A FUNCTIONAL CENTRAL LIMIT THEOREM FOR POLARON PATH MEASURES

VOLKER BETZ AND STEFFEN POLZER

Abstract. The application of the Feynman-Kac formula to Polaron models of quantum theory leads to the path measure of Brownian motion perturbed by a pair potential that is translation invariant both in space and time. An important problem in this context is the validity of a central limit theorem in infinite volume. We show both the existence of the relevant infinite volume limits and a functional central limit theorem in a generality that includes the Fröhlich polaron for all coupling constants. The proofs are based on an extension of a novel method by Mukherjee and Varadhan.

Keywords: Polaron, renewal process, functional central limit theorem, point process

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1. Introduction

The polaron is a model for a quantum particle interacting with a polar crystal. The interaction affects at least two quantities of physical interest. On the one hand, the ground state energy of the system is lowered when the interaction is increased. On the other hand, the particle needs to drag along the polarization when it moves, and thus appears heavier than it would be without the interaction: it has an effective mass larger than its bare mass. For a particular polaron model now known as the Fröhlich polaron, both of these effects were investigated by Feynman in [Fey55] using his newly invented path integral method. The central object in this context is the quantity

\[ z_t = \int \exp \left( \alpha \int_0^t \int_0^t \frac{e^{-|s-r|}}{|x_s - x_r|} \, ds \, dr \right) \mathcal{W}_0(dx), \]

where \( \mathcal{W}_0 \) is three-dimensional Brownian motion, and \( \alpha \geq 0 \) is the coupling constant determining the strength of the interaction between the particle and the polar crystal. For the ground state energy, Feynman argues that there exists \( E_\alpha \leq 0 \) such that \( z_t \sim e^{-E_\alpha t} \) as \( t \to \infty \), and that \( E_\alpha \) is then ground state energy of the polaron. For the effective mass, he finds that it can be obtained by pinning the Brownian motion at \( \bar{x} \) at the final time \( t \) and observing the dependence of \( z_t \) on \( \bar{x} \) under these circumstances. More precisely, when \( z_t(\bar{x}) \) is the value of the integral (1.1) with an additional factor of \( \delta(x_t - \bar{x}) \) in the integrand, then \( z_t \) should behave like \( \exp(-E_\alpha t - m_\alpha \bar{x}^2/2t) \) for large \( t \), where \( m_\alpha \) is the effective mass. Feynman then goes on to derive quantitative estimates for the ground state energy and the effective mass as functions of \( \alpha \). Among other things, he finds that \( E_\alpha \leq -\alpha - 1.23(\alpha/10)^2 \), and that \( m_\alpha \sim 202(\alpha/10)^4 \) for large \( \alpha \), and remarks that these results compare well with those obtained earlier by Pekar [Pek49] using the adiabatic approximation.

The arguments of Feynman, while ingenious, are almost completely non-rigorous. Since then, there have been many efforts to understand various aspects of the polaron problem in a mathematically rigorous way. Most fundamental among them is a firm connection between formula (1.1) and the underlying many-body quantum system. This connection
has emerged over the years in various forms, and it is difficult to track it back to a single source. We refer to [Møl06] for a review on functional analytic aspects of the polaron, and to [DS20] for an outline on the connection to probability theory. For the convenience of the reader, we also include a short overview on the topic as an appendix to the present paper.

The problem of the ground state energy was completely solved by Donsker and Varadhan in [DV83] at least for the (physically most relevant) limit of large \( \alpha \). Using their large deviation techniques, they find that both \( \psi(\alpha) := \lim_{t \to \infty} \frac{1}{t} \log z_t \) and \( g_0 := \lim_{\alpha \to \infty} \psi(\alpha)/\alpha^2 \) exist, and that \( g_0 \) can be calculated using the variational formula of Pekar [Pek49]. Lieb and Thomas [LT97] give a functional analytic proof of the same result that in addition yields explicit error estimates.

The problem of the effective mass turned out to be more difficult. Spohn [Spo87] observed that the mathematically rigorous connection between the effective mass and the quantity \( z_t \) should appear through a central limit theorem: for each \( t > 0 \), one can read (1.1) as the normalization of a probability measure \( \mathbb{P}_t \) on path space, namely the perturbation of Brownian motion by the exponentiated double integral; see formula (1.2) below. The task is then to prove the existence of a limiting probability measure \( \mathbb{P}_\infty \) as \( t \to \infty \), and a central limit theorem under diffusive rescaling of \( \mathbb{P}_\infty \). The emerging diffusion constant is the inverse of the effective mass. There are some details that need consideration, such as the sense in which the limit exists, given that the interaction is translation invariant. While the treatment in [Spo87] is still partly non-rigorous, recently Dybalski and Spohn [DS20] put the connection between a central limit theorem and the effective mass on solid mathematical ground.

This leaves the problem of actually showing the central limit theorem, and as a prerequisite the existence of an infinite volume measure. A main obstacle is the singularity in the integrand of the double integral that prevents e.g. uniform estimates in the paths. In this paper, we show for the first time that for all values of the coupling constant \( \alpha \), infinite volume measures related to (1.1) exist, and that a functional central limit theorem holds. We also give an explicit expression for the diffusion constant that is explicit enough to read off its strict positivity immediately.

Our results actually hold for more general models than the Fröhlich polaron. As discussed in the Appendix, various polaron models of quantum theory give rise to probability measures of the type

\[
\mathbb{P}_{\alpha,T}(dx) = \frac{1}{Z_{\alpha,2T}} \exp \left( \frac{\alpha}{2} \int_{-T}^{T} \int_{-T}^{T} w(|t-s|, x_t - x_s) \, ds \, dt \right) \mathcal{W}(dx),
\]  

(1.2)

where \( \alpha, T > 0 \), and where \( w \) depends on the specific model, \( Z_{\alpha,2T} \) is the normalizing constant, or partition function, and \( \mathcal{W} \) is the path measure of Brownian motion, although here we have to be a bit careful: the translation invariance of the interaction means that when we e.g. let \( \mathcal{W} \) be the distribution of Brownian motion started at time \(-T\) in the origin, then the distribution of \( x \mapsto x_0 \) under \( \mathbb{P}_{\alpha,T} \) can not be expected to converge. An elegant solution is to let \( \mathcal{W} \) be the measure of two-sided Brownian motion pinned to 0 at time 0, and to consider the family \( (\mathbb{P}_{\alpha,T})_{T \geq 0} \) on the \( \sigma \)-algebra \( \mathcal{A} \) generated by the increments \( X_{s,t}(x) = x_t - x_s \). This choice gets rid of the rather arbitrary pinning at \( t = 0 \), and the measures on the restricted \( \sigma \)-algebra will be shown to converge locally in the usual sense.

We call measures of the form (1.2) *Polaron path measures*. The precise conditions (A1)-(A3) that we impose on \( w \) will be stated at the beginning of Section 2. The most important
among them is the growth condition
\[ \limsup_{T \to \infty} \frac{Z_{\alpha,T}}{Z_{\alpha,2T}^2} < \infty \] (GC)
on the partition function. For suitable \( w \), we find a number of equivalent conditions to (GC), one of them being the existence and strict positivity of \( \lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha)T} \), where
\[ \psi(\alpha) = \lim_{T \to \infty} \frac{1}{T} \log Z_{\alpha,T} \]
is the free energy associated to (1.2). Under these conditions, we show the existence of an infinite volume limit \( \mathbb{P}_{\alpha,\infty} \) of the family \( (\mathbb{P}_{\alpha,T})_{T \geq 0} \). Under the additional condition that \( x \mapsto w(t,x) \) is quasiconcave for all \( t \geq 0 \) we show a functional central limit theorem for \( \mathbb{P}_{\alpha,\infty} \) under diffusive rescaling.

Central limit theorems for similar models have been proved before. [BS04] focuses on Polaron path measures that originate from quantum polaron models. The method rests on not integrating out the field, and instead working directly with a Markov process on an infinite dimensional state space; the central limit theorem is then obtained by studying the quantum field as seen from the moving particle, and by applying a suitable theory of Kipnis and Varadhan [KV86]. The method needs, among other assumptions, that \( w \) is bounded, and thus does not work for the Fröhlich polaron.

Gubinelli [Gub06] treats general Polaron path measures, and implements an approach that had already been proposed in [Spo87]: By cutting \([−T,T]\) into blocks of finite length and considering the space of continuous functions on each block, he obtains an one-dimensional system of infinite dimensional spins, where all spins interact with each other through the double integral, but in case of sufficiently fast decay of \( t \mapsto w(t,x) \), that interaction becomes weak for distant blocks. Dobrushins theory of one-dimensional spin systems [Dob68, Dob70] then guarantees existence and uniqueness of the infinite volume limit, and the central limit theorem is a consequence of sufficiently fast decay of correlations between distant blocks. The conditions on \( w \) are weaker than those from [BS04] in some aspects, but stronger in others. They also need uniformly bounded potentials and thus exclude the Fröhlich polaron. Also, the coupling constant \( \alpha \) needs to be sufficiently small for Dobrushins method to work.

The method of Mukherjee [Muk20] also relies on cutting the space of functions into blocks. He uses it to construct a discrete time Markov chain (with infinite dimensional state space) that describes the behaviour of the path in a given block given its behaviour in the previous block, and shows that this Markov chain fulfills the conditions for a generalized Perron-Frobenius argument. The central limit theorem is then derived as a consequence of the CLT for additive functionals of stationary Markov chains. The method works most cleanly when \( t \mapsto w(t,x) \) has compact support, in which case it can also handle some specific unbounded \( w \). It also works when \( \sup_{x} |w(t,x)| \leq Ct^{-2-\delta} \) for constants \( C, \delta > 0 \). Both conditions exclude the Fröhlich polaron.

Important progress was achieved recently by Mukherjee and Varadhan in the work [MV19] mentioned already above. The key idea is not technically difficult, but ingenious: for the case of the Fröhlich polaron, one expands the exponential in (1.2) into a Taylor series, exchanges the order of integration, and interprets (1.2) as a mixture of Gaussian measures, where the mixing measure is a point process on the space of (possibly overlapping) finite subintervals of \( \mathbb{R} \). The results of the paper are the existence of \( \mathbb{P}_{\alpha,\infty} \) and a central limit theorem, valid for either large enough or small enough \( \alpha \). The paper is strictly devoted to the Fröhlich polaron, and many calculations are specific to the precise form of
\[ w(t,x) = e^{-t}/|x|, \] in particular for writing it as a mixture of Gaussian measures. Also, the central limit theorem is proved as a 'diagonal limit', where the diffusive scaling is applied at the same time as \( T \) is sent to infinity.

We find that the method of [MV19] can be generalized significantly, and thereby becomes both conceptually simpler and more powerful. The central object of our theory is a Gibbs measure on marked partitions of the real line, and it is the infinite volume limit of this measure that needs to be understood in order to study limits and properties of the measure (1.2). Using the mixing properties of the limit measure, we prove a full functional central limit theorem, with the diffusive scaling applied after taking the infinite volume limit. Its proof relies on mixing properties of the limiting Gibbs measure on marked partitions and does not need \( P_{\alpha,T} \) to be a mixture of Gaussian measures. Another important result of our paper concerns the conditions under which the central limit theorem holds. [MV19] give a sufficient condition (see Condition (G) below) for the validity of their results, which is however not easy to check in general. We show that condition (G) is equivalent to several natural conditions, one of them being (GC). In particular, this allows us to treat the Fröhlich polaron for all coupling constants with very little additional effort. We also find that the critical parameter \( \lambda \) in (G) is equal to \( \psi(\alpha) - \alpha \), which leads to some intriguing relations between the Gibbs measure on marked partitions, the ground state energy \( E_\alpha(0) \) of the quantum polaron, and the overlap \( \langle \Omega, \Psi_\alpha \rangle \) between the Fock vacuum \( \Omega \) and the ground state \( \Psi_\alpha \) of the polaron. We do not explore these relations in depth in the present paper, but we present a few interesting calculations in Section 7 below.

The paper is organised as follows: in Section 2, we present our results and compare them to the results available in the literature. In Section 3 we give an outline of the proof and discuss where we follow the ideas of [MV19] and where we go beyond them. Section 4 is the main technical part of the paper: here we develop a general theory of Gibbs measures on (marked) partitions of the real line and discuss conditions for existence of infinite volume measures. Section 5 contains the proof of the central limit theorem. In Section 6 we prove a sufficient conditions for the validity of (GC), and in Section 7 we discuss how various quantities for polaron models are connected to the measures from Section 4.

2. Results

Consider the Polaron path measure (1.2) defined on \((C(\mathbb{R}, \mathbb{R}^d), \mathcal{A})\), where \( \mathcal{A} \) is the \( \sigma \)-algebra generated by the increment maps

\[ X_{s,t} : C(\mathbb{R}, \mathbb{R}^d) \to \mathbb{R}^d, \quad x \mapsto X_{s,t}(x) := x_{s,t} := x_t - x_s, \]
for \( s, t \in \mathbb{R} \), and where \( W \) is the distribution of Brownian increments (i.e. the restriction of the distribution of a two-sided Brownian motion to \( \mathcal{A} \)). For the function \( w \) and the parameter \( \alpha \) we make the following assumptions:

(A1): \( w \) is measurable, positive (allowing the value +\( \infty \)), and fulfills \( w(t,-x) = w(t,x) \) for all \( t > 0 \) and \( x \in \mathbb{R}^d \).

(A2): \( Z_{\alpha,T} < \infty \) for all \( T \geq 0 \).

(A3): \( \limsup_{T \to \infty} \frac{Z_{\alpha,2T}}{Z_{\alpha,T}} < \infty \), i.e. (GC) holds.

Remark 2.1. The positivity condition in (A1) can be relaxed: let \( f : [0,\infty) \to [0,\infty) \) be a measurable function with \( \int_0^\infty (1+t)f(t)\,dt < \infty \). The measures \( P_{\alpha,T} \) are invariant under replacing \( w \) with \( (t,x) \mapsto w(t,x) + f(t) \) in the double integral, and a short calculation
shows that (GC) holds after the transformation if and only if it held before the transformation. Therefore, it is enough to assume that there exists \( f \) with the above integrability condition such that \( w + f \) is positive. In other words, as long as \( w \) is bounded below and its negative part decays sufficiently fast at infinity uniformly in \( x \), our results hold. The well-definedness condition (A2) is trivial for bounded \( w \), but becomes an issue for unbounded \( w \) such as in the Fröhlich polaron. While Proposition 2.4 contains some sufficient conditions under which (A2) holds, we refer to the work of Bley and Thomas [BT17] for much more general, quantitative estimates on the partition functions of Feynman-Kac type perturbations of Brownian motion, as well as an overview over the literature on this topic.

On \((C(\mathbb{R}, \mathbb{R}^d), \mathcal{A})\), we say that a family \((\mathcal{P}_t)_{t>0}\) of probability measures converges to a probability measure \(\mathcal{P}\) locally in total variation if for all \(a < b\) the restriction of \(\mathcal{P}_t\) to

\[
\mathcal{A}_a^b := \sigma(X_{s,t} : s, t \in [a, b])
\]

converges in total variation to the restriction of \(\mathcal{P}\) to \(\mathcal{A}_a^b\) as \(t \to \infty\).

**Theorem 2.2** (Existence of infinite volume measure). Assume (A1)–(A3). Then there exists a measure \(\mathbb{P}_{a,\infty}\) on \((C(\mathbb{R}, \mathbb{R}^d), \mathcal{A})\) such that \(\mathbb{P}_{a,T} \to \mathbb{P}_{a,\infty}\) locally in total variation as \(T \to \infty\).

There is an explicit representation for \(\mathbb{P}_{a,\infty}\), which however requires us to introduce and explain several further notions. We refer to Theorem 3.3 below for details.

For stating the central limit theorem, let

\[
X_t := X_{0,t}, \quad \text{and} \quad X^n_t := \frac{1}{\sqrt{n}}X_{nt}
\]

for \(n \in \mathbb{N}\) and \(t \in \mathbb{R}\). Recall that a function \(u : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}\) is called quasiconcave if its superlevel sets \(u^{-1}([a, \infty])\), \(a \in \mathbb{R} \cup \{\infty\}\) are convex sets. An example for a quasiconcave function is \(x \mapsto w(t, x)\) with \(w\) as in the Fröhlich polaron.

**Theorem 2.3** (Functional central limit theorem). Assume (A1)–(A3), and assume that \(w(t, \cdot)\) is quasiconcave for all \(t > 0\). Then there exists a positive definite matrix \(\Sigma \in \mathbb{R}^{d \times d}\) with \(\Sigma \leq I_d\) (in the sense of quadratic forms) such that the distribution of \(X^n\) under \(\mathbb{P}_{a,\infty}\) converges weakly to the distribution of \(\sqrt{\Sigma}X\) under \(\mathcal{W}\) as \(n \to \infty\).

Again, there is an explicit representation for the covariance matrix \(\Sigma\) that we do not have the notation and notions to state yet. We refer to Theorem 3.4 below.

We now discuss sufficient conditions for (GC). Let \(H : D(H) \subseteq \mathcal{H} \to \mathcal{H}\) be a self-adjoint, lower bounded operator in some Hilbert space \(\mathcal{H}\) with inner product \(\langle \cdot, \cdot \rangle\) and let \(\Phi \in \mathcal{H}\). We say \((Z_{a,T})_{T \geq 0}\) is of spectral type with Hamiltonian \(H\) and state \(\Phi\) if

\[
Z_{a,T} = \langle \Phi, e^{-TH} \Phi \rangle
\]

for all \(T \geq 0\). Clearly, this implies the validity of condition (A2). If the bottom of the spectrum of \(H\) is an eigenvalue, any normalized eigenvector corresponding to that eigenvalue is called a ground state of \(H\). In this case, we say that \(H\) has a ground state. We denote by \(\nu_{H,\Phi}\) the spectral measure of \(H\) with respect to \(\Phi\), i.e. \(\nu_{H,\Phi}(A) = \langle \Phi, 1_A(H)\Phi \rangle\) for all \(A \in \mathcal{B}(\mathbb{R})\).

**Proposition 2.4.**

a) Assume that there exists a measurable function \(f : [0, \infty) \to [0, \infty)\) satisfying \(\int_0^{\infty} (1 + t) f(t) \, dt < \infty\) such that \(|w(t, x)| \leq f(t)\) for all \(t \geq 0\) and \(x \in \mathbb{R}^d\). Then assumption (A2)
and (A3) are fulfilled.

b) Assume that (A1) holds and that \((Z_{\alpha,T})_T \geq 0\) is of spectral type with Hamiltonian \(H\) and state \(\Phi\) where \(H : D(H) \subseteq \mathcal{H} \to \mathcal{H}\) is a self-adjoint, lower bounded operator and \(\Phi \in \mathcal{H}\). Then

\[
\psi(\alpha) = -\inf \text{supp}(\nu_{H,\Phi})
\]

and condition (A3) is satisfied if and only if \(-\psi(\alpha)\) is an eigenvalue of \(H\) and \(\Phi\) is non-orthogonal to the respective eigenspace. In particular, if \(H\) has a ground state \(\Psi\) with \(\langle \Phi, \Psi \rangle \neq 0\) then (A3) is satisfied and \(-\psi(\alpha)\) is the ground state energy.

c) Assume that \(w(t, \cdot) = \tilde{w}(t, \cdot)\) is rotationally symmetric and positive and that \(\tilde{w}(t, \cdot)\) is decreasing for all \(t \geq 0\). Let \(\tilde{w}_\beta(r) := \sup_{t \geq 0} e^{\beta t} \tilde{w}(t, r)\) for \(\beta, r \geq 0\). Assume there exist \(\beta > 0\) and \(p > 1\) such that

\[
\begin{align*}
\int_0^1 \tilde{w}_\beta(r)^p dr < \infty & \quad \text{in case } d = 1 \\
\int_0^1 r |\log(r)| \cdot \tilde{w}_\beta(r)^p dr < \infty & \quad \text{in case } d = 2 \\
\int_0^1 r \cdot \tilde{w}_\beta(r)^p dr < \infty & \quad \text{in case } d \geq 3.
\end{align*}
\]

Then Condition (A2) is fulfilled for all \(\alpha \geq 0\), \(\psi(\alpha)\) exists and is finite for all \(\alpha \geq 0\), and there exists an \(\alpha_0 > 0\) such Condition (A3) is fulfilled for all \(\alpha \leq \alpha_0\).

**Proof.** For the proof of a), notice that for all \(T > 0\)

\[
\int_{-T}^T \int_{-T}^T |w(t-s, X_{s,t})| ds dt \leq \int_{-T}^T \int_{-T}^T f(|t-s|) ds dt \leq 4T \int_0^\infty f(\tau) d\tau < \infty,
\]

and thus (A2) holds. The estimate

\[
\int_0^T \int_0^T |w(t-s, X_{s,t})| dt ds \leq \int_0^\infty \int_0^\infty f(\tau) d\tau ds = \int_0^\infty \tau f(\tau) d\tau < \infty
\]

leads to

\[
Z_{\alpha,2T} \leq Z_{\alpha,T}^2 \exp \left( \alpha \int_0^\infty \tau f(\tau) d\tau \right)
\]

for all \(\alpha, T > 0\). This shows (A3).

For b), let \(E := \inf \text{supp}(\nu_{H,\Phi})\). Since

\[
Z_{\alpha,T} = e^{-TE} \langle \Phi, e^{-T(H-E)\Phi} \rangle = e^{-TE} \int_{[E,\infty)} e^{-T(x-E)} \nu_{H,\Phi}(dx)
\]

holds for all \(T \geq 0\), we have

\[
\psi(\alpha) = \lim_{T \to \infty} \frac{1}{T} \log Z_{\alpha,T} = -E + \lim_{T \to \infty} \frac{1}{T} \log \int_{[E,\infty)} e^{-T(x-E)} \nu_{H,\Phi}(dx) = -E,
\]

the last equality holding because for each \(\varepsilon > 0\), we have

\[
1 \geq \int_{[E,\infty)} e^{-T(x-E)} \nu_{H,\Phi}(dx) \geq e^{-T\varepsilon} \int_{[E,E+\varepsilon)} \nu_{H,\Phi}(dx),
\]

with the integral on the right hand side being strictly positive. Thus

\[
\lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha)T} = \lim_{T \to \infty} \int_{[E,\infty)} e^{-T(x-E)} \nu_{H,\Phi}(dx) = \nu_{H,\Phi}(\{E\})
\]

by dominated convergence. Hence, the limit on the left hand side is positive if and only if \(E\) is an eigenvalue of \(H\) and \(\Phi\) is non-orthogonal to the corresponding eigenspace. If the limit is positive, \(\lim_{T \to \infty} Z_{\alpha,2T}/Z_{\alpha,T}^2 = (\lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha)T})^{-1}\) and (GC) is satisfied.
We will see in Theorem 3.2 that the other direction holds as well: If (GC) is satisfied then
\[ \lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha)T} \]
eeds to exist and is positive.
The proof of part c) is more involved and is given in Section 6 below. □

**Corollary 2.5.** Theorems 2.2 and 2.3 hold for the Fröhlich polaron, for all values of \( \alpha \geq 0 \).

**Proof.** For the Fröhlich polaron, \((Z_{\alpha,T})_{T \geq 0}\) is of spectral type where \( H = H_{\alpha}(0) \) is the
Fröhlich Hamiltonian at total momentum zero, and \( \Phi = \Omega \) is the Fock vacuum; for instance, set \( k = t = 0 \) in formulae (2.11) and (2.14) of [DS20] to see this. It is well known that
for all coupling constants \( \alpha > 0 \), \( H(0) \) has a spectral gap and an unique (up to a phase)
ground state \( \Psi_\alpha \), and that \( \langle \Psi_\alpha, \Omega \rangle \neq 0 \); again, we refer to [DS20]. We can therefore apply
Proposition 2.4 b) and obtain the result. □

The rather direct connection of Condition (GC) to quantum systems given in 2.4 b) has
some interesting implications: it means that independent criteria for the validity of (GC),
such as those from Proposition 2.4 c) or from Theorem 3.2 below, can be used to decide
about the existence of ground states in quantum systems. Used in the other direction, this
connection indicates that the sufficient condition from Proposition 2.4 a) is very close to
being also necessary for the case of bounded \( w \): for the massless Nelson model without
infrared cutoff it is known that \( w \) is bounded with \( w(t,0) \sim |t|^{-2} \), and that the Hamiltonian
has no ground state in Fock space; see Chapter 6.7 of [LHB11]. Therefore (GC) cannot
hold in this case.

Let us end this section by giving a more detailed comparison of our results with those
previously available. We will take Remark 2.1 into account, i.e. that the assumption of
positivity in (A1) can be weakened. Then Proposition 2.4 a) implies that (GC) holds for
all cases considered in [BS04] as well as those included in Assumption 2.1 of [Muk20]. For
the alternative Assumption 2.2 of [Muk20], Proposition 2.4 c) guarantees that (GC) holds
for small enough \( \alpha \) in all cases except the singular potential in one dimension. Also, note
that Assumption 2.2 of [Muk20] does not carry a restriction on the value of \( \alpha \).

For the central limit theorem, we require quasiconcavity of \( w \), which is not needed in
[BS04] and in the cases from [Muk20] where \( w \) is bounded. Quasiconcavity is used for
a short proof of both tightness in path space and finite variance, via Gaussian correla-
tion inequalities. We believe that there should be other methods of achieving these goals
whenever (GC) holds, but at the moment we do not have them.

Gubinelli [Gub06] also requires \( w \) to be bounded, but proves the existence of an infinite
volume limit (but not the central limit theorem) also in some cases where the criterion
of Proposition 2.4 a) does not hold. As discussed above in the context of the massless
Nelson model, we do not expect (GC) to hold in general in such cases. In this context, the
work [OS99] is also of interest, where in a non-translation invariant version of the
model with slow decay of \( t \mapsto w(x,t) \), the existence of at least two different infinite volume
limit measures is proved, depending on boundary conditions. Since we believe that (A1)
- (A3) should already be sufficient for a central limit theorem, and since such a central
limit theorem is not expected to hold in situations where correlations are so strong that
multiple Gibbs measures exist, we conjecture that in generic cases of bounded potentials
violating the assumptions of Proposition 2.4 a), (GC) will not hold.

The results in [MV19] were obtained using a method that is related in some aspects
to the proof we will give for Proposition 2.4 c), although other aspects differ significantly.
An important common feature is that in both methods, there is a restriction on the range of \( \alpha \) that can be treated: Proposition 2.4 c) only works for small \( \alpha \), while the results in
[MV19] are stated for large enough or small enough \( \alpha \). We believe that there is a gap
in the proof of Theorem 4.1 of [MV19]. Namely, it is unclear to us where the factor of \( \alpha + \lambda \) comes from in formula (4.18) of [MV19]; such a factor corresponds to the ‘dormant’ period at the beginning of a cluster, but if (4.18) were true, a dormant period would occur for each point in the cluster, which is not the case. Without the presence of that factor, however, the choice (4.19) for \( \lambda(\alpha) \) is no longer possible when \( \alpha \) is large, and this creates a gap in the proof of Theorem 4.8 for the case of large \( \alpha \). On the other hand, our Corollary 2.5 shows that the results of [MV19] are correct as stated and extends them to all \( \alpha \geq 0 \).

In conclusion, the extended method of Mukherjee and Varadhan in conjunction with (GC) is a powerful tool to study infinite volume limits and functional central limit theorems for Polaron path measures including the Fröhlich polaron for all coupling strengths. One restriction of the method is that \( w \) has to be bounded below; this is necessary for positivity properties of the measure on partitions. Consequently, e.g. the ‘anti-polaron’ \( w(t, x) = -e^{-|t|/|x|}, \) cannot be treated. Another restriction is the requirement of translation invariance. This excludes interesting cases, e.g. measures corresponding to particles interacting both with a quantized field and an external potential; see [Bet03, BHL+02, OS99], for instance.

3. Outline of the proof

Let \( \Delta = \{(s, t) \in \mathbb{R}^2 : s < t \} \). We denote by \((N(\Delta), N(\Delta))\) and \((N_f(\Delta), N_f(\Delta))\) the state spaces of point processes and finite point processes on \( \Delta \) respectively. In the following, we always assume that Assumptions (A1) and (A2) hold. We start by modifying the approach taken by Mukherjee and Varadhan for the Polaron [MV19]: By using the series expansion of the exponential function and exchanging the order of integration, we write the measure \( P_{\alpha, T} \) as a mixture of probability measures \( P_\xi, \xi \in N_f(\Delta) \) which have a product structure under non-intersecting “clusters”. Contrary to [MV19] (where the mixing measure is the distribution of a point process on \( \Delta \times (0, \infty) \)), our measures \( P_\xi \) are in general not Gaussian (not even for the Fröhlich polaron). We discuss the connection between both representations in Remark 4.14. It turns out to be useful to absorb a time damping factor into our Poisson point process. For this, we choose a measurable function \( v : [0, \infty) \times \mathbb{R}^d \to (0, \infty] \) and a probability density \( g : [0, \infty) \to (0, \infty) \) with finite first moment such that

\[
w(t, x) = g(t)v(t, x)
\]

for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \) (this representation is clearly not unique). Let \( \Gamma_{\alpha, T} \) be the distribution of a Poisson point process on \( \Delta \) with intensity measure

\[
\mu_{\alpha, T}(dsdt) := \alpha \cdot g(t - s)1_{\{-T<s<t<T\}}dsdt
\]

and \( c_{\alpha, T} := \mu_{\alpha, T}(\Delta) \). If \( \tau_1 \) is a random variable with density \( g \), then

\[
c_{\alpha, T} := \mu_{\alpha, T}(\Delta) = 2\alpha T - \alpha \int_0^{2T} \mathbb{P}(\tau_1 > s) \, ds
\]
which will be relevant at a later point). We then have for \( A \in \mathcal{A} \)
\[
\mathbb{P}_{\alpha,T}(A) = \frac{1}{Z_{\alpha,2T}} \int_A \mathcal{W}(dx) \exp \left( \alpha \int_{-T}^T ds \int_s^T dt \, g(t-s)v(t-s,x_{s,t}) \right)
\]
\[
= \frac{1}{Z_{\alpha,2T}} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Delta^n} \mu^{\otimes n}_{\alpha,T}(ds_1 dt_1, \ldots, ds_n dt_n) \left[ \int_A \mathcal{W}(dx) \prod_{i=1}^n v(t_i - s_i, x_{s_i,t_i}) \right]
\]
\[
= \frac{e^{\alpha \cdot T}}{Z_{\alpha,2T}} \int_{\mathbb{N}_f(\Delta)} \Gamma_{\alpha,T}(d\xi) \int_A \mathcal{W}(dx) \prod_{(s,t) \in \text{supp}(\xi)} v(t-s, x_{s,t}). \tag{3.3}
\]

Note that already here, we used the symmetry and nonnegativity conditions from Condition (A1). For \( \xi = \sum_{i=1}^n \delta_{(s_i,t_i)} \in \mathbb{N}_f(\Delta) \), we define
\[
F(\xi) := \mathbb{E}_\mathcal{W} \left[ \prod_{i=1}^n v(t_i - s_i, X_{s_i,t_i}) \right]
\]
and the measure \( \mathbb{P}_\xi \) on \( (C(\mathbb{R}, \mathbb{R}^d), \mathcal{A}) \) by
\[
\mathbb{P}_\xi(dx) := \frac{1}{F(\xi)} \prod_{i=1}^n v(t_i - s_i, x_{s_i,t_i}) \mathcal{W}(dx)
\]
(where \( 1/\infty := 0 \)). Additionally, we define the measure \( \Gamma_{\alpha,T} \) on \( (\mathbb{N}_f(\Delta), \mathcal{N}_f(\Delta)) \) by
\[
\Gamma_{\alpha,T}(d\xi) := \frac{e^{\alpha \cdot T}}{Z_{\alpha,2T}} F(\xi) \Gamma_{\alpha,T}(d\xi).
\]

We then have \( F(\xi) < \infty \) for \( \Gamma_{\alpha,T} \) almost all \( \xi \in \mathbb{N}_f(\Delta) \) (choose \( A = C(\mathbb{R}, \mathbb{R}^d) \) in (3.3)) and
\[
\mathbb{P}_{\alpha,T}(\cdot) = \int_{\mathbb{N}_f(\Delta)} \Gamma_{\alpha,T}(d\xi) \mathbb{P}_\xi(\cdot). \tag{3.4}
\]

In particular \( \mathbb{P}_\xi \) is a probability measure for \( \Gamma_{\alpha,T} \) almost all \( \xi \in \mathbb{N}_f(\Delta) \) and \( \Gamma_{\alpha,T} \) is a probability measure. We have now interchanged the task of proving convergence of the path measures \( \mathbb{P}_{\alpha,T} \) with proving convergence of the distributions of suitable point processes. Notice that
\[
Z_{\alpha,2T} := \int_{\mathbb{N}_f(\Delta)} \Gamma_{\alpha,T}(d\xi) F(\xi) = Z_{\alpha,2T} e^{-\alpha \cdot T} \tag{3.5}
\]
acts as a normalization constant in the definition of the reweighted measure \( \Gamma_{\alpha,T} \). One can view \( \xi = \sum_{i=1}^n \delta_{(s_i,t_i)} \in \mathbb{N}_f(\Delta) \) as a collection of intervals \([s_i, t_i], 1 \leq i \leq n \). We call \( \xi \) a cluster, if these intervals are overlapping in the sense that \( \bigcup_{i=1}^n [s_i, t_i] \) is an interval. For an interval \( I \subseteq \mathbb{R} \) we say that \( \xi \) falls into \( I \) if \([s_i, t_i] \subseteq I \) for all \( 1 \leq i \leq n \). Due to independence of increments under \( \mathcal{W} \), the measures \( \mathbb{P}_\xi \) have a product structure along non-intersecting clusters: Let \( I_1, \ldots, I_n \) be intervals such that for \( i \neq j \) the intersection \( I_i \cap I_j \) contains at most one point. Let \( \xi_1, \ldots, \xi_n \in \mathbb{N}_f(\Delta) \) be such that \( \xi_k \) falls into \( I_k \) for all \( 1 \leq k \leq n \) and let \( \xi = \sum_{k=1}^n \xi_k \). Then
\[
F(\xi) = \prod_{k=1}^n F(\xi_k).
\]
If we additionally assume that $F(\xi_k) < \infty$ for all $k \in \{1, \ldots, n\}$ then the processes of increments $(X_{s,t})_{s,t \in I_1}, \ldots, (X_{s,t})_{s,t \in I_n}$ are independent under $P_\xi$ and for all $k \in \{1, \ldots, n\}$

$$P_\xi \circ (X_{s,t})_{s,t \in I_k}^{-1} = P_{\xi_k} \circ (X_{s,t})_{s,t \in I_k}^{-1}.$$ 

In particular, for $a < b$, $\xi \in N_f(\Delta)$ and $A \in A_0$, the probability $P_\xi(A)$ only depends on all clusters of $\xi$ that intersect $[a, b]$. We denote by $R_a^b$ the restriction to all clusters that intersect $[a, b]$, that is for $\xi = \sum_{i \in I} \delta_{(s_i, t_i)} \in N(\Delta)$

$$R_a^b \xi := \sum_{i \in I_a^b(\xi)} \delta_{(s_i, t_i)}$$

with $J_a^b(\xi) := \max \{ J \subseteq I : \bigcup \{ [s_i, t_i] \cup [a, b] \text{ is an interval} \} \}$. Then for all $A \in A_0$

$$P_{\alpha,T}(A) = \int_{N_f(\Delta)} \hat{\Gamma}_{\alpha,T}(d\xi) P_{R_a^b \xi}(A). \quad (3.6)$$

In Section 4, we prove Theorem 2.2 i.e. the existence of an infinite volume measure by showing that for all $a < b$ the measures $\hat{\Gamma}_{\alpha,T} \circ (R_a^b)^{-1}$ converge in total variation as $T \to \infty$ provided that (GC) holds. We show:

**Proposition 3.1.** Assume that Conditions (A1)-(A3) hold. Then there exists a stationary (with respect to translations along the diagonal) measure $\hat{\Gamma}_{\alpha,\text{st}}$ on $(N(\Delta), N(\Delta))$ such that for all $a < b$ we have $\hat{\Gamma}_{\alpha,T} \circ (R_a^b)^{-1} \to \hat{\Gamma}_{\alpha,\text{st}} \circ (R_a^b)^{-1}$ in total variation as $T \to \infty$.

We give a short summary of the argument used. Consider a Poisson point process $\eta = \sum_{i=1}^\infty \delta_{(s_i, t_i)}$ with $s_1 < s_2 < \ldots$ on $\Delta$ with intensity measure $\alpha g(t-s)1_{\{T-T<s<T\}} \, ds \, dt$. Here $s_i$ and $t_i$ can be interpreted at the arrival and departure of the $i$-th customer of a $M/G/\infty$-queue (Poisson arrivals, general service time distribution, infinitely many servers) started empty at time $-T$ with arrival intensity $\alpha$ and service time distribution with density $g$. Conditionally on the event that no customer is present at $T$, the process of all customers arriving between $-T$ and $T$ has distribution $\Gamma_{\alpha,T}$. We can decompose the $M/G/\infty$-queue into successive busy cycles (a dormant period, in which no customers are present, followed by an active period in which at least one customer is present). We denote by $\xi_1$ the first cluster of the queue, i.e. the process of all customers arriving during the first busy cycle, shifted in time such that the first arrival is at time zero. We denote by $T_1$ the length of the first busy cycle, i.e. the sum of the first dormant and the first active period. As identified by Mukherjee and Varadhan in [MV19] for the Polaron measure, the condition

There exists a $\lambda \in \mathbb{R}$ such that $E_\alpha \left[ e^{-\lambda T_1} F(\xi_1) \right] = 1$ and $E_\alpha \left[ T_1 e^{-\lambda T_1} F(\xi_1) \right] < \infty \quad (G)$

is sufficient to conclude Proposition 3.1. In Section 4 we show this in much greater generality, not needing any specific properties of the distribution $\Xi_\alpha$. The argument is as follows: The condition $E_\alpha \left[ e^{-\lambda T_1} F(\xi_1) \right] = 1$ guarantees that $\hat{\Xi}_\alpha(d\xi) := \frac{1}{\alpha+\lambda} e^{-\lambda a(\xi)} F(\xi) \Xi_\alpha(d\xi)$ defines a probability measure (where $a(\xi)$ denotes the length of the active period of $\xi$). As the sum of all active and dormant periods intersecting $[-T, T]$ is $2T$ (up to a correction term that is irrelevant due to memorylessness) we can multiply $F$ with factors $e^{-\lambda a_i}$ and $e^{-\lambda d_j}$ for each active period $a_i$ and each dormant period $d_j$ intersecting $[-T, T]$ without changing the Gibbs measure $\hat{\Gamma}_{\alpha,T}$. As $F$ is multiplicative in the clusters, the measure $\hat{\Gamma}_{\alpha,T}$ can hence be obtained by simply reweighting the distribution of exponentially distributed dormant periods and $\Xi_\alpha$ distributed clusters: Starting in $-T$ we alternate independently drawn Exp$(\alpha + \lambda)$ distributed dormant periods with $\hat{\Xi}_\alpha$ distributed clusters. Conditionally on the event that no customer is present at $T$, the process of customers arriving between
−T and T has distribution $\hat{\Gamma}_{\alpha,T}$. The second condition $\mathbb{E}_\alpha \left[ T_1 e^{-\lambda T_1} F(\xi_1) \right] < \infty$ yields the finiteness of the expected value of the cluster length for the reweighted cluster distribution. This allows us to apply renewal theory to deduce the existence of a stationary version $\hat{\Gamma}_{\alpha,st}$ of the process obtained by alternating the reweighted distributions of dormant periods and active clusters. Due to the memorylessness property of the exponential distribution, the measure $\hat{\Gamma}_{\alpha,T}$ can alternatively be seen as the process of all customers under $\hat{\Gamma}_{\alpha,st}$ that arrive between $−T$ and $T$ conditionally on the boundary condition that the system is dormant at $−T$ and $T$. Unsurprisingly, locally the effect of the boundary condition vanishes as $T \to \infty$. Condition (G) is rather difficult to verify directly, but we find that it is equivalent to several natural conditions, including (GC). Remember that we denote by $\psi(\alpha) := \lim_{T \to \infty} \log(\mathbb{Z}_{\alpha,T}) / T \in \mathbb{R} \cup \{\infty\}$ the free energy (the limit exists by the version of Feketes lemma for measurable superadditive functions). We set $e^{-\infty} := 0$. In Section 4 we show:

**Theorem 3.2.** Assume (A1) and (A2) hold. Then the following conditions are equivalent:

1. There exists a $\lambda \in \mathbb{R}$ such that $\mathbb{E}_\alpha \left[ e^{-\lambda T_1} F(\xi_1) \right] = 1$ and $\mathbb{E}_\alpha \left[ T_1 e^{-\lambda T_1} F(\xi_1) \right] < \infty$
2. $\lim \sup_{T \to \infty} Z_{\alpha,2T}/Z_{\alpha,T}^2 < \infty$
3. $\lim \inf_{T \to \infty} \hat{\Gamma}_{\alpha,T}(\text{the system is dormant at } 0) > 0$
4. $\lim \inf_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha) T} > 0$.

If (1)-(4) hold then $\lambda = \psi(\alpha) - \alpha$ is the unique real number satisfying (G) and

$$\lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha) T} = \frac{1}{\psi(\alpha) \mathbb{E}_\alpha \left[ T_1 e^{-((\psi(\alpha) - \alpha) T_1) F(\xi_1)} \right]}.$$

We prove Theorem 3.2 through another application of renewal theory. More specifically, we use that the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(T) := e^{-(\psi(\alpha) - \alpha) T} \mathbb{Z}_{\alpha,T} \cdot \mathbb{P}_\alpha(\text{the } M/G/\infty\text{-queue is dormant at } T)$$

satisfies the renewal equation

$$f(T) = \mathbb{E} \left[ \mathbb{I}_{\{T_1 \leq T\}} e^{-(\psi(\alpha) - \alpha) T_1} F(\xi_1) f(T - T_1) \right] + e^{-\psi(\alpha) T} \quad \text{for all } T \geq 0.$$

Combining the previous considerations, we obtain

**Theorem 3.3** (Existence of infinite volume measure). Assume that (A1)-(A3) hold. Then there exists a measure $\mathbb{P}_{\alpha,\infty}$ on $(C(\mathbb{R}, \mathbb{R}^3), \mathcal{A})$ such $\mathbb{P}_{\alpha,T} \to \mathbb{P}_{\alpha,\infty}$ locally in total variation as $T \to \infty$. For any $a < b$ the restriction of $\mathbb{P}_{\alpha,\infty}$ to $\mathcal{A}_a^b$ is given by

$$\mathbb{P}_{\alpha,\infty}(A) = \int_{\mathbb{N}(\triangle)} \hat{\Gamma}_{\alpha,\text{st}}(d\xi) \mathbb{P}_{\alpha,\infty}(A)$$

for all $A \in \mathcal{A}_a^b$ where $\hat{\Gamma}_{\alpha,\text{st}}$ is the stationary distribution of the process obtained by alternating $\text{Exp}(\psi(\alpha))$ distributed dormant periods and the distribution

$$\hat{\Xi}_\alpha(d\xi) = \frac{\alpha}{\psi(\alpha)} e^{-(\psi(\alpha) - \alpha) a(\xi)} F(\xi) \Xi_\alpha(d\xi)$$

of active clusters.

For the Fröhlich Polaron path measure, Mukherjee and Varadhan [MV19] show (for $\alpha$ such that (G) holds) an ordinary central limit theorem in the “diagonal limit”, using that their counterparts $\mathbb{P}_{\xi,a}$ of our measures $\mathbb{P}_\alpha$ are Gaussian. We use a different approach that allows us to show Theorem 2.3, i.e. a full functional central limit theorem, for more general
potentials \( w \). Roughly speaking, we use the following rather natural arguments (assuming that \((GC)\) is satisfied):

- Let \( \tilde{\eta}_s \sim \tilde{\Gamma}_{\alpha,s} \) be a stationary version of the process obtained by alternating the reweighted dormant periods and clusters. Let \( s_1 < t_1 < \ldots < s_k < t_k \). For large \( n \) the processes \( P^{\eta_{s_1}}_{\eta_{t_1}} \tilde{\eta}_s \), \( \ldots \), \( P^{\eta_{t_k}}_{\eta_{s_k}} \tilde{\eta}_s \) are approximately independent due to the renewal structure. By the product structure of the measures \( \mathbf{P}_\xi \) along non-intersecting clusters, the increments \( X_{ns_1,nt_1}, \ldots, X_{ns_k,nt_k} \) are approximately independent under \( \mathbb{P}_{\alpha,\infty} \) for large \( n \).

- For \( a < b \) and large \( n \), approximately \( n(b - a)/\hat{E}_\alpha[\hat{T}_1] \) renewal points fall into \([na,nb]\). By the product structure, the increment \( \frac{1}{\sqrt{n}}X_{an,bn} \) is under \( \mathbb{P}_{\alpha,\infty} \) approximately the sum of \( n(b - a)/\hat{E}_\alpha[\hat{T}_1] \) independent increments.

- To prove an ordinary central limit theorem, we still need that the variance of an increment along a single renewal period is finite. For the functional central limit theorem, we additionally need tightness of the family of probability measures \( \{\mathbb{P}_{\alpha,\infty} \circ (X^n)^{-1} : n \in \mathbb{N}\} \). We assume that \( w(t,\cdot) \) is quasiconcave for all \( t > 0 \) and apply the Gaussian correlation inequality in order to show these two prerequisites.

While the assumption of quasiconcavity of \( w(t,\cdot) \) is satisfied for the Fröhlich polaron and several other cases, it is conceivable that at least the finite variance along a single renewal period (and thus the ordinary central limit theorem) can be obtained under weaker assumptions on \( w \). We do not pursue this any further here and settle for:

\textbf{Theorem 3.4 (Functional Central limit theorem).} \textit{Assume that (A1)-(A3) hold, and that \( w(t,\cdot) \) is quasiconcave for all \( t > 0 \). Then the distribution of \( X^n \) under \( \mathbb{P}_{\alpha,\infty} \) converges weakly to the distribution of \( \sqrt{\Sigma}X \) under \( \mathcal{W} \) where

\[
\Sigma := \frac{\hat{E}_\alpha[\Sigma(\hat{d}_1,\hat{T}_1)\]]}{\hat{E}_\alpha[\hat{T}_1]} \quad \text{with} \quad \Sigma(\hat{d}_1,\hat{T}_1) := \hat{d}_1\cdot I_d + \hat{E}_\alpha[X_0,\hat{\alpha}_1,\cdot, X_{0,\hat{\alpha}_1}^T],
\]

(3.7)

and \( (\hat{d}_1,\hat{T}_1) \sim \text{Exp}(\psi(\alpha)) \otimes \widehat{\Xi}_\alpha, \hat{\alpha}_1 \) is the length of the cluster \( \hat{\xi}_1 \) and \( \hat{T}_1 = \hat{\alpha}_1 + \hat{d}_1 \). Additionally, \( \Sigma \leq I_d \) in the sense of quadratic forms.}

Here \( \hat{d}_1I_d \) is the contribution of a dormant period on which the process is a Brownian motion. Equation (3.7) directly entails that \( \Sigma \) is positive definite. In order to show that \( \Sigma \leq I_d \), we use quasiconcavity of \( w(t,\cdot) \) and the Gaussian correlation inequality.

If we replace \( w \) with the potential \( (t,x) \mapsto w(t,x) + g(t) \) our partition function becomes \( Z_{\alpha,2c,T}e^{c_{\alpha,\cdot}} \). By Equation (3.2),

\[
e^{c_{\alpha,\cdot}} \sim e^{2cT - \alpha E[\tau_1]}
\]

(3.8)
as \( T \to \infty \). Thus, adding \( g \) to \( w \) does not change whether \((GC)\) holds or not. After the transformation of the potential, we have \( v > 1 \) and thus \( \mathbb{E}_\alpha[F(\xi_1)] > 1 \), which turns out to be beneficial for showing \((G)\) for sufficiently small \( \alpha \): Then, by the intermediate value theorem, the condition \( \mathbb{E}_\alpha[F(\xi_1)] < \infty \) is sufficient such that \((G)\) is satisfied. We apply this in Section 6 and prove Proposition 2.4 c). Besides giving a similar, slightly stronger, upper estimate on \( F \) (providing necessarily a different proof which does not rely on the specific choice of \( w \)) our approach is different from that used in [MV19], replacing the Markov chain argument with an application of the optional stopping theorem, and thus using an argument that does not need \( g \) to be the density of an exponential distribution. At the end of Section 6 (see Section 7) we will use the fact that \( \psi(\alpha) - \alpha \) is the unique
sition to (G) (assuming that (GC) holds) and perform a few heuristic calculations that connect the free energy to properties of the reweighted point process.

4. Gibbs measures on partitions of \([-T, T]\)

It will be notationally convenient to identify a point process on \(\triangle\) that alternates between dormant periods and active clusters with a random partition of \(\mathbb{R}\) into dormant and active intervals, each active interval being equipped with the corresponding cluster as a “state”. We will consider general systems which alternate between a dormant state and (possible multiple) active states. Starting dormant at time zero, the system switches into an active state after an exponentially distributed waiting time. After an independently drawn waiting time (whose distribution might depend on the state) the system turns dormant again. Iterating this procedure generates a partition of \([0, \infty)\), where each interval of the partition is in a certain state. For a visualization see Figure 1. By an application of renewal theory, there exists a stationary version of the process. We will use its distribution as a reference measure for a certain class of Gibbs measures under the dormant boundary condition. We will assume that dormant intervals do not contribute energy and that the Hamiltonian is additive in the active states. Given a growth condition on the partition function, we will see that the Gibbs measures can be obtained by simply reweighting the distributions of the independent components mentioned above. This yields the existence of an infinite volume measure.

Before continuing, we give a short summary of some standard results from renewal theory. A renewal process \((S_n)_{n \geq 0}\) is a sequence of random variables such that the so called interarrival times \((T_n)_{n \geq 1} := (S_n - S_{n-1})_{n \geq 1}\) form an iid sequence of a.s. positive random variables and such that \(S_0 = 0\) almost surely. For \(t \geq 0\) let \(N(t) := \#\{n : S_n \leq t\}\) be the number of renewal points in the interval \([0, t]\). The renewal function \(U : [0, \infty) \rightarrow [0, \infty)\), \(U(t) := \mathbb{E}[N(t)]\) is the expected number of renewal points (including the origin) in the interval \([0, t]\). Then it holds

**Theorem 4.1.** [Asm03, p. 140] Irrespective whether \(\mathbb{E}[T_1]\) is finite or infinite (setting \(1/\infty = 0\)) we have as \(t \rightarrow \infty\)

\[
\frac{N(t)}{t} \rightarrow \frac{1}{\mathbb{E}[T_1]} \quad \text{almost surely and} \quad \frac{U(t)}{t} \rightarrow \frac{1}{\mathbb{E}[T_1]}.
\]

Let \(\mu\) be a Radon measure on \([0, \infty)\) with \(\mu(\{0\}) = 0\). Let \(z : [0, \infty) \rightarrow \mathbb{R}\) be a locally bounded measurable function. Then the renewal equation

\[
Z(t) = \int_{[0,t]} Z(t-s) \mu(ds) + z(t) \quad \text{for all} \ t \geq 0
\]

has an unique locally bounded measurable solution \(Z_z : [0, \infty) \rightarrow \mathbb{R}\) given by the convolution of \(z\) with the renewal measure \(\sum_{n=0}^{\infty} \mu^n\). We will apply the renewal theorem in the following form:

**Theorem 4.2** (Renewal theorem). [ANT78] If \(\mu\) is a probability measure that is absolutely continuous with respect to the Lebesgue measure and \(g : [0, \infty) \rightarrow [0, \infty)\) is bounded and Lebesgue integrable with \(g(t) \rightarrow 0\) as \(t \rightarrow \infty\) then

\[
\lim_{t \rightarrow \infty} \sup_{|z| \leq g} \left| Z_z(t) - \frac{1}{\int_0^\infty s \mu(ds)} \int_0^{\infty} z(s) ds \right| = 0.
\]

For sub-probability measures, the following holds:
consider the point process
\[ \eta := \sum_{n=1}^{\infty} \delta_{(X_{2n-1}, X_{2n}, \chi_n)} \]
on \Delta \times \mathcal{X}, and denote its distribution by \( \Gamma_{\Xi}^{\alpha} \). The point process shall be interpreted as “on the interval \([X_{2n-1}, X_{2n})\) the system is in state \( \chi_n \) and on the interval \([X_{2n}, X_{2n+1})\) the system is dormant”. We call the renewal process \((S_n)_n := (X_{2n})_n\) the embedded renewal process of \( \eta \), see Figure 1 for a visualization. For \( n \in \mathbb{N} \) we denote by \( T_n := d_n + a_n \) the \( n \)-th interarrival time of embedded renewal process. For \( t > 0 \) let
\[ \tau_t := \inf \{ n : S_n \geq t \} \]
be the hitting time of \([t, \infty)\) of the embedded renewal process of \( \eta \). Notice that \( \tau_t \) coincides with the number of renewal points in \([0, t)\) (counting the origin as a renewal point). The distribution of \( T_1 \) is absolutely continuous with respect to the Lebesgue measure as \( a_1 \) and \( d_1 \sim \text{Exp}(\alpha) \) are independent. Since
\[ \mathbb{P}(\eta \text{ is dormant at } t) = \mathbb{E}[\mathbf{1}_{(T_1 \leq t)}\mathbb{P}(\eta \text{ is dormant at } t - T_1)] + \mathbb{P}(d_1 > t) \]
for all \( t \geq 0 \) the renewal theorem implies
\[ \lim_{t \to \infty} \mathbb{P}(\eta \text{ is dormant at } t) = \mathbb{E}[d_1]/\mathbb{E}[T_1]. \]
For $a \leq b$ let $M_{a,b} := \{(s,t,x) \in \Delta \times X : [s,t] \cap [a,b] \neq \emptyset\}$ and let

$$R^b_a : \mathbf{N}(\Delta \times X) \to \mathbf{N}(\Delta \times X), \quad \mu \mapsto \mu(\cdot \cap M_{a,b})$$

be the restriction to all marked intervals that intersect the interval $[a,b]$. Let us consider again the example of the process $\eta$ representing the $M/G/\infty$-queue. Then $R^0_a \eta$ contains not only all individuals that arrive or depart during $[a,b]$ but all clusters that intersect $[a,b]$. In a similar manner, we define $R^0_0$ to be the restriction to all marked intervals intersecting $[0,\infty)$. For $t > 0$ we define

$$\theta_t : \mathbf{N}(\Delta \times X) \to \mathbf{N}(\Delta \times X), \quad \sum_{i \in I} \delta_{(s_i,t_i,x_i)} \mapsto \sum_{i \in I} \delta_{(s_i,t_i-t_i,x_i)}$$

to be the translation by $-t$. We call a probability measure $\mathcal{P}$ on $(\mathbf{N}(\Delta \times X), \mathcal{N}(\Delta \times X))$ stationary, if $\mathcal{P} = \mathcal{P} \circ \theta^{-1}_t$ for all $t \in \mathbb{R}$. We call a point process on $\Delta \times X$ stationary if its distribution is stationary. By an application of renewal theory, we obtain the existence of a stationary version $\Gamma_{\alpha, st}$ of $\Gamma^\Xi_{\alpha}$.

**Proposition 4.4.** There exists a stationary point process $\eta^s$ on $\Delta \times X$ such that the distribution of $R^\infty_0 \theta_t \eta$ converges in total variation to the distribution of $R^\infty_0 \eta^s$ as $t \to \infty$. Additionally, we have

$$R^\infty_0 \eta^s \overset{d}{=} \gamma + \theta_{-T_0} \eta$$

where $(T_0, \gamma)$ is a $(0,\infty) \times \mathbf{N}_f(\Delta \times X)$ valued random variable independent of $\eta$ with

$$\mathbb{P}((T_0, \gamma) \in C) = \frac{1}{\mathbb{E}[T_1]} \int_0^{\infty} \mathbb{P}((T_1 - t, \theta_t R^T_1 \eta) \in C) \, dt \quad (4.1)$$

for all $C \in \mathcal{B}((0,\infty)) \otimes \mathcal{N}_f(\Delta \times X)$.

**Proof.** First, we show convergence of the distribution of $R^\infty_0 \theta_t \eta$ as $t \to \infty$. The proof follows the general idea: “Condition with respect to a suitable chosen initial condition and show convergence of the distribution of the initial condition”, see e.g. [DV.08, p.224 ff.]

If $A_t$ is the forward recurrence time of the embedded renewal process at $t$ and $\gamma_t := \theta_t R^A_t \eta$ is the re-shifted marked interval ending in $t + A_t$ then we have for all $B \in \mathcal{N}(\Delta \times X)$

$$\mathbb{P}(R^\infty_0 \theta_t \eta \in B) = \int_{(0,\infty) \times \mathbf{N}_f(\Delta \times X)} \mathbb{P}(\zeta + \theta_s \eta \in B) \, (\mathbb{P} \circ (A_t, \gamma_t)^{-1})(ds \, d\zeta)$$

Now for all $C \in \mathcal{B}((0,\infty)) \otimes \mathcal{N}_f(\Delta \times X)$ and $t \geq 0$

$$\mathbb{P}((A_t, \gamma_t) \in C) = \mathbb{E}[\mathbf{1}_{\{T_1 \leq t\}} \mathbb{P}((A_{t-T_1}, \gamma_{T_1}) \in C)] + \mathbb{P}((T_1 - t, \theta_t R^T_1 \eta) \in C).$$

By the renewal theorem (with the majorant $g(t) := \mathbb{P}(T_1 > t)$ for $t \geq 0$), the distribution of $(A_t, \gamma_t)$ converges in total variation to the distribution defined by the right hand side of (4.1) as $t \to \infty$. This implies the convergence of the distribution of $R^\infty_0 \theta_t \eta$ in total variation as $t \to \infty$. By construction, the limit $\mathcal{P}$ satisfies $\mathcal{P} \circ (R^\infty_0 \theta_t)^{-1} = \mathcal{P}$ for all $t > 0$ and can be extended to a stationary distribution on $\mathbf{N}(\Delta \times X)$.

We denote by $\Gamma^\Xi_{\alpha, st}$ the distribution of $\eta^s$. Below we will use $\Gamma^\Xi_{\alpha, st}$ as a reference measure for certain Gibbs measures. For $t > 0$ we denote by $\Gamma^\Xi_{\alpha, t}$ the distribution of $R^\Xi_{-t} \theta_t \eta$ conditionally on $\{\eta \text{ is dormant at } 2t\}$. Notice that, by the memorylessness property of the exponential distribution, $\Gamma^\Xi_{\alpha, t}$ is also the distribution of the process $R^\Xi_{-t} \eta^s$ conditionally on $\{\eta^s \text{ is dormant at } -t \text{ and } t\}$. Unsurprisingly, locally the effect of the boundary condition vanishes as $t \to \infty$. Given probability measures $(\mathcal{P}_t)_t$, $\mathcal{P}$ on $(\mathbf{N}(\Delta \times X), \mathcal{N}(\Delta \times X))$ we
say \( \mathcal{P}_t \to \mathcal{P} \) locally in total variation if, for any \( a < b \), we have \( \mathcal{P}_t \circ (R_a^b)^{-1} \to \mathcal{P} \circ (R_a^b)^{-1} \) in total variation as \( t \to \infty \).

**Proposition 4.5.** We have \( \Gamma_{a,t}^\Xi \to \Gamma_{a,s,t}^\Xi \) locally in total variation as \( t \to \infty \).

*Proof.* Let \( a < b \). To shorten notation, we denote for \( t > 0 \) by \( \sigma_t := \tau_{t+b} \) the hitting time of \([t+b, \infty)\) of the embedded renewal process of \( \eta \) and set

\[
N_t(\eta) := \begin{cases} 
0 & \text{if } \eta \text{ is dormant at } t \\
1 & \text{if } \eta \text{ is active at } t.
\end{cases}
\]

As a consequence of the renewal theorem, we have

\[
\mathbb{P}(N_t(\eta) = 0) \to \mathbb{E}[d_1]/\mathbb{E}[T_1] =: c
\]

as \( t \to \infty \). Let \( \varepsilon > 0 \) and let \( t_1 > 0 \) be such that for all \( t \geq t_1 \)

\[
\mathbb{P}(N_t(\eta) = 0) \geq c/2, \quad |\mathbb{P}(N_t(\eta) = 0) - c| \leq \varepsilon.
\]  

(4.3)

For \( t \geq 0 \) let \( s_t := b + t + \sqrt{t} \). Let \( t_2 \geq 0 \) be such that for all \( t \geq t_2 \) we have \( t - b - \sqrt{t} \geq t_1 \).

For \( t \geq t_2 \) we then have on \( \{S_{\sigma_t} \leq s_t\} \) that \( 2t - S_{\sigma_t} \geq t_1 \) holds. By conditioning with respect to the process up the first renewal after \( b+t \), we get for all \( A \in \mathcal{N}(\Delta \times \mathcal{X}) \) and \( t \geq t_2 \)

\[
\mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} \leq s_t, N_{2t}(\eta) = 0) - \mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} \leq s_t) \mathbb{P}(N_{2t}(\eta) = 0)
\]

\[
\leq \int_{\Omega} \mathbb{P}(d\omega) \left[ 1_{\{S_{\sigma_t} \leq s_t\}}(\omega) \mathbb{P}(N_{2t-S_{\sigma_t}(\omega)}(\eta) = 0) - \mathbb{P}(N_{2t}(\eta) = 0) \right]
\]

\[
\leq 2\varepsilon.
\]

On \( \{S_{\sigma_t} > s_t\} \) the forward recurrence time at \( t + b \) is at least \( \sqrt{t} \). By convergence of the distribution of the forward recurrence time there exists a \( t_3 \geq t_2 \) such that for all \( t \geq t_3 \)

\[
\mathbb{P}(S_{\sigma_t} > s_t) < \varepsilon.
\]

We then have for all \( t \geq t_3 \)

\[
\mathbb{P}(R^b_{a t} \theta t \eta \in A, N_{2t}(\eta) = 0) - \mathbb{P}(R^b_{a t} \theta t \eta \in A) \mathbb{P}(N_{2t}(\eta) = 0)
\]

\[
\leq \mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} > s_t, N_{2t}(\eta) = 0)
\]

\[
+ \mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} \leq s_t, N_{2t}(\eta) = 0) - \mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} \leq s_t) \cdot \mathbb{P}(N_{2t}(\eta) = 0)
\]

\[
+ \mathbb{P}(R^b_{a t} \theta t \eta \in A, S_{\sigma_t} > s_t) \cdot \mathbb{P}(N_{2t}(\eta) = 0)
\]

\[
\leq 4\varepsilon.
\]

and hence

\[
\mathbb{P}(R^b_{a t} \theta t \eta \in A | N_{2t}(\eta) = 0) - \mathbb{P}(R^b_{a t} \theta t \eta \in A) \leq \frac{4\varepsilon}{\mathbb{P}(N_{2t}(\eta) = 0)} \leq \frac{4\varepsilon}{c/2}.
\]

Combining this with Proposition 4.4 yields the claim. \( \square \)

We will now consider Gibbs measures with respect to the stationary distribution \( \Gamma_{a,s,t}^\Xi \) under dormant boundary conditions. We will assume that dormant intervals have no energy contribution and that the Hamiltonian is additive in the active states. Let \( E : \mathcal{X} \to \mathbb{R} \cup \{-\infty\} \) be measurable. We define \( \mathcal{H} : \mathcal{N}_f(\Delta \times \mathcal{X}) \to \mathbb{R} \cup \{-\infty\} \) by

\[
\mathcal{H}\left( \sum_{k=1}^n \delta_{(s_k, t_k, x_k)} \right) := \sum_{k=1}^n E(x_k) \quad \text{for} \quad \sum_{k=1}^n \delta_{(s_k, t_k, x_k)} \in \mathcal{N}_f(\Delta \times \mathcal{X}).
\]
Proposition 4.6. Assume that there exists a $\mu \in \mathbb{R}$ such that $\mathbb{E}[e^{-\mu T_1 - E(\chi_1)}] < \infty$. Then

$$t \mapsto Z_t := \int_{\mathcal{N}_f(\Delta \times \mathcal{X})} e^{-\mathcal{H}(\cdot)} \Gamma^\Xi_{\alpha,t}(d\zeta)$$

is locally bounded, more precisely there exists a $C > 0$ such that $Z_t \leq C e^\lambda$ for all $t \geq 0$.

Proof. By the dominated convergence theorem, there exists a $\lambda \in \mathbb{R}$ such that

$$c := \mathbb{E}[e^{-\lambda T_1 - E(\chi_1)}] \leq 1.$$

If $A_t$ denotes for $t \geq 0$ the forward recurrence time of the embedded renewal process of $\eta$ at time $t$ we have $T_1 + \ldots + T_{\tau_s} = t + A_t$. By the memorylessness property of the exponential distribution, $(A_t, \chi_s)$ is independent of $R_0^0 \eta$ conditionally on $\{\eta$ is dormant at $t\}$ and we have for all $t \geq 0$

$$Z_t e^{-\lambda t} = \frac{1}{c \cdot \mathbb{P}(\eta \text{ is dormant at } t)} \mathbb{E} \left[ \prod_{i=1}^{\tau_s} e^{-\lambda T_i - E(\chi_i)} \mathbb{1}_{\{\eta \text{ is dormant at } t\}} \right] \leq \frac{1}{c \cdot \mathbb{P}(\eta \text{ is dormant at } t)} \mathbb{E} \left[ \prod_{i=1}^{\tau_s} e^{-\lambda T_i - E(\chi_i)} \right].$$

As $(\prod_{i=1}^{\tau_s} e^{-\lambda T_i - E(\chi_i)})_{n \geq 1}$ is a supermartingale with respect to the filtration generated by $((d_n, a_n, \chi_n))_{n \geq 1}$ we get by an application of the optional stopping theorem and Fatou’s lemma

$$\mathbb{E} \left[ \prod_{i=1}^{\tau_s} e^{-\lambda T_i - E(\chi_i)} \right] \leq \liminf_{n \to \infty} \mathbb{E} \left[ \prod_{i=1}^{\tau_s \wedge n} e^{-\lambda T_i - E(\chi_i)} \right] \leq 1.$$

The claim follows as $\lim_{t \to \infty} \mathbb{P}(\eta \text{ is dormant at } t) > 0$. \hfill $\Box$

From now on, we will always assume that $t \mapsto Z_t$ is locally bounded. For $t > 0$ we define the probability measure $\widehat{\Gamma}^\Xi_{\alpha,t}$ by

$$\widehat{\Gamma}^\Xi_{\alpha,t}(d\zeta) = \frac{1}{Z_{2t}} e^{-\mathcal{H}(\cdot)} \Gamma^\Xi_{\alpha,t}(d\zeta).$$

To make contact with the mixing measure in the representation introduced in Section 3, consider $\mathcal{X} = \mathcal{N}_f(\Delta)$ and $(d_n), (a_n), (\chi_n)$ to be the sequences of dormant and active periods of the $M/G/\infty$ queue and $(\chi_n) = (\xi_n)_{n \geq 1}$ to be the sequence of clusters (i.e. the reshifted processes of customers arriving during the respective active periods) and $E = -\log(F)$ with

$$F(\sum_{i=1}^{n} \delta_{(s_i, t_i)}) := \mathbb{E}[W(\prod_{i=1}^{n} v(t_i - s_i, X_{s_i, t_i}))].$$

We give a condition under which the reweighted measure $\widehat{\Gamma}^\Xi_{\alpha,t}$ can be obtained by simply reweighting the distributions of the dormant periods $(d_k)$ and active periods and states $(a_k, \chi_k)$. If the reweighted active periods have finite expectation, this will in particular imply local convergence in total variation of the measures $\widehat{\Gamma}^\Xi_{\alpha,t}$ as $t \to \infty$.

Proposition 4.7. Assume $\lambda \in \mathbb{R}$ satisfies

$$\mathbb{E}[e^{-\lambda T_1 - E(\chi_1)}] = 1.$$
Then
\[ \hat{\Xi}_{\alpha,t} = \Gamma_{\alpha+\lambda,t} \]
where \( \hat{\Xi}(da \, d\chi) := \frac{\alpha}{\alpha + \lambda} e^{-\lambda \alpha - E(\chi)} \Xi(da \, d\chi) \)
for all \( t > 0 \). Additionally,
\[ Z_t = e^{\lambda t} \cdot \frac{\Gamma_{\alpha}(\text{the system is dormant at } t)}{\Gamma_{\alpha+\lambda}(\text{the system is dormant at } t)}. \]

Proof. First, notice that \( \hat{\Xi} \) indeed defines a probability measure as
\[ 1 = E[e^{-\lambda T_1 - E(\chi_1)}] = E[e^{-\lambda t_1} \cdot E[e^{-\lambda a_1 - E(\chi_1)}]] = \frac{\alpha}{\alpha + \lambda} E[e^{-\lambda a_1 - E(\chi_1)}]. \]
Let \( (\tilde{a}_n)_n \) be an iid sequence of \( \text{Exp}(\alpha + \lambda) \) distributed random variables. Let \( ((\tilde{a}_n, \tilde{\chi}_n))_n \) be an iid sequence of \( \hat{\Xi} \) distributed random variables, independent of \( (\tilde{a}_n)_n \), and \( \hat{\eta} \) be the process constructed in the same manner as in the beginning of the section. As usual, we denote for \( t \in \mathbb{R} \) by \( \tilde{\tau}_t \) and \( \tau_t \) the hitting times of \([t, \infty)\) of the embedded renewal processes of \( \hat{\eta} \) and \( \eta \) respectively. For \( n \in \mathbb{N}_0 \), define \( \delta_n := 2t - S_n \) and \( \delta_n := 2t - \hat{S}_n \). Then
\[ \{ \tau_{2t} = n + 1, \eta \text{ is dormant at } 2t \} = \{ S_n \leq 2t, d_{n+1} > \delta_n \}. \]
The distribution of \( \hat{Y} = (\tilde{a}_1, \ldots, \tilde{a}_{n+1}, \tilde{\chi}_1, \ldots, \tilde{\alpha}_n, \tilde{\chi}_n) \) has a density with respect to the distribution of \( Y = (a_1, \ldots, d_{n+1}, a_1, \chi_1, \ldots, a_n, \chi_n) \) given by
\[ (\tilde{a}_1, \ldots, \tilde{a}_{n+1}, \tilde{\chi}_1, \ldots, \tilde{\alpha}_n, \tilde{\chi}_n) \mapsto \frac{\alpha + \lambda}{\alpha} e^{-\lambda \hat{\alpha}_{n+1}} \prod_{i=1}^{n} e^{-\lambda (\hat{a}_i + \tilde{a}_i)} e^{-E(\hat{\chi}_i)}. \]
For measurable \( f : \mathbb{N}_f(\Delta \times \mathcal{X}) \to [0, \infty) \) we have that \( f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ \hat{S}_n \leq 2t, d_{n+1} > \delta_n \}} = h(\hat{Y}) \) for a suitable measurable function \( h \). By writing the following expected value as an integral with respect to the image measure of \( \hat{Y} \) and using that
\[ e^{-\lambda(T_1 + \ldots + T_n + d_{n+1})} = e^{-2\lambda t - \lambda(d_{n+1} - \delta_n)} \]
one obtains
\[ \mathbb{E}\left[f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ \hat{S}_n \leq 2t, d_{n+1} > \delta_n \}} \right] = \frac{\alpha + \lambda}{\alpha} e^{-2\lambda t} \mathbb{E}\left[f(R_{-t}^t \hat{\eta}) \mathbb{1}_{\{ S_n \leq 2t, d_{n+1} > \delta_n \}} e^{-\sum_{i=1}^{n} E(\chi_i)} e^{-\lambda (d_{n+1} - \delta_n)} \right]. \]
By the memorylessness property of the exponential distribution, we have for all \( t \geq 0 \)
\[ \mathbb{E}\left[e^{-\lambda (d_{n+1} - t)} \mathbb{1}_{\{ d_{n+1} > t \}} \right] = \frac{\alpha}{\alpha + \lambda} \mathbb{E}\left[\mathbb{1}_{\{ d_{n+1} > t \}} \right]. \]
By conditioning with respect to \((d_1, \ldots, d_n, a_1, \chi_1, \ldots, a_n, \chi_n)\) we hence obtain
\[ \mathbb{E}\left[f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ \hat{S}_n \leq 2t, d_{n+1} > \delta_n \}} \right] = e^{-2\lambda t} \mathbb{E}\left[f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ S_n \leq 2t, d_{n+1} > \delta_n \}} e^{-H(R_0^t \eta)} \right]. \]
By summing over all \( n \in \mathbb{N}_0 \) we get
\[ \mathbb{E}\left[f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ \hat{\eta} \text{ is dormant at } 2t \}} \right] = e^{-2\lambda t} \mathbb{E}\left[f(R_{-t}^t \theta \hat{\eta}) \mathbb{1}_{\{ \hat{\eta} \text{ is dormant at } 2t \}} e^{-H(R_0^t \eta)} \right] \]
and hence
\[ \int_{\mathbb{N}_f(\Delta \times \mathcal{X})} f(\zeta) \hat{\Xi}_{\alpha+t}(d\zeta) = e^{-2\lambda t} \mathbb{P}(\hat{\eta} \text{ is dormant at } 2t) \int_{\mathbb{N}_f(\Delta \times \mathcal{X})} f(\zeta) e^{-H(\zeta)} \Xi_{\alpha+t}(d\zeta) \]
which yields the claims. \( \square \)
As a corollary of Proposition 4.7 and Proposition 4.5 we obtain our first main result, namely a sufficient condition for the existence of an infinite volume measure for the Gibbs measures $\tilde{\Xi}_{\alpha,T}$. This implies the existence of the infinite volume measure for our path measures $\tilde{P}_{\alpha,T}$.

**Corollary 4.8.** Assume $\lambda \in \mathbb{R}$ satisfies

$$\mathbb{E}[e^{-\lambda T_1 - E(\chi_1)}] = 1 \quad \text{and} \quad \mathbb{E}[T_1 e^{-\lambda T_1 - E(\chi_1)}] < \infty.$$

Then $\tilde{\Xi}_{\alpha,t} \to \tilde{\Xi}_{\alpha + \lambda,\text{st}} =: \tilde{\Xi}_{\alpha,\text{st}}$ locally in total variation.

Note that Condition (G') specializes to (G) in the context of Section 3. We will now show that the existence of a real number $\lambda$ satisfying (G') is equivalent to several natural conditions, including exponential growth of the partition function with $t$.

**Proposition 4.9.** The limit $\varphi := \lim_{t \to \infty} \log(Z_t)/t$ exists in $\mathbb{R} \cup \{\infty\}$.

**Proof.** For $t > 0$ let

$$f(t) := \mathbb{E}[e^{-H(R_0^t\eta)}1_{\{\eta \text{ is dormant at } t\}}] = Z_t\mathbb{P}(\eta \text{ is dormant at } t).$$

By the memorylessness property of the exponential distribution, conditionally on the event $\{\eta \text{ is dormant at } t\}$ the process $R_0^t\eta$ is independent of $R_1^{t+s}\eta$ for all $s,t > 0$ and

$$\mathbb{P}(\theta_0 R_1^{t+s} \eta \in \cdot \mid \eta \text{ is dormant at } t) = \mathbb{P}(R_0^t \eta \in \cdot).$$

This yields for all $s,t > 0$

$$f(t + s) = f(t)f(s) + \mathbb{E}[e^{-H(R_0^t\eta)}1_{\{\eta \text{ is dormant at } t+s\}}1_{\{\eta \text{ is active at } t\}}]$$

which implies, by the version of Feketes lemma for measurable superadditive functions, $\log(f(t))/t \to \sup_{s>0} \log(f(s))/s$ as $t \to \infty$. Since $\lim_{t \to \infty} \mathbb{P}(\eta \text{ is dormant at } t) > 0$, the claim follows. \(\square\)

**Lemma 4.10.** Let $f : (0, \infty) \to (0, \infty)$ be a function such that $a := \lim_{t \to \infty} \log(f(t))/t$ exists in $\mathbb{R}$ and such that $\lim \sup_{t \to \infty} f(t)e^{-at} < \infty$. Then we have

$$\lim \inf_{t \to \infty} f(t)e^{-at} > 0 \iff \lim \sup_{t \to \infty} \frac{f(2t)}{f(t)^2} < \infty.$$

**Proof.** Define $g(t) := f(t)e^{-at}$ for $t > 0$. First, assume that $\lim \inf_{t \to \infty} g(t) > 0$. Then

$$\lim \sup_{t \to \infty} \frac{f(2t)}{f(t)^2} = \lim \sup_{t \to \infty} \frac{g(2t)}{g(t)^2} < \infty.$$

Now, assume that $\lim \sup_{t \to \infty} f(2t)/f(t)^2 < \infty$. Then there exists a $c > 1$ and a $T > 0$ such that $g(2t) \leq cg(t)^2$ for all $t \geq T$. Assume that $\lim \inf_{t \to \infty} g(t) = 0$ would hold. Then, there would exists a $t_0 \geq T$ such that $cg(t_0) < 1$. Set $t_k := 2^k t_0$ for $k \in \mathbb{N}$. Then we have with $b := (cg(t_0))^{1/t_0}$

$$g(t_k) \leq c^{2^k - 1}g(t_0)^{2^k} = \frac{1}{e^{t_k \log(b)}} < b^k$$

for all $k \in \mathbb{N}$. However, as $\log(g(t))/t \to 0$ as $t \to \infty$ and $\log(b) < 0$ there exists a $k \in \mathbb{N}$ such that $g(t_k) > e^{t_k \log(b)} = b^k$. \(\square\)

**Theorem 4.11.** Recall that $\varphi := \lim_{t \to \infty} \log(Z_t)/t \in \mathbb{R} \cup \{\infty\}$. The following conditions are equivalent:
(1) Condition \((G')\) holds, i.e. there exists a \(\lambda \in \mathbb{R}\) such that
\[
\mathbb{E}[e^{-\lambda T_1 - E(x_1)}] = 1 \quad \text{and} \quad \mathbb{E}[T_1 e^{-\lambda T_1 - E(x_1)}] < \infty.
\]

(2) \(\limsup_{t \to \infty} \frac{Z_{2t}}{Z_t^2} < \infty\)

(3) \(\liminf_{t \to \infty} \Gamma_{\alpha,t}^{\Xi}(\text{the system is dormant at } 0) > 0\)

(4) \(\liminf_{t \to \infty} Z_t e^{-\varphi t} > 0\).

If (1)-(4) hold, then \(\lambda = \varphi < \infty\) is the unique real number satisfying Condition 1 and
\[
\lim_{t \to \infty} Z_t e^{-\varphi t} = \frac{\alpha}{\alpha + \lambda} \frac{\mathbb{E}[T_1]}{\mathbb{E}[1]}.
\]

Proof. We will use the same notation as in the proof of Proposition 4.5. First, we will point out that Conditions (2) and (3) are equivalent. Using the same argument as in the proof of Proposition 4.9 we obtain
\[
\hat{\Gamma}_{\alpha,t}^{\Xi}(N_0 = 0) = \frac{1}{Z_{2t}} \mathbb{P}(N_{2t}(0) = 0)^2 \mathbb{E}[e^{-\hat{H}(R_0^0)^2} 1_{\{N_{2t}(0) = 0\}^2}] = \frac{1}{Z_{2t}} \mathbb{P}(N_{2t}(0) = 0)^2 \mathbb{P}(N_{2t}(0) = 0)^2 Z_{2t}^2.
\]

The equivalence of both conditions follows from
\[
\lim_{t \to \infty} \frac{\mathbb{P}(N_t(\eta) = 0)^2}{\mathbb{P}(N_{2t}(\eta) = 0)} = \frac{\mathbb{E}[d_1]}{\mathbb{E}[1]}.
\]

Similar to the proof of Proposition 4.10, one can see that Condition (2) implies \(\varphi < \infty\).

The equivalence of Conditions (2) and (4) follows from Lemma 4.10 since
\[
\log(Z_t \mathbb{P}(N_t(\eta) = 0))/t \leq \varphi
\]
holds by superadditivity (as seen in in the proof of Proposition 4.9). Finally, we show that Conditions (1) and (4) are equivalent. Assume that there exists a \(\lambda \in \mathbb{R}\) as in Condition (1). By Proposition 4.7, we then have
\[
Z_t e^{-\varphi t} = \frac{\hat{\mathbb{P}}(N_t(\eta) = 0)}{\mathbb{P}(N_t(\eta) = 0)} \to \frac{\hat{\mathbb{E}}_\alpha[d_1]/\mathbb{E}_\alpha[T_1]}{\mathbb{E}_\alpha[d_1]/\mathbb{E}_\alpha[T_1]} = \frac{\alpha}{\alpha + \lambda} \frac{\mathbb{E}[T_1]}{\mathbb{E}[1]}
\]
as \(t \to \infty\). As the limit is finite and positive, this additionally implies that \(\log(Z_t) - t\lambda\) converges to a real number as \(t \to \infty\) and hence, \(\lambda = \varphi < \infty\). For the other direction, assume that
\[
\liminf_{t \to \infty} Z_t e^{-\varphi t} > 0.
\]
Then in particular \(\varphi < \infty\). We define \(f : [0, \infty) \to (0, \infty)\) by
\[
f(t) := e^{-\varphi t} \mathbb{E}[e^{-\hat{H}(R_0^0)^2} 1_{\{N_t(\eta) = 0\}}] = Z_t e^{-\varphi t} \mathbb{P}(N_t(\eta) = 0).
\]

By the usual renewal argument
\[
f(t) = \mathbb{E}[\mathbb{I}_{\{T_1 \leq t\}} e^{-\varphi T_1 - E(x_1)} f(t - T_1)] + e^{-\varphi t} \mathbb{P}(d_1 > t)
\]
\[
= \mathbb{E}[\mathbb{I}_{\{T_1 \leq t\}} e^{-\varphi T_1 - E(x_1)} f(t - T_1)] + e^{-(\varphi + \alpha)t}.
\]

That is, \(f\) satisfies a renewal equation with respect to the image measure of \(T_1\) under the (possibly non probability-) measure \(e^{-\varphi T_1 - E(x_1)} d\mathbb{P}\). Notice that Equation (4.4) implies...
If \( f(t) \leq 1 \) for all \( t \geq 0 \) and thus \( \varphi > -\alpha \). Equation (4.5) in combination with Fatou’s Lemma implies
\[
\mathbb{E}\left[ \liminf_{t \to \infty} f(t) e^{-\varphi T_1 - E(\chi_1)} f(t - T_1) \right] \leq \liminf_{t \to \infty} f(t).
\]
(4.6)
Since
\[
\liminf_{t \to \infty} f(t) = \liminf_{t \to \infty} Z_t e^{-\varphi t} \mathbb{P}(N_t = 0) = \frac{\mathbb{E}[d_t]}{\mathbb{E}[T_1]} \liminf_{t \to \infty} Z_t e^{-\varphi t} > 0
\]
we can divide both sides of Inequality (4.6) by \( \liminf_{t \to \infty} f(t) \) and obtain
\[
\mathbb{E}[e^{-\varphi T_1 - E(\chi_1)}] \leq 1.
\]
If \( \mathbb{E}[e^{-\varphi T_1 - E(\chi_1)}] < 1 \) or \( \mathbb{E}[T_1 e^{-\varphi T_1 - E(\chi_1)}] = \infty \) would hold, then \( f \) (as the solution to the renewal equation (4.5)) would converge to zero by renewal theory (compare Proposition 4.13, Theorem 4.2). We thus conclude that Condition (1) holds.

We finish our treatment of the abstract alternating processes by first considering the case that \((G')\) is not satisfied and then giving two sufficient criteria for \((G')\).

**Proposition 4.12.** Assume \( \varphi < \infty \). Irrespective whether \((G')\) is satisfied, we have
\[
\varphi = \min \left\{ \lambda \in \mathbb{R} : \mathbb{E}[e^{-\lambda T_1 - E(\chi_1)}] \leq 1 \right\}.
\]
If \((G')\) does not hold then \( \lim_{t \to \infty} Z_t e^{-\varphi t} = 0 \).

**Proof.** Assume that \( \mathbb{E}[e^{-\varphi T_1 - E(\chi_1)}] > 1 \). Then there would exist a \( N \in \mathbb{N} \) such that
\[
\mathbb{E}[e^{-\varphi T_1 - E(\chi_1)} \mathbb{1}_{\{T_1 \leq N, E(\chi_1) > -N\}}] > 1.
\]
By the intermediate value theorem, there would exists a \( \lambda > 0 \) such that
\[
\mathbb{E}[e^{-(\lambda + \varphi) T_1 - E(\chi_1)} \mathbb{1}_{\{T_1 \leq N, E(\chi_1) > -N\}}] = 1.
\]
(4.7)
For \( t \geq 0 \) set
\[
f_N(t) := e^{-(\varphi + \lambda) t} \mathbb{E} \left[ \prod_{i=1}^{\tau_1-1} e^{-E(\chi_1)} \mathbb{1}_{\{T_i \leq N, E(\chi_i) > -N\}} \right].
\]
As in the proof of Theorem 4.11, one can derive a renewal equation for \( f_N \). Equation (4.7) in combination with the renewal theorem would imply convergence of \( f_N(t) \) to a positive real number as \( t \to \infty \). This, however, contradicts
\[
f_N(t) \leq e^{-(\varphi + \lambda) t} \mathbb{E} \left[ \prod_{i=1}^{\tau_1-1} e^{-E(\chi_1)} \mathbb{1}_{\{N_t(\eta) = 0\}} \right] = e^{-(\varphi + \lambda) t} Z_t \mathbb{P}(\eta \text{ is dormant at } t)
\]
for all \( t > 0 \). Thus \( \mathbb{E}[e^{-\varphi T_1 - E(\chi_1)}] < 1 \). If \( \mathbb{E}[e^{-\varphi T_1 - E(\chi_1)}] < 1 \) or \( \mathbb{E}[T_1 e^{-\varphi T_1 - E(\chi_1)}] = \infty \) holds then \( f \) as defined in the proof of Theorem 4.11 converges to zero and thus
\[
\lim_{t \to \infty} Z_t e^{-\varphi t} = 0.
\]
Similarly it can be shown, that \( \varphi \) is indeed the minimum: Assume there would exists a \( \varepsilon \in (0, \varphi + \alpha) \) such that
\[
\mathbb{E}[e^{-(\varphi - \varepsilon) T_1 - E(\chi_1)}] < 1.
\]
Then, by Proposition 4.3, we would have \( e^{\varepsilon t} f(t) \to 0 \) as \( t \to \infty \) i.e. exponential decay of \( t \mapsto Z_t e^{-\varphi t} \).

**Proposition 4.13.** The following two conditions are sufficient for \((G')\) to hold:
(1) There exists a $\mu \in \mathbb{R}$ such that
\[ 1 < \mathbb{E}[e^{-\mu T_1 - E(\chi_1)}] < \infty. \]

(2) We have $\varphi < \infty$ and there exists an $\epsilon > 0$ such that
\[ \mathbb{E}[e^{-(\varphi-\epsilon)T_1 - E(\chi_1)}] < \infty. \]

Proof. Assume there exists a $\mu \in \mathbb{R}$ such that $1 < \mathbb{E}[e^{-\mu T_1 - E(\chi_1)}] < \infty$. Then, by the intermediate value theorem, there exists a $\lambda > \mu$ such that $\mathbb{E}[e^{-\lambda T_1 - E(\chi_1)}] = 1$. Then $\mathbb{E}[T_1 e^{-\lambda T_1 - E(\chi_1)}] < \infty$ holds as well, as $t \in \mathcal{O}(e^{(\lambda-\mu)t})$ for $t \to \infty$. The second claim follows from the first one and Proposition 4.12.

We are now in the position to prove Theorem 3.2 and Theorem 3.3 by applying the previous results to the measures introduced in Section 3. We consider $X = \mathcal{N}_f(\Delta)$ and $(d_n)_n, (a_n)_n$ to be the sequences of dormant and active periods of the $M/G/\infty$-queue and $(\chi_n)_n = (\xi_n)_n$ to be the sequence of clusters of the queue. Additionally, we choose $E = -\log(F)$. Since the first active period $a_1 = a(\xi_1)$ is a function of the first cluster $\xi_1$, we reduce notation and denote by $\Xi_\alpha$ (as in Section 3) the distribution of $\xi_1$ (and not the joint distribution of $(a_1, \xi_1)$).

Proof of Theorem 3.2 and Theorem 3.3. The number of customers present at time $T > 0$ in the $M/G/\infty$-queue (started empty at time zero) is $\text{Poi}(\beta(\alpha, T))$ distributed, where
\[ \beta(\alpha, T) = \alpha \int_0^T \int_T^\infty g(t - s) \, ds \, dt = \alpha \int_0^T P(\tau_1 > r) \, dr. \]
Hence,
\[ \mathbb{E}_\alpha[T_1] = \frac{\mathbb{E}_\alpha[d_1]}{\lim_{T \to \infty} P_\alpha(\eta \text{ is dormant at } T)} = \frac{1}{\alpha} \lim_{T \to \infty} e^{-\beta(\alpha, T)} = \frac{e^{\alpha \mathbb{E}[\tau_1]}}{\alpha}. \]
As mentioned before
\[ e^{c_{\alpha, T}} \sim e^{2aT - \alpha \mathbb{E}[\tau_1]} = (\alpha \mathbb{E}[T_1])^{-1} e^{2aT} \]
as $T \to \infty$. In particular, since
\[ Z_{\alpha, 2T} := \int_{\mathcal{N}_f(\Delta)} \text{d} \Gamma_{\alpha, T}(\xi) F(\xi) = Z_{\alpha, 2T} e^{-c_{\alpha, T}} \]
we have
\[ \varphi(\alpha) := \lim_{T \to \infty} \log(Z_{\alpha, T})/T = \psi(\alpha) - \alpha. \]
In combination with Theorem 4.11 the above yields Theorem 3.2. We have
\[ \mathbb{E}_\alpha\left[ e^{-(\varphi(\alpha) - \alpha)T_1} F(\xi_1) \right] = 1, \quad \mathbb{E}_\alpha\left[ T_1 e^{-(\varphi(\alpha) - \alpha)T_1} F(\xi_1) \right] < \infty \]
and thus Corollary 4.8 and Equation (3.6) yield for all $a < b$
\[ \mathbb{P}_{\alpha, T}(A) \to \int_{\mathcal{N}(\Delta)} \hat{\Gamma}_{\alpha, st}(d\xi) \mathbb{P}_{R_\xi}(A) = \mathbb{P}_{\alpha, \infty}(A) \]
uniformly in $A \in \mathcal{A}_0^b$ as $T \to \infty$. Consistency of the family of measures $(\mathbb{P}_{\alpha, \infty})_{a < b}$ implies the existence of the measure $\mathbb{P}_{\alpha, \infty}$.

\[ \square \]
Remark 4.14. Let us assume that \( w(t, \cdot) = \tilde{w}(t, |\cdot|) \) is rotationally symmetric and positive and that \( r \mapsto \tilde{w}(t, \sqrt{r}) \) is completely monotone on \((0, \infty)\) with \( \tilde{w}(t, 0) = \lim_{r \to 0} \tilde{w}(t, r) \) for all \( t > 0 \). We will represent the measure \( \mathbb{P}_{\alpha,T} \) as a mixture of Gaussian measures. For the Polaron measure, this will make contact between our representation and the representation introduced in [MV19]. Additionally, this representation can be used in order to show that (under these stronger assumptions on \( w \)) the convergence of the finite dimensional distributions in the proof of the central limit theorem below is even in total variation. By Bernsteins Theorem, there exists for all \( t > 0 \) a Radon measure \( \tilde{\mu}_t \) on \([0, \infty)\) such that for all \( r \geq 0 \)
\[
\tilde{v}(t, \sqrt{2r}) = \int_{[0,\infty)} \tilde{\mu}_t(du) e^{-ur}
\]
where \( v(t, \cdot) := \tilde{v}(t, |\cdot|) \). With \( \mu_t := \tilde{\mu}_t \circ \sqrt{\cdot}^{-1} \) we have for all \( t > 0 \) and \( x \in \mathbb{R}^d \)
\[
v(t, x) = \int_{[0,\infty)} \mu_t(du) e^{-u^2|x|^2/2}.
\]
Hence, we can further rewrite for \( A \in \mathcal{A} \)
\[
P_\xi(A) = \frac{1}{F(\xi)} \int_A \mathcal{W}(dx) \prod_{i=1}^n v(t_i - s_i, x_{s_i,t_i})
= \frac{1}{F(\xi)} \int_{[0,\infty)^n} \bigotimes_{i=1}^n \mu_{t_i - s_i}(du_i) \int_A \mathcal{W}(dx) e^{-\frac{1}{2} \sum_{i=1}^n u_i^2 |x_{s_i,t_i}|^2}. \quad (4.8)
\]
We normalize the inner expression by marking a point process with distribution \( \tilde{\Gamma}_{\alpha,T} \) accordingly. Given a locally compact Polish space \( E \) we can identify a probability measure on \((\mathcal{N}_f(E), \mathcal{B}(\mathcal{N}_f(E)))\) with a symmetric probability measure on \( E^U := \bigcup_{n=0}^\infty E^n \). Let \( E_1 \) and \( E_2 \) be two locally compact Polish spaces and \( \kappa : E_1^U \times \mathcal{B}(E_2^U) \to [0, 1] \) be a probability kernel satisfying

\begin{enumerate}
  \item \( \kappa(x, E_2^n) = 1 \) for all \( x \in E_1^n, n \in \mathbb{N} \)
  \item \( \kappa(\sigma x, \sigma A) = \kappa(x, A) \) for all \( x \in E_1^n, A \in \mathcal{B}(E_2^n), \sigma \in S_n \) and \( n \in \mathbb{N} \)
\end{enumerate}

where \( \sigma x := (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \) for \( x \in E_1^n \), a permutation \( \sigma \in S_n \) and \( i \in \{1, 2\} \). Given a probability measure \( \mathcal{P} \) on \( \mathcal{N}_f(E_1) \) we can define the marked distribution \( \mathcal{P} \otimes \kappa \) in the following way: Draw a sample \( \sum_{i=1}^n \delta_{x_i} \) according to \( \mathcal{P} \), draw marks \( (y_1, \ldots, y_n) \) according to \( \kappa((x_1, \ldots, x_n), \cdot) \) and obtain \( \sum_{i=1}^n \delta_{(x_i, y_i)} \). That is, under the identification mentioned above,
\[(\mathcal{P} \otimes \kappa)(dx dy) = \mathcal{P}(dx)\kappa(x, dy).\]

We apply this to our case and define for \( \zeta = \sum_{i=1}^n \delta_{(s_i,t_i,u_i)} \in \mathcal{N}_f(\Delta \times [0, \infty)) \) the centered Gaussian measure \( \mathcal{Q}_\xi \) on \((C(\mathbb{R}, \mathbb{R}^d), \mathcal{A})\) by
\[
\mathcal{Q}_\xi(dx) = \frac{1}{\phi(\zeta)} \exp \left( -\frac{1}{2} \sum_{i=1}^n u_i^2 |x_{s_i,t_i}|^2 \right) \mathcal{W}(dx)
\]
where \( \phi(\zeta) \) is a normalization constant and \( \kappa : \Delta^U \times \mathcal{B}([0, \infty)^U) \to [0, 1] \) by
\[
\kappa(\xi, du) := \frac{\phi(\xi, u)}{F(\xi)} \bigotimes_{i=1}^n \mu_{t_i - s_i}(du_i)
\]
with \( \phi(\xi, u) := 0 \) for \( \xi \in \Delta^m, u \in [0, \infty)^m \) with \( n \neq m \). By Equation (4.8) we then obtain

\[
P_{\alpha,T}(\cdot) = \int_{N_j(\Delta \times [0,\infty))} (\widehat{\Gamma}_{\alpha,T} \otimes \kappa)(d\zeta) \mathcal{Q}_\zeta(\cdot).
\]

Notice that local convergence of the measures \( \widehat{\Gamma}_{\alpha,T} \) immediately implies local convergence of the measures \( \widehat{\Gamma}_{\alpha,T} \otimes \kappa \). Provided that (GC) is satisfied, one can obtain \( \widehat{\Gamma}_{\alpha,T} \otimes \kappa \) by marking the tilted clusters independently of each other. That is, starting in \(-T\) we alternate independently drawn \( \exp(\psi(\alpha)) \) distributed dormant periods with \( \hat{\Xi}_\alpha \otimes \kappa \) distributed marked clusters. Conditionally on the event that the system is dormant at \( T \), the process of marked customers arriving between \(-T\) and \( T \) has distribution \( \widehat{\Gamma}_{\alpha,T} \otimes \kappa \). In particular, the explicit form of the infinite volume measure is the same if we replace the measures \( P_\xi \) with the measures \( Q_\zeta \) and \( \hat{\Xi}_\alpha \) with \( \hat{\Xi}_\zeta \).

For the special case of the Polaron, i.e. \( w(t, x) := e^{-t/|x|} \) and \( w(t, 0) := \infty \) for \( t \geq 0 \) and \( x \in \mathbb{R}^3 \setminus \{0\} \), one chooses \( g(t) = e^{-t} \) for \( t \geq 0 \). Since

\[
\frac{1}{|x|} = \sqrt{\frac{2}{\pi}} \int_0^\infty du e^{-u^2|x|^2/2}
\]

for all \( x \in \mathbb{R}^3 \setminus \{0\} \), one obtains the representation derived in [MV19] of the Fröhlich polaron as a mixture of Gaussian measures.

5. Proof of the functional central limit theorem

We equip \( C(\mathbb{R}, \mathbb{R}^d) \) with the topology of locally uniform convergence. A function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\} \) is called quasiconcave (quasiconvex) if for all \( a \in \mathbb{R} \cup \{\infty\} \) the superlevel set \( f^{-1}([a, \infty]) \) is convex (the sublevel set \( f^{-1}((-\infty, a]) \) is convex). Provided that \( w(t, \cdot) \) is quasiconcave for all \( t > 0 \) and that Conditions (A1)-(A3) are met, we are going to show that the distribution of \( X^n \) under \( P_{\alpha,\infty} \) converges weakly to the distribution of a centered Gaussian process with stationary and independent increments. In case that \( w(t, \cdot) \) is rotationally symmetric for all \( t > 0 \), this Gaussian process is a rescaled Brownian motion.

We start by showing tightness of \( \{P_{\alpha,\infty} \circ (X^n)^{-1} : n \in \mathbb{N}\} \). For \( n \in \mathbb{N} \) we define the modulus of continuity by

\[
\omega_n : C(\mathbb{R}, \mathbb{R}^d) \times (0, \infty) \to [0, \infty), \quad \omega_n(x, \delta) := \sup_{s,t \in [-n,n], |s-t| < \delta} |x_{s,t}|.
\]

The Gaussian correlation inequality [Roy14] states that for all convex sets \( A_1, A_2 \subseteq \mathbb{R}^n \) that are symmetric about the origin and any centered Gaussian measure \( \mu \) on \( \mathbb{R}^n \)

\[
\mu(A_1 \cap A_2) \geq \mu(A_1) \mu(A_2).
\]

It is well known that this implies \( E\mu[f,g] \geq E\mu[f]E\mu[g] \) for all non-negative, symmetric, quasiconcave functions \( f, g \) and any centered Gaussian measure \( \mu \) on \( \mathbb{R}^n \). We generalize this and obtain the following proposition:

**Proposition 5.1.** Let \( X \) be a \( n \)-dimensional centered Gaussian vector.

1. If \( f_1, \ldots, f_k : \mathbb{R}^n \to [0, \infty] \) are symmetric (with respect to point reflections in the origin) and quasiconcave and \( f_{k+1} : \mathbb{R}^n \to [0, \infty] \) is symmetric and quasiconvex then

\[
E\left[\prod_{i=1}^{k+1} f_i(X)\right] \leq E\left[\prod_{i=1}^{k} f_i(X)\right] \cdot E[f_{k+1}(X)].
\]
(2) If \( f_1, \ldots, f_m : \mathbb{R}^n \to [0, \infty] \) are symmetric and quasiconcave then

\[
\mathbb{E} \left[ \prod_{i=1}^{m} f_i(X) \right] \geq \prod_{i=1}^{k} \mathbb{E} \left[ \prod_{j \in J_i} f_j(X) \right].
\]

for any partition \( \{1, \ldots, m\} = J_1 \cup \ldots \cup J_k \).

**Proof.** We only show the first statement, the proof of the second statement can be conducted similarly. As a direct consequence of the Gaussian correlation inequality, one obtains for \( A_1 \subseteq \mathbb{R}^n \) symmetric and convex and \( A_2 \subseteq \mathbb{R}^n \) symmetric such that \( A_2^c \) is convex

\[
\mathbb{P}(X \in A_1, X \in A_2) \leq \mathbb{P}(X \in A_1) \mathbb{P}(X \in A_2).
\]

We write

\[
f_i(X) = \int_0^\infty \mathbb{I}_{[0,f_i(X)]}(s) \, ds, \quad \text{for } 1 \leq i \leq k; \quad f_{k+1}(X) = \int_0^\infty \mathbb{I}_{[0,f_{k+1}(X)]}(s) \, ds.
\]

Now, for all \( s_1, \ldots, s_{k+1} > 0 \) the sets \( \bigcap_{i=1}^{k} f_i^{-1}(\{s_i, \infty\}) \) and \( (f_{k+1}^{-1}(\{s_{k+1}, \infty\}))^c \) are symmetric and quasiconcave. We hence get with the Gaussian correlation inequality

\[
\mathbb{E} \left[ \prod_{i=1}^{k+1} f_i(X) \right] = \int_0^\infty \cdots \int_0^\infty \mathbb{P}(f_1(X) \geq s_1, \ldots, f_{k+1}(X) > s_{k+1}) \, ds_1 \cdots ds_{k+1}
\]

\[
\leq \int_0^\infty \cdots \int_0^\infty \mathbb{P}(f_1(X) \geq s_1, \ldots, f_k(X) \geq s_k) \mathbb{P}(f_{k+1}(X) > s_{k+1}) \, ds_1 \cdots ds_{k+1}
\]

\[
= \mathbb{E} \left[ \prod_{i=1}^{k} f_i(X) \right] \cdot \mathbb{E}[f_{k+1}(X)].
\]

**Corollary 5.2.** Assume that (A1) holds and that \( w(t, \cdot) \) is quasiconcave for all \( t > 0 \). Then for all \( \xi \in \mathbb{N}_f(\Delta) \) with \( F(\xi) < \infty \) and all \( a < b \)

\[
\mathbb{E}_\xi \left[ |X_{a,b}|^2 \right] \leq \mathbb{E}_W \left[ |X_{a,b}|^2 \right] = d(b - a).
\]

**Proof.** For \( \xi = \sum_{i=1}^{k} \delta_{(s_i, t_i)} \in \mathbb{N}_f(\Delta) \) with \( F(\xi) < \infty \) we apply Proposition 5.1 to the quasiconvex function \( f_{k+1} : \mathbb{R}^{dk+d} \to [0, \infty) \), \( f_{k+1}(x_1, \ldots, x_{k+1}) := |x_{k+1}|^2 \), the quasiconcave functions \( f_1, \ldots, f_k : \mathbb{R}^{dk+d} \to [0, \infty] \) defined by

\[
f_i(x_1, \ldots, x_{k+1}) := v(t_i - s_i, x_i),
\]

for \( 1 \leq i \leq k \) and the Gaussian vector \( (X_{s_1, t_1}, \ldots, X_{s_k, t_k}, X_{a,b}) \). \( \square \)

**Remark 5.3.** Assume that (A1) and (A2) hold. If \( w(t, \cdot) \) is quasiconcave for all \( t > 0 \) then Proposition 5.1 implies that for \( \xi_1, \xi_2 \in \mathbb{N}_f(\Delta) \)

\[
F(\xi_1 + \xi_2) \geq F(\xi_1)F(\xi_2)
\]

i.e. in that sense \( \log(F) \) is superadditive. In particular, if (GC) holds, then the interaction energy between left and right half axis satisfies

\[
\mathbb{E}_W \left[ \exp \left( \alpha \int_{-\infty}^{0} \int_{0}^{\infty} w(t-s, X_{s,t}) \, dt \, ds \right) \right] < \infty. \tag{5.1}
\]
Proof.

We enumerate

\[
\begin{align*}
\mu_{\alpha,T}^1(dsdt) & := \alpha \cdot g(t-s) \mathbb{1}_{\{-T < s < t < 0\}} dsdt \\
\mu_{\alpha,T}^2(dsdt) & := \alpha \cdot g(t-s) \mathbb{1}_{\{0 < s < t < T\}} dsdt \\
\mu_{\alpha,T}^3(dsdt) & := \alpha \cdot g(t-s) \mathbb{1}_{\{-T < s < 0 < t < T\}} dsdt
\end{align*}
\]

respectively. Then

\[
Z_{\alpha,2T} = \mathbb{E}[F(\eta_1 + \eta_2 + \eta_3)] \\
\geq \mathbb{E}[F(\eta_1)]\mathbb{E}[F(\eta_2)]\mathbb{E}[F(\eta_3)] = Z_{\alpha,T}^2 \mathbb{E}[F(\eta_3)].
\]

By the same calculations as in Section 3 we have

\[
\mathbb{E}[F(\eta_3)] = e^{-b_{\alpha,T}} \mathbb{E}_\mathcal{W}\left[ \exp \left( \alpha \int_{-T}^0 \int_0^T w(t-s,X_{s,t}) dt ds \right) \right]
\]

with \(b_{\alpha,T} = \mu_{\alpha,T}^3(\Delta)\) and the statement follows by the monotone convergence theorem and boundedness of \(T \mapsto b_{\alpha,T}\). Notice that

\[
\frac{Z_{\alpha,T}^2}{Z_{\alpha,2T}} = \mathbb{E}_{\alpha,T}\left[ \exp \left( -\alpha \int_{-T}^0 \int_0^T w(t-s,X_{s,t}) dt ds \right) \right]
\]

Therefore, heuristically we would expect (GC) to hold if and only if (assuming an infinite volume measure \(\mathbb{P}_{\alpha,\infty}\) exists)

\[
\mathbb{E}_{\alpha,\infty}\left[ \exp \left( -\alpha \int_{-\infty}^0 \int_0^\infty w(t-s,X_{s,t}) dt ds \right) \right] > 0
\]

i.e. \(\mathbb{P}_{\alpha,\infty}(\int_{-\infty}^0 \int_0^\infty w(t-s,X_{s,t}) dt ds < \infty) > 0\); it would be interesting to have a rigorous proof (or counterexample) for this connection, as well as some understanding how it relates to Equation (5.1).

Lemma 5.4. Assume that (A1) holds and that \(w(t,\cdot)\) is quasiconcave for all \(t > 0\). Then for all \(n \in \mathbb{N}, \varepsilon, \delta > 0\) and \(\xi \in \mathcal{N}_f(\Delta)\) with \(F(\xi) < \infty\)

\[
\mathbb{P}_\xi(\{x \in C(\mathbb{R},\mathbb{R}^d) : \omega_n(x,\delta) > \varepsilon\}) \leq \mathcal{W}(\{x \in C(\mathbb{R},\mathbb{R}^d) : \omega_n(x,\delta) > \varepsilon\}).
\]

Proof. We enumerate

\([-n,n] \cap \mathbb{Q} = \{q_1,q_2,q_3,\ldots\}.
\]

Let \(\xi = \sum_{i=1}^m \delta_{(s_i,t_i)} \in \mathcal{N}_f(\Delta)\) with \(F(\xi) < \infty\) and \(k \in \mathbb{N}\). We define \(f_1,\ldots,f_m : \mathbb{R}^{d(k^2+m)} \rightarrow (0,\infty)\) by

\[
f_i(x_{11},x_{12},\ldots,x_{kk},y_1,\ldots,y_m) := v(t_i - s_i, y_i)
\]

for all \(1 \leq i \leq m\) and \(f_{m+1} : \mathbb{R}^{d(k^2+m)} \rightarrow [0,\infty)\) by

\[
f_{m+1}(x_{11},x_{12},\ldots,x_{kk},y_1,\ldots,y_m) := \begin{cases} 1 & \exists i,j \in \{1,\ldots,k\} : |x_{ij}| > \varepsilon \\
0 & \text{else}
\end{cases}
\]

and \(|q_i - q_j| < \delta\).

Then \(f_{m+1}\) is symmetric and quasiconvex and \(f_1,\ldots,f_m\) are symmetric and quasiconcave.

If we define

\[
Y := (X_{q_1,q_1},\ldots,X_{q_1,q_k}, X_{q_2,q_1},\ldots,X_{q_k,q_k}, X_{s_1,t_1},\ldots,X_{s_m,t_m})
\]
we hence get with the Gaussian correlation inequality
\[
P_{\xi}(\{x \in C(\mathbb{R}, \mathbb{R}^d) : (\exists i, j \in \{1, \ldots, k\} : |q_i - q_j| < \delta \text{ and } |x_{q_i, q_j}| > \varepsilon\})
\]
\[
= \frac{1}{F(\xi)} \mathbb{E}_W \left[ \prod_{i=1}^{m+1} f_i(Y) \right]
\]
\[
\leq \mathbb{E}_W[f_{m+1}(Y)]
\]
\[
= \mathcal{W}(\{x \in C(\mathbb{R}, \mathbb{R}^d) : (\exists i, j \in \{1, \ldots, k\} : |q_i - q_j| < \delta \text{ and } |x_{q_i, q_j}| > \varepsilon\}).
\]
Writing
\[
\{x \in C(\mathbb{R}, \mathbb{R}^d) : \omega_n(x, \delta) > \varepsilon\}
\]
\[
= \bigcup_{k=1}^{\infty} \{x \in C(\mathbb{R}, \mathbb{R}^d) : (\exists i, j \in \{1, \ldots, k\} : |q_i - q_j| < \delta \text{ and } |x_{q_i, q_j}| > \varepsilon\}
\]
and using continuity of measures from below yields the claim. \qed

**Lemma 5.5.** Assume that \(w(t, \cdot)\) is quasiconcave for all \(t > 0\) and that Conditions (A1)-(A3) are met. Then the family of measures \(\{\mathbb{P}_{\alpha, \infty} \circ (X^n)^{-1} : n \in \mathbb{N}\}\) is tight.

**Proof.** As \(\mathbb{P}_{\alpha, \infty}(X^n_0 = 0) = 1\) for all \(n \in \mathbb{N}\) the tightness is equivalent to
\[
\forall m \in \mathbb{N} \forall \eta > 0 \exists \varepsilon > 0 \exists \delta > 0 \forall n \in \mathbb{N} : \mathbb{P}_{\alpha, \infty}(\omega_m(X^n, \delta) > \varepsilon) < \eta.
\]
By Lemma 5.4, we have for \(n, m \in \mathbb{N}\) and \(\varepsilon, \delta > 0\)
\[
\mathbb{P}_{\alpha, \infty}(\omega_m(X^n, \delta) > \varepsilon) = \mathbb{P}_{\alpha, \infty}(\omega_{mn}(X, \delta n) > \sqrt{n} \varepsilon)
\]
\[
= \mathbb{E}_\alpha[\mathbb{P}_{R_0^{mn} \hat{\eta}_n}(\omega_{mn}(X, \delta n) > \sqrt{n} \varepsilon)]
\]
\[
\leq \mathbb{E}_\alpha[\mathcal{W}(\omega_{mn}(X, \delta n) > \sqrt{n} \varepsilon)]
\]
\[
= \mathcal{W}(\omega_{mn}(X, \delta n) > \sqrt{n} \varepsilon)
\]
\[
= \mathcal{W}(\omega_m(X, \delta) > \varepsilon)
\]
where we used Brownian scaling in the last step. The claim follows by the fact that a single probability measure on a Polish space is tight. \qed

In order to prove the functional central limit theorem, it is left to show convergence of the finite dimensional distributions. We will use the following lemma.

**Lemma 5.6.** Assume Conditions (A1)-(A3). Let \(s_1 < t_1 < s_2 < t_2 < \ldots < s_k < t_k\). Then, as \(n \to \infty\),
\[
\sup_{A_1, \ldots, A_k \in B(\mathbb{R}^d)} \left| \mathbb{P}_{\alpha, \infty}(X_{s_1, t_1}^n \in A_1, \ldots, X_{s_k, t_k}^n \in A_k) - \prod_{i=1}^{k} \mathbb{P}_{\alpha, \infty}(X_{s_i, t_i}^n \in A_i) \right| \to 0.
\]

**Proof.** Let \((\hat{T}_0, \hat{\gamma}), (\hat{d}_1, \hat{\xi}_1), (\hat{d}_2, \hat{\xi}_2), \ldots)\) be a sequence of independent \((0, \infty) \times \mathbb{N}_{\text{f}}(\Delta)\) valued random variables such that
- For all \(k \geq 1\) the random variable \((\hat{d}_k, \hat{\xi}_k)\) is \(\text{Exp}(\psi(\alpha)) \otimes \hat{\Xi}_\alpha\) distributed
- \((\hat{T}_0, \hat{\gamma})\) is as in Proposition 4.4 such that \(R_0^{\infty} \hat{\eta}_s \overset{d}{=} \hat{\gamma} + \theta_{-\hat{T}_0} \hat{\eta}\) (where \(\hat{\eta}\) denotes the process obtained by alternating dormant periods \(\hat{d}_k\) and active clusters \(\hat{\xi}_k\) (starting dormant) and \(\hat{\eta}_s\) denotes a stationary version of \(\hat{\eta}\)).
By stationarity, we may assume w.l.o.g. that $s_1 = 0$. Let $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^d)$. For $n \in \mathbb{N}$, we define $f_n, g_n : N_f(\triangle) \to [0, 1]$ by

$$f_n(\xi) := P_\xi(X_{0,t_1}^n \in A_1, \ldots, X_{s_{k-1},t_{k-1}}^n \in A_{k-1})$$

$$g_n(\xi) := P_\xi(X_{0,t_k-s_k}^n \in A_k).$$

For $j \geq 1$ let (as usual) $\hat{T}_j$ denote the sum of the $j$-th dormant and active period. Let $(\hat{S}_j^n)_{j \geq 0} := (\hat{T}_0, \hat{T}_0 + \hat{T}_1, \ldots)$ denote the embedded delayed renewal process of $R_0^n\hat{\eta}_n$ and, for $n \in \mathbb{N}$,

$$Y_n := \inf\{\hat{S}_j^n - nt_{k-1} : j \geq 0 \text{ such that } \hat{S}_j^n > nt_{k-1}\}$$

denote the forward recurrence time at time $nt_{k-1}$. Let $\varepsilon > 0$. By Proposition 4.4 and the convergence of the distribution of $Y_n$ as $n \to \infty$ (it does not matter that the renewal process is delayed by $\hat{T}_0$, see e.g. [Asm03, p. 155]), there exists a $N \in \mathbb{N}$ and a $T > 0$ such that for all $n \geq N$ and $t \geq T$

$$\hat{P}_\alpha(Y_n > T) < \varepsilon, \quad n(s_k - t_{k-1}) > 2T,$$

$$\|\hat{P}_\alpha \circ (R_0^n\hat{\eta}_n) - \hat{P}_\alpha \circ (R_0^n\eta)\| < \varepsilon.$$

As $n(s_k - t_{k-1}) - Y_n(\omega) > T$ for $\omega \in \{Y_n \leq T\}$ and $n \geq N$ and since

$$P_{\alpha,\infty}(X_{s_{k},t_{k}}^n \in A_k) = \hat{E}_\alpha[g_n(R_0^n(t_{k}-s_k)\hat{\eta}_n)]$$

conditioning with respect to the process up to the first renewal after $nt_{k-1}$ yields

$$\left| P_{\alpha,\infty}(X_{0,t_1}^n \in A_1, \ldots, X_{s_{k},t_{k}}^n \in A_k)
- P_{\alpha,\infty}(X_{0,t_1}^n \in A_1, \ldots, X_{s_{k-1},t_{k-1}}^n \in A_{k-1})P_{\alpha,\infty}(X_{s_{k},t_{k}}^n \in A_k) \right|
\leq 2\varepsilon + \int \hat{P}_\alpha(d\omega) \mathbf{1}_{\{Y_n \leq T\}}(\omega)f_n(R_0^n(t_{k}-s_k)\theta_\alpha(s_k - t_{k-1}) - Y_n(\omega)\hat{\eta}))$$

$$\leq 4\varepsilon$$

for all $n \geq N$. The claim follows by inductively applying this argument. \hfill \square

For a proof of the following theorem by Rényi [Rén63] (for $d = 1$) that can directly be generalized to higher dimensions we refer the reader to [Gut13, p. 346 f.].

**Theorem 5.7 (Anscombe-Rényi).** Let $(X_n)_{n}$ be an iid sequence of centered random vectors with $\mathbb{E}[|X_1|^2] < \infty$ and $(N_t)_t$ be a family of $\mathbb{N}$-valued random variables such that $N_t/t$ converges in probability to a constant $\theta > 0$ as $t \to \infty$. Then

$$\frac{1}{\sqrt{t}} \sum_{k=1}^{N_t} X_k \overset{d}{\to} \mathcal{N}(0, \theta \Sigma)$$

as $t \to \infty$ where $\Sigma := \mathbb{E}[X_1 \cdot X_1^T]$.

**Lemma 5.8.** Assume that $w(t, \cdot)$ is quasiconcave for all $t > 0$ and that (A1)-(A3) hold. Let $s_1 < t_1 < \ldots < s_k < t_k$. Then

$$P_{\alpha,\infty} \circ (X_{s_1,t_1}^n, \ldots, X_{s_k,t_k}^n)^{-1} \Rightarrow \bigotimes_{i=1}^{k} \mathcal{N}(0, (t_i - s_i)\Sigma)$$
as $n \to \infty$, where
\[
\Sigma := \frac{\mathbb{E}[\Sigma (d_1, \xi_1)]}{\mathbb{E} [\hat{T}_1]} \quad \text{with} \quad \Sigma (d_1, \xi_1) := d_1 I_d + \mathbb{E}_{\xi_1} [X_{0, \hat{a}_1} \cdot X_{0, \hat{a}_1}^T],
\]
and $(\hat{d}_1, \hat{\xi}_1) \sim \text{Exp}(\psi(\alpha)) \otimes \mathbb{E}_\alpha$, $\hat{a}_1$ is the length of the cluster $\hat{\xi}_1$ and $\hat{T}_1 = \hat{a}_1 + \hat{d}_1$.

**Proof.** By Lemma 5.6, it is sufficient to show that for all $a < b$
\[
\mathbb{P}_{\alpha, \infty} \circ (X_{a,b}^n)^{-1} \Rightarrow N(0, (b-a) \Sigma)
\]
as $n \to \infty$. By stationarity, we may assume w.l.o.g. that $a = 0$. In order to reduce notation we will additionally assume that $b = 1$.

Let $( (\hat{T}_0, \hat{\gamma}, Y^0), (\hat{d}_1, \hat{\xi}_1, Y^1), (\hat{d}_2, \hat{\xi}_2, Y^2), \ldots )$ be a sequence of independent $(0, \infty) \times \text{N}_f(\Delta) \times C(\mathbb{R}, \mathbb{R}^d)$ valued random variables such that

- For all $k \geq 1$ the random variable $(\hat{d}_k, \hat{\xi}_k)$ is $\text{Exp}(\psi(\alpha)) \otimes \mathbb{E}_\alpha$ distributed and $\hat{Y}^k$ has conditionally on $(\hat{d}_k, \hat{\xi}_k)$ distribution $\mathbb{P}_{\theta - \hat{d}_k, \hat{\xi}_k}$.
- $(\hat{T}_0, \hat{\gamma})$ is as in Proposition 4.4 such that $R_0^\infty \hat{\eta}_s \Rightarrow \hat{\gamma} + \theta - \hat{T}_0 \hat{\gamma}$ (where $\hat{\eta}$ denotes as usual the process obtained by alternating dormant periods $\hat{d}_k$ and active clusters $\hat{\xi}_k$ (starting dormant) and $\hat{\eta}_s$ denotes a stationary version of $\hat{\eta}$)
- $Y^0$ has conditionally on $(\hat{T}_0, \hat{\gamma})$ distribution $\mathbb{P}_{\hat{\gamma}}$.

In particular
\[
\mathbb{E}_\alpha \left[ \hat{Y}^1_{0, \hat{T}_1} \cdot (\hat{Y}^1_{0, \hat{T}_1})^T \right] = \mathbb{E}_\alpha [\Sigma (d_1, \xi_1)]
\]
where $\hat{T}_k$ denotes for $k \in \mathbb{N}$ as usual the sum of the $k$-th active and dormant period. Notice that the existence and finiteness of the second moments follows from
\[
\mathbb{E}_\alpha \left[ |\hat{Y}^1_{0, \hat{T}_1}|^2 \right] \leq d \cdot \mathbb{E}_\alpha [\hat{T}_1] < \infty
\]
(remember Corollary 5.2). Let $(\hat{S}_n)_{n \geq 0} = (0, \hat{T}_1, \hat{T}_1 + \hat{T}_2, \ldots)$ denote the embedded renewal process of $\hat{\eta}$. For $t \geq 0$ let
\[
\hat{X}_t := \sum_{k=1}^n \hat{Y}^k_{0, \hat{T}_k} + \hat{Y}^n_{0, \hat{T}_n - \hat{S}_n}
\]
where $n \in \mathbb{N}_0$ is such that $t \in [\hat{S}_n, \hat{S}_{n+1})$.

For $t \geq 0$ let $\hat{A}_t$ denote the forward recurrence time of the embedded renewal process of $\hat{\eta}$. By conditioning with respect to $(\hat{T}_0, \hat{\gamma}, Y^0)$ we get with $V_n := n - \hat{T}_0$ for $f : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded
\[
\mathbb{E}_{\alpha, \infty} [f(X^n_\theta)] = \mathbb{E}_\alpha \left[ \mathbb{E}_{R^n_\theta \hat{\eta}} [f(X^n_\theta)] \right]
\]
\[
= \int_{\hat{\Omega}} \mathbb{E}_\alpha (d\omega_1) \int_{\hat{\Omega}} \mathbb{E}_\alpha (d\omega_2) f \left( \frac{1}{\sqrt{n}} \hat{Y}^0_{0, \hat{T}_0(\omega_1) \wedge n}(\omega_1) \right.
\]
\[
+ \left. \mathbb{1}_{\{\hat{T}_0(\omega_1) < n\}} \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{\hat{\gamma}^k_{0, \hat{T}_k(\omega_2)}} \hat{Y}^k_{0, \hat{T}_k(\omega_2)}(\omega_2) - \frac{1}{\sqrt{n}} \hat{X}_{V_1(\omega_1), V_2(\omega_1) + \hat{A}_n(\omega_1)(\omega_2)} \right) \right).
\]

Now, fix $\omega_1 \in \hat{\Omega}$ and set $r := \hat{T}_0(\omega_1)$. We have by Theorem 4.1
\[
\frac{\hat{\gamma}_{n-r}}{n} \to \frac{1}{\mathbb{E}[\hat{T}_1]}.
\]
almost surely as $n \to \infty$. By Theorem 5.7

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{\hat{n}_n} \hat{Y}_{0, T_k}^k \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

as $n \to \infty$. For $\varepsilon, \delta > 0$ let $N \in \mathbb{N}$ be such that for all $n \geq N$

$$\|\hat{P}_\alpha \circ (X_{n-r, n-r+\hat{A}_{n-r}})^{-1} - \hat{P}_\alpha \circ (Y_{0, T_0}^0)^{-1}\| < \delta$$

and $\hat{P}_\alpha(|\hat{Y}_{0, T_0}^0| > \sqrt{n}\varepsilon) < \delta$. Then, for all $n \geq N$

$$\hat{P}_\alpha\left(\frac{1}{\sqrt{n}}|X_{n-r, n-r+\hat{A}_{n-r}}| > \varepsilon\right) < 2\delta.$$

Hence, by Slutsky’s theorem, the inner integral in Equation (5.2) converges for all $\omega \in \hat{\Omega}$ to $\int_{\mathbb{R}^d} \mathcal{N}(0, \Sigma)(dx)f(x)$. Thus, by dominated convergence,

$$\lim_{n \to \infty} \mathbb{E}_{\alpha, \infty}(f(X^n)) = \int_{\mathbb{R}^d} \mathcal{N}(0, \Sigma)(dx)f(x).$$

Proof of Theorem 3.4. We briefly give the argument why we could exclude the case $t_i = s_{i+1}$ for some $1 \leq i \leq k - 1$ in Lemma 5.8. Let $(n_k)_k$ be a strictly increasing sequence of natural numbers. By tightness, there exists a subsequence $(n_{k_j})_j$ and a measure $\mathcal{P}$ on $C(\mathbb{R}, \mathbb{R}^d)$ such that

$$\mathbb{P}_{\alpha, \infty} \circ (X^{n_{k_j}})^{-1} \Rightarrow \mathcal{P}$$

as $j \to \infty$. By the continuous mapping theorem and Lemma 5.8, for any $s_1 < t_1 < \ldots < s_k < t_k$

$$\mathcal{P} \circ (X_{s_1, t_1}, \ldots, X_{s_k, t_k})^{-1} \Rightarrow \hat{\mathcal{P}} \circ (X_{s_1, t_1}, \ldots, X_{s_k, t_k})^{-1}$$

where $\hat{\mathcal{P}}$ denotes the distribution of $\sqrt{\Sigma}X$ under $\mathcal{W}$. By approximation, we obtain for all $s_1 < t_1 \leq \ldots \leq s_k < t_k$

$$\mathcal{P} \circ (X_{s_1, t_1}, \ldots, X_{s_k, t_k})^{-1} \Rightarrow \hat{\mathcal{P}} \circ (X_{s_1, t_1}, \ldots, X_{s_k, t_k})^{-1}$$

i.e. $\mathcal{P} = \hat{\mathcal{P}}$. Hence, each subsequence of $(\mathbb{P}_{\alpha, \infty} \circ (X^n)^{-1})_n$ has a subsequence that converges weakly to $\hat{\mathcal{P}}$. This implies that $(\mathbb{P}_{\alpha, \infty} \circ (X^n)^{-1})_n$ converges weakly to $\hat{\mathcal{P}}$. It is left to show $\Sigma \leq I_d$. By the Gaussian correlation inequality in the form of Proposition 5.1, we have for all $x \in \mathbb{R}^d$

$$\mathbb{E}_{\xi_1}[(x, X_{0, \hat{d}_1})^2] \leq \mathbb{E}_{\mathcal{W}}[(x, X_{0, \hat{d}_1})^2] = \hat{a}_1|x|^2$$

and hence

$$\langle x, \hat{E}_\alpha[\Sigma(\hat{d}_1, \hat{\xi}_1)]x \rangle = \hat{E}_\alpha[\hat{d}_1]|x|^2 + \hat{E}_\alpha\left[\sum_{i,j=1}^{d} x_i x_j \mathbb{E}_{\hat{\xi}_1}[X_{0, \hat{d}_1}^i X_{0, \hat{d}_1}^j]\right]$$

$$= \hat{E}_\alpha[\hat{d}_1]|x|^2 + \hat{E}_\alpha\left[\mathbb{E}_{\hat{\xi}_1}[(x, X_{0, \hat{d}_1})^2]\right]$$

$$\leq \mathbb{E}[\hat{T}_1]|x|^2.$$

Remark 5.9. If $w(t, \cdot)$ is additionally rotationally symmetric for all $t > 0$ then $\Sigma$ is a multiple of the unit matrix, i.e. the limiting distribution is a rescaled Wiener measure.
In this section we assume that \( w(t, \cdot) \) is rotationally symmetric and positive (allowing the value \(+\infty\)) for all \( t \geq 0 \), and that the function \( \tilde{w} \) defined by \( \tilde{w}(\cdot, \cdot) = w(\cdot, | \cdot |) \) is decreasing in the second variable. Remember that we defined \( \tilde{w}_\beta = \sup_{t \geq 0} e^{\beta t} \tilde{w}(t, \cdot) \) for \( \beta > 0 \). Let \( \beta > 0 \) and \( p > 1 \) be such that the integrability condition in Proposition 2.4 c) holds. We choose \( g(t) = \beta e^{-\beta t} \) in the decomposition (3.1) and obtain
\[
v(t, x) = \frac{1}{\beta} e^{\beta t} w(t, x) \leq \frac{1}{\beta} \tilde{w}_\beta(|x|)
\]
for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \). Notice that \( \tilde{w}_\beta \) is decreasing and (as \( \tilde{w}_\beta \) additionally satisfies the integrability condition) that \( \tilde{w}_\beta(r) < \infty \) for all \( r \in (0, \infty) \). Without loss of generality we can assume that \( v(t, x) > 1 \) for all \( t \geq 0 \) and \( x \in \mathbb{R}^d \). The reason is that the function \( \tilde{w} \) defined by \( \tilde{w}(t, x) = w(t, x) + \beta e^{-\beta t} \) satisfies the assumptions of Proposition 2.4 c) if \( w \) does, and leads to \( v > 1 \). As mentioned at the end of Section 3, (GC) holds for \( \tilde{w} \) if and only if it holds for \( w \).

Let \( (\sigma_n)_{n \geq 0} \) be the iid sequence of \( \text{Exp}(\alpha) \) distributed interarrival times and let \( (\tau_n)_{n \geq 1} \) be the iid sequence of \( \text{Exp}(\beta) \) distributed service times (which is independent of \( (\sigma_n)_{n} \)) of our queue. For \( n \in \mathbb{N} \) let
\[
s_n := \sum_{k=0}^{n-1} \sigma_k, \quad t_n := s_n + \tau_n.
\]
That is, \( s_n \) is the arrival and \( t_n \) the departure of the \( n \)-th customer and \( \sigma_n \) is the time that passes between the arrival of the \( n \)-th and the \( n + 1 \)-th customer. Then the first cluster \( \xi_1 \) is given by
\[
\xi_1 = \sum_{i=1}^{N} \delta_{(s_i - s_1, t_i - s_1)}
\]
where
\[
N = \inf \{ n \in \mathbb{N} : \tau_k < s_{n+1} - s_k \text{ for all } 1 \leq k \leq n \}
\]
is the number of customers in the first cluster. Notice that \( N \) is a stopping time with respect to the filtration generated by \((\sigma_n, \tau_n)_{n \geq 1}\). We define the function
\[
h : (0, \infty) \times (0, \infty) \to (0, \infty], \quad h(t_1, t_2) := \mathbb{E}W[v(t_1, X_{t_2})].
\]

Lemma 6.1. We have
\[
\prod_{i=1}^{N} h(\tau_i, \tau_i) \leq F(\xi_1) \leq \prod_{i=1}^{N} h(\tau_i, \tau_i \land \tau_i).
\]

Proof. The lower estimate follows immediately from the Gaussian correlation inequality in the form of Proposition 5.1. For the upper bound we distinguish between two cases: If \( t_1 \leq t_2 \) we immediately get by independence of increments
\[
F(\xi_1) = \mathbb{E}W\left[ \prod_{i=1}^{N} v(\tau_i, X_{s_i, t_i}) \right] = h(\tau_1, \tau_1)\mathbb{E}W\left[ \prod_{i=2}^{N} v(\tau_i, X_{s_i, t_i}) \right].
\]
In case that \( t_1 > t_2 \) we have by independence of \( X_{s_1, s_2} \) and \((X_{s,t})_{s,t \geq s_2}\)
\[
F(\xi_1) = \int_{C(\mathbb{R}, \mathbb{R}^d)} \mathcal{W}(dx) \mathbb{E}W[v(\tau_1, X_{s_1, s_2} + x_{s_2, t_1}] \prod_{i=2}^{N} v(\tau_i, x_{s_i, t_i}).
\]
The assumptions on \(w\) imply \(\mathbb{E}_W[v(\tau_1, X_{s_1,s_2} + x)] \leq \mathbb{E}_W[v(\tau_1, X_{s_1,s_2})]\) for all \(x \in \mathbb{R}^d\) and thus

\[
F(\xi_1) \leq h(\tau_1, \sigma_1)\mathbb{E}_W\left[ \prod_{i=2}^{N} v(\tau_i, X_{s_i,t_i}) \right].
\]

Iterating this procedure completes the proof. \(\square\)

For the Fröhlich polaron path measure we choose \(\beta = 1\) and obtain by an integration in spherical coordinates

\[
h(t_1,t_2) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{t_2}}
\]

for all \(t_1,t_2 > 0\). Hence, our lower estimate for \(F\) becomes the lower estimate given in [MV19]. If one sorts the customers in the first cluster by the time of their departure, starting with the customer that departs last, and modifies our proof accordingly, one obtains an upper estimate for \(F\) that is sharper than the estimate given in [MV19].

**Lemma 6.2.** Assume that \(w, \beta\) and \(p\) satisfy the assumptions of Proposition 2.4 c). Then for all \(\alpha > 0\),

\[
\mathbb{E}_\alpha[h(\tau_1, \sigma_1 \wedge \tau_1)^p] \leq (\alpha + \beta)^{(d+2)/4}c(w,\beta,p),
\]

where the constant \(c(w,\beta,p)\) is independent of \(\alpha\).

**Proof.** By Jensens inequality,

\[
\mathbb{E}_\alpha[h(\tau_1, \sigma_1 \wedge \tau_1)^p] \leq \mathbb{E}_\alpha\left[ \mathbb{E}_W[v(\tau_1, X_{\sigma_1 \wedge \tau_1})^p] \right] \leq \beta^{-p} \cdot \mathbb{E}_\alpha\left[ \mathbb{E}_W[v(\tau_1, X_{\sigma_1 \wedge \tau_1})^p] \right].
\]

For \(a,b > 0\) we have

\[
\int_0^\infty dt \ t^{-d/2} \exp\left(-a/t - b \cdot t\right) = \int_0^\infty dt \ t^{d-3} \exp\left(-at^2 - b/t^2\right) = (b/a)^{(d-2)/4} K_{(d-2)/2}(2\sqrt{ab})
\]

(the second equality can be obtained by substituting \(t = (b/a)^{1/4}e^u\) where, for \(\gamma \in \mathbb{R}\) and \(x > 0\)

\[
K_\gamma(x) = \int_0^\infty e^{-x \cosh(u)} \cosh(\gamma u) \ du
\]

denotes the modified Bessel function of second kind. With appropriately chosen constants, \(c_d, \tilde{c}_d > 0\) we obtain

\[
\mathbb{E}_\alpha\left[ \mathbb{E}_W[\tilde{w}_\beta(\cdot^{|X_{\sigma_1 \wedge \tau_1}^{|p}})] \right] = c_d(\alpha + \beta) \int_0^\infty dt \ e^{-(\alpha + \beta)t} \int_0^\infty dr \ r^{d-1} \tilde{w}_\beta(r) e^{-r^2/2t} t^{-d/2}
\]

\[
= \tilde{c}_d(\alpha + \beta)^{(d+2)/4} \int_0^\infty dr \ r^{d-1} \tilde{w}_\beta(r) r^{-(d-2)/2} K_{(d-2)/2}(2(\alpha + \beta) \cdot r)
\]

\[
\leq \tilde{c}_d(\alpha + \beta)^{(d+2)/4} \int_0^\infty dr \ \tilde{w}_\beta(r) \cdot r^{d/2} K_{(d-2)/2}(\sqrt{2\beta} \cdot r). \tag{6.1}
\]

Now, as \(r \to \infty\), we have [AS72, p. 378]

\[
K_{(d-2)/2}(\sqrt{2\beta} \cdot r) \in O\left(r^{-1/2} e^{-\sqrt{2\beta} \cdot r}\right),
\]
and since \( \bar{w}_\beta \) is decreasing and thus bounded on \([1, \infty)\), the integral (6.1) is finite on \([1, \infty)\).

For \( r \to 0 \) we have \cite[p. 375]{AS72}

\[
K_{(d-2)/2}(\sqrt{2\beta} \cdot r) \in \begin{cases} \mathcal{O}( -\ln(r) ) & \text{for } d = 2 \\ \mathcal{O}( r^{-|d-2|/2} ) & \text{for } d \neq 2. \end{cases}
\]

The integrability assumptions in 2.4 c) then guarantee that the integral (6.1) is finite on \([0,1]\), too, proving the claim.

\[ \square \]

**Lemma 6.3.** Let \( (X_n)_n \) be an iid sequence of positive random variables adapted to some filtration \( (\mathcal{F}_n)_n \). Let \( X_{n+1} \) be independent of \( \mathcal{F}_n \) for all \( n \in \mathbb{N} \). Let \( \tau \) be an a.s. finite stopping time with respect to \( (\mathcal{F}_n)_n \). Assume that \( p > 1 \) is such that \( \mathbb{E}[X^p] < \infty \). Then

\[
\mathbb{E}\left[ \prod_{i=1}^\tau X_i \right] \leq \mathbb{E}\left[ \left( \mathbb{E}[X_1^p]^{q/p} \right)^{\tau/q} \right]^{1/q}
\]

where \( q \) denotes the conjugated Hölder index of \( p \).

**Proof.** For \( n \in \mathbb{N} \) define

\[
Y_n := \prod_{i=1}^n \frac{X_i^p}{\mathbb{E}[X_1^p]}.
\]

Then \( (Y_n)_n \) is a martingale with respect to \( (\mathcal{F}_n)_n \). By the optional stopping theorem, we have \( \mathbb{E}[Y_{\tau \wedge n}] = 1 \) for all \( n \in \mathbb{N} \). By Fatou’s Lemma

\[
\mathbb{E}[Y_\tau] \leq \liminf_{n \to \infty} \mathbb{E}[Y_{\tau \wedge n}] = 1.
\]

If we denote by \( q \) the conjugated Hölder index to \( p \) we get with Hölder’s inequality

\[
\mathbb{E}\left[ \prod_{i=1}^\tau X_i \right] = \mathbb{E}\left[ \mathbb{E}[X_1^p]^{\tau/p} \prod_{i=1}^\tau \frac{X_i}{\mathbb{E}[X_1^p]^{1/p}} \right] \\
\leq \mathbb{E}\left[ \mathbb{E}[X_1^p]^{\tau/q} \mathbb{E}[Y]\right]^{1/q} \\
\leq \mathbb{E}\left[ \mathbb{E}[X_1^p]^{\tau/q} \right]^{1/q}.
\]

**Proof of Proposition 2.4 c).** We first show that \( Z_{\alpha,T} < \infty \) for all \( \alpha, T > 0 \) and that \( \psi(\alpha) < \infty \) for all \( \alpha > 0 \). Notice that, by the series expansion of the exponential function and by a change of the order of integration (in the same manner as in the beginning of Section 3), one obtains Equation (3.5) even without assuming that \( \mathbb{P}_\alpha,T \) defines a probability measure (i.e. without assuming that \( Z_{\alpha,T} < \infty \)). Since \( T_1 \geq \sigma_0 + \ldots + \sigma_{N-1} + \tau_N \), Lemma 6.1 implies

\[
\mathbb{E}_\alpha[e^{-\mu T_1} F(\xi_1)] \leq \mathbb{E}_\alpha \left[ \prod_{i=1}^N e^{-\mu(\sigma_i \wedge \tau_i)} h(\tau_i, \sigma_i \wedge \tau_i) \right]. 
\]

(6.2)

for all \( \mu \geq 0 \). By Lemma 6.2, we have \( \mathbb{E}_\alpha[h(\tau_1, \sigma_1 \wedge \tau_1)] = \infty \). Hence, there exists a \( \mu \geq 0 \) such that \( \mathbb{E}_\alpha[e^{-\mu(\sigma_1 \wedge \tau_1)} h(\tau_1, \sigma_1 \wedge \tau_1)] \leq 1 \).

As in the proof of Lemma 6.3, an application of the optional stopping theorem to the supermartingale \( \left( \prod_{i=1}^n e^{-\mu(\sigma_i \wedge \tau_i)} h(\tau_i, \sigma_i \wedge \tau_i) \right)_{n \in \mathbb{N}} \) gives \( \mathbb{E}_\alpha[e^{-\mu T_1} F(\xi_1)] \leq 1 \) for this \( \mu \).
Figure 2. Expected value $E_{\alpha} \left[ e^{-\lambda T_1} F(\xi_1) \right]$ in dependency of $\alpha$ and $\lambda$ under the assumptions of Proposition 2.4 c) and assuming that $v > 1$. In the shaded region we do not know whether the expected value is finite or infinite.

By Proposition 4.6, there exist $C, \lambda > 0$ such that $Z_{\alpha, T} \leq C e^{\lambda T}$ for all $T \geq 0$. Since $e^{c_{\alpha,T}} \sim e^{2\alpha T - \alpha E[\tau_1]}$ this yield the existence of $\widetilde{C}, \widetilde{\lambda} > 0$ such that

$$Z_{\alpha, T} \leq \widetilde{C} e^{\widetilde{\lambda} T}$$

for all $T \geq 0$. By superadditivity

$$\psi(\alpha) = \lim_{T \to \infty} \frac{\log(Z_{\alpha, T})}{T} = \sup_{T > 0} \frac{\log(Z_{\alpha, T})}{T} \leq \tilde{\lambda}.$$  

For showing the validity of (GC), we show that (G) holds, and as we already know that (A1) and (A2) hold, we may then apply Theorem 3.2. Since we assumed $v > 1$, we have $E_{\alpha}[F(\xi_1)] > 1$ for all $\alpha > 0$, and by Proposition 4.13 it is thus sufficient to show that $E_{\alpha}[F(\xi_1)] < \infty$ for sufficiently small $\alpha$. By Lemma 6.1 it is sufficient to show that

$$E_{\alpha} \left[ \prod_{i=1}^{N} h(\tau_i, \sigma_1 \wedge \tau_i) \right] < \infty$$

for sufficiently small $\alpha$. Let $r_\alpha$ be the radius of convergence of the probability generating function of $N$. By Lemma 6.3, it is sufficient to show that

$$E_{\alpha} \left[ h(\tau_1, \sigma_1 \wedge \tau_1)^p \right] < \frac{p^p}{r_\alpha^p}$$  (6.3)

for sufficiently small $\alpha$. One can convince oneself (e.g. by looking at the known formula of the probability generating function of $N$ for this particular choice of $g$, see [GS95]) that $\lim_{\alpha \to 0} r_\alpha = \infty$. On the other hand, Lemma 6.2 shows that the left hand side of (6.3) remains bounded as $\alpha \to 0$. This shows the claim.  

\[ \square \]

Remark 6.4. Let us consider the situation from Proposition 2.4 c). We choose $\beta$ and $g$ as in the proof of Proposition 2.4 c) and assume, as in the proof, that $v > 1$. By Proposition
4.12 we have for all \( \alpha > 0 \)
\[
\psi(\alpha) - \alpha = \min \{ \lambda \in \mathbb{R} : \mathbb{E}_\alpha \left[ e^{-\lambda T_1} F(\xi_1) \right] \leq 1 \}.
\]
Using the known formula for the Laplace transform of an active period for this choice of \( g \), see [GS95], one obtains that the behavior of \( \mathbb{E}_\alpha \left[ e^{-\lambda T_1} F(\xi_1) \right] \) as a function of \( \alpha \) and \( \lambda \) is as depicted in Figure 2. We call \( \alpha \) “good” for \( w \) if (GC) is satisfied. Notice that \( \alpha \) is good for \( w \) if and only if 1 is good for \( \alpha w \). A small calculation shows (where we denote the dependency on \( w \) by another subscript)
\[
\partial_\alpha \hat{\Gamma}_{1,T,\alpha w} (\text{the system is dormant at } 0) \leq 0
\]
\[
\iff \mathbb{E}_{\hat{\Gamma}_{1,T,\alpha w}} [N_{\Delta}] \geq \mathbb{E}_{\hat{\Gamma}_{1,T,\alpha w}} [N_{\Delta} | \text{the system is dormant at } 0]
\]
where \( N_{\Delta} : N_f(\Delta) \to \mathbb{N}_0, \mu \mapsto \mu(\Delta) \) denotes the number of points in \( \Delta \). Hence, it seems plausible that \( \hat{\Gamma}_{1,T,\alpha w} (\text{the system is dormant at } 0) \) is decreasing in \( \alpha \) for all \( T > 0 \), which would imply (by Theorem 3.2) that the set of \( \alpha \) that are good for \( w \) is of the form \((0,b) \) or \((0,\infty) \) for some \( b \in (0,\infty) \). Then, if \( b < \infty \), the function \( \alpha \mapsto \lim_{T \to \infty} Z_{\alpha,T} e^{-\psi(\alpha)T} \) would be non-analytic (the limit exists irrespective whether (GC) is satisfied or not by Theorem 4.11 and Proposition 4.12).

7. Relations between \( \psi(\alpha) \) and \( \hat{\Gamma}_{\alpha,\text{st}} \)

In this short section we present a few formal calculations that relate the free energy \( \psi \) to certain expectations with respect to the tilted stationary measure \( \hat{\Gamma}_{\alpha,\text{st}} \). Although we exchange limits and integrals in an uncontrolled way in several places, we expect the resulting formulae to be correct. They show that there is an intricate relationship between \( \psi \) and the expected value of several natural random variables with respect to \( \hat{\Gamma}_{\alpha,\text{st}} \) which seems well worth exploring further in the future.

We assume that (A1)–(A3) are satisfied for all \( \alpha \) and hence
\[
\mathbb{E}_\alpha \left[ e^{-\psi(\alpha) - \alpha} T_1 F(\xi_1) \right] = 1
\]
for all \( \alpha > 0 \). Multiplying the potential by \( e^c \) for some \( c \in \mathbb{R} \) yields with the number \( N \) of customers in the first cluster
\[
\mathbb{E}_\alpha \left[ e^{-\psi(c\alpha) - \alpha} T_1 e^{cN} F(\xi_1) \right] = 1.
\]
Assuming that \( \psi \) is differentiable and that we may differentiate under the integral we obtain by differentiation with respect to \( c \)
\[
\alpha \psi'(\alpha) \mathbb{E}_\alpha \left[ T_1 e^{-\psi(\alpha) - \alpha} T_1 F(\xi_1) \right] = \mathbb{E}_\alpha \left[ N e^{-\psi(\alpha) - \alpha} T_1 F(\xi_1) \right]
\]
or short \( \alpha \psi'(\alpha) \mathbb{E}_\alpha [\hat{T}_1] = \mathbb{E}_\alpha [\hat{N}] \). In other words, \( \psi(\alpha) = 1/\mathbb{E}_\alpha [\hat{d}_1] \) determines the length of dormant periods in the reweighted process and \( \alpha \psi'(\alpha) \) determines the number of points per unit of time in the reweighted process. Notice that
\[
\psi'(\alpha) = \frac{\mathbb{E}_\alpha [\hat{N}] / \mathbb{E}_\alpha [\hat{T}_1]}{\mathbb{E}_\alpha [N] / \mathbb{E}_\alpha [T_1]} \quad (7.1)
\]
as \( \alpha = \mathbb{E}_\alpha [N] / \mathbb{E}_\alpha [T_1] \) (this can be obtained from the previous considerations by choosing \( w(t,x) = g(t) \) for \( t \geq 0 \) and \( x \in \mathbb{R}^d \)). For a constant \( c \geq 0 \) we add \( c \cdot g \) to \( w \) (i.e. we add \( c \) to \( v \)) and obtain
\[
\mathbb{E}_\alpha \left[ e^{-\psi(\alpha) + \alpha c - \alpha} T_1 \mathcal{E}_W \left[ \prod_{i=1}^N v(t_i - s_i, X_{s_i,t_i} + c) \right] \right] = 1.
\]
If we again assume that we may differentiate under the integral we obtain
\[ \alpha \mathbb{E}_\alpha \left[ T_1 e^{-(\psi(\alpha) - \alpha)T_1} F(\xi_1) \right] = \mathbb{E}_\alpha \left[ N e^{-(\psi(\alpha) - \alpha)T_1} \mathcal{F}(\xi_1) \right] \]
where
\[ \mathcal{F}(\xi_1) = \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}_\psi \left[ \prod_{i \neq j} \psi(t_i - s_i, X_{s_i, t_i}) \right]. \]
Combining both equalities yields
\[ \psi'(\alpha) = \frac{\mathbb{E}_\alpha \left[ N e^{-(\psi(\alpha) - \alpha)T_1} \mathcal{F}(\xi_1) \right]}{\mathbb{E}_\alpha \left[ N e^{-(\psi(\alpha) - \alpha)T_1} \mathcal{F}(\xi_1) \right]}. \quad (7.2) \]
In other words, \( \psi'(\alpha) \) is a measure for how much \( F \) changes on average if we randomly delete a point of \( \xi_1 \). If there exists a measurable function \( f : [0, \infty) \to (0, \infty) \) satisfying \( \int_0^\infty (1 + t) f(t) \, dt < \infty \) such that \( w(t, x) \leq f(t) \) for all \( x \in \mathbb{R}^d \) and \( t \geq 0 \) (i.e. if the assumptions of Proposition 2.4 a) are satisfied) then we may choose \( g = f/\Vert f \Vert_{L^1} \) and obtain \( v \leq C \) with \( C := \Vert f \Vert_{L^1} \). This means that \( F(\xi_1) \leq CF(\xi_1) \), and then (7.2) implies a linear upper bound for the growth of \( \psi(\alpha) \) with \( \alpha \). It should be noted, however, that this can be obtained by an elementary calculation, using \( e^{Cv_\alpha} \rightarrow e^{2\alpha CT - \alpha CE[\tau_1]} \). Similarly, lower linear bounds can be derived. For the Fröhlich polaron, \( \psi(\alpha) \sim g_0 \alpha^2 \) and thus \( \psi'(\alpha) \rightarrow \infty \) as \( \alpha \to \infty \) (by convexity of \( \psi \)). Combining this with Equation (7.1), we see that, as a consequence of the singularity of the potential, reweighting the point process leads to a relative increase of the number of customers per unit of time that diverges to \(+\infty\) as \( \alpha \to \infty \).

By Theorem 3.2 we have
\[ \mathbb{E}_\alpha[\hat{T}_1] = \mathbb{E}_\alpha[ T_1 e^{-(\psi(\alpha) - \alpha)T_1} F(\xi_1) ] = \frac{1}{\psi(\alpha) \lim_{T \to \infty} Z_{\alpha, T} e^{-\psi(\alpha)T}}. \]
As seen in the proof of Proposition 2.4 b) and Corollary 2.5, for the Fröhlich polaron this yields
\[ \mathbb{E}_\alpha[\hat{T}_1] = -\frac{1}{\mathbb{E}_\alpha(0)|\langle \Omega, \Psi_\alpha \rangle|^2} \]
where \( \Omega \) is the Fock-vacuum and \( \Psi_\alpha \) and \( \mathbb{E}_\alpha(0) \) are the ground state and the ground state energy of the Hamiltonian of the Fröhlich polaron at total momentum zero. Notice that
\[ |\langle \Omega, \Psi_\alpha \rangle|^2 = \mathbb{E}_\alpha[\hat{d}_1]/\mathbb{E}_\alpha[\hat{T}_1] \]
is the probability that we are dormant at a given point in time under the stationary measure \( \tilde{\omega}_{\alpha, st} \). We can also put the coupling parameter into the potential, to obtain
\[ \mathbb{E}_1 \left[ \alpha^N e^{-(\psi(\alpha) - 1)T_1} F(\xi_1) \right] = 1. \]
This identity might potentially be used in order to show analyticity of \( \psi \) using the implicit function theorem.

**APPENDIX: POLARON MODELS AND POLARON PATH MEASURES**

Here we give an overview over polaron models and their connection to Polaron path measures. For details and proofs we refer to [DS20] and [Møl06], see also Chapters 5 and 6 of [LHB11].

The polaron describes a \( d \)-dimensional quantum particle coupled to a scalar Bosonic field, e.g. the lattice vibrations of a polar crystal. Its Hamiltonian acts in the space \( L^2(\mathbb{R}^d) \otimes F \),
where $\mathcal{F}$ the Hilbert space for the Bose field, i.e. the symmetric Fock space over $L^2(\mathbb{R}^d)$. The Hamiltonian is given by

$$H = \frac{1}{2} p^2 + \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) \, dk + \sqrt{\alpha} \int_{\mathbb{R}^d} \frac{\hat{g}(k)}{\sqrt{2\omega(k)}} \left( e^{ik \cdot x} a(k) + e^{-ik \cdot x} a^*(k) \right) \, dk.$$ 

Here $a^*(k), a(k)$ are the creation and annihilation operators of the free Bose field, respectively, satisfying the canonical commutation relations $[a^*(k), a(k')] = \delta(k - k')$. The first term represents the momentum operator of the free particle, and the second term is the energy of the free field, which is the differential second quantization of the operator of multiplication with $\omega$. The energy-momentum relation $\omega$ of the Bose field is assumed to be nonnegative, strictly positive almost everywhere, continuous, and invariant under rotations. The third term implements the coupling between particle and field, $x$ being the position operator of the particle. The function $g$ is used to 'smear out' the coupling of the particle to the field. $\hat{g}$ is assumed to be rotation invariant and real-valued. The function $g(k) = \hat{g}(k)/\sqrt{2\omega(k)}$

is usually called the coupling function, and $\alpha$ is the coupling constant. Important special cases are the Fröhlich polaron where

$$\omega(k) = 1, \quad g(k) = (\sqrt{2\pi|k|})^{-1},$$

and the Nelson model where

$$\omega(k) = \sqrt{k^2 + m^2} \quad \text{with} \quad m \geq 0, \quad \text{and} \quad g(k) = 1_{[|k|, \Lambda]}(|k|) \omega(k)^{-1/2} \text{ with } 0 \leq \kappa < \Lambda < \infty.$$ 

$m$ is the mass of the Bosons, and $\kappa, \Lambda$ are the infrared and ultraviolet cutoffs, respectively. They restrict the interaction of the particle with the field modes to those modes with energy between $\kappa$ and $\Lambda$.

Under suitable assumptions on $\omega$ and $g$ (which are fulfilled for the two examples above), the operator $H$ is self-adjoint and bounded below. Since the coupling of the particle to the field is invariant under translations, $H$ commutes with the total momentum operator

$$P_{\text{tot}} = p + P_t = p + \int_{\mathbb{R}^d} k a^*(k) a(k) \, dk.$$ 

Therefore, $H$ admits a fiber decomposition $H = U^* \int_{\mathbb{R}^d} H(P) \, dP \, U$ with a suitable unitary operator $U$, where for each $P \in \mathbb{R}^d$ the operator

$$H(P) = \frac{1}{2} (P - P_t)^2 + \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) \, dk + \sqrt{\alpha} \int_{\mathbb{R}^d} g(k) (a(k) + a^*(k)) \, dk.$$ 

now acts only on Fock space. $H(P)$ is bounded below and self-adjoint whenever $H$ is, and the map that takes $P \in \mathbb{R}^d$ to the bottom $E(P)$ of the spectrum of $H(P)$ is the energy-momentum relation for the particle. By the rotation invariance of $\omega$ and $g$, $E(P) = E_t(|P|)$ only depends on $|P|$, and $E''_t(0)$ is the inverse of the effective mass of the particle interacting with the Bose field.

Polaron path measures are related to polaron models by a Feynman-Kac formula: the particle Hamiltonian $\frac{1}{2} p^2$ is the generator of Brownian motion $\mathcal{W}$, and the field Hamiltonian $\int \omega(k) a^*(k) a(k) \, dk$ is unitarily equivalent to the generator of an infinite dimensional Ornstein-Uhlenbeck process $\mathcal{G}$. Its probability distribution is supported on distribution-valued functions $s \mapsto \phi_s$, and its covariance function is given by

$$\mathbb{E}_{\mathcal{G}}(\phi_s(u)\phi_t(v)) = \int \hat{u}(k) \frac{1}{2\omega(k)} e^{-|t-s|\omega(k)\hat{v}(k)} \, dk.$$
for suitable test functions $u,v$. Much like in the ordinary Feynman-Kac formula, this allows to write matrix elements $\langle \Psi , e^{-tH} \Phi \rangle$ as integrals with respect to the measure $G \otimes W$. When $\Omega$ is the Fock vacuum and $f_1,f_2 \in L^2(\mathbb{R}^d)$, this leads to the equality
\[
\langle f_1 \otimes \Omega, e^{-tH} f_2 \otimes \Omega \rangle = \int dx f_1(x) \int W^x(dx) \int G(d\phi) \exp \left( -\sqrt{\alpha} \int_0^t \phi_s(\rho(-x_s)) \, ds \right) f_2(x_t),
\]
where $W^x$ is Brownian motion started at $x$. Since the exponent is linear in the field variable $\phi$, the Gaussian integral can be carried out explicitly, with the result
\[
\langle f_1 \otimes \Omega, e^{-tH} f_2 \otimes \Omega \rangle = \int dx f_1(x) \int W^x(dx) e^{\frac{x}{2} \int_0^t dr f_0^s ds \int w(r-s,x_r-x_s) f_2(x_t)}, \tag{7.3}
\]
where
\[
w(r-s,x_r-x_s) = \mathbb{E}_G(\phi_r(\cdot-x_r)\phi_s(\cdot-x_s)) = \int |g(k)|^2 e^{i(k \cdot (x_r-x_s))} e^{-\omega(k)|r-s|}.
\]
This establishes the connection between the Polaron models and Polaron path measures. The formal choice $f_1 = \delta(0)$ and $f_2 = 1$ in (7.3) corresponds to the matrix element $\langle \Omega, e^{-tH(0)} \Omega \rangle$, see e.g. [DS20], where also expressions for $\langle \Omega, e^{-tH(P)} \Omega \rangle$ are derived for arbitrary $P$.

For the Fröhlich polaron in three dimensions, an explicit computation leads to $w(t,x) = \frac{1}{|x|} e^{-|t|}$. For the massless Nelson model (i.e. $m = 0$) in three dimensions, one obtains
\[
w(t,x) = 4\pi \int_{\kappa}^A e^{-r|t|} \frac{\sin(r|x|)}{|x|} \, dr,
\]
When $\kappa = 0$, i.e. when the particle is allowed to interact with low energy field modes, this $w$ decays like $|t|^{-2}$ for large $t$.

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Volker Betz
FB Mathematik, TU Darmstadt
Email address: betz@mathematik.tu-darmstadt.de

Steffen Polzer
FB Mathematik, TU Darmstadt
Email address: polzer@mathematik.tu-darmstadt.de