GRISVARD’S SHIFT THEOREM NEAR $L^\infty$ AND YUDOVICH THEORY ON POLYGONAL DOMAINS

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ABSTRACT. Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain with boundary $\partial \Omega$ of class $C^{1,1}$ except at finitely many points $S_j$ where $\partial \Omega$ is locally a corner of aperture $\alpha_j \leq \frac{\pi}{2}$.

Improving on results of Grisvard [13, 14], we show that the solution $G_\Omega f$ to the Dirichlet problem on $\Omega$ with data $f \in L^p(\Omega)$ and homogeneous boundary conditions satisfies the estimates

$$\|G_\Omega f\|_{W^{2,p}(\Omega)} \leq C p \|f\|_{L^p(\Omega)}, \quad \forall 2 \leq p < \infty,$$

$$\|D^2 G_\Omega f\|_{\text{Exp}L^1(\Omega)} \leq C \|f\|_{L^\infty(\Omega)}.$$

The proof uses sharp $L^p$ bounds for singular integrals on power weighted spaces inspired by the work of Buckley [5]. Our results allow for the extension of the Yudovich theory [31, 32] of existence, uniqueness and regularity of weak solutions to the Euler equations on $\Omega \times (0,T)$ to polygonal domains $\Omega$ as above.

1. INTRODUCTION AND MAIN RESULTS

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected open set. Given $f \in H^{-1}(\Omega)$, we denote by $G_\Omega f \in H^1_0(\Omega)$ the unique variational solution to the Dirichlet problem on $\Omega$ with data $f$ and zero boundary conditions. The main feature of the solution to the Dirichlet problem, and, more generally, to elliptic boundary value problems with $L^p$ data, on a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary is the so-called regularity shift theorem, which, in the generality below, can be traced back to [2], and summarized into the estimate

$$\|G_\Omega f\|_{W^{m+2,p}(\Omega)} \leq C(m,p,\Omega)\|f\|_{W^m,p(\Omega)},$$

valid for all $f \in W^{m,p}(\Omega)$, $1 < p < \infty$, and $0 \leq m \leq M - 2$, under the assumption that $\partial \Omega$ be of class (say) $C^M$. A quick insight on the proof in the basic case $m = 0$ is as follows: one takes advantage of the representation

$$\partial_{x_j} \partial_{x_k} G_\Omega f(x) = \text{p.v.} \int_{\Omega} (\partial_{x_j} \partial_{x_k} G_\Omega)(x,y)f(y) \, dy := T_{j,k} f(x) \quad \forall f \in \mathcal{C}^\infty(\overline{\Omega}),$$

where $G_\Omega$ is the Green function of the domain $\Omega$. When $\partial \Omega \in C^2$, or even $C^{1,1}$, the derivatives $\partial_{x_j} \partial_{x_k} G_\Omega$ are Calderon-Zygmund kernels, whence the weak-$L^1(\Omega)$ boundedness of the $T_{j,k}$. The case $p = 2$ being available from Green’s identity, it follows from Marcinkiewicz interpolation, dualization, and symmetry of $G_\Omega$ that

$$\|T_{j,k}\|_{L^p(\Omega) \rightarrow L^p(\Omega)} \leq C_\Omega \max\{p, p'\}, \quad \forall 1 < p < \infty,$$

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so that $C(0, p, \Omega) \sim \max\{p, p'\}$; the lower order derivatives are easier. The bounds (1.1) are generally false for $p = \infty$, just as well as the $L^\infty$-boundedness of the Calderón-Zygmund singular integrals. However, the substitute inequality

\begin{equation}
(1.3) \quad \|D^2 G_\Omega f\|_{\text{Exp}L^1(\Omega)} \leq C(\Omega)\|f\|_{L^\infty(\Omega)},
\end{equation}

\text{Exp}L^1(\Omega) being the Orlicz space with Orlicz function $t \mapsto e^t - 1$ (see (3.1) below and [30], for instance, for further reference), follows by extrapolation on (1.2), using that $C(0, p, \Omega)$ grows linearly as $p \to \infty$. This is the same quantitative behavior predicted, via the John-Nirenberg inequality, by the more recent (and harder) BMO-type bounds for $T_{j,k}$ [6]. See [1, 12, 13] for a classical and comprehensive treatment of elliptic regularity theory on smooth domains and, for instance, [26] for its relationship with the classical Calderón-Zygmund theory of singular integrals.

The present article is concerned with the natural questions whether (1.2), or equivalently the endpoint (1.3), holds under significantly weaker assumptions that $\partial \Omega \in \mathcal{C}^{1,1}$. To begin with, we note that the approach outlined above for (1.2) fails, the reason being that, when $\partial \Omega$ is not of class $\mathcal{C}^{1,1}$, the second derivatives of the Green function $G_\Omega$ are no longer necessarily Calderón-Zygmund kernels. A beautiful counterexample by Jerison and Kenig [17] (see also [7]) rules out $\mathcal{C}^{0,1}$ and even $\mathcal{C}^1$ regularity; in short, there exists $\Omega \subset \mathbb{R}^2$ with $\mathcal{C}^1$ boundary and $f \in C^\infty(\Omega)$ with $D^2 G_\Omega f \not\in L^1(\Omega)$. However, for convex domains, inequality (1.2) holds in the range $1 < p \leq 2$, see [9] (and generally fails outside this range).

In the sequel, we focus on the extension (with suitable modifications) of (1.2), (1.3) to a subclass of the planar Lipschitz domains strictly wider than $\mathcal{C}^{1,1}$, which we term \textit{polygonal domains}. We say that $\Omega \subset \mathbb{R}^2$ is a polygonal domain if it is a bounded simply connected open set and $\partial \Omega$ is a piecewise $\mathcal{C}^{1,1}$ planar curve, with finitely many points $\{S_j\}_{j=1}^N$ of discontinuity for the tangent vector, and such that, in some neighborhood of each $S_j$, $\Omega$ coincides with the cone of vertex $S_j$ and aperture $\alpha_j \in (0, 2\pi)$.

Elliptic problems in polygonal and polyhedral domains, and more generally, in domains with point singularities, have been extensively studied: see for instance the monographs [4, 13, 20, 21] and references therein. Our starting point is the following accurate description of the solution $G_\Omega f$ to the Dirichlet problem with $f \in L^p(\Omega)$, for a polygonal domain $\Omega$ as above, borrowed from Grisvard's influential treatise [13]; see also [14].

\textbf{Theorem 0.} [13, Theorem 4.4.3.7] Let $\Omega$ be a polygonal domain with $\max \alpha_j < \pi$. For each

\begin{equation}
(1.4) \quad p \in (1, \infty), \quad p \not\in \overline{p_{\Omega}} := \left\{ p_{a_j} := \frac{2a_j}{2a_j - \pi} : a_j > \frac{\pi}{2} \right\},
\end{equation}

there exists $C(p, \Omega) > 0$ such that

\begin{equation}
(1.5) \quad \|G_\Omega f - \Pi_{S(p, \Omega)} G_\Omega f\|_{W^2,p(\Omega)} \leq C(p, \Omega)\|f\|_{L^p(\Omega)}.
\end{equation}

Here, $\Pi_{S(p, \Omega)}$ denotes the $H^1(\Omega)$-orthogonal projection on the subspace of singular solutions to the Dirichlet problem on $\Omega$

\begin{equation}
(1.6) \quad S(p, \Omega) := \text{span}\left\{ s_{j,k} : j = 1, \ldots, N, 1 \leq k \leq \frac{2a_j}{p} \right\}.
\end{equation}
Each singular solution \( s_{j,k} \) is given, using polar coordinates \((\rho, \theta)\) centered at \( S_j \), by

\[
(1.7) \quad s_{j,k}(\rho e^{i\theta}) = \begin{cases} 
\eta_{j,k}(\rho)\rho^{\frac{k}{a_j}} \sin \left( \frac{k\theta}{a_j} \right) & \text{if } k \notin \mathbb{N}, \\
\eta_{j,k}(\rho)\rho^{\frac{k}{a_j}} \left( \log \rho \right) \sin \left( \frac{k\theta}{a_j} \right) + \cos \left( \frac{k\theta}{a_j} \right) & \text{if } k \in \mathbb{N},
\end{cases}
\]

with \( \eta_{j,k} \) suitable smooth cutoff functions.

A closer look at the above statement tells us that

(A) when \( \max \alpha_j \leq \frac{\pi}{2} \), \( \overline{\alpha}_\Omega \) is empty, and the range of \( k \) in definition (1.6) is void for each \( j \) and for each \( 1 < p < \infty \), so that \( S(p, \Omega) = \emptyset \);

(B) \( S(p_1, \Omega) = S(p_2, \Omega) \) for each \( p_1 > p_2 > \max \overline{p}_\Omega \).

To unify notation, we set

\[
(1.8) \quad S(\Omega) := \bigcup_{p > \max \overline{p}_\Omega} S(p, \Omega).
\]

Note that, in case (A), we simply have \( S(\Omega) = \emptyset \), and Grisvard's Theorem 0 recovers exactly the case \( m = 0 \) of (1.1).

1.1. **Main results.** Our first main result is that, in short, the constant \( C(p, \Omega) \) in Theorem 0 grows linearly as \( p \to \infty \), and as a consequence the \( L^\infty \) endpoint bound (1.3) holds for polygonal domains as well, up to the projection on the space of singular solutions \( S(\Omega) \) (if any exist).

**Theorem 1.** Let \( \Omega \) be a polygonal domain with \( \max \alpha_j < \pi \). We have the estimates

\[
(1.9) \quad \| G_\Omega f - \Pi_{S(\Omega)} G_\Omega f \|_{W^{2,p}(\Omega)} \leq C_{\Omega,p} \| f \|_{L^p(\Omega)}, \quad \forall p \geq p_\Omega := \begin{cases} 
2, & \text{if } \overline{\alpha}_\Omega = \emptyset, \\
2 \max \overline{p}_\Omega, & \text{if } \overline{\alpha}_\Omega \neq \emptyset,
\end{cases}
\]

\[
(1.10) \quad \| D^2(G_\Omega f - \Pi_{S(\Omega)} G_\Omega f) \|_{\text{Exp}^{1,1}(\Omega)} \leq C'_{\Omega} \| f \|_{L^\infty(\Omega)},
\]

with \( C'_{\Omega} = e^{\alpha_{\Omega} + 1} C_{\Omega} \). The positive constant \( C_{\Omega} \) depends only on \( \{\alpha_j\}_{j=1}^N \) and on the piecewise \( C^{1,1} \) character of \( \partial \Omega \) away from the corners \( \{S_j\}_{j=1}^N \).

One important application of Theorem 1, which served as initial motivation for our investigation, and which constitutes the second main result of this article, is the extension of the theory of Yudovich [31, 32] to weak solutions of the planar Euler equations on polygonal domains \( \Omega \) as described earlier in the introduction, when \( \max \alpha_j \leq \frac{\pi}{2} \). As discussed above, in this case the projection \( \Pi_{S(\Omega)} \) is trivial, so that (1.9)-(1.10) coincide with the classical Calderón-Zygmund estimates employed by Yudovich to prove uniqueness and log-Lipschitz regularity of weak solutions with initial vorticity in \( L^\infty(\Omega) \) (or, more generally, unbounded vorticities with slow growth of the \( L^p \) norms as \( p \to \infty \); see Remark 2.1 below). Our results improve on the previous works [3, 22]: in Section 2, we provide a precise statement (Theorem 2), and additional context and references.

**Remark 1.1.** Tracking the constant \( C(p, \Omega) \) in Grisvard's original proof (which is not done explicitly in either [13] or [14]) yields quadratic growth, that is \( C(p, \Omega) = p^2 C(\Omega) \), which is not sufficient to recover (1.10). This is, in short, due to the fact that Grisvard's proof proceeds via two consecutive applications of a Calderón-Zygmund-like inequality in the vein of (1.2), each costing a \( p \) factor; we elaborate on this point in Remark 4.2.
We further remark that (1.10) was only known in the case where all angles $\alpha_j$ of $\Omega$ are of the form $\pi \frac{k}{k}$ for some integer $k \geq 2$ (whence $\mathbf{S}(\Omega) = \emptyset$) [3, Proposition 3.1], as a consequence of the stronger inequality

$$
(1.11) \quad \|D^2G_\Omega f\|_{\text{bmo}(\Omega)} \leq C(\Omega)\|f\|_{\text{bmo}(\Omega)}.
$$

The proof of (1.11) in [3] uses a reflection argument, which is unapplicable to the general case; thus the extension of (1.11), possibly up to projection on $\mathbf{S}(\Omega)$, to polygonal domains is still an open problem, to the best of our knowledge.

**Remark 1.2.** We want to point out that analogues of Theorem 0 above can be formulated in much greater generality: see [13, Theorem 5.2.2]. Therein, the Laplacian with Dirichlet boundary conditions can be replaced by suitable uniformly elliptic operators with nonhomogeneous Dirichlet, Neumann or oblique boundary conditions, possibly different on each side of the curvilinear polygon $\Omega$, under the assumption that the corresponding boundary value problem has a unique variational solution; furthermore, the requirement that the polygonal domain $\Omega$ coincides exactly with a cone of aperture less than $\pi$ in some neighborhood of each $S_j$ can be relaxed to $\partial \Omega$ being piecewise $C^1$ with finitely many jump discontinuities (at the points) $S_j$ of the normal vector with jump less than $\pi$. In this generality, the basis of the space of singular solutions is no longer given by (1.7).

Motivated by our application to the planar Euler equations, as well as for the sake of clarity and simplicity, we restricted ourselves to the Dirichlet problem and to domains with perfect corner singularities in our main result, Theorem 1. However, it will be clear from the proof how our techniques could extend to the more general setting of [13].

**1.2. Plan of the article and a word on the proof.** In Section 2, as we mentioned earlier in the introduction, we relate Theorem 1 with the planar Euler equations on polygonal domains. The proof of Theorem 1 is laid out in Sections 3 to 5. In Section 3, we first localize to the case of an infinite plane sector $\Sigma_\alpha$ of aperture $\alpha$. Then, for say $F \in C_0^\infty(\Sigma_\alpha)$, we observe the pointwise bound

$$
|D^2F(x)| \leq C|x|^{-1}\|\nabla F(x)\| + C|x|^{\frac{2\alpha}{\alpha - 2}}\left|\int_{\Sigma_\alpha} DK_\pi(x, y) \Delta F(y) \, dy\right| := R_1(x) + R_2(x),
$$

where $DK_\pi$ stands for the Jacobian matrix of the Biot-Savart kernel for the halfspace, by changing variables in the Biot-Savart law. A byproduct of one of the steps in Grisvard’s proof of (1.5) is that, whenever $F$ has no singular part,

$$
\|x|^{-1}\|\nabla F(x)\|_{L^p(\Sigma_\alpha)} \leq C(p, \alpha)\|\Delta F\|_{L^p(\Sigma_\alpha)};
$$

in Section 4, we reprove this bound following the same Kondratiev technique, but making sure that $C(p, \alpha) = C_\alpha p$ for $p$ larger than, and sufficiently far away from, the singular value $p_\alpha$ (see (3.2) below). We later note that the $L^p$ bounds on the $R_2$ part are equivalent to estimates for the singular integral with (Calderon-Zygmund) kernel $DK_\pi$ on the $L^p$ space with weight $x \to |x|^{2\delta(p-1)}$, with $\delta = 1 - \frac{\alpha}{p}$. Weighted bounds of this sort appear in the work of Buckley [5]: in Section 5, we carefully adapt his argument in order to obtain linear dependence on $p$ in the bounds that we need for Theorem 1.
**Notation.** We write
\[ \Sigma_\alpha = \{ x = \rho e^{i\theta} : \rho > 0, 0 < \theta < \alpha \} \subset \mathbb{R}^2 \]
for the open cone of aperture $0 < \alpha < 2\pi$ and vertex at the origin. In particular $\Sigma_\alpha$ coincides with the halfspace $\mathbb{R}^2_+$. Given a locally integrable function $w : \mathbb{R}^n \to (0, \infty)$, and a bounded measurable set $B \subset \mathbb{R}^n$, we denote
\[ w(B) = \int_B w(x) \, dx. \]
Also, for $A \subset \mathbb{R}^n$ open, we make use of the weighted spaces $L^p(A; w)$, $1 \leq p < \infty$, with norm
\[ \|f\|_{L^p(A; w)} = \left( \int_A |f(x)|^p w(x) \, dx \right)^{1/p}. \]
Throughout the article, we use the signs $\lesssim$, or $\sim$, to mean $\leq$, or $=$ respectively, up to an absolute multiplicative constant which may be different at each occurrence. The symbols $C_*$ will stand for positive constants, depending only on the argument(s) $\star$, allowed to implicitly vary from line to line as well.

2. **Uniqueness and Regularity of Solutions of the Planar Euler Equations**

We consider the Euler system set on $\Omega \subset \mathbb{R}^2$ in its vorticity-velocity formulation

\[
\begin{aligned}
\partial_t \omega(x, t) + (u \cdot \nabla) \omega(x, t) &= f(x, t), \quad x \in \Omega, \ t \in (0, T);
\omega(x, 0) &= \omega_0(x), \quad x \in \Omega.
\end{aligned}
\]

(2.1)

We refer the interested reader to e.g. the monographies [25, 29], the articles [3, 18, 27, 28] and references therein for a more comprehensive presentation.

Given $\omega_0 \in L^\infty(\Omega), f \in L^1((0, T); L^\infty(\Omega))$, a weak solution to (2.1) is a pair $(\omega, u)$ with

\[
\omega \in \mathcal{C}([0, T]; L^\infty(\Omega)) \cap L^\infty(\Omega \times (0, T)),
\]
\[
\omega(x, 0) = \omega_0(x), \quad x \in \Omega,
\]
\[
u(x, t) = \nabla \cdot G_\Omega(\omega(\cdot, t))(x), \quad x \in \Omega, t \in [0, T],
\]
satisfying the weak form of (2.1)

\[
\int_\Omega (\omega(x, t_2) - \omega(x, t_1)) \varphi(x) \, dx = \int_{t_1}^{t_2} \int_\Omega (\omega(x, t) u(x, t) \cdot \nabla \varphi(x) + f(x, t) \varphi(x)) \, dx \, dt,
\]

for all $0 \leq t_1 < t_2 \leq T$ and all $\varphi \in \mathcal{D}(\Omega)$. A consequence of the transport character of (2.1) is that any weak solution of (2.1) must satisfy

\[
\|\omega\|_{L^\infty(\Omega \times (0, T))} \leq Q(t) := \left( \|\omega_0\|_{L^\infty(\Omega)} + \|f\|_{L^1((0, T); L^\infty(\Omega))} \right) \quad \forall \ t \in [0, T].
\]

(2.2)

Now, assume that the domain $\Omega$ is such that estimate

\[
\|D^2 G_\Omega f\|_{\text{Exp} L^1(\Omega)} \leq C_\Omega \|f\|_{L^\infty(\Omega)},
\]
holds for all $f \in L^\infty(\Omega)$; we read from Theorem 1 that this is the case for a polygonal domain with $\max a_j \leq \frac{\pi}{2}$, since $S(\Omega) = \emptyset$ in (1.10). If $(\omega, u)$ is a weak solution to (2.1),
the uniform in time bound (2.2) then entails that $D\mathbf{u}$ is uniformly in time bounded in $\text{Exp}L^1(\Omega)$. This, in turn, implies that $\mathbf{u}$ is a log-Lipschitz vector field on $\Omega$ [1], that is

\begin{equation}
\|\mathbf{u}(t)\|_{LL(\Omega)} := \|\mathbf{u}(t)\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, 0 < |x-y| < \varepsilon^-1} \frac{|\mathbf{u}(x,t) - \mathbf{u}(y,t)|}{|x-y|\log(|x-y|)} \lesssim C'_\Omega Q(t); \tag{2.3}
\end{equation}

in particular, $\mathbf{u}$ generates a unique flow on $\Omega$. Therefore, arguing as in [3, Theorem 5.2] leads to the result we anticipated in the introduction.

**Theorem 2.** Let $\Omega$ be a polygonal domain with $\max \alpha_j \leq \frac{\pi}{2}$ and $C'_\Omega$ as in Theorem 1. Let $\omega_0 \in L^\infty(\Omega)$, $f \in L^1(0,T;L^\infty(\Omega))$, be given. Then, there exists a unique weak solution $(\omega, \mathbf{u})$ to (2.1), satisfying the estimate

\begin{equation}
|\mathbf{u}(t)|_{LL(\Omega)} \lesssim C'_\Omega (\|\omega_0\|_{L^\infty(\Omega)} + \|f\|_{L^1(0,T;L^\infty(\Omega))}) \quad \forall t \in [0,T]. \tag{2.4}
\end{equation}

**Remark 2.1.** As in [32], estimate (1.9) from Theorem 1 can be used to show (the existence and) uniqueness of weak solutions under weaker assumptions than $\omega_0 \in L^\infty(\Omega)$, $f \in L^1 L^\infty(\Omega)$, including unbounded initial vorticities (and forcing terms) with controlled blow-up of the $L^p$-norms as $p \to \infty$. For instance, one can take

$$\omega_0 \in \bigcap_{1 < p < \infty} L^p(\Omega) \quad \text{with} \quad \sup_{p > \varepsilon^2} \|\omega_0\|_{L^p(\Omega)} < \infty.$$  

A precise definition of the class of allowed data (usually referred to as Yudovich-type) can be found in [32, Section 5].

**Remark 2.2.** Theorem 2 was obtained in [3] in the case of angles $\alpha_j = \frac{\pi}{k}$, $k = 2,3,\ldots$, as a consequence of (1.11). In the recent preprint [22], the uniqueness part of Theorem 2 is proved under the more restrictive assumption that $\partial \Omega \in \mathcal{C}^{2,\varepsilon}$ (for some $\varepsilon > 0$) away from the corners. The methods employed therein are different in nature from ours and do not rely on elliptic estimates near $L^\infty(\Omega)$ like those of Theorem 1. We note that the techniques of [22] do not recover the log-Lipschitz regularity (2.4) of $\mathbf{u}$, and do not seem to extend to unbounded Yudovich-type data as in Remark 2.1.

3. Proof of Theorem 1: main reductions

We fix once and for all a polygonal domain $\Omega$ with $N$ corners $S_j$ of aperture $\alpha_j$, such that $\max \alpha_j < \pi$. It is convenient to denote $\Sigma := \Sigma_{\alpha_j} + S_j$; observe that, near each $S_j$, $\Omega$ coincides with $\Sigma_j$.

The next three subsections are devoted to the proof of (1.9). The estimate (1.10) is derived by first rewriting (1.9) in the weak-type form

$$|\{x \in \Omega : |\phi(x)| > \lambda\}| \leq \left(\frac{C_\Omega \lambda}{\lambda}\right)^p \|f\|_{L^p(\Omega)}, \quad \forall p \geq \rho_\Omega, \lambda > 0;$$

here and below, $\phi := D^2(G_\Omega f - \Pi_{\Omega\Sigma}(G_\Omega f))$. By linearity, we reduce to $\|f\|_{\infty} = 1$; we then have $\|f\|_{L^p(\Omega)} \leq |\Omega|$ for all $1 < p < \infty$, and the above display can be turned into

$$|\{x \in \Omega : |\phi(x)| > \lambda\}| \leq |\Omega| \exp\left(-\frac{1}{eC_\Omega}\right), \quad \forall \lambda > e \rho_\Omega C_\Omega.$$
by choosing \( p = e^{-1}(C_\Omega)^{-1} \). It is then an exercise in Orlicz spaces to show that
\[
\inf \left\{ t > 0 : \int \Omega \exp \left( \frac{\| \phi \|}{t} \right) dx \leq 1 + |\Omega| \right\} =: \| \phi \|_{\text{Exp}^1(\Omega)} \leq e^{p_\Omega + 1}C_\Omega = e^{p_\Omega + 1}C_\Omega \| f \|_{L^\infty(\Omega)},
\]
which is the claimed inequality (1.10).

**Remark 3.1.** When angles \( \alpha = \alpha_j \) with \( \frac{\pi}{2} < \alpha < \pi \) are present, the estimate (1.5) fails exactly for those values of \( p \in (2, \infty) \) given by
\[
\alpha := \frac{2\pi}{2\alpha - \pi},
\]
which we singled out into \( \frac{\partial}{\partial} \) in (1.4). The condition \( p \geq p_\Omega = 2 \max \frac{\alpha_j}{2} > 4 \) in Theorem 1 ensures both that \( S(p, \Omega) = S(\Omega) \) for \( p \) in this range (see (1.8) for notation), and that we are sufficiently far away from the values \( p_\alpha \), so that certain constants intervening in the estimates are uniformly bounded in \( p \). When \( \max \alpha_j \leq \frac{\pi}{2} \), \( \frac{\partial}{\partial} \) is empty, and this restriction is not necessary; however, the proof we give below for (1.9) yields a bound of the type \( C_\Omega \) only if \( p \geq \bar{p} > 2 \), with an additional constant depending on \( \bar{p} \). To unify notation, from now on we (re)define
\[
p_\Omega := \begin{cases} 4, & \max \alpha_j \leq \frac{\pi}{2}, \\ 2 \max \frac{\alpha_j}{2}, & \frac{\pi}{2} < \max \alpha_j < \pi; \end{cases}
\]
when \( \max \alpha_j \leq \frac{\pi}{2} \), once we have the cases \( p \geq 4 \) in hand, we recover the uniform estimate claimed in (1.9) in the range \( 2 \leq p < 4 \) by interpolation with the well-known case \( p = 2 \).

3.1. **Proof of (1.9): preliminaries.** For simplicity of presentation, we will rely on Grisvard’s Theorem 0 for the proof of the estimate (1.9) of Theorem 1. In particular, we gather from its proof in [13] that
\[
L^p(\Omega) = \{ f \in L^p(\Omega) : G_\Omega f \in W^{2,p} \cap \omega^{1,p}(\Omega) \} + \{ \Delta F : F \in S(p, \Omega) \}, \quad p \in [2, \infty) \cap \frac{\partial}{\partial},
\]
the sum being direct; moreover, the estimate (1.5) can be rewritten in a priori form as
\[
\| F \|_{W^{2,p}(\Omega)} \leq C(p, \Omega) \| \Delta F \|_{L^p(\Omega)} \quad \forall p \in [2, \infty) \cap \frac{\partial}{\partial}, \forall F \in W^{2,p} \cap \omega^{1,p}(\Omega),
\]
where \( C(p, \Omega) \) is a positive constant depending on \( p \) and \( \Omega \) (in particular, via the angles \( \alpha_j \)). Estimate (1.9) then follows once we show that \( C(p, \Omega) = C_\Omega p \) in (3.4) for all \( p \geq p_\Omega \), referring to (3.3). Therefore, in the sequel, we turn to the proof of family of a priori estimates
\[
\| F \|_{W^{2,p}(\Omega)} \leq C(p, \Omega) \| \Delta F \|_{L^p(\Omega)} \quad \forall p \leq p < \infty, \forall F \in W^{2,p} \cap \omega^{1,p}(\Omega).
\]

The proof of (3.5) begins with the derivation of further, preliminary, a priori estimates. Let then \( p \geq p_\Omega \), \( F \in W^{2,p} \cap \omega^{1,p}(\Omega) \) be given: since \( \Omega \) is bounded, we are allowed to take \( p = p_\Omega \) in (3.4), which yields
\[
\| F \|_{W^{2,p}(\Omega)} \leq C(p_\Omega, \Omega) \| \Delta F \|_{L^p(\Omega)} = C_\Omega \| \Delta F \|_{L^p(\Omega)} \leq C_\Omega \| \Delta F \|_{L^p(\Omega)}.
\]
Taking advantage of the Sobolev embedding \( W^{2,p} \cap \omega^{1,p}(\Omega) \subset C^1(\Omega) \), which holds under the condition that \( \partial \Omega \) be Lipschitz [10, 13], we have
\[
F \in C^1(\Omega), \quad \| F \|_{L^\infty(\Omega)} + \| \nabla F \|_{L^\infty(\Omega)} \leq C_\Omega \| \Delta F \|_{L^p(\Omega)}.
\]
We pause for a moment and further note that
\[ F(S_j) = 0, \nabla F(S_j) = 0, \quad \forall \ j = 1, \ldots, N. \]
The first part of (3.6) ensures that \( F(S_j), \nabla F(S_j) \) are well defined, and the first equality is obvious. The second equality follows from the fact that \( \nabla F(S_j) \) must be orthogonal to the normal vector of each side of the corner at \( S_j \), which span \( \mathbb{R}^2 \). This simple observation allows us to appeal to [13, Theorem 4.3.2.2], and obtain that the \textit{a priori} assumption \( F \in W^{2,p} \cap W^{1,p}_0(\Omega) \) entails the formally stronger property
\[ (3.7) \quad \| F \|_{p^2, p(\Omega)} := \| (\rho_\Omega)^{-2} F \|_{L^p(\Omega)} + \| (\rho_\Omega)^{-1} \nabla F \|_{L^p(\Omega)} + \| D^2 F \|_{L^p(\Omega)} < \infty \]
where \( \rho_\Omega: \Omega \to (0, \infty) \) is the distance to the singular set, namely \( \rho_\Omega(x) = \inf_{j} |x - S_j| \).

We are now free to assume (3.7) in the proof of (3.5), which occupies the next two subsections. In the upcoming Subsection 3.2, by means of a standard localization procedure, we reduce (3.5) to the analogous \textit{a priori} estimate for the infinite sector \( \Sigma_\alpha \), summarized in Proposition 3.2. The main line of the proof of Proposition 3.2 is then laid out in Subsection 3.3.

3.2. Localization to a single corner. In analogy with the norm appearing in (3.7) above, we will need the weighted norms
\[ (3.8) \quad \| F \|_{p^k, p(\Sigma_\alpha)} := \sum_{j=0}^{k} \| |x|^{-2+j} D^j F(x) \|_{L^p(\Sigma_\alpha)}, \quad k \in \{0, 1, 2\}, \ p \in (1, \infty). \]

**Proposition 3.2.** Let \( 0 < \alpha < \pi \). There exists a constant \( C_\alpha > 0 \) such that for each
\[ (3.9) \quad \begin{cases} p \in [4, \infty), & \text{if } 0 < \alpha \leq \frac{\pi}{2}, \\ p \in [2p_\alpha, \infty), & \text{if } \frac{\pi}{2} < \alpha < \pi, \end{cases} \]
and each \( F \in W^{2,p} \cap W^{1,p}_0(\Sigma_\alpha) \) with
\[ (3.10) \quad \| F \|_{p^2, p(\Sigma_\alpha)} < \infty, \]
there holds
\[ (3.11) \quad \| D^2 F \|_{L^p(\Sigma_\alpha)} \leq C_\alpha p \| \Delta F \|_{L^p(\Sigma_\alpha)}. \]

We defer the proof until Subsection 3.3, and turn to the task of recovering (3.5). Choose a positive \( r_0 < \frac{1}{2} \inf_{j \neq k} |S_j - S_k| \) and small enough so that
\[ \Omega_j := \Omega \cap \{ x \in \mathbb{R}^2 : |x - S_j| < r_0 \} = \Sigma_j \cap \{ x \in \mathbb{R}^2 : |x - S_j| < r_0 \}, \quad \forall \ j = 1, \ldots, N. \]
This choice guarantees \( \overline{\Omega_j} \cap \overline{\Omega_k} = \emptyset \) for \( j \neq k \). Let \( \Omega_0 \) be an open simply connected set with \( C^{1,1} \) boundary chosen so that
\[ \{ x \in \Omega : \inf_j |x - S_j| > \frac{r_0}{2} \} \subset \Omega_0 \subset \{ x \in \Omega : \inf_j |x - S_j| > \frac{r_0}{4} \}. \]
Let then \( \{ \mu_j \}_{j=0}^N \) be a smooth partition of \( 1_\Omega \) subordinated to the open cover \( \{ \Omega_j \}_{j=0}^N \), and write \( F_j = F \mu_j \). We see that \( F_j \) solves the Dirichlet problem
\[ (3.12) \quad \begin{cases} \Delta F_j = f_j \quad & \text{on } \Omega_j, \\ F_j = 0 \quad & \text{on } \partial \Omega_j, \end{cases} \]
\[ f_j = \mu_j \Delta F + \nabla F \cdot \nabla \mu_j + F \Delta \mu_j, \quad j = 0, \ldots, N. \]
The point of having (3.6) at our disposal is that
\[(3.13) \quad \|f_j\|_{L^p(\Omega_j)} \lesssim \|\mu_j\|_{C^2(\Omega)}(\|\Delta F\|_{L^p(\Omega)} + \|\nabla F\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)}) \leq C_\Omega \|\Delta F\|_{L^p(\Omega)}.\]

We estimate each summand $F_j$ separately. First, $\partial \Omega_0 \in C^{1,1}$, and the bound
\[(3.14) \quad \|D^2 F_0\|_{L^p(\Omega)} = \|D^2 F_0\|_{L^p(\Omega_0)} \leq C_\Omega p \|f_0\|_{L^p(\Omega_0)} \leq C_\Omega p \|\Delta F\|_{L^p(\Omega)},\]

simply follows from the standard Calderón-Zygmund theory as described in the introduction. For $j = 1, \ldots, N$, we use Proposition 3.2. We denote by $\overline{F}_j, \overline{f}_j : \Sigma_j \to \mathbb{R}$ the trivial extensions of $F_j$ (resp. $f_j$) by 0 on $\Sigma_j \setminus \Omega_j$. It is easy to see that $\overline{F}_j \in W^{2,p} \cap W_0^{1,p}(\Sigma_j)$, and $\Delta \overline{F}_j = \overline{f}_j$. Moreover, by localization of our assumption (3.7), we see that condition (3.10) holds for $F = \overline{F}_j$. Lastly, $p \geq 2p_\Omega \geq 2p_{\alpha_j}$ for each $j$ with $\alpha_j > \frac{\pi}{2}$. Thus, we may apply Proposition 3.2 and estimate
\[(3.15) \quad \|D^2 F_j\|_{L^p(\Omega)} = \|D^2 \overline{F}_j\|_{L^p(\Sigma_j)} \leq C_{\alpha_j} \|\overline{f}_j\|_{L^p(\Sigma_j)} \leq C_\Omega p \|\Delta F\|_{L^p(\Omega)},\]

making use of (3.13) in the last step. Finally, the bound (3.5) follows by summing up (3.6) (for the lower order derivatives), (3.14), and (3.15) over $j = 1, \ldots, N$. To complete the deduction of Theorem 1, we are thus left with proving Proposition 3.2; we do so in the next subsection.

3.3. Proof of Proposition 3.2. Before entering the actual proof of the Proposition, we point out that the behavior of the constant $C_\alpha$ intervening in (3.11) is as follows:
\[(3.16) \quad C_\alpha \lesssim \begin{cases} \varepsilon_\alpha^{-1}, & \alpha \neq \frac{\pi}{2} \\ 1 & \alpha = \frac{\pi}{2}, \end{cases} \quad \varepsilon_\alpha := \text{dist}(\alpha, \{0, \frac{\pi}{2}, \pi\}).\]

In particular, although the cases $\alpha \in \{\frac{\pi}{2}, \pi\}$ of the Proposition hold, the constant $C_\alpha$ blows up as $\alpha \to \frac{\pi}{2}, \pi$ (and as $\alpha \to 0$ too), and the proof below does not work when $\alpha = \frac{\pi}{2}$. In that case, estimate (3.11) can be obtained by a reflection argument in the vein of [3, Proposition 3.1]. We assume $\alpha \neq \frac{\pi}{2}$ for the remainder of the subsection (and of the article as well).

To prove Proposition 3.2, we must show that, for each $F \in W^{2,p} \cap W_0^{1,p}(\Sigma_\alpha)$ satisfying assumption (3.10), there holds, for $j = 1, 2$,
\[(3.17) \quad \|\partial_j \nabla F\|_{L^p(\Sigma_\alpha)} \leq C_\alpha p \|\Delta F\|_{L^p(\Sigma_\alpha)}, \quad \bar{p} < p < \infty.\]

From now on, we adopt the complex notation $x = (x_1, x_2) = x_1 + ix_2$, and we make use of the change of angle map
$$Z_\alpha : \Sigma_\alpha \to \Sigma_\pi, \quad Z_\alpha(x) = x^{\frac{\alpha}{\pi}};$$
with derivative and Jacobian
$$DZ_\alpha(x) = \frac{\pi}{\alpha} x^{\frac{\alpha}{\pi} - 1}; \quad \det DZ_\alpha(x) = \left(\frac{\pi}{\alpha} |x|^{\frac{\alpha}{\pi} - 1}\right)^2.$$

Let $K_\pi$ be the Biot-Savart kernel for the halfplane, which, by the image method, is
\[(3.18) \quad K_\pi(z, \zeta) = (2\pi)^{-1}\left(\frac{z - \zeta}{|z - \zeta|^2} - \frac{z - \overline{\zeta}}{|z - \overline{\zeta}|^2}\right)\perp, \quad z \neq \zeta \in \Sigma_\pi.\]
By conformality of $Z_\alpha : \Sigma_\alpha \to \Sigma_\pi$, the Biot-Savart law on $\Sigma_\alpha$, $f \mapsto u_f = \nabla \circ G_{Z_\alpha} f$ can be written as the fractional (in the sense of Hardy and Littlewood) integral
\[ u_f(x) := DZ_\alpha(x) \int_{\Sigma_\alpha} K_\pi(Z_\alpha(x), Z_\alpha(y)) f(y) \, dy =: DZ_\alpha(x) I_f(Z_\alpha(x)). \]

The core of the argument for (3.17) begins now: we use that $\partial_j \nabla F = \partial_j u_{DF}$, so that for $j = 1,2,$
\[ \partial_j \nabla F(x) = (\partial_j(DZ_\alpha(x))) I_{DF}(Z_\alpha(x)) + DZ_\alpha(x)D I_{DF}(Z_\alpha(x)) \partial_j Z_\alpha(x) \]
\[ = (\partial_j(DZ_\alpha(x))) D^{-1}u_{DF}(x) + DZ_\alpha(x)D I_{DF}(Z_\alpha(x)) \partial_j Z_\alpha(x) =: R_1(x) + R_2(x), \]
where $DI_{DF} : \Sigma_\pi \to \mathbb{C}^2$ is given by the (singular) integral
\[ DI_{DF}(z) = \int_{\Sigma_\alpha} (D_2 K_\pi)(z, Z_\alpha(y)) \Delta F(y) \, dy, \quad z \in \Sigma_\pi. \]

3.3.1. Estimating $\|R_1\|_{L^p(\Sigma_\alpha)}$. An easy computation yields
\[ |\partial_j(DZ_\alpha(x))| \leq 2(\frac{a}{\alpha})^2 \|x\|^{\frac{\alpha}{2}} - 2, \quad x \in \Sigma_\alpha, \]
and therefore
\[ |R_1(x)| \leq \frac{a}{\alpha} |x|^{1 - \frac{\alpha}{2}} |\partial_j(DZ_\alpha(x))| |u_{DF}(x)| \leq \frac{2a}{\alpha} |x|^{-1} |u_{DF}(x)| = \frac{2a}{\alpha} |x|^{-1} |\Delta F(x)|, \quad x \in \Sigma_\alpha. \]
The bound (3.17) for $R_1$ then follows immediately from the next lemma, comparing with the definition of the weighted norms in (3.8).

**Lemma 3.3.** Under the assumptions of Proposition 3.2, when $\alpha \neq \frac{\pi}{2}$,
\[ \|F\|_{L^1(\Sigma_\alpha)} \lesssim a \varepsilon^{-1} \|\Delta F\|_{L^p(\Sigma_\alpha)}. \]

Apart from the explicit behavior in $p$ of the constant in (3.21), Lemma 3.3 is contained in the proof of [13, Theorem 4.3.2.2]; for a different approach, see [14, Subsection 4.4]. The proof, along the same lines, though keeping track of the $p$-dependence in (3.21), is given in Section 4.

3.3.2. Estimating $\|R_2\|_{L^p(\Sigma_\alpha)}$. Bounding $R_2$ involves the definition of a suitable power weight on the halfspace $\Sigma_\pi$. We set
\[ \delta := 1 - \frac{a}{\alpha}, \quad w_{p,\delta}(z) = (\frac{a}{\alpha}|z|^\delta)^{2(p-1)}, \quad z \in \Sigma_\pi, \]
so that defining
\[ g(\zeta) := (\text{det}(DZ_\alpha(Z_\alpha)^{-1}(|\zeta|)))^{-1} \Delta F(Z_\alpha)^{-1}(\zeta) = \left(\frac{a}{\alpha}|\zeta|^\delta\right)^{-2} \Delta F(\zeta^{\frac{2}{\alpha}}), \quad \zeta \in \Sigma_\pi, \]
and changing variables $y = (Z_\alpha)^{-1}(\zeta)$,
\[ \|g\|_{L^p_{\delta}(\Sigma_\pi, w_{p,\delta})} = \left( \int_{\Sigma_\pi} |\Delta F(Z_\alpha)^{-1}(\zeta)|^p \left(\frac{a}{\alpha}|\zeta|^\delta\right)^{-2} d\zeta \right)^{\frac{1}{p}} = \|\Delta F\|_{L^p_{\delta}(\Sigma_\pi)}^p. \]
We introduce the singular integral operator on $\Sigma_\pi$ with kernel $D_2 K_\pi(z, \zeta)$, that is
\[ f(\zeta) \mapsto T f(\zeta) = \int_{\Sigma_\pi} D_2 K_\pi(z, \zeta) f(\zeta) d\zeta, \quad z \in \Sigma_\pi. \]
Note that $T$ is a standard convolution-type Calderón-Zygmund singular integral; its kernel can be read explicitly from (5.1) below. The change of variable $\zeta = Z_\alpha(y)$ in the integral (3.20) for $DI_{\Delta F}(Z_\alpha(x))$ leads us to the equality
\begin{equation}
(3.25) \quad DI_{\Delta F}(Z_\alpha(x)) = Tg(Z_\alpha(x)), \quad x \in \Sigma_\alpha.
\end{equation}
Taking advantage of the estimate
\[ |DZ_\alpha(x)||\partial_j Z_\alpha(x)| \leq \left( \frac{3}{\alpha} \right)^2 |x|^{2(\frac{3}{\alpha} - 1)} = \left( \frac{3}{\alpha} \right)^2 |Z_\alpha(x)|^\delta, \]
and then performing the change of variable $z = Z_\alpha(x)$, one sees that
\begin{equation}
(3.26) \quad \|R_2\|_{L^p(\Sigma_\alpha)} \leq \int_{\Sigma_\alpha} \left( \frac{4}{\alpha} |Z_\alpha(x)|^{\delta} \right)^{2p} |Tg(Z_\alpha(x))|^p \, dx
= \int_{\Sigma_\pi} |Tg(z)|^p \left( \frac{\pi}{\alpha} |z|^{\delta} \right)^{2(p-1)} \, dz = \|Tg\|_{L^p(\Sigma_{\pi}^\delta, w, p, \delta)}^{p}.
\end{equation}
We obtain the bound (3.17) on $R_2$, and thus conclude the main line of proof of Proposition 3.2, in view of the above display and of (3.23), making use of Lemma 3.4 below for the second inequality:
\[ \|R_2\|_{L^p(\Sigma_\alpha)} \leq \|Tg\|_{L^p(\Sigma_{\pi}^\delta, w, p, \delta)} \lesssim p \epsilon^{-1} \|g\|_{L^p(\Sigma_{\pi}^\delta, w, p, \delta)} = p \epsilon^{-1} \|\Delta F\|_{L^p(\Sigma_\alpha)}. \]

**Lemma 3.4.** Let $2 \leq p < \infty$, $w, p, \delta$ as in (3.22), and $T$ as in (3.24). Then
\[ \|T\|_{L^p(\Sigma_{\pi}^\delta, w, p, \delta)} \lesssim \frac{p}{\delta(1-\delta)} \lesssim p \epsilon^{-1}. \]

This lemma is an instance of a sharp (in terms of $p$) $L^p$ bound for Calderón-Zygmund singular integrals on (positive) power weighted spaces, see Proposition 5.1. Section 5 contains the statement and proof of this proposition, and the subsequent (easy) derivation of Lemma 3.4.

4. PROOF OF LEMMA 3.3

In this proof, we use complex polar notation for $(x_1, x_2) \in \Sigma_\alpha$, writing
\[ (x_1, x_2) = \rho(x_1, x_2)e^{i\theta(x_1, x_2)}; \]
unless otherwise specified, the differential operators $\nabla, D, \Delta$ are understood to be in cartesian coordinates. For $2 < p < \infty$, we write $q = p' \in (1, 2)$. As anticipated, we follow the same rough outline of [13, Theorem 4.3.2.2], beginning with the definitions of
\[ U(t, \theta) = e^{-\frac{2}{\alpha} t} F(e^{t+i\theta}), \quad h(t, \theta) = e^{\frac{2}{\alpha} t} \Delta F(e^{t+i\theta}), \quad (t, \theta) \in O_\alpha := \mathbb{R} \times (0, \alpha) \]
corresponding to the change of variables $\rho(x_1, x_2) = e^t$, $\theta(x_1, x_2) = \theta$. A simple computation shows that, for $k \in \{0, 1, 2\}$,
\begin{equation}
(4.1) \quad \|F\|_{p^k(p, \Sigma_\alpha)} \leq 4 \|U\|_{W^k(p, \Sigma_\alpha)} \leq 16 \|F\|_{p^k(p, \Sigma_\alpha)};
\end{equation}
in particular, using the rightmost inequality in (4.1) for $k = 2$, we learn from the assumption (3.10) that $U \in W^2(p(O_\alpha))$. Since $\|\Delta F\|_{L^p(O_\alpha)} = \|h\|_{L^p(O_\alpha)}$, inequality (3.21) for $F$ (and hence Lemma 3.3) will follow by coupling the estimate
\[ \|U\|_{W^1(p, O_\alpha)} \lesssim \alpha \epsilon^{-1} p \|h\|_{L^p(O_\alpha)} \]
with the leftmost inequality in (4.1) for \( k = 1 \). By construction, \( U \) and \( h \) satisfy the elliptic problem (4.2) below, and the above inequality is an instance of the following Lemma, which we will prove momentarily; this discussion completes the proof of Lemma 3.3.

**Lemma 4.1.** Let \( \alpha \neq \frac{\pi}{2} \), \( h \in L^p(O_a) \) be given and suppose that \( U \in W^{2,p}(O_a) \) solves the elliptic problem

\[
(4.2) \begin{cases}
(\partial_t + \frac{2}{q}I)^2 U(t,\theta) + \partial_\theta^2 U(t,\theta) = h(t,\theta), & (t,\theta) \in O_a; \\
U(t,0) = U(t,\alpha) = 0, & t \in \mathbb{R}.
\end{cases}
\]

Then, for all \( p \) in the range (3.9),

\[
(4.3) \|U\|_{W^{1,p}(O_a)} \lesssim \alpha \varepsilon_a^{-1} p \|h\|_{L^p(O_a)}.
\]

**Proof.** By a density argument, we can further assume that \( U \in W^{2,p}(O_a) \cap W^{2,2}(O_a) \); this justifies the use of the partial Fourier transform

\[
\hat{U}(\xi,\theta) := \int_{\mathbb{R}} U(t,\theta) e^{-i t \xi} dt, \quad \xi \in \mathbb{R}.
\]

For simplicity, denote \( \Xi = \Xi(\xi,\rho) = \frac{2}{q} + i \xi \). This is the point where we use condition (3.9), which guarantees that

\[
(4.4) \quad |\sin(\alpha \Xi)| \geq |\sin(\frac{2\alpha}{q}) \cosh(\alpha \xi)| \gtrsim \varepsilon_a, \quad \forall \xi \in \mathbb{R},
\]

referring to (3.16) for \( \varepsilon_a \). Hence, in the interval (3.9), the solvability condition \( \sin(\alpha \Xi) \neq 0 \) for all \( \xi \in \mathbb{R} \) of [13, Theorem 4.2.2.2] is fulfilled, and we have the explicit formulas

\[
(4.5) \quad \hat{U}(\xi,\theta) = \int_0^\alpha K(\xi,\theta,y) h(\xi,y) dy,
\]

\[
(4.6) \quad \hat{\partial_t U}(\xi,\theta) = \int_0^\alpha L(\xi,\theta,y) h(\xi,y) dy, \quad L(\xi,\theta,y) = i \xi K(\xi,\theta,y)
\]

\[
(4.7) \quad \hat{\partial_\theta U}(\xi,\theta) = \int_0^\alpha M(\xi,\theta,y) h(\xi,y) dy, \quad M(\xi,\theta,y) = (\partial_\theta K)(\xi,\theta,y)
\]

with

\[
K(\xi,\theta,y) = \frac{\sin(\Xi y) \sin(\Xi(y-a))}{\Xi \sin(\Xi a)}, \quad \theta \leq y, \quad K(\xi,\theta,y) = \frac{\sin(\Xi(y-a)) \sin(\Xi y)}{\Xi \sin(\Xi a)}, \quad \theta > y.
\]

Using the asymptotics (for \( |\xi| \) large) \( |\sin(\Xi x)| \sim \exp(|x\Xi|) \), and the bound from below (4.4) for \( |\xi| \lesssim 1 \), we verify that the bounds

\[
(4.8) \quad |\Xi| |K(\xi,\theta,y)| + |\xi \partial_\xi K(\xi,\theta,y)| \lesssim \varepsilon_a^{-1} \exp\left((y+\theta-2a)|\xi|\right) \lesssim \varepsilon_a^{-1},
\]

\[
(4.9) \quad |M(\xi,\theta,y)| + |L(\xi,\theta,y)| \lesssim \varepsilon_a^{-1} \exp\left((y+\theta-2a)|\xi|\right) \lesssim \varepsilon_a^{-1},
\]

\[
(4.10) \quad |\xi \partial_\xi M(\xi,\theta,y)| + |\xi \partial_\xi L(\xi,\theta,y)| \lesssim \varepsilon_a^{-1}(2a-y-\theta)|\xi| e^{(y+\theta-2a)|\xi|} \lesssim \varepsilon_a^{-1},
\]

hold uniformly in \( y, \theta \in (0,a) \) and \( \xi \in \mathbb{R} \). We show how the bound (4.8) implies

\[
(4.11) \quad \|U\|_{L^p(O_a)} \lesssim \alpha \varepsilon_a^{-1} p \|h\|_{L^p(O_a)},
\]

and the estimate on \( \|\nabla_{t,\theta} U\|_{L^p(O_a)} \) will follow by the same procedure, this time exploiting the inequalities (4.9)-(4.10). For fixed \( \theta, y \in (0,a) \), define the multiplier operator on \( \mathbb{R} \)

\[
T_{\theta,y} f(t) = \mathcal{F}^{-1}(\hat{f}(\xi) K(\xi,\theta,y))(t).
\]
By the Hörmander-Mihlin multiplier theorem with sharp constant (see e.g. [26]), indicating with \( \kappa(\theta, y) \) the supremum over \( \xi \in \mathbb{R} \) of the left hand side of (4.8), we have that

\[
\| T_{\theta, y} \|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \lesssim p \kappa(\theta, y) \leq p \left( \sup_{\theta, y \in (0, a)} \kappa(\theta, y) \right) \lesssim p \varepsilon^{-1}, \quad \forall 2 \leq p < \infty;
\]

we used the bound (4.8). Then, denoting by \( h_y : \mathbb{R} \to \mathbb{C} \) the function \( t \mapsto h_y(t) := h(t, y) \), inverse Fourier transformation of (4.5) yields

\[
U(t, \theta) = \int_0^\alpha T_{\theta, y}(h_y(t)) dy.
\]

We thus obtain, by applying in sequence Hölder's inequality, Minkowski's inequality, the bound (4.12) and Hölder's inequality again,

\[
\| U \|_{L^p(O_a)} \leq a^{\frac{1}{2}} \sup_{\theta \in (0, a)} \left( \int_{\mathbb{R}} \| U(t, \theta) \|^p dt \right)^{\frac{1}{p}} \leq a^{\frac{1}{2}} \sup_{\theta \in (0, a)} \int_0^\alpha \| T_{\theta, y}(h_y) \|_{L^p(\mathbb{R})} dy \lesssim a^{\frac{1}{2}} \varepsilon^{-1} \int_0^\alpha \| h_y \|_{L^p(\mathbb{R})} \| h \|_{L^p(O_a)} \varepsilon^{-1} \int_0^\alpha \| h_y \|_{L^p(\mathbb{R})} \| h \|_{L^p(O_a)};
\]

that is, we have proved (4.11). The proof of the lemma is complete. \( \square \)

**Remark 4.2.** The proof of inequality (1.5) in [13] (as well as in [14]) relies on the change of variable to the strip (familiarly known as Kondratiev’s technique, first appearing in [19]) for the bound on the second derivatives of \( F \) as well, more precisely on

\[
\| U \|_{W^{2, p}(O_a)} \leq C(p, \alpha) \| h \|_{L^p(O_a)}, \quad p > \max\{4, p_a\}
\]

for the solution to (4.2), which is then turned into \( P^{2, p}(\Sigma_a) \)-bounds for \( F \) via the equivalence (4.1). However, as anticipated in the introduction, the method of proof therein yields a constant \( C(p, \alpha) \sim C_a p^2 \). In fact, (4.13) is derived by bootstrap of (4.3), using the fact that \( U \) solves the elliptic problem

\[
\begin{aligned}
LU &= \tilde{h}, \\
U(t, 0) &= U(t, a) = 0, \\
\theta &= \Delta_{t, \theta} - \text{Id}, \quad \tilde{h} := h + \left( 1 - \frac{4}{q} \right) U - \frac{4}{q} \partial_t U.
\end{aligned}
\]

This entails (see [13, Theorem 4.2.2.2]) the estimate

\[
\| D_{t, \theta}^2 U \|_{L^p(O_a)} \leq C_a \| \tilde{h} \|_{L^p(O_a)};
\]

the linear growth in \( p \) is introduced by a further application of the Hörmander-Mihlin theorem to the symbol associated to \( D_{t, \theta}^2 \circ L^{-1} \). Recalling, from (4.3), that \( \| \tilde{h} \|_{L^p(O_a)} \leq C_a \| h \|_{L^p(O_a)}, \) the above display yields (4.13) with quadratic growth in \( p \).

5. Sharp weighted estimates and Proof of Lemma 3.4

In this section, we derive Lemma 3.4 from a sharp \( L^p \) bound for Calderón-Zygmund singular integral operators on power weighted spaces, Proposition 5.1 below.

**Proposition 5.1.** Let \( T \) be a Calderón-Zygmund singular integral operator of convolution type on \( \mathbb{R}^2 \), that is, \( T f = f \ast K \) is given by principal value convolution with an \( \mathbb{R}^m \)-valued
kernel $K$ satisfying the size and cancellation conditions:
\[
\|\hat{K}\|_{L^\infty(\mathbb{R}^2)} \lesssim 1,
\]
\[
|K(x)| \lesssim |x|^{-2}, \quad x \in \mathbb{R}^2 \setminus 0,
\]
\[
|K(x) - K(x - y)| \lesssim |y||x|^{-3}, \quad x \in \mathbb{R}^2 \setminus 0, |y| < |x|/2.
\]

For each $2 \leq p < \infty$, $0 < \delta < 1$, let $w_{p,\delta}(x) = |x|^{2\delta(p-1)}$. Then,
\[
\|T\|_{\mathscr{L}(L^p(\mathbb{R}^2; w_{p,\delta}))} \lesssim \frac{1}{\delta(1-\delta)p} \quad \forall 2 \leq p < \infty, \ 0 < \delta < 1.
\]

Our proposition is a re-elaboration of Buckley’s result [5, Theorem 2.14 (iii)] with focus on sharp dependence on $p$ rather than on the $A_p$ constant of the power weight. The proof, which is postponed to the concluding Subsection 5.3, will still rely on the theory of $A_p$ weights. Subsection 5.2 contains the basic definitions, as well as a preliminary result: a $p$-independent version of Buckley’s sharp $A_p$ bound for the Hardy-Littlewood maximal function [5, Theorem 2.5]. The upcoming Subsection 5.1 is devoted to the derivation of Lemma 3.4.

5.1. Proof of Lemma 3.4. We put ourselves into position for an application of Proposition 5.1. Begin by observing that, for $j = 1, 2$
\[
L_1(z, \zeta) = L_1(z - \zeta) := (2\pi)^{-1}\partial_{z_1} \left( \frac{(z - \zeta)^1}{|z - \zeta|^2} \right)
\]
\[
= (2\pi)^{-1}|z - \zeta|^{-4}( -2(z_1 - \zeta_2)(\zeta_1 - \zeta_2), |z - \zeta|^2 - 2(z_1 - \zeta_1)^2),
\]
is a Calderón-Zygmund convolution kernel on $\mathbb{R}^2$, that is, it satisfies the size and cancellation conditions of Proposition 5.1; the same holds for the analogously obtained $L_2(z, \zeta) := (2\pi)^{-1}\partial_{z_2} \left( \frac{(z - \zeta)^1}{|z - \zeta|^2} \right)$, and we call $T_L$ the linear operator on $\mathbb{R}^2$ given by principal value convolution with the (matrix valued) kernel $L(z - \zeta) := (L_1(z - \zeta), L_2(z - \zeta))$.

Recall that the operator $T$ defined in (3.24) is given by principal value convolution on $\Sigma_{\pi} \equiv \mathbb{R}^2_+$ with the kernel
\[
DK_{\pi}(z, \zeta) = L(z - \zeta) - L(z - \bar{\zeta}).
\]
Hence, defining $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$, $g_1(\zeta) := g(\zeta)1_{\zeta_2 > 0}, g_2(\zeta) := g(\bar{\zeta})1_{\zeta_2 < 0}$, one has the equality
\[
Tg(z) = T_L(g_1)(z) - T_L(g_2)(z), \quad \forall z \in \mathbb{R}^2_+.
\]
Referring to $w_{p,\delta}$ in (3.22), it is clear that $\|g_j\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})} = \|g\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})}$. Of course, Proposition 5.1 also applies to the operator $T_L$ with weight $w_{p,\delta}$, which is a constant multiple of $w_{p,\delta}$, so that
\[
\|Tg\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})} \lesssim \sum_{j=1}^{2} \|T_L(g_j)\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})} \lesssim \sum_{j=1}^{2} \|T_L(g_j)\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})} \lesssim C\delta \varepsilon \sum_{j=1}^{2} \|g_j\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})} \lesssim C\delta \varepsilon \|g\|_{L^p(\mathbb{R}^2_+; w_{p,\delta})},
\]
with $C\delta = \delta^{-1}(1-\delta^{-1})$. This concludes the proof of Lemma 3.4.
5.2. The class of $A_p$ weights. We refer the interested reader to [26, Chapter 5] and to the volume [11] for detailed accounts of the classical $A_p$ theory of singular integrals, and limit ourselves to what is needed for the proof of Proposition 5.1. We mention here that the $A_p$ theory has seen a surge of activity in the recent years, leading in particular to the proof of the sharp dependence of the $L^p$ operator norm of a general Calderón-Zygmund operator on the $A_p$ constant of the weight (not necessarily power), originally due to Hytönen [15]; the article [23] surveys the modern $A_p$ theory leading to Hytönen’s result and further developments.

Let $1 < p < \infty$. We say that a locally integrable $w : \mathbb{R}^n \rightarrow (0, \infty)$ is an $A_p$ weight if

$$w^*(x) := (w(x))^{-\frac{1}{p}}, \quad x \in \mathbb{R}^n,$$

is locally integrable as well and the $A_p$ characteristic of $w$

$$(5.2) \quad [w]_{A_p} := \left( \sup_{B \text{ balls } \subset \mathbb{R}^n} \frac{(w(B))^{\frac{1}{p}}(w^*(B))^{\frac{1}{p'}}}{|B|} \right)^p$$

is finite. Note that $[w]_{A_p} \geq [1_{\mathbb{R}^n}]_{A_p} = 1$, that $[w]_{A_p}$ is 0-homogeneous, namely $[cw]_{A_p} = [w]_{A_p}$ for all $c > 0$, and that (by Hölder’s inequality) $[w]_{A_q} \leq [w]_{A_p}$ whenever $q > p$. The weight $w^*$ is referred to as the dual weight of $w \in A_p$: from the definition, it is clear that

$$w \in A_p \implies w^* \in A_p', \quad [w]_{A_p}' = [w^*]_{A_p}^p.$$

Finally, it is easy to see that for any sublinear operator $T$

$$(5.3) \quad \|T\|_{\mathcal{L}(L^p(\mathbb{R}^n; w))} = \|T(w^*)\|_{L^p(\mathbb{R}^n; w^*) \rightarrow L^p(\mathbb{R}^n; w)}.$$

We are chiefly interested in power weights $w(x) = |x|^\eta$. A computation (see also [5, Lemma 1.4]) shows that $|x|^\eta$ is an $A_p$ weight if and only if $-2 < \eta < 2(p - 1)$. More precisely, when $\eta = 2\delta(p - 1)$ with $0 < \delta < 1$ (which is the case in our application: see (3.22)),

$$(5.4) \quad [x \mapsto |x|^{2\delta(p-1)}]_{A_p} \lesssim (1 - \delta)^{-(p-1)}.$$

In the remainder of this subsection, we state and prove a version of Buckley’s sharp $A_p$ bound for the Hardy-Littlewood maximal function [5, Theorem 2.5] with explicit constant, which we are going to use in the proof of Proposition 5.1.

**Proposition 5.2.** Indicate by $M$ the Hardy-Littlewood maximal function. Let $2 \leq p < \infty$, and $w$ be any (not necessarily power) $A_p$ weight. Then

$$(5.5) \quad \|M\|_{\mathcal{L}(L^p(\mathbb{R}^n; w))} \leq 4 \cdot 3^4 [w]_{A_p}^{\frac{1}{p-1}}.$$

**Proof.** We first give a proof of the weaker bound

$$(5.6) \quad \|M\|_{\mathcal{L}(L^p(\mathbb{R}^n; w))} \lesssim [w]_{A_p}^{\frac{1}{p-1}},$$

in the case of a power weight $w(x) = |x|^{2\delta(p-1)}$, which would suffice for our purpose of proving Proposition 5.1, albeit with a worse conditioned (quadratic, instead of linear) dependence on $1 - \delta$. The idea is to pair the sharp weak-type bound (see [5, eq. (2.6)])

$$(5.7) \quad \|M\|_{L^q(\mathbb{R}^n; w) \rightarrow L^{q,\infty}(\mathbb{R}^n; w)} \leq 9^q [w]_{A_q}^{\frac{1}{q-1}} \leq 9 [w]_{A_q}^{\frac{1}{q-1}} \quad \forall w \in A_q, 1 < q < \infty$$
with the Marcinkiewicz interpolation theorem for positive measures: for a sublinear operator $S$, $1 \leq p_0 < p_1 < \infty$, $0 < t < 1$, and $p^{-1} = (p_0)^{-1} + t((p_1)^{-1} - (p_0)^{-1})$, there holds

$$
(5.8) \quad \|S\|_{L^p(d\mu)} \leq 2 \left( \frac{p_1}{p(p_1-p)} \right)^{\frac{1}{2}} \left( \left\|S\right\|_{L^{p_0}(d\mu) \to L^{p_0\infty}(d\mu)} \right)^{(1-t)} \left( \left\|S\right\|_{L^{p_1}(d\mu) \to L^{p_1\infty}(d\mu)} \right)^{t}.
$$

Resorting to formula (5.4), $w(x) = |x|^{2\delta(p-1)} \in A_q$ for all $q$ with $\eta := \delta(p-1)(q-1)^{-1} < 1$, and $[w]_{A_q} \sim (1-\eta)^{-(q-1)}$. We choose $p_0 < p < p_1$ such that

$$
p_0 - 1 = \frac{2\delta}{\delta+1}(p-1), \quad \frac{1}{p} = \frac{1}{2p_0} + \frac{1}{2p_1}.
$$

The point of this choice is that, setting $\kappa := \frac{2\delta}{(\delta+1)p} + \frac{1}{p}$ and using $p \geq 2$,

$$
\left( \frac{p_1}{p(p_1-p)} + \frac{p_0}{p(p-p_0)} \right)^{\frac{1}{p}} = \left( \frac{2\kappa}{p(1-x)} \right)^{\frac{1}{p}} = \left( \frac{2p'(1+\delta)}{p(1-\delta)} \right)^{\frac{1}{p}} \leq 2 \left( \frac{1}{1-\delta} \right) \sim [w]\frac{1}{p-1},
$$

$$
[w]\frac{1}{A_p} \sim (1-\delta)^{-1} \sim [w]\frac{1}{A_p} \leq [w]\frac{1}{A_p} \leq [w]\frac{1}{A_p},
$$

and (5.6) follows by plugging estimates (5.7) for $q = p_0$ and $q = p_1$ into (5.8) and using the bounds of the last display.

To prove (5.5) in full, we employ an argument inspired by Lerner’s approach [24] of bounding the dyadic Hardy-Littlewood maximal operator by the composition of weighted maximal functions. This argument was pointed out to us by Kabe Moen (personal communication). The first step is to pass to the dyadic maximal function

$$
M^D f(x) := \sup_{x \in Q \in D} \frac{1}{|Q|} \int_Q |f| \, dy,
$$

where $D$ is the standard dyadic grid on $\mathbb{R}^2$, relying on the pointwise bound

$$
Mf(x) \leq 3^4 \sup_{D \subset \{D_1, \ldots, D_9\}} M^{D_j} f(x),
$$

$D_1, \ldots, D_9$ being suitable fixed shifts of $D$; see [24] for details. Thus, the inequality (5.5) will follow from the dyadic analogue

$$
(5.9) \quad \|M^D\|_{L^p(\mathbb{R}^2; w)} \leq p^{\gamma} p [w]\frac{1}{A_p} \leq 4[w]\frac{1}{A_p}, \quad w \in A_p, 2 \leq p < \infty.
$$

The advantage of working with the dyadic versions resides in having at disposal $L^p$-bounds for the weighted maximal functions which do not depend on the $A_p$ characteristic, via the doubling constant of the weight, unlike the non-dyadic version. The lemma below is proved in exactly the same way as the (dyadic) Hardy-Littlewood maximal theorem; that is, by first proving the weak $(1,1)$ bound and then interpolating with the trivial $L^\infty$ bound.

**Lemma 5.3.** Let $w$ be an $A_q$ weight for some $1 < q < \infty$ and consider the weighted dyadic maximal function

$$
M^D_w f(x) := \sup_{x \in Q \in D} \frac{1}{w(Q)} \int_Q |f| w \, dy.
$$

Then, for all $1 < p < \infty$, $\|M^D_w\|_{L^p(\mathbb{R}^2; w)} \leq p^\gamma$. 
The core of the proof of (5.9) lies in the pointwise bound

\[ M^p(f \ast w)(x) \leq [w]_{A_p}^{1/p} \cdot \left( M^p_w ((M^p_{\ast w} f)^{p}) w^{-1}(x) \right)^{1/p}. \]

Indeed, taking $L^p$-norms in (5.10) yields

\[ \|M^p(f \ast w^{\star})\|_{L^p(\mathbb{R}^2;w)} \leq [w]_{A_p}^{1/p} \cdot \left( M^p_w ((M^p_{\ast w} f)^{p}) w^{-1} \right)^{1/p}. \]

Using (5.3) and Lemma 5.3 (twice), the above inequality is readily turned into (5.9). The bound (5.10) is obtained by taking supremum over $x \in \mathbb{R}^2$, $Q \in \mathcal{D}$ with $x \in Q$ in

\[
\frac{1}{|Q|} \int_Q |f| w^* \, dy \leq [w]_{A_p}^{1/p} \left( \frac{|Q|}{w(Q)} \left( \int_Q |f| w^* \, dy \right)^{1/p} \right)^{1/p}
\]

\[
\leq [w]_{A_p}^{1/p} \left( \frac{1}{w(Q)} \int_Q (M^p_{\ast w} f)^{p} \, dy \right)^{1/p} = [w]_{A_p}^{1/p} \left( \frac{1}{w(Q)} \int_Q ((M^p_{\ast w} f)^{p}) w^{-1} \, dy \right)^{1/p};
\]

we made use of (5.2) to get the first inequality. This concludes the proof of (5.9), and, in turn, of Proposition 5.2.

\( \square \)

5.3. Proof of Proposition 5.1. One particular consequence of the size and cancellation conditions on $T$, which we are going to use below, is the unweighted bound

\[ \|T\|_{\mathcal{L}(L^p(\mathbb{R}^2))} \lesssim p, \quad 2 \leq p < \infty, \]

which follows from the standard Calderón-Zygmund techniques we outlined in the introduction for $T_{j,k}$. The same linear behavior in $p$ is actually true even for the maximal truncations of Calderón-Zygmund operators with (minimally) smooth kernel, not necessarily of convolution type. This can be obtained by pairing the $A_2$ bound of [16] with Rubio de Francia’s trick; see [8] (for instance) for a proof. In view of this fact, the argument below, with some small modifications, extends to (maximal truncations of) non-convolution type operators. For simplicity, we restricted ourselves to the translation-invariant case, which is what we need in Lemma 3.4.

In this proof, we write $w(x)$ for $w_{p,\delta}(x) = |x|^{2\delta(p-1)}$. Begin by assuming $\|f\|_{L^p(\mathbb{R}^2;w)} = 1$.

Setting

\[ A_j := \{ x \in \mathbb{R}^2 : |x| \leq 2^{j-1} \}, \quad B_j := \{ x \in \mathbb{R}^2 : 2^j \leq |x| < 2^{j+1} \}, \quad j \in \mathbb{Z}, \]

we define $f_{j,1} = f 1_{A_j}$, $f_{j,2} = f - f_{j,1}$, and split

\[ \|Tf\|_{L^p(\mathbb{R}^2;w)}^p \leq 2^p \sum_{j \in \mathbb{Z}} \|Tf_{j,1}\|^p + 2^p \sum_{j \in \mathbb{Z}} \|Tf_{j,2}\|^p = R_1 + R_2. \]

We first bound $R_2$ without any use of the $A_p$ constant. Using

\[ x \in B_j \implies 2^{j+1} - 2^j \leq w(x) \leq 2^{j+1} - 2^j, \]

We have

\[ \langle D_j \rangle \lesssim 2^{j \delta(p-1)} \]
in the first and last steps, we estimate
\begin{equation}
2^{-p} R_2 \leq \sum_{j \in \mathbb{Z}} 2^{(j+1)2\delta(p-1)} \int_{B_j} |Tf_{j,2}|^p \, dx \lesssim p^p \sum_{j \in \mathbb{Z}} 2^{(j+1)2\delta(p-1)} \int_{\mathbb{R}^2} |f_{j,2}|^p \, dx
\end{equation}
\begin{align*}
&= p^p \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^2} |f 1_{B_k}|^p \, dx \sum_{j \leq k+1} 2^{(j+1)2\delta(p-1)} \\
&\leq \frac{p^p}{2^{2\delta(p-1)}} \sum_{k \in \mathbb{Z}} \int_{B_k} |f|^p 2^{(k+1)2\delta(p-1)} \, dx \lesssim \frac{p^p}{2^{2\delta(p-1)}} 2^{2\delta(p-1)} \|f\|^p_{L^p(\mathbb{R}^2,w)}.
\end{align*}
We have made use of (5.11) to get the second inequality. To estimate $R_1$, we rely on the (classical) pointwise bound
\[ x \in B_j \implies |Tf_{j,1}(x)| \lesssim Mf_{j,1}(x) \leq Mf(x) \]
M being the Hardy-Littlewood maximal function, so that
\begin{equation}
2^{-p} R_1 \lesssim \sum_{j \in \mathbb{Z}} \int_{B_j} |Mf|^p w \, dx = \|Mf\|^p_{L^p(\mathbb{R}^2,w)} \lesssim [w]_{A_p}^p \|f\|^p_{L^p(\mathbb{R}^2,w)} \lesssim (1-\delta)^{-p}.
\end{equation}
The last two inequalities are an instance of Proposition 5.2 followed by (5.4). Summarizing (5.12), (5.13) and (5.14), we get
\[ 2^{-p} \|Tf\|^p_{L^p(\mathbb{R}^2,w)} \lesssim \left(\frac{1}{1-\delta}\right)^p + \frac{p^p}{2^{2\delta(p-1)}} 2^{2\delta(p-1)} \lesssim \frac{p^p}{2^{2\delta(p-1)(1-\delta)^p}} 2^{2\delta(p-1)} \|f\|^p_{L^p(\mathbb{R}^2,w)} \]
which, after raising to $1/p$-th power, yields the bound of Proposition 5.1.

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