Banach Contraction Principle for Cyclical Mappings on Partial Metric Spaces

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\textbf{Abstract.} In this paper, we prove that the Banach contraction principle proved by S. G. Matthews in 1994 on \textit{0–}complete partial metric spaces can be extended to cyclical mappings. However, the generalized contraction principle proved by D. Ilić, V. Pavlović and V. Rakočević in "Some new extensions of Banach’s contraction principle to partial metric spaces, Appl. Math. Lett. 24 (2011), 1326–1330" on complete partial metric spaces can not be extended to cyclical mappings. Some examples are given to illustrate the effectiveness of our results. Moreover, we generalize some of the results obtained by W. A. Kirk, P. S. Srinivasan and P. Veeramani in "Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4 (1) (2003),79–89". Finally, an Edelstein’s type theorem is also extended in case one of the sets in the cyclic decomposition is \textit{0–}compact.

\textbf{Keywords.} Partial metric space; Fixed point, Cyclic mapping, Banach contraction principle, 0-Compact set.

1 Introduction and Preliminaries

The partial metric spaces were first introduced in \cite{1} as a part of the study of non–symmetric topology, domain theory and denotational semantics of dataflow networks. In particular, the author established the precise relationship between partial metric spaces and the so–called weightable quasimetric spaces and proved a partial metric generalization of Banach contraction mapping theorem which is considered to be the core of many extended fixed point theorems; we refer the reader to the papers \cite{1,2,3,4,5,6,7,8,9,10}.

The widespread applications of the notion of partial metric spaces in programming theory have attracted the attention of many authors who recently published important results in the direction of generalizing this principle; see for instance \cite{11,12,13,14}. The contraction type conditions used in these generalizations, however, do not apparently reflect the structure of partial metric spaces. In the remarkable paper \cite{15}, the authors proved more appropriate contraction principle in partial metric spaces. Indeed, it is more convenient to call the contraction type condition used in this paper by partial contractive condition.

In this paper, we prove that the Banach contraction principle obtained in \cite{1} on \textit{0–}complete partial metric spaces can be extended to cyclical mappings. However, the
generalized contraction principle proved in [15] on complete partial metric spaces cannot be extended to cyclical mappings. Some examples are given to illustrate the effectiveness of our results. In addition to this, we generalize some of the results obtained in [16]. Finally, an Edelstein’s type theorem is also extended in case one of the sets in the cyclic decomposition is 0-compact.

We recall some definitions of partial metric spaces and state some of their properties. A partial metric space (PMS) is a pair \((X, p)\) (where \(\mathbb{R}^+\) denotes the set of all non-negative real numbers) such that

1. \(p(x, y) = p(y, x)\) (symmetry);
2. If \(0 \leq p(x, x) = p(x, y) = p(y, y)\) then \(x = y\) (equality);
3. \(p(x, x) \leq p(x, y)\) (small self-distances);
4. \(p(x, z) + p(y, y) \leq p(x, y) + p(y, z)\) (triangularity);

for all \(x, y, z \in X\).

For a partial metric \(p\) on \(X\), the function \(p^* : X \times X \to \mathbb{R}^+\) given by

\[
p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\]

is a (usual) metric on \(X\). Each partial metric \(p\) on \(X\) generates a \(T_0\) topology \(\tau_p\) on \(X\) with a base of the family of open \(p\)-balls \(\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}\), where \(B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}\) for all \(x \in X\) and \(\varepsilon > 0\).

**Definition 1.** [1]

(i) A sequence \(\{x_n\}\) in a PMS \((X, p)\) converges to \(x \in X\) if and only if \(p(x, x) = \lim_{n \to \infty} p(x, x_n)\).

(ii) A sequence \(\{x_n\}\) in a PMS \((X, p)\) is called a Cauchy if and only if \(\lim_{n,m \to \infty} p(x_n, x_m)\) exists (and finite).

(iii) A PMS \((X, p)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges, with respect to \(\tau_p\), to a point \(x \in X\) such that \(p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)\).

(iv) A mapping \(f : X \to X\) is said to be continuous at \(x_0 \in X\), if for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(f(B_p(x_0, \delta)) \subset B_p(f(x_0), \varepsilon)\).

**Lemma 1.** [1]

(a1) A sequence \(\{x_n\}\) is Cauchy in a PMS \((X, p)\) if and only if \(\{x_n\}\) is Cauchy in a metric space \((X, p^*)\).

(a2) A PMS \((X, p)\) is complete if and only if the metric space \((X, p^*)\) is complete. Moreover,

\[
\lim_{n \to \infty} p^*(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m).
\]
Lemma 2. Let $(X, p)$ be a partial metric space and let $T : X \to X$ be a continuous self-mapping. Assume $\{x_n\} \subset X$ such that $x_n \to z$ as $n \to \infty$. Then

$$\lim_{n \to \infty} p(Tx_n, Tz) = p(Tz, Tz).$$

Proof. Let $\epsilon > 0$ be given. Since $T$ is continuous at $z$ find $\delta > 0$ such that $T(B_p(z, \delta)) \subseteq B_p(Tz, \epsilon)$. Since $x_n \to z$ then $\lim_{n \to \infty} p(x_n, z) = p(z, z)$ and hence find $n_0 \in \mathbb{N}$ such that $p(z, z) \leq p(x_n, z) < p(z, z) + \delta$ for all $n \geq n_0$. That is, $x_n \in B_p(z, \delta)$ for all $n \geq n_0$. Thus $T(x_n) \in B_p(Tz, \epsilon)$ and so $p(Tz, Tz) \leq p(Tx_n, Tz) < p(Tz, Tz) + \epsilon$ for all $n \geq n_0$. This shows our claim.

A sequence $\{x_n\}$ is called 0–Cauchy if $\lim_{m,n \to \infty} p(x_n, x_m) = 0$. The partial metric space $(X, p)$ is called 0–complete if every 0–Cauchy sequence in $X$ converges to a point $x \in X$ with respect to $p$ and $p(x, x) = 0$. Clearly, every complete partial metric space is 0–complete. The converse need not be true; see [17] for more details.

Example 1. Let $X = \mathbb{Q} \cap [0, \infty)$ with the partial metric $p(x, y) = \max\{x, y\}$ where $\mathbb{Q}$ is the set of rationals. Then $(X, p)$ is a 0–complete partial metric space which is not complete.

Theorem 1. [1, 17] Let $(X, p)$ be a 0–complete partial metric space and $f : X \to X$ be such that

$$p(f(x), f(y)) \leq \alpha p(x, y) \forall x, y \in X \text{ and } \alpha \in [0, 1)$$

there exists a unique $u \in X$ such that $u = f(u)$ and $p(u, u) = 0$.

Let $\rho_p = \inf \{p(x, y) : x, y \in X\}$ and define $X_p = \{x \in X : p(x, x) = \rho_p\}$.

Theorem 2. [15] Let $(X, p)$ be a complete metric space, $\alpha \in [0, 1)$ and $T : X \to X$ a given mapping. Suppose that for each $x, y \in X$ the following condition holds

$$p(x, y) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\}.$$

Then

1. the set $X_p$ is nonempty;
2. there is a unique $u \in X_p$ such that $Tu = u$;
3. for each $x \in X_p$ the sequence $\{T^n x\}_{n \geq 1}$ converges with respect to the metric $p^s$ to $u$.

Definition 2. Let $A$ and $B$ be two nonempty closed subsets of a complete partial metric space $(X, p)$ such that $X = A \cup B$. A mapping $T : X \to X$ is called cyclical contraction if it satisfies

(C1) $T(A) \subseteq B$ and $T(B) \subseteq A$.
(C2) \( \exists 0 < \alpha < 1 : p(Tx, Ty) \leq \alpha p(x, y) \), \( \forall x \in A \) and \( \forall y \in B \).

If (C2) in Definition 2 is replaced by the condition

\[
(\text{PC2}) \exists 0 < \alpha < 1 : p(Tx, Ty) \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\} \quad \forall x \in A \text{ and } \forall y \in B.
\]

Then \( T \) is called a partial cyclical contraction.

**Remark 1.** The partial cyclical contractions reflects the real structure of partial metric space.

The proof of the following lemma can be easily achieved by using the partial metric topology.

**Lemma 3.** A subset \( A \) of a partial metric space is closed if and only if \( x_n \in A \) whenever \( x_n \to x \) as \( n \to \infty \).

**Definition 3.** A set \( A \) in a partial metric space \( (X, p) \) is called 0-compact if for any sequence \( \{x_n\} \) in \( A \) there exists a subsequence \( \{x_{k_n}\} \) and \( x \in A \) such that \( \lim_{n \to \infty} p(x_{k_n}, x) = p(x, x) = 0 \).

Clearly a closed subset of a 0-compact set is 0-compact.

**Lemma 4.** \([2, 4]\) Assume \( x_n \to z \) as \( n \to \infty \) in a PMS \((X, p)\) such that \( p(z, z) = 0 \). Then \( \lim_{n \to \infty} p(x_n, y) = p(z, y) \) for every \( y \in X \).

## 2 The Main Results

We start this section by a theorem that will motivate to obtain our main result for cyclic contraction mappings.

**Theorem 3.** Let \( (X, p) \) be a 0-complete partial metric space and \( T : X \to X \) be continuous such that

\[
p(Tx, T^2x) \leq \alpha p(x, Tx) \quad \forall x \in X, \quad \text{where} \quad \alpha \in (0, 1).
\]

Then there exists \( z \in X \) such that \( p(z, z) = 0 \) and \( p(Tz, z) = p(Tz, Tz) \).

**Proof.** Condition (3) implies that the sequence \( T^n(x) \) is 0-Cauchy for all \( x \in X \). Hence, there exists \( z \in X \) such that \( x_n = Tx_{n-1} \) converges to \( z \) and \( p(z, z) = 0 \). The conclusion that \( p(Tz, z) = p(Tz, Tz) \) follows by Lemma 2 (P2) and the inequality

\[
p(Tz, z) \leq p(Tz, x_{n+1}) + p(x_{n+1}, z).
\]

We observe that if the partial metric in Theorem 3 is replaced by a metric then we conclude that \( z \) is a fixed point. The following theorem is an extension of Theorem 1.1 in [16].
**Theorem 4.** Let $A$ and $B$ be two nonempty closed subsets of a $0$-complete partial metric space $(X, p)$ such that $X = A \cup B$, and suppose $T : X \to X$ be a cyclical contraction self–mapping of $X$. Then $T$ has a unique fixed point in $A \cap B$.

**Proof.** Condition (C1) implies that for any $x \in A \cup B$

$$p(Tx, T^2x) \leq \alpha p(x, Tx)$$

and this by (P4) implies that the sequence $\{T^n(x)\}$ is $0$-Cauchy for any $x \in X$. Consequently, $\{T^n(x)\}$ converges to some point $z \in X$ such that $p(z, z) = 0$. However, in view of (C2) an infinite number of terms of the sequence $\{T^n(x)\}$ lie in $A$ and an infinite number of terms lie in $B$. Then by Lemma 3 we conclude that $z \in A \cap B$, so $A \cap B \neq \emptyset$.

Now (C1) and (C2) imply that the map $T$ restricted to $A \cap B$ is contraction. Then the result follows by Theorem 1.

In what follows, we give an example showing that the generalization to partial metric space in Theorem 4 is proper.

**Example 2.** Let $X = \left[0, 1\right]$, $A = \left[0, \frac{1}{2}\right]$ and $B = \left[\frac{1}{2}, 1\right]$. Then $X = A \cup B$ and $A \cap B = \left\{\frac{1}{2}\right\}$. Provide $X$ with the partial metric

$$p(x, y) = \begin{cases} |x - y| & \text{if both } x, y \in [0, 1) \\ \max\{x, y\} & \text{otherwise.} \end{cases}$$

Then clearly $(X, p)$ is a complete partial metric space. Define $T : X \to X$ by $T(x) = \frac{1}{2}$ if $0 \leq x < 1$ and $T(1) = 0$. Then it can be easily checked that $T$ is a cyclical contraction with $\alpha = \frac{3}{4}$. Notice that the cyclical contractive condition of Theorem 4 is not satisfied when the partial metric $p$ is replaced by the usual absolute value metric.

The following example shows that Theorem 2 can not be extended for cyclical mappings when the cyclical contraction is replaced by a partial cyclical contraction.

**Example 3.** Let $A = [0, 1]$, $B = [3, 4] \cup \left\{\frac{3}{2}\right\}$ and $X = A \cup B$. Define $p : X \times X \to [0, \infty)$ by $p(x, y) = \max\{x, y\}$. Then $(X, p)$ is a complete partial metric space. Define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{3}{2}, & 0 \leq x < 1 \\ \frac{1}{2}, & x = \frac{3}{2} \\ \frac{x - 2}{2}, & 3 \leq x \leq 4. \end{cases}$$

It can be easily seen that

$$p(Tx, Ty) = \max\left\{\frac{3}{2}, \frac{y - 2}{2}\right\} = \frac{3}{2} \leq \max\{\alpha p(x, y), p(x, x), p(y, y)\} = y,$$

for any $x \in A$, $y \in B$ and any $\alpha \in (0, 1)$. However, $A \cap B = \emptyset$.

**Corollary 1.** Let $A$ and $B$ be two nonempty closed subsets of a complete partial metric space $(X, p)$ such that $X = A \cup B$. Let $f : A \to B$ and $g : B \to A$ be two functions such that $f(x) = g(x)$ for all $x \in A \cap B$ and
\[ p(f(x), g(y)) \leq \alpha p(x, y) \quad \forall x \in A \text{ and } y \in B, \quad (4) \]

where \( 0 < \alpha < 1 \). Then there exists a unique \( x_0 \in A \cap B \) such that
\[ f(x_0) = g(x_0) = x_0. \]

**Proof.** Apply Theorem 4 to the mapping \( T : A \cup B \to A \cup B \) defined by setting
\[ T(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}. \]

Observe that the assumption \( f(x) = g(x) \) for all \( x \in A \cap B \) implies that \( T \) is well defined. \( \square \)

**Remark 2.** In the metric space case, condition (4) implies that the map \( T \) is well defined.

Obviously Theorem 4 can be extended to the following version.

**Theorem 5.** Let \( \{A_i\}_{i=1}^k \) be nonempty closed subsets of a 0–complete partial metric space, and suppose that \( T : \bigcup_{i=1}^k A_i \to \bigcup_{i=1}^k A_i \) satisfies the following conditions (where \( A_{k+1} = A_1 \))
\[
(1) \quad T(A_i) \subseteq A_{i+1} \text{ for } 1 \leq i \leq k; \\
(2) \quad \exists \alpha \in (0, 1) \text{ such that } p(T(x), T(y)) \leq \alpha p(x, y) \forall x \in A_i, y \in A_{i+1} \text{ for } 1 \leq i \leq k.
\]

Then \( T \) has a unique fixed point.

**Proof.** One only need to observe that given \( x \in \bigcup_{i=1}^k A_i \), infinitely many terms of the Cauchy sequence \( \{T^n(x)\} \) lie in each \( A_i \). Thus \( \bigcap_{i=1}^k A_i \neq \emptyset \), and the restriction of \( T \) to this intersection is a contraction mapping. \( \square \)

**Remark 3.** It is of our belief that Theorem 4 can be extended to more general cyclical contraction mappings. However, it would be of more interest if the contractive type conditions are considered with control functions.

The following theorem is an extension of an Edelstein’s type to partial metric spaces.

**Theorem 6.** Let \( \{A_i\}_{i=1}^k \) be nonempty closed subsets of a partial metric space \( (X, p) \), at least one of which is 0–compact, and suppose that \( T : \bigcup_{i=1}^k A_i \to \bigcup_{i=1}^k A_i \) satisfies the following conditions (where \( A_{k+1} = A_1 \)).
\[
(1) \quad T(A_i) \subseteq A_{i+1} \text{ for } 1 \leq i \leq k; \\
(2) \quad p(T(x), T(y)) < p(x, y) \forall x \in A_i, y \in A_{i+1} \text{ for } 1 \leq i \leq k.
\]

Then \( T \) has a unique fixed point.
Proof. Let $A_1$ be 0-compact and $\delta = p(A_1, A_k) = \inf\{p(x, y) : x \in A_1, y \in A_k\}$. From the definition of $\delta$ there exist sequences $\{x_n\} \subset A_1$ and $\{u_n\} \subset A_k$ such that

$$p(x_n, u_n) \leq \delta + \frac{1}{n}.$$ 

By the 0-compactness of $A_1$ we may assume that there exists $x_0 \in A_1$ such that the limit $\lim_{n \to \infty} p(x_n, x_0) = p(x_0, x_0) = 0$. Then by the triangle inequality it follows that $\lim_{n \to \infty} p(x_0, u_n) = \delta$. Let $\delta > 0$. Then

$$p(T^{k+1}(x_0), T^{k+1}(u_n)) < \ldots < p(x_0, u_n). \tag{5}$$

Since the sequence $\{T^{k+1}(u_n)\}$ is in $A_1$ and $A_1$ is 0-compact, we may assume that there exists $z \in A_1$ such that $\lim_{n \to \infty} p(T^{k+1}(u_n), z) = p(z, z) = 0$. By (5) and Lemma 4 we conclude that $p(z, T^{k+1}(x_0)) \leq \delta$.

It follows that $p(T^{k-1}(z), T^{2k}(x_0)) < \delta$ but since $T^{k-1}(z) \in A_k$ and $T^{2k}(x_0) \in A_1$ we obtain a contradiction. Therefore, we conclude that $\delta = 0$ and $A_1 \cap A_k \neq \emptyset$. Thus, by assumption (1) of the theorem, $A_1 \cap A_2 \neq \emptyset$.

We now consider the sets $B_1 = A_1 \cap A_2$, $B_2 = A_2 \cap A_3$, $\ldots$, $B_k = A_k \cap A_1$. In view of assumption (1) these sets are all nonempty (and closed) and $B_1$ is 0-compact. Thus the assumptions (1) and (2) of the theorem hold for $T$ and the family $\{B_i\}_{i=1}^k$. By repeating the arguments just given we arrive at $B_1 \cap B_k \neq \emptyset$.

It follows that $A_1 \cap A_2 \cap A_3 \neq \emptyset$. Continuing step-by-step, we conclude that $A := \cap_{i=1}^k \neq \emptyset$. The uniqueness, however, follows from the fact that any fixed point of $T$ necessarily lies in $A := \cap_{i=1}^k$ which is clearly obtained by assumption (1). \qed

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