SPECTRAL PROPERTIES OF THE MÖBIUS FUNCTION AND A RANDOM MÖBIUS MODEL.

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Abstract. Assuming Sarnak conjecture is true for any singular dynamical process, we prove that the spectral measure of the Möbius function is equivalent to Lebesgue measure. Conversely, under Elliott conjecture, we establish that the Möbius function is orthogonal to any uniquely ergodic dynamical system with singular spectrum. Furthermore, using Mirsky Theorem, we find a new simple proof of Cellarosi-Sinai Theorem on the orthogonality of the square of the Möbius function with respect to any weakly mixing dynamical system. Finally, we establish Sarnak conjecture for a particular random model.

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1. Introduction

The Möbius function is defined for the positive integers \( n \) by

\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1; \\
(-1)^r & \text{if } n \text{ is the product of } r \text{ distinct primes;} \\
0 & \text{if not}
\end{cases}
\] (1.1)

It is of great importance in Number Theory because of its connection with the Riemann \( \zeta \)-function via the formulae

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)} \quad \text{with } \text{Re}(s) > 1.
\]

Furthermore, the estimate

\[
\left| \sum_{n=1}^{x} \mu(n) \right| = O(x^{\frac{1}{2}+\varepsilon}) \quad \text{as } x \to +\infty, \quad \forall \varepsilon > 0
\] (1.2)

is equivalent to the Riemann Hypothesis ([36, pp.315]).

The main goal of this note is to investigate the problem of spectral disjointness of the Möbius function with any dynamical sequence with zero topological entropy. The question is initiated by P. Sarnak in [33] [34]. In his three lectures notes [35], P. Sarnak makes the following conjecture.

**Sarnak’s conjecture.** The Möbius function is orthogonal to any deterministic sequence \((a_n)_{n \in \mathbb{N}}\), that is,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)a_n = 0.
\] (1.2)

The sequence \((a_n)\) is deterministic if it is generated by a deterministic topological dynamical system \((X, T)\), i.e. \(X\) is a compact space, \(T\) a continuous map from \(X\) onto \(X\) with topological entropy equal to zero and there exists a continuous function from \(X\) to the complex plan \(\mathbb{C}\) and a point \(x \in X\) for which \(a_n = f(T^n x)\) for all \(n\).

The Möbius function is orthogonal to any constant function [19]. This is follows from Kronecker’s Lemma combined with Landau’s observation:

\[
\sum_{n \geq 1} \frac{\mu(n)}{n} = 0.
\]

This last relation also implies that the orthogonality of the Möbius function to the function 1 (i.e. \(\frac{1}{x} \sum_{n \leq x} \mu(n) \to 0\) as \(x \to +\infty\)) is “equivalent” to the Prime Number Theorem (PNT for short), which says that the number \(\pi(x)\) of primes below \(x\) satisfies

\[
\lim_{x \to +\infty} \frac{\pi(x) \log(x)}{x} = 1.
\]

The orthogonality of the Möbius function to any sequence arising from a rotation dynamical system \((X, T)\) is the circle \(\mathbb{T}\) and \(Tx = x + \alpha, \alpha \in \mathbb{T}\) follows from the following inequality (Davenport [15])

\[
\max_{\theta \in \mathbb{T}} \left| \sum_{k \leq x} \mu(k)e^{ik\theta} \right| \leq \frac{x}{\log(x)^\varepsilon}, \quad \text{where } \varepsilon > 0.
\]
It is an easy exercise to establish, from Davenport’s estimate above combined with the spectral theorem and the standard ergodic argument \[16\], that the Möbius function is orthogonal to any sequence \( f(T^nx) \), for almost every point \( x \in X \) with \( f \in L^2(X) \) (we assume \( X \) is equipped with a probability measure in this case).

The case of the orthogonality of the Möbius function to any nilsequence \( X = G/\Gamma \), where \( G \) is a nilpotent Lie group, \( \Gamma \) is a lattice in \( G \) and \( T_g(\Gamma x) = \Gamma xg \) is covered by Green-Tao Theorem \[18\]. Recently, Bourgain-Sarnak and Ziegler in \[9\] extend the Green-Tao result and establish that the Möbius function is orthogonal to the horocycle flow. Indeed, applying Bourgain-Sarnak-Ziegler criterion, one can get a simple proof of Green-Tao Theorem \[40\]. Roughly speaking, Bourgain-Sarnak-Ziegler criterion implies Conjecture (1.2) is true when \( \mu(n) \) is replaced by any multiplicative function with module less than 1 provided that the mutual powers of the two dynamical systems are disjoint in the sense of Furstenberg. It is well known that this latter property holds in the case of the spectral disjointness of the mutual powers. Therefore, the spectral disjointness of different primes powers implies the disjointness of different primes which ensures by Bourgain-Sarnak-Ziegler criterion that the Sarnak Conjecture holds.

Exploiting this fact, el Abdalaoui-Lemanczyk-de-la-Rue obtain in the very recent work \[1\] a new proof of Bourgain Theorem \[7\] saying that the rank one maps with bounded parameters are orthogonal to Möbius. Precisely, the authors extend Bourgain Theorem to a large class of rank one maps.

We should point out here that the spectral disjointness of the mutual powers doesn’t holds in the case of Lebesgue spectrum.

There is a conjecture due to Elliott \[17\] saying that the spectral measure of \( \mu \) is exactly the Lebesgue measure up to a constant. As far as we know, this conjecture is still open. Here we are able to prove that, under Sarnak’s conjecture (plus one technical assumption) the spectral measure of Möbius function is equivalent to the Lebesgue measure.

Moreover we give a new simple proof of Sarnak and Cellarosi-Sinai result and assuming Elliott conjecture we establish that the Möbius function is orthogonal to any uniquely ergodic dynamical system with singular spectrum.

Our proof uses a spectral approach based on the classical methods introduced in \[13\] and intensively used to study the spectrum of arithmetical \( q \)-multiplicative functions \[21,12\]. In \[31,32\], M. Queffelec used the standard method of Riesz products to obtain more results on the spectrum of \( q \)-multiplicative functions. In a forthcoming paper \[2\], the first author and M. Lemańczyk give a new proof of all these results based on the Bourgain methods introduced in the context of generalized Riesz products associated to the spectrum of rank one maps \[8\]. In addition, they establish the orthogonality of Möbius to \( q \)-multiplicative functions and, as a consequence, to any Morse sequences \[3\].

**The random Möbius function.** In this paper, following Ng \[28\], we “simulate” randomly the behavior of the values of the Möbius function in the following way. Let \( \mathcal{P} \) be the set of prime numbers. A positive number \( n \) is square-free (quadratfrei) if \( p^2 \nmid n \) for every \( p \in \mathcal{P} \) (where \( a \nmid b \) means \( a \) does not divide \( b \), and \( a, b \in \mathbb{N} \) ). Denote the set of all square-free numbers by \( \mathbb{Q} \) and \( \omega(n) \) the number of distinct prime factors of \( n \). Thus, for any \( n \in \mathbb{Q} \), we have \( \mu(n) = 1 \) if \( \omega(n) \) is even and \( \mu(n) = -1 \) if \( \omega(n) \) is odd. To simulate randomly this behavior let \( \epsilon_n \) be a sequence...
of independent Rademacher random variables, indexed by \( n \in \mathbb{Q} \), that is
\[
\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}
\]
independently for each \( n \in \mathbb{Q} \).

The random M"obius function \( \mu_{\text{rand}} \) is then defined by
\[
\mu_{\text{rand}}(n) = \begin{cases} 
\epsilon_n & \text{if } n \in \mathbb{Q} \\
0 & \text{otherwise}
\end{cases}
\]

In this work we prove that almost surely the random M"obius function is orthogonal
to any topological dynamical system with zero entropy. Our main tool is based
on a concentration inequality due to Hoeffding and Azuma combined with a Borel-
Cantelli argument.

Remark 1.1. The probability that \( \mu_{\text{rand}} = \mu \) is zero.

The rest of the paper is organized as follows: in section 2, we recall some basic
facts from the spectral analysis of dynamical systems, we give the definition of
topological entropy and state our main results. In section 3, we prove our results
on the deterministic M"obius function and finally in section 4 we prove our result
for the random case. In the Appendix we added some results that, though they not
necessary for the proof (we only need a weaker version), may be useful for some
future generalizations.

2. Main results and some basic facts from spectral theory for
dynamical systems

A dynamical system is a pair \((X, T)\) where \( X \) is a space (equipped with a topology,
a metric, or a probability measure) and \( T \) is a measurable bijection \( T : X \to X \).
In this paper we will fix \( X \) to be a metric compact space and \( T \) a continuous
function. We will consider three types of dynamical systems: (a) dynamical system in
a measurable setting ((a) weakly mixing or (a)' uniquely ergodic) and (b) topo-
logical with zero entropy. Our main problem will be to study dynamical sequences
\( a_n = f(T^n x) \), or dynamical processes \( Y_n = f \circ T^n \), generated inside some dynamical
system via a continuous function \( f \).

2.1. Dynamical systems in a measurable setting. We endow the space \( X \)
with a probability measure structure \((\mathcal{A}, P)\). We also require \( T \) to be bimesurable
and to preserve \( P \), i.e. \( P(T^{-1}(A)) = P(A) \), for every \( A \in \mathcal{A} \). The dynamical system
is ergodic if the \( T \)-invariant set is trivial: \( P(T^{-1}(A) \triangle A) = 0 \Rightarrow P(A) \in \{0, 1\} \). \( T \)
induces an operator \( U_T \) in \( L^p(X) \) via \( f \mapsto U_T(f) = f \circ T \) called Koopman operator.
For \( p = 2 \) this operator is unitary and its spectral resolution induces a spectral
decomposition of \( L^2(X) \) [20]:
\[
L^2(X) = \bigoplus_{n=0}^{+\infty} C(f_i) \text{ and } \sigma_{f_1} >> \sigma_{f_2} >> \cdots
\]
where
- \( \{f_i\}_{i=1}^{+\infty} \) is a family of functions in \( L^2(X) \);
- \( C(f) = \text{span}\{U_T^n(f) : n \in \mathbb{Z}\} \) is the cyclic space generated by \( f \in L^2(X) \);
• }_{f} is the spectral measure on the circle generated by }_{f} via the Bochner-Herglotz relation
\begin{equation}
\hat{\sigma}_{f}(n) = \langle U_{T}^{n}f, f \rangle = \int_{X} f \circ T^{n}(x)f(x) dP(x);
\end{equation}

• for any two measures on the circle }_{\alpha} and }_{\beta}, }_{\alpha} >> }_{\beta} means }_{\beta} is absolutely continuous with respect to }_{\alpha}: for any Borel set, }_{\alpha}(A) = 0 \implies }_{\beta}(A) = 0.

The two measures }_{\alpha} and }_{\beta} are equivalent if and only if }_{\alpha} >> }_{\beta} and }_{\beta} >> }_{\alpha}.

We will denote measure equivalence by }_{\alpha} \sim }_{\beta}.

The spectral theorem ensures this spectral decomposition is unique up to isomorphisms. The maximal spectral type of }_{T} is the equivalence class of the Borel measure }_{\sigma_{f_{1}}}. The multiplicity function }_{MT} : }_{T} \rightarrow \{1, 2, \ldots, \} \cup \{+\infty\} is defined }_{\sigma_{f_{1}}} a.e. and
\begin{equation}
MT(z) = \sum_{n=1}^{+\infty} \# Y_{j}(z), \text{ where, } Y_{1} = }_{T} \text{ and } Y_{j} = \sup \frac{d\sigma_{f_{j}}}{d\sigma_{f_{1}}}, \forall j \geq 2.
\end{equation}

An integer }_{n} \in \{1, 2, \ldots, \} \cup \{+\infty\} is called an essential value of }_{MT} if }_{\sigma_{f_{1}}\{z \in }_{T} : }_{MT}(z) = n\} > 0. The multiplicity is uniform or homogenous if there is only one essential value of }_{MT}. The essential supremum of }_{MT} is called the maximal spectral multiplicity of }_{T}. The map }_{T}

• has simple spectrum if }_{L^{2}(X)} is reduced to a single cyclic space;
• has discrete spectrum if }_{L^{2}(X)} has an orthonormal basis consisting of eigenfunctions of }_{U_{T}} (in this case }_{\sigma_{f_{1}}} is a discrete measure);
• has Lebesgue spectrum (resp. absolutely continuous, singular spectrum) if }_{\sigma_{f_{1}}} is equivalent (resp. absolutely continuous, singular) to the Lebesgue measure.

The reduced spectral type of the dynamical system is its spectral type on the }_{L^{2}_{0}(X)} the space of square integrable functions with zero mean.

**Definition 2.1.** Two dynamical systems are called spectrally disjoint if their reduced spectral types are mutually singular.

In this paper we consider two types of measurable dynamical systems:

(a)’ }_{(X, T, A)} is uniquely ergodic if }_{X} is a compact metric space, }_{T} is a homeomorphism and there exists a unique ergodic probability measure }_{P}:

(a)” }_{(X, T, A, P)} is uniquely weakly mixing if there exists a unique ergodic probability measure }_{P} and }_{\sigma_{f_{1}}} is the sum of the Dirac measure on zero and a continuous measure.

With the Jewett result \([20]\) in mind, it is easy to see that we still deal with a large class of dynamical systems.

**2.2. Topological dynamical systems and topological entropy.** The pair }_{(X, T)} is called topological dynamical system if }_{X} is a compact metric space and }_{T} : }_{X} \rightarrow }_{X} is a homeomorphism.

There are different ways to define the topological entropy for such a system. Here we use the one introduced by Bowen \([10]\) based on }_{(m, \epsilon)}-spanning sets. For each }_{m} \in \mathbb{N} we define a new distance }_{d_{m}} given by
\begin{equation}
d_{m}(x, y) = \max_{0 \leq k \leq m-1} d(T^{k}x, T^{k}y), \quad x, y \in }_{X}.
\end{equation}
Two points in $X$ are $\varepsilon$-close with respect to the distance $d_m$ if their iterates under $T$ stay $\varepsilon$-close up to time $m - 1$. Note that the definition of $d_m$ depends on the transformation $T$. The open ball with respect to this metric

$$B_{d_m}(x, \varepsilon) = \{y \in X : \text{such that } d(T^k x, T^k y) < \varepsilon \forall 0 \leq k \leq m - 1\}$$

consist of all points whose trajectories up to time $m - 1$ remain $\varepsilon$-close to the finite orbit segment $\{x, Tx, \cdots, T^{m-1}x\}$. Since $X$ is compact, for any $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists a finite set of points $R(m, \varepsilon) \subset X$ of minimal cardinality $r_m(\varepsilon)$

$$R(m, \varepsilon) = \{x_1^{(m)}, \ldots, x_{r(m, \varepsilon)}^{(m)}\} \subset X,$$

such that

$$X = \bigcup_{j=1}^{r(m, \varepsilon)} B_{d_m}(x_j^{(m)}, \varepsilon).$$

The cardinality $r(m, \varepsilon)$ is then the minimal number of balls we need to describe all possible segments of trajectory of length $m$. The topological entropy of $(X, T)$ is defined by

$$h(T) = \lim_{\varepsilon \to 0} \limsup_{m \to +\infty} \frac{1}{m} \log [r(m, \varepsilon)].$$

A nice account on the topological entropy may be found in [37],[30]. With these definitions, the dynamical system $(X, T)$ has zero topological entropy if for any positive constant $\eta > 0$ there exists a positive constant $\varepsilon_0(\eta) > 0$ and a positive integer $m_0(\varepsilon) > 0$ such that

$$r(m, \varepsilon) < e^{m\varepsilon} \quad \forall \varepsilon < \varepsilon_0(\eta), \forall m > m_0(\varepsilon).$$

This property will be crucial to prove our result.

2.3. Spectral measure of a sequence. The notion of spectral measures for sequences is introduced by Wiener in his 1933 book [38]. More precisely, Wiener considers the space $S$ of complex bounded sequences $g = (g_n)_{n \in \mathbb{N}}$ such that

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_n + k \overline{g}_n = F(k)$$

exists for each integer $k \in \mathbb{N}$. The sequence $F(k)$ can be extended to negative integers by setting $F(-k) = \overline{F(k)}$.

It is well known that $F$ is positive definite on $\mathbb{Z}$ and therefore (by Herglotz-Bochner theorem) there exists a unique positive finite measure $\sigma_g$ on the circle $T$ such that the Fourier coefficients of $\sigma_g$ are given by the sequence $F$. Formally, we have

$$\widehat{\sigma_g}(k) \overset{\text{def}}{=} \int_T e^{-ikx} d\sigma_g(x) = F(k).$$

The measure $\sigma_g$ is called the spectral measure of the sequence $g$.

The orthogonality issue of the Möbius function may be connected to the spectral analysis of the sequence $\mu(n)$ on one side and the nature of the spectral type of
the dynamical sequence \( g_n = f(T^n x) \) on the other. Indeed, there exists a natural connection between the spectral measure and the spectral type of the dynamical system. For a uniquely ergodic system we have the following result.

**Lemma 2.2.** Let \((X, \mathcal{A}, P, T)\) be a uniquely ergodic topological dynamical system. Then, for any \( f \in C(X) \), for every \( x \in X \), the sequence \( g_n = f(T^n x) \) belongs to the Wiener space \( S \) and its spectral measure is exactly the spectral measure of the function \( f \) given by

\[
\hat{\sigma}_f(k) = \langle U_T^k(f), f \rangle = \int f \circ T^k(x) \cdot \overline{f(x)} dP(x),
\]

where \( U_T \) is a unitary operator on \( L^2(X) \) defined by \( f \mapsto U_T(f) = f \circ T \).

**Proof.** Let \( g_n = f(T^n x) \). Then, for any \( N > 1 \),

\[
\frac{1}{N} \sum_{n=0}^{N-1} g_{n+k} g_n = \frac{1}{N} \sum_{n=0}^{N-1} f(T^{n+k} x) f(T^n x) = \frac{1}{N} \sum_{n=0}^{N-1} [(f \circ T^k) \cdot \overline{f}] (T^n x).
\]

By the unique ergodicity of the system, the right-hand side converges

\[
\frac{1}{N} \sum_{n=0}^{N-1} [(f \circ T^k) \cdot \overline{f}] (T^n x) \xrightarrow{N \to \infty} \int (f \circ T^k)(y) \cdot \overline{f(y)} dP(y) = \hat{\sigma}_f(k).
\]

Then the sequence \((g_n)\) belongs to the Wiener space \( S \) and its spectral measure coincides with the spectral measure of \( f \)

\[
\hat{\sigma}_g(k) = \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} g_{n+k} g_n = \int (f \circ T^k)(y) \cdot \overline{f(y)} dP(y) = \hat{\sigma}_f(k).
\]

The proof of the lemma is complete. \( \square \)

**Remark 2.3.** In the case of any ergodic system (not necessarily uniquely ergodic), the result of Lemma 2.2 above still holds for almost every \( x \in X \), by Birkhoff theorem.

Therefore, the orthogonality issue of the Möbius function seems to be related to the computation of the Möbius spectral measure or, in other words, the self-correlations of the Möbius function. On the other hand, the weaker form of the Elliott conjecture [17] implies that the spectral measure of the Möbius function is up to a constant the Lebesgue measure on the circle, i.e.

**Conjecture 2.4** (of Elliott). [17]

\[
(2.9) \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) \mu(n + h) = \begin{cases} 
0 & \text{if } h \neq 0 \\
\frac{6}{\pi^2} & \text{if not}.
\end{cases}
\]

P.D.T.A. Elliott writes in his 1994’s AMS Memoirs that “even the simple particular cases of the correlation (when \( h = 1 \) in (2.9)) are not well understood. Almost surely the Möbius function satisfies (2.9) in this case, but at the moment we are unable to prove it.”

**Remark 2.5.** (a) \( (2.9) \) implies that the sequence \( \mu(n) \) belongs to the Wiener space \( S \) and its spectral measure is exactly (up to a normalization constant) the Lebesgue measure on the circle.
(b) An alternative way to state Elliott conjecture is the following \[17\]. Let \(a, b, A\) and \(B\) be integers for which \(aB \neq Ab\). Then
\[
\frac{1}{N} \sum_{n=1}^{N} \mu(an + b)\mu(An + B) \xrightarrow{N \to +\infty} 0,
\]

Elliott conjecture may be seen as a consequence of the \(L^1\)-flatness of the trigonometric polynomials with Möbius coefficients. A sequence of polynomials \(P_n(\theta)\) is \(L^1\)-flat if
\[
\left\| \frac{|P_n(\theta)|^2}{\|P_n\|_2^2} - 1 \right\|_1 \xrightarrow{n \to \infty} 0.
\]

Suppose the following sequence of polynomials with Möbius coefficients
\[
P_n(\theta) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mu(j) e^{ij\theta}
\]
satisfies
\[
\int_{0}^{2\pi} \frac{|P_n(\theta)|^2}{\|P_n\|_2^2} \frac{d\theta}{2\pi} - 1 \xrightarrow{N \to +\infty} 0.
\]

where the normalization constant
\[
\|P_n\|_2^2 = \int_{0}^{2\pi} |P_n(\theta)|^2 \frac{d\theta}{2\pi} = \frac{1}{n} \sum_{j=1}^{n} \mu^2(j) \xrightarrow{N \to +\infty} \frac{6}{\pi^2} > 0
\]

corresponds to the fraction of square free integers in the interval \([0, n]\). Then this sequence is \(L^1\)-flat and the spectral measure for \(\mu\) is the Lebesgue measure. Indeed
\[
\left| \frac{1}{n} \sum_{j=1}^{n} \mu(j)\mu(j+k) \right| = \left| \int_{0}^{2\pi} e^{-ik\theta} |P_n(\theta)|^2 \frac{d\theta}{2\pi} \right|
\leq \|P_n\|_2^2 \left( \int_{0}^{2\pi} \left| \frac{|P_n(\theta)|^2}{\|P_n\|_2^2} - 1 \right| + \int_{0}^{2\pi} e^{-ik\theta} \frac{d\theta}{2\pi} \right)
\]

Then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n-1} \mu(j)\mu(j+k) = 0 \quad \forall k \neq 0.
\]

In his lectures \[35\] P. Sarnak constructs a uniquely ergodic topological dynamical system called the Möbius flow. For this dynamical system the Möbius function is a generic point and the topological entropy of this system is positive.

We are now able to state our main results.

**Theorem 2.6 (Main result 1).** Assume the Möbius function is orthogonal to any singular dynamical process \(Y_n\) on a compact metric space\(^1\), and in the Möbius flow conditional expectation preserves the space of continuous functions. More precisely we assume
\[
(2.10) \quad \frac{1}{N} \sum_{n=1}^{N} Y_n(w) \mu(n) \xrightarrow{N \to +\infty} 0 \quad \forall w \in X.
\]

\(^1\)Note that the notion of singular process gives an alternative definition of deterministic sequence which is in general not equivalent to the notion of deterministic sequence based on the topological entropy.
Then the spectral measure of $\mu(n)$ is absolutely continuous with log integrable density $\rho$.

**Theorem 2.7 (Main result 2).** Assume that Elliott conjecture (2.9) holds. Then the Möbius function is orthogonal to any uniquely ergodic dynamical system with singular spectrum. Precisely

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \mu(n) \rightarrow 0 \quad \forall f \in C_0(X), \forall x \in X,$$

where $C_0(X)$ is the set of continuous functions on $X$ with zero mean.

**Theorem 2.8 (Main result 3: Sarnak and Cellarosi-Sinai).** The sequence $\mu^2(n)$ generated by the square of the Möbius function is spectrally orthogonal to the measure of any uniquely weakly mixing dynamical system $f(T^n x)$. More precisely

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \mu^2(n) \rightarrow 0 \quad \forall f \in C_0(X), \forall x \in X,$$

where $C_0(X)$ is the set of continuous functions on $X$ with zero mean.

**Remark 2.9.** Theorem 2.8 is essentially due to Sarnak [35] and Cellarosi-Sinai [11]. Indeed, in [35] and [11] the authors construct a dynamical system associated to the sequence $\mu^2(n)$. The dynamical system obtained is isomorphic to the translations on a compact Abelian group.

**Theorem 2.10 (Main result 4).** The random Möbius function is orthogonal to any topological dynamical system with zero topological entropy. More precisely, let $(X, T)$ be a topological dynamical system with zero topological entropy and $C(X)$ the set of continuous complex valued functions on $X$. Then there exists a subset $\Omega' \subset \Omega$ independent of $x$ and $f$ such that $P(\Omega') = 1$ and

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \mu_{\text{rand}}(n)(\omega) \rightarrow 0 \quad \forall \omega \in \Omega', \forall x \in X, \forall f \in C(X).$$

### 3. Spectral disjointness for the Möbius function

In this section we consider the (non random) Möbius function and prove Theorem 2.7, 2.8 and 2.10. Both proofs use the notion of affinity. Here we consider only uniquely ergodic dynamical systems so the spectral measure of the corresponding sequence is well defined. We will see in subsection 3.2 below that the spectral measure of $\mu^2(n)$ is also well defined. We start by giving a few preliminary results and definitions.

**3.1. Affinity and correlations.** The affinity between two finite measures is defined by the integral of the corresponding geometric mean. It is introduced and studied in a series of papers by Matusita [22, 23, 24] and it is also called Bahattacharyya coefficient [6]. It has been widely used in statistics literature as a useful tool to quantify the similarity between two probability distributions. It is symmetric in distributions and has direct relationships with error probability when classification or discrimination is concerned.

Let $M_1(\mathbb{T})$ be a set of probability measures on the circle $\mathbb{T}$ and $\mu, \nu \in M_1(\mathbb{T})$ two measures in this space. There exists a probability measure $\lambda$ such that $\mu$ and $\nu$
are absolutely continuous with respect to $\lambda$ (take for example $\lambda = \frac{\mu + \nu}{2}$). Then the 
affinity between $\mu$ and $\nu$ is defined by

$$G(\mu, \nu) = \int \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} d\lambda.$$  

This definition does not depend on $\lambda$. Affinity is related to the Hellinger distance as it can be defined as

$$H(\mu, \nu) = \sqrt{2(1 - G(\mu, \nu))}.$$  

Note that $G(\mu, \nu)$ satisfies (by Cauchy-Schwarz inequality)

$$0 \leq G(\mu, \nu) \leq 1.$$  

Remark 3.1. The definition of affinity can be extended to any pair of positive non trivial finite measures. Indeed any such measure becomes a probability measure if we divide by the (positive) normalization factor.

It is an easy exercise to see that $G(\mu, \nu) = 0$ if and only if $\mu$ and $\nu$ are mutually singular (denoted by $\mu \perp \nu$): this means $\mu$ assigns measure zero to every set to which $\nu$ assigns a positive probability, and vice versa. Similarly, $G(\mu, \nu) = 1$ holds if and only if $\mu$ and $\nu$ are equivalent: $\mu \ll \nu$ and $\nu \ll \mu$. Affinity can be used to compare sequences of measures via the following theorem. The proof may be found in [13].

Theorem 3.2 (Coquet-Kamae-Mandès-France [13]). Let $(P_n)$ and $(Q_n)$ be two sequences of probability measures on the circle weakly converging to the probability measures $P$ and $Q$ respectively. Then

$$\limsup_{n \to +\infty} G(P_n, Q_n) \leq G(P, Q).$$

As in the case of the affinity, this result can be generalized to any sequence of positive non trivial finite measures $P_n, Q_n$ converging weakly to two positive non trivial finite measures $P$ and $Q$.

We want to use the affinity and Theorem 3.2 above to estimate the orthogonality properties of pairs of sequences in the Wiener space $S$ (defined in subsection 2.3). To do that we will need to replace the sequence of Fourier coefficients $\frac{1}{n} \sum_{j=0}^{n-1} g_j e^{ijx}$ by a sequence of finite positive measures on the circle. For any $g \in S$, we introduce the sequence of functions

$$d\sigma_{g,n}(x) = \rho_{g,n}(x) \frac{dx}{2\pi}, \text{ where } \rho_{g,n}(x) = \left| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g_j e^{ijx} \right|^2.$$  

With this definition $\sigma_{g,n}$ defines a finite positive measure on the circle. Moreover we have the relation

$$\frac{1}{n} \sum_{j=0}^{n-1} g_j e^{ikx} = \int_0^{2\pi} e^{-ikx} d\sigma_{g,n}(x) + \Delta_{n,k} = \tilde{\sigma}_{g,n}(k) + \Delta_{n,k},$$

where

$$|\Delta_{n,k}| = \left| \frac{1}{n} \sum_{j=n-k}^{n-1} g_j e^{ikx} \right| \leq \frac{k}{n} \sup_j |g_j|^2 \to_{n \to \infty} 0.$$
Taking the limit we have

\[ \hat{\sigma}_g(k) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} g_{j+k} = \lim_{n \to \infty} \hat{\sigma}_{g,n}(k) \]

so the sequence of measures \( \sigma_{g,n} \) converges weakly to \( \sigma_g \).

To prove our theorems we will need the following result.

**Corollary 3.3.** Let \( g = (g_n)_{n \in \mathbb{N}}, h = (h_n)_{n \in \mathbb{N}} \in S \) two non trivial sequences i.e. \( \hat{\sigma}_g(0) > 0 \) and \( \hat{\sigma}_h(0) > 0 \). Then

\[ \limsup_{n \to \infty} \frac{1}{n} \left| \sum_{j=1}^{n} g_j h_j \right| \leq G(\sigma_g, \sigma_h). \tag{3.4} \]

**Proof.** Take \( P_n = \sigma_{g,n} \) and \( Q_n = \sigma_{h,n} \) (see (3.3)). Then

\[ \left| \frac{1}{n} \sum_{j=1}^{n} g_j h_j \right| = \frac{1}{n} \int_{\mathbb{T}} \sum_{j,k=1}^{n} g_j e^{ijx} \overline{h_k} e^{-ikx} dx \leq G(P_n, Q_n). \]

Applying (3.2) the result follows. \( \square \)

**Remark 3.4.** It may be possible that the bounded sequence \( g_n \) does not belong to the Wiener space \( S \) (see eq. 2.8) but we can always extract a subsequence \( (n_r) \) in (2.8) such that

\[ \lim_{r \to \infty} \frac{1}{n_r} \sum_{j=0}^{n_r-1} g_{j+k} \]

exists for each \( k \in \mathbb{N} \). In fact, define the sequence of finite positive measures \( (\sigma_{g,n})_{n \in \mathbb{N}} \) on the torus defined in (3.3). These measures are all finite and \( \sigma_{g,n}(\mathbb{T}) \leq \|g\|_\infty^2 = \sup_j |g_j|^2 \ \forall n \). Therefore they all belong to the subset of measures on the circle \( B(0, \|g\|_\infty^2) \). This subset is compact so there exists a subsequence \( (n_r) \) such that the sequence of probability measures \( (\sigma_{g,n_r})_{r \in \mathbb{N}} \) converge weakly to some probability measure \( \sigma_{g,(n_r)} \). The measure \( \sigma_{g,(n_r)} \) is called the spectral measure of the sequence \( g \) along the subsequence \( (n_r) \).

Theorems 3.2 can be extended to a lower bound on the absolutely continuous part of the spectral measure of a given sequence in the Wiener space. More precisely we have the following proposition.

**Proposition 3.5.** Let \( g \in S \) a non trivial sequence, i.e. \( \hat{\sigma}_g(0) > 0 \). Then

\[ \limsup_{n \to +\infty} \int_{\mathbb{T}} \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g_j e^{ijx} dx \leq \int \sqrt{\frac{d\sigma_{g,a}}{dx}} dx \]

where \( \frac{d\sigma_{g,a}}{dx} \) is the Radon-Nikodym derivative of the Lebesgue component of \( \sigma_g \).

**Proof.** Let \( n_k \) be a subsequence such that

\[ \limsup_{n \to +\infty} \int_{\mathbb{T}} \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} g_j e^{ijx} dx = \lim_{k} \int_{\mathbb{T}} \frac{1}{\sqrt{n_k}} \sum_{j=0}^{n_k-1} g_j e^{ijx} dx. \]
Put
\[ \xi_k = \left| \frac{1}{\sqrt{n_k}} \sum_{j=0}^{n_k-1} g_j e^{ijx} \right| dx. \]

Then there exists a subsequence of \( \xi_k \) (that we still denote by \( n_k \) to avoid heavy notation) such that \( \xi_k \) converges weakly to some positive finite measure \( \xi \). We will show later that for any Borel set \( A \) of \( T \) we have
\[ (3.6) \quad \xi(A) \leq \sqrt{\sigma_g(A) \lambda(A)}. \]

Now let \( E \) be a Borel set such that \( \lambda(E) = 1 \) and \( \sigma_{g,s}(E) = 0 \), where \( \sigma_{g,s} \) is the singular part of \( \sigma \) with respect to the Lebesgue measure. Then by (3.6) \( \xi \) is absolutely continuous and for any Borel set \( A \subseteq E \) we have
\[
\int_A d\xi \leq \sqrt{\int_A d\sigma_{g,a} dx} \sqrt{\int_A dx}.
\]

By a Martingale argument we deduce that for almost all \( x \)
\[
\frac{d\xi}{d\lambda}(x) \leq \sqrt{\frac{d\sigma_{g,a}}{dx}(x)}.
\]
From this the proof of the proposition follows. It remains to prove (3.6). Let \( w \) be a positive continuous function. Then by Cauchy-Schwarz inequality
\[
\int_T w(x) \left| \frac{1}{\sqrt{n_k}} \sum_{j=0}^{n_k-1} g_j e^{ijx} \right| dx \leq \left[ \int_T w(x) \left| \frac{1}{\sqrt{n_k}} \sum_{j=0}^{n_k-1} g_j e^{ijx} \right|^2 dx \right]^{1/2} \left[ \int_T w(x) dx \right]^{1/2}
\]
Letting \( k \) go to infinity we get
\[
\int_T w(x) d\xi \leq \left[ \int_T w(x) d\sigma_g \right]^{1/2} \left[ \int_T w(x) dx \right]^{1/2}
\]
Hence, by the density of subspace of continuous functions in \( L^2(\sigma_g + dx + \xi) \) the claim follows. The proof of the proposition in then complete.

3.2. The spectral measure for \( \mu^2(n) \). Before starting the proof of the two main theorems, we need to introduce some additional results on the spectrum of the sequence \( \mu^2(n) \).

**Theorem 3.6.** [Mirsky, 1948 [26] / As \( N \to \infty \), we have
\[
\sum_{n=0}^{N} \mu^2(n) \mu^2(n+k) = N \left[ \prod_{p \in \mathcal{P}} \left( 1 - \frac{2}{p^2} \right) \prod_{p^2 \mid k} \left( 1 + \frac{1}{p^2 - 2} \right) \right] + O \left( N^{\frac{2}{3}} \log^3(N) \right),
\]
where the \( O \)-constant may depend on \( k \), and we assume the empty product gives contribution 1 (this happens when \( k \) is square free).

From Mirsky Theorem [3.6] we deduce that the spectral measure of the square of the Möbius function \( \sigma_{\mu^2} \) is discrete. Precisely we have the following result.
Corollary 3.7. The spectral measure of the square of the M"obius function $\sigma_{\mu^2}$ is given by

\[
(3.7) \quad \sigma_{\mu^2} = \prod_{p \in P} \left( 1 - \frac{2}{p^2} \right) \left( \delta_0 + \sum_{l=1}^{\infty} \sum_{d \in \mathbb{Q}} \left\{ \frac{1}{d^2} \left( \prod_{p \in D(d)} \left( \frac{1}{p^2 - 2} \right) \sum_{j=0}^{d^2-1} \delta e^{2\pi i j x} \right) \right\} \right)
\]

where $D(d)$ is the set of distinct primes in the decomposition of $d \in \mathbb{Q}$, $d = \prod_{p \in D(d)} p$ and $\omega(d)$ is the cardinal of $D(d)$.

Proof. It is easy to see that for any $d, k \geq 1$ we have

\[
\sum_{j=0}^{d-1} e^{\frac{2\pi i j k}{d}} = \begin{cases} 0 & \text{if } d \nmid k \\ d & \text{if } d \mid k. \end{cases}
\]

To see it put $x = e^{\frac{2\pi i k}{d}}$ and observe that $x = 1$ if and only if $d \mid k$. Combining this fact with Mirsky Theorem 3.6 we have, for any $k \geq 0$,

\[
\frac{1}{N} \sum_{n=1}^{N} \mu^2(n+k)\mu^2(n) \longrightarrow \sigma_{\mu^2}
\]

where $\sigma_{\mu^2}$ is the finite measure on the circle given above (3.7). This completes the proof. □

3.3. Proof of Main results 1, 2 and 3.

Proof of Main result 3 (Theorem 2.8). The proof is a direct consequence of Mirsky Theorem 3.6 combined with Corollary 3.7 above.

Let $(X, \mathcal{A}, P, T)$ be a uniquely weakly mixing dynamical system. Thus, for any continuous function with zero mean it is well known from the classical Wiener Theorem [27] that the spectral measure of $f$ is continuous. Therefore it is orthogonal to any discrete measure. By lemma 2.2, this is true also for the spectral measure $\hat{\sigma}_\mu(0) > 0$. By Lemma 2.2, this is true also for the spectral measure $\hat{\sigma}_g$ of the sequence $g_n = f(T^nx)$ for every $x$. But the spectral measure of $\mu^2(n)$ is indeed discrete (by corollary 3.7 above), so $G(\sigma_{\mu^2}, \sigma_g) = 0$.

Now, applying 3.3 we have

\[
\frac{1}{N} \sum_{n=1}^{N} f(T^nx)\mu^2(n) \longrightarrow 0, \forall f \in C_0(X), \forall x \in X.
\]

This achieves the proof of the Theorem. □

Remark 3.8. This result is similar to Wiener-Wintner Theorem [39].

Proof of Main result 2 (Theorem 2.7). Let $(X, \mathcal{A}, T, P)$ be a uniquely ergodic dynamical system and suppose (2.7) holds. Then the sequence $\mu(n)$ belongs to the Wiener space $S$ and is non trivial $\hat{\sigma}_\mu(0) > 0$. By Lemma 2.2 the sequence $f(T^nx)$ also belongs to the Wiener space, it is non trivial since $\hat{\sigma}_f(0) > 0$ unless $f$ is zero everywhere, and its spectral measure coincides with the spectral measure of $f$.

Then Corollary 3.3 gives

\[
\limsup \left| \frac{1}{N} \sum_{n=1}^{N} f(T^nx)\mu(n) \right| \leq G(\sigma_f, \sigma_\mu)
\]
for any continuous function $f$, for any $x \in X$. But under our assumptions the spectral measure of $\mu$ is the Lebesgue measure and $\sigma_f$ is singular, therefore

$$G(\sigma_f, \sigma_\mu) = 0.$$  

This implies that

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \mu(n) \xrightarrow{N \to \infty} 0.$$  

The proof of the Theorem is complete.  

To prove this theorem it would be enough to assume that the spectral measure of $\mu$ is absolutely continuous with respect to the Lebesgue measure (Elliott conjecture implies it is exactly equal to it). Theorem 2.6 below provides the reverse statement.

**Proof of Main result 1 (Theorem 2.6).** In his Lectures [35] Sarnak constructs a dynamical system with positive entropy, called M"obius dynamical system, for which $\mu(n)$ is a generic point. Let us denote this dynamical system by $(X, \mathcal{B}, \mathbb{P}, S)$. Therefore, there exists a continuous function such that

$$\mu(n) = f(S^n x),$$

where $x$ is a generic point. Let $Y_n = f \circ S^n$. By Wold decomposition we can decompose $Y_n$ as

$$Y_n = Y_n^r + Y_n^s$$

where $Y_n^r$ (resp. $Y_n^s$) is a regular (resp. singular) process. By our assumptions, $Y_0^s$ (resp. $Y_0^r$) is a continuous function and

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) Y_n^s(x) \xrightarrow{n \to \infty} 0$$

hence,

$$\frac{1}{N} \sum_{n=1}^{N} [Y_n^s(x)]^2 \xrightarrow{n \to \infty} 0$$

since

$$\frac{1}{N} \sum_{n=1}^{N} Y_n^s(x) Y_n^r(x) \xrightarrow{n \to \infty} \int Y_0^s Y_0^r \, d\mathbb{P} = 0$$

by the ergodic theorem. Therefore we get (again by the ergodic theorem)

$$\int [Y_0^s(x)]^2 \, d\mathbb{P} = 0.$$  

This means that $Y_0^s = 0$. We conclude that the spectral measure of $\mu(n)$ is absolutely continuous with density $\rho$ and $\ln |\rho|$ is integrable [14, pp.28]. This completes the proof.
4. Spectral disjointness for the random Möbius function

We consider now the random version of the Möbius function defined in (1.4). In order to prove Theorem 2.10 we need to find a subset \( \Omega' \subset \Omega \) such that \( \mathbb{P}(\Omega') = 1 \) and

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \mu_{rand}(n) (w) \to_{N \to \infty} 0 \quad \forall w \in \Omega', \ \forall x \in X, \ \forall f \in C(X).
\]

The proof is based on a Borel-Cantelli argument. The main difficulty is to construct a set \( \Omega' \) that is independent of \( x \) and \( f \). Actually \( C_0(X) \) is compact so it is not hard to get rid of the \( f \) dependence. On the other hand, the \( x \) dependence is quite delicate. Though the space \( X \) is compact the orbits generated by \( T^n x \) may be too complicated to be controlled. The key to solve the problem is to replace \( f(T^n x) \mu_{rand}(n) \) in the sum by an average over a long segment of the orbit starting at \( T^n x \)

\[
f(T^n x) \mu_{rand}(n) \to \frac{1}{m} \sum_{j=0}^{m-1} f(T^{n+j} x) \mu_{rand}(n + j).
\]

The number of orbits can then be bounded using the zero topological entropy.

The rest of this section is devoted to the proof of the theorem. We start by recalling a few basic tools and proving some preliminary results. Finally in the last subsection we give the proof of Theorem 2.10.

4.1. Some preliminary results.

**Lemma 4.1** (Borel-Cantelli Lemma). Let \((X, \mathcal{B}, \nu)\) be a measure space and \( E_n \) be a countable collection of measurable sets. We have

\[
\sum_n \nu(E_n) < \infty \quad \Rightarrow \quad \nu(\limsup_{n \to +\infty} E_n) = 0,
\]

where

\[
\limsup_{n \to +\infty} E_n = \bigcap_n \bigcup_{k \geq n} E_k.
\]

This is a classical result in probability theory. In the following we will also need the following concentration inequality.

**Lemma 4.2.** Let \( x_1, \ldots, x_m \) be \( m \) independent random variables with symmetric distribution \( \mathbb{P}(x_j) = \mathbb{P}(-x_j), \ |x_j| \leq c_j \) for some positive constant \( c_j \) and \( \mathbb{E}(x_j) = 0 \ \forall j \). Then we have

\[
\mathbb{P} \left( \left| \sum_{j=1}^{m} x_j \right| > t \right) \leq 2e^{-\frac{t^2}{2 \sum_{j=1}^{m} c_j^2}}
\]

**Proof.** By Markov inequality

\[
\mathbb{P} \left( \left| \sum_{j=1}^{m} x_j \right| > t \right) \leq e^{-\lambda \mathbb{E} \left( e^{\lambda \left| \sum_{j=1}^{m} x_j \right|} \right)} \ \forall \lambda > 0
\]

\[
\leq e^{-\lambda \mathbb{E} \left( e^{\lambda \sum_{j=1}^{m} x_j} + e^{-\lambda \sum_{j=1}^{m} x_j} \right)}
\]

(4.5)
Since the $x_j$ are independent

$$E\left(e^{\pm \lambda \sum_{j=1}^{m} x_j}\right) = \prod_{j=1}^{m} E\left(e^{\pm \lambda x_j}\right) = \prod_{j=1}^{m} E\left(e^{(\cosh(\lambda x_j))}\right) \leq \prod_{j=1}^{m} \cosh(\lambda c_j) \leq \prod_{j=1}^{m} e^{\frac{(\lambda c_j)^2}{2}}$$

where we used the symmetry of the distribution. Then

$$P\left(\sum_{j=1}^{m} x_j > t\right) \leq 2e^{\sum_{j=1}^{m} \frac{(\lambda c_j)^2}{2}} e^{-\lambda t} \leq 2e^{-\frac{\lambda t^2}{2\sum_{j=1}^{m} c_j^2}}$$

where in the last line we replaced $\lambda = t/\sum_{j=1}^{m} c_j^2$. \hfill $\square$

**Remark 4.3.** This phenomenon of the concentration of probability measure can be interpreted as the absence of randomness for a large values. The bound (4.4) above is a special case of a more general concentration inequality due to Hoeffding and Azuma [4,25]. Indeed it can be generalized to the case of a sum of martingale differences. The proof is given in appendix [4].

**Lemma 4.4.** Let $X$ be a compact metric space, $T$ a continuous map on $X$, $f$ a complex valued continuous function on $X$. Assume the topological entropy of the dynamical system $(X,T)$ is zero. We introduce the $m$-sequence

$$\xi_m(T^n x) = (f(T^n x), f(T^{n+1} x), \ldots, f(T^{n+m} x)).$$

This is the segment of the dynamical sequence generated by $m$ iterations starting at $T^n x$. Then

1. for any $\delta > 0$ there exists a constant $\varepsilon_0(\delta)$ such that for any $\varepsilon < \varepsilon_0(\delta)$ we can find a minimal set of points

$$R(m, \varepsilon) = \left(x_1^{(m)}, \ldots, x_{r(m, \varepsilon)}^{(m)}\right)$$

such that each $m$-sequence $\xi_m(T^n x)$ is localized in a $\delta$-neighborhood of exactly one of these points. More precisely for any $n, x$ there exists an integer $1 \leq j_n, r \leq r(m, \varepsilon)$ such that

$$|f(T^{n+j} x) - f(T^{j} x_j^{(m)})| \leq \delta \quad \forall j = 0, \ldots, m - 1.$$

2. The number of points $r(m, \varepsilon)$ we need in order to localize all $m$-sequences does not grow too fast with $m$. Precisely, for any $\eta > 0$ there exists a constant $\varepsilon_1(\eta)$ and an integer $M_0(\eta) > 0$ such that

$$r(m, \varepsilon) < e^{m\eta} \quad \forall m > M_0(\eta), \forall \varepsilon < \varepsilon_1(\eta).$$

**Proof.** The set $X$ being compact, for each choice of $\varepsilon$ there exists a minimal set of points $R(m, \varepsilon) = \left(x_1^{(m)}, \ldots, x_{r(m, \varepsilon)}^{(m)}\right)$ such that

$$X = \bigcup_{j=1}^{r(m, \varepsilon)} B_{d_m}(x_j^{(m)}, \varepsilon),$$

where $d_m$ is the distance defined in (2.2). Moreover $f$ being continuous, for any $\delta > 0$ there exists a $\varepsilon_0(\delta) > 0$ such that

$$d(x, y) < \varepsilon_0(\delta) \Rightarrow |f(x) - f(y)| < \delta.$$
Now let \( \varepsilon < \varepsilon_0(\delta) \). Then for any \( n, x \) there is a \( 0 \leq j_{n,x} \leq r(m, \varepsilon) \) such that

\[
T^{n}x \in B_{d_{m}}(x_{j_{n,x}}^{(m)}, \varepsilon)
\]

so

\[
d \left( T^{n+j_{n,x}}x, T^{j_{n,x}}x^{(m)} \right) \leq \varepsilon \Rightarrow \left| f(T^{n+j_{n,x}}x) - f(T^{j_{n,x}}x^{(m)}) \right| \leq \delta \quad \forall j = 0, \ldots, m - 1.
\]

The second statement is a direct result of the definition of zero topological entropy. see (2.6) and (2.7).

Before going to the proof of the theorem we still need to introduce a few definitions.

**Definition 4.5.** We will call \( X_{n}(x, \cdot) \), \( Y_{n}^{m}(x, \cdot) \) the initial random variable and new one associated with a \( m \)-sequence

\[
X_{n}(x, w) = f(T^{n}x)\mu_{rand}(n)(w)
\]

(4.11)

\[
Y_{n}^{m}(x, w) = \frac{1}{m} \sum_{j=0}^{m-1} f(T^{j}x)\mu_{rand}(n+j)(w)
\]

(4.12)

Note that \( Y_{n}^{m} \) is a sum of bounded independent random variables with symmetric distribution then lemma 4.2 applies. Moreover they depend on the parameter \( n \) only through the random Moebius function. These facts will be crucial to control our bounds.

**4.2. Proof of Main result 4 (Theorem (2.10)).** We are now able to give the proof of our main result.

**Proof.** Using the definitions above the sum we need to estimate can be written as

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^{n}x)\mu_{rand}(n)(w) = \frac{1}{N} \sum_{n=0}^{N-1} X_{n}(x, w).
\]

(4.13)

Since we are interested only in the limit \( N \to \infty \) we can replace \( X_{n}(x, \cdot) \) by \( Y_{n}^{m}(T^{n}x, \cdot) \). Indeed

\[
\frac{1}{N} \sum_{n=0}^{N-1} X_{n}(x, w) = \frac{1}{N} \sum_{n=0}^{N-1} Y_{n}^{m}(T^{n}x, w) + R_{Nm}(x, w)
\]

(4.14)

where

\[
R_{Nm}(x, w) = \frac{1}{Nm} \sum_{j=0}^{m-1} \sum_{n=0}^{j-1} \left[ f(T^{n}x)\mu_{rand}(n)(w) - f(T^{n+N}x)\mu_{rand}(n+N)(w) \right]
\]

(4.15)

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^{n}x)\mu_{rand}(n)(w) \to 0 \quad \text{as} \quad N \to \infty
\]

for any fixed \( m \), since

\[
|R_{Nm}| \leq \frac{1}{Nm} \sum_{j=0}^{m-1} j\|f\|_{\infty} \leq \frac{m}{N}\|f\|_{\infty} \to 0 \quad \text{as} \quad N \to \infty.
\]
Therefore the difference between the two sequences in (4.14) has a well defined limit. We will consider the second sequence \( \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n \). Let us consider the \( \limsup_N \) and \( \liminf_N \) of this sequence. We have

\[
\left| \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(x, w) \right| \leq \limsup_N \left| \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(x, w) \right|
\]

\[
\left| \liminf_N \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(x, w) \right| \leq \limsup_N \left| \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(x, w) \right|
\]

We claim the right-hand side of these equations is zero. The rest of the proof is devoted to prove this claim. Therefore we need to study

(4.16) \[
\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} |Y^m_n(T^n x, w)|.
\]

In Lemma 4.6 below we will prove that for each \( n \in \mathbb{N} \), and each \( f \in C(X) \) there exists a subset \( \Omega'_{f,n} \subset \Omega \) such that \( \mathbb{P}(\Omega'_{f,n}) = 1 \) and

(4.17) \[
\limsup_{m} \sup_{x \in X} |Y^m_n(T^n x, w)| = 0 \quad \forall \; w \in \Omega'_{f,n}.
\]

Now we define \( \Omega'_f = \bigcap_{n \in \mathbb{N}} \Omega'_{f,n} \). This is still a measurable set, \( \mathbb{P}(\Omega'_f) = 1 \) and for each \( w \in \Omega'_f \) we have

\[
\limsup_{m} \sup_{x \in X} |Y^m_n(T^n x, w)| = 0 \quad \forall \; n \in \mathbb{N}.
\]

Then

\[
\lim_{m} Y^m_n(T^n x, w) = 0 \quad \forall \; n \in \mathbb{N}, \; \forall \; x \in X,
\]

and for any \( w \in \Omega'_f \) we have

\[
\lim_{m} \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(T^n x, w) = 0 \quad \forall \; N \geq 1, \; \forall \; x \in X.
\]

Exchanging the roles of \( m \) and \( N \) above we have

\[
\lim_{N} \frac{1}{m} \sum_{j=0}^{m-1} Y^N_j(T^j x, w) = 0 \quad \forall m \geq 1, \; \forall \; x \in X.
\]

Note that for any finite \( N \) and \( m \) we have

\[
\frac{1}{m} \sum_{j=0}^{m-1} Y^N_j(T^j x, w) = \frac{1}{m} \sum_{j=0}^{m-1} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{n+j}) \mu_{\text{rand}}(n+j)(w) = \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(T^n x, w),
\]

then if \( w \in \Omega'_f \) we have

\[
\lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} Y^m_n(T^n x, w) = 0. \quad \forall \; m \geq 1, \; \forall \; x \in X.
\]

Inserting this result in (4.14) we have

\[
\lim_{N} \frac{1}{N} \sum_{n=0}^{N-1} X_n(x, w) = 0. \quad \forall \; x \in X.
\]
for each \( w \in \Omega'_f \). To complete the proof we need to get rid of the \( f \) dependence in the set \( \Omega'_f \). But, since \( X \) is a compact set, the space \( C(X) \) of continuous functions on \( X \) is separable. Then, there exists a sequence of continuous functions \( (f_r)_{r \geq 0} \) dense in \( C(X) \). We define

\[
\Omega' = \bigcap_{r \in \mathbb{N}} \Omega'_{f_r}.
\]

This set satisfies \( \mathbb{P}(\Omega') = 1 \). Now let \( w \in \Omega' \) and \( f \in C(X) \). For any \( \delta > 0 \) we can find a function \( f_r \) in the sequence such that \( \|f - f_r\|_\infty < \delta \). Then

\[
\lim_N \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \mu_{\text{rand}}(n)(w) \leq \delta + \lim_N \frac{1}{N} \sum_{n=0}^{N-1} f_r(T^n x) \mu_{\text{rand}}(n)(w) = \delta.
\]

Letting \( \delta \to 0 \) we have for each \( w \in \Omega' \)

\[
\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \mu_{\text{rand}}(n)(w) \xrightarrow{N \to \infty} 0 \\ \forall x \in X, \forall f \in C(X).
\]

This completes the proof of the theorem. \( \square \)

Finally we give the proof of the bound (4.17), that we used in the proof above.

**Lemma 4.6.** Let

\[
Y^{(m)}_n(x, w) = \frac{1}{m} \sum_{j=0}^{m-1} f(T^j x) \mu_{\text{rand}}(n + j)(w).
\]

Then for each \( n \in \mathbb{N} \), and each \( f \in C(X) \) there exists a subset \( \Omega'_{f,n} \subset \Omega \) such that \( \mathbb{P}(\Omega'_{f,n}) = 1 \) and

\[
\limsup_{n \to \infty} \sup_{x \in X} |Y^{(m)}_n(T^n x, w)| = 0 \quad \forall w \in \Omega'_{f,n}.
\] (4.18)

**Proof.** Let \( \delta, \eta > 0 \) be two fixed positive constants. We will adjust later \( \eta \) as a function of \( \delta \). Let \( R(m, \varepsilon) \) as in (2.3) and \( \varepsilon < \min[\varepsilon_0(\delta), \varepsilon_1(\eta)] \) as in lemma 4.3 above. Then for any \( x, n, \varepsilon \) there exists a \( 0 \leq j_{n,x} \leq r(m, \varepsilon) \) such that

\[
|f(T^{n+j} x) - f(T^j x_{j_{n,x}})| \leq \delta \quad \forall j = 0, \ldots, m - 1.
\]

Inserting this bound in \( Y^{(m)}_n \) we have

\[
|Y^{(m)}_n(T^n x, w)| \leq |Y^{(m)}_n(T^n x, w) - Y^{(m)}_n(x_{j_{n,x}}, w)| + |Y^{(m)}_n(x_{j_{n,x}}, w)|
\]

\[
< \delta + \sup_{1 \leq k \leq r(m, \varepsilon)} |Y^{(m)}_n(x_k^{(m)}, w)|.
\]

The right-hand side does not depend on \( x \) so we also have

\[
\sup_{x \in X} |Y^{(m)}_n(T^n x, w)| < \delta + \sup_{1 \leq k \leq r(m, \varepsilon)} |Y^{(m)}_n(x_k^{(m)}, w)|.
\]

Now remark that \( Y^{(m)}_n(x_k^{(m)}, \cdot) \) is a sum of independent bounded random variables with zero average and symmetric distribution so lemma 4.2 applies.

\[
\mathbb{P}(|Y^{(m)}_n(x)| > t) = \mathbb{P}\left( \sum_{j=0}^{m-1} f(T^j x) \mu_{\text{rand}}(n + j) \right) > tm) \leq 2e^{-\frac{tm^2}{2\|f\|_\infty}}.
\] (4.19)
where we used $\sum_{j=0}^{m-1} c_j^2 \leq m \|f\|_2^2$. Then

$$
P \left( \sup_{1 \leq k \leq r(m,c)} |Y^m_n(x_k)| > t \right) \leq \sum_{k=1}^{r(m,c)} P (|Y^m_n(x_k)| > t) \leq 2 e^{\eta m} e^{-\frac{m^2}{2\|f\|_\infty^2}} \leq 2 e^{-m \frac{t^2}{2\|f\|_\infty^2}} \eta \tag{4.20}
$$

In the following we fix $t = 2\delta$ and $\eta = \frac{\delta^2}{2\|f\|_\infty^2}$. Then

$$
\sup_{x \in X} |Y^m_n(T^m x)| \leq \delta + t = 3\delta \quad \text{with probability larger than} \quad 1 - 2 e^{-m \frac{t^2}{2\|f\|_\infty^2}} \cdot
$$

Finally to get rid of the $m$ dependence, we consider

$$
P \left( \sup_{m \geq M} \sup_{x \in X} |Y^m_n(T^m x)| > 3\delta \right) \leq \sum_{m \geq M} P \left( \sup_{x \in X} |Y^m_n(T^m x)| > 3\delta \right) \leq 2 e^{-m \frac{t^2}{2\|f\|_\infty^2}} < \delta. \tag{4.21}
$$

The last inequality is true only for $M$ large enough. But, we can always choose $M(\delta)$ such that the sum above is bounded by $\delta$. Inserting all these results in the initial sum we find that $\forall M \geq M(\delta)$

$$
\sup_{m \geq M} \sup_{x \in X} |Y^m_n(T^m x)| \leq 3\delta \quad \text{with probability larger than} \quad 1 - \delta. \tag{4.22}
$$

with probability larger than $1 - \delta$. Let us take the sequence $\delta_q = \frac{\delta}{q^2}$ and let

$$
E_{q,n} = \{ \limsup_{m} \sup_{x \in X} |Y^m_n(T^m x)| > 3\delta_q \}.
$$

Then we have

$$
\sum_q P (E_{q,n}) \leq \sum_q \delta_q < \infty
$$

and by Borel-Cantelli lemma 4.1

$$
P \left( \limsup_{q} E_{q,n} \right) = 0.
$$

This means that

$$
1 = P \left( \liminf_{q} E_{q,n}^c \right) = P \left( \cup_{q \geq q_0} \cap_{q \geq q_0} E_{q,n}^c \right).
$$

We define

$$
\Omega_{n,f}' = \cup_{q \geq q_0} \cap_{q \geq q_0} E_{q,n}^c.
$$

Then $P(\Omega_f') = 1$ and for each $w \in \Omega_f'$ there exists a $q_0$ such that

$$
\limsup_{m} \sup_{x \in X} |Y^m_n(T^m x, w)| \leq 3\delta_q \quad \forall q \geq q_0.
$$

Then for all $w \in \Omega_{n,f}'$

$$
\limsup_{m} \sup_{x \in X} |Y^m_n(T^m x, w)| = 0.
$$
This concludes the proof of the lemma.

Appendix A. Concentration inequality

We include here a proof of a useful concentration inequality due to Hoeffding and Azuma [21, 25].

**Theorem A.1 (Hoeffding-Azuma Inequality).** Let \( \{Y_k, F_k\}_{k=1}^n \) be a martingale difference sequence (i.e., \( Y_k \) is \( F_k \)-measurable, \( E[|Y_k|] < +\infty \) and \( E[Y_k|F_{k-1}] = 0 \) a.s.

for every \( k \leq n \)). Assume that, for every \( k \in \{1, \ldots, n\} \), there exist numbers \( c_k \in \mathbb{R}_+ \) such that a.s. \( |Y_k| \leq c_k \). Then, for every \( t > 0 \),

\[
\mathbb{P}\left\{ \left| \sum_{k=1}^n Y_k \right| \geq t \right\} \leq 2e^{-\frac{t^2}{4\sum_{k=1}^n c_k^2}}
\]

To prove this theorem we will need the following lemma.

**Lemma A.2.** Let \( f \) be a convex smooth real function such that, for any \( n \in \mathbb{N} \), \( f^{(2n)}(0) = f^{(n)}(0) > 0 \), \( F \) a sub \( \sigma \)-algebra and \( Y \) a random variable such that almost surely \( E[Y|F] = 0 \) and \( |Y| \leq c \) for some constant \( c > 0 \). Then for any \( t > 0 \) we have

\[
E(f(tY)|F) \leq f\left(\frac{t^2c^2}{2}\right) \quad \text{a.s.}
\]

**Proof.** Since \( f \) is convex we have

\[
f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}, \forall 0 \leq \alpha \leq 1.
\]

Then, if \( |y| \leq c \), we can choose \( x_1 = c \) and \( x_2 = -c \), so

\[
f(y) \leq \frac{1}{2}\left\{1 + \frac{y}{c}\right\}f(c) + \frac{1}{2}\left\{1 - \frac{y}{c}\right\}f(-c).
\]

Taking a conditional expectation with respect to \( F \) on both sides, we get

\[
E(f(Y)|F) \leq \frac{1}{2}\left\{f(c) + f(-c)\right\}.
\]

Now expanding in a Taylor series, we have

\[
f(-c) + f(c) = \sum_{n=0}^{+\infty} \frac{f^{(2n)}(0)}{2n!}c^{2n}.
\]

But, for every integer \( m \geq 0 \),

\[
2n! \geq 2n(2n-2)(2n-4)\cdots 2 = 2^nn!
\]

then due to the our assumption \( f^{(2n)}(0) = f^{(n)}(0) > 0 \), we have

\[
f(-c) + f(c) \leq \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!}\left(\frac{c^2}{2}\right)^n = f\left(\frac{c^2}{2}\right),
\]

for any choice of \( c \in \mathbb{R} \). Finally

\[
E(f(Y)|F) \leq f\left(\frac{c^2}{2}\right).
\]

Replacing \( Y \) by \( tY \) and \( c \) by \( tc \) we obtain the result. □
Proof of Theorem [A.1] By Markov inequality we have

\[
\mathbb{P}\left\{ \sum_{k=1}^{n} Y_k \geq t \right\} = \mathbb{P}\left\{ \sum_{k=1}^{n} Y_k - t \geq 0 \right\} \leq e^{-\lambda t} \mathbb{E}\left( e^{\lambda \sum_{k=1}^{n} Y_k} \right),
\]

for every \( \lambda > 0 \). Now

\[
\mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n} Y_k \right) \right) = \mathbb{E}\left( \mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n} Y_k \right) \mid \mathcal{F}_{n-1} \right) \right)
\]

\[
= \mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n-1} Y_k \right) \mathbb{E}\left( \exp(\lambda Y_n) \mid \mathcal{F}_{n-1} \right) \right)
\]

where the last passage holds since \( \exp \left( \lambda \sum_{k=1}^{n-1} Y_k \right) \) is \( \mathcal{F}_{n-1} \)-measurable. Applying Lemma [A.2], we get

\[
\mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n} Y_k \right) \right) = \mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n-1} Y_k \right) \mathbb{E}\left( \exp(\lambda Y_n) \mid \mathcal{F}_{n-1} \right) \right)
\]

\[
\leq \mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n-1} Y_k \right) \right) \exp(\lambda^2 c_n^2/2)
\]

(A.2)

Hence, by repeatedly using the recursion in (A.2), it follows that

\[
\mathbb{E}\left( \exp \left( \lambda \sum_{k=1}^{n} Y_k \right) \right) \leq \prod_{k=1}^{n} \exp(\lambda^2 c_k^2/2) = \exp\left( \lambda^2 \sum_{k=1}^{n} c_k^2/2 \right).
\]

Inserting this result in (A.1) we have

\[
\mathbb{P}\left\{ \sum_{k=1}^{n} Y_k \geq t \right\} \leq \exp\left( -\lambda t + \lambda^2 \sum_{k=1}^{n} c_k^2/2 \right)
\]

for every \( \lambda > 0 \). Optimizing over the free parameter \( \lambda > 0 \) we obtain

\[
\mathbb{P}\left\{ \sum_{k=1}^{n} Y_k \geq t \right\} \leq \exp\left( -t^2/(2 \sum_{k=1}^{n} c_k^2) \right).
\]

Therefore, by substituting \(-Y\) by \( Y \) and using the same procedure, we get

\[
\mathbb{P}\left\{ \sum_{k=1}^{n} Y_k \leq -t \right\} \leq \exp\left( -t^2/(2 \sum_{k=1}^{n} c_k^2) \right).
\]

Hence

\[
\mathbb{P}\left\{ \left| \sum_{k=1}^{n} Y_k \right| \geq t \right\} \leq 2 \exp\left( -t^2/(2 \sum_{k=1}^{n} c_k^2) \right).
\]

This proves the theorem. \( \square \)

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