Chromatic Numbers of Stable Kneser Hypergraphs via Topological Tverberg-Type Theorems

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Kneser’s 1955 conjecture—proven by Lovász in 1978—asserts that in any partition of the $k$-subsets of $\{1, 2, \ldots, n\}$ into $n - 2k + 1$ parts, one part contains two disjoint $k$-sets. Schrijver showed that one can restrict to significantly fewer $k$-sets and still observe the same intersection pattern. Alon, Frankl, and Lovász proved a different generalization of Kneser’s conjecture for $r$ pairwise disjoint sets. Dolnikov generalized Lovász’ result to arbitrary set systems, while Kříž did the same for the $r$-fold extension of Kneser’s conjecture. Here we prove a common generalization of all of these results. Moreover, we prove additional strengthenings by determining the chromatic number of certain sparse stable Kneser hypergraphs, and further develop a general approach to establishing lower bounds for chromatic numbers of hypergraphs using a combination of methods from equivariant topology and intersection results for convex hulls of points in Euclidean space.

1 Introduction

Splitting $\binom{[n]}{k}$, the set of $k$-element subsets of $[n] = \{1, 2, \ldots, n\}$, into $n - 2k + 1$ parts, one of the parts must contain two disjoint $k$-sets. This was conjectured by Kneser [21] and proven by Lovász [25]. This statement about the intersection pattern of all $k$-element subsets of $[n]$ easily translates into a graph coloring problem; construct the Kneser graph $KG(k, n)$ on vertex set $\binom{[n]}{k}$ with an edge between any two vertices that correspond to disjoint $k$-sets. Kneser’s conjecture then says that the chromatic number $\chi(KG(k, n))$, 

Communicated by Prof. Benny Sudakov

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that is, the least number of colors needed to color the vertices such that any edge has endpoints of distinct colors, is at least \( n - 2(k - 1) \). A greedy coloring shows that this lower bound is optimal.

This has been generalized in several different ways. Firstly, Schrijver [35] showed that if one restricts the vertex set of \( KG(k, n) \) to stable \( k \)-element sets, that is, those \( k \)-sets \( \sigma \subset [n] \) that do not contain two successive elements in cyclic order, the chromatic number does not decrease. We denote the induced subgraph of stable \( k \)-sets by \( KG(k, n)_{2}^{\text{stab}} \). Secondly, Alon et al. [4] showed that in any partition of \( \binom{[n]}{k} \) into \( \lceil \frac{n-r(k-1)}{r-1} \rceil - 1 \) parts, there is one part that contains \( r \) pairwise disjoint \( k \)-sets, proving a conjecture of Erdős [15]. In the same way that Kneser’s question translates into a graph coloring problem Alon, Frankl, and Lovász’ result establishes a lower bound for the chromatic number of an associated hypergraph; for an integer \( r \geq 2 \) denoted by \( KG^r(k, n) \) the Kneser hypergraph with vertex set \( \binom{[n]}{k} \) and a hyperedge of cardinality \( r \) for any \( r \) sets \( \sigma_1, \ldots, \sigma_r \subset [n] \) of cardinality \( k \) that are pairwise disjoint.

The chromatic number \( \chi(H) \) of a hypergraph \( H \) is the least number of colors needed to color its vertices such that no hyperedge is monochromatic, that is, receives only one color. In this language Alon, Frankl, and Lovász showed that \( \chi(KG^r(k, n)) = \lceil \frac{n-r(k-1)}{r-1} \rceil \).

Thirdly, Dolnikov [13] gave general lower bounds for the chromatic number of graphs that arise from arbitrary set systems with vertices corresponding to sets and edges to disjoint sets. His bound specializes to Kneser’s conjecture for the system of \( k \)-subsets of \( [n] \). Kříž [22] extended this to the hypergraph setting, establishing bounds for chromatic numbers that specialize to the result of Alon, Frankl, and Lovász. For a system \( \mathcal{F} \) of subsets of \( [n] \) Kříž defines a combinatorial quantity, the \( r \)-colorability defect \( cd^r(\mathcal{F}) \), and shows that in general \( (r - 1)\chi(KG^r(\mathcal{F})) \geq cd^r(\mathcal{F}) \); see Section 4 for definitions.

Since then some effort has been invested into combining the 1st two generalizations into one common generalization of Kneser’s conjecture. Meunier [29] showed that the chromatic number of \( KG^r(k, n) \) does not decrease if one restricts to those vertices that correspond to almost stable \( k \)-sets, where a set \( \sigma \subset [n] \) is almost stable if it does not contain two successive elements of \( [n] \) but may contain 1 and \( n \). Alishahi and Hajiabolhassan [2] showed that \( KG^r(k, n)_{2}^{\text{stab}} \), the induced subhypergraph of \( KG^r(k, n) \) that only contains those vertices corresponding to stable \( k \)-sets, still satisfies \( \chi(KG^r(k, n)_{2}^{\text{stab}}) = \lceil \frac{n-r(k-1)}{r-1} \rceil \) unless \( r \) is odd and \( n \equiv k \mod r - 1 \).

Here we prove the common generalization of the results of Schrijver and of Alon, Frankl, and Lovász in full generality; see Theorem 3.6.
Theorem 1.1. For any integers $r \geq 2$, $k \geq 1$, $n \geq rk$ we have that

$$\chi(KG^r(k, n)_{2-stab}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$ 

Our methods are different from those of [29] and [2]. We use a combination of methods from equivariant topology, in particular, geometric transversality results, and the geometry of point sets in Euclidean space. This was already partially developed in the recent paper [17] of the author; see also Sarkaria [32, 33] for earlier related work.

In fact, it is possible to generalize these lower bounds from the family of $k$-subsets of $[n]$ to arbitrary systems $F$ of subsets of $[n]$, establishing Kříž' lower bound for the set system $F_{2-stab}$, those sets in $F$ that are stable; see Theorem 4.2.

Theorem 1.2. Let $F$ be a system of subsets of $[n]$ and $r \geq 3$. Then

$$\chi(KG^r(F_{2-stab})) \geq \left\lceil \frac{1}{r-1}cd^r(F) \right\rceil.$$ 

For $r = 2$ this lower bound only holds for the almost stable sets in $F$; see Theorem 4.3. With our methods that relate chromatic numbers of hypergraphs to intersections of convex hulls of point sets in Euclidean space, Kříž' result follows from putting points in strong general position, while for the theorem above we place points along the moment curve.

While $cd^r(F)$ is a purely combinatorial quantity, we define the topological $r$-colorability defect $tcd^r(F)$ that takes the topology of the set system $F$ into account and satisfies $tcd^r(F) \geq cd^r(F)$, where the gap can be arbitrarily large. We show the following improvement of Kříž' lower bound; see Theorem 4.6.

Theorem 1.3. For $r$ a power of a prime

$$\chi(KG^r(F)) \geq \left\lceil \frac{1}{r-1}tcd^r(F) \right\rceil.$$ 

Interestingly, while this result easily implies Kříž' result for arbitrary $r$, the improved lower bound only holds for prime powers $r$. We further extend the theorem above to proper subhypergraphs of $KG^r(F)$; see Theorems 4.7 and 4.9, where the former is a common generalization of the three extensions of Kneser’s conjecture mentioned above, which furthermore detects global topological structure of the set system $F$ in addition to the local combinatorial structure.
It has been conjectured by Ziegler [38] and Alon et al. [3] that the bound 
\[ \chi(KG^r(k,n)) = \lceil \frac{n-r(k-1)}{r-1} \rceil \] still holds when the hypergraph \( KG^r(k,n) \) is restricted to vertices corresponding to \( r \)-stable sets. A set \( \sigma \subset [n] \) is \( s \)-stable if any two elements of \( \sigma \) are at distance \( \geq s \) in the cyclic order of \([n]\). Schrijver’s result is the \( r=2 \) case of this conjecture, and Alon, Drewnowski, and Łuczak furthermore showed that if the conjecture holds for \( r=p \) and \( r=q \) then it also holds for their product \( r=pq \). Thus, the conjectured lower bound holds for \( r \) a power of two. Aside from this combinatorial reduction, no results for \( s \)-stable Kneser hypergraphs for \( s > 2 \) were known. Here we prove optimal lower bounds for chromatic numbers of \( s \)-stable Kneser hypergraphs \( \chi(KG^r(k,n)_{s\text{-stab}}) \) for any \( s \geq 2 \); see Corollary 4.10.

**Theorem 1.4.** Let \( s \geq 2 \) be an integer and \( r > 6s - 6 \) a prime power. Then for integers \( k \geq 1 \) and \( n \geq rk \) we have that
\[ \chi(KG^r(k,n)_{s\text{-stab}}) = \lceil \frac{n-r(k-1)}{r-1} \rceil. \]

We prove extensions of this bound to arbitrary set systems; see Theorem 4.9. Section 5 is devoted to proving further extensions of Kříž’ bound to arbitrary \( r \)-uniform hypergraphs, extending the results for generalized Kneser hypergraphs. There we briefly relate our methods to the standard “box complex approach”.

2 The Chromatic Number of Kneser Hypergraphs—A Simple Proof

As a toy example to showcase our method of relating chromatic numbers of hypergraphs to the geometry of point sets in Euclidean space and geometric transversality results, we give a new and simple proof that \( \chi(KG^r(k,n)) = \lceil \frac{n-r(k-1)}{r-1} \rceil \). In [17] the author gave a simple proof of the \( r=2 \) case, that is, Kneser’s original conjecture, by reducing it to the topological Radon theorem of Bajmoczy and Bárány [5] that for any continuous map \( f: \Delta_{d+1} \rightarrow \mathbb{R}^d \) there are two points in disjoint faces of the \((d+1)\)-simplex \( \Delta_{d+1} \) that are identified by \( f \). Generalizations to the hypergraph setting, \( r > 2 \), use an \( r \)-fold analog of the topological Radon theorem, the topological Tverberg theorem; for any continuous map \( f: \Delta_{(r-1)(d+1)} \rightarrow \mathbb{R}^d \) there are \( r \) pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( \Delta_{(r-1)(d+1)} \) such that \( f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset \) provided that \( r \) is a power of a prime. This theorem is due to Bárány et al. [7] for prime \( r \), and due to Özaydin [30] for the general case of prime powers. The same result fails for \( r \) with at least two distinct prime divisors [9, 16, 26]. In the following we will use that a generic map \( f: \Delta_{(r-1)(d+1)-1} \rightarrow \mathbb{R}^d \) does not identify points from \( r \) pairwise disjoint faces by a simple codimension count. If \( f \) is an affine
map then the correct notion of genericity for the vertex set is strong general position; see Perles and Sigron [31].

Let $K$ be a simplicial complex on vertex set $[n]$. A set $\sigma \subset [n]$ is called missing face of $K$ if $\sigma \notin K$ and every proper subset of $\sigma$ is a face of $K$. For an integer $r \geq 2$ the Kneser hypergraph $KG^r(K)$ of $K$ has as its vertex set the missing faces of $K$, and a hyperedge for any $r$ pairwise disjoint missing faces. More generally, for a simplicial complex $L \supset K$ the subhypergraph of $KG^r(K)$ induced by those vertices that correspond to missing faces of $K$ that are faces of $L$ is denoted by $KG^r(K, L)$. We will need the following result, which is a simple consequence of the topological Tverberg theorem.

**Theorem 2.1 ([17, Cor. 4.7]).** Let $d \geq 0$ be an integer, and let $r \geq 2$ be a prime power. Let $K$ be a simplicial complex on at most $N = (r - 1)(d + 1) + 1$ vertices such that there exists a continuous map $f: K \to \mathbb{R}^d$ with the property that for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $K$ we have that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$. Then $\chi(KG^r(K)) \geq \lceil \frac{N - 1}{r - 1} \rceil - d$.

As a simple consequence we obtain the following:

**Theorem 2.2 (Alon et al. [4]).** For any integers $r \geq 2$, $k \geq 1$, $n \geq r k$ we have that

$$\chi(KG^r(k, n)) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

**Proof.** Let $r \geq 2$ be a prime. Choose the integer $d$ such that $r(k - 1) - 1 \geq (r - 1)d > r(k - 2)$ and let $t = (r - 1)d - r(k - 2) - 1$. Let $K = \Delta_{n-1}^{(k-2)} * \Delta_{t-1}^{(k-2)}$, that is, $K$ is the simplicial complex on vertex set $[n + t]$ whose missing faces are precisely the $k$-element subsets of $[n]$. Let $f: K \to \mathbb{R}^d$ be a generic map.

Let $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint faces of $K$. They involve at most $r(k - 1) + t = (r - 1)(d + 1)$ vertices. Thus, the intersection $f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ is empty. Now the hypergraph $KG^r(K)$ of missing faces of $K$ is $KG^r(k, n)$, and thus $\chi(KG^r(k, n)) \geq \lceil \frac{n + t - 1}{r - 1} \rceil - d$. Write $\lceil \frac{n + t - 1}{r - 1} \rceil = \frac{n + t - 1 - \alpha}{r - 1}$ for some $0 \leq \alpha \leq r - 2$, and furthermore we have that $d = \frac{r(k - 2) + t + 1}{r - 1}$. Thus,

$$\chi(KG^r(k, n)) \geq \frac{n + t - 1 - \alpha}{r - 1} - \frac{r(k - 2) + t + 1}{r - 1} = \frac{n - r(k - 2) - (\alpha + 2)}{r - 1} \geq \frac{n - r(k - 1)}{r - 1}.$$

The case for general $r$ follows by a simple induction on the number of prime divisors as in [4].
The proof above provides us with two ways to fine-tune in order to obtain stronger results. First, the simplicial complex $K$ does not have a point of $r$-fold coincidence among its pairwise disjoint faces for codimension reasons. Adding a face to $K$ while preserving this property either deletes a missing face from $K$ and thus a vertex from $\text{KG}^r(K)$ while not decreasing its chromatic number, or increases the size of a missing face by one, which potentially deletes hyperedges from $\text{KG}^r(K)$ while preserving $\chi(\text{KG}^r(K))$.

Secondly, the lower bound for the chromatic number builds on a contradiction to the topological Tverberg theorem. Proper strengthenings are due to Vučić and Živaljević [37], Hell [18], Engström [14], and Blagojević et al. [10]. In Sections 3 and 4 we will leverage some of these results to obtain sparse subhypergraphs of $\text{KG}^r(k,n)$ that still have the same chromatic number, and extend this approach to general Kneser hypergraphs.

3 Lower Bounds for Chromatic Numbers via Geometric Transversality

In this section, we will extend the reasoning of the previous section to prove tight lower bounds for 2-stable and, for $r \geq 5$, $r \not\in \{6,12\}$, and for $r = 4$ for certain parameters, for 3-stable Kneser hypergraphs. The same methods also imply an approximation to Ziegler’s conjecture that $\chi(\text{KG}^r(k,n)_{r-\text{stab}}) = \chi(\text{KG}^r(k,n))$ for $n \geq rk$. The proofs in this section all follow the same general pattern that we already outlined for the proof of Theorem 2.2 and contain ad hoc constructions. We will present a more generalized framework that works for general Kneser hypergraphs and implies tight lower bounds for the chromatic number of $s$-stable Kneser hypergraphs, where $s$ grows linearly with $r$, in Section 4. In order to find sparse subhypergraphs of $\text{KG}^r(k,n)$ that still require at least $\lceil \frac{n-r(k-1)}{r-1} \rceil$ colors in any proper coloring, we first need to recall the following:

**Theorem 3.1 ([17, Theorem 4.5]).** Let $d, c \geq 0$ and $r \geq 2$ be integers, $K \subseteq L$ simplicial complexes such that for every continuous map $F: L \rightarrow \mathbb{R}^{d+c}$ there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $L$ such that $F(\sigma_1) \cap \cdots \cap F(\sigma_r) \neq \emptyset$. Suppose $\chi(\text{KG}^r(K,L)) \leq c$. Then for every continuous map $f: K \rightarrow \mathbb{R}^d$ there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $K$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

As the proof of Theorem 3.1 is quite simple, building on a method developed in [8], we briefly outline it at this point; suppose that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ for all pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $K$. Since the chromatic number of the hypergraph $\text{KG}^r(K,L)$ is at most $c$, it is possible to color the missing faces of $K$ that are contained in $L$ with
such that any collection of \( r \) pairwise disjoint missing faces receives at least two distinct colors. Extend the map \( f \) to a map \( F: L \rightarrow \mathbb{R}^{d+c} \) by mapping the barycenter of a missing face of \( K \) contained in \( L \) to the standard basis vector \( e_{d+i} \) if it is colored by \( i \). This mapping extends to all of \( L \), by first mapping barycenters of larger non faces of \( K \) to the same basis vector \( e_{d+i} \) as one of its subfaces, and then extending by linearity. It is simple to check that this new map \( F \) satisfies \( F(\sigma_1) \cap \cdots \cap F(\sigma_r) = \emptyset \) for all pairwise disjoint faces \( \sigma_1, \ldots, \sigma_r \) of \( L \). Certainly the last \( c \) coordinates of a point in \( F(\sigma_1) \cap \cdots \cap F(\sigma_r) \) cannot all be zero, since \( f \) does not identify points from \( r \) pairwise disjoint faces. But the last \( c \) coordinates of points in \( F(\sigma_1), F(\sigma_2), \ldots, F(\sigma_r) \) do not all have the same nonzero coordinates, since this would violate the coloring condition of the hypergraph \( KG'(K,L) \). This completes the outline of the proof. We now need the following:

1. A large simplicial complex \( K \) containing the \((k-2)\)-skeleton of \( \Delta_{n-1} \) such that there exists a continuous map \( f: K \rightarrow \mathbb{R}^d \) that does not identify points in \( r \) pairwise disjoint faces, where \( d = \lceil \frac{r(k-2)+1}{r-1} \rceil \).

2. A small simplicial complex \( L \supset K \) such that any continuous map \( f: L \rightarrow \mathbb{R}^{d-1} \) identifies points from \( r \) pairwise disjoint faces, where \( D = \lceil \frac{r-1}{r} \rceil \).

Concerning point 1 good candidates are (subcomplexes of) cyclic polytopes of dimension \( 2k-2 \). These polytopes arise as convex hulls of finitely many points on the moment curve \( \gamma(t) = (t, t^2, \ldots, t^d) \). In fact, we will use that the intersection combinatorics of convex hulls of point sets on the stretched moment curve are understood, provided that the points are sufficiently far apart; see Bukh et al. [11] (according to the authors of [11] this was independently observed by Bárány and Pór as well as Mabillard and Wagner). A partition of \( \{1, 2, \ldots, (r-1)(d+1)+1 \} \) into \( r \) parts \( X_1, \ldots, X_r \) is called colorful if for each \( 1 \leq k \leq d+1 \) the set \( Y_k = \{(r-1)(k-1) + 1, \ldots, (r-1)(k)+1\} \) satisfies \(|Y_k \cap X_i| = 1 \) for all \( i \). We say that a partition \( X_1 \sqcup \cdots \sqcup X_r \) of \( \{1, 2, \ldots, (r-1)(d+1)+1\} \) occurs as a Tverberg partition in a sequence \( x_1, \ldots, x_N \) of points in \( \mathbb{R}^d \) if there is a subsequence \( x_{i_1}, \ldots, x_{i_n} \) of length \( n = (r-1)(d+1) + 1 \) such that the sets \( \text{conv}\{x_{i_k} \mid k \in X_j\} \) all share a common point. We now have the following lemma:

**Lemma 3.2 (Bukh et al. [11]).** There are arbitrarily long sequences of points in \( \mathbb{R}^d \) such that the Tverberg partitions that occur are precisely the colorful ones.

These point sets are spread along the stretched moment curve. Let \( P \subset \mathbb{R}^d \) be a (sufficiently large) point set provided by the lemma above. Let \( X_1, \ldots, X_r \subset P \) be pairwise disjoint sets with \( |\bigcup_i X_i| \leq (r-1)(d+1) \). Then the intersection \( \text{conv} X_1 \cap \cdots \cap \text{conv} X_r \) is
necessarily empty, since otherwise we could find a point $p$ of $P \setminus \bigcup_i X_i$ and an index $j \in [r]$ such that adding $p$ to $X_j$ would not be a colorful partition, but $\text{conv} X_1 \cap \cdots \cap \text{conv} X_r \neq \emptyset$. Finding point sets with this particular property is much simpler than Lemma 3.2. We need point sets in strong general position, which is a generic property; see Perles and Sigron [31].

For point 2 above we will use the following proper extensions of the topological Tverberg theorem.

**Theorem 3.3 (Hell [18]).** Let $r \geq 5$ be a prime power, $d \geq 1$ an integer, and $N = (r - 1)(d + 1)$. Let $G$ be a vertex-disjoint union of cycles and paths on the vertex set of $\Delta_N$. Then for any continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$ there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$ and no $\sigma_i$ contains both endpoints of an edge of $G$. The same result holds for $r = 4$ if $G$ is a disjoint union of paths or even for $r = 3$ if $G$ is a disjoint union of edges.

**Theorem 3.4 (Blagojević et al. [10]).** Let $r \geq 2$ be a prime, $d \geq 1$ an integer, and $N = (r - 1)(d + 1)$. Let $C_1 \sqcup \cdots \sqcup C_m$ be a partition of the vertex set of $\Delta_N$ such that $|C_i| \leq r - 1$. Then for any continuous map $f : \Delta_N \rightarrow \mathbb{R}^d$ there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

Finally, since these results come with restrictions on the number of prime divisors of $r$, we need the following combinatorial reduction result:

**Lemma 3.5 (Alishahi and Hajiabolhassan [2]).** Let $r$, $s$, and $p$ be positive integers, where $r \geq s \geq 2$ and $p$ is a prime number. Assume that for any $n \geq rk$, $\chi(\text{KG}^p(k, n)_{s-stab}) = \lceil \frac{n-r(k-1)}{r-1} \rceil$. Then for any $n \geq prk$, we have $\chi(\text{KG}^{pr}(k, n)_{s-stab}) = \lceil \frac{n-pr(k-1)}{pr-1} \rceil$.

This allows us to prove the following:

**Theorem 3.6.** For any integers $r \geq 2$, $k \geq 1$, $n \geq rk$ we have that

$$\chi(\text{KG}^r(k, n)_{2-stab}) = \left\lfloor \frac{n - r(k - 1)}{r - 1} \right\rfloor.$$

**Proof.** According to Lemma 3.5 we can from now on assume that $r$ is a prime. We can also assume that $r \geq 3$, since the $r = 2$ case is due to Schrijver [35]. (In fact, the proof
at hand works with minor changes for $r = 2$; see [17].) Let $K$ be the simplicial complex on vertex set $[n]$ defined by: $\sigma \subset [n]$ is a missing face of $K$ if and only if $|\sigma| = k$ and if $i \in \sigma$ for some $i \in [n - 1]$ then $i + 1 \notin \sigma$, that is, the almost 2-stable $k$-sets are the missing faces of $K$. Choose the integer $d$ such that $r(k - 2) - 1 \geq (r - 1)(d + 1)$, that is, $d$ is the largest integer that satisfies $d \leq \frac{r(k - 1) - 1}{r - 1}$. Let $t = (r - 1)(d - r(k - 2) - 1)$. Let $p_1, \ldots, p_{n+t} \in \mathbb{R}^d$ be a sequence of points provided by Lemma 3.2. Now consider the complex $\overline{K} = K \ast \Delta_{t-1}$ for $t > 0$ or $\overline{K} = K$ if $t = 0$. Map the vertices $[n]$ of $K$ bijectively to $p_1, \ldots, p_n$ in the canonical order. Map the $t$ vertices of $\Delta_{t-1}$ to $p_{n+1}, \ldots, p_{n+t}$ in arbitrary order. Extending linearly onto faces yields an affine map $f: \overline{K} \rightarrow \mathbb{R}^d$. In fact, let us from here on not distinguish between the vertices of $\overline{K}$ and their images $p_1, \ldots, p_{n+t}$.

Let $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint faces of $\overline{K}$ and suppose that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$. W.l.o.g. the faces $\sigma_1, \ldots, \sigma_r$ involve precisely $(r - 1)(d + 1) + 1$ vertices by a codimension count. We now argue that for one of the faces $\sigma_i$ the intersection $\sigma_i \cap [p_1, \ldots, p_n]$ has cardinality at least $k$. Indeed, if all intersections $\sigma_i \cap [p_1, \ldots, p_n]$ had cardinality at most $k - 1$, then the $\sigma_i$ would involve at most $r(k - 1) + t = (r - 1)(d + 1)$ points, which is a contradiction to $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$. Thus, one of the faces $\sigma_j$ satisfies $\sigma_j \cap [p_1, \ldots, p_n] \geq k$.

If the intersection $f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ were nonempty, then by Lemma 3.2 the partition $\sigma_1 \cup \cdots \cup \sigma_r$ of the $(r - 1)(d + 1) + 1$ points $\bigcup \sigma_i$ would be colorful. In particular, no two vertices of $\sigma_i$ can be successive in the linear order of $[n + t]$. But then $\sigma_j$ cannot be a face of $\overline{K}$, which is a contradiction. Hence, the intersection $f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ is empty.

Denote the vertex set of $\Delta_{t-1}$ by $v_1, \ldots, v_t$. Let $L$ be the simplicial complex obtained by deleting the edge $(1, n)$ from the simplex on vertex set $[n] \cup \{v_1, \ldots, v_t\}$. Let $D = \lceil \frac{n+t-1}{r-1} \rceil$. Since $(r - 1)D + 1 \leq n + t$, we have by Theorem 3.3 that for any continuous map $F: L \rightarrow \mathbb{R}^{D-1}$ there are $r$ points in $r$ pairwise disjoint faces of $L$ whose images coincide. By Theorem 3.1 we have that $\chi(KG^r(\overline{K}, L)) \geq D - d$. Indeed, if $\chi(KG^r(\overline{K}, L)) \leq D - 1 - d$, then Theorem 3.1 would imply that $f$ identifies $r$ points from $r$ pairwise disjoint faces. Now $D = \frac{n+t-1-\alpha}{r-1}$ for some $0 \leq \alpha \leq r - 2$, and $d = \frac{r(k-2)+t+1}{r-1}$. Thus,

$$\chi(KG^r(\overline{K}, L)) \geq \frac{n+t-1-\alpha}{r-1} - \frac{r(k-2)+t+1}{r-1} = \frac{n - r(k - 2) - (\alpha + 2)}{r-1} \geq \frac{n - r(k - 1)}{r-1}.$$ 

The missing faces of $\overline{K}$ that are faces of $L$ are precisely the 2-stable $k$-sets in $[n]$. Thus, the hypergraph $KG^r(\overline{K}, L)$ is $KG^r(k, n)_{2-\text{stab}}$.

**Remark 3.7.** For the $r = 2$ case (i.e., Kneser graphs) Meunier [29] showed that for $n \geq sk$ the $s$-stable Kneser graph $KG^2(k, n)_{s-\text{stab}}$ can be properly colored with $n - s(k - 1)$
colors and conjectured that this is in fact the chromatic number. This was confirmed by Jonsson \cite{20} for $s \geq 4$ and $n$ sufficiently large, and by Chen \cite{12} for even $s$ and any $n$.

We can extend the reasoning above by using the full generality of Theorem 3.3. If, in addition, we are somewhat more careful how to order the points $p_1, \ldots, p_{n+t}$ we obtain a result for 3-stable Kneser hypergraphs:

**Theorem 3.8.** Let $r \geq 4$, $r \notin \{6, 12\}$, $k \geq 1$, and $n \geq rk$ be integers, and if $r = 4$ let either $n$ be even or $k \neq 2 \mod 3$. Then

$$\chi(K^r_n) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$  

**Proof.** We follow the same steps as in the proof of Theorem 3.6. First let $r \geq 5$ be a prime power. Recall that $K$ was defined as the simplicial complex on vertex set $[n]$ whose missing faces are precisely the almost 2-stable $k$-subsets of $[n]$. Let $\hat{K}$ be the simplicial complex on vertex set $[n]$ obtained from $K$ by also adding any subset $\sigma \subset [n]$ of cardinality at least $k$ and with $\{1, 3\} \subset \sigma$, or $\{2, 4\} \subset \sigma$, or $\{n - 2, n\} \subset \sigma$ to $K$ as a face. With $t$ defined as before (i.e., integer $d$ satisfies $(r(k - 1) - 1 \geq (r - 1)d > r(k - 2)$ and $t = (r - 1)d - r(k - 2) - 1$) let $\hat{K} = \hat{K} \ast \Delta_{t-1}$, where we identify the vertex set of $\Delta_{t-1}$ with $\{n+1, n+2, \ldots, n+t\}$.

Let $p_1, \ldots, p_{n+t} \in \mathbb{R}^d$ be a sequence of points provided by Lemma 3.2. Call the first $\lfloor \frac{t}{2} \rfloor$ points $q_1, \ldots, q_{\lfloor t/2 \rfloor}$, and the last $\lceil \frac{t}{2} \rceil$ points $q_{\lceil t/2 \rceil+1}, \ldots, q_t$. This leaves $n$ points of $p_1, \ldots, p_{n+t}$ in the middle that we will denote $p'_1, \ldots, p'_n$. Now define the facewise linear map $f: \hat{K} \rightarrow \mathbb{R}^d$ by mapping vertex $i \in [n]$ to $p'_i$ and vertex $i \in \{n+1, n+2, \ldots, n+t\}$ to $q_{i-n}$. Let us again not distinguish between vertices of $\hat{K}$ and their image points in $\mathbb{R}^d$.

Let $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint faces of $\hat{K}$ and suppose that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$. Let us again assume w.l.o.g. that $|\bigcup \sigma_i| = (r-1)(d+1)+1$. As before one of the faces $\sigma_j$ satisfies that $\sigma_j \cap \{p'_1, \ldots, p'_n\}$ has cardinality at least $k$. Now since $\sigma_1 \uplus \cdots \uplus \sigma_r$ is a colorful partition of those points in $p_1, \ldots, p_{n+t}$ that occur in one $\sigma_i$ the face $\sigma_j$ cannot contain two successive elements of $[n]$. It can also contain at most one element within the 1st $r$ points $p_1, \ldots, p_r$ and at most one element within the last $r$ points $p_{n+t-r+1}, \ldots, p_{n+t}$ by definition of colorful partition. Since $t \leq r - 2$ we have that $\{p'_1, \ldots, p'_4\} \subset \{p_1, \ldots, p_r\}$—that is, $4 + \lfloor \frac{t}{2} \rfloor \leq r$ since $r \geq 5$—and $\{p'_{n-2}, p'_n\} \subset \{p_{n+t-r+1}, \ldots, p_{n+t}\}$. Thus, $\{1, 3\} \notin \sigma_j$, $\{2, 4\} \notin \sigma_j$, and $\{n - 2, n\} \notin \sigma_j$, so $\sigma_j$ is a missing face of $\hat{K}$—a contradiction.

Let $G$ be the graph on vertex set $[n+t]$ with edges $\{3, 5\}, \{4, 6\}, \ldots, \{n-3, n-1\}$ and $\{1, n\}, \{2, n\}, \{1, n-1\}$. In particular, $G$ is a disjoint union of paths. Let $L$ be the simplicial
complex obtained from the simplex on vertex set \([n + t]\) by deleting any edge in \(G\). Now as in the proof of Theorem 3.6 and using Theorem 3.3 we get that any continuous map \(F: L \to \mathbb{R}^{D-1}\) identifies \(r\) points from \(r\) pairwise disjoint faces for \(D = \lceil \frac{n+t-1}{r-1} \rceil\). Then as before

\[
\chi(\text{KG}^r(K, \emptyset)) \geq \frac{n - r(k - 1)}{r - 1}.
\]

Now observe that \(\text{KG}^r(K, \emptyset) = \text{KG}^r(k, n)_{3\text{-stab}}\). In the argument above we used that \(r \geq 4\) to invoke Theorem 3.3 for the complex \(L\), which is obtained from a simplex by deleting the disjoint union of paths \(G\). The only point where we used that in fact \(r \geq 5\), was to ensure that \(4 + \lfloor \frac{t}{2} \rfloor \leq r\). Thus, the same proof works for \(r = 4\) as long as \(4 + \lfloor \frac{t}{2} \rfloor \leq 4\), or equivalently \(t = 3d - 4(k - 2) - 1 \leq 1\), which is true unless \(3d = 4(k - 1) - 1\), that is, unless \(k \equiv 2 \mod 3\). If \(r = 4\) and \(t = 2\), then for even \(n\) the following modification of the proof still works; add only those \(\sigma \subset [n]\) to \(K\) that contain \(\{1, 3\}\) or \(\{n - 2, n\}\) to obtain \(\hat{K}\). We then have to exclude faces containing \(\{2, 4\}\) via the graph \(G\), that is, we add the edge \(\{2, 4\}\) to \(G\). If \(n\) is even \(G\) is a path, so we can still apply Theorem 3.3. The case of general \(r\) now follows by an induction using Lemma 3.5.

Our methods also yield an approximation to the following conjecture of Ziegler [38] and Alon et al. [3].

**Conjecture 3.9.** For \(n \geq rk\) we have that

\[
\chi(\text{KG}^r(k, n)_{r\text{-stab}}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.
\]

Schrijver’s theorem [35] is the \(r = 2\) case of this conjecture. It also holds for \(r\) a power of two, since Alon et al. [3] found a combinatorial reduction that shows if Conjecture 3.9 holds for \(r = 2\) then it also holds for the product \(r = pq\). Denote by \(\text{KG}^r(k, n)_{2\text{-stab}}\) the subhypergraph of \(\text{KG}^r(k, n)\) induced by those vertices corresponding to almost 2-stable \(k\)-sets. Recall that a set \(\sigma \subset [n]\) is called almost 2-stable if \(|i - j| \neq 1\) for any \(i, j \in \sigma\).

**Theorem 3.10.** Let \(r \geq 2\) be a prime, and let \(k \geq 1\), and \(n \geq rk\) be integers. Let \(C_1 \cup \cdots \cup C_m\) be a partition of \([n]\) with \(|C_i| \leq r - 1\). Let \(H \subset \text{KG}^r(k, n)_{2\text{-stab}}\) be the subhypergraph that only contains those almost 2-stable \(k\)-sets \(\sigma \subset [n]\) as vertices that satisfy \(|\sigma \cap C_i| \leq 1\) for all \(i\). Then \(\chi(H) = \lfloor \frac{n - r(k - 1)}{r - 1} \rfloor\).
Proof. Repeat the proof of Theorem 3.6, but now for the complex $L$ use $C_1 \ast \ldots \ast C_m \ast \Delta_{t-1}$, the join of the sets in the partition of $[n]$ and a small simplex $\Delta_{t-1}$ with $t$ chosen as before. The existence of the required $r$-fold intersection point among pairwise disjoint faces is guaranteed by Theorem 3.4—now only for $r$ a prime. The hypergraph $H$ is $KG^r(K,L)$. ■

Theorem 3.10 implies that for certain parameters we find optimal lower bounds for the chromatic number of the subhypergraph $H \subset KG^r(k,n)$ whose vertices are induced by almost 2-stable $k$-sets that arise as unions of two $r$-stable sets of cardinality $k/2$; for $n$ divisible by $r-1$ let the $C_i$ be blocks of $r-1$ consecutive integers. If $k$ is even, say $k = 2\ell$, then any $k$-subset $\sigma \subset [n]$ with $|\sigma \cap C_i| \leq 1$ for all $i$ can be split into two $r$-stable $\ell$-subsets; if $i_1 < i_2 < \ldots < i_k$ are the elements of $\sigma$, split it into $[i_1, i_3, \ldots, i_{k-1}]$ and $[i_2, i_4, \ldots, i_k]$. These sets are $r$-stable since between $i_j$ and $i_j + 2$ (indices modulo $k$) the element $i_{j+1}$ is contained in some intermediate block $C_p$ of size $r-1$. We explicitly formulate this as a corollary.

Corollary 3.11. Let $r \geq 2$ be a prime, $n$ divisible by $r-1$, $k = 2\ell$ for some integer $\ell \geq 1$, and $n \geq rk$. Let $H \subset KG^r(k,n)$ be the subhypergraph induced by vertices corresponding to almost 2-stable $k$-sets that arise as the union of two $r$-stable $\ell$-sets whose elements alternate. Then $\chi(H) = \lceil \frac{n - r(k-1)}{r-1} \rceil$.

4 Eliminating Local Effects on the Colorability Defect

Let $\mathcal{F}$ be a family of subsets of $[n]$. The $r$-colorability defect $cd^r(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$cd^r(\mathcal{F}) = n - \max \left\{ \sum_{i=1}^{r} |A_i| : A_1, \ldots, A_r \subset [n] \text{ pairwise disjoint and } F \not\subset A_i \text{ for all } F \in \mathcal{F} \text{ and } i \in [r] \right\}.$$ 

The term colorability defect stems from the following rephrasing of the definition. Think of the sets $A_1, \ldots, A_r$ as color classes coloring part of the ground set $[n]$ of the hypergraph $\mathcal{F}$. The condition $F \not\subset A_i$ for all $F \in \mathcal{F}$ and $i \in [r]$ then just means that the $A_i$ define a proper $r$-coloring of all but $cd^r(\mathcal{F})$ vertices of $\mathcal{F}$. The quantity $cd^r(\mathcal{F})$ measures at least how many vertices need to be deleted from $\mathcal{F}$ to ensure that the remaining hypergraph has a proper $r$-coloring.

For an arbitrary set system $\mathcal{F}$ we denote by $KG^r(\mathcal{F})$ the generalized Kneser hypergraph whose vertices are the elements of $\mathcal{F}$ and whose hyperedges are $r$-tuples of
pairwise disjoint sets. This is only slightly more general than the hypergraphs \( KG^r(K) \) for a simplicial complex \( K \), where the vertices are determined by the missing faces of \( K \). For any system \( \mathcal{F} \) of subsets of \([n]\) such that for \( F \neq F' \in \mathcal{F} \) neither \( F \subset F' \) nor \( F' \subset F \), there is a unique simplicial complex \( K \) on vertex set \([n]\) whose missing faces are precisely the sets in \( \mathcal{F} \). Moreover, for an arbitrary system of sets \( \mathcal{F} \), if \( \mathcal{F}' \subset \mathcal{F} \) denotes the system of inclusion-minimal sets in \( \mathcal{F} \), then \( \chi(KG^r(\mathcal{F})) = \chi(KG^r(\mathcal{F}')) \). See [17, Lemma 4.1 and 4.2] for details. Hence, for the purposes of chromatic numbers we may and will tacitly assume that \( \mathcal{F} \) contains no two sets ordered by inclusion. Those set systems are precisely the systems that arise as missing faces of simplicial complexes.

A usual progression has been that a result that was first established for Kneser hypergraphs later gets extended to generalized Kneser hypergraphs of arbitrary set systems; Lovász established Kneser’s conjecture, that \( \chi(KG^2(k,n)) = n - 2(k - 1) = cd^2(KG^2(k,n)) \), which was extended to arbitrary set systems—\( \chi(KG^2(\mathcal{F})) \geq cd^2(KG^2(\mathcal{F})) \) —by Dolnikov [13]. In this language the generalization of Kneser’s conjecture to hypergraphs due to Alon, Frankl, and Lovász—Theorem 2.2—can be phrased as \( \chi(KG^r(k,n)) = \lceil \frac{1}{r - 1} cd^r(KG^r(k,n)) \rceil \). The extension to set systems is due to Kříž [22].

**Theorem 4.1 (Kříž [22]).**

\[
\chi(KG^r(\mathcal{F})) \geq \left\lceil \frac{1}{r - 1} cd^r(\mathcal{F}) \right\rceil.
\]

This raises the questions whether Schrijver’s result also holds in this generalized setting. For \( \mathcal{F} \) a system of subsets of \([n]\), denote by \( \mathcal{F}_{s-stab} \) the subsystem of \( s \)-stable sets in \( \mathcal{F} \). Then Schrijver’s theorem guarantees that \( \chi(KG^2(\binom{[n]}{k}_{2-stab})) \geq cd^2(\binom{[n]}{k}) \). For \( n = 5 \) let \( \mathcal{F} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\} \). Then \( cd^2(\mathcal{F}) = 1 \), while \( \chi(KG^2(\mathcal{F}_{2-stab})) = 0 \), so such an extension of Schrijver’s result to arbitrary set systems cannot hold in general. However, this extension holds in full generality in the hypergraph setting, see Theorem 4.2, and almost holds for \( r = 2 \), see Theorem 4.3.

**Theorem 4.2.** Let \( \mathcal{F} \) be a system of subsets of \([n]\) and \( r \geq 3 \). Then

\[
\chi(KG^r(\mathcal{F}_{2-stab})) \geq \left\lceil \frac{1}{r - 1} cd^r(\mathcal{F}) \right\rceil.
\]

**Proof.** First assume that \( r \geq 3 \) is a prime power. Let \( K \) be the simplicial complex on vertex set \([n]\) with missing faces \( \mathcal{F}_{2-stab} \), the collection of sets in \( \mathcal{F} \) that are almost 2-stable. In particular, the set system \( \mathcal{F}_{2-stab} \) still contains those sets in \( \mathcal{F} \) that have \( \{1, n\} \) as a subset. These particular sets will be excluded at a later stage in the proof,
when we invoke a topological Tverberg-type result for a complex $L$ that does not contain the edge $\{1, n\}$.

Denote by $K'$ the simplicial complex on vertex set $[n]$ with missing faces $F$. Let

$$M = \max \left\{ \sum_{i=1}^{r} |A_i| : A_1, \ldots, A_r \subset [n] \text{ pairwise disjoint and } F \not\subset A_i \text{ for all } F \in F \text{ and } i \in [r] \right\}.$$ 

In particular, the sets $A_i$ in the definition of $M$ above determine faces of the complex $K'$. Choose the integer $d$ as the least integer such that $M < (r - 1)(d + 1) + 1$, that is, $M + m = (r - 1)(d + 1) + 1$ for some $0 < m \leq r - 1$. Define the simplicial complex $\overline{K}'$ to be the join $K' \ast \Delta_{m-2}$ for $m \geq 2$ or $\overline{K}' = K'$ for $m = 1$. Then any strong general position map $f : \overline{K}' \longrightarrow \mathbb{R}^d$ satisfies $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\overline{K}'$. By definition of $M$ the faces $\sigma_i$ may in total only involve at most $M$ vertices of $K$. The complex $\overline{K}'$ has $m - 1$ additional vertices, so the faces $\sigma_i$ involve less than $M + m = (r - 1)(d + 1) + 1$ vertices, thus the sets $f(\sigma_1), \ldots, f(\sigma_r)$ do not have a common point of intersection by a codimension count.

Now let $K$ be the join $K \ast \Delta_m$, and let $\overline{f} : K \longrightarrow \mathbb{R}^d$ be a general position map that maps the vertices of $K$ to pairwise distinct points on the stretched moment curve according to Lemma 3.2. Suppose there were $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $K$ such that $\overline{f}(\sigma_1) \cap \cdots \cap \overline{f}(\sigma_r) \neq \emptyset$. Then w.l.o.g. the $\sigma_i$ involve precisely $(r - 1)(d + 1) + 1$ vertices, and necessarily at least one $\sigma_j$ is not contained in $K'$. Such a face $\sigma_j$ must contain two successive integers in $[n]$, in contradiction to Lemma 3.2.

Let $L$ be the simplicial complex on the same vertex set as $K$ that is obtained from the simplex by deleting the edge $\{1, n\}$. We now proceed similar to the proof of Theorem 3.6. Let $D = \lfloor \frac{n + m - 2}{r - 1} \rfloor$. Then by Theorem 3.3 any continuous map $F : L \longrightarrow \mathbb{R}^{D-1}$ identifies $r$ points from $r$ pairwise disjoint faces. Thus, by Theorem 3.1 we obtain the lower bound $\chi(KG^r(K, L)) \geq D - d$. It is left to show that $(r - 1)(D - d) \geq cd^r(F) = n - M$. Writing $D = \frac{n + m - 2 - \alpha}{r - 1}$ for some $0 \leq \alpha \leq r - 2$ we need to show that $n + m - 2 - \alpha + M - (r - 1) d \geq n$, which is equivalent to $M + m \geq (r - 1) d + \alpha + 2$. Since $M + m = (r - 1)(d + 1) + 1$ this follows from $\alpha \leq r - 2$.

The case for general $r$ follows by induction on prime power factors very similar to the proof of Kříž [23]. We have to be slightly more careful for those $r$ with precisely one two in their prime factorization. We essentially repeat Kříž' proof with a few changes. Let $r = pq$, where we have shown the claimed lower bound for the chromatic number for $KG^p(F_{2\text{-stab}})$ and for $KG^q(F_{2\text{-stab}})$ for any set system $F$ already. Let us moreover assume that $q \neq 2$. Write $t = \chi(KG^r(F_{2\text{-stab}}))$ and assume for contradiction that $t - 1 < \frac{1}{r - 1}cd^r(F)$, that is, $(r - 1)(t - 1) < cd^r(F)$. 


For any subset $E \subset [n]$ denote by $\mathcal{F}|_E$ the set system $\{F \cap E : F \in \mathcal{F}\}$. Let $\Gamma = \{E \subset [n] : \text{cd}^q(\mathcal{F}|_E) > (q - 1)(t - 1)\}$. Then $\text{cd}^P(\Gamma) > (p - 1)(t - 1)$. Otherwise we could find $A_1, \ldots, A_p \subset [n]$ with no subset in $\Gamma$ such that $|n| \setminus \bigcup_i A_i| \leq (p - 1)(t - 1)$. Since the $A_i$ are not in $\Gamma$ we know that $\text{cd}^q(\mathcal{F}|_{A_i}) \leq (q - 1)(t - 1)$, and so we can find $B_{i,1}, \ldots, B_{i,q} \subset A_i$ with no subset in $\mathcal{F}$ and $|A_i \setminus \bigcup_j B_{i,j}| \leq (q - 1)(t - 1)$. There are $pq = r$ sets $B_{i,j}$ in total and the complement of their union in $[n] \setminus \bigcup_i A_i$ has at most $p(q - 1)(t - 1) + (p - 1)(t - 1)$ elements, which is equal to $(r - p)(t - 1) + (p - 1)(t - 1) = (r - 1)(t - 1)$, which is a contradiction to $\text{cd}^r(\mathcal{F}) > (r - 1)(t - 1)$.

Given an arbitrary $(t - 1)$-coloring of $\mathcal{F}$ we need to show that there are $r$ pairwise disjoint sets of $\mathcal{F}_{2-\text{stab}}$ that receive the same color. By induction hypothesis we have that $\chi(\mathcal{F}|_E) > t - 1$ for any $E \in \Gamma$. Thus, we can find $q$ pairwise disjoint sets $X_{E,1}, \ldots, X_{E,q} \in (\mathcal{F}|_E)_{2-\text{stab}}$ all colored by $i_E$. Associating color $i_E$ to set $E$ defines a $(t - 1)$-coloring of $\Gamma$. Now since $\text{cd}^P(\Gamma) > (p - 1)(t - 1)$ we have by induction hypothesis (even for $p = 2$ by Theorem 4.1) that there are $p$ pairwise disjoint $E_1, \ldots, E_p \in \Gamma$ that receives the same color. Then $X_{E_1,1}, \ldots, X_{E_1,q}, \ldots, X_{E_p,1}, \ldots, X_{E_p,q}$ are $r$ pairwise disjoint sets in $\mathcal{F}_{2-\text{stab}}$ that all receive the same color, as desired.

For $r = 2$ the proof above still works as long as we replace the simplicial complex $L$ with the simplex on the vertex set of $\overline{K}$ and use the topological Radon theorem. We then obtain a lower bound for the chromatic number of generalized Kneser graphs, but instead of $\mathcal{F}_{2-\text{stab}}$ this lower bound holds for the set system $\mathcal{F}_{2-\text{stab}}$ that contains all almost 2-stable sets in $\mathcal{F}$.

**Theorem 4.3.** Let $\mathcal{F}$ be a system of subsets of $[n]$. Then $\chi(\mathcal{F}_{2-\text{stab}}) \geq \text{cd}^2(\mathcal{F})$.

**Remark 4.4.** It was pointed out by Frédéric Meunier that a weaker version of Theorem 4.2 for almost stable sets can also be deduced from recent work of Alishahi and Hajiabolhassan [1] and their notion of *alternation number* of a Kneser hypergraph.

Let $\mathcal{F} = \binom{[n]}{k}$ be the system of all $k$-subsets of $[n]$. Then $\text{cd}^r(\mathcal{F}) = n - r(k - 1)$, and thus Theorem 4.1 is an extension of Theorem 2.2 that $\chi(\mathcal{F}(k, n)) \geq \lceil \frac{n - r(k - 1)}{r - 1} \rceil$ to arbitrary set systems. However, these general lower bounds are significantly worse for stable Kneser hypergraphs; any set of $2k - 2$ consecutive integers in $[n]$ cannot contain a 2-stable $k$-set, and thus for $\mathcal{F}$ the collection of 2-stable $k$-sets in $[n]$ the colorability defect is $\text{cd}^r(\mathcal{F}) \leq n - 2r(k - 1)$. Tight lower bounds for the chromatic numbers of stable Kneser hypergraphs cannot be obtained from the general result of Theorem 4.1.
Here we extend Theorem 4.1 for $r$ a power of a prime to show that these “local effects” have no influence on the chromatic number of a general Kneser hypergraph. We do this in two ways; first, we define a topological analog $tcd^r(\mathcal{F})$ of the colorability defect $cd^r(\mathcal{F})$, which satisfies $tcd^r(\mathcal{F}) \geq cd^r(\mathcal{F})$ and still gives a lower bound for the chromatic number of $\chi(K_{G^r}(\mathcal{F}))$ for $r$ a prime power. The quantity $tcd^r(\mathcal{F})$ takes the global topology of the set system $\mathcal{F}$ into account, while $cd^r(\mathcal{F})$ is defined purely combinatorially. The 2nd way of eliminating local effects of a set system $\mathcal{F}$ that results in loose lower bounds for $\chi(K_{G^r}(\mathcal{F}))$ will be to extend the results above to the $s$-stable setting.

There is a unique simplicial complex $K$ on vertex set $[n]$, whose missing faces are the inclusion-minimal sets in $\mathcal{F}$. The complexes $K \ast \Delta_t$, for $t \geq 0$, have the same missing faces as $K$, and up to isomorphism these are all simplicial complexes whose missing faces are the inclusion-minimal sets in $\mathcal{F}$. To determine $tcd^r(\mathcal{F})$ we need to decide the minimal dimension $d$ of Euclidean space that these complexes can be continuously mapped to without points of $r$-fold coincidence among their pairwise disjoint faces; the topological $r$-colorability defect $tcd^r(\mathcal{F})$ of a system $\mathcal{F}$ of subsets of $[n]$ is defined as the maximum of $N - (r - 1)(d + 1)$, where $[N]$, for some $N \geq n$, is vertex set of a simplicial complex $K$ whose missing faces are precisely the inclusion-minimal sets in $\mathcal{F}$ and $d$ is the smallest dimension such that there is a continuous map $f: K \to \mathbb{R}^d$ that does not identify $r$ points from $r$ pairwise disjoint faces. A general position map $f: K \to \mathbb{R}^d$ shows that $cd^r(\mathcal{F}) \leq tcd^r(\mathcal{F})$.

**Lemma 4.5.** Let $\mathcal{F}$ be a system of subsets of $[n]$. Then $cd^r(\mathcal{F}) \leq tcd^r(\mathcal{F})$.

**Proof.** Let $K$ be the simplicial complex on vertex set $[n]$ with missing faces $\mathcal{F}$. Let

$$
M = \max \left\{ \sum_{i=1}^{r} |A_i| : A_1, \ldots, A_r \subset [n] \text{ pairwise disjoint and } F \not\subset A_i \text{ for all } F \in \mathcal{F} \text{ and } i \in [r] \right\}.
$$

In particular, the sets $A_i$ in the definition of $M$ above determine faces of the complex $K$. Choose the integer $d$ as the least integer such that $M < (r - 1)(d + 1) + 1$, that is, $M + m = (r - 1)(d + 1) + 1$ for some $0 < m \leq r - 1$. Define the simplicial complex $\overline{K}$ to be the join $K \ast \Delta_{m-2}$. Then any strong general position map $f: \overline{K} \to \mathbb{R}^d$ satisfies $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ for any $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\overline{K}$. By definition of $M$ the faces $\sigma_i$ may in total only involve at most $M$ vertices of $K$. The complex $\overline{K}$ has $m - 1$ additional vertices, so the faces $\sigma_i$ involve less than $M + m = (r - 1)(d + 1) + 1$, thus the sets $f(\sigma_1), \ldots, f(\sigma_r)$ do not have a common point of intersection by a codimension count.
By definition \( \text{tcd}^r(\mathcal{F}) \geq n + m - 1 - (r-1)(d+1) \) and this is at least \( n - M = \text{cd}^r(\mathcal{F}) \) since \( M + m = (r-1)(d+1) + 1 \).

\[\text{Theorem 4.6.}\] For \( r \) a power of a prime

\[ \chi(\mathcal{K}G^r(\mathcal{F})) \geq \left\lceil \frac{1}{r-1} \text{tcd}^r(\mathcal{F}) \right\rceil. \]

\[\text{Proof.}\] By definition of topological \( r \)-colorability defect there is a map \( f: K \rightarrow \mathbb{R}^d \) that does not identify \( r \) points from \( r \) pairwise disjoint faces, where \( K \) is a simplicial complex on \( N \) vertices whose missing faces are the sets in \( \mathcal{F} \) and \( \text{tcd}^r(\mathcal{F}) = N - (r-1)(d+1) \). For any continuous map \( F: \Delta_{N-1} \rightarrow \mathbb{R}^{D-1} \) there are \( r \) pairwise disjoint faces of \( \Delta_{N-1} \) whose images have a common point of intersection, where \( D = \lfloor \frac{N-1}{r} \rfloor \).

By Theorem 3.1 we have that \( \chi(\mathcal{K}G^r(K)) \geq D - d \). It remains to show that \( (r-1)(D - d) \geq \text{tcd}^r(\mathcal{F}) = N - (r-1)(d+1) \). This is equivalent to \( (r-1)\lfloor \frac{N-1}{r} \rfloor \geq N - (r-1) \), which is evidently true.

We make two remarks that easily follow from [17, Examples 4.9 and 4.11]. Theorem 4.6 fails for any \( r \) that is not a prime power. Let \( \mathcal{F} \) be \( \binom{[n]}{k} \), the system of \( k \)-subsets of \( [n] \), where \( n = rk - 1 \) for some integer \( r \geq 6 \) with at least two distinct prime divisors. The hypergraph \( \mathcal{K}G^r(\mathcal{F}) \) does not contain any hyperedges and thus has chromatic number \( \chi(\mathcal{K}G^r(\mathcal{F})) = 1 \). Let \( K \) be the complex on vertex set \( [n] \) with missing faces \( \mathcal{F} \), that is, \( K = \Delta_{n-1}^{(k-2)} \). Suppose that \( k-2 = (r-1)\ell \) for some integer \( \ell \geq 3 \). Then there is a map \( f: K \rightarrow \mathbb{R}^{r\ell} \) that does not identify \( r \) points from \( r \) pairwise disjoint faces [26].

One now easily checks that \( \text{tcd}^r(\mathcal{F}) \geq n - (r-1)(r\ell + 1) = r \), and thus \( \lfloor \frac{1}{r-1} \text{tcd}^r(\mathcal{F}) \rfloor \geq 2 \).

The gap \( \text{tcd}^r(\mathcal{F}) - \text{cd}^r(\mathcal{F}) \) can be arbitrarily large. For example, let \( K \) be a triangulation of a \( d \)-ball on \( n \) vertices and denote by \( \mathcal{F} \) the system of its missing faces. Since \( K \) embeds into \( \mathbb{R}^d \) we have that \( \text{tcd}^2(\mathcal{F}) \geq n - (d+1) \), while \( \text{cd}^2(\mathcal{F}) = n - 2(d+1) \), since the facets of \( K \) have no subset in \( \mathcal{F} \).

Let \( \mathcal{F} \) be a system of subsets of \( [n] \), and let \( C_1 \sqcup \cdots \sqcup C_m \) be a partition of \( [n] \) into \( m \) sets. We call a set \( F \in \mathcal{F} \) transversal if \( |F \cap C_i| \leq 1 \) for every \( i \). Denote by \( \mathcal{F}_{\text{transversal}} \) the system of all transversal sets in \( \mathcal{F} \) with respect to the partition \( C_1 \sqcup \cdots \sqcup C_m \).

\[\text{Theorem 4.7.}\] Let \( r \geq 2 \) be a prime, and let \( \mathcal{F} \) be a system of subsets of \( [n] \), and \( C_1 \sqcup \cdots \sqcup C_m \) a partition of \( [n] \) into sets of size at most \( r - 1 \). Then

\[ \chi(\mathcal{K}G^r(\mathcal{F}_{\text{transversal}})) \geq \left\lceil \frac{1}{r-1} \text{tcd}^r(\mathcal{F}) \right\rceil. \]
Proof. Repeat the proof of Theorem 4.6, but now apply Theorem 3.4 to the simplicial complex obtained from the simplex on the vertex set of $K$ by deleting all edges with both endpoints in one $C_i$.

Theorem 4.7 is wrong for any $r$ with at least two distinct prime divisors since the weaker Theorem 4.6 is wrong for those $r$. We conjecture that the primality of $r$ is required. Theorem 4.7 is a common generalization and extension of the results of Kříž (for $r$ a prime) and Schrijver as well as the hypergraph version of Schrijver’s result, Theorem 4.2. The proof of Theorem 4.2 shows that it is implied by Theorem 4.7 (for $r \geq 3$ a prime), where we bound $\text{tcd}^r$ from below by exhibiting an affine map that sends vertices to points along the moment curve, and sets $C_i$ of the partition have size one, except for one set that contains 1 and $n$. For the $r = 2$ setting we are not allowed to have sets $C_i$ of size two, and we can only restrict to almost 2-stable sets. Thus, Theorem 4.3 is also an easy corollary of Theorem 4.7.

We will now prove tight lower bounds for chromatic numbers of $s$-stable Kneser hypergraphs. For a graph $G$ and a vertex $v$ of $G$ denote by $N(v)$ the neighborhood of $v$, that is, all vertices at distance one from $v$. Similarly denote by $N^2(v)$ the set of vertices at distance precisely two from $v$. Engström proved the following Tverberg-type result for sparse subcomplexes of a simplex determined by a graph $G$ of forbidden edges.

**Theorem 4.8 (Engström [14]).** Let $r \geq 2$ be a prime power, $d \geq 1$ an integer, and $N = (r - 1)(d + 1)$. Let $G$ be a graph on the vertex set of $\Delta_N$ that satisfies $2|N(v)| + |N^2(v)| < r$ for every vertex $v$. Then for any continuous map $f: \Delta_N \to \mathbb{R}^d$ there are $r$ pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta_N$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$ and no $\sigma_i$ contains both endpoints of an edge of $G$.

Using this result we obtain a strengthening of Theorem 4.6:

**Theorem 4.9.** Let $s \geq 2$ be an integer and $r > 6s - 6$ a prime power. Then for any system $\mathcal{F}$ of subsets of $[n]$ we have that

$$\chi(KG_r^s(F_{stab})) \geq \left\lceil \frac{\text{tcd}^r(F)}{r - 1} \right\rceil.$$
with an edge \((i, j)\) precisely if \(i\) and \(j\) are at distance at most \(s - 1\) in the cyclic order on \([n]\). Notice that for every vertex \(v\) of \(G\) the neighborhood \(N(v)\) has cardinality \(2s - 2\) and \(N^2(v)\) has cardinality \(2s - 2\) as well. Let \(L\) be the simplicial complex obtained from the simplex on vertex set \([n]\) by deleting the edges of \(G\). The missing faces of \(K\) that are contained in \(L\) are precisely the \(s\)-stable sets in \(F\). By Theorem 4.8 any continuous map \(F: L \rightarrow \mathbb{R}^{D-1}\) identifies \(r\) points from \(r\) pairwise disjoint faces.

Since \(\text{tcd}_r\left(\binom{[n]}{k}\right) = n - r(k - 1)\)—this is the content of the proof of Theorem 2.2—we obtain the following as an immediate consequence:

**Corollary 4.10.** Let \(s \geq 2\) be an integer and \(r > 6s - 6\) a prime power. Then for integers \(k \geq 1\) and \(n \geq rk\) we have that

\[
\chi(KG^r(k, n)_{s-stab}) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.
\]

Certainly our methods can be pushed further with some additional effort. The special structure of the constraint graph for the Tverberg-type result needed to prove Corollary 4.10 could enable us in principle to get the same result for \(r \sim 2s\) by topological connectivity alone. Conjecture 3.9 is widely believed to be true. Here we conjecture its generalization to arbitrary set systems.

**Conjecture 4.11.** Let \(r \geq 3\) and let \(F\) be a set system. Then

\[
\chi(KG^r(F_{-stab})) \geq \left\lceil \frac{\text{cd}_r(F)}{r - 1} \right\rceil.
\]

5 Lower Bounds for Chromatic Numbers via Equivariant Topology

The methods presented in the preceding sections have the shortcoming that they only apply to hypergraphs that are represented as Kneser hypergraphs, with vertices in correspondence with missing faces of a simplicial complex and hyperedges that contain \(r\)-tuples of pairwise disjoint missing faces. Here we will remedy this shortcoming by lifting our construction to a box complex construction for hypergraphs. Our results turn out to again improve upon the general lower bounds obtained by Kříž. This section also aims to clarify how the methods of the preceding sections relate to the more classical box complex constructions.
Let $H$ be an $r$-uniform hypergraph. Let $\overline{H}$ be the simplicial complex that for any hyperedge $\sigma \in H$ contains all subsets of $\sigma$. Denote by

$$B(H) = \{A_1 \times \{1\} \cup \cdots \cup A_r \times \{r\} \mid A_i \subseteq V(H), A_i \cap A_j = \emptyset \text{ for } i \neq j,$$

and picking at most one element from each $A_i$ always results in a face of $\overline{H}$

the box complex of $H$. In particular, the vertex set of $B(H)$ consists of $r$ copies of the vertex set of $H$, provided that $H$ does not have isolated vertices. The requirement “picking at most one element from each $A_i$ always results in a face of $\overline{H}$” can be rephrased as “the join of the nonempty $A_i$ is a subcomplex of $\overline{H}$”, where we think of the $A_i$ as discrete (0-dimensional) complexes.

As an example consider the hypergraph $H$ on vertex set $\{1, 2, 3, 4\}$. The box complex $B(H)$ has twelve vertices; for vertex $i$ of $H$ the box complex has vertices $a_i, b_i, c_i$. Then $B(H)$ contains the triangles $a_1, b_2, c_3$ and $a_2, b_3, c_4$ and five symmetric copies of each obtained by permuting $a, b$, and $c$, as well as the tetrahedron $a_1, b_2, b_3, a_4$ and its symmetric copies.

Topological box complex approaches (or the more general Hom-complexes) to finding lower bounds for chromatic numbers of graphs are standard; see Matoušek and Ziegler [28] for a survey. Extensions to hypergraphs are not new either; see Lange and Ziegler [24], Iriye and Kishimoto [19], and Alishahi [1]. Our contribution is a linearization of the theory that also works for $r$ that are not powers of primes; see Theorem 5.2. Moreover, Theorem 5.4 explains the relation of our methods to the usual box complex approach.

**Lemma 5.1 (Volovikov [36]).** Let $p$ be a prime and $G = (\mathbb{Z}/p)^n$ an elementary abelian $p$-group. Suppose that $X$ and $Y$ are fixed-point free $G$-spaces such that $\tilde{H}^i(X; \mathbb{Z}/p) \cong 0$ for all $i \leq n$ and $Y$ is an $n$-dimensional cohomology sphere over $\mathbb{Z}/p$. Then there does not exist a $G$-equivariant map $X \rightarrow Y$.

We denote the standard representation of the symmetric group $S_r$ by $W_r$, that is,

$$W_r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r : \sum x_i = 0\}$$

with the action by $S_r$ that permutes coordinates. We denote by $K^{*r}$ the $r$-fold join of a simplicial complex $K$, whose faces are joins $\sigma_1 \ast \cdots \ast \sigma_r$ of $r$ faces of $K$. If the faces $\sigma_i$ are required to be $s$-wise disjoint, that is, any $s$ of them have no vertex in common, then the
resulting complex is denoted \( K_{\Delta(S)}^s \), the \( s \)-wise deleted \( r \)-fold join. We write \( K_{\Delta}^s \) for \( K_{\Delta(2)}^r \).

See Matoušek [27] for details and notation.

**Theorem 5.2.** If \( H \) is \( c \)-colorable then there exists an affine \( S_r \)-equivariant map \( B(H) \rightarrow W_r^{\oplus c} \setminus \{0\} \). In particular, if \( r \) is a prime power and \( B(H) \) is \([(r-1)(c-1)-1]\)-connected, then \( \chi(H) \geq c \).

**Proof.** Let \( f: V(H) \rightarrow [1, \ldots, c] \) be a proper coloring of \( H \). Think of color \( i \in [1, \ldots, c] \) as the standard unit vector \( e_i \in \mathbb{R}^c \). Denote by \( \Delta \) the simplex on vertex set \( V(H) \). By affinely extending onto the faces of \( \Delta \) we can think of \( f \) as a simplicial map \( f: \Delta \rightarrow \Delta_{c-1} \). The box complex \( B(H) \) is a subcomplex of \( \Delta_{\Delta}^s \). Thus, the \( r \)-fold join of \( f \) yields an \( S_r \)-equivariant map \( f: B(H) \rightarrow (\Delta_{c-1})^{sr} \). Since \( f \) is a proper coloring of \( H \), the image of \( f \) is contained in \((\Delta_{c-1})^{sr}_{\Delta(r)} \); otherwise, there is a vertex \( i \) of \( \Delta_{c-1} \) (equivalently, a color in \([1, \ldots, c]\)) and pairwise distinct vertices \( v_1, \ldots, v_r \in V(H) \) that are all colored by \( i \) and such that \( \{v_1, \ldots, v_r\} \) is a hyperedge of \( H \). But this is a contradiction to \( f \) being a proper coloring.

Now, \((\Delta_{c-1})_{\Delta(r)}^{sr} \cong (pt)^{sr}_{\Delta(r)} \cong (pt)^{sr}_{\Delta} \cong (\partial \Delta_{c-1}) \cong S^{(r-1)c-1} \). This join of boundaries of simplices is a polytope that can be realized with its full group of symmetries in \( W_{r}^{\oplus c} \cong \mathbb{R}^{(r-1)c} \) such that the origin is in its interior. The hypergraph \( H \) cannot be \((c-1)\)-colorable if there is no \( S_r \)-map \( B(H) \rightarrow W_{r}^{\oplus (c-1)} \setminus \{0\} \). In particular, such a map does not exist if \( r \) is a prime power and \( B(H) \) is \([(r-1)(c-1)-1]\)-connected by Lemma 5.1.

For this, observe that the action of \((\mathbb{Z}/p)^n \) on itself embeds it into \( S_{\mathbb{Z}}^n \) as a transitive subgroup. In particular, the induced action of \((\mathbb{Z}/p)^n \) on \( W_{r}^{\oplus (c-1)} \setminus \{0\} \) is fixed-point free; see Özaydin [30, Lemma 2.1] for details.

**Example 5.3.** Let \( r \geq 1 \) and \( c \geq 1 \) be integers. Let \( H \) be the hypergraph on vertex set \([n]\) for \( n = (r-1)(c-1) + 1 \) that has all \( r \)-element sets as hyperedges. Then the chromatic number of \( H \) is \( \chi(H) = c \). To see that \( c \) colors suffice, color the vertex set \([n]\) by splitting it into \( c-1 \) parts of size \( r-1 \) and color the last vertex by an additional color. The box complex \( B(H) \) is isomorphic to \((\Delta_{n-1})_{\Delta}^s \). An \( S_r \)-equivariant map \( B_r(H) \rightarrow W_{r}^{\oplus c} \setminus \{0\} \) exists if and only if \( r \) has two distinct prime divisors; see Özaydin [30]. Thus, for prime powers we get the lower bound \( \chi(H) \geq c \). This topological approach does not yield an immediate tight lower bound for the chromatic number when \( r \) not a power of a prime. However, there is no affine \( S_r \)-equivariant map \( B_r(H) \rightarrow W_{r}^{\oplus c} \setminus \{0\} \) for any \( r \). This follows from Bárány’s colorful Carathéodory’s theorem [6]; see Sarkaria [34]. Thus, the linearized version in Theorem 5.2 is properly stronger than the usual topological approach.
The following theorem explains the relation of the approach of Section 4 via geometric transversality results to the box complex approach of Theorem 5.2. The two approaches can be related using ideas of Sarkaria; see in particular [33, Proof of Theorem 1.3].

**Theorem 5.4.** Let $K \subset L$ be simplicial complexes and let $r$ be a prime. Further assume that there exists an $\mathbb{Z}/r$-equivariant map $f: K^r_\Delta \to S^N$, where the $N$-sphere $S^N$ has a free linear $\mathbb{Z}/r$-action. If $\chi(KG'(K,L)) \leq c$ then there exists a $\mathbb{Z}/r$-equivariant map $F: L^r_\Delta \to S^{N+(r-1)c}$, where the $\mathbb{Z}/r$-action on $S^{N+(r-1)c}$ is free.

**Proof.** Let $H$ be the hypergraph with vertices the nonfaces of $K$ contained in $L$ (recall that the vertices of $KG'(K,L)$ are the missing faces of $K$ in $L$, that is, the minimal nonfaces) and hyperedges corresponding to $r$ pairwise disjoint nonfaces. Then $KG'(K,L) \subset H$ and $\chi(KG'(K,L)) = \chi(H)$. Any vertex of $H$ is a nonface $\sigma$ of $K$ that contains some missing face $\tau$ of $K$; color $\sigma$ by the same color as $\tau$.

Denote by $\Sigma'$ the barycentric subdivision of the simplicial complex $\Sigma$. A vertex of $(L^r_\Delta)'$ corresponds to a join $\tau_1 \ast \cdots \ast \tau_r$ of pairwise disjoint faces in $L$. Think of a join of faces in $L$ as the (disjoint) union $\bigsqcup_i \tau_i \times \{i\}$. Denote by $\tau_K$ the face $\bigsqcup_{i: \tau_i \in K} \tau_i \times \{i\}$ of $K^r_\Delta$ and by $\tau_L$ the face $\bigsqcup_{i: \tau_i \notin K} \tau_i \times \{i\}$. Each vertex of $B(H)$ corresponds to $\tau \times \{i\}$, where $\tau \in L \setminus K$ and $i \in [r]$. Since the $\tau_i$ are pairwise disjoint, the face $\tau_L$ is naturally a face of $B(H)$, and thus a vertex of $B(H)'$. By Theorem 5.2 there is an (affine) $S_r$-equivariant map $h: B(H) \to W^{\oplus c}_r \setminus \{0\}$.

Think of $S^N$ as equivariantly embedded into $\mathbb{R}^{N+1}$. The map $F$ can now simply be defined as an affine map on $(L^r_\Delta)'$ by sending the vertex corresponding to the face $\tau_1 \ast \cdots \ast \tau_r \in L^r_\Delta$ to

$$(h(\tau_L), f(\tau_K)) \in W^{\oplus c}_r \oplus \mathbb{R}^{N+1},$$

where we consider $\tau_L$ as a vertex of $B(H)'$ and $\tau_K$ as a vertex of $(K^r_\Delta)'$, provided that $\tau_L$ and $\tau_K$ are nonempty. We define $h(\emptyset) = 0 = f(\emptyset)$. The map $F$ is $\mathbb{Z}/r$-equivariant and misses the origin of $W^{\oplus c}_r \oplus \mathbb{R}^{N+1}$. To see the latter, notice that it is not possible that both $\tau_K$ and $\tau_L$ are empty, and that a face of $(L^r_\Delta)'$ corresponds to a chain of faces $\tau^{(1)} \subset \tau^{(2)} \subset \cdots \subset \tau^{(m)}$ of $L^r_\Delta$, the corresponding nonempty $\tau^{(i)}_K$ form a face of $(K^r_\Delta)'$ (and thus $F$ does not hit 0 in $\mathbb{R}^{N+1}$ as long as $\tau^{(i)}_K \neq \emptyset$), and similarly the corresponding nonempty $\tau^{(i)}_L$ form a face of $B(H)'$ (and thus $F$ does not hit 0 in $W^{\oplus c}_r$ as long as $\tau^{(i)}_L \neq \emptyset$). The map $F$ as defined above thus retracts equivariantly to a map to the unit sphere $S^{N+(r-1)c}$ in $W^{\oplus c}_r \oplus \mathbb{R}^{N+1}$. ■
Since any map $K \to \mathbb{R}^d$ that does not identify $r$ points from $r$ pairwise disjoint faces induces a $\mathbb{Z}/r$-equivariant map $K^r_\Delta \to S(W_r^{(d+1)})$ as a retraction to the unit sphere of its $r$-fold join, the results of the previous sections are special cases of Theorem 5.4 for $r$ a prime. The theorem can be extended to prime powers by using fixed-point free actions of elementary abelian groups. The methods of [17] and the manuscript at hand can be seen as augmenting the box complex construction by first constructing an equivariant map from a sufficiently large symmetric subcomplex. We avoid having to deal with obstruction theory by inducing these equivariant maps from continuous maps $K \to \mathbb{R}^d$ that do not identify points from $r$ pairwise disjoint faces.

**Funding**

This work was supported by the National Science Foundation [DMS–1440140].

**Acknowledgments**

I would like to thank Frédéric Meunier for many helpful discussions about chromatic numbers of Kneser hypergraphs and Günter M. Ziegler for good comments on a draft of this manuscript. The comments of an anonymous referee helped improve the exposition. This material is based upon work supported by the National Science Foundation while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2017 semester.

**References**

[1] Alishahi, M. “Colorful subhypergraphs in uniform hypergraphs.” *Electron. J. Comb.* 24 (2017): P1.23.

[2] Alishahi, M. and H. Hajiabolhassan. “On the chromatic number of general Kneser hypergraphs.” *J. Combin. Theory Ser. B* 115 (2015): 186–209.

[3] Alon, N., L. Drewnowski, and T. Łuczak. “Stable Kneser hypergraphs and ideals in $\mathbb{N}$ with the Nikodym property.” *Proc. Amer. Math. Soc.* 137 (2009): 467–71.

[4] Alon, N., P. Frankl, and L. Lovász. “The chromatic number of Kneser hypergraphs.” *Trans. Amer. Math. Soc.* 298, no. 1 (1986): 359–70.

[5] Bajmóczy, E. G. and I. Bárány. “On a common generalization of Borsuk’s and Radon’s theorem.” *Acta Math. Hungar.* 34, no. 3 (1979): 347–50.

[6] Bárány, I. “A generalization of Carathéodory’s theorem.” *Discrete Math.* 40, nos. 2–3 (1982): 141–52.

[7] Bárány, I., S. B. Shlosman, and A. Szücs. “On a topological generalization of a theorem of Tverberg.” *J. Lond. Math. Soc.* 23 (1981): 158–64.

[8] Blagojević, P. V. M., F. Frick, and G. M. Ziegler. “Tverberg plus constraints.” *Bull. Lond. Math. Soc.* 46 (2014): 953–67.
[9] Blagojević, P. V. M., F. Frick, and G. M. Ziegler. “Barycenters of polytope skeleta and counterexamples to the topological Tverberg conjecture, via constraints.” J. Eur. Math. Soc., to appear (2018).
[10] Blagojević, P. V. M., B. Matschke, and G. M. Ziegler. “Optimal bounds for the colored Tverberg problem.” J. Eur. Math. Soc. 17, no. 4 (2015): 739–54.
[11] Bukh, B., P. S. Loh, and G. Nivasch. “Classifying unavoidable Tverberg partitions.” J. Comput. Geom. 8, no. 1 (2017): 174–205.
[12] Chen, P.-A. “On the multichromatic number of s-stable Kneser graphs.” J. Graph Theory 79, no. 3 (2015): 233–48.
[13] Dol’nikov, V. “A certain combinatorial inequality.” Siberian Math. J. 29, no. 3 (1988): 375–9.
[14] Engström, A. “A local criterion for Tverberg graphs.” Combinatorica 31, no. 3 (2011): 321–32.
[15] Erdős, P. “Problems and results in combinatorial analysis.” Colloq. Internat. Theor. Combin. Rome (1973): 3–17.
[16] Frick, F. “Counterexamples to the topological Tverberg conjecture.” Oberwolfach Rep. 12, no. 1 (2015): 318–21.
[17] Frick, F. “Intersection patterns of finite sets and of convex sets.” Proc. Amer. Math. Soc. 145, no. 7 (2017): 2827–42.
[18] Hell, S. “Tverberg’s theorem with constraints.” J. Combin. Theory Ser. A 115, no. 8 (2008): 1402–16.
[19] Iriye, K. and D. Kishimoto. “Hom complexes and hypergraph colorings.” Topology Appl. 160, no. 12 (2013): 1333–44.
[20] Jonsson, J. “On the Chromatic Number of Generalized Stable Kneser Graphs.” Manuscript, 2012, available at https://people.kth.se/jakobj/doc/submitted/stablekneser.pdf.
[21] Kneser, M. “Aufgabe 360.” Jahresber. Deutsch. Math.-Verein. 2 (1955): 27.
[22] Kříž, I. “Equivariant cohomology and lower bounds for chromatic numbers.” Trans. Amer. Math. Soc. 33 (1992): 567–77.
[23] Kříž, I. “A correction to, ‘Equivariant cohomology and lower bounds for chromatic numbers’.” Trans. Amer. Math. Soc. 352, no. 4 (2000): 1951–2.
[24] Lange, C. and G. M. Ziegler. “On generalized Kneser hypergraph colorings.” J. Combin. Theory Ser. A 114, no. 1 (2007): 159–66.
[25] Lovász, L. “Kneser’s conjecture, chromatic number, and homotopy.” J. Combin. Theory Ser. A 25, no. 3 (1978): 319–24.
[26] Mabillard, I. and U. Wagner. “Eliminating Higher-Multiplicity Intersections, I.” A Whitney Trick for Tverberg-Type Problems (2015): preprint arXiv:1508.02349.
[27] Matoušek, J. Using the Borsuk-Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry. 2nd ed. Universitext. Heidelberg: Springer, 2008.
[28] Matousek, J. and G. M. Ziegler. “Topological lower bounds for the chromatic number: a hierarchy.” Jahresber. Deutsch. Math.-Verein. 106 (2004): 71–90.
[29] Meunier, F. “The chromatic number of almost stable Kneser hypergraphs.” J. Combin. Theory Ser. A 118, no. 6 (2011): 1820–8.
[30] Özaydin, M. “Equivariant maps for the symmetric group.” (1987): preprint, 17 pages, http://digital.library.wisc.edu/1793/63829.
[31] Perles, M. A. and M. Sigron. “Tverberg partitions of points on the moment curve.” Discrete Comput. Geom. 57, no. 1 (2017): 56–70.
[32] Sarkaria, K. S. “A generalized Kneser conjecture.” J. Combin. Theory Ser. B 49, no. 2 (1990): 236–40.
[33] Sarkaria, K. S. “A generalized van Kampen-Flores theorem.” Proc. Amer. Math. Soc. 11 (1991): 559–65.
[34] Sarkaria, K. S. “Tverberg partitions and Borsuk-Ulam theorems.” Pacific J. Math. 196, no. 1 (2000): 231–41.
[35] Schrijver, A. “Vertex-critical subgraphs of Kneser-graphs.” Nieuw Archief voor Wiskunde 26 (1978): 454–61.
[36] Volovikov, A. Y. “On a topological generalization of the Tverberg theorem.” Math. Notes 59, no. 3 (1996): 324–6.
[37] Vučić, A. and R. T. Živaljević. “Note on a conjecture of Sierksma.” Discrete Comput. Geom. 9, no. 1 (1993): 339–49.
[38] Ziegler, G. M. “Generalized Kneser coloring theorems with combinatorial proofs.” Invent. Math. 147, no. 3 (2002): 671–91.