Survival probability of an immobile target surrounded by mobile traps

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Abstract. We study analytically, in one dimension, the survival probability $P_s(t)$ up to time $t$ of an immobile target surrounded by mutually noninteracting traps, each performing a continuous-time random walk (CTRW) in continuous space. We consider a general CTRW with symmetric and continuous (but otherwise arbitrary) jump length distribution $f(\eta)$ and arbitrary waiting time distribution $\psi(\tau)$. The traps are initially distributed uniformly in space with density $\rho$. We prove an exact relation, valid for all time $t$, between $P_s(t)$ and the expected maximum $E[M(t)]$ of the trap process up to time $t$, for rather general stochastic motion $x_{\text{trap}}(t)$ of each trap. When $x_{\text{trap}}(t)$ represents a general CTRW with arbitrary $f(\eta)$ and $\psi(\tau)$, we are able to compute exactly the first two leading terms in the asymptotic behavior of $E[M(t)]$ for large $t$. This allows us subsequently to compute the precise asymptotic behavior, $P_s(t) \sim a \exp[-bt^\theta]$, for large $t$, with exact expressions for the stretching exponent $\theta$ and the constants $a$ and $b$ for arbitrary CTRW. By choosing appropriate $f(\eta)$ and $\psi(\tau)$, we recover the previously known results for diffusive and subdiffusive traps. However, our result is more general and includes, in particular, superdiffusive traps as well as totally anomalous traps.

Keywords: driven diffusive systems (theory), exact results, stochastic particle dynamics (theory), extreme value statistics

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1. Introduction

The study of the survival probability of a target particle surrounded by diffusing traps is a classic problem in physical chemistry, dating back to Smoluchowski in 1916 [1]. Smoluchowski introduced this problem to compute the rate of chemical reactions in the diffusion-driven regime. The basic model and its variants [2,3] have since found a large number of applications in a wide variety of contexts, such as reaction–diffusion systems [4], chemical kinetics [5]–[11], predator–prey models in population dynamics [3,12] and also in a wide class of the so-called ‘walker persistence’ problems [13].

In the simplest setting of the model, known as the ‘target annihilation’ problem [14], the target particle is immobile at the origin and the noninteracting traps, initially distributed uniformly over space with a finite density $\rho$, each perform an unbiased continuous-time Brownian motion with a diffusion constant $D$. When any of these traps hits the target particle at the origin, the target particle gets annihilated. The average survival probability $P_s(t)$ of the target up to time $t$, where the average is over the initial positions of the traps distributed uniformly over space with density $\rho$, has been computed exactly [14] in all dimensions. In particular, in the simplest one-dimensional case, $P_s(t)$ decays with $t$ as a stretched exponential for all $t$,

$$P_s(t) = \exp \left[ -4\rho \sqrt{D t} \right]. \quad (1)$$

For $d = 2$, $P_s(t) \sim \exp[-\rho c_2 t/\ln t]$, while for $d > 2$, $P_s(t) \sim \exp[-\rho c_d t]$ for large $t$, where the constants $c_2$ and $c_d$ are known [14]. For $d > 2$, the target particle needs to have a finite size while for $d \leq 2$, the target can be considered to be a point particle.

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the original Smoluchowski version of the problem, the quantity $F(t) = -\ln[P_s(t)]$ has a
physical interpretation: it represents the net flux of traps to a static absorber at the origin
up to time $t$ [8]–[10]. For $d = 1$, the net flux $F(t) = 4\rho\sqrt{Dt/\pi}$ thus grows algebraically
$\sim t^{1/2}$ with time $t$.

The original Smoluchowski, or equivalently the ‘target annihilation’, problem has
been generalized in the literature in several ways, e.g., (a) when the target itself moves
stochastically (not necessarily diffusive) while the noninteracting traps perform normal
diffusion; (b) when the target is static but the traps, while still noninteracting, perform,
in general, nondiffusive stochastic motion; (c) when both the target and the traps move
nondiffusively in general; and (d) the famous Donsker–Varadhan problem [15], where the
target moves diffusively but the traps are immobile. In this paper, we will focus on cases
(a) and (b) only.

An interesting variant belonging to class (a) above is the ‘diffusive target annihilation’
problem where the target particle itself diffuses with a diffusion constant $D_0$ [16]. This
problem has seen a recent flurry of activity [17]–[21] and it has been proved rigorously that,
in one dimension, the average survival probability $P_s(t)$ of the diffusing target particle
has the same asymptotic stretched exponential decay as in equation (1) (and hence
is independent of $D_0$), though with considerable subleading corrections to the leading
behavior for intermediate $t$. Another variant in class (a) is the ‘ballistic target annihilation’
problem, where the target particle moves ballistically with a constant velocity $c$. In this
case, the survival probability decays faster, $P_s(t) \sim \exp[-\vartheta t]$ in all dimensions $d$, where
the inverse decay rate $\vartheta$ has been computed exactly for $d \leq 2$ and $d = 3$ [22].

The variants in class (b), where the target is static and the traps undergo subdiffusive
motion, or (c), where both the target and the traps undergo subdiffusive motion, have
been studied extensively by Yuste and collaborators [23]–[27].

Another recent extension belonging to class (b) above is to the case where the target
is static, each trap performs independent Brownian motion but resets to its own initial
position with a constant rate $r$ [28]. This problem has interesting implication in the
context of search problems where a team of searchers (traps) adopt the resetting strategy
to make the search of the target more efficient [28]. For $r = 0$, it reduces to the standard
target annihilation problem. For a nonzero $r$, the average survival probability in one
dimension was computed exactly and found to decay as a power law at late times [28],

$$P_s(t) \approx Cr^{-2\rho\sqrt{D/r}},$$

(2)

where $C$ is a constant. For short times $rt \ll 1$, the decay is stretched exponential [28], as
in equation (1).

In this paper, we study analytically the average survival probability of a target
particle in one dimension in another variant belonging to class (b). In this model, the
target remains immobile at the origin, as in the original target annihilation model. The
traps also remain noninteracting and are distributed initially in space with a uniform
density $\rho$. However, each trap, instead of performing a continuous-time Brownian motion
with a diffusion constant $D$, now performs a continuous-time random walk (CTRW)
with arbitrary jump length distribution and arbitrary waiting time distribution between
successive jumps. Our result for the asymptotic survival probability includes the cases
when the traps undergo subdiffusive as well as superdiffusive motion, thus generalizing
the previously known result for subdiffusive traps [23].
In the standard version of CTRW, a trap moves by making successive random jumps at random times \([29–31]\). More precisely, starting from its initial position, a trap waits a certain amount of random time \(\tau\) (drawn from a normalized distribution \(\psi(\tau)\)) and then jumps by a certain distance \(\eta\) (also a random variable drawn from a continuous and symmetric distribution \(f(\eta)\)) (see figure 2). At its new position it again waits for a random time drawn from \(\psi(\tau)\) (each time independently) and then jumps to another new position with a random distance drawn from \(f(\eta)\) and the process continues. Thus, CTRW has two ‘input’ distributions, one for time and one for space, namely \(\psi(\tau)\) and \(f(\eta)\). We also assume that these two distributions do not vary from trap to trap, that is, all trap motions are characterized by the same \(\psi(\tau)\) and \(f(\eta)\). In this model, the immobile target particle at the origin gets annihilated whenever any trap trajectory goes past the origin. Note that the particle at the origin gets annihilated even when a trap jumps over the origin. Thus the survival probability \(P_s(t)\) of the target particle in this case is just the probability that none of the traps jumps over the origin up to time \(t\). Our goal is to understand how the two arbitrary input distributions \(\psi(\tau)\) and \(f(\eta)\) affect the canonical stretched exponential decay of \(P_s(t)\) in equation (1), which is valid when each trap performs continuous-time Brownian motion.

The advantage of CTRW is that, by appropriately choosing the two input distributions \(\psi(\tau)\) and \(f(\eta)\), one can study various types of motions of a trap as special cases. For example, if one chooses \(\psi(\tau) = \delta(\tau - \Delta t)\), one generates a discrete-time jump process with jump length distribution \(f(\eta)\) where the jumps happen instantaneously at time steps separated by \(\Delta t\). If one further chooses a Gaussian jump length distribution \(f(\eta) = \exp[-\eta^2/2\langle \eta^2 \rangle]/\sqrt{2\pi\langle \eta^2 \rangle}\) and takes the limit \(\langle \eta^2 \rangle \rightarrow 0\), \(\Delta t \rightarrow 0\), but keeping the ratio \(\langle \eta^2 \rangle/(\Delta t) = 2D\) constant, one recovers the continuous-time Brownian motion. Similarly, by choosing a power-law distribution \(f(\eta) \sim |\eta|^{-\nu-1}\) for large \(|\eta|\) with \(0 < \mu \leq 2\), one can generate Lévy flights for a trap \([30–32]\). Besides, the CTRW also provides a natural framework to generate anomalous diffusion processes, namely subdiffusive and superdiffusive behavior, which occurs in many natural processes \([30,31]\). It would thus be interesting to know how the average survival probability \(P_s(t)\) of a target particle decays with time when each of the traps, though still noninteracting, performs CTRW with arbitrary waiting time and jump length distributions, \(\psi(\tau)\) and \(f(\eta)\) respectively.

Our main results are twofold:

(i) We will first derive a very general result for the average survival probability \(P_s(t)\) for any model belonging to class (b) above, i.e., when the target is immobile and each trap performs independently a stochastic motion \(x_i(t)\) starting from its initial position \(x_i(0)\). The stochastic evolution law of \(x_i(t)\) can be rather general. The only assumptions are (A) the process \(x_i(t)\) is invariant under a constant shift, i.e., \(x_i(t)\) and \(x_i(t) - c\) (where \(c\) is a constant) has the same evolution equation and (B) the process \(x_i(t) - x_i(0)\) is symmetric around the origin. Since the evolution law of the \(i\)th trap \(x_i(t) - x_i(0)\) does not depend explicitly on the index \(i\), we will generally denote it by \(x_{\text{trap}}(t) \equiv x_i(t) - x_i(0)\). For such a stochastic trap motion \(x_{\text{trap}}(t)\), we obtain a rather general result for the average survival probability as

\[
P_s(t) = \exp[-2\rho E[M(t)]],
\]
where $E[M(t)]$ is the expected value of the maximum of the process $x_{\text{trap}}(\tau)$ over the time interval $\tau \in [0, t]$, i.e,

$$M(t) = \max_{0 \leq \tau \leq t} \{x_{\text{trap}}(\tau)\}. \quad (4)$$

We emphasize that this relation is very general and holds whatever be the stochastic process that describes the motion of a trap, as long as the process $x_{\text{trap}}(t)$ satisfies the two properties (A) and (B) above. For example, if $x_{\text{trap}}(t)$ is a continuous-time Brownian motion, it is well known that the probability distribution of the maximum $M(t)$ of the process is a half-Gaussian $[33, 34]$, i.e., $P(M, t) = \exp[-M^2/4Dt]/\sqrt{\pi Dt}$ for $M \geq 0$. Hence the expected value $E[M(t)] = 2\sqrt{Dt}/\pi$. Plugging this result in the general relation in equation (3), one recovers the classical stretched exponential decay in equation (1).

(ii) The relation in equation (3) holds, in particular, when $x_{\text{trap}}(t)$ represents a CTRW with arbitrary $\psi(\tau)$ and $f(\eta)$. For such a general CTRW, we will first derive the exact asymptotic expression of $E[M(t)]$ for large $t$. Plugging this result in the general relation (3) then provides us with the exact asymptotic behavior of $P_s(t)$

$$P_s(t) \approx a \exp[-bt^\theta], \quad (5)$$

where the stretching exponent $\theta$, the constant $b$ and the amplitude $a$ are determined exactly for arbitrary $\psi(\tau)$ and $f(\eta)$ (symmetric and continuous) (the precise results are detailed in equation (36)). We also verify numerically our analytical predictions.

The rest of the paper is organized as follows. In section 2, we derive the main results of this paper: section 2.1 provides the derivation of the general relation (3), while in section 2.2, we derive $E[M(t)]$ for CTRW by extending known results for $E(M_n)$ in a discrete-time setting [33]. In section 3, these results are combined to obtain the asymptotic large $t$ behavior of the survival probability $P_s(t)$ for a CTRW trap with arbitrary $\psi(\tau)$ and $f(\eta)$. Our general result will recover, as special cases, the known expression for traps performing diffusion and subdiffusion. In that section, we also compare our analytical results to numerical simulations and obtain good agreement. Finally we conclude in section 4 with a summary and some open questions.

2. Derivation of the main results

2.1. Survival probability for general stochastic motion of a trap

We consider an immobile target at the origin in one dimension, surrounded by traps, initially distributed uniformly over space with density $\rho$ and each performing an independent stochastic evolution $x_i(t)$ starting from the initial position $x_i(0)$. We will denote the process $x_i(t) - x_i(0)$ by $x_{\text{trap}}(t)$ that starts at the origin and satisfies two properties (A) and (B) mentioned in section 1. In order to derive an expression for the survival probability of the target at the origin, we follow the strategy used by Bray and Blythe [18] for the special case when $x_{\text{trap}}(t)$ is a Brownian motion around the origin. We consider first a finite number $N$ of mobile traps starting at fixed initial positions $\{x(0)\} \equiv \{x_1(0), \ldots, x_N(0)\}$ on the interval $(-L/2, L/2)$, cf figure 1 and calculate the survival probability of the target up to time $t$ for this fixed set of initial positions.
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Figure 1. Schematic example of a realization where the target at the origin is immobile and there are 6 noninteracting traps around it undergoing stochastic motion. The traps are initially placed uniformly in the box \((-L/2, L/2)\). Eventually we are interested in the limit when the number of traps \(N \to \infty\), the box size \(L \to \infty\) but with the density of traps \(\rho = N/L\) fixed.

Figure 2. A typical trajectory \(x_{\text{trap}}(\tau)\) of a CTRW up to a total time \(t\). The process starts at time \(\tau = 0\) at the origin and the first interval \(\tau_1\) also starts at \(\tau = 0\). The trap waits at a point in space for a random time \(\tau\) drawn from a distribution \(\psi(\tau)\) and then jumps by a distance \(\eta\) drawn from a symmetric and continuous jump length distribution \(f(\eta)\). The successive time intervals \(\{\tau_1, \tau_2, \tau_n, \tau_{\text{last}}\}\) spanning the total time \(t\) are statistically independent.

Next we average this survival probability over the starting positions, where each \(x_i(0)\) is drawn independently from a uniform measure on \((-L/2, L/2)\), and take the limit \(N \to \infty, L \to \infty\) while keeping the density of traps \(N/L = \rho\) fixed.

Let \(p(t|x_i(0))\) denote the probability that a single trap, starting at the initial position \(x_i(0)\), does not cross the origin up to time \(t\). By independence of the \(N\) particles, the
survival probability of the target factorizes into a product of the probabilities $p(t|x_i(0))$

$$P_{s,N,L}(t|x(0)) = \prod_{i=1}^{N} p(t|x_i(0)).$$

(6)

Now, averaging over all starting positions with uniform distribution on the interval $(-L/2, L/2)$ and noting that particles were only distinguishable by their starting positions before averaging,

$$P_{s,N,L}(t) = \left\langle \prod_{i=1}^{N} p(t|x_i(0)) \right\rangle = \left[ \frac{1}{L} \int_{-L/2}^{L/2} dx p(t|x) \right]^{N}$$

$$= \left[ 1 - \frac{1}{L} \int_{-L/2}^{L/2} dx (1 - p(t|x)) \right]^{N}.$$

(7)

Taking the limit $L \to \infty, N \to \infty$, while keeping $N/L = \rho$ fixed and using the symmetry $p(t|x) = p(t|-x)$, we get

$$P_s(t) = \exp \left[ -2\rho \int_{0}^{\infty} dx (1 - p(t|x)) \right].$$

(8)

Note that, by definition,

$$p(t|x) = \text{Prob.}[x_i(\tau) > 0 \text{ for all } 0 \leq \tau \leq t | x_i(0) = x],$$

(9)

where $x_i(\tau)$ represents the stochastic trajectory of a trap, say the $i$th trap, starting at the initial position $x_i(0) = x$. Consider now the stochastic process $x_{\text{trap}}(\tau) = x_i(\tau) - x_i(0) = x(\tau) - x$. Then the probability $p(t|x)$ in equation (9) translates into the following probability for the process $x_{\text{trap}}(\tau)$

$$p(t|x) = \text{Prob.}[x_{\text{trap}}(\tau) > -x \text{ for all } 0 \leq \tau \leq t | x_{\text{trap}}(0) = 0].$$

(10)

Using the spatial inversion symmetry of the process $x_{\text{trap}}(\tau)$ around the origin, equation (10) can be written as

$$p(t|x) = \text{Prob.}[x_{\text{trap}}(\tau) < x \text{ for all } 0 \leq \tau \leq t | x_{\text{trap}}(0) = 0],$$

(11)

which, incidentally, is precisely the cumulative probability that the maximum $M(t) = \max_{0 \leq \tau \leq t} \{x_{\text{trap}}(\tau)\}$ of the process $x_{\text{trap}}(\tau)$ in the interval $\tau \in [0,t]$ is less than $x$, i.e.,

$$p(t|x) = \text{Prob.}[M(t) < x].$$

(12)

Thus the probability density of the maximum is $\partial_x p(t|x)$ with $x \geq 0$. The expected value of the maximum is then given by $E[M(t)] = \int_{0}^{\infty} x \partial_x p(t|x) dx$. Noting that $p(t|x) \to 1$ as $x \to \infty$, it is actually useful to rewrite this as $E[M(t)] = -\int_{0}^{\infty} x \partial_x [1 - p(t|x)] dx$ and then do integration by parts, giving

$$E[M(t)] = \int_{0}^{\infty} [1 - p(t|x)] dx.$$

(13)

Substituting this result in equation (8) immediately gives the general result in equation (3).

Let us make a quick remark here. For a random walk on a $d$-dimensional lattice, starting at the origin, the expected number of distinct sites visited by a walker up to time...
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t is simply [35], \( \sum_{\vec{r}} [1 - p(t|\vec{r})] \), where \( p(t|\vec{r}) \) is the probability that starting at \( \vec{r} \), the origin is not visited by the walker up to time \( t \). For such a lattice walk in one dimension, the expected number of distinct sites visited up to time \( t \) is thus \( \sum_{\vec{x}} [1 - p(t|x)] = 2 \sum_{x>0} [1 - p(t|x)] \). Hence, the quantity \( \sum_{x>0} [1 - p(t|x)] \) has the interpretation of half the average number of distinct sites visited by a walker up to time \( t \). The lattice version of the relation (13), where the integral is replaced by a sum over sites on the positive side, is consistent with this fact, since in one dimension the average maximum up to \( t \) is precisely half the average number of distinct sites visited up to \( t \), see [14,36] and references therein.

2.2. Expected maximum \( E[M(t)] \) for the CTRW

In this subsection, we focus on the case when \( x_{\text{trap}}(\tau) \) is a CTRW and derive the asymptotic large \( t \) expression for \( E[M(t)] \) of this process. The main idea behind this computation is as follows. Consider first a discrete-time random walk on a continuous line where the position \( x_n \) of the walker at step \( n \), starting at \( x_0 = 0 \), evolves at discrete integer time steps via the Markov rule
\[
x_n = x_{n-1} + \eta_n,
\]
where the \( \eta_n \) are independent and identically distributed (i.i.d) random variables, each drawn from a symmetric and continuous probability distribution \( f(\eta) \). Let \( M_n = \max_{0 \leq k \leq n} \{ x_k \} \) denote the maximum of this process till step \( n \). It turns out that an asymptotic expression for the expected maximum \( E[M_n] \) for large \( n \) can be derived explicitly for arbitrary symmetric and continuous jump length distribution \( f(\eta) \) [33]. Knowing this explicit result for the discrete-time case, one can then derive the corresponding expression of \( E[M(t)] \) for large \( t \) for a CTRW by using a standard method involving renewal theory. In fact, it turns out that this renewal method, originally due to Montroll and Scher [29], is rather general, in the sense that if one knows explicitly the expression for the expectation of any observable in the discrete-time jump process, then the corresponding expectation value for the CTRW can be derived in a straightforward manner, as illustrated below. This fact sometimes goes by the name ‘subordination property’ in the literature [31].

2.2.1. Subordination property. Let \( O_n \) be the expected value of any observable associated with the discrete-time process in equation (14). For a corresponding CTRW, these steps do not occur ‘at every tick of the clock’, but rather at random points in time separated by random variables \( \{ \tau_n \} \), called ‘waiting times’. The waiting times \( \tau_n \) are i.i.d variables each drawn from a waiting time density (WTD) denoted by \( \psi(\tau) \). For a fixed total time \( t \), the number of kicks, \( n \), is therefore a random variable and \( Q(n|t) \) denotes the probability that there are exactly \( n \) kicks in time \( t \). The continuous-time expectation value \( O(t) \) of the observable in CTRW is then related to its discrete counterpart \( O_n \) by the relation
\[
O(t) = \sum_{n=0}^{\infty} O_n Q(n|t).
\]

To compute \( Q(n|t) \) one can use the renewal theory, i.e., the successive intervals are statistically independent. Consider the interval \([0,t]\), broken up into \((n+1)\) intervals (see figure 2). The first \( n \) intervals are denoted by \( \tau_1, \tau_2, \ldots, \tau_n \) and the last interval

\(\tau_{\text{last}} = t - \sum_{i=1}^{n} \tau_i\), which is free of ‘kicks’. While each of the \(\tau_i\) (for \(i = 1, 2, \ldots, n\)) is drawn from \(\psi(\tau)\), the distribution of the last interval \(\tau_{\text{last}}\) is slightly different (see figure 2). The probability that an interval \(\tau\) is ‘kick’ free is simply \(\phi(\tau) = \int_{\tau}^{\infty} \psi(\tau') \, d\tau' = 1 - \int_{0}^{\tau} \psi(\tau') \, d\tau'\).

Thus, \(\tau_{\text{last}}\) is drawn from the density function \(\phi(\tau)\), while the preceding ones are each drawn independently from \(\psi(\tau)\). Then the probability \(Q(n|t)\) is given by the convolution

\[
Q(n|t) = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \, d\tau_1 \, d\tau_2 \cdots d\tau_n \, d\tau_{\text{last}} \, \psi(\tau_1) \psi(\tau_2) \cdots \psi(\tau_n) \phi(\tau_{\text{last}}) \times \delta(\tau_1 + \tau_2 + \cdots + \tau_n + \tau_{\text{last}} - t).
\]

(16)

Convolutions factorize under a Laplace transform. Denoting the Laplace transform of the WTD by \(\tilde{\psi}(u)\) and, noting that \(\int_{0}^{\infty} dt \, e^{-ut} \phi(t) = (1 - \tilde{\psi}(u))/u\), one gets

\[
\tilde{O}(u) \equiv \int_{0}^{\infty} dt \, e^{-ut} \, O(t) = \frac{1 - \tilde{\psi}(u)}{u} \sum_{n=0}^{\infty} O_n[\tilde{\psi}(u)]^n.
\]

(17)

Now, since \(0 < \tilde{\psi} < 1\) for \(u > 0\), the sum on the right-hand side of equation (17) can be interpreted as the generating function \(h(s) = \sum_{n=0}^{\infty} O_n \, s^n\) of \(O_n\) with \(s = \tilde{\psi}(u)\). Then the Laplace transform \(\tilde{O}(u)\) of the continuous-time quantity \(O(t)\) is related to the generating function \(h(s)\) of the discrete-time process via equation (17).

2.2.2. Typical displacement of a trap. As an example of the subordination method, let us consider a discrete-time process (14) where the jump length distribution \(f(\eta)\) has a power-law tail for large \(|\eta|\)

\[
f(\eta) \sim \frac{A_1}{|\eta|^{1+\nu}}
\]

(18)

with \(\nu > 0\). The variance of the jump length distribution \(\sigma^2 = \int_{-\infty}^{\infty} \eta^2 \, f(\eta) \, d\eta\) is finite if \(\nu > 2\) and infinite if \(\nu < 2\). For \(\nu < 2\), this process is the so called Lévy flights [30]–[32]. Let \(P_n(x)\) denote the distribution of the position of the trap after \(n\) steps, starting at the origin. It is then well known and is easy to show that \(P_n(x)\), for large \(n\) and for \(0 < \nu < 2\), has the scaling form [30]

\[
P_n(x) = \frac{1}{n^{1/\nu}} \, \mathcal{L}_\nu \left( \frac{x}{n^{1/\nu}} \right),
\]

(19)

where the scaling function, for large \(|z|\), decays as \(\mathcal{L}_\nu(z) \sim |z|^{-1-\nu}\) for \(0 < \nu < 2\). For \(\nu \geq 2\), the variance \(\sigma^2\) is finite and it follows from the central limit theorem that for large \(n\),

\[
P_n(x) \to \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp[-x^2/2\sigma^2 n].
\]

(20)

Thus, the typical displacement of the trap after step \(n\) scales, for large \(n\), as

\[
x_{\text{typ}}(n) \sim n^{1/\nu} \quad \text{for} \quad 0 < \nu \leq 2 \, \text{(superdiffusive)}
\]

\[
\sim n^{1/2} \quad \text{for} \quad \nu \geq 2 \, \text{(diffusive)}.
\]

(21)

(22)
Let us also consider a WTD $\psi(\tau)$ that has a power-law tail for large $\tau$

$$\psi(\tau) \to \frac{A}{\tau^{1+\alpha}}$$

(23)

with $\alpha > 0$. The mean waiting time $\mu = \int_0^\infty \tau \psi(\tau) \, d\tau$ is finite if $\alpha > 1$ and infinite if $\alpha < 1$. Correspondingly, the leading asymptotic behavior of the Laplace transform $\tilde{\psi}(u)$ of the WTD can be computed near $u \to 0$ (see, e.g., appendix 2 of [37]) as

$$\tilde{\psi}(u) \approx \begin{cases} 
1 - \mu u & \text{for finite } \mu \\
1 - A |\Gamma(-\alpha)| u^\alpha & \text{for infinite } \mu.
\end{cases}$$

(24)

Replacing $O_n$ by $x_{\text{typ}}(n)$ in (17) and using the above asymptotic expressions one can easily check that $x_{\text{typ}}(t)$, the typical displacement of the trap undergoing CTRW, behaves asymptotically in different ways depending on the following four cases

(i) both $\sigma$ and $\mu$ finite
(ii) $\sigma$ finite, $\mu$ infinite
(iii) $\sigma$ infinite but $\mu$ finite
(iv) both $\sigma$ and $\mu$ infinite

which, for power-law jump length distribution $f(\eta)$ in equation (18) and power-law WTD in equation (23) correspond to

(i) $\nu > 2$ and $\alpha > 1$
(ii) $\nu > 2$ and $\alpha < 1$
(iii) $\nu \leq 2$ and $\alpha > 1$
(iv) $\nu \leq 2$ and $\alpha < 1$.

This gives

$$x_{\text{typ}}(t) \sim \begin{cases} 
t^{1/2} & \text{case 1} \\
t^{\alpha/2} & \text{case 2} \\
t^{1/\nu} & \text{case 3} \\
t^{\alpha/\nu} & \text{case 4}
\end{cases}$$

(25)

where the cases refer to the four different situations mentioned above. Note, in particular, that by appropriately choosing the input exponents $\alpha$ and $\nu$, one can generate diffusive (case 1), subdiffusive (case 2) and superdiffusive (case 3) behavior, as well as a completely anomalous case (case 4), where one can get both subdiffusive and superdiffusive behavior by appropriately choosing $\alpha/\nu$.

As an aside, another recent example where this subordination property has been used concerns the statistics of records. For a discrete-time jump process in equation (14), the statistics of the number of records up to step $n$ turn out to be completely universal, i.e., independent of the jump length distribution $f(\eta)$ as long it is symmetric and continuous [38]. For instance, the average number of records up to step $n$ grows as $\langle R_n \rangle \approx \sqrt{4n/\pi}$ for large $n$ [38]. Using the subordination property mentioned above and by choosing $O_n = R_n$, the average number of records up to time $t$ for a CTRW with power-law WTD as in equation (23) has been computed recently [39].
2.2.3. Expected maximum of the trap motion. Having obtained the late time scaling of the typical displacement of a trap undergoing CTRW, we now turn to computing the expected maximum $E[M(t)]$ of the trap motion $x_{\text{trap}}(t)$. In order to use the general subordination technique mentioned above, we then need to know the behavior of $E[M_n]$ for large $n$ of the discrete-time process (14), where $M_n$ denote the maximum up to step $n$. Fortunately this can be done as illustrated below.

Computing explicitly the full distribution of $M_n$ for arbitrary $f(\eta)$ is, in general, a hard problem [33, 34]. However, there exists a general formula known as the Pollaczek–Spitzer formula [40, 41] which reads as [33]

$$\sum_{n=0}^{\infty} s^n E[e^{-\rho M_n}] = \frac{1}{\sqrt{1-s}} \phi(s, \rho),$$  \hspace{1cm} (26)

where

$$\phi(s, \rho) = \exp \left[ -\frac{\rho}{\pi} \int_{0}^{\infty} \frac{\ln(1-s \hat{f}(k))}{k^2 + \rho^2} \, dk \right],$$ \hspace{1cm} (27)

and $\hat{f}(k) = \int_{-\infty}^{\infty} f(\eta) e^{i k \eta} \, d\eta$ is the Fourier transform of $f(\eta)$. From the expression (26), the generating function of the expected maximum can, in principle, be obtained via the formula

$$h(s) = \sum_{n=0}^{\infty} s^n E[M_n] = -\frac{1}{\sqrt{1-s}} \frac{\partial \phi(s, \rho)}{\partial \rho} \bigg|_{\rho=0}.$$ \hspace{1cm} (28)

Replacing $O_n$ by $E[M_n]$ in the general formula (17), we then have for CTRW

$$\int_{0}^{\infty} d\xi e^{-ut} E[M(t)] = \frac{1 - \tilde{\psi}(u)}{u} h(\tilde{\psi}(u)).$$ \hspace{1cm} (29)

To extract the leading asymptotic behavior for large $t$, one needs to analyze the right-hand side of equation (29) in the limit $u \to 0$, or equivalently $h(s)$ in equation (28) in the limit $s \to 1$. Extracting the leading singularity of the right-hand side of (28) in the limit $s \to 1$ turns out to be rather nontrivial [33]. For jump length distributions with finite second moment $\sigma^2 = \int_{-\infty}^{\infty} \eta^2 f(\eta) \, d\eta$, the principal singular behavior of $h(s)$ near $s = 1$ can be extracted explicitly [33]

$$h(s) = \frac{\sigma}{\sqrt{2}} \frac{1}{(1-s)^{3/2}} + \frac{1}{\pi(1-s)} \int \frac{dk}{k^2} \ln \left[ \frac{2}{\sigma^2} \left( \frac{1 - \hat{f}(k)}{k^2} \right) \right] + O \left( \frac{1}{\sqrt{1-s}} \right).$$ \hspace{1cm} (30)

In case the when $\sigma^2$ is infinite, as for instance the case for Lévy flights in equation (18) with $\nu < 2$, where $\hat{f}(k) \approx 1 - |ak|^{\nu}$ to leading order for $k \to 0$, the asymptotic behavior of $h(s)$ near $s = 1$ turns out to be different from equation (30). For example, for $1 < \nu \leq 2$, one can show [33]

$$h(s) = \frac{a B(1/\nu, 1 - 1/\nu)}{\pi(1-s)^{1+1/\nu}} + \frac{1}{\pi(1-s)} \int_{0}^{\infty} \frac{dk}{k^2} \ln \left( \frac{1 - s \hat{f}(k)}{(ak)^{\nu}} \right) + O \left( \frac{1}{(1-s)^{1/\nu}} \right),$$ \hspace{1cm} (31)

where $B(a, b) = \Gamma[a]\Gamma[b]/\Gamma[a + b]$ is the standard Beta function.
Figure 3. The four different asymptotic power-law growths of $E[M(t)]$ for large $t$ in the four cases in the ($\alpha$–$\nu$) plane is summarized.

For the case $0 < \nu \leq 1$, it turns out $E[M_n]$ is infinite for any $n > 0$. Subsequently $E[M(t)]$ is also infinite for any finite $t$. Consequently, for $0 < \nu \leq 1$, the survival probability $P_s(t)$, via the relation (3), is 0 for any finite time $t$ (the target is killed instantaneously). This rather pathological case will not be discussed further.

We now need to substitute these asymptotic $s \to 1$ behaviors (for the two cases in equations (30) and (31)) in equation (29) and analyze the large $t$ behavior of $E[M(t)]$. To carry out the asymptotic analysis, one again needs to distinguish the four different cases mentioned in the previous subsection, namely (1) both $\sigma$ and $\mu$ finite (2) $\sigma$ finite but $\mu$ infinite (3) $\sigma$ infinite but $\mu$ finite (4) both $\sigma$ and $\mu$ infinite, where $\sigma^2$ and $\mu$ refer respectively to the second moment of the jump length distribution $f(\eta)$ and the first moment of the WTD $\psi(\tau)$. When both $f(\eta)$ and $\psi(\tau)$ have power-law tails, as in equations (18) and (23) respectively, the above four cases correspond to four regions in the ($\alpha$–$\nu$) plane

- (1) $\nu > 2$ and $\alpha > 1$
- (2) $\nu > 2$, $\alpha < 1$
- (3) $\nu < 2$, $\alpha > 1$
- (4) $\nu \leq 2$, $\alpha < 1$

As mentioned before, we will not discuss the pathological case $0 < \nu \leq 1$ and will restrict ourselves in the ($\alpha$–$\nu$) plane only for $\nu > 1$ and $\alpha > 0$ (see figure 3).

The next step is to plug the appropriate asymptotic expression for the Laplace transform of the WTD from equation (24) into the corresponding expression for the generating function of $E(M_n)$ in equations (30) and (31) and subsequently invert the Laplace transform in the asymptotic large $t$ limit. One needs to use the Tauberian theorem, cf [42,43], which states that a Laplace space behavior of $u^\rho$ for $u \ll 1$ corresponds to a $t^{\rho-1}$ behavior for $t \gg 1$ under inversion of the Laplace transform. It turns out that three constants appear in the final expressions for $E[M(t)]$, which we denote below as

$$K \equiv \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln \left[ \frac{2}{\sigma^2} \left( \frac{1 - \hat{f}(k)}{k^2} \right) \right], \quad (32)$$

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\[ I \equiv \frac{1}{\pi} \int_0^\infty \frac{dk}{k^2} \ln \left( \frac{1 - \hat{f}(k)}{(ak)^\nu} \right) \tag{33} \]

\[ J \equiv \frac{aB(1/\nu, 1 - 1/\nu)}{\pi^\nu(1 + \alpha/\nu)(\alpha A \Gamma(-\alpha))^{1/\nu}} \tag{34} \]

with \( B(a, b) \) being the standard Beta function. In the definition of equation (32), we assume \( \nu > 2 \) (finite), and in equation (33), we assume \( 1 < \nu < 2 \). We then obtain for large \( t \) the following asymptotic expressions for \( E[M(t)] \) in the four cases

\[
E[M(t)] \approx \begin{cases} 
\frac{\sqrt{2\sigma}}{\sqrt{\pi \mu}} t^{1/2} + K & \text{case 1} \\
\frac{\sigma}{\Gamma(1 + \alpha/2)\sqrt{2A\Gamma(-\alpha)}} t^{\alpha/2} + K & \text{case 2} \\
\frac{a\nu\Gamma(1 - 1/\nu)}{\pi(\mu)^{1/\nu}} t^{1/\nu} + I & \text{case 3} \\
Jt^{\alpha/\nu} + I & \text{case 4} 
\end{cases} \tag{35}
\]

where the cases refer to the four different situations mentioned above. Note that the result in case 1 also holds for arbitrary distributions \( f(\eta) \) and \( \psi(\tau) \) such that \( \sigma^2 \) and \( \mu \) are finite. These four different asymptotic power-law growths of \( E[M(t)] \) in the \((\alpha - \nu)\) plane are summarized in figure 3. Note that the leading asymptotic growth of \( E[M(t)] \) for large \( t \) is similar to that of the typical displacement in equation (25) in all four regions, except the prefactors of the power-law growth of \( E[M(t)] \) and \( x_{\text{typ}}(t) \) are different. As in the typical case, \( E[M(t)] \) shows the usual diffusive scaling to leading order in case 1, subdiffusive in case 2, superdiffusive in case 3, while in case 4 it scales like \( t^{\alpha/\nu} \). Interestingly, this totally anomalous case 4 of Lévy flights interspersed with long waiting times can exhibit diffusive scaling, for any choice of \( \alpha \) and \( 1 < \nu \leq 2 \) but with \( \alpha/\nu = 1/2 \). We have verified the analytical predictions in equation (35) via numerical simulations. The results are shown in figure 4, where different types of anomalous scaling behaviors are compared. Numerical results are in excellent agreement with analytical predictions.

3. Asymptotic survival probability: analytical predictions and numerical simulations

Plugging the expressions for \( E[M(t)] \) for the four cases considered above into the general relation in equation (3), we obtain the following large \( t \) behavior for the average survival probability \( P_s(t) \)

\[
P_s(t) \approx \begin{cases} 
\exp \left[ -2\rho \left( \frac{\sqrt{2\sigma}}{\sqrt{\pi \mu}} t^{1/2} + K \right) \right] & \text{case 1} \\
\exp \left[ -2\rho \left( \frac{\sigma}{\Gamma(1 + \alpha/2)\sqrt{2A\Gamma(-\alpha)}} t^{\alpha/2} + K \right) \right] & \text{case 2} \\
\exp \left[ -2\rho \left( \frac{a\nu\Gamma(1 - 1/\nu)}{\pi(\mu)^{1/\nu}} t^{1/\nu} + I \right) \right] & \text{case 3} \\
\exp \left[ -2\rho \left( Jt^{\alpha/\nu} + I \right) \right] & \text{case 4} 
\end{cases} \tag{36}
\]

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Figure 4. Expected maximum of a CTRW as function of $t$. On the log–log scale, the exponent of $t$ can be read off from the slope. (a) corresponds to the case of finite $\mu$, while (b) corresponds to infinite $\mu$. The bottom curve of (a) corresponds to case 1 where both $\mu$ and $\sigma$ are finite and the exponent is $1/2$ (see equation (35)). The top curve of (a) corresponds to case 3 where $\mu$ is finite but $\sigma$ is infinite—the exponent is $1/\nu$. Similarly, in (b), the bottom curve corresponds to case 2 (where $\mu$ is infinite but $\sigma$ is finite and the exponent is $\alpha/2$ as in equation (35)) while the top curve corresponds to case 4 (where both $\mu$ and $\sigma$ are infinite and the exponent is $\alpha/\nu$). Note, in particular, that in the bottom curve of (b), by choosing $\alpha/\nu = 1/2$ one obtains the diffusive behavior even though the process is totally anomalous. Simulations used a Lévy stable or standard normal density for the jump lengths and either exponential or Pareto distributed waiting times. Parameters are given in the figure.

where the constants $K$, $I$ and $J$ are given respectively in equations (32), (33) and (34). Thus, in all four cases, one finds generically the stretched exponential behavior for large $t$

$$P_s(t) \approx a \exp[-bt^\theta]$$

(37)

except that the amplitude $a$, the constant $b$ and the stretching exponent $\theta$ differs in the four cases. In the four different cases, the exponent $\theta$ is respectively (1) $\theta = 1/2$ (2) $\theta = \alpha/2$ with $\alpha < 1$ (3) $\theta = 1/\nu$ with $1 < \nu \leq 2$ and (4) $\theta = \alpha/\nu$ with $\alpha < 1$ and $1 < \nu \leq 2$. We also emphasize that our method yields an exact result even for the amplitude $a$ in front of the stretched exponential in equation (37). Usually this is much harder to obtain, as it corresponds to computing the subleading correction term in $E[M(t)]$. In the four respective cases, we get the amplitudes

(i) $a = \exp[-2\rho K]$
(ii) $a = \exp[-2\rho K]$
(iii) $a = \exp[-2\rho I]$
(iv) $a = \exp[-2\rho I]$

where the constants $K$ and $I$ are given respectively in equations (32) and (33).
3.1. Recovering known results for diffusive and subdiffusive traps

From our general result in equation (36), it is easy to recover the known results for the Brownian (diffusive) and subdiffusive traps first derived in [14] and [23] respectively. Consider first the Brownian case. To recover this result corresponding to Brownian motion of a trap, we need to focus on case 1 of our result in equation (36) and take the Brownian scaling limit where both mean-square jump length $\sigma^2$ and mean waiting time $\mu$ tend to zero, while their ratio $D = \sigma^2/2\mu$ stays constant. In order to take this limit cleanly, it is convenient to first make a change of variable $k \rightarrow \sigma k/\sqrt{2}$ in the expression for the constant $K$ in (32) to get

$$K = \frac{\sigma}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{k^2} \ln \left( \frac{1 - \hat{f}(\sqrt{2}k/\sigma)}{k^2} \right).$$  

(38)

Note that $K$ is independent of $\mu$. Since $\hat{f}(k) \rightarrow 0$ as $k \rightarrow \infty$, we can safely take the limit $\sigma \rightarrow 0$ in equation (38) and obtain $K = 0$. From case 1 of equation (36), we then recover the Brownian result: $P_s(t) \rightarrow \exp(-2\rho \sqrt{4Dt/\pi})$.

To recover the result for the continuous-time subdiffusive case in [23], we need to take the scaling limit of our result in equation (36) corresponding to case 2. As in the Brownian case, we need to take the $\sigma^2 \rightarrow 0$ limit. However, unlike in the Brownian case, here the mean waiting time $\mu$ is already infinite. So, the only way a sensible limit of line 2 in equation (36) can be reached is if $2A|\Gamma(-\alpha)|$ goes to zero while $\sigma^2 \rightarrow 0$, but their ratio $D_\alpha \equiv \sigma^2/(2A|\Gamma(-\alpha)|)$ remains fixed. This ratio $D_\alpha$ can be called the generalized diffusion constant, since it can be shown that it appears also in the amplitude of the mean-square displacement in case 2 (with $\alpha < 1$)

$$\langle x_{\text{trap}}^2(t) \rangle \approx \frac{2D_\alpha}{\Gamma(1+\alpha)} t^\alpha.$$  

(39)

This relation can be thought of as the generalization of the standard diffusion relation $\langle x_{\text{trap}}^2(t) \rangle = 2Dt$, with $D$ being the standard diffusion constant. Indeed equation (39) reduces to the standard diffusion relation when $\alpha \rightarrow 1$. Taking this scaling limit in line 2 of equation (36), we thus recover the result of Yuste and Lindenberg [23],

$$P_s(t) \rightarrow \exp \left[ -\frac{2\rho \sqrt{D_\alpha}}{\Gamma(1+\alpha/2)} t^{\alpha/2} \right].$$  

(40)

Our general result in equation (36), apart from recovering known results in the appropriate limits as shown above, also provides new results for the superdiffusive case (which can be achieved in cases 3 and 4). In addition, our result also provides not just the leading asymptotic behaviors in all cases, but also the subleading order corrections in the form of non-universal amplitudes.

3.2. Numerical simulations

To confirm our analytical results, we also performed numerical simulations. Due to the computational effort of simulating large systems, only a finite number of $N$ particles uniformly distributed over the interval $(0, L)$ with $N/L = \rho$ fixed were considered. The reason for only considering traps on one side of the origin is just computational
Survival probability of an immobile target surrounded by mobile traps

Figure 5. Left: for standard normal distributed jumps length with $\sigma^2 = 1$. This corresponds to case 1. The dotted line shows the analytical behavior. Right: for Lévy jump length distribution with Fourier transform $\hat{f}(k) = \exp[-(a|k|^\nu)]$ with the choice $\nu = 1.5$ and $a = 1$. While for the Gaussian case, the survival probability was averaged over $10^5$ realizations, for the Lévy distribution, between $10^6$ ($N = 10, L = 1000$) and $10^7$ (all others) realizations were used, since due to the possibly large jumps, only very few realizations survive up to long times. For the same reason, the density of traps is a factor of 10 lower than for the Gaussian case.

convenience. For better accuracy of comparison, figure 5 compares $-\ln(P_s(t))/t^\theta$ to numerical simulations with the exponent $\theta$ taking the value corresponding to the case considered, see equation (36). In all cases, $-\ln(P_s(t))/t^\theta$ is of the form $\rho a_1 + \rho a_2/t^\theta$, with constants $a_1$ and $a_2$ depending on the cases. The missing pre-factor 2 in front of the trap density $\rho$ is due to the asymmetry of having only traps on one side of the origin.

Simulations were then performed for increasing values of $N$ and $L$ to show that, as the effects of finite system size become less important, the simulated curves converge onto the analytical predictions, see figure 5. The simulations were performed with the WTD $\psi(\tau) = \delta(\tau - 1)$, thus leaving us with only cases 1 and 3 to distinguish. The influence of infinite mean waiting time on the expected value of the maximum of a CTRW has already been illustrated numerically in figure 4, so it suffices to study the survival probability of a target surrounded by traps performing a random walk or Lévy flight with finite $\mu$.

From figure 5, one sees that, with increasing size $L$, the numerical curves approach the analytically predicted behaviors shown by dotted lines.

4. Conclusion

To summarize, we have studied analytically, in one dimension, the survival probability $P_s(t)$ of an immobile target at the origin surrounded by a sea of independent mobile traps initially distributed uniformly with density $\rho$ on the line. We have shown that for a rather general stochastic motion $x_{\text{trap}}(t)$ of each trap, there is very simple general relation, $P_s(t) = \exp[-2\rho E[M(t)]]$, that relates the survival probability to the expected maximum of the process $x_{\text{trap}}(t)$. This general result recovers all earlier known special cases of $x_{\text{trap}}(t)$. In addition, we have presented a new result when $x_{\text{trap}}(t)$ is a continuous-time random walk (CTRW) with arbitrary jump length distribution $f(\eta)$ and with arbitrary

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waiting time distribution \( \psi(\tau) \). By choosing these two input distributions of CTRW one can generate a variety of stochastic motions of the trap, both subdiffusive and superdiffusive. For such a general CTRW, we were able to evaluate exactly the two leading terms in the asymptotic behavior of \( E[M(t)] \) for large \( t \) and, plugging this in our general relation, we were able to compute the two leading terms for the large \( t \) behavior of \( P_s(t) \). Generically, we found that, for large \( t \),

\[
P_s(t) \sim a \exp[-bt^\theta],
\]

where the exponent \( \theta \), as well as the two constants \( a \) and \( b \) are computed exactly in this paper when \( x_{\text{trap}}(t) \) is a CTRW with arbitrary \( f(\eta) \) and \( \psi(\tau) \). Previous results in the literature were known only for diffusive and subdiffusive \( x_{\text{trap}}(t) \) (cases 1 and 2 in figure 3).

Our result is more general and, in particular, the exact asymptotic expressions for the two remaining cases (the superdiffusive (case 3) and totally anomalous (case 4)) are, to our knowledge, new. In addition, our method allows us even to compute the non-universal amplitude \( a \) (and not just the stretching exponent \( \theta \)) exactly in all cases.

There are several future directions in which it may be possible to extend our results. Firstly, it would be interesting to generalize our result to higher dimensions. The main difficulty in higher dimensions is that it is difficult to obtain an explicit expression for the probability that a discrete-time random walker does not hit a static ball (target) of finite radius up to time \( t \) [8,9], though it may be possible to obtain some exact results in the special case of \( d = 3 \) [9–11]. It would be interesting to derive the result for \( P_s(t) \) in this more realistic \( d = 3 \) case.

Secondly, it would be interesting to derive the corresponding result for \( P_s(t) \), in one or higher dimensions, in the case when the target itself is mobile, e.g., when the target also performs a general CTRW as the trap. As mentioned in section 1, in special cases of diffusive and subdiffusive motion of the target it is known whether the leading asymptotic behavior (but not the subleading) of \( P_s(t) \) does not change from the situation when the target is static. Is this fact still true when the target performs a general CTRW? It would be interesting to prove or disprove this fact.

Finally, it would be interesting to find suitable applications of our results. In fact, the totally anomalous case 4 (see figure 3), for which we present new exact results, may at first glance seem to be only of academic interest, but has recently been realized in a more physical context, namely, in a model for the spread of epidemics by human travel [44]. The results presented above thus have a natural interpretation as the probability of staying on the ‘safe’ side of the propagating front of a disease up to time \( t \), and it would be interesting to explore in detail this interesting possible application in future.

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