Annihilation of S-wave quarkonia and the measurement of $\alpha_s$ *

Martin Gremm and Anton Kapustin

*California Institute of Technology, Pasadena, CA 91125

Abstract

We analyze the relativistic corrections to annihilation rates of S-wave quarkonia within the framework of NRQCD. We show that order $v^2$ corrections can be expressed in terms of the heavy quark pole mass and the quarkonium mass. The ratio of hadronic to radiative annihilation rates for $\eta_b$ and $\eta_c$ can therefore be predicted accurately. The contributions of color-octet operators to the hadronic decay rates of spin-triplet quarkonia are shown to be significant, even though they arise at order $v^4$ in the velocity expansion. We provide a rough estimate of the color-octet contributions and extract the value of $\alpha_s$ from the experimental data on $\Upsilon$ decays.

*Work supported in part by the U.S. Dept. of Energy under Grant no. DE-FG03-92-ER 40701.
I. INTRODUCTION

Since their discovery, heavy quarkonia have been considered an important testing ground for quantum chromodynamics [1]. By now it is well established that all qualitative features of quarkonia (e.g., a confining potential, a positronium-like spectrum, ratios of leptonic to hadronic widths) are in agreement with what we expect from QCD. However, in most cases we still do not have a fully quantitative description based on first principles. An important step towards such a description was made in Ref. [2], where a formalism of Nonrelativistic QCD (NRQCD) was proposed. It is based upon the observation that in a heavy quarkonium there are several widely separated momentum scales: the typical kinetic energy of the heavy quark, $Mv^2$, is much smaller than the inverse size of the quarkonium, $Mv$, which in turn is much smaller than the heavy quark mass $M$. NRQCD allows one to factor the annihilation and production rates for quarkonia into perturbatively calculable short-distance coefficients and nonperturbative long-distance matrix elements. This justifies the assumption of “naive factorization” for S-wave quarkonium. On the other hand, NRQCD elucidates the role of the higher Fock components of the quarkonium wavefunction and explains why naive factorization fails for P-wave states [2].

NRQCD provides a rigorous definition of long-distance matrix elements and thus allows, in principle, their calculation on the lattice. Still, given the immaturity of present day lattice simulations, one may ask what one can learn from quarkonia without plunging into a full-fledged lattice NRQCD computation. What we have in mind here is, first of all, a more accurate determination of $\alpha_s$ from the ratio of hadronic to electromagnetic widths [3]. Such a determination in particular could help to clarify the long-standing problem of a possible discrepancy between low-energy and high energy measurements of $\alpha_s$ [4].

In this paper we analyze the relativistic corrections to the annihilation rates of S-wave quarkonia (both spin-singlet and spin-triplet) and apply the results of this analysis to restrict the value of $\alpha_s$. The paper is organized as follows. In section 1 we show that order $v^2$ corrections to the color-singlet part of the annihilation rate can be expressed through
the mass of the quarkonium and the heavy quark pole mass. The hadronic widths of the
spin-triplet states, $\psi$ and $\Upsilon$, contain also a piece due to the annihilation of the quark-
antiquark pair in a color-octet state. In section II we provide a rough estimate of the latter
contribution based on the running of the color-octet matrix elements. These results are used
in section IV to extract $\alpha_s$ from the ratio of hadronic to leptonic widths of $\Upsilon(1S)$.

II. RELATIVISTIC CORRECTIONS TO S-WAVE QUARKONIUM
ANNIHILATION RATES IN NRQCD

The Lagrangian of NRQCD [2] is

$$\mathcal{L}_{\text{NRQCD}} = \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{heavy}} + \delta\mathcal{L},$$

(2.1)

where $\mathcal{L}_{\text{light}}$ is the usual QCD Lagrangian for gluons and light quarks. $\mathcal{L}_{\text{heavy}}$ is given by

$$\mathcal{L}_{\text{heavy}} = \psi^\dagger \left( iD_t + \frac{D^2}{2M} \right) \psi + \chi^\dagger \left( iD_t - \frac{D^2}{2M} \right) \chi,$$

(2.2)

with $\psi$ being an operator annihilating a heavy quark, and $\chi$ being an operator creating a
heavy antiquark. Both $\psi$ and $\chi$ belong to the fundamental representation of the color group
$SU(N_c)$. The last term, $\delta\mathcal{L}$, includes relativistic corrections to $\mathcal{L}_{\text{heavy}}$ and is of order $v^2$
compared to it.

The annihilation of the quarkonium is a short distance process (the characteristic mo-
mentum scale is of order $M$) which, in the framework of NRQCD, is described by adding
4-quark local operators to $\delta\mathcal{L}$. The corresponding term has the following structure:

$$\delta\mathcal{L}_{4\text{-fermion}} = \sum_n \frac{f_n(\Lambda)}{M^{d_n-4}} \mathcal{O}_n(\Lambda),$$

(2.3)

where $d_n$ is the canonical dimension of $\mathcal{O}_n$. The dimensionless coefficients $f_n(\Lambda)$ depend on
the Wilsonian cutoff $\Lambda$ needed to define NRQCD and can be calculated by matching the
NRQCD amplitudes generated by 4-fermion terms with annihilation contributions to the
scattering in full QCD. Following Ref. [2], we take $\Lambda \sim M$. 


First let us collect the expressions for the decay rates of S-wave quarkonia including the first relativistic corrections. According to Ref. [2], the inclusive decay rates of $\eta_c$ and $\eta_b$ to light hadrons and to two photons are given by

$$\Gamma(\eta_{c,b} \to \text{LH}) = \frac{2 \text{Im} f_1(1S_0)}{M^2} \langle \eta_{c,b}|O_1(1S_0)|\eta_{c,b}\rangle + \frac{2 \text{Im} g_1(1S_0)}{M^4} \langle \eta_{c,b}|P_1(1S_0)|\eta_{c,b}\rangle + O(v^4 \Gamma),$$

$$\Gamma(\eta_{c,b} \to \gamma\gamma) = \frac{2 \text{Im} f_{\gamma\gamma}(1S_0)}{M^2} \langle \eta_{c,b}|O_1(1S_0)|\eta_{c,b}\rangle + \frac{2 \text{Im} g_{\gamma\gamma}(1S_0)}{M^4} \langle \eta_{c,b}|P_1(1S_0)|\eta_{c,b}\rangle + O(v^4 \Gamma),$$

where $O_1(1S_0) = \psi^\dagger \chi^\dagger \psi$, $P_1(1S_0) = 1/2 \left[ \psi^\dagger \chi^\dagger (-i \frac{\tau^i}{2} D^i)^2 \psi + \text{h.c.} \right]$.

For the spin-triplet S-states, $\psi$ and $\Upsilon$, the situation is more complicated. The leading term and the order $v^2$ relativistic correction to the $\ell^+\ell^-$ decay rate are proportional to the expectation values of $O_1(3S_1) = \psi^\dagger \sigma \chi^\dagger \psi$ and $P_1(3S_1) = \frac{1}{2} \left[ \psi^\dagger \sigma \chi^\dagger (-i \frac{\tau^i}{2} D^i)^2 \psi + \text{h.c.} \right]$ respectively:

$$\Gamma(\Upsilon \to \ell^+\ell^-) = \frac{2 \text{Im} f_{ee}(3S_1)}{M^2} \langle \Upsilon|O_1(3S_1)|\Upsilon\rangle + \frac{2 \text{Im} g_{ee}(3S_1)}{M^4} \langle \Upsilon|P_1(3S_1)|\Upsilon\rangle.$$  \hspace{1cm} (2.5)

The decay rate of $\psi$ or $\Upsilon$ to light hadrons receives contributions from both color-singlet and color-octet components of the quarkonium wavefunction. The color-singlet component can only decay into three or more gluons. In contrast, the color-octet component can decay into two gluons or into a virtual gluon which then creates a quark-antiquark pair (see Fig. 1.) Hence, the color-octet contribution is of order $\alpha_s^2 v^4$ and may compete with the relativistic correction to the color-singlet rate which is of order $\alpha_s^3 v^2$. (We will see in the next section that this color-octet contribution is essential for explaining experimental data on $\Upsilon$ decays.)

Therefore the inclusive rate to light hadrons is

$$\Gamma(\Upsilon \to \text{LH}) = \frac{2 \text{Im} f_1(3S_1)}{M^2} \langle \Upsilon|O_1(3S_1)|\Upsilon\rangle + \frac{2 \text{Im} g_1(3S_1)}{M^4} \langle \Upsilon|P_1(3S_1)|\Upsilon\rangle + \Gamma^{(8)}(\Upsilon \to \text{LH}),$$

(2.6)

The color-octet part of the decay rate $\Gamma^{(8)}(\Upsilon \to \text{LH})$ receives contributions from three four-quark operators corresponding to the three diagrams in Fig. 1:
\[ \Gamma^{(8)}(\Upsilon \to \text{LH}) = \frac{2 \text{Im} \left( f_8^{(3)} P_0 + 5 f_8^{(3)} P_2 \right)}{M^4} \langle \Upsilon | O_8^{(3)} P_0 | \Upsilon \rangle + \frac{2 \text{Im} f_8^{(1)} S_0}{M^2} \langle \Upsilon | O_8^{(1)} S_0 | \Upsilon \rangle + \frac{2 \text{Im} f_8^{(3)} S_1}{M^2} \langle \Upsilon | O_8^{(3)} S_1 | \Upsilon \rangle. \] (2.7)

In the latter equation we have used heavy quark spin symmetry to reexpress the expectation value of \( O_8^{(3)} P_2 \) in terms of \( O_8^{(3)} P_0 \). The color-octet operators are defined as

\[ O_8^{(3)} P_0 = \frac{1}{3} \psi^\dagger T^a \left( -\frac{i}{2} \frac{\not{D}}{\not{D}} \cdot \sigma \right) \chi \psi, \]

\[ O_8^{(1)} S_0 = \psi^\dagger T^a \chi \psi, \]

\[ O_8^{(3)} S_1 = \psi^\dagger \sigma T^a \chi \psi. \] (2.8)

The short-distance coefficients in the expressions for the decay rates Eqs. (2.4-2.7) depend on the scheme adopted to define the operators. We wish to derive relations between the matrix elements of the operators \( P_1^{(1)} S_0 \), \( P_1^{(3)} S_1 \) and \( O_1^{(1)} S_0 \), \( O_1^{(3)} S_1 \), which can be used to express the decay rates including the first relativistic corrections through the leading order matrix elements. Since these relations also depend on the choice of the scheme, we will discuss this issue in some detail here. We will limit our discussion to the operators appearing in the decay rate of the \( \eta_{b,c} \). An identical argument can be made for the operators appearing in the decay rates of the \( \Upsilon \) and \( \psi \).

The vacuum saturation approximation provides the following estimate for the matrix element of \( P_1^{(1)} S_0 \)

\[ \langle \eta_{c,b} | P_1^{(1)} S_0 | \eta_{c,b} \rangle = \frac{1}{2} \text{Re} \langle \eta_{c,b} | \psi^\dagger \chi | 0 \rangle \langle 0 | \left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi | \eta_{c,b} \rangle. \] (2.9)

The operator with two derivatives in this expression is not defined unambiguously beyond tree level; for example, one is free to perform a shift

\[ \left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi \rightarrow \left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi \right)_\Lambda + C(\Lambda, M) \left( \chi^\dagger \psi \right)_\Lambda . \] (2.10)

Here \( \Lambda \) is the Wilsonian cutoff, and \( C(\Lambda, M) \) is a power series in \( \alpha_s \) starting with a term of order \( \alpha_s \). (In what follows we will omit the subscript \( \Lambda \), with the understanding that all
operators are regularized using the Wilsonian cutoff.) Among all possible definitions of the operator with two derivatives, only a subset satisfies the NRQCD velocity counting rules. According to these rules, the matrix element of the operator with two derivatives should scale as \( p^2 \) relative to the matrix element of \( \chi^\dagger \psi \) as \( p \to 0 \):

\[
\frac{\langle 0 | (iD)^2 \chi^\dagger \psi + \chi^\dagger (iD)^2 \psi | p, -p \rangle}{\langle 0 | \chi^\dagger \psi | p, -p \rangle} = \mathcal{O}(p^2). \tag{2.11}
\]

Imposing this condition removes the freedom to redefine the operator as in Eq. (2.10).

With the convention Eq. (2.11) it is particularly simple to determine the short-distance coefficient of the operator \( \mathcal{O}_1(1S_0) \) at next-to-leading order (NLO) by comparing the \( v^{-1} \) and \( v^0 \) contributions in full QCD and NRQCD. On the NRQCD side it is sufficient to calculate the annihilation of a quark-antiquark pair via the operator \( \mathcal{O}_1(1S_0) \). The choice Eq. (2.11) guarantees that there are no NLO contributions from the operator \( \mathcal{P}_1(1S_0) \) proportional to \( v^{-1} \) or \( v^0 \).

Other ways of defining the operators are, of course, possible. If the \( p \to 0 \) limit of the expression on the right-hand side of Eq. (2.11) is nonzero, the operator \( \mathcal{P}_1(1S_0) \) will contribute to the annihilation rate at order \( v^{-1} \) and \( v^0 \). In this case one needs to know the leading order short-distance coefficient of \( \mathcal{P}_1(1S_0) \) in order to determine that of \( \mathcal{O}_1(1S_0) \) at NLO. In fact, unless one requires operators with arbitrarily many derivatives to satisfy conditions similar to Eq. (2.11), they all have to be taken into account in the computation of the short-distance coefficient of \( \mathcal{O}_1(1S_0) \) to next-to-leading order.

In Ref. [2] it was assumed that the operators with two or more derivatives need not be taken into account when computing the NLO short-distance coefficient of \( \mathcal{O}_1(1S_0) \). In other words, it is implicit in Ref. [2] that the scaling behavior of the operators is given by the NRQCD counting rules or, equivalently, that Eq. (2.11) and similar equations for operators with more derivatives are satisfied. We adopt the same convention here.

The equations of motion for the quark fields to leading order in \( v^2 \) are

\[
\left( iD_t + \frac{D^2}{2M} \right) \psi = 0, \quad \left( iD_t - \frac{D^2}{2M} \right) \chi = 0. \tag{2.12}
\]
They can be used to trade the spatial derivatives in the operator with two derivatives in Eq. (2.9) for time derivatives acting on the quark fields:

\[
\left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi + A(\Lambda, M) \chi^\dagger \psi = 2M i\partial_t \left( \chi^\dagger \psi \right). \tag{2.13}
\]

Here A is a scheme dependent coefficient whose expansion in powers of \( \alpha_s \) starts, in general, with the term of order \( \alpha_s \). The term proportional to \( \chi^\dagger \psi \) has to be included in the relation Eq. (2.13), because \( \chi^\dagger \psi \) mixes into the operator with two derivatives under shifts as in Eq. (2.10). There are corrections to Eq. (2.13) at higher orders in \( v^2 \), but for our purposes it is sufficient to take into account only the terms shown.

Let us show that with the convention Eq. (2.11) \( A(\Lambda, M) \) is zero to all orders in \( \alpha_s \). Evaluating Eq. (2.13) between vacuum and a quark-antiquark state \( |p, -p\rangle \) yields:

\[
\langle 0 | \left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi | p, -p \rangle = \left( 2p^2 - A(\Lambda, M) \right) \langle 0 | \chi^\dagger \psi | p, -p \rangle. \tag{2.14}
\]

We have used the identity

\[
\langle 0 | i\partial_t \left( \chi^\dagger \psi \right) | p, -p \rangle = \langle 0 | \left[ \chi^\dagger \psi, H \right] | p, -p \rangle = \frac{p^2}{M} \langle 0 | \chi^\dagger \psi | p, -p \rangle, \tag{2.15}
\]

where \( H \) is the NRQCD Hamiltonian. The identity Eq. (2.15) holds to all orders in \( \alpha_s \) because the asymptotic state \( |p, -p\rangle \) is an eigenstate of \( H \) with eigenvalue \( p^2/M \). Dividing both sides of Eq. (2.14) by \( \langle 0 | \chi^\dagger \psi | p, -p \rangle \), taking the limit \( p \to 0 \) and using Eq. (2.11), one sees that \( A(\Lambda, M) = 0 \).

Using Eq. (2.13) with \( A = 0 \) yields the following relation:

\[
\langle \eta_{c,b} | P_1 (1S_0) | \eta_{c,b} \rangle = \frac{1}{2} \text{Re} \langle \eta_{c,b} | \psi^\dagger \chi | 0 \rangle \langle 0 | \left( (iD)^2 \chi \right)^\dagger \psi + \chi^\dagger (iD)^2 \psi | \eta_{c,b} \rangle \\
\quad = M \text{Re} \langle \eta_{c,b} | \psi^\dagger \chi | 0 \rangle \langle 0 | i\partial_t \left( \chi^\dagger \psi \right) | \eta_{c,b} \rangle \left( 1 + O(v^2) \right) \\
\quad = M E_{\eta_{c,b}} \langle \eta_{c,b} | O_1 (1S_0) | \eta_{c,b} \rangle \left( 1 + O(v^2) \right). \tag{2.16}
\]

Here \( E_{\eta_{c,b}} \) is the energy of the quarkonium state. Since in NRQCD the rest mass of the quarks is not included in the energy of a quarkonium state, we can express \( E_{\eta_{c,b}} \) in terms of the mass of the \( \eta_{c,b} \) and the quark pole mass.
\[ E_{n,c,b} = M_{n,c,b} - 2M. \] (2.17)

Eqs. (2.16) and (2.17) and similar relations for the operators in the spin-triplet decay rates allow us to express the relativistic corrections to the decay rates in terms of the leading order matrix elements and the ”binding energy” of the quarkonium \( M_{q\bar{q}} - 2M_q \). For example, the decay rate of the \( \Upsilon \) to light hadrons now takes the form

\[
\Gamma(\Upsilon \to LH) = \frac{2 \text{Im} f_1(3S_1)}{M^2} \langle \Upsilon | \mathcal{O}_1(3S_1) | \Upsilon \rangle \left( 1 + \frac{M_{\Upsilon} - 2M \text{Im} g_1(3S_1)}{M \text{Im} f_1(3S_1)} \right) + \Gamma^{(8)}(\Upsilon \to LH),
\] (2.18)

and similar expressions hold for the other decay rates Eqs. (2.4) and Eq. (2.5).

All coefficients in Eqs. (2.4) and Eqs. (2.5-2.7), except \( f_8(3P_0) \), \( f_8(3P_2) \), and \( g_1(3S_1) \), have been calculated to the necessary order in \( \alpha_s \) in Ref. [2]. The coefficients \( f_8(3P_0) \), \( f_8(3P_2) \) can be extracted from Eqs. (A9-A13) of Ref. [2]:

\[
\text{Im} f_8(3P_0) = \frac{3\pi(N_c^2 - 4)}{4N_c} \alpha_s^2(M), \quad \text{Im} f_8(3P_2) = \frac{\pi(N_c^2 - 4)}{5N_c} \alpha_s^2(M).
\] (2.19)

To extract \( g_1(3S_1) \) to leading order in \( \alpha_s \) we need to compute the three-gluon annihilation rate of a free quark-antiquark pair to order \( v^2 \) in their relative velocity. Fortunately, this computation has already been performed in the context of \( e^+e^- \) annihilation [3]. We have checked the results quoted in these papers. The annihilation rate of a free quark-antiquark pair in a spin triplet state to order \( v^2 \) turns out to be

\[
\Gamma (q\bar{q}(3S_1) \to 3g, v) = \Gamma (q\bar{q}(3S_1) \to 3g, 0) \left[ 1 - v^2 \frac{19\pi^2 - 132}{12\pi^2 - 108} + O(v^4) \right],
\] (2.20)

where \( \Gamma (q\bar{q}(3S_1) \to 3g, v) \) represents the annihilation rate of the quark-antiquark pair in a spin-triplet state, with \( v \) denoting the velocity of the quark in the center of mass frame.

Comparing with the corresponding amplitude in NRQCD, we obtain

\[
\frac{\text{Im} g_1(3S_1)}{\text{Im} f_1(3S_1)} = -\frac{19\pi^2 - 132}{12\pi^2 - 108} (1 + O(\alpha_s)).
\] (2.21)

One consequence of the last equation is that for the spin-triplet states the order \( v^2 \) relativistic correction to the hadronic rate is unexpectedly large. For the \( b \)-quark pole mass
in the range 4.6 − 4.9 GeV, the correction to the Υ(1S) decay rate can be as large as 25%, and still bigger for radially excited states. For \( m_0^{pole} \simeq 1.3 \) GeV, the correction to the \( J/\psi \) hadronic decay rate is about 150%. Its magnitude makes one question the usefulness of the nonrelativistic expansion for charmonium. For spin-singlet states, the relativistic corrections are of the expected size.

III. ESTIMATES OF THE COLOR-OCTET MATRIX ELEMENTS

In order to use our expressions for phenomenological applications estimates for the color-octet contributions are needed. Following Ref. [2], we can obtain very rough estimates by solving the renormalization group equations for the color-octet operators. To order \( v^4 \) and leading order in \( \alpha_s \) we find

\[
\frac{d}{d\Lambda} \langle \Upsilon | \mathcal{O}_8(1S_0) | \Upsilon \rangle = 0,
\]

\[
\frac{d}{d\Lambda} \langle \Upsilon | \mathcal{O}_8(3S_1) | \Upsilon \rangle = \frac{4(N_c^2 - 4)\alpha_s}{N_c\pi M^2} \langle \Upsilon | \mathcal{O}_8(3P_0) | \Upsilon \rangle,
\]

\[
\frac{d}{d\Lambda} \langle \Upsilon | \mathcal{O}_8(3P_0) | \Upsilon \rangle = \frac{4C_F\alpha_s}{81N_c\pi} (M_\Upsilon - 2M)^2 \langle \Upsilon | \mathcal{O}_1(3S_1) | \Upsilon \rangle,
\]

where

\[
\langle \Upsilon | \psi \sigma(-\frac{i}{2}\vec{D})^2 \chi \sigma(-\frac{i}{2}\vec{D})^2 \psi | \Upsilon \rangle = M^2 (M_\Upsilon - 2M)^2 \langle \Upsilon | \mathcal{O}_1(3S_1) | \Upsilon \rangle
\]

was used in Eqs. (3.1). We can express the matrix elements at the factorization scale \( \Lambda \sim M \) in terms of those at a low scale \( \Lambda \sim \Lambda_{QCD} \) by solving Eqs. (3.1). The color-octet operators mix between themselves and with color-singlet operators. Formally, the terms coming from the mixing with color-singlets are logarithmically enhanced. To get a rough estimate of the color-octet matrix elements, we assume that these terms dominate. This yields:

\[
\langle \Upsilon | \mathcal{O}_8(3S_1) | \Upsilon \rangle \approx \frac{8(N_c^2 - 4)C_F}{81N_c^2\pi^2} (M_\Upsilon - 2M)^2 \left( \frac{2\pi}{\beta_0} \ln \left( \frac{1}{\alpha_s(M)} \right) \right)^2 \langle \Upsilon | \mathcal{O}_1(3S_1) | \Upsilon \rangle,
\]

\[
\langle \Upsilon | \mathcal{O}_8(3P_0) | \Upsilon \rangle \approx \frac{4C_F}{81N_c\pi} (M_\Upsilon - 2M)^2 \frac{2\pi}{\beta_0} \ln \left( \frac{1}{\alpha_s(M)} \right) \langle \Upsilon | \mathcal{O}_1(3S_1) | \Upsilon \rangle.
\]
In the same spirit we set $\langle Y|O_8^{(1)}S_0|Y \rangle \approx 0$ since it does not acquire a logarithmically enhanced contribution.

In order to check whether these estimates are reasonable, we consider the following “ratio of ratios”:

$$R_{mn}(\Upsilon) = \frac{\Gamma(\Upsilon(mS) \rightarrow LH) / \Gamma(\Upsilon(mS) \rightarrow \ell^+ \ell^-)}{\Gamma(\Upsilon(nS) \rightarrow LH) / \Gamma(\Upsilon(nS) \rightarrow \ell^+ \ell^-)}.$$  \hspace{1cm} (3.5)

Substituting Eq. (2.7) and a similar expression for the dileptonic rate into Eq. (3.5) we get

$$R_{mn} = 1 - \frac{8.0}{M_m - M_n} M_m + \frac{\Gamma^{(8)}(\Upsilon(mS) \rightarrow LH)}{\Gamma(\Upsilon(mS) \rightarrow LH)} - \frac{\Gamma^{(8)}(\Upsilon(nS) \rightarrow LH)}{\Gamma(\Upsilon(nS) \rightarrow LH)} + O(v^4, \alpha_s v^2).$$  \hspace{1cm} (3.6)

Neglecting the color-octet contribution completely, we obtain $R_{12} \simeq 1.5$, in disagreement with the experimental value $R_{12} = 0.95 \pm 0.15$. In order to evaluate the color-octet contribution numerically, we need the pole mass of the $b$ quark. Various methods give $M$ in the range $4.6 - 4.9$ GeV corresponding to the theoretical value $R_{12} \simeq 0.95 - 1.53$. For the lower quark masses our estimates shift the value of $R_{12}$ much closer to the experimental number. We take this to be an indication that our estimates give reasonable order of magnitude values for the color-octet matrix elements.

IV. APPLICATION TO THE DETERMINATION OF $\alpha_s$

As we have seen in the previous section, the leading relativistic correction to the annihilation rates of $\eta_c$ and $\eta_b$ is expressed in terms of the $b$-quark pole mass. The latter can be extracted from the measurement of moments of the photon spectrum in the inclusive $b \rightarrow s\gamma$ decay, from inclusive semileptonic $b \rightarrow c$ decays, or from sum rules for quarkonia. Therefore the ratios of hadronic to radiative decay rates of $\eta_c$ and $\eta_b$ are ideal for determining $\alpha_s$. Unfortunately, these measurements are very hard (though not impossible) to do. 

‡There is still one problem on the theoretical side. Next-to-leading order (NLO) perturbative corrections to these ratios are very large, and one would like to know NNLO corrections to have some idea about the convergence of the perturbation series.
The $\Upsilon$s are much easier to study from the experimental point of view, but the theoretical interpretation is complicated by the presence of the color-octet contribution. We use the results of the previous section to estimate this contribution to the hadronic decay rate. For $M = 4.6 - 4.9$ GeV the color-octet contribution ranges from 0 to 9%. Therefore we take 9% as an estimate of the error from neglecting it. The order $v^2$ relativistic correction is evaluated with $M$ in the same range as above. We use the renormalization scale dependence of the NLO prediction to estimate the error from perturbative NNLO corrections.

Having adopted such estimates of theoretical uncertainties, we use the experimental value $\Gamma(\Upsilon(1S) \to \ell^+\ell^-) / \Gamma(\Upsilon(1S) \to LH) = 37.3 \pm 1.0$ [3], to determine $\alpha_s(M) = 0.154 - 0.218$. This corresponds to

$$\alpha_s(M_Z) = 0.097 - 0.117$$

(4.1)

at the scale $M_Z$. The higher value of $\alpha(M_Z)$ corresponds to the lower value of the $b$-quark pole mass. This range for $\alpha_s(M_Z)$ overlaps with the 1 $\sigma$ confidence interval of the LEP measurement $\alpha_s(M_Z) = 0.120 \pm 0.004$ [11]. The accuracy of our extraction being limited by theoretical uncertainties, the range in Eq. (4.1) should not be interpreted as a 1 $\sigma$ error. We do not quote here the values of $\alpha_s$ obtained from $\Upsilon(2S)$ decays because the theoretical uncertainties are much larger, and also because the accuracy of data on $\Upsilon(2S)$ is worse.

Further improvements in this determination of $\alpha_s$ would come from a more accurate extraction of the $b$-quark pole mass, and also from a NNLO perturbative calculation of the short-distance coefficient $\text{Im} f_1(3S_1)$. For example, knowledge of the pole mass to within 50 MeV would reduce the uncertainties roughly by a factor of two.

We have shown that order $v^2$ relativistic corrections to annihilation rates of the S-wave quarkonia can be expressed in terms of the quarkonium “binding energy”. For spin-singlet states this observation makes it possible to predict accurately the ratio of hadronic to radiative decay rates in terms of $\alpha_s$ and the heavy quark pole mass. However, a calculation of the NNLO perturbative contributions to the short-distance coefficients is necessary to ensure that perturbative corrections are under control. For spin-triplet states, which are much
more accessible experimentally, the color-octet component of the quarkonium wavefunction may contribute significantly to the hadronic annihilation rate, although the corresponding contributions are of order $v^4$ in the nonrelativistic expansion. Therefore, knowledge of the expectation values of color-octet operators is needed, if we want to predict the hadronic to leptonic ratio for spin-triplet quarkonia. We used crude estimates based on renormalization group equations to deduce the uncertainties due to the color-octet contributions. From the experimental data on the $\Upsilon(1S)$ decays we extract $\alpha_s(M_Z) = 0.097 - 0.117$, the major part of the uncertainty coming from the uncertainty in the $b$-quark pole mass. Further experimental and theoretical efforts are needed to obtain a better estimate of $\alpha_s$ from quarkonia decays.

ACKNOWLEDGMENTS

We thank Peter Cho, Adam Leibovich, Zoltan Ligeti, David Politzer and Mark Wise for helpful discussions.
REFERENCES

[1] T. Applequist and H. D. Politzer, Phys. Rev. Lett. 34 (1975) 43; see also W. Buchmüller, ed., Quarkonia, North-Holland, 1992.
[2] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D51 (1995) 1125 (erratum hep-ph/9407339).
[3] I. Hinchliffe, Quantum Chromodynamics, in: L. Montanet et al., Review of Particle Properties, Phys. Rev. D54 (1996) 77.
[4] M. Shifman, Mod. Phys. Lett. A10 (1995) 605.
[5] E.A. Kuraev, T.V. Kuhto, and Z.K. Silagadze, Sov. J. Nucl. Phys. 51 (1990) 1036; P. Labelle, G.P. Lepage, and U. Magnea, Order $m\alpha^8$ contributions to the decay rate of orthopositronium, hep-ph/9310208.
[6] L. Montanet et al., Review of Particle Properties, Phys. Rev. D54 (1996) 1.
[7] A. F. Falk, M. Luke, and M. J. Savage, Phys. Rev. D53 (1996) 6316, M. Gremm, A. Kapustin, Z. Ligeti, and M. B. Wise, Phys. Rev. Lett. 77 (1996) 20, M. Gremm and A. Kapustin, preprint CALT-68-2042, hep-ph/9603448.
[8] S. Narison, Phys. Lett. B341 (1994) 73, M. B. Voloshin, Int. J. Mod. Phys. A10 (1995) 2865.
[9] A. Kapustin and Z. Ligeti, Phys. Lett. B355 (1995) 318.
[10] R. Barbieri, G. Curci, E. d’Emilio, and E. Remiddi, Nucl. Phys. B154 (1979) 535.
[11] G. Altarelli, Status of precision tests of the Standard Model, hep-ph/9611239.
FIG. 1. Contributions of higher Fock states to the hadronic annihilation of spin-1 S-wave quarkonia. All diagrams shown here contribute to the rate at order $\alpha_s^2 v^4$. 