Formation of Power-law Energy Spectra in Space Plasmas by Stochastic Acceleration due to Whistler-Mode Waves

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Short title: FORMATION OF POWER-LAW ENERGY SPECTRA
**Abstract.** A non-relativistic Fokker-Planck equation for the electron distribution function is formulated incorporating the effects of stochastic acceleration by whistler-mode waves and Coulomb collisions. The stationary solution $f$ to the equation, subject to a zero-flux boundary condition, is found to be a generalized Lorentzian (or kappa) distribution, which satisfies $f \propto v^{-2(\kappa+1)}$ for large velocity $v$, where $\kappa$ is the spectral index. The parameter $\kappa$ depends strongly on the relative wave intensity $R$. Taking into account the critical energy required for resonance of electrons with whistlers, we calculate a range of values of $R$ for each of a number of different space plasmas for which kappa distributions can be expected to be formed. This study is one of the first in the literature to provide a theoretical justification for the formation of generalized Lorentzian (or kappa) particle distribution functions in space plasmas.
1. Introduction

In the natural space environment, e.g., planetary magnetospheres, and the solar wind (and actually many other kinds of astrophysical objects), plasmas are generally observed to possess a particle distribution function with a non-Maxwellian high-energy tail. The distribution function typically has a power-law tail in energy \[ \text{distribution} \propto (\text{particle energy})^{-(\kappa+1)} \], and frequently can be modeled by a generalized Lorentzian (kappa) distribution. For a variety of observational data, see Vasyliunas [1968], Gosling et al. [1981], Armstrong et al. [1983], Christon et al. [1988], Divine and Garrett [1983] and references contained in Summers and Thorne [1991], Collier [1993], and Mace and Hellberg [1995]. Associated with the generalized Lorentzian distribution has been developed the modified plasma dispersion function which is an effective tool for investigating waves and microinstabilities in space plasmas [Summers and Thorne, 1991; Mace and Hellberg, 1995; Summers et al., 1994, 1996]. Physical mechanisms that can produce power-law particle distributions include stochastic acceleration by plasma wave turbulence, and collisionless shocks. Kappa-like distribution functions which are Maxwellian at low energies and power law at high energies can also be produced by velocity-space Lévy flight probability distributions [e.g., Collier, 1993]. It is the process of stochastic acceleration by plasma wave turbulence with which we shall be concerned in this paper. Many researchers have studied stochastic acceleration of charged particles by various modes of plasma wave, e.g., Gurevich [1960], Kennel and Engelmann [1966], Melrose [1980, 1986], Steinacker and Miller [1992], and Schlickeiser [1997]. Hasegawa et al. [1985] showed that superthermal radiation can enhance velocity-space diffusion so as to produce a power-law distribution. However, these authors did not estimate values of the intensity of the radiation field and other plasma parameters necessary to obtain a reasonable value for the power law. The source of the free energy to excite such intense radiation was another question left unaddressed. Melrose [1980] pointed out that the equilibrium distribution is a power law only if the momentum diffusion
coefficient $D(v)$ is proportional to $v^{-1}$, where $v$ is the particle velocity. For example, for Langmuir waves, $D(v) \propto v^{-1}$; but this requires the turbulent spectrum $I(k)$ to be independent of the wave number $k$, which is an unlikely possibility [Melrose, 1980]. Stochastic acceleration by plasma waves is characteristically slow due to the diffusive nature of the scattering process. Nevertheless, the process may be applicable, for instance, in planetary magnetospheres in cases when typical electron acceleration times are of the order of tens of hours. We note that Schlickeiser [1997] has recently obtained a diffusion coefficient varying as $D(v) \propto v^{-1}$ when the charged particles are accelerated by whistler-mode waves. The momentum diffusion coefficient results from the resonant interaction between particles and whistler-electron cyclotron plasma waves. Similar formulae have also been obtained by Dermer et al. [1996]. Numerical calculations of the momentum diffusion coefficients for particle-whistler interaction were given by Steinacker and Miller [1992] and Pryadka and Petrosian [1997], but no analytical formulae were obtained for medium energetic electrons (10 keV to a few hundreds of keV), which limits their application by other authors. There are several physical processes that can drive the whistler instability. For example, anisotropies in the plasma velocity distribution can make whistlers unstable. Moreover, a kappa distribution can itself enhance the whistler-mode instability, as compared with the Maxwellian distribution [e.g., see Summers and Thorne, 1992; Xue et al., 1993; Mace, 1998].

In the following section, we shall develop a non-relativistic theory for the formation of kappa distributions by means of whistler-mode wave stochastic acceleration.

2. Theory

We assume a Kolmogorov-type magnetic turbulence power spectrum in the wave number for the whistler-mode waves, namely,

$$I = I_0 k_{||}^{-q},$$

(1)
with \( q > 1 \), where \( k_\parallel \geq k_{\text{min}} \), \( k_\parallel \) is the parallel wave number, and 
\[ I_0 = (q-1)(\delta B)B_0^2k_{\text{min}}^{q-1} \] is the energy density of the whistler turbulence. It is natural to assume that 
\( k_{\text{min}}V_A = \Omega_i \) for whistler waves, where \( V_A \) and \( \Omega_i \) are respectively the Alfvén speed and the non-relativistic ion gyrofrequency. In this paper we are concerned with the situation in which pitch-angle anisotropy is small and the particle distribution is isotropic. For higher energy electrons, it is generally the case that the time-scale for pitch angle diffusion is much less than that for momentum diffusion, and so the assumption of isotropy is justified. For lower energy electrons, where the pitch angle and momentum diffusion rates are comparable, we assume that isotropization by Coulomb collisions occurs. The momentum diffusion coefficient can then be averaged over the pitch angle. Herein, we employ the diffusion coefficient given by Schlickeiser [1997], namely,
\[
D = c^2\Omega_e \frac{\pi(q-1)}{8} \left( \frac{\delta B}{B_0} \right)^2 \beta_A^{2+q} \left( \frac{k_{\text{min}}c}{\Omega_i} \right)^{q-1} \left( \frac{m_p}{m_e} \right) J_W \beta^{-1},
\] (2)
where \( \Omega_e \) is the non-relativistic electron gyrofrequency, \( \beta_A = V_A/c, \beta = v/c, c \) is the speed of light, and \( J_W \) is a weakly varying function of \( v \). Henceforth in this paper, the Kolmogorov turbulent spectrum \( (q = 5/3) \) is adopted. It then follows that \( J_W \) is of order unity. In accordance with the whistler wave dispersion relation and the resonance condition [Melrose, 1986], we find that electrons can resonate with whistlers provided that \( \beta \geq (m_p/m_e)^{1/2}\beta_A \). The diffusion coefficient [2] is valid in the region where 
\( (m_p/m_e)^{1/2}\beta_A \leq \beta \leq (m_p/m_e)\beta_A \). However, lower-energy electrons can be accelerated by high wave-number whistlers as the electron cyclotron branches of the dispersion relation are approached [Petrosian, 1994].

Coulomb interactions result in both momentum diffusion and friction in the kinetic equation for the particle distribution function. As we are assuming the distribution to be isotropic, we can ignore the angular contribution to Coulomb collisions [Hinton, 1983], giving the equation,
\[
\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} v^2 \left( F_c f + \frac{1}{2} D_\parallel \frac{\partial f}{\partial v} \right),
\] (3)
where the frictional coefficient $F_c$ is given by

$$F_c = \nu_s v + \frac{D_{\perp c} - D_{\parallel c}}{v} + \frac{1}{2} \frac{dD_{\parallel c}}{dv}.$$  

(4)

$\nu_s$ is the slowing-down rate, and $D_{\perp c}$ and $D_{\parallel c}$ are the respective perpendicular and parallel diffusion coefficients. When the velocity of the particle $v$ is much larger than the thermal velocities of the electrons and ions [$v_e \ll v, v_i \ll v, v_{e,i} = (2T_{e,i}/m_{e,i})^{1/2}$], we have by Hinton [1983],

$$\nu_s = 3n_e \frac{\Gamma_e v}{v^3},$$  

(5)

$$D_{\perp c} = 2n_e \frac{\Gamma_e}{v},$$  

(6)

$$D_{\parallel c} = n_e \frac{\Gamma_e (v_e^2 + v_i^2)}{v^3} \simeq n_e \frac{\Gamma_e v_e^2}{v^3},$$  

(7)

where $n_e$ is the electron number density,

$$\Gamma_e = \frac{4\pi e^4 \log_{10} \Lambda}{m_e^2},$$

and $\log_{10} \Lambda \simeq 20$ is the Coulomb logarithm. Finally, modifying Equation (3) to include acceleration by whistler-mode waves, we obtain the following Fokker-Planck equation to describe the evolution of the distribution function:

$$\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} v^2 \left( F_c f + \frac{1}{2} D_{\parallel c} \frac{\partial f}{\partial v} \right) + \frac{1}{v^2} \frac{\partial}{\partial v} \left( v^2 D \frac{\partial f}{\partial v} \right),$$  

(8)

where $D$ is given by Equation (2). With a zero-flux boundary condition, the stationary solution of Equation (8) is given by

$$f = A \exp \left\{ - \int \frac{2n_e \Gamma_e v^{-2}}{n_e \Gamma_e v_e^2 v^{-3} + 2D} dv \right\}.$$  

(9)

Substituting the expression (2) for $D$ in Equation (8), we obtain

$$f = A \exp \left\{ - \int \frac{d[v^2/v_e^2]}{1 + \frac{v^2}{(\kappa+1)v_e^2}} \right\}.$$
\[ A = \frac{1}{(\pi \kappa)^{3/2} \theta^3} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)}, \]  

(10)

where

\[ \kappa + 1 = \frac{24e^4 n_e \log_{10} \Lambda}{\Omega_e m_p m_e c^2 R \beta_A^2 J W}, \]  

(11)

\( A \) is a constant of integration, \( R = (\delta B/B_0)^2 \) is the power of the wave turbulence, and \( \theta = [(\kappa + 1)/\kappa]^{1/2} \nu_e \). Normalizing the distribution using \( \int_0^{\infty} 4\pi v^2 f \, dv = 1 \) gives

\[ A = A \left[ 1 + \frac{\nu^2}{\kappa \theta^2} \right]^{-(\kappa+1)}, \]  

where \( \Gamma \) is the gamma function. The distribution (10) is formally a generalized Lorentzian (kappa) distribution, with the spectral index \( \kappa \) given by Equation (11).

Parenthetically, it can be noted that setting \( D = 0 \) in Equation (9) leads to the recovery of the Maxwellian distribution \( f = A \exp(-v^2/\nu_e^2) \). From Equation (11) we determine that \( \kappa \) is given by

\[ \kappa + 1 = 0.9 \times 10^{-25} n_e^{5/2} B_0^{-4} R^{-1} J_W^{-1}, \]  

(12)

where \( n_e \) is in \( \text{cm}^{-3} \), and \( B_0 \) in Gauss.

3. Discussion

In the foregoing, we have determined that stochastic acceleration by whistler-mode turbulence can produce a particle distribution with a power-law tail, i.e., \( f \propto v^{-(\kappa+1)} \). By Equation (12), we note that the value of the spectral index \( \kappa \) depends strongly on the relative wave intensity \( R \). In Table I, for various space plasmas, we have listed typical values of the parameters \( n_e, B_0, \beta_A \) and the thermal energy(temperature). The basic plasma parameters in this table for the solar wind were taken from Kivelson and Russell [1995], for the Earth’s plasma sheet from Christon et al. [1988], for the Jovian magnetosphere from Dessler [1983] and Divine and Garrett [1983], and for the Saturnian magnetosphere from Sittler et al. [1983]. Because only the energetic electrons with velocities satisfying \( \beta \geq (m_p/m_e)^{1/2} \beta_A \) can resonate with first-harmonic
whistlers, we list in Table 1 the critical energy $E_c$ corresponding to the critical velocity $\beta_c = (m_p/m_e)^{1/2}\beta_A$. If the critical energy $E_c$ is much larger than the electron thermal energy, then the whistlers can only accelerate non-thermal high energy particles, for which the Coulomb momentum diffusion coefficient $D_{||c}$ is much less than the acceleration diffusion coefficient. In this case, an injection mechanism is required to pre-accelerate the particle to the required critical energy $E_c$. It is straightforward to show that this situation applies, for instance, to solar flares. In fact, there is no observational evidence that electrons have a kappa distribution in solar flares. If the critical energy $E_c$ is of the same order as the thermal energy, then whistlers can accelerate the thermal particles into the high-energy tail of the distribution, and so a kappa distribution can be produced. This is essentially the case for all the space plasmas shown in Figure 1. For a specific plasma characterized by its electron number density $n_e$ and background magnetic field $B_0$, formula (12) gives the spectral index $\kappa$ produced by whistler turbulence of power $R$. In Figure 1, for each of the given space plasmas, we show the range of $R$-values associated with the range of $\kappa$-values that might be typically observed, namely $2 \leq \kappa \leq 8$. In addition, we also show the value of the critical energy $E_c$ for each plasma. The results imply that relatively weak whistler-mode turbulence could produce power-law spectra in the Earth’s plasma sheet and in the Jovian and Saturnian magnetospheres, whereas stronger turbulence is required in the solar wind.

In conclusion, we should point out both the merits and the limitations of this study. Of course, it is not the case that all electron power-law spectra are produced by whistler-mode turbulence, or that whistler-mode turbulence will always give rise to electron power-law spectra. Nevertheless, under certain restrictions, we have shown that the production of electron power-law energy spectra by stochastic acceleration due to whistlers is a viable possibility. Apart from the non-relativistic framework, limitations of the study include the assumptions of isotropy, a zero-flux boundary condition, and a momentum diffusion coefficient based on wave propagation strictly parallel to the
background magnetic field. The intrinsic value of the study is that it is one of the first in the literature to present a firm theoretical basis for the formation of generalized Lorentzian (or kappa) particle distribution functions in space plasmas.

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References

Armstrong, T. P., M. T. Paonessa, E. V. Bell II, and S. M. Krimigis, Voyager observations of Saturnian ion and electron phase space densities, *J. Geophys. Res.*, 88, 8893-8904, 1983.

Christon, S. P., D. G. Mitchell, D. J. Williams, L. A. Frank, C. Y. Huang, and T. E. Eastman, Energy spectra of plasma sheet ions and electrons from $\approx 50 \text{ eV/e}$ to $\approx 1 \text{ Mev}$ during plasma temperature transitions, *J. Geophys. Res.*, 93, 2562-2572, 1988.

Collier, M. R., On generating kappa-like distribution functions using velocity space Lévy flights, *Geophys. Res. Lett.*, 20, 1531-1534, 1993.

Dermer, C. D., J. A. Miller, and H. Li, Stochastic particle acceleration near accretion black holes, *Astrophys. J.*, 456, 106-109, 1996.

Dessler, A. J. (Ed.), *Physics of the Jovian Magnetosphere*, Cambridge University Press, New York, 1983.

Divine N., and H. B. Garrett, Charged particle distribution in Jupiter’s magnetosphere, *J. Geophys. Res.*, 88, 6889-6903, 1983.

Gosling, J. T., J. R. Asbridge, S. J. Bame, W. C. Feldman, R. D. Zwickl, G. Paschmann, N. Sckopke, and R. J. Hynds, Interplanetary ions during an energetic storm particle event: The distribution function from solar wind thermal energies to 1.6 Mev, *J. Geophys. Res.*, 86, 547-554, 1981.

Gurevich, A. V., On the amount of accelerated particles in an ionized gas under various accelerating mechanisms, *Sov. Phys. JETP*, 11, 1150-1157, 1960.

Hasegawa, A., K. Mima, and M. Duong-van, Plasma distribution function in a superthermal radiation field, *Phys. Rev. Lett.*, 54, 2608-2610, 1985.

Hinton, F. L., in *Basic Plasma Physics I*, edited by A. A. Galeev and R. N. Sudan, pp. 147, North-Holland Publishing Company, Amsterdam, 1983.

Kennel, C. F., and F. Engelmann, Velocity space diffusion from weak plasma turbulence in a magnetic field, *Phys. Fluids*, 9, 2377-2388, 1966.
Kivelson, M. G., and C. T. Russell (Eds.), *Introduction to Space Physics*, Cambridge University Press, New York, 1995.

Mace, R. L., Whistler instability enhanced by superthermal electrons within the Earth’s foreshock, *J. Geophys. Res.*, 103, 14,643-14,654, 1998.

Mace, R. L., and M. A. Hellberg, A dispersion function for plasmas containing superthermal particles, *Phys. Plasmas*, 2, 2098-2109, 1995.

Melrose, D. B., *Plasma Astrophysics: Nonthermal Processes in Diffuse Magnetized Plasmas*, Vol. 2, 60 pp., Gordon and Breach Science Publishers, New York, 1980.

Melrose, D. B., *Instabilities in Space and Laboratory Plasmas*, 234 pp., Cambridge University Press, New York, 1986.

Petrosian, V., in *High Energy Solar Phenomena*, AIP Conf. Proc. 294, edited by J.M. Ryan, and W.T. Vestrand, pp. 162, AIP, New York, 1994.

Pryadka, J. M., and V. Petrosian, Stochastic acceleration of low-energy electrons in cold plasmas, *Astrophys. J.*, 482, 774-781, 1997.

Schlickeiser, R., γ-ray evidence for galactic in situ electron acceleration, *Astron. Astrophys.*, 319, L5-L8, 1997.

Sittler, E. C., Jr., K. W. Ogilvie, and J. D. Scudder, Survey of low-energy plasma electrons in Saturn’s magnetosphere: Voyagers 1 and 2, *J. Geophys. Res.*, 88, 8847-8870, 1983.

Steinacker, J., and J. A. Miller, Stochastic gyroresonant electron acceleration in a low-beta plasma. I. Interaction with parallel transverse cold plasma waves, *Astrophys. J.*, 393, 764-781, 1992.

Summers, D., and R. M. Thorne, The modified plasma dispersion function, *Phys. Fluids, B3*, 1835-1847, 1991.

Summers, D., and R.M. Thorne, A new tool for analyzing microinstabilities in a space plasmas modeled by a generalized Lorentzian (kappa) distribution, *J. Geophys. Res.*, 97, 16827-16832, 1992.
Summers, D., S. Xue, and R. M. Thorne, Calculation of the dielectric tensor for a generalized Lorentzian (kappa) distribution function, Phys. Plasmas, 1, 2012-2025, 1994.

Summers, D., R. M. Thorne, and H. Matsumoto, Evaluation of the modified plasma dispersion function for half-integral indices, Phys. Plasmas, 3, 2496-2501, 1996.

Vasyliunas, V. M., A survey of low-energy electrons in the evening sector of the magnetosphere with OGO 1 and OGO 3, J. Geophys. Res., 73, 2839-2884, 1968.

Xue, S., R. M. Thorne, and D. Summers, Electromagnetic ion-cyclotron instability in space plasmas, J. Geophys. Res., 98, 17475-17484, 1993.

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Ref. Geophysical Research Letters 25, No.21, 4099, 1998.
Figure 1. The range of values of the relative wave intensity $R$, calculated from Equation (12), corresponding to the range of $\kappa$-values, $2 \leq \kappa \leq 8$, for space plasmas for which typical parameters and $E_c$-values are given in Table 1.
Table 1. Typical parameters for different space plasmas. Also given are values for the critical energy $E_c$ which is the minimum particle energy required for particle acceleration by whistler-mode waves.

| Plasma parameters | Solar wind$^a$ | Earth’s plasma sheet$^b$ | Jupiter$^c$ | Saturn$^d$ |
|-------------------|----------------|--------------------------|------------|------------|
| Density $n_e$ (cm$^{-3}$) | 7 | 1 | $2 \times 10^3$ | 0.14 |
| Magnetic field $B_0$ (G) | $7 \times 10^{-5}$ | $2 \times 10^{-4}$ | $1.8 \times 10^{-2}$ | $6 \times 10^{-5}$ |
| Alfvén speed parameter $\beta_A$ | $1.9 \times 10^{-4}$ | $1.5 \times 10^{-3}$ | $2.8 \times 10^{-3}$ | $1.1 \times 10^{-3}$ |
| Thermal energy (eV) | 12 | 1000 | 1000 | 300 |
| Critical energy $E_c$ (eV) | 17 | 1500 | 3500 | 500 |

$^a$At 1 AU, A.J. Hundhausen, in Kivelson and Russell [1995], page 92.

$^b$At L=14, Christon et al. [1988].

$^c$At L=6, e.g., Dessler [1983] and Divine and Garrett [1983].

$^d$At L=15, Sittler et al. [1983].
Figure 1

- Jovian magnetosphere
- Earth's plasma sheet
- Saturnian magnetosphere
- Solar wind