MULTIWISTED REAL SPECTRAL TripLES.

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ABSTRACT. We generalize the notion of spectral triple with reality structure to multitwisted real spectral triples, the class of which is closed under the tensor product composition. In particular, we introduce a multitwisted order one condition (characterizing the Dirac operators as an analogue of first-order differential operator). This provides a unified description of the known examples, which include conformally rescaled triples and (on the algebraic level) triples on quantum disc and on quantum cone, that satisfy twisted first order condition of [1, 2], as well as asymmetric tori, non-scalar conformal rescaling and noncommutative circle bundles. In order to deal with them we allow twists that do not implement automorphisms of the algebra of spectral triple.

RÉSUMÉ. Nous généralisons la notion de triple spectral avec structure de réalité à des triples spectraux réels „multitordus”, dont la classe est fermée sous la composition de produit tensoriel. En particulier, nous introduisons une condition d’ordre un multitordu (caractérisant les opérateurs de Dirac en tant qu’analogue d’un opérateur différentiel de premier ordre). Ceci fournit une description unifiée des exemples connus, qui incluent des triplets rééchelonnés de manière conforme (et en niveau algébrique) triple sur disque quantique et sur cône quantique, qui répondent au premier ordre [1, 2], ainsi que les tores quantiques asymétriques, le rééchelonnement conforme non scalaire et la non-commutative espace fibré avec la fibre d’un cercle. Afin de les traiter, nous permettons les torsions qui ne mettent pas en œuvre des automorphismes de l’algèbre du triple spectral.

1. INTRODUCTION

Spectral triples were introduced [3] as a setup to generalize differential geometry to noncommutative algebras that carries topological information and allows explicit analytic computations of index pairings [5]. The concept of real spectral triples [4] was motivated by successful applications to the Standard Model of particle physics and also by the quest for the equivalence in the commutative case with the geometry of spin manifolds, culminating in the reconstruction theorem [9]. The role of the real structure in noncommutative examples became evident in the relation between the classes of equivariant real spectral triples and the spin structures on noncommutative tori [17].

While the theory of real spectral triples gained more and more examples [6] [5] [14], some interesting noncommutative geometries did not fit into the original set of axioms for real spectral triples. Remarkable ones were the twisted (or modular) spectral triples on the curved noncommutative torus [8], intensively studied afterwards. A scheme to incorporate the above conformally rescaled noncommutative geometries in the framework of usual spectral triples though with twisted reality structure, together with a generalized first order condition was proposed in [1]. It was further studied in [2], where the relation between spectral triples with twisted real structure and real twisted spectral triples [16] was uncovered.

Yet even these generalizations do not embrace the recent examples of partially rescaled conformal torus [13] and spectral triples over a circle bundle with the Dirac operator compatible with a given connection [12]. Moreover, neither the class of spectral triples with a twisted first order condition nor the twisted spectral triples is closed under the tensor product composition of spectral triples. We propose here a further generalization, which complies with tensor products, allows for fluctuations and covers almost all known interesting examples.
2. MULTITWISTED REAL SPECTRAL TRIPLES

Consider a spectral triple \((A, H, D)\), where \(A\) is a \(*\)-algebra identified with a subalgebra of bounded operators \(B(H)\) on a Hilbert space \(H\), and \(D\) is a densely defined selfadjoint operator on \(H\) such that \(D\) has a compact resolvent and for each \(a \in A\) the commutator \([D, a]\) is bounded. Let \(J\) be an antilinear isometry on \(H\), such that \(J^2 = \pm 1\) and
\[
[a, JbJ^{-1}] = 0,
\]
in which case (with a slight abuse of terminology) we call \((A, H, D, J)\) a real spectral triple. If in addition there is a grading \(\gamma\) of \(H, \gamma^2 = 1\,\text{, such that} D\gamma = -\gamma D\) and \([\gamma, a] = 0\) for all \(a\) in \(A\), we call \((A, H, D, J, \gamma)\) a real even spectral triple.

**Definition 2.1.** We say that \((A, H, D, J)\) is **multitwisted** real spectral triple if there are \(N\) densely defined operators \(D_\ell, \ell = 1, \ldots, N\), with \(\sum_{\ell=1}^N D_\ell = D\) and for every \(\ell\) there exists an operator \(\nu_\ell \in B(H)\), with bounded inverse, such that for every \(a, b \in A\) the **multitwisted first-order condition**
\[
[D_\ell, a]J\nu_\ell(b)J^{-1} = J\nu_\ell^{-1}(b)J^{-1}[D_\ell, a],
\]
and that **multitwisted \(\epsilon'\) condition** holds if
\[
D_\ell J\nu_\ell = \epsilon' \nu_\ell JD_\ell, \quad \text{where} \quad \epsilon' = \pm 1
\]
and we call the multitwisted real spectral triple **regular** if
\[
\nu_\ell J\nu_\ell = J,
\]
for each \(\ell\).

**Remark 2.2.** We implicitly assume that the domain of the full Dirac operator \(D\) is contained in the domains of all operators \(D_\ell\) so the decomposition makes sense at least on this domain. However, in principle it is not required that individually each \(D_\ell\) is selfadjoint with compact resolvent and each \([D_\ell, a]\) is bounded as in order to obtain a spectral triple only the sum \(D\) of all \(D_\ell\) is required to have these properties.

**Remark 2.3.** The notion of a spectral triple with a twisted real structure in [1, 2] fits this definition as a special case when \(N = 1\) and \(\tilde{\nu}_1\) is an automorphism of \(A\), since then the multitwisted zero order condition (2.2) is equivalent with (2.1), and the relation (2.4), though it appears slightly different, is equivalent with the previous twisted order one condition by taking \(b = \tilde{\nu}_1(c)\). Definition 2.1 is however slightly more general as we do not assume that \(\tilde{\nu}_1\) are automorphisms of the algebra \(A\), and in order that the multitwisted first order condition (2.4) be satisfied for all one forms, that is if \(\omega_\ell = \sum_i a_i[D_\ell, b_i]\) then \(\omega_\ell J\tilde{\nu}_1(b)J^{-1} = J\tilde{\nu}_1^{-1}(b)J^{-1}\omega_\ell\), we require that besides (2.1) also \([a, J\tilde{\nu}_1(b)J^{-1}] = 0\) holds. Furthermore, the consistency with the \(A\)-bimodule structure of 1-forms requires also that \([a, J\tilde{\nu}_1^{-1}(b)J^{-1}] = 0\), however, thanks to the consistency under the adjoint operation in \(A\), if \(\nu_\ell^* = \nu_\ell\), it suffices to impose one of these two conditions.
2.1. **Properties of multitwisted real spectral triples.** The important feature of the spectral triples which are multitwisted real is that they are closed under the product.

**Proposition 2.4.** Let \((A', H', D', J', \gamma')\) and \((A'', H'', D'', J'')\) be multitwisted real spectral triples (the first one even and satisfying \(J'\gamma' = \gamma' J'\)), with \(D' = \sum_{j=1}^{N'} D_j'\) and \(D'' = \sum_{k=1}^{N''} D_k''\), for the twists \(\nu_j' \in B(H')\) and \(\nu_k'' \in B(H'')\), respectively. Then
\[
(A' \otimes A'', H' \otimes H'', D' \otimes id + \gamma' \otimes D'', J' \otimes J'')
\]
is a multitwisted real spectral triple with the Dirac operator decomposing as a sum of
\[
D_\ell = \begin{cases} 
D_\ell' \otimes id, & 1 \leq \ell \leq N' \\
\gamma' \otimes D_{\ell-N'}, & N' + 1 \leq \ell \leq N' + N'' 
\end{cases}
\]
for the twists
\[
\nu_\ell = \begin{cases} 
\nu_\ell' \otimes id, & 1 \leq \ell \leq N' \\
id \otimes \nu_{\ell-N'}, & N' + 1 \leq \ell \leq N' + N''.
\end{cases}
\]
Furthermore, if both triples satisfy the multitwisted zero or first order conditions, and are regular then this holds for their tensor product.

**Remark 2.5.** Note that the resulting spectral triple is not even (as a product of an even and an odd triple). All other cases of the product of even and odd spectral triples can be also considered and we postpone the full discussion till future work.

In [1] we have demonstrated that, with an appropriate definition of the fluctuated Dirac operator, a perturbation of \(D\) by a one form and its appropriate image in the commutant of the algebra \(A\) yields the Dirac operator with the same properties. This functorial property holds also in the multitwisted case.

**Proposition 2.6.** Assume that \((A, H, D)\) and \(J\) is a multitwisted real spectral triple with the operators \(D = \sum_{\ell=1}^{N} D_\ell\) satisfying the twisted zero and first order condition \((2.2), (2.4)\) and let \(\omega = \sum_{i} a_i [D_\ell, b_i]\) be a selfadjoint one-form. Then \((A, H, D_\omega)\), where \(D_\omega = D + \omega\), together with \(J\) is again a multitwisted real spectral triple satisfying the twisted zero \((2.2)\) and first order condition \((2.4)\), with \(D_\omega = \sum_{\ell=1}^{N} (D_\omega)_\ell\), where \((D_\omega)_\ell = D_\ell + \omega_\ell\) and \(\omega_\ell = \sum_{i} a_i [D_\ell, b_i]\), and with the same twists. Moreover, if \((A, H, D)\) is regular then so is \((A, H, D_\omega)\).

**Remark 2.7.** Note that the above construction does not preserve the multitwisted \(\epsilon'\)-condition \((2.5)\). To cure this problem, we modify the manner of fluctuations of \(D\).

**Proposition 2.8.** If \((A, H, D)\) and \(J\) is a multitwisted real spectral triple satisfying \((2.2), (2.4)\) and \(\omega\) is a one-form as in the proposition above, then with
\[
(D_\omega)'_\ell = D_\ell + \omega_\ell + \epsilon' \nu_\ell J(\omega_\ell)J^{-1} \nu_\ell,
\]
\((A, H, D_\omega')\) and \(J\), where \(D'_\omega = \sum_{\ell=1}^{N} (D_\omega')_\ell\), is a multitwisted spectral triple with the same twists, satisfying the conditions \((2.2), (2.4)\) and \((2.5)\). Moreover, for each \(a \in A\),
\[
[(D_\omega)'_\ell, a] = [(D_\omega)_\ell, a].
\]
Remark 2.9. Observe that, in principle, the above fluctuation of the Dirac operator may be an unbounded operator unless we assume that the sum, \( \sum_{\ell=1}^{N} \nu_{\ell} J(\omega_{\ell})J^{-1} \nu_{\ell} \), is bounded. To avoid problems with the other properties of the Dirac operator (like the compactness of the resolvent) we restrict possible fluctuations only to those which do not change these properties.

Remark 2.10. It is also worth noting that one can extend the possible fluctuations of the Dirac operator to the sums of partial fluctuations that is, fluctuating each of \( D_{\ell} \) by \( \omega_{\ell} \), which may differ each from other, provided that the resulting full Dirac operator \( D'_{\omega} \) is a bounded perturbation of \( D \).

The notion of a spectral triple with a twisted real structure in [1, 2] was largely motivated by spectral triples conformally rescaled by a positive element in \( JAJ \). We have a generalization of this construction as follows.

**Proposition 2.11.** Suppose \( (A, H, D) \) and \( J \) is a real spectral triple and \( D = \sum_{\ell=1}^{N} D_{\ell} \) such that each \( D_{\ell} \) satisfy the first order condition for every \( \ell = 1, \ldots, N \). Let \( k_{\ell} \) be positive elements from \( A \) with bounded inverses. Then \( (A, D, J, H) \), where

\[
(2.10) \quad \tilde{D} := (D_1)_{k_1} + \cdots + (D_N)_{k_N},
\]

with \( \tilde{D} \) satisfies a multitwisted zero \((2.2)\) first order condition \((2.4)\) with

\[
\nu_{\ell} = k_{\ell}^{-1} J k_{\ell} J^{-1},
\]

and is regular \((2.6)\).

3. Examples

### 3.1. Multiconformally rescaled spectral triples

A specific example of the above construction, motivated by the study of possible spectral triples over the noncommutative torus, which give rise to non-flat metric and non-vanishing curvature was given as the asymmetric torus in [13]. Let \( \partial_{1} \) and \( \partial_{2} \) denote the operators that extend the standard derivations of \( C^{\infty}(T_{\theta}^{2}) \) to \( H = L^{2}(T_{\theta}^{2}) \otimes \mathbb{C}^{2} \) as selfadjoint (unbounded) operators and \( J \) be usual antilinear isometry \( J \). Then for any positive invertible \( k_1, k_2 \in C^{\infty}(T_{\theta}^{2}) \), the Dirac operator

\[
(3.11) \quad \tilde{D} = J k_1 J^{-1} \sigma_{1} \partial_{1} J k_1 J^{-1} + J k_2 J^{-1} \sigma_{2} \partial_{2} J k_2 J^{-1},
\]

where \( \sigma_{1} \) and \( \sigma_{2} \) are the usual Pauli matrices, makes \((C^{\infty}(T_{\theta}^{2}), L^{2}(T_{\theta}^{2}) \otimes \mathbb{C}^{2}, \tilde{D}, J)\) a multitwisted real spectral triple satisfying by Proposition 2.11 all conditions including \((2.2), (2.4), (2.6)\). In [13] we considered a particular case with \( k_1 = 1 \) (which is not a product spectral triple). A four-dimensional generalization (of product type) with two different scalings was studied in [10].

### 3.2. Conformal rescaling without an automorphism

Consider the following situation, which further generalizes the construction from the subsection 3.1. Let \( (A, H, D, J) \) be a real spectral triple, which satisfies the usual first order condition and let \( k \in Cl_{D}(A) \) be an invertible element with bounded inverse. Note that unlike in the case of conformal rescaling by an element from \( A \), the automorphism of the algebra \( B(H) \), \( \varphi(x) = k^{-1} x k \), does not necessarily preserve the algebra \( A \), so \( k^{-1} a k \notin A \). Nevertheless, the following relation holds for \( D_{k} = J k J^{-1} D J k J^{-1} \):

\[
(3.12) \quad [D_{k}, a] J \varphi(b) J^{-1} = J \varphi^{-1}(b) J^{-1} [D_{k}, a],
\]
for any \( a, b \in A \), which is precisely (2.4). It is easy to see that also the (2.2) condition is satisfied.

Further generalization of the above situation to the case of multiconformal scaling provides new multitwisted real spectral triples. A geometric motivation comes from Dirac operators over noncommutative circle bundles \([11]\), instance of which is a three-dimensional noncommutative torus seen as the \( U(1) \) bundle over the two-dimensional noncommutative torus. Namely, for the \( U(1) \) action given as in \([11]\) a \( U(1) \) connection over \( C^\infty(T^3_\theta) \) is given by a one-form:

\[ \omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1, \]

where \( \omega_1, \omega_2 \in T^2_\theta \) are \( U(1) \)-invariant elements of the algebra \( C^\infty(T^3_\theta) \).

For any selfadjoint connection \( \omega \) (3.13) there exists a compatible in the sense of \([11]\) Dirac operator over \( A = C^\infty(T^3_\theta) \), which has the form

\[ D_\omega = \sigma^1 \partial_1 + \sigma^2 \partial_2 + J w J^{-1} \partial_3, \]

where \( J \) is the usual real structure on \( A \) and

\[ w = \sigma^3 - \sigma^1 \omega_1 - \sigma^2 \omega_2. \]

Although \( D_\omega \) does not satisfy a twisted first order condition, it can be decomposed as \( D_{(2)} + D_w \), with \( D_{(2)} = \sigma^1 \partial_1 + \sigma^2 \partial_2 \) and \( D_w = J w J^{-1} \partial_3 \), so that for each \( a, b \in A \),

\[ [D_{(2)}, a] J b J^{-1} = J b J^{-1} [D_{(2)}, a], \]
\[ [D_w, a] J \bar{\nu}(b) J^{-1} = J \bar{\nu}^{-1}(b) J^{-1} [D_w, a], \]

where \( \bar{\nu}(x) = w^{-\frac{1}{2}} x w^\frac{1}{2} \). Note that the condition (2.2) is also satisfied since \( w \) is in the completion of Clifford algebra and therefore \( J \bar{\nu}(a) J^{-1} \) is in the commutant of \( A \).

### 4. Conclusions and Outlook

The proposed here new notion of multitwisted real spectral triples has a few major advantages. Firstly, it is consistent with the usual definition of spectral triples (unbounded Fredholm modules) thus, allowing to use the power of Connes-Moscovici local index theorem. Secondly it vastly extends the realm of examples, covering almost all known spectral triples, including those motivated by geometrical constructions, like conformal rescaling or noncommutative principle fibre bundles. Moreover, it is closed under the tensor product operation.

In particular, the multitwisted first order condition may provide a better understanding of the notion of first order differential operators in noncommutative geometry, which we hope will allow to finer apprehend the examples arising from the quantum groups and quantum homogeneous spaces as constituting noncommutative manifolds.

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