Topics in Born-Infeld Electrodynamics

R. Kerner, A.L. Barbosa* and D.V. Gal’tsov†

L.P.T.L. - Tour 22, 4-ème étage,
Boîte 142, Université Paris-VI,
4, Place Jussieu, 75005 Paris

December 27, 2021

1 Introduction - a Short Glance at the History

Since the discovery of the electron by J.J.Thomson in 1899, physicists tried to develop models of finite energy charge concentrations that could describe the elementary electric charge. One of the ideas was to use non-linear generalizations of Maxwell’s theory, deviating from it only at very short distances and very strong fields in order to ensure a cut-off and to avoid singularity at \( r \to 0 \).

G. Mie (1) was first to introduce such a model, based on the assumption that the electric field \( E \) can not exceed the limiting value \( E_0 \), and that the repulsive force should be proportional to the expression

\[
F \sim \frac{E}{\sqrt{1 - \frac{E^2}{E_0^2}}}. \tag{1}
\]

In this model it was possible to find a nonsingular solution with finite energy and charge, and with the field \( E \) falling off as \( r^{-2} \) at great distances, but this solution was not covariant with respect to the Lorentz transformations.

In 1932 and in 1934 Born and Infeld have published by now celebrated version of non-linear electrodynamics, in which they proposed the following Lorentz-invariant Lagrangian:

\[
\mathcal{L} = \beta^2 \left[ \sqrt{\det \left( \delta^\mu_\lambda + \beta^{-1} F^\mu_\lambda \right)} \right]. \tag{2}
\]

*Permanent address: IFT, Universidade Estadual Paulista, Rua Pamplona 145, 01405 So Paulo, Brazil.
†Permanent address: Department of Theoretical Physics, Moscow State University, 119899 Moscow, Russia.
The constant $\beta$ appears for dimensional reasons, and plays the same rôle here as the limiting value of the electric field in G. Mie’s non-linear electrodynamics. When expressed in terms of two invariants of Maxwell’s tensor,

$$P = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{and} \quad S = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu},$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$, this Lagrangian can be written explicitly as

$$L_{BI} = \beta^2 \left[ 1 - \sqrt{1 + 2P - S^2} \right], \quad (3)$$

or as

$$L_{BI} = \beta^2 \left[ 1 - \sqrt{1 + \frac{1}{2\beta^2} (B^2 - E^2) - \frac{1}{16\beta^2} (E \cdot B)^2} \right]. \quad (4)$$

With the advent of Quantum Mechanics and Dirac’s equation for the electron, the interest in classical models of charged particles has considerably faded. However, in 1970 G. Boillat ([3] considered the Born-Infeld electrodynamics as an example of non-linear theory in order to study its propagation properties. Starting with the most general non-linear theory derived from an arbitrary Lagrangian depending on two Lorentz invariants of Maxwell’s tensor, $\mathcal{L}(P, S)$, he discovered that among all such non-linear theories, the Born-Infeld electrodynamics is the only one ensuring the absence of birefringence, i.e. propagation along a single light-cone, and the absence of shock waves. In this respect the Born-Infeld theory is unique (except for another singular and unphysical Lagrangian $\mathcal{L} = P/S$). A beautiful discussion of these properties can be found in I. Bialynicki-Birula’s paper ([4]); an interesting generalization of Born-Infeld theory in a curved space-time background can be found in the paper by L.N. Chang et al. ([5]). Let us remind very shortly how the birefringence phenomenon may occur in non-linear theories.

From the mathematical point of view, these theories are based on systems of second order partial differential equations, linear in highest derivatives, with coefficients which depend only on the fields (but not on their derivatives). The systems of this type can be reduced to a set of differential equations of first order via introduction of auxiliary fields, which are the independent linear combinations of the first partial derivatives of functions corresponding to the degrees of freedom of our system.

The differential system can be represented by means of a matrix whose entries contain the operators of partial derivation or multiplicative coefficients, acting on a vector-column representing auxiliary fields. If the vector-column $\mathbf{u}$ with the fields $\psi, \chi_i, E_i$ and $B_i$ contains $N$ elements, then let us denote by $\mathcal{A}$ the $N \times N$ matrix containing the partial derivatives and by $\mathcal{B}$ the $N \times N$ matrix containing the multiplicative factors. Then the field equations can be written as:

$$\mathcal{A}^\mu(\mathbf{u}) \partial_\mu \mathbf{u} + \mathcal{B}(\mathbf{u}) \mathbf{u} = 0. \quad (5)$$
If the hypersurface defined by the implicit equation
\[ \Sigma(x^\mu) = 0 \] (6)
is a surface of discontinuity, then the first derivatives of fields are discontinuous across this surface, whereas the fields themselves are continuous. So, when applied to the discontinuities across the hypersurface \( \Sigma \), the equation (6) reduces to
\[ (A^\mu \Sigma_\mu) \delta_1 u = 0, \] (7)
where \( \Sigma_\mu \equiv \partial_\mu \Sigma \), and \( \delta_1 u \) denotes the discontinuity of the first derivative across \( \Sigma \). By definition, for a characteristic surface one has \( \delta_1 u \neq 0 \), therefore, in order for (7) to hold, one must have
\[ \det(A^\mu \Sigma_\mu) = 0, \] (8)
on the surface of discontinuity. The characteristic equation (8) determines the surface whose generic equation is \( H(x, \Sigma_\mu) = 0 \), with \( H \) a homogeneous function of order \( N \) in \( \Sigma_\mu \). The Born-Infeld theory turns out to be completely exceptional since it obeys the corresponding condition of (3), namely \( \delta_0 H \equiv H|_+ - H|_-= 0 \).

Let us give an illustration of this principle on the simplest case: the scalar field wave equation in a two-dimensional space-time \((t, x)\):
\[ \partial_t^2 \phi - \partial_x^2 \phi = 0. \] (9)
(Partial derivatives in Lorentz indices, \((0, x, y, z)\) or \((0, 1, 2, 3)\), will be denoted by \( \partial_0, \partial_x \), etc.. According to the prescription, we can use as auxiliary fields \( \psi \) and \( \chi \) the first derivatives of the scalar field \( \phi \), \( \partial_0 \phi = \psi \) and \( \partial_x \phi = \chi \). Then by definition, the first derivatives of auxiliary fields are not independent, because we have, as \( \partial_0 (\partial_x \phi) = \partial_x (\partial_0 \phi) \), automatically \( \partial_0 \chi - \partial_x \psi = 0 \). On the other hand the dynamical equation (3) can be written as \( \partial_0 \psi - \partial_x \chi = 0 \). In the matrix notation of (3) these two equations can be combined to yield
\[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_x \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \] (10)
We then find
\[ A^\mu \Sigma_\mu = \begin{pmatrix} -\Sigma_x & \Sigma_0 \\ \Sigma_0 & -\Sigma_x \end{pmatrix}, \] (11)
and the characteristic equation \( \det(A^\mu \Sigma_\mu) = 0 \) can be written as
\[ \Sigma_0^2 - \Sigma_x^2 = 0, \] (12)
where \( \Sigma_0 \equiv \partial_0 \Sigma \) and \( \Sigma_x \equiv \partial_x \Sigma \). The last equation defines the characteristic surfaces \( \Sigma(t, x) \), which in this case are the light-cones in two space-time dimensions.
The same technique can be easily applied to the electromagnetic Maxwellian field. We have to solve a $6 \times 6$ matrix, because we have now six independent combinations of its first derivatives (the fields $E$ and $B$) appearing in the first-order Maxwell’s equations. As we know, the characteristic surfaces in four dimensions are given by

$$
\Sigma_{\mu} \Sigma^{\mu} = \Sigma_0^2 - \Sigma_x^2 - \Sigma_y^2 - \Sigma_z^2 = 0.
$$

The same is true for the Born-Infeld non-linear electrodynamics. A more exhaustive discussion of the propagation properties of various non-linear generalizations of the electromagnetism can be found in recent papers (\[6\], \[7\]).

An entirely new and unexpected impulse for the revival of interest in the Born-Infeld electrodynamics, and in its non-abelian generalizations, came from recent developments of the string and brane theories. The string Lagrangian in $(4 + D)$ dimensions, which defines a minimal surface in a $(4+D)$-dimensional Minkowskian space-time, is in fact a generalization of geodesic equation for a point-like particle.

Consider a two-dimensional surface with cylindrical topology, parametrized with one time-like and one space-like parameter, $\tau$ and $\sigma$, respectively, and embedded in a $(4+D)$-dimensional space-time. The embedding functions will be denoted by $X^\mu (\xi)$, or equivalently, as $X^\mu (\xi^a)$, with $\mu, \nu = 0, 1, 2, \ldots, 3 + D$, and $A, B, \ldots = 1, 2$ so that $\xi^1 = \tau$, $\xi^2 = \sigma$.

The exterior space-time metric $g_{\mu\lambda}$ induces the internal metric of the world-sheet spanned by the string,

$$
G_{AB} = g_{\mu\lambda} \partial_a X^\mu \partial_b X^\lambda. 
$$

Let $h^{AB} (\xi^c)$ be an arbitrary metric on the world-sheet; the variational principle introduced first by A. Polyakov reads then

$$
\delta S = -\frac{1}{4\pi\alpha'} \delta \int \int \sqrt{-h} h^{AB} G_{AB} d\tau d\sigma = 0. 
$$

Under the independent variations $\delta x^\mu$ and $\delta h^{AB}$ one gets the following equations:

$$
G_{AB} - \frac{1}{2} h_{AB} \left( h^{CD} G_{CD} \right) = 0, 
$$

$$
\frac{1}{16\lambda^2} \left[ \nabla_A \nabla_B x^\mu + \Gamma^\mu_{\lambda\rho} \partial_A x^\lambda \partial_B x^\rho \right] = 0.
$$

After dimensional reduction from 11 to 10 dimensions, auxiliary fields $A$ and $\phi$ do appear, and the total Lagrangian takes on the form that contains the Born-Infeld Lagrangian (\[[15, 16, 17]\]).

$$
\mathcal{L} = \frac{1}{2} D_\mu \Phi D^\mu \Phi + \beta^2 (1 - R) - \frac{\lambda}{2} (\Phi^* \Phi - \nu^2)^2 + \frac{1}{16\pi G} R 
$$

with $\Phi$ denoting scalar field, $R$ the Riemann curvature scalar, and $R$ given by

$$
R = \sqrt{1 + \frac{1}{2\beta^2} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{16\beta^4} (F^a_{\mu\nu} F^{\mu\nu}_a)^2}. 
$$
For dimensional reductions onto lower dimensions, the non-abelian generalizations of this Lagrangian are naturally produced.

In a pure Yang-Mills theory in flat space-time, with the usual Lagrangian density
\[ L_{YM} = -\frac{1}{4} g_{AB} F^A_{\mu \nu} F^B_{\mu \nu} \]
there are no finite energy static non-singular solutions describing a charged soliton. This fact can be explained by the conformal invariance of the theory, and the tracelessness of the energy-momentum tensor,
\[ T^\mu_\mu = -T_{00} + \sum_{i=1}^{3} T_{i i} = 0. \] (19)

Given the positivity of energy, i.e. \( T_{00} > 0 \), this means that the sum of principal pressures is positive, too, \( \sum T_{i i} > 0 \), which leads to the conclusion that the Yang-Mills “matter” is naturally subjected to repulsive forces only.

The presence of the Higgs field breaks the conformal invariance, which leads to the existence of ’t Hooft and Prasad-Sommerfield magnetic monopoles. In what follows, we are interested in soliton-like solutions arising in other non-linear theories, including non-abelian versions of Yang-Mills theories, which are no more conformally invariant.

2 Non-linear Electrodynamics from the Kaluza-Klein Theory

An interesting non-linear generalization of electrodynamics derived from the Kaluza-Klein theory in five dimensions has been proposed in (9, 10). It is based on the addition of the Gauss-Bonnet term, \( R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2 \), which in five dimensions is not a topological invariant, leading to non-trivial equations of motion of second order when added to the Einstein-Hilbert Lagrangian.

In a flat space-time and without the scalar field the Kaluza-Klein metric is
\[ g_{AB} = \begin{pmatrix} g_{\mu \nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{pmatrix}, \] (20)
where \( A, B = 0, 1, 2, 3, 5 \) and \( \mu, \nu = 0, 1, 2, 3 \) (or \( \mu, \nu = 0, x, y, z \) following the convention we have been using). The full Lagrangian is taken to be (see [9, 10]):
\[ \mathcal{L} = R + \gamma (R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2), \] (21)
with \( \gamma \) being a certain dimensional parameter characterizing the strength of the non-linearity. When expressed in four dimensions in terms of the Maxwell tensor, it becomes
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{3\gamma}{16} \left[ (F_{\mu \nu} F^{\mu \nu})^2 - 2 (F_{\mu \lambda} F_{\nu \rho} F^{\mu \nu} F^{\lambda \rho}) \right]. \] (22)
In terms of the invariants $P$ and $S$ this Lagrangian is given by $\mathcal{L} = 2P + \frac{3\gamma}{2}S^2$, which for the choice $\gamma = -\frac{2}{3}$ yields essentially the square of the Born-Infeld Lagrangian. The equations of motion are:

$$F_{\lambda\rho,\mu} + F_{\rho\mu,\lambda} + F_{\mu\lambda,\rho} = 0,$$

which correspond to the Bianchi identities and are geometrical equations valid independently of the Lagrangian chosen, and the dynamical equations resulting from the variational principle,

$$[F^\lambda{}_{\rho} - \frac{3\gamma}{2}(F_{\mu\nu}F^\mu{}^{\rho})]F^\lambda{}_{\rho} + \frac{3\gamma}{2}F_{\mu\rho}F_{\lambda}{}^{\mu}F_{\rho}{}^{\nu},,\lambda = 0.$$  (24)

The Lagrangian (22) is particularly simple when expressed in more familiar terms with the fields $E$ and $B$:

$$\mathcal{L} = \frac{1}{2}(B^2 - E^2) + \frac{3\gamma}{2}(E \cdot B)^2.$$  (25)

The equations of motion also display a clear physical meaning when expressed in terms of $E$ and $B$. The equation (24) becomes

$$\text{div } B = 0, \quad \text{rot } E = -\partial_0 B,$$  (26)

whereas the equations (23) become

$$\text{div } E = -3\gamma B \cdot \text{grad } (E \cdot B)$$

$$\text{rot } B = \partial_0 E + 3\gamma \left[ B\partial_0 (E \cdot B) - E \times \text{grad}(E \cdot B) \right],$$  (27)

which show how the density of charge and the current are created by the non-linearity of the field: indeed, we can introduce

$$\rho = -3\gamma B \cdot \text{grad } (E \cdot B)$$

$$j = 3\gamma \left[ B\partial_0 (E \cdot B) - E \times \text{grad}(E \cdot B) \right]$$  (28)

which satisfy the continuity equation

$$\partial_0 \rho + \text{div } j = 0.$$  (29)

The Poynting vector conserves its form known from the Maxwellian theory, but the energy density is modified:

$$S = E \times B, \quad \mathcal{E} = \frac{1}{2}(E^2 + B^2) + \frac{3\gamma}{2}(E \cdot B)^2,$$  (30)

with the continuity equation resuming the energy conservation satisfied by virtue of the equations of motion:

$$\partial_0 \mathcal{E} + \text{div } S = 0.$$  (31)
It can be easily proved that there is birefringence in this theory. One wave propagates in a Maxwellian way, the other possible wave solution propagates differently; in fact, it is delayed (see [6] for details).

The properties of possible stationary axisymmetric solutions, endowed with non-vanishing charge, intrinsic kinetic and magnetic moments, have been discussed in [9, 10]. In the theory based on the Gauss-Bonnet term in 5 dimensions, one can try to find axially-symmetric configurations displaying both finite electric charge and finite magnetic moment; also, a kinetic momentum can be expected, parallel to the magnetic moment.

In cylindrical coordinates $\rho, \varphi, z$ we expect the induced current density to be aligned on the $e_\varphi$-vector of the local frame, giving a current density circulating around the $z$-axis; the fields $E$ and $B$ should be contained in the $\rho - z$ planes orthogonal to $e_\varphi$. Recalling the fact that the lines of strength of $B$ must be closed, the best description of this configuration can be obtained using the toroidal curvilinear coordinates $(\mu, \eta, \varphi)$ defined as follows in terms of cylindrical coordinates:

$$\rho = \frac{a \cosh \mu}{\cosh \mu - \cos \eta}, \quad z = \frac{a \sin \eta}{\cosh \mu - \cos \eta}, \quad \varphi.$$

with $0 \leq \varphi < 2\pi$, $0 \leq \eta < 2\pi$, $0 \leq \mu \leq \infty$. The coordinate lines of $\varphi$ are concentric circles in the $(z = 0)$-plane, while the coordinate lines of the variable $\eta$ are excenetric tori concentrating around the circle $\rho = a$. We shall suppose that the lines of force of the magnetic field coincide with the coordinate curves given by $\varphi = \text{Const.}$ and $\mu = \text{Const.}$ The configuration we seek can be written as:

$$E = E_\rho e_\rho + E_z e_z = E_\mu e_\mu + E_\eta e_\eta;$$

$$B = B_\rho e_\rho + B_z e_z = B_\eta e_\eta; \quad j_{\text{ind}} = j_{\text{ind}} e_\varphi.$$ (32)

$$\mathbf{B} = \mathbf{rotA}, \quad B_\mu = 0,$$

and because here $\mathbf{E} = -\nabla V$, we have $\mathbf{A} = A(\mu, \eta) e_\varphi$, and $V = V(\mu, \eta)$.

Approximate solutions of this form have been found in ([9], [10]); here we shall only remind their essential features. At great distances, the fields $E$ and $B$ behave as if they were generated by a finite charge $Q$ and a finite magnetic dipole $\mathbf{m}$:

$$E_\infty \approx \frac{Q r}{4\pi r^3}, \quad B_\infty \approx \frac{\mathbf{m} \wedge \mathbf{r}}{4\pi r^3}.$$ (34)

The charge is concentrated around the circle $\rho = a$ and "smeared" in its vicinity; if it is chosen to be positive, there is a little "halo" of negative charge density farther away, imitating the vacuum polarization effect. The charge density’s fall-off is very rapid, behaving at short distances as $r^{-9}$, and then falling off exponentially; the same concerns the density of induced current $j_{\text{ind}}$ which falls off as $r^{-8}$. The induced current behaves as if it were produced by the charge density rotating around the $z$-axis with the speed of light.
Another interesting feature of this solution is its $Z_2 \times Z_2$ symmetry. Indeed, any such solution displaying the total energy (mass) $E$, the total charge $Q$, magnetic momentum $m$ and the total spin $s$ is followed by three similar solutions with the same energy, but either with the same charge, but with the spin and magnetic momentum in the opposite direction (both “down”), or another couple of solutions having the opposite charge, and spin and magnetic moment up or down, but always opposite to each other - just like with what we know about the electron and the positron. The following table shows the properties of the four solutions:

| Fields | Energy | Charge | m | Spin |
|--------|--------|--------|---|------|
| E, B   | $\mathcal{E}$ | $Q$    | $m$ | $s$  |
| E, −B  | $\mathcal{E}$ | $Q$    | $−m$ | $−s$ |
| −E, B  | $\mathcal{E}$ | $−Q$   | $m$  | $−s$ |
| −E, −B | $\mathcal{E}$ | $−Q$   | $−m$ | $s$  |

Tab 1. The symmetry properties of four solutions.

Unfortunately, these solutions present a mild singularity on the circle $\rho = a$, which can not be avoided. Its presence can be proved by using Poincaré’s lemma; the details can be found in (9), (10).

## 3 An SU(2)-Based Non-Abelian Generalization of Born-Infeld Theory

The superstring theory gives rise to one important modification of the standard Yang-Mills quadratic Lagrangian suggesting the action of the Born-Infeld (BI) type [15, 16, 17]. Because this modification breaks the scale invariance, the natural question arises whether in the Born–Infeld–Yang–Mills (BIYM) theory the non-existence of classical particle-like solutions can be overruled. Although a mere scale invariance breaking, being a necessary condition, by no means guarantees the existence of particle-like solutions, a detailed study ([11] has shown that the $SU(2)$ BIYM classical glueballs indeed do exist.

Non–Abelian generalization of the Born–Infeld action presents an ambiguity in specifying how the trace over the the matrix–valued fields is performed in order to define the Lagrangian [15, 18]. Here we adopt the version with the ordinary trace which leads to a simple closed form for the action. The BIYM action with the ordinary trace looks like a straightforward generalisation of the corresponding $U(1)$ action in the “square root” form

$$S = \frac{\beta^2}{4\pi} \int (1 - \mathcal{R}) \, d^4x, \quad (35)$$

where

$$\mathcal{R} = \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu}^a F_{\mu\nu}^a - \frac{1}{16\beta^4} (F_{\mu\nu}^a F_{\mu\nu}^a)^2}. \quad (36)$$
It is easy to see that the BI non-linearity breaks the conformal symmetry ensuring the non-zero trace of the stress–energy tensor
\[ T^\mu_\mu = R^{-1} \left[ 4\beta^2 (1 - R) - F^a_{\mu\nu} F^a_{\mu\nu} \right] \neq 0. \] (37)

This quantity vanishes in the limit \( \beta \to \infty \) when the theory reduces to the standard one. For the Yang-Mills field we assume the usual monopole ansatz
\[ A_0^a = 0, \quad A_i^a = \epsilon_{aik} \frac{n^k}{r} (1 - w(r)), \] (38)
where \( n^k = x^k / r, \ r = (x^2 + y^2 + z^2)^{1/2} \), and \( w(r) \) is the real-valued function. After the integration over the sphere in (35) one obtains a two-dimensional action from which \( \beta \) can be eliminated by the coordinate rescaling \( \sqrt{\beta} t \to t, \sqrt{\beta} r \to r. \) The following static action results then:
\[ S = \int L dr, \quad L = r^2 (1 - R), \] (39)
with
\[ R = \sqrt{1 + 2 \frac{w'^2}{r^2} + \frac{(1 - w^2)^2}{r^4}}, \] (40)

where prime denotes the derivative with respect to \( r. \) It is worth noticing that the non-linearity arises here because of the non-linear dependence of the tensor \( F^a_{\mu\nu} \) on the potentials \( A_\mu^a. \) The corresponding equation of motion reads
\[ \left( \frac{w'}{R} \right)' = \frac{w(w^2 - 1)}{r^2 R}. \] (41)

A trivial solution \( w \equiv 0 \) corresponds to the point-like magnetic BI-monopole with the unit magnetic charge (embedded \( U(1) \) solution). In the Born–Infeld theory it has a finite self-energy [19]. For time-independent configurations the energy density is equal to minus the Lagrangian, so the total energy (mass) is given by the integral
\[ M = \int_0^\infty (R - 1) r^2 dr. \] (42)

For \( w \equiv 0 \) one finds
\[ M = \int \left( \sqrt{r^4 + 1} - r^2 \right) dr = \frac{\pi^{3/2}}{3 \Gamma(3/4)^2} \approx 1.23604978. \] (43)

Looking now for the essentially non–Abelian solutions of finite mass, we observe that in order to assure the convergence of the integral (42), the quantity \( R - 1 \) must fall down faster than \( r^{-3} \) as \( r \to \infty. \) Thus, far from the core the BI
corrections have to vanish and the Eq. (41) should reduce to the ordinary Yang-Mills equation, equivalent to the following two-dimensional autonomous system [20, 21, 22, 23]:

\[ \dot{w} = u, \quad \dot{u} = u + (w^2 - 1)w, \quad (44) \]

where a dot denotes the derivative with respect to \( \tau = \ln r \). This dynamical system has three non-degenerate stationary points \((u = 0, w = 0, \pm 1)\), from which \( u = w = 0 \) is a focus, while two others \( u = 0, w = \pm 1 \) are saddle points with eigenvalues \( \lambda = -1 \) and \( \lambda = 2 \). The separatrices along the directions \( \lambda = -1 \) start at infinity and after passing through the saddle points go to the focus with the eigenvalues \( \lambda = (1 \pm i\sqrt{3})/2 \).

It has been proved in ([11]) that the only finite-energy configurations with non-vanishing magnetic charge are the embedded U(1) BI-monopoles. Indeed, such solutions should have asymptotically \( w = 0 \), which does not correspond to bounded solutions unless \( w \equiv 0 \). The remaining possibility is \( w = \pm 1, \dot{w} = 0 \) asymptotically, which corresponds to zero magnetic charge. Coming back to \( r \)-variable one finds from (41)

\[ w = \pm 1 + c + O(r^0), \quad (45) \]

where \( c \) is a free parameter. This gives a convergent integral ([12]) as \( r \to \infty \). The two values \( w = \pm 1 \) correspond to two neighboring topologically distinct Yang-Mills vacua.

Now consider local solutions near the origin \( r = 0 \). For convergence of the total energy ([12]), \( w \) should tend to a finite limit as \( r \to 0 \). Then using the Eq. (41) one finds that the only allowed limiting values are \( w = \pm 1 \) again. In view of the symmetry of (44) under reflection \( w \to \pm w \), one can take without loss of generality \( w(0) = 1 \). Then the following Taylor expansion can be checked to satisfy the Eq. (44):

\[ w = 1 - br^2 + \frac{b^2(4b^2 + 3)}{10(4b^2 + 1)} r^4 + O(r^6), \quad (46) \]

with \( b \) being (the only) free parameter.

As \( r \to 0 \), the function \( R \) tends to a finite value

\[ R = R_0 + O(r^2), \quad R_0 = 1 + 12b^2, \quad (47) \]

therefore it is not a solution of the initial system ([12]). What remains to be done is to find appropriate values of constant \( b \) leading to smooth finite-energy solutions by gluing together the two asymptotic solutions between \( 0 \) and \( \infty \).

It has been proved in ([11]) that any regular solution of the Eq. (41) belongs to the one-parameter family of local solutions (15) near the origin.

It follows that the global finite energy solution starting with (16) should meet some solution from the family (15) at infinity. Since both these local solutions are non-generic, one can at best match them for some discrete values of
parameters. This technique has been used first in ([21]).

For some precisely tuned value of \( b \) the solution will remain a monotonous function of \( \tau \) reaching the value \(-1\) at infinity (Fig.1). This happens for \( b_1 = 12.7463 \).

By a similar reasoning one can show that for another fine-tuned value \( b_2 > b_1 \) the integral curve \( w(\tau) \) which has a minimum in the lower part of the strip and then becomes positive will be stabilized by the friction term in the upper half of the strip \([-1, 1]\) and tend to \( w = 1 \). This solution will have two nodes. Continuing this process we obtain the increasing sequence of parameter values \( b_n \) for which the solutions remain entirely within the strip \([-1, 1]\) tending asymptotically to \((-1)^n\). The lower values \( b_n \) found numerically are given in Tab. 2.

| \( n \) | \( b \)            | \( M \)            |
|-------|-------------------|-------------------|
| 1     | \( 1.27463 \times 10^4 \) | \( 1.13559 \times 10^4 \) |
| 2     | \( 8.87397 \times 10^2 \)  | \( 1.21424 \)     |
| 3     | \( 1.87079 \times 10^4 \)  | \( 1.23281 \)     |
| 4     | \( 1.27455 \times 10^6 \)  | \( 1.23547 \)     |
| 5     | \( 2.65030 \times 10^7 \)  | \( 1.23595 \)     |

Tab 2. Parameters \( b, M \) for first five solutions.

4 An \( SU(2) \times U(1) \) Generalization of Born-Infeld Lagrangian and its Embedding in the Standard Electroweak Model

The Born-Infeld Lagrangian generalizes the usual Maxwell theory; however, since we know that this theory is a part of the non-abelian field theory which accounts for electromagnetic and weak interactions, a natural question can be asked: is the original abelian version of Born-Infeld theory just a “shadow” of a more complicated non-abelian analog of the Born-Infeld Lagrangian? If so, we should be able to compare the pure electromagnetic (abelian) BI-Lagrangian with what can be extracted from its non-abelian version based on the symmetry group \( SU(2) \times U(1) \) after defining physical fields as linear combinations of the \( U(1) \) and \( SU(2) \) gauge fields with the coefficients defined by a rotation with the Weinberg angle. The ultimate comparison is beyond the scope of this paper; we will show how the first few terms in the Taylor expansions of these two Lagrangians can be compared. Performing the expansion of the BI Lagrangian
and we obtain the following first few terms in the series up to the fourth order.

\[
L_{BI} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{32} \beta^{-2} (F_{\mu \nu} F^{\mu \nu})^2 + \frac{1}{32} \beta^{-2} (F_{\mu \nu} F^{\mu \nu})^2 \\
- \frac{1}{128} \beta^{-4} (F_{\mu \nu} F^{\mu \nu})^3 - \frac{1}{128} \beta^{-4} F_{\mu \nu} F_{\rho \sigma} F^{\rho \sigma} + \frac{5}{2048} \beta^{-6} (F_{\mu \nu} F^{\mu \nu})^4 \\
+ \frac{3}{1024} \beta^{-6} (F_{\mu \nu} F^{\mu \nu})^2 (F_{\mu \nu} F^{\mu \nu})^2 + \frac{1}{2048} \beta^{-6} (F_{\mu \nu} F^{\mu \nu})^4 .
\]

(48)

For non-abelian groups we shall use the same generalization of the Born-Infeld Lagrangian as in the previous section, \((55)\). Here we will construct the non-abelian Lagrangian for \(SU(2)\) and \(U(1)\), to be compared with \((48)\). We expand the series in powers of \(\beta^{-2}\) in terms of Lorentz invariants of the fields, the abelian ones, \(P\) and \(S\), and their non-abelian generalizations \(P'\) and \(S'\):

\[
P' \equiv F_{\mu \nu} F^{\mu \nu} , \quad S' \equiv F_{\mu \nu} F^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} ,
\]

(49)

with \(F_{\mu \nu} = F^a_{\mu \nu} J_a\), with \(a = 0\) for \(U(1)\) and \(a = 1..3\) for \(SU(2)\). With the invariants \((44)\) replacing the abelian ones, and taking all the traces in the Lagrangian, which in the non-abelian case takes value in the matrix algebra of the fundamental representation of \(SU(2) \times U(1)\) chosen here, we obtain

\[
L_{SU(2)U(1)} = 2a \ F^0_{\mu \nu} F^0_{\mu \nu} + \frac{1}{2} \ F^a_{\mu \nu} F^a_{\mu \nu} \\
+ \beta^{-2} M^2 \left[ b \ [2 F^0_{\mu \nu} F^0_{\rho \sigma} F^0_{\rho \sigma} + \frac{1}{8} F^a_{\mu \nu} F^a_{\rho \sigma} F^a_{\rho \sigma} \right] + c \ [2 F^0_{\mu \nu} F^0_{\rho \sigma} F^0_{\rho \sigma} + \frac{1}{8} F^a_{\mu \nu} F^a_{\rho \sigma} F^a_{\rho \sigma} ] + \ldots,
\]

(50)

where \(\beta\) is the BI-Lagrangian parameter, \(M\) the mass scale of the unified theory, and \(a, b, c, \ldots\) are complicated numerical coefficients coming from traces and representation-dependent. Introducing physical fields with linear combinations of the \(U(1)\) and the \(SU(2)\) gauge fields

\[
F^0_{\mu \nu} = F_{\mu \nu} \cos \theta - (\partial_{\mu} Z_{\nu} - \partial_{\nu} Z_{\mu}) \sin \theta ,
\]

(51)

\[
F^\alpha_{\mu \nu} = F_{\mu \nu} \sin \theta + (\partial_{\mu} Z_{\nu} - \partial_{\nu} Z_{\mu}) \cos \theta = +ig(W^\dagger_{\mu} W_{\nu} - W^\dagger_{\nu} W_{\mu}) ,
\]

(52)

\[
F^1_{\mu \nu} = \frac{1}{\sqrt{2}} \left[ (\partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu}) + (\partial_{\mu} W^\dagger_{\nu} - \partial_{\nu} W^\dagger_{\mu}) \right] \\
- ig[(W_{\nu} W^\dagger_{\nu}) (A_{\mu} \sin \theta + Z_{\mu} \cos \theta) \\
+ (W_{\mu} W^\dagger_{\mu}) (A_{\nu} \sin \theta + Z_{\nu} \cos \theta)] ,
\]

(53)
\[
F_{\mu\nu}^2 = \frac{i}{\sqrt{2}} \left[ (\partial_\mu W_\nu - \partial_\nu W_\mu) - (\partial_\mu W^\dagger_\nu - \partial_\nu W^\dagger_\mu) \right] \\
+ g ((W_\nu + W^\dagger_\nu)(A_\mu \sin \theta + Z_\mu \cos \theta) \\
- (W_\mu + W^\dagger_\mu)(A_\nu \sin \theta + Z_\nu \cos \theta) ,
\]

where \(A_\mu\) is the pure electromagnetic field, \(Z_\mu\) is the neutral boson, \(W^+_\mu\) and \(W^-_\mu\) are the charged \(W\)-bosons, we can now compare the two series, term by term, trying to fix the coefficients in order to make coincide as many terms as possible. With the Weinberg angle \(\theta\) we can identify the pure electromagnetic sector in (50), then evaluate the difference. Because of the lack of space, we show here only first terms of this expression:

\[
L_{SU(2)U(1)} - L_{EM} = a (2 \cos^2 \theta + \frac{1}{2} \sin^2 \theta) F_{\mu\nu} F^{\mu\nu} \\
+ \beta^{-2} M^2 \left[ b (2 \cos^4 \theta + \frac{1}{8} \sin^4 \theta + 3 \sin^2 \theta \cos^2 \theta) F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \\
+ c \left[ 2 \cos^4 \theta + \frac{1}{2} \sin^4 \theta + 3 \sin^2 \theta \cos^2 \theta \right] F_{\mu\nu} \tilde{F}^{\mu\nu} F_{\rho\sigma} \tilde{F}^{\rho\sigma} \right] \\
+ \beta^{-4} M^4 \left[ g \left[ \frac{3}{16} \sin^6 \theta + \frac{15}{8} \cos^2 \theta \sin^4 \theta + \frac{15}{2} \cos^4 \theta \sin^2 \theta + 2 \cos^6 \theta \right] + \ldots \right] .
\]

(55)

It is possible to show that the coefficient for the \(n\)-th order of \(\beta^{-2}\) is given by

\[
C_n (\theta) = \frac{1}{4^n} \left[ (1 - 2 \cot \theta)^{2n} + (1 + 2 \cot \theta)^{2n} \right] (\sin \theta)^{2n} ,
\]

and its derivative is given by

\[
C'_n (\theta) = \frac{n}{2^{2n-1}} \left[ \cot \theta (1 - 2 \cot \theta)^{2n} + (1 + 2 \cot \theta)^{2n} \\
+ 2 [(1 - 2 \cot \theta)^{2n-1} + (1 + 2 \cot \theta)^{2n-1}] (\csc \theta)^2 \right] (\sin \theta)^{2n} .
\]

(57)

It is interesting to examine the behaviour of these coefficients. Surprisingly enough, starting from \(n = 3\) they display a maximum, whose position converges to a certain value with growing \(n\); moreover, this position is very close to the established value of the Weinberg angle (satisfying \(\sin^2 \theta_W = 0.227 \pm 0.014\), corresponding to \(\theta_W = 28^\circ, 45\) or to 0.497 radians). The maxima were found solving \(C'_n (\theta) = 0\) for a given value of \(n\). We show below examples of the coefficients \(C_n\) starting from \(n = 3\), and the value of the angle (in radians) for the first maximum of \(C_n\)

\[
C_3 = \frac{1}{32} \sin^6 \theta + \frac{15}{8} \sin^4 \theta \cos^2 \theta + \frac{15}{2} \sin^2 \theta \cos^4 \theta + 2 \cos^6 \theta .
\]

(58)

The first maximum corresponds to \(C_3 = 2.0838\) for \(\theta = 0.34682\).

Later on, we have:
First maximum for \(C_4 = 2.4886\) at \(\theta = 0.43522\).
First maximum at \(C_5 = 3.0713\) for \(\theta = 0.45474\).
First maximum at \(C_6 = 3.8232\) for \(\theta = 0.4606\).
For \(n = 8\) we obtain
First maximum at \(C_8 = 5.9622\) for \(\theta = 0.46327\).
The value of the angle corresponding to the first maximum for higher order tends to \(\theta = 0.463648\) and remains constant for \(n = 50\) and higher.
The fact that the value of the mixing angle obtained for the first maximum of the so defined coefficients approaches the value of the Weinberg angle seems to be rather accidental; nevertheless, it is worth noticing.

Acknowledgments

A. L. Barbosa thanks FAPESP (São Paulo, Brazil) for financial support. D. V. Gal’tsov acknowledges support from RFBR under grant 00-02-16306.

References

[1] Mie, G., *Annalen der Physik*, 37, 511 (1912).
[2] Born, M., and Infeld, L., *Nature* 132, 970 (1932); also, *Proc. Roy. Soc.*, A 144, 425 (1934).
[3] Boillat, G., *J. Math. Phys.*, 11, 941 (1970).
[4] Bialynicki-Birula, I., in J. Lopuszanski’s *Festschrift, Quantum Theory of Particles and Fields*, Eds. B. Jancewicz and J. Lukierski, p. 31 - 42, World Scientific, Singapore (1983).
[5] Chang L.N., Macrae K.I. and Mansouri F., *Phys. Rev. D*, 13 (2), 235-243 (1976).
[6] Lemos, J.P.S., and Kerner, R., *Gravitation and Cosmology*, 6, 49-58 (2000).
[7] Gibbons, G.W., and Herdeiro, C., *Phys. Rev.*, D63, 064006 (2001).
[8] Müller-Hoissen, F., *Phys. Lett.*, B156, 315 - 320 (1985).
[9] Kerner, R., *Comptes Rendus Acad. Sci. Paris*, 304, série 2, 621-624 (1987).
[10] Kerner, R., in *Infinite dimensional Lie algebras and Quantum Field Theory*, proceedings of the Varna Summer School (Bulgaria) 1987, H.D. Doebner, J.D. Hennig and T.D. Palev eds., p.53 - 72, *World Scientific*, (1988).
[11] Gal’tsov, D.V., and Kerner, R., *Phys. Rev. Lett.*, 84, (26), 5955-5959, (2000).
[12] Gal’tsov, D.V., and Volkov, M.S., *Phys. Lett.*, B273, 255–259 (1991).
[13] Sudarsky, D., and Wald, R.M., *Phys. Rev.*, **D 46**, 1453–1447 (1992).

[14] Volkov, M.S., and Gal’tsov, D.V., *Phys. Rep.*, **C319**, 1–83 (1999), hep-th/9810070.

[15] Tseytlin, A., *Nucl. Phys.*, **B501**, 41 (1997).

[16] Gauntlett, J.P., Gomis, J., and Townsend, P.K., *JHEP*, **01**, 003 (1998).

[17] Brecher, D., and Perry, M.J., *Nucl. Phys.*, **B527**, 121 (1998).

[18] Grandi, N., Moreno, E.F., and Shaposhnik, F.A., Monopoles in non-Abelian Dirac-Born-Infeld theory, hep-th/9901073.

[19] Gibbons, G.W., Wormholes on the World Volume: Born-Infeld particles and Dirichlet p-branes, hep-th/9801106.

[20] Chernavskii, D.S., and Kerner, R., *J. Math. Phys.*, **9**, (1), 287 (1978).

[21] Kerner, R., *Phys. Rev.*, **D19**, (4), 1243 (1979).

[22] Protogenov, A.P., *Phys. Lett.*, **B87**, 80 (1979).

[23] Breitenlohner, P., Forgacs, P., and Maison, D., *Comm. Math. Phys.*, **163**, 141–172 (1994).