Gauge non-invariance of quark-quark interactions

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We discuss the gauge invariance of quark-quark potential in QCD in the perturbative treatment, and show that the gauge dependence of the quark-gluon interaction cannot be eliminated. Therefore, the quark-quark potential cannot properly be written in terms of physical variables in the perturbation theory. This is in contrast to the QED case in which the gauge invariance of the electron-photon interaction is guaranteed by the current conservation. As an example, we show that the different choices of the gauge fixing give different Coulomb interactions when the fermion current is not conserved. Also, we examine a possible existence of $QG$-state (one quark—one gluon) corresponding to the conserved color current, which should have a fractional electric charge.

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I. INTRODUCTION

The strong interaction is described by Quantum Chromodynamics, and hadrons are described in terms of quark and gluon dynamics 1. Quarks and gluons are confined inside hadrons since they have color charges. Each color current of quark and gluon is gauge dependent, and therefore they are not physical observables. The confinement of quarks and gluons is due to the non-abelian character of the gauge field theory, and the dynamics of quarks and gluons should be controlled by the kinematical constraints of the non-abelian nature of the gauge field theory. For example, the quark color current which is gauge dependent cannot come out from hadrons as the MIT bag model assumes 2. In this respect, the confinement should not simply be originated from the force between quarks and anti-quarks. In fact, it has recently been shown that Wilson’s area law between two quarks 3 is not physically justified and his area law is the result of an artifact of Wilson’s action. Indeed, Wilson’s action inevitably picks up unphysical contributions to the Wilson loop integral 4 which results in a fictitious area law. Therefore, there is no confinement which arises from a linear potential between quarks inside hadrons. In addition, it is, even more, difficult to define a force between quarks since the color charge of quarks is not a conserved quantity.

The gauge dependence of the quark color charge is well known. Yet, it is somewhat puzzling that one often finds discussions in field theory textbooks concerning the shape of the quark-quark potential in the perturbative calculations. The potential between quarks may be defined if the color charge of quarks is a conserved quantity. However, since the color charge of quarks is not conserved and is indeed gauge dependent in QCD, it is difficult to consider any interactions between quarks whose color charges are arbitrary time dependent numbers.

In this paper, we clarify that the gauge dependence of the quark-gluon interaction cannot be taken away by any kinds of transformation. Therefore, the quark-quark potential which is derived in the second and higher order perturbative calculations of the quark-gluon interaction cannot be a physical quantity since it depends on the gauge variable $\chi^a$ which is an arbitrary function of space and time.

The difficulty of evaluating interactions between quarks is also related to the non-existence of a free quark state since one cannot carry out any perturbative calculations which are always based on the S-matrix formulation. As is well known, one has to prepare free quark states as the initial and final states in the S-matrix evaluations. However, since there is no free quark state, one does not know how to proceed the S-matrix calculation in QCD as far as one wishes to be sufficiently careful.

For comparison, we briefly discuss the gauge dependence of the electron-photon interaction in the perturbation calculation. This is carefully examined in some of the field theory textbooks, and the potential between electrons is well defined because the gauge dependence of the electron-photon interaction is nicely eliminated under the vector current conservation. This is closely related to the fact that the electron charge is indeed conserved, and one can always define a free electron state as a trivially simple fact. In this respect, there is no problem in evaluating Feynman diagrams in a perturbation theory of QED.

Under the condition that the fermion current is not conserved, we show that the different choices of the gauge fixings (temporal gauge $A_0 = 0$ and Coulomb gauge $\nabla \cdot A = 0$ ) give different Coulomb interactions. This is quite serious since one cannot rely on the perturbative calculations of the quark-quark interactions since the color quark current is not conserved. The only way out of this dilemma may be to calculate the quark-quark interactions with the total Hamiltonian in a non-perturbative fashion since the total Hamiltonian is always gauge invariant. At the present stage, we do not know which...
of the shape of the quark-quark interactions may emerge from the non-perturbative treatment. We note that the MIT bag model assumes that the quark-quark interactions can be ignored and quarks are free inside hadrons and that the confinement must be realized by the boundary condition at the bag surface.

Here, we also discuss an old but new physical quantity, the conservation of the Noether current in QCD. As long as a physical quantity like charge is gauge invariant and is conserved, then there is a good chance that it may be observed. Here, we discuss a conserved current which is composed of one quark and one gluon. This is the only conserved color current of QCD which must play an important role for QCD dynamics. Since this current is connected to the QG-state, this state can be a physical observable since it is gauge independent and has a conserved charge. Therefore, this QG-state has a finite possibility of existence in nature even though it has a color charge and a fractional electric charge. Up to now, there is no observation of the fractional charge \[ \sum \mathfrak{g} \equiv \sum \delta_{ab} \] , but it does not mean that the object cannot be a physical observable, at least, theoretically.

II. QCD WITH SU(N) COLORS

The properties of QCD are well discussed in field theory textbooks, but we review gauge dependent parts in order to make later discussions easier to understand. In particular, we stress that the conservation law of the color current is the only conserved current in connection with the gauge fields in the theory. It is a well known fact that, in any field theory models without exception, a conserved charge has been playing a dominant role for understanding its dynamics. In this sense, it is very strange that the conserved color current has not been carefully considered up to now when QCD dynamics is investigated and evaluated, and we believe it should be definitely necessary to take into account the current conservation in some way or the other in order to obtain any important information of physical observables in hadron dynamics from QCD.

A. Lagrangian density of QCD

The Lagrangian density of QCD with SU(N) colors is described as

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - gT^a \gamma^\mu A_\mu - m_0)\psi - \frac{1}{2} \text{Tr}\{G_{\mu\nu}G^{\mu\nu}\},
\]

where \( m_0 \) denotes the quark mass, and the field strength \( G_{\mu\nu} \) is written as

\[
G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]
\]

where \( T^a \) denotes the generator of SU(N) group and \( n \) is related to \( N \) by \( n = N^2 - 1 \). Here, \( T^a \) satisfies the following commutation relations

\[
[T^a, T^b] = iC^{abc}T^c
\]

where \( C^{abc} \) denotes the structure constant of the group generators.

In eq.(2.1), Tr \{ \} means the trace of the group generators of SU(N), and we make use of the following identity as a normalization of the representation of the generators

\[
\text{Tr}\{T^a T^b\} = \frac{1}{2} \delta_{ab}.
\]

Therefore, the last term of eq.(2.1) can be rewritten as

\[
-\frac{1}{2} \text{Tr}\{G_{\mu\nu}G^{\mu\nu}\} = -\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu}
\]

where \( G_{\mu\nu}^a \) is described as

\[
G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gC_{abc} A_\mu^b A_\nu^c.
\]

B. Infinitesimal gauge transformation

The QCD Lagrangian density is invariant under the following infinitesimal local gauge transformation with \( \chi = \chi^a T^a \)

\[
\psi' = (1 - i\chi)\psi = (1 - igT^a \chi^a)\psi \quad (2.6a)
\]

\[
A'_\mu = A_\mu + ig[A_\mu, \chi] + \partial_\mu \chi
\]

where \( \chi \) is infinitesimally small. By defining

\[
D_\mu = \partial_\mu + igT^a A_\mu^a
\]

we can prove

\[
\bar{\psi}' i\gamma^\mu D'_\mu \psi' = \bar{\psi} i\gamma^\mu D_\mu \psi \quad (2.8a)
\]

\[
G'_{\mu\nu} = (1 - igT^a \chi^a)G_{\mu\nu}(1 + igT^a \chi^a).
\]

Therefore, we obtain

\[
\text{Tr}\{G'_{\mu\nu}G^{\mu\nu}\} = \text{Tr}\{(1 - igT^a \chi^a)G_{\mu\nu}G^{\mu\nu}(1 + igT^a \chi^a)\}
\]

\[
= \text{Tr}\{G_{\mu\nu}G^{\mu\nu}\}.
\]

This completes the proof of the local gauge invariance of eq.(2.1).
III. NOETHER CURRENT IN QCD

The QCD Lagrangian density is invariant under the following infinitesimal global gauge transformation

\[ \psi' = (1 - igT^a \theta^a) \psi \]  
\[ A^\alpha'_\mu = A^\alpha_\mu - gC^{abc} A^c_\mu \theta^a \]  

where \( \theta^a \) is an infinitesimally small constant. In this case, we obtain

\[ \delta \mathcal{L} = \mathcal{L}(\psi', \partial_\mu \psi', A^\alpha'_{\mu}, \partial_\mu A^\alpha_{\nu}) - \mathcal{L}(\psi, \partial_\mu \psi, A^\alpha_{\mu}, \partial_\mu A^\alpha_{\nu}) = 0. \]

By making use of the equations of motion for \( \psi \) and \( A^\alpha_{\mu} \), we obtain

\[ \delta \mathcal{L} = \left[ -ig(i\partial_\mu \bar{\psi} \gamma^\mu T^a \psi + i\bar{\psi} \gamma^\mu T^a \partial_\mu \psi) + gC^{abc}(\bar{\psi} G^{\mu\nu;b} A^c_\nu + G^{\mu\nu;b} \partial_\mu A^c_\nu) \right] \theta^a = 0. \]

Therefore, we easily see that

\[ \partial_\mu (\bar{\psi} \gamma^\mu T^a \psi + C^{abc} G^{\mu\nu;b} A^c_\nu) = 0. \]

This means that the Noether current

\[ I^{\mu,a} \equiv j^{\mu,a} + C^{abc} G^{\mu\nu;b} A^c_\nu \]

is indeed conserved. That is,

\[ \partial_\mu I^{\mu,a} = 0 \]

where the quark color current \( j^{\mu,a}_\mu \) is defined as

\[ j^{\mu,a}_\mu = \bar{\psi} \gamma^\mu T^a \psi. \]

Thus, the quark color current alone is not conserved, and therefore there is no conservation of the quark color charge. This is consistent with the fact that the color current of quarks is not a gauge invariant quantity.

IV. GAUGE NON-INVARINANCE OF INTERACTION LAGRANGIAN

The interaction Lagrangian density of QCD that involves quark color current is written as

\[ \mathcal{L}_I = -g j^{\mu,a}_\mu A^{\mu,a}. \]

Now, the interaction Lagrangian density \( \mathcal{L}_I \) is not gauge invariant, and therefore if one wishes to make any perturbation calculations involving the quark color currents, then one should check it in advance whether one can make the gauge invariant quark-quark interactions or not.

The interaction Lagrangian density is transformed into a new shape under the gauge transformation

\[ \mathcal{L}_I = -g j^{\mu,a}_\mu (A^{\mu,a} + \partial^\mu \chi^a) \]

where the second term is a gauge dependent term. In the same way as QED case which will be briefly reviewed in the next section, we can rewrite the second term by making use of the conserved current as

\[ -g j^{\mu,a}_\mu \chi^a = -g \partial^\mu (j^a_{\mu} \chi^a) + gC^{abc} \chi^a \partial^\mu G^{b} \psi^c. \]

The first term is a total divergence and thus does not contribute to perturbative calculations. However, there is no way to eliminate the second term which is a gauge dependent term.

Therefore, one sees that one cannot make any simple-minded perturbative calculations of quark-quark interactions in QCD, contrary to the QED case where the electron-electron interaction is well defined and calculated as we will see below. This means that one cannot define any potential between quarks, and this is of course consistent with the picture that the color charge of quarks are not gauge invariant, and therefore it is time dependent.

V. GAUGE INVARIANCE OF ELECTRON-PHOTON INTERACTION IN QED

In QED, the similar situation indeed happens \[7, 8\]. That is, the electron-photon interaction is, at a glance, gauge dependent since it transforms as,

\[ L_I = -e j^\mu_\mu A^\mu - e j^\mu_\mu \partial^\mu \chi. \]

However, eq. (5.1) can be rewritten as

\[ L_I = -e j^\mu_\mu A^\mu - e \partial^\mu (j^\mu_\mu \chi) + e (\partial^\mu j^\mu_\mu) \chi. \]

and one sees easily that the second term in the right hand side is a total divergence and thus does not contribute to perturbative evaluations. The third term vanishes as long as the current conservation holds,

\[ \partial^\mu j^\mu_\mu = 0. \]

Therefore, it is clear that the electron-photon interaction is gauge independent, and the potential between electrons is a well defined quantity in QED under the condition that the vector current is conserved. Therefore, one can well calculate electron-electron potentials in the perturbation theory, which are Coulomb potential or some others like hyperfine interactions.

VI. TWO CHOICES OF GAUGE FIXINGS

Now, we present the evaluation of the quark-quark interactions when there is no conservation of the quark current. Here, we suppress both the color degree and the explicit gluon-gluon vertex in the calculations since they are not relevant in the present discussions.
Since there is no conservation of the quark current, the equation of motion becomes
\[ \partial_{\mu} F^{\mu\nu} = g(j^{\nu} + X^{\nu}) \]  
(6.1)
where the extra current \( X^{\nu} \) is introduced by hand here, but, in QCD, this corresponds to the gluon currents. Therefore, we have \( \partial_{\mu}(j^{\mu} + X^{\mu}) = 0 \), but \( \partial_{\mu}j^{\mu} \neq 0 \). In this case, we can easily calculate the Coulomb interaction for the different gauge fixings, the Coulomb gauge \( \nabla \cdot A = 0 \) and the temporal gauge \( A_0 = 0 \), and show that the different choices of the gauge fixing give indeed different Coulomb interactions between quarks. This is consistent with the observation in section IV.

### A. Coulomb gauge

The Hamiltonian of quarks interacting with the gauge field can be written
\[ H = \bar{\psi}(-ig\gamma \cdot \nabla + m) \psi - gj \cdot \mathbf{A} + gj_0A_0 \]
\[ + \frac{1}{2} \left[ \mathbf{A}^2 - (\nabla A_0)^2 + \mathbf{B}^2 \right]. \]
(6.2)
We take the Coulomb gauge
\[ \nabla \cdot \mathbf{A} = 0. \]
(6.3)
Since the quark current is not conserved ( \( \partial_{\mu}j^{\mu} \neq 0 \) ), the equation of motion for the gauge field \( A_0 \) becomes from eq.(6.1)
\[ \nabla^2 A_0 = -g(j_0 + X_0). \]
(6.4)
This is just a constraint which can be easily solved, and we obtain
\[ A_0(r) = \frac{g}{4\pi} \int \frac{(j_0(r') + X_0(r')) d^3r'}{|r' - r|}. \]
(6.5)
Now, we can make use of the following relation
\[ \frac{1}{2} \int (\nabla A_0)^2 d^3r = -\frac{1}{2} \int (\nabla^2 A_0)A_0 d^3r \]
\[ = g^2 \frac{8\pi}{|r' - r|} \int \frac{(j_0(r') + X_0(r')) (j_0(r) + X_0(r)) d^3r' d^3r'}{|r' - r|} \]
(6.6)
where the surface integrals are set to zero. Therefore, the Coulomb interaction becomes
\[ H_C = \int gj_0 A_0(r) d^3r - \frac{1}{2} \int (\nabla A_0)^2 d^3r \]
\[ = g^2 \frac{8\pi}{|r' - r|} \int \frac{(j_0(r') - X_0(r')) (j_0(r) + X_0(r)) d^3r' d^3r'}{|r' - r|} \]
(6.7)
It is clear that if we set \( X_0 = 0 \), then we can recover the normal Coulomb interaction.

### B. Temporal gauge

Now, we take the \( A_0 = 0 \) and therefore the equation of motion for the gauge field becomes
\[ \nabla \frac{\partial \mathbf{A}}{\partial t} = -gj_0. \]
(6.8)
Here, there is still a gauge freedom left. Namely, \( \mathbf{E} = -\frac{\partial A}{\partial t} \) and \( \mathbf{B} = \nabla \times \mathbf{A} \) are invariant under the following transformation
\[ \mathbf{A} \rightarrow \mathbf{A} + \nabla \chi(r) \]
(6.9)
where \( \chi(r) \) depends only on the coordinate \( r \).

Therefore, we can write the vector field \( \mathbf{A} \) as
\[ \mathbf{A} = \mathbf{A}_T + \nabla \xi, \quad \text{with} \quad \nabla \cdot \mathbf{A}_T = 0. \]
(6.10)
In this case, the equation of motion for the gauge field becomes
\[ \nabla^2 \phi_0 = -g(j_0 + X_0), \quad \text{with} \quad \phi_0 \equiv \dot{\xi}. \]
(6.11)
Therefore, \( \frac{1}{2} \dot{\mathbf{A}}^2 \) term in the Hamiltonian can be written with \( \mathbf{E}_T = -\frac{\partial \mathbf{A}_T}{\partial t} \) where \( \mathbf{E}_T \) is the transverse electric field
\[ \frac{1}{2} \int \dot{\mathbf{A}}^2 d^3r = \frac{1}{2} \int \dot{\mathbf{E}}_T^2 d^3r + \int \mathbf{E}_T \cdot \nabla \phi_0 d^3r + \frac{1}{2} \int (\nabla \phi_0)^2 d^3r. \]
(6.12)
The second term in the right hand side vanishes and the third term is just the same as the normal Coulomb interaction. Therefore, the Coulomb interaction with the temporal gauge becomes
\[ H_C = \frac{g^2}{8\pi} \int \frac{(j_0(r') + X_0(r')) (j_0(r) + X_0(r)) d^3r d^3r'}{|r' - r|} \]
(6.13)
which is different from the Coulomb interaction from the Coulomb gauge fixing. Clearly again, if the extra-current \( X_0 \) is switched off, then we can recover the normal Coulomb interaction which, of course, agrees with the one that is obtained from the Coulomb gauge fixing.

Therefore, we see that the different choices of the gauge fixing give different Coulomb interactions when the current is not conserved. This confirms that the gauge invariance of the interaction term \( \mathcal{L}_I = -gj_\mu A^{\mu - \alpha} \) cannot be recovered for the nonabelian gauge field theory like QCD since the quark currents alone are not conserved. This means that the quark-quark interactions have some difficulties when one starts from the perturbation theory.

It is unfortunate that one gluon exchange potential which has been very popular up to now does not make sense if one starts to calculate it from the perturbation theory. We should find out some alternative expressions for interactions between quarks and gluons.
VII. QG–STATE

From eqs.(3.4) and (3.5), one sees that the color octet vector current of one quark and one gluon state (QG–state) is conserved. Therefore, the color charge $Q_I^a$ of the QG–state which is defined as

$$Q_I^a = \int I^{0,a} d^3x \tag{7.1}$$

is indeed a conserved quantity even though it is not a color singlet. In addition, the charge $Q_I^a$ of QG–state is a gauge invariant quantity.

Here, one may ask as to whether this QG–state can be observables or not. The answer is yes, that is, it may physically exist since it is gauge independent and a conserved quantity. At least, there is no reason to say that it cannot exist theoretically.

One may then ask whether this is a bound state or not. But we believe it is not an important question whether it may be a bound state or something else. It is for sure that, once it were made, then there is no way to decay since quark or gluon alone cannot exist by themselves. The only question, though rather serious, is that, in which way the QG–state can be created. Or in other words, what should be a probability of producing this object from any kind of hadronic or leptonic reactions?

For the above statement, one may claim that there is no fractional electric charge observed in nature \cite{5,6}, and therefore it should not exist. This may be right, but at the present stage, we cannot claim more than what it is like.

Since we prove that the one gluon exchange potential between quarks is not a well defined quantity, we should ask what is the potential between two nucleons. To this question, we can answer to some extent. Firstly, the most important nucleon-nucleon potential must be given by the one pion exchange process. This is quite reasonable since pion which is a color singlet object can propagate between nucleons and should indeed produce a one pion exchange potential which is a favorable object from experiments. In addition to the boson exchange interactions, the exchange of QG states between nucleons may give rise to a nucleon-nucleon potential which should be somewhat similar to the exchange force in atomic electrons. This may cause a repulsive interaction between nucleons and it should be rather short range interactions. However, this mechanism of the QG exchange force should be a future problem to be solved.

VIII. CONCLUSIONS

We have examined the gauge invariance of the quark-gluon interaction in QCD. In most of field theory text books, people have carefully discussed the gauge dependence of the electron-photon interaction in QED and confirmed that the apparent gauge dependence of the interaction can be well eliminated. However, for QCD, the gauge invariance of the quark gluon interaction in the context of the perturbation theory has not been examined carefully in any of field theory textbooks. This may be because the QCD case was treated just in the same as the QED case. Obviously, there is a serious difference between QCD and QED, which is essentially due to the nonabelian character of the gauge fields. Here, we have shown that the quark-gluon interaction in QCD is always gauge dependent and there is no way to eliminate the gauge dependence in the quark-gluon interaction. This means that one cannot define the quark-quark potential, at least, in a perturbative sense. Therefore, it should be meaningless to discuss the shape of the quark-quark potential in perturbative calculations even though one can treat a heavy quark in the non-relativistic kinematics in which the concept of potential itself is well defined.

However, we have not presented any alternative for the quark interactions with gluons, and this is a future problem to be solved.

Further, we point out that there is a conserved quantity which is composed out of one quark and one gluon QG–state, and this is gauge invariant and has a conserved charge. The possibility of its existence in nature should be carefully examined.

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