BSDEs with weak reflections and partial hedging of American options

Roxana Dumitrescu ∗ Romuald Elie † Wissal Sabbagh ‡ Chao Zhou §

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Abstract

We introduce a new class of Backward Stochastic Differential Equations with weak reflections whose solution \((Y, Z)\) satisfies the weak constraint \(\mathbf{E}[\Psi(\theta, Y_\theta)] \geq m\), for all stopping time \(\theta\) taking values between 0 and a terminal time \(T\), where \(\Psi\) is a random non-decreasing map and \(m\) a given threshold. We study the wellposedness of such equations and show that the family of minimal time \(t\)-values \(Y_t\) can be aggregated by a right-continuous process. We give a nonlinear Mertens type decomposition for lower reflected \(g\)-submartingales, which to the best of our knowledge, represents a new result in the literature. Using this decomposition, we obtain a representation of the minimal time \(t\)-values process. We also show that the minimal supersolution of such an equation can be written as a stochastic control/optimal stopping game, which is shown to admit, under appropriate assumptions, a value and saddle points. From a financial point of view, this problem is related to the approximative hedging for American options.

Key words: BSDEs with weak reflections, Partial hedging, American options, Optimal control, Optimal stopping, Stochastic game, Stochastic target.

AMS 1991 subject classifications: 93E20, 60J60, 47N10.

1 Introduction

The theory of reflected backward stochastic differential equations (RBSDEs for short) was first introduced by El Karoui et al. [13]. In this context, the first component of the BSDE solution is forced to stay above a given - so-called obstacle or reward - stochastic process.
In order to maintain the solution above the obstacle, the BSDE dynamics contains an extra increasing component, which is part of the solution. The uniqueness property for such equations is due to a Skorokhod type minimality condition. The first application of RBSDEs was related to pricing and hedging concerns of American options. Since then, a large number of applications to optimal stopping, optimal switching or stochastic games gave rise to a vast literature on this topic.

The valuation of an American option with payoff process \( (L_t)_{0 \leq t \leq T} \) requires to determine its optimal selling time and corresponding hedging strategy. Nevertheless, in realistic and hereby imperfect financial markets, a replicating strategy is often inaccessible. From the point of view of the seller, who wants to protect himself against his contractual obligation, a conservative approach consists in superhedging the American option, via the construction of an investment strategy generating enough capital to cover the payoff at any possible stopping time chosen by the option holder. Solving such problem consists in finding an initial data \( Y_0 \), a control \( Z \) and an additional increasing process \( K \) such that

\[
Y_t^Z = Y_0 + \int_0^t g(s, Y_s^Z, Z_s) ds - \int_0^t Z_s dW_s + K_t \quad (1.1)
\]

\[
Y_\tau^Z \geq L_\tau, \quad \mathbb{P}-a.s. \text{ for all stopping time } \tau \in [0, T], \quad (1.2)
\]

\[
\int_0^T (Y_t^Z - L_t) dK_t = 0. \quad (1.3)
\]

The driver function \( g \) contains in particular the discounting factors as well as some imperfections of the financial market. It may be non linear, whenever for example the lending and borrowing rates are different. \( Y_t \) interprets as the super-replication price of the American option at time \( t \), whereas \( Z \) corresponds to the optimal sur-replication strategy. Observe that the Skorkhod condition \((1.3)\) enforces to choose the minimal super-replicating price.

From a practical point of view, however, the cost of superhedging is fairly too high, so that the option seller needs to accept to take some risk. One alternative approach mainly developed so far for European options consists in replacing the too strong super-replicating \( \mathbb{P}\)-a.s. terminal condition by a weaker one. Namely, for European options,

\[
Y_T^Z \geq L_T \quad \text{is replaced by} \quad \mathbb{E}[\ell(Y_T^Z - L_T)] \geq m,
\]

where \( m \) stands for a given success threshold and \( \ell \) represents a non-decreasing loss function. From a financial point of view, this approach is referred to as quantile or efficient hedging, and was first discussed by Föllmer and Leukert [14, 15]. In particular, they explained how the so-called quantile hedging price for European option can be computed explicitly in a complete market, using duality arguments and Neyman-Pearson lemma. In a general Markovian setting, Bouchard et al. [4] provided a direct dynamic approach to tackle this question, via the introduction of an additional well-chosen state variable. Even in incomplete markets and for general loss functions, they characterize the pricing function as the solution of a non-linear parabolic second order differential equation, using tools developed in the context of stochastic target problems by Soner and Touzi [23]. Recently, Bouchard,
Elie and Reveillac [3] extended this approach to a possibly non-Markovian setting and introduced a new class of BSDEs, namely BSDEs with weak terminal condition, in which the terminal value $Y_T$ of the portfolio is required to satisfy a weak constraint of the form (1). This approach has been extended by Dumitrescu [10], allowing for the consideration of nonlinear risk measure constraints.

The seller of an American option using a quantile efficient hedging approach is hereby required to solve a BSDE with dynamics (1.1), but shall replace the too strong constraint (1.2) by a weaker one of the form

$$
\mathbb{E}[\ell(Y^Z_{\tau} - L_{\tau})] \geq m, \text{ for all stopping time } \tau \in [0, T].
$$

(1.4)

The main objective of this paper is to derive the well-posedness and main properties of BSDEs with such type of constraint (1.4) and discuss its connection with the efficient hedging of American options. Up to our knowledge, we provide the first dynamic probabilistic representation for the efficient price of American Options in continuous time. Let now mention some related works in the literature. Pérez [22] or Mulinacci [18] discuss the existence of an efficient hedge in such context. Dolinsky and Kifer [9] focus on the partial hedging of game options in a discrete time setting with transaction costs. In a Markovian setting, an obstacle version of the geometric dynamic programming principle of Soner and Touzi [23] is given in [5], and Bouchard et al [2] provided a probabilistic numerical algorithm for the computation of the quantile hedging of Bermudean options, using duality representations. Recently, Briand et al [6] followed a very different approach to study BSDEs of the form (1.1) together with a weaker version of (1.4) where the constraint only hold for deterministic times on $[0, T]$. In such a framework, no dynamic programming principle is available and the derived solution relates to stochastic differential equations of McKean-Vlasov type.

Trevino [24] considered the problem that the seller of an American option aims to control the shortfall risk by using a partial hedge. He is interested in the problems of partial hedging and of optimal exercise of an American option in an incomplete market in continuous time. In particular, Trevino [24] proposed an optimization problem which involves minimization over a family of stochastic integrals and maximization over the family of stopping times.

In this paper, we first formulate the notion of BSDEs with weak reflections, whose constraint takes the following general form

$$
\mathbb{E}_\tau[\Psi(\theta, Y^Z_{\theta})] \geq \mu, \text{ for all stopping time } \theta \in [\tau, T],
$$

(1.5)

where $\mu \in \mathcal{L}^2(\mathcal{F}_\tau)$ is the target success ratio at a given stopping date $\tau$ and $\Psi$ is a possibly random non-decreasing map. This representation allows of course to encompass the efficient pricing of American options presented above. We first observe that the minimal solution to this BSDE rewrites as the infimum over a family of solutions to classical reflected BSDEs with appropriate obstacle, that is $\inf\{Y^\alpha_0, \alpha \in V_0\}$, where $(Y^\alpha_t)$ is the first composant of
the solution of the reflected BSDE associated with obstacle \( \Phi(t, M^\alpha_t) \), with \( M^\alpha \) a martingale process. As it is usual in stochastic control, we study the dynamical counterpart

\[
\mathcal{Y}^\alpha(\tau) = \text{ess inf}\{Y_{\tau}^{\alpha'}, \alpha' \in V_0 \text{ s.t. } \alpha' = \alpha \text{ on } [0, \tau]\}. \tag{1.6}
\]

We derive a dynamic programming principle for this family, from which we deduce that the value \( (\mathcal{Y}^\alpha) \) is a \( \text{Ref}^{\gamma, \alpha} \)-submartingale family, where \( \text{Ref}^{\gamma, \alpha} \) is the nonlinear operator induced by the solution of the lower reflected BSDE with obstacle \( \Phi(t, M^\alpha_t) \) and driver \( g \). Using some fine results from the general theory of processes, we show that the value family \( (\mathcal{Y}^\alpha) \) can be aggregated by a right-continuous and left limited process \( (\mathcal{Y}^\alpha_t) \). Moreover, we show that any strong \( \mathcal{R}^{\gamma, \alpha} \)-submartingale admits a \( \mathcal{E}^\gamma \)-Mertens decomposition, which to the best of our knowledge, represents a new result in the literature. We propose an original proof, which does not use the classical penalization approach. Taking advantage of this decomposition, we show that, for each \( \alpha \), the value process \( (\mathcal{Y}^\alpha_t) \) has a backward SDE representation. Moreover, \( (\mathcal{Y}^\alpha_t) \) corresponds to the upper value of a stochastic control/optimal stopping game, which is shown to admit, under appropriate assumptions, a value and a saddle point.

The outline of the paper is the following. After presenting notations, we introduce the definition of supersolution of \textit{BSDE with weak reflections} in Section 2. In Section 3, we specialize our discussion to the minimal supersolution of the \textit{BSDE with weak reflections}. We first prove a dynamic programming principle and that we can aggregate the value family by a càdlàg process. In this Section we also provide a nonlinear Mertens decomposition of \( \text{Ref}^{\gamma, \xi} \)-submartingale processes, which is then used in order to give a representation of the value process. In Sections 4, we study a related stochastic control/optimal stopping problem, which is shown to admit a value and a saddle point.

**Notations** We first introduce a series of notations that will be used throughout the paper. Let \( d \geq 1 \) and \( T > 0 \) be fixed. We denote by \( W := (W_t)_{t \in [0, T]} \) a \( d \)-dimensional Brownian motion defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( \mathbb{P} \)-augmented natural filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]} \). The notation \( \mathbb{E} \) will stand for the expectation with respect to \( \mathbb{P} \). Hereafter, we define the following spaces:

- \( \mathbb{L}^p(U, \mathcal{G}) \) is the set of \( p \) integrable \( \mathcal{G} \)-measurable random variables with values in \( U \), \( p \geq 0 \), \( U \) a Borel set of \( \mathbb{R}^n \) for some \( n \geq 1 \) and \( \mathcal{G} \subset \mathcal{F} \). When \( U \) and \( \mathcal{G} \) can be clearly identified by the context, we omit them. This will be in particular the case when \( \mathcal{G} = \mathcal{F} \).

- \( \mathbb{H}_2 \) is the set of \( \mathbb{R}^d \)-valued \( \mathbb{P} \)-predictable processes \( \phi = (\phi_t)_{t \in [0, T]} \) such that

\[
\|\phi\|_{\mathbb{H}_2}^2 := \mathbb{E} \left[ \int_0^T |\phi_t|^2 \, dt \right] < \infty.
\]

- \( \mathbb{S}_2 \) is the set of real-valued optional processes \( \phi = (\phi_t)_{t \in [0, T]} \) such that

\[
\|\phi\|_{\mathbb{S}_2}^2 := \mathbb{E}[\text{ess sup}_{0 \leq \tau \leq T} |\phi_\tau|^2] < \infty.
\]
• $K_2$ is the set of real-valued non decreasing RCLL and $\mathbb{F}$-predictable processes $K = (K_t)_{t \in [0, T]}$ with $K_0 = 0$ and $\mathbb{E}[K_T^2] < \infty$.

• $T_0$ denotes the set of $\mathbb{F}$-stopping times $\tau$ such that $\tau \in [0, T]$ a.s. The notation $\mathbb{E}_\tau[.]$ stands for the conditional expectation given $\mathcal{F}_\tau$, $\tau \in T_0$.

• For $\theta$ in $T_0$, $T_\theta$ is the set of stopping times $\tau \in T_0$ such that $\theta \leq \tau \leq T$ $\mathbb{P}$-a.s.

2 BSDEs with weak reflections

2.1 Definition and Assumptions

Let us introduce the new mathematical object.

**Definition 2.1 (BSDEs with weak reflections)** Given a measurable map $\Psi : [0, T] \times \mathbb{R} \times \Omega \to U$, with $U \subset \mathbb{R} \cup \{-\infty\}$ and $\mu \in L_0(U, \mathcal{F}_T)$, we say that $(Y, Z) \in S_2 \times H_2$ is a supersolution of the BSDE with generator $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and weak reflections if for any $0 \leq t \leq s \leq T$,

$$Y_t \geq Y_s + \int_t^s g(s, Y_s, Z_s)ds - \int_t^s Z_s dW_s$$

(2.7)

$$\mathbb{E}_\tau[\Psi(\theta, Y_\theta)] \geq \mu \text{ for all } \theta \in T_\tau.$$ 

(2.8)

We would like to emphasize that the terminology BSDEs with weak reflections is due to the fact that, given a stopping time $\tau \in T_0$ and a threshold $\mu \in L_0$, the first component of the solution of the above BSDE, denoted here by $(Y_t)$, satisfies the condition $\mathbb{E}_\tau[\Psi(\theta, Y_\theta)] \geq \mu$ for all $\theta \in T_\tau$. The wellposedness of this BSDE is discussed in Remark 2.1.

Throughout the paper, we assume that $g$ satisfies

**Assumption 2.1** $g$ is a measurable map from $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$ to $\mathbb{R}$ and $g(., y, z)$ is $\mathbb{F}$-predictable, for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. There exists a constant $K_g > 0$ and a random variable $\chi_g \in L_2(\mathbb{R}^\times)$, such that

$$|g(t, 0, 0)| \leq \chi_g \; \mathbb{P} - \text{a.s.}$$

$$|g(t, y_1, z_1) - g(t, y_1, z_2)| \leq K_g(|y_1 - y_2| + |z_1 - z_2|) \; \mathbb{P} - \text{a.s.}$$

$$\forall (t, y_i, z_i) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d, i = 1, 2.$$

We also recall the definition of the conditional $g$-expectation.

**Definition 2.2 (Conditional $g$-expectation)** We recall that if $g$ is a Lipschitz driver and if $\xi$ is a square-integrable $\mathcal{F}_T$-measurable random variable, then there exists a unique solution $(X, \pi) \in S_2 \times H_2$ to the following BSDE

$$X_t + \int_t^T g(s, X_s, \pi_s)ds - \int_t^T \pi_s dW_s \text{ for all } t \in [0, T] \; \mathbb{a.s.}$$
For \( t \in [0, T] \), the non-linear operator \( \mathcal{E}_{t,T}^g : L^2(F_T) \rightarrow L^2(F_t) \) which maps a given terminal condition \( \xi \in L^2(F_t) \) to the first component at time \( t \) of the solution of the above BSDE (denoted by \( X_t \)) is called conditional \( g \)-expectation at time \( t \). It is also well-known that this notion can be extended to the case where the (deterministic) terminal time \( T \) is replaced by a general stopping time \( \tau \in T_0 \) and \( t \) is replaced by a stopping time \( S \) such that \( S \leq \tau \) a.s.

We now give the assumption on the map \( \Psi \).

**Assumption 2.2** For \( \text{Leb} \times d\mathbb{P} \)-a.e. \( (t, \omega) \in [0, T] \times \Omega \), the map \( y \in \mathbb{R} \mapsto \Psi(t, \omega, y) \) is nondecreasing, right-continuous, valued in \([0, 1] \cup \{-\infty\}\) and its left-continuous inverse \( \Phi(t, \omega, \cdot) \) satisfies \( \Phi : [0, T] \times \Omega \times [0, 1] \rightarrow [0, 1] \) is measurable.

By left-continuous inverse we mean the left-continuous map defined for \((t, \omega)\) fixed by

\[
\Phi(t, \omega, x) := \inf\{y \in \mathbb{R}, \Psi(t, \omega, y) \geq x\}
\]

which satisfies

\[
\Phi(t, \omega, \Psi(t, \omega, x)) \leq x \leq \Psi(t, \omega, \Phi(t, \omega, x)).
\]

**Remark 2.1** Let us discuss the wellposedness of the BSDE with weak reflections. Let \( \xi \) be a square integrable \( F_T \)-measurable random variable such that \( E_\tau[\xi] = \mu \) a.s. Due to the martingale representation theorem, there exists \( \beta \in H_2 \) such that \( M^\beta_T = \mu + \int_T^T \beta_s dW_s \). The solution \((Y^\beta_t, Z^\beta_t)\) of the reflected BSDE associated with the driver \( g \) and obstacle \( \Phi(t, M^\beta_t) \) (which exists under the above assumptions) is a supersolution of the BSDE with weak reflections. Note that, due to the weak constraints, we do not have uniqueness of the solution.

We now introduce the set \( \Theta(\tau, \mu) \) of \((\tau, \mu)\)-initial supersolutions, which is defined as follows:

\[
\Theta(\tau, \mu) := \{Y_\tau : (Y, Z) \text{ is a supersolution of } (2.7) \text{ and } (2.8)\}.
\]

The aim of this paper is to study the lower bound of the set \( \Theta(\tau, \mu) \), that is \( \text{ess inf} \Theta(\tau, \mu) \). We would like to emphasize once again the relation between this quantity and the price of an American corresponding to an approximative hedging, under the risk constraint \( E_\tau[\Psi(\theta, Y_\theta)] \geq \mu \) a.s., for all \( \theta \in T_\tau \).

### 2.2 Equivalent reformulation with "strong" constraints

Our main purpose now is to show that we can reformulate the problem into an equivalent one with "strong" constraints, similar to the case of the partial hedging problem for European options ( we refer to Bouchard, Elie, Touzi [4] in the Markovian framework or Bouchard, Elie, Reveillac [3] in the Non-Markovian setting).

For this, let \( V_{\tau,\mu} \) denote the set elements \( \alpha \in H_2 \) such that

\[
M^{\tau,\mu,\alpha}_T := \mu + \int_\tau^T \alpha_s dW_s \quad \text{takes values in } [0, 1].
\]

The main difficulty in our case is represented by the fact that, a priori, we can obtain an equivalent formulation in which the controlled martingale depends on the stopping time \( \theta \),
that is for each \( \theta \in \mathcal{T} \) there exists \( \alpha^\theta \in H_2 \) such that \( \mathbf{E}_\tau[\Phi(\theta, Y_\theta)] \geq \mu \) is equivalent to \( Y_\theta \geq \Phi(\theta, M^{\tau,\alpha}_\theta) \) a.s.

We see in the next Lemma that we can overcome this issue and obtain the existence of a controlled martingale independent on the stopping time \( \theta \).

**Lemma 2.3** Let \((Y_t)\) be an optional process belonging to \( S_2 \) and satisfying (2.7)-(2.8), \( \tau \) a stopping time belonging to \( T_0 \) and \( \mu \) a \( F_\tau \)-measurable random variable. Then the condition \( \mathbf{E}[\Phi(\theta, Y_\theta)|F_\tau] \geq \mu \) for all \( \theta \in \mathcal{T}_\tau \) is equivalent to the existence of a control \( \alpha \in V_{\tau,\mu} \) such that \( Y_\theta \geq \Phi(\theta, M^{\tau,\mu,\alpha}_\theta) \) a.s. for all \( \theta \in \mathcal{T}_\tau \).

**Proof.** For each \( \sigma \in T_0 \), we define the \( F_\sigma \)-measurable random variable

\[
V(\sigma) := \essinf_{\tau \in \mathcal{T}_\sigma} \mathbf{E}[\Psi(\tau, Y_\tau)|F_\sigma].
\]

By classical results of the general theory of processes, the family \((V(\sigma), \sigma \in T_0)\) is a submartingale family, which can be aggregated by an optional process \((V_t)\) admitting the Mertens decomposition:

\[
V_t := N_t + A_t + C_t,
\]

where \( N \) is a square integrable martingale, \( A \) is an increasing RCLL predictable process such that \( A_0 = 0 \) and \( C \) is a right-continuous adapted process, purely discontinuous satisfying \( C_{0^-} = 0 \).

Let us first show the first implication, that is:

\[
\mathbf{E}[\Psi(\theta, Y_\theta)|F_\tau] \geq \mu \quad \text{for all } \theta \geq \tau \implies \text{existence of a control } \alpha \in H_2 \quad \text{such that } \quad Y_\theta \geq \Phi(\theta, M^{\tau,\mu,\alpha}_\theta) \quad \text{a.s. for all } \theta \in \mathcal{T}_\tau.
\]

Since for all \( \theta \in \mathcal{T}_\tau \) we have \( \mathbf{E}[\Psi(\theta, Y_\theta)|F_\tau] \geq \mu \) a.s., we get that \( \essinf_{\theta \geq \tau} \mathbf{E}[\Psi(\theta, Y_\theta)|F_\tau] \geq \mu \) a.s. Hence, by using the definition of \( V \) (see (2.10)), we obtain

\[
V_\tau = N_\tau + A_\tau + C_{\tau^-} \geq \mu \quad \text{a.s.}
\]

We now fix \( \theta \geq \tau \). We have \( \Psi(\theta, Y_\theta) = \mathbf{E}[\Psi(\theta, Y_\theta)|F_\theta] \geq \essinf_{\sigma \geq \theta} \mathbf{E}[\Psi(\sigma, Y_\sigma)|F_\theta] = V_\theta \) a.s. This observation together with (2.11) imply

\[
\Psi(\theta, Y_\theta) \geq N_\theta + A_\theta + C_{\theta^-} = N_\tau + A_\tau + C_{\tau^-} + \int_\tau^\theta \alpha_s dW_s + A_\theta - A_\tau + C_\theta - C_{\tau^-}.
\]

Using the above inequality, (2.11) and the fact the processes \( A \) and \( C \) are non-decreasing, we obtain

\[
\Psi(\theta, Y_\theta) \geq M^{\tau,\mu,\alpha}_\theta \quad \text{a.s.}
\]

By applying now the map \( \Phi \) which is non-decreasing in its last variable, we finally derive

\[
Y_\theta \geq \Phi(\theta, M^{\tau,\mu,\alpha}_\theta) \quad \text{a.s.}
\]

The second implication is trivial.

\[\Box\]

Let us show the following result, which will be crucial in the sequel.
Proposition 2.4  Fix $\tau \in \mathcal{T}_0$, $\mu \in \mathcal{L}_0([0,1],\mathcal{F}_\tau)$. Then $(Y,Z) \in \mathbf{S}_2 \times H_2$ is a solution of the BSDE (2.7)-(2.8) if and only if $(Y,Z)$ satisfies (2.7) and there exists $\alpha \in \mathbf{V}_{\tau,\mu}$ such that $Y_\nu \geq \operatorname{ess sup}_{\theta \in \mathcal{T}_\nu} \mathcal{E}_{\nu,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)]$ a.s. for all $\nu \in \mathcal{T}_\tau$.

Proof. Let $(Y,Z)$ be a supersolution of BSDE (2.7)-(2.8). Then by Lemma 2.3, there exists $\tilde{\alpha} \in \mathbf{V}_{\tau,\mu}$ such that for all $\theta \in \mathcal{T}_\tau$ we have $\Psi(\theta, Y_\theta) \geq M^{\tau,\mu,\tilde{\alpha}}_\theta$. We now define $\theta^\alpha := \inf\{s \geq \tau, M^{\tau,\mu,\tilde{\alpha}}_s = 0\}$. Let us introduce the control $\alpha := \tilde{\alpha} 1_{[0,\theta^\alpha]}$, which clearly belongs to $\mathbf{V}_{\tau,\mu}$. Let us fix $\nu \in \mathcal{T}_\tau$. One can remark that for all $\theta \in \mathcal{T}_\nu$ we have $\Psi(\theta, Y_\theta) \geq M^{\tau,\mu,\bar{\alpha}}_\theta$ a.s. The monotonocity of the map $\Phi$ and the above inequality imply that:

$$Y_\theta \geq \Phi(\theta, M^{\tau,\mu,\bar{\alpha}}_\theta) \quad \text{a.s.}$$

By the comparison theorem for BSDEs, we get that for all $\theta \in \mathcal{T}_\nu$, we have $Y_\nu \geq \mathcal{E}^g_{\nu,\theta}[\Phi(\theta, M^{\tau,\mu,\bar{\alpha}}_\theta)]$ a.s. Now, by arbitrariness of $\theta \in \mathcal{T}_\nu$ we finally obtain:

$$Y_\nu \geq \mathcal{E}^g_{\nu,\theta}[\Phi(\theta, M^{\tau,\mu,\bar{\alpha}}_\theta)] \quad \text{a.s.} \quad (2.12)$$

Let us show the converse implication. For all $\nu \in \mathcal{T}_\tau$, we have $Y_\nu \geq \Phi(\nu, M^{\mu,\bar{\alpha}}_\nu)$ a.s. Hence we get $\Psi(\nu, Y_\nu) \geq M^{\tau,\mu,\bar{\alpha}}_\nu$ a.s. This implies that $(Y,Z)$ satisfies (2.7) and (2.8). □

Using the above results, we show in the following proposition how to relate the lower bound of the family $\Theta(\tau, \mu)$ to the value of a stochastic control/optimal stopping game. To this aim, we define the value function

$$Y(\tau, \mu) := \operatorname{ess inf}_{\alpha \in \mathbf{V}_{\tau,\mu}} \operatorname{ess sup}_{\theta \in \mathcal{T}_\nu} \mathcal{E}_{\tau,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)]. \quad (2.13)$$

Proposition 2.5 We have $\operatorname{ess inf} \Theta(\tau, \mu) = Y(\tau, \mu)$ a.s.

Proof. Let $Y_\tau \in \Theta(\tau, \mu)$. By Proposition 2.4, we obtain that $Y_\tau \geq \operatorname{ess sup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}_{\tau,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)]$ a.s., which clearly implies that

$$Y_\tau \geq \operatorname{ess inf}_{\alpha \in \mathbf{V}_{\tau,\mu}} \operatorname{ess sup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}_{\tau,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)] = Y(\tau, \mu) \quad \text{a.s.}$$

By arbitrariness of $Y_\tau$, we derive that $\operatorname{ess inf} \Theta(\tau, \mu) \geq Y(\tau, \mu) \quad \text{a.s.}$

Conversely, we have that $\operatorname{ess sup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}_{\tau,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)]$ belongs to $\Theta(\tau, \mu)$, which leads to

$$\operatorname{ess sup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}_{\tau,\theta}^g[\Phi(\theta, M^{\tau,\mu,\alpha}_\theta)] \geq \operatorname{ess inf} \Theta(\tau, \mu) \quad \text{a.s.}$$

By taking the essential infimum on $\alpha \in \mathbf{V}_{\tau,\mu}$, the result follows. □

In the sequel, we assume that the map $\Phi$ is continuous with respect to $t$ and $m$.

We now introduce the nonlinear operator $\mathcal{R}e^g_{\tau,\xi}$ defined through the solution of a nonlinear reflected BSDE with driver $g$ and lower obstacle $\xi$. 

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Definition 2.6 (The nonlinear operator $\text{Ref}^{g,\xi}$) Let $g$ be a Lipschitz driver and $\xi$ a RCLL process belonging to $S_2$. Let $L^{\xi,T}_2$ be the set of random variables $\zeta$ included in $L^2(F_T)$ such that $\zeta \geq \xi_T$ a.s. Then there exists a unique solution $(Y, Z, A) \in S_2 \times H_2 \times S_2$ to the following lower reflected BSDE

$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + A_T - A_t \text{ for all } t \in [0, T] \text{ a.s.}$$

$$Y_t \geq \xi_t \text{ a.s., } 0 \leq t \leq T$$

$$Y_T = \zeta \text{ a.s.}$$

$$\int_0^T (Y_s - \xi_s^-)dA_s = 0 \text{ a.s.}$$

For $t \in [0, T]$, the non-linear operator $\text{Ref}^{g,\xi}_{t,T}: L^{\xi,T}_2 \mapsto L^{\xi,t}_2$ is defined as follows:

$$\text{Ref}^{g,\xi}_{t,T}[\zeta] := Y_t,$$

where $Y$ is the first component at time $t$ of the solution of the above Reflected BSDE. This notion can be extended to the case where the (deterministic) terminal time $T$ is replaced by a general stopping time $\tau \in T_0$ and $t$ is replaced by a stopping time $S$ such that $S \leq \tau$ a.s.

Remark 2.7 Note that, due to the flow property of reflected BSDEs, the nonlinear operator $\text{Ref}^{g,\xi}$ is consistent.

Using the characterization of the first component of the solution of a nonlinear reflected BSDE as the value of an optimal stopping with nonlinear BSDEs, we obtain that $\mathcal{Y}(\tau, \mu)$ can be rewritten as follows:

$$\mathcal{Y}(\tau, \mu) = \text{ess inf} \text{Ref}^{g,\Phi^\alpha}_{\tau,T}[\Phi(T, M^{\tau,\mu,\alpha}_T)],$$

where $\Phi^\alpha$ corresponds to the obstacle process $\Phi(t, M^{\tau,\mu,\alpha}_t)$.

3 Properties and representation of the value family

In this section, we focus our study on $\mathcal{Y}(\tau, \mu)$ which is the lower bound of the set $\Theta(\tau, \mu)$. For ease of notations, we fix $m_0 \in [0, 1]$ and set

$$\begin{cases} M_t := M^{0,m_0,\alpha}_t, & V^\alpha_t := \{\alpha' \in V_{\tau,M^\alpha_t} : \alpha' = \alpha \text{ d}t \times dP \text{ on } [0, \tau] \}, \\
V_0 := V_{0,m_0} \text{ and } \gamma^\alpha(\tau) := \mathcal{Y}(\tau, M^\alpha_t) \text{ for } \alpha \in V_0, t \in [0, T] \text{ and } \tau \in T_0. \end{cases} \quad (3.14)$$

3.1 Properties of the value family

Let us first recall the definition of a $T_0$-admissible system.

Definition 3.1 A family $S = \{S(\tau), \tau \in T_0\}$ is admissible (or a $T_0$-system) if for all $\tau, \tau' \in T_0$

$$\begin{cases} S(\tau) \in L^0(F_\tau), \\
S(\tau) = S(\tau') \text{ a.s. on } \{\tau = \tau'\}. \end{cases} \quad (3.15)$$
Lemma 3.2 (Admissibility of the family $(\mathcal{Y}^{\alpha}(\tau))_{\tau \in \mathcal{T}}$) The family $(\mathcal{Y}^{\alpha}(\tau))_{\tau \in \mathcal{T}}$ is a square-integrable admissible family.

Proof. For each $S \in \mathcal{T}$, $\mathcal{Y}^{\alpha}(S)$ is an $\mathcal{F}_S$-measurable square-integrable random variable, due to the definitions of the conditional $g$-expectation and of the essential supremum and essential infimum. Let $S$ and $S'$ be two stopping times in $\mathcal{T}$. We set $B := \{S = S'\}$. We show that $\mathcal{Y}^{\alpha}(S) = \mathcal{Y}^{\alpha}(S')$ a.s. on $B$. Set $\theta_B := \theta 1_B + T 1_{B^c}$. We clearly have $\theta_B \in \mathcal{T}_{S'}$ and moreover $\theta_B = \theta$ a.s. on $B$, for all $\theta \in \mathcal{T}_S$. We also fix $\alpha' \in V^0_{S'}$ and set $\alpha'_B := \alpha 1_{[0,\theta]} + \alpha' 1_{\{S,T\}}1_B$. Clearly $\alpha'_B \in V^0_{S}$ and $\alpha'_B = \alpha'$ on $[S',T]$ on $B$. By using the fact that $S = S'$ on $B$, as well as several properties of the $g$-expectation, we obtain

$$1_B E^g_{S,\theta}[\Phi(\theta, M^{\alpha'}_\theta)] = 1_B E^g_{S',\theta}[\Phi(\theta, M^{\alpha'}_\theta)] = E^g_{S,\theta}[1_B \Phi(\theta, M^{\alpha'}_\theta)] = E^g_{S',\theta}[1_B \Phi(\theta, M^{\alpha'}_\theta)]$$

By taking the essential supremum on $\theta \in \mathcal{T}_S$ and then the essential infimum on $\alpha' \in V^0_{S'}$, we get $\mathcal{Y}^{\alpha}(S) \leq \mathcal{Y}^{\alpha}(S')$ a.s. By interchanging the roles of $S$ and $S'$, the converse inequality follows by the same arguments.

We now prove the existence of an optimizing sequence.

Lemma 3.3 Fix $\tau \in \mathcal{T}$, $\theta \in \mathcal{T}$, $m \in \mathcal{L}_0([0,1], \mathcal{F}_\tau)$ and $\alpha \in V_{\tau,\mu}$. Then there exists a sequence $(\alpha'_n) \subset V^0_{\tau,\mu} := \{\alpha' \in V_{\tau,\mu} : \alpha' 1_{[0,\theta]} = \alpha 1_{[0,\theta]}\}$ such that $\lim_{n \to \infty} \downarrow \text{Ref}^{g,\Phi_{\alpha}}_{\theta,\tau} [\Phi(T, M^{\tau,m,\alpha'}_T)] = \mathcal{Y}(\theta, M^{\tau,m,\alpha}_\theta)$ a.s.

Proof. In order to prove the result, we have to show that the family $\{ J(\alpha') := \text{Ref}^{g,\Phi_{\alpha'}}_{\theta,\tau} [\Phi(T, M^{\tau,m,\alpha'}_T)] , \alpha' \in V^0_{\tau,\mu} \}$ is direct downward. Fix $\alpha'_1, \alpha'_2 \in V^0_{\tau,\mu}$ and set $\alpha' := \alpha 1_{[0,\theta]} + 1_{[\theta,\tau]}(\alpha'_1 1_A + \alpha'_2 1_{A^c})$, where $A := \{J(\alpha'_1) \leq J(\alpha'_2)\} \in \mathcal{F}_\theta$, which implies that $\alpha' \in V^0_{\tau,\mu}$ and, since $A \in \mathcal{F}_\theta$,

$$J(\alpha') = \text{Ref}^{g,\Phi_{\alpha'}}_{\theta,\tau} [\Phi(T, M^{\tau,m,\alpha'}_T)1_A + \Phi(T, M^{\tau,m,\alpha'_2}_T)1_{A^c}]$$

$$= 1_A \text{Ref}^{g,\Phi_{\alpha'}}_{\theta,\tau} [\Phi(T, M^{\tau,m,\alpha'}_T)] + 1_{A^c} \text{Ref}^{g,\Phi_{\alpha'_2}}_{\theta,\tau} [\Phi(T, M^{\tau,m,\alpha'_2}_T)]$$

$$= \min(J(\alpha'_1), J(\alpha'_2)).$$

This gives the desired result.

Let us now introduce the notion of $\text{Ref}^{g,\xi}$-submartingale system (resp. a $\text{Ref}^{g,\xi}$-martingale system).

Definition 3.4 An admissible family $(X(\tau), \tau \in \mathcal{T})$ is said to be a $\text{Ref}^{g,\xi}$-submartingale family (resp. a $\text{Ref}^{g,\xi}$-martingale family) if for each $\tau \in \mathcal{T}$, $X(\tau) \in \mathcal{L}^{g,\xi}_2$ and if, for all $\tau, \sigma \in \mathcal{T}$ such that $\sigma \in \mathcal{T}_\tau$ a.s.,

$$X(\tau) \leq \text{Ref}^{g,\xi}_{\tau,\sigma} [X(\sigma)] \text{ a.s.} \quad (\text{resp. } X(\tau) = \text{Ref}^{g,\xi}_{\tau,\sigma} [X(\sigma)] \text{ a.s.})$$

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We now proceed to show that for each $\alpha \in V_0$, the family $(\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}_0)$ is a $\text{Ref}^{g, \Phi^\alpha}$-submartingale family. This a direct consequence of the following dynamic programming principle.

**Theorem 3.5 (Dynamic programming principle)** The value family satisfies the following dynamic programming principle: for all $(\tau_1, \tau_2, \alpha) \in \mathcal{T}_0 \times \mathcal{T}_0 \times V_0$ such that $\tau_1 \leq \tau_2$, we have:

$$
\mathcal{Y}^\alpha(\tau_1) = \text{ess inf}_{\alpha' \in V^\alpha_{\tau_1}} \text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} [\mathcal{Y}^\alpha(\tau_2)] \text{ a.s.}
$$

**(Proof.**

Let us first show that

$$
\mathcal{Y}^\alpha(\tau_1) \geq \text{ess inf}_{\alpha' \in V^\alpha_{\tau_1}} \text{ess sup}_{\alpha' \in V^\alpha_{\tau_1}, \theta \in \mathcal{T}_{\tau_1}} \mathcal{E}^\theta_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \mathbf{1}_{\theta \geq \tau_2} + \Phi(\theta, M^\alpha_{\tau_2}) \mathbf{1}_{\theta < \tau_2} \right] \text{ a.s.}
$$

Fix $\alpha' \in V^\alpha_{\tau_1}$. By the flow property for Reflected BSDEs we obtain:

$$
\text{Ref}^{g, \Phi^\alpha}_{\tau_1, T} \left[ \Phi(T, M^\alpha_T) \right] = \text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \text{Ref}^{g, \Phi^\alpha}_{\tau_2, T} \left[ \Phi(T, M^\alpha_T) \right] \right] \text{ a.s.}
$$

By the comparison theorem for Reflected BSDEs, we get:

$$
\text{Ref}^{g, \Phi^\alpha}_{\tau_1, T} \left[ \Phi(T, M^\alpha_T) \right] \geq \text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \right] \text{ a.s.}
$$

By arbitrariness of $\alpha' \in V^\alpha_{\tau_1}$, we finally obtain:

$$
\mathcal{Y}^\alpha(\tau_1) \geq \text{ess inf}_{\alpha' \in V^\alpha_{\tau_1}} \text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \right] \text{ a.s.}
$$

Conversely, we prove that

$$
\mathcal{Y}^\alpha(\tau_1) \leq \text{ess inf}_{\alpha' \in V^\alpha_{\tau_1}} \text{ess sup}_{\alpha' \in V^\alpha_{\tau_1}, \theta \in \mathcal{T}_{\tau_1}} \mathcal{E}^\theta_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \mathbf{1}_{\theta \geq \tau_2} + \Phi(\theta, M^\alpha_{\tau_2}) \mathbf{1}_{\theta < \tau_2} \right].
$$

Let $\alpha^\prime \in V^\alpha_{\tau_2}$ such that:

$$
\mathcal{Y}^{\alpha'}(\tau_2) = \lim_{n \to \infty} \text{Ref}^{g, \Phi^{\alpha^\prime}}_{\tau_2, T} \left[ \Phi(T, M^\alpha_T) \right] \text{ a.s.}
$$

The continuity of the reflected BSDEs with respect to its terminal condition gives:

$$
\text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \right] = \lim_{n \to \infty} \text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \text{Ref}^{g, \Phi^{\alpha^\prime}}_{\tau_2, T} \left[ \Phi(T, M^\alpha_T) \right] \right] \text{ a.s.}
$$

We set:

$$
\tilde{\alpha}^n_s := \alpha_s \mathbf{1}_{s < \tau_2} + \alpha^n_s \mathbf{1}_{s \geq \tau_2}.
$$

The two above relations and the consistency of the operator $\text{Ref}^{g, \Phi^\alpha}$ finally give:

$$
\text{Ref}^{g, \Phi^\alpha}_{\tau_1, \tau_2} \left[ \mathcal{Y}^{\alpha'}(\tau_2) \right] = \lim_{n \to \infty} \text{Ref}^{g, \Phi^{\alpha^n}}_{\tau_1, \tau_2} \left[ \Phi(T, M^\alpha_T) \right] \geq \mathcal{Y}^\alpha(\tau_1) \text{ a.s.}
$$

Now, by arbitrariness of $\alpha' \in V^\alpha_{\tau_1}$, the result follows. □
3.2 Aggregation results and $\mathcal{E}^g$-Mertens decomposition of $\text{Ref}^{g,\xi}$-submartingales

We now aim at proving that for each $\alpha \in \mathbf{V}_0$, the family $(\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}_0)$ can be aggregated by an optional process, that is it exists an optional process $(Y^\alpha_\tau)$ such that, for all stopping time $\tau \in \mathcal{T}_0$, it holds $\mathcal{Y}^\alpha(\tau) = Y^\alpha_\tau$ a.s. The existence of such a process is in general a delicate question and, so far, it has only be addressed in the case of $\mathcal{E}^g$-(super)submartingales. We thus show that this result can be extended to the case of $\text{Ref}^{g,\Phi}$-submartingales, with an operator $\text{Ref}^{g,\Phi}$ induced by the first composant of the solution of a lower reflected BSDE.

**Theorem 3.6 (Aggregation of the value family by an optional process)** For any $\alpha \in \mathbf{V}_0$, there exists an optional process $(Y^\alpha_\tau)$ which aggregates the family $(\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}_0)$, that is $\mathcal{Y}^\alpha(\tau) = Y^\alpha_\tau$ a.s., for all $\tau \in \mathcal{T}_0$.

**Proof.** Fix $\alpha \in \mathbf{V}_0$. Let $(\tau_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of stopping times such that $\tau_n \downarrow \tau$ a.s. The definition of $\mathcal{Y}^\alpha$ implies that

$$\mathcal{Y}^\alpha(\tau) \leq \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)] \quad \text{a.s., for all } n \in \mathbb{N}. \quad (3.19)$$

The nondecreasingness of the sequence $(\tau_n)_n$ together with the consistency of the operator $\text{Ref}^{g,\Phi^\alpha}$ yield

$$\text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)] = \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_{n+1}}[\text{Ref}^{g,\Phi^\alpha}_{\tau_{n+1},\tau_n}[\mathcal{Y}^\alpha(\tau_n)]] \geq \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_{n+1}}[\mathcal{Y}^\alpha(\tau_{n+1})] \quad \text{a.s.,}$$

where the last inequality follows by (3.19).

This implies that the sequence $(\text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)])_{n \in \mathbb{N}}$ is nondecreasing and thus it converges almost surely. Moreover,

$$\mathcal{Y}^\alpha(\tau) \leq \lim_{n \to \infty} \downarrow \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)] \quad \text{a.s.} \quad (3.20)$$

By Lebesgue’s theorem we have

$$\mathbb{E}[\mathcal{Y}^\alpha(\tau)] \leq \lim_{n \to \infty} \downarrow \mathbb{E}[\text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)]] \quad (3.21)$$

Now, since $\limsup_{n \to \infty} \mathcal{Y}^\alpha(\tau_n) \geq \Phi(\tau,M^\alpha_{\tau})$ a.s., we can apply the Fatou lemma for Reflected BSDEs (see Proposition 3.13 in [12]). We therefore obtain

$$\mathbb{E}[\mathcal{Y}^\alpha(\tau)] \leq \mathbb{E}[\limsup_{n \to \infty} \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)]] \leq \mathbb{E}[\limsup_{n \to \infty} \text{Ref}^{g,\Phi^\alpha}_{\tau,\tau_n}[\mathcal{Y}^\alpha(\tau_n)]] = \mathbb{E}[\limsup_{n \to \infty} \mathcal{Y}^\alpha(\tau_n)]. \quad (3.22)$$

This implies that the family $(-\mathcal{Y}^\alpha(\tau_n))_{n \in \mathbb{N}}$ satisfies

$$\mathbb{E}[-\mathcal{Y}^\alpha(\tau)] \geq \mathbb{E}[\lim_{n \to \infty} \inf (-\mathcal{Y}^\alpha(\tau_n))]. \quad (3.23)$$

Since the family $(-\mathcal{Y}^\alpha(\tau), \tau \in \mathcal{T}_0)$ is uniformly integrable, Theorem 12 in [8] gives the existence of an optional process $(Y^\alpha_\tau)$ such that $\mathcal{Y}^\alpha(\tau) = Y^\alpha_\tau$ a.s. for all $\tau \in \mathcal{T}_0$. Moreover, by the same Theorem, this process is right lower semicontinuous.

Let us introduce the notion of strong $\text{Ref}^{g,\xi}$-submartingale process.
Definition 3.7 (Strong Ref\(^g\)-\(\xi\)-submartingale process) An optional process \((Y_t)\) satisfying \(Y_\sigma \geq \xi_\sigma\) a.s. for all \(\sigma \in \mathcal{T}_0\) and such that \(E[\text{ess sup}_{\tau \in \mathcal{T}_0} Y_\tau^2] < \infty\) is said to be a strong Ref\(^g\)-\(\xi\)-submartingale if \(Y_S \leq \text{Ref}_{S,T} g,\xi [Y_T]\) a.s. on \(S \leq \tau\), for all \(S, \tau \in \mathcal{T}_0\).

We now show a \(\mathcal{E}g\)-Mertens decomposition of r.l.s.c. \(\text{Ref}\(^g\)-\(\xi\)-submartingales in the case of a r.u.s.c. obstacle, which represents, to the best of our knowledge, a new result in the literature. Moreover, our proof is simple, being based on some recent results on the theory of optimal stopping with \(g\)-conditional expectations.

Theorem 3.8 (\(\mathcal{E}g\)-Mertens decomposition of \(\text{Ref}\(^g\)-\(\xi\)-submartingales) Let \((Y_t)\) be a right lower semicontinuous process such that \(E[\text{ess sup}_{\tau \in \mathcal{T}_0} (Y_\tau)^2] < \infty\) and \((\xi_t)\) a right upper semicontinuous process such that \(E[\text{ess sup}_{\tau \in \mathcal{T}_0} (\xi_\tau)^2] < \infty\). The process \((Y_t)\) is a strong \(\text{Ref}\(^g\)-\(\xi\)-submartingale if and only if there exists two nondecreasing right-continuous predictable processes \(A, K \in S_2\) such that \(A_0 = 0\) and \(K_0 = 0\), two nondecreasing right-continuous adapted purely discontinuous processes \(C, C'\) in \(S_2\) with \(C_0 = 0\) and \(C'_0 = 0\) and a process \(Z \in \mathcal{H}_2\) such that a.s. for all \(t \in [0, T]\),

\[
Y_t = Y_T + \int_{\tau}^{T} g(s, Y_s, Z_s)ds + A_T - A_t + C_T - C_t - K_T + K_t - C'_T - C'_t, \quad (3.24)
\]

\(Y_t \geq \xi_t\) a.s., \(0 \leq t \leq T\).

\[
\int_{\tau}^{T} (Y_s - \xi_s)ds + (Y_\tau - \xi_\tau)(C_\tau - C_\tau) = 0 \text{ a.s. for all } \tau \in \mathcal{T}_0;
\]

\(dA_t \perp dK_t; \quad dC_t \perp dC'_t. \quad (3.25)\)

**Proof.** Fix \(S \in \mathcal{T}_0\). Since \((Y_t)\) is a strong \(\text{Ref}\(^g\)-\(\xi\)-submartingale, we derive that for each \(\tau \in \mathcal{T}_S\), we have \(Y_S \leq \text{Ref}_{S,T} g,\xi [Y_T]\) a.s. By definition of the operator \(\text{Ref}\(^g\)-\(\xi\)), thus we have \(Y_S \leq \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}g_{S', S' \wedge \tau} (Y_T 1_{S' \geq \tau} + \xi_{S'} 1_{S' < \tau})\).

By arbitrariness of \(\tau \in \mathcal{T}_0\), hence we get

\[
Y_S \leq \text{ess inf}_{\tau \in \mathcal{T}_S} \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}g_{S', S' \wedge \tau} (Y_T 1_{S' \geq \tau} + \xi_{S'} 1_{S' < \tau}) \quad \text{a.s.} \quad (3.26)
\]

Now, one can remark that we have

\[
Y_S = \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}g_{S, S' \wedge \tau} (Y_T 1_{S' \geq S} + \xi_{S'} 1_{S' > S}) \quad \text{a.s.}
\]

As \(S \in \mathcal{T}_S\), we deduce:

\[
Y_S \geq \text{ess inf}_{\tau \in \mathcal{T}_S} \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}g_{S', S' \wedge \tau} (Y_T 1_{S' \geq \tau} + \xi_{S'} 1_{S' \geq S}) \quad \text{a.s.} \quad (3.27)
\]

The inequalities (3.26) and (3.27) allow to conclude that

\[
Y_S = \text{ess inf}_{\tau \in \mathcal{T}_S} \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}g_{S, S' \wedge \tau} (Y_T 1_{S' \geq \tau} + \xi_{S'} 1_{\tau > S'}) \quad \text{a.s.}
\]

From the characterization theorem of the solution of a DRBSDE (associated with two obstacles supposed to be r.l.s.c., resp. r.u.s.c.) as the value function of a Generalized Dynkin
game (that is, $\bar{Y}_S = \text{ess inf}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S, \tau \wedge \sigma} [\xi_{\tau} \mathbf{1}_{\tau < \sigma} + \zeta_{\sigma} \mathbf{1}_{\sigma \leq \tau}]$, where $\bar{Y}$ is the first composant of the solution of the DRBSDE with driver $g$ and obstacles $(\xi_t)$ and $(\zeta_t)$, see Theorem 4.5 in [16]), we derive that the process $(Y_t)$ coincides with the solution of the doubly reflected BSDE associated with obstacles $(Y_t)$ and $(\xi_t)$. The result follows.

Let us now show the converse implication.

The reflected BSDE (3.24) can be seen as a reflected BSDE associated to the generalized driver $f(t, \omega, y, z) dt - dK_t - dC'_{t-}$.

Fix $\tau \in \mathcal{T}_S$. Using the flow property for reflected BSDEs and their representation as the value function of an optimal stopping problem, we get

$$Y_S = \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}^g_{S, S' \wedge \tau} [Y_{\tau} \mathbf{1}_{\tau \leq S'} + \xi_{S'} \mathbf{1}_{S' < \tau}] \text{ a.s.} \quad (3.28)$$

Using the comparison theorem for BSDEs with generalized driver, we deduce that

$$Y_S \leq \text{ess sup}_{S' \in \mathcal{T}_S} \mathcal{E}^g_{S, S' \wedge \tau} [Y_{\tau} \mathbf{1}_{\tau \leq S'} + \xi_{S'} \mathbf{1}_{S' < \tau}] \text{ a.s.}, \quad (3.29)$$

which implies that

$$Y_S \leq \text{Ref}^g_{S, \tau} [Y_{\tau}] \text{ a.s., for all } \tau \in \mathcal{T}_S. \quad \Box$$

We now show the existence of a RCLL process which aggregates the value family $(Y^\alpha)$.

**Theorem 3.9 (Existence of a RCLL aggregator process)** For any $\alpha \in V_0$, there exists a RCLL process $(Y^\alpha_t)$ which aggregates the family $(Y^\alpha(\tau), \tau \in \mathcal{T})$, that is $Y^\alpha(\tau) = Y^\alpha_t \text{ a.s., for all } \tau \in \mathcal{T}.$

**Proof.** Fix $\alpha \in V_0$. By Theorem 3.6, we get the existence of an optional process $(Y^\alpha_t)$ that aggregates the family $(Y^\alpha(\tau), \tau \in \mathcal{T}_0)$ and satisfies $E[\text{ess sup}_{\tau \in \mathcal{T}_0} (Y^\alpha_\tau)^2] < \infty$. Recall that, by Theorem 3.5, the process $(Y^\alpha_t)$ is a $\text{Ref}^g_{S, \tau}$-submartingale. We can thus use Theorem 3.8, which shows that $(Y^\alpha_t)$ admits a $\mathcal{E}^g$-Mertens decomposition, giving the existence of its left and right limits.

We thus define the process:

$$(Y^\alpha)^+ := \lim_{s \uparrow (t, T]} Y^\alpha_s, \quad t \in [0, T]. \quad (3.30)$$

In order to show that the process $Y^\alpha$ is indistinguishable of a RCLL process, we have to prove that

$$(Y^\alpha)^+ = Y^\alpha_t, \text{ a.s., for all } \tau \in \mathcal{T}_0. \quad (3.31)$$

Let us introduce $(\tau_n)_{n \in \mathbb{N}}$, a decreasing sequence of stopping times with values in $[0, T]$ such that $\tau_n \downarrow \tau$ a.s. as $n \to +\infty$. By the definition of the process $(Y^\alpha)^+$, we have

$$(Y^\alpha)^+_{\tau} = \lim_{n \to \infty} Y^\alpha_{\tau_n} \text{ a.s.} \quad (3.32)$$
The inequality \((Y_\alpha^\alpha)^{+} \geq \text{ess inf}_{\alpha' \in \mathcal{V}_\tau^\alpha} \text{Ref}^{{g, \Phi, \alpha'}}_{\tau, \Theta} \left[ \Phi(\theta, M_{\theta}') \right] = Y_\tau^\alpha\)

is clear by (3.20) and the continuity of the reflected BSDEs with respect to terminal time and terminal condition.

It remains to show that \((Y_\alpha^\alpha)^{+} \leq \text{ess inf}_{\alpha' \in \mathcal{V}_\tau^\alpha} \text{Ref}^{{g, \Phi, \alpha'}}_{\tau, T} \left[ \Phi(T, M_{T}') \right] = Y_T^\alpha\) a.s.

Fix \(\alpha' \in \mathcal{V}_\tau^\alpha\) and set

\[
\lambda_n := \left( \frac{M_{\tau_n}^\alpha}{M_{\tau_n}^\alpha} \wedge \frac{1 - M_{\tau_n}^\alpha}{1 - M_{\tau_n}^\alpha} \right) \mathbf{1}_{\{M_{\tau_n}^\alpha \notin [0,1]\}} \in [0,1].
\]

We set \(\alpha'_n := \alpha \mathbf{1}_{[0,\tau_n]} + \lambda_n \alpha' \mathbf{1}_{[\tau_n, T]}\). This implies that \(\alpha'_n\) belongs to \(\mathcal{V}_{\tau_n}^\alpha\),

Now, relation (3.32) together with the \(\mathcal{F}_\tau\)-measurability of \(\lim_{n \to \infty} Y_{\tau_n}^\alpha\) and the continuity of BSDEs with respect to the terminal time and terminal condition give:

\[
(Y_\alpha^\alpha)^{+} \leq \mathcal{E}_{\tau, \tau_n}^g \left[ \lim_{n \to \infty} Y_{\tau_n}^\alpha \right] = \lim_{n \to \infty} \mathcal{E}_{\tau, \tau_n}^g \left[ Y_{\tau_n}^\alpha \right] \text{ a.s.} \tag{3.33}
\]

By the optimal stopping theory, there exists an optimal stopping time \(\hat{\theta}_n \in \mathcal{T}_{\tau_n}\) for the optimal stopping problem \(\text{ess sup}_{\Theta \in \mathcal{T}_{\tau_n}} \mathcal{E}_{\tau_n, \Theta}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right]\). We thus derive

\[
\mathcal{E}_{\tau, \tau_n}^g \left[ Y_{\tau_n}^\alpha \right] \leq \mathcal{E}_{\tau, \tau_n}^g \left[ \text{ess sup}_{\Theta \in \mathcal{T}_{\tau_n}} \mathcal{E}_{\tau_n, \Theta}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] \right] = \mathcal{E}_{\tau, \tau_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] \text{ a.s.},
\]

where the first inequality follows by admissibility of the control \(\alpha'_n\). Furthermore, we get

\[
\mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] = \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] - \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] + \text{ess sup}_{\Theta \in \mathcal{T}_{\tau}} \mathcal{E}_{\tau, \Theta}^g \left[ \Phi(\theta, M_{\Theta}') \right] \text{ a.s.}
\]

Since \(\hat{\theta}_n \in \mathcal{T}_{\tau_n} \subset \mathcal{T}_{\tau}\), we have

\[
\mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] \leq \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] - \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] + \text{ess sup}_{\Theta \in \mathcal{T}_{\tau}} \mathcal{E}_{\tau, \Theta}^g \left[ \Phi(\theta, M_{\Theta}') \right] \text{ a.s.}
\]

Now, by using the a priori estimates with BSDEs we have:

\[
\mathbf{E} \left[ \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] - \mathcal{E}_{\tau, \hat{\theta}_n}^g \left[ \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right] \right]^2 \leq C \mathbf{E} \left[ \left( \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') - \Phi(\hat{\theta}_n, M_{\hat{\theta}_n}') \right)^2 \right]
\]

\[
\leq C \mathbf{E} \left[ \sup_{0 \leq t \leq T} \left( \Phi(t, M_{t}') - \Phi(t, M_{t}') \right)^2 \right].
\]

The convergence \(M_{t}' \to M_{t}'\) when \(n \to \infty\), together with Doob inequality, the uniform continuity of \(\Phi\) and Lebesgue’s Theorem implies that

\[
\mathbf{E} \left[ \sup_{0 \leq t \leq T} \left( \Phi(t, M_{t}') - \Phi(t, M_{t}') \right)^2 \right] \to 0 \text{ when } n \to \infty. \tag{3.37}
\]

Using (3.34), (3.35), (3.36), (3.37), taking the limit in \(n\) and then the essential infimum on \(\alpha' \in \mathcal{V}_{\tau}^\alpha\), the result follows.
3.3 A Backward SDE representation of the value process

In this subsection, we provide a Backward SDE representation of the value process \( \mathcal{Y}_t^\alpha \), for each \( \alpha \in \mathcal{V}_0 \). In order to do this, we first establish a Doob-Meyer decomposition of the value process \( \mathcal{Y}_t^\alpha \).

**Theorem 3.10 (Doob-Meyer decomposition of the value process)** For each \( \alpha \in \mathcal{V}_0 \), the process \( \mathcal{Y}_t^\alpha \) admits the following Doob-Meyer decomposition: there exist \( Z^\alpha \in \mathcal{H}_2 \) and two RCLL predictable processes \( A^\alpha \in \mathcal{K}_2 \) and \( K^\alpha \in \mathcal{K}_2 \) with \( A^\alpha_0 = 0 \) and \( K^\alpha_0 = 0 \) such that

\[
\mathcal{Y}_t^\alpha = \Phi(T, M_T^\alpha) + \int_t^T g(s, \mathcal{Y}_s^\alpha, Z_s^\alpha)ds + A_T^\alpha - A_t^\alpha - K_T^\alpha + K_t^\alpha, \quad (3.38)
\]

\[
\mathcal{Y}_t^\alpha \geq \Phi(t, M_t^\alpha) \text{ a.s., } 0 \leq t \leq T.
\]

\[
\int_0^T (\mathcal{Y}_s^\alpha - \Phi(s^-, M_{s^-}^\alpha))dA_s^\alpha = 0 \text{ a.s.}
\]

\[
dA_s^\alpha \perp dK_s^\alpha. \quad (3.39)
\]

**Proof.** By Theorems 3.6 and 3.5, we obtain that \( \mathcal{Y}_t^\alpha \) is a r.l.s.c. Ref\( g, \Phi^\alpha \)-submartingale. We can thus apply Theorem (3.8) and obtain the existence of the processes \( (Z^\alpha, A^\alpha, K^\alpha, C^\alpha, C^\alpha') \in \mathcal{H}_2 \times (\mathcal{K}_2)^4 \) such that (3.24) holds. Due to this equation, we have \( \Delta C_t^\alpha - \Delta C_t^\alpha' = - (\mathcal{Y}_t^\alpha - \mathcal{Y}_t^\alpha') \). Since by the previous Theorem the process \( \mathcal{Y}_t^\alpha \) is right-continuous, the process \( C = 0 \). The result follows.

We now show the following Backward SDE representation of the value process.

**Theorem 3.1 (Representation of the value process)** There exists a family \( (Z^\alpha, K^\alpha, A^\alpha)_{\alpha \in \mathcal{V}_0} \subseteq \mathcal{H}_2 \times \mathcal{K}_2 \times \mathcal{K}_2 \) such that, for all \( \alpha \in \mathcal{V}_0 \), we have

\[
\begin{aligned}
\mathcal{Y}_t^\alpha &= \Phi(T, M_T^\alpha) + \int_t^T g(s, \mathcal{Y}_s^\alpha, Z_s^\alpha)ds - \int_t^T Z_s^\alpha dW_s + K_T^\alpha - K_t^\alpha - A_T^\alpha + A_t^\alpha, \quad 0 \leq t \leq T; \\
\mathcal{Y}_t^\alpha &\geq \Phi(t, M_t^\alpha) \text{ a.s., } 0 \leq t \leq T; \\
\int_0^T (\mathcal{Y}_s^\alpha - \Phi(s^-, M_{s^-}^\alpha))dA_s^\alpha &= 0 \text{ a.s.; } dA^\alpha \perp dK^\alpha; \\
\text{ess inf}_{\alpha' \in \mathcal{V}_0^\alpha} \mathcal{E} &\left[ \int_T^T M_t^\alpha dA_t^\alpha \right] = 0 \text{ a.s., for all } \tau \in \mathcal{T}_0 \\
(\mathcal{Y}^\alpha, Z^\alpha, K^\alpha, A^\alpha) &\mathbf{1}_{[0, \tau]} = (\mathcal{Y}_0^\alpha, Z_0^\alpha, K_0^\alpha, A_0^\alpha) \mathbf{1}_{[0, \tau]}, \quad \forall \tau \in \mathcal{T}_0, \alpha \in \mathcal{V}_0^\alpha
\end{aligned}
\]

where for each \( \tau \in \mathcal{T} \) and \( \alpha' \in \mathcal{V}_0^\alpha \), the process \( (M_t^{\tau, \alpha'})_{t \geq \tau} \) represents the linearization process associated with \( \mathcal{Y}^{\alpha'}(\tau) \) and \( \mathcal{Y}^{\alpha'}(\tau) \) satisfying \( M_t^{\tau, \alpha'} = 1 \) a.s., with \( (Y_t^{\alpha'}, Y_t^{\alpha'}, Y_t^{\alpha'}) \) the solution of the reflected BSDE with driver \( g \) and obstacle \( \Phi(t, M_t^{\alpha'}) \). Moreover, \( (\mathcal{Y}^\alpha, Z^\alpha, A^\alpha, K^\alpha)_{\alpha \in \mathcal{V}_0} \) is the unique family satisfying the above BSDE.

**Proof.** First note that for \( (\alpha, \tau) \in \mathcal{V}_0 \times \mathcal{T}_0 \), we have \( V_{\alpha'}^\tau = V^\alpha_{\tau} \) on \([0, \tau]\) for \( \alpha' \in \mathcal{V}_0^\alpha \). The definition of \( \mathcal{Y} \) implies that \( \mathcal{Y}^\alpha \mathbf{1}_{[0, \tau]} = \mathcal{Y}^{\alpha'} \mathbf{1}_{[0, \tau]} \) for \( \alpha' \in \mathcal{V}_0^\alpha \). Fix \( \tau \in \mathcal{T}_0 \) and \( \alpha \in \mathcal{V}_0 \). By Theorem 3.10, we get the existence of \((Z^\alpha, K^\alpha, A^\alpha)\) such that

\[
\begin{aligned}
\mathcal{Y}_t^\alpha &= \Phi(T, M_T^\alpha) + \int_t^T g(s, \mathcal{Y}_s^\alpha, Z_s^\alpha)ds + A_T^\alpha - A_t^\alpha - K_T^\alpha + K_t^\alpha - \int_t^T Z_s^\alpha dW_s, \\
\mathcal{Y}_t^\alpha &\geq \Phi(t, M_t^\alpha) \text{ a.s. } 0 \leq t \leq T \\
\int_0^T (\mathcal{Y}_s^\alpha - \Phi(s^-, M_{s^-}^\alpha))dA_s^\alpha &= 0; \quad dA_s^\alpha \perp dK_s^\alpha. \quad (3.40)
\end{aligned}
\]
By the uniqueness of the representation of a semimartingale, we derive that \((Y^\alpha, Z^\alpha, K^\alpha, A^\alpha)1_{[0,\tau]} = (Y^{\bar{\alpha}}, Z^{\bar{\alpha}}, K^{\bar{\alpha}}, A^{\bar{\alpha}})1_{[0,\tau]}, \forall \tau \in \mathcal{T}_0, \alpha \in \mathcal{V}_\tau^{\bar{\alpha}}\). It remains to show the minimality condition satisfied by the process \(A^{\alpha'} - A^{\alpha'} - K^{\alpha'}\).

To do so, let us first consider an arbitrary control \(\bar{\alpha} \in \mathcal{V}_\tau^{\bar{\alpha}}\) and \((Y^{\bar{\alpha}}, Z^{\bar{\alpha}}, A^{\bar{\alpha}})\) the solution of the following reflected BSDE:

\[
\begin{align*}
Y^{\bar{\alpha}}_t &= \Phi(T, M^{\bar{\alpha}}_T) + \int_t^T g(s,Y^{\bar{\alpha}}_s, Z^{\bar{\alpha}}_s)ds - \int_t^T Z^{\bar{\alpha}}_s dW_s + A^{\bar{\alpha}}_t, \\
Y^{\bar{\alpha}}_T \geq \Phi(t, M^{\bar{\alpha}}_T) \text{ a.s.} & 0 \leq t \leq T, \\
\int_0^T (Y^{\bar{\alpha}}_s - \Phi(s, M^{\bar{\alpha}}_s)) dA^{\bar{\alpha}}_s = 0.
\end{align*}
\]

We now define the linearization process \(M^{\tau,\bar{\alpha}}\) such that \(M^{\bar{\alpha}}_\tau = 1\) and \(M^{\bar{\alpha}}_t = \exp\left(\int_t^\tau \beta_s dW_s + \int_t^\tau (\lambda_s - \frac{\beta_s^2}{2})ds\right)\), where

\[
\lambda_s := \frac{g(s,Y^{\bar{\alpha}}_s, Z^{\bar{\alpha}}_s) - g(s,Y^{\bar{\alpha}}_s, Z^{\bar{\alpha}}_s)}{Y^{\bar{\alpha}}_s - Z^{\bar{\alpha}}_s} 1\{Y^{\bar{\alpha}}_s - Z^{\bar{\alpha}}_s \neq 0\};
\]

\[
\beta_s := \frac{g(s,Y^{\bar{\alpha}}_s, Z^{\bar{\alpha}}_s) - g(s,Y^{\bar{\alpha}}_s, Z^{\bar{\alpha}}_s)}{|Z^{\bar{\alpha}}_s - Z^{\bar{\alpha}}_t|^2} (Z^{\bar{\alpha}}_s - Z^{\bar{\alpha}}_t) 1\{Z^{\bar{\alpha}}_s - Z^{\bar{\alpha}}_t \neq 0\}.
\]

Using a classical linearization procedure, we obtain:

\[
Y^{\bar{\alpha}}_\tau - Y^{\bar{\alpha}}_\tau = \mathbb{E}_\tau \left[ \int_\tau^T M^{\tau,\bar{\alpha}}_s (dA^{\bar{\alpha}}_s - dA^{\bar{\alpha}}_s + dK^{\bar{\alpha}}_s) \right] \text{ a.s.} \tag{3.41}
\]

We take now the ess inf on \(\bar{\alpha} \in \mathcal{V}_\tau^{\bar{\alpha}}\) and using the definition of the value function \(Y^{\bar{\alpha}}\), the minimality condition follows.

We now show the uniqueness of the family. Let \((\tilde{Y}^{\alpha}, \tilde{Z}^{\alpha}, \tilde{K}^{\alpha}, \tilde{A}^{\alpha})\) be a solution of (3.40). Notice that, by using the comparison theorem between BSDEs with \textit{generalized driver} and the characterization of the solution of a reflected BSDE as the solution of an optimal stopping problem, we deduce that

\[
Y^{\alpha}_t = \text{ess inf}_{\alpha' \in \mathcal{V}_t^{\alpha}} \text{ess sup}_{\theta \in \mathcal{T}_t} \mathcal{E}^\theta_{t,\theta} [\Phi(\theta, M^{\alpha'}_{\theta})] \geq \text{ess sup}_{\theta \in \mathcal{T}_t} \mathcal{E}^\theta_{t,\theta} [\Phi(\theta, M^{\alpha'}_{\theta})] = \tilde{Y}^{\alpha}_t \text{ a.s.} \tag{3.42}
\]

By using the same linearization procedure, we obtain

\[
Y^{\alpha'}_\tau - \tilde{Y}^{\alpha'}_\tau = \mathbb{E}_\tau \left[ \int_\tau^T M^{\tau,\alpha'}_s (dA^{\alpha'}_s - dA^{\alpha'}_s + dK^{\alpha'}_s) \right] \text{ a.s.} \tag{3.43}
\]

The minimality condition implies that \(\tilde{Y}^{\alpha'}_t = \text{ess inf}_{\alpha' \in \mathcal{V}_t^{\alpha}} Y^{\alpha'}_t\) a.s. Hence, the result follows.

\(\square\)

**Remark 3.2** Note that since in general the process \(A^{\alpha} - A^{\bar{\alpha}} - K^{\bar{\alpha}}\) is not non-decreasing, we cannot reduce to a formulation only involving \(A^{\alpha}, A^{\bar{\alpha}} \text{ and } K^{\bar{\alpha}}\), as in the case of non-reflected BSDEs with weak terminal condition. We point out that in the case when \(\Phi = -\infty\) and thus there is no reflection, the processes \(A^{\alpha} \text{ and } A^{\bar{\alpha}}\) become 0 for all \(\alpha \in \mathcal{V}_0\). Hence the minimality condition is indeed equivalent to

\[
\text{ess inf}_{\alpha' \in \mathcal{V}_\tau^{\alpha}} \mathbb{E}_\tau \left[ K^{\alpha'}_\tau - K^{\alpha'}_\tau \right] = 0 \text{ a.s.} \tag{3.44}
\]
4 BSDEs with weak reflections and a related game problem

In this section, we study a related game problem. We show that, given a threshold process \((M_t^\alpha)\), the minimal initial process \(Y^\alpha\) corresponds to the value of an optimal stopping problem. More precisely, we provide some conditions under which one can interchange the inf-sup and obtain the existence of a saddle point. This problem is in general non trivial, and the additional complexity in our case is due to the presence of the control \(\alpha\) in the obstacle \(\Phi(t, M_t^\alpha)\).

Let \(S \in T_0\) and \(\alpha \in V_0\). Define the first value function at time \(S\) as
\[
\mathcal{Y}^\alpha(S) := \text{ess inf}_{\alpha' \in V^S_0} \text{ess sup}_{\tau \in T_S} E[S,\tau][\Phi(\tau, M^\alpha_{\tau'})].
\] (4.45)

and the second value function at time \(S\) as
\[
\mathcal{Y}^\alpha(S) := \text{ess sup}_{\tau \in T_S} \text{ess inf}_{\alpha' \in V^S_0} E[S,\tau][\Phi(\tau, M^\alpha_{\tau'})].
\] (4.46)

By definition, we say that there exists a value function at time \(S\) for the game problem if \(\mathcal{Y}^\alpha(S) = \mathcal{Y}^\alpha(S)\) a.s.

We recall the definition of a \(S\)-saddle point.

**Definition 4.1 (S-saddle point)** Let \(S \in T_0\). A pair \((\tau^*_S, \alpha^*_S)\) \(\in T_S \times V\) is called a \(S\)-saddle point if

(i) \(\mathcal{Y}^\alpha(S) = \mathcal{Y}^\alpha(S)\) a.s.

(ii) The essential infimum in (4.45) is attained at \(\alpha^*_S\)

(iii) The essential supremum in (4.46) is attained at \(\tau^*_S\).

Let us now give the main result of this section.

**Theorem 4.2** 1. Assume that \(g(t, \omega, y, z) \geq 0\), for all \((t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d\) and suppose that \(\Phi\) is increasing with respect to \(t\) and convex with respect to \(m\). Then the game problem admits a value function, that is
\[
\mathcal{Y}^\alpha(S) = \mathcal{Y}^\alpha(S)\quad \text{a.s., for all } S \in T_0.
\] (4.47)

2. Assume that \(g(t, \omega, y, z) \leq 0\), for all \((t, \omega, y, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d\) and suppose that \(\Phi\) is decreasing with respect to \(t\) and concave with respect to \(m\). Then the game problem admits a value function, that is
\[
\mathcal{Y}^\alpha(S) = \mathcal{Y}^\alpha(S)\quad \text{a.s., for all } S \in T_0.
\] (4.48)

3. Under the additional assumption that \(g\) is convex with respect to \((y, z)\), there exists a \(S\)-saddle point for the game problem (4.48) in the sense of Definition 4.1.
Proof. 1. Fix $S \in \mathcal{T}_0$. First note that
\[
\text{ess sup}_{\theta \in \mathcal{T}_S} \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \leq \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \text{ess sup}_{\theta \in \mathcal{T}_S} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.}
\]

It remains to show the converse inequality.

Fix $\theta \in \mathcal{T}_S$ and $\alpha' \in \mathcal{V}_S^3$. By the flow property for nonlinear BSDEs, we get
\[
\mathcal{E}_{S,\theta}^g \left[ \mathcal{E}_{\theta,T}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \right] = \mathcal{E}_{S,\theta}^g \left[ \mathcal{E}_{\theta,T}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \right] \text{ a.s.}
\]

Applying the comparison theorem for BSDEs and using the assumption on the driver $g$, we derive
\[
\mathcal{E}_{S,\theta}^g \left[ \mathcal{E}_{\theta,T}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \right] \geq \mathcal{E}_{S,\theta}^g \left[ \mathbb{E} \left[ \Phi(T, M_{\theta}^{\alpha'}) | \mathcal{F}_\theta \right] \right] \text{ a.s.} \quad (4.49)
\]

The above relation, together with the properties of the map $\Phi$ and the conditional Jensen inequality implies that
\[
\mathcal{E}_{S,\theta}^g \left[ \mathbb{E} \left[ \Phi(T, M_{\theta}^{\alpha'}) | \mathcal{F}_\theta \right] \right] \geq \mathcal{E}_{S,\theta}^g \left[ \mathbb{E} \left[ \Phi(\theta, M_{\theta}^{\alpha'}) | \mathcal{F}_\theta \right] \right] \geq \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, \mathbb{E} [M_{\theta}^{\alpha'} | \mathcal{F}_\theta]) \right] \text{ a.s.} \quad (4.50)
\]

The martingale property of $M_{\theta}^{\alpha'}$ implies that
\[
\mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, \mathbb{E} [M_{\theta}^{\alpha'} | \mathcal{F}_\theta]) \right] = \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.} \quad (4.51)
\]

Combining (4.50) and (4.51), we get
\[
\mathcal{E}_{S,T}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \geq \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.}
\]

By taking first the essential suprema on $\theta \in \mathcal{T}_S$ and then the essential infima on $\alpha' \in \mathcal{V}_S^3$, it follows that
\[
\text{ess sup}_{\theta \in \mathcal{T}_S} \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \mathcal{E}_{S,\theta}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \geq \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \text{ess sup}_{\theta \in \mathcal{T}_S} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.}
\]

We clearly have
\[
\text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \mathcal{E}_{S,T}^g \left[ \Phi(T, M_{\theta}^{\alpha'}) \right] \leq \text{ess sup}_{\theta \in \mathcal{T}_S} \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.}
\]

The last two inequalities allow to conclude that
\[
\text{ess sup}_{\theta \in \mathcal{T}_S} \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \geq \text{ess inf}_{\alpha' \in \mathcal{V}_S^3} \text{ess sup}_{\theta \in \mathcal{T}_S} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \text{ a.s.}
\]

The result follows. \qed

Remark 4.3 We emphasize that the above results still hold under different assumptions on the map $\Phi$. Indeed, in the case of a positive driver $g$, one could consider the function $\Phi$ of the form $\Phi(t, \omega, m) = m + h(S_t)$, with $S$ a submartingale process and $h$ a convex function. In the case of a negative driver $g$, the proof still works for a function $\Phi$ of the form $\Phi(t, \omega, m) = m + h(S_t)$, with $S$ a supermartingale process and $h$ a concave function.
2. For the proof of this point, we mainly use the same ideas as for the previous proof. For sake of clarity, we give it below. Fix $S \in T_0$. Notice that

$$\esssup_{\theta \in T_S} \essinf_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \leq \essinf_{\alpha' \in \mathcal{V}_S^0} \esssup_{\theta \in T_S} \mathcal{E}_{S,\theta}^g \left[ \Phi(\theta, M_{\theta}^{\alpha'}) \right] \quad \text{a.s.}$$

Let us show the converse inequality.

Fix $\vartheta \in T_S$ and $\alpha' \in \mathcal{V}_S^0$. By the martingale property of $M_{\alpha'}$, we deduce

$$\mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] = \mathcal{E}_{S,S}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.} \quad (4.52)$$

Using (4.52), the properties of the function $\Phi$ and the conditional Jensen inequality, we derive

$$\mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] \geq \essinf_{\vartheta \in T_S} \mathcal{E}_{S,\vartheta}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.}$$

The assumption on the driver $g$ and the comparison theorem for BSDEs lead to

$$\mathcal{E}_{S,S}^g \left[ \mathbb{E}[\Phi(\vartheta, M_{\alpha'}^{\vartheta}) | \mathcal{F}_S] \right] \geq \essinf_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,\alpha'}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.}$$

By arbitrariness of $\vartheta \in T_S$, we have

$$\mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] \geq \essinf_{\vartheta \in T_S} \mathcal{E}_{S,\vartheta}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.}$$

Since the above inequality holds for any $\alpha' \in \mathcal{V}_S^0$, we obtain

$$\essinf_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] \geq \essinf_{\vartheta \in T_S} \esssup_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,\alpha'}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.} \quad (4.53)$$

Since $S \in T_S$ we deduce that

$$\essinf_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] \leq \essinf_{\vartheta \in T_S} \esssup_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,\alpha'}^g \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.} \quad (4.54)$$

From the last two inequalities we deduce

$$\esssup_{\vartheta \in T_S} \essinf_{\alpha' \in \mathcal{V}_S^0} \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \geq \essinf_{\alpha' \in \mathcal{V}_S^0} \mathcal{E}_{S,S}^g \left[ \Phi(S, M_{\theta}^{\alpha'}) \right] \geq \essinf_{\alpha' \in \mathcal{V}_S^0} \esssup_{\vartheta \in T_S} \left[ \Phi(\vartheta, M_{\alpha'}^{\vartheta}) \right] \quad \text{a.s.} \quad (4.55)$$

The result follows.

3. We now show the existence of a $S$-saddle point, under the additional assumption that $g$ is convex with respect to $(y, z)$, that is Assumption 4.3 holds.

**Assumption 4.3** For all $(\lambda, m_1, m_2, t, y_1, y_2, z_1, z_2) \in [0, 1] \times [0, 1]^2 \times [0, T] \times \mathbb{R}^2 \times (\mathbb{R}^d)^2$,

$$g(t, \lambda y_1 + (1 - \lambda) y_2, \lambda z_1 + (1 - \lambda) z_2) \leq \lambda g(t, y_1, z_1) + (1 - \lambda) g(t, y_2, z_2) \quad \text{a.s.}$$
We start by proving the existence of an optimal control \( \alpha_T^* \) for Problem 4.45. By Lemma 3.3, there exists a sequence of controls \( (\alpha^n) \) belonging to \( V^n_T \) such that

\[
\tilde{\mathcal{Y}}(S) = \lim_{n \to \infty} \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}^n)] \quad \text{a.s.} \tag{4.56}
\]

As the sequence \( (M^n_T) \) is bounded in \([0, 1]\), one can find sequences of nonnegative real numbers \( (\lambda^n_i) \) with \( \sum_{i \geq n} \lambda^n_i = 1 \), such that only a finite number of \( \lambda^n_i \) do not vanish, for each \( n \), and such that the sequence of convex combinations \( (\tilde{M}^n_T) \) given by

\[
\tilde{M}^n_T := \sum_{i \geq n} \lambda^n_i M_t^i
\]

converges a.s. to some \( \bar{M}_T \). By dominated convergence, the convergence holds in \( L_2 \), in particular \( \mathbb{E}[\bar{M}_T] = m_\alpha \) and the martingale representation theorem gives the existence of a control \( \bar{\alpha} \) such that \( \bar{M}_T = M_0^{\alpha_0, \bar{\alpha}} \). Due to the fact that \( (\tilde{M}^n_T) \) and \( \bar{M}_T \) are martingales, we obtain that, for all \( \theta \in \mathcal{T} \), \( \tilde{M}^n_T = \sum_{i \geq n} \lambda^n_i M_t^i \) a.s.

Moreover, since \( \Phi \) and \( g \) are convex, we have

\[
\sum_{i \geq n} \lambda^n_i \mathcal{E}^g_{T,\theta}[\Phi(\theta, M_t^i)] \geq \mathcal{E}^g_{T,\theta}[\Phi(\theta, \bar{M}_T)] \quad \text{a.s.} \tag{4.58}
\]

We thus obtain that

\[
\bar{Y}^n(S) := \sum_{i \geq n} \lambda^n_i \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, M_t^i)] \geq \esssup_{\theta \in \mathcal{T}} \left( \sum_{i \geq n} \lambda^n_i \mathcal{E}^g_{S,\theta}[\Phi(\theta, M_t^i)] \right) \geq \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] \quad \text{a.s.} \tag{4.59}
\]

Then (4.56) implies that \( \bar{Y}^n(S) \to \bar{Y}(S) \) a.s.

Let us now show that

\[
\esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] \to \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] \quad \text{a.s.} \tag{4.60}
\]

The a priori estimates on BSDEs give:

\[
\left| \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] - \esssup_{\theta \in \mathcal{T}} \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] \right| \leq \esssup_{\theta \in \mathcal{T}} \left| \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] - \mathcal{E}^g_{S,\theta}[\Phi(\theta, \bar{M}_T)] \right|
\]

\[
\leq C \esssup_{\theta \in \mathcal{T}} \mathbb{E}_S \left[ \left( \Phi(\theta, \bar{M}_T) - \Phi(\theta, \bar{M}_T) \right)^2 \right]^\frac{1}{2} \leq C \mathbb{E}_S \left[ \sup_{0 \leq t \leq T} \left( \Phi(t, \bar{M}_T) - \Phi(t, \bar{M}_T) \right)^2 \right]^\frac{1}{2} \quad \text{a.s.},
\]

with \( C \) a constant depending on \( T \) and the Lipschitz constant of the driver \( g \).

The Doob maximal inequality together with the uniform continuity of \( \Phi \) with respect to \( t \) and \( m \) imply the convergence to 0, up to a subsequence, of the RHS term of the above inequality. Hence, we obtain (4.60). From (4.59) and (4.60) we derive that \( \bar{\alpha} \) is an optimal control. \( \square \)
We can thus conclude that, under Assumption 1 of Theorem 4.2 and Assumption 4.3, the pair \((T, \bar{\alpha})\) is a \(S\)-saddle point. Under Assumption 2 of Theorem 4.2 and Assumption 4.3, the pair \((S, \bar{\alpha})\) is a \(S\)-saddle point.

We can easily observe that the existence of the value function of the game implies that we have the following representation of the minimal process \(Y^\alpha\).

**Corollary 4.1** Fix \(\theta \in \mathcal{T}_0\) and \(\alpha \in \mathcal{V}_0\). Then \(Y^\alpha_\theta\) corresponds to the value of the following optimal stopping problem

\[
Y^\alpha_\theta = \text{ess sup}_{\tau \in \mathcal{T}_0} X^\alpha_{\theta, \tau} \text{a.s.,}
\]

where \(X^\alpha_{\theta, \tau}\) corresponds to the minimal \(\theta\)-initial supersolution of the BSDE with weak terminal condition at time \(\tau\).

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