GEOMETRIC REPRESENTATION OF THE INFIMAX S-ADIC FAMILY

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Abstract. We construct geometric realizations for the infimax family of substitutions by generalizing the Rauzy-Canterini-Siegel method for a single substitution to the S-adic case. The composition of each countably infinite subcollection of substitutions from the family has an asymptotic fixed sequence whose shift orbit closure is an infimax minimal set $\Delta^+$. The subcollection of substitutions also generates an infinite Bratteli-Vershik diagram with prefix-suffix labeled edges. Paths in the diagram give the Dumont-Thomas expansion of sequences in $\Delta^+$ which in turn gives a projection onto the asymptotic stable direction of the infinite product of the Abelianization matrices. The projections of all sequences from $\Delta^+$ is the generalized Rauzy fractal which has subpieces corresponding to the images of symbolic cylinder sets. The intervals containing these subpieces are shown to be disjoint except at endpoints, and thus the induced map derived from the symbolic shift translates them. Therefore the process yields an Interval Translation Map (ITM), and the Rauzy fractal is proved to be its attractor.

1. Introduction

Substitution morphisms are an integral part of many areas of mathematics including dynamical systems, combinatorics, number theory and formal language theory. The books [29], [23], and [7] give a good sense of the diversity and depth of the field. The natural generalization of a single substitution to the composition of an infinite sequence of substitutions, termed S-adic systems by Ferenczi in [21], allows the modeling and analysis of a wider range of fundamental structures (see, for example, [5], [37], [4], and [8]).

In [11] an S-adic family was found to generate the solutions to the following problem. A one-sided sequence is called maximal if it is larger in the lexicographic order than all its shifts. Let $\mathcal{M}(\rho)$ denote all the maximal sequences with asymptotic digit frequency vector $\rho$. The infimum in the lexicographic order of $\mathcal{M}(\rho)$ is called the infimax sequence for $\rho$ and in [11] it was shown that it can be constructed using a specific S-adic family. In this paper we further study the properties of this infimax S-adic family. The family studied here is indexed by the positive integers $\mathbb{N}^+ = 1, 2, 3, \ldots$
with the substitution $\Lambda_n$ given by
\begin{align}
\Lambda_n : 
1 & \rightarrow 2 \\
2 & \rightarrow 3^{1^{n+1}} \\
3 & \rightarrow 3^{1^n}.
\end{align}

Note that the substitution $\Lambda_0$ is required for the full infimax problem: see Remark 7.4(b) below.

The collection of allowable lists of indices is $\Sigma_\infty^+ := (\mathbb{N}^+)^{\mathbb{N}^+}$. Given a list of indices $n \in \Sigma_\infty^+$, it is easy to check that the right one-sided sequence
\begin{equation}
\alpha := \lim_{k \rightarrow \infty} \Lambda_{n_1} \circ \Lambda_{n_2} \cdots \Lambda_{n_k}(3)
\end{equation}
exists. For a constant list of indices $n = nnn \ldots$, the sequence $\alpha$ is a fixed point of $\Lambda_n$. For a general $n \in \Sigma_\infty^+$, the sequence $\alpha$ is an asymptotic fixed point in the sense that for any one-sided sequence $s = s_0s_1 \ldots$ with $s_i \in \{1, 2, 3\}$,
\[\lim_{k \rightarrow \infty} \Lambda_{n_1} \Lambda_{n_2} \cdots \Lambda_{n_k}(s) = \alpha.\]

Returning to the infimax problem, for a given asymptotic digit frequency vector $\rho$, it was shown in [11] that a three dimensional continued fraction algorithm using $\rho$ as an input generates a list $n \in \Sigma_\infty^+$. The infimax for $\rho$ is then the corresponding $\alpha$ as defined in (1.2).

The main dynamical object of study is the orbit closure of $\alpha$ under the left shift, or $\Delta_n^+ := \text{Cl}(o^+(\alpha, S))$. Since $\Delta_n^+$ is always dynamically minimal, we call it an infimax minimal set. The study of $\Delta_n^+$ and its associated substitutions is facilitated by finding a geometric representation of $\Delta_n^+$ as defined in [29]. This means a concrete, geometrically defined model system $T : X \rightarrow X$ so that the dynamics of $\Delta_n^+$ under the shift are embedded in the dynamics of $T$ on $X$ in a nice way. Specifically, we seek a map $\Upsilon^+ : \Delta_n^+ \rightarrow X$ which is a conjugacy, $\Upsilon^+ \circ S = T \circ \Upsilon^+$ on the image of $\Upsilon^+$. In addition, the space $X$ is required to have a nice partition by which $T$-orbits are coded so that corresponding sequences are recovered as itineraries. Geometric representations have been found for many substitutions (see [34] for a summary). Depending on the circumstances, different requirements can be made on the map $\Upsilon^+$. From the ergodic theory perspective, one would like $\Upsilon^+$ to be measure preserving and injective almost everywhere with respect to the appropriate invariant measures. We work here in the topological category and so $\Upsilon^+$ is required to be continuous and, in fact, $\Upsilon^+$ will be a homeomorphism onto its image.

The geometric models for the S-adic infimax family given here are elements of a two-parameter family of interval translation maps (ITM). These maps are generalizations of interval exchange transformations in which the
images of intervals are allowed to overlap ([10]). The *attractor* of an ITM is the intersection of the forward iterates of the entire interval. For each $n \in \Sigma^+_{\infty}$, we show in Theorem 11.2 that the symbolic infimax minimal set $\Delta^+_n$ is conjugate via a map $\Upsilon^+$ to the attractor of a (slightly extended) ITM. The extension of the ITM is necessary to make it continuous as is often done with interval exchange maps ([28]).

**Theorem 1.1.** For each $n \in \Sigma^+_{\infty}$, the three symbol infimax minimal set has a geometric representation as the attractor of an Interval Exchange Map (ITM) on three intervals.

In addition, the conjugacy respects order structures: the lexicographic order on the symbolic minimal set is reversed under the conjugacy onto the ITM attractor inside the unit interval.

Previously Bruin and Troubetzkoy have shown that each $\Delta_n$ is isomorphic to the attractor of an ITM ([14], see also Section 5 in [10]). We extend these results obtaining a full homeomorphic conjugacy and in addition study the two-sided infimax minimal set. The starting point in [14] was the natural renormalizations of a family of ITM. The substitutions (1.1) then arise as the symbolic descriptors of this process. On the other hand, motivated by the infimax problem, we begin here with the substitutions and sequences themselves. Using generalizations of methods commonly used for single substitutions we find the ITM geometric representation as a direct and natural consequence of the structure of the generalized Rauzy fractal and its induced transformation under the shift.

The methods we use have their origin in Rauzy’s classic papers ([30], [32], [31]) and their subsequent development by many authors. Most significant and relevant here is the process laid out by Canterini and Siegel ([15], [16]) and from a somewhat different angle by Holton and Zamboni ([27], [26]). The paper [2] provides an excellent exposition of the process and related constructions for a very important special case. We adopt these single substitution methods to the S-adic case. While often the generalizations are reasonably straightforward, they differ in enough detail that an independent, self-sufficient treatment is required.

The main idea in this geometric representation process is projection onto the stable subspace of the Abelianizations of the substitution. For the family (1.1) the Abelianizations are

$$(1.3) \quad A_n := \begin{bmatrix} 0 & n + 1 & n \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$
Each $A_n$ has two eigenvalues outside the unit circle and one inside (proof of Lemma 52 in [12]) and so each substitution (1.1) is inverse-Pisot, unimodal and primitive. It is shown in Theorem 7.3 below that for each $n \in \Sigma^\infty$ the limit
\[
\lim_{k \to \infty} A_{n_1}A_{n_2} \ldots A_{n_k}
\]
has a well defined asymptotic (or generalized) one-dimensional stable direction $\vec{\ell}_n$. A similar argument shows the existence of an asymptotic two-dimensional unstable subspace. However, if the $n_i$ grow sufficiently fast there is not an asymptotic one-dimensional strongest unstable direction (Theorems 11 and 12 in [14] and Theorem 27 in [11]). This implies that in these cases the asymptotic digit frequency vector of $\alpha$ does not exist and, in addition, $\Delta^+_n$ is not uniquely ergodic ([14]).

To achieve the projection, Rauzy's original method was to embed the Abelianization of the sequence $\alpha$ in $\mathbb{R}^3$ and then project its vertices down to the stable subspace and take the closure. We adapt the alternative route developed by Canterini and Siegel ([15], [16]) and use the machinery of the Prefix-Suffix Automaton. This automaton is a particularly useful way of labeling and ordering a Bratelli diagram based on a single substitution. The sequence of edge labels in an infinite path naturally yield both the Dumont-Thomas prefix-suffix expansion of a symbolic sequence ([19], [3]) and a map to the stable subspace of the Abelianization. In the natural generalization to the S-adic case given in Example 3.5 of [5] and used below, each level of the diagram corresponds to a substitution from the list $\Lambda_{n_1}, \Lambda_{n_2}, \ldots$. The sequence of edge labels $(u_i, a_i, v_i)$ in an infinite path then yields the S-adic version of the Dumont-Thomas prefix-suffix expansion as in formula (5) in [5]

\[
\ldots \Lambda_{n_2}\Lambda_{n_1}(u_2)\Lambda_{n_1}(u_1)u_0.a_0v_0 \Lambda^{(1)}(v_1)\Lambda_{n_2}\Lambda_{n_1}(v_2) \ldots .
\]

With the appropriate alterations for the S-adic situation, this yields a map from the path space of the Bratelli diagram to $\Delta^+_n$ as well as a projection onto the asymptotic stable subspace of the infinite composition of the Abelianizations in ([1]). The inverse of the first map composed with the second map yields the map $\Upsilon^+$ from $\Delta^+_n$ to the real line given in Theorem 1.1 above. The image of $\Upsilon^+$ is sometimes called a generalized Rauzy fractal, and in our case it is always a Cantor set embedded in an interval.

The final ingredient in this geometric representation process is the subdivision of the Rauzy fractal into pieces corresponding to the images of symbolic cylinder sets under $\Upsilon^+$. The importance of these subpieces is that there is a natural translation induced on them by the shift map on $\Delta^+_n$. 
Thus the process yields a geometric representation as long as the subpieces are disjoint almost everywhere and so proving this disjointness is often the central problem in this process of geometric representation. For the infimax family, these subpieces are again Cantor sets and we show in Theorem 9.1 that their convex hulls (obviously intervals) can only intersect at their endpoints. The induced map on these interval convex hulls is the representing Interval Exchange map (ITM) and we then show that the Rauzy fractal is, in fact, the attractor of this ITM.

While the construction of an infimax minimal set requires infinitely many choices to designate the list of defining substitutions in (1.2), one of the striking features of the geometric representation is that it faithfully describes an infimax minimal set by specifying just two parameters in the family of ITM. The geometric representation has additional consequences. For example, Boshernitzan and Kornfeld note in §7 of [10] that after using the defining intervals of the ITM to code orbits it is straightforward to show that the number of distinct words of length $n$ in itineraries of orbits can grow at most at a polynomial rate. This implies that the ITM have zero topological entropy. Thus using the geometric representation all the infimax minimal sets also have zero entropy (this also follows from more general, more recent results; see Theorem 4.3 in [5]). Boshernitzan and Kornfeld ask whether this factor complexity growth rate is actually linear. In the special case of what are called infimax minimal sets here, Cassaigne and Nicolas showed that the factor complexity satisfies $p(j) \leq 3j$ ([18]).

In many of the geometric representations of substitutions in the literature the symbolic minimal set is represented by either an interval exchange map or a toral translation. This provides a great deal of information about the symbolic minimal set, in particular, about its spectrum. Compared to interval exchange maps, ITM are poorly understood and there is little known about their spectrum. The ITM in the representing family here have a rich variety of behaviours like non-unique ergodicity and thus provide a good model problem for future development: one can work jointly with the ITM and the symbolic, S-adic description.

There are a number of important features that are specific to the family studied here. The first is that a list of substitutions $\Lambda_{n_1}, \Lambda_{n_2}, \ldots$ when it acts on bi-infinite sequences yields asymptotic period-two points rather than a fixed point. As a consequence, the resulting Bratteli-Vershik diagram is not properly ordered: it has two maximal elements and one minimal element (for background on Bratteli-Vershik diagrams and adic transformations see
and for their use with substitutions [36] and [20]). This complicates the reading off the Dumont-Thomas expansion of a sequence from the labels on edges of the maximal paths and necessitates the eventual use of a map from $\Delta^+_n$ back to the path space in Section 5.

Also, the improper order implies that the Vershik map on the path space cannot be globally defined. As done in [36] and many subsequent papers, it can be defined almost everywhere but since we are working in the topological category, we require all maps to be globally defined. Our main objective is a map from $\Delta^+_n$ to $\mathbb{R}$, and so the path space is a convenient intermediate structure, but we never need to consider the dynamics on it. Thus the order on the Bratteli diagram and the Vershik map are not utilized here. However, the labeling of edges in the diagram by prefixes and suffixes is of crucial importance and so we have adapted the terminology “infinite prefix-suffix automaton” (IPSA) for the labeled Bratteli diagram corresponding to a list of substitutions indexed by $n \in \Sigma^+_\infty$.

Another special feature is the central role of the lexicographic order on the symbol space and its relation under the representation map to the usual linear order on the real line. Note that each substitution $\Lambda_n$ preserves the lexicographic order (Lemma 2 in [11]) and that the family arose as the solution to the infimax question which depends fundamentally on the lexicographic order. In the final analysis it is the relation of the orders on sequences and the reals which yields the fundamental fact of the disjointness of the subpieces of the Rauzy fractal.

Finally, in contrast to much of the existing work on Sadic systems, the infimax family contains infinitely many substitutions and further, our results hold for all sequences $n \in \Sigma^+_\infty$ rather than just a large, say full measure, subcollection.

While our main results concern the one-sided shift space, a number of steps in the representation process are technically simpler using two-sided infinite sequences. This has the added benefits of yielding useful results about the relationship between the two-sided version $\Delta_n$ and the one-sided version $\Delta^+_n$ of the infimax minimal set. For example, in Theorem 6.1 we show that the projection $\pi: \Delta_n \to \Delta^+_n$ is injective except on the forward orbits of the asymptotic period two points, or informally, the one-sided version is obtained by collapsing a single pair of orbits of the two-sided version. This in turn implies that the left shift has unique inverses in $\Delta^+_n$ except on the $\alpha$ defined in (1.2).
This paper deals primarily with the geometric representation of the $S$-adic family with $N = 3$ symbols: more detailed results about the infimax minimal sets and their languages are saved for a later paper. We remark on the $N \neq 3$ case in the last section.

2. Preliminaries

We start with some basic definitions about words, sequences and substitutions. The alphabet here will always be $\mathcal{A} = \{1, 2, 3\}$. The length of a finite word $w$ is denoted $|w|$, and the empty word $\epsilon$ has $|\epsilon| = 0$. A bi-infinite sequence $\underline{s}$ is an element of $\Sigma_3 = \mathcal{A}^\mathbb{Z}$ and is written with a decimal point between the zeroth and minus first symbols, $\underline{s} = \ldots s_{-2}s_{-1}.s_0s_1s_2 \ldots$. A right infinite sequence has the form $s_0s_1s_2\ldots$ and we use an under-arrow to indicate it $\underline{s} = s_0s_1\ldots$, and a left infinite sequence is written $\underline{s} = \ldots s_{-2}s_{-1}$. The collection of right infinite sequences is denoted $\Sigma_3^+$. The spaces $\Sigma_3$ and $\Sigma_3^+$ are given the topology induced by the metric $d(\underline{s}, \underline{s}') = 1/(1 + M)$ where $M = \min\{|i|: s_i \neq s'_i\}$. The left shift $S$ acting on $\Sigma_3$ is $S(\underline{s}) = \ldots s_{-2}s_{-1}s_0.s_1s_2\ldots$ and acting on $\Sigma_3^+$ is $S(\underline{s}) = .s_1s_2\ldots$. In the dynamics literature $\sigma$ is usually used for the shift and in the substitutions literature $\sigma$ usually denotes a substitution. To avoid confusion we refrain from using $\sigma$ altogether.

A pointed word is word with decimal point placed between two of its symbols or at the beginning or end of the word. The shift acts on pointed words as long as its action does not move the decimal point beyond the beginning or end of the word. A pointed one-sided sequence has the form $\ldots s_{-2}s_{-1}.s_0s_1s_2\ldots \epsilon$ or $\epsilon s_{-n} \ldots s_{-2}s_{-1}.s_0s_1s_2\ldots$. The empty symbol $\epsilon$ is included to indicate the end or beginning of the pointed one-sided sequence. The shift also acts on pointed one-sided sequences again with the proviso that the decimal point cannot move beyond the end.

A substitution $\Lambda$ is specified by assigning a nonempty word $\Lambda(a)$ to each symbol $a \in \mathcal{A}$. It acts on sequences, words and pointed objects yielding another object of the same type by respecting the decimal point, so, for example,

$$\Lambda(\epsilon s_{-2}s_{-1}).s_0s_1s_2\ldots = \epsilon\Lambda(s_{-2})\Lambda(s_{-1}).\Lambda(s_0)\Lambda(s_1)\Lambda(s_2)\ldots.$$ 

For a homeomorphism $h: X \to X$, the full orbit of a point $x$ is $o(x, h) = \{\ldots, h^{-2}(x), h^{-1}(x), x, h(x), h^2(x), \ldots\}$ and its forward orbit is $o^+(x, h) = \{x, h(x), h^2(x), \ldots\}$. When $f: X \to X$ is not injective, $o^+(x, f)$ is defined in the same way.
3. THE INFINITE PREFIX-SUFFIX AUTOMATON

3.1. Definitions. Fix a sequence $n \in \Sigma^+_{\infty}$. Our main object of study is the sequence of substitutions $\Lambda_{n_1}, \Lambda_{n_2}, \ldots$. For each $k > 0$, let
\[ \Lambda^{(k)} = \Lambda_{n_1} \circ \Lambda_{n_2} \cdots \circ \Lambda_{n_k}. \]

For a subcollection of indices $n_j, n_{j+1}, \ldots, n_{j+k}$ write $\Lambda_{n_j n_{j+1} \ldots n_{j+k}} = \Lambda_{n_j} \circ \Lambda_{n_{j+1}} \cdots \circ \Lambda_{n_{j+k}}$, and so $\Lambda^{(k)} = \Lambda_{n_1 n_2 \ldots n_k}$.

We now define the principal tool in this paper, the Infinite Prefix-Suffix Automaton (IPSA). A related automaton for a single substitution was contained in Rauzy’s classic papers ([30], [32], [31]) and similar constructions were present in other seminal works in other fields (see page 218 of [34] for some history). The automaton was formalized, extended and utilized in [15] and [16] and independently in a slightly different form in [26]. The version we use here is the S-adic generalization described in Example 3.5 of [5].

The Infinite Prefix-Suffix Automaton (IPSA) or Bratteli-Vershik diagram associated with $n \in \Sigma^+_{\infty}$ is an infinite directed graph built in levels. Each level of the graph contains three nodes or states 1, 2, 3 and the levels are indexed by 0, 1, 2, $\ldots$. There is a directed edge from state $a$ on level $n-1$ to state $b$ on level $n$ if and only if $\Lambda_n(b) = uav$ for some perhaps empty words $u$ and $v$. This edge is then labeled $(u, a, v)$. Note that that level zero of our diagram has three states and not a single root state as is common in the literature. See Figure 1.
It is easy to check that an infinite path $\Gamma$ in the IPSA is uniquely specified by its sequence of labels and we write $\Gamma = \Gamma_0 \Gamma_1 \ldots$ or $\Gamma = (u_0, a_0, v_0), (u_1, a_1, v_1), \ldots$ where for all $i$, $\Lambda_n(a_i) = u_{i-1} a_{i-1} v_{i-1}$. Note that the sequence of edges is indexed by $0, 1, 2, \ldots$ while the sequence of indices $n$ is indexed by $1, 2, 3, \ldots$. The collection of all infinite paths in the IPSA generated by an $2$ is denoted $\mathcal{P}_\mathcal{N}$ or just $\mathcal{P}$. The space $\mathcal{P}$ is a compact metric space under the metric $d(\Gamma, \Gamma') = 1/(N+1)$ where $N = \min\{i : \Gamma_i \neq \Gamma_i'\}$.

3.2. The Dumont-Thomas expansion; the word and sequence maps.

A finite path $\gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_k$ in the IPSA is said to have length $k+1$ and the collection of all such length $k+1$ paths is denoted $\mathcal{P}^{(k+1)}$. We now define the maps which assign a Dumont-Thomas expansion ([19], [3]) of a word or sequence to each finite or infinite path.

For each $k$ the word map assigns a finite pointed word to a length $k+1$ path via
\begin{equation}
W(\gamma) = \Lambda^{(k)}(u_k) \ldots \Lambda^{(1)}(u_1) u_0. a_0 v_0 \Lambda^{(1)}(v_1) \ldots \Lambda^{(k)}(v_k).
\end{equation}

Given an infinite path $\Gamma = \Gamma_0, \Gamma_1, \ldots$, for each $k$ let $\Gamma^{(k)}$ be the length $(k+1)$ path $\Gamma^{(k)} = \Gamma_0, \ldots, \Gamma_k$. The sequence map assigns a pointed sequence to each infinite path via
\begin{equation}
W(\Gamma) = \lim_{k \to \infty} W(\Gamma^{(k)}) = \ldots \Lambda^{(2)}(u_2) \Lambda^{(1)}(u_1) u_0. a_0 v_0 \Lambda^{(1)}(v_1) \Lambda^{(2)}(v_2) \ldots
\end{equation}

In most cases the sequence $W(\Gamma)$ will be bi-infinite, but in certain special cases studied in Section 3.3 it will be infinite in only one direction.

The first lemma describes in more detail how the word map gives a correspondence between paths in the IPSA and symbolic words. As an example, let $\Gamma_0 \Gamma_1 = (u_0, a_0, v_0), (u_1, a_1, v_1)$ be a length two path that terminates in the node labeled $a_2$. Thus from the definitions and keeping track of the decimal points we have $S^{j_1} \Lambda_n(\epsilon, a_1) = u_0. a_0 v_0$ and $S^{j_2} \Lambda_n(\epsilon, a_2) = u_1 a_1 v_1$ for some $0 \leq j_i < |\Lambda_n(a_i)|$. Evaluating the word map we have

$W(\Gamma_0 \Gamma_1) = \Lambda_n(v_1) u_0. a_0 v_0 \Lambda_n(v_1) \Lambda_n(v_1) S^{j_1} \Lambda_n(\epsilon, a_1) \Lambda_n(v_1)$

$= S^{j_1} \Lambda_n(u_1 a_1 v_1) = S^{j_1} \Lambda_n(S^{j_2} \Lambda_n(\epsilon, a_2)) = S^{j_1 + j_2} \Lambda^{(2)}(\epsilon, a_2)$

Thus for any length two path which terminates at the node $a_2$, its word map image is a shift of $\Lambda^{(2)}(\epsilon, a_2)$. In fact, Lemma 3.1 and Theorem 3.4 together will show that each shift of $\Lambda^{(2)}(\epsilon, a_2)$ corresponds to one and only one length two path terminating at $a_2$. In general, let $\mathcal{P}^{(k)}(a)$ be the subcollection of length-$k$ paths which terminate at node $a$ and so $\Lambda_{n_k}(a) = u_{k-1} a_{k-1} v_{k-1}$. The first lemma shows that for any path in $\mathcal{P}^{(k)}(a)$ its word map image is a shift of $\Lambda^{(k)}(\epsilon, a)$. Theorem 3.4 will show that the correspondence is
injective. Part (b) of the lemma shows how to replace the inner portion of a sequence map image with the form of the image of an initial segment of the path given in part (a).

Lemma 3.1. For all \( n \in \Sigma^+ \),
(a) if \( \gamma \in \mathcal{P}^g(a) \), then \( \mathcal{W}(\gamma) = S^j(\Lambda^g(\epsilon,a)) \) for some \( 0 \leq j < |\Lambda^g(a)| \).
(b) If \( \Gamma \in \mathcal{P} \), then for all \( k > 0 \) there exists \( 0 \leq j < |\Lambda^g(a_k)| \), so that
\[
\mathcal{W}(\Gamma) = \ldots \Lambda^{(k+1)}(u_{k+1}) \Lambda^g(u_k) S^j(\Lambda^g(\epsilon,a_k)) \Lambda^g(v_k) \Lambda^{(k+1)}(v_{k+1}) \ldots
\]
where \( S^j(\Lambda^g(\epsilon,a_k)) = \mathcal{W}(\Gamma^{(k-1)}) \).

Proof. We prove (a) by induction on \( k \), with the case \( k = 0 \) following from the definition of the IPSA. So assume the result is true for length \( k \) paths. Given \( \gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_k \) with \( a_{k+1} = a \), we first show that \( \mathcal{W}(\gamma) \) has the required form. Form the length \( k \) path \( \gamma' = \Gamma_1, \Gamma_2, \ldots, \Gamma_k \) in the IPSA of \( n' = n_2n_3 \ldots \). Using the inductive hypothesis on \( \mathcal{W}' \), the assignment associated with \( n' \), we have a \( 0 \leq j' < |\Lambda_{n_2\ldots n_k}(a)| \) with
\[
S^{j'}(\Lambda_{n_2\ldots n_k+1}(\epsilon,a)) = \mathcal{W}'(\gamma') = \Lambda_{n_2\ldots n_k}(u_k) \ldots \Lambda_{n_2}(u_2) u_1 a_1 v_1 \Lambda_{n_2}(v_2) \ldots \Lambda_{n_2\ldots n_k}(v_k).
\]
Thus
\[
\mathcal{W}(\gamma) = S^{[u_0]}(\Lambda_{n_1} \mathcal{W}'(\gamma')) = S^{[u_0]}(\Lambda_{n_1} S^{j'}(\Lambda_{n_2\ldots n_k}(\epsilon,a))) = S^{j}(\Lambda^g(\epsilon,a)),
\]
for some \( j \geq 0 \) where since we know that \( \mathcal{W}(\gamma) \) is a finite, pointed word, \( j \leq |\Lambda^g(a)| \). But since from the IPSA, \( a_0 = \epsilon \) is impossible, in fact \( j < |\Lambda^g(a)| \), finishing (a).

For (b) it follows immediately from the definition that for any \( k > 0 \),
\[
\mathcal{W}(\Gamma) = \ldots \Lambda^{(k)}(u_k) \mathcal{W}(\Gamma^{(k-1)}) \Lambda^{(k)}(v_k) \ldots,
\]
and so (b) then follows from (a). \( \square \)

Remark 3.2. Letting \( j' = |\Lambda^g(a_k)| - j \), we also have
\[
\mathcal{W}(\Gamma) = \ldots \Lambda^{(k+1)}(u_{k+1}) \Lambda^g(u_k) S^{-j'} \Lambda^g(a_k,\epsilon) \Lambda^g(v_k) \Lambda^{(k+1)}(v_{k+1}) \ldots
\]

3.3. Some special paths and sequences. The sequence map given in \([3.2]\) yields a two-sided sequence for most paths in \( \mathcal{P} \). The paths \( \Gamma \) for which \( \mathcal{W}(\Gamma) \) is a pointed one-sided sequence require individual consideration and fall into three classes. Recall that two paths \( \Gamma \) and \( \Gamma' \) are said to be tail equivalent if there exists a \( k \) so that \( \Gamma_i = \Gamma_i' \) for all \( i \geq k \). This obviously yields an equivalence relation on the path space \( \mathcal{P} \).
• The set $S_1$ will consist of the tail equivalence class of

$$
\Gamma_\beta := (31^{n_1}, 1, \epsilon) \prod_{i=1}^\infty ((\epsilon, 2, \epsilon), (31^{n_{2i+1}}, 1, \epsilon))
$$

(3.3)

or all $\Gamma = \Gamma_0, \ldots, \Gamma_w, \prod_{i=1}^\infty ((\epsilon, 2, \epsilon), (31^{n_{2i+1}}, 1, \epsilon))$

with $\Gamma_w \neq (31^{n_{w+1}}, 1, \epsilon)$ and $w$ even.

• The set $\hat{S}_1$ will consist of the tail equivalence class of

$$
\Gamma_\beta := \prod_{i=1}^\infty ((\epsilon, 2, \epsilon), (31^{n_{2i}}, 1, \epsilon))
$$

(3.4)

or all $\Gamma = \Gamma_0, \ldots, \Gamma_w, \prod_{i=1}^\infty ((\epsilon, 2, \epsilon), (31^{n_{2i+1}}, 1, \epsilon))$

with $\Gamma_w \neq (31^{n_{w+1}}, 1, \epsilon)$, and $w$ odd.

• The set $S_2$ will consist of the tail equivalence class of

$$
\Gamma_\alpha := \prod_{i=1}^\infty (\epsilon, 3, 1^n_i) \text{ or all } \Gamma = \Gamma_0, \ldots, \Gamma_w, \prod_{i=w+1}^\infty (\epsilon, 3, 1^n_i)
$$

(3.5)

with $\Gamma_w \neq (\epsilon, 3, 1^{n_{w+1}})$.

• Finally, $\mathcal{N} = \mathcal{P} \setminus (S_1 \cup \hat{S}_1 \cup S_2)$. Note that $\Gamma \in \mathcal{N}$ if and only if there exist arbitrarily large $k$ with $u_k \neq \epsilon$ and there exist arbitrarily large $k$ with $v_k \neq \epsilon$. Thus $\mathcal{N}$ consists of those paths for which $W(\Gamma)$ is a two-sided infinite sequence.

As mentioned in the introduction, we don’t make explicit use of the substitution induced order on the path space but its description will clarify what follows. On a path space derived from a general diagram, Vershik defines a partial order which restricts to a total order on each tail equivalence class. It is built from a total order on the outgoing edges at each vertex. Specifically, for two paths in the same tail equivalence class, the lesser one is the one with the lesser outgoing edge at the last vertex they disagree. The Vershik map on the path space sends each non-maximal path to its successor.

In the particular case of a Bratteli-Vershik diagram which is the Prefix-Suffix Automaton for a single substitution $\sigma$, it was noted in [15] and [26] that there is a natural total order on the outgoing edges from a vertex. If the vertex is labeled by the letter $b$, each outgoing edge is labeled $(u, a, v)$ where $\sigma(b) = uav$, and so declare $(u, a, v) < (u', a', v')$ if $|u| < |u'|$. When the path space is mapped onto the substitution minimal set via the analog
of (3.2), the Vershik map on the path space will conjugate (on a large set) with the shift map on the substitution minimal set.

For the IPSA of the infimax S-adic family considered here, one may define a partial order on the path space exactly as in the single substitution case. In this order the path $\Gamma_\alpha$ is the maximal element and the paths $\Gamma_\beta$ and $\hat{\Gamma}_\beta$ are minimal elements. Thus the sets $S_i$ just defined are the tail equivalence classes of the maximal and minimal elements. We therefore expect from Vershik’s construction that each set would correspond to an orbit of sequences under the shift. This is the content of Lemma 3.3. The corresponding sequences are built from the following:

$$\alpha = \lim_{k \to \infty} \Lambda^{(k)}(\epsilon.3)$$
$$\beta = \lim_{k \to \infty} \Lambda^{(2k)}(1.\epsilon) = \lim_{k \to \infty} \Lambda^{(2k-1)}(2.\epsilon)$$
$$\hat{\beta} = \lim_{k \to \infty} \Lambda^{(2k)}(2.\epsilon) = \lim_{k \to \infty} \Lambda^{(2k+1)}(1.\epsilon)$$

It is clear that the limits exist. Let $t = \beta \cdot \alpha$ and $\hat{t} = \hat{\beta} \cdot \alpha$. These are the S-adic analogs of the period two point in the single substitution case in the sense that

If $s_{-1} = 1$, then $\Lambda^{(2k)}(s) \to t$ and $\Lambda^{(2k+1)}(s) \to \hat{t}$
If $s_{-1} = 2$ or 3, then $\Lambda^{(2k+1)}(s) \to t$ and $\Lambda^{(2k)}(s) \to \hat{t}$

Lemma 3.3. For an $n \in \Sigma^+$,

(a) if $\Gamma \in S_1$ as in (3.3), there exists a $j < 0$ with $W(\Gamma) = S^j(\beta.\epsilon)$. In particular, $W(\Gamma_\beta) = S^{-1}(\beta.\epsilon)$.

(b) If $\Gamma \in S_1$ as in (3.4), there exists a $j < 0$ with $W(\Gamma) = S^j(\hat{\beta}.\epsilon)$. In particular, $W(\Gamma_\hat{\beta}) = S^{-1}(\hat{\beta}.\epsilon)$.

(c) If $\Gamma \in S_2$ as in (3.5), there exists a $j \geq 0$ with $W(\Gamma) = S^j(\epsilon.\alpha)$. In particular, $W(\Gamma_\alpha) = \epsilon.\alpha$.

Proof. We start with (c). If $\Gamma \in S_2$ as in (3.5), then $a_k = 3$ for all $k > w$. Thus by Lemma 3.1(b), there is some $j = j(k)$ with $W(\Gamma) = \ldots S^{j(k)}(\Lambda^{(k)}(\epsilon.3)) \ldots$. On the other hand, $u_k = \epsilon$ for all $k > w$, and so for those $k$, $j(k)$ is a constant $j$, and thus $W(\Gamma) = \lim_{k \to \infty} \epsilon S^j(\Lambda^{(k)}(\epsilon.3)) \ldots = S^j(\epsilon.\alpha)$. In particular since for $\Gamma_\alpha$ all $u_k = \epsilon$, $W(\Gamma_\alpha) = \epsilon.\alpha$.

The arguments for (a) and (b) are similar and we just give (a). If $\Gamma \in S_1$ as in (3.3), then $a_{2m} = 1$ for all $2m > w$. Thus by Remark 3.2 there is some $j = j(m) \geq 1$ with $W(\Gamma) = \ldots S^{-j(m)}\Lambda^{(2m)}(1.\epsilon) \ldots$. On the other hand, $v_k = \epsilon$ for all $k > w$, and so for those $k$, $j(k)$ is a constant $j$, and
thus $W(\Gamma) = \lim_{m \to \infty} \ldots S^{-j}(\Lambda^{(2m)}(1, \epsilon)) \epsilon = S^{-j}(\beta, \epsilon)$. In particular since for $\Gamma_\beta$ all $v_k = \epsilon$ and $a_0 = 1$, so $W(\Gamma_\beta) = S^{-1}(\beta, \epsilon)$. $\square$

With Lemma 3.3 in mind, now define
\[
\mathcal{S}\mathcal{S}_1 = \{S^j(\beta, \epsilon): j < 0\}
\]
(3.7)
\[
\mathcal{S}\mathcal{S}_1 = \{S^j(\beta, \epsilon): j < 0\}
\]
\[
\mathcal{S}\mathcal{S}_2 = \{S^j(\epsilon, \alpha): j \geq 0\}
\]

3.4. Injectivity of the word and sequence maps. The maps $W$ takes the prefix-suffix labels along a path and generates a word or sequence via the Dumont-Thomas expansion. The next theorem states that this assignment is injective.

Theorem 3.4. Given $n \in \Sigma_\infty^+$.

(a) For each $a \in \{1, 2, 3\}$, the word map $\gamma \mapsto W(\gamma)$ defined in (3.1) gives a bijection between $P^{(k)}(a)$ and
\[
\mathcal{S}(k)(a) := \{S^j(\Lambda^{(k)}(a)): 0 \leq j < |\Lambda^{(k)}(a)|\}.
\]
(b) The sequence map $\Gamma \mapsto W(\Gamma)$ defined in (3.2) is an injection on $\mathcal{N}$ and a bijection between $S_i$ and $\mathcal{S}\mathcal{S}_i$ for $i = 1, 1, 2$.

Remark 3.5. This theorem coupled with later results may be be viewed as a desubstitution or recognizability result. As in the proof of Lemma 3.1, given $n \in \Sigma_\infty^+$ and a path $\Gamma = \Gamma_0, \Gamma_1, \ldots \in P_n$, let $n' = n_2n_3\ldots$ and form the path $\Gamma' = \Gamma_1, \Gamma_2, \cdots \in P_{n'}$. If $W'$ is the sequence map associated with the path space $P_{n'}$, then directly from the definitions of the sequence maps
\[
S^{[n_0]}\Lambda_{n_1}(W'(\Gamma')) = W(\Gamma).
\]
Thus if $W(\Gamma)$ is a bi-infinite sequence (i.e. $\Gamma \in \mathcal{N}$), it is desubstituted under $\Lambda_{n_1}$ by $W'(\Gamma')$. By Theorem 3.4 the assignment $\Gamma \mapsto W(\Gamma)$ is injective, so the desubstitution is unique. To obtain a full result, we need in addition that $W(\Gamma) \in \Delta_n$ (Theorem 5.3) and for $\Gamma \in S_i$, we must extend $W(\Gamma)$ to a bi-infinite sequence in $\Delta_n$ as described above Theorem 5.3. Note that a sequence in $\Delta_n$ is desubstituted under $\Lambda_{n_1}$ by a sequence in $\Delta_{n'}$. One may continue, desubstituting under $\Lambda_{n_2}$ a sequence in $\Delta_{n'}$ by one in $\Delta_{n''}$, where $n'' = n_3n_4\ldots$, etc. Fisher studies this situation in [22] and constructs the S-adic analog of the subshift of finite type corresponding to a single substitution (see Section 7.2 in [34]).

The proof of Theorem 3.4 requires more detailed information on the word and sequence maps $W$. Since we are considering words and sequences of
Lemma 3.7. For all $\underline{n} \in \Sigma_\infty^+$:

(a) If $\Gamma \neq \overline{\Gamma}$ in $\mathcal{P}_{\underline{n}}$ or $\gamma \neq \overline{\gamma}$ in $\mathcal{P}^{(k)}$, then $\mathcal{W}(\Gamma) \neq \mathcal{W}(\overline{\Gamma})$ and $\mathcal{W}(\gamma) \neq \mathcal{W}(\overline{\gamma})$.

(b) If $\Gamma \neq \overline{\Gamma}$ in $\mathcal{P}_{\underline{n}}$ then $\mathcal{W}(\Gamma) \neq \mathcal{W}(\overline{\Gamma})$ at an index $j \geq 0$ except for the case of pairs of the form

\[
\Gamma = \Gamma_0, \ldots, \Gamma_{w-2}, (31^{n_w-\ell}, 1, 1^\ell), \Gamma_w, \ldots
\]

\[
\overline{\Gamma} = \Gamma_0, \ldots, \Gamma_{w-2}, (31^{n_w-\ell}, 1, 1^\ell), \Gamma_w, \prod_{i=1}^{\infty} ((\ell, 2, \epsilon), (31^{n_{w+2i}}, 1, \epsilon))
\]

with $\ell > \overline{\ell}$ or

\[
\Gamma = \Gamma_0, \ldots, \Gamma_{w-2}, (31^{n_w-\ell}, 1, 1^\ell), \Gamma_w, \ldots
\]

\[
\overline{\Gamma} = \Gamma_0, \ldots, \Gamma_{w-2}, (31^{n_w-\ell}, 1, 1^\ell), \Gamma_w, \prod_{i=1}^{\infty} ((\ell, 2, \epsilon), (31^{n_{w+2i}}, 1, \epsilon))
\]

with $\ell \geq \overline{\ell}$.

Proof. We prove (b) first. The proof is by induction on $k$, the smallest index with $\Gamma_k \neq \overline{\Gamma}_k$. To start assume that $\Gamma_0 \neq \overline{\Gamma}_0$. Now if $a_0 \neq \overline{a}_0$, then $\mathcal{W}(\Gamma_0) \neq \mathcal{W}(\overline{\Gamma}_0)$ at the index $j = 0$ and so we may assume $a_0 = \overline{a}_0$. Since just one edge emerges from state 2, the case $a_0 = \overline{a}_0 = 2$ is impossible, so we are left with two remaining cases.

Now if $a_0 = \overline{a}_0 = 3$ and $\Gamma_0 \neq \overline{\Gamma}_0$, the only possibilities are, say, $\Gamma_0 = (\epsilon, 3, 1^{n_1+1})$ which can only be followed by $\Gamma_1 = (\epsilon, 2, \epsilon)$, and $\overline{\Gamma}_0 = (\epsilon, 3, 1^{n_1})$ which can only be followed by $\overline{\Gamma}_1 = (\epsilon, 3, 1^m)$ with $m = n_2$ or $m = n_2 + 1$. In either case, $\mathcal{W}(\Gamma_0 \Gamma_1)_R = .31^{n_1+1}$ and $\mathcal{W}(\overline{\Gamma}_0 \overline{\Gamma}_1)_R = .31^{n_1}1\overline{\lambda}_{n_1}(1^m) = .31^{n_1}2^m$, and so $\mathcal{W}(\Gamma_0 \Gamma_1) \neq \mathcal{W}(\overline{\Gamma}_0 \overline{\Gamma}_1)$ at index $j = n_1 + 1 > 0$.

Now assume $a_0 = \overline{a}_0 = 1$ and we consider various subcases. First assume that $\Gamma_0$ and $\overline{\Gamma}_0$ both terminate at state 3. Thus $\Gamma_0 = (31^{n_1-\ell}, 1, 1^\ell)$ and
\[ \Gamma_0 = (31^{n_1-\ell}, 1, 1) \] where, say, \( \ell > \bar{\ell} \). Thus \( \mathcal{W}(\Gamma_0, \Gamma_1)_R = .1^{\ell+1}\Lambda^{(1)}(1^m) \) and \( \mathcal{W}(\Gamma_0, \Gamma_1)_R = .1^{\ell+1}\Lambda^{(1)}(1^m) \) with \( m, \bar{m} = n_2 \) or \( n_2 + 1 \). Since \( \Lambda^{(1)}(1) = 2 \), \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\bar{\Gamma}) \) at index \( j = \bar{\ell} + 1 > 0 \).

Now assume that \( \Gamma_0 \) and \( \Gamma_0 \) both terminate at state 2. Thus \( \Gamma_0 = (31^{n_1-\ell}, 1, 1) \) and \( \Gamma_0 = (31^{n_1-\ell}, 1, 1) \) where, say, \( \ell > \bar{\ell} \). If \( \bar{v}_k = \epsilon \) for all \( k \geq 1 \) then we are in the excluded case (3.8). Thus for some \( k \geq 1 \), \( \bar{v}_k = 1^m \) with \( m > 0 \). Thus \( \mathcal{W}(\Gamma_0, \Gamma_1)_R = .1^{\ell+1}\Lambda^{(1)}(1^m) \) and since \( \Lambda^{(1)}(1) = 2 \), \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\bar{\Gamma}) \) at index \( j = \bar{\ell} + 1 > 0 \).

For the last subcase, assume that \( \Gamma_0 \) terminates at 3 and \( \Gamma_0 \) terminates at 2. Thus \( \Gamma_0 = (31^{n_1-\ell}, 1, 1) \) and \( \Gamma_0 = (31^{n_1-\ell}, 1, 1) \). Of necessity \( v_1 = 1^m \) with \( m = n_2 \) or \( n_2 + 1 \), thus if \( \ell < \bar{\ell} \), then \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\bar{\Gamma}) \) at index \( j = \bar{\ell} + 1 > 0 \). Now assume that \( \ell \geq \bar{\ell} \). If \( \bar{v}_k = \epsilon \) for all \( k \geq 1 \) then we are in the excluded case (3.9). Thus for some \( \bar{k} \geq 1 \), \( \bar{v}_\bar{k} = 1^m \) with \( m > 0 \). In fact, \( \bar{k} \geq 2 \) because of necessity, \( \Gamma_1 = (\epsilon, 2, \epsilon) \). Since \( v_1 = 1^m \) with \( m = n_2 \) or \( n_2 + 1 \), \( \mathcal{W}(\Gamma)_R = .1^{\ell+1}\Lambda^{(1)}(1^m) = .1^{\ell+1}2 \ldots \) and \( \mathcal{W}(\Gamma)_R = .1^{\ell+1}\Lambda^{(1)}(1^m) = .1^{\ell+1}2 \ldots \) using the fact that \( \bar{k} \geq 2 \) and so \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\bar{\Gamma}) \) at index \( j = \bar{\ell} + 1 > 0 \). This finishes the initial inductive step of \( k = 0 \).

Now assume the result is true for \( k > 0 \). Let \( n' = n_2n_3 \ldots \) and consider the infinite paths in the associated IPSA, \( P_{n'} \), obtained by deleting the first edges of \( \Gamma \) and \( \bar{\Gamma} \) and so \( \Gamma' = \Gamma_1, \Gamma_2, \ldots \) and \( \bar{\Gamma}' = \bar{\Gamma}_1, \bar{\Gamma}_2, \ldots \). If \( \Gamma \) and \( \bar{\Gamma} \) are not of the form (3.8) or (3.9), then neither are \( \Gamma' \) and \( \bar{\Gamma}' \). Since \( \bar{\Gamma}' \) and \( \Gamma' \) have \( \bar{\Gamma}' = \Gamma' \), if \( \mathcal{W}' \) is the word map for \( P_{n'} \), then by the inductive hypothesis \( \mathcal{W}'(\Gamma') \neq \mathcal{W}'(\bar{\Gamma}') \) at some index \( j \geq 0 \) and let \( j' \) be the least such \( j \). Now note as in the proof of Lemma 3.1, \( S^{[n_o]}\Lambda_{n_1}\mathcal{W}'(\Gamma') = \mathcal{W}(\Gamma) \) and \( S^{[n_o]}\Lambda_{n_1}\mathcal{W}'(\bar{\Gamma}') = \mathcal{W}(\bar{\Gamma}) \). We claim that this implies that \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\bar{\Gamma}) \) at an index \( J \geq 0 \). We first show the result that \( \Lambda_{n_1}\mathcal{W}'(\Gamma') \neq \Lambda_{n_1}\mathcal{W}'(\bar{\Gamma}') \) at an index \( j \geq 0 \) and then argue that, in fact, \( j \geq |u_0| \).

Now if it was the case that \( \mathcal{W}'(\Gamma') \neq \mathcal{W}'(\bar{\Gamma}') \) at an index \( j' \geq 0 \) with either \( \mathcal{W}'(\Gamma')_{j'} = 1 \) or \( \mathcal{W}'(\bar{\Gamma}')_{j'} = 1 \), then the result is immediate since \( \Lambda_{n_1}(1) = 2 \) and both \( \Lambda_{n_1}(2) \) and \( \Lambda_{n_1}(3) \) begin with a 3. So the only case remaining is, say, \( \mathcal{W}'(\Gamma')_{j'} = 2 \) and \( \mathcal{W}'(\bar{\Gamma}')_{j'} = 3 \). Using Remark 3.6, the 3 must be followed by a 1. Since \( j' \geq 0 \) and \( \Lambda_{n_1}(2) = 31^{n_1+1} \) while \( \Lambda_{n_1}(31) = 31^{n_1}2 \), we have the result in this case also.

Next we show that \( j \geq |u_0| \). When \( k > 1 \),

\[
\Lambda_{n_1}\mathcal{W}'(\Gamma')_R = \Lambda_{n_1}(a_1)\Lambda^{(1)}(v_1)\ldots\Lambda^{(k-1)}(v_{k-1})\Lambda^{(k)}(v_k)\ldots
\]

\[
\Lambda_{n_1}\mathcal{W}'(\bar{\Gamma}')_R = \Lambda_{n_1}(a_1)\Lambda^{(1)}(v_1)\ldots\Lambda^{(k-1)}(v_{k-1})\Lambda^{(k)}(\bar{v}_k)\ldots
\]
and so whatever \( j \) is it satisfies
\[
j \geq |\Lambda_{n_1}(a_1)| + \sum_{i=1}^{k-1} |\Lambda^{(i)}(v_i)| = |u_0a_0v_0| + \sum_{i=1}^{k-1} |\Lambda^{(i)}(v_i)|.
\]

When \( k = 1 \), since \( \Gamma_0 = \Gamma_0 \), both \( \Gamma_1 \) and \( \Gamma_1 \), emerge from the same state \( a_1 = \bar{a}_1 \) and so
\[
\Lambda_{n_1}(\Gamma_0')_R = \Lambda_{n_1}(a_1)\Lambda^{(1)}(v_1) \ldots \\
\Lambda_{n_1}(\Gamma_1')_R = \Lambda_{n_1}(a_1)\Lambda^{(1)}(v_1) \ldots
\]
so \( j \geq |\Lambda_{n_1}(a_1)| = |u_0a_0v_0| \). Thus in every case, \( j > |u_0| \) and so \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\Gamma) \) at an index \( J = j - |u_0| \geq 0 \).

For (a), note that for \( \Gamma \) and \( \Gamma \) as in (3.8) or (3.9) have \( u_i = \bar{u}_i \) for \( i = 1, \ldots, w-2 \) and \( u_{w-1} \neq \bar{u}_{w-1} \), thus \( \mathcal{W}(\Gamma) \neq \mathcal{W}(\Gamma) \) at an index \( j < 0 \). \( \square \)

**Proof of Theorem 3.4.** For (a) first note the assignment is injective by Lemma 3.7(a) and the inclusion \( \mathcal{W}(\mathcal{P}^{(k)}(a)) \subset \mathcal{S}^{(k)}(a) \) follows from Lemma 3.1.
To finish part (a), by the definition of the IPSA, the Abelianizations and standard graph theory,
\[
\#(\mathcal{P}^{(k)}(a)) = \|A_{n_1 \ldots n_k}(\bar{e}a)\|_1 = |\Lambda^{(k)}(a)|,
\]
and so the assignment is onto.

To see that \( \mathcal{W} : \mathcal{S}_2 \to \mathcal{S}\mathcal{S}_2 \) is onto, pick \( S^j(\alpha_j) \) for some \( j \geq 0 \). Using part (a), for any \( k \) with \( 0 \leq j < |\Lambda^{(k)}(3)| \) there is a path \( \gamma = \Gamma_0, \ldots, \Gamma_k \) with \( a_k = 3 \) and \( \mathcal{W}(\gamma) = S^j\Lambda^{(k)}(3) \). Thus if we define the infinite path \( \Gamma = \Gamma_0, \ldots, \Gamma_k, \prod_{i=k+1}^{\infty}(e,3,1^{n_i}) \), then as in the proof of Lemma 3.3 \( \mathcal{W}(\Gamma) = S^j(\alpha_j) \). The proof that \( \mathcal{W} : \mathcal{S}_1 \to \mathcal{S}\mathcal{S}_1 \) and \( \mathcal{W} : \mathcal{S}_1 \to \mathcal{S}\mathcal{S}_1 \) are onto are similar. Since \( \mathcal{W}(\mathcal{N}), \mathcal{S}\mathcal{S}_1, \mathcal{S}\mathcal{S}_1 \) and \( \mathcal{S}\mathcal{S}_2 \) are all mutually disjoint, the injectivity of all the assignments follow from Lemma 3.7. \( \square \)

4. A TOPOLOGICAL LEMMA

Recall that our goal is to construct a geometric representation of the S-adic symbolic minimal sets defined in the introduction. The word map \( \mathcal{W} \) takes us from paths to sequences and so we need a map in the other direction. This requires an elementary, but technical topology lemma. For a subset \( Z_1 \) in a topological space \( Z_2 \), its set of limit points is denoted \( \text{LimPts}(Z_1) \)

**Lemma 4.1.** Assume \( X \) and \( Y \) are compact metric spaces with \( X = X_1 \sqcup X_2 \) and there exist injections \( f_1 : X_1 \to Y \) and \( f_2, f_3 : X_2 \to Y \) so that when
$Y_i = \text{image}(f_i)$ for $i = 1, 2, 3$ the $Y_i$ are disjoint with $Y_1$ dense in $Y$. Define a set-valued function $\mathcal{F}$ by

\[
\mathcal{F}(x) = \begin{cases} f_1(x) & \text{when } x \in X_1 \\
\{f_2(x), f_3(x)\} & \text{when } x \in X_2.
\end{cases}
\]

and assume that $\mathcal{F}$ has the property that if $x_n \to x \in X$, then $\text{LimPts}\{\mathcal{F}(x_n)\} \subseteq \mathcal{F}(x)$. Then $\mathcal{F}$ is onto in the sense that $Y = \bigcup_{x \in X} \mathcal{F}(x)$ and if $g : Y \to X$ is defined by $g(y) = f_i^{-1}(y)$ when $y \in Y_i$, then $g$ is continuous, injective on $Y_i$ and two-to-one on $Y_2 \cup Y_3$.

Proof. We first prove $Y = \bigcup_{x \in X} \mathcal{F}(x)$. Since $Y_1 = f_1(X_1)$ is dense in $Y$ by hypothesis, for all $y \in Y$ there is a sequence $\{x_n\} \subset X_1$ with $f_1(x_n) \to y$. Since $X$ is compact, there is an $x_0$ and a subsequence $x_{n_i} \subset X_1$ with $x_{n_i} \to x_0$. Thus $\mathcal{F}(x_0) \supset \text{LimPts}\{\mathcal{F}(x_{n_i})\} = \text{LimPts}\{f_1(x_{n_i})\} = y$ and so $y = f_i(x_0)$ for some $i = 1, 2, 3$ as required.

We now prove that $g$ is continuous. For $y \in Y_2$, let $\hat{y} = f_3 \circ f_2^{-1}(y)$ and for $y \in Y_3$, let $\hat{y} = f_2 \circ f_3^{-1}(y)$ and for $y \in Y_1$, $\hat{y} = y$. Note that in all cases, $g(y) = g(\hat{y})$ and that $\mathcal{F} \circ g(y) = \{y, \hat{y}\}$ and $g \circ \mathcal{F}(x) = \{x\}$.

Assume now that $y_n \to y \in Y$ and we show $g(y_n) \to g(y) \in X$. Since $X$ is compact, there is a subsequence and an $x_0 \in X$ with $g(y_n) \to x_0$. Thus $\mathcal{F}(x_0) \supset \text{LimPts}\{\mathcal{F} \circ g(y_n)\} = \text{LimPts}\{y_n, \hat{y}_n\}$ which contains $y$. Thus $y \in \mathcal{F}(x_0)$ and so $g(y) \in g \circ \mathcal{F}(x_0) = \{x_0\}$ as needed.

5. FROM THE INFIMAX MINIMAL SET TO THE IPSA

For $\underline{\alpha} \in \Sigma_3^+$, recall that its asymptotic period two points are $t = \underline{\beta} \cdot \underline{\alpha}$ and $\hat{t} = \underline{\beta} \cdot \underline{\alpha}$ defined in (3.6).

Definition 5.1. The infimax minimal set corresponding to a collection of substitutions $\underline{\alpha} \in \Sigma_3^+$ is $\Delta_{\underline{\alpha}} = \text{Cl}(o(t, S)) \subset \Sigma_3$.

It follows from Remark 23(b) in [11] that $t$ is almost periodic and thus $\Delta_{\underline{\alpha}}$ is, in fact, a minimal set. Further, by Theorem 16 in [11] it is aperiodic. It is standard that a compact minimal set is equal to the closure of the forward or backward orbits of any of its elements and so we have:

Lemma 5.2. For all $\underline{\alpha} \in \Sigma_3^+$, $\Delta_{\underline{\alpha}}$ is an aperiodic, minimal set and $\Delta_{\underline{\alpha}} = \text{Cl}(o(t, S)) = \text{Cl}(o^+(t, S)) = \text{Cl}(o^-(t, S))$.

Recall from the introduction that the strategy here is to produce a geometric representation of the infimax minimal set $\Delta_{\underline{\alpha}}$ as the attractor of an
interval word map using the path space $P$ as an intermediary. By Theorem 3.4(b) the word map $W$ gives an injection from paths to sequences. However, for the special paths in the $S_1$, the image path is a one-sided sequence. There is a natural way to make these sequence two-sided as follows. By Theorem 3.4(b), the image under $W$ of a path in $S_1$ is a one-sided sequence of the form $S^j(\beta, \epsilon)$ with $j < 0$ and so we redefine the word map image of paths in $S_1$ as $\hat{S}^j(t)$ for the same $j$. Similarly, we redefine the word map image of paths in $S_1$ as $\hat{S}^j(\hat{t})$ for $j < 0$. These are natural extensions because sequences of the form $S^j(t)$ and $S^j(\hat{t})$ are not in the image of $W$ and they clearly are in $\Delta_n$. In addition as we will prove below, the only sequences of the form $\beta_* \hat{t}$ and $\beta_* \hat{t}$ in $\Delta_n$ are $t$ and $\hat{t}$ respectively.

The complication comes with extending the word map images of paths in $S_2$. These images are of the form $S^j(\epsilon, \alpha)$ for $j \geq 0$. These one-sided sequences can be extended as both $S^j(t)$ and $S^j(\hat{t})$, so the natural extension of $W$ is not a single valued function. Fortunately, what we require for the geometric representation is a map going the other way, $\Delta_n \to P$, and this map can be constructed in the next theorem using Lemma 4.1 which was, of course, designed for just this purpose.

**Theorem 5.3.** Let $Z = \{S^j(t) : j \geq 0\} \cup \{S^j(\hat{t}) : j \geq 0\}$. There exists a continuous surjection $G : \Delta_n \to P$ which is an injection on $\Delta_n \setminus Z$ and two-to-one on $Z$ with $G(S^j(t)) = G(S^j(\hat{t}))$ for all $j \geq 0$.

**Proof.** We start by constructing the various spaces and maps needed for the application of Lemma 4.1. To distinguish the specific constructions from that general lemma we use uppercase letters to denote the corresponding maps. Let $X_1 = N \sqcup S_1 \sqcup S_1$, $X_2 = S_2$, and $Y = \Delta_n$. The definition of $F_1 : X_1 \to Y$ depends on the location of $\Gamma$. When $\Gamma \in N$, let $F_1(\Gamma) = W(\Gamma)$. When $\Gamma \in S_1$, by Theorem 3.4 we have $W(\Gamma) = S^j(\beta, \epsilon)$ for a unique $j < 0$ and using this $j$ define $F_1(\Gamma) = S^j(\beta, \alpha)$. Similarly, when $\Gamma \in S_1$, $W(\Gamma) = S^j(\hat{\beta}, \epsilon)$ for a unique $j < 0$ and define $F_1(\Gamma) = S^j(\hat{\beta}, \alpha) = S^j(\hat{t})$.

When $\Gamma \in X_2 = S_2$, again by Lemma 3.4 we have $W(\Gamma) = S^j(\epsilon, \alpha)$ for a unique $j \geq 0$ and using this $j$ define $F_2(\Gamma) = S^j(\beta, \alpha) = S^j(t)$ and $F_3(\Gamma) = S^j(\hat{\beta}, \alpha) = S^j(\hat{t})$.

We thus have as subsets of $Y = \Delta_n$, $F_1(S_1) = \{S^j(t) : j < 0\}$, $F_1(S_1) = \{S^j(\hat{t}) : j < 0\}$, $F_2(S_2) = \{S^j(t) : j \geq 0\}$, and $F_3(S_2) = \{S^j(\hat{t}) : j \geq 0\}$. Let $Y_1 = F_1(N) \sqcup F_1(S_1) \sqcup F_1(S_1)$, $Y_2 = F_2(S_2)$, and $Y_3 = F_3(S_2)$.

We now show that these spaces and maps have the required properties. Since $F_1(S_1) = o^-(t, S) \setminus \{t\}$, Lemma 5.2 shows that $F_1(S_1)$ is dense in
and thus so is $F_1(X_1)$. All the $F_i$ are injective by Theorem 3.4 and Lemma 3.7. That lemma also implies that $Y_1 \cap (Y_2 \cup Y_3) = \emptyset$. Now if $\Gamma, \Gamma' \in S_2$ with $\Gamma \neq \Gamma'$, by Lemma 3.7 again, $F_2(\Gamma) \neq F_3(\Gamma')$. Finally, for $\Gamma \in S_2$, $F_2(\Gamma) \neq F_3(\Gamma)$ since $\beta = \ldots 1.$ and $\hat{\beta} = \ldots 2.$ Thus all the $Y_i$ are disjoint.

We now check the required limit assertions. Two initial observations about paths in the IPSA will be needed. The first is that if $u \neq \epsilon$ is a prefix in an edge label, then $u = 3^i \ell$ with $\ell \geq 0$. Thus for all $k > 2$ there exists a word $W$ with $\Lambda(k)(u) = W \Lambda(\sigma)(1)$ and $\sigma = k$ or $k + 1$. The second is that if $v \neq \epsilon$ is a suffix in an edge label, then $v = 1^\ell$ with $\ell > 0$. Thus for all $k > 3$ there exists a word $W$ with $\Lambda(k)(v) = \Lambda(k-2)(3)W$.

Assume that $\Gamma(\alpha)$ is a sequence in $\mathcal{P}$ with $\Gamma(\alpha) \to \Gamma \in S_2$ with

$$\Gamma = \Gamma_0, \ldots, \Gamma_w, \prod_{i=w+1}^{\infty} (\epsilon, 3, 1^n),$$

and $\Gamma_w \neq (\epsilon, 3, 1^n)$. By Lemma 3.3(c) and Lemma 3.1(b), we know that for all $k > w$ there is a $j \geq 0$ (independent of $k$) so that

$$\mathcal{W}(\Gamma) = S^j(\epsilon, \alpha) = \epsilon S^j \Lambda(k)(\epsilon, 3) \Lambda(k)(v) \ldots$$

Now if $\Gamma_m \in S_2$ and $d(\Gamma_m, \Gamma) < 1/(w + 2)$, then $\Gamma_m = \Gamma$, so we may assume that $\Gamma_m \notin S_2$ for all sufficiently large $m$. Let $M(m)$ be such that $(\Gamma(\alpha)_i) = \Gamma_i$ for $i = 1, \ldots, M(m) - 1$ and $(\Gamma(\alpha)_i)M(m) \neq \Gamma M(m)$. If $M(m) > w$ this implies that $(\Gamma(\alpha)_i)M(m) = (\epsilon, 3, 1^{nM(m)+1})$, $(\Gamma(\alpha)_i)M(m)+1 = (\epsilon, 2, \epsilon)$, and $(\Gamma(\alpha)_i)M(m)+2 = 31^\ell$. Thus using the first observation above,

$$\mathcal{W}(\Gamma(\alpha)_i) = \ldots \Lambda(\sigma^i)M(m)(1)S^j \Lambda(M(m))(\epsilon, 3) \Lambda(M(m))(v M(m)) \ldots$$

with $\sigma(m) = M(m) + 1$ or $M(m) + 2$. Now of necessity $M(m) \to \infty$ as $m \to \infty$ and so $\sigma(m) \to \infty$ also and thus by Lemma 3.7(b), $\text{LimPts} \{\Gamma(\alpha)_i\} \subset \{S^j(\beta, \alpha) \cup S^j(\hat{\beta}, \alpha)\}$, as required.

Now assume that $\Gamma(\alpha)$ is a sequence in $\mathcal{P}$ with $\Gamma(\alpha) \to \Gamma \in S_1$ with

$$\Gamma = \Gamma_0, \ldots, \Gamma_w, \prod_{i=1}^{\infty} ((\epsilon, 2, \epsilon), (31^{n \ell + 1}, 1, \epsilon))$$

with $\Gamma_w \neq (31^{n \ell + 1}, 1, \epsilon)$ and $w$ is even. By Lemma 3.3 and Remark 3.2 we know that for all $2k > w$ there is a $j < 0$ (independent of $k$) so that

$$\mathcal{W}(\Gamma) = S^j(\beta, \epsilon) = \ldots \Lambda(2k)u_{2k}S^j \Lambda(2k)(1, \epsilon) \epsilon \ldots$$

As in the previous argument, we may assume that $\Gamma_m \notin S_1$ for all sufficiently large $m$. Let $M(m)$ be such that $(\Gamma(\alpha)_i) = \Gamma_i$ for $i = 1, \ldots, M(m) - 1$ and $(\Gamma(\alpha)_i)M(m) \neq \Gamma M(m)$. Since there is a unique outgoing edge from state
2 this implies that \( M(m) \) is even. Examining the IPSA to determine \( \Gamma_{M(m)} \) and its successors and then using the second observation above,

\[
\mathcal{W}(\Gamma_{(m)}) = \ldots \Lambda^{(M(m))}(u_{M(m)})S^{j} \Lambda^{(M(m))}(1, \epsilon)\epsilon\Lambda^{(\sigma)}(3) \ldots
\]

with \( \sigma(m) = M(m) - 1 \) or \( M(m) - 2 \). Now \( M(m) \) and \( \sigma(m) \to \infty \) as \( m \to \infty \) and thus by (3.6), \( \Gamma_{(m)} \to S_{j}^{\beta} \). The case \( \Gamma_{(m)} \to \Gamma \in S \) is similar.

Thus we have verified all the required properties to utilize Lemma 4.1 and so \( G : \Delta \to P \) defined by \( G(y) = F_{i}^{-1}(y) \) when \( y \in Y_{i} \) is continuous, injective on \( X_{1} \) and two-to-one as indicated on \( Z = Y_{2} \cup Y_{3} \).

6. The one-sided case

Theorem 5.3 describes the relation of the infimax minimal set \( \Delta_{\mathbb{N}} \) and the path space \( P \) using the map \( G \) which is, roughly speaking, the inverse of the extended word map. Theorem 5.3 shows the only non-injectivity of \( G \) is in sending points on the forward orbits of \( \hat{t} \) and \( \hat{t} \) to the same path. Recall now that \( t = \beta \leftarrow \alpha \rightarrow \) and \( \hat{t} = \hat{\beta} \leftarrow \hat{\alpha} \rightarrow \) and thus for \( j \geq 0 \), \( \pi(S^{j}(\hat{t})) = \pi(S^{j}(\hat{t})) \) where \( \pi : \Sigma_{3} \to \Sigma_{3}^{+} \) is defined as \( \pi(s) = s_{0}s_{1} \ldots \). This suggests that the path space \( P \) is, in fact, homeomorphic to the infimax minimal set restricted to right one-sided sequences defined as \( \Delta_{\mathbb{N}}^{+} = \text{Cl}(o^{+}(\alpha_{\gamma}, S)) \). This is the main content of next theorem.

The required homeomorphism is defined using notation as in the proof of Theorem 5.3 by \( H : P \to \Delta_{\mathbb{N}}^{+} \) when \( \Gamma \in X_{1} \) as \( H(\Gamma) = \pi \circ F_{1}(\Gamma) \) and for \( \Gamma \in X_{2} \), \( H(\Gamma) = \pi \circ F_{2}(\Gamma) \). Note that when \( \Gamma \in X_{2} \), \( \pi \circ F_{3}(\Gamma) = \pi \circ F_{2}(\Gamma) = H(\Gamma) \). Thus informally, \( H = \pi \circ F \). Note that it is standard that \( \Delta_{\mathbb{N}}^{+} = \pi(\Delta_{\mathbb{N}}) \).

**Theorem 6.1.**

(a) The map \( H : P \to \Delta_{\mathbb{N}}^{+} \) is a homeomorphism.

(b) The map \( \pi : \Delta_{\mathbb{N}} \to \Delta_{\mathbb{N}}^{+} \) is injective except for the case where \( \alpha = S^{j}(\hat{t}) \) and \( \hat{\alpha} = S^{j}(\hat{t}) \) for the same \( j \geq 0 \), in which case \( \pi(\alpha) = \pi(\hat{\alpha}) \).

(c) Each \( \alpha_{\gamma} \in \Delta_{\mathbb{N}}^{+} \) has a unique inverse under the shift \( S \) with the sole exception that \( S^{-1}(\alpha_{\gamma}) = \{1\alpha_{\gamma}, 2\alpha_{\gamma}\} \).

**Proof.** Since \( \pi \) is continuous, the limit assertions proved in Theorem 5.3 show that \( H \) is continuous. Since \( P \) is compact, it suffices to show \( H \) is injective. Using Lemma 3.7(b), \( \Gamma \neq \hat{\Gamma} \) implies \( H(\Gamma) \neq H(\hat{\Gamma}) \) except for the possibility of a pair \( \Gamma, \hat{\Gamma} \) as in (3.8) or (3.9). To finish part (a) we assume that there exists such a pair with \( H(\Gamma) = H(\hat{\Gamma}) \) and obtain a contradiction. In the remainder of the proof we will use the notation of Theorem 5.3.
First note that from its definition in (3.8) or (3.9), \( \Gamma \in S_1 \cup S_1^\hat{} \) and so via Remark 3.2 and the definition of \( F_1 \),
\[
F_1(\Gamma) = \ldots S^{-j}(A^w(\epsilon, a_w)) \alpha_j
\]
for some \( j > 0 \). Again via Remark 3.2 for that same \( j \),
\[
F_k(\Gamma) = \ldots S^{-j}(A^w(\epsilon, a_w)) \sigma_j
\]
for some one-sided sequence \( \sigma_j \), where \( k = 1 \) or 2 depending whether \( \Gamma \) is in \( X_1 \) or \( X_2 \). Thus if \( H(\Gamma) = H(\Gamma') \), left shifting by \( j > 0 \) we have \( \alpha_j = \sigma_j \).

Now \( F_1(\Gamma) \) and \( F_k(\Gamma) \) are both contained in \( \Delta^+_\alpha \) which is shift invariant, and thus \( S^j(F_1(\Gamma)) \) and \( S^j(F_k(\Gamma)) \) are also both contained in \( \Delta^+_\alpha \). Thus by Theorem 5.3 there are \( \Gamma, \Gamma' \in P \) with \( S^j(F_1(\Gamma)) \in F(\Gamma') \) and \( S^j(F_k(\Gamma)) \in F(\Gamma') \). Thus \( H(\Gamma') = H(\Gamma') = \alpha_j \). Now it follows from Lemma 3.3(c) that \( H(\Gamma_\alpha) = \alpha_j \), and also note that since \( \Gamma_\alpha = \prod_{i=1}^\infty (\epsilon, 3, 1^n) \), it could not be in a pair of type (3.8) or (3.9). Thus Lemma 3.7(b) implies that \( \Gamma' = \Gamma' = \Gamma_\alpha \) in which case \( \mathcal{W}(\Gamma) = \mathcal{W}(\Gamma) \) and so by Lemma 3.7(a), \( \Gamma = \Gamma \), a contradiction.

Part (b) follows from part (a), since if \( \pi \) is not injective outside of the special case given, then using Theorem 5.3 \( H \) could not be injective as was just proved. Part (c) follows directly from part (b).

\[
\text{\bf Remark 6.2.}
\]
(a) Theorem 6.1 says that a given right infinite sequence \((s)_R\) with \( s \in \Delta^+_\alpha \) has a unique left extension yielding a full sequence in \( \Delta^+_\alpha \) except when the right infinite sequence is in \( o^+(\hat{t}) = o^+(\hat{t}) \). In dynamical language, this says that the semiconjugacy \( \pi : (\Delta^+_\alpha, S) \to (\Delta^+_\alpha, S) \) is injective except on \( o^+(\hat{t}) \) and \( o^+(\hat{t}) \).

(b) Theorem 6.1 also says the noninjectivity of \( G \) is the same as that of \( \pi \), or more precisely, \( G = H^{-1} \circ \pi \).

7. Asymptotic stable projection

7.1. Preliminaries. As noted in the introduction, the first step in Rauzy’s method of geometric representation of a substitution minimal set is to find the stable direction of the Abelianization. For the analog in the S-adic case we need an asymptotic stable direction of the composition of an arbitrary sequence of the Abelianization matrices \( A_n \) defined in (1.3). The standard method for this type of proof uses the Hilbert metric and imitates Garret Birkhoff’s proof of the Perron-Frobenius theorem. We just give the standard definitions and results for the case of interest here, but they of course hold in a more general context.
The open octants in $\mathbb{R}^3$ are cones specified by a triple $(\delta_1, \delta_2, \delta_3)$ with each $\delta$ being $+$ or $-$ and $C_{\delta_1,\delta_2,\delta_3} = \{ \mathbf{v} \in \mathbb{R}^3 : \delta_1 v_1 > 0, \delta_2 v_2 > 0, \delta_3 v_3 > 0 \}$. Thus, for example, the positive cone is $C_{+,+,+}$. The projectivations of these cones is given by their intersection with the unit sphere, $S^3_{\delta_1,\delta_2,\delta_3} = C_{\delta_1,\delta_2,\delta_3} \cap S^3$.

The Hilbert projective pseudometric on $C_{+,+,+}$ is:

$$\rho(\alpha, \beta) = \log \max_{1 \leq i,j \leq k} \frac{\alpha_i \beta_j}{\alpha_j \beta_i}.$$ 

This pseudometric restricts to a metric on $S^3_{+,+,+}$ which is equivalent to the standard metric.

If $A$ is a strictly positive $3 \times 3$ matrix and $f_A : S^3_{+,+,+} \to S^3_{+,+,+}$ is defined as $f_A(\alpha) = \frac{A\alpha}{\|A\alpha\|}$, then Birkhoff’s Contraction Theorem ([9], [17]) says that

$$\rho(f_A(\alpha), f_A(\beta)) \leq \tau(d(A))\rho(\alpha, \beta),$$

where

$$d(A) = \max_{1 \leq i,j,k,l \leq 3} \frac{a_{ik}a_{jl}}{a_{il}a_{jk}} \geq 1.$$ 

and

$$\tau(d) = \frac{\sqrt{d} - 1}{\sqrt{d} + 1}.$$ 

7.2. The case of a single matrix. To clarify the argument for the existence of an asymptotic stable projection of the infinite composition of matrices, we first consider the case of a single matrix $A_n$ for some $n > 0$.

The matrix $A_n$ has three distinct eigenvalues $0 < \lambda_1 < 1 < -\lambda_2 < \lambda_3$ (proof of Lemma 52 in [12]). Let the corresponding left and right eigenvectors be $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since the eigenvalues are all distinct, we may choose the eigenvectors so that $\mathbf{u}^{(i)} \cdot \mathbf{v}^{(j)} = \delta_{ij}$, the Kronecker-delta. In particular, if $V = \text{span}(\mathbf{v}^{(2)}, \mathbf{v}^{(3)})$ (the unstable subspace), then $\mathbf{u}_1 \perp V$.

The projection down the unstable subspace onto the stable subspace is given by $\Phi(\mathbf{w}) = \mathbf{u}_1 \cdot \mathbf{w}$. Thus if $\mathbf{w}$ is written in right eigen-coordinates, $\mathbf{w} = \sum c_i \mathbf{v}_i$, then $c_1 = \Phi(\mathbf{w})$. Note that $\Phi$ is the stable eigen-covector of $A_n$ in that $\Phi(A_n \mathbf{w}) = \lambda_1 \Phi(\mathbf{w})$. Thus to accomplish a stable projection in an asymptotic Rauzy fractal construction we need the analog of the stable eigen-covector.

To find this covector in the infinite product case, we use the analog of Birkhoff’s proof of the Perron-Frobenius theorem which finds the strongest unstable eigenvector. However since we require a stable covector, we seek the unstable left eigenvector of $A_n^{-1}$ or the unstable right eigenvector of $A_n^{-T}$. Conjugating $A_n^{-T}$ by the involution $\tau$ which sends $\mathbf{e}_3 \to -\mathbf{e}_3$ and leaves $\mathbf{e}_1$
and $e_2$ fixed and we obtain a nonnegative matrix

$$B_n := \tau A_n^{-T} \tau = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & n & n+1 \end{pmatrix}.$$ 

A simple computation yields that $(B_n)^3 > 0$, and so using the Birkhoff’s contraction Theorem,

$$\frac{(B_n)^k(w)}{\|(B_n)^k(w)\|} \to \tau(\bar{u}_1)^T$$

for any $\bar{w} \in C_{+,+}$, where $\| \cdot \|$ is the two norm. Translating,

$$\frac{(A_n^{-T})^k(w)}{\|(A_n^{-T})^k(w)\|} \to \bar{\ell}$$

for any $\bar{w} \in C_{+,-}$, where $(\bar{\ell})^T$ is the unit left eigenvector of $A_n$ for eigenvalue $\lambda_1$ with $\bar{\ell} \in C_{+,-}$. For technical reasons that will be clear later, it will be easier to work with the unit left eigenvector of $A_n$ for eigenvalue $\lambda_1$ with $\bar{\ell} \in C_{-,+}$ and we then have

$$\frac{(A_n^{-T})^k(w)}{\|(A_n^{-T})^k(w)\|} \to \bar{\ell}$$

for any $\bar{w} \in C_{-,+}$.

Remark 7.1. A part of the process just described for the Abelianizations has a nonlinear analog. Treating the substitution $\Lambda_n$ as an homomorphism of the free group generated by $\{\delta_1, \delta_2, \delta_3\}$ it maps $\delta_1 \mapsto \delta_2, \delta_2 \mapsto \delta_3 \delta_1^{n+1}$, and $\delta_3 \mapsto \delta_3 \delta_1^n$. This is an automorphism with inverse $\delta_1 \mapsto \delta_3^{-1} \delta_2, \delta_2 \mapsto \delta_1$, and $\delta_3 \mapsto \delta_3 (\delta_2^{-1} \delta_3)^n$. Conjugating by the free group involution $\delta_1 \mapsto \delta_1, \delta_2 \mapsto \delta_2$, and $\delta_3 \mapsto \delta_3^{-1}$ yields $\delta_1 \mapsto \delta_3 \delta_2, \delta_2 \mapsto \delta_1$, and $\delta_3 \mapsto (\delta_3 \delta_2)^n \delta_3$ which may be viewed as a substitution. Because $\Lambda_n$ is inverse-Pisot, this new “inverse” substitution is Pisot. There are a great many results and constructions associated with Pisot substitutions (see [2] for a thorough case study and [3] for results on Pisot S-adic families). A natural question is whether there is any kind of “duality” so that the application of these methods to the infimax “inverse” substitutions can yield insights into the infimax family itself.

7.3. The infinite composition. A family of matrices $\{C_n\}$ is called eventually positive with constant $N$ if every product of $N$ matrices from the family is strictly positive, $C_{n_1}C_{n_2} \ldots C_{n_N} > 0$. Note that there are many similar notions in the literature under a variety of names most incorporating “primitive”, “positive” or “Perron-Fröbenius” in some manner. Given a list of indices $(n_1, n_2, \ldots) = \mathbf{n} \in \Sigma_+^\infty$ and family of $3 \times 3$ matrices $C_n$, the product of the first $k$ matrices is written $C^{(k)} = C_{n_1}C_{n_2} \ldots C_{n_k}$. 
The following is a standard result used in many areas of mathematics. It is an easy consequence of Birkhoff’s contraction theorem and the well-known fact that if \( A \) is a non-negative matrix which has no zero row, its induced map \( f_A : S^3_{+,+,+} \to S^3_{+,+,+} \) does not increase the Hilbert metric, or \( \rho(f_A(z_1), f_A(z_2)) \leq \rho(z_1, z_2) \), for all \( z_1, z_2 \in S^3_{+,+,+} \).

**Lemma 7.2.** Let \( C_n \geq 0 \) be an eventually positive family of non-negative matrices with constant \( N \) and each \( C_n \) has no zero row. Further, assume there is a \( \kappa < 1 \) such that \( \tau(C_{n_1}C_{n_2} \ldots C_{n_N}) < \kappa \) for all \( n \)-tuples \( (n_1, \ldots, n_N) \). Then for all \( n \in \Sigma^+_{\infty} \) there exist a \( z_n \in S^3_{+,+,+} \) such that for all \( \vec{w} \in C_{+,+,+} \),
\[
\lim_{k \to \infty} \frac{C^{(k)}(\vec{w})}{\|C^{(k)}(\vec{w})\|_2} \to z_n.
\]
In addition, if \( \vec{v}^{(k)} \) is the unit length positive right Perron-Frobenius eigenvector of the finite product \( C^{(k)} \) with \( k > N \), then \( \vec{v}^{(k)} \to z_n \).

Using this lemma we get the existence of the asymptotic stable covector.

**Theorem 7.3.** For all sequences \( n \in \Sigma^+_{\infty} \), there exists an \( \vec{\ell}_n \in S^3_{-, -, +} \) such that for all \( \vec{w} \in C_{-, -, +} \),
\[
\lim_{k \to \infty} \frac{(A^{(k)})^{-T}(\vec{w})}{\|(A^{(k)})^{-T}(\vec{w})\|} \to \vec{\ell}_n.
\]
In addition, if \( \Phi^{(k)} \) is the stable eigen-covector of the finite product \( A^{(k)} \), then \( \Phi^{(k)} \to \Phi_n \), where \( \Phi_n \) is defined as \( \Phi_n(\vec{w}) = \vec{\ell}_n \cdot \vec{w} \).

**Proof.** First note that
\[
D := B_cB_bB_a = \begin{pmatrix}
1 + a(b + 1) & 1 + (a + 1)(b + 1) \\
1 + (a + 1)(b + 1) & 1 + (a + 1)(b + 1) \\
(a + 1) & (a + 1) & (a + 1) \\
\end{pmatrix}
\]
and so the family \( \{B_n\} \) is eventually positive and also note that each \( B_n \) has no zero row. Thus to use Lemma 7.2 we must find a uniform \( \kappa < 1 \) with \( \tau(D) < \kappa \) for all \( a, b, c \).

Examining the formula for \( D \), we see that each term in the definition of \( d(D) \),
\[
(7.1) \quad \frac{D_{ik}D_{jl}}{D_{il}D_{jk}},
\]
has the property that the term \( a^{m_1}b^{m_2}c^{m_3} \) with \( m_1 + m_2 + m_3 \) maximal occurs for a unique triple \( (m_1, m_2, m_3) \) in both the numerator and denominator, and these maximal terms are the same in the numerator and denominator. In addition, every other term in the numerator and denominator, \( a^{m'_1}b^{m'_2}c^{m'_3} \), has
the property that $m'_i \leq m_i$ for all $i$. Dividing the numerator and denominator by the maximal terms shows that the quotient in (7.1) is bounded above by the maximum number of terms $a^{m'_1}b^{m'_2}c^{m'_3}$ in any numerator or denominator, which is 9 and so for all triples, $\tau(D) \leq (\sqrt{9} - 1)/(\sqrt{9} + 1) = 1/2$. □

Remark 7.4.

(a) The theorem does not assert the existence of an asymptotic stable eigenvalue or smallest Lyapunov exponent. Its existence can be obtained from the Multiplicative Ergodic Theorem for almost every sequence with respect to a shift invariant measure $\mu$ on $\Sigma_{\infty}^+$ as long as $B_n$ is $\mu$-log integrable.

(b) As mentioned in the introduction, the solution to the infimax problem for all asymptotic symbol frequency vectors requires the inclusion of the $n = 0$ substitution in the family \cite{11}. However when it is included the family is no longer eventually positive and the analysis requires many special cases and results without much significant additional payoff.

The next lemma is essential for a subsequent estimate.

Lemma 7.5. For all sequences $\underline{n} \in \Sigma_{\infty}^+$ and for all $k > 0$, $(A^{(k)})^T(\tilde{\underline{\ell}}_{\underline{n}}) \in C_{-, -, +}$.

Proof. Using Theorem 7.3

\begin{align}
(7.2) \quad (A^{(k)})^T(\tilde{\underline{\ell}}_{\underline{n}}) &= (A^{(k)})^T \left( \lim_{j \to \infty} \frac{(A^{(j)})^{-T}(\tilde{\underline{\ell}}_{\underline{n}})}{\|(A^{(j)})^{-T}(\tilde{\underline{\ell}}_{\underline{n}})\|} \right) \\
&= \lim_{j \to \infty} \frac{\|A_{n_{k+1}}^{-T} \cdots A_{n_j}^{-T}(\tilde{\underline{\ell}}_{\underline{n}})\|}{\|A_{n_{k+1}}^{-T} \cdots A_{n_j}^{-T}(\tilde{\underline{\ell}}_{\underline{n}})\|} \cdot \frac{A_{n_{k+1}}^{-T} \cdots A_{n_j}^{-T}(\tilde{\underline{\ell}}_{\underline{n}})}{\|A^{(j)} - T(\tilde{\underline{\ell}}_{\underline{n}})\|} \cdot \frac{\|A^{(j)} - T(\tilde{\underline{\ell}}_{\underline{n}})\|}{\|A_{n_{k+1}}^{-T} \cdots A_{n_j}^{-T}(\tilde{\underline{\ell}}_{\underline{n}})\|}
\end{align}

Applying Theorem \cite{73} to the sequence $\underline{n}' = n_kn_{k+1} \ldots$ we have that the far right hand term converges to some $\tilde{\underline{\ell}}_{\underline{n}'} \in C_{-, -, +}$. An easy induction shows that the left hand side is a nonzero vector. Taking norms of both sides then yields that the first term on the right hand side converges to a scalar $c > 0$. Thus $(A^{(k)})^T(\tilde{\underline{\ell}}_{\underline{n}}) = c\tilde{\underline{\ell}}_{\underline{n}'} \in C_{-, -, +}$, as required. □

8. FROM PATHS IN THE IPSA TO THE REAL LINE

For an unpointed, length $k$ word $w$, its Abelianization is

\[ P(w) = \sum_{i=0}^{k-1} e_{w_i} = (|w|_1, |w|_2, |w|_3) \in \mathbb{R}^3. \]
By convention \( P(\epsilon) = 0 \). The Abelianization of the substitution \( \Lambda_n \) is the matrix \( A_n \) in (1.3) and is defined by \((A_n)_{i,j} = |\Lambda_n(j)|_i \). The action of \( A_n \) on Abelianized words satisfies \( P \circ \Lambda_n = A_n \circ P \).

We now define a number of objects which depend on a given \( n \in \Sigma^+_\infty \), but we usually suppress the dependence on \( n \) for notational compactness. By Theorem 7.3, each \( n \) yields an asymptotic stable eigen-covector \( \Phi : \mathbb{R}^3 \to \mathbb{R} \) with \( \Phi(x) = \vec{\ell} \cdot x \) and \( \vec{\ell} \) as in the theorem. Let \( \phi = \Phi \circ P \) and so \( \phi(w_1w_2) = \phi(w_1) + \phi(w_2) \) and

\[
\phi(\Lambda_n(w)) = \vec{\ell} \cdot (A_n P(w)) = (A_n^T \vec{\ell}) \cdot P(w).
\]

Letting \( \vec{\ell}^{(k)} = (A^{(k)})^T \vec{\ell} \), we have for \( a = 1, 2, 3 \),

\[
\phi(\Lambda^{(k)}(a)) = ((A^{(k)})^T \vec{\ell}) \cdot e_a^{(k)},
\]

the \( a^{th} \) component of \( \vec{\ell}^{(k)} \). Note that according to Lemma 7.5 for all \( k \), \( \ell_1^{(k)}, \ell_2^{(k)} < 0 \) and \( \ell_3^{(k)} > 0 \).

Motivated by the way a path generates a sequence in the Dumont-Thomas expansion in the definition of \( W \) in (3.2), define \( \Psi_R, \Psi_L : \mathcal{P} \to \mathbb{R} \) by

\begin{align*}
\Psi_L(\Gamma) &= \phi(u_0) + \sum_{j=1}^{\infty} \phi(\Lambda^{(j)}(u_j)) \\
\Psi_R(\Gamma) &= \phi(a_0v_0) + \sum_{j=1}^{\infty} \phi(\Lambda^{(j)}(v_j))
\end{align*}

Remark 8.1. The infinite sum \( \sum \phi(s_i) \) obviously doesn’t converge. One of the many advantages of the Dumont-Thomas expansion is that it naturally yields a grouping of terms so that the projection onto the stable subspace is expressed as a convergent sum. A different but equivalent method was used by Holton and Zamboni in [27] and [26] for the single substitution case. If \( \mathcal{w} \) is the fixed point of substitution, for any sequence \( \mathcal{v} \in \text{Cl}(\mathcal{o}(\mathcal{w}, S)) \) by definition there are \( n_i \to \infty \) with \( S^{n_i}(\mathcal{w}) \to \mathcal{v} \). They show that with \( \phi \) the composition of Abelianization and projection as above, the sequence \( \phi(w_0w_1\ldots w_{n-1}) \) converges to a \( f(\mathcal{v}) \) yielding a uniformly continuous representation \( f : \text{Cl}(\mathcal{o}(\mathcal{w}, S)) \to \mathbb{C} \).

Theorem 8.2. For each \( n \in \Sigma^+_\infty \), \( \Psi_L \) and \( \Psi_R \) define continuous functions on \( \mathcal{P} \) with \( \Psi \geq 0 \) and \( \Psi_R = -\Psi_L \).
Proof. By definition $\vec{\ell}^{(k)} = A^T_{nk} \vec{\ell}^{(k-1)}$ and so using the formula for $A^T_{nk}$ and Lemma 7.5 we get

\begin{equation}
\ell_1^{(k)} = \ell_2^{(k-1)} < 0 \\
\ell_2^{(k)} = (n_k + 1)\ell_1^{(k-1)} + \ell_3^{(k-1)} < 0 \\
\ell_3^{(k)} = n_k \ell_1^{(k-1)} + \ell_3^{(k-1)} > 0.
\end{equation}

Thus

\begin{equation}
-n_k \ell_1^{(k-1)} < \ell_3^{(k-1)} < -(n_k + 1)\ell_1^{(k-1)},
\end{equation}

and so

\begin{equation}
|\ell_2^{(k-2)}| = |\ell_1^{(k-1)}| < \frac{\ell_3^{(k-1)}}{n_k} \leq \ell_3^{(k-1)}.
\end{equation}

Using the right hand inequality of (8.3),

\begin{equation}
0 < \frac{\ell_3^{(k)}}{\ell_3^{(k-1)}} = \frac{n_k \ell_1^{(k-1)} + \ell_3^{(k-1)}}{\ell_3^{(k-1)}} = 1 + \frac{n_k \ell_1^{(k-1)}}{\ell_3^{(k-1)}} < 1 + \frac{n_k \ell_1^{(k-1)}}{-(n_k + 1)\ell_1^{(k-1)}}
\end{equation}

\begin{equation}
\leq 1 - \frac{1}{2} = \frac{1}{2},
\end{equation}

and so when $k > 0$,

\begin{equation}
\ell_3^{(k)} < \frac{1}{2^{k-2}} \ell_3^{(0)}.
\end{equation}

Next we examine the terms making up the sum defining $\Psi_L$. Using the IPSA, we have that the possibilities for $u_k$ are

$$u_k = \epsilon, 3, 31, \ldots, 31^{n_k+1}.$$

Thus when $\phi(\Lambda^{(k)}(u_k)) \neq 0$, $\phi(\Lambda^{(k)}(u_k)) = \phi(\Lambda^{(k)}(31^m)) = \ell_3^{(k)} + m\ell_1^{(k)}$ with $0 \leq m \leq n_k+1$. Note that when $m = n_k+1$, $\ell_3^{(k)} + n_k+1\ell_1^{(k)} = \ell_3^{(k+1)}$, using (8.2). Thus since $\ell_1^{(k)} < 0$, when $\phi(\Lambda^{(k)}(u_k)) \neq 0$ and $0 < m \leq n_k+1$, we have

\begin{equation}
0 < \ell_3^{(k+1)} \leq \phi(\Lambda^{(k)}(u_k)) < \ell_3^{(k)},
\end{equation}

and when $m = 0$,

\begin{equation}
0 < \ell_3^{(k+1)} < \phi(\Lambda^{(k)}(u_k)) \leq \ell_3^{(k)}.
\end{equation}

The convergence, positivity and continuity of $\Psi_L$ then follow from (8.6).

The proof for $\Psi_R$ is similar. The possibilities for $v_k$ are

$$v_k = \epsilon, 1, 1^2, \ldots, 1^{n_k+1}.$$
Thus when \( \phi(\Lambda^k(v_k)) \neq 0 \), \( \phi(\Lambda^k(v_k)) = \phi(\Lambda^k(1^m)) = m\ell_1^{(k)} \) and note that when \( m = n_k + 1 \), then \( (n_k + 1)\ell_1^{(k)} = \ell_2^{(k+1)} - \ell_3^{(k+1)} \), using (8.2). Thus either \( v_k = \epsilon \) and then \( \phi(\Lambda^k(v_k)) = 0 \), or

\[
0 < |\ell_1^{(k)}| \leq |\phi(\Lambda^k(v_k))| \leq |\ell_2^{(k+1)}| + |\ell_3^{(k+1)}|,
\]

and convergence and continuity of \( \Psi_R \) again follow from (8.6) after using (8.4).

To show that \( \Psi_R = -\Psi_L \) note that Lemma 3.1(b) gives that for all \( k > 0 \),

\[
\Psi_L(\Gamma) + \Psi_R(\Gamma) = \phi(u_0a_0v_0) + \sum_{j=1}^{k} (\phi(\Lambda^j(u_j)) + \phi(\Lambda^j(v_j)))
\]

\[
+ \sum_{j=k+1}^{\infty} (\phi(\Lambda^j(u_j)) + \phi(\Lambda^j(v_j)))
\]

\[
= \phi(\Lambda^k(a_k)) + \sum_{j=k+1}^{\infty} (\phi(\Lambda^j(u_j)) + \phi(\Lambda^j(v_j))).
\]

But \( \phi(\Lambda^k(a_k)) = \ell_1^{(k)} \rightarrow 0 \) by (8.4) and (8.6) and since the sums defining \( \Psi_L \) and \( \Psi_R \) converge we have \( \Psi_L + \Psi_R = 0 \). \( \square \)

9. The Rauzy fractal

We first construct the Rauzy fractal corresponding to a list \( \underline{n} \in \Sigma^+_\infty \) in the real line as the image under \( \Psi_L \) of the path space \( \mathcal{P}_{\underline{n}} \). In the next section we compose \( \Psi_L \) with the map \( G \) from Section 5 to get the full representation of \( \Delta_{\underline{n}} \) into \( \mathbb{R} \).

The Rauzy fractal is \( \mathcal{R} = \Psi_L(\mathcal{P}) \). It has three natural subpieces \( \mathcal{R}_1, \mathcal{R}_2, \) and \( \mathcal{R}_3 \) defined by \( \mathcal{R}_a = \Psi_L(\mathcal{P}[a]) \) where \( \mathcal{P}[a] = \{ \Gamma \in \mathcal{P} : a_0 = a \} \), all the paths that begin at state \( a \). The conjugacy in Theorem 11.2 below implies that \( \mathcal{R} \) is a Cantor set.

As noted in the introduction, a central issue in the process of a geometric representation is the disjointness of the subpieces of the Rauzy fractal. The main result of this section is the convex hulls of the subpieces \( \mathcal{R}_a \) in \( \mathbb{R} \) are disjoint except perhaps at their endpoints. The argument proceeds by specifying the endpoints of the intervals using particular paths in \( \mathcal{P}_{\underline{n}} \). This allows us to specify values of the endpoints of the intervals in terms of the coordinates of the asymptotic stable covector \( \vec{\ell}_{\underline{n}} \) given by Theorem 7.3. Note that some of these paths have already been defined but are given new names.
here for clarity of exposition.

\[ \Gamma_{\text{min}} = \Gamma_\alpha = \prod_{i=1}^{\infty} (\epsilon, 3, 1^{n_i}), \]

\[ \Gamma_{3\text{max}} = (\epsilon, 3, 1^{n_{1+1}}) \prod_{i=1}^{\infty} ((\epsilon, 2, \epsilon), (3, 1, 1^{n_{2+i}})) \]

\[ \Gamma_{2\text{min}} = \Gamma_\beta = \prod_{i=1}^{\infty} ((\epsilon, 2, \epsilon), (3^{1^{n_{2i}}}, 1, \epsilon)) \]

\[ \Gamma_{2\text{max}} = \prod_{i=1}^{\infty} ((\epsilon, 2, \epsilon), (3, 1, 1^{n_{2i}})) \]

\[ \Gamma_{1\text{min}} = \Gamma_\beta = (3^{1^{n_{1}}}, 1, \epsilon) \prod_{i=1}^{\infty} ((\epsilon, 2, \epsilon), (3^{1^{n_{2i+1}}}, 1, \epsilon)) \]

\[ \Gamma_{\text{max}} = \prod_{i=1}^{\infty} ((3, 1, 1^{n_{2i-1}}), (\epsilon, 2, \epsilon)) \]

**Theorem 9.1.** Each subpiece \( R_a \) of the Rauzy fractal \( R \) is contained in an interval \( J_a \) and these intervals intersect in at most one point. Specifically, with the paths defined above,

\[
0 = \Psi_L(\Gamma_{\text{min}}) \leq R_3 \leq \Psi_L(\Gamma_{3\text{max}}) = -\ell_2 = \Psi_L(\Gamma_{2\text{min}}) \leq R_2
\]

\[
\leq \Psi_L(\Gamma_{2\text{max}}) = -\ell_1 = \Psi_L(\Gamma_{1\text{min}}) \leq R_1 \leq \Psi_L(\Gamma_{\text{max}}) = \ell_3 - \ell_2.
\]

Further, the paths that achieve the extremes are only those given above, or \( \Psi_L^{-1}(0) = \Gamma_{\text{min}}, \Psi_L^{-1}(R_3 \cap R_2) = \{\Gamma_{3\text{max}}, \Gamma_{2\text{min}}\}, \Psi_L^{-1}(R_2 \cap R_1) = \{\Gamma_{2\text{max}}, \Gamma_{1\text{min}}\}, \) and \( \Psi_L^{-1}(\ell_3 - \ell_2) = \Gamma_{\text{max}}. \)

**Proof.** For any prefix \( u_k \) of an arc label, \( \phi(\Lambda^{(k)}(u_k)) \geq 0 \) by (8.7) and (8.8). Since \( \Gamma_{\text{min}} \) is the unique path in \( \mathcal{P} \) with all prefixes equal to \( \epsilon \), it uniquely achieves \( 0 = \text{min}(\mathcal{R}) \). As for \( \text{max}(\mathcal{R}) \), note that any arc with a prefix label not equal to \( \epsilon \) must emerge from the state 1. Of these, the maximal contribution to \( \Psi_L \) comes from \( \phi(\Lambda^{(k)}(3)) \) and since by (8.7) and (8.8) again, \( \phi(\Lambda^{(k+1)}(3)) < \phi(\Lambda^{(k)}(3)) \), whatever path achieves \( \text{max}(\mathcal{R}) \) must begin at state 1. In addition, since a non-\( \epsilon \) prefix occurs at most on every other level, \( \text{max}(\mathcal{R}) \) is achieved uniquely by \( \Lambda_{\text{max}} \).

Using the logic of the previous paragraph \( \text{max}(\mathcal{R}_2) \) will be achieved by the unique path originating at state 2 that goes to state 1 in the least number of steps, namely, \( \Lambda_{2\text{max}} \). Similarly, \( \text{max}(\mathcal{R}_3) \) is uniquely achieved by \( \Lambda_{3\text{max}} \).

To see what achieves \( \text{min}(\mathcal{R}_1) \) and \( \text{min}(\mathcal{R}_2) \) we use suffix labels and the fact that \( \Psi_R = -\Psi_L \) from Theorem 8.2. Now all suffixes are \( \epsilon \) or \( 1^n \) and any path \( \Gamma \) emerging from state 1 by definition has \( a_0 = 1 \) and so
Ψ_R(Γ) = ℓ_1 + ... Since ℓ_1 < 0 by Theorem 7.3, min(ROI) would be uniquely achieved by a path with only ε as a suffix, and this path is Γ_{1min}. Similarly, min(ROI_2) is uniquely achieved by Γ_{2min}.

We now compute the values of the extrema. Since Γ_{1min} and Γ_{2min} have all suffixes equal to ε, it follows immediately that Ψ_R(Γ_{2min}) = φ(2) = ℓ_2 = -Ψ_L(Γ_{2min}). Similarly, Ψ_L(Γ_{1min}) = -ℓ_1.

On the other hand, computing Φ_L(Γ_{1min}) from the definition yields

$$Φ_L(Γ_{1min}) = ∑_i=0^∞ φ(λ_1^{(2i)}(31^{n_{2i+1}})) = ∑_i=0^∞ ℓ_3^{(2i)} + n_{2i+1} ℓ_1^{(2i)} = ∑_i=0^∞ ℓ_3^{(2i+1)}$$

using (8.2) in the last equality. Thus -ℓ_1 = ∑_i=0^∞ ℓ_3^{(2i+1)} Similarly, -ℓ_2 = Ψ_L(Γ_{2min}) = ∑_i=1^∞ ℓ_3^{(2i)}.

Now observe that also from the definition, Φ_L(Γ_{3max}) = ∑_i=1^∞ φ(λ_1^{(2i)}(3)), so Φ_L(Γ_{3max}) = Ψ_L(Γ_{2min}) = -ℓ_2. Similarly, Φ_L(Γ_{2max}) = Ψ_L(Γ_{1min}) = -ℓ_1. Finally, Φ_L(Γ_{max}) = ∑_i=0^∞ φ(λ_1^{(2i)}(3)) = ∑_i=0^∞ ℓ_3^{(2i)} = ℓ_3 - ℓ_2.

Remark 9.2. For a single substitution Λ_n, or equivalently when n = nnn ..., the vector ℓ is the left eigenvector corresponding to the stable eigenvalue λ_1 of A_n. Using the eigenvector equation and the characteristic polynomial we have that

$$ℓ_1 = -\frac{λ_1 ℓ_3}{1 - λ_1^3} \quad \text{and} \quad ℓ_2 = -\frac{λ_2^2 ℓ_3}{1 - λ_1^3}$$

Since (∆^T) i ℓ = λ_i ℓ, the summation formulas from the previous proof -ℓ_1 = ∑_i=0^∞ ℓ_3^{(2i+1)} and -ℓ_2 = ∑_i=1^∞ ℓ_3^{(2i)} are geometric series for (9.1).

10. FROM SEQUENCES TO THE REAL LINE

We now construct the desired real valued-map on the symbolic system Δ_n by composing the maps constructed in Sections 5 and 8.

Definition 10.1. Let Υ = Ψ_L ∘ G : Δ_n → [0, ∞) and Υ⁺ = Ψ_L ∘ H⁻¹ : Δ_n⁺ → [0, ∞).

Remark 10.2.

(a) By Remark 6.2, G = H⁻¹ ∘ π and so Υ = Ψ_L ∘ H⁻¹ ∘ π = Υ⁺ ∘ π.
(b) From the definition, F_1(Γ_β) = S⁻¹(Γ) and so using Theorem 9.1, Υ(S⁻¹(Γ)) = Ψ_L(Γ_β) = -ℓ_1. Similarly, Υ(S⁻¹(Γ_β)) = -ℓ_2. Also, F_2(Γ_α) = 1 and F_3(Γ_α) = 1, so again using Theorem 9.1, Υ(Γ_α) = Υ(Γ_α) = 0.

The map Ψ_L and thus the map Υ is defined via a summation utilizing a very specific grouping into words of a sequence s ∈ Δ_n. The shifted sequence
$S(\mathbf{s})$ could depend on a quite different grouping. The next lemma confirms that the effect of the shift is as expected.

**Lemma 10.3.** For all $\mathbf{s} \in \Delta_{\mathbf{2}}$, $\Upsilon(S(\mathbf{s})) = \Upsilon(\mathbf{s}) + \phi(s_0)$ and for all $\mathbf{s} \in \Delta_{\mathbf{2}}^+$, $\Upsilon^+(S(\mathbf{s})) = \Upsilon^+(\mathbf{s}) + \phi(s_0)$.

**Proof.** We first prove the result for $\Upsilon$. Examining the IPSA, there are no paths with $\mathcal{W}(\Gamma)_R = \epsilon$ and exactly two paths with $\mathcal{W}(\Gamma)_R = a\epsilon$ with $a \neq \epsilon$, namely, $\Gamma_\beta$ and $\Gamma_\beta^\prime$, with $\mathcal{W}(\Gamma_\beta)_R = 1\epsilon$ and $\mathcal{W}(\Gamma_\beta^\prime)_R = 2\epsilon$. We consider $\Gamma_\beta$, the other case is similar. Letting $\mathbf{s} = S^{-1}(t)$, by Remark 10.2(b), $\Upsilon(\mathbf{s}) = -\ell_1 = -\phi(s_0)$. On the other hand, using Remark 10.2(b) again since $S(\mathbf{s}) = t$ and $G(t) = \Gamma_\alpha$, we have $\Upsilon(S(\mathbf{s})) = 0 = \Upsilon(\mathbf{s}) + \phi(s_0)$, proving the result in this case.

Now assume $\mathbf{s} \neq S^{-1}(t), S^{-1}(\hat{t})$ and so $\Gamma = G(\mathbf{s})$ satisfies $\mathcal{W}(\Gamma)_i \neq \epsilon$ for $i = 0, 1$. Using Lemma 3.1(b), this implies that by picking $k$ large enough we can ensure that

$$\mathcal{W}(\Gamma) = \ldots \Lambda^{(k+1)}(u_{k+1}) \Lambda^{(k)}(u_k) S^j(\Lambda^{(k)}(\epsilon.a_k)) \Lambda^{(k)}(v_k) \Lambda^{(k+1)}(v_{k+1}) \ldots$$

with $j < |\Lambda^{(k)}(a_k)| - 1$. Now by Lemma 3.1(a), there exists a length $(k+1)$-path $\gamma$ with $\mathcal{W}(\gamma) = S^{j+1}(\Lambda^{(k)}(a_k))$. So letting $\Gamma' = \gamma, \Gamma_{k+1}, \Gamma_{k+2}, \ldots$ we have

$$\mathcal{W}(\Gamma') = \ldots \Lambda^{(k+1)}(u_{k+1}) \Lambda^{(k)}(u_k) S^{j+1}(\Lambda^{(k)}(\epsilon.a_k)) \Lambda^{(k)}(v_k) \Lambda^{(k+1)}(v_{k+1}) \ldots$$

Thus $\mathcal{W}(\Gamma') = S(\mathcal{W}(\Gamma))$ and $\Psi_L(\Gamma') = \Psi_L(\Gamma) + \phi(s_0)$. But using the definition of $G$, $G(S(\mathbf{s})) = \Gamma'$ and so $\Upsilon(S(\mathbf{s})) = \Upsilon(\mathbf{s}) + \phi(s_0)$, as required.

Since by Remark 10.2(a), $\Upsilon = \Upsilon^+ \circ \pi$ and $\Delta_{\mathbf{2}}^+ = \pi(\Delta_{\mathbf{2}})$, the result for $\Upsilon^+$ also follows. \hfill $\Box$

**Definition 10.4.**

(a) Let $\mathbf{w} = \mathcal{W}(\Gamma_{2\max})$ and $\hat{\mathbf{w}} = \mathcal{W}(\Gamma_{3\max})$.

(b) Endow $\Delta_{\mathbf{2}}^+$ with the lexicographic order. Recall that a map $h : X \to Y$ between two linearly ordered spaces is *weakly order reversing* if $x < y$ implies $h(x) \geq h(y)$ and *strictly order reversing* if $x < y$ implies $h(x) > h(y)$.

There are two sources of non-injectivity for $\Upsilon = \Psi_L \circ G$. The first is a consequence of the noninjectivity of $\Psi_L$ at the overlaps of the $\mathcal{R}_\alpha$ given in Theorem 9.1 (cases (2) and (3) in Theorem 10.5 below). The second is a consequence of the noninjectivity of $G$. This noninjectivity is the same as that of $\pi$ given in Theorem 6.1(b) (case (1) in Theorem 10.5 below) because by Remark 6.2(b), $G = H^{-1} \circ \pi$ and $H$ is injective.
Theorem 10.5.

(a) The map \( \Upsilon : \Delta_\mathbb{R} \rightarrow \mathbb{R} \) is continuous and injective except for the following cases:

1. \( \Upsilon(S^j(t)) = \Upsilon(S^j(\hat{t})) \) for \( j \geq 0 \).
2. \( \Upsilon(S^j(t)) = \Upsilon(S^{j+1}(w)) \) for \( j < 0 \).
3. \( \Upsilon(S^j(\hat{t})) = \Upsilon(S^{j+1}(\hat{w})) \) for \( j < 0 \).

(b) The map \( \Upsilon^+ : \Delta_\mathbb{R}^+ \rightarrow \mathbb{R} \) is continuous and weakly order reversing. It is strictly order reversing except for cases (2) and (3) above, or more precisely, \( \Upsilon^+(\pi \circ S^j(t)) = \Upsilon^+(\pi \circ S^{j+1}(w)) \) for \( j < 0 \) and \( \Upsilon^+(\pi \circ S^j(\hat{t})) = \Upsilon^+(\pi \circ S^{j+1}(\hat{w})) \) for \( j < 0 \).

Proof. The continuity of \( \Psi_L \) is proved in Theorem 8.2 that of \( G \) in Theorem 5.3 and that of \( H \) in Theorem 6.1 thus the continuity of \( \Upsilon \) and \( \Upsilon^+ \) follows.

We prove the rest of (b) first. Since \( \Delta_\mathbb{R}^+ = \pi(\Delta_\mathbb{R}) \), each sequence in \( \Delta_\mathbb{R}^+ \) can be expressed as \( \pi(s) \) for some \( s \in \Delta_\mathbb{R} \). So assume that \( \pi(s) \neq \pi(s') \) with \( s, s' \in \Delta_\mathbb{R} \) and let \( k \geq 0 \) be the smallest index with \( s_k \neq s'_k \) and assume \( s_k > s'_k \). Thus \( \Upsilon(S^k(s)) \in \mathcal{R}_{s_k} \) and \( \Upsilon(S^k(s')) \in \mathcal{R}_{s'_k} \). Therefore by Lemma 9.1, \( \Upsilon(S^k(s)) \leq \Upsilon(S^k(s')) \) with equality only when \( S^k(s) = w \) and \( S^k(s') = t \) or \( S^k(s) = \hat{w} \) and \( S^k(s') = \hat{t} \). Now if \( \Upsilon(S^k(s)) < \Upsilon(S^k(s')) \), using Lemma 10.3 this implies that \( \Upsilon(s) + \phi(s_0s_1 \ldots s_{k-1}) < \Upsilon(s') + \phi(s'_0s'_1 \ldots s'_{k-1}) \). But by assumption \( s_i = s'_i \) for \( i = 0, \ldots, k-1 \), and so \( \Upsilon(s) < \Upsilon(s') \) as required.

Remark 10.2(a) says that \( \Upsilon = \Upsilon^+ \circ \pi \) and so Theorem 6.1(b) implies that the only noninjectivity that \( \Upsilon \) possesses in addition to that of \( \Upsilon^+ \) comes from case (1), finishing the proof of (a).

\[ \square \]

Remark 10.6. Using Remark 10.2(b) and Theorem 10.5 we have \( \Upsilon^{-1}(0) = \{t, \hat{t}\} \), \( \Upsilon^{-1}(-\ell_1) = \{S^{-1}(t), w\} \), and \( \Upsilon^{-1}(-\ell_2) = \{S^{-1}(\hat{t}), \hat{w}\} \).

Theorem 10.5 gives us detailed information on the geometric representation maps \( \Upsilon \) and \( \Upsilon^+ \) defined from the infimax minimal sets into the reals. The next step is to connect these maps to the dynamics on their domain and range.

Lemma 10.3 tells us the required dynamics on the image of \( \Upsilon \) in order to mirror the shift on \( \Delta_\mathbb{R} \). Specifically, if \( [a] \subset \Delta_\mathbb{R} \) is the cylinder set \( \{s \in \Delta_\mathbb{R} : s_0 = a\} \), then \( \mathcal{P}[a] = G([a]) \) with \( G \) defined in Theorem 5.3 and so each component of the Rauzy fractal \( \mathcal{R}_a \) is \( \mathcal{R}_a = \Upsilon([a]) \) for \( a = 1, 2, 3 \). Thus by Lemma 10.3 the shift map on \( \Delta_\mathbb{R} \) corresponds to translation by \( \phi(a) = \ell_a \) on \( \mathcal{R}_a \).
However, recall that the $\mathcal{R}_a$ are not disjoint, rather by Lemma 9.1, $\mathcal{R}_3 \cap \mathcal{R}_2 = \{-\ell_2\}$ and $\mathcal{R}_2 \cap \mathcal{R}_1 = \{-\ell_1\}$. Thus to define the translation we remove the overlap points and let $\mathcal{R}' = \mathcal{R} \setminus \{-\ell_2, \ell_1\}$ and $T : \mathcal{R}' \rightarrow \mathcal{R}$ is defined via

$$T(x) = x + \ell_a \text{ when } x \in \mathcal{R}_a.$$ 

We also remove the corresponding points from $\Delta_n$ and let $\Delta' = \Delta_n \setminus (\Upsilon^{-1}(-\ell_2) \cup \Upsilon^{-1}(-\ell_1))$ and so $\mathcal{R}' = \Upsilon(\Delta')$. Thus $\Upsilon \circ S = T \circ \Upsilon$ restricted to $\Delta'$.

To get a full dynamical conjugacy we have to be able to iterate this relation which requires excluding the full orbits of $\Upsilon^{-1}(-\ell_2)$ and $\Upsilon^{-1}(-\ell_1)$). Thus we let $\Delta'' = \Delta_n \setminus (\sigma(t, S) \cup \sigma(\hat{t}, S) \cup \sigma(w, S) \cup \sigma(\hat{w}, S))$ and $\mathcal{R}'' = \Upsilon(\Delta'')$. Now note that $S(\Delta'') = \Delta''$ and $T(\mathcal{R}'') = \mathcal{R}''$ and so we have a full conjugacy between $\Delta''$ under the shift $S$ to $\mathcal{R}''$ under the translations $T$. Finally, using Theorem 10.5(b) we can see that $\Upsilon^+$ now gives a strictly order reversing conjugacy from $\pi(\Delta'')$ to $\mathcal{R}''$. Thus we have proved:

**Corollary 10.7.**

(a) Restricted to $\Delta'$ we have $\Upsilon \circ S = T \circ \Upsilon$.
(b) $\Upsilon$ restricted to $\Delta''$ gives a topological conjugacy from $(\Delta'', S)$ to $(\mathcal{R}'', T)$.
(c) $\Upsilon^+$ restricted to $\pi(\Delta'')$ gives a strictly order reversing topological conjugacy from $(\pi(\Delta''), S)$ to $(\mathcal{R}'', T)$.

The last stage of the geometric representation process in the next section embeds $\mathcal{R}''$ densely in the attractor of a specific class of interval maps.

**11. The geometric representation as the attractor of an ITM**

We now define the two-parameter family of ITM which can occur in the geometric representations of the infimax S-adic family. Define $T_{\mu_1, \mu_2} : I \rightarrow I$ with $I = [0, 1]$ and $0 < \mu_1 < \mu_2 < 1$ by

$$T_{\mu_1, \mu_2}(x) = x + 1 - \mu_1 \text{ for } x \in [0, \mu_1)$$
$$T_{\mu_1, \mu_2}(x) = x - \mu_1 \text{ for } x \in [\mu_1, \mu_2)$$
$$T_{\mu_1, \mu_2}(x) = x - \mu_2 \text{ for } x \in [\mu_2, 1].$$

See Figure 2. This family is topologically conjugate to the one studied by Bruin and Troubetzkoy in [14]. Specifically, using the conjugacy $x \mapsto 1 - x$ their map $T_{\alpha, \beta}$ is conjugate to $T_{\mu_1, \mu_2}$ with $\beta = \mu_1$ and $\alpha = \mu_2$. The dynamical object of interest is the attractor

$$\Omega_{\mu_1, \mu_2} = \bigcap_{i=0}^{\infty} T_{\mu_1, \mu_2}^i(I).$$
The maps $T_{\mu_1, \mu_2}$ are not continuous, but there is a standard method to construct a continuous extension going back to Keane’s work on IET (28). The extension is simplest to describe if we restrict to the case of interest here. So we assume that $T_{\mu_1, \mu_2}$ is such that for all $n > 0$, $T_{\mu_1, \mu_2}^n(\mu_1), T_{\mu_1, \mu_2}^n(\mu_2)$, and $T_{\mu_1, \mu_2}^n(0)$ are not equal to either $\mu_1$ nor $\mu_2$ (the importance of zero here is that $0 = \lim_{x \to \mu_1^+} T_{\mu_1, \mu_2}(x) = \lim_{x \to \mu_2^+} T_{\mu_1, \mu_2}(x)$).

Informally the extension is accomplished by splitting the backward orbits of $\mu_1$ and $\mu_2$ into a pair of orbits. More formally, the extension is built by extending the linear order on $I$ to a disjoint union of $I$ and a second copy of the backward orbits of $\mu_1$ and $\mu_2$. Let

$$\tilde{I} = I \sqcup \bigcup_{j=1,2} \bigcup_{i=0}^{\infty} T_{\mu_1, \mu_2}^{-i}(\mu_j).$$

Thus $\tilde{I}$ contains two copies of any $y \in \bigcup_{j=1,2} \bigcup_{i=0}^{\infty} T_{\mu_1, \mu_2}^{-i}(\mu_j)$. Denote the version in $I$ itself as $y$ and its copy as $\tilde{y}$. Extend the usual linear order on $I$ to one on $\tilde{I}$ by declaring that $y < \tilde{y}$.

Endowing $\tilde{I}$ with the order topology makes it a compact metric space. The map $T_{\mu_1, \mu_2}$ naturally extends to a continuous $\tilde{T}_{\mu_1, \mu_2} : \tilde{I} \to \tilde{I}$ with $\tilde{T}_{\mu_1, \mu_2}(\mu_1) = \tilde{T}_{\mu_1, \mu_2}(\mu_2) = 0$ and if $y \in I$ has $T_{\mu_1, \mu_2}^n(y) = \mu_i$ with $n > 0$, then
Define the extended attractor
\[ \hat{\Omega}_{\mu_1, \mu_2} = \bigcap_{i=0}^{\infty} \hat{T}_{\mu_1, \mu_2}^i(\hat{I}). \]

Given a \( n \in \Sigma_{x,0}^+ \), define \( T_n = T_{\mu_1, \mu_2} \) with \( \mu_1 = -\ell_2/(\ell_3 - \ell_2) \) and \( \mu_2 = -\ell_1/(\ell_3 - \ell_2) \) and let \( \hat{T}_n = \hat{T}_{\mu_1, \mu_2} \) with \( \hat{\ell} = (\ell_1, \ell_2, \ell_3) \) as in Theorem 7.3.

**Remark 11.1.** When \( n = 0^\infty \), Remark 9.2 yields that the parameters of the corresponding ITM are \( \mu_1 = \lambda_i^2 \) and \( \mu_2 = \lambda_i \) with \( \lambda_i \) the stable eigenvalue of the Abelianization matrix \( A_n \).

We now rescale the geometric representation maps \( \Upsilon \) and \( \Upsilon^+ \) so that their image is \( I \) and then extend them so their range is in \( \hat{I} \). The map \( \Upsilon_1^+ : \Delta \to I \) is defined as \( \Upsilon_1^+ = \Upsilon^+/(\ell_3 - \ell_2) \), while the extended range map \( \hat{\Upsilon}^+ : \Delta \to \hat{I} \) is defined by \( \hat{\Upsilon}^+ = \Upsilon_1^+ \) except for the preimages of the new points \( \hat{y} \in \hat{I} \). To deal with this case, note that using Theorem 10.5 for \( j > 0 \), \( T_n^{j-1} \Upsilon^+ \pi S^{-j}(\hat{\ell}) = T_n^{j-1} \Upsilon^+ \pi S^{-j+1}(\hat{w}) = \mu_1 \). So when \( y = \Upsilon^+ \pi S^{-j}(\hat{\ell}) \) let \( \hat{\Upsilon}^+ \pi S^{-j}(\hat{\ell}) = y \) and \( \hat{\Upsilon}^+ \pi S^{-j+1}(\hat{w}) = \hat{y} \). Similarly, when \( y = \Upsilon^+ \pi S^{-j}(\hat{\ell}) \) let \( \hat{\Upsilon}^+ \pi S^{-j}(\hat{\ell}) = y \) and \( \hat{\Upsilon}^+ \pi S^{-j+1}(\hat{w}) = \hat{y} \). Finally, let \( \hat{\Upsilon} = \hat{\Upsilon}^+ \circ \pi \).

The extension of \( \Upsilon \) and its range not only allows us to extend \( T \) to the continuous \( \hat{T} \), it also eliminates the noninjectivity of \( \Upsilon \) given in (2) and (3) in Theorem 10.5(a). Since the noninjectivity in (1) is eliminated by the passage to \( \hat{\Upsilon}^+ \) defined on \( \Delta_+^n \) we have that \( \hat{\Upsilon}^+ \) is a homeomorphism onto its image which we show to be \( \Omega_n \). Using the commutivity from Corollary 10.7 we see that \( \hat{\Upsilon}^+ \) is a topological conjugacy.

There are three intervals in the order topology on \( \hat{I} \) that will be used to code the dynamics on the extended attractor. These are
\[ \hat{I}_3 = [0, \mu_1], \hat{I}_2 = [\hat{\mu}_1, \mu_2], \text{ and } \hat{I}_3 = [\hat{\mu}_2, 1]. \]

For a point \( x \in \hat{\Omega}_n \) its itinerary under \( \hat{T} \) with respect to the partition \( \hat{I}_1, \hat{I}_2 \) and \( \hat{I}_3 \) is \( s_0s_1 \ldots \) with \( s_i = a \) if and only if \( \hat{T}^i(x) \in \hat{I}_a \).

The next theorem formalizes the statement in Theorem 1.1 of the introduction: the map \( \hat{\Upsilon}^+ \) provides a conjugacy of the infimax minimal set \( \Delta_+^n \) to the attractor of an extended ITM and further, the partition just given codes the dynamics exactly as in the symbolic minimal set.

**Theorem 11.2.**

(a) \( \hat{\Upsilon}_2^n : (\Delta_+^n, S) \to (\hat{\Omega}_n, \hat{T}_n) \) is a strictly order reversing topological conjugacy and for each \( s \in \Delta_+^n \) the itinerary under \( \hat{T}_n \) of \( \hat{\Upsilon}_2^n(s) \) with respect to the partition \( \hat{I}_1, \hat{I}_2 \) and \( \hat{I}_3 \) is \( s \).
(b) \( \tilde{\Upsilon}_n : (\Delta_n, S) \rightarrow (\tilde{\Omega}_n, \tilde{T}_n) \) is a topological semiconjugacy with its only noninjectivity being that \( \tilde{\Upsilon}_n^{-1}(\tilde{T}_n^j(0)) = \{S^j(\ell), S^j(\hat{\ell})\} \) for \( j \geq 0 \).

Proof. By Theorem 10.5(a), \( \Upsilon^+_1 \) is continuous and except for the cases (2) and (3) given there it is strictly order reversing. The extensions \( \tilde{\Upsilon}^+ \) and \( \tilde{I} \) are exactly designed to make \( \tilde{\Upsilon}^+ \) continuous and strictly order reversing for the points in cases (2) and (3). The commutativity \( \tilde{\Upsilon}^+ \circ S = \tilde{T}_{\mu_1, \mu_2} \circ \tilde{\Upsilon}^+ \) follows from Corollary 10.7 and the construction of the extensions. This gives the semiconjugacy of \( \Delta_n^+ \) onto the image \( \tilde{\Upsilon}^+(\Delta_n^+) \). This image is a minimal set clearly contained in the attractor \( \tilde{\Omega}_n \). Now by Proposition 3 in [14], if \( \tilde{T}_n \) had a periodic orbit the attractor would consist of periodic orbits and we know from Theorem 16 in [11] that \( \Delta_n^+ \) is aperiodic. Thus \( \tilde{T}_n \) is aperiodic and so by Theorem 2.4 in [33] it has exactly one minimal set and so \( \tilde{\Upsilon}^+(\Delta_n^+) = \tilde{\Omega}_n \). The coding assertion follows from the conjugacy and the fact as noted above Corollary 10.7 that \( R_a = \Upsilon([a]) \) for \( a = 1, 2, 3 \) where \( [a] \subset \Delta_n \) is the cylinder set \( \{s \in \Delta_n : s_0 = a\} \). That completes the proof of part (a). For part (b), since \( \tilde{\Upsilon} = \tilde{\Upsilon}^+ \circ \pi \), the result for \( \tilde{\Upsilon} \) also follows using Theorem 6.1(b). \( \square \)

Using the conjugacy we can now transfer what we know about the dynamics and topology of the infimax minimal set to the attractor \( \tilde{\Omega}_n \) of the extended ITM \( \tilde{T}_n \). Using Theorem 6.1(b) and the conjugacy we get:

**Corollary 11.3.**

(a) Restricted to the extended attractor \( \tilde{\Omega}_n \), the extended self-map \( \tilde{T}_n \) is injective but for the sole exception that \( \tilde{T}_n^{-1}(0) = \{\mu_1, \mu_2\} \).

(b) Both \( \Omega_n \) and \( \tilde{\Omega}_n \) are Cantor sets

12. **The infimax family for other \( N \)**

The analog of (1.1) with \( N \) symbols generates the solution to the digit frequency infimax problem for sequences with elements from \( \{1, 2, \ldots, N\} \). Bruin shows using renormalization that the attractor of an ITM on \( N \) intervals is isomorphic to an S-adic minimal set using (1.1) with \( N \) symbols ([13]). While we focus here on \( N = 3 \), the generalization of the geometric representation for larger \( N \) is fairly straightforward but heavier in computation and indices.

It has long been known that the \( N = 2 \) infimax problem is solved by Sturmian sequences ([24], [35]). The \( n = 2 \) infimax family of substitutions, \( \lambda_n : 1 \mapsto 21^{n+1}, 2 \mapsto 1^n \) for \( n \in \mathbb{N}^+ \), generate these Sturmian sequences for irrational digit frequency ratios. There are other well known S-adic families
that generate all Sturmians (for a survey, see [1]): the $N = 2$ family just generates the infimax Sturmian for each frequency ratio. The geometric representation in this case differs from $N > 2$ in that the Rauzy fractal is an entire interval and its two subpieces are intervals with abutting endpoints. The induced map on the subpieces switches them and so the geometric representation is an IET on two intervals which can be interpreted as a circle homeomorphism. The identification of Sturmian sequences with itineraries of a circle map goes back to Morse and Hedlund. In addition, when $N = 2$ the substitution induced order on the Bratteli-Vershik diagram is proper and so the Vershik map can be globally defined.

The Sturmian sequences on two symbols have a host of nice properties ([1]) and there is much literature devoted to their generalization to more symbols (see, for example, [6]). There is not a single generalization for all the Sturmian properties, but rather many generalizations, each of which possesses one or a few of the Sturmians’ nice properties. The infimax $S$-adic family for $N > 2$ generalizes the infimax property of the Sturmians and their geometric representations, rather than being an IET on more intervals, is an ITM on $N$ intervals.

Acknowledgements. Our thanks to the referee for many useful comments.

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