THE TWO DIMENSIONAL HANNAY-BERRY MODEL

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Abstract

The main goal of this paper is to construct the Hannay-Berry model of quantum mechanics, on a two dimensional symplectic torus. We construct a simultaneous quantization of the algebra of functions and the linear symplectic group $\Gamma = \text{SL}_2(\mathbb{Z})$. We obtain the quantization via an action of $\Gamma$ on the set of equivalence classes of irreducible representations of Rieffel's quantum torus $\mathcal{A}_h$. For $h \in \mathbb{Q}$ this action has a unique fixed point. This gives a canonical projective equivariant quantization. There exists a Hilbert space on which both $\Gamma$ and $\mathcal{A}_h$ act equivariantly. Combined with the fact that every projective representation of $\Gamma$ can be lifted to a linear representation, we also obtain linear equivariant quantization.

0 Introduction

0.1 Motivation

In the paper “Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating”, published in 1980 (cf. [HB]), the physicists J. Hannay and M.V. Berry explore a model for quantum mechanics on the 2-dimensional torus. Hannay and Berry suggested to quantize simultaneously the functions on the torus and the linear symplectic group $\Gamma = \text{SL}_2(\mathbb{Z})$. They found (cf. [HB], [Me]) that the theta subgroup $\Gamma_\Theta \subset \Gamma$ is the largest that one can quantize and asked (cf. [HB], [Me]) whether the quantization of $\Gamma$ satisfy a multiplicativity property (i.e., is a linear representation of the group). In this paper we want to construct the Hannay-Berry’s model for the bigger group of symmetries, i.e., the whole symplectic group $\Gamma$. The central question is whether there exists a Hilbert space on which a deformation of the algebra of functions and the linear symplectic group $\Gamma$ both act in a compatible way.
0.2 Results

In this paper we give an affirmative answer to the existence of the quantization procedure. We show a construction (Theorem 0.3, Corollary 0.4 and Theorem 0.5) of the canonical equivariant quantization procedure for rational Planck constants. It is unique as a projective quantization (see definitions below). We show that the projective representation of \( \Gamma \) can be lifted in exactly 12 different ways to a linear representation (to obey the multiplicativity property). These are the first examples of such equivariant quantization for the whole symplectic group \( \Gamma \). Our construction slightly improves the known constructions \([HB, Me, KR1]\) for which the group of quantizable elements is \( \Gamma_\Theta \subset \Gamma \) and gives a positive answer to the Hannay-Berry question on the linearization of the projective representation of the group of quantizable elements. (cf. \([HB, Me]\)). Previously it was shown by Mezzadri and Kurlberg-Rudnick (cf. \([Me, KR1]\)) that one can construct an equivariant quantization for the theta subgroup, in case when the Planck constant is of the form \( \hbar = \frac{1}{N}, \ N \in \mathbb{N} \).

0.2.1 Classical torus

Let \((T, \omega)\) be the two dimensional symplectic torus. Together with its linear symplectomorphisms \( \Gamma \simeq \text{SL}_2(\mathbb{Z}) \) it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let \( T = W/\Lambda \) where \( W \) is a two dimensional real vector space, i.e., \( W \simeq \mathbb{R}^2 \) and \( \Lambda \) is a rank two lattice in \( W \), i.e., \( \Lambda \simeq \mathbb{Z}^2 \). We obtain the symplectic form on \( T \) by taking a non-degenerate symplectic form on \( W \):

\[
\omega : W \times W \rightarrow \mathbb{R}.
\]

We require \( \omega \) to be integral, namely \( \omega : \Lambda \times \Lambda \rightarrow \mathbb{Z} \) and normalized, i.e., \( \text{Vol}(T) = 1 \).

Let \( \text{Sp}(W, \omega) \) be the group of linear symplectomorphisms, i.e., \( \text{Sp}(W, \omega) \simeq \text{SL}_2(\mathbb{R}) \). Consider the subgroup \( \Gamma \subset \text{Sp}(W, \omega) \) of elements that preserve the lattice \( \Lambda \), i.e., \( \Gamma(\Lambda) \subseteq \Lambda \). Then \( \Gamma \simeq \text{SL}_2(\mathbb{Z}) \). The subgroup \( \Gamma \) is the group of linear symplectomorphisms of \( T \).

We denote by \( \Lambda^* \subseteq W^* \) the dual lattice:

\[
\Lambda^* := \{ \xi \in W^* | \xi(\Lambda) \subset \mathbb{Z} \}.
\]

The lattice \( \Lambda^* \) is identified with the lattice \( T^\vee := \text{Hom}(T, \mathbb{C}^*) \) of characters of \( T \) by the following map:

\[
\xi \in \Lambda^* \mapsto e^{2\pi i \langle \xi, \cdot \rangle} \in T^\vee.
\]

The form \( \omega \) allows us to identify the vector spaces \( W \) and \( W^* \). For simplicity we will denote the induced form on \( W^* \) also by \( \omega \).
0.2.2 Equivariant quantization of the torus

We will construct a particular type of quantization procedure for the functions. Moreover this quantization will be equivariant with respect to the action of the “classical symmetries” \( \Gamma \):

**Definition 0.1** By Weyl quantization of \( A \) we mean a family of \( C \)-linear, \(*\)-morphisms \( \pi_h : A \to \text{End}(\mathcal{H}_h) \), \( h \in \mathbb{R} \), where \( \mathcal{H}_h \) is a Hilbert space, s.t. the following property holds:

\[
\pi_h(\xi + \eta) = e^{\pi i h w(\xi,\eta)} \pi_h(\xi) \pi_h(\eta)
\]

for all \( \xi, \eta \in \Lambda^* \) and \( h \in \mathbb{R} \).

This type of quantization procedure will obey the “usual” properties (cf. [D4]):

\[
||\pi_h(fg) - \pi_h(f)\pi_h(g)||_{\mathcal{H}_h} \to 0, \quad \text{as} \quad h \to 0,
\]

\[
||i \frac{\pi_h(f) - \pi_h(g)}{h}||_{\mathcal{H}_h} \to 0, \quad \text{as} \quad h \to 0.
\]

where \( \{,\} \) is the Poisson brackets on functions.

**Definition 0.2** By equivariant quantization of \( T \) we mean a quantization of \( A \) with additional maps \( \rho_h : \Gamma \to \text{U}(\mathcal{H}_h) \) s.t. the following equivariant property (called Egorov’s identity) holds:

\[
\rho_h(B)^{-1} \pi_h(f) \rho_h(B) = \pi_h(f \circ B) \quad (0.2.1)
\]

for all \( h \in \mathbb{R} \), \( f \in A \) and \( B \in \Gamma \). Here \( \text{U}(\mathcal{H}_h) \) is the group of unitary operators on \( \mathcal{H}_h \). If \( (\rho_h, \mathcal{H}_h) \) is a projective (respectively linear) representation of the group \( \Gamma \) then we call the quantization projective (respectively linear).

The idea of the construction is as follows: We use a ”deformation” of the algebra \( A \) of functions on \( T \). We define an algebra \( A_h \), usually called the two dimensional non-commutative torus (cf. [Ri]). If \( h = \frac{M}{N} \in \mathbb{Q} \), then we will see that all irreducible representations of \( A_h \) have dimension \( N \). We denote by \( \text{Irr}(A_h) \) the set of equivalence classes of irreducible algebraic representations of the quantized algebra. We will see that \( \text{Irr}(A_h) \) is a set ”equivalent” to a torus.

The group \( \Gamma \) naturally acts on a quantized algebra \( A_h \) and hence on the set \( \text{Irr}(A_h) \). Let \( h = \frac{M}{N} \) with \( \text{gcd}(M, N) = 1 \). The following holds:

**Theorem 0.3 (Canonical equivariant representation)** There exists a unique (up to isomorphism) \( N \)-dimensional irreducible representation \( (\pi_h, \mathcal{H}_h) \) of \( A_h \) for which its equivalence class is fixed by \( \Gamma \).
This means that:
\[ \pi_h \cong \pi_h^B \]
for all \( B \in \Gamma \).

Since the canonical representation \((\pi_h, \mathcal{H}_h)\) is irreducible, by Schur’s lemma we get the canonical projective representation of \( \Gamma \) compatible with \( \pi_h \):

**Corollary 0.4 (Canonical projective representation)** There exists a unique projective representation \( \rho_p : \Gamma \to \text{PGL}(\mathcal{H}_h) \) s.t.:
\[ \rho_p(B)^{-1}\pi_h(f)\rho_p(B) = \pi_h(f \circ B) \]
for all \( f \in \mathcal{A} \) and \( B \in \Gamma \).

**Remark.** Corollary 0.4 is an improvement to the known constructions (cf. [HB, Mc, KR1]) which has the group \( \Gamma_{\Theta} := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ab = cd = 0 \ (2) \} \) as the group of quantizable elements.

Using a result of Coxeter-Moser [CM] about the structure of the group \( \Gamma \) we get:

**Theorem 0.5 (Linearization)** The projective representation \( \rho_p \) can be lifted to a linear representation in exactly 12 different ways.

**Remark.** The existence of the linear representation \( \rho_p \) in Theorem 0.5 answers Hannay-Berry’s question (cf. [HB, Mc]) on the multiplicativity of the map \( \rho_p \).

**Summary.** For \( \hbar \in \mathbb{Q} \) let \((\rho_h, \pi_h, \mathcal{H}_h)\) be the canonical (projective) equivariant quantization of \( T \). We can endow the space \( \mathcal{H}_h \) with a canonical unitary structure s.t. \( \pi_h \) is a \( * \)-representation and \( \rho_h \) is unitary. This “family” of \(*\)-representations of \( \mathcal{A}_h \) is by definition a Weyl quantization of the functions on the torus. The above results show the existence of a canonical projective equivariant quantization of the torus, and the existence of a linear equivariant quantization of the torus.

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1 Construction

We consider the algebra $\mathcal{A} := C^\infty(T)$ of smooth complex valued function on the torus and the dual lattice $\Lambda^* := \{\xi \in V^* | \xi(\Lambda) \subset \mathbb{Z}\}$. Let $\langle , \rangle$ be the pairing between $W$ and $W^*$. The map $\xi \mapsto s(\xi)$ where $s(\xi)(x) := e^{2\pi i \langle x, \xi \rangle}$, $x \in T$ and $\xi \in \Lambda^*$ defines a canonical isomorphism between $\Lambda^*$ and the group $T^\vee := \text{Hom}(T, \mathbb{C}^*)$ of characters of $T$.

1.1 The quantum tori

Fix $\hbar \in \mathbb{R}$. The Rieffel’s quantum torus (cf. [Ri]) is the non-commutative algebra $A_\hbar$ defined over $\mathbb{C}$ by generators $\{s(\xi), \xi \in \Lambda^*\}$, and relations:

$$s(\xi + \eta) = e^{\pi i \hbar \omega(\xi, \eta)} s(\xi) s(\eta)$$

for all $\xi, \eta \in \Lambda^*$.

Note that the lattice $\Lambda^*$ serves, using the map $\xi \mapsto s(\xi)$, as a basis for the algebra $A_\hbar$. This induces an identification of vector spaces $A_\hbar \simeq A$ for every $\hbar$. We will use this identification in order to view elements of the (commutative) space $A$ as members of the (non-commutative) space $A_\hbar$.

1.2 Weyl quantization

To get a Weyl quantization of $A$ we use a specific one-parameter family of representations (see subsection 1.4 below) of the quantum tori. This defines an operator $\pi_\hbar(\xi)$ for every $\xi \in \Lambda^*$. We extend the construction to every function $f \in A$ using the Fourier theory. Suppose:

$$f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$$

is its Fourier expansion. Then we define its Weyl quantization by:

$$\pi_\hbar(f) := \sum_{\xi \in \Lambda^*} a_\xi \pi_\hbar(\xi).$$

The convergence of the last series is due to the rapid decay of the Fourier coefficients of the function $f$.

1.3 Projective equivariant quantization

The group $\Gamma = \text{SL}_2(\mathbb{Z})$ acts on $\Lambda$ preserving $\omega$. Hence $\Gamma$ acts on $A_\hbar$ and the formula of this action is $s^B(\xi) := s(B\xi)$. Given a representation $(\pi_\hbar, \mathcal{H}_\hbar)$ of $A_\hbar$
and an element $B \in \Gamma$, define $\pi_h^B(s(\xi)) := \pi_h(s^{B^{-1}}(\xi))$. This formula induces an action of $\Gamma$ on the set $\text{Irr}(A_h)$ of equivalence classes of irreducible algebraic representations of $A_h$.

**Lemma 1.1** All irreducible representations of $A_h$ are $N$-dimensional.

Now, suppose $(\pi_h, A_h, H_h)$ is an irreducible representation for which its equivalence class is fixed by the action of $\Gamma$. This means that for any $B \in \Gamma$ we have $\pi_h \simeq \pi_h^B$, so by definition there exists an operator $\rho_h(B) \in \text{GL}(H_h)$ such that:

$$\rho_h(B)^{-1}\pi_h(\xi)\rho_h(B) = \pi_h(B\xi)$$

for all $\xi \in \Lambda^*$. This implies the Egorov identity (0.2.1) for any function. Now, since $(\pi_h, H_h)$ is an irreducible representation then by Schur’s lemma for every $B \in \Gamma$ the operator $\rho_h(B)$ is uniquely defined up to a scalar. This implies that $(\rho_h, H_h)$ is a projective representation of $\Gamma$.

### 1.4 The canonical equivariant quantization

In what follows we consider only the case $\hbar \in \mathbb{Q}$. We write $h$ in the form $h = \frac{M}{N}$ with $\gcd(M, N) = 1$.

**Proposition 1.2** There exists a unique $\pi_h \in \text{Irr}(A_h)$ which is a fixed point for the action of $\Gamma$.

### 1.5 Unitary structure

Note that $A_h$ becomes a $\ast$-algebra using the formula $s(\xi)^\ast := s(-\xi)$. Let $(\pi_h, H_h)$ be the canonical representation of $A_h$.

**Remark 1.3** There exists a canonical (unique up to scalar) unitary structure on $H_h$ for which $\pi_h$ is a $\ast$-representation.

### 1.6 Realization

Choosing a symplectic basis for $\Lambda^*$ we get the identifications $\Lambda^* \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $\Gamma = \text{SL}_2(\mathbb{Z})$. We will consider the realization on the Hilbert space:

$$H := L^2(\mathbb{Z}/N\mathbb{Z}).$$
1.6.1 Formula for $\pi$

The representation $\pi$ is given by:

$$[\pi(m, n)f](x) = \alpha(m, n)\psi(nx)f(x + m),$$

where $\alpha(m, n) := (-1)^{M(m+n)} e^{\pi i h mn}$ and $\psi(t) = e^{2\pi i h t}$ on $\mathbb{Z}/N\mathbb{Z}$.

1.6.2 Formula for $\rho$

The projective representation $\rho$ is described by the following formulas:

$$\left[\rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}f\right](x) = Q(x)f(x),$$

where $Q(x) := (-1)^ex e^{\pi i hx^2}$, with $e := MN \pmod{2}$, and:

$$\left[\rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}f\right](x) = \hat{f}(x),$$

where $\hat{f}$ denote the Fourier transform:

$$\hat{f}(x) := \frac{1}{\sqrt{N}} \sum_{y \in \mathbb{Z}/N\mathbb{Z}} f(y)\psi(yx).$$

2 Proofs

2.1 Proof of Lemma 1.1

Suppose $(\pi_h, \mathcal{H}_h)$ is an irreducible representation of $A_h$.

**Step 1.** First we show that $\mathcal{H}_h$ is finite dimensional. $A_h$ is a finite module over $Z(A_h) = \{s(N\xi), \xi \in \Lambda^*\}$ which is contained in the center of $A_h$. Because $\mathcal{H}_h$ has at most countable dimension (as a quotient space of $A_h$) and $\mathbb{C}$ is uncountable then by Kaplansky’s trick (cf. [MR]) $Z(A_h)$ acts on $\mathcal{H}_h$ by scalars. Hence dim $\mathcal{H}_h < \infty$.

**Step 2.** We show that $\mathcal{H}_h$ is N-dimensional. Choose a basis $(e_1, e_2)$ of $\Lambda^*$ s.t. $\omega(e_1, e_2) = 1$. Suppose $\lambda \neq 0$ is an eigenvalue of $\pi_h(e_1)$ and denote by $\mathcal{H}_\lambda$ the corresponding eigenspace. We have the following commutation relation
\[ \pi_h(e_1)\pi_h(e_2) = \gamma \pi_h(e_2)\pi_h(e_1) \text{ where } \gamma := e^{-2\pi i \frac{M}{N}}. \] Hence \( \pi_h(e_2) : \mathcal{H}_{\gamma/\lambda} \to \mathcal{H}_{\gamma/\lambda+1} \), and because \( \gcd(M,N) = 1 \) then \( \mathcal{H}_{\gamma/\lambda} \neq \mathcal{H}_{\gamma/\lambda} \) for \( 0 \leq i \neq j \leq N-1 \). Now, let \( v \in \mathcal{H}_\lambda \) and recall that \( \pi_h(e_2)^N = \pi_h(Ne_2) \) is a scalar operator. Then the space span\{\( v, \pi_h(e_2)v, \ldots, \pi_h(e_2)^{N-1}v \}\} is \( N \)-dimensional \( A_h \)-invariant subspace hence it equals \( \mathcal{H}_h \).}

### 2.2 Proof of Proposition 1.2

Let us show the existence of a unique fixed point for the action of \( \Gamma \) on \( \text{Irr}(A_h) \).

Suppose \( (\pi_h, \mathcal{H}_h) \) is an irreducible representation of \( A_h \). By Schur’s lemma for every \( \xi \in \Lambda^* \) the operator \( \pi_h(N\xi) \) is a scalar operator, i.e., \( \pi_h(N\xi) = q_{\pi_h}(\xi) \cdot I \). We have \( \pi_h(0) = I \) and hence \( q_{\pi_h}(\xi) \neq 0 \) for all \( \xi \in \Lambda^* \). Thus to any irreducible representation we have attached a scalar function \( q_{\pi_h} : \Lambda^* \to \mathbb{C}^* \). Consider the set \( Q_h \) of twisted characters of \( \Lambda^* \):

\[
Q_h := \{ q : \Lambda^* \to \mathbb{C}^*, \; q(\xi + \eta) = (-1)^{M_\Lambda \omega(\xi,\eta)}q(\xi)q(\eta) \}.
\]

The group \( \Gamma \) acts naturally on this space by \( q^{B}(\xi) := q(B^{-1}\xi) \). It is easy to see that we have defined a map \( q : \text{Irr}(A_h) \to Q_h \) given by \( \pi_h \mapsto q_{\pi_h} \) and it is obvious that this map is compatible with the action of \( \Gamma \). We use the space of twisted characters in order to give a description for the set \( \text{Irr}(A_h) \):

**Lemma 2.1** The map \( \pi_h \mapsto q_{\pi_h} \) is a \( \Gamma \)-equivariant bijection:

\[
q : \text{Irr}(A_h) \longrightarrow Q_h.
\]

Now, Proposition 1.2 follows from the following claim:

**Claim 2.2** There exists a unique \( q_s \in Q_h \) which is a fixed point for the action of \( \Gamma \).

**Proof of Lemma 2.1**

**Step 1.** The map \( q \) is surjective. Denote by \( T := \text{Hom}(\Lambda^*, \mathbb{C}^*) \) the group of complex characters of \( \Lambda^* \). We define an action of \( T \) on \( \text{Irr}(A_h) \) and on \( Q_h \) by \( \pi_h \mapsto \chi \pi_h \) and \( q \mapsto \chi^Nq \), where \( \chi \in T \), \( \pi_h \in \text{Irr}(A_h) \) and \( q \in Q_h \). The map \( q \) is clearly a \( T \)-equivariant map with respect to these actions. Since \( q \) is \( T \)-equivariant, it is enough to show that the action of \( T \) on \( Q_h \) is transitive. Suppose \( q_1, q_2 \in Q_h \). By definition there exists a character \( \chi_1 \in T \) for which \( \chi_1 q_1 = q_2 \). Let \( \chi \) be one of the \( N \)'s roots of \( \chi_1 \) then \( \chi^N q_1 = q_2 \).

**Step 2.** The map \( q \) is one to one. Suppose \( (\pi_h, \mathcal{H}_h) \) is an irreducible representation of \( A_h \). It is easy to deduce from the proof of Lemma 1.1 (Step 2)
that for $\xi \notin N\Lambda^*$ we have $\text{tr}(\pi_h(\xi)) = 0$. But we know from character theory that an isomorphism class of a finite dimensional irreducible representation of an algebra is recovered from its character. This completes the proof of Lemma 2.1. ■

**Proof of Claim 2.2**. **Uniqueness.** Fix $q \in Q_h$. The map $\chi \mapsto \chi q$ give a bijection of $T$ with $Q_h$. But the trivial character $1 \in T$ is the unique fixed point for the action of $\Gamma$ on $T$.

**Existence.** Choose a basis $(e_1, e_2)$ of $\Lambda^*$ s.t. $\omega(e_1, e_2) = 1$. This allows to identify $\Lambda^*$ with $\mathbb{Z} \oplus \mathbb{Z}$. It is easy to see that the function:

$$q_o(m, n) = (-1)^{MN(mn+m+n)}$$

is a twisted character which is fixed by $\Gamma$. This completes the proof of Claim 2.2 and of Proposition 1.2. ■

### 2.3 Proof of Theorem 0.5

The theorem follows from the following proposition:

**Proposition 2.3** Fix a projective representation $\rho_p : \Gamma \longrightarrow \text{GL}(\mathcal{H}_h)$. Then it can be lifted to a linear representation in exactly 12 ways.

**Proof.** Existence. We want to find constants $c(B)$ for every $B \in \Gamma$ s.t. $\rho_h := c(\cdot)\rho_p$ is a linear representation of $\Gamma$. This is possible to carry out due to the following fact:

**Lemma 2.4 ([CM])** The group $\Gamma$ is isomorphic to the group generated by three letters $S$, $B$ and $Z$ subjected to the relations: $Z^2 = 1$ and $S^2 = B^3 = Z$.

Lemma 2.4 $\Rightarrow$ Existence. We need to find constants $c_z, c_B, c_S$ so that the operators $\rho_h(Z) := c_z\rho_p(Z)$, $\rho_h(B) := c_B\rho_p(B)$, $\rho_h(S) := c_S\rho_p(S)$ will satisfy the identities:

$$\rho_h(Z)^2 = I, \quad \rho_h(B)^3 = \rho_h(Z), \quad \rho_h(S)^2 = \rho_h(Z).$$

This can be done by taking appropriate scalars.

Now, fix one lifting $\rho_h$. Then for the collection of operators $\rho_h(B)$ which lifts $\rho_p$ define a function $\chi(B)$ by $\rho_h(B) = \chi(B)\rho_h(B)$. It is obvious that $\rho_h$ is a representation if and only if $\chi$ is a character. Thus liftings corresponds to characters. By Lemma 2.4 the group of characters $\Gamma^\vee := \text{Hom}(\Gamma, \mathbb{C}^*)$ is isomorphic to $\mathbb{Z}/12\mathbb{Z}$. ■

9
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