MANY-BODY LOCALIZATION IN THE DROPLET SPECTRUM OF THE RANDOM XXZ QUANTUM SPIN CHAIN

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ABSTRACT. We study many-body localization properties of the disordered XXZ spin chain in the Ising phase. Disorder is introduced via a random magnetic field in the $z$-direction. We prove a strong form of dynamical exponential clustering for eigenstates in the droplet spectrum: For any pair of local observables separated by a distance $\ell$, the sum of the associated correlators over these states decays exponentially in $\ell$, in expectation. This exponential clustering persists under the time evolution in the droplet spectrum. Our result applies to the large disorder regime as well as to the strong Ising phase at fixed disorder, with bounds independent of the support of the observables.

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1. INTRODUCTION

1.1. Many-Body Localization. Understanding the structure and complexity of the eigenstates of “typical” local Hamiltonians is one of the central problems in Condensed Matter Physics and Quantum Complexity Theory. The concept of disorder induced localization has been introduced in the seminal work of Anderson [6], who suggested a mechanism responsible for the absence of diffusion of waves in disordered media. This mechanism is well understood by now in the single particle case, both physically and mathematically. In random Schrödinger operators, localization manifests itself as pure point spectrum, with the corresponding eigenvectors exhibiting exponentially fast spatial decay, for almost all configurations of the environment.

It turns out that many manifestations of single-particle Anderson localization remain valid if one consider a fixed number of interacting particles, e.g., [14, 5, 33]. The methodology used in these works is unfortunately inadequate to study the thermodynamic limit of an electron gas in a random environment, i.e., an infinite volume limit in which the number of electrons grows proportionally to the volume. In the landmark paper [3] it was suggested that some hallmarks of localization indeed survive the passage to a true many-body system. This has sparked extensive efforts in the physics community to understand this phenomenon, known as many-body localization (MBL), see, e.g., [7, 9, 12, 43, 46, 49, 50]. This compilation, far from being comprehensive, lists only a few salient works closely related to the current paper. Definitions and heuristic arguments that lay a general foundation for an analytical approach to MBL were introduced...

Date: July 31, 2017.
A. K. was supported in part by the NSF under grant DMS-1001509.
in [25], based on earlier work on gapped systems (e.g., [24]), by identifying their possible counterparts in mobility gapped systems.

Investigating the combined effects of disorder and interactions on large quantum systems is difficult, because the many-body quantum states are extremely complicated objects. Even an approximate description of such a state, in general, requires the introduction of an exponential number of parameters (in terms of the system’s size). As a consequence, the basic questions related to the behavior of disordered many-body quantum systems are still subject of debate in the physics community. However, a clearer phenomenological picture has by now been drawn in the so called *fully many-body localized* regime, e.g., [13, 26, 27, 47].

Mathematical proofs of accepted MBL characteristics such as zero-velocity Lieb-Robinson bounds, rapid decay of correlations, as well as area laws for the bipartite entanglement of eigenstates, have generally been restricted to quasi free systems where MBL properties can be reduced to Anderson localization of an effective one-particle Hamiltonian. Examples are the XY spin chain in random transversal field (see [1] for a recent review), the disordered Tonks-Girardeau gas [45], and systems of quantum harmonic oscillators [39, 40]. The mean field theory in the Hartree-Fock approximation for random systems was studied in [16].

Very few rigorous results exist for models where this type of reduction is not possible, and their scope is rather limited. One can mention the exponential clustering property for the ground state of the André-Aubry quasi-periodic model [46, 47].

A model which has received considerable attention in the physics literature is the XXZ chain in a random field. In particular, numerical evidence suggests that this model exhibits a many-body localization-de-localization transition in the weak disorder regime, e.g. [3, 35, 44]. What makes this model accessible to rigorous analysis is that it is particle number preserving in the case of a field in z-direction. This allows the reduction to an infinite system of discrete N-body Schrödinger operators on the fermionic subspaces of $Z^N$ [42, 41, 20]. In particular, for the Ising phase of the XXZ chain, in the absence of an external field, the low energy states above the ground state are characterized by a droplet regime. In this regime, as can be explained by an attractive particle interaction in the associated Schrödinger operators, spins form a droplet, i.e., a single cluster of down spins (in the normalization chosen below) in a sea of up spins.

In this paper we prove that the XXZ chain in a random field exhibits one of the expected hallmarks of localization, namely a strong form of the exponential clustering property, in the droplet spectrum. In concrete terms, we show exponential clustering of the averaged correlations of local observables in all eigenstates with energies in the droplet spectrum. We actually establish a stronger dynamical exponential clustering property: this exponential clustering is preserved under the time evolution in the droplet spectrum. While our analysis relies on establishment of the localization properties of the associated Schrödinger operators, the main result requires a set of new ideas on how to translate these properties to the global theory for the spin chain.

Our proof works in the Ising phase for two distinct regimes, the case of random field with large disorder (described by a parameter $\lambda > 0$), as well as in the strong Ising phase (described by a parameter $\Delta > 1$). In fact, our proof works in the parameter region described by

$$\left( \Delta, \lambda \right) \in (1, \infty) \times (0, \infty) \quad \text{satisfying} \quad \lambda \sqrt{\Delta - 1} \min \left\{ 1, (\Delta - 1) \right\} \geq K, \quad (1.1)$$

where the finite constant $K > 0$ is system size independent. The underlying physical mechanism is that the droplets formed by the spins can be considered as single quasi particles, which become localized in the presence of disorder. Crucial for the rigorous analysis is that this can be quantified with bounds independent of the size of the droplets.

In a sequel to this work [17], we investigate further dynamical manifestations of localization in the droplet spectrum of the random XXZ chain, including non-spreading of information, zero-velocity Lieb-Robinson bounds, and general dynamical clustering.

After circulating an early draft of this work we learned about work in progress concerning the N-body Schrödinger operators associated with the random XXZ model, now available in [10].
we consider the Ising phase of the XXZ chain, i.e., we assume
\[ \Delta \] the random field
configuration (or vacuum vector
\( H \) of
\[ = \left( \omega \right) \]
which is self-adjoint on
\[ \otimes \]
where \( \sigma^{x,y,z} \) are the standard Pauli matrices and \( \Delta \) is a positive parameter. Also,
\[ N_i = \frac{1}{2}(1 - \sigma^z_i) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]
is the projection onto the down-spin state (or local number operator) at site \( i \). The positive parameter \( \lambda \) describes the strength of the disordered longitudinal magnetic field \( B_\omega \). We choose the random parameters \( \omega = (\omega_i)_{i \in \mathbb{Z}} \) as independent identically distributed random variables with distribution \( \mu \), assuming that \( \text{supp} \mu = [0, \omega_{\text{max}}] \) for some finite \( \omega_{\text{max}} \) and that \( \mu \) is absolutely continuous with bounded density \( \rho \). Thus the random field \( B_\omega \) is non-negative. We have normalized both \( H_0 \) and \( B \) so that the ground state energy of \( H \) is \( E_0 = 0 \), independent of the random parameters \( \omega \), with ground state given by the all-spins up configuration (or vacuum vector).

As special cases one gets the Heisenberg chain for \( \Delta = 1 \) and the Ising chain in the limit \( \Delta \to \infty \). Here we consider the Ising phase of the XXZ chain, i.e., we assume \( \Delta > 1 \). The factor \( \frac{1}{2} \) in our normalization is reminiscent of the fact that many physics papers use the spin matrices \( S^{x,y,z} = \frac{1}{2} \sigma^{x,y,z} \) in the definition of the Hamiltonian.

We will generally work with restrictions of \( H \) to finite intervals \([-L, L], L \in \mathbb{N}, \) i.e.,
\[ H^{(L)} = \sum_{i=-L}^{L-1} h_{i,i+1} + \lambda \sum_{i=-L}^{L} \omega_i N_i + \beta(\mathcal{N}_- + \mathcal{N}_+), \]
which is self-adjoint on \( \otimes_{-L}^{L} \mathbb{C}^2 \). In the droplet boundary term \( \beta(\mathcal{N}_- + \mathcal{N}_+) \) (compare [42]) we choose \( \beta \geq \frac{1}{2}(1 - \frac{1}{\Delta}) \).

The choice of the boundary condition in (1.5) is due to our methods of proof, as it gives us a convenient positivity property. While it is likely that the case \( \beta < \frac{1}{2}(1 - \frac{1}{\Delta}) \) can be covered by similar methods, we heavily use particle number conservation of the model (see Section 2.1), leaving more general boundary terms out of our reach at the moment.

As we will see from the more rigorous definition of the infinite volume Hamiltonian \( H \) as a random operator in Section 2.1, its spectrum is given with probability one by \( \sigma(H) = \{0\} \cup [1 - \frac{1}{\Delta}, \infty) \), see also [20]. Our main goal here is to show that \( H \) is many-body localized in the droplet spectrum \( I_1 = [1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})] \) if \( \Delta \) is sufficiently large or \( \lambda \) is sufficiently large. This will be expressed in terms of exponential clustering of the eigenstates of the finite chain \( H^{(L)} \) for energies in \( I_1 \), uniform in \( L \).

We will see that for each \( L \) the spectrum of \( H^{(L)} \) is almost surely simple, so that its normalized eigenvectors can be labeled as \( \psi_E, E \in \sigma(H^{(L)}) \). Given a finite subset \( J \subset [-L, L] \cap \mathbb{Z} \), a local observable \( X \) with support \( J \) is an operator on \( \otimes_{j \in J} \mathbb{C}^2 \), considered as an operator on \( \otimes_{-L}^{L} \mathbb{C}^2 \) by acting as the identity on spins not in \( J \); we write \( J = \text{supp} X \). A useful concept that quantifies how close a many body state \( \psi \) is to a product state is its correlator:
\[ R_{X,Y}(\psi) := |\langle \psi, XY\psi \rangle - \langle \psi, X\psi \rangle \langle \psi, Y\psi \rangle|, \]
where \( X \) and \( Y \) are local observables. The rapid decay of correlators for gapped ground states was first discovered by Hastings [24], and led to the introduction of definitions and heuristic arguments in [25] that lay a general foundation for an analytical approach to MBL.
Localizatìon of the finite chain $H^{(L)}$ in an interval $I$ actually leads to the rapid decay of the sum of the correlators of all eigenstates with eigenvalues in $I$, $\sum_{E \in \sigma(H^{(L)}) \cap I} R_{X,Y}(\psi_E)$. To study this and more general correlators, note that the correlator of a simple eigenvector $\psi_E$ can be rewritten as

$$R_{X,Y}(\psi_E) = \left| \text{tr} \left( P^{(L)}_E X \bar{P}^{(L)}_E Y P^{(L)}_E \right) \right|,$$

where $P^{(L)}_E = \sum_{\lambda \in \sigma(E)} |\lambda\rangle \langle \lambda|$, with $P^{(L)}_E = \chi_F(H^{(L)})$ and $\bar{P}^{(L)}_E = 1 - P^{(L)}_E$ for $F \subset \mathbb{R}$. This suggests defining the correlator of an energy set $F \subset \mathbb{R}$ by

$$R_{X,Y}(F) = \left| \text{tr} \left( P^{(L)}_F X \bar{P}^{(L)}_F Y P^{(L)}_F \right) \right| = \left| \text{tr} \left( P^{(L)}_F XY P^{(L)}_F \right) - \text{tr} \left( P^{(L)}_F X P^{(L)}_F Y P^{(L)}_F \right) \right|. \quad (1.8)$$

In addition, given a finite energy interval $I$, we let $\mathcal{F}_I$ denote the collection of all partitions $F = \{F_q\}_{q \in \mathcal{Q}}$ of $I$ into finitely many disjoint intervals (of any kind), and define

$$\mathcal{R}_{X,Y}(I) = \sup_{F \in \mathcal{F}_I} R_{X,Y}(F), \quad \text{where } R_{X,Y}(F) = \sum_{q \in \mathcal{Q}} R_{X,Y}(F_q) \text{ for } F = \{F_q\}_{q \in \mathcal{Q}} \in \mathcal{F}_I. \quad (1.9)$$

Note that, since $\sigma(H^{(L)}) \cap I$ is a finite set, $\mathcal{R}_{X,Y}(I)$ is a measurable function.

In one particle localization, the localization properties of the eigenfunctions in an energy interval $I$ (closely related to the concept of exponential clustering in many body systems) lead to dynamical localization (non spreading of the wave packets under the time evolution) in the same energy window. As we shall see, this phenomenon persists in the many body setting as well, in the form of exponential clustering in an energy interval under the Heisenberg dynamics restricted to the same interval. Specifically, the time evolution of a local observable $X$ in the energy window $I$ for a finite chain $H^{(L)}$ in an interval $I$ is given by

$$\tau^I_t(X) = (\tau^I_t)^{(L)}(X) = e^{itH^{(L)}_I} X e^{-itH^{(L)}_I}, \quad \text{with } H_I^{(L)} = P_I^{(L)} H^{(L)}. \quad (1.10)$$

Our main result is given in the following theorem, where $\mathbb{E}$ denotes the expectation with respect to the random variables $\omega$. We fix $0 < \delta < 1$, and let

$$I_{1,\delta} = \left[ 1 - \frac{1}{\delta}, (2 - \delta)(1 - \frac{1}{\delta}) \right], \quad (1.11)$$

meaning that we set a fixed, but arbitrarily small, distance from the upper end of the droplet spectrum $I_1$.

**Theorem 1.1** (Dynamical exponential clustering in the droplet spectrum). There exists a constant $K > 0$ with the following property: If the parameters $(\Delta, \lambda)$ are in the region described by (1.1), there exist constants $C < \infty$ and $m > 0$ such that, letting $I = I_{1,\delta}$, for all local observables $X$ and $Y$ with $\text{max supp } X < \text{min supp } Y$ (or vice versa) we have, uniformly in $L$,

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \mathcal{R}_{\tau^I_t(X),Y}(I) \right) \leq C \|X\| \|Y\| e^{-m \text{dist(supp } X, \text{ supp } Y)}. \quad (1.12)$$

In particular, we have the special cases

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \sum_{E \in \sigma(H^{(L)}) \cap I} R_{\tau^I_t(X),Y}(\psi_E) \right) \leq C \|X\| \|Y\| e^{-m \text{dist(supp } X, \text{ supp } Y)} \quad (1.13)$$

and

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} R_{\tau^I_t(X),Y}(I) \right) \leq C \|X\| \|Y\| e^{-m \text{dist(supp } X, \text{ supp } Y)}. \quad (1.14)$$

The left hand side in (1.13) is the average of the supremum in time of the sum of the correlators of all eigenstates with eigenvalues in the droplet spectrum $I_{1,\delta}$. A special case of Theorem 1.1 is exponential clustering of stationary correlations (corresponding to the choice $t = 0$ in its statement). The constants $K, C, m$ depend on the distribution $\mu$ and on the parameter $\delta$. Note that the bound (1.12) depends only on the distance between the supports of $X$ and $Y$, without pre-factors depending on the sizes of these supports—a particularly strong feature of our model and the methods used to prove the result. Theorem 1.1 extends
exponential clustering bounds previously only known for stationary ground state correlations of gapped systems (e.g., [38]), and also covers dynamical correlations at arbitrary time.

In view of a result by Brandao and Horodecki [12], this behavior strongly suggests that droplet states satisfy an area law for the bipartite entanglement as well. However, an additional analysis is required to properly account for the use of disorder averages in our context.

2. Strategy

An important special case of the local observables $X$ and $Y$ are the local number operators $N_i$ in (1.4). Their correlators in a normalized eigenvector $\psi_E$ of $H^{(L)}$ satisfy

$$R_{N_i,N_j}(\psi_E) = |\langle \psi_E, N_i(I - |\psi_E\rangle\langle \psi_E|)N_j\psi_E \rangle| \leq \|N_i\psi_E\| \|N_j\psi_E\|.$$  

(2.1)

The next theorem establishes Theorem 1.1 for the special case of $X = N_i, Y = N_j, \text{and } t = 0$.

Given an interval $I$, we set $G_I = \{ g : \mathbb{R} \to \mathbb{C} \text{ Borel measurable, } |g| \leq \chi_I \}$. We also let $\| \|$ denote the trace norm.

Theorem 2.1 (Exponential localization in the droplet spectrum). There exists a constant $K > 0$ with the following property: If the parameters $(\Delta, \lambda)$ are in the region described by (1.1), there exist constants $C < \infty$ and $m > 0$ such that

$$\mathbb{E} \left( \sum_{E \in \sigma(H^{(L)}) \cap I_{1,\delta}} \|N_i\psi_E\| \|N_j\psi_E\| \right) \leq Ce^{-m|i-j|} \text{ for all } -L \leq i, j \leq L,$$  

(2.2)

and

$$\mathbb{E} \left( \sup_{g \in G_{I_{1,\delta}}} \|N_i g(H^{(L)})N_j\|_1 \right) \leq Ce^{-m|i-j|} \text{ for all } i, j \in [-L, L],$$  

(2.3)

uniformly in $L$.

Note that (2.3) is an immediate consequence of (2.2), because, since the spectrum of $H^{(L)}$ is almost surely simple, we have $\|N_i F_E N_j\|_1 = \|N_i\psi_E\| \|N_j\psi_E\|$.

Theorem 2.1 will be pivotal in the proof of Theorem 1.1 in its general form (see Section 3 for details), and can be viewed as our main technical result.

2.1. Results for Fixed Particle Number. The crucial property of the XXZ chain to be exploited for the proof of Theorem 2.1 is particle number conservation. Let

$$\mathcal{N}^{(L)} = \sum_{i=-L}^{L} N_i$$  

(2.4)

be the total (down) spin number operator. Then $[H^{(L)}, \mathcal{N}^{(L)}] = 0$, and $\mathcal{N}^{(L)}$ has eigenvalues $\lambda_N = N, N = 0, 1, \ldots, 2L+1$, with eigenspaces spanned by all the spin basis states with $N$ down spins, called the $N$-particle sector (or $N$-magnon sector, as one may equivalently use the eigenspaces of the total magnetization operator $S^z = \sum_i \sigma_i^z$).

The restriction of the XXZ chain to the $N$-particle sector is unitarily equivalent to an $N$-body discrete Schrödinger operator restricted to the fermionic subspace. Due to working with the one-dimensional XXZ chain, this operator can also be directly expressed as a Schrödinger-type operator over the (induced) subgraph of ordered lattice points $A_N = \{ x = (x_1, \ldots, x_N) \in \mathbb{Z}^N : x_1 < x_2 < \ldots < x_N \}$ of $\mathbb{Z}^N$, and, in the finite volume case, over $A_N^{(L)} = \{ x \in \mathbb{Z}^N : -L \leq x_1 < \ldots < x_N \leq L \}$.

For the infinite volume case this is carried out in detail in [20]. The argument there can be adjusted to the finite volume case (1.5), including the boundary condition, which we need here. As [20] uses adjacency operators while we will consider graph Laplacians, some care is needed in this derivation due to the fact...
that the graphs $\mathcal{X}_N^{(L)}$ have non-constant nearest neighbor degree. As a result, the restriction of $H^{(L)}_N$ to the $N$-particle sector is unitarily equivalent to the self-adjoint operator on $l^2(\mathcal{X}_N^{(L)})$ given by

$$H^{(L)}_N = -\frac{1}{2\lambda} \mathcal{L}_N^{(L)} + (1 - \frac{1}{\lambda}) \bar{W} + \lambda V_\omega + (\beta - \frac{1}{2}(1 - \frac{1}{\lambda})) \chi^{(L)}.$$  

(2.5)

Here $\mathcal{L}_N^{(L)}$ is the graph Laplacian on $\mathcal{X}_N^{(L)}$,

$$\left(\mathcal{L}_N^{(L)}\psi\right)(x) = \sum_{y \in \mathcal{X}_N^{(L)} \atop |x-y|=1} (\psi(y) - \psi(x)),$$

(2.6)

with $|x-y| := \sum_{i=1}^N |x_i - y_i|$ denoting graph distance,

$$\bar{W}(x) = N - \# (j : x_{j+1} = x_j + 1) = 1 + \# (j : x_{j+1} \neq x_j + 1),$$

(2.7)

and the $N$-body random potential $V_\omega$ is given by

$$(V_\omega \psi)(x) = \left(\sum_{j=1}^N \omega_{x,j}\right) \psi(x).$$

(2.8)

Finally, in the last term of (2.5) we have $\chi^{(L)} = \chi_-^{(L)} + \chi_L^{(L)}$, where $\chi_-^{(L)}$ and $\chi_L^{(L)}$ denote the indicator functions of the left and right boundaries

$$\{(x_1, \ldots, x_N) \in \mathcal{X}_N^{(L)} : x_1 = -L\} \quad \text{and} \quad \{(x_1, \ldots, x_N) \in \mathcal{X}_N^{(L)} : x_N = L\}$$

(2.9)

of $\mathcal{X}_N^{(L)}$ within $\mathcal{X}_N$. The exact value $\beta - \frac{1}{2}(1 - \frac{1}{\lambda})$ of the pre-factor in (2.5) is due to the fact that part of the boundary term $\beta(N_L - N_L)$ in (1.5) is absorbed into the restricted graph Laplacian $-\frac{1}{2\lambda} \mathcal{L}_N^{(L)}$. This explains our assumption $\beta \geq \frac{1}{2}(1 - \frac{1}{\lambda})$, as this assures that the last term in (2.5) is non-negative.

These considerations yield the unitary equivalence

$$H^{(L)} \cong \bigoplus_{N=0}^{2L+1} H^{(L)}_N,$$

(2.10)

where one identifies the standard basis vectors $\phi_x := \delta_x \in l^2(\mathcal{X}_N^{(L)})$, $x \in \mathcal{X}_N^{(L)}$, of up-down spin configurations over $[-L, L]$, with down-spins in the positions $x_1 < \ldots < x_N$ and up-spins elsewhere. Here $H^{(L)}_0$ denotes the zero operator on a one-dimensional Hilbert space, representing the all up-spins ground state of $H^{(L)}$.

The identity (2.10) also provides a convenient way to rigorously define the infinite XXZ Hamiltonian as the direct sum $H = \bigoplus_{N=0}^{\infty} H_N$, where

$$H_N = -\frac{1}{2\lambda} \mathcal{L}_N + (1 - \frac{1}{\lambda}) \bar{W} + \lambda V_\omega,$$

(2.11)

as an operator on $l^2(\mathcal{X}_N)$. Boundedness of the random variables $\omega_i$ assures that each $H_N$ is bounded and self-adjoint. Their norms grow linearly in $N$, so that $H$ becomes an unbounded self-adjoint operator on the direct sum of these Hilbert spaces. On infinite spin configurations with finitely many down-spins, which form an operator core of the direct sum, $H$ acts formally by the expression in (1.2).

We will be interested in the droplet regime of the XXZ chain. To describe this, first set $V_\omega = 0$ and consider the infinite volume unperturbed operators

$$H_{N,0} = -\frac{1}{2\lambda} \mathcal{L}_N + (1 - \frac{1}{\lambda}) \bar{W}$$

(2.12)

in $l^2(\mathcal{X}_N)$. These operators are purely absolutely continuous due to their invariance under the translations $T_N(x_1, \ldots, x_N) = (x_1 + 1, \ldots, x_N + 1)$ on $\mathcal{X}_N$, interpreted as a shift of the center of mass (see [20] for a review of this and the other properties of $H_{N,0}$ discussed in the following). $H_{N,0}$ can be explicitly diagonalized via the Bethe ansatz for arbitrary $N$. In particular, in the Ising phase $\Delta > 1$ the term $(1 - \frac{1}{\lambda}) \bar{W}$
represents an attractive next-neighbor interaction (each pair of particles occupying neighboring sites lowers the energy by \((1 - \frac{1}{\Delta})\)). This leads to a “droplet band” \(\delta_N\) at the bottom of the spectrum in each \(N\)-particle sector. The corresponding generalized eigenfunctions are concentrated at the one-dimensional “edge”

\[
X_{N,1} = \{ x = (x_1, x_1 + 1, \ldots, x_1 + N - 1) : x_1 \in \mathbb{Z} \}
\]  

(2.13)
of \(X_N\), along which they are quasi-periodic Bloch waves, and decay exponentially into all \(N - 1\) bulk directions of \(X_N\) (representing the separation distance of particle clusters). Thus droplet states are \(N\)-particle states with all particles packed into \(N\) neighboring sites, up to exponentially small tails. In the spin chain this corresponds to states which are exponentially close to a single droplet of \(N\) neighboring down-spins in a sea of up-spins, compare [42].

The droplet bands are explicitly given by

\[
\delta_N = \left[ \tanh(\rho) \cdot \frac{\cosh(N\rho) - 1}{\sinh(N\rho)}, \tanh(\rho) \cdot \frac{\cosh(N\rho) + 1}{\sinh(N\rho)} \right],
\]

(2.14)
where \(\cosh(\rho) = \Delta\). They form a decreasing sequence of intervals, the first few given by

\[
\delta_1 = [1 - \frac{1}{2\Delta}, 1 + \frac{1}{2\Delta}], \quad \delta_2 = [1 - \frac{1}{4\Delta}, 1], \quad \delta_3 = [1 - \frac{1}{2\Delta}, 1 - \frac{1}{2\Delta + \Delta}],
\]

(2.15)
which for \(N \to \infty\) contract monotonically into the single point \(\delta_\infty = \left\{ \sqrt{1 - \frac{1}{2\Delta}} \right\}\). For \(\Delta > 3\) these bands are strictly separated from the rest of the spectrum of \(H\), which consists of the ground state energy 0 and an infinite number of additional bands of bulk spectrum, corresponding to non-trivial scattering channels of the \(N\)-body operator \(H_{N,0}\), all contained in \([2(1 - \frac{1}{\Delta}), \infty)\).

Adding the positive random field \(V_\omega\) will enlarge the spectral bands upwards. Indeed, the \(H_N\) are ergodic with respect to the shifts \(T_N\) and have almost sure spectrum \(\Sigma_N = \sigma[H_{N,0}] + [0, N\omega_{\max}]\). Thus, as \(N\) is arbitrarily large, in the almost sure spectrum \(\Sigma = \{0\} \cup \bigcup_{N=1}^{\infty} \Sigma_N = \{0\} \cup [1 - \frac{1}{\Delta}, \infty)\) of \(H\) all spectral gaps (with the exception of the ground state gap) will be closed. Still, as the random potential \(V_\omega\) is non-negative, one expects that all generalized eigenfunctions to energies in the droplet spectrum \([1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})]\) of \(H\) will remain localized along the edge \(X_{N,1}\) of the graph. Rigorously establishing this, with bounds uniform in the particle number \(N\), will provide us with the main technical ingredient for the proof of Theorem 2.1.

Technically, we will accomplish the latter by proving localization of a suitable form of many-body eigenfunction correlators for the finite volume operators \(H_N^{(L)}\) in the droplet spectrum. These are defined as follows: Given a finite interval \(I \subset \mathbb{R}\) and a pair of indices \(i, j \in \mathbb{Z}\), the corresponding eigenfunction correlator is given by

\[
Q_N^{(L)}(i, j; I) = \sum_{E \in \sigma(H_N^{(L)}) \cap I} \| Q_i P_E Q_j \|_1,
\]

(2.16)
where \(\| \cdot \|_1\) denotes the trace class norm, \(P_E\) denote the spectral projection of \(H_N^{(L)}\) onto \(E\), and \(Q_i = Q_i^{(N,L)}\) is the indicator function of

\[
S_{\{i\}} = S_{\{i\}}^{(N,L)} := \{ x \in X_N^{(L)} : x_j = i \text{ for some } j \in \{1, \ldots, N\}\},
\]

(2.17)
i.e., the set of all those lattice sites at which the random potential depends on the random variable \(\omega_i\).

As we already mentioned in Section 1.2, the spectrum of \(H^{(L)}\) is almost surely simple. It allows us to label normalized eigenvectors as \(\psi_E, E \in \sigma(H^{(L)})\). Since each eigenvector lies in a fixed \(N\)-particle sector, we have \(\| Q_i P_E Q_j \|_1 = \| Q_i \psi_E \| \| Q_j \psi_E \|\) almost surely. We remark that \(Q_i = Q_i^{(N,L)}\) is the restriction of the local number operator \(N_i\) to the \(N\)-particle sector, as can be seen by the action of \(N_i\) on the product basis vectors

\[
e_\alpha = e_{\alpha_{-L}} \otimes \cdots \otimes e_{\alpha_L}, \quad \alpha \in \{0, 1\}^{\{-L, -L+1, \ldots, L\}},
\]

(2.18)
where \(e_0 = (1, 0)^t\) and \(e_1 = (0, 1)^t\).
It follows that, almost surely,

\[ \sum_{N=1}^{\infty} Q_N^{(L)}(i, j; I) = \sum_{E \in \sigma(H^{(L)}) \cap I} \| N_i \psi_E \| \| N_j \psi_E \|. \]  

(2.19)

Thus the estimate (2.2) can be reformulated as

\[ \sum_{N=1}^{\infty} \mathbb{E}(Q_N^{(L)}(i, j; I_1, \delta)) \leq C e^{-m|i-j|} \text{ for all } -L \leq i, j \leq L, \]  

(2.20)

a more convenient form of the bound that will be used in the proof of Theorem 2.1 in Section 8.

Let us note that if one defines

\[ \hat{Q}_N^{(L)}(i, j; I) = \sup \left\{ \| Q_i g(H_N^{(L)}) Q_j \|_1 \mid \text{supp } g \subset I, |g| \leq 1 \right\}, \]  

(2.21)

it is easy to see that \( \hat{Q}_N^{(L)}(i, j; I) \leq Q_N^{(L)}(i, j; I) \) (however, with the exception of the case \( N = 1 \), these two quantities are not equal). Thus, for each fixed \( N \), Theorem 2.1 yields exponential decay of \( \mathbb{E}(\hat{Q}_N^{(L)}(i, j; I_1, \delta)) \) in \(|i - j|\), uniform in \( L \). By known methods available in the literature on Anderson localization, this decay translates into exponential decay for the corresponding eigenfunction correlators of the infinite volume operators \( H_N \). The latter property can then be used to deduce that all \( H_N \) (and thus their direct sum, the infinite spin chain Hamiltonian \( H \)) have pure point spectrum in \( I_{1, \delta} \). Since these arguments are fairly standard and the result is only marginally related to the presentation here, we skip a more detailed discussion.

Casting (2.2) in the form (2.20) allows us to reduce the proof of Theorem 2.1 to establishing decay properties of the Green’s functions associated with the operators \( H_N^{(L)} \). In fact, our proofs of these results will also hold for the infinite volume operators \( H_N \), which we include because they are of some independent interest.

The Green’s function analysis will be done separately along the edge \( \lambda_{N,1} \) (and its finite volume analogs \( \lambda_{N,1}^{(L)} := \lambda_{N,1} \cap \mathcal{X}_N^{(L)} \)) and within the bulk \( \lambda_{N,1}^b := \mathcal{X}_N \setminus \lambda_{N,1} \) (and \( \lambda_{N,1}^{(L)} := \lambda_{N,1}^b \cap \lambda_{N,1}^{(L)} \)). In the bulk we have the following (purely deterministic) Combes-Thomas-type bound, which will be proven in Section 4.

**Theorem 2.2** (Combes-Thomas bound in the bulk). Consider \( (\Delta, \lambda) \in (1, \infty) \times (0, \infty) \), and let \( \hat{H}_{N,1} \) denote the restriction of \( H_N \) to \( \ell^2(\lambda_{N,1}) \). Then there exist constants \( C = C(\Delta) < \infty \) and \( \eta = \eta(\Delta) > 0 \), independent of \( \lambda \) and \( N \), such that

\[ \| \chi_A(\hat{H}_{N,1} - E - i\epsilon)^{-1} \chi_B \| \leq C e^{-\eta \text{dist}_1(A,B)}, \]  

(2.22)

for all \( N \in \mathbb{N}, E \in I_{1, \delta}, \epsilon \in \mathbb{R}, \) and subsets \( A \) and \( B \) of \( \lambda_{N,1}^b \).

The same bound holds for the restrictions \( \hat{H}_{N,1}^{(L)} \) of \( H_N^{(L)} \) to \( \ell^2(\lambda_{N,1}^{(L)}) \), with constants uniform in \( N \) and \( L \).

Here \( \text{dist}_1(A, B) = \inf_{x \in A, y \in B} |x - y| \) and \( \chi_A \) and \( \chi_B \) are the indicator functions of \( A \) and \( B \).

In fact, we will need and prove results more general than Theorem 2.2 in Section 4 but the above version captures the essence of the type of Combes-Thomas bounds which will be used in this work.

Along the edge we will prove exponential decay of fractional moments of the Green’s function in Section 5 as in the following theorem.

**Theorem 2.3** (Fractional moment estimate on the edge). There exists a constant \( K > 0 \) with the following property: If the parameters \( (\Delta, \lambda) \) are in the region described by (1.1), there exist constants \( C = C(\Delta) < \infty \) and \( \xi = \xi(\Delta) > 0 \) (depending only on \( \Delta \)), such that

\[ \mathbb{E} \left( \left| \left\langle \phi_u, (H_N - E - i\epsilon)^{-1} \phi_v \right\rangle \right|^{\frac{3}{2}} \right) \leq \frac{C}{\sqrt{\lambda_A}} e^{-\xi\|u - v\|}, \]  

(2.23)

for all \( N \in \mathbb{N}, E \in I_{1, \delta}, \epsilon > 0, \) and \( u, v \in \lambda_{N,1} \).

Moreover, the bound (2.23) also holds for the operators \( H_N^{(L)} \), uniformly in \( L \), where \( \epsilon = 0 \) is included.
Here \( \|u - v\| := \max \{|u_i - v_i| : 1 \leq i \leq N\} \) is the \( \infty \)-distance.

It is possible to combine Theorems 2.2 and 2.3 into a global decay bound for the fractional moments. However, this would essentially require to use the \( \infty \)-distance (coming from the edge bound), which is insufficient to handle the bulk contributions in the application to eigencorrelators in Theorem 2.1 (note that in the bulk and for high dimension \( N \), the 1-distance is typically much larger than the \( \infty \)-distance). Instead, we will directly combine Theorems 2.2 and 2.3 (and variants of them) to prove Theorem 2.1 in Section 8 after providing Wegner-type estimates and other technical tools in Sections 6 and 7.

2.2. Outline of Contents. In Section 5 we will link Theorem 2.1 to our main result on exponential many-body clustering. As noted above, the key idea is to reduce the analysis for general local observables \( X \) and \( Y \) and general times \( t \) to the special case given by 2.1.

The remaining part of the paper, Sections 6–8, is devoted to the derivation of the technical results stated in Theorem 2.1–2.3 and, in the case of Theorems 2.2 and 2.3, to some further extensions of these statements used in the proofs.

The key observation behind our arguments there is that for energies in the droplet band \( I_1 \) only the one-quasi-particle sector \( \mathcal{X}_{N,1} \) (the ”edge” of \( \mathcal{X}_N \)) is classically accessible for all values of \( N \). So, a naive removal of the classically forbidden region for each \( N \) maps the problem to the well studied one dimensional Anderson model (with a correlated random potential). The latter is characterized by complete spectral and dynamical localization. As it turns out, the underlying localization length is uniformly bounded in \( N \), while the density of states in the droplet band rapidly decreases with \( N \). These features play an instrumental role in understanding the many body properties of the XXZ model in the sequel.

The rigorous passage from the whole \( \mathcal{X}_N \) space to the edge \( \mathcal{X}_{N,1} \) is implemented by means of Schur complementation (known in the physics literature as the Feshbach map). It allows to express the edge-restricted Green’s function \( G_E \) of the full operator \( H_N \) at energy \( E \) in terms of the Green’s function of an effective Hamiltonian \( K_E \) defined on \( \mathcal{X}_{N,1} \). Similar techniques have been used frequently before, explicitly or implicitly, for example in works on random surface potentials (e.g. [28, 29]) or on random potentials restricted to sublattices [18].

The operator \( K_E \) is comprised of two parts: The first one is simply the restriction of \( H_N \) to the edge as in the naive description, while the second part encodes the influence of the bulk. The technical difficulties associated with the addition of the second term are two-fold: On one hand, it is non-local and non-linear (in \( E \)), on the other, it is statistically dependent on the randomness associated with the first term. Both issues can have potentially fatal consequences as far as localization is concerned: Non-locality allows for hopping between distant sites while strongly correlated randomness can amplify the effect of resonances, suppressed in the non correlated case.

A significant portion of this paper is devoted to the handling of these two issues. The non-locality of \( K_E \) is tackled with tools developed in Section 4, namely Combes-Thomas bounds for \( H_N \) in the bulk, and for a (properly) modified \( H_N \) on the whole \( \mathcal{X}_N \). They allow us to show that \( K_E \) is in fact quasi-local for energies in the droplet band, i.e., its kernel exhibits rapid spatial decay. Combes-Thomas estimates are widely used in localization theory; the novel element in our context is the uniform control (in \( N \)) of the associated decay rate. We then modify the fractional moment proof of localization initially developed by Aizenman and Molchanov in the one particle context to apply to the one quasiparticle Hamiltonian \( K_E \) in Section 5 resulting in rapid spatial decay for its Green’s function in expectation.

It turns out that Combes-Thomas estimates are also useful in controlling correlations of the random variables; they allow us to effectively decouple Hamiltonians associated with well separated domains in Section 7. A Wegner estimate for \( K_E \), which provides a preliminary result pertaining to this decoupling (and is of some independent value on its own right), is established in Section 5 using a strategy from [48] to handle the non-linearity of \( K_E \). Part of the decoupling procedure in Section 7 specifically Lemma 7.1 can also be seen as a form of the Wegner estimate which is uniform in the correlations within the random potential, reminiscent of a Wegner estimate in [31].
The main decoupling result of Section 7, Lemma 7.2 follows [19] to show that we can extract the conclusions of the energy interval multiscale analysis as in [15] from the fractional moment estimate of Section 5. We then use ideas from proofs of dynamical localization for random Schrödinger operators as in [22] to derive Theorem 2.1 in Section 8. An interesting feature of the argument used there is that small $N$ are treated using the decoupling between “boxes” of Lemma 7.2, while large $N$ are handled via a large deviation argument reflecting the smallness of the density of states of $H_N$ in $I_1$.

3. Exponential Clustering

We start by showing that our main result, Theorem 1.1 is a consequence of the $N$-particle eigencorrelator bounds of Theorem 2.1.

We note that $H^{(L)}$ almost surely has simple spectrum. A simple analyticity based argument for this can be found in Appendix A of [2] (the argument is presented there for the XY chain, but it holds for every random operator of the form $H_0 + \sum_{k=-L}^{L} \omega_k \mathcal{N}_k$ in $\otimes_{k=-L}^{L} \mathbb{C}^2$, as in our case). Thus, almost surely, all its normalized eigenstates can be labeled as $\psi_E$ with the corresponding eigenvalue $E$ of $H^{(L)}$. Moreover, $\psi_E$ belongs to one of the $N$-particle sectors of $H^{(L)}$.

Given a local observable $X$ with support $\mathcal{S}_X$, we define projections $P^{(X)}_{\pm}$ by

$$P^{(X)}_+ = \bigotimes_{j \in \mathcal{S}_X} (1 - N_j), \quad \text{and} \quad P^{(X)}_- = 1 - P^{(X)}_+ \quad (3.1)$$

Note that $P^{(X)}_{\pm}$ are supported on $\mathcal{S}_X$, $P^{(X)}_+$ projects onto the basis states $e_\alpha$ (see (2.18)) with no particles in $\mathcal{S}_X$, i.e., $\alpha_j = 0$ for all $j \in \mathcal{S}_X$, and $P^{(X)}_-$ projects onto states with at least one particle in $\mathcal{S}_X$. In particular, we have

$$P^{(X)}_- \leq \sum_{i \in \mathcal{S}_X} N_i. \quad (3.2)$$

In addition, $[P^{(X)}_{\pm}, \mathcal{N}] = 0$ (where $\mathcal{N} = \mathcal{N}^{(L)}$) and

$$X = \sum_{a,b \in \{+,-\}} X^{a,b}, \quad \text{where} \quad X^{a,b} = P^{(X)}_a X P^{(X)}_b, \quad (3.3)$$

all of which are supported on $\mathcal{S}_X$. Moreover, since $P^{(X)}_-$ is a rank one projection on $\mathcal{S}_X$, we must have

$$X^{+,+} = \zeta P^{(X)}_+, \quad \text{where} \quad \zeta \in \mathbb{C}, \ |\zeta| \leq \|X\|. \quad (3.4)$$

Lemma 3.1. Let $I \subset \mathbb{R}$ be an interval, set $\sigma_I = \sigma(H^{(L)}) \cap I$, and assume that all eigenvalues $E$ in $\sigma_I$ are simple. Then for any two local observables $X$ and $Y$ with $\mathcal{S}_X \cap \mathcal{S}_Y = \emptyset$, we have (see (1.9))

$$\sup_{t \in \mathbb{R}} R_{\tau_I t}(X,Y) (I) \leq C \|X\| \|Y\| \sum_{E \in \sigma_I} \|P_{-}^{(X)} \psi_E\| \|P_{-}^{(Y)} \psi_E\|. \quad (3.5)$$

Proof. Let $F \subset I$. It follows from (1.8) and (3.3) that

$$R_{\tau_I t}(X,Y) (F) \leq \sum_{a,b,c,d \in \{+,-\}} R_{\tau_I t}(X^{a,b},Y^{c,d}) (F). \quad (3.6)$$

Let $\psi$ be a many body state with $\|\psi\| = 1$ and $\mathcal{N} \psi = N \psi$, where $N \in \mathbb{N}$. Then for $Z^{a,b} = \tau_I t (X^{a,b})$ or $Z^{a,b} = Y^{a,b}$, we have (using, in particular, that $\mathcal{E}^{\pm iH^{(L)}}$ leaves the particle sectors invariant)

$$\chi_{[0,N-1]}(\mathcal{N}) Z^{+,+} \psi = Z^{+,+} \psi \quad \text{and} \quad \chi_{[N+1,\infty]}(\mathcal{N}) Z^{-,+} \psi = Z^{-,+} \psi. \quad (3.7)$$

Since it follows from (1.8) that (we mostly omit $L$ from the notation)

$$R_{\tau_I t}(X,Y) = \sum_{E \in \sigma_F} \langle \psi_E, \tau_I t (X) \tilde{P}_F Y \psi_E \rangle, \quad (3.8)$$
where $\sigma_F = \sigma(H^{(L)}) \cap F$, we conclude that
\begin{equation}
R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)} = R_{\tau_F}^{(X^{-,+},Y^{-,+})(F)} = 0.
\end{equation}
(3.9)

In addition, it follows from (3.8), (3.4) and $P_+^{(X)} = 1 - P_-^{(X)}$ that for any observable $Z$ we have
\begin{equation}
R_{\tau_F}^{(X^{+,+}),Z}(F) = |\zeta| R_{\tau_F}^{(P_-^{(X)}),Z}(F), \quad \text{where} \quad \zeta \in \mathbb{C}, \; |\zeta| \leq \|X\|.
\end{equation}
(3.10)

Furthermore, for any two local observables $Z$ and $W$ we have
\begin{equation}
R_{\tau_F}^{(Z),W}(F) = R_{\tau_F}^{(W^*),Z^*}(F).
\end{equation}
(3.11)

Thus, using (3.9) – (3.11), one can reduce estimating all sixteen terms in (3.6) to estimating correlators of the form
\begin{equation}
\begin{align*}
&\quad R_{\tau_F}^{(X^{-,+},Y^{+,+})(F)}, \quad R_{\tau_F}^{(X^{+,+},Y^{-,+})(F)}, \quad R_{\tau_F}^{(X^{-,+},Y^{-,+})(F)}, \\
&\quad R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)}, \quad R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)}, \quad R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)}, \quad R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)},
\end{align*}
\end{equation}
(3.12)

where $X$ and $Y$ are local observables with disjoint supports with $X^{+,+} = Y^{+,+} = 0$.

The three cases in (3.12) can be treated together. If $a, b \in \{+, -, \}$, we have
\begin{equation}
\begin{aligned}
R_{\tau_F}^{(X^{+,+},Y^{+,+})(F)} &\leq \sum_{E \in \sigma_F} \left| \langle \psi_E, \tau_F^{\dagger} \left( X^{-b} H \bar{P}_F Y^{a} \bar{\psi}_E \right) \rangle \right| \\
&\leq \|X\| \|Y\| \sum_{E \in \sigma_F} \left| P_-^{(X)} \bar{\psi}_E \right| \left| P_-^{(Y)} \bar{\psi}_E \right|. \tag{3.14}
\end{aligned}
\end{equation}

It remains to consider the two cases in (3.13), which are of the form $R_{\tau_F}^{(X^{a,-},Y^{-,b})(F)}$ with $a, b \in \{+, -, \}$, and can be treated together. We have
\begin{equation}
\begin{aligned}
R_{\tau_F}^{(X^{a,-},Y^{-,b})(F)} &\leq \left| \text{tr} \left( P_F \tau_F^{\dagger} \left( X^{a,-} H \bar{P}_F Y^{b} - P_F \right) \right) \right| + \left| \text{tr} \left( P_F \tau_F^{\dagger} \left( X^{a,-} H \bar{P}_F Y^{b} - P_F \right) \right) \right|
\end{aligned}
\end{equation}
(3.15)

where we used $\bar{P}_F \bar{P}_F = \bar{P}_F$.

We estimate the first term by
\begin{equation}
\begin{aligned}
\left| \text{tr} \left( P_F \tau_F^{\dagger} \left( X^{a,-} \right) \bar{P}_F Y^{b} P_F \right) \right| &\leq \sum_{E \in \sigma_F} \left| \langle \psi_E, X^{a,-} \bar{P}_F Y^{b} \bar{\psi}_E \rangle \rangle \right| \\
&\leq \sum_{E \in \sigma_F} \sum_{E' \in \sigma_1} |\langle \psi_E, X^{a,-} \bar{\psi}_E \rangle| |\langle \bar{\psi}_E, Y^{b, \bar{\psi}_E} \rangle|.
\end{aligned}
\end{equation}
(3.16)

For the second term, we use
\begin{equation}
\begin{aligned}
\left| \text{tr} \left( P_F \tau_F^{\dagger} \left( X^{a,-} \right) \bar{P}_F Y^{b} P_F \right) \right| &= \left| \text{tr} \left( e^{iH\tau} P_F X^{a,-} \bar{P}_F Y^{b} P_F \right) \right| \\
&\leq \left| \text{tr} \left( e^{iH\tau} P_F X^{a,-} P_+^{(Y)} \bar{P}_F Y^{b} P_F \right) \right| + \left| \text{tr} \left( e^{iH\tau} P_F X^{a,-} P_-^{(Y)} \bar{P}_F Y^{b} P_F \right) \right|
\end{aligned}
\end{equation}
(3.17)

We have $X^{a,-} P_+^{(Y)} \bar{P}_F Y^{b} = - P_-^{(Y)} X^{a,-} \bar{P}_F Y^{b}$, since $P_+^{(Y)} P_-^{(Y)} = 0$ and $X^{a,-} P_+^{(Y)} = P_+^{(Y)} X^{a,-}$ (due to their disjoint supports), so we get
\begin{equation}
\begin{aligned}
\left| \text{tr} \left( e^{iH\tau} P_F X^{a,-} P_+^{(Y)} \bar{P}_F Y^{b} P_F \right) \right| &\leq \sum_{E \in \sigma_F} \sum_{E' \in \sigma_1} |\langle \psi_E, P_+^{(Y)} X^{a,-} \bar{\psi}_E \rangle| |\langle \bar{\psi}_E, Y^{b, \bar{\psi}_E} \rangle|.
\end{aligned}
\end{equation}
(3.18)
where we have used the Cauchy-Schwarz and Bessel’s inequalities. It thus follows from (3.15)-(3.21) that

\[ \left| \text{tr} \left( e^{itH} P_F X^{a,-} P_Y^{-} P_I Y^{-,b} P_F \right) \right| \leq \left| \text{tr} \left( e^{itH} P_F P_Y^{-} X^{a,-} P_I P_Y^{+} Y^{-,b} P_F \right) \right| + \left| \text{tr} \left( e^{itH} P_F P_Y^{-} X^{a,-} P_I Y^{-,b} P_Y^{+} P_F \right) \right|. \]  

(3.19)

We have

\[ \left| \text{tr} \left( e^{itH} P_F P_Y^{-} X^{a,-} P_I P_Y^{+} Y^{-,b} P_F \right) \right| = \left| \text{tr} \left( e^{itH} P_F P_Y^{-} X^{a,-} P_I Y^{-,b} P_Y^{+} P_F \right) \right| \leq \sum_{E \in \sigma_F} \sum_{E' \in \sigma_I} \left| \langle \psi_E, P_Y^{-} X^{a,-} \psi_{E'} \rangle \right| \left| \langle \psi_{E'}, Y^{-,b} P_Y^{+} \psi_{E'} \rangle \right|, \]  

(3.20)

where we have again exploited the fact that \( X \) and \( Y \) have disjoint supports and \( X^{a,-} P_I P_Y^{+} Y^{-,b} = -X^{a,-} P_I Y^{-,b} P_Y^{+} \). In addition,

\[ \left| \text{tr} \left( e^{itH} P_F P_Y^{-} X^{a,-} P_I Y^{-,b} P_Y^{+} P_F \right) \right| \leq \| X \| \| Y \| \sum_{E \in \sigma_F} \left\| P_Y^{-} \psi_{E'} \right\| \left\| P_Y^{+} \psi_{E'} \right\|. \]  

(3.21)

Now let \( \mathcal{F} = \{ F_q \}_{q \in \mathcal{Q}} \). Then for any operators \( Z_1, Z_2 \) we have

\[ \sum_{q \in \mathcal{Q}} \sum_{E \in \sigma_F} \sum_{E' \in \sigma_I} \left| \langle \psi_E, Z_1 \psi_{E'} \rangle \right| \left| \langle \psi_{E'}, Z_2 \psi_{E'} \rangle \right| \leq \sum_{E' \in \sigma_I} \left( \sum_{E \in \sigma_F} \left| \langle \psi_E, Z_1 \psi_{E'} \rangle \right|^2 \right)^{1/2} \left( \sum_{E \in \sigma_I} \left| \langle \psi_{E'}, Z_2 \psi_{E'} \rangle \right|^2 \right)^{1/2} \leq \sum_{E' \in \sigma_I} \| Z_1 \psi_{E'} \| \| Z_2 \psi_{E'} \|, \]  

(3.22)

where we have used the Cauchy-Schwarz and Bessel’s inequalities. It thus follows from (3.15)-(3.21) that

\[ R_{t_I}^{(X^{a,-}, Y^{-,b})}(\mathcal{F}) = \sum_{q \in \mathcal{Q}} R_{t_I}^{(X^{a,-}, Y^{-,b})}(F_q) \leq 4 \| X \| \| Y \| \sum_{E \in \sigma_I} \left\| P_Y^{-} \psi_{E'} \right\| \left\| P_Y^{+} \psi_{E'} \right\|. \]  

(3.23)

We conclude from (3.14) and (3.23) that, for local observables \( X, Y \) with \( X^{+,+} = Y^{+,+} = 0 \),

\[ \sup_{t \in \mathbb{R}} R_{t_I}^{(X, Y)}(\mathcal{F}) \leq C \| X \| \| Y \| \sum_{E \in \sigma_I} \left\| P_Y^{-} \psi_{E} \right\| \left\| P_Y^{+} \psi_{E} \right\|. \]  

(3.24)

We are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Let \( I = I_{1,\delta} \). Given two local observables \( X \) and \( Y \) with \( \max \mathcal{S}_X < \min \mathcal{S}_Y \) (or vice-versa), it follows from Lemma 3.1 that, almost surely,

\[ \sup_{t \in \mathbb{R}} \mathcal{R}_{t_I}^{(X, Y)}(I) \leq C \| X \| \| Y \| \sum_{E \in \sigma_I} \left\| P_Y^{-} \psi_{E} \right\| \left\| P_Y^{+} \psi_{E} \right\|. \]  

(3.25)

In view of (3.22), we have

\[ \left\| P_Y^{(Z)} \psi_{E} \right\| \leq \sum_{i \in S_Z} \| \mathcal{N}_i \psi_{E} \| \quad \text{for all } E \in \sigma_I, \quad \text{where } Z = X, Y. \]  

(3.26)

Thus, if the parameters \( (\Delta, \lambda) \) are in the region described by (1.1) as in Theorem 2.1 it follows from this theorem that

\[ \mathbb{E} \left( \sup_{t \in \mathbb{R}} \mathcal{R}_{t_I}^{(X, Y)}(I) \right) \leq C \| X \| \| Y \| \sum_{i \in \mathcal{S}_X, j \in \mathcal{S}_Y} e^{-m|j-i|}. \]  

(3.27)
This estimate implies (1.12), since given \( J, K \subset \mathbb{Z} \) with \( \max J < \min K \) we have
\[
\sum_{j \in J, k \in K} e^{-m|j-k|} \leq \sum_{j \in \mathbb{Z}, j \leq 0} e^{-m(k-j)} = e^{-m \text{dist}(J,K)} \sum_{j \in \mathbb{Z}, k \geq 0} e^{-m(k-j)}
\]
\[
= e^{-m \text{dist}(J,K)} \left( \sum_{k=0}^{\infty} e^{-mk} \right)^2 = (1 - e^{-m})^{-2} e^{-m \text{dist}(J,K)}.
\] (3.28)

Theorem 1.1 is proven. \( \square \)

4. COMBES-THOMAS BOUNDS

A key feature to be exploited in our analysis is that the “local dimension” of the graph \( \mathcal{X}_N \), when measured by the number of next neighbors of a vertex, gradually increases from being one-dimensional at the edge \( \mathcal{X}_{N,1} \) to \( N \)-dimensional in the deep bulk (where \( \mathcal{X}_N \) locally looks like \( \mathbb{Z}^N \)). This can be made precise with the following definitions.

Let
\[
\mathcal{X}_{N,k} = \left\{ x \in \mathcal{X}_N : \bar{W}(x) \leq k \right\}, \quad \bar{\mathcal{X}}_{N,k} = \mathcal{X}_N \setminus \mathcal{X}_{N,k},
\] (4.1)
with corresponding decomposition of \( \mathcal{H} \) into subspaces \( \mathcal{H}_k = \ell^2(\mathcal{X}_{N,k}) \) and \( \bar{\mathcal{H}}_k = \ell^2(\bar{\mathcal{X}}_{N,k}) \). Let \( P_k \) and \( \bar{P}_k \) be the orthogonal projections onto \( \mathcal{H}_k \) and \( \bar{\mathcal{H}}_k \), respectively. Equivalently, \( P_k \) is the projection onto states \( \phi_x \) where the configuration \( x = (x_1, \ldots, x_N) \) consists of at most \( k \) “clusters” (i.e. connected components of nearest neighbors) and \( \bar{P}_k \) projects onto states with at least \( k + 1 \) clusters. For an operator \( A \) on \( \ell^2(\mathcal{X}_N) \), we will denote by \( A_k = P_k A \bar{P}_k \) and \( \bar{A}_k = \bar{P}_k A \bar{P}_k \) the restrictions of \( A \) to \( \mathcal{H}_k \) and \( \bar{\mathcal{H}}_k \).

For \( k = 1 \) this gives the one-dimensional edge \( \mathcal{X}_{N,1} \) defined in (2.13) above. Note that a vertex \( x = (x_1, x_1 + 1, \ldots, x_1 + N - 1) \) in \( \mathcal{X}_{N,1} \) has only two nearest neighbors in \( \mathcal{X}_N \), \( (x_1 - 1, x_1 + 1, \ldots, x_1 + N - 1) \) and \( (x_1, \ldots, x_1 + N - 2, x_1 + N) \). Similarly, the vertices \( x \) in \( \mathcal{X}_{N,k} \setminus \mathcal{X}_{N,k-1} \), consisting of \( k \) clusters, have exactly \( \bar{W}(x) = 2k \) nearest neighbors. Thus the transitional region of \( \mathcal{X}_N \) between its edge and deep bulk has geometric structure highly uniform in \( N \). As this region determines the bottom of the spectrum of \( H \) in the Ising phase, we will be able to prove operator bounds with constants uniform in \( N \).

Most prominently, this geometric structure of the graphs \( \mathcal{X}_N \) will be exploited in the following Combes-Thomas bound suitable for our model. This result holds not only in the (single) droplet spectrum \( I_{1,\delta} \), but for the larger “\( k \)-droplet intervals”
\[
I_{k,\delta} := \left[ 1 - \frac{1}{N}, (k + 1 - \delta)(1 - \frac{1}{N}) \right].
\] (4.2)

The restrictions \( (H_N)_k \) of \( H_N \) to \( \mathcal{H}_k \) satisfy the lower bound
\[
(H_N)_k \geq (k + 1) \left( 1 - \frac{1}{N} \right).
\] (4.3)

Indeed, as the negative Laplacian and the random potential are non-negative, one has the lower bound \( H_N \geq \left( 1 - \frac{1}{N} \right) \bar{W} \), which yields (4.3) after restricting to \( \mathcal{H}_k \). Thus Combes-Thomas bounds for \( (H_N)_k \) to energies \( E \in I_{k,\delta} \) are a standard fact. However, the constants in these bounds generally depend on the dimension of the graph, compare, e.g., [30, Section 11.2], [34, Appendix B]. That we can prove an \( N \)-independent bound here is due to the special structure of the graphs \( \mathcal{X}_N \).

It will be convenient for the latter purposes to formulate and prove this result in a greater generality. To this end, we need to recollect a few facts from graph theory: A graph is an ordered pair \( G = (V, E) \) comprising a set \( V \) of vertices together with a set \( E \) of edges. A subgraph of a graph \( G \) is another graph formed from a subset of the vertices and edges of \( G \). The vertex subset must include all endpoints of the edge subset, but may also include additional vertices. An induced subgraph of a graph is a subgraph formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset.
a graph $G = (\mathcal{V}, \mathcal{E})$ we define its Laplacian as an operator on $\ell^2(\mathcal{V})$, acting as

$$\left(\mathcal{L}^{(G)} \psi\right)(x) = \sum_{y \in \mathcal{V}} \left(\psi(y) - \psi(x)\right), \quad \psi \in \ell^2(\mathcal{V}).$$  \hfill (4.4)$$

The operator $-\mathcal{L}^{(G)}$ is positive definite. For an operator $H$ on a graph $G = (\mathcal{V}, \mathcal{E})$ and $\mathcal{V}' \subset \mathcal{V}$, we will denote by $H_{\mathcal{V}'}$ the restriction of $H$ to $\mathcal{V}'$, i.e. $H_{\mathcal{V}'} = P_{\mathcal{V}'} H P_{\mathcal{V}'}$, where $P_{\mathcal{V}'}$ is the indicator of the set $\mathcal{V}'$. If $\mathcal{V}$ is a potential, we will usually simply write $V$ for $V_{\mathcal{V}}$. In particular, we usually write $P_k$ and $\bar{P}_k$ for $(P_k)_{\mathcal{V}}$ and $(\bar{P}_k)_{\mathcal{V}}$. If $G' = (\mathcal{V}', \mathcal{E}')$ is an induced subgraph for $G = (\mathcal{V}, \mathcal{E})$, then the restriction $(\mathcal{L}^{(G)})_{\mathcal{V}'}$ of $\mathcal{L}^{(G)}$ to $G'$ satisfies

$$-\left(\mathcal{L}^{(G)}\right)_{\mathcal{V}'} \geq -\mathcal{L}^{(G')}.$$

An immediate consequence of this relation is $|4.3|$ with $G = \mathcal{X}_N$ and $G' = \mathcal{X}_{N,k}$. Given subsets $A$ and $B$ of $\mathcal{X}_N$, we set $\text{dist}_1(A, B) := \inf_{x \in A, y \in B} |x - y|$.

**Proposition 4.1 (Combes-Thomas bound).** Let $G = (\mathcal{V}, \mathcal{E})$ be a subgraph of $\mathcal{X}_N$, let $V$ be a bounded potential on $\mathcal{X}_N$, consider the self-adjoint operator

$$H_N^{(G)} = -\frac{1}{2\Delta} \mathcal{L}^{(G)} + \left(1 - \frac{1}{\Delta}\right) \bar{W}_{\mathcal{V}} + V_{\mathcal{V}} \quad \text{on} \quad \ell_2(\mathcal{V}).$$  \hfill (4.6)$$

Let $z \notin \sigma(H_N^{(G)})$ with

$$\left\|\bar{W}^{\frac{1}{2}}(H_N^{(G)} - z)^{-1}\bar{W}^{\frac{1}{2}}\right\| \leq \frac{1}{\kappa_z} < \infty.$$  \hfill (4.7)$$

Then for all subsets $A$ and $B$ of $\mathcal{V}$ we have

$$\left\|\chi_A \left(H_N^{(G)} - z\right)^{-1} \chi_B\right\| \leq \left\|\chi_A \bar{W}^{\frac{1}{2}}(H_N^{(G)} - z)^{-1}\bar{W}^{\frac{1}{2}}\chi_B\right\| \leq \frac{2}{\kappa_z} e^{-\eta_z \text{dist}_1(A, B)},$$

where $\eta_z = \log \left(1 + \frac{\kappa_z \Delta}{2}\right)$.

This proposition implies the following Combes-Thomas bounds for $H_N$ in the bulk. We formulate our results in enough generality so they apply not only to the operator $H_N$ on $\ell^2(\mathcal{X}_N)$ given in (2.11) and the finite volume operators $H_N^{(L)}$ on $\ell^2(\mathcal{X}_N^{(L)})$ defined in (2.5), but also to similarly defined operators on subgraphs of $\mathcal{X}_N$ and $\mathcal{X}_N^{(L)}$, and their restrictions. For the Combes-Thomas bounds, this can be accomplished by proving bounds on subgraphs which are uniform with respect to the addition of a nonnegative potential, as in the following corollary. Note that the constants in the Combes-Thomas bounds are independent of the disorder parameter $\lambda$.

**Corollary 4.2 (Combes-Thomas bound in the bulk).** Let $G = (\mathcal{V}, \mathcal{E})$ be a subgraph of $\mathcal{X}_N$, and consider the operator

$$H_N^{(G)} = -\frac{1}{2\Delta} \mathcal{L}^{(G)} + \left(1 - \frac{1}{\Delta}\right) \bar{W}_{\mathcal{V}} + \lambda(V_\omega)_{\mathcal{V}} + Y \quad \text{on} \quad \ell_2(\mathcal{V}),$$  \hfill (4.9)$$

where $Y$ is a nonnegative bounded potential on $\mathcal{V}$. Then for all $k \in \mathbb{N}$, $E \in I_k, \delta \in \mathbb{R}$, and subsets $A$ and $B$ of $\mathcal{X}_{N,k} \cap \mathcal{V}$, we have

$$\left\|\chi_A \left(H_N^{(G)} - E - i\epsilon\right)^{-1}_k \chi_B\right\| \leq \frac{1}{k^{1+}} \left\|\chi_A \bar{W}^{\frac{1}{2}} \left(H_N^{(G)} - E - i\epsilon\right)^{-1}_k \bar{W}^{\frac{1}{2}} \chi_B\right\| \leq C e^{-\eta \text{dist}_1(A, B)},$$

with the inverse taken on $\bar{H}_k \cap \ell_2(\mathcal{V})$, where

$$C = C(\Delta, \delta) = \frac{4\Delta}{\delta(\Delta-1)} \quad \text{and} \quad \eta = \eta(\Delta, k, \delta) = \log \left(1 + \frac{\delta(\Delta-1)}{4(k+1)}\right).$$

(4.10)
In addition, for all $E \in I_{1,\delta}$, $\epsilon \in \mathbb{R}$, and subsets $A$ and $B$ of $\mathcal{V}$,
\[ \| \chi_A (H_N^{(G)} + P_1 - E - i\epsilon)^{-1} \chi_B \| \leq \| \chi_A \tilde{W}^{\frac{1}{2}} (H_N^{(G)} + P_1 - E - i\epsilon)^{-1} \tilde{W}^{\frac{1}{2}} \chi_B \| \leq C' e^{-\eta' \text{dist}_1(A,B)}, \]
where
\[ C' = C' (\Delta, \delta) = \frac{8\Delta}{D(\Delta_1)} \quad \text{and} \quad \eta' = \eta' (\Delta, \delta) = \log \left( 1 + \frac{\delta (\Delta_1 - 1)}{\delta} \right). \]

Proof of Proposition 4.7: We will write $H := H^{(G)}$ and $R_z := (H - z)^{-1}$ and use notation $\tilde{W}$ also for its restriction $\tilde{W}_{\mathcal{V}}$ to $\mathcal{V}$. Consider a distance function $\rho_A$ on $\mathcal{X}_N$ defined as $\rho_A(x) = \text{dist}_1 (A, x)$. For $\eta \in \mathbb{R}$, let $H_{\eta}$ be a dilation of $H$ with respect to $\rho_A$, i.e., $H_{\eta} = e^{-\eta \rho_A} H e^{\eta \rho_A}$, and set $K_{\eta} := e^{-\eta \rho_A} H e^{\eta \rho_A} - H$.

A straightforward computation yields
\[ (K_{\eta} \psi) (x) = \frac{1}{2\pi} \sum_{y \in \mathcal{V}; \langle x,y \rangle \in \mathcal{E}} \left( e^{\eta (\rho_A(y) - \rho_A(x))} - 1 \right) \psi(y) \]  \hspace{1cm} (4.14)
for $\psi \in l^2 (\mathcal{V})$. If we denote $f_{\eta} (x,y) := e^{\eta (\rho_A(y) - \rho_A(x))} - 1$, then it follows from the definition of $\rho_A$ that $|f_{\eta} (x,y)| \leq e^{\eta} - 1$ for $|x - y| = 1$. We note that for any $x \in \mathcal{X}_N$,
\[ \sum_{w \in \mathcal{X}_N \atop |x - w| = 1} 1 = 2\tilde{W} (x), \]
so for a subgraph $G$ of $\mathcal{X}_N$ we have
\[ \sum_{w \in \mathcal{V} \atop \langle x,w \rangle \in \mathcal{E}} 1 \leq 2\tilde{W} (x). \]  \hspace{1cm} (4.16)

Thus (4.14) and (4.15) imply
\[ \langle \tilde{W}^{-1/2} K_{\eta} \tilde{W}^{-1/2} \psi, \tilde{W}^{-1/2} K_{\eta} \tilde{W}^{-1/2} \psi \rangle = \frac{1}{4\pi} \sum_{x,y,w \in \mathcal{V}; \langle x,y \rangle \in \mathcal{E}, \langle x,w \rangle \in \mathcal{E}} \tilde{W}^{-1}(x) \tilde{W}^{-1/2}(y) \tilde{W}^{-1/2}(w) f_{\eta} (x,w) f_{\eta}(x,y) \psi(y) \bar{\psi}(w) \]
\[ \leq \frac{1}{4\pi} (e^{\eta} - 1)^2 \sum_{x,y,w \in \mathcal{V}; \langle x,y \rangle \in \mathcal{E}, \langle x,w \rangle \in \mathcal{E}} \tilde{W}^{-1}(x) \tilde{W}^{-1}(y) |\psi(y)|^2 \]
\[ \leq \frac{1}{2\pi} (e^{\eta} - 1)^2 \sum_{x,y \in \mathcal{V}; \langle x,y \rangle \in \mathcal{E}} \tilde{W}^{-1}(y) |\psi(y)|^2 \leq \frac{1}{4\pi} (e^{\eta} - 1)^2 \| \psi \|^2, \]
where the final line uses (4.16) and that $\tilde{W}^{-1}(y) \leq 2\tilde{W}^{-1}(x)$ for adjacent sites $x$ and $y$. This implies the bound
\[ \left\| \tilde{W}^{-1/2} K_{\eta} \tilde{W}^{-1/2} \right\| \leq \frac{1}{2\pi} (e^{\eta} - 1). \]  \hspace{1cm} (4.18)
If $\kappa_2$ is as in (4.7), and
\[ \frac{1}{\Delta} (e^{\eta} - 1) \kappa_2^{-1} \leq \frac{1}{2}, \quad \text{i.e.,} \quad \eta \leq \log \left( 1 + \frac{\kappa_2 \Delta}{2} \right), \]  \hspace{1cm} (4.19)
we have
\[ \tilde{W}^{\frac{1}{2}} (H_{\eta} - z)^{-1} \tilde{W}^{\frac{1}{2}} \]
\[ = \left( \tilde{W}^{\frac{1}{2}} (H - z)^{-1} \tilde{W}^{\frac{1}{2}} \right) \sum_{j=0}^{\infty} \left( \left( \tilde{W}^{\frac{1}{2}} K_{\eta} \tilde{W}^{\frac{1}{2}} \right) \left( \tilde{W}^{\frac{1}{2}} (H - z)^{-1} \tilde{W}^{\frac{1}{2}} \right) \right)^j, \]  \hspace{1cm} (4.20)
Proof. In the special case $\epsilon = 0$, we have
\[ \| \tilde{W}^{1/2} (H_{\eta} - z)^{-1} \tilde{W}^{1/2} \| \leq \frac{1}{2k_{x}}. \] (4.21)

Now let $A, B \subset V$ and $\eta = \log \left( 1 + \frac{k_{x} + 1}{2} \right)$. We have
\[
\| \chi_{A} \tilde{W}^{1/2} (H - z)^{-1} \tilde{W}^{1/2} \chi_{B} \| = \| \chi_{A} e^{\eta \rho A} \tilde{W}^{1/2} (H_{\eta} - z)^{-1} \tilde{W}^{1/2} e^{-\eta \rho A} \chi_{B} \|
\leq \| \tilde{W}^{1/2} (H_{\eta} - z)^{-1} \tilde{W}^{1/2} \| \| e^{-\eta \rho A} \chi_{B} \| \leq \frac{1}{2k_{x}} \| e^{-\eta \text{dist}_{1}(A,B)} \|,
\] (4.22)

To prove Corollary 4.2 we will use the following lemma. If $V \subset X$, we set $V_{k} = V \cap X_{N,k}$ and $\tilde{V}_{k} = V \setminus V_{k} = V \cap \tilde{X}_{N,k}$.

Lemma 4.3. Let $G = (V, \mathcal{E})$ be a subgraph of $X_{N, Y}$ a nonnegative potential on $V$, and $H_{N}^{(G)}$ as in (4.9). Then, for $k \in \{1, 2, \ldots, N - 1\}$, we have
\[ \tilde{H}_{k}^{(G)} \geq (k + 1) \left( 1 - \frac{1}{\Delta} \right) \text{ on } \ell_{2}(\tilde{V}_{k}). \] (4.23)

Moreover, if $E \in I_{k, \delta}$, then $E \notin \sigma(\tilde{H}_{k}^{(G)})$ and for all $\epsilon \in \mathbb{R}$ we have
\[ \| \tilde{W}^{1/2} (\tilde{H}_{k}^{(G)} - E - i\epsilon)^{-1} \tilde{W}^{1/2} \| \leq \frac{2(k + 1)}{\delta(1 - \frac{1}{\Delta})}. \] (4.24)

In the special case $\epsilon = 0$ we have
\[ \| \tilde{W}^{1/2} (\tilde{H}_{k}^{(G)} - E)^{-1} \tilde{W}^{1/2} \| \leq \frac{k + 1}{\delta(1 - \frac{1}{\Delta})}. \] (4.25)

Proof. It follows from (4.9) that
\[ H_{0}^{(G)} \geq (1 - \frac{1}{\Delta}) W \text{ on } \ell_{2}(V), \] (4.26)

so
\[ \tilde{H}_{k}^{(G)} \geq (1 - \frac{1}{\Delta}) W \geq (k + 1) \left( 1 - \frac{1}{\Delta} \right) \text{ on } \ell_{2}(V_{k}), \] (4.27)

and (4.23) follows by restriction to $\ell^{2}(\tilde{V}_{k})$.

If $E \in I_{k, \delta}$, we have
\[
\tilde{H}_{k}^{(G)} - E \geq \left( 1 - \frac{1}{\Delta} \right) W - \frac{E}{k_{x} + 1} W \geq \left( 1 - \frac{1}{\Delta} \right) W - (1 - \frac{\delta}{k_{x} + 1}) (1 - \frac{1}{\Delta}) W
\geq \frac{\delta}{k_{x} + 1} (1 - \frac{1}{\Delta}) W \geq \delta (1 - \frac{1}{\Delta}) > 0,
\] (4.28)

so $E \notin \sigma(\tilde{H}_{k}^{(G)})$ and
\[ W^{-1/2} (\tilde{H}_{k}^{(G)} - E) W^{-1/2} \geq \frac{\delta}{k_{x} + 1} (1 - \frac{1}{\Delta}), \] (4.29)

which implies (4.25).

If $\epsilon \neq 0$, we have, using (4.28),
\[ 0 \leq \text{Re} \left( (\tilde{H}_{k}^{(G)} - E - i\epsilon)^{-1} \right) \leq \left( (\tilde{H}_{k}^{(G)} - E)^{-1} \right), \]
\[ 0 \leq \frac{1}{|\epsilon|} \text{Im} \left( (\tilde{H}_{k}^{(G)} - E - i\epsilon)^{-1} \right) \leq \left( (\tilde{H}_{k}^{(G)} - E)^{-1} \right). \] (4.30)

Thus (4.24) follows immediately from (4.25) and the triangle inequality for norms. □
Proof of Corollary 4.2. The estimate (4.10) follows immediately from (4.8) and (4.24). To prove (4.12), note that \( P_1 \) is a bounded potential. If \( E \in I_{1, \delta} \), we have
\[
P_1 \left( (1 - \frac{1}{N}) \tilde{W} + P_1 - E \right) P_1 \geq P_1 \left( \delta \left( 1 - \frac{1}{N} \right) + \frac{1}{\delta} \right) P_1 \]
\[
= P_1 \left( \delta \left( 1 - \frac{1}{N} \right) + \frac{1}{\delta} \right) \tilde{W} P_1, \tag{4.31}
\]
and
\[
\bar{P}_1 \left( (1 - \frac{1}{N}) \tilde{W} + P_1 - E \right) \bar{P}_1 \geq \bar{P}_1 \left( \left( 1 - \frac{1}{N} \right) \tilde{W} - \frac{1}{2} E \tilde{W} \right) \bar{P}_1 \]
\[
\geq \bar{P}_1 \left( \frac{\delta}{2} \left( 1 - \frac{1}{N} \right) \right) \tilde{W} \bar{P}_1, \tag{4.32}
\]
and hence
\[
H_N^{(G)} + P_1 - E \geq \delta \left( 1 - \frac{1}{N} \right) \tilde{W} \geq \frac{\delta}{2} \left( 1 - \frac{1}{N} \right) > 0. \tag{4.33}
\]
As in Lemma 4.3, we conclude that
\[
\left\| \tilde{W}^{1/2} \left( H_N^{(G)} + P_1 - E \right)^{-1} \tilde{W}^{1/2} \right\| \leq \frac{2}{\delta \left( 1 - \frac{1}{N} \right)}, \tag{4.34}
\]
and, for all \( \varepsilon \in \mathbb{R} \),
\[
\left\| \tilde{W}^{1/2} \left( H_N^{(G)} + P_1 - E - i\varepsilon \right)^{-1} \tilde{W}^{1/2} \right\| \leq \frac{4}{\delta \left( 1 - \frac{1}{N} \right)}. \tag{4.35}
\]
The estimate (4.12) follows immediately from (4.8) and (4.35).

5. Fractional moment estimate on the edge

In this section we prove Theorem 2.3 using an iterative argument where geometric resolvent identities will be used to decouple random parameters. In principle, this is the familiar strategy in the fractional moments method. However, these resolvent expansions will lead away from the edge into the deep bulk, due to the long range correlations in the \( N \)-body random potential \( \mathcal{V} \). We then use (deterministic) exponential Green’s function decay in the bulk, i.e., the above Combes-Thomas bounds, to return the path of the iteration to the edge. Somewhat similar arguments have been used before in applications of the fractional moments method to random surface potentials, e.g. [11, 28].

The iteration will naturally lead to graphs with varying geometries, due to the repeated removal of hoppings in the geometric resolvent identities. A convenient way to handle this is to consider the family of all Hamiltonians \( H_N^{(G)} \) as in (4.9), where now for convenience we let \( G = (\mathcal{V}, \mathcal{E}, Y) \), where \( (\mathcal{V}, \mathcal{E}) \) is a subgraph of \( \mathcal{X}_N \), and \( Y \) is a nonnegative bounded potential on \( \mathcal{V} \), and to maximize the relevant quantity over all possible \( G \). Note that given such an operator \( H_N^{(G)} \) and \( A \subset \mathcal{V} \), then the decoupled operator
\[
H_N^{(G,A)} = \chi_A H_N^{(G)} \chi_A + (1 - \chi_A) H_N^{(G)} (1 - \chi_A) \tag{5.1}
\]
is again of the form given in (4.9), for an appropriate subgraph \( (\mathcal{V}, \mathcal{E}_A) \) of \( (\mathcal{V}, \mathcal{E}) \) and nonnegative bounded potential \( Y_A \) on \( \mathcal{V} \). All our results in Section 4 hold for the operators \( H_N^{(G)} \), with bounds uniform in \( G \).

We continue to use \( \| \cdot \| \) for \( \| \cdot \|_\infty \) and \( \| \cdot \| \) for \( \| \cdot \|_1 \) on \( \mathcal{X}_N \) in the sequel. We order \( \mathcal{X}_{N,1} \) by
\[
u < v \iff u_1 < v_1. \tag{5.2}
\]
Given \( u \in \mathcal{X}_N \) and \( k \in \mathbb{Z} \), we use the notation \( u + k = (u_1 + k, \ldots, u_N + k) \in \mathcal{X}_N \).

Given \( x, y \in \mathcal{X}_{N,1} \), we let (for a fixed \( s \in (0,1) \))
\[
\tau(y, x) := \sup_{G} \sup_{E \in I_{1, \delta}} \sup_{\epsilon > 0} \mathbb{E} \left( \| \phi_y, (H_N^{(G)} - E - i\epsilon)^{-1} \phi_x \|^s \right). \tag{5.3}
\]
Here the supremum is taken over all \( G = (\mathcal{V}, \mathcal{E}, Y) \), where \((\mathcal{V}, \mathcal{E})\) is a subgraph of \( \mathcal{X}_N \), and \( Y \) is a nonnegative bounded potential on \( \mathcal{V} \). (Or, what is equivalent, the supremum is taken over all Hamiltonians \( H_N^{(G)} \) as in (4.9).)

Theorem 5.3 follows from the following theorem.

**Theorem 5.1.** Fix \( s \in (0, 1) \). There exists a constant \( C(s, \mu) > 0 \), depending only on \( s \) and \( \mu \), independent of \( N \), such that if \( \Delta > 1 \) and \( \lambda > 0 \) satisfy

\[
\lambda (\delta(\Delta - 1))^{\frac{1}{2}} \min \left\{ 1, (\delta(\Delta - 1))^{\frac{1}{4} + \frac{1}{2}} \right\} \geq C(s, \mu), \tag{5.4}
\]

then for all \( N \in \mathbb{N} \) we have

\[
\tau(x, y) \leq C_N \frac{e^{-m\|x-y\|}}{x} \quad \text{for all} \quad x, y \in \mathcal{X}_{N,1}, \tag{5.5}
\]

where

\[
C_N = C(s, \mu) \left( 1 + \frac{8}{\delta(\Delta - 1)} \right)^s \quad \text{and} \quad m = s \log \left( 1 + \frac{\delta(\Delta - 1)}{8} \right), \tag{5.6}
\]

with \( C(s, \mu) \) a constant depending only on \( s \) and \( \mu \).

The following result is the first step in the iterative argument. It establishes a discrete Gronwall-type bound, which will then be used to prove Theorem 5.1.

**Lemma 5.2.** Let \( s \in (0, 1) \). There is a constant \( C_{s,\mu} \) depending only on \( s \) and the probability distribution \( \mu \), such that, setting \( C_N \) and \( m \) as in (5.6), then for all \( N \in \mathbb{N} \) and \( x, y \in \mathcal{X}_{N,1} \), if \( x < y \) we have

\[
\tau(y, x) \leq C_N \left( e^{-m\|y-x\|} + \sum_{w>x} e^{-m(||w-x||)} \tau(y, w) + e^{-m} \sum_{w \leq x-N} e^{-m(||x-N-w||)} \tau(y, w) \right), \tag{5.7}
\]

and if \( y < x \), we have

\[
\tau(y, x) \leq C_N \left( e^{-m\|y-x\|} + e^{-m} \sum_{w \geq x+N} e^{-m||w-(x+N)||} \tau(y, w) + \sum_{w<x} e^{-m(||w-x||)} \tau(y, w) \right). \tag{5.8}
\]

Finally, there exists a constant \( C'_{s,\mu} = C'_{s,\mu} \) depending only on \( s \) and the probability distribution \( \mu \), such that

\[
\tau(y, x) \leq C'_{s,\mu} \quad \text{for all} \quad x, y \in \mathcal{X}_{N,1}. \tag{5.9}
\]

**Proof.** For a given \( x \in \mathcal{X}_{N,1} \), let \( Q_x = Q_{\{x\}} \) denote the indicator of the set \( S_x = S_{\{x\}} \) as in (2.17), and let \( Q_{\overline{x}} = I - Q_x \). We set \( H_N^{(G,x)} = H_N^{(G,S_x)} \). Note that \( Q_x H_N^{(G)} Q_{\overline{x}} \) is a random operator independent of \( \omega_{x_1} \).

We will also need a comparison operator \( \tilde{H}_N^{(G,x)} \), which we define, using the terminology introduced at the beginning of Section 4 as

\[
\tilde{H}_N^{(G,x)} = H_N^{(G,x)} + P_1. \tag{5.10}
\]

We write \( z = E + j e \) for \( E \in I_1 \) and \( e > 0 \) and note that we have the estimate (4.12), where the constants \( C \) and \( m > 0 \), given in (4.13), are independent of \( N \) and \( z \).

We will also need

**Lemma 5.3.** Let \( s \in (0, 1) \). There is a constant \( C_{s,\mu} \) depending only on \( s \) and the probability distribution \( \mu \), such that, for all \( z = E + j e \), \( E \in I_1 \), \( e > 0 \), and \( x, w \in \mathcal{X}_{N,1} \),

\[
\mathbb{E}_{x_1} \left| \phi_w \left( H_N^{(G,x)} - z \right)^{-1} Q_x H_N^{(G)} Q_{\overline{x}} \left( H_N^{(G)} - z \right)^{-1} \phi_x \right|^s \leq C \frac{e^{-m(||x-w||)}}{x} \quad \text{if} \quad w > x \quad \text{if} \quad x-N \geq w \quad \text{if} \quad w \in (x-N,x) \tag{5.11}
\]

where \( C \) and \( m \) are given in (5.6).
Here we use the conditional expectation $\mathbb{E}_{x_1} (\ldots) = \int_{\mathbb{R}} \cdots \rho(\omega_{x_1}) d\omega_{x_1}$ for averaging over the random variable $\omega_{x_1}$. Below similar notation will also be used for averaging over several of the $\omega_j$.

We postpone the proof of Lemma 5.3 until after the completion of the proof of Lemma 5.2. Consider the case $y > x$. In particular, $y \notin S_x$, so that by the first resolvent identity
\[
\begin{align*}
\left\langle \phi_y, \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle &= -\left\langle \phi_y, \left( H_N^{(G,x)} - z \right)^{-1} \tilde{Q}_x H_N^{(G)} Q_x \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle .
\end{align*}
\]
Next, using the first resolvent identity once again, we get
\[
\begin{align*}
\left\langle \phi_y, \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle &= -\left\langle \phi_y, \left( \tilde{H}_N^{(G,x)} - z \right)^{-1} \tilde{Q}_x H_N^{(G)} Q_x \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle \\
&\quad - \left\langle \phi_y, \left( H_N^{(G,x)} - z \right)^{-1} P_1 \left( \tilde{H}_N^{(G,x)} - z \right)^{-1} \tilde{Q}_x H_N^{(G)} Q_x \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle .
\end{align*}
\]
Let us consider both terms on the right hand side. To estimate the first one, we use (5.11) to get
\[
\mathbb{E}_{x_1} \left| \left\langle \phi_y, \left( \tilde{H}_N^{(G,x)} - z \right)^{-1} \tilde{Q}_x H_N^{(G)} Q_x \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle \right|^8 \leq \frac{C}{\lambda^8} e^{-m||y-x||} .
\]
For the second term, we expand $P_1 = \sum_{w \in \Lambda_{N,1}} |\phi_w\rangle \langle \phi_w|$ and notice that only $w \notin S_x$ yield non-zero contributions. Using (5.11) again, we can bound
\[
\begin{align*}
\mathbb{E}_{x_1} \left| \left\langle \phi_y, \left( H_N^{(G,x)} - z \right)^{-1} P_1 \left( \tilde{H}_N^{(G,x)} - z \right)^{-1} \tilde{Q}_x H_N^{(G)} Q_x \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle \right|^8 &
\leq \frac{C}{\lambda^8} \sum_{w > x} e^{-m||x-w||} \left| \left\langle \phi_y, \left( H_N^{(G,x)} - z \right)^{-1} \phi_w \right\rangle \right|^8 \\
&\quad + \frac{C}{\lambda^8} \sum_{w \leq x - N} e^{-m||x-N-w||} \left| \left\langle \phi_y, \left( H_N^{(G,x)} - z \right)^{-1} \phi_w \right\rangle \right|^8 .
\end{align*}
\]
This implies that $\tau$ satisfies (5.7). The case $y < x$ is dealt with in the same manner, giving (5.8).

Finally, the bound (5.9) is a consequence of the weak-$L^1$-bounds in [4], which we summarize in Lemma 5.4 below. Given $x, y \in \Lambda_{N,1}$, let $Q_{xy}$ be the indicator of the set $S_x \cup S_y$ and $\tilde{Q}_{xy} = 1 - Q_{xy}$. We have
\[
Q_{xy} \left( H_N^{(G)} - z \right)^{-1} \phi_x = Q_{xy} \left( H_N^{(G)} - z \right)^{-1} \tilde{Q}_x \tilde{Q}_y \phi_x \\
= \left( T_{xy} - \lambda Q_x \omega_{x_1} - \lambda Q_y \omega_{y_1} \right)^{-1} \tilde{Q}_x \tilde{Q}_y \phi_x ,
\]
where the Schur complement $T_{xy}$ is the restriction of
\[
- \left( H_N^{(G)} - z - \lambda Q_x \omega_{x_1} - \lambda Q_y \omega_{y_1} \right) + H_N^{(G)} \tilde{Q}_x \tilde{Q}_y \left( H_N^{(G)} - z \right) \tilde{Q}_y \tilde{Q}_x \right)^{-1} \tilde{Q}_x H_N^{(G)}
\]
to $\ell_2(S_x \cup S_y)$, and thus an $\omega_{x_1}$- and $\omega_{y_1}$-independent dissipative operator with strictly positive imaginary part (as $\epsilon > 0$). Thus
\[
\begin{align*}
\left| \left\langle \phi_y, \left( H_N^{(G)} - z \right)^{-1} \phi_x \right\rangle \right| &= \left| \left\langle \phi_y, Q_y Q_{xy} \left( H_N^{(G)} - z \right)^{-1} Q_{xy} \phi_x \right\rangle \right| \\
&= \left\| K_1 Q_y (T_{xy} - \lambda Q_x \omega_{x_1} - \lambda Q_y \omega_{y_1})^{-1} \tilde{Q}_x \tilde{Q}_y K_2 \right\|_2 ,
\end{align*}
\]
where $K_1 = |\phi_y\rangle \langle \phi_y|$ and $K_2 = |\phi_x\rangle \langle \phi_x|$. Thus we can use Lemma 5.4(ii) below and the standard layer-cake integration argument to conclude the fractional moments bound
\[
\mathbb{E}_{x_1,y_1} \left| \left\langle \phi_y, \left( H_N^{(G)} - E - i\epsilon \right)^{-1} \phi_x \right\rangle \right|^8 \\
= \int \left\| K_1 Q_y (T_{xy} - \lambda Q_x \omega_{x_1} - \lambda Q_y \omega_{y_1})^{-1} \tilde{Q}_x \tilde{Q}_y K_2 \right\|_2^8 \rho(\omega_{x_1}) \rho(\omega_{y_2}) d\omega_{x_1} d\omega_{y_2} \leq \frac{C}{\lambda^4} ,
\]
with a constant $C$ only depending on $s$ and $\mu$. This yields (5.9).

We still have to prove Lemma 5.3.

\textbf{Proof of Lemma 5.3} Note that since $x \in S_x$, we have

$$Q_x \left( H^{(G)}_N - z \right)^{-1} \phi_x = Q_x \left( H^{(G)}_N - z \right)^{-1} Q_x \phi_x = - (T_x - \lambda \omega_{x_1})^{-1} \phi_x,$$

(5.20)

where the Schur complement $T_x$ is the restriction of

$$- \left( H^{(G)}_N - z - \lambda \omega_{x_1} \right) + H^{(G)}_N \tilde{Q}_x \left( \tilde{H}^{(G)}_N - z \right) \left( \tilde{Q}_x \right)^{-1} \tilde{Q}_x H^{(G)}_N$$

(5.21)

to $\ell_2(S_x)$, and thus an $\omega_{x_1}$-independent dissipative operator with strictly positive imaginary part.

Using (5.20), the left hand side of (5.11) is equal to $E_x \| K_1(T_x - \lambda \omega_{x_1})^{-1} K_2 \|^2$ with $K_2 = |\phi_x \rangle \langle \phi_x|$. This yields (5.8).

We have $\| K_2 \|_2 = 1$ and $\| K_1 \|_2 = \| Q_x H^{(G)}_N \tilde{Q}_x \left( \tilde{H}^{(G,x)}_N - \tilde{z} \right)^{-1} \phi_x \|$. The latter is non-zero only if $w \in S^c_x$, i.e. either $w > x$ or $w \leq x - N$.

We estimate $\| K_1 \|_2$ by

$$\| K_1 \|_2 \leq \| Q_x H^{(G)}_N \tilde{Q}_x \left( \tilde{H}^{(G,x)}_N - \tilde{z} \right)^{-1} \phi_w \|$$

(5.24)

where $\tilde{Q}_x$ is the indicator of the set $\tilde{S}_x := \{ y \in V : \operatorname{dist}_1(y, S_x) \leq 1 \}$.

Since the number of nearest neighbors of $y \in S_x$ that belong to $S^c_x$ is at most two, we conclude that $\| Q_x H^{(G)}_N \tilde{Q}_x \| \leq 2 \frac{1}{N} = \frac{2}{N}$. The last term in (5.24) can be estimated by the Combes-Thomas bound (4.12) in Corollary 4.2 which yields

$$\| \tilde{Q}_x \left( \tilde{H}^{(G,x)}_N - \tilde{z} \right)^{-1} \chi_{\{w\}} \| \leq C' e^{-\eta' \operatorname{dist}_1(w, S_x)},$$

(5.25)

where $C'$ and $\eta'$ are given in (4.13), so

$$\| K_1 \|_2 \leq C'' e^{-\eta' \operatorname{dist}_1(w, S_x)}, \quad \text{where} \quad C'' = \frac{C'}{\Delta} = \frac{\frac{8}{(\Delta-1)}}. \quad (5.26)$$

Since

$$\operatorname{dist}_1 \left( w, S_x \right) \geq \begin{cases} \| (x - w) \| - 1 & \text{if} \quad w > x \\ \| (x - N) - w \| & \text{if} \quad x - N \geq w \end{cases},$$

(5.27)

we obtain

$$\| K_1 \|_2 \leq \begin{cases} C'' e^{-\eta' \| (x - w) \| - 1} & \text{if} \quad w > x \\ C'' e^{-\eta' \| (x - N) - w \|} & \text{if} \quad x - N \geq w \end{cases}. \quad (5.28)$$

The desired bound (5.11) now follows from (5.23).

We can now prove Theorem 5.1.

\begin{thebibliography}{99}
\bibitem{Ref1}
\bibitem{Ref2}
\bibitem{Ref3}
\bibitem{Ref4}
\end{thebibliography}
Proof of Theorem 5.1] Fix \( y \in \mathcal{X}_{N,1} \), let \( C_{\tau} \), \( C_{\tau}' \), and \( m \) be as in Lemma 5.2. Let \( g(x) = e^{-m\|y-x\|} \) and \( f(x) = \tau(y,x) \) for \( x \in \mathcal{X}_{N,1} \). We will denote by \( \chi_S \) the indicator function of a set \( S \). It follows from Lemma 5.2 that \( f(x) \leq C_{\tau}' \lambda^{-s} \) for all \( x \in \mathcal{X}_{N,1} \) and \( f \) satisfies the integral inequality

\[
    f(x) \leq \frac{C_{\tau}}{\lambda^x} (g + hf)(x) \quad \text{for all} \quad x \in \mathcal{X}_{N,1}, x \neq y,
\]

where \( h \) is the operator on \( l^2(\mathcal{X}_{N,1}) \) whose kernel \( h(x, w) \) is given by

\[
    h(x, w) = \chi(x,\infty)(w)e^{-m\|w-x\|} + \chi_{(\infty,x-N)}(w)e^{-m\|x-N-w\|}
\]

if \( x < y \), and

\[
    h(x, w) = e^{-m} \chi_{[x+N,\infty)}(w)e^{-m\|w-(x+N)\|} + \chi_{(\infty,x)}(w)e^{-m\|w-x\|},
\]

if \( y < x \). Iterating \((5.29)\) \( k \) times, we get

\[
    f(x) \leq \sum_{j=0}^{k} \frac{C_{\tau}^{j+1}}{\lambda^{(j+1)x}} (h^j g)(x) + \frac{C_{\tau}^{k+1}}{\lambda^{(k+1)x}} \left( h^{k+1} f \right)(x).
\]

To bound the first term, let \( \hat{g}(x) = e^{-\frac{m}{2}\|x-y\|} \) and consider the case \( x < y \). Then

\[
    (h \hat{g})(x) = \sum_{w>x} e^{-m\|w-x\|} e^{-\frac{m}{2}\|w-y\|} + e^{-m} \sum_{w \leq x-N} e^{-m\|(x-N)-w\|} e^{-\frac{m}{2}\|x-y\|}
\]

and

\[
    \leq \sum_{w>x} e^{-\frac{m}{2}\|x-y\|} e^{-\frac{m}{2}\|w-y\|} + e^{-m} \sum_{w \leq x-N} e^{-m\|(x-N)-w\|} e^{-\frac{m}{2}\|x-y\|} \leq \hat{C} \hat{g}(x),
\]

where

\[
    \hat{C} = \sum_{w>x} e^{-\frac{m}{2}\|w-x\|} + e^{-m} \sum_{w \leq x-N} e^{-m\|(x-N)-w\|}
\]

\[
    \leq (k+1) e^{-\frac{m}{2}\|w-x\|} + 2 \sum_{r=1}^{\infty} e^{-\frac{m}{2}r} = 2 e^{-\frac{m}{2}} \left( 1 - e^{-\frac{m}{2}} \right)^{-1}.
\]

The same can be verified for \( x > y \). By induction we find that \( (h^j g)(x) \leq C_{\tau}^j \hat{g}(x) \).

On the other hand, by the a-priory bound \((5.29)\) we have \( \|f\|_{\infty} \leq C_{\tau}' \lambda^{-s} \), and

\[
    e^{-m\|w-x\|} + e^{-m} \sum_{w \leq x-N} e^{-m\|(x-N)-w\|} \leq 2 \sum_{r=1}^{\infty} e^{-mr} = e^{-m} \left( 1 - e^{-m} \right)^{-1},
\]

with a similar bound in the case given by \((5.31)\). Thus, setting \( \tilde{C} = \left( e^{\frac{m}{2}} - e^{-\frac{m}{2}} \right)^{-1} \), the last term in \((5.32)\) can be estimated by

\[
    \left( \tilde{h}^{k+1} f \right)(x) \leq \frac{C_{\tau}'}{\lambda^x} \tilde{C}^{k+1} e^{-\frac{m}{2}(k+1)},
\]

again proven inductively. Putting these two bounds together, we get

\[
    f(x) \leq \frac{C_{\tau}}{\lambda^x} \sum_{j=0}^{k} \left( \frac{C_{\tau} \tilde{C}}{\lambda^x} \right)^j \hat{g}(x) + \frac{C_{\tau}'}{\lambda^x} \frac{C_{\tau}^{k+1} \tilde{C}^{k+1}}{\lambda^{(k+1)x}} \tilde{C}^{k+1} e^{-\frac{m}{2}(k+1)}.
\]

We have

\[
    C_{\tau} \tilde{C} = \frac{2C_{s,\mu} \left( \frac{8}{\delta(\Delta-1)} \right)^{\ast}}{\left( \frac{1}{\delta(\Delta-1)} \right)^{\ast} - 1} \quad \text{and} \quad C_{\tau}' \tilde{C} = \frac{C_{s,\mu} \left( \frac{8}{\delta(\Delta-1)} \right)^{\ast}}{\left( \frac{1}{\delta(\Delta-1)} \right)^{\ast} - \left( \frac{1}{\delta(\Delta-1) - 1} \right)^{\ast}},
\]

so

\[
    \max \left\{ C_{\tau} \tilde{C}, C_{\tau}' \tilde{C} \right\} \leq C_{s,\mu} \max \left\{ \left( \delta(\Delta-1) \right)^{(1+s)}, \left( \delta(\Delta-1) \right)^{-\frac{s}{2}} \right\},
\]

where the constant \( C_{s,\mu}' \) depends only on \( s \) and \( \mu \).
Taking \((\Delta, \lambda)\) so the right hand side of (5.39) is \(\leq \frac{\lambda^2}{2}\), we get
\[
f(x) \leq \frac{2C_n^f}{\lambda^2} \tilde{g}(x) + \frac{C_n^f}{\lambda^2} e^{\frac{\lambda^2}{2}(k+1)}
\]  (5.40)

The choice \(k + 1 = \|x - y\|\) yields
\[
\tau(y, x) = f(x) \leq \frac{C_n^f}{\lambda^2} e^{\frac{\lambda^2}{2}\|x-y\|},
\]  (5.41)

where the constant \(C_n^f\) depends only on \(s\) and \(\mu\).

Finally, for completeness, we state the weak-\(L^1\)-bounds which we have used above, see Lemma 3.1 and Proposition 3.2 of [4]. Here \(\|\cdot\|_2\) denotes Hilbert-Schmidt norm and \(|\cdot|\) Lebesgue measure in dimension one and two, respectively.

**Lemma 5.4** ([4]). There exists a universal constant \(C < \infty\) such that the following holds:

(i) For arbitrary separable Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}_1\), arbitrary maximally dissipative operators \(A\) with strictly positive imaginary part, and arbitrary Hilbert-Schmidt operators \(M_1 : \mathcal{H} \to \mathcal{H}_1\) and \(M_2 : \mathcal{H}_1 \to \mathcal{H}\),
\[
\left|\{v \in \mathbb{R} : \|M_1(A - v)^{-1}M_2\|_2 > t\}\right| \leq C\|M_1\|_2\|M_2\|_2 \frac{1}{t}.
\]  (5.42)

(ii) Moreover, for \(A\), \(M_1\) and \(M_2\) as above and arbitrary non-negative operators \(U_1\) and \(U_2\),
\[
\left|\left\{(u, v) \in [0, 1]^2 : \|M_1U_1^{1/2}(A - v_1U_1 - v_2U_2)^{-1}U_2^{1/2}M_2\|_2 > t\right\}\right| \leq 2C\|M_1\|_2\|M_2\|_2 \frac{1}{t}.
\]  (5.43)

6. Weigner estimate for the droplet spectrum

6.1. “Boxes” in \(\mathcal{X}^N\). A useful concept in the theory of Schrödinger operators on \(L^2(\mathbb{Z}^d)\) are their restrictions to finite boxes \([-M, M]^d \cap \mathbb{Z}^d\), \(M \in \mathbb{N}\), or translates of this set. Information concerning the number of eigenvalues on a small interval for such restrictions as well as their independence play an important role in the analysis of the Anderson model on \(\mathbb{Z}^d\). Let us now introduce the counterpart of this concept for the \(N\)-body Hamiltonians \(H_N\) analyzed here. In fact, much of the following considerations depend on an interplay between two types of “boxes”, one of them, denoted by \(\Lambda_M\), being boxes along the edge \(\mathcal{X}^N_{1,1}\), and the other, denoted by \(S_M\), being boxes (or rather unions of “strips”) in the full lattice \(\mathcal{X}^N\), which will be used to create required independence properties.

More precisely, throughout the remainder of this work we will consider finite volume operators \(H^{(L)}_N\) defined in (2.5). Thus all the point sets to be introduced will generally be considered as subsets of \(\mathcal{X}^N_L\), without always specifying this in our notation.

Finite boxes along the edge are given by the subsets
\[
\Lambda_M(x) = [x - M, x + M] = \{(y_1, y_1 + 1, \ldots, y_1 + N - 1) : x - M \leq y_1 \leq x_1 + M\}
\]  (6.1)
of \(\mathcal{X}^N_{1,1}\), for some \(x \in \mathcal{X}^N_{1,1}\), where we use the convention introduced before Lemma 5.2. For such \(\Lambda_M(x)\), we define a subset of \(\mathbb{Z}\) as
\[
\Psi(\Lambda_M(x)) = \{x_1 - M, \ldots, x_1 + M\},
\]  (6.2)
i.e. the first components of sites in \(\Lambda_M(x)\), thus introducing a one-to-one correspondence between intervals in \(\mathbb{Z}\) and “intervals” in \(\mathcal{X}^N_{1,1}\). For a given set \(\Psi \subset \mathbb{Z}\), we define a subset \(S_\Psi\) of \(\mathcal{X}^N\) as
\[
S_\Psi := \{u \in \mathcal{X}^N : u_j \in \Psi\} \text{ for some } j \in \{1, \ldots, N\},
\]  (6.3)
meaning that \(S_\Psi\) contains all the sites \(u\) on which the random potential \(V_\omega(u)\) depends on one of the random parameters \(\omega_j\), \(j \in \Psi\). Note that \(S_x = S_{\{x_1\}}\) for the notation used in Section 5.

In particular,
\[
S_M(x) := S_{\Phi(\Lambda_M(x))} = S_{\{x_1 - M, \ldots, x_1 + M\}}
\]  (6.4)
are the lattice sites at which the potential depends on the random parameters \(\omega_j\) with \(x - M \leq j \leq x_1 + M\). Note that
\[
(S_M(x))_1 := S_M(x) \cap \mathcal{X}^N_{1,1} = [x - (M + N - 1), x + M]
\]  (6.5)
Lemma 6.1. Let $C_W = C_W(\delta, \Delta) = \left(1 + \frac{\sqrt{2}}{\delta(\Delta - 1)}\right)$. Then
\[
P\{x_{1-M}, \ldots, x_{1+M}\} \{\sigma(H_{S_M(x)}^{(L)}) \cap I \neq \emptyset\} \leq C_W \lambda^{-1}\|\rho\|_{\infty}(2M + 1)(2M + N)|I| \tag{6.6}
\]
for all $x \in X_{N,1}$, $N, M \in \mathbb{N}$ and subintervals $I \subset I_{1,\delta}$ with $|I| \leq 2\delta \left(\frac{1}{\Delta} - \frac{1}{\Delta'}\right)/C_W(\delta, \Delta)$.

The main idea behind our proof of Lemma 6.1 is to reduce it to a Wegner estimate for the Schur complement of $H_{S_M(x)}^{(L)}$ with respect to the configuration space $S_{\Psi(\lambda_M(x))}$ into edge and bulk. To this end, let $Q = Q_{(S_M(x))}$ be the indicator function of $(S_M(x))$, and $Q = I - Q$ in $\ell^2(S_M(x))$.

Let $E$ be the center of a subinterval $I \subset I_{1,\delta}$ and set $D = H_{S_M(x)}^{(L)} - E$, $A = QDQ$, $B = QDQ$ and $V = QDQ = -QLQ / 2\Delta$, where $L$ is the Laplacian defined in (2.6). Thus
\[
H_{S_M(x)}^{(L)} - E = \begin{pmatrix} A & V \\ V^* & B \end{pmatrix}.
\tag{6.7}
\]

$B$ is invertible with $\|B^{-1}\| \leq (\delta(1 - 1/\Delta))^{-1}$ by (4.3) and $\|V\| \leq \sqrt{2}/\Delta$, which uses the fact that each $v \in X_{N,1}$ has at most two nearest neighbors.

Thus $H_{S_M(x)}^{(L)} - E$ is invertible if and only if the Schur complement
\[
K_E = D / B = Q(H_{S_M(x)}^{(L)} - E)Q - VB^{-1}V^*
\tag{6.8}
\]
is invertible and $Q(H_{S_M(x)}^{(L)} - E)^{-1}Q = K_E^{-1}$. With this we can reduce Lemma 6.1 to the following Wegner estimate for $K_E$:

Lemma 6.2. For all $E \in I_{1,\delta}$, $M, N \in \mathbb{N}$, and $\epsilon > 0$ we have
\[
P\{x_{1-M}, \ldots, x_{1+M}\} \{\text{dist}(\{0\}, \sigma(K_E)) < \epsilon\} \leq 2\epsilon\lambda^{-1}\|\rho\|_{\infty}(2M + 1)(2M + N)\tag{6.9}
\]

We will first use this to complete the proof of Lemma 6.1. It suffices to show the claim for $|I|$ sufficiently small. If $\text{dist}(\{0\}, \sigma(K_E)) \geq \epsilon$, then $\|K^{-1}\| \leq 1/\epsilon$. That $K$ is invertible implies that $D = H_{S_M(x)}^{(L)} - E$ is invertible and, by a standard fact, that
\[
D^{-1} = \begin{pmatrix} K_E^{-1} & -K_E^{-1}VB^{-1} \\ -B^{-1}V^*K_E^{-1} & B^{-1} + B^{-1}V^*K_E^{-1}VB^{-1} \end{pmatrix}.
\tag{6.10}
\]

Thus $\|D^{-1}\|$ can be bounded in terms of the norms of the four matrix elements and we can conclude form the bounds provided above that that for $\epsilon \leq \delta(1 - 1/\Delta)$ we have $\|D^{-1}\| \leq C(\delta, \Delta) = \left(1 + \frac{\sqrt{2}}{\delta(\Delta - 1)}\right)$. Therefore $\sigma(H_{S_M(x)}^{(L)} - E) \cap (-\epsilon/C(\delta, \Delta), \epsilon/C(\delta, \Delta)) = \emptyset$. Using Lemma 6.2 we conclude that, if $|I| \leq 2\delta \left(\frac{1}{\Delta} - \frac{1}{\Delta'}\right)/C(\delta, \Delta)$, then $\sigma(H_{S_M(x)}^{(L)}) \cap I = \emptyset$ holds with conditional probability at least $1 - C(\delta, \Delta)\lambda^{-1}\|\rho\|_{\infty}(2M + 1)(2M + N)|I|/2$, yielding Lemma 6.1.

To prove Lemma 6.2 we use the following lemma from [48]. To state it, let $\mu$ be a probability measure on $\mathbb{R}$ and, for any $\epsilon > 0$, $s(\mu, \epsilon) := \sup\{\mu(\alpha, \beta) : \beta - \alpha \leq \epsilon\}$. Note that in the case that $\mu$ has a bounded density $\rho$, as in our application, we have $s(\mu, \epsilon) \leq \|\rho\|_{\infty}\epsilon$.

Also let $J$ be a finite index set and denote by $\mu^J$ the $J$-fold product measure of $\mu$ on $\mathbb{R}^J$. 
Lemma 6.3. Consider a monotone function $\Phi$ on $\mathbb{R}^J$ which satisfies $\Phi(q + te) - \Phi(q) \geq t$ for $e = (1, 1, \ldots, 1) \in \mathbb{R}^J$ and all $t > 0$. Then, for any open interval $I \subset \mathbb{R}$, we have

$$\mu^J\{q : \Phi(q) \in I\} \leq |J|s(\mu, |I|).$$  \hfill (6.11)

Proof of Lemma 6.2. Let $E_1 \leq E_2 \leq \ldots$ be the at most $|\{S_M(x)\}_1| = 2M + N$ eigenvalues of $K_E$ (fewer if $[x - (M + N - 1), x + M]$ is not fully contained in $\lambda_x^{(L)}$, counted with multiplicity, which we consider as functions of $\tilde{\omega} = (\omega_{x_1-M}, \ldots, \omega_{x_1+M})$, with all other $\omega_i$ fixed. Let $e = (1, 1, \ldots, 1)$ as in Lemma 6.3 with $J = 2M + 1$. Note that $-V B^{-1} V^* x$ in (6.8) is monotone increasing in all components of $\tilde{\omega}$ in quadratic form sense. Thus, for all $t > 0$,

$$K_E(\tilde{\omega} + te) - K_E(\tilde{\omega}) \geq Q(H_{S_M(x)}(\tilde{\omega} + te) - H_{S_M(x)}(\tilde{\omega}))Q = \lambda Q(V_{\tilde{\omega} + t e} - V_{\tilde{\omega}})Q \geq \lambda t I,$$  \hfill (6.12)

where we have used in the last step that each site $u$ in $\{S_M(x)\}_1$ has at least one component in $\{x_1 - M, \ldots, x_1 + M\}$, so that $V_{\tilde{\omega} + t e}(u) - V_{\tilde{\omega}}(u) \geq 1$. Thus we have $E_n(\tilde{\omega} + te) - E_n(\tilde{\omega}) \geq \lambda t$ for all $t > 0$ and all $n$ by the min-max principle and can apply Lemma 6.3 with $\Phi = \lambda^{-1} E_n$ to get

$$\mathbb{P}_{\{x_1 - M, \ldots, x_1 + M\}}(\tilde{\omega} : E_n(\tilde{\omega}) \in (-\epsilon, \epsilon)) \leq 2 \epsilon \lambda^{-1} \|\rho\|_\infty (2M + 1).$$  \hfill (6.13)

Using this for each one of the eigenvalues $E_n$ yields (6.9).
(i) \( \tilde{H}_{S_M(x)}^{(L)} \) does not depend on the random variables \( \omega_{\ell}, \ell = j - M, \ldots, j + M \), and \( \tilde{H}_{S_M(y)}^{(L)} \) does not depend on \( \omega_{\ell}, \ell = i - M, \ldots, i + M \).

(ii) We have
\[
\text{dist}(E, \sigma(\tilde{H}_{S_M(x)}^{(L)})) \leq C_1 e^{-\frac{\eta_1 M}{2}}
\]
for all \( E \in \sigma_{1,\frac{1}{2}}(H_{S_M(x)}^{(L)}) \), where
\[
\eta_1 = \log \left( 1 + \frac{\delta (\Delta - 1)}{\nu} \right) \quad \text{and} \quad C_1 = \frac{128}{\eta_1 \delta^2 (\Delta - 1)^2}.
\]

(iii) Let \( 0 < \varepsilon \leq \varepsilon_1 = \delta \left( 1 - \frac{1}{\Delta} \right) / (4C_W (\delta / 4, \Delta)) \). We have
\[
\mathbb{P} \left\{ \text{dist} \left( \sigma_{1,\frac{1}{2}}(\tilde{H}_{S_M(x)}^{(L)}), \sigma_{1,\frac{1}{2}}(\tilde{H}_{S_M(y)}^{(L)}) \right) \leq \varepsilon \right\} \leq C C_W \lambda^{-1} \| \rho \|_{\infty} M^3 \varepsilon,
\]
where \( C \) is an independent constant and \( C_W = C_W (\frac{\delta}{4}, \Delta) \) is as in Lemma \ref{lem:localization}.

(iv) Suppose \( C_1 e^{-\frac{\eta_1 M}{2}} < \frac{\delta}{4} \left( 1 - \frac{1}{\Delta} \right) \) and \( 0 < \varepsilon \leq \frac{\delta (1 - 1 / \Delta)}{4C_W (\delta / 4, \Delta)} - 2C_1 e^{-\frac{\eta_1 M}{2}} \). Then
\[
\mathbb{P} \left\{ \text{dist} \left( \sigma_{1,\frac{1}{2}}(H_{S_M(x)}^{(L)}), \sigma_{1,\frac{1}{2}}(H_{S_M(y)}^{(L)}) \right) \leq \varepsilon \right\} \leq C C_W \lambda^{-1} \| \rho \|_{\infty} M^3 \left( \varepsilon + 2C_1 e^{-\frac{\eta_1 M}{2}} \right).
\]

**Proof.** Part (i) follows immediately from (7.4) and (6.3). We will abbreviate \( S_x := S_M(x), H_{S_x} := H_{S_M(x)}^{(L)} \) and \( \tilde{H}_{S_x} := \tilde{H}_{S_M(x)}^{(L)} \). To prove part (ii) we proceed by Schur complementation with respect to the decomposition of \( S_x \) into \( (S_x)_1 := S_x \cap \mathcal{X}_N,1 \) and \( S_x \cap \mathcal{X}_N,1 \). Let \((\cdot)_1\) and \((\cdot)_{-1}\) denote the corresponding restrictions, define
\[
K_E = \left( H_{S_x} - E - H_{S_x} (H_{S_x} - E)_{-1} H_{S_x} \right)_{1},
\]
and let \( \tilde{K}_E \) be defined analogously, with \( H_{S_x} \) replaced by \( \tilde{H}_{S_x} \). We first observe that it follows from (4.3) that for any \( E \in I_{1,\frac{1}{2}} \) the operators \( K_E \) and \( \tilde{K}_E \) are well defined and bounded.

Using the resolvent identity we get
\[
K_E - \tilde{K}_E = P_{1,x} \tilde{H}_{S_x} P_{1,x} \left( \tilde{H}_{S_x} - E \right)_{-1} Q_{x,y} \lambda V (H_{S_x} - E)_{1}^{-1} P_{1,x} H_{S_x} P_{1,x},
\]
where \( P_{1,x} \) is the indicator function of \((S_x)_1\), and \( \tilde{P}_{1,x} = \chi_{S_x} - P_{1,x} \). Letting \( H_0 = H - \lambda V \), we have
\[
\lambda V (H_{S_x} - E)_{1}^{-1} = \tilde{P}_{1,x} - ((H_0)_{S_x} - E)_{1} (H_{S_x} - E)_{1}^{-1},
\]
and hence
\[
K_E - \tilde{K}_E = -P_{1,x} \tilde{H}_{S_x} \tilde{P}_{1,x} \left( \tilde{H}_{S_x} - E \right)_{1}^{-1} Q_{x,y} ((H_0)_{S_x} - E)_{1} (H_{S_x} - E)_{1}^{-1} \tilde{P}_{1,x} H_{S_x} P_{1,x},
\]
where we used that \( Q_{x,y} \chi_{(S_x)_1,2} = 0 \) for \((S_x)_1,2 = \{ u \in S_x \cap \mathcal{X}_N,2; \text{ dist}_1 (u, (S_x)_1) = 1 \} \), and
\[
Q_{x,y} P_{1,x} H_{S_x} P_{1,x} = Q_{x,y} \chi_{(S_x)_1,2} \tilde{P}_{1,x} H_{S_x} P_{1,x}.
\]
Clearly,
\[
Q_{x,y} ((H_0)_{S_x} - E)_{1} = Q_{x,y} (H_0)_{S_x} - E)_{1} \tilde{Q}_{x,y},
\]
where \( \tilde{Q}_{x,y} \) is the indicator of the set \( \{ u \in \mathcal{X}_N : \text{ dist}_1(u, \text{ supp } Q_{x,y}) \leq 1 \} \). Next, we observe that it follows from (2.5), (1.11), and (7.6) that
\[
\| (H_0)_{S_x} - E \| \leq 2N + 2\beta \leq M,
\]
and, since vertices in $X_{Ni}$ have at most two next neighbors, we have
\[ \| \tilde{P}_{1,x} H^{} S_{x} P_{1,x} \| = \| \tilde{P}_{1,x} \tilde{H}^{} S_{x} P_{1,x} \| \leq \sqrt{\frac{2}{\Delta}}. \tag{7.18} \]

Combining (7.14), (7.16), (7.17), and (7.18) we obtain the bound
\[ \| K_E - \tilde{K}_E \| \leq \frac{2M}{\Delta^2} \| \chi(s_x)_{1,2}^{-1} \tilde{H}^{} S_{x} - E \| Q_{x,y} \| \| \tilde{Q}_{x,y} (\tilde{H}^{} S_{x} - E)^{-1} \chi(s_x)_{1,2} \|. \tag{7.19} \]

Using (4.10) in Corollary 4.2, we have
\[ \| \chi(s_x)_{1,2}^{-1} \tilde{H}^{} S_{x} - E \| \leq C_1 e^{-\eta_1} \| \text{dist}(E S_{x}^N) \| \leq C_1 e^{-\eta_1} (M + 1), \tag{7.20} \]

where $C_1 = C(\Delta, \frac{\delta}{2})$ and $\eta_1 = \eta(\Delta, 1, \frac{\delta}{2})$ are as in (4.11), and we used (7.5). Moreover, $\| \tilde{Q}_{x,y} (\tilde{H}^{} S_{x} - E)^{-1} \chi(s_x)_{1,2} \|$ satisfies the same bound. Thus
\[ \| K_E - \tilde{K}_E \| \leq \frac{2M}{\Delta^2} C_2 e^{2\eta_1 M} \leq \frac{4}{\eta_1 \Delta^2} C_2 e^{-\frac{2}{3} \eta_1 M} \leq \frac{2}{\eta_1 \Delta^2} C_2 e^{-\frac{4}{3} \eta_1 M}. \tag{7.21} \]

Now, suppose $E \in \sigma_{1_\frac{\delta}{4}} (H^{} S_{x})$, so $0 \in \sigma (K_E)$. Letting $C' = \frac{2}{\eta_1 \Delta^2} C_2 e^{-\frac{4}{3} \eta_1 M}$, we deduce from (7.21) that $[-C', C'] \cap \sigma (K_E) = \emptyset$, and hence $\frac{1}{\Delta} \leq \| K_E^{-1} \| \leq \| (\tilde{H}^{} S_{x} - E)^{-1} \|$, which implies (7.7).

To prove (iii), note that $\tilde{H}^{} S_{M(x)} = (\tilde{H}^{} S_{M(x)} + P_{1,x}) - P_{1,x}$ and $\sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(x)} + P_{1,x}) = \emptyset$. Since $\text{tr} P_{1,x} = |(S^M(x))_1| = 2M + N$, we conclude that $\tilde{H}^{} S_{M(x)}$ has at most $2M + N$ eigenvalues in $I_{1_\frac{\delta}{4}}$. Thus it follows from Lemma 6.1 and part (i) that
\[ \mathbb{P} \left\{ \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(x)}), \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(y)}) \right) \leq \varepsilon \right\} \leq 2C_W \lambda^{-1} \| \rho \|_{\infty} (2M + 1)(2M + N)^2 \varepsilon \leq C C_W \lambda^{-1} \| \rho \|_{\infty} M^3 \varepsilon, \tag{7.22} \]

where $C_W = C_W (\frac{\delta}{2}, \Delta)$ is as in Lemma 6.1.

Finally, we prove part (iv). Under the hypotheses, for $E \in \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(x)})$ we have, for $u = x, y$,
\[ \text{dist}(E, \sigma(\tilde{H}^{} S_{M(u)})) = \text{dist}(E, \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(u)})), \tag{7.23} \]

so, using (7.7), we get
\[ \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(x)}), \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(y)}) \right) \leq \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(x)}), \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(x)}) \right) + \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(x)}), \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(y)}) \right) \]
\[ + \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(y)}), \sigma_{1_\frac{\delta}{4}} (\tilde{H}^{} S_{M(y)}) \right) \leq \text{dist} \left( \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(x)}), \sigma_{1_\frac{\delta}{4}} (H^{} S_{M(y)}) \right) + 2C_1 e^{-\frac{4}{3} \eta_1 M}. \tag{7.24} \]

Thus (7.10) follows from (7.9) \hfill \Box

From now on we assume (5.4) is satisfied, so we have the conclusions of Theorem 5.1.

For the pairs $i, j \in \mathbb{Z}$ and corresponding $x, y \in X_{Ni}$ as defined at the beginning of this section, let $Q_i$ and $Q_j$ be the indicator functions of the subsets $S_i \cap X_N(L)$ and $S_j \cap X_N(L)$, respectively, defined in (6.3).
In what follows, we will use the shorthand notations $\Lambda_x$ and $S_x$ for $\Lambda_M(x)$ and $S_M(x)$, defined in (6.1) and (6.4), and similarly for $y$. For $S = S_x$ or $S_y$, we set (recall (5.1))

$$H_S^{(L)} = Q_S H_N^{(L)} Q_S + Q_S H_N^{(L)} Q_S$$

and $G_E^S = \left( H_S^{(L)} - E \right)^{-1}$. (7.25)

We also let

$$\Gamma_S = H_N^{(L)} - H_S^{(L)} = \chi_{\partial S} H_S \chi_{\partial S},$$

where

$$\partial S = \left\{ u \in S : |u - v|_1 = 1 \text{ for some } v \in \Lambda_N \setminus S \right\},$$

$$\partial^+ S = \left\{ v \in \Lambda_N \setminus S : |u - v|_1 = 1 \text{ for some } u \in S \right\},$$

and note that $\|\Gamma_S\| \leq \frac{N}{2}$. In this section we let $H_S = H_S^{(L)}$.

**Lemma 7.2.** Let $i, j \in \mathbb{Z}$, $x, y \in \Lambda_{N, 1}$ with $x_1 = i$ and $y_1 = j$, and assume

$$M = M(i, j) : = \left\lfloor \frac{|i-j|}{4} \right\rfloor = \left\lfloor \frac{\|x-y\|}{4} \right\rfloor \geq 8N + 2\beta, \text{ where } \beta \text{ is given in (1.5).}$$

(7.27)

Fix $s \in (0, 1)$, and set $\tilde{m} = \frac{m}{90}$, with $m$ as in Lemma 5.2, i.e.,

$$\tilde{m} = \frac{s}{90} \log \left( 1 + \frac{\delta (\Delta - 1)}{8} \right).$$

(7.28)

Then there exists $\tilde{M} < \infty$, independent of $N$, such that for $M \geq \tilde{M}$, outside an event of probability less than $e^{-\tilde{M}}$ one can decompose $I_{1, \delta} = I_x \cup I_y$, so that for every $E \in I_x$ we have

$$\left\| G_E^S \right\| \leq e^{\tilde{m} M} \text{ and } \max_{w \in \Lambda_x : \|w-x\| \geq M/2} \left\| Q_i G_E^S \phi_w \right\| \leq e^{-\tilde{m} M},$$

(7.29)

which imply

$$\left\| Q_i G_E^S \Gamma_S \right\| \leq \frac{1}{\tilde{m} (\Delta - 1)} M^3 e^{-\tilde{m} M},$$

(7.30)

with similar estimates for $E \in I_y$.

**Proof.** The argument will follow closely [19] Proof of Proposition 5.1, using Lemma 7.1.

Fix $s \in (0, 1)$. Let $E \in I_{1, \delta}$. We first observe that we have

$$\max_{w \in \Lambda_x : \|w-x\| \geq \frac{M}{4}} \mathbb{E} \left\| Q_i G_E^S \phi_w \right\|^8 \leq e^{-\frac{5}{18}M} = e^{-10\tilde{m} M}, \text{ with } \tilde{m} = \frac{m}{90},$$

(7.31)

for $M \geq \tilde{M}$, where $\tilde{M}$ does not depend on $N$ and $m$ is given in (5.6). The same bound holds if one replaces $x$ by $y$ and $i$ with $j$ in (7.31). Indeed, introducing $H_N^{(S_y)} = H_N^{(S_x)} + P_{1, x}$, and denoting the corresponding resolvent by $\hat{G}_E^S$, we obtain, via the resolvent identity, that for $w \in \Lambda_x$ we have

$$\left\| Q_i G_E^S \phi_w \right\|^8 \leq \left\| Q_i \hat{G}_E^S \phi_w \right\|^8 + \left\| Q_i \hat{G}_E^S P_{1, x} G_E^S \phi_w \right\|^8.$$
estimate in the form (4.12) and Theorem 5.1 we obtain
\[
\mathbb{E} \left\| Q_i G_E^{S_x} \phi_w \right\|^s \leq (C')^s e^{-m \|x-w\|-(N-1)} + (C')^s C \sum_{u \in \Lambda_x} e^{-m \|x-u\|-(N-1)} e^{-\frac{m}{2} \|u-w\|}
\]
\[
\leq C^s \left(1 + \frac{C}{\mathcal{X}}\right) e^{mN} \left( e^{-m \|x-w\|} + \sum_{u \in \Lambda_x} e^{-m \|x-u\|} e^{-\frac{m}{2} \|u-w\|} \right)
\]
\[
\leq C^s \left(1 + \frac{C}{\mathcal{X}}\right) e^{mN} \left( e^{-m \|x-w\|} + e^{-\frac{m}{2} \|x-w\|} \sum_{u \in \Lambda_x} e^{-\frac{m}{2} \|x-u\|} \right)
\]
\[
\leq C^s \left(1 + \frac{C}{\mathcal{X}}\right) e^{mN} \left( e^{-\frac{m}{2} \|x-w\|} \left( 1 + \frac{1+e^{-\frac{m}{2}}}{1-e^{-\frac{m}{2}}} \right) = \frac{2CC^s}{1-e^{-\frac{m}{2}}} e^{mN} e^{-\frac{m}{2} \|x-w\|} \right)
\]
\[
\leq C^s \left(1 + \frac{C}{\mathcal{X}}\right) \frac{2}{1-e^{-\frac{m}{2}}} e^{-\frac{m}{2} M} \leq e^{-\frac{m}{2} M}, \quad (7.33)
\]
where \(C'\) is given in (4.13), \(C\) and \(m\) are given in (5.6), and we used (7.27) and \(\|w - x\| \geq \frac{M}{2}\). The last inequality holds for \(M \geq \tilde{M}\), where \(\tilde{M}\) does not depend on \(N\) (but depends on \(\lambda, \Delta, \delta, s\).

For \(x\), define the sets
\[
\Delta^x := \left\{ E \in I_{1,\delta} : \max_{u \in \Lambda_2 : \|u-x\| \geq M/2} \left\| Q_i G_E^{S_x} \phi_w \right\| > e^{-\tilde{m} M} \right\},
\]
\[
\tilde{\Delta}^x := \left\{ E \in I_{1,\delta} : \max_{u \in \Lambda_2 : \|u-x\| \geq M/2} \left\| Q_i G_E^{S_x} \phi_w \right\| > e^{-2\tilde{m} M} \right\}, \quad (7.34)
\]
and consider the event
\[
\tilde{B}_x = \left\{ \Delta^x \right\} > e^{-5\tilde{m} M}, \quad (7.35)
\]
where \(|J|\) denotes the Lebesgue measure of the set \(J \subset \mathbb{R}\).

If \(\tilde{B}_x\) holds, then
\[
\int_{I_{1,\delta}} \sum_{u \in \Lambda_2 : \|u-x\| \geq M/2} \left\| Q_i G_E^{S_x} \phi_w \right\|^s \, dE \geq \int_{I_{1,\delta}} \max_{u \in \Lambda_2 : \|u-x\| \geq M/2} \left\| Q_i G_E^{S_x} \phi_w \right\|^s > e^{-5\tilde{m} M} e^{-2\tilde{m} M} > e^{-7\tilde{m} M}. \quad (7.36)
\]
Using (7.31) and Chebyshev’s inequality, we obtain
\[
P\{\tilde{B}_x\} < Me^{7\tilde{m} M} e^{-10\tilde{m} M} = Me^{-3\tilde{m} M} \leq e^{-2\tilde{m} M}, \quad (7.37)
\]
for \(M\) large.

We now consider the set
\[
W_x = \left\{ E \in I_{1,\delta} : \text{dist} \left( E, \sigma(H^{(L)}_{S_x}) \right) \leq e^{-\tilde{m} M} \right\}
\]
\[
= \left\{ E \in I_{1,\delta} : \text{dist} \left( E, \sigma_{\frac{1}{2}}(H^{(L)}_{S_x}) \right) \leq e^{-\tilde{m} M} \right\}, \quad (7.38)
\]
for sufficiently large \(M\). We claim that in the complementary event \(\tilde{B}_x^c\) we have \(\Delta^x \subset W_x\). Indeed, suppose \(E \in \Delta^x \setminus W_x\). Then there exists \(w \in \Lambda_x\) with \(\|w-x\| \geq M/2\) such that
\[
\left\| Q_i G_E^{S_x} \phi_w \right\| > e^{-\tilde{m} M}. \quad (7.39)
\]
If $|E - E'| \leq 2e^{-5\hat{m}M}$, we have
\[ \text{dist}\left(E', \sigma(H^{(L)}_{S_x})\right) \geq e^{-\hat{m}M} - 2e^{-5\hat{m}M} \geq \frac{1}{2}e^{-\hat{m}M}. \] (7.40)

Combining (7.39) and (7.40), we get
\[ \|Q_t G_{E'}^S \phi_w\| \geq \|Q_t G_{E}^S \phi_w\| - |E - E'| \left\| G_{E'}^S \right\| \left\| G_{E}^S \right\| > e^{-\hat{m}M} - 4e^{-5\hat{m}M}e^{2\hat{m}M} > e^{-2\hat{m}M}. \] (7.41)

We infer that $[E - 2e^{-5\hat{m}M}, E + 2e^{-5\hat{m}M}] \cap \hat{I}_{1,\delta} \subset \hat{\Delta}^x$, and hence conclude that $\left|\hat{\Delta}^x\right| \geq e^{-5\hat{m}M}$, and hence the event $\hat{B}_x$ holds.

Thus on $B_x^c$ we have $I_x = I_{1,\delta} \setminus W_x \subset I_{1,\delta} \setminus \Delta^x$, so (7.29) holds for all $E \in I_x$. Moreover, the same results hold with $y$ substituted for $x$. To finish the proof we need to estimate the probability of the event
\[ B_{\text{res}} = \{W_x \cap W_y \neq \emptyset\} \subset \left\{ \text{dist}\left(\sigma_{I_{1,\delta}}(H^{(L)}_{S_x}), \sigma_{I_{1,\delta}}(H^{(L)}_{S_y})\right) \leq 2e^{-\hat{m}M}\right\}. \] (7.42)

Since (7.27) implies (7.6), it follows from Lemma 7.1(iv) that, for large $M$, we have
\[ \mathbb{P}\{B_{\text{res}}\} \leq 2CCW\lambda^{-1}\|\rho\|_{\infty} M^3 \left( e^{-\hat{m}M} + 2C_1 e^{-\frac{3}{2}\eta M} \right) \leq e^{-4\hat{m}M}. \] (7.43)

Consider now the event $\mathcal{E} = \hat{B}_x^c \cap \hat{B}_y^c \cap B_{\text{res}}$. We have
\[ \mathbb{P}\{\mathcal{E}\} \geq 1 - \left( 2e^{-2\hat{m}M} + e^{-\frac{3}{2}\hat{m}M} \right) \geq 1 - e^{-\frac{\hat{m}M}{2}}. \] (7.44)

and $I_{1,\delta} = I_x \cup I_y$ on $\mathcal{E}$.

Now, suppose (7.29) holds for $E \in I_x$. We have
\[ Q_t G_{E}^S \Gamma_{S_x} = Q_t \hat{G}_{E}^S \Gamma_{S_x} + Q_t G_{E}^S P_{1,x} \hat{G}_{E}^S \Gamma_{S_x}. \] (7.45)

Since
\[ \text{dist}_{\infty}\left(S_{(i)}, \partial_{-} S_x\right) \geq \text{dist}_{\infty}\left(S_{(i)}, \partial_{+} S_x\right) - 1 \geq M, \] (7.46)
and $\|\Gamma_{S_x}\| \leq \frac{N}{\Delta x}$ (recall (7.26)), it follows from (4.12) in Lemma 4.2 that
\[ \left\|Q_t \hat{G}_{E}^S \chi_{\partial_{-} S_x}\right\| \leq C'e^{-\eta' M}, \] where $C'$ and $\eta'$ are given in (4.13). (7.47)

In addition,
\[ \left\|Q_t G_{E}^S P_{1,x} \hat{G}_{E}^S \chi_{\partial_{-} S_x}\right\| \leq \sum_{w \in \Lambda_x} \left\|Q_t G_{E}^S \phi_w\right\| \left\|\chi_{\partial_{-} S_x} \hat{G}_{E}^S \phi_w\right\|. \] (7.48)

If $w \in \Lambda_x$, it follows from (7.29) and (4.12) in Lemma 4.2 plus (4.33) and (7.28), that
\[ \left\|Q_t G_{E}^S \phi_w\right\| \left\|\chi_{\partial_{-} S_x} \hat{G}_{E}^S \phi_w\right\| \leq \begin{cases} e^{\hat{m}M} C'e^{-\eta' M} & \text{if } \|x - w\| \leq \frac{M}{2} \\ e^{-\hat{m}M} 2(\delta (1 - \frac{1}{\Delta}))^{-1} & \text{if } \|x - w\| \geq \frac{M}{2} \\ \frac{8\Delta}{(\Delta - 1)} e^{-\hat{m}M}. \end{cases} \] (7.49)

We conclude that
\[ \left\|Q_t G_{E}^S P_{1,x} \hat{G}_{E}^S \chi_{\partial_{-} S_x}\right\| \leq \frac{M}{\Delta x} (2M + 1) \frac{8\Delta}{(\Delta - 1)} e^{-\hat{m}M} \leq \frac{3M^2}{(\Delta - 1)} e^{-\hat{m}M}. \] (7.50)

Combining the estimates, and taking $\hat{M}$ sufficiently large, we get
\[ \left\|Q_t G_{E}^S \Gamma_{S_x}\right\| \leq \left( \frac{8\Delta}{(\Delta - 1)} e^{-\eta M} + \frac{3M^2}{(\Delta - 1)} e^{-\hat{m}M} \right) \|\Gamma_{S_x}\| \leq \frac{1}{\delta(\Delta - 1)} M^3 e^{-\hat{m}M}. \] (7.51)

The lemma is proven. \qed
In view of the equivalence of \((2.2)\) and \((2.20)\), Theorem \(2.1\) follows from the following theorem.

**Theorem 8.1.** Fix \(s \in (0, 1)\), and assume \(\Delta > 1\) and \(\lambda > 0\) satisfy the condition \((5.4)\) of Theorem \(5.7\). Let \(Q_N^{(L)}(i, j; I_{\delta})\) be the eigenfunction correlators defined in \((2.16)\). There exist constants \(C < \infty\) and \(m > 0\) such that

\[
\sum_{N=1}^{\infty} E \left( Q_N^{(L)}(i, j; I_{\delta}) \right) \leq C e^{-m|i-j|} \quad \text{for all} \quad -L \leq i, j \leq L,
\]

uniformly in \(L\).

In the context of many-body localization, \(Q_N^{(L)}(i, j; I)\) is the relevant eigenfunction correlator for \(H_N^{(L)}\) corresponding to a finite interval \(I \subset \mathbb{R}\) and a pair of indices \(i, j \in \mathbb{Z}\). Our proof follows the outlines of the proofs of dynamical localization for random Schrödinger operators given in \([21, 22, 23, 32]\). The key ingredients are Lemma \(7.2\) and the a priori bound on eigenfunction correlators derived in Lemma \(8.2\) below.

For a given \(N \in \mathbb{N}\), let \(P_1\) be the indicator function of \(\mathcal{X}_{N,1}^{(L)} := \mathcal{X}_{1,1}^{(L)} \cap \mathcal{X}_{N,1}^{(L)}\), and set \(\tilde{H}_N^{(L)} = H_N^{(L)} + P_1\). Note that, due to working in finite volume, all spectra are finite, and that \(\tilde{H}_N^{(L)}\) has no spectrum below \(2(1 - \frac{1}{\delta})\). Let \(G_z\) and \(\tilde{G}_z\) be the corresponding resolvents.

Let \(i, j \in \mathbb{Z}\) and \(x, y \in \mathcal{X}_{N,1}\) with \(x_1 = i\) and \(y_1 = j\). Let \(Q_i\) and \(Q_j\) be the indicator functions of the subsets \(S_{\{i\}} \cap \mathcal{X}_{N}^{(L)}\) and \(S_{\{j\}} \cap \mathcal{X}_{N}^{(L)}\), respectively, defined in \((6.3)\). Given an energy interval \(I\), set \(\sigma_I = \sigma \left( H_N^{(L)} \right) \cap I\). We have

\[
Q_N^{(L)}(k, k; I) = \sum_{E \in \sigma_I} \| Q_k P_E Q_k \|_1 = \| Q_k P_I Q_k \|_1 \quad \text{for all} \quad k \in \mathbb{Z}.
\]

Moreover

\[
\sum_{E \in \sigma_I} \| Q_i P_E Q_j \|_1 \leq \sum_{E \in \sigma_I} \| Q_i P_E \|_2 \| Q_j P_E \|_2 = \sum_{E \in \sigma_I} \| Q_i P_E Q_j \|_1 \leq \frac{1}{2} \| Q_j P_E Q_j \|_1 \leq \frac{1}{2} \| Q_j P_E Q_j \|_1.
\]

Thus it follows from \((2.16)\) and \((8.3)\) that that

\[
Q_N^{(L)}(i, j; I) \leq \sqrt{Q_N^{(L)}(i, i; I)Q_N^{(L)}(j, j; I)}.
\]

The following lemma provides an a priori bound for the interval \(I_{\delta}\).

**Lemma 8.2.** For all \(i \in \mathbb{Z}\) and \(N \in \mathbb{N}\) we have

\[
Q_N^{(L)}(i, i; I_{\delta}) \leq \| P_{I_{\delta}} Q_i \|_1 \leq \tilde{C} N, \quad \text{where} \quad \tilde{C} = \frac{16\Delta}{\delta(\Delta-1)} \left( \frac{8}{\delta(\Delta-1)} + 2 \right).
\]

In particular,

\[
Q_N^{(L)}(i, j; I_{\delta}) \leq \tilde{C} N.
\]

**Proof:** Let \(\Gamma\) be the circle of radius \(\frac{1}{2}(1 - \frac{1}{\delta})\) in \(\mathbb{C}\), centered at the midpoint of \(I_{\delta}\), so that \(\Gamma\) encloses \(I_{\delta}\) and \(\text{dist}(z, I_{\delta})\) as well as \(\text{dist}(z, \sigma(\tilde{H}_N^{(L)}))\) are both at least \(\frac{\delta}{2}(1 - \frac{1}{\delta})\) for any \(z \in \Gamma\). We obtain, using \(\text{dist} \left( I_{\delta}, \sigma \left( \tilde{H}_N^{(L)} \right) \right) \geq \delta(1 - \frac{1}{\delta})\),

\[
P_{I_{\delta}} = \frac{1}{2\pi i} \int_{\Gamma} G_z dz = \frac{1}{2\pi i} \int_{\Gamma} \left( G_z - \tilde{G}_z \right) dz = \frac{1}{2\pi i} \int_{\Gamma} G_z P_1 \tilde{G}_z dz.
\]
Therefore,
\[
\|P_{1,i}Q_i\|_1 \leq \frac{1}{2} \left(1 - \frac{1}{M} \right) \max_{z \in I} \|P_{1,i} \partial_z\| \|P_{1,i} \hat{G}_{z}Q_i\|_1 \leq \frac{1}{2} \max_{z \in I} \|P_{1,i} \hat{G}_{z}Q_i\|_1. \tag{8.8}
\]
For \(z \in \Gamma\) we have the norm bound \(\|\hat{G}_{z}\| \leq \frac{2}{\delta(1-\frac{1}{M})}\). Also, for \(u \in \mathcal{X}^{(L)}_{N,1}\) we have \(\text{dist}(u, S_{\{i\}}) \geq \|x-u\|-(N-1)\) and thus it follows from the Combes-Thomas bound (4.12) that
\[
\|Q_{i} \hat{G}_{z} \phi_{u}\| \leq C' \min \left(1, e^{-\eta'\|x-u\|-(N-1)}\right), \tag{8.9}
\]
where \(C'\) and \(\eta'\) are given in (4.13). Here \(x = (i, \ldots, i + N - 1)\), as in Section 7. We conclude that
\[
\|P_{1} \hat{G}_{z}Q_{i}\|_1 \leq \sum_{u \in \mathcal{X}^{(L)}_{N,1}} \|P_{1} \hat{G}_{z} \phi_{u}\| \leq C' \left(\sum_{\|x-u\|>(N-1)} e^{-\eta'\|x-u\|-(N-1)} + 2N\right)
\]
\[
\leq 2C' \left((e^{\eta'} - 1)^{-1} + N\right) = 8\delta (\frac{8}{\delta(N-1)}) \left(\frac{8}{\delta(N-1)} + N\right), \quad (8.10)
\]
which yields (8.5).

**Proof of Theorem (8.7)** Let \(E \in I_{1,\delta}\). We have \(\left(H_{N}^{(L)} - E\right) P_E = 0\). Note that if \(i \in \mathcal{Z}\), \(x \in \mathcal{X}_{N,1}\) with \(x_1 = i, M \geq 2N, S_x = S_M(x)\), and \(E \notin \sigma(H_{S_x}^{(L)})\), we have \(Q_i P_E = -Q_i G_{E} S_x \Gamma_{S_x} P_E\). (8.11)

We now suppose (7.27) is satisfied, so we can use Lemma 7.2. Thus, for \(M \geq \tilde{M}\), outside an event of probability \(\leq e^{-\frac{1}{2}M}\), we have \(I_{1,\delta} = I_x \cup I_y\), which can be chosen so \(I_x \cap I_y = \emptyset\), and (7.29) and (7.30) hold for \(E \in I_x\), with similar inequalities for \(E \in I_y\). Take \(E \in I_x\). Then \(E \notin \sigma(H_{S_x}^{(L)})\), and (8.11) holds.

Let \(k \in \mathcal{Z}\) and \(u \in \mathcal{X}_{N,1}\) with \(u_1 = k\), where we allow for \(k = i\), and let \(Q_k\) be the indicator function of the set \(S_{\{k\}} \cap \mathcal{X}^{(L)}_{N}\). Then, using (8.11) and (7.30), we get
\[
\|Q_i P_E Q_k\|_1 \leq \|Q_i G_{E} S_x \Gamma_{S_x}\| \|\chi_{\partial \ast S_x} P_E Q_k\|_1 \leq \frac{1}{\delta(N-1)} M^3 e^{-\tilde{m}N} \|\chi_{\partial \ast S_x} P_E Q_k\|_1. \tag{8.12}
\]
Since \(\partial \ast S_x \subset S_{\{i-M\}-1} \cup S_{\{i+M\}}\), we have \(\chi_{\partial \ast S_x} \leq Q_{i-M} - Q_{i+M+1}\), and hence
\[
\|\chi_{\partial \ast S_x} P_E Q_k\|_1 \leq \|Q_{i-M} - Q_{i+M+1}\| P_E Q_k\|_1 \leq \|Q_{i-M} - P_E Q_k\|_1 + \|P_{i+M} P_E Q_k\|_1 \\
\leq \|P_{i-M} P_E Q_k\|_2 + \|P_{i-M} P_E Q_k\|_2 + \|P_{i+M} P_E Q_k\|_2 \\
\leq \|P_{i-M} P_E Q_k\|_2^2 + \|P_{i+M} P_E Q_k\|_2^2. \tag{8.13}
\]
We conclude, using Lemma 8.2 that
\[
\sum_{E \in \sigma_{I_x}} \|\chi_{\partial \ast S_x} P_E Q_k\|_1 \leq 2\tilde{C} N \leq \frac{4 \delta}{\delta(N-1)} M. \tag{8.14}
\]
Combining (8.12) and (8.13) we get
\[
\sum_{E \in \sigma_{I_x}} \|Q_i P_E Q_k\|_1 \leq \frac{\tilde{C}}{\delta(N-1)} M^4 e^{-\tilde{m}M}. \tag{8.15}
\]
Since the same estimates hold for \(E \in I_y\), with \(Q_j\) substituted for \(Q_i\), we conclude that outside an event of probability \(\leq e^{-\frac{1}{2}M}\), we have
\[
Q_N^{(L)}(i,j; I_{1,\delta}) = \sum_{E \in \sigma_{I_x}} \|Q_i P_E Q_j\|_1 + \sum_{E \in \sigma_{I_y}} \|Q_j P_E Q_i\|_1 \leq \frac{\tilde{C}}{\delta(N-1)} M^4 e^{-\tilde{m}M} \leq C_1 e^{-\frac{1}{2}M}, \tag{8.16}
\]
where the last inequality holds for \(M\) large.
Lemma 4.2 holds for \( S \) and hence, in particular, \( E / S \), we have

\[
\mathbb{E} \left( Q_N^{(L)}(i, j; I_{1, \delta}) \right) \leq C_1 e^{-\frac{\bar{\eta}}{7} M} + C_2 M e^{-\frac{\bar{\eta}}{7} M} \leq C_2 e^{-\frac{\bar{\eta}}{7} |i-j|}. \tag{8.17}
\]

In particular, for sufficiently large \( M \), say \( M \geq \hat{M} \), we have

\[
\sum_{N=1}^{\left\lfloor \frac{|i-j| - 2\beta + 1}{8} \right\rfloor} \mathbb{E} \left( Q_N^{(L)}(i, j; I_{1, \delta}) \right) \leq C_2 \frac{|i-j|}{32} e^{-\frac{\bar{\eta}}{7} |i-j|} \leq C_3 e^{-\frac{\bar{\eta}}{70} |i-j|}. \tag{8.18}
\]

Now let \( N > N_1 \) for some \( N_1 \in \mathbb{N} \). Let \( \tilde{\mu} = \mathbb{E} \{ \omega_0 \} \), and assume \( N \tilde{\mu} > 2 \). Then the standard large deviations estimate gives

\[
P \{ V_\omega(x) < 1 \} \leq P \{ V_\omega(x) < N \tilde{\mu} \} \leq e^{-c_\mu N}, \tag{8.19}
\]

where \( c_\mu \) is a constant depending only on the probability distribution \( \mu \). It implies that, setting \( C' = e^{2c_\mu / \tilde{\mu}} \), we have \( P \{ V_\omega(x) < 1 \} \leq C e^{-c_\mu N} \) for all \( N \in \mathbb{N} \). If \( i \in \mathbb{Z}, x \in \mathcal{X}_{N, 1} \) with \( x_1 = i \), taking \( M = 2N \), \( S_x = S_M(x) \), then

\[
P \{ i-M-N+1, i-M-N+2, \ldots, i+M+N-1 \} \{ V_\omega(x) < 1 \text{ for some } x \in \Lambda_x \}
\leq (2M + 1) C e^{-c_\mu N} = C(4N + 1) e^{-c_\mu N}. \tag{8.20}
\]

Thus, outside an event of probability less than \( C(4N + 1) e^{-c_\mu N} \) we have

\[
P_{1,x} V_\omega P_{1,x} \geq P_{1,x}, \tag{8.21}
\]

and hence, in particular, \( E \notin \sigma(H_{S_x}^{(L)}) \). Moreover, in view of (8.21), the Combes-Thomas estimate (4.12) in Lemma 4.2 holds for \( H_{S_x}^{(L)} \). Thus

\[
\left\| Q_i G_{E} S_y \Gamma_{S_x} \right\| \leq \left\| Q_i G_{E} S_y \chi_{\partial S_x} \right\| \| \Gamma_{S_x} \| \leq \frac{N}{2} C_{\tilde{\mu}} e^{-\eta' \tilde{\mu} M} \leq \frac{8N}{\delta(\Delta-1)} e^{-180 \bar{\eta} \tilde{\mu} N}, \tag{8.22}
\]

where \( C' \) and \( \eta' \) are given in (4.13), and we used (7.46) and (7.28). Thus, using (8.11), and proceeding as in (8.16), we get

\[
Q_N^{(L)}(i, j; I_{1, \delta}) \leq \frac{16C}{\delta(\Delta-1)} N^2 e^{-180 \bar{\eta} \tilde{\mu} N}. \tag{8.23}
\]

It follows, once again using Lemma 8.2 and (8.20), that

\[
\mathbb{E} \left( Q_N^{(L)}(i, j; I_{1, \delta}) \right) \leq \frac{16C}{\delta(\Delta-1)} N^2 e^{-180 \bar{\eta} \tilde{\mu} N} + \tilde{C} C_{\mu} N(4N + 1) e^{-c_\mu N} \leq \tilde{C} e^{-\bar{m} N}, \tag{8.24}
\]

where \( \bar{m} = \frac{1}{2} \min \{ c_\mu, 180 \bar{\eta} \} \), and \( \tilde{C} \) is a constant independent of \( N \). We conclude that

\[
\sum_{N=N_1+1}^{\infty} \mathbb{E} \left( Q_N^{(L)}(i, j; I_{1, \delta}) \right) \leq \sum_{N=N_1+1}^{\infty} \tilde{C} e^{-\bar{m} N} \leq C_4 e^{-\frac{\bar{m}}{5} N_1}. \tag{8.25}
\]

If \( |i - j| > 4 \left( \frac{\hat{M}}{2} + 1 \right) \), we take \( N_1 = \left\lfloor \frac{|i-j|}{32} - \frac{2\beta + 1}{8} \right\rfloor + 1 \), obtaining

\[
\sum_{N=\left\lfloor \frac{|i-j|}{32} - \frac{2\beta + 1}{8} \right\rfloor + 1}^{\infty} \mathbb{E} \left( Q_N^{(L)}(i, j; I_{1, \delta}) \right) \leq C_4 e^{-\frac{\bar{m}}{50} |i-j|}. \tag{8.26}
\]
Combining (8.18) and (8.26), we get that for $|i - j| > 4 \left( \hat{M} + 1 \right)$ we have
\[
\sum_{N=1}^{\infty} \mathbb{E} \left( Q_{N}^{(L)}(i, j; I_{1, \delta}) \right) \leq C_{5} e^{-\bar{m}|i-j|}, \tag{8.27}
\]
for all $-L \leq i, j \leq L \in \mathbb{Z}$, with $\bar{m} = \min \left\{ \frac{m}{110}, \frac{m}{100} \right\}$, with $C_{5}$ a constant.

Finally, we consider the case when $|i-j| \leq 4 \left( \hat{M} + 1 \right)$. In this case we take $N_{1} = |i-j|$, and using Lemma 8.2 and (8.25), we conclude that
\[
\sum_{N=1}^{\infty} \mathbb{E} \left( Q_{N}^{(L)}(i, j; I_{1, \delta}) \right) \leq \hat{C} |i-j|^{2} + C_{4} e^{-\frac{m}{4}|i-j|}
\leq 16\hat{C} \left( \hat{M} + 1 \right)^{2} + C_{4} e^{-\frac{m}{2}|i-j|} \leq C_{6} e^{-\frac{m}{2}|i-j|}, \tag{8.28}
\]
for some constant $C_{6}$.

The proof is complete. \hfill \Box

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