Replica approach to the KPZ equation with the half Brownian motion initial condition

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Abstract

We consider the one-dimensional Kardar–Parisi–Zhang (KPZ) equation with the half Brownian motion initial condition, studied previously through the weakly asymmetric simple exclusion process. We employ the replica Bethe ansatz and show that the generating function of the exponential moments of the height is expressed as a Fredholm determinant. From this, the height distribution and its asymptotics are studied. Furthermore, using the replica method we also discuss the multi-point height distribution. We find that some good properties of the deformed Airy functions play an important role in the analysis.

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1. Introduction

Surface growth phenomena appear widely in nature and have attracted much attention in non-equilibrium physics. In 1986, Kardar, Parisi and Zhang proposed a stochastic differential equation which describes surface growth with local interaction [1]. For a one-dimensional case, the Kardar–Parisi–Zhang (KPZ) equation is given by

\begin{equation}
\frac{\partial h(x,t)}{\partial t} = \lambda \left( \frac{\partial h(x,t)}{\partial x} \right)^2 + v \frac{\partial^2 h(x,t)}{\partial x^2} + \sqrt{D} \eta(x,t). \tag{1.1}
\end{equation}

Here $h(x,t)$ represents the height of the surface at position $x \in \mathbb{R}$ and time $t \geq 0$. The first term represents the effect of nonlinearity and the second one describes the smoothing effect of the surface. $\eta(x,t)$ represents randomness described by the Gaussian white noise with covariance,

\begin{equation}
\langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t'). \tag{1.2}
\end{equation}

The parameters $\lambda$, $v$ and $D$ determine their strengths. For the one-dimensional KPZ equation (1.1), it has been shown by the dynamical renormalization group that the height fluctuation scales as $O(t^{1/3})$ as $t$ goes to infinity [1]. The exponent $1/3$ coincides with the ones found in Monte Carlo simulations of many stochastic models of surface growth. The
KPZ equation is accepted as a prototypical equation describing the universality class called the KPZ universality class.

Our understanding of the KPZ universality class has deepened in the last decade. Not merely the exponent but the height distribution functions have been computed based on intriguing connections with the random matrix theory [2]. For the totally asymmetric simple exclusion process (TASEP), which is an exactly solvable model in the KPZ universality class, the current distribution function for the step initial condition has been obtained exactly and it was found that in the long-time limit, it converges to the GUE (Gaussian unitary ensemble) Tracy–Widom distribution function [5], the largest eigenvalue distribution in the random Hermitian matrix [2].

An interesting finding from the studies of distribution functions is that they detect the difference of initial conditions, which the scaling exponent of the height cannot. For example, in the surface growth model called the polynuclear growth (PNG) model, it has been recognized that the GUE Tracy–Widom distribution describes the height distribution in the droplet growth [3], while in the flat initial condition, it is described by the GOE (Gaussian orthogonal ensemble) Tracy–Widom distribution [4]. Furthermore, generalizations of these results to the multi-point distribution function have also been studied and the universal processes, the Airy_3 [6] and Airy_1 [7, 8] processes, have been obtained. For recent progress on this topic, see [9–11].

Since 2010, the studies on the KPZ equation and KPZ universality class have entered a new stage [12]. First as an experimental progress, high-accuracy measurements for the exponents and the height distribution in the KPZ growth problem using turbulent liquid crystal have been performed and the GUE Tracy–Widom distribution function was observed as well as the critical exponents [13]. Second, we have begun to understand exactly the height distribution function of the KPZ equation itself. In [14–17], the height distribution at a single position was computed for the narrow-wedge initial condition,

\[
\frac{\lambda}{2\nu} h(x, t = 0) = -\frac{|x|}{\delta}, \quad \delta \to 0, \quad (1.3)
\]

from which the surface grows to a parabolic shape. In the long-time limit, the fluctuation around the macroscopic shape is described by the GUE Tracy–Widom distribution. The analysis of [14–17] is based on recent progress on the current distribution of the (partially) ASEP [18, 19] and a fact that, in the weak asymmetry limit, the stochastic time evolution of the current of the ASEP can be mapped to that of the height described by the KPZ equation [20]. For more recent developments, see [21–26].

A more direct approach without relying on the results on the ASEP has also been developed recently [27–29]. The idea was first proposed in [30], in which the author used the interesting relation that the exponential moment of the height is represented as the dynamics of the one-dimensional δ-function Bose gas with attractive interaction, which is solved by the Bethe ansatz [31, 32]. In the original work [30], only the ground state contribution was considered and the dynamical exponent 1/3 was obtained. In [28, 29], the authors succeeded in taking into account the whole contribution of the eigenstates and obtained the Fredholm determinant representation of the generating function of the exponential moment for the narrow-wedge initial condition (1.3). More recently, a multi-point generating function is also discussed [33, 34].

The replica method is quite attractive since we expect to be able to calculate various quantities for the one-dimensional KPZ equation directly and easily. In particular, we expect that it would be a powerful tool to understand the universality in the KPZ equation. There are two recent progresses in this direction: in [35], the authors discussed a process characterizing the renormalization fixed point of the KPZ universality class and the Fredholm determinant
expression of its transition probability was obtained; in [36], the authors obtained the exact height distribution for the flat initial condition and clarified its convergence to the GOE Tracy–Widom distribution in the long-time limit.

In this paper, we pursue the potential of the replica method by applying it to the half Brownian-motion initial condition depicted in figure 1. This is one of the typical spatially extended initial conditions and is written in terms of the single-valued function $h(x, t)$ as

$$ \frac{\lambda}{2\nu} h(x, t = 0) = \begin{cases} \frac{x}{\delta}, & \delta \to 0, \quad x < 0, \\ \alpha B(x), & x \geq 0. \end{cases} $$

(1.4)

Here $B(x)$ represents the one-dimensional standard Brownian motion with $B(0) = 0$ and $\alpha = (2\nu)^{-3/2}\lambda^{1/2}$. For this initial condition, the macroscopic shape expected by solving (1.1) without the noise term is

$$ h(x, t) \sim \begin{cases} -x^2/2\lambda t, & x \leq 0, \\ 0, & 0 < x. \end{cases} $$

(1.5)

The main interest in this paper is the fluctuation around this macroscopic shape. We expect that a one-dimensional Brownian motion describes the fluctuation in the positive region ($x > 0$) since it is known to be a stationary measure of the KPZ equation. On the other hand, in the negative region ($x < 0$) where the parabolic growth is observed, the situation is the same as the narrow-wedge case. Thus, the fluctuation property around the origin shows a crossover behavior between the two typical growths.

This initial condition has already been considered in [38]. There the analysis is based on the result for the ASEP with the step Bernoulli initial condition [37] and naturally leads to an expression for the height distribution in the form of contour integral. We take the replica method and mainly treat the generating function of the exponential moment. Accordingly our formula is somewhat different from the one in [38] and is expressed in terms only of real quantities. Furthermore, the replica approach allows us to discuss the multi-point distribution function with the help of the factorization assumption by [33, 34] and properties of the deformed Airy functions discussed in the appendix.

This paper is arranged as follows. In the next section, we state our main results. Briefly explaining the relation between the KPZ equation and the $\delta$-Bose gas in section 3, we give a derivation of our main result, the Fredholm determinant expression of the generating function (theorem 1 in section 2), in section 4. In section 5, we discuss another expression of theorem 1 stated as proposition 2 in section 2, which is useful for the compact expression of the height distribution function described by theorem 3 in section 2. In section 6, we discuss the multi-point height distribution. In section 7, we consider the interpretation of our result as the free energy distribution of a directed polymer in random media. Concluding remarks are given in the last section.
2. Model and main results

2.1. The generating function

The KPZ equation (1.1) is in fact ill-defined as it is. As time goes on, the height profile approaches the stationary one described by the Brownian motion, for which the nonlinear term in the KPZ equation is not well defined. A proper prescription is proposed in [20], which we follow here. In this scheme, one defines the height of the KPZ equation by

\[ h_{\nu,\lambda,D}(x,t) = \frac{2\nu}{\lambda} \log \left( Z_{\nu,\lambda,D}(x,t) \right), \tag{2.1} \]

using the solution \( Z_{\nu,\lambda,D}(x,t) \) of the stochastic heat equation,

\[ \frac{\partial Z_{\nu,\lambda,D}(x,t)}{\partial t} = \nu \frac{\partial^2 Z_{\nu,\lambda,D}(x,t)}{\partial x^2} + \frac{\lambda \sqrt{D}}{2\nu} \eta(x,t) Z_{\nu,\lambda,D}(x,t), \tag{2.2} \]

which is a well-defined stochastic partial differential equation of Itô-type. This is called the Cole–Hopf solution of the KPZ equation because equation (2.2) is related to the KPZ equation through the (inverse of) the Cole–Hopf transformation (2.1). We will give more explanations in subsection 3.1 below. Hereafter we investigate the properties of these regularized quantities \( h_{\nu,\lambda,D}(x,t) \) and \( Z_{\nu,\lambda,D}(x,t) \) with the initial conditions (1.4). In the following, we denote by \( \langle \cdot \cdot \cdot \rangle \) the average over both \( \eta(x,t) \) in the KPZ equation (1.1) and \( B(x) \) in the initial condition (1.4).

We are interested in the distribution of the height \( h_{\nu,\lambda,D}(x,t) \). It is well established, known as the KPZ scaling, that the fluctuation of the height scales like \( O(t^{1/3}) \) and nontrivial correlations are seen in the \( x \) direction with scale \( O(t^{2/3}) \) when \( t \) is large. Let us define a parameter \( \gamma \) which scales as \( O(t^{1/3}) \) and a rescaled space coordinate \( X \) by

\[ \gamma = (\alpha^4 \nu t)^{1/3}, \quad x = \frac{2\gamma^2 X}{\alpha^2}, \tag{2.3} \]

with \( \alpha \) given below (1.4). We introduce the scaled height \( \tilde{h}(X) \) by

\[ \frac{\lambda}{2\nu} h_{\nu,\lambda,D} \left( \frac{2\gamma^2 X}{\alpha^2}, t \right) = \frac{\gamma^3}{12} - \gamma X^2 + \gamma \tilde{h}(X). \tag{2.4} \]

Here the second term corresponds to the macroscopic shape in (1.5); it is an interesting aspect of the KPZ equation that one has also to take into account the first term to focus on the height fluctuations.

To study the distribution, it is often useful to consider the generating function of the moments \( \langle Z_{\nu,\lambda,D}(x,t)^N \rangle, N = 0, 1, 2, \ldots \). We define \( G_\gamma(s; X) \) as

\[ G_\gamma(s; X) = \sum_{N=0}^\infty \frac{(-e^{-\gamma s})^N}{N!} \left( Z_{\nu,\lambda,D} \left( \frac{2\gamma^2 X}{\alpha^2}, t \right) \right)^N e^{s X^2-N\gamma^3} = (e^{-e^{(s+i\gamma)}(x^2-x)}}. \tag{2.5} \]

By using the replica method, one can express the moment \( \langle Z_{\nu,\lambda,D}(x,t)^N \rangle \) of our problem in the language of the \( \delta \)-function Bose gas, which is a well-known exactly solvable model. One can further perform the summation over \( N \) to obtain the Fredholm determinant representation of the generating function.

**Theorem 1.** \( G_\gamma(s; X) \) is expressed as the Fredholm determinant with the kernel acting on \( L^2(\mathbb{R}) \),

\[ G_\gamma(s; X) = \det(1 - P_0 K_X P_0). \tag{26} \]
Here $P_x$ represents the projection onto $(s, \infty)$ and the kernel of $K_X$ is given by
\[
K_X(\omega_j, \omega_k) = \sum_{n=1}^{\infty} (-1)^{n-1} \int_{-\infty}^{\infty} \frac{dq}{\pi} e^{\pi i (\omega_j + \omega_k) - 2i q (\omega_j - \omega_k) - \frac{n^2}{4} q^2 + \frac{\gamma}{2} n^2 - \gamma n}s \frac{\Gamma(i q - \frac{\gamma}{2} - \frac{n}{2})}{\Gamma(i q - \frac{\gamma}{2} + \frac{n}{2})},
\]
(2.7)

where $\Gamma(x)$ is the gamma function and $c_n$ satisfies $c_n > X/\gamma + n/2$.

Here for an operator on $L^2(\mathbb{R})$ with kernel $K(x, y)$, the Fredholm determinant is defined by
\[
\det(1 - K) = \sum_{M=0}^{\infty} (-1)^M M! \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_M \det(K(x_j, x_k))_{j,k=1}^{M},
\]
(2.8)
where the right-hand side is assumed to converge.

A derivation of this result will be given in section 4 after the explanation of the relation of the 3rd moment of $Z_{v,k,D}(x, t)$ with $N$-particle dynamics of the $\delta$-function Bose gas in section 3. For the narrow-wedge case (1.3), where $Z_{v,k,D}(x, 0) = \delta(x)$, the corresponding generating function has been obtained in the same form as (2.6) [28, 29]. But the kernel $K_X(\omega_j, \omega_k)$ is replaced by
\[
K_{nw}(\omega_j, \omega_k) = \sum_{n=1}^{\infty} (-1)^{n-1} \int_{-\infty}^{\infty} \frac{dq}{\pi} e^{\pi i (\omega_j + \omega_k) - 2i q (\omega_j - \omega_k) - \frac{n^2}{4} q^2 + \frac{\gamma}{2} n^2 - \gamma n}s \frac{\Gamma(i q - \frac{\gamma}{2} - \frac{n}{2})}{\Gamma(i q - \frac{\gamma}{2} + \frac{n}{2})},
\]
(2.9)
which is independent of the position $X$.

One can replace the kernel by the one written in the form of products of two deformed Airy functions.

**Proposition 2.** In (26), the kernel $P_x K_X P_0$ can be replaced by $P_0 \tilde{K}_X P_0$ where
\[
\tilde{K}_X(\xi_j, \xi_k) = \int_{-\infty}^{\infty} dy \text{Ai}^T(\xi_j + y, \frac{1}{\gamma}, -\frac{X}{\gamma}) \text{Ai}_r(\xi_k + y, \frac{1}{\gamma}, -\frac{X}{\gamma}) \frac{e^{\gamma y}}{e^{\gamma x} + e^{\gamma x}},
\]
(2.10)

Here $\text{Ai}^T(a, b, c)$ and $\text{Ai}_r(a, b, c)$ are defined by
\[
\text{Ai}^T(a, b, c) = \frac{1}{2\pi} \int_{\Gamma_{\tilde{\xi}}} dz e^{i z^2 + \frac{1}{2} z^2} \Gamma(ibz + c),
\]
(2.11)
\[
\text{Ai}_r(a, b, c) = \frac{1}{2\pi} \int_{\Gamma_{\tilde{\xi}}} dz e^{i z^2 + \frac{1}{2} z^2} \frac{1}{\Gamma(-ibz + c)}.
\]
(2.12)

In (2.11), $\Gamma_{\tilde{\xi}}$ represents the contour from $-\infty$ to $\infty$ and, along the way, passing below the pole $z_p = ic/b$.

The functions $\text{Ai}^T(a, b, c)$ and $\text{Ai}_r(a, b, c)$ have appeared in [38]. Note that these become the ordinary Airy functions if the gamma function factors in the integrand are eliminated in (2.11) and (2.12). For the narrow-wedge case, the corresponding kernel is the one where $\text{Ai}^T(x, 1/\gamma, -X/\gamma)$ and $\text{Ai}_r(x, 1/\gamma, -X/\gamma)$ are replaced by the Airy functions. To obtain the kernel in proposition 2 from that in theorem 1, one has to find some generalizations of formulas utilized in [28, 29]. They and a derivation of (2.10) will be given in section 5. Some properties of the deformed Airy functions are summarized in the appendix.
2.2. The height distribution function

The information of all the moments is enough to extract that of the probability distribution function. Applying the discussions in [29, 33], we can find a formula of the height distribution from the generating function \( G_\gamma(s, X) \). Let \( G_\gamma(s, X) \) be

\[
F_\gamma(s; X) = \text{Prob} \left( \frac{\lambda}{2 \nu} h(x, t) + \frac{\nu}{12} + \gamma X^2 \leq \gamma s \right) = \text{Prob}(\tilde{h}_t \leq s). \tag{2.13}
\]

By using the Fredholm determinant (2.6), \( F_\gamma(s; X) \) can be expressed as follows [29, 33]:

\[
F_\gamma(s; X) = 1 - \int_{-\infty}^{\infty} du \, e^{-\epsilon(s-u)} \, g_\gamma(u; X). \tag{2.14}
\]

Here

\[
g_\gamma(u; X) = \frac{1}{2\pi i} \left( \det \left( 1 - P_0 K^+_X P_0 \right) - \det \left( 1 - P_0 K^-_X P_0 \right) \right), \tag{2.15}
\]

where \( K^\pm(x, y) \) is kernel (2.7) or (2.10) in which the term \( e^{-\epsilon} \) is replaced by \(-e^{iu} \pm i\epsilon\) with \( \epsilon > 0 \) being infinitesimal.

Using (2.10), and the relation \( 1/(x \pm i\epsilon) = \mathcal{P}(1/x) \mp i\pi \delta(x) \), where \( \mathcal{P} \) denotes the Cauchy principal value, we can easily find that \( K^\pm_\gamma \) in (2.15) is represented as

\[
K^\pm_\gamma(\xi_j, \xi_k) = \mathcal{P} \int_{-\infty}^{\infty} dy \, e^{i\xi_j y} \, \frac{1}{\gamma y_i} \, \frac{1 - X}{\gamma y_i} \, A_i \left( \xi_k + y, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i}, 1 - e^{i\gamma(y-u)} \right)
\]

\[
\mp i\pi \, A_i \left( \xi_j + u, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) \, A_i \left( \xi_k + u, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right).
\tag{2.16}
\]

Substituting this expression into (2.15) and using the basic properties of the determinant, we eventually arrive at the expression in terms of the Fredholm determinant.

Theorem 3.

\[
F_\gamma(s; X) = 1 - \int_{-\infty}^{\infty} du \, e^{-\epsilon(s-u)} \, g_\gamma(u; X). \tag{2.17}
\]

Here \( g_\gamma(u; X) \) is expressed as a difference between two Fredholm determinants,

\[
g_\gamma(u; X) = \det \left( 1 - P_u (B^\gamma_\gamma - P^\gamma_A) P_u \right) - \det \left( 1 - P_u B^\gamma_A P_u \right), \tag{2.18}
\]

where

\[
B^\gamma_A(\xi_1, \xi_2) = \int_0^\infty dy \, A_i \left( \xi_1 + y, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) \, A_i \left( \xi_2 + y, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right)
\]

\[
+ \int_0^\infty dy \, \frac{1}{e^{i\gamma y} - 1} \left( A_i \left( \xi_1 + y, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) - A_i \left( \xi_2 + y, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) \right)
\]

\[
- A_i \left( \xi_1, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) \, A_i \left( \xi_2, \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right).
\tag{2.19}
\]

\[
P^\gamma_A(\xi_1, \xi_2) = A_i \left( \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right) \, A_i \left( \frac{1}{\gamma y_i}, -\frac{X}{\gamma y_i} \right).
\tag{2.20}
\]

In [38], another expression using a contour integral was obtained, whereas ours (2.17) are represented as the convolution with the Gumbel distribution. For numerical analysis, this form would be convenient. Actually, from this expression, we can readily draw the picture of the probability density function \( dF_\gamma(s; X) / ds \) as in figure 2. In this figure, the graphs are drawn approximating the Fredholm determinant by a finite-dimensional matrix determinant.
using a simple discretization. For more precise estimation, the method in [39, 40] is available and actually it was applied to the narrow-wedge case [41].

The distribution function (2.17) has a similar form as the narrow-wedge case obtained in [14–17]: if we replace the functions $A_{i1}(x)/\Gamma_1(x)$ and $A_{i2}(x)/\Gamma_1(x)$ by the ordinary Airy function in (2.19) and (2.20), the distribution function (2.17) becomes the one for the narrow-wedge initial condition.

We also consider the long-time limit ($t \to \infty$) of the distribution function. Let us remember a basic fact that the probability distribution function is in general written as the expectation, $\text{Prob}(X \leq s) = \mathbb{E}(\Theta(s - X))$, where $\Theta(y)$ is the step function, $\Theta(y) := 1(y \geq 0), 0(y < 0)$. Noting $\lim_{a \to \infty} \exp[-e^{-Xa}] = \Theta(x)$, we take the $t \to \infty$ limit in (2.5) and see

$$\lim_{\gamma \to \infty} \text{Prob}(\tilde{h}_t \leq s) = \lim_{\gamma \to \infty} G_{\gamma}(s; X).$$ (2.21)

Combining this relation with theorem 1 and proposition 2, one obtains

$$\lim_{\gamma \to \infty} \text{Prob}(\tilde{h}_t \leq s) = \text{det} (1 - P_f K_X P_s);$$ (2.22)

the kernel is given by

$$K_X(\xi_j, \xi_k) := \lim_{\gamma \to \infty} \tilde{K}_X(\xi_j - s, \xi_k - s)$$

$$= \int_0^{\infty} dy A_i(\xi_j + y) A_i(\xi_k + y)$$

$$+ A_i(\xi_k) \left( e^{-X^2} - \int_0^{\infty} dy e^{-Xy} A_i(\xi_j + y) \right).$$ (2.23)

This distribution function was obtained in the study on the PNG model with an external source [42, 43] and the TASEP with the step Bernoulli initial condition [44]. At $X = 0$, another expression using the solution to the Painlevé differential equation was given in [45]. In figure 2, we illustrate the picture of the probability density function at the origin $X = 0$ as a solid curve. The derivation of (2.22) will be given in section 6 including the discussion of the multi-point height distribution.
2.3. Multi-point distribution function

The replica approach allows us to discuss a multi-point height distribution function. We introduce the \( n \)-point generating function

\[
G_n^\prime (\{s\}_n, \{X\}_n) = \langle e^{-\sum_j \theta_j (x_j - s_j)} \rangle,
\]

where we abbreviated \( s_1, \ldots, s_n \) and \( X_1, \ldots, X_n \) as \( \{s\}_n \) and \( \{X\}_n \), respectively, and we set \( X_1 < X_2 < \cdots < X_n \).

In [33, 34], the authors proposed the ‘factorization assumption’ for the \( n \)-point generating function for the narrow-wedge initial condition. In this paper, we show that if we employ the same approximation to the case of the half Brownian motion initial condition, we obtain the following result:

\[
G_n^\prime (\{s\}_n, \{X\}_n) = \det (1 - Q),
\]

and the kernel \( Q(x, y) \) is given by

\[
Q(u_1, u_{n+1}) = \int_{-\infty}^{\infty} du_2 \cdots du_n \langle u_1 | e^{(X_1 - X_2)H} | u_2 \rangle \cdots \langle u_n | e^{(X_n - X_1)H} L_n | u_{n+1} \rangle \Phi (\{u - x\}_n),
\]

where \( H \) is the Airy Hamiltonian \( H = -\frac{d^2}{dx^2} + u \), and

\[
\Phi (\{x\}_n) = \frac{\sum_{j=0}^{\infty} e^{-\gamma x_j}}{1 + \sum_{j=1}^{\infty} e^{-\gamma x_j}},
\]

\[
L_n (x, y) = \int_0^\infty dw \text{Ai}_1 (w) \left( w + x, \frac{1}{\gamma_1}, \frac{X_1}{\gamma_1} \right) \text{Ai}_1 (w + y, \frac{1}{\gamma_1}, -\frac{X_1}{\gamma_1}),
\]

with \( j \geq 1 \). In (2.25), we put the symbol \( \gamma \) in order to represent explicitly the fact that this is an expression after using the factorization approximation. The factor \( \langle x | e^{(X_j - X_k)H} | y \rangle \) in (2.26) can be expressed as

\[
\langle x | e^{(X_j - X_k)H} | y \rangle = \int_{-\infty}^{\infty} dz e^{\gamma x z} \text{Ai}_1 (x + z, \frac{1}{\gamma_1}, -\frac{X_1}{\gamma_1}) \text{Ai}_1 (y + z, \frac{1}{\gamma_1}, -\frac{X_1}{\gamma_1}),
\]

see (A.1). For the one-point case \( n = 1 \), however, the method in [33, 34] is exact. Actually, when \( n = 1 \), we find that the Fredholm determinant is equivalent to kernel (2.10): in the one-point case, kernel (2.26) becomes

\[
Q(u_1, u_2) = L_1 (u_1, u_2) \Phi (u_1 - s_1). \tag{2.30}
\]

We divide it into \( Q = Q_1 Q_2 \), where

\[
Q_1 (u, w) = \Phi (u - s) \text{Ai}_1 (w + u, \frac{1}{\gamma_1}, -\frac{X_1}{\gamma_1}) \chi_0 (w), \tag{2.31}
\]

\[
Q_2 (w, u) = \chi_0 (w) \text{Ai}_1 (w + u, \frac{1}{\gamma_1}, -\frac{X_1}{\gamma_1}), \tag{2.32}
\]

where \( \chi_0 (\xi) = 1(s \leq \xi < \infty), 0(-\infty < \xi < s) \) and note the property of the Fredholm determinant \( \det (1 - Q_1 Q_2) = \det (1 - Q_2 Q_1) \). We easily find that \( Q_2 Q_1 (\xi_1, \xi_2) = \chi_0 (\xi_2) K_1 (\xi_2, \xi_1) \chi_0 (\xi_1) \), where \( K_1 (\xi_2, \xi_1) \) is given in (2.10).

Because an approximation is involved in the derivation, expression (2.25) is most likely not exact when \( n \geq 2 \). A validity of employing this approximation is that, for the narrow-wedge case, the generating function becomes the multi-point distribution function for the Airy process in the long-time limit which is expected from the universality. In this paper, we will
see that the situation is similar for the case of the half Brownian motion initial condition. In the long-time limit, the multi-point generating function is equivalent to the multi-point height distribution function,

$$\lim_{t \to \infty} G_{i_j}((s)_{a}, (X)_{a}) = \lim_{t \to \infty} \text{Prob}(\tilde{h}_t(X_j) \leq s_j, j = 1, \ldots, n).$$  \hfill (2.33)

From (2.25), we find

$$\lim_{t \to \infty} G_{i_j}^2((s)_{a}, (X)_{a}) = \det(1 - P_tK_{12}P_t)$$

\[:= \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \prod_{i=1}^{M} \left( \sum_{n_i=1} \int_{-\infty}^{\infty} d\xi_i \chi_n(\xi_i) \right) \det(K_{12}(X_{n_i}, \xi; X_{n_i}, \xi))_{1 \leq k \leq M}, \]

where $\chi_n(\xi)$ is defined below (2.32) and

$$K_{12}(X_1, \xi_1; X_2, \xi_2) = K_2(X_1, \xi_1; X_2, \xi_2) + Ai(\xi_2) \left( e^{-\frac{X_1^2}{3} + X_1 \xi_1} - \int_0^{\infty} dy e^{-X_2 y} Ai(\xi_2 + y) \right).$$  \hfill (2.35)

$$K_2(X_j, \xi_j; X_k, \xi_k) = \begin{cases} \int_0^{\infty} dy e^{-w(X_j - X_k)} Ai(\xi_j + w)Ai(\xi_k + w), & j \geq k, \\ - \int_{-\infty}^{0} dy e^{-w(X_j - X_k)} Ai(\xi_j + w)Ai(\xi_k + w), & j < k. \end{cases}$$  \hfill (2.36)

The Fredholm determinant with this kernel has appeared in the PNG model and the TASEP [42–44]. Note that for the one-point case $n = 1$, it reduces to (2.22).

As pointed out in conjecture 9 in [38], it is plausible that in the long-time limit, the Fredholm determinant $\det(1 - P_tK_{12}P_t)$ describes the multi-point distribution considering the exact result of the one-point case. Thus, (2.34) strongly suggests that the factorization approximation becomes exact in the long-time limit in both narrow-wedge and half Brownian motion initial conditions. In section 6, we briefly discuss the derivation of these two equations (2.25) and (2.34).

3. KPZ equation and $\delta$-Bose gas

3.1. Feynman–Kac formula

In subsection 2.1, we mentioned that the KPZ equation (1.1) is not well defined as it is, and defined its solution by (2.1) and (2.2). Here we explain this using a limit of a modified version of the KPZ equation,

$$\frac{\partial h_t(x, t)}{\partial t} = \frac{\lambda}{2} \left( \frac{\partial h_t(x, t)}{\partial x} \right)^2 + \nu \frac{\partial^2 h_t(x, t)}{\partial x^2} + \sqrt{D} \eta_t(x, t) - \frac{1}{2} \left( \frac{\lambda \sqrt{D}}{2\nu} \right)^2 C_\kappa(0).$$  \hfill (3.1)

This looks almost the same as the original (1.1), but there are two differences. First, $\eta_t(x, t)$ is still a Gaussian noise but its covariance in the $x$-direction is now smeared as

$$\langle \eta_t(x, t)\eta_t(x', t') \rangle = C_\kappa(x - x') \delta(t - t'),$$  \hfill (3.2)

where $C_\kappa(x)$ is written as $C_\kappa(x) = C(x\kappa)$ and $C(x)$ is a smooth, even and positive function such that $\int_{-\infty}^{\infty} C(x) = 1$. Second, the constant velocity term $- (\lambda / \nu) C_\kappa(x\kappa) C_\kappa(0) / 2$ is added. Note that the above properties of $C_\kappa$ imply $\lim_{x \to \infty} C_\kappa(x) = \kappa(x)$ so that the noise $\eta_t$ tends to
the white noise in the original KPZ equation. An apparent problem is that the additional term in (3.1) diverges as δ(0) but in fact this term is necessary for considering a meaningful limit of equation (3.1). To see this, let us apply the Cole–Hopf transformation,

\[ Z_\kappa (x, t) = \exp \left( \frac{\gamma}{2v} \eta_\kappa (x, t) \right), \]

to (3.1). By Itô’s formula, (3.1) becomes

\[ \frac{dZ_\kappa (x, t)}{dt} = \nu \frac{\partial^2 Z_\kappa (x, t)}{\partial x^2} + \frac{\lambda \sqrt{D}}{2v} \eta_\kappa (x, t) Z_\kappa (x, t), \]

which is a well-defined stochastic differential equation of Itô type. Clearly the solution (2.1) of equation (2.2) as \( \kappa \to \infty \) [20]. Hence, the Cole–Hopf solution (2.1) with (2.2) can be interpreted as a limit of the solution to equation (3.1):

\[ Z_{\kappa, \nu, D} (x, t) = \lim_{\kappa \to \infty} Z_\kappa (x, t). \]

Note that if the constant velocity term is absent in (3.1), there appears an additional term in (3.4), or equation (3.4) should be understood as representing a stochastic differential equation of Stratonovich type, which is not well defined in the limit \( \kappa \to \infty \).

The solution of (3.4) can be represented as the Feynman–Kac formula [46],

\[ Z_\kappa (x, t) = \mathbb{E}_x \left[ \exp \left( \frac{\lambda \sqrt{D}}{2v} \int_0^t \eta_\kappa (b(2vs), t - s) \, ds \right) Z_\kappa (b(t), 0) \right] e^{-\frac{1}{2} \left( \frac{\lambda \sqrt{D}}{2v} \right)^2 C_\kappa (0) t}, \]

where \( \mathbb{E}_x \) represents the averaging over the standard Brownian motion \( b(s) \), \( 0 < s < t \), with \( b(0) = x \) and the initial condition is

\[ Z_\kappa (x, t = 0) = \begin{cases} 0, & x < 0, \\ \exp (b(x)), & x \geq 0. \end{cases} \]

Here \( B(x) \) represents the one-dimensional Brownian motion with \( B(0) = 0 \). Some readers may find the following path-integral expression more intuitive:

\[ Z_\kappa (x, t) = \int_0^\infty \left. dy \int_{x(0)=y}^{\xi(x)=x} D[\xi(\tau)] \exp \left( -S[\xi(x)] + \alpha B(y) - \frac{1}{2} \left( \frac{\lambda \sqrt{D}}{2v} \right)^2 C_\kappa (0) t \right) \right] d\xi(x, \tau), \]

where the action \( S[\xi(x)] \) is given by

\[ S[\xi(x)] = \int_0^t \left( \frac{1}{4v} \left( \frac{d\xi(x)}{d\tau} \right)^2 - \frac{\lambda \sqrt{D}}{2v} \eta_\kappa (x(\tau), \tau) \right). \]

We find that \( Z_{\kappa, \nu, D} (x, t) \) defined in (2.1) is written in terms of \( Z_{1,1,1}(x, t) \) with specific parameters \( \nu = 1/2 \), \( \lambda = D = 1 \) as follows. First using the Feynman–Kac formula (3.6), one has

\[ Z_{\kappa, \nu, D} \left( x, \frac{t}{2v} \right) = \lim_{\kappa \to \infty} \mathbb{E}_x \left[ \exp \left( \frac{\lambda \sqrt{D}}{2v} \int_0^{\frac{t}{2v}} \eta_\kappa (b(2vs), \frac{t}{2v} - s) \, ds \right) \right] e^{-\frac{1}{2} \left( \frac{\lambda \sqrt{D}}{2v} \right)^2 C_\kappa (0) \frac{t}{2v}} \]

\[ \times \left[ Z^{(0)} (b(t), \alpha) \right], \]

where we rewrite \( Z_\kappa (x, 0) \) as \( Z^{(0)} (x, \alpha) \) in order to represent explicitly the dependence on \( \alpha \) and the independence of \( \kappa \). From the properties of \( C_\kappa \) written below (3.2), we find the scaling relations of \( C_\kappa \) and \( \eta_\kappa (x, t) \),

\[ C_\kappa (x) = a C_\kappa / \alpha (x), \]

(3.11)
\[ \eta_k(x, t) = (ab)^{1/2} \eta_{k/a}(ax, bt), \]  
(3.12)

where \( a, b \in \mathbb{R} \) and equality (3.12) holds in the sense of distribution. By use of (3.12) with \( a = 1, b = 1/2 \), (3.10) becomes

\[
\lim_{k \to \infty} \mathbb{E}_{\kappa} \left( \exp \left[ \alpha \int_0^t \eta_k(b(s), t-s) \, ds - \frac{\alpha^2}{2} C_k(0)t \right] Z^{(0)}(b(t), \alpha) \right).
\]  
(3.13)

Next one remembers the scaling property of the Brownian motion,

\[ ab_t(s) = b_{ac}(a^2 s), \]  
(3.14)

where the equality holds again in the sense of distribution and we put the initial position \( x \) of \( b(s) \) explicitly. By (3.12) and (3.14) with \( a = \alpha^2 \) and \( b = \alpha^4 \), equation (3.13) can be rewritten as

\[
\lim_{k \to \infty} \mathbb{E}_{\alpha^2 z} \left( \exp \left[ \int_0^{\alpha^4 t} \eta_{k/\alpha^2}(b(s), \alpha^4 t - s) \, ds - \frac{\alpha^4}{2} C_{k/\alpha^2}(0)t \right] Z^{(0)}(\alpha^{-2} b(\alpha^4 t), \alpha) \right).
\]  
(3.15)

At last noting from (3.7) and (3.14) that

\[ Z^{(0)}(\alpha^{-2} x, \alpha) = Z^{(0)}(x, \alpha = 1), \]  
\[ Z^{(0)}(x, \alpha = 1), \]  
(3.16)

we find that (3.15) is nothing but \( Z_{1,1}^{1,1}(\alpha^2 x, \alpha^4 t) \).

Thus, we have established

\[
\frac{\eta}{2v} h_{v,1,D} \left( x, \frac{t}{2v} \right) = h_{1,1}(\alpha^2 x, \alpha^4 t),
\]  
(3.17)

In what follows, we restrict our discussions to \( h_{1,1}(x, t) \) and \( Z_{1,1}^{1,1}(x, t) \) (hereafter we omit the indices \( 1, 1 \)). We remark that, for these special parameter values, (2.3) reads

\[ \gamma_t = \left( \frac{t}{2} \right)^{1/2}, \quad x = 2\gamma_t^2 X. \]  
(3.18)

### 3.2. The \( \delta \)-function Bose gas

Next we consider the replica partition function \( \langle Z_N(x, t) \rangle \) \( (N = 0, 1, 2, \ldots) \). First we perform the Gaussian average over \( \eta_k(x, t) \) using the path integral representation (3.8) and (3.2), to find

\[
\langle e^{\sum_{j=1}^N C_k(x_j - x_k)} \rangle = e^{\frac{1}{2} \sum_{j=1}^N C_k(x_j - x_k)}.
\]  
(3.19)

The right-hand side of this equation includes the self-interaction term \( \sum_{j=1}^N C_k(x_j - x_k) = NC_k(0) \), which becomes divergent in the limit \( \kappa \to \infty \). This term, however, cancels out the last term in (3.8), and thus we can take the limit \( \kappa \to \infty \). We obtain

\[
\langle Z_N(x, t) \rangle = \lim_{\kappa \to \infty} \langle Z_N(x, t) \rangle
\]

\[
= \prod_{j=1}^N \int_0^\infty dy_j \int_{x_j(0) = y_j}^x D[x_j(\tau)] \exp \left[ - \frac{1}{2} \left( \frac{dx_j(\tau)}{d\tau} \right)^2 - \sum_{j \neq k=1}^N \delta(x_j(\tau) - x_k(\tau)) \right] \exp \left( \sum_{k=1}^N B(\gamma_k) \right),
\]  
(3.20)

where \( \langle \cdots \rangle \) in the last factor indicates the remaining average over the Brownian motion \( B(\gamma) \).
The right-hand side of this equation represents the imaginary-time dynamics of the $\delta$-function Bose gas with attractive interaction with the Hamiltonian $H_N$,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k} \delta(x_j - x_k),$$

(3.21)

in terms of which the replica partition function $\langle Z^N(x, t) \rangle$ can be written as

$$\langle Z^N(x, t) \rangle = \langle x | e^{-H_N t} | \Phi \rangle.$$  

(3.22)

Here $|x\rangle$ represents the state with all $N$ particles being at the position $x$ and $|\Phi\rangle$ represents the initial state of the $\delta$-function Bose gas. One can perform the average over the Brownian motion $B(y)$ and the dependence of $|\Phi\rangle$ on $x_1, \ldots, x_N$ can be explicitly calculated as

$$\langle x_1, \ldots, x_N | \Phi \rangle = \frac{1}{N!} \sum_{P \in S_N} \exp \left( \sum_{k=1}^{N} B(x_P(k)) \right) \prod_{j=1}^{N} \int_{-\infty}^{\infty} dz_j e^{-z_j^2/2\pi} \Theta(x_P(j) - x_P(j-1)) \Theta(x_P(j) - x_P(j-1)),$$

(3.23)

Here $S_N$ denotes the set of permutations of order $N$, $\Theta(x)$ is the step function and we set $x_{P(0)} = 0$. Since we are considering a Boson system, the above function is taken to be symmetric in $x_1, \ldots, x_N$.

The eigenvalues and eigenfunctions of the $\delta$-Bose gas can be constructed by using the Bethe ansatz $[27-29, 31, 32]$. Let $|\Psi_\alpha\rangle$ and $E_\alpha$ be the eigenstate and eigenvalue of $H_N$,

$$H_N |\Psi_\alpha\rangle = E_\alpha |\Psi_\alpha\rangle.$$  

(3.24)

By the Bethe ansatz, they are given as

$$\langle x_1, \ldots, x_N | \Psi_\alpha \rangle = C_\alpha \sum_{P \in S_N} \sgn P \prod_{1 \leq j < k \leq N} (z_{P(j)} - z_{P(k)} + i \sgn(x_j - x_k)) \exp \left( i \sum_{l=1}^{N} z_{P(l)} x_l \right),$$

(3.25)

where $C_\alpha$ is the normalization constant, for which a formula is given in (3.27) below.

For the $\delta$-Bose gas with attractive interaction, the quasimomenta $z_j (1 \leq j \leq N)$ which label the state, are in general complex numbers. $z_j (1 \leq j \leq N)$ are divided into $M$ groups where $1 \leq M \leq N$. The orth group consists of $n_\alpha$ quasimomenta $z'_j$ which share the common real part $q_\alpha$. Note that $\sum_{\alpha=1}^{M} n_\alpha = N$. The quasimomenta in each group line up with regular intervals with unit length along the imaginary direction. Using $q_\alpha$ and $n_\alpha (1 \leq \alpha \leq M)$, we represent $z_j (1 \leq j \leq N)$ as

$$z_j = q_\alpha + i \left( n_\alpha + 1 - 2r_\alpha \right), \quad \text{for} \quad j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha,$$

(3.26)

where $1 \leq \alpha \leq M$ and $1 \leq r_\alpha \leq n_\alpha$. The normalization constant $C_\alpha$, which is taken to be a positive real number, and the eigenvalue $E_\alpha$ are given by $[28]$

$$C_\alpha = \left( \prod_{\alpha=1}^{M} n_\alpha \prod_{1 \leq j < k \leq N} \frac{1}{|z_j - z_k + i|^2} \right)^{1/2},$$

(3.27)
$$E_c = \frac{1}{2} \sum_{j=1}^{N} \zeta_j^2 = \frac{1}{2} \sum_{a=1}^{M} n_a \eta_a^2 - \frac{1}{24} \sum_{a=1}^{M} (n_a^3 - n_a).$$  (3.28)

There is a problem of completeness of these states. This has not been resolved rigorously yet but the fact that one can recover the height distribution of the KPZ equation for the narrow-wedge initial condition is strong affirmative evidence. Here we proceed assuming its validity and will see the consistency with the computations based on the ASEP. This provides further evidence that the above Bethe states are in fact complete.

4. Generating function

In this section, we give a derivation of formula (26) in theorem 1. The replica partition function \( \langle Z^N(x, t) \rangle \) (3.22) can be written as

$$\langle Z^N(x, t) \rangle = \int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_N \langle x| e^{-H_{\text{tot}}}|y_1, \ldots, y_N\rangle \langle y_1, \ldots, y_N|\Phi \rangle. \quad (4.1)$$

Expanding the propagator \( \langle x| e^{-H_{\text{tot}}}|y_1, \ldots, y_N \rangle \) by the Bethe eigenstates of the \( \delta \)-Bose gas (3.25), we have

$$\langle Z^N(x, t) \rangle = \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left( \int_{-\infty}^{\infty} \sum_{a=1}^{M} \frac{dq_a}{2\pi} \delta_{\Sigma \alpha \eta_a = 1} \langle \Psi_j | \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N |\Phi \rangle \right) \langle \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N |\Phi \rangle \quad (4.2)$$

Here we want to perform the integrations over \( y_j (1 \leq j \leq N) \),

$$\prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \langle \Psi_j | \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N |\Phi \rangle \quad (4.3)$$

using (3.23) and (3.25). But this is not allowed for the moment because the integrations over \( q_{a} (1 \leq \alpha \leq M) \) must be performed before those over \( y_{j} (1 \leq j \leq N) \). To see this explicitly, one uses (3.23) and notes the symmetry of the eigenfunction, \( \langle \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N \rangle = \langle \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N \rangle \), to find that the right-hand side of (4.2) is represented as

$$\langle Z^N(x, t) \rangle = \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left( \int_{-\infty}^{\infty} \sum_{a=1}^{M} \frac{dq_a}{2\pi} \delta_{\Sigma \alpha \eta_a = 1} \langle \Psi_j | \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N |\Phi \rangle \right) \langle \Psi_{y_1}, \ldots, \Psi_{y_N} | y_1, \ldots, y_N |\Phi \rangle \quad (4.4)$$

It is clear that the integrand on the right-hand side of this equation is not integrable on \( y_j (1 \leq j \leq M) \) due to the factor \( \exp((2N - 2j + 1)y_j) \), while it is integrable on \( q_{a} (1 \leq \alpha \leq M) \) thanks to the factor \( \exp(-q_{a}y_j^2/2) \) in \( e^{-E_{\delta}} \).

4.1. Deformation of contours

To exchange the order of integrations, we note

**Lemma 4.** In \( (4.4) \), we can deform the contour of \( q_{a} (1 \leq \alpha \leq M) \) to \( \mathbb{R} - ic \) where \( c \) is an arbitrary real constant.
Proof. In (4.4), we change the variables from \( q_\alpha \) to \( u_\alpha (1 \leq \alpha \leq M) \) such that
\[
u_\alpha = q_\alpha - q_{\alpha+1} \quad (1 \leq \alpha \leq M-1), \quad u_M = q_M. \tag{4.5}\]
The singularity in \( u_\alpha \) of the integrand in (4.4) comes only from the factor \(|C_\alpha|^2\). Note that it depends only on \( u_1, \ldots, u_{M-1} \) and is independent of \( u_M \). Thus, there are no poles of \( u_M \) in (4.4), and we can deform the contour of \( u_M \) to \( \mathbb{R} - ic \). Noting that
\[
\prod_{\alpha=1}^{M-1} \int_{-\infty}^{\infty} \frac{du_\alpha}{2\pi} \times \int_{\mathbb{R} - ic} \frac{du_M}{2\pi} = \prod_{\alpha=1}^{M} \int_{\mathbb{R} - ic} \frac{du_\alpha}{2\pi}, \tag{4.6}\]
we find that the statement holds. \( \square \)

Using lemma 4, we deform the contours of \( q_\alpha (1 \leq \alpha \leq M) \) to \( \mathbb{R} - ic \) where \( c \) is a positive constant so large that the factor \( \exp \left( -i \sum_{j=1}^{N} \frac{1}{2} \frac{\alpha(q_j)}{2j+1} y_j + \frac{1}{2} \sum_{j=m}^{N} (2N-2j+1)y_j \right) \) in the integrand of (4.4) converges when \( y_j > 0 \) goes to infinity. Here \( z_j = q_{\alpha(j)} + \frac{1}{2} \alpha(q_{\alpha(j)} + 1 - 2r(j)) \) but one has to be careful about the notation. This is a usual complex conjugate of \( z_j \) before lemma 4, i.e. when \( q_{\alpha(j)} \) is real. But now it is not because after the deformation of the contour, \( q_{\alpha(j)} \) has an imaginary part.

After this deformation, we can perform the integrations of \( y_j (1 \leq j \leq N) \) before those of \( q_\alpha (1 \leq \alpha \leq M) \). Equation (4.2) can now be expressed as
\[
\langle Z^N(x, t) \rangle = \sum_{M=1}^{N} \frac{1}{M!} \prod_{\alpha=1}^{M} \left( \int_{\mathbb{R} - ic} \frac{dq_\alpha}{2\pi} \sum_{n=1}^{\infty} \delta_{\sum_{j=1}^{N} n_j \alpha} \langle x | \Psi_\alpha \rangle \langle \Psi_\alpha | \Phi \rangle e^{-Ezt} \right). \tag{4.7}\]
Here \( \langle x | \Psi_\alpha \rangle \) is given by (3.25) with \( x_1 = \cdots = x_N = x \) and \( \langle \Psi_\alpha | \Phi \rangle \) is computed as
\[
\langle \Psi_\alpha | \Phi \rangle = \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \langle \Psi_\alpha | y_1, \ldots, y_N \rangle \langle y_1, \ldots, y_N | \Phi \rangle
\]
\[
= N! C_\alpha \sum_{P \in \mathcal{S}_N} \text{sgn} P \prod_{l=1}^{N} \int_{\mathbb{R} - ic} dy_j e^{-i(z_j + \frac{1}{2} (2N-2j+1)y_j)} \sum_{1 \leq j < k \leq N} (z_j^P - z_k^P + i) \frac{1}{-i(z_j^P + \cdots + z_{N-l+1}^P) + i^{l/2}}. \tag{4.8}\]
In (4.7), we take the imaginary part \( c \) of \( q_\alpha \) in such a way that we can perform the integrations of \( y_j (1 \leq j \leq N) \) in (4.8), i.e. \( \text{Re}(-i(z_j^P + \cdots + z_{N-l+1}^P) + i^{l/2}) < 0 \) for any \( l (1 \leq l \leq N) \). For example, if we fix \( c \) such that \( c > N/2 + \max_{\alpha} n_\alpha / 2 \), the above condition is satisfied.

4.2. Combinatorial identities

For further analysis of the integrand in (4.7), we need two combinatorial identities for \( \langle x | \Psi_\alpha \rangle \) and \( \langle \Psi_\alpha | \Phi \rangle \). The first one is for \( \langle x | \Psi_\alpha \rangle \). One has
\[
\sum_{P \in \mathcal{S}_N} \text{sgn} P \prod_{1 \leq j < k \leq N} (w_{j,k} - w_{P(j,k)} + i f(j,k)) = N! \prod_{1 \leq j < k \leq N} (w_j - w_k) \tag{4.9}\]
for any complex variables \( w_j (1 \leq j \leq N) \) and \( f(j,k) \). This identity was derived as lemma 1 in [33].
The next one for the term $\langle \Psi_\gamma \mid \Phi \rangle$ is

**Lemma 5.** For any complex numbers $w_j (1 \leq j \leq N)$ and $a$,

$$\sum_{P \in S_N} \text{sgn} P \prod_{1 \leq j < k \leq N} (w_{P(j)} - w_{P(k)} + a) \prod_{m=1}^{N} \frac{1}{w_{P(N)} + \cdots + w_{P(N-m+1)} + m^2a/2} = \prod_{1 \leq j < k \leq N} (w_j - w_k) \prod_{m=1}^{N} \frac{1}{w_m + a/2}. \quad (4.10)$$

**Proof.** A similar identity appears in the context of the ASEP with the step Bernoulli initial condition and has been proved in section III of [19]. Here we follow the same strategy using mathematical induction. Let us call the left-hand side and the right-hand side of (4.10) $\psi_N$ and $\phi_N$, respectively, i.e.

$$\psi_N(w_1, \ldots, w_N) := \sum_{P \in S_N} \text{sgn} P \prod_{1 \leq j < k \leq N} (w_{P(j)} - w_{P(k)} + a) \prod_{m=1}^{N} \frac{1}{w_{P(N)} + \cdots + w_{P(N-m+1)} + m^2a/2}, \quad (4.11)$$

$$\phi_N(w_1, \ldots, w_N) := \prod_{1 \leq j < k \leq N} (w_j - w_k) \prod_{m=1}^{N} \frac{1}{w_m + a/2}. \quad (4.12)$$

We want to show that $\psi_N = \phi_N$. We easily see that it holds for the case $N = 1$. Let us assume that it holds for the $N - 1$ case. In (4.11), we first sum over all permutations with $P(1) = l$ fixed and then sum over $l$. Noting $\prod_{1 \leq j < k \leq N} (w_l - w_{P(k)} + a) = \prod_{1 \leq j \neq l} (w_l - w_k + a)$, we see

$$\psi_N(w_1, \ldots, w_N) = \frac{1}{\sum_{j=1}^{N} w_j + N^2a/2} \sum_{l=1}^{N} (-1)^{l+1} \prod_{j \neq l} (w_l - w_j + a) \times \psi_{N-1}(w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_N)$$

$$= \frac{1}{\sum_{j=1}^{N} w_j + N^2a/2} \sum_{l=1}^{N} (-1)^{l+1} \prod_{j \neq l} (w_l - w_j + a) \times \phi_{N-1}(w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_N), \quad (4.13)$$

where we used the assumption of the mathematical induction, $\psi_{N-1} = \phi_{N-1}$, in the second equality. The identity $\psi_N = \phi_N$ is now equivalent to

$$\sum_{l=1}^{N} (-1)^{l+1} \prod_{j \neq l} (w_l - w_j + a) \frac{\phi_{N-1}(w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_N)}{\phi_N(w_1, \ldots, w_N)} = \sum_{j=1}^{N} w_j + \frac{N^2a}{2}. \quad (4.14)$$

From definition (4.12) of $\phi_N$, we obtain

$$\phi_{N-1}(w_1, \ldots, w_{l-1}, w_{l+1}, \ldots, w_N) = \frac{w_l + a/2}{\prod_{j=1}^{l-1} (w_l - w_j) \cdot \prod_{j=l+1}^{N} (w_l - w_j)} \frac{\phi_N(w_1, \ldots, w_N)}{w_l + a/2}$$

$$= (-1)^{l-1} \prod_{j \neq l}(w_l - w_j) \phi_N(w_1, \ldots, w_N). \quad (4.15)$$
Substituting this equation into (4.14), one sees that it is now enough to show
\[
\sum_{l=1}^{N} (w_l + a/2) \prod_{j \neq i} \left( 1 + \frac{a}{w_i - w_j} \right) = \sum_{j=1}^{N} w_j + \frac{N^2 a}{2}.
\]  
(4.16)

This is proved as follows. One notes that the left-hand side can be represented by a contour integral,
\[
\frac{1}{2\pi i a} \int_{C_R} dz (z + a/2) \prod_{j=1}^{N} \left( 1 + \frac{a}{z - w_j} \right),
\]  
(4.17)

where \(C_R\) is a contour enclosing the origin anticlockwise with radius \(R\) taken to be so large that the contour surrounds all the poles \(w_j (1 \leq j \leq N)\) in the integrand. The contour integration can be performed as follows. Expanding the product in the integrand, one finds
\[
\frac{1}{2\pi i a} \int_{C_R} dz \left[ \left( z + a/2 \right) + \sum_{l=1}^{N} \frac{a}{z - w_l} \left( z + a/2 \right) + \sum_{1 \leq l < m \leq N} \frac{a^2}{(z - w_l)(z - w_m)} \left( z + a/2 \right) + \cdots \right]
\]
\[
+ a^n \prod_{j=1}^{N} \frac{1}{z - w_j} \left( z + a/2 \right).
\]  
(4.18)

Further expanding the second and third terms in the integrand in \(1/z\), we see
\[
\frac{a}{z - w_l} \left( z + a/2 \right) = a + \frac{a(w_l + a/2)}{z} + O \left( \frac{1}{z^2} \right)
\]
\[
\frac{a^2}{(z - w_l)(z - w_m)} \left( z + a/2 \right) = \frac{a^2 z}{z} + O \left( \frac{1}{z^2} \right).
\]  
(4.19)

The higher terms in (4.18) are of order \(O(1/z^2)\) and thus do not contribute to the contour integral. Hence, (4.17) is calculated as
\[
\frac{1}{2\pi i a} \int_{C_R} dz (z + a/2) \prod_{j=1}^{N} \left( 1 + \frac{a}{z - w_j} \right)
\]
\[
= \frac{1}{2\pi i a} \int_{C_R} dz \left( \sum_{l=1}^{N} \frac{a(w_l + a/2)}{z} + \sum_{1 \leq l < m \leq N} \frac{a^2}{z} \right)
\]
\[
= \sum_{l=1}^{N} \left( w_l + a/2 \right) + \sum_{l < m} a = \sum_{j=1}^{N} w_j + \frac{N^2 a}{2};
\]  
(4.20)

thus, we obtain (4.16). \(\square\)

Using identities (4.9) and (4.10) in (3.25) and (4.8), respectively, we obtain
\[
\langle x | \Psi_j \rangle = N! C_z \prod_{1 \leq j < k \leq N} (z_j - z_k) e^{\sum_{k=1}^{N} z_kx},
\]  
(4.21)

\[
\langle \Psi_j | \Phi \rangle = i^{-N} N! C_z \prod_{1 \leq j < k \leq N} (z_j^* - z_k^*) \prod_{l=1}^{N} \frac{1}{z_l^* + i/2}.
\]  
(4.22)

Thus using these relations and (3.27), we find that the factor \(\langle x | \Psi_j \rangle \langle \Psi_j | \Phi \rangle\) in (4.7) becomes
\[
\langle x | \Psi_j \rangle \langle \Psi_j | \Phi \rangle = N! \prod_{a=1}^{M} \left( \frac{n_a}{n_a} \right)^2 \prod_{1 \leq j < k \leq N} \frac{|z_j - z_k|^2}{|z_j - z_k - i|^2} \prod_{l=1}^{N} \left| \frac{e^{ix}}{z_l^* - 1/2} \right|^2.
\]  
(4.23)
Here we used the fact that $z_j^* - z_k^*$ is the complex conjugate of $z_j - z_k$ although $z_j^*$ is not that of $z_j$ as mentioned below in lemma 4. The common imaginary part $c$ of $q_{\omega(j)}$ and $q_{\omega(k)}$ cancels out by subtraction.

We want to rewrite this equation in terms of $q_a$ and $n_a$ in (3.26). For the last factor of this equation, we easily find

$$\prod_{l=1}^N \frac{e^{iz_l^*} - 1/2}{1} = \prod_{a=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)}.$$

From (3.27) and (4.21), we know that the remaining factors in (4.23) are represented by $|\langle 0 | \Psi_z \rangle|^2$. For this quantity, the following result was obtained in appendix B in [28]:

$$|\langle 0 | \Psi_z \rangle|^2 = N! \prod_{a=1}^M \frac{(n_a)!^2}{n_a} \prod_{1 \leq j < k \leq N} \frac{|z_j - z_k|^2}{|z_j - z_k - i|^2} = N! \prod_{a=1}^M \prod_{1 \leq a < b \leq M} \frac{|q_a - q_b - \frac{1}{2}(n_a - n_b)|^2}{|q_a - q_b - \frac{1}{2}(n_a + n_b)|^2}. \tag{4.25}$$

From (4.23)–(4.25), we obtain

$$\langle x | \Psi_z \rangle \langle \Psi_z | \Phi \rangle = N! \prod_{a=\beta}^M \frac{|q_a - q_\beta - \frac{1}{2}(n_a - n_\beta)|^2}{|q_a - q_\beta - \frac{1}{2}(n_a + n_\beta)|^2} \prod_{a=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)}. \tag{4.26}$$

We can further deform (4.26) to an expression in terms of a determinant by using Cauchy’s determinant formula

$$\prod_{a < \beta}^M (a_a - a_\beta)(b_a - b_\beta) = (-1)^{\frac{M(M-1)}{2}} \det \left( \frac{1}{a_a - b_\beta} \right), \tag{4.27}$$

and a few basic properties of the determinant. We find

$$\langle x | \Psi_z \rangle \langle \Psi_z | \Phi \rangle = 2^M N! \det \left( \frac{1}{n_j + n_k + 2i(q_j - q_k)} \right) \prod_{j,k=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)}$$

$$= 2^M N! \prod_{a=1}^M \left( \int_0^\infty d\omega_a \right) \det (e^{-i \omega_a (n_j + n_k + 2i(q_j - q_k))}) \prod_{j,k=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)}$$

$$= 2^M N! \prod_{a=1}^M \left( \int_0^\infty d\omega_a \right) \det (e^{i \omega_a (q_j - n_j - q_k + n_k)} - 2i q_j (q_j - q_k)) \prod_{j,k=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)}$$

$$= 2^M N! \prod_{a=1}^M \left( \int_0^\infty d\omega_a \right) \det (e^{i \omega_a (q_j - n_j + q_k - n_k)} - 2i q_j (q_j - q_k)) \prod_{j,k=1}^M e^{i\omega_a q_a t} \prod_{r=1}^{n_a} \frac{1}{i q_a + \frac{1}{2}(n_a - 2r)} \tag{4.28},$$

where in the last equality, we used a simple fact

$$\det (A_{j=1}^b B_{j=1}^a) = \det ((a_j b_k)^{j,k}). \tag{4.29}$$
4.3. Fredholm determinant representation of the generating function

From (3.28), (4.7) and (4.28), we obtain an expression of \( (Z^N(x,t)) \) in terms of the determinant

\[
(Z^N(x,t)) = \sum_{M=1}^{\infty} \frac{2^M N!}{M!} \prod_{a=1}^{M} \left( \sum_{n_a=1}^{\infty} \int_{\mathbb{R} - i c} dq_a \frac{e^{-\frac{\gamma}{2} q_a^2 + \frac{\gamma}{2} x n_a^2}}{2\pi} \right) \delta_{\lambda_a, n_a} N \times \det \left( \frac{e^{i q_j q_k + \frac{\gamma}{2} q_j^2 + \frac{\gamma}{2} x n_j^2 - n_j (\alpha_j + \alpha_k) - 2i q_j (\alpha_j - \alpha_k)}}{\prod_{r=1}^{N} i q_j + \frac{\gamma}{2} (n_j - 2r)} \right)_{j,k=1}^{M} = \sum_{M=1}^{\infty} \frac{N!}{M!} \prod_{a=1}^{M} \left( \int_{\mathbb{R} - i c} dq_a \sum_{n_a=1}^{\infty} \delta_{\lambda_a, n_a} N \right) \times \det \left( \frac{e^{i q_j q_k + \frac{\gamma}{2} q_j^2 + \frac{\gamma}{2} x n_j^2 - n_j (\alpha_j + \alpha_k) - 2i q_j (\alpha_j - \alpha_k)}}{\prod_{r=1}^{N} i q_j + \frac{\gamma}{2} (n_j - 2r)} \right)_{j,k=1}^{M}.
\]

(4.30)

The imaginary part \( c \) of the contour of \( q \) is discussed below (4.7) and we find it sufficient to satisfy the condition \( c > N/2 + \max \alpha a \). Note that in (4.30), the contour satisfying this condition passes below the poles \( q = j - n_a/2 (j = 0, 1, \ldots, [n_a/2]) \) where \( [a] \) is the largest integer which is smaller than \( n_a \) and that we can deform the contour as long as it satisfies this property. Thus, in (4.30), we can relax the condition to \( c > \max \alpha a \).

Substituting this equation into (2.5), we eventually obtain the Fredholm determinant representation of the generating function (2.6). Remembering (3.18), we have

\[
G_{\gamma}(x; X) = 1 + \sum_{N=1}^{\infty} \frac{(-e^{-\gamma})^N}{N!} \prod_{a=1}^{M} \left( \int_{\mathbb{R} - i c} dq_a \sum_{n_a=1}^{\infty} \delta_{\lambda_a, n_a} N \right) \times \det \left( \frac{e^{i q_j q_k + \frac{\gamma}{2} q_j^2 + \frac{\gamma}{2} x n_j^2 - n_j (\alpha_j + \alpha_k) - 2i q_j (\alpha_j - \alpha_k)}}{\prod_{r=1}^{N} i q_j + \frac{\gamma}{2} (n_j - 2r)} \right)_{j,k=1}^{M} = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \int_{\mathbb{R} - i c_a} dq_a \delta_{\lambda_a, n_a} \times \frac{e^{i q_j q_k + \frac{\gamma}{2} q_j^2 + \frac{\gamma}{2} x n_j^2 - n_j (\alpha_j + \alpha_k) - 2i q_j (\alpha_j - \alpha_k)}}{\prod_{r=1}^{N} i q_j + \frac{\gamma}{2} (n_j - 2r)} \right)_{j,k=1}^{M},
\]

(4.31)

with \( c_n > n/2 \). Shifting the variable \( q \) to \( q + i X/\gamma \), we obtain

\[
G_{\gamma}(x; X) = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!} \prod_{a=1}^{M} \int_{\mathbb{R} - i c_a} dq_a \times \det \left( \sum_{n=1}^{\infty} (-1)^{n-1} \int_{\mathbb{R} - i c_a} dq_a \frac{e^{-\frac{\gamma}{2} (n_j + \alpha_k) - 2i q_j (\alpha_j - \alpha_k) - \frac{\gamma}{2} X^2/\gamma})}{\prod_{r=1}^{N} i q_j + \frac{\gamma}{2} (n_j - 2r)} \right)_{j,k=1}^{M},
\]

(4.32)

with \( c_n > X/\gamma + n/2 \). Applying the relation

\[
\prod_{r=1}^{n} i q_r - X/\gamma + \frac{\gamma}{2} (n - 2r) = \frac{\Gamma(i q - \frac{\gamma}{2} + \frac{\gamma}{2})}{\Gamma(i q - \frac{\gamma}{2} - \frac{\gamma}{2})}
\]

(4.33)

to (4.32), we finally obtain (26).
5. Another expression of the kernel

In this section, we give a derivation of expression (2.10) in proposition 2. We use the following two relations.

Lemma 6.

(a) We set \( a \in \mathbb{R} \) and \( m, n \geq 0 \). When \( \text{Im} q < -n/2 + a \), we have

\[
\frac{\Gamma(\text{i}q + a - \frac{n}{2})}{\Gamma(\text{i}q + a + \frac{n}{2})} e^{\frac{\text{i}q}{2}} = \int_{-\infty}^{\infty} dy \, \text{Ai}^\Gamma (y, \frac{1}{2m}, \text{i}q + a) e^{\text{im}y}, \quad (5.1)
\]

where

\[
\text{Ai}^\Gamma (a, b, c) = \frac{1}{2\pi} \int_{\Gamma} dz \, e^{\text{iaz} + \text{iz}^2/3} \frac{\Gamma(\text{ib}z + c)}{\Gamma(-\text{ib}z + c)}
\]

with \( \Gamma \) defined in (2.12) below.

(b) For \( u, v, x \in \mathbb{R} \) and \( w \geq 0 \), we have

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, \text{Ai}^\Gamma (\text{i}p^2 + v, \text{i}w + u) e^{\text{ip}x}
\]

\[
= \frac{1}{2} \text{Ai}^\Gamma (2^{-\frac{3}{4}}(v + x), 2^\frac{3}{4} w, u) \text{Ai}^\Gamma (2^{-\frac{3}{4}}(v - x), 2^\frac{3}{4} w, u), \quad (5.3)
\]

where \( \text{Ai}^\Gamma (a, b, c) \) and \( \text{Ai}^\Gamma (a, b, c) \) are defined by (2.11) and (2.12), respectively.

Proof.

(a) The right-hand side of (5.1) is written as

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dy \, \int_{\Gamma + i\text{mn}} \frac{dz}{\Gamma(-\frac{1}{2m} + iq + a)} e^{(z - \text{imn})y + \frac{1}{2}y^2} e^{\frac{\text{i}q}{2}}. \quad (5.4)
\]

Here we used the fact that the contour of \( z \) can be deformed from \( \Gamma_{2m(a - q)} \) to \( \mathbb{R} + i\text{mn} \)

since the imaginary part of the poles \( 2m(\text{i}a - q + i r), r = 0, 1, 2, \ldots \) of the integrand

is larger than \( mn \) when the condition \( \text{Im}q < a - n/2 \) is satisfied. In this equation, we change the variable \( z \) on the right-hand side to \( y_2 = z - i\text{mn} \) and obtain

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dy \, \int_{-\infty}^{\infty} \frac{dy_2}{\Gamma(-\frac{1}{2m} + iq + a)} e^{(y_2 + i\text{mn})y + \frac{1}{2}y^2} e^{\frac{\text{i}q}{2}} \int_{\Gamma} dz_2 \delta(y_2)
\]

\[
= \int_{-\infty}^{\infty} dy_2 \frac{\Gamma(\text{i}q + a - \frac{n}{2} + i\text{mn})}{\Gamma(\text{i}q + a + \frac{n}{2} - i\text{mn})} e^{\frac{3}{4}y_2^2} \frac{\Gamma(\text{i}q + a + \frac{n}{2} - \frac{1}{2})}{\Gamma(\text{i}q + a + \frac{n}{2} + \frac{1}{2})} e^{\frac{1}{4}y_2^2} = \int_{-\infty}^{\infty} dy_2 \frac{\Gamma(\text{i}q + a - \frac{n}{2})}{\Gamma(\text{i}q + a + \frac{n}{2})} e^{\frac{1}{4}y_2^2}. \quad (5.5)
\]

(b) The left-hand side of (5.3) reads

\[
\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp \, \int_{\Gamma_{p,y+\frac{3}{4}}} dz \, e^{(p^2 + v)z + \frac{3}{4}y^2 + ipx} \frac{\Gamma(\text{i}w(z + p) + u)}{\Gamma(-\text{i}w(z - p) + u)} e^{\frac{3}{4}w(z + p)^2} \quad (5.6)
\]

By applying the change of variables \( p = (z_1 - z_2)/2^{3/2} \) and \( z = (z_1 + z_2)/2^{3/2} \), we obtain

the desired expression. \( \square \)

Using lemma 6, we can obtain (2.10). Applying (a) of the lemma with \( a = -X/\gamma_t \) and \( m = \gamma_t/2^{3/2} \) to (2.7), one has

\[
K_X(\omega_1, \omega_2) = \sum_{n=1}^{\infty} (-1)^{n-1} \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \text{Ai}^\Gamma (y_1, \frac{1}{2\gamma_t}, \text{i}q - \frac{X}{\gamma_t}) e^{-2\text{i}q(y_1 - y_2)}
\]

\[
\times e^{-n(y_1 + \text{m}(\text{i}q^2 + y_2 - 2^{-\frac{3}{2}}y_1))}. \quad (5.7)
\]
At this point, the contour of \( q \) can be replaced by \( \mathbb{R} \) since the contour of \( z \) in the definition of \( A^I_l(a, b, c) \) (5.2) passes below the singularity of the Gamma function. Changing the variables \( q, \omega_j \) and \( y_1 \) to \( p = 2^{1/3} y q, \xi_j = 2^{1/3} \omega_j \) and \( y = y_1/2^{2/3} - p^2/2^{2/3} - (\xi_j + \xi_k)/2 \), we see

\[
K(x, \omega, \xi) \, d\omega = K(x, \xi) \, dx,
\]

where

\[
\tilde{K}_X(\xi, \xi_k) = \frac{2i}{\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dp \, A^I_l(p^2 + 2^{1/3} y + \frac{\xi_j + \xi_k}{2}, \frac{1}{2^{1/3} y}, \frac{1}{2^{1/3} y}) e^{-\frac{i(\xi_j - \xi_k)}{y}}.
\]

Thus, we can apply (2) of Lemma 6 to this equation with \( v = 2^{2/3} y + (\xi_j + \xi_k)/2^{1/3}, \quad w = 1/(2^{1/3} y), \quad u = -X/y, \quad x = (\xi_j - \xi_k)/2^{1/3} \) and arrive at (2.10).

6. Multi-point distribution

6.1. Generating function

We consider the \( n \)-point generating function defined by (2.24) and derive result (2.25). Equation (2.24) can be expanded in terms of the replica partition function \((Z(x_1, t) \cdots Z(x_n, t))\),

\[
G(y) \{s_n \}, \{X_n \} = 1 + \sum_{N=1}^{\infty} \frac{(-1)^N e^{\frac{\alpha}{N}}}{N!} \times \sum_{l_1, \ldots, l_N = 1}^{N} e^{-\gamma \sum_{j=1}^{N} (s_j - X_j)} \left| Z(2y^2 X_{l_1}, t) \cdots Z(2y^2 X_{l_N}, t) \right|.
\]

As we discussed in Sections 3 and 4, we can express the replica partition function in terms of the Bethe states (3.25)–(3.28) of the \( \delta \)-Bose gas. Specifically, we can then generalize (4.7) to the case where \( x \)'s are distinct and we obtain

\[
(Z(x_1, t) \cdots Z(x_n, t)) = \sum_{M=1}^{N} \frac{1}{M!} \prod_{a=1}^{M} \left( \int_{R-ic} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} \delta_{\sum_{a=1}^{M} n_a N} \langle x_1, \ldots, x_n | \Psi_e \rangle \langle \Psi_e | \Phi \rangle e^{-E_{\text{int}}}. \right.
\]

Thus, the generating function is written as

\[
G(y) \{s_n \}, \{X_n \} = 1 + \sum_{N=1}^{\infty} \frac{(-1)^N e^{\frac{\alpha}{N}}}{N!} \prod_{a=1}^{N} \left( \int_{R-ic} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} \delta_{\sum_{a=1}^{M} n_a N} A_{z} B_{z}, \right.
\]

where

\[
A_z = e^{-E_z} \langle 0 | \Psi_e \rangle \langle \Psi_e | \Phi \rangle,
\]

\[
B_z = \sum_{l_1, \ldots, l_N = 1}^{N} e^{-\gamma \sum_{j=1}^{N} (s_j - X_j)} \left| \frac{2y^2 X_{l_1}, \ldots, 2y^2 X_{l_N}}{\langle 0 | \Psi \rangle} \right|.
\]

Note that in the two factors \( A_z \) and \( B_z \), only \( A_z \) depends on the initial condition and we can readily use (3.28) and (4.28) for \( A_z \). On the other hand, the factor \( B_z \), which is irrelevant to specific initial conditions, has already appeared in the narrow-wedge initial condition and the following approximation has been proposed in [33, 34].
For $B_\zeta$, we have to estimate the equation with the following type:

$$
\sum_{l_1, \ldots, l_N} \langle x_{l_1}, \ldots, x_{l_n} | \Psi_\zeta \rangle \prod_{j=1}^N e^{i f_j}.
$$

(6.6)

Here $g_j (j = 1, \ldots, n)$ are variables which depend on $j$. Although the eigenfunction $\langle x_{l_1}, \ldots, x_{l_n} | \Psi_\zeta \rangle$ is given in (3.25), the following compact form given in [28] is convenient for our current discussion:

$$
\langle y_1, \ldots, y_N | \Psi_\zeta \rangle = \sum_{P \in \Omega_N} A_P (y_1, \ldots, y_N) \exp \left( i \sum_{a=1}^M q_a \sum_{c \in \Omega_\alpha (P)} y_c - \frac{1}{4} \sum_{a=1}^M \sum_{c, c' \in \Omega_\alpha (P)} |y_c - y_{c'}| \right),
$$

(6.7)

where $A_P, \Omega_\alpha (P)$ and the prime in the summation are defined as follows. the amplitude $A_P$ is represented as

$$
A_P (y_1, \ldots, y_N) = \text{sgn } P \cdot C_P \prod_{a=1}^M \prod_{k=1}^{\Omega_\alpha (P)} (z_{P_{ij}} - z_{P_{ik}} + i \text{sgn } (y_j - y_k)).
$$

(6.8)

The cluster $\Omega_\alpha (P)$ is defined by $\Omega_\alpha (P) = \{ a \alpha (P(a)) = \beta \}$ by the use of the cluster counting function $\alpha : [1, \ldots, N] \to [1, \ldots, M]$ such that $\alpha (a) = \beta$, for $\sum_{j=1}^{\beta - 1} b_j < a \leq \sum_{j=1}^{\beta} b_j$. The prime means the summation over permutations which keep the order inside each cluster, i.e. the permutations satisfying the condition $j < k \to P(j) < P(k)$ for $j$ and $k$ such that $\alpha (P(j)) = \alpha (P(k)) (1 \leq j, k \leq N)$.

Thus, (6.6) is expressed as

$$
\sum_{l_1, \ldots, l_N} \langle x_{l_1}, \ldots, x_{l_n} | \Psi_\zeta \rangle \prod_{j=1}^N e^{i f_j} = \sum_{l_1, \ldots, l_N} \sum_{P \in \Omega_N} A_P (x_{l_1}, \ldots, x_{l_n}) e^{\phi ([l], P)},
$$

(6.9)

where $\phi ([l], P)$ represents the phase

$$
\phi ([l], P) = i \sum_{a=1}^M q_a \sum_{c \in \Omega_\alpha (P)} y_c - \frac{1}{4} \sum_{a=1}^M \sum_{c, c' \in \Omega_\alpha (P)} |y_c - y_{c'}| + \sum_{j=1}^N g_{l_j}.
$$

(6.10)

The difficulty in (6.9) is that the summations over $l_1, \ldots, l_N$ and $P$ are coupled. In [33, 34], the authors proposed an approximation that the two summations factorize,

$$
\sum_{l_1, \ldots, l_N} \sum_{P \in \Omega_N} A_P (y_1, \ldots, y_N) e^{\phi ([l], P)} \sim \sum_{l_1, \ldots, l_N} \sum_{P \in \Omega_N} A_P (y_1, \ldots, y_N) e^{\phi ([l], \ast)},
$$

(6.11)

where $\phi ([l], \ast) = \phi ([l], P = (1, 2, \ldots, N))$. Under this assumption, we can perform the two summations using the relations

$$
\sum_{P \in \Omega_N} A_P (y_1, \ldots, y_N) = \langle 0 | \Phi_\zeta \rangle,
$$

(6.12)

which comes from identity (4.9), and

$$
\sum_{l_1, \ldots, l_N} e^{\phi ([l], \ast)} = \prod_{a=1}^M e^{-\frac{i}{2} \sum_{k=1}^n |y_{l_k} - y_{l_{k'}}| b_k} \left( \sum_{\lambda=1}^n e^{\nu \lambda + i \nu X} \right)^{n_\lambda},
$$

(6.13)

which is obtained as (3.13) in [34]. From (6.11)–(6.13), and setting $y_j = 2y_j^2 X_j, g_j = s_j - X_j^2$, we obtain the approximated form of $B_\zeta (6.5)$,

$$
B_\zeta \sim \sum_{a=1}^M e^{-\frac{i}{2} \sum_{k=1}^n |y_{l_k} - y_{l_{k'}}| b_k} \left( \sum_{\lambda=1}^n e^{\nu \lambda + i \nu X} \right)^{n_\lambda}.
$$

(6.14)
Substituting the above equation into (6.3) and following the same procedure as in section 4.3, we have

\[ G_L^\gamma (\{s\}_n, \{X\}_n) = 1 + \sum_{M=1}^{\infty} \frac{1}{M!} \left( \prod_{a=1}^{M} \int_{\mathbb{R} - i\epsilon} \frac{d\alpha}{2\pi} \sum_{n_a=1}^{\infty} \right) \det \left( \frac{1}{\frac{1}{2} (\alpha a + \beta b) + i(\alpha a - \beta b)} \right) \times \prod_{a=1}^{M} \prod_{r=1}^{n_a} \left( i_q a + \frac{1}{2} (\alpha a - 2r) \right) \times \prod_{a=1}^{M} e^{\gamma_n a / 12 - \frac{1}{2} \sum_{i=1}^{n} |X_j - X_i| b_i h_i} \left( -e^{\frac{\gamma n}{\gamma + 1}} \sum_{i=1}^{n} e^{-\gamma_j (X_i + \gamma X_j + 2i r_j) q_i X_j} \right)_{n_a} . \] (6.15)

Here the index \( \sharp \) indicates that we use approximation (6.14). We find that the above equation has a similar structure to the corresponding one in the narrow-wedge case in [34]. All the differences are described in just two points: the contour of \( q \) includes the imaginary term \( -i\epsilon \) and the factor \( \prod_{a=1}^{M} \prod_{r=1}^{n_a} (\gamma a + \frac{1}{2} (\alpha a - 2r)) \) is added. Thus, we can just follow section 4 in [34] if we replace equations (4.14) and (4.19) in [34] by (1) and (2) in lemma 6, respectively. As a result, we find the Fredholm determinant expression of \( G_L^\gamma (\{s\}_n, \{X\}_n) \):

\[ G_L^\gamma (\{s\}_n, \{X\}_n) = \det(1 - L) , \] (6.16)

where

\[ L(z, z') = \chi_0(z) \chi_0(z') \exp \left( -\frac{1}{2} \sum_{j,k=1}^{n} |X_j - X_k| \partial_j \partial_k - \gamma \sum_{j=1}^{n} X_j^2 \partial_j - (\partial_z - \partial_{z'}) \sum_{j=1}^{n} X_j \partial_j \right) \times \int_{-\infty}^{\infty} du \ Ai^\gamma \left( u + z, \frac{1}{\gamma' - 1}, 0 \right) Ai_{\gamma'} \left( u + z', \frac{1}{\gamma' - 1}, 0 \right) \Phi(\{u - s\}_n) . \] (6.17)

Here \( \chi_0(x) \) is given below (2.32), \( Ai^\gamma (a, b, c) \) and \( Air (a, b, c) \) are defined by (2.11) and (2.12), respectively, and for \( \Phi(\{x\}_n) \), see (2.27). This expression of the kernel corresponds to equation (4.25) in [34] for the narrow-wedge initial condition. The only difference is that the product of the ordinary Airy functions \( Ai(u + z) Ai(u + z') \) is replaced by \( Ai^\gamma (u + z, 1/\gamma - 1, 0) Air (u + z', 1/\gamma - 1, 0) \) in (6.17).

We can further deform expression (6.17) following the discussion in section 4.2 in [34].

The equation corresponding to equation (4.38) in [34] becomes

\[ L(z, z') = \int_{-\infty}^{\infty} du_1 \cdots du_n |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma |L_0 e^{(X_0 - X_2)^H} |u_1 \rangle \langle u_1| e^{(X_0 - X_2)^H} |u_2 \rangle \times \cdots \times |u_{n-1} \rangle \langle u_{n-1}| e^{(X_0 - X_2)^H} |u_n \rangle \langle u_n| e^{(X_0 - X_2)^H} L_0 |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H |u_s \rangle \langle u_s| , \] (6.18)

with \( X_0 = 0 \). Here \( H \) is the Airy Hamiltonian given below (2.26), and \( \Phi(\{x\}_n) \) and \( L_0 \) are defined by (2.27) and (2.28), respectively. The states \( |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H \rangle \) and \( |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H \rangle \) are defined by the relations \( \langle u |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H \rangle = Ai^\gamma (z + u, b, c) \) and \( \langle u |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H \rangle = Air (z + u, b, c) \), respectively. In this kernel, we put \( L = Q_2 Q_1 \), where

\[ Q_2 (z', u_1) = |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma |L_0 e^{-X_2^H} |u_1 \rangle , \] (6.19)

\[ Q_1 (u_1, z) = \int_{-\infty}^{\infty} du_2 \cdots du_n |u_1 \rangle e^{(X_0 - X_2)^H} |u_2 \rangle \times \cdots \times |u_{n-1} \rangle e^{(X_0 - X_2)^H} |u_n \rangle \times \langle u_n| e^{X_2^H} L_0 |Ai_{\gamma', 1/\gamma - 1, -X_0 / \gamma'}^\gamma H |u_s \rangle \langle u_s| . \] (6.20)
Noting the relation of the Fredholm determinant, \( \det(1 - Q_2 Q_1) = \det(1 - Q_1 Q_2) \), and the biorthogonality (A.2) of the deformed Airy function, we obtain \( G^\gamma_j([s]_n, [X]_n) = \det(1 - Q) \) where

\[
Q(x, y) := Q_1 Q_2(x, y) = \int_{-\infty}^{\gamma} du_2 \ldots du_n [u_1] e^{x_1 - x_2 H} [u_2] \ldots [u_n] e^{x H} L^2_{0} e^{-x H} |y| \Phi(|u - s|_n). \tag{6.21}
\]

The factor \( \langle x | e^{x - x H} |y \rangle \) is represented in terms of the deformed Airy functions as (2.29), which will be shown in the appendix. Using (A.2) and (ii) of the appendix, we find \( L^2_0 = L_0 \) and \( e^{-x H} = e^{-x H} L_1 \), and thus we finally obtain (2.26). The second-last factor \( \langle x | e^{H} L_j |y \rangle \) in (2.26) is written as

\[
\langle x | e^{H} L_j |y \rangle = \int_{0}^{\infty} dz e^{-iz} \Ai^{-} \left( x + z, \frac{1}{\gamma}, \frac{-x_j + t}{\gamma} \right) \Ai^{+} \left( y + z, \frac{1}{\gamma}, \frac{-x_i}{\gamma} \right). \tag{6.22}
\]

This readily follows for \( t < 0 \) from (A.4) but one has to be careful when \( t > 0 \). The integrand is divergent as \( z \to -\infty \) and hence \( \langle x | e^{H} \rangle \) in (2.29) is not well defined. But in \( \langle x | e^{H} L_j |y \rangle \), \( L_j \) projects the range of integration to the positive side only and (6.22) is still valid.

### 6.2. Long-time limit

Here we consider the long-time limit of the generating function \( G^\gamma_j([s]_n, [X]_n) \) (2.26). We first introduce the limiting version of \( L_j(x, y) \) (2.28),

\[
L_j(x, y) := \lim_{\gamma \to \infty} L_j(x, y) = \int_{0}^{\infty} dw \Ai^{-}(x + w, X_j) \Ai^{+}(y + w, X_j), \tag{6.23}
\]

where \( \Ai^{-}(x + w, X_j) \) is the limit of the Gamma-deformed Airy functions (2.11) and (2.12),

\[
\Ai^{-}(x, y) := \lim_{\gamma \to \infty} \frac{1}{\gamma} \Ai^{+} \left( x, \frac{1}{\gamma}, \frac{-y}{\gamma} \right) = \frac{1}{2\pi} \int_{\Gamma_{-y}} dz e^{iz + \frac{z^2}{4}}. \tag{6.24}
\]

Here the contour \( \Gamma_{-y} \) in (6.24) is given below (2.12). Noting

\[
\lim_{t \to \infty} \Phi(|u - s|_n) = 1 - \prod_{j=1}^{n} (1 - \Theta(s_j - u_j)) \tag{6.26}
\]

where \( \Theta(x) = 1(x \geq 0), 0(x < 0) \), we find that the long-time limit of (2.26) becomes

\[
\lim_{t \to \infty} G^\gamma_j([s]_n, [X]_n) = \det(1 - L_1 + \tilde{P}_1 e^{x_1 - x_2 H} \ldots \tilde{P}_{n-1} e^{x_{n-1} - x_1 H} \tilde{P}_n e^{x_n - x_1 H} L_j). \tag{6.27}
\]

Here \( \tilde{P}_i \) represents the projection operator onto \((-\infty, s)\). To this equation we apply the result of the appendix in [34] with \( K^0_j = \delta_{ij} K \) replaced by \( K^0_j = \delta_{ij} L_j \). We eventually obtain the Fredholm determinant with the matrix kernel \( M \):

\[
\lim_{t \to \infty} G^\gamma_j([s]_n, [X]_n) = \det(1 - M), \tag{6.28}
\]

where

\[
M_{ij} (\xi_1, \xi_2) = \begin{cases} x_i (\xi_1) |\xi_1| e^{x_1 - x_2 H} L_1 |\xi_1| \chi_u (\xi_2), & \text{for } j \geq k, \\ -x_j (\xi_1) |\xi_1| e^{x_1 - x_2 H} (1 - L_1) |\xi_1| \chi_u (\xi_2), & \text{for } j < k. \end{cases} \tag{6.29}
\]

Here \( \chi_u (x) \) is given in (2.34) below.
We find that this matrix kernel is equivalent to $K_{12}(X_j, \xi_1; X_k, \xi_2)$. By taking the limit $\gamma_1 \to \infty$ in (6.22), we easily find that the factor $(\xi_1 | e^{i(X_1 - X_2)^T \mathcal{L}_k} | \xi_2)$ in the case $j \geq k$ of (6.29) is expressed as
\[
\langle \xi_1 | e^{i(X_1 - X_2)^T \mathcal{L}_k} | \xi_2 \rangle = \int_0^\infty dz e^{-(X_1 - X_2)^T \mathcal{A} \xi_1} \mathcal{A}^{\dagger} (\xi_1 + z, X_j) \mathcal{A}^{\dagger} (\xi_2 + z, X_k) \]
\[= \int_{-\infty}^\infty dw_1 \int_{-\infty}^\infty dw_2 e^{iw_1 \xi_1 + iw_2 \xi_2} \frac{e^{iw_2 + X_k}}{(iw_1 - X_j)(iw_1 + iw_2 - X_j + X_k)}. \tag{6.30}
\]
Here we used (6.24) and (6.25) in the second equality. Noting
\[
\frac{e^{iw_2 + X_k}}{(iw_1 - X_j)(iw_1 + iw_2 - X_j + X_k)} = \frac{1}{iw_1 - X_j} - \frac{1}{iw_1 + iw_2 - X_j + X_k}, \tag{6.31}
\]
we finally find that the factor can be represented as the Airy function. For $j \geq k$, we obtain
\[
\langle \xi_1 | e^{i(X_1 - X_2)^T \mathcal{L}_k} | \xi_2 \rangle = \int_0^\infty dw e^{-w(X_1 - X_2)} \mathcal{A}(\xi_1 + w) \mathcal{A}(\xi_2 + w) + \mathcal{A}(\xi_2) \left( e^{-X_1^T \mathcal{X} \xi_2} \int_0^\infty dw e^{-X_1 w} \mathcal{A}(\xi_1 + w) \right). \tag{6.32}
\]
Similarly for the case $j < k$ in (6.29), we obtain
\[
\langle \xi_1 | e^{i(X_1 - X_2)^T \mathcal{L}_k} | \xi_2 \rangle = -\int_{-\infty}^0 dw e^{-w(X_1 - X_2)} \mathcal{A}(\xi_1 + w) \mathcal{A}(\xi_2 + w) + \mathcal{A}(\xi_2) \left( e^{-X_1^T \mathcal{X} \xi_2} \int_0^\infty dw e^{-X_1 w} \mathcal{A}(\xi_1 + w) \right). \tag{6.33}
\]
Hence, we finally find $M_{j,k}(\xi_1, \xi_2) = K_{12}(X_j, \xi_1; X_k, \xi_2)$.

7. Directed polymer interpretation

Our result can also be rephrased for the free energy distribution of the (1+1)-dimensional directed polymer in random media. First let us rewrite the path integral expression of $Z_{\nu, \beta, D}(x, \tau)$ (3.8) as
\[
Z_{\beta, \gamma, D}(x, \tau) = \int_0^\infty dy \int_{x(0)=y}^{x(\tau)=x} D[x(\tau)] \exp \left[ -\beta (H[x] + \mu B(y)) \right], \tag{7.1}
\]
where
\[
H[x] = \int_0^{\tau} d\tau \left( \gamma \left( \frac{dx}{d\tau} \right)^2 - \sqrt{D} \eta(x(\tau), \tau) \right). \tag{7.2}
\]
The parameters $\beta, \gamma$ and $\mu$ are defined as
\[
\beta = \frac{\lambda}{2v}, \quad \gamma = \frac{1}{\lambda}, \quad \mu = \frac{\alpha}{\beta} = \sqrt{\frac{D}{2v}} \tag{7.3}
\]
in terms of those of the KPZ equation (1.1).

Expression (7.1) can be interpreted as the partition function of a directed polymer in random media with inverse temperature $\beta$. On the two-dimensional $(x, \tau)$ $(x \in \mathbb{R}$ and $\tau \geq 0)$ plane, the configuration of a polymer is represented as a function $x(\tau)(0 \leq \tau \leq \tau)$. Here it is assumed that $x(\tau)$ is a single-valued function, which means that the polymer is ‘directed’ for the $\tau$ coordinate. For a configuration $x(\tau)(0 \leq \tau \leq \tau)$, the bulk energy $H[x]$ (7.2) is assigned. The first term of $H[x]$ represents the elastic energy of the polymer whose strength is
determined by γ, and the second term represents a random potential energy with D adjusting its strength.

The half Brownian motion initial condition (1.4) in the KPZ equation corresponds to the following boundary condition in the language of the random directed polymer. At the boundary \( \tau = 0 \), the position of the polymer \( x(0) \) can take any positive value \( y \geq 0 \) and, depending on the position \( y \), the random boundary energy \( \mu B(y) \) is assigned to the polymer. Note that the other edge \( x(t) \) is pinned at \( x \). As a statistical mechanical model, \( \mu \) in (7.1) is an independent parameter, but when translated from the KPZ equation one has to keep in mind that it is written as \( \mu = \sqrt{\beta \gamma D} \).

We want to discuss the low-temperature behavior of the directed polymer for a fixed \( t \). When the temperature is sufficiently low, the configuration which gives the minimum energy is dominant for the free energy. When the position \( x \) of the edge \( x(t) \) of the polymer is positive, the other position of the edge \( y \) tends to be close to \( x \) because otherwise the polymer obtains excess bulk elastic energy. In this case, it is dominated by the boundary energy \( B(y) \). Thus, in the positive region \( x > 0 \), the free energy fluctuation is described by the Gaussian. On the other hand, when \( x < 0 \), the edge \( x(0) \) tends to be pinned at the origin. In this case, the free energy fluctuation is determined only by the bulk energy (note that \( B(y = 0) = 0 \)). In between around \( x = 0 \), one observes the crossover distribution, which is exactly what have been computed in this paper.

Concretely, from (3.3), \( -h_{\beta, \gamma, D} \) corresponds to the free energy \( f_{\beta, \gamma, D}(x, t) \) of the random directed polymer and hence (2.13) is translated as

\[
\text{Prob}\left( \beta f_{\beta, \gamma, D}(x, t) - \frac{\gamma_1^3}{12} + \gamma_1 X^2 \geq \gamma_1 \right) = F_{\gamma_1}(s; X),
\]

(7.4) where \( F_{\gamma_1}(s; X) \) is given in (2.17) and \( \alpha \) and \( \gamma_1 \) are written in terms of \( \gamma \) and \( \beta \) as \( \alpha = (\beta^3 \gamma D)^{1/2} \) and \( \gamma_1 = (\beta^3 \gamma D^3 t/2)^{1/3} \), respectively. Based on (7.4), one can obtain various information about the free energy of the directed polymer. For instance, one sees that as \( \beta \to \infty \), the macroscopic free energy scales as \( O(\beta^3) \), the fluctuations of the bulk free energy are \( O(\beta^{2/3}) \). To obtain a nontrivial distribution in the low-temperature limit \( \beta \to \infty \), one has to scale the space direction as \( X = O(\beta^{1/3}) \). The free energy distribution in this limit is given by (2.22) since \( \beta \to \infty \) implies \( \gamma_1 \to \infty \). This describes the crossover between boundary and bulk free energy fluctuation, i.e. between Gaussian and the GUE Tracy–Widom distribution.

**8. Conclusion**

In this paper, we have considered the KPZ equation (1.1) with the half Brownian motion initial condition (1.4). Using the Bethe ansatz results of the one-dimensional attractive \( \delta \)-Bose gas, we have obtained a Fredholm determinant expression for the generating function of exponential moments of the height (theorem 1). Thanks to this result and proposition 2, the compact representation of the probability distribution of the height was obtained (theorem 3). We have also found an expression for the multi-point generating function employing an approximation proposed in [33, 34].

These results are expressed in terms of the deformed Airy functions (2.11) and (2.12). If we change these functions to the ordinary Airy function, we recover the results for the narrow–wedge initial condition obtained in [14–17]. The deformed Airy functions have some good properties, which are discussed in the appendix. They satisfy the biorthogonality relation. Their time evolution by the Airy Hamiltonian is again given by the same functions with a parameter modified. We remark that similar relations of the multiple Hermite polynomials played an important role in the study of the PNG model with external source [42, 43]. In
the long-time limit, the kernel of the Fredholm determinant becomes the rank-1 perturbation of the Airy kernel. It is not clear whether the higher rank perturbations studied in \[47\] have corresponding finite-time generalizations.

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**Appendix. Properties of the deformed Airy functions**

In this appendix, we pick up a few properties of the gamma-deformed Airy functions (2.11) and (2.12), which are necessary for the discussions in section 6. Let \(t \leq 0\) and \(H\) be the Airy Hamiltonian \(H = -\partial^2/\partial x^2 + x\). We have the following relations.

(i) The deformed Airy function representation of the propagator

\[
\langle x | e^{tH} | y \rangle = \int_{-\infty}^{\infty} dw \ e^{-tw} \Ai_{\gamma}(w + x) \Ai_{\gamma}(w + y).
\]  

Note that when we set \(t = 0\) in (A.1), we obtain the biorthogonality relation of the deformed Airy function,

\[
\int_{-\infty}^{\infty} dw \ Ai_{\gamma}(x + w, b, c)Ai_{\gamma}(y + w, b, c) = \delta(x - y).
\]  

Using (A.1), we easily obtain the following relations.

(ii) ‘Time evolution’ by the Airy Hamiltonian

\[
e^{tH} \Ai_{\gamma}^{\pm}(x + w, b, c) = e^{-tw} \Ai_{\gamma}^{\pm}(x + w, b, c - bt),
\]  

\[
e^{tH} \Ai_{\gamma}^{-}(x + w, b, c) = e^{-tw} \Ai_{\gamma}^{-}(x + w, b, c + bt).
\]  

When we set \(b = 1/\gamma_t\) and \(c = -X/\gamma_t\) and take the limit \(\gamma_t \to \infty\), these relations (A.1), (A.3) and (A.4) become those for \(\Ai_{\gamma}^{-}(x, y)\) (6.24) and \(\Ai_{\gamma}^{+}(x, y)\) (6.25),

\[
\langle x | e^{tH} | y \rangle = \int_{-\infty}^{\infty} dz \ e^{-tz} \Ai_{\gamma}^{\pm}(x + z, X) \Ai_{\gamma}^{-}(y + z, X).
\]  

\[
e^{tH} \Ai_{\gamma}^{-}(x + w, X) = e^{-tw} \Ai_{\gamma}^{-}(x + w, X - t),
\]  

\[
e^{tH} \Ai_{\gamma}^{+}(x + w, X) = e^{-tw} \Ai_{\gamma}^{+}(x + w, X + t).
\]  

**Proof.**

(i) The factor \(\langle x | e^{tH} | y \rangle\) can be represented in terms of the Airy function,

\[
\langle x | e^{tH} | y \rangle = \int_{-\infty}^{\infty} dw \ e^{-tw} \Ai(w + x)\Ai(w + y).
\]  

Thus, it is enough to show that

\[
\int_{-\infty}^{\infty} dw \ e^{-tw} \Ai_{\gamma}(x + w, b, c - bt) \Ai_{\gamma}^{\pm}(y + w, b, c)
\]  

\[= \int_{-\infty}^{\infty} dw \ e^{-tw} \Ai(w + x)\Ai(w + y).
\]  

26
In definition (2.11) of $Ai^T (a, b, c)$, we find that in the case $c > 0$, the contour integral $\Gamma_{it} / b$ can be replaced by $R$, while in the case $c \leq 0$, we can divide the contour integral into the integral on $R$ and the contributions from the poles at $(ic + m)/b$, $m = 0, 1, 2, \ldots$, of $\Gamma(ibz + c)$. Thus, (2.11) can be represented as

$$
Ai^T (a, b, c) = \begin{cases} 
\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz e^{iz^2 + iz^3} \Gamma(ibz + c), & c > 0, \\
\frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{iz^2 + iz^3} \Gamma(ibz + c) + \sum_{n=0}^{\infty} \frac{(-1)^n}{Nb} e^{-\frac{(a+nb)^2}{b} + \frac{2i(a+nb)c}{b}}, & c \leq 0.
\end{cases}
$$

(A.10)

Here $n$ is the maximum integer satisfying $c + n < 0$.

Thus, for the case $c > 0$, using (2.12) and the first relation in (A.10), we find that the left-hand side of (A.9) is represented as

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 e^{iw(z_1 + z_2 + it) + i(z_1^2 + z_2^2) + \frac{1}{2}(z_1^3 + z_2^3)} \frac{\Gamma(ibz_2 + c)}{\Gamma(-ibz_1 + c - bt)}
$$

Here we deformed the contour of $z_1$ to $R - it$. This deformation is valid when $t \leq 0$. Note that in this equation given above, the part $z_1 + z_2 + it$ becomes real. Thus, one has

$$
\int_{R - it} \frac{dz_1}{\Gamma(-ibz_1 + c - bt)} \int_{-\infty}^{\infty} dz_2 \delta(z_1 + z_2 + it) e^{iz_1(z_1 + z_2 + it) + \frac{1}{2}(z_1^2 + z_2^3)} \frac{\Gamma(ibz_2 + c)}{\Gamma(-ibz_1 + c - bt)}
$$

A useful property of the Gamma functions is eliminated because of the delta function. We easily find that the last expression of this equation is equal to the right-hand side of (A.9) from the integral representation of the Airy function,

$$
Ai(x) = \int_{-\infty}^{\infty} dz e^{izx + iz^3}.
$$

(A.13)

For $c < 0$, on the other hand, the terms coming from the pole contributions in (A.10) are added. However, we find that these terms do not affect the result since

$$
\int_{-\infty}^{\infty} dw e^{-tw} Ai^T (x + w, b, c) e^{-\frac{z_1 x}{b} - \frac{b}{z_1} + \frac{b}{z_1} + i z_1^3}
$$

$$
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dz_1 \frac{e^{i(z_1 + z_2^2 + it)w - \frac{(a+nb)^2}{b} + \frac{2i(a+nb)c}{b}}}{\Gamma(-ibz_1 + c - bt)}
$$

$$
= \int_{-\infty}^{\infty} dz_1 \delta \left( z_1 + \frac{1}{b} + it \right) \frac{e^{-\frac{z_1^2 x + k_1 x}{b}}}{\Gamma(-ibz_1 + c - bt)}
$$

$$
= \frac{e^{i\frac{z_1^2 x + k_1 x}{b}}}{\Gamma(-N)} = 0.
$$

(A.14)
Here we consider only (A.3) and omit the proof of (A.4) since it can be shown by using the same strategy as (A.3). The left-hand side is given by

\[
\langle x | e^{tH} | Ai_{w,b,c} \rangle = \int_{-\infty}^{\infty} dy \langle x | e^{tH} | y \rangle Ai \Gamma (y + w, b, c).
\]  

(A.15)

Substituting (A.1) into this equation, we readily obtain the right-hand side of (A.3),

\[
\langle x | e^{tH} | Ai_{w,b,c} \rangle = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy e^{-tz} Ai \Gamma (x + z, b, c - bt) \\
\times Ai \Gamma (y + z, b, c) Ai \Gamma (y + w, b, c) = e^{-tw} Ai \Gamma (x + w, b, c - bt).
\]  

(A.16)

Here in the second equality, we used the biorthogonality relation (A.2).  

References

[1] Kardar M, Parisi G and Zhang Y C 1986 Dynamic scaling of growing interfaces Phys. Rev. Lett. 56 889–92
[2] Johansson K 2000 Shape fluctuations and random matrices Commun. Math. Phys. 209 437–76
[3] Prähofer M and Spohn H 2000 Statistical self-similarity of one-dimensional growth processes Physica A 279 342–52
[4] Prähofer M and Spohn H 2000 Universal distributions for growth processes in 1+1 dimensions and random matrices Phys. Rev. Lett. 84 4882–5
[5] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel Commun. Math. Phys. 159 151–74
[6] Prähofer M and Spohn H 2002 Scale invariance of the PNG droplet and the Airy process J. Stat. Phys. 108 1071–106
[7] Sasamoto T 2005 Spatial correlations of the 1D KPZ surface on a flat substrate J. Phys. A: Math. Gen. 38 L549–56
[8] Borodin A, Ferrari P L, Prähofer M and Sasamoto T 2007 Fluctuation properties of the TASEP with periodic initial configuration J. Stat. Phys. 129 1055–80
[9] Sasamoto T 2007 Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques J. Stat. Mech. P07007
[10] Kriecherbauer T and Krug J 2010 A pedestrian’s view on interacting particle systems, KPZ universality, and random matrices J. Phys. A: Math. Theor. 43 403001
[11] Ferrari P 2010 From interacting particle systems to random matrices J. Stat. Mech. P10016
[12] Sasamoto T and Spohn H 2010 The 1+1-dimensional Kardar–Parisi–Zhang equation and its universality class J. Stat. Mech. P11013
[13] Takeuchi K A and Sano M 2010 Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals Phys. Rev. Lett. 104 230601
[14] Sasamoto T and Spohn H 2010 One-dimensional Kardar–Parisi–Zhang equation: an exact solution and its universality Phys. Rev. Lett. 104 230602
[15] Sasamoto T and Spohn H 2010 Exact height distributions for the KPZ equation with narrow wedge initial condition Nucl. Phys. B 834 523–42
[16] Sasamoto T and Spohn H 2010 The crossover regime for the weakly asymmetric simple exclusion process J. Stat. Phys. 140 299–31
[17] Amir G, Corwin I and Quastel J 2011 Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions Commun. Pure Appl. Math. 64 466–537
[18] Tracy C A and Widom H 2008 Integral formulas for the asymmetric simple exclusion process Commun. Math. Phys. 279 815–44
[19] Tracy C A and Widom H 2009 Asymptotics in ASEP with step initial condition Commun. Math. Phys. 290 129–54
[20] Bertini L and Giacomin G 1997 Stochastic Burgers and KPZ equations from particle systems Commun. Math. Phys. 183 571–607
[21] Ferrari P and Frings R 2011 Finite time correlations in KPZ growth models arXiv:1104.2129
[22] Quastel J and Remenik D 2011 Finite variation of the crossover Airy2 process with respect to Brownian motion, arXiv:1105.0952
[23] O’Connell N 2009 Directed polymers and the quantum Toda lattice, arXiv:0910.0069
[24] O’Connell N and Warren J 2011 A multi-layer extension of the stochastic heat equation, arXiv:1104.3509
[25] Goncalves P and Jara M 2010 Universality of KPZ equation, arXiv:1003.4478
[26] Goncalves P and Jara M 2010 Crossover to KPZ equation, arXiv:1004.2726
[27] Dotsenko V 2010 Bethe ansatz derivation of the Tracy–Widom distribution for one-dimensional directed polymers Europhys. Lett. 90 20003
[28] Dotsenko V 2010 Replica Bethe ansatz derivation of the Tracy–Widom distribution of the free energy fluctuations in one-dimensional directed polymers J. Stat. Mech. P07010
[29] Calabrese P, Le Doussal P and Rosso A 2010 Free-energy distribution of the directed polymer at high temperature Europhys. Lett. 90 20002
[30] Kardar M 1987 Replica Bethe ansatz studies of two-dimensional interfaces with quenched random impurities Nucl. Phys. B 290 582–602
[31] Lieb E H and Liniger W 1963 Exact analysis of an interacting Bose gas: I. The general solution and the ground state Phys. Rev. 130 1605–16
[32] McGuire J B 1964 Study of exactly soluble one dimensional N body problems J. Math. Phys. 5 622–36
[33] Prolhac S and Spohn H 2011 Two-point generating function of the free energy for a directed polymer in a random medium J. Stat. Mech. P01031
[34] Prolhac S and Spohn H 2011 The one-dimensional KPZ equation and the Airy process J. Stat. Mech. P03020
[35] Corwin I and Quastel J 2011 Renormalization fixed point of the KPZ universality class, arXiv:1103.3422
[36] Calabrese P and Le Doussal P 2011 An exact solution for the KPZ equation with flat initial conditions Phys. Rev. Lett. 106 250603
[37] Tracy C A and Widom H 2009 On ASEP with step Bernoulli initial condition J. Stat. Phys. 137 825–38
[38] Corwin I and Quastel J 2010 Crossover distributions at the edge of the rarefaction fan, arXiv:1006.1338
[39] Bornemann F 2010 On the numerical evaluation of Fredholm determinants Math. Comput. 79 871–915
[40] Bornemann F 2010 On the numerical evaluation of distributions in random matrix theory: a review Markov Process. Relat. Fields 16 803–66
[41] Prolhac S and Spohn H 2011 The height distribution of the KPZ equation with sharp wedge initial condition: numerical evaluations Phys. Rev. E 84 011119
[42] Imamura T and Sasamoto T 2004 Fluctuations of the one-dimensional polynuclear growth model with external sources Nucl. Phys. B 699 503–44
[43] Imamura T and Sasamoto T 2005 Polynuclear growth model with external source and random matrix model with deterministic source Phys. Rev. E 71 041106
[44] Corwin I, Ferrari P L and Péché S 2010 Limit processes for TASEP with shocks and rarefaction fans J. Stat. Phys. 140 232–67
[45] Baik J and Rains E M 2000 Limiting distributions for a polynuclear growth model with external sources J. Stat. Phys. 100 523–41
[46] Bertini L and Cancrini N 1995 The stochastic heat equation: Feynman–Kac formula and intermittence J. Stat. Phys. 78 1377–401
[47] Baik J, Ben Arous G and Péché S 2006 Phase transition of the largest eigenvalue for non-null complex sample covariance matrices Ann. Probab. 33 1643–97