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DECREASING PROPERTIES OF TWO RATIOS DEFINED BY
THREE AND FOUR POLYGAMMA FUNCTIONS

FENG QI

Dedicated to my elder brother, Can-Long Qi, and his family

Abstract. In the paper, by virtue of convolution theorem for the Laplace
transforms, with the aid of three monotonicity rules for the ratios of two func-
tions, of two definite integrals, and of two Laplace transforms, in terms of the
majorization, and in the light of other analytic techniques, the author presents
decreasing properties of two ratios defined by three and four polygamma func-
tions.

1. Motivations

In the literature [1, Section 6.4], the function
\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0 \]
and its logarithmic derivative \( \psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)} \) are called Euler’s gamma
function and digamma function respectively. Moreover, the functions \( \psi'(z), \psi''(z), \psi'''(z), \) and \( \psi^{(4)}(z) \) are known as the trigamma, tetragamma, pentagamma, and
hexagamma functions respectively. As a set, all the derivatives \( \psi^{(k)}(z) \) for \( k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) are known as polygamma functions.

Recall from Chapter XIII in [13], Chapter 1 in [36], and Chapter IV in [40] that,
if a function \( f(x) \) on an interval \( I \) has derivatives of all orders on \( I \) and satisfies
\((-1)^nf^{(n)}(x) \geq 0\) for \(x \in I\) and \(n \in \mathbb{N}_0\), then we call \(f(x)\) a completely monotonic function on \(I\).

In [27] Theorem 1.1 and [28] Theorem 3], among other things, the function
\[ [\psi'(x)]^2 + \lambda \psi''(x) \]
was proved to be completely monotonic on \((0, \infty)\) if and only if \(\lambda \leq 1\). In [10] Theorem 1, it was proved that, among the functions
\[ f_{m,n}(x) = [\psi^{(m)}(x)]^2 + \psi^{(n)}(x), \quad m, n \in \mathbb{N}, \quad x \in (0, \infty), \]
(1) the functions
\[ f_{1,2}(x) = [\psi'(x)]^2 + \psi''(x) \]
and
\[ f_{m,2n-1}(x) = [\psi^{(m)}(x)]^2 + \psi^{(2n-1)}(x) \]
are completely monotonic on \((0, \infty)\), but the complete monotonicity of \(f_{m,2n-1}(x)\) is trivial;
(2) the functions
\[ f_{m,2n}(x) = [\psi^{(m)}(x)]^2 + \psi^{(2n)}(x) \]
for \((m, n) \neq (1, 1)\) are not monotonic and does not keep the same sign on \((0, \infty)\).

For \(k \in \mathbb{N}\) and \(x \in (0, \infty)\), let
\[ \mathcal{F}_{k, \eta_k}(x) = \psi^{(2k)}(x) + \eta_k [\psi^{(k)}(x)]^2 \quad \text{and} \quad \mathfrak{G}_{k, \vartheta_k}(x) = \frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)] \vartheta_k}. \]

In [18] Theorem 3.2, the author proved the following conclusions:
(1) if and only if \(\eta_k \geq \frac{1}{2} \frac{(2k)!}{(k-1)!k!}\), the function \(\mathcal{F}_{k, \eta_k}(x)\) is completely monotonic on \((0, \infty)\);
(2) if and only if \(\eta_k \leq 0\), the function \(\mathcal{F}_{k, \eta_k}(x)\) is completely monotonic on \((0, \infty)\);
(3) if and only if \(\vartheta_k \geq 2\), the function \(\mathfrak{G}_{k, \vartheta_k}(x)\) is decreasing on \((0, \infty)\);
(4) if and only if \(\vartheta_k \leq \frac{2k+1}{k+1}\), the function \(\mathfrak{G}_{k, \vartheta_k}(x)\) is increasing on \((0, \infty)\);
(5) the following limits are valid:
\[
\lim_{x \to 0^+} \mathfrak{G}_{k, \vartheta_k}(x) = \begin{cases} 
\frac{(2k)!}{[(k)!]^2 (k+1)}, & \vartheta_k = \frac{2k+1}{k+1} \\
0, & \vartheta_k > \frac{2k+1}{k+1} \\
-\infty, & \vartheta_k < \frac{2k+1}{k+1} 
\end{cases}
\]
and
\[
\lim_{x \to \infty} \mathfrak{G}_{k, \vartheta_k}(x) = \begin{cases} 
\frac{(2k-1)!}{[(k-1)!]^2}, & \vartheta_k = 2 \\
-\infty, & \vartheta_k > 2 \\
0, & \vartheta_k < 2 
\end{cases}
\]
(6) the double inequality
\[
-\frac{1}{2} \frac{(2k)!}{(k-1)!k!} < \frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^2} < 0
\]
is valid on \((0, \infty)\) and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.
Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$. A $n$-tuple $\alpha$ is said to strictly majorize $\beta$ (in symbols $\alpha \succ \beta$) if $(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq (\beta_1, \beta_2, \ldots, \beta_n)$, $\sum_{i=1}^{k} \alpha[i] \geq \sum_{i=1}^{k} \beta[i]$ for $1 \leq k \leq n - 1$, and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$, where $\alpha[1] \geq \alpha[2] \geq \cdots \geq \alpha[n]$ and $\beta[1] \geq \beta[2] \geq \cdots \geq \beta[n]$ are rearrangements of $\alpha$ and $\beta$ in a descending order. See [12] p. 8, Definition A.1 or closely related texts and references in the papers [6, 35, 37, 47].

**Theorem 1.1 ([13] Theorem 3.1).** Let $p, q, m, n \in \mathbb{N}_0$ satisfying $(p, q) \succ (m, n)$ and let

$$F_{p,m,n,q,c}(x) = \begin{cases} |\psi^{(m)}(x)||\psi^{(n)}(x)| - c|\psi^{(p)}(x)|, & q = 0 \\ |\psi^{(m)}(x)||\psi^{(n)}(x)| - c|\psi^{(p)}(x)||\psi^{(q)}(x)|, & q \geq 1 \end{cases}$$

for $c \in \mathbb{R}$ and $x \in (0, \infty)$. Then

1. for $q \geq 0$, if and only if

$$c \leq \left\{ \begin{array}{ll} \frac{(m-1)!(n-1)!}{(p-1)!}, & q = 0 \\ \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, & q \geq 1 \end{array} \right.$$ the function $F_{p,m,n,q,c}(x)$ is completely monotonic in $x \in (0, \infty)$;

2. for $q \geq 1$, if and only if $c \geq \frac{m!n!}{p!q!}$, the function $-F_{p,m,n,q,c}(x)$ is completely monotonic in $x \in (0, \infty)$;

3. the double inequality

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} < 0 \quad (1.1)$$

for $m, n \in \mathbb{N}$ and the double inequality

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} < \frac{m!n!}{p!q!} \quad (1.2)$$

for $m, n, p, q \in \mathbb{N}$ with $p > m \geq n > q \geq 1$ and $m + n = p + q$ are valid on $(0, \infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.

In [19] Remark 6.2, the preprint of the formally published paper [13], the author guessed that,

1. for $m, n \in \mathbb{N}$, the function

$$Q_{m,n}(x) = \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} \quad (1.3)$$

should be decreasing on $(0, \infty)$;

2. for $m, n, p, q \in \mathbb{N}$ such that $(p, q) \succ (m, n)$, the function

$$Q_{m,n,p,q}(x) = \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} \quad (1.4)$$

should be decreasing on $(0, \infty)$.

It is clear that $Q_{k,k}(x) = \psi_{k,2}(x)$ for $k \in \mathbb{N}$, which is decreasing on $(0, \infty)$.

In this paper, we aim to confirm these two guesses. We also supply an alternative proof of Theorem 1.1.
2. Lemmas

The following lemmas are necessary in this paper.

**Lemma 2.1** ([2] pp. 10–11, Theorem 1.25). For \( a, b \in \mathbb{R} \) with \( a < b \), let \( f(x) \) and \( g(x) \) be continuous on \([a, b]\), differentiable on \((a, b)\), and \( g(x) \neq 0 \) on \((a, b)\). If the ratio \( \frac{f(x)}{g(x)} \) is increasing on \((a, b)\), then both \( \frac{f(x)-f(a)}{g(x)-g(a)} \) and \( \frac{f(x)-f(b)}{g(x)-g(b)} \) are increasing in \( x \in (a, b)\).

**Lemma 2.2** ([1] p. 260, 6.4.1]). The integral representation

\[
\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1-e^{-zt}} \, dt
\]  

is valid for \( \Re(z) > 0 \) and \( n \geq 1 \).

**Lemma 2.3** (Convolution theorem for the Laplace transforms [40] pp. 91–92]). Let the functions \( f_k(t) \) for \( k = 1, 2 \) be piecewise continuous in arbitrary finite intervals included in \((0, \infty)\). If there exist some constants \( M_k > 0 \) and \( c_k \geq 0 \) such that the inequalities \( |f_k(t)| \leq M_k e^{c_k t} \) for \( k = 1, 2 \) are valid, then

\[
\int_0^\infty \left[ \int_0^t f_1(u) f_2(t-u) \, du \right] e^{-st} \, dt = \int_0^\infty f_1(u) e^{-su} \, du \int_0^\infty f_2(v) e^{-sv} \, dv.
\]

**Lemma 2.4** ([14] Lemma 4) and ([15] Section 3]). Let the functions \( A(x) \) and \( B(x) \neq 0 \) be defined on \((0, \infty)\) such that their Laplace transforms exist. If the ratio \( \frac{A(x)}{B(x)} \) is increasing, then the ratio \( \frac{\int_0^\infty A(x) e^{-st} \, dx}{\int_0^\infty B(x) e^{-st} \, dx} \) is decreasing on \((0, \infty)\).

**Lemma 2.5.** Let

\[
g(t) = \begin{cases} 
t \frac{t}{1-e^{-t}}, & t \neq 0; \\
1, & t = 0.
\end{cases}
\]

Then the following conclusions are valid.

1. The function \( g(t) \) is infinitely differentiable on \((-\infty, \infty)\), increasing from \((-\infty, \infty) \) onto \((0, \infty)\), convex on \((-\infty, \infty)\), and logarithmically concave on \((-\infty, \infty)\).

2. For fixed \( s \in (0, 1) \), the ratio \( \frac{g(t)}{g(st)} \) is decreasing in \( t \) from \((0, \infty)\) onto \((0, 1)\).

3. For \( s \in (0, \frac{1}{2}) \) and \( t \in (0, \infty) \), the mixed second-order partial derivative

\[
\frac{\partial^2 \ln[g(st)g((1-s)t)]}{\partial s \partial t} > 0.
\]

**Proof.** The differentiability, monotonicity, and convexity of \( g(t) \) come from utilization of [22] Lemma 2.3].

Direct computation yields

\[
[\ln g(t)]'' = [\ln g(-t)]'' = -\frac{e^{2t} - e^t (t^2 + 2) + 1}{(e^t - 1)^2 t^2}
\]

\[
= -\frac{1}{(e^t - 1)^2 t^2} \sum_{k=4}^{\infty} (2k) - (k-1)k - 2 \frac{t^k}{k!} < 0
\]

on \((0, \infty)\). Hence, the function \( g(t) \) is logarithmically concave on \((-\infty, \infty)\). See also the proof of [19] Lemma 2.3].

It is straightforward that

\[
\lim_{t \to 0} \frac{g'(t)}{g(st)} = \lim_{t \to 0} \frac{g'(t)}{g(st)} = \frac{1}{s} = 1
\]
and
\[ \lim_{t \to \infty} \frac{g^*(t)}{g(st)} = \frac{\lim_{t \to \infty} [g(t)/t]^s}{\lim_{t \to \infty} [g(st)/st]^s} = \frac{t^s}{s^s t^{-s}} = 1 = \lim_{t \to \infty} \frac{1}{t^{1-s}} = 0. \]

The first derivative of the ratio \( \frac{g^*(t)}{g(st)} \) is
\[\frac{d}{dt} \left( \frac{g^*(t)}{g(st)} \right) = \frac{sg^*(t)}{g(st)} \left[ \frac{g'(t)}{g(t)} - \frac{g'(st)}{g(st)} \right].\]

Hence, for arriving at decreasing property of the ratio \( \frac{g^*(t)}{g(st)} \), it is sufficient to show that the ratio \( \frac{g'(t)}{g(t)} \) is decreasing on \((0, \infty)\). For this, it is sufficient to show that the function \( g(t) \) is logarithmically concave on \((0, \infty)\). This requirement has been verified in last paragraph.

By Lemma 2.1 straightforward differentiation gives
\[
\frac{\partial \ln[g(st)g((1-s)t)]}{\partial t} = \frac{1}{t} \left[ t + 2e^t + 2 - e^t[(1-s)t + 2] - e^{(1-s)t}(t + 2) \right] - \left[ e^t - e^{2st} \right] = 0,
\]
and
\[
\frac{(e^t - e^{2st})(1 - s)t + 1)}{(e^t - e^{2st})}' = \frac{1}{2} - \frac{e^{(1-2s)t} - (1 - 2s)t - 1}{2t}.\]

This means that the partial derivative \( \frac{\partial \ln[g(st)g((1-s)t)]}{\partial t} \) is increasing in \( s \in [0, \frac{1}{2}] \). This means that the partial derivative \( \frac{\partial \ln[g(st)g((1-s)t)]}{\partial t} \) is increasing in \( s \in [0, \frac{1}{2}] \) for fixed \( t > 0 \). As a result, the inequality (2.2) is valid. The proof of Lemma 2.1 is complete. \( \square \)

Lemma 2.6 ([18] Lemma 2.1). For \( k \in \mathbb{N} \), we have the limits
\[ \lim_{x \to 0^+} [x^k \psi^{(k-1)}(x)] = (-1)^k (k - 1)! \quad (2.3) \]
and
\[ \lim_{x \to \infty} [x^k \psi^{(k)}(x)] = (-1)^{k-1} (k - 1)!. \quad (2.4) \]

Lemma 2.7. For \( m, n, p, q \in \mathbb{N} \) such that \((p, q) \succ (m, n)\), the function
\[ \frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}} \]
is increasing in \( s \in (0, \frac{1}{2}) \).

Proof. Direct computation yields
\[
\frac{s^{m-1}(1-s)^{n-1} + (1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1} + (1-s)^{p-1}s^{q-1}} = \frac{\left(\frac{1}{2} - 1\right)^n + \left(\frac{1}{2} - 1\right)^m}{\left(\frac{1}{2} - 1\right)^q + \left(\frac{1}{2} - 1\right)^p} = \frac{z^{n-1} + z^{m-1}}{z^{q-1} + z^{p-1}} = \frac{z^n + z^m}{z^q + z^p}.
\]
and

\[
\frac{d}{dz} \left( \frac{z^b + z^c}{1 + z^a} \right) = \frac{(bz^b + cz^c)(1 + z^a) - az^a(z^b + z^c)}{z(1 + z^a)^2}
\]

\[
= \frac{bz^b(1 - z^{2c}) + cz^c(1 - z^{2b})}{z(1 + z^a)^2}
\]

< 0,

where \( z = \frac{1}{a} - 1 \in (1, \infty) \) and \( a = p - q > b = m - q \geq c = n - q > 0 \) with \( a = b + c \).

The proof of Lemma 2.7 is complete. □

**Lemma 2.8.** Let the functions \( U(x) \), \( V(x) \) > 0, and \( W(x, t) > 0 \) be integrable in \( x \in (a, b) \). If the ratios \( \frac{\partial W(x, t)}{\partial t} \) and \( \frac{U(x)}{V(x)} \) are both increasing or both decreasing in \( x \in (a, b) \), then the ratio

\[
R(t) = \frac{\int_a^b U(x)W(x, t) \, dx}{\int_a^b V(x)W(x, t) \, dx}
\]

is increasing in \( t \); if one of the ratios \( \frac{\partial W(x, t)}{\partial t} \) and \( \frac{U(x)}{V(x)} \) are increasing and the other is decreasing in \( x \in (a, b) \), then the ratio \( R(t) \) is decreasing in \( t \).

**Proof.** Direct differentiation gives

\[
R'(t) = \left[ \int_a^b U(x) \frac{\partial W(x, t)}{\partial t} \, dx \int_a^b V(x)W(x, t) \, dx 
- \int_a^b U(x)W(x, t) \, dx \int_a^b V(x) \frac{\partial W(x, t)}{\partial t} \, dx \right]
\]

\[
\left[ \int_a^b V(x)W(x, t) \, dx \right]^2
\]

\[
= \left[ \int_a^b U(x) \frac{\partial W(x, t)}{\partial t} \, dx \int_a^b V(x)W(x, t) \, dx 
- \int_a^b U(x)W(x, t) \, dx \int_a^b V(x) \frac{\partial W(x, t)}{\partial t} \, dx \right]
\]

\[
\left[ \int_a^b V(x)W(x, t) \, dx \right]^2
\]

\[
= \int_a^b \int_a^b U(x)V(y)W(x, t)W(y, t) \left[ \frac{\partial W(x, t)}{\partial t} \right] - \frac{\partial W(x, t)}{\partial t} \right] \, dx \, dy
\]

\[
\int_a^b \int_a^b V(x)W(x, t) \, dx \left[ \int_a^b V(x)W(x, t) \, dx \right]^2
\]

\[
= \frac{2 \int_a^b V(x)W(x, t) \, dx}{\left[ \int_a^b V(x)W(x, t) \, dx \right]^2}.
\]

The proof of Lemma 2.8 is complete. □

**Lemma 2.9** ([27] p. 161, Theorem 12b). A function \( f(x) \) is completely monotonic on \((0, \infty)\) if and only if

\[
f(x) = \int_0^\infty e^{-xt} \, d\sigma(t), \quad x \in (0, \infty), \tag{2.5}
\]

where \( \sigma(s) \) is non-decreasing and the integral in (2.5) converges for \( x \in (0, \infty) \).

3. Decreasing property of a ratio defined by three polygamma functions

In this section, we prove that the function \( Q_{m,n}(x) \) defined in (1.3) is decreasing.
By Lemma 2.4, we only need to prove the ratio
\[ Q_m,n(x) = \frac{\int_0^\infty t^{m+n} g(t) e^{-xt} dt}{\int_0^\infty t^{m-1} g(t) e^{-xt} dt} \]

in (1.3) is decreasing on \((0, \infty)\), as proved in Lemma 2.5 in this paper. As a result, the function
\[ Q_m,n(x) \]
is valid on \((0, \infty)\) and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

**Proof.** By virtue of the integral representation (2.1), we can rearranged \(Q_m,n(x)\) as
\[ Q_m,n(x) = -\frac{\int_0^\infty t^{m+n-1} g(t) e^{-xt} dt}{\int_0^\infty t^{m-1} g(t) e^{-xt} dt} \int_0^\infty t^{n-1} g(t) e^{-xt} dt \]
By Lemma 2.3, we obtain
\[ Q_m,n(x) = -\frac{\int_0^\infty t^{m+n-1} g(t) e^{-xt} dt}{\int_0^\infty \int_0^t u^{m-1} (t-u)^{n-1} g(u) (t-u) du} e^{-xt} dt \]
By Lemma 2.4, we only need to prove the ratio
\[ P_m,n(t) = \frac{\int_0^t u^{m-1} (t-u)^{n-1} g(u) g((1-s)t) ds}{\int_0^t u^{m-1} (1-s)^{n-1} g(s) g((1-s)t) ds} \]
is decreasing on \((0, \infty)\). Hence, it suffices to show that the ratio \(\frac{g(st)}{g(t)}\) for fixed \(s \in (0, 1)\) is increasing in \(t \in (0, \infty)\). This increasing property of \(\frac{g(st)}{g(t)}\) has been proved in Lemma 2.5 in this paper. As a result, the function \(Q_m,n(x)\) defined in (1.3) is decreasing on \((0, \infty)\).

Making use of the limits (2.3) and (2.4) in Lemma 2.6 yields
\[ \lim_{x \to 0^+} Q_m,n(x) = \lim_{x \to 0^+} \frac{\int x^{m+n+1} \psi^{(m+n)}(x)}{\int x^{m+n+1} \psi^{(m)}(x)} = 0 \]
and
\[ \lim_{x \to \infty} Q_m,n(x) = \lim_{x \to \infty} \frac{\int x^{m+n} \psi^{(m+n)}(x)}{\int x^{m+n} \psi^{(m)}(x)} = \frac{(-1)^{m+n-1} (m+n-1)!}{(m-1)! (n-1)!} \]
The proof of Theorem 3.1 is complete.

**4. Decreasing Property of a Ratio Defined by Four Polygamma Functions**

In this section, we prove that the function \(Q_{m,n,p,q}\) defined in (1.4) is decreasing.

**Theorem 4.1.** For \(m,n,p,q \in \mathbb{N}\) with the majorizing relation \((p,q) \succ (m,n)\), the ratio \(Q_{m,n,p,q}(x)\) defined in (1.4) is decreasing from \((0, \infty)\) onto the interval
Proof. By the limits (2.3) and (2.4) in Lemma 2.6, we obtain
\[\lim_{x \to 0^+} Q_{m,n,p,q}(x) = \frac{m!n!}{p!q!},\]
and
\[\lim_{x \to \infty} Q_{m,n,p,q}(x) = \frac{m!n!}{p!q!} m^{m+1}n^{n+1} p^{p+1} q^{q+1} e^{-st} dt \int_0^\infty t^{m+1} g(t) e^{-xt} dt \int_0^\infty t^{n+1} g(t) e^{-xt} dt.\]
Utilizing Lemma 2.3 gives
\[Q_{m,n,p,q}(x) = \frac{1}{m!n!} \left[ \int_0^t r^{m-1} (t-u)^{n-1} g(u) g(t-u) du \right] e^{-xt} dt \int_0^\infty t^{m+1} g(t) e^{-xt} dt \int_0^\infty t^{n+1} g(t) e^{-xt} dt.\]
Employing Lemma 2.4 tells us that, it suffices to prove the increasing property in t of the ratio
\[\frac{\int_0^t u^{m-1} (t-u)^{n-1} g(u) g(t-u) du}{\int_0^t u^{p-1} (t-u)^{q-1} g(u) g(t-u) du} = \frac{\int_0^{1/2} s^{m-1} (1-s)^{n-1} g(st) g((1-s)t) ds}{\int_0^{1/2} s^{p-1} (1-s)^{q-1} g(st) g((1-s)t) ds} = \frac{\int_0^{1/2} \phi_{m,n}(s) \phi(t) ds}{\int_0^{1/2} \phi_{p,q}(s) \phi(t) ds},\]
where
\[\phi_{\ell,j}(s) = s^{j-1}(1-s)^{j-1} + (1-s)^{j-1}s^{j-1} \text{ and } \varphi(s, t) = g(st) g((1-s)t). \quad (4.1)\]
Lemma 2.7 implies that the ratio \(\phi_{m,n}(s)\) is increasing in \(s \in (0, 1/2)\) for \((p, q) \succ (m, n)\). Further making use of the inequality (2.2) in Lemma 2.5 and utilizing Lemma 2.8 reveal that the function \(Q_{m,n,p,q}(x)\) is decreasing in \(x \in (0, \infty)\). The proof of Theorem 4.1 is complete. \(\square\)
5. An alternative proof of Theorem 1.1

In this section, we supply an alternative proof of Theorem 1.1. For \( q = 0 \), we have

\[
F_{p, m, n, 0, c}(x) = \left| \psi^{(m)}(x) \right| \left| \psi^{(n)}(x) \right| - c \left| \psi^{(p)}(x) \right| \left| \psi^{(q)}(x) \right|
\]

\[
= \int_0^\infty t^{m-1} g(t) e^{-xt} dt \int_0^\infty t^{n-1} g(t) e^{-xt} dt - c \int_0^\infty t^{p-1} g(t) e^{-xt} dt \int_0^\infty t^{q-1} g(t) e^{-xt} dt
\]

\[
= \int_0^\infty \left[ \int_0^t u^{m-1}(t-u)^{n-1} g(u) g(t-u) du - c t^{m+n-1} g(t) \right] e^{-xt} dt
\]

\[
= \int_0^\infty \int_0^1 s^{m-1}(1-s)^{n-1} g(st) g((1-s)t) ds \left[ s^{m+n-1} g(t) e^{-xt} dt \right]
\]

where we used the integral representation (2.1) and Lemma 2.3. From the second property in Lemma 2.3, it follows that the function

\[
\int_0^1 s^{m-1}(1-s)^{n-1} g(st) g((1-s)t) g(t) ds
\]

is decreasing in \( t \in (0, \infty) \) and has the limits

\[
\int_0^1 s^{m-1}(1-s)^{n-1} g(st) g((1-s)t) g(t) ds \rightarrow \begin{cases} 
\int_0^1 s^{m-1}(1-s)^{n-1} ds, & t \to 0; \\
\infty, & t \to \infty.
\end{cases}
\]

Consequently, basing on Lemma 2.9, we see that, if and only if

\[
c \leq \int_0^1 s^{m-1}(1-s)^{n-1} ds = B(m, n) = \frac{(m-1)! (n-1)!}{(m+n-1)!},
\]

the function \( F_{p, m, n, 0, c}(x) \) is completely monotonic on \((0, \infty)\).
where we used the integral representation (2.1) and Lemma 2.3. Employing the inequality (2.2) in Lemma 2.5 and applying Lemmas 2.7 and 2.8 reveal that the function
\[
\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] g((1-s)t) \, ds
\]
\[
\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] g((1-s)t) \, ds
\]
is increasing in \( t \in (0, \infty) \), where \( \phi_{i,j}(s) \) and \( \varphi(s,t) \) are defined in (4.1). It is easy to see that
\[
\lim_{t \to 0} \int_0^{1/2} \phi_{m,n}(s) \varphi(s,t) \, ds = \int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] \, ds
\]
\[
\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] \, ds = B(m, n) / B(p, q) = (m-1)!/(n-1)!
\]
\[
(p-1)!/(q-1)!
\]
Since \( \lim_{t \to \infty} g(t) = 1 \), we acquire
\[
\lim_{t \to \infty} \int_0^{1/2} \phi_{m,n}(s) \varphi(s,t) \, ds = \int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] \, ds
\]
\[
\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] \, ds = B(m+1, n+1) / B(p+1, q+1) = m!/p!q!
\]
Combining these with Lemma 2.9 concludes that,
1. if and only if
\[
c \leq \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!},
\]
the function \( F_{p,m,n,q,c}(x) \) is completely monotonic in \( x \in (0, \infty) \);
2. if and only if \( c \geq \frac{m!}{p!q!} \), the function \( -F_{p,m,n,q,c}(x) \) is completely monotonic in \( x \in (0, \infty) \).

The proof of Theorem 1.1 is complete.

6. Remarks

Finally, we list several remarks on our main results and their proofs.

Remark 6.1. The papers [3, 4, 7, 8, 42] are related to Theorem 1.1. Theorems 3.1 and 4.1 in this paper are related to some results reviewed and surveyed in [15, 26] and closely related references therein.

Remark 6.2. Lemma 2.9 in this paper generalizes the second item in [19, Lemma 2.3], which reads that the function \( \frac{G(t)}{\varphi(t)} \) is decreasing from \((0, \infty)\) onto \((0, 1)\).

Remark 6.3. Taking \( W(x, t) = e^{-xt} \) in Lemma 2.8 gives
\[
\frac{\partial W(x, t)}{\partial t} / W(x, t) = \frac{\partial e^{-xt}}{\partial t} / e^{-xt} = -x,
\]
which is decreasing in $x \in (-\infty, \infty)$. Further setting $U(x) = A(x)$, $V(x) = B(x)$, and $(a, b) = (0, \infty)$ in Lemma 2.8 leads to Lemma 2.1, which was established in Lemma 4. This means that Lemma 2.8 in this paper is a generalization of Lemma 2.4. Lemma 2.8 has been announced in Remark 7.2.

Remark 6.4. From the majorizing relation $(n + 2, n) > (n + 1, n + 1)$, we see that Theorem 4.1 in this paper generalizes a conclusion in Theorem 2, which states that the function $\frac{\psi^{(n+1)}(x)^2}{\psi^{(n)}(x)}$ for $n \geq 1$ is decreasing from $(0, \infty)$ onto the interval $(\frac{n}{n+2}, \frac{n+1}{n+2})$.

Remark 6.5. Direct differentiation gives
$$Q'_{m,n}(x) = \frac{\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x) - \psi^{(m+n)}(x)\psi^{(m)}(x)\psi^{(n)}(x)^\tau}{(\psi^{(m)}(x)\psi^{(n)}(x))}.$$

The decreasing property of $Q_{m,n}(x)$ in Theorem 3.1 implies that the inequality
$$\psi^{(m+n)}(x)\psi^{(m)}(x)\psi^{(n)}(x)^\tau - \psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x) > 0,$$
equivalently,
$$\frac{\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(m+n)}(x)\psi^{(m)}(x)\psi^{(n)}(x)} > 0,$
is valid on $(0, \infty)$ for $m, n \in \mathbb{N}$.

We guess that, for $m, n \in \mathbb{N}$, the function
$$\psi^{(m+n)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$
should be completely monotonic in $x \in (0, \infty)$.

Generally, one can discuss necessary and sufficient conditions on $\Omega_{m,n} \in \mathbb{R}$ such that the function
$$\psi^{(m+n)}(x)\psi^{(m)}(x)\psi^{(n)}(x) - \Omega_{m,n}\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$$
and its opposite are respectively completely monotonic on $(0, \infty)$.

Remark 6.6. It is immediate that
$$Q'_{m,n,p,q}(x) = \frac{\left[\psi^{(m)}(x)\psi^{(n)}(x)^\tau\right] \left[\psi^{(p)}(x)\psi^{(q)}(x)\right]}{(\psi^{(m)}(x)\psi^{(n)}(x))^2}.$$

The decreasing property of $Q_{m,n,p,q}(x)$ in Theorem 4.1 implies that the inequality
$$\left[\psi^{(m)}(x)\psi^{(n)}(x)^\tau\right] \left[\psi^{(p)}(x)\psi^{(q)}(x)\right] - \left[\psi^{(m)}(x)\psi^{(n)}(x)^\tau\right] \left[\psi^{(p)}(x)\psi^{(q)}(x)\right] > 0,$$
equivalently,
$$\frac{\psi^{(p)}(x)\psi^{(q)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} > 0,$$
is valid on $(0, \infty)$ for $(p, q) > (m, n)$.

We guess that, for $(p, q) > (m, n)$, the function
$$\psi^{(m)}(x)\psi^{(n)}(x)\psi^{(p)}(x)\psi^{(q)}(x)$$
should be completely monotonic in $x \in (0, \infty)$.

Generally, for $(p, q) > (m, n)$, one can discuss necessary and sufficient conditions on $\Omega_{m,n,p,q} \in \mathbb{R}$ such that the function
$$\left[\psi^{(m)}(x)\psi^{(n)}(x)^\tau\right] \left[\psi^{(p)}(x)\psi^{(q)}(x)\right] - \Omega_{m,n,p,q}\left[\psi^{(m)}(x)\psi^{(n)}(x)^\tau\right] \left[\psi^{(p)}(x)\psi^{(q)}(x)\right]$$
and its opposite are respectively completely monotonic on $(0, \infty)$.
Remark 6.7. For \( n \geq 2 \) and two nonnegative integer tuples \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n \), let

\[
P_{\alpha,\beta;C_{\alpha,\beta}}(x) = \prod_{r=1}^{n} \psi^{(\alpha_r)}(x) - C_{\alpha,\beta} \prod_{r=1}^{n} \psi^{(\beta_r)}(x)
\]

and

\[
Q_{\alpha,\beta}(x) = \prod_{r=1}^{n} \psi^{(\alpha_r)}(x) / \prod_{r=1}^{n} \psi^{(\beta_r)}(x)
\]
on \((0, \infty)\), where we denote \( \psi^{(0)}(x) = -1 \) for our own convenience. It is clear that

\[
P_{(2k,0),(k,k);C_{(2k,0),(k,k)}}(x) = \mathcal{F}_{k,-C_{(2k,0),(k,k)}}(x), \quad Q_{(2k,0),(k,k)}(x) = \mathcal{O}_{k,2}(x), \quad Q_{(m+n,0),(m,n)}(x) = Q_{m,n}(x), \quad Q_{(m,n),(p,q)}(x) = Q_{m,n;p,q}(x).
\]

We guess that, if \( \alpha \succ \beta \), the function \( Q_{\alpha,\beta}(x) \) is increasing from \((0, \infty)\) onto the interval

\[
\left( \prod_{r=1}^{n} \alpha_r ! \prod_{r=1}^{n} (\alpha_r - 1) ! \right) \left/ \left( \prod_{r=1}^{n} \beta_r ! \prod_{r=1}^{n} (\beta_r - 1) ! \right) \right.
\]

Generally, for \( \alpha \succ \beta \), one can discuss necessary and sufficient conditions on \( C_{\alpha,\beta} \in \mathbb{R} \) such that the function \( P_{\alpha,\beta;C_{\alpha,\beta}}(x) \) and its opposite are respectively completely monotonic on \((0, \infty)\).

Remark 6.8. Gurland’s ratio

\[
T(s,t) = \frac{\Gamma(s) \Gamma(t)}{\Gamma((s + t)/2)^2}
\]

was firstly defined in \([11]\). In appearance, we can regard the functions \( Q_{m,n}(x) \) and \( Q_{m,n;p,q}(x) \) defined in \((1.3)\) and \((1.4)\) as analogues of Gurland’s ratio \( T(s,t) \). In \([32, 38]\), there existed a detailed survey and review of Gurland’s ratio \( T(s,t) \) and related results. In \([46]\), the functions \( T(\frac{1}{p}, \frac{2}{p}) \) and \( T(\frac{1}{p}, \frac{5}{p}) \) with their statistical backgrounds were mentioned.

Remark 6.9. The ratios of finitely many gamma functions and polygamma functions have applications in differential geometry, manifolds, statistics, probability, and their intersections. See, for example, the papers \([5, 9, 33, 34, 48]\).

Remark 6.10. As a generalization of decreasing property of real functions of one variable, one can consider (logarithmically) complete monotonicity and completely monotonic degrees. For details, please refer to \([14, 15, 31, 39, 41, 43, 49]\) and the review article \([26]\).

Remark 6.11. This paper is a revised version of the electronic preprint \([16]\) and is the eighth one in a series of articles including \([17, 18, 20, 21, 22, 23, 24, 25, 29]\).

7. Declarations

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References

[1] M. Abramowitz and I. A. Stegun (Eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, Conformal Invariants, Inequalities, and Quasiconformal Maps, John Wiley & Sons, New York, 1997.

[3] H. Alzer and J. W. Wells, Inequalities for the polygamma functions, SIAM J. Math. Anal. 29 (1999), no. 5, 1122–1128; available online at https://doi.org/10.1137/S0036141097325071

[4] N. Batir, On some properties of digamma and polygamma functions, J. Math. Anal. Appl. 328 (2007), no. 1, 452–465; available online at https://doi.org/10.1016/j.jmaa.2006.05.065.

[5] A. L. Brigant, S. C. Preston, and S. Puechmorel, Fisher–Rao geometry of Dirichlet distributions, Differential Geom. Appl. 74 (2021), 101702, 16 pages; available online at https://doi.org/10.1016/j.difgeo.2020.101702.

[6] Y.-M. Chu and X.-H. Zhang, Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave, J. Math. Kyoto Univ. 48 (2008), no. 1, 229–238; available online at https://doi.org/10.1215/kjm/1250280982.

[7] P. Gao, Some completely monotonic functions involving the polygamma functions, J. Inequal. Appl. 2019, Paper No. 218, 9 pages; available online at https://doi.org/10.1186/s13660-019-2172-3.

[8] P. Gao, Some monotonicity properties of gamma and q-gamma functions, ISRN Math. Anal. 2011, Art. ID 375715, 15 pages; available online at https://doi.org/10.5402/2011/375715.

[9] B.-N. Guo and F. Qi, On the increasing monotonicity of a sequence originating from computation of the probability of intersecting between a plane couple and a convex body, Turkish J. Anal. Number Theory 3 (2015), no. 1, 21–23; available online at http://dx.doi.org/10.12961/tjant-3-1-5.

[10] B.-N. Guo, F. Qi, and H. M. Srivastava, Some uniqueness results for the non-trivially complete monotonicity of a class of functions involving the polygamma and related functions, Integral Transforms Spec. Funct. 21 (2010), no. 11, 849–868; available online at https://doi.org/10.1080/10652461003748112.

[11] J. Gurland, An inequality satisfied by the gamma function, Skand Aktuarietidskr 39 (1956), 171–172; available online at https://doi.org/10.1080/03661238.1956.1041949.

[12] A. W. Marshall, I. Olkin, and B. C. Arnold, Inequalities: Theory of Majorization and its Applications, 2nd Ed., Springer Verlag, New York/Dordrecht/Heidelberg/London, 2011; available online at http://dx.doi.org/10.1007/978-0-387-68276-1.

[13] D. S. Mitrinovic, J. E. Pečarić, and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht-Boston-London, 1999; available online at https://doi.org/10.1007/978-94-017-1433-5.

[14] F. Qi, Bounds for completely monotonic degree of a remainder for an asymptotic expansion of the trigamma function, Arab J. Basic Appl. Sci. 28 (2021), no. 1, 314–318; available online at https://doi.org/10.1080/25765299.2021.1962060.

[15] F. Qi, Completely monotonic degree of a function involving trigamma and tetragamma functions, AIMS Math. 5 (2020), no. 4, 3391–3407; available online at https://doi.org/10.3934/math.2020219.

[16] F. Qi, Decreasing monotonicity of two ratios defined by three or four polygamma functions, HAl preprint (2020), available online at https://hal.archives-ouvertes.fr/hal-02998414.

[17] F. Qi, Decreasing property and complete monotonicity of two functions constituted via three derivatives of a function involving trigamma function, Math. Slovaca 72 (2022), in press; available online at https://doi.org/10.31219/osf.io/wbh2q.

[18] F. Qi, Lower bound of sectional curvature of Fisher–Rao manifold of beta distributions and complete monotonicity of functions involving polygamma functions, Results Math. 76 (2021), no. 4, Article 217, 16 pages; available online at https://doi.org/10.1007/s00025-021-01530-2.

[19] F. Qi, Lower bound of sectional curvature of manifold of beta distributions and complete monotonicity of functions involving polygamma functions, Preprints 2020, 2020110315, 22 pages; available online at https://doi.org/10.20944/preprints202011.0315.v1.

[20] F. Qi, Necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic, Math. Inequal. Appl. 24 (2021), no. 3, 845–856; available online at https://doi.org/10.7153/mia-2021-24-58.

[21] F. Qi, Necessary and sufficient conditions for a difference defined by four derivatives of a function containing trigamma function to be completely monotonic, Appl. Comput. Math. 21 (2022), accepted; available online at https://doi.org/10.31219/osf.io/66czx.
[43] Z.-H. Yang and J.-F. Tian, *A class of completely mixed monotonic functions involving the gamma function with applications*, Proc. Amer. Math. Soc. **146** (2018), no. 11, 4707–4721; available online at https://doi.org/10.1090/proc/14199

[44] Z.-H. Yang and J.-F. Tian, *Monotonicity and inequalities for the gamma function*, J. Inequal. Appl. **2017**, Paper No. 317, 15 pages; available online at https://doi.org/10.1186/s13660-017-1591-9

[45] Z.-H. Yang and J.-F. Tian, *Monotonicity rules for the ratio of two Laplace transforms with applications*, J. Math. Anal. Appl. **470** (2019), no. 2, 821–845; available online at https://doi.org/10.1016/j.jmaa.2018.10.034

[46] Z.-H. Yang, B.-Y. Xi, and S.-Z. Zheng, *Some properties of the generalized Gaussian ratio and their applications*, Math. Inequal. Appl. **23** (2020), no. 1, 177–200; available online at https://doi.org/10.7153/mia-2020-23-15

[47] H.-P. Yin, X.-M. Liu, J.-Y. Wang, and B.-N. Guo, *Necessary and sufficient conditions on the Schur convexity of a bivariate mean*, AIMS Math. **6** (2021), no. 1, 296–303; available online at https://doi.org/10.3934/math.2021018

[48] J. F. Zhao, P. Xie, and J. Jiang, *Geometric probability for pairs of hyperplanes intersecting with a convex body*, Math. Appl. (Wuhan) **29** (2016), no. 1, 233–238. (Chinese)

[49] L. Zhu, *Completely monotonic integer degrees for a class of special functions*, AIMS Math. **5** (2020), no. 4, 3456–3471; available online at https://doi.org/10.3934/math.2020224

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