ON SOFIC APPROXIMATIONS OF PROPERTY (T) GROUPS

GÁBOR KUN

Abstract. We prove Bowen’s conjecture that every sequence of finite graphs that locally converges to the Cayley graph of a countably infinite group with Kazhdan Property (T) is essentially a vertex-disjoint union of expander graphs. We characterize graph sequences that are essentially a vertex-disjoint union of expander graphs in terms of the Markov operator. We show that a sequence of 2-dimensional complexes that locally converges to the Cayley complex of a finitely presented Property (T) group is not 1-hyperfinite: This gives an alternative to the construction of Freedman and Hastings.

1. Introduction

A sequence of finite graphs is locally convergent if for every \( r \) the isomorphism class of a rooted \( r \)-ball centered at a vertex chosen uniformly at random converges in distribution. In particular, a sequence of finite (labeled) graphs converges to a Cayley graph of a finitely generated group if for every \( r \) the portion of vertices such that the \( r \)-ball centered at this vertex is isomorphic to the \( r \)-ball in the Cayley graph goes to one. Our main result is the proof of a conjecture of Lewis Bowen [5]. We prove the following.

**Theorem 1.** Let \( \Gamma \) be a countably infinite Property (T) group and \( \{G_n\}_{n=1}^{\infty} \) a sequence of finite, bounded degree graphs that locally converges to a Cayley graph of \( \Gamma \). Then there exists a \( \gamma > 0 \) and a sequence of finite \( d \)-regular graphs \( \{G'_n\}_{n=1}^{\infty} \) such that

1. \( V(G_n) = V(G'_n) \)
2. \( \lim_{n \to \infty} \frac{|E(G_n) \Delta E(G'_n)|}{|V(G_n)|} = 0 \)
3. For every \( n \) the graph \( G'_n \) is a vertex-disjoint union of \( \gamma \)-expander graphs.

This research was supported by Marie Curie IIF Fellowship Grant 627476, by ERC Consolidator Research Grant No. 648017 and by National Research, Development and Innovation Office (NKFIH) Grant ERC-HU-15 118286.
A finitely generated group is called sofic if for any of its labeled Cayley graphs there is a sequence of finite labeled graphs that locally converges to the Cayley graph. (Note that if this holds for one Cayley graph of a group then it will hold for any.) Sofic groups were introduced by Gromov [11], see also Weiss [17]. Many classical conjectures are known to hold for the class of sofic groups: Gottschalk’s conjecture (Gromov [11]), Kaplansky’s direct finiteness conjecture (Elek, Szabó [6]) and Connes’ embedding conjecture (Elek, Szabó [7]). It is a major open problem if every group is sofic, though it is widely believed that non-sofic groups exist. Theorem 1 is expected to be a useful tool in the solution of this fundamental problem. See Pestov [14] for a survey on sofic groups.

Our theorem reveals a highly unexpected graph theoretical phenomenon: The local statistics of a sparse graph may (robustly) witness expansion properties. So far in every known case when the local statistics determines the global behavior of a sparse graph the graph is hyperfinite. (A sequence of finite graphs is hyperfinite if for every \( \varepsilon > 0 \) there is a \( k > 0 \) such that by removing \( \varepsilon |V(G)| \) edges all, but finitely many graphs in the sequence can be broken into components of size at most \( k \).) For example, every minor-closed class is known to be hyperfinite (Benjamini, Schramm, Shapira [4]), and a major result of Schramm states that every sequence of graphs that locally converges to a Cayley graph of an amenable group is hyperfinite [16]. Theorem 1, our main result is a natural counterpart of Schramm’s theorem.

Our theorem is the first one for sparse graphs that implies quasirandom global structure under local conditions. The notion of quasirandomness is at the heart of Szemerédi’s regularity lemma and the limit theory of dense graphs: Note that for dense graphs quasirandomness can be implied by local conditions. See the book of Lovász on graph limits for the details [12].

Graph sequences that are essentially a vertex-disjoint union of expander graphs (with uniform expansion) can not be characterized by local conditions: Graphs with large girth may be expander graphs but may also be very far from a disjoint union of expander graphs. (Here we consider the so-called edit distance: The edit distance of two graphs on the same set of vertices is the size of the symmetric difference of the edge sets.)

However, we can give a characterization in terms of the Markov operator. \( M \) denotes the Markov operator, and \( \| \cdot \| \) denotes the \( L^2 \) norm with respect to the uniform probability distribution. Theorem 2 is the main tool in the proof of Theorem 1.
Theorem 2. Let $\{G_n\}_{n=1}^{\infty}$ be a sequence of $d$-regular graphs. Then the following are equivalent:

1. There exists an $\varepsilon > 0$ such that for every $\delta > 0$ for all, but finitely many $n$ and for every function $f : V(G_n) \rightarrow [0; 1]$ the inequality $\|M^2f - Mf\| \leq (1 - \varepsilon}\|Mf - f\| + \delta$ holds.
2. There exists a $\gamma > 0$ and a sequence of $d$-regular graphs $\{G'_n\}_{n=1}^{\infty}$ such that $V(G_n) = V(G'_n)$, $\lim_{n \rightarrow \infty} \frac{|E(G_n) \Delta E(G'_n)|}{|V(G_n)|} = 0$ and every $G'_n$ is a vertex-disjoint union of $\gamma$-expanders.

Note that if we had no $\delta$ in (1) then $G_n$ would be a vertex-disjoint union of expander graphs with second eigenvalue less than $(1 - \varepsilon)$.

Here we have to make clear the connection of our decomposition to similar decompositions used to attack Khot’s Unique Games Conjecture: In order to improve the best known algorithm [1] for the Unique Games Conjecture one needs only a positive portion of the edges to be the edge set of a disjoint union of expander graphs, while the expanders in our decomposition consist of essentially every edge, but such decompositions exist for a very special class of graphs only.

Theorem gives an ergodic decomposition theorem for certain non-separable probability measure spaces with an invariant group action: Given a sequence of finite labeled graphs that locally approximates the labeled Cayley graph of $\Gamma$, the ultraproduct of these graphs will admit an almost free, measure-preserving $\Gamma$-action. The support of almost every ergodic measure in the decomposition of the ultraproduct space will be almost an ultraproduct of expanders. Hence these supports can be disjoint in the decomposition. See [5, 8] on ultraproducts of graphs.

We apply Theorem to study embeddings of graph sequences converging to the Cayley graph and sequences of 2-dimensional CW complexes converging to the Cayley complex of a Property (T) group. An $r$-ball in a CW complex centered at a 0-cell $x$ will be the subcomplex spanned by the 0-cells at distance at most $r$ from $x$, where the distance is the graph distance in the 1-skeleton. A sequence of CW complexes is locally convergent if for every $r$ the isomorphism class of a rooted $r$-ball centered at a 0-cell chosen uniformly at random converges in distribution.

Theorem 3. Consider a sequence of finite CW complexes $\{X_n\}_{n=1}^{\infty}$ that converges to a Cayley complex $\mathcal{C}$ of a countably infinite Property (T) group. And consider a sequence of finite 1-dimensional CW complexes $\{Y_n\}_{n=1}^{\infty}$ and continuous mappings $\{f_n : X_n \rightarrow Y_n\}_{n=1}^{\infty}$. Then for every
If \( n \) is large enough then there exists a 1-cell in \( X_n \) whose image intersects the image of at least \( K \) other 1-cells.

This is an alternative construction of 2-dimensional non-hyperfinite simplicial complexes in the sense of Freedman and Hastings [9]. Informally speaking, a 2-dimensional simplicial complex is 1-hyperfinite if for every \( \varepsilon > 0 \) it admits after the removal of an \( \varepsilon \)-portion of the 0-cells and the higher dimensional cells containing these a continuous mapping to a 1-dimensional complex such that the pre-image of every point has bounded diameter (depending on \( \varepsilon \)).

**Corollary 4.** Consider a sequence of finite CW complexes that converges to a Cayley complex of a finitely presented infinite Property (T) group, and a sequence \( \{X_n\}_{n=1}^{\infty} \) of simplicial complexes obtained by the subdivision of the 2-faces of the CW complexes into a bounded number of simplices. Then \( \{X_n\}_{n=1}^{\infty} \) is not 1-hyperfinite.

Theorem 3 has a surprising graph theoretical corollary, too. This answers a question of L. M. Lovász [13].

**Corollary 5.** Consider a sequence of finite graphs \( \{G_n\}_{n=1}^{\infty} \) that converges to a Cayley graph corresponding to a finite presentation of a Property (T) group \( \Gamma \). The sequence \( \{G_n\}_{n=1}^{\infty} \) admits no L-Lipschitz embedding for any \( L \) into a sequence of graphs with bounded degree and girth greater than \( rL \), where \( r \) is the length of the shortest relation in the presentation of \( \Gamma \).

**Proof.** Consider a sequence of large girth graphs, an integer \( L \) and a sequence of L-Lipschitz mappings. Let \( \{X_n\}_{n=1}^{\infty} \) denote the following sequence of 2-dimensional CW-complexes: The 1-skeleton of \( X_n \) is \( G_n \) and for every cycle of length at most \( r \) there is a 2-cell on this cycle. Let \( \{Y_n\}_{n=1}^{\infty} \) denote the sequence of 1-dimensional CW-complexes. And let \( \{f_n : X_n \rightarrow Y_n\}_{n=1}^{\infty} \) a sequence of continuous mappings extending the sequence of the L-Lipschitz mappings: Every 1-cell of \( X_n \) will be mapped to the shortest path connecting the image of its two 0-subcells. The image of every short cycle will be null-homotopic by the girth condition, hence the mapping can be extended to 2-cells. Theorem 3 can be applied: The theorem follows, since the degrees are bounded.

\[\square\]

\(^1\)An ineffective version of the corollary can be obtained using that Property (T) groups admit no treeable almost free action [2] and that every subrelation (refinement) of a treeable relation is treeable [10]. This holds also in the case when \( \Gamma \) is not finitely presented.
In particular, a sequence of graphs that locally converges to a Property (T) group admits no coarse embedding into any large girth sequence: This gives a new proof of Theorem 9.1. of Mendel and Naor [15].

2. Definitions

We deal with sequences \( \{G_n\}_{n=1}^\infty \) of finite, undirected, \( d \)-regular graphs. Finite graphs will be equipped with the uniform measure, where the measure of every vertex is \( 1/|V(G)| \). Given a subset \( S \subseteq V(G) \) we denote by \( |S| \) the measure of \( S \). Let \( \| \cdot \|_2 \) and \( \| \cdot \|_1 \) denote the \( L_2 \) and \( L_1 \) norms, respectively. \( M_G \) will denote the Markov operator of the graph \( G \). We will usually drop the subscript. We say that a \( d \)-regular, finite graph \( G \) is a \( \gamma \)-expander if for every \( S \subseteq V(G) \), where \( |S| \leq |V(G)|/2 \) we have
\[
|E(S, V(G) \setminus S)| \geq \gamma d |S|.
\]
We say that the sequence \( \{G_n\}_{n=1}^\infty \) is essentially a disjoint union of expanders if every \( G_n \) can be turned into a vertex-disjoint union of \( d \)-regular \( \delta \)-expander graphs on the same set of vertices after removing and adding \( o(|V(G_n)|) \) edges.

Given an integer \( r \) and a graph \( G \) we will consider the following probability distribution on isomorphism classes of rooted graphs: Pick a vertex uniformly at random and consider the isomorphism class of the rooted \( r \)-ball centered and rooted at \( x \). We call this sequence of probability distributions the local statistics of \( G \). A sequence of graphs is locally convergent if this probability measure converges in distribution for every \( r \).

A graphing is a graph equipped with an extra measurable structure, a generalization of finite graphs (with the uniform measure). The vertex set of a graphing is a probability measure space, the edge set is measurable in the product measure, and the edge set of a graphing is a countable union of measure-preserving, partially defined mappings. The notion of local statistics and local convergence extends to graphings. Every sequence of bounded degree graphs admits a locally convergent subsequence. The most convenient limit objects are the weak limit and the ultraproduct. These are graphings. See Lovász [12] on limits of graphs. The edges of the graph may have an asymmetric labeling by the generating set of a (finitely generated) group. Every unlabeled graph will be undirected in this paper. The notion of local statistics and local convergence extends to labeled graphs and graphings.

We define the local statistics for finite CW complexes. Given a CW complex \( X \), a 0-cell \( x \) and an integer \( r \) the ball \( B(x, r) \) will be the following CW complex: the set of 0-cells will be the set of 0-cells in \( X \) at distance at most \( r \) from \( x \), where we consider the graph distance in
the 1-skeleton. And $B(x, r)$ will be the subcomplex spanned by these 0-cells: A cell of $X$ will be in $B(x, r)$ if all of its 0-subcells are in $B(x, r)$. Given an integer $r$ and a CW complex $X$ we will consider the following probability distribution on isomorphism classes of rooted graphs: Pick a 0-cell uniformly at random and consider the isomorphism class of the rooted $r$-ball centered and rooted at $x$. A sequence of CW complexes is locally convergent if this probability measure converges in distribution for every $r$.

Given a finitely generated group with its presentation the Cayley complex is a 2-dimensional CW complex: The 1-skeleton of the Cayley complex is the Cayley graph, and there are 2-cells for every cycle that belongs to a relation in the presentation. In other words, the Cayley complex is the universal cover of the presentation complex.

We say that the finitely generated group $\Gamma$ has Kazhdan Property (T) if there is a finite set of generators $S$ and an $\varepsilon > 0$ such that for every Hilbert space $\mathcal{H}$ and $\pi : \Gamma \to U(\mathcal{H})$ unitary representation of $\Gamma$ either $\pi$ has a non-zero, invariant vector, or for $A = \sum_{s \in S} \pi(s)/|S|$ the inequality $\|Ah\| \leq (1 - \varepsilon)\|h\|$ holds for every $h \in \mathcal{H} \setminus \{0\}$. See the book of Bekka, de La Harpe and Valette on Property (T) [3]. We will use the following consequence of the Kazhdan Property.

**Lemma 6.** Let $\Gamma$ be a finitely generated group with Kazhdan Property (T) with respect to $\varepsilon > 0$ and a finite set of generators $S \subseteq \Gamma$. Let $\mathcal{H}$ be a Hilbert space, $\pi : \Gamma \to U(\mathcal{H})$ a unitary representation of $\Gamma$. Set $A = \sum_{s \in S} \pi(s)/|S|$. Then for every $h \in \mathcal{H}$ the inequality $\|A^2h - Ah\| \leq (1 - \varepsilon)\|Ah - h\|$ holds.

**Proof.** Consider the set of fixed points $F = \{h \in \mathcal{H} : \pi(\gamma)h = h \ \forall \gamma \in \Gamma\}$. $F$ is a closed subspace of $\mathcal{H}$. The orthogonal complement of $F$, $F^\perp$ is a closed subspace invariant under $\pi(\gamma)$ for every $\gamma$, since $\pi$ is a unitary representation. Hence $F^\perp$ is invariant under $A$. These yield that $(Ah - h) \in F^\perp$ for every $h \in \mathcal{H}$.

The restriction of the representation $\pi$ to $F^\perp$ induces a representation that does not have any fixed point but 0, hence $\|Ag\| \leq (1 - \varepsilon)\|g\|$ holds for every $g \in F^\perp$. We conclude that $\|A^2h - Ah\| = \|A(Ah - h)\| \leq (1 - \varepsilon)\|Ah - h\|$. □

3. The proof of Theorem [2]

**Lemma 7.** Let $G$ be a finite $d$-regular graph and $M$ its Markov operator. Consider the set $S \subseteq V(G)$ and the function $f : V(G) \to [0; 1]$. Assume that $\|f\|_1 = |S|$ and $\|f - \chi_S\| < \frac{|S|^\frac{1}{d}}{6}$. Then there exists a
set of vertices $U \subseteq \text{supp}(f)$ such that $|U| < 2|S|$, $|U \cap S| > \frac{3|S|}{4}$ and

$$||\chi_U - M\chi_U||_1 \leq 4d^\frac{1}{2}72^\frac{1}{2}|S|^\frac{1}{2}||f - Mf||^\frac{1}{2}.$$  

Proof. Pick a number $t \in I = (\frac{1}{2}; \frac{3}{4})$ uniformly at random and set

$U_t = \{v \in V(G) : f(v) > t\}$. The size of $U_t$ is $< 2|S|$ for every $t \in I$, since $f(u) > \frac{1}{2}$ for every $u \in U_t$ and $||f||_1 = |S|$.

We show that the size of $U_t \cap S$ is at least $\frac{|S|}{2}$: For every $t \in I$ and every $u \notin U_t$ the inequality $f(u) < \frac{2}{3}$ holds. Hence $|S \setminus U_t| > |S| - |S| = \frac{|S|}{2}$.

We will use for every $t \in I$ that

$$||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1 \geq \frac{1}{d} \sum_{x,y} m_{xy} f(x) > f(y), (x,y) \in E(G) (t - f(y)).$$

Integrate this over $I$ in order to get

$$|I||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1 \geq \frac{1}{d} \int_{\{x,y \in G \}} (f(x) - f(y)) \cap I^2/2.$$  

Finally, the expected value of $||\chi_{U_t} - M\chi_{U_t}||_1$ is at most

$$\sum_{(x,y) \in E(G)} \frac{|f(x) - f(y)| \cap I}{d|I|} \leq \frac{2}{d|I|} \sum_{f(x) > \frac{1}{2}, (x,y) \in E(G)} |f(x) - f(y)| \cap I^2 \leq$$

$$\frac{2}{d|I|} (2|S|d^\frac{1}{2} (2d|I||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1) \frac{1}{2} =$$

$$\frac{2}{d|I|} (2|S|d^\frac{1}{2} (2d|I||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1) \frac{1}{2} \leq$$

$$\frac{4|S|^\frac{1}{2}d^\frac{1}{2}}{|I|^\frac{1}{2}} ||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1 \leq$$

$$\frac{4|S|^\frac{1}{2}d^\frac{1}{2} ||(f - Mf)\chi_{\{x : f(x) > \frac{1}{2}\}}||_1 \frac{1}{2} \leq \frac{4|S|^\frac{1}{2}2^\frac{1}{2}d^\frac{1}{2}}{|I|^\frac{1}{2}} ||f - Mf||^\frac{1}{2} =$$

$$4d^\frac{1}{2}72^\frac{1}{2}|S|^\frac{1}{2} ||f - Mf||^\frac{1}{2}.$$  

There exists a $t \in I$ that the required inequality holds for $U = U_t$.  

Lemma 8. Let $G$ be a finite $d$-regular graph, $k > 0$ an integer, $0 < c < c' < 1$ and $\alpha, \delta > 0$. Assume that for every $S \subseteq V(G)$ the inequality $||M^{k+1}\chi_S - M^{k}\chi_S|| < c||M\chi_S - \chi_S|| + \delta$ holds. Then there exists a set of vertices $B \subseteq V(G)$ of size $\leq \frac{\alpha^2 d^2}{\delta^2 (c - c')^2} (d^{2k+2} + 1)$ such that for every $S \subseteq V(G) \setminus B$ either $||M\chi_S - \chi_S|| < \alpha||\chi_S||$ or $||M^{k+1}\chi_S - M^{k}\chi_S|| < c'||\chi_S||$ holds.

Proof. Let $B'$ be a maximal subset of $V(G)$ under containment such that $||M^{k+1}\chi_{B'} - M^{k}\chi_{B'}|| \geq c'||M\chi_{B'} - \chi_{B'}||$ and $||M\chi_{B'} - \chi_{B'}|| \geq \alpha||\chi_{B'}||$. Set $B = N_{2k+2}(B')$. 


First, let $S \subseteq V(G) \setminus B$ and consider the set $S \cup B'$. Note that 
\[ \|M^{k+1}\chi_{S \cup B'} - M^k\chi_{S \cup B'}\|^2 = \|M^{k+1}\chi_S - M^k\chi_S\|^2 + \|M^{k+1}\chi_{B'} - M^k\chi_{B'}\|^2, \]
and 
\[ \|M\chi_{S \cup B'} - \chi_{S \cup B'}\|^2 = \|M\chi_S - \chi_S\|^2 + \|M\chi_{B'} - \chi_{B'}\|^2, \]
since $S$ and $B'$ are far from each other. The choice of $B'$ implies either $\|M\chi_S - \chi_S\| < \alpha\|S\|$ or $\|M^{k+1}\chi_S - M^k\chi_S\| < c'\|M\chi_S - \chi_S\|$.

Now we prove the required upper bound on $|B|$. Note that $\|\chi_B\| \leq (d^{2k+2} + 1)\|\chi_{B'}\|$. We know that $\|M^{k+1}\chi_{B'} - M^k\chi_{B'}\| < c\|M\chi_{B'} - \chi_{B'}\| + \delta$. On the other hand, $\|M^{k+1}\chi_{B'} - M^k\chi_{B'}\| \geq c\|M\chi_{B'} - \chi_{B'}\|$ and $\|M\chi_{B'} - \chi_{B'}\| \geq \alpha\|\chi_{B'}\|$. We can conclude that $\alpha(c' - c)\|\chi_{B'}\| \leq \delta$, the lemma follows. 

**Proof.** (of Theorem 2) The implication $(2) \rightarrow (1)$ is straightforward: We know the inequality $\|M^2f - Mf\| \leq (1 - \varepsilon)\|Mf - f\|$ if $\varepsilon > 0$ is smaller than the eigenvalue gap, while $\delta$ will bound the error term coming from edges added and removed. Now we prove $(2) \rightarrow (1)$. The following straightforward consequence of $(1)$ will be handy:

**Claim 1:** For every $\delta > 0$ and integer $k > 0$ for all, but finitely many $n$ and every $S \subseteq V(G_n)$ the inequality $\|M^{k+1}\chi_S - M^k\chi_S\| \leq (1 - \varepsilon)^k\|M\chi_S - \chi_S\| + \delta$ holds.

**Claim 2:** For every $\alpha, \beta > 0$, integer $k > 0$ and $c' > (1 - \varepsilon)^k$ for all, but finitely many $n$ there exists $B_n^k \subseteq V(G_n)$ such that $|B_n^k| < \beta$ and for every $S \subseteq V(G_n) \setminus B_n^k$ either $\|\chi_S - M\chi_S\| < \alpha\|\chi_S\|$ or the inequality $\|M^{k+1}\chi_S - M^k\chi_S\| \leq c'\|M\chi_S - \chi_S\|$ holds.

**Proof.** The claim follows by Lemma 3.

**Claim 3:** There exists a $\gamma > 0$ such that for every $\delta > 0$ for all, but finitely many $n$ there exists a partition of $V(G_n) = \bigcup_{i=1}^{a_n} P_{n,i}$ such that 
\[ \sum_{i=0}^{a_n} \|M\chi_{P_{n,i}} - \chi_{P_{n,i}}\|_1 < \delta, \|P_{n,0}\| < \delta \]
and for every $1 \leq i \leq a_n$ and $S \subseteq P_{n,i}$, where $|S| \leq |P_{n,i}|/2$ the inequality $\|M\chi_S - \chi_S\|_1 \geq \gamma\|\chi_S\|_1$ holds.

**Proof.** Set $\gamma = \frac{\varepsilon^2}{36d}$. Choose $\alpha, \beta > 0$ and a positive integer $k$ later. The following process will give the required partition of $V(G_n)$. Set $P_{n,0} = B_n^k$, and for $i \geq 0$ proceed as follows: If there is a set $S \subseteq V(G_n) \setminus \bigcup_{j=0}^{i} P_{n,j}$ such that $\|M\chi_S - \chi_S\|_1 < \gamma|S|$ then consider such a set of minimum size. If $\|M\chi_S - \chi_S\|_1 < \alpha|S|$ then set $P_{n,i+1} = S$, else apply Lemma 7 to $S$ and $f = M^k\chi_S$. Both conditions of the lemma hold: clearly $\|f\|_1 = \|\chi_S\|_1$. And $\|\chi_S - M^k\chi_S\| \leq \sum_{i=1}^{k} \|M^{i-1}\chi_S - M^i\chi_S\| <
\[ \| \chi_S - M\chi_S \| / \varepsilon \leq \left( \| \chi_S - M\chi_S \| d \right)^{\frac{1}{2}} < \frac{(\gamma|d|)^{\frac{1}{2}}}{\varepsilon} = \frac{|S|^\frac{1}{2}}{6}. \] (The previous last inequality follows from the fact that on its support \(|\chi_S - M\chi_S|\) is between 1/d and 1.) This yields the desired bound.

We obtain a set \( U \): set \( P_{n,i+1} = U \setminus \cup_{j=0}^i P_{n,j} \). The expansion condition holds for the subsets of \( P_{n,i+1} \), since \(|P_{n,i+1}| < 2|S|\), and \( S \) is a subset of \( V(G_n) \setminus \cup_{j=0}^i P_{n,j} \) of minimum size with small expansion. In what follows we denote \( S \) by \( S_{n,i+1} \) and \( U \) by \( U_{n,i+1} \). The total expansion of the sets in the partition is

\[ \sum_{i=0}^{n_a} \| M\chi_{P_{n,i}} - \chi_{P_{n,i}} \| \leq 2\beta + 2 \sum_{i=1}^{a_n} \max \{ \alpha|P_{n,i}|, 4|S_{n,i}| \} \left( \| \chi_{S_{n,i}} - M\chi_{S_{n,i}} \| \right)^{\frac{1}{2}}. \]

The first inequality uses the bound on \( P_{n,0} \), Lemma \( \Box \) and the fact that even though \( P_{n,i} \) may have large boundary than \( U_{n,i} \), the extra edges belong to the boundary of \( P_{n,j} \) for a \( j < i \), hence we can count this distribution to the boundary of \( P_{n,j} \) by a loss of a multiplicative factor 2. The second inequality holds since \(|S_{n,i}| < 2|P_{n,i}|, |\chi_{S_{n,i}} - M\chi_{S_{n,i}}|\) is at most 1 and its \( L_1 \) norm is at most \( 4|P_{n,i}| \). The right hand side is less than \( \delta \) if \( \alpha, \beta \) and \( c' > (1 - \varepsilon)^k \) in Claim 2 are small enough. \( \Box \)

**Claim 4:** There is a \( \gamma' > 0 \) such that for every \( \alpha > 0 \) for all, but finitely many \( n \) the graph \( G_n \) can be turned into a vertex-disjoint union of \( \gamma' \)-expanders by adding and removing an \( \alpha \)-portion of the edges.

**Proof.** If \( d \) is odd then we may assume that every class \( P \) has even size. We can get such a partition slightly changing the classes (by a single vertex at most) and adding and removing a small number of edges. After these changes the conditions of Claim 3 will still hold. The constants may get worse: Denote the new \( \gamma \) by \( \gamma_0 \).

First we remove every edge between different classes. We will make a surgery on (almost) every class separately, while the edge set of a few classes will be simply replaced by an expander. For example, \( P_{n,0} \) will be replaced.

Consider a class \( P \). Let \( B \) denote the set of vertices in \( P \) adjacent to a vertex not in \( P \). Set \( r = 6/\gamma_0 \). Find a matching \( \mathcal{M} \) in \( P \) such that the distance of any two edges in \( \mathcal{M} \) is greater than \( 2r \), no vertex of \( B \) is adjacent to the an endpoint of an edge in \( \mathcal{M} \), and the size of \( \mathcal{M} \) equals to the half of the number of the edges leaving \( P \). If there is no such \( \mathcal{M} \) than \(|E(P, P^c)|\) is too large: This can hold for very few \( P \) (o(|V(G_n)|))
in total size) only: the edge set of every such $P$ will be replaced by an expander. Now assume that such a matching $\mathcal{M}$ exists and let $M$ denote the set of vertices covered by $\mathcal{M}$. Consider a matching $\mathcal{N}$ between the vertices of $B$ with multiplicity $d$ minus their degree, i.e., the number of its neighbors not in $P$ and $M$. Let $Q$ denote the following graph: $V(Q) = P$, $E(Q) = \mathcal{N} \cup \{(x, y) \in E(G) : x, y, P \in P \} \setminus \mathcal{M}$.

$Q$ is $d$-regular. We may assume that $Q$ has diameter at least $2r$: Otherwise we have an expansion constant (possibly small). We prove that $Q$ is a $\gamma' = \frac{2\gamma_0}{d} = \frac{1}{dr}$-expander. Consider a subset of vertices $S \subseteq P$ of size at most $|P|$. If $|S \cap (B \cup M)| < 2\gamma_0|S|/3$ then $|E_Q(S, P \setminus S)| \geq |E_{G_n}(S, V(G_n) \setminus S)| - |S \cap (B \cup M)| \geq \gamma_0d/3|S|$. 

If $|S \cap (B \cup M)| \geq 2\gamma_0|S|/3$ and $|S \cap M| \leq \gamma_0|S|/3$ then $|E_Q(S, P \setminus S)| \geq |E_Q(S, P \setminus S) \cap \mathcal{N}| \geq |S \cap B| - |S \cap M| \geq \gamma_0|S|/3$: 

Finally, assume that $|S \cap M| > \gamma_0|S|/3$. Consider the ball $B(x, r)$ for every $x \in S \cap M$: At least half of these balls contains less than $r$ vertices of $S$, and hence at least one edge in $|E_Q(S, P \setminus S)|$. We can conclude that $|E_Q(S, P \setminus S)| \geq \gamma_0|S|/6$. 

\[\square\]

4. The proof of Bowen’s conjecture

Consider a sequence of functions $\{f_n : V(G_n) \rightarrow [0; 1]\}_{n=1}^{\infty}$. We suffice to show by Theorem 2 that there is an $\varepsilon > 0$ that for every $\delta > 0$ and for every sequence of functions $\{f_n : V(G_n) \rightarrow [0; 1]\}_{n=1}^{\infty}$ the inequality $\|M^2f_n - Mf_n\| \leq (1 - \varepsilon)\|Mf_n - f_n\| + \delta$ holds for all, but finitely many $n$. We choose $\varepsilon > 0$ such that Lemma 4 holds. We will prove by contradiction. Suppose that the inequality fails for infinitely many $n$. We may suppose that it fails for every $n$.

Consider the graph sequence $\{G_n\}_{n=1}^{\infty}$ equipped with the function $\{f_n\}_{n=1}^{\infty}$, and a weak limit of these. We may suppose that the limit exists. This is a graphing on a probability measure space with a function. The connected component of almost every point is isomorphic to the Cayley graph of $\Gamma$: Let $X$ denote the probability measure space consisting of these points. We do not have a natural $\Gamma$-action on $X$, hence we will consider the following product space. Let $Y$ denote the set of automorphisms of the rooted Cayley graph: this is a probability measure space, too. The elements of $Y$ are in one-to-one correspondence with a (consistent) labeling of the Cayley graph. The product space $Z = X \times Y$ is a probability measure space. Consider the following graph on $Z$: the vertices $(x, y)$ and $(x', y')$ are adjacent if $x$ is adjacent to $x'$ and the labeling $y$ can be obtained from the labeling $y'$ by moving the root $x'$ to $x$. Every connected component of this graphing
is isomorphic to the Cayley graph, and the labeling induces a natural, measure-preserving Γ-action on \( Z \), and hence on \( L^2(Z) \).

Let \( f \) denote the weak limit of the functions \( f_n \) on \( X \), this naturally induces a function \( f^* \) on \( Z \). We know by Lemma 6 that for our set of generators and the corresponding \( \varepsilon > 0 \) for every such probability measure space \( Z \), every measure-preserving, free action of \( \Gamma \) on \( Z \) and \( g \in L^2(Z) \) the inequality \( \| M^2g - Mg \| \leq (1 - \varepsilon)\| Mg - g \| \) holds, where \( M \) denotes the “average of the actions of the generators”. Clearly \( (Mf)^* = Mf^* \), and \( Mf \) is the weak limit of the sequence \( \{ Mf_n \}_{n=1}^\infty \). The theorem follows by choosing \( g = f^* \).

5. The proof of Theorem 3

Lemma 9. Consider a sequence of finite \( d \)-regular graphs \( \{ G_n \}_{n=1}^\infty \). Assume that \( \{ G_n \}_{n=1}^\infty \) locally converges to the Cayley graph of a finitely generated infinite group \( \Gamma \) with Property (T). Consider a positive integer \( L \), a prime \( p \) and a mapping \( \varphi_n : E(G_n) \to \{-L, \ldots, L\} \subseteq \mathbb{Z}_p \). Assume the followings.

1. \( \varphi((x,y)) = -\varphi((y,x)) \) for every edge \( (x,y) \).
2. Given an integer \( l \) consider the sum of \( \varphi_n \) over every cycle of length \( l \). Assume that for every \( l \) the portion of cycles with nonzero sum goes to zero as \( n \) goes to infinity.
3. Given an integer \( t > 0 \) let \( s^n_0, \ldots, s^n_t \) be a random walk on \( G_n \) chosen uniformly at random. Assume that the limit distribution \( \lim_{t \to \infty} \lim_{n \to \infty} \sum_{i=1}^t \varphi_n((s^n_{i-1}, s^n_i)) \) exists, where we consider the statistical distance, and it is equal to the uniform distribution on \( \mathbb{Z}_p \).

Then \( p \leq 1 + \frac{4L}{\gamma} \), where \( \gamma > 0 \) is given in Theorem 4.

Proof. We may assume that the sequence \( \{ G_n \}_{n=1}^\infty \) is expander. The mapping \( \varphi_n \) induces a \( p \)-fold covering of \( G_n \), denote the covering graph by \( C_n \). The vertices of the graph \( C_n \) are pairs \( (x,y) \), where \( x \in G_n \) and \( y \in \mathbb{Z}_p \), and a pair of vertices \( (x,y) \) and \( (x',y') \) is an edge if \( x \) and \( x' \) are adjacent and \( y - y' = \varphi_n((x,x')) \). Now \( \mathbb{Z}_p \) embeds into the automorphism group of \( C_n \) acting on the second coordinate. The sequence \( \{ C_n \}_{n=1}^\infty \) converges still to the same Cayley graph, this is the point where we use (2). Hence by Theorem 1 it is essentially a disjoint union of expanders. It is easy to see that it is either essentially an expander or essentially a disjoint union of \( p \) expanders isomorphic to \( G_n \), where the \( \mathbb{Z}_p \)-action is essentially permuting these subgraphs.

First suppose that (an infinite subsequence of) \( \{ C_n \}_{n=1}^\infty \) is essentially an expander sequence: there is a \( \gamma > 0 \) given by Theorem 4 such
that for every subset $S \subseteq C_n$, where $|S| < \frac{1}{2}$ we have $|E(S, S^c)| \geq \gamma d|S| + o(1)$. For every $z \in \mathbb{Z}_p$ the equality $|f^{-1}(\{z\})| = 1/p$ holds. Consider the set $S = f^{-1}(\{1, \ldots, \frac{p-1}{2}\})$. Clearly $|S| = \frac{p-1}{2p}$. Since $|E(S, S^c)| \leq \frac{2dL}{p}$, we can conclude that $p \leq 1 + \frac{4L}{\gamma}$.

Now suppose that (an infinite subsequence of) $\{C_n\}_{n=1}^\infty$ is essentially a disjoint union of $p$ expanders isomorphic to $G_n$. Consider one of these subgraphs essentially isomorphic to $G_n$. Let $f_s$ denote the restriction of $f$ to this subgraph, and consider the probability measure on it. We show that $|f_s^{-1}(\{z\})| = 1/p + o(1)$ for every $z \in \mathbb{Z}_p$. The probability that for a random walk of length $t$ we have $f(s_0) \equiv f(s_t)$ converges to $1/p$. On the other hand, for a uniform random walk of length $2t$ in our component this is equal to $\sum_{z \in \mathbb{Z}_p} ||M^t \chi_{f_s^{-1}(\{z\})}||^2 \geq \sum_{z \in \mathbb{Z}_p} |f_s^{-1}(\{z\})|^2 \geq 1/p$. The second inequality should be (asymptotically) an equality for every component, hence $|f_s^{-1}(\{z\})| = 1/p + o(1)$.

Consider the set $S = f^{-1}(\{1, \ldots, \frac{p-1}{2}\})$. Since $|E(S, S^c)| \geq \gamma d|S| + o(1)$ and $|E(S, S^c)| \leq \frac{2dL}{p} + o(1)$ we conclude again that $p \leq 1 + \frac{4L}{\gamma}$. □

Proof. (of Theorem 3) We prove by contradiction. Consider a sequence of finite CW complexes $\{X_n\}_{n=1}^\infty$ that converges to a Cayley complex $C$ of a finitely generated infinite Property (T) group, a sequence of finite 1-dimensional CW complexes $\{Y_n\}_{n=1}^\infty$, continuous mappings $\{f_n : X_n \to Y_n\}_{n=1}^\infty$ and an integer $K$. Suppose for a contradiction that the image of every 1-cell in $X_n$ intersects the image of at most $K$ other 1-cells. We think of $Y_n$ as a graph and use the graph theoretical terminology. We may assume that the image of every 0-cell is a vertex and the image of every 1-cell is a path, since we can change $f$ by a deformation to achieve these.

We choose a prime $p$ later. We assign an element of $\{-1, +1\} \subset \mathbb{Z}_p$ independently at random (with probability $\frac{1}{2}$ each) to a subset of edges of $Y_n$: We will call such edges weighted. To every nonempty set of 1-cells in $X_n$ whose image intersects in an edge at least we will assign such a weighted edge in $Y_n$, and to every such set of 1-cells we will assign a different edge. The image of every 1-cell contains less than $2^K$ weighted edges.

Let $G_n$ denote the 1-skeleton of $X_n$, $G_n$ is a graph. We show that the image of a long uniform random walk in $G_n$ under $f_n$ contains many weighted edges with high probability or there is a point whose pre-image intersects many 1-cells (if the image of every 1-cell in $X_n$ intersects the image of at most $K$ other 1-cells).

Claim: For every $k$ there is a $t$ such that either the image of a uniform random walk of length at least $t$ in $G_n$ (with a uniformly random
starting vertex) contains at least $k$ weighted edges with probability at least $(1 - 1/k)$ if $n$ is large enough (depending on $k, K$).

**Proof.** Choose a starting vertex $s_0$. Consider the image of the walk under $f_n$. And consider the path between $f_n(s_0)$ and $f_n(s_t)$ obtained by the removal of the cycles of this walk in $Y_n$. If this path has at most $k$ weighted edges then it can be covered by the image of at most $k$ 1-cells. If there is no point whose pre-image intersects at least $K$ 1-cells then there are less than $K^{k+1}$ possible images of 0-cells reachable via such a short sequence from $f_n(s_0)$, and the probability that the endpoint is the pre-image of any of these 0-cells can be arbitrary small if $t$ and $n$ is large enough. □

Consider the mapping $\varphi_n : E(G_n) \to \mathbb{Z}_p$ that assigns to every edge of $G_n$ (1-cell of $X_n$) the sum of these weights over the edges in its image (with orientation and multiplicity).

The values assigned to the weighted edges are chosen uniformly at random from $\{-1, +1\} \subset \mathbb{Z}_p$ independently. The sum over every path will be distributed identically to the endpoint of a random walk on the cycle of length $p$, where the number of steps equals to the number of weighted edges on the path. This will converge to the uniform distribution as the length goes to infinity. And the value for paths with disjoint images will be independent, hence we have a concentration. Note that so far we have only used that $X_n$ is large enough and connected.

Choose a prime $p > \frac{4K}{\gamma}$. For every length $l$ the portion of cycles with nonzero sum can be arbitrary small if $n$ is large enough, since $C$ is simply connected and the sum over null-homotopic cycles is zero. Hence condition (2) of Lemma 9 is satisfied. The distribution of the sum over uniform random walks in condition (3) can be arbitrarily close to uniform on $\mathbb{Z}_p$ if $n$ is large enough, since $X_n$ is large enough and connected. Lemma 9 gives a contradiction, the theorem follows. □

**References**

[1] Arora, Sanjeev, Boaz Barak, and David Steurer. "Subexponential algorithms for unique games and related problems." Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on. IEEE, 2010.

[2] Adams, S. R., and R. J. Spatzier. "Kazhdan groups, cocycles and trees." American Journal of Mathematics (1990): 271-287.

[3] Bekka, Bachir, Pierre de La Harpe, and Alain Valette. Kazhdan's property (T). Vol. 11. Cambridge university press, 2008.

[4] Benjamini, Itai, Oded Schramm, and Asaf Shapira. "Every minor-closed property of sparse graphs is testable." Proceedings of the fortieth annual ACM symposium on Theory of computing. ACM, 2008.
[5] Lewis Bowen, "Ergodic decompositions of sofic approximations", manuscript, 2011.
[6] G. Elek and E. Szabó, Sofic groups and direct finiteness, Journal of Algebra 280 (2004) 426-434.
[7] G. Elek and E. Szabó, Hyperlinearity, essentially free actions and L2-invariants. The sofic property, Math. Ann. 332 (2005) 421-441.
[8] G. Elek and B. Szegedy, A measure-theory approach to the theory of dense hypergraphs, Advances in Mathematics 231 (2012) 1731–1772.
[9] Freedman, Michael H., and Matthew B. Hastings. ”Quantum systems on non-k-hyperfinite complexes: A generalization of classical statistical mechanics on expander graphs.” Quantum Information & Computation 14.1-2 (2014): 144-180.
[10] D. Gaboriau, Cout des relations d’equivalence et des groupes, (in French) [Cost of equivalence relations and of groups], Inventiones Mathematicae, 139 (2000), no. 1, 41–98.
[11] M. Gromov, Endomorphisms of symbolic algebraic varieties, J. Eur. Math. Soc. 1 (1999) no. 2, 109-197.
[12] Lovász, László. Large networks and graph limits. Vol. 60. American Mathematical Soc., 2012.
[13] Lovász, László Miklós, Question at Open Problem Session, Graph Theory Meeting, 10-16 January, 2016, MFO, Oberwolfach.
[14] Pestov, Vladimir G. ”Hyperlinear and sofic groups: a brief guide.” Bulletin of Symbolic Logic 14.04 (2008): 449-480.
[15] Mendel, Manor, and Assaf Naor. ”Nonlinear spectral calculus and super-expanders.” Publications mathmatiques de l’IHS 119.1 (2014): 1-95.
[16] O. Schramm. Hyperfinite graph limits. Electron. Res. Announc. Math. Sci., 15 (2008), pp. 1723.
[17] B. Weiss, Sofic groups and dynamical systems (Ergodic theory and harmonic analysis, Mumbai, 1999) Sankhyā Ser. A. 62 (2000) no. 3, 350-359.

E-mail address: kungabor@renyi.hu