An algorithm for computing the integral closure

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In this paper we describe an algorithm for computing the normalization for certain rings. This quite general algorithm is essentially due to Grauert and Remmert [3], [4], and seems to have escaped the attention of the computer algebra specialists until now. As I am not a computer algebra specialist myself, I do not know whether this algorithm is fast. Grauert and Remmert proved a normality criterium in order to give a simple and nice proof of a theorem of Oka, who proved that the non-normal points of an analytic space is an analytic space itself. We just reformulate their results, so that it better suits our purposes.

To fix the notation, let $R$ be a Noetherian and reduced ring, $\tilde{R}$ the integral closure (also called normalization) of $R$. We consider the set:

$$NNL := \{ p \in \text{Spec}(R) : R_p \text{ is not normal} \}$$

Here $NNL$ stands for non-normal locus. Let $I$ be an ideal of $R$ containing a nonzerodivisor. We have canonical inclusions:

$$R \subset \text{Hom}_R(I, I) \subset \tilde{R}$$

The first inclusion is the map which sends an element of $R$ to multiplication with this element. The second inclusion is sending $\phi \in \text{Hom}_R(I, I)$ to $\phi(f)$ for any element $f \in I$ which is a nonzerodivisor of $R$. It is easily checked that the map is independent of the choice of $f$. That we in fact land in $\tilde{R}$ can be found in any textbook which has included integral closure as a topic.

**Theorem 1.** [3] pp. 220-221, [4], pp. 125-127. Assume that the ideal $I$ contains a nonzerodivisor, and has the following property:

$$NNL \subset V(I)$$

where $V(I) = \{ p \in \text{Spec}(R) : I \subset p \}$ denotes, as usual, the zero set of $I$. Suppose moreover that $I$ has the property

$$\text{Hom}_R(I, I) = \text{Hom}_R(I, R) \cap \tilde{R} \quad (*)$$

Then one has the following normality criterium:

$$R = \text{Hom}_R(I, I) \iff R \text{ is normal}$$

**Proof.** The implication $\Leftarrow$ is trivial. To prove the converse, let $h = \sum g \in \tilde{R}$. Consider the following ideal in $R$

$$\{ g \in R : hg \in R \}$$
Its zero set is called the "pole set" of $h$:

$$P(h) := \{ p \in \text{Spec}(R) : h \notin R_p \}$$

It is immediate that $P(h) \subset NNL$. Let $J$ be the ideal of $P(h)$. There exists a $c > 0$ such that $hJ^c \subset R$, by the Nullstellensatz. By the Nullstellensatz again $\sqrt{J} \subset J$. Therefore there exists a $d > 0$ such that $hI^d \subset R$. Let $d$ be minimal with this property. We claim $d = 1$.

Suppose the converse, i.e. $d > 1$. Then there exists an $a \in I^{d-1}$ with $ha \notin R$. Furthermore $ha \in \tilde{R}$ and $(ha)I \subset R$. By assumption:

$$R = \{ h \in \tilde{R} : hI \subset I \} = \{ h \in \tilde{R} : hI \subset I \}$$

so that $(ha) \in I$. Therefore $ha \in R$ after all, a contradiction. 

We have to find an ideal which satisfies condition $(\ast)$. This is provided by:

**Theorem 2.** Every radical ideal $I$ containing a nonzerodivisor satisfies condition $(\ast)$.

**Proof.** The proof is in [3] and [4], but because it is so nice and simple we give it here. Let $h \in \tilde{R}$, so we have an equation:

$$h^n = a_0 + a_1h + \ldots + a_{n-1}h^{n-1}; \quad a_i \in R$$

If $hI \subset R$, then we have for all $f \in I$:

$$(hf)^n = a_0f^n + a_1h + \ldots + a_{n-1}h^{n-1}f \in I$$

As $I$ is supposed to be reduced it follows that $hf \in I$, and that is what we had to prove. 

These results give rise to the following algorithm:

**ALGORITHM**

INPUT: A reduced noetherian ring $R$.
OUTPUT: The normalization $\tilde{R}$ of $R$.
STEP 1: Determine an ideal $I$ with $NNL \subset V(I)$.
STEP 2: Compute the radical $\sqrt{I}$ of $I$. Put $I := \sqrt{I}$.
STEP 3: Take an $f \in I$, and compute $J := \text{Ann}(f)$. If $J = 0$, GOTO STEP 5.
STEP 4: Put $R := R/(f) \oplus R/J$ and GOTO STEP 1.
STEP 5: Compute $\text{Hom}_R(I, I)$. If $R = \text{Hom}_R(I, I)$ then put $\tilde{R} := R$ and STOP.
STEP 6: Set $R := \text{Hom}_R(I, I)$ and GOTO STEP 1.

This algorithm stops exactly when the normalization $\tilde{R}$ is finitely generated as an $R$-module, so for example for affine rings, due to a classical result of E. Noether.

Some remarks are in order.

1. To determine an ideal $I$ with $NNL \subset V(I)$ one can take any $I$ which contains the non-regular locus of $R$. 

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2. Algorithms for computing the radical of an ideal are described in [2] and probably will take the longest in this algorithm.

3. In step 4, \( R \subset R/(f) \oplus R/J \) is indeed an inclusion, as \((f) \cap J = 0\) follows from the assumption \( R \) is reduced. This extension is also finite. To see this, take a \( g \in J \) with \( f + g \) a nonzerodivisor (prime avoidance). The extension:

\[
R \subset R/(f) \oplus R/(g) \cong R[X]/(X^2 - X, X(f + g) - f)
\]

obviously is finite. As \( R \subset R/(f) \oplus R/J \) is a quotient of \( R/(f) \oplus R/(g) \) (in fact it is equal) it is a finite extension too.

4. I think it should be possible to extend this algorithm to an algorithm which computes the primary decomposition for a radical ideal.

We finish with describing the ring structure of \( \text{Hom}_R(I, I) \), which essentially is due to Catanese, see [1]. Take generators \( u_0 := 1, u_1, \ldots, u_t \) of \( \text{Hom}_R(I, I) \) as \( R \)-module. Consider the map:

\[
R \cdot X_0 \oplus R \cdot X_1 \oplus \cdots \oplus R \cdot X_t \xrightarrow{\phi} \text{Hom}_R(I, I)
\]

\[
X_i \mapsto u_i
\]

Computing the kernel of the map \( \phi \) gives "linear equations":

\[
L_i = \sum_{j=0}^{t} \alpha_{ij} X_j = 0 \quad \alpha_{ij} \in R; \ i = 1, \ldots, s
\]

Because \( \text{Hom}_R(I, I) \) is a ring, we have that \( u_i u_j \) for all \( 1 \leq i \leq j \leq t \) is in \( \text{Hom}_R(I, I) \) again. We therefore can find elements \( \beta_{ijk} \in R \) such that:

\[
u_i u_j = \sum_{k=0}^{t} \beta_{ijk} u_k
\]

giving us \( \frac{t(t+1)}{2} \) "quadratic equations":

\[
Q_{ij} := X_i X_j - \sum_{k=0}^{t} \beta_{ijk} X_k
\]

For the easy proof of the following theorem we refer to Catanese [1].

**Theorem 3.** Put \( X_0 = 1 \), and consider the ideal \( J \subset R[X_1, \ldots, X_t] \) generated by the \( L_i, i = 1, \ldots, s \) and the \( Q_{ij} \) for \( 1 \leq i \leq j \leq t \). Then we have a ring isomorphism:

\[
\text{Hom}_R(I, I) \cong R[X_1, \ldots, X_t]/J
\]

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References

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[4] H. Grauert and R. Remmert: *Coherent Analytic Sheaves* Grundl. **265**, Springer Verlag, Berlin etc. 1984