \mathcal{D}|\);
- \(\sum_{i=1}^n p_i \cdot \mathcal{L}_i = \sum_{j=1}^m q_j \cdot \mathcal{F}_j\).

**Proof.** By induction on \(|M \Rightarrow \mathcal{D}| + |M \Rightarrow \mathcal{E}|\). If one of the derivations ends with \(t_1\) then there is nothing to prove. Otherwise, both derivations \(M \Rightarrow \mathcal{D}\) and \(M \Rightarrow \mathcal{E}\) end with the rule \(t_2\).
\[
\begin{align*}
M &\rightarrow M_1, M_2 \\
M &\Rightarrow \frac{1}{2} \cdot D_1 + \frac{1}{2} \cdot D_2
\end{align*}
\]

\[
\begin{align*}
M &\rightarrow N_1, N_2 \\
N_1 &\Rightarrow \delta_1 \\
N_2 &\Rightarrow \delta_2
\end{align*}
\]

Clearly, if \( M_1, M_2 \) is equal to \( N_1, N_2 \) (modulo sort) then we apply the induction hypothesis and we are done. So let us suppose that \( M_1, M_2 \) and \( N_1, N_2 \) are different. We have four cases:

- If \( M_1 = M_2 \) and \( N_1 = N_2 \) then by Lemma 5 there exists \( L \) such that \( M_1 \rightarrow L \) and \( N_1 \rightarrow L \). By using the rule \( t1 \) we get \( L \Rightarrow L \), so \( M_1 \Rightarrow L \). By induction hypothesis on \( M_1 \Rightarrow \mathcal{D}_1 \) and \( M_1 \Rightarrow L \) there exist \( \mathcal{L}_1, \ldots, \mathcal{L}_n \) and \( \mathcal{K} \) such that, for all \( i \leq n \), \( P_i \Rightarrow \mathcal{L}_i \), \( L \Rightarrow \mathcal{K} \), \( \max_{i=1}^n (|P_i \Rightarrow \mathcal{L}_i|) \leq |M \Rightarrow L|, |L \Rightarrow \mathcal{K}| \leq |M \Rightarrow \mathcal{D}_1| \), and \( \sum_{i=1}^n p_i \cdot \mathcal{L}_i = \mathcal{K} \). Similarly, we have that there exists \( \mathcal{F}_1, \ldots, \mathcal{F}_m, \mathcal{H} \) such that, for all \( i \leq m \), \( Q_i \Rightarrow \mathcal{F}_i \), \( L \Rightarrow \mathcal{H} \), \( \max_{i=1}^m (|Q_i \Rightarrow \mathcal{F}_i|) \leq |M \Rightarrow L|, |L \Rightarrow \mathcal{H}| \leq |M \Rightarrow \mathcal{E}| \), and \( \sum_{i=1}^m q_i \cdot \mathcal{F}_i = \mathcal{H} \). We obtain \( |L \Rightarrow \mathcal{K}| + |L \Rightarrow \mathcal{H}| \leq |M \Rightarrow \mathcal{D}_1| + |M \Rightarrow \mathcal{E}| \). Let \( \mathcal{H} = r_1 \cdot R_1 + \ldots + r_h \cdot R_h \) and \( \mathcal{K} = s_1 \cdot S_1 + \ldots + s_k \cdot S_k \). We apply the induction hypothesis and we obtain that there exist \( \mathcal{R}_1, \ldots, \mathcal{R}_h, \mathcal{F}_1, \ldots, \mathcal{F}_k \) such that \( R_i \Rightarrow \mathcal{R}_i \) and \( S_j \Rightarrow \mathcal{F}_j \) for all \( i \leq h \) and \( j \leq k \). Moreover, \( \max_{i=1}^h (|R_i \Rightarrow \mathcal{R}_i|) \leq |L \Rightarrow \mathcal{K}|, \max_{j=1}^k (|S_j \Rightarrow \mathcal{F}_j|) \leq |L \Rightarrow \mathcal{H}|, \) and \( \sum_{i=1}^h r_i \cdot \mathcal{R}_i = \sum_{j=1}^k s_j \cdot \mathcal{F}_j \). Notice that the cardinality of \( \mathcal{D} \) and \( \mathcal{K} \) may differ but for sure they have the same terms with non zero probability. Similar, \( \mathcal{E} \) and \( \mathcal{H} \) have the same terms with non zero probability. By using Lemma 9 and using the transitive property of equality we obtain that \( \sum_{i=1}^h r_i \cdot \mathcal{R}_i = \sum_{j=1}^k s_j \cdot \mathcal{F}_j \Rightarrow \sum_{j=1} m q_j \cdot \mathcal{F}_j \). Moreover, we have

\[
\max_{i=1}^h (|P_i \Rightarrow \mathcal{R}_i|) \leq |L \Rightarrow \mathcal{K}| \leq |M \Rightarrow \mathcal{E}|
\]

\[
\max_{j=1}^m (|Q_j \Rightarrow \mathcal{F}_j|) \leq |L \Rightarrow \mathcal{H}| \leq |M \Rightarrow \mathcal{D}_1|.
\]

- If \( M_1 \neq M_2 \) and \( N_1 \neq N_2 \) then by Lemma 6 there exists \( L_1, L_2 \) such that \( M_1 \Rightarrow L_1, M_2 \Rightarrow L_2 \) and \( N_1 \Rightarrow L_1, L_2 \). W.l.o.g. we can assume that \( \mathcal{R}_1 = 2p_1 \cdot P_1 + \ldots + 2p_{o-1} \cdot P_{o-1} + p_o \cdot P_o + \ldots + p_n \cdot P_n \) and \( \mathcal{R}_2 = p_0 \cdot P_0 + \ldots + p_{h-1} \cdot P_{h-1} + p_h + 2p_{h+1} + \ldots + 2p_n \cdot P_n \), where \( 1 \leq o \leq t \leq n \). By using the induction rule, we associate with every \( L_i \) a distribution \( \mathcal{D}_i \) such that \( L_1 \Rightarrow \mathcal{D}_1 \) and \( L_2 \Rightarrow \mathcal{D}_2 \). Let \( \mathcal{D}_1 = r_1 \cdot R_1 + \ldots + r_h \cdot R_h \) and \( \mathcal{D}_2 = s_1 \cdot S_1 + \ldots + s_k \cdot S_k \). So, we have, for all \( i, M_i \Rightarrow \mathcal{D}_i \) and \( M_i \Rightarrow \mathcal{E} \), \( N_1 \Rightarrow \mathcal{E} \) and \( N_1 \Rightarrow \frac{1}{2} \cdot \mathcal{D}_1 + \frac{1}{2} \cdot \mathcal{D}_2 \). By applying the induction hypothesis on all the three cases we have that there exist \( \mathcal{L}_1, \ldots, \mathcal{L}_n, \mathcal{F}_1, \ldots, \mathcal{F}_m, \mathcal{K}, \mathcal{L}, \mathcal{F}, \mathcal{H} \) such that \( P_1 \Rightarrow \mathcal{L}_1, \ldots, P_n \Rightarrow \mathcal{L}_n, Q_1 \Rightarrow \mathcal{F}_1, \ldots, P_m \Rightarrow \mathcal{F}_m, L_1 \Rightarrow \mathcal{K}, L_2 \Rightarrow \mathcal{K} \), \( L_1 \Rightarrow \mathcal{E} \), \( L_2 \Rightarrow \mathcal{E} \). Moreover:

1. \( \max_{1 \leq i \leq t} (|P_i \Rightarrow \mathcal{L}_i|) \leq |M_1 \Rightarrow \mathcal{D}_1|, |L_1 \Rightarrow \mathcal{K}| \leq |M_1 \Rightarrow \mathcal{D}_1|, \) and \( \sum_{i=1}^t p_i \cdot \mathcal{L}_i + \sum_{i=t+1}^n p_i \cdot \mathcal{L}_i = \mathcal{K} \).
2. \( \max_{o \leq i \leq n} (|P_i \Rightarrow \mathcal{L}_i|) \leq |M_2 \Rightarrow \mathcal{D}_2|, |L_2 \Rightarrow \mathcal{K}| \leq |M_2 \Rightarrow \mathcal{D}_2|, \) and \( \sum_{i=o}^t p_i \cdot \mathcal{L}_i + \sum_{i=t+1}^n 2p_i \cdot \mathcal{L}_i = \mathcal{K} \).

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3. \( \max_{j=1}^m (|Q_j \Rightarrow \mathcal{F}_j|) \leq |N_1 \Rightarrow \frac{1}{2} \cdot \mathcal{P}_1 + \frac{1}{2} \cdot \mathcal{P}_2|, \max(|L_1 \Rightarrow \mathcal{R}|, |L_2 \Rightarrow \mathcal{R}|) \leq |N_1 \Rightarrow \mathcal{E}|, \) and \( \sum_{j=1}^m q_j \cdot \mathcal{F}_j = \frac{1}{2} \cdot \mathcal{R} + \frac{1}{2} \cdot \mathcal{E}. \)

Notice that \( |L_1 \Rightarrow \mathcal{R}| + |L_1 \Rightarrow \mathcal{E}| < |M \Rightarrow \mathcal{R}| + |M \Rightarrow \mathcal{E}|. \) Moreover, notice also that the following inequality holds: \( |L_2 \Rightarrow \mathcal{R}| + |L_2 \Rightarrow \mathcal{E}| < |M \Rightarrow \mathcal{R}| + |M \Rightarrow \mathcal{E}|. \) We are allowed to apply, again, induction hypothesis and have a confluent distribution for both cases. Lemma 9 then allows us to connect the first two main derivations and by transitivity of property of equality we have the thesis.

- The case \( M_1 = M_2 \) and \( N_1 \neq N_2 \) and the case \( M_1 \neq M_2 \) and \( N_1 \neq N_2 \) are proven similarly by using, respectively, Lemma 6 and Lemma 7. □

Finally, we are ready for the following proof:

**Proof of Theorem 1.** Since \( \Rightarrow \subseteq \Rightarrow, \) we have \( M \Rightarrow \mathcal{G} \) and \( M \Rightarrow \mathcal{E}. \) By Lemma 10, \( \mathcal{G} = \mathcal{E}. \) □

### B Proofs of Section 4

**Proof of Lemma 1.** By induction on the structure of \( \mathcal{D}. \) Point 1 and point 3 are straightforward. Concerning point 2, we consider the most interesting case where \( \mathcal{D} \) has been obtained from a derivation \( \mathcal{D}' \) by applying the rule \( \text{sp} \) with \( sp\text{-rank} \ k. \) By using the induction hypothesis, we have:

\[
\begin{align*}
w(\mathcal{D}, r) &= r \cdot (w(\mathcal{D}', r) + k) + 1 \\
&\leq r \cdot (r^{d(\mathcal{D}') \cdot w(\mathcal{D}', 1) + k} + 1 \\
&\leq r \cdot (r^{d(\mathcal{D}') \cdot w(\mathcal{D}', 1)} + r^{d(\mathcal{D}') \cdot k} + r^{d(\mathcal{D}')}) + 1 \\
&\leq r^{d(\mathcal{D}') + 1} \cdot (w(\mathcal{D}', 1) + k + 1) = r^{d(\mathcal{D})} \cdot w(\mathcal{D}, 1).
\end{align*}
\]

□

The following lemmas can be easily proved by inspecting the rules of PSTA.

**Lemma 11** (Generation).

1. If \( \mathcal{D} \triangleleft \Gamma \vdash \lambda x.M : \sigma \) then \( \sigma = \forall \tilde{\alpha}.((A \rightarrow B)\langle D_1/\beta_1, \ldots, D_n/\beta_n \rangle) \) and \( \mathcal{D} \) is some \( \mathcal{D}' \triangleleft \Gamma, x : A \vdash M' : B \) followed by \( \rightarrow \text{II} \) and a sequence of \( \forall \text{I}, \forall \text{E}, \) and \( m \) where \( \tilde{\alpha} = \alpha_1, \ldots, \alpha_k, \) for some \( k \geq 0. \)

2. If \( \mathcal{D} \triangleleft \Gamma \vdash \lambda x.M : \sigma \) then \( \sigma = \forall \tilde{\alpha}.((!\tau \rightarrow A)\langle D_1/\beta_1, \ldots, D_n/\beta_n \rangle) \) and \( \mathcal{D} \) is some \( \mathcal{D}' \triangleleft \Gamma, x : !\tau \vdash M' : A \) followed by \( \rightarrow \text{II} \) and a sequence of \( \forall \text{I}, \forall \text{E}, \) and \( m \) where \( \tilde{\alpha} = \alpha_1, \ldots, \alpha_k, \) for some \( k \geq 0. \)

3. If \( \mathcal{D} \triangleleft \Gamma \vdash MN : \sigma \) then \( \sigma = \forall \tilde{\alpha}.(A\langle D_1/\beta_1, \ldots, D_n/\beta_n \rangle) \) and \( \mathcal{D} \) is some \( \mathcal{D}' \triangleleft \Gamma \vdash M' : \tau \rightarrow A \) and \( \mathcal{D}'' \triangleleft \Gamma'' \vdash N' : \tau \) followed by \( \rightarrow \text{II} \) and a sequence of \( \forall \text{I}, \forall \text{E}, \) and \( m, \) where \( \tilde{\alpha} = \alpha_1, \ldots, \alpha_k \) and \( k \geq 0. \)

4. If \( \mathcal{D} \triangleleft \Gamma \vdash \text{copy}^V N \) as \( x_1, x_2 \in \langle M_1, M_2 \rangle : \sigma, \) then \( \sigma = B_1 \& B_2 \) and \( \mathcal{D} \) is \&I followed by a sequence of applications of the rule \( m. \)
5. If $\mathcal{D} \triangleright \Gamma \vdash \text{proj}(M) : \sigma$ then $\sigma = \forall \alpha.(B_1/\beta_1, \ldots, D_n/\beta_n)$, and $\mathcal{D}$ is $D' \triangleright \Gamma : M' : B_1 \& B_2$ followed by $\&E$ and a sequence of $\forall I$, $\forall E$, and $m$, where $\alpha = \alpha_1, \ldots, \alpha_k$, for some $k \geq 0$.

6. If $\mathcal{D} \triangleright \Gamma \vdash !M : \sigma$ then $\sigma = !\sigma'$, $\Gamma$ is an strictly exponential context, and $\mathcal{D}$ is sp, followed by some applications of the rule $m$.

**Lemma 12.**

1. If $\mathcal{D} \triangleright \Gamma \vdash M : !\sigma$ then $\mathcal{D}$ has been obtained from a derivation $\mathcal{D}'$ by applying the rule sp, followed by some applications of the rule $m$. Hence, $\Gamma'$ is a strictly exponential context and $M = !M'$, for some $M'$.

2. If $\mathcal{D} \triangleright \Gamma, x : A \vdash M : \tau$ then $x$ is $s$-linear in $M$.

3. If $\mathcal{D} \triangleright \Gamma, x : !\sigma \vdash M : \tau$ then either $x : !\sigma$ has been introduced by a sp rule or by a $m$ rule.

Following Gaboardi and Ronchi [10], we prove a “weighted” formulation of the substitution property. Since we work with two kinds of types, namely the linear types (i.e. those with form $A$) and the strictly exponential ones (i.e. those with form $\sigma$), we split the task: first, we consider a substitution theorem for linear types; then, we generalize the statement to arbitrary types.

**Lemma 13 (Weighted linear substitution).** Let $r \geq 1$. If $\mathcal{D}_1 \triangleright \Gamma, x : A \vdash M : \tau$ and $\mathcal{D}_2 \triangleright \Delta \vdash N : A$, then there exists a derivation $\mathcal{S}(\mathcal{D}_1, \mathcal{D}_2)$ such that:

- $S(\mathcal{D}_1, \mathcal{D}_2) \triangleright \Gamma, \Delta \vdash M[N/x] : \tau$,
- $w(S(\mathcal{D}_1, \mathcal{D}_2), r) \leq w(\mathcal{D}_1, r) + w(\mathcal{D}_2, r)$.

**Proof.** By **Lemma 12.2**, $x$ is $s$-linear in $M$, i.e. $x$ occurs exactly once in $M$ and this occurrence is out of the scope of both a $!$-operator and a $s$-operator. The statement is proved by induction on $\mathcal{D}_1$. The cases were the last rule is $ax$, $\neg \&I$, $\neg \&E$, $\&E$, $\forall I$, $\forall E$, and $m$ are easy. Now, suppose $\mathcal{D}_1$ is of the form:

$\begin{array}{c}
\mathcal{D} \\
\triangleright \Gamma, x : A \vdash P : B \quad x_1 : B \vdash Q_1 : C_1 \quad x_2 : B \vdash Q_2 : C_2 \quad V : B
\end{array}$ & $\begin{array}{c}
\mathcal{D}' \\
\triangleright \Gamma, x : A \vdash \text{copy}^V P [x_1, x_2] \text{ as } x_1, x_2 \text{ in } \langle Q_1, Q_2 \rangle : C_1 \& C_2
\end{array}$ & $\begin{array}{c}
\mathcal{D}'' \\
\triangleright \Gamma, x : A \vdash \text{copy}^V P [x_1, x_2] \text{ as } x_1, x_2 \text{ in } \langle Q_1, Q_2 \rangle : C_1 \& C_2
\end{array}$ & $\begin{array}{c}
\mathcal{D}''' \\
\triangleright \Gamma, x : A \vdash \text{copy}^V P [x_1, x_2] \text{ as } x_1, x_2 \text{ in } \langle Q_1, Q_2 \rangle : C_1 \& C_2
\end{array}$

so that $\tau = C_1 \& C_2$ and $M = \text{copy}^V P$ as $x_1, x_2$ in $\langle Q_1, Q_2 \rangle$. By induction hypothesis, there exists $S(\mathcal{D}', \mathcal{D}_2) \triangleright \Gamma, \Delta \vdash P[N/x] : B$ such that $w(S(\mathcal{D}', \mathcal{D}_2), r) \leq w(\mathcal{D}', r) + w(\mathcal{D}_2, r)$. We define $S(\mathcal{D}_1, \mathcal{D}_2)$ with conclusion:

$\begin{array}{c}
\Gamma, \Delta \vdash \text{copy}^V P[N/x] \text{ as } x_1, x_2 \text{ in } \langle Q_1, Q_2 \rangle : C_1 \& C_2
\end{array}$

as the derivation obtained by applying $\&I$ to $S(\mathcal{D}', \mathcal{D}_2), \mathcal{D}'', \mathcal{D}'''$. Moreover, by using the induction hypothesis, we have:

$w(S(\mathcal{D}_1, \mathcal{D}_2), r) = w(S(\mathcal{D}', \mathcal{D}_2), r) + w(\mathcal{D}'', r) + w(\mathcal{D}'''r) + w(\mathcal{D}'', r) + w(\mathcal{D}'''r) + 2$

$\leq w(\mathcal{D}', r) + w(\mathcal{D}''r) + w(\mathcal{D}'', r) + w(\mathcal{D}'''r) + w(\mathcal{D}_2, r) + 2$

$= w(\mathcal{D}_1, r) + w(\mathcal{D}_2, r)$.

Last, since $A$ is a linear type, the last rule of $\mathcal{D}_1$ cannot be sp. □
Lemma 14 (Weighted substitution). Let \( r \geq \text{rk}(\mathcal{D}_1) \). If \( \mathcal{D}_1 \vdash \Gamma, x : \sigma \vdash M : \tau \) and \( \mathcal{D}_2 \vdash \Delta \vdash N : \sigma \), then there exists a derivation \( S(\mathcal{D}_1, \mathcal{D}_2) \) such that:

- \( S(\mathcal{D}_1, \mathcal{D}_2) \vdash \Gamma, \Delta \vdash M\{N/x\} : \tau \),
- \( w(S(\mathcal{D}_1, \mathcal{D}_2), r) \leq w(\mathcal{D}_1, r) + w(\mathcal{D}_2, r) \).

Proof. Since \( \sigma = \text{ls} A \), for some linear type \( A \) and some \( q \geq 0 \), we reason by induction on \( q \). If \( q = 0 \) then, by Lemma 12.2, \( x \) is \( s \)-linear in \( M \), i.e. \( x \) occurs exactly once in \( M \) and this occurrence is out of the scope of both a \( ! \)-operator and a \( \text{d} \)-operator. This means that \( M\{N/x\} = M[N/x] \), and we can apply Lemma 13. Suppose now that \( \sigma = !\sigma' \). On the one hand, by Lemma 12.1 we have that \( \Delta \) is strictly exponential, \( N = !P \), and \( \mathcal{D}_2 \) is composed by a subderivation \( \mathcal{D}_2^* \) of the form:

\[
\frac{D_2^*}{\mathcal{D}_2} \quad \frac{D_2^*}{\mathcal{D}_2}
\]

with \( \text{sp}\)-rank \( h \) and such that \( \Delta' = y_1 : \sigma_1, \ldots, y_m : \sigma_m \), followed by a sequence of \( t \geq 0 \) rules with \( m \)-rank, respectively, \( k_1, \ldots, k_t \) recovering \( \Delta \vdash !P : !\sigma' \). On the other hand, by applying Lemma 12.3, the assumption \( x : !\sigma' \in \mathcal{D}_1 \vdash \Gamma, x : !\sigma' \vdash \tau \) has been obtained by applying either the rule \( \text{sp} \) or the rule \( \text{m} \). We just consider the latter case, the former being similar. W.l.o.g. we can suppose that such an instance of \( m \) is the last rule of \( \mathcal{D}_1 \), since we can always permute an application of \( m \) downward obtaining a derivation of the same judgement. Then, \( \mathcal{D}_1 \) has the following form:

\[
\frac{D_1^*}{\mathcal{D}_1} \quad \frac{D_1^*}{\mathcal{D}_1}
\]

with \( m \)-rank \( k \) and such that \( M = M'[d(x)/x_1, \ldots, d(x)/x_n] \). If \( k = 0 \) then \( S(\mathcal{D}_1, \mathcal{D}_2) \) is \( D_1^* \) followed by some applications of the \( m \) rule with \( m \)-rank 0 in order to recover the context \( \Delta \), which is strictly exponential by Lemma 12.1. In this case, we have \( w(S(\mathcal{D}_1, \mathcal{D}_2), r) = w(\mathcal{D}_1, r) \). Otherwise, by using the induction hypothesis, we can build the following derivations:

\[
S^1 \triangleq S(D_2', D_1') \vdash \Gamma, \Delta, x_2 : \sigma', \ldots, x_n : \sigma' \vdash M'[P'/x_1] : \tau \\
S^2 \triangleq S(D_2', S(D_2', D_1')) \vdash \Gamma, \Delta, x_3 : \sigma', \ldots, x_n : \sigma' \vdash M'[P'/x_1, P'/x_2] : \tau \\
\vdots \\
S^n \triangleq S(D_2', S(D_2', \ldots, S(D_2', D_1'))) \vdash \Gamma, \Delta', \ldots, \Delta' \vdash M'[P'/x_1, \ldots, P'/x_n] : \tau
\]

such that \( w(S^1, r) \leq w(D_2', r) + w(D_1', r) \) and, for all \( 1 \leq i < n \), \( w(S^{i+1}, r) \leq w(D_2', r) + w(S^i, r) \leq w(D_1', r) + (i + 1) \cdot w(D_2', r) \). Then, \( S(\mathcal{D}_1, \mathcal{D}_2) \) can be obtained from \( S^n \) by applying a sequence of \( h \) applications of the rule \( m \) with \( m \)-rank \( k \); and a sequence of \( t \) applications of the rule \( m \) with \( m \)-rank, respectively, \( k_1, \ldots, k_t \), in order to get \( \Delta \) from \( \Delta', \ldots, \Delta' \). This means that \( S(\mathcal{D}_1, \mathcal{D}_2) \vdash \Gamma, \Delta \vdash M'[P/x_1, \ldots, P/x_n] : \tau \) and, by definition of surface-preserving substitution:

\[
M'[P/x_1, \ldots, P/x_n] = (M'[z/x_1, \ldots, z/x_n])\{P/z\}
\]
= ((M'[z/x_1, \ldots, z/x_n])[d(x)/z]))\{!P/x\} \ x \notin \text{FV}(M')
= (M'[d(x)/x_1, \ldots, d(x)/x_n])\{!P/x\}
= M\{N/x\}.

By using the induction hypothesis, we finally have:

\[ w(S(D_1, D_2), r) = w(S^n, r) + k \cdot h + \sum_{i=1}^{t} k_i \]
\[ \leq w(D'_1, r) + k \cdot w(D'_2, r) + k \cdot h + \sum_{i=1}^{t} k_i \]
\[ \leq w(D'_1, r) + r \cdot w(D'_2, r) + r \cdot h + \sum_{i=1}^{t} k_i \]
\[ \leq w(D_1, r) + (r \cdot (w(D'_2, r) + h) + 1 + \sum_{i=1}^{t} k_i) \]
\[ = w(D_1, r) + (w(D'_2, r) + \sum_{i=1}^{t} k_i) \]
\[ \leq w(D_1, r) + w(D_2, r). \]

This concludes the proof. \(\square\)

We are now able to prove the weighted version of the Subject reduction property:

**Proof of Theorem 2.** The proof is by induction on the definition of the one-step reduction relation. We have several cases, and we consider the most interesting ones:

- If \( M = (\lambda x. N)!P \rightarrow N\{!P/x\} = M_1 \) then, by applying Lemma 11.2 and Lemma 11.3, \( D \) contains a derivation \( D^* \) of the form:

\[
\begin{array}{c}
\Gamma', x : \tau \vdash N' : A \\
\Gamma' \vdash (\lambda x. N')!P' : A
\end{array}
\]

possibly followed by a sequence of applications of the rules \( \forall I, \forall E, \) and \( m \).

Let \( t \geq 0 \) be the number of applications of the rule \( m \), and let \( k_1, \ldots, k_t \) be their respective \( m \)-rank. By applying Lemma 14, there exists a derivation \( S(D', D'') \) such that \( S(D', D'') \triangleleft \Gamma', \Gamma'' \vdash N'\{!P'/x\} : A \). We define \( D_1 = D_2 \) as the derivation obtained by applying to \( S(D', D'') \) a sequence of applications of the rules \( \forall I, \forall E, \) and \( m \) in order to obtain \( \Gamma' \vdash N\{!P/x\} : \sigma \) as a concluding judgement. By Lemma 14, we have:

\[
w(D_1, r) = w(S(D', D''), r) + \sum_{j=1}^{t} k_j \leq w(D', r) + w(D'', r) + \sum_{j=1}^{t} k_j \]
\[
< w(D', r) + w(D'', r) + \sum_{j=1}^{t} k_j + 2 = w(D, r).
\]
• If \( M = \text{proj}(M_1, M_2) \rightarrow M_1, M_2 \) then, by applying Lemma 11.4 and Lemma 11.5, \( \sigma = \forall \alpha_1. (B'(D_1/\beta_1, \ldots, D_n/\beta_n)) \), where \( \alpha = \alpha_1, \ldots, \alpha_k \), for some \( k \geq 0 \). Moreover, \( D \) is a derivation \( D^* \) of the form:

\[
\begin{array}{c}
D' \\
\vdash M_1 : B \\
\vdash M_2 : B \\
\vdash \langle M_1, M_2 \rangle : B \land \land \\
\vdash \text{proj}(M_1, M_2) : B \\
\end{array}
\]

followed by a sequence of applications of the rules \( \forall I, \land E \), and \( m \). Then, we define \( D_1 \) (resp. \( D_2 \)) as the derivation \( D' \) (resp. \( D'' \)) followed by the same sequence of rules \( \forall I, \land E \), and \( m \), the latter being of \( m \)-rank 0 and introducing the context \( \Gamma \). By definition of weight, we have: \( w(D_1, r) = w(D', r) \) and similarly for \( D_2 \).

• If \( M = \text{copy}^U V \) as \( x_1, x_2 \) in \( \langle Q_1, Q_2 \rangle \rightarrow \langle Q_1[V/x_1], Q_2[V/x_2] \rangle = M_1 = M_2 \) then, by Lemma 11.4, \( \sigma = B_1 \land B_2 \) and \( D \) is a derivation \( D^* \) of the form:

\[
\begin{array}{c}
D' \\
\Gamma' \vdash V : A \\
\vdash x_1 : A \vdash Q_1 : B_1 \\
\vdash x_2 : A \vdash Q_2 : B_2 \\
\vdash U : A \\
\vdash \text{copy}^U V \text{ as } x_1, x_2 \text{ in } \langle Q_1, Q_2 \rangle : B_1 \land B_2 \\
\end{array}
\]

followed by a sequence of applications of the rule \( m \). Since \( \Gamma' \) is \( \forall I \)-lazy by definition, it is \( ! \)-free, and hence all types in \( \Gamma' \) are linear. Then, since \( V \) is closed, Lemma 12.2 implies \( \Gamma' = \emptyset \). Therefore, the applications of the rule \( m \) below \( D^* \) are all of \( m \)-rank 0, so that \( w(D, r) = w(D', r) \). By applying Lemma 13 twice, there exist two derivations \( S(D', D'') \vdash Q_1[V/x_1] : B_1 \) and \( S(D', D'''') \vdash Q_2[V/x_2] : B_2 \) such that \( w(S(D', D''), r) \leq w(D', r) + w(D'', r) \) and \( w(S(D', D'''), r) \leq w(D', r) + w(D'''', r) \). We define \( D_1 = D_2 \) as the following derivation:

\[
\begin{array}{c}
S(D', D'') \\
\vdash Q_1[V/x_1] : B_1 \\
\vdash Q_2[V/x_2] : B_2 \\
\vdash \langle Q_1[V/x_1], Q_2[V/x_2] \rangle : B_1 \land B_2 \\
\end{array}
\]

By Remark 1 we can safely assume that \( U \) has largest size among the values with type \( A \). Moreover, \( D' \) and \( D''' \) have no application of the rules \( sp \) and \( m \) so that, by Lemma 1.3, \( w(D', r) = |V| \leq |U| = w(D'''', r) \). Therefore:

\[
w(D_1, r) = w(S(D', D''), r) + w(S(D', D'''), r) + 1 \\
\leq 2 \cdot w(D', r) + w(D'', r) + w(D'''', r) + 1 \\
\leq w(D', r) + w(D'', r) + w(D'''', r) + 1 \\
< w(D', r) + w(D', r) + w(D'''', r) + 2 \\
= w(D', r) = w(D, r).
\]

This concludes the proof.
Proof of Lemma 2. The proof is by induction on \(|\pi'| + |\pi''|\). If the last rule of \(\pi'\) is s1 then \(M\) is a surface normal form, and the last rule of \(\pi''\) must be s1. In this case, \(|\pi'| = 0 = |\pi''|\). If the last rule of \(\pi'\) is s2, then \(M\) is not a surface normal form, so that the last rule of \(\pi''\) is s2. Hence, \(\pi'\) and \(\pi''\) have the following forms:

\[
\begin{align*}
M \rightarrow M'_1, M'_2 & \quad \pi'_1 : M'_1 \Rightarrow \mathcal{D}'_1 \quad \pi'_2 : M'_2 \Rightarrow \mathcal{D}'_2 \quad \text{s2} \\
M \rightarrow M''_1, M''_2 & \quad \pi''_1 : M''_1 \Rightarrow \mathcal{D}''_1 \quad \pi''_2 : M''_2 \Rightarrow \mathcal{D}''_2 \quad \text{s2}
\end{align*}
\]

We have several possibilities depending on \(M'_1, M'_2, M''_1, M''_2\). We just consider the case where they are all distinct. By applying Lemma 7 there exist \(N_1, N_2, N_3, N_4\) such that \(M'_1 \rightarrow N_1, N_2, M'_2 \rightarrow N_3, N_4\) and \(\exists i \in \{1, 2\}\) such that \(M''_i \rightarrow N_1, N_3, M''_{i+1} \rightarrow N_2, N_4\). Let us suppose \(i = 1\). By Theorem 2 \(N_1, N_2, N_3\) and \(N_4\) are all typable in PSTA. Moreover, since each typable term can be associated with exactly one surface distribution by Theorem 1 and Theorem 2, for all \(1 \leq j \leq 4\), we have \(\rho_j : N_j \Rightarrow \mathcal{E}_j\), for some \(\rho_j\) and \(\mathcal{E}_j\). Then, we can construct the following derivations:

\[
\begin{align*}
M'_1 \rightarrow N_1, N_2 & \quad \rho_1 : N_1 \Rightarrow \mathcal{E}_1 \quad \rho_2 : N_2 \Rightarrow \mathcal{E}_2 \quad \text{s2} \\
\rho'_1 : M'_1 \Rightarrow \mathcal{D}'_1 \\
M'_2 \rightarrow N_3, N_4 & \quad \rho_3 : N_3 \Rightarrow \mathcal{E}_3 \quad \rho_4 : N_4 \Rightarrow \mathcal{E}_4 \quad \text{s2} \\
\rho'_2 : M'_2 \Rightarrow \mathcal{D}'_2 \\
M''_1 \rightarrow N_1, N_3 & \quad \rho_1 : N_1 \Rightarrow \mathcal{E}_1 \quad \rho_3 : N_3 \Rightarrow \mathcal{E}_3 \quad \text{s2} \\
\rho''_1 : M''_1 \Rightarrow \mathcal{D}''_1 \\
M''_2 \rightarrow N_2, N_4 & \quad \rho_2 : N_2 \Rightarrow \mathcal{E}_2 \quad \rho_4 : N_4 \Rightarrow \mathcal{E}_4 \quad \text{s2} \\
\rho''_2 : M''_2 \Rightarrow \mathcal{D}''_2
\end{align*}
\]

By applying the induction hypothesis we have:

\[
|\pi'| = \max(|\pi'_1|, |\pi'_2|) + 1 \\
= \max(|\rho'_1|, |\rho'_2|) + 1 \\
= \max(\max(|\rho_1|, |\rho_2|) + 1, \max(|\rho_3|, |\rho_4|) + 1) + 1 \\
= \max(\max(|\rho_1|, |\rho_3|) + 1, \max(|\rho_2|, |\rho_4|) + 1) + 1 \\
= \max(|\rho''_1|, |\rho''_2|) + 1 \\
= \max(|\pi''_1|, |\pi''_2|) + 1 = |\pi''|.
\]

The remaining cases are similar.

\(\square\)

C Proofs of Section 5

In this section we give a detailed proof of the Probabilitic Polytime Completeness Theorem for PSTA (Theorem 4). The basic scheme of the proof is taken
from Gaboardi and Ronchi Della Rocca [10], and consists in encoding PTMs configurations, transitions between configurations, the initialization of a PTM, and its output extraction. By putting everything together, we are able to represent in PSTA a pPTM. Before giving the complete encoding, we shall first show how to define in PSTA natural numbers and polynomials.

C.1 Numerals and polynomial completeness

Gaboardi and Ronchi Della Rocca stressed in [10] that the presence of the multiplexor, i.e. rule \( m \), makes the encoding of a Turing Machine “non-uniform” in STAR. If we consider for example the standard type for natural numbers \( \text{N} \equiv \forall \alpha.(!\alpha \to \alpha) \to \alpha \to \alpha \), a term \( \text{succ} \) implementing the usual successor function with type \( \text{N} \to \text{N} \) is unknown. This is why the usual data types are represented in PSTA by indexed families of types.

**Definition 15** (Indexed numerals). For all \( i \geq 1 \), the indexed type \( \text{N}_i \) and the indexed numerals \( n_i \) of type \( \text{N}_i \) are defined as follows:

\[
\text{N}_i \equiv \forall \alpha.(!\alpha \to \alpha) \to \alpha \to \alpha \\
n_i \equiv \lambda f. \lambda x. (d^i(f).n.(d^i(f)x)\ldots) \quad n \in \mathbb{N}
\]

when \( i = 1 \), we shall write \( \text{N} \) (resp. \( n \)) in place of \( \text{N}_1 \) (resp. \( n_1 \)).

**Definition 16.** Let \( i, j \geq 1 \). The indexed successor \( \text{succ}_i \) of type \( \text{N}_i \to \text{N}_{i+1} \), the indexed addition of type \( \text{N}_i \to \text{N}_j \to \text{N}_{\max(i,j)+1} \), and the indexed multiplication of type \( \text{N}_i \to !\text{N}_j \to \text{N}_{i+j} \) are definable in PSTA as follows:

- \( \text{succ}_i \equiv \lambda n. \lambda f. \lambda x. (d^{i+1}(f)(n !d^{i+1}(f)x)) \);
- \( \text{add}_{i,j} \equiv \lambda n. \lambda m. \lambda f. \lambda x. n (d^{\max(i,j)+1}(f))(m (d^{\max(i,j)+1}(f)x)) \);
- \( \text{mult}_{i,j} \equiv \lambda n. \lambda m. \lambda f. n !_!(m (d^{i+j}(f))) \).

Successor, addition, and multiplication in **Definition 16** can be composed to obtain all polynomials.

**Theorem 8** (Representing polynomial functions [10]). Let \( p : \mathbb{N} \to \mathbb{N} \) be a polynomial in the variable \( x \) and \( \deg(p) \) be its degree. There is \( p \) such that:

\[
x : !\deg(p) \mathbb{N} \vdash p : \mathbb{N}^{2 \deg(p)+1}.
\]

Booleans and indexed strings of booleans are defined, respectively, in (14) and (16). The function associating with each string of booleans its length is defined for all \( i \geq 1 \) as follows:

\[
\text{len}_i \equiv \lambda s. \lambda z. \lambda f. s !f(\lambda x. \lambda y. \text{let} \ E_B x \ be \ I \ in \ fy)
\]

with type \( \text{S}_i \to \text{N}_i \), where \( \text{E}_B \) is as follows:

\[
E_B \equiv \lambda z. \lambda f. \lambda y. \text{let} \ z \ II \ be \ x \otimes y \ in \ (\text{let} \ y \ be \ I \ in \ x) : \text{B} \to \text{I}
\]
C.2 Encoding the pPTM

In this subsection we show how to encode a pPTM in PSTA and how to simulate its computation by means of the relation \( \Rightarrow \) in Definition 5. One of the key steps toward completeness is to prove that every PTM transition function is definable in PSTA, and its encoding is in (15).

A configuration can be represented by a tuple divided up in three parts: the first one represents the left hand-side of the tape with respect to the head; the second one represents the right part of the tape starting with the cell scanned by the head; finally, the third part represents the state of the machine. W.l.o.g., we shall assume that the left part of the tape is represented in reversed order, that the alphabet is composed by the two symbols 0 and 1, and that the final states are in \( \{0, 1\} \), and 1 that the states 0 and 1.

Definition 17 (Indexed configuration). For all \( i, k \geq 1 \), we define the indexed type \( \text{PTM}^k_i \) and the indexed configuration \( \text{config}_i \) of type \( \text{PTM}^k_i \) as follows:

\[
\text{PTM}^k_i \triangleq \forall \alpha. \exists! (B \rightarrow \alpha \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \alpha)^2 \otimes B^k)
\]

\[
\text{config}_i \triangleq \lambda c. (d^i(c) b^i_0 \otimes \cdots \otimes d^i(c) b^i_{n_i}) \otimes (d^i(c) b'^i_0 \otimes \cdots \otimes d^i(c) b'^i_{n'_i}) \otimes Q.
\]

where \( M \circ N \triangleq \lambda z. M(N z) \), \( Q \triangleq q_1 \otimes \cdots \otimes q_k \), and \( b^i_0, \ldots, b^i_n, b'^i_0, b'^i_{n'}, q_1, \ldots, q_k \)

are in \( \{0, 1\} \). We define the terms:

\[
d^i(c) b^i_0 \otimes \cdots \otimes d^i(c) b^i_{n_i}, \quad d^i(c) b'^i_0 \otimes \cdots \otimes d^i(c) b'^i_{n'_i}, \quad Q \triangleq q_1 \otimes \cdots \otimes q_k
\]

represent, respectively, the left and the right part of the tape, where \( d^i(c) b^i_k \) is the scanned symbol, and the current state \( Q = (q_1, \ldots, q_k) \).

Following Mairson and Terui [19], in order to define the PTM transition from a configuration to another we consider two distinct phases. In the first one, the PTM configuration is decomposed to extract the first symbol of each part of the tape. In the second phase, depending on the transitions function, these symbols are combined to reconstruct the tape after the transition step. Thus, we require an intermediate type, denoted \( \text{ID}_i^k \), and defined for all \( i, k \geq 1 \) as follows:

\[
\forall \alpha. \exists! (B \rightarrow \alpha \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \alpha)^2 \otimes (B \rightarrow \alpha \rightarrow \alpha) \otimes (B \rightarrow \alpha \rightarrow \alpha) \otimes (B \otimes B^k))
\]

and the decomposition phase is defined by the term \( \text{decom}_i \) of type \( \text{PTM}^k_i \rightarrow \text{ID}_i^k \) below:

\[
\text{decom}_i \triangleq \lambda m. \lambda c. \lambda l. \lambda r. \lambda q i m(F[d^i(c)]) \text{ be } l \otimes r \otimes q \text{ in } (\text{let } l(I \otimes (\lambda x. \text{let } E_B x \text{ be } I \text{ in } I) \otimes 0) \text{ be } s_{l} \otimes c_{l} \otimes b^i_0 \text{ in } (\text{let } r(I \otimes (\lambda x. \text{let } E_B x \text{ be } I \text{ in } I) \otimes 0) \text{ be } s_{r} \otimes c_{r} \otimes b'^i_0 \text{ in } (s_{l} \otimes s_{r} \otimes c_{l} \otimes b^i_0 \otimes c_{r} \otimes b'^i_0 \otimes q)) (19)
\]

where \( F[x] \triangleq \lambda b. \lambda z. \text{let } z \text{ be } g \otimes h \otimes i \text{ in } (h i \otimes g) \otimes x \otimes b \) and \( E_B \) is as in (18).

The behaviour of \( \text{decom}_i \) is to decompose a configuration in such a way as to extract the symbols of the tape which determine, together with the current state, the structure of the next configuration:

\[
\text{decom}_i(\lambda c. (d^i(c) b^i_0 \otimes C[b^i_1, \ldots b^i_{n_i}]) \otimes (d^i(c) b'^i_0 \otimes C[b'^i_1, \ldots b'^i_{n'_i}]) \otimes Q)
\]
is the encoding of the transition function to combine the symbols we put aside in order to return a distribution of the next configurations. For example, if the deterministic transition functions \( \delta \) where \( \text{ID} \) is defined by

\[
\text{ID} \equiv \text{indexed transition step}
\]

\[
\text{ID} = \text{indexed initial configuration}
\]

Definition 19

The initial configuration of a \( \text{PTM} \) is a configuration in the initial state \( \pi \) as in (15), and:

\[
\text{if } x \text{ then } M_1 \text{ else } M_2 \equiv \pi_1(x M_1 M_2)
\]

Then, the behaviour of \( \text{com} \), depending on \( \delta_P \) and on the current state, is to combine the symbols we put aside in order to return a distribution of the next configurations. For example, if the deterministic transition functions \( \delta_0 \) and \( \delta_1 \) defining \( \delta_P \) are such that \( \delta_0 ((b_0, Q)) = (Q', b', \text{right}) \) and \( \delta_1 ((b_0, Q)) = (Q', b', \text{left}) \), then:

\[
\text{com}_i (C[b_1, \ldots, b_n] \otimes C[b'_1, \ldots, b'_m] \otimes d^i(c) \otimes b^i \otimes d^i(c) \otimes b^i \otimes Q)
\]

\[
\Rightarrow \frac{1}{2} \cdot \lambda \epsilon. (d^i(c) b^i \otimes d^i(c) b^i \otimes C[b_1, \ldots, b_n] \otimes C[b'_1, \ldots, b'_m] \otimes Q')
\]

\[
+ \frac{1}{2} \cdot \lambda \epsilon. C[b_1, \ldots, b_n] \otimes (d^i(c) b^i \otimes d^i(c) b^i \otimes C[b'_1, \ldots, b'_m]) \otimes Q'.
\]

where \( C[b_1, \ldots, b_n] \equiv d^i(c) b^i \otimes d^i(c) b^i \otimes C[b_1, \ldots, b_n] \equiv d^i(c) b^i \otimes d^i(c) b^i \).

By combining the above terms we obtain an entire \( \text{PTM} \) transition step.

Definition 18 (Indexed transition step). *Let* \( i, k \geq 1 \). *The indexed transition step is defined by* \( \text{tr}_i \equiv \text{com} \circ \text{decom} \), *with type* \( \text{PTM}_i^k \rightarrow \text{PTM}_i^k \) *in* \( \text{PTA} \).

The initial configuration of a \( \text{PTM} \) is a configuration in the initial state \( Q_0 = (q_1, \ldots, q_k) \) with the head at the beginning of a tape filled by \( 0 \)'s. Then, we need a term that, taking a numeral \( \underline{n} \) as input, gives the encoding of the initial configuration with tape of length \( n \) as output.

Definition 19 (Indexed initial configuration). *For all* \( i, k \geq 1 \), *the indexed initial configuration* \( \text{init}_i \), *of type* \( \text{NT}_i \rightarrow \text{PTM}_i^k \) *is defined as follows:

\[
\text{init}_i \equiv \lambda u. \lambda c. (\lambda z. c \otimes (\lambda z. n^i(d^i(c) \otimes z)) \otimes Q_0).
\]

The \( \text{PTM} \) needs now to be initialized with the given input string, by writing it on its tape. The term representing the initialization requires the term \( \text{decom} \), in (19).
Definition 20 (Indexed initialization). For all \( i, k \geq 1 \), the indexed initialization is defined by 
\[ \text{in}_i \triangleq \lambda s. \lambda m. s!(\lambda b. Tb \circ \text{decom}_i) m \text{ of type } S \rightarrow \text{PTM}^k_i \rightarrow \text{PTM}^k_i, \]
where:
\[
T \triangleq \lambda b. \lambda m. \lambda c. \text{let } m \ (l^i d^i(c)) \text{ be } l \circ r \circ c_l \circ b_l \circ c_r \circ b_r \circ q \text{ in } \\
\text{let } E_B b_r \text{ be } I \text{ in } Rbq \ (l \circ r \circ c_l \circ b_l \circ c_r)
\]
\[
R \triangleq \lambda b'. \lambda q'. \lambda p. \text{let } p \text{ be } l \circ r \circ c_l \circ b_l \circ c_r \text{ in } (c_r b' \circ c_l b \circ l) \circ r \circ q'
\]
where \( E_B \) is as in (18).

Last, we need to extract the output string from the final configuration.

Definition 21 (Indexed extraction). For all \( i, k \geq 1 \), we define the indexed extraction \( \text{ext}^S_i \) of type \( \text{PTM}^k_i \rightarrow S_i \) as the following term:
\[
\text{ext}^S_i \triangleq \lambda m. \lambda c. \text{let } m \ (l^i d^i(c)) \text{ be } l \circ r \circ q \text{ in } (\text{let } E_B q \text{ be } I \text{ in } l \circ r).
\]
where \( E_B \) has type \( B^k \rightarrow 1 \), and can be constructed from (18).

By putting everything together, we are now able to encode a pPTM in PSTA:

Proof of Theorem 4. Let \( \mathcal{P} \) be a PTM running in polynomial time \( p : \mathbb{N} \rightarrow \mathbb{N} \) and in polynomial space \( q : \mathbb{N} \rightarrow \mathbb{N} \), with \( \deg(p) = d_1 \) and \( \deg(q) = d_2 \). We set \( \lceil p \rceil = 2d_1 + 1 \) and \( \lceil q \rceil = 2d_2 + 1 \). By Theorem 8 and Lemma 14 we have that the following judgements are derivable in PSTA:
\[
\begin{align*}
\text{s}_p : l^d_1 S \vdash P : N_{\lceil p \rceil} \\
\text{s}_q : l^d_2 S \vdash Q : N_{\lceil q \rceil}
\end{align*}
\]
(21)

where \( P \triangleq p \ (l_1 \text{d}_1 \text{d}_1 \text{len}_1 d^1(s_p))/x \), \( Q \triangleq q \ (l_1 \text{d}_1 \text{d}_1 \text{len}_1 d^1(s_q))/x \), and \( \text{len}_1 \) is defined in (17). Again, by repeatedly applying Lemma 14 we can compose the terms in Definitions 18, 19, 20, and 21 to obtain a derivation in PSTA of the following judgement:
\[
\text{s}' : S, p : N_{\lceil p \rceil}, q : N_{\lceil q \rceil} \vdash \text{ext}^S_{\lceil q \rceil}(p (l_1 \text{d}_1 \text{d}_1 \text{len}_1 d^1(s_p)))(\text{in}_{\lceil q \rceil} \text{s}'(\text{init}_{\lceil q \rceil} q)) : S_{\lceil q \rceil}.
\]
(22)

By two further applications of Lemma 14, we can compose (21) and (22) to obtain the following:
\[
\text{s}' : S, s_p : l^d_1 S, s_q : l^d_2 S \vdash \text{ext}^S_{\lceil q \rceil}(P (l_1 \text{d}_1 \text{d}_1 \text{len}_1 d^1(s_p)))(\text{in}_{\lceil q \rceil} \text{s}'(\text{init}_{\lceil q \rceil} Q)) : S_{2d_2 + 1}.
\]

By repeatedly applying rule \( m \), and by applying rule \( \rightarrow \text{II} \), we obtain the term:
\[
\vdash_{\text{PTA}} P : l_{\max(d_1, d_2, 1)} + 1 S \rightarrow S_{2d_2 + 1}.
\]

One can check that both point i and point ii hold. \( \square \)