New derivation of quantum equations
from classical stochastic arguments

H. Bergeron
LURE bat. 209D Centre Universitaire Paris-Sud -BP34- 91898 Orsay cedex, FRANCE
(Dated: October 24, 2018)

Abstract
In a previous article [H. Bergeron, J. Math. Phys. 42, 3983 (2001)], we presented a method to obtain a continuous transition from classical to quantum mechanics starting from the usual phase space formulation of classical mechanics. This procedure was based on a Koopman-von Neumann approach where classical equations are reformulated into a quantumlike form. In this article, we develop a different derivation of quantum equations, based on purely classical stochastic arguments, taking some technical elements from the Bohm-Fényes-Nelson approach. This study starts from a remark already noticed by different authors [M. Born, Physics in My Generation (Pergamon Press, London, 1956); E. Prugovečki, Stochastic Quantum Mechanics and Quantum Spacetime (Reidel, Dordrecht, 1986)], suggesting that physical continuous observables are stochastic by nature. Following this idea, we study how intrinsic stochastic properties can be introduced into the framework of classical mechanics. Then we analyze how the quantum theory can emerge from this modified classical framework. This approach allows us to show that the transition from classical to quantum formalism (for a spinless particle) does not require real postulates, but rather soft generalizations.

PACS numbers: 03.65.Ta, 02.50.Ey

*Electronic address: herve.bergeron@lure.u-psud.fr
I. INTRODUCTION

Since the beginning of quantum mechanics, different links have been developed between classical and quantum frameworks, in order to overcome the main difficulties due to the difference of formalisms, and to get a better understanding of their interplay. These attempts can be roughly divided into three families.

One family is represented by the Wigner-Weyl (WW) approach, based on the reformulation of quantum mechanics into phase space thanks to a continuous map. Particularly, the Wigner-Weyl transformation allows one to recover the semi-classical limit of quantum mechanics. Moreover quantum mechanics appears as a deformation of the Abelian function algebra in phase-space where the standard multiplication is replaced by the so-called $\ast_h$-product (deformation quantization). But the Wigner-Weyl transformation is not the unique way to obtain such a continuous map. The use of coherent states (specially the coherent states of the Galilei group) allows one to obtain the same kind of correspondence.

Another family is represented by the Koopman-von Neumann (KvN) approach, based on a reformulation of classical mechanics into the Hilbert space language. This leads to a quantumlike theory (but always classical) that can be directly compared to quantum mechanics. Particularly, we can study how this quantumlike framework must be modified in order to rebuild a true quantum theory.

These two attempts are in fact two sides of the same problem, namely finding a unified mathematical framework for classical and quantum mechanics. Among the numerous publications on this subject, and in addition to the authors already mentioned, we can quote the works of G. Mackey and E. Prugovecki.

The third attempt is very different, and it is represented by the Bohm-Fényes-Nelson (BFN) approach. Above all, this approach is centered on the problem of the interpretation of quantum mechanics. The leading idea is that quantum phenomena are due to some stochastic effects that take place on a classical background where the idea of trajectory (obtained from equations of motion) remain valid. The Planck constant $\hbar$ becomes a “measure” of the stochastic effects. Although this interpretation can lead to disagreements with quantum mechanics, especially in the case of non-interacting subsystems, this point of view displays remarkable properties and many articles...
were devoted to this subject. In particular, different authors studied the relations between
the BFN approach and the theory of deformation quantization \[39, 40, 41, 42, 43, 44]\. N. C.
Dias and J. N. Prata study in their article \[44\] the relations between quasi-distributions (the
phase space representation of quantum states in the deformation quantization approach)
and the Bohm distributions in phase space. As we will see in the appendix, our point of
view allows us to find a similar relation.

Now, let us situate our article. In a previous work \[22\] we have already explored to
what extent the KvN approach can be used to rebuild quantum mechanics using physical
arguments. The starting point of our study was not the abrupt data of the KvN formalism,
but rather a physical reasoning about the necessary stochastic properties of classical observ-
able. Namely we followed M. Born’s remark \[45\] by noticing that the true mathematical
representation of any physical measurement is an expectation value associated with some
probability distribution, because a real result of a physical experiment always contains some
uncertainties. Therefore, physical continuous observables are stochastic by nature. This
important remark is also the starting point of E. Prugovečki’s monograph \[15\]. Then the
picture of classical mechanics based on a complete determinism is an extrapolation of the real
information obtained from experiments. But if this picture cannot be proved from classical
measurements, it cannot be invalidated any more, in particular because classical equations
are compatible with (or based on) this picture. This explains the difficulty to introduce, in a
consistent way, intrinsic stochastic properties into the classical framework. Nevertheless this
stage is necessary, if we want to find a continuous transition from the classical framework to
the quantum one. The KvN approach is one solution. It associates to each particle a square
integrable function \(\psi(\vec{p}, \vec{q})\) on phase space such as the probability distribution \(\rho = |\psi(\vec{p}, \vec{q})|^2\)
always exhibits uncertainties. We mean that the limit \(\rho \to \delta(\vec{p} - \vec{p}_0, \vec{q} - \vec{q}_0)\) is not allowed,
because there is no \(\psi\) such as \(\psi = \sqrt{\delta}\). Then pure classical states (\(\delta\) distributions) can be
approximated, but never reached. Then this formalism is a better representation of the real
information obtained from measurements.

In this article we want to explore another possibility to introduce intrinsic stochastic
properties into the classical framework, and also a complete different form of continuous
transition toward quantum mechanics. Some parts of the article possess common technical
features with the BFN approach, but the leading ideas are very different, since we assume
from the beginning that physical observables are intrinsically stochastic at each given time.
Then all interpretations in terms of trajectories (deduced from our field equations) can only have a mathematical meaning.

One of the main ideas is based on the analysis of the physical role of the action minimization principle. This principle allows one to obtain dynamical equations in a very large variety of situations. But this does not mean that all physical equations of evolution are deductible from some action minimization.

Particularly, the equations of evolution of a pure state \((\vec{p}_0(t), \vec{q}_0(t))\) can be deduced from an action minimization, but we show in the section III A that the Liouville equation cannot be obtained from a minimization principle. Nevertheless, any solution of the Liouville equation is a linear superposition of pure states, where the coefficients are the probabilities (due to the observer). This suggests that the action minimization principle only works for intrinsic dynamical equations that do not include probabilities due to some observer. This leads to the idea that the solutions of the Liouville equation, only parametrized by quantities that verify some action minimization, are intrinsic dynamical solutions, that is intrinsic stochastic distributions.

This allows us to define the \(q\)-stochastic pure states, intrinsic stochastic distributions, analogous of the usual classical pure states. These distributions are classical Bohm distributions usually interpreted as the classical limit of quantum pure states \([44, 46]\). They are parametrized by two fields, a probability distribution \(n(\vec{q}, t)\) and an action field \(S(\vec{q}, t)\), such as the equations verified by \(n\) and \(S\) are solutions of the variational principle \(\delta \int L d^3q dt = 0\), where \(L\) is the Lagrangian. [Of course these solutions can be classically interpreted in terms of trajectories as in the BFN approach, but in our point of view these trajectories only possess a mathematical meaning.] The properties of these states are analyzed in the section III C.

After the introduction of intrinsic stochastic properties into classical mechanics, we analyze to what extent we can perform a transition to quantum mechanics [Sec. IV]. This transition is obtained by a new hypothesis, namely we must assume that the field \(S\) exhibits intrinsic stochastic properties. Then the classical Lagrangian \(L\) must be replaced by some averaged Lagrangian \(L_m\).

We show in the section IV how \(L_m\) can be obtained, and how it leads to the Schrödinger equation. This derivation of the Schrödinger equation is technically close to the one already published by M. J. W. Hall [47] and M. Reginatto [47, 48, 49] (based on the Fisher informa-
tion), but our arguments are very different. This procedure also is different from the ones proposed by M. Davidson [50] or by G. Kaniadakis [51].

Then we analyze in the section VI how all the quantum framework can completely be rebuilt from the previous results. In particular, we show that quantum axioms appear as abstract generalizations of classical calculations.

II. CLASSICAL MECHANICS IN PHASE SPACE

We recall in few lines the framework of classical mechanics [52, 53].

A. Phase space and classical dynamics

Phase space is the set of pairs \((\vec{p}, \vec{q})\) of momenta and positions, where \((\vec{p}, \vec{q})\) represents the (pure) state of the system. Physical observables are functions \(f(\vec{p}, \vec{q})\) on phase space. If we call \(M\) the \(\mathbb{R}^3\) space manifold, phase space is the cotangent bundle \(TM^*\). It possesses a natural geometry (namely a symplectic geometry) that allows us to define the Poisson brackets (PB), \(\{f, g\}\) of two functions by

\[
\{f, g\} = \partial_{\vec{p}} f \partial_{\vec{q}} g - \partial_{\vec{q}} g \partial_{\vec{p}} f.
\] (1)

Dynamics on phase space is defined by the Hamiltonian equations

\[
\frac{d}{dt} \vec{q} = \partial_{\vec{p}} H(\vec{q}, \vec{p}, t),
\] (2a)

\[
\frac{d}{dt} \vec{p} = -\partial_{\vec{q}} H(\vec{q}, \vec{p}, t),
\] (2b)

where \(H\) is the Hamiltonian (eventually time dependent).

B. Dynamics and statistics

These equations (2) correspond to the ideal case of a particle perfectly localized in phase space and we can represent this situation by the probability distribution \(\rho(\vec{p}, \vec{q}, t) = \delta(\vec{p} - \vec{p}_0(t)) \delta(\vec{q} - \vec{q}_0(t))\). If we build a general distribution \(\rho\) as superposition of \(\delta\) by defining \(\rho = \sum_i P_i \delta(\vec{p}_i(t), \vec{q}_i(t))\), we find that \(\rho\) verifies the Liouville equation:

\[
\frac{\partial \rho}{\partial t} = -\{H, \rho\}.
\] (3)
We notice for the following that this equation can be written as

\[ \partial_t \rho + \partial_\vec{q} (\rho \partial_\vec{p} H) - \partial_\vec{p} (\rho \partial_\vec{q} H) = 0 \]  

(4)

So we say classically that Eq. (3) describes the evolution of any probability distribution \( \rho \).

Starting from a distribution \( \rho \) that verifies the equation (3), we can look at the evolution of the expectation value \( \langle f \rangle_t \) of an observable \( f(\vec{p}, \vec{q}, t) \) defined as

\[ \langle f \rangle_t = \int_{\mathbb{R}^6} f(\vec{p}, \vec{q}, t) \rho(\vec{q}, \vec{p}, t) d^3 \vec{p} d^3 \vec{q}. \]  

(5)

We find:

\[ \frac{d}{dt} \langle f \rangle_t = \langle \frac{\partial f}{\partial t} \rangle_t + \langle \{H, f\} \rangle_t. \]  

(6)

Applied to the special case of the two fundamental observables \( \vec{p} \) and \( \vec{q} \), Eq. (6) gives:

\[ \frac{d}{dt} \langle \vec{q} \rangle_t = \langle \partial_\vec{p} H \rangle_t, \]  

(7a)

\[ \frac{d}{dt} \langle \vec{p} \rangle_t = - \langle \partial_\vec{q} H \rangle_t. \]  

(7b)

III. DEFINITION OF STOCHASTIC CLASSICAL PURE STATES

A. Action minimization and the Liouville equation

As indicated above, classical pure states are defined as points \( (\vec{p}_0, \vec{q}_0) \) of phase space corresponding to the “\( \delta \)” distribution \( \rho_0(\vec{p}, \vec{q}) = \delta (\vec{p} - \vec{p}_0) \delta (\vec{q} - \vec{q}_0) \).

We remark that the equations of evolution (2) for such a pure state (equivalent to the Liouville equation for \( \rho_0 \)) are given by an action minimization \( (\delta \int L dt = 0) \) applied to the Lagrangian \( L \) depending on the parametric fields \( \vec{q}_0(t) \) and \( \vec{p}_0(t) \)

\[ L \left( \vec{q}_0, \frac{d\vec{q}_0}{dt}, \vec{p}_0, \frac{d\vec{p}_0}{dt}, t \right) = \vec{p}_0 \cdot \frac{d\vec{q}_0}{dt} - H(\vec{q}_0, \vec{p}_0, t). \]  

(8)

In other words, usual pure states are special cases of parametrized distributions \( \rho \), solutions of the Liouville equation, such that the dynamical equations verified by the parameters are the consequences of some action minimization.
So we can wonder if other parametrized solutions of the Liouville equation exhibiting the same property exist.

We must first verify that the Liouville equation itself cannot be obtained from some action minimization [action based on a Lagrangian only depending on the field \( \rho \) and on the coordinates].

To prove that, let us define a Lagrangian \( \mathcal{L}(\rho, \partial_\alpha \rho, \vec{p}, \vec{q}, t) \) [with \( \alpha = \vec{p}, \vec{q} \) or \( t \)] associated with the minimization condition \( \delta \int \mathcal{L} d^3 \vec{p} d^3 \vec{q} dt = 0 \). Then the equation in \( \rho \) is given by

\[
\partial_t \frac{\partial \mathcal{L}}{\partial (\partial_t \rho)} + \partial_{\vec{p}} \frac{\partial \mathcal{L}}{\partial (\partial_{\vec{p}} \rho)} + \partial_{\vec{q}} \frac{\partial \mathcal{L}}{\partial (\partial_{\vec{q}} \rho)} = \frac{\partial \mathcal{L}}{\partial \rho}.
\] (9)

If this equation (9) reduces to the Liouville equation (3), this implies that Eq. (9) must not contain second derivatives of \( \rho \) in the variables \( t, \vec{p} \) or \( \vec{q} \). Then we deduce

\[
\frac{\partial \mathcal{L}}{\partial (\partial_t \rho)} = f(\rho, \vec{p}, \vec{q}, t),
\] (10a)

\[
\frac{\partial \mathcal{L}}{\partial (\partial_{\vec{p}} \rho)} = g(\rho, \vec{p}, \vec{q}, t),
\] (10b)

\[
\frac{\partial \mathcal{L}}{\partial (\partial_{\vec{q}} \rho)} = h(\rho, \vec{p}, \vec{q}, t).
\] (10c)

So we have

\[
\mathcal{L} = F(\rho, \vec{p}, \vec{q}, t) + f \partial_t \rho + g \partial_{\vec{p}} \rho + h \partial_{\vec{q}} \rho.
\] (11)

With this Lagrangian, the equation (9) becomes

\[
\partial_t f(Y) + \partial_{\vec{p}} g^i(Y) + \partial_{\vec{q}} h^i(Y) = \partial_\rho F(Y),
\] (12)

where \( Y = (\rho, \vec{p}, \vec{q}, t) \).

Obviously this equation is not a differential equation in \( \rho \), so the Liouville equation cannot be obtained from some action minimization based on a Lagrangian only depending on \( \rho \) and the coordinates. [In fact we can obtain the Liouville equation from a Lagrangian that contains two unknown fields (\( \rho \) and another one), but there is no reason to introduce some supplementary field.]
1. Conclusion

We have proved that only subsets of solutions of the Liouville equation are the consequences of some action minimization. This result is important because a general solution of the Liouville equation can be written as a convex linear superposition of pure states where the coefficients are interpreted as probabilities only due to the observer (the uncertainties are not intrinsic).

The fact that pure states are derived from some action minimization, while a general solution of the Liouville equation cannot be obtained by this procedure seems to show that the action minimization principle discriminates the solutions that are intrinsically dynamical from those that contain some (or too many) external probabilities (due to the observer).

If we follow M. Born [45] and E. Prugovečki [15] by assuming that dynamical variables are intrinsically stochastic [exact values must be seen as a mathematical extrapolation] we see that the action minimization can be naturally lifted up into a general procedure to specify the solutions of the Liouville equation that are dynamical and intrinsically stochastic (probabilities not due to the observer). We will call them stochastic pure states. If these distributions exist, they constitute a new starting point (similar to usual pure states) to rebuild a version of classical mechanics that includes the idea of intrinsic stochastic properties.

This is one of the main ideas of this article, and this explains why we are looking for the solutions of the Liouville equation that verify this condition.

B. Solutions of the Liouville equation derived from an action minimization

In the remainder we focus on two families of solutions (the proofs are given below) defined as follows.

The first family (I) depends on the fields \( n(\vec{q},t) \) and \( S(\vec{q},t) \) and corresponds to classical Bohm distributions, usually interpreted as the classical limit of quantum pure states [44, 46]:

\[
\rho_I(\vec{p},\vec{q},t) = n(\vec{q},t)\delta(\vec{p} - \partial_{\vec{q}}S(\vec{q},t)).
\]  

The second family (II) depends on the fields \( n(\vec{p},t) \) and \( S(\vec{p},t) \):

\[
\rho_{II}(\vec{p},\vec{q},t) = n(\vec{p},t)\delta(\vec{q} - \partial_{\vec{p}}S(\vec{p},t)).
\]
a. **Remark**  At first sight, these families can be unified and generalized, by introducing some canonical transformation. But if we want to parametrize distributions with fields that possess a direct physical meaning, these fields must be defined on spaces that possess a material physical realization (spaces where the Galilei group directly acts). Then, only the first family (I) based on spacetime really is fundamental. The second family (II) is its canonical conjugate. Other families obtained after some canonical transformation that mixes $\vec{p}$ and $\vec{q}$ must be seen as purely mathematical solutions with no concrete physical interest.

So the remainder of the article is essentially based on the analysis of the solutions (I). Solutions (II) are analyzed at the end of the article.

1. **Fields equations**

We begin first by the equations verified by the fields $n(\vec{q},t)$ and $S(\vec{q},t)$. If we write the Liouville equation (4) with the formula (13), we obtain

$$
\partial_t n\delta - n\partial^2_{\vec{q}}S\partial_{\vec{p}}\delta + \partial_{\vec{q}}(n\partial_{\vec{p}}H\delta) = \partial_{\vec{p}}(n\partial_{\vec{q}}H\delta),
$$

(15)

where $\delta = \delta(\vec{p} - \partial_{\vec{q}}S)$.

Using the well-known identity verified by $\delta$ distributions $f(\vec{p},\vec{q})\delta(\vec{p} - \partial_{\vec{q}}S) = f(\partial_{\vec{q}}S,\vec{q})\delta(\vec{p} - \partial_{\vec{q}}S)$, and after the development of derivatives, we obtain

$$
\partial_t n\delta - n\partial^2_{\vec{q}}S\partial_{\vec{p}}\delta + \partial_{\vec{q}}(n\partial_{\vec{p}}H(\partial_{\vec{q}}S,\vec{q},t))\delta =
$$

$$
n\partial_{\vec{p}}H(\partial_{\vec{q}}S,\vec{q},t)\partial^2_{\vec{q},i}S\partial_{\vec{p},i}\delta + n\partial_{\vec{q}}H(\partial_{\vec{q}}S,\vec{q},t)\partial_{\vec{p}}\delta,
$$

(16)

where the summation on $i$ and $j$ is implicit.

By collecting the terms in $\delta$ we obtain the equation for $n(\vec{q},t)$ that expresses the local conservation law

$$
\partial_t n + \partial_{\vec{q}}(n\partial_{\vec{p}}H(\partial_{\vec{q}}S,\vec{q},t))) = 0.
$$

(17)

By collecting the terms in $\partial_{\vec{p}}\delta$, we obtain the equation verified by $S(\vec{q},t)$

$$
\partial_{\vec{q}}(\partial_t S + H(\partial_{\vec{q}}S,\vec{q},t)) = 0.
$$

(18)
This equation can be simplified (up to a function of \( t \)), and we obtain the Hamilton-Jacobi equation
\[
\partial_t S + H(\partial_q S, \vec{q}, t) = 0. \tag{19}
\]

On the other hand we can verify that the equations (17) and (19) are deduced from the variational principle \( \delta \int L d^3 \vec{q} dt = 0 \) where \( L \) is defined as
\[
L = n(\partial_t S + H(\partial_q S, \vec{q}, t)). \tag{20}
\]

a. Remark The definition of \( \rho_I \) shows that the field \( n \) is a probability distribution on the ordinary space while \( S \) is homogeneous to an action, then the integral \( \int L d^3 \vec{q} dt \) also is homogeneous to an action. In addition, the classical Lagrangian \( L \) is often used to describe a flow of classical particles, but in our approach it describes the stochastic properties of a unique object, as we will see below.

C. \textit{q}-stochastic pure states

As previously mentioned, the distributions of type (I) can be interpreted as the classical limit of quantum pure states \([44, 46]\). We also notice that the classical pure state \( \rho(\vec{p}, \vec{q}, t) = \delta(\vec{p} - \vec{p}_0(t)) \delta(\vec{q} - \vec{q}_0(t)) \) correspond to the limit \( n(\vec{q}, t) \to \delta(\vec{q} - \vec{q}_0(t)) \). Then pure states are limit cases of solutions of type (I). Moreover, if \( \rho(\vec{p}, \vec{q}, t) \) is equal to \( n(\vec{q}, t) \delta(\vec{p} - \partial_q S) \), then for each value of \( \vec{q} \) only one value of \( \vec{p} \) [equal to \( \partial_q S(\vec{q}, t) \)] is possible. So (as in the BFN approach) the function \( S \) specifies a family of classical trajectories, solutions of the Hamiltonian equations. These trajectories verify
\[
\frac{d\vec{q}}{dt} = \partial_\vec{p} H(\partial_q S, \vec{q}, t), \tag{21a}
\]
\[
\vec{p} = \partial_q S(\vec{q}, t). \tag{21b}
\]

The function \( n \) that represents the probability distribution on the \( q \)-space, also specifies the probability law of the trajectories \([21]\) [since the data of \( \vec{q} \) is sufficient to specify \( \vec{p} \)]. But these trajectories only have a mathematical meaning in our approach, since the position of the particle is assumed to be stochastic by nature.

So the distributions \( \rho_I(\vec{p}, \vec{q}, t) = n(\vec{q}, t) \delta(\vec{p} - \partial_q S) \) represent some extension of pure states where only the condition of exact localization in the ordinary space is relaxed, but a sharp
localization is always possible. Moreover, as usual pure states, the solutions (1) are the consequences of an action minimization. We call the solutions of the family (1) \textit{q-stochastic pure states}.

1. Conclusion

Following the idea developed in the section III A 1, we assume in the remainder that these \textit{q}-stochastic pure states are intrinsic (classical) stochastic states, stochastic analogous of pure states. So a particle must be associated with a \textit{q}-stochastic pure state at each given time.

On the other hand, convex linear superpositions of these \textit{q}-stochastic pure states always give a general probability distribution that verifies the Liouville equation.

We deduce that classical statistical mechanics can be rebuilt starting from the Lagrangian (20) and the definition of the \textit{q}-stochastic pure states on phase space. But the usual formula for pure states must be modified, and this leads to the following equations.

D. Marginal laws and expectation values for \textit{q}-stochastic pure states

1. The marginal laws

For a probability distribution \(\rho\) on phase space, the marginal laws that gives the distributions \(\mu(\vec{q}, t)\) on the \(q\)-space and \(\nu(\vec{p}, t)\) on the \(p\)-space are given by

\[
\mu(\vec{q}, t) = \int_{\mathbb{R}^3} \rho(\vec{p}, \vec{q}, t) d^3\vec{p}; \quad \nu(\vec{p}, t) = \int_{\mathbb{R}^3} \rho(\vec{p}, \vec{q}, t) d^3\vec{q}.
\] (22)

For a \textit{q}-stochastic pure state we obtain

\[
\mu(\vec{q}, t) = n(\vec{q}, t); \quad \nu(\vec{p}, t) = \int_{\mathbb{R}^3} \delta(\vec{p} - \partial_{\vec{q}} S(\vec{q}, t)) n(\vec{q}, t) d^3\vec{q}.
\] (23)

2. The expectation values

For any observable \(f(\vec{p}, \vec{q})\) the expectation value \(\langle f \rangle_t\) is given by Eq. (5), then

\[
\langle f \rangle_t = \int_{\mathbb{R}^3} f(\partial_{\vec{q}} S(\vec{q}, t), \vec{q}) n(\vec{q}, t) d^3\vec{q}.
\] (24)
The normalization condition becomes

\[ 1 = \langle 1 \rangle_t = \int_{\mathbb{R}^3} n(q, t) d^3q. \]  

(25)

The expectation values of position and momentum are

\[ \langle \vec{q} \rangle_t = \int_{\mathbb{R}^3} \vec{q} n(q, t) d^3q, \]  

(26a)

\[ \langle \vec{p} \rangle_t = \int_{\mathbb{R}^3} \partial \vec{q} S(q, t) n(q, t) d^3q. \]  

(26b)

The expectation values of energy and angular momentum are given by

\[ \langle H \rangle_t = \int_{\mathbb{R}^3} H(\partial \vec{q} S(q, t), \vec{q}) n(q, t) d^3q, \]  

(27a)

\[ \langle \vec{l} \rangle = \langle \vec{q} \wedge \vec{p} \rangle_t = \int_{\mathbb{R}^3} \vec{q} \wedge \partial \vec{q} S(q, t) n(q, t) d^3q. \]  

(27b)

The evolution of the expectation values of \( \vec{p} \) and \( \vec{q} \) follows Eqs. (28), then

\[ \frac{d}{dt} \langle \vec{q} \rangle_t = \int_{\mathbb{R}^3} \partial \vec{p} H(\partial \vec{q} S(q, t), \vec{q}) n(q, t) d^3q, \]  

(28a)

\[ \frac{d}{dt} \langle \vec{p} \rangle_t = -\int_{\mathbb{R}^3} \partial \vec{q} H(\partial \vec{q} S(q, t), \vec{q}) n(q, t) d^3q. \]  

(28b)

IV. THE TRANSITION FROM CLASSICAL TO QUANTUM MECHANICS

A. The hypothesis

In the previous definition of \( q \)-stochastic pure states, stochastic properties only appear through the probability distribution \( n(q, t) \). The stochastic properties of \( \vec{p} \) are only due to those of \( \vec{q} \), because the well-defined function \( S(q, t) \) specifies \( \vec{p} \) for each \( q \).

But if we assume that \( S \) is an intrinsic stochastic field (for each given time), what are the consequences? This is the starting point of the transition from classical to quantum mechanics.

The field \( S \) can be essentially randomized starting from two different hypothesis. The first hypothesis consists in assuming that \( S \) is a random field on the set of solutions of the
Hamilton-Jacobi equation (equation derived from the Lagrangian). The second hypothesis consists in assuming that $S$ is a complete random field that does not follow any law.

The first hypothesis is not really consistent because the Hamilton-Jacobi equation is derived from the Lagrangian $L$ defined in Eq. (20). So, if we assume that $S$ is an intrinsic random field, then the values of $L$ also are randomized, and there is no reason to assume that the Hamilton-Jacobi equation is always valid. So, only the second hypothesis appears satisfactory, and we adopt this point of view in the remainder.

B. Consequences on the classical idea of trajectory

We have seen that the $q$-stochastic pure state $(n, S)$ is associated with a family of trajectories given by the equations (21). If we use the standard Hamiltonian $H = \frac{1}{2m}\vec{p}^2 + V(\vec{q})$, this leads to the equations

$$\frac{d\vec{q}}{dt} = \frac{1}{m}\partial_q S(\vec{q}, t), \quad (29a)$$

$$\vec{p} = \partial_q S(\vec{q}, t). \quad (29b)$$

If we assume that $S$ is a complete random field, and if we give a physical meaning to the trajectories (this is not our case) then the previous equations imply that all trajectories are possible (not restricted to the Hamiltonian ones). Moreover for each given position $\vec{q}$, the momentum $\vec{p}$ becomes a complete random variable. So, on one hand we recover the intuitive idea used by Feynmann to introduce the Feynmann path integral, and on the other hand we have some kind of Heisenberg uncertainty principle.

C. The quantization

If $S(\vec{q}, t)$ is a random field at each given time, then only the expectation values of $S$ (or functions of $S$) have a physical meaning. So the physical field becomes the averaged field $S_m(\vec{q}, t) = \langle S(\vec{q}, t) \rangle$. The Lagrangian $L = n (\partial_t S + H(\partial_q S, \vec{q}, t))$ is a physical quantity, so it must be replaced by $L_m = \langle L \rangle$. It is natural to say that the averaged Lagrangian $L_m$ only depends on the fields $n$ and $S_m$, because we assume that the stochastic properties are intrinsic (they are not due to the action of some unknown external dynamical field, or due to some observer effect).
Moreover, following our leading idea of section III A 1, if the stochastic properties of $S$ are intrinsic, the action minimization principle must always be valid, but now it must be applied to the Lagrangian $L_m$ (then we assume implicitly that $L_m$ depends on the fields $n$ and $S_m$ through their values and their first derivatives). The equations deduced from the new Lagrangian are the quantized equations.

M. J. W. Hall and M. Reginatto \[47, 48, 49\] obtained a derivation of the Schrödinger equation from a procedure technically close to this one.

V. THE SCHRODINGER EQUATION

The principle of quantization previously developed depends explicitly on the form of the Hamiltonian. So we assume in the remainder that $H$ is the classical Hamiltonian

$$H = \frac{1}{2m} \left( \vec{p} - e \vec{A}(\vec{q}, t) \right)^2 + V(\vec{q}, t). \quad (30)$$

where $m$ is the mass and $e$ the charge of the particle.

The Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = n \left( \partial_t S + \frac{1}{2m} \left( \partial_{\vec{q}} S - e \vec{A}(\vec{q}, t) \right)^2 + V(\vec{q}, t) \right). \quad (31)$$

We notice that $\mathcal{L}$ is gauge invariant under the transformation

$$S \to S + e \varphi, \vec{A} \to \vec{A} + \partial_{\vec{q}} \varphi, V \to V - e \partial_t \varphi \quad (32)$$

A. The new Lagrangian $L_m$

The Lagrangian $L_m = \langle \mathcal{L} \rangle$ introduced in the previous section becomes

$$L_m = n \left( \partial_t S_m + \frac{1}{2m} \left\langle \left( \partial_{\vec{q}} S - e \vec{A}(\vec{q}, t) \right)^2 \right\rangle + V(\vec{q}, t) \right), \quad (33)$$

where $S_m = \langle S \rangle$.

Moreover

$$\left\langle \left( \partial_{\vec{q}} S - e \vec{A}(\vec{q}, t) \right)^2 \right\rangle = \left( \partial_{\vec{q}} S_m - e \vec{A}(\vec{q}, t) \right)^2 + \Sigma^2, \quad (34)$$

where $\Sigma = \sqrt{\langle (\partial_{\vec{q}} S - \partial_{\vec{q}} S_m)^2 \rangle}$ is the standard uncertainty.

We deduce that

$$L_m = \mathcal{L}(n, S_m) + \frac{n}{2m} \Sigma^2, \quad (35)$$

and the new Lagrangian $L_m$ only is indeterminate through the unknown uncertainty $\Sigma$. 

14
B. Obtaining $\Sigma$

We list first the natural hypothesis that allows us to restrict the possible forms of $\Sigma$.

(i) The dimensional analysis shows that $\Sigma$ is homogeneous to an action divided by a length.

(ii) If we assume that the Lagrangian $L_m$ only depends on $n, S_m$, their first derivatives and the spacetime coordinates, then $\Sigma$ has the same dependences. Since $L$ only depends on the coordinates through the external fields $A$ and $V$, it is natural to assume that this property is preserved in $L_m$. But $\Sigma$ represents intrinsic stochastic effects (not due to external fields), consequently it cannot depend on the external fields, and then it does not depend on the coordinates. So $\Sigma$ is a function of $n, S_m$ and their first derivatives.

(iii) Since $\Sigma$ represents the effect of stochastic properties of $S$ at each given time, it is logical to assume that $\Sigma$ does not depend on time derivatives of $n$ and $S$.

(iv) The Lagrangian $L$ is invariant under gauge transformations, and this symmetry is related to the conservation of particles, so we assume that the gauge invariance is unbroken, and then $\Sigma$ must be gauge invariant.

These simple remarks are sufficient to obtain the “plausible” form of $\Sigma$.

From (ii) and (iii), $\Sigma$ only depends on $(n, S_m, \partial_q n, \partial_q S_m)$. But from (iv) we know that $\Sigma$ must be invariant under the transformation $S_m \rightarrow S_m + e\varphi$. These two conditions are verified only if $\Sigma$ does not depend on $S_m$. Then $\Sigma$ is a function of $n$ and $\partial_q n$.

On the other hand, from (i) $\Sigma$ is homogeneous to an action divided by a length, but it is impossible to build a quantity with this dimension only using $n$ and $\partial_q n$, because $n$ is homogeneous to $1/L^3$ and $\partial_q n$ is homogeneous to $1/L^4$ (where $L$ is a length). So we have to introduce some new constants.

Since we are looking for a quantity homogeneous to $a/L$ where “$a$” is an action, the simplest idea is to introduce some unit of action “$a$” and eventually some other undimensional (real) constant that we call “$\beta$”.

It is very easy to check that the simplest expression of $\Sigma^2$ verifying these constraints is given by

$$\Sigma^2 = a^2 \left[ \left( \frac{\partial_q n}{n} \right)^2 + \beta^2 n^{2/3} \right]. \quad (36)$$
As we will see below the new constant “a” must be adjusted to $\hbar/2$, where $\hbar$ is the usual reduced Planck constant.

1. Conclusion

The uncertainty $\Sigma$ is given by the formula

$$\Sigma^2 = \frac{\hbar^2}{4} \left[ \left( \frac{\partial q_i n}{n} \right)^2 + \beta^2 n^{2/3} \right], \quad (37)$$

where $\beta$ is some undimensional parameter.

We point out that this procedure gives a classical stochastic meaning to the Planck constant $\hbar$ (as in the BFN approach), since this constant only is related to some classical uncertainty. Moreover the limit $\hbar \to 0$ means that the uncertainty $\Sigma$ vanishes and then $S$ cannot be any more a stochastic field.

On the other hand, we remark that our knowledge of the stochastic properties of $S$ are limited to the following expectation values

$$\langle S(\vec{q}, t) \rangle = S_m(\vec{q}, t), \quad (38a)$$

$$\langle (\partial_q S)^2 \rangle = (\partial_q S_m)^2 + \frac{1}{4} \hbar^2 \left[ \left( \frac{\partial q_i n}{n} \right)^2 + \beta^2 n^{2/3} \right]. \quad (38b)$$

Nevertheless, from the knowledge of $\langle (\partial_q S)^2 \rangle$, we can reasonably assume that the expectation value of $\partial_{q_i} S \partial_{q_j} S$ verifies

$$\langle \partial_{q_i} S \partial_{q_i} S \rangle = \partial_{q_i} S_m \partial_{q_i} S_m + \frac{1}{4} \hbar^2 \left[ \frac{\partial q_i n \partial q_i n}{n^2} + \frac{1}{3} \delta_{ij} \beta^2 n^{2/3} \right]. \quad (39)$$

We suppose that this condition is verified in the remainder.

As we will see below in the section VII the partial knowledge of the $S$ stochastic properties selects the classical calculations that can be performed, and this is a key point to show that we need some new axiom.
C. The Schrödinger equation

We have the following Lagrangian $L_m$

$$L_m = n \left( \partial_t S_m + \frac{1}{2m} \left( \partial_q S_m - e \vec{A} \right)^2 + \frac{\Sigma^2}{2m} + V \right). \quad (40)$$

We introduce the complex wave function $\psi(\vec{q}, t)$ defined as

$$\psi(\vec{q}, t) = \sqrt{n} \exp \left( \frac{i}{\hbar} S_m \right). \quad (41)$$

We find after some algebra

$$L_m = \frac{i\hbar}{2} \left( \psi \partial_t \bar{\psi} - \bar{\psi} \partial_t \psi \right) + \frac{\hbar^2}{2m} D_q \bar{\psi}. D_q \psi$$

$$+ \frac{\beta^2 \hbar^2}{8m} |\psi|^{10/3} + V |\psi|^2, \quad (42)$$

where $\bar{\psi}$ represents the complex conjugate and $D_q \psi = \partial_q \psi + i(e/\hbar) \vec{A} \psi$.

The minimization $\delta \int L_m d^3 \vec{q} dt = 0$ can be done on the fields $n$ and $S_m$, or equivalently on $\psi$ and $\bar{\psi}$. This leads to the nonlinear Schrödinger equation

$$i\hbar \partial_t \psi = \frac{1}{2m} \left( -i\hbar \partial_q - e \vec{A} \right)^2 \psi + \frac{5\beta^2 \hbar^2}{24m} |\psi|^{4/3} \psi + V \psi. \quad (43)$$

If we assume that $\beta = 0$, we recover the usual Schrödinger equation. In the remainder, we restrict ourselves to this special case.

D. Conclusion

This procedure allows us to recover the Schrödinger equation from classical stochastic arguments and the action minimization principle. Moreover the semi-classical definition of the complex wave function appears naturally, but its meaning is related to averaged fields.

In addition, $\psi$ is by construction the new version of the $q$-stochastic pure state. Then if a particle is represented by a $q$-stochastic pure state in classical mechanics, it must be represented by $\psi$ in the quantum theory [but the wave function is just a convenient mathematical representation of the pair $(n, S_m)$].

We also notice that Bohm trajectories that are built from the action field $S_m$ can only be mathematical objects in our approach (since $S_m$ is an averaged field).
VI. THE QUANTUM FRAMEWORK

In this section we want to show how quantum “axioms” can be recovered in a very natural way. From the previous section we already know that a particle must be represented by a wave function that verifies the Schrödinger equation. Then it remains to study the relations linking the wave function, the (classical) observables and the statistical techniques.

A. Expectation values of the usual observables

The expectation values for \( q \)-stochastic pure states defined in the section III D were dependent on the fields \( n \) and \( S \). But if we assume that \( S \) is a stochastic field, these expectation values must be also taken on \( S \). This leads to the following equations for the usual observables.

The normalization condition [Eq. (25)] is unchanged

\[
1 = \int_{\mathbb{R}^3} n(\vec{q}, t) d^3\vec{q} = \int_{\mathbb{R}^3} |\psi|^2 d^3\vec{q}.
\]

(44)

Then a particle must be represented by a normalized wave function.

The expectation value of position [Eq. (26a)] gives

\[
\langle \vec{q} \rangle_t = \int_{\mathbb{R}^3} \vec{q} n(\vec{q}, t) d^3\vec{q} = \int_{\mathbb{R}^3} \vec{q} |\psi|^2 d^3\vec{q}.
\]

(45)

This expression can be generalized to any function of position

\[
\langle f(\vec{q}) \rangle_t = \int_{\mathbb{R}^3} f(\vec{q}) n(\vec{q}, t) d^3\vec{q} = \int_{\mathbb{R}^3} f(\vec{q}) |\psi|^2 d^3\vec{q}.
\]

(46)

The expectation value of momentum [Eq. (26b)] becomes (taking into account vanishing integrals)

\[
\langle \vec{p} \rangle_t = \int_{\mathbb{R}^3} \partial_q S_m n(\vec{q}, t) d^3\vec{q} = -i\hbar \int_{\mathbb{R}^3} \bar{\psi} \partial_q \psi d^3\vec{q}.
\]

(47)

We also find (taking into account Eq. (39) with the condition \( \beta = 0 \))

\[
\langle p_i^2 \rangle_t = \int_{\mathbb{R}^3} \langle (\partial_q S)^2 \rangle n(\vec{q}, t) d^3\vec{q} = \hbar^2 \int_{\mathbb{R}^3} \partial_q \bar{\psi} \partial_q \psi d^3\vec{q}.
\]

(48)

The expectation value for the energy [Eq. (27a)] gives (with the condition \( \beta = 0 \))

\[
\langle H \rangle_t = \int_{\mathbb{R}^3} \left[ \frac{\hbar^2}{2m} \vec{D}_q \bar{\psi} \cdot \vec{D}_q \psi + V(\vec{q}, t) |\psi|^2 \right] d^3\vec{q}.
\]

(49)
Finally, the expectation value for angular momentum [Eq. (27b)] becomes

\[
\langle \vec{l} \rangle_t = \int_{\mathbb{R}^3} \bar{q} \wedge \partial_q S_{m,n}(\bar{q}, t) d^3 \bar{q} = -i\hbar \int_{\mathbb{R}^3} \bar{\psi} \bar{q} \wedge \partial_q \psi d^3 \bar{q}.
\] (50)

So we recover the usual quantum formula for the main classical observables (that are in fact the generators of the Galilei group). But for the time being, we have not introduced any quantum axiom about observables.

\[\text{a. Remark} \] The previous calculations do not prove the identity of classical and quantum expectation values for any observable.

Because our approach only specifies the expectation values of \(S\) and \((\partial_q S)^2\), we cannot calculate expectation values with classical formula for observables that are more than quadratic in \(\vec{p}\). This means, for example, that we cannot calculate the uncertainty on energy. Facing this situation, two attitudes are possible:

(i) In a classical point of view, we must say that some important properties of \(S\) are unknown and then this theory is incomplete. So we have to add some new information about \(S\) by external means.

(ii) In a quantum point of view, we must believe in the efficiency of the action minimization principle and think that all the elements that we have, are exactly that we need. The only way out is to assume a breakdown of classical techniques. So we have to look for new consistent statistical definitions, compatible with the formula that have been already found, that allow us to calculate the classically undefined quantities. At first sight this point of view can appear very formal, but it works.

Before going further in our approach, we open a parenthesis to study the necessary conditions on the stochastic properties of \(S\) that can be deduced from the hypothesis of a complete simultaneous validity of quantum and classical expectation value calculations.

1. **Compatibility conditions between classical and quantum calculations**

We have seen in Eq. (23) the marginal laws \(\mu(q, t)\) and \(\nu(\bar{p}, t)\) for a \(q\)-stochastic pure state. If we take into account the stochastic properties of \(S\), we obtain the marginal laws
\( \mu_C(\vec{q}, t) \) and \( \nu_C(\vec{p}, t) \) that must be used for calculations with classical formula:

\[
\mu_C(\vec{q}, t) = n(\vec{q}, t);
\]
\[
\nu_C(\vec{p}, t) = \int_{\mathbb{R}^3} \langle \delta(\vec{p} - \partial_q S(\vec{q}, t)) \rangle n(\vec{q}, t) d^3 \vec{q}.
\]  
(51)

On the other hand, for a given wave function \( \psi \), we can find the distribution laws \( \mu_Q(\vec{q}, t) \) and \( \nu_Q(\vec{p}, t) \) deduced from quantum mechanics

\[
\mu_Q(\vec{q}, t) = |\psi(\vec{q}, t)|^2; \quad \nu_Q(\vec{p}, t) = \left| \hat{\psi}(\vec{p}, t) \right|^2,
\]  
(52)

where \( \hat{\psi} \) is the Fourier transform of \( \psi \) defined as

\[
\hat{\psi}(\vec{p}, t) = (2\pi \hbar)^{-3/2} \int_{\mathbb{R}^3} \exp(-i\vec{\xi} \cdot \vec{q}) \psi(\vec{q}, t) d^3 \vec{q}.
\]  
(53)

If we take into account the definition of the wave function [Eq. 111], and if we want the classical and quantum distributions to be identical, we must have \( \mu_C = \mu_Q \) and \( \nu_C = \nu_Q \). The condition \( \mu_C = \mu_Q \) is verified since \( \mu_C = \mu_Q = n \), then it remains the compatibility condition \( \nu_C = \nu_Q \) that gives

\[
\int_{\mathbb{R}^3} \langle \delta(\vec{p} - \partial_q S) \rangle |\psi(\vec{q}, t)|^2 d^3 \vec{q} = \left| \hat{\psi}(\vec{p}, t) \right|^2.
\]  
(54)

This condition is a very strong statement about the stochastic properties of \( S \). We analyze its consequences in the following lines.

First of all, we can take the Fourier transform of the previous equation, and we obtain

\[
\int_{\mathbb{R}^3} \langle e^{-i\vec{\xi} \cdot \partial_q S} \rangle |\psi|^2 d^3 \vec{q} = \int_{\mathbb{R}^3} e^{-i\vec{\xi} \cdot \vec{p}} \left| \hat{\psi} \right|^2 d^3 \vec{p},
\]  
(55)

where the vector \( \vec{\xi} \) is a mathematical parameter (independent of \( \hbar \)).

Then, the right-hand side of this equation can be transformed, leading to

\[
\int_{\mathbb{R}^3} e^{-i\vec{\xi} \cdot \vec{p}} \left| \hat{\psi} \right|^2 d^3 \vec{p} = \int_{\mathbb{R}^3} \hat{\psi} \left( \vec{q} + \frac{\hbar}{2} \vec{\xi} \right) \psi \left( \vec{q} - \frac{\hbar}{2} \vec{\xi} \right) d^3 \vec{q}.
\]  
(56)

We deduce that the expectation value \( \langle \exp \left(-i\vec{\xi} \cdot \partial_q S \right) \rangle \) must verify

\[
\int_{\mathbb{R}^3} \left\langle e^{-i\vec{\xi} \cdot \partial_q S} \right\rangle |\psi|^2 d^3 \vec{q} =
\int_{\mathbb{R}^3} \hat{\psi} \left( \vec{q} + \frac{\hbar}{2} \vec{\xi} \right) \psi \left( \vec{q} - \frac{\hbar}{2} \vec{\xi} \right) d^3 \vec{q}.
\]  
(57)
This last equation is the compatibility condition we are looking for.

At first sight, we can think that the solution is given by the relation

\[
\langle e^{-i\vec{\xi} \partial_{\vec{q}}} \rangle = \frac{1}{|\psi(q)|^2} \bar{\psi} \left( \vec{q} + \frac{\hbar}{2} \vec{\xi} \right) \psi \left( \vec{q} - \frac{\hbar}{2} \vec{\xi} \right),
\]

but this expression is not satisfactory, although it possesses some right properties. This particular point will be analyzed in the appendix, since it does not interfere with the remainder of the article. In particular, we will show that this expression introduces a connection between Bohm distributions and Wigner functions. This point is in relation with the article [44] already mentioned.

We now return to our approach, and we analyze to what extend the previous expressions of expectation values [Eqs. (45) to (50)] can imply the quantum axioms related to observables.

**B. The quantized observables**

We first introduce the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^3) \) equipped with the inner product \( \langle \psi | \varphi \rangle = \int_{\mathbb{R}^3} \bar{\psi}(\vec{q}) \varphi(\vec{q}) d^3 \vec{q} \), and we define the self-adjoint operators \( \vec{Q} \) and \( \vec{P} \) as usually

\[
\vec{Q}(\varphi)(\vec{q}) = \vec{q} \varphi(\vec{q}) \quad \text{and} \quad \vec{P}(\varphi)(\vec{q}) = -i\hbar \partial_{\vec{q}} \varphi.
\]

**1. The quantum observables of position and momentum**

The expectation values of \( \vec{p} \) and \( \vec{q} \) given by the equations (47) and (45) can be written with the Dirac formalism as

\[
\langle \vec{p} \rangle = \langle \psi | \vec{P} | \psi \rangle \quad \text{and} \quad \langle \vec{q} \rangle = \langle \psi | \vec{Q} | \psi \rangle.
\]

In a same way, from the equations (46) and (48) we obtain

\[
\langle p_1^2 \rangle = \langle \psi | P_1^2 | \psi \rangle \quad \text{and} \quad \langle q_1^2 \rangle = \langle \psi | Q_1^2 | \psi \rangle.
\]

Then we notice that not only the expectation values but also the uncertainties for \( \vec{p} \) and \( \vec{q} \) correspond to the usual quantum formula. This implies that the Heisenberg uncertainty principle is obtained from classical stochastic arguments and without any quantum axiom. This point is directly related to the articles of M. J. W. Hall and M. Reginatto [47, 48, 49] already mentioned. But this does not mean that we do not need quantum axioms at all.
The equation (46) becomes

\[ \langle f(q) \rangle = \langle \psi | f(\mathbf{Q}) | \psi \rangle, \tag{62} \]

and this shows that the probability distribution of \( q \) is the function \( |\psi(q)|^2 \).

On the other hand, all the previous calculations do not directly specify the probability distribution of the momentum \( \mathbf{p} \) for the state \( \psi \). This means that we need external arguments to find this distribution.

If we introduce the eigenvectors \( |p\rangle \) of the operator \( \mathbf{P} \) defined as

\[ |p\rangle = (2\pi \hbar)^{-3/2} \int_{\mathbb{R}^3} \exp[(i/\hbar)\mathbf{p} \cdot \mathbf{q}] |q\rangle \, d^3q, \tag{63} \]

we find:

\[ \langle \mathbf{p} \rangle = \int_{\mathbb{R}^3} \mathbf{p} |\langle \mathbf{p} | \psi \rangle|^2 \, d^3p \text{ and } \langle p_i^2 \rangle = \int_{\mathbb{R}^3} p_i^2 |\langle \mathbf{p} | \psi \rangle|^2 \, d^3p. \tag{64} \]

This means that the calculations of expectation values and uncertainties are compatible with the interpretation of \( |\langle \mathbf{p} | \psi \rangle|^2 \) as being the probability distribution of momentum.

2. The angular momentum

We define the self-adjoint operator \( \mathbf{L} \) as

\[ \mathbf{L} = \mathbf{Q} \wedge \mathbf{P}. \tag{65} \]

From Eq. (50) we deduce

\[ \langle \mathbf{l} \rangle = \langle \mathbf{q} \wedge \mathbf{p} \rangle = \langle \psi | \mathbf{L} | \psi \rangle. \tag{66} \]

Taking into account the expectation value of \( \partial_i S \partial_j S \) [Eq. 39] we also obtain

\[ \langle l_i^2 \rangle = \langle \psi | \mathbf{L}_i^2 | \psi \rangle. \tag{67} \]

Then the uncertainty \( \Delta l_i \) on each component \( l_i \) is given by the quantum formula. Moreover we know by classical arguments that the uncertainty \( \Delta l_i \) vanishes when the expectation value \( \langle l_i \rangle \) is in fact an exact value. But the condition \( \Delta l_i = 0 \) means that \( \psi \) is an eigenvector of \( \mathbf{L}_i \) and that \( \langle l_i \rangle \) is the correspondent eigenvalue. From quantum algebraic calculations, we deduce that the possible values of the angular momentum components are quantized.
This result is obtained without any quantum axiom. But this is not sufficient to specify the $l_i$ probability distribution.

Then we can develop a reasoning similar to the momentum case, and deduce that the quantum formula of the probability distribution is compatible with these results.

3. The energy

We define the self-adjoint operators $H$ as

$$H = \frac{1}{2m} \left( \vec{P} - e\vec{A}(\vec{Q},t) \right)^2 + V(\vec{Q},t).$$

(68)

From Eq. (49) we deduce

$$\langle H \rangle = \langle \psi | H | \psi \rangle.$$  

(69)

As mentioned above in the section VI A, our knowledge of the stochastic properties is not sufficient to specify the uncertainty for $H$, but this quantity certainly exists. So we need to develop some new idea, based on the results that we have already obtained.

4. Axioms for the quantized observables

The previous calculations deal with the main physical observables, so it is natural to try to generalize them axiomatically.

First, we can assume that any physical observable "$a$" is represented by a self-adjoint operator "$A$" on the Hilbert space.

Second, the expectation value of "$a$" is given by $\langle a \rangle = \langle \psi | A | \psi \rangle$.

These two axioms seem reasonable from the previous results, but they are not sufficient to specify:

- the probability distribution of the observable "$a$",
- the operator $B$ associated with the observable $f(a)$.

So we need some new arguments to reply to these questions. The momentum and angular momentum studies give the method.

Since $A$ is a self-adjoint operator, from the spectral theorem it possesses the splitting

$$A = \int a |a,\alpha\rangle \langle a,\alpha| \mu(a,\alpha)d\alpha d\alpha,$$

(70)
where the kets $|a, \alpha\rangle$ are the eigenvectors corresponding to the eigenvalue $a$, and $\mu$ is some positive density such that

$$\int |a, \alpha\rangle \langle a, \alpha| \mu(a, \alpha) \, da \, d\alpha = 1. \quad (71)$$

Then we can write

$$\int \langle \psi|a, \alpha\rangle \langle a, \alpha|\psi\rangle \mu(a, \alpha) \, da \, d\alpha = \langle \psi|\psi\rangle = 1, \quad (72a)$$

$$\langle a \rangle = \langle \psi|A|\psi\rangle = \int a \langle \psi|a, \alpha\rangle \langle a, \alpha|\psi\rangle \mu(a, \alpha) \, da \, d\alpha. \quad (72b)$$

If we define $\rho(a) = \int |\langle a, \alpha|\psi\rangle|^2 \mu(a, \alpha) \, da \, d\alpha$, the equations (72) show that $\rho(a)$ can be classically interpreted as the probability distribution of the observable “$a$”. This constitutes the only new ingredient that we need.

Then the expectation value of $f(a)$ is given by

$$\langle f(a) \rangle = \int f(a) \rho(a) \, da = \langle \psi|f(A)|\psi\rangle. \quad (73)$$

We deduce that the operator associated with $f(a)$ must be $f(A)$.

With this new definition, we can calculate the uncertainty $\Delta a$ on “$a$”

$$\Delta a^2 = \langle a^2 \rangle - \langle a \rangle^2 = \langle \psi|A^2|\psi\rangle - \langle \psi|A|\psi\rangle^2. \quad (74)$$

Moreover, if $\rho_\psi(a)$ is the probability distribution of the observable “$a$” for the state $\psi$, then the domain $\text{Supp}(\rho_\psi)$ of $\mathbb{R}$ where $\rho_\psi(a)$ does not vanish is by definition the domain of possible values of “$a$” corresponding to the state $\psi$. If we bring together the subsets $\text{Supp}(\rho_\psi)$ for all $\psi$, we must obtain the full set of possible values of “$a$” . From the definition of $\rho_\psi$ we deduce that the set of possible values is the spectrum of the operator $A$.

a. Conclusion If we look at the concrete observables, this extended framework does not change the expectation values and the uncertainties for the position and the momentum (expressions obtained from classical formula), and these observables retain a continuous spectrum. Only the probability distributions of momentum have a new definition. Then we do not need quantum axioms to recover the Heisenberg uncertainty principle.

We find without quantum axioms that the possible values of the angular momentum are quantized, but we need one axiom to prove that the energy is quantized (in suitable
situations). Nevertheless the expectation values can always be calculated using classical techniques.

So we remark that the quantum axiom adds new information (about probability distributions) but do not invalidate the classical calculations that can be done from our knowledge of the \( S \) stochastic properties.

So if we have to specify what the quantum axiom about observables really is, we must say that it is a new interpretation of classical expectation value formula, leading to a new definition of the observable probability distributions. This means that classical expectation value formula hide some new possible statistical definition that can be taken into account, or completely ignored. Nevertheless, this reasoning does not give the physical meaning of this new interpretation.

We conclude this remark by noticing that the famous “correspondence principle” is not needed in our approach.

In the list of usual quantum axioms, it remains two points that we need to study. They concern the definition of a general probability distribution in the quantum framework and the famous collapse of the wave function.

C. Quantum densities and the collapse of the wave function

1. Quantum densities

We mentioned in the section \[\text{IIIC}\] that a general classical distribution \( \rho \) on phase space is a convex linear superposition of \( q \)-stochastic pure states \( \{ \nu_\alpha \} \)

\[
\rho = \sum_\alpha p_\alpha \nu_\alpha, \tag{75}
\]

with \( \sum_\alpha p_\alpha = 1 \).

We deduce that the classical expectation values of \( f(\vec{p},\vec{q}) \) verify

\[
\langle f(\vec{p},\vec{q}) \rangle_\rho = \sum_\alpha p_\alpha \langle f(\vec{p},\vec{q}) \rangle_{\nu_\alpha}. \tag{76}
\]

If each \( q \)-stochastic pure state \( \nu_\alpha \) is represented in the new framework by a wave function \( \psi_\alpha \), and if the observable \( f \) corresponds to the operator \( F \), we deduce that the expectation
value of \( f \) (in the new framework) is given by

\[
\langle f \rangle = \sum_{\alpha} p_{\alpha} \langle \psi_{\alpha} | F | \psi_{\alpha} \rangle.
\] (77)

Moreover if we define the operator \( D \) as

\[
D = \sum_{\alpha} p_{\alpha} | \psi_{\alpha} \rangle \langle \psi_{\alpha} |,
\] (78)

we have

\[
\langle f \rangle = Tr(D.F).
\] (79)

On the other hand, \( D \) is a positive operator and

\[
Tr(D) = \sum_{\alpha} p_{\alpha} = 1.
\] (80)

We deduce that a general statistical situation is represented by a positive trace class operator \( D \) such that \( Tr(D) = 1 \).

Reciprocally, any positive trace class operator \( D \) such that \( Tr(D) = 1 \) describes a statistical situation, because it can be split in

\[
D = \sum_{n} p_{n} | \psi_{n} \rangle \langle \psi_{n} |,
\]

where \( p_{n} \geq 0 \) and \( \sum_{n} p_{n} = 1 \).

2. The collapse of the wave function

In the spirit of our approach, the collapse of the wave function is just a quantum translation of conditional probabilities. So let us briefly recall what conditional probabilities in the classical framework are.

If we have a random variable \( "a" \) described by the probability distribution \( \rho(a) \) and if we do some experiment \( \mathcal{E} \) that specifies that the only possible values of \( "a" \) belong to some subset \( \Delta \) of \( \mathbb{R} \), the new probability distribution \( \rho_{\mathcal{E}}(a) \) after the experiment is given by the conditional probability law

\[
\rho_{\mathcal{E}}(a) = \frac{\chi_{\Delta}(a) \rho(a)}{\int_{\Delta} \rho(x) dx},
\] (81)

where \( \chi_{\Delta} \) is defined by \( \chi_{\Delta}(x) = 1 \) if \( x \in \Delta \) and \( \chi_{\Delta}(x) = 0 \) elsewhere.

We assume now that \( "a" \) is a physical observable represented by the self-adjoint operator \( A \) and we look at a particle in the (pure) state \( \psi \). The reasoning of the section \( \text{VI B 4} \) shows that the probability distribution \( \rho(a) \) (in the usual sense) that describes the stochastic properties of \( "a" \) is given by

\[
\rho(a) = \int |\langle a, \alpha | \psi \rangle|^2 \mu(a, \alpha) d\alpha,
\] (82)
where the notations are those of the section VI B 4.

If we do some experiment \( \mathcal{E} \) that specifies that the range of values of “\( a \)” is some subset \( \Delta \) of \( \mathbb{R} \), we deduce that after the experiment the probability law is the conditional probability \( \rho_{\mathcal{E}}(a) \) given by the equation (81).

But after the experiment the particle must always be described by some normalized wave function \( \psi_{\mathcal{E}} \) since the particle is associated with a wave function for each given time (the experiment is assumed to be perfect). Then \( \psi_{\mathcal{E}} \) must verify

\[
\rho_{\mathcal{E}}(a) = \int \left| \langle a, \alpha | \psi_{\mathcal{E}} \rangle \right|^2 \mu(a, \alpha) d\alpha,
\]

with

\[
\rho_{\mathcal{E}}(a) = \frac{\chi_\Delta(a)}{\int_\Delta \rho(x) dx} \int \left| \langle a, \alpha | \psi \rangle \right|^2 \mu(a, \alpha) d\alpha.
\]

This condition is fulfilled for \( \psi_{\mathcal{E}} \) defined as

\[
|\psi_{\mathcal{E}}\rangle = \frac{1}{\sqrt{\langle \psi | \Pi_\Delta | \psi \rangle}} \Pi_\Delta |\psi\rangle,
\]

where \( \Pi_\Delta \) is the projector

\[
\Pi_\Delta = \int_{a \in \Delta} |a, \alpha\rangle \langle a, \alpha | \mu(a, \alpha) d\alpha da.
\]

Then we recover the collapse of the wave function.

So in our approach, the collapse of the wave function appears as a mathematical tool that translates the (classical) conditional probability process into quantum language. But it is well-known that the logical consequences in the quantum framework are very different from those in the classical framework.

We end this article by a last section devoted to the canonical conjugate approach mentioned in Sec. III B.

VII. THE CANONICAL CONJUGATE APPROACH

As indicated in the section III B, we can start the reasoning from another family of stochastic pure states that we call \( p \)-stochastic pure states defined as

\[
\rho(\vec{p}, \vec{q}, t) = n(\vec{p}, t) \delta (\vec{q} - \partial_p S(\vec{p}, t)).
\]

We briefly summarize the different steps of the study, similar to those already developed.
Assuming that $\rho$ verifies the Liouville equation, we obtain the equations of evolution for $n$ and $S$

\[
\partial_t n - \partial_{\vec{p}} (n \partial_\vec{p} H(\vec{p}, \partial_\vec{p} S)) = 0, \quad (88a)
\]

\[
\partial_t S - H(\vec{p}, \partial_\vec{p} S) = 0. \quad (88b)
\]

These equations are solutions of the minimization condition $\delta \int \mathcal{L} d^3\vec{p} dt = 0$, with $\mathcal{L}$ defined as

\[
\mathcal{L} = n (\partial_t S - H(\vec{p}, \partial_\vec{p} S)). \quad (89)
\]

Each $p$-stochastic pure state is associated with a family of solutions of the Hamiltonian equations verifying

\[
\vec{q} = \partial_\vec{p} S(\vec{p}, t), \quad (90a)
\]

\[
\frac{d\vec{p}}{dt} = -\partial_\vec{q} H(\vec{p}, \partial_\vec{p} S). \quad (90b)
\]

The field $n(\vec{p}, t)$ is a probability distribution on the $p$-space, and it also specifies the probability law of these trajectories.

The normalization condition becomes

\[
1 = \langle 1 \rangle_t = \int_{\mathbb{R}^3} n(\vec{p}, t)d^3\vec{p}. \quad (91)
\]

The expectation values of position and momentum are given by

\[
\langle \vec{q} \rangle_t = \int_{\mathbb{R}^3} n(\vec{p}, t)\partial_\vec{p} S d^3\vec{p}, \quad (92a)
\]

\[
\langle \vec{p} \rangle_t = \int_{\mathbb{R}^3} \vec{p} n(\vec{p}, t)d^3\vec{p}, \quad (92b)
\]

moreover the expectation values of energy and angular momentum are

\[
\langle H \rangle_t = \int_{\mathbb{R}^3} H(\vec{p}, \partial_\vec{p} S)n(\vec{p}, t)d^3\vec{p}, \quad (93a)
\]

\[
\langle \vec{q} \wedge \vec{p} \rangle_t = \int_{\mathbb{R}^3} \partial_\vec{p} S \wedge \vec{p} n(\vec{p}, t)d^3\vec{p}. \quad (93b)
\]

Then we start the quantization process by assuming that $S$ is a stochastic field, and we define the expectation value $S_m(\vec{p}, t) = \langle S(\vec{p}, t) \rangle$. The following step is the calculation of
the Lagrangian expectation value $\mathcal{L}_m$, where the Hamiltonian $H$ is always the classical one. This leads to the equation

$$\mathcal{L}_m = n \left( \frac{1}{2m} \left\langle \left( \vec{p} - e\vec{A}(\partial_\vec{p}S, t) \right)^2 \right\rangle + \left\langle V(\partial_\vec{p}S, t) \right\rangle \right).$$  \hspace{1cm} (94)$$

But unlike the $q$-stochastic case, we generally need a complete knowledge of the stochastic properties of $S$ to calculate this expectation value. So the procedure can only be continued in very special situations.

The conclusion is that the roles of $\vec{p}$ and $\vec{q}$ cannot be reversed in our reasoning, because the symmetry in ($\vec{p}, \vec{q}$) of the classical mechanics formulation in phase space only is a mathematical appearance. This symmetry does not take into account the fact that the classical Hamiltonian is not any function of $\vec{p}$ and $\vec{q}$, but a specific one that reflects the physical meaning of the dynamical variables.

VIII. CONCLUSION

All the elements of the quantum framework are rebuilt, only starting from the classical framework and some stochastic ingredients. Physical arguments and natural generalizations of classical formula only are needed.

This shows that the usual axiomatic approach to quantum mechanics can partially be bypassed, allowing us to obtain a picture of quantum mechanics closer to classical (statistical) mechanics, even if the quantum equations are finally the usual ones (but we notice that the linear Schrödinger equation is not the unique possibility). Moreover, different key elements of the quantum formalism (the Heisenberg uncertainty principle, the correspondence principle, the quantization of angular momentum) are obtained from classical stochastic reasonings.

Nevertheless we remark that one axiom (about probability distribution of observables) always is necessary. If the necessity of this axiom clearly appears on a logical level, its physical interpretation from classical framework remains obscure.

APPENDIX A

This appendix is devoted to the study of the compatibility condition introduced in the section [VI A 1]. We recall that the stochastic field $S$ must verify the equation [57], and a
possible solution is given by
\[ \langle e^{-i\vec{\xi}\partial_q S} \rangle = \frac{1}{|\psi(\vec{q})|^2} \tilde{\psi} \left( \vec{q} + \frac{\hbar}{2} \vec{\xi} \right) \psi \left( \vec{q} - \frac{\hbar}{2} \vec{\xi} \right). \] (A1)

This expression possesses the right symmetry for the complex conjugation, and we want to see to what extent it is relevant. Then we study the development of the expectation value in power of \( \vec{\xi} \) to the second order. We have
\[ \langle e^{-i\vec{\xi}\partial_q S} \rangle = 1 - i\xi^i \langle \partial_i S \rangle - \frac{1}{2} \xi^i \xi^j \langle \partial_i S \partial_j S \rangle + o(\xi^2). \] (A2)

Then, we can do the same development for \( \psi \)
\[ \psi \left( \vec{q} - \frac{\hbar}{2} \vec{\xi} \right) = \psi(\vec{q}) - \frac{\hbar}{2} \xi^i \partial_i \psi(\vec{q}) \\
+ \frac{\hbar^2}{8} \xi^i \xi^j \partial^2_{ij} \psi(\vec{q}) + o(\xi^2). \] (A3)

We deduce
\[ \frac{\tilde{\psi} \left( \vec{q} + \frac{1}{2}\hbar \vec{\xi} \right) \psi \left( \vec{q} - \frac{1}{2}\hbar \vec{\xi} \right)}{|\psi(\vec{q})|^2} = 1 + \xi^i M_i \\
+ \frac{1}{2} \xi^i \xi^j M_{ij} + o(\xi^2), \] (A4)

with
\[ M_i = \frac{1}{2 |\psi(\vec{q})|^2} \hbar \left( \psi \partial_i \tilde{\psi} - \tilde{\psi} \partial_i \psi \right), \] (A5a)
\[ M_{ij} = \frac{1}{8 |\psi(\vec{q})|^2} \hbar^2 \left( \tilde{\psi} \partial^2_{ij} \psi + \psi \partial^2_{ij} \tilde{\psi} - \partial_i \tilde{\psi} \partial_j \psi - \partial_i \psi \partial_j \tilde{\psi} \right). \] (A5b)

By identification of the formula (A2) and (A4) we obtain
\[ \langle \partial_i S \rangle = \frac{i}{2 |\psi(\vec{q})|^2} \hbar \left( \psi \partial_i \tilde{\psi} - \tilde{\psi} \partial_i \psi \right), \] (A6a)
\[ \langle \partial_i S \partial_j S \rangle = \left( \partial_i \tilde{\psi} \partial_j \psi + \partial_i \psi \partial_j \tilde{\psi} \right) \tilde{\psi} \partial^2_{ij} \psi - \psi \partial^2_{ij} \tilde{\psi} \) \times \frac{1}{4 |\psi(\vec{q})|^2} \hbar^2. \] (A6b)

If we take into account the definition of the wave function as
\[ \psi = \sqrt{n} \exp \left( \frac{i}{\hbar} S_m \right), \] (A7)
the expectation values of the equations (A6) become

\[ \langle \partial_i S \rangle = \partial_i S_m, \quad (A8a) \]

\[ \langle \partial_i S \partial_j S \rangle = \partial_i S_m \partial_j S_m - \frac{1}{4} \hbar^2 \partial^2_{ij} \ln(n). \quad (A8b) \]

Then we recover the expectation value of \( S \), but we find

\[ \langle \partial_i S \partial_j S \rangle = \partial_i S_m \partial_j S_m + \frac{1}{4} \hbar^2 \partial_i n \partial_j n - \frac{1}{4} \hbar^2 \partial^2_{ij} n. \quad (A9) \]

So we do not recover the right expectation value, due to the term in \( \partial^2_{ij} n \). Moreover the standard uncertainty is not always a positive number. But the supplementary term disappears in the integral \( \int n \langle (\partial_q S)^2 \rangle \, d^3 \vec{q} \), then it does not contribute to the integral form of the compatibility condition specified by the equation (57).

We end this appendix by noticing the relation that exists between the proposed solution and the Wigner functions. Namely the expression of \( \langle \exp \left( -i \vec{\xi} \cdot \partial_q S \right) \rangle \) implies that \( n(\vec{q}) \langle \delta(\vec{p} - \partial_q S) \rangle \) is equal to

\[ \int_{R^3} \exp \left[ (i/\hbar) \vec{p} \cdot \vec{x} \right] \psi(\vec{q} + \vec{x}/2) \psi(\vec{q} - \vec{x}/2) \frac{d^3 \vec{x}}{(2\pi\hbar)^3}. \quad (A10) \]

So we find that \( n(\vec{q}) \langle \delta(\vec{p} - \partial_q S) \rangle \) is the Wigner function associated with the projector \(|\psi><\psi|\). But it is well-known that this Wigner function is not always positive, then it cannot be equal to a probability distribution. Nevertheless this shows the existence of some relation between the averaged classical Bohm distributions and the Wigner functions (quasi-distributions), and this is directly related to the article of N. C. Dias and J. N. Plata [44].

The conclusion of this appendix is that the local solution proposed for the compatibility condition is not satisfactory, even if it is not so very far from the solution we are looking for.

[1] H. Weyl, Z. Phys. 40, 1 (1927).
[2] E. Wigner, Phys. Rev. 40, 749 (1932).
[3] G. A. Baker, Phys. Rev. 109, 2196 (1958).
[4] J. E. Moyal, Proc. Cambridge Philos. Soc. 45, 99 (1949).
[5] N. L. Balazs and B. K. Jennings, Phys. Rep. 104, 347 (1984).
[6] M. Hillery, R. F. O’Connel, M. Scully, and E. Wigner, Phys. Rep. 106, 121 (1984).
[7] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Ann. Phys. (Paris) 110, 111 (1978).
[8] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, Ann. Phys. (Paris) 111, 61 (1978).
[9] M. Flato, A. Lichnerowicz, and D. Sternheimer, Composito Mathematica 31, 41 (1975).
[10] A. M. Perelomov, Commun. Math. 26, 222 (1972).
[11] J. R. Klauder, Phys. Rev. D 19, 2349 (1979).
[12] J. R. Klauder, Coherent States. Applications in Physics and Mathematical Physics (World Scientific, Singapore, 1985).
[13] A. M. Perelomov, Generalized Coherent States and Their Applications (Springer, Berlin, 1986).
[14] H. Bergeron and A. Valance, J. Math. Phys 36, 1572 (1995).
[15] E. Prugovecki, Stochastic Quantum Mechanics and Quantum Spacetime (Reidel, Dordrecht, 1986).
[16] B. O. Koopman, Proc. Natl. Acad. Sci. USA 17, 315 (1931).
[17] J. von Neumann, Ann. Math. 33, 587 (1932).
[18] J. von Neumann, Ann. Math. 33, 789 (1932).
[19] E. Gozzi, M. Reuter, and W. D. Thacker, Phys. Rev. D 40, 3363 (1989).
[20] E. Gozzi and M. Reuter, Phys. Lett. B 233, 383 (1989).
[21] D. Mauro, Int. J. Mod. Phys. A 17, 1301 (2002).
[22] H. Bergeron, J. Math. Phys. 42, 3983 (2001).
[23] G. Mackey, The Mathematical Foundations of Quantum Mechanics (Benjamin, New York, 1963).
[24] G. Mackey, Induced Representations of Groups and Quantum Mechanics (Benjamin, New York, 1968).
[25] E. Prugovecki, Quantum Geometry (Kluwer, Dordrecht, 1992).
[26] E. Prugovecki, Principles of Quantum General Relativity (World Scientific, Singapore, 1995).
[27] D. Bohm, Phys. Rev. 85, 166 (1952).
[28] D. Bohm, The undivided universe: an ontological interpretation of quantum theory (Routledge and Kegan, London, 1993).
[29] D. Bohm and J. P. Vigier, Phys. Rev. 96, 208 (1954).
[30] I. Fényes, Z. Phys. 132, 81 (1952).
[31] E. Nelson, Phys. Rev. 150, 1079 (1966).
[32] E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton University Press, Princeton, 1967).
[33] E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton, 1985).
[34] L. de la Pena-Auerbach, Phys. Lett. 24A, 603 (1967).
[35] L. de la Pena-Auerbach, Phys. Lett. 27A, 594 (1968).
[36] P. Blanchard, S. Golin, and M. Serva, Phys. Rev. D 34, 3732 (1986).
[37] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, 1987).
[38] M. Correggi and G. Morchio, Ann. Phys. 296, 371 (2002).
[39] P. R. Holland, A. Kyprianidis, Z. Marie, and J. P. Vigier, Phys. Rev. A 33, 4380 (1986).
[40] T. Takabayasi, Prog. Theor. Phys. 11, 341 (1954).
[41] C. R. Leavens and R. S. Mayato, Phys. Lett. A 280, 163 (2001).
[42] G. G. Polavieja, Phys. Lett. A 220, 303 (1996).
[43] N. C. Dias and J. N. Prata, Phys. Lett. A 291, 355 (2001).
[44] N. C. Dias and J. N. Prata, Phys. Lett. A 302, 261 (2002).
[45] M. Born, *Physics in My Generation* (Pergamon Press, London, 1956).
[46] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, 1993).
[47] M. J. W. Hall and M. Reginatto, Fortschritte der Physik 50, 646 (2002).
[48] M. Reginatto, Phys. Rev. A 58, 1775 (1998).
[49] M. Reginatto, Phys. Lett. A 249, 355 (1998).
[50] M. Davidson, J. Math. Phys. 20, 1865 (1979).
[51] G. Kaniadakis, Physica A 307, 172 (2002).
[52] L. Landau and E. Lifchitz, *Mécanique* (Editions Mir, Moscou, 1974).
[53] H. Golstein, *Classical Mechanics* (Addison-Wesley, MA, 1980).