A LERAY REGULARIZED ENSEMBLE-PROPER ORTHOGONAL DECOMPOSITION METHOD FOR PARAMETERIZED CONVECTION-DOMINATED FLOWS

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Abstract. Partial differential equations (PDEs) are often dependent on input quantities which are inherently uncertain. To quantify this uncertainty, these PDEs must be solved over a large ensemble of parameters. Even for a single realization this can a computationally intensive process. In the case of flows governed by the Navier-Stokes equations, an efficient method has been devised for computing an ensemble of solutions. To further reduce the computational cost of this method, an ensemble proper orthogonal decomposition (POD) method was recently proposed.

The main contribution of this work is the introduction of POD spatial filtering for ensemble-POD methods. The POD spatial filter makes possible the construction of the Leray ensemble-POD model, which is a regularized reduced order model for the numerical simulation of convection-dominated flows. The Leray ensemble-POD model employs the POD spatial filter to smooth (regularize) the convection term in the Navier-Stokes equations and greatly diminishes the numerical inaccuracies produced by the ensemble-POD method in the numerical simulation of convection-dominated flows. Specifically, for the numerical simulation of a convection-dominated two-dimensional flow between two offset cylinders, we show that the Leray ensemble-POD method yields accurate results, whereas the ensemble-POD is highly inaccurate.

The second contribution of this work is a new numerical discretization of the variable viscosity ensemble algorithm in which the average viscosity is replaced with the maximum viscosity. It is shown that this new numerical discretization is significantly more stable than those in current use. Furthermore, error estimates for the novel Leray ensemble-POD algorithm with this new numerical discretization are also proven.

Key words. Navier-Stokes equations, ensemble computation, proper orthogonal decomposition, Leray regularization, POD differential filter

1. Introduction. The mathematical models used in realistic applications often times rely on input quantities which are subject to a degree of uncertainty. Some of these quantities include the initial conditions, forcing functions, model coefficients, and the boundary conditions. In order to develop robust models the impact of this uncertainty must be quantified.

Common approaches for recovering accurate solutions of these models are the Monte Carlo and stochastic collocation methods. These algorithms all require the underlying model to be solved over an ensemble of parameters. Depending upon the problem the spatial resolution required for accurate realizations of the model can render these approaches computationally intractable. In particular, realizations for flow models such as the incompressible Navier-Stokes equations (NSE) can take on the order of weeks. In this work, we are interested in computing ensembles of solutions for the NSE with uncertainty present in the initial conditions, viscosities, and body forces. Specifically, for \( j = 1, \ldots, J \), we have

\[
\begin{align*}
  u^j_t + u^j \cdot \nabla u^j - \nu_j \Delta u^j + \nabla p^j &= f^j(x, t) \quad \forall x \in \Omega \times (0, T) \\
  \nabla \cdot u^j &= 0 \quad \forall x \in \Omega \times (0, T) \\
  u^j &= 0 \quad \forall x \in \partial \Omega \times (0, T) \\
  u^j(x, 0) &= u^{j, 0}(x) \quad \forall x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), is an open regular domain.

Historically, the solution of the NSE for each parameter has been treated as a separate problem. Recently new algorithms have been developed...
that allow for simultaneous calculations at each time step. Specifically the focus of these algorithms has been to use the same linear system for each right hand side. Taking advantage of this problem structure, efficient block solvers, such as block CG [7], block QMR [9], and block GMRES [10] can then be utilized.

To further improve the efficiency of these ensemble algorithms, reduced order models (ROMs) [21, 38] were recently utilized [17, 18]. Specifically, the proper orthogonal decomposition (POD) method was used to extract the dominant (most energetic) modes from a high-resolution numerical simulation, and the NSE were projected onto these POD modes to obtain an ensemble-POD model. In [17, 18], it was shown that the ensemble-POD model significantly decreased the computational cost of the standard ensemble methods, without compromising their numerical accuracy. We note, however, that the numerical investigation of the ensemble-POD model in [17, 18] was restricted to low Reynolds numbers. It is well known that, for convection-dominated flows, standard ROMs generally yield inaccurate results, usually in the form of spurious numerical oscillations (see, e.g., [13, 46, 45]). To mitigate these ROM inaccuracies, several numerical stabilization techniques have been proposed over the years (see, e.g., [1, 2, 4, 5, 28, 37, 41, 6, 8, 42, 43]). Regularized ROMs (Reg-ROMs) are recently proposed stabilized ROMs for the numerical simulation of convection-dominated flows, both deterministic [39, 44, 47] and stochastic [22]. These Reg-ROMs use explicit ROM spatial filtering to regularize (smooth) various ROM terms and thus increase the numerical stability of the resulting ROM. This idea was first used by the great Jean Leray [33] in the mathematical study of the NSE and later on in, e.g., [12, 31] to develop regularized models for the numerical simulation of turbulent flows [12, 31]. In the ROM arena, Reg-ROMs were also successfully used in the numerical simulation of convection-dominated flows. For example, the Reg-ROMs used in the numerical simulation of a 3D flow past a circular cylinder at a Reynolds number $Re = 1000$ produced accurate results in which the spurious numerical oscillations of standard ROMs were significantly decreased [44].

In this paper, we put forth ROM spatial filtering and Reg-ROMs as a means to mitigate the numerical inaccuracies that are generally produced by the ensemble-POD method when this is applied to convection-dominated flows. Specifically, we propose and investigate the Leray ensemble-POD method, which replaces the convective field in the nonlinearity of the standard ensemble-POD method with its spatially filtered version. For the spatial filter in the Leray ensemble-POD method, we use the POD differential filter [44, 46]. In Section 3 we also propose a new numerical discretization of the variable viscosity ensemble algorithm in which the average viscosity is replaced with the maximum viscosity. We show that this new numerical discretization is significantly more stable than those in current use. Furthermore, we prove error estimates for the new Leray ensemble-POD algorithm with this new numerical discretization. Finally, in Section 6 we test the new Leray ensemble-POD method in the numerical simulation of the two-dimensional flow between offset circles used in [18, 17]. To this end, we compare the new Leray ensemble-POD method with the standard ensemble-POD method and a fine resolution numerical simulation, which is used as a benchmark.

2. Notation and preliminaries. We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the $L^2(\Omega)$ norm and inner product, respectively, and by $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^{k}}$ the $L^p(\Omega)$ and Sobolev $W^{k}_p(\Omega)$ norms, respectively. $H^k(\Omega) = W^{k}_2(\Omega)$ with norm $\| \cdot \|_{k}$. For a function $v(x,t)$
that is well defined on $\Omega \times [0, T]$, we define the norms

$$\|v\|_{2,s} := \left( \int_0^T \|v(\cdot, t)\|^2_s \, dt \right)^{\frac{1}{2}}$$

and

$$\|v\|_{\infty,s} := \sup_{0 \leq t \leq T} \|v(\cdot, t)\|_s.$$  

The space $H^{-1}(\Omega)$ denotes the dual space of bounded linear functionals defined on $H^1_0(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$; this space is equipped with the norm

$$\|f\|_{-1} = \sup_{0 \neq v \in X} \frac{(f, v)}{\|v\|} \quad \forall f \in H^{-1}(\Omega).$$

The solutions spaces $X$ for the velocity and $Q$ for the pressure are respectively defined as

$$X := [H^1_0(\Omega)]^d = \{v \in [L^2(\Omega)]^d : \nabla v \in [L^2(\Omega)]^d \times d \text{ and } v = 0 \text{ on } \partial\Omega\}$$

and

$$Q := L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}.$$  

A weak formulation of (1.1) is given as follows: for $j = 1, \ldots, J$, find $u^j : (0, T] \to X$ and $p^j : (0, T] \to Q$ such that, for almost all $t \in (0, T]$, satisfy

$$\begin{cases}
(u^j_t, v) + (u^j \cdot \nabla u^j, v) + \nu_j (\nabla u^j, \nabla v) - (p^j, \nabla \cdot v) = (f^j, v) & \forall v \in X \\
(\nabla \cdot u^j, q) = 0 & \forall q \in Q \\
u^j(x, 0) = u^{j,0}(x).
\end{cases} \quad (2.1)$$

The subspace of $X$ consisting of weakly divergence-free functions is defined as

$$V := \{v \in X : (\nabla \cdot v, q) = 0 \quad \forall q \in Q\} \subset X.$$  

We denote conforming velocity and pressure finite element spaces based on a regular triangulation of $\Omega$ having maximum triangle diameter $h$ by $X_h \subset X$ and $Q_h \subset Q$. We assume that the pair of spaces $(X_h, Q_h)$ satisfy the discrete inf-sup (or $LBB_h$) condition required for stability of finite element approximations; we also assume that the finite element spaces satisfy the approximation properties

$$\inf_{v_h \in X_h} \|v - v_h\| \leq Ch^{s+1} \quad \forall v \in [H^{s+1}(\Omega)]^d$$

and

$$\inf_{v_h \in X_h} \|\nabla(v - v_h)\| \leq C h^s \quad \forall v \in [H^{s+1}(\Omega)]^d$$

and

$$\inf_{q_h \in Q_h} \|q - q_h\| \leq Ch^s \quad \forall q \in H^s(\Omega),$$

where $C$ is a positive constant that is independent of $h$. The Taylor-Hood element pairs ($P^s$-$P^{s-1}$), $s \geq 2$, are one common choice for which the $LBB_h$ stability condition and the approximation estimates hold \cite{14, 19}.

To ensure the uniqueness of the NSE solution and ensure that standard finite element error estimates, we make the following regularity assumptions on the data and true solution:

**Assumption 2.1.** In (1.1) we assume that $u^0 \in V$, $f^j \in L^2(0, T; L^2(\Omega))$, $w^j \in L^\infty(0, T; H^{s+1}(\Omega)) \cap H^1(0, T; H^{s+1}(\Omega)) \cap H^2(0, T; L^2(\Omega))$, and $p \in L^\infty(0, T; Q \cap H^k(\Omega)).$
Using the regularity assumptions above and assuming a sufficiently small $\Delta t$, the following error estimate can be proven for the full discretization of (2.1) with Taylor-Hood elements and the Crank-Nicolson time-discretization:

$$
\|u(t^N) - u_h^N\|^2 + \nu \Delta t \sum_{n=1}^M \|\nabla (u(t^n) - u_h^n)\|^2 \leq C \left(h^{2m} + \Delta t^4\right),
$$

(2.2)

where $C$ is independent of $h$ and $\Delta t$.

We define the trilinear form

$$
b(w, u, v) = (w \cdot \nabla u, v) \quad \forall u, v, w \in [H^1(\Omega)]^d
$$

and the explicitly skew-symmetric trilinear form given by

$$
b^*(w, u, v) := \frac{1}{2}(w \cdot \nabla u, v) - \frac{1}{2}(w \cdot \nabla v, u) \quad \forall u, v, w \in [H^1(\Omega)]^d,
$$

which satisfies the bounds:

$$
b^*(w, u, v) \leq C_b \|\nabla w\| \|\nabla u\| (\|v\| \|\nabla v\|)^{1/2} \quad \forall u, v, w \in X
$$

(2.3)

$$
b^*(w, u, v) \leq C_b' (\|w\| \|\nabla w\|)^{1/2} \|\nabla u\| \|\nabla v\| \quad \forall u, v, w \in X.
$$

(2.4)

We also define the discretely divergence-free space $V_h$ as

$$
V_h := \{v_h \in X_h : (\nabla \cdot v_h, q_h) = 0 \forall q_h \in Q_h\} \subset X.
$$

In most cases, and for the Taylor-Hood element pair in particular, $V_h \not\subset V$, i.e., discretely divergence-free functions are not weakly divergence-free.

**Definition 2.2.** Let $t^n = n\Delta t$, $n = 0, 1, 2, \ldots, N$, where $N := T/\Delta t$, denote a partition of the interval $[0, T]$. For $j = 1, \ldots, J$ and $n = 0, 1, 2, \ldots, N$, let $u^{j,n}(x) := u^j(x, t^n)$. Then, the ensemble mean is defined, for $n = 0, 1, 2, \ldots, N$, by

$$
< u >^n := \frac{1}{J} \sum_{j=1}^J u^{j,n}.
$$

The full space and time model which we will base our method off of is similar to the one used in [16] [15]. For $j = 1, \ldots, J$, given $u_h^{j,0} \in X_h$ and $u_h^{j,1} \in X_h$, for $n = 0, 1, 2, \ldots, N - 1$ find $u_h^{j,n+1} \in X_h$ and $p_h^{j,n+1} \in Q_h$ satisfying

$$
\left(\frac{u_h^{j,n+1} - u_h^{j,n}}{\Delta t}, v_h\right) + b^*(< u_h >^n, u_h^{j,n+1}, v_h) + b^*(u_h^{j,n} - < u_h >^n, u_h^{j,n}, v_h) + \nu_{\text{max}}(\nabla u_h^{j,n+1}, \nabla v_h) + (\nu_j - \nu_{\text{max}})(\nabla u_h^{j,n}, \nabla v_h)
$$

$$
- (p_h^{j,n+1}, \nabla \cdot v_h) = (f^{j,n+1}, v_h) \quad \forall v_h \in X_h
$$

$$
\nabla \cdot u_h^{j,n+1} = 0 \quad \forall q_h \in Q_h.
$$

The major difference between the two algorithms the use of maximum value of the viscosities $\nu_{\text{max}}$ rather than the average $< \nu >$ resulting in a superior stability condition.

3. Proper Orthogonal Decomposition Ensemble Based Models.
3.1. Proper Orthogonal Decomposition. In this subsection we briefly describe the POD method and apply it to the previously stated ensemble algorithm. A more detailed description of this method can be found in [29].

Given a positive integer \( N_S \), let \( 0 = t_0 < t_1 < \cdots < t_{N_S} = T \) denote a uniform partition of the time interval \([0,T] \). For \( j = 1, \ldots, J_S \), we select \( J_S \) different initial conditions \( u^{i,0}(x) \), viscosities \( \nu^j \), and forcing functions \( f^j \) denoted by \( u^{j,m}_{h,S}(x) \in X_h \), \( j = 1, \ldots, J_S, m = 1, \ldots, N_S \), the finite element approximation to (1.1) evaluated at \( t = t_m, m = 1, \ldots, N_S \). We then define the space spanned by the \( J_S(N_S + 1) \) discrete snapshots as

\[
X_{h,S} := \text{span}\{u^{j,m}_{h,S}(x)\}_{j=1,m=0}^{J_S,N_S} \subset V_h \subset X_h.
\]

Denoting by \( \tilde{u}^{j,m}_{h} \) the vector of coefficients corresponding to the finite element function \( u^{j,m}_{h,S}(x) \), we define the \( K \times J_S(N_S + 1) \) snapshot matrix \( A \) as

\[
A = (\tilde{u}^{1,0}_{h,S}, \tilde{u}^{1,1}_{h,S}, \ldots, \tilde{u}^{1,N_S}_{h,S}, \tilde{u}^{2,0}_{h,S}, \tilde{u}^{2,1}_{h,S}, \ldots, \tilde{u}^{2,N_S}_{h,S}, \ldots, \tilde{u}^{J_S,0}_{h,S}, \tilde{u}^{J_S,1}_{h,S}, \ldots, \tilde{u}^{J_S,N_S}_{h,S}),
\]

i.e., the columns of \( A \) are the finite element coefficient vectors corresponding to the discrete snapshots. The POD method then seeks a low dimensional basis

\[
X_R := \text{span}\{\varphi_i\}_{i=1}^R \subset X_{h,S} \subset V_h \subset X_h
\]

which can approximate the snapshot data. This basis can be determined by solving the constrained minimization problem

\[
\min_{\varphi} \sum_{k=1}^{J_S} \sum_{l=0}^{N_S} \left\| u_{h,s}^{k,l} - \sum_{j=1}^{R} (u_{h,s}^{k,l}, \varphi_j) \varphi_j \right\|^2 \text{ subject to } (\varphi_i, \varphi_j) = \delta_{ij} \text{ for } i,j = 1, \ldots, R,
\]

where \( \delta_{ij} \) denotes the Kronecker delta. Defining the correlation matrix \( C = A^T M A \) where \( M \) denotes the finite element mass matrix, this problem can then be solved by considering the eigenvalue problem

\[
C \tilde{a}_i = \lambda_i \tilde{a}_i.
\]

It can then be shown the POD basis functions will be given by

\[
\tilde{\varphi}_i = \frac{1}{\sqrt{\lambda_i}} \tilde{a}_i, \quad i = 1, \ldots, R.
\]

We now define the POD \( L^2 \) projection we will need for the ensuing stability and error analysis.

**Definition 3.1 (POD \( L^2 \) projection).** Let \( P_r : L^2(\Omega) \to X_R \) such that

\[
(u - P_r u, \varphi) = 0 \quad \forall \varphi \in X_R.
\]

Next we give a POD inverse estimate. Let \( S_R = (\nabla \varphi_i, \nabla \varphi_j)_{L^2} \) be the POD stiffness matrix and let \( \| \cdot \|_2 \) denote the matrix 2-norm.

**Lemma 3.2 (POD inverse estimate).**

\[
\| \nabla \varphi \| \leq \| S_R \|_2^{\frac{1}{2}} \| \varphi \| \quad \forall \varphi \in X_R.
\]
3.2. Ensemble-POD Algorithm. Using this POD basis we can now construct the ensemble-POD algorithm. The construction is similar to the full finite element approximation except we seek a solution in the POD space $X_R$ using the basis $\{\varphi_i\}_{i=1}^R$. The fully discrete algorithm can be written as:

\[
\begin{align*}
\left(\frac{u_R^{j,n+1} - u_R^{j,n}}{\Delta t}, \varphi\right) + b^*(< u_R >^n, u_R^{j,n+1}, \varphi) + b^*(u_R^{j,n} - < u_R >^n, u_R^{j,n}, \varphi) \\
+ \nu_{\text{max}} \nabla u_R^{j,n+1}, \nabla \varphi) + (u_j - \nu_{\text{max}})(\nabla u_R^{j,n}, \nabla \varphi) = (f_j^{j,n+1}, \varphi) \quad \forall \varphi \in X_R.
\end{align*}
\]  

(3.4)

We note that because $X_R \subset V_h$ the POD basis is discretely divergence-free by construction. Therefore, there is no pressure term present in (3.2). In recent works constructing a basis for the pressure space in addition to the velocity space has been investigated. The interested reader should consult [3].

3.3. Leray Ensemble-POD Algorithm. To construct the Leray ensemble-POD algorithm we use the ROM differential filter.

DEFINITION 3.3 (ROM differential filter). $\forall v \in X$ let $v_R$ be the unique element of $X_R$ such that

\[
\delta^j(\nabla v_R^t, \nabla \varphi) + (v_R^t, \varphi) = (v, \varphi) \quad \forall \varphi \in X_R.
\]  

(3.5)

Here $\delta$ is known as the filtering radius. The differential filter was first developed by Germano [11] for large eddy simulations. It was introduced in the ROM setting in [39] and expanded further in [41, 47].

Incorporating this into the ensemble framework the fully discrete Leray ensemble-POD algorithm can be written as:

\[
\begin{align*}
\left(\frac{u_R^{j,n+1} - u_R^{j,n}}{\Delta t}, \varphi\right) + b^*(< u_R >^n, u_R^{j,n+1}, \varphi) + b^*(u_R^{j,n} - < u_R >^n, u_R^{j,n}, \varphi) \\
+ \nu_{\text{max}} \nabla u_R^{j,n+1}, \nabla \varphi) + (u_j - \nu_{\text{max}})(\nabla u_R^{j,n}, \nabla \varphi) = (f_j^{j,n+1}, \varphi) \quad \forall \varphi \in X_R.
\end{align*}
\]  

(3.6)

4. Stability Analysis. In this section we present a result pertaining to the stability of the Leray ensemble-POD algorithm. A stability bound for the ensemble-POD algorithm for a fixed viscosity was proven in Theorem 4.2 in [17], while a stability bound for an ensemble-FE algorithm with variable viscosity was proven in Theorem 2.1 in [16]. The stability bound proven in this section is less restrictive than the bound proven in [16] due to the use of $\nu_{\text{max}}$ in the algorithm as opposed to $< \nu >$.

THEOREM 4.1. Consider algorithm (3.6); define $0 \leq \epsilon \leq 1$ such that

\[
\max_{1 \leq j \leq J} \frac{|\nu_j - \nu_{\text{max}}|}{\nu_{\text{max}}} = 1 - \epsilon
\]  

(4.1)

and assume the following condition holds for $j = 1 \ldots J$:

\[
\frac{C_D^2 \Delta t}{\nu_{\text{max}}} \|S_R\| \|\nabla(u_R^{j,n} - < u_R >^n)\|^2 \leq \epsilon.
\]  

(4.2)

Then, for any $N \geq 1$

\[
\begin{align*}
\frac{1}{2} \|u_R^{j,n}\|^2 + \frac{\nu_{\text{max}} \Delta t}{2} \|\nabla u_R^{j,n}\|^2 + \frac{\epsilon \nu_{\text{max}} \Delta t}{4} \sum_{n=0}^{N-1} \|u_R^{j,n+1}\|^2 \\
\leq \sum_{n=0}^{N-1} \frac{\Delta t}{\nu_{\text{max}}} \|f_j^{j,n+1}\|^2 + \frac{1}{2} \|u_R^0\|^2 + \frac{\nu_{\text{max}} \Delta t}{2} \|\nabla u_R^0\|^2 \text{ notation } C_{\text{stab}}.
\end{align*}
\]  

(4.3)
Proof. Setting $\varphi = u_R^{j,n+1}$ and using the skew-symmetry of the trilinear term we have

$$\frac{1}{2} \|u_R^{j,n+1}\|^2 - \frac{1}{2} \|u_R^{j,n}\|^2 + \frac{1}{2} \|u_R^{j,n+1} - u_R^{j,n}\|^2 + \nu_{\text{max}} \Delta t \| \nabla u_R^{j,n+1} \|^2$$

$$+ \Delta t (u_R^{j,n} - u_R^{j,n}) = \text{(4.4)}$$

Now applying Young’s inequality on the right hand side we have

$$\frac{1}{2} \|u_R^{j,n+1}\|^2 - \frac{1}{2} \|u_R^{j,n}\|^2 + \frac{1}{2} \|u_R^{j,n+1} - u_R^{j,n}\|^2 + \nu_{\text{max}} \Delta t \| \nabla u_R^{j,n+1} \|^2$$

$$+ \Delta t (u_R^{j,n} - u_R^{j,n}) \leq \frac{\alpha \Delta t \nu_{\text{max}}}{4} \| \nabla u_R^{j,n+1} \|^2$$

$$+ \frac{\Delta t}{\nu_{\text{max}}} \| f_{j+1} \|^2 + \frac{\beta \Delta t \nu_{\text{max}}}{4} \| \nabla u_R^{j,n+1} \|^2 + \frac{\Delta t (|\nu_j - \nu_{\text{max}}|^2)}{2 \nu_{\text{max}}} \| \nabla u_R^{j,n} \|^2. \quad \text{(4.5)}$$

Since both $\frac{\beta \Delta t \nu_{\text{max}}}{4} \| \nabla u_R^{j,n+1} \|^2$ and $\frac{\Delta t (|\nu_j - \nu_{\text{max}}|)}{2 \nu_{\text{max}}} \| \nabla u_R^{j,n+1} \|^2$ need to be absorbed into $\nu_{\text{max}} \Delta t \| u_R^{j,n+1} \|^2$ we minimize the quantity $\frac{\beta \Delta t \nu_{\text{max}}}{4} + \frac{\Delta t (|\nu_j - \nu_{\text{max}}|)}{2 \nu_{\text{max}}}$ by selecting $\beta = \frac{2 |\nu_j - \nu_{\text{max}}|}{\nu_{\text{max}}^2}$. It then follows that

$$\frac{1}{2} \|u_R^{j,n+1}\|^2 - \frac{1}{2} \|u_R^{j,n}\|^2 + \frac{1}{2} \|u_R^{j,n+1} - u_R^{j,n}\|^2 + \nu_{\text{max}} \Delta t \| \nabla u_R^{j,n+1} \|^2$$

$$+ \Delta t (u_R^{j,n} - u_R^{j,n}) \leq \frac{\alpha \Delta t \nu_{\text{max}}}{4} \| \nabla u_R^{j,n+1} \|^2$$

$$+ \frac{\Delta t}{\nu_{\text{max}}} \| f_{j+1} \|^2 + \frac{\Delta t (|\nu_j - \nu_{\text{max}}|)}{2 \nu_{\text{max}}} \| \nabla u_R^{j,n+1} \|^2. \quad \text{(4.6)}$$

Next we bound the trilinear term using (2.3) and (3.3), obtaining

$$- \Delta t (u_R^{j,n} - u_R^{j,n})$$

$$\leq C_b \Delta t \| \nabla u_R^{j,n} \|^2$$

$$\leq C_b \Delta t \| \nabla u_R^{j,n} \|^2 \leq C_b \Delta t \| \nabla u_R^{j,n} \|^2.$$ 

Then using Young’s inequality we obtain

$$- \Delta t (u_R^{j,n} - u_R^{j,n})$$

$$\leq \frac{C_b^2 \Delta t^2}{2} \| \nabla u_R^{j,n} \|^2.$$ 

Combining like terms we then have

$$\frac{1}{2} \|u_R^{j,n+1}\|^2 - \frac{1}{2} \|u_R^{j,n}\|^2 + \nu_{\text{max}} \Delta t \left( 1 - \frac{\alpha}{4} \frac{|\nu_j - \nu_{\text{max}}|}{2 \nu_{\text{max}}} \right) \| \nabla u_R^{j,n+1} \|^2$$

$$\leq \frac{\Delta t}{\alpha \nu_{\text{max}}} \| f_{j+1} \|^2 + \frac{C_b^2 \Delta t^2}{2} \| \nabla u_R^{j,n} \|^2.$$

$$+ \frac{\Delta t (|\nu_j - \nu_{\text{max}}|)}{2 \nu_{\text{max}}} \| \nabla u_R^{j,n+1} \|^2.$$
Rearranging terms it follows that
\[
\frac{1}{2} \| u_R^{n+1} \|^2 - \frac{1}{2} \| u_R^n \|^2 + \nu_{\text{max}} \Delta t \left( \left( 1 - \frac{\alpha}{4} \right) \frac{|v_j - v_{\text{max}}|}{2\nu_{\text{max}}} \right) \| \nabla u_R^{n+1} \|^2 - \left( \frac{|v_j - v_{\text{max}}|}{2\nu_{\text{max}}} + \frac{C^2 \Delta t}{2\nu_{\text{max}}} \| S_R \| \frac{1}{2} \| u_R^{n} - \nabla u_R^{n} \|^2 \right) \leq \frac{\Delta t}{\alpha \nu_{\text{max}}} \| f_{j+1} \|^2. 
\]
(4.10)

Using the fact that \( \max_{1 \leq j \leq J} \frac{|v_j - v_{\text{max}}|}{v_{\text{max}}} = 1 - \epsilon \) for \( 0 \leq \epsilon \leq 1 \) and taking \( \alpha = \epsilon \) we have
\[
\frac{1}{2} \| u_R^{n+1} \|^2 - \frac{1}{2} \| u_R^n \|^2 + \nu_{\text{max}} \Delta t \left( \left( 1 - \frac{\epsilon}{4} \right) \frac{1}{2} \| \nabla u_R^{n+1} \|^2 - \frac{1}{2} \| \nabla u_R^n \|^2 \right) \leq \frac{\Delta t}{\nu_{\text{max}} \epsilon} \| f_{j+1} \|^2. 
\]
(4.11)

Now using assumption (4.12), (4.11) we have
\[
\frac{1}{2} \| u_R^{n+1} \|^2 - \frac{1}{2} \| u_R^n \|^2 + \nu_{\text{max}} \Delta t \left( \frac{1}{2} \| \nabla u_R^{n+1} \|^2 - \frac{1}{2} \| \nabla u_R^n \|^2 \right) + \nu_{\text{max}} \Delta t \frac{\epsilon}{4} \| u_R^n \|^2 \leq \frac{\Delta t}{\nu_{\text{max}} \epsilon} \| f_{j+1} \|^2. 
\]
(4.12)

Summing up (4.12) from 0 to \( N - 1 \) yields (4.3).

**Remark 4.2.** The term \( \epsilon \) in the above theorem measures the relative uncertainty present in the viscosities. In practice, the amount of uncertainty present in the viscosities can be one or two orders of magnitude. In this case \( \epsilon \approx O(10^{-1}) \) or \( O(10^{-2}) \).

**5. Error analysis.** We next provide an error analysis for Leray ensemble-POD solutions. First, we present several results obtained in [17], which we use in the analysis. We also use the following notation:

**Definition 5.1 (Generic Constant \( C \)).** Let \( C \) be a generic constant that can depend on \( j, \nu, \) but not on \( h, \Delta t, R, \lambda_i, \epsilon, \nu_{\text{max}}, \delta, C_{\text{stab}}, C_{b} \).

The following lemma is similar to Lemma 5.1 in [17].

**Lemma 5.2.** \( L^2(\Omega) \) norm of the error between snapshots and their projections onto the POD space. We have
\[
\frac{1}{J_S (N_S + 1)} \sum_{j=1}^{J_S} \sum_{m=0}^{N_S} \left\| u_{h,S}^{j,m} - \sum_{i=1}^{R} (u_{h,S}^{j,m}, \varphi_i) \varphi_i \right\|^2 = \sum_{i=R+1}^{J_S (N_S + 1)} \lambda_i 
\]
and thus for \( j = 1, \ldots, J_S \),
\[
\frac{1}{N_S + 1} \sum_{m=0}^{N_S} \left\| u_{h,S}^{j,m} - \sum_{i=1}^{R} (u_{h,S}^{j,m}, \varphi_i) \varphi_i \right\|^2 \leq \frac{1}{J_S} \sum_{i=R+1}^{J_S (N_S + 1)} \lambda_i. 
\]

The following lemma is similar to Lemma 5.2 in [17].

**Lemma 5.3.** \( H^1(\Omega) \) norm of the error between snapshots and their projections in the POD space. We have
\[
\frac{1}{J_S (N_S + 1)} \sum_{j=1}^{J_S} \sum_{m=0}^{N_S} \left\| \nabla \left( u_{h,S}^{j,m} - \sum_{i=1}^{R} (u_{h,S}^{j,m}, \varphi_i) \varphi_i \right) \right\|^2 = \sum_{i=R+1}^{J_S (N_S + 1)} \lambda_i \left\| \nabla \varphi_i \right\|^2 
\]
and thus, for \( j = 1, \ldots, J_S \),

\[
\frac{1}{N_S + 1} \sum_{m=0}^{N_S} \left\| \nabla \left( u_{j,m}^{i,m} - \sum_{i=1}^{R} (u_{j,m}^{i,m}, \varphi_i) \varphi_i \right) \right\|^2 \leq J_S \sum_{i=R+1}^{J_S (N_S+1)} \lambda_i \| \nabla \varphi_i \|^2.
\]

The following lemma is similar to Lemma 5.3 in [17] (see also Lemma 3.3 in [23]).

**Lemma 5.4.** [Error in the projection onto the POD space]

Consider the partition \( 0 = t_0 < t_1 < \cdots < t_{N_S} = T \) used in Section 3. For any \( u \in H^1(0,T; [H^{s+1}(\Omega)]^d) \), let \( u^m = u(\cdot, t_m) \). Then, the error in the projection onto the POD space \( X_R \) satisfies the estimates

\[
\frac{1}{N_S + 1} \sum_{m=0}^{N_S} \| u^{j,m} - P_R u^{j,m} \|^2 \leq C \left( h^{2s+2} + \Delta t^4 \right) + J_S \sum_{i=R+1}^{J_S (N_S+1)} \lambda_i
\]

\[
\frac{1}{N_S + 1} \sum_{m=0}^{N_S} \| \nabla (u^{j,m} - P_R u^{j,m}) \|^2 \leq (C + h^2 \| S_R \|_2) h^{2s} + (C + \| S_R \|_2) \Delta t^4 + J_S \sum_{i=R+1}^{J_S (N_S+1)} \| \nabla \varphi_i \|^2 \lambda_i.
\]

We assume the following estimates are also valid, as done in [23].

**Assumption 5.5.** Consider the partition \( 0 = t_0 < t_1 < \cdots < t_{N_S} = T \) used in Section 3. For any \( u \in H^1(0,T; [H^{s+1}(\Omega)]^d) \), let \( u^m = u(\cdot, t_m) \). Then, the error in the projection onto the POD space \( X_R \) satisfies the estimates

\[
\| u^{j,m} - P_R u^{j,m} \|^2 \leq C \left( h^{2s+2} + \Delta t^4 \right) + J_S \sum_{i=R+1}^{J_S (N_S+1)} \lambda_i
\]

\[
\| \nabla (u^{j,m} - P_R u^{j,m}) \|^2 \leq (C + h^2 \| S_R \|_2) h^{2s} + (C + \| S_R \|_2) \Delta t^4 + J_S \sum_{i=R+1}^{J_S (N_S+1)} \| \nabla \varphi_i \|^2 \lambda_i.
\]

Next we need to make an assumption on the regularity of \( u^{j,m}_R \) in order to establish an estimate for the ROM filtering error. We note that this assumption is consistent with our regularity assumption [2.1]

**Assumption 5.6.** We assume that \( \Delta u^{j,m}_R \in L^2 \)

We now state an estimate for the ROM filtering error which is a simple extension of Lemma 4.3 in [17].

**Lemma 5.7.** [ROM filtering error estimates] If \( \Delta u^{j,m}_R \in L^2 \), then the following
estimate holds:

\[ \delta^2 \| \nabla (u^{j,m}_R - \bar{u}^{j,m}_R) \|^2 + \| u^{j,m}_R - \bar{u}^{j,m}_R \|^2 \leq C \left( C \left( h^{2s+2} + \Delta t^4 \right) + J_S \sum_{i=R+1}^{J_S(N_S+1)} \lambda_i \right) \]
\[ + C\delta^2 \left( (C + h^2 \| S_R \|_2) h^{2s} + (C + \| S_R \|_2) \Delta t^4 + J_S \sum_{i=R+1}^{J_S(N_S+1)} \| \nabla \varphi_i \|^2 \lambda_i \right) \]
\[ + C\delta^4 \| \Delta u^{j,m}_R \|^2. \]  

(5.1)

Lastly we state a result for the stability of the ROM filtered variables proven in Lemma 4.4 in [47].

**Lemma 5.8. [ROM stability estimates]** For \( u \in X \), we have

\[ \| \varpi \| \leq \| u \| \]
\[ \| \nabla \varpi \| \leq \| S_R \|_2^{\frac{1}{2}} \| u \|. \]  

(5.2)

For \( u \in X_R \), we have

\[ \| \nabla \varpi \| \leq \| \nabla u \|. \]  

(5.3)

Let \( e^{j,n} = u^{j,n} - u^{j,n}_R \) denote the error between the true solution and the POD approximation; then, we have the following error estimates.

**Theorem 5.9.** Consider the Leray ensemble-POD algorithm and the partition \( 0 = t_0 < t_1 < \cdots < t_{N_S} \) used in Section 3. Suppose for any \( 0 \leq n \leq N_S \), the stability conditions from Theorem 4.7 and all previously stated regularity assumptions hold. Then for any \( 1 \leq N \leq N_S \), there is a positive constant \( C \), such that the following bound holds:
Using (3.6) we split the error

\[ e^{j,n} = u^{j,n} - u^{j,n}_R = (u^{j,n} - P_R u^{j,n}) = (P_R u^{j,n} - u^{j,n}_R) + \eta^{j,n} + \xi^{j,n}_R, \quad j = 1, \ldots, J. \]

Subtracting (3.6) from (5.5) as well as adding and subtracting the terms

\[
\begin{align*}
\frac{1}{2} ||e^{j,N}||^2 + \nu_{\text{max}} \left( \frac{\nu_{\text{max}}}{2} ||\nabla e^{j,N}||^2 + \frac{\epsilon}{4} \nu_{\text{max}} ||\nabla e^{j,N}||^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} ||e^{j,n+1}||^2 \right) \\
\leq \exp \left( C \frac{\nu_{\text{max}} \Delta t}{\epsilon} + C \frac{|\nu_j - \nu_{\text{max}}|^2}{\nu_{\text{max}}} + 2C |||S|||_2^{-\frac{1}{2}} + \frac{C C_{\nu_{\text{max}}}}{\epsilon \nu_{\text{max}}} \Delta t \right) \\
+ \frac{CC_{\nu_{\text{max}}}}{2 \epsilon \nu_{\max}} \left( C (h^{2s+2} + \Delta t^4) + JS \sum_{i=R+1}^{J_\delta(N_{\delta}+1)} ||\nabla \varphi_i||^2 \lambda_i \right) \\
+ \frac{C C_{\nu_{\text{max}}} \Delta t^2}{\epsilon \nu_{\text{max}}} \left( C \frac{|\nu_j - \nu_{\text{max}}|^2}{\nu_{\text{max}}} + C \Delta t |||S|||_2^{-\frac{1}{2}} + \frac{C h^{2s}}{d \epsilon \nu_{\text{max}}} ||p_j||_{L^2}\| \Delta t^2 \| \nu_{\text{max}} \right) \\
+ \left( C + CN \nu_{\text{max}} \Delta t \right) \left( C (h^{2s+2} + \Delta t^4) + JS \sum_{i=R+1}^{J_\delta(N_{\delta}+1)} ||\nabla \varphi_i||^2 \lambda_i \right) \\
+ \left( C + CN \nu_{\text{max}} \Delta t \right) \left( C \frac{h^{2s+2} + \Delta t^4}{2} + JS \sum_{i=R+1}^{J_\delta(N_{\delta}+1)} ||\nabla \varphi_i||^2 \lambda_i \right).
\end{align*}
\]

(5.4)

Proof.

The weak solution of the NSE \( \tilde{u}^j \) satisfies

\[
\left( \frac{u^{j,n+1} - u^{j,n}}{\Delta t}, \varphi \right) + b^*(u^{j,n+1}, u^{j,n+1}, \varphi) + \nu_j (\nabla u^{j,n+1}, \nabla \varphi) - (p^{j,n+1}, \nabla \cdot \varphi) = (f^{j,n+1}, \varphi) + \text{Intp}(u^{j,n+1}; \varphi)
\]

(5.5)

where

\[
\text{Intp}(u^{j,n+1}; \varphi) = \left( \frac{u^{j,n+1} - u^{j,n}}{\Delta t} - u^j(t^{(n+1)}), \varphi \right).
\]

(5.6)

We split the error

\[
e^{j,n} = u^{j,n} - u^{j,n}_R = (u^{j,n} - P_R u^{j,n}) + (P_R u^{j,n} - u^{j,n}_R) = \eta^{j,n} + \xi^{j,n}_R,
\]

(5.7)
$$\nu_{\max}(\nabla u_j^{n+1}, \nabla \varphi) \quad \text{and} \quad \nu_j - \nu_{\max}(\nabla u_j^{n+1}, \nabla \varphi) \quad \text{we have}$$

\[
\frac{\partial}{\partial t} \left( \nabla \xi_R^{j,n+1} \right) + \nu_j - \nu_{\max}(\nabla u_j^{n+1}, \nabla \varphi) + (\nu_j - \nu_{\max})(\nabla (u_j^{n+1} - u_j^{n}), \nabla \varphi) \\
+ (\nu_j - \nu_{\max})(\nabla \xi_R^{j,n}, \nabla \varphi) + b^*(u_j^{n+1}, u_j^{n+1}, \varphi) \\
- b^*(u_j^{n} - <u_R>_n, u_j^{n+1}, \varphi) - b^*(u_j^{n+1} - <u_R>_n, u_j^{n}, \varphi) \\
- (\nu_j - \nu_{\max})(\nabla \xi_R^{j,n}, \nabla \varphi) + Intp(u_j^{n+1}; \varphi).
\]

(5.8)

Setting $\varphi = \xi_R^{j,n+1}$ rearranging the nonlinear terms by adding and subtracting $b^*(u_j^{n} - <u_R>_n, u_j^{n+1}, \xi_R^{j,n+1})$, and using the fact that $(\eta^{j,n+1} - \eta^{j,n}, \xi_R^{j,n+1}) = 0$ by the definition of the $L^2$ projection we have

\[
\frac{1}{\Delta t} \left( \frac{1}{2} \xi_R^{j,n+1} \right) - \frac{1}{\Delta t} \left( \frac{1}{2} \xi_R^{j,n} \right) + \nu_{\max} \left( \xi_R^{j,n+1} \right) \\
= -(\nu_j - \nu_{\max})(\nabla (u_j^{n+1} - u_j^{n}), \nabla \xi_R^{j,n+1}) - (\nu_j - \nu_{\max})(\nabla \xi_R^{j,n}, \nabla \xi_R^{j,n+1}) \\
- \nu_{\max}(\nabla \eta^{j,n+1}, \nabla \xi_R^{j,n+1}) - (\nu_j - \nu_{\max})(\nabla \xi_R^{j,n}, \nabla \xi_R^{j,n+1}) \\
+ b^*(u_j^{n}, u_j^{n+1}, \xi_R^{j,n+1}) - b^*(u_j^{n+1} - <u_R>_n, u_j^{n+1} - u_j^{n}, \xi_R^{j,n+1}) \\
- b^*(u_j^{n+1}, u_j^{n+1}, \xi_R^{j,n+1}) + (\nu_j - \nu_{\max})(\nabla \xi_R^{j,n+1}, \nabla \xi_R^{j,n+1}) + Intp(u_j^{n+1}, \xi_R^{j,n+1}).
\]

(5.9)

We bound the viscous terms in a similar manner to Theorem 3.1 of [10]

\[
-(\nu_j - \nu_{\max})(\nabla (u_j^{n+1} - u_j^{n}), \nabla \xi_R^{j,n+1}) \leq \frac{\Delta t}{4\epsilon} \left( \frac{\nu_j - \nu_{\max}}{\nu_{\max}} \right)^2 \left( \int_{t^n}^{t^{n+1}} \|\nabla u_{j,t}\|^2 dt \right) + \epsilon \nu_{\max} \|\nabla \xi_R^{j,n+1}\|^2,
\]

(5.10)

\[
-\nu_{\max}(\nabla \eta^{j,n+1}, \nabla \xi_R^{j,n+1}) \leq \frac{\nu_{\max}}{4\epsilon} \|\nabla \eta^{j,n+1}\|^2 + \epsilon \nu_{\max} \|\nabla \xi_R^{j,n+1}\|^2,
\]

(5.11)

\[
-(\nu_j - \nu_{\max})(\nabla \eta^{j,n}, \nabla \xi_R^{j,n+1}) \leq \frac{1}{4\epsilon} \left( \frac{\nu_j - \nu_{\max}}{\nu_{\max}} \right)^2 \|\nabla \eta^{j,n}\|^2 + \epsilon \nu_{\max} \|\nabla \xi_R^{j,n+1}\|^2,
\]

(5.12)

\[
-(\nu_j - \nu_{\max})(\nabla \xi_R^{j,n}, \nabla \xi_R^{j,n+1}) \leq \frac{|\nu_j - \nu_{\max}|}{2} \|\nabla \xi_R^{j,n}\|^2 + \frac{|\nu_j - \nu_{\max}|}{2} \|\nabla \xi_R^{j,n+1}\|^2.
\]

(5.13)
We next rewrite the second nonlinear term on the right hand side of (5.9).
\[
\begin{align*}
&b^*(u_R^{j,n} - <u_R>^n, u_R^{j,n+1} - u_R^{j,n}, \xi_R^{j,n+1}) \\
&= -b^*(u_R^{j,n} - <u_R>^n, \eta^{j,n+1} - \psi^{j,n}, \xi_R^{j,n+1}) \\
&\quad + b^*(u_R^{j,n} - <u_R>^n, u_R^{j,n+1} - u_R^{j,n}, \xi_R^{j,n+1}) \\
&= -b^*(u_R^{j,n} - <u_R>^n, \eta^{j,n+1} + \xi_R^{j,n+1}) \\
&\quad + b^*(u_R^{j,n} - <u_R>^n, \psi^{j,n}, \xi_R^{j,n+1}) \\
&\quad + b^*(u_R^{j,n} - <u_R>^n, \xi_R^{j,n}, \xi_R^{j,n+1}) \\
&\quad + b^*(u_R^{j,n} - <u_R>^n, u_R^{j,n+1} - u_R^{j,n}, \xi_R^{j,n+1}).
\end{align*}
\]
(5.14)

As done in Theorem 3.1 of [10], using Young’s inequality, (2.3), (2.4), and (3.3), we derive the estimates
\[
\begin{align*}
-b^*(u_R^{j,n} - <u_R>^n, \eta^{j,n+1}, \xi_R^{j,n+1}) \leq \frac{C_{b^*} \nu_{\text{max}}^{-1}}{4\epsilon} \|\nabla (u_R^{j,n} - <u_R>^n)\|^2 \|\nabla \eta^{j,n+1}\|^2 + \epsilon \nu_{\text{max}} \|\nabla \xi_R^{j,n+1}\|^2, \quad (5.15) \\
b^*(u_R^{j,n} - <u_R>^n, \eta^{j,n}, \xi_R^{j,n+1}) \leq \frac{C_{b^*} \nu_{\text{max}}^{-1}}{4\epsilon} \|\nabla (u_R^{j,n} - <u_R>^n)\|^2 \|\nabla \eta^{j,n}\|^2 + \epsilon \nu_{\text{max}} \|\nabla \xi_R^{j,n+1}\|^2, \quad (5.16) \\
b^*(u_R^{j,n} - <u_R>^n, u_R^{j,n+1} - u_R^{j,n}, \xi_R^{j,n+1}) \leq \frac{CC_{b^*} \nu_{\text{max}}^{-1}}{4\epsilon} \|\nabla (u_R^{j,n} - <u_R>^n)\|^2 + \epsilon \nu_{\text{max}} \|\nabla \xi_R^{j,n+1}\|^2. \quad (5.17)
\end{align*}
\]

By skew-symmetry, inequality (2.4) and the inverse inequality (3.3), we have
\[
\begin{align*}
b^*(u_R^{j,n} - <u_R>^n, \xi_R^{j,n}, \xi_R^{j,n+1}) &\leq C_{b^*} \|\nabla (u_R^{j,n} - <u_R>^n)\| \|\nabla \xi_R^{j,n}\| \sqrt{\|\xi_R^{j,n+1} - \xi_R^{j,n}\| \|\nabla (\xi_R^{j,n+1} - \xi_R^{j,n})\|} \\
&\leq C_{b^*} \|\|S_R\|\|^{1/4} \|\nabla (u_R^{j,n} - <u_R>^n)\| \|\nabla \xi_R^{j,n}\| \sqrt{\|\xi_R^{j,n+1} - \xi_R^{j,n}\|} \\
&\leq \frac{1}{2\Delta t} \|\xi_R^{j,n+1} - \xi_R^{j,n}\|^2 + \left( \frac{C_{b^*} \Delta t}{2} \|\|S_R\|\|^{1/2} \|\nabla (u_R^{j,n} - <u_R>^n)\| \right)^2 \|\nabla \xi_R^{j,n}\|^2. \quad (5.18)
\end{align*}
\]

Bounding the other two nonlinear terms we add and subtract the terms $b^*(u_R^{j,n}, u_R^{j,n+1}, \xi_R^{j,n+1})$ and $b^*(u_R^{j,n}, u_R^{j,n+1}, \xi_R^{j,n+1})$. It then follows from (2.3)
\[
\begin{align*}
&-b^*(u_R^{j,n+1}, u_R^{j,n+1}, \xi_R^{j,n+1}) + b^*(u_R^{j,n}, u_R^{j,n+1}, \xi_R^{j,n+1}) \\
&= -b^*(u_R^{j,n} - u_R^{j,n}, u_R^{j,n+1}, \xi_R^{j,n+1}) - b^*(u_R^{j,n}, \eta^{j,n+1}, \xi_R^{j,n+1}) \\
&\quad - b^*(u_R^{j,n+1} - u_R^{j,n}, u_R^{j,n+1}, \xi_R^{j,n+1}).
\end{align*}
\]
(5.19)

Now by Young’s inequality, (2.4), the stability analysis, i.e.
\[ \|u^{j,n}_{R}\|^2 \leq C_{\text{stab}}, \text{and the assumption } w^j \in L^\infty(0,T,H^1(\Omega)) \text{ we have} \]
\[
 b^*(u^{j,n}_{R},\eta^{j,n+1},\xi^{j,n+1}) \leq C_b \|\nabla u^{j,n}_{R}\| \frac{\Delta t}{2} \|\nabla u^{j,n}_{R}\| \frac{\Delta t}{2} \|\nabla \eta^{j,n+1}\| \|\nabla \xi^{j,n+1}\| \\
 \leq \frac{C_{\text{stab}} C_b^2}{4\tilde{c}} \nu_{\max}^{-1} \|\nabla u^{j,n}_{R}\| \|\nabla \eta^{j,n+1}\|^2 + \tilde{c} \nu_{\max} \|\nabla \xi^{j,n+1}\|^2, \tag{5.20}
\]

as well as
\[
 b^*(u_j^{j,n+1} - u_j^{j,n},\xi^{j,n+1}) = \]
\[
 -b^*(\epsilon^{j,n},u_j^{j,n+1},\xi^{j,n+1}) - b^*(u_j^{j,n} - u_j^{j,n+1},\xi^{j,n+1}). \tag{5.22}
\]

Bounding the second term
\[
 -b^*(u_j^{j,n} - u_j^{j,n+1},\xi^{j,n+1}) = \\
 \leq C_b \|u_j^{j,n} - u_j^{j,n+1}\| \|\nabla (u_j^{j,n} - u_j^{j,n+1})\| \|\nabla u_j^{j,n+1}\| \|\nabla \xi^{j,n+1}\| \\
 \leq \frac{C_b^2}{4\tilde{c}} \nu_{\max}^{-1} \|u_j^{j,n} - u_j^{j,n+1}\| \|\nabla (u_j^{j,n} - u_j^{j,n+1})\| \|\nabla u_j^{j,n+1}\|^2 + \tilde{c} \nu_{\max} \|\nabla \xi^{j,n+1}\|^2. \tag{5.23}
\]

Then after decomposing \(\epsilon^{j,n} = \eta^{j,n} + \xi^{j,n}\) again using Young’s inequality and the assumption \(w^j \in L^\infty(0,T,H^1(\Omega))\)

\[
 -b^*(\eta^{j,n},u_j^{j,n+1},\xi^{j,n+1}) \leq \frac{CC_b^2}{4\tilde{c}} \nu_{\max}^{-1} \|\nabla \eta^{j,n}\|^2 + \tilde{c} \nu_{\max} \|\nabla \xi^{j,n+1}\|^2 \tag{5.24}
\]

and

\[
 -b^*(\xi^{j,n},u_j^{j,n+1},\xi^{j,n+1}) \leq C_b \|\nabla \xi^{j,n}\| \|\nabla \xi^{j,n}\| \|\nabla u_j^{j,n+1}\| \|\nabla \xi^{j,n+1}\| \\
 \leq C_b \left( \alpha \|\nabla \xi^{j,n}\|^2 + \frac{1}{4\alpha} \|\nabla \xi^{j,n}\| \|\xi^{j,n}\| \right) \\
 \leq C_b \left( \alpha \|\nabla \xi^{j,n}\|^2 + \frac{1}{4\alpha} \left( \beta \|\nabla \xi^{j,n}\|^2 + \frac{1}{\beta} \|\xi^{j,n}\|^2 \right) \right) \\
 = \nu_{\max} \|\nabla \xi^{j,n}\|^2 + \frac{13\tilde{c}}{4} \nu_{\max} \|\nabla \xi^{j,n}\|^2 + \frac{C_b^2 C^4}{52 \nu_{\max}} \|\eta^{j,n}\|^2. \tag{5.25}
\]

For the pressure term since \(\xi^{j,n+1} \in X^R \subset V^h\) it follows for \(q_h \in Q^h\)

\[
 (p_j^{j,n+1}, \nabla \cdot \xi^{j,n+1}) = (p_j^{j,n+1} - q_h^{j,n+1}, \nabla \cdot \xi^{j,n+1}) \leq \tilde{c} \nu_{\max} \|\nabla \xi^{j,n+1}\|^2 + \frac{\nu_{\max}^{-1}}{4\tilde{c}} \|p_j^{j,n+1} - q_h^{j,n+1}\|^2. \tag{5.26}
\]

For the last term we have

\[
 \text{Intp}(u_j^{j,n+1},\xi^{j,n+1}) \leq \frac{\|w_j^{j,n+1} - u_j^{j,n}\|}{\Delta t} \tag{5.27}
\]

\[
 \leq \frac{C \Delta t}{4\tilde{c}} \nu_{\max}^{-1} \tilde{t} + \tilde{c} \nu_{\max} \|\nabla \xi^{j,n+1}\|^2.
\]
Now combining (5.10) - (5.27), (5.9) becomes

\[
\frac{1}{\Delta t} \left( \frac{1}{2} \| \xi_{i,n+1} \|^2 - \frac{1}{2} \| \xi_{i,n} \|^2 \right) + \nu_{\text{max}} \| \nabla \xi_{i,n+1} \|^2 - \frac{C_b^2 \Delta t}{2} \| \mathcal{S}_R \|^2 \| \nabla (u_{R>0}^i - u_R) \|^2 \| \nabla \xi_{i,n} \|^2 \\
- \frac{13\tilde{\epsilon}}{4} \nu_{\text{max}} \| \nabla \xi_{i,n+1} \|^2 - \frac{13\tilde{\epsilon}}{4} \nu_{\text{max}} \| \nabla \xi_{i,n} \|^2 \\
- \frac{|\nu_j - \nu_{\text{max}}|}{2} \| \nabla \xi_{i,n+1} \|^2 - \frac{|\nu_j - \nu_{\text{max}}|}{2} \| \nabla \xi_{i,n} \|^2 \\
\leq \frac{C_b^2 C^4}{52 \nu_{\text{max}}^3 \tilde{\epsilon}^3} \| \xi_{i,n} \|^2 + \frac{C \Delta t}{4 \tilde{\epsilon}} \| \xi_{i,n} \|^2 + \frac{\nu_{\text{max}}}{4 \tilde{\epsilon}} \| \nabla \eta_{i,n+1} \|^2 \\
+ \frac{CC_b^2 \Delta t}{4 \tilde{\epsilon} \nu_{\text{max}}} \| \nabla (u_{R>0}^i) \|^2 \\
\frac{1}{4 \tilde{\epsilon} \nu_{\text{max}}} \| \xi_{i,n+1} \|^2 - \frac{q_{\text{stab}}^2}{4} \| \xi_{i,n} \|^2 + \frac{C \Delta t}{4 \tilde{\epsilon} \nu_{\text{max}}} \\
+ \frac{CC_b^2 \Delta t}{4 \tilde{\epsilon} \nu_{\text{max}}} + \frac{C_b^2}{4 \tilde{\epsilon} \nu_{\text{max}}} \| |\nabla (u_{R>0}^i - u_R)| \| \| \nabla u_{i,n+1} \| \| \nabla \eta_{i,n} \|^2 \\
+ \frac{1}{4 \tilde{\epsilon}} \| \nu_j - \nu_{\text{max}} \|^2 \| \nabla \eta_{i,n} \|^2 + \frac{C_b^2 \nu_{\text{max}}}{4 \tilde{\epsilon} \nu_{\text{max}}} \| \nabla (u_{R>0}^i) \|^2 \| \nabla \eta_{i,n} \|^2 \\
+ \frac{C_b^2 \nu_{\text{max}}}{4 \tilde{\epsilon} \nu_{\text{max}}} \| \nabla \eta_{i,n} \|^2 \| \nabla \eta_{i,n} \|^2 \\
+ \frac{C_b^2}{4 \tilde{\epsilon} \nu_{\text{max}}} \| \nabla \eta_{i,n} \|^2.
\]

(5.28)

The terms on the LHS of (5.28) (except first) can be rearranged as follows:

\[
\left( \nu_{\text{max}} - \frac{13\tilde{\epsilon}}{4} \nu_{\text{max}} - \frac{|\nu_j - \nu_{\text{max}}|}{2} \right) \| \nabla \xi_{i,n+1} \|^2 \\
- \left( \frac{13\tilde{\epsilon}}{4} \nu_{\text{max}} + \frac{|\nu_j - \nu_{\text{max}}|}{2} \right) + \frac{C_b^2 \Delta t}{2} \| \mathcal{S}_R \|^2 \| \nabla (u_{R>0}^i - u_R) \|^2 \| \nabla \xi_{i,n} \|^2.
\]

(5.29)

Choosing \( \tilde{\epsilon} = \frac{\epsilon}{13} \) and using (4.1) in (5.29), (5.28) yields
\[
\frac{1}{\Delta t} \left( \frac{1}{2} \| \xi_{j,n+1}^a \|^2 - \frac{1}{2} \| \xi_{j,n}^a \|^2 \right) \\
+ \left( \frac{\nu_{\text{max}}}{2} + \frac{\epsilon \nu_{\text{max}}}{4} \right) \left( \| \nabla \xi_{j,n+1}^a \|^2 - \| \nabla \xi_{j,n}^a \|^2 \right) \\
+ \nu_{\text{max}} \left( \frac{\epsilon}{2} - \frac{C_b^2 \Delta t}{2\nu_{\text{max}}} \| \mathcal{S}_R \| \frac{1}{2} \| \nabla (u_{j,n}^R - <u_R>_n) \|^2 \right) \| \nabla \xi_{j,n}^a \|^2 \\
\leq \frac{13}{52} C_b^4 \frac{C^4}{\nu_{\text{max}}^3} \frac{\Delta t}{\epsilon} \| \xi_{j,n}^a \|^2 \\
+ \frac{13}{4 \epsilon} C \frac{\Delta t}{\nu_{\text{max}}} \| \nabla (u_{j,n}^R - <u_R>_n) \|^2 \\
+ \frac{13}{4 \epsilon} \frac{\Delta t}{\nu_{\text{max}}} \| \nabla \xi_{j,n+1}^a \|^2 + \frac{13}{4 \epsilon} \frac{\Delta t}{\nu_{\text{max}}} \| \nabla \xi_{j,n}^a \|^2 \\
+ \frac{13}{4 \epsilon} \frac{\Delta t}{\nu_{\text{max}}} \| \nabla \xi_{j,n}^a \|^2. \\
\]

(5.30)
It follows from the stability condition (4.2) that

\[
\nu_{\text{max}} \left( \frac{\epsilon - C_\nu^2 \Delta t}{2 \nu_{\text{max}}} \| S_R \| \right) \frac{1}{2} \| \nabla (u_j^{i,n} - u_R^j) \|^2 \geq C \nu_{\text{max}} \geq 0 \quad (5.31)
\]

Now we use (5.31), sum (5.30) from \( n = 0 \) to \( N - 1 \), multiply both sides by \( \Delta t \), and absorb constants. Since \( u_R^j = \sum_{i=1}^{R} (u_j^{i,0}, \varphi_i) \), we have \( \| \xi_{j,0}^j \|^2 = 0 \) and \( \| \nabla \xi_{j,0}^j \|^2 = 0 \). It then follows from (5.30) that we have

\[
\frac{1}{2} \| \xi_{j,n}^j \|^2 + \frac{\nu_{\text{max}}}{2} \| \nabla \xi_{j,n}^j \|^2 + \frac{\epsilon}{4} \nu_{\text{max}} \| \nabla \xi_{j,n}^j \|^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} \| \nabla \xi_{j,n}^j \|^2
\]

\[
\leq \Delta t \sum_{n=0}^{N-1} \left\{ C_\nu^4 \frac{C_\nu^4}{\epsilon^2 \nu_{\text{max}}} \| \xi_{j,n}^j \|^2 + \frac{C \nu_j - \nu_{\text{max}}}{\epsilon} \frac{\| \nabla (u_j^{i,n} < u_R > n) \|^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C \nu_j - \nu_{\text{max}}}{\epsilon} \frac{\| \nabla (u_j^{i,n} < u_R > n) \|^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C \nu_j - \nu_{\text{max}}}{\epsilon} \frac{\| \nabla (u_j^{i,n} - u_R^j) \|^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C \nu_j - \nu_{\text{max}}}{\epsilon} \frac{\| \nabla (u_j^{i,n} - u_R^j) \|^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 \right\} + \frac{C C_\nu^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C C_\nu^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C C_\nu^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2 + \frac{C C_\nu^2}{\nu_{\text{max}}} \| \nabla \eta_{j,n}^j \|^2
\]

Now using assumption 5.5, lemma 5.8, and the stability result from theorem 4.1, i.e. \( \frac{\nu_{\text{max}}}{2} \sum_{n=0}^{N-1} \| \nabla \xi_{j,n}^j \|^2 \leq C_{\text{stab}} \), we have

\[
\frac{\Delta t}{\epsilon} C_\nu^2 C_{\text{stab}} \sum_{n=0}^{N-1} \| \nabla u_{j,n}^j \| \| \nabla \eta_{j,n}^j \|^2 \leq C_{\text{stab}}
\]

\[
\leq \Delta t C_\nu^2 C_{\text{stab}} \frac{1}{\epsilon} \sum_{n=0}^{N-1} \| \nabla \xi_{j,n}^j \| \| \nabla \eta_{j,n}^j \|^2 \quad (5.33)
\]

\[
\left( \epsilon + \nu_{\text{max}}^2 \| S_R \| \right) \frac{1}{2} \| \nabla \xi_{j,n}^j \| \| \nabla \eta_{j,n}^j \|^2 \leq C_{\text{stab}} \left( \sum_{n=0}^{N-1} \| \nabla \xi_{j,n}^j \| \| \nabla \eta_{j,n}^j \|^2 \right)
\]

\[
= \frac{\Delta t}{\epsilon} C_\nu^2 C_{\text{stab}} \left( \sum_{n=0}^{N-1} \frac{1}{2} \| \nabla \xi_{j,n}^j \|^2 \right)
\]

\[
= C_\nu^2 C_{\text{stab}} \frac{2}{\epsilon} \sum_{n=0}^{N-1} \| \nabla \xi_{j,n}^j \|^2 \| \nabla \eta_{j,n}^j \|^2 \quad (5.34)
\]
Now combining everything, absorbing constants, invoking the discrete Gronwall's inequality, using Assumption 2.1 and the stability estimate (4.2) (5.32) becomes

It then follows that

\[
\frac{\Delta t}{\epsilon \nu_{\max}} C C_{b, C_{\text{stab}}}^2 \sum_{n=0}^{N-1} \| \nabla u_{j,n}^{i,n} \| \| \nabla \eta_{j,n}^{i,n+1} \|^2 \leq \left( \frac{C C_{b, C_{\text{stab}}}^2}{2 \epsilon \nu_{\max}} + \frac{2 C C_{b, C_{\text{stab}}}^2}{\epsilon^2 \nu_{\max}} \right) \times \left( C + h^2 \| S_R \|_2 \right) + J S \sum_{i=R+1}^{J_S (N_S + 1)} \| \nabla \varphi_i \|^2 \lambda_i \right). \tag{5.35}
\]

Next using lemma 5.4 and Assumptions 2.1 and 5.6

\[
\frac{C C_{b, C_{\text{stab}}}^2}{\epsilon \nu_{\max}} \Delta t \sum_{n=0}^{N-1} \| \nabla u_{j,n}^{i,n+1} \|^2 \frac{1}{\delta} \left[ C \left( C (h^{2s+2} + \Delta t^4) + J_S \sum_{i=R+1}^{J_S (N_S + 1)} \lambda_i \right) 
+ C \delta^2 \left( (C + h^2 \| S_R \|_2) h^{2s} + (C + \| S_R \|_2) \Delta t^4 + J_S \sum_{i=R+1}^{J_S (N_S + 1)} \| \nabla \varphi_i \|^2 \lambda_i \right) 
+ C \delta^4 \| \Delta u_{j,n}^{i,n} \|^2 \right] \leq \frac{C C_{b, C_{\text{stab}}}^2}{\epsilon \nu_{\max}} \Delta t \sum_{n=0}^{N-1} \| \nabla u_{j,n}^{i,n+1} \|^2 \frac{1}{\delta} \left[ C \left( C (h^{2s+2} + \Delta t^4) + J_S \sum_{i=R+1}^{J_S (N_S + 1)} \lambda_i \right) 
+ C \delta^2 \left( (C + h^2 \| S_R \|_2) h^{2s} + (C + \| S_R \|_2) \Delta t^4 + J_S \sum_{i=R+1}^{J_S (N_S + 1)} \| \nabla \varphi_i \|^2 \lambda_i \right) 
+ C \delta^4 \| \Delta u_{j,n}^{i,n} \|^2 \right]. \tag{5.36}
\]

Next using theorem 4.4 we have

\[
\frac{\Delta t}{\epsilon \nu_{\max}} \| \nabla (u_{j,n}^{i,n} - \overline{u_R^i < u_R >}) \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}} \| \nabla (u_{j,n}^{i,n} - \overline{u_R^i < u_R >}) \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}}. \tag{5.37}
\]

Therefore we can bound the quantities

\[
\frac{\Delta t}{\epsilon \nu_{\max}} \| \nabla (u_{j,n}^{i,n} - \overline{u_R^i < u_R >}) \|^2 \| \eta_{j,n}^{i,n+1} \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}} \| \eta_{j,n}^{i,n+1} \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}} \| \eta_{j,n}^{i,n+1} \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}} \| \eta_{j,n}^{i,n+1} \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}} \| \eta_{j,n}^{i,n+1} \|^2 \leq C \Delta t \| \| S_R \|_2^{-\frac{1}{2}} \| \eta_{j,n}^{i,n+1} \|^2 \leq C \| \| S_R \|_2^{-\frac{1}{2}}. \tag{5.38}
\]

Now combining everything, absorbing constants, invoking the discrete Gronwall’s inequality, using Assumption 5.5 and the stability estimate (4.2) (5.32) becomes
The mesh utilized contains 14,590 degrees of freedom and is given in figure 6.1. We consider the two-dimensional flow between offset cylinders used in [17]. The domain is a disk with a smaller off-center disc inside. Let \( r_1 = 1, r_2 = 0.1, c_1 = 1/2, \) and \( c_2 = 0; \) then, the domain is given by

\[
\Omega = \{ (x, y) : x^2 + y^2 \leq r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 \geq r_2^2 \}.
\]

The mesh utilized contains 14,590 degrees of freedom and is given in figure 5.1. We discretize in space via the \( P^2 - P^1 \) Taylor-Hood element pair. The no-slip, no-penetration boundary conditions are imposed on both cylinders. In our test problems the flow

\[
\frac{1}{2} \| \xi_R^{j,n} \|^2 + \frac{\nu_{\text{max}}}{2} \| \nabla \xi_R^{j,n} \|^2 + \frac{\epsilon}{4} \nu_{\text{max}} \| \nabla \xi_R^{j,n} \|^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} \| \nabla \xi_R^{j,n+1} \|^2 \\
\leq \exp \left( C \frac{h^4 T}{\epsilon^4 \nu_{\text{max}}^3} \right) \left[ \frac{C \Delta t}{\epsilon \nu_{\text{max}}} \| \xi_j - \xi_{\text{max}} \|^2 + 2C \| \mathcal{S}_R \|_2 \frac{1}{\sqrt{C}} + \frac{C C_{\delta}^2 \Delta t}{\epsilon \nu_{\text{max}}} \right] \\
+ \frac{C C_{\delta}^2 \Delta t}{\epsilon \nu_{\text{max}}} \left( C \left( h^4 + \Delta t^4 \right) + JS \sum_{i=R+1}^{JS(N_S+1)} \| \nabla \varphi_i \|^2 \lambda_i \right) \\
\left( C + h^2 \| \mathcal{S}_R \|_2 \right) h^2 + \left( C + \| \mathcal{S}_R \|_2 \right) \Delta t^4 + JS \sum_{i=R+1}^{JS(N_S+1)} \| \nabla \varphi_i \|^2 \lambda_i \\
+ \frac{C C_{\delta}^2 \Delta t^2}{\epsilon \nu_{\text{max}}} \frac{\| \nabla \xi_j - \nabla \xi_{\text{max}} \|^2}{\nu_{\text{max}}} + \frac{C \Delta t^2}{\epsilon \nu_{\text{max}}} \| \mathcal{S}_R \|_2 \frac{1}{\sqrt{C}} + \frac{C h^2}{\epsilon \nu_{\text{max}}} \| \nabla \varphi_i \|^2 \lambda_i + \frac{C \Delta t^2}{\epsilon \nu_{\text{max}}} \right].
\]

By the triangle inequality we have \( \| e^{j,n} \|^2 \leq 2(\| \xi_R^{j,n} \|^2 + \| \eta^{j,n} \|^2) \) from which it follows

\[
\frac{1}{2} \| \xi_R^{j,n} \|^2 + \frac{\nu_{\text{max}}}{2} \| \nabla \xi_R^{j,n} \|^2 + \frac{\epsilon}{4} \nu_{\text{max}} \| \nabla \xi_R^{j,n} \|^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} \| e^{j,n+1} \|^2 \\
\leq \| \eta^{j,n} \|^2 + \| \nabla \eta^{j,n} \|^2 + \frac{\epsilon}{2} \nu_{\text{max}} \| \nabla \eta^{j,n} \|^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} \| e^{j,n+1} \|^2 \]

\[
+ \| \xi_R^{j,n} \|^2 + \| \nabla \xi_R^{j,n} \|^2 + \frac{\epsilon}{2} \nu_{\text{max}} \| \nabla \xi_R^{j,n} \|^2 + C \nu_{\text{max}} \Delta t \sum_{n=0}^{N-1} \| \xi_R^{j,n+1} \|^2.
\]

Now applying inequality (5.39) and Assumption 5.5 the result follows. \( \Box \)

6. Numerical Experiments. In this section we provide numerical experiments for the Leray ensemble-POD algorithm (3.6) demonstrating the efficacy of this approach. All computations will be done using the FEniCS software suite [34] and all meshes generated via the built in meshing package mshr.

6.1. Problem Setting. For the numerical experiments we consider the two-dimensional flow between offset cylinders used in [17]. The domain is a disk with a smaller off-center disc inside. Let \( r_1 = 1, r_2 = 0.1, c_1 = 1/2, \) and \( c_2 = 0; \) then, the domain is given by

\[
\Omega = \{ (x, y) : x^2 + y^2 \leq r_1^2 \text{ and } (x - c_1)^2 + (y - c_2)^2 \geq r_2^2 \}.
\]
will be driven by the counterclockwise rotational body force

\[ f(x, y, t) = \left( -4y(1 - x^2 - y^2), 4x(1 - x^2 - y^2) \right)^T. \]

This flow displays interesting structures which interact with the inner circle. Specifically the flow rotates about the origin and interacts with the immersed cylinder forming a Von Kármán vortex street.

6.2. Numerical Results. In this experiment we demonstrate the improved accuracy and stability of the Leray ensemble-POD algorithm. In order to generate the POD basis we use two different viscosities \( \nu_1 = .0016 \) and \( \nu_2 = .002 \). The initial conditions will be generated by solving a steady Stokes problem using the previously defined counterclockwise rotational body force. We run a finite element code utilizing a linearly implicit backwards Euler method for each viscosity from \( t_0 = 0 \) to \( T = 6 \) with fixed time step \( \Delta t = .01 \). At time \( T = 3.0 \) we begin taking snapshots every .04 seconds. In Figure 6.2 we show the decay of the singular values for the snapshot matrix.

To illustrate the accuracy of the Leray ensemble-POD algorithm we compare it against the ensemble-POD algorithm using the same viscosities from the offline stage. The computations are carried out over the time interval \( t_0 = 3 \) to \( T = 6 \) with fixed time step \( \Delta t = .01 \) and \( r = 10 \) reduced basis functions. The initial condition at \( T = 3 \) is the \( L^2 \) projection of the FE solution at \( T = 3.0 \) into the POD space The filtering length for the Leray ensemble-POD algorithm is taken to be \( \delta = .025 \). The filtering length is selected as the value of \( \delta \) which allows the average kinetic energy of Leray ensemble-POD to most closely match the average kinetic energy of the benchmark solution. We purposefully utilize a small number of basis functions to demonstrate the situation where the ROM does not allow for all spatial scales to be resolved. To
determine the accuracy of our methods the average of the solutions from the implicit backwards Euler method for $\nu_1$ and $\nu_2$ will be used as a benchmark.

In Figure 6.3 we compare the average kinetic energy evolution of the Leray ensemble-POD and ensemble-POD against our benchmark solution. It can be seen that the ensemble-POD fails to match the kinetic energy of the benchmark solution, while the Leray ensemble-POD approximates it reasonably well. In Figure 6.4 we compare the evolution of the error in the $L^2$ norm of Leray ensemble-POD and ensemble-POD algorithms. The Leray ensemble-POD has a significantly smaller error than the ensemble-POD algorithm. In Figure 6.5 we plot the average POD mode evolution for ensemble-POD versus Leray ensemble-POD. We see that the oscillations in the POD modes are damped for Leray ensemble-POD.

7. Conclusions. In this work, a Leray regularized ensemble-POD method is developed for the incompressible Navier-Stokes equations with perturbations in the forcing function, initial conditions, and viscosities. The proposed algorithm is first ensemble-POD approach designed to work for higher Reynolds number flows. The stability and convergence of the finite element discretization of the Leray ensemble-POD model are proven. In the numerical simulation of two-dimensional flow past two offset cylinders, it is shown that the Leray ensemble-POD model is significantly more accurate than the standard ensemble-POD model.

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For $3 \leq t \leq 6$, the average energy of the Leray ensemble-POD, ensemble-POD, and Implicit Euler method.

\[ \text{Fig. 6.3.} \]

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Fig. 6.4. For $3 \leq t \leq 6$, the $L^2$ error evolution of the Leray ensemble-POD and ensemble-POD algorithms with $r = 10$ basis functions.

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