1. INTRODUCTION

Radiation mediated shocks (RMSs) are shocks in which the downstream (DS) energy density is dominated by radiation rather than by particle thermal energy, and in which the fast upstream (US) plasma approaching the shock is decelerated by scattering of photons, generated in the DS and propagating into the US, by the fast US electrons. RMSs are expected to occur in a variety of astrophysical flows. The shock waves propagating through, and expelling, the envelopes of massive stars undergoing core collapse supernova (SN) explosions, are non-relativistic (NR) RMSs (Colgate 1974a, 1974b; Weaver 1976). Relativistic RMSs (RRMS) may play an important role, in, e.g., gamma-ray bursts (GRBs), trans-relativistic supernovae, and pulsar accretion flows.

1. Gamma ray bursts. Within the framework of the collapsar model of GRBs (e.g., Woosley 1993; Zhang et al. 2003), a highly relativistic jet driven by the collapsed core of a massive star penetrates through the stellar envelope. The shock that decelerates the jet is expected to be a highly RRMS (Mészáros & Waxman 2001; Aloy et al. 2000).

2. Trans-relativistic SNe. Several recent SN events, that were identified in very early stages of the explosion, have been shown to deposit a significant fraction, \( \sim 1\% \), of the explosion energy in mildly relativistic, \( \gamma \beta \gtrsim 1 \), ejecta (Soderberg et al. 2006, and references therein). The existence of mildly relativistic ejecta components suggests that a mildly RRMS shock traversed the outer envelope of the progenitor.

3. Pulsar accretion flows. Accretion onto the polar cap of a pulsar is expected to produce a mildly RRMS which is approximately stationary in the neutron star frame (Burnard et al. 1991; Becker 1988, and references therein).

NR RMSs were studied in detail in (Weaver 1976), describing photon propagation using the diffusion approximation and describing the radiation field using two parameters, photon effective temperature and density. These approximations hold for slow shocks, \( v/c < 0.2 \), for which relativistic effects are negligible and the Thomson optical depth of the shock deceleration region is large, \( \Delta \sim \Gamma_0^2 (\Delta t \sim 1 \text{ neglecting the contribution of pairs}) \) due to Klein–Nishina suppression of the scattering cross section. A high-energy photon component, narrowly beamed in the DS direction, with a nearly flat power-law-like spectrum, \( \nu I_\nu \propto v^\phi \), and an energy cutoff at \( \sim \Gamma_0^2 m_e c^2 \) carries a fair fraction of the energy flux at the end of the deceleration region. An approximate analytic model of RRMS, reproducing the main features of the numerical results, is provided.

Key words: gamma-ray burst: general – radiation mechanisms: non-thermal – shock waves

Online-only material: color figures
the breaking out of SN shocks from the stellar envelopes of Blue Super Giants and Wolf-Rayet stars, which reach \( v/c > 0.1 \) for reasonable stellar parameters, may include a hard component with photon energies reaching tens or even hundreds of keV. This may account for the X-ray outburst associated with SN2008D (Soderberg et al. 2008), and possibly for other SN-associated outbursts with spectra not extending beyond few 100 keV (e.g., XRF060218/SN2006aj; Campana et al. 2006).

In this paper, we derive exact numerical solutions for the steady state structure of RRMS, propagating into a cold US plasma of protons and electrons, for shock Lorentz factors \( \lesssim 30 \) and US proper densities \( \ll 10^{25} \text{ cm}^{-3} \). The solutions are obtained using a novel iteration method for self-consistently solving the energy, momentum, and particle conservation equations along with the equation of radiation transport. We assume that the electrons, positrons, and protons may be described as a fluid, along with the equation of radiation transport. We assume that the electrons, positrons, and protons are cold. The validity of these assumptions is discussed in detail in Section 2.3. The radiation mechanisms that are taken into account include Compton scattering, pair production and annihilation, and bremsstrahlung emission and absorption. Other radiation mechanisms, e.g., double Compton scattering, are shown to have a minor effect on our results.

The paper is organized as follows. In Section 2, we review the physics of RMS, analyze qualitatively the shock structure, and motivate the main assumptions. In Section 3, we write down the conservation and transport equations that are numerically solved, in physical and dimensionless forms. In Section 4, we present the numerical iteration scheme used to obtain the solutions and apply it to several test cases. In Section 5, we present the numerical solutions of the shock structure and spectrum. In Section 6, we give a simple analytic description of the structure of the shock, which reproduces the main results of the numerical calculations. In Section 7 we present, for completeness, a preliminary detailed numerical solution of an NR RMS, and compare it with previously known results. In Section 8, we summarize the main results and discuss their implications.

Throughout this paper, the subscripts \( u \) and \( d \) are used to denote US and DS values, respectively. The term “shock frame” refers to the frame at which the shock is at rest (and in which the flow is stationary), and the term “rest frame” refers to the local rest frame of the plasma, i.e., the frame at which the plasma is (locally) at rest. \( n \) stands for number density of a species of particles, and if not mentioned otherwise refers to protons. A summary of the notations repeatedly used in this paper appears in Appendix A.

2. THE PHYSICS OF RMS

In this section, we discuss the physics of RMS. In Section 2.1, we define RMS, write down the global requirements that must be satisfied by a physical system in order to allow the formation of RMS, and derive the asymptotic DS conditions. We then give a short summary of NR RMS in Section 2.2, writing down the assumptions, describing the physical mechanisms at play, and quoting order of magnitude estimates for the shock width and temperature. The results of Section 2.2 may be found in earlier papers (Zel’dovich & Raizer 1966; Weaver 1976; KBW10). The physics of RRMS is discussed in Section 2.3. We highlight the main differences between the relativistic and the NR cases, and describe the assumptions under which the analysis of subsequent sections is carried out.

2.1. Introduction to RMS

2.1.1. Radiation Domination

Consider a steady state shock traveling with velocity \( c\beta_u \) through an infinitely thick, cold plasma of protons and electrons, with US rest-frame density \( n_u \). The thermal and radiation pressures in the asymptotic far DS, which are determined by conservation laws and thermal equilibrium, are given by \( 2\pi a_BT_d^2/3 \), respectively, where \( n_d \) and \( T_d \) are the far DS proton density and temperature, and \( a_{BB} = \pi^2/15(\hbar c)^3 \) is the Stefan–Boltzmann energy density coefficient. The radiation pressure grows much faster than the thermal pressure as a function of \( \beta_u \), and at high enough \( \beta_u \) the DS pressure is dominated by the radiation,

\[
\beta_u \gg \left( n_u/a_{BB} \right)^{1/6} \left( m_p c^2 \right)^{1/2} \approx 3 \times 10^{-4} \left( \frac{n_u}{10^{20} \text{ cm}^{-3}} \right)^{1/6}.
\]

(1)

A shock satisfying this condition is hence radiation dominated.

2.1.2. Global Requirements from a System Through which an RMS Propagates

In order to sustain a quasi-steady state RMS, the system which the shock traverses has to be larger than the shock width. The width of the deceleration region, \( L_{\text{dec}} \), is \( \beta_u^{-1} \) Thomson optical depths for NR shocks (see Section 2.2) and, as we show in this paper, is \( \sim 1 \) Thompson optical depths for relativistic shocks. Hence, systems which RMS traverse much satisfy

\[
L \gg L_{\text{dec}} = (\sigma_T n \beta)^{-1},
\]

(2)

where \( L \) is the size of the system, \( n \) is the proton density, and \( \beta \) is the shock velocity. The minimum total energy and mass of such systems are

\[
E \sim \frac{\beta^2}{2} m_p c^2 n L^3 > \frac{m_p c^2}{2\sigma_T^2 n^3} \approx 3 \times 10^{29} n_{20}^{-2} \beta^{-1} \text{ erg}
\]

(3)

and

\[
M > L^3 n m_p = \frac{m_p}{\sigma_T^2 n^3 \beta^3} \approx 7 \times 10^{50} \left( \frac{n}{\text{ cm}^{-3}} \right)^{-2} \left( \frac{\beta}{0.2} \right)^{-3} \text{ g}
\]

(4)

respectively, where \( n = 10^{20} \text{ cm}^{-3} \). For example, Equation (3) implies that at typical interstellar medium densities, \( n \ll 10^4 \), a solar mass rest energy cannot drive an RMS. In such cases, the shock would be mediated by other mechanism, e.g., collective plasma processes.

2.1.3. Far DS Conditions

In the far DS, which is in thermal equilibrium, the conditions are completely determined by conservation of energy, momentum, and particle fluxes. In the NR and ultrarelativistic limits, the solution reduces to the expressions (e.g., KBW10)

\[
T_d \approx \left( \frac{21 n_u \beta_u^2 m_p c^2}{8 a_{BB}} \right)^{1/4} \approx 0.41 n_u^{1/4} \beta_u^{1/2} \text{ keV},
\]

(5)

and

\[
T_d \approx \left( \frac{2I_u^2 n_u m_p c^2}{a_{BB}} \right)^{1/4} \approx 0.385 T_u^{1/4} n_u^{1/4} \text{ keV},
\]
thermal equilibrium is approximately maintained throughout the shock. The shape of the temperature profile is largely determined by the photon production in one diffusion length into the DS (the first $\beta_d^{-1}$ optical depths of the DS region, henceforth the immediate DS). In this region, the photons mediating the shock are produced. If a photon density of $\sim \alpha_{BB} T_d^2 / 3$ is produced ($L_T \lesssim L_{\text{diff}}$), the flow will stay close to thermal equilibrium, and the temperature profile, which can be extracted directly from an analytic pressure profile, essentially follows the velocity profile. Otherwise, when $L_T \gg L_{\text{diff}}$, the velocity transition of the shock ends without reaching the far DS equilibrium temperature. The radiation pressure $n_{\gamma,\text{eff}} T$ reaches its DS value as soon as the velocity is close to the DS velocity. The immediate DS temperature for NR RMS (KBW10; for more exact numerical values see Weaver1976) is roughly given by

$$\beta_u = \frac{7 \sqrt{3}}{10} \left( \frac{1}{2} \alpha_e \Lambda_{\text{eff}} g_{\text{eff}} \right) \left( \frac{m_e}{m_p} \right)^{1/4} \left( \frac{T_s}{m_e c^2} \right)^{1/8} \approx 0.2 \Lambda_{\text{eff}}^{1/4} \left( \frac{g_{\text{eff}}}{2} \right)^{1/4} \left( \frac{T_s}{10 \text{ keV}} \right)^{1/8},$$

where $\Lambda_{\text{eff}} \approx 10 \Lambda_{\text{eff}}$. DS of this region, the density of photons $n_{\gamma,\text{eff}}$ grows with distance as more and more photons are being generated and advected with the flow and saturates at the equilibrium black body photon density $\sim \alpha_{BB} T_d^2 / 3$. Accordingly, $T$ is decreasing throughout the DS. The thermalization length is roughly given by (KBW10)

$$L_T \beta_u n_{e} \sigma_T \sim \frac{1}{100 \alpha_e \Lambda_{\text{eff}} g_{\text{eff}}} \sqrt{m_e c^2 T_d m_p c^2},$$

where $\beta_d \approx 1/3$, respectively, where $n_d = 10^{15} n_{d,15} \text{ cm}^{-3}$ and $\Gamma_v = (1 - \beta_d^2)^{-1/2}$ is the US Lorentz factor. Note that the condition for a radiation dominated DS, Equation (1), can be obtained by comparing $T_d$ with $\beta_d^2 m_p c^2$.

### 2.2. Non-relativistic RMS

We next focus on NR RMS. By NR shocks, we refer to shocks in which neither the protons nor the electrons move with relativistic bulk or thermal velocities throughout the shock. In particular, this implies that the temperature is always much smaller than $m_e c^2$.

#### 2.2.1. Assumptions

The discussion below of NR RMS is valid under the following assumptions (e.g., Weaver1976). (1) The electron fluid and the ion fluid move together with the same velocity. (2) The pressure is dominated by radiation throughout the shock transition. (3) The optical depth is dominated by Compton scattering implying that the diffusion equation can be used to approximate the spatial transport of the radiation. (4) The Compton $\gamma$ parameter is much larger than 1 throughout the flow, implying that the radiation spectrum is Wien like. (5) The main source of photon production is thermal bremsstrahlung.

#### 2.2.2. Non-relativistic RMS Shock Structure

A schematic cartoon of the velocity and temperature profiles of NR RMS is shown in Figure 1. For NR RMS, the width $L_{\text{dec}}$ of the velocity transition region is comparable to the distance $L_{\text{diff}} \sim (\beta_u n_e \sigma_T)^{-1}$ over which a photon can diffuse against the flow before being advected with the flow. The region of the shock profile over which the temperature changes before it reaches $T_d$ can be extended to distances that are much larger than $L_{\text{diff}}$. For high shock velocities, $\beta_d > 0.07 n_{15}^{1/30} (\Lambda_{\text{eff}} g_{\text{eff}})^{4/15}$, where $g_{\text{eff}}$ is the Gaunt factor, $\Lambda_{\text{eff}} \sim \log[T/(h \nu_{\text{min}})]$, and $\nu_{\text{min}}$ is the lowest frequency of photons emitted by the plasma which may be upscattered to $3T_T$ prior to being absorbed (absorption is dominated by bremsstrahlung self-absorption for far DS values), the length required to produce the DS photon density is much larger than the deceleration scale. For lower shock velocities,
where \( n_e = 10^{19} n_{19} \text{ cm}^{-3} \), is much lower than unity given that \( n_{e}/n_e \) is not very large, see Section 2.3.3.

The second assumption we make regarding the plasma, the existence of an effective temperature, is somewhat more subtle. In principle, the electrons and positrons can have a general distribution function. However, most of the shock is characterized by a strong dominance of radiation energy density over particle thermal energy density. This leads to the electrons and positrons being “held” in momentum space by the radiation, since each scattering changes the energy of the electron considerably, if it departs significantly from the average photon energy. The only way to maintain a very non-thermal electron spectrum is by having a radiation spectrum which is not dominated by a typical photon energy, e.g., a power law. Our numerical results show that the radiation energy density is dominated by photons of limited energy range in the rest frame of the plasma, and that when a high-energy photon tail appears, the photons populating this tail have a very low cross section for interaction with electrons or other photons. This supports the assumption that the energy distribution of electrons and positrons may be characterized by some typical “thermal” energy, greatly simplifying the calculations.

A note is in place here regarding Coulomb collisions. The effective cross section for Coulomb collisions of electrons on protons is \( \sigma \sim e^4/\varepsilon^2 \), where \( \varepsilon_k \) is the electron kinetic energy. When the energy of the electron is of the order of \( m_e c^2 \), the cross section is similar to \( \sigma_T \), the Thomson cross section. This implies that Coulomb collisions in RRMS play a marginal role in equilibrating the motion of particles in the plasma, as the photon density inside the shock is typically of the order of the electron density. Unlike plasma instabilities, this process cannot account for the equilibration of the distribution function of the particles. At low energies, i.e., NR RMS, Coulomb collisions may become dominant (see Weaver 1976) due to a much larger effective cross section.

### 2.3.2. Velocity and Temperature Transition Regions’ Widths

#### Velocity transition

The line of arguments presented in KBW10, leading to the results given in Section 2.2 for estimating the velocity transition width cannot be directly extended to relativistic shocks since Klein–Nishina (KN) corrections to the Compton scattering cross section depend on the priori unknown photon frequency and plasma temperature, which vary throughout the transition region. Note that pair production and relativistic corrections to the cross sections become important already at NR US energies \( \varepsilon = \beta^2 m_e c^2 \sim 2 \times 100 \text{ MeV} \), since the temperature of the plasma within the deceleration region reaches a considerable fraction of \( m_e c^2 \) for this value of \( \varepsilon \). The production of pairs also changes the simple estimate, since it changes both the scatterers’ number density and the shock optical depth. Finally, an additional complication is introduced by the strong dependence of the scattering mean free path on the photon’s direction of propagation, expected due to the relativistic velocity of the plasma.

#### Thermalization length

The thermalization length can be estimated in a way similar to the NR case, since the width of the temperature transition is much larger than the deceleration width (see Equation (7)), and the scale is set by the lowest temperature, i.e., \( T_d \), which is NR. The thermalization can then be estimated in an NR method, leading to Equation (9). Since \( \varepsilon = (T_d - 1) m_p c^2 \) and \( T_d \) is many orders of magnitude smaller, this width is always very large in terms of Thomson optical depths.

### 2.3.3. Immediate DS

A rough estimate of the average temperature in the first few optical depths of the immediate DS of RRMS is given in KBW10. The assumption used is that the electron–positron pairs and the radiation are in Compton-pair equilibrium (CPE). This assumption is valid since the velocity is \( \approx c/3 \), and since the y parameter arising from mildly relativistic temperatures is large, as shown below. The numerical calculations are not based on this assumption, and its self-consistency is discussed in Section 6.1.

Following the NR RMS analysis, \( T_r \) is estimated by equating the number density of photons produced by bremsstrahlung and by inverse Compton emission of thermal pairs with the number density of photons needed to carry the energy flux at the end of the deceleration region.

Assuming that the number density of pairs is much larger than that of protons, and neglecting double Compton emission, the equilibrium ratio of photon to electron–positron number densities and photon production with diffusion is combined in detail in KBW10. One can see that if \( T \gtrsim 200 \text{ keV} \) there would be too many photons generated per lepton. Much lower temperatures lead to insufficient photon production, as can be deduced from the NR case. The conclusion is that for relativistic shocks,

\[
T_s \sim 200 \text{ keV}.
\]

The weak dependence of the immediate DS temperature on parameters is due to the rapid increase of pair density with \( T \) at \( T \sim m_e c^2 \).

#### Subsonic region

The US flow is “supersonic,” in the sense that the plasma velocity, \( \beta_0 c \), is larger than the plasma speed of sound, \( \beta_0 a \). The production of a large number of pairs in the immediate DS, \( n_e/n_p \gg 1 \), and the heating of the plasma at this region to relativistic temperatures, \( T_r \sim 0.4 m_e c^2 \), implies a “subsonic” flow, \( \beta_{ss} > \beta \), in the immediate DS. The large number of pairs implies that the average plasma particle mass is close to \( m_e \), for which the temperature, \( T_r \sim 0.4 m_e c^2 \), gives a speed of sound which is close to its highly relativistic value, \( \beta_0 = 1/\sqrt{3} \) (see Appendix C for a detailed calculation of the speed of sound). \( \beta_{ss} = 1/\sqrt{3} \) is larger than the plasma velocity in the immediate DS, which is close to its far DS value, \( \beta_f \leq 1/3 \).

Note that we are referring here to the plasma speed of sound neglecting the (dominant) contribution of the radiation to the pressure. This speed of sound describes the propagation of (small) disturbances in the plasma on length (time) scales which are short compared to the mean free path (time) for electron–photon collisions. As explained in Section 2.3.1, see Equation (10), collective plasma modes are expected to lead to a fluid like behavior of the plasma on length and timescales much shorter than the electron–photon collision mean free path.

In the far DS, the flow becomes supersonic again, \( \beta_{ss} < \beta_d \). This implies that for relativistic shocks the flow crosses two sonic points, accompanied by singularities of the differential conservation equations (Equations (14) and (15)):

1. At the first sonic point, the flow changes from supersonic to subsonic. This is a hydrodynamically unstable point which results in a hydrodynamic shock. A steady state hydrodynamic flow cannot smoothly cross a sonic point going from supersonic to subsonic velocities because DS of the sonic point, US-going characteristics converge to the sonic point (e.g., Zel’dovich & Raizer 1966), infinitely steepening a continuous profile at that point and resulting in a shock. We show below that while most of the deceleration
of the plasma is continuous, a (sub)shock across which the velocity jump is small, \( \delta(\Gamma \beta) \sim 0.1 \) (see Figure 7), is indeed required to exist at the end of the deceleration region. This subshock must be mediated by the same processes that are assumed to isotropize the particle’s velocities in the fluid rest frame on a scale much shorter than the radiation mean free path (e.g., plasma instabilities, see Equation (10)).

2. At the second sonic point, the flow passes from a subsonic to a supersonic region. This is a stable point which has no special significance, and is simply a part of the thermalization tail of the shock.

2.3.4. Structure

The structure of RRMS differs from that of NR RMS. The main differences are as follows.

1. The deceleration length is much larger than the naive estimate: the length, measured in Thomson optical depths of \( e^{-\tau} \), grows with the US Lorentz factor \( \Gamma_u \), in a manner faster than linear (for NR RMS it is \( \sim \beta_u^{-1} \)).

2. Pair production has a significant contribution to the deceleration of the plasma.

3. As explained above, a “hydrodynamic” subshock (possibly mediated by plasma instabilities) across which the velocity jump is small, \( \delta(\Gamma \beta) \sim 0.1 \), is required to exist at the end of the deceleration region.

4. The radiation is highly anisotropic, and exhibits a high-energy tail with a typical cutoff energy of \( \sim \Gamma^2 m_e c^2 \).

2.3.5. Scaling of the Profile with Density

The velocity, density and pressure profiles of RMS (as a function of optical depth \( \tau = \sigma_T \chi n_e \)) are independent of the US density. The scaling of the temperature depends on whether or not thermal equilibrium is sustained. In case it is (if \( L_T \ll L_{\text{diff}} \)), the temperature scales with density as \( T \propto n_e^{1/4} \). Alternatively, when \( L_T \gg L_{\text{diff}} \), in the shock regions where the temperature is much higher than its equilibrium value, the temperature profile does not scale with density, \( T \propto n_e^{0} \), see, for example, Equation (8). To see this, note that in these regions bremsstrahlung absorption is negligible, while Compton scattering and bremsstrahlung emission are both two body processes that scale similarly with \( n_e \). The conservation and radiation transfer equations are invariant under the scaling of the radiation intensity, densities, and length scales across the shock by \( n_e^{1/2}, n_e^{1}, \) and \( n_e^{-1} \), respectively. The ratio between densities of different species (e.g., positron to proton density ratio) becomes invariant as well. This scaling is shown explicitly later, in Section 3.1. Bremsstrahlung absorption may still be important at low frequencies and affect the structure through logarithmic corrections to the effective photon production rate. In the far DS, where \( T \) approaches its equilibrium value, absorption will no longer be negligible.

2.3.6. From Radiation Domination to Radiation Mediation

For shocks satisfying Equation (1), the far DS pressure is dominated by radiation. It is not a priory trivial that the pressure is dominated by radiation in the velocity transition region of such shocks since photons that are generated in the DS are able to diffuse US over a finite distance only. This question is relevant to NR and to RRMS and was not previously addressed to our knowledge. We next illustrate that under a wide range of conditions, the radiation does indeed dominate the pressure in the velocity transition region in both cases.

Radiation domination to radiation mediation, NR RMS.

Consider a hypothetical shock, having a DS energy density dominated by radiation, in which the velocity transition is mediated by some mechanism other than radiation. In the absence of radiation, the temperature immediately behind the velocity transition would be \( T \sim \varepsilon = 0.5 \beta u c^2 \). Photons generated in this region can diffuse US to a characteristic distance of \( L_{\text{diff}} = (\beta n_e \sigma_T)^{-1} \). Under these assumptions, the energy in photons that are produced by bremsstrahlung is much larger than the available thermal energy,

\[
\frac{e_y}{e_{\text{th}}} \approx \frac{Q_{\text{Br}} T L_{\text{diff}}}{\beta c n_e \varepsilon} > \frac{\alpha_e}{\beta^2} \sqrt{\frac{m_e c^2}{\varepsilon}} \approx 8 \alpha_e m_p \left( \frac{m_e c^2}{\varepsilon} \right)^{3/2} \gg 1,
\]

where \( Q_{\text{Br}} \) is the photon production rate by bremsstrahlung at its energy \( \sim T \), and where we used \( \beta = \beta_u/4 \), appropriate for NR shocks which are not radiation mediated and \( \varepsilon \sim T \ll m_e c^2 \). Note, that by definition, the condition (12) is roughly equivalent to demanding that the temperature \( T \) in Equation (8) be smaller than \( \varepsilon \).

This implies that a shock with a radiation dominated DS and negligible radiation in the velocity transition region cannot exist if the transition region is smaller than \( L_{\text{diff}} \). Once the pressure is dominated by photons, they will also mediate the shock.

Radiation domination to radiation mediation, RRMS. We next show that radiation domination implies radiation mediation in RRMS. Let us assume the contrary, i.e., that the energy density in the deceleration region is not dominated by radiation and that deceleration is therefore not mediated by radiation. In this case, the plasma reaches a temperature \( \sim \Gamma_{\text{sh}} m_e c^2 \) at the end of the deceleration region, and then gradually thermalizes as it flows further into the DS. Since \( T_d < m_e c^2 \) (see Equation (6)), let us consider the point in the DS where the temperature reaches \( T \sim 10 \text{ MeV} \). Since the velocity at this point already reached its DS value, \( \beta \sim \beta_d \leq 1/3 \), photon transport is well described in this region by the diffusion approximation, with diffusion length \( L_{\text{diff}} \approx 3(n_e \sigma_T)^{-1} \). An electron crossing this diffusion length produces a large number of \( \sim 10 \text{ MeV} \) photons,

\[
\frac{n_y}{n_e} \approx \frac{L_{\text{diff}}}{n_e \sigma_T Q_{\gamma,\text{eff}}} \approx \frac{\alpha_e \beta u_{\text{rel}} A_{\text{eff}}}{\beta^2} \approx 10 \frac{\sigma_{\text{eff}, \gamma}(10 \text{ MeV}) \Lambda_{\text{eff}}^3 (3 \beta_d)^{-2}}{100}
\]

(using conservative estimates for the Gaunt factor and the logarithmic correction). This ratio is much larger than its CPE value \( n_{\gamma,\text{eff}} \sim n_e \), expected at \( T \gg m_e c^2 \) (see also discussion in Svensson 1984). For such a high ratio of photons to electrons, a photon will produce a pair on another photon on a timescale much shorter than its scattering timescale. Such a deviation from equilibrium on a length scale \( \sim 3 \) scattering optical depths is not self-consistent. We conclude therefore that the temperature cannot significantly exceed \( m_e c^2 \) at the point where the deceleration is complete. This implies, in turn, that most of the energy at the end of the deceleration must be carried out by radiation.

3. RMS EQUATIONS AND BOUNDARY CONDITIONS

In this section, we write down the equations of RMS that are numerically solved based on assumptions 1–4 given in Section 2.3.1. In Section 3.1, we write down the hydrodynamic and radiation transfer equations in physical and dimensionless form and define the variables that are solved for. In Section 3.2,
we provide expressions for radiation scattering (Compton), production and absorption (bremsstrahlung), and pair production and annihilation. A summary of all the equations in dimensionless form is given in Section 3.2.4. The boundary conditions are described in Section 3.3.

3.1. Hydrodynamic and Radiation Transfer Equations

The equations governing the structure of a steady planar shock propagating along the z-direction are

\[
\frac{d}{dz_{\text{sh}}} T_{z_{\text{sh}}}^{\text{sc}} = 0, \quad (14)
\]

\[
\frac{d}{dz_{\text{sh}}} T_{z_{\text{sh}}}^{\text{ev}} = 0, \quad (15)
\]

\[
n_p = n_{p,\text{u}} \frac{\Gamma u \beta_n}{\Gamma \beta}, \quad (16)
\]

\[
\frac{d(\Gamma \beta n_\gamma)}{dz_{\text{sh}}} = \frac{Q_\nu}{c}, \quad (17)
\]

\[
\mu_{sh} \frac{d I_{\nu}(\mu_{sh})}{dz_{\text{sh}}} = \eta_{sh}(\mu_{sh}, v_{sh}) - I_{\nu}(\mu_{sh}) \chi_{sh}(\mu_{sh}, v_{sh}). \quad (18)
\]

The first three equations describe the conservation of energy, momentum, and proton number. The forth equation describes the production and annihilation of positrons, and the fifth equation describe the transport of photons. \(z_{\text{sh}}\) is the shock-frame distance along the shock propagation direction, \(\beta c\) is the plasma velocity in the shock frame, \(\Gamma = 1/\sqrt{1-\beta^2}\) is the corresponding Lorentz factor, \(n_p\) is the proper proton density, \(n_\gamma\) is the proper positron density, \(I_{\nu}(\mu_{sh})\) is the shock-frame specific intensity at (shock frame) frequency \(v_{sh}\) and direction \(\mu_{sh} = \cos \theta_{sh}\) (\(\theta_{sh}\) is the azimuthal angle with respect to \(z\)), and \(T_{z_{\text{sh}}}^{\text{ev}}\) is the shock-frame energy-momentum tensor. \(Q_\nu = \partial n_\gamma/\partial t\) is the net positron production rate (production minus annihilation), which is frame independent. \(\eta\) and \(\chi\) are the emissivity and absorption coefficients, respectively, and are functions of the plasma parameters and of the local radiation field described in Section 3.2. We use \(\{v, I, \mu, \eta, \chi\}\) to denote quantities measured in the plasma rest frame, and add a subscript “sh” to denote values of these quantities measured in the shock frame.

The energy and momentum are carried by the plasma and the radiation,

\[
T_{sh}^{\alpha\beta} = T_{sh,\text{pl}}^{\alpha\beta} + T_{sh,\text{rad}}^{\alpha\beta}, \quad (19)
\]

where the subscripts pl and rad refer to the plasma and radiation contributions, respectively. The radiation part of \(T_{sh}^{\alpha\beta}, T_{sh,\text{rad}}^{z_{\text{sh}}}, T_{sh,\text{rad}}^{ev}\) is given by

\[
F_{\text{rad}} = \int d\Omega \mu d\nu I_{\nu}(\mu), \quad (20)
\]

\[
P_{\text{rad}} = c^{-1} \int d\Omega \mu^2 d\nu I_{\nu}(\mu) \quad (21)
\]

(see Appendix D for rules of transformation between restframe and shock-frame measured quantities). As mentioned in Section 1, we assume that the protons, electrons, and positrons may be described as a fluid of single velocity \(c \beta(z_{\text{sh}})\), that the energy distribution of the electrons and positrons is thermal, with temperature \(T(z_{\text{sh}})\), and that the protons are cold. Under these assumptions,

\[
T_{\text{pl},\text{sh}}^{z_{\text{sh}}} = \Gamma^2 \beta (e_{\text{pl}} + P_{\text{pl}}), \quad (22)
\]

\[
T_{\text{pl},\text{sh}}^{\alpha\beta} = \Gamma^2 \beta T_{\text{pl},\text{sh}}^{\alpha\beta} + \Gamma^2 \beta P_{\text{pl}} T_{\text{pl},\text{sh}}^{\alpha\beta}, \quad (23)
\]

\[
T_{\text{pl},\text{sh}}^{z_{\text{sh}}} = \Gamma^2 \beta T_{\text{pl},\text{sh}}^{z_{\text{sh}}} + \Gamma^2 \beta P_{\text{pl}} T_{\text{pl},\text{sh}}^{z_{\text{sh}}}, \quad (24)
\]

\[
T_{\text{pl},\text{sh}}^{\alpha\beta} = \Gamma^2 \beta T_{\text{pl},\text{sh}}^{\alpha\beta} + \Gamma^2 \beta P_{\text{pl}} T_{\text{pl},\text{sh}}^{\alpha\beta}, \quad (25)
\]

\[
f(T) = \frac{1}{2} \tanh \left( \frac{\ln(T/m_e c^2) + 0.3}{1.93} \right) + \frac{3}{2}. \quad (26)
\]

This approximation in Equations (24) and (26) describes the equation of state (energy as a function of temperature and density) of relativistic Maxwell–Boltzmann (Maxwell–Jüttner) distributed plasmas, to an accuracy better than \(\sim 2 \times 10^{-3}\) for all temperatures, as shown in Figure 2.

3.1.1. Dimensionless Equations

We define the following dimensionless quantities:

\[
\hat{T} = \frac{T}{m_e c^2}, \quad \hat{\nu} = \frac{\nu c}{m_e}, \quad \hat{x} = \frac{x}{n_+ c}, \quad \hat{z}_{\text{sh}} = \frac{z_{\text{sh}}}{c n_+}, \quad \hat{\rho} = \frac{\Gamma(1+\beta)n_+(n_++n_\gamma)\sigma T z_{\text{sh}}}{m_e}, \quad \hat{I} = \frac{I}{\Gamma_\nu \Gamma_n \hat{\rho} n_+(m_p/m_e)hc}. \quad (27)
\]

With these definitions, and using the explicit forms of \(T_{\text{sh}}^{\alpha\beta}\) derived above, the energy and momentum conservation equations take the form

\[
\frac{\Gamma}{\Gamma_n} \left[ 1 + (1 + 2x_+ \frac{m_e}{m_p}) \left[ 1 + \hat{T} \left( 1 + \frac{3}{2} f(\hat{T}) \right) \right] \right] + 2\pi \hat{F}_{\text{rad},\text{sh}} = 1 + \frac{\hat{m}_e}{m_p}, \quad (28)
\]
\[
\gamma \text{corrections are due to double Compton scattering (photon pair production and annihilation. Other processes, which may be written as}
\]
\[
\gamma_e = \frac{1 + (1 + 2\chi_e)}{m_p} \left[ 1 + \hat{T} \left( \frac{1}{\Gamma \beta^2} + 1 + \frac{3}{2} f(\hat{T}) \right) \right],
\]
\[
\hat{T}_{\text{rad},sh} = 1 + \frac{m_e}{\beta_p},
\]
\[
\hat{\Gamma}_{\text{rad},sh} = \frac{2\pi \Gamma_{\beta_2} \beta_1 n c m_e}{\hat{\Gamma}},
\]
are the scaled energy and momentum fluxes of the radiation field.

The emissivity and absorption coefficients are the sum of the contributions due to the various processes considered
\[
\hat{\eta}(\mu, \hat{\nu}) = \sum \hat{\eta}_{\text{proc}}(\mu, \hat{\nu}),
\]
\[
\hat{\chi}(\mu, \hat{\nu}) = \sum \hat{\chi}_{\text{proc}}(\mu, \hat{\nu}).
\]

The transformation relations for the scaled emissivity and absorption are
\[
\hat{\eta} = \frac{\eta}{(1 + \beta)\sigma_T(n_e + n_+) \hat{\Gamma}},
\]
\[
\hat{\chi} = \frac{\chi}{(1 + \beta)\sigma_T(n_e + n_+)}.
\]

Finally, the equation describing the evolution of pair density may be written as
\[
\frac{dx_+}{dt_{\text{sh}}} = \dot{Q}_+,
\]
where the scaled rate of pair production is
\[
\dot{Q}_+ = \frac{Q_+}{\Gamma^2(1 + \beta)n_p(n_e + n_+)\sigma_T c}.
\]

We describe next the various radiative processes included.

3.2. Radiation Mechanisms

The radiative processes we take into account are Compton scattering, bremsstrahlung emission and absorption, and two photon pair production and annihilation. Other processes, which we neglect, do not modify the results significantly. The leading corrections are due to double Compton scattering ($\gamma + e \rightarrow 2\gamma + e$), three photon pair production ($e^+e^- \rightarrow 3\gamma$), and pair production on nuclei. Other processes, such as muon and pion pair production and synchrotron emission, are less significant.

3.2.1. Compton Scattering

The contribution of Compton scattering to $\eta$ and $\chi$ is
\[
\eta_{\pm}(\mu, \hat{\nu}) = (n_e + n_+) \int d\Omega d\nu' \frac{d\sigma_{\pm}}{d\nu' d\Omega} (\nu, \Omega \rightarrow \nu', \Omega'),
\]
where the total cross section,
\[
\sigma_{\pm}(\nu, T) = \int d\Omega d\nu' \frac{d\sigma_{\pm}}{d\nu' d\Omega} (\nu, \Omega \rightarrow \nu', \Omega'),
\]
may be written as
\[
\sigma_{\pm}(\nu, T) = \frac{2}{\hat{T}} \left[ \frac{1 + \zeta}{\zeta^3} \left\{ \frac{2\eta\zeta + 1}{1 + 2\zeta} - \ln(1 + 2\zeta) \right\} + \frac{1 + 3\zeta}{2\zeta} - \frac{1 + 3\eta}{(1 + 2\zeta)^2} \right].
\]

Here, $\zeta \equiv \hat{T}(1 + 2\hat{T})$ (see, e.g., Rybicki & Lightman 1979). The normalized emissivity and absorption are
\[
\hat{\eta}_{\pm}(\hat{\nu}, \hat{T}) = \frac{1}{\Gamma(1 + \beta)} \times \int d\Omega d\hat{\nu'} \frac{d\sigma_{\pm}}{d\hat{\nu}' d\Omega}(\hat{\nu}', \hat{\Omega}' \rightarrow \hat{\nu}, \hat{\Omega} \hat{I}_{\nu}'),
\]
\[
\hat{\chi}_{\pm}(\hat{\nu}, \hat{T}) = \frac{1}{\Gamma(1 + \beta)} \hat{\sigma}_{\pm}(\hat{\nu}, \hat{T}),
\]
where $\hat{\sigma} \equiv \sigma / \sigma_T$ and the transformations between the shock frame and plasma rest-frame values of $\nu, \mu$ are given in Appendix E.

Since using the exact form of the differential cross section for Compton scattering greatly increases the computational resources demands, we use instead an approximation described in Appendix B. In particular, we assume isotropic scattering in the rest frame of the plasma, i.e., $d\sigma_{\pm}(\nu, \Omega \rightarrow \nu', \Omega')$ independent of $\Omega'$.

3.2.2. Pair Production and Annihilation

Pair annihilation. The photon emission arising from annihilation of pairs has the form
\[
\eta_0 = \frac{1}{4\pi} \hat{\eta}_0 h \nu = \frac{h\nu n_e n_+ \sigma_T c f_{\pm}(\nu, T)}{4\pi} r_\pm(T),
\]
where $r_\pm$ is a dimensionless function of $T$ accounting for the rate of annihilation and $f_{\pm}$ is the spectral distribution of the photons, where
\[
\int f_{\pm}(\nu, T) d\nu = 1.
\]

The approximation we use for $f_{\pm}$ is based on the analysis of Zdziarski (1980), who fits an analytic function to the results of Monte Carlo calculations. For the annihilation rate, we use, based on Svensson (1982),
\[
r_\pm(T) = \frac{3}{4} \left[ 1 + \frac{2\hat{T}^2}{\ln(2\eta_0 \hat{T} + 1.3)} \right]^{-1},
\]
where $\eta_0$ is the photon density and $\hat{T}$ is the shock frame temperature.
where \( \eta_E = e^{-\gamma_E} \approx 0.5616 \), and \( \gamma_E \approx 0.5772 \) is Euler’s constant. The normalized emissivity is given, based on Equation (45), by

\[
\hat{n}_\parallel (\hat{v}, \Omega) = \left[ (1 - \beta \mu_{ab})^2 \right] \hat{n}_{\parallel,ab}(\hat{v}_{\parallel b}, \Omega_{ab}) (x_e + 1)x_e \hat{f}_\parallel (\hat{v}, \hat{T}) r_{\pm}(T) m_e = \frac{1}{2\pi(2x_e + 1)} \Gamma_\alpha \Gamma_\beta (1 + \beta)(1 + 2x_e + 1). \tag{48}
\]

The annihilations rate in Equation (17) is simply

\[
\hat{Q} = -\frac{1}{2} n_e n_\gamma \sigma_T c r_{\pm}(T), \tag{49}
\]

and the scaled contribution to Equation (37) is

\[
\hat{Q}_+ = -\frac{x_e(x_e + 1)r_{\pm}(T)}{2\Gamma_e^2 \beta (1 + \beta)(1 + 2x_e + 1)}. \tag{50}
\]

**Pair production.** The two photon pair production contribution to the absorption in the transfer equation is

\[
\chi_{\nu,\gamma\gamma}(\mu) = \int \sigma_{\gamma\gamma}(\nu, \nu', \mu, \Omega') \times \frac{I_\nu(\Omega')}{c h \nu'(1 - \cos \theta_1)} \Theta(1 - \cos \theta_1) \Theta(2 \nu' - 2\nu', d \Omega d \nu', \nu, \nu', \mu, \Omega') \tag{51}
\]

where \( \theta_1 \) is the angle between \( \mu \) and \( \Omega' \). The scaled absorption can be written as

\[
\hat{\chi}_{\nu,\gamma\gamma}(\mu) = \frac{\Gamma_\alpha \beta (1 + \beta)(2x_e + 1)}{(1 + \beta)(1 + 2x_e + 1)} \int \sigma_{\gamma\gamma}(\nu, \nu', \mu, \Omega') \times \frac{I_\nu(\Omega')}{\nu'} \Theta(1 - \cos \theta_1) \Theta(1 - \cos \theta_1) d \Omega d \nu'. \tag{52}
\]

(\( \hat{\chi} \) should be calculated at the same frame for which \( \hat{I} \) is given). For the cross section, we use (e.g., Padmanabhan 2000)

\[
\sigma_{\gamma\gamma}(s) = \frac{3}{8} \frac{\sigma_T}{s} \left[ \left( \frac{2 + \frac{1}{s} - 1}{s^2} \right) \cosh^{-1} s^{1/2} \right. \tag{53}
\]

where

\[
s = \frac{1}{2} \frac{\nu}{\nu} (1 - \mu \mu'). \tag{54}
\]

is the center of momentum energy squared. To shorten the computing time, we integrate over \( \phi' \) assuming that \( \sigma_{\gamma\gamma}(s) \) changes slowly with \( \phi' \) and that \( \Theta(\nu \nu'(1 - \cos \theta_1) - 2\nu') \) has the same value for most \( \phi' \) values, obtaining approximately

\[
(1 - \cos \theta_1) \phi' = 1 - \mu \mu'. \tag{55}
\]

To find the positron production rate \( \hat{Q}_+ \), we use the rate of photon loss to this process,

\[
\hat{Q}_+ = -\frac{1}{2} \hat{n}_\gamma = \frac{1}{2} \int \frac{I_\nu(\mu)}{\nu} \hat{\chi}_{\nu,\gamma\gamma}(\mu) d \nu d \Omega. \tag{56}
\]

The scaling of the production rate follows

\[
\hat{Q}_+ = \frac{\Gamma_\alpha m_e}{2m_e} \int \frac{I_\nu(\mu)}{\nu} \hat{\chi}_{\nu,\gamma\gamma}(\mu) d \nu d \Omega. \tag{57}
\]

**3.2.3. Bremsstrahlung**

Bremsstrahlung emission includes contributions from \( e^-p \) and \( e^+p \) encounters, as well as from \( e^-e^- \), \( e^+e^+ \), and \( e^-e^+ \) encounters, which become important sources of photon production at high temperatures. The emission can be expressed by (Svensson 1982)

\[
\hat{n}_{\parallel,\gamma}(\Omega, v) = \frac{1}{\pi^2} \sqrt{2} \alpha_e \sigma_T m_e^2 c^2 n_e^2 e^{-\nu v / T} \sqrt{T} \nu, \tag{58}
\]

where \( \alpha_e \) is the fine structure constant, and

\[
\lambda_{\Omega}(x_e, T) = (1 + x_e) \lambda_{\gamma p} + \left[ x_e^2 + (1 + x_e)^2 \right] \lambda_{\gamma e} + x_e(1 + x_e) \lambda_{\gamma \gamma}, \tag{59}
\]

is a numerical factor accounting for the presence of electron–positron pairs and for relativistic corrections at high temperature. We use a prescription for bremsstrahlung emission based on Skibo et al. (1995; note that there is an errata correction to this paper), which gives a general fit for the Gaunt factor as a function of temperature, positron density, and the emitted frequency. The transformation between the different notations is \( \lambda_{\Omega} = \frac{2}{\pi^2} g_v, \) where \( g_v \) is the Gaunt factor as given in Skibo et al. (1995).

The emissivity resulting from Equation (57) is

\[
\eta_{\parallel,\gamma}(s) = n_e^2 e^{-\nu v / T} \frac{2}{\pi^2} \alpha_e \sigma_T m_e^2 c^2 n_e^2 e^{-\nu v / T} \sqrt{T} \nu. \tag{60}
\]

The normalized emissivity then reads

\[
\frac{\Gamma(1 - \beta \mu_{ab})^2 \hat{n}_{\parallel,\gamma}(\hat{v}_{\parallel b}, \Omega_{ab})}{\hat{n}_{\parallel,\gamma}(\hat{v}, \Omega)} \hat{n}_{\parallel,\gamma}(\hat{v}, \Omega) = \frac{\alpha_e m_e^2}{\pi^2 \Gamma_\alpha \Gamma_\beta (1 + \beta)(1 + 2x_e + 1)} \sqrt{2} \times \frac{e^{-\nu v / T}}{\sqrt{T}} \lambda_{\Omega}. \tag{61}
\]

Minimal \( v \). Coulomb screening suppresses bremsstrahlung emission at impact parameters larger than the Debye length \( \lambda_D = \sqrt{T / 4 \pi e^2 n_e + n_e}, \) implying a low-energy cutoff for bremsstrahlung emission (Weaver 1976)

\[
\epsilon_{sc} \simeq \frac{\gamma e \beta_{e,th} h c}{\lambda_D}, \tag{62}
\]

where \( \gamma e, th \) is the Lorentz factor associated with the random ("thermal") motion of the electrons, and \( \beta_{e,th} \) is the associated velocity (in units of \( c \)). Setting \( \gamma e, th \simeq 1 + 3T / m_e c^2 \) we get for the NR case \( T \ll m_e c^2 \)

\[
\epsilon_{sc, NR} \simeq 2.87 \times 10^{-6} n_{e,15}^{1/2} (1 + 2x_e + 1)^{1/2} \keV, \tag{63}
\]

and for the relativistic case \( T \gg m_e c^2 \)

\[
\epsilon_{sc, rel} \simeq 9.12 \times 10^{-10} n_{e,15}^{1/2} (1 + 2x_e + 1)^{1/2} \left( \frac{T}{\keV} \right)^{3/2} \keV, \tag{64}
\]

where \( n_e = n_{e,15} 10^{15} \cm^{-3} \).

We note that since our calculation explicitly describes upscattering and bremsstrahlung self-absorption, there is no need to introduce (as was done, for example, in Weaver 1976) a cutoff to the bremsstrahlung emission at low frequencies, for which the flow dynamical timescale or the self-absorption timescale
is shorter than the time required for a low-energy photon to be
upscattered to $T$.

*Bremssstrahlung self-absorption.* Using Kirchhoff’s law and the
calculated value of $\eta_{\nu, ff}$ in the rest frame of the plasma, we have
\[
\chi_{\nu, ff} = \frac{\eta_{\nu, ff}}{B_\nu(T)} \text{ (cm$^{-1}$)},
\]  
where
\[
B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}
\]  
is Plank’s spectrum.
The normalized Plank spectrum is
\[
\hat{B}_\nu = \frac{2m_e^4c^3}{h^3m_e^2} \frac{1}{\hat{\nu}^3} \frac{\hat{\nu}^3}{\hat{\nu}^3/\hat{T} - 1},
\]  
and the normalized absorption is
\[
[\Gamma(1 - \beta\mu_{\sh})]^{-1} \hat{\chi}_{\nu,sh}(\hat{\nu}_{\sh}) = \frac{\hat{\chi}_{\nu,ff}}{\hat{B}_\nu(T)} B_\nu(T) \frac{\alpha_n \chi(\nu)}{\sqrt{2\pi^2m_e^3c^3}} \eta_n(1 - e^{-\hat{\nu}/\hat{T}}) \hat{\nu}^3 \sqrt{\hat{T}}
\]  
(67)

3.2.4. Summary

To summarize: we use Equations (28), (29), (32), and (37), to
determine the variables $\hat{T}(\tau_\nu)$, $\beta(\tau_\nu)$, $x_{\nu}(\tau_\nu)$, and $\hat{I}_{\nu,sh}(\mu_{\sh})(\tau_\nu)$.
The contributions of the radiative processes to the transfer equation (Equation (32)) are given by Equations (43), (44), (48), (52), (60), and (67). The contributions to $\hat{Q}$ in the positron fraction equation (Equation (37)) are given by Equations (50) and (56).

3.3. Boundary Conditions

We obtained solutions of the equations given above over a
finite optical depth range around the shock transition that
satisfies the following requirements.

1. The solution includes a subsonic region DS of a supersonic
region with continuous radiation field $I_{\nu,sh}(\mu_{\sh})$ and positron
flux across the subshock separating the two regions (see
Section 2.3.3).
2. The radiation momentum flux in the last several photon
mean free paths away from the shock transition in the US
region is negligible compared to the far US electron momentum
flux.
3. The width of the subsonic region is sufficiently large
comparable to the photon mean free path, such that the
solution is insensitive to the precise boundary conditions
that are applied at the DS edge, while remaining short
enough as to avoid reaching the second supersonic region
which exists DS of the subsonic region.
The boundary conditions in the far US are $I_{\nu,sh}(\mu_{\sh}) > 0$, $z_{\sh} = -\infty = 0$, i.e., no incident radiation at the US. (In practice, we use an effective “reflector” in the US end of the calculation, to avoid numerical fluctuations and shorten the iteration time. It does not affect the shock structure) In addition, the positron position is taken as 0 at the US boundary.
The boundary conditions at the far DS are given by thermal
equilibrium. Since the calculation does not reach the far DS, we use a boundary condition in the DS which corresponds to
isotropy of the radiation field in the rest frame of the far DS.
This is done by equating the intensity and spectrum of US-going radiation at the DS boundary to that of the DS-going radiation. For numerical reasons, we multiply the reflected radiation by
a factor which is close to unity, this has a negligible effect on
the shock structure. In addition, we impose an upper limit on
the photon energy of the reflected radiation, typically $3m_e c^2$.
The physical reasoning for this upper limit is that high-energy
photons that cross this point in the DS either scatter and lose
most of their energy (as $\hat{T} \ll 1$ at that point and further away),
or more likely, produce an $e^+ e^-$ pair that is swept DS with the
flow.

4. THE NUMERICAL METHOD

We briefly describe below the numerical method we use for
solving the equations.

4.1. Iteration Scheme

We start with an initial guess for the shock profile, $S^0 = \{\hat{T}^0(\tau_\nu), \beta^0(\tau_\nu), x^0(\tau_\nu), \hat{I}_{\nu,sh}^0(\mu_{\sh}, \tau_\nu)\}$, and modify it iteratively until a solution of the equations is obtained. The iterations are
performed as follows.
1. Compute $\hat{\eta}(S^0)$ and $\hat{\chi}(S^0)$ using the profile $S^0$.
2. Integrate directly the transfer eq., Equation (32), using
$\hat{\eta}(S^0)$ and $\hat{\chi}(S^0)$, to obtain $\hat{I}_{\nu,sh}^{+1}(\mu_{\sh}, \tau_\nu)$.
3. Use Equations (28), (29), and (37) with the new radiation
field, $\hat{I}_{\nu,sh}^{+1}(\mu_{\sh}, \tau_\nu)$, to obtain the new profile $S^{+1}$.

Usage of “partial iterations,” where $S^{+1}$ is replaced with a
weighted average of $S^0$ and $S^{+1}$, was required in order to achieve convergence and stability.
At any given $\tau_\nu$, the energy and momentum conservation
equations, Equations (28) and (29), have a supersonic and a
subsonic solution for $\beta$ and $\hat{T}$ given $x_\nu$, $\hat{I}_{\nu,sh}(\mu_{\sh})$. The position of the subshock (see Section 2.3.3) was set to $\tau_\nu = 0$, US of
which the supersonic solution was chosen and DS of which, the
subsonic solution was chosen.
We significantly reduced the computational time of the
calculation, by separating the spatial grid into two regions, and
reforming the above iterations on each. The DS-going photons
on the DS boundary of the first region were used as a boundary
condition for the second region and vice versa. We preformed macro iterations in which we updated these boundary conditions until a self-consistent profile was obtained across the border between the regions. The calculations presented in this paper
were separated at the point $\tau_\nu = 0$, the sonic point. Separating at different points in the vicinity of $\tau_\nu = 0$ leads to similar results.

4.2. Discretization

We use a discrete approximation of $I_{\nu,sh,\nu}(\mu_{\sh})$,
\[
I_{\nu,sh,\nu}(\mu_{\sh}) = \sum I_{\nu,sh,ij}(\nu_{\sh,i}, \nu_{\sh,j+1}) f_{\nu,i}(\nu_{\sh,i}, \nu_{\sh,j+1}) f_{\mu,i}(\mu_{\sh,i}, \mu_{\sh,j+1})
\]  
(68)
where $f_{\nu,i}(x, x_1, x_2) = \Theta(x - x_1)\Theta(x_2 - x)$ is the top hat function and $\Theta$ is the step function. The distribution of $\nu_{\sh,i}$ is logarithmic
in the range $\nu_{\text{min}}$ to $\nu_{\text{max}}$. Typical values are $h\nu_{\text{max}} = 10^7 m_e c^2$ and $h\nu_{\text{min}} = 10^{-8} m_e c^2$. The distribution of $\mu_{\sh,i}$ is set to account
for relativistic beaming of the radiation in the shock frame as well as for a relatively isotropic component in all frames,
from US to DS. This is achieved by a logarithmic separation of $\mu_{sh}$ in the US direction between $\mu_{sh} = 0$ and $\mu_{sh} = 1$, with $1 - \max(\mu_{sh}) < \Gamma_u^{-2}$. The $\mu_{sh} < 0$ directions are chosen as the zeros of a Legendre polynomial, the same as the common Gaussian quadrature. A short Matlab code implementing the $\mu_{sh}$ discretization is given in Appendix D. A typical division is shown in Figure 3. We note that in order to account correctly for the relativistic beaming using Gaussian quadrature, for instance, shown in Figure 3. We note that in order to account correctly for the relativistic beaming using Gaussian quadrature, for instance, would require a much larger number of azimuthal directions for high values of $\Gamma_u$. The convergence of the solutions with respect to the resolution is demonstrated in Section 5.4.

4.3. Test Problems

The numerical scheme and its implementation were tested thoroughly to ensure the results are valid. The tests verified a correct description of the different radiation mechanisms in steady state problems including, e.g., Compton scattering with pair production, bremsstrahlung emission with self-absorption. We present here only two of the tests, demonstrating the suitability of the numerical scheme for dealing with repeated Compton scattering and pair production and annihilation. The test results are compared with analytic solutions or Monte Carlo simulations and are shown to reproduce them well.

4.3.1. Comptonization in a Cloud of Low and Medium Optical Depths

A thin, stationary planar layer of plasma with Thomson optical depth $\tau_T$ in the $z$-direction (perpendicular to the symmetry plane) and a given temperature $T$ is irradiated at one end, $\tau_s = 0$, by a $\delta$ function in $v$, directed along the $z$-axis,

$$I_\nu(\mu > 0, \tau = 0) = I_0\delta(v - v_0)\delta(\mu - 1).$$

(69)

At $\tau_s = \tau_T$ a free boundary condition, $I(\mu < 0, \tau_s = \tau_T) = 0$, is applied. In order to reach the steady state solution for the radiation field, the iteration scheme of the radiative transfer equation is used until the radiation field converges.

The results of these calculations are compared with an independent Monte Carlo simulation of the setups using the same approximate Compton kernel, as described in Section 3.2.1.

The specific photon flux escaping through the free boundary at $\tau_s = \tau_T$,

$$j_\nu = \int_0^1 \frac{I_\nu}{\nu} \mu d\mu,$$

(70)

was calculated for two cases with $\hat{T} = 1$, one with $\tau_T = 1$, $v_0 = 10^{-8}$ and the other with $\tau_T = 0.01$, $v_0 = 10^{-4}$. The resulting spectra are shown in Figures 4 and 5 for $\Gamma_T = 1$ and $\tau_T = 0.01$, respectively. In each figure, the results of the code (blue pluses) and the Monte Carlo simulation (black lines) are shown. As can be seen, there is an excellent agreement between the two independent methods for calculating the spectrum of escaping photons.

Figure 3. $\Gamma_u = 20$, distribution of 18 $\mu$’s, shown in three different frames: shock frame (bold blue circles), US frame (green ×’s), and $\Gamma = 10$ frame (red pluses). Note that the beaming toward the US is accounted for, as can be seen in the shock frame.

(A color version of this figure is available in the online journal.)

Figure 4. Specific photon flux leaving a cloud of plasma with $\hat{T} = 1$ and width $\tau_T = 1$. The radiation entering the cloud has a single frequency $v_0 = 10^{-8}$. The results of the code are marked with blue pluses and the Monte Carlo results are shown as a black line.

(A color version of this figure is available in the online journal.)

Figure 5. Specific photon flux leaving a cloud of plasma with $\hat{T} = 1$ and width $\tau_T = 0.01$. The radiation entering the cloud has a single frequency $v_0 = 10^{-8}$. The results of the code are marked with blue pluses and the Monte Carlo results are shown as a black line.

(A color version of this figure is available in the online journal.)
the two values obtained is given in Table 1 for different

temperatures.

Table 1
Equilibrium Values of \( x_\nu \), Balancing the Pair Production and Annihilation Rates at Different Temperatures

| \( \hat{T} \) | \( x_{\nu\text{analytic}} \) | \( x_{\nu\text{num}} \) |
|-------------|-----------------|-----------------|
| 0.3         | 550             | 425             |
| 0.5         | 541             | 500             |
| 0.8         | 421             | 421             |
| 1.5         | 259             | 266             |
| 10          | 43              | 43              |

4.3.2. Pair Quasi-equilibrium for Given \( T \)

This test checks the numerical description of the (integral) pair production and annihilation. We use a setup with a given Wien spectrum of the radiation field

\[
I_\nu(\mu) \propto \hat{\nu}^2 e^{-\hat{\nu}/\hat{T}}. \tag{71}
\]

For a given \( \hat{T} \), we find the equilibrium value of \( x_\nu = n_+ / n_\nu \) for which the positron production and annihilation rates cancel each other analytically and numerically. A comparison between the two values obtained is given in Table 1 for different temperatures.

Note that \( x_\nu \) does not necessarily grow with \( \hat{T} \), since we use different densities \( n \) for convenience. We obtain an accuracy of a few % except for very low temperature, where higher resolution is needed in order to account for the exponential cutoff near \( \hat{\nu} = 1 \). The resolution used here is \( v_{\nu\text{ref}}/v_\nu = 1.4, N_\nu = 12 \).

5. NUMERICAL RESULTS

In this section, we present the numerical results, solving Equations (14)–(18) self-consistently for different values of the US Lorentz factor \( \Gamma_u \). We divide the presentation of the results into two parts: the structure (Section 5.1) and the radiation spectrum (Section 5.2). The structure is the spatial distribution of integral parameters such as temperature, velocity (or Lorentz factor), pair density, and radiation pressure. The spectrum is the distribution of radiation intensity at different angles and photon energies (at given locations across the shock), measured in a specific reference frame. Two important frames of reference are the shock frame, in which the solution is a steady state solution, and the local rest frame of the plasma, which is useful for understanding the interaction between the radiation and the plasma.

5.1. Structure

The values of \( \Gamma \beta, \hat{T} \) and \( x_\nu \) for \( \Gamma_u = 6, 10, 20, \) and 30 are shown in Figures 6–11 as functions of the Thomson optical depth for US-going photons \( \tau_u \) (defined in Equation (27)) or \( \tau_\nu/\Gamma_u \). Figures zoomed on the DS region (\( \tau_\nu \geq 0 \)) are separately given. The results are calculated for \( n_\nu = 10^{15} \text{cm}^{-3} \), over regimes where bremsstrah lung absorption is negligible (i.e., they are in the low density limit, see Section 2.3.5).

The shock profiles can be divided into four regions.

1. Far upstream. The velocity is constant, while the radiation intensity and positron fraction grow exponentially until they hold a significant fraction of the energy and momentum of the flow.
2. The velocity transition. Here, the flow decelerates considerably, reaching a velocity close to the DS velocity. For RRMS, this regime is bound by a subshock.
3. Immediate downstream. In the first \( \beta_\nu^{-1} \) optical depths behind the velocity transition the flow approximately stays at constant velocity, while the plasma and radiation are in CE. A gradual cooling by bremsstrahlung emission and inverse Compton scattering takes place. This region produces the radiation that diffuses US and decelerates the incoming plasma.
4. Far downstream. Further than approximately \( \beta_\nu^{-1} \) optical depths into the DS, from where most photons cannot diffuse US. From this point on, a slow thermalization takes place accompanied by a slow decline in the plasma temperature and photon energies, ending when the temperature reaches the DS temperature. The decline in temperature leads first to a decrease in positron number, until the pair density becomes negligible compared to that of the original electrons (\( x_+ < 1 \)) at \( T \sim 50 \text{ keV} \). Then the thermalization...
continues until bremsstrahlung absorption takes over and thermal radiation at equilibrium is established.

We do not solve the equations in the fourth region since the solution there is straightforward (the radiation is isotropic and in equilibrium with the plasma). Also, note that since the far DS is supersonic, a second sonic point is expected in RRMS. This, however, is a stable point with no special physical significance.

Figures 6 and 7 show, for different values of $\Gamma_u$, the structure of the relativistic velocity $\Gamma \beta$ across the shock. It can be seen that the deceleration length in units of $\tau_*$ is much larger than unity and grows with $\Gamma_u$ in a manner faster than linear. A subshock is obtained at the sonic point, with a discontinuous deceleration of $\delta(\Gamma \beta) \sim 0.1$. Behind the subshock, the velocity approaches its far DS value in a few Thomson optical depths. The last optical depth is affected by the boundary conditions imposed on the right-hand side. This effect will be discussed in Section 5.4.2.

Figures 8 and 9 show, for different values of $\Gamma_u$, the structure of the temperature $\hat{T}$ across the shock. The far US shows an exponential growth of $\hat{T}$ as a function of $\tau_*$. The temperature then saturates at a maximum which is approximately linear in $\Gamma_u$, and then decreases toward the subshock. Behind the subshock the temperature jumps, reaching a value of $\hat{T}_{\text{jump}} \sim 0.5$, which grows with $\Gamma_u$, and then cools with a typical distance of a few Thomson optical depths ($\tau_*$).

Figures 10 and 11 show, for different values of $\Gamma_u$, the structure of the positron to proton number ratio, $x_+$. The growth of $x_+$ as a function of $\tau_*$ when approaching the subshock is super exponential, and its value reaches a maximum a few optical depths behind the subshock. The maximal value is approximately linear in $\Gamma_u$ (see Figure 12). Figure 13 shows $x_+ \hat{T}$ across the shock, which represents the pressure of the positrons and their relative importance in setting the speed of sound in the plasma, compared to the protons. The value of $x_+ \hat{T}$ goes above a few hundreds at the subshock for $\Gamma_u \geq 6$.

Figure 14 shows the ratio of thermal energy flux carried by electrons and positrons to the radiation energy flux, $F_{\text{th}}$, versus $\Gamma \beta/(\Gamma_u \beta_u)$. The energy flux (“taken” from the protons)
Figure 12. Maximum value of $x_+$ for different $\Gamma_u$ values. The approximation $x_{+,\text{max}} = 142\Gamma_u$ is accurate to better than 10% in the range we investigated. (A color version of this figure is available in the online journal.)

Figure 13. Rest-frame normalized positron pressure $x_+\hat{T}$ vs. $\tau_u/\Gamma_u$ for different values of $\Gamma_u$, from the US to the subshock. (A color version of this figure is available in the online journal.)

is dominated by thermal and rest mass energy flux of the electrons and positrons during most of the transition rather than by radiation energy flux. The energy is transferred to the radiation when the flow approaches the DS velocity, and the two fluxes are comparable around the subshock. Comparing the results at a fixed point (e.g., $\Gamma = \Gamma_u/2$), this ratio grows with $\Gamma_u$.

Figures 15 and 16 show the relativistic velocity $\Gamma\beta$, the temperature $\hat{T}$, and $x_+$ as a function of the scaled distance $\tilde{z}_\text{sh} = \Gamma_u\nu_\text{T}^\tau\tilde{z}_\text{sh}$, for $\Gamma_u = 10$. These figures illustrate that the shock width is comparable to the US Thomson mean free path, as $\tilde{z}_\text{sh}$ is approximately measured in these units.

5.2. Spectrum

Figures 17–26 show the radiation spectrum at different points along the shock profile for the cases $\Gamma_u = 10$ and $\Gamma_u = 30$. The normalization of the intensity and frequency is given in Equation (27). The points of interest are as follows.

1. The upstream—where $\Gamma = 0.99\Gamma_u$. At this point, we show the spectrum in the rest frame of the plasma (Figures 17 and 18 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

2. The transition—where $\Gamma = \Gamma_u/2$. At this point, we show the spectrum in the rest frame of the plasma (Figures 19 and 20 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively), and in the shock frame (Figures 21 and 22 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

3. The immediate DS—one Thomson optical depth ($\tau_e = 1$) DS of the subshock. At this point, we show the spectrum in the shock frame (Figures 25 and 26 for $\Gamma_u = 10$ and $\Gamma_u = 30$, respectively).

We now give a short description of the main characteristics of the spectrum at different locations across the shock. An extensive analysis and an analytic description of the results are given in Section 6.
Figure 16. Same as Figure 15, showing only the DS region.
(A color version of this figure is available in the online journal.)

Figure 17. Plasma rest-frame radiation spectrum $\hat{\nu} \hat{I}_r$ vs. $\hat{\nu}$, for $\Gamma_u = 10$ in the US ($\Gamma = 9.9$).
(A color version of this figure is available in the online journal.)

Figure 18. Plasma rest-frame radiation spectrum $\hat{\nu} \hat{I}_r$ vs. $\hat{\nu}$, for $\Gamma_u = 30$ in the middle of the transition ($\Gamma = 15$).
(A color version of this figure is available in the online journal.)

Figure 19. Plasma rest-frame radiation spectrum $\hat{\nu} \hat{I}_r$ vs. $\hat{\nu}$, for $\Gamma_u = 10$ in the middle of the transition ($\Gamma = 5$).
(A color version of this figure is available in the online journal.)

Figure 20. Plasma rest-frame radiation spectrum $\hat{\nu} \hat{I}_r$ vs. $\hat{\nu}$, for $\Gamma_u = 30$ in the middle of the transition ($\Gamma = 15$).
(A color version of this figure is available in the online journal.)

Figure 21. Shock-frame radiation spectrum $\hat{\nu} \hat{I}_s$ vs. $\hat{\nu}$, for $\Gamma_u = 10$ within the transition region ($\Gamma = 5$).
(A color version of this figure is available in the online journal.)
Figure 22. Shock-frame radiation spectrum $\tilde{\nu}_{sh} \tilde{I}_{sh,\delta_{sh}}$ vs. $\tilde{\nu}_{sh}$ for $\Gamma_u = 30$ within the transition region ($\Gamma = 15$).
(A color version of this figure is available in the online journal.)

Figure 23. Solid angle weighted rest-frame radiation spectrum, $\tilde{\nu}_{re} \tilde{I}_{re,\delta_{re}} (1 - \mu_{re}^2)$, for $\Gamma_u = 30$ within the transition region ($\Gamma = 15$).
(A color version of this figure is available in the online journal.)

Figure 24. Solid angle weighted shock-frame radiation spectrum, $\tilde{\nu}_{sh} \tilde{I}_{sh,\delta_{sh}} (1 - \mu_{sh}^2)$, for $\Gamma_u = 30$ within the transition region ($\Gamma = 15$).
(A color version of this figure is available in the online journal.)

Figure 25. Shock-frame radiation spectrum $\tilde{\nu}_{sh} \tilde{I}_{sh,\delta_{sh}}$ vs. $\tilde{\nu}_{sh}$ for $\Gamma_u = 10$ in the immediate DS ($\tau_* = 1$).
(A color version of this figure is available in the online journal.)

Figure 26. Shock-frame radiation spectrum $\tilde{\nu}_{sh} \tilde{I}_{sh,\delta_{sh}}$ vs. $\tilde{\nu}_{sh}$ for $\Gamma_u = 30$ in the immediate DS ($\tau_* = 1$).
(A color version of this figure is available in the online journal.)

Figure 27. Integrated spectrum in the shock frame, $\tilde{\nu}_{sh} \int \tilde{I}_{sh,\delta_{sh}} (\mu_{sh}) d\mu_{sh}$, at different depths inside the immediate DS ($\tau_* > 0$) for $\Gamma_u = 30$.
(A color version of this figure is available in the online journal.)
1. **US.** The rest-frame spectrum (Figures 17 and 18) is strongly dominated by a photon component beamed in the US direction, with a typical energy of $\sim 3\Gamma_u m_c c^2$, and a much weaker, isotropic component with energy $\sim \Gamma_u m_c c^2$. In the shock frame (not shown here), the dominant component is beamed in the DS direction, with characteristic energy $\sim \Gamma_u^2 m_c c^2$. There is also a weaker and not strongly beamed US-going component with energy somewhat higher than $m_c c^2$.

2. **Transition region.** The radiation in this region is extremely anisotropic in both the shock frame and the rest frame of the plasma. In the rest frame (Figures 19 and 20), the radiation is dominated by a high-energy component beamed in the US direction, with a typical energy of $h\nu \approx \Gamma m_c c^2$, where $\Gamma$ is the local Lorentz factor. An isotropic component, which is much weaker in intensity and with typical photon energy similar to the beamed component, also exists. In the shock frame (Figures 21 and 22), the spectrum is composed of a dominant narrowly beamed component in the DS direction with typical photon energy $h\nu \approx \Gamma_m^2 m_c c^2$, and of a much weaker intensity of US-going photons with typical energy of $h\nu \sim m_c c^2$.

The spectrum in both frames contains highly beamed components. In order to estimate the amount of energy carried by the beams, we show in Figures 23 and 24 the intensity $I$ multiplied by $1 - \mu^2$, which for $1 - |\mu| \ll 1$ is proportional to the solid angle. In the rest frame, the $h\nu_{\gamma\gamma} \approx \Gamma m_c c^2$ component dominates the total energy, while in the shock frame the energy carried by the US-going photons is comparable to that of DS-going $h\nu_{\gamma\gamma} \approx \Gamma^2 m_c c^2$ photons.

3. **Immediate DS.** Figures 25 and 26 show that the spectrum is composed of two components: a relatively isotropic component with $h\nu \sim m_c c^2$, and a component narrowly beamed into the DS direction with energy reaching $h\nu \sim \Gamma_u^2 c^2$.

Figures 27 and 28 show the spectrum integrated over $\mu$ in the immediate DS. The integrated spectrum is dominated by photons of energies $\sim m_c c^2$, but includes a significant high-energy tail. The high-energy component holds $10\%-20\%$ of the total energy flux of the radiation and is analyzed in Section 6.5.

5.3. **Compton Scattering and Pair Production Optical Depths**

The dominant mechanisms affecting the radiation in the transition region are Compton scattering and photon–photon pair production. To determine the relative importance of the two processes and obtain a handle on some of the important physical features of the deceleration mechanism, we examine the optical depth for US-going and DS-going photons in the transition region, for the cases $\Gamma_u = 10$ and $\Gamma_u = 30$. Figures 29 and 30 show the cumulative optical depths for US-going photons leaving the subshock and reaching the point where $\Gamma = \Gamma_u / 2$ as a function of shock-frame frequency. It is clear that many of the photons with $\bar{v}_{sh} \gtrsim 1$ will be scattered. These photons constitute the majority of the photon flux leaving the subshock.
the immediate DS in the US direction. It can be seen that most of the shock profile, up to $\Gamma \sim 0.9 \Gamma_u$, has a total optical of $\sim 5$ for these photons, most of it due to Compton scattering, and order unity optical depth due to photon–photon pair production.

Figures 33 and 34 show the cumulative optical depths for DS-going photons, starting from the point $\Gamma = 3$ in the transition and reaching the subshock, as a function of $\hat{\nu}_{sh}$. Comparing the results for $\Gamma_u = 30$ and $\Gamma_u = 10$, we find that the optical depth due to both scattering and photon–photon pair production are very similar for both values of $\Gamma_u$, suggesting a common structure and a common US-going photon spectrum in this region.

Figures 35 and 36 show the cumulative optical depths for DS-going photons, starting from the point $\Gamma = \Gamma_u/2$ in the transition and reaching the subshock, as a function of $\hat{\nu}_{sh}$. As was shown earlier, the shock-frame radiation in the transition region is dominated by photons with energy $\sim \Gamma^2 \gamma m_e c^2$ propagating toward the DS. The figures illustrate that the optical depth for these photons to reach the immediate DS is less than unity. On the other hand, photons with energies around the pair production threshold in the shock frame, $0.1 \lesssim \hat{\nu}_{sh} \lesssim 10$, will suffer a strong attenuation due to pair production.

5.4. Numerical Convergence

5.4.1. Resolution

The solution of the equations is obtained using iterations, as described in Section 4.1. Iterations are continued until the changes in integral quantities ($T$, $\Gamma$, $x_a$, $P_{rad}$, etc.) are less than $\sim 1\%$ between successive iterations. The resolution used for the solutions presented in the preceding sub-sections is given in Table 2. Using a Matlab code parallelized on four CPUs of Intel Xeon E5440 at 2.83 GHz, the $\Gamma_u = 30$ calculations take about 10 days to converge. The scaling of computational time is approximately $N^2 \nu^2 N_{\mu}^2 N_T$ where $N_{\nu}$, $N_{\mu}$, and $N_T$ are the numbers of frequencies, azimuthal directions, and spatial points on the
computational grid, respectively. The required memory for that run is approximately 5 Gb, and scales like $N_{\nu} N_{\mu} N_{t}$.

We found that the solutions are modified by $\sim 1\%$ when the resolution in $\tau_s$ is increased from $\delta\tau_s = 0.2$ to $\delta\tau_s = 0.1$, and therefore concluded that solutions obtained with either resolution are satisfactory. The convergence of the solutions with respect to the resolution in $\nu$, $\mu$, and $\mu_{sh}$ was tested using solutions with lower and higher resolutions for $\Gamma_a = 10$. We used several properties of the solution to quantify the convergence. The solution properties we checked were as follows.

1. $T_{\text{jump}}$. The temperature immediately behind the subshock.
2. The maximal $x_*$ value.
3. $P_{\text{sh,jump}}$. The value of the radiation pressure in the shock frame at the subshock.
4. $-\tau_s (\Gamma \beta = 5)$, the normalized optical depth US of the subshock at which the Lorentz factor drops by $\frac{1}{2}$.
5. $-\tau_{s,\text{nl}}$. The normalized optical depth US of the subshock at which the US evolution becomes nonlinear (see detailed explanation in Section 6.3).

The value of $\tau_{s,\text{nl}}$ is very sensitive to small changes in resolution, since it is set by the exponential decay of the number of photons arriving from the immediate DS. However, its exact value does not affect significantly the structure of the deceleration region. We use it here merely as a stringent test of numerical convergence.

The changes in the values of the test parameters as a function of resolution are given in Figure 37. The results were obtained using lower and higher resolutions in $\nu$ and $\mu$, and are presented as a function of $N_{\nu} \times N_{\mu}$, the product of the number of discrete values chosen for $\mu$ and for $\nu$.

The numerical error around the nominal resolution used in our calculations is few percent, except for the most sensitive parameter, $\tau_{s,\text{nl}}$, for which the numerical error is around 10%.

5.4.2. Changes in the Length of the DS

In order to verify that the boundary conditions imposed on the DS edge of the shock do not have a significant effect on the final results, around the subshock and in the shock transition region, we compare the results shown above to the results obtained with a solution including a longer DS region behind the subshock. We are limited in extending the DS because of numerical problems, caused by the proximity to a second sonic point. For this reason, we extend only the DS of the calculation for $\Gamma_a = 30$, from $\tau_s = 7$ in the calculations presented above to $\tau_s = 10$. The changes in integral quantities resulting from this modification of the DS region length are of order of a percent. The only quantity which changes by a larger amount, $\sim 3\%$, is $\tau_{s,\text{nl}}$. The temperature and velocity profiles obtained in the two calculations are compared in Figures 38 and 39.

6. SIMPLIFIED ANALYTIC MODELLING OF RRMS STRUCTURE

The key to understanding the structure of RRMS lies in the understanding of the behavior in the immediate DS. In our qualitative analysis of the immediate DS of RRMS, Section 2.3.3, we have argued that the immediate DS photon–electron–positron plasma should be close to CPE. This enabled us to demonstrate that the temperature in the immediate DS is expected to be $T_s \sim 0.4m_{\text{e}}c^2$, and that the immediate DS should be subsonic. These results are consistent with the numerical results presented in Section 5. We first discuss in some detail in Section 6.1 the accuracy of the CPE approximation for the description of the immediate DS.

Once the immediate DS is understood, a simple estimate of the spectrum of photons emanating from this region in the US direction leads to an understanding of the transition (deceleration) region, and of the asymptotic (far) US. These are discussed in Sections 6.2 and 6.3.

The flow DS of the immediate DS is smooth and NR. As the plasma flows away from the shock transition, it slowly produces the photon density needed for thermal equilibrium, eventually reaching the asymptotic DS thermal equilibrium conditions.
This “thermalization” phase is discussed in Section 6.4. Finally, Figure 38. Summary of numerical convergence tests. Results obtained with different resolutions in $\nu$ and $\mu$ are shown for $\Gamma_u = 10$, as function of $N_{\nu} \times N_{\mu}$, the product of the number of discrete values chosen for $\mu$ and for $\nu$. The y-axis shows the values of $T_{\text{jump}}$, $x_{\mu, \text{max}}$, $P_{\text{sh,jump}}$, $-\tau_*(\Gamma \beta = 5)$, and $-\tau_{\text{id}}$ (see the text for definitions), divided by their values obtained using the resolution given in Table 2. The different sets of $(N_{\mu}, N_{\nu})$ shown are [8, 26], [8, 41], [13, 16], [20, 41], and [13, 60], while the reference is [13, 41]. (A color version of this figure is available in the online journal.)

Figure 39. $\Gamma \beta$ vs. $\tau_*$ for $\Gamma_u = 30$, obtained using a DS optical depth of $\tau_* = 7$ (dashed line) and $\tau_* = 10$ (solid line). (A color version of this figure is available in the online journal.)

\[
\frac{n_{\parallel}}{n_{\nu, \text{eff}}} = \left. \int_0^\infty d\chi \chi^2 e^{-\sqrt{\chi^2 + T_*^2}} \right|_{\text{eq}} = \frac{K_2(\hat{T}^{-1})}{\hat{T}^2},
\]

where $K_2$ is the order 2 second kind modified Bessel function and $n_{\parallel} = n_\nu + n_\mu$ is the number density of leptons. The effective number of photons and the average energy per photon were calculated using the spectrum around the maximum of $I_\nu$ in the rest frame of the plasma, $v_{\text{peak}}$. Specifically, the numerical values shown for $n_{\nu, \text{eff}}$ and $\langle \hat{\nu} \rangle$ are the number of photons in the energy range $[v_{\text{peak}}/10, 10v_{\text{peak}}]$ and their average energy, respectively. The values of $\langle \hat{\nu} \rangle$ and $\langle \nu_{\parallel, \text{eff}} \rangle$ are shown as functions of $\hat{T}$ in the vicinity of the subshock. Immediately DS of the subshock these values are far from those expected for CPE, and they approach the CPE values away from the subshock. The figures show a systematic deviation from CPE. This is expected, since the high-energy DS photon beam, discussed in detail in Section 6.5, carries a significant fraction of the energy and is very weakly coupled to the plasma, due to suppression of the cross sections at high photon energy.

The lower panels of Figure 40 compare the values of $\Gamma$, $P_{\text{rad,sh}}$, and $x_\perp$ obtained in the numerical solution, with those obtained under the CPE approximation (note that under the CPE approximation the conservation equations, Equations (14)–(16), allow one to determine $\Gamma$, $P_{\text{rad,sh}}$, and $x_\perp$ as a function of $T$). Here too, the solution deviates from the CPE predictions immediately DS of the subshock, and approaches the CPE prediction away from it. We conclude that the CPE approximation yields estimates of the global flow variables ($\Gamma$, $P_{\text{rad,sh}}$, and $x_\perp$) which are accurate in the immediate DS to within tens of percent.

6.1. Immediate DS

Let us examine the accuracy of the CPE approximation, relating the temperature, the number of positrons, and the spectrum of the photons. In the top panel of Figure 40, we compare the average photon energy $\langle \hat{\nu} \rangle_{\text{eff}}$ and the $n_{\parallel}/n_{\nu, \text{eff}}$ ratio obtained in the immediate DS of the $\Gamma_u = 20$ solution, with those expected at CPE, $\langle \hat{\nu} \rangle_{\text{eff}} = 3\hat{T}$, and

\[
T/(m_e c^2) \begin{array}{c}{\text{Long DS}} \text{ (solid line)} \end{array} \end{array}
\]

\[
T/(m_e c^2) \begin{array}{c}{\text{Short DS}} \text{ (dashed line)} \end{array} \end{array}
\]

This “thermalization” phase is discussed in Section 6.4. Finally, we discuss in Section 6.5 the high-energy photon “beam” propagating from the transition region into the DS, and comment on the behavior in the $\Gamma_u \to \infty$ limit in Section 6.6.

6.2. The Transition Region

The transition or deceleration region is the region in which the energy and momentum flux of the US plasma are transferred to the radiation and to the $e^+e^-$ pairs. The behavior in the $\Gamma_u \gg \Gamma \gg 1$ regime may be understood using the following arguments.

1. The photons decelerating the plasma originate in the immediate DS and have a shock-frame energy of $\sim m_e c^2$.\]
and a rest-frame energy of $\sim \Gamma m_e c^2$. Since the immediate DS temperature is $T_s \sim 0.4 m_e c^2$ (see Section 6.1), the characteristic shock-frame energy of these photons is $h\nu \sim 3T_s \sim m_e c^2$. Across the transition region, these photons dominate the energy density in the rest frame of the plasma, where their energy is $\sim \Gamma m_e c^2$ (see, e.g., Figure 23).

2. The US-going photons decelerate the plasma by Compton scattering, and by pair production interactions with photons, that are generated within the transition region either by bremsstrahlung emission or by inverse Compton scattering (US-going photons that are backscattered by the DS flow). The three processes similarly contribute to the deceleration, as explained in point 8 below.

3. $T \sim \Gamma m_e c^2$. Both Compton scattering and photon–photon pair production generate electrons/positrons with characteristic energy $\sim \Gamma m_e c^2$, driving the plasma temperature to $\sim \Gamma m_e c^2$.

4. Pairs produced in the deceleration region drift with the plasma all the way to $\Gamma \sim 1$ without annihilating, due to the high temperatures that reduce the annihilation cross section ($\propto \log 2\hat{T}/\hat{T}$) and to the $\propto \Gamma^{-2}$ suppression of the collision rate.
5. The plasma rest-frame energy density is dominated by pairs rather than protons \((n_e + n_p)(m_e c^2 + 3T) > n_p m_p c^2\). Once \(\Gamma \ll \Gamma_u\), most of the energy flux is carried by radiation and pairs. The pairs carry a significant fraction of the energy flux (see Figure 13), hence their energy in the rest frame dominates over the protons rest mass.

6. A significant deceleration of a fluid element, \(\Gamma \rightarrow \Gamma/2\), requires that the number of Compton/pair production interactions occurring within it be similar to the number of leptons within it. The relative velocity between the two frames (\(\Gamma, \Gamma/2\)) is \(\beta' = 0.6\). Acquiring this velocity requires a momentum transfer of \(~2T/\beta\) to each lepton (recall, that for \(\Gamma \ll \Gamma_u\), the plasma energy density is dominated by pairs). This is similar to the momentum transfer by Compton scattering or pair production interaction of a typical US-going photon, for a plasma rest-frame temperature of \(~2\Gamma m_e c^2\).

7. The optical depth for typical US-going photons between \(\Gamma \rightarrow \Gamma/2\) is \(\Delta \tau \sim 1\). The number flux of US-going photons is similar to the sum of number fluxes of typical DS-going photons and pairs (pairs are DS going). The similarity between the number densities and the fact that DS-going pairs undergo \(~1\) interaction between \(\Gamma \rightarrow \Gamma/2\) implies that US-going photons roughly interact once as well.

8. The Thompson optical depth in the range \(\Gamma \rightarrow \Gamma/2\), is roughly \(\Delta \tau \sim \Gamma^2\), whether the deceleration is due to Compton scattering or due to pair creation on bremsstrahlung generated photons. This implies \(\tau_{\gamma}(\Gamma) \sim \Gamma^2\). In the range \(\Gamma \rightarrow \Gamma/2\) there is \(~1\) interaction per lepton crossing (see point 6). The Thomson optical depth required for a single Compton scattering is \(\delta \tau_{\gamma,\text{scat}} \sim \Gamma^2 \sim \Gamma^2\) due to the KN correction to the cross section. Similarly, the Thomson optical depth required for a single pair production on a “returning” (DS scattered) photon is \(\delta \tau_{\gamma,\text{ret}} \sim \Gamma^2 n_{\gamma,\text{ret}} / n_l \sim \Gamma^2\), where \(n_{\gamma,\text{ret}} \sim n_l\) is the number density of returning photons. Bremsstrahlung generated photons with energy \(hv \sim m_e c^2 / \Gamma\) have a large optical depth for pair creation on the US-going typical photons, since they do not suffer a suppression to the cross section. The number of these photons, produced up to a given point in the transition region is given by

\[
\delta \tau_{\gamma,\text{ff}} = \frac{Q_{\gamma,\text{ff}} \delta \xi_{\text{sh}}}{c},
\]

where \(Q_{\gamma,\text{ff}}\) is the production rate of photons that are able to upscatter to \(hv \sim m_e c^2 / \Gamma\) in the rest frame of the plasma, and \(\delta \xi_{\text{sh}}\) is the shock-frame distance over which \(\Delta \tau\) changes significantly. The Thomson optical depth required for producing enough photons to decelerate the plasma, \(\delta \tau_{\gamma,\text{ff}} \sim n_l\), is thus

\[
\delta \tau_{\gamma,\text{ff}} \sim \frac{\Gamma^2}{\alpha_e \tilde{g}(T, \tilde{\nu}) \Lambda_{\text{US}}},
\]

where \(10 \lesssim \tilde{g}(T, \nu) \lesssim 20\) is the Gaunt factor at high temperatures and low frequencies \(hv/(m_e c^2) \sim \Gamma^{-1}\) and \(\Lambda_{\text{US}} \sim 5\) is a logarithmic correction accounting for photons that are produced at low energies and upscatter to the required energy by the available \(~1\) number of Compton scatterings on the thermal electrons.

Two conclusions can be drawn. First, the Thomson cross section needed to decelerate a Lorentz factor \(\Gamma \sim \tau_{\gamma} \sim \Gamma^2\). Second, all three processes discussed in this point are comparable. Simply taking \(\Gamma^2 = -\tau_{\gamma}\) (where \(\tau_{\gamma}\) is measured from the subshock) results in a qualitatively good fit to the numerical results, as can be seen in Figure 41. It is evident that the deceleration, when approaching the subshock, has a universal structure for different \(\Gamma_u\) values.

The following additional properties are implied by the above considerations.

1. \(x_+ \sim (\Gamma_u / \Gamma^2) \times (m_p / m_e) / 8\) when \(\Gamma \ll \Gamma_u\). This follows from conservation of momentum flux, and the significance of pairs in the flux.

2. Most of the photons resulting from Compton scattering will propagate to the immediate DS without undergoing further interactions. The optical depth for scattering of photons originating from scattering into the DS direction is negligible since they have shock-frame energy of \(~\Gamma^2 m_e c^2\) and suffer a \(~\Gamma^{-2}\) attenuation in interaction rate. The optical depth for pair production is of order unity. This can be seen by the fact that the cross section and target photons for pair production are similar to the inverse Compton cross section and target photons of the \(e^+e^-\) in the deceleration region. In fact, the total optical depth is \(\lesssim 1\), as can be seen in Figure 36.

6.3. Far US

As was shown in Section 5.2, the radiation, as seen in the rest frame of the far US plasma, is strongly dominated by a beamed \([\mu_{\text{rest}} \approx -1 + 1 / \Gamma_{u}^2]\), radiation field with photons of typical energy of several times \(\Gamma_u m_e c^2\). To understand the main physical properties of this region, it is useful to approximate the radiation field as a delta function in energy and direction, going in the US direction. The asymptotic solution for such a radiation field can be easily found to be an exponential growth of the parameters \(P_{\text{rad},\text{sh}}, \Gamma_{\text{rad},\text{sh}}, \Gamma_u \beta_n - \Gamma \beta,\) and \(T\), with the same exponent, \(\lambda_{\text{sh}}\), and an exponential growth of \(x_+\) with an exponent \(2\lambda_{\text{sh}}\) (see Sagiv 2006). An approximate value for \(\lambda_{\text{sh}}\)
assumption with Equations (76) and (77), we obtain
\[ \frac{d\hat{T}}{d\tau_e} = -2 \sqrt{\frac{\pi}{8}} \frac{g_{\text{eff}}(\hat{T}) \Lambda_{\text{eff}}(\hat{T})}{\beta_d \sqrt{\hat{T}}} e^{-1/\hat{T}}. \] (78)

The flow reaches \( n_{\gamma,e} \approx n_\gamma (x_+ \approx 1) \) when \( \hat{T} \approx 0.06 \), with a weak dependence on \( \Gamma_u \). From Equation (78), we see that the length scale is set by the lowest temperatures in this range, and reasonable parameters yield \( \approx 10^5 \) optical depths required for the positrons to annihilate.

In the \( x_+ < 1 \) region, a gradual increase in photon number lowers the temperature until thermal equilibrium is reached. The length scale for this process is
\[ L_T = \beta_d c \frac{n_{\gamma,\text{eq}}}{Q_{\text{eff},d}}, \] (79)

where \( n_{\gamma,\text{eq}} \approx a_{\text{eq}} T_3^3/2.8 \) is the thermal equilibrium photon density and \( Q_{\text{eff},d} \approx g_{\text{eff},d} a_{\text{eff},d} n_\gamma \sigma_T / \sqrt{\hat{T}} \) is the photon generation rate in the DS. An estimate for \( L_T \) yields
\[ \frac{L_T}{(\sigma_T n_\gamma)^{-1}} \approx 3 \times 10^6 (3\beta_d) \left( \frac{g_{\text{eff},d} \Lambda_{\text{eff},d}}{10} \right)^{-1} 1^{3/4} a_{\text{eff},d} n_\gamma^{1/8}. \] (80)

In this region, the temperature drops as a power law \( T(\tau_e) \propto (\tau_e - \tau_0)^{-2} \), as was shown in KBW10.

6.5. The High-energy Photon Component Beamed in the DS Direction

As can be seen in Figures 25 and 26, the immediate DS has a high-energy photon component beamed in the DS direction. We use below the simplified analysis presented in Section 6.2 to derive the characteristics of the spectrum of this beam.

The photons in this beam originated from the immediate DS, propagated into the transition region and then were Compton scattered once before returning to the DS. Photons that were scattered at a point with Lorentz factor \( \Gamma \) return to the DS with an energy boosted to \( \sim \Gamma^2 m_e c^2 \) and within a beaming angles \( \theta \sim \Gamma^{-1} \).

Denote the shock-frame intensity of US-going photons with typical energies \( I_0(\tau_e) \). Conclusion 7 in Section 6.2, leads to the equation
\[ I_0(\Gamma) = \varepsilon_\Gamma I_0(\Gamma/2), \] (81)

where \( \varepsilon_\Gamma \sim 1/3 \) is related to the exact total optical depth for typical US-going photons from \( \Gamma/2 \) to \( \Gamma \). Assuming that the fraction of photons that scatter is constant with \( \Gamma \), the resulting intensity emitted at \( \Gamma \), \( I_\Gamma(\Gamma) \) will be
\[ I_\Gamma(\Gamma) \approx 4\varepsilon_\Gamma I_0(\Gamma/2), \] (82)

since the photons gain a factor of \( \sim \Gamma^2 \) to their energy when scattered at \( \Gamma \). The scattered photons are beamed into a cone with an opening angle \( \Gamma^{-1} \) in the DS direction. Since the losses of the scattered photons on the way to the immediate DS are less than a factor of two and depend weakly on the angle and energy of the photon, we find that the spectrum of the high-energy beam \( I_B \) can be approximately described as
\[ \hat{\nu}_{\text{sh}} I_B(\hat{\nu}_{\text{sh}}, \theta_{\text{sh}}) \propto \hat{\nu}_{\text{sh}}^{a_1} \Theta(\hat{\nu}_{\text{sh}}^{-1} - \hat{\nu}_{\text{sh}}^{1/2}) \Theta(\hat{\nu}_{\text{max}} - \hat{\nu}_{\text{sh}}), \] (83)

where \( \hat{\nu}_{\text{max}} \approx \Gamma^3 \) and \( a_1 \approx \log_2(4\varepsilon_\Gamma)/2 \) is close to zero, and is equal to zero when \( 4\varepsilon_\Gamma = 1 \).
We next verify that this analysis complies with the numerical results (results shown for $\Gamma_u = 20$ calculation). Figure 43 shows the shock-frame intensity of a beam with $\theta_{sh} \approx 10^{-2}$ with different $\nu_{sh}$ along the shock, versus $\Gamma^2/\nu_{sh}$. We see that the intensity is mostly contributed by the part in the flow in which $\Gamma^2/\nu_{sh} \approx 0.2$ (where the slope in $I_{sh}$ is maximal), as the physical picture requires. Figure 44 shows the shock-frame intensity immediately after the subshock, at different $\nu_{sh}$, as a function of $\theta_{sh}\nu_{sh}^{1/2}$. We see that the structure of the beam is such that the different energies are beamed according to Equation (83).

6.6. The $\Gamma_u \to \infty$ Limit

Based on the results for $\Gamma_u \lesssim 30$ and the analysis above, it appears that for $\Gamma_u \to \infty$, $(T, \Gamma)$ approach asymptotic profiles in the regime where $\Gamma \ll \Gamma_u$ and $\xi \gg 1$. In particular, $T \sim m_e c^2$ in the immediate DS and $\Gamma \sim \tau^2$ in the transition region. However, we have also seen that the high-energy beam becomes more dominant as $\Gamma_u$ grows. The structure of the shock, particularly the immediate DS, may be different if the high-energy beam becomes the dominant carrier of momentum and energy of the radiation. Unfortunately, a full calculation of very high $\Gamma_u$ shocks is beyond our current numerical capabilities, and requires further investigation.

7. NR RMS REVISITED

In this section, we briefly describe a preliminary application of the code to NR shocks. Our numerical scheme was designed and optimized for the solution of the relativistic problem, and is not efficient and easy to use for NR problems. The main difficulties are (1) solving the momentum and energy conservation equations for the velocity and temperature of the plasma is problematic due to the negligible contributions of the thermal energy and pressure. (2) Radiation field convergence requires a large number of iterations, roughly one iteration per single Compton scattering, implying $\propto \beta^{-2}$ iterations.

A different scheme for finding the plasma temperature and velocity and a different boundary condition in the far DS were used for solving the NR problem:

1. The temperature was set to the local CE value calculated from the radiation field and the velocity was found by solving the momentum conservation only. This approximation is justified in the case where $n_\gamma/n_e \gg 1$, where $n_\gamma$ is the number density of photons, which holds in the transition region (when the energy density of the radiation is a fair fraction of the flow) and the DS of an NR RMS. Convergence required that the temperature be set to a value that is slightly smaller than the actual CE value.

2. The following DS boundary condition was used. The radiation field in the US direction $I_{sh}^\parallel (\mu_{sh} < 0, \tau_s = \max(\tau))$, was set to represent the radiation field at a chosen point in the DS. This was done by assuming a Wien spectrum with a temperature lower than $T_s$ (see Equation (8)), and an intensity that satisfies the equilibrium at the DS velocity as expected in the DS well behind the velocity transition.

The radiation transport is solved similarly to the relativistic case.

We find a preliminary solution for a shock with US energy per proton $\varepsilon = 50$ MeV and a very low density ($n_u = 10^6$ cm$^{-3}$), which ensures that bremsstrahlung absorption remains unimportant until after the velocity has already reached its DS value. In the calculation shown here, absorption is everywhere unimportant, since it does not reach the DS temperature. We stress that the resulting solution contains a limited region of optical depth $\sim \beta^{-1}$ below the velocity transition, hence the temperature profile may not correctly represent the actual solution.

Figure 45 shows the structure $(\Gamma^2, \Gamma^2, \Gamma^2)$ of a shock with $\varepsilon = 50$ MeV as a function of $\tau_s$. The dotted black line is the analytic solution for $\Gamma^2$ obtained by Weaver (1976, Equation (5.10)), using $\sigma_C = 0.56c_\gamma$ for the average Compton cross section, suitable for $h\nu \approx 0.5m_e c^2$ typical photon energy in the transition region. Weaver’s solution deviates from the numerical solution near the immediate DS. This is due to the lower average photon frequency there, compared to that in the transition region, which leads to an increase in $\sigma_C$ in the (more accurate) numerical calculation.

Examining the radiation spectra obtained in our numerical calculations, Figure 46, we are able to verify the validity of two of Weaver’s assumptions. First, it is clear that the spectrum at each point along the shock is dominated by photons within a narrow energy range. Second, the anisotropy of the radiation is of order $\beta$, which is the expected anisotropy due to diffusion of the radiation.
To conclude, the preliminary solution found using our numerical scheme is consistent with Weaver’s results. In addition, the detailed spectra support the validity of Weaver’s approximations regarding the radiation spectrum. The fact that the results for NR shocks are in agreement with previous work supports the validity of the numerical scheme.

8. DISCUSSION

We have calculated and analyzed the structure of RRMSs. A qualitative discussion of the shock physics was presented in Section 2, including analytic estimates for the length scales of NR RMS (see Figure 1 for a schematic shock structure description) and of the immediate DS temperatures of both NR RMS (Equation (8)) and RRMS (Equation (11)). We have also shown (in Section 2.3.3) that the immediate DS of RRMS is expected to be subsonic, and concluded that the structure of RRMS must include two sonic points.

In Section 3, we derived a dimensionless form of the equations describing the conservation and transport equations determining the structure of the shock, and described in detail the radiative processes included in our treatment and the approximations we used. In Section 4, we presented a novel iteration scheme for numerically solving the equations, and demonstrated its validity by applying it to several test cases. In Section 5, we have presented numerical solutions for the profiles and radiation spectra of RRMS, for upstream Lorentz factors $\Gamma_u$ in the range of $\sim 30$. The main results obtained are described below.

1. Structure and radiation spectrum. In Section 5.1, we showed that the structure of RRMS can be divided into four regions, from US to DS: the far US, the transition region, the immediate DS, and the far DS. The far US is characterized by a velocity close to the US velocity and a radiation energy–momentum flux much smaller than that of the US plasma. The transition region is where the
velocity ($\Gamma \beta$) changes significantly, approaching $\Gamma \beta \sim 1$, while the momentum and energy fluxes are transferred to the $e^+e^-$ pairs and to the radiation. In both regions, the radiation spectrum (shown in Section 5.2) is dominated in the plasma rest frame by US-going photons with energy of a few times $\Gamma m_e c^2$. In the shock frame, the radiation is dominated by DS-going photons, beamed into a cone with opening angle $\sim \Gamma^{-1}$, and a typical energy $\Gamma^2 m_e c^2$. In the far US, the temperature grows exponentially with $\tau_{\nu}$ toward the DS (Figure 42), until it reaches $\sim m_e c^2$ ($\tau_{\nu}$ is the Thomson optical depth for photons moving toward the US). The temperature then continues to grow at a slower rate until it reaches $T/(m_e c^2) \sim \Gamma$ in the transition region, and then decreases, approximately following the deceleration, $T/(m_e c^2) \sim \Gamma$ (Figure 8). The transition region ends at a subshock, possibly mediated by plasma instabilities, with a velocity jump of $\delta(\Gamma \beta) \sim 0.1$ and a slight increase in temperature, to $0.4 < T/(m_e c^2) < 0.9$ for $6 < \Gamma_u < 30$ (see Figures 7 and 9). The immediate DS, following the subshock, is characterized by a small change of velocity, approaching the DS value within $\sim 2$ Thomson optical depths, and a temperature that decreases on a scale of a few Thomson optical depths to $T/(m_e c^2) \sim 0.25$. The ratio of positron density to proton density in the immediate DS reaches a maximum of $\sim 140 \Gamma u$ (see Figure 11), approximately when the temperature crosses $T/(m_e c^2) \sim 0.3$, and then decreases. The radiation spectrum in the immediate DS is dominated by a relatively isotropic component with $h\nu \sim 3T$, but a fraction of $10\% - 20\%$ of the energy flux is carried by a high-energy photon tail, strongly beamed toward the DS, with a cutoff at $\sim T^2 m_e c^2$ and a nearly flat spectrum, $\nu F_\nu \propto \nu^0$ (see Figures 26–28).

2. **Comparison with KBW10.** Our numerical results are consistent with, and therefore confirm the validity of, the approximate analytic analysis of KBW10. In particular, the immediate DS temperatures derived here are consistent with those derived in KBW10. This supports the conclusion of KBW10 that SN shock breakouts may account for the X-ray outburst associated with SN2008D, and possibly for other SN-associated outbursts with spectra not extending beyond few 100 keV (e.g., XRF060218/SN2006aj).

3. **Optical depths due to Compton scattering and pair production.** In Section 5.3, we showed that the optical depth of typical photons ($h\nu \sim m_e c^2$) leaving the subshock in the US direction is a few. The optical depth is provided by both Compton scattering and pair production, the latter having a somewhat smaller contribution (see Figures 31 and 32). Photons with much smaller energies are scattered close to the immediate DS and do not reach the transition region (see Figures 29 and 30). Typical DS-going photons from the transition region (with shock-frame energy $\sim \Gamma^2 m_e c^2$) undergo very few interactions on the way to the immediate DS (see Figures 33–36).

4. **The importance of $e^+e^-$ pairs.** In Figure 13, we show that the pairs produced along the shock transition and in the immediate DS play an important role in decelerating the US plasma. The energy flux removed from the protons is dominated by pairs over radiation during most of the transition, and the ratio between the two becomes larger as $\Gamma_u$ grows. The pair energy flux is dominated by thermal energy flux since the transition region temperatures are relativistic ($T > m_e c^2$).

We find several characteristics of the structure of RRMS, which are qualitatively different from those of NR RMS.

1. The Thomson optical depth of the transition region is much larger than unity, is dominated by pairs, and grows with $\Gamma_u$ in a manner faster than linear. However, the actual (KN corrected) optical depth (including pair production) for a typical photon crossing the shock is of order of a few.

2. The temperatures of the pair plasma within the transition region are relativistic, $T > m_e c^2$.

3. The relativistic shock structure includes a sonic point crossing, in which the flow changes from supersonic to subsonic. We find that this sonic point must be a subshock mediated by processes not included in our calculation, which operate on a scale much shorter than the radiation mean free path (e.g., plasma instabilities, see Equation (10)).

4. $e^+e^-$ pairs carry most of the energy and momentum flux in the transition region.

5. In RRMS, a fair fraction of the energy density in the immediate DS is carried by a non-thermal tail of high-energy photons, where in the DS of NR RMS the radiation is in CE with the plasma.

We developed in Section 6 an analytical understanding of the key features of the shock structure and radiation spectrum. Several points should be highlighted.

1. **Immediate DS.** The key to understanding the structure and radiation spectrum of RRMS is the understanding of the immediate DS. The immediate DS of RRMS is close to CPE (see Figure 40), which, due to the fast increase of the number of pairs with temperature, sets the temperature to a large fraction of $m_e c^2$ (KBW10). The large amount of positrons and the high temperature imply a relativistic speed of sound in matter $\beta_{\nu a} \sim 1/\sqrt{3}$, and combined with the low velocity in this region that quickly approaches DS value $\beta_{\nu} \lesssim 1/3$, a subsonic regime is inevitable. The immediate DS acts as the supplier of photons directed toward the US, which decelerate the incoming plasma through Compton scattering and pair production.

2. **Deceleration region.** For $\Gamma \ll \Gamma_u$, we find $\Gamma (-\tau_{\nu}) \approx \sqrt{-\tau_{\nu}}$ (see Figure 41), where the subshock is located at $\tau_{\nu} = 0$. This behavior is due mainly to the KN scaling of the cross sections, and to the fact that the optical depth for US-going photons is of order few. This approximation follows closely the numerical results up to $\Gamma \approx \Gamma_u/2$.

3. **High-energy photon beam.** The immediate DS has a high-energy photon component narrowly beamed in the DS direction, with a nearly flat power-law-like spectrum, $\nu I_\nu \propto \nu^0$ and an energy cutoff at $\sim T^2 m_e c^2$. The photons in this beam originated from the immediate DS, propagated into the transition region and then were Compton scattered once, before returning to the DS. Photons that were scattered at a point with Lorentz factor $\Gamma$ return to the DS with an energy boosted to $\sim \Gamma^2 m_e c^2$ and within a beaming angle $\theta \sim \Gamma^{-1}$. An approximate description of the resulting spectral and azimuthal structure of the beam is given in Equation (83). The total optical depth for these photons to reach the immediate DS is small, and they carry $10\% - 20\%$ of the energy flux in the immediate DS. The beam is only stopped far into the DS, producing pairs on low-energy photons.

4. **Far US.** In the far US, $P_{rad,sh,}\ F_{rad,sh,}\ \Gamma_u \beta_{\nu a} - \Gamma \beta$, and $T$, all grow exponentially with $\tau_{\nu}$ with the same exponent, $\lambda_{\nu a}$.
5. Thermalization length scale. The thermalization length is much longer than the shock transition, both in terms of Thomson optical depth and in real distance. Thermal equilibrium is reached ~10^6 Thomson optical depths into the DS.

Finally, we showed for completeness in 7 the preliminary results of a detailed calculation of the structure of an NR RMS including full radiation transport. The results are consistent with previously published ones, and support the validity of the approximations used for solving the problem in earlier work.

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APPENDIX A
NOTATIONS FREQUENTLY USED IN THE PAPER

A.1. Subscripts, Superscripts, and Miscellanea

as: asymptotic upstream behavior
US: upstream
DS: downstream
CE: Compton equilibrium
CPE: Compton-pair equilibrium
d: asymptotic downstream (postshock) value
dec: deceleration
p: proton value
NR: non-relativistic
pl: plasma value
rad: radiation field value
sh: shock-frame value
rest: plasma rest-frame value
u: asymptotic upstream (preshock) value
γ': photon value
+: positron value
∧ (hat): normalized units

A.2. Symbols

d_{bb}: radiation constant
σs: Compton scattering cross section
F_{rad} (erg cm^{-2} s^{-1}): radiation energy flux
h (erg s): Planck’s constant
I(Ω, ν) (erg cm^{-2} s^{-1} str^{-1} Hz^{-1}): specific intensity of radiation field
η(Ω, ν) (erg cm^{-3} s^{-1} str^{-1} Hz^{-1}): emissivity coefficient
ℓ (cm): photon mean free path
n_e, n_p, n_i, n_y_eff (cm^{-3}): number density of electrons, positrons, ions (protons), and typical photons
n_a (cm^{-3}): upstream proton (and electron) number density
P (erg cm^{-3}): pressure
P_{rad} (erg cm^{-3}): radiation pressure
Q_x (cm^{-3} s^{-1}): net rate of positron production
T (erg): electrons and positron temperature
T ≡ T/m_e c^2
T_0, T_{zz} (erg cm^{-3}): components of stress–energy tensor (energy and momentum fluxes, respectively)
x_+ = n_+ / n_i: positron fraction
z (cm): length along flow direction
χ(Ω, ν) (cm^{-1}): absorption coefficient
α_: fine structure constant
β ≡ √1 − Γ^{-2}: flow velocity (units of c)
Γ: flow Lorentz factor
Γ_a: upstream flow Lorentz factor
γ_{sh}: Lorentz factor associated with random motion of e^+ and e^{-}
de = 1 − Γβ / Γ_a β: asymptotic deceleration parameter
ε_sc (erg): radiation emission cutoff energy due screening
ζ: Riemann’s zeta function
η ≡ exp(−νE) = 0.5616 (where νE ≃ 0.5772 is Euler’s constant)
λ_D (cm): Debye length [≡ √T/4π e^2 (n_e + n_i)]
μ: cosine of angle relative to positive z-axis (flow direction)
ν (Hz): photon frequency
ν: total Compton scattering cross section
σ_s (cm^2): total Compton scattering cross section
σ_{γγ} (cm^2): cross section for γγ → e^+e^- pair production
σ_T (cm^2): Thomson cross section (8πr_0^2 / 3)
τ_D: Thomson optical depth for upstream-going photons, given by 
τ_D = ∫ f(1 + β)(n_e + n_i) σ_T dζ_{sh}.
τ_a = 0 at the subshock and grows toward the downstream.

APPENDIX B
COMPTON SCATTERING APPROXIMATION

In order to reduce (significantly) the computing time, we use an approximate Compton Scattering Kernel (CSK) that represents the physically important features of the exact CSK. We make the approximation that the scattering is isotropic in the plasma frame, and write the differential cross section as

\[ \frac{dσ_s}{dΩ dν} (ν, Ω → ν', Ω') = \frac{1}{4π} σ_s(ν, T) f_d(ν, T, ν'). \]  

(B1)

Here, σ_s is the total cross section given in Equation (42), and f_d is the spectral redistribution function of the photons. We require scattering to conserve photon number and require f_d to satisfy

\[ \int_0^∞ f_d(ν, T, ν') dν' = 1, \]  

(B2)

and

\[ \int_0^∞ f_d(ν, T, ν') ν' dν' = ν_0(ν, T), \]  

(B3)

where ν_0 is the average frequency of scattered photons. The approximations used for ν_0 and f_d are given below. We use different approximations for low (NR) temperatures and for high (relativistic) temperatures, with a transition temperature \( T_{m0} = 0.25 \). We use a smooth interpolation between the two temperature regimes (over a ~10% interval in \( T \)).

B.1. Low T (\( T < T_{m0} \))

Average energy shift. We chose ν_0 to produce the correct average energy shift for \( ν \ll 4T \) and for \( ν \gg 4T \), and no energy shift for NR Compton equilibrium, \( ν = 4T \). We use

\[ \frac{ν_0}{ν} = \min \left( 1 + 4T(4T + 1) - ν(ν + 1) \right) \left( 1 + a_ν ν \right)^2, \]  

(B4)
for \( \hat{\nu} < 4\hat{T} \), and

\[
\frac{\hat{v}_0}{\hat{v}} = \frac{1}{1 + \log \left( \frac{\hat{v} + 1}{4\hat{T} + 1} \right)} \tag{B5}
\]

for \( \hat{v} > 4\hat{T} \). \( a_\nu(\hat{T}) \) is determined by requiring that for a Wien spectrum, the energy gain of photons with energy less than \( 4\hat{T} \),

\[
P_{\text{gain}}(\hat{T}, a) \propto \int_0^{4\hat{T}} d\hat{\nu} \hat{\nu}^3 e^{-\hat{\nu}/\hat{T}} \sigma(\hat{v}, \hat{T})(\hat{v}_0(\hat{\nu}, \hat{T}, a_\nu) - \hat{v}),
\]

be equal to the energy loss of higher energy photons,

\[
P_{\text{loss}}(\hat{T}) \propto \int_{4\hat{T}}^{\infty} d\hat{\nu} \hat{\nu}^3 e^{-\hat{\nu}/\hat{T}} \sigma(\hat{v}, \hat{T})(\hat{v}_0(\hat{\nu}, T) - \hat{v}). \tag{B7}
\]

We use a fourth-order polynomial for \( a(\log \hat{T}) \),

\[
a_\nu(\hat{T}) = -0.003763 \log(\hat{T})^4 - 0.0231 \log(\hat{T})^3
- 0.01922 \log(\hat{T})^2 - 0.129 \log(\hat{T}) + 3.139,
\]

which is accurate to better than a percent, and set \( a_\nu(\hat{T} < 0.01) = a_\nu(\hat{T} = 0.01) \).

**Photon redistribution.** We choose a photon redistribution function that follows the shape of a thermal spectrum with a target temperature \( \hat{T}_{\text{tar}} = \hat{v}_0(\hat{\nu}, \hat{T})/4 \),

\[
f_d(\hat{\nu}, \hat{T}, \hat{v}) = A\hat{\nu}^3 e^{-\hat{\nu}/\hat{T}_{\text{tar}}},
\]

where

\[
A = \left( \int_0^{\infty} \hat{\nu}^3 e^{-\hat{\nu}/\hat{T}_{\text{tar}}} d\hat{\nu} \right)^{-1} = \frac{1}{6\hat{T}_{\text{tar}}^4}. \tag{B10}
\]

**High \( \hat{T} \)—Klein–Nishina corrections.** For \( \hat{v} > 1/(4\hat{T}) \), in the Klein–Nishina regime, we choose

\[
f_d(\hat{\nu}, \hat{T}, \hat{v}) = \frac{\hat{v}_0^2 e^{-\hat{\nu}/\hat{T}} \sigma(\hat{v'}, \hat{T})}{a_\nu(\hat{T}) \hat{T}^3 \sigma_T}. \tag{B11}
\]

The value of \( a_\nu \) is chosen so that the integral over \( f_d(\hat{\nu}, \hat{T}, \hat{v}) d\hat{\nu} \) is 1. This form ensures that a Wien spectrum with a relativistic temperature is unchanged by scattering of electrons with the same temperature. We use a fourth-order polynomial approximation,

\[
a_\nu(\hat{T}) = -0.0046111 \log(\hat{T})^4 + 0.0071973 \log(\hat{T})^3
+ 0.09079 \log(\hat{T})^2 - 0.3166 \log(\hat{T}) + 0.3146
\]

and set \( a(\hat{T}) = a(5)(\hat{T}/5)^{-1.7} \) for \( \hat{T} > 5 \). This approximation is accurate to better than a percent for temperatures below \( m_e c^2 \), and to better than 10% everywhere.

**Low \( \hat{v} \)—inverse Compton.** In order that a power-law spectrum of the form \( I_\nu \propto \nu^{\gamma} \) retains its form after scattering, and in order to reproduce the ultrarelativistic limit of the energy boost, \( 16\hat{T}^2 \), we choose

\[
f_d(\hat{\nu}, \hat{T}, \hat{v}) \propto \sqrt{\nu e^{-\frac{v}{4\hat{T}^2}} \Theta(v' - v)}. \tag{B13}
\]

We use a cutoff at \( 8\hat{T} \) to avoid overproducing photons at high frequencies, and normalize accordingly.

**Appendix C**

**Plasma Speed of Sound**

Below is a short derivation of a general formula for the speed of sound in a plasma of electrons, protons and \( e^+e^- \) pairs, neglecting the thermal pressure of the protons (valid for \( T \ll m_p c^2 \)). Here, \( n_1 \) is number density of leptons (electrons+positrons), and \( n_p = n_1/(2x_e + 1) \). For convenience, we use \( \hat{\rho} \equiv p/m_e c^2, \hat{e} \equiv e/m_e c^2 \). Let \( s \) be the entropy per lepton. Rewriting Equations (24) and (25), we have

\[
\hat{\rho} = n_1 \hat{T}, \tag{C1}
\]

\[
\hat{e} = n_1 \left( 1 + \hat{f}(\hat{T}) + \frac{m_p}{2x_e + 1} \right), \tag{C2}
\]

where we define

\[
\hat{f}(\hat{T}) = \frac{3}{2} f(\hat{T}) \hat{T}, \tag{C3}
\]

where \( f(\hat{T}) \) was defined in Equation (24). The derivative with respect to \( \hat{T} \) at constant \( s \) is denoted by \( \hat{.} \). Constant entropy per lepton implies

\[
\left( \frac{\hat{\rho}}{n_1} \right)' = -\hat{\rho} \left( \frac{1}{n_1} \right)' \tag{C4}
\]

We therefore have

\[
\hat{f}' = \frac{\hat{T} n_1}{n_1}, \tag{C5}
\]

and

\[
\hat{\rho}' = n_1 \hat{f}' + n_1 \tag{C6}
\]

\[
\hat{e}' = n_1 \hat{f}' + \left( 1 + \hat{f} + \frac{m_p}{2x_e + 1} \right) \frac{n_1 \hat{f}'}{\hat{T}}. \tag{C7}
\]

The speed of sound is finally given by

\[
\beta_{s\mu} = c_{s\mu}/c = \sqrt{\frac{\hat{\rho}'}{\hat{e}'}} = \sqrt{\frac{1 + 1/\hat{f}'}{1 + \hat{f}' + \frac{m_p}{2x_e + 1} + \hat{T}'}}. \tag{C8}
\]

This equation can be easily verified to obey the asymptotic NR and ultrarelativistic limits.

**Appendix D**

**Matlab Code for Discretization of \( \mu_{s\mu} \)**

The following Matlab code is used to get the discretized values of \( \mu_{s\mu} \). The inputs are \( N_{\text{mu}} \) (the approximate number of \( \mu \) values required) and \( G_u \), which stands for \( \Gamma_u \). The output contains a vector of \( \mu_{s\mu} \) values and its length \( N_{\text{mu\_out}} \), which is \( \geq N_{\text{mu}} \). This function calls the function Gauss_for_mus(n) which returns the zeros of Legendre polynomial of the nth order. function \( [\mu_{s\mu\_out}, N_{\text{mu\_out}}] = \text{get_mus}(N_{\text{mu}}, G_u) \)

\[
b_u = \text{sqrt}(1-G_u^2); \tag{D1}
\]

\[
N_L = \text{floor}(N_{\text{mu}}/2); \tag{D2}
\]

\[
N_R = N_{\text{mu}} - N_L; \tag{D3}
\]
\[
\mu_{\text{rest\_max}} = 1 - 1/N_R;
\]
\[
\mu_{\text{sh\_max}} = (\mu_{\text{rest\_max}} + b_u) / (1 + \mu_{\text{rest\_max}} b_u);
\]
\[
\text{one\_minus\_mu\_min} = 1 - \mu_{\text{sh\_max}};
\]
\[
\% \mu > 0
\]
\[
q = (1/\text{one\_minus\_mu\_min})^{1/N_R};
\]
\[
\text{for } l_R = 1:N_R,
\]
\[
\mu_{\text{sh}(1)} = 1 - \text{one\_minus\_mu\_min} q^{(1_R-1)};
\]
\[
\text{while } \mu_{\text{sh}(1_R)} > 1.5/N_L;
\]
\[
\mu_{\text{sh}(1_R)} = \mu_{\text{sh}(1_R)} - 1/N_L;
\]
\[
\% \mu < 0;
\]
\[
[\mu_{\text{sh\_L}}, \text{del\_mu\_no\_use}] = \text{Gauss\_for\_mus}(2*N_L);
\]
\[
\mu_{\text{sh\_L}} = \mu_{\text{sh\_L}}(1:N_L)^{'};
\]
\[
\mu_{\text{sh\_out}} = \text{sort}(\mu_{\text{sh\_L}});
\]
\[
N_{\text{mu\_out}} = \text{length}(\mu_{\text{sh\_L}});
\]

**APPENDIX E**

**FRAME TRANSFORMATIONS**

Below are some useful transformation rules relating the values of \(\nu, I, \eta,\) and \(\chi\) measured in the shock and plasma rest frames:

\[
\mu_{\text{sh}} = \frac{\mu + \beta}{1 + \beta \mu}, \quad \mu = \frac{\mu_{\text{sh}} - \beta}{1 - \beta \mu_{\text{sh}}}, \quad (E1)
\]
\[
\nu_{\text{sh}} = \nu (1 + \beta \mu), \quad \nu = \nu_{\text{sh}} (1 - \beta \mu_{\text{sh}}), \quad (E2)
\]

\[
\frac{I(\mu, \nu)}{I_{\text{sh}}(\mu_{\text{sh}}, \nu_{\text{sh}})} = \left( \frac{\nu}{\nu_{\text{sh}}} \right)^3, \quad (E3)
\]
\[
\frac{\eta(\mu, \nu)}{\eta_{\text{sh}}(\mu_{\text{sh}}, \nu_{\text{sh}})} = \left( \frac{\nu}{\nu_{\text{sh}}} \right)^2, \quad (E4)
\]
\[
\frac{\chi(\mu, \nu)}{\chi_{\text{sh}}(\mu_{\text{sh}}, \nu_{\text{sh}})} = \left( \frac{\nu}{\nu_{\text{sh}}} \right)^{-1}. \quad (E5)
\]

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