Abstract. Hyperplane codes are a class of convex codes that arise as the output of a one
layer feed-forward neural network. Here we establish several natural properties of stable
hyperplane codes in terms of the polar complex of the code, a simplicial complex associated
to any combinatorial code. We prove that the polar complex of a stable hyperplane code
is shellable and show that most currently known properties of the hyperplane codes follow
from the shellability of the appropriate polar complex.

Contents

1. Introduction 1
2. Background 2
3. Obstructions for hyperplane codes 5
4. The main results 10
5. Discussion 11
6. Algebraic signatures of a hyperplane code 12
7. Proofs of Theorem 4 and Theorem 5 16

1. Introduction

Combinatorial codes, i.e. subsets of the Boolean lattice, naturally arise as outputs of
neural networks. A codeword $\sigma \subseteq [n] \overset{\text{def}}{=} \{1, \ldots, n\}$ represents an allowed subset of co-
active neurons, while a code is a collection $\mathcal{C} \subseteq 2^{[n]}$ of codewords. Combinatorial codes in
a number of areas of the brain are often convex, i.e. they arise as an intersection pattern
of convex sets in a Euclidean space $[18, 21, 25]$. The combinatorial code of a one-layer
feedforward neural network is also convex, as it arises as the intersection patterns of half-
spaces $[14, 26]$. It is well-known that a two-layer feedforward network can approximate any
measurable function $[11, 20]$, and thus may produce any combinatorial code. In contrast, the
codes of one-layer feedforward networks are not well-understood. The intersection lattices of
affine hyperplane arrangements have been studied in the oriented matroid literature $[1, 2, 4]$. However, combinatorial codes contain less detailed information than oriented matroids, and
the precise relationship is not clear. We are motivated by the following question: How
can one determine if a given combinatorial code is realizable as the output of a one-layer
feedforward neural network?

We study stable hyperplane codes, codes that arise from the intersection patterns of half-
spaces that are stable under certain small perturbations. The paper is organized as follows.
Relevant background and definitions are provided in Section 2. In Section 3, we establish
a number of obstructions that prevent a combinatorial code from being a stable hyperplane
code. In Section 4, we show that all but one of the currently known obstructions to being a stable hyperplane code are subsumed by the condition that the polar complex of the
code, defined in Section 2.3, is shellable. Lastly, in Section 6 we show how techniques from commutative algebra can be used to computationally detect the presence of these obstructions.

2. Background

2.1. Stable Hyperplane Codes. We call a collection \(\mathcal{U} = \{U_i\}\) of \(n\) subsets \(U_i \subseteq X\) of a set \(X\) an arrangement \((\mathcal{U}, X)\). Note that we do not require that \(\bigcup_{i \in [n]} U_i = X\).

**Definition 2.1.** For \(\sigma \subseteq [n]\), let \(A_{\sigma}^\mathcal{U}\) denote the atom of \((\mathcal{U}, X)\)

\[ A_{\sigma}^\mathcal{U} = \left(\bigcap_{i \in \sigma} U_i \right) \setminus \bigcup_{j \not\in \sigma} U_j \subseteq X, \quad \text{where} \quad A_{\varnothing}^\mathcal{U} = X \setminus \bigcup_{i \in [n]} U_i. \]

When \((\mathcal{U}, X)\) is clear from context, \(\mathcal{U}\) will be suppressed from the notation. The code of the arrangement \((\mathcal{U}, X)\) is defined as

\[ \text{code}(\mathcal{U}, X) \overset{\text{def}}{=} \{\sigma \subseteq [n] \text{ such that } A_{\sigma}^\mathcal{U} \neq \varnothing\} \subseteq 2^{[n]}. \]

A realization of a code \(\mathcal{C}\) is an arrangement \((\mathcal{U}, X)\) such that \(\mathcal{C} = \text{code}(\mathcal{U}, X)\). The simplicial complex of the code, denoted \(\Delta(\mathcal{C})\), is the closure of \(\mathcal{C}\) under inclusion:

\[ \Delta(\mathcal{C}) \overset{\text{def}}{=} \{\tau \mid \tau \subseteq \sigma \text{ for some } \sigma \in \mathcal{C}\}. \]

Note that for \(\mathcal{C} = \text{code}(\mathcal{U}, X)\), the simplicial complex of the code is equal to the nerve of the corresponding cover:

\[ \Delta(\text{code}(\mathcal{U}, X)) = \text{nerve}(\mathcal{U}) \overset{\text{def}}{=} \left\{\sigma \subseteq [n] \mid \bigcap_{i \in \sigma} U_i \neq \varnothing\right\}. \]

A natural class of codes that arises in the context of neural networks is the class of hyperplane codes [14]. A hyperplane is a level set \(H = \{x \in \mathbb{R}^d \mid w \cdot x - h = 0\}\) of a linear function. An oriented hyperplane partitions \(\mathbb{R}^d\) into three pieces: \(\mathbb{R}^d = H^+ \cup H \cup H^-,\) where \(H^\pm\) are the open half-spaces, e.g. \(H^+ = \{x \in \mathbb{R}^d \mid w \cdot x - h > 0\}\).

**Definition 2.2.** A code \(\mathcal{C} \subseteq 2^{[n]}\) is a hyperplane code if there exists an open convex subset \(X \subseteq \mathbb{R}^d\) and a collection \(\mathcal{H} = \{H_1^+, \ldots, H_n^+\}\) of open half-spaces such that \(\mathcal{C} = \text{code}(\{H_i^+ \cap X\}, X)\). With a slight abuse of notation, we denote this arrangement of subsets of \(X\) by \((\mathcal{H}, X)\), thus code(\(\mathcal{H}, X\)) = code(\(\{H_i^+ \cap X\}, X\)), with atoms defined as before:

\[ A_{\sigma}^\mathcal{H} = X \cap \left(\bigcap_{i \in \sigma} H_i^+\right) \setminus \bigcup_{j \not\in \sigma} H_j^+ \].

Hyperplane codes are produced by one-layer feedforward neural networks [14], where the convex set \(X\) is often the positive orthant \(\mathbb{R}^d_{\geq 0}\). A well-behaved subset of hyperplane codes are the stable hyperplane codes. Informally, these are codes that are preserved under small perturbations of the hyperplanes and the convex set \(X\). These perturbations correspond to perturbations of the parameters of the neural network [26], i.e. the vectors \((w_i, h_i) \in \mathbb{R}^d \times \mathbb{R}\) defining hyperplane \(H_i\) for each \(i = 1, \ldots, n\) in our context. Thus, we restrict our attention to the class of stable hyperplane codes.
Figure 1. (a) A stable arrangement \((H, X)\) with atoms labeled by their corresponding codewords. (b) The polar complex \(\Gamma(\text{code}(H, X))\), defined in Section 2.3.

**Definition 2.3.** An arrangement \((H, X)\) is stable if \(X\) is open and convex, and the hyperplanes have generic intersections in \(X\), that is if \(X \cap H_\sigma \overset{\text{def}}{=} X \cap (\bigcap_{i \in \sigma} H_i) \neq \emptyset\), then \(\dim H_\sigma = d - |\sigma|\). A code \(C \subseteq 2^n\) is a stable hyperplane code if there exists a stable arrangement \((H, X)\) such that \(C = \text{code}(H, X)\).

Stable arrangements are robust to noise in the sense that all atoms have nonzero measure.

**Lemma 2.4.** If \((H, X)\) is a stable arrangement, then every nonempty atom \(A_\sigma^H\) has a nonempty interior.

**Proof.** Let \(A_\sigma\) be a nonempty atom of the stable arrangement \((H, X)\) and consider a point \(x \in A_\sigma\). Let \(\tau = \{j \mid x \in H_j\}\) index the set of hyperplanes that contain \(x\). Then \(x\) has an open neighborhood \(V \subset X \cap (\bigcap_{i \in \sigma} H_i^+) \cap (\bigcap_{j \notin \sigma \cup \tau} H_j^-)\). By genericity, the set \(\{w_i \mid i \in \tau\}\) is linearly independent. Therefore, there exists some \(v \in \mathbb{R}^d\) such that \(w_i \cdot v < 0\) for all \(i \in \tau\).

For sufficiently small \(\varepsilon > 0\), \(y = x + \varepsilon v \in V\); therefore for any \(i \in \tau\),

\[
  w_i \cdot y - h_i = w_i \cdot (x + \varepsilon v) - h_i = w_i \cdot \varepsilon v < 0,
\]

and thus \(y \in X \cap (\bigcap_{i \in \sigma} H_i^+) \cap (\bigcap_{j \notin \sigma} H_j^-)\), which is the interior of \(A_\sigma\). \(\square\)

**Example 2.5.** The code \(C_1 = \{1, 12, 123, 2, 23\}\) is a stable hyperplane code; a realization is illustrated in Figure 1(a). To avoid notational clutter, we adopt the convention of writing sets without brackets or commas, so the set \(\{1, 2\}\) is written 12.

2.2. Bitflips and stable hyperplane codes. The abelian group \((\mathbb{Z}_2)^n\) acts on \(2^n\) by “flipping bits” of codewords. Each generator \(e_i \in (\mathbb{Z}_2)^n\) acts by flipping the \(i\)-th bit, i.e.

\[
e_i \cdot \sigma \overset{\text{def}}{=} \begin{cases} \sigma \cup i & \text{if } i \notin \sigma \\ \sigma \setminus i & \text{if } i \in \sigma. \end{cases}
\]

This action extends to the action of \((\mathbb{Z}_2)^n\) on codes, with \(g \cdot C = \{g \cdot \sigma \mid \sigma \in C\}\). The group \((\mathbb{Z}_2)^n\) also acts on oriented hyperplane arrangements. Here each generator \(e_i\) acts by reversing the orientation of the \(i\)-th hyperplane:

\[
e_i \cdot H_i^+ \overset{\text{def}}{=} \begin{cases} H_i^+ & \text{if } i \neq j \\ H_i^- & \text{if } i = j. \end{cases}
\]
One might hope that applying bitflips commutes with taking the code of a hyperplane arrangement, but this is not true for arbitrary hyperplane codes.

**Example 2.6.** Consider \( H_1^+, H_2^+, H_3^+ \subseteq \mathbb{R}^2 \), with \( H_1^+ = \{ x+y > 0 \} \), \( H_2^+ = \{ x-y > 0 \} \), and \( H_3^+ = \{ x > 0 \} \), illustrated in Figure 2(a). By inspection, \( C_2 = \text{code}(\mathcal{H}, \mathbb{R}^2) \) has codewords \( \{ \emptyset, 1, 13, 123, 23, 2 \} \). Meanwhile,

\[
\text{code}(e_3 \cdot \mathcal{H}, \mathbb{R}^2) = \{ 3, 13, 1, 12, 2, 23, \emptyset \} = e_3 \cdot \text{code}(\mathcal{H}, \mathbb{R}^2) \cup \{ \emptyset \}.
\]

The “extra” codeword \( \emptyset \) appears because after flipping hyperplane \( H_3 \), the origin no longer belongs to the same atom as the points to its left, and thus produces a new codeword, see Figure 2(b).

Nevertheless, the group action does commute with taking the code of a stable hyperplane arrangement.

**Proposition 2.7.** If \((\mathcal{H}, X)\) is a stable arrangement, then for every \( g \in (\mathbb{Z}_2)^n \), \((g \cdot \mathcal{H}, X)\) is also a stable arrangement and

\[
\text{code}(g \cdot \mathcal{H}, X) = g \cdot \text{code}(\mathcal{H}, X).
\]

**Proof.** Since the action of \((\mathbb{Z}_2)^n\) does not change the hyperplanes \( H_i \) (only their orientation) nor the set \( X \), the stability is preserved. By Lemma 2.4, each atom of \((\mathcal{H}, X)\) has a nonempty interior; this interior is not changed by reorientation of the hyperplanes. Thus, atoms are neither created nor destroyed by reorienting hyperplanes in a stable arrangement; only their labels change, and \(\text{code}(g \cdot \mathcal{H}, X) = g \cdot \text{code}(\mathcal{H}, X)\). \(\square\)
2.3. The polar complex. The invariance \(\Gamma\) of the class of stable hyperplane codes under the \((\mathbb{Z}_2)^n\) action makes it natural to consider a simplicial complex whose structure is preserved by bitflips. Note that the simplicial complex of the code is insufficient for this purpose: for any nontrivial code \(C \subseteq 2^n\) with a nonempty codeword, the simplicial complexes of the codes in the \((\mathbb{Z}_2)^n\)-orbit of \(C\) will include the full simplex on \(n\) vertices, regardless of the structure of \(\Delta(C)\).

We denote by \([n]\) \(\defeq\{1, \ldots, n\}\) and \([\bar{n}]\) \(\defeq\{\bar{1}, \ldots, \bar{n}\}\) two separate copies of the vertex set. Given a code \(C \subseteq 2^n\), define the polar complex, \(\Gamma(C)\), as a pure \((n-1)\)-dimensional simplicial complex on vertex set \([n]\union[\bar{n}]\) with facets in bijection with the codewords of \(C\).

**Definition 2.8.** Let \(C \subseteq 2^n\) be a combinatorial code. For every codeword \(\sigma \in C\) denote

\[
\Sigma(\sigma) \defeq \{i \mid i \in \sigma\} \union \{\bar{i} \mid i \notin \sigma\} = \sigma \union [\bar{n}] \setminus \sigma
\]

and define the polar complex of \(C\) as

\[
\Gamma(C) \defeq \Delta(\{\Sigma(\sigma) \mid \sigma \in C\}).
\]

Continuing Example 2.5, the polar complex of \(C_1 = \{1, 12, 123, 2, 23\}\) is given by \(\Gamma(C_1) = \Delta(\{123, 123, 123, 123, 123\})\). It is depicted in Figure 2(b) as a subcomplex of the octahedron. Note that the polar complex \(\Gamma(2^{[3]})\) consists of the eight boundary faces of the octahedron. Generally, the polar complex of the code consisting of all \(2^n\) codewords on \(n\) vertices is the boundary of the \(n\)-dimensional cross-polytope.

The polar complex of code \(C_2\) in Example 2.6 is depicted in Figure 2(c). Note that it follows from Theorem 4 in Section 4 that \(C_2\) is not a stable hyperplane code, i.e. it can not be realized by a stable arrangement \((\mathcal{H}, X)\). This is because the complex \(\Gamma(C_2)\) is not shellable. In contrast, while Figure 2(b) depicts a non-stable arrangement, the code of that arrangement has a stable realization depicted in Figure 2(d).

The action of the bitflips \((\mathbb{Z}_2)^n\) on the Boolean lattice induces an action on the facets of the polar complex, so that \(g \cdot \Sigma(\sigma) = \Sigma(g \cdot \sigma)\). In particular, \(\Gamma(g \cdot C) = g \cdot \Gamma(C)\), and the complex \(\Gamma(g \cdot C)\) is isomorphic to \(\Gamma(C)\). The Stanley-Reisner ideal of \(\Gamma(C)\) is closely related to the neural ideal, defined in \([9]\); this will be elaborated in Section 6. Moreover, in the case of stable hyperplane codes, \(\Gamma(C)\) has a simple description as the nerve of a cover:

**Lemma 2.9.** If \(C = \text{code}(\mathcal{H}, X)\) is the code of a stable hyperplane arrangement, then

\[
(2) \quad \Gamma(C) = \text{nerve}(\{H^+_i \intersection X, H^-_i \intersection X\}_{i \in [n]}),
\]

where the sets \(H^-_i \intersection X\) are indexed by \(i \in [\bar{n}]\).

**Proof.** Consider a maximal face \(\Sigma(\sigma) \in \Gamma(C)\). By Lemma 2.4, each atom \(A_{\sigma}\) has a nonempty interior given by \(X \intersection \bigcap_{i \in \sigma} H^+_i \intersection \bigcap_{j \in \sigma} H^-_j\), hence \(\Sigma(\sigma) \in \text{nerve}(\{H^+_i \intersection X, H^-_i \intersection X\}_{i \in [n]})\).

Likewise, if \(F\) is maximal in the complex \(\text{nerve}(\{H^+_i \intersection X, H^-_i \intersection X\}_{i \in [n]})\), the subset consisting of unbarred vertices in \(F\) is a codeword as the corresponding atom is nonempty. \(\square\)

3. Obstructions for hyperplane codes

Here we describe all currently known hyperplane obstructions, properties of a combinatorial code that are necessary for it to be realized by a stable hyperplane arrangement.
3.1. Local obstructions and bitflips. A larger class of codes that arises in the neuroscience context are the open convex codes \([7, 9, 14]\). A code \(C \subset \mathbb{R}^d\) is called open convex if there exists a collection \(\mathcal{U}\) of \(n\) open and convex sets \(U_i \subseteq X \subseteq \mathbb{R}^d\), such that \(C = \text{code}(\mathcal{U}, X)\). Not every combinatorial code is convex. One obstruction to being an open convex code stems from an analogue of the nerve lemma \([3]\), recently proved in \([6]\); see also \([22]\).

Recall the link of a face \(\sigma\) in a simplicial complex \(\Delta\) is the subcomplex defined by

\[
\text{link}_\sigma \Delta \stackrel{\text{def}}{=} \{\nu \in \Delta \mid \sigma \cap \nu = \emptyset, \sigma \cup \nu \in \Delta\}.
\]

When \(\sigma \not\in \text{code}(\mathcal{U}, X)\), yet \(\sigma \in \text{nerve}(\mathcal{U})\), the subset \(U_\sigma \stackrel{\text{def}}{=} \bigcap_{i \in \sigma} U_i\) is covered by the collection of sets \(\{U_j \cap U_\sigma\}_{j \not\in \sigma}\). It is easy to see that in this situation,

\[
\text{link}_\sigma \text{nerve} \mathcal{U} = \text{nerve}(\{U_j \cap U_\sigma\}_{j \not\in \sigma}),
\]

see e.g. \([7, 8, 14]\).

**Definition 3.1.** A pair of faces \((\sigma, \tau)\) of a simplicial complex \(\Delta\) is a free pair if \(\tau\) is a facet of \(\Delta\), \(\sigma \subsetneq \tau\), and \(\sigma \not\subset \tau'\) for any other facet \(\tau' \neq \tau\). The simplicial complex

\[
\text{del}_\sigma \Delta \stackrel{\text{def}}{=} \{\nu \in \Delta \mid \nu \supset \sigma\}
\]

is called the collapse of \(\Delta\) along \(\sigma\), and is denoted as \(\Delta \setminus_\sigma \text{del}_\sigma \Delta\). If a finite sequence of collapses of \(\Delta\) results in a new complex \(\Delta'\), we write \(\Delta \setminus \Delta'\). If \(\Delta \setminus \{\}\), we say \(\Delta\) is collapsible.

Note that the irrelevant simplicial complex \(\{\emptyset\}\), consisting of a single empty face, is not collapsible, as there is no other face properly contained in \(\emptyset\). However, the void complex \(\{\}\) with no faces is collapsible.

**Lemma 3.2** (\([6, \text{Lemma } 5.9]\), \([22]\)). For any collection \(\mathcal{U} = \{U_1, \ldots, U_n\}\) of open convex sets \(U_i \subset \mathbb{R}^d\) whose union \(\bigcup_{i \in [n]} U_i\) is also convex, its nerve, \(\text{nerve}(\mathcal{U})\), is collapsible.

**Corollary 3.3** (\([6, \text{Theorem } 5.10]\)). Let \(C = \text{code}(\mathcal{U}, X)\) with each \(U_i \subseteq X \subseteq \mathbb{R}^d\) open and convex. Then \(\text{link}_\sigma \Delta(C)\) is collapsible for every nonempty \(\sigma \in \Delta(C) \setminus C\).

The last observation provides a “local obstruction” for a code \(C\) being an open convex code: if a non-empty \(\sigma \in \Delta(C) \setminus C\) has a non-collapsible link, then \(C\) is nonconvex. It had been previously known (see, for example, \([14, \text{Theorem } 3]\)) that \(\text{link}_\sigma \Delta(C)\) is contractible under the hypotheses of Corollary 3.3. Since collapsibility implies contractibility but not vice versa, we refer to a face \(\sigma \in \Delta(C) \setminus C\) with non-collapsible link as a strong local obstruction; if \(\text{link}_\sigma \Delta(C)\) is non-contractible, we refer to \(\sigma\) as a weak local obstruction.

Half-spaces are convex, thus local obstructions to being a convex code are also obstructions to being a hyperplane code. Therefore Proposition 2.7 implies a much stronger statement. Not only are local obstructions in \(C\) forbidden, we must also exclude local obstructions in \(g \cdot C\) for all bitflips \(g \in (\mathbb{Z}_2)^n\), since \(g \cdot C\) is also a stable hyperplane code. We make this precise below.

**Definition 3.4.** Let \(g \in (\mathbb{Z}_2)^n\) and \(\tau \subseteq [n]\) be a pair such that \(\text{link}_\tau \Delta(g \cdot C)\) is not collapsible (respectively, contractible) and \(\tau \not\in g \cdot C\). Then \((g, \tau)\) is called a strong (resp. weak) bitflip local obstruction.
Theorem 1 (Bitflip local property). Suppose \( C \) is a stable hyperplane code. Then \( C \) has no strong bitflip local obstructions.

Proof. Halfspaces are convex, thus \( C \) has no strong local obstructions. By Proposition 2.7, \( g \cdot C \) is a stable hyperplane code for all \( g \in (\mathbb{Z}_2)^n \). Hence, \( g \cdot C \) has no strong local obstructions. \( \square \)

The nomenclature of “weak” and “strong” local obstructions signifies that a code with no strong local obstructions has no weak local obstructions, but generally not vice-versa. In particular, a stable hyperplane code also has no weak bitflip local obstructions.

Example 3.5. The code \( C_3 = \{ \emptyset, 2, 3, 4, 12, 13, 14, 23, 24, 123, 124, 3, 4, 13, 14 \} \) is realizable by open convex sets in \( \mathbb{R}^2 \) (see Figure 3), and thus it cannot have local obstructions to convexity. Flipping bit 2 yields

\[
e_2 \cdot C_3 = \{ 2, \emptyset, 23, 24, 1, 123, 124, 3, 4, 13, 14 \}.
\]

The new simplicial complex \( \Delta(e_2 \cdot C_3) \) has facets 123 and 124. The edge 12 is not in the code and \( \text{link}_{12} \Delta(e_2 \cdot C_3) \) is two vertices; therefore, \( (e_2, 12) \) is a bitflip local obstruction and \( C_3 \) is not a stable hyperplane code.

It is worth highlighting an essential feature of the polar complex that makes it a natural tool for studying hyperplane codes, in light of the bitflip local property. For every \( g \in (\mathbb{Z}_2)^n \), the simplicial complex \( \Delta(g \cdot C) \) is isomorphic to an induced subcomplex of \( \Gamma(C) \): Let \( \sigma \) denote the support of \( g \) and define

\[
\Gamma(C)_{|([n] \setminus \sigma) \cup \sigma} \overset{\text{def}}{=} \{ F \in \Gamma(C) \mid F \subseteq ([n] \setminus \sigma) \cup \sigma \}.
\]

Then \( \Gamma(C)_{|[n] \setminus \sigma} \approx \Delta(g \cdot C) \), with the isomorphism given by “ignoring the bars,” i.e. \( i \mapsto i \) for \( i \in [n] \setminus \sigma \) and \( j \mapsto j \) for \( j \in \sigma \). Thus we can find bitflip local obstructions directly in the polar complex as follows.

Proposition 3.6. Let \( C \subseteq 2^n \) be a code, \( g \in (\mathbb{Z}_2)^n \) with \( \sigma \) its support, and let \( \tau \subseteq [n] \). Then \( (g, \tau) \) is a bitflip local obstruction for \( C \) if and only if

\[
g \cdot \tau \sqcup [n] \setminus g \cdot \tau \not\in \Gamma(C) \quad \text{and} \quad \text{link}_{g \cdot \tau} \Gamma(C)_{|([n] \setminus \sigma) \cup \sigma} \text{ is not collapsible}.
\]

Proof. Note that \( g \cdot \tau \sqcup [n] \setminus g \cdot \tau \not\in \Gamma(C) \) if and only if \( \tau \not\in g \cdot C \). The complex \( \Gamma(C)_{|([n] \setminus \sigma) \cup \sigma} \) is isomorphic to \( \Delta(g \cdot C) \), and

\[
\text{link}_{\tau} \Delta(g \cdot C) \cong \text{link}_{g \cdot \tau} \Gamma(C)_{|([n] \setminus \sigma) \cup \sigma}.
\]

Hence, the conditions of the proposition are equivalent to the conditions of Definition 3.4. \( \square \)
3.2. Spherical Link Obstructions. Here we introduce another obstruction that can be detected via the polar complex of stable hyperplane codes. We use the following notation to aid our discussion. For a face \( F \in \Gamma(C) \), we write \( F = F^+ \cup F^- \) to denote the restrictions of \( F \) to \([n]\) and \([\overline{n}]\). The support of \( F \) is \( F = F^+ \cup F^- \), the set of (barred or unbarred) vertices appearing in it.

For stable arrangements \((\mathcal{H}, X)\), Lemmas 2.4 and 2.9 allow us to translate between faces of \( \Gamma(\text{code}(\mathcal{H}, X)) \) and convex subsets of \( X \) as follows: The face \( F = F^+ \cup F^- \in \Gamma(C) \) corresponds to the open convex set

\[
R_F = X \cap \left( \bigcap_{i \in F^+} H_i^+ \right) \cap \left( \bigcap_{j \in F^-} H_j^- \right).
\]

Note that for a facet \( F = \sigma \cup ([n] \setminus \sigma) \) of the polar complex, \( R_F \) is precisely the interior of the atom \( A_\sigma \). In addition, it is easy to see that \( \text{link}_F \Gamma(C) = \Gamma(C') \) for some \( C' \subseteq 2^{[n]} \setminus \mathcal{E} \). Therefore, we consider the topology of the covered subset of \( R_F \). We show the positive and negative halfspaces indexed by the complement of \( F \) will cover either all of \( R_F \) or all but a linear subspace of \( R_F \). The following proposition describes the combinatorics of the nerve of this cover.

**Proposition 3.7.** Let \((\mathcal{H}, X)\) be a stable arrangement, and let \( R_F \) be a nonempty region with \(|F| < n\). Then \((\{H_i^+ \cap R_F\}_{i \notin \mathcal{E}, F}, R_F)\) is a stable arrangement. Moreover, the complex nerve \(\{H_i^+ \cap R_F\}_{i \notin \mathcal{E}, F}\) is either collapsible or is the polar complex of the full code on the vertices \([n] \setminus F\), i.e. nerve \(\{H_i^+ \cap R_F\}_{i \notin \mathcal{E}, F}\) = \(\Gamma\left(2^{[n]} \setminus \mathcal{E}\right)\).

**Proof.** Denote \( \nu \overset{\text{def}}{=} [n] \setminus F \). First we verify the arrangement \(\{H_i^+ \cap R_F\}_{i \in \nu, F}, R_F\) is stable. The region \( R_F \) is open and convex, and intersections of hyperplanes in \( R_F \) lie in \( X \), so they already satisfied the genericity condition.

Consider \( H_\nu \cap R_F \): if it is empty, then the union of the positive and negative open halfspaces indexed by \( \nu \) is all of the convex set \( R_F \), and so by Lemma 3.2, the nerve is collapsible. If \( H_\nu \cap R_F \neq \emptyset \), by stability, we have \( \dim H_\nu = d - |\nu| \). In this case, the linear independence of \( \{w_i \mid i \in \nu\} \) ensures all of the \( 2^{|\nu|} \) intersection patterns of halfspaces, i.e. the nerve is \(\Gamma(2^\nu) = \Gamma(2^{[n]} \setminus \mathcal{E})\).

**Definition 3.8.** Let \( F \in \Gamma(C) \) be a non-maximal face such that \( \text{link}_F \Gamma(C) = \Gamma(C') \) for some \( C' \subseteq 2^{[n]} \setminus \mathcal{E} \). We call \( F \) a sphere link obstruction.

By Lemma 2.9, we have \( \text{link}_F \Gamma(C) = \text{nerve}(\{H_i^+ \cap R_F, H_i^- \cap R_F\}_{i \notin \mathcal{E}}) \). This, together with the Proposition 3.7 imply

**Theorem 2** (Sphere link property). **Suppose \( C \) is a stable hyperplane code. Then \( C \) has no sphere link obstructions.**

**Example 3.9.** Continuing Example 2.6, we consider the polar complex \( \Gamma(C_2) \) for the unstable arrangement \((\mathcal{H}, X)\) in Figure 2(a). This complex is illustrated in Figure 2(c). The face \( \emptyset \) is a sphere link obstruction: \( \text{link}_{\emptyset} \Gamma(C_2) = \Gamma(C_2) \), and this complex is neither the complex \( \Gamma(2^{[3]}) \), which would have 8 facets, nor is it collapsible. Therefore, \( C_2 \) is not a stable hyperplane code.
3.3. Chamber Obstructions. The intuition behind the third obstruction in this section concerns maximal hyperplane intersections. If a collection \( \{H_i\}_{i \in \sigma} \) of hyperplanes intersects in a point (\( \dim H_\sigma = 0 \)), then that point has fixed position relative to other hyperplanes. In particular, there cannot be two distinct regions defined by the other hyperplanes that contain that point. More generally, if \( H_\sigma \neq \emptyset \) is a maximal non-empty intersection, then it intersects only one atom of the arrangement \( \{H_j\}_{j \not\in \sigma} \) of the remaining hyperplanes.

**Definition 3.10.** The geometric chamber complex of a hyperplane arrangement \( \mathcal{H} \) relative to an open convex set \( X \), cham(\( \mathcal{H}, X \)), is the set of \( \sigma \subseteq [n] \) such that \( H_\sigma \cap X \neq \emptyset \). By convention, \( H_\emptyset = \mathbb{R}^d \) so \( \emptyset \in \text{cham}(\mathcal{H}, X) \) for all \( (\mathcal{H}, X) \).

The combinatorial chamber complex of a code \( C \), denoted cham(\( C \)), is given by the set of \( \sigma \subseteq [n] \) such that there exists \( T \in \Gamma(C) \) with \( T = [n] \setminus \sigma \) and \( \text{link}_T \Gamma(C) = \Gamma(2^{\sigma}) \). We call such a subset \( T \) a chamber of \( \sigma \).

Both cham(\( \mathcal{H}, X \)) and cham(\( C \)) are simplicial complexes: the former because for any \( i \in \sigma \), \( H_{\sigma \cup \{i\}} \supseteq H_\sigma \); the latter because if \( \text{link}_T \Gamma(C) = \Gamma(2^{\sigma}) \) then \( \text{link}_{T \cup \{i\}} \Gamma(C) = \Gamma(2^{\sigma \cup \{i\}}) \). For stable hyperplane codes, the facets of these simplicial complexes correspond to maximal hyperplane intersections.

**Example 3.11.** Returning to the stable code \( C_1 \) from Example \( [2,5] \) the maximal faces of cham(\( C_1 \)) are 2 and 13. This is because \( \text{link}_{13} \Gamma(C_1) = \Gamma(2^{\{2\}}) \) and \( \text{link}_3 \Gamma(C_1) = \Gamma(2^{\{1,3\}}) \). By inspection, these are also maximal faces of the geometric chamber complex cham(\( \mathcal{H}, X \)) for the arrangement in Figure \( [1] \).

**Proposition 3.12.** For a stable arrangement \( (\mathcal{H}, X) \), the associated chamber complexes coincide, cham(\( \mathcal{H}, X \)) = cham(code(\( \mathcal{H}, X \))). Moreover, for \( C = \text{code}(\mathcal{H}, X) \), each facet \( \sigma \) of cham(\( C \)) has a unique chamber \( T \in \Gamma(C) \).

**Proof.** Let \( (\mathcal{H}, X) \) be a stable pair and set \( C = \text{code}(\mathcal{H}, X) \). Suppose \( \sigma \in \text{cham}(\mathcal{H}, X) \), so \( H_\sigma \cap X \neq \emptyset \). Then, for any atom \( A_\tau \) of the arrangement \( \{H_i^+ \cap X\}_{i \in \sigma} \) such that \( H_\sigma \cap A_\tau \neq \emptyset \), the set \( T = \tau \cup [n] \setminus \sigma \) is a chamber of \( \sigma \), hence \( \sigma \in \text{cham}(C) \). For the reverse containment, suppose \( \sigma \in \text{cham}(C) \) has chamber \( T \). Then

\[
\Gamma(2^{\sigma}) = \text{link}_T \Gamma(C) = \Gamma(\text{code}(\{H_i^+ \cap R_T\}_{i \not\in \sigma}, R_T)),
\]

meaning the hyperplanes \( \{H_i\}_{i \in \sigma} \) partition \( R_T \) into the maximal number of regions, i.e. it is a central arrangement. Thus \( H_\sigma \cap R_T \neq \emptyset \) and therefore \( H_\sigma \cap X \neq \emptyset \) and \( \sigma \in \text{cham}(\mathcal{H}, X) \).

Now consider \( \sigma \) a facet of cham(\( C \)). Because \( C = \text{code}(\mathcal{H}, X) \), the intersection of hyperplanes \( H_\sigma \cap X \) does not meet any other hyperplanes inside \( X \). Therefore, it is interior to only one atom of the arrangement \( \{H_j^+\}_{j \not\in \sigma}, X \); the face in \( \Gamma(C) \) corresponding to this atom is the unique chamber \( T \).

We reformulate Proposition \( [3.12] \) into our third and final obstruction to hyperplane codes.

**Definition 3.13.** Let \( \sigma \subseteq [n] \) be a maximal face of cham(\( C \)) such that there exist two faces \( T_1 \neq T_2 \in \Gamma(C) \) with \( \text{link}_{T_1} \Gamma(C) = \text{link}_{T_2} \Gamma(C) = \Gamma(2^{\sigma}) \). Then we call \( \sigma \) a chamber obstruction.

**Theorem 3 (Single chamber property).** Suppose \( C = \text{code}(\mathcal{H}, X) \) is a stable hyperplane code. Then \( C \) has no chamber obstructions.
Example 3.14. The code \( C_3 \) from Example 3.5 also has a chamber obstruction, in the form of \( \sigma = \{1, 2\} \). There are two faces \( \{3, 4\} \) and \( \{3, 4\} \) with link in \( \Gamma(C_3) \) equal to the full polar complex on \( \{1, 2\} \). One can check that this is maximal in \( \text{cham}(C_3) \), creating a chamber obstruction.

4. The main results

Our main results consist of showing that (i) the polar complex of a stable hyperplane code is shellable and (ii) shellability of \( \Gamma(C) \) implies \( C \) has none of the obstructions thus far considered, except possibly the strong bitflip obstruction. First, we define shellability.

**Definition 4.1.** Let \( \Delta \) be a pure simplicial complex of dimension \( d \) and \( F_1, \ldots, F_t \) an ordering of its facets. The ordering is a **shelling order** if, for \( i > 1 \), the complex

\[
\Delta(\{F_i\}) \cap \Delta(\{F_1, \ldots, F_{i-1}\})
\]

is pure of dimension \( d - 1 \). A simplicial complex is **shellable** if its facets permit a shelling order.

A shelling order constructs a simplicial complex one facet at a time in such a way that each new facet intersects the previous complex along maximal boundary faces. For example, the polar complex \( \Gamma(C_1) \) illustrated in Figure 1(b) is shellable. A shelling order is given by \( 1\bar{2}\bar{3}, 12\bar{3}, 123, \bar{1}\bar{2}\bar{3}, \bar{1}\bar{2}3 \), illustrated in Figure 4.

An equivalent characterization of shellability can be formulated in terms of what elements of the new simplex do **not** belong to the previous complex (see, e.g. [27, Chapter III]). This characterization will be useful for the proof of Theorem 4.

**Lemma 4.2** ([12, Proposition 6.13]). Let \( \Delta \) be a simplicial complex and \( F_1, \ldots, F_t \) an ordering of its facets. The ordering is a shelling order if and only if the sequence of complexes \( \Delta_i = \Delta(\{F_1, \ldots, F_i\}) \) satisfies the property that the collection of faces \( \Delta_i \setminus \Delta_{i-1} \), for each \( i = 2, \ldots, t \), has a unique **minimal element**. This element is denoted \( r(F_i) \) and called the associated minimal face of \( F_i \).

The facets of \( \Gamma(C) \) correspond to codewords of \( C \), thus a shelling order of \( \Gamma(C) \) corresponds to an ordering of the codewords. We construct such an order in Section 7.1 to prove the following theorem.

**Theorem 4.** Let \( C \subseteq 2^{[n]} \) be a stable hyperplane code. Then \( \Gamma(C) \) is shellable.

It turns out that the structure of shellable polar complexes does not allow for many of the obstructions thus far considered.
Theorem 5. Let $\mathcal{C} \subseteq 2^{[n]}$ be a combinatorial code such that $\Gamma(\mathcal{C})$ is shellable. Then,
1. $\mathcal{C}$ has no weak bitflip local obstructions,
2. $\mathcal{C}$ has no sphere link obstructions, and
3. $\mathcal{C}$ has no chamber obstructions.

Theorem 5 is proven in Section 7.2. Note the conclusion of Theorem 5 refers to weak local obstructions, highlighting the gap between the notions of collapsibility and contractibility.

5. Discussion

Hyperplane codes are a special class of convex codes that naturally arise as the output of a one-layer feedforward network [14]. Hyperplane codes are a proper subclass of the class of open convex codes. We set out to find obstructions to being a hyperplane code, while focusing on stable hyperplane codes. There are two reasons for primarily considering the stable hyperplane codes: (i) they are ‘generic’ in that they are stable to small perturbations, and (ii) they allow the action of the group of bitflips ($\mathbb{Z}_2^n$). The second property makes it natural to consider the polar complex $\Gamma(\mathcal{C})$ of a code, because the combinatorics of the polar complex captures all the bitflip-invariant properties of the underlying stable hyperplane code. We have established the following relationships among the properties of the polar complex of the code. First, necessary conditions for $\mathcal{C}$ being a stable hyperplane code:

$$\Gamma(\mathcal{C}) \text{ is shellable} \iff \mathcal{C} \text{ is a stable hyperplane code} \implies \left\{ \begin{array}{l}
\mathcal{C} \text{ has no strong bitflip obstructions,} \\
\mathcal{C} \text{ has no sphere link obstructions,} \\
\mathcal{C} \text{ has no chamber obstructions.}
\end{array} \right.$$ 

We have also established that almost all currently known necessary conditions follow from the shellability of the polar complex:

$$\Gamma(\mathcal{C}) \text{ is shellable} \implies \left\{ \begin{array}{l}
\mathcal{C} \text{ has no weak bitflip obstructions,} \\
\mathcal{C} \text{ has no sphere link obstructions,} \\
\mathcal{C} \text{ has no chamber obstructions.}
\end{array} \right.$$ 

Note that the shellability of the polar complex implies the lack of weak bitflip obstructions, while a stable hyperplane code lacks strong bitflip obstructions. It is currently an open problem if the gap between the strong and the weak versions of the local obstructions is indeed a property of shellable polar complexes. Alternatively, codes with shellable polar complexes may also lack the strong bitflip obstructions. An example of a code whose polar complex is shellable, but has the strong bitflip obstruction would provide a negative answer to the following open question: Is shellability of the polar complex equivalent to the code being a stable hyperplane code?

What makes a code a stable hyperplane code is still an open question. It seems likely that the shellability of the polar complex is not the only necessary condition for a code to be a stable hyperplane code. From a computational perspective, deciding if a given pure simplicial complex is shellable is known to be an NP-hard problem [15]. This likely means that answering the question of whether a given code is produced by a one-layer network may

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1See e.g. Example 3.5 and Figure 3.
2In particular, the appropriate link in Definition 3.4 is contractible, but not collapsible.
be not computationally feasible. Ruling out that a given code is a hyperplane code may be less computationally intensive however, as it can rely on computing the Betti numbers of the free resolution of the Stanley-Reisner ideal of the polar complex, as illustrated in the following section.

6. Algebraic signatures of a hyperplane code

Given a code \( \mathcal{C} \), how can we rule out that \( \mathcal{C} \) is a stable hyperplane code? In this section, we show how the tools from computational commutative algebra can be used to detect sphere link obstructions via Stanley-Reisner theory.

6.1. The neural ideal and the Stanley-Reisner ideal. The connections between neural codes and Stanley-Reisner theory were first developed in [9], and later expanded upon in [10], [13], and [17]. The key observation is that a code \( \mathcal{C} \subseteq 2^{[n]} \) can be considered as a set of points in \((\mathbb{F}_2)^n\), and the vanishing ideal \( I_\mathcal{C} \) of that variety is a “pseudo-monomial ideal” with many similarities to a monomial ideal. In this section, we show that this connection can be made more explicit via the polar complex.

First, we state necessary prerequisites about the neural ring. Let \( \mathbb{F}_2 \) denote the field with two elements, and consider the polynomial ring \( R \) defined as \( \mathbb{F}_2[x_1, \ldots, x_n] \). A polynomial \( f \in R \) can be considered as a function \( f : 2^n \to \mathbb{F}_2 \) by defining \( f(\sigma) \) as the evaluation of \( f \) with \( x_i = 1 \) for \( i \in \sigma \) and \( x_i = 0 \) for \( i \notin \sigma \). Polynomials of the form
\[
x_\sigma(1-x)^\tau = \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1-x_j),
\]
where \( \sigma, \tau \subseteq [n] \), are said to be pseudo-monomials. Note that the pseudo-monomial \( x_\sigma(1-x)^{[n]\setminus \sigma} \) evaluates to 1 if and only if the support of \( x \) equals \( \sigma \); such a pseudo-monomial is called the indicator function of \( \sigma \).

**Definition 6.1** ([9]). The vanishing ideal of a code \( \mathcal{C} \subseteq 2^{[n]} \) is the ideal of polynomials that vanish on all codewords of \( \mathcal{C} \),
\[
I_\mathcal{C} \overset{\text{def}}{=} \{ f \in R \mid f(\sigma) = 0 \text{ for all } \sigma \in \mathcal{C} \}.
\]

The neural ideal of \( \mathcal{C} \) is the ideal generated by indicator functions of non-codewords,
\[
J_\mathcal{C} \overset{\text{def}}{=} \langle x_\sigma(1-x)^{[n]\setminus \sigma} \mid \sigma \notin \mathcal{C} \rangle.
\]

The Boolean ideal of \( \mathcal{C} \) is the ideal generated by the Boolean relations, pseudo-monomials with \( \sigma = \tau = i \),
\[
\mathcal{B} \overset{\text{def}}{=} \langle x_i(1-x_i) \mid i \in [n] \rangle.
\]

**Lemma 6.2** ([9, Lemma 3.2]). Let \( \mathcal{C} \) be a neural code. Then \( I_\mathcal{C} = J_\mathcal{C} + \mathcal{B} \).

Pseudomonomials in the vanishing ideal \( I_\mathcal{C} \) correspond to relations of the form \( \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j \) among sets in any cover realizing \( \mathcal{C} \).

**Lemma 6.3** ([9, Lemma 4.2]). Let \( \mathcal{C} = \text{code}(\mathcal{U}, X) \) be a combinatorial code. Then
\[
x_\sigma(1-x)^\tau \in I_\mathcal{C} \iff \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j,
\]
where by convention $\bigcap_{i \in \emptyset} U_i = X$ and $\bigcup_{j \in \emptyset} U_j = \emptyset$.

In particular, the generators of $B$ correspond to the tautological relations $U_i \subseteq U_i$. The neural ideal records the non-tautological relations.

**Definition 6.4** ([9]). A pseudo-monomial $f \in J_C$ is said to be minimal if there is no other pseudo-monomial $g \in J_C$ that divides $f$. The canonical form of $J_C$, denoted $CF(J_C)$, is the set of all the minimal pseudo-monomials in $J_C$.

The elements of the canonical form correspond to the minimal nontrivial relations $\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j$. We will see that the canonical form of $J_C$ and the Boolean relations also correspond to the generating set of the Stanley-Reisner ideal of $\Gamma(C)$. We make these relationships explicit in Lemma 6.7 and Corollary 6.8.

The Stanley-Reisner correspondence associates to any simplicial complex on $n$ vertices an ideal generated by square-free monomials in a polynomial ring in $n$ variables [27]. The construction of the polar complex is seen to be particularly natural when considering its associated Stanley-Reisner ideal. For the unbarred vertices, we set the corresponding variables via $i \mapsto x_i$; for the barred vertices, we associate $\bar{i} \mapsto y_i$. The Stanley-Reisner ideal of $\Gamma(C)$ is the ideal in $S \overset{\text{def}}{=} \mathbb{F}_2[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by the squarefree monomials indexed by non-faces of $\Gamma(C)$.

**Definition 6.5.** Let $C \subseteq 2^{[n]}$ be a combinatorial code. The Stanley-Reisner ideal of the polar complex is given by

$$I_{\Gamma(C)} = \langle x^\sigma y^\tau \mid \sigma \cup \tau \not\in \Gamma(C) \rangle \subseteq S.$$  

**Example 6.6.** Consider the code $C_1 = \{1, 12, 123, 2, 23\}$ from Example 2.5. The corresponding variety in $\mathbb{F}_2^3$ is $\{100, 110, 111, 010, 011\}$ with canonical form given by

$$CF(J_{C_1}) = \{(1 - x_1)(1 - x_2), x_3(1 - x_2)\}.$$  

The polar complex of $C_1$ is given by

$$\Gamma(C_1) = \Delta(\{1 \bar{2} \bar{3}, 12 \bar{3}, 123, \bar{1}2 \bar{3}, \bar{1}23\}).$$

The minimal nonfaces of $\Gamma(C_1)$ are $\{1 \bar{1}, 2 \bar{2}, 3 \bar{3}, \bar{1}2, 23\}$. This gives the Stanley-Reisner ideal

$$I_{\Gamma(C_1)} = \langle x_1 y_1, x_2 y_2, x_3 y_3, y_1 y_2, x_3 y_2 \rangle.$$  

The first three monomials in this list correspond to the Boolean relations, while the last two can be compared to the canonical form.

The intuition intimated by Example 6.6 holds true in general.

**Lemma 6.7.** For any nonempty combinatorial code $C \subseteq 2^{[n]}$, the Stanley-Reisner ideal of the polar complex is induced by the canonical form and the Boolean relations. That is,

$$x^\sigma y^\tau \in I_{\Gamma(C)} \iff x^\sigma (1 - x)^\tau \in I_C.$$  

and so

$$I_{\Gamma(C)} = \langle x^\sigma y^\tau \mid x^\sigma (1 - x)^\tau \in CF(J_C) \rangle + \langle x_i y_i \mid i \in [n] \rangle.$$
Proof of Lemma 6.7. Consider a square-free monomial \( x^\sigma y^\tau \in S \). By definition, \( x^\sigma y^\tau \in I_{\Gamma(C)} \) if and only if \( \sigma \sqcup \tau \) is a nonface of \( \Gamma(C) \). The set \( \sigma \sqcup \tau \) is a nonface of \( \Gamma(C) \) if and only if any codeword in \( C \) which contains \( \sigma \) is not disjoint from \( \tau \), that is, \( C \) satisfies the following property:

\[
\text{for all } \alpha \in C, \quad \sigma \subseteq \alpha \implies \alpha \cap \tau \neq \emptyset. \tag{5}
\]

If \( C \) satisfies (5), the pseudomonomial \( x^\sigma (1-x)^\tau \) vanishes on all of \( C \), as \( x^\sigma \) evaluates to 0 on any codeword not containing \( \sigma \), and \( (1-x)^\tau \) evaluates to 0 on any codeword not disjoint from \( \tau \), e.g. any codeword containing \( \sigma \). Conversely, if \( x^\sigma (1-x)^\tau \) vanishes on all of \( C \), every codeword that contains \( \sigma \) must not be disjoint from \( \tau \), so \( C \) satisfies (5). Therefore, \( x^\sigma (1-x)^\tau \in I_C \). Thus we have established (3) and (4) follows, as any pseudomonomial in \( I_C \) is divisible either by \( x^i (1-x^i) \) for some \( i \), or by an element of the canonical form \( CF(J_C) \). \( \square \)

The following is an immediate corollary of Lemma 6.3 and Lemma 6.7.

**Corollary 6.8.** Let \( C = \text{code}(U, X) \subseteq 2^{[n]} \) and \( I_{\Gamma(C)} \) the Stanley-Reisner ideal of the polar complex of \( C \). Then

\[
x^\sigma y^\tau \in I_{\Gamma(C)} \iff \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j.
\]

### 6.2. Sphere link obstructions and multigraded free resolutions.

In Section 3.2, we showed that \( \text{link}_{\Sigma}(\Gamma(C)) \) is either empty, collapsible, or is isomorphic to a sphere of dimension \( n - |\Sigma| - 1 \) when \( C \) is a stable hyperplane code. One consequence of this fact is that if a stable hyperplane realization of \( C \) exists, then a lower bound on the dimension of the realizing space is

\[
d \geq \max_{\Sigma \in \Gamma(C)} \left\{ (n - |\Sigma|) \mid \text{link}_{\Sigma}(\Gamma(C)) \sim S^{n-|\Sigma|-1} \right\}.
\]

However, this may not be the true lower bound.

![Figure 5](image.png)

**Figure 5.** (a) Realization of \( C_4 = \{ \emptyset, 1, 2, 3 \} \) in \( \mathbb{R}^2 \). Though sphere link dimension is 1, minimal realization dimension is 2. (b) The polar complex \( \Gamma(C_4) \). The only non-collapsible links are of the form \( \text{link}_{\Sigma}(\Gamma(C)) \)

**Example 6.9.** Consider the code \( C_4 = \{ \emptyset, 1, 2, 3 \} \) consisting of four words; this can be realized by hyperplanes in \( \mathbb{R}^2 \) as in Figure 5. Still, the polar complex \( \Gamma(C) \) has facets \( 123, 123, 123, 123 \), which has spherical links only at \( \Sigma = \{i, j\} \) for \( i \neq j \in \{1, 2, 3\} \). This might lead us to infer that the minimal realizing dimension is \( n - |\Sigma| = 3 - 2 = 1 \); however, it is easy to prove that it is impossible to realize by hyperplanes in \( \mathbb{R}^1 \).

Another consequence of the sphere link property (Theorem 2) relates to algebraic properties of the Stanley-Reisner ring. The dual version of Hochster’s formula relates the multigraded minimal free resolution of the Stanley-Reisner ideal to the simplicial homology of the
corresponding complex. A full exposition of minimal free resolutions is beyond the scope of this article, so we give a brief description and direct the reader to [24, Chapter 1] for more information.

The **multidegree** of a monomial \( (\prod_{i=1}^{n} x_i^{a_i} \prod_{j=1}^{n} y_j^{b_j}) \in S \) is the vector of exponents \( (a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{N}^{2n} \). When the exponents are all 0 or 1, we identify the the multidegree with its support as a subset of \([n] \cup [\overline{n}]\). The **coarse degree** of a monomial is the sum of the exponents \( \sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j \in \mathbb{N} \). For a homogeneous ideal \( I \subset S \), a **minimal free resolution of \( S/I \)** is an exact sequence of free modules that terminates in \( S/I \rightarrow 0 \). Each module in the minimal free resolution of \( S/I \) can be **multigraded** so that each map in the resolution preserves multidegree. The **multigraded Betti number** of \( S/I \), \( \beta_{i,\sigma} = \beta_{i,\sigma}(S/I) \), is the rank of the free module in position \( i \) in the free resolution and with multidegree \( \sigma \). Importantly for our purposes, these Betti numbers can be explicitly computed with Macaulay2 [16] and similar computational algebra software.

**Lemma 6.10** (Hochster’s formula, dual version [24, Corollary 1.40]). For \( \Gamma(C) \) the polar complex of a code \( C \subseteq 2^{[n]} \) and \( \Sigma \) a face of \( \Gamma(C) \),

\[
\beta_{i+1,\Sigma^c}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{i-1}(\text{link}_\Sigma \Gamma(C); k).
\]

Here \( \Sigma^c = ([n] \cup [\overline{n}]) \setminus \Sigma \) denotes the complement of \( \Sigma \) in the vertex set of \( \Gamma(C) \), and \( \Gamma(C)^\vee \) denotes the Alexander dual simplicial complex, \( \Gamma(C)^\vee \overset{\text{def}}{=} \{ F^c \mid F \notin \Gamma(C) \} \).

We use this lemma to detect sphere link obstructions.

**Proposition 6.11.** Let \( C \) be a stable hyperplane code with polar complex \( \Gamma(C) \). Then, \( \beta_{i,\sigma}(S/I_{\Gamma(C)^\vee}) = 0 \) for all \( i \geq 1 \) except:

\[
\begin{cases}
\beta_{1,\Sigma^c}(S/I_{\Gamma(C)^\vee}) = 1 & \text{if } \Sigma \text{ is a facet.} \\
\beta_{n-|\Sigma|+1,\Sigma^c}(S/I_{\Gamma(C)^\vee}) = 1 & \text{if } \text{link}_{\Sigma} \Gamma(C) \sim S^{n-|\Sigma|-1}.
\end{cases}
\]

**Proof.** Inserting \( i = 0 \) and \( \Sigma \) a facet into the dual version of Hochster’s formula yields

\[
\beta_{1,\Sigma^c}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{-1}(\text{link}_\Sigma \Gamma(C); k).
\]

The right-hand side is equal to 1, since the link of a facet is the irrelevant simplicial complex, which gives a generator of \((-1)-\)homology. This gives the first equation from the Proposition.

Setting \( i = n - |\Sigma| \) and \( \Sigma \) a face of \( \Gamma(C) \):

\[
\beta_{n-|\Sigma|+1,\Sigma^c}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{n-|\Sigma|-1}(\text{link}_\Sigma \Gamma(C); k).
\]

The right-hand side is 1 precisely when the link is a sphere of the right dimension. In all other cases, the link is collapsible (Proposition 3.7) or equal to the void complex (links of non-faces), so the reduced homology is zero. \( \square \)

This proposition provides an algebraic signature of stable hyperplane codes.

**Example 6.12.** We again consider the code from Example 3.5. First, we translate into its polar complex \( \Gamma(C_3) \), which has eleven facets for its eleven codewords. Then we compute the Stanley-Reisner ideal of its Alexander dual, and the Betti numbers associated to a minimal free resolution (e.g. using Macaulay2).
The table below is a condensed representation of the Betti numbers of \( I_\Gamma(C_3) \), where the \((i,j)\)-th entry is \( \beta_{j,i+j} \) under the coarse grading.

|   | 0  | 1  | 2  | 3  | 4  |
|---|----|----|----|----|----|
| 0 | 1  |    |    |    |    |
| 1 |    | 11 | 16 | 6  |    |
| 2 |    |    |    |    | 1  |
| 3 |    |    |    |    | 2  |
| 4 |    |    |    |    |    |
| 5 |    |    |    |    |    |

The value of \( \beta_{1,4} \) counts the codewords, which are facets of \( \Gamma(C) \). The remaining entries of row 3 indicate links with the appropriate dimension. Rows 4 and 5, under the multigrading, point to the following nonzero Betti numbers:

\[
\beta_{2,234134} = 1, \quad \beta_{3,234134} = 1, \quad \beta_{3,1234134} = 1, \quad \beta_{4,1234134} = 1.
\]

Note that the multigrading of each Betti number corresponds to the link of its complement; specifically, \( 234134 \mapsto 1 \overline{2} \), \( 2341234 \mapsto 1 \), \( 1234134 \mapsto \overline{2} \), and \( 12341234 \mapsto \emptyset \). These entries give us the following sphere link obstructions to \( \Gamma(C_3) \) being the polar complex of a stable hyperplane code.

1. \( \text{link}_{12} \Gamma(C_3) = \Delta(\{34, \overline{34}\}) \), which has two connected components and hence nontrivial reduced homology of rank 1.
2. \( \text{link}_{1} \Gamma(C_3) = \Delta(\{234, 234, 234, 234, 234\}) \sim S^1 \), which has the wrong dimension.
3. \( \text{link}_{2} \Gamma(C_3) = \Delta(\{134, 134, 134, 134, 134\}) \sim S^1 \), which also has the wrong dimension.
4. \( \text{link}_{\emptyset} \Gamma(C_3) = \Gamma(C_3) \) has nontrivial homology, but \( C_3 \neq 2^{[4]} \).

Each of these indicates the presence of a sphere link obstruction. Thus, \( C \) cannot be a stable hyperplane code.

7. **Proofs of Theorem 4 and Theorem 5**

7.1. **Shellability.** The proof of Theorem 4 is organized as follows. First, we prove it in the special case \( X = \mathbb{R}^d \). To extend the proof to the general case, we prove stable hyperplane codes can be realized by a pair \((\mathcal{H}, \mathcal{P})\) with \( \mathcal{P} \) the interior of a convex polyhedron with bounding hyperplanes \( \mathcal{B} \) such that \( (\mathcal{H} \cup \mathcal{B}, \mathbb{R}^d) \) is a stable arrangement. Lastly, we use links to consider \( \mathcal{P} \) as a region in \( \mathbb{R}^d \), reducing to the special case.

**Lemma 7.1.** If \((\mathcal{H}, \mathbb{R}^d)\) has generic intersections, then \( \Gamma(\text{code}(\mathcal{H}, \mathbb{R}^d)) \) is shellable.

**Proof.** Let \( C = \text{code}(\mathcal{H}, \mathbb{R}^d) \) with \( k = |C| \) the number of codewords. Without loss of generality, the \( w_i \) defining the hyperplanes \( H_i \) are unit vectors that span \( \mathbb{R}^d \). Recall the notation

\[
R_F = \bigcap_{i \in F^+} H_i^+ \cap \bigcap_{j \in F^-} H_j^-
\]

for \( F \in \Gamma(C) \). Our proof proceeds by induction on \( d \), the ambient dimension. An example of the \( d = 2 \) case is illustrated in Figure 6.

The base case \( d = 1 \) is straightforward and guides the intuition for the general case. We order the codewords of \( C \) in a natural way based on their atoms, and show the corresponding
ordering of facets of $\Gamma(\mathcal{C})$ is a shelling order. Each half-space $H_i^+$ is defined by an inequality of the form $x > h_i$ or $-x > h_i$ (i.e. $w_i = \pm 1$ for all $i$). Each atom $A_\sigma$ has a nonempty interior $(a_\sigma, b_\sigma) \subseteq \mathbb{R}$. With the exception of the atom which is unbounded from below (for which $a_\sigma = -\infty$), each $a_\sigma = h_\sigma i$ for some $i_\sigma \in [n]$. Order the codewords $\sigma_1, \ldots, \sigma_k$ in increasing order of $a_\sigma$. We claim this is a shelling order: when we add facet $\Sigma(\sigma)$ to our simplicial complex, this is the first time a facet contains $i_\sigma$ if $w_i = 1$, otherwise it’s the first time a facet contains $\overline{i_\sigma}$. In other words, $\sigma$ is the first codeword in this order which contains $i$ if $w_i = 1$ or the first codeword which does not contain $i$ if $w_i = -1$; all later atoms lie on the same side of the hyperplane $H_i$. See Figure 6(d). Thus, every facet of $\Gamma(\mathcal{C})$ has an associated minimal face and this ordering is a shelling order.

Now consider $d > 1$. Denote by $\Omega(\mathcal{H})$ the set of points where $d$ hyperplanes intersect. We choose a generic “sweep” direction, a vector $u \in \mathbb{R}^d$ which satisfies the following properties:

![Diagram](image)

**Figure 6.** An example of the shelling order construction in the $d = 2$ case. (a) The atoms discovered at time $t_0$, i.e. the atoms $A_\sigma$ with $m(\sigma) = -\infty$. Note the four atoms of $(\mathcal{H}, \mathbb{R}^2)$ which intersect $H(t_0)$ partition it into four intervals. (b) As $t$ increases, $H(t)$ slides to the right, encountering atoms one at a time. The shaded atom is newly discovered. (c) Uniqueness of $r(\Sigma(\sigma))$ follows because $e_3 \cdot \sigma_6$ and $e_1 \cdot \sigma_6$ have already been discovered. (d),(e),(f) The inductive step and next two steps of the shelling order. The associated minimal face is highlighted with a large mark (panel (d)) or a dashed line (panels (e),(f)). (d) The polar complex $\Gamma(\text{code}(\mathcal{L}, H(t_0)))$. Ordering the four codewords discovered in panel (a) from top to bottom yields $r(\overline{123}) = 3$. (e) Facet 123 is added when $H(t)$ contains the intersection $H_1 \cap H_2$ (panel (b)), thus $r(123) = 12$. (f) Atom $A_{12}$ is discovered when $H(t)$ contains $H_1 \cap H_3$ (panel (c)). Thus $r(123) = 13$. 

(i) $u$ is not in the span of any $(d - 1)$-element subset of $\{w_1, \ldots, w_n\}$.
(ii) For every pair of distinct points $x, y$ in $\Omega(\mathcal{H})$, $u$ is not in the orthogonal complement $(x - y)^\perp$.

Such a $u$ exists because we exclude finitely many subsets of measure zero from $\mathbb{R}^d$. We use $u$ to define a sliding hyperplane $H(t)$ and its corresponding “discovery time” function $m : \mathcal{C} \to \mathbb{R} \cup \{-\infty\}$,

$$H(t) = \{x \in \mathbb{R}^d \mid u \cdot x - t = 0\}$$

$$m(\sigma) = \inf \{u \cdot x \mid x \in A_\sigma\}$$

In the $d = 1$ case, $m(\sigma) = a_\sigma$ and thus induces a total order on codewords. For the $d > 1$ case, the goal is once again to use $m$ to order the codewords. To do this, (1) we order the codewords with $m(\sigma) = -\infty$ inductively, then (2) we show $m$ is injective on the remaining codewords, and lastly, (3) we show every facet has an associated minimal face.

(1) By construction, $\mathcal{H} \cup \{H^+(t)\}$ is a stable arrangement in $\mathbb{R}^d$ for all but finitely many values of $t$, specifically, the values where $H(t)$ contains a point in $\Omega(\mathcal{H})$. Let $t_0$ be a constant less than all of these values (see Figure 6(a) for an illustration). Property (i) ensures $H_i^+ \cap H(t_0) \neq \emptyset$ for all $i$, so in particular $L \equiv \{H_i^+ \cap H(t_0)\}_{i \in [n]}$ is a stable arrangement in $H(t_0) \cong \mathbb{R}^{d-1}$. By inductive hypothesis, $\Gamma(\text{code}(L, H(t_0)))$ is shellable. Each nonempty atom of the arrangement $(L, H(t_0))$ is the intersection of an atom of $(\mathcal{H}, \mathbb{R}^d)$ with $H(t_0)$, and the corresponding codewords are precisely those with $m(\sigma) = -\infty$. Thus, we have an ordering for these codewords which is an initial segment of a shelling of $\Gamma(\mathcal{C})$ (Figure 6(d)).

(2) Let $\sigma \in \mathcal{C}$ be a codeword with $m(\sigma) > -\infty$. The function $f(x) = u \cdot x$ is minimized along a face of the (closure of) polyhedron $R_{\Sigma(\sigma)}$. Property (ii) ensures this face is a vertex, which is an element of $\Omega(\mathcal{H})$. Property (iii) ensures $f|_{\Omega(\mathcal{H})}$ is injective. Therefore, $m$ induces a total order on codewords $\sigma$ with $m(\sigma) > -\infty$. Let $\sigma_1, \ldots, \sigma_k$ be the ordering of codewords of $\mathcal{C}$ obtained appending this ordering to the order from (1). We will show each facet has an associated minimal face to complete the proof.

(3) Denote $\Gamma_i \equiv \Gamma(\{\sigma_1, \ldots, \sigma_i\})$ for $i = 1, \ldots, k$. From (1), $r(\Sigma(\sigma_i))$ is defined whenever $m(\sigma_i) = -\infty$. Let $\sigma_i$ be a codeword with $t_i = m(\sigma_i) > -\infty$, meaning there is a vertex of $R_{\Sigma(\sigma_i)}$ minimizing $f$. This vertex is an element of $\Omega(\mathcal{H})$, i.e. it is the intersection $H_{\alpha_i}$ of $d$ hyperplanes (see Figure 6(b)). For $F \in \Gamma(\mathcal{C})$ and $\alpha \subseteq [n]$, we denote

$$F|\alpha \equiv F \cap (\alpha \cup \overline{\alpha})$$

the subset of $F$ with support $\alpha$. We claim $r(\Sigma(\sigma_i)) = \Sigma(\sigma_i)|\alpha_i$ (see Figure 6(e) and (f)). The region $R_{\Sigma(\sigma_i)|\alpha_i}$ is a cone supported by $H(t_i)$, so this is the first codeword in our order with this exact combination of “on” and “off” vertices indexed by $\alpha_i$. Thus, $\Sigma(\sigma_i)|\alpha_i \in \Gamma_i \setminus \Gamma_{i-1}$.

Now consider $F = \Sigma(\sigma_i)|\beta \in \Gamma_i \setminus \Gamma_{i-1}$. Suppose, for the sake of contradiction, $\beta \not\supseteq \alpha_i$, i.e. there is some $\ell \in \alpha_i \setminus \beta$. Then $F \subseteq \Sigma(\mathbf{e}_\ell \cdot \sigma_i)$. Note $\mathbf{e}_\ell \cdot \sigma_i \in \mathcal{C}$ since, by genericity, all $2^d$ possible regions around the point $H_{\alpha_i}$ produce codewords. However, since $H(t_i)$ intersects the interior of $R_{\Sigma(\mathbf{e}_\ell \cdot \sigma_i)}$, we have $m(\mathbf{e}_\ell \cdot \sigma_i) < m(\sigma_i)$ and therefore $\Sigma(\mathbf{e}_\ell \cdot \sigma_i) \in \Gamma_{i-1}$ (see Figure 6(c)). We reach a contradiction, as this implies $F \in \Gamma_{i-1}$. Therefore, $r(\Sigma_i) = \Sigma_i|\alpha_i$ is the unique minimal face in $\Gamma_i \setminus \Gamma_{i-1}$. This completes the proof.
We now prove that a stable hyperplane code is a subset of codewords of a stable hyperplane arrangement in \( \mathbb{R}^d \).

**Lemma 7.2.** If \( \mathcal{C} \) is a stable hyperplane code, then \( \mathcal{C} \) can be realized by a stable pair \( (\mathcal{H}, \mathcal{P}) \) such that \( \mathcal{P} = \bigcap_{j \in [m]} B_j^+ \) is an open polytope with bounding hyperplanes \( \mathcal{B} \) such that \( \mathcal{H} \cup \mathcal{B} \) has generic intersections in \( \mathbb{R}^d \).

**Proof.** Let \( (\mathcal{H}, X) \) be a stable pair realizing \( \mathcal{C} \). By Lemma 2.4, we can perturb the hyperplanes \( \mathcal{H} \) to an arrangement \( \mathcal{H}' \) while preserving the atoms of the arrangement \( (\mathcal{H}, X) \), i.e. \( \text{code}(\mathcal{H}', X) = \text{code}(\mathcal{H}, X) \). Thus, \( \mathcal{C} \) has a realization \( (\mathcal{H}', X) \) such that \( \mathcal{H}' \) has generic intersections outside of \( X \) as well.

Applying Lemma 2.4 again, we can choose a point \( p_\sigma \) in the interior of \( A_{\mathcal{H}'}^\sigma \) for every \( \sigma \in \mathcal{C} \).

Let \( P \) be the interior of the convex hull of the set of points \( \{ p_\sigma \mid \sigma \in \mathcal{C} \} \); by perturbing the points slightly we may assume \( P \) is full-dimensional. Let \( \mathcal{B} = \{ B_{n+1}^+, \ldots, B_{n+m}^+ \} \) denote the bounding hyperplanes of this polytope, i.e. \( \mathcal{P} = \bigcap_{j=n+1}^{n+m} B_j^+ \). Since \( \mathcal{P} \subseteq X \), we conclude \( \text{code}(\mathcal{H}', \mathcal{P}) \subseteq \text{code}(\mathcal{H}', X) \). Since we chose a points \( p_\sigma \) for every codeword of \( \mathcal{C} \), \( \sigma \in \mathcal{C} \) implies \( A_{\mathcal{H}'}^\sigma \cap P \neq \emptyset \) and therefore \( \text{code}(\mathcal{H}', X) \subseteq \text{code}(\mathcal{H}', \mathcal{P}) \). Thus we have \( \mathcal{C} = \text{code}(\mathcal{H}', \mathcal{P}) \) and \( (\mathcal{H}', \mathcal{P}) \) is a stable arrangement.

The hyperplanes in \( \mathcal{H}' \cup \mathcal{B} \) do not necessarily have generic intersections. Again, we apply Lemma 2.4: one can perturb each hyperplane in \( \mathcal{B} \) to hyperplanes \( \mathcal{B}' \), so that these hyperplanes have generic intersections, yet the appropriate code is preserved, i.e. \( \mathcal{C} = \text{code}(\mathcal{H}', \mathcal{P}) = \text{code}(\mathcal{H}', \mathcal{P}') \), where \( \mathcal{P}' \) is the open polyhedron \( \mathcal{P}' = \bigcap_{B \in \mathcal{B}'} B^+ \). This completes the proof. \( \square \)

We extend Lemma 7.1 to the general case with the following standard lemma [5].

**Lemma 7.3 ( [5, Proposition 10.14]).** Let \( \Delta \) be a shellable simplicial complex. Then \( \text{link}_\sigma \Delta \) is shellable for any \( \sigma \in \Delta \), with shelling order induced from the shelling order of \( \Delta \).

**Proof of Theorem 4.** By Lemma 7.2 \( \mathcal{C} \) can be realized as \( \mathcal{C} = \text{code}(\mathcal{H}, \mathcal{P}) \) with

\[
\mathcal{P} = \bigcap_{j=n+1}^{n+m} B_j^+
\]

an open polyhedron such that the arrangement \( \mathcal{H} \cup \mathcal{B} \) has generic intersections in \( \mathbb{R}^d \). Set \( \mathcal{C}' = \text{code}(\mathcal{H} \cup \mathcal{B}, \mathbb{R}^d) \), a code on vertex set \( [n+m] \). By Lemma 7.1 \( \Gamma(\mathcal{C}') \) is shellable. Set \( F = \{ n+1, \ldots, n+m \} \in \Gamma(\mathcal{C}') \). Then we have

\[
\text{link}_F \Gamma(\mathcal{C}') = \Gamma \left( \text{code} \left( \mathcal{H}, \bigcap_{j=n+1}^{n+m} B_j^+ \right) \right) = \Gamma(\mathcal{C}).
\]

By Lemma 7.3, as the link of a shellable complex, \( \Gamma(\mathcal{C}) \) is shellable. \( \square \)

**7.2. Obstructions following from shellability.** In general, shellable simplicial complexes are homotopy-equivalent to a wedge sum of spheres, where the number and dimension of the spheres correspond to the facets with \( r(F) = F \) in some shelling order [23]. Here we prove a stronger version of this statement for the polar complex of a code, which will be used throughout the proofs of all parts of Theorem 5.
Lemma 7.4. If $\Gamma(C)$ is shellable, then either $C = 2^{[n]}$ or $\Gamma(C)$ is collapsible.

Proof. We induct on the number of codewords of $C$. Let $F_1, \ldots, F_t$ be a shelling order of $\Gamma(C)$, with $\sigma_1, \ldots, \sigma_t$ the corresponding order of codewords in $C$. For ease of notation, let $C' = \{\sigma_1, \ldots, \sigma_{t-1}\}$ denote the first $t - 1$ codewords in this shelling order. By construction, $\Gamma(C')$ is shellable. Therefore, we can apply Lemma 7.4 to conclude $\Gamma(C')$ is collapsible.

By definition, $r(F_t)$ is the unique minimal element of the collection $\Gamma(C) \setminus \Gamma(C')$ and hence the only facet that contains $r(F_t)$ is $F_t$. If $r(F_t) \not\subset F_t$, then $(r(F_t), F_t)$ is a free pair, and $\Gamma(C) \setminus r(F_t)$ is shellable.

In the case $r(F_t) = F_t$, we claim we must have $C = 2^{[n]}$. Suppose not, for the sake of contradiction, and let $\tau \in 2^{[n]} \setminus C$. Note $\Gamma(2^{[n]} \setminus \{\tau\})$ is homeomorphic to a closed $(n - 1)$-ball (as it is a sphere missing top-dimensional open disc). Since $\Gamma(C')$ is a collapsible subcomplex of a simplicial complex, $\Gamma(C')$ is homotopy-equivalent to the quotient space $\Gamma(C)/\Gamma(C')$ (see [19, Proposition 0.17 and Proposition A.5]). Because $r(F_t) = F_t$, the boundary of the $(n - 1)$-simplex $\Delta(\{F_t\})$ is contained in $\Gamma(C')$, and therefore $\Gamma(C)/\Gamma(C')$ is homotopy equivalent to $S^{n-1}$. We reach a contradiction, as $\Gamma(C) \subseteq \Gamma(2^{[n]} \setminus \{\tau\})$, but there is no embedding $S^{n-1} \hookrightarrow \mathbb{R}^{n-1}$ (see, e.g. [19, Corollary 2B.4]). Therefore, in this case we have $C = 2^{[n]}$. \hfill $\square$

To prove Theorem 5.1, we need one more lemma. Note that this lemma concerns the contractibility of certain subcomplexes, hence it can only be used to show $C$ has no weak local obstructions.

Lemma 7.5 ([8, Lemma 4.4]). Let $\Delta$ be a simplicial complex on vertex set $V$. Let $\alpha, \beta \in \Delta$ with $\alpha \cap \beta = \emptyset$, $\alpha \cup \beta \subseteq V$, and link$_{\alpha}(\Delta|_{\alpha \cup \beta})$ not contractible. Then there exists $\alpha' \in \Delta$ such that (i) $\alpha' \supseteq \alpha$, (ii) $\alpha' \cap \beta = \emptyset$, and (iii) link$_{\alpha'}(\Delta)$ is not contractible.

Proof of Theorem 5.1. Assume that the polar complex $\Gamma(C)$ is shellable. To show that $C$ has no weak local obstructions, first suppose $\tau \in \Delta(C)$ and link$_{\tau}(\Delta(C))$ is not contractible. We will show $\tau \in \mathbb{C}$. Note that $\Delta(C) = \Gamma(C)|_{[n] \setminus \{\tau\}}$, thus we apply Lemma 7.5 to the pair $\alpha = \tau \cup \emptyset$, $\beta = ([n] \setminus \tau) \cup \emptyset$ in the polar complex $\Gamma(C)$: there exists a face $T \in \Gamma(C)$ such that (i) $T = T^+ \cup \overline{T^-} \supseteq \tau \cup \emptyset$, (ii) $T \cap (([n] \setminus \{\tau\} \cup \emptyset) = \emptyset$, and (iii) link$_{\tau}(\Gamma(C))$ is not contractible. Statements (i) and (ii) together imply $T^+ = \tau$. Statement (iii) together with Lemma 7.4 implies link$_{\tau}(\Gamma(C)) = \Gamma(2^{[n] \setminus \tau})$. Therefore this link contains the facet $F$ consisting of all barred vertices in $[n] \setminus \tau$. Thus $T \cup F = \tau \cup [n] \setminus \tau$ is a face of $\Gamma(C)$ and therefore $\tau \in \mathbb{C}$; hence $\tau$ cannot be a local obstruction.

For any $g \in (\mathbb{Z}_2)^n$, the above argument extends to $g \cdot C$ verbatim, since $\Gamma(g \cdot C) = g \cdot \Gamma(C)$, and $g \cdot \Gamma(C)$ is also shellable. Thus, $C$ has no bitflip local obstructions. \hfill $\square$

Proof of Theorem 5.2. Assume that the polar complex $\Gamma(C)$ is shellable. Links of $\Gamma(C)$ are polar complexes of a code on a smaller set of vertices, and links of shellable complexes are shellable (Lemma 7.3). Therefore, we can apply Lemma 7.4 to conclude link$_F(\Gamma(C))$ is either collapsible or $\Gamma(2^{[n] \setminus \tau})$ for any $F \in \Gamma(C)$. Thus, no face $F$ can be a sphere link obstruction. \hfill $\square$

We use one final lemma to prove Theorem 5.3, which concerns faces of simplicial complexes with collapsible links.
**Lemma 7.6.** Let $\Delta$ be a simplicial complex with $\alpha \in \Delta$ such that $\text{link}_\alpha \Delta$ is collapsible. Then $\Delta \setminus \text{del}_\alpha \Delta$.

**Proof.** Let $(\sigma_1, \tau_1), \ldots, (\sigma_k, \tau_k)$ be the sequence of free pairs along which $\Delta_1 = \text{link}_\alpha \Delta$ is collapsed (in particular, $\sigma_k = \emptyset$), resulting in the sequence of simplicial complexes

$$\text{link}_\alpha \Delta = \Delta_1 \setminus \sigma_1 \setminus \sigma_2 \setminus \cdots \setminus \sigma_k \setminus \alpha \Delta_{k+1} = \{\}.$$

Consider the sequence $(\sigma_1 \cup \alpha, \tau_1 \cup \alpha), \ldots, (\sigma_k \cup \alpha, \tau_k \cup \alpha)$ in $\Delta$. We claim, $(\sigma_1 \cup \alpha, \tau_1 \cup \alpha)$ is a free pair: $\sigma_1 \cup \alpha \subsetneq \tau_1 \cup \alpha$ and $\tau_1 \cup \alpha$ is a facet of $\Delta$. If $\sigma_1 \cup \alpha \subseteq \tau'$ for some facet $\tau'$, then $\tau' \setminus \alpha$ is a facet of $\text{link}_\alpha \Delta$ which contains $\sigma_1$, hence $\tau' = \tau$. This argument can be repeated for the pair $(\sigma_2 \cup \alpha, \tau_2 \cup \alpha)$ in $\text{del}_{\sigma_1 \cup \alpha} \Delta$, and so on, to show that this is a sequence of free pairs in $\Delta$. Thus, we have a sequence of collapses

$$\Delta \setminus \sigma_1 \cup \alpha \cdots \setminus \sigma_k \cup \alpha \text{del}_{\sigma_k \cup \alpha} \Delta.$$

Since $\sigma_k \cup \alpha = \alpha$, we have $\Delta \setminus \text{del}_\alpha \Delta$. \qed

**Proof of Theorem 3.3.** Assume the polar complex $\Gamma(C)$ is shellable. We demonstrate that if $\sigma \in \text{cham}(C)$ has more than one chamber, then $\sigma$ is not maximal.

Suppose $T_1 \neq T_2$ are chambers of $\sigma$, that is,

$$\text{link}_{T_1} \Gamma(C) = \text{link}_{T_2} \Gamma(C) = \Gamma(2^\sigma).$$

We will proceed by induction on $k = |T_1 \setminus T_2| > 0$. Since $T_1 = T_2 = [n] \setminus \sigma$, $k$ is the number of indices where one $T_i$ has a barred vertex and the other does not.

For the base case $k = 1$, suppose $T_1 \setminus T_2 = i$. Then

$$\text{link}_{T_1 \cap T_2} \Gamma(C) = \Gamma(2^{\sigma \cup \{i\}})$$

so $\sigma \cup i \in \text{cham}(C)$ and $\sigma$ is not maximal.

Now suppose $|T_1 \setminus T_2| = k > 1$. We produce a face $F$ such that $\text{link}_F \Gamma(C) = \Gamma(2^\sigma)$ and $|T_1 \setminus F| < k$, giving the induction step. Let $T = T_1 \cap T_2$, and consider $\text{link}_{T} \Gamma(C)$. This is a shellable subcomplex of $\Gamma(2^{[n]})$; denote its corresponding code by $C'$. Let $T'_1 = T_1 \setminus T$ and $T'_2 = T_2 \setminus T$; by design these are disjoint with $|T'_1 \setminus T'_2| = |T'_1| = |T'_2| = k$ and $\text{link}_{T'_i} \Gamma(C') = \Gamma(2^\sigma)$ for $i = 1, 2$. Because they are disjoint, $\text{star}_{T'_1} \Gamma(C') \cup \text{star}_{T'_2} \Gamma(C')$ is a suspension of $\Gamma(2^\sigma)$ hence homotopy equivalent to $S^{n-1}$.

Consider a face $F' \in \Gamma(C')$ such that $F' = T'_1$. By construction, $\text{link}_{F'} \Gamma(C')$ is a subcomplex of $\Gamma(2^\sigma)$. If $\text{link}_{F'} \Gamma(C') \neq \Gamma(2^\sigma)$, then the link is collapsible by Lemmas 7.3 and 7.4. Lemma 7.6 implies that $\Gamma(C')$ collapses to $\text{del}_{F'} \Gamma(C')$.

There are $2^k - 2$ faces $F' \neq T_1, T_2$ with $F' = T'_1$. If none of these $F'$ had $\text{link}_{F'} \Gamma(C') = \Gamma(2^\sigma)$, this would lead to a contradiction: we would have a sequence of collapses

$$\Gamma(C') \setminus \text{star}_{T'_1} \Gamma(C') \cup \text{star}_{T'_2} \Gamma(C').$$

Since $\Gamma(C)$ is shellable, by Lemma 7.3 it is homotopy equivalent to $S^{n-1}$ or is contractible. Collapsing preserves homotopy type, so we reach a contradiction.

Therefore, for one of these $F'$ we must have $\text{link}_{F'} \Gamma(C') = \Gamma(2^\sigma)$. Thus $\text{link}_{F' \cup T} \Gamma(C) = \Gamma(2^\sigma)$ and so we have another face in $\Gamma(C)$ whose link yields $\Gamma(2^\sigma)$, namely $F = F' \cup T$. Since $|T_1 \setminus F| < k$, by induction $\sigma$ is not maximal in $\text{cham} C$. Therefore, if $\sigma$ is maximal in $C$, it must have a unique chamber, and thus $C$ has no chamber obstructions. \qed
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