Cosmological dynamics in tomographic probability representation

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Abstract

The probability representation for quantum states of the universe in which the states are described by a fair probability distribution instead of wave function (or density matrix) is developed to consider cosmological dynamics. The evolution of the universe state is described by standard positive transition probability (tomographic transition probability) instead of the complex transition probability amplitude (Feynman path integral) of the standard approach. The latter one is expressed in terms of the tomographic transition probability. Examples of minisuperspaces in the framework of the suggested approach are presented. Possibility of observational applications of the universe tomographs are discussed.

1 Introduction

Recently the tomographic probability approach to describe the states of the universe in quantum cosmology was suggested. In the framework of this approach the quantum state of the universe is associated with the standard positive probability distribution (function or functional). The probability distribution contains
the same information on the universe quantum state that the wave function of the universe [2] [3] [4] or the density matrix of the universe [5], [6]. The latter can be presented in different forms, e.g. in form of a Wigner function [7] considered in [8] in a cosmological context. In fact the tomographic probability distribution describing the state of the universe is a symbol of a density operator [9] [10] and the tomographic symbols of the operators realize one of the variants of the star-product quantization scheme widely used [11] to study the relation of classical and quantum pictures [12], which can also be applied to study the relation of classical and quantum descriptions of the universe in quantum cosmology. One of the important ingredients of such descriptions is the evolution of the state. In quantum mechanics such evolution is completely described by means of a complex transition probability amplitude from an initial state to a final one. This probability amplitude (propagator) can be presented in the form of a Feynman path integral containing the classical action. In quantum mechanics in the probability representation using the tomographic approach the state evolution can be associated with the standard transition probability. It contains also information on the transition probability amplitude related to the probability by integral transform induced by the Radon transform relating the density matrix (Wigner function) with the quantum tomographic probability [13], [14], [15], [16].

In our previous work [1] we suggested to associate the state of the universe in quantum cosmology with the tomographic probability (or tomogram). The aim of our paper is to consider now in the framework of the suggested probability representation of the universe state in quantum cosmology also the cosmological dynamics and to express this dynamics in terms of a positive transition probability connecting initial and final tomograms of the universe. Another goal of the work is to discuss a possible experimental approach to observe the tomogram of the universe at its present stage and try to extract some information on the tomogram of the initial state of the universe. The idea of this attempt is based on the fact that the same tomogram describes the state of a classical system and its quantum counterpart. In this sense in the probability representation of the quantum state there is no such dramatic difference between the classical and quantum pictures as the difference between wave function (or density matrix) and classical probability distribution (or trajectory) in the classical phase space. Due to this one can try to study the cosmological dynamics namely in the tomographic probability representation.

In order to illustrate the idea we will use the same simple example of the universe description by means of the minisuperspace discussed, e.g. in [4], [17]. In these minisuperspaces the quantum cosmological dynamics in operative form
is reduced to the dynamics of formal quantum systems described by Hamiltonians of the types of oscillator, free motion and free falling particles. In view of this one can apply the same recent obtained results on description of such systems by tomographic probabilities [18] to the cosmological dynamics.

The paper is organized as follows. In the next section we will review the cosmology in terms of a homogeneous (and isotropic) metric with a time dependent parameter the expansion factor of the universe. In section 3, we review the tomographic approach to evolution of the quantum system. In section 4 we consider the examples of the minisuperspace described by the reduced Hamiltonians.

## 2 Cosmology

Classically a homogeneous and isotropic universe is described by one of the following metrics

\[
\text{ds}^2 = -c^2 dt^2 + \frac{a^2}{1 - kr^2} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)
\]

where the parameter \( k \) can be \( k > 0 \), \( k = 0 \) and \( k < 0 \) being related to a closed universe, a flat universe and an open universe respectively.

When the gravitational source is a perfect fluid, described by the energy-momentum tensor, the Einstein equations with the metric (1) are in the second order form

\[
\ddot{a} = -\frac{4\pi G}{3} \left( \rho + 3P \right)
\]

which represents the dynamic equation and

\[
\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3} \rho
\]

which is a constraint, i.e. it defines the manifold of allowed initial conditions. It takes a simple computation to show that there are no secondary constraints. It constitutes an “invariant relation”, according to Levi-Civita.

From equations (2) and (3) the first order equation

\[
\dot{\rho} = -3\frac{\dot{a}}{a} (\rho + P)
\]

can be derived by taking the time derivative of (3). It can be used alternatively in a system with equation (3).
The system of equations (2) and (3) or (3) and (4) are not complete, they must be completed by an equation of state $P = P(\rho)$ which is generally linear, $P = (\gamma - 1)\rho$ ($\gamma = 1$ is the so-called matter fluid, $\gamma = 4/3$ is the radiation fluid and so on).

Equation (4) together with an equation of state, is important for our purpose because it shows that the lefthand side of equations (2) and (3) can be expressed as a function of $a$ and represents a force in these equations, if we treat them as equations for a “point” particle as a result we have

$$\rho = \frac{\rho_0 a_0^{3\gamma}}{a^{3\gamma}}$$  \hspace{1cm} (5)

when the equation of state is linear.

It is possible to derive the cosmological model from a point particle Lagrangian, where the expansion factor $a$ takes the part of the particle coordinate. Let us introduce the following Lagrangian

$$\mathcal{L} = 3a\dot{a}^2 - 3ka - 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}.$$  \hspace{1cm} (6)

The gravitational part is formally derived by substituting directly metric (1) into the (field) general relativistic action $\int \sqrt{-g} R$ and the material part is obtained by putting a corresponding potential term $\Phi(a) = 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}$, in the case of a fluid source.

Equation (2) follows from the variation of the Lagrangian (6).

From equation (6) the conjugate momentum of $a$ is

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = 6a\dot{a}.$$  \hspace{1cm} (7)

Equation (3) is a constraint which is equivalent to the vanishing of the “energy function” $E_L$ associated to the Lagrangian

$$E_L = 3a\dot{a}^2 + 3ka - 8\pi G\rho_0 a_0^{3\gamma} a^{3(1-\gamma)}.$$  \hspace{1cm} (8)

An alternative way to describe cosmology with a cosmological fluid, with $\Lambda = 0$, was introduced by Lemos [17] and Faraoni [19].

They showed that equations (2) and (3) can be transformed in equations similar to the harmonic oscillator ones. By passing to the conformal time $\eta$, defined by the relation

$$d\eta = \frac{dt}{a(t)},$$
and with the change of variables

\[ w = a^\chi \]  \hspace{1cm} (9)

where

\[ \chi = \frac{3}{2} \gamma - 1 \]

equation (2) takes the form

\[ w'' + k\chi^2 w = 0. \]  \hspace{1cm} (10)

Let us consider now a cosmological model where the source is originated by a scalar field, which satisfies the Klein-Gordon equation specialized to a homogeneous and isotropic universe (which substitutes equation (4))

\[ \ddot{\varphi} + 3 \frac{\dot{a}}{a} \dot{\varphi} + V'(\varphi) = 0 \] \hspace{1cm} (11)

where \( V(\varphi) \) is the potential for the scalar field.

It is possible to derive the cosmological equations (2) and (3), where \( \rho_\varphi = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \) and \( P_\varphi = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \) from a Lagrangian \( \mathcal{L} = \mathcal{L}(a, \dot{a}, \varphi, \dot{\varphi}) \).

\[ \mathcal{L} = 3a\dot{a}^2 - 3ka - 8\pi Ga^3 \left( \frac{1}{2} \dot{\varphi}^2 - V(\varphi) \right). \] \hspace{1cm} (12)

where the minisuperspace becomes a two dimensional space with coordinates \( a \) and \( \varphi \).

From the previous Lagrangian (12) we compute the conjugate momenta of \( a \) and \( \varphi \):

\[ p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = 6a\dot{a}. \] \hspace{1cm} (13)

\[ p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 8\pi Ga^3 \dot{\varphi}. \] \hspace{1cm} (14)

Equation (3) is now a constraint which is equivalent to the vanishing of the “energy function” \( E_\mathcal{L} \) associated to the Lagrangian

\[ E_\mathcal{L} = 3a\dot{a}^2 + 3ka - 8\pi Ga^3 \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right). \] \hspace{1cm} (15)

If \( V(\varphi) \) takes the form

\[ V(\varphi) = \frac{1}{2} (2 - \gamma) \rho_\varphi^0 \exp(-\sqrt{24\pi G\gamma \varphi}), \] \hspace{1cm} (16)
with constant $\gamma$, then $\rho_\varphi$ and $P_\varphi$ satisfy the equation of state \[20\]
\[
P_\varphi = (\gamma - 1)\rho_\varphi
\] (17)
and also in this case, the evolution of universe can be described by equation (10). In [24] there are other examples in which the cosmological models with a scalar field can be described by equations similar with (10), the tomographic version of these models has already been discussed in [1].

3 Evolution in minisuperspace in the framework of tomographic probability representation

We will discuss below the evolution of a universe in the framework of the minisuperspace model discussed in the previous section. Thus the state of the universe is described by a wave function $\Psi(x, t)$. This wave function evolves in time from its initial value $\Psi(x, t_0)$ and this evolution can be described by a propagator $G(x, x', t, t_0)$

\[
\Psi(x, t) = \int G(x, x', t, t_0)\Psi(x', t_0)dx'.
\] (18)

The propagator can be obtained using path integration over classical trajectories of the exponential of the classical action $S$

\[
G(x, x', t, t_0) = \int D[x(t)]e^{\frac{iS[x(t)]}{\hbar}}.
\] (19)

In our previous work [1] we discussed the properties of the new representation (tomographic probability representation) of the quantum states of the universe.

In this representation (which we discuss below in the framework of a minisuperspace model) the wave function of the universe $\Psi(x, t)$ or the density matrix of the universe

\[
\rho(x, x', t) = \Psi(x, t)\Psi^*(x', t)
\] (20)

can be mapped onto the standard positive distribution $\mathcal{W}(X, \mu, \nu, t)$ of the random variable $X$ depending on the two real extra parameters $\mu$ and $\nu$ and the time $t$. The map is given by the formula (we take $\hbar = 1$)

\[
\mathcal{W}(X, \mu, \nu, t) = \frac{1}{2\pi|\nu|} \int \rho(y, y', t)e^{\frac{i(y^2 - y'^2)}{2\nu}} - i\frac{X}{\nu}(y - y')dydy'.
\] (21)
In fact, equation (21) is the fractional Fourier transform \([21] [22]\) of the density matrix. The map has inverse and the density matrix can be expressed in terms of the tomographic probability representation as follows

\[
\rho(x, x', t) = \frac{1}{2\pi} \int \mathcal{W}(Y, \mu, x - x', t) e^{i(Y - \mu \nu(x + x'))} dY d\mu. \tag{22}
\]

The expression (21) can be given in an invariant form \([23]\)

\[
\mathcal{W}(X, \mu, \nu, t) = \langle \delta(X - \mu \hat{q} - \nu \hat{p}) \rangle \tag{23}
\]

Here \(\langle \rangle\) means trace with the density operator \(\hat{\rho}(t)\) of the universe state, \(\hat{q}\) and \(\hat{p}\) are the operators of position (universe expansion factor) and the conjugate moment respectively. From equation (23) some properties of the tomogram \(\mathcal{W}(X, \mu, \nu, t)\) are easily extracted. First, the universe tomogram is a normalized probability distribution, i.e.

\[
\int \mathcal{W}(X, \mu, \nu, t) dX = 1 \tag{24}
\]

if the universe density operator is normalized (i.e. \(Tr \hat{\rho}(t) = 1\)). Second, the tomogram of the universe state has the homogeneity property \([26]\)

\[
\mathcal{W}(\lambda X, \lambda \mu, \lambda \nu, t) = \frac{1}{|\lambda|} \mathcal{W}(X, \mu, \nu, t) \tag{25}
\]

The tomogram can be related with such quasidistribution as the Wigner function \(W(q, p, t)\) \([7]\) used in the phase space representation of the universe states in \([8]\).

The relation reads

\[
\mathcal{W}(X, \mu, \nu, t) = \int W(q, p, t) \delta(X - \mu q - \nu p) \frac{dq dp}{2\pi} \tag{26}
\]

which is the standard Radon transform of the Wigner function. The physical meaning of the tomogram \(\mathcal{W}(X, \mu, \nu, t)\) is the following. One has in the phase space the line

\[
X = \mu q + \nu p \tag{27}
\]

which is given by the zero of the delta-function argument in equation (26). The real parameters \(\mu\) and \(\nu\) can be given in the form

\[
\mu = s \cos \theta \quad \nu = s^{-1} \sin \theta. \tag{28}
\]

Here \(s\) is a real squeezing parameter and \(\theta\) is a rotation angle. Then the variable \(X\) is identical to the position measured in the new reference frame in the universe.
phase-space. The new reference frame has new scaled axis \( s q \) and \( s^{-1} p \) and after the scaling the axis are rotated by an angle \( \theta \). Thus the tomogram implies the probability distribution of the random position \( X \) measured in the new (scaled and rotated) reference frame in the phase-space. The remarkable property of the tomographic probability distribution is that being a fair positive probability distribution, it contains a complete information of the universe state contained in the density operator \( \hat{\rho}(t) \) which can be expressed in terms of the tomogram as \[25\]

\[
\hat{\rho}(t) = \frac{1}{2\pi} \int \mathcal{W}(X, \mu, \nu, t) e^{i(X - \mu \hat{q} - \nu \hat{p})} dX d\mu d\nu
\] (29)

Formulae \[23\] and \[29\] can be treated with the tomographic star-product quantization schemes \[10\] used to map the universe quantum observables (operators) onto functions (tomographic symbols) on a manifold \( (X, \mu, \nu) \). The tomographic map can be used not only for the description of the universe state by probability distributions, but also to describe the evolution of the universe (quantum transitions) by means of the standard real positive transition probabilities (alternative to the complex transition probability amplitudes). The transition probability

\[
\Pi(X, \mu, \nu, t, X', \mu', \nu', t_0)
\]

is the propagator which gives the tomogram of the universe \( \mathcal{W}(X, \mu, \nu, t) \), if the tomogram at the initial time \( t_0 \) is known, in the form

\[
\mathcal{W}(X, \mu, \nu, t) = \int \Pi(X, \mu, \nu, t, X', \mu', \nu', t_0) \mathcal{W}(X', \mu', \nu', t_0) dX' d\mu' d\nu'.
\] (30)

The positive transition probability describing the evolution of the universe has the obvious nonlinear properties used in classical probability theory, namely

\[
\Pi(X, \mu, \nu, t, X', \mu', \nu', t_0) = \int \Pi(X, \mu, \nu, t, X_2, \mu_2, \nu_2, t_2) \times \Pi(X_2, \mu_2, \nu_2, t_2, X_1, \mu_1, \nu_1, t_1) dX_2 d\mu_2 d\nu_2.
\] (31)

They follow from the associativity property of the evolution maps. This nonlinear relation is analogous to the nonlinear relation of the complex quantum propagators of the universe wave function

\[
G(x_3, x_1, t_3, t_1) = \int G(x_3, x_2, t_3, t_2) G(x_2, x_1, t_2, t_1) dx_2.
\] (32)

Both relations \[31\] and \[32\] imply that the state of the universe evolves from the initial one to the final one through all intermediate states. The remarkable
fact is that this quantum evolution of the universe state can be associated with the standard positive transition probabilities like in classical dynamics. This is connected with the existence of the invertible relations of the tomographic and quantum propagators [16][18]. If one denotes

$$K(X, X', Y, Y', t) = G(X, Y, t)G^*(X', Y', t),$$

then the quantum propagator may be given in the following form

$$K(X, X', Y, Y', t) = \frac{1}{(2\pi)^2} \int \frac{1}{|Y'|} \exp \left\{ i \left( Y - \mu \frac{(X + X')}{2} \right) - i \frac{Z - Z'}{\nu'} Y' \right\}$$

$$+ i \frac{Z^2 - Z'^2}{2\nu'} \mu' \right\} \Pi(Y, \mu, X - X', 0, X', \mu', \nu', t) d\mu d\mu' dY dY' d\nu'. \quad (34)$$

This relation can be reversed. Thus the propagator for the tomographic probability can be expressed in terms of the Green function $G(x, y, t)$ as follows (we take $t_0 = 0$)

$$\Pi(X, \mu, \nu, X', \mu', \nu', t) = \frac{1}{4\pi} \int k^2 G(a + \frac{k\nu}{2}, y, t)G^*(a - \frac{k\nu}{2}, z, t) \delta(y - z - k\nu')$$

$$\times \exp \left[ ik(X' - X + \mu q - \mu' \frac{y + z}{2}) \right] dk dy dz dq. \quad (35)$$

The relation can be used to express the tomographic propagation in terms of the Feynmann path integral using the formula for the quantum propagator (19) where the classical action is involved. It means that the positive transition probabilities (35) can be reexpressed in terms of the double path integral (with four extra usual integrations).

The discussed relations demonstrate that the quantum universe evolution can be described completely using only positive transition probabilities.

Standard complex transition probability amplitudes (and Feynman path integral) can be reconstructed using this transition probability by means of equation (34).
4  Evolution of the universe in the oscillator model framework

As we have shown the equation for the universe evolution in the conformal time picture (10) can be cast in the form of an oscillator equation. The oscillator has the frequency $\omega^2 = \pm k \chi^2$.

For $k = 0$ one has the model of free motion. For $k < 0$ one has the model of an inverted oscillator and for $k > 0$ one has the standard oscillator as solution of the equation (10). Since the problem of gravity quantization is not established with complete rigor, we assume below that the quantum behavior of the universe in the framework of the considered minisuperspace model is described by the quantum behavior of the oscillator. Though the connection (9) of the expansion factor $a(\eta)$ with the classical observable $w$ which obeys to oscillator motion provides constraints on the ranging domain of this variable, we assume in the quantum picture of the variable to lie on the real line $R$. In such approach we apply the tomographic probability representation, developed in the last section, to quantum states of the universe in the framework of the oscillator model. We will denote in the quantum description the variable as $q$ ($w \to q$) and the conformal time as $t$ ($\eta \to t$). Thus the tomographic probability $W(X, \mu, \nu, t)$ of the universe state obeys the evolution equation (11) for the potential energy $V(q)$ in the form

$$\frac{\partial W(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial W(X, \mu, \nu, t)}{\partial \nu} + i \left[ V \left( - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} - \frac{i \nu}{2} \frac{\partial}{\partial X} \right) - V \left( \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{i \nu}{2} \frac{\partial}{\partial X} \right) \right] W(X, \mu, \nu, t) = 0,$$

(36)

where the operator $(\partial/\partial X)^{-1}$ is defined by the relation

$$\left( \frac{\partial}{\partial X} \right)^{-1} \int f(y) e^{iyX} dy = \int \frac{f(y)}{(iy)} e^{iyX} dy.$$

(37)

The propagator of this equation $\Pi(X, \mu, \nu, t, X', \mu', \nu')$ satisfies equation (36) with the extra term

$$\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + i \left[ V \left( - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} - \frac{i \nu}{2} \frac{\partial}{\partial X} \right) \right]$$
\[-V \left( -\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{i\nu}{2} \frac{\partial}{\partial X} \right) \right] \Pi = \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t). \tag{38}\]

For the considered model the general equation for the universe tomogram evolution takes the simple form of a first order differential equation

\[
\frac{\partial W}{\partial t} - \mu \frac{\partial W}{\partial \nu} + \omega^2 \nu \frac{\partial W}{\partial \mu} = 0. \tag{39}\]

Analogously for the propagator of the tomographic equation for the universe in the framework of the oscillator model one has

\[
\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + \omega^2 \nu \frac{\partial \Pi}{\partial \mu} = \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t). \tag{40}\]

The solution to this equation can be found to be in the case \(k > 0\)

\[
\Pi^{\text{osc.}}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta'(-\mu \cos \omega t + \omega \nu \sin \omega t) \]
\[
\times \delta\left( \nu' - \nu \cos \omega t - \frac{\mu}{\omega} \sin \omega t \right). \tag{41}\]

In the limit \(k = 0\) (free motion) the equation for the tomogram (39) becomes

\[
\frac{\partial \mathcal{W}(X, \mu, \nu, t)}{\partial t} - \mu \frac{\partial \mathcal{W}(X, \mu, \nu, t)}{\partial \nu} = 0. \tag{42}\]

The corresponding propagator solution reads

\[
\Pi^{\text{free}}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta'(-\mu \cos \omega t - \omega \nu \sin \omega t) \]
\[
\times \delta\left( \nu' - \nu \cos \omega t - \frac{\mu}{\omega} \sin \omega t \right). \tag{43}\]

Finally for the case \(k < 0\) the propagator has the form corresponding to a repulsive oscillator

\[
\Pi^{\text{rep}}(X, \mu, \nu, t, X', \mu', \nu') = \delta(X - X')\delta(\mu' - \mu)\delta(\nu' - \nu - \mu t). \tag{44}\]

Thus we got the dynamics of the universe given by the transition probabilities \(\Pi^{\text{osc.}}, \Pi^{\text{free}}\) and \(\Pi^{\text{rep}}\) for the three cases \(k > 0\), \(k = 0\) and \(k < 0\) respectively. One can see that this dynamics is compatible with the dynamics calculated in the
standard representation of the complex Green function (quantum propagator). For \( k = 1 \) the form of the Green function reads

\[
G^{osc.}(X, X', t) = \sqrt{\frac{\omega}{2\pi}} \sin \omega t \exp \left\{ \frac{i\omega}{2} \left[ \cot \omega t \left( X^2 + X'^2 \right) - \frac{2XX'}{\sin \omega t} \right] \right\} \tag{45}
\]

For the case of the free motion model the Green function can be obtained by the limit \( \omega \to 0 \) in this expression and one has

\[
G^{free}(X, X', t) = \sqrt{\frac{1}{2\pi}} t \exp \left\{ i \frac{(X - X')^2}{2t} \right\} \tag{46}
\]

and for the repulsive oscillator model one has

\[
G^{rep.}(X, X', t) = \sqrt{\frac{\omega}{2\pi \sinh \omega t}} \exp \left\{ \frac{i\omega}{2} \left[ \coth \omega t \left( X^2 + X'^2 \right) - \frac{2XX'}{\sinh \omega t} \right] \right\} \tag{47}
\]

All these three universe cases can be discussed using the Green function in terms of the Feynman path integral.

Thus the expression (45) is given by the formula

\[
G(X, X', t) = \int e^{i \int_0^t \left[ \frac{\dot{x}^2(t)}{2} - \frac{\omega^2 x^2(t)}{2} \right] dt} D[x(t)] \tag{48}
\]

The integral in the exponent of the path integral provides the classical action for the oscillator

\[
S^{cl.}(X, X', t) = \int_0^t \left[ \frac{\dot{x}^2(t)}{2} - \frac{\omega^2 x^2(t)}{2} \right] dt \tag{49}
\]

where the trajectories start at \( t = 0 \) at \( X' \) and end at time \( t \) at the point \( X \). The classical action satisfies the Hamilton-Jacobi equation

\[
\frac{\partial S^{cl.}(q, q', t)}{\partial t} + \mathcal{H} \left( q, p = -\frac{\partial S^{cl.}(q, q', t)}{\partial q} \right) = 0 \tag{50}
\]

where \( \mathcal{H} \) is the Hamiltonian

\[
\mathcal{H} = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} \tag{51}
\]

For the free motion model one has
\[ G^{\text{free}}(X, X', t) = \int e^{i \int_0^t \frac{x'^2(t)}{2} dt} D[x(t)]. \] (52)

The path integral is integrated and the result (46) contains in the exponent term the classical action
\[ S^{(f)}(X, X', t) = \frac{(X - X')^2}{2t}. \] (53)

which is solution of the Hamilton-Jacobi equation with the Hamiltonian
\[ \mathcal{H} = \frac{p^2}{2}. \] (54)

For the repulsive model one has the same structure of path integral and the result of path integration is expressed in terms of the classical action
\[ S^{\text{rep.}}(X, X', t) = \frac{\omega}{2} \left[ \coth \omega t \left( X^2 + X'^2 \right) - \frac{2XX'}{\sinh \omega t} \right], \] (55)

which is solution of the Hamilton-Jacobi equation with the Hamiltonian
\[ \mathcal{H}^{\text{rep.}} = \frac{p^2}{2} - \frac{\omega^2 q^2}{2}. \] (56)

It is remarkable that all the obtained propagators complex Green functions or path integrals are related with the propagators in probability representation by means of equations (31) and (32).

Thus the universe evolution can be described in the oscillator model of minisuperspace for \( k > 0 \), \( k = 0 \) and \( k < 0 \) by means of the standard transition probabilities expressed as propagators \( \Pi^{\text{osc.}}, \Pi^{\text{free}} \) and \( \Pi^{\text{rep.}} \), respectively.

## 5 Conclusions

To conclude we discuss the main results of the work. In addition to what suggested in [1], the probability representation of the universe quantum states for which the states (e.g. of the universe in a minisuperspace model) are described by the standard positive probability distribution, we introduce the description of the universe dynamics by means of standard transition probabilities.

The transition probabilities are determined as propagators (integral kernels) providing the evolution of the universe tomograms. It is shown that there is a relation of the standard propagator determining the quantum evolution of the universe
wave function to the tomographic propagator. This relation permits to reconstruct
the complex propagator for the wave function in terms of the positive propagator
for the universe tomogram. Also, one can express the propagator for the tomogram
in terms of the propagator for the wave function of the universe.

These relations between the propagators mean that the Feynmann path integral
formulation or the universe properties (in quantum gravity) contains the same
information that the probability representation of the quantum states of the uni-
verse including the universe quantum evolution. As the simplest example of the
suggested transition probability picture, we considered the minisuperspace model
for which classical and quantum evolution is described by the harmonic vibrations
in conformal time [17], [19], [24], [27]. The specific property of this minisuper-
space model is that the tomographic propagators for both classical and quantum
universe tomograms coincide. This fact provides some possibility to connect ob-
servations related to the classical epoch of the universe and its purely initial
quantum state. In the framework of the suggested approach (and in the frame-
work of the considered oscillator model), the universe evolution can be studied
using specific properties of the tomographic propagator. If one considers tomo-
grams and their evolution in classical mechanics [14] the specific property of
the linear systems (e.g. oscillator model) is that the tomographic propagators in
quantum and classical domains are in one-to-one correspondence and are given in
the same carrier space, therefore we may say that the difference of the quantum
and classical evolution is solely in the initial conditions, the quantum and classical
pictures differ by constraints which must satisfy quantum tomograms. They must
satisfy uncertainty relations. The choice of initial conditions (initial tomogram
of the universe) in correspondence with the uncertainty relation provides a possi-
bility to avoid the singularity of the metric, which is unavoidable in the classical
picture. But the following evolution of the universe coded by the tomographic
propagator is the same (for the oscillator model).

Due to this result, one can extract from the present observational classical data
conclusions over the cosmological evolution. Evolving backwards in time the
present situation by means of the “true” quantum or the classical propagators, we
may find discrepancies between the initial conditions at minus infinity.

We are going to discuss this aspect in a future work.
References

[1] V. I. Manko, G. Marmo and C. Stornaiolo, “Radon transform of Wheeler-De Witt equation and tomography of quantum states of the universe,” to appear on Gen.Rel.Grav., gr-qc/0307084.

[2] S. W. Hawking, Nucl. Phys B239, 257 (1984)

[3] B. S. DeWitt Phys. Rev. 160, 1113, (1983); J. A Wheeler in Battelle Rencontres, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[4] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960, (1983)

[5] D. N. Page Phys. Rev. D 34, 2267, (1986)

[6] S. W. Hawking Phys. Scr. T15, 151 (1987)

[7] E. Wigner, Phys. Rev. 40, 749 (1932)

[8] F. Antonsen, “Deformation Quantisation of Gravity”, gr-qc/9712012

[9] O. V. Manko, V.I. Manko, G. Marmo, Phys. Scr. 62, 446, (2000).

[10] O. V. Manko, V. I. Manko, G. Marmo, J. Phys. A 35, 699, (2002)

[11] F. Bajen, M. Flato, M. Fronsdal, C. Lichnerowicz, D. Sternheimer, Lett. Math. Phys. 1, 521 (1975)

[12] G. Esposito, G. Marmo, G. Sudarshan, “From Classical to Quantum Mechanics An Introduction to the Formalism, Foundations and Applications” Cambridge University Press (2004).

[13] S. Mancini, V.I.Manko, P.Tombesi, J.Opt.B: Quantum and Semiclass. Opt., 7, 615 (1995)

[14] O. V. Manko, V. I. Manko, J. Russ. Laser Res. 18, 407 (1997)

[15] V. I. Manko, R.V. Mendes, Phys. Lett. A263, 53 (1999)

[16] V.I. Manko, R. V. Mendes, Physica D145, 330 (2000)

[17] N. A. Lemos, J. Math. Phys. 37 (1996) 1449 [arXiv:gr-qc/9511082].

[18] O. V. Man’ko, Theor. Math. Phys. 121, 285 (1999)

15
[19] V. Faraoni, Am. J. Phys. 67 (1999) 732 [arXiv:physics/9901006].
[20] C. Stornaiolo, Phys. Lett. A 189 (1994) 351.
[21] M. A. Manko, J. Russ. Laser Res. 21, 421 (2000)
[22] M. A. Manko, J. Russ. Laser Res. 22, 168 (2001)
[23] M. A. Manko, V. I. Manko, R. V. Mendes J Phys A 34, 8321 (2001)
[24] S. S. Gousheh, H. R. Sepangi, Phys. Lett. A272, 304 (2000)
[25] G. M. D’Ariano, S. Mancini, V. I. Manko and P. Tombesi, Quantum Semi-
class. Opt 8, 1017, (1996), quant-ph/9606034
[26] V. I. Manko, L. Rosa, P. Vitale Phys. Lett B 439, 328 (1998)
[27] V. G. Lapchinsky and V. A. Rubakov, Teor. Mat. Fiz. 33 (1977) 364.