A Whitney type theorem for surfaces: characterising graphs with locally planar embeddings

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August 10, 2020

Abstract
We prove that a graph $G$ embeds $r$-locally planarly in a pseudo-surface if and only if a certain matroid associated to the graph $G$ is co-graphic. This extends Whitney’s abstract planar duality theorem from 1932.

1 Introduction
In 1932 Whitney initiated\footnote{While I see Whitney’s abstract duality theorem as the point of birth of matroid theory, Whitney’s axioms for matroids were only published in 1935, see \[23\]. Independently, matroids were introduced by Nakasawa in 1935, see \[16\].} the systematic study of matroids by proving that a graph can be embedded in the plane if and only if its cycle matroid is co-graphic [22, Theorem 29]. A pseudo-surface is obtained from a surface by identifying finitely many points. While every surface is an example of a pseudo-surface, every embedding of a graph in a pseudo-surface can be extended to an embedding in a genuine surface. We extend Whitney’s duality theorem to embeddings of graphs in general pseudo-surfaces that are just planar locally; meaning that, every cycle of bounded length is generated by the face boundaries. Analogous to the cycle matroid of a graph $G$, we define the local matroid of $G$ whose circuits are those generated by cycles of $G$ of bounded length. We prove the following extension of Whitney’s duality theorem.

**Theorem 1.1.** Given a parameter $r \in \mathbb{N} \cup \{\infty\}$, a graph embeds $r$-locally planarly in a pseudo-surface if and only if its $r$-local matroid is co-graphic.
Remark 1.2. Although every embedding of a graph in a pseudo-surface can be extended to an embedding in a genuine surface, Theorem 1.1 with ‘surface’ in place of ‘pseudo-surface’ is not true as such extensions may lose $r$-local planarity. However, we completely understand all graphs where this phenomenon arises. Indeed, such graphs are not $r$-locally 2-connected; that is, they have a vertex $v$ such that the punctured ball $B_{r/2}(v) - v$ is disconnected. Corollary 5.2 below is a variant of Theorem 1.1 for embeddings in genuine surfaces for $r$-locally 2-connected graphs.

Checking whether a given matroid is graphic can be checked in quadratic time. So a consequence of Theorem 1.1 is that there is a polynomial time algorithm that checks whether a graph has a locally planar embedding.

Locally planar embeddings. The idea of using embeddings that locally look like plane embeddings as a tool has been introduced by Robertson and Seymour as part of their Graph Minors project [17]. Thomassen proved that LEW embeddibility, a particular type of locally planar embeddibility, can be tested in polynomial time, and in fact such embeddings have always minimum genus [21]. Since then locally planar embeddings (and their cousins ‘face width’ and ‘edge width’) have become an essential part of Topological Graph Theory [14].

For example, they are used in the recent project of three-colouring triangle-free graphs on surfaces by Dvořák, Král and Thomas [7]. Very often proving a result for locally planar embeddings is already considered a major step worth publication [3, 6, 11, 12, 13]. For example DeVos and Mohar generalised a theorem of Thomassen by proving that graphs with large edge-width are 5-list-colourable [6].

Duality in surfaces. Abrams and Slilaty have a whole series of papers in which they explore the relationship between surface embeddings, dual graphs and matroids. Based on foundational work of Zaslavsky [24], in [19], Slilaty describes toridal duality and projective duality in terms of lift matroids and bias matroids. In [20] and [1], Abrams and Slilaty characterise embeddability of a graph $G$ in the projective plane via representability of the cycle matroid of $G$ as a bias matroid of a signed graph. In [2], they prove a theorem describing duality in pseudo-surfaces. This inspired the first half of

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2 An example is obtained from a large square-grid by identifying two interior vertices that are far apart. This graph has no locally planar embedding in the torus but in the pseudo-surface obtained from the sphere by identifying two points.

3 See [8] for an overview of algorithms and a short proof that this problem is polynomial.

4 According to [15]

5 According to [21]. LEW embeddings were introduced by Hutchingson in [10].
the proof of Lemma 3.4. Slilaty [20] asked the following: ‘Planarity of graphs precisely determines the intersection of the class of cographic matroids with the class of graphic matroids. But what happens when [the graph] $G$ is nonplanar?’

We derive Theorem 1.1 as a corollary of a general duality theorem for surfaces, Theorem 5.1 below, which provides an answer to Slilaty’s question. Indeed, given a graph $G$ and a set $S$ of its cycles, an embedding of $G$ in a surface is $S$-facial if every cycle in $S$ is generated by the face boundaries.

**Example 1.3.** Every embedding is an $S$-facial embedding, where $S$ is the set of cycles generated by the face boundaries. For $S$ equal to the cycle space of $G$, $S$-facial embeddings of $G$ are just plane embeddings.

In Theorem 5.1, we show that (under certain necessary conditions) $G$ has an $S$-facial embedding if and only if a certain matroid is co-graphic.

## 2 Duals of $S$-facial embeddings

In this section we prove Lemma 2.13 below, which is used in the proof of the main result stated in the Introduction. This lemma expresses duality in surfaces in terms of matroids.

Fix a field $k$ to be one of the finite fields $\mathbb{F}_2$ or $\mathbb{F}_3$.

**Remark 2.1.** To investigate embeddings in general surfaces, take $k = \mathbb{F}_2$. For embeddings in orientable surfaces, take $k = \mathbb{F}_3$.

Given a graph $G$ embedded in a surface, a cycle of $G$ is *facially generated* if it is a sum of face boundaries over the field $k$, where all coefficients of the sum are plus one.

**Example 2.2.** Cycles that are face boundaries are facially generated. More generally cycles bounding discs are facially generated. The double torus has a cycle that separates its two handles; this cycle is facially generated. On the other hand, cycles cutting a handle are not facially generated. Contractible cycles are facially generated over $\mathbb{F}_2$. The converse is not true as the double torus has a cycle that is facially generated (that is homologically trivial over $\mathbb{F}_2$) but not contractible (that is homotopically trivial).

Given a graph $G$ and a subspace $S$ of its cycle space over the field $k$, we say that an embedding of the graph $G$ is $S$-facial if all cycles of the graph $G$ that are in the vector space $S$ are facially generated.

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This last condition is trivial for $k = \mathbb{F}_2$. 3
Example 2.3. If the vector space $S$ is equal to the cycle space of a graph $G$, then $S$-facial embeddings of $G$ are simply plane embeddings of $G$.

Any embedding of a graph $G$ in a surface is $S$-facial, where $S$ is a subspace of the vector space generated over $\mathbb{F}_2$ by the face boundaries. If the embedding is in an oriented surface, the faces generate the same cycles over the fields $\mathbb{F}_2$ and $\mathbb{F}_3$, and in fact the set $S$ generated by the faces is in this case the cycle space of a regular matroid.

Remark 2.4. In an attempt to simplify notation, we will sometimes consider sets of $k$-vectors whose coordinates are the edges of a graph $G$, such as the subspace $S$ defined below, simply as sets of edge sets of $G$. Formally, we identify the set $S$ with the set containing the supports of the vectors in $S$. Usually, we indicate this by writing things like ‘the set $S$’.

Fix a graph $G$, and a subspace $S$ of its cycle space, and an $S$-facial embedding $\iota$ of the graph $G$ in a surface $\Sigma$. A hole of the $S$-facial embedding $\iota$ is a face of the embedding whose boundary is not in the set $S$.

Example 2.5. $S$-facial embeddings in which all face boundaries are in the set $S$ do not have any holes.

An area (bounded by $S$) is a sum of faces of the embedding $\iota$ such that the sum of these face boundaries is in the vector space $S$ and all coefficients of the sum are plus one.

Example 2.6. A cycle is facially generated if and only if it bounds an area.

We say that a set of holes is fenced in by an area $A$ if it is equal to the set of all holes contained in the area $A$ (that is more formally, that are contained in the sum for the area $A$). We say that an area $A$ bounded by $S$ is minimal if $A$ contains at least one hole and there is no other area bounded by $S$ fencing in a proper nonempty subset of the set of holes fenced in by $A$.

We say that a set $S$ is fencing in for the embedding $\iota$ if the sets of holes fenced in by any two minimal areas bounded by $S$ are either identical or disjoint. A set $S$ that is fencing in for any embedding, we simply call fencing in.

Example 2.7. If the vector space $S$ is equal to the circuit space of the graph $G$, then it is fencing in, as there are no holes.

Lemma 4.11 below says that for any $r \in \mathbb{N}$ if the vector space $S$ is generated by the cycles of $G$ of length at most $r$, then it is fencing in.
Example 2.8. Here we give an example of a set $S$ that is not fencing in. The graph $G$ is the ladder with eight vertices, embedded in the plane as indicated in Figure 1. Let $S$ be the set generated by the two cycles of length six. Then all the three cycles of length four of the ladder are holes. And every area contains an even number of these 4-cycles. As for any pair of the three four-cycles, there is an area containing them, the set $S$ is not fencing in.

![Figure 1: An embedding of the ladder with eight vertices in the plane.](image)

We say that a subspace $S$ of the cycle space of $G$ has a cyclic generating set if there is a set of cycles of $G$ generating $S$.

Example 2.9. Trivially, the vector space $S$ generated by the cycles of length at most $r$ has a cyclic generating set.

Given a subspace $S$ of the cycle space of a graph $G$ over the field $k$, the $S$-local matroid for $G$ is the matroid with ground set $E(G)$ whose circuits are the minimal nonempty supports of elements of $S$.

Remark 2.10. By definition, the $S$-local matroid is $k$-represented.

Example 2.11. In this paper we are mostly interested in $S$-local matroids, where the subspace $S$ is the set of circuits generated by circuits of length at most $r$. We refer to this matroid as the $r$-local matroid of the graph $G$.

Given the field $k$, we say that an embedding of a graph $G$ is $k$-admissible if the sum over all face boundaries of the embedding with coefficient plus one is identically zero, when evaluated over the field $k$.

Example 2.12. For $k = \mathbb{F}_2$, every embedding is $k$-admissible. For $k = \mathbb{F}_3$, the $k$ admissible embeddings such that all faces are discs are precisely the embeddings in orientable surfaces.

Lemma 2.13. Let $G$ be a graph, $S$ be a nonempty subspace of the cycle space of $G$ over $k$ with a cyclic generating set. Let $\iota$ be an $S$-facial $k$-admissible embedding of $G$. Assume that $S$ is fencing in for $\iota$. Let $H$ be the dual graph of the embedding $\iota$. 

Then there is a quotient of the graph \( H \) whose cycle matroid is equal to the dual of the \( S \)-local matroid of \( G \).

**Remark 2.14.** Lemma 2.13 may be regarded as an analogue of the statement that the cycle matroids of dual plane graphs are dual to one another. Indeed, if the embedding \( \iota \) does not have any holes, the proof below will show that the cycle matroid of the dual graph \( H \) is equal to the dual of the \( S \)-dual matroid for \( G \).

**Proof of Lemma 2.13** We start with the construction of a quotient \( Q \) of the dual graph \( H \). We say that two holes are *related* if they are contained in a common minimal area bounded by \( S \).

**Sublemma 2.15.** Every hole is contained in an area bounded by \( S \).

**Proof.** By assumption, the set \( S \) contains a cycle. As the embedding is \( S \)-facial, there is an area bounded by this cycle of \( S \). Denote this area by \( A \). Let \( X \) be the set of all faces of the embedding \( \iota \). As the embedding \( \iota \) is \( k \)-admissible by assumption, the sum over the face boundaries of \( X \) with coefficient plus one is identically zero over the field \( k \). As every hole is contained in one of the areas \( A \) or \( X - A \), every hole is contained in an area.

By Sublemma 2.15, ‘being related’ is a reflexive relation (that is, all holes are related to themselves). It is symmetric by definition. So as the set \( S \) is fencing in, this relation is transitive, so it defines an equivalence relation on the set of holes. Each hole is a vertex of the graph \( H \). We obtain the quotient \( Q \) from the graph \( H \) by identifying any two vertices of holes that are related.

It remains to show that the bond matroid of the graph \( Q \) is equal to the \( S \)-local matroid for the graph \( G \). We denote the bond matroid of the graph \( Q \) by \( M(Q) \) and the \( S \)-local matroid by \( M(S) \).

**Sublemma 2.16.** Every cycle \( o \) of \( G \) in \( S \) is in the circuit space of \( M(Q) \).

**Proof.** As the embedding \( \iota \) is \( S \)-facial, the cycle \( o \) of \( S \) is facially generated. Let \( A' \) be an area whose boundary is the cycle \( o \). Let \( A \) be the set of vertices of the graph \( H \) contained in the area; meaning that, their faces get the coefficient plus one in the area \( A' \), and let \( B \) be the set of other vertices of the graph \( H \). In the graph \( H \), the cycle \( o \) is the cut consisting of the edges from \( A \) to \( B \). Indeed, even with coefficients taken into account, the cycle \( o \) is the sum over the atomic cuts of the vertices in \( A \). Moreover, the set of holes fenced in by a minimal area is either completely contained in \( A \)
or $B$ as the set $S$ is fencing in by assumption. Hence the cut $o$ of the graph $H$ induces a cut of the quotient $Q$. To summarise, the edge set $o$ is a cut of the graph $Q$, and thus in the circuit space of the matroid $M(Q)$.

**Sublemma 2.17.** For every atomic cut of $Q$, its characteristic vector is in $S$.

*Proof.* There are two types of vertices of $Q$, those that are vertices of the graph $H$ and those that come from identifying holes from an equivalence class. Atomic cuts for vertices of the first type come from faces whose boundary is in the set $S$. Hence (characteristic vectors of) these atomic cuts are in the vector space $S$. Now let $q$ be a vertex of the quotient $Q$ coming from identifying the holes fenced in by a minimal area bounded by $S$. Let $D$ be a minimal area fencing in precisely the holes of $q$. Let $o$ be the boundary of the area $D$. Let $D = \sum F_i$ be a representation of the area $D$ as a sum of faces $F_i$. Let $X$ be the sum of the terms $\partial(F_i)$ for faces $F_i$ that are not holes, where $\partial(F_i)$ denotes the boundary of the face $F_i$. By definition of holes, the difference $o - X$ is in the vector space $S$. The difference $o - X$ is equal to the sum of the face boundaries of holes – with all coefficients being plus one – in the equivalence class for $q$. So the (characteristic vector of the) atomic cut at $q$ is a sum of the characteristic vectors of atomic cuts of $H$ for the holes in $q$. So it is a vector in the vector space $S$.

Now we show that the $k$-represented matroids $M(Q)$ and $M(S)$ are isomorphic. For that we check that they have the same circuit space. As the set $S$ has a cyclic generating set, the circuit space $S$ of the matroid $M(S)$ is generated by a set of cycles of the graph $G$. By [Sublemma 2.16](#) all these cycles are in the circuit space of the matroid $M(Q)$. So the circuit space of $M(S)$ is a subspace of the circuit space of the matroid $M(Q)$. As the circuit space of the matroid $M(Q)$ is generated by the atomic cuts of the graph $Q$ (over the integers and hence also over the field $k$), by [Sublemma 2.17](#) the circuit space of $M(Q)$ is a subspace of $S$. Thus the two matroids $M(Q)$ and $M(S)$ are isomorphic, completing the proof.

### 3 Constructing embeddings from abstract duals

In this section, we prove [Lemma 3.4](#) below, which is used in the proof of the main result stated in the Introduction. In this lemma we construct an embedding of a graph $G$ in a surface using an abstract dual graph.
We say that a subspace $S$ of the cycle space of a graph $G$ is *locally connected* if for every vertex $v$ of $G$ there is no vector supported at a non-empty proper subset of the atomic cut at $v$ that is orthogonal to the vector space $S$.

**Example 3.1.** If $k = \mathbb{F}_2$, $G$ is locally connected if and only if for every vertex $v$ of $G$ no non-empty proper subset of the atomic cut at $v$ intersects all sets of $S$ evenly.

**Definition 3.2.** Given a graph $G$ with a vertex $v$ and an integer $s$, the *ball* of radius $s$ around the vertex $v$ is the induced subgraph of $G$, whose vertices are those of distance at most $s$ from $v$ and without all edges joining two vertices of distance precisely $s$. Similarly, given a half-integer $s + \frac{1}{2}$, the *ball* of radius $s + \frac{1}{2}$ around the vertex $v$ is the induced subgraph of $G$, whose vertices are those of distance at most $s$ from $v$. We denote the ball of radius $s$ around $v$ by $B_s(v)$. Given a parameter $r$, a vertex $v$ is an $r$-local cutvertex if the punctured ball $B_{r/2}(v) - v$ is disconnected.

**Lemma 3.3.** Assume $G$ has no $r$-local cutvertex. The subspace of the cycle space of $G$ generated by the cycles of length at most $r$ is locally connected.

*Proof.* Let $S$ denote the subspace of the cycle space of $G$ generated by the cycles of length at most $r$. Suppose for a contradiction, there is a vector $\vec{X}$ supported at a non-empty proper subset $X$ of an atomic cut of a vertex $v$ that is orthogonal to all vectors in the vector space $S$ over the field $k$. Let $a$ be a neighbour of the vertex $v$ that is incident with an edge of $X$, and $b$ be a neighbour of the vertex $v$ that is incident with an edge incident with $v$ that is not in $X$.

As the vertex $v$ is no $r$-local cutvertex, the punctured ball $B_{r/2}(v) - v$ contains a path from the vertex $a$ to the vertex $b$. This path together with the vertex $v$ is a cycle. Denote it by $o$. By construction, the cycle $o$ intersects the set $X$ precisely once. (In the case $a = b$, the cycle $o$ has size two and is chosen to contain a single edge of the set $X$.) By [5, Lemma 4.3] and [5, Lemma 3.1] the cycle $o$ is generated by cycles of length at most $r$. As the cycle $o$ is not orthogonal to the vector $\vec{X}$, one of the generating cycles has to be non-orthogonal to the vector $\vec{X}$. As this generating cycle is in the set $S$, we get the desired contradiction. \qed

**Lemma 3.4.** Let $G$ be a 2-connected graph and $S$ be a subspace of the cycle space of $G$ over $k$. Assume that $S$ is locally connected. Assume that the dual $M^*$ of the $S$-local matroid of $G$ is graphic.
Then the graph $G$ has an $S$-facial $k$-admissible embedding $\iota$ such that the dual graph of $\iota$ has a quotient whose cycle matroid is $M^\ast$. Moreover, all faces of the embedding $\iota$ are discs.

Proof. Let $H$ be a graph representing the dual matroid $M^\ast$ of the $S$-local matroid of $G$. For later reference, we stress that we do pick the graph $H$ such that it does not have a cutvertex; this is always possible. We also assume that the graph $H$ has no isolated vertices. The edge set of the graph $H$ is in bijection with the edge set of the graph $G$.

Sublemma 3.5. For every vertex $g \in G$ its atomic cut forms a cycle in the graph $H$.

Proof. Let $a(g)$ be the atomic cut of a vertex $g \in G$. By construction, the characteristic vector of the edge set $a(g)$ is orthogonal to the vector space $S$, as this is a subspace of the cycle space of $G$. As the set $S$ is locally connected by assumption, no (characteristic vector of a) proper nonempty subset of the atomic cut $a(g)$ is orthogonal to the vector space $S$.

As the bonds of the graph $H$ are given by the minimal non-empty supports of vectors of the vector space $S$, the characteristic vector of the edge set $a(g)$ is in the cycle space of the graph $H$, and no proper nonempty subset of it is. Hence the edge set $a(g)$ is a circuit of the matroid $M^\ast$, and thus the edge set of a cycle of the graph $H$.

Now we construct an embedding of the graph $H$ into a pseudo-surface; here a pseudo-surface is obtained from a compact 2-manifold by identifying finitely many points, see Figure 2.

![Figure 2](image)

Figure 2: A pseudo-surface obtained from the torus by identifying two points.

\[\text{The relation between 2-manifolds and surfaces is as follows. Surfaces are connected compact 2-manifolds.}\]
Starting with the geometric realisation of the graph $H$, for each vertex $g \in G$ we attach a disc at its incident edges, which form a cycle of the graph $H$ by Sublemma 3.5. By $\Sigma$ we denote the topological space obtained from the graph $H$ by attaching these discs.

**Sublemma 3.6.** A neighbourhood around any vertex of $H$ in $\Sigma$ is obtained from finitely many disjoint discs by identifying their midpoints.

*Proof.* Let $h$ be a vertex of the graph $H$. We construct an auxiliary graph whose vertex set is the set of edges incident with the vertex $h$. Two of these edges are adjacent in the auxiliary graph if there is a disc attached at both edges. As in the construction of the topological space $\Sigma$ all discs are attached at cycles of the graph $H$ each disc containing the vertex $h$ gives rise to precisely one edge of the auxiliary graph.

As each edge of the graph $H$ is an edge of the graph $G$, it is contained in precisely two of the attached discs. Thus every vertex of this auxiliary graph has degree two. Hence this auxiliary graph is a vertex-disjoint union of cycles. By construction of the graph $H$, the vertex $h$ is incident with an edge and hence this auxiliary contains at least one cycle. Hence a neighbourhood around any vertex of $H$ in $\Sigma$ is obtained from finitely many disjoint discs by identifying their midpoints.

As each edge of the graph $H$ is also an edge of the graph $G$, it is contained in precisely two of the attached discs. Hence by Sublemma 3.6 the topological space $\Sigma$ is a pseudo-surface and all identification points of $\Sigma$ are vertices of the graph $H$. Let $\Lambda$ be the unique compact 2-manifold such that the pseudo-surface $\Sigma$ is a quotient of $\Lambda$ obtained by identifying finitely many points.

We construct a graph $K$, which has the graph $H$ as a quotient as follows, and that embeds in the compact 2-manifold $\Lambda$. For that, we locally cut each vertex $h$ of $H$ that is an identification point of the pseudo-surface $\Sigma$ into one copy for each of its preimages in the compact 2-manifold $\Lambda$. A new vertex obtained by locally cutting $h$ is incident with those edges incident with $h$ that are embedded in the corresponding disc (formally, the edges incident with the vertex $h$ are partitioned into discs by Sublemma 3.6, and we have one copy of the vertex $h$ for each of these discs. This copy is incident with the edges meeting this disc).

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8We remark that Sublemma 3.6 entails the statement that there is always at least one disc as neighbourhoods of a point can never be empty.

9See [5] for a definition.
Now we construct an embedding of the graph $G$ into the compact 2-manifold $\Lambda$. We embed each vertex of the graph $G$ into its disc. Then we embed half of each edge incident with a vertex $v$ within the disc for $v$ as a path joining $v$ and the midpoint of the edge of $H$ corresponding to that edge within the disc for $v$. The two halves of an edge meet at this midpoint of the edge of $H$. Hence this defines an embedding of the graph $G$.

Note that in the compact 2-manifold $\Lambda$, the graphs $G$ and $K$ are duals. By construction the dual graph $K$ of the embedding into the compact 2-manifold $\Lambda$ has the graph $H$ as a quotient; and this graph $H$ has the cycle matroid $M'^*$, which in turn is the dual matroid of the $S$-local matroid for $G$. As the graph $G$ is connected by assumption, the 2-manifold $\Lambda$ is connected, and hence a surface.

Having completed the construction of the embedding of the graph $G$ into the surface $\Lambda$, we show that this embedding is $S$-facial. Let $o$ be a cycle of the graph $G$ that is in the set $S$. As the set $S$ is a subspace of the cycle space of the graph $G$, the edge set $o$ is a circuit of the $S$-local matroid for $G$. By construction of the graph $H$, the edge set $o$ is a bond of the graph $H$. So $o$ is a cut of the graph $K$, as it has $H$ as a quotient. So $o$ considered as a cut of $K$ is generated by the atomic cuts of $K$. As these atomic cuts are the faces of the embedding of $G$ in $\Lambda$, the cycle $o$ of the graph $G$ is generated by the faces of the embedding with all coefficients in $\{0, 1\}$; that is, it is facially generated. Hence the constructed embedding of $G$ is $S$-facial.

Next, to see $k$-admissibility, consider the sum over all face boundaries of faces of the embedding of the graph $G$ in the 2-manifold $\Lambda$ over the field $k$. This sum\textsuperscript{10} is equal to the sum over the atomic cuts of the graph $K$, and hence evaluates to zero. Hence the embedding of $G$ in $\Lambda$ is $k$-admissible.

Finally, we check the ‘Moreover’-part; that is, we show that all faces of the embedding of the graph $G$ into the 2-manifold $\Lambda$ are discs. These faces are indexed by the vertices of the dual graph $K$. Let $x$ be an arbitrary vertex of $K$. The boundary of the face for $x$ consists of those edges that in the graph $K$ are incident with the vertex $x$. By construction of the embedding of the graph $G$, this face boundary is a closed trail. It remains to show that this closed trail traverses every vertex $v$ of $G$ at most once. By Sublemma 3.5 the atomic cut at $v$ intersects the atomic cut at $x$ in at most two edges. Thus the face boundary for $x$ can traverse the vertex $v$ at most once. Hence all faces of the embedding of $G$ are discs.

\textsuperscript{10}For the field $k = F_3$, the signs of these vectors come from the representation of the cycle matroid of the graph $H$. Indeed, this gives a representation of the cycle matroid of $K$ by taking for the vector at an atomic cut just the restriction of the corresponding vector for the graph $H$. 

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4 Fencing in of holes

In this section we prove Lemma 4.11 below, which is used in the proof of the main result mentioned in the Introduction. This lemma says that the set $S_r$, defined in the next sentence, is fencing in. Given a graph $G$, we denote by $S_r$ the vector space generated by the cycles of $G$ of length at most $r$.

Throughout this section we abbreviate ‘area bounded by $S_r$’ simply by ‘area’. And we fix a graph $G$ that is $S_r$-locally embedded in a surface. Additionally, for $k = \mathbb{F}_3$ assume that the embedding is in an oriented surface.

Lemma 4.1. Let $A$ be a sum of faces with empty boundary. Then either $A = 0$ or $A$ contains all faces with the same coefficient.

Proof. Assume that the sum $A$ is non-zero at some face $f$. As the boundary is identically zero, every face adjacent to $f$ must appear in the sum $A$. Continuing like this inductively, we deduce that $A$ contains all faces, as surfaces are connected.

If $k = \mathbb{F}_2$, then we are done. So we may assume that $k = \mathbb{F}_3$. Here we have the additional assumption that graph embeds in an oriented surface. Hence the face boundaries containing each edge traverse that edge in opposite direction. Using this, we do not only get that all faces must appear in the sum $A$ but also that they must appear in the sum with the same coefficient.

We say that an area is good if its boundary is a geodesic cycle of $G$ of length at most $r$. Recall that an area is a sum of faces with all coefficients equal to plus one such that its boundary is in the vector space $S_r$. The sum of two areas whose faces are disjoint is again an area. Similarly, if one area $A$ includes an area $B$, then $A - B$ is an area.

Lemma 4.2. Every area is a sum of good areas with coefficients in $\{-1, +1\}$.

Proof. Let $A$ be an area and let $\alpha$ be its boundary. As $\alpha$ is in the vector space $S_r$, by definition, it is generated by cycles of length at most $r$, which in turn are generated by geodesic cycles of length at most $r$. Let $\alpha_1, \ldots, \alpha_n$ be such a generating set. As the embedding is $S_r$-local, each cycle $\alpha_i$ bounds an area. Let $B$ be the sum of all these areas over the field $k$. The boundary of the sum $B$ is the vector $\alpha$.

Consider the sum $A - B$. Its boundary is identically zero. There are precisely two ways in which this could happen.

Firstly, $A = B$ and we are done. Otherwise, by Lemma 4.1 the sum $A - B$ consists of all faces with the same coefficient.
If $k = \mathbb{F}_2$, modify the area bounded by $\alpha_1$ by adding $A - B$. This is again an area with the same boundary. Then the resulting modified term for $B$ is equal to $A$, and we are done.

Hence it remains to consider the case $k = \mathbb{F}_3$. Let $X$ be the sum over all areas with coefficient one. As the sum $X$ is equal to $A - B$ up to a constant, $X$ has empty boundary. Let $A_i'$ be the area bounded by $\alpha_i'$ chosen above. Then $A_i' = X - A_i$ is an area whose boundary $-\alpha_i$. We distinguish two cases.

**Case 1:** $X = -(A - B)$. Let $B'$ be the sum of the areas $A_i'$ with $'-A_1'$ in place of the area $'A_1'$. Then $B' = B - X$. So $B' = A$, which gives the desired representation of $A$ as a sum.

**Case 2:** not Case 1. Then $X = A - B$. As we were done otherwise immediately, we assume that $n \geq 2$. Let $B'$ be the sum of the areas $A_i'$ with $'-A_i'$ in place of the area $'A_i'$ for $i = 1, 2$. Then $B' = B - 2X$. So $B' = A$, which completes the proof. \[\square\]

**Corollary 4.3.** Assume there is an area containing a hole. Then there is a good area containing a hole.

**Proof.** Let $A$ be an area containing a hole $h$. By **Lemma 4.2** the area $A$ is a sum of good areas. Hence one of these summands must contain the hole $h$. \[\square\]

**Lemma 4.4.** Let $G$ be a graph embedded in a surface $\Sigma$. Let $\ell$ be a natural number. Let $A$ be a closed subset of $\Sigma$ whose boundary is a geodesic cycle of $G$ of length at most $\ell$. Let $o$ be a cycle included in $A$ of length at most $\ell$. Then there is a family of geodesic cycles of length at most $\ell$ included in the set $A$ such that their sum evaluates to $o$ over $\mathbb{Z}$.

**Proof.** We denote the bounding cycle of the set $A$ by $\alpha$. Clearly the cycle $o$ is a sum of geodesic cycles of length at most $\ell$. Let $\mathcal{F}$ be the subfamily of these generating cycles that either are included in the closed subset $A$, or intersect its boundary $\alpha$. As the boundary $\alpha$ is a geodesic cycle, paths outside the set $A$ whose endvertices are on the cycle $\alpha$, can be replaced by subpaths of the cycle $\alpha$ that are not longer. Now we manipulate all the geodesic cycles in the family $\mathcal{F}$ in this way. Consider the sum of this modified family. This sum is included in the set $A$. And in the interior of $A$ it coincides with the cycle $o$. So its difference to $o$ is supported at the boundary $\alpha$ of $A$. As this difference must be in the cycle space, it must be a multiple of the boundary

\[\text{This just says that } A \text{ is an area. We write it this way to stress that here we consider } A \text{ as a topological object with its boundary rather than a combinatorial object.}\]
cycle $\alpha$. Hence subtracting a suitable multiple of $\alpha$ from the modified family gives a way to write the cycle $o$ as a sum of geodesic cycles of length at most $\ell$ that are included in $A$. 

We say that an area $A$ includes an area $B$ if if all faces contained in $B$ are also contained in $A$; that is, the sum $B$ is a subsum of the sum $A$.

**Lemma 4.5.** Let $A$ be a good area. Let $o$ be a cycle included in $A$ of length at most $r$. Then there is an area included in $A$ bounded by $o$.

**Proof.** As the cycle $o$ has length at most $r$ and the embedding is $S_r$-local, the cycle $o$ is the boundary of an area. Denote that area by $B$. Let $C$ be the sum (with coefficient one) of those faces of $B$ that are not included in the area $A$. As the boundary of the area $B$ is included in the area $A$, also the boundary of the sum $C$ must be included in the area $A$. Thus the boundary of the sum $C$ can only take nonzero values at the bounding cycle of the area $A$. As the only such vectors in the cycle space are multiples of the bounding cycle of the area $A$, the boundary of the sum $C$ has the form $x \cdot \alpha$, where $x$ is a coefficient in the field $k$ and $\alpha$ is the bounding cycle of the area $A$.

If $x = 0$, then the area $B - C$ is bounded by $o$, and we are done.

If $x = -1$, then the sum $X = A - (B - C)$ has the boundary $-o$; that is, the cycle $o$ with the reverse orientation. So $X$ is the desired area.

Hence it remains to show that the case $x = 1$ and $k = \mathbb{F}_3$ is not possible. Then the sum $X = A - C$ has empty boundary. As the set $A$ is nonempty and disjoint from $C$ by construction, by [Lemma 4.1](#) the sum $X$ takes the same non-zero value at all faces. As the area $A$ is bounded by a cycle, there is a face in $A$ and one outside. The sum $X$ takes the value plus one at the face inside and the value minus one outside. This is a contradiction, completing the proof. 

**Lemma 4.6.** Let $\Sigma$ be a surface and let $B$ be a closed subset whose boundary $b$ is a circle. Let $\gamma$ be a curve intersecting the circle $b$ in finitely many points but not in its endvertices. Then precisely one of the endpoints of the curve $\gamma$ is in the set $B$ if and only if the curve $\gamma$ intersects the cycle $b$ oddly.

**Proof:** by induction on the number of intersection points of the curve $\gamma$ and the boundary $b$. 

**Lemma 4.7.** Assume there is a good area containing a hole. Then there is a good area that is fencing in.
Proof. By assumption, there is a good area containing a hole. Pick a good area containing as few holes as possible but at least one. Call it $A$. It remains to prove that the area $A$ is fencing in.

Suppose not for a contradiction. Then there is an area $C$ such that it contains a hole of the area $A$ and avoids a hole of $A$.

**Sublemma 4.8.** There is a good area $B$ that contains a hole of $A$ and avoids a hole of $A$.

*Proof.* By Lemma 4.2 the area $C$ is a sum of good areas. Let $C'$ be the sum of such good areas that contain some hole of $A$. Suppose for a contradiction that all summands of $C'$ take the same coefficient at all holes of $A$. Then in the sum $C'$ all holes appear with the same multiplicity. By construction, each holes of $A$ appears with the same multiplicity in the sums $C'$ and $C$. This is a contradiction as in $C$ two holes of $A$ have a different multiplicity. Hence there is a good area that has different multiplicities at holes of $A$. As areas can only have multiplicity one and zero at holes, there is a a good area that contains a hole of $A$ and avoids a hole of $A$. \qed

By Sublemma 4.8 there is a good area $B$ that contains a hole of $A$ and avoids a hole of $A$. Pick a hole $h_1$ that is in both areas $A$ and $B$, and a hole $h_2$ that is only in the area $A$. As the area $A$ is bounded by a cycle, it is connected. So there is a curve $\gamma$ included in the closed set $A$ from the hole $h_1$ to the hole $h_2$ such that it does not intersect the bounding cycle of the good area $A$. By modifying the curve $\gamma$ locally if necessary, we may assume, and we do assume, that $\gamma$ intersects the bounding cycle of the good area $B$ only in finitely many points.

Denote the bounding cycle of the good area $A$ by $a$. Denote the bounding cycle of the good area $B$ by $b$.

**Sublemma 4.9.** The cycles $a$ and $b$ intersect in at least two points.

*Proof.* As the curve $\gamma$ joins a hole in $B$ with a hole outside, the curve $\gamma$ has to intersect the cycle $b$; in particular $b$ contains a point in the interior of the area $A$. Suppose for a contradiction that the cycle $b$ is included in the area $A$. Then by Lemma 4.5 there is an area $B'$ included in the area $A$ and bounded by $b$. Consider the sum $B - B'$. This sum has empty boundary. By Lemma 4.1 either this sum is identically zero or it contains all faces with the same multiplicity. So either the areas $B$ and $B'$ agree or they partition the faces. In either case, the area $B'$ contains fewer holes than $A$ but precisely one of $h_1$ and $h_2$. Note that the area $B'$ is good. This is a contradiction to the choice of the area $A$.\[15\]
So the cycle \( b \) contains a point outside the area \( A \). Applying Lemma 4.6 to two subpaths of the cycle \( b \) between vertices inside \( A \) and outside \( A \), yields that the cycles \( a \) and \( b \) have to intersect in at least two points. \( \square \)

**Sublemma 4.10.** There is a subpath \( P \) of the cycle \( b \) that intersects the cycle \( a \) precisely in its endvertices and that intersects the curve \( \gamma \) in an odd number of points.

*Proof.* For each intersection point \( y \) of the curves \( b \) and \( \gamma \) pick the unique subpath of the cycle \( b \) that contains the point \( y \) and that intersects the cycle \( a \) precisely in its endvertices; this is well defined by Sublemma 4.9. Denote that path by \( P_y \). Two paths \( P_y \) are either identical or intersect at most in their endvertices. If all paths \( P_y \) intersected the curve \( \gamma \) evenly, then the cycle \( b \) would intersect the curve \( \gamma \) evenly. This is not possible as precisely one of the holes \( h_1 \) and \( h_2 \) is in the area \( B \) by Lemma 4.6. Hence one path \( P_y \) must intersect the curve \( \gamma \) oddly. Pick such a path for \( P \). \( \square \)

Let \( P \) be a path as in Sublemma 4.10. Denote the two endvertices of the path \( P \) by \( x \) and \( y \). Let \( Q \) be a shortest path between the vertices \( x \) and \( y \) included in the geodesic boundary cycle \( a \) of the good area \( A \). As the length of the geodesic path \( Q \) cannot be longer than any \( x \)-\( y \)-path included in the bounding cycle \( b \) of the good area \( B \), the length of the cycle \( PQ \) is at most the length of the bounding cycle of \( B \); and thus at most \( r \).

As the path \( P \) contains an interior point of the area \( A \), and it intersects its boundary only in its endvertices, the connected path \( P \) is included in the area \( A \). So the cycle \( PQ \) is included in the area \( A \).

By Lemma 4.4 the cycle \( PQ \) can be written as a sum of geodesic cycles of length at most \( r \) that are included in \( A \). As the cycle \( PQ \) intersects the curve \( \gamma \) oddly, one of the generating geodesic cycles has to intersect the curve \( \gamma \) oddly. Pick such a geodesic cycle and denote it by \( x \).

By Lemma 4.5 there is an area \( X \) included in the area \( A \) whose boundary is the geodesic cycle \( x \). By construction the area \( X \) is good. As the curve \( \gamma \) intersects the cycle \( x \) oddly, the area \( X \) contains precisely one of the holes \( h_1 \) and \( h_2 \) by Lemma 4.6. Thus the area \( X \) contains a hole. As all holes of the area \( X \) are holes of the area \( A \) but \( X \) does not contain the hole \( h_1 \) or \( h_2 \), the existence of the good area \( X \) is a contradiction to the choice of the area \( A \). Thus the area \( A \) must be fencing in. \( \square \)

We introduce the following notation to state the next lemma for the fields \( \mathbb{F}_2 \) and \( \mathbb{F}_3 \) simultaneously. An \( \mathbb{F}_3 \)-oriented embedding is an embedding in
an oriented surface. Every embedding of a graph in a surface is $\mathbb{F}_2$-oriented.

**Lemma 4.11.** The vector space $S_r$ generated by the cycles of length at most $r$ over $k$ is fencing in (for every $S_r$-facial $k$-oriented embedding).

**Proof.** We prove this by induction on the number of holes. The induction starts where there is no area containing holes; then the statement is true as there are no minimal areas. So suppose there is an area containing holes. By Corollary 4.3 there is also a good area that contains a hole. By Lemma 4.7 there is an area that is bounded by a geodesic cycle and is fencing in. Denote that area by $A$ and its bounding cycle by $\alpha$.

Now construct a new embedding from the old embedding by deleting all faces contained in the area $A$ – including all vertices of the graph $G$ that are in the interior of the area $A$ – and attaching a disc at the cycle $\alpha$. Denote the new embedded subgraph of $G$ by $G'$.

As the area $A$ contains a hole, this new embedding of this subgraph has strictly fewer holes. We denote the vector space generated by cycles of length at most $r$ in the graph $G'$ by $S'_r$. Hence by induction, the set $S'_r$ for this new embedded graph $G'$ is fencing in.

Areas of the graph $G'$ are also areas of the graph $G$ after removing the newly added disc with boundary $\alpha$ of the embedded graph $G'$ if necessary. The following lemma allows us to transform areas of the graph $G$ into areas of the subgraph $G'$.

**Sublemma 4.12.** Assume there is a face not contained in the area $A$. For every area $B$ of $G$, there is an area $B'$ that is an area for the graphs $G$ and $G'$ such that one of $B'$ or $B' + A$ contains the same holes as the area $B$.

**Proof.** Let $\hat{B}$ be the sum (with coefficients one) over all faces of the area $B$ that are contained in the area $A$. We want to show that the sum $\hat{B}$ is an area; that is, its boundary is in the set $S'_r$. Denote the boundary of the area $B$ by $b$.

Our first step will be to construct from the boundary $b$ a candidate for the boundary of the sum $\hat{B}$, as follows. As the boundary $\alpha$ of the area $A$ is a geodesic cycle, paths outside the set $A$ whose endvertices are on the cycle $\alpha$, can be replaced by subpaths of the cycle $\alpha$ that are not longer. Applying this to the boundary $b$, gives a closed walk $\gamma$ of length at most $r$ that agrees with the boundary $b$ on the interior of the area $A$ and is included in $A$. By Lemma 4.5 there is an area $C$ bounded by $\gamma$ and included in $A$. 

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Now consider the difference $\hat{B} - C$. Its boundary is a difference of the boundaries of $\hat{B}$ and $C$. As these two boundaries are included in $A$ and agree on the interior of $A$, the boundary of $\hat{B} - C$ is supported on the boundary of the area $A$. As the boundary is in the cycle space, it must be a multiple of the bounding cycle $\alpha$ of $A$. Thus there is a coefficient $x$ in $k$ such that $\hat{B} - C + x \cdot A$ has empty boundary. As there is a face outside the area $A$ by assumption, this sum is not supported on all faces. Hence by Lemma 4.1, we have that

$$\hat{B} - C + x \cdot A = 0$$

If $k = \mathbb{F}_2$, then $\hat{B} = C - x \cdot A$ is an area. Then $B' = B - \hat{B}$ is an area that uses no faces of the area $A$. As the area $A$ is fencing in, the area $\hat{B}$ contains either all holes of $A$ or none. So $B'$ has the desired properties. So it remains to consider the case that $k = \mathbb{F}_3$.

If $x = 0$, then $\hat{B} = C$, so $\hat{B}$ is an area. Arguing as above, we conclude that $B' = B - \hat{B}$ is the desired area.

If $x = 1$, then $-\hat{B} = A - C$. As $A - C$ is an area and the sum $\hat{B}$ has never the coefficient minus one, we conclude that $\hat{B} = 0$. So $B' = B$ is the desired area.

If $x = -1$, then $\hat{B} = A + C$. Then faces contained in the area $C$ have coefficient minus one on the right. As $\hat{B}$ has no such coefficients, the area $C$ must be empty. So $\hat{B} = A$. Then $B' = B - A$ is the desired area.

Having finished the proof of Sublemma 4.12, we show that $S_r$ is fencing in. Let $B$ be a minimal area. If the area $B$ has the same holes as the area $A$, then it is fencing in, and we are done. So we may assume, and we do assume, that the area $B$ contains a hole that is not contained in the minimal area $A$. As the area $B$ is minimal, it cannot contain any hole of the area $A$, as $A$ is fencing in. By Sublemma 4.12, there is an area $B'$ of $G'$ such that one of $B'$ or $B' + A$ contains the same holes as the area $B$. Let $C'$ be a minimal area of $G'$ containing a hole of the area $B$. By induction, the area $C'$ is fencing in for $G'$. In particular, all its holes are contained in the area $B'$. Let $C''$ be the area obtained from $C'$ by setting the coefficient of the newly added disc with boundary $\alpha$ of the embedded graph $G'$ to zero. Note that $C''$ is an area of the graph $G$ with the same holes as the area $C'$ of $G'$.

So the area $C''$ contains no hole outside the minimal area $B$. By minimality of $B$, the areas $B$ and $C''$ contain the same holes. By replacing the area $B'$ by $C''$ if necessary, we assume that $B = B'$ and $B'$ is fencing in for the graph $G'$.

Now let $D$ be an arbitrary area of the graph $G$. By Sublemma 4.12, there is an area $D'$ of $G'$ such that one of $D'$ or $D' + A$ contains the same
holes as the area $D$. As the area $B'$ is fencing in for $G'$, the area $D'$ either contains no hole of $B'$ or all holes of $B'$. As the area $B$ contains no hole of the area $A$, the area $D$ either contains no hole of $B$ or all holes of the area $B$. As the area $D$ was arbitrary, the area $B$ is fencing in. This completes the induction step.

5 Abstract duality for surfaces

In this section we deduce the main result stated in the Introduction from the lemmas proved in the previous sections.

First we prove the following analogue of Whitney’s abstract duality theorem for general surfaces.

**Theorem 5.1.** Let $G$ be a 2-connected graph, and $S$ be a nonempty subspace of its cycle space over $\mathbb{F}_2$ with a cyclic generating set. Assume $S$ is locally connected and fencing in.

Then $G$ has an $S$-facial embedding into a surface if and only if the $S$-local matroid for $G$ over $\mathbb{F}_2$ is co-graphic.

**Proof.** Since the set $S$ is fencing in and has a nonempty cyclic generating set by assumption, by [Lemma 2.13] if $G$ has an $S$-facial embedding into a surface, then the $S$-local matroid for $G$ is co-graphic.

Since the set $S$ is locally connected by assumption, by [Lemma 3.4] if the $S$-local matroid for $G$ is co-graphic, then $G$ has an $S$-facial embedding.

We say that an embedding of a graph $G$ in a (pseudo-) surface is $r$-locally planar if all cycles of length at most $r$ are generated by the face boundaries over the field $k$. Note that a graph is $r$-locally planar if and only if it is $S_r$-facial, where $r$ is the vector space generated by the cycles of $G$ of length at most $r$.

**Corollary 5.2.** Any $r$-locally 2-connected graph $G$ has an $r$-locally planar embedding if and only if the $r$-local matroid for $G$ over $\mathbb{F}_2$ is co-graphic.

**Proof.** It suffices to prove this theorem for every connected component of the graph $G$. Hence we may assume, and we do assume, that the graph $G$ is connected, and so 2-connected by the local connectivity assumption. If the graph $G$ has no cycle of length at most $r$, any embedding is $r$-locally planar, so there is nothing to prove. So we assume that the graph $G$ has a cycle of length at most $r$.

Let $S$ be the set of all edge sets generated by the cycles of the graph $G$ of length at most $r$. By assumption the set $S$ is nonempty. Clearly $S$ is a
subspace of the cycle space of $G$ with a cyclic generating set. By [Lemma 4.11] the set $S$ is fencing in. As $G$ is $r$-locally 2-connected, by [Lemma 3.3] the set $S$ is locally connected. By definition, an $S$-facial embedding is simply an $r$-locally planar embedding. The $r$-local matroid is identical to the $S$-local matroid.

Hence by [Theorem 5.1] the graph $G$ has an $r$-locally planar embedding if and only if the $r$-local matroid for $G$ is co-graphic.

Actually, [Corollary 5.2] can be extended beyond the locally 2-connected case. In order to prove this, we do some preparation. A pseudo-manifold is a topological space obtained from a compact 2-dimensional manifold by identifying finitely many points.

**Example 5.3.** A connected pseudo-manifold is a pseudo-surface.

We refer to the finitely many identification points of a pseudo-manifold as *singularities*. Given a graph $G$ embedded in a pseudo-manifold $\Sigma$, a vertex $v$ embedded to a singularity of $\Sigma$ is an $r$-local singularity if the point $v$ in $\Sigma$ has a neighbourhood $O$ such that two edges incident with $v$ intersect the same connected component of $O - v$ if and only if their endvertices aside from $v$ are in the same connected component of the punctured ball $B_{r/2}(v) - v$. An embedding of a graph in a pseudo-manifold is $r$-nice if its singularities are all $r$-local singularities.

**Lemma 5.4.** Let $G$ be a graph and $G'$ be the graph obtained from $G$ by $r$-locally cutting a vertex $v$ of $G$. Then $G$ embeds $r$-locally planarly $r$-nicely in a pseudo-manifold with $v$ being an $r$-local singularity if and only if $G'$ embeds $r$-locally planarly $r$-nicely in a pseudo-manifold.

**Proof.** First assume that $G'$ embeds $r$-locally planarly $r$-nicely in a pseudo-manifold. We obtained an embedding of the graph $G$ by identifying all slices of the vertex $v$. This defines an embedding of the graph $G$ in a pseudo-manifold with $v$ being an $r$-local singularity. This embedding is $r$-nice by construction. This embedding is $r$-locally planar as every non-facially generated cycle of $G$ is either a non-facially generated cycle of $G'$ or not a cycle of $G'$ at all. In the first case, the cycle has length more than $r$ by the $r$-local planarity of the embedding of $G'$. Cycles of the second type have length at least $r + 1$ by definition of $r$-local cutting. So the embedding of $G$ has the desired properties.

Next assume that $G$ embeds $r$-locally planarly $r$-nicely in a pseudo-manifold with $v$ being an $r$-local singularity. We obtain an embedding of the graph $G'$ by replacing the vertex $v$ by its slices (formally we delete the
vertex \( v \) and then take the completion of the remainder. The newly added points are precisely the slices of \( v \). As the vertex \( v \) is an \( r \)-local singularity, this defines an embedding of the graph \( G' \). This embedding is \( r \)-nice by construction. As every non-facially generated cycle of the graph \( G' \) is a non-facially generated cycle of the graph \( G \), all these cycles have more than \( r \). Hence this embedding is \( r \)-locally planar.

**Corollary 5.5.** A graph \( G \) has an \( r \)-locally planar \( r \)-nice embedding in a pseudo-surface if and only if all its \( r \)-local blocks have \( r \)-locally planar embeddings in surfaces.

**Proof.** This statement with ‘embeddings in surfaces’ replaced by ‘\( r \)-nice embeddings in pseudo-surfaces’ follows directly from Lemma 5.4 by induction on the number of \( r \)-local cutvertices. Finally note that as \( r \)-local blocks do not have \( r \)-local cutvertices, \( r \)-nice embeddings of such graphs in pseudo-surfaces are always embeddings in genuine surfaces.

Next we prove the following extension of Corollary 5.2.

**Theorem 5.6.** Any graph \( G \) has an \( r \)-locally planar \( r \)-nice embedding in a pseudo-surface if and only if the \( r \)-local matroid for \( G \) over \( \mathbb{F}_2 \) is co-graphic.

**Proof.** Extending the well-known block-cutvertex theorem, in [5] it is shown that any graph has an edge-disjoint decomposition into its \( r \)-local blocks. Each of these \( r \)-local blocks is \( r \)-locally 2-connected. Moreover, the \( r \)-local matroid of \( G \) is the direct sum over the \( r \)-local matroids of the blocks; indeed, analogous to the fact that cutting at cutvertices does not change the cycle matroid of \( G \), locally cutting at \( r \)-local cutvertices does not change the \( r \)-local matroid of \( G \).

Applying Corollary 5.2 to each \( r \)-local block separately yields that all \( r \)-local blocks have an \( r \)-locally planar embedding if and only if the \( r \)-local matroid for \( G \) is co-graphic. So the statement follows by Corollary 5.5.

**Proof of Theorem 1.1.** In the Introduction we abbreviated the ‘\( r \)-local matroid for \( G \) over \( \mathbb{F}_2 \)’ simply by the ‘\( r \)-local matroid for \( G \)’. The other difference between Theorem 1.1 and Theorem 5.6 is that the term ‘\( r \)-nice’ is omitted in the statement in the Introduction. It is an easy exercise to make an embedding in a pseudo-surface \( r \)-nice (first note that the only vertices that can be mapped to singularities are \( r \)-local cutvertices. Then take the rotation system of the old embedding and at each \( r \)-local cutvertex \( v \) replace
the rotator by one rotator for every slice of \( v \). This rotator at a slice is obtained from the original rotator at \( v \) by restricting to the edges incident with the slice. This defines an embedding in a pseudo-surface that is \( r \)-locally planar and \( r \)-nice by construction.)

So Theorem 1.1 is just a restatement of Theorem 5.6. \( \square \)

Analogous to Theorem 5.1 and Theorem 5.6 we prove the following oriented analogues.

**Theorem 5.7.** Let \( G \) be a 2-connected graph, and \( S \) be a nonempty subspace of its cycle space over \( \mathbb{F}_3 \) with a cyclic generating set. Assume \( S \) is locally connected and fencing in.

Then \( G \) has an \( S \)-facial embedding into an orientable surface if and only if the \( S \)-local matroid for \( G \) over \( \mathbb{F}_3 \) is co-graphic.

**Proof.** The proof is the same as the proof for Theorem 5.1 where we take the field \( k \) to be \( \mathbb{F}_3 \) in place of \( \mathbb{F}_2 \). Additionally only note that a \( k \)-admissible embedding such that all faces are disc, defines an orientation of that surface. Hence the embedding \( \iota \) constructed in Lemma 3.4 is into an oriented surface. \( \square \)

**Theorem 5.8.** Any graph \( G \) has an \( r \)-locally planar \( r \)-nice embedding in an orientable pseudo-surface if and only if the \( r \)-local matroid for \( G \) over \( \mathbb{F}_3 \) is co-graphic.

**Proof.** The proof is the same as the proof for Theorem 5.6 where we take the field \( k \) to be \( \mathbb{F}_3 \) in place of \( \mathbb{F}_2 \). \( \square \)

### 6 Concluding remarks

An LEW embedding of a graph \( G \) is an embedding in a surface such that all faces are strictly shorter than the non-contractible cycles of the embedding. Every LEW embedding is an \( r \)-locally planar embedding, where \( r \) is the maximum length of a face of the embedding. Locally planar embeddings are more general than LEW embeddings in the following ways. Firstly, faces in locally planar embeddings can have arbitrary size. Secondly, while being contractible is a special case of being facially generated, the converse is not true. So from the point of view of algebraic topology, local planarity is phrased in terms of the more general homological notion, while LEW embeddings rely on the more restricted homotopical definition.
Of particular interest is the relation between locally planar embeddings and minimum genus embeddings. In [21] Thomassen, proved that LEW embeddings are minimum genus embeddings.

**Example 6.1.** As explained above, locally planar embeddings include LEW embeddings. Building on this, we construct a large class of locally planar embeddings that are not LEW embeddings but still of minimum genus. For that, take a LEW embedding of a graph $G$. Let $r$ be the maximum length of a face of this embedding. Pick a face $f$ of the embedding arbitrarily. Pick an arbitrary planar graph that has $f$ has a face, and $f$ is a geodesic cycle in that graph. Now glue that planar graph along the face $f$ onto the graph $G$. Call that new graph $H$. The embedding of the graph $G$ extends to an embedding of the graph $H$ by embedding the new planar graph within the face $f$. This embedding is $r$-locally planar.

And it is a minimum genus embedding of the graph $H$, as the genus of $H$ cannot be smaller than that of the subgraph $G$.

**Open Question 6.2.** Can you characterise when locally planar embeddings are minimum genus embeddings?

In the following we give a rough estimate how far locally planar embeddings are from minimum genus embeddings. We denote by $def(G)$ the genus deficit of an $r$-locally planar embedding of $G$ (that is, the difference between the genus of the embedding and the optimal genus), by $\alpha$ the rank of the $r$-local matroid for $G$, by $g(G)$ the girth of $G$ and by $E$ the edge-number of $G$.

**Observation 6.3.** Assume a connected graph $G$ has an $r$-locally planar embedding. Then:

$$def(G) \leq \alpha + \frac{2E}{g(G)} - E - 1$$

**Proof.** Fix an $r$-locally planar embedding of $G$. Let $H$ be the dual graph of that embedding. Let $F$ be the number of faces for that embedding; that is, the number of vertices of the graph $H$. By [Lemma 2.13] and [Lemma 4.11] $H$ has a quotient $H'$ whose cycle matroid is the dual of the $r$-local matroid of $G$. So $|V(H')| = E - \alpha + 1$. So $F \geq E - \alpha + 1$.

Let $H^*$ be a dual graph of $G$ in some optimal genus embedding. Let $F^*$ be the number of faces for an optimal embedding. Using the definition of the average degree of the graph $H^*$, we estimate: $F^* \leq \frac{2E}{g(G)}$.

By Euler’s Formula, the genus deficit is equal to $F^* - F$. Plugging in the two expressions for $F$ and $F^*$ gives the desired result.  ∎
exploring classes of matroids that can arise as $S$-local matroids further is an exciting direction for future research. Indeed, inspired by a theorem of Seymour [13] and their work on the Matroid Minors Project, Geelen, Gerards and Whittle introduced the class of quasi-graphic matroids [9], see [4] for further examples of quasi-graphic matroids. It seems natural to compare the conditions in [Theorem 5.1] to those for quasi-graphic matroids. In particular the following seems to be of interest.

**Open Question 6.4.** Are there natural classes of quasi-graphic matroids that are cographic and $S$-local matroids for graphs $G$?

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