At Most 43 Moves, At Least 29
Optimal Strategies and Bounds for Ultimate Tic-Tac-Toe

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Abstract. Ultimate Tic-Tac-Toe is a variant of the well known tic-tac-toe (noughts and crosses) board game. Two players compete to win three aligned “fields”, each of them being a tic-tac-toe game. Each move determines which field the next player must play in.

We show that there exist a winning strategy for the first player, and therefore that there exist an optimal winning strategy taking at most 43 moves; that the second player can hold on at least 29 rounds; and identify any optimal strategy’s first two moves.

1 Introduction

Ultimate Tic-Tac-Toe (U3T) is a two-player zero-sum game with perfect information. U3T is played on a board containing 9 fields arranged in a 3 × 3 grid. The nine fields are indexed from 0 to 8 as shown in Figure 1. Each field itself is further divided into 3 × 3 = 9 spots. Spots are also indexed from 0 to 8 using the same indexing system. In other words, the board contains 81 spots uniquely identified as (i, j), where i is the field and j the spot.

Fig. 1. Indexing for fields and spots. The board has 9 fields (left). Each field contains 9 spots, and we write (i, j) for the j-th spot of the i-th field (right).

1 The game is also known as super or meta tic-tac-toe, we chose to follow what seems to be the most common denomination.
Players alternately place one mark in a free spot, in a field determined according to the rules explained below which we call the active field. By convention, the first player (Xavier) uses X marks and the second (Olivia) uses O marks. The active field is constrained by the previous player’s move:

- If the previous player’s move was \((i, j)\), and there are free spots in field \(j\), then the active field is \(j\);
- If the previous player’s move was \((i, j)\), and there are no free spots in field \(j\), then the current player can freely choose the active field. The same applies if there is no previous move (beginning of the game).

When a player aligns three of their marks (in line, column or diagonal fashion) in a field we say they win that field. Any further action in that field will not change this status and is it not possible to win an already won field. When a player aligns three won fields (in line, column, or diagonal fashion) they win the game. It may happen that a field is filled without any player winning it, in which case we say it is a draw. Similarly, the full board may be filled without a winner for the game.

By design, there are at most 81 moves in a game of U3T. Figure 2 shows an example ongoing game where the first move, by Xavier, was \((4, 4)\) in the center of the board. Next move is Olivia’s in the field 0 (top-left corner of the board).

![U3T Board](image)

**Fig. 2.** Example of a U3T board. Xavier has won the field 0. Last move was \(X_{16}\) and it is now Olivia’s turn to play, in the active field 0. Note that although Olivia can align three \(O\) in this field, this field has already been won by Xavier and such a move would therefore have no effect.

2 The players’ initials remind the marks X, O.
2 A winning strategy for Xavier

In conventional tic-tac-toe, the first player cannot lose if they play perfectly. If the second player also plays perfectly, then the game ends in a draw. In U3T the situation is different and the first player has a simple winning strategy. Such a strategy seems to have been known before\(^3\) but to the best of the authors’ knowledge it is the first time that such a strategy is formally described, analysed, and shown to always succeed. Several results are known about the structure of winning U3T strategies [GJ16]. Xavier’s winning strategy has three phases: an openin\(\)g, a middlegame, and an endgame.

2.1 Opening

1. First move: play \((4, 4)\) (centre spot of the centre field). Olivia is therefore constrained to play in the central field, say \((4, j)\).
2. Next seven moves: Xavier plays \((j, 4)\) (centre spot of the field \(j\)). Olivia must again play in the central field.
3. After Olivia played her eighth move, the middle field is full as in Figure 3, and it is Xavier’s turn.

![Typical board after 16 turns using the winning strategy. It is Xavier’s turn.](https://tinyurl.com/ULTTT)

2.2 Middlegame

It is now Xavier’s turn, and one field hasn’t been played yet. We assume that field to be field 0 (the proof is similar if another field is chosen).

1. Xavier plays \((0, 0)\), forcing Olivia to play in field 0:
   - (a) If Olivia plays \((0, k)\) with \(k \in \{1, 2, 3, 5, 6, 7\}\) then Xavier plays \((k, 0)\), forcing Olivia back to field 0;

\(^3\) We found an implementation dating before 2013: [https://tinyurl.com/ULTTT](https://tinyurl.com/ULTTT)
(b) If Olivia plays \((0,k)\) with \(k \in \{4,8\}\), then Xavier plays \((8,0)\). Indeed this is possible because field 4 is full and Xavier can choose 8 as the active field. If Xavier has already played this position, then \((8,8)\) is played instead, and the *middlegame* phase is over.

Observe that Xavier cannot be sent twice from the field 0 to the same field, and all the spots Xavier might want to use are free at the beginning of the strategy.

The game is not over yet: there is at most one field with three X’s (field 8), and Olivia has only played in two fields. Moreover, since there is a finite number of spots in the first field, we know that after at most 17 turns, we are in the situation where Xavier plays \((8,8)\). Therefore Xavier wins field 8 after at most 17 turns. The situation is analogous to that of Figure 4.

![Fig. 4. Typical board after 25 turns into the winning strategy.](image)

### 2.3 Endgame

Now the last step of the strategy: whichever field \(k\) is active, when it is Xavier’s turn, if \((k,0)\) can be played then it is played; otherwise \((k,8)\) is played. Eventually, Xavier wins.

### 3 Strategy analysis

Consider the following properties:

- **P1** For all \(i\) in \(A := \{1, 2, 3, 5, 6, 7\}\) if \((i,0)\) and \((i,8)\) are taken by Xavier, then both \((0,i)\) and \((8,i)\) are taken by Olivia. Note that in this case, Xavier has won field \(i\), because he has already taken \((i,4)\) at the beginning of the game.
Fig. 5. The grid after the endgame. Players may keep playing after Xavier wins the game.

- **P2** \( \forall i \in A \), if only one of \((i, 0), (i, 8)\) is taken by Xavier it is \((i, 0)\), and only one of \((0, i)\) and \((8, i)\) are taken by Olivia, and if it is in field 8, then Olivia plays in field 0.
- **P3**: \( \forall i \in A \), if neither \((i, 0)\) nor \((i, 8)\) are taken by Xavier, then neither \((0, i)\) nor \((8, i)\) are taken by Olivia
- **P4**: Olivia must play in either field 0 or 8, and the field where she plays is not full. Assuming P1, the fields 0 and 8 cannot be full at the same time before the game ends (if both field 0 and 8 are full, Xavier has won fields 1,2,3,5,6,7 and 8 creating a winning situation with fields 2,5 and 8)
- **P5**: Olivia plays in field 0 if and only if there exists a unique \( j \in A \) such that \((8, j)\) is taken by Olivia but \((0, j)\) is not. If Olivia is in field 8 there exist no \( j \in A \) such that \((8, j)\) is taken by Olivia but \((0, j)\) is not.
We say that \( i \) verifies P5 if \( i \in A \) and \((8, i)\) is taken by Olivia but \((0, i)\) is not. P5 means Olivia plays in field 0 if and only if there exists a unique \( i \in A \) such that \( i \) verifies P5, and Olivia plays in field 8 if and only if no \( i \in A \) verifies P5.
- **P6**: Olivia has only played in fields 0, 4, 8 and Xavier has not played in \((0, i)\) and \((8, i)\) \( \forall i \in A \).

**Lemma 1.** Properties P1–P6 hold at the beginning of the endgame phase.

**Proof.** At the beginning of the endgame phase, for every \( i \in A \), \((8, i)\) is empty. Moreover, if \((0, i)\) is taken by Olivia, then Xavier has already taken \((i, 0)\). This confirms that P1, P2 and P3 hold at the beginning of the endgame phase. Similarly, Olivia must play in field 8, which isn’t full, thus P4 and P5 hold. Finally, P6 is also verified by design of the middlegame phase.

**Lemma 2.** If the properties P1–P6 hold and it is Xavier’s turn, then they hold at the next round.

**Proof.** Assume that the properties are true. We exhaust the possibilities:
If Olivia must play in field 0, and plays in spot \( i \), then we have \( i \in A \) since spots 0, 4, 8 are already taken (cf. middlegame). By hypothesis of induction, \((i, 0)\) or \((i, 8)\) is empty.

- If \((i, 0)\) is empty, then Xavier plays in this spot.

Let’s prove P1, P2 and P3: The properties hold for all for all \( j \in A \) except for \( i \) because no concerned spot was taken. Moreover, by induction with P2, P3 and P6, we know that \((0, i), (8, i), (i, 0)\) and \((i, 8)\) were empty before the round. At the end of the round, only one spot is taken by Xavier in field \( i \), and it is \((i, 0)\). \((0, i)\) has been taken by Olivia, and \((8, i)\) is still empty. All in all, P1, P2 and P3 are verified at the end of the round.

Here is the proof for P4 and P5: By induction on P5 there is a unique \( j \in A \) that verifies P5 at the beginning of the round. We have \( j \neq i \), because we have shown above that \((8, i)\) is empty at the beginning of the round. Moreover, \( i \) does not verify P5 at the end of the round either. Since only the situation of \( i \) could have changed during the round regarding P5, the property is still verified at the end of the round (because only \( j \) verifies P5 at this point). As a corollary, field 0 (where Olivia has to play next round) is not full as \((0, j)\) is empty. This gives P4 at the end of the round.

- If \((i, 0)\) is already taken, by P1, \((i, 8)\) is empty. Indeed, at the beginning of the round, the situation is either the one described in P2 or P3 (because \((0, i)\) is empty), in both of which \((i, 8)\) is empty.

Thus \( i \) verifies P5 at the beginning of the round (and is the only one to do so by definition of P5). Xavier plays in \((i, 8)\).

Since at the end of the step \((0, j), (8, j)\) are taken by Olivia, \((j, 0)\) and \((j, 8)\) by Xavier we have P1, P2 and P3 verified for \( i \) at the end of the round. P1, P2 and P3 remain true for other elements of \( A \), because the spots concerned didn’t change during the round.

P5 is a simple conclusion from the unicity of \( j \) at the last step, i.e. if one \( i \) verifies P5, it must be \( j \) (otherwise this \( i \) would have verified P5 at the last step), and since \( j \) does not verify P5, P5 is proved.

With this established, field 8 is full only if field 0 is also full. If this is the case, the game has ended. Else, P4 is verified.

- If Olivia must play in field 8, and decides to play in spot \( i \) with \( i \in A \) (spots 0, 4 and 8 are already taken in this field, so Olivia has to do this choice)

  - If \((i, 0)\) is empty then Xavier plays in \((i, 0)\).

One can see that if some \( j \in A \) verifies P5 at the end of the round, then if \( j \neq i \), it verified P5 at the step before, which is contradictory by induction hypothesis. By induction on P2, P3 and P6, at the beginning
of the round, \((i,0), (i,8), (0,i)\) and \((8,i)\) are empty, and thus at the end of the round, \((i,0)\) is taken by Xavier, \((8,i)\) is taken by Olivia and the two others are still empty, thus \(i\) verifies \(P5\), and this proves \(P5\). \(P4\) is trivial as \((0,i)\) is empty.

\(P1\), \(P2\) and \(P3\) hold for every \(j \neq i, j \in A\) by induction (because the situation did not change), and also for \(i\) (we proved above that we are in the situation of \(P1\)).

- If \((i,0)\) is not free, then \((i,8)\) must be free because \(P1, P2\) and \(P3\) are verified at the beginning of the round. That ensures we are in the situation described by \(P2\) before Olivia plays, that is \((i,0)\) and \((0,i)\) are taken, and \((i,8)\) and \((8,i)\) are free. According to the strategy, Xavier plays in \((i,8)\).

At the end of the round, \(P1, P2\) and \(P3\) hold. Indeed the situation is the one described in \(P1\) for \(i\). Indeed, spots \((i,0)\) and \((i,8)\) are taken by Xavier and spots \((0,i)\) and \((8,i)\) are taken by Olivia. For \(j \neq i, j \in A\), nothing changed during the round, so \(j\) does not bring a contradiction to \(P1, P2\) or \(P3\).

\(P4\) also holds because Olivia was sent in field 8. If the field is full, the game is finished because \(P1\) is verified, and the proof ends.

Note that \(i\) does not verify \(P5\) because \((0,i)\) is taken, and by induction on \(P5\), no other \(j \in A\) verifies \(P5\) either. This proves \(P5\) at the end of the round.

Finally, in all cases, \(P6\) holds trivially at the end of the round. \(\Box\)

**Corollary 1.** Following the strategy described above, properties \(P1-P6\) hold as long as Xavier does not win the game. If Xavier does not win, we can construct an infinite sequence of rounds. Since the game terminates in at most 81 moves, this is impossible, and therefore Xavier eventually wins the game.

### 4 Results on an optimal winning strategy

#### 4.1 Upper bound

**Lemma 3.** An optimal winning strategy for Xavier takes at most 43 moves.

**Proof.** We have shown that the strategy described in Section 2 is winning, we just need to show that it takes at most 43 moves, and therefore that any optimal strategy takes at most 43 moves.

Indeed, at turn 43 only one cross X is missing in some field \(h\) with \(h \in A\). Thus, if Olivia sent Xavier in field 0 at the end of the opening, either the line composed of field 6,7,8 or the column composed of field 2,5,8 is won by
Xavier. Similarly, if Olivia plays every spot before the spot 8 in field 0 during middlegame, and then plays every spot except 5 and 7 in field 8, and then plays (8, 5), she loses in 43 turns. On the other hand, if Olivia sent Xavier in field 1 at the end of the opening, either the line composed of field 0, 3, 6 or the column composed of field 2, 5, 8 is won by Xavier. If Olivia plays every spot before the spot 7 in field 1 during middlegame, and then plays every spot except 6 and 8 in field 8, and then plays (7, 6), she loses in 43 turns. In each case, after 43 turns Xavier has won. □

Remark 1. It seems that our winning strategy hinges on Xavier winning field 4, which essentially makes the opening sequence the bottleneck.

4.2 Lower bound

Lemma 4. Olivia can resist Xavier’s optimal strategy for at least 29 rounds.

We show this by providing an explicit strategy for Olivia, which ensures she does not lose before turn 29. The strategy, which we call LBS, is as follows: when Olivia plays in field $j$, she chooses the spot $i$ where she plays in this order:

1. If Olivia is the first to play in field $j$, she chooses $i = j$.
2. Otherwise, Xavier already has a $X$ in field $j$. Olivia chooses $i$ such that there is no $X$ in field $i$ and $(j, i)$ is free. If no such $i$ exists, she picks $i$ at random among the free spots.

If Olivia is sent in a field that is already full, she plays in a random field $j$ according to the strategy above.

Lemma 5. At turn 18 (that is after Xavier and Olivia play 9 moves), if Olivia plays the LBS strategy, there is one and only one $X$ in each field.

Proof (of Lemma 5). Suppose that at turn 18, there exists a field $j$ where there are no $X$. Then, Olivia did not play in field $j$ either before turn 18, else she would have sent Xavier in field $j$ at turn 17 or before. It follows that in every field, spot $j$ was free before turn 17.

Moreover, at turn 17, Xavier had 9 $X$ in 8 fields, so one field has at least 2 $X$. Let $k$ be such a field. At some point in the game, before turn 17, Olivia sent Xavier in field $k$, but there already was an $X$ in this field. However, Olivia could play in spot $j$ (because it was free for every field at this moment) and send Xavier to a field where there is no $X$.

That means Olivia did not follow the LBS strategy at some point in the game, so there is a contradiction. □

4 Note that if Xavier was not the first to play in field $j$, then Olivia was, and decided to send Xavier back to field $j$.

5 Note that this claim is the exact contradiction of the assumption of the theorem, because there are 9 $X$ in the grid at turn 18.
Remark 2. This proof can easily be adapted to account for the situation where some fields are full (let $E$ be the set of these fields) and all other fields have exactly the spots of $E$ taken. If there are $\alpha$ full fields, let $\beta = 9 - \alpha$ be the number of “free” fields. Then Olivia can force Xavier to play exactly one $X$ in the $\beta$ “free” fields for his next $\beta$ turns.

Proof (of Lemma 4). By Lemma 5 after 18 turns, there is exactly one $X$ in each field. To win, Xavier needs to win 3 fields, in which at least 3 $X$ are required. That means Xavier has to play at least 6 times after turn 18. As a result, Xavier can not win before turn 29. □

4.3 Optimal strategy’s first two moves

We call the nine spots $(i, i)$ on the board doubles.

Lemma 6. Xavier’s optimal strategy’s first move is a double.

Proof. The idea of the proof is that if the first move of Xavier is not a double, Olivia can force Xavier’s moves between two fields, which ensures Xavier doesn’t win, for long enough.

Suppose that the first move of Xavier is $(i, j)$ with $i \neq j$. Then Olivia plays in $(j, j)$. Now Olivia uses a strategy similar to the one used by Xavier with field $j$ as field 0 and field $j$ as field 8: wherever Olivia get sent, if she can play in spot $j$ she does so, otherwise she plays in spot $i$. This process ends when fields $i$ and $j$ are full.

Since the strategy is essentially identical to Xavier’s endgame in the strategy described in Section 2, it will not be detailed further here.

There are 16 $X$ and 16 $O$ played, and Xavier can play wherever he wants. Now Olivia can reuse the LBS which will guarantee that during the 14 next turns, Xavier will have at most one $X$ in each field except fields $i$ and $j$.

Thus after 46 turns, $X$ has not won the game. Therefore by Lemma 3 Xavier’s strategy cannot be optimal. □

An illustration of Olivia’s strategy is given in Figure 6.

Lemma 7. If Olivia first move is $(i, j)$, then Xavier must play $(j, i)$

Proof. Let $(i, i)$ be Xavier’s first move, that Olivia plays $(i, j)$ and that the Xavier’s second move is $(j, k)$ with $k \neq i$. In a first time, let’s assume $k \neq j$.

Olivia has a strategy to block Xavier in fields $k$ and $i$, similar to the one described in the previous proof. Wherever Olivia get sent, if she can play in spot $k$ she does so otherwise she plays in spot $i$. This process will end with field $i$ and field $j$ full, (the only difference with the previous proof is the case of spot $(k, j)$ and spots $(i, k)$ and $(k, i)$).

There are 15 $X$ and 15 $O$ played, and Xavier can play wherever he wants. Now Olivia can reuse the LBS which will guarantee that during the 14 next turns, Xavier will have at most two $X$ in each field except fields $i$ and $j$. 9
Fig. 6. Assume that Xavier’s first move was \((0, 8)\), then by following the strategy of Lemma 6 Olivia ensures that Xavier doesn’t win the game for at least 46 turns.

Thus after 44 turns, X has not won the game: by Lemma 3 Xavier’s strategy is not optimal.

Finally, if \(k = j\) then Olivia can reuse the strategy described in the previous proof (with \(i, j\) as given), ensuring that Xavier doesn’t win in less than 46 steps, which shows that Xavier’s strategy is not optimal either □

Figure 7 shows a typical board after this strategy has been used, for \(i = 0\) and \(j = 4\) and \(k = 8\).

Fig. 7. A typical board following the strategy of Lemma 7 if the moves were \((0, 0)\), \((0, 4)\), \((4, 8)\).

5 Conclusion

We described a winning strategy for the first player that wins in at most 43 moves, and a strategy for the second player that doesn’t lose for at least 29 moves.
Thus unlike the standard game of tic-tac-toe the first player has a substantial advantage: he wins even against a perfect adversary as long as he doesn’t make a mistake. Our strategies, for both players, may in turn help design more efficient AI players for this game beating the constructions from [Sis16][CDX18].

References

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