THE SPECIAL CONCAVE TORIC DOMAIN FOR THE ROTATING KEPLER PROBLEM

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INTRODUCTION

In this paper, I am going to introduce a bounded component for the rotating Kepler problem when the critical energy value is less than or equal to $-\frac{3}{2}$. We will see this bounded component after applying the Ligon-Schaaf regularization and the Levi-Civita regularization respectively on the rotating Kepler problem. This is one of my Ph.D. results. This result for the first time proved in my thesis and I use the results to compute the several ECH capacities for the RKP. In the paper [1], you can see the final results and new methods for ECH capacities computation.

Here I give a special concave toric domain for the RKP [2] and I will use it to compute the ECH capacities for the rotating Kepler problem for energies less than or equal to the critical energy value $-\frac{3}{2}$ [1]. Since the RKP is just the Kepler problem in the rotating coordinate system and its importance lies the fact that it rises as a limiting case of the restricted 3-body problem, the following result are interesting to understand symplectic embedding for the restricted 3-body problem.

THE ROTATING KEPLER PROBLEM

The rotating Kepler problem is the Kepler problem in rotating coordinate system. It is also a limit case of the restricted three body problem, where the mass of one primaries is zero. The Hamiltonian of the Kepler problem is

$$ H : T^*(\mathbb{R}^2 \setminus \{0\}) \longrightarrow \mathbb{R} $$

$$ H(q,p) = \frac{1}{2} |p|^2 - \frac{1}{|q|}, $$

and the Hamiltonian of the angular momentum is

$$ L : T^*\mathbb{R}^2 \longrightarrow \mathbb{R} $$

$$ (q,p) \mapsto q_1p_2 - q_2p_1, $$

generate the rotation flow around the origin. Therefore, the Hamiltonian of the rotating Kepler problem is

$$ K : T^*(\mathbb{R}^2 \setminus \{0\}) \longrightarrow \mathbb{R} $$

$$ K(q,p) = \frac{1}{2} |p|^2 - \frac{1}{|q|} + q_1p_2 - q_2p_1, \quad (q,p) \in T^*(\mathbb{R}^2 \setminus \{0\}), $$

the Hamiltonian system $K = H + L$ is an integrable system in the sense of Arnold-Liouville. [2,3].

Lemma 0.1. The angular momentum is preserved under the flow of $X_H$. Thus $H$ and $L$ Poisson commute.

Proof. The standard $SO(2)$ action acts Hamiltonianly on $T^*\mathbb{R}^2$ with the momentum map $L$. Thus the Hamiltonian for the central force is $SO(2)$-invariant, so the Noether theorem implies the results. \qed
Since $H$ and $L$ Poisson commute, we can write
\[(0.7)\] \[\{K, L\} = \{H, L\} + \{L, L\} = 0.\]

Consider the Hamiltonian of the rotating Kepler problem as
\[(0.8)\] \[K(q, p) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) + \frac{-1}{|q|} - \frac{1}{2}|q|^2\]
where we define the effective potential
\[(0.9)\] \[U : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, \quad U(q) = -\frac{1}{|q|} - \frac{1}{2}|q|^2.\]
Thus we can write
\[(0.10)\] \[K(q, p) = \frac{1}{2}((p_1 - q_2)^2 + (p_2 + q_1)^2) + U(q).\]

**Lemma 0.2.** The effective potential $U$ of the rotating Kepler problem has a unique critical value $-\frac{3}{2}$ and its critical set constant of a circle of radius 1 around the origin.

**Proof.** Proof in [2]. □

Via the projection map $\pi_{\text{crit}(K)}(q_1, q_2) = (q_1, q_2, q_2, -q_1)$, where $(q, p) \mapsto q$, the critical points of $K$ and $U$ are bijection. The critical value of $K$ coincides with the critical value of $U$ at the same critical points.

The RKP has a unique critical value at $-\frac{3}{2}$. Below this critical value the energy hypersurface has two connected components. The projection of one of the components is bounded in configuration space where the projection of the other one is unbounded.

Consider the Runge-Lenz vector
\[(0.11)\] \[A^2 = 1 + 2cL^2,\]
such that whose length corresponds to the eccentricity of corresponding Kepler ellipse.

If we substitute the Hamiltonian [0.5] on the above equation, we have the following inequality,
\[(0.12)\] \[0 \leq 1 + 2H(K - H)^2 = 1 + 2K^2H - 4KH^2 + 3H^3 =: p(K, H).\]
The equality $p(K, H) = 0$ holds if and only if the eccentricity of the corresponding periodic orbit vanishes, i.e. when periodic orbits are circular.

Denote the Kepler ellipse $\varepsilon_\tau : [0, \pi] \to \mathbb{R}$ where $\pi$ is period. With this relation, we can obtain a solution for the RKP as
\[(0.13)\] \[\varepsilon_\tau^R = e^{i\theta} \varepsilon_\tau(t)\]
which is not longer period.

On the other hand, the angular momentum $L$ generate the rotation in the q-plane and the p-plane. Thus we have two cases of the orbits.

i) $\varepsilon_\tau$ is a circular. In this case, $\varepsilon_\tau^R$ is a periodic unless it is a critical point when $\tau = 2\pi$.

ii) $\varepsilon_\tau$ is not circle. In this case, it is a proper ellipse or a collision orbit that looks like a line.

If we consider $\varepsilon_\tau$ as an ellipse orbit. The resonance relation satisfies in
\[(0.14)\] \[2\pi l = \tau k,\]
for some positive integer $k$ and $l$.

**Lemma 0.3.** Periodic orbits in the rotating Kepler problem of the second kind satisfy the following rotational symmetry
\[(0.15)\] \[\varepsilon_\tau^R(t + \tau) = e^{2\pi il/k} \varepsilon_\tau^R(t).\]
Proof. The resonance condition gives us the equality \( \tau = 2\pi l/k \) and therefore we have
\[
\varepsilon_\tau^R(t + \tau) = e^{it + i\tau} \varepsilon_\tau(t + \tau) = e^{2\pi i l/k} e^{it} \varepsilon_\tau(t) = e^{2\pi il/k} \varepsilon_\tau^R(t).
\] \hfill (0.16)

If we fix \( K \), the function
\[
p_K := p(K, \cdot)
\]
is a cubic polynomial in \( H \). Now consider \( H \) is fixed, we define the following function
\[
p^H := p(\cdot, K)
\]
which is a quadratic polynomial in \( K \). Note that, \( K = -\frac{3}{2} \) is a unique critical value.

Let \( K > -\frac{3}{2} \) and denote the root of the cubic polynomial by \( R^1(K), R^2(K), R^3(K) \) in \( \mathbb{R} \) with order \( R^1(K) < R^2(K) < R^3(K) \). We have the following equalities in [2] when \( K = -\frac{3}{2} \)
\[
R^1(-\frac{3}{2}) = -2, R^2(-\frac{3}{2}) = R^3(-\frac{3}{2}) = -\frac{1}{2}.
\] \hfill (0.19)

If \( K > -\frac{3}{2} \), we can extend \( R^1 \) to a continuous function on the whole real line such that \( R^1(K) \) be unique real root of \( p_K \).

The circular orbits exist only if it holds
\[
1 + 2HL^2 = 0.
\] \hfill (0.20)

The second kind of periodic orbits of the RKP are positive eccentricity respectively rotating collision orbits.

A Kepler ellipse in the inertial system becomes an orbit in the rotating or synodical system.

Since the period of the rotating coordinate system is \( 2\pi \), if the orbit in the rotating system is periodic of the period of the ellipse should be
\[
\tau = \frac{2\pi l}{k},
\]
where \( k \) and \( l \) relatively prime.

Lemma 0.4. The minimum period \( \tau \) of a Kepler ellipse only depends on the energy of a periodic orbit with
\[
c_{k,l} = -\frac{1}{2} \left( \frac{k}{l} \right)^{\frac{3}{2}}.
\] \hfill (0.22)

For the fix Jacobi energy \( K \), the angular momentum is \( L = K - H \) and we can write
\[
A^2 = 1 + 2HL^2.
\] \hfill (0.23)

Therefore we can determine periodic orbit of the second kind corresponding to relatively prime positive integer \( k, l \), if we know the energy \( c \).

If we consider the Sun-Jupiter system, we can given an astronomically description of the periodic orbit of the second kind as follow.

We denote the torus corresponding to the integers \( k \) and \( l \) by \( T_{k,l} \). Thus using the function \( p^H \), \( 3, 3, 10 \), for a periodic orbit of type \((k, l)\) or equivalently \( T_{k,l} \), we have the following relations
\[
L_{k,l} = \sqrt{-\frac{1}{2K_{k,l}}} = \left( \frac{l}{k} \right)^{\frac{3}{2}}.
\] \hfill (0.24)
\[
c_{k,l} = c_{k,l} - L_{k,l} = -\frac{1}{2} \left( \frac{k}{l} \right)^{\frac{3}{2}} - \left( \frac{l}{k} \right)^{\frac{3}{2}} = -\left( \frac{l}{k} \right)^{\frac{3}{2}} \left( \frac{k + 2l}{2l} \right)
\] \hfill (0.25)
\[
c_{k,l}^+ = c_{k,l} + L_{k,l} = -\frac{1}{2} \left( \frac{k}{l} \right)^{\frac{3}{2}} + \left( \frac{l}{k} \right)^{\frac{3}{2}} = \left( \frac{l}{k} \right)^{\frac{3}{2}} \left( -\frac{k + 2l}{2l} \right)
\] \hfill (0.26)
Thus the energy of a periodic orbit of type \((k, l)\) is
\[ c \in \left( c_{k,l}^-, c_{k,l}^+ \right). \]

Using the above notation, we can write the following relation for the periodic orbit of type \((k, l)\) is interior or exterior

(i) If \(k = l = 1\), the critical value of the RKP is \(c_{k,l}^- = \frac{3}{2}\) and the exterior and interior direct orbits both collapse to the critical point.

(ii) If \(k > l\), then \(|L_{k,l}| < 1\) and the direct orbit is interior.

(iii) If \(k < l\), then \(|L_{k,l}| > 1\) and the direct orbit is exterior.

**Theorem A:** We consider the energy value below \(-\frac{3}{2}\). In this case, the bounded component of the RKP after the Ligon-Schaaf regularization combined with the Levi-Civita regularization becomes a special concave toric domain.

To get the goal of the theorem A, we need to be familiar with the Ligon-Schaaf and the Levi-Civita regularizations in more details.

1. **Regularization**

1.1. **The Ligon-Schaaf Regularization.** The Ligon-Schaaf regularization helps us to regularize the collisions. The Ligon-Schaaf regularization is a symplectomorphism that maps the solutions of the planar Kepler problem to the geodesics on the sphere \(S^2\). Moreover, in contrast to the Belbruno-Moser-Osipov regularization \([9][10]\), the Ligon-Schaaf regularization do not change time and one can think of the Ligon-Schaaf regularization a global variant of Delaunay variables.

Here we consider the negative energy of the system and dimension \(n = 2\). Any way, the Ligon-Schaaf symplectomorphism works for positive and dimension \(n\) as same as the Belbruno-Moser-Osipov regularization.

Given the form \(y \mapsto \langle x, y \rangle\) on \(\mathbb{R}^2\) where \(\langle x, y \rangle\) is the standard inner product. Using this from, we can identify the phase space \(P\), i.e. the cotangent bundle of \(\mathbb{R}^2 \setminus \{0\}\) with the set of \((q, p)\) such that \(q \in \mathbb{R}^2, q \neq 0\) and \(p \in \mathbb{R}^2\).

Now we denote an open subset of \(P\) which lives on the negative part of the energy with
\[ P_- = \{ (q, p) \in P \mid H(q, p) < 0 \}. \]

Take the angular momentum \(L = q_1p_2 - q_2p_1\) and write the vector \(A\) as \(A = (A_1, A_2)\). Then we can have the following equalities
\[ \{L, A_1\} = -A_2 \]
\[ \{L, A_2\} = A_1 \]
\[ \{A_1, A_2\} = -2HL. \]

If we define the eccentricity vector by
\[ \eta := \nu A, \]
where \(\nu := \frac{1}{(-2H)^{\frac{1}{2}}}\). Hence we can write the Poisson bracket relation 1.2 in terms of \(\eta\) as follow
\[ \{L, \eta_1\} = -\eta_2 \]
\[ \{L, \eta_2\} = \eta_1 \]
\[ \{\eta_1, \eta_2\} = L. \]

If we think of \(L\) as \(\eta_3\). We can recover precisely the Lie algebra of \(SO(3)\).

We define \(J = (L, \eta_1, \eta_2)\) from \(P_-\) to the dual of the Lie algebra \(SO(3)\) as the momentum map of an infinite small Hamiltonian action of \(SO(3)\) on \(P_-\). Note that if we assume the subalgebra \(SO(2)\), then we can extend this infinitesimal action to the standard infinite rotation.
The Ligon-Schaaf regularization describe how we can map the solutions of the Kepler problem to the geodesics on the sphere $S^2$ in $\mathbb{R}^3$ such that the rotation group $SO(3)$ acts naturally.

First we define the phase space for the geodesics on the sphere $S^2$.

**Definition 1.1.** The cotangent bundle of $S^2$ can be identify with vectors $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $<x, x> = 1$ and $<x, y> = 0$. The zero section corresponds to the element $(x, 0)$ where $<x, x> = 1$. We denote by $T$ the complement of the zero section.

We can define the angular momentum map of the infinitesimal Hamiltonian action $SO(3)$ on $T$ by

$$\tilde{J} : (x, y) \rightarrow x \wedge y. \quad (1.5)$$

From the above notation we can show that the image of the Kepler solutions are geodesics with time rescaled under the Ligon-Schaaf map factor that depend only on the energy. In other words, the Kepler solutions are mapped to the solution curves of the the Delaunay Hamiltonian which is defined as follow

$$\tilde{H}(x, y) = -\frac{1}{2} \cdot \frac{1}{|y|^2} = -\frac{1}{2} \cdot \frac{1}{|J|^2} \quad (1.6)$$

where $(x, y) \in T$.

Now we use the above notations and give the Ligon-Schaaf regularization is a symplectomorphism that maps the phase space $P_-$ into the phase space $T$ and denote it by $\Phi = \Phi_{LS}$ and define it as

$$\Phi = \Phi_{LS} : P_- \rightarrow T \quad (1.7)$$

$$\Phi(q, p) := ((\sin \phi A) + \cos \phi B, \nu(\cos \phi)A + \nu(\sin \phi)B), \quad (1.8)$$

where

$$A = A(q, p) := ((|q|^{-1}q - <q, p)p, \nu^{-1} < q, p >), \quad (1.9)$$

$$B = B(q, p) := (\nu^{-1}|q|p, |p|^2 |q| - 1), \quad (1.10)$$

and

$$\phi = \Phi_{LS}(q, p) := \nu^{-1} < q, p >. \quad (1.11)$$

The Ligon-Schaaf symplectomorphism has the following properties which are useful to compute the solution of the Kepler problem on the sphere $S^2$.

(i) Let $e_3$ be the third standard basis vector in $\mathbb{R}^3$, which is the north pole of the sphere $S^2$. Then $\Phi$ is an analytic diffeomorphism from $P_-$ onto the open subset $T_-$ of $T$ consisting of all $(x, y) \in T$ such that $x \neq e_3$.

(ii) $\Phi$ is a symplectomorphism.

(iii) If $\gamma$ is a solution curve of the Kepler vector field $X_H$ in $P_-$, then $\Phi \circ \gamma$ is a solution curve of the Delaunay vector field $X_{\tilde{H}}$ in $T$.

(iv) It holds that $J = \tilde{J} \circ \Phi$.

The Ligon-Schaaf symplectomorphism helps us to define the action of $g$ on $P_-$ as an action on $T$. Let the action $g \in SO(3)$ and denote the obvious action $g$ on $T$ by $g_T$ and the action $g$ on $P_-$ by $g_{P_-}$. Hence we define

$$g_{P_-}(q, p) := \Phi^{-1} \circ g_T \circ \Phi(q, p), \quad (q, p) \in P_- \quad (1.12)$$

This is a well-define action. For the map $\Phi$ the identity $J = \tilde{J} \circ \Phi$ holds.

**Proposition 1.2.** Suppose $\Phi$ is a map from $P_-$ to $T$. $\Phi$ satisfies $J = \tilde{J} \circ \Phi$ if and only if there exists an $\mathbb{R}/2\pi \mathbb{Z}$-valued function $\phi$ on $P_-$ such that $\Phi = \Phi_{\phi}$.

**Proof.** In paper [5].
1.2. The Levi-Civita Regularization. The Levi-Civita regularization is a double cover so that your energy hypersurface becomes $S^3$ instead of $\mathbb{R}P^3$. In the language of physics, we can explain this double cover of the geodesic flow on $S^2$ as a Hamiltonian flow of two uncoupled harmonic oscillators.

We denote the Levi-Civita regularization by $\mathcal{L}$ which is a 2:1 maps from $\mathbb{C}^2 \setminus \{0\}$ to $T^* S^2 \setminus S^2$ as follows

\[(1.13) \quad \mathcal{L}: \mathbb{C}^2 \setminus (\mathbb{C} \times \{0\}) \longrightarrow T^* \mathbb{C} \setminus \mathbb{C} \]

\[(1.14) \quad (u, v) \mapsto (\frac{u}{\bar{v}}, 2|v|^2) \]

where $\bar{v}$ is the complex conjugate of $v$.

This regularization works only for a 2-dimensional space, i.e. $\mathbb{C}^2$. Note that, there is a higher dimension as Levi-Civita but we do not consider here [11]. We consider a 2-dimensional space and discuss the Levi-Civita transformation. We extend the Levi-Civita regularization $\mathcal{L}$ to the cotangent bundle $T^* S^2$ as follows

\[(1.15) \quad \mathcal{L}: \mathbb{C}^2 \setminus \{0\} \longrightarrow T^* S^2 \setminus S^2 \]

where $\mathbb{C}$ is assumed to be a chart of $S^2$ via stereographic projection as the north pole. The above extension gives us a covering map with degree 2. (See [2] for more details).

**Lemma 1.3.** A closed hypersurface $\Sigma T^* S^2$ is fiberwise star-shaped if and only if $\mathcal{L}^{-1} \Sigma \subset \mathbb{C}^2$ is star-shaped.

**Corollary 1.4.** There exists a diffeomorphism between a fiberwise star-shaped hypersurface in $T^* S^2$ and the projective space $\mathbb{R}P^3$ if $\mathcal{L} \Sigma \subset \mathbb{C}^2$ is star-shaped.

Note that, a star-shaped hypersurface in $\mathbb{C}$ is a diffeomorphic to the 3-dimensional sphere $S^3$ which is a twofold cover of $\mathbb{R}P^3$.

**Example 1.5.** We apply the Levi-Civita regularization to the Kepler problem. Thus we have a new Hamiltonian with respect to $u$ and $v$ as follows

\[(1.16) \quad H(u, v) = \frac{|u|^2}{2|v|^2} - \frac{1}{2} |v|^2 - c, \]

where $c$ is the energy value.

The above relation gives us the following definition

\[(1.17) \quad H'(u, v) := |v|^2 H(u, v) = \frac{1}{2} (|u|^2 - c |v|^2 - 1) \]

and for the energy zero we have the level set

\[(1.18) \quad \Sigma := H^{-1}(0) = H'^{-1}(0) \]

This is a 3-dimensional sphere for a negative energy $c$.

The Hamiltonian flow of $H'$ on $\Sigma$ is just a parametrization of the Hamiltonian flow $H$ on $\Sigma$. The new Hamiltonian flow is periodic and physically it is the flow of two uncoupled harmonic oscillators.

2. The special concave toric domain for the rotating Kepler problem

In this section, I am going to introduce a special concave toric domain for the rotating Kepler problem. The special concave toric domain is in particular a concave toric domain and the precise meaning what is special we will see later. Then I would mention all the work by Hutchings and his collaborators which shows that the concavity of a toric domain has many applications to the computation of ECH capacities and symplectic embedding questions. Finally you can see first important ingredient in my project to understand symplectic embedding questions for the restricted three-body problem for small mass ratios.

To compute this concave toric domain, we use the stereographic projection and transfer the cotangent bundle of $\mathbb{R}^2$ to the cotangent bundle of $S^2$. 
Since the Ligon-Schaaf symplectomorphism interchanges the Hamiltonian of the KP with the Delaunay Hamiltonian. Therefore, we get the solution of the Kepler problem as geodesics on the cotangent bundle $T^*S^2$.

On the other hand, angular momentum generates rotation. Therefore, the Hamiltonian of the RKP is obtained by adding angular momentum to the Hamiltonian of the Kepler problem. We know that the Ligon-Schaaf symplectomorphism interchanges the angular momentum on the plane with a component of the angular momentum on the sphere. Therefore, the Ligon-Schaaf symplectomorphism pulls back the Hamiltonian of the RKP to a Hamiltonian defined on the cotangent bundle $S^2$ minus its zero section. The Levi-Civita map which is a 2:1 map between $\mathbb{C}^2$ minus the origin and the cotangent bundle of $S^2$ minus the zero section.

Now we assume the phase space $T$ of the geodesic solutions of the RKP. Then by Levi-Civita regularization, we will map them to the space $\mathbb{C}^2$. This maps gives us a double cover such that we can define a special concave toric domain which is a appropriate concave toric domain for the RKP.

2.1. Construction. Here we consider the above notation and definitions, to compute the concave toric domain of the RKP in some steps.

Given the unit sphere $S^2$ and denote the north pole of it in $\mathbb{R}^3$ with $N = (0,0,1)$. Take a point $x = (x_1, x_2, x_3)$ on $S^2$ and a covector on the tangent space of $S^2$ at $x$ with $y = (y_1, y_2, y_3)$ such that $x \neq N$, $x \cdot x = 1$ and $x \cdot y = 0$.

Now we use the stereographic transformation and map the cotangent bundle of the space $\mathbb{R}^2$ to the cotangent bundle of the sphere $S^2$. In other words, we have

\begin{align}
T^*\mathbb{R}^2 & \rightarrow T^*S^2 \\
(q,p) & \mapsto (x,y).
\end{align}

such that the following equalities hold

\begin{align}
x_k &= \frac{2q_k}{(q^2 + 1)} , \quad x_3 = \frac{(q^2 - 1)}{(q^2 + 1)} \\
y_k &= \frac{(q + 1)p_k}{2} - (q \cdot p)q_k , \quad y_3 = q \cdot p
\end{align}

where $k = 1, 2$.

These are canonical transformations in the sense that the symplectic form $\Sigma^2_{k=1} dq_k \wedge dp_k$ and the restriction of $\Sigma^3_{k=1} dx_k \wedge dy_k$ to $T^*S^2$ match. Given the Delaunay Hamiltonian

\begin{equation}
\tilde{H}(x,y) = -\frac{1}{||y||^2},
\end{equation}

where $||.||$ is the norm respect to the round geometry of $S^2$. Note that the Hamiltonian flow of the Delaunay Hamiltonian is a reparametrized geodesic flow on $S^2$.

Applying the stereographic projection $\Phi^*$ to the Delaunay Hamiltonian becomes

\begin{equation}
\tilde{H}(q,p) = -\frac{2}{(|q| + 1)^2 |p|^2}.
\end{equation}

The property

\begin{equation}
\Phi^*_L, S H = \tilde{H},
\end{equation}

of the Ligon-Schaaf symplectomorphism guarantees the Ligon-Schaaf symplectomorphism maps the Hamiltonian vector field of the Kepler problem to the Hamiltonian vector field of the Delaunay Hamiltonian.

The Ligon-Schaaf interchanges angular momentum in $\mathbb{R}^2$ with the first component of the angular momentum on $S^2$. Therefore, applying the Ligon-Schaaf symplectomorphism and the stereographic projection on the Hamiltonian of the RKP becomes

\begin{equation}
K(q,p) = \tilde{H}(q,p) + L(q,p) = -\frac{2}{(|q| + 1)^2 |p|^2} + q_1p_2 - q_2p_1.
\end{equation}
If we let $q$ and $p$ as a complex numbers, i.e. $q = q_1 + iq_2$ and $p = p_1 + ip_2$. We can write
\begin{equation}
K(q, p) = \tilde{H}(q, p) + L(q, p) = -\frac{2}{(|q| + 1)^2|p|^2} + \text{Im}(\bar{q} \cdot p).
\end{equation}

We know that the Levi-Civita transformation is a 2:1 map which up to a constant factor is a symplectic when we think of $\mathbb{C}^2$ as $T^*\mathbb{C}$. It pulls back the geodesics flow on $S^2$ to the flow of two uncoupled oscillator.

We apply the Levi-Civita regularization. To this purpose, we apply $\frac{u}{v}$ and $2v^2$ in the relation (2.8) instead of $q$ and $p$ respectively. Then we get the following identity
\begin{equation}
\tilde{H}(u, v) + L(u, v) = -\frac{2}{(|\frac{u}{v}| + 1)^2(2v^2)^2} + \text{Im}((\frac{\bar{u}}{v} \cdot 2v^2)
= -\frac{2}{2(|u|^2 + |v|^2)^2} + 2\text{Im}((\frac{\bar{u}}{v}v)
= -\frac{1}{2(|u|^2 + |v|^2)^2} + 2(u_1v_2 - u_2v_1).
\end{equation}

We introduce the function
\begin{equation}
\mu : T^*\mathbb{C} \longrightarrow [0, \infty) \times \mathbb{R} \subset \mathbb{R}^2
\end{equation}
\begin{equation}
(u, v) \mapsto \begin{cases}
\frac{1}{2}(|u|^2 + |v|^2)
\frac{u_1v_2 - u_2v_1}.
\end{cases}
\end{equation}
This is the momentum map of the torus action on $T^*\mathbb{C}$.

Note that in view of the elementary inequality
\begin{equation}
|ab| \leq \frac{1}{2}(a^2 + b^2),
\end{equation}
follows that $|\mu_2| \leq \mu_1$. Therefore, componentwise we define
\begin{equation}
\mu_1 := \frac{|u|^2 + |v|^2}{2}, \quad \mu_2 := u_1v_2 + u_2v_1.
\end{equation}
If we define the Hamiltonian of the RKP with $K$ and using the above notation and definitions, we have the following proposition.

**Proposition 2.1.** Given the Ligon-Schaaf symplectomorphism and the Levi-Civita regularization, the pull back of $K$ becomes
\begin{equation}
\mathcal{L}^*\Phi_{LS}^*(K) = -\frac{1}{8\mu_1^2} + 2\mu_2.
\end{equation}

**Proof.** This follows from the discussion above. \qed

We show that the symplectic manifold $\mathbb{C} \oplus \mathbb{C}$ and the cotangent bundle $T^*\mathbb{C}$ are symplectomorphic.

**Proposition 2.2.** There exists a linear symplectomorphism between the symplectic manifold $\mathbb{C} \oplus \mathbb{C}$ and the cotangent bundle $T^*\mathbb{C}$. In other words, we have the linear symplectomorphism
\begin{equation}
S : (\mathbb{C} \oplus \mathbb{C}, \omega_0) \longrightarrow (T^*\mathbb{C}, \omega_1).
\end{equation}

**Proof.** Consider the symplectic form on $T^*\mathbb{C}$ as
\begin{equation}
\omega_1 = du_1 \wedge dv_1 + du_2 \wedge dv_2.
\end{equation}
Let $(z_1, z_2) \in \mathbb{C}^2$ such that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We define the following linear map
\begin{equation}
S : \mathbb{C}^2 \longrightarrow T^*\mathbb{C}
\end{equation}
as
\begin{align*}
u_1 &\rightarrow \frac{1}{\sqrt{2}}(y_1 - y_2) \\
u_2 &\rightarrow \frac{1}{\sqrt{2}}(x_1 + x_2) \\
v_1 &\rightarrow \frac{1}{\sqrt{2}}(x_2 - x_1) \\
v_2 &\rightarrow \frac{1}{\sqrt{2}}(y_1 + y_2)
\end{align*}

To prove that $S$ interchanges the symplectic forms $\omega_0$ and $\omega_1$ we compute using (2.18). Thus we have
\begin{align*}
S^*(\omega_1) &= S^*(du_1 \wedge dv_1 + du_2 \wedge dv_2) \\
&= \left( \frac{1}{\sqrt{2}}(dy_1 - dy_2) \wedge \frac{1}{\sqrt{2}}(dx_2 - dx_1) \right) + \left( \frac{1}{\sqrt{2}}(dx_1 - dx_2) \wedge \frac{1}{\sqrt{2}}(dy_1 + dy_2) \right) \\
&= dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \\
&= \omega_0.
\end{align*}

We extend the function (2.14) to $T^*\mathbb{C} \setminus \{0\}$ and define
\begin{align*}
\tilde{K} : T^*\mathbb{C} \setminus \{0\} &\rightarrow \mathbb{R} \\
\tilde{K} : = -\frac{1}{8\mu_1^2} + 2\mu_2.
\end{align*}

Using the above function gives us a Concave Toric Domain for the RKP on a coordinate system which is rotated in view of the proposition (2.2).

We make the following rotations. Denote the first quarter in $\mathbb{R}^2$ by $Q := [0, \infty) \times [0, \infty)$ and define
\begin{align*}
Q_1 \setminus \{0\} : = \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| < x\},
\end{align*}
where $Q_1 \setminus \{0\}$ is the first quadrant of $\mathbb{R}^2$ that rotated by 45 degree clockwise.

Assume $\Omega \subset Q$ is close in the first quarter in $\mathbb{R}^2$. A toric domain is defined by
\begin{align*}
X_\Omega : = \nu^{-1}(\Omega),
\end{align*}
where
\begin{align*}
\nu = (\nu_1, \nu_2) : \mathbb{C}^2 &\rightarrow \mathbb{R}^2 \\
(\nu_1, \nu_2) &\mapsto (\pi|z_1|^2, \pi|z_2|^2).
\end{align*}

Note that $\nu$ is a momentum map for the torus action
\begin{align*}
(\nu_1, \nu_2)(z_1, z_2) = (e^{i\theta_1} z_1, e^{i\theta_2} z_2)
\end{align*}
on $\mathbb{C}^2$. We define the symplectic 4-manifold with boundary as
\begin{align*}
X_\Omega : = \{z = (z_1, z_2) \in \mathbb{C}^2 : \pi(|z_1|^2, |z_2|^2) \in \Omega\}.
\end{align*}

**Recall** : The Concave Toric Domain according Hutchings [12] is defined as follow.

**Definition 2.3.** We say that a toric domain $X_\Omega$ is a concave toric domain if $\Omega$ is a closed region bounded by the horizontal segment from $(0, 0)$ to $(a, 0)$, the vertical segment from $(0, 0)$ to $(0, b)$ and graph of a convex function $f : [0, a] \rightarrow [0, b]$ with $f(0) = b$ and $f(a) = 0$, where $a > 0$ and $b > 0$.

Now we define a Special Concave Toric Domain as follow.
**Definition 2.4.** A concave toric domain $X_\Omega \subset \mathbb{C}^2$ is called special if the function $f$ satisfies the additional property $f'(t) \geq -1$ for $t \in [0,a]$.

We can compare the Hutchings CTD and the SCTD for the RKP.

Define $\bar{S}$ by

$$\bar{S} : Q \rightarrow Q_{\frac{1}{2}}$$

which is a clockwise 45 degree rotation composed with a $\frac{1}{\sqrt{2\pi}}$ dilation.

Using the above notation, we can obtain following relations between momentum maps $\nu$ and $\mu$ for the torus actions $\mathbb{C}^2$ and $T^*\mathbb{C}^2$.

$$\bar{S}(\frac{1}{2\pi}(\nu_1 + \nu_2)) = \mu_1$$

$$\bar{S}(\frac{1}{2\pi}(\nu_1 - \nu_2)) = \mu_2.$$ 

Using these equalities gives us the following commutative diagram

$$\begin{aligned}
\mathbb{C} \oplus \mathbb{C} & \xrightarrow{S} T^*\mathbb{C} \\
\nu \downarrow & \quad \downarrow \mu
\end{aligned}$$

We define alternatively a concave toric domain for $\Omega'$

$$\Omega' := \bar{S}(\Omega) \subset Q_{\frac{1}{2}},$$

by

$$X_{\Omega'} = \mu^{-1}(\Omega') = S(X_\Omega)$$

in $T^*\mathbb{C}$.

From here, we assume that the concave toric domain is a subset of $T^*\mathbb{C}$ instead of $\mathbb{C}^2$ and we think of $\Omega$ is a closed subset of $Q_{\frac{1}{2}}$ and miss the prime.

Using the new convention, the special concave toric domain can be defined as follows.

**Remark 2.5.** Using the above identification of $\mathbb{C}^2$ and $T^*\mathbb{C}$, a toric domain $X_\Omega$ is special concave toric domain if and only if there exists a convex function $g : [a,b] \rightarrow \mathbb{R}$, $0 < a < b < \infty$, such that $\Omega \subset Q_{\frac{1}{2}}$ is bounded by the segment $\{(t,t) : t \in [0,a]\}, \{(t,-t) : t \in [0,b]\}$ and the graph of the convex function $g$.

**Remark 2.6.** In the following, we are working with $\Omega \subset Q_{\frac{1}{2}}$. If $\Omega$ satisfies the conditions of remark 2.5 we refer to $X_\Omega := \mu^{-1}(\Omega)$ as a special concave toric domain.

Assume $c \leq -\frac{3}{2}$, we define a closed subset of $Q_{\frac{1}{2}}$ by

$$\mathcal{K}_c := \mu(\tilde{K}^{-1}(-\infty,c)) \subset Q_{\frac{1}{2}}.$$ 

Note that if $c < -\frac{3}{2}$ then $\mathcal{K}_c$ has two connected components, one bounded and one unbounded, i.e. we write

$$\mathcal{K}_c = \mathcal{K}_c^b \cup \mathcal{K}_c^u,$$

for $\mathcal{K}_c^b$ the bounded connected component and $\mathcal{K}_c^u$ the unbounded connected component.

For $c = -\frac{3}{2}$ the two sets become connected at singularity which is the point $\left(\frac{1}{2}, -\frac{1}{2}\right)$.
Theorem 2.7. For $c \leq -\frac{3}{2}$, we have

\begin{equation}
\tilde{K}^{-1}(-\infty, c) = X_{K_c}^\circ \cup X_{K_c}^\circ \subset T^*\mathbb{C}
\end{equation}

and $X_{K_c}$ is a special concave toric domain.

Proof. After all these transformations this now follows immediately from 2.20, since the function

$$x \mapsto \frac{1}{16x^2}$$

is convex. \qed

We can see the graphs of the SCTD for the energies $c \leq -\frac{3}{2}$, $c = -\frac{3}{2}$ and $c > -\frac{3}{2}$ in the following figures.

![Figure 1. The direct and the retrograde orbits for an energy $c < -\frac{3}{2}$](image1)

![Figure 2. The direct orbit for the energy $c = -\frac{3}{2}$](image2)
Figure 3. There is no direct orbit for energy $c > \frac{3}{2}$

3. Construction of a new tree:

Here we are going to introduce a new tree that is useful to compute the slopes of the SCTD which is rotated by 45 degree. We consider the Stern-Brocot tree and explain the new tree. To this goal, first we gave a recall of the Stern-Brocot tree.

3.1. The Stern-Brocot tree. The Stern-Brocot tree was introduced by Moritz-Stern 1858 and Achille Brocot 1861. The Stern-Brocot tree is a complete infinity binary tree whose nodes are labelled by a unique rational number.

There are more information about this tree and the Calkin-Wilf tree and the relations of them in [2].

We use induction and a mediant method and give the Stern-Brocot tree. You can find another way to obtain the Stern-Brocot tree via Calkin-Wilf tree in [2].

Definition 3.1. A mediant is a fraction such that its numerator is the sum of the numerators of two other fraction and its denominator is the sum of the denominators of two other fraction.

The Stern-Brocot tree starts at level -1 from the pesodofractions $\frac{0}{1}$ and $\frac{1}{0}$. We use the previous level and the mediants to generate the terms of a new level. Then we write the terms increasingly on a line and generate the new level of the Stern-Brocot tree. In other words, we can write the above induction as follow

Stage -1: We start with the auxiliary labels $\frac{0}{1}$ and $\frac{1}{0}$ lowest to highest terms. Stage -1, we do not really consider as part of the tree but this level is used in the inductive constructive of the tree.

Stage 0: The root is $\frac{1}{1}$ which can be interpreted as mendiant of stage -1.

Stage 1: We add the mediant of the boundaries.

Stage n+1: We add the mediants of all consecutive fraction in the tree including the boundaries from the lowest to highest.

Therefore we have the following tree.
For convenience, we define a labelling for the nodes of the new tree. We start from a root like \( \frac{a}{b} \) and try to write the next level according to the previous one such that the lowest node is the first and the highest one is the last node. In the new level each root like \( \frac{a}{b} \) has two children such that the left child is less than \( \frac{a}{b} \) and the right child is bigger than \( \frac{a}{b} \). We called the left child even and denote it with zero and we called the right child odd and denote it with 1. We use these notations to find the nodes on the Stern-Brocot tree and also determine the place of new tree nodes.

3.2. The New Tree: Using the above notations, we explain the new tree which is useful to find the slopes and critical energy values of tori and asteroids in the SCTD. Consider the labelling of the Stern-Brocot tree and let the nodes of the Stern-Brocot tree by the fractional number \( \frac{k}{l} \). We write the node \( \frac{k}{l} \) as a matrix \( \begin{bmatrix} k \\ l \end{bmatrix} \). Since we want to have the new tree on the rotated coordinate by 45 degree, we multiply the matrix of the node \( \frac{k}{l} \) by \( \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \) which corresponds to a rotation by 45 degree and a dilation by \( \sqrt{2} \) in the coordinate system. Note that multiplicity and dilation do not influence the slope. Namely,

\[
\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k \\ l \end{bmatrix} = \frac{k + l}{-k + l}.
\]

Now we replace the node \( \frac{k}{l} \) in the Stern-Brocot tree by the node \( \frac{k + l}{-k + l} \). The follow the above method for the all nodes of the Stern-Brocot tree to get the new tree. The new tree is
Note that the nodes of the new tree are the slopes of the tori $T_{k,l}$ in the SCTD. Therefore, the slopes in the SCTD determined uniquely by a rational number from the new tree.

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