Some Norm Identities Characterizing Inner-Product

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Abstract: In this paper a sequence of identities indexed by two integers $n > k \geq 2$ is given. Each of these identities hold in a normed space if and only if its norm is given by an inner product.

1. Introduction: Investigating norm identities that are satisfied only by norms induced by inner products dates back to the late 19th century (see [1] introduction). Fréchet [2] showed that a normed space is an inner product space if and only if

$$\|x + y + z\|^2 = \|x + y\|^2 + \|x + z\|^2 + \|y + z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2$$

for all $x, y, z$, a condition coinciding with our identity (7) for $n = 3, k = 2$. Jordan and von Neumann [3] showed that the norm is induced by an inner product if and only if the parallelogram law $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ holds for all $x, y$. We use this result in proving the sufficiency of our conditions. For a survey of characterizations of norms induced by inner products see [1], or Chapter 4 of [4].

2. Main Results:

Let $I$ be indexed set, and $x_i$ where $i \in I$ be elements of a vector space $\mathcal{H}$ then for $A \subseteq I$ we denote by $x_A$ the sum of vectors $x_i$, $i \in A$. i.e. $x_A = \sum_{i \in A} x_i$. We also use $\bar{n}$ to denote the set \{1,2,\ldots, n\}. For a finite set $A$ we use $|A|$ to denote the cardinality of $A$ and $\chi_A$ the characteristic function of $A$. i.e. $\chi_A: A \to \{0,1\}$ such that

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

We recall the familiar identity for inner product norms

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2Re(x, y) \quad (1)$$

Or equivalently

$$2Re(x, y) = \|x + y\|^2 - \|x\|^2 - \|y\|^2 \quad (2)$$
**Lemma 2.1:** For an inner product space, the equality
\[ \|x_1 + x_2 + \cdots + x_n\|^2 = \sum_{|A|=2, A \subseteq \mathbb{R}} \|x_A\|^2 - (n - 2) \sum_{i=1}^{n} \|x_i\|^2 \] (3)
holds for all positive integer \( n \geq 2 \).

**Proof:**
By induction on \( n \). For \( n = 2 \) the equality is simply \( \|x_1 + x_2\|^2 = \|x_1 + x_2\|^2 \).
Suppose the equality holds for \( n \) then, from (1)
\[ \|x_1 + x_2 + \cdots + x_{n+1}\|^2 = \|x_1 + x_2 + \cdots + x_n\|^2 + \|x_{n+1}\|^2 + 2Re\langle x_1 + \cdots + x_n, x_{n+1} \rangle \] (4)
But
\[ 2Re\langle x_1 + \cdots + x_n, x_{n+1} \rangle = \sum_{i=1}^{n} 2Re \langle x_i, x_{n+1} \rangle \]
\[ = \sum_{i=1}^{n} (\|x_i + x_{n+1}\|^2 - \|x_i\|^2 - \|x_{n+1}\|^2) \] (5)
\[ = \sum_{|A|=2, A \subseteq \mathbb{R}+1, n+1 \in A} \|x_A\|^2 - \sum_{i=1}^{n} \|x_i\|^2 - n \|x_{n+1}\|^2 \]
We also have from our induction hypothesis that
\[ \|x_1 + x_2 + \cdots + x_n\|^2 = \sum_{|A|=2, A \subseteq \mathbb{R}} \|x_A\|^2 - (n - 2) \sum_{i=1}^{n} \|x_i\|^2 \] (6)
Substituting (5) and (6) in (4), we obtain
\[ \|x_1 + x_2 + \cdots + x_{n+1}\|^2 = \left( \sum_{|A|=2, A \subseteq \mathbb{R}} \|x_A\|^2 - (n - 2) \sum_{i=1}^{n} \|x_i\|^2 \right) + \|x_{n+1}\|^2 \]
\[ + \left( \sum_{|A|=2, A \subseteq \mathbb{R}+1, n+1 \in A} \|x_A\|^2 - \sum_{i=1}^{n} \|x_i\|^2 - n \|x_{n+1}\|^2 \right) \]
\[ = \sum_{|A|=2, A \subseteq \mathbb{R}+1} \|x_A\|^2 - (n - 1) \sum_{i=1}^{n+1} \|x_i\|^2. \]
Thus, the relation (3) is true for \( n + 1 \).
By induction principle, it is true for all \( n \in \mathbb{N}, n \geq 2 \). ■
Lemma 2.2: In an inner product space, the identity

\[
\binom{n-2}{k-2} \|x_1 + \cdots + x_n\|^2 = \sum_{|A|=k, A \subseteq \bar{n}} \|x_A\|^2 - \binom{n-2}{k-1} \sum_{i=1}^n \|x_i\|^2
\]

holds for all \( n > k \geq 2 \) and all \( x_1, \cdots, x_n \).

Proof:

For \( k = 2 \), this is just lemma 2.1. For \( k > 2 \) we have by Lemma 2.1 that for each \( A \subseteq \bar{n} \) with \( |A| = k \),

\[
\|x_A\|^2 = \sum_{|B|=2, B \subseteq A} \|x_B\|^2 - (k - 2) \sum_{i \in A} \|x_i\|^2
\]

Thus, the righthand side of (7) is equal to

\[
\sum_{|A|=k, A \subseteq \bar{n}} \left( \sum_{|B|=2, B \subseteq A} \|x_B\|^2 - (k - 2) \sum_{i \in A} \|x_i\|^2 \right) - \binom{n-2}{k-1} \sum_{i=1}^n \|x_i\|^2
\]

For a set \( B \) of size 2, the number of \( k \)-subset \( A \subseteq \bar{n} \) containing \( B \) is equal to the number of \( (k - 2) \)-subsets \( C \) of \( \bar{n} \setminus B \) (the correspondence being \( C \rightarrow C \cup B \)) so is \( \binom{n-2}{k-2} \).

Thus,

\[
\sum_{|A|=k, A \subseteq \bar{n}} \sum_{|B|=2, B \subseteq A} \|x_B\|^2 = \binom{n-2}{k-2} \sum_{|B|=2, B \subseteq \bar{n}} \|x_B\|^2.
\]

Similarly,

Each \( \|x_i\|^2 \) where \( 1 \leq i \leq n \), occurs in \( \sum_{|A|=k, A \subseteq \bar{n}} \sum_{i \in A} \|x_i\|^2 \) as many times as there are \( k \)-subsets \( A \subseteq \bar{n} \) that contain \( i \) which equals the number of \( (k - 1) \)-subsets \( C \) of \( \bar{n} \setminus \{i\} \) (the correspondence being \( C \rightarrow C \cup \{i\} \)) so occurs \( \binom{n-1}{k-1} \) times.

Thus,

\[
\sum_{|A|=k, A \subseteq \bar{n}} \sum_{i \in A} \|x_i\|^2 = \binom{n-1}{k-1} \sum_{i=1}^n \|x_i\|^2.
\]

Substituting in (8), we get that the righthand side of (7) is

\[
\binom{n-2}{k-2} \sum_{|B|=2, B \subseteq \bar{n}} \|x_B\|^2 - (k - 2) \binom{n-1}{k-1} \sum_{i=1}^n \|x_i\|^2 - \binom{n-2}{k-1} \sum_{i=1}^n \|x_i\|^2
\]

\[
= \binom{n-2}{k-2} \sum_{|B|=2, B \subseteq \bar{n}} \|x_B\|^2 - (k - 1) \binom{n-1}{k-1} - \binom{n-1}{k-1} + \binom{n-2}{k-1} \sum_{i=1}^n \|x_i\|^2
\]

Using the identities

\[
t \binom{s}{t} = s \binom{s-1}{t-1} = \frac{s!}{(t-1)!(s-t)!} \quad \text{ and } \quad -\binom{x}{y} + \binom{x-1}{y} = -\binom{x-1}{y-1}
\]
we can simplify the coefficient of the $\sum_{i=1}^{n} \|x_i\|^2$ in (9) to
\[
-(k - 1) \binom{n - 1}{k - 1} - \binom{n - 2}{k - 2} + \binom{n - 2}{k - 1} = -(n - 1) \binom{n - 2}{k - 2} - \binom{n - 2}{k - 2}
\]
\[
= -(n - 2) \binom{n - 2}{k - 2}.
\]

Thus (9), i.e. The righthand side of (7) equals
\[
\binom{n - 2}{k - 2} \left( \sum_{|A|=2, A \subseteq \mathbb{S}} \|x_A\|^2 - (n - 2) \sum_{i=1}^{n} \|x_i\|^2 \right) = \binom{n - 2}{k - 2} \|x_1 + \cdots + x_n\|^2
\]
(where the last equality is by Lemma 2.1) Establishing the desired result.

Having shown that the existence of an inner product gives us the identity (7) above we show that the existence of an inner product follows if (7) holds for some $n > k \geq 2$ and all $x_1, \cdots, x_n$. We give a direct proof then a shorter deduction using a result by Reznick [5] stated below

**Lemma 2.3**: If $\mathcal{H}$ is a normed space whose norm satisfies (7) for some $n > k \geq 2$ and all $x_1, \cdots, x_n$ in $\mathcal{H}$ then the norm in $\mathcal{H}$ is induced by an inner product.

**Proof:**

We show that (7) implies the parallelogram law. Suppose that $n > k \geq 2$ are fixed integers and that
\[
\binom{n - 2}{k - 2} \|x_1 + \cdots + x_n\|^2 = \sum_{|A|=k, A \subseteq \mathbb{S}} \|x_A\|^2 - \binom{n - 2}{k - 1} \sum_{i=1}^{n} \|x_i\|^2
\]
for all $x_1, \cdots, x_n$ in $\mathcal{H}$. For $x, y \in \mathcal{H}$ let $x_1 = x, x_2 = y, x_3 = -y$ and $x_j = 0$ for $3 < j$. By our choice of the $x_i$'s the value of $x_A$, hence $\|x_A\|^2$, is determined by $A \cap \{1,2,3\}$. So, we split the first sum on the righthand side into sums based on this intersection:

1) For sets $A$ with $\{1,2,3\} \cap A = \{1,2,3\}$ we have $\|x_A\|^2 = \|x\|^2$. The number of such sets is equal to the number of ways of choosing the $k - 3$ elements of $A \setminus \{1,2,3\}$ from $\mathbb{N} \setminus \{1,2,3\}$ which is $\binom{n - 3}{k - 3}$ so the sum over these is $\binom{n - 3}{k - 3} \|x\|^2$.

2) For sets $A$ with $\{1,2,3\} \cap A = \{i,j\}$ a two-element set $\|x_A\|^2 = \|x_i + x_j\|^2$. The number of such sets is the number of ways of choosing the $k - 2$ elements of $A \setminus \{i,j\}$ from $\mathbb{N} \setminus \{1,2,3\}$ which is $\binom{n - 3}{k - 2}$.

Thus, the sum of these corresponding to $\{1,2\}, \{1,3\}$ and $\{2,3\}$ respectively is
\[
\binom{n - 3}{k - 2} \|x+y\|^2 + \binom{n - 3}{k - 2} \|x-y\|^2 + \binom{n - 3}{k - 2} \|0\|^2.
\]

3) For sets $A$ with $\{1,2,3\} \cap A = \{i\}$ a singleton we have $\|x_1 + \cdots + x_n\|^2 = \|x_i\|^2$. The number of such sets is the number of ways of choosing the $k - 1$ elements of $A \setminus \{i\}$
from \(\bar{n}\{1,2,3\}\) which is \(\binom{n-3}{k-1}\). So, the sum of these corresponding to \{1\}, \{2\} and \{3\} respectively is

\[
\binom{n-3}{k-1} \|x\|^2 + \binom{n-3}{k-1} \|y\|^2 + \binom{n-3}{k-1} \|y\|^2 = \binom{n-3}{k-1} \|x\|^2 + 2 \binom{n-3}{k-1} \|y\|^2
\]

Of course, \(x_A = 0\) for all other \(A\)'s (those with \(A \cap \{1,2,3\} = \emptyset\) ), so these can be ignored. Thus, the first sum on the righthand side of (7) is

\[
\left( \binom{n-3}{k-3} + \binom{n-3}{k-1} \right) \|x\|^2 + \left( \binom{n-3}{k-3} - \binom{n-2}{k-3} \right) \|x\|^2
\]

Noting that \(\sum_{i=1}^{n} \|x_i\|^2 = \|x\|^2 + 2 \|y\|^2\), we see that the righthand side of (7) is

\[
\left( \binom{n-3}{k-3} + \binom{n-3}{k-1} - \binom{n-2}{k-1} \right) \|x\|^2 + 2 \|y\|^2 \left( \binom{n-3}{k-1} - \binom{n-2}{k-1} \right)
\]

\[+ \left( \binom{n-3}{k-2} \right) \|x+y\|^2 + \|x-y\|^2\]

Substituting the above expression for the right hand side of (7) then subtracting \(\binom{n-2}{k-2} \|x\|^2\) from both sides of (7) gives us

\[
0 = \left( \binom{n-3}{k-3} + \binom{n-3}{k-1} - \binom{n-2}{k-1} - \binom{n-2}{k-2} \right) \|x\|^2
\]

\[+ 2 \|y\|^2 \left( \binom{n-3}{k-1} - \binom{n-2}{k-1} \right) + \binom{n-3}{k-2} \|x+y\|^2 + \|x+y\|^2\).

But the coefficient of \(\|y\|^2\) is

\[2 \left( \binom{n-3}{k-1} - \binom{n-2}{k-1} \right) = -2 \binom{n-3}{k-2}\]

and that of \(\|x\|^2\) is

\[
\left( \binom{n-3}{k-1} - \binom{n-2}{k-1} \right) + \left( \binom{n-3}{k-3} - \binom{n-2}{k-2} \right) = - \binom{n-3}{k-2} - \binom{n-3}{k-2} = -2 \binom{n-3}{k-2}
\]

Thus (7) for these values of \(x_1, \cdots, x_n\) is equivalent to

\[
0 = -2 \binom{n-3}{k-2} \left( \|x\|^2 + \|y\|^2 \right) + \binom{n-3}{k-2} \left( \|x+y\|^2 + \|x+y\|^2 \right)
\]

On dividing by \(\binom{n-3}{k-2}\) which is nonzero since \(n-3, k-2 \geq 0\), we obtained the parallelogram law. Thus, the norm is induced by an inner product. \(\blacksquare\)
As noted above Lemma 2.3 can be deduced from a theorem by Reznick. We state this theorem as it appears in [1] (1.17) (see [5] for a more general version) with minor re-indexing.

**Theorem 2.4:** If for some \( m \geq 1 \) and \( n \geq 2 \) there exist constants \( a_k \neq 0, 0 \leq k \leq m \) and pairwise linearly independent vectors \( \overline{c}_k = (c_k(1), \ldots, c_k(n)) \) such that for each \( 1 \leq i \leq n \) there is a \( k \) with \( c_k(i) \neq 0 \) and

\[
\sum_{k=0}^{m} a_k \|c_k(1)x_1 + \cdots + c_k(n)x_n\|^2 = 0 \quad \text{for all } x_1, \ldots, x_n \tag{10}
\]

Then the norm is induced by an inner product.

To apply this theorem, we let \( a_0 = -1, \overline{c}_0 = \chi_{\overline{n}} = (1, \ldots, 1) \) and for all \( A \subseteq \overline{n} \) with \( |A| = k \) we let \( a_A = 1, \overline{c}_A = \chi_A \) and for all \( i \in \overline{n} \) we let \( a_i = -(\frac{n-2}{k-1}), \overline{c}_i = \chi_{\{i\}} \). Note that the vectors

\[
\overline{c}_0, \overline{c}_A: |A| = k, \overline{c}_i: 1 \leq i \leq n
\]

are clearly pairwise linearly independent and the coefficients \( a_0, a_A, |A| = k, a_i: 1 \leq i \leq n \) are nonzero and for each \( 1 \leq i \leq n \) we have \( c_i(i) = 1 \neq 0 \). Finally, note that, with these notations, (7) becomes

\[
a_0\|c_0(1)x_1 + \cdots + c_0(n)x_n\|^2 + \sum_{|A|=k,A \subseteq \overline{n}} a_A\|c_A(1)x_1 + \cdots + c_A(n)x_n\|^2
\]

\[
+ \sum_{i=1}^{n} a_i\|c_i(1)x_1 + \cdots + c_i(n)x_n\|^2 = 0
\]

which has the form (10). Thus if (7) holds for all \( x_1, \ldots, x_n \) then the norm is induced by an inner product.

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