Modular Forms as Classification Invariants of 4D $\mathcal{N}=2$ Heterotic–IIA Dual Vacua

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**Abstract**

We focus on 4D $\mathcal{N}=2$ string vacua described both by perturbative Heterotic theory and by Type IIA theory; a Calabi–Yau three-fold $X_{\text{IIA}}$ in the Type IIA language is assumed to have a regular K3-fibration. It is well-known that one can assign a modular form $\Phi$ to such a vacua by counting perturbative BPS states in Heterotic theory or collecting Noether–Lefschetz numbers associated with the K3-fibration of $X_{\text{IIA}}$. In this article, we expand the observations and ideas (using gauge threshold correction) in the literature and formulate a modular form $\Psi$ with full generality for the class of vacua above, which can be used along with $\Phi$ for the purpose of classification of those vacua. Topological invariants of $X_{\text{IIA}}$ can be extracted from $\Phi$ and $\Psi$, and even a pair of diffeomorphic Calabi–Yau’s with different Kähler cones may be distinguished by introducing the notion of “the set of realizable $\Psi$’s”. We illustrated these ideas by simple examples.
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1 Introduction

The duality between the Heterotic string theory and the Type IIA string theory has been known for a long time. The duality with SO(5, 1) symmetry and 16 supersymmetry charges—the duality at 6D—comes with just one piece of moduli space \([\text{Isom}(4, 16) \setminus \text{SO}(4, 20) / \text{SO}(4) \times \text{SO}(20)] \times \mathbb{R}_{>0}\), and its various aspects are understood very well [1]. The duality with SO(3, 1) symmetry and 8 supersymmetry charges [2]—the duality at 4D—is less understood. The moduli space of Heterotic–IIA dual 4D vacua forms a complicated network of branches. It is desirable that those individual branches are characterized both in the language of Heterotic string and Type IIA string, and the dictionary between the branch-characterizing data on both sides are understood. At the moment, we do not have one for the 4D Heterotic–IIA duality as clear as Batyrev’s dual polyhedra for mirror symmetry.

For a systematic approach, we need to find invariants characterizing the branches of the moduli space. A lattice pair \(\tilde{\Lambda}_S \oplus \Lambda_T\) that fits into \(\Pi_{4,20}\) is assigned for those branches [3] [4] [5] [6], and a modular form \(\Phi\) of certain type that depends on \(\Lambda_S\) is also assigned [7] [8]. It is known, however, that there are physically distinct branches of vacua that cannot be distinguished by the triple of invariants \((\Lambda_S, \Lambda_T, \Phi)\). The primary purpose of this article is to introduce more invariants by using modular forms to improve the state of affairs.

From the perspective of pure mathematics, this task is equivalent to classification of Calabi–Yau three-folds with a K3-fibration. The modular forms introduced in this article can be used therefore for study of such a geometry classification. It should be mentioned, however, that we consider only regular K3-fibrations in this article.

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1 An exception is for the case \(\Lambda_S = U\) (see main text for what it is), when the duality lifts to the Heterotic–F-theory duality at 6D.
The organization of this article is as follows. We begin in section 2.1 with a short review on the Heterotic–Type IIA duality and a summary of technical limitations on the class of vacua to be considered in this article. The modular form $\Phi$ is parametrized by low-energy BPS indices, which are bounded from below as we argue in section 2.2 in the Heterotic language, the bounds come from the quantization of the level of current algebra and the spin under the $SU(2)$ action. In sections 2.3 and 2.4, those bounds and the modular nature are combined to constrain the possibilities of $\Phi$ and the Euler number of a Calabi–Yau three-fold $X_{\text{IIA}}$ that compactifies IIA theory.

We formulate in section 3 a modular form $\Psi$ for $X_{\text{IIA}}$ and a little more data, and use it to define new invariants for $X_{\text{IIA}}$ in addition to $(\Lambda_S, \Lambda_T, \Phi)$. Section 3.1.1 includes the basic definition of $\Psi$. This modular form appears in the integrand of 1-loop gauge threshold correction. In section 3.1.2, we comment on the restrictions on the degree of freedom of $\Psi$ that comes from some physical constraints. We introduce a map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ in section 3.1.3, which extracts from $(\Phi, \Psi)$ the full information in specifying the diffeomorphism class of a Calabi–Yau three-fold $X_{\text{IIA}}$. Combining this map with the degree-of-freedom study of $\Phi$ and $\Psi$ in sections 2 and 3.1.2, we can use modular nature of $\Phi$ and $\Psi$ to obtain non-trivial results in the diffeomorphism classes of real six-dimensional manifolds realized by Calabi–Yau three-folds. We also propose to use the notion of “the set of $\Psi$’s realized by Higgs cascades” or of “the set of $\Psi$’s associated with curve classes” as an invariant that resolves a diffeomorphic pair of Calabi–Yau three-folds with different cone of curves. In section 3.2, all those ideas are illustrated by using simple examples.

In section 4 we comment on some open questions.

The appendix A contains basics about (vector-valued) modular forms and explicit Fourier expansions of those in the main text. In the appendix B we review the lattice unfolding method and the embedding trick of Borcherd’s [9], presented in a form we need for threshold calculations in Heterotic theory. The embedding trick is used in explicitly evaluating the integrals, for example in the case of $\Lambda_S = \langle +2 \rangle$ in the appendix B.3.1.

\section{Coarse Classification}

\subsection{A Brief Review}

Let us first review what is known in the literature about the classification using the new supersymmetry index / the generating function of the Noether–Lefschetz number.
2.1.1 Heterotic Description: the New Supersymmetry Index

A Heterotic string compactification to 3+1-dimensions has an unbroken $\mathcal{N} = 2$ supersymmetry (8 supersymmetry charges), if and only if the right-mover of the internal worldsheet CFT contains $N = 4$ superconformal algebra (SCA) with central charge $\hat{c} = 6$ and $N = 2$ free SCA with $\hat{c} = 3$ corresponding to a flat space of one complex dimension [11]. We restrict our attention in this article only to compactifications without an NS5-brane or its generalizations discussed in §5 of [11].

Let $\rho$ be the number of free chiral bosons in the left-mover in such a compactification. There are vertex operators of the form $e^{ip_L X_L + ip_R X_R}$ in the CFT, where $X_L$ and $X_R$ are the $\rho + 2$ chiral bosons in the left mover and right mover; the set of $U(1)$ charges $\{(p_L, p_R)\} = \tilde{\Lambda}_S$ forms a lattice with the quadratic form given by $p^2_R/2 - p^2_L/2$, so its signature is $(+, -) = (2, \rho)$. This lattice $\tilde{\Lambda}_S$ should be even, $p^2_R/2 - p^2_L/2 \in \mathbb{Z}$ for any element of $\tilde{\Lambda}_S$, since the contribution of the state $e^{ip_L X_L + ip_R X_R}$ to the partition function should be invariant under $T : \tau \rightarrow \tau + 1$. The $U(1)$ charge of any worldsheet operator should lie in $\tilde{\Lambda}_S^\perp$, the dual lattice of $\tilde{\Lambda}_S$. Note that we deal with the case $\tilde{\Lambda}_S$ is not necessarily unimodular, so that the discriminant group $G_S := \tilde{\Lambda}_S^\perp/\tilde{\Lambda}_S$ may be non-trivial. We assume, however, that $\tilde{\Lambda}_S$ is a primitive sublattice of $\Pi_{4,20} = U^{\oplus 4} \oplus E_8[-1]^{\oplus 2}$; the orthogonal complement $[\tilde{\Lambda}_S^\perp \subset \Pi_{4,20}]$ is denoted by $\Lambda_T$.

For example, when we compactify Heterotic theory on $K3 \times T^2$ with instantons in $g \subset E_8 \times E_8$, the lattice $\tilde{\Lambda}_S$ is equal to $U^{\oplus 2} \oplus W$, where $W = [g^\perp \subset E_8[-1]^{\oplus 2}]$.

The Hilbert space of the internal CFT can be decomposed using the action of the free boson algebra and the $N = 4$ right-mover SCA [12, 7]:

$$\mathcal{H}^{\text{int}} = \bigoplus_{(w, h, I)} \mathcal{H}^{(22-\rho, 0)}_{w, h, I} \otimes \mathcal{H}^{(\rho, 3)}_{w} \otimes \mathcal{H}^{(0, 6)}_{h, I},$$

where superscripts show the central charge $(c, \check{c})$. The rank-$(\rho + 2)$ $U(1)$ charges in $\tilde{\Lambda}_S$ are denoted by $w$, and $\mathcal{H}^{(\rho, 3)}_{w}$ consists of states of the charge $w$ with any free bosonic/fermionic oscillator excitations. A pair $(\tilde{h}, I)$ of conformal weight and $SU(2)$ spin labels a unitary irreducible representation $\mathcal{H}^{(0, 6)}_{h, I}$ of $N = 4$ SCA. The spectrum of the free sector $\mathcal{H}^{(\rho, 3)}_{w}$ is

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2 We use the convention $\alpha' = 2$.

3 We assume all the charge $w \in \tilde{\Lambda}_S^\perp$ is realized by some state. The Type IIA counterpart of this assumption is that the pairing $(H^2(X; \mathbb{Z})/\mathbb{Z}D_s) \times [H_2(X; \mathbb{Z})]^\text{vert} \rightarrow \mathbb{Z}$ is represented by the unit matrix; see page [3] for notations.

4 The even unimodular lattice of signature $(1,1)$ is denoted by $U$. We use the same notation $\mathcal{R}$ for one of ADE types, its Lie algebra, and its root lattice with positive definite signature in this article.
specified by the central charge $p^C_R : \tilde{\Lambda}_S^c \to \mathbb{C}$, which appears in the 4D $\mathcal{N} = 2$ supersymmetry algebra; At the Heterotic string perturbative level, $p^C_R$ is governed by the Coulomb branch moduli space

$$D(\tilde{\Lambda}_S) := \mathbb{P} \left\{ \mathcal{U} \in \tilde{\Lambda}_S \otimes \mathbb{C} \mid (\mathcal{U}, \mathcal{U})_{\tilde{\Lambda}_S} = 0, (\mathcal{U}, \overline{\mathcal{U}})_{\tilde{\Lambda}_S} > 0 \right\},$$

which constitutes the special geometry along with the dilaton complex scalar $s := 4\pi i S$; the weak coupling limit is $s_2 \gg 1$. The rest of spectrum information (i.e. the spectrum of $H_{w,(\bar{h}, \bar{I})}$ for each possible $(w; \bar{h}, \bar{I})$) depends also on the hypermultiplet moduli space.

The new supersymmetry index $\text{[13, 7]}$ of the internal CFT is defined by

$$Z_{\text{new}}(\tau, \bar{\tau}) := \frac{-i}{\eta(\tau)^2} \text{Tr}_{\text{R sector}}^{(22, 9)} \left[ e^{\pi i F_R} F_R q^{L_0 - \frac{c}{24}} \bar{q}^{-L_0 - \frac{\bar{c}}{24}} \right],$$

where $F_R := 2(\tilde{J}_3^{i=6})_0 + (\tilde{J}_3^{i=3})_0$ is the total $U(1)$ current in the right-mover; $\tilde{J}_3^{i=6}$ is the $SU(2)$ Cartan operator in $N = 4$ SCA and $\tilde{J}_3^{i=3}$ the $U(1)$ current in $N = 2$ SCA. The important point is that this index does not depend on any continuous deformations of the hypermultiplet moduli, but on that of vector multiplet moduli. Furthermore, it is used in computing the 1-loop correction $\Delta_{\text{grav}}$ to the gravitational coupling $\sqrt{-g}R^2$ in the 4D effective theory;

$$\Delta_{\text{grav}} = \int \frac{d\tau_1 d\tau_2}{\tau_2} (\mathcal{B}_{\text{grav}} - b_{\text{grav}}), \quad \mathcal{B}_{\text{grav}} = Z_{\text{new}} \hat{E}_2,$$

where the integration is over the fundamental region of $\text{SL}(2; \mathbb{Z})$ in the upper complex half plane (of the torus world sheet complex structure $\tau$), and $\hat{E}_2 := E_2 - \frac{3}{\pi^2}$ is a non-holomorphic modular form of weight $(2, 0)$. The constant $b_{\text{grav}}$ is set to the $q^0$ coefficient of $\mathcal{B}_{\text{grav}}$, to cut off the IR divergent 1-loop contributions and brings the massless degrees of freedom back into the path integration in low energy effective theory. From the modular invariance of the integrand, $Z_{\text{new}}$ has weight $(-1, 1)$.

The action of free boson algebra on $\mathcal{H}^\text{int}$ leads to the following decomposition

$$Z_{\text{new}}(\tau, \bar{\tau}) = \sum_{\gamma \in G_S} \theta_{\tilde{\Lambda}_S[-1] + \gamma}(\tau, \bar{\tau}) \frac{\Phi_4(\tau)}{\eta(\tau)^{24}},$$

$$\theta_{\tilde{\Lambda}_S[-1] + \gamma}(\tau, \bar{\tau}) := \sum_{w \in \tilde{\Lambda}_S + \gamma} q^{\rho_S^2(w)/2} \bar{q}^{\rho_R^2(w)/2},$$

The real and imaginary components of $s$ are denoted by $s_{1, 2}$. Similar notations ($t_2$, $\tau_{1, 2}$, $\rho_2$ etc.) are used throughout this article.

To be more precise, only the trace part is called as the new supersymmetric index of the internal CFT. The $1/\eta^2$ factor is included within $Z_{\text{new}}$ here because the trace part appears in $\Delta_{\text{grav}}$ in the combination $\mathcal{B}_{\text{grav}}$; the $1/\eta^2$ factor is from the 4D Minkowski part in the light-cone gauge.
where $\theta_{\tilde{\Lambda}_S[-1]+\gamma}(\tau, \bar{\tau})$ is the Siegel modular form, which describe the dependence on the vector multiplet moduli. $\theta_{\tilde{\Lambda}_S[-1]} = \sum_{\gamma \in G_S} e_{\gamma} \theta_{\tilde{\Lambda}_S[-1]+\gamma}$ is a vector valued modular form in $\text{Mod}((\rho/2, 1), \rho_{\tilde{\Lambda}_S[-1]} = \rho_S^{\gamma})$, while $\Phi \in \text{Mod}(11 - \rho/2, \rho_{\tilde{\Lambda}_S})$. (See appendix A.1 for our notations.) In particular, $\Phi$ is holomorphic at cusps, i.e. has no negative power of $q$ in its expansion. Transformation law under $T : \tau \rightarrow \tau + 1$ fixes the fractional part of power of $q$, so $\Phi_{\gamma}/\eta^{24}$ can be expanded as

$$\Phi_{\gamma}(\tau) = \sum_{\nu \in \gamma^2/2 + \mathbb{Z}} c_{\gamma}(\nu) q^\nu, \quad c_{\gamma}(\nu) = 0 \quad \text{for} \quad \nu < -1. \quad (7)$$

Non-zero contributions to $Z_{\text{new}}$ comes only from “BPS states” whose right-mover is given by the (Ramond) ground states of $\tilde{c} = 3$ sector and the short representations of $N = 4$ SCA in $\tilde{c} = 6$ sector. There are two short representations of $N = 4$ SCA: vector-type and hyper-type. Their quantum numbers $(\tilde{h}, I)$ are given by:

| R-sector | NS-sector |
|----------|-----------|
| vector-type: | (1/4, 1/2) | (0, 0) |
| hyper-type: | (1/4, 0) | (1/2, 1/2) |

Their contributions can be written as

$$\Phi_{\gamma}/\eta^{24} = -2 \times \frac{1}{\eta^2} \text{Tr}_{H^{(22,0)}_{\gamma,V}} \left[q^{L_0 - \frac{c}{24}}\right] + 1 \times \frac{1}{\eta^2} \text{Tr}_{H^{(22,0)}_{\gamma,H}} \left[q^{L_0 - \frac{c}{24}}\right], \quad (8)$$

where $H_{\gamma,(\tilde{h}, \tilde{I})} := \bigoplus_{w} H_{w,(\tilde{h}, \tilde{I})}$ and

$$H_{\gamma,V} := H_{\gamma,(1/4, 1/2)} = H_{\gamma,(0, 0)}, \quad (9)$$

$$H_{\gamma,H} := H_{\gamma,(1/4, 0)} = H_{\gamma,(1/2, 1/2)}. \quad (10)$$

The second equality in each line comes from the spectral flow of $N = 4$ SCA that brings the representations in R-sector to those in NS-sector. The coefficient $-2, +1$ in (8) corresponds to the Witten index of the representation $(1/4, 1/2), (1/4, 0)$:

$$\text{Tr}_{(1/4, 1/2)}(-1)^{2(J_3^{\text{even}})_{\gamma}} = -2, \quad \text{Tr}_{(1/4, 0)}(-1)^{2(J_3^{\text{even}})_{\gamma}} = +1. \quad (11)$$

It follows from (8), in particular, all the Fourier coefficients $c_{\gamma}(\nu)$ are integers, so we have only discrete choice of $\Phi$. In fact, since $\Phi/\eta^{24}$ has negative weight as a modular form, it can

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7 In fact, we impose stronger assumption. See (14).
8 Note that $c_{\gamma}(\nu) = c_{-\gamma}(\nu)$ is required by properties of the Weil representation.
be uniquely specified by the coefficients of negative power of $q$

$$n_\gamma := -2n_\gamma^V + n_\gamma^H = c_\gamma([\gamma^2/2]_{\text{frac}} - 1), \quad \gamma \in G_S. \quad (12)$$

Here the fractional part $[x]_{\text{frac}}$ of $x \in \mathbb{R}$ is defined by $[x]_{\text{frac}} \equiv x \mod \mathbb{Z}$ and $0 \leq [x]_{\text{frac}} < 1$. $n_\gamma^{V/H}$ is the number of states of conformal weight $[\gamma^2/2]_{\text{frac}}$ in $H^{V/H}_\gamma$.

We can deduce $n_{\gamma=0} = -2$ from the supersymmetry constraints. First, the uniqueness of the ground state forces $n_{0}^{V} = 1$: $n_{0}^{V}$ is the number of states in $H^{V}_0$ with conformal weight 0. Such states, tensored with the right-moving Ramond ground states (in $c = 3$ sector) and the highest weight state of $(\tilde{h}, \tilde{I}) = (1/4, 1/2)$ (in $\tilde{c} = 6$ sector) gives the exactly required number of gravitino states for 4d $\mathcal{N} = 2$ supersymmetry. If $n_{0}^{H} > 0$, the 4d effective theory would have $\mathcal{N} = 2 + n_{0}^{H}$ supersymmetry by similar way. Since we focus on the case with only 8 supercharges, $n_{0}^{H}$ should be zero, so $n_{0} = -2$. For the explanation of $n_{0} = -2$ from the Type IIA perspective, see section 2.1.2.

$n_\gamma$ for $\gamma \neq 0$ also has simple relation with spacetime effective theory: for each $w \in \gamma \subset \tilde{\Lambda}_S$ such that $-2 \leq w^2 < 0$, \[9\] there exist $n_\gamma^{V}$ BPS vector multiplets and $n_\gamma^{H}$ BPS half-hypermultiplets, both of which have $U(1)$-charge $w$ and BPS mass $p_{R}^{2}(w)$. $w^2 < 0$ implies that these states will be massless at some points in the (weak coupling) Coulomb branch moduli space $D(\Lambda_S)$. For this reason, we call $n_\gamma$’s the low-energy BPS indices in this article.

In this paper, we impose some technical constraints on $\tilde{\Lambda}_S$ and $\Phi$ for simplicity. First, we only consider the case

$$\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S \quad (13)$$

for some even lattice $\Lambda_S$ of signature $(1, \rho - 1)$. Note that the direct summand $U[-1]$ does not contribute to the discriminant group: $G_S = \tilde{\Lambda}_S^\vee/\tilde{\Lambda}_S = \Lambda_S^\vee/\Lambda_S$. The factor $U[-1]$ corresponds to $H^0(K3, \mathbb{Z}) \oplus H^4(K3, \mathbb{Z})$ of the generic fibre of K3-fibred Calabi-Yau manifold that compactifies Type IIA theory. See also section 2.1.2 \[13\]

\[9\] In fact, we assume some $n_\gamma$ to be zero. See the comments around (14).

\[10\] This condition comes from the left-right matching of conformal weights.

\[11\] If $2\gamma \neq 0 \in G_S$, these $n_\gamma^H + n_{-\gamma}^H = 2n_\gamma^H$ half-hypermultiplets are combined to become $n_\gamma^H$ full hypermultiplets.

\[12\] In section 2.2, we show that there is non-trivial constraints about $\gamma$ for BPS massless vector multiplet states to exist, coming from charge and level quantization conditions.

\[13\] $\tilde{\Lambda}_S$ that does not have $U[-1]$ as a direct summand may correspond to some non-geometric background of Type IIA, such as mirror-folds, etc.
Second, we assume that if $\gamma^2/2 \equiv 0 \mod \mathbb{Z}$ and $\gamma \neq 0$ then $n_\gamma = c_\gamma(-1) = 0$. In other words, $\Phi$ has expansion $-2q^0 e_0 + (\text{strictly higher in } q)$. We denote this condition as

$$\Phi \in \text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S}).$$

We define $h_{\min}(\gamma)$ so that the Fourier expansion of $\Phi_\gamma$ begins with $O(q^{h_{\min}(\gamma)})$ (or higher):

$$h_{\min}(\gamma) \in \gamma^2/2 + \mathbb{Z}, \quad h_{\min}(0) = 0, \quad 0 < h_{\min}(\gamma) \leq 1 \text{ for } \gamma \neq 0.$$  \hspace{1cm} (15)

We denote

$$G_S^\prec = \{ \gamma \in G_S \mid h_{\min}(\gamma) < 1 \}, \quad d^\prec = |G_S^\prec/\pm|.$$  \hspace{1cm} (16)

After all, the modular form $\Phi$ is specified by at least $d^\prec - 1$ integers

$$n_{|\gamma|} = n_{\pm \gamma} \geq -2, \quad |\gamma| \in G_S^\prec/\pm, \quad \gamma \neq 0.$$  \hspace{1cm} (17)

Sometimes, the modular properties predict linear relations among the low-energy BPS indices $\{n_{|\gamma|}\}$; see section 2.3.2.

### 2.1.2 Type IIA Description: the Generating Function of the Noether–Lefschetz Numbers

In this article, we consider Type IIA string compactified on a non-singular Calabi–Yau threefold $X = X_{\text{IIA}}$ that has K3-fibration over $\mathbb{P}^1 = \mathbb{P}^1_{\text{IIA}}, \pi : X \to \mathbb{P}^1$; complexified Kähler parameters may be analytically continued out of a geometric phase, but otherwise we remain in a geometric phase. \hspace{1cm} (10)

This restriction means, in particular, that we do not treat T-folds or mirror folds. \hspace{1cm} (15)

This restriction corresponds to (13) in the Heterotic side. We also assume that

$$h^{1,0}(X) = h^{2,0}(X) = 0.$$  \hspace{1cm} (18)

The effective theory on 3+1-dimensions has stricly $\mathcal{N} = 2$ supersymmetry, not more, not less.

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14 This condition corresponds to the the claim in the IIA side. Even when there is a Heterotic construction that is not subject to (14), there would probably be no Type IIA dual with a geometric phase. See also section 2.1.2.

15 This is an extra non-trivial condition only in a lattice $\Lambda_S$ with a non-zero isotropic element $\gamma$. Examples of such lattices include $\Lambda_S = \langle 2n^2m \rangle$. Let $e$ be a generator of the rank-1 abelian group $\mathbb{Z}$ underlying $\Lambda_S$. $(e,e)_{\Lambda_S} = 2n^2m$. Elements such as $\gamma = e/n + \Lambda_S \in G_S$ give $\gamma^2/2 = m \equiv 0 \mod \mathbb{Z}$.

16 Both $R^4_{\pi_*\mathbb{Z}}$ and $R^0_{\pi_*\mathbb{Z}}$ are trivial rank-1 local systems on $\mathbb{P}^1$. 

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K3-fibres in $\pi : X \to \mathbb{P}^1$ may degenerate at isolated points on the base $\mathbb{P}^1$. Degeneration of K3-fibration is classified (by allowing base change locally) into Type I, Type II, and Type III \cite{16}. When a K3-fibration $\pi : X \to \mathbb{P}^1$ only has degeneration classified as Type I\textsuperscript{17}, such a K3-fibration is said to be regular. In this article, we only consider regular K3-fibrations, because that is when one can find Heterotic dual descriptions without (generalization of) NS5-branes \cite{11}.

Let $\pi : X \to \mathbb{P}^1$ be a regular K3-fibration. Then the cohomology groups of $X$ have the following filtration:

$$H^2(X; \mathbb{Z}) \supset \mathbb{Z}D_s \supset \{0\},$$

$$H_2(X; \mathbb{Z})/\mathbb{Z}D_s =: \Lambda_S,$$  \hspace{1cm} (19)

$$H_2(X; \mathbb{Z}) \supset [H_2(X; \mathbb{Z})]^{\text{vert}} \supset \{0\},$$  \hspace{1cm} (20)

where $D_s$ is the total K3-fibre divisor class, and $[H_2(X; \mathbb{Z})]^{\text{vert}}$ is the subgroup generated by curves that are projected to points on $\mathbb{P}^1$. The free abelian group $\Lambda_S$ is a subgroup of the Neron–Severi lattice $L_S$ of $X$, the fibre K3 surface over a generic point $p \in \mathbb{P}^1$. So, an intersection form is introduced on $\Lambda_S$ by restricting the intersection form of $L_S$; $\Lambda_S$ is now regarded as a lattice. The natural pairing between $\Lambda_S$ and $[H_2(X; \mathbb{Z})]^{\text{vert}}$ is non-degenerate, and we have an isomorphism $[H_2(X; \mathbb{Z})]^{\text{vert}} \cong_{ab} \Lambda_S^\vee$ as abelian groups. We reserve $\rho$ for the rank of $\Lambda_S$, not for $L_S$. $\Lambda_S$ [resp. $L_S$] is a primitive sublattice of $\Pi_{3,19} \cong H^2(K3; \mathbb{Z})$; the orthogonal complement lattice is denoted by $\Lambda_T$ [resp. $L_T$].

For a regular K3-fibration $\pi : X \to \mathbb{P}^1$, one can think of a generating function $\Phi$ of the number of Noether–Lefschetz points on the base $\mathbb{P}^1_{\text{IA}}$. When it is defined appropriately (see below), it is known to be a modular form \cite{8}. First, there is a holomorphic map

$$t_\pi : \mathbb{P}^1_{\text{IA}} \longrightarrow D(\Lambda_T)/\Gamma_T,$$  \hspace{1cm} (21)

where $\Gamma_T := \text{Ker}(\text{Isom}(\Lambda_T) \to \text{Isom}(G_T))$, because $D(\Lambda_T)/\Gamma_T$ is the coarse moduli space of $\Lambda_S$-polarized K3 surfaces. At points on the base $\mathbb{P}^1_{\text{IA}}$ where the $t_\pi$-image hit the Noether–Lefschetz divisor $D_{NL(F_\perp)}$ of $D(\Lambda_T)$ for $F_\perp \in \Lambda_T^\vee$, the transcendental cycles $(F_\perp + \Lambda_S^\vee) \cap \Pi_{3,19}$ become algebraic.

Think of the Heegner divisor \cite{8}

$$\Phi_{\text{pre}} := \sum_{F_\perp \in \Lambda_T^\vee} D_{NL(F_\perp)} q^{-F_\perp^2/2} \epsilon[F_\perp] / \Gamma_T \in \text{Pic}(D(\Lambda_T)/\Gamma_T)[q^{1/N}] \otimes \mathbb{C}[G_T],$$  \hspace{1cm} (22)

\textsuperscript{17}We avoid saying “only has Type I fibres” here, because such expressions as “Type I fibre” are reserved only for the central fibres after the local geometry of $\pi : X \to \mathbb{P}^1$ around a degeneration point $p \in \mathbb{P}^1$ is brought into a Kulikov model by a base change around $p$. 


where \( \{e_{\gamma}\}_{\gamma \in G_T} \) is the set of formal basis elements of the vector space \( \mathbb{C}[G_T] \), \( q = e^{2\pi i \tau} \) a formal variable, and \( N \) the level of the quadratic discriminant form of the lattice \( \Lambda_T \). \( F_\perp \in \Lambda_T^\vee \) in the summation is either \( F_\perp = 0 \) or arbitrary \( F_\perp \in \Lambda_T^\vee \) with \( (F_\perp)^2 < 0 \); this makes sense because the Noether–Lefschetz loci in \( D(\Lambda_T) \) exist only for negative definite \( F_\perp \)'s in \( \Lambda_T^\vee \).

An extra term \( e_0 q^0 D_{NL(0)}/\Gamma_T \) term is included in the definition of \( \Phi_{\text{pre}} \); we do not provide a description of the divisor “\( D_{NL(0)}/\Gamma_T \)” (see [8] for details), but all the necessary properties are provided later on. Note that in this definition \( \Phi_{\text{pre}} \) does not have a term proportional to \( e_0 q^0 \) for a non-zero isotropic \( \gamma \in G_T \).

\( \Phi_{\text{pre}} \) is independent of \( \tau_\pi \). Given \( \tau_\pi \), we obtain a \( \mathbb{C}[G_T] \)-valued function of the formal variable \( q = e^{2\pi i \tau} \) by pairing \( \Phi_{\text{pre}} \) with the image of the base \( \mathbb{P}^{1}_{\text{IIA}} \) mapped into \( \Gamma_T^\perp ) = \Phi_{\text{pre}} \cdot \tau_\pi (\mathbb{P}^{1}) = \sum_{\gamma \in G_T} \sum_{\gamma \in G_T} N_{L[F_\perp,\gamma]} q^{-(F_\perp)^2/2} e_\gamma \) (23)

Define \( N_{L_{\nu,\gamma}} = \sum_{(F_\perp)_{\gamma} = \nu} N_{L[F_\perp,\gamma]} \); it is the Fourier coefficients of \( \Phi_{\gamma} = \sum_{\nu} N_{L_{\nu,\gamma}} q^\nu \). Ref. [8] arrives at a statement (by using earlier math results in [17], but not relying on the duality with Heterotic string)

\[ \Phi \in \text{Mod}_0 \left( 11 - \frac{\rho}{2}, \rho_{\Lambda_S} \right). \] (24)

See footnote [18] for why \( \Phi \) is in \( \text{Mod}_0 \), not in \( \text{Mod} \).

The coefficient \( N_{L_{\nu,\gamma}} \) with \( \gamma = 0 \) and \( \nu = 0 \) in \( \Phi \) does not describe the number of Noether–Lefschetz points on \( \mathbb{P}^{1}_{\text{IIA}} \). Following the definition of the divisor “\( D_{NL(0)}/\Gamma_T \)” in [8], one arrives at

\[ N_{L_{0,0}} = \deg_{\mathbb{P}^{1}_{\text{IIA}}} (R^2 \pi_*(\mathcal{O}_X)) = -2, \] (25)

where we used \( h^{0,3}(X) = 1 \) and \( h^{0,2}(X) = 0 \) at the last equality. So, as a consequence of [18], we have \( \Phi \sim -2q^0 e_0 + (\text{strictly higher in } q) \).

---

18 This leads to the subtle fact that \( \Phi \) lies in \( \text{Mod}_0 \) rather than \( \text{Mod} \) in [24].

19 Precisely, this procedure is only well-defined for smooth fibrations. In the case of Calabi–Yau \( X \) with finitely many nodal singular K3-fibres (these degenerations are classified as Type I), one has to take double cover of \( X \) and resolve the conifold singularities, so that non-singular fibration is obtained. One can apply the procedure to this fibration and divide the modular form by 2. We denote as \( \Phi \) the modular form defined in this way. See [18] [19] for details.

20 If \( X = K3 \times T^2 \), where \( h^{0,3}(X) = 1 \), \( R^2 \pi_*(\mathcal{O}_X) = \mathcal{O}_{T^2} \). So, \( N_{L_{0,0}} = \deg_{T^2}(\mathcal{O}_{T^2}) = 0 \). This is consistent with the Heterotic string description \( (n_0 = -2 + n_{\rho}^0 = 0 \) in the \( N = 4 \) supersymmetry situation).
2.1.3 Heterotic-Type IIA Duality and Effective Theory

When branches of moduli space of Heterotic theories and Type IIA theories associated with a pair of primitive sublattices $\tilde{\Lambda}_S$ and $\Lambda_T$ of $\text{II}_4,20$ are identified, both descriptions should give the same modular form:

$$\{ \Phi^\text{Het}_\gamma(q) \}_{\gamma \in G_S} = \{ \Phi^\text{IIA}_\gamma(q) \}_{\gamma \in G_T}. \quad (26)$$

The isomorphism $G_S \cong G_T$ as Abelian groups is the one specifying the embedding relation $$(\tilde{\Lambda}_S \oplus \Lambda_T) \subset \text{II}_4,20 \subset (\tilde{\Lambda}_S^\vee \oplus \Lambda_T^\vee).$$

That is because all the Fourier coefficients of $\Phi/\eta^{24}$ determine physical quantities in the $\mathcal{N} = 2$ supersymmetric effective theory on 3+1-dimensions; the helicity supertrace is defined by

$$\Omega(0, 0; (1, w), q_0) := -2\text{Tr}_{\mathcal{H}(0, 0, (1, w), q_0)} \left[ (-1)^{2J_3} (J_3)^2 \right] \quad (27)$$
on the Hilbert space $\mathcal{H}(0, 0, (1, w), q_0)$ of particles on $\mathbb{R}^{3,1}$ with a given pure electric charge under the $(\rho+2) \ U(1)$ gauge fields; $q_0 \in \mathbb{Z}$ and $w \in \Lambda_S^\vee$. $J_3$ is the 3-component of the angular momentum of this space-time. In Heterotic language $^{20}$

$$\Omega(0, 0; (1, w), q_0) = c^\text{Het}_{[v]} \left( v^2 / 2 \right), \quad v = 1 e^0 + w + q_0 e^4 \in \tilde{\Lambda}_S^\vee. \quad (28)$$

In Type IIA language, we apply the electromagnetic duality transformation in the effective theory for the one of the $(\rho+1) \ U(1)$ gauge fields originating from the Ramond–Ramond 3-form field, the one associated with the base 2-form; then a D4-brane wrapped on the fibre class along with a 2-form $F \in H^2(K3; \mathbb{Z})$ and $N \geq 0$ units of anti-D0-brane gives rise to a particle on $\mathbb{R}^{3,1}$ with a pure electric charge (Mukai vector on the K3 surface)

$$v = e^0 + F_\parallel + q_0(F, N)e^4 \in \tilde{\Lambda}_S^\vee, \quad q_0(F, N) = 1 - N + \frac{F^2}{2}. \quad (29)$$

Here, $F_\parallel$ is the projection of $F \in H^2(K3; \mathbb{Z})$ to $i^*(H^2(X; \mathbb{Z})) \cong \Lambda_X^\vee$, and $i : (\text{a fibre K3}) \hookrightarrow X$. The helicity supertrace is $^{21, 22, 23, 24}$ (and $^{25}$)

$$\Omega(0, 0; (1, F_\parallel), q_0) = c^\text{IIA}_{[v]} \left( v^2 / 2 \right). \quad (30)$$

$^{21}$More generally, there is such a unique isomorphism $\phi_M : G_L \cong G_{L'}$ for any mutually orthogonal pair $L$, $L'$, of primitive sublattices of a unimodular lattice $M$. In this article, elements of $G_L$ and $G_{L'}$ are identified without referring to $\phi_M$ (e.g., in $^{213}$).
The value of $v^2/2$ in the set of states above exhausts all $\nu \in h_{\text{min}}(\gamma) + \mathbb{Z}_{\geq 0}$, so all the Fourier coefficients of $\Phi^{\text{Het}}/\eta^{24}$ and $\Phi^{\text{IIA}}/\eta^{24}$ should be the same. The Coulomb branch moduli space $D(\tilde{\Lambda}_S)$ in (2) is parametrized by $t \in \Lambda_S \otimes \mathbb{C}$ with $(s_2, s_2) > 0$, which is interpreted as Narain moduli in Heterotic description and as complexified Kähler parameter in Type IIA description.

\[
\mathcal{U}(t) = e^0 + e^4 \left( \frac{t}{2} \right) + t,
\]  

(31)

where $\tilde{\Lambda}_S = U[-1] \oplus \Lambda_S$ is regarded as the Mukai lattice of $\Lambda_S$-polarized K3 surface, and $e^0$ and $e^4$ generate the $U[-1] \cong ab H^0(K3; \mathbb{Z}) \oplus H^4(K3; \mathbb{Z}) \cong ab \mathbb{Z}e^0 \oplus \mathbb{Z}e^4$ factor. The central charge of a BPS state with the electric charge $v \in \tilde{\Lambda}_S^\vee$ is proportional to $(\rho_R(v) \propto (\mathcal{U}(t), v))$. We will focus only on the $s_2 \gg 1$ and $s_2 \gg t_2$ region of the special geometry throughout in this article; that is the weak coupling region in the Heterotic description, and the large base $\mathbb{P}^1$ region in the Type IIA description.

To wrap up, moduli space of Heterotic–Type IIA dual vacua with 4D $\mathcal{N} = 2$ supersymmetry form branches, and each branch is labeled by a pair of lattices $\tilde{\Lambda}_S$ and $\Lambda_T$, and a modular form $\Phi \in \text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$. It is known that one can find a $\mathbb{C}$-basis $\{\phi_i\}$ of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ so that all the Fourier coefficients of $\phi_i(\tau)$ of $\phi_i = \sum_{\gamma \in G_S} e_{i,\gamma}(\tau)$ are integers. So, $\Phi$ must be in the free abelian group within $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ whose rank is the same as the dimension of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$. This free abel group is denoted by $\text{Mod}_0^\mathbb{Z}(11 - \rho/2, \rho_{\Lambda_S})$.

Without relying on explicit constructions (such as toric complete intersection), we can therefore hope to use properties of the free abelian group in $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ to derive some properties of $\Lambda_S$-polarized K3-fibred Calabi–Yau three-folds.

### 2.2 Conditions for $n^V_\gamma = 1$

As seen in the section 2.1.1, $n^V_\gamma$ is the number of BPS vector multiplets of fixed charge $w \in \gamma$ subject to $-2 < w^2 < 0$. Since charged massless gauge boson should be a part of

22 The Gopakumar–Vafa invariants of vertical curve classes $\beta \in [H_2(X; \mathbb{Z})]^{\text{vert}} \cong \Lambda_S^\vee$ are equal to $c_\gamma(\beta^2/2)$, and are also physical in the effective field theory because of its appearance in the prepotential $\{K_{\text{IIA}}\}$; see $[2, 8, 26]$. Not all the coefficients $c_\gamma(\nu)$ of $\Phi$ correspond to those Gopakumar–Vafa invariants for some $\Lambda_S$ (e.g., $\Lambda_S = (+2)$).

23 That is when we expect little contributions to low-energy physics from the NS5-branes in Heterotic string and D-branes wrapped on cycles that are mapped surjectively to $\mathbb{P}^1_{\text{IIA}}$. BPS states of such origins are not used in defining the modular form $\Phi$.

24 Heterotic string non-perturbative effects modify the infrared dynamics and the moduli space, as in Seiberg–Witten theory, but they are all known story. 28
non-abelian gauge bosons for some compact Lie group, its multiplicity must be one for each possible charge. This implies that \( n^V_\gamma = 0, 1 \) for any nonzero \( \gamma \in G_S \), so that \( n_\gamma \geq -2 \). Let us consider what happens when \( n^V_\gamma = 1 \).

Suppose \( n^V_{\gamma_0} = 1 \) for given nonzero \( \gamma_0 \in G_S \). Fix \( w_0 \in \gamma_0 \subset \tilde{\Lambda}_S \) such that \(-2 \leq (w_0, w_0) < 0\). Let us consider the points in the vector moduli space \( D(\tilde{\Lambda}_S) \) where the states of charge \( w_0 \) become massless: \( p_R(w_0) = 0 \). At these points, the massless vector bosons of charge \( w_0 \) should be a part of non-abelian gauge bosons. In the language of worldsheet theory, this gauge symmetry are described by some current algebra carried by the left-mover. Therefore we have \( SU(2) \) current algebra (that may be a sub-algebra of larger current algebra) that consists of

\[
J^\pm(z) := e^{\pm ip_L(w_0) \cdot X_L} \mathcal{O}^\pm, \quad J^3(z) := \frac{i}{-(w_0, w_0)} p_L(w_0) \cdot \partial X_L,
\]

where \( X_L \) is the internal free bosons in \( c = \rho \) sector, \( \mathcal{O}^\pm \) are the vertex operators of conformal weight \( \left(\gamma_0, \gamma_0\right) / 2 \) in \( \mathcal{H}_{\gamma_0,V} \). Note that \( J^\pm \) have conformal weight 1 as well as \( J^3 \), by the assumption \(-2 \leq (w_0, w_0) < 0\). Normalization of \( J^3 \) is set so that we have OPE

\[
J^3(z) J^\pm(0) \sim \frac{\pm 1}{z} J^\pm(0), \quad J^3(z) J^3(0) \sim -\frac{1/(w_0, w_0)}{z^2}.
\]

This means that\(^{25}\) the level of current algebra is \( k = -2/(w_0, w_0) \).

Let us consider some constraints from unitarity. First, the level \( k \) needs to be a positive integer. Second, any state (with \( U(1) \)-charge \( w \in \tilde{\Lambda}_S^\vee \)) should have half-integral spin in terms of this \( SU(2) \): this results in

\[
\frac{(w_0, w)}{(w_0, w_0)} \in \frac{1}{2} \mathbb{Z} \quad \forall w \in \tilde{\Lambda}_S^\vee,
\]

because that\(^{26}\) is the charge of \( e^{i(p_L(w) \cdot X_L + p_R(w) \cdot X_R)} \) under \( J_3 \). These mean the following constraint\(^{27}\) for \( \gamma_0 \in G_S = \tilde{\Lambda}_S^\vee / \tilde{\Lambda}_S \):

\[
\frac{(\gamma_0, \gamma_0)}{2} \equiv -\frac{1}{k} \mod \mathbb{Z} \quad \text{and} \quad k \gamma_0 \equiv 0 \in G_S, \quad \text{for some } k \in \mathbb{Z}_{\geq 2}.
\]

---

\(^{25}\) In other words, \( \mathcal{O}^\pm \) are from the coset \( SU(2)_k/U(1)_k \). This coset model is known as parafermion theory with \( \mathbb{Z}_k \) symmetry.

\(^{26}\) This condition is required for all \( w \in \tilde{\Lambda}_S^\vee \), because we assumed in section 2.1 (footnote 3) that there exists at least one state of charge \( w \) for any \( w \in \gamma \subset \tilde{\Lambda}_S^\vee \).

\(^{27}\) It follows from these constraints that, for any nonzero isotropic \( \gamma \) in \( G_S \), we get \( n^V_{\gamma_0} = 0 \). The assumption \( n_{\gamma_0} = 0 \) in 14—automatic in Type IIA (footnote 18)—further implies \( n^V_{\gamma_0} = 0 \) then.
The possibility $k = 1$ does not have to be included here, because $k = 1$ would simply mean $\gamma_0 = 0$ in $H$, where we know that $n^{\perp}_{\gamma_0 = 0} = 1$ from the beginning.

Not all $\gamma \in G_S$ satisfy the conditions $H$, but those that satisfy $H$ are not extremely rare. Table 1 shows the list of such $\gamma$’s for some of the $\rho = 1$ lattices, $\Lambda_S = \langle +2n \rangle \cong_{ab} \mathbb{Z} e$. In the case of lattices of the form $\Lambda_S = U \oplus \langle -2m \rangle =:_{ab} U \oplus \mathbb{Z} e$, at least $\langle \gamma_0 = \frac{1}{2m} e, k = 4m \rangle$ and $\langle \gamma_0 = \frac{2}{2m} e, k = m \rangle$ satisfy $H$.

There are a couple of different behaviours in the Type IIA geometry $X_{\text{IIA}}$ that correspond to appearance of a massless non-abelian gauge boson in the low-energy effective field theory. Let $v = re^0 + q_0 e^4 + F_\parallel \in \Lambda_X^*$ be the U(1) charge of such a gauge boson (as in (29)).

Suppose $r\gamma_0 = 0$. Then the Calabi-Yau $X_{\text{IIA}}$ should have a curve class $F_\parallel \in \Lambda_X^* \cong [H_2(X_{\text{IIA}}; \mathbb{Z})]^{\text{vert}}$ realized algebraically over a generic point of the base $\mathbb{P}_\text{IIA}^1 (B_\infty$ in §3.2 of [9]); this is because

$$c_{\gamma_0}(F_\parallel^2/2) = NL_{1+(F_\parallel)^2/2,\gamma_0} < 0, \quad -1 < \frac{F_\parallel^2}{2} < 0 \quad (36)$$

is possible only when $\nu_\gamma(\mathbb{P}^1)$ stays within a Noether–Lefschetz divisor $(\sum_{F_L} D_{\text{NL}(F_L)})/\Gamma_T$ with the sum ranging over those with $F_\parallel^2 + F_\perp^2 = -2$. The algebraic curve is $F = (F_\parallel + F_\perp) \in \Pi_{3,19}$, which must be a $(-2)$ curve. The vector boson on the spacetime $\mathbb{R}^{3,1}$ is massless when this $(-2)$ curve collapses to zero volume. $F_\perp$ must be nonzero since we think of a case $\gamma_0 \neq 0 \in G_S$. Then there must be non-trivial monodromy on $F_\perp \in \Lambda_T$ so that $F$ is in $L_S$, but not in $\Lambda_S$. $\Lambda_S$ is a proper subset of $L_S$ then.

In the case of $\Lambda_S = U \oplus \langle -2m \rangle$ and $\gamma_0 = \pm e/m$ (the level is $k = m$), the following interpretation seems to work: in terms of lattice, $L_S \cong U \oplus A_1^{\pm m}$ and the lattice $\langle -2m \rangle \subset \Lambda_S$ is embedded diagonally in $A_1^{\pm m}$; in terms of geometry, each K3 fibre of $X \to \mathbb{P}^1$ has $m$ points of $A_1$ singularity, and those singular points forms a curve in $X$ that is a $m$-fold cover over the base $\mathbb{P}^1$. The gauge kinetic term of the vector field is $ms$, which agrees nicely with the the gauge kinetic function $ks$ for gauge fields associated with the level-$k$ current algebra in Heterotic constructions [29].

There should also be geometry / lattice interpretation along the line of $\Lambda_S \subset L_S$ also for the case $\Lambda_S = U \oplus \langle -2m \rangle$ and $\gamma_0 = \pm e/(2m)$, but we have not been able to find a functioning

| $n$ | $(j, k)$ | 1 | (2, 2) | 3 | 4 | (±4, 5) | (±4, 3), (±6, 2) | $\cdots$ |
|-----|----------|-----|--------|-----|-----|----------|----------------|-----|

Table 1: list of $(\gamma_0, k) = (\pm \frac{1}{2m} e, k)$ satisfying the condition $H$ when $\Lambda_S = \langle +2n \rangle$
interpretation yet.

Suppose instead that \( rq_0 \neq 0 \). The massless vector boson is then a D4-D2-D0 bound state, not just of D2- and D0-branes \((r \neq 0)\). The Kähler parameter \( t \) must be of order unity for the vector boson to become massless (for moderate choices of \( r, q_0 \), and \( F_\parallel \)), so the base \( \mathbb{P}^1_{\text{IA}} \) may be large \((\text{Im}(s) \gg 1)\), but the fibre K3 is not safely in the large radius geometric regime. In any case of \( \Lambda_S = \langle +2n \rangle \) with \( \gamma_0 \in G_S \) satisfying \((35)\), the \( U(1) \) charge \( w_0 \) for such a massless vector boson should be the one of this category, because \( F_\parallel^2/2 \) is positive definite in \( \Lambda_S = \langle +2n \rangle \).

### 2.3 Examples of \( \Phi \)

One can list up modular forms \( \Phi \in \text{Mod}^Z_0(11 - \rho/2, \rho_{\Lambda_S}) \) that satisfies \( n_0 = -2 \) and the lower bounds \( n_\gamma \geq -2 \) or \( n_\gamma \geq 0 \) depending on \( \gamma \in G_S \), as seen in section 2.2. The easiest and well-known case is when \( \Lambda_S \) is unimodular:

\[
(\tilde{\Lambda}_S, \Lambda_T) = (U^{\oplus 2}, U^{\oplus 2} \oplus E_8^{\oplus 2}[-1]), \quad (U^{\oplus 2} \oplus E_6[-1], U^{\oplus 2} \oplus E_8[-1]), \quad (37)
\]

The Picard number \( \rho \) is 2, 10, 18 for each case and \( \Phi \) should be a scalar-valued modular form of weight \( (22 - \rho)/2 \) with \( n_0 = -2 \). So \( \Phi = -2E_4E_6 \) and \(-2E_6 \) for the first and second case, respectively. There is no candidate of \( \Phi \) for the third; \( \tilde{\Lambda}_S = U^{\oplus 2} \oplus E_8[-1]^{\oplus 2} \) (i.e., zero instantons in \( E_8 \times E_8 \) in Heterotic string) cannot be realized at least in our setup reviewed in section 2.1.

#### 2.3.1 Cases \( \Lambda_S = \langle +2 \rangle, \langle +4 \rangle, \text{ and } \langle +6 \rangle \)

We attempt at assessing how well/poorly the combination of the modular invariance, integrality of the BPS indices, and their lower bounds explains possible topological choices of K3-fibration of Calabi–Yau three-folds for \( \Lambda_S \) that are not unimodular. In section 2.3.1 we first work on the cases \( \Lambda_S = \langle +2 \rangle, \langle +4 \rangle, \text{ and } \langle +6 \rangle \), where a set of independent generators of \( \text{Mod}_0(21/2, \rho_{\Lambda_S}) \) is known \([16, 24, 8, 30]\), as summarized in the appendix \( A.2.1 \). The list of \( \Phi \)'s determined in this way is compared against a list of explicitly constructed Calabi–Yau three-folds with those \( \Lambda_S \)-polarized K3-fibration \([24]\).

Let us illustrate the procedure using the case \( \Lambda_S = \langle +2 \rangle \) as an example. To prepare a notation, let \( e \) be the generator of the free abelian group \( \langle +2n \rangle =: a_{ab} \mathbb{Z}e \), and the formal basis elements of \( \mathbb{C}[G_S] \) [resp. the low-energy BPS indices] are denoted by \( e_\gamma = e_j/2n \) [resp.
\[ n_\gamma = n_{j/2n} \] for short, when \( \gamma = (j/2n + Z)e \in G_S \) for some \( j \in \{0, 1, \cdots, 2n - 1 \} \). Now, one can choose \( \phi(i) = [\theta_{+2}, E_{10-2}i], i = 0, 1 \) as a \( \mathbb{C} \)-basis of the vector space \( \text{Mod}_0(21/2, \rho_{(-2)}) \) \[24, 30\]. The modular form \( \Phi \) is parametrized by

\[
\Phi = -2(\phi(0) + \phi(1)/2) - \frac{n_{1/2}}{4}\phi(1),
\]

\[
= e_0 \left( -2 + (300 - 56n_{1/2})q + (217200 - 13680n_{1/2})q^2 + \cdots \right)
\]

\[
+ e_{1/2} \left( n_{1/2}q^{3/2} + (2496 + 360n_{1/2})q^{5/2} + (665600 + 30969n_{1/2})q^{7/2} + \cdots \right);
\]

this linear combination is chosen so that the coefficients of \( e_0 \) and \( e_{1/2}q^{1/4} \) be \( n_0 = -2 \) and \( n_{1/2} \) respectively \[28\]. \( n_{1/2} \) should be non-negative integer. See section 2.2. This condition indicates that the Euler number \( \chi = -c_0(0) = -[\Phi_0/\eta^{24}]_{q^0} \) of IIA geometry can be written as

\[ \chi(X_{\text{IIA}}) = -252 + 56n_{1/2}, \quad n_{1/2} \in \mathbb{Z}_{\geq 0}; \] \[40\]

the value is quantized and bounded from below. It is also bounded from above, because \( h^{2,1}(X_{\text{IIA}}) \geq 0 \) (and hence \( \chi(X_{\text{IIA}}) \geq 2h^{1,1} = 4 \)) in the Type IIA language \[30\]. So,

\[ n_{1/2} \in \{0, 1, 2, 3, 4\}. \] \[41\]

As a result, we see that when \( \Lambda_S = \langle 2 \rangle \), there are only finite possibilities for \( \Phi, \chi \), and BPS indices \( n_{|\gamma|} \).

Similar procedure can be worked out for \( \Lambda_S = \langle 4 \rangle, \langle 6 \rangle \). See appendix \[A.2.1\] for detail. Here we just cite the results for Euler number in the table \[2\]. \( \Lambda_S = \langle 4 \rangle, \langle 6 \rangle \) also allow only finite possibilities for \( \chi, \{n_\gamma\} \) and \( \Phi \).

We give some comments on comparison with the result of \[24\]. The authors of \[24\] scanned combinatorial data in toric complete intersection construction and made some explicit examples of Calabi-Yau three-folds with \( \langle 2n \rangle \)-polarized regular K3 fibration: Table 1 in \[24\]. All of their examples satisfy the integrality of \( n_\gamma \) and bound of \( n_\gamma, \chi \) as described

\[28\]Here, \([-,-]_i \) is the Rankin-Cohen bracket. See the appendix \[A.1\].

\[29\]Because \( \Phi \) was parametrized by \( \Phi = -2(\phi(0) + \phi(1)/2) + n/4\phi(1) \) in \[24, 30\], the parameter \( n \) should be interpreted as \(-n_{1/2}\).

\[30\]In the Heterotic language, \( c_0(0) = (252 - 56n_{1/2}) \) is \( \dim_{\mathbb{C}}(\mathcal{H}_{0,(1/2,1/2)}|_{L_0=1}) - 2 \dim_{\mathbb{C}}(\mathcal{H}_{0,(0,0)}|_{L_0=1}) - 4 \), where the last term \(-4\) is from \( \eta^{-2} \) of the first term in \[28\]. By definition, \( \dim_{\mathbb{C}}(\mathcal{H}_{0,(0,0)}|_{L_0=1}) =: \rho \), and the assumption that the Heterotic construction in consideration has a Type IIA dual in the geometric phase implies that \( \dim_{\mathbb{C}}(\mathcal{H}_{0,(1/2,1/2)}|_{L_0=1}) \geq 2 \) because the Type IIA compactifications has at least one hypermultiplet containing the dilaton. So, \( c_0(0) \geq -2(\rho + 1) \).
Table 2: Euler number $\chi$ in terms of the low-energy BPS indices $n_i$. $n_{2/4}$ and $n_{3/6}$ are no less than $-2$, and all the rest are non-negative.

| $\Lambda_S$ | $\chi(X)$ |
|-------------|-----------|
| $(+2)$      | $\chi = -252 + 56n_{1/2}$ |
| $(+4)$      | $\chi = -168 + 128n_{1/4} + 14n_{2/4}$ |
| $(+6)$      | $\chi = -148 + 108n_{1/6} + 54n_{2/6} + 2n_{3/6}$ |

above, of course, but do not exhaust all possibilities under these conditions. For example, when $\Lambda_S = (+2)$, their list include examples for $\chi = -252, -196, -140, -84$, which correspond to $n_{1/2} = 0, 1, 2, 3$ in the parametrization above, but not for $\chi = -28$ and $n_{1/2} = 4$. The discussion above indicates that there cannot exist Calabi–Yau three-folds with $\Lambda_S = (+32)$ polarized regular K3 fibration (whether toric complete intersection or not) with $\chi < -252$, or $-252 < \chi < -196$, etc. But, we did not find theoretical reason to rule out the case $n_{1/2} = 4$ and $\chi = -28$.

Similar story holds for $\Lambda_S = (+4)$ or $(+6)$; the Table 1 in [24] show geometric realizations for $n_{1/4} = 0, n_{2/4} = -2, 0, 2, 4, 6$ and $n_{1/6} = n_{2/6} = 0, n_{3/6} = 0, 8, 10, 14, 16, 20$. But there exist other values of integers $(n_{1/4}, n_{2/4})$ or $(n_{1/6}, n_{2/6}, n_{3/6})$ satisfying the bounds $n_{1/2}, n_{1/4}, n_{1/6}, n_{2/6} \geq 0, n_{2/4}, n_{3/6} \geq -2$.

The absence of examples with odd $n_{2/4}$ and $n_{3/6}$ in geometry constructions is presumably explained as follows. The BPS index $n_{2/4}$ [resp. $n_{3/6}$] in the case of $\Lambda_S = (+4)$ [resp. $(+6)$] are $NL_{4/8,2/4}$ [resp. $NL_{9/12,3/6}$]. To the Noether–Lefschetz number $NL_{4/8,2/4}$, for example, both $F_{\perp} = (2/4)e'$ and $F_{\perp} = (2/4)e'$ give rise to separate contributions, $NL_{[F_{\perp}],2/4}$ where $[F_{\perp}] = F_{\perp} \Gamma_T$; here, $e'$ is the generator of $(-4) = a_{ab} Z e'$ in $\Lambda_T \cong (-4) \oplus U^{\oplus 2} \oplus E_8[-1]^{\oplus 2}$. They are separate contributions, because the two $F_{\perp}$ shown above are not in a common orbit of $\Gamma_T$. Their Noether–Lefschetz divisors $D_{NL_{[F_{\perp}]}}$ in $D(\Lambda_T)/\Gamma_T$ are the same, so the sum of the two contributions is twice the single contribution, $NL_{[F_{\perp}],2/4}$ (cf. [8]). This argument is adapted in an obvious way to the case of $\Lambda_S = (+6)$ and $n_{3/6}$.

31 In the argument here, we discuss only the contributions $NL_{[F_{\perp}],2/4}$ only from the $\Gamma_T$-orbits $[e/2]$ and $[-e/2]$; even if the $\Gamma_T$-orbit decomposition of $\{F_{\perp} \in e/2 + \Lambda_T \mid -(F_{\perp})^2/2 = 1/2 = \nu\}$ contains another pair represented by $x$ and $-x$ in $e/2 + \Lambda_T$, the same argument as in the main text still works. If there is a $\Gamma_T$-orbit that contains both $x_0$ and $-x_0$ in $e/2 + \Lambda_T$, the trial argument in the main text breaks down, however.

32 The authors do not have confidence to say that the divisor–curve intersection number $NL_{[F_{\perp}],2/4}$ is definitely an integer, because $D(\Lambda_T)/\Gamma_T$ has orbifold singularity associated with K3-surfaces of a non-trivial group of purely non-symplectic automorphisms. If $NL_{[F_{\perp}],2/4} \in \mathbb{Z}$, then $2NL_{[F_{\perp}],2/4} \in 2\mathbb{Z}$.

33 This argument is not applicable to the case $\Lambda_S = (+2)$. That is because $F_{\perp} = (1/2)e'$ and $F_{\perp} = -(1/2)e'$ are in one orbit under $\Gamma_T$. That is consistent with the fact that Ref. [24] found geometric constructions of
We could not rule out \( n_{1/4}, n_{1/6}, n_{2/6} \) that are non-zero or \( n_{3/6} = 2, 4, 6 \). We may have missed some additional physical/mathematical constraints or it is possible that some of such Calabi–Yau three-folds may exist, maybe outside of the scanned range of the combinatorial data in [21], or as those that do not allow their realization by complete intersections in toric varieties.

### 2.3.2 Linear Relations on the Spectrum of Local Effective Field Theory

As stated already in section 2.1.1, the classification invariant \( \Phi \) is in the free abelian group whose rank is the \( \dim \mathcal{C}(\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})) \), and is also completely determined by the low-energy BPS indices \( \{ n|\gamma| \}_{\pm \gamma \in G_S^0} \) (because the weight \(-1 - \rho/2\) of \( \Phi/\eta^{24} \) is strictly negative for any \( \rho = 1, 2, \ldots, 20 \)). This implies immediately that there is a linear relation among \( n_\gamma \)'s when \( \dim \mathcal{C}(\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})) \) is strictly less than \( d^\prec = |G_S^0/\{\pm 1\}| \). As is clear from the Heterotic description, the low-energy BPS indices \( n_\gamma \)'s with \( \gamma \neq 0 \) are the multiplicities of fields with purely electric charge under the \((\rho + 2)\) gauge bosons. So, such a linear relation is that of the spectrum of Lagrangian-based effective field theory on \( \mathbb{R}^3,1 \) with vector-like matter representations; it cannot be related to the 4D triangle anomaly (possibly to 6D box anomaly if \( \Lambda_S \supset U \)), but it originate from the modular invariance of \( \Phi \).

Examples of the lattice \( \Lambda_S \) with such a prediction are found by using the dimension formula \((143, 144)\) if \( \rho < 18 \) (so \( 11 - \rho/2 > 2 \)). Within rank-1 \( \Lambda_S \)'s, the lattice \((+2n) = (+14)\) is the first example, where \( \dim \mathcal{C}(\text{Mod}_0(21/2, \rho_{(14)})) = 7 \), less than \( d^\prec = d = 8 \). We found that

\[
14n_{1/4} + 8n_{2/4} - 13n_{3/4} - 6n_{4/4} - 6n_{5/4} - 6n_{6/4} + n_{7/4} + 28 = 0;
\]

(42)

X for \( \Lambda_S = (+2) \) with odd \( n_{1/2} \)'s.

34 \( n_{1/4} \geq 2 \) and \( n_{1/6} \geq 2 \) are ruled out because \( \chi(X) \) would be larger than \( 2h^{1,1} \) then.

35 The procedure explained in section 4 and exemplified in the appendix B.1.3 is one of the ways to find a constraint on \( n_\gamma \)'s.

36 The modular form \( \Phi \) can be defined in Type IIA compactification on a Calabi–Yau \( X \) that does not necessarily have a K3-fibration; it is the generating function of the helicity supertrace for states originating from a D4-brane wrapped on a divisor \( P \) of \( X \). It is then in \( \text{Mod}(11 - r/2, \rho_{\Lambda}) \), where the lattice \( \Lambda \) is the sublattice of \( H^2(P) \) corresponding to the image \( \iota_P(H^2(X)) \) of the embedding \( \iota_P : P \hookrightarrow X; r := \text{rank} (\Lambda) \). See [31]. Here, the weight \(-1 - r/2\) of \( \Phi/\eta^{24} \) is always strictly negative, and the \( n_\gamma \)'s are the multiplicities of states whose central charge may vanish at a positive Kähler parameter, even in this more general set-up [25, §3]. So, the same argument as in the main text also holds; whenever \( \dim \mathcal{C}(\text{Mod}(11 - r/2, \rho_{\Lambda})) \) is strictly less than \( |G_{\Lambda}/\{\pm 1\}| \), there is a linear relation among the low-energy BPS indices.

For a general divisor \( P \) in \( X \), however, such a linear relation is among the multiplicities of states whose \( U(1) \) charges are not necessarily mutually local. So, it cannot be regarded as a prediction on a spectrum of a Lagrangian-based local effective field theory. In the set-up discussed in the main text, \( P \) is the total fibre class \( D_s \), where \( P \cdot P = 0 \), and \( r = h^{1,1}(X) - 1 \), not \( h^{1,1}(X) \). This property makes all the states from a D4-brane on \( P \) free from magnetic charge (obvious in Heterotic description from the start).
see appendix A.2.2 for necessary details. In a series of $\rho = 3$ lattices $\Lambda_S = U \oplus \langle -2m \rangle$, the $m = 2$ case already has a prediction, because $\dim_C(\text{Mod}_0(19/2, \rho_{-4})) = 2$ and $d^c = d = 3$.

$$n_{1/4} = 8n_{2/4} + 96. \quad (43)$$

Details are found in the appendix A.2.1. In both of the series $\Lambda_S = \langle +2n \rangle$ and $\Lambda_S = U \oplus \langle -2m \rangle$, we confirmed that the dimension of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ lies strictly below $d^c$ and also above zero for large $n$’s and $m$’s, by evaluation of the dimension formula (143, 144). Swampland surely exists within the space of local effective field theories\footnote{The lattice $\Lambda_S$ is characterized within the language of local effective field theory on $\mathbb{R}^{3,1}$; it appears in the prepotential (81).} if we restrict our attention to the class of Heterotic–Type IIA dual vacua reviewed in section 2.1.

It is also found that the vector spaces $\text{Mod}_0(21/2, \rho_{(+2n)})$ and $\text{Mod}_0(19/2, \rho_{(-2m)})$ continue to have strictly positive dimensions\footnote{There are more modular forms of a fixed weight and for $\Gamma(4n)$ for large $n$. So, that is not surprising.} for large $n$ and $m$ (by numerically evaluating (143, 144)). So, the modular invariance of $\Phi$ and the integrality of its coefficients alone do not rule out existence of Calabi–Yau three-folds $X$ with $\Lambda_S = \langle +2n \rangle$-polarized K3-fibration for an arbitrary large $n$, or also with $\Lambda_S = U \oplus \langle -2m \rangle$ for an arbitrary large $m$.

### 2.3.3 Cases with $\rho = 20$ and $\rho = 19$

Here, we have a look at a few families of choices ($\tilde{\Lambda}_S, \Lambda_T$) with $\rho = 20$ and $19$. In some of them, we will see that the vector space $\text{Mod}(11 - \rho/2, \rho_{\Lambda_S})$ is empty, and that there cannot be such a lattice-polarized regular K3-fibration in a Calabi–Yau three-fold, so studies from both sides agree nicely.

When $\rho = 20$, the lattice $\Lambda_T$ is rank-2, positive definite, and even. One can see that the vector space $\text{Mod}(1, \rho_{\Lambda_S})$ is empty in the following way. For any $\Phi \neq 0$ in this vector space, $\theta_{\Lambda_T} \cdot \Phi$ must be a scalar-valued weight-2 modular form starting with $-2 + O(q)$. Because there is no such weight-2 modular form, the vector-valued modular form $\Phi$ should have been zero.

One can also arrive at almost the same conclusion independently by using geometry available in Type IIA language. If a Calabi–Yau three-fold $X_{\text{IIA}}$ has a K3-fibration with a generic fibre having $\rho = 20$ Neron–Severi lattice, the fibre K3 surface has a fixed complex structure over the entire base $\mathbb{P}^1$, so $X_{\text{IIA}}$ must be of the form $(\rho = 20 \text{ K3}) \times \mathbb{P}^1_{\text{IIA}}$. This is not a Calabi–Yau three-fold, so there should not be such a K3-fibred Calabi–Yau three-fold. It
should be noted, however, that this second argument does not rule out non-geometric phase Type IIA constructions in a $\rho = 20$ case, and hence the first argument is stronger.

Similar arguments also rule out a family of $\rho = 19$ cases

$$\tilde{\Lambda}_S = U^{\oplus 2} \oplus E_8[-1]^{\oplus 2} \oplus \langle -2n \rangle, \quad \Lambda_T = U \oplus \langle +2n \rangle,$$

(44)

where $n \in \mathbb{Z}_{>0}$. The first argument for the $\rho = 20$ cases can be repeated by replacing $\theta_{\Lambda_T}$ with $\theta_{+2n}$, to see that the vector space $\text{Mod}(3/2, \rho_{\Lambda_S})$ is empty. So there is no suitable $\Phi$ in this case.

The absence of such a $\Phi$ (with $n_0 = -2$) is also understandable in geometry language. The Fourier coefficients of $\Phi$ are the intersection numbers of the Noether–Lefschetz divisors in $D(\Lambda_T)/\Gamma_T$ (they are points in the $\rho = 19$ cases), and the image of the holomorphic map $\iota_\pi : \mathbb{P}^1_{\text{IIA}} \to D(\Lambda_T)/\Gamma_T$ determined by the K3 fibration map $\pi : X_{\text{IIA}} \to \mathbb{P}^1_{\text{IIA}}$. The modulai space $D(\Lambda_T)/\Gamma_T = \mathcal{H}/\Gamma_0(n)$ contains the large complex structure limit point; if $\iota_\pi$ is surjective, then there are points in the base $\mathbb{P}^1_{\text{IIA}}$ where the K3-fibration is not regular. All the $\rho = 19$ K3-fibration studied in [33]. If $\iota_\pi$ were to be a constant map, then $X_{\text{IIA}}$ would not be a Calabi–Yau three-fold (see the second argument for the $\rho = 20$ case).

The argument for the absence of $\Lambda_S$-polarized regular K3-fibred Calabi–Yau three-folds holds true for all the $\rho = 19$ cases, not necessarily for the $\Lambda_T = U \oplus \langle +2n \rangle$ cases discussed above. The authors do not have a proof yet that $\text{Mod}(3/2, \rho_{\Lambda_S})$ is empty for more general $\Lambda_T$’s in the $\rho = 19$ case, however.

The consequence that the dimension of $\Phi$ is smaller for larger $\rho$ is understandable intuitively in itself. K3 fibration is specified, after all, by specifying a map from the base $\mathbb{P}^1$ to the period domain $D(\Lambda_T)$; less complicated geometry $D(\Lambda_T)$ allows less variety in the map from $\mathbb{P}^1$ to $D(\Lambda_T)$.

### 2.4 Lower Bound on Euler Number

We have seen in Table 2 that the Euler number $\chi(X_{\text{IIA}})$ of the Calabi–Yau three-fold $X_{\text{IIA}}$ that admits $\Lambda_S$-polarized K3-fibration is given by a linear sum of the low-energy BPS in dices \{\n_{|\gamma|}\}, for a few choices of $\Lambda_S$. In Table 2, all the coefficients of \{\n_{|\gamma|}\} are positive, from

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30 Because we only think of Type IIA compactification in the Im($s$) $\gg$ 1 regime, this premise almost says that the base $\mathbb{P}^1$ is in the geometric phase. But we still need to find an appropriate technical language to extend this argument to cover constructions without entirely geometric $X_{\text{IIA}}$.

40 In the case of $\Lambda_T = U \oplus \langle +2n \rangle$, $\text{Isom}(\Lambda_T) \cong \Gamma_0(n)$ (see [32]), but $\Gamma_T \not\cong \Gamma_0(n)$.
which it follows that $\chi(\text{XIIA})$ is bounded from below. Actually, this is true for any choice of $\Lambda_S$, as we see below. Suppose $\phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^x)$. Then

$$\phi \cdot \frac{\Phi}{\eta^{24}} = -2[\phi]q^0 \frac{E_4^2 E_6}{\eta^{24}}$$

(45)

because $\phi \cdot \Phi$ must be a scalar-valued SL(2; $\mathbb{Z}$) modular form of weight-14 with the leading coefficient $[\phi \cdot \Phi]q^0 = -2[\phi]q^0$. By comparing the coefficients of the $q^0$ term on both sides, we obtain one linear relation of $\{n_{\gamma}\}$s and $c_0(0) = -\chi(\text{XIIA})$ for one $\phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^x)$.

A nontrivial theta can be constructed by a theta function for a suitable lattice. The $(1, 7 + \rho)$ lattice $\Lambda_S \oplus E_8[-1]$ has a primitive sublattice isometric to $U$. Since $U$ is unimodular, the orthogonal complement $L := [U^\perp \subset \Lambda_S \oplus E_8[-1]]$ satisfies

$$\Lambda_S \oplus E_8[-1] \cong U \oplus L.$$  

(46)

This lattice $L$ has signature $(0, 6 + \rho)$ and shares same discriminant form $(G_S, (-,-)_{G_S})$ with $\Lambda_S$, so the lattice theta function $\theta_L[-1]$ is in $\text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^x)$. The linear relation for $\phi = \theta_L[-1]$ is

$$\sum_{b \in L^\vee} c_{[b]}(b^2/2) = 0.$$  

(47)

Since $L[-1]$ is positive definite and $c_{s}(\nu) = 0$ for $\nu < -1$, the lhs is a finite sum. This leads to

$$\chi(X) = -c_0(0) = \sum_{b \in L^\vee} n_{[b]} + \sum_{b^2 = -2} c_{[b]}(-1) = \sum_{b \in L^\vee} n_{[b]} - 2 \cdot \#(\text{roots of } L).$$  

(48)

In the last equality, we have used $c_{s}(-1) = 0$ for $s \neq 0 \in G_S$ (see [14]). The linear coefficient of $n_{|\gamma|}$ on the rhs is positive for any $|\gamma| \neq 0 \in G_S/\{\pm 1\}$, because we used a lattice theta function

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1 There is a more general version of this argument. A linear relation among Fourier coefficients of arbitrary $\Phi' \in \text{Mod}(2k + 11 - \rho/2, \rho_{\Lambda_S})$ may be obtained by using $\phi \in \text{Mod}(2k' + 1 + \rho/2, \rho_{\Lambda_S}^x)$; the combination $\phi \cdot \Phi'$ must be a scalar-valued SL(2; $\mathbb{Z}$) modular form of weight-(2$k + 2k' + 12$).

2 The same relation is obtained from $0 = \oint_{T \sim \infty} dT \phi \cdot (\Phi/\eta^{24}) = [\phi \cdot (\Phi/\eta^{24})]q^0$, because $\phi \cdot \Phi/\eta^{24}$ is of weight 2 (e.g., [24]).

3 To see this, choose an element $x \in \Lambda_S$ with positive norm $2n$; there exists such $x$ for some $n > 0$, since $\Lambda_S$ is indefinite. One can choose a primitive element $y \in E_8[-1]$ with norm $-2n$, and also an element $z \in E_8[-1]$ such that $(y, z) = 1$. Now, the sublattice spanned by $x + y, z \in \Lambda_S \oplus E_8$ is isometric to $U$.

4 For example, when $\Lambda_S = U \oplus W$ for some even lattice $W$ of signature $(0, \rho - 2)$, there is always an obvious embedding of $U$ into $\Lambda_S \oplus E_8[-1] = U \oplus W \oplus E_8[-1]$. $L = W \oplus E_8[-1]$, so $\theta_L[-1] = \theta_W[-1]E_4$.  

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21
Table 3: Lower bounds on $\chi(X)$ for $X$’s that have a regular $\Lambda_S = \langle +2n \rangle$-polarized K3-fibration. Two lower bounds are listed for $\langle 2n \rangle = \langle 14 \rangle$ because we can construct two inequivalent lattices $L$.

| $2n$ | $\chi \geq$ |
|------|----------|
| 2    | $-252$  |
| 4    | $-196$  |
| 6    | $-152$  |
| 8    | $-112$  |
| 10   | $-124$  |
| 12   | $-124$  |
| 14   | $-144$  |
| 14   | $-92$   |

for $\phi$. The lower bounds on $\{n_\gamma\}$’s imply a lower bound on $\chi(X)$:

$$\chi \geq -2 \cdot \# \{b \in L^\vee ; -2 \leq b^2 < 0\}.$$  \hspace{1cm} (49)

If there appear no higher-level current algebras (i.e. there are no $\gamma \in G_S$ that satisfy the condition (35)) then $n_\gamma \geq 0$ for $\gamma \neq 0 \in G_S$, and

$$\chi \geq -2 \cdot \#(\text{roots of } L).$$  \hspace{1cm} (50)

It is not necessary to work out a basis of $\text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S})$ (where $\Phi$ is in) in deriving such relations as those in Table 2.

The relation (48) reproduces those in the Table 2 by using $\phi = \theta_{L[-1]}$ for $L$ determined as in footnote 43. We applied the same procedure for some $\Lambda_S = \langle +2n \rangle$ not covered in section 2.3 and obtained the result summarized in Table 3; calculations that led to Table 3 are found in the appendix A.2.2. In the cases of $\Lambda_S = U \oplus W$ for some even lattice $W$, the relation (48) for $\phi = \theta_{W[-1]}E_4$ yields

$$\chi(X_{\text{IIA}}) = \left( \sum_{b \in W^\vee \mid b^2 > -2n[b]} + (-2) \times |\{b \in W \mid b^2 = -2\}| - 480 \right).$$  \hspace{1cm} (51)

In particular, $\chi = -960$ when $W = E_8[-1]$.

All the lower bounds of $\chi(X_{\text{IIA}})$ for individual $\Lambda_S$ are safely above the absolute lower bound for all Calabi–Yau three-folds $X_{\text{IIA}}$ [33]

$$\chi \geq -\frac{5}{3} e^{4\pi} \sim -5 \times 10^5.$$  \hspace{1cm} (52)

The bound (52) was derived [33] by exploiting the modular invariance of the fundamental string partition functions of Type II compactification (with $X_{\text{IIA}}$ as the target space), while the bound (49) is due to the fact that the generating function $\Phi$ of the helicity supertraces of BPS D4–D2–D0 brane bound states on $X_{\text{IIA}}$ is a modular form, and the fact that the new supersymmetry index $Z_{\text{new}}$ of the fundamental string in the Heterotic description must be a modular form of weight (-1,1).
One also finds from the relation (48) that the low-energy BPS indices also have an upper bound. This is because

\[ \chi(X_{\text{IIA}}) = 2h^{11} - 2h^{21} \leq 2h^{11} = 2(\rho + 1). \] (53)

This is a generalization of the same observation made already in section 2.3.1. Note, however, that all of the coefficients \( n_\gamma \) do not necessarily appear in the equation (48); in the case of \( \Lambda_S = U \oplus \langle -2n \rangle =: ab U \oplus \mathbb{Z}e, \) for example, the linear relation (48) for \( \phi = \theta_{(+2n)}E_4 \) has contributions only from \( \gamma = [\frac{x}{2n}e] \in G_S \) for \( 0 \leq x \leq \sqrt{4n} \). These are only \( \mathcal{O}(\sqrt{n}) \) coefficients among \( \mathcal{O}(n) \) low-energy BPS indices. So, we cannot use this argument to claim that only a finite set of the low-energy BPS indices \( \{n_\gamma\} \) corresponds to a geometric phase in Type IIA description.

### 3  Finer Classifications

The modular form \( \Phi \) (new supersymmetry index (Het)/ NL number generating function (IIA)) is not enough discrete data for classification of families (branches) of moduli space of Het–IIA dual vacua. Let us take the \( \Lambda_S = U \) case as an example. In the Heterotic language, all the \( K^3 \times T^2 \) compactifications with the 24 instantons on K3 distributed by \((12 - n, 12 + n)\) to the two weakly coupled \( E_8 \) gauge groups share the same \( \Phi = (\Phi_4 \Phi_6) \) for all \( 0 \leq n \leq 2 \), but they form three distinct branches of moduli space. In the Type IIA language, the modular form \( \Phi \) determines the Gopakumar–Vafa invariants of all the vertical curve classes of elliptic-K3 fibred Calabi–Yau three-folds \( X_{\text{IIA}} \), but the classical trilinear intersection numbers of the divisors are not \( \{7\} \) (for precise statements, see section 3.1.3). \( X_{\text{IIA}} \) can be any one of the elliptic fibrations over \( F_n \) with \( n = 0, 1, 2 \). Presence of such multiple branches of moduli space sharing \( \Phi \) has been reported also for the case of \( \Lambda_S = \langle +2 \rangle \) and \( \langle +4 \rangle \) (see also \[11\]).

We introduce invariants of branches of moduli space of Het–IIA dual vacua that can distinguish those sharing a common \( \Phi \). That is done by developing observations and ideas that are found in the literatures. Those invariants do not rely on supergravity approximation or explicit construction of geometries, but use modular forms.

#### 3.1  The Idea

Consider a branch of moduli space of Het–IIA dual vacua, where we have special geometry and hypermultiplet moduli space of fixed dimensions \( (h^{1,1}(X_{\text{IIA}})) \)- and \( (h^{2,1}(X_{\text{IIA}}) + 1) \)-dimensions,
respectively, if the branch contains a geometric phase in the Type IIA language). We call it the original branch. It often comes with special loci in the hypermultiplet moduli space where non-abelian gauge symmetry $\mathcal{R}$ is enhanced in the effective theory on $\mathbb{R}^{3,1}$; one ventures into other branches of moduli space by turning on non-zero Coulomb vevs in $\mathcal{R}$. Modular forms denoted by $\Psi$ and $\Phi$ are assigned to such a symmetry-enhanced branch (see below for more); the idea is to use the set of such modular forms as an invariant of the original branch. We will see in this section that the set of $\Psi$’s or the set of $\Phi$’s distinguish multiple branches sharing the same $\tilde{\Lambda}_S$, $\Lambda_T$, and $\Phi$; moreover, the modular form $\Psi$ or $\Phi$ of even just one symmetry-enhanced branch attached to the original branch already improves classification by the modular form $\Phi$ alone.

### 3.1.1 Higgs Cascades and Modular Forms

Let us first assign two modular forms $\Phi$ and $\Psi$ for a symmetry-enhanced branch. We will discuss in section 3.1.3 the information of target-space geometry that we can extract from such a modular form $\Psi$, or from the set of $\Psi$’s associated with all the symmetry-enhanced branches.

We restrict our attention to the case of $\mathcal{R}$ as one of ADE types, and its non-abelian gauge bosons are given by left-mover level $k = 1$ current algebra in the Heterotic language. The lattice $\mathcal{R}[-1]$ is chosen within $\Lambda_T$; now we introduce lattices

$$\Lambda_T := [\mathcal{R}[-1] \subset \Lambda_T], \quad \tilde{\Lambda}_S := [\Lambda_T \subset \Pi_{4,20}]$$

for the symmetry-enhanced branch. The lattice $\tilde{\Lambda}_S$ is $\tilde{\Lambda}_S \oplus \mathcal{R}[-1]$ or its extension.

It is assumed here that the symmetry-enhanced branch is also realized without NS5-branes and the likes in the Heterotic description, or without a degeneration of K3 fibre classified as Type II or III in the Type IIA description. A related discussion is found at the end of section 3.2.2.

Under this assumption, there must be a modular form

$$\Phi \in \text{Mod}_0(11 - \rho/2, \rho_{\Lambda_S});$$

it describes the BPS indices of the Heterotic description and the Noether–Lefschetz numbers of the K3-fibre in the Type IIA description in the symmetry-enhanced branch, just like $\Phi$.

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45 It is possible for a gauge group with level $k > 1$ to enhance, but we do not use it as a probe in this article.
does for the original branch. Here $\rho := \rho + \text{rank}(R)$, and $\rho_{\Delta s}$ the representation of $\text{Mp}(2,\mathbb{Z})$ associated with the lattice $\Lambda_{\Delta s}$.

The modular form $\Phi$ of the symmetry-enhanced branch should be related to $\Phi$ of the original branch in the following way. At the entrance of the symmetry-enhanced branch (so the Coulomb branch moduli still stays within the subset $D(\tilde{\Lambda}_{\Delta})$ of $D(\Delta_{\Delta})$), the non-abelian symmetry $R$ remains unbroken. Since this vacuum belongs to both of the original and symmetry-enhanced branch, the new supersymmetry index $Z_{\text{new}}$ of the both branches should be equal at this point:

$$\sum_{\gamma \in G_{\Delta s}} \theta_{\Lambda_{\Delta s}[-1] + \gamma} \Phi_\gamma = \sum_{\gamma \in G_{\Delta s}} \theta_{\Lambda_{\Delta s}[-1] + \gamma} \Phi_\gamma.$$  \hfill (56)

Here $G_{\Delta s} = \tilde{\Lambda}_{\Delta s}/\tilde{\Lambda}_{\Delta s}$. Denoting $G_0 = \tilde{\Lambda}_{\Delta s}/(\tilde{\Lambda}_{\Delta s} \times R[-1]) \subset G_{\Delta s} \oplus G_R$ and $[\gamma, \delta] = (\gamma, \delta) + \tilde{\Lambda}_{\Delta s} \in G_{\Delta s}$ for $(\gamma, \delta) \in G_0$, we can rewrite the lhs of the above as

$$\sum_{(\gamma, \delta) \in G_0} \theta_{\Lambda_{\Delta s}[-1] + \gamma} \Phi_{\gamma, \delta} [\gamma, \delta].$$  \hfill (57)

Therefore, the modular form $\Phi$ of the symmetry-enhanced branch should reproduce $\Phi$ of the original branch through

$$\Phi_\gamma = \sum_{(\gamma, \delta) \in G_0} \theta_{\gamma, \delta} \Phi_{\gamma, \delta}.$$  \hfill (58)

In the case of $\tilde{\Lambda}_{\Delta s} = \tilde{\Lambda}_{\Delta} \oplus R$, this simplifies to \hfill (38)

$$\Phi_\gamma = \sum_{\delta \in G_R} \theta_{\gamma, \delta} \Phi_{\gamma, \delta}.$$  \hfill (59)

The modular form $\Psi$ is associated with 1-loop\hfill (46) threshold correction $\Delta_R$ to the coupling constant of the enhanced non-abelian gauge group $R$. It is given by \hfill (39)

$$\Delta_R \delta^{IJ} = \int_{\mathbb{H}/\text{SL}_2 \mathbb{Z}} \frac{d\tau_1 d\tau_2}{\tau_2^2} \left( B^{IJ} - b_R \delta^{IJ} \right),$$  \hfill (60)

$$B^{IJ} = \frac{-i}{\eta^2} \text{Tr}_{R} \left( e^{\pi i J_0 \tilde{J}_0} - \frac{e^{\pi i J_0 \tilde{J}_0}}{24} q^{\tilde{J}_0} - \frac{e^{\pi i J_0 \tilde{J}_0}}{24} \left( \frac{Q^I Q^J}{2} - \frac{\delta^{IJ}}{8 \pi \tau_2} \right) \right).$$

\hfill 46 The corrections to special geometry that is regarded as 1-loop contributions in Heterotic string language are those that neither diverge (tree in Het) nor vanish (non-perturbative in Het) in the large base limit.
where the Heterotic string is used as a language. Here, $I, J \in \{1, \cdots, \text{rank}(\mathcal{R})\}$ label $\text{rank}(\mathcal{R})$ left-moving free bosons $X^I$, and $Q^I$ is the zero-mode momentum in the expansion $X^I(z) = x^I + Q^I \ln z + \text{oscillators}$. $Q^I$ works as the Cartan charge operator of $\mathcal{R}$. $b_\mathcal{R} = \left( \sum_{\text{hyper}} 2T_{\text{rep}} \right) - 2T_\mathcal{R}$ is the 1-loop beta function of the probe gauge group $\mathcal{R}$. Contracting the indices $I, J$, we arrive at

$$
\Delta_\mathcal{R} = \int_{\mathcal{H}/\text{SL}_2\mathbb{Z}} \frac{d\tau_1 d\tau_2}{\tau_2} (B_\mathcal{R} - b_\mathcal{R}),
$$

$$
B_\mathcal{R} = \frac{1}{24} \sum_{\gamma \in G_S} \theta_{\Lambda_S[-1]+\gamma} \frac{\Phi_2 \hat{E}_2 - \Psi_\gamma}{\eta^{24}},
$$

$$
\Psi_\gamma = -\frac{24}{\text{rank}(\mathcal{R})} \sum_{(\gamma, \delta) \in G_0} (\partial^S \partial_{\mathcal{R}+\delta}) \Phi_{[\gamma, \delta]}.
$$

Here $\partial^S$ is the Ramanujan–Serre derivative (see the appendix A.1). Immediately from (55, 63),

$$
\Psi \in \text{Mod}_0(13 - \rho/2, \rho_{\Lambda_S}).
$$

Obviously the modular form $\Psi/\eta^{24}$ (for not necessarily unimodular $\Lambda_S$) is the generalization of $\Psi/\eta^{24} \in \text{Span}_\mathbb{C}\{E_4^3, E_6^3\}/\eta^{24}$ for $\Lambda_S = U$ and $\Psi/\eta^{24} = (-2E_4^2)/\eta^{24}$ for $\Lambda_S = U \oplus E_8[-1]$.

The modular form $\Psi$ of the symmetry-enhanced branch captures only a part of information in $\Phi$, because $\Psi$ can be determined from $\Phi$ as in (63). In fact, when there is a chain of symmetry enhancements $\mathcal{R}_1 \subsetneq \mathcal{R}_2 \subsetneq \cdots$ accompanied by chain of tunings in the hypermultiplet moduli space, the chain of the invariants $(\tilde{\Lambda}_S, \Lambda_T, \Phi)$ all reproduce one common modular form $\Psi$ through (63). This is because the corrections to the gauge coupling constants remain unchanged by continuous change in the hypermultiplet vevs in $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{R}^{3,1}$, an observation implicit already in [7]. For this reason, the modular form a $\Phi$ is assigned to each symmetry-enhanced branch, but $\Psi$ to a chain of symmetry-enhanced branches attached to the original branch (such a chain is called a Higgs cascade).

We will see in section 3.1.3 that the modular form $\Psi$ for one Higgs cascade attached to the original branch—an arbitrarily chosen cascade is fine—specifies the diffeomorphism class of $X_{1\text{A}}$ of the original branch (of the individual geometric phase chambers of the original branch, to be more precise); the modular form $\Phi$ alone does not have enough information for this purpose in general. Furthermore, the set of $\Psi$’s for the set of Higgs cascades attached

\footnote{normalized so $X^I(z)X^J(w) \sim -\delta^{IJ}\ln(z - w)$}
to the original branch can also be used as a classification invariant of the original branch. This viewpoint is sometimes useful for distinguishing two different branches of moduli space with the same diffeomorphism class of $\Lambda_S$-polarized K3-fibred Calabi–Yau three-folds. The modular forms $\Phi$ are more useful in capturing the network of symmetry-enhanced branches.

### 3.1.2 The Space of the Modular Form $\Psi$’s

For a given branch of Het–IIA dual moduli space characterized by $(\tilde{\Lambda}_S, \Lambda_T, \Phi)$, the modular form $\Psi$ of a Higgs cascade of the original branch is not completely arbitrary element of the vector space $(64)$. We will derive a few constraints on $\Psi$ in the following.

#### General Constraints

First, recall the definition:

$$
\frac{\theta_{\tilde{\Lambda}_S[-1]} \cdot (\Phi E_2 - \Psi)}{\eta^{24}} \delta_{IJ} = 12 \cdot \frac{-i}{\eta^2} \text{Tr}_{\mathcal{R}} (c, \tilde{c}) = (22, 9) \left[ e^{\pi i \tilde{\Lambda}_S J_0 q^L_0 - \tilde{q} q^L_0 - \frac{\pi}{2} Q^I Q^J \right] \tag{65}
$$

for $1 \leq I, J \leq \text{rank}(\mathcal{R})$. Choose a basis of the left-moving free bosons $X^I$ so that roots of a fixed subalgebra $\mathfrak{su}(2) \subset \mathcal{R}$ have charges only in $I = 1$; $Q^1 = \pm \sqrt{2} \delta^1_1$. Now set $I = J = 1$ in the above equation. Since all the states in the Hilbert space with a definite charge under $\mathcal{R}$ have $Q^I = 1$, contributions from states with a charge $(w, Q^1)$ under $\tilde{\Lambda}_S \oplus \mathfrak{su}(2)$ and those with a charge $(w, -Q^1)$ add up to be an integer. This implies that

$$d_{\gamma}(\nu) \in 12\mathbb{Z}, \quad \frac{\Phi_\gamma E_2 - \Psi_\gamma}{\eta^{24}} =: \sum_{\nu \in h_{\min}(\gamma) + \mathbb{Z}} d_{\gamma}(\nu) q^\nu \tag{66}
$$

As an immediate consequence, all the Fourier coefficients $c^{\Psi}_\gamma(\nu)$ in

$$
\frac{\Psi}{\eta^{24}} = \sum_{\gamma \in G_S} e_\gamma \sum_{\nu \in Q} c^{\Psi}_\gamma(\nu) q^\nu \tag{67}
$$

are all integers.

In other words, this comes just from the properties of lattice theta functions of simple Lie algebra: Using the relations $(63, 65)$, we see that

$$
\frac{\Phi_\gamma E_2 - \Psi_\gamma}{\eta^{24}} = 12 \frac{2}{\text{rank}(\mathcal{R})} \sum_{(\gamma, \delta) \in G_0} q^{\frac{\partial}{\partial q} \theta_{\mathcal{R} + \delta}} \Phi^{(\gamma, \delta)} \eta^{24}. \tag{68}
$$

Defining $a^{(\mathcal{R})}_\delta(\nu)$ by

$$
\sum_{\nu \in \delta^2/2 + \mathbb{Z}} a^{(\mathcal{R})}_\delta(\nu) q^\nu := \frac{2}{\text{rank}(\mathcal{R})} q^{\frac{\partial}{\partial q} \theta_{\mathcal{R} + \delta}}, \tag{69}
$$

27
we have
\[ d_\gamma(\nu) = 12 \sum_{(\gamma, \delta) \in G_0} \sum_{\nu' + \nu'' = \nu} a_\delta(\nu') \zeta_{\gamma, \delta}(\nu''). \tag{70} \]

The integrality of \(a_\delta(\nu)\) can be seen in essentially the same way as the discussion above (see also the appendix \[\text{A.2.3}\]). So we have \(d_\gamma(\nu) \in 12\mathbb{Z}\), because the BPS indices \(\zeta_{\gamma, \delta}(\nu)\) are also integers.

As we see later, consistency in the low-energy effective field theory on \(\mathbb{R}^{3,1}\) implies that \(d_0(0) \in 24\mathbb{Z}_{\geq 0}\). We have not tried much to think whether this condition can be derived directly from consistency of string theory.

Not all the \(d_\gamma(\nu)\)'s are arbitrary integers divisible by 12 (or 24). It follows immediately\(^{48}\) from the fact \(a_0^{(R)}(0) = 0\) that
\[ d_\gamma(-1) = 0, \quad \forall \gamma \in G_S, \quad \text{s.t.} \quad (\gamma, \gamma)/2 = 0 \in \mathbb{Q}/\mathbb{Z}. \tag{71} \]

We can say a little more. Define \(m_\gamma := d_\gamma([\gamma^2/2]_{\text{frac}} - 1)\) for \((\Phi E_2 - \Psi)\), like we defined \(n_\gamma\) for \(\Phi\):
\[ m_\gamma = 12 \sum_{(\gamma, \delta) \in G_0} a_\delta(\nu_\delta) n_{[\gamma, \delta]}. \tag{72} \]

Here only \(\delta \in G_R\) such that \([\gamma, \gamma)/2]_{\text{frac}} > \nu_\delta\) can contribute to the sum. In particular,
\[ m_\gamma = 0 \quad \forall \gamma \in G_S, \quad \text{s.t.} \quad [\gamma^2/2]_{\text{frac}} \leq 1/4, \tag{73} \]
regardless of \(R\), because \((\delta = e_1, R = A_1)\) gives the smallest\(^{49}\) \(\nu_\delta\) among all possibilities for \((\delta, R)\). This supercedes the condition (71). See the appendix \[\text{A.2.3}\] for list of values of \(\nu_\delta\) for various \(R\)'s.

Ref. \[\text{7}\] sets a constraint that Heterotic string tachyon states should have zero contribution to the gauge threshold correction \(\Delta_R\) (because they are not charged under \(R\)) in relating \(\Psi\) to \(\Phi\), and this reasoning was enough to determine \(\Psi\) completely in the case of \(\Lambda_S = U \oplus E_8[-1]\). The relation (71) and (73) are kinds of generalizations of this argument. On the other hand, \(m_\gamma\) for \(\gamma\) satisfying \([\gamma^2/2]_{\text{frac}} \geq 1/4\) can be non-trivial in many cases. An easiest example is for \(\Lambda_S = U + A_1[-1]\) and enhancement of symmetry \(R\) so that \(A_1 + R\) is contained in one weakly coupled \(E_8\) of the Heterotic string theory.

\(^{48}\) This is because \(\nu_\delta := \min\{x^2/2 \mid x \in \delta \subset R^\vee\}\) is strictly positive for \(\delta \neq 0\).

\(^{49}\) This is because \(A_1\) is Lie subalgebra of any \(R\).
Let us introduce a space denoted by Mod$^\Phi_0 (13 - \rho/2, \rho \Lambda_S)$ for a given $\Phi$, which is the set of modular forms $\Psi$ satisfying $d_\gamma (\nu) \in 12\mathbb{Z}$, $d_0 (0) \in 24\mathbb{Z}_{\geq 0}$, and (73). When the modular form $\Psi$ is for a Higgs cascade attached to the original branch with $\Phi$, then $\Psi$ must be in this set. For example, in the case of $\Lambda_S = U$ (where $\Phi = -2E_4E_6$), the space Mod$^\Phi_0 (12, \rho U)$ consists of $\Psi = -E_3^4 - E_2^6 + (288 - d(0))\eta^{24}$ with $d(0) \in 24\mathbb{Z}_{\geq 0}$.

**Field-theory Argument for $d_0 (0) \in 24\mathbb{Z}_{\geq 0}$:** We realize that $d_0 (0) \in 24\mathbb{Z}$, not just in $12\mathbb{Z}$, by remembering that it is related to the 1-loop beta function $b_R$ through $b_R = d_0 (0) / 24$ (eg. (7)); this relation itself is obvious also from the expression (70): $d_0 (0) = \sum_{(0, \delta)_{\rho \in G_0}} a^{(\mathcal{R})} (\nu_\delta) \mu_{0, \delta} (-\nu_\delta) \rightarrow -2T_R + \sum_{\text{halfhyp}} T_\delta = b_R, \quad (74)$

where we made a replacement $a^{(\mathcal{R})} (1) \rightarrow 2T_R$ and $a^{(\mathcal{R})} (\nu_\delta) \rightarrow 2T_\delta$ for $\nu_\delta < 1$, and interpreted $\mu_{0, \delta} = \mu_{0, -\delta}$ as the (effective) number of half-hyper multiplets in the representation. We see $b_R \in \mathbb{Z}/2$ comes from $d_0 (0) \in 12\mathbb{Z}$. But $b_R$ can be in $1/2 + \mathbb{Z}$ only when $\mathcal{R} = A_1$ and there are odd number of hypers in the fundamental representation, which is not allowed because it would cause the SU(2) global anomaly. So $b_R \in \mathbb{Z}$, and $d_0 (0) \in 24\mathbb{Z}$.

In addition, $b_R$ should be non-negative because there must be plenty of matter fields to Higgs the gauge symmetry $\mathcal{R}$ completely; the Higgsing brings the symmetry-enhanced branch back to the original branch. In the case of $\Lambda_S = U$ with the probe symmetry set in one of the two weakly coupled $E_8$ of the Heterotic string, for example, it is known that there are $I = 10 + 12^{-2}d(0)$ instantons in the rest of the $E_8$. The condition that $d(0) \geq 0$ corresponds to the fact that at least 10 instantons on K3 are necessary to break $E_8$ completely.

As a side remark, one notices (see the appendix A.2.3) that the coefficients $a^{(\mathcal{R})} (\nu)$ for smaller values of $\nu$ are divisible by 2 in $D_{r \geq 4}$, by 6 = $2T_{27}$ in $\mathcal{R} = E_6$, 12 = $2T_{56}$ in $\mathcal{R} = E_7$, and by 60 = $2T_{248}$ in $\mathcal{R} = E_8$, although the authors have not know a proof that this property may persist for arbitrary large values of $\nu$. So, a modular form $\Psi$ is such that all the $d_\gamma (\nu)$’s are divisible not just by 12, but by 24 [resp. 12 x 6, 12 x 12, or 12 x 60] if the Higgs cascade to which $\Psi$ is assigned has an enhanced symmetry $\mathcal{R}$ as large as $D_{r \geq 4}$ [resp. $E_6$, $E_7$, or $E_8$].

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50 Remember that we pose the question how large an enhanced symmetry can be here within Heterotic compactifications without 5-branes, or within Type IIA compactification where the K3-fibration remains regular.
Extra Degrees of Freedom  The set $\text{Mod}^\Phi_0(13-\rho/2, \rho_{\Lambda_S})$ is parametrized by finite number of $d_\gamma(\nu)$’s in $12\mathbb{Z}$. Those with $\nu < 1$—the $m_\gamma$’s—are enough in the case of $\rho > 2$, and those of $q^\nu$ with $\nu < 2$ are enough if $\rho = 1, 2$, because $\Psi/\eta^{24}$ [resp. $\Psi/\eta^{48}$] has negative weight when $\rho > 2$ [resp. $\rho = 1, 2$]. Those $d_\gamma(\nu)$’s (or equivalently the Fourier coefficients $c_\gamma(\nu)$’s) may be subject to some linear constraints, just like we discussed for $\Phi$ in section 2.3.2.

In the case $\rho = 2$, the remaining freedom in the space $\text{Mod}^\Phi_0(0, \rho_{\Lambda_S})$ not specified by the $m_\gamma$’s is in the free abelian group $\text{Mod}^\mathbb{Z}(k = 0, \rho_{\Lambda_S})$. This abelian group is equivalent to that of a ($\tau$-independent) vector $\phi = \{\phi_\gamma = \Delta d_\gamma(0)\} \in 12\mathbb{Z}[G_S/\{\pm 1\}]$ invariant under $\rho_{\Lambda_S}(g)$ for any $g \in \text{Mp}(2, \mathbb{Z})$.

It is enough to make sure that $\{\phi_\gamma\}$ is invariant under $\rho_{\Lambda_S}(T)$ and $\rho_{\Lambda_S}(S)$. The invariance under $\rho_{\Lambda_S}(T)$ implies that $\phi_\gamma \neq 0$ only if $(\gamma, \gamma)/2 \in \mathbb{Z}$. Imposing the invariance under the $\rho_{\Lambda_S}(S)$, it follows that non-zero $\phi_\gamma$ is possible only when $G_S$ contains a non-zero isotropic element $\gamma \neq 0 \in G_S$ or $G_S = \{0\}$ (so $\Lambda_S = U$). For most of rank-2 $\Lambda_S$, therefore, there is no extra degree of freedom for $\Psi/\eta^{24}$.

Suppose that $G_S$ contains an isotropic subgroup $H$ and that $|H| = \sqrt{|G_S|}$ Whenever there is such a subgroup $H$, the vector $\phi = \{\phi_\gamma = 1$ if $\gamma \in H$, $\phi_\gamma = 0$ otherwise$\}$ is invariant under $\rho_{\Lambda_S}(S)$. So, there is one independent extra degree of freedom in the form of $\Delta \Psi = x\phi_0\eta^{24}$, with a parameter $x$. For example, in the case of $\Lambda_S = U[N]$ for some integer $N > 1$, there are two isotropic subgroups $H \cong \mathbb{Z}_N$ in $G_S = \mathbb{Z}_N \times \mathbb{Z}_N$, so there are at least two extra degrees of freedom for $\Psi/\eta^{24}$ besides $n_{|\gamma|}$ and $d_\gamma(\nu)$’s. We have neither been able to prove that all the invariant vectors of a Weil representation are written as linear combinations of vectors associated with isotropic subgroups $H$ with $|H| = \sqrt{|G_S|}$, nor to find a counter example.

In the case of $\rho = 1$, namely, $\Lambda_S = (+2n)$ for some $n \in \mathbb{N}_{>0}$, the $2(n+1)$ integers $\{\Delta m_\gamma\}$ and $\{\Delta d_\gamma(0 < \nu < 1)\}$ must be enough to parametrize $\Psi$ for a given $\Phi$. They are often redundant, however.\footnote{51 Consider lifting $\phi \in \text{Mod}(k = 0, \rho_{\Lambda_S})$ to a modular curve $\overline{H}/\Gamma(N)$ where $\Gamma(N)$ is in the kernel of the representation $\rho_{\Lambda_S}$. The lift $\phi$ should be $\mathbb{C}$-valued functions on the compact curve, so it has to be $\tau$-independent.} We do not have a general theory about how many linear constraints exist within $\{\Delta(m_\gamma)\}$ without relying on a case-by-case analysis. About $\{\Delta d_\gamma(0 < \nu < 1)\}$, at least we know that there is one degree freedom not captured by $\{m_\gamma\}$’s; we stay within the $\text{Mod}^\Phi_0(25/2, \rho_{(+2n)})$ does not grow as fast as $\sim (2n)$.\footnote{52 A subgroup $H$ of a discriminant group $G$ is isotropic, if the restriction of the discriminant quadratic form on $H$ is trivial.}
set \( \text{Mod}^\Phi(25/2, \rho_{(2n)}) \) under a change by

\[
\frac{\Delta \Psi}{\eta^{24}} \propto -(\Delta d_0(0))\theta_{(2n)}.
\] (75)

With a case-by-case analysis, it is possible to find out which of those \( d_\gamma(\nu < 1) \)'s are linearly independent, when an explicit basis of the vector space \( \text{Mod}(13 - \rho/2, \rho_{(2n)}) \) is available. An alternative is to find a basis of \( \text{Mod}(k', \rho_{(-2n)}) \) with \( k' \equiv \rho/2 + 1 \mod 2 \), and find linear constraints on \( \Delta d_\gamma(\nu < 1) \) as in the discussion in footnote \[41\]. A basis of \( \text{Mod}(k', \rho_{(-2n)}) \) can be worked out by using the vector space of \( \langle +2n \rangle \)-polarized Jacobi form of weight \( (k' + 1)/2 \) (see \[49\] or appendix \( \Lambda.1 \)). For any \( \phi \in \text{Mod}(7/2, \rho_{(-2)}) \) and \( \phi' \in \text{Mod}(9/2, \rho_{(-2)}) \), for example, the coefficients \( a, b \) and \( a', b' \) in

\[
\phi \cdot \eta^{24} = \frac{aE_4 + bE_4E_6^2}{\eta^{24}}, \quad \phi' \cdot \eta^{24} = \frac{a'E_4^3E_6 + b'E_6^3}{\eta^{24}}
\] (76)

are determined by \( \phi \) and the small number of \( \{\Delta d_\gamma(\nu < 1)\} \)'s by comparing the coefficients of the \( q^{-1} \) and \( q^0 \) terms; comparison of the coefficients of the \( q^1 \) term yields a linear constraint on \( \{d_\gamma(\nu < 1)\} \) and \( d_\gamma(\nu = 1) \) for isotropic \( \gamma \)'s.

### 3.1.3 Modular Forms and Topological Invariants

Let \( X \) [resp. \( \overline{X} \)] be a Calabi–Yau three-fold with a regular \( \Lambda_S \)-polarized [resp. \( \Lambda_S \)-polarized] K3 fibration, and suppose that \( X \) with some cycles collapsed is regarded as a limit of complex structure of \( X \) in a way a complex codimension-2 singularity of type-\( \mathcal{R} \) emerges along a curve \( C_\mathcal{R}; \lim_{\text{cpx str}} X = \lim_{\text{Kahler}} \overline{X} \). The modular form \( \Phi \) and \( \Phi \) assigned for \( X \) and \( \overline{X} \) determine such information as Noether–Lefschetz numbers of \( X \) and \( \overline{X} \), but there are also some topological invariants of \( X \) and \( C_\mathcal{R} \) that can be determined from the Fourier coefficients of \( \Phi \) and \( \Psi \) \[7\].

We start off with quickly reviewing traditional calculation of the matching between the data \( \Phi \) and \( \Psi \) and the low-energy effective theory, and proceed to discuss how we can use such modular forms for classification of such \( \Lambda_S \)-polarized K3-fibrations.

### A Quick Summary of the Matching

The low-energy effective theory on \( \mathbb{R}^{3,1} \) has a prepotential \( F \), gravitational coupling \( F_1 \), and the gauge kinetic function \( f_\mathcal{R} \) of the enhanced symmetry \( \mathcal{R} \), when the Type IIA string is compactified on \( \lim_{\text{cpx str}} X = \lim_{\text{Kahler}} \overline{X} \). Those
functions of the effective theory in the $\text{Im}(s) \gg 1$ limit:

$$\mathcal{F}^{\text{pert}} = \frac{s}{2} (t, t) + f^{(1)}, \quad F_1^{\text{pert}}, \quad f_R^{\text{pert}} = s + 4\pi i h^{(1)}$$

are determined from the microscopic data $\Delta_R$ and $\Delta_{\text{grav}}$ through the relations:

$$4\pi \text{Re}(F_1^{\text{pert}}) = 24s_2 + \frac{1}{4\pi} \left(24V_{GS} + \Delta_{\text{grav}} - b_{\text{grav}} \hat{K}\right),$$

$$4\pi \text{Re}(h^{(1)}) = \frac{1}{4\pi} \left(V_{GS} + \Delta_R - b_R \hat{K}\right),$$

$$V_{GS} = \frac{4\pi}{(t_2, t_2)} \text{Im}\left[(1 - it_2^2 \partial_{t_2}) f^{(1)}\right].$$

Those low-energy functions are of the following form because we already assume a geometric phase Type IIA compactification:

$$\mathcal{F}^{\text{pert}} = \frac{1}{2} s (t, t) \Lambda_S + \frac{d_{abc} t^a t^b t^c}{3!} - \frac{\zeta(3)}{(2\pi i)^3} \frac{\chi}{2} + \frac{1}{(2\pi i)^3} \sum_{\beta_{\text{eff}}} n_{\beta_{\text{eff}}} \text{Li}_3(e^{2 \pi i (\beta_{\text{eff}}, t)}),$$

$$4\pi i F_1^{\text{pert}} = 24s + (c_2)_a t^a = \frac{2}{2\pi i} \sum_{\beta_{\text{eff}}} \left((n_{\beta_{\text{eff}}}^0 + 12n_{\beta_{\text{eff}}}^1) \text{Li}_1(e^{2 \pi i (\beta_{\text{eff}}, t)})\right),$$

$$s + 4\pi i h^{(1)} = s + d'_a t^a - \sum_{\beta_{\text{eff}}} \frac{A_{\beta_{\text{eff}}}}{2\pi i} \text{Li}_1(e^{2 \pi i (\beta_{\text{eff}}, t)}).$$

Here, a component description $\{t^{a=1,\ldots,p}\}$ is given to $t \in \Lambda_S \otimes \mathbb{C}$ by choosing an integral basis $\{D_s, D_{a=1,\ldots,p}\}$ of $H^2(X, \mathbb{Z})$ consistent with the filtration structure in $\mathbb{C}$; the divisors $\{D_{a=1,\ldots,p}\}$ modulo +$\mathbb{Z}D_s$ may be regarded as a basis of $\Lambda_S$. The complexified Kähler

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54 We consider only the cases where the curve $C_R$ covers the base $\mathbb{P}^1$ just once.

55 See footnote 14.

56 $\hat{K} = -\ln(t_2, t_2) + \text{const.}$ is the (Heterotic string) tree-level Kähler potential of the non-dilaton vector-multiplet scalars.

57 In $\mathcal{N} = 2$ field theory on $\mathbb{R}^{3,1}$, the prepotential itself is not physical (e.g., [20]); different choices of $(2n_V + 2)$-tuple of symplectic sections $(X_f, F_I)$ related by a Sp$(2n_V + 2; \mathbb{Z})$ duality transformation may have different prepotentials (a prepotential does not exist for some frames). The prepotential here is for a frame where a D2-brane wrapped on any real 2-dimensional cycle (in Type IIA language) is treated as an electrically charged particle in $\mathbb{R}^{3,1}$. cf section 2.1.3.

58 For a Calabi–Yau three-fold, the structure of K3-fibration $\pi : X_{\text{IIA}} \to \mathbb{P}^1_{\text{IIA}}$ is in one-to-one with a divisor class $D_s$ of $X_{\text{IIA}}$ satisfying $D_s^2 = 0$ and $\int_{X_{\text{IIA}}} D_s \cdot e_2(TX_{\text{IIA}}) = 24$. The divisor class charcterized in this way is the topological class of the K3 fibre over a generic point in the base $\mathbb{P}^1$. Choice of a divisor $D_a$ has ambiguity $D_a \to D_a + \delta n'_a D_s$ with $\delta n'_a \in \mathbb{Z}$. In terms of the coefficients, this corresponds to $s \to s - \delta n'_a t^a$ and $t^a$ unchanged.
of $X$ is $t_{CY} = sD_s + t^a D_t = sD_s + t$ when $e^{2\pi i s}$ corrections are ignored (as we will everywhere in this article).

The sums of exponential terms run over effective vertical curve classes $\beta_{\text{eff}}$, because we retain only the terms that remain non-zero in the large base ($\text{Im}(s) \gg 1$) region of the moduli space. $n'_\beta$, is Gopakumar–Vafa invariant. $A_\beta$ is related to $r = 0$ Gopakumar–Vafa invariants of $X_{\text{IIA}}$. It is well-known that the matching relations (78, 79, 80) determine those parameters in terms of the coefficients of $\Phi$ and $\Psi$ as

$$\chi = -c_0(0), \quad n^0_w = c_{[w]}(w^2/2), \quad A_w = \frac{d_{[w]}(w^2/2)}{12}, \quad n^1_w = \frac{\tilde{c}_{[w]}(w^2/2) - c_{[w]}(w^2/2)}{12}$$

(84)

for any $w \in [H_2(X; \mathbb{Z})]^{\text{vert}} \subset \Lambda_\chi^*$ that is effective; $\tilde{c}_\nu(\nu)$ is the Fourier coefficient $[E_2 \Phi_\nu/\eta^{24}]_{q^\nu}$. We have nothing to add or discuss about them in this article, however.

The non-exponential part of them captures topological invariants of $X_{\text{IIA}}$ and the curve $C_R$ of enhanced singularity of type $R$ in $\text{lim}_{\text{cpx str}} X = \text{lim}_{\text{Kahler}} X$; $\chi = \chi(X)$ is the Euler number, and

$$d_{abc} t^a t^b t^c = \int_{X_{\text{IIA}}} t \wedge t \wedge t, \quad t = t^a D_a,$$

$$24 s + (c_2)_a t^a = \int_{X_{\text{IIA}}} c_2(X_{\text{IIA}}) \wedge t_{CY} = 24 s + t^a \int_X c_2(T X) D_a,$$

$$s + d'_a t^a = \langle t_{CY}, C_R \rangle = s + \langle t, C_R \rangle.$$

(85-87)

The coefficients $d'_a$ can also be regarded as trilinear intersection numbers in $X$ among $D_a$ and a pair of exceptional divisors that emerge after resolving the type $R$ singularity.

Those invariants are determined by (78, 79, 80) as

$$\mathcal{F}_{\text{cub}} := \frac{1}{2} s(t, t) + \frac{d_{abc} t^a t^b t^c}{3!} = \frac{1}{2} \tilde{s}(t, t) + \frac{1}{3!} P_3(t),$$

$$4\pi i (F_1)_{\text{nonexp}} = 24 s + (c_2)_a t^a = 24 \tilde{s} + P_1(t),$$

$$s + 4\pi i (h^{(1)})_{\text{nonexp}} = s + d'_a t^a =: \tilde{s},$$

(88-90)

where $P_3(t)$ and $P_1(t)$ are polynomials of $t$ given by the integrals over the fundamental region.

---

59 Here we have used $\tilde{s} = s + d'_a t^a$ for convenience. Since $d'_a$ is an integer, using $(\tilde{s}, t^a)$ instead of $(s, t^a)$ corresponds to the integral basis change $D_a \to D_a - d'_a D_s$. 

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33
\[ P_3(t_2) := -\frac{t_2^3}{32\pi \sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \frac{\theta_{\Lambda_S}(\tau, \bar{\tau}; t_2)}{\eta^{24}} \Phi(\tau) \hat{E}_2(\tau, \bar{\tau}) - \Psi(\tau), \] (91)

\[ P_1(t_2) := \frac{1}{4\pi \sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \frac{\theta_{\Lambda_S}(\tau, \bar{\tau}; t_2)}{\eta^{24}} \Psi(\tau). \] (92)

See appendix \[B\] for details of the integrals; evaluation method for the case \(\Lambda_S\) has a non-trivial null element is reviewed in appendix \[B.1\]. Appendix \[B.3\] explains how to reduce a case of \(\Lambda_S\) without such an element to cases with such an element.

**Discussion 1** For a given \(X\), choose any Higgs cascade attached to the branch of moduli space \(60\) of Type IIA compactification of \(X\). We see in the following that the pair of modular forms \(\Phi\) and \(\Psi\) contains complete information in specifying the diffeomorphism class of \(X\).

Let us recall Wall’s theorem \[41\], which states that the set of diffeomorphism classes of real six-dimensional, simply-connected, spin, oriented manifolds with a torsion free cohomology and a given set of Betti numbers \(b_2\) and \(b_3\) are in one-to-one with the set

\[ \{(\mu, p_1) \mid \mu \in \text{Hom}^{\text{Sym}}(H^2 \times H^2 \times H^2, \mathbb{Z}), \ p_1 : \text{Hom}(H^2, \mathbb{Z}), \ (a), \ (b) \}/\sim, \] (93)

where \(H^2 \cong \mathbb{Z}^{\oplus b_2}\) and

(a) \(\mu(x, x, y) + \mu(x, y, y) \equiv 0 \mod 2\) for \(\forall x, y \in H^2\),

(b) \(4\mu(x, x, x) - p_1(x) \equiv 0 \mod 24\) for \(\forall x \in H^2\);

the relation is \((\mu, p_1) \sim (\mu', p_1')\) if and only if there is an isomorphism \(\phi : H^2 \to H^2\) such that \((\mu', p_1') = (\mu, p_1) \cdot \phi\). For a manifold \(X\), the trilinear symmetric form \(\mu\) is the wedge product of \(H^2(X; \mathbb{Z})\), and \(p_1\) the linear form \(\int_X p_1(TX) \wedge x\) for \(x \in H^2(X; \mathbb{Z})\).

Its subset of interest in this article is those where \(H^2\) contains an element \(D_s\) of the property described in footnote \[58\]. It is given by

\[ \text{Diff}_{\Lambda_S} := \{(d, c_2^{\Lambda_S}, \chi) \mid d : \text{Hom}^{\text{Sym}}(\Lambda_S \times \Lambda_S \times \Lambda_S, \mathbb{Z}), \ c_2^{\Lambda_S} : \text{Hom}(\Lambda_S, \mathbb{Z}), \ \chi \in \mathbb{Z} \} / \sim_{\Lambda_S}, \] (94)

where \(\Lambda_S = \mathbb{Z}^{\oplus \rho}\) and

\[ Mathematically, the relation \[60\] can be regarded as the definition of the modular form \(\Psi\) of a Higgs cascade; \(\Phi\) is defined as in \[8\] (and reviewed in section \[2.1.2\]).\]
(a') \( d_{aabb} + d_{aabb} \equiv 0 \mod 2 \) for any \( a, b \in \{1, \cdots, \rho\} \),
(b') \( 4d_{aa} + 2(c_2)_a \equiv 0 \mod 24 \) for any \( a \in \{1, \cdots, \rho\} \);

d_{abc} and \( (c_2)_a \) for \( a, b, c = 1, \cdots, \rho \) are the component description of \( d \) and \( c_2^{A_S} \) for some basis of \( \Lambda_S \); the relation \( \delta n^{[61]} \) given by setting \( (d, c_2^{A_S}) \sim_{\Lambda_S} (d', c_2^{A_S'}) \) if and only if they become identical for some combination of isometries of \( \Lambda_S \) and the basis changes in footnote \( [58] \). The modular form \( \Phi \) of \( X \) determines the combinations\( [62] \)

\[
\frac{P_3(t_2)}{8} - \left(\frac{t_2}{t_2}\right) P_1(t_2) = \frac{|t_2|^3}{32\pi\sqrt{2}} \int \frac{d\tau_1 d\tau_2}{\tau_2^{3/2}} \frac{\Phi \hat{E}_2}{\eta^{24}},
\]

and hence the combinations

\[
d'_{abc} := d_{abc} - \frac{[(c_2)_a c_{bc}^{A_S} + \text{cycl.}]}{24}.
\]

They remain invariant under the shifts \( D_a \rightarrow D_a + (\delta n'_a) D_s \) with \( \delta n'_a \in \mathbb{Q} \) for a basis \( \{D_S, D_{a=1,\cdots,\rho}\} \in H^2 \otimes \mathbb{Q} \). So the modular form \( \Phi \) of a Calabi–Yau three-fold \( X \) determines an element of

\[
\text{Diff}_{\Lambda_S}(\mathbb{Q}) := \left\{ d'_{abc} := (d_{abc} - [(c_2)_a c_{bc}^{A_S} + \text{cycl.}]/24) \mid (a'), (b') \right\} / \text{Isom}(\Lambda_S) \times \{ \chi \in \mathbb{Z} \}.
\]

There may be a pair of three-folds \( X \) and \( X' \) sharing the same modular form \( \Phi \) that are not diffeomorphic to each other. They must have the same combination \( d_{abc} - [(c_2)_a c_{bc}^{A_S} + \cdots]/24 \) but \( (d_{abc}, (c_2)_a) \) of \( X \) may be converted to that of \( X' \) only by allowing the shifts with \( (\delta n'_a + Z) \neq 0 \in \mathbb{Q}/\mathbb{Z} \).

\[
\text{Diff}_{\Lambda_S}^d := \left\{ (d_{abc} \in \mathbb{Z}, (c_2)_a \in \mathbb{Z}) \mid (a'), (b'), \text{ fixed } d'_{abc}, \right\} / \sim_{\Lambda_S}.
\]

With just the modular form \( \Psi \) of one arbitrary chosen Higgs cascade of \( X \) (along with \( \Phi \)), however, the dictionary \( [58] [59] [60] \) determines \( (d_{abc}, (c_2)_a) \) precisely with the relation \( \sim_{\Lambda_S} \), because the integrality of \( d'_a \) allows only the shifts \( s \rightarrow s - \delta n'_a t^a \) with \( \delta n'_a \in \mathbb{Z} \).

To summarize, the modular forms \( \Phi \) and \( \Psi \) may be seen as information of the spectrum of BPS states of string theory, or that of Noether–Lefschetz numbers and curve counting invariants, but they also carry full information the diffeomorphism class of the original manifold

\( \delta n^{[61]} \) Note that an element \( D_s \in H^2 \) with the property in footnote \( [58] \) is mapped by \( \phi : H^2 \cong H^2 \) for the relation \( (\mu, p_1) \sim (\mu', p'_1) \) to an element \( \phi(D_s) \in H^2 \) that also has the same property.

\( \delta n^{[62]} \) It also determines \( \chi(X) \). Under the assumption that \( X \) is a Calabi–Yau three-fold, now all the Betti numbers are specified by \( \rho \) and \( \chi(X) \).
The way to extract has already been described.

\[
[\text{Mod}_0^Z(11 - \frac{\rho}{2}; \rho_{\Lambda S})]^\S \supset [\text{Mod}_0^Z(11 - \frac{\rho}{2}; \rho_{\Lambda S})]^\text{r.mfd} \longrightarrow \text{Diff}_{\Lambda S}(Q) \longrightarrow \text{Diff}_{\Lambda S}
\]

\[
\text{Mod}_0^\Phi(13 - \frac{\rho}{2}; \rho_{\Lambda S}) \supset [\text{Mod}_0^\Phi(13 - \frac{\rho}{2}; \rho_{\Lambda S})]^\text{r.mfd} \longrightarrow \text{Diff}_{\Lambda S}^{d'}
\]

(99)

The following disccussions explain why some of the mapps are drawn in double lines.

For a given three-fold \(X\), there may be multiple Higgs cascades attached to the original branch of the moduli space, and hence multiple modular form \(\Psi\)'s. The arrow from \(\{X\}'s\) to \(\text{Mod}_0^\Phi(12 - \rho/2; \rho_{\Lambda S})\) in (99) is shown in a double line because of that. Those \(\Psi\)'s should yield the same element in \(\text{Diff}_{\Lambda S}^{d'}\). It follows that the difference \(\Delta \Psi\) must be such that the resulting \(\Delta P_1(t)\) is of the form \(24(\Delta d'_a) t^a\) with \((\Delta d'_a) \in \mathbb{Z}\).

One may also ask which subset of the diffeomorphism classes \(\text{Diff}_{\Lambda S}\) of real six-dimensional manifolds are realized under the restriction that \(X\) is a Calabi–Yau three-fold. Because we do not know well the set of such Calabi–Yau three-folds, \(\{X\}'s\), one may think of applying the procedure of assigning \((d'_{abc}, \chi)\) to an abstract general element \(\Phi \in \text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda S})\). First, it is not true that the resulting \(d'_{abc}\) can be interpreted as some \(d_{abc} \in \mathbb{Z}\) and \((c_2)_a \in \mathbb{Z}\), if we just require that \(\Phi \in \text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda S})\) is subject to the inequalities we derived in section 2; see an example in the appendix B.1.3. So, the subset of the \(\Phi\)'s whose \(d'_{abc}\) backed by integer \((d'_{abc}, (c_2)_a)\) subject to \((a,b)\) is denoted by \([\text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda S})]^\text{r.mfd}\). Similarly, not a general element of \(\Psi \in \text{Mod}_0^\Phi(13 - \rho/2; \rho_{\Lambda S})\) yields an element of \(\text{Diff}_{\Lambda S}^{d'}\) (see section 3.2), so those that fall into \(\text{Diff}_{\Lambda S}^{d'}\) forms a subset denoted by \([\text{Mod}_0^\Phi(13 - \rho/2; \rho_{\Lambda S})]^\text{r.mfd}\). We have the map

\[
\text{diff}_{\text{coarse}} : [\text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda S})]^\text{r.mfd} \longrightarrow \text{Diff}_{\Lambda S}(Q),
\]

\[
\text{diff}_{\text{fine}} : [\text{Mod}_0^\Phi(13 - \rho/2; \rho_{\Lambda S})]^\text{r.mfd} \longrightarrow \text{Diff}_{\Lambda S}^{d'},
\]

(100)

(101)

see (99). The set of diffeomorphism classes represented by Calabi–Yau three-folds must be within the image of the map \((\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})\) in \(\text{Diff}_{\Lambda S}\).

---

\(^{63}\) A cautionary remark is that the diffeomorphism class of \(X\) may not be contained in the image of \(\text{diff}_{\text{fine}}\), if \(X\) does not have any symmetry-enhanced branch of \(X\) with a regular K3-fibration.
The map \( (\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}}) \) can be worked out by dealing with purely mathematical objects. In setting up the relation between the modular forms and diffeomorphism classes, however, we have combined two physics observations under the Heterotic–Type IIA string duality; the parameters \( d_{abc} \) and \( (c_2)_a \) in the low-energy effective theory is determined i) by the topology of the target space \( X \) in Type IIA string compactification, and ii) also by 1-loop integrals in the Heterotic string where the integrands are modular forms. The \( \mathcal{R} \)-independence of \( \Psi \) in a given Higgs cascade may well be proved purely in math, although physics reasonings are enough; the claim that the difference among the \( \Psi \)'s from different Higgs cascades of a given original branch disappear in the image of \( \text{diff}_{\Lambda_S}^{d'} \) also relies on physics reasonings.

**Discussion 2** For a given \( X \) with its modular form \( \Phi \), one may specify a curve class \( C \in H_2(X; \mathbb{Z}) \) and ask whether complex structure of \( X \) can be tuned to have singularity of some type \( \mathcal{R} \) along \( C \) (and its resolution is still a regular K3-fibration). In general, there is no guarantee that a modular form \( \Psi \) exists in \( \text{Mod}^{\Phi}_0(13 - \rho/2; \rho_{\Lambda_S}) \) so that the RHS of (88, 89, 90) reproduce all the input data on the left-hand sides. In such cases, we learn that complex structure cannot be of \( X \) cannot be tuned in that way.

**Discussion 3** Suppose that a pair of Calabi–Yau three-folds \( X \) and \( X' \) have a diffeomorphism between them, but not a holomorphic one-to-one map. Type IIA string compactifications over \( X \) and \( X' \) form two different branches of moduli space then. Such a pair of branches of moduli space cannot be distinguished by the invariants \( (\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}}) \).

Instead of finding the modular form \( \Psi \) for one Higgs cascade of the branch of \( X \) and specify \( \text{diff}_{\text{fine}}(\Psi) \), we can specify the subset of \[ \text{Mod}^{\Phi}_0(13 - \rho/2; \rho_{\Lambda_S}) \] mfd of all the \( \Psi \)'s of the Higgs cascades attached to the original branch of \( X \). An example of such a pair is discussed in section 3.2.1.

That idea of extracting an invariant of a branch of moduli space is faithful to the way we analyze the moduli space by using the low-energy effective field theory. In practice, however, it is not easy to work out all the possible ways to tune complex structure of a manifold to obtain singularity. A close alternative to the idea of using the set of \( \Psi \)'s of all the Higgs cascades is i) to think of all the holomorphic curves in \( H_2(X; \mathbb{Z}) \), ii) apply the reasoning in Discussion 2 to eliminate some of those curves, and finally, iii) to extract the set of \( \Psi \)'s for those remaining curves. The latter set of \( \Psi \)'s contain the former set of \( \Psi \)'s.

The latter idea detects difference in the Kähler cone, or in the cone of curves. Let
\( f : X \rightarrow X' \) be a diffeomorphism; if \( C \in H_2(X; \mathbb{Z}) \) is in the cone of curves of \( X \), but \( f_*(C) \) is not in that of \( X' \), then the modular form \( \Psi \) for \( C \) may be in the set of \( \Psi \)'s for \( X \), but not in the set for \( X' \). In the example of section 3.2.2, we discuss this latter set of \( \Psi \)'s.

**Discussion 4** The positive cone \((t_2,t_2) > 0\) of \( \Lambda_S \otimes \mathbb{R} \) may contain multiple chambers separated by walls orthogonal to some elements in \( \Lambda_S^\vee \); some of those chambers correspond to three-folds with different topology [42]. The modular form \( \Phi \) remains the same on both sides of the wall, but the integral in (95) does not necessarily yield the same polynomial on both sides of the walls (see the appendix [B.2]), and hence not necessarily the same \( \{(d'_{abc})\} \); the arrow from \([\text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda_S})]^r.mfd\) to \( \text{Diff}_{\Lambda_S}(\mathbb{Q}) \) in (99) is shown in a double line because of that. The diffeomorphism classes represented by those \( \{(d'_{abc})\} \) in \( \text{Diff}_{\Lambda_S}(\mathbb{Q}) \) are not necessarily the same \(^{64}\) Note also that the combinations \( \{(d'_{abc})\} \) extracted from \( \Phi \)'s in \([\text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda_S})]^{\mathbb{Q}} \) for a single chamber do not necessarily show symmetry under \( \text{Isom}(\Lambda_S) \) for the quotient in (97).

There are not many things we can say with confidence about a symmetry-enhanced branch available on one side of the wall continues to exist on the other side of the wall. We think it is likely, however, if \( \lim_{\text{cpx. str}} X \) has \( A_1 \) singularity along a curve \( C \subset X \), then a flop transition on \( X \) along a curve disjoint from \( C \) yields a threefold \( X' \) that continues to have a symmetry-enhanced phase with singularity along \( C \). Even in such cases, the map \( \text{diff}_{\text{fine}} \) depends on the choice of a chamber, because the integrals \( P_1 \) and \( P_3 \) have singularity along the walls.

Such invariants as \( \text{diff}_{\text{coarse}}(\Phi) \) and \( \text{diff}_{\text{fine}}(\Psi) \) are not assigned to branches of Coulomb- and hyper moduli space, but for branches of individual chambers-and-hyper moduli space.

**Discussion 5** For Heterotic string compactifications reviewed in section 2.1.1 the invariants such as the set of \( \Psi \)'s and the set of \( \Phi \)'s are assigned for (individual chambers of the) branches of moduli space. Their Type IIA dual do not necessarily have a geometric phase, so we may think of \( \Phi \) in \( \text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda_S}) \) than in \( [\text{Mod}_0^Z(11 - \rho/2; \rho_{\Lambda_S})]^r.mfd \). Those invariants are given in terms of the CFT of the fundamental string, and are well-defined, without relying on Heterotic supergravity approximation, or a geometric phase.

Those invariants beyond \( (\Lambda_S, \Lambda_T, \Phi) \) generalize the integer \( n \) of the \((12 + n, 12 - n)\) instanton number distribution in the case of \( \Lambda_S = U \), and detect difference among branches of moduli space.

\(^{64}\) In fact, the wall crossing formula makes it possible to compute how much the images are different in \( \text{Diff}_{\Lambda_S}(\mathbb{Q}) \).
Heterotic string moduli space already present at perturbative level (with corrections of order $(e^{2\pi i s})$ ignored); see also [24, 11].

**Remark:** This is a small side remark before closing this section [3.1]. The modular forms $\Phi^{(R_i)}$ for a chain of symmetry enhancement $\{0\} \subset R_1 \subset R_2 \cdots$ also contain how many hypermultiplet moduli need to be tuned to have the enhanced symmetries in the chain:

$$\Delta h^{-2,1}_R := h^{-2,1}(X) - h^{-2,1}(X^{(R)}) = h^{1,1}(X) - (\chi(X) - \chi(X^{(R)}))/2,$$

$$= -\text{rank}(R) + [c_0(0) - \omega^{(R)}(0)]/2. \quad (102)$$

This $\Delta h^{-2,1}_R$ was used in [11] to distinguish four different branches of moduli space that share the same $\Lambda_S = \langle +2 \rangle$ and the modular form $\Phi$ (those that are discussed in section [3.2.2]). But it is enough to have $\Phi^{(R)}$ without $\Delta h^{-2,1}_R$ as an invariant of branches of the moduli space.

### 3.2 Examples

Let us see in simple examples how $\Phi$, $\Psi$ and “the set of possible $\Psi$” as an invariant work in distinguishing different branches of the moduli space.

#### 3.2.1 $\Lambda_S = U$

Let us begin with a traditional example, $\Lambda_S = U$. The maps $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ are worked out first.

We know that $\Phi = -2E_4E_6$ and a general element of $\text{Mod}_0^\Phi(12, \rho U)$ is $\Psi = -(E_4^3 + E_6^2) + (288 - 24b_R)\eta^{24}$, where we use $b_R = d(0)/24 \in \mathbb{Z}_{\geq 0}$ as the parameter. It is known that the integrals $P_3(t)$ and $P_1(t)$ for those $\Phi$ and $\Psi$ are given by [52, 7, 38, ..

$$\frac{1}{3!}P_3(t) = \frac{2}{3!}\rho^3 + \frac{n' - 2}{2}\rho^2 u + \frac{n' u}{2}, \quad (103)$$

$$P_1(t) = -4\rho + 12(2 + n')(\rho + u), \quad (104)$$

where $n' := 2 - 12^{-2}d(0) = 2 - b_R/6$. Here, the component description of $t \in \Lambda_S = U$ is that of (104) associated with an obvious null element $z \in U$. The expressions above is for the chamber $0 \leq \rho_2 \leq u_2$; those for the other chamber $0 \leq u_2 \leq \rho_2$ are obtained by exchanging $\rho$ and $u$. As we will take a quotient by $\text{Isom}(U)$, which includes the $\rho \leftrightarrow u$ exchange, it is enough to focus on the $0 \leq \rho_2 \leq u_2$ chamber in the following.
One can see that

\[ [\text{Mod}^\mathbb{Z}_0(10, \rho_U)]^{\text{r.mfd}} = [\text{Mod}^\mathbb{Z}_0(10, \rho_U)]^{\mathbb{B}^2} = \{ \Phi = (-2E_4E_6) \}, \quad (105) \]

\[ [\text{Mod}^\mathbb{Z}_0(12, \rho_U)]^{\text{r.mfd}} = \{ b = 0, 6, \ldots \} \subset \{ b \in \mathbb{Z}_{\geq 0} \} = \text{Mod}^\mathbb{Z}_0(12, \rho_U); \quad (106) \]

after working out details by using the expressions of \( P_3 \) and \( P_1 \) above. The restriction on the value of \( b \) is from the integrality of \( d_{abc} \); once \( n' \in \mathbb{Z} \) is imposed, then the condition \((a')\) and \((b')\) are automatically satisfied in this \( \rho_{\Lambda S} = U \) case. Corresponding to the shift \( s \to s + (\Delta n'_a)t^a \) with \( \Delta n'_a \in \mathbb{Z} \) is \( \Delta n' = 2, \Delta b_R = -12 \), which mods out \([\text{Mod}^\mathbb{Z}_0(12, \rho_U)]^{\text{r.mfd}}\) in passing to \( \text{Diff}^\mathbb{Z}_{\Lambda S} \) by the map \( \text{diff}_{\text{fine}} \).

The image of the map \( (\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}}) \) must be in

\[ \text{Diff}_{\Lambda S = U} = \left\{ (d_{\rho pp}, d_{u uu}, N_\rho, N_u) \in \mathbb{Z}^{\otimes 4} \right\} / \{ \rho \leftrightarrow u \} \times \{ \nu = 0, 1 \} \times \{ \chi \in \mathbb{Z} \} \quad (107) \]

\[ (c_2)_\rho = -2d_{\rho pp} + 12N_\rho + 24\delta n'_\rho, \quad d_{\rho pu} = \nu + 2\delta n'_\rho; \quad (108) \]

\[ (c_2)_u = -2d_{u uu} + 12N_u + 24\delta n'_u, \quad d_{pu u} = \nu + 2\delta n'_u. \quad (109) \]

Both of \( \nu = 0, 1 \) of \( \text{Diff}^\mathbb{Z}_{\Lambda S} \) are realized by the images of even \( n' \) and odd \( n' \). Only just one element of \( \text{Diff}^\mathbb{Z}_{\Lambda S} (\mathbb{Q}) = \mathbb{Z}^{\otimes 4} / (\rho \leftrightarrow u) \times \{ \chi \in \mathbb{Z} \} \) is in the image of \( \text{diff}_{\text{coarse}} \), however. It is the element represented by \( d_{\rho pp} = 2, d_{u uu} = 0, N_\rho = 4 + \nu, N_u = 4 + \nu, \) and \( \chi = -480 \). The modular forms \( \Phi \) behind the scene indicates that the diffeomorphism classes realized in the form of Calabi–Yau three-folds are significantly less \( ^{65} \).

As is well-known, there are Calabi–Yau three-folds for both of even \( n' \) and odd \( n' \). Think of a Weierstrass-model elliptic fibration over the Hirzebruch surface \( F_n \) that is Calabi–Yau, and denote it by \( X^{(n)} \); we denote by \( D_7 \) the zero-section divisor of \( X^{(n)} \). The base surface \( F_n \) is a \( \mathbb{P}^1 \)-fibration over \( \mathbb{P}_{\text{IA}}^1 \), where the \( D_f \) is the \( \mathbb{P}^1 \)-fibre class, and the two sections denoted by \( D_+ \) and \( D_- \) have self-intersection \( +n \) and \( -n \), respectively. The pull-back of the divisors \( D_f, D_+, \) and \( D_- \) to \( X^{(n)} \) are denoted by \( D_s, D_3, \) and \( D_4 \), respectively. \( D_3 \sim D_4 + nD_4 \).

\(^{65}\)In the case of \( \Lambda S = U \), this is not surprising, because a simple argument \cite{37, 38} shows that \( X^{(n)} \) with \( n = 0, 1, 2 \) (explained shortly in the main text) are all the possibilities. First, a \( U \)-polarized K3-fibration \( X \) over \( \mathbb{P}_{\text{IA}}^1 \) must be an elliptic fibration over a surface, which itself must be a \( \mathbb{P}^1 \)-fibration over \( \mathbb{P}_{\text{IA}}^1 \). So, the base surface is \( F_n \); once the base is fixed, then the Weierstrass model with \( f \) in \( \mathcal{O}(-4K) \) and \( g \) in \( \mathcal{O}(-6K) \) determines \( X^{(n)} \) without an extra topological freedom. Those with \( n > 2 \) are ruled out, because the lattice \( \Lambda S \) is strictly larger than \( U \) for \( n > 2 \).
Some of the triple intersection numbers are
\[ D_7 \cdot \begin{pmatrix} D_3 \cdot D_3 & D_3 \cdot D_s \\ D_s \cdot D_3 & D_s \cdot D_s \end{pmatrix} = \begin{pmatrix} +n & 1 \\ 1 & 0 \end{pmatrix} \tag{110} \]
\[ D_s \cdot \begin{pmatrix} (D_7 + D_3)^2 & (D_7 + D_3) \cdot D_3 \\ D_3 \cdot (D_7 + D_3) & D_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{111} \]
which means that we can use \{D_s, (D_7 + D_3), D_3\} as a basis of \(H^2(X^{(n)}; \mathbb{Z})\) in a way that \{(D_7 + D_3) + \mathbb{Z}D_3, (D_3) + \mathbb{Z}D_s\} becomes a basis of \(\Lambda_S = U\). We use the parametrization \(t_{\text{CY}} = sD_s + \rho(D_7 + D_3) + uD_3\),
\[
\int_X t_{\text{CY}}^3 = s\rho u + \frac{2}{3!}\rho^3 + \frac{n-2}{2}\rho^2u + \frac{n}{2}\rho u^2, \tag{112}
\]
\[
\int_X c_2(TX) = 24s - 4\rho + 12(2+n)(\rho + u). \tag{113}
\]

Now, think of a symmetry-enhanced limit \(\lim_{\text{cpx str}} X^{(n)}\) of this \(X^{(n)}\) so a singularity of type \(R\) emerges in the fibre of \(D_- \subset F_n\). Then \((f_R)_{\text{nonexp}} = s = \tilde{s}\). The modular form \(\Psi\) for this Higgs cascade must be the one for \(n' = 2 - 12^{-2}d(0) = n\), because \(s\rho u + P_3(t; n' = n)/3!\) and \(24\tilde{s} + P_1(t; n' = n)\) reproduces the topological invariants \((112, 113)\) of \(X^{(n)}\). In particular, the set \(\text{Diff}^d_{\Lambda_S} \simeq \mathbb{Z}/2\mathbb{Z}\) is realized by the diffeomorphism classes of \(X^{(n=0)} \sim X^{(n=2)}\) and \(X^{(n=1)}\).

It is possible to find a broader class of symmetry-enhanced limits of \(X^{(n)}\) by using F-theory. Let \(C'\) be \(H^2(F_n; \mathbb{Z})\) represented by an irreducible curve (we call it an irreducible curve class). Now, choose \(f\) and \(g\) of the Weierstrass model \(y^2 = x^3 + fx + g\) so
\[
f = -3c^2 + \sigma a + \mathcal{O}(\sigma^2), \quad g = 2c^3 - c\sigma a + \mathcal{O}(\sigma^2), \tag{114}
\]
where \(c \in \Gamma(F_n; \mathcal{O}(-2K_{F_n}))\), \(a \in \Gamma(F_n; \mathcal{O}_{F_n}(-4K_{F_n} - C'))\), and \(\sigma \in \Gamma(F_n; \mathcal{O}_{F_n}(C'))\) \[45\]; in this limit, the three-fold \(X^{(n)}\) has a singularity of type \(R = A_1\) along a curve \(C\) in the fibre of the \(\sigma = 0\) curve in \(F_n\). Think of \(C'\)'s of the form \[46\] \(C' \sim D_- + mD_f\) labeled by \(m \in \mathbb{Z}\), so that \(C' \cdot D_f = 1\) in \(F_n\), and \(C \cdot D_s = +1\) in \(X^{(n)}\). For this class of symmetry-enhanced limits (Higgs cascades), we have \[47\] \((f_R)_{\text{nonexp}} = s + m(\rho + u) = \tilde{s}\). Therefore, we find that the modular

\[\text{We have discussed the invariant } \Psi \text{ for the Higgs cascades with } C' = D_- (m = 0); \text{ the choice } C' = D_+ (m = n) \text{ corresponds to placing the probe gauge group } R \text{ in the other weakly coupled } E_8 \text{ in the Heterotic language, but there are more varieties (m) in the symmetry-enhancement limits (e.g., } A_1).\]

\[\text{In this construction of } \lim_{\text{cpx str}} X^{(n)}, \text{ the curve } C \text{ of } A_1 \text{ singularity is along } (x, y) = (h, 0) \text{ in the elliptic fibre, so it does not touch the zero section divisor } D_7. \text{ So } D_7 \cdot C = 0.\]
form $\Psi$ of this Higgs cascade must be that of $n' = n - 2m$ ($b_R/6 = 2m + 2 - n$), which we find by requiring that the topological invariants \((12, 13)\) of $X^{(n)}$ must be reproduced by $\tilde{s} \rho u + P_3$ and $24 \tilde{s} + P_1(t)$, respectively.

The class $C'$ is represented by a curve when $m \geq 0$; the curve is not irreducible for a choice $m = 1$ in the case of $n = 2$, however, because the divisor $D_+ + D_f$ has $D_-$ as a base locus then. There is also an upper bound, $m \leq 8 + 4n$; when the divisor class $-4K_{F_n} - C'$ is not effective, $X^{(n)}$ is singular in the fibre of any points on $F_n$ because $f = -3c^2$ and $g = 2c^3$. The effectiveness is translated into the upper bound. So, we have found a class of Higgs cascades attached to the branch of the moduli space of $X^{(n)}$ whose invariants are

$$X^{(n=2)} : \{ \Psi_{b_R/6=2m+2-2} \mid m = 0, 2, 3, \ldots, 16 \} \subset [\text{Mod}^\Phi_0(12, \rho_U)]^{r, \text{r.mfd}},$$

$$X^{(n=0)} : \{ \Psi_{b_R/6=2m+2-0} \mid m = 0, 1, \ldots, 8 \} \subset [\text{Mod}^\Phi_0(12, \rho_U)]^{r, \text{r.mfd}},$$

$$X^{(n=1)} : \{ \Psi_{b_R/6=2m+2-1} \mid m = 0, 1, \ldots, 12 \} \subset [\text{Mod}^\Phi_0(12, \rho_U)]^{r, \text{r.mfd}}.$$  

The difference among $\Psi$’s for one given $X^{(n)}$ is precisely of the form we expected in Discussion 1. The set of $\Psi$’s of $X^{(n=2)}$ and $X^{(n=0)}$ are not the same, however, reflecting the fact that this pair of three-folds have a diffeomorphism but not a holomorphic one-to-one map between them, and the Kähler cones are not identical when $H^2(X^{(2)}; \mathbb{R})$ and $H^2(X^{(0)}; \mathbb{Z})$ are identified by using the diffeomorphism between them.

One will wonder if there are other symmetry-enhancement limits of $X^{(n)}$. At least we can rule out cases where singularity of type $R$ emerges along a curve $C$ satisfying $C \cdot D_ = 1$ and $C \cdot D_7 \neq 0$ (Discussion 2). To see this, suppose that there is such a limit. Then $(f_R)_{\text{nonexp}} = s + m_\rho \rho + m_u u = \tilde{s}$ with $m_\rho \neq m_u$; no choice of $\Psi$ from $[\text{Mod}^\Phi_0(12, \rho_U)]^{r, \text{r.mfd}}$ for the polynomials $P_3(t)$ and $P_1(t)$ can reproduce $F_{\text{cub}}$ and $F_1$ appropriate for $X^{(n)}$, and hence the assumption must be wrong.\(^{68}\) We do not have an argument to rule out that there are other limits of $X^{(n)}$ for symmetry-enhancements with $C \cdot D_s = C \cdot D_7 = 0$ that cannot be obtained in the form \((124)\).

3.2.2 $\Lambda_S = \langle +2 \rangle$

Consider the case $\Lambda_S = \langle +2 \rangle \cong ab \mathbb{Z}$ now, where the modular form $\Phi$ is in \((38)\). The modular form $\Psi \in \text{Mod}^\Phi_0(25/2, \rho_{(+2)})$ is parametrized by $m_9 = d_0(-1), m_{1/2} = d_{1/2}(-3/4)$, and one more, because $\dim_{\mathbb{C}} \text{Mod}^\Phi_0(25/2, \rho_{(+2)}) = 3$. We can use the $d_0(0) = 24b_R$ as the

\(^{68}\) This argument still allows a limit of $X^{(n)}$ whose singularity resolution $\bar{X}^{(n)}$ does not have a regular K3-fibration over $\mathbb{P}_{\text{IIB}}^1$. 

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third parameter. Just one among them, \( d_0(0) \in 24\mathbb{Z}_{\geq 0} \), is the free parameter, while \( m_0 = m_{1/2} = 0 \) because of (123). So, a general element \( \Psi \) in \( \text{Mod}^\phi_0(25/2, \rho^{(+2)}) \) is given by

\[
\Psi = \frac{E_6}{E_4} \Phi - (d_0(0) + 1440) \theta_{(+2)} \eta^{24} \\
= e_0 \left[ -2 + (348 - d_0(0) - 56n_{1/2}) q + (-280656 + 22d_0(0) + 27984n_{1/2}) q^2 + \cdots \right] \\
+ e_{1/2} \left[ n_{1/2} q^{1/2} + (-384 - 2d_0(0) - 384n_{1/2}) q^5 \right. \\
\left. + (-1122304 + 48d_0(0) - 77103n_{1/2}) q^{9} \cdots \right].
\]

We computed the integrals \( P_3 \) and \( P_1 \) in (91, 92) for those \( \Phi \) in (38) and \( \Psi \) in (118) parametrized by \( n_{1/2}, b_R \in \mathbb{Z}_{\geq 0} \); details are left to the appendix (133) and only the result is shown here:

\[
\frac{1}{3!} P_3(t) = \frac{(4 - b_R - n_{1/2})}{3!} (t^{a=1})^3, \quad P_1(t) = (52 - 4b_R - 10n_{1/2}) (t^{a=1}),
\]

where \( t \in \Lambda_S \otimes \mathbb{C} \) is parametrized by \( t = (t^{a=1})e \) with \( t^{a=1} \in \mathbb{C} \). So, the dictionary (88, 89, 90) yields

\[
d_{111} + 6\delta n'_a = 4 - b_R - n_{1/2}, \quad (c_2)_1 + 24\delta n'_1 = (52 - 4b_R - 10n_{1/2})
\]

for some \( \delta n'_1 \in \mathbb{Z} \). By comparing this with

\[
\text{Diff}_{\Lambda_S^{(+2)}} = \{ N \in \mathbb{Z} \} \times \{ \nu \in \{0, 1, 2, 3, 4, 5\} \},
\]

\[
d_{111} = \nu + 6(\delta n'_1), \quad (c_2)_1 = 12N - 2\nu + 24(\delta n'_1),
\]

we find that

\[
[\text{Mod}_0^\mathbb{Z}(21/2, \rho^{(+2)})]^{\text{mfd}} = [\text{Mod}_0^\mathbb{Z}(21/2, \rho^{(+2)})]^{\text{gfd}} = \{ n_{1/2} = 0, 1, 2, 3, 4 \},
\]

\[
[\text{Mod}_0^\mathbb{Z}(25/2, \rho^{(+2)})]^{\text{mfd}} = \{ b_R = 0, 2, 4, \cdots \} \subsetneq \{ b_R = 0, 1, 2, \cdots \} = \text{Mod}_0^\mathbb{Z}(25/2, \rho^{(+2)}).
\]

Within the set \( \text{Diff}_{\Lambda_S}(\mathbb{Q}) = \{ (3/2)(d_{111} - (c_2)_1)/4 = (\nu - 2N) \in \mathbb{Z} \} \times \{ \chi \in \mathbb{Z} \} \), the image of \( \text{diff}_{\text{coarse}} \) consists of \( \{ (\nu - 2N, \chi) = (n_{1/2} - 6, -252 + 56n_{1/2}) \mid n_{1/2} = 0, 1, 2, 3, 4 \} \). For a given \( d'_{111} \), all the three elements of \( \text{Diff}_{\Lambda_S}^{\text{fine}} \simeq \mathbb{Z}/3\mathbb{Z} = \{ \Delta \nu/2 = \Delta N = 0_{+3\mathbb{Z}}, 1_{+3\mathbb{Z}}, 2_{+3\mathbb{Z}} \} \) are realized by the image of the map \( \text{diff}_{\text{fine}} \) of \( \{ b_R = 0, 2, 4, \cdots \}/\{ \Delta b_R = 6 \} \).

---

69 \( d_{1/2}(1/4) \) must be linearly dependent with the other three.
70 The modular form \( J \) of (140) corresponds to \( \Psi = -J\eta^{24} \) with \( n_{1/2} = 0 \) and \( d_0(0) = 300 \).
71 One can employ the “embedding trick” to apply the lattice unfolding method.
Table 4: Topological invariants of Calabi–Yau three-folds with a regular \((+2)\)-polarized K3-fibration quoted from [24]. Corresponding parameters \((n_{1/2}, b_R + 6\mathbb{Z})\) of the modular forms \(\Phi\) and \(\Psi\) are shown on the right.

Some diffeomorphism classes of three-folds with a \((+2)\)-polarized regular K3-fibration are constructed by using toric technique, and are listed up in [24, Table 1] Those in the list must have the invariants \(\Phi\) and \(\Psi + \{\Delta \Psi\}\) specified in Table 4. Such choices as \((n_{1/2}, b_R + 6\mathbb{Z}) = (1, 4_{+6\mathbb{Z}}), (2, 4_{+6\mathbb{Z}}), (3, 2_{+6\mathbb{Z}}), (3, 4_{+6\mathbb{Z}})\), and all of \((4, 2\mathbb{Z}/6\mathbb{Z})\) are in the images of the map \((\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})\) but are not found in the Table; we have not made an effort to search in a larger Calabi–Yau topology database.

There is at least one pair of three-folds in the same diffeomorphism class but a holomorphic one-to-one map between them may or may not exist, also in the \(\Lambda_S = \langle +2 \rangle\) case. The three-folds denoted by \(M^{n=2}_{\langle +2 \rangle}\) with \(n = 2, 1, 0, -1\) in [11] are in the diffeomorphism classes with \(n_{1/2} = 0\) for all of them, and \(b_R + 6\mathbb{Z} = 0_{+6\mathbb{Z}}, 2_{+6\mathbb{Z}}, 4_{+6\mathbb{Z}},\) and \(0_{+6\mathbb{Z}},\) respectively; the pair \(M^{n=2}_{\langle +2 \rangle}\) and \(M^{n=-1}_{\langle +2 \rangle}\) are in the same class. Unlike the diffeomorphic pair \(X^{(n=2)}\) and \(X^{(n=0)}\) in the \(\Lambda_S = U\) case, however, the Kähler cones of \(M^{n=2}_{\langle +2 \rangle}\) and \(M^{n=-1}_{\langle +2 \rangle}\) are identical. To see this, let \(\{D_s^{(n)}, D_a^{(n)}\}_{a=1}^3\) be the basis of \(H^2(M^{(n)}_{\langle +2 \rangle}; \mathbb{Z})\) characterized by \((D_a^{(n)})^3 = 2n\), and \(\{\Sigma_B^{(n)}, \Sigma_F^{(n)}\}\) its dual basis of \(H_2(M_{\langle +2 \rangle}; \mathbb{Z})\). Toric techniques are used to find that \(\Sigma_B^{(n)}\) and \(\Sigma_F^{(n)}\) generate the cone of curves for \(n = 0, 1, 2\), while the generators should be \((\Sigma_B^{(n)} + n\Sigma_F^{(n)})\) and \(\Sigma_F^{(n)}\) for \(n = 0, -1\). It is not hard to find that the diffeomorphism \(f: M^{n=2}_{\langle +2 \rangle} \rightarrow M^{n=-1}_{\langle +2 \rangle}\) maps \(\Sigma_F^{(2)}\) to \(\Sigma_F^{(-1)}\) and \(\Sigma_B^{(2)}\) to \(\Sigma_B^{(-1)} - \Sigma_F^{(-1)}\), so the cone of curves are identical indeed.

For the three-folds \(M^{(n)}_{\langle +2 \rangle}\) with \(n = 2, 1, 0, -1\), the Discussion 2 cannot rule out a possibility for any \(C\) in the cone of curves that complex structure of \(M^{(n)}_{\langle +2 \rangle}\) can be tuned to have

\begin{table}[h]
| \(\chi\) | \(d_{111}\) | \((c_2)_1\) | \(n_{1/2}\) | \(b_R\) mod \(6\mathbb{Z}\) |
|-------|---------|----------------|--------|----------------|
| −252  | 4       | 52             | 0      | 0              |
| 2     | 44      |                | 2      |                |
| 0     | 36      |                | 4      |                |
| −196  | 3       | 42             | 1      | 0              |
| 1     | 34      |                | 2      |                |
| −140  | 2       | 32             | 2      | 0              |
| 0     | 24      |                | 2      |                |
| −84   | 1       | 22             | 3      | 0              |

Table 1: Discrete parameters

\footnote{Here we cited only the case \(\Lambda_S = \langle +2 \rangle\). They treat also the case \(\Lambda_S = \langle +4 \rangle\) and \(\langle +6 \rangle\).}
singularity of type $\mathcal{R}$ along $C$. For a curve class $C = \Sigma_{B}^{(n)} + m\Sigma_{F}^{(n)}$ (with $m \in \mathbb{Z}_{\geq 0}$ for $n \geq 0$), we have $f_{\mathcal{R}} = s + m(t^{a=1})$. The dictionary (120) with $n_{1/2} = 0$ and $-\delta n'_{a=1} = m$ reads $2n = 4 - b_{\mathcal{R}} + 6m$. So, the modular form $\Psi$ is that of $b_{\mathcal{R}} = 6m + 4 - 2n$, if it is possible to tune complex structure, and the corresponding Higgs cascade exists for this class $C = \Sigma_{B}^{(n)} + m\Sigma_{F}^{(n)}$. The easy-going version of the set of $\Psi$'s (using the cone of curves) is

$$M_{(+)2}^{n} : \{\Psi_{b_{\mathcal{R}}=6m+4-2n} \mid m \geq 0\} \subset \text{Mod}^{\Phi}_{0}(25/2, \rho_{(+)2})_{r_{\text{mfd}}}.$$  

$M_{(+)2}^{n}$ with different $n \mod 3$ are also distinguished in this way. For further attempt at finding difference between $M_{(+)2}^{(2)}$ and $M_{(+)2}^{(-1)}$, see discussion at the end of this section.

**A Look at $\mathcal{R}$-dependence**  The analysis up to this point relied on $\Psi$ of a Higgs cascade as a whole, so it is independent of the choice of the symmetry $\mathcal{R}$. Now let us use $\Phi$ and look into the information which type of singularity may develop in a given manifold.

Suppose that a three-fold $M_{(+)2}$ is in the diffeomorphism class characterized by $n_{1/2}$ and $b_{\mathcal{R}} + 6Z$. Unless $(b_{\mathcal{R}} + 6Z) = 0 \in \mathbb{Z}/6\mathbb{Z}$, any Higgs cascade attached to the branch of Type IIA compactification on $M_{(+)2}$ do not lead to an enhancement of singularity of type $\mathcal{R} = E_{6}$ (or higher) whose resolution $\overline{M}_{(+)2}$ has a regular K3-fibration.

As a test for whether an enhancement of singularity of a given type $\mathcal{R}$, one may ask whether an appropriate modular form $\Phi^{(\mathcal{R})}$ can be found. For example, take $\Delta_{S} = (+2) \oplus A_{3}[-1]$. Then the modular form $\Phi$ of the hypothetical branch of Type IIA on $M$ is in the form of

$$\Phi = \frac{-68 + 6n_{1} + n_{2} + n_{3}}{72} E_{6}\theta_{D_{5}} \otimes \theta_{(+2)} + \frac{-52 - n_{2} - n_{3}}{72} E_{4}\tilde{\partial}^{S}\theta_{D_{5}} \otimes \theta_{(+2)}$$

$$+ \frac{8 - 4n_{1} - 3n_{2} + 6n_{3}}{48} E_{4}\theta_{D_{5}} \otimes \tilde{\partial}^{\prime S}\theta_{(+2)} + \frac{-8 + n_{2} - 2n_{3}}{16} (\tilde{\partial}^{S})^{2}\theta_{D_{5}} \otimes \tilde{\partial}^{\prime S}\theta_{(+2)}$$  

$\tilde{\partial}^{S}$ is a shorthand: for modular form $F$ of weight $k$, $\tilde{\partial}^{S}F := (\partial^{S}F)/(-\frac{1}{12}) = (\frac{1}{24} \frac{\partial}{\partial \tau} - \frac{k}{12} E_{2})F/(-\frac{1}{12})$.
parametrized by $n_{1,2,3} \in \mathbb{Z}$ and $n_0 = -2$. This parametrization is for an obvious reason:

\[
\Phi = e_0 \otimes e_0 (-2 + (324 - 56n_1 - 8n_2 - 6n_3)q + \cdots) \\
+ e_0 \otimes e_1 (n_1q^{1/4} + \cdots) \\
+ (e_1 + e_3) \otimes e_0 (n_2q^{5/8} + \cdots) \\
+ (e_1 + e_3) \otimes e_1 ((16n_1 - 2n_2 + 8n_3 + 96)q^{7/8} + \cdots) \\
+ e_2 \otimes e_0 (n_3q^{1/2} + \cdots) \\
+ e_2 \otimes e_1 ((8 + 10n_1 + 4n_2 - 6n_3)q^{3/4} + \cdots) .
\]

(127)

By the discussion in section 2.2 the coefficients $n_{1,2,3} \in \mathbb{Z}$ need to satisfy

\[
n_1, n_2 \geq 0, \quad \text{and} \quad n_3, 16n_1 - 2n_2 + 8n_3 + 96, 8 + 10n_1 + 4n_2 - 6n_3 \geq -2.
\]

(128)

\[\Phi\] and \[\Psi\] is given by \[\Phi = \theta_{A_3} \cdot \Phi\] and \[\Psi = \tilde{\theta}^S \theta_{A_3} \cdot \Phi\]. Then we have

\[
n_{1/2} = n_1, \quad b_R = -8 + n_2 + n_3.
\]

(129)

For $n_{1/2} = 0$ and $b_R = 0,2,4$, for example, there are solutions $(n_1, n_2, n_3)$ to (128). We do not have to rule out a possibility that there exists tuning of complex structure of $M^{n=1,0}_{(+2)}$ so it develops singularity of type $A_3$ along a curve $\Sigma_B$, and its resolution $\hat{M}$ has a regular K3-fibration. Indeed, such enhancement can be realized as we see below.

The Calabi–Yau three-folds $M^{n=2,1,0,-1}_{(+2)}$ are given by a hypersurface surface equation

\[
X_1^2 + F^{(6)}(X_{2,3,4}) = 0.
\]

(130)

\[
F^{(6)} = a_0X_4^6 + b_0X_4^4(X_2X_3) + c_0X_4^2(X_2X_3)^2 + d_0(X_2X_3)^3 \\
+ (a_1X_4^5X_2 + a_2X_4^4X_2^2 + \cdots + a_6X_2^6) \\
+ (b_1X_3^2X_2^2 + b_2X_2X_3^3 + b_3X_3X_4^2 + b_4X_4^5)X_3 + (c_1X_4X_2 + c_2X_4^2)X_3^2 \\
+ (a'_1X_4X_3 + a'_2X_4^2X_3^2 + \cdots + a'_6X_3^6) \\
+ (b'_1X_3^2X_3^2 + b'_2X_2X_3^3 + b'_3X_3X_4^2 + b'_4X_4^5)X_2 + (c'_1X_4X_3 + c'_2X_4^2)X_2^2.
\]

(131)

Homogenous coordinates $X_1$ and $X_{2,3,4}$ of the toric ambient space are subject to the $\mathbb{C}^\times$ action $X_1 \to X_1\lambda^3$ and $X_{2,3,4} \to X_{2,3,4}\lambda$ for projectivization. The coefficients $a_{i=1,\ldots,6}, b_{i=1,\ldots,4}, c_{i=1,2}$,

74 Remembering that $SU(4) \times SO(10)$ fits into $E_8$, it is a reasonable idea to try to construct a basis by using $\theta_{(+2)} \otimes \theta_{D_5}$, derivatives, and the Eisenstein series $E_4$ and $E_6$. The dimension formula indicates that the vector space is of 4-dimensions for the weight and type for this $\Phi$.

75 Of course, there is no guarantee that there exists an actual enhancement $\Phi$ when we can construct a candidate for modular form $\Phi$.
has pseudotype II degeneration in the fibre of those points. So, this failure in finding a
failure in finding a Higgs cascade for \( \xi \) equation is consistent with \( b_R + 6Z = 2 + 4 + 6Z \) that does not allow interpretation as a 1-loop beta function of \( R = E_7 \) gauge group. On the other hand, singularity of type \( R = A_3 \) enhances along \( C_R = \Sigma^{(a)}_B \) by setting \( a_{3,...,6} \) and \( b_{3,4} \) to zero, and we can see by using toric data (just like in \([11]\)) that their \( M \) has a regular K3-fibration, as announced earlier.

Given the diffeomorphism \( f : M^{(2)}_{(+2)} \to M^{(-1)}_{(+2)} \) the symmetry-enhancement branch of \( M^{(2)}_{(+2)} \) with singularity along \( C_R = \Sigma^{(2)}_B \) should be compared with the symmetry-enhancement branch of \( M^{(-1)}_{(+2)} \) with singularity along \( \Sigma^{(-1)}_B - \Sigma^{(-1)}_F \). For singularity of type \( R = E_7 \) for the latter, we can tune \( a'_{4,...,6}, b', b_0, a_1, b_1 \), and \( a_2 \) to zero, for example, because the hypersurface equation is \( \xi_1^3 + b_2 \xi_3^2 \xi_2 + a_3 \xi_3^3 \simeq 0 \) near the curve \( \xi_2 \xi_3^2 = 0 \); this is not difficult to see by using toric language. So, this Higgs cascade for \( M^{(-1)}_{(+2)} \) and \( C_R = \Sigma^{(-1)}_B - \Sigma^{(-1)}_F \) must have the same modular form \( \Psi \) as the Higgs cascade for \( M^{(2)}_{(+2)} \) and \( C_R = \Sigma^{(2)}_B \). Those two symmetry-enhancement branches are different, because \( a_3 \) is a section of \( O(3) \) on \( \mathbb{P}^{1}_{IIA} \) in the former, whereas \( d_0 \) that of \( O(0) \) on \( \mathbb{P}^{1}_{IIA} \) in the latter, and the three-fold \( M \) for the former does not have a regular K3-fibration. There is still a pair of branches of Heterotic–IIA dual

| \( a_i, a'_i \) | \( b_i, b'_i \) | \( c_i, c'_i \) | \( d_0 \) |
|---|---|---|---|
| \( 12 - 2i + (i - 4)(2 - n) \) | \( 8 - 2i + (i - 2)(2 - n) \) | \( 4 - 2i + i(2 - n) \) | \( 2(2 - n) \) |

Table 5: The “coefficients” \( a_i, b_i, \) etc. in \([131]\) are sections of \( O_{\mathbb{P}^1}(\deg) \), with the degree “deg” specified in this table.

\( a'_{i=1,...,6}, b'_{i=1,...,4}, c'_{i=1,2}, \) and \( a_0, b_0, c_0 \) and \( d_0 \) are regarded sections of appropriate line bundles of the base \( \mathbb{P}^1 \); those line bundles should have the degree specified in Table 5 for construction of \( M^{n}_{(+2)} \). More details are found in \([11]\).

Singularity of type \( R = E_7 \) develops in \( M^{n}_{(+2)} \) for any one of \( n = 2, 1, 0, -1 \), when all of the sections \( a_{2,...,6}, b_{2,...,4}, \) and \( c_{1,2} \) are set to zero. The singularity is along the curve
\[
(x_1, x_4, x_3) = (0, 0, 0), \quad \text{which is in the class } C_R = \Sigma^{(n)}_B; \quad \text{it is of type } E_7 \text{ because } x_1^2 + b_1 x_4^2 x_3 + d_0 x_3^3 \simeq 0 \text{ is in the direction of the K3 fibre.}
\]

It is only in the case of \( n = 2 \), however, that the K3-fibration in the three-fold \( M \) after the tuning remains regular; for \( n \neq 2 \), the coefficient \( d_0 \) is a section of a line bundle of the base \( \mathbb{P}^1 \) of positive degree. The section \( d_0 \) vanishes at some points in the base, and the three-fold \( M \) has pseudo-Type II degeneration in the fibre of those points. So, this failure in finding a tuning of complex structure for \( M^{n=1.0}_{(+2)} \) is consistent with \( b_R + 6Z = 2 + 4 + 6Z \) that does not allow interpretation as a 1-loop beta function of \( R = E_7 \) gauge group. On the other hand, singularity of type \( R = A_3 \) enhances along \( C_R = \Sigma^{(a)}_B \) by setting \( a_{3,...,6} \) and \( b_{3,4} \) to zero, and we can see by using toric data (just like in \([11]\)) that their \( M \) has a regular K3-fibration, as announced earlier.

Given the diffeomorphism \( f : M^{(2)}_{(+2)} \to M^{(-1)}_{(+2)} \) the symmetry-enhancement branch of \( M^{(2)}_{(+2)} \) with singularity along \( C_R = \Sigma^{(2)}_B \) should be compared with the symmetry-enhancement branch of \( M^{(-1)}_{(+2)} \) with singularity along \( \Sigma^{(-1)}_B - \Sigma^{(-1)}_F \). For singularity of type \( R = E_7 \) for the latter, we can tune \( a'_{4,...,0}, b', b_0, a_1, b_1 \), and \( a_2 \) to zero, for example, because the hypersurface equation is \( \xi_1^3 + b_2 \xi_3^2 \xi_2 + a_3 \xi_3^3 \simeq 0 \) near the curve \( \xi_2 \xi_3^2 = 0 \); this is not difficult to see by using toric language. So, this Higgs cascade for \( M^{(-1)}_{(+2)} \) and \( C_R = \Sigma^{(-1)}_B - \Sigma^{(-1)}_F \) must have the same modular form \( \Psi \) as the Higgs cascade for \( M^{(2)}_{(+2)} \) and \( C_R = \Sigma^{(2)}_B \). Those two symmetry-enhancement branches are different, because \( a_3 \) is a section of \( O(3) \) on \( \mathbb{P}^{1}_{IIA} \) in the former, whereas \( d_0 \) that of \( O(0) \) on \( \mathbb{P}^{1}_{IIA} \) in the latter, and the three-fold \( M \) for the former does not have a regular K3-fibration. There is still a pair of branches of Heterotic–IIA dual

\textsuperscript{76}We used inhomogeneous coordinates \( x_1 = X_1/X_3^3, x_4 = X_4/X_2, \) and \( x_3 = X_3/X_2 \).

\textsuperscript{77}Now, the inhomogeneous coordinates are \( \xi_1 = X_1/X_3^4, \xi_{2,3} = X_{2,3}/X_4 \).
vacua that cannot be distinguished by $\Lambda_S$, $\Lambda_T$, $\Phi$, and the set of $\Psi$’s.

4 Open Questions

Practical questions remain: how small are the subspaces $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2; \rho \Lambda_S)]^{r,mfd}$ and $[\text{Mod}_0^\Phi(13 - \rho/2; \rho \Lambda_S)]^{r,mfd}$ within $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2; \rho \Lambda_S)]$ and $[\text{Mod}_0^\Phi(13 - \rho/2; \rho \Lambda_S)]$, respectively. We worked on this question for $\Lambda_S = U, \langle +2 \rangle$, and $U \oplus \langle -2 \rangle$ in this article, but not for general $(\Lambda_S, \Lambda_T)$ that fits into $\Pi_{3,19}$. For example, it is doable (at least in theory) to study whether the subspaces remain non-empty for the series $\Lambda_S = \langle +2n \rangle$ with large $n$.

The image of the map $(\text{diff}_{\text{coarse}}, \text{diff}_{\text{fine}})$ restricts possible diffeomorphism classes of real six-dimensional manifolds that can be realized by Calabi–Yau three-folds with $\Lambda_S$-polarized regular K3-fibrations. This method has been applied only for $\Lambda_S = U$ and $\langle +2 \rangle$, and found that the image of $\text{diff}_{\text{coarse}}$ is much smaller than the set $\text{Diff}_{\Lambda_S}(\mathbb{Q})$ for both $\Lambda_S$, and the image of $\text{diff}_{\text{fine}}$ is all of $\text{Diff}_{\Lambda_S}^{d', \chi}$ in the image of $\text{diff}_{\text{coarse}}$. One can find out whether that remains to be true for various different $\Lambda_S$’s, by working out the images of $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2; \rho \Lambda_S)]^{r,mfd}$ and $[\text{Mod}_0^\Phi(13 - \rho/2; \rho \Lambda_S)]^{r,mfd}$.

A few theoretical questions can also be put down. There are two possibilities for a pair of modular forms $\Phi$ and $\Psi$ that are not in the subsets $[\text{Mod}_0^\mathbb{Z}(11 - \rho/2; \rho \Lambda_S)]^{r,mfd}$ and $[\text{Mod}_0^\Phi(13 - \rho/2; \rho \Lambda_S)]^{r,mfd}$. One is that there are more theoretical constraints of string theory that we failed to capture in sections 2 and 3 and such a $(\Phi, \Psi)$ is in conflict with those constraints. The other is that such a modular form $(\Phi, \Psi)$ is for a branch of moduli space whose Type IIA description does not involve a geometric phase. It remains to be an open question how to determine the boundary between those two possibilities in the space of $(\Phi, \Psi)$.

We have already seen that the image of the map $(\text{Diff}_{\text{coarse}}, \text{Diff}_{\text{fine}})$ is small compared with the set $\text{Diff}_{\Lambda_S}$. But not all the diffeomorphism classes of real six-dimensional manifold in the image are guaranteed to be realized as a Calabi–Yau three-fold. By taking advantage of large data base of topology of Calabi–Yau three-folds, one may try to get the feeling how much fraction of the diffeomorphism classes in the image of $(\text{Diff}_{\text{coarse}}, \text{Diff}_{\text{fine}})$ are indeed guaranteed to be realized by Calabi–Yau three-folds. Such a study may provide hints in considering the “determining the boundary” issue above.

In pure mathematics literatures, some inequalities on topological invariants of Calabi–Yau three-folds have been derived (e.g., [17, §2]). It is beyond the scope of this article to study how those inequalities are related to the the bounds that we discussed in this article.
Also in physics approach, various integer parameters are likely not just bounded from below, but also from above (for maintaining strictly a given lattice \( \Lambda_S \)). But we have not given enough thoughts on how this intuition is related or unrelated to the bounds and classifications discussed in this article.

A few ideas are also available in improving the effort of introducing invariants for classification of the branches of moduli space of Het–IIA dual vacua. We just introduced the idea of using the set of \( \Psi \) of all the Higgs cascade as an invariant of a branch (than just using its image by \( \text{Diff}_{\text{fine}} \)) in this article; more knowledge in the cone of curves (and tuning of complex structure to have certain singularity along a curve class) would make it possible to compute the set of \( \Psi \)'s for general \( \Lambda_S \), not just for \( \Lambda_S = U \). Also, a part of the idea (using \( \Delta h_{2,1}^R \)) for invariants in the case \( \Lambda_S = \langle +2 \rangle \) in [11] has been incorporated as a part of \( \Phi^{(R)} \) for general \( \Lambda_S \) in this article, but a bit more idea beyond \( \Delta h_{2,1}^{2,1} \) in [11] has not been generalized to other \( \Lambda_S \)'s, or brought into the language of world-sheet CFT in this article.

Finally, as a reminder, K3-fibration of a Calabi–Yau three-fold was assumed to be regular in this article. Classifications of Calabi–Yau three-folds with a non-regular K3-fibration (and their Heterotic duals) should be considered separately from this article. Furthermore, we also set some other technical limitations in section 2.1 on the class of Heterotic–IIA dual vacua to study in this article. Structure of branches of all those vacua is yet to be figured out.

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**A Modular forms**

**A.1 Notations and basic facts**

In this subsection we explain our notations and some basic facts about modular forms.
Metaplectic Group  Metaplectic group $\text{Mp}(2, \mathbb{Z})$ is defined by

$$\text{Mp}(2, \mathbb{Z}) = \left\{ \left( A, f(\tau) \right) \bigg| A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \ f(\tau)^2 = c\tau + d \right\}. \quad (132)$$

$f$ is a holomorphic function in the upper half plane $\mathcal{H} \subset \mathbb{C}$. $f$ specifies the choice of sign $\pm \sqrt{c\tau + d}$, so $\text{Mp}(2, \mathbb{Z})$ is a double covering of $\text{SL}(2, \mathbb{Z})$. The multiplication of two elements in the group $\text{Mp}(2, \mathbb{Z})$ is defined by

$$(A, f(\tau))(B, g(\tau)) = (AB, f(B \cdot \tau)g(\tau)). \quad (133)$$

The group $\text{SL}(2, \mathbb{Z})$ is generated by two elements

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (134)$$

which satisfy $S^2 = (ST)^3 = -1$. Similarly, $\text{Mp}(2, \mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \sqrt{\bar{\tau}}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, 1, \quad (135)$$

which satisfy $S^2 = (ST)^3 = Z$, where

$$Z = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, i, \quad Z^4 = 1. \quad (136)$$

Vector valued modular form  Let $k^\pm \in \frac{1}{2}\mathbb{Z}$, and $\rho : \text{Mp}(2, \mathbb{Z}) \to \text{GL}(V)$ be a representation on a vector space $V$. A real analytic (but not necessarily holomorphic) function $F : \mathcal{H} \to V$ is called a (vector valued) modular form of weight $(k^+, k^-)$ and type $\rho$ if $F$ satisfies the modular transformation laws

$$F(A \cdot \tau) = \sqrt{c\tau + d}^{2k^+} \sqrt{c\bar{\tau} + d}^{2k^-} \rho(A, f)F(\tau), \quad (A, f) \in \text{Mp}(2, \mathbb{Z}). \quad (137)$$

A modular form $F(\tau, \bar{\tau})$ is said to be almost holomorphic, if it is a polynomial of $(1/\tau_2)$ with $\tau$-dependent coefficients and has finite values at all the cusp points. The vector space of almost holomorphic modular forms of weight $(k^+, k^-)$ and type $\rho$ is denoted by $\text{Mod}((k^+, k^-), \rho)$. The vector space of truely holomorphic modular forms—no $\bar{\tau}$ dependence—in $\text{Mod}((k^+, 0), \rho)$ is denoted by $\text{Mod}(k^+, \rho)$. We mainly consider the case where $\rho$ is a Weil representation (explained shortly) in this article. A subspace $\text{Mod}_0(k, \rho)$ of $\text{Mod}(k, \rho)$ is also defined below.
**Weil representation** Let \( M \) be an even lattice of signature \((b^+, b^-)\), and \( G_M = M^r/M \) the discriminant group. Define \( \mathbb{C}[G_M] = \text{span}_\mathbb{C}\{e_\gamma \mid \gamma \in G_M\} \) by using a formal symbol \( e_\gamma \).

The Weil representation \( \rho_M : \text{Mp}(2, \mathbb{Z}) \to \text{GL}(\mathbb{C}[G_M]) \) is defined by

\[
\rho_M(T)e_\gamma = e_\gamma \mathbb{E}\left(\frac{(\gamma, \gamma)}{2}\right),
\]

\[
\rho_M(S)e_\gamma = \sum_{\delta \in G_M} e_\delta \frac{1}{\sqrt{|G_M|}} \mathbb{E}\left(-\frac{\text{sgn}(M)}{8} - (\delta, \gamma)\right),
\]

where \( \mathbb{E}(x) = e^{2\pi ix} \). The element \( Z \) acts as \( \rho_M(Z)e_\gamma = e^{-\gamma} \mathbb{E}(-\text{sgn}(M)/4) \). The relation \( S^2 = (ST)^3 \) holds because of Milgram’s formula:

\[
\sum_{\gamma \in G_M} \mathbb{E}\left((\gamma, \gamma)/2\right) = \sqrt{|G_M|} \mathbb{E}\left(\text{sgn}(M)/8\right).
\]

If two even lattices \( M_1, M_2 \) are primitive sublattices of certain unimodular lattice \( L \) and are orthogonal to each other inside \( L \otimes \mathbb{R} \), then \( \rho_{M_1} \) and \( \rho_{M_2} \) are the dual (contragradient) representation of \( \text{Mp}(2, \mathbb{Z}) \) of each other. For example \( \rho_{\Lambda_S} \) and \( \rho_{\Lambda_T} \) in the main text are dual.

**Subspaces of Mod\((k, \rho)\) of interest** A modular form \( \Phi \in \text{Mod}(k, \rho_M) \) has Fourier expansion

\[
\Phi(\tau) = \sum_{\gamma \in G_M} e_\gamma \sum_{\nu \in \gamma^2/2+\mathbb{Z}} x_\gamma(\nu) q^\nu, \quad x_\gamma(\nu) = 0 \text{ for } \nu < 0.
\]

Here \( q = e^{2\pi i \tau} \). \( \gamma^2/2 \) denotes the quadratic form over \( G_M \). We define \( \text{Mod}_0(k, \rho) \) by imposing cusp condition at isotropic (but nonzero) \( \gamma \):

\[
\text{Mod}_0(k, \rho) = \{ \Phi \in \text{Mod}(k, \rho) \mid x_\gamma(0) = 0 \text{ if } \gamma \neq 0 \text{ and } \gamma^2/2 = 0 \}.
\]

In this article, the Fourier coefficients \( x_\gamma(\nu) \) are denoted by \([\Phi_\gamma]_{q^\nu} \).

**Dimensional formula** The dimension of the vector space \( \text{Mod}(k, \rho_M) \) is determined by the formula in the case of \( k > 2 \) and \( k/2 \equiv \text{sgn}(M)/4 \text{ mod } \mathbb{Z} \) [17]:

\[
\dim_{\mathbb{C}}(\text{Mod}(k, \rho_M)) = d + \frac{dk}{12} - \alpha\left(\mathbb{E}\left(\frac{k}{4}\right) \rho_M(S)\right) - \alpha\left(\mathbb{E}\left(\frac{k}{6}\right) \rho_M(ST)^{-1}\right) - \alpha(\rho_M(T)),
\]
where $d := \dim \mathbb{C}[G_M]^+$, and $\alpha(\rho_M(g)) := \sum_{i=1}^{d} \beta_i$ when $\{\beta_i\}$'s are the complex phases of the eigenvalues of the representation matrix, $\rho_M(g) : \mathbb{C}[G_M]^+ \sim \mathbb{E}(\text{diag}(\beta))$, set within the range $0 \leq \beta_i < 1$. The restriction $k > 2$ of this formula is due to the fact that $\dim \mathbb{C} \text{Mod}(k, \rho_M^x) = \dim \mathbb{C} \text{Mod}(2 - k, \rho_M)$ is not necessarily zero for $k \leq 2$.

The dimension of the subspace $\text{Mod}_0(k, \rho_M)$ is \[\dim \mathbb{C}(\text{Mod}_0(k, \rho_M)) = \dim \mathbb{C}(\text{Mod}(k, \rho_M)) - \# \{ \pm \gamma \in G_M/\{\pm 1\} | \gamma \neq 0, \gamma^2/2 = 0 \}. \tag{144}\]

**Siegel theta function** Let $M$ be an even lattice of signature $(b^+, b^-)$. Fix a point $v$ in Grassmannian $Gr(M) = Gr(M \otimes \mathbb{R}; b^+)$, i.e. a pair of positive/negative definite $b^+$-dimensional subspace of $M \otimes \mathbb{R}$, orthogonal to each other, and define $v_{\pm} : M \otimes \mathbb{R} \to M \otimes \mathbb{R}$ by the orthogonal projections. The *Siegel theta function* is defined by

$$\theta_M(\tau, \bar{\tau}; v) = \sum_{\gamma \in M^\vee/M} e_\gamma \sum_{\lambda \in M + \gamma} q^{v_\pm^2(\lambda)/2} \bar{q}^{v_\pm^2(\lambda)/2}, \quad q := e^{2\pi i r}. \tag{145}$$

Here $v_\pm^2(\lambda) := |(v_{\pm}(\lambda), v_{\pm}(\lambda))|$ are both non-negative. $\theta_M(\tau, \bar{\tau}; v)$ is a $\mathbb{C}[G_M]$-valued modular form of weight $(b^+/2, b^-/2)$ and type $\rho_M$.

**Eisenstein series**

\[E_2 = 1 - 24 \left( q + 3q^2 + \cdots \right) = 1 - 24 \sum_{n=1}^{\infty} q^n \sigma_1(n) = 1 - 24 \sum_{m=1}^{\infty} \frac{mq^m}{1 - q^m}, \tag{146}\]

\[E_4 = 1 + 240 \left( q + 9q^2 + \cdots \right) = 1 + 240 \sum_{n=1}^{\infty} q^n \sigma_3(n) = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1 - q^m}, \tag{147}\]

\[E_6 = 1 - 504 \left( q + 33q^2 + \cdots \right) = 1 - 504 \sum_{n=1}^{\infty} q^n \sigma_5(n) = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1 - q^m}. \tag{148}\]

$E_4(\tau)$ and $E_6(\tau)$ are modular forms of weight 4 and 6, respectively, for $\text{SL}(2; \mathbb{Z})$, but $E_2(\tau)$ is not modular (it is Mock modular). The space of scalar valued modular form can be identified with the polynomial ring $\mathbb{C}[E_4, E_6]$.

The $q^0$-term vanishes in the combination

$$E_4^3 - E_6^2 = (1 + 3 \cdot 240q + \cdots) - (1 - 2 \cdot 504q - \cdots) = 1728q + \cdots = 1728\eta^{24}, \tag{149}$$

where

$$\eta = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{150}$$
Ramanujan–Serre derivative and Rankin–Cohen bracket  For a modular form $F \in \text{Mod}(k, \rho)$, the Ramanujan–Serre derivative $\partial^S : \text{Mod}(k, \rho) \to \text{Mod}(k+2, \rho)$ is defined by

$$\partial^S F := \left( \frac{1}{2\pi i} \frac{\partial}{\partial \tau} - \frac{k}{12} E_2 \right) F = \eta^{2k} q \partial_q \left( \frac{F}{\eta^{2k}} \right).$$

(151)

Vector-valued modular forms and their Ramanujan–Serre derivatives can be multiplied to produce yet another vector-valued modular form. The Rankin–Cohen bracket $[F, G]_n$ of a pair of modular forms $F \in \text{Mod}(w_F, \rho_F)$ and $G \in \text{Mod}(w_G, \rho_G)$ and $n \in \mathbb{N}$ is

$$[F, G]_n := \frac{1}{(2\pi i)^n} \sum_{r+s=n} (-1)^r \left( \begin{array}{l} w_F + n - 1 \\ s \end{array} \right) \left( \begin{array}{l} w_G + n - 1 \\ r \end{array} \right) \frac{\partial^r F \partial^s G}{\partial^r \tau \partial^s \tau}. \quad (152)$$

For example, $[F, G]_0 = FG$, and $[F, G]_1 = w_F F q(\partial_q G) - w_G q(\partial_q F)G$. It is known that $[F, G]_n \in \text{Mod}(w_F + w_G + 2n, \rho_F \otimes \rho_G)$.

Lattice-polarized Jacobi forms and vector valued modular forms  For a positive definite even lattice $M$, and $k \in \frac{1}{2} \mathbb{Z}$, a holomorphic function $\phi : \mathcal{H} \times (\mathbb{M} \otimes \mathbb{R} \mathbb{C}) \ni (\tau, z) \mapsto \mathbb{C}$ is said to be a weight-$k$ index-$M$ Jacobi form, if it satisfies

$$\phi \left( a\tau + b, \frac{c\tau + d}{c\tau + d} \right) = (c\tau + d)^k \mathbb{E} \left( \frac{c(z, z)}{2(c\tau + d)} \right) \phi(\tau, z),$$

(153)

$$\phi(\tau, z + \mu + \lambda \tau) = \mathbb{E} \left( -\left( \lambda, z \right) - \tau \frac{(\lambda, \lambda)}{2} \right) \phi(\tau, z),$$

(154)

where the bilinear form $(-, -)$ of $M$ has been extended linearly to $\mathbb{M} \otimes \mathbb{R} \mathbb{C}$. The classical definition of a weight-$k$ index-$m$ Jacobi form is regarded as that of a weight-$k$ index-$M$ Jacobi form with the lattice $M = \langle +2m \rangle$.

To any Jacobi form $\phi(\tau, z)$ of weight-$k$ and index-$M$, one can assign a vector valued modular form of weight-$(k-1/2)$ and type $\rho_M^\vee$. That is through

$$\phi(\tau, z) = \sum_{x \in G_M} \left( \sum_{\lambda \in x + M} q^{(\lambda, \lambda)} e^{2\pi i (\lambda, z)} \right) f_x(\tau) \iff \sum_{x \in G_M} e_x f_x(\tau),$$

(155)

and this is one-to-one. The modular form $\sum_x e_x f_x(\tau)$ is holomorphic at the cusps if and only if the expansion $\phi(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{\lambda \in M^\vee} c(n, \lambda) q^n e^{2\pi i (\lambda, z)}$ has support in $n \geq (\lambda, \lambda)/2$. See [48, 49] for more information.
A.2 Explicit Examples

Some of the modular forms used in the main text are written down explicitly here.

A.2.1 Explicit Basis of \( \text{Mod}_0(11 - \rho/2, \rho_{A_S}) \)

The case \( \Lambda_S = \langle +2 \rangle \): (cf. [24, 30]) The vector space \( \text{Mod}_0(21/2, \rho_{(+2)}) \) is 2-dimensional over \( \mathbb{C} \). One can use \( \phi_{(i)} := [\theta_{(+2)}, E_{10-2i}]_i \) with \( i = 0, 1 \) as a basis. Their Fourier expansions are

\[
\phi_{(0)} = e_0 \left( 1 - 262q - 135960q^2 + \cdots \right) + e_{1/2} \left( 2q^{1/4} - 528q^{3/4} - 270862q^{5/4} + \cdots \right), \\
\phi_{(1)} = e_0 \left( 0 + 224q + 54720q^2 + \cdots \right) + e_{1/2} \left( -4q^{1/4} - 1440q^{3/4} - 123876q^{5/4} + \cdots \right),
\]

where \( \{e_0, e_{1/2}\} \) is the basis of \( \mathbb{C}[G_S] \).

The Case \( \langle +4 \rangle \): (cf. [24, 30]) The vector space \( \text{Mod}_0(21/2, \rho_{(+4)}) \) is of 3-dimensions, and is known to be generated by \( \phi_{(i)} := [\theta_{(+4)}, E_{10-2i}]_i \) with \( i = 0, 1, 2 \). Their Fourier expansions are

\[
\phi_{(0)} = 1e_0 + q^{1/8}(e_{1/4} + e_{3/4}) + 2q^{1/2}e_{2/4} - 264qe_0 - 263q^{9/8}(e_{1/4} + e_{3/4}) + \cdots, \\
\phi_{(1)} = -q^{1/8}(e_{1/4} + e_{3/4}) - 8q^{1/2}e_{2/4} + 240qe_0 - 249q^{9/8}(e_{1/4} + e_{3/4}) + \cdots, \\
\frac{64}{21}\phi_{(2)} = +q^{1/8}(e_{1/4} + e_{3/4}) + 32q^{1/2}e_{2/4} - 576qe_0 + 1017q^{9/8}(e_{1/4} + e_{3/4}) + \cdots.
\]

The modular form \( \Phi \) for a Heterotic–IIA dual vacuum is parametrized by the low-energy BPS indices \( n_0 = -2 \), and \( n_{1/4}, n_{1/2} \). It must be

\[
\Phi = -2\phi_{(0)} + \frac{(n_{1/2} - 32n_{1/4} - 60)}{24} \phi_{(1)} + \frac{(n_{1/2} - 8n_{1/4} - 12)}{24} \frac{64}{21} \phi_{(2)},
\]

\[
e_0 \left( -2 + (216 - 14n_{1/2} - 128n_{1/4})q + (153900 - 57344n_{1/4} - 568n_{2/4})q^2 + \cdots \right) \\
+ (e_{1/4} + e_{3/4}) \left( n_{1/4}q^{1/4} + (640 - 7n_{1/4} + 32n_{1/2})q^{3/4} + (273028 - 272n_{1/4} + 544n_{2/4})q^{5/4} + \cdots \right) \\
+ e_{2/4} \left( n_{1/2}q^{1/2} + (10032 + 4864n_{1/4} - 188n_{1/2})q^{3/2} + \cdots \right).
\]

It follows that

\[
\chi(X_{IIA}) = 48 - NL_{1,0} = -168 + 128n_{1/4} + 14n_{2/4}.
\]
is parametrized by the low-energy BPS indices \( n_0 = -2 \), and \( n_\gamma \) with \( \gamma = 1/6, 2/6, 3/6 \):

\[
\frac{\Phi}{\eta^{1/4}} = F^{-1} + n_{1/6}F^{-11/12} + n_{2/6}F^{-8/12} + n_{3/6}F^{-3/12},
\]  

(164)

where we can use the following [30]

\[
F^{-1} = \frac{-2}{q} e_0 + 148e_0 + 336q^{1/4}(e_{1/6} + e_{5/6}) + 2730q^{3/4}(e_{2/4} + e_{4/6}) + 35360q^{9/4}e_{3/6} + \cdots,
\]

\[
F^{-11/12} = \frac{q^{1/2}}{q} (e_{1/6} + e_{5/6}) - 108e_0 - 134q^{1/2}(e_{1/6} + e_{5/6}) + 924q^{3/2}(e_{2/4} + e_{4/6}) + 20196q^{9/2}e_{3/6} + \cdots,
\]

\[
F^{-8/12} = \frac{q^{1/2}}{q} (e_{2/4} + e_{4/6}) - 54e_0 + 56q^{1/2}(e_{1/6} + e_{5/6}) + 214q^{3/2}(e_{2/4} + e_{4/6}) - 3024q^{9/2}e_{3/6} + \cdots,
\]

\[
F^{-3/12} = \frac{q^{1/2}}{q} e_{3/6} - 2e_0 + 3q^{1/2}(e_{1/6} + e_{5/6}) - 6q^{3/2}(e_{2/4} + e_{4/6}) + 14q^{9/2}e_{3/6} + \cdots.
\]  

(165)

It follows that

\[
\chi(X_{\text{IIA}}) = -c_0(0) = -148 + (108n_{1/6} + 54n_{2/6} + 2n_{3/6}).
\]  

(166)

All the details so far in the appendix A.2.1 are used in section 2.3.1.

**The Case \( \Lambda_S = U \oplus (-4) \):** There are two linearly independent holomorphic Jacobi forms of weight 10 and index 2 (see [49]). One can work out a basis of the 2-dimensional vector space \( \text{Mod}_0(19/2, \rho(-4)) \) by using those holomorphic Jacobi forms. The modular form \( \Phi \) in this vector space is parametrized by \( n_0 = -2 \) and \( n_{2/4} \), as

\[
\Phi = e_0 \left( -2 + (336 - 18n_{2/4})q + (116340 - 16n_{2/4})q^2 + \cdots \right) + (e_{1/4} + e_{3/4}) \left( (96 + 8n_{2/4})q^{1/2} + (66976 + 120n_{2/4})q^{3/2} + (2539488 - 1368n_{2/4})q^{5/2} + \cdots \right) + e_{2/4} \left( n_{2/4}q^{3/2} + (10192 - 120n_{2/4})q^{5/2} + (771456 + 900n_{2/4})q^{7/2} + \cdots \right).
\]  

(167)

We can read out \( n_{1/4} = 8n_{2/4} + 96 \) from \( \Phi \) above; this is used in section 2.3.2.

**A.2.2 Some Lower Bounds on \( \chi(X) \) for \( \Lambda_S = (+2n) \)**

In section 2.4, we derive an expression for \( \chi(X) \) for Calabi–Yau three-folds \( X \) that have a regular \( \Lambda_S \)-polarized K3-fibration, by using \( \phi \in \text{Mod}(3 + \rho/2, \rho_{\Lambda_S}^\vee) \) whose Fourier coefficients at lower powers of \( q \)'s are all positive. In this appendix A.2.2, we leave details of \( \phi = \theta_L[-1] \) in some of the \( \rho = 1 \) cases [78] and the expression for \( \chi(X) \) that follows.

[78] In the \( \rho = 1 \) cases (\( \Lambda_S = (+2n) \)), we can use knowledge on the vector space of weight-4 index-\( n \) holomorphic Jacobi forms to construct a basis of \( \text{Mod}(7/2, \rho_{(+2n)}^\vee) \). So, it is also possible to find \( \phi \)'s with positive Fourier coefficients from this vector space, instead of finding lattices \( L \) and using \( \phi = \theta_L[-1] \).
\begin{table}[h]
\centering
\begin{tabular}{c c}
\hline
$\lambda_S$ & $\chi(X_{\text{IIA}})$ \\
\hline
$\langle +2 \rangle$ & $\chi = -252 + 56n_{1/2}$ \\
$\langle +4 \rangle$ & $\chi = -196 + 128n_{1/4} + 14(n_{2/4} + 2)$ \\
$\langle +6 \rangle$ & $\chi = -152 + 108n_{1/6} + 54n_{2/6} + 2(n_{3/6} + 2)$ \\
$\langle +8 \rangle$ & $\chi = -112 + 112n_{1/8} + 56n_{2/8} + 16n_{3/8}$ \\
$\langle +10 \rangle$ & $\chi = -124 + 88n_{1/10} + 66n_{2/10} + 24n_{3/10} + 2(n_{4/10} + 2) + 32n_{5/10}$ \\
$\langle +12 \rangle$ & $\chi = -124 + 96n_{1/12} + 60n_{2/12} + 32n_{3/12} + 6(n_{4/12} + 2) + 96n_{5/12} + 10(n_{6/12} + 2)$ \\
$\langle +14 \rangle$ & $\chi = -144 + 56n_{1/14} + 54n_{2/14} + 54n_{3/14} + 2n_{4/14} + 2n_{5/14} + 54n_{6/12}$ \\
$\langle +14 \rangle$ & $\chi = -92 + 84n_{1/14} + 70n_{2/14} + 28n_{3/14} + 14n_{4/14} + 0n_{5/14} + 42n_{6/12} + 2(n_{7/14} + 2)$ \\
\hline
\end{tabular}
\caption{Euler number $\chi$ in terms of BPS indices $n_\gamma$. $n_\gamma$ or $(n_\gamma + 2)$ above is non-negative. For example, $\chi \geq -196$ when $\lambda_S = \langle +4 \rangle$. The last two lines show an example where Weyl-inequivalent lattices $L[-1] \subset E_8$ give different expressions of $\chi$ in terms of $\{n_\gamma\}$.}
\end{table}

In the case of $\lambda_S = \langle +2n \rangle = \langle 2n \rangle$, choose a primitive element $y' \in E_8$ of norm $2n$. Then the positive definite $L[-1]$ can be taken as $\langle y' \rangle \subset E_8$, the prescription in footnote 3. In the $n = 1$ case, $L[-1] \cong E_7$; the relation (48) for $\phi = \theta_{E_7}$ reproduces $\chi(X) = -252 + 56n_{1/2}$. Similarly, the expressions of $\chi(X)$ for $n = 2, 3$ in section 2.3 (Table 2) are also reproduced from (48) by choosing $L[-1]$ as above. Here, we write $n_{x/2n} = n_{x/2n}^{-}$ for $\langle (x/2n) e \rangle \in G_{L[-1]}$.

We apply the same procedure to the $n = 4, 5, 6$ cases. It turns out that we can use the lattice theta functions $\phi = \theta_{L[-1]}$ shown in the following to obtain an expression for $\chi(X)$:

\begin{align}
\theta^n_{L[-1]}(\tau) &= e_0(1 + 56q) + (e_{1/2} + e_{3/2})(56q^{15/16}) + (e_{1/3} + e_{5/3})(28q^{3/4}) + (e_{1/4} + e_{7/4})(8q^{7/16}) \\
&\quad + e_{1/8}(0q^0 + 70q) + O(q^{1+\epsilon}), \\
\theta^n_{L[-1]}(\tau) &= e_0(1 + 60q) + (e_{1/2} + e_{3/2})(44q^{19/20}) + (e_{1/3} + e_{5/3})(33q^{4/5}) + (e_{1/4} + e_{7/4})(12q^{11/20}) \\
&\quad + (e_{1/8} + e_{7/8})(q^{1/5}) + e_{1/16}(32q^{3/4}) + O(q^{1+\epsilon}), \\
\theta^n_{L[-1]}(\tau) &= e_0(1 + 46q) + (e_{1/2} + e_{3/2})(48q^{23/24}) + (e_{1/3} + e_{5/3})(30q^{5/6}) + (e_{1/4} + e_{7/4})(16q^{7/8}) \\
&\quad + (e_{1/8} + e_{7/8})(3q^{1/3}) + (e_{1/16} + e_{7/16})(48q^{23/24}) + e_{1/32}(10q^{1/2}) + O(q^{1+\epsilon}).
\end{align}

The relation (48) for those $\phi = \theta_{L[-1]}$ are shown in Table 2. For all of $\lambda_S = \langle +2n \rangle$ with $n = 1, 2, 3, 4, 5, 6$, we know from the dimension formula that all the $(n + 1)$ Fourier coefficients $\{c_\gamma\}([\gamma, \gamma]/2\text{frac} - 1) \mid \pm \gamma \in G_S/\{\pm 1\}$ of $\Phi/\eta^{24}$ are linearly independent for $\Phi \in \text{Mod}(21/2, \rho_{(+2n)})$; so there cannot be any other linear expressions of $\chi(X) = -c_0(0)$ in terms of those independent $c_\gamma = n_\gamma$’s. So, it is enough just to find one $L[-1]$.

In the $n = 7$ case, there must be one linear relation among the 8 coefficients $n_{|\gamma|}$ with $|\gamma| \in G_S/\{\pm 1\}$, because the dimension formula indicates that the vector space $\text{Mod}(21/2, \rho_{(+14)})$
(and $\text{Mod}_0(21/2, \rho_{(+14)})$ is of 7-dimensions. So, an expression of the form $\chi(X) = -c_0(0) = \sum_{|\tau|} \kappa_1|\tau| \kappa_2|\tau|$ is not expected to be unique. Indeed, we can think of two choices of

$$y' = 3e_1' + 2e_2' + e_3' \in E_8, \quad y' = 3e_1' + (e_2' + \cdots + e_6') \in E_8; \quad (171)$$

here, the lattice $E_8$ is expressed as the abelian group $\mathbb{Z}^{\oplus 8} \oplus (1/2, \ldots, 1/2) + \mathbb{Z}^{\oplus 8}$, and the intersection form on this is given by setting $(i, j) = \langle e_i, e_j \rangle = \pm 0$ for the generators $e_i$ of the $i$-th factor of $\mathbb{Z}^{\oplus 8}$. The lattices $L[-1] \subset E_8$ are worked out for both choices, and it turns out that the corresponding lattice theta functions

$$\theta_{\mathcal{L}[-1]}(\tau) = e_0(1 + 72q) + (e_{1/2} + e_{12/14})(28q^{27/28}) + (e_{1/14} + e_{2/14})(27q^{6/7}) + (e_{1/14} + e_{7/14})(27q^{19/28})$$

$$+ (e_{1/14} + e_{10/14})(q^{3/7}) + (e_{3/14} + e_{5/14})(q^{3/28}) + (e_{6/14} + e_{12/14})(7q^{5/7} + e_{14/14} (0q^{1/4}) + \mathcal{O}(q^{1+\epsilon})$$

and

$$\theta_{\mathcal{L}[-1]}(\tau) = e_0(1 + 44q) + (e_{1/14} + e_{12/14})(42q^{27/28}) + (e_{3/14} + e_{12/14})(5q^{6/7}) + (e_{3/14} + e_{11/14})(14q^{19/28})$$

$$+ (e_{3/14} + e_{10/14})(7q^{3/7}) + (e_{3/14} + e_{14/14})(0q^{3/28}) + (e_{9/14} + e_{12/14})(21q^{5/7} + e_{14/14} (2q^{1/4}) + \mathcal{O}(q^{1+\epsilon}),$$

are not the same. Two expressions for $\chi$ is obtained from the relation (18), and are shown in Table 2.

Multiple $\phi \in \text{Mod}_0(3 + \rho/2, \rho_\Lambda \Lambda)$’s result in multiple expressions for $\chi(X)$ in terms of $\{n_\gamma \mid \pm \gamma \in G_S / \{\pm 1\}\}$’s, as we have seen above in the $\Lambda_S = \langle +14 \rangle$ case. When we form a linear combinations of such $\phi$’s so that the leading Fourier coefficient vanishes (i.e., a cusp form), then we obtain linear relations among $\{n_\gamma\}$’s discussed in section 2.3.2. Multiple expressions for $\chi(X)$ are consistent because of the linear relations among those low-energy BPS indices.

### A.2.3 The First Derivative of Some Lattice Theta Functions

For any one of ADE types, $\mathcal{R}$, the coefficients $a_\delta^{(\mathcal{R})}(\nu)$ in

$$\frac{2}{\text{rank}(\mathcal{R})} q^{-\partial_\delta} \theta_\mathcal{R} =: \sum_{\delta \in G_\mathcal{R}} e_\delta a_\delta^{(\mathcal{R})}(\nu) q^{\nu} \quad (172)$$

are all integers; $\mathcal{R}$ also stands for the positive definite lattice of type $\mathcal{R}$. To see that they are integers, we use Lemma 6.1 and 6.2 of [50] for the lattice $\mathcal{R}$. We have a formula

$$a_\delta^{(\mathcal{R})}(\nu) = \sum_{b \in \mathcal{R}^\nu \cap \langle b, b \rangle_{\mathcal{R}}/2 = \nu} \left( \frac{\langle b, b \rangle_{\mathcal{R}}}{\langle b, b \rangle_{\mathcal{R}}} \right)^2 \quad (173)$$

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for any \( a \in \mathcal{R} \otimes \mathbb{R} \). By using any root in \( \mathcal{R} \) as the vector \( a \), we see that any pair \( b \) and \(-b\) in the sum \[^{79}\text{?}\] contributes by

\[
\frac{((b, a)_{\mathcal{R}})^2 + ((-b, a)_{\mathcal{R}})^2}{(a, a)_{\mathcal{R}}} \in \mathbb{Z}.
\]

(174)

The values of \( a_\delta^{(\mathcal{R})}(\nu) \) for \( \nu < 1 \) are recorded here.

\[
\begin{align*}
\mathcal{R} & \quad \sum_{\delta \in G_{\mathcal{R}}} \sum_{\nu} \nu < 1 e_\delta a_\delta^{(\mathcal{R})}(\nu) q^\nu
\end{align*}
\]

(175)

\[
\begin{align*}
A_1 &: = e_1 q^{1/4}, \\
A_2 &: = (e_1 + e_2) q^{1/3}, \\
A_3 &: = (e_1 + e_3) q^{3/8} + e_2 2q^{1/2}, \\
A_4 &: = (e_1 + e_4) q^{2/5} + (e_2 + e_3) 3q^{3/5}, \\
A_5 &: = (e_1 + e_5) q^{5/12} + (e_2 + e_4) 4q^{2/3} + e_3 6q^{3/4}, \\
A_6 &: = (e_1 + e_6) q^{3/7} + (e_2 + e_5) 5q^{5/7} + (e_3 + e_4) 10q^{6/7}, \\
A_7 &: = (e_1 + e_7) q^{7/16} + (e_2 + e_6) 6q^{3/4} + (e_3 + e_5) 15q^{15/16}, \\
A_8 &: = (e_1 + e_8) q^{4/9} + (e_2 + e_7) 7q^{7/9}, \\
A_r &\geq 8 \quad = (e_1 + e_r) q^{r\nu + C_0} + (e_2 + e_{r-1}) (r - 1)q^{\nu C_2}, \\
D_r &\geq 2m \quad = (e_{20} + e_{10}) 2^{r-3} q^{r/8} + e_{11} 2q^{1/2} \quad \text{ignore the } q^{r/8} \text{ term for } r \geq 8, \\
D_r &\geq 2m+1 \quad = (e_1 + e_3) 2^{r-3} q^{r/8} + e_2 2q^{1/2} \quad \text{ignore the } q^{r/8} \text{ term for } r > 8, \\
E_6 &: = (e_1 + e_2) 6q^{2/3}, \\
E_7 &: = e_1 12q^{3/4}.
\end{align*}
\]

(176) - (188)

All those \( a_\delta(\nu) \)'s recorded here are equal to \( 2T_R \), where \( T_R \) is the Dynkin index of the fundamental representation \( R \) of the algebra \( \mathcal{R} \) associated with \( \delta \in G_{\mathcal{R}} \).

## B Evaluation of Integrals by Lattice Unfolding

We need evaluation of integrals \( \Delta_{\text{grav}} \) in \([41, 3]\) and \( \Delta_{\mathcal{R}} \) in \([61]\), just like in Ref. \([52, 7]\). The evaluation method in \([52]\) (with extension by \([7]\)), however, is applicable immediately only for lattices \( \Lambda_S \) of the form \( \Lambda_S = U \oplus W \) with a signature \((0, \rho - 2)\) lattice \( W \). Instead, we

\[^{79}\text{?}\] \( b = 0 \) does not form a pair, but \( a_0(0) = 0 \) obviously.
rely on the evaluation method presented in [9]; here, we describe the outline of the evaluation method in [9], and quote results relevant to this article (a review is also found in [51, §3]). The method is applied to cases of our interest in the appendices B.1.3 and B.3.1; some of those results are used in the main text. The embedding trick is presented in a more general form in the appendix B.3 than in the original [9, §8], we use this general form in the calculation in B.3.1.

A class of integrals considered in [9] was

\[ I_M(v, F) := \int_{SL(2,\mathbb{Z})\backslash\mathcal{H}} \frac{d\tau_1 d\tau_2}{\tau_2^2} \frac{\theta_M(\tau, \bar{\tau}; v) F(\tau, \bar{\tau})}{b^+ / 2}, \quad (189) \]

where \( M \) is an even lattice of signature \((b^+, b^-)\), and \( v \) a point in \( Gr(M) := Gr(M \otimes \mathbb{R}; b^+) \).

One point \( v \in Gr(M) \) has the same information as isometries \((p_R, p_L) : M \to \mathbb{R}^{b_+, b_-} \mod \text{SO}(b_+; \mathbb{R}) \times \text{SO}(b_-; \mathbb{R})\) when the lattice \( M \) has an indefinite intersection form; in the case \( M \) is negative definite \((b_+ = 0)\), \( v \) has empty information, and \( \theta_M(\tau, \bar{\tau}; v) \) is the theta function \( \theta_{M[-1]}(\tau) \). The other factor in the integrand \( F(\tau, \bar{\tau}) \) is a \( \mathbb{C}[M^\vee / M] \)-valued modular form of weight \((b^+ - b^-, 0)\) and type \( \rho_M \); it is allowed to have \( \bar{\tau} \)-dependence only through some negative powers of \( \tau_2 \),

\[ F(\tau, \bar{\tau}) = \sum_{\gamma \in M^\vee / M} \sum_{\nu \in \mathbb{Q}, \nu \geq \nu_{\min}} \sum_{k=0}^{k_{\max}} c^{(k)}(\nu) q^{\nu} \left( \frac{-3}{\pi \tau_2} \right)^k, \quad c^{(k)}(\nu) \in \mathbb{C}. \quad (190) \]

Those integrals with \( M = \tilde{\Lambda}_S \) are of immediate relevance, because

\[ \Delta^{\text{grav}} = I_{\tilde{\Lambda}_S} \left( p, \frac{\Phi \tilde{E}_2}{\eta^{24}} \right), \quad \Delta^{\text{gauge}} = \frac{1}{24} I_{\tilde{\Lambda}_S} \left( p, \frac{\Phi \tilde{E}_2 - \Psi}{\eta^{24}} \right), \quad (191) \]

where we used the right mover and left mover momenta \( p = (p_R, p_L) \) in the Heterotic description to refer to a choice of \( v \in Gr(\tilde{\Lambda}_S) \). We can just take \( \nu_{\min} = -1 \) and \( k_{\max} = 1 \), and both \( \Delta^{\text{grav}} \) and \( \Delta^{\text{gauge}} \) are within the class of integral [189] introduced above.

In evaluating the integral \( I_M(v, F) \), Ref. [9] relates it to an integral in the same class, but with a lattice \( M' \) of \( b_+ (M') = b_+ (M) - 1, \), \( v' \in Gr(M') \), and \( F' \) characterized\(^{82}\) as in [190].

\(^{80}\) When the integral shows some divergence, we understand the integral as regularization by subtracting the integrand by const \( \tau_2^{b^+/2 - 2} \) (equivalent of integrating in IR degrees of freedom) \(^{52}\) or replacing \( d\tau_2 / \tau_2^2 \rightarrow d\tau_2 / \tau_2^{2 + x} \) as in [9].

\(^{81}\) We have to treat cases \( \nu_{\min} < -1 \) when using the embedding trick in the appendix B.3.

\(^{82}\) The value of \( k_{\max} \) does not change under this reduction, but the value of \( \nu_{\min} \) may not be the same as before if one applies the embedding trick (see section B.3).
with $M$ replaced by $M'$, as we review in the appendices B.1 and B.3). This procedure is called the lattice unfolding method. At the end, we are left with evaluating integrals of the form (189, 190) for a negative definite lattice $M''$. Since $\theta_{M''}(\tau)$ is holomorphic, the integral $I_{M''}(F'')$ can be regarded as that for 0-dimensional lattice $I_{\{0\}}(\theta_{M''}(\tau)F'')$. This type of integral can be evaluated by simple partial integrals. When $F'(\tau) = \phi(\tau) \hat{E}_m^2$ with $\phi$ some scalar valued modular form of weight $-2m$, the formula is [53, 52]

$$I_{\{0\}}(\phi \hat{E}_m^2) = \frac{\pi}{3(m+1)}[\phi E_n^{m+1}]_q^0. \quad (192)$$

### B.1 Lattice Unfolding Formula

When the lattice $M$ with a signature $(b_+, b_-)$ has a nonzero element $z$ with norm $z^2 = 0$, a lattice $M' := [z^\perp \subset M]/\mathbb{Z}z$ has signature $(b_+ - 1, b_- - 1)$. For $v \in Gr(M)$, $v' \in Gr(M')$ is determined by the $(b_+ - 1)$-dimensional vector subspace of the $b_+$-dimensional positive definite subspace corresponding to $v$ orthogonal to $z$. Discussions in [9] rewrite $I_M(v; F)$ as a sum of $I_{M'}(v', F')$ for an appropriately chosen $F'$ and additional terms that are completely determined in terms of $v$ and $F$.

Since $U[-1]$ has a nonzero null vector, evaluation of $I_{\bar{\Lambda}_S}$ with $\bar{\Lambda}_S = U[-1] \oplus \Lambda_S$ can be reduced to that of $I_{\Lambda_S}$. When the signature $(1, \rho - 1)$ lattice $\Lambda_S$ also has a nonzero null vector, we can again reduce $I_{\Lambda_S}$ to an integral for smaller lattice $[z^\perp \subset \Lambda_S]/\mathbb{Z}z$ of signature $(0, \rho - 2)$. Since this lattice is negative definite, we can apply the formula (192).

When the lattice $\Lambda_S$ does not have a nonzero null vector, we can use the embedding trick (explained in appendix B.3) to think of $I_{\bar{\Lambda}_S}(v', F')$ as $I_{\tilde{M}}(\bar{v}, \tilde{F})$ for some lattice $\tilde{M}$ and some modular form $\tilde{F}$. Here $\tilde{M}$ contains $\Lambda_S$ as a sublattice and $b_+(\tilde{M}) = b_+(\Lambda_S)$, and it can be chosen so that it has a nonzero null element. In this way, the integrals for all $\Lambda_S$ can be reduced to integrals with some negative definite lattice.

#### B.1.1 From $\bar{\Lambda}_S$ to $\Lambda_S$

In the lattice unfolding process from the lattice $M = \bar{\Lambda}_S$ to $M' = \Lambda_S$, the positive definite $(b_+ = 2)$-plane $v \in Gr(M)$ is specified by the real and imaginary parts $\bar{u}$ in (31). Let $\{e^0, e^4\}$ be a basis of $U[-1]$ with the intersection form $(e^0, e^0) = (e^4, e^4) = 0$ and $(e^0, e^4) = -1$; they are the generators of $H^0(K3; \mathbb{Z})$ and $H^1(K3; \mathbb{Z})$, as in section 2.1.3. When we choose $z = -e^4$, $M'$ is $\Lambda_S$ and the positive definite $(b_+(M') = 1)$-plane $v'$ in $M' \otimes \mathbb{R}$ is the imaginary
part of $\mathcal{U}$, namely $\mathbb{R}t_2$. Following [9 Thm. 7.1], one finds that
\begin{equation}
I_{\tilde{\Lambda}}(v, F) = \frac{|t_2|}{\sqrt{2}} I_{\Lambda}(v', F) + c_{0}^{(0)}(0) \left[ - \log(2\pi t_2^2) + \text{const.} \right] - \frac{6\zeta(3)}{\pi^2 t_2^2} + \sum_{0 \neq w \in \Lambda_S^\vee} \left[ 2c_{[w]}^{(0)}(w^2/2) \text{Li}_1(e^{2\pi i(w \cdot t)}) - \frac{24}{t_2^2} c_{[w]}^{(1)}(w^2/2) \text{Li}_3(e^{2\pi i(w \cdot t)}) \right].
\end{equation}

Here $(w \cdot t) = (w, t_1) + i |(w, t_2)|$ and $\text{Li}_3(e^{iz}) = \text{Li}_3(e^{iz}) + \text{Im}(z)\text{Li}_2(e^{iz})$. The modular form $F$ is used as $F'$ in the first term on the right hand side. The details of the second and the third line are necessary when working out the matching calculation [84], but are not relevant to the matching [88, 89, 90] that we need to discuss in this article.

The integral $I_{\tilde{\Lambda}}(v, F)$ as a function of $v \in \text{Gr}(\tilde{\Lambda}_S)$ has singularity only along real codimension $(b_+(\tilde{\Lambda}_S) = 2)$ subspace, whereas $I_{\Lambda}$ as a function of $v' \in \text{Gr}(\Lambda_S)$ has singularity (“wall-crossing”) along real codimension $(b_+ (\Lambda_S) = 1)$ walls; $I_{\tilde{\Lambda}}(v, F)$ has logarithmic singularities at the locus $\mathcal{U}(w) = 0$ for some $w \in \tilde{\Lambda}_S^\vee$, while $I_{\Lambda}(v', F)$ has conical singularities at $(w, t_2) = 0$ for some $w \in \Lambda_S^\vee$. This implies that there is (partial) cancellation of singularity between $I_{\Lambda}(v', F')$ and the third line of (193) so the sum of them remains non-singular a t the codimension-1 walls (except the codimension-2 points) [84]. See section [B.2] for the wall-crossing formula of $I_{\Lambda}$. For more information, see [9 §6].

### B.1.2 $I_{\Lambda}$ for $\Lambda_S$ with a Null Element

In the rest of the appendix B.1, we discuss evaluation of $I_{\Lambda}(v', F)$ for $\Lambda_S$ that has a nonzero null element $z$. Any Neron–Severi lattice $\Lambda_S$ of a K3 surface with $\rho \geq 5$ are known to have such a nonzero null element, and the same is also true for lattices $\Lambda_S = U \oplus W'$ for some even lattice $W'$ with signature $(0, \rho - 2)$.

Let $z$ be a nonzero primitive null element of $\Lambda_S$, and $N$ be the GCD of the values of $(z, \lambda) \in \mathbb{Z}$ of $\lambda \in \Lambda_S$. We then choose $\tilde{z} \in \Lambda_S$ so that $(z, \tilde{z}) = N$, and a free abelian subgroup $W \subset \Lambda_S$ such that $z \perp W$ and $\Lambda_S \cong ab \mathbb{Z}z \oplus \mathbb{Z}\tilde{z} \oplus W$; here $\cong_{ab}$ stands for an

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83 Again, $k_{\text{max}} \leq 1$ is assumed. The value of “const” in the expression is regularization dependent.

84 The equation (193) holds even when $t_2$ does not have large norm (although [9 Thm. 7.1] proves the corresponding claim only when the norm $(t_2, t_2)$ is sufficiently large), because both hands of (193) are well-defined even for small $t_2$ and they remain (real-)analytic inside any single chamber, so they must be the same.
isomorphism between abelian groups. The same $W$ also denotes the lattice $[z^\perp \subset \Lambda_S]/\mathbb{Z}z$. The complexified Kähler parameter $t \in \Lambda_S \otimes \mathbb{C}$ can be parametrized by

$$t = uz + \rho \bar{z} + a, \quad u, \rho \in \mathbb{C}, \quad a \in W \otimes \mathbb{C}. \quad (194)$$

Here is additional preparation. Pairing $(\cdot, z) : \Lambda_S \to \mathbb{Z}$ induces $(\cdot, z) : G_S \to \mathbb{Z}_N$, where $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$. The kernel of this homomorphism $\ker(\cdot, z)$ includes the subgroup $\langle z/N \rangle \cong \mathbb{Z}_N$ of $G_S$ generated by $z/N + \Lambda_S$. We have a natural isomorphism $j : \ker(\cdot, z)/\langle z/N \rangle \cong G_W$ of finite abelian groups, which preserves the quadratic form. The subset $j^{-1}(\lambda) \subset \ker(\cdot, z)$ for $\lambda \in G_W$ is of the form $\delta_0 + \langle z/N \rangle$ for some $\delta_0$ that depends on $\lambda$. The linear map

$$\mathbb{C}[G_S] \to \mathbb{C}[G_W], \quad \begin{cases} e_{\delta \in \ker(\cdot, z)} \mapsto e_{j(\delta)}, \\ e_{\bar{\delta} \notin \ker(\cdot, z)} \mapsto 0, \end{cases} \quad (195)$$

is compatible with the Weil representation [9, Thm. 5.3]. For a modular form $F$ of type $\rho_S$, we denote by $F_W$ the modular form of type $\rho_W$ obtained by composing $F : \mathcal{H} \to \mathbb{C}[G_S]$ and this homomorphism.

Now Thm. 7.1, Lemma 7.3, and Thm. 10.2 of Ref. [9] can be used to rewrite the integrals $I_{\Lambda_S}(v', F)$ in terms of $u$, $\rho$, $a$, and the coefficients $c_{\gamma}^{(k)}(\nu)$ of $F$. We just quote the results for the case $k_{\text{max}} \leq 1$; when $t_2$ lies in the fundamental chamber\cite{85,86} $C_z$,

$$\frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', F) = \frac{t_2^2}{2N\rho_2} I_W(F_W) + 2\pi \sum_{b \in W^\vee} \sum_{\delta \in j^{-1}(b)} \left( N\rho_2 c_\delta^{(0)}(b^2/2)B_2(\cdots) + \frac{2N^3\rho_2^3 c_\delta^{(1)}(b^2/2)B_4(\cdots)}{t_2^2} \right),$$

where the argument of the functions $B_2$ and $B_4$ are $(b, a_2/(N\rho_2)) + N^{-1}(\delta, \bar{z})$. The first term $I_W(F_W)$ can be evaluated by the formula (192).

\textit{The positive cone $\{t_2 \in \Lambda_S \otimes \mathbb{R} | (t_2, t_2) > 0\}$ is divided into chambers based on which subset of $\Pi := \{w \in \Lambda_S^* | 2\nu_{\text{min}} \leq w^2 < 0\}$ is characterized as $\{w \in \Pi | (w, t_2) > 0\}$. We call a chamber $C$ a fundamental chamber, if it contains a region $|(z, t_2)| \ll |t_2|$ (equivalently $N|\rho_2| \ll |t_2|$). There may be multiple fundamental chambers in general.}

\textit{This formula is guaranteed to be valid in fundamental chambers. For evaluating the integral in other chambers, one needs to use the wall-crossing formula (207). See (192) for details.}

\textit{The function $B_m(x)$ is defined (for $m > 0$) by}

$$B_m(x) = -m! \sum_{n \neq 0} \frac{E(n\pi x)}{(2\pi i n)^m} = -\frac{m!}{(2\pi i)^m} (\text{Li}_m(E(x)) + (-1)^m\text{Li}_m(E(-x))). \quad (197)$$

It satisfies $B_m(x + 1) = B_m(x)$ and $B_m(-x) = (-1)^mB_m(x)$ by definition. Its value for $0 < x < 1$ is given by the usual Bernoulli polynomial $B_m(x)$; e.g. $B_2(x) = x^2 - x + 1/6$, $B_4(x) = x^4 - 2x^3 + x^2 - 1/30$.}
When one expand the rhs, there appear some terms with $\rho^2$ in the denominator, but they should cancel out because of consistency with wall-crossing behavior of $I_{\Lambda_S}$ (see [9, Thm. 10.3]). After this cancellation is taken care of, the integral $I_{\Lambda_S}(v', F) \times |t_2|^{1+2k_{\text{max}}}$ should become a chamber-wise homogeneous function of the components of $t_2$ of degree $(1 + 2k_{\text{max}})$.

The integrals of our interest are those in (191), and we find it useful to assign notations for the following combinations (the same as (92, 91)):

$$P_1(t_2) := \frac{1}{4\pi} \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', \Psi^{-24}), \quad P_3(t_2) := \frac{-t_2^3}{32\pi} \frac{|t_2|}{\sqrt{2}} I_{\Lambda_S}(v', (\Phi \hat{E}_2 - \Psi)\eta^{-24}); \quad (198)$$

$P_1$ is a chamber-wise polynomial of degree 1, $P_3$ of degree 3. They are used to determine some of the functions of the low-energy effective theory in (88, 89).

**B.1.3 Examples: $\Lambda_S = U \oplus W$**

The polynomials $P_3$ and $P_1$ have simpler expressions when $\Lambda_S$ is of the form $U \oplus W$ for some even and negative definite lattice $W$.

$$\frac{1}{3!} P_3(t) = \frac{2}{3!} \rho^3 + \frac{n' - 2}{2} \rho^2 u + \frac{n'}{2} \rho u^2 + \frac{n'}{2} (u + \rho)(a, a) - \frac{u + \rho}{2} \sum_{b \in W^\vee} d_{[b]} (b^2/2) (b, a)^2 + \frac{2u \rho + (a, a)}{4} \sum_{(b, a_2) > 0} d_{[b]} (b^2/2) (b, a)$$

$$+ \frac{1}{3!} \frac{1}{2} \sum_{b \in W^\vee \atop (b, a_2) > 0} c_{[b]} (b^2/2) (b, a)^3, \quad (199)$$

$$P_1(t) = -4 \rho + 12 (2 + n')(u + \rho) - \sum_{b \in W^\vee \atop (b, a_2) > 0} c_{[b]} (b^2/2) (b, a) \quad (200)$$

in the fundamental chambers, where we have taken care of cancellation referred in the appendix B.1.2. The parameter $n'$ extracts a combination of $d_\gamma(\nu)$ through $(2 - n') := 144^{-1} \sum_b d_{[b]} (b^2/2)$.

---

88 Roughly, the claim is proved by observing that different choices of $z$ put different variable (e.g. $u_2$) in the denominator and may cause different type of singularity. But the wall-crossing is just a polynomial so the fractional terms must vanish.

89 This expression is for fundamental chambers that contain the regions with $(b, a_2) < \rho_2$ for $b$'s appearing in the sum.

90 When the probe gauge group $R$ is in one weakly coupled $E_8^{(2)}$ in the Heterotic language, $W \subset U^{\oplus 2} \oplus E_8^{(1)}$, and $[R^\perp \subset E_8^{(2)}]$ has $I = 12 - n'$ instantons on K3 (of the K3 $\times T^2$ internal space), then the modular form $\Psi$ should be the one with $d_\gamma(\nu)$'s that are related to $n'$ in this way.
An element of \( \Phi \in \text{Mod}^\mathbb{Z}_6(11 - \rho/2, \rho_{\Lambda_S})^{\mathbb{Z}_2} \) and \( \Psi \in \text{Mod}^\Phi(13 - \rho/2, \rho_{\Lambda_S}) \) is in \([\text{Mod}^\mathbb{Z}_6(11 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}\) and \([\text{Mod}^\Phi(13 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}\), respectively, if the following conditions should be satisfied. For integrality of the coefficients \( d_{abc} \) of the \( \rho^2u/2 \) and \( \rho u^2/2 \) terms in \( P_3 \),

\[
\left( 2 - \frac{1}{12} \sum_{b \in W^\vee} d_{b}(b^2/2) \right) =: n' \in \mathbb{Z} \tag{201}
\]

The \( upa \) term and \( aaa/3! \) term have integer coefficients, only if

\[
\sum_{(b,a) > 0}^{(b,a_2) > 0} d_{b}(b^2/2) b \in 2W^\vee \tag{202}
\]

and

\[
\sum_{(b,a) > 0}^{(b,a_2) > 0} c_{[b]}(b^2/2)(b, r_1)(b, r_2)(b, r_3) \in 2\mathbb{Z} \quad \text{for } r_{1,2,3} \in W, \tag{203}
\]

respectively. The condition (a') is translated into

\[
\sum_{(b,a) > 0}^{(b,a_2) > 0} c_{[b]}(b^2/2)(b, r_1)(b, r_2)(b, r_1 + r_2) \equiv 0 \quad (\text{mod } 4) \quad \text{for } r_{1,2} \in W, \tag{204}
\]

and the condition (b') to

\[
\sum_{(b,a) > 0}^{(b,a_2) > 0} (c_{b}(b^2/2)(b, r)^3 - c_{[b]}(b^2/2)(b, r)) \equiv 0 \quad (\text{mod } 12) \quad \text{for } r \in W, \tag{205}
\]

or equivalently, to

\[
\sum_{(b,a) > 0}^{(b,a_2) > 0} c_{[b]}(b^2/2)(b, r)((b, r) + 1)((b, r) - 1) \equiv 0 \quad (\text{mod } 12) \quad \text{for } r \in W. \tag{206}
\]

The special case \( W = \emptyset \) has been treated in section 3.2.1. The \( W = A_1[-1] \) case yields an example where \([\text{Mod}^\mathbb{Z}_6(11 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}\) is a proper subset of \([\text{Mod}^\mathbb{Z}_6(11 - \rho/2; \rho_{\Lambda_S})]^{\mathbb{Z}_2}\). To see this, note that the modular forms \( \Phi \) and \( \Psi \) are parametrized by \( n_{1/2} \in \mathbb{Z}_{\geq 0} \) and \( m_{1/2} \in 12\mathbb{Z} \) (after using \( \dim C(\text{Mod}_0(23/2, \rho_{(-2)})) = 2 \) and imposing \( n_0 = -2 \) and \( m_0 = 0 \)). The integrality of \( n' = 2 - [56 - n_{1/2} + m_{1/2}]/12 \) is translated into \( n_{1/2} \in 8 + 12\mathbb{Z} \), so \([\text{Mod}^\mathbb{Z}_6(11 - \rho/2; \rho_{\Lambda_S})]^{\text{r.mfd}}\) is strictly smaller. The four other conditions above follow automatically for \( n' \in \mathbb{Z} \).

The difference among \( \Psi \)'s that is irrelevant in \( \text{Diff}^d_{\Lambda_S} \) after the map \( \text{diff}_{\text{fine}} \) is \( \Delta n' \in 2\mathbb{Z} \) and \( \sum_{b \in W^\vee}^{(b,a_2) > 0} \Delta d_b(b^2/2)b \in 24W^\vee \). So, the set \( \text{Diff}^d_{\Lambda_S} \) is \( \mathbb{Z}/2\mathbb{Z} \) for any \( W \).
### B.2 Wall-crossing Behavior

We give some comments on wall-crossing behavior. See section 6 of [9] for details. $I_{\Lambda_S}(v, F)$ as a function of $v = \mathbb{R}t_2 \in Gr(\Lambda_S)$ shows conical singularity at $(w, t_2) = 0$ for some $w \in \Lambda_S^\vee$. These points in $Gr(\Lambda_S)$ forms real-codimension-1 walls that separate $Gr(\Lambda_S)$ into many chambers; $I_{\Lambda_S}$ is analytic in each chamber but shows jump from its analytic continuation when $t_2$ crosses a wall. Let $I_{\Lambda_S}(v, F; C)$ be the analytic continuation of the restriction of $I_{\Lambda_S}(v, F)$ to a chamber $C$. Then the difference of them for two different chamber $C_1$ and $C_2$ is given by

$$I_{\Lambda_S}(v; C_1) - I_{\Lambda_S}(v; C_2) = \sum_{(w, C_1) > 0 \atop (w, C_2) < 0} \left\{ c_w^{(0)}(w^2/2)(-8\sqrt{2}\pi)(w, v) + c_w^{(1)}(w^2/2)(-32\sqrt{2}\pi)(w, v)^3 \right\}.$$  

(207)

Here $(w, C) > 0$ means $(w, t_2) > 0$ for any $t_2 \in C$. Note that the difference as shown above is a polynomial of $v$. Using this fact, [9, Thm. 10.3] shows that $I_{\Lambda_S}(v, F)$ gives chamber-wise polynomial of $v$ with degree at most 3 in our case (in fact, without even degree terms).

As a corollary, we obtain wall-crossing formulas for $P_1(t)$ and $P_3(t)$, that are directly relevant to the topological invariants of $X_{\text{IIA}}$:

$$\frac{1}{3!}P_3(t; C_1) - \frac{1}{3!}P_3(t; C_2) = \sum_w \left\{ \frac{c_w(w^2/2)}{3!}(w, t)^3 + \frac{(t, t)}{2} \frac{d_w(w^2/2)}{12}(w, t) \right\},$$  

(208)

$$P_1(t; C_1) - P_1(t; C_2) = -\sum_w 2c_w^w(w^2/2)(w, t).$$  

(209)

The summation over $w$ is same as in (207).

### B.3 Embedding Trick

Even when $\Lambda_S$ has no nonzero null elements, one can evaluate the integral $I_{\Lambda_S}(v, F)$ by embedding $\Lambda_S$ into a larger lattice(s) $M$ with a nonzero null element so that $I_{\Lambda_S}(v, F)$ is equal to (linear combination of) $I_M(v, G)$ for suitable modular form $G$. One can then apply the lattice unfolding method to compute it. This method is called “embedding trick” in [9]. In this section, we explain the original embedding trick and its slight modifications. We also treat concrete example (the case $\Lambda_S = (+2)$).

---

9 The $w$ that gives singularity satisfies $-2 \leq w^2 < 0$; $w^2 < 0$ comes from $t_2^2 > 0$, and $w^2/2 \geq \nu_{\min} = -1$ comes from $c_{\lambda}(\nu) = 0$ for $\nu < \nu_{\min}$. 

---

65
The original embedding trick is as follows [9, Thm. 8.1]. To begin with, choose a pair of (negative definite) Niemeier lattices $M_1, M_2$ with different number of roots:

$$r_1 - r_2 \neq 0, \quad r_i := \text{the number of roots in } M_i. \tag{210}$$

Then we obtain

$$1 = \left( \bar{\theta}_{M_1}(\tau) - \bar{\theta}_{M_2}(\tau) \right) \frac{1}{(r_1 - r_2)\eta(\tau)^{24}}; \tag{211}$$

this is because the rhs is a scalar-valued modular form of weight 0 with the term $q^{-1}$ vanishing and the coefficient of $q^0$ normalized. By inserting this equation to the integrand of $I_{\Lambda_S}(v, F)$, we get

$$I_{\Lambda_S}(v, F) = I_{\Lambda_S \oplus M_1}(v, G) - I_{\Lambda_S \oplus M_2}(v, G), \tag{212}$$

where $G = F [(r_1 - r_2)\eta^{24}]^{-1}$ and $v \in Gr(\Lambda_S)$ is regarded as the same positive definite $(b_+ = 1)$-dimensional subspace $v \in Gr(\Lambda_S) \subset Gr(\Lambda_S \oplus M_i)$. Since $\Lambda_S \oplus M_i$ has $U$ as direct summand ([9, §8]), we can apply the lattice unfolding formula to evaluate the right hand side.

There are a few points to keep in mind. First, $G$ has a pole of higher order at cusps than $F$ (i.e. $\nu_{\min}(G) = \nu_{\min}(F) - 1 = -2$). Second, in many cases (e.g., the appendix [B.3.1]), we need to care about wall-crossings of $I_{\Lambda_S \oplus M_i}$ for walls between $v$ and the region in $Gr(\Lambda_S \oplus M_i)$ where the unfolding formula (197) is valid.

This embedding trick adds Niemeier lattice to the original lattice and increases the rank of lattice by as many as rank $M_i = 24$; sometimes it is troublesome to handle such big lattice and consider all the relevant wall-crossings. But actually, it suffices to add smaller lattice: choose sublattices $N_i \subset M_i$ such that $N_1 \cong N_2 (\cong: N)$. Then by decomposing the theta function of $M_i$ as

$$\bar{\theta}_{M_i}(\tau) = \sum_{\delta \in N_i^\vee/N_i} \bar{\theta}_{N_i + \delta} \bar{\theta}_{N_i^\perp + \delta}(\tau), \tag{213}$$

we can rewrite (211) to

$$1 = \sum_{\delta \in N_i^\vee/N} \bar{\theta}_{N_i + \delta}(\tau) h_\delta(\tau), \quad h_\delta(\tau) = \frac{\bar{\theta}_{N_i^\perp + \delta}(\tau) - \bar{\theta}_{N_i^\perp + \delta}(\tau)}{(r_1 - r_2)\eta(\tau)^{24}}. \tag{214}$$
\{h_\delta\} is a modular form of type \(\rho_N\). Inserting this equation to the integrand of \(I_{\Lambda_S}(v, F)\), we get

\[
I_{\Lambda_S}(v, F) = I_{\Lambda_S \oplus N}(v, G), \quad G_{\gamma, \delta} = F_{\gamma} h_\delta.
\]

(215)

Here \(G\) is a modular form of type \(\rho_{\Lambda_S \oplus N}\). Note that also in this case \(\nu_{\min}(G)\) is less than \(\nu_{\min}(F) = -1\) (but still \(\nu_{\min}(G) \geq -2\)).

After all, an important point for embedding trick is to find a decomposition

\[
1 = \sum_{\delta \in N'/N} \bar{\theta}_{N+\delta}(\tau) h_\delta(\tau)
\]

(216)

for some lattice \(N\) and modular form \(h\), not necessarily associated with Niemeier lattices. There are many choices. For example, let \(N = \langle -2m \rangle\) and \(\varphi(\tau, z)\) be a weak Jacobi form of weight 0 and index \(m\), satisfying \(\varphi(\tau, z = 0) \neq 0\). We can theta-expand \(\varphi\) using a modular form \(h\):

\[
\varphi(\tau, z) = \sum_{\delta \in \mathbb{Z}^2} \bar{\theta}_{-2m+\delta}(\tau, z) h_\delta(\tau).
\]

(217)

Since a modular form of weight 0 and holomorphic at cusps is necessarily just a constant, setting \(z = 0\) in the above equation leads to the required decomposition (216) up to some normalization. If \(m \geq 2\), there are multiple choices for such \(\varphi\).

So, even for a given lattice \(\Lambda_S\), there are multiple different ways to use the embedding trick, \(I_{\Lambda_S}(v, F) = \sum_i I_{\tilde{M}_i}(v, G_i)\). There is no unique choice for \(\tilde{M} = \Lambda_S \oplus N\); even for a given \(N\), the choice of \(G\) is not necessarily unique. One can use just any version of the embedding trick, so practical calculation of one’s interest is easier.

**B.3.1 Example: \(\Lambda_S = \langle +2 \rangle\)**

Let us compute \(I_{\Lambda_S=\langle +2 \rangle}(F)\) for \(F = \Psi \eta^{-24}, (\Phi \hat{E}_2 - \Psi) \eta^{-24}\) using the embedding trick. They contribute to (191) through (193); as a reminder, \(\Phi\) and \(\Psi\) to be used here (for \(\Lambda_S = \langle +2 \rangle\)) are those in (38) and (118).

As a practical implementation of the embedding trick, we choose \(N = \langle -2 \rangle\), so \(\tilde{M} = \langle +2 \rangle \oplus \langle -2 \rangle\), and \(G_{\tilde{M}} = \mathbb{Z}_2 \times \mathbb{Z}_2\). We will use the notation \(\mathbb{Z}_2 \cong \{0, 1\}\) (instead of \(\{0, 1/2\}\))

---

92 We can choose \(N\) to be rank-1 for any \(\Lambda_S\). This is because \(\Lambda_S\) has necessarily an element of norm \(2m\) for some positive integer \(m\); the lattice \(\Lambda_S \oplus N\) with \(N = \langle -2m \rangle\) has a nonzero null element then.

93 When \(F\) is holomorphic (e.g. \(\Psi \eta^{-24}\)), \(I_{\langle +2n \rangle}(F)\) can be also calculated by using [11] Cor 9.6, which uses Zagier’s modular form and Stokes’ theorem.
in this appendix [B.3.1]. Let us use the weight-0 index-1 (i.e., index-(+2)) Jacobi form \( \varphi(\tau, z) \) normalized so that \( \varphi(\tau, z = 0) = 1 \), and determine a vector-valued modular form \( h \) through [155 217]. Now
\[
I_{(+2)}(v, \Psi \eta^{-24}) = I_{M}(v, G), \quad I_{(+2)}(v, (\Phi \tilde{E}_2 - \Psi) \eta^{-24}) = I_{M}(v, \tilde{G}),
\]
(218)
where \( G = \sum_{\delta, \gamma \in \mathbb{Z}_2} e_{\delta \gamma} h_{\delta} \Psi_{\gamma} \eta^{-24} \) and \( \tilde{G} = \sum_{\delta, \gamma \in \mathbb{Z}_2} e_{\delta \gamma} h_{\delta} (\Phi_{\gamma} \tilde{E}_2 - \Psi_{\gamma}) \eta^{-24} \). Because
\[
h(\tau) = \frac{e_0}{12} \left( 10 + 108q^{4/4} + \mathcal{O}(q^{8/4}) \right) + \frac{e_1}{12} \left( \frac{1}{q^{1/4}} - 64q^{3/4} + \mathcal{O}(q^{7/4}) \right),
\]
(219)
\( \nu_{\min}(G) \) and \( \nu_{\min}(\tilde{G}) \) are both \(-5/4\), rather than \(-1\).

Let us describe the chamber structure in \( \tilde{M} \otimes \mathbb{R} \). To start, we introduce a parametrization of the space \( \{ \tilde{t}_2 \in \tilde{M} \otimes \mathbb{R} \} \) as in [194]. We denote a generating element of \( \Lambda_S = \left< +2 \right> \) and \( N = \left< -2 \right> \) by \( v_{+2} \) and \( v_{-2} \), respectively. In a basis \( (z, \tilde{z}) = (v_{+2} + v_{-2} - v_{-2}) \), the intersection form of \( \tilde{M} \) is given by
\[
\begin{pmatrix}
0 & 2 \\
2 & -2
\end{pmatrix}.
\]
(220)
So we can use \( z \) as the null element for the lattice unfolding method; now \( \tilde{t}_2 = u_2 z + \rho_2 \tilde{z} \).

Next, the walls of interest are of the form \( (\tilde{t}_2, \lambda) = 0 \) for some \( \lambda \in \tilde{M} \), with \( \nu_{\min} \leq \lambda^2/2 < 0 \). This condition is equivalent to \( 0 < m^2 - n^2 \leq 5 \) for \( \lambda = nv_{+2}/2 + mv_{-2}/2 \in \tilde{M} \), \( n, m \in \mathbb{Z} \). So, there are only finite number of solutions \( (n, m) \):
\[
(n, m) = (0, \pm 1) \text{ and } (0, \pm 2), \ (\pm 1, \pm 2), \ (\pm 2, \pm 3).
\]
(221)
See Table 7 for the list of those walls in \( \tilde{M} \otimes \mathbb{R} \).

Their limit toward the wall \( \rho_2 = u_2 \) yield the integrals for \( v = \mathbb{R} v_{+2} \subset \Lambda_S \otimes \mathbb{R} \), because \( \tilde{t}_2 = v_{+2}^0 + v_{+2} \) implies \( u_2 = \rho_2 = t_{+2}^0 = 1 \).

The integrals (218) are evaluated, first, in the fundamental chamber \( C_0 \) by using the lattice unfolding formula (197). The fundamental chamber is the region \( 0 < 3\rho_2 < u_2 \). The result is
\[
P_1(\tilde{t}_2; C_0) = (2u_2 - \rho_2) \left( 10 - b_R - \frac{7}{2} n_{1/2} \right) + \rho_2 \frac{(248 - 18b_R - 29n_{1/2})}{6},
\]
(222)
\[
P_3(\tilde{t}_2; C_0) = (2u_2 - \rho_2)^2 \rho_2 \frac{(-b_R - 3n_{1/2})}{4}
+ (2u_2 - \rho_2)^3 \rho_2 \frac{(20 - 3b_R + 5n_{1/2})}{4} + (\rho_2)^3 \frac{1 - 7n_{1/2}}{3};
\]
(223)
\[ (n, m) \quad \lambda^2/2 \quad (\lambda, t_2 = u_2 z + \rho_2 \bar{z}) \quad \text{wall at } u_2/\rho_2 = \]

| \( \pm(2, -3) \) | \(-5/4 \) | \( \pm(5u_2 - 3\rho_2) \) | 3/5 |
| \( \pm(1, -2) \) | \(-3/4 \) | \( \pm(3u_2 - 2\rho_2) \) | 2/3 |
| \( \pm(0, 1) \) | \(-1/4 \) | \( \mp(u_2 - \rho_2) \) | 1 |
| \( \pm(0, 2) \) | \(-1 \) | \( \mp 2(u_2 - \rho_2) \) | |
| \( \pm(1, 2) \) | \(-3/4 \) | \( \mp(u_2 - 2\rho_2) \) | 2 |
| \( \pm(2, 3) \) | \(-5/4 \) | \( \mp(u_2 - 3\rho_2) \) | 3 |

Table 7: The list of walls where \( I_M(\mathbb{R}\tilde{t}_2, G) \) and \( I_M(\mathbb{R}\tilde{t}_2, \tilde{G}) \) are singular. They are sorted in the order of their slopes in the \( \rho_2-u_2 \) plane. They are all within the positive cone \( (2u_2-\rho_2)\rho_2 > 0 \) of the lattice \( \tilde{M} \).

in using the formula (197) for the lattice \( \tilde{M} \) and the null element \( z = (v_2 + v_2) \), the lattice \( W \) is \( \{0\} \), and \( N = 2 \); the subgroup \( \ker(P, z) = \{0, (z/2)_{+, \rho} \} = \{(0, 0), (1, 1)\} \subset G_M \), which is also equal to \( j^{-1}(0) \) for \( 0 \in G_W \). Relevant Fourier coefficients of \( G \) and \( \tilde{G} \) are computed from (218) and (38118).

The fundamental region \( C_0 \) is separated from the \( \Lambda_S \otimes \mathbb{R} \subset \tilde{M} \otimes \mathbb{R} \) locus by two walls \( u_2 = 3\rho_2 \) and \( u_2 = 2\rho_2 \). The integrals in the chamber \( \rho_2 \leq u_2 \leq 2\rho_2 \) are obtained by adding the following terms to (222) (223),

\[
\Delta P_1 = -\frac{2}{6} (u_2 - 3\rho_2) + \frac{10n}{6} (u_2 - 2\rho_2), \tag{224}
\]

\[
\Delta P_3 = \frac{2}{12} (u_2 - 3\rho_2)^3 - \frac{10n_{1/2}}{12} (u_2 - 2\rho_2)^3, \tag{225}
\]

because of the wall crossing formula (207). The first and second terms are associated with the walls at \( u_2 = 3\rho_2 \) and \( u_2 = 2\rho_2 \), respectively. Relevant Fourier coefficients are \([G_{10}]_{+} = -2/12, \ [G_{01}]_{-} = 10n/12, \ [h_1\Phi \eta^{-24}]_{q} = -2/12, \ [h_0\Phi \eta^{-24}]_{q} = 10/12\).

The integrals \( P_1(\tilde{t}_2) \) and \( P_3(\tilde{t}_2) \) for the argument of real interest, \( \tilde{t}_2 = (t_2^{a=1})v_2 + \in (\Lambda_S \otimes \mathbb{R}) \subset (\tilde{M} \otimes \mathbb{R}), \) is obtained by taking the limit \( u_2 \to t_2^{a=1}, \rho_2 \to t_2^{a=1} \) in the their expressions in the chamber \( \rho_2 \leq u_2 \leq 2\rho_2 \). Therefore,

\[
P_1(t_2^{a=1}) = \left[ \left( 10 - 6bR - \frac{7}{2}n_{1/2} + \frac{248 - 18bR - 29n_{1/2}}{6} \right) + \frac{4}{6} - \frac{10n_{1/2}}{6} \right] (t_2^{a=1}),
\]

\[
= (52 - 4bR - 10n_{1/2})(t_2^{a=1}), \tag{226}
\]

\[
P_3(t_2^{a=1}) = \left[ \left( \frac{20 - 4bR + 2n_{1/2}}{4} + \frac{1 - 7n_{1/2}}{3} \right) + \frac{2(-8)}{12} - \frac{10}{12}n_{1/2} \right] (t_2^{a=1})^3,
\]

\[
= (4 - bR - n_{1/2})(t_2^{a=1})^3. \tag{227}
\]
Those two results are used in section 3.2.2.

References

[1] C. Vafa and E. Witten, “A Strong coupling test of S duality,” Nucl. Phys. B 431 (1994) 3 [hep-th/9408074]. J. A. Harvey and A. Strominger, “The heterotic string is a soliton,” Nucl. Phys. B 449 (1995) 535 [Nucl. Phys. B 458 (1996) 456] [hep-th/9504047]. C. Vafa, “Gas of d-branes and Hagedorn density of BPS states,” Nucl. Phys. B 463 (1996) 415 [hep-th/9511088]. M. Bershadsky, C. Vafa and V. Sadov, “D-branes and topological field theories,” Nucl. Phys. B 463 (1996) 420 [hep-th/9511222]. “Instantons on D-branes,” Nucl. Phys. B 463 (1996) 435 [hep-th/9512078].

[2] S. Kachru and C. Vafa, “Exact results for N=2 compactifications of heterotic strings,” Nucl. Phys. B 450 (1995) 69 [hep-th/9505105].

[3] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, “Second quantized mirror symmetry,” Phys. Lett. B 361 (1995) 59 [hep-th/9505162].

[4] A. Klemm, W. Lerche and P. Mayr, “K3 Fibrations and heterotic type II string duality,” Phys. Lett. B 357 (1995) 313 [hep-th/9506112].

[5] C. Vafa and E. Witten, “Dual string pairs with N=1 and N=2 supersymmetry in four-dimensions,” Nucl. Phys. Proc. Suppl. 46 (1996) 225 [hep-th/9507050].

[6] P. S. Aspinwall and J. Louis, “On the ubiquity of K3 fibrations in string duality,” Phys. Lett. B 369 (1996) 233 [hep-th/9510234].

[7] J. A. Harvey and G. W. Moore, “Algebras, BPS states, and strings,” Nucl. Phys. B 463 (1996) 315 [hep-th/9510182].

[8] D. Maulik and R. Pandharipande, “Gromov–Witten theory and Noether–Lefschetz theory,” arXiv:0705.1653.

[9] R. Borcherds, “Automorphic forms with singularities on Grassmannians,” Invent. Math. 132 (1998) 491, arXiv:alg-geom/9609022.

[10] T. Banks and L. J. Dixon, “Constraints on String Vacua with Space-Time Supersymmetry,” Nucl. Phys. B 307 (1988) 93. T. Banks, L. J. Dixon, D. Friedan and E. J. Martinec, “Phenomenology and Conformal Field Theory Or Can String Theory Predict the Weak Mixing Angle?,” Nucl. Phys. B 299 (1988) 613.
[11] A. P. Braun and T. Watari, “Heterotic-Type IIA Duality and Degenerations of K3 Surfaces,” JHEP 1608 (2016) 034 [arXiv:1604.06437 [hep-th]].

[12] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain and T. R. Taylor, “Perturbative prepotential and monodromies in N=2 heterotic superstring,” Nucl. Phys. B 447 (1995) 35 [hep-th/9504034].

[13] S. Cecotti, P. Fendley, K. A. Intriligator and C. Vafa, “A New supersymmetric index,” Nucl. Phys. B 386 (1992) 405 [hep-th/9204102].

[14] T. Eguchi, H. Ooguri, A. Taormina and S. K. Yang, “Superconformal Algebras and String Compactification on Manifolds with SU(N) Holonomy,” Nucl. Phys. B 315 (1989) 193.

[15] C. M. Hull, “A Geometry for non-geometric string backgrounds,” JHEP 0510 (2005) 065 [hep-th/0406102]; for recent developments in the context of Heterotic–Type IIA duality, see also C. Hull, D. Israel and A. Sarti, “Non-geometric Calabi-Yau Backgrounds and K3 automorphisms,” JHEP 1711, 084 (2017) [arXiv:1710.00853 [hep-th]], Y. Gautier, C. M. Hull and D. Isral, “Heterotic/type II Duality and Non-Geometric Compactifications,” JHEP 1910, 214 (2019) [arXiv:1906.02165 [hep-th]].

[16] V. S. Kulikov, “Degenerations of K3 surfaces and Enriques surfaces,” Math. USSR Izvestija, 11 (1977) 957. [Russian original Izv. Akad. Nauk SSSR Ser. Mat. Tom 41 (1977)]

[17] R. Borcherds, “The Gross–Kohnen–Zagier theorem in higher dimensions,” Duke Math. J. 97 (1999) 219.

[18] A. Gholampour and A. Sheshmani. “Donaldson-Thomas Invariants of 2-Dimensional sheaves inside threefolds and modular forms,” arXiv:1309.0050.

[19] V. Bouchard, T. Creutzig, D. E. Diaconescu, C. Doran, C. Quigley and A. Sheshmani, “Vertical D4D2D0 Bound States on K3 Fibrations and Modularity,” Commun. Math. Phys. 350 (2017) no.3, 1069 [arXiv:1601.04030 [hep-th]].

[20] A. Dabholkar, F. Denef, G. W. Moore and B. Pioline, “Exact and asymptotic degeneracies of small black holes,” JHEP 0508 (2005) 021 [hep-th/0502157].

[21] M. Bershadsky, C. Vafa and V. Sadov, “D-branes and topological field theories,” Nucl. Phys. B 463 (1996) 420 [hep-th/9511222].

[22] R. Gopakumar and C. Vafa, “M theory and topological strings. 1.,” hep-th/9809187
R. Gopakumar and C. Vafa, “M theory and topological strings. 2.,” hep-th/9812127
M. Dedushenko and E. Witten, “Some Details On The Gopakumar-Vafa and Ooguri-Vafa Formulas,” Adv. Theor. Math. Phys. 20 (2016) 1 [arXiv:1411.7108 [hep-th]].

[23] S. H. Katz, A. Klemm and C. Vafa, “M theory, topological strings and spinning black holes,” Adv. Theor. Math. Phys. 3 (1999) 1445 [hep-th/9910181].

[24] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, JHEP 0505 (2005) 023 [hep-th/0410018].

[25] F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” JHEP 1111 (2011) 129 [hep-th/0702146].

[26] A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger, “Noether–Lefschetz theory and the Yau–Zaslow conjecture,” J. Amer. Math. Soc. 23 (2010) 1013.

[27] W. McGraw, “The rationality of vector valued modular forms associated with the Weil representation” Math. Ann. 326 105–122 (2003).

[28] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, “Nonperturbative results on the point particle limit of N=2 heterotic string compactifications,” Nucl. Phys. B 459 (1996) 537 [hep-th/9508155].

[29] P. H. Ginsparg, ”Gauge and Gravitational Couplings in Four-Dimensional String Theories,” Phys. Lett. B197 (1987) 139.

[30] B. Haghighat and A. Klemm, “Solving the Topological String on K3 Fibrations,” JHEP 1001 (2010) 009 [arXiv:0908.0336 [hep-th]].

[31] J. M. Maldacena, A. Strominger and E. Witten, “Black hole entropy in M theory,” JHEP 9712 (1997) 002 [hep-th/9711053].

[32] I. Antoniadis and H. Partouche, “Exact monodromy group of N=2 heterotic superstring,” Nucl. Phys. B 460, 470 (1996) [hep-th/9509009].

[33] C. F. Doran, A. Harder, A.Y. Novoseltsev, and A. Thompson, “Calabi-Yau threefolds fibred by high rank lattice polarized K3 surfaces,” Math. Z. (2019). [arXiv:1701.03279 [math.AG]].

[34] J. H. Bruinier, “Borcherds products on O(2,ℓ) and Chern classes of Heegner divisors,” (2000) Springer.

[35] S. Hellerman and C. Schmidt-Colinet, “Bounds for State Degeneracies in 2D Conformal Field Theory,” JHEP 1108 (2011) 127 [arXiv:1007.0756 [hep-th]].
[36] G. Lopes Cardoso, G. Curio and D. Lust, “Perturbative couplings and modular forms in N=2 string models with a Wilson line,” Nucl. Phys. B 491 (1997) 147 [hep-th/9608154].

[37] S. Stieberger, “(0,2) heterotic gauge couplings and their M theory origin,” Nucl. Phys. B 541 (1999) 109 [hep-th/9807124].

[38] M.C.N.Cheng, X.Dong, J.Duncan, J.Harvey, S.Kachru and T.Wrase, “Mathieu Moonshine and N=2 String Compactifications,” JHEP 1309 (2013) 030 [arXiv:1306.4981 [hep-th]].

[39] V. Kaplunovsky and J. Louis, “On Gauge couplings in string theory,” Nucl. Phys. B 444 (1995) 191 [hep-th/9502077].

[40] B. de Wit, V. Kaplunovsky, J. Louis and D. Lust, “Perturbative couplings of vector multiplets in N=2 heterotic string vacua,” Nucl. Phys. B 451 (1995) 53 [hep-th/9504006].

[41] C. T. Wall, “Classification problems in differential topology V. On certain 6-manifolds.” Inv. Math. 1 (1966) 355.

[42] P. S. Aspinwall, B. R. Greene and D. R. Morrison, “Multiple mirror manifolds and topology change in string theory,” Phys. Lett. B 303 (1993) 249 [hep-th/9301043]; “Calabi-Yau moduli space, mirror manifolds and space-time topology change in string theory,” Nucl. Phys. B 416 (1994) 414 [AMS/IP Stud. Adv. Math. 1 (1996) 213 [hep-th/9309097].

[43] D. R. Morrison and C. Vafa, “Compactifications of F theory on Calabi-Yau threefolds. 2.,” Nucl. Phys. B 476 (1996) 437 [hep-th/9603161].

[44] D. R. Morrison and C. Vafa, “Compactifications of F theory on Calabi-Yau threefolds. 1.,” Nucl. Phys. B 473 (1996) 74 [hep-th/9602114].

[45] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, “Geometric singularities and enhanced gauge symmetries,” Nucl. Phys. B 481 (1996) 215 [hep-th/9605200].

[46] T. Kawai, “String duality and modular forms,” Phys. Lett. B 397 (1997) 51 [hep-th/9607078].

[47] A. Kanazawa, “Study of Calabi–Yau geometry,” Ph.D thesis, the University of British Columbia, 2014.

[48] M. Eichler and D. Zagier, “The theory of Jacobi forms,” Birkhäuser, 1985.
[49] A. Dabholkar, S. Murthy and D. Zagier, “Quantum Black Holes, Wall Crossing, and Mock Modular Forms,” arXiv:1208.4074 [hep-th].

[50] R. Borcherds, “Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products,” Invent. Math. 120 (1995) 161.

[51] M. Marino and G. W. Moore, “Counting higher genus curves in a Calabi-Yau manifold,” Nucl. Phys. B 543 (1999) 592 [hep-th/9808131].

[52] L. J. Dixon, V. Kaplunovsky and J. Louis, “Moduli dependence of string loop corrections to gauge coupling constants,” Nucl. Phys. B 355 (1991) 649.

[53] W. Lerche, A. N. Schellekens and N. P. Warner, “Lattices and Strings,” Phys. Rept. 177 (1989) 1.