Fractal structure of a white cauliflower

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(Dated: February 2, 2008)

Abstract

The fractal structure of a white cauliflower is investigated by box-counting method of its cross-section. The capacity dimension of the cross-section is $1.88 \pm 0.02$ independent of directions. From the result, we predict that the capacity dimension of the cauliflower is about 2.8. The vertical cross-section of the cauliflower is modeled into a self-similar set of a rectangular tree. We discuss the condition of the fractal object in the tree, and show that the vertical cross-section has an angle of 67 degrees in our model.

PACS numbers: 61.43.H, 87.15

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I. INTRODUCTION

A cauliflower is a variety of cabbage with an edible head of condensed flowers and flower stems. It is a form of cabbage in the mustard family, consisting of a compact terminal mass of greatly thickened, modified, and partially developed flower structures, together with their embracing fleshy stalks. This terminal cluster has been known as a typical fractal among living organisms [1].

It is clear that the dimensionality of a fractal is the most fundamental concept of fractal analysis. Nevertheless, the fractal dimension of a white cauliflower has not known yet. Grey and Kjems discussed the fractal dimension of a cauliflower, and they just suggested the possibility that the fractal dimension of a cauliflower could be larger than that of a broccoli [2]. Romera et. al. suggested a mathematical model of a cauliflower as a sequence of a baby Mandelbrot set [3], and its fractal dimension was not obtained, either.

In this article, we measure the fractal dimension of the cross-section of a white cauliflower. Then, the fractal dimension of the bulk cauliflower is introduced. Next, we create a mathematical model for the cauliflower, and compare it with our experimental measurement. The condition of creating the fractal in our mathematical model is discussed, too.

II. FRACTAL DIMENSION OF A CROSS-SECTION

There are many definitions of fractal dimensions, and the basic concept in our model has almost the property as follows: Let us introduce that \( \delta \) is a measurement scale, and that the measurement is \( M_\delta(F) \), then the fractal dimension \( D \) of a set \( F \) is determined by the power law

\[
M_\delta(F) \sim \delta^{-D}.
\]

If \( D \) is a constant as \( \delta \to 0 \), \( F \) has a dimension of \( D \).

The term “capacity dimension” or “box-counting dimension” is most widely used, because it can be easily applied to fractal objects. It is defined as

\[
D = \frac{\log N_\delta(F)}{-\log \delta},
\]

where \( N_\delta(F) \) is the smallest number of sets of diameter at most \( \delta \) which can cover \( F \) [4, 5].

Fractal objects embedded in one- or two-dimension are easy to measure relatively compared to other higher dimensions, but most of fractals in nature that are embedded in three
dimensions is difficult to measure even in the capacity dimension. However, the dimensions of a cross-section is generally known to be related with that of the bulk. Let us think of a bulk of three dimensions, and assume the dimension of its cross-section is two, that is, it is $2/3$ of the bulk. We can propose an ansaz from the above property: Let $D_c$ be a fractal dimension of a cross-section embedded in two dimensions. Then, the fractal dimension of the bulk imbedded in three dimensions can be written as

$$D = \frac{3}{2N} \sum_{i=1}^{N} D_{ci},$$

(3)

where $i$ represents every possible cross-section. As the fractal dimension of the cross-section is independent of directions, Eq. (3) is simply written as $D = (3/2)D_c$.

III. EXPERIMENTAL MEASUREMENT

We prepared several white cauliflowers of about 300g in mass and 15cm in diameter. We cut half of them in horizontal (H) direction, and the other half in vertical (V) direction. Then, we scanned the cross-section by a scanner and read the image into black and white in Fig. 1. Here, the two figures by the two perpendicular directions look totally different.

Next, we read the image into matrix of numbers. The numbers of the matrix are the reflectivity of each pixels. The black backgrounds produces 0’s, and the white images produce large numbers. Let us call the non-zero number 1 for convenience. The size of the matrix is about $M_1 \times M_2 = 1400 \times 1200$. Note that the size of a pixel is an order of 100µm.

In order to measure the fractal dimension of the cross-sections in Eq. (2), we count the non-zero numbers by the box-counting method. It becomes the smallest number of sets to cover the white images by 100µm $\times$ 100µm. In a half reduction procedure of the matrix, a $4 \times 4$ component becomes 1 if it contains any non-zero number, otherwise it becomes 0. Then, the number of 1’s becomes the smallest number of sets to cover the white images by 200µm $\times$ 200µm. And so on. In the reducing steps in half, that is $M_1/2^n \times M_2/2^n$, $n = 1, 2, 3..., the conversion from the reduced matrices to images is plotted in Fig. 2 for the first four steps. We see that the basic structure of the cross-sections remains unchanged.

Fig. 3(a) is the log-log plot for the H-direction, and Fig. 3(b) is for the V-direction. We plotted it for a couple of different cauliflower samples. The slopes are the capacity dimensions of the cross-sections. Surprisingly, we found the two slopes for different directions are very
similar. It is $1.88 \sim 1.90$. Repeating the procedure for different white cauliflowers, we observed similar values of $D_c$ as $1.88 \pm 0.02$ for the two different directions. Therefore, from Eq. (3) we predict that the capacity dimension of the white cauliflower is about 2.8.

IV. MATHEMATICAL MODELING

Converting a fractal found in nature into a corresponding mathematical model is a hope of a theorist, but most of the work is far away from the real world or extremely complicated in analysis. The modeling of nature should be as simple as it can, and at the same time, it should contain the basic structure of nature. Furthermore, if some physical quantities of the model can match the real systems, it will guarantee more credits to the mathematical model.

We modeled the cross-section of a cauliflower in V-direction as a rectangular tree that has three equilateral sides in Fig. 4. This is the simplest model of a self-similar set that has a single scale factor $s$. Note that the cross-section of a broccoli in V-direction has modeled into a self-similar set of triangular tree or Pythagoras tree [6].

The the scale factor $s$ of the rectangular tree in Fig. 4 have the following relation

$$s = \frac{1}{2 \cos \phi + 1},$$  \hspace{1cm} (4)

where $\phi$ is the angle of the rectangular tree. Downsizing the tree by a ratio $s$, it creates three branches. Therefore, the fractal dimension of Fig. 4 is

$$D_c = \frac{\log 3}{- \log s},$$  \hspace{1cm} (5)

The areas of the rectangular tree have the following series

$$1, 3s^2, 9s^4, 27s^6, \ldots (3s^2)^n, \ldots$$  \hspace{1cm} (6)

where $n = 0, 1, 2, 3, \ldots$. Because the fractal is the limit of the series, it should converge as $n$ increases. The condition of convergence of the series is $3s^2 < 1$, or $s < 0.577$. It is clear from $D_c < 2$. This condition corresponds to $\phi < 68.5^\circ$ by Eq. (4).

Since the $D_c$ in the cauliflower is 1.88, the scale factor $s$ and the angle $\phi$ in the model is obtained as 0.56 and 67$^\circ$ from Eqs. (4) and (5). We recognize that the cross-section in V-direction of the cauliflower is pretty close to two dimensions.
As the $\phi$ approaches 68.5°, the dimension of the tree goes to two because the limit bents to cover the two dimensional surface. On the other hand, as the $\phi$ approaches 0°, the dimension of the tree goes to one because the limit goes to a long rod. The head of the real cauliflower is rounded not like the model in Fig. 4. It gives a possibility that the real cauliflower is not a self-similar set of single scale factor, but a set of multi-scale factors.

V. SUMMARY

We measured the fractal dimension of the cross-section of a white cauliflower by the direct scanning method. It was 1.88 ± 0.02, and almost independent of the directions of the cross-sections. From these results, we predict that the fractal dimensions of the bulk cauliflower is about 2.8.

We created a mathematical model for the V-direction of a cauliflower with only one scale factor. It is a rectangular tree of three equilateral sides. We suggested the condition of creating fractals in our model, and we showed that the angle of the model is 67 degrees and the scale factor is 0.56 comparing with experiment. This method of a scanning cross-sections and mathematical model from a polygonal tree can be widely applied to many complex bulk fractals in nature.

Acknowledgments

We send our special thanks to K.S. Kim for useful discussions and J. K. Ahn for graphic assistance.

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FIG. 1: The scanned images of the two cuts of the cauliflower. (a) Horizontal direction, (b) Vertical direction.

FIG. 2: Converted images of the cross-section by the half reduction procedure. The upper one is the horizontal direction (a), and the bottom one is the vertical direction (b). (1) $M_1/2 \times M_2/2$, (2) $M_1/4 \times M_2/4$, (3) $M_1/8 \times M_2/8$, and (4) $M_1/16 \times M_2/16$.

FIG. 3: The log-log plots of the two perpendicular directions for two different cauliflower samples. The slopes are $D'_c$s of the white cauliflower. (a) Horizontal direction, (b) Vertical direction.

FIG. 4: Self-similar set of a rectangular tree with only one scale factor: $s = 0.56$. or $\phi = 67^\circ$
\[ \begin{align*}
\log(N) & \quad -\log(\delta) \\
(a) &
\end{align*} \]

(a)
(b) $\log(N) - \log(\delta)$
