I. INTRODUCTION

Much of the extensive research in the last few years has focused on topological phases of matter and their classifications. A prime example is the one-dimensional topological superconducting phase of paired fermions, which is characterized by Majorana zero modes at the edges. Such zero modes actually resemble the ones found in the cores of vortices in two-dimensional topological superconductors, and have been shown useful for quantum information processing. However, it has been argued that one-dimensional fermion systems with interactions and no extra symmetry than the intrinsic fermion parity can only realize two topologically distinct phases. In order to search for a universal quantum computation platform and fully understand topological excitations in strongly interacting electron systems, more exotic parafermion excitations have been investigated in an effectively one-dimensional system, which exists at the edges of a two-dimensional fractionalized topological state and cannot be realized in a strictly one-dimensional system.

For various correlated low-dimensional gapped systems, it has been known that matrix product states (MPS) and their high dimensional generalizations, tensor network states or projective entangled pair states (PEPS), have been proven increasingly successful. The framework of MPS and PEPS naturally provides an efficient method to classify topologically ordered phases, symmetry protected topological phases, and the long-range ordered phases with spontaneous symmetry breaking. However, it is not straight forward to extend the MPS representation to the one-dimensional parafermion systems. Recently, the fermionic MPS have been successfully constructed by using the language of super vector space, and all possible topological phases with additional symmetries in terms of Majorana fermions have been classified within the matrix product representation. By generalizing the concepts of fermion parity and associated Fock space, the present authors have proposed a general framework to construct the MPS of \( \mathbb{Z}_p \) parafermions in the Fock representation, and the corresponding parent Hamiltonians have been also derived. Therefore, the road has been cleared to classify all possible \( \mathbb{Z}_p \) parafermion gapped phases within the framework of the MPS.

In this paper, we first review the Fock space of parafermions and then construct the parafermionic MPS. From the analysis of irreducibility of these parafermionic MPS, we provide the complete classification of all possible gapped phases without extra symmetry, compared to the previous classification based on the edge fractionalization. More importantly, we find that the various irreducible forms for local matrices of MPS spanned different kinds of graded algebras characterize distinct parafermionic gapped phases. The local matrices of MPS describing topological phases span the non-trivial graded algebras with characteristic graded centers, resulting in the degeneracies of the full transfer matrix spectra and entanglement spectra in the thermodynamic limit. Meanwhile, the spontaneous symmetry breaking phases correspond to the trivial semisimple \( \mathbb{Z}_p/n \) graded algebras (\( n \) is a divisor of \( p \)), which can be further reduced to the trivial simple \( \mathbb{Z}_p/n \) graded algebras. The trivial phase corresponds to trivial simple \( \mathbb{Z}_p \) graded algebra without non-trivial center. Furthermore, we also found that the topological order is closely related to the non-trivial center of the graded algebra, giving rise to the degeneracy of the full transfer matrix spectrum and the existence of parafermion edge zero modes.

In Sec. II, we briefly review the Fock space of parafermions and present the construction of the \( \mathbb{Z}_p \) parafermionic MPS. Then, we outline the general classification framework and the detailed classifications for \( \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6, \) and \( \mathbb{Z}_p \) parafermionic gapped phases are considered separately in Sec. III. In Sec. IV, the general irreducible forms for various phases of the \( \mathbb{Z}_p \) parafermions are summarized, and the topological order in the form of
MPS is discussed. Conclusion and outlook are given in Sec. V. The related concepts used in the classification scheme are listed in the appendix.

II. PARAFERMIonic MPS

A. Fock space of parafermions

In order to keep our discussion self-contained, we first briefly review the parafermions and their Fock space. It has been known that, from the $\mathbb{Z}_p$ spin degrees of freedom of the clock models, the parafermions are defined by a generalized Jordan-Wigner transformation as

$$\chi_{2l-1} = \left( \prod_{k \leq l} \tau_k \right) \sigma_l, \quad \chi_{2l} = -e^{i\pi/p} \left( \prod_{k \leq l} \tau_k \right) \sigma_l,$$

where the $\mathbb{Z}_p$ spin operators satisfy the following relations

$$\sigma_i^p = \tau_i^p = 1, \quad \sigma_i \tau_m = \omega_p^{\delta_i_m} \tau_m \sigma_i, \quad (2)$$

with $\omega_p = e^{i2\pi/p}$. So the algebras of the parafermions are

$$\chi_i^p = 1, \quad \chi_i^{p-1} = \chi_i^1, \quad \chi_i \chi_m = \omega_p \chi_m \chi_i \quad (3)$$

for $l < m$. These are the generalized Clifford $\mathbb{Z}_p$ graded algebras, and the parafermions are referred to as the Weyl parafermions, because it was first introduced by Weyl\cite{29}. It has been noticed that the second quantized description of the Weyl parafermions is given by the Fock parafermions\cite{30}. So the basis of $\mathbb{Z}_p$ Fock parafermions can be assumed as $| i_1 i_2 \cdots i_L \rangle$, where $i_1, i_2, \cdots, i_L \in \mathbb{Z}_p \equiv \{0, 1, 2, \cdots, p-1\}$ are the respective occupation numbers of the single particle orbitals. The general structure of the Fock space is defined by

$$\mathbb{V}_F = \bigoplus_{M=0}^{L(p-1)} \text{Span} \left\{ |i_1 i_2 \cdots i_L\rangle, \sum_{l=1}^L i_l = M \right\} \quad (4)$$

In the following the abbreviated notation $|i_i\rangle = |0 \cdots i_i \cdots 0\rangle$ denotes the single-particle states. In order to encode the parafermion statistics into the Fock space, the graded tensor product $\otimes_g$ building many-body states is introduced as

$$\langle i_1 i_2 \cdots i_L | = |i_1\rangle \otimes_g |i_2\rangle \cdots \otimes_g |i_L\rangle, \quad |i_1 i_2 \cdots i_L | = |i_1\rangle \otimes_g |i_2\rangle \cdots \otimes_g |i_L\rangle \langle i_1 | \langle i_2 | \cdots \langle i_L |, \quad (5)$$

which describes the graded structure of Hilbert space mathematically. The crucial ingredient of the graded tensor product is the following isomorphism mapping $\mathcal{F}$:

$$\mathcal{F}(|i_i\rangle \otimes_g |j_m\rangle) = \omega_p^{\delta_{i_m}} |j_m\rangle \otimes_g |i_i\rangle, \quad (6)$$

$$\mathcal{F}(|i_i\rangle \otimes_g |j_m\rangle) = \omega_p^{\delta_{i_m}} |j_m\rangle \otimes_g |i_i\rangle, \quad (6)$$

for $l < m$. The isomorphism $\mathcal{F}$ exchanges two nearby local Fock states, and the whole Fock space is a graded vector space, which is a generalization of super vector space of the fermions\cite{23}. Thus parafermion statistics is encoded into the Fock space by the isomorphism, which becomes crucial for the construction of the MPS wave functions.

Since a contraction is necessary for tensor networks, the homomorphism $\mathcal{C}$ has to be defined via a mapping $\mathbb{V}_F \otimes_g \mathbb{V}_F \rightarrow \mathbb{C}$:

$$\mathcal{C}(\langle i_i | \otimes_g |j_j\rangle) = \langle i_i | \otimes_g \delta_{i_i j_j}, \quad (7)$$

which is nothing but the inner product and orthonormal.

From the above propositions, the $p$-exclusion principle can be derived as

$$(|i_i = 1\rangle)^{\otimes_p} \equiv |i_i = p\rangle = 0. \quad (8)$$

So the dimension of the Fock space of parafermions is determined as $p^L$. The creation and annihilation operators of Fock space can also be introduced, and their commutation relations have been derived\cite{30}. Furthermore, the local charge operator can be defined by $Q_l = -e^{i\pi/p} \chi_{2l-1} \chi_{2l}$, and the global one is accordingly given by $Q = \prod_l Q_l$, determining the charge of the Fock basis as\cite{235}

$$Q | I \rangle = Q | i_1 i_2 \cdots i_L \rangle = \omega_p^{\sum_{l=1}^L i_l} | i_1 i_2 \cdots i_L \rangle. \quad (9)$$

Then the charge of the Fock state $| I \rangle$ can be calculated as $| I \rangle = (\sum_{i=1}^L i_i) \mod p$, while the charge of the bar $| \bar{I} \rangle$ is given by $-| I \rangle$. It should be emphasized that only the many-body states which are superpositions of the Fock states with the same charge have well-defined charges.

B. MPS for $\mathbb{Z}_p$ parafermions

To construct the MPS with physical degrees of freedom of dimension $d$, we have to introduce two auxiliary virtual degrees of freedom of dimension $D$. Two virtual degrees of freedom form a maximally entangled state on the neighboring sites, while the virtual degrees of freedom on the same site are mapped to the physical degree of freedom. In the Fock space of parafermions, we can write down the local tensor as

$$A[i] = \sum_{\alpha \beta l} A[i]^{[\alpha \beta l]} (\alpha_l \otimes_g |i_i\rangle \otimes_g (\beta_{l+1}|, \quad (10)$$

where $A[i] \in \mathbb{V}_I \otimes_g \mathbb{H}_l \otimes_g \mathbb{V}_{l+1}^*$, $l$ denotes the site index, $|i_i\rangle$ stands for the physical state with the charge $|i_i\rangle \in \mathbb{Z}_p$, and $(\alpha_l), (\beta_{l+1})$ stand for the virtual states with the charges $(\alpha_l), (\beta_{l+1}) \in \mathbb{Z}_p$, respectively.

Since the charge symmetry acts locally on the tensor networks, we impose the constraint that all the local tensors $A[i]$ must have well-defined charges. This enforces that the local matrices $A[i]^{[\alpha \beta l]}$ as the components of local tensors have well-defined charges as well.
Then we choose the simplest convention that all local tensors $A^{[i]}$ are charge-0, so that the different orders of the tensors $A^{[i]}$ do not induce any phases and the total charges of the tensor networks are independent of the system size\(^{25}\). The charges of the local matrices $A^{[i]} = \sum_{\alpha} a^{[i]}_\alpha \otimes g(\beta_{i+1})$ are given by $(\alpha - \beta) \mod p$, shown in Table.\(^1\) To ensure that the local tensors are charge-0, the matrices $A^{[i]}$ must have the following block structures under the basis with the well-defined $\mathbb{Z}_p$ charges:

\[
A^{[i]} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = 0;
\]

\[
A^{[i]} = \begin{bmatrix}
0 & 0 & \cdots & a^{[i]}_0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = 1;
\]

\[
A^{[i]} = \begin{bmatrix}
0 & 0 & \cdots & a^{[i]}_0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = p-1,
\]

where $a^{[i]}_r$ with $r \in \mathbb{Z}_p$ are matrices in the sub-blocks with smaller virtual dimensions. Actually the structures of the local matrices are determined by the fact that the Fock space is a graded vector space. Moreover, the charges of the local matrices can be revealed by the representation of charge operator $Q_p = \text{diag}(1, \omega_p, \omega_p^2, \cdots, \omega_p^{p-1})$ as

\[
(Q_p \otimes 1)^{-1} A^{[i]} (Q_p \otimes 1) = \omega_p^{|i|} A^{[i]}.
\]

For the later discussion, it is useful to introduce another $p \times p$ matrix

\[
Y_p = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

as the regular representation of generator of $\mathbb{Z}_p$ symmetry. It will be frequently used and played a significant role in the following classification.

So the parafermionic MPS can be constructed by taking graded tensor product of local tensors and contracting the virtual bonds between nearby local tensors with the homomorphism $C$. The contraction does not affect the charges of the tensor networks, because the charge of $|\alpha_i \otimes g(\alpha_j)|$ is zero. Therefore, the general parafermionic MPS is expressed as

\[
|\psi\rangle = C(C_a \otimes g A^{[1]} \otimes g A^{[2]} \otimes g \cdots \otimes g A^{[L]}) = \sum_{i_1, \cdots, i_L} (C_t^T A^{[i_1]} \cdots A^{[i_L]}) |i_1, \cdots, i_L\rangle,
\]

where $C_a = \sum_{\gamma_0} C_{a, \gamma_0} (\gamma_1 \otimes g | \delta_L)$ is the closure tensor and different choices of $C_a$ just result in the different charges of the closed MPS wave function\(^{25}\). It should be emphasized that, unlike the Majorana fermion chains, the periodic boundary condition can not reconcile with the algebras of parafermions, so the periodic boundary condition for parafermion chains does not exist. How to define the Hamiltonian for closed boundary conditions has been specifically discussed\(^{25,31}\).

### III. CLASSIFICATION OF $\mathbb{Z}_p$ PARAFERMIONIC MPS WITH IRREDUCIBILITY

Irreducibility is the most important property for a general MPS, because the irreducible MPS determines the major physical properties of the system. Irreducible forms of bosonic and fermionic MPS have been constructed, the concept of irreducibility of fermionic MPS is quite different from that of bosonic MPS. For the fermionic MPS, there are two types of irreducible fermionic MPS\(^{25}\), one is called even type with the local matrices with the block structures:

\[
A^{[i]} = \begin{bmatrix}
a^{[i]}_0 & 0 & 0 & \cdots & 0 \\
0 & a^{[i]}_1 & 0 & \cdots & 0 \\
0 & 0 & a^{[i]} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = 1,
\]

where the sub-block matrices can not be equal under gauge transformations. These irreducible matrices span the even type simple $\mathbb{Z}_2$ graded matrix algebra with the center consisting of multiples of the identity. So it is as simple as the ungraded algebra. While the other is called the odd type which can be gauge transformed into

\[
A^{[i]} = \begin{bmatrix}
a^{[i]}_0 & 0 & 0 & \cdots & 0 \\
0 & a^{[i]}_1 & 0 & \cdots & 0 \\
0 & 0 & a^{[i]} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a^{[i]}_{p-1}
\end{bmatrix}, \quad |i| = 1.
\]
Then we can be further expressed into a more compact form:

\[ A^{[i]} = Y_2^{[i]} \otimes a^{[i]}, \ |i| = 0, 1, \]  

where \( Y_2 \) has been defined in Eq. (13). Thus \( A^{[i]} \) of irreducible MPS span an odd type simple \( Z_2 \) graded algebra with the center consisting of multiples of \( \mathbb{1} \) and \( Y_2 \), characterizing the non-trivial topological phase with unpaired Majorana zero modes at the edges of one-dimensional systems. These two types of irreducible MPS represent the \( Z_2 \) classification of one-dimensional interacting fermionic systems.

Expanding the analysis of irreducibility, we can derive all irreducible \( Z_p \) parafermionic MPS, and the different algebras spanned by the local matrices correspond to distinct gapped phases. Therefore, we can establish a complete classification of all one-dimensional parafermionic gapped phases, including the topological phases and conventional spontaneous symmetry breaking phases. The symmetry protected topological phases are not included, because no extra symmetry than the parafermionic charge is involved and \( Z_p \) symmetry is not sufficient to support symmetry protected topological phases. The general method is as follows. We first assume that all \( A^{[i]} \) have an irreducible invariant subspace with the corresponding orthogonal projector \( P_0 \). Then we analyze all the commutation relations between the operators \( P_0 \) and \( Q^r_p \), where \( r \) is a positive integer. \( P_0 \) and \( Q^r_p \) generate a space which contains all \( Z_p \) charge sectors. Finally, by using the invariant subspace projectors containing all charge sectors, we can determine the irreducible structures of \( A^{[i]} \) and the corresponding algebras spanned by them. The different algebras associate with different gapped phases. In the following, several important cases are carried out in detail.

### A. Irreducibility of \( Z_3 \) parafermion MPS

Now we first derive two types of irreducible MPS for \( Z_3 \) parafermion chains. Without the lost of generality, we assume that there is an irreducible invariant subspace projector \( P_0 \) of local matrices \( A^{[i]} \), i.e.

\[ A^{[i]} P_0 = P_0 A^{[i]} P_0. \]  

Because \( A^{[i]} \) have a definite charge, \( Q_3^{-1} A^{[i]} Q_3 = \omega^{[i]} A^{[i]} \), where \( Q_3 \) is the \( Z_3 \) charge matrix, we can further derive

\[ A^{[i]} P_1 = P_1 A^{[i]} P_1, A^{[i]} P_2 = P_2 A^{[i]} P_2, \]

where \( P_1 = Q_3 P_0 Q_3^{-1} \) and \( P_2 = Q_3^{-1} P_0 Q_3 \) are also invariant subspace projectors. Since \( P_0 \) is already associated with an irreducible invariant space, \( P_0, P_1 \) and \( P_2 \) must be either the same or mutually orthogonal, otherwise it will contradict with the fact that \( P_0 \) is already associated with an irreducible invariant space. So there are two different situations, we discuss them separately.

1. \( [P_0, Q_3] = 0 \)

It can be simply determined that \( P_0 = P_1 = P_2 \), and \( P_0 \) contains all three charge sectors, indicating that \( A^{[i]} \) in \( P_0 \) irreducible invariant subspace preserve the \( Z_3 \) charge symmetry. Thus all \( A^{[i]} \) in this invariant subspace will have the initial structures shown in Eq. (11) and span a trivial simple \( Z_3 \) graded algebra.

2. \( [P_0, Q_3] \neq 0 \)

In this situation, \( P_0, P_1 \) and \( P_2 \) are mutually orthogonal projectors, the corresponding invariant subspaces do not contain all \( Z_3 \) charge sectors. For the parafermionic MPS, the \( Z_3 \) charge symmetry can never be broken and the MPS can not be reduced, since the invariant spaces do not contain all charge sectors. The reduced matrices also break the \( Z_3 \) graded structures of local matrices and no longer span a \( Z_3 \) graded algebra. Thus, the concept of irreducibility should be reformulated. Notice that \( [P_0 + P_1 + P_2, Q_3] = 0 \), and the total invariant space is the complete, leading to \( P_0 + P_1 + P_2 = \mathbb{1} \). The idempotency requires \( P_0^2 = P_0, P_1^2 = P_1 \) and \( P_2^2 = P_2 \). From these constraints, the invariant subspace projectors can be derived as

\[ P_0 = \frac{1}{3} \left[ \begin{array}{c c} \mathbb{1} & U_1 \ U_1^\dagger \ U_2 \\ U_1^\dagger \ U_1 & \mathbb{1} & U_2 \end{array} \right], \]

where \( U_1 \) and \( U_2 \) are unitary block matrices with the same dimensions. Since \( A^{[i]} P_j = P_j A^{[i]} P_j \), we can obtain

\[ A^{[i]} = \begin{cases} \begin{array}{ccc} a_0^{[i]} & 0 & 0 \\ 0 & U_1 a_0^{[i]} U_1 & 0 \\ 0 & 0 & U_2 a_0^{[i]} U_2 \end{array} \end{cases}, & |i| = 0; \]

\[ A^{[i]} = \begin{cases} \begin{array}{ccc} 0 & a_0^{[i]} & 0 \\ 0 & 0 & U_1 a_0^{[i]} U_2 \\ U_2 a_0^{[i]} U_1 & 0 & 0 \end{array} \end{cases}, & |i| = 1; \]

\[ A^{[i]} = \begin{cases} \begin{array}{ccc} 0 & 0 & a_0^{[i]} \\ 0 & U_1 a_0^{[i]} U_1 & 0 \\ U_2 a_0^{[i]} U_2 & 0 & 0 \end{array} \end{cases}, & |i| = 2. \]  

The gauge transformation \( G = \mathbb{1} \oplus U_1 \oplus (U_1 U_2) \) can be used to rewrite them in the standard forms. After substituting \( a_0^{[i]} \) for \( a_0^{[i]} \) if \( |i| = 0 \), \( a_0 U_1^{[i]} \) if \( |i| = 1 \), and \( a_0 U_2^{[i]} U_1^{[i]} \) if \( |i| = 2 \), we can express the local matrices into more compact form:

\[ A^{[i]} = Y_3^{[i]} \otimes a^{[i]}. \]

To obtain the irreducible MPS, it should be guaranteed that \( A^{[i]} \) must have no irreducible invariant subspace commuting with \( Q_3 \). If there was such an invariant subspace corresponding to the projector \( P \), it should have the form \( \tilde{P} = \text{diag}(P_0, P_1, P_2) \). According to \( A^{[i]} \tilde{P} = \tilde{P} A^{[i]} P \), it further satisfies

\[ a^{[i]} \tilde{P}_0 = \tilde{P}_0 a^{[i]} P_0, a^{[i]} \tilde{P}_1 = P_1 a^{[i]} P_1, a^{[i]} \tilde{P}_2 = P_2 a^{[i]} P_2, \]

\[ (21) \]
for $\forall |i| = 0$. To exclude such a situation, we must impose the necessary condition that the “charge-0” subalgebra spanned by all $\{a^{[i]}| \cdots a^{[i]}\}$ with $\forall p \in \mathbb{N}$ and $\sum_{i=1}^{p} |i| = 0$ is simple. In the following, when we mention the “charge-0” subalgebra, it has the same definition, but it is the simple matrix algebra with different dimension.

So the local matrices $A[^i]$ are irreducible if $A[^i]$ can be gauge transformed into $Y[^i]_3 \otimes a[^i]$ and the “charge-0” sub-algebra is a simple matrix algebra. These conditions imply that $A[^i]$ span a non-trivial simple $\mathbb{Z}_3$ graded algebra. The graded center consists of multiples of $1, Y_3$, and $Y_3^2$. Taking the trivial simple $\mathbb{Z}_3$ graded algebra of MPS into consideration, we obtain the conclusion that a $\mathbb{Z}_3$ parafermion MPS is irreducible iff $A[^i]$ span a simple $\mathbb{Z}_3$ graded algebra.

**B. Topological order in $\mathbb{Z}_3$ parafermion MPS**

The characteristic properties of the parafermionic MPS can be found in the transfer matrix

$$E = \sum_i A[^i] \otimes \bar{A}[^i].$$

(22)

For the trivial algebra MPS, the irreducible matrices $A[^i]$ span a simple algebra. The corresponding transfer matrix forms a completely positive map\(^{23}\), whose eigenvalue spectrum is real and non-negative, and the largest eigenvalue is non-degenerate.

However, for the non-trivial type algebra MPS, the transfer matrix can be expressed as

$$E = \sum_i \left[ Y[^i]_3 \otimes a[^i] \right] \otimes \left[ Y[^i]_3 \otimes \bar{a}[^i] \right].$$

(23)

By supposing $\bar{\sigma}_R$ as the right eigenvector of the sub-block transfer matrix $\overline{E} = \sum_i a[^i] \otimes \bar{a}[^i]$ with the real eigenvalue $\lambda$, i.e., $\sum_i a[^i] \bar{\sigma}_R a[^i] = \lambda \bar{\sigma}_R$, it can be easily verified that $\sigma[^j]_R = Y[^j]_3 \otimes \bar{\sigma}_R$ with $|j| = 0, 1, 2$ are three eigenvectors of the transfer matrix $T$ with the same eigenvalue $\lambda$. It can be further proved that all eigenvalues of the transfer matrix $E$ have at least three-fold degeneracy. The details are given in the Sec. IV.B. The largest eigenvalue and the corresponding eigenvectors stem from the sub-block transfer matrix $\overline{E}$, so the three-fold degeneracy of the transfer matrix spectrum reflects the existence of unpaired parafermion edge zero modes, characterizing the topological order in one dimension. In contrast, the largest eigenvalue of the transfer matrix of a symmetry protected topological state is non-degenerate.

Moreover, according to the holographic principle, the left and right dominant eigenvectors of the transfer operator determine the reduced density matrix in the thermodynamic limit\(^{24,13}\). We can study the entanglement spectrum via a bipartition of the parafermionic MPS. Here we merely consider the non-trivial MPS. Supposing that the left and right dominant eigenvectors of the sub-block transfer matrix $\overline{E}$ are given by $\tilde{\sigma}_L$ and $\tilde{\sigma}_R$, the transfer matrix $E$ has three left and three right dominant eigenvectors $\sigma[^j]_L = Y[^j]_3 \otimes \tilde{\sigma}_L$ and $\sigma[^j]_R = Y[^j]_3 \otimes \tilde{\sigma}_R$, respectively, displayed in Fig. 1(a) and Fig. 1(b). Notice that the tensors $v_R \otimes v_R^\dagger$ and $v_L \otimes v_L^\dagger$, fixing the double layer tensor network must be charge zero, where $v_R$ and $v_L$ are the right and left boundary vectors of MPS, respectively, as shown in Fig. 1(c). The entanglement Hamiltonian $H_E$ in the thermodynamic limit is thus determined by the left and right charge zero fixed-points as

$$e^{H_E} = (1 \otimes \tilde{\sigma}_L^\dagger) (1 \otimes \tilde{\sigma}_R^\dagger) = 1 \otimes \tilde{\sigma}_L^\dagger \tilde{\sigma}_R^\dagger.$$  

(24)

Hence the entanglement spectrum have at least three-fold degeneracy, fully determined by the structure of $A[^i]$.

To summarize, there are only two types of irreducible $\mathbb{Z}_3$ parafermionic MPS. One type corresponds to the local matrices spanning a trivial simple $\mathbb{Z}_3$ graded algebra, so the dominant eigenvector of the transfer matrix is unique and the entanglement spectrum is not necessarily degenerate. The other type corresponds to the non-trivial simple $\mathbb{Z}_3$ graded algebra spanned by local matrices. The full transfer matrix spectrum has at least three-fold degenerate eigenvalues, and so does the entanglement spectrum in the thermodynamic limit. The degeneracy of the transfer matrix spectrum implies the existence of the unpaired $\mathbb{Z}_3$ parafermion zero modes. Actually such analysis can be generalized to all $\mathbb{Z}_p$ parafermionic MPS for topological phases, and the necessary degeneracy of the transfer matrix spectrum as well as the degeneracy of the entanglement spectrum in thermodynamic limit just depend on the structure of $A[^i]$.

**C. Irreducibility of $\mathbb{Z}_4$ parafermion MPS**

Beside the topological and the trivial phases, there is a spontaneous symmetry breaking phase in the classifi-
for local matrices contains four different charge sectors, the forms of the discussion is divided into three different situations.

1. \( [P_0, Q_4] = 0 \)

Since the irreducible invariant subspace projector \( P_0 \) contains four different charge sectors, the forms of the local matrices \( A^{[i]} \) are given by Eq. (14), and the irreducible \( A^{[i]} \) span a trivial simple \( Z_4 \) graded algebra. Both the transfer matrix spectrum and the entanglement spectrum are not necessarily degenerate.

2. \( [P_0, Q_4] \neq 0 \) but \( [P_0, Q_4^2] = 0 \)

In this case we have only two orthogonal projectors \( P_0 \) and \( P_1 \) for two irreducible invariant subspaces. According to \( Q_4 P_0 Q_4^{-1} + P_1 = \mathbb{I} \) and \( P_0^2 = P_0 \), \( P_0 \) can be determined as

\[
P_0 = \frac{1}{2} \begin{bmatrix}
1 & 0 & U_1 & 0 \\
0 & 1 & 0 & U_2 \\
U_1^\dagger & 0 & 1 & 0 \\
0 & U_2^\dagger & 0 & 1
\end{bmatrix}
\]

where \( U_1 \) and \( U_2 \) are unitary block matrices and both the charge-0 and charge-1 sectors have the same dimension as the charge-2 and charge-3 sectors. The relation \( A^{[i]} P_j = P_j A^{[i]} P_1 \) and the gauge transformation \( G = \mathbb{I} \oplus \mathbb{I} \oplus U_1 \oplus U_2 \) lead to

\[
A^{[i]} = \text{diag} \left( a_0^{[i]}, a_1^{[i]}, a_0^{[i]}, a_1^{[i]} \right) \times \left( Y_4^{[i]} \otimes \mathbb{I} \right)
\]

where we have substituted \( a_1^{[i]} \) for \( a_1^{[i]} U_1^\dagger \) if \( |i| = 1 \), \( a_0^{[i]} \) for \( a_0^{[i]} U_1^\dagger \), \( a_1^{[i]} \) for \( a_1^{[i]} U_2^\dagger \) if \( |i| = 2 \), and \( a_0^{[i]} \) for \( a_0^{[i]} U_2^\dagger \) if \( |i| = 3 \). It is further required that the dimensions of the four charge sectors are the same. Then, by permuting the order of basis, \( (0, 1, 2, 3) \rightarrow (0, 2, 1, 3) \), these local matrices can display an even \( Z_2 \) graded structure:

\[
A^{[0]} = \begin{bmatrix}
a_0^{[0]} & 0 & 0 & 0 \\
0 & a_0^{[0]} & 0 & 0 \\
0 & 0 & a_1^{[0]} & 0 \\
0 & 0 & 0 & a_1^{[0]}
\end{bmatrix}
\]

\[
A^{[1]} = \begin{bmatrix}
0 & a_0^{[1]} & 0 & 0 \\
0 & 0 & a_0^{[1]} & 0 \\
0 & 0 & 0 & a_1^{[1]} \\
0 & 0 & 0 & a_1^{[1]}
\end{bmatrix}
\]

\[
A^{[2]} = \begin{bmatrix}
0 & a_0^{[2]} & 0 & 0 \\
0 & 0 & a_0^{[2]} & 0 \\
0 & 0 & 0 & a_1^{[2]} \\
0 & 0 & 0 & a_1^{[2]}
\end{bmatrix}
\]

\[
A^{[3]} = \begin{bmatrix}
0 & a_0^{[3]} & 0 & 0 \\
0 & 0 & a_0^{[3]} & 0 \\
0 & 0 & 0 & a_1^{[3]} \\
0 & 0 & 0 & a_1^{[3]}
\end{bmatrix}
\]

Since we have \( [P_0, Q_4^2] = 0 \), the invariant subspace \( P_0 \) contains the even and odd \( Z_2 \) parity sectors. In addition, \( A^{[i]} \) must not contain an invariant subspace whose projector commutes with the operator \( Q_4 \). Similar to the \( Z_3 \) case, it is necessary that the “charge-zero” subalgebras of matrix algebras spanned by \( a_0^{[i]} \) and \( a_1^{[i]} \) are simple, respectively. Then the local matrices \( A^{[i]} \) span a simple algebra in the \( Z_4 \) graded sense, but it is semisimple in the \( Z_2 \) graded sense. Because the semisimple algebra can split into irreducible ones, we can reduce the local matrices and break the \( Z_4 \) symmetry down to \( Z_2 \) symmetry. To make it explicit, another gauge transformation

\[
G' = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & 1 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

changes the local matrices into the canonical form:

\[
A^{[i]} = \omega_4^{[i]} d^{[i]} \oplus d^{[i]},
\]

\[
d^{[0]} = \begin{bmatrix}
a_0^{[0]} \\
0 \\
ad_0^{[0]}
\end{bmatrix}, |i| = 0, 2;
\]

\[
d^{[1]} = \begin{bmatrix}
a_1^{[1]} \\
0 \\
ad_1^{[1]}
\end{bmatrix}, |i| = 1, 3.
\]

The block diagonal form of \( A^{[i]} \) represents a spontaneous symmetry breaking phase. One may question why it is not a topological phase? The answer is that it is impossible to transform \( A^{[i]} \) into \( Y_4^{[i]} \otimes d^{[i]} \), which is required in the topological phase. The reducing process actually mixes the charge-0 and charge-2 sectors, as well as the charge-1 and charge-3 sectors. In this sense, the \( Z_4 \) symmetry breaks down to the \( Z_2 \) symmetry. After reducing, the sub-block matrices \( d^{[i]} \) span an even type simple \( Z_2 \) graded algebra, shown in Eq. (15). So there is no Majorana zero edge modes. The two degenerate ground states are purely resulted from the spontaneous symmetry breaking, and can be transformed into each other via the representation of the \( Z_4 \) charge symmetry generator \( U_4 = \text{diag}(1, \omega_4, \omega_4^2, \omega_4^3) \). Therefore, the two parts in the direct sum are connected by \( \sum_i U_{ij} d^{[j]} = \omega_4^{[i]} d^{[i]} \). Moreover, the parafermions \( \chi_i \) can form bosons \( \chi_i^2 \) and its anti-particles \( \overline{\chi}_i^3 \). But the Majorana fermions can not be obtained from \( Z_4 \) parafermions, so there is no non-trivial topological phase with Majorana edge zero modes in this classification.

3. \( [P_0, Q_4] \neq 0 \) and \( [P, Q_4^2] \neq 0 \)

Following the general procedure of deriving the irreducible MPS, the standard form of the local matrices can be written as

\[
A^{[i]} = Y_4^{[i]} \otimes a^{[i]}, |i| = 0, 1, 2, 3,
\]

where the “charge-zero” sub-algebra is a simple matrix algebra and the local matrices span a non-trivial simple \( Z_4 \) graded algebra with the non-trivial center consisting
of multiples of $1$, $Y_1$, $Y_2$, and $Y_3$. So the minimal four-fold degeneracy of the transfer matrix spectrum and the entanglement spectrum can be found, implying the existence of $\mathbb{Z}_4$ parafermion edge zero modes. Thus, this kind of irreducible MPS corresponds to the non-trivial topological phase.

D. Classification of $\mathbb{Z}_6$ parafermion MPS

In this case, there exist more than one topological phases, exhibiting a richer physics than $\mathbb{Z}_3$ and $\mathbb{Z}_4$ parafermions. From the irreducibility of $\mathbb{Z}_6$ parafermion MPS, we first assume that there is an irreducible invariant subspace projector $P_0$, and then divide the discussion into four different situations.

1. $[P_0, Q_6] = 0$

This means that the irreducible invariant subspace $P_0$ contains all six charge sectors. The matrices in the invariant subspace $P_0$ span the trivial simple $\mathbb{Z}_6$ graded algebra, corresponding to the trivial phase.

2. $[P_0, Q_6] \neq 0$ but $[P_0, Q_6^2] = 0$

The above two relations automatically lead to $[P_0, Q_6^3] \neq 0$. Considering the constraints $P_0 + Q_6 P_0 Q_6^{-1} = \mathbb{1}$ and $P_0^2 = P_0$, we can express

$$P_0 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & U_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & U_2 & 0 \\ 0 & 0 & 1 & 0 & U_3 \\ U_1^\dagger & 0 & 0 & 1 & 0 \\ 0 & U_2^\dagger & 0 & 0 & 1 \\ 0 & 0 & U_3^\dagger & 0 & 1 \end{bmatrix},$$

where $U_1$, $U_2$ and $U_3$ are unitary matrices required by the idempotency. Applying $P_0$ to $A[i]$, we obtain

$$A[i] = \text{diag} \left(a_0[i], a_1[i], a_2[i], a_0[i], a_1[i], a_2[i] \right) \times \left(Y_6[i] \otimes \mathbb{1} \right)$$

with some redefinitions. Via permuting the basis $(0, 1, 2, 3, 4, 5, 6) \rightarrow (0, 2, 4, 1, 3, 5)$, we can rewrite $A[i]$ into the standard form on the $\mathbb{Z}_2$ parity basis:

$$A[i] = \begin{bmatrix} d[i] & 0 \\ 0 & U^\dagger d[i] U \end{bmatrix}, |i| = 0, 2, 4;$$

$$A[i] = \begin{bmatrix} 0 & d[i] \\ U^\dagger d[i] U^\dagger & 0 \end{bmatrix}, |i| = 1, 3, 5,$n

where

$$U = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$d[i] = \begin{bmatrix} a_0[i] & 0 & 0 \\ 0 & a_2[i] & 0 \\ 0 & 0 & a_1[i] \end{bmatrix}, |i| = 0, 1;$$

$$d[i] = \begin{bmatrix} 0 & a_0[i] & 0 \\ 0 & 0 & a_2[i] \\ a_1[i] & 0 & 0 \end{bmatrix}, |i| = 2, 3;$$

$$d[i] = \begin{bmatrix} 0 & 0 & a_0[i] \\ a_2[i] & 0 & 0 \\ 0 & a_1[i] & 0 \end{bmatrix}, |i| = 4, 5. \quad (34)$$

Note that the dimensions of the six charge sectors must be the same. Such a form $A[i]$ actually satisfies the odd type $\mathbb{Z}_2$ graded algebra shown in Eq. (16), so these matrices can be further transformed via the gauge transformation $\mathbb{1} \otimes U$ into the standard form

$$A[i] = Y_2[i] \otimes d[i]. \quad (35)$$

Since $[P_0, Q_6^2] = 0$ means that $P_0$ containing three $\mathbb{Z}_3$ charge sectors is irreducible, $d[i]$ span a trivial simple $\mathbb{Z}_3$ graded algebra. Moreover, it should be emphasized that there exists no invariant subspace whose projector commutes with the operator $Q_6$. To guarantee such a situation, all the “charge-zero” subalgebras of matrix algebras spanned by $a_0[i], a_1[i]$, and $a_2[i]$ should be simple, respectively. This determines $A[i]$ to span a type of the simple $\mathbb{Z}_6$ graded algebra, which is the similar to the odd-type $\mathbb{Z}_2$ graded algebra. The graded center of this non-trivial algebra consists of multiples of $\mathbb{1}$ and $Y_2$.

Because the $\mathbb{Z}_6$ symmetry can not be broken for $\mathbb{Z}_6$ parafermions, the $\mathbb{Z}_6$ graded structure can not be removed, and the local matrices are not reduced. From Eq. (35), we can show that the full transfer matrix spectrum have two-fold degeneracy, implying that the entanglement spectrum is at least two-fold degeneracy in the thermodynamic limit and there exist unpaired Majorana zero edge modes. Thus, this phase is a $\mathbb{Z}_6$ symmetric non-trivial topological phase without any symmetry breaking, but it shares the same property as the $\mathbb{Z}_2$ non-trivial topological phase.

3. $[P_0, Q_6] \neq 0$ and $[P_0, Q_6^2] \neq 0$ but $[P_0, Q_6^3] = 0$

Similar to the above case, the standard form of $A[i]$ can be written as

$$A[i] = Y_3[i] \otimes d[i],$$

$$d[i] = \begin{bmatrix} a_0[i] & 0 \\ 0 & a_1[i] \end{bmatrix}, |i| = 0, 1, 2;$$

$$d[i] = \begin{bmatrix} 0 & a_0[i] \\ a_1[i] & 0 \end{bmatrix}, |i| = 3, 4, 5. \quad (36)$$
Provided that the “charge-zero” sub-algebras spanned by $a_0^{[i]}$ and $a_1^{[i]}$ are simple, $A^{[i]}$ will span a type of simple $Z_8$ graded algebra, which is the same as the non-trivial simple $Z_3$ graded algebra with the non-trivial center. Eq. (36) determines that the eigenvalue spectrum of the transfer matrix have at least three-fold degeneracy, indicating that there are three-fold degenerate entanglement spectrum in the thermodynamic limit and unpaired $Z_3$ parafermion edge zero modes. Thus, the resulting MPS represents another $Z_6$ symmetric non-trivial topological phase with the same property as the $Z_3$ non-trivial topological phase.

4. $[P_0, Q_8] \neq 0$, $[P_0, Q_8^2] \neq 0$ and $[P_0, Q_8^3] \neq 0$

In this case, the form of matrices $A^{[i]}$ can be expressed as the standard form

$$A^{[i]} = Y_6^{[i]} \otimes a_1^{[i]},$$

(37)

which span another type of non-trivial simple $Z_6$ graded algebra provided the “charge zero” sub-algebra spanned by $a_0^{[i]}$ is simple. The graded center consists of multiples of the regular representation of $Z_6$ group. Because of the six-fold degeneracy of the transfer matrix spectrum, this case corresponds to a topological non-trivial phase with $Z_6$ parafermion edge zero modes, yielding six-fold degeneracy of the entanglement spectrum in the thermodynamic limit.

To summarize, there exist three topologically distinct phases and one trivial phase. Their local matrices of the irreducible MPS form different types of simple $Z_6$ graded algebra with the distinct centers. These topological phases are characterized by the Majorana zero modes, $Z_3$ parafermion zero modes, and $Z_6$ parafermion zeros modes at the edges of systems, respectively.

E. Classification of $Z_8$ parafermion MPS

The reason why we are interested in this case is that it is alleged that there exist two distinct spontaneous symmetry breaking phases from the $Z_8$ to $Z_4$ symmetry: one has a two-fold ground state degeneracy purely due to the symmetry breaking, and another has a four-fold ground state degeneracy resulting from both spontaneous symmetry breaking and topological order.26 Here we can carefully examine these results from the irreducibility perspective of the MPS. To gain more information about related phases, especially the symmetry breaking phases, we divide our discussion into four situations.

1. $[P_0, Q_8] = 0$

The matrices associated to the irreducible invariant subspace projector $P_0$ contain eight $Z_8$ charge sectors, corresponding to a trivial simple $Z_8$ graded algebra and thus a trivial phase.

2. $[P_0, Q_8] \neq 0$ but $[P_0, Q_8^2] = 0$

It implies that the invariant subspace denoted by the projector $P_0$ does not contain all eight $Z_8$ charge sectors, but it contains only four $Z_4$ charge sectors. The commutation relation $[P_0, Q_8^3] = 0$ indicates that $Z_2$ parity sectors are also contained in $P_0$ as well. Then the resulting MPS is trivial from both $Z_4$ and $Z_2$ symmetry point of view. Since the $P_0$ invariant space is irreducible, we can not break the symmetry down to $Z_2$. The local matrices span a type of simple $Z_8$ graded algebra which is the same as the trivial semisimple $Z_4$ graded algebra, provided that the charge-0 subalgebras are simple. Hence it is reducible and the $Z_8$ symmetry spontaneously breaks down to $Z_4$, attributing a two-fold degeneracy. Because the trivial graded algebra has no non-trivial graded center, the corresponding MPS describes a pure symmetry breaking phase. Actually the spontaneous symmetry breaking is related to the phenomenon of boson condensation, since we have $|x_1^0, x_2^4\rangle = 0$.

3. $[P_0, Q_8] \neq 0$ and $[P_0, Q_8^2] \neq 0$ but $[P_0, Q_8^3] = 0$

Then the invariant subspace denoted by the projector $P_0$ contains neither eight $Z_8$ charge sectors nor four $Z_4$ charge sectors, but it only contains two $Z_2$ charge sectors, leading to a trivial MPS from the $Z_2$ symmetry point of view. Since $P_0^2 = P_0$ and

$$P_0 + Q_8 P_0 Q_8^{-1} + Q_8^2 P_0 Q_8^{-2} + Q_8^3 P_0 Q_8^{-3} = \mathbb{1},$$

(38)

we can find that $P_0$ is

$$P_0 = \begin{pmatrix}
1 & 0 & U_1 & 0 & U_2 & 0 & U_3 & 0 \\
0 & 1 & 0 & U_4 & 0 & U_5 & 0 & U_6 \\
U_1^* & 0 & 1 & 0 & U_1^* U_2 & 0 & U_1^* U_3 & 0 \\
1 & 0 & U_4^* & 0 & 1 & 0 & U_4^* U_5 & 0 \\
\frac{1}{4} & U_2^* & 0 & U_1^* U_2 & 0 & 1 & 0 & U_2^* U_3 & 0 \\
0 & U_5^* & 0 & U_5^* U_4 & 0 & 1 & 0 & U_5^* U_6 & 0 \\
U_3^* & 0 & U_3^* U_4 & 0 & U_3^* U_2 & 0 & 1 & 0 & 0 \\
0 & U_6^* & 0 & U_6^* U_4 & 0 & U_6^* U_5 & 0 & 1 & 0
\end{pmatrix},$$

where $U_i$ are unitary matrices with the same dimension. According to $A^{[i]} P_0 = P_0 A^{[i]} P_0$ and with some proper substitutions, we can express the local matrices as

$$A^{[i]} = \text{diag}(a_0^{[i]}, a_1^{[i]}, a_0^{[i]}, a_1^{[i]}, a_0^{[i]}, a_1^{[i]}, a_0^{[i]}, a_1^{[i]}) \times (Y_8^{[i]} \otimes \mathbb{1}).$$

(39)

By permuting the basis, $(0, 1, 2, 3, 4, 5, 6, 7) \rightarrow (0, 2, 4, 6, 1, 3, 5, 7)$, we can explicitly show that the matrices $A^{[i]}$ have an even $Z_2$ graded structure as

$$A^{[i]} = \begin{pmatrix}
Y_4^{[i]}/2 \otimes a_0^{[i]} & 0 & 0 \\
0 & Y_4^{[i]}/2 \otimes a_1^{[i]} & 0 \\
0 & 0 & Y_4^{([i]-1)/2} \otimes a_0^{[i]} \\
0 & 0 & 0 & Y_4^{([i]-1)/2} \otimes a_1^{[i]}
\end{pmatrix},$$

(40)

where $|| = 0, 2, 4, 6$;

$$A^{[i]} = \begin{pmatrix}
Y_4^{[i+1]}/2 \otimes a_1^{[i]} & 0 & 0 \\
0 & Y_4^{[i+1]}/2 \otimes a_0^{[i]} & 0 \\
0 & 0 & Y_4^{([i]+1)/2} \otimes a_1^{[i]} \\
0 & 0 & 0 & Y_4^{([i]+1)/2} \otimes a_0^{[i]}
\end{pmatrix},$$

(40)

where $|| = 1, 3, 5, 7$, from above equation one can exclude the possibility of existence of topological order. Since there are no non-trivial graded center but four irreducible invariant spaces,
we can reduce the simple $Z_8$ graded algebra to the even-type simple $Z_2$ graded algebras, and the $Z_8$ symmetry is broken down to the $Z_2$ symmetry, attributing four-fold degenerate ground states. Then Eq. (10) can be transformed into a direct sum of four sets of even simple algebra $Z_2$ matrices via the gauge transformation

$$G = \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 \\
ω^1_2 & ω^1_2 & ω^1_2 & ω^2_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 \\
ω^2_2 & ω^2_2 & ω^2_2 & ω^2_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0
\end{bmatrix},$$

(41)

which mixes the even $Z_8$ charge sectors (0, 2, 4, 6) as well as the odd $Z_8$ charge sectors (1, 3, 5, 7), separately. After this transformation, the local matrices are rewritten as

$$A[i] = (ω^2_8[i]d[i]) ⊕ (ω^1_8[i]d[i]) ⊕ (ω^3_8[i]d[i]) ⊕ d[i],$$

$$d[i] = \begin{bmatrix}
a^{[1]}_i \\
a^{[2]}_i \\
a^{[3]}_i \\
a^{[4]}_i
\end{bmatrix}, |i| = 0, 2, 4, 6;$$

$$d[i] = \begin{bmatrix}
a^{[1]}_i \\
a^{[2]}_i \\
a^{[3]}_i \\
0
\end{bmatrix}, |i| = 1, 3, 5, 7.$$  

(42)

Notice that the four-fold degeneracy is only contributed by the spontaneous symmetry breaking, and there exists no topological order.

4. $[P_0, Q_8] ≠ 0$, $[P_0, Q_8^2] ≠ 0$ and $[P_0, Q_8^2] ≠ 0$

In this case, the irreducible invariant subspace $P_0$ only contains one charge sector. The irreducible matrices is given by

$$A[i] = Y_8^{-i} ⊗ a[i],$$

(43)

which span a non-trivial simple $Z_8$ graded algebra with the non-trivial center consisting of multiples of regular representation of $Z_8$, provided the “charge-zero” sub-algebra is a simple algebra. The transfer matrix spectrum has eight-fold degeneracy and entanglement spectrum in the thermodynamic limit should have at least eight-fold degeneracy, corresponding to the $Z_8$ symmetric topological phase. Hence there is only one topological phase.

**IV. GENERAL RESULTS FOR $Z_p$ PARAFERMION MPS**

**A. Irreducibility and classification**

Summarizing the above several examples, we can obtain the general classification for all $Z_p$ parafermion phases. First we assume an irreducible invariant subspace projector $P_0$ for all matrices $A[i]$. Then we consider the commutation relations between $Q_p$ and $P_0$, where $r$ is the divisor of $p$. It can be proved that the number of different cases denoted by commutation relations between $Q_p^r$ and $P_0$ is the number of divisor $p$. Each case is labelled by the smallest divisor $n \in \{r\}$ such that $[Q_p^n, P_0] = 0$. Together with the idempotency constrain, the structures of all irreducible invariant subspace projectors as well as that of $A[i]$ can be determined.

Actually, Eq. (11) can be written in a more concise form:

$$A[i] = \text{diag}(a^{[1]}_i, a^{[2]}_i, \cdots, a^{[p-1]}_i) × (Y_p^{[i]} ⊗ I).$$  

(44)

For the case $[P_0, Q_p] = 0$, $a^{[i]}_s$ with $s \in Z_p$ are not equal for all $i$ under gauge transformations and redefinitions, and all $A[i]$ span the trivial simple $Z_p$ graded algebra. The MPS generated by the matrices of Eq. (44) belong to the trivial phase.

However, for the case $[P_0, Q_p] ≠ 0$ with $r < n$ and $[P_0, Q_p^2] = 0$, the relation $a^{[i]}_s × (s + p/n) \mod p$ satisfies under gauge transformations and redefinitions. There exist $p/n$ unequal sub-block matrices $a^{[i]}_s (s \in Z_p/n)$. Then, there are two different situations, depending on whether $p/n$ and $p/n$ are mutually prime or not. If $p$ and $p/n$ are mutually prime, by using a charge-preserving gauge transformation represented by a permutation matrix, the local matrices can be transformed into

$$A[i] = Y_p^{-i} × d[i],$$

$$d[i] = \text{diag}(a^{[1]}_0, \cdots, a^{[p/n-1]}_0) × (Y_{p/n}^{[i]} ⊗ I).$$  

(45)

Under the condition that the “charge-0” sub-algebras are simple, all $A[i]$ are irreducible and span a non-trivial simple $Z_p$ graded algebra with a non-trivial center. The MPS generated by Eq. (45) indicate a $Z_p$ symmetric topological phase with unpaired $Z_n$ parafermion zero edge modes. This topological phase is characterized by the $n$-fold degenerate transfer matrix spectrum and entanglement spectrum.

In the case where $n$ and $p/n$ are not mutually prime, it is impossible that the local matrices can be transformed into the form of Eq. (45). The reason is that we can write $Y_p \sim Y_{p/n} ⊗ Y_n$ only if $n$ and $p/n$ are mutually prime. Actually, we can express $Y_p \sim Q_{p/n} ⊗ Y_n$, where $Q_{p/n} = \text{diag}(1, ω^1_p, ω^2_p, \cdots, ω^{n-1}_p)$, so the local matrices can be transformed via a gauge transformation into

$$A[i] = \tilde{Q}_{p/n}^{[i]} × d[i].$$  

(46)

However, the gauge transformation breaks the $Z_p$ charge symmetry but preserves the $Z_{p/n}$ charge symmetry. Provided that the “charge-0” sub-algebras are simple, all $A[i]$ span a simple $Z_p$ graded algebra, which is the same as the trivial semisimple $Z_{p/n}$ graded algebra. And it can be reduced into the trivial simple $Z_{p/n}$ graded algebra. So this situation corresponds to the phases where the $Z_p$ symmetry is spontaneously broken down to $Z_{p/n}$.
So a conclusion can be drawn that the number of phases is equal to the number of divisors of \( p \), and every divisor \( n \) uniquely labels a different gapped phase. The topological phases including the trivial phase are labeled by \( n \) satisfying that \( n \) and \( p/n \) are mutually prime. The different parafermion gapped phases have one-to-one correspondence to the different \( \mathbb{Z}_p \) graded algebras. A more concise summary is shown in Table. [11]

**B. Degeneracy of transfer matrix spectrum**

For the symmetry breaking phases, the degenerate ground states can be transformed with each other by acting the \( \mathbb{Z}_p \) symmetry generator \( U = \text{diag}(1, \omega^1, \omega^2, \ldots, \omega^{p-1}) \) several times, i.e.,

\[
\sum_i U_i d^{[i]} = \omega^{[i]} d^{[i]},
\]

if we act the \( \mathbb{Z}_p \) charge operator \( n \) times, we will go back to the original state, since it is \( \mathbb{Z}_p/n \) symmetric. Thus the local matrices is shown in Eq. (10). Therefore, the transfer matrices for the whole ground state subspace of symmetry breaking phases are given by

\[
\mathcal{E} = \sum_i \left[ \left( \bigoplus_{r=0}^{n-1} \omega^{[i]}_p d^{[i]} \right) \otimes \left( \bigoplus_{r=0}^{n-1} \omega^{[i]}_p d^{[i]} \right) \right]. (47)
\]

Actually, because all \( Y_n^r \) with \( r \in \mathbb{Z}_n \) can be diagonalized simultaneously, \( A^{[i]} = Y_n^{[i]} \otimes d^{[i]} \sim Q_n^{[i]} \otimes d^{[i]} \), the transfer matrices for topological phases can be transformed into

\[
\mathcal{E}' = \sum_i \left[ \left( \bigoplus_{r=0}^{n-1} \omega^{[i]}_n d^{[i]} \right) \otimes \left( \bigoplus_{r=0}^{n-1} \omega^{[i]}_n d^{[i]} \right) \right]. (48)
\]

These two expressions are very similar but the phase factors are different. Their eigenvalue spectra are equivalent to those of the following matrices:

\[
\bigoplus_{r,r'=0}^{n-1} \omega^{[i]}_p (r-r') d^{[i]} \otimes d^{[i]}, \quad \bigoplus_{r,r'=0}^{n-1} \omega^{[i]}_n (r-r') d^{[i]} \otimes d^{[i]}. (49)
\]

When \( r = r' \), \( \mathcal{E} (r, r) = \sum_i d^{[i]} \otimes d^{[i]} \) defines a block transfer matrix, whose largest eigenvalue is non-degenerate and the spectrum is real and non-negative. Thus the real eigenvalues are \( n \)-fold degenerate for both cases. On the other hand, \( \mathcal{E} (r, r') = \sum_i \omega^{[i]}_p (r-r') d^{[i]} \otimes d^{[i]} \) and \( \mathcal{E}' (r, r') = \sum_i \omega^{[i]}_n (r-r') d^{[i]} \otimes d^{[i]} \) for \( r \neq r' \) denote mixed transfer matrices, and their eigenvalues are complex, and the magnitudes of eigenvalues are smaller than unity.

However, for topological phases, taking the advantage of the fact that \( r \in \mathbb{Z}_n \) and the periodicity of \( \omega_n \) is also \( n \), there are \( n \) possible values of \( \{ r, r' \} \) for a fixed value of \( r - r' \). While for the symmetry breaking phases, \( \mathcal{E} (r, r') \) does not have such a property. Therefore, the complex eigenvalues of the transfer matrix are also \( n \)-fold degenerate for topological phases, while the complex eigenvalues of the transfer matrix of the symmetry breaking phase are not necessarily \( n \)-fold degenerate. Different from those features displayed in the spontaneous symmetry breaking phases, the degeneracy of the full transfer matrix spectrum is the unique characteristic property for topological phases. This degeneracy is a clear evidence of the existence of parafermion zero edge modes. In contrast, the degeneracy of entanglement spectrum can appear for both the topological order phases and the symmetry protected topological phases.

**C. Understanding topological order in MPS formalism**

It is known that all one-dimensional bosonic gapped systems can support short-range entanglement without any intrinsic topological order. However, the one-dimensional fermion and parafermion systems can probably have the topological order. Actually, since the statistics is not well-defined in one dimension, the parafermion chains only emerge at the edges of a two-dimensional fractionalized topological states [12–14], and the topological order is inherited from the bulk of fractional topological insulators. So these phases are distinct from the symmetry protected topological phases, and it is more appropriate to recognize them as invertible topological orders.36

Previously such a topological order is characterized by strong zero edge modes. A strong zero edge mode carrying the \( \mathbb{Z}_p \) charge is defined by an operator localized at the edges, which commutes with the model Hamiltonian.37 But such a strong zero edge mode can be easily washed away when an arbitrary small perturbation is introduced into the fixed point model Hamiltonian.38,39 Nevertheless, even in the absence of the strong zero edge modes, the gapped phases still display topological nature, and the weak edge modes commuting with the ground state subspace exist. It is more proper describing the topological order from the ground state wave functions rather than the model Hamiltonians.

In the view point of the fermionic/parafermionic MPS, we have understood that different topological phases correspond to the non-trivial simple \( \mathbb{Z}_p \) graded algebras with different non-trivial centers. The matrix \( Y_n \) features the non-trivial graded structure and acts as the fractionalized charge operators, characterizing the topological order. In fact the matrix \( Y_n \) has more profound indications. It can also be regarded as the gauge symmetry of the local tensors, namely, \( (Y_n \otimes 1) A^{[i]} (Y_n \otimes 1)^{-1} = A^{[i]} \), which plays a crucial role and becomes the necessary condition of the topological order. It can be further verified that the non-trivial algebra MPS are the G-injective MPS.36 Actually, the properties of the topological order in fermionic/parafermionic MPS are similar to those found in PEPS in two dimensional systems. We believe that our formalism provides the proper way to describe the topological order in one dimension.
TABLE II: Comparison of different types of gapped phases of $\mathbb{Z}_p$ parafermions characterized by the integer $n$.

| Phase                    | Topological | Symmetry breaking | Trivial |
|--------------------------|-------------|-------------------|---------|
| Label $n$                | $n$ and $p/n$ are coprime | $n$ and $p/n$ are not coprime | $n = 1$ |
| Transfer matrix          | $n$-fold degenerate spectrum | $n$-fold degenerate real part of spectrum (for whole ground space) | Non-degenerate largest eigenvalue |
| spectrum                 |             |                   |         |
| Algebra                  | Non-trivial simple $\mathbb{Z}_p$ graded | Trivial semisimple $\mathbb{Z}_p/n$ graded | Trivial simple $\mathbb{Z}_p$ graded |

V. CONCLUSION AND OUTLOOK

Using the graded tensor product, we have encoded the parafermion statistics into the Fock space, and identified it as a graded vector space. Then, based on the Fock space, we have constructed the general MPS for all one-dimensional gapped phases for $\mathbb{Z}_p$ parafermions without extra symmetry. We have also investigated several specific examples, covering all possible gapped phases. From the analysis of irreducibility of MPS, it has been found that all parafermion gapped phases can be classified by MPS. By identifying algebras spanned by the irreducible local matrices, we also find different phases have one-to-one correspondence to different simple $\mathbb{Z}_p$ graded algebras. We have further analyzed the properties of the corresponding transfer matrix spectra and entanglement spectra. The topological phases can be identified by the unique property that the full transfer matrix spectra are $n$-fold degenerate ($n \leq p$).

Our formalism can be easily generalized to the classification with extra symmetries, including the general on-site symmetries and time-reversal symmetry. Moreover, some exact solvable models can also be designed within our MPS formalism and the characteristic properties of various phases can be more easily calculated. And the renormalization group of fermionic/parafermionic MPS can be developed as well. Finally, our present formulation may be useful to investigate the fermionic/parafermionic PEPS in more than one spatial dimension. Thus the present theoretical framework and forthcoming results will greatly enrich our understanding of low-dimensional strongly correlated many-body systems.

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Appendix: Some useful concepts

In the content of our paper, we frequently mention the graded algebra, irreducibility, simplicity and semisimplicity. In this appendix, we will list the definitions of these concepts, which are summarized from the literature.

Graded algebra. An algebra $A$ is said to be $\mathbb{Z}_p$ graded if there is a decomposition of the underlying vector space $A = \mathbb{Z}^p_{\mathbb{Z}_p}$ such that $A^n A^m = A^{n+m}$. If $A \in A^n$, then $A$ is said to be homogeneous of degree $n$.

Irreducibility. If the local tensor has a block upper triangular form:

$$A^{[i]} = \begin{pmatrix} a_0^{[i]} & a_1^{[i]} \\ 0 & a_2^{[i]} \end{pmatrix}. \quad (50)$$

The MPS generated by $A^{[i]}$ doesn’t depend on $a_0^{[i]}$. In fact, there exist a subspace $S_i$ which is invariant under the action, $A^{[i]} S_i \subset S_i$. Then we can choose $a_0^{[i]} = 0$ and assume $S_i$ doesn’t contain any other invariant subspace. Denoting $P_i (Q_i = 1 - P_i)$ as the orthogonal projector onto $S_i (S_i^\perp)$, we have

$$A^{[i]} P_i = P_i A^{[i]} P_i, \quad Q_i A^{[i]} = Q_i A^{[i]} Q_i. \quad (51)$$

Then $P_i A^{[i]} P_i$ generates the irreducible MPS. Actually, any MPS can be gauge-transformed into a direct sum of irreducible MPS. For parafermion systems, the irreducibility is different. The invariant subspace must be a graded vector space, projected by $P_i$ commuting with charge matrix. If it contain other subspaces which is not a graded vector space, we can not reduce it.

Simplicity and semisimplicity. The simplicity and semisimplicity are usually defined in terms of (finite-dimensional) representations, i.e., the vector spaces over the field on which the algebra acts linearly. An algebra is called semisimple if any representation of a subgroup has a complementary representation. Any representation of a semisimple algebra splits into irreducible ones. A semisimple algebra is called simple if it has a unique irreducible representation. Any semisimple algebra is a direct sum of simple algebras. For example, the group algebra splits as a direct sum of algebra spanned by its irreducible representations. Actually, the algebra spanned by local matrices for topological and symmetry breaking phases is the simple $\mathbb{Z}_p$ graded algebra, but they are semisimple without $\mathbb{Z}_p$ grading, corresponding to the interpretation for phases of clock spin chains.

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