Homotopical variations and high-dimensional Zariski-van Kampen theorems

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Abstract
In 1933, van Kampen described the fundamental groups of the complements of plane complex projective algebraic curves. Recently, Chéniot-Libgober proved an analogue of this result for higher homotopy groups of the complements of complex projective hypersurfaces with isolated singularities. Their description is in terms of some “homotopical variation operators”. We generalize here the notion of “homotopical variation” to (singular) quasi-projective varieties. This is a first step for further generalizations of van Kampen’s theorem. A conjecture, with a first approach, is stated in the special case of non-singular quasi-projective varieties.

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Introduction
Our topic is best understood in the general frame of Lefschetz type theorems. Let $X := Y \setminus Z$, where $Y$ is an algebraic subset of complex projective space $\mathbb{P}^n$, with $n \geq 2$, and $Z$ an algebraic subset of $Y$ (such an $X$ is called an (embedded) quasi-projective variety). Let $\mathcal{L}$ be a projective hyperplane of $\mathbb{P}^n$. The non-singular quasi-projective version of the Lefschetz Hyperplane Section Theorem, proved by Hamm-Lê and Goresky-MacPherson (cf. [HL1] and [GM1,2]), asserts that if $X$ is non-singular and $\mathcal{L}$ generic, then the natural maps (between homology and homotopy groups, respectively)

$$H_q(\mathcal{L} \cap X) \to H_q(X) \quad \text{and} \quad \pi_q(\mathcal{L} \cap X, \ast) \to \pi_q(X, \ast)$$

are bijective for $0 \leq q \leq d - 2$ and surjective for $q = d - 1$, where $d$ is the least (complex) dimension of the irreducible components of $Y$ not contained in $Z$. In
the special case $Y = \mathbb{P}^n$, that is for the complement $X = \mathbb{P}^n \setminus Z$ of a projective variety, the bounds can be improved by $c - 1$, where $c$ is the least codimension of the irreducible components of $Z$ (cf. [C2]): then the above maps are bijective for $0 \le q \le n + c - 3$ and surjective for $q = n + c - 2$ (there is no improvement if $c = 1$).

The question now arises of determining the kernel of these maps in dimension $d - 1$ (resp. $n + c - 2$ for a complement). For this purpose, it is classical (at least when $Z = \emptyset$) to consider $\mathcal{L}$ as a member of a pencil $\mathcal{P}$ of hyperplanes of $\mathbb{P}^n$ with axis a generic $(n - 2)$-plane $\mathcal{M}$. The sections of $X$ by all the hyperplanes of such a pencil are isotopic to the one by $\mathcal{L}$ with the exception of the sections by a finite number of exceptional hyperplanes $(\mathcal{L}_i)_i$. For each $i$, there are some homomorphisms

$$\text{var}_{i,q}: H_q(\mathcal{L} \cap X, \mathcal{M} \cap X) \to H_q(\mathcal{L} \cap X), \text{ for all } q,$$

called “homological variation operators”, defined by patching each relative cycle on $\mathcal{L} \cap X$ modulo $\mathcal{M} \cap X$ with its transform by monodromy around $\mathcal{L}_i$ (cf. [C3]). These are defined even if $X$ is singular. In loc. cit., the first author showed that if $X$ is non-singular, then the kernel of the natural epimorphism

$$(E) \quad H_{d-1}(\mathcal{L} \cap X) \to H_{d-1}(X)$$

is equal to the sum of the images of the homological variation operators $(\text{var}_{i,d-1})_i$ associated to the exceptional members $(\mathcal{L}_i)_i$ of the pencil. In the case of a complement $X = \mathbb{P}^n \setminus Z$, the same is true for the epimorphism

$$(E') \quad H_{n+c-2}(\mathcal{L} \cap (\mathbb{P}^n \setminus Z)) \to H_{n+c-2}(\mathbb{P}^n \setminus Z)$$

with $n + c - 2$ in place of $d - 1$. Combined with the Hyperplane Section Theorem quoted above, this gives a natural isomorphism

$$(1) \quad H_{d-1}(\mathcal{L} \cap X) / \sum_i \text{Im} \text{var}_{i,d-1} \sim H_{d-1}(X)$$

and, in the case of a complement,

$$(1') \quad H_{n+c-2}(\mathcal{L} \cap (\mathbb{P}^n \setminus Z)) / \sum_i \text{Im} \text{var}_{i,n+c-2} \sim H_{n+c-2}(\mathbb{P}^n \setminus Z)$$

(which coincides with a special case of (1) when $c = 1$).

This is different from the classical point of view of the Second Lefschetz Theorem (cf. [Lef], [Wa] and [AF]) where the kernel is expressed in terms of “vanishing cycles”, that is $(d - 1)$-cycles of $\mathcal{L} \cap X$ which vanish when $\mathcal{L}$ tends to some $\mathcal{L}_i$. But the classical theorem applies only to the case where $X$ is non-singular closed (i.e., $Z = \emptyset$) while formula (1) applies to both the closed and non-closed cases, provided $X$ is non-singular. Thus isomorphisms (1) and (1’) are generalizations of the Second Lefschetz Theorem.
But isomorphisms (1) and (1') may also be considered as generalized homological forms of the classical Zariski-van Kampen theorem on curves. Recall that this theorem expresses the fundamental group of the complement of an algebraic curve in $\mathbb{P}^2$ by generators and relations. The generators are loops in a generic line, around its intersection points with the curve. The relations are obtained by considering a generic pencil containing this line: each loop must be identified with its transforms by monodromy around the exceptional members of the pencil (cf. [Za], [vK] and [C1]).

The first true (homotopical) generalization of van Kampen’s theorem to higher dimensions was given by Libgober (cf. [Li]). It applies to the $(n-1)$-st homotopy group of the complement of a hypersurface with isolated singularities in $\mathbb{C}^n$ behaving well at infinity. In this case, if $n \geq 3$, the fundamental group is $\mathbb{Z}$ and the $(n-1)$-st homotopy group is the first higher non-trivial one as explained in [Li], and at the same time the first not preserved by hyperplane section. Libgober also showed how the projective case can be deduced from the affine case. The projective version of Libgober’s theorem can be stated in the previous frame as follows. With the notation above, assume that $Y = \mathbb{P}^n$ with $n \geq 3$, and $Z = H$ where $H$ is an algebraic hypersurface of $\mathbb{P}^n$ having only isolated singularities. Consider a generic pencil of hyperplanes of $\mathbb{P}^n$ as above. Let $* \in M \cap (\mathbb{P}^n \setminus H)$ be a base point in the axis of the pencil. For each $i$, we denote by

$$h_{i,q}: \pi_q(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *) \rightarrow \pi_q(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *), \text{ for all } q \geq 1,$$

the isomorphism induced by a geometrical monodromy around $\mathcal{L}_i$ leaving fixed the points of $M \cap (\mathbb{P}^n \setminus H)$. Then there is a natural isomorphism

$$\pi_{n-1}(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *)/\left(\text{Im } (h_{1,n-1} - \text{id}), \text{Im } D_{1,n-1}, \ldots, \text{Im } (h_{N,n-1} - \text{id}), \text{Im } D_{N,n-1}\right) \sim \pi_{n-1}(\mathbb{P}^n \setminus H, *),$$

(2)

where $N$ is the number of the exceptional hyperplanes and where each

$$D_{i,n-1}: \pi_{n-2}(\mathcal{L}_i \cap (\mathbb{P}^n \setminus H), *) \rightarrow \pi_{n-1}(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *)/\text{Im } (h_{i,n-1} - \text{id})$$

is some homomorphism called “degeneration operator”.

Later in [CL], Chéniot and Libgober showed that the combination (ordinary variations by monodromies - degeneration operators) of [Li] is related to some homotopical variation

$$\mathcal{V}AR_{i,n-1}: \pi_{n-1}(\mathcal{L} \cap (\mathbb{P}^n \setminus H), M \cap (\mathbb{P}^n \setminus H), *) \rightarrow \pi_{n-1}(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *)$$

defined from the homological variation $\text{var}_{i,n-1}$ of [C3] acting in a pencil supported by the universal covering of $\mathbb{P}^n \setminus H$. In particular, (2) yields a natural isomorphism

$$\pi_{n-1}(\mathcal{L} \cap (\mathbb{P}^n \setminus H), *)/\sum_i \text{Im } \mathcal{V}AR_{i,n-1} \sim \pi_{n-1}(\mathbb{P}^n \setminus H, *).$$

(3)
Isomorphism (3) can also be deduced from (1) applied to the universal covering of \( \mathbb{P}^n \setminus H \) (cf. [CL]).

Isomorphism (3) provides a homotopy analogue of isomorphisms (1) and (1') in the special case of complements of projective hypersurfaces with isolated singularities. Thus, this high-dimensional Zariski-van Kampen theorem is also a homotopy version of the Second Lefschetz Theorem for this special case. But the definition of homotopical variation operators in [CL] (as well as the definition of degeneration operators in [Li]) relies heavily upon the special topology of complements of hypersurfaces with isolated singularities and does not make sense in a more general setting. The need then arises for an alternative definition which could lead to a more general homotopy analogue of isomorphisms (1) and (1'). It must be said that in a more general situation, the considered homotopy group may not be the first higher non-trivial one but it remains the first not preserved by generic hyperplane section. The aim of this article is to give such an alternative definition and to state a reasonable conjecture generalizing (3) (and (2)) with a first approach toward its proof.

We introduce here new homotopical variation operators,

\[ \text{VAR}_{i,q}: \pi_q(L \cap X, M \cap X, \ast) \to \pi_q(L \cap X, \ast), \quad \text{for all } q \geq 1, \]

which extend those of Chéniot-Libgober and have several advantages with respect to them. Our definition is purely homotopical, that is, it does not go through homology. This frees it from the special topology of the case considered by Chéniot and Libgober, which was precisely required to express homotopy groups with the help of homology groups. In fact our definition is valid for any (singular) quasi-projective variety \( X := Y \setminus Z \). Our operators coincide with those of Chéniot-Libgober when the latter are defined. More precisely, we show that, when \( X := \mathbb{P}^n \setminus H \) and \( q = n - 1 \), with \( n \geq 3 \), and \( H \) is an algebraic hypersurface having only isolated singularities, then \( \text{VAR}_{i,n-1} = \text{VAR}_{i,n-1} \). In particular, isomorphism (3) is valid with \( \text{VAR}_{i,n-1} \) in place of \( \text{VAR}_{i,n-1} \).

We also show that our homotopical operators are linked by Hurewicz homomorphisms with the homological operators \( \text{var}_{i,q} \) of [C3] and that they are equivariant under the action of the fundamental group.

It is then natural to ask whether isomorphism (1) when \( X \) is non-singular and isomorphism (1') are true for homotopy groups, with our homotopical variation operators instead of the homological ones. We show easily that the kernel of the homotopy analogue of epimorphism \( (E) \) (resp. \( (E') \)) contains the images of operators \( \text{VAR}_{i,d-1} \) (resp. \( \text{VAR}_{i,n+c-2} \)). Thus, there are well defined epimorphisms

\[
\pi_d(X, \ast) / \sum_i \text{Im VAR}_{i,d-1} \to \pi_d(X, \ast) \quad \text{if } d \geq 3, \\
\pi_1(X, \ast) / \bigcup_i \text{Im VAR}_{i,1} \to \pi_1(X, \ast) \quad \text{if } d = 2,
\]

where \( \bigcup_i \text{Im VAR}_{i,1} \) denotes the normal subgroup generated by \( \bigcup_i \text{Im VAR}_{i,1} \).
and a well-defined epimorphism

\[(4') \quad \pi_{n+c-2}(L \cap (\mathbb{P}^n \setminus Z), \ast) / \sum \text{Im} \text{VAR}_{i,n+c-2} \to \pi_{n+c-2}(\mathbb{P}^n \setminus Z, \ast)\]

when \( n + c \geq 4 \) (if \( c = 1 \), \( 4' \) is only a special case of \( 4 \)). The question is whether these are isomorphisms. Notice that when \( n \geq 3 \) and \( X = \mathbb{P}^n \setminus H \) is the complement of a hypersurface \( H \) with isolated singularities, \( 4 \) and \( 4' \) coincide with isomorphism \( 3 \) since our operators extend those of Chéniot-Libgober. Also, when \( n = 2 \) and \( X := \mathbb{P}^2 \setminus C \) is the complement of a plane curve, the second row of \( 4 \) coincides with the map that van Kampen’s theorem asserts to be an isomorphism, as we shall see in Section 4.

We conjecture that the answer to our question is almost always yes:

**Conjecture.** Epimorphism \( 4 \), where \( X \) is non-singular, and epimorphism \( 4' \), unless \( Z = \emptyset \), are isomorphisms.

We are far from having a proof of this conjecture. We shall only give a very small step in its direction in the last section.

A detailed exposition from the origins to nowadays of the questions mentioned in this introduction, like the Lefschetz Hyperplane Section Theorem, the Second Lefschetz Theorem or the van Kampen Theorem on curves, can be found in [E3].

The content of this article is as follows. Section 1 is devoted to some basic facts on generic pencils and monodromies. In Section 2, we recall the definition of the homological variation operators \( \text{var}_{i,q} \) of [C3] which are used to define the homotopical variation operators \( \text{VAR}_{i,n-1} \) of [CL]. The latter will be described in Section 3. In Sections 4 and 5, we introduce the generalized homotopical variation operators \( \text{VAR}_{i,q} \), we give their elementary properties and we prove that they coincide with the Chéniot-Libgober operators when the latter are defined. We also prove that there are epimorphisms \( 4 \) and \( 4' \) as stated above. Finally, in Section 6, we give a first approach to the conjecture we have formulated.

**Notation 0.1.** Throughout the paper, homology groups are singular homology groups with integer coefficients, unless there is an explicit statement of the contrary. We shall note the homology class in a space \( A \) of an (absolute) cycle \( z \) by \([z]_A\) and the homology class in \( A \) modulo a subspace \( B \) of a relative cycle \([z']_{A,B}\). If there is no ambiguity, we shall omit the subscripts. If \((A,B)\) is a pointed pair with base point \(* \in B\), we shall denote by \( F^q(A,B,*) \) the set of relative homotopy \( q \)-cells of \( A \) modulo \( B \) based at \(*\). These are maps from the \( q \)-cube \( I^q \) to \( A \) with the face \( x_q = 0 \) sent into \( B \) and all other faces sent to \(*\) (as in [St, §15]). We designate by \( F^q(A,*) \) the set of absolute homotopy \( q \)-cells of \( A \) based at \(*\), that is maps from \( I^q \) to \( A \) sending the boundary \( \partial I^q \) of \( I^q \) to \(*\). Given \( f \in F^q(A,B,*) \) (resp. \( F^q(A,*) \)), the homotopy class of \( f \) in \( A \) modulo \( B \) based at \(*\) (resp. in \( A \) based at \(*\)) will be denoted by \([f]_{A,B,*}\) (resp. \([f]_{A,*}\)). Again, if there is no ambiguity, we shall omit the subscripts.
1. Generic pencils and monodromies

Let $X := Y \setminus Z$, where $Y$ is a non-empty closed algebraic subset of $\mathbb{P}^n$, with $n \geq 2$, and $Z$ a proper closed algebraic subset of $Y$. Take a Whitney stratification $\mathcal{S}$ of $Y$ such that $Z$ is a union of strata (cf. [Wh], [LT]), and consider a projective hyperplane $\mathcal{L}$ of $\mathbb{P}^n$ transverse to (the strata of) $\mathcal{S}$ (the choice of such a hyperplane is generic). Denote by $d$ the least dimension of the irreducible components of $Y$ not contained in $Z$.

Notice that, when $X$ is non-singular, the application of the Lefschetz Hyperplane Section Theorems mentioned in the introduction is valid for $\mathcal{L}$. Indeed, the genericity of the hyperplane required for these theorems can be specified as its transversality to a Whitney stratification of $Z$ (cf. [HL2, Appendix], [GM2, end of the proof of II.5.1], [C2, Corollaire 1.2]).

Now, consider a pencil $\mathcal{P}$ of hyperplanes of $\mathbb{P}^n$ having $\mathcal{L}$ as a member and the axis $\mathcal{M}$ of which is transverse to $\mathcal{S}$ (the choice of such an axis is generic inside $\mathcal{L}$). All the members of $\mathcal{P}$ are transverse to $\mathcal{S}$ with the exception of a finite number of them $(\mathcal{L}_i)_i$, called exceptional hyperplanes, for which, nevertheless, there are only a finite number of points of non-transversality, all of them situated outside of $\mathcal{M}$ (cf. [C2]). If necessary, one may take the liberty of considering some ordinary members of $\mathcal{P}$, different from $\mathcal{L}$, as exceptional ones.

We parametrize the elements of $\mathcal{P}$ by the complex projective line $\mathbb{P}^1$ as usual. Let $\lambda$ be the parameter of $\mathcal{L}$ and, for each $i$, let $\lambda_i$ be the parameter of $\mathcal{L}_i$. For each $i$, take a small closed semi-analytic disk $D_i \subset \mathbb{P}^1$ with centre $\lambda_i$ together with a point $\gamma_i$ on its boundary. Choose the $D_i$ mutually disjoint. Take also the image $\Gamma_i$ of a simple real-analytic arc in $\mathbb{P}^1$ joining $\lambda$ to $\gamma_i$ and such that: (i) $\Gamma_i \cap D_i = \{\gamma_i\}$; (ii) $\Gamma_i \cap \Gamma_{i'} = \{\lambda\}$ if $i \neq i'$; (iii) $\Gamma_i \cap D_{i'} = \emptyset$ if $i \neq i'$. Then, set $K_i := \Gamma_i \cup D_i$.

Finally, consider a loop $\omega_i$ in the boundary $\partial K_i$ of $K_i$ starting from $\lambda$, running along $\Gamma_i$ up to $\gamma_i$, going once counter-clockwise around the boundary of $D_i$ and coming along $\Gamma_i$ back to $\lambda$.

**Notation 1.1.** – For any subsets $G \subset \mathbb{P}^n$ and $E \subset \mathbb{P}^1$, note

$$G_E := \bigcup_{\mu \in E} G \cap \mathcal{P} (\mu),$$

where $\mathcal{P} (\mu)$ is the member of $\mathcal{P}$ with parameter $\mu$.

The monodromies around each $\mathcal{L}_i$, more precisely above each $\omega_i$, proceed from the following lemma.

**Lemma 1.2 (cf. [C3, Lemma 4.1]).** – For each $i$, there is an isotopy $H: (\mathcal{L} \cap X) \times I \to X_{\partial K_i}$ such that:




(i) \( H(x, 0) = x, \) for every \( x \in \mathcal{L} \cap X; \)

(ii) \( H(x, t) \in X_{\omega_i(t)} \) for every \( x \in \mathcal{L} \cap X \) and every \( t \in I; \)

(iii) for every \( t \in I, \) the map \( \mathcal{L} \cap X \to X_{\omega_i(t)}, \) defined by \( x \mapsto H(x, t), \) is a homeomorphism;

(iv) \( H(x, t) = x, \) for every \( x \in \mathcal{M} \cap X \) and every \( t \in I. \)

As usual, \( I \) is the unit interval \([0, 1].\)

The terminal homeomorphism

\[
  h: \mathcal{L} \cap X \to \mathcal{L} \cap X
\]

of \( H, \) defined by \( h(x) := H(x, 1), \) of course leaves \( \mathcal{M} \cap X \) pointwise fixed. Such a homeomorphism \( h \) is called a geometric monodromy of \( \mathcal{L} \cap X \) relative to \( \mathcal{M} \cap X \) above \( \omega_i. \)

Another choice of loop \( \omega_i \) within the same homotopy class \( \overline{\omega}_i \) in \( \mathbb{P}^1 \setminus \bigcup_i \lambda_i \) and another choice of isotopy \( H \) above \( \omega_i \) as in Lemma 1.2 would give a geometric monodromy isotopic to \( h \) within \( \mathcal{L} \cap X \) by an isotopy leaving \( \mathcal{M} \cap X \) pointwise fixed. Thus, the isotopy class of \( h \) in \( \mathcal{L} \cap X \) relative to \( \mathcal{M} \cap X \) is wholly determined by the homotopy class \( \overline{\omega}_i \) of \( \omega_i \) in \( \mathbb{P}^1 \setminus \bigcup_i \lambda_i. \)

2. Homological variation operators

Fix an index \( i, \) and consider a geometric monodromy \( h \) of \( \mathcal{L} \cap X \) relative to \( \mathcal{M} \cap X \) above \( \omega_i. \) Denote by \( S_q(\mathcal{L} \cap X) \) the abelian group of singular \( q \)-chains of \( \mathcal{L} \cap X \) with integer coefficients, and by \( h_q: S_q(\mathcal{L} \cap X) \to S_q(\mathcal{L} \cap X) \) the chain homomorphism induced by \( h. \) Since \( h \) leaves \( \mathcal{M} \cap X \) pointwise fixed (cf. Lemma 1.2), it is easy to see that for every relative \( q \)-cycle \( z \) of \( \mathcal{L} \cap X \) modulo \( \mathcal{M} \cap X, \) the variation by \( h_q \) of \( z, \) that is the chain \( h_q(z) - z, \) is an absolute \( q \)-cycle of \( \mathcal{L} \cap X \) (cf. [C3, Lemma 4.6]). Moreover, one has the following lemma.

**Lemma 2.1** (cf. [C3, Lemma 4.8]). – The correspondence

\[
  \text{var}_{i,q}: H_q(\mathcal{L} \cap X, \mathcal{M} \cap X) \to H_q(\mathcal{L} \cap X)
  \]

\[
  [z]_{\mathcal{L} \cap X, \mathcal{M} \cap X} \mapsto [h_q(z) - z]_{\mathcal{L} \cap X}
\]

gives a well-defined homomorphism which depends only on the homotopy class \( \overline{\omega}_i \) of \( \omega_i \) in \( \mathbb{P}^1 \setminus \bigcup_i \lambda_i. \)

This means that another choice of loop \( \omega_i \) within the same homotopy class \( \overline{\omega}_i \) in \( \mathbb{P}^1 \setminus \bigcup_i \lambda_i \) and another choice of monodromy \( h \) above \( \omega_i \) as in Lemma 1.2 would give a homomorphism equal to \( \text{var}_{i,q}. \)

Homomorphism \( \text{var}_{i,q} \) is called a homological variation operator associated to \( \overline{\omega}_i. \)

These operators were used by the first author in [C3] to prove the following result.
Theorem 2.2 (cf. [C3, Proposition 5.1]). – If $X$ is non-singular, then there is a natural isomorphism

$$H_{d-1}(\mathcal{L} \cap X) \bigg/ \sum_i \text{Im } \text{var}_{i,d-1} \xrightarrow{\sim} H_{d-1}(X).$$

In the special case of the complement of a projective algebraic set, i.e., if $X := \mathbb{P}^n \setminus Z$, this can be improved into an isomorphism

$$H_{n+c-2}(\mathcal{L} \cap X) \bigg/ \sum_i \text{Im } \text{var}_{i,n+c-2} \xrightarrow{\sim} H_{n+c-2}(X),$$

where $c$ is the least of the codimensions of the irreducible components of $Z$.

3. Homotopical variation operators on the complements of projective hypersurfaces with isolated singularities

Throughout this section, we work under the following hypotheses.

Hypotheses 3.1. – We assume that $Y = \mathbb{P}^n$, with $n \geq 3$, and $Z = H$, where $H$ is a (closed) algebraic hypersurface of $\mathbb{P}^n$, with degree $k$, having only isolated singularities. Thus, $X = \mathbb{P}^n \setminus H$. We also assume that $S$ is the Whitney stratification the strata of which are: $\mathbb{P}^n \setminus H$, the singular part $H_{\text{sing}}$ of $H$, and the non-singular part $H \setminus H_{\text{sing}}$ of $H$. Being transverse to $S$ then means avoiding the singularities of $H$ and being transverse to the non-singular part of $H$. Observe that $\mathcal{M} \cap X \neq \emptyset$. We fix a base point $* \in \mathcal{M} \cap X$.

In [CL], a $k$-fold (unramified) holomorphic covering

$$p: X' \to X$$

is constructed, where $X' := Y' \setminus Z'$ is a (pathwise) connected quasi-projective variety in $\mathbb{P}^{n+1}$. In fact, $X'$ is the global Milnor fibre of the cone of $\mathbb{C}^{n+1}$ corresponding to $H$. Moreover, it is shown that there is a Whitney stratification $S'$ of $Y'$ preserving $Z'$ and a pencil $\mathcal{P}'$ in $\mathbb{P}^{n+1}$ with axis $\mathcal{M}'$ transverse to $S'$ such that

$$p^{-1}(\mathcal{M} \cap X) = \mathcal{M}' \cap X'$$

and

$$p^{-1}(\mathcal{P}(\mu) \cap X) = \mathcal{P}'(\mu) \cap X'$$

for every $\mu \in \mathbb{P}^1$, the member $\mathcal{P}'(\mu)$ of $\mathcal{P}'$ with parameter $\mu$ being transverse to $S'$ if and only if $\mathcal{P}(\mu)$ is transverse to $S$. Recall that $\mathcal{L} = \mathcal{P}(\lambda)$, and put $\mathcal{L}' := \mathcal{P}'(\lambda)$. Then, for each $i$, pencil $\mathcal{P}'$ gives rise to a homological variation operator

$$\text{var}_{i,n-1}': H_{n-1}(\mathcal{L}' \cap X', \mathcal{M}' \cap X') \to H_{n-1}(\mathcal{L}' \cap X')$$

associated to $\bar{\omega}_i$, defined as in section 2.
Given an index $i$ and a base point $\bullet \in p^{-1}(\ast)$, one can then consider the following diagram:

$$
\begin{array}{ccc}
H_{n-1}(L' \cap X', M' \cap X') & \xrightarrow{\text{var}'_{i,n-1}} & H_{n-1}(L' \cap X') \\
\chi & & \hat{\chi}
\end{array}
$$

(3.2)

$$
\begin{array}{ccc}
\pi_{n-1}(L' \cap X', M' \cap X', \bullet) & \xrightarrow{\tilde{\pi}} & \pi_{n-1}(L' \cap X', \bullet) \\
\pi & & \\
\pi_{n-1}(L \cap X, M \cap X, \ast) & \xrightarrow{\pi} & \pi_{n-1}(L \cap X, \ast),
\end{array}
$$

where $\tilde{\pi}$ and $\pi$ are induced by $p$ and where $\chi$ and $\hat{\chi}$ are Hurewicz homomorphisms. Now, by a general property of covering projections, $\tilde{\pi}$ is an isomorphism (cf. [Sp, Theorem 7.2.8]). Moreover, homomorphism $\hat{\chi}$ too is an isomorphism as a consequence of the special fact that $X$ is the complement of a projective hypersurface $H$ with isolated singularities. Indeed, $L \cap H$ is then a non-singular hypersurface of $L \simeq \mathbb{P}^{n-1}$, so that

$$
\pi_1(L \cap X, \ast) \simeq \mathbb{Z}/k\mathbb{Z} \quad \text{and} \quad \pi_q(L \cap X, \ast) = 0 \quad \text{for} \quad 2 \leq q \leq n - 2
$$

(this range may be empty) (cf. [Li, Lemma 1.1]). Knowing that $L' \cap X'$ is pathwise connected (cf. [CL, Lemma 2.9]), these facts imply that $L' \cap X'$ is $(n - 2)$-connected, and $\hat{\chi}$ is then an isomorphism by the Hurewicz Isomorphism Theorem.

Thus, for each $i$, and for every $\bullet \in p^{-1}(\ast)$, there is a homomorphism

$$
\mathcal{VAR}_{i,n-1}: \pi_{n-1}(L \cap X, M \cap X, \ast) \to \pi_{n-1}(L \cap X, \ast)
$$

defined by the composition

$$
\pi \circ \hat{\chi}^{-1} \circ \text{var}'_{i,n-1} \circ \chi \circ \tilde{\pi}^{-1}
$$

in diagram (3.2) (cf. [CL, Section 5]).

One shows easily that homomorphism $\mathcal{VAR}_{i,n-1}$ does not in fact depend on the choice of the base point $\bullet \in p^{-1}(\ast)$.

Homomorphism $\mathcal{VAR}_{i,n-1}$ is called a homotopical variation operator associated to $\bar{\omega}_i$.

These operators were used in [CL] to prove the following result.

**Theorem 3.3** (cf. [CL, Theorem 7.1]). Under Hypotheses 3.1, there is a natural isomorphism

$$
\pi_{n-1}(L \cap X, \ast) / \sum_i \text{Im} \mathcal{VAR}_{i,n-1} \sim \pi_{n-1}(X, \ast).
$$
4. Generalized homotopical variation operators

In this section, \(X := Y \setminus Z\) is again a (possibly singular) quasi-projective variety as in Sections 1 and 2. We assume further that \(\mathcal{M} \cap X \neq \emptyset\) and we fix a base point \(*\) in \(\mathcal{M} \cap X\). Observe that the condition \(\mathcal{M} \cap X \neq \emptyset\) is equivalent to \(\dim X \geq 2\). We also fix an index \(i\), and consider a geometric monodromy \(h\) of \(L \cap X\) relative to \(\mathcal{M} \cap X\) above \(\omega_i\) (cf. Section 1).

Let \(q\) be an integer \(\geq 1\). Since \(h\) leaves \(\mathcal{M} \cap X\) pointwise fixed, if \(f \in F^q(L \cap X, \mathcal{M} \cap X, *)\) (cf. Notation 0.1), then the map \(f \perp (h \circ f)\) defined on \(I^q := [0,1]^q\) by

\[
\begin{align*}
f \perp (h \circ f)(x_1, \ldots, x_q) := \begin{cases} f(x_1, \ldots, x_{q-1}, 1 - 2x_q), & 0 \leq x_q \leq \frac{1}{2}, \\ h \circ f(x_1, \ldots, x_{q-1}, 2x_q - 1), & \frac{1}{2} \leq x_q \leq 1,
\end{cases}
\end{align*}
\]

is in \(F^q(L \cap X, *)\). Notice that the reversion of \(f\) and its concatenation with \(h \circ f\) are performed on the variable transverse to the free face. This would in general not make sense but here it does because \(f\) and \(h \circ f\) have the same boundary.

**Lemma 4.1.** – The correspondence

\(\text{VAR}_{i,q}: \pi_q(L \cap X, \mathcal{M} \cap X, *) \rightarrow \pi_q(L \cap X, *)\)

\((f)_{L \cap X, \mathcal{M} \cap X, *} \mapsto (f \perp (h \circ f))_{L \cap X, *}\)

gives a well-defined map which depends only on the homotopy class \(\tilde{\omega}_i\) of \(\omega_i\) in \(\mathbb{P}^1 \setminus \bigcup_i \lambda_i\). If \(q \geq 2\), it is a homomorphism.

The independence assertion follows from the remark we made just after Lemma 1.2. That the map is well-defined is straightforward and one checks easily that it is a homomorphism if \(q \geq 2\) (the sum of homotopy cells being performed as in [St, §15]).

We shall call map \(\text{VAR}_{i,q}\) a **generalized homotopical variation operator associated to \(\tilde{\omega}_i\)**. This terminology is justified by Theorem 5.1 below which asserts that, in the case where the homotopical variation operators of [CL] are defined (cf. Section 3), the latter coincide with our generalized operators.

We remark that if our operators are applied to absolute cells of \(L \cap X\) or if their result is considered as relative cells of \(L \cap X\) modulo \(\mathcal{M} \cap X\), then they act as what can be called ordinary variations by monodromy. More precisely:

**Observation 4.2.** – Let \(\text{incl}_q: \pi_q(L \cap X, *) \rightarrow \pi_q(L \cap X, \mathcal{M} \cap X, *)\) be the natural map. Then,

(i) \(\text{VAR}_{i,q}(\text{incl}_q(x)) = -x + h_q(x)\) for all \(x \in \pi_q(L \cap X, *)\);
if $q \geq 2$, $\text{incl}_q(\text{VAR}_{i,q}(y)) = -y + \tilde{h}_q(y)$ for all $y \in \pi_q(L \cap X, M \cap X, \ast)$; where $h_q$ and $\tilde{h}_q$ are the automorphisms of $\pi_q(L \cap X, \ast)$ and $\pi_q(L \cap X, M \cap X, \ast)$ respectively induced by $h$.

The right-hand sides of the equalities are written additively though the first group is not a priori commutative if $q = 1$ nor is the second one if $q = 2$; the order of operations must then be respected.

Observation 4.2 relies on the same reasons as those which allow the sum of two homotopy cells to be performed indiscriminately on any variable not transverse to the (possible) free face and which make the sum commutative in high dimension. Its detailed proof is left to the reader.

Notice that when $n = 2$ and $X$ is the complement $\mathbb{P}^2 \setminus C$ of a plane projective curve, $M \cap X$ is reduced to a single point and the observation above shows that the epimorphism (4), second row, of the introduction (the existence of which will be justified by Lemma 4.8 below) coincides with the map that van Kampen’s theorem asserts to be an isomorphism.

Operator $\text{VAR}_{i,q}$ is linked to the homological variation operator $\text{var}_{i,q}$ of Section 2 by Hurewicz homomorphisms. This is stated in the next lemma.

**Lemma 4.3.** – The following diagram is commutative:

$$
\begin{array}{ccc}
H_q(L \cap X, M \cap X) & \xrightarrow{\text{var}_{i,q}} & H_q(L \cap X) \\
\chi \uparrow & & \hat{\chi} \uparrow \\
\pi_q(L \cap X, M \cap X, \ast) & \xrightarrow{\text{VAR}_{i,q}} & \pi_q(L \cap X, \ast),
\end{array}
$$

where $\chi$ and $\hat{\chi}$ are Hurewicz homomorphisms.

**Proof.** Since homotopy cells are defined on cubes, it is convenient to use cubical singular homology theory (cf. [HW], [M]), which is equivalent to ordinary (simplicial) singular theory (cf. [HW, Section 8.4]). So, let us first introduce some notation. For any pair of spaces $(U, V)$, with $V \subset U$, we shall denote by $H_q^c(U, V)$ the $q$-th cubical singular relative homology group of $(U, V)$, by $[\cdot]_U^c$ the homology classes in this group, and by $S_q^c(U, V)$ the group of $q$-dimensional cubical chains of the pair $(U, V)$. Given a (continuous) map $g: (U, V) \to (U', V')$, we shall denote by $g_q: S_q^c(U, V) \to S_q^c(U', V')$ the induced cubical chain homomorphism. A similar notation is used for the absolute case.

Let $f$ be a representative of an element of $\pi_q(L \cap X, M \cap X, \ast)$. We have

$$
\begin{align*}
\chi((f) \varepsilon_{L \cap X, M \cap X, \ast}) &= [f_q(\iota)]_{L \cap X, M \cap X}^c \\
\hat{\chi}((f \perp (h \circ f)) \varepsilon_{L \cap X, \ast}) &= [(f \perp (h \circ f))_q(\iota)]_{L \cap X}^c,
\end{align*}
$$

where $\iota: I^q \to I^q$ is the identity map (cf. [HW, 8.8.4]). Since the expression for $\text{var}_{i,q}$ (given by Lemma 2.1) remains valid in cubical theory (by [HW, 8.4.7]...
and the paragraph before 8.4.10], it then suffices to prove that the following equality holds in $H_\mathcal{L}^c(\mathcal{L} \cap X)$:

\begin{equation}
[h_q(f_q(t)) - f_q(t)]^c_{\mathcal{L} \cap X} = [(f \perp (h \circ f))_q(t)]^c_{\mathcal{L} \cap X}.
\end{equation}

For this purpose, consider the singular $q$-cubes $\sigma_1, \sigma_2$: $I^q \to I^q$ in $I^q$ defined by

\[
\begin{align*}
\sigma_1(x_1, \ldots, x_q) &:= (x_1, \ldots, x_{q-1}, \frac{1-x_q}{2}), \\
\sigma_2(x_1, \ldots, x_q) &:= (x_1, \ldots, x_{q-1}, \frac{1+x_q}{2}).
\end{align*}
\]

Observe that $-\sigma_1 + \sigma_2$ is a relative cycle of $I^q$ modulo $\hat{I}^q$.

**Lemma 4.5.** The following equality holds in $H_\mathcal{L}^c(I^q, \hat{I}^q)$:

\[
[-\sigma_1 + \sigma_2]^c_{I^q, \hat{I}^q} = [\iota]^c_{I^q, \hat{I}^q}.
\]

**Proof.** Let $\sigma$: $I^{q+1} \to I^q$ be the singular $(q+1)$-cube in $I^q$ defined by

\[
\sigma(x_1, \ldots, x_{q+1}) := \left\{ \begin{array}{ll}
(x_1, \ldots, x_{q-1}, \frac{2x_{q+1}+x_q-1}{2}), & x_{q+1} \geq -\frac{x_q}{2} + 1, \\
(x_1, \ldots, x_{q-1}, \frac{1}{2}), & \frac{x_q}{2} \leq x_{q+1} \leq -\frac{x_q}{2} + 1, \\
(x_1, \ldots, x_{q-1}, \frac{1-x_q+2x_{q+1}}{2}), & x_{q+1} \leq \frac{x_q}{2}.
\end{array} \right.
\]

The boundary operator

\[
\partial: S_{q+1}^c(I^q, \hat{I}^q) \to S_q^c(I^q, \hat{I}^q),
\]

applied to $\sigma$, satisfies

\[
\partial \sigma = (-1)^{q+1} \sigma_1 - (-1)^q \sigma_2 - (-1)^q \iota.
\]

Indeed, the face of $\sigma$ of index $(q, 0)$ is degenerated and all other non-mentioned faces are in $\hat{I}^q$. The equality in the statement of Lemma 4.5 follows.

Now, since $f \perp (h \circ f)$ maps $(I^q, \hat{I}^q)$ into $(\mathcal{L} \cap X, *)$, this lemma implies

\[
[(f \perp (h \circ f))_q(t)]^c_{\mathcal{L} \cap X} = [(f \perp (h \circ f))_q(-\sigma_1 + \sigma_2)]^c_{\mathcal{L} \cap X},
\]

and since

\[
(f \perp (h \circ f)) \circ \sigma_1 = f \quad \text{and} \quad (f \perp (h \circ f)) \circ \sigma_2 = h \circ f,
\]

one sees immediately that

\[
[(f \perp (h \circ f))_q(-\sigma_1 + \sigma_2)]^c_{\mathcal{L} \cap X} = [(h \circ f)_q(t) - f_q(t)]^c_{\mathcal{L} \cap X}.
\]

This completes the proof of (4.4) and, consequently, the proof of Lemma 4.3.
Operator \( \text{VAR}_{i,q} \) also satisfies the following equivariance property.

**Lemma 4.6.** – If \( \gamma \in F^1(\mathcal{M} \cap X, *) \) and \( f \in F^q(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \), then
\[
\text{VAR}_{i,q}(\langle \gamma \rangle_{\mathcal{M} \cap X, *} \cdot \langle f \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *}) = \langle \gamma \rangle_{\mathcal{L} \cap X, *} \cdot \text{VAR}_{i,q}(\langle f \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *})
\]
where \( \cdot \) denotes equally the action of \( \pi_1(\mathcal{M} \cap X, *) \) on \( \pi_q(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \) or the action of \( \pi_1(\mathcal{L} \cap X, *) \) on \( \pi_q(\mathcal{L} \cap X, *) \).

**Proof.** Let \( \gamma^- \) be the inverse loop of \( \gamma \) and
\[
K_{\gamma^-} : (I^q, \hat{I}^q) \times I \to (\mathcal{L} \cap X, \mathcal{M} \cap X)
\]
be a \( \gamma^- \)-homotopy starting at \( f \) (i.e., \( K_{\gamma^-}(x, 0) = f(x) \) for every \( x \in I^q \), \( K_{\gamma^-}(x, t) = \gamma^-(t) \) for every \( x \in \hat{I}^q \) \( \backslash \{x_q = 0\} \) and every \( t \in I \), and \( K_{\gamma^-}(x, t) \) is \( \mathcal{M} \cap X \) for every \( x \in \{x_q = 0\} \) and every \( t \in I \)). Denote by \( g \) the element of \( F^q(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \) defined by \( g(x) := K_{\gamma^-}(x, 1) \). One has
\[
\langle \gamma \rangle_{\mathcal{L} \cap X, *} \cdot \langle f \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *}, \quad \langle g \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *},
\]
Since \( h \) leaves \( \mathcal{M} \cap X \) pointwise fixed, the map
\[
K_{\gamma^-} : (I^q, \hat{I}^q) \times I \to (\mathcal{L} \cap X, \mathcal{M} \cap X)
\]
defined by
\[
K_{\gamma^-}((x_1, \ldots, x_q), t) := \begin{cases} K_{\gamma^-}((x_1, \ldots, x_{q-1}, 1 - 2x_q), t), & 0 \leq x_q \leq \frac{1}{2}, \\ h \circ K_{\gamma^-}((x_1, \ldots, x_{q-1}, 2x_q - 1), t), & \frac{1}{2} \leq x_q \leq 1, \end{cases}
\]
is a \( \gamma^- \)-homotopy from \( f \perp (h \circ f) \) to \( g \perp (h \circ g) \) such that \( K_{\gamma^-}(x, t) = \gamma^-(t) \) for every \( (x, t) \in \hat{I}^q \times I \). In other words,
\[
\langle \gamma \rangle_{\mathcal{L} \cap X, *} \cdot \text{VAR}_{i,q}(\langle f \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *}) = \text{VAR}_{i,q}(\langle g \rangle_{\mathcal{L} \cap X, \mathcal{M} \cap X, *}).
\]
Lemma 4.6 then follows from (4.7).

Finally we prove a lemma which justifies the existence of the epimorphisms (4) and \((4')\) of the introduction. But this lemma is valid for every \( q \geq 1 \) and even if \( X \) is singular.

**Lemma 4.8.** – The image of operator \( \text{VAR}_{i,q} \) is contained in the kernel of the natural map \( \pi_q(\mathcal{L} \cap X, *) \to \pi_q(X, *) \).

**Proof.** A representative of an element of the image of \( \text{VAR}_{i,q} \) is of the form \( f \perp (h \circ f) \) with \( f \in F^q(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \). Let \( H \) be an isotopy giving rise to \( h \) as in Lemma 1.2. One defines a homotopy \( I^q \times I \to X \) from \( f \perp (h \circ f) \) to the constant map equal to * by
\[
((x_1, \ldots, x_q), t) \mapsto \begin{cases} H(f(x_1, \ldots, x_{q-1}, 1 - 2x_q), 2t), & 0 \leq t \leq \frac{1}{2}, \\ h \circ f(x_1, \ldots, x_{q-1}, 2x_q - 1), & 0 \leq t \leq \frac{1}{2}, \\ h \circ f(x_1, \ldots, x_{q-1}, 1 - 2x_q), & \frac{1}{2} \leq t \leq 1, \\ h \circ f(x_1, \ldots, x_{q-1}, 2t - 1), & \frac{1}{2} \leq t \leq 1, \\ h \circ f(x_1, \ldots, x_{q-1}, 2x_q - 1), & \frac{1}{2} \leq t \leq 1, \end{cases}
\]
for \( 0 \leq t \leq 1 \).
By the first half of this homotopy, the lower part of the cell undergoes the monodromy \( h \) while remaining attached to the upper part; at the end of this process the two half cells become opposite and the second half of the homotopy collapses them together.

**Remark.** – Since \( \operatorname{Im} H \subset X_{\partial K_i} \), the proof above shows in fact that the image of \( \operatorname{VAR}_{i,q} \) is contained in the kernel of the natural map \( \pi_q(\mathcal{L} \cap X, *) \to \pi_q(X_{\partial K_i}, *) \).

### 5. The Link Between \( \mathcal{VAR}_{i,n-1} \) and \( \operatorname{VAR}_{i,n-1} \) – Main Result

Throughout this section, we work under Hypotheses 3.1.

**Theorem 5.1.** – Under Hypotheses 3.1, the homotopical variation operator \( \mathcal{VAR}_{i,n-1} \) of Chéniot-Libgober which is then well-defined (cf. Section 3) coincides with the generalized homotopical variation operator \( \operatorname{VAR}_{i,n-1} \) (defined in Section 4).

Before proving this theorem, observe that, together with Theorem 3.3, it implies the following result.

**Theorem 5.2.** – Under Hypotheses 3.1, there is a natural isomorphism

\[
\pi_{n-1}(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \xrightarrow{\sum_i \operatorname{Im} \operatorname{VAR}_{i,n-1}} \pi_{n-1}(L \cap X, *).
\]

Of course, Theorem 5.2 is equivalent to Theorem 3.3 and to the projective version of [Li, Theorem 2.4].

We now turn to the proof of Theorem 5.1.

**Proof of Theorem 5.1.** Consider the diagram obtained from diagram (3.2) by completing its lower row with the homomorphism

\[
\pi_{n-1}(\mathcal{L} \cap X, \mathcal{M} \cap X, *) \xrightarrow{\operatorname{VAR}_{i,n-1}} \pi_{n-1}(\mathcal{L} \cap X, *)
\]

and its middle row with the homomorphism

\[
\pi_{n-1}(\mathcal{L}' \cap X', \mathcal{M}' \cap X', \bullet) \xrightarrow{\operatorname{VAR}'_{i,n-1}} \pi_{n-1}(\mathcal{L}' \cap X', \bullet)
\]

defined from \( \omega_i \) as \( \operatorname{VAR}_{i,n-1} \) but with pencil \( P' \) and the point \( \bullet \) instead of pencil \( P \) and the point \( * \). We have to show that this new diagram is commutative. But its lower square is indeed commutative since, given a geometric monodromy \( h \) of \( \mathcal{L} \cap X \) relative to \( \mathcal{M} \cap X \) above \( \omega_i \), there exists a geometric monodromy \( h' \) of \( \mathcal{L}' \cap X' \) relative to \( \mathcal{M}' \cap X' \) above \( \omega_i \) such that

\[ p \circ h' = h \circ p \]
(cf. [CL, Remark 4.2]). As to the upper square, it commutes thanks to Lemma 4.3.

6. A conjecture generalizing the van Kampen theorem to non-singular quasi-projective varieties

In this section, we come back to the general hypotheses of Section 1, so that $X := Y \setminus Z$ is again a (possibly singular) quasi-projective variety in $\mathbb{P}^n$ with $n \geq 2$ as in Sections 1, 2 and 4. We also assume that $M \cap X \neq \emptyset$; as this condition is equivalent to $\dim X \geq 2$, it will be automatically fulfilled when $d \geq 2$. We fix a base point $*$ in $M \cap X$.

The conjecture we made in the introduction can be specified as follows.

**Conjecture 6.1.** Under the hypotheses of Section 1 and if $X$ is non-singular, there are natural isomorphisms

\[
\pi_{d-1}(L \cap X,*)/\sum_i \text{Im VAR}_{i,d-1} \sim \pi_{d-1}(X,*), \quad \text{if } d \geq 3,
\]

\[
\pi_1(L \cap X,*)/\bigcup_i \text{Im VAR}_{i,1} \sim \pi_1(X,*) \quad \text{if } d = 2,
\]

involving the generalized homotopical variation operators $\text{VAR}_{i,q}$ defined in Section 4, with $\bigcup_i \text{Im VAR}_{i,1}$ denoting the normal subgroup of $\pi_1(L \cap X,*)$ generated by $\bigcup_i \text{Im VAR}_{i,1}$.

In the special case $Y = \mathbb{P}^n$, so that $X = \mathbb{P}^n \setminus Z$, and provided $Z \neq \emptyset$, there is a natural isomorphism

\[
\pi_{n+c-2}(L \cap X,*)/\sum_i \text{Im VAR}_{i,n+c-2} \sim \pi_{n+c-2}(X,*), \quad \text{if } n + c \geq 4,
\]

where $c$ is the least of the codimensions of the irreducible components of $Z$ (notice that $n + c - 2 = d - 1$ when $c = 1$).

If proved, this conjecture would extend Theorem 5.2 (and hence Theorem 3.3 and the projective version of [Li, Theorem 2.4] reported in the introduction as isomorphism (2)) and would also extend the classical Zariski-van Kampen theorem on curves as remarked in Section 4. It would give a complete homotopy analogue of Theorem 2.2 and thus would gather in a generalized form the Zariski-van Kampen theorem with a homotopical version of the Second Lefschetz Theorem.

We now give a first little approach to this conjecture.

**First approach to Conjecture 6.1.** By Lemma 4.8, the subgroups by which the quotients are taken are contained in the kernels of the corresponding natural maps (which are epimorphisms by the Lefschetz Hyperplane Section Theorems, as mentioned in the introduction). The reverse inclusion which would lead to the conclusion is much more difficult and not proved at the moment. We are simply giving below, via the following lemma, a first little step in this direction.
Lemma 6.2. – If $X$ is non-singular and $d \geq 2$, there is a natural epimorphism
\[ \pi_d(X_K, \mathcal{L} \cap X, *) \to \pi_d(X, \mathcal{L} \cap X, *), \]
where $K$ is the union of the $K_i$ (cf. Section 1).

In the special case where $Y = \mathbb{P}^n$ and $Z \neq \emptyset$, there is a natural epimorphism
\[ \pi_{n+c-1}(X_K, \mathcal{L} \cap X, *) \to \pi_{n+c-1}(X, \mathcal{L} \cap X, *), \]
where $c$ is as in Conjecture 6.1.

This lemma is a weak homotopical analogue of [C3, Corollary 3.4]. It shows that (with the same hypotheses of course) the kernels of the natural maps
\[
\begin{align*}
\pi_{d-1}(\mathcal{L} \cap X, *) & \to \pi_{d-1}(X, *) & \text{and} & \quad \pi_{d-1}(\mathcal{L} \cap X, *) & \to \pi_{d-1}(X_K, *) \\
\text{(resp.} & \quad \pi_{n+c-2}(\mathcal{L} \cap X, *) & \to \pi_{n+c-2}(X, *) & \text{and} & \quad \pi_{n+c-2}(\mathcal{L} \cap X, *) & \to \pi_{n+c-2}(X_K, *) \text{)}
\end{align*}
\]
are the same. So, with the remarks above, Conjecture 6.1 reduces to the following one.

Conjecture 6.3. – Under the hypotheses of Conjecture 6.1, the kernel of the natural map $\pi_{d-1}(\mathcal{L} \cap X, *) \to \pi_{d-1}(X_K, *)$ is contained in $\sum_i \text{Im } \text{VAR}_{i, d-1}$ if $d \geq 3$ and in $\bigcup_i \text{Im } \text{VAR}_{i, 1}$ if $d = 2$.

In the special case where $Y = \mathbb{P}^n$ and $Z \neq \emptyset$, the kernel of the natural map $\pi_{n+c-2}(\mathcal{L} \cap X, *) \to \pi_{n+c-2}(X_K, *)$ is contained in $\sum_i \text{Im } \text{VAR}_{i, n+c-2}$ if $n + c \geq 4$.

To complete this section, it remains to prove Lemma 6.2.

Proof of Lemma 6.2. By the homotopy sequence of the triple
\[(X, X_K, \mathcal{L} \cap X),\]
it suffices to prove that the pair $(X, X_K)$ is $d$-connected (resp. $(n+c-1)$-connected).

We start by noticing that the pair $(\mathcal{L} \cap X, \mathcal{M} \cap X)$ is $(d-2)$-connected (resp. $(n+c-3)$-connected). This is shown by applying the Lefschetz Hyperplane Section Theorem for non-singular quasi-projective varieties (resp. for complements) to the section of $\mathcal{L} \cap X$ by the hyperplane $\mathcal{M}$ of $\mathcal{L}$. To check the required hypotheses and verify that the conclusion is the announced one, we refer to the beginning of the proof of [C3, Corollary 3.4]. We just point out here that the hypothesis $Z \neq \emptyset$ is crucial to ensure that the codimension of $\mathcal{L} \cap Z$ in $\mathcal{L}$ is still $c$.

Thus, to show that $(X, X_K)$ is $d$-connected (resp. $(n+c-1)$-connected), it is enough to prove the following result which in fact holds even if $X$ is singular.
Lemma 6.4. – For this lemma, \( X \) may be singular. Let \( k \) be an integer \( \geq 0 \). If \((L \cap X, M \cap X)\) is \( k \)-connected, then \((X, X_K)\) is \((k + 2)\)-connected.

This is a weak homotopy analogue of [C3, Lemma 3.9]. In its proof, the homology excision property is replaced by a much more restrictive homotopy excision theorem, and the Eilenberg-Zilber theorem and Künneth formula by a criterion on the connectivity of the product of two pairs of spaces.

Proof of Lemma 6.4. Let \( \widetilde{P}^n \) be the blow up of \( P^n \) along \( M \), which is defined by
\[
\widetilde{P}^n := \{(x, \mu) \in P^n \times P^1 \mid x \in P(\mu)\}.
\]
It is a compact analytic submanifold of \( P^n \times P^1 \) with dimension \( n \).

The projections of \( P^n \times P^1 \) onto its two factors, when restricted to \( \widetilde{P}^n \), give two proper analytic morphisms
\[
\varphi: \widetilde{P}^n \to P^n \quad \text{and} \quad \pi: \widetilde{P}^n \to P^1.
\]

For any subsets \( G \subset P^n \) and \( E \subset P^1 \), write
\[
\widetilde{G} := \varphi^{-1}(G) \quad \text{and} \quad \widetilde{G}_E := \widetilde{G} \cap \pi^{-1}(E).
\]
One must not confuse \( \widetilde{G}_E \) with \( \widetilde{G} \); we have
\[
\widetilde{G}_E = \widetilde{G} \cup (\widetilde{G} \cap M) = \widetilde{G} \cup (G \times P^1).
\]

For simplicity, we also set
\[
L := L \cap X \quad \text{and} \quad M := M \cap X.
\]

By stratifying suitably \( \tilde{X} \) and then applying the First Isotopy Theorem of Thom-Mather (cf. [Th], [Ma]) one obtains the following lemma.

Lemma 6.5 (cf. [C2, (11.1.5)]). – The restriction of \( \pi \) to \( \tilde{X} \setminus \bigcup_i \tilde{X}_{\lambda_i} \) is a topological locally trivial fibration over \( P^1 \setminus \bigcup_i \lambda_i \) with typical fibre \( \tilde{X}_\lambda \) homeomorphic to \( L \). Moreover, this bundle has \( M \times (P^1 \setminus \bigcup_i \lambda_i) \) as a trivial subbundle of it.

The proof is now along lines very similar to [La, 8.3]. Decompose \( P^1 \) into two closed semi-analytic hemispheres \( D_+ \) and \( D_- \) such that: (i) \( D_+ \cap D_- = S^1 \), where \( S^1 \) is a 1-sphere; (ii) \( K \) is contained in \( D_+ \); (iii) \( \lambda \in S^1 \); (iv) \( D_- \) is contained in some coordinate neighbourhood of the fibre bundle \( \tilde{X} \setminus \bigcup_i \tilde{X}_{\lambda_i} \) for a trivialization which preserves the subbundle \( M \times (P^1 \setminus \bigcup_i \lambda_i) \). This choice of \( D_- \) implies that the pairs
\[
(L \times D_-, L \times S^1 \cup M \times D_-) \quad \text{and} \quad (\tilde{X}_{D_-}, \tilde{X}_{S^1} \cup \tilde{M}_{D_-})
\]
are homeomorphic. Now, consider the following sequence of pairs of spaces:
\[
(L \times D_-, L \times S^1 \cup M \times D_-) \simeq (\tilde{X}_{D_-}, \tilde{X}_{S^1} \cup \tilde{M}_{D_-}) \overset{\text{exc}}{\leftrightarrow} (\tilde{X}, \tilde{X}_{D_-} \cup \tilde{M}) \hookrightarrow (\tilde{X}, \tilde{X}_K \cup \tilde{M}) \overset{\chi}{\hookrightarrow} (X, X_K).
\]
Since \((L, M)\) and \((D_-, S^1)\) are respectively \(k\)-connected and 1-connected, Exercise 9, p. 95 of [W] shows that \((L \times D_-, L \times S^1 \cup M \times D_-)\), and consequently \((\tilde{X}_{D_-, S^1} \cup \tilde{M}_{D_-})\), are \((k + 2)\)-connected.

Next, for excision \(\text{exc}\), we might use the Homotopy Excision Theorem of Blakers-Massey (cf. [BM], [Sw, 6.21]) but we do not need its full force. The elementary proof of [La, (8.2.2)] which is about a relative homotopy can be adapted to an excision and gives the following result. Let \((A, B) \hookrightarrow (C, D)\) be an excision, that is an inclusion of topological pairs such that \(A \setminus B = C \setminus D\). Suppose that \((A, B)\) is a relative CW-complex and that \(A\) and \(D\) are closed in \(C\). If \((A, B)\) is \(m\)-connected for some integer \(m \geq 0\), then \((C, D)\) is also \(m\)-connected. Now, since \((\tilde{X}_{D_-}, \tilde{X}_{S^1} \cup \tilde{M}_{D_-})\) is triangulable (cf. [Lo]) and since \(\tilde{X}_{D_-}\) and \(\tilde{X}_{D_+} \cup \tilde{M}\) are closed in \(\tilde{X}\), this result can be applied to excision \(\text{exc}\) and shows that

\[(\tilde{X}, \tilde{X}_{D_+} \cup \tilde{M})\text{ is also } (k + 2)\text{-connected.}\]

Since \(\tilde{X} \cap \pi^{-1}(D_+ \setminus \bigcup \lambda_i)\) is a fibre bundle (cf. Lemma 6.5) and \(K \setminus \bigcup \lambda_i\) is a strong deformation retract of \(D_+ \setminus \bigcup \lambda_i\), the First Homotopy Covering Theorem [St, §11.3] shows that \(\tilde{X} \cap \pi^{-1}(K \setminus \bigcup \lambda_i)\) is a strong deformation retract of \(\tilde{X} \cap \pi^{-1}(D_+ \setminus \bigcup \lambda_i)\). Moreover, since the \(\lambda_i\) are interior points of \(K\), the deformation retraction may be in fact extended so that \(\tilde{X}_K\) is a strong deformation retract of \(\tilde{X}_{D_+}\). As furthermore \(\tilde{X}_K\) is also a strong deformation retract of \(\tilde{X}_K \cup \tilde{M}_{D_+}\), one deduces that the pair \((\tilde{X}_{D_+}, \tilde{X}_K \cup \tilde{M}_{D_+})\) is \(q\)-connected for all \(q \geq 0\). By the theorem of Blakers-Massey [Sw, Theorem 6.21], applied to the excision

\[(\tilde{X}_{D_+}, \tilde{X}_K \cup \tilde{M}_{D_+}) \hookrightarrow (\tilde{X}_{D_+} \cup \tilde{M}, \tilde{X}_K \cup \tilde{M}),\]

the same property holds for \((\tilde{X}_{D_+} \cup \tilde{M}, \tilde{X}_K \cup \tilde{M})\). Then, the homotopy sequence of the triple

\[(\tilde{X}, \tilde{X}_{D_+} \cup \tilde{M}, \tilde{X}_K \cup \tilde{M}),\]

together with (6.6), implies that \((\tilde{X}, \tilde{X}_K \cup \tilde{M})\) is \((k + 2)\)-connected.

Now, the same holds for \((X, X_K)\) since

\[\varphi: (\tilde{X}, \tilde{X}_K \cup \tilde{M}) \rightarrow (X, X_K)\]

is a relative homeomorphism (cf. [La, (8.2.2)]). This completes the proof of Lemma 6.4 and, consequently, the proof of Lemma 6.2.

Conjecture 6.3, to which Conjecture 6.1 is thus reduced, remains of course the hard part of the work.

The generalized homotopical variation operators introduced here are also certainly a first step for further generalizations of van Kampen’s theorem to singular quasi-projective varieties, the influence of the singularities being measured by the (local or global) rectified homotopical depth (cf. [G], [HL1,2], [GM1,2], [E1,2]).
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