Monopole Scattering with a Twist

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Abstract

By imposing certain combined inversion and rotation symmetries on the rational maps for $SU(2)$ BPS monopoles we construct geodesics in the monopole moduli space. In the moduli space approximation these geodesics describe a novel kind of monopole scattering. During these scattering processes axial symmetry is instantaneously attained and, in some, monopoles with the symmetries of the regular solids are formed. The simplest example corresponds to a charge three monopole invariant under a combined inversion and $90^\circ$ rotation symmetry. In this example three well-separated collinear unit charge monopoles coalesce to form first a tetrahedron, then a torus, then the dual tetrahedron and finally separate again along the same axis of motion. We explicitly construct the spectral curves in this case and use a numerical ADHMN construction to compute the energy density at various times during the motion. We find that the dynamics of the zeros of the Higgs field is extremely rich and we discover a new phenomenon; there exist charge $k$ $SU(2)$ BPS monopoles with more than $k$ zeros of the Higgs field.

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1 Introduction

Recently it has been shown that BPS multi-monopoles exist which have the symmetries of the regular solids [5, 8, 9]. The subject of this paper is to investigate particularly symmetric examples of multi-monopole scattering during which these monopole configurations are attained. We work within the moduli space approximation in which monopole dynamics is approximated by motion along the geodesics in the moduli space.

The moduli space of charge \( k \) multi-monopoles is equivalent to the space of degree \( k \) rational maps. In Section 2 we review this equivalence and by imposing twisted inversion symmetries on rational maps identify some one-dimensional submanifolds of these maps. Since we arrive at these submanifolds by imposing symmetry they are totally geodesic submanifolds of the monopole moduli space and since they are one-dimensional they are geodesics. They correspond to scattering processes in the moduli space approximation, which we refer to as twisted line scatterings.

The simplest example of a twisted line scattering involves a charge three monopole which is symmetric under a combined inversion and 90° rotation. The tetrahedral monopole is formed twice during this 3-monopole scattering. In Section 3 we investigate this case in more detail by constructing the associated spectral curves and Nahm data. Using a numerical ADHMN construction we display surfaces of constant energy density at various times during the motion. We find that the scattering is particularly novel and complicated. The motion of the zeros of the Higgs field is intriguing. There are bifurcation points where the number of zeros of the Higgs field changes. This is discussed in Section 4.

2 Monopoles and rational maps

The Bogomolny equation for \( SU(2) \) BPS monopoles in \( \mathbb{R}^3 \) is

\[
D_i \Phi = -\frac{1}{2} \epsilon_{ijk} F_{jk}
\]  

where \( D_i = \frac{\partial}{\partial x_i} + [A_i, \) is the covariant derivative, \( A_i \) the \( su(2) \)-valued gauge potential and \( F_{jk} \) the gauge field. The Higgs field, \( \Phi \), is an \( su(2) \)-valued scalar field satisfying the boundary condition

\[
\|\Phi\| = 1 - \frac{k}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty
\]  

where \( r = |x|, \|\Phi\|^2 = -\frac{1}{2} \text{tr} \Phi^2 \) and \( k \) is a positive integer, known as the magnetic charge. We shall refer to a monopole with magnetic charge \( k \) as a \( k \)-monopole. The energy density, \( \mathcal{E} \), of a monopole is given by

\[
\mathcal{E} = -\frac{1}{2} \text{tr}(D_i \Phi)(D_i \Phi) - \frac{1}{4} \text{tr}(F_{ij} F_{ij}).
\]  

By substituting from the Bogomolny equation this can be rewritten in the more convenient form [10],

\[
\mathcal{E} = \Delta \|\Phi\|^2.
\]
The energy is the integral of \( E \) over all space and is equal to \( 8\pi k \).

It was proved by Donaldson [4] that \( k \)-monopoles are equivalent to rational maps \( R : \mathbb{C} \rightarrow \mathbb{C} \mathbb{P}^1 \) given by

\[
R(z) = \frac{p(z)}{q(z)}
\]

where \( q \) is a monic polynomial in \( z \) of degree \( k \) and \( p \) is a polynomial in \( z \) of degree less than \( k \), which has no factor in common with \( q \). This is a 1-1 correspondence. The rational map description of monopoles is very simple and elegant and provides a convenient description of the monopole moduli space. Unfortunately the rational map does not describe the detailed properties of a monopole corresponding to a particular point in the moduli space.

In the rational map description one breaks the \( SO(3) \) rotation symmetry of the problem by decomposing \( \mathbb{R}^3 \) into \( \mathbb{R} \times \mathbb{C} \). To do this one direction is chosen to be special and space is decomposed into this direction and its orthogonal plane. Choosing, say, the positive \( x_3 \)-axis as the special direction then the \( z \)-coordinate in the rational map is the complex coordinate \( z = x_1 + ix_2 \). The only rotation symmetries which survive in the rational map description are those which preserve this decomposition of space.

Consider the case of a single monopole. For \( k = 1 \) the most general rational map of this type is

\[
R = \frac{\lambda e^{i\chi}}{z - c}
\]

where \( \lambda \in \mathbb{R}^+, \chi \in S^1, c \in \mathbb{C} \). This describes a monopole with position \( (x_1, x_2, x_3) \) and phase angle \( \chi \), where \( x_1 + ix_2 = c \) and \( x_3 = \frac{1}{2} \log \lambda \). For general \( k \) there is a similar interpretation of \( R \) but only if the separations of the roots of \( q \) are large compared to the values of \( p \) at those roots. This corresponds to all the \( k \) monopoles being well-separated in the \( x_1x_2 \)-plane. In [5] the concept of a strongly centred monopole was introduced. Roughly, a monopole is strongly centred if its total phase is unity and the centre of mass of the monopole is the origin. In terms of the rational maps a monopole is strongly centred if and only if the roots of \( q \) sum to zero and the product of \( p \) evaluated at each of the roots of \( q \) is equal to unity. Strongly centred monopoles form a totally geodesic submanifold of the monopole moduli space. Here we shall be concerned only with monopoles which are strongly centred.

In this Section we require the following two results on the rational maps of cyclically symmetric and inversion symmetric monopoles:

(i) A monopole is invariant under a rotation by \( \Theta \) around the \( x_3 \)-axis if

\[
R(e^{i\Theta}z) \cong R(z)
\]

where the equivalence means that the two rational maps are equal up to multiplication by a phase.

(ii) Let \( I \) denote the inversion operation

\[
I : (x_1, x_2, x_3) \rightarrow (x_1, x_2, -x_3)
\]
then a monopole is inversion symmetric if
\[ p(z)^2 = 1 \mod q(z). \]  \hfill (2.9)

Note that, in general, the inversion of a monopole with rational map \( p(z)/q(z) \) is the monopole with rational map \( Ip(z)/q(z) \) where \( Ip(z) \) is uniquely determined by \( p(z)Ip(z) = 1 \mod q(z) \). Also, for consistency, we have followed \cite{5} in referring to the transformation \hfill (2.8) as inversion. In group character tables this is usually referred to as a reflection and denoted \( \sigma_h \).

Let \( C_n \) denote the operation of rotation around the \( x_3 \)-axis through an angle \( 2\pi/n \). It is given by
\[ C_n : (x_1, x_2, x_3) \mapsto (x_1 \cos \frac{2\pi}{n} + x_2 \sin \frac{2\pi}{n}, -x_1 \sin \frac{2\pi}{n} + x_2 \cos \frac{2\pi}{n}, x_3). \]  \hfill (2.10)

We now generalise the inversion operation \( I \) to a twisted inversion \( I_{2n} \) given by composing inversion and rotation as
\[ I_{2n} = I \circ C_{2n}. \]  \hfill (2.11)

Note that \( (I_{2n})^2 = C_n \) so that \( I_{2n} \) symmetry implies \( C_n \) rotation symmetry.

We now discuss charge \( k \) strongly centred monopoles, with \( k > 2 \), which are invariant under the twisted inversion symmetry \( I_{2l} \), where \( l \) is an integer satisfying \( k - 1 \geq l > k/2 \). As noted above, \( I_{2l} \) symmetry implies \( C_l \) symmetry, so we first impose this condition. By result (i) this requires the rational map to have the form
\[ R = \frac{c + bz^l}{z^k - (z^l - a)} \]  \hfill (2.12)

for some complex constants \( a, b, c \). The requirement of \( I_{2l} \) symmetry for this rational map gives the constraint
\[ (c - bz^l)(c + bz^l) = 1 \mod z^k - (z^l - a). \]  \hfill (2.13)

This can only be satisfied if \( a = 0 \) and \( c = \pm 1 \). We can set \( c = 1 \) by a choice of phase. We arrive at the family of rational maps
\[ R = \frac{1 + bz^l}{z^k} \]  \hfill (2.14)

parameterized by the complex number \( b \). The rational maps in this family are strongly centred. This family defines a surface of two real dimensions in the \( k \)-monopole moduli space, which we denote by \( \Sigma_k^l \). It is a totally geodesic submanifold as it is the fixed point set of a symmetry. \( \Sigma_k^l \) is a surface of revolution; the phase of \( b \) corresponds to the orientation about the \( x_3 \)-axis. We may impose a reflection symmetry on the rational map so that \( b \) is real. This gives a geodesic in \( \Sigma_k^l \) corresponding to the generator of the surface of revolution. Geodesic flow then corresponds to \( b \) increasing monotonically from \( b = -\infty \) to \( b = +\infty \). If \( b = 0 \) then \hfill (2.14) is the rational map of the axisymmetric charge \( k \) monopole, with the \( x_3 \)-axis as the axis of symmetry.
In [1] pp. 25-26 it is argued that for monopoles strung out in well separated clusters along, or nearly along, the $x_3$-axis the first term in a large $z$ expansion of the rational map $R(z)$ for some phase choice will be $e^{2x}/z^L$ where $L$ is the charge of the topmost cluster and $x$ is its elevation above the plane. In an earlier paper [9] we extended this and argued that if the next highest cluster has charge $M$ and is $y$ above the plane then the first two terms in the large $z$ expansion of the rational map will be given by

$$R(z) \sim \frac{e^{2x}}{z^L} + \frac{e^{2y}}{z^{2L+M}} + \ldots$$

(2.15)

Writing (2.14) in the form

$$R = \frac{b}{z^{k-l}} + \frac{1}{z^k}$$

(2.16)

we deduce that as $b \to \pm \infty$ the rational map (2.14) describes axisymmetric monopoles of charge $k-l$ at the positions $(0,0,\pm \frac{1}{2} \log |b|)$ and an axisymmetric charge $2l-k$ monopole at the origin.

In the moduli space approximation [12, 15] the dynamics of $k$-monopoles corresponds to geodesic motion in the $k$-monopole moduli space. Atiyah and Hitchin [1] studied geodesics on two surfaces of revolution for the case $k = 2$. Hitchin, Manton and Murray [5] have investigated cyclically symmetric monopoles for $k > 2$ and obtained surfaces of revolution which are different from those obtained here and describe a completely different type of monopole scattering. We shall now describe the novel monopole scattering which results from geodesic motion along the generator of $\Sigma_k'$.

The simplest example is when $k = 3$, in which case we must have $l = 2$. The rational map is

$$R = \frac{1 + b z^2}{z^3}.$$  

(2.17)

where $-\infty < b < \infty$. Setting $k = 3$ and $l = 2$ in the above cluster decomposition we can interpret the geodesic as the following scattering event. At large negative times there are three well-separated monopoles which are all located on the $x_3$-axis. One monopole is stationary at the origin, with a second monopole located on the positive $x_3$-axis and a third monopole on the negative $x_3$-axis. The second and third monopoles are equidistant from the origin and are moving towards the stationary monopole. For large positive times the situation is similar, but now the two monopoles which are on the positive and negative $x_3$-axis are moving away from the monopole at the origin. We see that all the monopoles remain on the $x_3$-axis throughout the motion, including when they merge. This is true of all the geodesic motions along generators on the surfaces $\Sigma_k'$. They all describe monopole scattering along a line. The initial configuration is of two $(k-l)$-monopoles approaching a $(2l-k)$-monopole at the origin along the positive and negative $x_3$-axis and the final configuration is of two $(k-l)$-monopoles receding along the positive and negative $x_3$-axis leaving a $(2l-k)$-monopole at the origin. The one-parameter family of rational maps given by (2.14) is invariant, up to a phase change, under $b \to -b$ and $z \to e^{i\pi/l}z$. This means that the outgoing configurations are always like the incoming configurations but twisted by $\pi/l$ about the $x_3$-axis.
It is interesting that the scattering angle is zero in each case; such zero angle scattering behaviour is normally associated with systems which remain integrable even when time dependence is introduced, rather than non-integrable monopole dynamics where we expect phenomena such as right angle scattering.

As noted above, when \( b = 0 \) we obtain the axisymmetric \( k \)-monopole. The scattering of monopoles through axisymmetric configurations is a common occurrence which has been well studied and has analogues in lower dimensions. However, in a twisted line scattering there is a subtlety in the formation of the axisymmetric monopole. In all previously known examples the axisymmetric \( k \)-monopole occurs in a \( C_k \) symmetric monopole scattering, with the axis of symmetry of the torus perpendicular to the plane in which the monopoles approach. In a twisted line scattering the axis of symmetry of the torus is the line along which the monopoles approach. The torus formed is perpendicular to that which one might naively expect to find.

The \( I_4 \) symmetry we have imposed to obtain the surface \( \Sigma_3^2 \) is a combined inversion and 90° rotation symmetry. A tetrahedron with vertices at \((x_1, x_2, x_3)\) given by

\[
\{ (+d, +d, +d), (+d, -d, -d), (-d, -d, +d), (-d, +d, -d) \}
\] (2.18)

has this symmetry. Here \( d \) is arbitrary and the replacement \( d \rightarrow -d \) gives the dual tetrahedron. There is a 3-monopole with tetrahedral symmetry \( \mathbb{T} \) so for some value, \( b = b_c \) say, (2.17) is the rational map of the tetrahedral monopole. When \( b = -b_c \) the tetrahedral monopole is again formed, but this time in the orientation dual to the previous one. So, although the asymptotic in and out monopole states may suggest a simple scattering process, the dynamics must be relatively complicated since a tetrahedral monopole then an axisymmetric monopole then another tetrahedral monopole are all formed during the scattering. In Section 3 we shall investigate in greater detail the \( k = 3, l = 2 \) twisted line scattering and display energy density plots which allow us to see exactly how these various configurations are attained during the motion.

We shall now describe how the other recently discovered monopoles with the symmetries of the regular solids may be produced in monopole scatterings. In addition to the tetrahedral 3-monopole there is a 4-monopole with octahedral symmetry \( \mathbb{O} \) resembling a hollow cube \( \mathbb{C} \), a 5-monopole with octahedral symmetry resembling a hollow octahedron and a 7-monopole with icosahedral symmetry resembling a hollow dodecahedron \( \mathbb{D} \).

Orient a cube so that the \( x_3 \)-axis goes through two opposite vertices and the centre of the cube. This cube then has \( I_6 \) symmetry. The implied \( C_3 \) symmetry is a rotation of the two bases of the tetrahedra which are inscribed within the cube. The cubic 4-monopole is therefore contained within the rational maps

\[
R = \frac{1 + bz^3}{z^4}
\] (2.19)

defining the surface \( \Sigma_4^3 \). The cubic 4-monopole is formed twice during the twisted line scattering associated with the generator on \( \Sigma_4^3 \) where two unit charge monopoles approach along the \( x_3 \)-axis towards a charge two axisymmetric monopole at the origin. The charge four torus is also formed between the formation of the two cubes.
There is a similar scattering through the octahedral 5-monopole. The surface $\Sigma^5_5$ also describes monopoles with $I_6$ symmetry. An octahedron orientated so that two of its triangular faces are parallel to the $x_1x_2$-plane is invariant under $I_6$. Thus geodesic motion along the generator of $\Sigma^3_5$ describes two 2-monopoles approaching from the positive and negative $x_3$-axis a single monopole at the origin; the monopoles then coalesce to form the octahedral 5-monopole which deforms further into the toroidal 5-monopole and then into the octahedral 5-monopole rotated through $\pi/3$ before separating again into two 2-monopoles, receding along the $x_3$-axis and leaving a single monopole at the origin.

A dodecahedron with two faces parallel to the $x_1x_2$-plane has $I_{10}$ symmetry. The dodecahedral 7-monopole occurs during the geodesic scattering on the surface $\Sigma^5_7$ of rational maps

$$R = \frac{1 + bz^5}{z^7}.$$  \hfill (2.20)

The scattering involves two axisymmetric 2-monopoles approaching from the positive and negative $x_3$-axis a axisymmetric charge three monopole at the origin. The dodecahedral monopole is formed, followed by the axisymmetric charge seven monopole, then the dodecahedral monopole rotated $\pi/5$ relative to the previous one. Finally two 2-monopoles separate out again along the $x_3$-axis leaving a charge three monopole behind.

Scattering geodesics which include the tetrahedral and cubic monopoles have been obtained previously \cite{5} by imposing cyclic $C_3$ and dihedral $D_4$ symmetries respectively. A scattering geodesic containing the octahedral 5-monopole also exists \cite{9} resulting from imposition of $D_4$ symmetry. A further scattering geodesic through the cubic monopole can be obtained by imposition of tetrahedral symmetry on four monopoles \cite{8}. All these examples are very different from the twisted line scatterings which we present here. Furthermore, only the last example, that of 4-monopole scattering with tetrahedral symmetry, has been studied in great detail with the evolution of the energy density examined. The twisted line scatterings appear to be more complicated than any of those above and one really needs to examine energy density surfaces to understand these processes.

### 3 The charge three twisted line scattering

We now investigate the 3-monopoles along the $\Sigma^3_3$ geodesic in more detail in order to plot their energy densities. We use constructions similar to those in \cite{5, 8, 9} and these papers should be consulted for a more detailed explanation. We exploit two different formulations of the monopole problem; spectral curves and Nahm data.

Monopoles correspond to certain algebraic curves, called spectral curves, in the tangent bundle to the Riemann sphere $T\mathbb{CP}^1$. We let $\zeta$ be the standard inhomogeneous coordinate on the base space and $\eta$ the fibre coordinate. Monopoles of charge $k$ correspond to curves of the form

$$\eta^k + \eta^{k-1}a_1(\zeta) + \ldots + \eta^2a_{k-1}(\zeta) + \ldots + \eta a_k(\zeta) + a_k(\zeta) = 0.$$  \hfill (3.1)
where, for $1 \leq r \leq k$, $a_r(\zeta)$ is a polynomial in $\zeta$ of maximum degree $2r$ and satisfies the reality condition

$$a_r(\zeta) = (-1)^r \zeta^{2r} a_r\left(-\frac{1}{\zeta}\right). \quad (3.2)$$

A general algebraic curve in $\mathbb{T}\Phi\mathbb{P}^1$ satisfying these conditions will not necessarily correspond to a monopole. It is difficult to demonstrate a particular algebraic curve is the spectral curve of a monopole. We will circumvent these difficulties by using the ADHMN construction. Before we do so we will construct a candidate algebraic curve for $I_4$ symmetric strongly centred monopoles.

The transformations of $(\eta, \zeta)$ corresponding to real $O(3)$ transformations of space can be calculated. For example, rotation through an angle $\phi$ around the $x_3$-axis is given by

$$R_\phi : (\eta, \zeta) \rightarrow (e^{i\phi} \eta, e^{i\phi} \zeta), \quad (3.3)$$

and inversion, $I$, is given by

$$I : (\eta, \zeta) \rightarrow \left(\frac{-\bar{\eta}}{\zeta^2}, \frac{1}{\zeta}\right). \quad (3.4)$$

General $SO(3)$ transformations will correspond to Möbius transformations of $\zeta$ which may be calculated. This allows us to impose symmetry on candidate algebraic curves. Strong centring can easily be imposed [5] on the spectral curve. For strongly centred monopoles

$$a_1(\zeta) = 0. \quad (3.5)$$

We can now construct the candidate algebraic curve. It is

$$\eta^3 + f \eta \zeta^2 + ig(\zeta^5 - \zeta) = 0 \quad (3.6)$$

where the constant coefficients $f$ and $g$ are real. The reality of $g$ corresponds to choosing an orientation around the $x_3$-axis, analogous to choosing $b$ real in (2.14).

To show that there is a one-parameter set of $f$ and $g$ values such that this algebraic curve is the spectral curve of a monopole we employ the ADHMN construction. This is an alternative approach to the construction of monopoles which nicely complements the spectral curve formulation. The ADHMN construction [13, 7] is an equivalence between $k$-monopoles and Nahm data $(T_1, T_2, T_3)$, which are three $k \times k$ matrices depending on a real parameter $s \in [0, 2]$ and satisfying the following:

(i) Nahm’s equation

$$\frac{dT_i}{ds} = \frac{1}{2} \epsilon_{ijk}[T_j, T_k], \quad (3.7)$$

(ii) $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0$ and $s = 2$. 

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(iii) the matrix residues of \((T_1, T_2, T_3)\) at each pole form the irreducible \(k\)-dimensional representation of \(SU(2)\),

(iv) \(T_i(s) = -T_i^\dagger(s)\),

(v) \(T_i(s) = T_i^\dagger(2-s)\).

Equation (3.7) is equivalent to a Lax pair. This gives an associated algebraic curve which is given in terms of the Nahm data as

\[
\det(\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2) = 0.
\] (3.8)

By differentiating and substituting from (3.7) we can see that this curve is independent of \(s\). This algebraic curve is, in fact, the spectral curve.

We now construct Nahm data invariant under the \(I_4\) transformation. The Nahm data are an \(\mathbb{R}^3 \otimes \text{sl}(k, \mathbb{C})\) valued function of \(s\), which transform under the rotation group \(SO(3)\) as

\[
3 \otimes \text{sl}(k) \cong 3 \otimes (2k - 1 \oplus 2k - 3 \oplus \cdots \oplus 3)
\cong (2k + 1 \oplus 2k - 1 \oplus 2k - 3 \oplus \cdots \oplus 5 \oplus 3 \oplus 1). 
\] (3.9)

where \(r\) denotes the unique irreducible \(r\)-dimensional representation of \(su(2)\). Thus for charge three monopoles the Nahm data transform as the representation

\[
(\mathbb{Z}_u \oplus 5\mathbb{Z}_m \oplus 3\mathbb{Z}_l) \oplus (\mathbb{Z}_u \oplus 5\mathbb{Z}_m \oplus 1\mathbb{Z}_l)
\] (3.10)

where the subscripts \(u, m\) and \(l\), standing for upper, middle and lower, are a notational convenience to allow us to distinguish between isomorphic representations with different pedigrees. The most convenient way of finding \(I_4\)-invariant vectors in these representations is to represent \(su(2)\) on homogeneous polynomials over \(\mathbb{C}\text{P}^1\). Write \(X, Y\) and \(H\) for the basis of \(SU(2)\) satisfying the commutation relations

\[
[X, Y] = H; \quad [H, X] = 2X; \quad [H, Y] = -2Y.
\] (3.11)

The \((r + 1)\)-dimensional \(su(2)\) representation \(r + 1\) is defined on degree \(r\) homogeneous polynomials by the identification

\[
X = \zeta_1 \frac{\partial}{\partial \zeta_0}; \quad Y = \zeta_0 \frac{\partial}{\partial \zeta_1}; \quad H = -\zeta_0 \frac{\partial}{\partial \zeta_0} + \zeta_1 \frac{\partial}{\partial \zeta_1}.
\] (3.12)

It is easy to see that the degree four polynomial

\[
p_4(\zeta_1, \zeta_0) = \zeta_0^2 \zeta_1^2
\] (3.13)

is invariant under \(U(1)\) transformations around the \(x_3\)-axis. It has two of its zeros on each of the two poles of the Riemann sphere. The degree four polynomial

\[
q_4(\zeta_1, \zeta_0) = \zeta_1^4 + \zeta_0^4
\] (3.14)
is invariant under $D_4$ transformations; its zeros are arranged with $C_4$ symmetry around the equator of the Riemann sphere. Furthermore, up to a choice of orientation, the one-parameter family of homogeneous polynomials
\begin{equation}
\zeta_1^4 + ih\zeta_1^2\zeta_0^2 + \zeta_0^4
\end{equation}
constitute all the $I_4$-invariant vectors in $5\mathfrak{u}$. Similarly
\begin{equation}
p_6(\zeta_1, \zeta_0) = \zeta_1^3\zeta_0^3
\end{equation}
and
\begin{equation}
q_6(\zeta_1, \zeta_0) = \zeta_1\zeta_0(\zeta_1^4 - \zeta_0^4)
\end{equation}
are degree six polynomials invariant under $I_4$ and
\begin{equation}
p_2(\zeta_1, \zeta_0) = \zeta_1\zeta_0
\end{equation}
is a degree two $U(1)$ invariant.

Not all of these polynomials correspond to inhomogeneous polynomial coefficients in the candidate algebraic curve (3.6). The polynomial $p_2$ is absent since strongly centring the monopole forces the coefficient of $\eta^2$ to vanish. Furthermore not all invariant polynomials over $\mathbb{C}P^1$ lift to invariant polynomials over the entire tangent bundle.

We now construct a set of invariant Nahm triplets with closed commutation relations and corresponding to the candidate spectral curve (3.6) and thus to an $I_4$-symmetric 3-monopole. Such a set can be found by constructing the triplets corresponding to the $D_4$-invariant vectors in $\mathfrak{su}$ and $\mathfrak{su}_0$, the $U(1)$ invariant in $\mathfrak{su}$ and the $SO(3)$ invariant $\mathfrak{l}$.

We use the usual scheme \cite{5, 8, 9} to construct Nahm triplets corresponding to the given homogeneous polynomials. Polarizing $q_6$ gives
\begin{equation}
\xi_1^2 \otimes 2\xi_1^3\xi_0 + 2\zeta_1\xi_0 \otimes 5(\zeta_1^4 - \zeta_0^4) + \xi_0^2 \otimes 20\xi_1\zeta_0^3
\end{equation}
\begin{equation*}
\sim (5(\zeta_0 \frac{\partial}{\partial \zeta_1})^4\zeta_1^4, 5\zeta_1^4 - \frac{5}{24}(\zeta_0 \frac{\partial}{\partial \zeta_1})^4\zeta_1^4, \frac{5}{6}(\zeta_0 \frac{\partial}{\partial \zeta_1})^3\zeta_1^4).
\end{equation*}

We make the identification
\begin{equation}
\zeta_1^2r \leftrightarrow X^r; \quad \frac{\partial}{\partial \zeta_1} \leftrightarrow \text{ad}Y
\end{equation}
and derive the Nahm triplet
\begin{equation}
(5\text{ad}Y X^2, 5X^2 - \frac{5}{24}\text{ad}Y^4 X^2, \frac{5}{6}\text{ad}Y^3 X^2).
\end{equation}

We choose the explicit basis
\begin{equation}
X = \begin{bmatrix}
0 & 0 & 0 \\
-i\sqrt{2} & 0 & 0 \\
0 & -i\sqrt{2} & 0
\end{bmatrix}; \quad Y = \begin{bmatrix}
0 & i\sqrt{2} & 0 \\
0 & 0 & i\sqrt{2} \\
0 & 0 & 0
\end{bmatrix}; \quad H = \begin{bmatrix}
-2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 2
\end{bmatrix},
\end{equation}
and substitute (3.22) into (3.21). For convenience we perform the Nahm isospace basis transformation

\[(S_1, S_2, S_3) = \left( \frac{1}{2} S'_1 + S'_3, -i \frac{1}{2} S'_1 + i S'_3, -i S'_2 \right). \tag{3.23} \]

Relative to this basis the \(SO(3)\)-invariant Nahm triplet corresponding to the \(1\) representation in (3.9) is given by \((\rho_1, \rho_2, \rho_3)\) where

\[
\rho_1 = X - Y; \quad \rho_2 = i(X + Y); \quad \rho_3 = iH, \tag{3.24}
\]

and the \(\mathfrak{su}(D)_4\)-invariant Nahm triplet (3.21) is

\[
W_1 = \begin{bmatrix}
0 & \sqrt{2} & 0 \\
-\sqrt{2} & 0 & -\sqrt{2} \\
0 & \sqrt{2} & 0
\end{bmatrix}; \quad W_2 = \begin{bmatrix}
0 & i \sqrt{2} & 0 \\
i \sqrt{2} & 0 & -i \sqrt{2} \\
0 & -i \sqrt{2} & 0
\end{bmatrix}; \quad W_3 = \begin{bmatrix}
0 & 0 & 2 \\
0 & 0 & 0 \\
-2 & 0 & 0
\end{bmatrix}. \tag{3.25}
\]

The \(U(1)\)-invariant polynomial \(p_4\) polarizes as

\[
\xi_1^2 \otimes 2 \zeta_0^2 + 2 \xi_1 \xi_0 \otimes 4 \zeta_1 \zeta_0 + \xi_0^2 \otimes 2 \xi_1^2 \sim \left( \xi_0 \frac{\partial}{\partial \xi_1} \right)^2 \xi_1^2, 2 \left( \xi_0 \frac{\partial}{\partial \xi_1} \right) \xi_1^2, \xi_1^2 \right) \leftrightarrow (\text{ad}^2 Y, 2 \text{ad} Y X, X).
\]

After performing the transformation (3.23) the \(\mathfrak{su}(D)_4\) \(U(1)\)-invariant Nahm triplet is

\[
Y_1 = \begin{bmatrix}
0 & -i \sqrt{2} & 0 \\
i \sqrt{2} & 0 & -i \sqrt{2} \\
0 & -i \sqrt{2} & 0
\end{bmatrix}; \quad Y_2 = \begin{bmatrix}
0 & -\sqrt{2} & 0 \\
\sqrt{2} & 0 & -\sqrt{2} \\
0 & \sqrt{2} & 0
\end{bmatrix}; \quad Y_3 = \begin{bmatrix}
4i & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -4i
\end{bmatrix}. \tag{3.26}
\]

To construct the \(D_4\) invariant in \(\mathfrak{su}(m)\) we first construct the invariant in \(\mathfrak{su}(n)\). Polarizing \(q_4\) gives, to a constant multiple,

\[
\xi_1^2 \otimes \zeta_1^2 + \xi_0^2 \otimes \zeta_0^4 \sim \xi_1^2 \otimes \zeta_1^2 + \frac{1}{4} \left( \xi_0 \frac{\partial}{\partial \xi_1} \right)^2 \xi_1^2 \otimes \left( \xi_0 \frac{\partial}{\partial \xi_1} \right)^2 \zeta_1^2 = \left[ 1 + \frac{1}{24} \left( \xi_0 \frac{\partial}{\partial \xi_1} \otimes 1 + 1 \otimes \xi_0 \frac{\partial}{\partial \xi_1} \right)^4 \right] \xi_1^2 \otimes \zeta_1^2. \tag{3.27}
\]

We now map this invariant into \(\mathfrak{su}(m)\) by replacing the \(\mathfrak{su}(n)\) highest weight vector \(\xi_1^2 \otimes \zeta_1^2\) with the \(\mathfrak{su}(m)\) highest weight vector

\[
\xi_1^2 \otimes \zeta_0 \zeta_1^3 - \xi_0 \xi_1 \otimes \zeta_1^4. \tag{3.28}
\]

Expanding this we derive

\[
\xi_1^2 \otimes \left( -6 \left( \xi_0 \frac{\partial}{\partial \xi_1} \right) \zeta_1^4 \right) + 2 \xi_1 \xi_0 \otimes \left( 12 \zeta_1^4 + \frac{1}{2} \left( \xi_0 \frac{\partial}{\partial \xi_1} \right)^4 \zeta_1^4 \right) + \xi_0^2 \otimes \left( - \left( \xi_0 \frac{\partial}{\partial \xi_1} \right) \zeta_1^4 \right)
\]
\[
\sim (-6 \left( \zeta_0 \frac{\partial}{\partial \zeta_1} \right) \zeta_4^4, 12 \zeta_1^4 + \frac{1}{2} \left( \zeta_0 \frac{\partial}{\partial \zeta_1} \right)^4 \zeta_1^4, -\frac{1}{2} \left( \zeta_0 \frac{\partial}{\partial \zeta_1} \right)^3 \zeta_1^4)
\]
\[
\leftrightarrow (-6 \text{ad} Y X^2, [12 + \frac{1}{2} \text{ad} Y^4] X^2, -\frac{1}{2} \text{ad} Y^3 X^2). \quad (3.29)
\]

After performing the transformation \((3.23)\) the \(5\text{D}\)-invariant Nahm triplet is
\[
Z_1 = \begin{bmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 0 & -i \sqrt{2} & 0 \\ i \sqrt{2} & 0 & i \sqrt{2} \\ 0 & i \sqrt{2} & 0 \end{bmatrix}; \quad Z_3 = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}. \quad (3.30)
\]

The Nahm data are
\[
T_i(s) = x(s) \rho_i + y(s) Y_i + z(s) Z_i + w(s) W_i. \quad (3.31)
\]

They give monopoles with the required symmetry. We now know the matrices \(\rho_i, Y_i, Z_i\) and \(W_i\) and can substitute \(T_i(s)\) into Nahm’s equation \((3.7)\) so that it reduces to equations for \(x, y, z\) and \(w\).

To solve Nahm’s equation it is convenient to replace the variables \(x, y, z, w\) by the new variables \(\alpha, \beta, \theta, \phi\) defined via
\[
3x = \alpha \cos \phi + 2 \beta \cos \theta \\
3y = -\alpha \cos \phi + \beta \cos \theta \\
3z = \alpha \sin \phi + \beta \sin \theta \\
3w = \alpha \sin \phi - 2 \beta \sin \theta. \quad (3.32)
\]

Then the Nahm data take the simple form
\[
T_1 = i \beta \sqrt{2} \begin{bmatrix} 0 & e^{-i\theta} & 0 \\ e^{i\theta} & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 \end{bmatrix}; \quad T_2 = \beta \sqrt{2} \begin{bmatrix} 0 & e^{i\theta} & 0 \\ -e^{-i\theta} & 0 & -e^{i\theta} \\ 0 & -e^{i\theta} & 0 \end{bmatrix}; \quad T_3 = 2\alpha \begin{bmatrix} i \cos \phi & 0 & -\sin \phi \\ 0 & 0 & 0 \\ \sin \phi & 0 & -i \cos \phi \end{bmatrix}. \quad (3.33)
\]

With these data, Nahm’s equation becomes
\[
\dot{\alpha} = -2\beta^2 \cos(2\theta - \phi) \\
\dot{\beta} = -2\alpha \beta \cos(2\theta - \phi) \\
\dot{\theta} = 2\alpha \sin(2\theta - \phi) \\
\dot{\phi} = -2\beta^2 \alpha^{-1} \sin(2\theta - \phi). \quad (3.34)
\]

The spectral curve, calculated using \((3.8)\), is of the required form \((3.6)\) with the constants \(f\) and \(g\) given by
\[
f = 16(\beta^2 - \alpha^2), \quad g = 32\alpha \beta^2 \sin(2\theta - \phi). \quad (3.35)
\]
It is convenient, although at first it may appear a strange choice, to trade-in the variables \( f \) and \( g \) for two new variables \( a \) and \( \kappa \) defined by
\[
f = -2\kappa^2 (a^2 + 4\epsilon)^{1/3} \quad , \quad g = 2\kappa^3 a
\]
where \( \epsilon = \pm 1 \) allows two choices for this transformation. Using these we solve the equations for \( \alpha \) and \( \beta \) as
\[
\alpha(s) = \frac{\kappa}{2} \sqrt{\wp(\kappa s) + (a^2 + 4\epsilon)^{1/3}} \tag{3.37}
\]
\[
\beta(s) = \frac{\kappa}{2} \sqrt{\wp(\kappa s) - \frac{1}{2}(a^2 + 4\epsilon)^{1/3}} \tag{3.38}
\]
where \( \wp \) is the Weierstrass elliptic function satisfying
\[
\wp'^2 = 4\wp^3 - 3(a^2 + 4\epsilon)^{2/3}\wp + 4\epsilon \tag{3.39}
\]
and prime denotes differentiation with respect to the argument. The equation for \( \phi \) then becomes
\[
\dot{\phi} = \frac{-\kappa a}{2(\wp + (a^2 + 4\epsilon)^{1/3})} \tag{3.40}
\]
and \( \theta \) is related to \( \phi \) via
\[
\sin(2\theta - \phi) = \frac{\kappa^3 a}{16\alpha\beta^2}. \tag{3.41}
\]

Let us now check that the above data satisfy the Nahm boundary conditions and thus correspond to a monopole. For \( \alpha \) and \( \beta \) to be finite for \( s \in (0, 2) \) and to have simple poles at \( s = 0 \) and \( s = 2 \) requires that \( \kappa \) is half the real period of the elliptic function given by \eqref{eq:3.39}. This fixes the value of \( \kappa \) for given \( a \) and \( \epsilon \). It can easily be checked from the above equations that both \( \theta \) and \( \phi \) are finite for \( s \in [0, 2] \). Next we need to check that \( R_i \), the matrix residues of the poles of \( T_i \), form the irreducible representation of \( SU(2) \). As \( s \to 0 \) then, by \eqref{eq:3.37},
\[
\alpha \sim \frac{1}{2s} \tag{3.42}
\]
so that \( iR_3 \) has eigenvalues \( \{0, \pm 1\} \), independently of the value of \( \phi \) at \( s = 0 \). This demonstrates that the representation formed by the matrix residues is the irreducible one. A similar argument applies for the pole at \( s = 2 \). We have proved that the above Nahm data correspond to a monopole. Conveniently this is done without having to explicitly solve \eqref{eq:3.40} which would require performing an elliptic integral of the third kind.

In summary, we have shown that
\[
\eta^3 - 6(a^2 + 4\epsilon)^{1/3}\eta\zeta^2 + 2i\kappa^3 a(\zeta^5 - \zeta) = 0 \tag{3.43}
\]
is the spectral curve of a 3-monopole with \( I_4 \) symmetry for all \( a \in \mathbb{R} \) and \( \epsilon = \pm 1 \), provided \( 2\kappa \) is the real period of the elliptic curve
\[
y^2 = 4x^3 - 3(a^2 + 4\epsilon)^{2/3}x + 4\epsilon. \tag{3.44}
\]
We shall now analyse this family of spectral curves and examine some particular cases in detail. For this we shall need the following integral representation for $\kappa$,

\[
\kappa = \int_{x_0}^{\infty} \frac{dx}{\sqrt{4x^3 - 3(a^2 + 4\epsilon)^{2/3}x + 4\epsilon}}
\]

(3.45)

where $x_0$ is the largest real root of $4x^3 - 3(a^2 + 4\epsilon)^{2/3}x + 4\epsilon = 0$.

First we shall consider the case $\epsilon = -1$. Then $\kappa$ is finite for all finite $a$. If $a = 0$ then the Weierstrass function given by (3.39) is degenerate, since there are two equal real zeros of the elliptic curve (3.44). This is the case of infinite imaginary period and in this case $\wp$ may be written in terms of trigonometric functions as

\[
\wp(z) = 2^{-1/3} \left( \frac{3}{\sin^2(2^{-1/6} \sqrt{3}z)} - 1 \right).
\]

(3.46)

The real period of this function must be $2\kappa$, hence

\[
\kappa = \frac{\pi}{2^{5/6} \sqrt{3}}.
\]

(3.47)

Substituting these values into (3.43) gives the curve

\[
\eta^3 + \pi^2 \eta \zeta^2 = 0
\]

(3.48)

which is the spectral curve of the axisymmetric 3-monopole [6].

If $a = \pm 2$ then

\[
\kappa = \int_1^{\infty} \frac{dx}{2\sqrt{x^3 - 1}} = \frac{\Gamma(1/6)\Gamma(1/3)}{4\sqrt{3\pi}}
\]

(3.49)

and (3.43) becomes

\[
\eta^3 \pm i \frac{\Gamma(1/6)^3 \Gamma(1/3)^3}{48\sqrt{3\pi}^{3/2}} (\zeta^5 - \zeta) = 0
\]

(3.50)

which is the spectral curve of the tetrahedral monopole [5] in two different orientations.

Now consider the limit $a \to \infty$. An asymptotic expansion of the integral representation (3.43) gives

\[
\kappa \sim \frac{\Gamma(1/4)^2}{4a^{1/3}3^{1/4} \sqrt{\pi}}
\]

(3.51)

so that (3.43) tends to the limiting spectral curve

\[
\eta^3 - \frac{\Gamma(1/4)^4 \sqrt{3}}{8\pi} \eta^2 \kappa^2 + i \frac{\Gamma(1/4)^6}{32\pi^{3/2}3^{3/4}} (\zeta^5 - \zeta) = 0.
\]

(3.52)

This is a new explicit spectral curve which describes three monopoles which are not well-separated. We shall see later that the energy density of this monopole configuration has a complicated and unusual structure. It resembles a twisted figure-of-eight, with the bottom
loop at right angles to the top loop. The same monopole configuration is obtained in the limit $a \to -\infty$, but this time rotated by $90^\circ$ around the $x_3$-axis.

For $\epsilon = -1$ we have seen that, although the parameter $a$ ranges over the whole real line, this maps out only a finite segment in the 3-monopole moduli space i.e there are no solutions representing well-separated monopoles for $\epsilon = -1$. From the rational map approach of Section 2 we know that a one-parameter family of solutions exists which includes well-separated monopoles. This implies that another branch of solutions exists which continues the $\epsilon = -1$ branch discussed so far. This is indeed true and is given by setting $\epsilon = 1$. Let us now consider this case.

It is immediately clear from the general spectral curve (3.43) and the integral representation (3.45) that the values

$$\epsilon = -1, a \to \infty \text{ and } \epsilon = 1, a \to \infty$$

(3.53)

describe the same spectral curve and hence the same monopole solution. This is the twisted figure-of-eight solution which describes three monopoles close together. The same is true for $a \to -\infty$ with $\epsilon = \pm 1$.

Next consider the limit $a \to 0$, with $\epsilon = 1$. In this limit the elliptic curve (3.44) has a double zero at the largest positive real root. Hence from (3.45) $\kappa$ tends to infinity logarithmically with $a$ in this limit. The spectral curve (3.43) is then asymptotic to

$$\eta^3 - 4^{1/3}6\kappa^2\eta\xi^2 = 0.$$ 

(3.54)

The product of three spectral curves describing unit charge monopoles with centres $(x_1, x_2, x_3)$ given by

$$\{(0, 0, 0), (0, 0, -c), (0, 0, +c)\}$$

(3.55)

is

$$\eta^3 - 4c^2\eta\xi^2 = 0.$$ 

(3.56)

The curve (3.54) has this form with $c = 2^{-1/6}\sqrt{3}\kappa$.

In summary, the full family of monopole solutions is described, in our coordinates, by a family of spectral curves which has three connected segments given by $\epsilon = 1, a > 0; \epsilon = -1, a \in \mathbb{R}; \epsilon = 1, a < 0$.

The Higgs field $\Phi$ can be reconstructed from the Nahm data using the ADHMN algorithm. This we have done using a numerical implementation of the ADHMN algorithm introduced by the authors in a previous paper [8]. The energy density can then be calculated by using the formula (2.4). The numerical algorithm requires the Nahm data at $s$-values on a grid. Since we do not have a convenient analytic expression for $\phi$ the equation (3.40) is solved numerically, but this is a simple task.
Table 1. Parameter values for scattering shown in Fig. 1

| image | $\epsilon$ | a   |
|-------|-----------|-----|
| 1     | +1        | 0.300 |
| 2     | +1        | 0.600 |
| 3     | +1        | 1.000 |
| 4     | -1        | 4.000 |
| 5     | -1        | 2.010 |
| 6     | -1        | 2.000 |
| 7     | -1        | 1.995 |
| 8     | -1        | 1.800 |
| 9     | -1        | 0.000 |
| 10    | -1        | -1.800 |
| 11    | -1        | -1.995 |
| 12    | -1        | -2.000 |
| 13    | -1        | -2.010 |
| 14    | -1        | -4.000 |
| 15    | +1        | -1.000 |
| 16    | +1        | -0.600 |
| 17    | +1        | -0.300 |

Fig. 1 displays a surface of constant energy density $E = 0.18$ for 17 different members of the family of monopoles. Table 1 gives the values of the parameters $\epsilon$ and $a$ for each image. Since these monopole configurations all lie on the generator of the surface $\Sigma^2_3$ we can now describe the monopole scattering corresponding to motion along that generator in more detail than before. At large negative times (1) there are three well-separated monopoles. One monopole is stationary at the origin, a second monopole is approaching along the positive $x_3$-axis and a third is approaching along the negative $x_3$-axis. As the monopoles merge (2) the one in the centre twists as it attempts to align with both the top and bottom monopoles. The energy tries to flow towards the centre but gets squeezed out sideways to form the twisted figure-of-eight shape (4). This is the configuration given by the explicit spectral curve \((3.32)\). The energy continues to flow towards the $x_1x_2$-plane, but now it has more of a sideways motion, which leads to the formation of the tetrahedral monopole (6). The diagonal movement of the energy density pulls the tetrahedron apart (7) into a buckled torus (8), which then straightens out to form the axisymmetric charge three torus (9) at time zero. The energy continues to flow in the same direction so that the torus buckles in the opposite sense (10). The motion at positive times goes backwards through the configurations just described, except that the monopoles are inverted so that

\footnote{Fig 1. is not available in the hep-th version of this paper. A hard copy is available on request to P.M.Sutcliffe@ukc.ac.uk, or it can be viewed at URL http://www.ukc.ac.uk/IMS/maths/people/P.M.Sutcliffe/preprints.html}
the tetrahedral monopole (12) formed at positive time is dual to the one (6) formed at negative time etc.

4 The zeros of the Higgs field

The monopole dynamics which occurs here is very novel and unlike any previously known. In this section we shall discuss a surprising aspect of the charge three twisted line scattering process. It appears that during this scattering process the number of zeros of the Higgs field is not conserved. The total number of zeros of the Higgs field counted with their multiplicity is $k$ for a $k$-monopole and these zeros are not always isolated, but may coalesce to form zeros of higher multiplicity. For example, in the case of the toroidal 3-monopole there is a single zero but it has multiplicity three. What is surprising about the charge three twisted line scattering geodesic is that there are intervals when the total number of zeros exceeds three.

The scattering process passes from three well-separated monopoles through the tetrahedral configuration to the toroidal configuration. When there are three well-separated monopoles the Higgs field has exactly three zeros. The axisymmetric monopole has all three Higgs zeros at the origin and it is clear that the only way to arrange three points with tetrahedral symmetry is to put all three points at the origin. Thus if the tetrahedral monopole has three zeros of the Higgs field then they must all be at the origin, as in the case of the axisymmetric monopole. Moreover throughout twisted line scattering the imposed symmetry means that if there are three zeros of the Higgs field then one must be at the origin with the other two on the $x_3$-axis and equidistant from the origin. However, numerical investigations reveal that there are no zeros of the Higgs field on the $x_3$-axis (except at the origin) for all the monopole solutions between the tetrahedral monopole (Fig 1 (5)) and the axisymmetric monopole (Fig 1 (9)). So, if the number of Higgs zeros remains three, then we are forced to the surprising conclusion that the zeros of the Higgs field must stick at the origin for a finite period of time. It turns out that this is not in fact what is happening, and the true description is even more fascinating.

The above argument fails because it assumes that the number of zeros of the Higgs field is always three for a 3-monopole solution of the Bogomolny equation. The basis for such an assumption is that no $k$-monopole solution has been presented which had greater than $k$ zeros of the Higgs field and furthermore in the analogous case of abelian Higgs vortices at critical coupling the vortex number not only gives the total number of zeros counted with their multiplicity but also bounds the total number of zeros \cite{10} pp 76-78. However, what we have found is that some of the monopoles in our one-parameter family have more than three zeros.\footnote{We are extremely grateful to Werner Nahm for suggesting this possibility} In fact, at different points on the one-parameter family the number of zeros can be one, three, five or seven.

The first approach we take is to compute the winding number, $Q(r_0)$, of the normalized Higgs field on a two-sphere of radius $r_0$, centred at the origin. This integer winding number...
counts the number of zeros of the Higgs field counted with multiplicity inside this two-
sphere. By definition \( Q(\infty) = k \) for a \( k \)-monopole. The numerical scheme used to compute the winding number is described in the appendix. The results obtained are integer valued to within six decimal places, so we shall give all our results as integers.

First we consider the tetrahedral monopole and compute that \( Q(1.0) = +3 \). This is a good check on our numerical scheme as we require that the winding number is equal to three when \( r_0 \) is sufficiently large. Now if all three Higgs zeros were at the origin then the winding number would equal three for all positive values of \( r_0 \). However we find the result that \( Q(0.2) = -1 \) ie. locally around the origin the field configuration is that of an anti-zero. Therefore between the sphere of radius \( r_0 = 0.2 \) and the sphere of radius \( r_0 = 1.0 \) there must be (at least) four zeros, which each have an associated local winding number of +1. Let us now look for these extra zeros by plotting the components of the Higgs field.

Write the Higgs field in terms of Pauli matrices as
\[
\Phi = i\sigma_1 \varphi_1 + i\sigma_2 \varphi_2 + i\sigma_3 \varphi_3 \quad (4.1)
\]
and plot the individual components \( \varphi_1, \varphi_2, \varphi_3 \). It is easier to locate a zero of the Higgs field by searching for where all three components are zero rather than simply looking at the single quantity \( ||\Phi|| \). The task is made simple by the presence of tetrahedral symmetry. Fig 2(a) shows the components of the Higgs field along the line \( x_1 = x_2 = x_3 = L \), for \(-0.4 \leq L \leq 0.4\). It is clear that all three curves have a zero at \( L = 0 \) and \( L \approx -0.38 \). By tetrahedral symmetry, similar curves are obtained along each of the other three diagonals. Hence the numerical evidence suggests that there are four positive zeros (ie. each corresponding to a winding of +1) on the vertices of a regular tetrahedron and an anti-zero (ie. corresponding to a winding of -1) at the origin. Therefore the tetrahedral 3-monopole is a solution in which the Higgs field has both positive multiplicity and negative multiplicity zeros but nonetheless saturates the Bogomolny energy bound.

An obvious question is whether the positive zeros lie along the directions of the vertices of the tetrahedron (where the energy density is maximal) or along the directions of the faces of the tetrahedron (where a surface of constant energy density has holes). Fig 2(b) shows a plot of the energy density along the line \( x_1 = x_2 = x_3 = L \), for \(-3 \leq L \leq 3\). Clearly the zeros lie along the lines joining the origin to the vertices of the tetrahedron. However the zeros are not as far from the origin as the points of maximal energy density. The zeros occur at \( L \approx -0.38 \) whereas the energy density takes its maximum value at \( L \approx -1 \). It is interesting to note that the location of the anti-zero appears to coincide with a local minimum of the energy density.

We are now in a position to describe the motion of the zeros of the Higgs field. We have roughly sketched this motion in Fig 3. When the monopoles are well separated there are three zeros of the Higgs field, Fig 3(a). One is at the origin and the other two are equidistant above and below the origin along the \( x_3 \)-axis. Obviously in the asymptotic limit of infinite separation each of these zeros lies at the centre of a 1-monopole. At some critical point as the zeros approach there is a bifurcation. Each of the zeros above and below the origin split into three zeros, two with positive multiplicity and one with negative
multiplicity. In this way the number of zeros counted with their multiplicity is conserved locally. Unfortunately neither the precise details of this bifurcation nor the precise point at which it occurs is discernable numerically but it is certain that as the matter begins to coalesce there are seven zeros of the Higgs field, Fig 3(b), one at the origin of positive unit multiplicity, two above and below the origin on the $x_3$-axis of negative unit multiplicity and four further positive multiplicity zeros away from the $x_3$-axis. These four zeros move consistently with the twisted line symmetry, the two above the $x_1x_2$-plane are separated along a line parallel to the line $x_1 = -x_2$ and $x_3 = 0$ and the two below, parallel to the line $x_1 = x_2$ and $x_3 = 0$.

As the monopoles continue to coalesce the anti-zeros approach the origin, Fig 3(c). They leave behind the four zeros which are off the $x_3$-axis. They reach the origin at the tetrahedral configuration, Fig. 3(d). At the tetrahedral configuration there are five zeros, a single zero of negative unit multiplicity at the origin and four with positive multiplicity arranged in a tetrahedron. The four positive multiplicity zeros then move towards the origin, Fig 3(e) and finally reach the origin to give a single multiplicity three zero, Fig 3(f).

We have already presented numerical evidence to support our claim for the configuration of Higgs zeros at the parameter value $a = 2$ (the tetrahedral monopole). Now we shall give similar numerical results to support our claims for the configuration of Higgs zeros prior to the formation of the tetrahedral monopole, but after the splitting of the Higgs zeros.

The case considered is for the parameter value $a = 2.05$. First we compute some winding numbers. The results are that

$$Q(0.2) = +1, \ Q(0.5) = -1, \ Q(0.7) = +3 \quad (4.2)$$

These results are consistent with a positive zero at the origin, two negative zeros on the $x_3$-axis and four positive zeros which are further from the origin than the negative zeros. The positions of the zeros can be located, in the same manner as in the tetrahedral case, by plotting the components of the Higgs field. By the imposed symmetries each of the zeros must lie on a line where $x_1 = \pm x_2$. Therefore we plot the components of the Higgs field along the line $x_1 = x_2 = L$, with $x_3$ fixed. Fig 4(a) shows such a plot with $x_3 = -0.425$. This clearly shows a zero on the $x_3$-axis (ie. $L = 0$). Fig 4(b) shows a similar plot for $x_3 = -0.605$. It can be seen that there are two zeros, which are a distance $L \approx 0.17$ from the $x_3$-axis. These results are in agreement with the winding number calculations, which placed bounds on the distances of each of the zeros from the origin.

It is difficult to give a physical interpretation of these negative multiplicity zeros of the Higgs field. A natural interpretation would be that the configuration contains anti-monopoles as well as monopoles, but such a statement can not be made rigorous unless a suitable definition of magnetic charge density is found. The standard definition of the magnetic field of a monopole relies upon a consideration of the asymptotic field of the configuration far from the monopole, where the non-abelian $SU(2)$ gauge symmetry is broken to a $U(1)$ symmetry that can be identified with the abelian gauge symmetry of electromagnetism. In the twisted line scattering configurations the negative multiplicity zeros remain close to the positive multiplicity zeros and so a definition of magnetic field that is valid only in the asymptotic region is useless.
The existence of anti-zeros raises a number of mathematical questions. For example, can the presence of an anti-zero be seen from the spectral curve or rational map of a monopole? It seems likely that the appearance and disappearance of anti-zeros has a signature in the space of rational maps or in the space of spectral curves. Unfortunately our numerical results cannot pin-point the exact values of the parameter $a$ at which anti-zeros appear or disappear in the 3-monopole twisted line scattering. One possibility is that such an event is associated with the elliptic curve (3.44) being singular. The discriminant $\Delta$ of the elliptic curve is
\[ \Delta = 27a^2(a^2 + 8\epsilon). \] (4.3)
This vanishes when $a = 0$ and when $a = \pm \sqrt{8}, \epsilon = -1$. The first singular curve corresponds to the toroidal monopole and we know that anti-zeros disappear and appear at this point. It is consistent with our numerical results that the second singular curve $a = \sqrt{8}, \epsilon = -1$ (and its inverted partner) corresponds to the splitting point where anti-zeros appear (disappear). It is also interesting to note that a naive parameter count demonstrates that the motion of the Higgs zeros and anti-zeros cannot be independent.

5 Conclusion

By imposing twisted inversion symmetries on multi-monopoles we have been able to present some interesting new examples of geodesic monopole scattering. These scattering events help to explain how the recently discovered monopoles with the symmetries of the regular solids are formed from individual monopoles as they merge and deform. The scattering processes are quite exotic and reveal new features which are not yet fully understood.

The discovery that there exist $k$-monopole solutions with greater than $k$ zeros of the Higgs field is unexpected. It is also surprising to find a large family of scattering events in which the scattering angle is zero. It seems that the behaviour of higher charge monopoles is not completely typified by that of charge two monopoles and the behaviour of three-dimensional solitons is not typified by that of two-dimensional solitons. It remains to be seen whether these results concerning monopole scattering are relevant to the dynamics of other three-dimensional topological solitons such as skyrmions.

On a more speculative note, it seems likely that the creation of extra Higgs zeros is connected with points in the monopole moduli space related to singular elliptic curves. Singular points of moduli spaces are relevant to phase transitions in string theory and are a central feature of the Seiberg-Witten [14] treatment of duality in $N = 2$ supersymmetric Yang-Mills theory, where they are associated with the appearance of massless monopoles. Perhaps some insight into these issues may be obtained from a deeper study of the BPS monopole moduli space at points where anti-zeros occur.

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of this paper. CJH thanks the EPSRC for a research studentship and the British Council for a FCO award. PMS thanks the EPSRC for a research fellowship.
A Appendix: Computation of winding numbers

In this appendix we give the details of the numerical scheme used to compute the winding number $Q$, of the normalised Higgs field on a two-sphere of radius $r_0$, centred at the origin.

First of all, we discretise the above two-sphere into the $n^2$ lattice points given by

\begin{align}
x_1 &= r_0 \sin(\pi i/n) \cos(2\pi j/n) \\
x_2 &= r_0 \sin(\pi i/n) \sin(2\pi j/n) \\
x_3 &= r_0 \cos(\pi i/n)
\end{align}

with $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, n-1$. Then using the numerical ADHMN construction the Higgs field is computed at each of these lattice points. Write the Higgs field in terms of Pauli matrices as

$$\Phi = i\sigma_1 \varphi_1 + i\sigma_2 \varphi_2 + i\sigma_3 \varphi_3.$$  

(A2)

Then, providing the Higgs field is not identically zero, we can define the unit 3-vector $\psi$ as

$$\psi = (\varphi_1, \varphi_2, \varphi_3) \frac{1}{\sqrt{\varphi_1^2 + \varphi_2^2 + \varphi_3^2}}.$$  

(A3)

So there is a unit 3-vector defined at each lattice point on the two-sphere. Fig 5 shows four such lattice points, which are numbered 1 to 4. They correspond to the lattice points $(i, j), (i+1, j), (i, j-1), (i+1, j-1)$, for some integers $i$ and $j$. Let the unit 3-vectors defined at these points be denoted by $\psi_1, \psi_2, \psi_3, \psi_4$ respectively.

Now we can make use of the work of Berg and Lüscher \[2\] who have defined a lattice topological charge for the $O(3)$ $\sigma$-model. Let $Q_{ij}$ be the lattice topological charge density

$$Q_{ij} = (A_{123} + A_{134})/4\pi$$  

(A4)

where $A_{123}$ denotes the signed area of the spherical triangle with vertices $\psi_1, \psi_2, \psi_3$. Explicitly the formula is

$$\exp\left(\frac{iA_{123}}{2}\right) = \frac{1 + \psi_1 \cdot \psi_2 + \psi_2 \cdot \psi_3 + \psi_3 \cdot \psi_1 + i\psi_1 \cdot (\psi_2 \times \psi_3)}{\sqrt{2(1 + \psi_1 \cdot \psi_2)(1 + \psi_2 \cdot \psi_3)(1 + \psi_3 \cdot \psi_1)}}.$$  

(A5)

$4\pi Q$ is defined to be the total signed area of the surface obtained by gluing together all these elementary spherical triangles i.e.

$$Q = \sum_{i=0}^{n-1} \sum_{j=1}^{n} Q_{ij}.$$  

(A6)

By the geometrical interpretation of $Q$ it is clear that it is integer valued. Moreover, it is topological in the sense that local continuous deformations of the lattice field $\psi$ do not change the winding number $Q$ (providing certain exceptional configurations are excluded) \[2\].
Figure captions

Fig. 1: Surface of constant energy density $\mathcal{E} = 0.18$ at increasing times.

Fig. 2(a): Components of the Higgs field of the tetrahedral monopole along the line $x_1 = x_2 = x_3 = L$, for $-0.4 \leq L \leq 0.4$.

Fig. 2(b): Energy density of the tetrahedral monopole along the line $x_1 = x_2 = x_3 = L$, for $-3 \leq L \leq 3$.

Fig. 3: Schematic representation of the motion of the zeros of the Higgs field.

Fig. 4(a): Components of the Higgs field of the monopole with parameter $a = 2.05$, along the line $x_3 = -0.425$, $x_1 = x_2 = L$, for $-0.4 \leq L \leq 0.4$.

Fig. 4(b): As Fig. 4(a) but with $x_3 = -0.605$.

Fig. 5: A vector field on a lattice $S^2$. 
References

[1] M.F. Atiyah and N.J. Hitchin, ‘The geometry and dynamics of magnetic monopoles’, Princeton University Press, 1988.

[2] B. Berg and M. Lüscher, ‘Definition and statistical distributions of a topological number in the lattice O(3) σ-model’, Nucl. Phys. B190, 412 (1981).

[3] R. Bielawski, ‘Monopoles, particles and rational functions’, McMaster preprint, 1994. 147 (1990).

[4] S.K. Donaldson, ‘Nahm’s equations and the classification of monopoles’, Commun. Math. Phys. 96, 387 (1984).

[5] N.J. Hitchin, N.S. Manton and M.K. Murray, ‘Symmetric monopoles’, Nonlinearity, 8, 661 (1995).

[6] N.J. Hitchin, ‘Monopoles and geodesics’, Commun. Math. Phys. 83, 579 (1982).

[7] N.J. Hitchin, ‘On the construction of monopoles’, Commun. Math. Phys. 89, 145 (1983).

[8] C.J. Houghton and P.M. Sutcliffe, ‘Tetrahedral and cubic monopoles’, Cambridge preprint DAMTP 95-13, to appear in Commun. Math. Phys.

[9] C.J. Houghton and P.M. Sutcliffe, ‘Octahedral and dodecahedral monopoles’, Cambridge preprint DAMTP 95-20, to appear in Nonlinearity.

[10] A. Jaffe and C. Taubes, ‘Vortices and monopoles’, Boston, Birkhäuser, 1980.

[11] F. Klein, ‘Lectures on the icosahedron’, London, Kegan Paul, 1913.

[12] N.S. Manton, ‘A remark on the scattering of BPS monopoles’, Phys. Lett. 110B, 54 (1982).

[13] W. Nahm, ‘The construction of all self-dual multimonomes by the ADHM method’, in Monopoles in quantum field theory, eds. N.S. Craigie, P. Goddard and W. Nahm, World Scientific, 1982.

[14] N. Seiberg and E. Witten, ‘Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory’, Nucl. Phys. B426, 19 (1994).

[15] D. Stuart, ‘The geodesic approximation for the Yang-Mills-Higgs equation’, Commun. Math. Phys. 166, 149 (1994).

[16] R.S.Ward ‘A Yang-Mills-Higgs monopole of charge 2’, Commun. Math. Phys. 79, 317 (1981).
Fig. 2

(a) Components of the Higgs field $L^2$

(b) Energy density of the Higgs field
Fig. 3
components of the Higgs field

Fig. 4
Fig. 5