Adaptive output regulation based on internal model principle for systems, with uncertain disturbances and reference signals

Zhihao Yao\textsuperscript{a}, Kota Akaike\textsuperscript{a} and Ikuro Mizumoto\textsuperscript{b}

\textsuperscript{a}Graduate School of Science and Technology, Kumamoto University, Kumamoto, Japan; \textsuperscript{b}Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto, Japan

**ABSTRACT**

An output regulation problem under the existence of disturbances is one of the main objectives to design the control system for mechanical systems, and the strategy based on the internal model principle (IMP) is one of the attractive ways to suppress the disturbances. In this paper, we propose an almost strictly positive real (ASPR) based adaptive output feedback control with adaptive internal model for output regulation for systems with uncertain disturbances and reference signals. The stability of the obtained adaptive control system is analyzed, and the effectiveness of the proposed method is confirmed through numerical simulations.

**1. Introduction**

A system is said to be almost strictly positive real (ASPR) if there exists a static output feedback such that the resulting closed-loop control system is strictly positive real (SPR) [1, 2]. The ASPR characteristic of the system allows us to design an adaptive output feedback type control system with very simplified structure, and from the SPR property of the resulting closed-loop system, the obtained output feedback control system has strong robustness with respect to disturbances and systems uncertainties. However, since the most practical systems do not have ASPR characteristic, one has to alleviate the ASPR condition imposed on the considered controlled system in order to design the ASPR-based control system. One of the most common and simple strategies to alleviate the ASPR condition is the introduction of a parallel feedforward compensator (PFC) [3–6]. The PFC is introduced in parallel with the non-ASPR system so as to render the resulting augmented system with the PFC ASPR, and then one can design the ASPR-based control system to the ASPR augmented system. Unfortunately, however, since the control system is designed for the obtained ASPR augmented system, the control performance might degrade from the effect of PFC output even if the control performance for the augmented system can be perfect. Therefore, how to reduce the effect on the practical output while keeping the ASPR-ness of the augmented system has been an important issue.

On the other hand, an output regulation problem under the existence of disturbances is one of the main objectives to design the control system for mechanical systems including Hydraulic/ Pneumatic servo control systems, electrical motor control systems, magnetic levitation control systems, and so on. The strategy based on the internal model principle (IMP) is a common and an attractive way to suppress disturbances and to attain the output regulation in the case where the models of the disturbance and reference signal are known. However, in the case where the models of the disturbances and reference signals are uncertain, the perfect regulation is not ensured anymore. With this in mind, adaptive output regulation design schemes based on IMP have attracted a great deal of attention from the early 90’s. However, most of them supposed that either the controlled system or the exosystem that generates disturbances and reference signals is known [7–9].

Taking this into consideration, an adaptive output feedback control based on IMP for output regulation for uncertain systems with uncertain disturbances and reference signals has been proposed in [10]. In [10], the uncertain model of disturbances and reference signals is adaptively estimated and the stability of the obtained control system is guaranteed via ASPR-based adaptive output feedback with an appropriate PFC. However, the results in [10] only showed the boundedness of all the signals in the resulting control system. In this paper, we expand the analysis of the resulting control system and show that the convergence of the output tracking error to a small range will be attained with appropriate setting of the design parameters in the adaptive algorithms according to the convergence of the augmented output tracking error. To clarify the conditions which maintain the tracking error small is a very important

**CONTACT**

Ikuro Mizumoto ikuro@gpo.kumamoto-u.ac.jp Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto 860-8555, Japan

© 2021 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
**2. Problem statement**

Consider the following $n$th order LTI system:

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t) + d_d(t) \\
y(t) &= c^T x(t) + d(t),
\end{align*}
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t), y(t) \in \mathbb{R}$ are the input and output of the system. $d_d(t)$ and $d(t)$ are unknown disturbances. It is supposed that the reference signal $y_m(t)$, for which the output $y(t)$ of the system is required to follow, and disturbances $d_d(t)$ and $d(t)$ are generated by the following unknown exosystem:

$$
\begin{align*}
\dot{\omega}(t) &= A_d \omega(t) \\
y_m(t) &= c_m^T \omega(t) \\
d_d(t) &= B_d \omega(t) \\
d(t) &= c_d^T \omega(t)
\end{align*}
$$

(2)

with appropriate unknown vectors and matrix $c_m, c_d \in \mathbb{R}^{n \times m}$. Where, $A_d \in \mathbb{R}^{m \times m}$ is an unknown matrix having all its eigenvalues on the imaginary axis and having the following characteristic polynomial:

$$
D_{IM}(\lambda) = \det(\lambda I - A_d) = \lambda^m + \alpha_m \lambda^{m-1} + \cdots + \alpha_1 \lambda + \alpha_0.
$$

(3)

The order $m$ of the exosystem given in (2) is also assumed to be unknown, but we suppose that an upper bound of order $m$ is known.

The considered system (1) is supposed to satisfy the following assumptions:

**Assumption 2.1:** The system (1) has a relative degree of $\gamma \geq 2$, which is not necessarily to be known.

**Assumption 2.2:** $c^T A^{-1} b > 0$, i.e. the high frequency gain of the system (1) is positive.

**Assumption 2.3:** The output $y(t)$ and the reference signal $y_m(t)$ are both available for measurement.

**Figure 1.** Augmented system with a PFC.

**Assumption 2.4:** Considering an extended system with a proper pre-internal model compensator:

$$
G_{IM}(s) = \frac{N_{IM}(s)}{D_{IM}(s)}
$$

(4)

with unknown $D_{IM}(s)$ defined in (3) and any given stable polynomial: $N_{IM}(s)$ of order $m$ of the form:

$$
N_{IM}(s) = s^m + \beta_{m-1}s^{m-1} + \cdots + \beta_1 s + \beta_0,
$$

(5)

a PFC: $H(s)$, which can make the resulting augmented system ASPR, is known (see Figure 1).

The objective in this paper is to design an adaptive control system that has the output $y(t)$ track the reference signal $y_m(t)$ generated by an unknown exosystem given in (2) for unknown systems with unknown disturbances generated by the unknown exosystem in (2) using only the output signal under Assumptions 2.1-2.4.

**Remark 2.1:** To satisfy Assumption 2.4, an approximated or nominal model for both the controlled system and the exosystem might be known. However, it is not necessary to know the exact model. The models are just for designing the PFC and the obtained PFC is allowed to be conservative one.

**3. Ideal error system**

**3.1. Error system with an internal model filter**

Let’s consider the following internal model filter given in Assumption 2.4:

$$
u_f(t) = G_{IM}(s)[u(t)] = \frac{N_{IM}(s)}{D_{IM}(s)}[u(t)],
$$

(6)

where the notation $y_v(t) = W(s)[v(t)]$ represents the output $y_v(t)$ of the system $W(s)$ with the input $v(t)$. That is, $\nu_f(t)$ given in (6) represents the output of the filter $G_{IM}(s)$ with input $u(t)$.

Defining the output error by $e(t) = y(t) - y_m(t)$, the error system from $u(t)$ to $e(t)$ shown in Figure 2 can be represented as follows by taking into account the fact that $G_{IM}(s)$ is proper and the system has relative degree
This augmented system (9) is ASPR. In this case, by designing the controller as

\[ u^*(t) = -k^e e_a(t) \]

with an ideal feedback gain \( k^e \) which make the resulting closed-loop system SPR, we can easily guarantee that \( \|x_{ea}(t)\| \) converges to zero as \( t \to \infty \), i.e., \( \|e_z(t)\|, \|e_f(t)\| \) and \( e(t), y_f(t) \) converge to zero.

### 4. Adaptive control system design

Since the system is uncertain and the model of the exosystem is also uncertain, we are not able to design the ideal control system given in the previous section. Here, we consider designing adaptive controller to solve the problem.

Now, from the relation given in (6):

\[ u_f(t) = G_{IM}(s)[u(t)] = \frac{N_{IM}(s)}{D_{IM}(s)}[u(t)], \quad (10) \]

we have

\[ u(t) = G_{IM}^{-1}(s)[u_f(t)] = \frac{D_{IM}(s)}{N_{IM}(s)}[u_f(t)]. \quad (11) \]

Note that \( N_{IM}(s) \) is a given and known stable polynomial.

Express the relation (11) by the following state space form:

\[ \dot{z}_c(t) = A_c z_c(t) + b_c u_f(t) \]

\[ u(t) = \theta^T z_c(t) + u_f(t) \quad (12) \]

with

\[ A_c = \begin{bmatrix} 0 & I_{m-1} \\ -\beta_0 & -\beta_1 & \cdots & -\beta_{m-1} \end{bmatrix}, \quad b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

\[ \theta^T = [\alpha_0 - \beta_0, \alpha_1 - \beta_1, \cdots, \alpha_{m-1} - \beta_{m-1}] \]

As mentioned above, since the pre-compensator \( G_{IM}(s) \) (or \( D_{IM}(s) \)) is unknown, we cannot generate the input signal \( u_f(t) \) of the controlled system \( G(s) \) from \( u(t) \) using the ideal compensator \( G_{IM}(s) \). Therefore we consider directly designing \( u_f(t) \) from (12) as follows:

\[ u_f^*(t) = u^*(t) - \theta^T z_c(t) \]

\[ u^*(t) = -k^e e_a(t). \quad (13) \]

However, \( \theta \) and \( k^e \) are also unknown, so that we consider obtaining \( \theta(t) \) and \( k(t) \) adaptively and design
where \( \hat{\Theta}(t) \) and \( k(t) \) are adjusted by the following parameter adjusting laws:

\[
\begin{align*}
\dot{k}(t) & = \gamma \varepsilon_{az}(t)\varepsilon_{a}(t) - \sigma_{k} k(t) \\
\dot{\Theta}(t) & = \Gamma \varepsilon_{a}(t)\varepsilon_{a}(t) - \sigma_{\Theta} \dot{\Theta}(t) - \sigma_{\Theta2} \| \varepsilon_{a}(t) \|^{2} \hat{\Theta}(t)
\end{align*}
\]

with positive constants \( \gamma, \sigma_{k}, \sigma_{\Theta}, \sigma_{\Theta2} > 0 \) and positive definite matrix \( \Gamma = \Gamma^{T} > 0 \). \( \varepsilon_{az}(t) \) and \( \varepsilon_{z}(t) \) are given by

\[
\begin{align*}
\varepsilon_{az}(t) & = \varepsilon(t) + \frac{1}{\rho_{z}} \varepsilon_{z}(t) \\
\varepsilon_{z}(t) & = \varepsilon(t) + \frac{1}{\rho_{z}} \varepsilon_{z}(t) \\
\end{align*}
\]

and \( P_{c} = P_{c}^{T} > 0 \) is a solution of the following Lyapunov equation for any symmetric positive definite matrix \( Q_{c} \):

\[
A_{c}^{T} P_{c} + P_{c} A_{c} = -Q_{c} < 0.
\]

Since \( \varepsilon_{a}(t) \)-system given in (12) is stable, such a \( P_{c} \) exists.

5. Stability analysis

5.1. Boundedness of the signals

Represent the obtained control system by

\[
\begin{align*}
u_{f}(t) & = u(t) - \theta^{T}(t)\varepsilon(t) - \rho_{z} \varepsilon_{z}(t) \\
u(t) & = -k(t)\varepsilon_{a}(t)
\end{align*}
\]

From Equation (14), we have

\[
u_{f}(t) = -k(t)\varepsilon_{a}(t) - \theta^{T}(t)\varepsilon(t) - \rho_{z} \varepsilon_{z}(t)
\]

as the control input applied to the controlled system \( G(s) \). Since \( u(t) \) with ideal internal model compensator can be expressed as

\[
u(t) = u_{f}(t) + \theta^{T}(t)\varepsilon(t)
\]

from (12), the virtual input \( u_{p}(t) \) for the controlled system is given from Equations (20) and (21):

\[
u_{p}(t) = -k(t)\varepsilon_{a}(t) - \Delta \theta^{T}(t)\varepsilon(t) - \rho_{z} \varepsilon_{z}(t)
\]

where \( \Delta \theta^{T}(t) = \theta^{T}(t) - \dot{\theta}(t) \).

Thus, defining \( \eta_{yp}(t) = H(s)[u_{p}(t)] \), we have

\[
\eta_{yp}(t) = G_{a}(s)[u_{p}(t)] + H(s)[u(t) - u_{p}(t)]
\]

and

\[
\Delta \eta_{yp} = G_{a}^{-1}(s)H(s)[\Delta \theta^{T}(t)\varepsilon(t) + \rho_{z} \varepsilon_{z}(t)].
\]

Since \( G_{a}^{-1}(s)H(s) \) is stable and proper, \( \Delta \eta_{yp} \) can be represented as

\[
\begin{align*}
\dot{x}_{a}(t) & = A_{a}x_{a}(t) + b_{a}(\Delta \theta^{T}(t)\varepsilon(t) + \rho_{z} \varepsilon_{z}(t)) \\
\dot{y}_{yp} & = c_{a}^{T}x_{a}(t) + d_{a}(\Delta \theta^{T}(t)\varepsilon(t) + \rho_{z} \varepsilon_{z}(t))
\end{align*}
\]

with a stable matrix \( A_{a} \).

From (23), the error system of the augmented system can be represented by

\[
\begin{align*}
\dot{x}_{ea}(t) & = A_{ea}x_{ea}(t) + b_{ea}(u_{p} + \Delta \eta_{yp}) \\
\dot{e}_{a}(t) & = c_{ea}^{T}x_{ea}(t).
\end{align*}
\]

Since this error system (25) is ASPR having relative degree of 1 and minimum-phase, it can be transformed into the following canonical form [12]:

\[
\begin{align*}
\dot{e}_{a}(t) & = a_{ea}e_{a}(t) + c_{ea}^{T}\eta_{a}(t) + b_{ea}(u_{p} + \Delta \eta_{yp}) \\
\dot{\eta}_{a}(t) & = A_{\eta a}\eta_{a}(t) + b_{\eta a}e_{a}(t)
\end{align*}
\]

with the stable zero dynamics: \( \dot{\eta}_{a}(t) = A_{\eta a}\eta_{a}(t) \).

Now, consider the following positive definite function \( V(t) \):

\[
V(t) = V_{1}(t) + V_{2}(t) + V_{3}(t) + V_{4}(t) + V_{5}(t) + V_{6}(t)
\]

\[
V_{1}(t) = \frac{1}{2}e_{a}(t)^{2}
\]

\[
V_{2}(t) = \eta_{a}(t)^{T}P_{\eta a}\eta_{a}(t)
\]

\[
V_{3}(t) = \frac{b_{ea}}{2\gamma}\Delta k(t)^{2}
\]
\[ V_4(t) = \frac{b_{ea}}{2} \Delta \dot{\theta}(t)^T \Gamma^{-1} \Delta \dot{\theta}(t) \]
\[ V_5(t) = \chi_n^T(t) P_n x_n(t) \]
\[ V_6(t) = \frac{b_{ea}}{2 \rho_p} z_c^T(t) P_c z_c(t), \]  
(27)

where \( \Delta k(t) = k(t) - k^* \) and \( k^* \) is the ideal feedback gain to be determined later. \( P_{na}, P_n, P_c \) are symmetric positive definite matrices such that the following Lyapunov equations are satisfied for any symmetric positive definite matrices \( Q_{na}, Q_n, Q_c \), respectively.

\[ A_{na}^T P_{na} + P_{na} A_{na} = -Q_{na} < 0 \]  
(28)
\[ A_n^T P_n + P_n A_n = -Q_n < 0 \]  
(29)
\[ A_c^T P_c + P_c A_c = -Q_c < 0 \]  
(30)

Taking the time derivative of \( V_1(t) \), we have

\[ \dot{V}_1(t) = \dot{e}_a(t) \cdot e_a(t) \]
\[ = [a_{ea} e_a(t) + T_n \eta_a(t)] e_a(t) \]
\[ + b_{ea} (u_p(t) + \Delta y_f(t)) e_a(t). \]  
(31)

Since the \( u_p \) can be expressed as

\[ u_p(t) = -k(t)e_a(t) - \Delta \theta^T(t) z_c(t) - \rho_z z_{cp}(t) \]

from (22), it follows that

\[ \dot{V}_1(t) = a_{ea} e_a(t)^2 + T_n \eta_a(t) e_a(t) \]
\[ - b_{ea} e_a(t) \Delta \theta^T(t) z_c(t) - b_{ea} e_a(t) \rho_z z_{cp}(t) \]
\[ - b_{ea} e_a(t)^2 \Delta k(t) - b_{ea} e_a(t)^2 k^* \]
\[ + b_{ea} e_a(t) \Delta y_f(t) \]
\[ = a_{ea} e_a(t)^2 + T_n \eta_a(t) e_a(t) \]
\[ - b_{ea} e_a(t) \Delta \theta^T(t) z_c(t) - b_{ea} e_a(t)^2 \Delta k(t) \]
\[ - b_{ea} e_a(t)^2 k^* + b_{ea} e_a(t) c_n(t) x_n(t) \]
\[ + b_{ea} e_a(t) (d_n - 1) \rho_z e_a(t) \]
\[ + (d_n - 1) b_{ea} e_a(t) \rho_z z_{cp}(t). \]  
(32)

Moreover, since \( \eta_a(t) = A_{na} \eta_a(t) + b_{na} e_a(t) \), the time derivatives \( V_2(t) \) leads to

\[ \dot{V}_2(t) = \ddot{\eta}_a(t) P_{na} \eta_a(t) + \eta_a^T(t) P_{na} \dot{\eta}_a(t) \]
\[ = (A_{na} \eta_a(t) + b_{na} e_a(t))^T P_{na} \eta_a(t) \]
\[ + \dot{\eta}_a(t) P_{na} (A_{na} \eta_a(t) + b_{na} e_a(t)) \]
\[ = \dot{\eta}_a^T(t) A_{na}^T P_{na} \eta_a(t) + b_{na}^T e_a(t) P_{na} \eta_a(t) \]
\[ + \eta_a^T(t) P_{na} A_{na} \eta_a(t) + \eta_a^T(t) P_{na} b_{na} e_a(t) \]
\[ = -\eta_a^T(t) Q_{na} \eta_a(t) + 2 e_a(t) b_{na}^T P_{na} \eta_a(t). \]  
(33)

According to the adjusting law given in (15), the time derivatives \( V_3(t), V_4(t) \) lead to

\[ \dot{V}_3(t) = \frac{b_{ea}}{y} \Delta \dot{k}(t) \Delta k(t) \]
\[ = \frac{b_{ea}}{y} \left[ \gamma \left( e_a(t) + \frac{1}{\rho_{cp}} z_{cp}(t) \right) e_a(t) - \sigma_k k(t) \right] \Delta k(t) \]
\[ = \frac{b_{ea}}{y} \left[ \gamma e_a^T(t) \Delta k(t) + \frac{\gamma}{\rho_{cp}} z_{cp}(t) e_a(t) \Delta k(t) - \sigma_k \Delta k^2(t) - \sigma_k k^* \Delta k(t) \right] \]
\[ = b_{ea} e_a^2(t) \Delta k(t) + b_{ea} z_{cp}(t) e_a(t) \Delta k(t) \]
\[ - b_{ea} \sigma_k \Delta k^2(t) - \frac{b_{ea} \sigma_k k^*}{y} \Delta k(t), \]  
(34)

\[ \dot{V}_4(t) = \frac{b_{ea}}{2} \left[ \Delta \theta^T(t) \Gamma^{-1} \Delta \theta(t) + \Delta \theta^T(t) \Gamma^{-1} \Delta \dot{\theta}(t) \right] \]
\[ = \frac{b_{ea}}{2} \left[ e_a^T(t) \Gamma \Gamma^{-1} \Delta \theta(t) \left( e_a(t) + \frac{1}{\rho_{cp}} z_{cp}(t) \right) - \sigma_{\theta_1} \dot{\theta}(t) \Gamma^{-1} \Delta \theta(t) \right. \]
\[ - \sigma_{\theta_2} \left| z_c(t) \right|^2 \dot{\theta}(t) \Gamma^{-1} \Delta \theta(t) \]
\[ + \Delta \theta^T(t) \Gamma^{-1} \Delta \theta(t) \left( e_a(t) + \frac{1}{\rho_{cp}} z_{cp}(t) \right) - \Delta \theta^T(t) \Gamma^{-1} \dot{\theta}(t) \]
\[ - \Delta \theta^T(t) \Gamma^{-1} \sigma_{\theta_2} \left| z_c(t) \right|^2 \dot{\theta}(t) \Gamma^{-1} \left| \dot{\theta}(t) \right|^2 \]
\[ = b_{ea} e_a(t) \Delta \theta^T(t) z_c(t) \]
\[ + b_{ea} z_{cp}(t) \Delta \theta^T(t) z_c(t) \]
\[ - b_{ea} (\sigma_{\theta_1} + \sigma_{\theta_2} \left| z_c(t) \right|^2) \]
\[ \times (\Delta \theta^T(t) \Gamma^{-1} \Delta \theta(t) + \Delta \theta^T(t) \Gamma^{-1} \dot{\theta}(t) \right) \],

(35)

Moreover we have for \( V_5(t) \) and \( V_6(t) \) that

\[ \dot{V}_5(t) = x_n^T(t) P_n x_n(t) + b_n^T \Delta \theta^T z_c(t) + \rho_z z_{cp}(t) \]
\[ = (A_{na} x_n(t) + b_{na} \Delta \theta^T z_c(t) + \rho_z z_{cp}(t))^T P_n x_n(t) \]
\[ + x_n^T(t) P_n (A_{na} x_n(t) + b_{na} \Delta \theta^T z_c(t) + \rho_z z_{cp}(t)) \]
\[ + x_n^T(t) P_n A_{na} x_n(t) + b_{na}^T \Delta \theta^T z_c(t) + \rho_z z_{cp}(t) \]
\[ = x_n^T(t) A_{na}^T P_n x_n(t) \]
\[ + (\Delta \theta^T z_c(t) + \rho_z z_{cp}(t))^T b_{na}^T P_n x_n(t) \]
\[ + x_n^T(t)P_nA_nx_n(t) \\
+ x_n^T(t)P_nb_n(\Delta \theta^T(t)z_c(t) + \rho_z z_{cp}(t)) \\
= -x_n^T(t)Q_nx_n(t) + 2x_n^T(t)P_nb_n(\Delta \theta^T(t)z_c(t)) \\
+ 2x_n^T(t)P_nb_n\rho_z z_{cp}(t), \]  
\( \text{(36)} \)

and

\[ V_6(t) = \frac{b_{ea}}{2\rho_{cp}} [z_{ea}^T(t)P_zz_{ea}(t) + z_{cp}^T(t)P_zz_{cp}(t)] \]

\[ = \frac{b_{ea}}{2\rho_{cp}} [(A\dot{z}_{ea}(t) + b_c u_f(t))^T z_{ea}(t) \\
+ z_{cp}^T(t)P_{1e}(A\dot{z}_{ea}(t) + b_c u_f(t))] \\
= \frac{b_{ea}}{2\rho_{cp}} [z_{ea}(t)A^T_1P_zz_{ea}(t) + b_c^T P_zz_{ea}(t)u_f(t) \\
+ z_{cp}^T(t)P_{1e}A\dot{z}_{ea}(t) + z_{cp}^T(t)P_{1e}b_c u_f(t)] \\
= \frac{b_{ea}}{2\rho_{cp}} [-z_{ea}^T(t)Q_{1e}z_{ea}(t) + 2z_{cp}^T(t)P_{1e}b_c \\
\times [-k(t)e_a(t) - \dot{\theta}^T(t)z_{ea}(t) - \rho_z z_{cp}(t)]] \\
= \frac{b_{ea}}{2\rho_{cp}} [-z_{ea}^T(t)Q_{1e}z_{ea}(t) + 2z_{cp}^T(t) \\
\times [-k(t)e_a(t) - \dot{\theta}^T(t)z_{ea}(t) - \rho_z z_{cp}(t)]] \\
= \frac{b_{ea}}{2\rho_{cp}} z_{ea}^T(t)Q_{1e}z_{ea}(t) - \frac{b_{ea}}{\rho_{cp}} z_{cp}^T(t)e_a(t) \Delta k(t) \\
- \frac{b_{ea}}{\rho_{cp}} k^* z_{cp}(t)e_a(t) - \frac{b_{ea}}{\rho_{cp}} z_{cp}(t)\Delta \theta^T(t)z_{ea}(t) \\
- \frac{b_{ea}}{\rho_{cp}} \theta^* T z_{ea}(t)z_{ea}(t) - \frac{b_{ea}}{\rho_{cp}} \rho_z z_{cp}(t). \]  
\( \text{(37)} \)

Consequently, the time derivative of \( V(t) \) is obtained as

\[ \dot{V}(t) = a_{ea} e_a(t)^2 + \kappa^T \eta \kappa(t)e_a(t) \\
- b_{ea} e_a(t)k^* + b_{ea} e_a(t)c_{1a}^T x_n(t) \\
+ b_{ea} e_a(t)d_n(\Delta \theta^T(t)z_{ea}(t)) \\
+ (d_n - 1)b_{ea} e_a(t)\rho_z z_{cp}(t) \\
- \eta_{1a}^T(t)Q_{1a}\eta(t) + 2e_a(t)b_{1a}^T P_{1a} \eta_a(t) \\
- \frac{b_{ea}}{\gamma} \Delta k^2(t) - \frac{b_{ea}}{\gamma} k^* \Delta k(t) \\
- \frac{b_{ea}}{\gamma} (\sigma_{1a} + \sigma_{2a} \Vert z_{ea}(t) \Vert^2) \\
\times (\Delta \theta^T(t) \Gamma^{-1} \hat{\Delta} \theta + \Delta \theta^T(t) \Gamma^{-1} \theta^*) \\
- x_n^T(t)Q_nx_n(t) + 2x_n^T(t)P_n b_n(\Delta \theta^T(t)z_{ea}(t)) \\
+ 2x_n^T(t)P_nb_n \rho_z z_{cp}(t) \\
- \frac{b_{ea}}{2\rho_{cp}} z_{ea}(t)Q_nz_{ea}(t) \\
- \frac{b_{ea}}{\rho_{cp}} k^* z_{cp}(t)e_a(t) \\
- \frac{b_{ea}}{\rho_{cp}} \theta^* T z_{ea}(t)z_{ea}(t) - \frac{b_{ea} \theta^* T z_{ea}(t)z_{ea}(t)}{\rho_{cp}} - \frac{b_{ea} \rho_z z_{cp}(t)}{\rho_{cp}^2}. \]  
\( \text{(38)} \)

Thus, the time derivative of \( V(t) \) can be evaluated as

\[ \dot{V}(t) \leq -\left( \frac{b_{ea}}{2\rho_{cp}} \left( 1 - \frac{1}{2\delta_{7}} \right) \right) k^* - a_{ea} \\
- \frac{b_{ea}}{\delta_2} - \frac{b_{ea}d_n}{\delta_3} - \frac{1}{2\delta_4} - \frac{b_{ea}(d_n - 1)\rho_z}{\delta_8} \Vert e_a(t) \Vert^2 \\
- \lambda_{\min}[Q_{pq}] - \delta_1 \Vert e_{pq}^T \Vert^2 - 2\delta_6 \Vert b_{pq}^T \Vert^2 \Vert P_{pq} \Vert^2 \\
\times \Vert \eta_{pq}(t) \Vert^2 \\
- \lambda_{\min}[Q_n] - b_{ea} \sigma_2 \Vert e_n^T \Vert^2 - 2\delta_6 \Vert P_n \Vert^2 \Vert b_n \Vert^2 \\
- 2\delta_8 \rho_z \Vert P_{pq} \Vert^2 \Vert b_{pq} \Vert^2 \\
\times \Vert x_n \Vert^2 \\
- \frac{b_{ea} \sigma_{1a}}{2\gamma} \Delta k(t)^2 \\
- \frac{b_{ea} \sigma_{1a}(\lambda_{\min}[\Gamma^{-1}] - \delta_5) \Vert \Delta \theta(t) \Vert^2}{2\delta_5} \\
- \left( \frac{b_{ea} \sigma_{1a}(\lambda_{\min}[\Gamma^{-1}] - \delta_5) - b_{ea}d_n \delta_3 - \frac{1}{2\delta_6}}{\lambda_{\min}[Q_{pq}]} \right) \rho_{cp} \sigma_{1a} \sigma_2 \frac{1}{2\delta_5} \Vert \Gamma^{-1} \Vert^2 \Vert \theta^* \Vert^2 \\
- \delta_7 k^* \| P_c b_c \|^2 - \frac{1}{2\delta_8} \| \theta^* \|^2 \]  
\times \| z_c(t) \|^2 \\
- \left( \frac{b_{ea}}{\rho_z} - \frac{b_{ea}}{\rho_{cp}} \delta_z - \delta_7 b_{ea}(d_n - 1)\rho_z - \frac{\rho_z}{2\delta_9} \right) \| z_{cp} \|^2 \\
+ \frac{b_{ea} \sigma_{1a}}{4\delta_5} \| \Gamma^{-1} \|^2 \| \theta^* \|^2 \\
+ \frac{b_{ea} \sigma_{1a}}{2\gamma} k^* \| \kappa \|^2, \]  
where \( \delta_1 \) to \( \delta_9 \) are any positive constants. Finally, considering a sufficiently large \( k^* \), it can be confirmed by setting a large \( \rho_{cp} \), small \( \rho_z \) and \( Q_c \) with a large \( \lambda_{\min}[Q_n] \) that there exist appropriate \( \delta_1, Q_{pq} \) and \( Q_n \) such that

\[ \frac{b_{ea}}{\rho_{cp}} \left( 1 - \frac{1}{2\delta_7 \rho_{cp}} \right) k^* - a_{ea} - \frac{b_{ea} \sigma_{1a}}{\delta_2} - \frac{b_{ea} d_n}{\delta_3} - \frac{1}{2\delta_4} - \frac{b_{ea}(d_n - 1)\rho_z}{\delta_8} \]  
\[ = \alpha_1 > 0 \]
\[
\begin{split}
\lambda_{\min}[Q_{\alpha}] - \delta_1 \|e_{\alpha}^T\|^2 - 2\delta_4 \|b_{\alpha}^T\|^2 \|P_{\alpha}\|^2 &= \alpha_2 > 0 \\
\lambda_{\min}[Q_n] - b_{\alpha}\epsilon_{\delta_2} \|e_{\alpha}^T\|^2 - 2\delta_6 \|P_n\|^2 \|b_n\|^2 \\
-2\delta_9 \rho_z \|P_n\|^2 \|b_n\|^2 &= \alpha_3 > 0 \\
\frac{b_{\alpha}\epsilon\lambda_{\min}[Q_n]}{2\gamma} &= \alpha_4 > 0 \\
b_{\alpha}\epsilon\sigma_{\alpha_1}(\lambda_{\min}[\Gamma^{-1}] - \delta_5) &= \alpha_5 > 0 \\
b_{\alpha}\epsilon\sigma_{\alpha_2}(\lambda_{\min}[\Gamma^{-1}] - \delta_5) - b_{\alpha}\epsilon\sigma\delta_3 - \frac{1}{2\delta_6} &= \alpha_6 > 0 \\
b_{\alpha}\epsilon\sigma_{\alpha_3}[Q_{\alpha}] - \frac{b_{\alpha}\epsilon\sigma_{\alpha_2}}{4\delta_5} \|\Gamma^{-1}\|^{2\|\theta^\alpha\|^2} \\
- \frac{b_{\alpha}\epsilon\delta\lambda_{\min}[Q_n]}{2\rho_p} \|P_{\alpha}\|^2 \|b_{\alpha}\|^2 \\
- \frac{b_{\alpha}\epsilon\delta\|\theta^\alpha\|^2}{4\rho_p \rho_z} &= \alpha_7 > 0 \\
\frac{b_{\alpha}\epsilon\sigma\rho_z}{\rho_p} - \frac{b_{\alpha}\epsilon\sigma\delta_3 - \delta_b b_{\alpha}(d_n - 1)\rho_z}{\frac{\rho_z}{2\delta_9}} &= \alpha_8 > 0.
\end{split}
\]

Therefore, we obtain
\[
\begin{split}
\dot{V}(t) &\leq -\alpha_1 \|e_{\alpha}(t)\|^2 - \alpha_2 \|\eta_{\alpha}(t)\|^2 \\
&\quad - \alpha_3 \|x_{\alpha}(t)\|^2 - \alpha_4 \|\theta^\alpha(t)\|^2 - \alpha_5 \|\Delta\theta^\alpha(t)\|^2 \\
&\quad - \alpha_6 \|z_{\alpha}(t)\|^2 - \alpha_7 \|z_{\alpha}(t)\|^2 \\
&\quad - \alpha_8 \|z_{\alpha}(t)\|^2 + R \\
&\leq - \alpha_1 \|e_{\alpha}(t)\|^2 - \alpha_2 \|\eta_{\alpha}(t)\|^2 \\
&\quad - \alpha_3 \|x_{\alpha}(t)\|^2 - \alpha_4 \|\theta^\alpha(t)\|^2 - \alpha_5 \|\Delta\theta^\alpha(t)\|^2 \\
&\quad - \alpha_6 \|z_{\alpha}(t)\|^2 - \alpha_7 \|z_{\alpha}(t)\|^2 + R,
\end{split}
\]

where
\[
R = \frac{b_{\alpha}\epsilon\sigma_{\alpha_1}}{4\delta_5} \|\Gamma^{-1}\|^2 \|\theta^\alpha\|^2 + \frac{b_{\alpha}\epsilon\sigma_k}{2\gamma} k^2.
\]

Thus the time derivative of \(V(t)\) can be evaluated by
\[
\dot{V}(t) \leq -\alpha V(t) + R
\]
with
\[
\alpha = \min \left( \frac{\alpha_1}{\lambda_{\max}[P_{\alpha}]}, \frac{\alpha_3}{\lambda_{\max}[P_n]}, \frac{2\alpha_4\sqrt{T}}{b_{\alpha}, \frac{b_{\alpha}\epsilon\sigma_{\alpha_1}}{2\gamma}, \frac{2\alpha_5}{b_{\alpha}\epsilon\lambda_{\max}[\Gamma^{-1}]}, \frac{2\alpha_6}{b_{\alpha}\epsilon\lambda_{\max}[\Gamma^{-1}], \frac{2\rho_p \rho_z}{b_{\alpha}\epsilon\lambda_{\max}[P_{\alpha}]}}, \right).
\]

Consequently, we have the following theorem concerning the boundedness of all the signals in the control system:

**Theorem 5.1.** By setting a large \(\rho_{cp}\), small \(\rho_z\) and \(Q_c\) with a large \(\lambda_{\min}[Q_n]\), there exists a sufficiently large ideal feedback gain \(k^*\) such that all the signals in the control system are bounded.

### 5.2. Convergence analysis of the tracking error

We have the following theorem concerning the convergence of \(|e_{\alpha}(t)|\) and \(|k(t)e_{\alpha}(t)|\):

**Theorem 5.2.** For the ideal feedback gain \(k^*\) satisfying the condition in Theorem 5.1, the mean square of \(|e_{\alpha}(t)|\) converges to a range within \(\delta_{ke} = \frac{k^*}{\alpha_1}\) and the mean value of \(|k(t)e_{\alpha}(t)|\) converges to a range within \(\delta_{ke} = \frac{k^*}{\sqrt{\alpha_1}k^{1/2}}\).

That is we have
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T |e_{\alpha}(t)|^2 dt \leq \frac{R^*}{\alpha_1} = \delta_{ke} \\
\lim_{T \to +\infty} \frac{1}{T} \int_0^T |k(t)e_{\alpha}(t)| dt \leq \frac{1}{\sqrt{\alpha_1}} \frac{R^*}{\sqrt{\alpha_4}} = \delta_{ke}.
\]

**Proof.** Firstly, we consider the mean square of \(|e_{\alpha}(t)|\) by setting \(\delta_5\) so as to be
\[
\alpha_5 = \frac{1}{2} b_{\alpha}\epsilon\sigma_{\alpha_1}\lambda_{\min}[\Gamma^{-1}],
\]
and \(\delta_5 = \frac{1}{2}\lambda_{\min}[\Gamma^{-1}],\) it follows in (40) that
\[
R \leq R^* = \frac{b_{\alpha}\epsilon\sigma_{\alpha_1}[\Gamma^{-1}]^2}{2\lambda_{\max}[\Gamma^{-1}]} \|\theta^\alpha\|^2 + \frac{b_{\alpha}\epsilon\sigma_k}{2\gamma} k^2.
\]

Then we have from (40) that
\[
\dot{V}(t) \leq -\alpha_1 |e_{\alpha}(t)|^2 + R^*.
\]

Integrating this from 0 to \(T\), we have
\[
\frac{V(T) - V(0)}{T} \leq -\alpha_1 \int_0^T |e_{\alpha}(t)|^2 dt + R^*.
\]

Finally, as \(T \to \infty\), it follows that
\[
0 \leq -\lim_{T \to +\infty} \frac{\alpha_1}{T} \int_0^T |e_{\alpha}(t)|^2 dt + R^*,
\]
and thus the mean square of \(|e_{\alpha}|\) can be evaluated as
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T |e_{\alpha}(t)|^2 dt \leq \frac{R^*}{\alpha_1}.
\]

By setting the designed parameters \(\sigma_{\alpha_1}, \sigma_{\alpha_2}\) to satisfy
\[
\frac{b_{\alpha}\epsilon\sigma_{\alpha_1}[\Gamma^{-1}]^2}{2\lambda_{\max}[\Gamma^{-1}]} \|\theta^\alpha\|^2 < \sigma_{\alpha_2} (\text{any positive constant})\) and
\[
\frac{\sigma_{\alpha_2}}{\sqrt{\alpha_1}} < k^{-1/2}\]
and considering a sufficient large \(k^*\), we can get
\[
\frac{R^*}{\alpha_1} \leq \delta_{ke}
\]
for a small constant \(\delta_{ke} \in O(k^{-1})\), and
thus the mean square of $|e_a(t)|$ converges to a small range within $\delta_{ea}$.

Similarly, for $k(t)$ we can obtain from (47) that
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T k(t)^2 \, dt \leq \frac{R^*}{\alpha_4}. \tag{48}
\]

From Schwarz’s inequality, we have from (47) and (48) that
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T |k(t)e_a(t)| \, dt \leq \frac{1}{\sqrt{\alpha_1 \alpha_4}} R^*, \tag{49}
\]
where
\[
\frac{R^*}{\sqrt{\alpha_4}} = \frac{h_{ea} \sigma_{\theta 1} \|\Gamma^{-1}\|^2}{2\lambda_{\min}[\Gamma^{-1}]} \sqrt{\frac{2\gamma}{h_{ea} \sigma_k}} + \sqrt{\frac{2\gamma}{2\gamma}}.
\]

Now, consider a case where the designed parameters $\gamma, \sigma_k, \Gamma, \sigma_{\theta 1}$ in the adaptive controller are set so as to satisfy
\[
\sqrt{\frac{\sigma_k}{\gamma}} \leq k^{*-4}, \tag{50}
\]
\[
\frac{h_{ea} \sigma_{\theta 1} \|\Gamma^{-1}\|^2}{2\lambda_{\min}[\Gamma^{-1}]} \leq \sqrt{\frac{\sigma_k}{\gamma}},
\]
for a considered $k^*$, it follows from Theorem 5.2 that
\[
\delta_{ea} = O(k^*-1),
\]
\[
\delta_{ke} = O \left( k^{*-1} \right).
\]
Therefore, considering sufficiently large $k^*$ and setting designed parameters to satisfy (50), $|e_a(t)|$ and $|k(t)e_a(t)|$ converge to a small range. Thus, since a small $|k(t)e_a(t)|$ leads to small $|y(t)|$ from (19), we can conclude that $e(t)$ also converges to a small range under the condition that the designed parameters satisfy (50).

Remark 5.1: $f(x) \in O(x)$ means that there exists positive constant $a$ and $\delta$ such that
\[
|f(x)| \leq ax \quad \text{for} \quad 0 < x < \delta. \tag{51}
\]

6. Validation through numerical simulation

In order to confirm the effectiveness of the proposed method, the proposed method is validated through numerical simulations. Consider the following second-order linear systems:
\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2.25 & -0.06 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + d_1(t)
\]
\[
y(t) = \begin{bmatrix} 2.25 & 0 \end{bmatrix} x(t) + d(t),
\]
where $d(t), d_1(t)$ are unknown disturbances and given as follows:
\[
d(t) = 2\sin t
\]
\[
d_1(t) = \begin{cases} 
0 & (0 \leq t < 20) \\
[\sin t \ 0]^T & (20 \leq t < 70) \\
[0 \ 0]^T & (70 \leq t \leq 120). 
\end{cases} \tag{52}
\]

And consider the following PFC to make the augmented system ASPR:
\[
H(s) = \frac{1}{s + 5}. \tag{53}
\]

In the controller, $N_{IM}$ is set as
\[
N_{IM}(s) = s^3 + \frac{17}{3} s^2 + \frac{41}{4} s + 6. \tag{54}
\]

The designed parameters for the adaptive adjusting laws are given by
\[
\gamma = 2.0 \times 10^9, \quad \sigma_k = 1.0 \times 10^{-6}, \quad \Gamma = 5.0 \times 10^8 \\
\sigma_{\theta 1} = 1.0 \times 10^{-3}, \quad \sigma_{\theta 2} = 3 \times 10^2, \\
\rho_{\phi} = 1.0 \times 10^{12} \\
\rho_z = 1.0 \times 10^{-5}. \tag{55}
\]

Figure 5 to Figure 6 shows the results of the proposed method and the fixed internal model system separately under the uncertain disturbances. The fixed internal model system is designed with the following fixed pre-internal model compensator:
\[
G_{IM}(s) = \frac{s^3 + \frac{17}{3} s^2 + \frac{41}{4} s + 6}{s^3 + 4s}. \tag{56}
\]

Figure 7 shows the error comparison of two methods. Compared with the fixed internal model system, the result with the proposed method maintains the better control performance under the existence of the uncertain disturbances.

Figures 8 and 9 shows the results of the proposed method and the fixed internal model system separately with the same reference signal:
\[
y_m(t) = \begin{cases} 
0 & (0 \leq t < 2\pi) \\
\sin 2t & (2\pi \leq t < 20\pi) \\
\sin 0.5t & (20\pi \leq t < 40\pi) \\
\sin 0.2t & (40\pi \leq t \leq 200). \tag{57}
\end{cases}
\]

Figure 10 shows the error comparison of two methods. The result of the fixed internal model system is better with the exact value of the sin wave reference from $2\pi$ sec to $20\pi$ sec, however the output will no longer track the reference signal well even though the proposed adaptive control system can maintain the better tracking performance when the sin wave reference changes.

Remark 6.1: In the simulation, we set large $\gamma$ and $\Gamma$, and small $\sigma_k$ and $\sigma_{\theta 1}$ in order to show and validate the theoretical behaviour of the designed control system.
Figure 5. Adaptive control system. (a) Output of the plant. (b) Input of the plant. (c) Adaptive feedback gain $k(t)$. (d) Adaptive gain $\theta(t)$.

Figure 6. Fixed internal model system. (a) Output of the plant. (b) Input of the plant.

Figure 7. Error comparison.

according to Theorem 5.2. However, in a practical situation with output noise, it will be required to set larger $\sigma_k$ and $\sigma_{\theta 1}$ to maintain robustness of the designed adaptive control system.

7. Conclusion

In this paper, we proposed an adaptive output regulation for linear systems with uncertain disturbances and reference signals. In the proposed method, output
feedback-based adaptive internal model control strategy is provided to handle the systems with uncertain disturbances. It was also shown that the boundedness of all signals in the proposed adaptive output regulation control was maintained via ASPR-based adaptive
output feedback for ASPR augmented system with a PFC. Moreover, the convergence of the output tracking error signal was analysed. The effectiveness of the proposed method was confirmed through numerical simulation for an unknown second-order system with uncertain disturbances and reference signals.

Disclosure statement
No potential conflict of interest was reported by the author(s).

Notes on contributors

Zhizhao Yao He received his BE degree in Vehicle Engineering from Chongqing Jiaotong University, China in 2015. He joined Chongqing Changan Automobile Co., Ltd from 2015 to 2017. He is currently a Master course student at Kumamoto University. His research interests include output feedback-based adaptive control and its applications.

Kota Akaike He received his MS degree in Mechanical Engineering from Kumamoto University, Japan in 2020. He is currently a PhD student at Kumamoto University. His research interests include robust and adaptive control for nonlinear systems.

Ikuro Mizumoto He received his BE degree, the ME degree and the Dr. Eng. degree, all in mechanical engineering from Kumamoto University, Kumamoto, Japan, in 1989, 1991 and 1996, respectively. Since 1991 he has been with Kumamoto University, where he is currently a Professor in the Robot, Control and Instrumentation Group in Faculty of Advanced Science and Technology. His research interests over last decade have been adaptive control system design, robust adaptive control and output feedback-based control for nonlinear systems and their applications.

References

[1] Bar-Kana I. Positive realness in multivariable continuous-time systems. J Franklin Inst. 1991;328(4):403–418.
[2] Kaufman H, Barkana I, Sobel K. Direct adaptive control algorithms. 2nd ed. New York: Springer; 1997.
[3] Bar-kana I. Parallel feedforward and simplified adaptive control. Int J Adapt Control Signal Process. 1987;1(2):95–109.
[4] Iwai Z, Mizumoto I. Realization of simple adaptive control by using parallel feedforward compensator. Int J Control. 1994;59(6):1543–1565.
[5] Mizumoto I, Iwai Z. Simplified adaptive model output following control for plants with unmodelled dynamics. Int J Control. 1996;64(1):61–80.
[6] Fradkov AL. Shunt output feedback adaptive controller for nonlinear plants. Proc. of 13th IFAC World Congress; San-Francisco; Vol. K, 1996 July. p. 367–372.
[7] Feg G, Palaniswami M. Unified treatment of internal model principle based adaptive control algorithms. Int J Control. 1991;54(4):883–901.
[8] Nikiforov VO. Adaptive servocompensation of input disturbances. Proc. of the 13th IFAC World Congress; San-Francisco, USA: Vol. K, July 1996. p. 175–180.
[9] Marino R, Tomei P. Output regulation of linear systems with adaptive internal model. Proc. of the 40th IEEE CDC; Orlando, Florida, USA: 2001 December. p. 745–749.
[10] Yao ZH, Akaike K, Mizumoto I. ASPR based Adaptive Output Regulation for System with Uncertain Disturbances. Proc. of the SICE Annual Conference 2020; Chiang Mai, Thailand (online): 2020. p. 1186–1191.
[11] Mizumoto I, Iwai Z. Adaptive pid control system design based on aspr property of systems. Chapter in book, ‘Advances in PID Control’, IsTech. 2011.
[12] Isidori A. Nonlinear control systems. 3rd ed. London: Springer; 1995.