Confined Willmore energy and the Area functional

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Abstract

We consider minimization problems of functionals given by the difference between the Willmore functional of a closed surface and its area, when the latter is multiplied by a positive constant weight $\Lambda$ and when the surfaces are confined in the closure of a bounded open set $\Omega \subset \mathbb{R}^3$. We explicitly solve the minimization problem in the case $\Omega = B_1$. We give a description of the value of the infima and of the convergence of minimizing sequences to integer rectifiable varifolds, depending on the parameter $\Lambda$. We also analyze some properties of these functionals and we provide some examples. Finally we prove the existence of a $C^{1,\alpha} \cap W^{2,2}$ embedded surface that is also $C^\infty$ inside $\Omega$ and such that it achieves the infimum of the problem when the weight $\Lambda$ is sufficiently small.

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Introduction

If $\Sigma \subset \mathbb{R}^3$ is a smooth immersed surface and $H$ is its mean curvature vector, that we define with norm equal to the absolute value of the arithmetic mean of the principal curvatures, we define the Willmore energy of $\Sigma$ as:

$$W(\Sigma) := \int_{\Sigma} |H|^2 \, d\sigma$$

where $\sigma$ is the area measure on $\Sigma$. The operator $W$ is called Willmore functional. Surfaces will be usually denoted by $\Sigma$ and will be always compact and without boundary, but not necessarily connected.

The variational study of this functional has been revived in 1965 with the work of T. Willmore ([36] and [37]). He found that round spheres are the only global minimizers for $W$ and then he introduced the study of the minimization problem subject to constraints of topological type, such as fixing the genus of the surfaces; the celebrated Willmore Conjecture is related to these kind of problems, and it has been proved in [17]. In the last decades a number of properties about the functional itself have been proved, and the ones we will use are recalled in Section 1. The minimization problem at fixed genus has also been solved in a couple of works ([34] and then [2]), developing also a theory of which we will make use in the following.

In this work we are going to study the following functional:

$$W_\Lambda(\Sigma) := W(\Sigma) - \Lambda|\Sigma|,$$

where $\Lambda > 0$ is fixed and $|\Sigma|$ denotes the area of $\Sigma$. Moreover, we will always consider surfaces $\Sigma \subset \Omega$ with $\Omega \subset \mathbb{R}^3$ open and bounded with $\partial \Omega$ of class $C^2$. Also, by a rescaling property shown in Section 1, we will usually take $\Omega \subset B_1(0)$ (so that $\text{diam}(\Omega) \leq 1$).

With the above assumptions we show that the minimization problem

$$(P)_{\Omega, \Lambda} : \quad \min\{W_\Lambda(\Sigma) : \Sigma \subset \Omega\}$$

achieves the infimum when the weight $\Lambda$ is sufficiently small.
sets a non trivial competition between the Willmore and the Area terms. We define
\[ C_\Lambda := \inf \{ W_\Lambda(\Sigma) : \Sigma \subset \Omega \}. \] (4)
We also give here the following definitions, that will be useful later on:
\[ \Lambda_\Omega := \inf \{ \Lambda > 0 : C_\Lambda = -\infty \}, \quad \mathcal{W}(\Sigma) := \frac{W(\Sigma)}{|\Sigma|}, \quad \Lambda_\Omega := \inf \{ \mathcal{W}(\Sigma) : \Sigma \subset \Omega \}. \] (5)

Variational problems of a similar type, that is problems involving the area, have already been studied. There is a complete treatment of the minimization problem of the Willmore energy with fixed area for surfaces of genus zero ([19] and [20]) and with fixed isoperimetric ratio for surfaces of arbitrary fixed genus ([31] and [12]). This kind of works found interesting comparisons with works about the shape of organic corpuscles ([32]). The link with the quantities \( W \) and \( \Lambda_\Omega \) defined above resembles the Cheeger Problem, which is actually strongly related to the existence of confined surfaces with prescribed mean curvature vector ([15]). It would be interesting to study related problems for curves in dimension two, for which there are already remarkable results about the variational problems of functionals depending on the curvature of the curve in the same way the Willmore energy depends on the curvature of the surface (in [3] and [7] confined elastic curves are considered, while in [8] [11] and [22] relaxed notions of the elastic energy are studied).

In the next statement we sum up our main results in the case of a general domain \( \Omega \).

**Theorem A.** Under the above assumptions on \( \Omega \), denoting \( C_\Lambda : (0, +\infty) \to [-\infty, +\infty) \) the function that associates to \( \Lambda \) the infimum of \((P)_{\Omega,\Lambda}\), it holds:

(i) \( C_\Lambda \) is a concave, continuous, non negative, strictly decreasing function on an interval \((0, \Lambda_\Omega]\) for some \( \Lambda_\Omega \in [4, 1/\epsilon_2]\) where \( \Lambda_\Omega, \epsilon_2 \) depend on \( \Omega \). Moreover \( \lim_{\Lambda \to 0^+} C_\Lambda = 4\pi, C_{\Lambda_\Omega} \geq 0 \) and \( C_\Lambda = -\infty \) for all \( \Lambda > \Lambda_\Omega \).

(ii) If \( \Lambda \in (0, \Lambda_\Omega) \) there exists a sequence \( (\Sigma_n^\Lambda) \) that is minimizing for the functional \( W_\Lambda \) and such that it converges in the sense of varifolds to a varifold \( V \) that is integer rectifiable and has generalized mean curvature square integrable with respect to the weight measure of \( V \).

(iii) If \( \Lambda \) is sufficiently small, depending only on \( \Omega \), the limit varifold in item (ii) is actually a \( C^{1,\alpha} \cap W^{2,2} \) embedded surface \( \Sigma \) with multiplicity one and it is such that \( W_\Lambda(\Sigma) = C_\Lambda \). Moreover it holds that \( \Sigma \cap \Omega \) is of class \( C^\infty \).

Next we state the result concerning the case \( \Omega = B_1 \), where \( B_1 \) is the standard unit ball of \( \mathbb{R}^3 \).

**Theorem B.** If \( \Omega = B_1 \) the minimization problem \((P)_{B_1,\Lambda}\) admits a solution if and only if \( \Lambda \leq 1 \), in which case the minimum is \( 4\pi(1 - \Lambda) \). If \( \Lambda < 1 \) the unique minimizer is the unit sphere \( S^2 \). Moreover for all \( \Lambda > 1 \) the infimum of the problem is \( -\infty \).

The paper is organized as follows. In Section 1 we state some classical properties of the Willmore functional and of the \( W_\Lambda \) energy. In Section 2 we prove the first two items of Theorem A and we prove Theorem B. Section 3 is devoted to the proof of item (iii) of Theorem A, that is essentially a regularity issue. In this work we adopt a very classical method, today named Simon’s ambient approach ([31]), that is well applicable in our setting. We will mainly highlight the differences that arise with respect to [34], that is taking care of the area term and of the presence of the boundary \( \partial \Omega \). By now we just mention that this method is based on the direct proof of the regularity of a set contained in \( \mathbb{R}^3 \) from information about the boundedness of its second fundamental form and it has already been used in other works linked to the Willmore energy ([14], [18], [30] and [31]). It is very remarkable a more modern method, called parametrization approach, essentially due to Riviè re and presented for example in [26], [27] and [28]. This method is based on the formulation of suitable spaces of parametrizations of surfaces, where abstract techniques of calculus of variations are applicable. Notable applications are contained in the already cited [12] and [19]. We think that applying this method to our problem could give very good results and it can certainly be a future project to improve our current results following this way.

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## 1 Basic Properties

We are going to collect some useful properties about the Willmore functional that we will use later on. The symbol \( \mathcal{V}_2(\Omega) \) denotes the set of \( 2 \)-rectifiable integer varifold defined in \( \mathbb{R}^3 \) with support contained in \( \Omega \). The convergence in \( \mathcal{V}_2(\Omega) \) is the classical convergence of varifolds in \( \mathbb{R}^3 \). The symbol \( \mu_{\nu} \) will always denote the Radon measure on \( \Omega \) induced by the varifold \( V \in \mathcal{V}_2(\Omega) \). We recall that \( \Omega \subset \mathbb{R}^3 \) is open, bounded and with \( \partial \Omega \) of class \( C^2 \). For the general notation and results about varifolds see Appendix A. Let us start with an important observation.
**Remark 1.1** (Semicontinuity). Let us consider a sequence \( V_k \in V_2(\Omega) \) that converges to \( V \in V_2(\Omega) \) in the sense of varifolds. Assume that for each \( k \) there exists the generalized mean curvature \( H_k \) of \( V_k \) such that

\[
W(V_k) := ||H_k||_{L^2(\Omega, \nu_{V_k})}^2 \leq C_0 < +\infty,
\]

with \( C_0 \) independent of \( k \).

Then applying convergence of Radon measures and using the continuity of the first variation with respect to the varifold convergence, we have that \( V \) has generalized mean curvature \( H_V \) such that

\[
\liminf_k W(V_k) \geq W(V).
\]

We note also that, since \( \Omega \) is compact, we have that \( M(V_k) \to M(V) \), where \( M \) denotes the mass of a varifold. Thus:

\[
\liminf_k W(\Lambda(V_k)) \geq W(\Lambda)(V),
\]

\[
\liminf_k W(V_k) \geq W(V).
\]

For further details see [29], where it is also shown the more involved lower semicontinuity property under convergence of currents.

Now we state a couple of fundamental properties of the Willmore energy.

**Theorem 1.2** (Conformal Invariance, [37]). Let \( \Sigma \subset \mathbb{R}^3 \) be an immersed surface in the 3-dimensional Euclidean space. Suppose \( \Sigma \subset \Omega \), with \( \Omega \subset \mathbb{R}^3 \) open. If \( F : \Omega \to F(\Omega) \) is a conformal transformation, then:

\[
W(\Sigma) = W(F(\Sigma)).
\]

**Remark 1.3.** We recall that, by Liouville’s Theorem, conformal transformations of the Euclidean \( \mathbb{R}^3 \) are just compositions of translations, dilatations, orthogonal transformations and spherical inversions (for an interesting proof see [22]).

**Theorem 1.4** (Lower Bound for Immersed Surfaces, [2]). Let \( \Sigma \) be an immersed surface and \( \xi \in \Sigma \) be a point with multiplicity \( k \). If \( I : \mathbb{R}^3 \setminus \{\xi\} \to \mathbb{R}^3 \setminus \{\xi\} \) is the standard spherical inversion about the sphere \( S^2_\xi(\xi) \), then:

\[
W(I(\Sigma \setminus \{\xi\})) = W(\Sigma) - 4\pi k.
\]

**Remark 1.5.** We immediately get from Theorem 1.4 that if a surface \( \Sigma \) has a point with multiplicity \( k \), then

\[
W(\Sigma) \geq 4\pi k.
\]

A similar argument holds for a varifold \( V \in V(\Omega) \) with square integrable generalized mean curvature in the sense that it holds:

\[
\theta(x) \leq \frac{W(V)}{4\pi} \quad \mu_V\text{-almost every } x,
\]

where \( \theta \) is the multiplicity function of \( V \) and \( \mu_V \) is the Radon measure given by \( V \) on \( \mathbb{R}^3 \) (see [14], Appendix A). In particular we get that if \( W(V) < 8\pi \), then the varifold has multiplicity 1 \( \mu_V\)-almost everywhere.

Now we state some results relating the Willmore and the Area functionals.

**Lemma 1.6** ([34]). If \( \Sigma \subset \mathbb{R}^3 \) is a connected surface, then:

\[
\sqrt{\frac{\text{Area}(\Sigma)}{W(\Sigma)}} \leq \text{diam}\Sigma \leq \sqrt{\frac{2}{\pi}}\sqrt{\text{Area}(\Sigma)W(\Sigma)},
\]

where \( \text{diam}\Sigma \) is the diameter of \( \Sigma \) and \( C \) is a constant independent of \( \Sigma \).

Another fundamental inequality is the following:

**Lemma 1.7** (Willmore vs Area Inequality, [20]). Let \( \Omega \subset B_1 \subset \mathbb{R}^3 \) and let \( V \in V_2(\Omega) \) such that there exists the generalized mean curvature \( H_V \in L^2(\Omega, \mu_V) \). Then:

\[
W(V) := \int_{\Omega} H_V^2 \ d\mu_V \geq M(V),
\]

with equality if and only if \( \mu_V = kH_2^2 \mathbb{S}^2 \) and \( \mathbb{S}^2 \subset \Omega \), with \( k \in \mathbb{N}_{>0} \).
Remark 1.8. The inequality proved in Lemma 1.7 can be specialized in the case of $\Omega \subset B_{1/2}$. If $V \in V_2(\tilde{\Omega})$, by a simple scaling argument and using the conformal invariance of $W$ one gets that

$$W(V) \geq 4M(V).$$

(9)

From these results we establish some very useful inequalities, as stated in the following.

Corollary 1.9. If $\Sigma \subset \tilde{\Omega}$ is a connected surface, then

$$W(\Sigma) \geq |\Sigma| \left(1 - \frac{\Lambda}{(\text{diam}\Sigma)^2}\right),$$

$$W(\Sigma) \geq W(\Sigma)(1 - \Lambda) - \frac{\Lambda}{C^2} |\Sigma|,$$

where $C$ is the constant in Lemma 1.6.

If $\Omega \subset B_1$ then

$$W(\Sigma) \geq 1.$$
2 Compactness and properties of $C_\Lambda$

This section is devoted to the proof of items i), ii) of Theorem A and of Theorem B. Let us start with a significant example.

Example 2.1. If $\Lambda > 1$ then
\[ \inf_{\Sigma \subset \Omega} \mathcal{W}_\Lambda(\Sigma) = -\infty. \]

In fact let us define the sequence of surfaces
\[ D^k = S^2_{r_1} \cup \cdots \cup S^2_{r_k} \subset \Omega, \]
\[ r_i = \frac{1}{\sqrt{\Lambda}} + \frac{i - 1}{k} \left( 1 - \frac{1}{\sqrt{\Lambda}} \right), \quad i = 1, \ldots, k, \]
that is a surface made of $k$ concentric spheres with minimum radius $r_1 = 1/\sqrt{\Lambda}$, $r_i < r_{i+1}$ for $i = 1, \ldots, k - 1$ and maximum radius $r_k < 1$ (since $\Lambda > 1$). We have:
\[ \mathcal{W}_\Lambda(D^k) = 4\pi k - 2 \sum_{i=1}^k 4\pi r_i^2 \]
\[ = 4\pi \left( k - k - \Lambda \sum_{i=1}^k \left( 1 - \frac{1}{\sqrt{\Lambda}} \right)^2 \frac{1}{k^2} (i - 1)^2 + \frac{2}{k\sqrt{\Lambda}} (1 - \frac{1}{\sqrt{\Lambda}}) (i - 1) \right) \]
\[ \leq -4\pi \Lambda \left( 1 - \frac{1}{\sqrt{\Lambda}} \right) \frac{1}{k} \left( \frac{k(k+1)}{2} - k \right) \to -\infty \quad k \to +\infty, \]
where we strongly used the fact that $\Lambda > 1$.

Now we see that we can actually connect together the rounds of the previous example in a way in which we are able to obtain the same conclusion also in the case in which the problem is restricted to connected surfaces. We are going to see this in a general way as stated in next lemma.

Lemma 2.2. If there exists an embedded surface $\Sigma \subset \Omega$ such that $\mathcal{W}_\Lambda(\Sigma) < 0$, then there exists a sequence of embedded surfaces $\Sigma_n$ such that $\mathcal{W}_\Lambda(\Sigma_n) \to -\infty$. In particular $C_\Lambda = -\infty$.

Moreover if $\Sigma$ is connected, the surfaces $\Sigma_n$ can be taken connected.

Proof. We are going to reproduce the idea of Example 2.1 with the surface $\Sigma$ in the hypothesis of the statement. Let us fix $\epsilon \in (0, 1)$. First we notice that it may occur that $\Sigma_\epsilon := \Sigma \cap \partial \Omega = \emptyset$, and so we suppose we are in this situation (the case $\Sigma_\epsilon = \emptyset$ will be a simplier by-product of this case). Let us fix for each connected component $\Sigma_\alpha$ (note that $\Sigma_\alpha$ is compact) a field $N^\alpha \in \mathcal{N}(\Sigma_\alpha)$ such that $N^\alpha$ point inside $\Omega$ for each $\alpha$, where $\mathcal{N}(\Sigma_\alpha)$ denotes the normal bundle of $\Sigma_\alpha$. Now fix an open neighborhood $U^\alpha \subset \Sigma$ of each $\Sigma_\alpha$ such that $\text{dist}(p, \Sigma_\alpha) < \epsilon$ for each $p \in U^\alpha$ and $U^\alpha \cap U^\beta = \emptyset$ for all $\alpha \neq \beta$. Let for all $\alpha$ the functions $\phi^\alpha \in C^\omega_c(\Sigma)$ such that $\phi^\alpha(p) = 1$ for all $p \in \Sigma_\alpha$ and $\phi^\alpha(p) = 0$ for all $p \in \Sigma \setminus U^\alpha$.

Now mapping:
\[ U^\alpha \ni p \mapsto p + \delta N^\alpha(p) \phi^\alpha(p) \in \Omega, \]
we obtain a new embedded surface $\Sigma' \subset \Omega$ such that for an appropriate choice of $\delta$ above sufficiently small we have:
\[ \mathcal{W}_\Lambda(\Sigma') = \mathcal{W}_\Lambda(\Sigma) + \epsilon. \]

Since $\Sigma'$ is compact and embedded, it is orientable, so there exists a field $N \in \mathcal{N}(\Sigma')$ that orients the surface. For $M \in \mathbb{R}$ sufficiently big we can consider the surface:
\[ \Sigma_M' := \left\{ p + \frac{1}{M} N(p) : p \in \Sigma' \right\} \subset \Omega \quad \text{s.t.} \quad \mathcal{W}_\Lambda(\Sigma_M') = \mathcal{W}_\Lambda(\Sigma') + \epsilon. \]

Now we are going to connect together $\Sigma'$ with $\Sigma_M'$ in order to obtain the first term $\Sigma_1$ of the desired sequence $(\Sigma_n)$. Select $\hat{p} \in \Sigma'$ and consider the corresponding $\hat{p}_M = \hat{p} + \frac{1}{M} N(\hat{p}) \in \Sigma_M'$. Letting $\bar{q}$ the middle point between $\hat{p}$ and $\hat{p}_M$, there exists $\delta_0$ such that $(\Sigma' \cup \Sigma_M') \cap B_{\delta_0}(\bar{q})$ is diffeomorphic to the disjoint union of two 2-dimensional discs. Operating a blow up procedure by a factor $\Gamma$ sufficiently big on $(\Sigma' \cup \Sigma_M') \cap B_{\delta_0}(\bar{q})$ we obtain a surface $C^\infty$-close to the disjoint union of two 2-dimensional discs. By removing appropriate sets $\Gamma D_1$ and $\Gamma D_2$ diffeomorphic to a disc from each disconnected component, we see that we can connect the remaining surfaces (diffeomorphic to a disjoint union of two 2-dimensional annular surfaces) with a modification $\Gamma \hat{C}$ of the catenoid that is $C^2$ close to the standard catenoid.
and such that \( \Lambda(|D_1 \cup D_2| - |\tilde{C}|) \leq \epsilon \) and \( W(\tilde{C}) \leq \epsilon \) (for an explicit construction see [38]). Rescaling back in \( \Omega \) and using the dilatation invariance we see that we have obtained a connected embedded surface \( \Sigma_1 \subset \Omega \) such that:

\[
W_\Lambda(\Sigma_1) = W_\Lambda(\Sigma') + W_\Lambda(\Sigma_1'') + W(\tilde{C}) + \Lambda(\vert D_1 \cup D_2 \vert - |\tilde{C}|) - W(D_1 \cup D_2) + \Lambda(D_1 \cup D_2)
\]

\[
\leq 2W_\Lambda(\Sigma) + 5\epsilon.
\]

Now we can clearly iterate the procedure obtaining \( \Sigma_2 \), and in this case, by arbitrariness on the value of \( \epsilon \), we can take a value \( \epsilon^2 \). Thus, using the notation above with an additional index 1 to distinguish from the previous quantities, we get a connected embedded surface \( \Sigma_2 \subset \Omega \) such that:

\[
W_\Lambda(\Sigma_2) = W_\Lambda(\Sigma_1) + W_\Lambda(\Sigma_1, M_1) + W(\tilde{C}_1) + \Lambda(\vert D_{1,1} \cup D_{2,1} \vert - |\tilde{C}_1|) - W(D_{1,1} \cup D_{2,1})
\]

\[
\leq 2(2W_\Lambda(\Sigma) + 5\epsilon) + 3\epsilon^2
\]

\[
= 2^2W_\Lambda(\Sigma) + 5(2\epsilon) + 3\epsilon^2.
\]

So iterating the procedure taking \( \epsilon^n \) when constructing \( \Sigma_n \) we obtain:

\[
W_\Lambda(\Sigma_n) \leq 2^nW_\Lambda(\Sigma) + 5(2^n - 1)\epsilon + 3 \sum_{i=2}^{n} 2^n - i \epsilon^i
\]

\[
\leq 2^nW_\Lambda(\Sigma) + 5(2^n - 1)\epsilon + 3 \sum_{i=2}^{n} \frac{2^n - 2^i}{1 - \epsilon} \to -\infty \quad \text{as } n \to \infty,
\]

being \( W_\Lambda(\Sigma) < 0 \).

The previous discussion allows us to solve completely Problem (P) in the ball \( B_1 \):

**Theorem 2.3 (Solution of (P)\(_{B_1, \Lambda} \))**. If \( \Omega = B_1 \) the minimization problem \( (P)\(_{B_1, \Lambda} \) admits a solution if and only if \( \Lambda \leq 1 \), in which case the minimum is \( 4\pi(1 - \Lambda) \). If \( \Lambda < 1 \) the unique minimizer is the unit sphere \( S^2 \). Moreover for all \( \Lambda > 1 \) the infimum of the problem is \( -\infty \).

**Proof.** The last part of the statement is a consequence of Example 2.1 and Lemma 2.2 in fact for \( \Lambda > 1 \) we have \( W_\Lambda(S^1) < 0 \).

If we consider \( \Lambda \leq 1 \), applying Lemma 2.2 we get \( W_\Lambda(\Sigma) > |\Sigma|(1 - \Lambda) > 0 \), hence as for the minimization problem we can restrict ourselves to connected surfaces. Moreover \( W_\Lambda(\Sigma) \geq W(\Sigma)(1 - \Lambda) \geq 4\pi(1 - \Lambda) = W(\Lambda(S^2)) \). So \( S^2 \) is a minimizer.

If \( \Lambda < 1 \), the uniqueness of the minimizer follows having \( W_\Lambda(\Sigma) \geq W(\Sigma)(1 - \Lambda) \geq |\Sigma|(1 - \Lambda) \) for all \( \Sigma \), with equality if and only if \( \Sigma = S^2 \).

**Remark 2.4 (Upper Bound for \( \Lambda_\Omega \)).** Combining Example 2.1 with the proof of Lemma 2.2 we see that if there exist two balls \( B_r(p), B_{r-\delta}(p) \) such that \( B_r(p) \setminus B_{r-\delta}(p) \subset \Omega \) for some \( \delta > 0 \) then a minimizing sequence of connected surfaces \( (\Sigma_n) \) can be realized inside \( B_r(p) \setminus B_{r-\delta}(p) \) and thus for all \( \Lambda > 1 \) we have \( C_\Lambda = -\infty \).

By rescaling invariance this means that if two balls \( B_r(p), B_{r-\delta}(p) \) are such that \( B_r(p) \setminus B_{r-\delta}(p) \subset \Omega \) for a \( \delta > 0 \), then for all \( \Lambda > \frac{1}{r^2} \) we have \( C_\Lambda = -\infty \) (in other words \( \Lambda_\Omega \leq \frac{1}{r^2} \)).

Now we turn our attention to the study of the general Problem (P)\(_{\Omega, \Lambda} \). When no other is specified, \( \Omega \) is assumed to be open, with boundary of class \( C^2 \) and

\[
\Omega \subset B_{\frac{1}{2}}.
\]

By the rescaling invariance of Remark 1.11 we can do this without losing any information.

Let us first make a simple observation.

**Remark 2.5 (Monotonicity).** It is very important to keep in mind a simple monotonicity relation about functional \( W_\Lambda \):

\[
\Lambda_1 > \Lambda_2 \quad \Rightarrow \quad W_{\Lambda_1}(\Sigma) < W_{\Lambda_2}(\Sigma) \quad \text{and} \quad C_{\Lambda_1} \leq C_{\Lambda_2}.
\]

Moreover let us define a useful parameter:

\[
\alpha_\Omega := \sup \{ r > 0 : \exists \delta > 0, \exists B_r(p), B_{r-\delta}(p) \text{ s.t. } B_r(p) \setminus B_{r-\delta}(p) \subset \tilde{\Omega} \},
\]

so that by Remark 2.3 we have \( \Lambda_\Omega \leq \frac{1}{\alpha_\Omega} \).

**Lemma 2.6.** If \( \Lambda > \Lambda_\Omega \) then \( C_\Lambda = -\infty \).
Corollary 2.9. Combining the information collected up to now we have the following consequence.

Proof. By definition of $\Lambda_\Omega$ there is $\lambda \in (\Lambda_\Omega, \Lambda)$ such that $C_\lambda = -\infty$. By monotonicity for all $\Sigma \subset \Omega$ we have:

$$W_\lambda(\Sigma) > W_\lambda(\Sigma),$$

so $C_\lambda = -\infty$. □

Now we are able to give a first result about compactness.

Theorem 2.7 (Compactness for $\Lambda < \Lambda_\Omega$). If $\Lambda < \Lambda_\Omega$ and if $\Sigma_n$ is minimizing for $W_\lambda$, then $\Sigma_n$ converges (up to subsequence) to a varifold $V \in \mathcal{V}_2(\Omega)$ with generalized mean curvature $H_V \in L^2(\Omega, \nu_V)$.

Proof. Let us take a minimizing sequence $\Sigma_n$ such that $W_\lambda(\Sigma_n) \leq C + \frac{1}{n}$. Suppose that $|\Sigma_n| \to \infty$. Let $\lambda \in (\Lambda, \Lambda_\Omega)$, then:

$$0 \leq C_\lambda \leq W_\lambda(\Sigma_n) = W_\lambda(\Sigma_n) - (\lambda - \Lambda)|\Sigma_n| \to -\infty,$$

that is impossible. So we have that there exists $L$ such that $|\Sigma_n| \leq L$ for all $n$, and being $(\Sigma_n)$ a minimizing sequence then there also exists $C_0$ such that $W(\Sigma_n) \leq C_0$ for all $n$. Moreover, denoting by $H_n$ the mean curvature vector of $\Sigma_n$ and calling again $\Sigma_n$ the varifold associated to $\Sigma_n$, for all $W \subset \subset \Omega_\epsilon := \epsilon$-neighborhood of $\Omega$ we have (see Appendix A):

$$||\delta\Sigma_n|| (W) = \sup_{|X| \leq 1, \supp(X) \subset W} \left|\delta\Sigma_n(X)\right| = \sup_{|X| \leq 1, \supp(X) \subset W} \left|\int_{\Omega_n} \langle X, H_n \rangle \, d\mu_{\Sigma_n}\right| \leq \sqrt{L C_0} \quad \forall n.$$

So by compactness of varifolds ([1], Appendix A) we get the existence of a limit $V \in \mathcal{V}_2(\Omega)$ of a subsequence $(\Sigma_{n_k})$ in the sense of varifolds. By lower semicontinuity we have that $V$ has mean curvature $H_V \in L^2(\Omega, \nu_V)$. □

Remark 2.8. From the proof of Theorem 2.7 it is useful to remember that if a sequence $(\Sigma_n)$ of uniformly bounded surfaces has both uniformly bounded Willmore energy $W(\Sigma_n)$ and area $|\Sigma_n|$, then such sequence is precompact with respect to varifold convergence.

Combining the information collected up to now we have the following consequence.

Corollary 2.9. The number $\Lambda_\Omega$ is contained in the interval $[4, \frac{1}{\epsilon_1}]$.

Proof. First recall that we already observed that $\Lambda_\Omega \leq \frac{1}{\epsilon_1}$ (observe that $\epsilon_1 \leq \frac{1}{2}$ since $\Omega \subset B_1/2$). Now let us blow up $\Omega$ by a factor 2; we get $2\Omega \subset B_1$ and $W_\lambda(\Sigma) \geq |\Sigma|(1 - \Lambda) \geq 0$ for all $\Sigma \subset 2\Omega$ and for all $\Lambda \leq 1$ by Lemma 1.17. Thus we get $\Lambda_2\Omega \geq 1$, then rescaling back to $\Omega$ we get $\Lambda_\Omega \geq \frac{1}{(1/2)^2} = 4$.

Without further assumptions on $\Omega$ we will see that we are not able to identify $\Lambda_\Omega$ among its possible values (Example 2.14). We need some results first.

Remark 2.10. Let us consider two parameters $\Lambda, (\Lambda + \epsilon) \in (\Lambda, \Lambda_\Omega)$, and call $(\Sigma_n^\Lambda)$ and $(\Sigma_n^{\Lambda+\epsilon})$ two corresponding minimizing sequences. By the proof of Theorem 2.7 we know that areas $|\Sigma_n^\Lambda|, |\Sigma_n^{\Lambda+\epsilon}|$ are uniformly bounded; assume that there exist the limits of the sequences of their areas. It holds that:

$$\lim_n |\Sigma_n^\Lambda| \leq \lim_n |\Sigma_n^{\Lambda+\epsilon}| < +\infty,$$

$$\lim_n |\Sigma_n^{\Lambda+\epsilon}| \leq \lim_n \frac{\Lambda - 4}{\Lambda} |\Sigma_n^\Lambda| + \frac{1}{\Lambda} W(\Sigma_n^{\Lambda+\epsilon}),$$

$$C_{\Lambda+\epsilon} = \lim_n W_{\Lambda+\epsilon}(\Sigma_n^{\Lambda+\epsilon}) \leq \lim_n W_\Lambda(\Sigma_n^\Lambda) - \epsilon |\Sigma_n^\Lambda| < C_{\Lambda}.$$  

Let us prove such inequalities separately. First we have:

$$\lim_n W_{\Lambda+\epsilon}(\Sigma_n^{\Lambda+\epsilon}) \leq \lim_n W_{\Lambda+\epsilon}(\Sigma_n^\Lambda) = \lim_n W_\Lambda(\Sigma_n^\Lambda) - \epsilon |\Sigma_n^\Lambda| + \epsilon |\Sigma_n^{\Lambda+\epsilon}| - \epsilon |\Sigma_n^{\Lambda+\epsilon}|$$

$$\leq \lim_n W_\Lambda(\Sigma_n^{\Lambda+\epsilon}) - \epsilon |\Sigma_n^\Lambda| + \epsilon |\Sigma_n^{\Lambda+\epsilon}| - \epsilon |\Sigma_n^{\Lambda+\epsilon}|$$

$$= \lim_n W_{\Lambda+\epsilon}(\Sigma_n^{\Lambda+\epsilon}) + \epsilon |\Sigma_n^{\Lambda+\epsilon}| - |\Sigma_n^\Lambda|. $$

Since $|\Sigma_n^\Lambda|, |\Sigma_n^{\Lambda+\epsilon}|$ are uniformly bounded we get ([13]). Moreover from $\lim_n W_\Lambda(\Sigma_n^\Lambda) \leq \lim_n W_\Lambda(\Sigma_n^{\Lambda+\epsilon})$ we get

$$\Lambda \lim_n |\Sigma_n^{\Lambda+\epsilon}| \leq \lim_n W(\Sigma_n^{\Lambda+\epsilon}) - W(\Sigma_n^\Lambda) + \Lambda |\Sigma_n^\Lambda| \leq \lim_n W(\Sigma_n^{\Lambda+\epsilon}) + (\Lambda - 4) |\Sigma_n^\Lambda|,$$
where in the second inequality we used Remark 1.8. Hence we got (15). Finally
\[
\lim_{n} W_{A+\varepsilon}(\Sigma_{n}^{A}) \leq \lim_{n} W_{A+\varepsilon}(\Sigma_{n}^{A}) = \lim_{n} W_{A}(\Sigma_{n}^{A}) - \varepsilon |\Sigma_{n}^{A}|
\]
that will give (16) once we prove that $|\Sigma_{n}^{A}| \geq \delta > 0$ for all $n$. But in fact if $S$ is any sphere contained in $\Omega$ we have that $W_{A}(S) < 4\pi$ and thus $C_{A} < 4\pi$; therefore if by contradiction $\lim_{n} |\Sigma_{n}^{A}| = 0$ we would have $C_{A} = \lim_{n} W_{A}(\Sigma_{n}^{A}) \geq 4\pi$ that gives a contradiction.

Now we are able to complete the characterization of the infima $C_{A}$, recall that
\[
\Lambda_{\Omega} := \inf\{A > 0 : C_{A} = -\infty\}.
\]

**Theorem 2.11** (Properties of $C_{A}$). The function $C_{A} : \mathbb{R}_{>0} \to [-\infty, +\infty)$ that associates to the parameter $\Lambda$ the corresponding infimum $C_{A}$ has the following properties:

(i) $C_{0} := \lim_{\Lambda \to 0} C_{A} = 4\pi$ independently of $\Omega$, and $C_{\Lambda_{0}} \geq 0$,

(ii) for $\Lambda \in (0, \Lambda_{0}]$ the function $C_{A}$ is continuous, nonnegative, concave and strictly decreasing. Moreover for all $\varepsilon < \Lambda_{\Omega}$ there exists $\delta = \delta(\varepsilon) < 0$ such that the derivative $C'_{A} \leq \delta < 0$ for almost all $\Lambda \in (\Lambda_{\Omega} - \varepsilon, \Lambda_{\Omega}]$ (i.e. where it exists), and $\delta(\varepsilon)$ cannot decrease as $\varepsilon$ decreases,

(iii) $C_{A} = -\infty$ for each $\Lambda > \Lambda_{\Omega}$.

**Proof.** For any $\Sigma$, since $|\Sigma| \leq W(\Sigma)$ and $W(\Sigma) \geq 4\pi$, we have:
\[
W_{A}(\Sigma) \geq W(\Sigma)(1 - \Lambda) \geq 4\pi(1 - \Lambda) \to 4\pi \quad \Lambda \to 0,
\]
then $C_{0} \geq 4\pi$. Now take $r$ sufficiently small such that $S_{r}^{2} \subset \bar{\Omega}$, then:
\[
C_{A} \leq W_{A}(S_{r}^{2}) = 4\pi - 4\pi r^{2}\Lambda \to 4\pi \quad \Lambda \to 0,
\]
thus $C_{0} \leq 4\pi$, and we got (i).

For $\Lambda \in (0, \Lambda_{0})$ we already know from Equation (16) that in this interval the function is positive and strictly decreasing, thus it is differentiable almost everywhere and it has at most a finite number of jump-type discontinuities.

Now we have:
\[
\forall \Sigma \quad W_{A_{0}}(\Sigma) = \lim_{\Lambda \to A_{0}^{-}} W_{A}(\Sigma),
\]
thus:
\[
C_{A_{0}} = \inf_{\Sigma \subset \Omega} W_{A_{0}}(\Sigma) = \inf_{\Sigma \subset \Omega} \lim_{\Lambda \to A_{0}^{-}} W_{A}(\Sigma) \geq \inf_{\Sigma \subset \Omega} \lim_{\Lambda \to A_{0}^{-}} C_{A} \geq 0,
\]
since $C_{A} \geq 0$ for all $\Lambda < \Lambda_{0}$.

Now we can prove continuity from the left. Take $A_{0} \in (0, \Lambda_{0})$ and suppose by contradiction that there exists $\eta > 0$ such that $\lim_{\Lambda \to A_{0}^{-}} C_{A} \geq C_{A_{0}} + \eta$. Calling $(\Sigma_{n})$ a minimizing sequence for the functional $W_{A_{0}}$, we have:
\[
C_{A_{0}} = \lim_{n} W_{A_{0}}(\Sigma_{n}) = \lim_{\Lambda \to A_{0}^{-}} W_{A}(\Sigma_{n}) \geq \lim_{\Lambda \to A_{0}^{-}} C_{A} \geq C_{A_{0}} + \eta,
\]
that is impossible.

We can also consider continuity from the right. Take $A_{0} \in (0, \Lambda_{0})$ and suppose by contradiction that there exists $\eta > 0$ such that $\lim_{\Lambda \to A_{0}^{+}} C_{A} \leq C_{A_{0}} - \eta$. Call $(\Sigma_{n}^{A})$ a minimizing sequence for the functional $W_{A}$. If $A_{1} \in (A_{0}, \Lambda_{1})$, using Equation (14) we know that $\lim_{n} |\Sigma_{n}^{A}| \leq \lim_{n} |\Sigma_{n}^{A_{1}}| \leq L_{1}$ for any $A \in (A_{0}, A_{1})$. Therefore
\[
C_{A_{0}} \geq \eta + \lim_{\Lambda \to A_{0}^{+}} \frac{W_{A}(\Sigma_{n})}{n} = \eta + \lim_{\Lambda \to A_{0}^{+}} \frac{W_{A_{0}}(\Sigma_{n}^{A}) + (A_{0} - A)|\Sigma_{n}^{A}|}{n}
\]
\[
\geq \eta + C_{A_{0}} + \lim_{\Lambda \to A_{0}^{+}} (A_{0} - A) \lim_{n} |\Sigma_{n}^{A}|
\]
\[
\geq \eta + C_{A_{0}} + \lim_{\Lambda \to A_{0}^{+}} (A_{0} - A) \lim_{n} |\Sigma_{n}^{A_{1}}|
\]
\[
\geq \eta + C_{A_{0}} + L_{1} \lim_{\Lambda \to A_{0}^{+}} (A_{0} - A) = \eta + C_{A_{0}},
\]
for some constant $L_{1} \geq \lim_{n} |\Sigma_{n}^{A_{1}}|$, but that is impossible.

We can also check continuity from the left in $\Lambda_{\Omega}$. In fact let us consider a sequence $\Lambda_{n} \to \Lambda_{\Omega}^{-}$, then the functions
\[ W_{\Lambda_n} : \mathbb{V} := \{ V \in \mathcal{V}_2(\Omega) : \exists H V \in L^2(\Omega,\mu_V) \} \to \mathbb{R} \text{ converge uniformly to the function } W_{\Lambda_0} : \mathbb{V} \to \mathbb{R} \text{ with respect to the } F\text{-metric of } \mathcal{V}_2(\Omega) \text{ on bounded sets (i.e. bounded in mass), that is:} \]
\[
\begin{align*}
\sup_{V \in \mathbb{V}} |W_{\Lambda_n}(V) - W_{\Lambda_0}(V)| = \sup_{V \in \mathbb{V}, \mathbb{M}(V) \leq K} |(\Lambda_n - \Lambda_0)\mathbb{M}(V)| \to 0 \quad n \to \infty,
\end{align*}
\]
for all \( K > 0 \). Hence we can swap the limit with the infimum in the following relation.
\[
\begin{align*}
\lim_{\Lambda_n \to \Lambda_0^-} \inf_{\Sigma \subset \Omega, |\Sigma| \leq K} W_{\Lambda_n}(\Sigma) = \inf_{\Sigma \subset \Omega, |\Sigma| \leq K} \lim_{\Lambda_n \to \Lambda_0^-} W_{\Lambda_n}(\Sigma) = \inf_{\Sigma \subset \Omega, |\Sigma| \leq K} W_{\Lambda_0}(\Sigma),
\end{align*}
\]
for all \( K > 0 \). Hence:
\[
\begin{align*}
\lim_{K \to \infty} \lim_{\Lambda_n \to \Lambda_0^-} \inf_{\Sigma \subset \Omega, |\Sigma| \leq K} W_{\Lambda_n}(\Sigma) = C_{\Lambda_0}.
\end{align*}
\] (17)

If we are able to swap the first two limits in (17), we are done. Let
\[
C_{K,n} := \inf_{\Sigma \subset \Omega, |\Sigma| \leq K} W_{\Lambda_n}(\Sigma) \geq 0
\]
Since \( C_{K,n} \) is decreasing in the two indexes and the numbers \( C_{K,n} \) are greater than or equal of zero we have
\[
\inf K C_{K,n} = \lim_{K \to \infty} C_{K,n} \leq \lim_{K \to \infty} C_{K,m} = \inf K C_{K,m} \quad \forall n > m,
\]
and then
\[
\inf K \inf n = \lim \inf K C_{K,n} = \lim \lim K C_{K,n}.
\]
Similarly we get
\[
\inf K \inf n = \lim \lim K C_{K,n},
\]
and thus
\[
\lim K \inf n = \lim \lim K C_{K,n} = \inf K C_{K,n}.
\]
Using (17) we conclude that
\[
C_{\Lambda_0} = \lim_{K \to \infty} \lim_{\Lambda_n \to \Lambda_0^-} K C_{K,n} = \lim_{\Lambda_n \to \Lambda_0^-} \lim_{K \to \infty} K C_{K,n} = \lim_{\Lambda_n \to \Lambda_0^-} C_{\Lambda_0}.
\]
Now using Equation (10) and reminding that \( C_{\Lambda} \) is differentiable for almost all \( \Lambda \), we see that for almost all \( \Lambda \) the function \( f_\Lambda : \epsilon \mapsto C_{\Lambda + \epsilon} \) is such that \( f_\Lambda'(\epsilon) \leq -\lim_{n \to \infty} |\Sigma|_n^\Lambda < 0 \) for almost all \( \epsilon \) for which \( f_\Lambda \) is defined. Using now Equation (14), we get \( f_\Lambda'(\epsilon) \leq f_\Lambda'(0) < 0 \) for almost all \( \epsilon \) for which the relation is defined. Again by Equation (14) we see that \( \delta \) cannot decrease as \( \epsilon \) decreases, thus we have completed the proof of (ii). We already know that (iii) is true by Lemma 2.6 thus we completed the proof of the theorem.

For the convenience of the reader let us recall the definitions:
\[
\mathcal{W}(\Sigma) := \frac{\mathcal{W}(\Sigma)}{|\Sigma|}, \quad \Lambda_0 := \inf\{ \mathcal{W}(\Sigma) : \Sigma \subset \Omega \}.
\] (18)

We want to prove some results that motivate the connection between \( W_{\Lambda_0} \) and \( \mathcal{W} \).

**Proposition 2.12.** It holds that
\[
\Lambda_0 = \Lambda_{\Omega}.
\] (19)

Moreover

(i) if a minimizing sequence \( (\Sigma_n) \) for the functional \( \mathcal{W} \) satisfies that \( |\Sigma_n| \leq L \) for any \( n \), then it is also minimizing for the functional \( W_{\Lambda_0} \) and \( C_{\Lambda_0} = 0 \),

(ii) if a minimizing sequence \( (\Sigma_n) \) for the functional \( W_{\Lambda_0} \) satisfies that \( |\Sigma_n| \leq L \) for any \( n \) and if \( C_{\Lambda_0} = 0 \), then \( \Sigma_n \) is also a minimizing for \( \mathcal{W} \).

**Proof.** For all \( \Sigma \) it holds:
\[
\mathcal{W}(\Sigma) = \frac{W_{\Lambda_0}(\Sigma)}{|\Sigma|} + \Lambda_0 \geq \frac{C_{\Lambda_0}}{|\Sigma|} + \Lambda_0 \geq \Lambda_{\Omega},
\]
Corollary 2.13. If $W_{\Lambda}$ is coercive and $C_{\Lambda_{0}} = 0$, then $W$ is coercive. In particular, if $\Omega = B_{1}$ the functional $W$ is coercive.

Proof. The proof immediately follows by item ii) in Proposition 2.12 and by Theorem 2.13. 

In the following examples we show that it is not possible in general to identify the value of $\Lambda_{0}$ in the interval $[4, 1/\varepsilon_{1}^{2}]$.

Example 2.14. Let us illustrate three examples.

(i) A first simple example is $\Omega = B_{1/2} \setminus B_{1/2-\epsilon}$ with $0 < \epsilon < 1/2$, for which we have $4 = \Lambda_{0} = \frac{1}{\varepsilon_{1}}$.

(ii) Now we construct an example in which $4 < \Lambda_{0} = \frac{1}{\varepsilon_{1}}$. Let us consider $\Omega = B_{r} \cup B_{3/4}(\{x + \frac{1}{2} \delta, 0, 0\})$. In this case $\text{diam}(\Omega) = 1$ but clearly $\epsilon_{0} = r < 1/2$ and $\Lambda_{0} = \frac{1}{\varepsilon_{1}} > 4$.

(iii) We can also construct an example in which $\Lambda_{0} < \frac{1}{\varepsilon_{1}}$. Let us denote by $E_{a,c} = \{(x, y, z) \in \mathbb{R}^{3} : x^{2}/a^{2} + y^{2}/a^{2} + z^{2}/c^{2} = 1\}$. Let us fix $c = 1/2$ and $\epsilon - \eta < a < c$ for $\eta > 0$ sufficiently small such that $W(E_{a,1/2}) \leq 10$ (this is possible since when $a = 1/2$ we would obtain a sphere). Now we consider $\Omega$ as the volume enclosed by $E_{a,1/2}$ except the volume enclosed by $\{p - \delta \nu(p) : p \in E_{a,1/2}\}$ with $\nu$ outer normal of $E_{a,1/2}$ and $\delta << 1$. For $\delta$ sufficiently small we get that $\epsilon_{0} = \delta_{0} = 2$ and: $$W_{\Lambda}(E_{a,1/2}) \leq 10 - \Lambda|E_{a,1/2}| < 10 - 4\pi a^{4/3} \frac{1}{2^{2/3}},$$

where we used the isoperimetric inequality $(4\pi)^{1/3}2^{2/3}|A|^{2/3} \leq |\partial A|$ for $A \subset \mathbb{R}^{3}$. Finally we observe that for $\delta$ sufficiently small there exists $\Lambda < 1/\delta^{2}$ such that $W_{\Lambda}(E_{a,1/2}) = 0$. This implies $\Lambda_{0} < 1/\delta^{2} = 1/\epsilon_{1}^{4}$ as desired.

The following examples point out the strong dependence of the problems on the geometry of the domain as it is taken unbounded. The scenery seems to become somehow chaotic, in the sense that we did not find spontaneous hypotheses on an unbounded $\Omega$ under which general conclusions can be derived.

Example 2.15. Let us take: $$\Omega = \{(x, y, z) \in \mathbb{R}^{3} | x^{2} + y^{2} < 1\}.$$ Let us consider a sequence of surfaces $\Sigma_{n}$ that is $C^{1}$-close to $\Sigma_{n} = S_{1} \cup C_{n} \cup S_{2}$ where $C_{n}$ is a cylinder or radius $1$ and height $n$, while $S_{1}$ and $S_{2}$ are the two hemispheres of the standard $S^{2}$ translated in a way in which $\Sigma_{n}$ becomes an admissible surface. We can arrange: $$W_{\Lambda}(\Sigma_{n}) \leq 4\pi(1 - \Lambda) + \left(\frac{1}{4} - \Lambda\right)2\pi n + \delta,$$

for some $\delta > 0$, where the first term is the energy of the two hemispheres and the second one is due to the cylinder. Then $W_{\Lambda}(\Sigma)$ converges to $-\infty$ as $n$ increases if $\Lambda > 1/4$, so $\Lambda_{0} \leq 1/4$. Being $\Omega$ unbounded we cannot use the results obtained above, and it is also interesting to notice that the direct method consisting of taking a minimizing sequence and proving its convergence in the sense of varifolds is no longer applicable, since in this case we apparently have no tools in order to uniformly estimate the area of the sequence.
Considering different unbounded domains \( \Omega \) the situation may degenerate, as shown in the next example.

**Example 2.16.** Let us take:
\[
\Omega = \{(x, y, z) \in \mathbb{R}^3 : |z| < 1\}.
\]
In this case we will see that the problems become immediately trivial. Let us consider the sequence of surfaces \( \Sigma_n \) \( C^1 \)-close to \( \sigma_n \cup \sigma_n^2 \cup T_n \), where \( \sigma_n \) are two discs of radius \( n \) with center \((0,0,-1)\) or \((0,0,1)\) lying on the opposite sides of \( \partial \Omega \), and \( T_n \) is the subset with positive Gaussian curvature of the torus given by the rotation of a circumference of radius \( 1 \) at a distance \( n \) from the axis \( z \). We can estimate for some \( \delta > 0 \) that:
\[
W_\Lambda(\Sigma_n) \leq Cn - 2\pi An^2 + \delta \to -\infty \quad n \to \infty,
\]
for all \( \Lambda > 0 \). So there is not a minimum for \( W_\Lambda \) and the infimum of the problem is \(-\infty\).

Let us conclude with a further example.

**Example 2.17.** It is not true in general that the boundary \( \partial \Omega \) of a bounded convex domain is a minimizer for Problem \((P)_{\Omega;\Lambda}\) for all \( \Lambda \leq \Lambda_0 \). Take for example \( \partial \Omega \) \( C^1 \)-close to \( S_1 \cup C_\rho \cup S_2 \) where \( S_1 \) and \( S_2 \) are translations of the two hemispheres of the standard \( S^2 \) and \( C_\rho \) is a cylinder of radius one and height \( \rho \). \( \Omega \) is the bounded set with such boundary. We can arrange that:
\[
W_\Lambda(\partial \Omega) \geq W_\Lambda(S^2) + \left( \frac{1}{4} - \Lambda \right) 2\pi h - \delta > W_\Lambda(S^2),
\]
for some \( \delta > 0 \) for each \( \Lambda < \frac{1}{4} - \frac{\pi}{2\pi h} \). Being \( S^2 \subset \Omega \) we see that the boundary cannot be a minimizer for \( W_\Lambda \) for all \( \Lambda \leq \Lambda_0 \).

### 3 Regularity

In this section we prove statement \((iii)\) of Theorem A. We adopt the convention that if \( L \) is a plane in \( \mathbb{R}^3 \), then we write \( u = (u_1, u_2, u_3) \in C^\infty(\bar{A}; L^\perp) \), where \( A \subset L \), if \( u(x) \in L^\perp \forall x \in A \). In this case we write:
\[
\text{graph } u = \{x + u(x)|x \in A\}. \tag{20}
\]
Let us first recall the two main tools that we will use in the proof.

**Lemma 3.1 (Graphical Decomposition, [34]).** Let \( \Sigma \) be a compact surface without boundary with \( 0 \in \Sigma \). Then for any \( \beta > 0 \) there exists \( \epsilon_0 \) (independent of \( \Sigma, \rho \)) such that if \( \epsilon \in (0, \epsilon_0], |\Sigma \cap \bar{B}_\rho| \leq \beta \rho^2 \) and \( \int_{\Sigma \cap \bar{B}_\rho} |A| \leq \epsilon \rho \), then the following holds.

There are disjoint closed sets \( P_1, ..., P_N \subset \Sigma \) such that:
\[
\sum_{j=1}^N \text{diam } P_j \leq C \epsilon^{1/2} \rho
\]
and
\[
\Sigma \cap B_{\rho/2} \setminus \left( \bigcup_{j=1}^N P_j \right) = \left( \bigcup_{i=1}^M \text{graph } u_i \right) \cap B_{\rho/2},
\]
where \( u_i \in C^\infty(\bar{A}_i; L_i^\perp) \), with \( L_i \) plane, \( A_i \) smooth bounded connected open in \( L_i \) of the form \( A_i = A_i^0 \setminus (\bigcup_{k} d_{i,k}) \) where \( A_i^0 \) is simply connected and \( d_{i,k} \) are closed disjoint discs in \( L_i \) not intersecting \( \partial A_i^0 \) and also \( \sum_{i,k} \text{diam } d_{i,k} \leq C \epsilon^{1/2} \rho \), where \( \sum_{i,k} |d_{i,k}| \leq C \epsilon \rho^2 \).

Moreover graph \( u_i \) is connected and:
\[
\sup_{A_i} \frac{|u_i|}{\rho} + \sup_{A_i} |D u_i| \leq C \epsilon^{1/6}.
\]
If we also have \( \int_{B_\rho} |A| \leq \epsilon^2 \), then in addition to the above conclusions it holds that for every \( \sigma \in (\rho/4, \rho/2) \) such that \( \partial B_\sigma \) intersects \( \Sigma \) transversely and \( \partial B_\sigma \cap \cup_j P_j = \emptyset \), we have:
\[
\Sigma \cap \bar{B}_\sigma = \bigcup_{i=1}^M D_{\sigma,i},
\]
where each \( D_{\sigma,i} \) is homeomorphic to a disc and graph \( u_i \cap B_\sigma \subset D_{\sigma,i} \). Also \( D_{\sigma,i} \setminus \text{graph } u_i \) is a union of a subcollection of the \( P_j \) and each \( P_j \) is homeomorphic to a disc.
Lemma 3.2 (Comparison, [31]). Let \( L \) be a plane, \( x_0 \in L, u \in C^\infty(U; L^2) \) where \( U \subset L \) is an open neighborhood of \( L \cap \partial B_\rho(x_0) \) and assume \( |Du| \leq C \) on \( U \). Then there exists a function \( w \in C^\infty(B_\rho(x_0) \cap L; L^2) \) such that
\[
w = u, \quad \partial_\nu w = \partial_\nu u \quad \text{on} \quad \partial B_\rho(x_0) \cap L,\]
\[
\left\| w \right\|_{L^\infty(B_\rho(x_0) \cap L)} \leq c(n) \left( \left\| u \right\|_{L^\infty(\partial B_\rho(x_0) \cap L)} + |Du|_{L^\infty(\partial B_\rho(x_0) \cap L)} \right),
\]
\[
\left\| Dw \right\|_{L^\infty(B_\rho(x_0) \cap L)} \leq c(n) |Du|_{L^\infty(\partial B_\rho(x_0) \cap L)},
\]
\[
\int_{(B_\rho(x_0) \cap L)} |D^2 w|^2 \leq c(n) \rho \int_{\text{graph} \ (w\mid L \cap \partial B_\rho(x_0)))} |A|^2 \, dH^1,
\]
where \( \partial_\nu \) denotes the normal outward derivative and \( A \) is the second fundamental form of graph \((w)\).

Now we can prove the regularity result. We will make use of arguments in [34], so we will mainly focus on the differences that arise in our problem, namely the additional area term and the confinement in \( \Omega \). The feeling is that this method is very well applicable for functionals given by the sum of the Willmore energy and some lower order term.

Theorem 3.3 (Regularity). If \( \Lambda > 0 \) is sufficiently small (depending on \( \Omega \)), then there exists an embedded surface \( \Sigma \subset \Omega \) of class \( C^{1,\alpha} \cap W^{2,2} \) such that \( \mathcal{W}_\Lambda(\Sigma) = C_\Lambda \). Moreover the surface \( \Sigma \cap \Omega \) is of class \( C^\infty \).

Let us briefly illustrate the strategy of the proof of Theorem 3.3. We are going to consider a minimizing sequence \((\Sigma_n)\) for the functional \( \mathcal{W}_\Lambda \) with \( \Lambda < \Lambda_0 \) converging to some varifold \( V \). For \( \Lambda \) small enough we will have that \( \mathcal{W}(\Sigma_n) \leq 8\pi - \delta \) and \( \delta \) has multiplicity 1. The analysis of the regularity of the support of \( V \) is divided into two steps. We can distinguish finitely many points \( \xi_1, \ldots, \xi_P \in \text{supp} \, V \), called bad points, that are points that can accumulate energy in the limit. First we will study the regularity at points \( p \in \text{supp} \, V \setminus \{\xi_1, \ldots, \xi_P\} \), in fact around such points we will be able to apply the Graphical Decomposition Lemma 3.1. The graphical decomposition will be applied to any \( \Sigma_n \) of the minimizing sequence around the same chosen point \( p \in \text{supp} \, V \setminus \{\xi_1, \ldots, \xi_P\} \); in such a way we will be able to replace controlled pieces of \( \Sigma_n \) with graphs by Lemmas 3.2. The minimizing property of the sequence \((\Sigma_n)\) thus yields inequalities by comparing \((\Sigma_n)\) with the modified sequence. This will lead to the decay inequality [28], that readily implies \( C^{1,\alpha} \) regularity around the good point \( p \). Then a bootstrap argument based on the elliptic equation satisfied by critical points gives \( C^\infty \) regularity of \( \text{supp} \, V \setminus \{\xi_1, \ldots, \xi_P\} \) inside \( \Omega \) and \( C^{1,\alpha} \cap W^{2,2} \) of \( \text{supp} \, V \setminus \{\xi_1, \ldots, \xi_P\} \) in \( \Omega \). The study of the regularity around a chosen bad point \( \xi \) is similar in the spirit, but more careful. By uniform bounds one can identify around \( \xi \) a ball \( B_r(\xi) \) such that the convergence \( \Sigma_n \to V \) is smooth outside such ball; controlling the oscillation of the tangent planes in suitable annular regions around \( \Sigma_n \cap \partial B_r(\xi) \), we will replace part of \( \Sigma_n \cap B_r(\xi) \) with suitable controlled discs, and we will argue again by comparison with the original minimizing sequence. This yields estimates completely analogous to the ones of the first case, and one derives regularity around the bad point as well.

Proof of Theorem 3.3. For \( \Lambda < \Lambda_0 \), let us consider a minimizing sequence \((\Sigma_n)\) for \( \mathcal{W}_\Lambda \) converging in the sense of varifolds to \( V \in \mathcal{V}(\Omega) \). For a given \( \varepsilon > 0 \), we say that a point \( \xi \in \mathbb{R}^3 \) is a bad point if
\[
\lim_{\rho \searrow 0} \liminf_{n \to +\infty} \int_{\Sigma_n \cap B_\rho(\xi)} |A_n|^2 > \varepsilon^2,
\]
where \( A_n \) is the second fundamental form of \( \Sigma_n \) and \( |A_n| \) is its norm. If a point \( \xi \in \mathbb{R}^3 \) is not a bad point we then call it a good point. Now we show that there is only a finite number of bad points.

Fix some \( \Lambda < \Lambda_0 \). We know from Remark 2.10 that there exists sequences \((\Sigma_n^\Lambda)\) and \((\Sigma_n^\lambda)\) that are minimizing respectively for the parameters \( \Lambda \) and \( \lambda \) and they converge in the sense of varifolds, and then:
\[
\lim_n |\Sigma_n^\Lambda| \leq \lim_n |\Sigma_n^\lambda| \equiv m(\Lambda).
\]

In the following we choose \( \Lambda < \min \{\Lambda_0, 4\pi / m(\Lambda)\} \), so that \( \Lambda \lim_n |\Sigma_n^\lambda| < 4\pi \). Hence, since \( C_\Lambda < 4\pi \), we get that \( \mathcal{W}(\Sigma_n^\lambda) \leq 8\pi - \delta \) for \( n \) big enough and some \( \delta > 0 \). This implies that \( \Sigma_n^\lambda \) is embedded by Theorem 1.4 for \( n \) big enough and that the genus of \( \Sigma_n \) is bounded, in fact the minimum Willmore energy at genus \( g \) is less than \( 8\pi \) and converges to \( 8\pi \) as \( g \to \infty \) (see [13]).

The above discussion has also another consequence: let \( g_n \) be the genus of \( \Sigma_n \), then \( g_n \in \{0, 1, \ldots, g\} \) for some \( \tilde{g} \in \mathbb{N} \) big enough. Hence there is a convergent subsequence \( g_{n_j} \). This means that \( g_{n_j} \) is constant for \( j \) big enough. Then replacing \( \Sigma_n \) with \( \Sigma_{n_j} \) we get a minimizing sequence that has definitely constant genus. Hence we can assume without loss of generality that \( \Sigma_n \) has fixed genus \( g \) for all \( n \).
Another consequence is that, since by lower semicontinuity we have $W(V) < 8\pi$, then $V$ has multiplicity $1 \mu_V$-almost everywhere by Remark 3.2.

We can apply Gauss-Bonnet Theorem to get:

$$\frac{1}{4} \int_{\Sigma_n} |A_n|^2 = W(\Sigma_n) - \frac{\pi}{2}(2 - 2g),$$

with $g$ the genus of $\Sigma_n$ (the same for all $n$). Being $\Sigma_n$ minimizing, we have that $\int_{\Sigma_n} |A_n|^2$ is bounded. So if $N$ is the number of bad points related to $\epsilon > 0$, we get:

$$N \epsilon^2 \leq \liminf_n \int_{\Sigma_n} |A_n|^2,$$

giving an upper bound on $N$ in term of $\epsilon$.

Moreover, by modifying the minimizing sequence with small perturbations, we can assume that $\Sigma_n \subset \Omega$ for every $n$ without loss of generality.

The monotonicity formula ([34] Equation (1.3), or Appendix B in [21]) implies that for any $n, p \in \Sigma_n$.

$$\frac{\left|\Sigma_n \cap B_p(p)\right|}{\rho^2} \leq \frac{3}{2} W(\Sigma_n),$$

for any $\rho > 0$ and any $p \in \Sigma_n$. Hence we can take $\beta = 12\pi > \frac{3W(\Sigma_n)}{2}$ in Lemma 3.3 and let $\epsilon_0$ be the corresponding number given by such lemma.

Let us fix an arbitrary $\epsilon \in (0, \epsilon_0)$; from now on we will call $\xi_1, ..., \xi_p$ the bad points related to such $\epsilon$.

For any $\xi \in supp(V) \setminus \{\xi_1, ..., \xi_p\}$ we can select $\rho(\xi, \epsilon) > 0$ such that for all $\rho' \leq \rho(\xi, \epsilon)$ we have $\int_{\Sigma_n \cap B_{\rho}(\xi)} |A_n|^2 \leq \epsilon^2$ for infinitely many $n$; hence the last part of Lemma 3.3 is applicable to $\Sigma_n$ in $B_{\rho}(\xi)$ for infinitely many $n$. Also by (23) we get that there exists $r \in (0, \rho')$ such that $|\partial B_r(p) \cap \Sigma_n| \leq 3\beta r$. Taking $\rho = \min\{\rho', r\}$ we can Lemma 3.1 for $n$ large enough with $\theta$ small enough fixed (independent of $n, \epsilon, \xi$). We deduce that only one of the discs $D^{(n)}_j$, given by Lemma 3.1, can intersect the ball $B_{\theta r}(\xi)$. Also, for infinitely many $n$ we know that there exist a plane $L_n$ containing $\xi$ and a $C^\infty(\Omega_n)$ function $u_n : \Omega_n \to L_n^\perp$ such that:

$$\frac{|u_n|}{\rho} + |Du_n| \leq Ce^{1/6},$$

$$(\text{graph } u_n \cup_j P_{n,j}) \cap B_{\rho}(\xi) = D^{(n)}_j \cap B_{\rho}(\xi),$$

where each $P_{n,j}$ is diffeomorphic to a closed disc disjoint from graph $(u_n|_{\Omega_n})$ and $\sigma \in (\theta r/2, \theta r)$ is independent of $n$.

Now let us consider $C_\sigma(\xi) := \{x + y|x \in B_{\rho}(\xi) \cap L_n, y \in L_n^\perp\}$; by the Selection Principle 3.2 there exists a set $T \subset (\theta r/2, \theta r)$ of measure $\geq \theta r/8$ such that for each $\sigma \in T$ we have $\partial C_\sigma(\xi) \cap P_{n,j} = \emptyset$ for infinitely many $n$. Hence for infinitely many $n$ we can apply Lemma 3.2 on $D^{(n)}_j \cap B_{\rho}(\xi)$ to get a function $w_n$ on $B_{\sigma}(\xi) \cap L_n$ such that:

$$\int_{L_n \cap B_{\rho}(\xi)} |D^2 w_n|^2 \leq C\sigma \int_{\Gamma_n} |A_n|^2 d\mathcal{H}^1,$$

with $\Gamma_n = \text{graph } (w_n|_{L_n \cap \partial B_{\rho}(\xi)})$ (the integration on subsets of planes $L_n$ is always understood with respect to the Lebesgue measure on such planes).

Let $\hat{A}_n$ be the second fundamental form of graph $w_n$, in particular we have:

$$\int_{\text{graph}(w_n)} |\hat{A}_n|^2 \leq C\sigma \int_{\Gamma_n} |A_n|^2 d\mathcal{H}^1.$$

Note that by the estimates in Lemma 3.2 by choosing $\sigma$ sufficiently small (depending on $\xi$), we can assume graph $w_n \subset \Omega$. Then the $C^{1,1}$ surface $\Sigma_n := (\Sigma_n \setminus (D^{(n)}_j \cap B_{\rho}(\xi))) \cup \text{graph } w_n$ is such that $W_\lambda(\Sigma_n) \leq W_\lambda(\Sigma_n) + \epsilon_n$ for some $\epsilon_n \searrow 0$. Now we argue like in 3.3, except that here we have to control the area term in the energy.

Since $\Sigma_n$ has the same genus of $\Sigma_n$, by the Gauss-Bonnet Theorem we also get:

$$\epsilon_n + \int_{\Sigma_n} \left(\frac{1}{4} |\hat{A}_n|^2 - \Lambda\right) \geq \int_{\Sigma_n} \left(\frac{1}{4} |A_n|^2 - \Lambda\right).$$
So we have that:

\[
\int_{D(\rho) \cap B(\xi)} \frac{1}{4} |A_n|^2 \leq \epsilon_n + \int_{\text{graph}(u_n)} \frac{1}{4} |\tilde{A}_n|^2 \ d\mathcal{H}^2 + \Lambda \left( \int_{D(\rho) \cap B(\xi)} \ d\mathcal{H}^2 - \int_{\text{graph}(u_n)} \ d\mathcal{H}^2 \right).
\]

Using Equations (24), let us estimate:

\[
\int_{D(\rho) \cap B(\xi)} d\mathcal{H}^2 \leq \int_{\pi_{\mathcal{L}}(\text{graph}(u_n) \cap B(\xi))} \sqrt{1 + |Du_n|^2} \ d\mathcal{L}^2 + \sum_j |P_{n,j}|
\]

\[
\leq \sqrt{1 + \epsilon\theta^3/\sigma} |\pi_{\mathcal{L}}(\text{graph}(u_n) \cap B(\xi))| + C\rho^2 \leq (\sqrt{1 + \epsilon\theta^3/\sigma} + C) \rho^2 =: \frac{1}{4} a \rho^2.
\]

Hence:

\[
\int_{D(\rho) \cap B(\xi)} |A_n|^2 \leq 4\epsilon_n + a\rho^2 + \int_{\text{graph}(u_n)} |\tilde{A}_n|^2 \leq 4\epsilon_n + a\rho^2 + C\sigma \int_{\Gamma_n} |A_n|^2 d\mathcal{H}^1.
\]

That is:

\[
\int_{\Sigma_n \cap B(\xi)} |A_n|^2 \leq 4\epsilon_n + a\rho^2 + C \int_{\partial(D(\rho) \cap B(\xi))} |A_n|^2 d\mathcal{H}^1.
\]

(25)

Since \( \sigma \) was selected arbitrarily from the set \( T \) of measure \( \geq \theta \rho/8 \) in the interval \( (\theta \rho/2, \theta \rho) \) we can arrange that:

\[
\int_{\partial(D(\rho) \cap B(\xi))} |A_n|^2 d\mathcal{H}^1 \leq \frac{4}{\sigma} \int_{\Sigma_n \cap B(\xi)} |A_n|^2
\]

for infinitely many \( n \). So using Equation (26), for all \( \rho \leq \theta \rho(\epsilon, \xi) \) we get:

\[
\int_{\Sigma_n \cap B(\xi)} |A_n|^2 \leq 4\epsilon_n + a\rho^2 + C \int_{\Sigma_n \cap B(\xi)} |A_n|^2.
\]

Adding \( C \) times the left side we obtain:

\[
\int_{\Sigma_n \cap B(\xi)} |A_n|^2 \leq \epsilon_n + a\rho^2 + \gamma \int_{\Sigma_n \cap B(\xi)} |A_n|^2,
\]

where \( \gamma = \frac{C}{1 + t} \in (0, 1) \) and we named \( \epsilon_n \) and \( a \) respectively the quantities \( \frac{4}{t+1} \epsilon_n \) and \( \frac{a}{t+1} \).

Defining:

\[
\psi(\rho, \xi) := \liminf_n \int_{\Sigma_n \cap B(\xi)} |A_n|^2,
\]

(26)

we get the following decay relation:

\[
\psi\left(\frac{\rho}{2}, \xi\right) \leq \gamma \psi(\rho, \xi) + a\rho^2.
\]

(27)

Now let us observe that if \( \xi \in \text{supp}(V) \setminus \{\xi_1, \ldots, \xi_P\} \), we can take:

\[
\rho(\xi, \epsilon) = \frac{\rho(\xi_0, \epsilon)}{2}
\]

for all \( \xi \in \text{supp}(V) \cap B(\xi_0) \). Hence, fixed \( \xi_0 \in \text{supp}(V) \setminus \{\xi_1, \ldots, \xi_P\} \), Equation (27) holds for all \( \xi \in \text{supp}(V) \cap B(\xi_0) \) and for all \( \rho \leq \theta(\rho(\xi, \epsilon)/2 := \rho_0 \). The constant \( C \) defining \( \gamma \) is the one given by Lemma 3.2 so we can choose it arbitrarily big in order to get \( \gamma = \frac{C}{1 + t} \in (1/2, 1) \) and \( \alpha = \frac{a}{1 + t} \in (0, 1/8) \). Hence given \( \xi_0 \in \text{supp}(V) \setminus \{\xi_1, \ldots, \xi_P\} \) we can apply Corollary (4.8) to get:

\[
\psi(\rho, \xi) \leq C \left(\frac{\rho}{\rho_0}\right)^{\beta} \psi(\rho_0, \xi) \leq C \left(\frac{\rho}{\rho_0}\right)^{\beta} \psi(\rho(\xi_0, \epsilon), \xi_0)
\]

\[
\forall \xi \in \text{supp}(V) \cap B(\xi_0), \forall \rho \leq \rho_0 := \theta(\rho(\xi_0, \epsilon)/2,
\]

(28)

for some \( C > 0, \beta \in (0, 1) \), where second inequality holds since \( \psi(\rho_0, \xi) \leq \psi(\rho(\xi_0, \epsilon), \xi_0) \). Hence we ultimately got the key decay relation on the second fundamental form (the same of Equation (3.2) in [44]). So following the same arguments in [44] (page 301) one gets that the varifold \( V \) has a multiplicity 1 tangent plane at
each point \( \xi \in \text{supp}(V) \cap B_p(\xi_0) \) with a normal vector \( N(\xi) \) such that \( \|N(\xi_1) - N(\xi_2)\| \leq C|\xi_1 - \xi_2|^{\alpha} \) for all admissible \( \xi_1, \xi_2 \). Also this means that if \( U \) is a sufficiently small neighborhood of \( \xi_0 \), we have:

\[
\mu_V \cap U = \mathcal{H}^2 \cap (\Sigma \cap U),
\]

where \( \Sigma \) is a \( C^{1,\alpha} \) surface. Moreover from (38) one gets \( \int_{\Sigma \cap B_p(\xi)} \mathcal{H}^2 \leq C\rho^\alpha \) for \( \xi \) that is not a bad point, and this decay implies that \( \Sigma \) is a \( C^{1,\alpha} \cap W^{2,2} \) surface away from the bad points \( \xi_1, ..., \xi_P \).

Now we improve the regularity of \( \Sigma \) up to \( C^\infty \) around points contained in \( \Omega \) and different from the bad ones. This will be one of the main differences with (34) page 310-311, or Theorem 3.4 in [21] for a detailed proof), and hence in particular we get that \( \Sigma \) is connected.

At this point we know that \( \supp(V) = \Sigma \cup \{\xi_1, ..., \xi_P\} \), with \( \Sigma \) that is a \( C^{1,\alpha} \cap W^{2,2} \) surface (and \( C^\infty \) in \( \Omega \setminus \{\xi_1, ..., \xi_P\} \)).

From now on we will rename \( \Sigma \) the union \( \Sigma \cup \{\xi_1, ..., \xi_P\} \), so that \( \supp(V) = \Sigma \), keeping in mind that the regularity of \( \Sigma \) is achieved away from the bad points.

Since we chose \( \Lambda < \Lambda_0 \) we know that \( C_\Lambda > 0 \) and thus we can assume that \( \Sigma_n \) is connected for any \( n \). Together with boundedness of the Willmore energy, this implies that the sets \( \Sigma_n \) converge to \( \Sigma \) in the Hausdorff distance \( d_\mathcal{H} \) (see (34) page 310-311, or Theorem 3.4 in [21] for a detailed proof), and hence in particular we get that \( \Sigma \) is connected.

Now we are going to derive the regularity also in neighborhoods of the bad points. By the very same arguments of (34), pages 313-316, one can find distinct points \( y_1, ..., y_{M+P} \in \Sigma \) with \( y_{M+i} = \xi_i \) for \( i = 1, ..., P \) and radii \( \tau_k \) for \( k = 1, ..., M + P \) such that \( \Sigma \subseteq \bigcup_{k=1}^{M+P} B_{\tau_k}(y_k) \) and for each \( k \neq l \) we have that \( \partial B_{\tau_k}(y_k) \cap \Sigma \) and \( \partial B_{\tau_l}(y_l) \cap \Sigma \) are either disjoint or intersect transversely and:

\[
\partial B_{\tau_k}(y_k) \cap \partial B_{\tau_l}(y_l) \cap \partial B_{\tau_m}(y_m) \cap \Sigma = \emptyset
\]

to distinct \( k, l, m \). Moreover the curves \( \Gamma_l := \left( \Sigma \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right) \right) \cap \partial B_{\tau_l}(y_l) \) for \( l = 1, ..., M + P \) divide \( \Sigma \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right) \) into polygonal regions \( R_1, ..., R_Q \). And letting for all \( l = 1, ..., Q \)

\[
R_l := \left\{ x + z | x \in R_l, z \in (T_x \Sigma)^\perp, |z| \leq \theta \frac{\delta}{4}\right\},
\]

for some \( \delta \), it turns out that \( \Sigma_n \cap R_l \) is \( C^{1,\alpha} \)-diffeomorphic to \( R_l \) and then

\[
\Sigma_n \setminus \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k)
\]
is \( C^{1,\alpha}\)-diffeomorphic to \( \Sigma \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right) \) for \( n \) big enough (up to subsequence).

So we can now take surfaces \( \tilde{\Sigma}_n \) such that:

\[
\tilde{\Sigma}_n \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right), \quad \Sigma_n \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right), \quad \Sigma \setminus \left( \bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k) \right)
\]

are \( C^{1,\alpha}\)-diffeomorphic for all \( n \), and:

\[
\tilde{\Sigma}_n \cap V_k = \Sigma_n \cap V_k \quad \forall k = M + 1, \ldots, M + P
\]

for some neighborhood \( V_k \) such that \( B_{\tau_k}(y_k) \subset V_k \subset B_{2\tau_k}(y_k) \), and:

\[
\tilde{\Sigma}_n \setminus \left( \bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k) \right) = \Sigma \setminus \left( \bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k) \right) \quad \forall n,
\]

and also:

\[
\int_{\tilde{\Sigma}_n \setminus (B_{2\tau_k}(y_k) \setminus B_{\tau_k}(y_k))} |\vec{A}_n|^2 \leq C\epsilon^2,
\]

where \( \vec{A}_n \) is the second fundamental form of \( \tilde{\Sigma}_n \).

By (30) and the minimizing property of \( \Sigma_n \) we have:

\[
\int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |H_n|^2 \leq \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |\vec{H}_n|^2 + \Lambda |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))| + \Lambda |\tilde{\Sigma}_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))| + \epsilon_n,
\]

with \( \epsilon_n \to 0 \). Then by (31) and (32) we obtain:

\[
\int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |H_n|^2 \leq \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |H|^2 + \epsilon_n + C\epsilon^2 + \\
+ \Lambda (\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))) - |\tilde{\Sigma}_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))|).
\]

Using the hypotheses on the surfaces \( \tilde{\Sigma}_n \), let us estimate the quantity:

\[
|\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))| - |\tilde{\Sigma}_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))| \\
\leq |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| - |\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| + \\
+ \sum_{k=M+1}^{M+P} |\Sigma_n \cap B_{\tau_k}(y_k) \setminus (B_{\tau_k}(y_k) \cup V_k)| + \\
- |\bigcup_{k=M+1}^{M+P} \tilde{\Sigma}_n \cap B_{2\tau_k}(y_k) \setminus (B_{2\tau_k}(y_k) \cup V_k)| \\
\leq |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| - |\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| + \\
+ \sum_{k=M+1}^{M+P} |\Sigma_n \cap B_{2\tau_k}(y_k) \setminus (B_{\tau_k}(y_k) \cup V_k)| + \\
\leq |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| - |\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| + \\
+ \sum_{k=M+1}^{M+P} |\Sigma_n \cap B_{2\tau_k}(y_k)|.
\]

Since this is true for all \( \epsilon > 0 \), we get:

\[
\int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |H_n|^2 \leq \int_{\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{\tau_k}(y_k))} |H|^2 + \epsilon_n + \\
+ \Lambda \left( |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| - |\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2\tau_k}(y_k))| + \\
+ \sum_{k=M+1}^{M+P} |\Sigma_n \cap B_{2\tau_k}(y_k)| \right).
\]
Since we already know that by the convergence of varifolds we also have \(|\Sigma_n| \rightarrow M(\Sigma) = |\Sigma|\), then we get:

\[
\limsup_n \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |H_n|^2 \leq \int_{\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |H|^2 + \\
+ \limsup_n \Lambda \left( |\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))| - |\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))| \right) + \\
+ \sum_{k=M+1}^{M+P} |\Sigma_n \cap B_{2r_k}(y_k)|,
\]

Hence finally:

\[
\lim\limsup_{\sigma \searrow 0} \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |H_n|^2 \leq \lim_{\sigma \searrow 0} \int_{\Sigma \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |H|^2 = \int_{\Sigma} |H|^2.
\]  (33)

Combining this with the natural lower semicontinuity of the Willmore functional under varifold convergence, we establish that:

\[
|H_n|^2 H^2 \downarrow_{\Sigma} |H|^2 H^2 \downarrow_{\Sigma}
\]  (34)

as measures on the domain \(\mathbb{R}^3 \setminus \{\xi_1, ..., \xi_P\}\).

Moreover, with the above notation, we have by the Gauss-Bonnet Theorem that:

\[
\int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |\tilde{H_n}|^2 - \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |H_n|^2 \\
= \frac{1}{4} \left( \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |\tilde{A_n}|^2 - \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |A_n|^2 \right),
\]

then same conclusions hold for the second fundamental form, that is:

\[
\lim\limsup_{\sigma \searrow 0} \int_{\Sigma_n \setminus (\bigcup_{k=M+1}^{M+P} B_{2r_k}(y_k))} |A_n|^2 \leq \int_{\Sigma} |A|^2,
\]  (35)

and:

\[
|A_n|^2 H^2 \downarrow_{\Sigma} |A|^2 H^2 \downarrow_{\Sigma}
\]  (36)

as measures on the domain \(\mathbb{R}^3 \setminus \{\xi_1, ..., \xi_P\}\).

Finally we prove the claimed regularity of the varifold \(\Sigma\), that is regularity in the bad points. According to the above discussion, let us sum up some useful results. For each \(\delta > 0\) sufficiently small there is \(\sigma \in (\delta/2, \delta)\) such that:

\[
\limsup_n \int_{\Sigma_n \setminus (\bigcup_{i=1}^P B_{2r}(\xi_i) \setminus B_\sigma(\xi_i))} |A_n|^2 \leq \delta^2,
\]  (37)

\[
\Sigma \setminus \left( \bigcup_{i=1}^P B_{\delta}(\xi_i) \right) \text{ is } C^{1,\alpha}-\text{diffeomorphic to } \Sigma \setminus \left( \bigcup_{i=1}^P B_{\sigma}(\xi_i) \right),
\]  (38)

\[
\left| W\left( \Sigma_n \setminus \left( \bigcup_{i=1}^P B_\delta(\xi_i) \right) \right) - W(\Sigma) \right| \leq \delta^2.
\]  (39)

In particular choosing appropriate \(\delta_n \searrow 0\) and then \(\sigma_n \in (\delta_n/2, \delta_n)\), for all \(i = 1, ..., P\) we have:

\[
\lim_n W\left( \Sigma_n \setminus \left( \bigcup_{i=1}^P B_\sigma(\xi_i) \right) \right) = W(\Sigma).
\]  (40)

By Equation (33) we have that for \(\sigma\) small enough \(\Sigma \cap \{B_\sigma(\xi_i) \setminus B_\delta(\xi)\text{ is } C^1\text{-close to an annulus } L_1 \cap B_\sigma(\xi_i) \setminus B_\sigma(\xi_i)\text{.}\}

Hence we can take a smooth compact surface \(\tilde{\Sigma}\) such that, for suitable points \(y_1, ..., y_P \in \tilde{\Sigma}\) and sufficiently small \(\sigma\), \(\tilde{\Sigma} \setminus (\bigcup_{i=1}^P B_{\sigma}(y_i)) \text{ is } C^{1,\alpha}-\text{diffeomorphic to } \Sigma \setminus (\bigcup_{i=1}^P B_\sigma(\xi_i))\text{ and such that, for } \sigma = \sigma_n \text{ as above small enough, it is}

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possible to replace $\tilde{\Sigma} \cap B_{\sigma_n}(y_i)$ by a slight deformation of $\Sigma \cap B_{\sigma_n}(\xi_i)$ followed by a rigid motion to give $(\Sigma \cap B_{\sigma_n}(\xi_i))^*$ such that the surface

$$\Sigma_n := \left( \Sigma \setminus \left( \bigcup_{i=1}^{P} B_{\sigma_n}(y_i) \right) \right) \cup \left( \bigcup_{i=1}^{P} (\Sigma \cap B_{\sigma_n}(\xi_i))^* \right)$$

is $C^{1,\alpha} \cap W^{2,2}$ and

$$W_\Lambda((\Sigma \cap B_{\sigma_n}(\xi_i))^*) \leq W_\Lambda(\Sigma \cap B_{\sigma_n}(\xi_i)) + \epsilon_n \quad \epsilon_n \searrow 0. \quad (41)$$

Using the minimizing property of $\Sigma_n$ and then (41), we have:

$$W_\Lambda(\Sigma_n) = W_\Lambda(\Sigma \cap \left( \bigcup_{i} B_{\sigma_n}(\xi_i) \right)) + W_\Lambda \left( \Sigma \setminus \left( \bigcup_{i} B_{\sigma_n}(\xi_i) \right) \right) \leq W_\Lambda(\tilde{\Sigma}_n) + \epsilon_n \quad (42)$$

Hence:

$$W_\Lambda \left( \Sigma \setminus \left( \bigcup_{i} B_{\sigma_n}(\xi_i) \right) \right) \leq W_\Lambda \left( \tilde{\Sigma} \setminus \left( \bigcup_{i} B_{\sigma_n}(y_i) \right) \right) + (P + 1)\epsilon_n,$$

and by (40) we get:

$$W_\Lambda(\Sigma) \leq W_\Lambda(\tilde{\Sigma}). \quad (43)$$

Analogously, using Gauss-Bonnet Theorem on the first inequality in (42), being $\Sigma_n$ and $\tilde{\Sigma}_n$ dieromorphic, we find:

$$\int_{\Sigma} \left( |A|^2 - \Lambda \right) \leq \int_{\Sigma} \left( |\hat{A}|^2 - \Lambda \right). \quad (44)$$

Constructing $\tilde{\Sigma}$ taking a small perturbation of $\Sigma$ (so that no bad points lie on $\partial \Omega$) and replacing $\Sigma \setminus B_{\sigma}(\xi)$ with the graph of the function given by Lemma 3.2 by (41) we get the estimate

$$\int_{\Sigma \cap B_{\sigma}(\xi)} |A|^2 \leq C \rho^n + \Lambda(|\Sigma \cap B_{\sigma}(\xi)| - |\tilde{\Sigma} \cap B_{\sigma}(\xi)|) \leq C \rho^n,$$

for sufficiently small $\rho$ for some $\alpha > 0$. Hence actually:

$$\int_{\Sigma \cap B_{\sigma}(y)} |A|^2 \leq C \rho^\alpha$$

for $\rho$ small enough and for all $y \in \Sigma$, now bad points included. And by classical arguments similar to the ones applied above in the case of good points one can show that this imply that $\Sigma$ is a $C^{1,\alpha} \cap W^{2,2}$ surface globally (and $\Sigma \cap \Omega$ is of class $C^\infty$). In particular, arguing by approximation, we have $W_\Lambda(\Sigma) \geq C_\Lambda$ and then by lower semicontinuity $W_\Lambda(\Sigma) = C_\Lambda$ and by the upper bound on the Willmore energy we also conclude that $\Sigma$ is embedded. \hfill \Box

We conclude this section with some observations on the proof of Theorem 3.3.

Remark 3.4 (Smallness of $\Lambda$). The fundamental hypothesis of Theorem 3.3 is to take the weight $\Lambda$ sufficiently small. Observe that if $C_{\Lambda_0} = 0$ and if there exists a minimizing sequence $(\Sigma_n^{A_0})$ for $W_{\Lambda_0}$ with equibounded areas $|\Sigma_n^{A_0}|$ (i.e. if $W$ is coercive, by Corollary 2.13), then in the proof of Theorem 3.3 we can take

$$\Lambda < \frac{4\pi}{m(\Lambda_0)}, \quad (45)$$

with

$$m(\Lambda_0) = \lim_{n} |\Sigma_n^{A_0}|, \quad (46)$$

This estimate is sufficient for completing the proof. Of course the value of $m(\Lambda_0)$ depends on the chosen sequence $\Sigma_n^{A_0}$. It is interesting to notice that in the case of $\Omega = B^{\alpha}_2$, where we know that the sphere $S^{\alpha}_2$ of radius 1/2 ia a minimizer for $A_0B^{\alpha}_2 = 4$, we have $m(4) = |S^{\alpha}_2| = \pi$; hence the estimate (45) gives $\Lambda < 4$, that is precisely the critical parameter $A_0B^{\alpha}_2 = 4$, so in this case the estimate is sharp, excluding only the limit case of $W_{\Lambda_0}$.

It could be a future development to prove or disprove the convergence to an enough regular surface for greater parameters $\Lambda$ and in particular for the critical value $\Lambda_0$, perhaps using the more modern theory of [28].
Remark 3.5 (Regularity of the limit surface). We derived the existence of a globally $C^{1,\alpha} \cap W^{2,2}$ surface $\Sigma$ that it is actually $C^\infty$ inside $\Omega$, so if we know that $\Sigma \subset \Omega$ then $\Sigma$ is actually a smooth surface and hence a classical solution of the Problem $(P)_{\Omega,A}$. Also, $\partial \Omega$ is of class $C^2$ by hypothesis, so on each relatively open set $A \subset (\Sigma \cap \partial \Omega)$, the surface is actually $C^2$.

However, we want to notice here that it is not obvious that $\Sigma$ is globally $C^2$. In fact the smoothness of the surface inside $\Omega$ is obtained by the Elliptic Regularity Lemma A.2 used on Equation (29), that is an equation given by the first variation of a functional, so it is something like $\frac{\partial}{\partial t} F(w + t\varphi)|_{t=0} = 0$ for the appropriate functional $F$. While this calculation is possible inside $\Omega$, on $\partial \Omega$ this leads only to a variational inequality of the fourth order subject to an obstacle boundary condition (given by the boundary of $\Omega$), for which the development of a regularity theory is quite more difficult. Very remarkable results are proved in [5], where it is studied the variational inequality of the bilaplacian $\Delta^2$ subject to obstacle boundary conditions; here it is proved that in dimension 2 (that is also our case) the solution in $C^2$. Of course our case is different, since the elliptic operator is nonlinear (recall (29)), but it is likely that we could achieve the same conclusion, having then $\Sigma$ of class $C^2$ globally and $C^\infty$ inside $\Omega$ (hence getting a classical minimizer for the variational problem). This can be another possible development of the work, having also an interest itself in the theory of regularity for elliptic problems.

A Appendix

In the Appendix we collect the technical results used in the proof of Theorem 3.3 and some basic facts about varifold theory.

Lemma A.1 ([34]). Let $\Sigma$ be a compact surface without boundary, let $B_\rho$ be an open ball such that $\partial B_\rho$ intersects $\Sigma$ transversely and $\Sigma \cap B_\rho$ contains disjoint subsets $\Sigma_1, \Sigma_2$ with $\Sigma_1 \cap \partial B_\rho \neq \emptyset$, $\partial \Sigma_2 \subset \partial B_\rho$ and $|\partial \Sigma_j| \leq \beta \rho$ for $j = 1, 2$, where $\theta \in (0, \frac{1}{2})$ and $\beta > 0$. Then

$$W(\Sigma) \geq 8\pi - C\beta \theta,$$

with $C$ independent of $\Sigma, \beta, \theta$.

Lemma A.2 (Selection Principle, [31]). If $\delta > 0$, if $I \subset \mathbb{R}$ is a bounded interval and if $A_j \subset I$ is a measurable set with measure $\geq \delta$ for each $j = 1, 2, ..., $ then there exists a set $S \subset I$ of measure $\geq \delta$ such that each $x \in S$ lies in $A_j$ for infinitely many $j$.

Here we have the results leading to the decay estimate (28). Results of this kind are standard, however usually stated under more general forms; since we needed only the following more simple decay estimates, we prove such inequalities here for the convenience of the reader.

Lemma A.3. Let $f : (0, x_0] \to [0, +\infty)$ such that $f(x_0) > 0$, if is non decreasing and:

$$f\left(\frac{x}{2}\right) \leq \gamma f(x)$$

for all $x \in (0, x_0]$ for some $\gamma \in (0, 1)$. Then there are $C > 0, \beta \in (0, 1)$ such that:

$$f(x) \leq C\left(\frac{x}{x_0}\right)^{\beta} f(x_0)$$

for all $x \in (0, x_0]$.

Proof. For all $x \in (0, x_0/2]$ there is $n$ such that $2^n x := x' \in (x_0/2, x_0]$. Then:

$$f(x) \leq \gamma^n f(x') = 2^{-n \log_2(1/\gamma)} f(x') = \left(\frac{x}{x'}\right)^{\log_2(1/\gamma)} f(x')$$

$$= \left(\frac{x}{x_0}\right)^{\log_2(1/\gamma)} \left(\frac{x_0}{x'}\right)^{\log_2(1/\gamma)} f(x') \leq \frac{1}{\gamma} \left(\frac{x}{x_0}\right)^{\log_2(1/\gamma)} f(x_0).$$

For all $x \in (x_0/2, x_0]$:

$$f(x) = \frac{1}{\gamma} \left(\frac{1}{2}\right)^{\log_2(1/\gamma)} f(x) \leq \frac{1}{\gamma} \left(\frac{x}{x_0}\right)^{\log_2(1/\gamma)} f(x_0).$$

Now if $\log_2(1/\gamma) < 1$ we are done, otherwise, since $x/x_0 \leq 1$ for all $x$, we can choose an arbitrary $\beta \in (0, 1)$ and we have $(x/x_0)^{\log_2(1/\gamma)} \leq (x/x_0)^\beta$.  \qed
Corollary A.4. Let \( f : (0, x_0] \to [0, +\infty) \) such that \( f(x_0) > 0, f \) is non decreasing and:

\[
f\left(\frac{x}{2}\right) \leq \gamma f(x) + \alpha x^2
\]

for all \( x \in (0, x_0] \) for some \( \gamma \in (1/2, 1), \alpha \in (0, 1/8) \). Then there are \( C > 0, \beta \in (0, 1) \) such that:

\[
f(x) \leq C \left(\frac{x}{x_0}\right)^\beta f(x_0)
\]

for all \( x \in (0, x_0] \).

Proof. Let \( h(x) = f(x) + x^2 \). We have:

\[
h\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right) + \frac{x^2}{4} \leq \gamma f(x) + \left(\alpha + \frac{1}{4}\right)x^2 \leq \gamma h(x).
\]

Applying Lemma A.3 and taking \( a > 0 \) such that \( x_0^2 \leq af(x_0) \) we obtain:

\[
f(x) \leq h(x) \leq K \left(\frac{x}{x_0}\right)^\beta h(x_0) = K \left(\frac{x}{x_0}\right)^\beta \left(f(x_0) + x_0^2\right) \leq C \left(\frac{x}{x_0}\right)^\beta f(x_0),
\]

with \( C = K(1 + a) \). \( \square \)

Lemma A.5 (Elliptic Regularity, [34]). Let \( \beta, \gamma, L > 0, B^2 = \{ x \in \mathbb{R}^2 : |x| < 1 \} \) and let

\[
u = (u^1, ..., u^m) \in W^{2,2}(B^2; \mathbb{R}^m) \cap C^{1,\gamma}(B^2; \mathbb{R}^m)
\]

be such that \(|\nu| + |Du| \leq 1 \) and:

\[
\int_{B^2 \cap \{ |x-x_0| < \rho \}} |D^2 u|^2 \leq \beta \rho^{2\gamma}
\]

for each \( \xi \in B^2 \) and \( \rho < 1 \). Moreover suppose that \( u \) is a weak solution of the system:

\[
D_j D_k (A_{ijr}^{rs}(x, u, Du) D_i D_r u^{\beta \delta}) + D_j B_{ij}^\alpha(x, u, Du, D^2 u) + B^\alpha_\alpha(x, u, Du, D^2 u) = 0
\]

where \( A_{ijr}^{rs} = A_{ijr}^{rs}(x, z, p) \) and \( B_{ij}^\alpha(x, z, p, q) \) satisfy:

\[
\sum_{i,j,r,s,\alpha,\beta} A_{ijr}^{rs} c_{ij} c_{rs} \geq L^{-1} \sum_{i,j,\alpha} |c_{ij\alpha}|^2,
\]

\[
|A_{ijr}^{rs}(x, z, p)| \leq L, \quad |D_{(x,z,p)} A_{ijr}^{rs}(x, z, p)| \leq L,
\]

\[
|B_{ij}^\alpha(x, z, p, q) + |D_{(x,z,p)} B_{ij}^\alpha(x, z, p, q)| \leq L(1 + |q|^2),
\]

\[
|D_{(x,z,p)} B_{ij}^\alpha(x, z, p, q)| \leq L(1 + |q|),
\]

for all \(|z| + |p| \leq 1 \) where \( D_P F \) means the tensor of all first derivatives with respect to the variables \( P \).

Then \( u \in W^{1,2}_{\text{loc}}(B^2) \cap C^{2,\alpha} \).

Finally, we list some facts about theory of varifolds that we used in the work.

Theorem A.6 (Compactness of Varifolds, [1] and [33]). Let \( V_n = \nu(M_n, \theta_n) \) be a sequence of 2-rectifiable varifolds in \( U \subset \mathbb{R}^3 \) open such that:

1. \( \sup_n \mu_{\nu_n}(W) + ||\delta V_n||(W) < +\infty \) \( \forall W \subset \subset U \),

2. \( \exists \Theta(V_n, x) \geq 1 \) on \( U \setminus A_n : \mu_{\nu_n}(A_n \cap W) \to 0 \) \( \forall W \subset \subset U \),

where \( \mu_{\nu} \) denotes the Radon measure on \( U \) induced by a varifold \( V \) and \( \delta V \) is its first variation, and where \( \Theta(V, x) := \lim_{y \to 0} \mu_{\nu}(B(x,y)) \). Then there exists a subsequence \( V_{n_k} \) converging to a rectifiable varifold \( V \) with locally bounded first variation with the properties that:

\[
\exists \Theta(\mu_{\nu}, x) \geq 1 \mu_{\nu} - \text{ae in } U,
\]

\[
\liminf_n ||\delta V_n||(W) \geq ||\delta V||(W) \quad \forall W \subset \subset U.
\]

Moreover if each \( V_{n_k} \) is integer, then \( V \) is integer too.
Remark A.7. It is very important to observe that if in Theorem A.6 the varifolds $V_n$ are integer, then the hypothesis (2) is automatically satisfied (with sets such that $\mu_{V_n}(A_n) = 0$).

Also, we remind the concept of $F$-metric ([23], page 66) used in the proof of Theorem 2.11, defined as follows.

**Definition A.8.** The $F$-metric on $V_2(U)$, that is the set of 2-rectifiable integer varifolds with support contained in the open $U \subset \mathbb{R}^3$, is defined as:

$$F(V,W) = \sup \{ V(f) - W(f) : f \in C_c(G_n(\mathbb{R}^{n+k})), |f| \leq 1, \text{Lip}(f) \leq 1 \}.$$  \hspace{1cm} (47)

And we have the useful:

**Lemma A.9** ([23], page 66). In sets $V_2(U) \cap \{ V : M(V) \leq C < +\infty \}$ with $U \subset \mathbb{R}^{n+k}$ open, the convergence of varifolds is equivalent to the convergence in the $F$-metric.

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