1. Introduction

Almost 80 years have passed since the landmark paper of Einstein-Podolsky-Rosen (EPR) [1] on a paradoxical manifestation of quantum correlations which Schrödinger later termed quantum steering [2, 3], yet the topic is more timely than ever. From one-sided device independent entanglement verification [4] and quantum key distribution [5, 6] to signifying secure quantum teleportation [7] and performing entanglement-assisted subchannel discrimination [8], Einstein’s scrutinized notion of steering finds increasingly many applications in non-classical tasks after its recent formulation as a distinct type of asymmetric nonlocality by Wiseman and co-workers [4, 9], thus making it a subject of intense research [10].

Steering, in a modern quantum information language [4, 9], can be understood as the task of two distant parties, say Alice and Bob, in which Alice tries to convince Bob that the quantum state $\hat{\rho}_{AB}$ they share is entangled, by remotely creating quantum ensembles on Bob’s site that could not have been created without shared entanglement. Given that Bob does not trust Alice and her announced measurements, we say that Alice can steer Bob’s state (and thus convince Bob), or equivalently that the state $\hat{\rho}_{AB}$ is “$A \rightarrow B$” steerable, if and only if (iff) the probabilities of all possible joint measurements cannot be written in the factorizable form [4]:

$$P(A, B|a, b, \hat{\rho}_{AB}) = \sum_{\lambda} P_\lambda P(A|a, \lambda) P(B|b, \hat{\rho}_\lambda),$$  \hspace{1cm} (1)

where the lower-case letters $a \in M_A$ and $b \in M_B$ denote local observables for Alice and Bob, while $A$ and $B$ their corresponding outcomes. Violation of (1) implies the failure of a local hidden state model to explain the measurement statistics. As one can see from Eq. (1), steering is an asymmetric form of nonlocality that sits in-between entanglement [11] and Bell nonlocality [12–14]. Not all entangled states are steerable, and not all steerable states are Bell nonlocal.

In order for steering to be useful one should first be able to detect it in experiments [15–22]. The first attempt to create an experimental criterion that captures the essence of the EPR paradox [10] in a continuous variable setting was made in the 80’s by M. Reid [23], whose criterion is commonly known as Reid’s criterion and which was shown later to be only a special case of an EPR-steering test in the sense of (1) [24]. Today our knowledge about the detection and distribution of steering has significantly advanced [25–27], with a plethora of effective criteria derived [24, 28–30] and phenomena like steering monogamy identified in multi-party scenarios [27]. Besides a yes/no answer to the question of steerability given by various steering criteria, however, one is interested in how much a state is steerable for practical purposes. Only quite recently, the quantification of steering was put forward by researchers [8, 31, 32] to assess how much a quantum state’s statistics deviate from (1), and thus how useful it can be for tasks that use steering as their resource [33]. Measures of steering have been proposed both for discrete [8, 31] and continuous variable [32] systems but, while providing insightful characterizations of steerability, they are either inaccessible to analytical computation or defined solely for special classes of states and measurements.
In this paper we present a general and accessible approach to the quantification of steerability for bipartite two-mode continuous variable states. We examine recent experimental criteria for steering \cite{24}, the so-called EPR-Steering criteria whose applicability extends to all (Gaussian and non-Gaussian) states, and analyze their maximal violation by optimal local quadrature observables for Alice and Bob, in order to capture the largest possible departure from (1) for a given state. Hence we define (in Section 2) a suitable measure of steering for an arbitrary two-mode state, and we prove that it admits an analytically computable lower bound that captures the degree of steerability of the given state by Gaussian measurements. The lower bound coincides with the Gaussian steering measure introduced in a previous work \cite{32}, whose usefulness is here generalized from the Gaussian domain to arbitrary states. We prove Gaussian states to be in fact extremal \cite{34}, as they are minimally steerable among all states with the same covariance matrix, according to the measure proposed in this paper. As a corollary of our analysis, we show (in Section 3) a necessary and sufficient condition for steerability of Gaussian states under Gaussian measurements obtained by Wiseman et al., based on covariance matrices \cite{11} \cite{9}, remains valid as a sufficient steering criterion for arbitrary non-Gaussian states, and amounts to Reid’s criterion \cite{10} \cite{23} when optimal Gaussian local observables are chosen for the latter. We conclude (in Section 4) with a summary of our results and an outlook of currently open questions motivated by the present analysis.

2. A steering measure for two-mode states

In general \cite{35}, a measure of steering should quantify how much the correlations of a quantum state depart from the expression in Eq. (1). Since a manifestation of these correlations can be observed by suitable steering criteria, one can approximate the degree of departure by evaluating the maximum violation of a given steering criterion as revealed by optimal measurements. One expects that the higher the departure (i.e., the amount of correlations), the more useful the state will be in tasks that use quantum steering as a resource.

In this paper we consider an arbitrary state \( \hat{\rho}_{AB} \) of a two-mode continuous variable system, which is assumed with vanishing first moments without any loss of generality. The relevant steering criteria to our work will be the so-called multiplicative variance EPR-steering criteria \cite{24}, of which Reid’s criterion \cite{23} is a special case. Following \cite{10} \cite{23} \cite{24}, let us consider a situation where Bob measures two canonically conjugate observables on his subsystem, \( \hat{x}_B, \hat{p}_B \), with corresponding outcomes \( X_B, P_B \), and Alice tries to guess Bob’s outcomes based on the outcomes of measurements on her own subsystem. If, say, the outcome of Alice’s measurement is \( X_A \), corresponding to a local observable \( \hat{x}_A \), we can denote by \( \hat{x}_A \) Alice’s steering of Bob’s measurement outcome \( X_B \). The average inference variance of \( X_B \) given Alice’s estimator \( \hat{x}_A \) is defined by

\[
\Delta^2_{\text{inf}} X_B = \left( \langle X_B - \hat{x}_A \rangle^2 \right),
\]

where the average is taken with respect to the joint probability distribution \( P(X_A, X_B) \) and over all outcomes \( X_A, X_B \). One can show \cite{10} that the optimal estimator minimizing the inference variance \( \Delta^2_{\text{inf}} X_B \) is the mean \( \hat{x}_A = \langle X_B \rangle_{X_A} \) evaluated on the conditional distribution \( P(X_B|X_A) \). Substituting in (2) we obtain the minimal inference variance of \( X_B \) by measurements on \( A \),

\[
\Delta^2_{\text{min}} X_B = \sum_{X_A} P(X_A) \Delta^2(X_B|X_A),
\]

where \( \Delta^2(X_B|X_A) \) is the conditional variance of \( X_B \), calculated from \( P(X_B|X_A) \). Clearly, from the properties stated above, it holds that \( \Delta^2_{\text{inf}} X_B \geq \Delta^2_{\text{min}} X_B \). Similarly we can define an inference variance \( \Delta^2_{\text{inf}} P_B \) for \( \hat{p}_B \) and its corresponding minimum \( \Delta^2_{\text{min}} P_B \) given respectively by analogous formulas to (2) and (3), but conditioned on \( P_A \) instead of \( X_A \). In \cite{10} \cite{24} it was shown that a bipartite state \( \hat{\rho}_{AB} \) shared by Alice and Bob is steerable by Alice, i.e., “\( A \to B \)” steerable, if the condition

\[
\Delta^2_{\text{min}} X_B \Delta^2_{\text{min}} P_B \geq 1,
\]

is violated.

It is important to note that although we have implicitly considered Alice to perform homodyne measurements of quadrature operators on her subsystem, the EPR-steering criterion (1) is valid without any assumption on the Hilbert space of Alice’s subsystem, as no commutation relation between \( \hat{x}_A \) and \( \hat{p}_A \) is assumed (Bob just needs to identify two distinctly labelled measurements) \cite{23}. In an experimental setting, Alice’s measurements should of course be chosen suitably to maximize the violation of (1). The criterion is therefore applicable to arbitrary states and measurements, while it captures steerability manifested in correlations up to second order for Bob and arbitrary order for Alice, as it can be seen from (2), since the estimator \( \hat{x}_A \) can be any function of \( X_A \) in general.

One immediately sees that the product of variances in (1) is not invariant under local unitary operations by Alice and Bob, thus a state may be detected as more or less steerable if some local change of basis is implemented. In order to capture steerability in an invariant way, one can consider the maximum violation of (1) that a quantum state \( \hat{\rho}_{AB} \) can exhibit, by minimizing the product \( \Delta^2_{\text{inf}} X_B \Delta^2_{\text{inf}} P_B \) over all local unitaries \( U_{\text{local}} = U_A \otimes U_B \) for \( A \) and \( B \) applied to the state.

We then propose to quantify the “\( A \to B \)” steerability of an arbitrary two-mode CV state \( \hat{\rho}_{AB} \) via the measure

\[
S^{A \to B} (\hat{\rho}_{AB}) = \max \left\{ 0, -\frac{1}{2} \ln F \right\},
\]

where
The measure naturally quantifies the amount of violation of an optimized multiplicative variance EPR-steering criterion of the form \( \bar{\sigma}_{AB} \) for an arbitrary state \( \hat{\rho}_{AB} \). As one would expect from any proper quantifier of quantum correlations, the measure enjoys local unitary invariance by definition, and it vanishes for all states which are not “\( A \rightarrow B \)” steerable.

Calculating \( S^{A \rightarrow B} \) in an analytical manner for an arbitrary state is undoubtedly a difficult task. In general, given a quantum state, the minimization in \( F \) involves both Gaussian and non-Gaussian local unitaries for Alice and Bob, which correspond to violations of \( \bar{\sigma}_{AB} \) by Gaussian and non-Gaussian measurements, respectively. It is possible, though, to obtain a computable lower bound to \( S^{A \rightarrow B} \), if one constrains the optimization to Gaussian unitaries only. The lower bound, presented in the next subsection, will then provide a quantitative indication of the “\( A \rightarrow B \)” steerability of \( \hat{\rho}_{AB} \) that can be demonstrated by Gaussian measurements on Alice’s subsystem.

2.A. Lower bound

A short introduction of the reader to Gaussian states is first intended [36]. An arbitrary bipartite Gaussian state \( \hat{\rho}_{AB} \) is solely determined by its second moments, i.e., it is fully specified up to local displacements by the covariance matrix (CM) \( \sigma_{AB} \), which can be written in the block form

\[
\sigma_{AB} = \begin{pmatrix} A & C \\ C^{T} & B \end{pmatrix}.
\]  

(7)

Here, \( A \) and \( B \) are the marginal CMs corresponding to the reduced states of Alice and Bob respectively, while \( C \) encodes intermodal correlations. For two-mode states, \( A, B, \) and \( C \) are \( 2 \times 2 \) matrices. The matrix elements of \( \sigma_{AB} \), defined by \( \sigma_{AB} \)\(_{ij} = \Tr[(\hat{R}_i^{\dagger}\hat{R}_j)\hat{\rho}^B_{AB}] \), are expressed via the vector \( \hat{R} = (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B)^T \) that conveniently groups the phase-space operators \( \hat{x}_{A(B)}, \hat{p}_{A(B)} \) for each mode. The canonical commutation relations these operators satisfy can be compactly expressed as \( [\hat{R}_j, \hat{R}_k] = i(\Omega_{AB})_{jk} \), where \( \Omega_{AB} = \Omega_A \oplus \Omega_B \) is the symplectic matrix, with \( \Omega_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) [36]. The CM of any (Gaussian or non-Gaussian) physical state needs to satisfy the bona fide condition

\[
\sigma_{AB} + i(\Omega_A \oplus \Omega_B) \geq 0.
\]  

(8)

Gaussian operations are defined as those which preserve the Gaussianity of the states they act upon.

To obtain a lower bound for the steering measure \( S^{A \rightarrow B} (\hat{\rho}_{AB}) \) in terms of second moments, we will show that, for arbitrary states \( \hat{\rho}_{AB} \) with corresponding CM \( \sigma_{AB} \), the product of inference variances \( \Delta_\inf X_B \Delta_\inf P_B \), defined as in [2], acquires its minimum value when \( \sigma_{AB} \) is expressed in the so-called standard form

\[
\bar{\sigma}_{AB} = \begin{pmatrix} \bar{A} \\ \bar{C}^{T} \end{pmatrix} \begin{pmatrix} \bar{C} \\ \bar{B} \end{pmatrix},
\]  

(9)

in which the submatrices \( \bar{A} = \text{diag}(a, a) \), \( \bar{B} = \text{diag}(b, b) \), and \( \bar{C} = \text{diag}(c_1, c_2) \) take a diagonal form. The standard form can always be obtained for any state by suitable local unitary operations [37] [38] and is unique up to a sign flip in \( c_1 \) and \( c_2 \), as its elements can be recast as functions of four local invariants of the CM [39].

Let us begin by considering a steerable \( \hat{\rho}_{AB} \) that violates [4], so that \( S^{A \rightarrow B} (\hat{\rho}_{AB}) > 0 \). We use the fact that \( \Delta_\inf X_B \geq \Delta_\inf X_B \), when a linear estimator \( \hat{X}_{est}(X_A) = g_a X_A \) is used in [2]; after minimizing the inference variance over the real number \( g_a \), we find

\[
\Delta_\inf ^2 X_B = \langle X_B^2 \rangle - \langle X_B \rangle^2/\langle X_A^2 \rangle \text{ [10]}
\]

Similar considerations hold for the inference variance of momentum, where an estimator of the form \( \hat{P}_{est}(P_A) = g_p P_A \) will give

\[
\Delta_\inf ^2 P_B = (P_B^2) - \langle P_B P_A \rangle^2/\langle P_A^2 \rangle \text{ after optimizing over } g_p.
\]

Since a linear estimator is optimal for inferring the variance in the case of Gaussian states [10] [23], but not anymore in the general case, the inequality \( \Delta_\inf ^2 X_B \Delta_\inf ^2 P_B \geq \Delta_\inf ^2 X_B \Delta_\inf ^2 P_B \) will be true for all states (with equality on Gaussian states). Hence, \( F \) in [5] can be upper bounded as follows,

\[
F = \min_{\{U_g\}|\{U_{ng}\}} \Delta_\inf ^2 X_B \Delta_\inf ^2 P_B \leq \Delta_\inf ^2 X_B \Delta_\inf ^2 P_B \leq \Delta_\inf ^2 X_B \Delta_\inf ^2 P_B
\]  

(10)

where we have decomposed the set of local unitaries \( \{U_{local}\} \) into Gaussian \( \{U_g\} \) and non-Gaussian \( \{U_{ng}\} \) ones. The product of inference variances in [10] is intended as evaluated from the optimal linear estimator as detailed above [10], namely

\[
\Delta_\inf ^2 X_B \Delta_\inf ^2 P_B = (\langle X_B^2 \rangle - \langle X_B X_A \rangle^2/\langle X_A^2 \rangle)(P_B^2) - \langle P_B P_A \rangle^2/\langle P_A^2 \rangle.
\]  

(11)

Since an upper bound on \( F \) will give us the desired lower bound on \( S^{A \rightarrow B} \), what remains is to compute this upper bound, i.e., the rightmost quantity in [10], which only depends on the CM elements of the state. Note that the product of inference variances [11], using linear estimators, defines what is well-known in the literature as Reid’s criterion [23],

\[
\Delta_\inf ^2 X_B \Delta_\inf ^2 P_B \geq 1,
\]  

(12)

whose violation is sufficient to detect “\( A \rightarrow B \)” steerability of a general two-mode state based on second order moments.
Local Gaussian units (that do not give rise to displacements) acting on states \( \hat{\rho}_{AB} \) translate on the level of CMs as local symplectic transformations \( S_{\text{local}} = S_A \oplus S_B \). Acting by congruence: \( \sigma_{AB} \rightarrow S_{\text{local}} \sigma_{AB} S_{\text{local}}^T \). In order to compute \( \min_{S_{\text{local}}} \Delta^2_{\text{inf}} X_B \Delta^2_{\text{inf}} P_B \), we can, without loss of generality, consider a CM \( \tilde{\sigma}_{AB} \) in standard form, apply an arbitrary local symplectic operation \( S_{\text{local}} \) to it, then evaluate \( \Delta^2_{\text{inf}} X_B \Delta^2_{\text{inf}} P_B \) on the transformed CM \( S_{\text{local}} \tilde{\sigma}_{AB} S_{\text{local}}^T \) and finally minimize this quantity over all possible matrices \( S_{A(B)} \). To perform the minimization we parametrize the matrix elements of \( S_{A(B)} \) in the following convenient way,

\[
S_{A(B)} = \begin{pmatrix}
1 & v_{A(B)} \\
1-u_{A(B)} v_{A(B)} & u_{A(B)} w_{A(B)} & (1-u_{A(B)} v_{A(B)}) w_{A(B)} \\
0 & u_{A(B)} w_{A(B)} & u_{A(B)} w_{A(B)}
\end{pmatrix}
\]

(13)

where the symplectic condition \( S_{A(B)} \Omega_{A(B)} S_{A(B)}^T = \Omega_{A(B)} \) has been taken into account and the real variables \( u_{A(B)}, v_{A(B)}, w_{A(B)} \) are now independent of each other. Performing the (unconstrained) minimization over the variables \( u_{A(B)}, v_{A(B)} \) we were able to obtain analytically the global minimum of the product (11) with respect to Gaussian observables,

\[
\min_{\{u,v\}} \left[ \Delta^2_{\text{inf}} X_B \Delta^2_{\text{inf}} P_B \right] = \det M_{\sigma}^B,
\]

(14)

which also constitutes the upper bound for \( \mathcal{F} \) in (10). Here the local symplectic invariant \( \det M_{\sigma}^B = (b - \frac{c^2}{2}) (b - \frac{\tilde{c}^2}{2}) \) is the determinant of the Schur complement of \( A \) in \( \sigma_{AB} \), defined for any two-mode CM (7) as

\[
M_{\sigma}^B = B - C^T A^{-1} C.
\]

(15)

The minimum (14) can be obtained from every state using the following parameters that determine the local symplectic operations (13),

\[
(u_{A}, v_{A}, u_{B}, v_{B}) = \left( \frac{c_{1B} v_{B}}{c_{2B}}, \frac{-\tilde{a}_{B} + c_{1B}^2}{\tilde{a}_{B} - c_{2B}^2} \frac{c_{1B} v_{B}}{c_{1B}}, \frac{-\tilde{a}_{B} + c_{1B}^2}{\tilde{a}_{B} - c_{2B}^2} v_{B}, v_{B} \right),
\]

\( \forall v_{B}, w_{A(B)} \). It is evident from (14) that the minimum product of inference variances (11) is achieved, in particular, when evaluated for a standard form CM \( \tilde{\sigma}_{AB} \).

Substituting \( \mathcal{F} \leq \det M_{\sigma}^B \) in (3), a lower bound for the proposed steering measure of an arbitrary two-mode state \( \hat{\rho}_{AB} \) is obtained,

\[
S_{A \rightarrow B} (\hat{\rho}_{AB}) \geq G_{A \rightarrow B} (\sigma_{AB}) \),
\]

(16)

where we recognize the Gaussian steering measure introduced in [32],

\[
G_{A \rightarrow B} (\sigma_{AB}) = \max \left\{ 0, -\frac{1}{2} \ln \det M_{\sigma}^B \right\}.
\]

(17)

The lower bound \( G_{A \rightarrow B} \) solely depends on local symplectic invariant quantities that uniquely specify the CM of the state. As is known [39], these invariant quantities can be expressed back with respect to the original elements of the CM which one can measure in laboratory, e.g., via homodyne tomography [41]. Henceforth, the lower bound that we obtained is both analytically computable and also, experimentally accessible in a routinely fashion for any (Gaussian or non-Gaussian) state, since only moments up to second order are involved.

In the following we discuss some useful properties that the steering measure \( S_{A \rightarrow B} \) and its lower bound \( G_{A \rightarrow B} \) satisfy, and show how these results can be used to link and generalize existing steering criteria.

### 2.B. Properties

In a recent work [32] the present authors introduced a measure of EPR-steering for multi-mode bipartite Gaussian states that dealt with the problem of “how much a Gaussian state can be steered by Gaussian measurements”. This measure \( G_{A \rightarrow B} \) was defined as the amount of violation of the following criterion by Wiseman et al. [4, 9],

\[
\sigma_{AB} + i (0_A \oplus \Omega_B) \geq 0.
\]

(18)

Violation of (18) gives a necessary and sufficient condition for “A → B” steerability of Gaussian states by Gaussian measurements. We recall from the original papers [4, 9], where the details can be found, that for two modes the condition (18) is violated iff \( \det M_{\sigma}^B < 1 \), hence equivalently iff \( G_{A \rightarrow B} (\sigma_{AB}) > 0 \), where the Gaussian steering measure is defined in (17). In a two-mode continuous variable system, a non-zero value of Gaussian steering \( G_{A \rightarrow B} > 0 \) detected on a CM \( \sigma_{AB} \), which implies a non-zero value of the more general measure \( S_{A \rightarrow B} > 0 \) due to (16), constitutes therefore not only a necessary and sufficient condition for the steerability by Gaussian measurements of the Gaussian state \( \hat{\rho}_{AB} \) defined by \( \sigma_{AB} \), but also a sufficient condition for the steerability of all (non-Gaussian) states \( \hat{\rho}_{AB} \) with the same CM \( \sigma_{AB} \).

While \( S_{A \rightarrow B} \) is hard to study in complete generality, its lower bound however has been shown to satisfy a plethora of valuable properties. In [32] we showed that Gaussian steering acquires for two modes a form of coherent information [42],

\[
G_{A \rightarrow B} (\sigma_{AB}) = \max \{0, S (A) - S (\sigma_{AB})\},
\]

with the Renyi-2 entropies \( S (\sigma) = \frac{1}{2 \ln (\det \sigma)} \) replacing the standard von Neumann ones. Thanks to this connection \( G_{A \rightarrow B} (\sigma_{AB}) \) was shown to satisfy various properties that we repeat here without proof: (a) \( G_{A \rightarrow B} (\sigma_{AB}) \) is convex and additive; (b) \( G_{A \rightarrow B} (\sigma_{AB}) \) is monotonically decreasing under Gaussian quantum operations on the (untrusted) party Alice; (c) \( G_{A \rightarrow B} (\sigma_{AB}) = E (\sigma_{AB}^A) \) for \( \sigma_{AB}^A \) pure, and, (d) \( G_{A \rightarrow B} (\sigma_{AB}) \leq E (\sigma_{AB}) \) for \( \sigma_{AB} \) mixed, where \( E \) denotes the Gaussian Renyi-2 entropy measure of entanglement [43]. In the light of the recently developed resource theory of steering [33] properties (a) and
(b) should be satisfied by any proper measure of steering, while properties (c) and (d) should be satisfied by any quantifier that respects the hierarchy of quantum correlations. This affirms that our proposed quantifier is on the right track. The asymmetry of steering was also studied since $\mathcal{G}^{A\rightarrow B} \neq \mathcal{G}^{B\rightarrow A}$ in general. It was found [32] that for two-mode CMs the maximum absolute asymmetry between the two steering directions is $\ln 2$, which in terms of measure proposed in this paper yields the bound $\max(\rho_{AB}) \left| S^{A\rightarrow B} - S^{B\rightarrow A} \right| \geq \ln 2$. Thus $\ln 2$ is established as the maximum steering asymmetry of arbitrary states revealed by Gaussian measurements, while non-Gaussian ones can in general lead to even more asymmetry. The present paper, thus, validates all the already established properties of $\mathcal{G}^{A\rightarrow B}$ as an indicator of steerablearity by Gaussian measurements, and extends them to arbitrary states.

Interestingly, [16] suggests that by accessing only the second moments of an arbitrary state, one will not overestimate its steerablearity according to our measure. We can make this observation rigorous by showing that the steering quantifier $S^{A\rightarrow B}$ satisfies an important extremality property as formalized in [34]. Namely, the Gaussian state $\hat{\rho}_{AB}$ defined by its CM $\sigma_{AB}$ minimizes $S^{A\rightarrow B}$ among all states $\hat{\rho}_{AB}$ with the same CM $\sigma_{AB}$. This follows by recalling that the Reid product (11), which appears in (10), is independent from the (Gaussian versus non-Gaussian) nature of the state, and that linear inference estimators are globally optimal for Gaussian states as mentioned above [10]. This entails that the middle term in (10) can be recast as

$$\min_{\{U_G\} \cup \{U_{nG}\}} \left( \Delta_{\inf}^{2} X_B \Delta_{\inf}^{2} P_B \right) \hat{\rho}_{AB}$$

where, for the sake of clarity, we have explicitly indicated the states on which the variances are calculated: $\hat{\rho}_{AB}$ denotes an arbitrary two-mode state, and $\hat{\rho}_{G_{AB}}$ corresponds to the reference Gaussian state with the same CM.

Therefore, combining Eqs. (5), (10), (16), and (19), we can write the following chain of inequalities for the \(A \rightarrow B\) steerablearity of an arbitrary two-mode state $\hat{\rho}_{AB}$.

$$S^{A\rightarrow B} (\hat{\rho}_{AB}) \geq S^{A\rightarrow B} (\hat{\rho}_{G_{AB}}) \geq g^{A\rightarrow B} (\sigma_{AB}) \, . \quad (20)$$

The leftmost inequality in (20) embodies the desired extremality property [34] for our steering measure. This is very relevant in a typical experimental situation, where the exact nature of the state $\hat{\rho}_{AB}$ is mostly unknown to the experimentalist. Then, thanks to (20) we rest assured that, by assuming a Gaussian nature of the state under scrutiny, the experimentalist will never overestimate the EPR-steering correlations between Alice and Bob as quantified by the measure defined in [5]. In general, the exact value of $S^{A\rightarrow B}$ can be estimated arbitrarily well by choosing estimators in [2] of suitably higher order and form.

Finally, coming to operational interpretations for our proposed steering quantifier $S^{A\rightarrow B}$, we show that it is connected to the figure of merit of semi-device independent quantum key distribution [6], that is, the secret key rate. In the conventional entanglement-based quantum cryptography protocol [44], Alice and Bob share an arbitrary two-mode state $\hat{\rho}_{AB}$, and want to establish a secret key given that Alice does not trust her devices. By performing local measurements (typically homodyne detections) on their modes, and a direct reconciliation scheme (where Bob sends corrections to Alice) they can achieve the secret key rate [6]

$$K \geq \max \left\{ 0, \ln \left( \frac{2}{\epsilon \sqrt{\Delta_{\inf}^{2} X_B \Delta_{\inf}^{2} P_B}} \right) \right\} \, . \quad (21)$$

Notice that the secret key rate depends on the expression in (11), which is not unitarily invariant. Therefore, it can be optimized over local unitary operations. In the case where $\Delta_{\inf}^{2} X_B \Delta_{\inf}^{2} P_B$ takes its minimum value for the given shared $\hat{\rho}_{AB}$, the lower bound on the corresponding optimal key rate $K_{\text{opt}}$ can be readily expressed in terms of the \(A \rightarrow B\) steering measure, yielding

$$K_{\text{opt}} \geq \max \left\{ 0, S^{A\rightarrow B} (\hat{\rho}_{AB}) + \ln 2 - 1 \right\} \, . \quad (22)$$

Thus, $S^{A\rightarrow B}$ quantifies the best guaranteed key rate for a given state. If a reverse reconciliation protocol is used (in which Alice sends corrections to Bob) the quantifier $S^{B\rightarrow A}$ of the inverse steering direction enters (22) instead. Thus, one sees that the asymmetric nature of steering correlations can play a decisive role in communication protocols that rely on them as resources. In the cryptographic scenario discussed, if the shared state $\hat{\rho}_{AB}$ is only one-way steerable, say $A \rightarrow B$, then a reverse reconciliation protocol that relies on $S^{B\rightarrow A}$ is not possible. A looser lower bound to the key rate (22) can also be expressed in terms of $\mathcal{G}^{A\rightarrow B}$ by using (16), in case one wants to study the advantage that Gaussian steering alone gives for the key distribution, or one just wants to get an estimate.

3. Reid, Wiseman, and a stronger steering test

Finally, we discuss the implications of our work on existing EPR-steering criteria [4] [23]. The second order EPR-steering criteria by Reid [12] and Wiseman et al. [18], are perhaps the most well-known ones for continuous variable systems. Although a comparison between them has been issued before in a special case (two-mode Gaussian states in standard form) [9], they appear to exhibit quite distinct features in general [24]. On one hand, Wiseman et al.'s criterion [18], defined only in the Gaussian domain, is invariant under local symplecticities and provides a necessary and sufficient condition for steerablearity of Gaussian states under Gaussian
Fig. 1. (Color online) We illustrate the performance of Reid’s criterion and Wiseman et al.’s EPR-steering criteria for the steering detection of a pure two-mode squeezed state with squeezing $r$, with CM transformed from the standard form by the application of a local symplectic transformation parameterized as in (13), with $w_{A(B)} = v_{A(B)}/(1 + v_{A(B)}^2)$, $w_{A(B)} = 1 + v_{A(B)}^2$ (in the plot, we choose $v_A = 0.16$ and $v_B = 0.19$). The criteria are represented by their figures of merit, namely the product of conditional variances (dashed blue line) for Reid’s criterion (12) and the determinant $\det M_B$ (solid orange line) for Wiseman et al.’s criterion (18). The two-mode squeezed state is steerable for all $r > 0$, but the aforementioned criteria detect this steerability only when their respective parameters give a value smaller than unity (straight black line). As one can see, we have $\det M_B < 1$ for all $r > 0$ and independently of any local rotations, while Reid’s criterion detects steerability only for a small range of squeezing degrees and is highly affected by local rotations. If the state is sufficiently rotated out of the standard form, the unoptimized Reid’s criterion will not be able to detect any steering at all.

4. Conclusion

In conclusion, we introduced a quantifier of EPR-steering for arbitrary bipartite two-mode continuous-variable states, that can be estimated both experimentally and theoretically in an analytical manner. Gaussian states were found to be extremal with respect to our measure, minimizing it among all continuous variable states with fixed second moments [34]. By further restricting to Gaussian measurements, we obtained a computable lower bound for any (Gaussian or non-Gaussian) two-mode state, that was shown to satisfy a plethora of good properties [32]. The measure proposed in this paper is seen to naturally quantify the guaranteed key rate of semi-device independent quantum key distribution [6]. Finally, this work generalizes and sheds new light on existing steering criteria based on quadrature measurements [1, 23].

Nevertheless many questions still remain, complementing the ones posed previously in [32]. To begin with, it would be worthwhile to extend the results presented here to multi-mode states and see whether a connection similar to (14) still holds. We also leave for further research the possibility that our quantifier (or its lower bound) may enter in other figures of merit for protocols that consume steering as a resource, like the tasks of secure quantum teleportation and entanglement-assisted Gaussian subchannel discrimination with one-way measurements [8]. Moreover, the proved connection of the measure with entropic quantities in the purely Gaussian scenario could be an instance of a more general property that we believe is worth investigating, possibly making the link with the degree of violation of more powerful (nonlinear) entropic steering tests [28, 29].

Finally, it is presently unknown whether the rightmost inequality in (20) is tight; namely, whether or not...
non-Gaussian unitaries in the minimization of [5] can give rise to higher steerability of Gaussian states, compared to optimal Gaussian unitaries. This is related to the open question, first posed in [4], of whether or not there exist steerable Gaussian states which nonetheless cannot be steered by Gaussian measurements; so far, such states have not been found even by resorting to nonlinear steering criteria [28]. On one hand, one would expect that Gaussian measurements are optimal for steering Gaussian states, since Gaussian operations and decompositions are indeed optimal for (provably a large class of) two-mode Gaussian states when entanglement and discord-type correlations are considered [42,50]. On the other hand, non-Gaussian measurements are always required to violate any Bell inequality on Gaussian states [51,52] by virtue of their positive Wigner function, hence Gaussian measurements are in contrast completely useless for that task. Since steering is the ‘missing link’ which sits just below nonlocality and just above entanglement in the hierarchy of quantum correlations [4,9], pinning down precisely the role of Gaussian measurements for steerability of Gaussian states would be particularly desirable. Here, we dare to conjecture that $S_{\hat{A}\rightarrow\hat{B}}(\rho_{AB}) = G_{\hat{A}\rightarrow\hat{B}}(\sigma_{AB})$, that is, that the general measure of EPR-steering introduced in this paper would reduce exactly to the measure of Gaussian steering proposed in [32], for all two-mode Gaussian states; this would signify the optimality of Gaussian measurements for steerability of Gaussian states. However, a proof or disproof of this tempting hypothesis is beyond our current capabilities, and is left here as a future challenge to the community.

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