Existence and Ulam-Hyers-Rassias Stability of Stochastic Differential Equations with Random Impulses

Wenxuan Lang\textsuperscript{a}, Sufang Deng\textsuperscript{b}, Xiao-Bao Shu\textsuperscript{a}, Fei Xu\textsuperscript{c}

\textsuperscript{a}College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P.R. China
\textsuperscript{b}School of Mathematics and Finance, Xiangnan University, Chenzhou, Hunan 423000, P.R. China
\textsuperscript{c}Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada

Abstract. In this paper, we investigate the existence and Ulam-Hyers-Rassias stability of solutions for stochastic differential equations with random impulses. Based on the Krasnoselskii’s fixed point theorem, we perform investigations on the existence of solutions to the system of stochastic differential equations with random impulses. We apply the integral inequality of Gronwall type to the equations and study their Ulam-Hyers-Rassias stability.

1. Introduction

In 1940, Ulam [32] raised the question concerning the Hyers-Ulam stability of group homomorphisms. This question was first solved by Hyers [11] on Banach spaces. Rassias [22] introduced the concept of unbounded Cauchy difference and extended the Hyers’ Theorem to the case of approximate linear mappings. Since then, the theory of Hyers-Ulam stability attracted the attention of many researchers. There are many interesting results on Hyers-Ulam stability have been gained in [1–6, 8, 12–16, 18, 21, 23, 25, 27, 31] and the references therein. Recently, the Hyers-Ulam stability of stochastic differential equations has been investigated by many authors. Zhao [35] studied the Hyers-Ulam stability for a class of first order stochastic differential equations using the Itô formula. Sathiyaaraj et al. [26] studied the Ulam’s stability results for nonlinear Hilfer fractional stochastic differential systems in finite dimensional stochastic setting. Guo et al. [9] studied the Hyers-Ulam stability of the almost periodic solution to the fractional differential equation with impulse and fractional Brownian motion under nonlocal condition based on the basic theory of Hyers-Ulam stability.

Stochastic differential equations have wide applications in describing random or uncertain factors, and as such have advantages over ordinary differential equations in modeling real-world systems with uncertainty. A variety of results on the theory of Stochastic differential equations have been obtained, see [10, 19, 34] and the references therein.

Impulsive differential equations also attracted the attention of many researchers. The theory and application of impulsive differential equations have become an area of active investigation in the literature.

2010 Mathematics Subject Classification. 34K50, 34K45, 34D20, 47H10

Keywords. Stochastic differential equations, Random impulses, Ulam-Hyers-Rassias Stability, Krasnoselskii’s fixed point, Gronwall inequality

Received: 19 January 2018; Revised: 30 December 2020; Accepted: 03 April 2021

Communicated by Dragan S. Djordjević

Email addresses: dengsf093@163.com (Sufang Deng), sxb2210163.com (Xiao-Bao Shu), fxu.feixu@gmail.com (Fei Xu)
Great progress has been obtained [29, 30]. Impulses may take place at random moments or fixed moments. Differential equations with these two types of impulses behave differently. The solutions to differential equations with random impulses are stochastic processes, and the solutions to differential equations with impulses at fixed moments are piecewise continuous functions. Wu and Zhou [33] studied the existence and uniqueness of solutions to stochastic differential equations with random impulses and Markovian switching under non-lipschitz conditions using the Bihari inequality. Li et al. [20] studied the existence of upper and lower solutions to a second-order random impulsive differential equation. Shu et al. [28] studied the existence and exponential stability in mean square of mild solutions to second-order neutral stochastic functional differential equations with random impulses in Hilbert space using Mönch fixed point theorem and integral inequality.

Motivated by the above works, we study the existence and Ulam-Hyers-Rassias stability of solutions for stochastic differential equations with random impulses in this article. As far as we know, such type of equations has not been investigated in the literature. We obtain sufficient conditions for the existence of solutions to stochastic differential equations with random impulses using the Krasnoselskii’s fixed point. By using the integral inequality of Gronwall type, we study the Ulam-Hyers-Rassias stability for such equations in the mean square.

The rest of this paper is organized as follows. In Section 2, we present some notations and necessary preliminaries. In Section 3, we give the main results of this article. We prove the existence of solutions to the stochastic differential equations with random impulses and the Ulam-Hyers-Rassias stability of this type of equations in mean square.

2. Preliminaries

Suppose that \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) is a complete probability space with a filtration that satisfies the usual conditions. That is to say, it is increasing and right-continuous. We notice that \(\mathcal{F}_0\) contains all \(P\)-zero measure set. We use \(\| \cdot \|\) to denote the Euclidean space in \(\mathbb{R}^n\) and \(\| \cdot \|_M\) to denote some norm in \(\mathbb{R}^{m \times m}\) that is compatible with \(\| \cdot \|\), i.e., for any \(A \in \mathbb{R}^{m \times m}\) and \(Y \in \mathbb{R}^n\), \(\|AY\| \leq \|A\|_M\|Y\|\). We note that \(\| \cdot \|_{\mathcal{L}^2(\mathbb{P})} = \sqrt{\mathbb{E}(\| \cdot \|^2)}\), where \(\mathbb{E}(Y)\) is the mathematical expectation of random variable \(Y\). Let \(T > t_0\) be any fixed time to be determined later and let \(L^2_P([t_0, T], \mathbb{R}^n)\) be the space of stochastic processes \(f(t, \omega)\) such that each \(f(t, \omega)\) is adapted to the filtration \(\mathcal{F}_t\), and \(\mathbb{E}\left(\int_{t_0}^{T} |f(t)|^2 dt\right) < \infty\).

Now, we consider the following three random sequences on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\):

(i) Let \(\omega_0\) be a random sequence defined on \(D_i \equiv (0, d_i], 0 < d_i < +\infty\), where \(\omega_i\) and \(\omega_j\) are independent of each other for \(i \neq j, i, j = 1, 2, \ldots\).

(ii) Let \(\tau_i, i \geq 1\) be a \(\mathcal{F}_t\)-adapted random sequence that satisfies

\[
\tau_i = t_0 + \sum_{j=1}^{i} \omega_j,
\]

where \(t_0 \in \mathbb{R}\) is a given number. Hence, \(\{\tau_i, i \geq 1\}\) is a random sequence with independent increments and \(t_0 < \tau_1 < \tau_2 < \tau_3 < \cdots\).

(iii) Suppose that \(\{\xi_j\}\) is a series of \(m\)-dimensional random vectors independent of \(\{\tau_i, i \geq 1\}\) that satisfies \(\xi_j \in \mathcal{F}_{\tau_j}^{(2)}\) for \(j = 1, 2, \ldots\).

Consider the following stochastic differential equation with random impulses (RISDEs)

\[
\begin{align*}
\left\{ 
\begin{array}{l}
dX_t = b(t, X_t) dt + \sigma(t, X_t) dB_t, \\
X_{t_0} = l(X_{\tau_i}, \xi_i) \\
X_{t_0} = x_0,
\end{array}
\right.
& \quad t \geq t_0, t \neq \tau_i \\
& \quad i = 1, 2, \ldots, \\
& \quad (1)
\end{align*}
\]

where \(b : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times k}, l_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, i = 1, 2, \ldots, \) is \(k\)-dimensional Wiener process, \(X_{\tau_i} = \lim_{t \to \tau_i - 0} X_t\) in \(L^2(\mathbb{P})\), and \(x_0\) is a \(\mathcal{F}_{t_0}\)-measurable, \(\mathbb{R}^n\)-valued random variable such that \(\mathbb{E}(x_0^2) < \infty\).
We use \( \mathcal{F}_t^{(1)} \) to denote the minimal \( \sigma \)-field generated by \( \{B_s, s \leq t\} \), and use \( \mathcal{F}_t^{(2)} \) to denote the minimal \( \sigma \)-field containing all events \( \{\xi_i, 1 \leq i \leq n\} \), where \( \mathcal{F}_t = \mathcal{F}_t^{(1)} \cup \mathcal{F}_t^{(2)} \). It is assumed that \( \{\xi_i\}, \mathcal{F}_t^{(1)} \) and \( x_0 \) are mutually independent. Let \( L^2_{ad}([t_0, T], \mathbb{R}^n) \) be the Banach space equipped with the norm

\[
\|X_t\| = \sup_{t \in [t_0,T]} \left( \mathbb{E}\|X_t\|^2 \right)^{1/2}.
\]

**Remark 2.1.** Obviously, \( \omega_i = \tau_i - \tau_{i-1} \) is the \( i \)-th waiting time of impulses. Since RISDEs (1) does not require \( \sum_{i=1}^{T+1} \mathbb{P}(\omega_i) = +\infty \), the solutions to RISDEs (1) may have infinite number of impulses with positive probability in any given interval \([t_0, T]\), where \( T > 0 \) is a given number.

The function \( \delta \) is defined as

\[
\int_{-\infty}^{\infty} \delta(s)ds = \begin{cases} 
0, & t < 0, \\
1, & t \geq 0.
\end{cases}
\]

It thus follows that RISDEs (1) can be expressed as

\[
\begin{align*}
\begin{cases} 
\frac{dX_t}{dt} &= b(t, X_t)dt + \sigma(t, X_t)dB_t + \sum_{i=1}^{\infty} (I_t(X_{\tau_i}, \xi_i) - X_{\tau_i^-})dB(t - \tau_i), & t \geq t_0, \\
X_{t_0} &= x_0,
\end{cases}
\end{align*}
\]

which can be considered as a stochastic control system with random impulses. It is easy to see that RISDEs (1) extends stochastic differential equations with jumps.

**Definition 2.2.** \( \mathbb{R}^n \)-valued stochastic process \( \{X_t, t \in [t_0, T]\} \) is called a solution to RISDEs (1) with respect to the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq t_0}, \mathbb{P})\) if

1. \( X_t \) is right-continuous and has limit on the left;
2. \( X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) \) is \( \mathcal{F}_t \)-adapted;
3. \( X_{t_0} = x_0 \) and for every \( t_0 \leq t \leq T, \)

\[
\begin{align*}
\begin{cases} 
X_t &= x_0 + \int_{t_0}^{\tau_1} b(s, X_s)ds + \int_{t_0}^{\tau_1} \sigma(s, X_s)dB_s, & t \in [t_0, \tau_1), \\
X_t &= I_t(X_{\tau_i}, \xi_i) + \int_{\tau_i}^{\tau_{i+1}} b(s, X_s)ds + \int_{\tau_i}^{\tau_{i+1}} \sigma(s, X_s)dB_s, & t \in [\tau_i, \tau_{i+1}), i \geq 1.
\end{cases}
\end{align*}
\]

Next, we consider the Ulam’s type stability concepts for RISDEs (1).

**Definition 2.3.** RISDEs (1) is Ulam-Hyers stable in the mean square with respect to \( \varepsilon > 0 \) if there exists a constant \( \kappa > 0 \) such that for each solution \( Y_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) \) of the inequality

\[
\begin{align*}
\begin{cases} 
\|Y_t - x_0 - \int_{t_0}^{\tau_1} b(s, Y_s)ds - \int_{t_0}^{\tau_1} \sigma(s, Y_s)dB_s\| \leq \varepsilon, & t \in [t_0, \tau_1), \\
\|Y_t - I_t(Y_{\tau_i}, \xi_i) - \int_{\tau_i}^{\tau_{i+1}} b(s, Y_s)ds - \int_{\tau_i}^{\tau_{i+1}} \sigma(s, Y_s)dB_s\| \leq \varepsilon, & t \in [\tau_i, \tau_{i+1}), i \geq 1,
\end{cases}
\end{align*}
\]

there exists a solution \( X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) \) of RISDEs (1) such that

\[
\mathbb{E}\|Y_t - X_t\|^2 \leq \kappa \varepsilon, \quad t \in [t_0, T].
\]

**Definition 2.4.** RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square with respect to \( \varphi(t) \) if there exists a constant \( M_\varphi > 0 \) such that for each solution \( Y_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) \) of the inequality

\[
\begin{align*}
\begin{cases} 
\|Y_t - x_0 - \int_{t_0}^{\tau_1} b(s, Y_s)ds - \int_{t_0}^{\tau_1} \sigma(s, Y_s)dB_s\| \leq \varphi(t), & t \in [t_0, \tau_1), \\
\|Y_t - I_t(Y_{\tau_i}, \xi_i) - \int_{\tau_i}^{\tau_{i+1}} b(s, Y_s)ds - \int_{\tau_i}^{\tau_{i+1}} \sigma(s, Y_s)dB_s\| \leq \varphi(t), & t \in [\tau_i, \tau_{i+1}), i \geq 1,
\end{cases}
\end{align*}
\]

there exists a solution \( X_t \in L^2_{ad}([t_0, T], \mathbb{R}^n) \) of RISDEs (1) such that

\[
\mathbb{E}\|Y_t - X_t\|^2 \leq M_\varphi \varphi(t), \quad t \in [t_0, T],
\]

where \( M_\varphi \) is a constant that does not depend on \( Y_t \).
Remark 2.5. It is clear that Definition 2.4 $\implies$ Definition 2.3.

Lemma 2.6. Let $x(t) \in C([t_0, T], \mathbb{R}^n)$ be a solution to the inequality

$$x(t) \leq \varphi(t) + b \int_{t_0}^{t} x(s)ds + \beta_k x(t_k), \quad t_k \in [t_0, T], k \geq 1,$$

where $\varphi(t) \in C([t_0, T], \mathbb{R}^n)$, $\varphi(t)$ is nondecreasing and $b, \beta_k > 0$. Then,

$$x(t) \leq \varphi(t)(1 + \beta_k)\exp\left(\int_{t_0}^{t} b(s)ds\right).$$

More integral inequalities of Gronwall type can be found in [17, 24].

Lemma 2.7. (Krasnoselskii’s Fixed Point Theorem [7]) Let $B$ be a nonempty closed convex set of a Banach space $(X, \|\cdot\|)$. Suppose that $P$ and $Q$ map $B$ into $X$ such that

1. $Px + Qy \in B$ whenever $x, y \in B$;
2. $P$ is a contraction mapping;
3. $Q$ is compact and continuous.

Then there exists $z \in B$ such that $z = Pz + Qz$.

3. Main Results

In this section, we present and prove the existence and Ulam-Hyers-Rassias stability of the solutions to RISDEs (1). Before developing our main results, we introduce the following hypotheses.

(H1) The function $b : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and there exists a constant $L_b > 0$ such that, for any $x, y \in \mathbb{R}^n, t \in [t_0, T],$

$$\|b(t, x) - b(t, y)\|^2 \leq L_b\|x - y\|^2, \quad \|b(t, x)\|^2 \leq L_b(1 + \|x\|^2).$$

(H2) The function $\sigma : [t_0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$ satisfies the following conditions.

(i) For each $x \in \mathbb{R}^n$, $\sigma(\cdot, x) : [t_0, T] \rightarrow \mathbb{R}^{n \times k}$ is measurable and for each $t \in [t_0, T]$, $\sigma(t, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times k}$ is continuous.

(ii) There exists a constant $L_\sigma > 0$ such that, for any $x, y \in \mathbb{R}^n, t \in [t_0, T],$

$$\|\sigma(t, x) - \sigma(t, y)\|^2 \leq L_\sigma\|x - y\|^2, \quad \|\sigma(t, x)\|^2 \leq L_\sigma(1 + \|x\|^2).$$

(H3) The functions $I_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, i = 1, 2, \ldots$, are continuous and there exists a constant $L > 0$ such that, for any $x, y \in \mathbb{R}^n, u \in \mathbb{R}^m,$

$$\|I_i(x, u) - I_i(y, u)\|^2 \leq L\|x - y\|^2.$$

Now, we establish the following existence results via Krasnoselskii’s fixed point theorem.

Theorem 3.1. If assumptions (H1)-(H3) are satisfied, then the system (1) has at least one mild solution defined on $[t_0, T]$ provided that

$$\rho := \max\{3\omega_1(L_b + L_\sigma), 3(L + \omega_{t+1}(\omega_{t+1}L_b + L_\sigma))\} < 1, \quad \ell \geq 1,$$

and

$$\eta := \max\omega_\ell^2 L_b < 1, \quad \ell \geq 1.$$
Step 2.

Letting $B_r = \{X_t \in L^2_{ad}([0, T], \mathbb{R}^n) : ||X_t||^2 \leq r\}$, which stands for a closed ball with radius $r > 0$ and center at $X_t$, $t \in [0, T]$.

Here,

\[
 r \geq \left\{ \frac{3E||x_0||^2 + \omega_1(\omega_1 L_\beta + L_\sigma)}{1 - \rho}, \frac{3\omega_1(\omega_1 L_\beta + L_\sigma)}{1 - \rho} \right\}, \ell \geq 1.
\]

We then Transform problem (1) into a fixed point problem and define an operator $F : L^2_{ad}([0, T], \mathbb{R}^n) \rightarrow L^2_{ad}([0, T], \mathbb{R}^n)$ by

\[
 FX_t = \begin{cases} 
 x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, & t \in [0, \tau_1), \\
 I_t(X_{\tau_1}, \xi_1) + \int_{\tau_1}^t b(s, X_s)ds + \int_{\tau_1}^t \sigma(s, X_s)dB_s, & t \in [\tau_1, \tau_{\ell + 1}), \ell \geq 1.
\end{cases}
\]

Defining the operators $P$ and $Q$ on $B_r$ yields

\[
P X_t = \begin{cases} 
 x_0 + \int_0^t b(s, X_s)ds, & t \in [0, \tau_1), \\
 \int_{\tau_1}^t b(s, X_s)ds, & t \in [\tau_1, \tau_{\ell + 1}), \ell \geq 1,
\end{cases}
\]

and

\[
 Q X_t = \begin{cases} 
 \int_0^t \sigma(s, X_s)dB_s, & t \in [0, \tau_1), \\
 I_t(X_{\tau_1}, \xi_1) + \int_{\tau_1}^t \sigma(s, X_s)dB_s, & t \in [\tau_1, \tau_{\ell + 1}), \ell \geq 1.
\end{cases}
\]

For better readability, we divide the proof into 4 steps.

**Step 1.** For any $X_t \in B_r, t \in [0, T]$, we prove that $FX_t = PX_t + QX_t \in B_r$.

Case 1. For each $t \in [0, \tau_1)$, we have

\[
 E||PX_t + QX_t||^2 \leq 3E||x_0||^2 + 3E||\int_0^t b(s, X_s)ds||^2 + 3E||\int_0^t \sigma(s, X_s)dB_s||^2
\]

\[
 \leq 3E||x_0||^2 + 3(t - t_0) \int_{t_0}^t E||b(s, X_s)||^2ds + 3 \int_{t_0}^t E||\sigma(s, X_s)||^2ds
\]

\[
 \leq 3E||x_0||^2 + 3\omega_1^2 L_\beta (1 + r) + 3\omega_1 L_\sigma (1 + r) \leq r.
\]

Case 2. For each $t \in [\tau_1, \tau_{\ell + 1}), \ell \geq 1$, we have

\[
 E||PX_t + QX_t||^2 \leq 3E||I_t(X_{\tau_1}, \xi_1)||^2 + 3E||\int_{\tau_1}^t b(s, X_s)ds||^2 + 3E||\int_{\tau_1}^t \sigma(s, X_s)dB_s||^2
\]

\[
 \leq 3E||X_{\tau_1}||^2 + 3(t - \tau_1) \int_{\tau_1}^t E||b(s, X_s)||^2ds + 3 \int_{\tau_1}^t E||\sigma(s, X_s)||^2ds
\]

\[
 \leq 3Lx + 3\omega_1^2 L_\beta (1 + r) + 3\omega_1 L_\sigma (1 + r) \leq r.
\]

The above argument shows that $PX_t + QX_t \in B_r$.

**Step 2.** We prove that $P$ is a contraction mapping.

Case 1. For $X_t, Y_t \in B_r$ and $t \in [0, \tau_1)$, we have

\[
 E||PX_t - PY_t||^2 \leq (t - t_0) \int_{t_0}^t E||b(s, X_s) - b(s, Y_s)||^2ds \leq \omega_1^2 L_\beta ||X_t - Y_t||^2.
\]

Case 2. For $X_t, Y_t \in B_r$ and $t \in [\tau_1, \tau_{\ell + 1}), \ell \geq 1$, we have

\[
 E||PX_t - PY_t||^2 \leq (t - \tau_1) \int_{\tau_1}^t E||b(s, X_s) - b(s, Y_s)||^2ds \leq \omega_1^2 L_\beta ||X_t - Y_t||^2.
\]
Step 3. We prove that 

\[ \mathbb{E}[|P_X - P_Y|^2] \leq \eta \|X_t - Y_t\|^2 < \|X_t - Y_t\|^2. \]

The above inequalities imply that \( P \) is contraction on \( B_r \).

**Step 4.** We prove that \( Q \) is continuous on \( B_r \).

Let \( \{X^n_t\} \subset B_r \), with \( X^n_t \to X_t \), as \( n \to \infty \), \( t \in [t_0, T] \). By (H2) and (H3), we get 

(i) \( \sigma_t(X^n_t) \to \sigma_t(X_t) \), \( n \to \infty \),

(ii) \( \mathbb{E}[|I_r(X^n_{\tau_t}, \xi_t) - I_r(X_{\tau_t}, \xi_t)|^2] \to 0 \), \( n \to \infty \), \( \ell \geq 1 \).

It thus follows from the dominated convergence theorem that for each \( t \in [t_0, \tau_1] \),

\[ \mathbb{E}[|Q^n_t - Q_t|^2] \leq \int_{t_0}^{t} \mathbb{E}[|\sigma(s, X^n_s) - \sigma(s, X_s)|^2] ds \to 0, \quad \text{as} \ n \to \infty. \]

For each \( t \in [\tau_\ell, \tau_{\ell+1}] \), \( \ell \geq 1 \), we have

\[ \mathbb{E}[|Q^n_t - Q_t|^2] \leq 2\mathbb{E}[|I_r(X^n_{\tau_t}, \xi_t) - I_r(X_{\tau_t}, \xi_t)|^2] + 2 \int_{\tau_t}^{t} \mathbb{E}[|\sigma(s, X^n_s) - \sigma(s, X_s)|^2] ds \to 0, \quad \text{as} \ n \to \infty. \]

Combining the above arguments, we obtain that \( Q \) is continuous on \( B_r \).

**Step 4.** We prove that \( Q \) is compact.

First, we verify that \( Q(B_r) \) is uniformly bounded.

From Step 1, we derive that \( \mathbb{E}[|QX_t|^2] \leq r \) for \( t \in [t_0, T] \).

Second, we prove that \( Q \) maps bounded sets into equicontinuous sets of \( B_r \).

For \( t \in [t_0, T] \), \( t_0 < t_1 < t_2 < T \), \( X_t \in B_r \), we get

\[ \mathbb{E}[|QX_{t_1} - QX_{t_2}|^2] = \mathbb{E}\left[ \left| \int_{t_0}^{t_1} \sigma(s, X_s) dB_s - \int_{t_0}^{t_2} \sigma(s, X_s) dB_s \right|^2 \right] \leq \int_{t_0}^{t_2} \mathbb{E}[|\sigma(s, X_s)|^2] ds \to 0, \quad \text{as} \ t_2 \to t_1. \]

This implies that \( Q \) maps bounded sets into equicontinuous sets of \( B_r \).

Finally, we show that \( Q \) maps \( B_r \) into a precompact set for every \( t \in [t_0, T] \).

Let \( 0 < \epsilon < t < T \), we define

\[ U^\epsilon_t = \{ Q^\epsilon X_t : X_t \in B_r \}. \]

The set \( U^\epsilon_t = \{ Q^\epsilon X_t : X_t \in B_r \} \) is relatively compact in \( L^2_{ad}([t_0, T], \mathbb{R}^n) \) for each \( \epsilon \) with \( \epsilon \in (0, t) \).

Case 1. For interval \( t \in [t_0, \tau_1] \), (7) reduces to

\[ U^\epsilon_t = \left\{ \int_{t_0}^{t-\epsilon} \sigma(s, X_s) dB_s : X_t \in B_t \right\}. \]

Then

\[ \mathbb{E}[|QX_t - Q^\epsilon X_t|^2] = \mathbb{E}\left[ \left| \int_{t_0}^{t} \sigma(s, X_s) dB_s - \int_{t_0}^{t-\epsilon} \sigma(s, X_s) dB_s \right|^2 \right] \leq \int_{t_0}^{t-\epsilon} \mathbb{E}[|\sigma(s, X_s)|^2] ds \]

\[ \leq \epsilon L_{\sigma}(1 + r). \]

(8)

Case 2. For interval \( t \in [\tau_\ell, \tau_{\ell+1}] \), \( \ell \geq 1 \), (7) reduces to

\[ U^\epsilon_t = \left\{ I_r(X_{\tau_t}, \xi_t) + \int_{\tau_t}^{t-\epsilon} \sigma(s, X_s) dB_s : X_t \in B_r \right\}. \]
Then
\[
\mathbb{E}[|QX_t - Q^c X_t|^2] = \mathbb{E}[|\int_{t_\tau}^t \sigma(s, X_s)dB_s - \int_{t_\tau}^{t-\epsilon} \sigma(s, X_s)dB_s|^2] \leq \int_{t-\epsilon}^t \mathbb{E}[|\sigma(s, X_s)|^2]ds
\]
\[
\leq e L_\sigma (1 + r).
\]

As \( \epsilon \to 0 \), the right hand side of inequalities (8) and (9) tend to zero. Hence, there are relatively compact sets arbitrary close to set \( U_t = \{QX_t : X_t \in B_t\} \) and \( U_t \) is relatively compact in \( L^2_{ad}([t_0, T], \mathbb{R}^n) \). Thus, by the Arzela-Ascoli theorem, we know that \( Q \) is compact.

By Lemma 2.7, we know that \( FX_t = PX_t + QX_t \) has a fixed point in \( B_t \). The proof is thus complete. \( \square \)

Next, we verify the following Ulam-Hyers-Rassias stable via Gronwall lemma approach.

**Theorem 3.2.** Assume that assumptions (H1)-(H3) are satisfied and \( \varphi : [t_0, T] \to \mathbb{R}^+ \) is a continuous nondecreasing function. Then RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square.

**Proof.** Let \( Y_t \) be a solution of inequality (4) and let \( X_t \) be the solution to RISDEs (1).

Case 1. For interval \( t \in [t_0, \tau_1] \), we have

\[
\mathbb{E}[|Y_t - X_t|^2] = \mathbb{E}[|Y_t - x_0 - \int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t \sigma(s, Y_s)dB_s|^2]
\]
\[
\leq 3\mathbb{E}[|Y_t - x_0 - \int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t \sigma(s, Y_s)dB_s|^2]
\]
\[
+ 3\mathbb{E}[\int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t b(s, X_s)ds]^2
\]
\[
+ 3\mathbb{E}[\int_{t_0}^t \sigma(s, Y_s)dB_s - \int_{t_0}^t \sigma(s, X_s)dB_s]^2.
\]

From Definition 2.4 and the fact that \( \varphi : [t_0, T] \to \mathbb{R}^+ \) is a continuous function, we get

\[
\mathbb{E}[|Y_t - x_0 - \int_{t_0}^t b(s, Y_s)ds - \int_{t_0}^t \sigma(s, Y_s)dB_s|^2] \leq \varphi(t).
\]

It follows from (H1) and (H2) that

\[
\mathbb{E}[|Y_t - X_t|^2] \leq 3\varphi(t) + 3(t - t_0) \int_{t_0}^t \mathbb{E}[|b(s, Y_s) - b(s, X_s)|^2]ds
\]
\[
+ 3 \int_{t_0}^t \mathbb{E}[|b(s, Y_s) - b(s, X_s)|^2]ds
\]
\[
\leq 3\varphi(t) + 3(\omega_1 L_b + L_o) \int_{t_0}^t \mathbb{E}[|Y_s - X_s|^2]ds.
\]

According to Lemma 2.6, we obtain

\[
\mathbb{E}[|Y_t - X_t|^2] \leq 3\varphi(t) \exp\left(3(\omega_1 L_b + L_o)T\right) \leq 3\varphi(t) \exp(3\omega_1(\omega_1 L_b + L_o)).
\]

Thus,

\[
\mathbb{E}[|Y_t - X_t|^2] \leq M_{\varphi_1} \varphi(t),
\]

where \( M_{\varphi_1} = 3\exp(3\omega_1(\omega_1 L_b + L_o)). \)
Case 2. For interval \( t \in [\tau_{\ell}, \tau_{\ell+1}), \ell \geq 1 \), we have
\[
\mathbb{E}[|Y_t - X_t|^2] = \mathbb{E}[|Y_t - I_t(X_{\tau_{\ell}}, \xi_{\ell}) - \int_{\tau_{\ell}}^t b(s, Y_s) ds - \int_{\tau_{\ell}}^t \sigma(s, Y_s) dB_s|^2]
\leq 4\mathbb{E}[|Y_t - I_t(Y_{\tau_{\ell}}, \xi_{\ell}) - \int_{\tau_{\ell}}^t b(s, Y_s) ds - \int_{\tau_{\ell}}^t \sigma(s, Y_s) dB_s|^2]
+ 4\mathbb{E}[|I_t(X_{\tau_{\ell}}, \xi_{\ell})|]
+ 4\mathbb{E}\left[\int_{\tau_{\ell}}^t |b(s, Y_s) ds - b(s, X_s)|^2ds\right]
+ 4\mathbb{E}\left[\int_{\tau_{\ell}}^t |\sigma(s, Y_s) dB_s - \sigma(s, X_s) dB_s|^2\right].
\]
From Definition 2.4 and the fact that \( \varphi : [t_0, T] \to \mathbb{R}^+ \) is a continuous function, we have
\[
\mathbb{E}[|Y_t - I_t(Y_{\tau_{\ell}}, \xi_{\ell}) - \int_{\tau_{\ell}}^t b(s, Y_s) ds - \int_{\tau_{\ell}}^t \sigma(s, Y_s) dB_s|^2 \leq \varphi(t).
\]
By (H1)-(H3), we get
\[
\mathbb{E}[|Y_t - X_t|^2] \leq 4\varphi(t) + 4L\mathbb{E}[|Y_{\tau_{\ell}} - X_{\tau_{\ell}}|^2] + 4(t - \tau_{\ell}) \int_{\tau_{\ell}}^t \mathbb{E}[|b(s, Y_s) ds - b(s, X_s)|^2ds]
+ 4\int_{\tau_{\ell}}^t \mathbb{E}[|\sigma(s, Y_s) - \sigma(s, X_s)|^2ds]
\leq 4\varphi(t) + 4L\mathbb{E}[|Y_{\tau_{\ell}} - X_{\tau_{\ell}}|^2] + 4(\omega_{t+1} L_b + L_G) \int_{\tau_{\ell}}^t \mathbb{E}[|Y_s - X_s|^2ds].
\]
It follows from Lemma 2.6 that
\[
\mathbb{E}[|Y_t - X_t|^2] \leq 4\varphi(t)(1 + 4L)\exp\left(\int_{\tau_{\ell}}^t 4(\omega_{t+1} L_b + L_G)ds\right) \leq 4\varphi(t)(1 + 4L)\exp\left(4\omega_{t+1}(\omega_{t+1} L_b + L_G)\right).
\]
Hence,
\[
\mathbb{E}[|Y_t - X_t|^2] \leq M_{\varphi,2} \varphi(t),
\]
where \( M_{\varphi,2} = 4(1 + 4L)\exp\left(4\omega_{t+1}(\omega_{t+1} L_b + L_G)\right)\).
Let \( M_{\varphi} = \max\{M_{\varphi,1}, M_{\varphi,2}\} \). Then for all \( t \in [t_0, T] \), we obtain
\[
\mathbb{E}[|Y_t - X_t|^2] \leq M_{\varphi} \varphi(t).
\]
Summarizing the above, RISDEs (1) is Ulam-Hyers-Rassias stable in the mean square. This completes the proof. \( \square \)

**Remark 3.3.** Under the assumption of Theorem 3.2, we consider RISDEs (1) and inequality (3). One can repeat the same process to verify that RISDEs (1) is Ulam-Hyers stable in the mean square.

**Acknowledgments**

S. Deng was supported by the Scientific Research Foundation of Xiangnan University (No.2020XJ19). The authors thank the anonymous referee and the handling editor for their valuable suggestions and comments.

**References**

[1] M. R. Abdollahpour, R. Aghayaria, M. Th. Rassias, Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions, J. Math. Anal. Appl., 437 (2016) 605-612.
[2] M. R. Abdollahpour, M. Th. Rassias, Hyers-Ulam stability of hypergeometric differential equations, Aequat. Math., 93 (2019) 691-698.

[3] J. Aczel, J. Dhombres, Functional Equations in Several Variables, Cambridge University Press, 1989.

[4] D. G. Bourgin, Classes of transformations and bordering transformations, Bull. Amer. Math. Soc., 57 (1951) 223-237.

[5] S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abb. Math. Sem. Univ. Hamburg, 62 (1992) 59-64.

[6] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, 2002.

[7] J. Dabas, A. Chauhan, M. Kumar, Existence of the mild solutions for impulsive fractional equations with infinite delay, Int. J. Differ. Equ., 2011, Article ID 793023.

[8] Z. Gajda, On stability of additive mappings, Internat. J. Math. Math. Sci., 14 (1991) 431-434.

[9] Y. C. Guo, M. Q. Chen, X.-B. Shu, F. Xu, The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm, Stoch. Anal. Appl., 2020, https://doi.org/10.1080/07362994.2020.1824677.

[10] L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays, Acta. Appl. Math., 111 (2010) 303-317.

[11] D. H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci., 27 (1941) 222-224.

[12] D. H. Hyers, G. Isac, Th. M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, 1998.

[13] D. H. Hyers, Th. M. Rassias, Approximate homomorphisms, Aeq. Math., 44 (1992) 125-153.

[14] W. Jian, Some further generalization of the Ulam-Hyers-Rassias stability of functional equations, J. Math. Anal. Appl., 263 (2001) 406-423.

[15] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis, Springer, 2011.

[16] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, 2009.

[17] R. Kruse, M. Scheutzow, A discrete stochastic gronwall lemma, Math. Comput. Simulat. 143 (2018) 149-157.

[18] Y.-H. Lee, S.-M. Jung, M. Th. Rassias, On an n-dimensional mixed type additive and quadratic functional equation, Appl. Math. Comput., 228 (2014) 13-16.

[19] S. Li, L. X. Shu, X.-B. Shu, F. Xu, Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays, Stochastics, 91 (2019) 857-872.

[20] Z. H. Li, X.-B. Shu, F. Xu, The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem, AIMS Math., 5 (2020) 6189-6210.

[21] C. Park, M. Th. Rassias, Additive functional equations and partial multipliers in C∗-algebras, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales - Serie A: Matematicas, 113 (2019) 2261-2275.

[22] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, P. Am. Math. Soc., 72 (1978) 297-300.

[23] Th. M. Rassias, Functional Equations and Inequalities, Kluwer Academic Publishers, 2000.

[24] V. Sree Hari Rao, Integral inequalities of gronwall type for distributions, J. Math. Anal. Appl. 72 (1979) 545-550.

[25] P. K. Sahoo, Pl. Kannappan, Introduction to Functional Equations, CRC Press, 2011.

[26] T. Sathiyaraj, J. R. Wang, P. Balasubramaniam, Ulam’s stability of Hilfer fractional stochastic differential systems, Eur. Phys. J. Plus, 134 (2019).

[27] W. Smajdor, Note on a Jensen type functional equation, Publ. Math. Debrecen, 63 (2003) 703-714.

[28] L. X. Shu, X.-B. Shu, Q. X. Zhu, F. Xu, Existence and exponential stability of mild solutions for second-order neutral stochastic functional differential equation with random impulses, J. Appl. Anal. Comput., 11 (2021) 1-22.

[29] X.-B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, Nonlinear Anal., 74 (2011) 2003-2011.

[30] X.-B. Shu, Y. Shi, A study on the mild solution of impulsive fractional evolution equations, Appl. Math. Comput., 273 (2016) 465-476.

[31] T. Trif, On the stability of a functional equation deriving from an inequality of Popovicu for convex functions, J. Math. Anal. Appl., 272 (2002) 604-616.

[32] S. M. Ulam, A Collection of Mathematical Problems, Interscience Publishers, New York, 1968.

[33] S. Wu, B. Zhou, Existence and uniqueness of stochastic differential equations with random impulses and markovian switching under non-lipschitz conditions, Acta Math. Sin., 27 (2011) 519-536.

[34] Z. Yan, X. Yan, Existence of solutions for impulsive partial stochastic neutral integro-differential equations with state-dependent delay, Collect. Math., 64 (2013) 235-250.

[35] X. K. Zhao, Mean square Hyers-Ulam stability of stochastic differential equations driven by Brownian motion, Adv. Differ. Equ., 271 (2016).