Turbulence and acoustic waves in compressible flows

J. Cerretani
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad universitaria, 1428 Buenos Aires, Argentina.

P. Dmitruk* 
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires and IFIBA, CONICET, Ciudad universitaria, 1428 Buenos Aires, Argentina.

October 29, 2018

Abstract

In this work, direct numerical simulations of the compressible fluid equations in turbulent regimes are performed. The behavior of the flow is either dominated by purely turbulent phenomena or by the generation of sound waves in it. Previous studies suggest that three different types of turbulence may happen at the low Mach number limit in polytropic flows: Nearly incompressible (NI), modally equipartitioned compressible (MEC) and compressible wave (CW). The distinction between these types of turbulence is investigated here applying different kinds of forcing. Scaling of density fluctuations with Mach number, comparison of the ratio of transverse velocity fluctuations to longitudinal fluctuations and spectral decomposition of fluctuations are used to distinguish the nature of these solutions. From the study of the spatio-temporal spectra and correlation times we quantify the contribution of the waves to the total energy of the system. Also, in the dynamics of a compressible flow, three associated correlation times are considered: the non-linear time of local interaction between scales, the sweeping time or non-local time of large scales on small scales, and the time associated with acoustic waves (sound). We observed that different correlation times dominate depending on the wave number ($k$), the Mach number and the type of forcing.

1 Introduction

The theory of compressible turbulent flows has not been developed as much as incompressible turbulence flow models. For this reason, the theoretical work focused almost exclusively on the study of incompressible turbulence (1; 2; 3; 4). One of the aspects of interest that motivates us to study turbulence in compressible flows are density fluctuations, naturally absent in the incompressible case.

Compressible flows can be classified in several ways, the most common uses the Mach number (M) as a parameter. The problems of incompressible flows are governed only by two unknowns, these are the pressure and velocity that are obtained from the equations that describe the conservation of the mass and the linear momentum, where the density of the fluid is constant. However, in the compress-
The equations of motion for low mach number flows have been recently described (9; 10), with an equation of state, generalizing the equations of Klainerman and Majda (11; 12). In these formalisms the flow becomes almost incompressible, since the solutions expressed as powers of the Mach number are solutions of the incompressible equations.

The works of Klainerman, Majda (11; 13) and Kreiss (14), allow us to understand how the solutions of compressible flow equations approach, for small mach values, the solutions of incompressible flows. In all cases the pressure variations must scale as the Mach number squared. In this paper, we will refer to the models of Klainerman and Majda (11; 12) as theory of nearly incompressible turbulence (NI). For weakly compressible flows (Mach numbers less than unity), the theoretical model of nearly-incompressible flows (NI) (10; 15) has not been studied numerically for a three-dimensional system. In particular, the analysis of the scaling of the fluctuations of density with the Mach number is of interest for the characterization of the flow.

Turbulence studies (16) for Mach numbers less than about 0.3 infer that one consequence of the NI turbulence theory is that many of the incompressible turbulence characteristics survive in low Mach number flows. Furthermore, simulations of compressible MHD flows showed similarities, especially for low Mach numbers (17; 18; 19; 20; 21; 15), with the physics of compressible and incompressible flows.

When the nearly incompressible asymptotic expansion NI breaks down, the flow enters in a new regime (13; 16). The theoretical work on this new regime of density and velocity fluctuations is described by the Kraichnan (22) model. The compressible hydrodynamic turbulence model of Kraichnan assumes the existence of strong acoustic waves that are energetically equipartitioned with vorticity movements in each scale. We will refer to this description as modally equipartitioned compressible (MEC) turbulence. Kraichnan arrives at a different scaling of density and velocity fluctuations with the Mach number. As seen, in the turbulence model NI, the density fluctuations scale quadratically with the Mach number. On the other hand, in the MEC turbulence model, the flow, with larger fluctuations of the initial density, become strongly compressible and the fluctuations of the density scale linearly with the Mach number. (23).

It is also possible to consider a third type of flow, in which acoustic waves are favored over vortical movements. This third possibility to describe compressible turbulence, which we call compressible wave turbulence (CW), is dominated by acoustic waves and longitudinal velocity fluctuations.

2 Hydrodynamic equations

We consider mass balance (continuity equation) and momentum balance (Navier-Stokes equations) for a compressible flow, with a polytropic state equation. In dimensionless units the equations are:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$ (1)

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \mathbf{\omega} - \frac{1}{2} \nabla \left( \mathbf{u} \cdot \mathbf{u} + \frac{2}{M^2(\gamma - 1)} \rho^{\gamma-1} \right)$$

$$\quad - \frac{1}{\rho} \nu_c \nabla^2 \mathbf{u} + \frac{1}{\rho} \left( \xi_c + \frac{1}{3} \nu_c \right) \nabla (\nabla \cdot \mathbf{u}) + \mathbf{f}$$ (2)

$$p = \rho^\gamma$$ (3)

Here the dimensional units are $t_0$ for time $t$, $u_0$ for velocity $u$, $\rho_0$ for density $\rho$ and $p_0$ for pressure $p$. Length scale unit $l_0 = u_0 t_0$ is used for gradients. Dimensionless viscosities $\nu_c = \xi_c$ are considered as inverse of Reynolds number $Re_c = u_0 l_0 / \nu$, where $\nu$
is the viscosity. Also \( \omega = \nabla \times \mathbf{u} \) is the vorticity and \( \mathbf{f} \) the field of external forces.

In the polytropic state equation we assume \( \gamma = 5/3 \), which corresponds to the isentropic flow for an ideal gas (24). Finally, the nominal Mach number is \( M = u_0/C_s \), with \( C_s \) the speed of sound.

### 3 Energy spectrum, frequency spectrum and correlation functions

The wave number associated with scales of size \( l \) is defined as \( k = \frac{2\pi}{l} \). We can transform the velocity field \( \mathbf{u} \) from the physical space to the Fourier space \( \mathbf{u}_k \). Through this transformation it is possible to perform calculations on the energy of the system expressing the distribution of energy between the vortices in terms of their spectrum. The spectrum \( E(k) = \frac{1}{2} |\mathbf{u}_k|^2 \) gives us the energy associated with each wave number regardless of its spatial distribution.

We also consider the wavenumber-frequency spectrum \( E_{ij}(k, \omega) \), defined as (25):

\[
E_{ij}(k, \omega) = \frac{1}{2} \langle \mathbf{u}_i^*(k, \omega) \mathbf{u}_j(k, \omega) \rangle
\]

where \( \mathbf{u}_i(k, \omega) \) is the Fourier transform in time and space of the \( i \)-component of the velocity field \( \mathbf{u}(x, t) \), and where the asterisk indicates the complex conjugate.

The information of the time scales for each spatial mode and decorrelation times can be obtained from the temporal correlation function, given by the following relation (1):

\[
\Gamma_{ij}(k, \tau) = \frac{\langle \hat{\mathbf{u}}_i^*(k, \tau) \hat{\mathbf{u}}_j(k, t + \tau) \rangle}{\langle |\mathbf{u}_i(k, \tau) \mathbf{u}_j(k, t)| \rangle}
\]

where \( \hat{\mathbf{u}}_i(k, \tau) \) is the Fourier transform of the \( i \)-component of the velocity field, the brackets indicate the time average, and only the real part is used. The correlation time \( \tau_D(k) \) is defined as the time at which the \( \Gamma \) function falls to \( 1/e \) from its initial value.

Although this definition is arbitrary, there are no quantitative differences between these different definitions except for a multiplicative factor of order one in the values of the correlation times.

### 4 Characteristic times

From the equations of motion and arguments of scale, it is possible to estimate different characteristic times of interest for our system. The turnover time of an eddy can be defined as \( \tau_{nl} \sim [ku(k)]^{-1} \) where \( k \) is the wave number and \( u(k) \) is the amplitude of the velocity due to fluctuations at the scale \( \sim 1/k \). It is the typical time in which a structure of size \( \sim l \) suffers significant distortion due to the relative movement of its components. For the velocity scale \( u \sim u_{rms} (kL)^{-1/3} \), the nonlinear time scales in the inertial range, for a Kolmogorov prediction, can be written approximately as:

\[
t_{nl} = C_{nl} \left[ u_{rms} L^{-1/3} \left( \sqrt{k_+^2 + k_-^2} \right)^{2/3} \right]^{-1}
\]

where \( C_{nl} \) is a dimensionless constant of unit order and \( u_{rms} \) is defined as:

\[
u_{rms} = \left( \langle |\mathbf{u}|^2 \rangle \right)^{1/2}
\]

It is reasonable then that \( \tau_{nl} \) is also the typical time of energy transfer of scales \( \sim l \) to smaller scales.

The physics of the temporal correlation depends on other time scales, such as the characteristic sweep time on the scale \( \sim 1/k \), which can be expressed as:

\[
t_{sw} = C_{sw} \left( u_{rms} \sqrt{k_+^2 + k_-^2} \right)^{-1}
\]

This time corresponds to the advection of small-scale structures by large-scale flow. Where \( C_{sw} \) is another dimensionless constant of unit order.

Finally, the non-linear time of large scales \( \tau_{NLG} \) independent of \( k \), which should be distinguished from the non-linear time \( \tau_{nl} \), will be used to estimate the time scale of energy transfer between scales. This time is defined as:

\[
\tau_{NLG} = \frac{l_f}{u_{rms}}
\]

where \( l_f \) is the energy injection scale through the forcing.
5 Theoretical Background

In this section, we present the theoretical background of turbulent compressive flows of low Mach number with respect to the formation of sound waves and approximations to near incompressibility.

Klainerman and Majda postulate a set of dynamic equations that allows to describe flows of low Mach number. These equations are formed by incompressible hydrodynamic equations, a non-propagational density fluctuation field and other velocity and density fluctuations that obey acoustic equations. Since the dominant part of the asymptotic solution as the Mach number tends to zero, under a specific order of deficit and velocity fluctuations, are identical to the solutions of the pure incompressible equations, these equations are called: “nearly incompressible” (NI) hydrodynamics equations.

To obtain the solutions of nearly incompressible flow, the equations of compressible flows must be expanded in powers of Mach number $(M)$. In this way we can separate the slow convective movements from the acoustics, treating the convective quantities as $O(1)$. A condition that arises from this model is that the pressure variations must be of order $O(M^2)$, this is due to the longitudinal velocity fluctuations $(\nabla \cdot u = 0)$ and density variations $(\delta \rho = O(M))$ would violate the required pressure scale.

Then, in the NI model the fluctuations, in units of convective velocity $|u| = O(1)$, must be $U_T \equiv |\nabla \times u| = O(1)$ and $U_L \equiv |\nabla \cdot u| = O(M^2)$. Also, due to the polytropic relationship, a constant average density $\rho = O(1)$ is required, and the fluctuations must be $\delta \rho = O(M^2)$.

The NI model depends on assuming an almost incompressible medium. If we relax this condition and allows $\nabla \cdot u = O(1)$, then the medium can allow the formation of acoustic waves and vorticity movements with the same intensity. This situation is addressed by the Kraichnan (22) model where $U_L$ and $U_T$ are both of order $O(1)$ and the density fluctuations of order $O(M)$. We will refer to this type of turbulence as modally equipartitioned compressible (MEC) turbulence.

Since even with very low Mach numbers a flow with zero vorticity can be initiated, i.e. pure longitudinal acoustic movements, it is possible to consider a third possibility to describe the compressible turbulence. In this third possibility, which we call compressible wave turbulence (CW), acoustic waves are favored over vorticity movements. This is an alternative to the nearly incompressible model (NI), which favors vorticity movements, or to the modally equipartitioned compressible (MEC) turbulence model, which treats the two types of movements with the same intensity.

For this model there are no theoretical basis on the scaling of $U_L$ and $U_T$ nor for $\delta \rho$. However, previous computational work by Ghosh and Matthaeus (23) in 2-D domains of 64 x 64 showed that $\delta \rho = O(M)$. This order suggests that the CW turbulence obeys the MEC turbulence density scales. On the other hand for most of the reported simulation it was observed that $U_L$ is above $U_T$ suggesting that a smaller amount of vortical energy has been created.

6 Numerical simulations

We carried out 15 direct numerical simulations runs of the compressible fluid equations in turbulent regimes, in periodic cubic domains with a linear spatial resolution of $256^3$ grid points. The equations 1 and 2 were solved using a pseudospectral method, and evolved in time with a second order Runge-Kutta scheme.

A random forcing was used. This forcing has a parameter that allows us to determine its compressibility, so we can go from a pure incompressible forced system to a pure compressible forcing controlling this factor. The forcing is given by the equation:

$$\mathbf{F}_k = i \mathbf{k} \times \hat{\psi}_1 - i \mathbf{k} \psi_2$$

where

$$\psi_1 = (1 - a) f_1(k)$$
$$\psi_2 = a f_2(k)$$

Here $f_i(k)$ is a function that is 1 if $k$ is within the forced wavenumber range and is 0 out of that range.
By modifying the value of the parameter a the fluid can be compressibly excited \((a = 1)\), generating thrusts parallel to the direction of movement and, alternatively, the incompressible modes are generated using \((a = 0)\) pushing in the direction perpendicular to the direction of fluid movement. In addition, a temporal correlation is introduced that allows us to adjust the correlation time of the forcing. The forcing is applied for wave numbers between \(k_{\text{min}} = 0.9\) and \(k_{\text{max}} = 1.8\).

For the runs, random initial conditions were used for the velocity field and null density fluctuations in the whole space. A random forcing described by the equation 10 was used. By means of this forcing we carry out three groups of simulations: one with purely compressible forcing, another with purely incompressible forcing and finally a mixed forcing that injects energy in the same amount to compressible and incompressible modes. In addition, for each of these three groups, five simulations were performed for different values of the Mach number.

The amplitude of the force remained almost constant for all simulations \((f_o \sim 0.1)\) so that the energy converges to values of order \(O(1)\). The correlation frequency of the forcing (inverse of the correlation time) was set to \(0.5\).

The viscosity of the system (inverse of the nominal Reynolds number) was set to \(\nu_c = \xi_c = 0.001\) in such a way that the dissipation scale is within the resolution range of all the simulations. The actual Reynolds number depends on the rms value of the velocity for each run.

The values of the Mach number used are in the range \([0.15, 0.55]\).

In the table 1 the parameters of the simulations are detailed.

### Results

The figure 1 shows the evolution of kinetic energy, defined as \(E_c(t) = \langle |u|^2 \rangle_x\), for the simulations specified in the table 1.

The energy decays initially until \(t \sim 20\), when a statistically steady state is reached with \(E_c \sim 0.2\). At this point, the cascade of energy excite the dissipation scales and the system begins to dissipate energy significantly.

![Figure 1: Kinetic energy as a function of time (simulations of the table 1).](image)

Table 1: Simulation data with 256^3 grid points. The notation I, M, and C refer to the type of forcing (I = incompressible forcing - M = mixed forcing - C = compressible forcing). The average Reynolds number value \((R_e)\) that the simulation reaches in its steady state is indicated. It also indicated the dissipation rate \(\epsilon\) and dissipation scale \(k_\eta\). The value of the Mach number (M) used in each simulation is specified. \(\Delta T_{nl} = T/\tau_{NLG}\) is the total number of non-linear times considered in each run.

| run | \(R_e\) | \(\epsilon\) | \(k_\eta\) | M | \(\tau_{NLG}\) | \(\Delta T_{nl}\) |
|-----|--------|---------|--------|----|--------------|------------|
| I1  | 528    | 0.015   | 62     | 0.15 | 1.9          | 42         |
| I2  | 522    | 0.015   | 62     | 0.25 | 1.9          | 42         |
| I3  | 535    | 0.016   | 63     | 0.35 | 1.9          | 43         |
| I4  | 533    | 0.015   | 62     | 0.45 | 1.9          | 43         |
| I5  | 521    | 0.014   | 61     | 0.55 | 1.9          | 42         |
| M1  | 408    | 0.006   | 49     | 0.15 | 2.5          | 33         |
| M2  | 399    | 0.005   | 45     | 0.25 | 2.7          | 30         |
| M3  | 397    | 0.005   | 47     | 0.35 | 2.5          | 32         |
| M4  | 397    | 0.004   | 45     | 0.45 | 2.5          | 32         |
| M5  | 420    | 0.005   | 47     | 0.55 | 2.4          | 34         |
| C1  | 268    | 3.6E-5  | 14     | 0.15 | 3.7          | 21         |
| C2  | 330    | 5.2E-5  | 15     | 0.25 | 3.0          | 26         |
| C3  | 329    | 1.2E-4  | 18     | 0.35 | 3.0          | 26         |
| C4  | 280    | 8.3E-5  | 17     | 0.45 | 3.6          | 22         |
| C5  | 189    | 3.0E-5  | 13     | 0.55 | 5.3          | 15         |
allows us to infer, through an early analysis, the existence of waves in the fluid and that this is strongly dependent on the Mach number and the applied force.

To determine the stationary range, the system was allowed to evolve until the enstrophy is stabilized around an asymptotic value (see Fig: 2).

Figure 2: Enstrophy as a function of time (simulations of the table 1).

We can observe in the simulations $C_i$ (compressible forcing) that the enstrophy is practically null when the steady state is reached.

Analysis of spatio-temporal spectra and correlation times were made in a time window $I_t = [t_0, t_f] = [28, 108]$. This corresponds to between 15 and 43 non-linear times.

### 7.1 Spectral analysis

Before proceeding to the analysis of the wave number and frequency spectrum, and to the study of the correlation time for each mode, we present the spectra in the space of wave numbers, as is usually done to characterize the turbulent flows. In addition to being useful for characterizing simulations, these spectra will also be important to identify the behavior of different modes depending on what dynamical times are expected to be dominant.

We analyze the energy spectra $E(k)$ for the different types of forcing. This spectrum is compared with the law of scales, $k^{-5/3}$, predicted by the Kolmogorov theory for incompressible, isotropic and homogeneous turbulent flows (26; 27). A fit is made on the spectrum $E(k)$ and the power law is determined in each case. To find the characteristic exponent, a linear adjustment of minimum squares is applied to the data in a logarithmic scale (restricted to the wave numbers belonging to the inertial range).

In the figure 3 the energy spectra for the simulations with incompressible, mixed and compressible forcing, for a Mach value of 0.25, are shown.

![Figure 3: Energy spectra for simulations with Mach 0.25 averaged in the time window $I_t$. Above the spectra, the slopes corresponding to Kolmogorov law and those corresponding to the linear adjustment are shown.](image)

Although the Kolmogorov law is only valid for incompressible flows, the simulations with low Mach number and incompressible forcing show an exponent $\alpha = -1.66 \pm 0.06$ (for the simulation I2) compatible with the prediction of Kolmogorov $k^{-5/3}$ for isotropic and homogeneous turbulence. In addition, the spectrum presents a similar profile for all Mach number values. The rest of the Mach number values used presented a similar behavior.

The linear adjustment for the compressive forcing simulation C2 returns a value of $\alpha = -2.26 \pm 0.03$. These spectra present a more pronounced decay compared to Kolmogorov law. This should be expected since the energy cascade has an additional channel from eddies to acoustic modes that are then damped, as the model of Zank and Matthaeus predicts.

These spectra also show a greater preponderance of the large scales compared to the small scales, a result that is consistent with the low values of en-
strophy reported for these simulations in the previous section and with results of structure formation that we will see later.

Also, all the spectra present a similar profile for all values of the Mach number. These simulations reflect that, at low Mach numbers, the power law of the spectra does not present a remarkable dependence with the value of the Mach number. This situation could become different for Mach values greater than unity, where we move away from nearly incompressibility.

### 7.2 Axisymmetric spectrum

To continue with the analysis of the energy spectral distribution, the level curves of the axisymmetric spectra of kinetic energy are presented below as a function of $k_x$ and $k_y$, defined by the equation:

$$
e(k_x, k_y) = \left\langle u^2(k_x, k_y) \right\rangle_t$$  \hspace{1cm} (12)

In figure 4 the axisymmetric spectra are presented for the simulations with Mach 0.25 and for the different types of forcing.

The logarithmic grayscale indicates the intensity of $e(k_x, k_y)$ in each point of the Fourier space.

For the I2 simulation it is possible to observe a semicircle profile of the contour lines, consistent with previous simulations for incompressible, isotropic and homogeneous flows.

On the other hand, for the simulations C2, and to a lesser extent for M2, the spectra show higher energy near the points with $k_x = k_y$. Especially for Mach values less than 0.3 the axisymmetric spectra show a remarkable deformation around the line $k_x = k_y$.

These effects have not been observed in simulations or previous experiments under conditions of near incompressibility. The reason for the increase in energy in the axisymmetric spectra near the points with $k_x = k_y$ requires a more detailed analysis that we plan to study in future work.

This behavior is then reflected in the formation of a number of significant large-scale structures, absent in the rest of the runs.

Also in figure 5 for the axisymmetric spectra, we can see the effect of changing the Mach number for all the simulations with compressible forcing.

---

\(a\) I2

\(b\) M2

\(c\) C2

Figure 4: Axisymmetric spectra averaged over the time window \([t_i, t_f] = [28, 108]\). The logarithmic grayscale indicates the spectral amplitude at each point of the plane $k_x, k_y$. 

7.3 Spatial-temporal spectra

In this section we analyze the spatio-temporal spectra of the velocity fields $u_x$ and $u_y$, in order to detect the presence (or absence) of sound waves. To perform this type of analysis, the fields must be stored with very high frequency cadence in order to resolve the waves in time and space; in particular, $dt = 5 \cdot 10^{-3}$ was taken as the temporal sampling rate.

Since the spatio-temporal spectra depend on $k = (k_x, k_y, k_z)$ and on the frequency $\omega$, to visualize the spectra more conveniently, some of the components of $k$ are fixed; also, the spectra are normalized for each wave number $k$.

Figure 6 shows the spectra $E_{xx}(k_x, \omega)$ with $k_y = k_z = 0$. In figure 7 we show the spectra $E_{xx}(k_y, \omega)$ with $k_x = k_z = 0$ for Mach number $= 0.25$ and the different forcing types.

The spectra $E_{xx}(k_x, \omega)$, for incompressible forcing have higher energy around the dispersion relation of sound waves only for low $k$, in particular less than 20. For higher values of $k$ the energy is concentrated below the sweeping relation. That is, for simulations with incompressible forcing, for small $k$ the propagation of sound waves dominates, while for high $k$ turbulent effects are dominant. As the
value of the Mach number is increased, the sound waves take on greater “protagonism” and the concentration of energy below the sweeping relation decreases, although it is maintained to some extent.

For compressible forcing, the spectra $E_{xx}(k_x, \omega)$ have more energy around the dispersion relation of the sound and this is true for every value of $k$. In a similar way to the mixed forcing case, in this case even for high $k$ the spectra present higher energy around the dispersion relation of the sound. It is observed that for high $k$ the spectrum presents a wider profile, and tends to focus above the dispersion relation of the sound. In $E_{xx}(k_x, \omega)$ energy is not observed around the sweeping relation.

For the spectra $E_{xx}(k_y, \omega)$, for all runs, the dispersion relation of the sound is absent (the propagation of the sound occurs only in the longitudinal component), so that the system is dominated by pure turbulence. The concentration of energy does not present intensity changes when modifying the Mach number in these spectra.

### 7.4 Correlation times

The relevant information for temporal scales can also be obtained from the temporal decorrelation function. In this section we present the correlation times described by the equation 5. Similar to the previous section, to study the correlation times associated with sound waves we look at $\Gamma_{xx}(k_x, \tau)$ with $k_y = k_z = 0$. To study the rest of turbulence characteristic times we analyze the correlation times from $\Gamma_{xx}(k_y, \tau)$ with $k_x = k_z = 0$. Correlation times are defined as the time when the $\Gamma$ function decays to $1/e$ of its initial value.

![Figure 7: Space-time power spectrum $E_{xx}(k_y, \omega)$. The straight line shows the sweeping relation.](image)

![Figure 8: Correlation times $T_d$ for $\Gamma_{xx}(k_x, \tau)$. The curve (−) shows the characteristic time associated with sound waves, the line (:) is the characteristic time of sweeping.](image)
teristic times \( T d_x(k_y) \) with \( k_x = k_z = 0 \) for Mach 0.25 and the different forcing types.

In \( T d_x(k_x) \), for incompressible forcing, the correlation times show a jump. For low \( k \) the system is dominated by the characteristic times associated with sound waves, while for high \( k \) dominate the characteristic times associated with sweeping. This is consistent with the spatio-temporal spectra analyzed in the previous section, where a similar behavior was observed. For higher Mach values this effect is shielded and the characteristic times of the acoustic waves are dominant, even for large \( k \) values. The values of \( k \) for which this change in behavior occurs appear to be strongly correlated with the Mach number. We assume that this behavior could become an effect of turbulent structure formation.

7.5 Density fluctuations

To corroborate the predictions described by the NI, MEC and CW theories the density fluctuations and its dependence on the Mach number are calculated. Density fluctuations were calculated at different times (after the stabilization time \( t_o \)) and for different values of Mach number (\( M = 0.15 - 0.25 - 0.35 - 0.45 - 0.55 \)).

The values shown, for each value of Mach number, consist of the average of the measurements at each time and the error bars to the dispersion of them.

In addition, an adjustment of type \( \alpha M^\beta \) was made and the exponent \( \beta \) was compared with that predicted by the different theories.

In figures 11, 12 and 13 the density fluctuations are shown as a function of the Mach number for the
simulations with incompressible, mixed and compressible forcing, respectively.

Figure 11: Density fluctuations vs Mach number, for incompressible force.

Figure 12: Density fluctuations vs Mach number, for mixed forcing.

Figure 13: Density fluctuations vs Mach number, for compressible forcing.

The value of the adjustment $\beta = 2.11 \pm 0.09$ for the simulations with incompressible forcing is compatible with the predictions of the NI theory, where it is predicted that density fluctuations must scale with the Mach number squared. Through this analysis we can determine that for nearly incompressible systems with random initial conditions and incompressible forcing, we remain within the assumptions given by the nearly incompressible theory (NI).

The value of the adjustment $\beta = 1.00 \pm 0.04$ for the simulations with mixed forcing is compatible with the predictions of the MEC theory, where density fluctuations must scale with the Mach number. Finally, the density fluctuations for compressible forcing seem to have a linear dependence with the Mach number for $M < 0.5$, then this dependence breaks down. These results, for Mach less than 0.5, are consistent with previous works (23) where it was found, by means of simulations in two-dimensions, that the density fluctuations scale linearly with the Mach number.

7.6 Velocity fluctuations

Similar to the density fluctuations, the NI, MEC, and CW theories have predictions for the ratio of longitudinal and perpendicular velocities. The values of $U_T$, and $U_L$ vs Mach number for the incompressible forcing can be seen in Fig: 14 and 15.

As we can see $U_T$ is always $O(1)$ and is at least one order of magnitude above $U_L$. Using a least square adjustment we determine that $U_L$ scales as $M^{1.87 \pm 0.06}$. This is consistent with the scaling of $\nabla \cdot u \sim U_T U_L \sim M^2$ predicted by the NI theory.
Next we present the velocity fluctuations for mixed forcing in Fig: 16 and 17.

In this case, \( U_T \) is always \( O(1) \) and \( U_L \) scales as \( M^{0.73 \pm 0.05} \). For these simulations, \( U_L \) and \( U_T \) are almost always in the same order.

Finally, we present the velocity fluctuations for compressive forcing in Fig: 18 and 19.

For these simulations we find that both \( U_L \) and \( U_T \) does not present dependence on the Mach number and \( U_L \) always dominates the dynamics of the system, being always at least an order of magnitude above \( U_T \).

### 7.7 Structures

We show a cross section of the velocity field (See Fig: 20) and density field (See Fig: 21) for a value of Mach number of 0.25 and for the different types of forcing.

As we can see, the simulations with compressible forcing generate a number of significant large-scale structures, producing highly localized areas of high density. The velocity field follows a structure similar to that of the density field. We attribute this
to the generation of shock waves by increasing the compressibility of the forcing in the fluid.

Figure 20: Cross section of the velocity field.

On the other hand, for mixed forcing type, some structures can be detected, although these are much less significant than those observed for compressible forcing. For incompressible forcing, the velocity and density field are homogeneous.

Figure 21: Cross section of the density field.

8 Conclusions

The studies between the limits of the near incompressible turbulence and the purely incompressible turbulence are an important step in the advance of the theory towards a complete treatment of the compressible flows.

Using models based on the recent advances in the mathematical structure of low-Mach fluids, we have described three perspectives on the nature of the near incompressible polytropic hydrodynamics.

We observe the generation of sound waves in the fluid, consistent with the known dispersion relation. We also observe different behaviors of the fluid at different scales, being the turbulence dominant in the small length scales (large $k$) and coexisting waves and turbulence at other scales, depending on the type of forcing applied. The concentration of energy around the dispersion relation of acoustic waves increases with the compressibility of the forcing.

For the energy spectra $E(k)$, we find that they are compatible with the Kolmogorov prediction $k^{-5/3}$ for isotropic and homogeneous turbulence when using an incompressible forcing. For mixed and compressible forcing the spectra show a more pronounced energy decay compared to Kolmogorov law. We attribute this to extra dissipation channels. This seems plausible considering that in addition to the energy cascade from the large eddies to the small ones, there is also transfer from eddies to acoustic modes that are then quickly damped. The spectra present a similar profile for all values of the Mach number.

In addition, the axisymmetric spectra of kinetic energy were studied. These spectra show an energy concentration near the points with $k_x = k_y$, which becomes more evident as the compressibility of the forcing increases. These effects have not been observed in simulations or previous experiments under conditions of near incompressibility. The spectral anisotropy by increasing the compressibility of the forcing requires a more detailed analysis that is considered for study in future work. This effect is then reflected in the formation of large-scale structures, producing highly localized areas of high density. We attribute this to the generation of shock waves by increasing the compressibility in the fluid.

The temporal correlation function for each of the runs was studied. It was observed that the correlation times have, for an incompressible force, a change in behavior for high wave numbers ($k$) where the associated sweeping times predominate, while for low $k$ the time associated with acoustic
waves dominates. This change in behavior strongly depends on the Mach number used. We assume that this behavior could become an effect of turbulent structure formation. On the other hand, for mixed and compressible forcing, the acoustic correlation times dominate the system in all the scales.

Finally, to corroborate the predictions described by NI and MEC theories and the antecedents observed in previous CW turbulence studies, density fluctuations and dependence on the Mach number was calculated, as well as the longitudinal and perpendicular velocity fluctuations. We observe consistency with the predictions of the NI, MEC and CW theories according to the type of forcing used, where systems with incompressible forcing seem to obey the scaling of fluctuations of density and velocities of the NI theory, while systems with mixed forcing obey the scaling of the MEC theory. The systems with compressible forcing seem to obey the turbulence CW previous studies for Mach values less than 0.5. The velocity fluctuations seem not to be dependent on the Mach number.

The results of this work provide a certain degree of predictive power, at least within the framework of the specific model we have used. For example, by measuring fluctuations in density, the system can be unequivocally classified into one of the three groups we have described, and then making predictions about the rest of unmeasured magnitudes.

Acknowledgements

The authors acknowledge support from PIP Grant No. 11220150100324CO and UBACYT Grant No. 20020170100508BA.

References

[1] G. K. Batchelor. Theory of homogeneous turbulence. Cambridge U.P., 1970.

[2] R H Kraichnan and D Montgomery. Two-dimensional turbulence. Reports on Progress in Physics, 43(5):547, 1980.

[3] A. S. Monin and A. M. Yaglo. Statistical fluid mechanics: mechanics of turbulence. MIT Press, Cambridge, 2, 1971.

[4] J. R. Herring and J. C. McWilhams. Lecture notes on turbulence. World Scientific, Teaneck, NJ, 1989.

[5] M. J. Lighthill. On sound generated aerodynamically, i. general theory. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 211(1107):564–587, mar 1952.

[6] I. Proudman. The generation of noise by isotropic turbulence. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 214(1116):119–132, aug 1952.

[7] Robert Rubinstein and Ye Zhou. The frequency spectrum of sound radiated by isotropic turbulence. Physics Letters A, 267(5-6):379–383, mar 2000.

[8] L. D. Landau and E. M. Lifshitz. Fluid mechanics. Pergamon, Oxford, 1959.

[9] William H Matthaeus and Michael R Brown. Nearly incompressible magnetohydrodynamics at low mach number. The Physics of fluids, 31(12):3634–3644, 1988.

[10] GP Zank and WH Matthaeus. The equations of nearly incompressible fluids. i. hydrodynamics, turbulence, and waves. Physics of Fluids A: Fluid Dynamics, 3(1):69–82, 1991.

[11] Sergiu Klainerman and Andrew Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. Communications on Pure and Applied Mathematics, 34(4):481–524, jul 1981.

[12] Sergiu Klainerman and Andrew Majda. Compressible and incompressible fluids. Communications on Pure and Applied Mathematics, 35(5):629–651, sep 1982.

[13] A. Majda. Compressible fluid flow and systems of conservation laws in several space variables. Springer-Verlag, New York, 1984.

[14] Heinz-Otto Kreiss. Problems with different time scales for partial differential equations. Communications on Pure and Applied Mathematics, 33(3):399–439, 1980.
[15] S Ghosh and WH Matthaeus. Relaxation processes in a turbulent compressible magnetofluid. *Physics of Fluids B: Plasma Physics*, 2(7):1520–1534, 1990.

[16] Thierry Passot and Annick Pouquet. Numerical simulation of compressible homogeneous flows in the turbulent regime. *Journal of Fluid Mechanics*, 181:441–466, 1987.

[17] John V Shebalin and David Montgomery. Turbulent magnetohydrodynamic density fluctuations. *Journal of plasma physics*, 39(2):339–367, 1988.

[18] RB Dahlburg, JP Dahlburg, and JT Mariska. Helical magnetohydrodynamic turbulence and the coronal heating problem. *Astronomy and Astrophysics*, 198:300–310, 1988.

[19] RB Dahlburg, JM Picone, and JT Karpen. Growth of correlation in compressible two-dimensional magnetofluid turbulence. *Journal of Geophysical Research: Space Physics*, 93(A4):2527–2532, 1988.

[20] RB Dahlburg and JM Picone. Evolution of the orszag–tang vortex system in a compressible medium. i. initial average subsonic flow. *Physics of Fluids B: Plasma Physics*, 1(11):2153–2171, 1989.

[21] M Scholer, T Terasawa, and F Jamitzky. Reconnection and fluctuations in compressible mhd: A comparison of different numerical methods. *Computer Physics Communications*, 59(1):175–184, 1990.

[22] Robert H. Kraichnan. On the statistical mechanics of an adiabatically compressible fluid. *The Journal of the Acoustical Society of America*, 27(3):438–441, may 1955.

[23] S Ghosh and WH Matthaeus. Low mach number two-dimensional hydrodynamic turbulence: Energy budgets and density fluctuations in a polytropic fluid. *Physics of Fluids A: Fluid Dynamics*, 4(1):148–164, 1992.

[24] Chandrasekhar. An introduction to the study of stellar structure. *Courier Corporation*, 2, 1957.

[25] R. Lugones, P. Dmitruk, P. D. Mininni, M. Wan, and W. H. Matthaeus. On the spatio-temporal behavior of magnetohydrodynamic turbulence in a magnetized plasma. *Physics of Plasmas*, 23(11):112304, nov 2016.

[26] Geoffrey Ingram Taylor. Statistical theory of turbulence. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 151(873):421–444, 1935.

[27] Geoffrey Ingram Taylor. The spectrum of turbulence. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 476–490, 1938.