A Two Potential Formula and its Application to Proton-Proton Scattering

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Within the framework of potential scattering theory we derive an analytical two-potential formula for the on-shell partial wave scattering amplitude. This formula embodies a large number of possible applications, including long range Coulomb forces as well as short distance singular potentials. As an example illustrating the use of the formula we analyze the determination of the strong proton-proton scattering s-wave phase shift from the experimentally determined Coulomb phase when the one pion exchange and two pion exchange chiral potentials are taken into account and analyze the relevant scales of the problem.

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I. INTRODUCTION

The two-potential formalism, developed in the fifties by Watson \cite{1} and Gell-Mann and Goldberger \cite{2}, relates the scattering due to the sum of two different potentials and has a widespread use in scattering theory. The usual example is the treatment of Coulomb distortion for strongly interacting particles. The problem is to determine the total scattering amplitude \( T \) from a potential constructed as the sum of two potentials \( V = V_S + V_L \) in terms of the scattering amplitude \( T_S \) due only to the potential \( V_S \). The result can be found in a straightforward manner using the Lippmann-Schwinger equation

\[
T = V + V G_0 T ,
\]

with \( V \) the potential operator and \( G_0 = (E - H_0)^{-1} \) the resolvent of the free Hamiltonian. The outgoing boundary condition corresponds to \( E \to E + i0^+ \). The \( T \)-matrix can then be expressed as \( \textsuperscript{1} \)

\[
T = T_S + (1 + T_S G_0) \hat{T}_L (G_0 T_S + 1) ,
\]

where \( T_S \) is the short distance \( T \)-matrix, and \( \hat{T}_L \) the long distance one distorted by short range effects

\[
T_S = V_S + V_S G_0 T_S , \quad (3)
\]

\[
\hat{T}_L = V_L + V_L G_S \hat{T}_L , \quad (4)
\]

with \( G_S = G_0 + G_0 V_S G_S \) the full propagator for \( V_S \).

While the result above solves the problem, it does not explicitly relate the on-shell scattering amplitudes of the full and short distance potentials. The reason is that the Lippmann-Schwinger equation involves the off-shell behaviour of the potentials, which allows to treat nonlocal potentials quite straightforwardly, but precisely because of this it is hard to profit specifically from the simplifying features which arise in the interesting and quite frequent case of local potentials arising e.g. in a particle exchange picture. For the local case, a coordinate space formulation of the scattering problem is more convenient (an effective field theory example is provided by Ref. \cite{3}).

In this paper we derive a two-potential formula, which relates the phase shifts (i.e. the on-shell scattering matrix) of the full and short range potentials \( V \) and \( V_S \), and which is based on two assumptions: (i) the potentials are local and (ii) the short range potential dominates at short distances. Our result will be amenable to rather detailed analytical study, hence enlarging the class of situations one may cover. The connection to momentum space renormalization with counterterms of the Lippmann-Schwinger equation is also analyzed. This is particularly enlightening in the case of singular potentials and their renormalization, a subject of recent interest (see e.g. \cite{3}). As an illustrative application we show how our two-potential formula may be used to deduce the strong proton-proton phase shifts from the experimentally measured ones when long distance corrections coming from one and two pion exchange contributions are taken into account.

II. TWO POTENTIAL FORMULA

We consider the non-relativistic scattering of two particles by a spherically symmetric potential \( V \) which can be decomposed into the following two pieces

\[
V(r) = V_L(r) + V_S(r) , \quad (5)
\]
where $V_L$ and $V_S$ respectively represent the long and short distance components of the interaction. We will assume that the short range potential $V_S$ dominates at short distances $r = r_c$, i.e.
\[ V_S(r_c) \gg V_L(r_c), \] (6)
for $r_c$ small enough. The system can be described by solving the reduced Schrödinger equation (for simplicity, we only consider here the s-wave case)
\[ -u_k'' + 2\mu \left[ V_L(r) + V_S(r) \right] u_k(r) = k^2 u_k(r), \] (7)
with $u_k(r)$ the reduced wave function, $\mu$ the reduced mass of the two body system and $k$ the center-of-mass momentum. The phase shift can then be obtained by matching the reduced wave function $u_k$ to the usual asymptotic boundary condition for $r \to \infty$
\[ u_k(r) \to \cot \delta \sin kr + \cos kr, \] (8)
where we have assumed that the long range potential $V_L$ decays faster than $1/r^2$ at large distances, so phase shifts are well defined. We also consider the corresponding scattering problem for which only the short range potential $V_S$ is present. In such a case, the reduced Schrödinger equations reads
\[ -u_k^{S''} + 2\mu V_S(r) u_k^S(r) = k^2 u_k^S(r), \] (9)
with $u_k^S$ the short reduced wave function. The phase shift can be extracted from the asymptotic behaviour of $u_k^S$
\[ u_k^S(r) \to \cot \delta^S \sin kr + \cos kr, \] (10)
for $r \to \infty$.

The problem is to relate the full phase shift $\delta(k)$ with the short phase shift $\delta^S(k)$. For that purpose, we will assume that at short enough distances $r = r_c$ the full and short reduced wave function $u_k$ and $u_k^S$ are approximately equal, $u_k(r_c) \approx u_k^S(r_c)$. The previous approximation can be restated in terms of the logarithmic derivatives of the reduced wave functions
\[ \frac{u_k'(r_c)}{u_k(r_c)} = \frac{u_k^S'(r_c)}{u_k^S(r_c)}. \] (11)
This expression will hold true when the condition expressed in Eq. (6) is fulfilled. We now make use of the superposition principle to represent the full solution as the following linear combination
\[ u_k(r) = \cot \delta J_k(r) - Y_k(r), \] (12)
where $J_k$ and $Y_k$ are solutions of Eq. (7), subjected to the asymptotic conditions
\[ J_k(r) \to \sin kr, \] (13)
\[ Y_k(r) \to -\cos kr, \] (14)
for $r \to \infty$. By matching the logarithmic derivatives we arrive at our final expression
\[ \cot \delta(k) = \frac{\mathcal{A}(k, r_c) \cot \delta^S(k) - \mathcal{B}(k, r_c)}{\mathcal{C}(k, r_c) \cot \delta^S(k) - \mathcal{D}(k, r_c)}, \] (18)
where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are defined as
\[ \mathcal{A}(k, r_c) = W(J_k^S, Y_k)_{|r=r_c}, \] (19)
\[ \mathcal{B}(k, r_c) = W(Y_k^S, Y_k)_{|r=r_c}, \] (20)
\[ \mathcal{C}(k, r_c) = W(J_k^S, J_k)_{|r=r_c}, \] (21)
\[ \mathcal{D}(k, r_c) = W(Y_k^S, J_k)_{|r=r_c}, \] (22)
with $W(f, g)_{|r=r_c} = f(r_c)g'(r_c) - f'(r_c)g(r_c)$ the Wronskian between different wave functions evaluated at the cut-off radius $r = r_c$. The bilinear structure is reminiscent of the Möbius transformation invariance discussed at length in Ref. [3] in the context of the renormalization group analysis with boundary conditions. It should be noted that although the matching of log-derivatives in order to obtain long range correlations is not new, one nice example being the Landau-Smorodinsky derivation of the effective range expansion [3], or the treatment of hadronic atoms in Ref. [8], its use in combination with the superposition principle in order to derive direct relations between phase shifts is less common, and it has only been partially exploited in some effective field theory r-space computations [8, 9, 10].

In passing we also note that Eq. (18) cannot be derived from the Lippmann-Schwinger equation. The reason is that the two-potential formula depends on the explicit formulation of the following: (i) the superposition principle via Eqs. (12) and (13), and (ii) the short distance boundary condition for the Schrödinger equation, Eq. (11). These two conditions are included in the Lippmann-Schwinger equation, but implicitly, in a way that they cannot be directly handled, impeding the derivation of the previous formula (but allowing the derivations of formulas relating the full off-shell scattering amplitudes, like the two-potential trick, Eq. (2)).

The two-potential formula also holds in certain cases for non-local potentials. The necessary condition for its application is that the non-local potential does not involve derivatives of order higher than two, e.g. potentials of the type
\[ V_S^{NL} = \{ \nabla^2, f_S(r) \}, \] (23)
where $\{,\}$ represents the anti-commutator. In such a case the short distance boundary condition for the

2 We are not necessarily assuming a regular solution of the Schrödinger equation, i.e. $V_S$ can contain zero-range pieces.
Schrödinger equation can be expressed as the log-derivative of the wave function, Eq. (11). For non-local potentials involving higher derivatives, the two-potential formula can still be applied when the cut-off radius \( r_c \) is larger than the range at which the non-localities appear.

III. COULOMB SCATTERING

The case where the long range potential is of Coulomb type requires a special treatment as the usual asymptotic behaviour described in Eq. (5) does not apply. For definiteness, we analyze here the Coulomb repulsion between two unit charge particles in the s-wave. The full system is now described by the following equation

\[
-u_k^{(S)}'' + 2\mu \left[ V_S(r) + \frac{\alpha}{r} \right] u_k^{(S)}(r) = k^2 u_k^{(S)}(r),
\]

Eq. (15).

where we have added the \( C \) superscript for labelling the Coulomb solution, and \( \alpha \) represents the fine structure constant. The correct asymptotics for \( u_k^{(C)} \) is given by

\[
u_k^{(C)}(r) \rightarrow \cot \delta^C F_0(\eta, \rho) + G_0(\eta, \rho),
\]

with \( \delta^C \) the Coulomb modified phase shift, and \( F_0(\eta, \rho) \) and \( G_0(\eta, \rho) \) the usual s-wave Coulomb wave functions (see for example [10]), which depend on the parameters \( \eta = 1/(\kappa a_B) \) and \( \rho = kr; a_B = 1/(\mu \alpha) \) is the Bohr radius of the two particle system. \( F_0 \) and \( G_0 \) are solutions of the reduced Schrödinger equation for the Coulomb potential \( V_C(r) = \alpha/r \), with the asymptotic behaviour

\[
F_0(\eta, \rho) \rightarrow \sin(\kappa r - \eta \log 2kr + \sigma_0),
\]

\[
G_0(\eta, \rho) \rightarrow \cos(\kappa r - \eta \log 2kr + \sigma_0),
\]

where \( \sigma_0 \) is a phase shift defined as \( \sigma_0 = \arg \Gamma(1 + i\eta) \). As in the previous case, we can use the superposition principle to rewrite the full solution

\[
u_k^{(C)}(r) \rightarrow \cot \delta^C F_k^C(r) - G_k^C(r),
\]

with \( F_k(r) \) and \( G_k(r) \) solutions of Eq. (24) subjected to the asymptotic boundary conditions

\[
F_k^C(r) \rightarrow F_0(\eta, \rho),
\]

\[
G_k^C(r) \rightarrow -G_0(\eta, \rho).
\]

The short range system is described by Eq. (10), and the short range wave function \( u_k^{(S)} \) is again parametrized by Eq. (15).

After matching logarithmic derivatives we find

\[
\cot \delta^S(k) = \frac{A(k, r_c) \cot \delta^S(k) - B(k, r_c)}{C(k, r_c) \cot \delta^S(k) - D(k, r_c)}
\]

where \( A, B, C \) and \( D \) are now defined as

\[
A(k, r_c) = W(J_k^S, G_k^C)|_{r=r_c}
\]

\[
B(k, r_c) = W(Y_k^S, G_k^C)|_{r=r_c}
\]

\[
C(k, r_c) = W(J_k^S, F_k^C)|_{r=r_c}
\]

\[
D(k, r_c) = W(Y_k^S, F_k^C)|_{r=r_c}
\]

in complete analogy with Eq. (18). Previous relationships between Coulomb and short distance scattering can be found for some specific cases in Refs. [11, 12, 13].

A. Contact Short Range Potential

A simple application of the previous formula corresponds to a situation where the short range potential is zero for distances greater than the cut-off radius \( r_c \)

\[
V_S(r) = 0 \quad \text{for} \quad r > r_c,
\]

while, for distances shorter than \( r_c \), the potential is very strong. The previous potential corresponds to a \( \delta \)-type contact interaction regularized at the length scale \( r_c \). In such a case, the \( F_k^C(r) \) and \( G_k^C(r) \) wave functions are equal to their asymptotic behaviour for \( r \geq r_c \), and by taking into account their behaviour at small radii

\[
F_k^C(r) \rightarrow k C(\eta) \left[ r + \frac{r^2}{a_B} + O(r^3) \right],
\]

\[
G_k^C(r) \rightarrow -\frac{1}{C(\eta)} \left[ 1 + 2 \frac{r}{a_B} \left( \log r + 2 \gamma_E - 1 + h(\eta) \right) \right] + O(\eta^2),
\]

with \( \gamma_E \) the Euler-Mascheroni constant, and \( C(\eta) \) and \( h(\eta) \) defined as

\[
C^2(\eta) = \frac{2\pi \eta}{e^{2\pi \eta} - 1},
\]

\[
h(\eta) = \eta^2 \sum_{n=1}^{\infty} \frac{1}{n(\eta^2 + n^2)} - \log \eta - \gamma_E,
\]

the relationship given by Eq. (31) can be evaluated explicitly, yielding

\[
k \cot \delta^S(k) = C^2(\eta) k \cot \delta^C(k) + \frac{2 h(\eta)}{a_B}
\]

\[
- \frac{2}{a_B} \left[ \log \frac{a_B}{2a_B} - 2 \gamma_E \right] + O(r_c),
\]

where terms linear in the cut-off radius and higher powers of \( r_c \) have been ignored. As can be seen the previous expression is logarithmic divergent with respect to \( r_c \), but can be regularized if we take into account the corresponding expression for \( k = 0 \), which is similar to the well known relationship between strong and Coulomb scattering length from Blatt and Jackson [14, 15]

\[
- \frac{1}{\alpha_S} = - \frac{1}{\alpha_C} - \frac{2}{a_B} \left[ \log \frac{a_B}{2a_B} - 2 \gamma_E \right] + O(r_c).
\]

The expression above diverges in exactly the same way as Eq. (11). Subtracting the \( k = 0 \) expression to Eq. (11), and taking the \( r_c \rightarrow 0 \) limit, we arrive at the following
The expected error of this formula can be estimated reintroducing the cut-off \( r_c \) and interpreting it as the neglected range \( R_S \) of the short range potential \( V_S \), yielding a relative error of \( \mathcal{O}(r_c/a_B) = \mathcal{O}(R_S/a_B) \).

The corresponding formula for attractive Coulomb interaction may be of interest for the treatment of pionic atoms, and can be obtained by taking \( \eta = -1/(ka_B) \) negative, and making the following substitution

\[
h(\eta) \to \Re \left[ \psi(i\eta) - \log(-i\eta) \right]
\]

with \( \psi \) the digamma function.

Finally, the corresponding formula for p-wave repulsive Coulomb interaction can be worked out analogously to the s-wave case. The treatment of the divergences is nonetheless more involved: there is an additional logarithmic divergence proportional to \( k^2 \), due to the interplay between the Coulomb potential and the centrifugal barrier. The outcome is that two subtractions are needed in order to have finite results in the \( r_c \to 0 \) limit. The final formula is rather simple to summarize

\[
k^3 \cot \delta_S^p(k) + \frac{1}{\alpha_{1,S}} - \frac{1}{2} r_{1,S} k^2 = C_1^p(\eta) k^3 \cot \delta_C^p(k) + k^3 (1 + \eta^2) 2 \eta h(\eta) + \frac{1}{\alpha_{1,C}} - \frac{1}{2} r_{1,C} k^2,
\]

where \( C_1^p(\eta) = (1 + \eta^2)C^2(\eta) \), \( \alpha_{1,S} \) and \( \alpha_{1,C} \) are the p-wave short and Coulomb scattering volumes, and \( r_{1,S} \) and \( r_{1,C} \) the p-wave short and Coulomb effective ranges. The previous formula has less predictive power than the corresponding one for s-waves as a consequence of the extra subtraction needed to regularize it. A possible application is nucleon-alpha scattering [12].

### IV. APPLICATION TO PROTON-PROTON SCATTERING

Now we apply the previous results for Coulomb scattering to the specific case of proton-proton (pp) scattering. We consider the strong pp interaction as the short range potential \( V_S \), while the Coulomb repulsion between the protons plays the role of the long range potential \( V_L \).

#### A. Pionless theory

We first consider the simplifying case in which the pion exchange interactions between the protons are neglected and the pp potential consists on contact interactions only, i.e. the pionless theory, characterized by a short distance boundary condition. In such a case the two potential formula given by Eq. (13) applies. The previously mentioned relationship can be better understood by noticing the relationship with the strong and Coulomb effective range expansions [17], i.e.

\[
k \cot \delta_S = -\frac{1}{a_S} + \frac{1}{2} r_{S,k}^2 + \sum_{n=2}^{\infty} v_{n,S} k^{2n}, \quad (46)
\]

\[
k \cot \delta_C^p = 2 \frac{h(\eta)}{a_B} = \frac{1}{2} r_{C,k}^2 + \sum_{n=2}^{\infty} v_{n,C} k^{2n}, \quad (47)
\]

meaning that, with the exception of the scattering length, which explicitly depends on the regularization scale \( r_c \), see Eq. (12), the strong and Coulomb effective range parameters for pp scattering are equal in the present approximation

\[
r_S = r_C \quad \text{and} \quad v_{n,S} = v_{n,C} \quad \text{for} \quad n \geq 2,
\]

where \( r_{S,C} \) is the effective range, \( v_{2,S(C)} \) the shape parameter, etc. If we compare the previous results with the parameters obtained with the Nijmegen II potential [18], we observe a small discrepancy

\[
r_S = 2.84 \text{ fm} \quad r_C = 2.76 \text{ fm},
\]

giving a 3% relative difference between the strong and Coulomb parameters. According to the error estimation of the previous section, we should expect a relative error of \( R_S/a_B \), with \( R_S \) the range of the strong pp interaction, given by one pion exchange, \( R_S = R_{\pi^0} = 1/m_{\pi^0} \), where \( m_{\pi^0} \) the neutral pion mass, yielding the result \( R_{\pi^0}/a_B = M_p \alpha/(2m_{\pi^0}) \approx 2.5 \% \) (\( M_p \) is the proton mass) in agreement with the previous discrepancy.

The corresponding results for the strong pp phase shifts, obtained from the Coulomb pp ones for the Nijmegen II potential [18] by means of Eq. (13), are shown in Fig. [11] The agreement between the strong pp Nijmegen II phase shifts and the expected ones computed from the

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3 Instead of the Nijmegen II values, it is also possible to use the well-established experimental value for the Coulomb pp effective range \( r_C = 2.82(14) \) fm, and the model dependent strong pp one \( r_S = 2.84(4) \) fm (see Ref. [14]), although the conclusions do not change appreciably.

4 The contributions to the strong-Coulomb effective range difference from vacuum polarization [20], or from the modified Coulomb potential of Ref. [21], are expected to be much smaller than the strong (pionic) corrections. The magnetic moment interaction [22] does not contribute for s-wave proton-proton scattering.
moved, which is in agreement with the next-to-leading order calculation of Kong and Ravndal \[24\], but disagrees with the next-to-next-to-leading order results of Ref. \[27\]. It is also implied in our treatment that the pionless treatment of pp scattering can be made almost scale independent if, apart from the usual counterterms \(C_0 + C_2(p^2 + m^2) + O(p^4)\), a counterterm contribution proportional to \(\alpha\) is included in the computations, i.e. \(\alpha D_{\alpha}^2\). This observation is closely related with the results of Ref. \[28\], where the necessity of a strong and Coulomb version of \(C_0\) was discussed. The previous \(D_{\alpha}^2\) counterterm fixes the difference between the strong and Coulomb scattering length, so the price to pay in order to remove the log scale dependence in Eq. \[12\] is the impossibility to relate the two, as both scattering lengths become input parameters. This seems to be in contradiction with Kong and Ravndal \[23, 24\], who argue that the \(C_2\) counterterm stabilizes the scattering length (see also related discussions in Refs. \[27, 29, 30\]). This counterterm is analogous to the \(m^2 D_{\alpha}^2\) counterterm needed to renormalize Weinberg power counting at leading order \[31, 32\]. They are both due to the similar behaviour of the Coulomb and Yukawa potential at short distances.

Of course, these conclusions are based on our coordinate space analysis with cut-off regularization. Dimensional regularization with PDS yields different results \[27\], as Coulomb corrections to the effective range appear at next-to-next-to-leading order. These corrections depend on the off-shell behaviour of the \(O(p^4)\) counterterms, which in the cut-off approach seems to be under control as long as non-localities and off-shell ambiguities happen below \(r_c\). A more pessimistic view is presented by Gegelia in Ref. \[28\], where it is argued from the renormalization group behaviour of the counterterms in PDS that it needs to be a strong and Coulomb version of each counterterm (to absorb the \(\alpha\)-divergences), meaning that in the end it is impossible to relate strong and Coulomb observables. On the contrary, the renormalization group analysis with cut-off regularization of Birse and Barford \[33\] seems to support the idea that the Coulomb log-divergences can be absorbed in just one counterterm \[5\]. The observations of Gegelia \[28\] can be considered as an extension to any scattering observable of the results of Refs. \[34, 35\] about the difficulty of obtaining model independent strong scattering lengths from Coulomb ones due to short range ambiguities of the wave function. The previous seem to be in contradiction with usual requirement of short distance independence of physical results in effective field theory. In fact, as was shown in Ref. \[36\], further constraints on the short range ambiguities not considered in \[34, 35\] can notice-

The results of Ref. \[33\] does not exclude the existence of Coulomb corrections to all counterterms, neither do our results if supplemented by additional subtractions. It is just that they are not needed in order to have scale independent results.

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**FIG. 1:** (Upper panel) Strong pp phase shifts computed from the Coulomb pp phase shifts (Nijmegen II potential \[18\]) by using the zero range strong-Coulomb correlation of Eq. \[18\] and its corresponding extensions when including the OPE and chiral TPE (N^2LO) potentials with a cut-off radius of \(r_c = 0.1\) fm. (Lower panel) Difference between the Nijmegen phase shifts and those obtained with the strong-Coulomb correlations.

B. Comparison with other approaches

The above result may be relevant to the effective field theory (EFT) formulation of low-energy pp scattering done by Kong and Ravndal \[23, 24\] based on the power divergence subtraction (PDS) regularization scheme of Refs. \[25, 26\]. The admitted intricacy of the momentum space formalism in those works contrasts with the much shorter and transparent discussion of the coordinate space renormalization presented above. In particular, Eq. \[18\] implies that there are no Coulomb corrections to the effective range once the cut-off is removed, which is in agreement with the next-to-leading order calculation of Kong and Ravndal \[24\], but disagrees with the next-to-next-to-leading order results of Ref. \[27\]. It is also implied in our treatment that the pionless treatment of pp scattering can be made almost scale independent if, apart from the usual counterterms \(C_0 + C_2(p^2 + m^2) + O(p^4)\), a counterterm contribution proportional to \(\alpha\) is included in the computations, i.e. \(\alpha D_{\alpha}^2\). This observation is closely related with the results of Ref. \[28\], where the necessity of a strong and Coulomb version of \(C_0\) was discussed. The previous \(D_{\alpha}^2\) counterterm fixes the difference between the strong and Coulomb scattering length, so the price to pay in order to remove the log scale dependence in Eq. \[12\] is the impossibility to relate the two, as both scattering lengths become input parameters. This seems to be in contradiction with Kong and Ravndal \[23, 24\], who argue that the \(C_2\) counterterm stabilizes the scattering length (see also related discussions in Refs. \[27, 29, 30\]). This counterterm is analogous to the \(m^2 D_{\alpha}^2\) counterterm needed to renormalize Weinberg power counting at leading order \[31, 32\]. They are both due to the similar behaviour of the Coulomb and Yukawa potential at short distances.

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ably reduce the model dependence of strong parameters, in a better agreement with EFT expectations. Finally, we should also stress that we are only trying to separate strong from Coulomb corrections in non-relativistic quantum mechanics. A complete formulation on the separation of strong and electromagnetic effects is only possible in the context of quantum field theory, see Ref. [57] for a modern discussion on the subject.

We should nonetheless remember that cut-off regularization is a physical regularization, in the sense that the cut-off $r_c$ can be interpreted as a physical scale. From this point of view, the meaning of the log-divergence in the relationship between the strong and Coulomb scattering length is straightforward: it represents the expected error of the strong scattering length in the pionless approximation, which scales as $\log(R_S/a_B)$ (instead of $R_S/a_B$, as in the other parameters), yielding a very large, about $\sim 350\%$, expected relative error (to be compared with the one for the effective range $\sim 2.5\%$). One can also argue that the extra counterterm $D_\omega$ is not needed, as the inclusion of the higher order components of the potential will reduce the scale dependence.

C. Strong-Coulomb Correlations with Chiral TPE Potentials

The two potential formula makes it possible to obtain the (experimentally inaccessible) strong pp phase shifts from the (experimentally accessible) Coulomb ones. While a complete analysis should of course include vacuum polarization [20], modifications to the Coulomb potential [21] and even $2\pi \gamma$ exchange [38, 39, 40], the interesting issue is that one can obtain model independent strong phase shifts, provided that we employ a model independent strong pp potential $V_S$ and model independent Coulomb phase shifts. Here, we will do so with an eye put on the relevant scales in the problem, an aspect which our two-potential treatment can address in a rather clean way.

For the previous purpose we use the potentials of chiral perturbation theory [41] as the short distance potential $V_S$ of Eq. (31). These potentials include TPE effects and can be expressed as an expansion in powers of $Q$

$$V_S(r) = V_S^{(0)}(r) + V_S^{(2)}(r) + V_S^{(3)}(r) + \mathcal{O}(Q^4), \quad (50)$$

where $Q$ represents either the pion mass or the momentum of the protons. We also use the Nijmegen PW A [12], which is a model-independent extraction of the pp s-wave Coulomb [6] phase shifts from a large proton-proton scattering database. With that information, we can obtain the strong pp phase shift and its error from the PW A

$$\delta_{\text{PWA}}^S(k) \pm \Delta \delta_{\text{PWA}}^S \rightarrow (V_c, r_c) \rightarrow \delta^S(k) \pm \Delta \delta^S, \quad (51)$$

and analyze the resulting cut-off dependence, which is an important issue, as for large coordinate space cut-offs the higher order pieces of the chiral potential are not resolved.

It should be noted here that a complete model independent separation between strong and electromagnetic contributions is not always possible, specially if short distance electromagnetic effects are included. One example is the inclusion of nucleon form factor corrections to the magnetic moment interaction in the proton-proton PWA of Ref. [13]. Another example is proton finite size corrections to the Coulomb potential. The formalism presented here clearly separates between what we define as the strong and electromagnetic potential. That does not necessarily mean that exact model independence has been achieved, specially if corrections like the ones mentioned above are added, or that strong and electromagnetic effects have been actually separated, specially as electromagnetic corrections to the proton mass or to the coupling constants have not been included.

The specific procedure we will apply is analogous to the one followed in the pionless case, i.e., we do not directly use the strong-Coulomb two-potential formula, Eq. (34), but rather perform a subtraction of the equivalent two-potential formula for the scattering lengths, and then check for cut-off independence of the results. This choice also allows for a better comparison between the pionless correlation given in Eq. (43) and the corresponding improvements when the strong physics are included explicitly.

In the present calculation we are only going to consider the chiral potentials up to the $Q^3$ order, i.e. next-to-next-to-leading-order (N$^2$LO). At this order the finite range piece of the chiral potential consists of one-pion exchange and chiral two-pion exchange. An interesting feature of the chiral two-pion exchange potentials is that they are highly singular, diverging as $\sim 1/r^6$ at N$^2$LO. In harmony with previous findings [7, 8, 9], this divergence will become rather unimportant: the two potential formula shows a smooth cut-off dependence for singular chiral potentials [8]. In any case, we stress that our main concern is to analyze the minimal short distance cut-off $r_c$ for which

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6 It is important to notice that the pp phase shifts in the Nijmegen PW A are not Coulomb phase shifts, but electromagnetic phase shifts. By that it is meant that the pp phase shifts are defined with respect to the asymptotic solutions of the full electromagnetic potential used in their analysis, which consists on improved Coulomb, vacuum polarization and the magnetic moment interaction (see Ref. [12] for details). As our current analysis is not intended to be complete, we will ignore most of these details and simply assume that the long range potential is the usual Coulomb potential, and that the full electromagnetic phase shifts roughly coincide with the Coulomb ones, $\delta_{\text{EM}}^S(k) \approx \delta_{\text{PWA}}^S(k)$.

7 The most singular (non-contact) piece of $V^{(0)}(\nu)$ will behave as $1/r^{3+\nu}$ in coordinate space and as $|\vec{q}|^{3\nu}f(|\vec{q}|/m_a)$ in momentum space, being $\vec{q}$ the momentum exchanged between the nucleons and $f$ a non-polynomial function.

8 In fact, the singular chiral two-pion exchange potentials yield smoother results than the OPE potential. In the current regu-
which lie between the LO (OPE) and the N potential, and for compactness we skip the NLO results. The bands represent the error coming from the original results. Only the long range piece of the chiral potentials is considered, and the corresponding counterterms are ignored: they are equivalent to a boundary condition for the Schrödinger equation, and are therefore already implicitly included in the two-potential formula. We take \( g_A = 1.29 \), \( m_{\pi^0} = 134.98 \text{ MeV} \) and \( f_\pi = 92.4 \text{ MeV} \), which according to the definitions of Ref. [42] gives an \( f_{pp\pi}^2 = 0.0755 \) for the scaling mass \( m_s = m_{\pi^0} = 139.57 \text{ MeV} \). The previous chiral pp potential explicitly depend on three chiral couplings, \( c_1 \), \( c_3 \) and \( c_4 \), which appear at \( \mathcal{O}(Q^3) \) in the expansion of the potential, and which relate nucleon-nucleon and nucleon-pion scattering. We take for these chiral couplings the values obtained in Ref. [46] from analyzing the pp data alone, i.e. \( c_1 = -0.76(7) \text{ GeV}^{-1} \), \( c_3 = -4.78(11) \text{ GeV}^{-1} \), and \( c_4 = 3.92(52) \text{ GeV}^{-1} \). As can be seen, for a cut-off above \( r_c = 1.2 \text{ fm} \), one cannot distinguish, within uncertainties, between lower and higher order computations, i.e. it does not matter whether pions are included in \( V_S \) or not. Actually cut-offs below \( r_c = 0.8 \text{ fm} \) are needed in order to fully distinguish the chiral two-pion exchange contributions within the accuracy of the phase shifts. This result is not entirely surprising as could be anticipated from considering the two-pion exchange related Compton wavelength scale \( \lambda_{\gamma\gamma} = 1/2m_{\pi^0} \sim 0.7 \text{ fm} \). A more complete analysis should include vacuum polarization, 2\( \pi \gamma \) and \( \gamma\gamma \) exchange effects, which will affect the the precise values of the strong phase shifts but will hardly change the observation on the relevant scales. The same remarks also apply to the error analysis, which should include the error in the subtracted strong and Coulomb scattering lengths and the theoretical uncertainties in the chiral potential itself, like, for example, the error in the determination of the chiral couplings.

We can also compare the extracted strong effective ranges for the different cases considered. In this case, it is used the Coulomb pp phase shift from the Nijmegen II potential as input for the two-potential formula and the resulting strong phase shifts are shown in Fig. [II]. For the regularization scale \( r_c = 0.1 \text{ fm} \), we obtain

\[
\begin{align*}
    r_{S,\text{contact}} & = 2.78 \text{ fm}, \\
    r_{S,\text{OPE}} & = 2.63 \text{ fm}, \\
    r_{S,\text{TPE}} & = 2.87 \text{ fm},
\end{align*}
\]

higher order effects can be distinguished from lower order ones, rather than the specific cut-off dependence of the results.

The results for \( E_{\text{lab}} = 50 \text{ MeV} \) and \( E_{\text{lab}} = 200 \text{ MeV} \) can be seen in Fig. 2. By TPE we refer to the \( N^2\text{LO} \) chiral potential, and for compactness we skip the NLO results, which lie between the LO (OPE) and the \( N^2\text{LO} \) results. The bands represent the error coming from the original pp Coulomb phase shift in the PWA of Ref. [12]. For the polarization scheme, OPE shows a mild log-divergence at distances of \( 10^{-4} \text{ fm} \). This divergence can be eliminated by using a different, and more complex, subtraction prescription, but for the purposes of the present discussion it is not particularly important what happens at such small scales.
Finally, we note that despite the TPE potential becomes highly singular at short distances, diverging as \( \sim 1/r^6 \), nothing dramatic happens, making the limit \( r_c \to 0 \) innocuous precisely when the TPE effects become visible, i.e. for \( r_c \leq 0.8 \text{fm} \). This particular feature is a specific merit of our two potential formula which provides a clean separation between scales and implements in an extended distorted wave fashion the renormalization program carried out in previous works (see e.g. [3, 4]).

V. CONCLUSIONS

The two potential formalism provides a framework where forces of different origin and ranges may be disentangled rather explicitly. We have proposed a coordinate space formulation which re-states the result in a rather transparent way and fully exploits the boundary value character as well as the superposition principle of the scattering problem. Our result allows for a detailed investigation of the relevant scales built into the problem. This is particularly enlightening in the case of singular potentials and their renormalization, a subject of recent interest. We have exemplified our approach by discussing its consequences for the proton-proton system, where the electromagnetic and strong forces contribute to the scattering process, as a method to extract the strong phase shifts in a model independent fashion. We have only considered one channel scattering. The extension to higher partial waves, as well as coupled channels, is straightforward but cumbersome, see Appendix A. Such an extended formalism might allow to discuss further interesting applications of the present ideas to similar problems where a scale separation of different forces would be necessary.

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APPENDIX A: EXTENSION TO HIGHER PARTIAL WAVES AND COUPLED CHANNELS

1. Higher Partial Waves

The extension of our two potential formula to higher partial waves is straightforward. The full two-body system is described by the corresponding reduced Schrödinger equation for the \( l \)-wave

\[
-u''_{k,l} + \left[ 2\mu (V_L(r) + V_S(r)) + \frac{l(l+1)}{r^2} \right] u_{k,l}(r) = k^2 u_{k,l}(r),
\]

where \( u_{k,l}(r) \) is the \( l \)-wave reduced wave function, \( \mu \) the reduced mass and \( k \) the center-of-mass momentum. The asymptotics of \( u_{k,l} \) for \( r \to \infty \) is given by

\[
\lim_{r \to \infty} u_{k,l}(r) = \cot \delta_l \hat{j}_l(kr) - \hat{y}_l(kr),
\]

where \( \hat{j}_l \) and \( \hat{y}_l \) are the reduced spherical Bessel functions, defined as \( \hat{j}_l(x) = x j_l(x) \) and \( \hat{y}_l(x) = x y_l(x) \). We only consider here the case of a long range potential \( V_L \) decaying faster than \( 1/r^2 \) at large distances. By making use of the superposition principle we rewrite the full solution as

\[
\lim_{r \to \infty} u_{k,l}(r) = \cot \delta_l \hat{j}_l(kr) - \hat{y}_l(kr),
\]

where \( J_{k,l} \) and \( Y_{k,l} \) are solutions of Eq. (A1) subjected to the asymptotic boundary conditions \( J_{k,l}(r) \to \hat{j}_l(kr) \) and \( Y_{k,l}(r) \to \hat{y}_l(kr) \) for \( r \to \infty \). The corresponding scattering problem for which only the short range potential \( V_S \) is present is described by the reduced Schrödinger equation

\[
-u''_{k,l} + \left[ 2\mu V_S(r) + \frac{l(l+1)}{r^2} \right] u_{k,l}^S(r) = k^2 u_{k,l}^S(r),
\]

with \( u_{k,l}^S \) the short \( l \)-wave reduced wave function. The short distance phase shift is obtained from the asymptotic behaviour of \( u_{k,l}^S \)

\[
u_{k,l}^S(r) \to \cot \delta_l^S \hat{j}_l(kr) - \hat{y}_l(kr),
\]

for \( r \to \infty \). We rewrite \( u_{k,l}^S \) as

\[
u_{k,l}^S(r) = \cot \delta_l J_{k,l}^S(r) - Y_{k,l}^S(r),
\]

with \( J_{k,l}^S \) and \( Y_{k,l}^S \) solutions of Eq. (A4) obeying the asymptotic boundary conditions \( J_{k,l}^S(r) \to \hat{j}_l(kr) \) and \( Y_{k,l}^S(r) \to \hat{y}_l(kr) \).
As usual we match the logarithmic derivatives of $u_{k,l}(r)$ and $u^S_{k,l}(r)$ at the cut-off radius $r = r_c$, yielding

$$\cot h_l(k) = \frac{A_l(k, r_c) \cot \delta^S_l(k) - B_l(k, r_c)}{C_l(k, r_c) \cot \delta^S_l(k) - D_l(k, r_c)},$$

(A7)

where $A_l, B_l, C_l$ and $D_l$ are defined as

$$A_l(k, r_c) = W(J^S_{k,l}, Y_{k,l}) |_{r=r_c},$$

(A8)

$$B_l(k, r_c) = W(Y^S_{k,l}, Y_{k,l}) |_{r=r_c},$$

(A9)

$$C_l(k, r_c) = W(J^S_{k,l}, J_{k,l}) |_{r=r_c},$$

(A10)

$$D_l(k, r_c) = W(Y^S_{k,l}, J_{k,l}) |_{r=r_c},$$

(A11)

in analogy with the $s$-wave case. In principle the use of the previous formula is straightforward as long as a finite cut-off is employed in the computation. On the contrary, if one tries to remove the cut-off, some divergences may appear, mostly related to the centrifugal barrier. Therefore a detailed analytical or numerical study of the divergences will be necessary in order to obtain a finite result in the $r_c \to 0$ limit.

### 2. Coupled Channels

The extension to the coupled channel case is direct to obtain if an adequate notation is used. We will consider the general case of $N$ coupled channels. They can be described by the following Schrödinger equation, which in compact notation reads

$$-\mathbf{u}'' + \left[2\mu (\mathbf{V}_L(r) + \mathbf{V}_S(r)) + \frac{\mathbf{L}^2}{r^2}\right] \mathbf{u}(r) = k^2 \mathbf{u}(r),$$

(A12)

where the wave function $\mathbf{u}$ is now an $N \times N$ matrix, each column representing a linearly independent solution. The long and short range potentials $\mathbf{V}_L$ and $\mathbf{V}_S$ are also $N \times N$ matrices (the non-diagonal terms relating the different channels), and $\mathbf{L}^2$ is the angular momentum matrix, which is diagonal and given by

$$\mathbf{L}^2 = \text{diag}(l_1(l_1+1), l_2(l_2+1), \ldots, l_N(l_N+1)), (A13)$$

being $l_1, l_2, \ldots, l_N$ the orbital angular momentum of each channel. In principle there are $2N$ linearly independent solutions (two per channel), but regularity conditions at the origin reduce this number to $N$. This is why the wave function can be represented by an $N \times N$ matrix. We have also added the simplifying assumption that there are no inelasticities, meaning that in Eq. (A12) the source of the coupling is either tensor forces or dipole-dipole interactions.

The asymptotic behaviour of the wave function matrix $\mathbf{u}$ is given by the following expression

$$\mathbf{u}(r) \to \mathbf{j}(kr) \mathbf{M}(k) - \mathbf{y}(kr),$$

(A14)

for $r \to \infty$, where $\mathbf{j}$ and $\mathbf{y}$ are diagonal matrices given by

$$\mathbf{j}(kr) = \text{diag}(\hat{j}_{1}(kr), \hat{j}_{2}(kr), \ldots, \hat{j}_{N}(kr)), (A15)$$

$$\mathbf{y}(kr) = \text{diag}(\hat{y}_{1}(kr), \hat{y}_{2}(kr), \ldots, \hat{y}_{N}(kr)), (A16)$$

with $\hat{j}_l$ and $\hat{y}_l$ the reduced spherical Bessel functions as defined in the previous section. The matrix $\mathbf{M}(k)$ is the analogous of cot $\delta$ for coupled channels and is related to the $S$-matrix by $\mathbf{M}(k) = i(S(k)+1)/(S(k)-1)$ with $I$ the identity matrix. It is a symmetric matrix and contains $N(N+1)/2$ independent scattering parameters or phase shifts. By making use of the superposition principle, we can rewrite the wave function matrix as

$$\mathbf{u}(r) = \mathbf{J}(k) \mathbf{M}(k) - \mathbf{Y}(k),$$

(A17)

where $\mathbf{J}_l$ and $\mathbf{Y}_l$ are solutions of Eq. (A12) which asymptotically behave as $\mathbf{J}_l(r) \to \mathbf{j}(kr)$ and $\mathbf{Y}_l(r) \to \mathbf{y}(kr)$.

The corresponding Schrödinger equation for the short wave function is

$$-\mathbf{u}'' + \left[2\mu \mathbf{V}_S(r) + \frac{\mathbf{L}^2}{r^2}\right] \mathbf{u}(r) = k^2 \mathbf{u}(r),$$

(A18)

where, in analogy to the full case, the short wave function matrix can be written as

$$\mathbf{u}(r) = \mathbf{J}^S(k) \mathbf{M}^S(k) - \mathbf{Y}^S(k),$$

(A19)

with $\mathbf{J}^S_l$ and $\mathbf{Y}^S_l$ solutions of Eq. (A18) subjected to the asymptotic boundary condition $\mathbf{J}^S_l(r) \to \mathbf{j}(kr)$ and $\mathbf{Y}^S_l(r) \to \mathbf{y}(kr)$ for $r \to \infty$.

For obtaining the corresponding two potential formula one needs to match the logarithmic derivatives of the wave functions, which for the coupled channel case means

$$\mathbf{u}(r_c)(\mathbf{u}(r_c))^{-1} = \mathbf{u}^{S'}(r_c)(\mathbf{u}^{S'}(r_c))^{-1}.$$  

(A20)

Using the following Wronskian relation

$$\mathbf{u}^{T}(r_c)\mathbf{u}^{T'}(r_c) = \mathbf{u}^{T'}(r_c)\mathbf{u}^{T}(r_c),$$

(A21)

where the $T$ superscript denotes the transpose, the boundary condition given by Eq. (A20) can be rewritten as

$$\mathbf{u}^{T}(r_c)\mathbf{u}^{T'}(r_c) = \mathbf{u}^{T'}(r_c)\mathbf{u}^{S'}(r_c),$$

(A22)

an expression which does not involve the inverse of the wave functions. If we rewrite $\mathbf{u}$ and $\mathbf{u}^S$ in terms of Eq. (A17) and (A19), we arrive at our final expression

$$\mathbf{M}(k) = (\mathbf{A}(k, r_c) \mathbf{M}^S(k) - \mathbf{B}(k, r_c)) \times$$

$$\left[\mathbf{C}(k, r_c) \mathbf{M}^S(k) - \mathbf{D}(k, r_c)\right]^{-1}$$

(A23)

10 The reason why we write $\mathbf{J}_l(k)$ instead of $\mathbf{M}(k)$ in Eq. (A17) is because if $\mathbf{u}$ is a solution to the Schrödinger equation (A12) and $\mathbf{A}$ a constant $N \times N$ matrix, then $\mathbf{u} \mathbf{A}$ is also a solution (but this is not the case for $\mathbf{u} \mathbf{A}$).
with $A$, $B$, $C$ and $D$ defined as

\begin{align}
A(k, r_c) &= -W(Y_k^T, J_k^S)\big|_{r=r_c}, \\
B(k, r_c) &= -W(Y_k^T, Y_k^S)\big|_{r=r_c}, \\
C(k, r_c) &= -W(J_k^T, J_k^S)\big|_{r=r_c}, \\
D(k, r_c) &= -W(J_k^T, Y_k^S)\big|_{r=r_c},
\end{align}

where the Wronskian is given by $W(F, G)|_{r_c} = F'(r_c)G(r_c) - F(r_c)G'(r_c)$.

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