SOME PROPERTIES OF CAYLEY SIGNED GRAPHS ON FINITE
ABELIAN GROUPS

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ABSTRACT. Let $\Sigma = (\Gamma, \sigma)$ is a signed graph (or sigraph in short), where $\Gamma$ is a underlying graph of $\Sigma$ and $\sigma : E \rightarrow \{+,-\}$ is a function. Consider $\Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^\alpha_1} \times \mathbb{Z}_{p_2^\alpha_2} \times \cdots \times \mathbb{Z}_{p_k^\alpha_k}, \Phi)$, where all $p_1, p_2, \ldots, p_k$ are distinct prime factors and $\Phi = \varphi_{p_1} \times \varphi_{p_2^\alpha_2} \times \cdots \times \varphi_{p_k^\alpha_k}$. For any positive integer $n$, $\varphi_n = \{\ell | 1 \leq \ell < n, \gcd(\ell, n) = 1\}$. Motivated by [13], we will investigate balancing in $\Sigma$ and $L(\Sigma)$, clusterability and sign-compatibility of $\Sigma$.

1. Introduction

Let $(G, \cdot)$ be a group and $S = S^{-1}$ be a non empty subset of $G$ not containing the identity element $e$ of $G$. The Cayley graph $\text{Cay}(G, S)$ is the simple graph having vertex set $G$ and edge set $\{\{v, vs\} | v \in G, s \in S\}$. To find enough information about Cayley graphs, refer to books Biggs [5] and Godsil, Royle [7].

A signed graph (or sigraph in short) is an ordered pair $\Sigma = (\Gamma, \sigma)$ where $\Gamma = (V, E)$ is a graph called the underlying graph of $\Sigma$ and $\sigma : E \rightarrow \{+,-\}$ is a function. $\Sigma$ is all-positive (all-negative) if all its edges are positive (negative). Moreover, it is said to be homogeneous if it is either all-positive or all-negative and heterogeneous otherwise. $d^-(v)$ ($d^+(v)$) represents the number of negative (positive) edges incident at $v$ in $\Sigma$. A marked sigraph is an ordered pair $\Sigma_\mu = (\Sigma, \mu)$, where $\Sigma = (\Gamma, \sigma)$ is a sigraph and $\mu : V(\Sigma) \rightarrow \{+,-\}$ is a function, called a marking of $\Sigma$ (see [15, 16, 17]).

Behzad and Chartrand [3] defined line sigraph $L(\Sigma)$ as the sigraph in which the edges of $\Sigma$ are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in $\Sigma$ have a vertex in common, any such edge ef is defined to be negative whenever both $e$ and $f$ are negative edges in $\Sigma$. A positive cycle in $\Sigma$ is a cycle which contains even number of negative edges. A sigraph $\Sigma$ is a balanced sigraph if every cycle in $\Sigma$ is positive. The first classification of balanced graphs was done by Harary in 1953 [8].

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A clusterable sigraph is a sigraph in which its vertex set is partitioned into pairwise disjoint subsets in such a way that whenever there are positive edges they lie in the same subset and negative edges lie in the different subsets. These subsets are called clusters (see [6]). A sigraph is called sign-compatible [11] if it is possible to mark the vertices of the sigraph in such a manner that for every negative edge there are negatively marked vertices on its both ends and for any positive edge its ends are not assigned negative, sign-incompatible otherwise.

Let Σ = (Γ, σ) is a sigraph whose underlying graph is Γ = Cay(\(Z_{p_1} \times Z_{p_2}^{\alpha_1} \times \cdots \times Z_{p_k}^{\alpha_k}\), \(\Phi\)) where \(p_1, p_2, \ldots, p_k\) are distinct prime factors and \(\Phi = \varphi_{p_1} \times \varphi_{p_2}^{\alpha_1} \times \cdots \times \varphi_{p_k}^{\alpha_k}\) and also for an edge \(ab\) of Σ, the function \(\sigma\) is defined as follows [13].

\[
\sigma(ab) = \begin{cases} 
+ & \text{if } a \in \Phi \text{ or } b \in \Phi, \\
- & \text{otherwise.}
\end{cases}
\]

By \(Z_n\) we denote the abelian group of order \(n\). For \(p_1 = 2\), the signed graph Σ, is a disconnected sigraph with two connected components, say Σ₁ and Σ₂, where \(V(Σ_1) = \{(1, v) | v \text{ is odd}\} \cup \{(0, v) | v \text{ is even}\}\) and \(V(Σ_2) = \{(0, v) | v \text{ is odd}\} \cup \{(1, v) | v \text{ is even}\}\). Since \(1, 1 \in \Phi\) and \((0, 0)\) is adjacent to \((1, 1)\), so, by definition of \(\sigma\), the edge \((0, 0)(1, 1)\) is positive. Also, since the vertices of \(V(Σ_2)\) are not in \(\Phi\) and \((0, 1)\) is adjacent to \((1, 0)\), hence \((0, 1)(1, 0)\) is negative edge). Note that the tree has no cycle and using Theorem 1.6 [18], we can conclude that each tree is balanced.

In this article some properties of the signed graph Σ, such as balancing, clusterability and sign-compatibility, are investigated.

2. BALANCE IN SIGRAPHS Σ AND LINE SIGRAPHS Σ

In this chapter, we examined the number of positive and negative edges for Σ = (Γ, σ), where Γ = Cay(\(Z_{p_1} \times Z_{p_2}^{\alpha_1} \times \cdots \times Z_{p_k}^{\alpha_k}\), \(\Phi\)) and one of the prime factors say \(p_i\) is 2 and \(\alpha_j \geq 1\) for any \(j = 1, 2, \ldots, k\). Moreover we find the number of positive and negative edges of \(Γ_1 = \text{Cay}(Z_p \times Z_p^{\alpha}, \Phi)\), where \(\alpha \geq 1, p \geq 3, \) and \(Γ_2 = \text{Cay}(Z_p \times Z_{pq}, \Phi)\), where \(p, q \geq 3\) are prime numbers. Next we study the balanced property of Σ and \(L(Σ)\).

When \(Γ = \text{Cay}(Z_2 \times Z_2, \Phi)\), sigraph Σ has two connected components. One of the components has a positive edge and the other component has a negative edge. (According to the explanation given in the last paragraph of the introduction, \(V(Σ_1) = \{(0, 0), (1, 1)\}\) and \(V(Σ_2) = \{(0, 1), (1, 0)\}\). Since \((1, 1) \in \Phi\) and \((0, 0)\) is adjacent to \((1, 1)\), so, by definition of \(σ\), the edge \((0, 0)(1, 1)\) is positive. Also, since the vertices of \(V(Σ_2)\) are not in \(\Phi\) and \((0, 1)\) is adjacent to \((1, 0)\), hence \((0, 1)(1, 0)\) is negative edge). Note that the tree has no cycle and using Theorem 1.6 [18], we can conclude that each tree is balanced. Since Σ is the union of two
trees, so it is balanced. Also \(L(\Sigma)\) has only two vertices so we may consider it is a balanced graph. Therefore in the proof of Theorems 2.6, 2.8 this case is not considered.

**Proposition 2.1.** Let \(\Sigma = (\Gamma, \sigma)\), where \(\Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}, \Phi)\) and one of the prime factors is 2. Then \(\Sigma\) has \(|\Phi|^2\) positive edges.

**Proof.** Since one of \(p_i\) is 2, so without loss of generality we may assume that \(p_1 = 2\). In this case \(\Sigma\) is a disconnected sigraph with exactly two connected components \(\Sigma_1 = (\Gamma_1, \sigma)\) and \(\Sigma_2 = (\Gamma_2, \sigma)\), where \(V(\Sigma_1) = \{(1, v)\mid v \text{ is odd}\} \cup \{(0, v)\mid v \text{ is even}\}\) and \(V(\Sigma_2) = \{(0, v)\mid v \text{ is odd}\} \cup \{(1, v)\mid v \text{ is even}\}\). Note that \(\Sigma\) is \(|\Phi|\)-regular hence \(|E(\Sigma_1)| = |E(\Sigma_2)| = \frac{1}{4} |\Phi| p_1^{\alpha_1+1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\).

Suppose \((u, v)(u', v')\) is an arbitrary edge of \(\Sigma_1\), where \((u, v) \in \{(1, v)\mid v \text{ is odd}\}\) and \((u', v') \in \{(0, v)\mid v \text{ is even}\}\). Since \(\Phi \subseteq \{(1, v)\mid v \text{ is odd}\}\) and by the definition of \(\sigma\) the edge that has at least one end in \(\Phi\) is positive, so a number of edges in \(\Sigma_1\) are positive. Also each of the members of \(\Phi\) is adjacent to the \(|\Phi|\) vertices of \(\{(0, v)\mid v \text{ is even}\}\). Hence we have \(|\Phi|^2\) positive edges. Note that if \(\Gamma = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^e}, \Phi)\) then \(\Phi = \{(1, v)\mid v \text{ is odd}\}\). Therefore \(\Sigma_1\) is all-positive component of sigraph \(\Sigma\) (since the edges of \(\Sigma_1\) have one end in \(\{(1, v)\mid v \text{ is odd}\}\) and the other in \(\{(0, v)\mid v \text{ is even}\}\), so by definition of \(\sigma\) are all positive). Also \(|E(\Sigma_1)| = \frac{1}{4} |\Phi| p_1^{\alpha_1+1} = |\Phi| 2^{\alpha_1-1} = |\Phi|^2\), since the number of odd numbers in \(\{1, 2, 3, \ldots, 2^{\alpha_1}\}\) is equal to \(2^{\alpha_1-1} = |\Phi|\).

Now suppose that \((u, v)(u', v') \in E(\Sigma_2)\), where \((u, v) \in \{(0, v)\mid v \text{ is odd}\}\) and \((u', v') \in \{(1, v)\mid v \text{ is even}\}\). Clearly \(u\) and \(v'\) are multiples of 2. So \((u, v)\) and \((u', v')\) are not in \(\Phi\). This implies that \(\Sigma_2\) is the all-negative component of \(\Sigma\) (since the vertices of \(\Sigma_2\) are not in \(\Phi\), so all the edges in \(\Sigma_2\) are negative).

Next assume that \(p_1 \geq 3\). According to the definition of a Cayley graph, vertex \((0, 0)\) is adjacent to all vertices of \(\Phi\). Also each vertex \(\Phi\) is adjacent to \(|\Phi| - 1\) of other vertices of \(\Sigma\). By definition of \(\sigma\), all edges between these vertices are positive and their number is equal to \(|\Phi| + |\Phi|(|\Phi| - 1)| = |\Phi|^2\). We represent the set of all these vertices with \(V_1\) such that \(V_1 = \{(0, v)\mid v \text{ is even}\} \cup \{(u, v)\mid u \in \{1, 2, \ldots, p_1 - 1\} \text{ and } v \text{ is not odd-multiple of prime factors}\}\). Consider the vertices of \(V_2\), where

\[
V_2 = \{(0, v)\mid v \text{ is odd}\} \cup \{(u, v)\mid u \in \{1, 2, \ldots, p_1 - 1\} \text{ and } v \text{ is odd-multiple of prime factors}\}.
\]

Thus \(V = V_1 \cup V_2\). Assume that \((u, v)\) and \((u', v')\) are arbitrary vertices of \(V_2\). Since \(v\) and \(v'\) are both odd so \(v - v'\) is even, hence \((u, v) - (u', v') \notin \Phi\). Therefore none of the vertices \(V_2\) are
adjacent to each other. Let’s assume now \((u, v) \in V_2\) is adjacent to \((u', v') \in V_1\). Because \(v\) is an odd, so \(v'\) should be even. Thus \((u', v') \notin \Phi\). Moreover it is easy to see that the vertices of \(V_2\) are not in \(\Phi\). This implies that all edges incident with the vertices of \(V_2\) are negative and their number is \(|V_2| \times |\Phi| = \frac{n|\Phi|}{2} - |\Phi|^2\); \(n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}\).

\[\square\]

**Example 2.2.** Let \(\Sigma = (\text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times 3, \Phi), \sigma)\) and \(\Sigma' = (\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_2 \times 3, \Phi), \sigma)\) which are shown in Figures 1 and 2, respectively. Positive edges with straight line and negative edges are shown with dash.

![Figure 1. Two connected components of Σ, left Σ₁, right Σ₂](image)

**Lemma 2.3.** Let \(\Sigma = (\Gamma, \sigma)\), where \(\Gamma = \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_p^\alpha, \Phi)\), \(\alpha \geq 1\) and \(p \geq 3\) is prime number. Then the number of negative edges is \(p^{\alpha-1}|\Phi|\).

**Proof.** Assume that \((u, v)(u', v')\) is an arbitrary edge of \(\Sigma\). By the definition of \(\sigma\), we have negative edge only when \(u, v', u', v\) are multiples of \(p\). Without loss of generality we suppose that \(u, v'\) are multiples of \(p\). Since in \(\mathbb{Z}_p\) only zero is multiple of \(p\), hence \(u = 0, v \in \varphi_{p^\alpha}\) and also \(u' \in \varphi_p\) and \(v'\) is multiple of \(p\) less than \(p^\alpha\). We know that \(\varphi_{p^\alpha} = p^{\alpha-1}(p - 1)\) so \((u, v)\) can have \(p^{\alpha-1}(p - 1)\) cases. Moreover \(u' \in \{1, 2, \ldots, p - 1\}\) and we can consider \(p^{\alpha-1}\) cases for \(v'\), because the number of multiples of \(p\) less than \(p^\alpha\) is equal to \(p^{\alpha-1}\). Therefore \((u', v')\) can have \(p^{\alpha-1}(p - 1)\) cases. This implies that \(\Sigma\) has \((p^{\alpha-1}(p - 1))^2 = p^{\alpha-1}||\varphi_p||\varphi_{p^\alpha}| = p^{\alpha-1}|\Phi|\) negative edges. \(\square\)

**Example 2.4.** Let \(\Sigma = (\text{Cay}(\mathbb{Z}_3 \times \mathbb{Z}_3, \Phi), \sigma)\). This sigraph, which is shown in figure 3, has 4 negative edges.
Remark 1. Suppose \( \lambda \) is the length of the longest sequence of consecutive integers in \( \mathbb{Z}_n \), each of which shares a prime factor with \( n \). Assume that \( n = pq \), where \( p, q \) are prime numbers and \( p, q \geq 3 \). In \( \mathbb{Z}_{pq} \) we have \( \lambda = 2 \). Now we prove that \( \lambda \) appears exactly in two places.

Let \( kp \) and \( k'q \) be the first consecutive integers in \( \mathbb{Z}_{pq} \), which are multiples of \( p \) and \( q \), respectively. It is clear that \( pq - k'q \) and \( pq - kp \) are the final consecutive integers in \( \mathbb{Z}_{pq} \). Suppose that \( fp \) and \( tq \) are other consecutive integers in \( \mathbb{Z}_{pq} \), where \( f > k \) and \( t > k' \). Then \( (fp - tq) - (k'q - kp) = 0 \) and therefore \( (f + k)p = (t + k')q \). Hence there exists an integer \( r \) with \( f + k = rq \) and \( t + k' = rp \). Since \( 1 \leq k < f \leq q - 1 \) and \( 1 \leq k' < t \leq p - 1 \), we have
Theorem 2.6. Let \( f = q - k \) and \( t = p - k' \). Therefore \( fp \) and \( tq \) are the final consecutive integers in \( \mathbb{Z}_{pq} \).

Proof. Let’s consider the arbitrary edge \((u, v)(u', v')\) of \( \Sigma \). Only when we have a negative edge that the vertices \((u, v)\) and \((u', v')\) are not in \( \Phi \). Also this is possible if at each of the vertices \((u, v), (u', v')\) at least one of there elements multiples of one of the prime factors \( p \) or \( q \). The following cases are the only possible cases for the formation of negative edges. Note that we show nonzero multiples of \( p \) and \( q \) by \( \alpha p \) and \( \beta q \), respectively.

Case(i) \((u, v) \in \{(0, v)\mid v \in \varphi_{pq}\}, (u', v') \in \{(u', v')\mid u' \in \{1, 2, \ldots, p - 1\}, v' = 0 \text{ or } v' = \alpha p, v - v' \neq \beta q \text{ or } v' = \beta q, v - v' \neq \alpha p \}\).

Clearly there are \( |\varphi_{pq}| \) cases for \((u, v)\). For \((u', v')\) if \( v' = 0 \) then we have \( p - 1 \) cases and if \( v' = \alpha p, v - v' \neq \beta q \) then by Remark [1] we have \( (p - 1)(q - 2) \) cases and also if \( v' = \beta q, v - v' \neq \alpha p \) then we have \( (p - 1)(p - 2) \) cases for \((u', v')\). Therefore in this case, the number of negative edges is \( |\varphi_{pq}|((p - 1) + (p - 1)(q - 2) + (p - 1)(p - 2)) = |\Phi|(p + q - 3) \).

Case(ii) \((u, v) \in \{(0, v)\mid v = \alpha p\}, (u', v') \in \{(u', v')\mid u' \in \{1, 2, \ldots, p - 1\}, v' = \beta q\}.

In this case we have \( q - 1 \) cases for \((u, v)\) and \( (p - 1)(p - 1) \) cases for \((u', v')\). Hence there are \((q - 1)(p - 1)(p - 1) = |\Phi| \) cases for \((u, v)(u', v')\).

Case(iii) \((u, v) \in \{(0, v)\mid v = \beta q\}, (u', v') \in \{(u', v')\mid u' \in \{1, 2, \ldots, p - 1\}, v' = \alpha p\}.

Similar to Case(ii), there are \((p - 1)(p - 1)(q - 1) = |\Phi| \) cases for \((u, v)(u', v')\).

Case(iv) \((u, v) \in \{(u, v)\mid u \in \{1, 2, \ldots, p - 1\}, v = \alpha p\}, (u', v') \in \{(u', v')\mid u' \in \{1, 2, \ldots, p - 1\}, u' \neq u, v' = \beta q\}.

There are \( p - 1, q - 1, p - 2 \) for \( u, v, u' \) and \( v' \), respectively. Therefore we have \((p - 1)(q - 1)(p - 2)(p - 1) = |\Phi|(p - 2) \) cases for \((u, v)(u', v')\).

By considering all the above cases, the number of negative edges in \( \Sigma \) is equal to:

\[ |\Phi|(p + q - 3) + |\Phi| + |\Phi|(|\Phi|p - 2) = |\Phi|(2p + q - 3). \]

\[ \square \]

Theorem 2.6. Let \( \Sigma = (\Gamma, \sigma) \), where \( \Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_k^{\alpha_k}}, \Phi) \). Then \( \Sigma \) is balanced if and only if one of the prime factors is 2.

Proof. Necessity: Let \( \Sigma \) is balanced. On contrary, assume that \( p_i \geq 3 \) for any \( i = 1, 2, \ldots, k \). Since \( p_i \geq 3 \) hence both numbers 1 and 2 belong to \( \varphi_{p_1} \) and \( \varphi_{p_1^{\alpha_1}p_2^{\alpha_2} \ldots p_k^{\alpha_k}} \). This implies that \( \Sigma \) contains the cycle \( C = ((0, 1), (1, 2), (2, 0), (0, 1)) \). Note that \((1, 2) \in \Phi \) hence \((0, 1)(1, 2) \) and \((1, 2)(2, 0) \) are positive edges. Moreover \((0, 1) \) and \((2, 0) \) are not in \( \Phi \), so \((2, 0)(0, 1) \) is negative.
edge. We conclude that \( C \) is a negative cycle. Therefore, by definition of balanced sigraph, \( \Sigma \) is unbalanced. Which is in contradiction with the assumption.

**Sufficiency:** Suppose one of the prime factors is 2. Assume first that \( p_1 = 2 \). Then by proof of Proposition 2.7, \( \Sigma \) has exactly two connected components \( \Sigma_1 \) and \( \Sigma_2 \) where \( \Sigma_2 \) is all-negative. We show that every cycle in \( \Sigma_2 \) is positive. Suppose \( C = (x_1, x_2, \ldots, x_m, x_1) \) is an arbitrary cycle in \( \Sigma_2 \). Without loss of generality, let \( x_1 \in \{(0, v)|v is odd\} \). Since neither of the vertices of \( \{(0, v)|v is odd\} \) is not adjacent to each other, so \( x_2, x_m \in \{(1, v)|v is even\} \).

By continuing this process, it is easy to see that \( C \) contains an even number of negative edges.

Which implies that \( \Sigma_2 \) is balanced.

Now we prove that \( \Sigma_1 \) is also balanced. Let \( \Gamma = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^m}, \Phi) \). Then \( \Phi \) is consists of all the members of \( \{(1, v)|v is odd\} \) and since \( \{(1, v)|v is odd\} \subseteq V(\Sigma_1) \) Thus \( \Sigma_1 \) is all-positive. Clearly in this case \( \Sigma_1 \) is balanced. Assume that \( \Gamma = \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_{2^{a_1}p_{a_2} \cdots p_{a_k}}, \Phi) \). Consider the arbitrary cycle \( C' = (x'_1, x'_2, \ldots, x'_n, x'_1) \). The edges in \( \Sigma_1 \) are such that each edge has a vertex in \( \{(0, v)|v is even\} \) and a vertex in \( \{(1, v)|v is odd\} \). Now without loss of generality assume that \( x'_1 \in \{(0, v)|v is even\} \) hence \( x'_2 \in \{(1, v)|v is odd\} \). Hence we have two following cases.

If \( x'_2 \notin \Phi \) then all edges joint to \( x'_2 \) will be positive. So \( x'_1x'_2 \) and \( x'_2x'_3 \) are positive. But if \( x'_2 \in \Phi \), because \( x'_1 \notin \Phi \), then \( x'_1x'_2 \) is negative. Since \( x'_3 \in \{(0, v)|v is even\} \) hence \( x'_2x'_3 \) is also negative. This implies that after each negative edge in \( C' \) we will have another negative edge. Therefore, the number of negative edges in \( C' \) is even. Thus \( \Sigma_1 \) is balanced.

Next consider the case where \( p_1 \geq 3 \). In this case \( \Sigma \) is connected sigraph. Suppose \( C'' \) is an arbitrary cycle in \( \Sigma \). If all the edges in \( C'' \) are positive then the cycle is positive. Now assume that \( C'' \) contains a negative edge \( (u, v)(u', v') \). Using the proof of Proposition 2.7 and Figure 2 of Example 2.2, \( (u, v)(u', v') \) has a vertex in \( V_1 \) and other vertex in \( V_2 \). Without loss of generality suppose that \( (u, v) \in V_1 \) and \( (u', v') \in V_2 \). Because neither of the vertices of \( V_2 \) is adjacent to each other, \( (u', v') \) must be connected to one of the vertices of \( V_1 \), for example \( (u'', v'') \). Since all the edges that have a vertex in \( V_2 \) are negative, thus \( (u', v')(u'', v'') \) is negative edge. This implies that, in order to form cycle \( C'' \), for each negative edge in \( C'' \), another negative edge is required. We conclude that the number of negative edges in \( C'' \) is even. Therefore \( \Sigma \) is balanced. \( \square \)

**Theorem 2.7.** \[ \] For a sigraph \( S \), its line sigraph \( L(S) \) is balanced if and only if the following conditions hold:
1) for any cycle $Z$ in $S$,
(a) if $Z$ is all-negative, then $Z$ has even length;
(b) if $Z$ is heterogeneous, then $Z$ has even number of negative sections with even length;

2) for $v \in S$, if $d(v) > 2$, then there is at most one negative edge incident at $v$ in $S$.

**Theorem 2.8.** Let $\Sigma = (\Gamma, \sigma)$, where $\Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1^{\alpha_1}}, \Phi)$. Then $L(\Sigma)$ is balanced if and only if one of the prime factors is 2.

**Proof.** **Necessity:** Let $L(\Sigma)$ is balanced. Assume that the conclusion is false. Suppose $p_i \geq 3$ for any $i = 1, 2, \ldots, k$. It is easy to see that vertex $(0,1)$ is adjacent to vertices $(1,0), (2,0), \ldots, (p_i - 1,0)$. By definition $\Phi$, the vertices $(0,1), (1,0), (2,0), \ldots, (p_i - 1,0)$ are not in $\Phi$. Hence $d^-((0,1)) \geq 2$. By the Theorem 2.7 condition (ii) does not hold. Therefore $L(\Sigma)$ is unbalanced. Which is in contradiction with the assumption. Thus one of the prime factors is 2.

**Sufficiency:** Suppose one of the prime factors is 2. By Theorem 2.6, $\Sigma$ is balanced so, according to Lemma 1.5[18] and Theorem 1.6[18], $\Sigma$ is switching equivalent to a graph whose all edges are positive. Hence $\Sigma$ can be considered all-positive. Therefore $L(\Sigma)$ is balanced. □

3. Clustering and sign-compatibility of $\Sigma$

In this chapter two properties namely clusterability and sign-compatibility for $\Sigma$ are investigated.

**Theorem 3.1.** [6] Let $S$ be any signed graph. Then $S$ has a clustering if and only if $S$ contains no cycle having exactly one negative line.

**Theorem 3.2.** [11] A sigraph $S$ is sign-compatible if and only if $S$ does not contain a sub-sigraph that is isomorphic to either of the following two sigraphs formed from the path graph $P_4 = (x,u,v,y)$; $S_1$ with both edges $xu$ and $vy$ negative and the edge $uv$ positive and $S_2$ obtained from $S_1$ by identifying the vertices $x$ and $y$ (see Figure 4).

![Forbidden subsigraphs for a sign-compatible sigraph.](image)
**Theorem 3.3.** Let \( \Sigma = (\Gamma, \sigma) \), where \( \Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \cdots \mathbb{Z}_{p_k}, \Phi) \). Then \( \Sigma \) is clusterable if and only if \( \Sigma \) is balanced.

**Proof.** **Necessity:** Suppose \( \Sigma \) is clusterable. Assume the conclusion is false. Suppose \( \Sigma \) is unbalanced. Using Theorem 2.6, \( p_i \geq 3 \) for any \( i = 1, 2, \ldots, k \). Because \( p_i \geq 3 \) so \( (2, 2) \in \Phi \) and also it is adjacent to vertices \((0, 1)\) and \((1, 0)\). We now consider the cycle \( C = ((0, 1), (2, 2), (1, 0), (0, 1)) \) in \( \Sigma \). By the definition of \( \sigma \), we have \( \sigma((0, 1)(2, 2)) = + \) and \( \sigma((1, 0)(0, 1)) = - \). Hence \( C \) is a cycle with exactly one negative edge. Therefore, according to Theorem 3.1, \( \Sigma \) cannot be clusterable, a contradiction to the hypothesis. Hence one of the prime factors is 2 so \( \Sigma \) is balanced.

**Sufficiency:** Suppose \( \Sigma \) is balanced. Thus all cycles of \( \Sigma \) are positive. So the number of negative edges in them is even. Hence it cannot include the cycle with a single negative edge. Therefore, according to Theorem 3.1, \( \Sigma \) is clusterable. \( \square \)

**Theorem 3.4.** Let \( \Sigma = (\Gamma, \sigma) \), where \( \Gamma = \text{Cay}(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \cdots \mathbb{Z}_{p_k}, \Phi) \) and \( p_i \geq 2 \) for any \( i = 1, 2, \ldots, k \). Then \( \Sigma \) is sign-compatible.

**Proof.** Assume \( \Sigma \) is sign-incompatible. Then by Theorem 3.2, \( \Sigma \) contains a subgraph isomorphic to either \( S_1 \) or \( S_2 \). Let \( P = ((u, v), (u', v'), (u'', v''), (u''', v''')) \) is a path in \( \Sigma \) such that \( \sigma((u, v)(u', v')) = \sigma((u'', v'')(u''', v''')) = - \) and \( \sigma((u', v')(u'', v''')) = + \). According to the definition of \( \sigma \), since \((u', v')(u'', v''')\) is positive edge hence two cases may occur.

If \((u', v')\) and \((u'', v'')\) belong to \( \Phi \) then all the edges on \( P \) are positive, a contradiction to hypothesis. Now if one of the vertices \((u', v')\) and \((u'', v'')\) belong to \( \Phi \). Without loss of generality assume that \((u', v') \in \Phi \) and \((u'', v'') \notin \Phi \). Then we have \( \sigma((u, v)(u', v')) = ((u', v')(u'', v''')) = + \) and \( \sigma((u'', v'')(u''', v''')) = - \). Which is contradiction the hypothesis. Thus \( \Sigma \) does not contain a subgraph isomorphic to \( S_1 \).

Now let \( \Sigma \) contains a subgraph isomorphic to \( S_2 \). As proved for \( S_1 \), according to the definition of \( \sigma \), no positive edge can be placed between two negative edges. Hence \( \Sigma \) does not contain a subgraph isomorphic to \( S_2 \). Therefore \( \Sigma \) is sign-compatible. \( \square \)

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