The Sachs-Wolfe Effect: Gauge Independence and a General Expression

Jai-chan Hwang\textsuperscript{(a,c)} and Hyerim Noh\textsuperscript{(b,c)}

\textsuperscript{(a)} Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Taegu, Korea
\textsuperscript{(b)} Korea Astronomy Observatory, San 36-1, Whaam-dong, Yusung-gu, Daejeon, Korea
\textsuperscript{(c)} Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 85740 Garching bei München, Germany

We address two points concerning the Sachs-Wolfe effect: (i) the \textit{gauge independence} of the observable temperature anisotropy, and (ii) a gauge-invariant expression of the effect considering \textit{the most general situation} of hydrodynamic perturbations. The first result follows because the gauge transformation of the temperature fluctuation at the observation event only contributes to the isotropic temperature change which, in practice, is absorbed into the definition of the background temperature. Thus, we proceed without fixing the gauge condition, and express the Sachs-Wolfe effect using the gauge-invariant variables.

PACS number(s): 98.70.Vc, 98.80.Hw

1. The excess noise in the radio sky discovered by Penzias and Wilson in 1965\textsuperscript{[3]} was immediately recognized as the remnant of the early hot stage in our universe. We call it the cosmic microwave background radiation (CMBR). Soon after its discovery, in a fundamental paper published in 1967\textsuperscript{[4]} Sachs and Wolfe pointed out that the CMBR should show the temperature change which, in practice, is absorbed into the definition of the background temperature. Thus, we proceed without fixing the gauge condition, and express the Sachs-Wolfe effect using the gauge-invariant variables.

PACS number(s): 98.70.Vc, 98.80.Hw

We address two points concerning the Sachs-Wolfe effect: (i) the \textit{gauge independence} of the observable temperature anisotropy, and (ii) a gauge-invariant expression of the effect considering \textit{the most general situation} of hydrodynamic perturbations. The first result follows because the gauge transformation of the temperature fluctuation at the observation event only contributes to the isotropic temperature change which, in practice, is absorbed into the definition of the background temperature. Thus, we proceed without fixing the gauge condition, and express the Sachs-Wolfe effect using the gauge-invariant variables.

PACS number(s): 98.70.Vc, 98.80.Hw
$T$ as a scalar quantity, the perturbed part changes as $\delta T = \delta T - T \xi^t$, and considering $T \propto a^{-1}$, we have Eq. (2); Some of the fundamental gauge conditions we can recognize in Eq. (2) are: the uniform-curvature gauge ($\varphi \equiv 0$), the zero-shear gauge ($\chi \equiv 0$), the comoving gauge ($v \equiv 0$), the uniform-density gauge ($\delta \equiv 0$), and the uniform-temperature gauge ($\delta T \equiv 0$). Each one of these gauge conditions fixes the temporal gauge transformation property completely (i.e., $\xi^t = 0$), and, thus, each variable in these gauge conditions is equivalent to a corresponding gauge-invariant combination. The synchronous gauge imposes $\alpha = 0$ and fails to fix the gauge mode completely; i.e., we still have $\xi^t = \xi^t(x)$.

We proposed to write the gauge-invariant variables as:

$$\delta_v \equiv \delta + 3(1 + w) \frac{aH}{k} v, \quad \varphi_x \equiv \varphi - H \chi, \quad \alpha_x \equiv \alpha - \dot{\chi},$$

$$v_x \equiv v - \frac{k}{a} \chi, \quad \varphi_v \equiv \varphi - \frac{aH}{k} v, \quad \text{etc.}$$

(4)

$\delta_v$ becomes $\delta$ in the comoving gauge ($v \equiv 0$), etc. In this manner, using Eq. (2), we can systematically construct the corresponding gauge-invariant combination for any variable based on a gauge condition which fixes the temporal gauge transformation property completely. A given variable evaluated in different gauges can be considered as different variables, and they show different behaviors in general.

The background universe is described by:

$$H^2 = \frac{8\pi G}{3} \mu - \frac{K}{a^2} + \Lambda, \quad \dot{\mu} = -3H(\mu + p),$$

(5)

where $K$ and $\Lambda$ are the three-space curvature and the cosmological constant, respectively. Later, it is convenient to have the following equations, derived in [13]:

$$\frac{k^2 - 3K}{a^2} \varphi_x \equiv 4\pi G \mu \delta_v,$$

(6)

$$\dot{\varphi}_x + H \varphi_x = -4\pi G (\mu + p) \frac{a}{k} v_x - 8\pi G H \sigma,$$

(7)

$$\dot{\alpha}_x = -\dot{\varphi}_x - 8\pi G H \sigma,$$

(8)

where $\sigma(x,t)$ indicates the anisotropic pressure.

3. The CMBR has a blackbody distribution and the photons are redshifted during their travel from last scattering to the observer. After the last scattering, the photons are effectively collision-free and non-self-gravitating, thus follow the geodesic path in the given (perturbed) metric. The null vector tangent to the geodesic $x^a(\lambda)$ with an affine parameter $\lambda$ is $K^a = dx^a/d\lambda$. We define the null energy-momentum four-vector $k^a$ to the perturbed order as: $k^0 \equiv a^{-1}(\dot{\nu} + \delta \nu)$ and $k^a \equiv -\dot{\nu} a^{-1}(\dot{\varphi}_a + \delta \varphi_a)$. The temperatures of the CMBR at two different points ($O$ and $E$) along a single null-geodesic ray in a given observational direction is, [2],

$$\frac{T_O}{T_E} = \frac{(k^a u_a)_O}{(k^b u_b)_E},$$

(9)

where $O$ is the observed event here and now, and $E$ is the emitted event at the intersection of the ray and the last scattering surface. $u_a$ at $O$ and $E$ are the local four-velocities of the observer and the emitter, respectively. In the large angular scale we are considering (larger than the horizon size at the last scattering era which subtends about $2\sqrt{\bar{T}b}$ degree by an observer today) the detailed dynamics at last scattering is not important. The physical processes of last scattering are important in the small angular scale where we need to solve the Boltzmann equations for the photon distribution function, [3].

The observed temperature along the single ray may depend on the location of the observer $x_O$ (cosmic variance), and the direction of the observed ray $e_O$. Similarly as in Eq. (3) we may decompose the observed temperature along the single ray into the background and perturbed parts as

$$T(x_O,t_O;e_O) = \bar{T}(x_O,t_O) + \delta T(x_O,t_O;e_O).$$

(10)

Although we used similar notations in Eqs. (3) and (10), it is desirable to notice the difference: Eq. (3) decomposed the temperature at spacetime points, whereas Eq. (10) decomposed the observed temperature along different directions $e_O$ at observer’s location $x_O$. Up to this point, the decomposition in Eq. (10) still has arbitrariness as the one in Eq. (2). In the observations, however, we often take the background temperature as an averaged temperature all around the sky at the observer’s location, i.e., $\bar{T}(x_O,t_O) \equiv \langle \delta T(x_O,t_O;e_O) \rangle_{e_O}$. In this way the arbitrariness is fixed, and the remaining $\delta T\vert_O$ over the sky apparently coincides with the angular variation of observed temperature. Thus, $\delta T\vert_O$ should be independent of the gauge condition (imposed at the observer’s spacetime position). Let us explain this last point below. In the temporally evolving background, $\bar{T} = \bar{T}(t)$, $\delta T$ is a gauge dependent quantity. The gauge dependence of $\delta T$ should be considered in handling fluctuations at the last scattering era $E$. However, for $\delta T$ evaluated at the observation event $O$, the effect of the gauge transformation $H \xi^t(x,t)$ evaluated at $O$ will show no angular dependence, thus can be absorbed into our definition of the background temperature, and is irrelevant for the temperature anisotropy; thus, the observable temperature anisotropy is a concept independent of the gauge condition used [4]. Equivalently, since $H \xi^t\vert_O$ terms cancel, the difference of observed temperatures in two different directions is gauge-invariant.

Perturbation analyses of the null equation ($k^a k_a = 0$), the geodesic equation ($k^a \dot{x}_a = 0$), and Eq. (3) provide the equations we need. To the background order, we have: $\dot{T} \propto \dot{\nu} \propto a^{-1}$, $\dot{e}_a e_a = 1$, and $\dot{e}^{\alpha} = \dot{e}^{\alpha} \mathbf{e}^\beta$. To the perturbed order, we have [for convenience, we consider the contributions from three perturbation types separately as $\delta T\vert_O = \delta T^{(s)}\vert_O + \delta T^{(v)}\vert_O + \delta T^{(\nu)}\vert_O$]:

$$\frac{\delta T^{(s)}}{T} \Bigg|_O = \frac{\delta T^{(v)}}{T} \Bigg|_E - \frac{1}{k^a e^a} \frac{\partial \delta T^{(\nu)}}{\partial E} \Bigg|_E$$
suggestive form are the Newtonian velocity and potential fluctuations, $v_K$ and the perturbed potential and the perturbed velocity density perturbation variable in the comoving gauge (Newtonian behaviors in the pressureless limit: the density perturbation). In hydrodynamic perturbation based on Einstein gravity, it is known that only certain variations in Eq. (11) are gauge-invariant. In the literature, one can evaluate Eq. (11) in any gauge condition with the same ‘observable’ anisotropy. In this sense the observable temperature anisotropy on the LHS of Eq. (11) is gauge-independent.

Absorbing the isotropic contributions to $\hat{T}(x_O, t_O)$, we have

$$\frac{\delta T^{(s)}}{T}\bigg|_O = -\frac{1}{k}v_X e^{\alpha}\bigg|_O + \frac{1}{k}v_X e^{\alpha}\bigg|_E + \int_E^O (\alpha - \varphi_X') dy. \quad (15)$$

The RHS is apparently gauge-invariant. In the literature, the four terms on the RHS are often called: the Doppler effect due to the observer’s movement, the Doppler effect due to the movement of the photon-emitting plasma along the line-of-sight, the Sachs-Wolfe (SW) effect, and the integrated Sachs-Wolfe (ISW) effect, respectively.

Now, we re-express the SW and the ISW terms using $\varphi_X$ which has the close analogy with the Newtonian gravitational potential. In order to relate the temperature fluctuation with the coexisting matter at $E$, we take an ansatz

$$\frac{\delta T}{T} \bigg|_O = \frac{\delta}{3(1 + w)} + \epsilon_T - \frac{2}{\pi G}(\varphi_X + 8\pi G\sigma)$$

where $\epsilon_T(x, t)$ is apparently gauge-invariant and can be regarded as the deviation of the temperature fluctuation from the adiabaticity with the coexisting matter fluctuation: we may call it the entropic temperature fluctuation $\epsilon_T$. By considering $\epsilon_T$ we can handle the effects from the multi-component hydrodynamic situation.

Using Eqs. (11) we can express the SW and ISW terms in Eq. (13) using $\varphi_X$

$$\frac{\delta T^{(s, SW, ISW)}}{T} \bigg|_O = \left\{ -1 + \frac{H^2}{4\pi G(\mu + p)} \left( \varphi_X + 8\pi G\sigma \right) + \frac{H^2}{4\pi G(\mu + p)} \left( \frac{\varphi_X}{H} + \frac{k^2 - 3K}{3a^2H^2} \varphi_X + \epsilon_T \right) \right\} \bigg|_E$$

In this form, we considered the general $K$, $\Lambda$, and $p(\mu)$ in the background, and the general $\epsilon(x, t)$ (the entropic pressure), $\sigma$, and $\epsilon_T$ in the perturbation. In an ideal fluid (thus, $\epsilon = 0 = \sigma$), the general super-sound-horizon scale solution for $\varphi_X$ is presented in [13].

$$\varphi_X(x, t) = 4\pi GC(x) \frac{H}{a} \int_0^t \frac{a(\mu + p)}{H^2} dt + \frac{H}{a} d(x), \quad (16)$$

where $C(x)$ and $d(x)$ are integration constants indicating the relatively growing and decaying modes, respectively.
Remarkably, this solution is valid on scales larger than Jeans scale for the general $K$, $\Lambda$, and generally time-varying $p(\mu)$. In the near flat case (thus, ignoring $K$ terms), we have a powerful conserved quantity in the super-sound-horizon scale: $\varphi_\nu(x,t) = C(x)$, with the vanishing leading decaying mode. The structural seed originated from the quantum fluctuation during the inflation era provides the initial condition for $C(x)$ and it is conserved during the super-sound-horizon scale evolution independently of changing equation of state, changing gravity theories, and the horizon crossing.

For $K = 0 = \Lambda$ and $w = $ constant, the growing mode of $\varphi_\chi$ in Eq. (18) remains constant (we ignored $e$ and $\sigma$). Thus, ignoring the decaying mode, we have

$$\frac{\delta T^{(s,SW)}}{T}\big|_O = \left\{ \left[ -\frac{1 + 3w}{3(1+w)} + \frac{2}{9(1+w)} \left( \frac{k}{aH} \right)^2 \right] \varphi_\chi + e_T \right\} \big|_E,$$

(19)

and the ISW term vanishes. The large observed angular scale corresponds to the superhorizon scale at the time of last scattering, and the effect from $(k/aH)^2$ term becomes subdominant. Thus, in the large angular scale, assuming the pressureless era at $E$, and ignoring $e_T|_E$, we finally have

$$\frac{\delta T^{(s,SW)}}{T}\big|_O = -\frac{1}{3} \varphi_\chi|_E = \frac{1}{3} \delta \Phi|_E,$$

(20)

which is the commonly quoted result derived in [4]. Notice, however, the various levels of assumptions used to have Eq. (20): we assumed, a single component, pressureless ($p = 0$), adiabatic ($e_T = 0$), ideal fluid ($e = 0 = \sigma$), with $K = 0 = \Lambda$, and vanishing tensor mode at $E$ for the SW term, and along the ray’s path from $E$ to $O$ for the ISW term.

5. Equation (17) expresses the SW and the ISW effects in the very general situation. Besides this, we also have two Doppler terms in Eq. (14) and the vector and tensor contributions in Eqs. (12, 13). These altogether contribute to the observed temperature anisotropy, [3]. Attempts to explain the result in Eq. (20) in pedagogic ways, e.g., [2], usually involve gauge dependent interpretations [24] with limited implications, and should be read with due caution. Works in the literature start by fixing a certain gauge condition [24, 25] or by using combinations of the gauge-invariant variables [12, 13]. The final results for the observed temperature anisotropy are bound to be the same as ours, because, as we have shown, the concept is observationally gauge-independent.

We thank Profs. K. Subramanian and A. Mészáros for useful discussions, and Prof. S. D. M. White for careful comments and invitation to MPA. We wish to acknowledge the financial support of the Korea Research Foundation. HN was supported by the DFG fellowship (Germany) and the KOSEF (Korea).
tional field we cannot distinguish the gravitational shift from the Doppler shift in an invariant way.\[3\]

[17] Although Eq. (16) is motivated by the fact that $\dot{T} \propto a^{-1}$, and $\dot{\mu} \propto a^{-3(1+w)}$ for $w = \text{constant}$, we can still regard $w$ as the generally time varying one.

[18] In the multi-component situation we regard the fluid quantities as the collective ones. In the two component situation with dust (or dark matter, $d$) and radiation ($r$), it is convenient to introduce $S \equiv \delta_d - \frac{3}{4} \delta_r$, which indicates the entropy perturbation. If we take $\delta \mu = \delta \mu_d + \delta \mu_r = 0$ as the condition for an isocurvature mode, we can show $\epsilon_T = -S/(3 + 4 \mu_r/\mu_d)$.

[19] J. Hwang, Phys. Rev. D 53, 762 (1996).

[20] A similar expression using the covariant approach has been recently derived in Eq. (3.17) of \[21\].

[21] A. Challinor and A. Lasenby, Phys. Rev. D 58, 023001 (1998).

[22] M. White and W. Hu, Astron. Astrophys. 321, 8 (1997).

[23] In the literature we find many analyses based on either the synchronous gauge ($\alpha \equiv 0$) \[24\] or the zero-shear gauge ($\chi \equiv 0$) \[25\]. The often mentioned ‘gauge dependent’ wisdoms are the following [we assume the same conditions used to get Eq. (20)]: (i) In the zero-shear gauge, using Eqs. (6-8), we can show $\delta T_\chi/T = \frac{2}{3} \phi_\chi$, thus after compensation with $\alpha_\chi = -\phi_\chi$ in Eq. (15) we have Eq. (20). (ii) In a pressureless medium the synchronous gauge is effectively the same as the comoving gauge ($v \equiv 0$). Evaluating Eq. (16) in the comoving gauge, and using Eq. (6) we can show $\delta T_v/T = \frac{1}{3} \delta v = \frac{2}{3} (k/aH)^2 \phi_\chi$ at $E$, and thus negligible compared with $\phi_\chi$ at $E$. drilled shift in an invariant way, $T$.

[17] Although Eq. (16) is motivated by the fact that $T \propto a^{-1}$, and $\mu \propto a^{-3(1+w)}$ for $w = \text{constant}$, we can still regard $w$ as the generally time varying one.

[18] In the multi-component situation we regard the fluid quantities as the collective ones. In the two component situation with dust (or dark matter, $d$) and radiation ($r$), it is convenient to introduce $S \equiv \delta_d - \frac{3}{4} \delta_r$, which indicates the entropy perturbation. If we take $\delta \mu = \delta \mu_d + \delta \mu_r = 0$ as the condition for an isocurvature mode, we can show $\epsilon_T = -S/(3 + 4 \mu_r/\mu_d)$.

[19] J. Hwang, Phys. Rev. D 53, 762 (1996).

[20] A similar expression using the covariant approach has been recently derived in Eq. (3.17) of \[21\].

[21] A. Challinor and A. Lasenby, Phys. Rev. D 58, 023001 (1998).

[22] M. White and W. Hu, Astron. Astrophys. 321, 8 (1997).

[23] In the literature we find many analyses based on either the synchronous gauge ($\alpha \equiv 0$) \[24\] or the zero-shear gauge ($\chi \equiv 0$) \[25\]. The often mentioned ‘gauge dependent’ wisdoms are the following [we assume the same conditions used to get Eq. (20)]: (i) In the zero-shear gauge, using Eqs. (6-8), we can show $\delta T_\chi/T = \frac{2}{3} \phi_\chi$, thus after compensation with $\alpha_\chi = -\phi_\chi$ in Eq. (15) we have Eq. (20). (ii) In a pressureless medium the synchronous gauge is effectively the same as the comoving gauge ($v \equiv 0$). Evaluating Eq. (16) in the comoving gauge, and using Eq. (6) we can show $\delta T_v/T = \frac{1}{3} \delta v = \frac{2}{3} (k/aH)^2 \phi_\chi$ at $E$, and thus negligible compared with $\phi_\chi$ at $E$. drilled shift in an invariant way, $T$.