Tau function of the CKP hierarchy and nonlinearizable virasoro symmetries

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Abstract

We introduce a single tau function that represents the C-type Kadomtsev–Petviashvili (CKP) hierarchy in a generalized Hirota ‘bilinear’ equation. The actions on the tau function by additional symmetries for the hierarchy are also calculated, which involve strictly more than a central extension of the $wC∞$-algebra. As an application, for Drinfeld–Sokolov hierarchies associated to affine Kac–Moody algebras of type C, we obtain a formula to compute the obstacles in linearizing their Virasoro symmetries and hence prove the Virasoro symmetries to be nonlinearizable when acting on the tau function.

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1. Introduction

The Kadomtsev–Petviashvili (KP) hierarchy together with its subhierarchies of types B and C [2, 3], abbreviated as the BKP and the CKP hierarchies respectively, have attracted much research interest in areas of mathematical physics. These hierarchies can be represented equivalently as Lax equations of pseudo-differential operators or as bilinear equations. For instance, the CKP hierarchy considered in the present paper is defined by the following bilinear equation

\[ \text{res}_z w(t; z)w(t'; -z) = 0, \]

where $w$ is the so-called wave function depending on the time variable $t = (t_1, t_3, t_5, \ldots)$ and a parameter $z$, and $\text{res}_z \sum_i f_i z^i = f_{-1}$ for any formal Laurent series in $z$.

For the CKP hierarchy, in contrast to the KP and the BKP cases, there does not seem to exist a single tau function that represents (1.1) in the form of Hirota bilinear equations, though it was pointed out by Date \textit{et al} [2] that a tau function may be constructed from the action of bosonic fields on the vacuum vector in a Fock space. The idea in [2] was later developed by \textit{et al}. [2]
van de Leur et al [16]. They introduced a series of fermionic operators besides the bosonic fields, and constructed a tau function depending on both time variables \( t_k \) and certain odd Grassmannian parameters. In fact it is a tau function of a generalization of the CKP hierarchy, i.e., a system of bilinear equations like (1.1) of wave functions labelled with an ‘odd number of Odd Partitions with Distinct parts’ (see equation (2.38) in [16]). In particular, their wave function \( w_1 \) (with its expansion (2.35)–(2.37) in [16]) solves the bilinear equation (1.1) of the CKP hierarchy, but it is related to a series of tau functions rather than only one.

In this paper we introduce a tau function \( \tau(t) \) of the CKP hierarchy by making use of its Hamiltonian densities, considering the fact that the hierarchy carries a series of bi-Hamiltonian structures reduced from those for the KP hierarchy [5]. This tau function will be shown to be related to the wave function via the following formula

\[
w(t; z) = \left( 1 + \frac{1}{z} \frac{\partial}{\partial t_1} \log \frac{G(z) \tau(t)}{\tau(t)} \right)^{1/2} \frac{G(z) \tau(t)}{\tau(t)} e^{\xi(t; z)},
\]

where

\[
G(z) = \exp \left( - \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{2}{k} \frac{\partial}{\partial t_k} \right), \quad \xi(t; z) = \sum_{k \in \mathbb{Z}_{\text{odd}}} t_k z^k.
\]

Observe that (1.2) is different from those formulae relating wave and tau functions of integrable hierarchies in the literature, namely, now there is a square-root factor depending on the tau function. By substituting this formula into (1.1), the CKP hierarchy is recast to a generalized Hirota ‘bilinar’ equation (see equation (3.22)).

As the tau function of the CKP hierarchy is introduced, we continue to study the action on it by the additional symmetries. Recall that the additional symmetries for the CKP hierarchy were constructed by He, Tian, Foerster and Ma [9] with the help of certain Orlov–Schulman operators [19]. They also showed that these additional symmetries acting on the wave function \( w(t; z) \) form a centreless \( \mathfrak{wC}_\infty \)-algebra. However, their actions on the tau function of the hierarchy still need to be clarified. As observed by Adler et al [1] in the context of matrix integrals, lifting the actions of additional symmetries on the wave function to that of the tau function results in a central extension of the \( \mathfrak{wC}_\infty \)-algebra sort. Such phenomenon were confirmed for the KP and the BKP hierarchies as well as for the two-dimensional Toda lattice and the two-component BKP hierarchies [1, 4, 15, 20, 23]. In contrast to those, a counter example is found, which is the CKP hierarchy (1.1). More exactly, for the CKP hierarchy when the additional symmetries act on its tau function given in (1.2), it implies not only a central extension of the \( \mathfrak{wC}_\infty \)-algebra but also some non-trivial ‘tails’ given by polynomials in at-least-second-order derivatives of \( \log \tau \) with respect to the time variables. As far as we are aware, such counter examples have not been considered in the literature before, and whether their property can be illustrated in matrix models is unclear.

This paper is also motivated by the study of nonlinearizable Virasoro symmetries for integrable hierarchies proposed recently by one of the authors in [24] when considering Drinfeld–Sokolov hierarchies. Recall that for every affine Kac–Moody algebra with an arbitrary vertex of the Dynkin diagram marked (only the case of the zeroth vertex is considered below), Drinfeld and Sokolov [6] constructed an integrable hierarchy of Korteweg–de Vries (KdV) type. These hierarchies are applied to various areas of mathematical physics [8, 14, 21, 22]. For instance, the Drinfeld–Sokolov hierarchy associated to each simply-laced affine Kac–Moody algebra was conjectured by Dubrovin and Zhang [8] to coincide with the topological integrable hierarchies constructed from semi-simple Frobenius manifolds corresponding to ADE-type Coxeter groups. By now this conjecture has been proved in [7, 24], see also [8, 17, 23]. One of the main steps in the proof is to check the linearization of Virasoro
symmetries acting on the tau function. In fact, in [24] we obtained a general formula to compute how the tau function is acted on by Virasoro symmetries for all Drinfeld–Sokolov hierarchies, and confirmed the linearization property for each hierarchy associated to an either simply-laced or twisted [12] affine Kac–Moody algebra. In contrast, Drinfeld–Sokolov hierarchies associated to affine Kac–Moody algebras of the other types were claimed [24] not to have linearized Virasoro symmetries. We will prove this claim for the hierarchies of type $C_n^{(1)}$, that is

**Theorem 1.1.** For the Drinfeld–Sokolov hierarchy associated to affine Kac–Moody algebra of type $C_n^{(1)}$ with $n \geq 2$, the Virasoro symmetries acting on the tau function (see (5.14)) are nonlinearizable. More precisely, the obstacles in linearizing the Virasoro symmetries, which were introduced in [24], can be computed via formula (5.34).

This theorem shows that the Drinfeld–Sokolov hierarchy of type $C_n^{(1)}$ with $n \geq 2$ does not belong to the set of topological hierarchies associated to semi-simple Frobenius manifolds [8]. The proof of it is based on the observation that each such hierarchy is equivalent to a certain reduction of the CKP hierarchy and that its Virasoro symmetries can be reduced accordingly from the additional symmetries for the latter.

To achieve the above results, we will arrange this paper as follows. In the following section we recall the definition of the CKP hierarchy and its additional symmetries. In section 3 we introduce a tau function of the CKP hierarchy by using its Hamiltonian densities and then represent the hierarchy as a ‘bilinear’ equation of the tau function. The actions on the tau function by the additional symmetries are considered in section 4. Section 5 consists of two parts. The first part is devoted to a brief review of the construction of Drinfeld–Sokolov hierarchies from affine Kac–Moody algebras of type C as well as their tau function and Virasoro symmetries; in the second part, these hierarchies and their Virasoro symmetries are reconstructed from a $2n$-reduction of the CKP hierarchy and its additional symmetries, which provides an alternative way to compute the obstacles in linearizing the Virasoro symmetries considered in [24]. A summary will be given in the final section.

2. The CKP hierarchy and its additional symmetries

Let $A$ be an algebra of smooth functions of a spatial coordinate $x$, and $D = d/dx$ be a derivation on $A$. The algebra of pseudo-differential operators is the following linear space

$$\mathcal{D} = \left\{ \sum_{i<\infty} f_i D^i \mid f_i \in A \right\} \quad (2.1)$$

equipped with a product defined by

$$f D^i \cdot g D^j = \sum_{r \geq 0} \binom{i}{r} f D^r (g) D^{i-r}, \quad f, g \in A.$$

For any operator $A = \sum_i f_i D^i \in \mathcal{D}$, its non-negative part, negative part, residue and adjoint operator are given respectively by

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad \text{res } A = f_{-1}, \quad A^* = \sum_{i} (-D)^i f_i. \quad (2.2)$$

These notions will be frequently used in the following.

Assume a pseudo-differential operator

$$L = D + \sum_{i \geq 1} v_i D^{-i} \in \mathcal{D} \quad (2.3)$$
satisfies $L^* = -L$. Note that each coefficient $v_{2j}$ is a differential polynomial in the functions $v_1, v_3, \ldots, v_{2j-1}$. The CKP hierarchy is defined by the following Lax equations:

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \quad k \in \mathbb{Z}_{+}^{\text{odd}},$$

(2.4)

which form a system of evolutionary equations of the vector function $v = (v_1, v_3, \ldots)$ depending on $t = (t_1, t_3, t_5, \ldots)$. Clearly $\partial/\partial t_1 = \partial/\partial x$; henceforth we simply assume $t_1 = x$.

The operator $L$ can be represented in a dressing form as

$$L = \Phi D \Phi^{-1},$$

(2.5)

where $\Phi$ is a pseudo-differential operator:

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \Phi^* = \Phi^{-1}.$$  

(2.6)

Note that the dressing operator $\Phi$ is determined up to multiplication to the right by an arbitrary operator of the form (2.6) with constant coefficients. With the help of the dressing operator, the CKP hierarchy (2.4) can be redefined by the Sato equations:

$$\frac{\partial \Phi}{\partial t_k} = -(L^k)_- \Phi, \quad k \in \mathbb{Z}_{+}^{\text{odd}}.$$  

(2.7)

Let $\xi(t; z) = \sum_{k \in \mathbb{Z}^{\text{odd}}} t_k z^k$ with some parameter $z$. Introduce a wave function

$$w(t; z) = \Phi e^{\xi(t; z)} = \phi(t; z)e^{\xi(t; z)},$$

(2.8)

where

$$\phi(t; z) = 1 + \sum_{i \geq 1} a_i z^{-i}$$  

(2.9)

(The convention $D^i e^{tz} = z^i e^{tz}$ for any integer $i$ is adopted). The dual wave function reads

$$w^*(t; z) = (\Phi^{-1})^* e^{-\xi(t; z)} = \phi(t; -z)e^{-\xi(t; z)} = w(t; -z).$$  

(2.10)

The CKP hierarchy (2.7), or (2.4), is equivalent to the following bilinear equation [2]:

$$\text{res}_z w(t; z)w(t'; -z) = 0.$$  

(2.11)

Here $\text{res}_z \sum_i f_i z^i = f_{-1}$ for any formal Laurent series $\sum_i f_i z^i$ in $z$.

For the CKP hierarchy, He et al [9] constructed its additional symmetries by using the following Orlov–Schulman [19] operator:

$$M = \Phi \Gamma \Phi^{-1}, \quad \Gamma = \sum_{k \in \mathbb{Z}_{+}^{\text{odd}}} k t_k D^{k-1}.$$  

Clearly $[L, M] = 1$.

**Remark 2.1.** Strictly speaking, the operator $M$ does not belong to the algebra $D$ in (2.1) for $M$ may contain infinitely many terms with positive power in $D$. A method to resolve this problem was given in [23], that is to assign certain degrees to $t_k$ and extend $D$ to be the so-called algebra of pseudo-differential operators of the first type (see [17]). In this way $L$ and $M$ are contained in a common algebra so that the product between them makes sense.

Given any pair of integers $(m, l)$ with $m \geq 0$, let

$$A_{ml} = M^m L^l - (-1)^l L^m M^m.$$  

(2.12)
In particular, one can check

\[ A_{0l} = \begin{cases} 0, & l \text{ even;} \\ 2L^l, & l \text{ odd;} \end{cases} \]

\[ A_{1l} = \begin{cases} -lL^{l-1}, & l \text{ even;} \\ 2ML^l + lL^{l-1}, & l \text{ odd.} \end{cases} \]

Note also \( A^{\ast}_{ml} = -A_{ml} \), hence there are constants \( c_{q,r}^{ml,m'l'} \) such that

\[ [A_{ml}, A_{m'l'}] = \sum_{q,r} c_{q,r}^{ml,m'l'} A_{qr}. \] (2.13)

In other words, all operators \( A_{ml} \) generate a centreless \( w_{C_{\infty}} \)-algebra. As a matter of fact, only those \( A_{ml} \) with odd indices \( l \) are linearly independent, and the above structure constants are uniquely determined by letting \( c_{q,r}^{ml,m'l'} = 0 \) for even \( r \). For example, one has

\[ c_{0,0}^{0l,0l'} = c_{0l,0l'}^{0l,l+1} = 0, \quad c_{0,2i+1,1,2j+1}^{0l,2i+1,1,2j+1} = 2(2i + 1)\delta_{q,0}\delta_{r,2(i+j)+1}, \]

\[ c_{1,2i+1,1,2j+1}^{1l,2i+1,1,2j+1} = 4(i-j)\delta_{q,1}\delta_{r,2(i+j)+1}. \]

The following equations are well defined:

\[ \frac{\partial}{\partial \Phi} = -(A_{ml}) \Phi, \quad m \geq 0, \ l \in \mathbb{Z}. \] (2.14)

These flows are assumed to commute with \( \partial/\partial x \).

**Proposition 2.2 ([9]).** The flows (2.14) commute with those in (2.7) that compose the CKP hierarchy. Moreover, the vector fields \( \partial/\partial s_{ml} \) acting on the dressing operators \( \Phi \) (or on the wave function \( w(t; z) \)) satisfy

\[ \left[ \frac{\partial}{\partial \Phi}, \frac{\partial}{\partial s_{ml}} \right] = -\sum_{q,r} c_{q,r}^{ml,m'l'} \frac{\partial}{\partial s_{qr}}. \] (2.15)

This proposition means that, equations (2.14) define a set of symmetries, named as additional symmetries, for the CKP hierarchy. These additional symmetries acting on the wave function form a centreless \( w_{C_{\infty}} \)-algebra.

Introduce a generating function of operators as

\[ Y(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1}(A_{m,m+l}) \] (2.16)

with parameters \( \lambda \) and \( \mu \). He et al [9] obtained the following (see the case of the KP hierarchy studied in [18])

**Proposition 2.3.** The generator (2.16) can be represented as

\[ Y(\lambda, \mu) = w(t; -\lambda)D^{-1}w(t; \mu) + w(t; \mu)D^{-1}w(t; -\lambda). \] (2.17)

### 3. Tau function of the CKP hierarchy

We aim to introduce a tau function of the CKP hierarchy. To this end let us first rewrite the hierarchy (2.4) into the form of Hamiltonian systems.

Given an arbitrary positive integer \( n \), the operator \( L \) in (2.3) satisfies \( (L^{2n})^{\ast} = L^{2n} \). Assume \( F \) to be a formal functional depending on \( L \):

\[ F = \int f(v, \partial_x v, \partial_x^2 v, \ldots) \, dx \in A/\partial_x A. \] (3.1)
Its variational derivative with respect to $L_{2n}$ is defined to be a pseudo-differential operator $P$ such that

$$\delta F = \int \text{res}(P \delta L_{2n}) \, dx, \quad P^* = -P.$$ 

Let $F_P$ denote the functional whose variational derivative with respect to $L_{2n}$ is $P \in \mathcal{D}$. For such functionals, there is a pair of compatible Poisson brackets that are reduced from those in the bi-Hamiltonian representations of the KP hierarchy (see [5] and references therein):

$$\{F_P, F_Q\}_1^n = \int \text{res}(P([N \mathcal{Q}_{2n} + [Q, L_{2n}^2]]) \, dx, \quad (3.2)$$

$$\{F_P, F_Q\}_2^n = \int \text{res}(P(-(L_{2n}^2 Q)_{2n} + L_{2n}^2 (Q L_{2n}^2))) \, dx. \quad (3.3)$$

Then the CKP hierarchy (2.4) can be represented in a bi-Hamiltonian recursive form as

$$\frac{\partial F}{\partial t_k} = \{F, H_k\}_{2n} = \{F, H_k\}_{2}, \quad k \in \mathbb{Z}_{\text{odd}}, \quad (3.4)$$

where $F$ is an arbitrary functional of the form (3.1), and the Hamiltonians are

$$H_k = \frac{2n}{k} \int \text{res} L^k \, dx. \quad (3.5)$$

The Hamiltonian densities in (3.5) are tau-symmetric [8]. That is to say, they define a closed 1-form

$$\omega = \sum_{k \in \mathbb{Z}_{\text{odd}^+}} \text{res} L^k \, dt_k,$$

hence there locally is a smooth function $\tau(t)$ such that

$$d\left(2 \partial_x \log \tau\right) = \omega. \quad (3.6)$$

More precisely, we have

**Definition 3.1.** Given any solution of the CKP hierarchy (2.4), by the tau function we mean a smooth function $\tau(t)$ that satisfies

$$\frac{\partial^2 \log \tau}{\partial t_k \partial t_l} = \frac{1}{2} \partial_x^{-1} \text{res} [(L^k)^*, L^l], \quad k, l \in \mathbb{Z}_{\text{odd}^+}. \quad (3.7)$$

Here on the right-hand side of (3.7) the integration constants are taken to be zero (the residue of any commutator of pseudo-differential operators is a total derivative in $x$).

Observe that $\log \tau$ is determined up to addition of a linear function of the time variables.

In order to relate the tau function to the wave function of the CKP hierarchy, we introduce the following shift operator

$$G(t; z) = \exp \left( - \sum_{k \in \mathbb{Z}_{\text{odd}^+}} \frac{2}{k} z^k \frac{\partial}{\partial t_k} \right).$$

We also write $G(z) = G(t; z)$ to avoid any confusion.

**Proposition 3.2.** For the CKP hierarchy, the tau function in (3.7) and the wave function (2.8) are related via the following formula

$$w(t; z) = \left(1 + \frac{1}{z} \partial_z \log G(z) \frac{\tau(t)}{\tau(t)} \right)^{1/2} \frac{G(z) \tau(t)}{\tau(t)} e^{\xi(t; z)}. \quad (3.8)$$

To prove this proposition, we need the following two lemmas.
Lemma 3.3. Recall the dressing operator (2.6) for the CKP hierarchy. It holds that

\[ a_1(t) = -2 \partial_i \log \tau (t). \]

(3.9)

**Proof.** One takes the residue of equation (2.7), then the lemma follows from the definition of the tau function in (3.6).

□

Lemma 3.4. Let \( \psi (t; \lambda) = \phi (t; \lambda)G(\lambda)\phi (t; -\lambda) \) (recall \( \phi \) in (2.9)) with \( \lambda \) being a parameter. Then it satisfies

(i) \( \psi (t; \lambda) = 1 + \frac{1}{2\lambda} (1 - G(\lambda))a_1(t); \)

(ii) \( 2 \partial_i \log \phi (t; \lambda) - \partial_i \log \psi (t; \lambda) = (1 - G(\lambda))a_1(t). \)

(3.10)

(3.11)

**Proof.** According to the bilinear equation (2.11), we have

\[
0 = \text{res}_{i} \phi (t; z)e^{z(t;z)}G(\lambda)\left( \phi (t; -z)e^{-z(t;z)} \right)
\]

\[
= \text{res}_{i} \phi (t; z) G(\lambda) \phi (t; -z) \frac{1+z/\lambda}{1-z/\lambda} \bigg|_{z=\lambda}
\]

\[
= \lambda \left( \phi (t; z)G(\lambda)\phi (t; -z) \left( 1 + \frac{z}{\lambda} \right) \right) \bigg|_{z=\lambda}
\]

\[
= 2\lambda \phi (t; \lambda)G(\lambda)\phi (t; -\lambda) - 2\lambda(1 - G(\lambda))a_1(t),
\]

(3.12)

in the third equality of which the subscript ‘-’ means to take the negative-power part of a series in \( z \). Thus the first formula (3.10) is valid.

Secondly, the bilinear equation (2.11) also yields

\[
0 = \text{res}_{i} \left( \partial_i (\phi (t; z)e^{z(t;z)}) \right) G(\lambda) \left( \phi (t; -z)e^{-z(t;z)} \right)
\]

\[
= \text{res}_{i} \left( z\phi (t; z) + \partial_{z,i} \phi (t; z) \right) G(\lambda) \phi (t; -z) \frac{1+z/\lambda}{1-z/\lambda} \bigg|_{z=\lambda}
\]

\[
= \lambda \left( \left( z\phi (t; z) + \partial_{z,i} \phi (t; z) \right) G(\lambda) \phi (t; -z) \left( 1 + \frac{z}{\lambda} \right) \right) \bigg|_{z=\lambda}
\]

\[
= 2\lambda^2 \psi (t; \lambda) + 2\lambda \partial_i \phi (t; \lambda) \cdot G(\lambda) \phi (t; -\lambda) - 2\lambda^2
\]

\[- 2\lambda(1 - G(\lambda))a_1(t) - (1 + G(\lambda))\partial_i a_1(t) - \partial_i a_1(t).
\]

(3.13)

On the other hand, it follows from (2.6) that \( 2a_2(t) = a_1(t)^2 - \partial_i a_1(t) \), hence

\[
2(1 + G(\lambda))a_2(t) = ((1 - G(\lambda))a_1(t))^2 + 2a_1(t)G(\lambda)a_1(t) - (1 + G(\lambda))\partial_i a_1(t).
\]

(3.14)

Substituting (3.10) and (3.14) into (3.13), one deduces

\[
2\lambda \partial_i \phi (t; \lambda) \cdot G(\lambda) \phi (t; -\lambda)
\]

\[
= \lambda (1 - G(\lambda))a_1(t) + \frac{1}{2} ((1 - G(\lambda))a_1(t))^2 + \frac{1}{2}(1 - G(\lambda))\partial_i a_1(t)
\]

\[
= \lambda \psi (t; \lambda)(1 - G(\lambda))a_1(t) + \lambda \partial_i \phi (t; \lambda).
\]

(3.15)

Divide both sides by \( \lambda \psi (t; \lambda) \), then we obtain (3.11). The lemma is proved.

□
Proof of proposition 3.2. Substituting (3.9) into (3.10) and (3.11), we have respectively

\[
\varphi(t; \lambda) = 1 + \frac{1}{\lambda} \left( (G(\lambda) - 1) \partial_x \log \tau(t) = 1 + \frac{1}{\lambda} \partial_x \log \frac{G(\lambda) \tau(t)}{\tau(t)} \right),
\]

(3.16)

\[
\phi(t; \lambda) = \sqrt{\varphi(t; \lambda)} \frac{G(\lambda) \tau(t)}{\tau(t)}.
\]

(3.17)

These, with \( \lambda \) replaced by \( z \), lead to the equality (3.8) by virtue of the definition of the wave function (2.8). Therefore the proposition is proved. □

Remark 3.5. In [16], van de Leur et al found a formula that represents the wave function of the CKP hierarchy to some fermionic tau functions. This formula reads (see equations (2.25), (2.35) and (2.37) in [16])

\[
w(t; z) = G(-z) \left( \tau(t) + \sum_{i \geq 2} \tau_{(2i-1,1)}(t) z \right) \tau_0(t) e^{\xi(t; z)}.
\]

(3.18)

On the other hand, our formula (3.8) can be rewritten as

\[
w(t; z) = \frac{G(z) \left( \tau(t) \sqrt{1 + z^{-1}(1 - G(-z)) \partial_x \log \tau(t)} \right)}{\tau(t)} e^{\xi(t; z)}.
\]

(3.19)

Note that \( (1 - G(-z)) \partial_x \log \tau(t) = O(1/z) \). By comparing the coefficients of \( w(t; z) e^{-\xi(t; z)} \) given in (3.18) and (3.19), we obtain

\[
\partial_x \log \tau_0(t) = -\partial_x \log \tau(t),
\]

(3.20)

hence \( \tau_0(t) \tau(t) = \text{const} \), accordingly, \( \tau_{(2i-1,1)}(t) \) with \( i \geq 2 \) can be recursively represented by \( \tau(t) \) and its derivatives. Thus the construction in [16] (see also [2]) in fact provides a fermionic interpretation for the tau function \( \tau(t) \) defined via Hamiltonian densities. □

Denote

\[
X(t; z) = e^{\xi(t; z)} G(t; z).
\]

(3.21)

Now we achieve the main result of the present section by substituting (3.8) into (2.11).

Theorem 3.6. The CKP hierarchy (2.4) is equivalent to the following 'bilinear' equation of the tau function:

\[
\text{res}_z \sqrt{\varphi(t; z) \varphi(t'; -z)} X(t; z) \tau(t) X(t'; z) \tau(t') = 0,
\]

(3.22)

where the function \( \varphi(t; z) \) is given in (3.16).

Observe the difference between (3.22) and those Hirota bilinear equations of the usual sense (for example, the KP and the BKP hierarchies [3]) in the literature: now there is a square-root factor given by the tau function!

At the end of this section, we note that \( \varphi(t; z) = 1 + O(1/z^2) \) as \( z \to \infty \), and that it satisfies

\[
G(-z) \varphi(t; z) = \varphi(t; -z).
\]

(3.23)

These properties will be employed in the forthcoming section.

4. Additional symmetries represented via the tau function

In this section we want to represent the additional symmetries (2.14) for the CKP hierarchy via the tau function introduced above. Our main tool is served by vertex operators.
For any positive odd integer \(k\), denote
\[ p_k = 2 \frac{\partial}{\partial t_k}, \quad p_{-k} = k t_k. \]
Clearly \([p_k, p_l] = 2 k \delta_{k,-l}\) with \(k, l \in \mathbb{Z}^{\text{odd}}\). Introduce a vertex operator
\[ X(t; \lambda, \mu) = \exp \left( \sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{p_k}{k} \lambda^k = \sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{p_k}{k} \mu^k \right) : \]
where the normal-order product ‘:’ means to place \(p_{k>0}\) to the right of \(p_{k<0}\), and \(\lambda\) and \(\mu\) are parameters. Without any confusion we will simply write \(X(\lambda, \mu) = X(t; \lambda, \mu)\).

**Lemma 4.1.** The tau function in (3.7) of the CKP hierarchy satisfies the following equality
\[ \frac{\partial}{\partial t} \frac{X(\lambda, \mu) \tau(t)}{\tau(t)} = \frac{\mu - \lambda}{\mu + \lambda} w(t; \mu) w(t; -\lambda) \left( \lambda G(\mu) \sqrt{\psi(t; -\lambda)} \phi(t; -\mu) + \mu G(-\lambda) \sqrt{\phi(t; \lambda)} \psi(t; \mu) \right), \]
where the functions \(w(t; \mu)\) and \(\psi(t; \lambda)\) are given by (3.8) and (3.16) respectively.

**Proof.** We write
\[ X(\lambda, \mu) = X(t; \lambda, \mu) = e^{-\xi(t; \lambda)+\xi(t; \mu)} G(-\lambda) G(\mu). \]
Recalling (3.21), it is straightforward to verify
\[ \begin{align*}
X(t; z) X(t; \lambda, \mu) &= \frac{1 - \mu/z}{1 + \mu/z} \frac{1 + \lambda/z}{1 - \lambda/z} e^{\xi(t; z)+\xi(t; \lambda)+\xi(t; \mu)} G(z) G(-\lambda) G(\mu), \\
X(t; \lambda, \mu) X(t; z) &= \frac{1 - z/\mu}{1 + z/\mu} \frac{1 + z/\lambda}{1 - z/\lambda} e^{\xi(t; z)+\xi(t; \lambda)+\xi(t; \mu)} G(z) G(-\lambda) G(\mu).
\end{align*} \]
The bilinear equation (3.22) yields
\[ \begin{align*}
0 &= \text{res}_{z} X(t; \lambda, \mu) \sqrt{\phi(t; z) \phi(t'; z)} X(t; z) \tau(t) X(t'; -z) \tau(t') \\
&= \text{res}_{z} \sqrt{G(-\lambda) G(\mu) \phi(t; z) \cdot \phi(t'; z)} X(t; \lambda, \mu) X(t; z) \tau(t) X(t'; -z) \tau(t') \\
&= \text{res}_{z} \sqrt{G(-\lambda) G(\mu) \phi(t; z) \cdot \phi(t'; z)} X(t; z) X(t; \lambda, \mu) \tau(t) X(t'; -z) \tau(t') \\
&\quad - \text{res}_{z} (a(z, \lambda, \mu) e^{\xi(t; z)+\xi(t; \lambda)+\xi(t; \mu)}) \sqrt{G(-\lambda) G(\mu) \phi(t; z) \cdot \phi(t'; -z)} \\
&\quad \times X(t; z) G(-\lambda) G(\mu) \tau(t) X(t'; -z) \tau(t')
\end{align*} \]
(4.3)
with
\[ a(z, \lambda, \mu) = \frac{1 - \mu/z}{1 + \mu/z} \frac{1 + \lambda/z}{1 - \lambda/z} \frac{1 - z/\mu}{1 + z/\mu} \frac{1 + z/\lambda}{1 - z/\lambda}. \]
The function \(a(z, \lambda, \mu)\) can be rewritten as
\[ a(z, \lambda, \mu) = \frac{1 + \lambda/z}{1 + \mu/z} (z - \mu) \delta(z, \lambda) + \frac{\mu}{\lambda} \frac{1 - z/\mu}{1 - z/\lambda}, \]
where \(\delta(z, \lambda) = (z(1 - \lambda/z))^{-1} + (\lambda(1 - z/\lambda))^{-1}\) is the Dirac delta function such that \(\text{res}_{z} f(z) \delta(z, \lambda) = f(\lambda)\) for any Laurent series \(f(z)\), see [5]. Thus by taking \(t' = t\) in (4.3) and using (3.23) we have
\[ \begin{align*}
\text{res}_{z} \sqrt{G(-\lambda) G(\mu) \phi(t; z) \cdot \phi(t; -z)} &= \frac{2(\lambda - \mu)}{1 + \mu/\lambda} e^{-\xi(t; \lambda)+\xi(t; \mu)} \sqrt{G(-\lambda) G(\mu) \phi(t; \lambda) \cdot \phi(t; -\lambda)} \\
&\quad \times \frac{G(\mu) \tau(t) G(-\lambda) \tau(t)}{\tau(t)} \\
&= \frac{2(\lambda - \mu)}{1 + \mu/\lambda} e^{-\xi(t; \lambda)+\xi(t; \mu)} \sqrt{G(-\lambda) G(\mu) \phi(t; \lambda) \cdot \phi(t; -\lambda)} \\
&\quad \times \frac{G(\mu) \tau(t) G(-\lambda) \tau(t)}{\tau(t)}
\end{align*} \]
\[
\begin{align*}
+ \frac{2\mu(\lambda - \mu)}{\lambda + \mu} w(t; \mu) w(t; -\lambda) \\
+ 2\mu(\lambda - \mu) \sqrt{G(-\lambda)} \sqrt{\phi(t; -\lambda)} \phi(t; -\mu) \\
- 2\mu(\lambda - \mu) \sqrt{G(\mu)} \sqrt{\phi(t; \mu)} \phi(t; -\mu) \\
= \frac{2\lambda(\lambda - \mu)}{\lambda + \mu} \sqrt{G(\mu)} \sqrt{\phi(t; -\lambda)} \phi(t; -\mu) \\
+ 2\lambda(\lambda - \mu) \sqrt{G(-\lambda)} \sqrt{\phi(t; \mu)} \phi(t; -\mu).
\end{align*}
\] (4.5)

Recall \(\phi(t; z) = 1 + O(1/z^2)\), hence the left-hand side of this equation is
\[
\text{lhs} = \text{res}_{z} \frac{G(z) X(\lambda, \mu) \tau(t)}{\tau(t)} \\
= -2\partial_x (X(\lambda, \mu) \tau(t)) + X(\lambda, \mu) \tau(t) \partial_x \tau(t).
\] (4.6)

Substitute it into (4.5), then we obtain (4.2). The lemma is proved. \(\square\)

One expands the vertex operator (4.1) formally as
\[
X(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} W_{l}^{(m)}
\] (4.7)

with
\[
W_{l}^{(m)} = \text{res}_{\lambda} (\lambda^{m+l} - 1) \partial_{\mu} X(\lambda, \mu).
\]

For instance, it is straightforward to calculate
\[
W_{l}^{(0)} = \delta_{l0}, \quad W_{l}^{(1)} = p_l, \quad W_{l}^{(2)} = \sum_{i+j=l} p_i p_j : -(l+1) p_i \\
W_{l}^{(3)} = \sum_{i+j+k=l} p_i p_j p_k : -\frac{3}{2} (l+2) \sum_{i+j=l} p_i p_j : +(l+2)(l+1) p_i.
\]

Here for convenience we assume \(p_i = 0\) for even \(i\).

**Proposition 4.2.** For the CKP hierarchy, the additional symmetries (2.14) acting on the tau function are given by the following formula
\[
\sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l-1} \partial_x \left( \frac{1}{\tau(t)} \partial \tau(t) \right) = w(t; \mu) w(t; -\lambda) \left( 1 - \frac{\lambda}{\mu + \lambda} G(\mu) \frac{\phi(t; -\lambda)}{\phi(t; -\mu)} + \frac{\mu}{\mu + \lambda} G(-\lambda) \frac{\phi(t; \mu)}{\phi(t; \lambda)} \right).
\] (4.8)

**Proof.** Let
\[
Z(\lambda, \mu) = \frac{1}{\mu - \lambda} (X(\lambda, \mu) - 1) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-m-l} W_{l}^{(m+1)}.
\] (4.9)
Lemma 4.1 implies
\[ \frac{\partial}{\partial x} Z(\lambda, \mu) \tau(t) = w(t; \mu) \left( \frac{\lambda}{\mu + \lambda} G(\mu) \sqrt{\frac{\psi(t; -\lambda)}{\psi(t; -\mu)}} + \frac{\mu}{\mu + \lambda} G(-\lambda) \sqrt{\frac{\psi(t; \mu)}{\psi(t; \lambda)}} \right). \]  
(4.10)

Hence to show (4.8) we only need to verify
\[
\sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \lambda^{m - l - 1} \frac{\partial}{\partial x} \frac{\partial}{\partial \log \tau(t)} \frac{\partial}{\partial s_{m,m+l}} \left( \frac{\partial}{\partial s_{m,m+l}} \right) = w(t; \mu) w(t; -\lambda). 
\]  
(4.11)

In fact, according to (2.14) and (3.9) one has
\[
\text{res} A_{m,m+l} = - \frac{\partial a_1}{\partial s_{m,m+l}} = 2 \frac{\partial}{\partial x} \frac{\partial}{\partial \log \tau(t)} \frac{\partial}{\partial s_{m,m+l}}. 
\]

Thus the equality (4.11) is deduced by taking the residue of the operator (2.17). Therefore the proposition is proved. \( \blacksquare \)

Corollary 4.3. The additional symmetries (2.14) with \( m = 0 \) and 1 can be represented as follows
(i) for \( l \in \mathbb{Z} \),
\[ \frac{\partial}{\partial s_{l,2i}} \tau = \left( W^{(1)}_l + \delta_{l0} c^{(1)} \right) \tau; \]  
(4.12)

(ii) for \( i \in \mathbb{Z} \),
\[ \frac{\partial}{\partial s_{l,2i+1}} \tau = \frac{1}{2} W^{(2)}_{2i} \tau; \]  
(4.13)
\[ \frac{\partial}{\partial s_{l,2i+1}} \left( \frac{1}{2} W^{(2)}_{2i} + \delta_{l0} c^{(2)} \right) \tau = \begin{cases} 0, & i \leq 0; \\ \tau \partial_x^{-1} T_{2i+1}, & i \geq 1, \end{cases} \]  
(4.14)
in which
\[ T_{2i+1} = -\frac{1}{2} \text{res}_{\lambda^{2i+1}} G(\lambda) \tau G(-\lambda) \tau \frac{\psi(t; -\lambda)}{\psi(t; \lambda)} \frac{\psi(t; \lambda)}{\psi(t; -\lambda)} \left( N(\lambda) - \partial_x \right) \psi(t; \lambda) \]  
(4.15)

in which
\[ N(\lambda) = \sum_{k \in \mathbb{Z}^{\text{odd}}} \frac{2}{\lambda^{2k+1}} \frac{\partial}{\partial t_k}. \]

Here \( c^{(1)} \) and \( c^{(2)} \) are certain constants that arise from a central extension of the \( w_{\mathbb{C}} \)-algebra.

**Proof.** Clearly the right-hand side of (4.8) vanishes whenever \( \mu = \lambda \), which implies the first assertion. Let us proceed to show the second one.

It is easy to see
\[ [\partial_\tau, G(\lambda)] = N(\lambda) G(\lambda), \quad [\partial_\tau, G(-\lambda)] = -N(\lambda) G(-\lambda). \]

Denote
\[ \chi(t; \lambda, \mu) = 1 - \frac{\lambda}{\mu + \lambda} G(\mu) \sqrt{\frac{\psi(t; -\lambda)}{\psi(t; -\mu)}} - \frac{\mu}{\mu + \lambda} G(-\lambda) \sqrt{\frac{\psi(t; \mu)}{\psi(t; \lambda)}}. \]  
(4.16)
One has $\chi(t; \lambda, \lambda) = 0$ and
\[ \partial_\mu \bigg|_{\mu = \lambda} \chi(t; \lambda, \mu) = \frac{1}{4} G(\lambda) \partial_\lambda \log \varphi(t; -\lambda) - \frac{1}{4} G(-\lambda) \partial_\lambda \log \varphi(t; \lambda) \]
\[ = \frac{1}{4} \left( -N(\lambda) \log \varphi(t; \lambda) - \partial_\lambda \right) \log \varphi(t; -\lambda) \]
\[ = -\frac{1}{4} \left( N(\lambda) \log \varphi(t; \lambda) \right) \log \varphi(t; -\lambda). \] (4.17)

Hence
\[ \partial_x \left( \frac{1}{\tau(t)} \partial \tau(t) - \frac{1}{2} \frac{W_{\lambda+1}(t)}{\tau(t)} \right) \]
\[ = \text{res}_{\lambda^{l+1}} \partial_\mu \bigg|_{\mu = \lambda} \left( w(t; \mu) w(t; -\lambda) \chi(t; \lambda, \mu) \right) \]
\[ = -\frac{1}{4} \text{res}_{\lambda^{l+1}} w(t; \lambda) w(t; -\lambda) \]
\[ \times \left( N(\lambda) \log \varphi(t; \lambda) \varphi(t; -\lambda) - \partial_\lambda \log \frac{\varphi(t; \lambda)}{\varphi(t; -\lambda)} \right). \] (4.18)

Since (4.17) is an even function in $\lambda$, then (4.18) vanishes whenever $l$ is odd. Namely, the equality (4.13) is verified. Recall $\varphi(t; \lambda) = 1 + O(1/\lambda^2)$, then (4.18) vanishes when $l \leq 0$. If $l = 2i$ is even, we simplify the right-hand side of (4.18) to
\[ -\frac{1}{4} \text{res}_{\lambda^{2i+1}} w(t; \lambda) w(t; -\lambda) \left( N(\lambda) \log \varphi(t; \lambda) - \partial_\lambda \log \varphi(t; \lambda) \right), \]
and substitute into it with
\[ w(t; \lambda) = \sqrt{\varphi(t; \lambda)} \frac{G(\lambda) \tau(t)}{\tau(t)} e^{\xi(t; z)} \]
(see (2.8) and (3.17)), thus we recast it to $T_{2i+1}$ and verify (4.14). The corollary is proved. \(
\) \(\Box\)

**Example 4.4.** Let us illustrate how to compute $T_{2i+1}$ in the tails of (4.14). For this purpose, we expand
\[ G(\lambda) = 1 + \sum_{j \geq 1} \sigma_j(-\tilde{\partial}) \frac{1}{\lambda^j}, \quad \tilde{\partial} = \left( \frac{2}{3} \partial_{t_1}, 0, \frac{2}{3} \partial_{t_3}, 0, \ldots \right), \]
where $\sigma_j$ are elementary Schur polynomials defined by
\[ \exp \left( \sum_{j \geq 1} q_j z^j \right) = \sum_{j \geq 0} \sigma_j(q_1, q_2, \ldots) z^j. \]

Denote $\theta = \log \tau$, and $\theta_i = \partial_\theta \theta_i = \partial_\theta \partial_\theta \theta_i$ etc. (recall $x = t_1$). One has
\[ \varphi(t; \lambda) = 1 + \frac{1}{\lambda} (G(\lambda) - 1) \theta_1 = 1 + \sum_{j \geq 1} \frac{1}{\lambda^{j+1}} \sigma_j(-\tilde{\partial}) \theta_i = 1 + O(1/\lambda^2), \] (4.19)
\[ (N(\lambda) - \partial_\lambda) \varphi(t; \lambda) = \frac{1}{\lambda^2} (G(\lambda) - 1) \theta_1 - \frac{1}{\lambda} N(\lambda) \theta_1 \]
\[ = \sum_{j \geq 1} \left( \frac{1}{\lambda^{2j+1}} \left( \sigma_{2j-1}(-\tilde{\partial}) - 2 \frac{\partial}{\partial t_{2j-1}} \right) \theta_i + \frac{1}{\lambda^{2j+2}} \sigma_{2j}(-\tilde{\partial}) \theta_i \right) = O(1/\lambda^3), \] (4.20)
\[
\frac{G(\lambda) \tau}{\tau} = \frac{G(-\lambda) \tau}{\tau} = \exp \left( \frac{1}{\lambda} \sigma_2(-\partial) \theta \right) = \exp \left( 2 \sum_{i \geq 1} \frac{1}{\lambda^2} \sigma_2(-\partial) \theta \right) = 1 + O(1/\lambda^2). \tag{4.21}
\]

Substitute them into (4.15), then it is straightforward to obtain
\[
T_3 = -\theta_{xxx}, \quad T_5 = -\partial_x \left( \frac{2}{3} \theta_{x}\theta + \frac{1}{3} \theta_{xxxx} + 4 \theta_x^2 \right). \tag{4.22}
\]

Generally, for \( i \geq 1 \),
\[
T_{2i+1} = -\frac{1}{2} \sigma_2(-\partial) \theta_x + \text{nonlinear terms in derivatives of } \theta. \tag{4.23}
\]

Note that the nonlinear terms are trivial whenever \( i = 1 \).

**Lemma 4.5.** For every \( i \geq 1 \), it holds that \( T_{2i+1} = \partial_x \tilde{T}_{2i} \) for some polynomial
\[
\tilde{T}_{2i} \in \mathbb{C} \left[ \frac{\partial^{m_1 + \ldots + m_s}}{\partial t_1^{k_1} \cdots \partial t_s^{k_s}} \theta; \ m_1 + \ldots + m_s \geq 2 \right]. \tag{4.24}
\]

Moreover, each polynomial \( \tilde{T}_{2i} \) is homogeneous of degree \( 2i \) if we assign
\[
\deg \left( \frac{\partial^{m_1 + \ldots + m_s}}{\partial t_1^{k_1} \cdots \partial t_s^{k_s}} \theta \right) = m_1 k_1 + \ldots + m_s k_s.
\]

The validity of this lemma will be verified below, though not in so direct a way.

With the same method as in corollary 4.3, from (4.8) one can calculate \( \partial \tau / \partial s_{m_1, m_2} \) for \( m \geq 2 \). However, the formulae turn out complicated.

Now we conclude that, for the CKP hierarchy, when lifting the actions of additional symmetries on the wave function to the actions on the tau function, one has not only a central extension of the \( w_{C} \) algebra but also certain non-trivial ‘tails’ such as that given by \( T_{2i+1} \). This is the main difference between the additional symmetries for the CKP hierarchy and those as for the KP hierarchy considered before [1, 4, 15, 20, 23]. Why the CKP hierarchy is so special is not yet clear.

### 5. Virasoro symmetries for Drinfeld–Sokolov hierarchies of type C

In the celebrated work [6], Drinfeld and Sokolov associated an integrable hierarchy of KdV type to each affine Kac–Moody algebra \( \hat{\mathfrak{g}} \). Instead of getting into their general construction, in this section we only consider the case that \( \mathfrak{g} \) is of type \( C_n^{(1)} \). More exactly, we want to apply the above results for the CKP hierarchy to study the Virasoro symmetries for the Drinfeld–Sokolov hierarchy of type \( C_n^{(1)} \). Our inspiration is from the reduction relation of the Lie algebras \( C_n \to C_n^{(1)} \) underlying these hierarchies [11, 12].

#### 5.1. Drinfeld–Sokolov hierarchies of type C

For any integer \( n \geq 2 \) fixed, the simple Lie algebra \( \hat{\mathfrak{g}} \) of type \( C_n \) can be realized as
\[
\hat{\mathfrak{g}} = \mathfrak{sp}(2n) \times \mathfrak{c} \times \mathfrak{c},
\]
where \( S = \text{diag}(1, -1, 1, -1, \ldots, 1, -1) \) and the superscript ‘T’ means the transpose relative to the secondary diagonal [6]. Then one realizes the Kac–Moody algebra \( \mathfrak{g} \) of affine type \( \hat{\mathfrak{g}} \) as
\[
\mathfrak{g} = \hat{\mathfrak{g}}[\lambda, 1/\lambda] \odot \mathbb{C} \mathfrak{c} \odot \mathbb{C} \mathfrak{d}
\]
with \( c \) being the canonical central element and \( d \) the scaling element. In more detail, let \( e_{i,j} \) be the \( 2n \times 2n \) matrix whose \((i, j)\)-entry takes value 1 and any other entry vanishes, then a set of Weyl generators of \( \mathfrak{g} \) can be chosen as follows \([6, 12]\):

\[
e_{i,j} = e_{i,j} + e_{2n-i+1, 2n-j}, \quad (1 \leq i \leq n-1),
\]

\[
e_n = e_{n+1,n}, \quad e_0 = \lambda e_{1,2n},
\]

\[
f_{i,j} = e_{i,j} + e_{2n-i-1, 2n-j+1}, \quad (1 \leq i \leq n-1),
\]

\[
f_n = e_{n+1,n}, \quad f_0 = \lambda^{-1} e_{2n,1},
\]

\[
\alpha^\lor_i = [e_i, f_i]^\lor, \quad (1 \leq i \leq n),
\]

\[
\alpha^\lor_0 = e_{1,1} - e_{2n,2n} + c.
\]

Denote \( \Lambda = \sum_{i=0}^n e_i \). The elements \( \Lambda^k \) with \( k \in \mathbb{Z}^{\text{odd}} \) generate the principal Heisenberg subalgebra of \( \mathfrak{g} \). Moreover, they satisfy

\[(\Lambda^k | \Lambda^l) = 2n \delta_{k,-l}, \quad k, l \in \mathbb{Z}^{\text{odd}}\]

for the standard invariant bilinear (Killing) form \((\cdot | \cdot)\) on \( \mathfrak{g} \). Note that \( 2n \) is the Coxeter number.

Introduce a matrix operator

\[\mathcal{L} = D + \Lambda + q\]

with \( D = d/dx \) and \( q \) being a smooth function of \( x \) that takes value in the Borel subalgebra of \( \hat{\mathfrak{g}} \) generated by \( \alpha^\lor_i \) and \( f_i \) with \( 1 \leq i \leq n \). The nilpotent subalgebra, say \( \mathfrak{n} \), generated by \( f_i \) with \( 1 \leq i \leq n \), induces a group of gauge transformations of \( \mathcal{L} \) as

\[
\mathcal{L} \mapsto e^{ad N} \mathcal{L}, \quad N \in \mathfrak{n}.
\]

The Drinfeld–Sokolov hierarchy associated to \( \mathfrak{g} \) is defined as

\[
\frac{\partial \mathcal{L}}{\partial t_k} = [\mathcal{L}^{(k/2n)}_+/\mathcal{L}], \quad k \in \mathbb{Z}^{\text{odd}}^{\text{odd}}
\]

modulo the gauge transformations (5.8). Here \( \mathcal{L}^{(k/2n)}_+ \), depending on \( \Lambda^k \), are certain \( \mathfrak{g} \)-valued differential polynomials in \( q \) such that the right-hand side of (5.9) takes value in the Borel subalgebra of \( \hat{\mathfrak{g}} \). See [6] (also [24]) for details.

For the equivalence class of \( \mathcal{L} \) with respect to the transformations (5.8), a representative element can be chosen as

\[\mathcal{L}^{\text{can}} = D + \Lambda + q^{\text{can}}, \quad q^{\text{can}} = -\sum_{i=1}^n \frac{u_i}{2} (e_{2i,1} + e_{2n-2i+1,2n})
\]

with scalar functions \( u_i \). According to the theory of [6], the canonical operator (5.10) induces a scalar differential operator

\[\mathcal{L} = D^{2n} + \frac{1}{2} \sum_{i=1}^n (u_i + r_i) D^{2n-2i} + D^{2n-2i} (u_i + r_i)
\]

where \( r_i = r_i (u_1, \ldots, u_{i-1}) \) are differential polynomials in their arguments and particularly \( r_1 = 0 \). Hence the hierarchy (5.9) is converted to the following system of Lax equations

\[
\frac{\partial \mathcal{L}}{\partial t_k} = [\mathcal{L}^{(k/2n)}_+/\mathcal{L}], \quad k \in \mathbb{Z}^{\text{odd}}^{\text{odd}}.
\]
The Drinfeld–Sokolov hierarchy (5.9) of type $C_n^{(1)}$ carries a bi-Hamiltonian structure [6]. In [24] a set of Hamiltonian densities were selected appropriately such that they define a tau function, say, $\tilde{\tau}$ (to be distinguished from the previous notation $\tau$ of the CKP hierarchy). With the same method as in [24] (see equation (5.13) there), we have

$$\partial^2 x \log \tilde{\tau} = \left(-\Lambda \mid q^{cn}\right) \frac{u_1}{2n}. \quad (5.13)$$

**Theorem 5.1 ([24]).** The Virasoro symmetries for the Drinfeld–Sokolov hierarchy (5.9), i.e., (5.12), can be represented as

$$\frac{\partial \tilde{\tau}}{\partial \beta_j} = V_j \tilde{\tau} + \tilde{\tau} O_j, \quad j = -1, 0, 1, 2, \ldots. \quad (5.14)$$

Here

$$V_j = \frac{1}{4n} \sum_{k \in \mathbb{Z}_{odd}} : \tilde{p}_k \tilde{p}_{2n-j-k} : + \delta_{j0} cn \quad (5.15)$$

with

$$\tilde{p}_k = \frac{\partial}{\partial t_k}, \quad \tilde{p}_{-k} = k t_k, \quad k \in \mathbb{Z}_{odd},$$

and $cn$ being a constant; the terms $O_j$ are differential polynomials in second-order derivatives of $\log \tilde{\tau}$ with respect to the time variables, and, in particular, $O_{-1} = O_0 = 0$.

We write the operators $V_j$ explicitly as

$$V_{-1} = \frac{1}{2n} \sum_{k \in \mathbb{Z}_{odd}} (k + 2n) t_k \frac{\partial}{\partial t_k} + \frac{1}{4n} \sum_{k+l=2n} k l t_k t_l. \quad (5.16)$$

$$V_0 = \frac{1}{2n} \sum_{k \in \mathbb{Z}_{odd}} k t_k \frac{\partial}{\partial t_k} + cn, \quad (5.17)$$

$$V_j = \frac{1}{4n} \sum_{k+l=2n} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2n} \sum_{k \in \mathbb{Z}_{odd}} k t_k \frac{\partial}{\partial t_{k+2n}}, \quad j \geq 1. \quad (5.18)$$

where all indices $k$ and $l$ lie in $\mathbb{Z}_{odd}^+$. Choose

$$cn = \frac{n}{24} \left(1 + \frac{1}{2n^2}\right), \quad (5.19)$$

then $V_j$ satisfy the Virasoro commutation relation (see, for example, [13])

$$[V_i, V_j] = (i - j)V_{i+j}, \quad i, j \geq -1. \quad (5.20)$$

The terms $O_j$ in (5.14) are called obstacles in linearizing Virasoro symmetries in [24]. The Virasoro symmetries are said to be linearized if all such $O_j$ vanish, which is a crucial property of an integrable hierarchy of topological type [8]. We remark that all Drinfeld–Sokolov hierarchies associated to ADE-type affine Kac–Moody algebras, either untwisted or twisted, possess linearized Virasoro symmetries [24], see also [8, 10, 23]. However, for the Drinfeld–Sokolov hierarchies of type C, it was unknown whether these obstacles $O_j$ vanish or not, since it is not easy to compute them starting from the original definition in [24].
Such obstacles will be calculated in an alternative fashion in the forthcoming subsection, in consideration of the fact that (5.12) is indeed a subhierarchy of the CKP hierarchy (2.4).

5.2. Nonlinearizable Virasoro symmetries

Given an integer \( n \geq 2 \), unless otherwise stated the pseudo-differential operator (2.3) is henceforth assumed to satisfy

\[
(L^{2n})_+ = 0. \tag{5.21}
\]

Under this constraint, the CKP hierarchy (2.4) is reduced to the hierarchy (5.12) with \( L = L^{2n} \), and the bilinear equation (2.11) becomes

\[
\text{res}_z x^{2n} w(t; z)w(t'; -z) = 0, \quad j \geq 0. \tag{5.22}
\]

Meanwhile, the Poisson brackets (3.2) and (3.3) admit the constraint (5.21), hence one rederives the bi-Hamiltonian structure for the Drinfeld–Sokolov hierarchy achieved in [6]. The Hamiltonians are also given by the formulae (3.5), thus the tau function \( \tau \) of the CKP hierarchy can be reduced to a tau function of the hierarchy (5.12).

**Proposition 5.2.** For the Drinfeld–Sokolov hierarchy (5.12) of type \( C^{(1)}_n \), the tau functions \( \tau \) reduced from that of the CKP hierarchy and \( \tilde{\tau} \) as recalled in the preceding subsection satisfy

\[
\tau^2 = \tilde{\tau} \quad \text{(up to a factor of the form } \exp(\sum c_k t_k) \text{ with constant } c_k). \tag{5.23}
\]

**Proof.** The proof is similar to that of propositions 5.2 and 5.4 in [24] for Drinfeld–Sokolov hierarchies of types A and D. According to (3.6), (5.11) and (5.13), we have

\[
\partial^2 \log \tau^2 = \text{res } L = \text{res } L^{1/2n} = \frac{u_1}{2n} = \partial^2 \log \tilde{\tau}.
\]

Hence

\[
\partial^2 \left( \frac{\partial^2 \log \tau^2}{\partial t_k \partial t_l} - \frac{\partial^2 \log \tilde{\tau}}{\partial t_k \partial t_l} \right) = 0, \quad k, l \in \mathbb{Z}_+^{\text{odd}}.
\]

Note that the terms in parentheses are differential polynomials in the coefficients of \( L \), namely, in \((u_1, u_2, \ldots, u_n)\), hence their difference vanishes indeed. It follows that \( \tau^2 = \tilde{\tau} \). The proposition is proved. \( \square \)

Part of the additional symmetries (2.14) for the CKP hierarchy are compatible with the constraint (5.21). In fact, for \( j \geq -1 \), one has

\[
\left( \frac{\partial L^{2n}}{\partial x_{1,2n+1}} \right)_- = \left[ -(A_{1,2n+1}), L^{2n} \right]_-= 4n(L^{2n})_{-} = 0.
\]

Denote \( s_j = 4ns_{1,2n+1} \), then

\[
\frac{\partial L}{\partial s_j} = \frac{1}{4n} \left[ -(A_{1,2n+1}), L \right], \quad j \geq -1 \tag{5.24}
\]

are symmetries for the reduced hierarchy (5.12). Moreover, proposition 2.2 implies that these symmetries satisfy the Virasoro commutation relation

\[
\left[ \frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j} \right] = (j - i) \frac{\partial}{\partial s_{i+j}}, \quad i, j \geq -1 \tag{5.25}
\]
when acting on $L$ or on the dressing operator $\Phi$ given as in (2.6). According to (4.14) we have
\[
\frac{\partial \tau}{\partial s_j} = L_j \tau + \frac{\tau}{4n} T_{2n+1}, \quad j \geq -1,
\]
where
\[
L_j = \frac{1}{8n} W^{(2)}_{2n+1} + \frac{c_n}{2}.
\]
Explicitly, one has
\[
L_{-1} = \frac{1}{2n} \sum_{k \in \mathbb{Z}^{2n+1}_{\text{odd}}} (k + 2n) t_k \frac{\partial}{\partial t_k} + \frac{1}{8n} \sum_{k+l=2n} k l t_k t_l,
\]
\[
L_0 = \frac{1}{2n} \sum_{k \in \mathbb{Z}^{2n+1}_{\text{odd}}} k t_k \frac{\partial}{\partial t_k} + c_n,
\]
\[
L_j = \frac{1}{2n} \sum_{k+l=2n} \frac{\partial^2}{\partial t_k \partial t_l} + \frac{1}{2n} \sum_{k \in \mathbb{Z}^{2n+1}_{\text{odd}}} k t_k \frac{\partial}{\partial t_{k+2n}}, \quad j \geq 1.
\]
Here the constant $c_n/2$ (see (5.19)) in $L_0$ is chosen for the validity of the following theorem.

**Theorem 5.3.** For the Drinfeld–Sokolov hierarchy (5.12) of type $C^{(1)}_n$, the Virasoro symmetries (5.26) and (5.14) coincide. More precisely, acting on the tau function $\tilde{\tau}$ it holds that
\[
\frac{\partial \tilde{\tau}}{\partial s_j} = \frac{\partial \tilde{\tau}}{\partial \beta_j}, \quad j \geq -1.
\]

**Proof.** For $j \geq -1$, we write $L = L^{(2)}_j + L^{(1)}_j + L^{(0)}_j$, where $L^{(v)}_j$ is the part of the $v$th order derivations in $L_j$. For instance,
\[
L^{(0)}_0 = \frac{c_n}{2}, \quad L^{(2)}_1 = \frac{1}{2n} \sum_{k+l=2n} \frac{\partial^2}{\partial t_k \partial t_l}.
\]
Similarly we write $V_j = V^{(2)}_j + V^{(1)}_j + V^{(0)}_j$ for $V_j$ given in (5.15). It is easy to see
\[
L^{(2)}_j = 2 V^{(2)}_j, \quad L^{(1)}_j = V^{(1)}_j, \quad L^{(0)}_j = \frac{1}{2} V^{(0)}_j.
\]
Since $\tilde{\tau} = \tau^2$, then
\[
V_j \tilde{\tau} = 4 \tau V^{(2)}_j + 2 \tau V^{(1)}_j + \tau V^{(0)}_j - 2 \tau^2 V^{(2)}_j \log \tau
= 2 \tau L_j \tau - \tilde{\tau} V^{(2)}_j \log \tilde{\tau}.
\]
Comparing (5.26) and (5.14), we have
\[
\partial_x \left( \frac{\partial \log \tilde{\tau}}{\partial s_j} - \frac{\partial \log \tilde{\tau}}{\partial \beta_j} \right) = \partial_x \left( \frac{2}{\tau} (L_j \tau + \frac{\tau}{4n} \partial^{-1} T_{2n+1}) - \frac{1}{\tilde{\tau}} (V_j \tilde{\tau} + \tilde{\tau} O_j) \right)
= \frac{1}{n} T_{2n+1} - \partial_x \left( O_j - V^{(2)}_j \log \tilde{\tau} \right).
\]
The left-hand side depends linearly on $\log \tilde{\tau}$, so does the right-hand side. Observe (4.23) and recall that $O_j$ are differential polynomials in second-order derivatives of $\log \tilde{\tau}$, then the right-hand side of (5.32) must be of the form $\partial_x R_j \log \tilde{\tau}$ for some linear operator $R_j \in \mathbb{C}[\partial/\partial t_1, \partial/\partial t_2, \ldots]$. Thus acting on $\log \tilde{\tau}$ one has
\[
\frac{\partial}{\partial s_j} = \frac{\partial}{\partial \beta_j} + R_j, \quad j \geq 0,
\]
where \( R_{-1} = R_0 = 0 \). In fact, all \( R_j \) must vanish by virtue of the Virasoro commutation relations for the symmetries \( \partial/\partial s_j \) and for \( \partial/\partial \beta_j \) respectively. Therefore

\[
\frac{\partial \log \tilde{\tau}}{\partial s_j} = \frac{\partial \log \tilde{\tau}}{\partial \beta_j}, \quad j \geq -1.
\] (5.33)

The theorem is proved. \( \square \)

From the proof we also know that each \( T_{2n+1} \) is a total derivative of some differential polynomial in the second-order derivatives of \( \log \tilde{\tau} \) with respect to the time variables. Hence we obtain an alternative representation for the obstacles that were introduced from the Kac–Moody–Virasoro algebra in [24].

**Corollary 5.4.** The obstacles \( O_j \) in (5.14) can be represented as

\[
O_j = \frac{1}{2n} \left( \sigma_{2n+1}^{-1} + \frac{1}{2} \sum_{k \neq 2n} \frac{\partial^2 \log \tilde{\tau}}{\partial s_k \partial t_l} \right), \quad j \geq 1,
\] (5.34)

where \( T_{2n+1} \) are given in (4.15) with \( \tau = \exp(\frac{1}{2} \log \tilde{\tau}) \).

**Corollary 5.5.** For the Drinfeld–Sokolov hierarchy of type \( C_n(1) \) with \( n \geq 2 \), the obstacles \( O_j \neq 0 \) when \( j \geq 1 \).

**Proof.** Substitute (4.23) into (5.34), then the part linear in \( \log \tilde{\tau} \) of \( O_j \) with \( j \geq 1 \) is

\[
O_j^{(1)} = \frac{1}{4n} \left( \sigma_{2n+1}^{-1} - \frac{1}{2} \sum_{k \neq 2n} \frac{\partial^2 \log \tilde{\tau}}{\partial s_k \partial t_l} \right) \log \tilde{\tau}
\]

\[
= \frac{1}{4n} \sum_{k \neq 2n} \left( 1 - \frac{1}{k} \right) \frac{\partial^2 \log \tilde{\tau}}{\partial s_k \partial t_l}
\]

\[
= \frac{1}{8n} \sum_{k_1, m_1, \ldots, k_r, m_r, m_r \geq 3; k_1 < \ldots < k_r} \left( \prod_{i=1}^{r} \frac{1}{m_i!} \left( \frac{2}{k_i} \frac{\partial}{\partial s_{k_i}} \right)^{m_i} \right) \log \tilde{\tau}.
\] (5.35)

In particular, taking \( j = 1 \) one derives \( O_1^{(1)} \neq 0 \) hence \( O_1 \neq 0 \). Here we adopt the fact that the flows \( \partial/\partial t_k \) with \( k = 1, 3, \ldots, 2n-1 \) in the hierarchy (5.12) are independent so the linear part \( O_1^{(1)} \) cannot be cancelled by the omitted nonlinear part (see remark 5.6).

Furthermore, provided \( O_j = 0 \) for some \( j > 1 \), it follows that \( O_{j-1} = 0 \) due to the commutation relation between \( \partial/\partial s_{j-1} \) and \( \partial/\partial s_j \). Then step by step one deduces \( O_1 = 0 \), which is a contradiction. Therefore the theorem is proved. \( \square \)

**Proof of theorem 1.1.** The theorem is a combination of corollaries 5.4 and 5.5. \( \square \)

**Remark 5.6.** The condition \( n \geq 2 \) in corollary 5.5 is essential. Otherwise, suppose \( n = 1 \), then the reduced hierarchy (5.12) with

\[
\mathcal{L} = D^2 + u
\]

is nothing but the KdV hierarchy, or equivalently, the Drinfeld–Sokolov hierarchy associated to the affine Kac–Moody algebra of type \( A_1(1) \). As is known, the Virasoro symmetries for the KdV hierarchy are linearizable (see, for example, [6, 21, 24]).
In fact, according to (5.34) and (4.22), one has
\[
O_1 = \frac{1}{2} \hat{\partial}_x^{-1} T_3 + \frac{1}{4} \hat{\partial}_x^2 \log \tilde{\tau} = -\frac{1}{4} \hat{\partial}_x^2 \log \tilde{\tau} + \frac{1}{4} \hat{\partial}_x^2 \log \tilde{\tau} = 0,
\]
(5.36)
\[
O_2 = \frac{1}{2} \hat{\partial}_x^{-1} T_5 + \frac{1}{2} \hat{\partial}_x^2 \log \tilde{\tau} = \frac{1}{3} \hat{\partial}_x \hat{\partial}_x \log \tilde{\tau} = \frac{1}{12} \hat{\partial}_x^4 \log \tilde{\tau} - \frac{1}{2} \left( \hat{\partial}_x^2 \log \tilde{\tau} \right)^2.
\]
(5.37)
But the function \( u = 2 \hat{\partial}_x^2 \log \tilde{\tau} \) satisfies the KdV equation
\[
\frac{\partial u}{\partial t_3} = \frac{1}{4} \hat{\partial}_x^3 u + \frac{3}{2} \frac{\partial u}{\partial x}.
\]
One rewrites this equation in terms of \( \log \tilde{\tau} \) then achieves \( O_2 = 0 \). Furthermore, the Virasoro commutation relation for the symmetries \( \partial/\partial s_j \) implies \( O_j = 0 \) for all \( j \geq 3 \). Thus the linearization of Virasoro symmetries for the KdV hierarchy is derived again, which agrees with the result in the literature.

Proof of lemma 4.5. The independence of the above flows \( \partial/\partial t_k \) with \( k = 1, 3, \ldots, 2n - 1 \) implies that the term \( T_2n+1 \) for the reduced hierarchy (5.12) has the same expression as that for the CKP hierarchy (2.4). Thus \( T_2n+1 \) in the latter case is also a total derivative of the differential polynomial in second-order derivatives of \( \log \tilde{\tau} = 2 \log \tau \) with respect to \( t_1, t_3, \ldots, t_{2n-1} \).

Since \( n \) can be arbitrarily chosen, then such a property is possessed by every \( T_{2i+1} \) with \( i \geq 1 \) for the CKP hierarchy. In other words, \( T_{2i+1} \) is a total derivative of the polynomial in at-least-second-order derivatives of \( \log \tau \) with respect to the time variables. The homogeneity of \( T_{2i+1} \) is easily observed from the definition (4.15). Therefore lemma 4.5 is proved.

6. Conclusion

We have defined a single tau function of the CKP hierarchy from its Hamiltonian densities. With the help of this tau function, the CKP hierarchy is represented into a generalized Hirota bilinear equation (3.22), the form of which is different from that of the KP or of the BKP hierarchy. Furthermore, we have shown that the actions on the tau function by additional symmetries involve strictly more than a central extension of the \( \mathfrak{wC}_n \)-algebra. It is interesting to develop similar skills to deal with the generalization [16] of the CKP hierarchy that contains both normal and super variables. An answer to it must enrich our knowledge of integrable hierarchies and their applications.

By reducing additional symmetries for the CKP hierarchy, the Virasoro symmetries for the Drinfeld–Sokolov hierarchy associated to affine Kac–Moody algebra of type \( \mathfrak{C}_n^{(1)} \) with \( n \geq 2 \) are rederived. The Virasoro symmetries coincide with those constructed in [24], and are proved to be nonlinearizable when acting on the tau function, which implies that this Drinfeld–Sokolov hierarchy is not of topological type in the sense of [8]. In the proof we obtain a formula (5.34) to calculate the obstacles \( O_j \). This formula, with its two sides arising from different contexts, still needs to be better understood. We plan to study it in future work.

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