CHARACTERIZATION OF SHIFT-ININVARIANT CLOSED $L^2$-FORMS FOR LARGE SCALE INTERACTING SYSTEMS ON THE EUCLIDEAN LATTICE

KENICHI BANNAI and MAKIKO SASADA

ABSTRACT. We rigorously formulate and prove for a relatively general class of interactions the characterization of shift-invariant closed $L^2$-forms for a large scale interacting system on a Euclidean lattice $(\mathbb{Z}^d, \mathcal{B}^d)$. Such characterization of closed forms has played an essential role in proving the diffusive scaling limit of nongradient systems. The universal expression in terms of conserved quantities was sought from observations for specific models, but a precise formulation or rigorous proof up until now had been elusive. Our result is based on the universal characterization of shift-invariant closed local forms studied in our previous article [1]. In the present article, we show that the same universal structure also appears for $L^2$-forms. The essential assumptions are: (i) the set of states on each vertex is a finite set, (ii) the measure on the configuration space is the product measure, and (iii) there is a certain uniform spectral gap estimate for the mean field version of the interaction. Our result is applicable for generalized exclusion processes, multi-species exclusion processes, and more general lattice gas models.

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1. INTRODUCTION

In the last three decades, the theory of the hydrodynamic behavior of interacting particle systems has been well developed and a rigorous understanding of the various models has been achieved. However, we still do not have a sufficiently universal understanding of the diffusive scaling limit for nongradient models. In particular, the origin of the universality of specific expressions in the variational formula for diffusion coefficient in terms of conserved quantities was a mystery. There are not many results for models with more than one conserved quantity,
and it was not certain how common this universality is in practice for models with multiple conserved quantities.

In our previous article [1], we discovered that there is indeed a universal structure for a certain general class of models which leads to a universal characterization of shift-invariant closed local forms in terms of a group action on the configuration space and conserved quantities, which are defined concretely. If this is extended to $L^2$-form as well, this should also appear in the variational formula for the diffusion coefficient, and it can be said that the extension to $L^2$-form allows us to capture this universal structure of nongradient models.

The purpose of this article is to formulate and prove for relatively general interactions having a uniformly bounded spectral gap the characterization of shift-invariant closed $L^2$-forms for a large scale interacting system on an Euclidean lattice $(\mathbb{Z}^d, \mathbb{E}^d)$, when the set of states on each vertex of the Euclidean lattice is a finite set. We let $S$ be a finite nonempty set, the typical example being $S = \{0, \ldots, \kappa\}$ for some integer $\kappa > 0$ with base point $* = 0$, and we let

$$S^{\mathbb{Z}^d} := \prod_{x \in \mathbb{Z}^d} S$$

be the configuration space on $\mathbb{Z}^d$. A local function $f : S^{\mathbb{Z}^d} \to \mathbb{R}$ is any function on $S^{\mathbb{Z}^d}$ whose value depends on the states of the configuration at only a finite number of vertices, and we denote by $C_{\text{loc}}(S^{\mathbb{Z}^d})$ the $\mathbb{R}$-linear space of local functions on $S^{\mathbb{Z}^d}$.

We define an interaction to be a map $\phi : S \times S \to S \times S$ satisfying certain symmetry condition which expresses the interaction between states on adjacent vertices. Models such as the multi-species exclusion process and the generalized exclusion process may be described in a unified manner using the interaction. Let

$$\mathbb{E}^d := \left\{ (x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d \mid \sum_{j=1}^d |x_j - y_j| = 1 \right\},$$

where $x_j$ and $y_j$ for $j = 1, \ldots, d$ denotes the $j$-component of $x$ and $y$. Then the pair $(\mathbb{Z}^d, \mathbb{E}^d)$ defines a graph which we call the Euclidean lattice, and any $e = (x, y) \in \mathbb{E}^d$ referred to as a directed edge, or a directed bond, expresses an arrow from $x$ to $y$.

For any configuration $\eta = (\eta_x) \in S^{\mathbb{Z}^d}$ and directed edge $e = (x, y) \in \mathbb{E}^d$, we denote by $\eta^e$ the configuration obtained by interacting the states on the vertices of $e$. In other words, $\eta^e$ and $\eta$ coincides outside the $x$ and $y$ components, and if we denote by $\eta^e_x$ and $\eta^e_y$ the $x$ and $y$ components of $\eta^e$, then we have $(\eta^e_x, \eta^e_y) = \phi(\eta_x, \eta_y)$, where $\eta_x$ and $\eta_y$ are the $x$ and $y$ components of $\eta$. We call such $\eta^e$ a transition of $\eta$. For any local function $f$ and directed edge $e \in \mathbb{E}^d$, we define the differential $\nabla_e f$ to be the local function given for any $\eta \in S^X$ by

$$\nabla_e f(\eta) := f(\eta^e) - f(\eta).$$

In [1], we constructed the space of uniformly local functions $C_{\text{unif}}(S^{\mathbb{Z}^d})$, a class of functions which gives a rigorous framework to consider certain infinite sums of normalized local functions which appear in the formulation of the characterization of closed forms. This space allows for
the consideration of infinite sums such as

\[ \Gamma_f := \sum_{x \in \mathbb{Z}^d} \tau_x(f), \]

which in [13, p.144] is stated “does not make sense”. Here, \( f \) is any normalized local function and \( \tau_x \in G \) denotes the translation by \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \). The differential \( \nabla_e \) extends \( \mathbb{R} \)-linearly to a differential \( \nabla_e : C_{\text{unit}}(S^{\mathbb{Z}^d}) \to C_{\text{loc}}(S^{\mathbb{Z}^d}) \) giving the differential \( \partial \Gamma_f = (\nabla_e \Gamma_f)_{e \in E^d} \).

One key ingredient in the formulation of the characterization of closed forms are certain forms defined using the conserved quantities. We define the \textit{conserved quantity} to be a normalized function \( \xi : S \to \mathbb{R} \) satisfying

\[ \xi(\eta_1') + \xi(\eta_2') = \xi(\eta_1) + \xi(\eta_2) \]

for any \( (\eta_1, \eta_2) \in S \times S \), where \( (\eta_1', \eta_2') = \phi(\eta_1, \eta_2) \) (see Definition 2.3). Our key insight is that we may characterize conserved quantities simply as invariants preserved by interaction on adjacent vertices. If we denote by \( \text{Consv}^\phi(S) \) the \( \mathbb{R} \)-linear space of conserved quantities, then the number of independent conserved quantities of the interaction may be deduced from the dimension \( c_\phi := \dim \text{Consv}^\phi(S) \) of this space.

Any conserved quantity defines for any \( x \in \mathbb{Z}^d \) a local function \( \xi_x(\eta) := \xi(\eta_x) \) for any \( \eta = (\eta_x) \in S^{\mathbb{Z}^d} \), and the infinite sum

\[ \xi_x := \sum_{x \in \mathbb{Z}^d} \xi_x \]

defines a uniformly local function on \( S^{\mathbb{Z}^d} \) which satisfies \( \nabla_e \xi = 0 \). We say that two configurations \( \eta = (\eta_x) \) and \( \eta' = (\eta'_x) \) have the \textit{same conserved quantity}, if their components differ only at a finite set of vertices \( \Lambda \subset \mathbb{Z}^d \), and for any conserved quantity \( \xi \), we have \( \sum_{x \in \Lambda} \xi(\eta_x) = \sum_{x \in \Lambda} \xi(\eta'_x) \).

We say that an interaction is \textit{faithfully quantified}, if there exists a path in the configuration space between any two distinct configurations with the same conserved quantity.

A \textit{local form} is a system \( (\omega_e)_{e \in E^d} \) of local functions \( \omega_e : S^{\mathbb{Z}^d} \to \mathbb{R} \) for \( e \in E^d \) satisfying certain minor conditions. A \textit{path} in a configuration space is a sequence of transitions, and we define the integration \( \int_\gamma \omega \) of a form \( \omega = \{\omega_e\} \) with respect to the path \( \gamma \) to be the sum of values \( \omega_e(\eta) \) over the transitions \( (\eta, \eta') \) which appear in the path. We say that a path is \textit{closed} if the configuration at the beginning and the ending of the path coincide, and we say that a local form \( \omega \) is \textit{closed}, if the integral of \( \omega \) vanishes for any closed path in the configuration space.

We say that a local form is \textit{shift-invariant}, if it is invariant with respect to the translation \( \tau_x \) for any \( x \in \mathbb{Z}^d \). Our notion of shift-invariant closed local forms for the case of the generalized exclusion process coincides with the notion of germs of closed forms given in [13, §Appendix 4]. The main result of our article concerns the \textit{characterization of shift-invariant closed forms}, which is a generalization of [13, Theorem 4.14].

We first review the result of our previous article [11]. We fix a basis \( \xi^{(1)}, \ldots, \xi^{(c_\phi)} \) of \( \text{Consv}^\phi(S) \). The main result of [11] for the case of the Euclidean lattice is given as follows.

**Theorem 1 (11 Theorem 5).** Consider the Euclidean lattice \( (\mathbb{Z}^d, E^d) \) with an action of \( G = \mathbb{Z}^d \) given by translation. Let \( \phi \) be a faithfully quantified interaction, and assume in addition that...
\( \phi \) is simple if \( d = 1 \). Then for any shift-invariant local form \( \omega = (\omega_e)_{e \in E} \in \prod_{e \in E} C_{\text{loc}}(S^{2d}) \), there exists unique constants \( a_{ij} \in \mathbb{R} \) for \( i = 1, \ldots, c_\phi \) and \( j = 1, \ldots, d \), and a local functions \( f \in C_{\text{loc}}(S^{2d}) \) such that for any \( e \in E \), we have

\[
\omega_e = \nabla_e \left( \sum_{x \in \mathbb{Z}^d} \tau_x(f) + \sum_{i=1}^{c_\phi} \sum_{j=1}^{d} a_{ij} \sum_{x \in \mathbb{Z}^d} x_j \xi_x^{(i)} \right)
\]

in \( C_{\text{loc}}(S^{2d}) \), where \( \xi_x^{(i)} \) is the local function defined for any \( x \in \mathbb{Z}^d \) as \( \xi_x^{(i)}(\eta) := \xi^{(i)}(\eta_x) \) for \( \eta = (\eta_x) \in S^{2d} \).

Here, an interaction \( \phi \) is simple if and only if \( c_\phi = 1 \), and the monoid generated by \( \xi(S) \) in \( \mathbb{R} \) for any non-zero \( \xi \) is isomorphism to \( \mathbb{N} \) or \( \mathbb{Z} \). In this article, we will use Theorem 1 to prove the characterization for shift-invariant closed \( L^2 \)-forms. We fix a shift-invariant rate \( r = (r_e)_{e \in \mathbb{B}^d} \in \prod_{e \in \mathbb{B}^d} C_{\text{loc}}(S^{2d}) \) such that \( r_e(\eta) > 0 \) for any \( e \in \mathbb{B}^d \) and \( \eta \in S^{2d} \) encoding the information of the frequency of the transition on each directed edge. Such data may correspond to the stochastic process given by the generator

\[
L f(\eta) = \sum_{e \in \mathbb{B}^d} r_e(\eta) \nabla_e f(\eta) = \sum_{e \in \mathbb{B}^d} r_e(\eta) (f(\eta^e) - f(\eta)).
\]

From the perspective of the application to the hydrodynamic limit for nongradient models, it is natural to consider a probability measure \( \mu \) on \( S^{2d} \) which is reversible for the rate \( r = (r_e)_{e \in \mathbb{B}^d} \), but for our main theorem of this article, we do not need the reversibility. So, we take a probability measure \( \nu \) supported on \( S \), and consider the product measure \( \mu := \nu^{\otimes \mathbb{Z}^d} \) on \( S^{2d} \) independently form the rate.

We let \( L^2(\mu)_{r_e} \) be the weighted \( L^2 \)-space \( L^2(r_e d\mu) \) of functions on \( S^{2d} \) with a norm given by \( \| f \|_{r_e}^2 := E_{\mu} [r_e f^2] \). Then the space of local functions \( C_{\text{loc}}(S^{2d}) \) is a dense subspace of \( L^2(\mu)_{r_e} \). We rigorously extend the notion of closed forms to \( L^2 \)-forms. Our main result is as follows.

**Theorem 2** (=Theorem 3.9). Let the assumptions be as in Theorem 3.7 Furthermore, suppose \( \nu \) has a uniformly bounded spectral gap for the interaction \( (S, \phi) \) (in the sense of Definition 3.6). Then for any shift-invariant closed \( L^2 \)-form \( \omega = (\omega_e)_{e \in \mathbb{B}^d} \in \prod_{e \in \mathbb{B}^d} L^2(\mu)_{r_e} \), there exists unique constants \( a_{ij} \in \mathbb{R} \) for \( i = 1, \ldots, c_\phi \) and \( j = 1, \ldots, d \), and a sequence of local functions \( f_n \in C_{\text{loc}}(S^{2d}) \), such that for any \( e \in E \), we have

\[
\omega_e = \lim_{n \to \infty} \nabla_e \left( \sum_{x \in \mathbb{Z}^d} \tau_x(f_n) + \sum_{i=1}^{c_\phi} \sum_{j=1}^{d} a_{ij} \sum_{x \in \mathbb{Z}^d} x_j \xi_x^{(i)} \right)
\]

in \( L^2(\mu)_{r_e} \).

To study the diffusive scaling limit, such as the hydrodynamic limit and the equilibrium fluctuation, for nongradient models, the above characterization is necessary in the proof regardless of whether we use the entropy method or the relative entropy method, and so has been established for several specific models. The models below satisfy \( c_\phi = 1 \) unless explicitly stated otherwise.
The first results in the analysis of non-gradient models were obtained for the Ginzburg-Landau model in [27] by Varadhan, where the local state space $S = \mathbb{R}$ with one conserved quantity ($c_\phi = 1$), and for the two-color simple exclusion process [21] by Quastel, where $S = \{0, 1, 2\}$ with two conserved quantities ($c_\phi = 2$). However, the latter dynamics has some degeneracy, which means it does not satisfy the faithfully quantified condition in our terminology, and because of this, the characterization is shown only for some limited class of forms. Later, as more typical non-gradient models with a finite local state space $S$ and product reversible measures, the characterization as well as the diffusive scaling limits are established for the generalized exclusion process [13, 14], the exclusion process [7], the lattice gas with energy ($c_\phi = 2$) [18], and two-species exclusion process with annihilation and creation mechanism [22], for which our results are all applicable. Among non-gradient models with non-product reversible measures, there is an only rigorous result for the characterization and the hydrodynamic limit in [28] where the exclusion process with certain mixing condition was studied. For models with a continuous local state space $S$, besides the Ginzburg-Landau model mentioned above, as far as we know, the characterization theorem is only established for some energy conserving stochastic model [10], anharmonic oscillators with stochastic noise [20], and an active exclusion process ($c_\phi = \infty$ in some sense) [6]. There is also a preprint [17] on this direction. The generalization of non-gradient method to nonreversible models are also known, and the most essential part of the proof, namely the characterization of closed forms, is typically reduced to the reversible case (cf. [12, 20, 31]).

There are many important non-gradient models even with product reversible measures for which the diffusive scaling limits are not established, including the multi-species simple exclusion process (cf. [11]), energy Ginzburg-Landau model (cf. [5, 16]) and stochastic energy exchange models (cf. [8, 24]), even though the required spectral gap estimate is obtained for some of them. Also, regardless of the specific interaction $(S, \phi)$, models defined on crystal lattices, such as the hexagonal lattice and the diamond lattice, are typically non-gradient and its diffusive scaling limit is so far completely open (cf. [26]). Related to this problem, the generalization of the non-gradient method from a nearest neighbor interaction version to a finite range interaction version on $\mathbb{Z}^d$ are often claimed to be “straightforward”, but there is no clearly written proof in the literature. Here, we mean by the finite range interaction that the jumps of particles can be finite range, but do not mean the reversible measures have a finite range interaction. We find that there is an obstacle for the generalization to the finite range interaction version in general, in the proof of the locality of the boundary term, which is discussed in Section 5 in this article (see Remark 5.2). Hence, in this article, we restrict ourselves to the nearest neighbor interaction on the Euclidean lattice $\mathbb{Z}^d$, or in our terminology the model on the locale $(\mathbb{Z}^d, E)$. The extension of the locale to a general crystal lattice $(X, E)$ is an important problem and we are preparing an article for this extension with a sufficient condition on the locale $(X, E)$ for Theorem 2 to hold. With this in mind, we write the assertions and proofs in this article generally as possible with respect to locale. In the long term, we also hope that our main result may be extended to the case where $S$ is countable infinite, continuous, and $\nabla$ is a differential operator, an integral operator, and so on under suitable conditions.

As discussed, there have been many results on specific models, but there was no general framework for the non-gradient method nor the characterization of shift-invariant closed forms. The
biggest factor that prevented full generalization was that the dimension and concrete expression of “closed but not exact forms” were obtained by calculating simultaneous linear equations in each model “on a case-by-case basis,” and there was no unified understanding of them. Such concrete calculations are impractical when the dimension of the conserved quantities is large. The main reason we have succeeded in a complete generalization is that we have completely solved this problem in a previous article [1], as a problem independent of probability measures.

The major difference is the treatment of the boundary term specifically for each model, but in this article, we just show that the biggest factor that prevented full generalization was that the dimension and concrete expression of “closed but not exact forms” were obtained by calculating simultaneous linear equations in each model “on a case-by-case basis,” and there was no unified understanding of them. Such concrete calculations are impractical when the dimension of the conserved quantities is large. The main reason we have succeeded in a complete generalization is that we have completely solved this problem in a previous article [1], as a problem independent of probability measures.

In this subsection, we will introduce the data $((X, E), (S, \phi), r)$ which will be used to construct our model. We will then introduce the notion of co-local functions and forms on the configuration space with transition structure of our model.

2. Co-local Functions and Forms

In this section, we introduce the data $((X, E), (S, \phi), r)$ which will be used to construct our model. We will then introduce the notion of co-local functions and forms on the configuration space with transition structure of our model.

2.1. The Large Scale Interacting System. In this subsection, we will introduce the data $((X, E), (S, \phi), r)$ which we use to model the large scale interacting system.

We define a directed graph to be the pair $(X, E)$ consisting of a set $X$ which we call the set of vertices and $E \subset X \times X$, which we call the set of directed edges. We let $o, t: E \to X$ be the projections to the first and second components of $X \times X$, which we call the origin and the terminus of a directed edge. We say that a graph $(X, E)$ is symmetric, if the opposite $\bar{e} := (t(e), o(e)) \in E$ for any $e = (o(e), t(e)) \in E$, locally finite if $E_x := \{e \in E \mid o(e) = x\}$ is finite for any $x \in X$, and simple if $o(e) \neq t(e)$ for any $e \in E$.

We define a path on $(X, E)$ to be a finite sequence $\vec{p} := (e_1, \ldots, e_N)$ of edges satisfying $t(e_i) = o(e_{i+1})$ for any $0 < i < N$. We say that $\vec{p}$ is a path from $o(\vec{p}) := o(e_1)$ to $t(\vec{p}) := t(e_N)$ of length $\text{len}(\vec{p}) := N$. For any $x, x' \in X$, we let $P(x, x')$ the set of paths from $x$ to $x'$. We define the distance from $x$ to $x'$ by

$$d_X(x, x') := \inf_{\vec{p} \in P(x, x')} \text{len}(\vec{p})$$

if $P(x, x') \neq \emptyset$, and $d_X(x, x') := \infty$ otherwise. For any $Y \subset X$, we say that $Y$ is connected, if for any $y, y' \in Y$, there exists a path from $y$ to $y'$ consisting of edges whose origin and target are all elements of $Y$.

Definition 2.1. We define a locale to be a connected symmetric locally finite simple symmetric directed graph. The locale represents the space underlying our large scale interacting system.
The most important example of a locale is the Euclidean lattice \((\mathbb{Z}^d, \mathbb{E}^d)\) given in §1. Let \((X, E)\) be a graph. For any \(Y \subset X\), if we let \(E_Y := E \cap (Y \times Y)\), then the pair \((Y, E_Y)\) defines a graph. A subgraph of \((X, E)\) is any graph of the form \((Y, E_Y)\) for some \(Y \subset X\). In particular, if \((X, E)\) is a locale and the set \(Y \subset X\) is connected, then \((Y, E_Y)\) is a locale. A sublocale of \((X, E)\) is any locale of the form \((Y, E_Y)\) for some connected \(Y \subset X\).

As in §1, we define the set of states \(S\) to be a finite nonempty set, and we define the configuration space of \(S\) on \((X, E)\) to be the set

\[
S^X := \prod_{x \in X} S.
\]

An interaction is defined to be a map \(\phi: S \times S \to S \times S\), satisfying \(\bar{\phi} \circ \phi(\eta_1, \eta_2) = (\eta_1, \eta_2)\) for any \((\eta_1, \eta_2) \in S \times S\) such that \(\phi(\eta_1, \eta_2) \neq (\eta_1, \eta_2)\), where we let

\[
\bar{\phi} := \iota \circ \phi \circ \iota
\]

for the bijection \(\iota: S \times S \to S \times S\) obtained by exchanging the components of \(S \times S\). The pair \((S, \phi)\) describes the set of all possible states on a single vertices and the possible interactions between adjacent vertices.

**Example 2.2.** Let \(S = \{0, 1, 2, \ldots, \kappa\}\) for some integer \(\kappa > 0\).

(a) The map \(\phi: S \times S \to S \times S\) obtained by exchanging the components of \(S \times S\) defines an interaction giving the multi-species exclusion process.

(b) The map given by

\[
\phi(\eta_1, \eta_2) = \begin{cases} 
(\eta_1 - 1, \eta_2 + 1) & \eta_1 > 0, \eta_2 < \kappa \\
(\eta_1, \eta_2) & \text{otherwise}
\end{cases}
\]

is also an interaction, giving the generalized exclusion process.

See [1] Example 2.18] for other examples of interactions.

We next introduce a certain invariant called the conserved quantity associated to the pair \((S, \phi)\).

**Definition 2.3.** We fix an element \(* \in S\) which we call the base state. We define a conserved quantity, to be any map \(\xi: S \to \mathbb{R}\) satisfying \(\xi(*) = 0\) and

\[
\xi(\eta_1') + \xi(\eta_2') = \xi(\eta_1) + \xi(\eta_2)
\]

for any \((\eta_1, \eta_2) \in S \times S\), where \((\eta_1', \eta_2') := \phi(\eta_1, \eta_2)\).

We denote by \(\text{Consv}^\phi(S)\) the set of all conserved quantities, which has a natural structure of an \(\mathbb{R}\)-linear space. Given another base state \(*' \in S\), we have a canonical \(\mathbb{R}\)-linear isomorphism \(\text{Consv}^\phi(S) \cong \text{Consv}^{\phi'}(S)\) mapping any \(\xi \in \text{Consv}^\phi(S)\) to \(\xi' := \xi - \xi(*)\). Hence \(\text{Consv}^\phi(S)\) is independent up to canonical isomorphism of the choice of the base state. For this reason, we will simply denote \(\text{Consv}^\phi(S)\) by \(\text{Consv}(S)\). The dimension

\[
c_{\phi} := \dim_{\mathbb{R}} \text{Consv}(S)
\]

is independent of the choice of the base state, and describes the number of independent conserved quantities of the system.
The data \((X,E), (S,\phi)\) above gives the geometric data underlying our interacting system. We next give some construction associated to this data. For any locale \((X,E)\) and the pair \((S,\phi)\), we define the configuration space by

\[
S^X := \prod_{x \in X} S,
\]

and we call any \(\eta \in S^X\) a configuration. For any element \(\eta = (\eta_x) \in S^X\) and \(e \in E\), we denote by \(\eta^e\) the element \((\eta^e_x) \in S^X\) such that

\[
\eta^e_x := \begin{cases} 
\eta_x & x \neq o(e), t(e) \\
\eta'_x & x = o(e), t(e),
\end{cases}
\]

where \((\eta'_o(e), \eta'_t(e)) = \phi(\eta_o(e), \eta_t(e))\). We define a subset \(\Phi_X\) of \(S^X \times S^X\) by

\[
\Phi_X := \{ (\eta, \eta^e) \mid \eta \in S^X, e \in E \} \subset S^X \times S^X.
\]

Then the pair \((S^X, \Phi_X)\) is a graph, which we call the configuration space with transition structure.

**Lemma 2.4.** The graph \((S^X, \Phi_X)\) is a symmetric directed graph.

**Proof.** This is proved in \([1\text{, Lemma 2.5}]\), and can be seen from the fact that for any \(\eta \in S^X\) and \(e \in E\), if \(\eta^e \neq \eta\), then from the condition on \(\phi\), we have \((\eta^e)^\phi = \eta\), hence \((\eta^e, \eta) = (\eta^e, (\eta^e)^\phi) \in \Phi_X\).

The configuration space \(S^X\) describes all of the possible configuration of states of our system, and the transition structure describes all of the possible transitions which may occur at an instance in time. A condition that we often consider for the pair \((S,\phi)\) is the following.

**Definition 2.5.** We say that an interaction \((S,\phi)\) is faithfully quantified, if it satisfies the following property: For any finite locale \((X,E)\) and any configurations \(\eta, \eta' \in S^X\), if \(\sum_{x \in X} \xi(\eta_x) = \sum_{x \in X} \xi(\eta'_x)\) for any conserved quantity \(\xi \in \text{Consv}^\phi(S)\), then there exists a path \(\vec{y}\) from \(\eta\) to \(\eta'\) in \(S^X\).

*Faithfully quantified* is equivalent to the condition that the associated stochastic process on any finite locale \((X,E)\) with fixed conserved quantities are irreducible. This condition plays an important role in the proof of the main theorem.

**Example 2.6.** The interactions in Example \([2.2]\) are both faithfully quantified. We let \(\ast = 0\) be the base state of \(S = \{0, \ldots, \kappa\}\).

(a) For the multi-color exclusion process, we have \(c_\phi = \kappa\), and a basis of \(\text{Consv}^\phi(S)\) is given by \(\xi^{(1)}, \ldots, \xi^{(\kappa)} : S \to \mathbb{R}\) such that \(\xi^{(i)}(s) = \delta_{si}\) for \(i = 1, \ldots, \kappa\), where \(\delta_{si} := 1\) if \(s = i\) and \(\delta_{si} := 0\) if \(s \neq i\).

(b) For the generalized exclusion process, we have \(c_\phi = 1\), and a basis of \(\text{Consv}^\phi(S)\) is given by \(\xi : S \to \mathbb{R}\) such that \(\xi(s) = s\).

See \([1\text{, Example 2.18}]\) for other examples of faithfully quantified interactions.
Next, in order to prescribe the frequency of occurrence of a transition, we introduce certain stochastic data called the rate. For any \( \Lambda \subset X \) let \( C(S^\Lambda) := \text{Map}(S^\Lambda, \mathbb{R}) \) be the \( \mathbb{R} \)-linear space of \( \mathbb{R} \)-valued maps on \( S^\Lambda \). The natural projection \( S^X \rightarrow S^\Lambda \) induces an injection \( C(S^\Lambda) \hookrightarrow C(S^X) \), which allows us to view any \( f \in C(S^\Lambda) \) as a function on \( S^X \) whose value on \( \eta = (\eta_x) \) depends only on the components for \( x \in \Lambda \). A local function on \( S^X \) is any function in \( C(S^\Lambda) \) for a finite \( \Lambda \subset X \).

**Definition 2.7.** We define the rate of the system \(((X, E), (S, \phi))\) to be a family of local functions \( r = (r_e)_{e \in E} \) such that if \( \eta = \eta^e \), then we have \( r_e(\eta) = r_\bar{e}(\eta^e) \).

Formally, we consider the stochastic process given by the generator

\[
Lf(\eta) = \sum_{e \in E} r_e(\eta) \nabla_e f(\eta) = \sum_{e \in E} r_e(\eta)(f(\eta^e) - f(\eta))
\]

associated to the data \(((X, E), (S, \phi), r)\) and are interested in its space-time scaling limit. In this article, we do not need the existence or any property of this stochastic process, so we do not assume that such a process exists. Note that the value \( r_e(\eta) \) for \( \eta = \eta^e \) does not affect the generator \( [3] \). The second condition of Definition \( 2.7 \) is a convention to make the reversibility condition \( [4] \) automatically holds if \( \eta = \eta^e \).

We let \( \mathcal{F} \) be the set of subsets of \( S \), which is a \( \sigma \)-algebra. For any \( \Lambda \subset X \), we let \( \mathcal{F}_\Lambda := \mathcal{F}^{\otimes \Lambda} \) be the product \( \sigma \)-algebra on \( S^\Lambda \) obtained from \( \mathcal{F} \), which coincides with the Borel \( \sigma \)-algebra for the topological space \( S^\Lambda \). We also denote again by \( \mathcal{F}_\Lambda \) the sub-\( \sigma \)-algebra of \( \mathcal{F}_X \) generated by the projection \( S^X \rightarrow S^\Lambda \). We let \( \mu \) be a probability measure on \((S^X, \mathcal{F}_X)\), and for any \( \Lambda \subset X \), we denote again by \( \mu \) the measure on \((S^\Lambda, \mathcal{F}_\Lambda)\) given as the push-forward of \( \mu \) with respect to the projection \( S^X \rightarrow S^\Lambda \). We define the notion of a reversible measure.

**Definition 2.8.** Consider the data \(((X, E), (S, \phi), r)\). We say that a probability measure \( \mu \) on \( S^X \) is reversible for the rate \( r = (r_e)_{e \in E} \), if for any \( e \in E \) and finite \( \Lambda \subset X \) such that \( r_e, r_\bar{e} \in C(S^\Lambda) \), we have

\[
\mu(\eta)r_e(\eta) = \mu(\eta^e)r_\bar{e}(\eta^e)
\]

for any \( \eta \in S^\Lambda \).

For a rate \( r \), if there exists a probability measure \( \mu \) on \( S^X \) which is reversible for \( r \), then the stochastic process defined by the generator \( [3] \) (if it is well-defined) becomes a reversible process for the probability measure \( \mu \).

In terms of application to the diffusive scaling limit for non-gradient models, it is natural to consider a reversible measure \( \mu \), which obviously depends on the rate \( r \). However, since we can prove the main result without the reversibility condition, and instead require \( \mu \) to be a product measure, we choose a probability measure \( \nu \) on \( S \), satisfying \( \nu(\{s\}) > 0 \) for any \( s \in S \) independently form the rate \( r \), and consider the product measure \( \mu := \nu^{\otimes X} \).

Though we do not use it in the rest of the article, we remark that for such \( \mu \), we can characterize the class of rates \( r \) such that the measure \( \mu \) is reversible for the data \(((X, E), (S, \phi), r)\). Actually, the existence of such rate can be seen from Lemma \( 2.9 \) below.
Lemma 2.9. Suppose $\mu = \nu^\otimes_X$ as defined above. For any $e \in E$, let $\nu^e(\eta) := \nu(\eta_{o(e)})\nu(\eta_{t(e)})$ for any $\eta = (\eta_e) \in S^X$. We let $r^\nu = (r^\nu_e)_{e \in E}$ be the system of functions $r^\nu_e : S^X \to \mathbb{R}$ defined as

$$r^\nu_e(\eta) := \left\{ \begin{array}{ll}
\sqrt{\nu^e(\eta^e)/\nu^e(\eta)} & \eta^e \neq \eta \\
\sqrt{\nu^e(\eta^e)/\nu^e(\eta)} & \eta^e = \eta
\end{array} \right.$$

for any $e \in E$. Then $\mu$ is reversible for the data $((X, E), (S, \phi), r^\nu)$.

Proof. Note that $\nu^e(\eta) = \nu^\bar{e}(\eta)$ for any $\eta \in S^X$. Consider the case $\eta^e = \eta$. If $\eta^\bar{e} = \eta$, then we have

$$r^\nu_e(\eta^e) = r^\nu_e(\eta) = \sqrt{\nu^\bar{e}(\eta^e)/\nu^\bar{e}(\eta)} = \sqrt{\nu^e(\eta^e)/\nu^e(\eta)} = r^\nu_e(\eta).$$

If $\eta^\bar{e} \neq \eta$, then

$$r^\nu_e(\eta^e) = r^\nu_e(\eta) = \sqrt{\nu^\bar{e}(\eta^e)/\nu^\bar{e}(\eta)} = \sqrt{\nu^e(\eta^e)/\nu^e(\eta)} = r^\nu_e(\eta).$$

This shows that $r = (r^\nu_e)$ is a rate. To prove (4), consider any finite $\Lambda \subset X$ such that $\{o(e), t(e)\} \subset \Lambda$, and let $\eta \in S^\Lambda$. If $\eta^e = \eta$, then (4) follows immediately from the definition. Otherwise, if $\eta^e \neq \eta$, then $(\eta^e)^\bar{e} = \eta \neq \eta^e$. Hence by definition, we have

$$r^\nu_e(\eta) = \sqrt{\nu^e(\eta^e)/\nu^e(\eta)}, \quad r^\nu_e(\eta^e) = \sqrt{\nu^e(\eta)/\nu^e(\eta^e)},$$

which shows that $\nu^e(\eta)r^\nu_e(\eta) = \nu^e(\eta^e)r^\nu_e(\eta^e)$. Since $\mu$ is the product measure $\mu = \nu^\otimes_X$, this proves (4) as desired. $\square$

Using the rate $r^\nu$ of Lemma 2.9, we may characterize a class of rates $r^\ast$ such that $\mu$ is reversible for $r^\ast$.

Lemma 2.10. Let $r^\nu = (r^\nu_e)$ be the rate of Lemma 2.9. For a rate $r = (r_e)$, define $r^\ast = (r^\ast_e)$ by the relation $r_e = r^\nu_e r^\nu_{\bar{e}}$ for any $e \in E$. Then $\mu$ is reversible for the rate $r$ if and only if $r^\ast_e(\eta) = r^\nu_e(\eta)$ for any $e \in E$ and $\eta \in S^X$.

Proof. It immediately follows from that $r^\nu$ satisfies (4). $\square$

The following constants will be used in the calculation of the bound of norms later.

Definition 2.11. For an interaction $(S, \phi)$ and a probability measure $\nu$ supported on $S$, we define

$$C_{\phi, \nu} := \sup_{(\eta_1, \eta_2) \in S^2} \max_{(\eta_1', \eta_2') \in \phi(\eta_1, \eta_2)} \left\{ \frac{\nu(\eta_1')\nu(\eta_2')}{\nu(\eta_1)\nu(\eta_2)}, \frac{\nu(\eta_1)\nu(\eta_2)}{\nu(\eta_1')\nu(\eta_2')} \right\}$$

where $(\eta_1', \eta_2') = \phi(\eta_1, \eta_2)$. Since $S$ is finite and $\nu$ is supported on $S$, $1 \leq C_{\phi, \nu} < \infty$.

2.2. Uniformly Local Functions. We will study the property of the configuration space with transition structure using a class of functions which we call the uniformly local functions. This is a variant of the uniformly local function originally defined in [1] §3. In this subsection, we will review the definition of co-local functions and forms, as well as the definition of uniformly local functions. Detailed proofs of the results of this section are given in [2] §2 and §3.

Let $(X, E)$ be a locale, and let $\mathcal{J}$ be the set of finite subsets of $X$. For any $\Lambda \in \mathcal{J}$, as in §2.1 we let $C(S^\Lambda) = \text{Map}(S^\Lambda, \mathbb{R})$ be the $\mathbb{R}$-linear space of $\mathbb{R}$-valued functions on $S^\Lambda$. We may regard any $f \in C(S^\Lambda)$ as a function on $S^X$ whose value depends only on the components of $\eta = (\eta_e)$...
for \( x \in \Lambda \). For any \( \Lambda, \Lambda' \in \mathcal{F} \) such that \( \Lambda \subseteq \Lambda' \), the natural projection \( S^{\Lambda'} \to S^\Lambda \) induces an inclusion \( C(S^\Lambda) \hookrightarrow C(S^{\Lambda'}) \), which form a directed system with respect to inclusions in \( \mathcal{F} \).

**Definition 2.12.** We define the *space of local functions* by

\[
C_{\text{loc}}(S^X) := \lim_{\Lambda \in \mathcal{F}} C(S^\Lambda) = \bigcup_{\Lambda \in \mathcal{F}} C(S^\Lambda),
\]

where the direct system is taken with respect to the inclusions \( C(S^\Lambda) \hookrightarrow C(S^{\Lambda'}) \).

Next, let \( \mu \) be a probability measure on \( (S^X, \mathcal{F}_X) \). For any \( \Lambda \subset X \), we denote again by \( \mu \) the push forward of the measure \( \mu \) with respect to the natural surjection \( S^X \to S^\Lambda \). For any \( \Lambda \subset X \), we define \( C(S^\Lambda_{\mu}) \) to be the \( \mathbb{R} \)-linear space of \( \mathbb{R} \)-valued measurable functions on \( S^\Lambda \). If \( \Lambda \) is finite, then we have \( C(S^\Lambda_{\mu}) = C(S^\Lambda) \), since we have assumed that \( S \) is finite. The natural projection \( S^X \to S^\Lambda \) induces an inclusion \( C(S^\Lambda) \hookrightarrow C(S^\Lambda_{\mu}) \).

For any \( \Lambda, \Lambda' \in \mathcal{F} \) such that \( \Lambda \subseteq \Lambda' \) and any integrable function \( f \in C(S^{\Lambda'}) \), we let \( \pi^\Lambda f := E_\mu[f | \mathcal{F}_\Lambda] \) be the *conditional expectation* associated to the projection \( \text{pr}_\Lambda : S^{\Lambda'} \to S^\Lambda \). If \( \Lambda, \Lambda' \in \mathcal{F} \), then \( \pi^\Lambda \) induces the homomorphism

\[
\pi^\Lambda : C(S^{\Lambda'}) \to C(S^\Lambda).
\]

Note that for any \( \Lambda \subset \Lambda' \subset \Lambda'' \) and integrable function \( f \in C(S^{\Lambda''}) \), the tower property of the conditional expectation gives \( \pi^\Lambda f = \pi^\Lambda(\pi^{\Lambda'} f) \), which implies that \( \pi^\Lambda \) satisfies the compatibility condition for the projective system. Hence as in [2, Definition 1.4], we may define the space of co-local functions \( C_{\text{col}}(S^X_{\mu}) \) as follows.

**Definition 2.13.** We define the space of *co-local functions* by

\[
C_{\text{col}}(S^X_{\mu}) := \lim_{\Lambda \in \mathcal{F}} C(S^\Lambda),
\]

where the projective limit is taken with respect to the projections \( \pi^\Lambda \). We call any element in \( C_{\text{col}}(S^X_{\mu}) \) a co-local function.

By definition, a co-local function \( f = (f^\Lambda) \in C_{\text{col}}(S^X_{\mu}) \) is a system of functions \( f^\Lambda \in C(S^\Lambda) \) for \( \Lambda \in \mathcal{F} \) such that \( \pi^\Lambda f^{\Lambda'} = f^\Lambda \) for any \( \Lambda \subset \Lambda' \). In other words, co-local functions are *Martingales* indexed by \( \Lambda \in \mathcal{F} \) for the filtration \( (\mathcal{F}_\Lambda)_{\Lambda \in \mathcal{F}} \). Any integrable function \( f \in C(S^X_{\mu}) \) defines the co-local function \( (\pi^\Lambda f) \in C_{\text{col}}(S^X_{\mu}) \). This gives an injection

\[
C_{\text{loc}}(S^X) \hookrightarrow C_{\text{col}}(S^X_{\mu}).
\]

When \( \mu \) is a product measure \( \mu = \nu^\otimes_X \) where \( \nu \) is a probability measure supported on \( S \), the co-local functions may be expanded uniquely as an infinite sum of certain local functions.

**Proposition 2.14** ([2, Proposition 2.3]). Suppose \( \mu \) is a product measure \( \mu = \nu^\otimes_X \) on \( S^X \), where \( \nu \) is a probability measure supported on \( S \). For any \( \Lambda \in \mathcal{F} \), let

\[
C_\Lambda(S^X_{\mu}) := \{ f \in C(S^\Lambda) \mid \pi^\Lambda f \equiv 0 \text{ if } \Lambda \notin \Lambda' \}.
\]
Then for any co-local function \((f^\Lambda) \in C_{\text{col}}(S^X_\mu)\), there exists a unique family of functions \(f_\Lambda \in C_\Lambda(S^X_\mu)\) such that

\[(6) \quad f^\Lambda = \sum_{\Lambda'' \subset \Lambda} f_{\Lambda''}\]

for any \(\Lambda \in \mathcal{F}\).

**Proof.** See the proof of [2, Proposition 2.3] for details. We let \(f_\emptyset := f^\emptyset = E_\mu[f]\) for the case \(\Lambda = \emptyset\), and we define inductively with respect to the number of elements of \(\Lambda\) as

\[f_\Lambda := f^\Lambda - \sum_{\Lambda'' \subset \Lambda} f_{\Lambda''}\]

for any \(\Lambda \in \mathcal{F}\). To prove that \(f_\Lambda \in C_\Lambda(S^X_\mu)\) and satisfies the desired property, we use the fact proved in [2, Lemma 2.2] that for \(\Lambda \in \mathcal{F}\) and \(\Lambda', \Lambda'' \subset \Lambda\), we have

\[\pi^{\Lambda'}(\pi^{\Lambda''} f) = \pi^{\Lambda'\cap\Lambda''} f\]

for any \(f \in C(S^\Lambda)\), which is true as \(\mu\) is product. \(\square\)

In the rest of the article, we always assume that \(\mu\) is a product measure \(\mu = \nu^\otimes X\) for some probability measure \(\nu\) on \(S\) supported on \(S\).

For any co-local function \(f = (f^\Lambda)\), we will often write \(f = \sum_{\Lambda' \in \mathcal{F}} f_{\Lambda'}\) to imply that (6) holds for any \(\Lambda \in \mathcal{F}\). We let \(C^0_{\text{col}}(S^X_\mu) \coloneqq \{f \in C_{\text{col}}(S^X_\mu) \mid f_\emptyset = f^\emptyset = 0\}\), which we call the space of co-local functions of mean zero.

We define the **diameter** of \(\Lambda \subset X\) by \(\text{diam}(\Lambda) \coloneqq \sup_{x, x' \in \Lambda} d_X(x, x')\). For any constant \(R \geq 0\), denote by \(\mathcal{F}_R\) the set consisting of \(\Lambda \in \mathcal{F}\) such that \(\text{diam}(\Lambda) \leq R\). As in [2, Definition 2.4], we define the space of uniformly local functions as follows.

**Definition 2.15.** We say that a co-local function \(f\) is **uniformly local**, if there exists \(R \geq 0\) such that the expansion of Proposition 2.14 is given by

\[(7) \quad f = \sum_{\Lambda \in \mathcal{F}_R} f_\Lambda.\]

We call the infimum of such \(R\) the **diameter** of \(f\). We denote by \(C_{\text{unif}}(S^X_\mu)\) the \(\mathbb{R}\)-linear space of uniformly local functions, and by \(C^0_{\text{unif}}(S^X_\mu)\) the subspace of **normalized** uniformly local functions satisfying \(f_\emptyset = f^\emptyset = 0\).

We will next give an algebraic construction of the space of uniformly local function. Namely, we will review the construction of the space \(C^0_{\text{unif}}(S^X_\mu)\) denoted \(C^0_{\text{unif}}(S^X)\) in [1] Definition 3.5]. We will show in Proposition 2.19 that there exists a canonical isomorphism which we call the **renormalization** between \(C^0_{\text{unif}}(S^X_\mu)\) and \(C^0_{\text{unif}}(S^X_\mu)\).

We fix a point \(\ast \in S\) which we call the **base point**, and we denote by \(\ast \in S^X\) the configuration whose components are all at base state. For any configuration \(\eta \in S^X\), we define the support of \(\eta\) by \(\text{Supp}(\eta) \coloneqq \{x \in X \mid \eta_x \neq \ast\}\), and we let

\[S^X_\ast \coloneqq \{\eta \in S^X \mid |\text{Supp}(\eta)| < \infty\}.\]
We define $C(S^X) := \text{Map}(S^X, \mathbb{R})$ to be the $\mathbb{R}$-linear space of maps from $S^X$ to $\mathbb{R}$.

For any $\Lambda \in \mathcal{F}$, the projection $S^X \to S^\Lambda$ induces an inclusion $C(S^\Lambda) \hookrightarrow C(S^X)$, hence we may view $C_{\text{loc}}(S^X)$ as an $\mathbb{R}$-linear subspace of $C(S^X)$. For any $\Lambda, \Lambda' \in \mathcal{F}$ such that $\Lambda \subset \Lambda'$, we have an inclusion $\iota_{\Lambda'}: S^\Lambda \hookrightarrow S^{\Lambda'}$ given by extending any configuration $\eta \in S^\Lambda$ by $\eta_x := *$ for $x \in \Lambda' \setminus \Lambda$. This induces a projection $\iota^\Lambda: C(S^\Lambda) \to C(S^\Lambda)$.

**Definition 2.16.** For any $\Lambda \in \mathcal{F}$, we define the space of local functions with exact support $\Lambda$ by

$$C_\Lambda(S^X) := \{ f \in C(S^X) \mid \iota^\Lambda f \equiv 0 \text{ if } \Lambda \not\subset \Lambda' \}.$$ 

We have an expansion of functions in $C(S^X)$ via local functions with exact support as follows.

**Proposition 2.17.** Let $(f_{\Lambda})_{\Lambda \in \mathcal{F}}$ be a family of functions $f_\Lambda \in C_\Lambda(S^X)$. Then

$$f := \sum_{\Lambda \in \mathcal{F}} f_\Lambda$$

defines a function in $C(S^X)$. Conversely, for any function $f \in C(S^X)$, there exists a unique family of functions $f_\Lambda^* \in C_\Lambda(S^X)$ such that

$$f = \sum_{\Lambda \in \mathcal{F}} f_\Lambda^*.$$ 

**Proof.** The first statement is [1, Lemma 3.2], and follows from the fact that $f(\eta)$ for any $\eta \in S^X$ is a finite sum since $f_\Lambda(\eta) \neq 0$ only if $\Lambda \subset \text{Supp}(\eta)$. The second statement is [1, Proposition 3.3], noting that

$$C_\Lambda(S^X) = \{ f \in C(S^\Lambda) \mid f(\eta) = 0 \text{ if } \exists x \in \Lambda \text{ such that } \eta_x = * \}.$$ 

For any $f \in C(S^X)$, we let $f_\emptyset^* := \iota^\emptyset f = f(\bullet)$ for the case $\Lambda = \emptyset$. We then define $f_\Lambda^*$ by induction on the order of $\Lambda$ as

$$f_\Lambda^* := \iota^\Lambda f - \sum_{\Lambda' \supset \Lambda} f_{\Lambda'}^*$$

for any $\Lambda \in \mathcal{F}$. This construction gives the desired property. \hfill $\square$

As in [1, Definition 3.5], we define the space of uniformly local functions as follows.

**Definition 2.18.** We say that a function $f \in C(S^X)$ is uniformly local, if there exists $R \geq 0$ such that the expansion of Proposition 2.17 is given by

$$f = \sum_{\Lambda \in \mathcal{F}_R} f_\Lambda^*.$$ 

We call the infimum of such $R$ the diameter of $f$. We denote by $C_{\text{unif}}(S^X)$ the set of all uniformly local functions in $C(S^X)$, and by $C_{\text{unif}}^0(S^X)$ the subspace of normalized uniformly local functions satisfying $f_\emptyset^* = \iota^\emptyset f = f(\bullet) = 0$.

Let $\xi \in \text{Conv}^\phi(S)$ be a conserved quantity. Then for any $x \in X$, the function $\xi_x : S \to \mathbb{R}$ given by $\xi_x(\eta) := \xi(\eta_x)$ for any $\eta \in S^X$ defines a function in $C_{\{x\}}(S^X)$. By Proposition 2.17, the infinite sum

$$\xi_X := \sum_{x \in X} \xi_x$$
defines a uniformly local function of diameter 0 on $S^X$.

We define the renormalization as follows.

**Proposition 2.19.** We have a canonical isomorphism which we call the renormalization

$$\mathcal{R} : C^0_\text{unif}(S^X_\ast) \cong C^0_\text{unif}(S^X_\mu).$$

**Proof.** Let $f \in C^0_\text{unif}(S^X_\ast) \subset C^0(S^X_\ast)$. Then there exists $R \geq 0$ such that

$$f = \sum_{\Lambda'' \in \mathcal{J}_R} f^\ast_{\Lambda''}.$$

By Proposition 2.14, for each $\Lambda'' \in \mathcal{J}_R$, there is the unique expansion of $f^\ast_{\Lambda''}$ considered as a co-local functions as

$$f^\ast_{\Lambda''} = \sum_{\Lambda' \subset \Lambda''} (f^\ast_{\Lambda''})_{\Lambda'}$$

where $(f^\ast_{\Lambda''})_{\Lambda'} \in C_{\Lambda'}(S^X_\mu)$. Now, for each $\Lambda' \in \mathcal{J}$ such that $\Lambda' \neq \emptyset$, we construct $\mathcal{R}(f)_{\Lambda'}$ by

$$\mathcal{R}(f)_{\Lambda'} := \sum_{\Lambda'' \in \mathcal{J}_R} (f^\ast_{\Lambda''})_{\Lambda'}.$$

Note that since the number of $\Lambda'' \in \mathcal{J}_R$ such that $\Lambda' \subset \Lambda''$ is finite, $\mathcal{R}(f)_{\Lambda'}$ gives a function in $C_{\Lambda'}(S^X_\mu)$. We define $\mathcal{R}(f) \in C_{\text{col}}(S^X_\mu)$ by

$$\mathcal{R}(f) = \sum_{\Lambda' \in \mathcal{J}^0} \mathcal{R}(f)_{\Lambda'},$$

where $\mathcal{J}^0 := \mathcal{J} \setminus \{\emptyset\}$. Note that $\mathcal{R}(f)_{\Lambda'} = 0$ if $\text{diam}(\Lambda') > R$, hence $\mathcal{R}(f)$ is a function in $C^0_\text{unif}(S^X_\mu)$ as desired. This construction implies that for any $f \in C_{\text{loc}}(S^X)$, we have $\mathcal{R}(f^\ast) = f^\mu$ where $f^\ast := f - f(\ast) \in C^0(S^X_\ast)$ and $f^\mu := f - E_\mu[f]$. We may construct the inverse isomorphism of $\mathcal{R}$ in a similar manner, proving that (9) is an isomorphism.

Given two base states $\ast, \ast' \in S$, one may also construct a canonical isomorphism $C^0_\text{unif}(S^X_\ast) \cong C^0_\text{unif}(S^X_\ast')$ by $f \mapsto f - f(\ast')$, where $\ast'$ is the configuration in $S^X_\ast$ whose components are all at base state $\ast'$. In what follows, we will denote $C^0_\text{unif}(S^X_\ast)$ simply as $C^0_\text{unif}(S^X)$, and will regard any element $f \in C^0_\text{unif}(S^X)$ also as elements in $C^0_\text{unif}(S^X_\ast)$ or $C^0_\text{unif}(S^X_\ast')$ through the renormalization isomorphisms.

**2.3. Uniformly Local Forms.** In this subsection, we next consider the space of forms on the configuration space with transition structure. For any connected $\Lambda \in \mathcal{J}$, the graph $(\Lambda, E_\Lambda)$ is a finite locale. We let $(S^\Lambda, \Phi_\Lambda)$ be the associated configuration space with transition structure, where

$$\Phi_\Lambda := \{ (\eta, \eta^e) \mid \eta \in S^\Lambda, e \in E_\Lambda \}.$$

For any $\varphi = (\eta, \eta^e) \in \Phi_\Lambda$, let $\overline{\varphi} := (\eta^e, \eta)$, which by Lemma 2.4 is an element in $\Phi_\Lambda$. 
Definition 2.20. We define the space of forms on $S^\Lambda$ by

$$C^1(S^\Lambda) := \text{Map}^{\text{alt}}(\Phi_\Lambda, \mathbb{R}),$$

where $\text{Map}^{\text{alt}}(\Phi_\Lambda, \mathbb{R}) := \{\omega \in \text{Map}(\Phi_\Lambda, \mathbb{R}) \mid \omega(\bar{\varphi}) = -\omega(\varphi)\}.$

We have a natural inclusion

$$(10) \quad C^1(S^\Lambda) \hookrightarrow \prod_{e \in E_\Lambda} C(S^\Lambda)$$

given by $\omega \mapsto (\omega_e)_{e \in E_\Lambda}$, where $\omega_e \in C(S^\Lambda)$ is the map given by $\omega_e(\eta) := \omega(\eta, \eta^e)$ for any $e \in E_\Lambda$ and $\eta \in S^\Lambda$. The image of any $\omega \in C^1(S^\Lambda)$ with respect to the inclusion $(10)$ consists of $(\omega_e)$ satisfying $\omega_e(\eta) = 0$ if $\eta^e = \eta$, $\omega_e(\eta) = -\omega_e(\eta^e)$ if $\eta^e \neq \eta$, and $\omega_e(\eta) = \omega_e'(\eta)$ if $\eta^e = \eta^e'$. \hfill \Box

Lemma 2.21. For any $\Lambda, \Lambda' \in \mathcal{F}$ such that $\Lambda \subset \Lambda'$, the projection $\pi^\Lambda : C(S^{\Lambda'}) \to C(S^\Lambda)$ induces via $(10)$ an homomorphism

$$\pi^\Lambda : C^1(S^{\Lambda'}) \to C^1(S^\Lambda).$$

Proof. This follows from [2] Lemma 3.4], [2] Lemma 3.5], and the fact that we have assumed that $\mu$ is the product measure. \hfill \Box

Using Lemma 2.21 we define the space of co-local forms as follows.

Definition 2.22. We define the space of co-local forms on $S^\Lambda$ by

$$C^1_{col}(S^X_\mu) := \lim_{\Lambda \to \mathcal{F}} C^1(S^\Lambda),$$

where the project limit is taken with respect to the projection $\pi^\Lambda$.

The inclusion $(10)$ gives the inclusion $C^1_{col}(S^X_\mu) \hookrightarrow \prod_{e \in E} C_{col}(S^X_\mu)$. We define the space of local forms $C^1_{loc}(S^X)$ by

$$(11) \quad C^1_{loc}(S^X) := C^1_{col}(S^X_\mu) \cap \left(\prod_{e \in E} C_{loc}(S^X)\right).$$

For any $\Lambda \in \mathcal{F}$, we define the differential

$$(12) \quad \partial^\Lambda : C(S^\Lambda) \to C^1(S^\Lambda)$$

to be the $\mathbb{R}$-linear homomorphism given by $\partial^\Lambda f(\varphi) := f(\tau(\varphi)) - f(\sigma(\varphi))$. The image of $\partial^\Lambda f(\varphi)$ with respect to the inclusion $(10)$ coincides with $\nabla_e f(\eta)_{e \in E_\Lambda}$, where $\nabla_e f(\eta) := f(\eta^e) - f(\eta)$ for any $\eta \in S^\Lambda$. Since the measure $\mu$ is a product measure, we have the following.

Proposition 2.23. The projection $\pi^\Lambda$ is compatible with the differentials $\partial^\Lambda$ on $C^0(S^\Lambda)$. Hence the differential on each $C(S^\Lambda)$ induces the differential

$$\partial : C_{col}(S^X_\mu) \to C^1_{col}(S^X_\mu).$$

Proof. This follows from [2] Proposition 3.7], noting that a product measure is ordinary, as shown in [2] Lemma 3.4]. \hfill \Box
We say that a path \( \mathcal{P} \) for any closed path \( \mathcal{P} \), we define the closed forms on \( \Lambda \) for any \( \mathcal{P} \). Since the projection \( \pi^\Lambda \) on forms is induced from the embedding \( \{10\} \), the projection is also compatible with \( \nabla_\mathcal{P} \) for any \( \mathcal{P} \in E_\Lambda \). This implies that \( \nabla_\mathcal{P} \) induces a differential
\begin{equation}
\nabla_\mathcal{P} : \text{col}(S^X_\mu) \rightarrow \text{col}(S^X_\mu)
\end{equation}
such that we have \( \partial f = (\nabla_\mathcal{P} f)_{\mathcal{P} \in E} \in C^1(\text{col}(S^X_\mu)) \subset \prod_{\mathcal{P} \in E} \text{col}(S^X_\mu) \) for any \( f \in \text{col}(S^X_\mu) \).

We next define the notion of closed forms. Let \( \Lambda \in \mathcal{F} \), and let \( (S^\Lambda, \Phi_\Lambda) \) be a configuration space with transition structure for a pair \( (S, \phi) \). Recall that a path in \( (S^\Lambda, \Phi_\Lambda) \) is a sequence of transitions \( \gamma = (\varphi^1, \ldots, \varphi^N) \) in \( \Phi_\Lambda \) such that \( t(\varphi^i) = o(\varphi^{i+1}) \) for any integer \( 0 < i < N \). For any form \( \omega \in C^1(S^\Lambda) \), we define the integration of \( \omega \) with respect to the path \( \gamma \) by
\[ \int_\gamma \omega := \sum_{i=1}^N \omega(\varphi^i). \]
We say that a path \( \gamma \) is closed, if \( t(\gamma) = o(\gamma) \), where \( o(\gamma) = o(\varphi^1) \) and \( t(\varphi^N) = t(\gamma) \). As in \( \{11\} \) Definition 2.14, we define the closed forms on \( (S^\Lambda, \Phi_\Lambda) \) as follows.

**Definition 2.24.** We say that a form \( \omega \in C^1(S^\Lambda) \) is closed, if
\[ \int_\gamma \omega = 0 \]
for any closed path \( \gamma \) in \( (S^\Lambda, \Phi_\Lambda) \).

We denote by \( Z^1(S^\Lambda) \) the \( \mathbb{R} \)-linear space of closed forms in \( C(S^\Lambda) \). The notion of closed forms is compatible with the projection \( \pi^\Lambda \).

**Lemma 2.25.** The projection \( \pi^\Lambda \) induces an \( \mathbb{R} \)-linear homomorphism
\[ \pi^\Lambda : Z^1(S^\Lambda') \rightarrow Z^1(S^\Lambda) \]
for any \( \Lambda, \Lambda' \) in \( \mathcal{F} \) such that \( \Lambda \subset \Lambda' \).

**Proof.** This is \( \{2\} \) Lemma 3.10, since a product measure is ordinary by \( \{2\} \) Lemma 3.4. \( \square \)

**Definition 2.26.** We define the space of closed co-local forms \( Z^1_{\text{col}}(S^X_\mu) \) by
\[ Z^1_{\text{col}}(S^X_\mu) := \lim_{\Lambda \in \mathcal{F}} Z^1(S^\Lambda), \]
where the limit is the projective limit with respect to the projection \( \pi^\Lambda \).

By \( \{11\} \) Lemma 2.14, we see that for any \( \Lambda \in \mathcal{F} \) and \( f \in C(S^\Lambda) \), we have \( \partial_\Lambda f \in Z^1(S^\Lambda) \), where \( \partial_\Lambda \) is the differential defined in \( \{12\} \). Moreover, by \( \{11\} \) Lemma 2.15, the differential \( \partial_\Lambda : C^0(S^\Lambda) \rightarrow Z^1(S^\Lambda) \) is surjective and induces the isomorphism
\begin{equation}
C^0(S^\Lambda)/\ker \partial_\Lambda \cong Z^1(S^\Lambda).
\end{equation}
The compatibility of the differential \( \partial_\Lambda \) with the projection \( \pi^\Lambda \) gives the following.
**Proposition 2.27.** The differential $\partial : C^0_{\text{col}}(S^X_\mu) \rightarrow C^1_{\text{col}}(S^X_\mu)$ of Proposition 2.23 induces the isomorphism

$$C^0_{\text{col}}(S^X_\mu)/H^0_{\text{col}}(S^X_\mu) \cong Z^1_{\text{col}}(S^X_\mu),$$

where $H^0_{\text{col}}(S^X_\mu) := \lim_{\Lambda \in \mathcal{F}} \text{Ker} \partial_\Lambda$.

**Proof.** This is [2, Proposition 3.13], noting that a product measure is ordinary by [2, Lemma 3.4]. The proof follows from the fact that the projective system $\{\text{Ker} \partial_\Lambda\}$ satisfies the Mittag-Leffler condition, which follows immediately from the fact that $\text{Ker} \partial_\Lambda$ is finite dimensional for any $\Lambda \in \mathcal{F}$, since we have assumed that $S$ is finite. \hfill $\square$

Next, we consider the differential of uniformly local forms. Again, fix a base state $\ast \in S$. We let $C^1(S^X_\ast) := \text{Map}^\text{alt}(\Phi^*_X, \mathbb{R})$, where $\Phi^*_X := \Phi_X \cap (S^X_\ast \cap S^X)$. Similarly to the case of (10), there exists an embedding

$$C^1(S^X_\ast) \hookrightarrow \prod_{e \in E} C(S^X_\ast)$$

mapping $\omega \in C^1(S^X_\ast)$ to $(\omega_e)_{e \in E}$, where $\omega_e(\eta) := \omega((\eta, \eta^e))$ for any $e \in E$ and $\eta \in C(S^X_\ast)$. We define the differential

$$\partial : C^0(S^X_\ast) \rightarrow C^1(S^X_\ast)$$

by $\partial f(\varphi) := f(t(\varphi)) - f(o(\varphi))$. Again, we have $\partial f|_{e} = \nabla_e f$ through the inclusion (15), where $\nabla_e f(\eta) = f(\eta^e) - f(\eta)$ for any $\eta \in S^X_\ast$. Then space of local forms defined in (20) coincides with

$$C^1_{\text{loc}}(S^X) := C^1(S^X_\ast) \cap \left(\prod_{e \in E} C_{\text{loc}}(S^X_\ast)\right),$$

hence we may view a local form to be an element in both $C^1(S^X_\ast)$ and $C^1(S^X_\mu)$.

We next define the notion of uniformly local forms. For any $x \in X$ and $R \geq 0$, we let $B(x, R) := \{x' \in X \mid d_X(x, x') \leq R\}$ be the ball with center $x$ and radius $R \geq 0$, and for any $e \in E$, we let $B(e, R) := B(o(e), R) \cup B(t(e), R)$. Following [1, Definition 3.10], for any $R \geq 0$, we let

$$C^1_R(S^X) := C^1_{\text{loc}}(S^X) \cap \left(\prod_{e \in E} C(S^{B(e, R)})\right).$$

We define the space of uniformly local forms by

$$C^1_{\text{unif}}(S^X) := \bigcup_{R \geq 0} C^1_R(S^X) \subset C^1_{\text{loc}}(S^X).$$

**Lemma 2.28.** If $f \in C^0_{\text{unif}}(S^X_\ast)$, then we have $\partial f \in C^1_{\text{unif}}(S^X)$.

**Proof.** Since $\nabla_e f^\ast \neq 0$ if and only if $\{o(e), t(e)\} \cap \Lambda \neq \emptyset$, hence

$$\nabla_e f = \sum_{\Lambda \in \mathcal{F}, \{o(e), t(e)\} \cap \Lambda \neq \emptyset} \nabla_e f^\ast_\Lambda$$

is a local function since $f$ is uniformly local implies that the right hand side is a finite sum. If the diameter of $f$ is $R$, then $\partial f \in C^1_{R+1}(S^X)$, hence $\partial f$ is a uniformly local form. \hfill $\square$
Lemma 2.29. The isomorphism $\mathcal{R}$ of (9) is compatible with the differential. In other words, we have a commutative diagram

$$
\begin{array}{ccc}
C^0(\mathcal{S}^X) & \xrightarrow{\partial} & C^1(\mathcal{S}^X) \\
\mathcal{R} & & \mathcal{R} \\
C^0(\mathcal{S}_\mu^X) & \xrightarrow{\partial} & C^1(\mathcal{S}^X).
\end{array}
$$

Proof. Since $\partial$ is linear, and since any $f \in C^0(\mathcal{S}^X)$ by Proposition 2.17 may be expressed as a sum of $f^*_\Lambda$ for $\Lambda \in \mathcal{I}$, it is sufficient to prove our assertion for $f^*_\Lambda \in C(\mathcal{S}_\mu^\Lambda)$. By definition of $\mathcal{R}$, we have

$$
\mathcal{R}(f^*_\Lambda) = \sum_{\Lambda \subset \Lambda', \Lambda \neq \emptyset} (f^*_\Lambda)_{\Lambda''} = f^*_\Lambda - E_\mu[f^*_\Lambda].
$$

Our assertion now follows from the fact that the differential $\partial$ of a constant is zero. \qed

Lemma 2.29 assures that for any normalized uniformly local function $f \in C^0(\mathcal{S}^X)$, the differential $\partial f$ is well-defined in $C^1(\mathcal{S}^X)$ regardless of the normalization of $f$.

Definition 2.30. We define the space of uniformly local closed form by

$$Z^1_{\text{unif}}(\mathcal{S}^X) := Z^1_{\text{col}}(\mathcal{S}_\mu^X) \cap C^1_{\text{unif}}(\mathcal{S}^X).$$

2.4. Norms on Functions and Forms. In this subsection, we define norms on the spaces of co-local functions and forms. Again, we let $\mu = \nu^X$ be the product measure on $\mathcal{S}^X$. For any measurable function $f \in C(\mathcal{S}_\mu^X)$, we let

$$\|f\|^2_{\mu} := E_\mu[f^2] = \int_{\mathcal{S}^X} f^2 d\mu.
$$

Definition 2.31. We define the space of $L^2$-functions $L^2(\mu)$ by

$$L^2(\mu) := \{ f \in C(\mathcal{S}^X) | \|f\|_{\mu} < \infty \} / \{ f \in C(\mathcal{S}^X) | \|f\|_{\mu} = 0 \}.$$

The space $L^2(\mu)$ is a Hilbert space for the inner product

$$\langle f, g \rangle_{\mu} := E_\mu[fg].
$$

In particular, $L^2(\mu)$ is complete for the topology given by the norm $\| \cdot \|_{\mu}$. Moreover, any function $f \in L^2(\mu)$ is integrable for the measure $\mu$. The conditional expectation $\pi^\Lambda$ satisfies the following properties.

Lemma 2.32. For any $\Lambda \in \mathcal{I}$, the homomorphism $\pi^\Lambda$ is an orthogonal projection of $L^2(\mu)$ to $C(\mathcal{S}^\Lambda)$ with respect to the inner product (17).

Proof. The assertions follow directly from the definition of the conditional expectation. Detailed proofs are given in [2] Lemma 2.1]. \qed

For any $f \in L^2(\mu)$, the system $(f^\Lambda)_{\Lambda \in \mathcal{I}}$ for $f^\Lambda := \pi^\Lambda f$ defines an element in $C_{\text{col}}(\mathcal{S}_\mu^X)$. This shows that we have a homomorphism $L^2(\mu) \to C_{\text{col}}(\mathcal{S}_\mu^X)$. The martingale convergence theorem for $L^2$-bounded martingales gives the following.
**Theorem 2.33.** Let 

$$ C_{L^2}(S^X_\mu) := \{(f^\Lambda)_{\Lambda \in \mathcal{F}} \in C_{\text{col}}(S^X_\mu) \mid \sup_{\Lambda \in \mathcal{F}}\|f^\Lambda\|_\mu < \infty\}. $$

Then $C_{L^2}(S^X_\mu)$ coincides with the image of $L^2(\mu)$ in $C_{\text{col}}(S^X_\mu)$.

**Proof.** See [2, Theorem 5.2] and [2, Corollary 5.2] for a detailed proof. □

In what follows, we will identify $L^2(\mu)$ with $C_{L^2}(S^X_\mu)$. We next introduce norms on the space of forms $C_{\text{col}}^1(S^X_\mu)$. We let $r = (r_e)_{e \in E}$ be a rate as in Definition 2.7. For any $e \in E$, we define a weighted $L^2$-norm for $f$ in $C(S^X_\mu)$ as follows.

$$ \|f\|_{r_e}^2 := E_\mu[r_ef^2] = \int_{S^X} r_e f^2 d\mu. \tag{18} $$

If the rate $r = (r_e)$ is the trivial rate satisfying $r_e = 1$ for any $e \in E$, then the norm $\|\cdot\|_{r_e}$ coincides with the norm $\|\cdot\|_\mu$ of (16).

**Definition 2.34.** We define the space of $L^2$-functions $L^2(\mu)_{r_e}$ by

$$ L^2(\mu)_{r_e} := \{f \in C(S^X_\mu) \mid \|f\|_{r_e} < \infty\} / \{f \in C(S^X_\mu) \mid \|f\|_{r_e} = 0\}. $$

The space $L^2(\mu)_{r_e}$ is a Hilbert space for the inner product

$$ \langle f, g \rangle_{r_e} := E_\mu[r_efg]. \tag{19} $$

Since $r_e$ is a local function, it is bounded from above and below by some positive constants. Hence the norms $\|\cdot\|_{r_e}$ and $\|\cdot\|_\mu$ are equivalent, and we have $L^2(\mu)_{r_e} = L^2(\mu)$. Again using the inclusion $C_{\text{col}}^1(S^X_\mu) \hookrightarrow \prod_{e \in E} C_{\text{col}}(S^X_\mu)$ of (10), we may define the space of $L^2$-forms as follows.

**Definition 2.35.** We define the space of $L^2$-forms $C_{L^2}^1(S^X_\mu)$ by

$$ C_{L^2}^1(S^X_\mu) := C_{\text{col}}^1(S^X_\mu) \cap \left( \prod_{e \in E} L^2(\mu)_{r_e} \right), \tag{20} $$

which is independent from the choice of the rate $r = (r_e)$.

The following constants will be used in the calculation of the bound of norms.

**Definition 2.36.** Let $r = (r_e)$ be a rate for the system $((X, E), (S, \phi))$.

(a) For any $e \in E$, we define the rate bound to be the infimum $M_{r_e}$ of constants $M > 1$ satisfying

$$ M^{-1} < r_e(\eta) < M \tag{21} $$

for any $\eta \in S^X$. Such $M_{r_e}$ exists since $r_e$ is a local function.

(b) For any $e \in E$, we define the transition bound to be the infimum $A_{r_e}$ of constants $A > 1$ satisfying

$$ A^{-1} < \frac{\pi^\Lambda r_e(\eta)\mu(\eta)}{\pi^\Lambda r_{\bar{e}}(\eta^e)\mu(\eta^e)} < A \tag{22} $$

for any $\Lambda \in \mathcal{F}$ such that $e \in E_\Lambda$ and $\eta \in S^\Lambda$. Such $A_{r_e}$ exists since $r_e$ and $r_{\bar{e}}$ are local functions.
Remark 2.37. (a) If the measure $\mu$ is reversible for the rate $r$, then by (4) of Definition 2.8, we have $A_{r_{\bar{e}}} = 1$ for any $e \in E$.
(b) If the rate $r = (r_e)$ is the trivial rate given by $r_e \equiv 1$ for any $e \in E$, then $A_{r_e} = C_{\phi, \nu}$ where $C_{\phi, \nu}$ is the constant defined in Definition 2.11.
(c) For any rate $r = (r_e)$ and $e \in E$, we see from the definition that $A_{r_e} \leq M_{r_e} M_{r_e} C_{\phi, \nu}$.

Let $\Lambda \in \mathcal{F}$ and let $(\Lambda, E_{\Lambda})$ be the corresponding finite locale. For any $f \in C(S^\Lambda)$ and $e \in E_{\Lambda}$, let $\nabla_e f$ be the function defined by $\nabla_e f(\eta) = f(\eta^e) - f(\eta)$ for any $\eta \in S^{\Lambda}$. We may prove the following.

Lemma 2.38. Let $\Lambda \in \mathcal{F}$ and let $e \in E_{\Lambda}$. Then for any $f \in C(S^\Lambda)$, we have

$$A_{r_{\bar{e}}}^{-1} \|\nabla_e f\|^2_{r_{\bar{e}}} \leq \|\nabla_e f\|^2_{r_{\bar{e}}} \leq A_{r_e} \|\nabla_e f\|^2_{r_e}.$$  

In particular, if the measure $\mu$ is reversible for the rate $r = (r_e)$, then we have

$$\|\nabla_e f\|_{r_{\bar{e}}} = \|\nabla_e f\|_{r_e}.$$  

Proof. We have

$$\|\nabla_e f\|^2_{r_{\bar{e}}} = \sum_{\eta \in S^\Lambda} (f(\eta^e) - f(\eta))^2 \pi^\Lambda r_e(\eta) \mu(\eta) = \sum_{\eta \in S^\Lambda} (f(\eta^e) - f(\eta))^2 \pi^\Lambda r_e(\eta) \mu(\eta).$$  

$$= \sum_{\eta \in S^\Lambda, \eta^e \#(\eta^e)} (f(\eta^e) - f((\eta^e)^\bar{e}))^2 \pi^\Lambda r_e(\eta) \mu(\eta)$$  

$$= \sum_{\eta \in S^\Lambda, \eta^e \#(\eta^e)} (f(\eta') - f((\eta')^e))^2 \pi^\Lambda r_e((\eta')^e) \mu((\eta')^e)$$  

$$\leq A_{r_e} \sum_{\eta \in S^\Lambda} (f(\eta') - f((\eta')^e))^2 \pi^\Lambda r_e(\eta') \mu(\eta') = A_{r_e} \|\nabla_e f\|^2_{r_e}.$$  

This gives the second inequality. The first inequality follows by replacing $e$ by $\bar{e}$. Our assertion for the reversible case follows from the fact that $A_{r_e} = 1$ if $\mu$ is reversible for $r$.

We may prove the continuity of $\nabla_e$ as follows.

Lemma 2.39. For any $\Lambda \in \mathcal{F}$, $f \in C(S^\Lambda)$ and $e \in E_{\Lambda}$, we have

$$\|\nabla_e f\|^2_{r_{\bar{e}}} \leq 4M_{r_e} C_{\phi, \nu} \|f\|^2_{\mu}.$$  

In particular, $\nabla_e$ gives a continuous map from $L^2(\mu)$ to $L^2(\mu)_{r_{\bar{e}}}$. 
Proof. By definition, we have
\[
\|\nabla f\|_{r_e}^2 = \sum_{\eta, \eta' \in S^\Lambda, \eta' \neq \eta} (f(\eta') - f(\eta))^2 \pi^\Lambda r_e(\eta) \mu(\eta) \leq M_{r_e} \sum_{\eta, \eta' \in S^\Lambda} (f(\eta') - f(\eta))^2 \mu(\eta)
\]
\[
\leq 2M_{r_e} \sum_{\eta, \eta' \in S^\Lambda, \eta' \neq \eta} (f(\eta')^2 \mu(\eta) + f(\eta)^2 \mu(\eta))
\]
\[
\leq 2M_{r_e} C_{\phi, r} \sum_{\eta, \eta' \in S^\Lambda, \eta' \neq \eta} (f(\eta')^2 \mu(\eta) + f(\eta)^2 \mu(\eta))
\]
\[
= 2M_{r_e} C_{\phi, r} \left( \sum_{\eta \in S^\Lambda} f(\eta)^2 \mu(\eta) + \sum_{\eta \in S^\Lambda, \eta' \neq \eta} f(\eta)^2 \mu(\eta) \right) \leq 4M_{r_e} C_{\phi, r} \|f\|_{\mu}^2,
\]
where the first inequality follows from the definition of the rate bound \(M_{r_e}\), the second inequality follows from Schwartz inequality, the third inequality follows from the definition of \(C_{\phi, r}\) and the last inequality follows from the fact that \(\{\eta' | \eta \in S^\Lambda, \eta' \neq \eta\} = \{\eta | \eta \in S^\Lambda, \eta' \neq \eta\} \subset S^\Lambda\).

Next, we introduce the notion of the **spectral gap**, which plays an important role in the assumption for our main theorem. Let \((\Lambda, E_\Lambda)\) be a finite locale, with interaction \((S, \phi)\) and a product measure \(\mu = \gamma^{\otimes \Lambda}\). For any \(\Lambda \in \mathcal{F}\), we denote by \(\| \cdot \|_{\Lambda}\) the norm on
\[
K^0(S^\Lambda) := C(S^\Lambda) / \text{Ker } \partial_\Lambda
\]
induced from the norm \(\| \cdot \|_{\mu}\) on \(C(S^\Lambda)\). In general, \(\|h\|_{\Lambda} \leq \|h\|_{\mu}\), and we have \(\|h\|_{\Lambda} = \|h\|_{\mu}\) if and only if \(h \in (\text{Ker } \partial_\Lambda)^\perp\), where \((\text{Ker } \partial_\Lambda)^\perp\) is the orthogonal complement of \(\text{Ker } \partial_\Lambda\) for the inner product \(\langle \cdot, \cdot \rangle_{\mu}\).

To define the spectral gap, we first give a simple lemma.

**Lemma 2.40.** For any finite locale \((\Lambda, E_\Lambda)\), there exists a constant \(C > 0\) depends on the locale \((\Lambda, E_\Lambda)\) and the rate \(r = (r_e)\) such that for any \(f \in K^0(S^\Lambda)\),
\[
\|f\|_{\Lambda}^2 \leq C \sum_{e \in E_\Lambda} \|\nabla f\|_{r_e}^2.
\]

**Proof.** Since \((\text{Ker } \partial_\Lambda)^\perp\) is a finite dimensional space and the quadratic form \(f \mapsto \sum_{e \in E_\Lambda} \|\nabla f\|_{r_e}^2\) does not degenerate on \((\text{Ker } \partial_\Lambda)^\perp\), we have
\[
\inf_{f \in (\text{Ker } \partial_\Lambda)^\perp} \sum_{e \in E_\Lambda} \|\nabla f\|_{r_e}^2 > 0.
\]
This implies the assertion. \(\square\)

To state our assumption for the main theorem, we only need to introduce the notion of the spectral gap associated to the trivial rate \(r_e \equiv 1\) as follows.
Definition 2.41. For an interaction \((S, \phi)\), a finite locale \((\Lambda, E_\Lambda)\) and a probability measure \(\nu\) on \(S\) supported on \(S\), we define the spectral gap \(C_{SG,(\Lambda, E_\Lambda)}\) (which also depend on \((S, \phi)\) and \(\nu\)) to be the maximum of \(C > 0\) satisfying
\[
\|f\|_\Lambda^2 \leq C^{-1} \sum_{e \in E_\Lambda} \|\nabla_f\|_\mu^2
\]
for any \(f \in K^0(S^\Lambda)\) where \(\mu = \nu^\otimes \Lambda\).

Finally, we introduce a norm on the space of \(L^2\)-forms \(C^1_{L^2}(S^X_\mu)\).

Definition 2.42. We define the norm \(\| \cdot \|_{r,sp}\) to be the norm on \(C^1_{L^2}(S^X_\mu)\) defined by
\[
\|\omega\|_{r,sp} := \sup_{e \in E} \|\omega_e\|_{r,e},
\]
and we let \(C^1_{L^2}(S^X_\mu)_{r,sp}\) be the subspace of \(C^1_{L^2}(S^X_\mu)\) given as
\[
C^1_{L^2}(S^X_\mu)_{r,sp} := \{\omega \in C^1_{L^2}(S^X_\mu) \mid \|\omega\|_{r,sp} < \infty\}.
\]
Furthermore, we let \(Z^1_{L^2}(S^X_\mu)_{r,sp} := C^1_{L^2}(S^X_\mu)_{r,sp} \cap Z^1_{col}(S^X_\mu)\).

We remark that although \(C^1_{unif}(S^X) \subset C^1_{L^2}(S^X_\mu)\), we have in general
\[
C^1_{unif}(S^X) \not\subset C^1_{L^2}(S^X_\mu)_{r,sp}.
\]

Certain homogeneity condition such as shift-invariance is necessary for an uniformly local form to be an element of \(C^1_{L^2}(S^X_\mu)_{r,sp}\). We will simply denote \(\| \cdot \|_{r,sp}\) by \(\| \cdot \|_{sp}\), if the rate \(r\) is the trivial rate such that \(r_e = 1\) for any \(e \in E\).

3. The Main Theorem

In this section, we introduce the norms on invariant forms, and state our main theorem.

3.1. Closed and Exact Shift-Invariant \(L^2\)-Forms. In this subsection, in order to formulate our main theorem, we introduce the space of closed and exact shift-invariant \(L^2\)-forms on \(S^X\). We will also introduce the notion of a spectral gap.

In this subsection, we assume that the locale \((X, E)\) has an action of a group \(G\). An action of \(G\) on \((X, E)\) gives for any \(\tau \in G\) a bijection \(\tau : X \to X\) inducing a bijection \(\tau : E \to E\) on the set of directed edges, such that the identity element of \(G\) induces the identity map of \(X\), and the operation of elements of \(G\) maps to the composition of bijections. We will assume that the action of \(G\) is free, and that the set \(X/G\) of orbits of \(X\) with respect to the action of \(G\) is a finite set. In this case, the set \(E/G\) of orbits of \(E\) with respect to the action of \(G\) is also a finite set. The most typical example of such locale is given by the Euclidean lattice \((\mathbb{Z}^d, \mathbb{B}^d)\), with action of \(G = \mathbb{Z}^d\) given by translation. A locale which is a topological crystal in the sense of [25] §6.3, also referred to as a crystal lattice, satisfies this condition for some finitely generated abelian group \(G\).
The action of $G$ on $(X, E)$ induces an action of $G$ on $S^X$ given by mapping $\eta = (\eta_x)_{x \in X} \in \prod_{x \in X} S$ to $\tau(\eta) := (\eta_{\tau^{-1}(x)})_{x \in X}$ for any $\tau \in G$. For any subset $\Lambda \subset X$, the element $\tau \in G$ induces an $\mathbb{R}$-linear isomorphism

$$\tau : C(S^\Lambda) \rightarrow C(S^{\tau(\Lambda)})$$

given by $\tau(f)(\eta) = f(\tau^{-1}(\eta))$ for any $f \in C(S^\Lambda)$ and $\eta \in S^{\tau(\Lambda)}$. Furthermore, the action of $G$ defines a map of graphs $\tau : (S^\Lambda, \Phi_\Lambda) \rightarrow (S^{\tau(\Lambda)}, \Phi_{\tau(\Lambda)})$, which induces an $\mathbb{R}$-linear isomorphism

$$\tau : C^1(S^\Lambda) \rightarrow C^1(S^{\tau(\Lambda)})$$

given by $\tau(\omega)(\varphi) := \omega(\tau^{-1}(\varphi))$ for any $\omega \in C^1(S^\Lambda)$ and $\varphi \in \Phi_\Lambda$. Consider the image of $\omega$ in $\prod_{e \in E} C(S^\Lambda)$ through the embedding $[10]$. For any $\tau \in G$, we have

$$\tau(\omega)_e(\eta) = \tau(\omega)(\varphi) = \omega(\tau^{-1}(\varphi)) = \omega_{\tau^{-1}(e)}(\tau^{-1}(\eta)) = \tau(\omega_{\tau^{-1}(e)})(\eta),$$

where $\varphi = (\eta, \eta') \in \Phi_\Lambda$. This shows that we have $\tau(\omega)_e = \tau(\omega_{\tau^{-1}(e)})$ for any $e \in E$.

Since $\mu = \gamma^\odot_X$ on $S^X$, we have $\mu(A) = \mu(\tau(A))$ for any $A \in F_X$ and $\tau \in G$. In other words, $\mu$ is invariant with respect to the action of $G$. The action of $G$ is compatible with the projection $\pi^\Lambda$, hence we have an action of $G$ on $C_{\text{col}}(S^X_\mu)$ and $C^1_{\text{col}}(S^X_\mu)$. Since $\mu$ is invariant with respect to the action of $G$, we have

$$\|f\|_\mu = \|\tau(f)\|_\mu$$

for any $f \in C_{L^2}(S^X_\mu)$ and $\tau \in G$, hence $G$ also acts on $L^2(\mu)$.

Next, we consider rates which are invariant with respect to the action of $G$.

**Definition 3.1.** We say that a rate $r = (r_e)_{e \in E}$ is invariant with respect to the action of $G$, if

$$r_\tau = \tau(r)_e = \tau(r_{\tau^{-1}(e)})$$

for any $\tau \in G$ and $e \in E$.

Let $\| \cdot \|_{r, \text{sp}}$ be the norm on $C^1_{L^2}(S^X_\mu)$ defined in Definition [2.42] given by

$$\|\omega\|_{r, \text{sp}} = \sup_{e \in E} \|\omega_e\|_{r_e}.$$ 

If the rate $r = (r_e)$ is invariant with respect to the action of $G$, then for any $\tau \in G$, we have

$$\|\tau(\omega)_e\|_{r_\tau}^2 = \|\tau(\omega_{\tau^{-1}(e)})\|_{r_e}^2 = E_\mu[r_e \tau(\omega_{\tau^{-1}(e)})^2] = E_\mu[\tau(r_{\tau^{-1}(e)}\omega_{\tau^{-1}(e)})^2]$$

$$= \|\omega_{\tau^{-1}(e)}\|_{r_{\tau^{-1}(e)}\mu}^2,$$

where the last equality follows from the fact that $\mu$ is invariant with respect to the action of $G$. This shows that $\|\omega\|_{r, \text{sp}} < \infty$ if and only if $\|\tau(\omega)\|_{r, \text{sp}} < \infty$, hence $G$ also acts on the $\mathbb{R}$-linear space $C^1_{L^2}(S^X_\mu)_{r, \text{sp}} \subset C^1_{L^2}(S^X_\mu)$ of Definition [2.42].

For any $\mathbb{R}$-linear space $V$ with an action of $G$, we let $V^G := \{v \in V \mid \tau(v) = v \forall \tau \in G\}$ be the shift-invariant subspace of $V$. 

Lemma 3.2. If a rate \( r = (r_e) \) is invariant with respect to the action of \( G \), then the inclusion \( C^1_{L^2}(S^X)_{r, sp} \subset C^1_{L^2}(S^X) \) induces the identity
\[
C^1_{L^2}(S^X)_{r, sp} = C^1_{L^2}(S^X)^G
\]
on the shift-invariant parts. Moreover, the norms induced by \( \| \cdot \|_{r, sp} \) on \( C^1_{L^2}(S^X)^G \) are equivalent for any rate \( r = (r_e) \) which is invariant with respect to the action of \( G \).

Proof. Let \( \omega = (\omega_e)_{e \in E} \in C^1_{L^2}(S^X)^G \) be any shift-invariant \( L^2 \)-form. For any \( \tau \in G \), consider \( \varphi = (\eta, \eta^e) \in \Phi_X \). Since
\[
\tau(\omega_e)(\eta) = \tau(\omega)(\varphi) = \omega(\tau^{-1}(\varphi)) = \omega_{\tau^{-1}(e)}(\tau^{-1}(\eta)) = \tau(\omega_{\tau^{-1}(e)})(\eta),
\]
the shift-invariance of \( \omega \) shows that \( \omega_e = \tau(\omega_{\tau^{-1}(e)}) \) for any \( e \in E \). Then we see that
\[
\|\omega_{\tau^{-1}(e)}\|_{r_{\tau^{-1}(e), \mu}}^2 = E_\mu[r_{\tau^{-1}(e)}(\omega_{\tau^{-1}(e)})^2] = E_\mu[\tau(r_{\tau^{-1}(e)}(\omega_{\tau^{-1}(e)})^2)] = \|\omega_e\|_{r_e}^2,
\]
where the center equality follows from the fact that the probability measure \( \mu \) is invariant with respect to the action of \( G \). This shows that
\[
\|\omega\|_{r, sp} = \sup_{e \in E} \|\omega\|_{r_e} = \sup_{e \in E_0} \|\omega\|_{r_e},
\]
where \( E_0 \) is the set of representatives of the orbits of \( E \) with respect to the action of \( G \). Since \( E_0 \) is finite, we see that \( \omega \in C^1_{L^2}(S^X)^G \) as desired. The equivalence of norms follow from the fact that the norms \( \| \cdot \|_{r_e} \) are equivalent to the norm \( \| \cdot \|_\mu \) on \( L^2(\mu) \) for any \( e \in E_0 \). \( \square \)

If we let \( E_0 \) be the set of representatives of the orbits of \( E \) with respect to the action of \( G \), then the projection to the \( E_0 \)-component induces an injection
\[
C^1_{L^2}(S^X)^G \hookrightarrow \prod_{e \in E_0} L^2(\mu_e).
\]
The inner product \( \langle \cdot, \cdot \rangle \) on \( L^2(\mu)_e \) of (19) induces via linearity an inner product \( \langle \cdot, \cdot \rangle_r \) on the product, which in turn induces an inner product on \( C^1_{L^2}(S^X)^G \). We let \( \| \cdot \|_r \) be the norm on \( C^1_{L^2}(S^X)^G \) induced from the inner product \( \langle \cdot, \cdot \rangle_r \), given as
\[
\|\omega\|_r^2 := \langle \omega, \omega \rangle_r = \sum_{e \in E_0} \langle \omega_e, \omega_e \rangle_{r_e} = \sum_{e \in E_0} \|\omega_e\|_{r_e}^2
\]
for any \( \omega \in C^1_{L^2}(S^X)^G \). Since \( \omega \) is invariant with respect to the action of \( G \), the inner products \( \langle \cdot, \cdot \rangle_r \) and the induced norm \( \| \cdot \|_r \) is independent of the choice of \( E_0 \). The relation
\[
\|\omega\|_{r, sp}^2 \leq \|\omega\|_r^2 \leq |E_0| \|\omega\|_{r, sp}^2
\]
shows that the norms \( \| \cdot \|_r \) and \( \| \cdot \|_{r, sp} \) are equivalent on \( C^1_{L^2}(S^X)^G \), hence induces the same topology. Note that since this topology is the topology of a finite sum of Hilbert spaces \( L^2(\mu)_e \), for a sequence \( \{\omega_n\}_{n \in \mathbb{N}} \) in \( C^1_{L^2}(S^X)^G \), we have
\[
\omega = \lim_{n \to \infty} \omega_n
\]
if and only if \( \omega_e = \lim_{n \to \infty} \omega_{n, e} \in L^2(\mu)_e \) for any \( e \in E \).
Next, we consider the compatibility of the action of the group with the differential.

**Lemma 3.3.** The action of $G$ is compatible with the differential 

$$\partial_{\Lambda} : C(S^\Lambda) \to C^1(S^\Lambda),$$ 

namely $\partial_{\tau \Lambda} (\tau f) = \tau (\partial_{\Lambda} f)$ for any $f \in C(S^\Lambda)$ and $\tau \in G$.

**Proof.** This is [2] Lemma 4.1, and follows from the definition of the action of $G$. \qed

Since the action of $G$ is compatible with the projection $\pi^\Lambda$, the differential 

$$\partial : C^0_{\text{col}}(S^\Lambda) \to C^1_{\text{col}}(S^\Lambda)$$

is also compatible with the action of $G$. By Proposition [2.27] the space of closed co-local forms $Z^1_{\text{col}}(S^\Lambda)$ coincides with the image of $\partial$, hence we see that $Z^1_{\text{col}}(S^\Lambda)$ also has an action of $G$. This induces an action of $G$ on $Z^1_{L^2}(S^\Lambda) = Z^1_{\text{col}}(S^\Lambda) \cap C^1_{L^2}(S^\Lambda)$.

**Definition 3.4.** We denote the space of shift-invariant closed $L^2$-forms by 

$$\mathcal{C}_\mu \coloneqq Z^1_{L^2}(S^\Lambda)^G,$$

and the space of exact forms by 

$$\mathcal{E}_\mu \coloneqq \overline{\partial (C^0_{\text{unif}}(S^\Lambda)^G)},$$

where the bar indicates the closure of $\partial (C^0_{\text{unif}}(S^\Lambda)^G)$ with respect to the topology of $C^1_{L^2}(S^\Lambda)^G$.

Our main theorem concerns the characterization of forms in $\mathcal{C}_\mu$ as a sum of an exact form in $\mathcal{E}_\mu$ and certain forms obtained as differentials of uniformly local functions. For this, we first prove that $\mathcal{E}_\mu \subset \mathcal{C}_\mu$.

**Lemma 3.5.** The space of shift-invariant closed $L^2$-forms $\mathcal{C}_\mu$ is closed in $\prod_{e \in E / G} L^2(\mu)$. In particular, $\mathcal{C}_\mu$ is a Hilbert space equipped with the inner product induced from $\prod_{e \in E / G} L^2(\mu)$ and we have $\mathcal{E}_\mu \subset \mathcal{C}_\mu$.

**Proof.** Suppose we have an sequence $\{\omega_n\}_{n \in \mathbb{N}}$ in $\mathcal{C}_\mu = Z^1_{L^2}(S^\Lambda)^G$ such that 

$$\lim_{n \to \infty} \omega_n = \omega$$

for some element $\omega = (\omega_e) \in \prod_{e \in E} L^2(\mu)$. Since the measure $\mu$ is invariant with respect to the action of $G$, we see that $\omega$ is also invariant with respect to the action of $G$. In order to prove our assertion, by definition of closed co-local forms, it is sufficient to prove that $\pi^\Lambda \omega \in Z^1(S^\Lambda)$ for any $\Lambda \in \mathcal{F}$. The convergence (24) is equivalent to the condition that 

$$\lim_{n \to \infty} \omega_{n,e} = \omega_e$$

in $L^2(\mu)$ for any $e \in E$, which implies that $\lim_{n \to \infty} \pi^\Lambda \omega_{n,e} = \pi^\Lambda \omega_e$ in $C(S^\Lambda)$ for any $e \in E$ and $\Lambda \in \mathcal{F}$. Since $\omega_n \in Z^1_{L^2}(S^\Lambda)^G$, for any $\Lambda \in \mathcal{F}$, we have $\pi^\Lambda \omega_n \in Z^1(S^\Lambda)$. We may take $F_n^\Lambda \in (\ker \partial_{\Lambda})^\perp$ such that $\partial_{\Lambda} F_n^\Lambda = \pi^\Lambda \omega_n$. Lemma [2.40] and the fact that $\pi^\Lambda \omega_n \to \pi^\Lambda \omega$ imply that $F_n^\Lambda$ converges to some $F^\Lambda$ in $(\ker \partial_{\Lambda})^\perp$. By the continuity of $\partial_{\Lambda}$ which follows from Lemma [2.39] we have $\partial_{\Lambda} F^\Lambda_{\infty} = \pi^\Lambda \omega$. This shows that $\pi^\Lambda \omega \in Z^1(S^\Lambda)$ as desired. \qed
Next, we introduce the notion of the \textit{uniformly bounded spectral gap}, which is an important assumption for our main theorem.

We say that a locale \((\Lambda, E_\Lambda)\) is \textit{complete}, if it is complete as a graph. In other words, the set of edges satisfies \(E_\Lambda = (\Lambda \times \Lambda) \setminus \Delta_\Lambda\), where \(\Delta_\Lambda \setminus \Delta_\Lambda \setminus \{ (x, x) \mid x \in \Lambda \}\). Since a locale by definition is locally finite, a complete locale is always a finite locale.

\textbf{Definition 3.6.} We say that the data \(((S, \phi), \nu)\) has a \textit{uniformly bounded spectral gap}, if there exists a constant \(C_{SG} > 0\) such that for any complete locale \((\Lambda, E_\Lambda)\)

\[ C_{SG, (\Lambda, E_\Lambda)} \geq C_{SG} |\Lambda| \]

\textbf{Example 3.7.} There are some interactions having a \textit{uniformly bounded spectral gap}.

(a) Multi-species exclusion process introduced in \([2, 2]\) with any probability measure \(\nu\) supported on \(S\) has the uniformly bounded spectral gap, as shown in \([4]\).

(b) Generalized exclusion process introduced in \([2, 2]\) with a probability measure \(\nu\) supported on \(S\) given as \(\nu(\eta = m) = \frac{1 - \rho}{(1 - \rho)^m}\) for some \(\rho > 0\) has the uniformly bounded spectral gap. It can be shown by using Theorem 2.1 of \([3]\) and computations in Section 3 of \([19]\).

(c) More generally, Theorem 2.1 of \([3]\) and Theorem 3 of \([23]\) is applicable for our model. Hence, if the spectral gap has the corresponding simple averages dynamics of \((S, \nu)\) (cf. \([3]\) on the complete graph with 3 sites is enough large, we can conclude that the interaction \(((S, \phi), \nu)\) has the uniformly bounded spectral gap.

\subsection{The Statement of the Main Theorem.}

In this subsection, we state our main theorem. Consider the Euclidean lattice \((\mathbb{Z}^d, \mathbb{B}^d)\). We fix an interaction \((S, \phi)\) which is faithfully quantified in the sense of Definition \([2.5]\). We will also assume that \(\mu\) is a product measure \(\mu = \gamma^\otimes \mathcal{X}\) such that \(((S, \phi), \nu)\) has a uniformly bounded spectral gap in the sense of Definition \([3.6]\). We let \(r = \tau(r_{\epsilon_0})\) be a rate for the system \(((\mathbb{Z}^d, \mathbb{B}^d), (S, \phi))\) as introduced in Definition \([2.7]\).

The Euclidean lattice has a free action of the group \(G = \mathbb{Z}^d\) acting via translation. For any \(x \in \mathbb{Z}^d\), we denote by \(\tau_x \in G\) the translation by \(x\), given by \(\tau_x(y) = x + y\) for any \(y \in \mathbb{Z}^d\). Note that since \(\mu\) is the product measure on \(S^{\mathbb{Z}^d}\), it is invariant with respect to the action of \(G\). We assume that the rate is invariant with respect to the action of \(G\), in other words that we have \(r_e = \tau(r_{\tau^{-1}(e)})\) for any \(e \in \mathbb{B}^d\) and \(\tau \in G\).

We let \(\mathcal{C}_\mu\) and \(\mathcal{E}_\mu\) be the space of closed and exact shift-invariant \(L^2\)-forms on \(S^{\mathbb{Z}^d}\) given in Definition \([3.4]\). In other words, we let \(\mathcal{C}_\mu := Z_{L^2}(S^{\mathbb{Z}^d})_G\) and

\[ \mathcal{E}_\mu := \delta(C^\otimes_{\text{unif}}(S^{\mathbb{Z}^d})_G) \subseteq \mathcal{C}_\mu, \]

where the inclusion follows from Lemma \([3.5]\).

\textbf{Lemma 3.8.} For any conserved quantity \(\xi \in \text{ConsV}^\circ(S)\) and \(j = 1, \ldots, d\), let

\[ \mathfrak{a}_\xi^j := \sum_{x \in \mathbb{Z}^d} x_j \xi_x, \]
where \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \). Then for any \( y = (y_1, \ldots, y_d) \in \mathbb{Z}^d \), we have
\[
(1 - \tau_y) \Psi^i_x = y_j \sum_{x \in X} \xi_x.
\]
In particular, we have \( \partial \Psi^i_x \in \mathcal{Z}^1_{\text{unif}}(S^d)^G \subset \mathcal{C}_\mu \).

**Proof.** By definition, \( \Psi^i_x = \sum_{x \in X} x_j \xi_x \) is a uniformly local function with diameter \( R = 0 \), hence
\[
\partial \Psi^i_x \in \mathcal{Z}^1_{\text{unif}}(S^d) = \mathcal{Z}^1_{\text{col}}(S^d)^G \cap \mathcal{C}^1_{\text{unif}}(S^d)
\]
by Lemma 2.28. For any \( y = (y_1, \ldots, y_d) \in \mathbb{Z}^d \), we have
\[
(1 - \tau_y) \Psi^i_x = \sum_{x \in X} x_j \xi_x - \sum_{x \in X} x_j \xi_{x+y} = \sum_{x \in X} x_j \xi_x - \sum_{x \in X} (x_j - y_j) \xi_x = y_j \sum_{x \in X} \xi_x
\]
for \( j = 1, \ldots, d \). Since \( \partial \xi_x = 0 \), the compatibility of the group action with respect to the differential gives \( (1 - \tau_y) \Psi^i_x = 0 \). Our assertion follows from the fact that any \( \tau \in G \) is of the form \( \tau = \tau_y \) for some \( y \in \mathbb{Z}^d \). \( \square \)

We let \( c_\phi = \text{Cons}_{\phi}(S) \), and we fix a basis \( \xi^{(1)}, \ldots, \xi^{(c_\phi)} \) of \( \text{Cons}_{\phi}(S) \). Denote by
\[
\mathcal{H} := \text{Span}_\mathbb{R} \{ \Psi^i_{\xi^{(i)}} \mid i = 1, \ldots, c_\phi, j = 1, \ldots, d \}
\]
the \( \mathbb{R} \)-linear subspace of \( \mathcal{C}^1_{\text{unif}}(S^d) \) spanned by \( \Psi^i_{\xi^{(i)}} \) for \( i = 1, \ldots, c_\phi \) and \( j = 1, \ldots, d \). Our main result is as follows.

**Theorem 3.9.** Let \( (\mathbb{Z}^d, \mathbb{E}^d) \) be the Euclidean lattice with action of \( G = \mathbb{Z}^d \) by translation. We assume that \( (S, \phi) \) is an interaction which is faithfully quantified, and that \( ((S, \phi), \nu) \) has a uniformly bounded spectral gap in the sense of Definition 3.7. We assume in addition that \( \phi \) is simple if \( d = 1 \). If we let \( \mathcal{C}_\mu \) and \( \mathcal{C}_\mu \) be the spaces of closed and exact shift-invariant \( L^2 \)-forms, then we have a decomposition
\[
\mathcal{C}_\mu \cong \mathcal{C}_\mu \oplus \partial \mathcal{H}.
\]

Theorem 3.9 for the case of the exclusion process was proved in [7, Theorem 4.1], for the case of the generalized exclusion process in [13, Theorem 4.14], and for the case of lattice gas with energy in [18]. Our result generalizes such results to general faithfully quantified interactions satisfying the spectral gap estimate.

Note by Lemma 3.2 the topology hence the closure is independent of the choice of the various norms \( \| \cdot \|_r, \| \cdot \|_{r, \text{sp}} \) on \( \mathcal{C}^1_{L^2}(S^d)^G \). Hence it is sufficient to prove the theorem for the case of trivial rate, such that \( r_e \equiv 1 \) for any \( e \in E \). The rough strategy of the proof is as follows. Assume the conditions of Theorem 3.9. Given a closed shift-invariant \( L^2 \)-form \( \omega \in \mathcal{C}_\mu = \mathcal{Z}^1_{L^2}(S^d)^G \), we follow the strategy originally due to Varadhan, and construct from \( \omega \) certain sequence of shift-invariant uniformly local functions \( \{ \Psi_n \}_{n \in \mathbb{N}} \) in \( \mathcal{C}^0_{\text{unif}}(S^d)^G \) such that we have
\[
\lim_{n \to \infty} \partial \Psi_n = \omega + \omega^\dagger
\]
for some \( \omega^\uparrow \in C := Z^1_{\text{unif}}(S^d) \). In the original strategy by Varadhan, the key point was to use the explicit description of the interaction \((S, \phi)\) and obtain explicit conditions which must be satisfied by \( \omega^\uparrow \) to show that \( \omega^\uparrow \) may be described in terms of linear sums of the differential of \( \Psi_{\xi(i)}^j \). Once we have this, if we let \( \omega_\Psi := \lim_{n \to \infty} \partial \Psi_n \), then this would show that \( \omega \) splits as
\[
(25) \quad \omega = \omega_\Psi + (-\omega^\downarrow) \in \mathcal{E} \oplus \partial \mathcal{H}.
\]

However, in our case of a general interaction, especially for such case of the multi-color exclusion process when \( c_\phi > 1 \), such explicit calculation are much too complicated to follow through via brute calculation. Also, these direct computations do not explain the reason why we have the differential of \( \Psi_{\xi(i)}^j \) universally. Instead, we will use a general decomposition theorem for \( C \), which was proved in our previous article [1]. We let \( C = Z^1_{\text{unif}}(S^d) \) and \( \mathcal{E} := \partial(C^0_{\text{unif}}(S^d)G) \) be the spaces of closed and exact shift-invariant uniformly local forms. The main result of [1] is the following.

**Theorem 3.10** ([1] Theorem 5). Let the assumptions be as in Theorem 3.9. If we let \( C \) and \( \mathcal{E} \) be the spaces of closed and exact shift-invariant uniformly local forms, then we have a decomposition
\[
C \cong \mathcal{E} \oplus \partial \mathcal{H},
\]
where \( \partial \mathcal{H} \) is an \( \mathbb{R} \)-linear space of dimension \( c_\phi d \) with basis \( \partial \Psi_{\xi(i)}^j \) for \( i = 1, \ldots, c_\phi \) and \( j = 1, \ldots, d \).

By Theorem 3.10 we see that the closed form \( \omega^\uparrow \in C \) of (25) may be decomposed as
\[
\omega^\uparrow = \omega' + \omega^\downarrow \in \mathcal{E} \oplus \partial \mathcal{H}.
\]

Since \( \mathcal{E} \subset \mathcal{E}_\mu \), we have \( \omega_\Psi - \omega' \in \mathcal{E}_\mu \). This would give a decomposition
\[
\omega = (\omega_\Psi - \omega') + (-\omega^\downarrow) \in \mathcal{E}_\mu \oplus \partial \mathcal{H}
\]
as desired.

### 3.3. Construction of a Convergent Sequence.
Let the notations be as in §3.1. In particular, in this subsection, we consider the data \(((X, E), (S, \phi), \mu)\) and trivial rate \( r = (r_e)_{e \in E} \) such that \( r_e \equiv 1 \) for any \( e \in E \). We assume that \((X, E)\) has a free action of a group \( G \) such that \( X/G \) is finite, but \((X, E)\) is not necessarily \((\mathbb{Z}^d, \mathbb{B}^d)\).

Let \( \omega \in \mathcal{E}_\mu := Z^1_{L^2}(S^d) \) be a shift-invariant closed \( L^2 \)-form. The purpose of this section is to construct a certain sequence of uniformly local functions \( \{\Psi_n\} \) associated to \( \omega \), which will be used to prove Theorem 3.9. We fist introduce the notion of an approximation of a locale \( X \).

**Definition 3.11.** We say that a system of connected sets \((\Lambda_n)_{n \in \mathbb{N}}\) in \( \mathcal{F} \) is an approximation of \( X \), if \( \Lambda_n \subset \Lambda_{n+1} \) for any \( n \in \mathbb{N} \) and we have \( X = \bigcup_{n \in \mathbb{N}} \Lambda_n \). In this case, we denote \( \Lambda_n \uparrow X \).

Next, we introduce some conditions on the approximation. For any \( \Lambda \subset X \), we define the boundary \( \partial \Lambda \) of \( \Lambda \) by
\[
\partial \Lambda := \{ e \in E \mid o(e) \in \Lambda, t(e) \notin \Lambda \}.
\]

**Definition 3.12.** Let \( \Lambda_n \uparrow X \) be an approximation.
(a) We say that \( \Lambda_n \uparrow X \) is an approximation by fundamental domains, if \( \Lambda_n \) for any \( n \in \mathbb{N} \) is a fundamental domain for a subgroup \( G^{(n)} \subset G \).

(b) We say that \( \Lambda_n \uparrow X \) has a negligible boundary, if

\[
\lim_{n \to \infty} |\partial \Lambda_n| / |\Lambda_n| = 0.
\]

The archetypical approximation is the following approximation for the Euclidean lattice.

**Example 3.13.** Let \((X, E)\) be the Euclidean lattice \((\mathbb{Z}^d, \mathbb{B}^d)\), with an action of \( G = \mathbb{Z}^d \) given via translation. For each \( n \in \mathbb{N} \), we let

\[
\Lambda_n := [-n, n]^d = \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid \forall j \ |x_j| \leq n\}.
\]

Then \( \Lambda_n \uparrow X \) is an approximation by fundamental domains, since \( \Lambda_n \) is a fundamental domain for the subgroups \( G^{(n)} := (2n+1)\mathbb{Z}^d \subset G \). Since \( |\Lambda_n| = (2n+1)^d \) and \( |\partial \Lambda_n| = 2d(2n+1)^{d-1} \), we see that \( \Lambda_n \uparrow X \) has a negligible boundary.

For the rest of this section, we assume that \( X \) has an approximation by fundamental domains \( \Lambda_n \uparrow X \) which has a negligible boundary, and we fix such an approximation. The order of the boundary \( \partial \Lambda \) may be approximated by the perimeters defined as follows. For any \( \Lambda \subset X \) and integers \( \ell > 0 \), we define the \( \ell \)-perimeter of \( \Lambda \) by

\[
\Lambda^{[\ell]} := \{ x \in \Lambda \mid d_X(x, \Lambda^c) \leq \ell \},
\]

where \( \Lambda^c := X \setminus \Lambda \). For any \( x \in X \) the set \( E_x := \{ e \in E \mid o(e) = x \} \) is a finite set since the graph \((X, E)\) is locally finite. We define the degree of \( X \) by

\[
\deg(X) := \sup_{x \in X} |E_x|.
\]

If we let \( X_0 \) be the fundamental domain of \( X \) for the action of \( G \), then the fact that \( X/G \) is finite implies that

\[
\deg(X) = \sup_{x \in X} |E_x| = \sup_{x \in X_0} |E_x| < \infty.
\]

Moreover, since \( X \) is an infinite connected graph, we have \( \deg(X) \geq 2 \).

**Lemma 3.14.** We have the following inequalities.

\[ (a) \ |\Lambda^{[1]}| \leq |\partial \Lambda| \leq \deg(X)|\Lambda^{[1]}|. \]

\[ (b) \ |\Lambda^{[\ell]}| \leq |\Lambda^{[1]}| \deg(X)^\ell \text{ for any integer } \ell > 0. \]

**Proof.** Note that we have a map \( \partial \Lambda \to \Lambda^{[1]} \) given by \( e \mapsto o(e) \), which is surjective. This shows that \( |\Lambda^{[1]}| \leq |\partial \Lambda| \). In addition, we have

\[
|\partial \Lambda| = |\{ e \in E \mid o(e) \in \Lambda, t(e) \notin \Lambda \}| \leq |\{ e \in E_x \mid x \in \Lambda^{[1]} \}| \leq \deg(X)|\Lambda^{[1]}|,
\]

which proves (a). For the second assertion, we first prove that

\[
|x \in \Lambda \mid d_X(x, \Lambda^c) = N| \leq |\Lambda^{[1]}| \deg(X)^{N-1}.
\]
For \( x \in \Lambda \), if \( d_{\Lambda}(x, \Lambda^c) = N \), then there exists a path \( \vec{p} \) in \( \Lambda \) of length \( N - 1 \) such that \( o(\vec{p}) \in \Lambda^{[1]} \) and \( t(\vec{p}) = x \). Since the number of distinct paths of length \((N - 1)\) starting from a fixed vertex is bounded by \( \deg(X)^{N-1} \), we obtain (27). Note that

\[
\sum_{N=1}^{\ell} \deg(X)^{N-1} \leq (\deg X - 1) \sum_{N=1}^{\ell} \deg(X)^{N-1} \leq \deg(X)^{\ell} - 1 \leq \deg(X)^{\ell},
\]

where the first inequality follows from the fact that \( \deg(X) \geq 2 \). Our assertion for \((b)\) follows by taking the sum of (27) over \( N = 1, \ldots, \ell \).

We may use the perimeter \( \Lambda^{[\ell]} \) to give an equivalent condition for having a negligible boundary.

**Lemma 3.15.** An approximation \( \Lambda_n \uparrow X \) has a negligible boundary if and only if

\[
\lim_{n \to \infty} \frac{|\Lambda_n^{[\ell]}|}{|\Lambda_n|} = 0
\]

for \( \ell = 1 \). This is equivalent to the condition that (28) holds for any integer \( \ell > 0 \).

**Proof.** The statement follows immediately from Lemma 3.14.

We next introduce the *tempered enlargement* of an approximation, defined as follows.

**Definition 3.16.** We say that an approximation \( \Sigma_n \uparrow X \) is a *tempered enlargement* of \( \Lambda_n \uparrow X \), if \( \Lambda_n \subset \Sigma_n \) for any \( n \in \mathbb{N} \), and we have

\[
\sup_n \frac{|\partial \Lambda_n|^2}{|\Lambda_n|^2} \frac{|\Sigma_n|}{|\Lambda_n\setminus\Lambda_n|} \text{diam}(\Sigma_n)^2 < \infty.
\]

**Lemma 3.17.** Suppose there exists a tempered enlargement \( \Sigma_n \uparrow X \) of an approximation \( \Lambda_n \uparrow X \). Then the approximation \( \Lambda_n \uparrow X \) has a negligible boundary.

**Proof.** Suppose \( \Lambda_n \uparrow X \) is an approximation with a tempered enlargement \( \Sigma_n \uparrow X \). Let

\[ M := \sup_n \frac{|\partial \Lambda_n|^2}{|\Lambda_n|^2} \frac{|\Sigma_n|}{|\Lambda_n\setminus\Lambda_n|} \text{diam}(\Sigma_n)^2. \]

Then we have

\[
\frac{|\partial \Lambda_n|^2}{|\Lambda_n|^2} \leq M \frac{|\Sigma_n \setminus \Lambda_n|}{|\Lambda_n| \text{diam}(\Sigma_n)^2} \leq \frac{M}{\text{diam}(\Sigma_n)^2} \to 0 \quad (n \to \infty)
\]
as desired.

**Lemma 3.18.** Let \( \Lambda_n \uparrow X \) be the approximation by fundamental domains of the Euclidean lattice given in Example 3.13. If we let \( \Sigma_n := \Lambda_{2n} \), then \( \Sigma_n \uparrow X \) is a tempered enlargement of \( \Lambda_n \uparrow X \).

**Proof.** Note that \( |\Lambda_n| = (2n + 1)^d \), \( |\partial \Lambda_n| = 2d(2n + 1)^{d-1} \), \( |\Sigma_n| = |\Lambda_{2n}| = (4n + 1)^d \) and \( \text{diam}(\Sigma_n) = \text{diam}(\Lambda_{2n}) = d(4n + 1) \). This shows that

\[
\frac{|\partial \Lambda_n|^2}{|\Lambda_n|^2} \frac{|\Sigma_n|}{|\Lambda_n\setminus\Lambda_n|} \text{diam}(\Sigma_n)^2 = \frac{(2d)^2(2n + 1)^{2d-2}}{(2n + 1)^{2d}} \frac{(4n + 1)^d}{(4n + 1)^d - (2n + 1)^d} d^2(4n + 1)^2.
\]
Hence, we have
\[ \lim_{n \to \infty} \frac{\partial \lambda_n}{\lambda_n} \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 = \frac{4^{d+2}d^4}{(4d - 2d)}, \]
which proves the inequality (39) as desired.

In what follows, we fix an approximation by fundamental domains \( \Lambda_n \uparrow X \), and we suppose that a tempered enlargement \( \Sigma_n \uparrow \Lambda_n \) exists. Let \( \omega \in \mathcal{C}_\mu = Z_{L^2}(S^{X}_\mu)^G \). Since \( \omega \) is a closed co-local form, there exists \( F \in C^0_{\text{col}}(S^{X}_\mu) \) such that \( \partial F = \omega \). We will use \( F \) to construct the uniformly local functions \( \Psi_n \) as follows.

**Definition 3.19.** We fix an approximation by fundamental domains \( \Lambda_n \uparrow X \) with a tempered enlargement \( \Sigma_n \uparrow X \). For any \( \omega \in \mathcal{C}_\mu = Z_{L^2}(S^{X}_\mu)^G \), let \( F = (F^\Lambda) \in C^0_{\text{col}}(S^{X}_\mu) \) such that \( \partial F = \omega \), and we let \( F_n \) be the projection of \( F \Sigma_n \) to \( (\text{Ker } \partial_{\Sigma_n}) \). We let
\[ \Psi_n := \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G} \tau(\pi^\Lambda F_n), \]
where \( [G : G^{(n)}] \) denotes the index of the subgroup \( G^{(n)} \subset G \). By construction, \( \Psi_n \in C^0_{\text{unif}}(S^{X}_\mu)^G \).

We will separate \( \Psi_n \) into a sum for which, the differential of the first converging to \( \omega \), and the differential of the second converting to the boundary term \( \omega^\dagger \). We have
\[ \partial \Psi_n = \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G} \tau(\partial(\pi^\Lambda F_n)). \]

The main difficulty in calculation of the differential stems from the fact that for a general co-local local function \( f \in C^0_{\text{col}}(S^{X}_\mu) \), we may have \( \partial(\pi^\Lambda f) \neq \partial_{\Lambda}(\pi^\Lambda f) \). For any \( f \in C^0_{\text{col}}(S^{X}_\mu) \) and \( \Lambda \in \mathcal{F} \), we define the boundary differential by
\[ \partial^\dagger f := \partial(\pi^\Lambda f) - \partial_{\Lambda}(\pi^\Lambda f). \]

Since \( \partial(\pi^\Lambda f) = (\partial \Lambda(\pi^\Lambda f)) \) for any \( e \in E_{\Lambda} \) and \( \partial \Lambda(\pi^\Lambda f) = (\partial_{\Lambda}(\pi^\Lambda f)) \) for \( e \in E_{X \setminus \Lambda} \), we have \( \partial^\dagger f = 0 \) if and only if \( e \in \partial \Lambda \cup \partial \Lambda \). Note that for \( f \in \text{Ker } \partial \), we have \( \pi^\Lambda f \in \text{Ker } \partial \Lambda \). However, in general, \( \partial^\dagger f \neq 0 \), since in general, we have \( \text{Ker } \partial \Lambda \neq \text{Ker } \partial \).

**Definition 3.20.** Let the notations be as in Definition 3.19. For any \( n \in \mathbb{N} \), let
\[ \omega_n := \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G} \tau(\partial \Lambda_n(\pi^\Lambda F_n)), \quad \omega^\dagger_n := \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G} \tau(\partial^\dagger \Lambda_n(\pi^\Lambda F_n)). \]

By (31), we have \( \partial \Psi_n = \omega_n + \omega^\dagger_n \). We call \( \{\omega_n\}_{n \in \mathbb{N}} \) the boundary forms.

**Proposition 3.21.** For the sequence \( \{\omega_n\}_{n \in \mathbb{N}} \) of Definition 3.20 we have \( \omega_n \in C^1_{\text{loc}}(S^{X})^G \) and
\[ \lim_{n \to \infty} \omega_n = \omega \]
in \( C_{L^2}(S^{X}_\mu)^G \).
Proof. We first shows that the sum defines a co-local form in $C^1_{\text{col}}(S^X)^G$. Note that $\partial_{\Lambda_n}(\pi^{\Lambda_n}F_n) = \pi^{\Lambda_n}(\partial_{\Sigma_n}F_n) = \pi^{\Lambda_n}\omega_{\Sigma_n} = \omega^{\Lambda_n}$. We have $\omega^{\Lambda_n} = (\omega^{\Lambda_n}_e) \in C^1(S^X)$. In particular, $\omega^{\Lambda_n}_e \neq 0$ if and only if $e \in E_{\Lambda_n}$. For any $\tau \in G$, we have $\tau(\omega^{\Lambda_n}_e) = \tau(\omega^{\Lambda_1}_{\tau^{-1}(e)})$, hence

$$\omega_{n,e} := \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G} \tau(\omega^{\Lambda_n}_e) = \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G, \tau^{-1}(e) \in E_{\Lambda_n}} \tau(\omega^{\Lambda_n}_{\tau^{-1}(e)})$$

is a finite sum for any $e \in E$, hence $\omega_n$ defines an element in $C^1_{\text{loc}}(S^X)$. The fact that $\omega_n$ is $G$-invariant follows from the fact that it is the sum over the $G$-translates of $\omega^{\Lambda_n}$. For the convergence, note that

$$\omega_e - \omega_{n,e} = \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G, \tau^{-1}(e) \in E_{\Lambda_n}} (\omega_e - \tau(\omega^{\Lambda_n}_{\tau^{-1}(e)})) + \left[\frac{[G : G^{(n)}] - |Ge \cap E_{\Lambda_n}|}{[G : G^{(n)}]}\right] \omega_e$$

where $Ge = \{\tau e \mid \tau \in G\}$. Since $\omega$ is $G$-invariant, we have $\tau(\omega^{\Lambda_n}_{\tau^{-1}(e)}) = \tau(\omega^{\Lambda_n}_e) = \omega^{\Lambda_n}_e$. From the triangular equality for norms, we have

$$\|\omega_e - \omega_{n,e}\| \leq \frac{1}{[G : G^{(n)}]} \sum_{\tau \in G, \tau^{-1}(e) \in E_{\Lambda_n}} \|\omega_e - \tau(\omega^{\Lambda_n}_{\tau^{-1}(e)})\| + \left[\frac{[G : G^{(n)}] - |Ge \cap E_{\Lambda_n}|}{[G : G^{(n)}]}\right] \|\omega_e\|.$$ 

Noting that $|Go(e) \cap \Lambda_n| = [G : G^{(n)}]$, we have

$$\left[\frac{[G : G^{(n)}] - |Ge \cap E_{\Lambda_n}|}{[G : G^{(n)}]}\right] \leq \frac{|\partial\Lambda_n|}{[G : G^{(n)}]}.$$ 

Hence, noting $|[G : G^{(n)}]| = |\Lambda_n|$, the second term of the right hand side of (35) converges to zero if $n \to \infty$. Finally, we evaluate the first term of the right hand side of (35). For any $\varepsilon > 0$, there exists $\Lambda \in \mathcal{F}$ such that

$$\|\omega_e - \omega^\Lambda_e\| < \varepsilon.$$ 

Let $H_n := \{\tau \in G \mid \Lambda \subset \tau\Lambda_n\}$. Then

$$\sum_{\tau \in G, \tau^{-1}(e) \in E_{\Lambda_n}} \|\omega_e - \omega^{\Lambda_n}_{\tau^{-1}(e)}\| \leq \sum_{\tau \in H_n, \tau^{-1}(e) \in E_{\Lambda_n}} \|\omega_e - \omega^{\Lambda_n}_{\tau^{-1}(e)}\| + \sum_{\tau \in G \setminus H_n, \tau^{-1}(e) \in E_{\Lambda_n}} \|\omega_e - \omega^{\Lambda_n}_{\tau^{-1}(e)}\| 

\leq \sum_{\tau \in H_n, \tau^{-1}(e) \in E_{\Lambda_n}} \|\omega_e - \omega^\Lambda_e\| + |(G \setminus H_n)e \cap E_{\Lambda_n}| \|\omega_e\| 

< |H_n e \cap E_{\Lambda_n}| \varepsilon + \{\tau \in G \mid o(e) \in \tau(\Lambda_n), (\tau(\Lambda_n))^c \cap \Lambda = \emptyset\} \|\omega\|_{sp}.$$
where for the first inequality, we use \( \| \omega_e - \omega^\tau(\Lambda_n) \|_{\mu} \leq \| \omega_e - \omega^\Lambda \|_{\mu} \) if \( \Lambda \subseteq \tau \Lambda_n \) and \( \| \omega_e - \omega^\tau(\Lambda_n) \|_{\mu} \leq \| \omega_e \|_{\mu} \) for any \( \tau \). Here, we let \( \ell := \max_{e \in \Lambda} d_X(o(e), x) \). Then we have
\[
\frac{1}{[G : G^{(n)}]} \sum_{\tau \in \Gamma} \| \omega_e - \omega^\tau(\Lambda_n) \|_{\mu} < \epsilon + \frac{|\Lambda_n|^{\ell-1}}{[G : G^{(n)}]} \| \omega \|_{sp},
\]
which shows that the first term of the right hand side of (55) is \( \leq \epsilon \) when \( n \to \infty \). This shows that we have \( \lim_{n \to \infty} \| \omega - \omega_n \|_{sp} = 0 \) as desired. \( \square \)

The key point in the proof of Theorem 3.9 is proving that the sequence of boundary forms \( \{ \omega^\dagger_n \}_{n \in \mathbb{N}} \) converges weakly to some local form \( \omega^\dagger \) in \( C \). We will prove in \( \S 4 \) certain bounds of norms of differentials of local functions which will be used to uniformly bound the norms of \( \omega^\dagger_n \). The existence of the bound implies that there exists a subsequence of \( \omega^\dagger_n \) which converges weakly to some \( \omega^\dagger \in \mathcal{C}_\mu \). Then in \( \S 5 \) we show that \( \omega^\dagger \) in fact is a local form in \( C \). We will use this fact to prove Theorem 3.9

### 4. Bounds of Norms of Differentials

In this section, we prove some general results concerning norms on spaces of local functions and forms. More precisely, assuming that the rate \( r = (r_e)_{e \in E} \) is trivial, we prove the Moving Particle Lemma (Lemma 4.1) and the Boundary Estimate (Proposition 4.3), which gives the relation between certain differentials and operators on local functions. We will then in Proposition 4.8 use these results to bound the norms of the boundary forms \( \{ \omega^\dagger_n \} \) defined in Definition 3.20.

We fix a data \(( (X, E), (S, \phi)) \) and a probability measure \( \mu = \nu \otimes X \) on \( S^X \), where \( \nu \) is a probability measure on \( S \) supported on \( S \). We assume that \( \phi \) is faithfully quantified in the sense of Definition 2.5

#### 4.1. Moving Particle Lemma.

In this subsection, we prove the Moving Particle Lemma (Lemma 4.1), which bounds the change of potential induced by interactions between distant vertices in terms of the change of potentials induced by interactions on adjacent vertices of the locale. Let \((X, E)\) be a locale. For any \( \eta \in S^X \) and \( x, y \in X \) such that \( x \neq y \), consider the configuration
\[
\eta^{x \to y} \in S^X
\]
whose components outside \( x, y \in X \) coincides with that of \( \eta \), and \((\eta^{x \to y}_x, \eta^{x \to y}_y) := \phi(\eta_x, \eta_y) \). If \( x = y \), then we let \( \eta^{x \to y} := \eta \). For any local function \( f \in C(S^X) \) and \( x, y \in X \), we let \( f^{x \to y} \) be the local function given by \( f^{x \to y}(\eta) := f(\eta^{x \to y}) \) for any \( \eta \in X \). Furthermore, we let
\[
\nabla_{x \to y} f := f^{x \to y} - f.
\]
Note that by definition, \( \nabla_{x \to y} f = 0 \) if \( x = y \). The Moving Particle Lemma is the following.

**Lemma 4.1 (Moving Particle Lemma).** *There exists a constant \( C_{MP} > 0 \) depending only on \(( (S, \phi), \nu) \) such that for any finite locale \(( \Lambda, E_\Lambda) \), vertices \( x, y \in \Lambda \), and function \( f \in C(S^\Lambda) \), we have
\[
\| \nabla_{x \to y} f \|_{\mu}^2 \leq C_{MP} \operatorname{diam}(\Lambda)^2 \| \partial_\Lambda f \|_{sp}^2,
\]
where \( \operatorname{diam}(\Lambda) \) is the diameter of \( \Lambda \).*
where
\[ \|\omega\|_{sp} := \sup_{e \in E_\Lambda} \|\omega_e\|_\mu \]
for any \( \omega \in C^1(S^\Lambda) \), which corresponds to the norm \( \| \cdot \|_{r,sp} \) when \( r \) is the trivial rate.

We will give the proof of Lemma 4.1 at the end of this section. We first define the notion of an exchange. Let \( \eta \in S^X \). For any \( x, y \in X \), we define the exchange
\[ \eta^{x,y} \in S^X \]
to be the configuration whose components coincides with that of \( \eta \) outside \( x, y \in X \), and \( (\eta^{x,y}_x, \eta^{x,y}_y) := (\eta_y, \eta_x) \). For any local function \( f \in C_{loc}(S^X) \) and \( x, y \in X \), we let \( \nabla_{x,y} f \) be the local function in \( C_{loc}(S^X) \) given by
\[ \nabla_{x,y} f(\eta) := f(\eta^{x,y}) - f(\eta). \]
Note that by definition, \( \nabla_{x,y} f = 0 \) if \( x = y \).

**Lemma 4.2.** There exists a constant \( \bar{C}_{\phi,v} > 0 \) depending only on \( ((S, \phi), v) \) such that for any finite locale \( (\Lambda, E_\Lambda) \), \( x, y \in \Lambda \) and \( f \in C(S^\Lambda) \), we have
\[ \| \nabla_{x,y} f \|_\mu^2 \leq \bar{C}_{\phi,v} \| \nabla_{x,y} f \|_\mu^2. \]

**Proof.** Let \( x, y \in \Lambda \). By the tower property of the conditional expectation, we have
\[ \| \nabla_{x,y} f \|_\mu^2 = E_\mu[(\nabla_{x,y} f)^2] = E_\mu[E_\mu[(\nabla_{x,y} f)^2|F_{\Lambda\setminus \{x,y\}}]], \]
\[ \| \nabla_{x,y} f \|_\mu^2 = E_\mu[(\nabla_{x,y} f)^2] = E_\mu[E_\mu[(\nabla_{x,y} f)^2|F_{\Lambda\setminus \{x,y\}}]], \]
hence it is sufficient to prove our result for the case when the locale is \( (\{x, y\}, E_{\{x, y\}}) \) with \( E_{\{x, y\}} = \{ (x, y), (y, x) \} \subset \{x, y\} \times \{x, y\} \). Consider two quadratic forms on \( C(S^{\{x,y\}}) \) given by
\[ D_{x,y}(f) := E_\mu[(\nabla_{x,y} f)^2], \]
\[ D_{x,y}(f) := E_\mu[(\nabla_{x,y} f)^2]. \]
We fist prove that \( D_{x,y}(f) = 0 \) implies \( D_{x,y}(f) = 0 \). If this holds true, then our assertion follows since we may take
\[ \bar{C}_{\phi,v} := \sup_{f \in C(S^\Lambda), D_{x,y}(f) \neq 0} \frac{D_{x,y}(f)}{D_{x,y}(f)}, \]
which is finite since \( C(S^\Lambda) \) is a finite dimensional space. Suppose \( f \in C(S^{\{x,y\}}) \) satisfied \( D_{x,y}(f) = 0 \), namely that \( E_\mu[(\nabla_{x,y} f)^2] = 0 \), which implies that \( \nabla_{x,y} f = 0 \), hence \( f(\eta^{x,y}) = f(\eta) \) for any \( \eta \in S^{\{x,y\}} \). By Lemma 2.38 for the case \( r \) is trivial (see also Remark 2.37(b)), we have \( \| \nabla_{y,x} f \|_\mu^2 \leq C_{\phi,v} \| \nabla_{x,y} f \|_\mu^2 = 0 \), which shows that \( \nabla_{y,x} f = 0 \). Any \( \eta = (\eta_x, \eta_y) \in S \times S \) has the same conserved quantity as \( \eta^{x,y} = (\eta_y, \eta_x) \). Since \( (S, \phi) \) is faithfully quantified, there exists a path \( \gamma \) in \( S^{\{x,y\}} = S \times S \) from \( \eta \) to \( \eta^{x,y} \). This implies that \( \eta^{x,y} \) is obtained from \( \eta \) by executing \( e = (x, y) \) and \( \tilde{e} = (y, x) \) a finite number of times, which shows that \( f(\eta^{x,y}) = f(\eta) \), hence \( \nabla_{x,y} f = 0 \). This proves that \( D_{x,y}(f) = 0 \). \( \square \)

We may now prove Lemma 4.1.
**Proof of Lemma 4.1.** For any $x, y \in \Lambda$, since $\Lambda$ is connected, there exists a path $\overline{p} = (e_1, \ldots, e_N)$ from $x$ to $y$ such that $N' = \text{len}(\overline{p}) \leq \text{diam}(\Lambda)$. For any $\eta \in S^\Lambda$, let $\eta^0 := \eta$ and $\eta^i := (\eta^{i-1})^c(e_{i-1}), t(e_i)$ for any $i = 1, \ldots, N - 1$. Furthermore, let $\eta^N := (\eta^{N-1})^c_N$, and $\eta^i = (\eta^{i-1})^c(e_{N-1}), t(e_{N-1})$ for $i = N + 1, \ldots, 2N - 1$. Then by construction, we have $\eta^{x \to y} = \eta^{2N-1}$. This shows that we have

$$\nabla_{x,y}f = \left(\sum_{i=1}^{N-1} \nabla_{o(e_i), t(e_i)}f\right) + \nabla_{e_N}f + \left(\sum_{i=N+1}^{2N-1} \nabla_{o(e_i), t(e_i)}f\right).$$

By Schwartz’s inequality, we have

$$\|\nabla_{x,y} f\|_\mu^2 \leq 3(N-1)\left(\sum_{i=1}^{N-1} \|\nabla_{o(e_i), t(e_i)}f\|_\mu^2\right) + 3\|\nabla_{e_N}f\|_\mu^2 + 3(N-1)\left(\sum_{i=N+1}^{2N-1} \|\nabla_{o(e_i), t(e_i)}f\|_\mu^2\right)$$

$$\leq 3N \max\{1, \overline{C}_\phi, \nu\} \sum_{i=1}^{2N-1} \|\nabla_{e_i}f\|_\mu^2 \leq 6N^2 \max\{1, \overline{C}_\phi, \nu\} \sup_{e \in E_\Lambda} \|\nabla_e f\|_\mu^2$$

$$= 6N^2 \max\{1, \overline{C}_\phi, \nu\} \|\partial_{\Lambda} f\|_{sp}^2,$$

where the second inequality follows by Lemma 4.2. If we let $C_{MP} = 6 \max\{1, \overline{C}_\phi, \nu\} > 0$, then this gives the desired constant. \qed

### 4.2. Boundary Estimates

In this subsection, we prove Proposition 4.3 which gives the bound of differentials of local functions with respect to edges which intersect the boundary of the locality. We let $(X, E)$ be a locale.

**Proposition 4.3 (Boundary Estimate).** There exists a constant $C_{BE} > 0$ depending only on $(S, \phi)$ and the probability measure $\nu$ on $S$ satisfying the following property. Consider any $\Lambda, \Sigma \in \mathcal{F}$ such that $\Lambda \subset \Sigma$. For any $h \in C(S^\Sigma)$ and $e \in \partial_{\Lambda} \cap E_\Sigma$, we have

$$\|\nabla_e (\pi^\Lambda h)\|_\mu^2 \leq \frac{C_{BE}}{|\Sigma \setminus \Lambda|} \left(\|h\|_\mu^2 + \sum_{y \in \Sigma \setminus \Lambda} \|\nabla_{e \to y}h\|_\mu^2\right).$$

Similarly,

$$\|\nabla_{x \to (\pi^\Lambda h)}\|_\mu^2 \leq \frac{C_{BE}}{|\Sigma \setminus \Lambda|} \left(\|h\|_\mu^2 + \sum_{x \in \Sigma \setminus \Lambda} \|\nabla_{x \to t(e)}h\|_\mu^2\right).$$

We will give the proof of Proposition 4.3 at the end of this subsection. We let $\Lambda'' := \Sigma \setminus \Lambda$. For any $\beta \in S$, we let $\text{Fr}_\beta : S^{\Lambda''} \to \mathbb{N}$ be the map

$$\text{Fr}_\beta := \sum_{y \in \Lambda''} 1_{\{\eta_y = \beta\}},$$

where $1_{\{\eta_y = \beta\}}$ for $y \in \Lambda''$ is the indicator function of $\{\eta \in S^{\Lambda''} \mid \eta_y = \beta\} \subset S^{\Lambda''}$. By construction, $\text{Fr}_\beta$ calculates the frequency of the occurrence of $\beta$ in the components of $\eta$. Since $E_\mu[1_{\{\eta_y = \beta\}}] = \nu(\beta)$ for any $y \in \Lambda''$, we have $E_\mu[\text{Fr}_\beta] = |\Lambda''|\nu(\beta)$. If we let

$$(37) \quad \text{Fr}_\beta^* := \nu(\beta)^{-1} |\Lambda''|^{-1} \text{Fr}_\beta,$$
then we have $E_{\mu}[\text{Fr}_{\beta}^*] = 1$. Moreover, since $\text{Fr}_{\beta}^*$ is independent of $\mathcal{F}_\Lambda$, we have

$$\pi^\Lambda \text{Fr}_{\beta}^* = E[\text{Fr}_{\beta}^*|\mathcal{F}_\Lambda] = E_{\mu}[\text{Fr}_{\beta}^*] = 1.$$  

By a direct computation, we have

$$\pi^\Lambda (\text{Fr}_{\beta}^* - \pi^\Lambda \text{Fr}_{\beta}^*)^2 \leq \frac{1 - \nu(\beta)}{\nu(\beta)|\Lambda'|}.$$  

The following bound will be used in the proof of Proposition 4.3.

**Lemma 4.4.** For any random variables $g$, $h$ and $\sigma$-algebra $\mathcal{G}$, we have

$$(E[g|\mathcal{G}]E[h|\mathcal{G}] - E[gh|\mathcal{G}])^2 \leq E[(g - E[g|\mathcal{G}])^2|\mathcal{G}]E[(h - E[h|\mathcal{G}])^2|\mathcal{G}].$$

**Proof.** By the Schwartz’s inequality for conditional expectations, we have

$$(XY|\mathcal{G})^2 \leq E[X^2|\mathcal{G}]E[Y^2|\mathcal{G}]$$

for any random variable $X$ and $Y$. If we let $X = g - E[g|\mathcal{G}]$ and $Y = h - E[h|\mathcal{G}]$, then

$$E[XY|\mathcal{G}] = E[(g - E[g|\mathcal{G}]) (h - E[h|\mathcal{G}])|\mathcal{G}]$$

$$= E[(g - E[g|\mathcal{G}])h - gE[h|\mathcal{G}] + E[g|\mathcal{G}]E[h|\mathcal{G}]|\mathcal{G}]$$

$$= E[g|\mathcal{G}]E[h|\mathcal{G}] - E[g|\mathcal{G}]E[h|\mathcal{G}] + E[g|\mathcal{G}]E[h|\mathcal{G}]$$

$$= (E[g|\mathcal{G}] - E[g|\mathcal{G}]E[h|\mathcal{G}] - E[g|\mathcal{G}]E[h|\mathcal{G}]),$$

where we have used the fact that $E[g|\mathcal{G}]$ and $E[h|\mathcal{G}]$ are $\mathcal{G}$-measurable, and that $E[fX|\mathcal{G}] = fE[X|\mathcal{G}]$ for any random variables $f$ and $X$ such that $f$ is $\mathcal{G}$-measurable. Our assertion now follows from (39). $\square$

**Lemma 4.5.** Let $\beta \in S$ and let $\text{Fr}_{\beta}^*$ be the function given in (37). For any function $h \in C(S^\Sigma)$, we have

$$((\pi^\Lambda \text{Fr}_{\beta}^*)(\pi^\Lambda h)(\eta) - \pi^\Lambda (\text{Fr}_{\beta}^* h)(\eta))^2 \leq \frac{1 - \nu(\beta)}{\nu(\beta)|\Lambda'|} \pi^\Lambda (h - \pi^\Lambda h)^2(\eta).$$

**Proof.** By Lemma 4.4, we have

$$((\pi^\Lambda \text{Fr}_{\beta}^*)(\pi^\Lambda h)(\eta) - \pi^\Lambda (\text{Fr}_{\beta}^* h)(\eta))^2 \leq \pi^\Lambda (\text{Fr}_{\beta}^* - \pi^\Lambda \text{Fr}_{\beta}^*)^2 \pi^\Lambda (h - \pi^\Lambda h)^2(\eta).$$

Combining this inequality with (38), we see that

$$((\pi^\Lambda \text{Fr}_{\beta}^*)(\pi^\Lambda h)(\eta) - \pi^\Lambda (\text{Fr}_{\beta}^* h)(\eta))^2 \leq \frac{1 - \nu(\beta)}{\nu(\beta)|\Lambda'|} \pi^\Lambda (h - \pi^\Lambda h)^2(\eta)$$

as desired. $\square$

**Lemma 4.6.** Let $h \in C(S^\Sigma)$. Then for any $e \in \partial \Lambda \cap E_\Sigma$ and $\eta \in S^\Sigma$ such that $(\eta_{o(e)}, \eta_{t(e)}) = (\alpha, \beta) \in S \times S$, we have

$$\pi^\Lambda (\text{Fr}_{\beta}^* h)(\eta^e) = \frac{1}{\nu(\beta)|\Lambda'|} \sum_{y \in \Lambda''} \pi^\Lambda (1_{\eta_y = \beta} h_{o(e) \rightarrow y})(\eta),$$

where $(\alpha', \beta') = \phi(\alpha, \beta)$. 
Proof. The statement is trivial if \((\alpha', \beta') = (\alpha, \beta)\) so we assume that \((\alpha', \beta') \neq (\alpha, \beta)\). By definition of \(\text{Fr}^*_\beta\) and the conditional expectation \(\pi^\Lambda\), we have

\[
\pi^\Lambda(\text{Fr}^*_\beta, h)(\eta^\epsilon) = \frac{1}{\nu(\beta')|\Lambda''|} \sum_{y \in \Lambda''} \pi^\Lambda(1_{\{\eta_y = \beta'\}} h)(\eta^\epsilon)
\]

\[
= \frac{1}{\nu(\beta')|\Lambda''|} \sum_{y \in \Lambda''} \mu(\eta^\epsilon|\Lambda) \sum_{\eta' \in S^\Sigma} h(\eta') \mu(\eta').
\]

In the above sum, the condition \(\eta'_|\Lambda = \eta^\epsilon|\Lambda\) and \(\eta_y = \beta'\) ensures that \(\eta' = (\eta'_x)\) satisfies \(\eta'_x = \eta_x\) for \(x \in \Lambda \setminus \{o(e)\}\), \(\eta'_o(e) = \alpha'\), and \(\eta'_y = \beta'\). Since we have assumed that \((\alpha', \beta') \neq (\alpha, \beta)\), we have \((\alpha, \beta) = \overline{\sigma}(\alpha', \beta')\) by the definition of the interaction. Hence, for each \(y \in \Lambda''\) and \(\eta' \in S^\Sigma\) satisfying \(\eta'_|\Lambda = \eta^\epsilon|\Lambda\), \(\eta'_y = \beta'\), if we let \(\overline{\eta} = (\overline{\eta}_x)\) be the configuration such that \(\overline{\eta}_x = \eta'_x\) for \(x \neq o(e), y\), and \((\overline{\eta}_o(e), \overline{\eta}_y) := (\alpha, \beta)\), then \(\overline{\eta}\) is the unique element in \(S^\Sigma\) satisfying \(\overline{\eta}|\Lambda = \eta|\Lambda\), \(\overline{\eta}_y = \beta\), and \(\eta' = \overline{\eta}^{(o(e)-y)}\). This gives

\[
\pi^\Lambda(\text{Fr}^*_\beta, h)(\eta^\epsilon) = \frac{1}{\nu(\beta')|\Lambda''|} \sum_{y \in \Lambda''} \mu(\eta^\epsilon|\Lambda) \sum_{\overline{\eta} \in S^\Sigma} h(\overline{\eta}^{(o(e)-y)} \mu(\overline{\eta}^{(o(e)-y)})
\]

\[
= \frac{1}{\nu(\beta')|\Lambda''|} \sum_{y \in \Lambda''} \mu(\eta|\Lambda) \sum_{\overline{\eta} \in S^\Sigma} h_o(e-y)(\overline{\eta}) \mu(\overline{\eta})
\]

\[
= \frac{1}{\nu(\beta')|\Lambda''|} \sum_{y \in \Lambda''} \pi^\Lambda(1_{\{\eta_y = \beta\}} h_o(e-y))(\eta)
\]

as desired. We have used the equalities \(\mu(\eta|\Lambda) \nu(\alpha') = \mu(\eta^\epsilon|\Lambda) \nu(\alpha)\) and \(\mu(\overline{\eta}^{(o(e)-y)} \nu(\alpha) \nu(\beta') = \mu(\overline{\eta}) \nu(\alpha') \nu(\beta')\), which follows from the fact that \(\mu\) is the product measure \(\mu = \nu^{\oplus \Sigma}\). □

We are now ready to prove the boundary estimate.

Proof of Proposition 4.3. Let \(f = \pi^\Lambda h\). By definition, we have

\[
\|\nabla_c f\|_\mu^2 = \sum_{\eta \in S^{\Lambda^{\cup\{e\}}}} (f(\eta^\epsilon) - f(\eta))^2 \mu(\eta) = \sum_{(\alpha, \beta) \in \mathbb{S} \times S} \sum_{\eta \in S^{\Lambda^{\cup\{e\}}}} (f(\eta^\epsilon) - f(\eta))^2 \mu(\eta).
\]

Note that for any \(\eta \in S^{\Lambda^{\cup\{e\}}}\) such that \((\eta_o(e), \eta_{l(e)}) = (\alpha, \beta)\), noting that \((\pi^\Lambda \text{Fr}^*_\beta, (\pi^\Lambda \text{Fr}^*_\beta) = 1\), we may decompose the value \((f(\eta^\epsilon) - f(\eta))\) as

\[
f(\eta^\epsilon) - f(\eta) = (\pi^\Lambda \text{Fr}^*_\beta) f(\eta^\epsilon) - (\pi^\Lambda \text{Fr}^*_\beta) f(\eta) = (\pi^\Lambda \text{Fr}^*_\beta) f(\eta^\epsilon) - \pi^\Lambda (\text{Fr}^*_\beta, h)(\eta^\epsilon)
\]

\[
+ \pi^\Lambda (\text{Fr}^*_\beta, h)(\eta^\epsilon) - \pi^\Lambda (\text{Fr}^*_\beta, h)(\eta)
\]

\[
+ \pi^\Lambda (\text{Fr}^*_\beta, h)(\eta) - (\pi^\Lambda \text{Fr}^*_\beta) f(\eta).
\]
By Schwartz’s inequality, we have

\[(f(\eta^c) - f(\eta))^2 \leq 3((\pi^A Fr_{\beta^c}^*f(\eta^c) - \pi^A Fr_{\beta^c}h(\eta^c))^2 + 3(\pi^A Fr_{\beta}^*h(\eta^c) - \pi^A Fr_{\beta}h(\eta))^2 + 3(\pi^A Fr_{\beta}h(\eta) - (\pi^A Fr_{\beta})^2f(\eta))^2.\]  

We bound each of the terms on the right hand side of (41) with respect to the sum over \((\alpha, \beta) \in S \times S\) and \(\eta \in S^{\Lambda \cup \{t(e)\}}\) such that \((\eta_{\alpha(e)}, \eta_{\beta(e)}) = (\alpha, \beta)\). Concerning the first and the third terms of the right hand side, by Lemma 4.3 we see that

\[
\begin{align*}
((\pi^A Fr_{\beta}^*)(\pi^A h)(\eta^c) - \pi^A (Fr_{\beta}^* h)(\eta^c))^2 &\leq \frac{1 - \nu(\beta')}{\nu(\beta')|\Lambda''|} \pi^A (h - \pi^A h)^2(\eta^c) \\
((\pi^A Fr_{\beta}^*)(\pi^A h)(\eta) - \pi^A (Fr_{\beta}^* h)(\eta))^2 &\leq \frac{1 - \nu(\beta)}{\nu(\beta)|\Lambda'|} \pi^A (h - \pi^A h)^2(\eta).
\end{align*}
\]

For the first term, note that by the tower property, \(\pi^A (h - \pi^A h)^2 = \pi^A (\pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2)\). The definition of \(\pi^A\) gives

\[
\sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{(\eta_{\alpha(e)}, \eta_{\beta(e)}) = (\alpha, \beta)} \frac{1 - \nu(\beta')}{\nu(\beta')} \pi^A (h - \pi^A h)^2(\eta^c) \mu(\eta) \\
\leq \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{(\eta_{\alpha(e)}, \eta_{\beta(e)}) = (\alpha, \beta)} \frac{1}{\nu(\beta')} \pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2(\eta') \mu(\eta') \\
= \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{(\eta_{\alpha(e)}, \eta_{\beta(e)}) = (\alpha, \beta)} \pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2(\eta') \mu(\eta') \\
\leq C_{\phi, \nu} \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{(\eta_{\alpha(e)}, \eta_{\beta(e)}) = (\alpha, \beta)} \pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2(\eta') \mu(\eta') \\
= C_{\phi, \nu} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{\eta' \in S^{\Lambda \cup \{t(e)\}}} \pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2(\eta') \mu(\eta') \\
\leq C_{\phi, \nu} \sum_{\eta \in S^{\Lambda \cup \{t(e)\}}} \sum_{\eta' \in S^{\Lambda \cup \{t(e)\}}} \pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2(\eta') \mu(\eta') \\
\leq C_{\phi, \nu} |S|^2 E_{\mu} [\pi^{\Lambda \cup \{t(e)\}} (h - \pi^A h)^2] = C_{\phi, \nu} |S|^2 E_{\mu} [(h - \pi^A h)^2] \\
\leq C_{\phi, \nu} |S|^2 E_{\mu} [h^2] = C_{\phi, \nu} |S|^2 \|h\|^2_{\mu}.
\]
For the third term, we have

\[
\sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda(t(e))}} \frac{1 - \nu(\beta)}{\nu(\beta)} \pi^\Lambda(h - \pi^\Lambda h)^2(\eta) \mu(\eta) \\
= \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda(t(e))}} (1 - \nu(\beta)) \pi^\Lambda(h - \pi^\Lambda h)^2(\eta|\underline{\Lambda}) \mu(\eta|\underline{\Lambda}) \\
\overset{(*)}{=} \sum_{\alpha \in S} \sum_{\eta \in S^{\Lambda}} (|S| - 1) \pi^\Lambda(h - \pi^\Lambda h)^2(\eta) \mu(\eta) \\
= (|S| - 1) E_\mu[\pi^\Lambda(h - \pi^\Lambda h)^2] \leq (|S| - 1) E_\mu[h^2] = (|S| - 1)\|h\|_\mu^2,
\]

where \((*)\) follows from the fact that \(\sum_{\beta \in S}(1 - \nu(\beta)) = |S| - 1\). Finally, we consider the second term of \((41)\). By Lemma 4.6 for any \(\eta \in S^{\Lambda(t(e))}\) such that \((\eta_0(e), \eta_t(e)) = (\alpha, \beta)\), we have

\[
(\pi^\Lambda(F_{\beta^*} h)(\eta^c) - \pi^\Lambda(F_{\beta^*} h)(\eta))^2 = \frac{1}{\nu(\beta)^2 |\Lambda'|^2} \left( \pi^\Lambda \left( \sum_{y \in \Lambda''} 1_{\{y = \beta\}}(h_{\eta_0(e) \rightarrow y} - h) \right)(\eta) \right)^2.
\]

By Schwartz’s inequality,

\[
\left| \sum_{y \in \Lambda''} 1_{\{y = \beta\}}(h_{\eta_0(e) \rightarrow y} - h) \right| \leq \sqrt{\sum_{y \in \Lambda''} 1_{\{y = \beta\}} \sum_{y \in \Lambda''} (h_{\eta_0(e) \rightarrow y} - h)^2},
\]

so by applying Schwartz’s inequality for conditional expectation, we have

\[
\left( \pi^\Lambda \left( \sum_{y \in \Lambda''} 1_{\{y = \beta\}}(h_{\eta_0(e) \rightarrow y} - h) \right)(\eta) \right)^2 \leq \pi^\Lambda \left( \sum_{y \in \Lambda''} 1_{\{y = \beta\}} \right)(\eta) \pi^\Lambda \left( \sum_{y \in \Lambda''} (h_{\eta_0(e) \rightarrow y} - h)^2 \right)(\eta) \\
= |\Lambda''| \nu(\beta) \pi^\Lambda \left( \sum_{y \in \Lambda''} (h_{\eta_0(e) \rightarrow y} - h)^2 \right)(\eta).
\]
Hence
\[
\sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \setminus \{(e)\}}} \frac{1}{\nu(\beta)2|\Lambda''|^2} \left( \pi^\Lambda \left( \sum_{y \in \Lambda''} 1_{\{\eta_y = \beta\}} (h_{\alpha(e) \to y} - h) (\eta) \right) \right)^2 \mu(\eta)
\]
\[
\leq \frac{1}{|\Lambda''|} \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \setminus \{(e)\}}} \frac{1}{\nu(\beta)} \pi^\Lambda \left( \sum_{y \in \Lambda''} (h_{\alpha(e) \to y} - h)^2 (\eta) \right) \mu(\eta)
\]
\[
= \frac{1}{|\Lambda''|} \sum_{(\alpha, \beta) \in S \times S} \sum_{\eta \in S^{\Lambda \setminus \{(e)\}}} \pi^\Lambda \left( (h_{\alpha(e) \to y} - h)^2 (\eta) \right) \mu(\eta)
\]
\[
= \frac{|S|}{|\Lambda''|} \sum_{y \in \Lambda''} \sum_{\alpha \in S} \sum_{\eta \in S^{\Lambda \setminus \{(e)\}}} \pi^\Lambda \left( (h_{\alpha(e) \to y} - h)^2 (\eta) \right) \mu(\eta)
\]
\[
= \frac{|S|}{|\Lambda''|} \sum_{y \in \Lambda''} \|\nabla_{\alpha(e) \to y} h\|_\mu^2.
\]
Applying the above inequalities to (40), we see that
\[
\|\nabla f\|_\mu^2 \leq \frac{3}{|\Lambda''|} \left( \left( C_{\phi, \nu} |S|^2 + |S| - 1 \right) \|h\|_\mu^2 + |S| \sum_{y \in \Lambda''} \|\nabla_{\alpha(e) \to y} h\|_\mu^2 \right).
\]
Since $C_{\phi, \nu}$ and $|S|$ depends only on $((S, \phi), \nu)$, we see that if we let $C_{BE} := 3(C_{\phi, \nu} |S|^2 + |S|) > 0$, then $C_{BE}$ satisfies the required property.

The second inequality in the statement is shown in the same way. \hfill \Box

4.3. Convergence of the Boundary Sequence. We now return to the setting of §3.3. We will assume in addition that $(S, \phi)$ is faithfully quantified, and that $((S, \phi), \nu)$ has a uniformly bounded spectral gap.

We first prove the following simple estimate.

**Lemma 4.7.** If the data $((S, \phi), \nu)$ has a uniformly bounded spectral gap, then for any complete locale $(\Lambda, E_\Lambda)$ and function $h \in C(S^\Lambda)$, we have
\[
(42) \quad \|h\|_{\Lambda}^2 \leq C^{-1}_{SG, \Lambda} \|\partial_{\Lambda} h\|_{sp}^2.
\]

**Proof.** By definition,
\[
\|h\|_{\Lambda}^2 \leq C^{-1}_{SG, \Lambda} \sum_{e \in E_\Lambda} \|\nabla_e h\|_\mu^2 \leq C^{-1}_{SG, \Lambda} \sum_{e \in E_\Lambda} \|\nabla_e h\|_{sp}^2.
\]
Then, $\|\nabla_e h\|_{\mu}^2 \leq \|\partial_{\Lambda} h\|_{sp}^2$ for any $e \in E_\Lambda$ and $|E_\Lambda| \leq |\Lambda|^2$ implies the assertion. \hfill \Box

Then, the Moving Particle Lemma gives the following.
Proposition 4.8. If the data \((S, \phi, \nu)\) has a uniformly bounded spectral gap, then for any \(\Sigma \in \mathcal{F}\) and \(h \in C(S^\Sigma)\), we have
\[
\|h\|_\Sigma^2 \leq C_{SG}^{-1} C_{MP} |\Sigma| \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2.
\]

Proof. For the locale \((\Sigma, E_\Sigma)\), let \(\bar{\Sigma} := \Sigma \cup E_\Sigma := (\Sigma \times \Sigma) \setminus \Delta_\Sigma\). Then \((\bar{\Sigma}, E_\bar{\Sigma})\) by construction is a complete graph. Note that \(\text{Ker} \partial_{\bar{\Sigma}} \subset \text{Ker} \partial_{\Sigma}\), hence we have \(\|h\|_{\bar{\Sigma}} \leq \|h\|_\Sigma\) for any \(h \in C(S^{\bar{\Sigma}})\).

By Lemma 4.7 and the definition of the norm \(\| \cdot \|_{sp}\), we have
\[
\|h\|_{\Sigma} \leq \|h\|_{\bar{\Sigma}} \leq C_{SG}^{-1} |\bar{\Sigma}| \left(\sup_{e \in \bar{\Sigma}} \| (\partial_{\bar{\Sigma}} h) e \|_\mu \right) = C_{SG}^{-1} |\Sigma| \left(\sup_{x, y \in \Sigma} \| \nabla_{x \rightarrow y} h \|_\mu^2 \right).
\]

Here, we have used the fact that \(\nabla_{x \rightarrow y} h = 0\) if \(x = y\). Moreover, by the Moving Particle Lemma (Lemma 4.1), we have
\[
\|h\|_{\Sigma} \leq C_\phi |\Sigma| \left( C_{MP} \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2 \right) = C_{SG}^{-1} C_{MP} |\Sigma| \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2.
\]

\(\square\)

Remark 4.9. The assumption that \((S, \phi, \nu)\) has a uniformly bounded spectral gap is essentially used only Proposition 4.8 (and Lemma 4.7 for this proposition). Namely, the statements in the rest of the article assuming the uniformly bounded spectral gap may hold if Proposition 4.8 holds, including our main theorem. In the literature of the nongradient method, the condition on the spectral gap is typically given in the form of Proposition 4.8 for the sequence \(\{\Lambda_n\}\) of Example 3.13.

Using Proposition 4.8, we may prove the following bound for the boundary differential \(\partial_{\Lambda}^\perp\).

Proposition 4.10. Suppose the interaction \((S, \phi)\) is faithfully quantified and has a uniformly bounded spectral gap. Then there exists a constant \(C > 0\) depending only on \((S, \phi, \nu)\) such that for any \(\Lambda, \Sigma \in \mathcal{F}\) satisfying \(\Lambda \subset \Sigma\) and \(h \in (\text{Ker} \partial_\Sigma)^\perp\), we have
\[
\|\partial_{\Lambda}^\perp h\|_{sp}^2 \leq C \frac{|\Sigma|}{|\Sigma \setminus \Lambda|} \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2.
\]

Proof. Suppose \(h \in (\text{Ker} \partial_\Sigma)^\perp\). Then \(\|h\|_\Sigma = \|h\|_{\mu}\). By the boundary estimate Proposition 4.3 and the spectral gap estimate Proposition 4.8 for any \(\mu \in \partial \Lambda\), we have
\[
\|\nabla_e (\pi^\Lambda h)\|_\mu^2 \leq C_{BE} \frac{|\Sigma \setminus \Lambda|}{|\Sigma \setminus \Lambda|} \left( \|h\|_\Sigma^2 + \sum_{y \in \Sigma \setminus \Lambda} \|\nabla_{e \rightarrow y} h\|_{\mu}^2 \right)
\]
\[
\leq C_{BE} \frac{|\Sigma \setminus \Lambda|}{|\Sigma \setminus \Lambda|} \left( C_{SG}^{-1} C_{MP} |\Sigma| \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2 + \sum_{y \in \Sigma \setminus \Lambda} \|\nabla_{e \rightarrow y} h\|_{\mu}^2 \right).
\]

Combining this with Lemma 4.1, we see that
\[
\|\nabla_e (\pi^\Lambda h)\|_\mu^2 \leq C_{BE} \frac{|\Sigma \setminus \Lambda|}{|\Sigma \setminus \Lambda|} \left( C_{SG}^{-1} C_{MP} |\Sigma| \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2 + C_{MP} |\Sigma \setminus \Lambda| \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2 \right)
\]
\[
\leq C \frac{|\Sigma|}{|\Sigma \setminus \Lambda|} \text{diam}(\Sigma)^2 \|\partial_{\Sigma} h\|_{sp}^2
\]
for some constant $C > 0$. More precisely, noting that $|\Sigma \setminus \Lambda| \leq |\Sigma|$, we may take $C := C_{BE}C_{MP}(C_{SG}^{-1} + 1) > 0$. The same equality also holds for $e \in \partial \Lambda$. Our assertion follows from this inequality and the definition of $\partial^\uparrow_{\Lambda}$.

We will use the bound of Proposition 4.10 to uniformly bound the norms of the boundary forms \{\omega^+_n\} of Definition 3.20. Consider a closed $G$-invariant $L^2$-form $\omega \in \mathfrak{c}_\mu = C^1_L(S^X) \cap Z^1_{col}(S^X)_G$. We let $\Lambda_n \uparrow X$ and $F_n$ as in Definition 3.19 and we recall that

$$\omega^+_n = \frac{1}{[G : G(n)]} \sum_{\tau \in G} \tau(\partial^\uparrow_{\Lambda_n} F_n),$$

Let $G^e_n := \{\tau \in G \mid e \in \tau(\partial \Lambda_n)\}$. Since $\tau(\partial^\uparrow_{\Lambda_n} F_n) \neq 0$ if and only if $e \in \tau(\partial \Lambda_n)$ or $\bar{e} \in \tau(\partial \Lambda_n)$, if we let

\begin{align*}
\omega^+_{n,e} &:= \frac{1}{[G : G(n)]} \sum_{\tau \in G^e_n} \nabla_e(\tau(F_n)), \\
\omega^-_{n,e} &:= \frac{1}{[G : G(n)]} \sum_{\tau \in G^e_n} \nabla_e(\tau(F_n)),
\end{align*}

then we have $\omega^+_n = \omega^+_{n,e} + \omega^-_{n,e}$. We denote $\omega^\pm_{n,e}$ to mean $\omega^+_{n,e}$ or $\omega^-_{n,e}$. We next prove the existence of a uniform bound of the norms of \{\omega^\pm_{n,e}\} for any $e \in E$.

**Proposition 4.11.** For any $e \in E$, let $\omega^\pm_{n,e}$ be the forms defined in (43). Then for any $n \in \mathbb{N}$, we have $\omega^\pm_n \in \prod_{e \in E/G} L^2(\mu)$, and for any $e \in E$, we have

$$\sup_n \|\omega^\pm_{n,e}\|_\mu < \infty.$$ 

This implies in particular that \{\omega^\pm_{n,e}\} contains a subsequence which converges weakly to an element $\omega^\pm_e$ in $L^2(\mu)$.

**Proof.** From the triangular inequality for norms and the fact that the measure is invariant under the action of $G$, we have

$$[G : G(n)] \|\omega^\pm_{n,e}\|_\mu \leq \sum_{\tau \in G^e_n} \|\tau(\partial^\uparrow_{\Lambda_n} F_n)\|_\mu = \sum_{\tau \in G^e_n} \|\tau((\partial^\uparrow_{\Lambda_n} F_n)_{\tau^{-1}(e)})\|_\mu = \sum_{\tau \in G^e_n} \|((\partial^\uparrow_{\Lambda_n} F_n)_{\tau^{-1}(e)})\|_\mu \leq \|\partial \Lambda_n\| \|\partial^\uparrow_{\Lambda_n} F_n\|_{sp}.$$ 

This shows that

$$\|\omega^\pm_{n,e}\|_\mu \leq \frac{|\partial \Lambda_n|}{[G : G(n)]} \|\partial^\uparrow_{\Lambda_n} F_n\|_{sp}.$$ 

Combining this with Proposition 4.10 noting that $\partial^\uparrow_{\Lambda_n} F_n = \omega_{\Sigma_n}$ and $\|\omega_{\Sigma_n}\|_{sp} \leq \|\omega\|_{sp}$, we have

$$\|\omega^\pm_{n,e}\|_\mu \leq C \frac{|\partial \Lambda_n|^2}{[G : G(n)]^2} \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \diam(\Sigma_n)^2 \|\omega\|_{sp}^2.$$ 

Hence our assertion follows from the fact that $\Sigma_n \uparrow X$ is a tempered enlargement of $\Lambda_n \uparrow X$. The last assertion follows from the fact that any bounded sequence in a Hilbert space has a weakly convergent subsequence. The statement for $\omega^\pm_{n,e}$ may be proved in a similar manner. \qed
In what follows, for each \( e \in E \), we take subsequences of \( \{ \omega^+_n \}_{n \in \mathbb{N}} \) which converges weakly to \( \omega^+_e \) in \( L^2(\mu) \). Since \( \omega^+_n \) and \( \omega^-_n \) by construction is invariant with respect to the action of \( G \), the local functions \( \omega^{+}_{n,e} = \omega^{+}_{n,e} + \omega^{-}_{n,e} \) depends only on the class of \( e \) in the set of orbits \( E/G \), which is finite. Hence we may take subsequences \( \{ \omega^+_n \}_{n \in \mathbb{N}} \) such that \( \{ \omega^+_n \}_{n \in \mathbb{N}} \) converges weakly to \( \omega^+_e \) in \( L^2(\mu) \) for each \( e \in E \). Since the structure of the Hilbert space on \( C^1_{L^2}(S^X_\mu)^G \) is induced from the inclusion \( C^1_{L^2}(S^X_\mu)^G \hookrightarrow \prod_{e \in E/G} L^2(\mu) \), weakly convergence on each \( e \in E \) implies that \( \{ \omega^+_n \} \) is itself weakly convergent. We let \( \omega^+_e := (\omega^+_e) \in C^1_{L^2}(S^X_\mu)^G \) be the weak limit of \( \omega^+_n \).

**Proposition 4.12.** We have \( \omega^+_e \in \mathcal{E}_\mu = Z^1_{L^2}(S^X_\mu)^G \).

**Proof.** By the definition of \( \omega^+_n \) and (33), the form \( \omega^+_e \) is the weak limit of a subsequence of the sequence

\[
\omega^+_n = \partial \Psi_n - \omega_n,
\]

where \( \omega_n \) is defined as in Definition 3.20. By Proposition 3.21, we see that \( \lim_{n \to \infty} \omega_n = \omega \) strongly in \( C^1_{L^2}(S^X_\mu)^G \). If we let

\[
\omega_n := \partial \Psi_n - \omega,
\]

then by construction, \( \omega_n \in \mathcal{E}_\mu = Z^1_{L^2}(S^X_\mu)^G \). Since a subsequence of \( \omega^+_n \) converges weakly to \( \omega^+_e \), a subsequence of \( \omega_n \) also converges to \( \omega^+_e \). This shows that \( \omega^+_e \) is in the weak closure of \( Z^1_{L^2}(S^X_\mu)^G \).

Since by [30] Theorem V.1.11, the weak closure of a linear subspace of a Hilbert space coincides with its strong closure, this implies that \( \omega^+_e \) is in the strong closure of \( Z^1_{L^2}(S^X_\mu)^G \). Our assertion now follows from Lemma 3.5 which asserts that \( Z^1_{L^2}(S^X_\mu)^G \) is closed in \( \prod_{e \in E/G} L^2(\mu) \). \( \Box \)

5. Proof of the Varadhan Decomposition

5.1. Proof of Localizy. Let the notations be as in \[4.3\] In this subsection, assuming that the locale is a Euclidean lattice \( (\mathbb{Z}^d, \mathbb{E}^d) \), we prove that the form \( \omega^+_e \) in Proposition 4.12 obtained as the limit of the boundary forms \( \{ \omega^+_n \} \) is in fact a local form.

We let \( \Lambda_n \uparrow X \) be the approximation by fundamental domains given in Example 3.13. Let

\[
G^e_n := \{ \tau \in G \mid e \in \tau(\partial \Lambda_n) \},
\]

and for \( \tilde{e} \in E \), let \( G^{e,\tilde{e}}_n := G^e_n \cap G^{\tilde{e}}_n \). \( e \) is such that \( t(e) = o(e) + 1_j \) or \( t(e) = o(e) - 1_j \), where \( 1_j \) is the element in \( \mathbb{Z}^d \) with 1 in the \( j \)-th component and 0 in the other components. Let \( o(e)_j \) be the \( j \)-th component of \( o(e) \), and \( Y^e := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid x_j \leq o(e)_j \} \) if \( t(e) = o(e) + 1_j \), and \( Y^{\tilde{e}} := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d \mid x_j \geq o(e)_j \} \) otherwise. In other words, if we splits the vertices in \( \mathbb{Z}^d \) into two parts via the hyperplane perpendicular to \( e \) containing \( o(e) \), then \( Y^e \) consists of vertices in the part which is opposite to the direction of \( e \).

**Lemma 5.1.** Let \( \Lambda_n \uparrow X \) be the approximation of Example 3.13. For any \( e \in E \) and \( \tilde{e} \in E_{Y^e} \), we have

\[
\lim_{n \to \infty} \frac{|G^{e,\tilde{e}}_n|}{|G^e_n|} = 0.
\]
Proof. Note that \( |G_n^e| = \frac{1}{2d} |\partial \Lambda_n| = (2n + 1)^{d-1} \). If \( \tilde{e} \) is in the same direction as that of \( e \), then \( G_n^{e,\tilde{e}} = \emptyset \). If \( \tilde{e} \) is in the exactly opposite direction as that of \( e \), then \( G_n^{e,\tilde{e}} = \emptyset \) for \( n \) sufficiently large. Otherwise, \( G_n^{e,\tilde{e}} \) counts the positions of shifts of \( \Lambda_n \) touching both \( e \) and \( \tilde{e} \) at their origins, hence has \( (d-2) \)-dimension of freedom. This shows that \( |G_n^{e,\tilde{e}}| \leq (2n + 1)^{d-2} \), which proves our assertion. \( \square \)

Remark 5.2. Lemma 5.1 is not generalized straightforwardly to models with a finite range interaction, namely models on a locale \((\mathbb{Z}^d, E)\) where \( E \) is not necessarily \( \mathbb{B}^d \). If \( \mathbb{B}^d \subset E \), to prove the next proposition, we only need the estimates in Lemma 5.1 for \( e, \tilde{e} \in \mathbb{B}^d \), so we can generalized the main result to this case. However, without the condition \( \mathbb{B}^d \subset E \), the generalization is not obvious. We will give a proof of the main result for models with a general finite range interaction in a forthcoming paper.

Having this estimate, we may now prove that \( \omega^\dagger \) is local. The proof uses Proposition 5.4 which will be proved in §5.2.

Proposition 5.3. We have \( \omega^\dagger \in C^1_\text{loc} (S^X) \). In other words, the functions \( \omega^\dagger_e \) for any \( e \in E \) are local functions.

Proof. It is sufficient to prove that \( \omega^+_{\tilde{e}} \) and \( \omega^-_{\tilde{e}} \) are local functions for any \( e \in E \). We first prove that \( \omega^+_{\tilde{e}} \) is a local function. By construction, for any \( n \in \mathbb{N} \), the function \( \omega^+_{n,e} \) is a function in \( C(S^{\Lambda^\cup \{t(e)\}}) \). Let \( \Lambda := \{o(e), t(e)\} \) and \( \Lambda' \subset Y^e \setminus \{o(e)\} \) be a finite set. By Proposition 5.4, it is sufficient to prove that

\[
\nabla_{\tilde{e}} (\pi^{\Lambda \cup \Lambda'} \omega^+_{n,e}) = 0
\]

for any \( \tilde{e} \in E_{\Lambda'} \). By Lemma 2.39, the differential \( \nabla_{\tilde{e}} \) is continuous for local functions. Also, \( C(S^{\Lambda \cup \Lambda'}) \) is a finite dimensional space and the weak topology and the strong topology are same for a finite dimensional Hilbert spaces. Hence, noting that

\[
\| \nabla_{\tilde{e}} (\pi^{\Lambda \cup \Lambda'} \omega^+_{n,e}) \|_\mu = \| \pi^{\Lambda \cup \Lambda'} (\nabla_{\tilde{e}} (\omega^+_{n,e})) \|_\mu \leq \| \nabla_{\tilde{e}} \omega^+_{n,e} \|_\mu,
\]

it is sufficient to prove that

\[
\lim_{n \to \infty} \| \nabla_{\tilde{e}} \omega^+_{n,e} \|_\mu = 0.
\]

Since \( \tilde{e} \in E_{Y^e} \), we have \( \{o(e), t(e)\} \cap \{o(\tilde{e}), t(\tilde{e})\} = \emptyset \). This shows that

\[
\nabla_{\tilde{e}} \omega^+_{n,e} = \nabla_{\tilde{e}} \nabla_{\tilde{e}} \left( \frac{1}{[G: G^{(n)}]} \sum_{\tau \in G_n^e} \tau (F^{\Lambda_n}) \right) = \nabla_{\tilde{e}} \nabla_{\tilde{e}} \left( \frac{1}{[G: G^{(n)}]} \sum_{\tau \in G_n^e} \tau (F^{\Lambda_n}) \right).
\]

By Lemma 2.39 for any local function \( f \), we have

\[
\| \nabla_{\tilde{e}} f \|_\mu^2 \leq 4C_{\phi,\nu} \| f \|_\mu^2,
\]

where \( C_{\phi,\nu} \geq 1 \) is the constant given in Definition 2.11. Hence we have

\[
\| \nabla_{\tilde{e}} \omega^+_{n,e} \|_\mu \leq \frac{2C_{\phi,\nu}^{1/2}}{[G: G^{(n)}]} \sum_{\tau \in G_n^e} \nabla_{\tilde{e}} (\tau (F^{\Lambda_n})) \|_\mu \leq \frac{2C_{\phi,\nu}^{1/2}}{[G: G^{(n)}]} \sum_{\tau \in G_n^e} \| \nabla_{\tilde{e}} (\tau (F^{\Lambda_n})) \|_\mu.
\]
Furthermore, if $\tilde{e} \in E_{\tau(\Lambda_n)}$, then we have $\nabla_{\tilde{e}}(\tau(F^{\Lambda_n})) = \omega_{\tilde{e}}^{\Lambda_n}$ hence $\|\nabla_{\tilde{e}}(\tau(F^{\Lambda_n}))\|_{\mu} \leq \|\omega_{\tilde{e}}\|_{\mu} \leq \|\omega\|_{\text{sp}}$, and if $\{o(\tilde{e}), t(\tilde{e})\} \cap \tau(\Lambda_n) = \emptyset$, then we have $\nabla_{\tilde{e}}(\tau(F^{\Lambda_n})) = 0$. This shows that

$$\sum_{\tau \in G_n^e} \|\nabla_{\tilde{e}}(\tau(F^{\Lambda_n}))\|_{\mu} \leq |G_n^e| \|\omega\|_{\text{sp}} + \sum_{\tau \in G_n^e} \|\nabla_{\tilde{e}}(\tau(F^{\Lambda_n}))\|_{\mu}. \tag{44}$$

Since $|G_n^e| \leq |\partial \Lambda_n|$, the first term of (44) satisfies

$$\lim_{n \to \infty} \frac{|G_n^e|}{[G : G^{(n)}]} \leq \lim_{n \to \infty} \frac{|\partial \Lambda_n|}{|\Lambda_n|} = 0.$$  

For the second term of (44), the fact that the measure is invariant under the action of the group, we see that $\|\nabla_{\tilde{e}}(\tau(F^{\Lambda_n}))\|_{\mu} = \|\nabla_{\tau^{-1}(\tilde{e})}(F^{\Lambda_n})\|_{\mu}$. By Proposition 4.10 we have

$$\|\nabla_{\tau^{-1}(\tilde{e})}(F^{\Lambda_n})\|_{\mu}^2 \leq \|\partial_{\Lambda_n} F^{\Lambda_n}\|_{\text{sp}}^2 \leq C \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 \|\omega^\Sigma_n\|_{\text{sp}}^2 \leq C \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 \|\omega\|_{\text{sp}}^2.$$

Hence we have

$$\lim_{n \to \infty} \sup \|\nabla_{\tilde{e}}\omega_{n,e}^+\|^2 \leq 4CC_{\phi,Y} \lim_{n \to \infty} \sup \frac{|G_n^{e,e'}|}{[G : G^{(n)}]^2} \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 \|\omega\|_{\text{sp}}^2 \leq 4CC_{\phi,Y} \lim_{n \to \infty} \sup \frac{|G_n^{e,e'}|^2}{|G_n^e|^2} \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 \|\omega\|_{\text{sp}}^2.$$

Since $\Sigma_n \uparrow X$ is a tempered enlargement of $\Lambda_n \uparrow X$, noting that $|G_n^e| \leq |\partial \Lambda_n|$ and $[G : G^{(n)}] = |\Lambda_n|$, we see that

$$\sup_n \frac{|G_n^e|^2}{[G : G^{(n)}]^2} \frac{|\Sigma_n|}{|\Sigma_n \setminus \Lambda_n|} \text{diam}(\Sigma_n)^2 \|\omega\|_{\text{sp}}^2 < \infty.$$

Hence, Lemma 5.1 implies that

$$\lim_{n \to \infty} \sup \|\nabla_{\tilde{e}}\omega_{n,e}^+\|_{\mu} = 0$$

as desired. The proof of the locality for $\omega_{\tilde{e}}^+$ is obtained in a similar fashion, by replacing $G_n^e$ with $G_n^{e'}$. This gives our assertion.

\[ \square \]

5.2. Criterion for Locality. In this section, we give a criterion for a $L^2$-function to be a local function.

**Proposition 5.4.** Let $\Lambda \in \mathcal{F}$, and let $Y \subset X$ be an infinite sub-locale such that $\Lambda \cap Y = \emptyset$. For any $f \in L^2(\mu)$, suppose that $f$ is $\tau_{\Lambda \cap Y}$-measurable, and we have

$$\nabla_{\tilde{e}}(\pi^{\Lambda \cap \Lambda'} f) = 0$$

for any finite $\Lambda' \subset Y$ and $\tilde{e} \in E_{\Lambda'}$. Then we have $\pi^{\Lambda} f = f$. In particular, $f$ is a local function.
We will give the proof of Proposition 5.4 at the end of this section. We first review the 0-1 Law of Hewitt and Savage for exchangeable \(\sigma\)-algebra. For any countable set \(I\), we say that \(\varphi\) is a finite permutation of \(I\), if \(\varphi\) is a bijection \(\rho : I \to I\) such that the set \(\{x \in I \mid \varphi(x) \neq x\}\) is a finite set. We denote by \(\mathcal{G}_I\) the set of finite permutations of \(I\). Let \((\Omega, \mathcal{F}, \mu)\) be a probability space. For a fixed \(I\), let \((v_x)_{x \in I}\) be a family of independent and identically distributed random variables on \(\Omega\) parametrized by \(I\) with values in a measure space \(E\). We denote by \(\sigma((v_x)_{x \in I})\) the sub \(\sigma\)-algebra of \(\mathcal{F}\) generated by \((v_x)_{x \in I}\). By definition, for any \(A \in \sigma((v_x)_{x \in I})\), there exists measurable \(B \in E^I\) such that \(A = \{(v_x) \in B\}\). For any finite permutation \(\varphi \in \mathcal{G}_I\), we let \(A^\varphi := \{(v_{\varphi(x)}) \in B\}\). We define the exchangeable \(\sigma\)-algebra \(\mathcal{E}\) of \(\sigma((v_x)_{x \in I})\) by

\[
\mathcal{E} := \{A \in \sigma((v_x)_{x \in I}) \mid A^\varphi = A \forall \varphi \in \mathcal{G}_I\}.
\]

We say that a \(\sigma\)-algebra \(\mathcal{F}\) is trivial for a measure \(\mu\), or simply \(\mu\)-trivial, if we have \(\mu(A) \in \{0, 1\}\) for any \(A \in \mathcal{F}\). The following Corollary 5.5 is a key for our proof of Proposition 5.4. See for example [15, Corollary 12.19] for a proof.

**Corollary 5.5 (0-1 Law of Hewitt and Savage).** Let \(I\) be a countable set and let \((v_x)_{x \in I}\) be a family of independent and identically distributed random variables on \((\Omega, \mathcal{F}, \mu)\) parametrized by \(I\). Then the exchangeable \(\sigma\)-algebra \(\mathcal{E}\) of (45) is \(\mu\)-trivial.

We now return to the case where our probability space is \((S^X, \mathcal{F}_X, \mu)\). Suppose \(Y \subset X\) is an infinite locale, and consider an approximation \(\Lambda_n \uparrow Y\). Fix a basis \(\xi^{(1)}, \ldots, \xi^{(c_\varphi)}\) of \(\text{Consv}^\varphi(S)\). For any \(n \in \mathbb{N}\), we have a measurable map \(\xi_{\Lambda_n} : S^X \to \mathbb{R}^{c_\varphi}\) given by

\[
\eta \mapsto \left(\sum_{x \in \Lambda_n} \xi^{(1)}(\eta_x), \ldots, \sum_{x \in \Lambda_n} \xi^{(c_\varphi)}(\eta_x)\right).
\]

We let

\[
\mathcal{G}_n := \sigma(\xi_{\Lambda_n}, \mathcal{F}_{Y \setminus \Lambda_n}) \subset \mathcal{F}_X
\]

be the \(\sigma\)-algebra generated by the measurable functions \(\xi_{\Lambda_n}\) and \(\mathcal{F}_{Y \setminus \Lambda_n}\). Then we have a natural inclusion \(\mathcal{G}_n \supset \mathcal{G}_{n+1}\) for any \(n \in \mathbb{N}\).

In order to give the proof of Proposition 5.4, we first prove the following lemma.

**Lemma 5.6.** Let \(g \in C(S^{A \cup \Sigma})\), such that \(A, \Sigma \in \mathcal{F}\) and \(A \cap \Sigma = \emptyset\). We let \(Y\) be an infinite locale such that \(\Lambda \cap Y = \emptyset\) and \(\Sigma \subset Y\). Consider an approximation \(\Lambda_n \uparrow Y\). Then we have

\[
\lim_{n \to \infty} E[g|\sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)] = E[g|\mathcal{F}_\Lambda]
\]

in \(L^2\), where \(\mathcal{G}_n := \sigma(\xi_{\Lambda_n}, \mathcal{F}_{Y \setminus \Lambda_n})\) as in (46).

**Proof.** Since \(A\) and \(\Sigma\) are finite, \(S^{A \cup \Sigma} = S^\Lambda \times S^\Sigma\) is also finite. Let \(g \in C(S^{A \cup \Sigma})\). Then, \(g \in C(S^{A \cup \Sigma})\) is a linear sum of the indicator functions \(1_{\Lambda \times \Sigma} = 1_{\Lambda} 1_{\Sigma}\) on \(S^{A \cup \Sigma} = S^\Lambda \times S^\Sigma\), which defines a function in \(L^2(\mu)\). It is sufficient to prove our statement when \(g\) is such function. Since \(1_{\Lambda}\) is \(\mathcal{F}_\Lambda\)-measurable,

\[
E[1_{\Lambda} 1_{\Sigma}|\sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)] = 1_{\Lambda} E[1_{\Sigma}|\sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)].
\]
Since $\mathcal{F}_\Lambda$ is independent from $\mathcal{F}_Y$, which includes $\sigma(\eta_{n_2})$ and $\mathcal{G}_n$, 

$$E[\eta_2 | \sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)] = E[\eta_2 | \mathcal{G}_n],$$

whose proof can be found for example in Chapter 9 of [29]. Thus, $E[\eta_1, \eta_2 | \sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)] = E[\eta_1]E[\eta_2 | \mathcal{G}_n]$ and similarly $E[\eta_1 \eta_2 | \mathcal{F}_\Lambda] = E[\eta_1]E[\eta_2]$. By the convergence theorem of backward martingales, we have 

$$\lim_{n \to \infty} E[\eta_2 | \mathcal{G}_n] = E[\eta_2 | \mathcal{G}_\infty]$$

in $L^2(\mu)$ where $\mathcal{G}_\infty = \cap_n \mathcal{G}_n$. Hence in order to prove our theorem, it is sufficient to prove that $E[\eta_2 | \mathcal{G}_\infty] = E[\eta_2]$ in $L^2(\mu)$.

Let $\text{pr}_x : S^X \to S$ be the projection $\text{pr}_x(\eta) = \eta_x$ for any $\eta \in S^X$, which we view as a random variable on $S^X$, and we consider the system of random variables $(\text{pr}_x)_x \in Y$ of $S^X$. Since $\mathcal{F}_Y = \sigma(\text{pr}_x, x \in Y)$, we have

$$\mathcal{G}_n \subset \mathcal{F}_Y.$$

For any finite permutation $\varrho$ of $Y$, if we take $m$ sufficiently large, then we have $\varrho(x) = x$ for any $x \in Y \setminus \Lambda_m$. This implies that $\varrho$ induces a permutation of the finite set $\Lambda_m$. For any $\Lambda \in \mathcal{G}_m$, by the construction, we have $A^\varrho = A$ since the conserved quantity $\xi_{\Lambda_m}$ is invariant under permutations of the components of $\Lambda_m$. This implies in particular that $\mathcal{G}_\infty = \cap_{n \in \mathbb{N}} \mathcal{G}_n \subset \mathcal{C}$, where $\mathcal{C}$ is the exchangeable $\sigma$-algebra corresponding to the random variables $(\text{pr}_x)_x \in Y$. By the 0-1 law of Hewitt-Savage given in Corollary 5.5, we see that $\mathcal{C}$ hence the $\sigma$-algebra $\mathcal{G}_\infty$ is $\mu$-trivial. Since $E[\eta_2 | \mathcal{G}_\infty]$ is measurable for $\mathcal{G}_\infty$, we see that we have $E[\eta_2 | \mathcal{G}_\infty] = E[E[\eta_2 | \mathcal{G}_\infty]] = E[\eta_2]$ almost surely. This proves that

$$\lim_{n \to \infty} E[\eta_1 \eta_2 | \sigma(\mathcal{F}_\Lambda, \mathcal{G}_n)] = \lim_{n \to \infty} 1_{\eta_1} E[\eta_2 | \mathcal{G}_n] = 1_{\eta_1} E[\eta_2 | \mathcal{G}_\infty] = 1_{\eta_1} E[\eta_2 | \mathcal{F}_\Lambda]$$

in $L^2(\mu)$ as desired. \hfill \Box

We next review the implication of the faithfully quantified condition.

**Lemma 5.7.** Assume that $(S, \phi)$ is faithfully quantified. Let $\Lambda \in \mathcal{I}$. Suppose $f : S^X \to \mathbb{R}$ is a local function which satisfies $\nabla e f = 0$ for any $e \in E_\Lambda$. Then, for any $\Lambda' \in \mathcal{I}$ such that $\Lambda \subset \Lambda'$ and $f \in C(S^{\Lambda'})$, $f$ is measurable with respect to $\sigma(\xi_\Lambda, \mathcal{F}_{\Lambda' \setminus \Lambda})$.

**Proof.** Let $\Lambda' \in \mathcal{I}$ satisfy $\Lambda \subset \Lambda'$ and $f \in C(S^{\Lambda'})$. Suppose we have $\eta, \eta' \in S^{\Lambda'}$ such that $\eta_x = \eta'_x$ for $x \in \Lambda' \setminus \Lambda$ and $\xi_\Lambda(\eta) = \xi_\Lambda(\eta')$. Since $(S, \phi)$ is faithfully quantified, there exists a path $\tilde{\varphi} = (\varphi_1, \ldots, \varphi_n)$ from $\eta|_\Lambda$ to $\eta'|_\Lambda$. We let $e = (e_1, \ldots, e_n)$ be the sequence of edges $e_i \in E_\Lambda$ defining the path $\tilde{\varphi}$. If we let $\eta_0 = \eta$ and $\eta_i = \eta_{e_{i-1}}^e$ for $i = 1, \ldots, n$, then $\varphi' = (\varphi_1', \ldots, \varphi_n')$ for $\varphi'_i = (\eta_i, \eta_{i+1})$ defines a path from $\eta$ to $\eta'$. Since $\nabla e f = 0$ for $e \in E_\Lambda$, we have $f(\eta_i) = f(\eta_{i+1})$ for any $i = 0, \ldots, n - 1$. This shows that $f(\eta) = f(\eta')$, hence we see that $f$ is measurable with respect to $\sigma(\xi_\Lambda, \mathcal{F}_{\Lambda' \setminus \Lambda})$. \hfill \Box

We now give the proof of Proposition 5.4.
Proof of Proposition 5.4. Since $f \in L^2(\mu)$, it is sufficient to prove that $\langle f, g \rangle = \langle \pi^A f, g \rangle$ for any local function $g \in C_{loc}(S^X)$. If we note that $\langle \pi^A f, g \rangle = \langle \pi^A f, \pi^A g \rangle = \langle f, \pi^A g \rangle$, then we see that it is sufficient to prove that

$$\langle f, g \rangle = \langle f, \pi^A g \rangle$$

for any local function $g$. We fix a local function $g$. Since $Y$ is a connected infinite graph, there exists an approximation $\Lambda_n \uparrow Y$ by finite connected sets. Since $f$ is $\mathcal{F}_{\Lambda \cup Y}$-measurable, we see that $f = \pi^{\Lambda \cup Y} f$ and

$$\langle f, g \rangle = \langle f, \pi^{\Lambda \cup Y} g \rangle.$$ (47)

Since $g$ is a local function, if we let $g \in C(S^W)$ for some $W \in \mathcal{F}$, then $(\Lambda \cup Y) \cap W = (\Lambda \cup \Lambda_n) \cap W$ for $n$ sufficiently large. Hence for such $n$, since $\mu$ is a product measure and $(Y \setminus \Lambda_n) \cap W = \emptyset$, we have

$$E[g|\mathcal{F}_{\Lambda \cup Y}] = E[g|\sigma(\mathcal{F}_{\Lambda \cup \Lambda_n}, \mathcal{F}_{Y \setminus \Lambda_n})] = E[g|\mathcal{F}_{\Lambda \cup \Lambda_n}]$$

and so

$$\langle f, \pi^{\Lambda \cup \Lambda_n} g \rangle = \langle f, \pi^{\Lambda \cup \Lambda_n} g \rangle.$$ (48)

Note that for $\Lambda_n$ and any $e \in E_{\Lambda_n}$, our condition implies that we have

$$\nabla_e (\pi^{\Lambda \cup \Lambda_n} f) = 0,$$

hence by Lemma 5.7, this shows that $\pi^{\Lambda \cup \Lambda_n} f$ is measurable for $\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})$. Since $\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n}) \subset \sigma(\mathcal{F}_{\Lambda \cup \Lambda_n})$, we have

$$\langle f, \pi^{\Lambda \cup \Lambda_n} g \rangle = \langle \pi^{\Lambda \cup \Lambda_n} f, \pi^{\Lambda \cup \Lambda_n} g \rangle = \langle E[f|\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})], \pi^{\Lambda \cup \Lambda_n} g \rangle = \langle E[f|\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})], E[g|\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})] \rangle = \langle f, E[g|\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})] \rangle.$$ (47)

Since $(Y \setminus \Lambda_n) \cap W = \emptyset$ and $\mu$ is product again, we have $E[g|\sigma(\mathcal{F}_{\Lambda}, \xi_{\Lambda_n})] = E[g|\sigma(\mathcal{F}_{\Lambda}, \mathcal{G}_n)]$ where $\mathcal{G}_n = \sigma(\xi_{\Lambda_n}, \mathcal{F}_{Y \setminus \Lambda_n})$ as in (46). Combining this equality with (47) and (48), we have

$$\langle f, g \rangle = \langle f, \pi^{\Lambda \cup Y} g \rangle = \langle f, \pi^{\Lambda \cup \Lambda_n} g \rangle = \langle f, E[g|\sigma(\mathcal{F}_{\Lambda}, \mathcal{G}_n)] \rangle.$$ (48)

Hence in order to prove our assertion, it is sufficient to prove that we have

$$\lim_{n \to \infty} E[g|\sigma(\mathcal{F}_{\Lambda}, \mathcal{G}_n)] = E[g|\mathcal{F}_{\Lambda}] =: \pi^A g$$

in $L^2(\mu)$. This is simply Lemma 5.6 applied to $\pi^{\Lambda \cup Y} g$, which is a local function in $C(S^{\Lambda \cup \Sigma})$ for $\Sigma := W \cap Y \subset Y$. □

5.3. Proof of the Main Theorem. We will now prove our main theorem. Let the notations be as in §5.2. In particular, we let $(X, E) = (\mathbb{Z}^d, \mathbb{E})$ be the Euclidean lattice with action of $G = \mathbb{Z}^d$ given by translation. We take an approximation $\Lambda_n \uparrow X$ of $X = \mathbb{Z}^d$ with tempered enlargement $\Sigma_n \uparrow X$ given in Lemma 5.18. We assume that $(S, \phi)$ is faithfully quantified, and that $((S, \phi), \nu)$ has a uniformly bounded spectral gap.
In what follows, for $\Lambda \in \mathcal{F}$, let $C^0(S^\Lambda) := \{f \in C(S^\Lambda) \mid E_\mu[f] = 0\}$ and $K^0(S^\Lambda) = C^0(S^\Lambda)/\text{Ker} \partial_\Lambda$. If we let $K^0_{\text{col}}(S^X) := \lim_{\rightarrow_\Lambda} K^0(S^\Lambda)$, then Proposition \ref{prop:isomorphism} gives the isomorphism

$$K^0_{\text{col}}(S^X_G) \cong Z^1_{\text{col}}(S^X_G).$$

Let $1_j \in \mathbb{Z}^d$ be the element with 1 in the $j$-th component and 0 in the other components. We may construct an $\mathbb{R}$-linear homomorphism

$$\delta : K^0_{\text{col}}(S^X_G) \rightarrow \bigoplus_{j=1}^d H^0_{\text{col}}(S^X), \quad f \mapsto ((1 - \tau_1) f).$$

By construction, any element of $C^0_{\text{col}}(S^X_G)$ is in $\text{Ker} \delta$. The map $\delta$ is well-defined on $K^0_{\text{col}}(S^X_G)$ since $H^0_{\text{col}}(S^X) \subset C^0_{\text{col}}(S^X_G)$ by \cite{2} Lemma 4.7]. We have the following.

**Lemma 5.8.** The map $\delta$ of (49) induces an isomorphism

$$\mathcal{H} \cong \bigoplus_{j=1}^d \text{Conv}^\phi(S) \subset H^0_{\text{col}}(S^X).$$

**Proof.** By Lemma \ref{lem:image} we have $(1 - \tau_1)\Psi = \sum_{x \in X} \xi_x(i)$, which shows that $\delta$ gives an inverse isomorphism of the map mapping $\sum_{j=1}^d \sum_{i=1}^{c_\phi} a_{ij} \xi_x(i)$ in $\bigoplus_{j=1}^d \text{Conv}^\phi(S)$ to $\sum_{j=1}^d \sum_{i=1}^{c_\phi} a_{ij} \Psi(i).$ This shows that (50) is in fact an isomorphism.

We may now prove the following.

**Lemma 5.9.** Let $\mathcal{E}_\mu = \partial(C^0_{\text{unif}}(S^X_G)).$ Then we have

$$\mathcal{E}_\mu \cap \partial \mathcal{H} = \{0\}.$$

**Proof.** Let $f \in \mathcal{H}$ such that $\partial f \in \mathcal{E}_\mu$. By definition of the closure, there exists a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ in $C^0_{\text{unif}}(S^X_G)$ such that $\lim_{n \rightarrow \infty} \partial f_n = \partial f$ in $Z^1_{L^2}(S^X_G)$. This implies that for any $\Lambda \in \mathcal{F}$, we have

$$\lim_{n \rightarrow \infty} \partial_\Lambda f_n^\Lambda = \lim_{n \rightarrow \infty} (\partial f_n)^\Lambda = (\partial f)^\Lambda = \partial_\Lambda f^\Lambda$$

in $C^1(S^\Lambda)$. Lemma \ref{lem:limit} implies that $\lim_{n \rightarrow \infty} \bar{f}_n^\Lambda = \bar{f}_G^\Lambda$ in $K^0(S^\Lambda)$, where the bar denotes the image of a function in $K^0(S^\Lambda)$. By taking the images of $\bar{f}_n$ and $\bar{f}$ with respect to the map $\delta$, we see that $\delta(\bar{f}_n) = 0$ since $f_n$ is invariant as a function with respect to the action of $G$. Since $\delta(\bar{f}_G)^\Lambda = \lim_{n \rightarrow \infty} \delta(\bar{f}_n)^\Lambda = 0$ for any $\Lambda \in \mathcal{F}$, we see that $\delta(\bar{f}) = 0$. By Lemma \ref{lem:image} the homomorphism $\delta$ is injective on $\mathcal{H}$. Hence $f \in \mathcal{H}$ and $\delta(\bar{f}) = 0$ implies that $f = 0$, which proves that $\partial f = 0$ as desired.

We may now prove Theorem \ref{thm:characterization}.

**Proof of Theorem** \ref{thm:characterization} Let $\omega = (\omega^\Lambda) \in \mathcal{E}_\mu = Z^1_{L^2}(S^X_G) = C^1_{L^2}(S^X) \cap Z^1_{\text{col}}(S^X_G).$ We let $\Psi_n$ be the shift invariant uniformly local function of Definition \ref{def:shift_invariant} Then by Proposition \ref{prop:isomorphism} and
Proposition 4.11, we see that $\Psi_{\mu}$ converges weakly to $\omega \Psi := \omega + \omega^\dagger$ in $C^\mu$. Since weak closure of a subspace of a Hilbert space coincides with its strong closure, this implies that

$$\omega \Psi = \omega + \omega^\dagger \in C^\mu = \partial(C^0_{\text{unif}}(S^X)^G).$$

By Proposition 5.3, the form $\omega^\dagger$ is a local form in $C$. By Theorem 3.10, there exists $\omega' \in E = \partial(C^0_{\text{unif}}(S^X)^G) \subset C^\mu$ and $\omega^\dagger \in \partial H$ such that

$$\omega^\dagger = \omega' + \omega^\dagger.$$

Note that $(\omega \Psi - \omega') \in C^\mu$. By Lemma 5.9, we see that $\omega = (\omega \Psi - \omega') + (-\omega^\dagger)$ is a direct sum, hence this gives a decomposition

$$\omega = (\omega \Psi - \omega') + (-\omega^\dagger) \in C^\mu \oplus \partial H$$

as desired. □

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*Email address: bannai@math.keio.ac.jp*

*Email address: sasada@ms.u-tokyo.ac.jp*

*Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama 223-8522, Japan.*

*Department of Mathematics, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 606-8502, Japan.*

*Mathematical Science Team, RIKEN Center for Advanced Intelligence Project (AIP), 1-4-1 Nihonbashi, Chuo-ku, Tokyo 103-0027, Japan.*