HYPERKÄHLER NAHM TRANSFORMS

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ABSTRACT. Given two hyperkähler manifolds \( M \) and \( N \) and a quaternionic instanton on their product, a hyperkähler Nahm transform can be defined, which maps quaternionic instantons on \( M \) to quaternionic instantons on \( N \). This construction includes the case of Nahm transform for periodic instantons on \( \mathbb{R}^4 \), the Fourier-Mukai transform for instantons on K3 surfaces, as well as the Nahm transform for ALE instantons.

1. Introduction

The Nahm transform is an instance of “geometrical Fourier transform”, originally introduced by W. Nahm \([19]\) as an extension of the ADHM method to construct time-invariant instantons (alias monopoles) on \( \mathbb{R}^4 \). This idea was subsequently applied, by P. Braam & P. van Baal \([8]\), to study instantons on flat 4-tori. Nowadays, it is clearly understood that these constructions are special examples of a much wider framework, which provides an unified description of correspondences between solutions of the anti-self-duality equations that are invariant under dual subgroups of translations of \( \mathbb{R}^4 \) (a detailed exposition can be found in \([14]\)).

An alternative approach to the Nahm transform is provided by the so-called Fourier-Mukai transform in algebraic geometry. In this case one obtains correspondences between moduli spaces of stable sheaves over abelian or K3 surfaces \([3]\).

Yet another generalization of the Nahm transform can be defined for instantons over asymptotically locally Euclidean (ALE) 4-manifolds \([5]\). ALE spaces are diffeomorphic to minimal resolutions of \( \mathbb{C}^2/\Gamma \), where \( \Gamma \) is a finite subgroup of \( SU(2) \), and are endowed with complete hyperkähler Riemannian metric. The moduli spaces of instantons over ALE spaces were thoroughly studied by Nakajima \([18]\) and by Kronheimer & Nakajima \([16]\). Contrary to the previous examples, the Nahm transform on ALE spaces may fail to be invertible.

It turns out that all these various species of Nahm transforms can be adequately described in the setting of hyperkähler geometry. Actually, for hyperkähler manifolds of any dimension, it is possible to generalize the notion of instanton by defining the so-called quaternionic instantons \([17, 20]\). For these objects, one can prove a generalized Ward correspondence, which relates quaternionic
instantons on $M$ to some holomorphic vector bundle on the twistor space of $M$. Now, given two hyperkähler manifolds $M$ and $N$ and a quaternionic instanton $Q$ on $M \times N$, a hyperkähler Nahm transform is obtained roughly according to the following recipe: take a quaternionic instanton on $M$ pull it back to $M \times N$, twist it by $Q$, and push it down to $N$. The existence of the hyperkähler Ward correspondence ensures (under mild hypotheses) that the result of these operations is a quaternionic instanton on $N$.

In this expository paper we intend to review, from the unifying hyperkähler viewpoint, a number of results on Nahm transforms. Section 2 is devoted to the description of the Ward correspondence for quaternionic instantons. In Sections 3 and 4 we present the hyperkähler Nahm transform, discussing a few fundamental examples which point to an interesting new conjecture.

2. Hyperkähler Ward correspondence

We shall briefly describe the correspondence between quaternionic instantons on a hyperkähler manifold and certain holomorphic bundles over its twistor space (for a more general treatment valid for quaternionic Kähler manifolds see [11, 30]).

Let $(M, g)$ be a (possibly non-compact) $4n$-dimensional hyperkähler manifold (i.e. the holonomy group $H$ of the Riemannian metric $g$ is contained in $Sp(n)$). We denote by $\{I_k\}_{k=1,2,3}$ the triple of complex structures and by $\{\omega_k\}_{k=1,2,3}$ the triple of Kähler structures. We recall that the complex non-degenerate 2-form $\Omega = \omega_2 + i\omega_3$ is holomorphic w.r.t. $I_1$. The essential tool in the study of hyperkähler geometry is the twistor space, which encodes the information about the $\mathbb{C}P^1$ family of complex structures on the hyperkähler manifold. The twistor space $Z$ of $M$ is the sphere bundle associated with the rank 3 vector bundle $P \times_H sp(1)$, where $P$ is the holonomy bundle of $M$; we denote by $p : Z \to M$ and by $q : Z \to \mathbb{C}P^1$ the natural projections. On $Z$ there is a complex structure (different from the product structure), and an anti-holomorphic involution $\tau$, preserving the fibers of $p$, the so-called twistor lines. The differential of $\tau$ acts in the following way:

$$d\tau(v, z) = (\bar{v}, d\bar{v}(z)) \quad \forall (v, z) \in T_u Z = T_{p(u)} M \oplus T_{q(u)} \mathbb{C}P^1,$$

where $\iota : \mathbb{C}P^1 \to \mathbb{C}P^1$ is the antipodal map.

For any point $x \in M$, the endomorphism $\Phi = \sum_{k=1}^3 I_k \otimes I_k$ of $\Lambda^2 T^*_x M$ satisfies the identity $\Phi^2 = 2\Phi + 3\text{Id}$, so it has eigenvalues 3 and $-1$. The eigenspace $\mathcal{E}_x$ corresponding to the eigenvalue 3 is isomorphic to $sp(n)$ and can be characterized as

$$\mathcal{E}_x = \bigcup_{u \in p^{-1}(x)} \Lambda^{1,1}_u T^*_x M,$$
where $\Lambda^{1,1}_u T^* M$ is the space of the 2-forms of type $(1,1)$ w.r.t. the complex structure associated to the point $q(u) \in \mathbb{C}P^1$.

A quaternionic instanton on $M$ is a pair $(E, \nabla)$, where $E$ is a complex smooth vector bundle on $M$ and $\nabla$ a connection whose curvature at any point $x \in M$ takes values in $\mathfrak{c}_x \otimes \text{End} E_x$. When $\dim M = 4$, this definition corresponds to the usual one, since anti-self-dual connections are precisely the 2-forms which are of type $(1,1)$ w.r.t. all complex structures compatible with the hyperkähler metric \[1\].

Let $(E, \nabla)$ be a quaternionic instanton on $M$ such that the connection $\nabla$ is compatible with a given hermitian metric $h$ on $E$. It is easy to check that the pull-back connection $p^* \nabla$ induces a complex structure on the pull-back bundle $F = p^* E$ endowed with the pull-back hermitian metric $p^* h$. Moreover, we can define a positive real form $\beta : F \to F^*$, i.e. an antilinear anti-holomorphic bundle isomorphism covering the involution $\tau : Z \to Z$. We set

$$\beta(v)(w) = p^* h(v, \bar{\tau}(w)),$$

where $\bar{\tau} : F \to F$ is the lifting of $\tau$ to $F$. The following generalized Ward correspondence holds \[4, 20\].

**Theorem 1.** Gauge equivalence classes of rank $k$ hermitian quaternionic instantons on the hyperkähler manifold $M$ are in one-to-one correspondence with isomorphism classes of rank $k$ holomorphic bundles on the twistor space $Z$ of $M$, which are holomorphically trivial along the fibers of $p : Z \to M$ and carry a positive real form.

**Remark.** When $M$ is compact, irreducible quaternionic $SU(n)$-instantons correspond to holomorphic bundles which are $\mu$-stable of degree zero w.r.t. any Kähler structure induced on $M$ by the hyperkähler structure \[21\]. In particular, a line bundle fails to be a quaternionic instantons if its first Chern class is not orthogonal to the cohomology classes of the Kähler forms $\{\omega_k\}_{k=1,2,3}$.

3. **The Hyperkähler Nahm Transform**

Let us now suppose we are given the following set of data: two hyperkähler manifolds $M$, $N$ and an hermitian quaternionic instanton $(Q, \nabla_Q)$ on the product $M \times N$. Let $(Q_\xi, \nabla|_{Q_\xi})$ denote the restriction of $(Q, \nabla_Q)$ to $M \times \{\xi\}$, for any $\xi \in N$. If $(E, \nabla)$ is a vector bundle with connection on $M$, we can define the family of connections

$$\nabla_\xi = \nabla \otimes 1_{Q_\xi} + 1_E \otimes \nabla|_{Q_\xi}$$

on the bundle family $E \otimes Q_\xi$, for any $\xi \in N$. 


The twistor space $Z_{M \times N}$ of $M \times N$ is the fiber product along $\mathbb{CP}^1$ of the twistor spaces $Z_M$ and $Z_N$ of $M$ and $N$. So, we have the commutative diagram:

\[
\begin{array}{cccc}
Z_M & \xrightarrow{p_M} & Z_{M \times N} & \xrightarrow{p_N} & Z_N \\
\downarrow{p_M} & & \downarrow{q} & & \downarrow{p_N} \\
M & \leftarrow & \pi_M M \times N & \leftarrow & N \\
& & & & \\
\end{array}
\]

where the horizontal arrows are holomorphic maps while the vertical arrows are only smooth maps. By our assumptions, the pull-back of $(Q, \nabla_Q)$ to $Z_{M \times N}$ is a holomorphic bundle, which is holomorphically trivial along the fibers of $p : Z \to M$ and carries a positive real form.

Any hyperkähler manifold carries a standard spin structure; the spinor bundles $S^\pm$ are trivial and the projectivization of $S^+$ coincides with the twistor space. By coupling the connection $\nabla_\xi$ with the Dirac operators on $S^\pm$, we get the family of Dirac operators:

\[
D^\pm_\xi : L^2_k(E \otimes Q_\xi \otimes S^\pm) \to L^2_{k-1}(E \otimes Q_\xi \otimes S^\mp),
\]

parametrized by the points of $N$. If $M$ is non-compact, one has to be more careful, and consider suitable weighted Sobolev space completions of the spaces of $C^\infty$ sections in order to guarantee that the Dirac operators $D^\pm_\xi$ are Fredholm, but we do not want to enter in these details here (for an example, see Section 4.3 below).

**Theorem 2.** Let $(E, \nabla)$ be an hermitian quaternionic instanton on $M$ and assume that the index $\hat{E} = \text{Ind}(D^\pm)$ of the family of Dirac operators $D^\pm_\xi$ is a smooth vector bundle on $N$. Then $\hat{E} \to N$ admits an hermitian quaternionic instanton $\hat{\nabla}$.

**Proof.** The index bundle $\hat{E}$ is endowed with the index connection $\hat{\nabla}$, obtained by projecting the trivial connection on $L^2_k(E \otimes Q_\xi \otimes S^\pm)$ down to $\ker D^\pm_\xi$. It is easy to check that the pull-back of $(\text{Ind}(D^\pm), \hat{\nabla})$ to $Z_N$ is a holomorphic bundle holomorphically trivial along the fibers of $p : Z \to M$ and carrying a positive real form. \qed

Alternatively, when $M$ and $N$ are compact, we can use the machinery of higher direct images of coherent sheaves to reformulate the index construction above.

Let us fix a complex structure $I_z$ on $M$; we denote by $M_z$ the resulting complex manifold. If $(E, \nabla)$ is quaternionic instanton, we denote by $\mathcal{E}_z$ its associated
sheaf of holomorphic sections w.r.t. the given complex structure. We say that $(E, \nabla)$ satisfies the odd (resp. even) IT condition if

$$H^k(M, \mathcal{E}_z \otimes \mathcal{Q}_\xi) = 0 \quad \text{for } k \text{ even (resp. odd)} \text{ and all } \xi \in N.$$ 

The definition is well-posed, since the cohomology groups $H^k(M, \mathcal{E}_z \otimes \mathcal{Q}_\xi)$ do not depend on the choice of the complex structure [21]. Consequently, we shall drop the subscript $z$ in what follows.

By general results (see [2]), if $(E, \nabla)$ satisfies the odd IT condition, then one has the identification of holomorphic vector bundles

$$-\text{Ind} D^+ = \bigoplus_k R^k \pi_\ast N^\ast (\pi^\ast_M \mathcal{E} \otimes \mathcal{Q}).$$

(when the even IT condition is satisfied, one takes $D^-$ instead). Let $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{Q}}$ denote the sheaves of holomorphic sections of the pull-back bundles $p^\ast_M E$ and $q^\ast \mathcal{Q}$, respectively. It can be proved (being careful, since the vertical maps of the diagram [1] are not holomorphic maps) that the pull-back holomorphic bundle $p^\ast_N (-\text{Ind} D^+)$ coincides with $\bigoplus_k R^k \rho_\ast N^\ast (\rho^\ast_M \tilde{\mathcal{E}} \otimes \tilde{\mathcal{Q}})$. Summing up, we get the following result.

**Theorem-Definition 3.** Let $M$, $N$ be two compact hyperkähler manifolds, and let $(Q, \nabla_Q)$ be an hermitian quaternionic instanton over the product $M \times N$. Assume that $(E, \nabla)$ is an hermitian quaternionic instanton on $M$ satisfying the odd IT condition. Then, $\bigoplus_k R^k \pi_\ast N^\ast (\pi^\ast_M \mathcal{E} \otimes \mathcal{Q})$ is a quaternionic instanton on $M$, which is called the hyperkähler Nahm transform of $(E, \nabla)$ and is denoted by $(\hat{E}, \hat{\nabla})$.

**Remark.** Given the hermitian quaternionic instanton $(Q, \nabla_Q)$, the correspondence between quaternionic instantons on $M$ and quaternionic instantons on $N$ is not in general invertible. In particular, the hyperkähler Nahm transform of an irreducible quaternionic instanton may fail to be irreducible.

### 4. Hyperkähler Nahm transforms for instantons

Let $M$ be a 4-dimensional hyperkähler manifold (not necessarily compact) and let $N$ be a connected component of the moduli of (irreducible) instantons on $N$ (with some “framing” at infinity if $M$ is non-compact); i.e. each point $\xi \in N$ can be regarded as an irreducible anti-self-dual connection $\nabla_\xi$ on a fixed vector bundle $E \to M$. Then $N$ is also a hyperkähler manifold. Recall that one can define a *universal bundle with connection* over the product $M \times N$ in the following way [11, 13]. Let $\mathcal{A}$ denote the set of all connections on $E$, and let $\mathcal{G}$ denote the group of gauge transformations (i.e. bundle automorphisms). Again, if $M$ is non-compact, one must use the right analytical
framework to define $A$ and $G$, see Section 4.3 below. Moreover, let $G$ denote the structure group of $E$, so that $E$ can be associated with a principal $G_E$-bundle $P$ over $M$ by means of some representation $\rho : G \to \mathbb{C}^n$, where $n = \text{rank } E$. The gauge group $G$ acts on $E \times A$ by $g(p, A) = (g(p), g(A))$; This action has no fixed points, and it yields a principal $G$-bundle $E \times A \to Q$, where $Q = E \times A/G$.

The structure group $G$ also acts on $E \times A$, and since this action commutes with the one by $G$, $G$ acts on $Q$. Moreover, the $G$-action on $Q^* = E \times A^*/G$ has no fixed points, where $A^*$ denotes the set of irreducible connections on $F$.

We end up with a principal $G$-bundle $Q^* \to M \times (A^*/G)$, and we denote by $\tilde{P}$ the associated vector bundle $Q^* \times_{\rho} \mathbb{C}^n$. Since $N$ is a submanifold of $A^*/G$, we define the Poincaré bundle $P \to M \times N$ as the restriction of $\tilde{P}$.

The principal $G$-bundle $Q^*$ has a natural connection $\tilde{\omega}$, constructed as follows. The space $E \times A^*$ has a Riemannian metric which is equivariant under $G \times G$, so that it descends to a $G$-equivariant metric on $Q^*$. The orthogonal complements to the orbits of $G$ yields the connection $\tilde{\omega}$. Passing to the associated vector bundle $\tilde{P}$ and restricting it to $M \times N$ gives a connection $\omega$ on the Poincaré bundle $P$. The pair $(P, \omega)$ is universal in the sense that $(P, \omega)|_{M \times \{\xi\}} \simeq (E, \nabla_{\xi})$.

**Theorem 4.** If $M$ is 4-dimensional hyperkähler manifold and $N$ is a connected component of the moduli of instantons on $N$, then the universal connection $\omega$ on the Poincaré bundle $P$ is an hermitian quaternionic instanton.

**Proof.** For any complex structure on $M$ compatible with its hyperkähler metric, the curvature of the universal connection is of type $(1, 1)$ with respect to induced complex structure on the $M \times N$. $\square$

As usual, the Weitzenböck formula can be used to guarantee that $\ker D_{\xi}^+$ is trivial for all $\xi \in N$, so that the index bundle is indeed a smooth vector bundle. Hence, in view of Theorem 2, we may define a hyperkähler Nahm transform that maps instantons on $M$ into quaternionic instantons on $N$. In particular, if $N$ is also 4-dimensional, we get a Nahm transform taking instantons on $M$ into instantons on $N$.

It is worth noting that if $N$ is non-compact, then one must also do some extra work to determine whether the Nahm transformed quaternionic instanton has finite energy.

Furthermore, if $M$ can also be regarded as a moduli space of (quaternionic) instantons on $N$, one can define a Nahm transform in the reverse direction, taking (quaternionic) instantons on $N$ into instantons on $M$. It is then natural to ask whether such transforms are the inverse of one another. Based on the known examples of the hyperkähler Nahm transform (see below) we pose the following general conjecture.
**Conjecture.** Let $M$ and $N$ be two connected 4-dimensional hyperkähler manifolds. If $N$ is a component of the moduli of instantons on $M$ and $M$ is a component of the moduli of instantons on $N$, then:

- $M$ is diffeomorphic to $N$;
- the hyperkähler Nahm transform is invertible.

Whenever $M$ and $N$ are compact and algebraic w.r.t. at least one complex structure, then the second claim of the previous conjecture is actually true, in view of Theorem-Definition 3 and of [9, Theorem 1.1].

Let us now analyze some interesting applications of Theorem 4, mentioning a few examples in which the conjecture is expected to be true.

4.1. **Algebraic tori.** The simplest example of hyperkähler Nahm transform is for $M$ being an algebraic torus of complex dimension $2k$ and $N$ being its dual, regarded as a moduli space of flat connections on $M$; $\mathcal{P}$ is the usual Poincaré bundle. Then the hyperkähler Nahm transform is just the usual Fourier-Mukai transform.

In particular, for $M$ being a flat 4-torus, the hyperkähler Nahm transform coincides with the usual Nahm transform for instantons on $T^4$ [8, 11], so that the conjecture is true in this case ($M = T^4$ and $N = (T^4)^*$).

4.2. **K3 surfaces.** The first example of a non-flat hyperkähler Nahm transform was described in [3]. Let $M$ be an algebraic K3 surface, which meets the following requirements:

1) $M$ admits a Kähler form $\omega$ whose cohomology class $H$ satisfies $H^2 = 2$;
2) $M$ admits a holomorphic line bundle $L$ whose Chern class $\ell = c_1(L)$ is such that $\ell \cdot H = 0$ and $\ell^2 = -12$;
3) if $D$ is the divisor of a nodal curve on $M$, one has $D \cdot H > 2$.

We say that $M$ is a reflexive K3 surface.

Now let $N$ be the moduli space of instantons of rank 2 with determinant line bundle $L$ (so that $c_1 = \ell$) and $c_2 = -1$ over $M$; it can be shown that $N$ is isomorphic to $M$ as a complex algebraic variety [3]. Moreover, any instanton $E$ on $M$, whose dual is not a point of $N$, satisfies the odd IT condition. Thus, since both $M$ and $N$ are hyperkähler manifolds, we get the following result [3].

**Theorem 5.** Let $(E, \nabla)$ be an irreducible instanton on $M$ such that at least one of the following conditions is satisfied: $\text{rank} E \neq 2$, $\det E \neq L^*$, $c_2(E) \neq 1$. Then, the Nahm transform of $(E, \nabla)$ is an irreducible instanton on $N$ having the same degree. Moreover, the correspondence is invertible.

This proves the conjecture for $M$ being reflexive K3 surface and $N$ being the moduli space of instantons on $M$ described above.
It should be pointed out that, also in this case, the hyperkähler Nahm transform coincides with Fourier-Mukai transform of coherent sheaves on $M$.

4.3. ALE spaces. Let $\Gamma$ be a finite subgroup of $\mathrm{SU}(2)$. Let $M$ be the minimal resolution of the quotient $\mathbb{C}^2/\Gamma$; it can be proved that $M$ carries a hyperkähler metric $g$, which is asymptotically locally Euclidean (ALE) in the following sense. Some open neighborhood $V$ of infinity in $M$ has a finite covering $\tilde{V}$ diffeomorphic to $\mathbb{R}^4 \setminus B(0,R)$, for some $R > 0$, and in the induced coordinates $x_i$ the metric $g$ is required to satisfy the relation

$$g_{ij}(x) = \delta_{ij} + a_{ij} \quad \text{with} \quad |\partial^p a_{ij}(x)| = O(|x|^{-(4-p)}) \quad , \quad p \geq 0$$

The moduli spaces of instantons over ALE spaces have been studied in detail by Nakajima [18], and by Kronheimer & Nakajima [16]. Let $E \to M$ be a complex (smooth) vector bundle of rank $n$ and trivial determinant (i.e. an $\mathrm{SU}(n)$ vector bundle). In order to define a suitable notion of connections that are “framed at infinity”, we fix a group homomorphism $\rho : \Gamma \to \mathrm{SU}(n)$; this homomorphism will be identified with a flat $\mathrm{SU}(n)$ connection over $S^3/\Gamma$. By taking coordinates $x_i$ on $V \subset X$ as before, we can extend the function $r(p) = |x(p)|$ to a positive $r$ function on all of $M$. Given a connection $A_0$, a weighted Sobolev norm $\| \cdot \|_{l,2,\delta}$ on the space of $k$-forms $\Omega^k(E)$ is defined as follows:

$$\|\alpha\|_{l,2,\delta} = \sum_{j=0}^l \|r^{j-(\delta+2)}\nabla_{A_0}^j \alpha\|_{L^2}$$

for an integer $l \geq 0$ and $\delta \in \mathbb{R}$. We denote the completion of the space $\Omega^k(E)$ in this norm by $W^{l,2}_\delta(E \otimes \Lambda^k \pi^* \pi_0)$. Now fix $l > 2$. We say that a connection $A$ on $E$ is asymptotic to $\rho$ in $W^{l,2}_\delta$ if there is a gauge such that $A = A_0 + \alpha$, where the restriction of $(E, A_0)$ to

$$\{t\} \times S^3/\Gamma \subset (R, \infty) \times S^3/\Gamma \simeq V$$

is the flat bundle with connection $\rho$, for all $t > R$ and $\|\alpha\|_{l,2,-2} < \infty$.

We denote by $\mathcal{A}_X^l(\rho)$ the space of such connections. The space of anti-self-dual connections on $E$ asymptotic to $\rho$ and having topological charge $k$ is described as follows:

$$\mathcal{A}_{X,asd}^l(E, \rho, k) = \left\{ A \in \mathcal{A}_X^l(\rho) \mid A \text{ is anti-self-dual and } \frac{1}{8\pi^2} \int_X \|F_A\|^2 = k \right\}$$

The corresponding moduli space is given by the quotient

$$\mathcal{M}_M(E, \rho, k) = \mathcal{A}_{X,asd}^l(E, \rho, k)/\mathcal{G}_0^{l+1}$$
where $G_{0}^{l+1}$ is the gauge group of automorphisms of $E$ converging to the identity, that is:

$$
G_{0}^{l+1} = \left\{ s \in W_{-1}^{l+1,2}(\text{End}(E)) \mid \|s - 1_E\|_{l+1,2,,-1} < \infty \right\}.
$$

The following fundamental result is due to Nakajima [20]:

**Theorem 6.** Each non-empty, non-compact 4-dimensional component of the moduli space $\mathcal{M}_M(E, \rho, k)$ is a complete hyperkähler ALE space.

In other words, every such component of the moduli space $\mathcal{M}_M(E, \rho, k)$ is diffeomorphic to a minimal resolution of $\mathbb{C}^2/\hat{\Gamma}$ for some discrete subgroup $\hat{\Gamma} \subset SU(2)$, which might be, in general, distinct from $\Gamma$. However, a complete classification (due to Nakajima) of all possible subgroups $\hat{\Gamma}$ can be achieved [6]. Roughly speaking, Nakajima has shown that the Dynkin diagram associated with $\hat{\Gamma}$ has to be a subgraph of the Dynkin diagram associated with $\Gamma$. As a consequence of this classification, one gets two important results (the latter was already proved in [16] by different methods).

**Theorem 7.** Let $M$ and $N$ be two ALE spaces. If $N$ is diffeomorphic to a 4-dimensional component of the moduli space of instantons on $M$ and the same holds for $M$ w.r.t. $N$, then $M$ is diffeomorphic to $N$.

In other words, the first part of the conjecture is true.

**Theorem 8.** Let $E$ be a rank 2 complex vector bundle over an ALE space $M$. Let $\iota : \Gamma \hookrightarrow SU(2)$ be the inclusion map, with $|\Gamma|$ denoting the order of $\Gamma$. Then $\hat{M} = \mathcal{M}_M(E, \iota, |\Gamma|^{-1})$ is isomorphic to $M$ as a hyperkähler manifold.

Let us now consider the hyperkähler manifolds $M$, $N = \hat{M}$ and an instanton $(F, \nabla)$ over $M$. The following facts are true:

1) the universal Atiyah-Singer bundle $(Q, \nabla_Q)$ equipped with the universal connection is a quaternionic instanton by Theorem 4 and [12];

2) the index $\text{Ind}D^{-}$ of the family of Dirac operators parametrized by $\hat{M}$ (cf. Section 3) is a finite-dimensional smooth, vector bundle on $\hat{M}$, that we shall denote by $\hat{F}$ [16];

3) the transformed instanton $\hat{\nabla}$ has finite energy [3, 6].

Therefore, hyperkähler Nahm transform takes instantons on $M$ into instantons on $\hat{M}$, and vice versa. Using the equivalent Fourier-Mukai formulation, one can expect to show that the transform is invertible, thus proving also the second part of the conjecture [3, 6].
4.4. **Further perspectives.** Let us now consider the case $M = \mathbb{T}^2 \times \mathbb{R}^2$; let $(x, y)$ denote flat coordinates in $\mathbb{T}^2$, and $(r, \theta)$ denote polar coordinates in $\mathbb{R}^2$. As it was shown in \cite{7}, the moduli space $\mathcal{M}_{(k, \xi, \mu, \alpha)}$ of $SU(2)$ instantons $A$ on $M$ which are asymptotic to

$$A_0 = d + i \begin{pmatrix} a_0 & 0 \\ 0 & -a_0 \end{pmatrix},$$

with

$$a_0 = \lambda_1 dx + \lambda_2 dy + (\mu_1 \cos \theta - \mu_2 \sin \theta) \frac{dx}{r} + (\mu_1 \sin \theta + \mu_2 \cos \theta) \frac{dy}{r} + \alpha d\theta$$

is a smooth hyperkähler manifold of real dimension $8k - 4$, where

$$k = \frac{1}{8\pi^2} \int_M |F_A|^2$$

is the instanton charge. In particular, setting $k = 1$, one also shows that $N = \mathcal{M}_{(1, \xi, \mu, \alpha)} \simeq \mathbb{T}^2 \times \mathbb{R}^2$ with the flat metric, whenever it is non-empty. Thus there is a hyperkähler Nahm transform taking instantons on $M$ into instantons on $N$; it would be interesting to determine whether the Nahm transformed instanton satisfy $|F_A| \sim O(r^{-2})$ (which is equivalent to the asymptotic condition above, see \cite{7}).

Moreover, $M$ can also be regarded as a moduli space of instantons on $N$, so there is a hyperkähler Nahm transform taking instantons on $N$ into instantons on $M$, giving some evidence for the conjecture being true when $M = \mathbb{T}^2 \times \mathbb{R}^2$.

Finally, it would also be interesting to consider the hyperkähler Nahm transform for instantons over $S^1 \times \mathbb{R}^3$ (alias calorons) and $M = \mathbb{T}^3 \times \mathbb{R}$. Even more interesting would be to consider different gravitational instantons (other than ALE spaces), like asymptotically locally flat (ALF) 4-manifolds, or the ALG gravitational instantons constructed by Cherkis & Kapustin in \cite{10}. The moduli spaces of instantons over these spaces, however, are much less understood.

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