On the power of Ambainis’s lower bounds

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Abstract

The polynomial method and the Ambainis’s lower bound (or Alb, for short) method are two main quantum lower bound techniques. While recently Ambainis showed that the polynomial method is not tight, the present paper aims at studying the power and limitation of Alb’s. We first use known Alb’s to derive \(\Omega(n^{1.5})\) lower bounds for Bipartiteness, Bipartiteness Matching and Graph Matching, in which the lower bound for Bipartiteness improves the previous \(\Omega(n)\) one. We then show that all the three known Ambainis’s lower bounds have a limitation \(\sqrt{N \cdot \min\{C_0(f), C_1(f)\}}\), where \(C_0(f)\) and \(C_1(f)\) are the 0- and 1-certificate complexity, respectively. This implies that for some problems such as Triangle, \(k\)-Clique, and Bipartite/Graph Matching which draw wide interest and whose quantum query complexities are still open, the best known lower bounds cannot be further improved by using Ambainis’s techniques. Another consequence is that all the Ambainis’s lower bounds are not tight. For total functions, this upper bound for Alb’s can be further improved to \(\min\{\sqrt{C_0(f)C_1(f)}, \sqrt{N \cdot CI(f)}\}\), where \(CI(f)\) is the size of max intersection of a 0-and a 1-certificate set. Again this implies that Alb’s cannot improve the best known lower bound for some specific problems such as And-Or Tree, whose precise quantum query complexity is still open. Finally, we generalize the three known Alb’s and give a new Alb style lower bound method, which may be easier to use for some problems.

1 Introduction

Quantum computing has received a great deal of attention in the last decade because of the potentially high speedup over the classical computation. Among others, query model is extensively used in studying quantum complexity, partly because it is a natural quantum analog of classical decision tree complexity, and partly because many known quantum algorithms fall into this framework, including Simon’s algorithm [26], Shor’s period finding [27], Grover’s searching algorithm [19] and many others [10, 11, 16, 18, 20]. In the query model, the input is accessed by querying an oracle, and the goal is to minimize the number of queries made. We are most interested in double sided bound-error computation, where the output is correct with probability at least 2/3 for all inputs. We use \(Q_2(f)\) to denote minimal number of queries for computing \(f\) with double sided bound-error. For more details on quantum query model, we refer to [4, 14] as excellent surveys.

Two main lower bound techniques for \(Q_2(f)\) are the polynomial method [9] and Ambainis’s lower bounds [8], the latter of which is also called quantum adversary method. Many lower bounds have recently been achieved by applying the polynomial method [1, 9, 23, 25, 28] and Ambainis’s lower bounds [2, 3, 5, 15, 31]. Recently, Aaronson even used Ambainis’s lower bound technique to achieve lower bounds for some classical algorithms [2]. Given the usefulness of the two methods, it

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is interesting to know how tight they are. In a recent work \[3\], Ambainis proved that polynomial method is not tight, by showing a function with polynomial degree \(\Omega(M^{1.321})\). So a natural question is the power of Ambainis’s lower bounds. We show that all known Ambainis’s lower bounds are not tight either, among other results.

There are several known versions of Ambainis’s lower bounds, among which the three Ambainis’s theorems are widely used partly because they have simple forms and are thus easy to use. The first Alb’s two are given in \[3\] as follows.

**Theorem 1 (Ambainis, \[3\])** Let \(f : \{0,1\}^N \to \{0,1\}\) be a function and \(X,Y\) be two sets of inputs s.t. \(f(x) \neq f(y)\) if \(x \in X\) and \(y \in Y\). Let \(R \subseteq X \times Y\) be a relation s.t.

1. \(\forall x \in X\), there are at least \(m\) different \(y \in Y\) s.t. \((x,y) \in R\).
2. \(\forall y \in Y\), there are at least \(m'\) different \(x \in X\) s.t. \((x,y) \in R\).
3. \(\forall x \in X\), \(\forall i \in [N]\), there are at most \(l\) different \(y \in Y\) s.t. \((x,y) \in R\) and \(x_i \neq y_i\).
4. \(\forall y \in Y\), \(\forall i \in [N]\), there are at most \(l'\) different \(x \in X\) s.t. \((x,y) \in R\) and \(x_i \neq y_i\).

Then \(Q_2(f) = \Omega(\sqrt{\frac{mm'}{l_{\max}}})\).

**Theorem 2 (Ambainis, \[3\])** Let \(f : I^N \to \{0,1\}\) be a Boolean function where \(I\) is a finite set, and \(X,Y\) be two sets of inputs s.t. \(f(x) \neq f(y)\) if \(x \in X\) and \(y \in Y\). Let \(R \subseteq X \times Y\) satisfy

1. \(\forall x \in X\), there are at least \(m\) different \(y \in Y\) s.t. \((x,y) \in R\).
2. \(\forall y \in Y\), there are at least \(m'\) different \(x \in X\) s.t. \((x,y) \in R\).

Denote

\[ l_{x,i} = |\{y : (x,y) \in R, x_i \neq y_i\}|, \quad l_{y,i} = |\{x : (x,y) \in R, x_i \neq y_i\}| \]

\[ l_{\max} = \max_{x,y,i: (x,y) \in R, i \in [N], x_i \neq y_i} l_{x,i}l_{y,i}. \]

Then \(Q_2(f) = \Omega(\sqrt{\frac{mm'}{l_{\max}}})\).

Obviously, Theorem 2 generalizes Theorem 1. In \[5\], Ambainis gave another (weighted) way to generalize Theorem 1. We restate it in a form similar to Theorem 1.

**Definition 1** Let \(f : I^N \to \{0,1\}\) be a Boolean function where \(I\) is a finite set. Let \(X,Y\) be two sets of inputs s.t. \(f(x) \neq f(y)\) if \(x \in X\) and \(y \in Y\). Let \(R \subseteq X \times Y\) be a relation. A weight scheme for \(X,Y,R\) consists three weight functions \(w(x,y) > 0\), \(u(x,y,i) > 0\) and \(v(x,y,i) > 0\) satisfying

\[ u(x,y,i)v(x,y,i) \geq w^2(x,y) \]  

for all \((x,y) \in R\) and \(i \in [N]\) with \(x_i \neq y_i\). We further denote

\[ w_x = \sum_{y: (x,y) \in R} w(x,y), \quad w_y = \sum_{x: (x,y) \in R} w(x,y) \]

\[ u_{x,i} = \sum_{y: (x,y) \in R, x_i \neq y_i} u(x,y,i), \quad u_{y,i} = \sum_{x: (x,y) \in R, x_i \neq y_i} u(x,y,i), \]

\[ v_{x,i} = \sum_{y: (x,y) \in R, x_i \neq y_i} v(x,y,i), \quad v_{y,i} = \sum_{x: (x,y) \in R, x_i \neq y_i} v(x,y,i). \]

**Theorem 3 (Ambainis, \[5\])** Let \(f : I^N \to \{0,1\}\) where \(I\) is a finite set, and \(X \subseteq f^{-1}(0)\), \(Y \subseteq f^{-1}(1)\) and \(R \subseteq X \times Y\). Let \(w, u, v\) be a weight scheme for \(X,Y,R\). Then

\[ Q_2(f) = \Omega(\sqrt{\min_{x \in X \cap [N]} \frac{w_x}{u_{x,i}}} \cdot \min_{y \in Y \cap [N]} \frac{w_y}{v_{y,i}})} \]
Let us denote by $Alb_1(f)$, $Alb_2(f)$ and $Alb_3(f)$ the best lower bound for function $f$ achieved by Theorem 1, 2 and 3, respectively\(^1\). Note that in the four $Alb$’s, there are many parameters $(X, Y, R, u, v, w)$ to be set. By setting these parameters in an appropriate way, one can get lower bounds of quantum query complexity for many problems. In particular, we consider the following three graph properties\(^2\).

1. **Bipartiteness:** Given an undirected graph, decide whether it is a bipartite graph.
2. **Graph Matching:** Given an undirected graph, decide whether it has a perfect matching.
3. **Bipartite Matching:** Given an undirected bipartite graph, decide whether it has a perfect matching.

We show by using $Alb_2$ that all these three graph properties have a $\Omega(n^{1.5})$ lower bound, where $n$ is the number of vertices. For Bipartiteness, this improves the previous result of $\Omega(n)$ lower bound by Laplante and Magniez\(^2\) and Durr et al\(^1\).

Since $Alb_2$ and $Alb_3$ generalize $Alb_1$ in different ways, it is interesting to compare them and see which is more powerful. It turns out that $Alb_2(f) \leq Alb_3(f)$.

However, even $Alb_3$ has a limitation: we show that $Alb_3(f) \leq \sqrt{N \cdot \min\{C_0(f), C_1(f)\}}$, where $C_0(f)$ and $C_1(f)$ are the 0- and 1-certificate complexity, respectively. This has two immediate consequences. First, it gives a negative answer to the open problem whether $Alb_2$ or $Alb_3$ is tight, because for Element Distinctness, we know that $Q_2(f) = \Theta(N^{2/3})$ while on the other hand we have $\sqrt{N \cdot \min\{C_0(f), C_1(f)\}} = \sqrt{2N}$.

Second, for some problems whose precise quantum query complexities are still unknown, our theorem implies that the best known lower bound cannot be further improved by using Ambainis’s lower bound techniques, no matter how we choose the parameters in the $Alb$ theorems. For example Triangle/$k$-ClOQUE ($k$ is constant) are the problems to decide whether an $n$-node graph contains a triangle/$k$-node clique. It is easy to get a $\Omega(n)$ lower bound for both of them, and by our theorem, this is the best possible by using Ambainis’s lower bound techniques. Also the $\Omega(n^{1.5})$ lower bound for Bipartiteness, Bipartite Matching and Graph Matching cannot be further improved by $Alb$’s because the $C_1(f) = O(n)$ for all of them.

Further, if $f$ is a total function, then the above upper bound for $Alb$’s can be further tightened in two ways. The first one is that $Alb_3(f) \leq \sqrt{N \cdot CI(f)}$, where $CI(f)$ is the the size of the largest intersection of a 0-certificate set and a 1-certificate set, so $CI(f) \leq C_\ast(f)$. The second approach leads to another result $Alb_3(f) \leq \sqrt{C_0(f)C_1(f)}$. Both the results imply that for AND-OR Tree, a problem whose quantum query complexity is still open\(^5\), the current best $\Omega(\sqrt{N})$ lower bound cannot be further improved by using Ambainis’s lower bounds.

It is also natural to consider combining the different approaches that $Alb_2$ and $Alb_3$ use to generalize $Alb_1$, and get a further general one. Based on this idea, we give a new and more generalized lower bound theorem, which we call $Alb_4$. Compared with $Alb_3$, this may be easier to use.

**Related work**

Recently, Szegedy independently shows that $Alb_3(f) \leq \sqrt{N \cdot C_\ast(f)}$ in general and $Alb_3(f) \leq \sqrt{C_0(f)C_1(f)}$ in a different way\(^9\). He also shows in \(^30\) that $Alb_3$ by Ambainis\(^5\), $Alb_4$ in the present paper\(^2\), and another quantum adversary method proposed in \(^13\) are equivalent.

The theorem $Alb_3(f) \leq \sqrt{N \cdot C_\ast(f)}$ is also obtained by Laplante and Magniez by using Kolmogorov complexity\(^2\).

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\(^1\)To make the later results more precise, we actually use $Alb_1(f)$ to denote the value inside the $\Omega()$ notation. For example, $Alb_1(f) = \max_{X,Y,R} \sqrt{\frac{\mathbb{F}}{m}}$.

\(^2\)In this paper, all the graph property problems are given by adjacency matrix input.
2 Old Ambainis’s lower bounds

In this section we first use Alb₂ to derive \( \Omega(n^{1.5}) \) lower bounds for Bipartiteness, Bipartite Matching and Graph Matching, then show that Alb₂ has actually at least the same power as Alb₂.

**Theorem 4** All the three graph properties Bipartiteness, Bipartite Matching and Graph Matching have \( Q_\ell(f) = \Omega(n^{1.5}) \).

**Proof** 1. Bipartiteness. The proof is very similar to the one for proving \( \Omega(n^{1.5}) \) lower bound of Graph Connectivity by Durr et al [13]. Without loss of generality, we assume \( n \) is even, because otherwise we can use the following argument on arbitrary \( n - 1 \) (out of total \( n \)) nodes and leave the \( n^{th} \) node isolated. Let

\[
X = \{G : G \text{ is composed of a single } n\text{-length cycle}\},
\]

\[
Y = \{G : G \text{ is composed of two cycles each with length being an odd number between } n/3 \text{ and } 2n/3\},
\]

and

\[
R = \{(G, G') \in X \times Y : \exists \text{ four nodes } v_1, v_2, v_3, v_4 \text{ s.t. the only difference between graphs } G \text{ and } G' \text{ is that } (v_1, v_2), (v_3, v_4) \text{ are edges in } G \text{ but not in } G' \text{ and } (v_1, v_3), (v_2, v_4) \text{ are edges in } G' \text{ but not in } G\}.
\]

Note that a graph is bipartite if and only if it contains no cycle with odd length. Therefore, any graph in \( X \) is a bipartite graph because \( n \) is even, and any graph in \( Y \) is not bipartite graph because it contains two odd-length cycles. Then all the remaining analysis is the same as calculation in the proof for Graph Connectivity (undirect graph and matrix input) in [13], and finally

\( Alb_2(\text{ Bipartiteness }) = \Omega(n^{1.5}) \).

2. Bipartite Matching. Let \( X \) be the set of the bipartite graphs like Figure 1(a) where \( \tau \) and \( \sigma \) are two permutations of \( \{1, \ldots, n\} \), and \( \frac{n}{6} \leq k \leq \frac{2n}{3} \). Let \( Y \) be the set of the bipartite graphs like Figure 1(b), where \( \tau' \) and \( \sigma' \) are two permutations of \( \{1, \ldots, n\} \), and also \( \frac{n}{6} \leq k' \leq \frac{2n}{3} \). It is easy to see that all graphs in \( X \) have no matching, while all graphs in \( Y \) have one.

![Figure 1: X and Y](image)

Let \( R \) be the set of all pairs of \((x, y) \in X \times Y\) as in Figure 2, where graph \( y \) is obtained from \( x \) by choosing two horizontal edges \((\tau(i), \sigma(i)), (\tau(j), \sigma(j))\), removing them, and adding two edges \((\tau(i), \sigma(j)), (\tau(j), \sigma(i))\).

Now it is not hard to calculate the \( m, m', l_{\max} \) in Alb₂. For example, to get \( m \) we study \( x \) in two cases. When \( \frac{n}{6} \leq k \leq \frac{2n}{3} \), any edge \((\tau(i), \sigma(i))\) where \( i \in [k - n/3, k] \) has at least \( n/6 \) choices for edge \((\tau(j), \sigma(j))\) because the only requirement for choosing is that \( k' \in [n/3, 2n/3] \) and \( k' = j - i \). The case when \( \frac{n}{6} \leq k \leq \frac{2n}{3} \) can be handled symmetrically. Thus \( m = \Theta(n^2) \). Same argument yields \( m' = \Theta(n^2) \). Finally, for \( l_{\max} \), we note that if the edge \( e = (\tau(i), \sigma(i)) \) for some \( i \), then \( l_{x,e} = O(n) \)
and \( l_{y,e} = 1 \); if the edge \( e = (\tau(i), \sigma(j)) \) for some \( i, j \), then \( l_{x,e} = 1 \) and \( l_{y,e} = O(n) \). For all other edges \( e \), \( l_{x,e} = l_{y,e} = 0 \). Putting all together, we have \( l_{\text{max}} = O(n) \). Thus by Theorem 2, we know that \( \text{Alb}_2(\text{Bipartite Matching}) = \Omega(n^{1.5}) \).

3. Graph Matching. This can be easily shown either by using the same \((X, Y, R)\) as the proof for Bipartiteness, because a cycle with odd length has no matching, or by noting that Bipartite Matching is a special case of Graph Matching. □

It is interesting to note that we can also prove the above theorem by \( \text{Alb}_3 \). For example, for Bipartite Matching, We choose \( X, Y, R \) in the same way, and let \( w(x, y) = 1 \) for all \((x, y) \in R\).

Let \( u(x, y, e) = 1/\sqrt{n} \) if \( e \) is a horizontal edge \((\tau(i), \sigma(i))\) in \( x \), and \( u(x, y, e) = \sqrt{n} \) if \( e = (\tau(i), \sigma(j)) \) or \( e = (\tau(j), \sigma(i)) \) in \( x \). Thus \( u_{x,e} = \Theta(\sqrt{n}) \) for all edge \( e \), it is the same for \( v_{y,e}, \) thus \( w_x/u_{x,e} = \Theta(n^{1.5}), \) \( w_y/v_{y,e} = \Theta(n^{1.5}), \) and \( Q_2(f) = \Omega(n^{1.5}) \) by \( \text{Alb}_3 \).

This coincidence is not accident. Actually it turns out that we can always show a lower bound by \( \text{Alb}_3 \) provided that it can be shown by \( \text{Alb}_2 \).

**Theorem 5** \( \text{Alb}_2(f) \leq \text{Alb}_3(f) \).

**Proof** For any \( X, Y, R \) in Theorem 2, we set the weight functions in Theorem 3 as follows. Let \( w(x, y) = 1 \), \( u(x, y, i) = \sqrt{l_{\text{max}}/l_{x,i}} \) and \( v(x, y, i) = \sqrt{l_{\text{max}}/l_{y,i}} \). It’s easy to check that

\[
u(x, y, i)v(x, y, i) = \frac{l_{\text{max}}}{l_{x,i}l_{y,i}} \geq 1 = w(x, y)
\]

Now that \( u(x, y, i) \) is independent on \( y \), so we have \( u_{x,i} = l_{x,i}u(x, y, i) = \sqrt{l_{\text{max}}} \). Symmetrically, it follows that \( v_{y,i} = \sqrt{l_{\text{max}}} \). Thus, by denoting \( m_x = |\{y : (x, y) \in R\}| \) and \( m_y = |\{x : (x, y) \in R\}| \), we have

\[
\min_{x,i} \frac{w_x}{u_{x,i}} \min_{y,i} \frac{w_y}{v_{y,i}} = \min_{x,i} \frac{m_x}{\sqrt{l_{\text{max}}}} \min_{y,i} \frac{m_y}{\sqrt{l_{\text{max}}}} = \frac{m}{\sqrt{l_{\text{max}}} \sqrt{l_{\text{max}}}} = \frac{m \sqrt{l_{\text{max}}}}{l_{\text{max}}}
\]

which means that for any \( X, Y, R \) in Theorem 2, the lower bound result can be also achieved by Theorem 3. □

### 3 Limitations of Ambainis’s lower bounds

In this section, we show some bounds for the \( \text{Alb} \)’s in terms of certificate complexity. We consider Boolean functions.
3.1 A general limitation for Ambainis’s lower bounds

In this subsection, we give an upper bound for $Alb_4(f)$, which implies a limitation of all the three known Ambainis’s lower bound techniques.

**Theorem 6** $Alb_3(f) \leq \sqrt{N \cdot C_-(f)}$, for any $N$-ary Boolean function $f$.

**Proof** Actually we prove a stronger result: for any $(X,Y,R,u,v,w)$ as in Theorem 3,

$$\min_{(x,y)\in R, i\in [N]} \frac{w_x w_y}{u_{x,i} v_{y,i}} \leq NC_-(f).$$

With out loss of generality, we assume that $C_-(f) = C_0(f)$, and $X \subseteq f^{-1}(0)$ and $Y \subseteq f^{-1}(1)$. We can actually further assume that $R = X \times Y$, because otherwise we just let $R' = X \times Y$, and set new weight functions as follows.

$$u'(x,y,i) = \begin{cases} u(x,y,i) & (x,y) \in R \\ 0 & otherwise \end{cases}, \quad v'(x,y,i) = \begin{cases} v(x,y,i) & (x,y) \in R \\ 0 & otherwise \end{cases},$$

$$w'(x,y) = \begin{cases} w(x,y) & (x,y) \in R \\ 0 & otherwise \end{cases}.$$

Then it is easy to see that it satisfies (1), so it is also a weight scheme. And for these new weight functions, we have $u'_{x,i} = \sum_{y: (x,y) \in R, i \neq y} u'(x,y,i) = \sum_{y: (x,y) \in R, i \neq y} u(x,y,i) = u_{x,i}$ and similarly $v'_{y,i} = v_{y,i}$ and $w'_x = w_x, w'_y = w_y$. It follows that $\frac{w_x w_y}{u_{x,i} v_{y,i}} = \frac{w'_x w'_y}{u'_{x,i} v'_{y,i}}$, thus we can use $(X', Y', R', u', v', w')$ to derive the same lower bound as we use $(X, Y, R, u, v, w)$.

So now we suppose $R = X \times Y$ and We prove that $\exists x \in X, y \in Y, i \in [N], s.t.$

$$w_x w_y \leq N \cdot C_0(f) u_{x,i} v_{y,i},$$

Suppose the claim is not true. Then for all $x \in X, y \in Y, i \in [N]$, we have

$$w_x w_y > N \cdot C_0(f) u_{x,i} v_{y,i}.$$

We first fix $i$ for the moment. And for each $x \in X$, we fix a smallest certificate set $CS_x$ of $f$ on $x$. Clearly $|CS_x| \leq C_0(f)$. We sum (2) over $\{x \in X : i \in CS_x\}$ and $\{y \in Y\}$. Then we get

$$\sum_{x \in X : i \in CS_x} w_x \sum_{y \in Y} w_y > N \cdot C_0(f) \sum_{x \in X : i \in CS_x} \sum_{y \in Y} u_{x,i} v_{y,i}. \quad (3)$$

Note that $\sum_{y \in Y} w_y = \sum_{x \in X, y \in Y} w(x,y) = \sum_{x \in X} w_x$, and that $\sum_{y \in Y} v_{y,i} = \sum_{x \in X, y \in Y : x \neq y} v(x,y,i) = \sum_{x \in X} v_{x,i}$ where $v_{x,i} = \sum_{y \in Y : x \neq y} v(x,y,i)$. Inequality (3) turns to

$$\sum_{x \in X : i \in CS_x} w_x \sum_{x \in X} w_x > N \cdot C_0(f) \sum_{x \in X : i \in CS_x} u_{x,i} \sum_{x \in X} v_{x,i} \geq N \cdot C_0(f) \sum_{x \in X : i \in CS_x} u_{x,i} \sqrt{\sum_{x \in X} v_{x,i}} \geq N \cdot C_0(f) \sqrt{\sum_{x \in X : i \in CS_x} u_{x,i} v_{x,i}}^2.$$

\footnote{Note that the function values of $u', v', w'$ are zero when $(x,y) \neq R$, which does not conform to the definition of weight scheme. But actually Theorem 3 also holds for $u \geq 0, v \geq 0, w \geq 0$ as long as $u_{x,i}, v_{y,i}, w_x, w_y$ are all strictly positive for any $x, y, i$. This can be seen from the proof of $Alb_4$ in Section 4.}
due to Cauchy-Schwartz Inequality. We further note that
\[ u_{x,i}v_{x,i} = \sum_{y \in Y: x_i \neq y_i} u(x, y, i) \sum_{y \in Y: x_i \neq y_i} v(x, y, i) \]
\[ \geq \left( \sum_{y \in Y: x_i \neq y_i} \sqrt{u(x, y, i)v(x, y, i)} \right)^2 = \left( \sum_{y \in Y: x_i \neq y_i} w(x, y) \right)^2 = (w_{x,i})^2 \]
where we define \( w_{x,i} = \sum_{y \in Y: x_i \neq y_i} w(x, y) \). Thus
\[ \sum_{x \in X} w_x \sum_{i \in CS_x} w_x > N \cdot C_0(f) \left( \sum_{x \in X} w_{x,i} \right)^2 \] (4)

Now we sum (4) over \( i = 1, ..., N \), and note that
\[ \sum_{i} \sum_{x \in X, i \in CS_x} w_x = \sum_{x \in X} \sum_{i \in CS_x} w_x \leq C_0(f) \sum_{x \in X} w_x \]
because \( |CS_x| \leq C_0(f) \) for each \( x \). We have
\[ (\sum_{x \in X} w_x)^2 > N \sum_{i=1}^N \left( \sum_{x \in X, i \in CS_x} w_{x,i} \right)^2 \]
By the arithmetic-square average inequality \( N(a_1^2 + ... + a_N^2) \geq (a_1 + ... + a_N)^2 \), we have
\[ (\sum_{x \in X} w_x)^2 > \left( \sum_{x \in X, i \in [N]: \ i \in CS_x} w_{x,i} \right)^2 = \sum_{x \in X, i \in [N], y \in Y: \ i \in CS_x, x_i \neq y_i} w(x, y)^2 = \left( \sum_{x \in X, y \in Y: \ i \in [N]: \ i \in CS_x, x_i \neq y_i} w(x, y) \right)^2 \]
But by the definition of certificate, we know that for any \( x \) and \( y \) there is at least one index \( i \in CS_x \) s.t. \( x_i \neq y_i \). Therefore, we derive an inequality
\[ (\sum_{x \in X} w_x)^2 > \left( \sum_{x \in X, y \in Y} w(x, y) \right)^2 = (\sum_{x \in X} w_x)^2 \]
which is a contradiction, as desired. □

We add some comments about this upper bound of \( Alb_3 \). First, this bound looks weak at first glance because the \( \sqrt{N} \) factor seems too large. But in fact it is necessary at least for partial functions. Consider the problem of INVERT A PERMUTATION\(^4\), where \( C_0(f) = C_1(f) = 1 \) but even the \( Alb_2(f) = \Omega(\sqrt{N}) \).

Second, the quantum query complexity of ELEMENT DISTINCTNESS is known to be \( \Theta(N^{2/3}) \). The lower bound part is obtained by Shi\(^5\) (for large range) and Ambainis\(^6\) (for small range); the upper bound part is obtained by Ambainis\(^7\). Observe that \( C_1(f) = 2 \) thus \( \sqrt{NC_1(f)} = \Theta(N) \), we derive the following interesting corollary from the above theorem.

**Corollary 7 Alb_3 is not tight.**

We make some remarks on the quantity \( \sqrt{N \cdot C_\sim(f)} \) to end this subsection. A function \( f \) is symmetric if \( f(x_1, ..., x_N) = f(x_{\sigma(1)}, ..., x_{\sigma(n)}) \) for any input \( x \) and any permutation \( \sigma \) on \( [N] \). In\(^8\), Beal et al prove that \( Q_2(f) = \Theta(\sqrt{N(N-\Gamma(f))}) \) by using Paturi’s result \( \widetilde{deg}(f) = \Theta(\sqrt{N(N-\Gamma(f))}) \)
\(^2\). Here \( \Gamma(f) = min\{2k - n + 1: f_{k} \neq k_{k+1}, 0 \leq k \leq n - 1\} \). It is not hard to show that \( \Gamma(f) = N - \Theta(C_\sim(f)) \) for symmetric function \( f \). Thus we know that both \( \tilde{deg}(f) \) and \( Q_2(f) \) can also be described in terms of certificate complexity as \( \Theta(\sqrt{N \cdot C_\sim(f)}) \).

\(^4\)The original problem is not a Boolean function, but we can define a Boolean-valued version of it. Instead of finding the position \( i \) with \( x_i = 1 \), we are to decide whether \( i \) is odd or even. The original proof of the \( \Omega(\sqrt{N}) \) lower bound still holds.
3.2 Two better upper bounds for total functions

It turns out that if the function is total, then the upper bound can be further tightened. We introduce a new measure which basically characterizes the size of intersection of a 0- and 1-certificate sets.

**Definition 3** For any function \( f \), if there is a certificate set assignment \( CS : \{0, 1\}^N \rightarrow 2^{|N|} \) such that for any inputs \( x, y \) with \( f(x) \neq f(y) \), \( |CS_x \cap CS_y| \leq k \), then \( k \) is called a candidate certificate intersection complexity of \( f \). The minimal candidate certificate intersection complexity of \( f \) is called the certificate intersection complexity of \( f \), denoted by \( CI(f) \). In other words, \( CI(f) = \min_{CS} \max_{x, y : f(x) \neq f(y)} |CS_x \cap CS_y| \).

Now the improved theorem is as follows. Note that by the above definition we know \( CI(f) \leq C_-(f) \), thus the following theorem really improves Theorem 7 for total functions.

**Theorem 8** \( Alb_2(f) \leq \sqrt{N \cdot CI(f)} \), for any \( N \)-ary total Boolean function \( f \).

**Proof** Again, we prove a stronger result that for any \((X, Y, R, u, v, w)\) in Theorem 3,

\[
\min_{(x, y) \in R, i \in |N|} \frac{w_x w_y}{u_{x, i} v_{y, i}} \leq N \cdot CI(f).
\]

Similar to the proof for Theorem 6, we assume without loss of generality that \( R = X \times Y \) and for all \( x \in X, y \in Y \), we have

\[
w_x w_y > N \cdot CI(f) u_{x, i} v_{y, i}.
\]

and we shall show a contradiction. Now first fix \( i \) and sum \((\Box)\) over \( \{x \in X : i \in CS_x\} \) and \( \{y \in Y : i \in CS_y\} \). We get

\[
\sum_{x \in X, y \in Y : i \in CS_x \cap CS_y} w_x w_y \geq N \cdot CI(f) \sum_{x \in X : i \in CS_x} u_{x, i} \sum_{y \in Y : i \in CS_y} v_{y, i}
\]

\[
\geq N \cdot CI(f) \sum_{x \in X, y \in Y : i \in CS_x \cap CS_y, x \neq y} u(x, y, i)
\]

\[
\sum_{x \in X, y \in Y : i \in CS_x \cap CS_y, x \neq y} v(x, y, i)
\]

\[
\geq N \cdot CI(f) \frac{\sum_{x \in X, y \in Y : i \in CS_x \cap CS_y, x \neq y} u(x, y, i)v(x, y, i)}{2}
\]

\[
\geq N \cdot CI(f) \left( \sum_{x \in X, y \in Y : i \in CS_x \cap CS_y, x \neq y} w(x, y) \right)^2
\]

Now sum over \( i = 1, ..., N \), we get

\[
\sum_{x \in X, y \in Y : i \in |N| : i \in CS_x \cap CS_y} w_x w_y \geq N \cdot CI(f) \sum_{i = 1}^{N} \left( \sum_{x \in X, y \in Y : i \in CS_x \cap CS_y, x \neq y} w(x, y) \right)^2
\]

\[
\geq CI(f) \left( \sum_{x \in X, y \in Y : i \in |N| : i \in CS_x \cap CS_y, x \neq y} w(x, y) \right)^2
\]

Note that for total function \( f \), if \( f(x) \neq f(y) \), there is at least one position \( i \in CS_x \cap CS_y \) s.t. \( x_i \neq y_i \). Thus

\[
\sum_{x \in X, y \in Y : i \in |N| : i \in CS_x \cap CS_y, x_i \neq y_i} w(x, y) \geq \sum_{x \in X, y \in Y} w(x, y)
\]

On the other hand, by the definition of \( CI(f) \), we have

\[
\sum_{x \in X, y \in Y : i \in |N| : i \in CS_x \cap CS_y} w_x w_y \leq CI(f) \sum_{x \in X, y \in Y} w_x w_y = CI(f) \left( \sum_{x \in X, y \in Y} w(x, y) \right)^2
\]

Therefore we get a contradiction

\[
CI(f) \left( \sum_{x \in X, y \in Y} w(x, y) \right)^2 > CI(f) \left( \sum_{x \in X, y \in Y} w(x, y) \right)^2
\]

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as desired. □

In [31], Ambainis proposed the open problem AND-OR Tree. In the problem, there is a complete binary tree with height $2n$. Any node in odd levels is labelled with AND and any node in even levels is labelled with OR. The $N = 4^n$ leaves are the input variables, and the value of the function is the value that we get at the root, with value of each internal node calculated from the values of its two children in the common AND/OR interpretation. The best quantum lower bound is $\Omega(\sqrt{N})$ and best quantum upper bound is no more than the best classical (randomized) one $O((1+\frac{1}{\log N})^n)$. Note that $C_-(\text{AND-OR Tree}) = 2^n = \sqrt{N}$ and thus $\sqrt{NC_-(f)} = N^{3/4}$. Using the above theorem, we know that we cannot improve best known lower bounds of AND-OR Tree by $Alb$’s.

**Corollary 9** Alb$_4$(AND-OR Tree) $\leq \sqrt{N}$.

**Proof** It is sufficient to prove that there is a certificate assignment $CS$ s.t. $|CS_x \cap CS_y| = 1$ for any $f(x) \neq f(y)$. In fact, by a simple induction, we can prove that the standard certificate assignment satisfies this property. The base case is trivial. For the induction step, we note that for an AND connection of two subtrees, the 0-certificate set of the new larger tree can be chosen as any one of the two 0-certificate sets of the two subtrees, and the 1-certificate set of the new larger tree can be chosen as the union of the two 1-certificate sets of the two subtrees. As a result, the intersection of the two new certificate sets is not enlarged. The OR connection of two subtrees is analyzed in the same way. Thus the intersection of the final 0- and 1-certificate sets is of size 1. □

We can also tighten the $\sqrt{N \cdot C_-(f)}$ upper bound in another way and get the following result, which also implies Corollary 9.

**Theorem 10** Alb$_3(f) \leq \sqrt{C_0(f)C_1(f)}$, for any total Boolean function $f$.

**Proof** For any $(X, Y, R, u, v, w)$ in Theorem 3, we assume without loss of generality that $X \subseteq f^{-1}(0), Y \subseteq f^{-1}(1)$ and $R = X \times Y$. We are to prove $\exists x, y, i, j \text{ s.t. } w_x w_y \leq C_0(f)C_1(f)u_{x,i}v_{y,j}$. Suppose this is not true, i.e. for all $x \in X, y \in Y, i, j \in [N]$, $w_x w_y > C_0(f)C_1(f)u_{x,i}v_{y,j}$. First fix $x, y$ and sum over $i \in CS_x$ and $j \in CS_y$. Since $|CS_x| \leq C_0(f), |CS_y| \leq C_1(f)$, we have

$$w_x w_y > \sum_{i \in CS_x} u_{x,i} \sum_{j \in CS_y} v_{y,j}$$

Now we sum over $x \in X$ and $y \in Y$,

$$\sum_{x \in X} w_x \sum_{y \in Y} w_y > \sum_{x \in X, i \in CS_x} u_{x,i} \sum_{y \in Y, j \in CS_y} v_{y,j} \sum_{x \in X, y \in Y, i \in [N]: x_i \neq y_j} v(x, y, i)$$

Since $f$ is total, there is at least one $i_0 \in CS_x \cap CS_y$ s.t. $x_{i_0} \neq y_{i_0}$. Thus

$$\sum_{x \in X} w_x \sum_{y \in Y} w_y > \sum_{x, y \in Y, i \neq y_{i_0}} u(x, y, i_0) \sum_{x, y \in Y} v(x, y, i_0) \geq (\sum_{x, y \in Y} u(x, y, i_0) v(x, y, i_0))^2$$

which is a contradiction. □

Finally, we remark that these two improved upper bounds of Alb$_3(f)$ are not always tight. In [31], Yao and Zhang prove that two graph properties SCORPION and SINK both have $Q_2(f) = \Theta(\sqrt{n})$. But both $\sqrt{C_0(f)C_1(f)}$ and $\sqrt{N \cdot C_-(f)}$ are $\Theta(n)$. 

9
4 A further generalized Ambainis’s lower bound

While Alb2 and Alb3 use different ideas to generalize Alb1, it is natural to combine both and get a further generalization. The following theorem is a result in this direction. This theorem to Theorem 3 is as Theorem 2 to Theorem 1. The proof is similar to the ones in [3, 5], with inner products substituted for density operators to make it look easier\(^5\).

**Theorem 11** Let \( f : I^N \to \{0,1\} \) where \( I \) is a finite set, and \( X,Y \) be two sets of inputs s.t. \( f(x) \neq f(y) \) if \( x \in X \) and \( y \in Y \). Let \( R \subseteq X \times Y \). Let \( w,u,v \) be a weight scheme for \( X,Y,R \). Then

\[
Q_2(f) = \Omega(\sqrt{\frac{\min_{(x,y) \in R, i \in [N], x \neq y} w_x w_y}{u_{x,i} v_{y,i}}})
\]

**Proof** The query computation is a sequence of operations \( U_0 \to O_x \to U_1 \to \ldots \to U_T \) on some fixed initial state, say \( |0\rangle \). Note that here \( T \) is the number of queries. Denote \( |\psi^k_x\rangle = U_k \ldots U_1 O_x U_0 |0\rangle \). Note that \( |\psi^0_x\rangle = |0\rangle \) for all input \( x \). Because the computation is correct with high probability \((1-\epsilon)\), for any \((x,y) \in R\), the two final states have to have some distance to let the measurement distinguish them. In other words, we can assume that \(|\langle \psi^T_x |\psi^T_y \rangle| \leq c \) for some constant \( c < 1 \). Now suppose that

\[
|\psi^k_x\rangle = \sum_{i,a,z} \alpha_{i,a,z} |i,a,z\rangle, \quad |\psi^k_y\rangle = \sum_{i,a,z} \beta_{i,a,z} |i,a,z\rangle
\]

where \( i \) is for the index address, \( a \) is for the answer, and \( z \) is the workspace. Then the oracle works as follows.

\[
O_x |\psi^k_x\rangle = \sum_{i,a,z} \alpha_{i,a,z} |i,a \oplus x_i,z\rangle = \sum_{i,a,z} \alpha_{i,a \oplus x_i,z} |i,a,z\rangle
\]

\[
O_y |\psi^k_y\rangle = \sum_{i,a,z} \beta_{i,a,z} |i,a \oplus y_i,z\rangle = \sum_{i,a,z} \beta_{i,a \oplus y_i,z} |i,a,z\rangle
\]

So we have

\[
\langle \psi^k_x |\psi^k_y \rangle = \sum_{i,a,z} \alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z}
\]

\[
= \sum_{i,a,z} \alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z} + \sum_{i,a,z:x \neq y} \alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z} - \sum_{i,a,z:x \neq y} \alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z}
\]

Thus

\[
1 - c = 1 - |\langle \psi^T_x |\psi^T_y \rangle| = \sum_{k=1}^{T} (|\langle \psi^k_x |\psi^k_y \rangle| - |\langle \psi^k_x |\psi^k_y \rangle|)
\]

\[
\leq \sum_{k=1}^{T} |\langle \psi^k_x |\psi^k_y \rangle - \langle \psi^k_x |\psi^k_y \rangle|
\]

\[
= \sum_{k=1}^{T} \sum_{i,a,z:x \neq y} (|\alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z} - \alpha^*_{i,a \oplus x_i,z} \beta_{i,a \oplus y_i,z}|)
\]

Summing up the inequalities for all \((x,y) \in R\), with weight \( w(x,y) \) multiplied, yields

\[
(1 - c) \sum_{(x,y) \in R} w(x,y)
\]

\[
\leq \sum_{k=1}^{T} \sum_{(x,y) \in R} \sum_{i,a,z:x \neq y} w(x,y) (|\alpha_{i,a \oplus x_i,z} | \beta_{i,a \oplus y_i,z}| + |\alpha_{i,a \oplus x_i,z}| |\beta_{i,a \oplus y_i,z}|)
\]

\[
\leq \sum_{k=1}^{T} \sum_{(x,y) \in R} \sum_{i,a,z:x \neq y} \sqrt{u(x,y,i) v(x,y,i)} (|\alpha_{i,a \oplus x_i,z} | \beta_{i,a \oplus y_i,z}| + |\alpha_{i,a \oplus x_i,z}| |\beta_{i,a \oplus y_i,z}|)
\]

by (1). We then use inequality \(2AB \leq A^2 + B^2\) to get

\[
\sqrt{u(x,y,i) v(x,y,i)} |\alpha_{i,a \oplus x_i,z}| |\beta_{i,a \oplus y_i,z}| \leq \frac{1}{2} (u(x,y,i) v(x,y,i)) \sqrt{\frac{u_{x,i}}{u_{y,i}} \frac{w_x}{w_y} |\alpha_{i,a \oplus x_i,z}|^2 + v(x,y,i)) \sqrt{\frac{u_{x,i}}{u_{y,i}} \frac{w_x}{w_y} |\beta_{i,a \oplus y_i,z}|^2},
\]

\(^5\)This idea was also used in some other papers such as [20].
\[ \sqrt{u(x, y, i)v(x, y, i)}|\alpha_{x,a,z}|^2 + v(x, y, i) \sqrt{u(x, y, i)v(x, y, i)}|\beta_{x,a,z}|^2, \]

Denote \( A = \min_{x,y,i: (x,y) \in R, x \neq y} \frac{|w_xw_y|}{w_xv_y} \). Note that

\[ \sum_{y: (x,y) \in R, x \neq y} u(x, y, i) = u_{x,i}, \quad \sum_{x: (x,y) \in R, x \neq y} v(x, y, i) = v_{y,i} \]

by the definition of \( u_{x,i} \) and \( v_{y,i} \), we have

\[ \frac{1}{2} \sum_{k=1}^{T} \left( \sum_{x \in X} \sqrt{\frac{|w_x|}{w_xw_y}} |w_x||\alpha_{x,a,z}|^2 + \sum_{y \in Y} \sqrt{\frac{|w_y|}{w_xw_y}} |w_y||\beta_{x,a,z}|^2 \right) \]

\[ \leq \frac{1}{2} \sum_{k=1}^{T} \left( \sum_{x \in X} \sqrt{1/A}w_x \sum_{i,a,z} |\alpha_{x,a,z}|^2 + \sum_{i,a,z} |\beta_{x,a,z}|^2 \right) \]

\[ = \sqrt{1/A} \sum_{x \in X} w_x + \sum_{y \in Y} w_y \]

\[ = 2T \sqrt{1/A} \sum_{(x,y) \in R} w(x, y) \]

by noting that \( \sum_{x} w_x = \sum_{y} w_y = \sum_{(x,y) \in R} w(x, y) \). Therefore, \( T = \Omega(\sqrt{A}) \).

We denote by \( Alb_4(f) \) the best possible lower bound for function \( f \) achieved by this theorem. It is easy to see that \( Alb_4 \) generalizes \( Alb_3 \). \( Alb_4 \) may be easier to use than \( Alb_3 \). However, according to Szegedy’s recent result [30], \( Alb_3, Alb_4 \) and the quantum adversary method proposed by Barnum, Saks and Szegedy in [13] are all equivalent.

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