On $L_\infty$-Morphisms of Cyclic Chains

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Abstract. Recently the first two authors (Cattaneo and Felder in 2008) constructed an $L_\infty$-morphism using the $S^1$-equivariant version of the Poisson Sigma Model. Its role in the deformation quantization was not entirely clear. We give here a “good” interpretation and show that the resulting formality statement is equivalent to formality on cyclic chains as conjectured by Tsygan and proved recently by several authors (Dolgushev et al. in 2008; Willwacher in 2008).

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1. Introduction and Structure

We begin by drawing the big picture; precise definitions will be given below.

1.1. BIG PICTURE ON COCHAINS

Let $M$ be a smooth $d$-dimensional manifold and $A = C^\infty(M)$ ($A_c = C_c^\infty(M)$) the commutative algebras of smooth (compactly supported) functions. We denote by $T^\bullet$ the differential graded Lie algebra (DGLA) of multivector fields and by $C^\bullet(A)$ the multidifferential Hochschild complex. Kontsevich’s famous Formality Theorem asserts that there is an $L_\infty$-quasi-isomorphism of DGLAS

$U_K : T^\bullet \rightarrow C^\bullet(A)$.

Next, assume that $M$ is orientable$^1$ and pick a volume form $\Omega$. This endows $T^\bullet$ with an additional differential $\text{div}_\Omega$, the divergence, that is compatible with the

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$^1$This is actually not necessary, but we will assume it for simplicity.
Schouten bracket on \( T^\bullet \). We will denote the DGLA \((T^\bullet[[u]], u \text{ div}_{\Omega^1}, \cdot, \cdot)\) for short by \( T^\bullet[[u]] \). Here, \( u \) is a formal parameter of degree +2. There is a morphism of DGLAs
\[
T^\bullet[[u]] \xrightarrow{u=0} T^\bullet.
\]
We denote the composition of this morphism with \( U_K \) also by \( U_K \) for simplicity.

1.2. BIG PICTURE ON CHAINS

Let us turn to homology. Denote the negatively graded Hochschild (chain) complex by \( C^\bullet(A) = C^\bullet(A, A) \). It is a mixed complex, with the Hochschild differential \( b \) of degree +1 and with the Rinehart (or Connes) differential \( B \) of degree −1. The cohomology \( H_\bullet(A) \) of \( C^\bullet(A) \) with respect to the differential \( b \) is the de Rham complex \((\Omega^-\bullet(M), d)\), which we view as a bicomplex with vanishing first differential.

\( C^\bullet(A) \) also carries a compatible DGLA module structure over the Hochschild cochains \( C^\bullet(A) \). Pulling back this module structure along \( U_K \), we obtain an \( L_\infty \)-module structure over multivector fields \( T^\bullet \). The Hochschild Formality Theorem on chains \([4,7,8]\) states that there is a quasi-isomorphism of \( L_\infty \)-modules over \( T^\bullet \)
\[
\mathcal{V}: C_\bullet(A) \to \Omega^-\bullet(M).
\]
Actually, this morphism is compatible with the additional differentials \( B \) and \( d \) on both sides. Hence, we obtain an \( L_\infty \)-quasi-isomorphism
\[
\mathcal{V}: (C_\bullet(A)[[u]], b + uB) \to (\Omega^-\bullet(M)[[u]], ud).
\]
This last statement is known as the Cyclic Formality Theorem on chains \([5,8,10]\).

1.3. DUAL PICTURE

Recall that \( A = C^\infty(M) \). The following statement is a particularly simple case of van den Bergh duality \([9]\) (note the negative grading on the left)
\[
H_\bullet(A, A) \cong H^{d+\bullet}(A, \Omega^d(M)).
\]
Concretely, the left-hand side is \( \Omega^-\bullet(M) \), and the right-hand side is \( VT^{d+\bullet} := T^{d+\bullet} \otimes \Omega^d(M) \). The isomorphism from right to left is by contraction. Note that we can pull back the de Rham differential along this isomorphism, obtaining a differential “\( \text{div} \)” on \( VT^\bullet \). Note in particular that this differential \( \text{div} \) does not depend on a choice of volume form, in contrast to the \( \text{div}_{\Omega} \) defined before.

The dualized Hochschild formality theorem on chains states that there is a quasi-isomorphism of \( L_\infty \)-modules
\[
\mathcal{V}^*: VT^\bullet \to C^\bullet(A, \Omega^d).
\]
The dualized cyclic formality theorem states that this morphism is compatible with
the additional differentials div on the left and the (adjoint of the) Connes differ-
ential \( B \) on the right.

We will only consider such morphisms that are differential operators in each
argument. In this case there is a canonical way to obtain an adjoint morphism
\( V^* \) from the “direct” one \( V \) and vice versa. Concretely, there is a pairing between
\( C^\bullet (A, \Omega^d) \) and \( C_\bullet (A_c) \) given by
\[
\langle \phi, a_0 \otimes \cdots \otimes a_n \rangle = \int_M a_0 \phi(a_1, \ldots, a_n)
\]
and a pairing between \( V T^\bullet \) and \( \Omega^\bullet_c (M) \) given by
\[
\langle \gamma \Omega, \alpha \rangle = \int_M (\iota_\gamma \alpha) \Omega.
\]
Here, \( A_c \) and \( \Omega^\bullet_c (M) \) are the functions and forms with compact support and the
insertion \( \iota_\gamma \) is defined such that \( \iota_\gamma \gamma_1 \wedge \gamma_2 = \iota_\gamma 1 \iota_\gamma 2 \). One can see that to any direct mul-
tidifferential \( L_\infty \) morphism \( V \) there is a unique morphism \( V^* \) such that
\[
\langle \gamma \Omega, V(a_0 \otimes \cdots \otimes a_n) \rangle = \pm \langle V^*(\gamma \Omega), a_0 \otimes \cdots \otimes a_n \rangle.
\]
It follows that the direct and adjoint (multidifferential) formality statements are
equivalent.

Remark 1. (on quantization) The cohomology \( H^0 (A, \Omega^d) \) is important because it
classifies smooth traces on \( A_c \), i.e., top degree differential forms \( \Omega \) such that the
functional \( f \mapsto \int_M f \Omega \) is a trace on \( A_c \). Of course, in the current commuta-
tive setting, these are just all top degree differential forms. However, due to dual
Hochschild formality we can quantize. Let \( A_* \) be the algebra \( C^\infty (M) [[\hbar]] \) with the
Kontsevich star product \( [6] \) associated to a Poisson structure \( \pi \). The relevant coho-
mology is then \( H^0 (A_*, \Omega^d) \cong \{ \omega \in \Omega^d (M) [[\hbar]] | \text{div}_\omega \pi = 0 \} \). The quantized bimodule
structure on \( \Omega_*^d = \Omega^d [[\hbar]] \) is defined such that for all \( a, b \in A_c, \omega \in \Omega_*^d \)
\[
\int_M a \cdot (L_b \omega) = \int_M (a \star b) \cdot \omega = \int_M b \cdot (R_a \omega).
\]

1.4. OTHER MODULE STRUCTURES

The cyclic chain formality morphisms above are quasi-isomorphisms of \( L_\infty \-
modules over \((T^\bullet, 0, [\cdot, \cdot])\)_s). One may be tempted to replace this latter DGLA by
its “cyclic” counterpart \((T^\bullet[[u]], u \text{ div}_\Omega, [\cdot, \cdot])_s\), and ask whether the above formal-
ity statements remain true. Of course, if we use the module structures obtained via
pulling back along the DGLA morphism
\[
T^\bullet[[u]]_{u=0} \to T^\bullet
\]
the new formality statements will be equivalent to the original ones. However, one may try to change the module structures. We will only consider changing the module structure on the classical (differential forms) side.\footnote{One can also “naturally” change the action on the Hochschild side, but we do not discuss it here.} We show in Section 2.3 that there is a whole family of DGLA actions $L^{(t)}$ reducing to the original Lie derivative action for $t = 0$. However, all these module structures will be shown to be $L_\infty$-quasi-isomorphic in Proposition 2.

1.5. MEANING OF THE PSM MORPHISM

Using the $S^1$-equivariant version of the Poisson Sigma Model (PSM) the first two authors [1] recently constructed an $L_\infty$-morphism $\mathcal{V}_{\text{PSM, orig}}$, the “PSM morphism”. This paper is devoted to clarify the meaning of this morphism. To do this, we will reinterpret $\mathcal{V}_{\text{PSM, orig}}$ slightly, yielding a morphism $\mathcal{V}_{\text{PSM}}^*$. Concretely, we introduce a new complex $E^\bullet$ which is quasi-isomorphic (as bicomplex and $C^\bullet(A)$-module) to $C^\bullet(A, \Omega^d)$. The morphism $\mathcal{V}_{\text{PSM}}^*$ can then be understood as an adjoint cyclic chain formality morphism on

$$\mathcal{V}_{\text{PSM}}^*: T^\bullet[[u]] \cong VT^\bullet[[u]] \to E^\bullet[[u]].$$

Here, the action of $T^\bullet[[u]]$ on the left is the adjoint action, on the middle it is the (dual of the) action $L^{(1)}$, and on the right it is the action defined through pullback via $U_K$. The isomorphism on the left is defined by choosing a volume form.

1.6. ORGANISATION OF THE PAPER

The remainder of the paper is divided into two parts:

1. In the first part, we introduce the structures involved, i.e., the Hochschild and cyclic chain and cochain complexes. Here, there are two novel aspects: (i) We introduce the natural “extended” complex $E^\bullet$ mentioned above that allows us to give a nice interpretation of the PSM morphism and (ii) we introduce the aforementioned family $L^{(t)}$ of $T^\bullet[[u]]$-actions on differential forms that was (to our knowledge) not studied before.

2. In the second part, we define $\mathcal{V}_{\text{PSM}}^*$ and prove the formality statement made above.

2. Part I: The Objects of Study

In this section, we define the different complexes that will be related to each other through formality morphisms. Each complex can either constitute a differential graded Lie algebra (DGLA) or serve as a module over one of the DGLAs. We
will indicate the roles in the titles of each subsection. Of course, every DGLA is also a module over itself.

2.1. MULTIVECTOR FIELDS $T^\bullet$ (DGLA)

The algebra of multivector fields on $M$, $T^\bullet$, is the algebra of smooth sections of $\wedge^\bullet TM$. There is a Lie bracket $[\cdot, \cdot]_S$ on $T^{\bullet+1}(M)$, the Schouten bracket, extending the Lie derivative and making $T^\bullet$ a Gerstenhaber algebra. More concretely,

$[v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_n]_S = \sum_{i=0}^n \sum_{j=0}^{n} (-1)^{i+j} [v_i, w_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_n.$

Assume now that $M$ is oriented, with volume form $\Omega$. Contraction with $\Omega$ defines an isomorphism $T^\bullet \to \Omega^{d-*}(M)$. The divergence operator $\text{div}_\Omega$ on $T^\bullet$ is defined as the pull-back of the de Rham differential $d$ on $\Omega^{\bullet}(M)$ under this isomorphism. Concretely

$\iota_{\text{div}_\Omega \gamma} \Omega = dt \gamma \Omega.$

One can check that $\text{div}_\Omega$ is a derivation with respect to the Schouten bracket, i.e.,

$\text{div}_\Omega [\gamma_1, \gamma_2]_S = [\text{div}_\Omega \gamma_1, \gamma_2]_S + (-1)^{k_1-1} [\gamma_1, \text{div}_\Omega \gamma_2]_S.$

Introducing a new formal variable $u$ of degree +2, the complex $T^{\bullet+1}(M)[[u]]$ is a DGLA with differential $u \text{div}_\Omega$ and bracket the $u$-linear extension of the Schouten bracket.

Hence we have two DGLAs, $T^{\bullet+1}(M)$ and $T^{\bullet+1}(M)[[u]]$, related by a DGLA morphism

$T^{\bullet+1}(M)[[u]] \xrightarrow{u=0} T^{\bullet+1}(M).$

This morphism in particular allows us to view any $T^{\bullet+1}(M)$-module also as $T^{\bullet+1}(M)[[u]]$-module.

2.2. HOCHSCHILD COCHAINS $C^\bullet(A)$ (DGLA)

The normalized multidifferential Hochschild complex $C^\bullet(A)$ is the complex of $\bullet$-differential operators, which vanish upon insertion of a constant function in any of their arguments. E.g., $C^1(M)$ are differential operators $D$ such that $D1 = 0$.

\footnote{Actually $\text{div}_\Omega$ is a BV operator generating $[\cdot, \cdot]_S$ for any volume form $\Omega$.}
$C^{\bullet+1}(A)$ is a differential graded Lie algebra with the Gerstenhaber bracket

$$[\phi, \psi]_G(a_1, \ldots, a_{p+q-1}) = \phi(\psi(a_1, \ldots, a_q), a_{q+1}, \ldots, a_{p+q-1}) +$$
$$+ (-1)^{q-1} \phi(a_1, \psi(a_2, \ldots, a_{q+1}), a_{q+2}, \ldots, a_{p+q-1}) \pm$$
$$\pm \cdots +$$
$$+ (-1)^{(p-1)(q-1)} \phi(a_1, \ldots, \psi(a_p, \ldots, a_{p+q-1})) -$$
$$- (-1)^{(p-1)(q-1)} (\phi \leftrightarrow \psi)$$

for $\phi \in C^p(A), \psi \in C^q(A)$, and the Hochschild differential

$$b^H = [m_0, \cdot]_G.$$ 

Here, $m_0 \in C^2(A)$ is the usual (commutative) multiplication of functions.

2.3. THE DIFFERENTIAL FORMS $\Omega^\bullet(M)$ (MODULE)

Let $\Omega^\bullet = \Omega^\bullet(M)$ be the graded algebra of differential forms on $M$, with negative grading. Let $d = d_{dR}$ be the de Rham differential. Denote the insertion operators by $\iota_{\gamma}$. They take a form and contract it with the multivector field $\gamma$. The signs are such that

$$\iota : T^\bullet \to \text{End}(\Omega^\bullet)$$

$$\gamma \mapsto \iota_{\gamma}$$

is a morphism of graded algebras. For example, for a function $f$, $\iota_f$ is multiplication by $f$, for a vector field $\xi$, $\iota_{\xi}$ is a derivation of the DGA $\Omega^\bullet$ and for any multivector fields $\gamma$, $\nu$, $\iota_{\gamma} \wedge \nu = \iota_{\gamma} \iota_{\nu}$. The Lie derivative $L$ is:

$$L_{\gamma} := [d, \iota_{\gamma}].$$

It satisfies the following relation, which can alternatively be taken as the definition of the Schouten bracket.

$$\iota_{[\gamma, \nu]} = [\iota_{\gamma}, L_{\nu}] = (-1)^{|\gamma|} [L_{\gamma}, \iota_{\nu}]$$

It follows that $L$ forms a representation of the differential graded Lie algebra $T^{\bullet+1}$. Here and everywhere in the paper the degrees $|\gamma|$ are such that $\gamma \in T^{|\gamma|+1}$.

Next consider module structures on $(\Omega^\bullet[[u]], u \text{div}_\Omega)$ over the DGLA $(T^\bullet[[u]], [-, -], u \text{div}_\Omega)$. Let us introduce a family of actions $L_{\gamma}^{(t)}$ as follows. Let $S^{(t)}$ be the $u$-scaling operation on multivector fields given by

$$S^{(t)}\gamma = S^{(t)} \left( \sum_{j \geq 0} u^j \gamma_j \right) = \sum_{j \geq 0} (tu)^j \gamma_j.$$
Let further
\[ \imath(t) = \imath_S(t) \gamma. \]

The family of DGLA actions is then given by
\[ L(t) = \frac{1}{u} \left( [ud, \imath(t)] + \imath(t) u \text{div} \gamma \right) = \sum_{j \geq 0} (ut)^j \left( L\gamma_j + t \text{div} \gamma_j \right) \]
where \( \gamma = \sum_{j \geq 0} u^j \gamma_j \in T^*[[u]] \).

**Proposition 2.** For any \( t \in \mathbb{C} \), \( L(t) \) defines a DGLA module structure on \( \Omega^*[[u]] \). Furthermore all these module structures are \( L_\infty \)-isomorphic to each other.

**Proof.** To show that the \( L(t) \) are indeed DGLA actions, compute
\[
\left[ ud, L(t) \right] = \sum_{j \geq 0} (ut)^j t \left[ ud, \imath(t) \text{div} \gamma_j \right] = \sum_{j \geq 0} (ut)^j+1 L\text{div} \gamma_j = L u \text{div} \gamma(t).
\]
Furthermore,
\[
\left[ L(t), L(t) \right] = \sum_{j, k \geq 0} (ut)^j k \left[ L\gamma_j + t \text{div} \gamma_j, L\gamma_k + t \text{div} \gamma_k \right] = \sum_{j, k \geq 0} (ut)^j k \left( L\gamma_j \gamma_k + t(-1)^j \gamma_j \text{div} \gamma_k + t \text{div} \gamma_j \gamma_k \right) = \sum_{j, k \geq 0} (ut)^j k \left( L\gamma_j \gamma_k + t \text{div} \gamma_j \gamma_k \right) = L(t) \gamma, \nu].
\]

Next we construct a family of \( L_\infty \) isomorphisms \( H(t) \) relating \( L(t) \) and \( L(0) \). These isomorphisms will be solutions of a differential equation
\[ \dot{H}(t) = h(t) H(t) \]
for some family of infinitesimal morphisms (\( L_\infty \)-derivations) \( h(t) \). In fact, the \( h(t) \) will have vanishing 0th Taylor component and will all commute, so that one can explicitly write down the solution
\[ H(t) = \exp \left( \int_0^t h(t) dt \right). \]
The \( h^{(t)} \) will have only a single non-vanishing Taylor coefficient of degree one, which we denote (admittedly slightly confusing) by

\[
h^{(t)}_1(\gamma; \alpha) = -(-1)^{|\gamma|} h^{(t)}_{\gamma \alpha}.
\]

One finds that the \( L_\infty \)-derivation property is equivalent to the following two conditions for \( h^{(t)}_\gamma \).

\[
-\frac{d}{dt} L^{(t)}_\gamma = \left[ ud, h^{(t)}_\gamma \right] + h^{(t)}_u \text{div}_\Omega \gamma
\]

\[
h^{(t)}_{[\gamma, v]} = \left[ h^{(t)}_\gamma, L^{(t)}_v \right] + (-1)^{|\gamma|} \left[ L^{(t)}_\gamma, h^{(t)}_v \right]
\]

All higher \( L_\infty \) relations are trivially satisfied.

We claim that

\[
h^{(t)}_\gamma = -\frac{1}{u} \frac{d}{dt} t^{(t)}_\gamma
\]

satisfies these equations.\(^4\) Compute

\[
\frac{d}{dt} L^{(t)}_\gamma = \left( \frac{1}{u} \right) \left[ ud, \frac{d}{dt} t^{(t)}_\gamma \right] + \left( \frac{1}{u} \right) \frac{d}{dt} t^{(t)}_u \text{div}_\Omega \gamma =
\]

\[
= - \left[ ud, h^{(t)}_\gamma \right] - h^{(t)}_u \text{div}_\Omega \gamma.
\]

In second order,

\[
\left[ h^{(t)}_\gamma, L^{(t)}_v \right] + (-1)^{|\gamma|} \left[ L^{(t)}_\gamma, h^{(t)}_v \right] = - \sum_{j, k} (tu)^{j+k} \left( (j/t) t^{[\gamma, v]}_j + (k/t) t^{[\gamma, v]}_k \right) =
\]

\[
= - \frac{d}{dt} \sum_{j, k} (tu)^{j+k} t^{[\gamma, v]}_j
\]

\[
= h^{(t)}_{[\gamma, v]}.
\]

In the special case \( t = 0 \) the action becomes

\[
L^{(0)}_\gamma \alpha = L_{\gamma_0} \alpha
\]

and in the case \( t = 1 \)

\[
L^{(1)}_\gamma \alpha = L_{\gamma} \alpha + t \text{div}_\Omega \gamma \alpha.
\]

Inserting \( t = 1 \) into the formula for \( H^{(t)} \) from the preceding proof we obtain a quasi-isomorphism between these two structures:

\[
H^{(1)} = e^{\int_0^1 h^{(t)} dt} = e^{-t/\gamma}.
\]

\(^{4}\) Note that the expression on the right is well defined since \( \frac{d}{dt} t^{(t)}_\gamma \sim O(u) \).
Here, $\iota^+$ is the (pre-) $L_\infty$ derivation with a single non-vanishing Taylor coefficient in degree 1 given by:

$$(\gamma, \alpha) \mapsto \iota^+_\gamma \alpha := \iota^{(1)}_\gamma \alpha - \iota^{(0)}_\gamma \alpha.$$  

Concretely, the $n$th component of $H^{(1)}$ reads

$$H^{(1)}_n(\gamma_1, \ldots, \gamma_n) = \pm \frac{1}{u^n} \iota^+_\gamma \cdot \cdots \cdot \iota^+_\gamma.$$  

### 2.4. MULTIVECTOR FIELD VALUED TOP FORMS $VT^*$ (MODULE)

We define the multivector field valued top forms

$$VT^* := \Omega^d (M; \wedge^* TM).$$

There is a natural non-degenerate pairing

$$\langle \cdot, \cdot \rangle : VT^* \otimes \Omega^*_c (M) \to \mathbb{C}$$

$$\langle v \Omega, \alpha \rangle = \int_M \Omega (t_v \alpha).$$

Its obvious $u$-bilinear extension allows dualizing the DGLA-module structures $L$ and $L^{(t)}$ on $\Omega^*(M)[[u]]$ discussed above to DGLA-module structures on $VT^*[[u]]$. We denote these dual module structures also by $L^{(t)}$ and hope that no confusion arises. Concretely, in our sign conventions the differential, temporarily called $\delta$, and action are defined such that

$$\langle \delta (v \Omega), \alpha \rangle = -(-1)^{|v|} \langle v \Omega, u \alpha \rangle$$

$$\langle L^{(t)}_\gamma (v \Omega), \alpha \rangle = -(-1)^{|v||\gamma|} \langle v \Omega, L^{(t)}_\gamma \alpha \rangle.$$  

**Lemma 3.** The DGLA module structure $L^{(t)}$ on $VT^*[[u]]$ is given explicitly by the following data: The differential is $\delta = u \text{div}$ with

$$\text{div} (v \Omega) := (\text{div}_\Omega v) \Omega.$$  

The action is

$$L^{(t)}_\gamma (v \Omega) = \sum_{j \geq 0} (tu)^j \left( [\gamma, v]_S \Omega + (-1)^{|\gamma|} (1-t) (\text{div}_\Omega \gamma \wedge v) \Omega \right)$$

where $\gamma = \sum_{j \geq 0} u^j \gamma_j$.  

Proof. Note first that
\[ \int_M (\iota_\gamma \alpha) \Omega = \int_M \alpha \wedge \iota_\gamma \Omega. \]

It follows that
\[ \langle \delta(\nu/\Omega), \alpha \rangle = -(-1)^{\nu} u \int_M (\iota_\nu \alpha) \Omega = -(-1)^{\nu} \int_M (\iota_\nu \alpha) \Omega = \]
\[ = (-1)^{\nu + \nu} u \int_M \alpha \iota_\nu \Omega = u \int_M \alpha \iota_{\text{div}_\Omega} \nu \Omega = \]
\[ = u \int_M (\iota_{\text{div}_\Omega} \nu \alpha) \Omega = \langle u \text{div}(\nu \Omega), \alpha \rangle. \]

In the fourth line we used that everything is zero unless \( |\alpha| = |\gamma| \). Furthermore, note that by a small computation
\[ \int_M (\iota_\nu \alpha) \Omega = -\int_M (\iota_{\text{div}_\Omega} \nu \alpha) \Omega. \]

Hence we obtain
\[ \langle L^{(t)}_{\nu, \gamma_j} (\nu \Omega), \alpha \rangle = -(-1)^{\nu \gamma_j} u^j \int_M (t^i L_{\gamma_j} \alpha + t^{j+1} \iota_{\text{div}_\Omega} \gamma_j \alpha) \Omega = \]
\[ = -(-1)^{\nu \gamma_j} (tu)^j \int_M \left( (t^{[\nu, \gamma_j]}_{\lambda} + (-1)^{\nu + \nu} \gamma_j \Omega_{\gamma_j} \iota_\nu \alpha + \]
\[ + (-1)^{\nu + \nu} \gamma_j t_{\text{div}_\Omega} \gamma_j \wedge \nu \alpha \right) \Omega = \]
\[ = -(-1)^{\nu \gamma_j} (tu)^j \int_M \left( (t^{[\nu, \gamma_j]}_{\lambda} + (-1)^{\nu + \nu} \gamma_j t_{\text{div}_\Omega} \gamma_j \wedge \nu \alpha + \right. \]
\[ + \left. (-1)^{\nu + \nu} \gamma_j t_{\text{div}_\Omega} \gamma_j \wedge \nu \alpha \right) \Omega = \]
\[ = -(-1)^{\nu \gamma_j} (tu)^j \int_M \left( -(-1)^{\nu \gamma_j} t_{[\nu, \gamma_j]}_{\lambda} - \right. \]
\[ - (-1)^{\nu + \nu} \gamma_j t_{\text{div}_\Omega} \gamma_j \wedge \nu \alpha + (-1)^{\nu + \nu} \gamma_j t_{\text{div}_\Omega} \gamma_j \wedge \nu \alpha \right) \Omega = \]
\[ = \left( (tu)^j (\nu_{[\gamma_j, \nu]} + (-1)^{\nu \gamma_j} (1 - t) \text{div}_\Omega \gamma_j \wedge \nu) \Omega, \alpha \right). \]

In view of the PSM morphism, the most interesting case is \( t = 1 \). Here, the action is the push-forward of the adjoint action along the isomorphism
\[ T^*[[u]] \rightarrow VT^*[[u]] \]
\[ \gamma \mapsto \gamma \otimes \Omega. \]
2.5. THE HOCHSCHILD CHAINS (MODULE)

The (normalized) Hochschild chain complex of a unital algebra \( A \) is the complex
\[
C_{\bullet}(A) = A \otimes \bar{A}^{\otimes \bullet}
\]
where \( \bar{A} = A/C \cdot 1 \). In our case, i.e., \( A = C^\infty(M) \), we interpret the tensor products as
\[
A^{\otimes (\bullet + 1)} = \text{jets}_\Delta C^\infty(M^{\bullet + 1})
\]
and accordingly for \( A \otimes \bar{A}^{\otimes \bullet} \). Here \( \text{jets}_\Delta C^\infty(M^{\bullet + 1}) \) are the \( \infty \)-jets at the diagonal \( \Delta \subset M^{\bullet + 1} \).\(^5\) The complex \( C_{\bullet}(A) \) is equipped with a differential \( b_H \) such that
\[
b_H(a_0 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n \pm \cdots + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.
\]

The normalized Hochschild cochain complex acts on the normalized chain complex through the (DGLA) action
\[
L_D(a_0 \otimes \cdots \otimes a_n) = \sum_{j=n-d+1}^{n} (-1)^{n+j+1} D(a_{j+1}, \ldots, a_0, \ldots) \otimes a_{d+j-n} \otimes \cdots \otimes a_j + \sum_{i=0}^{d-1} (-1)^{(d-1)(i+1)} a_0 \otimes \cdots \otimes a_i \otimes D(a_{i+1}, \ldots, a_{i+d}) \otimes \cdots \otimes a_n. \tag{1}
\]
In particular \( b_H = L_{m_0} \).

2.6. THE CYCLIC CHAINS (MODULE)

The normalized Hochschild chain complex is equipped with an additional differential \( B \) of degree \(-1\) discovered by Rinehart and rediscovered by Connes.
\[
B(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n} (-1)^{j+1} a_j \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{j-1}
\]
One can check that this differential (graded) commutes with the action (1) above, and hence anticommutes with \( b_H \). Introducing an additional formal variable \( u \) of degree \( +2 \), one defines the negative cyclic chain complex as
\[
(C_{\bullet}(A)[[u]], b_H + uB).
\]
Its homology is called the negative cyclic homology. Other cyclic homology theories can be obtained from the negative cyclic complex by tensoring with an appropriate \( \mathbb{C}[u] \)-module and will not receive specialized treatment in this paper.

\(^5\)See also [8], Remark 3.1.1.
2.7. HOCHSCHILD COMPLEX: SHEAF VERSION $E^\bullet$ (MODULE)

Consider the sheaf $D^\bullet(M)$ of $\bullet$-differential operators. For example, $D^1(M)$ is the sheaf of differential operators. It is a complex with the Hochschild differential$^6$

$$(b\Phi)(a_0, \ldots, a_n) = \pm (\Phi(a_0 a_1, a_2, \ldots, a_n) - \Phi(a_0, a_1 a_2, \ldots, a_n)) \pm \cdots - (-1)^n \Phi(a_0, a_1, \ldots, a_{n-1} a_n) + (-1)^n \Phi(a_n a_0, a_1, \ldots, a_{n-1})$$

Also, note that there is an action of the cyclic group(oid) on $D^\bullet(M)$ generated by

$$(\sigma \Phi)(a_0, \ldots, a_n) = (-1)^n \Phi(a_1, a_2, \ldots, a_n, a_0).$$

There is a canonical flat connection $\nabla$ on $D^\bullet(M)$, compatible with the differential and the cyclic action. It is given by the de Rham differential:

$$(\nabla \Phi)(a_0, \ldots, a_n) = d(\Phi(a_0, \ldots, a_n)).$$

DEFINITION 4. The extended Hochschild cochain complex is the total complex

$$E^\bullet = \bigoplus_{p+q-d=\bullet} (\Gamma(D^{p+1}(M) \otimes_{C^\infty(M)} \Omega^q(M)), b + \nabla).$$

The normalized extended Hochschild complex $E^\bullet_{\text{norm}}$ is the subcomplex of multidifferential operators $\Phi$ such that

$$\Phi(a_0, \ldots, a_{j-1}, 1, a_{j+1}, \ldots, a_n) = 0$$

for all $a_0, \ldots, a_n$ and all $j = 1, \ldots, n$.

There is an action on $E^\bullet$ of the multidifferential operators, now considered as a sheaf of DG Lie algebras with differential $d_H$, by the formula dual to (1), i.e.,

$$(L_D \Phi)(a_0, \ldots, a_n) =$$

$$= -(-1)^{|D||\Phi|} \sum_{j=n-d+1}^n (-1)^{n(j+1)} \Phi(D(a_{j+1}, \ldots, a_0, \ldots), a_{d+j-n}, \ldots, a_j) +$$

$$+ \sum_{i=0}^{n-d} (-1)^{(d-1)(i+1)} \Phi(a_0, \ldots, a_i, D(a_{i+1}, \ldots, a_{i+d}), \ldots, a_n).$$

In terms of this action, the differential can be written as $b = L_m$ where $m$ is the multiplication cochain.

The complex $E^\bullet$ is just another complex computing Hochschild cohomology with values in $\Omega^d(M)$, as the following proposition shows.

$^6$Note that this is not the $b_H$ from above, there is no $a_0 \Phi(a_1, a_2, \ldots, a_n)$-term.
Proposition 5. The embedding $C^\bullet(A, \Omega^d(M)) \to E^\bullet$ given by

\[ \Phi \mapsto ((a_0, \ldots, a_n) \mapsto a_0 \Phi(a_1, \ldots, a_n)) \]

is a quasi-isomorphism.

We will benefit from the following elementary result.

Lemma 6. Let $(K_{p,q})_{0 \leq p \leq n, q \in \mathbb{Z}}$ be a double complex with differential $d_1 + d_2$, where

\[ d_1 : K_{p,q} \to K_{p+1,q}, \quad d_2 : K_{p,q} \to K_{p,q+1}. \]

Then the following holds:

1. If the $d_1$-cohomology is concentrated in bottom degree $p = 0$, then the inclusion of the $d_1$-closed, $p$-degree 0 elements

\[ \left\{ k \in K^{0,\bullet} | d_1 k = 0 \right\} \hookrightarrow K^{\bullet,\bullet} \]

is a quasi-isomorphism.

2. If the $d_1$-cohomology is concentrated in top degree $p = n$, then the projection onto the top $p$ degree elements modulo exact elements

\[ K^{\bullet,\bullet} \to K^{n,\bullet}/d_1 K^{n-1,\bullet} \]

is a quasi-isomorphism.

Proof. At least the first statement is probably familiar to the reader. The proof of the second statement is essentially dual to the proof of the first.

Proof of Proposition 5. It is more or less obvious that the above map is a map of complexes. It remains to be shown that it is a quasi-isomorphism.

Let us compute the cohomology of $E^\bullet$ with respect to $\nabla$, i.e., the first term in the spectral sequence associated to $E^\bullet$. We claim that it is concentrated in the top form-degree $d = \dim M$, and every class has exactly one representative in the image of the above quasi-isomorphism. To show this, consider the spectral sequence associated to the following filtration:

\[ F_p E = \left\{ \Gamma(\Phi \in D^\bullet(M) \otimes C^\infty(M) \Omega^k(M)) | k = 0, 1, \ldots \right\} \]

where $\text{ord}_0 \Phi$ is the order of $\Phi$ as a differential operator in the first “slot” (i.e., the slot in which $a_0$ is inserted). One can check that $\nabla F_p E \subset F_p E$. The first term in the spectral sequence is the associated graded, i.e., multidifferential operators with values in $\Lambda^\bullet T^* M \otimes S^\bullet TM$. The differential $d_0$ is, in local coordinates, the operator $d_0 = \sum_i (dx_i \wedge \cdot) \otimes (\partial_i \cdot)$, multiplying the $\Lambda^\bullet T^* M$-part by $dx^i$ and the $S^\bullet TM$-part
by $\partial_i$. The cohomology is concentrated in form degree $d$ and operator degree 0. Probably the quickest way to see this is to note that the complex $\wedge^d T^* M \otimes S^d T M$ with the above differential is isomorphic to the Koszul complex of $S^d T M$, the isomorphism being given by contracting the first factor with a section of $\wedge^d T M$. The spectral sequence degenerates at this point by (form-)degree reasons. This means that any $\nabla$-cohomology class has exactly one representative of form degree $d$ and of differential operator degree 0 in the first slot. This proves the above claim, and hence the proposition.

2.8. CYCLIC COCHAINS: SHEAF VERSION (MODULE)

DEFINITION 7. The extended cyclic complex is the complex $(E^\bullet)^\sigma$ of invariants under the cyclic action. The extended cyclic $(b, B)$-complex is the complex $E^\bullet_{\text{norm}}[[u]]$ with differential $b + uB$, where $B$ is Connes’ $B$.

For an orientable manifold, this complex computes the cyclic cohomology.

PROPOSITION 8. For $M$ orientable, the cohomology of the extended cyclic complexes $(E^\bullet)^\sigma$ and $E^\bullet_{\text{norm}}[[u]]$ is the cyclic cohomology of $C^\infty(M)$.

Proof. Consider again the spectral sequence and compute the $\nabla$-cohomology of the two complexes. As in the last proof, the first term of the spectral sequence for $E^\bullet_{\text{norm}}[[u]]$ is, as a vector space, isomorphic to $D^\bullet_{\text{norm}}[[u]]$, the isomorphism being given in the last proposition. One can see more or less by the definitions that the differentials $b, B$ are mapped to $b_H, B$ under this isomorphism.

For the case of $(E^\bullet)^\sigma$, note that $\nabla$ commutes with the action of the cyclic group. It follows that taking the $\nabla$-cohomology commutes with taking cyclic invariants. The result then follows as in the proof of the last proposition. □

3. Part II: The Meaning of the PSM Morphism

3.1. THE ORIGINAL PSM MORPHISM

Let $M$ be orientable and choose a volume form $\Omega$. The original PSM morphism $\psi_{\text{PSM, orig}}$ is an $L_\infty$-morphism of modules over the DG Lie algebra $(T^*[u], u \, \text{div} \Omega, [\cdot, \cdot]_S)$, constructed by the first two authors in [1] using essentially an equivariant version of the Poisson sigma model. The two modules it relates are the cyclic chains and the multivector fields.

$$\psi_{\text{PSM, orig}} : (C^\bullet(A, A)[[u]], b + uB) \to (T^*[u], u \, \text{div} \Omega).$$

The module structure on the left is given by pulling back the $C^\bullet(A)$-action along $U_K$. The module structure on the right is the trivial module structure (!). We copy the following proposition from [1].
PROPOSITION 9. The morphism $\psi_{PSM,\text{orig}}$ is a morphism of $L_\infty$-modules (but not a quasi-isomorphism).

3.2. THE (REINTERPRETED) PSM MORPHISM $\psi_{PSM}^*$

Here, we give a new interpretation of the above morphism. The (reinterpreted) PSM morphism $\psi_{PSM}^*$ is a quasi-isomorphism of $L_\infty$-modules over the DGLA $(T^\bullet[[u]],u\text{ div}_\Omega,[\cdot,\cdot])$. However, the two modules are the multivector-field-valued top forms, which can be identified with $T^\bullet[[u]]$ using the volume form, and the extended cyclic complex $E^\bullet_{\text{norm}}[[u]]$.

$$\psi_{PSM}^*: (T^\bullet[[u]], u\text{ div}_\Omega) \cong (VT^\bullet[[u]], ud) \to (E^\bullet_{\text{norm}}[[u]], \nabla + b + uB).$$

The DGLA module structure on the very left is the adjoint one, in contrast to the trivial one above, and on the middle $L^{(1)}$. The $L_\infty$-module structure on the right is defined via pullback of the DGLA action of $C^\bullet(A)$ via the (Kontsevich) $L_\infty$-morphism $U_K$.

The reinterpreted morphism is constructed from the original one as follows:

$$\psi_{PSM}^* (\gamma_1, \ldots, \gamma_m)(\gamma)(a_0, \ldots, a_n) = \iota_{\psi_{PSM,\text{orig}}} (\gamma_1, \ldots, \gamma_m, u\gamma; a_0, \ldots, a_n) \Omega.$$ 

THEOREM 10. The morphism $\psi_{PSM}^*$ is a quasi-isomorphism of $L_\infty$-modules.

Proof. The fact that it is an $L_\infty$-morphism is an easy consequence of Proposition 9 and the previous observation that for any multivector field $v$

$$\iota_{\text{div} v} \Omega = d\iota_v \Omega.$$ 

It remains to be shown that the 0th Taylor component is an isomorphism on cohomology. In view of Lemma 6 it is sufficient to show that the composition with the projection onto the top form degree part modulo the image of $\nabla$ is a quasi-isomorphism. Explicit computation yields that the 0th Taylor component is

$$\gamma \mapsto \pm((a_0, \ldots, a_k) \mapsto a_0 \gamma(a_1, \ldots, a_k) \Omega) + \text{(lower form degree)}.$$ 

The first part is the HKR morphism, known to be a quasi-isomorphism, and the remainder does not matter due to the projection onto top form degree components. 

The statement of Theorem 10 can be seen as a dualized version of B. Tsygan’s negative cyclic formality conjecture. The more precise relation is as follows. Note that the complex $E^\bullet$ is bigger than the dual (in an appropriate sense) of $C_\bullet(A)$. Concretely, it also contains non-top-degree differential forms. These forms do not show up in cohomology, but are needed to interpret the “white vertices”, see [1], occurring in the original PSM morphism. The “true” dual of $C_\bullet(A)$ occurs after
projecting $E^\bullet$ to top forms, modulo the image of $\nabla$. This projection in particular kills all white vertices occurring in $V^\bullet_{PSM}$, but leaves it as a quasi-isomorphism due to the proof of Proposition 8. By the remarks in Section 1.3, one can dualize this quasi-isomorphism again and obtain another solution of B. Tsygan’s formality conjecture.

Appendix A. Our Sign Conventions

There are many signs involved in the discussions above. Since sign computations are typically lengthy and boring, we did not explain them all. However, we list here the underlying conventions for the reader who believes $1 \neq -1$ and wants to check.

Let $g^\bullet$ be a graded vector space. An $L_\infty$-algebra structure on $g^\bullet$ is a degree 1 coderivation $Q$ on the cofree (graded) cocommutative coalgebra without counit cogenerated by $g^{\bullet+1}$, i.e., $S^+g^{\bullet+1}$, satisfying $Q^2 = 0$. Any such coderivation is determined by its Taylor coefficients $Q_n(x_1, \ldots, x_n) = \pi Q(x_1, \ldots, x_n)$ where $\pi$ is the projection on $g^{\bullet+1} \subset S^+g^{\bullet+1}$. If $g$ carries the structure $(d, [\cdot, \cdot])$ of a DGLA, we associate to it an $L_\infty$-structure by the following convention (others are possible)

$$Q_1(x) = dx \quad Q_2(x_1, x_2) = -(−1)^{|x_1|}[x_1, x_2].$$

An $L_\infty$-module structure on the graded vector space $M^\bullet$ is a coderivation $\tilde{Q}$ lifting $Q$ on the cofree comodule $Sg^{\bullet+1} \otimes M^\bullet$. Again, it is determined by its Taylor coefficients $\pi_M \circ \tilde{Q}$. We identify (by convention) a DGLA module $(M^\bullet, \delta, L)$ over the DGLA $g$ with the $L_\infty$-module

$$\tilde{Q}_0(m) = \delta m \quad \tilde{Q}_1(x; m) = -(−1)^{|x|}L_x m.$$

Next let $\hat{M}^\bullet$ be another graded vector space and $\langle \cdot, \cdot \rangle$ be a nondegenerate pairing between $\hat{M}^\bullet$ and $M^\bullet$. This allows us to endow $\hat{M}^\bullet$ with an $L_\infty$-structure $\hat{Q}^\bullet$ defined by

$$\langle \hat{Q}_n^\bullet(x_1, \ldots, x_n; \hat{m}), m \rangle = -(−1)^{|\hat{m}|(n+1+\sum_j |x_j|)} \langle \hat{m}, \hat{Q}_n(x_1, \ldots, x_n; m) \rangle.$$
Finally, let us describe the signs involved in Section 3. Let $Q$ be the coderivation determining the $L_\infty$-algebra structure on $T^*[[u]]$. Then the (adjoint) $L_\infty$-module structure on $T^*[[u]]$ is simply given by

$$\tilde{Q}_n(x_1, \ldots, x_n; x) = Q_{n+1}(x_1, \ldots, x_n, x).$$

Let $\tilde{P}$ determine the $L_\infty$ module structure on $C_\bullet(A, A)[[u]]$. Then the module structure on $E^\bullet_{\text{norm}}[[u]]$ is determined by the coderivation $\tilde{O}$, defined such that for a map $\lambda : C_\bullet(A, A)[[u]] \to T^*[[u]]$:

$$\tilde{O}_n(x_1, \ldots, x_n; \iota_\lambda(\cdot)/\Omega_1) = -(-1)^{|\lambda|(n+1+\sum_j |x_j|)}l_{\lambda}(\tilde{P}_n(x_1, \ldots, x_n; \cdot))/\Omega_1 + \delta_{n,0} \nabla \iota_\lambda(\cdot)/\Omega_1.$$

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