Can modified gravity prevent the gravitational collapse to black hole?

Emilio Santos

Abstract

It is shown that the theoretical arguments for the unavoidable collapse of massive stars, which are valid within standard general relativity, are not valid in extended gravity. In particular a calculation is reported of neutron stars within a theory obtained from an Einstein-Hilbert action involving a function of $R^2 - 1/2 R_{\mu\nu} R^{\mu\nu}$ added to the Ricci scalar $R$. The calculation suggests that within that theory there are stable neutron stars with arbitrarily high baryon number.

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I. Introduction

Claims for the existence of black holes rest upon observational evidence combined with theoretical arguments. Here I analyze whether the theoretical arguments, valid within general relativity (GR), remain valid when GR is modified, for instance by adding other terms to the Ricci scalar in the Einstein-Hilbert action.

Black holes have a long and interesting history. The realization of a Schwarzschild singularity was taken as a serious possibility in 1930, after the work of Chandrasekhar. He proved that white dwarf stars of high enough mass (and low enough temperature) are unstable against gravitational collapse. As is known today those stars do not collapse directly to black holes but suffer a supernova explosion, a core remaining in the form of a neutron star which is believed to evolve towards a black hole if the mass is large. The idea of collapsing stars had a strong opposition on the part of leading astrophysicists of the time, in particular Eddington who assumed that some effect, not yet known, would prevent the collapse.

The next important date was 1939 when Einstein, then in Princeton, and Oppenheimer in California wrote three important papers which I shall comment in the following. All three papers dealt with spherically symmetric bodies, which might be studied using standard coordinates with metric

$$ds^2 = \exp(\alpha(r))dr^2 + r^2d\Omega^2 - \exp(\beta(r))dt^2.$$  (1)

In these coordinates the GR hydrostatic equilibrium equation is

$$\frac{dp}{dr} + 2\frac{q-p}{r} + \frac{k(m + 4\pi r^3 \rho)}{8\pi r^2 - 2k mr} (\rho + p) = 0, m \equiv \int_0^r 4\pi r^2 \rho dr,$$  (2)

where $k$ is $8\pi$ times the Newton constant, $r$ the radial coordinate, and $\rho, p$ and $q$ are the mass density, the radial pressure and the transverse pressure, respectively. Oppenheimer and Volkoff (OV) solved eq. (2) with the assumption of local isotropy, that is $p = q$, and the equation of state, $p = p(\rho)$, appropriate for a fluid of free (non-interacting) neutrons. The result was an one-parameter family of equilibrium configurations where the total mass, $M \equiv m(\infty)$, increases with the central density until a maximum about 0.7 solar masses (the OV limit) and then decreases. The relevant conclusion was that there are no equilibrium configurations of free neutrons with mass above the OV limit. Later on it has been proved that for interacting neutrons a
limit still exists although it is somewhat higher, about 2 solar masses. Thus the question was, what happens to the core of a white dwarf after a supernova explosion when the core has a mass above the OV limit?. In order to answer this question Oppenheimer and Snyder studied the evolution of a spherical cloud of dust. The result was that whatever is the initial configuration the whole cloud collapses to a black hole. This suggests that any neutron star with a mass above the OV limit would suffer a collapse, qualitatively similar to the cloud of dust. Later calculations have confirmed this assumption, which has lead to the current wisdom that all observed compact (cold) objects with mass above the OV limit are black holes.4

Einstein studied a model of star consisting of a spherical distribution of particles each one moving in a circle around a central point. Thus the matter in the star may be considered as a fluid with extreme local anisotropy, that is nil radial pressure, \( p = 0 \). If this is put in the hydrostatic equilibrium eq.(2) we get

\[ q(r) = \frac{km(r)}{8\pi r - 2km(r)} \rho(r), \tag{3} \]

which has a solution for any (spherical) density distribution. In fact, given any function \( \rho(r) \) we may get \( m(r) \) via the second eq.(2) whence eq.(3) gives \( q(r) \). Actually those functions \( \rho(r) \) which lead to a pressure violating the constraint \( q(r) \leq \rho(r) \) should be excluded because this would mean that some of the constituent particles move with a velocity greater than that of light, \( c \). On the other hand it is easy to prove that the solutions of eq.(3) with \( q(r) < \rho(r) \) are stable against homologous collapse, which may be represented by a family of configurations depending on a parameter \( \lambda \) such that

\[ r = \lambda x, \rho(\lambda x) = \lambda^{-3} \rho(x), q(\lambda x) = \lambda^{-3} p(x), m(\lambda x) = m(x). \]

That is we should consider a sequence of configurations fulfilling

\[ \frac{q(\lambda x)}{\rho(\lambda x)} = \frac{q(x)}{\rho(x)} = \frac{km(x)}{8\pi \lambda x - 2km(x)}, \]

with decreasing values of \( \lambda \). It is obvious that the ratio \( q/\rho \) will increase indefinitely for \( \lambda \to km(x)/(4\pi x) \). Then the homologous collapse should stop before the constraint \( q(r) \leq \rho(r) \) is violated. From this result Einstein concluded that the existence of a limiting velocity, \( c \), prevents the collapse
to a Schwarzschild singularity, and he even asserted that for any star model (i.e. not necessarily fulfilling \( p = 0 \)) this would be the case.\(^5\)

As is well known today Einstein was wrong, because in the real world particles interact with each other and the interactions tend to produce local isotropy. Actually between local isotropy, \( p = q \), assumed by OV\(^2\) in their neutron star calculation and extremal local anisotropy, \( p = 0 \), as in Einstein model, there are intermediate cases. Indeed calculations have been performed which prove that for some amount of local anisotropy, \( q \neq p \), the OV limit may be surpassed in neutron stars.\(^6\) However it is easy to prove that stars in stable equilibrium should have local isotropy, which in particular shows that Einstein’s star model is unstable. In fact, let us assume an equilibrium configuration with density distribution \( \rho(r) \), \( p(r) \) and \( q(r) \) being the radial and transverse pressures respectively. Now, in the absence of external electric or magnetic fields a fluid of neutrons with given baryon number density, \( n(r) \), has the minimal energy density for an isotropic velocity distribution, that is \( p(r) = q(r) \). As a consequence if we start with an equilibrium star configuration presenting local anisotropy and change it to another configuration with the same \( n(r) \) but local isotropy everywhere, then the energy density decreases at every point where previously \( q \neq p \). In summary we pass from an equilibrium configuration with local anisotropy to another configuration, maybe out of equilibrium, with the same baryon number but a smaller total mass. Consequently the equilibrium star configuration is not stable.

Many authors have been reluctant to accept the existence of a spacetime singularity in the line of Eddington and Einstein. There are good reasons for that. In fact according to \( GR \) the metric of spacetime is ruled by second order differential equations and therefore it is natural to ask that the metric coefficients are twice derivable everywhere, with no singularity. The current solution to the problem is to assume that the singularity which appears in classical \( GR \) may be cured by quantum effects. But the good solution would be that quantum effects could modify \( GR \) in such a way that stars never collapse. On this line there is a recent proposal of “gravastars”, that is stars with a core where the vacuum, with equation of state \( p = -\rho \), provides a strong repulsive pressure able to stop the collapse.\(^7\) In this paper I propose a more natural solution, namely that the correct gravitational theory is an extended \( GR \), derived from a modified Einstein-Hilbert action,\(^8\) and that in such gravity theory stars never collapse. The purpose of this paper is to provide arguments suggesting that this possibility is worth to be seriously
The most popular stars in Newtonian gravity are polytropes, that is stars consisting of a fluid with equation of state

\[ p = K \rho^\gamma, \]

where \( K \) and \( \gamma \) are constant. It is well known that polytropes are stable (unstable) if \( \gamma \) is greater (smaller) than \( 4/3 \). It is the case that the most important types of Newtonian (or slightly relativistic) stars where gravitational collapse is currently assumed are (massive cold) white dwarfs and supernovae stars and both may be treated as close to polytropes with \( \gamma = 4/3 \). Thus these stars are at the limit of stability and it is believed that they become unstable due to GR corrections. In fact for a slightly relativistic star eq.\(^{(2)}\) may be written, assuming local isotropy, that is \( p = q \),

\[
\frac{dp}{dr} = -\frac{k (m + 4\pi r^3 p)}{8\pi r^2 - 2kmr} (\rho + p) \simeq -\frac{km\rho}{8\pi r^2} (1 + \frac{4\pi r^3 p}{m} + \frac{p}{\rho} + \frac{km}{4\pi r}).
\]

(4)

The main term in the right side correspond to the Newtonian approximation and the remaining 3 terms are the GR corrections. It is easy to see that every GR correction contributes to increase the gravitational field with respect to the Newtonian one, therefore inducing instability towards collapse. In the following I show that corrections derived from \( f(R) \) gravity have precisely the opposite effect.

\( F(R) \) gravity derives from the Einstein-Hilbert action

\[
S = \frac{1}{2k} \int d^4x \sqrt{-g} (R + F) + S_{\text{mat}},
\]

(5)

where \( R \) is the Ricci scalar and \( F(R) \) is an arbitrary function or \( R \), which I shall assume twice derivable. The action eq.\(^{(5)}\) gives the field equation\(^{10}\)

\[
(1 + F_1(R)) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R + F(R)) = \nabla_\mu \nabla_\nu F_1 - g_{\mu\nu} F_1 + kT_{\mu\nu},
\]

(6)

where \( F_1 \equiv dF/dR \) and \( R_{\mu\nu} \) \(( T_{\mu\nu} \) is the Ricci (stress-energy) tensor. The divergence of eq.\(^{(6)}\) leads to

\[
k\nabla^\mu T_{\mu\nu} = R_{\mu\nu} \nabla^\mu F_1 (R) + (\nabla_\nu - \nabla_\mu) F_1 (R).
\]

(7)
For a static spherical body in the metric eq. (11) only the component with \( \nu = 1 \) of the tensor eq. (7) gives a nontrivial result, that is (for local isotropy \( p = q \))

\[
\frac{dp}{dr} + \frac{1}{2} \beta' (\rho + p) = k^{-1} F_2 (R) R' \left( R_1^1 + \frac{1}{2} \beta'' - \frac{1}{2} \alpha'' - \frac{2}{r^2} \right),
\]

(8)

where \( F_2 \equiv d^2 F/dR^2, R' = dR/dr, \beta' \equiv d\beta/dr, \beta'' \equiv d^2 \beta/dr^2, \alpha'' \equiv d^2 \alpha/dr^2 \).

Now let us assume that the deviation from GR is not large, that is the right side of eq. (8) is in some sense small (note that the left side equated to zero just gives the GR eq. (2)). Then for a slightly relativistic star it is consistent to neglect the terms \( \alpha'' \) and \( \beta'' \) in the right hand side and we get

\[
\frac{dp}{dr} + \frac{1}{2} \beta' (\rho + p) \simeq -\frac{1}{2} \beta' (\rho + p) + k^{-1} F_2 (R) R' \left( R_1^1 - \frac{2}{r^2} \right).
\]

Similarly we have, from Einstein’s equation,

\[
R_1^1 = \frac{1}{2} k (\rho - p) \simeq \frac{1}{2} k \rho,
\]

where I have taken into account that \( p \ll \rho \) for an almost Newtonian star. But in addition we have

\[
\frac{4}{3} \pi \rho r^3 < m, \frac{km}{r^3} << \frac{2}{r^2},
\]

the former inequality being a consequence of the fact that the density decreases with increasing radius and the latter is true for slightly relativistic stars where \( km \ll 4\pi r \). Thus we may neglect the term with \( R_1^1 \) whence, taking eq. (11) into account eq. (8), gives finally the following effective gravitational field

\[
g \equiv \frac{1}{\rho} \frac{dp}{dr} \simeq -\frac{km}{8\pi r^2} (1 + \frac{4\pi r^3 p}{m} + \frac{p}{\rho} + \frac{km}{4\pi r}) - 2F_2 (R) \frac{1}{r^2} \frac{dp}{dr}. \]

(9)

The relevant result is that the latter term, which is the correction derived from \( f(R) \) gravity, is always positive because \( d\rho/dr < 0 \) and the inequality \( F_2 (\rho) > 0 \) holds true for any viable \( f(R) \) theory. In conclusion any modification of GR via f(R) theory tends to compensate for the instability created by GR in slightly relativistic stars. Whether this is enough to prevent the gravitational collapse of white dwarfs or supermassive stars may depend on
the form and size of the function $F(R)$, but I will not pursue the study here. In summary, although I cannot claim that $f(R)$ theory always prevents the gravitational collapse of white dwarfs and supermassive stars, it is true that their collapse is not unavoidable in $f(R)$ theory, at a difference with standard GR.

### III. Neutron stars in quartic gravity

Our second example will be the case of neutron stars in quartic gravity, where the Lagrangian $F$ of eq. (5) is

$$F = aR^2 + b R_{\mu\nu} R^{\mu\nu}, \quad a = -b/2 = 1 \text{km}^2.$$  \hfill (10)

The gravity theory resulting from this choice is not viable for, at least, two reasons. Firstly it contradicts Solar System and terrestrial observations or experiments, which require $\sqrt{a}$ and $\sqrt{-b}$ not greater than a few millimeters. Secondly the weak field limit of the theory presents ghosts. The difficulties may be solved if we assume that the theory resting upon eqs. (5) and (10) is an approximation, valid for fields of the order of those appearing inside or near neutron stars, of a theory giving a negligible correction to GR for weak fields. For instance, we might assume that eq. (10) is an approximation of the theory resting upon the Lagrangian

$$F = aR^2 + b R_{\mu\nu} R^{\mu\nu} - c \log \left(1 + \frac{a}{c}R^2 + \frac{b}{c}R_{\mu\nu} R^{\mu\nu}\right).$$  \hfill (11)

In particular if we choose $c \sim 1/(10^{20} \text{ km}^2)$ eq. (11) becomes

$$F \simeq \frac{1}{2c} \left(aR^2 + b R_{\mu\nu} R^{\mu\nu}\right)^2 \sim 10^{-32} R \text{ for the Solar System,}$$

$$F \simeq aR^2 + b R_{\mu\nu} R^{\mu\nu} \sim 10^{-3} R \text{ for neutron stars,}$$  \hfill (12)

and the relative error due to the terms neglected in going from eq. (11) to eq. (12) is smaller than $10^{-12}$ in both cases. I have taken into account that $R^2 \sim R_{\mu\nu} R^{\mu\nu} \sim (k\rho)^2$ and that typical densities are $\rho \sim 10^4 \text{ kg/m}^3$ in the Solar System and $\rho \sim 10^{18} \text{ kg/m}^3$ in neutron stars. I conclude that, in this example, the correction to GR would be negligible for the Solar System (and even more so for the cosmology at present time), but quite important for neutron stars. In any case it is irrelevant for the purposes of this paper the question whether the theory is viable or not. Indeed I do not pretend
to study neutron stars with the “correct” gravity theory which is unknown, but to provide a counterexample to the conjecture that stable neutron stars cannot exist with more than twice the baryon number of the Sun. The counterexample proves that such a limit does not hold true in extended gravity theories.

In the following I report on the results obtained from the calculation of noninteracting neutron stars using the gravity theory defined in eq.(10). The field equation is

$$G_{\mu\nu} + a \left[ -GG_{\mu\nu} + \frac{1}{4}g_{\mu\nu}(G^2 - G_{\lambda\sigma}G^{\lambda\sigma}) - C_G G_{\sigma\nu} \right]$$

$$+ \frac{1}{2}a \left[ G_{\mu\nu} - \nabla_\sigma \nabla_\nu G^\sigma_{\mu} - \nabla_\sigma \nabla_\mu G^\sigma_{\nu} + G_{\mu\nu} \right] = kT_{\mu\nu},$$

written in terms of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ rather than the Ricci tensor $R_{\mu\nu}$. When the components of $G_{\mu\nu}$ are written in terms of the parameters $\alpha$ and $\beta$ of the metric and their derivatives we get a system of 3 independent fourth order differential equations (one of the four components of the tensor eq.(13) is not independent because in spherical symmetry $G_{22} = G_{33}$.) The constraints that $\alpha$ and $\beta$ are finite at the origin and go to zero at infinity fix all the initial or boundary conditions except one, so that the solution of the 3 (coupled, nonlinear) ordinary differential equations provides a one-parameter family depending of the central value of the component $G^1_{1}$. For simplicity in the calculation I have used as equation of state the following

$$\rho_{\text{mat}} = 3p_{\text{mat}} + C\rho_{\text{mat}}^{3/5}, \quad C = 8.17 \times 10^7 \text{ kg}^{2/5}\text{ m}^{-6/5},$$

which has the correct behaviour for a fluid of free neutrons both at high and low densities. The constant $C$ is so chosen that a general relativistic calculation gives the same result as the original Oppenheimer and Volkoff for the maximum mass stable star (see Table 2 below).

A relevant quantity is the baryon number of the star, $N$, which may be calculated from the baryon number density $n(r)$ via

$$N = \int_0^R \frac{n(r)r^2dr}{\sqrt{1 - km(r)/(4\pi r)}},$$

The number density $n$ may be related to the mass density and the pressure
from the solution of the equation

\[ p_{\text{mat}} = n \frac{d\rho_{\text{mat}}}{dn} - \rho_{\text{mat}}, \]

which follows from the first law of thermodynamics. Inserting eq. (14) here we get a differential equation which may be easily solved with the condition \( \rho_{\text{mat}}/n \to \mu \) for \( \rho \to 0 \), \( \mu \) being the neutron mass. I get

\[ n = C^{5/8} \rho_{\text{mat}}^{3/5} \left( 4\rho_{\text{mat}}^{2/5} + C \right)^{3/8}. \] (16)

The details of the calculation may be seen elsewhere. In Table 1 I give the results obtained. For comparison in Table 2 the results of a calculation within GR are presented.

**Table 1. Calculation within extended gravity theory.** The parameter \( k \) is 8\( \pi \) times Newton’s constant. The central matter density, \( \rho_{\text{mat}}(0) \), is in units \( \rho_c \equiv 7.2 \times 10^{18} \text{ kg/m}^3 \) and similar units are used for the products of \( k^{-1} \) times the components of the Einstein tensor, \( G^\nu_{\mu} \) at the star center. Central baryon number density, \( n(0) \), is in units \( \rho_c/\mu \), \( \mu \) being the neutron mass. The star radius, \( R \), is in kilometers, the mass, \( M \), in solar masses and the baryon number, \( N \), in units of solar baryon number. I report also the dimensionless fractional surface red shift, \( \Delta \lambda/\lambda = 1/\sqrt{1 - 2M/R} - 1 \), and the percent binding energy, \( BE = 100(N - M)/N \). An expressions like 1.6E2 means \( 1.6 \times 10^2 \).

| \( k^{-1}G^1_{1}(0) \) | 0.01 | 0.1 | 1 | 10 | 100 | 1000 | 10000 |
|------------------------|------|-----|---|----|-----|------|-------|
| \( k^{-1}G^2_{4}(0) \) | 0.18 | 0.82 | 4.5 | 34 | 3.1E2 | 3.0E3 | 3.0E5 |
| \( \rho_{\text{mat}}(0) \) | 1.6E2 | 2.5E3 | 4.3E4 | 7.8E5 | 1.6E7 | 3.2E8 | 6.3E9 |
| \( n(0) \) | 56 | 4.5E2 | 3.7E3 | 3.3E4 | 3.2E5 | 3.0E6 | 2.8E7 |
| \( R \) | 10.7 | 6.7 | 4.0 | 2.7 | 2.1 | 2.2 | 2.2 |
| \( M \) | 0.67 | 0.80 | 0.60 | 0.39 | 0.264 | 0.268 | 0.292 |
| \( N \) | 0.73 | 0.94 | 1.03 | 1.13 | 1.44 | 2.00 | 2.63 |
| \( BE \) | 8.9% | 15% | 41% | 65% | 82% | 87% | 89% |
| \( \Delta \lambda/\lambda \) | 0.106 | 0.22 | 0.34 | 0.31 | 0.23 | 0.23 | 0.26 |

**Table 2. Calculation within general relativity.** The meaning of all symbols is as in Table 1, but here the matter pressure and density equal \( k^{-1} \) times the appropriate components of Einstein’s tensor.
\[ k^{-1}G_1^1(0) = \rho(0) \]

\[ -k^{-1}G_4^4(0) = \rho(0) \]

\[
\begin{array}{cccccccc}
R & M & N & BE & \Delta \lambda/\lambda \\
11.9 & 0.67 & 0.69 & 0.094 & 0.01 & 0.04 & 0.1 & 1 & 10 & 100 & 1000 \\
9.6 & 0.72 & 0.74 & 0.13 & 0.04 & 0.08 & 0.2 & 2 & 10 & 100 & 1000 \\
8.3 & 0.70 & 0.73 & 0.15 & 0.06 & 0.10 & 0.3 & 3 & 10 & 100 & 1000 \\
5.8 & 0.55 & 0.55 & 0.18 & 0.10 & 0.20 & 0.4 & 4 & 10 & 100 & 1000 \\
5.2 & 0.39 & 0.36 & 0.13 & -0.73 & -8.1 & -8.0 & -5.7 & 10 & 100 & 1000 \\
6.6 & 0.40 & 0.37 & 0.11 & -8.0 & -8.0 & -8.0 & -8.0 & 10 & 100 & 1000 \\
6.6 & 0.44 & 0.42 & 0.12 & -8.0 & -8.0 & -8.0 & -8.0 & 10 & 100 & 1000 \\
\end{array}
\]

IV. Discussion

The most dramatic difference between the calculations within extended gravity and within GR is in the variation of the baryon number with the central density. In GR there is a maximum about \( N = 0.74 \) times the baryon number of the Sun for some central density, \( \rho(0) \), and after that it decreases with increasing \( \rho(0) \) (which gives the OV limit for the baryon number). In our calculation with modified gravity \( N \) increases monotonically with the central density, which suggests that there is no limit. Of course for the extremely high densities which appear in Table 1 the individual neutrons would not exist and a fluid of different particles would appear, but we have used the same equation of state (of free neutrons) throughout. It is unlikely that a change of the equation of state could produce a very different qualitative behaviour, taking into account that for high density the equation should have the form \( p \approx \rho/3 \) in any case. The binding energy is also dramatically different in both calculations. Finally the gravitational surface redshift surpasses 0.3 whilst in GR it is always smaller than 0.2, which seems to agree better with observations. The comparison with observations is however not too relevant because the results do not take into account the interaction between neutrons.

If the “correct” gravity theory gives results qualitatively similar to those in Table 1, the evolution of any massive star would end in a compact star with relatively small mass but a very large baryon number. The essential conclusion of the paper is that the theoretical arguments supporting the existence of black holes, which rest upon standard GR, may be no longer valid in extended gravity theories. If it turns out that GR must be modified (possibly by quantum effects), then it might be the case that the alleged observational evidence for black holes should be revised.
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