On the fourth power moment of $\Delta(x)$ and $E(x)$ in short intervals

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Abstract

Let $\Delta(x)$ and $E(x)$ be error terms of the sum of divisor function and the mean square of the Riemann zeta function, respectively. In this paper their fourth power moments for short intervals of Jutila’s type are considered. We get an asymptotic formula for $U$ in some range.

1 Introduction

Let $d(n)$ denote the Dirichlet divisor function and $\Delta(x)$ denote the error term of the sum $\sum_{n \leq x} d(n)$ for a large real variable $x$:

$$\Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1),$$

where $\gamma$ is Euler’s constant. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$(1.1) \quad \Delta(x) \ll x^{131/416} (\log x)^{26947/8320},$$

proved by Huxley [4]. It is conjectured that

$$(1.2) \quad \Delta(x) = O(x^{1/4+\varepsilon}),$$

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which is supported by the classical mean square result

\begin{equation}
\int_1^T \Delta^2(x) dx = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)} T^{3/2} + O(T\log^5 T)
\end{equation}

proved by Tong [13]. For results of higher power moments of \( \Delta(x) \), see [2, 5, 7, 14, 15, 16, 17, 18].

Define the function \( E(t) \) by

\begin{equation}
E(t) := \int_0^t \left| \zeta\left(\frac{1}{2} + iu\right) \right|^2 du - t \log(t/2\pi) - (2\gamma - 1)t,
\end{equation}

where \( t \geq 2 \).

Many properties of \( E(t) \) are parallel to those of \( \Delta(x) \). Huxley [3] proved

\begin{equation}
E(t) = O(t^{72/227} \log^{629/227} t).
\end{equation}

The conjectured bound is

\begin{equation}
E(t) = O(t^{1/4+\varepsilon}),
\end{equation}

which is supported by the mean square formula

\begin{equation}
\int_2^T E^2(t) dt = \frac{2\zeta^4(3/2)}{3\zeta(3)\sqrt{2\pi}} T^{3/2} + O(T\log^5 T)
\end{equation}

proved by Meurman [11]. For higher power moments of \( E(t) \), see [2, 5, 7, 14, 15, 16, 17, 18, 19].

Jutila [8] first studied the mean square of the difference \( \Delta(x+U) - \Delta(x) \) for short intervals. He proved that if \( T \geq 2 \) and \( 1 \leq U \ll T^{1/2} \ll H \leq T \), then

\begin{equation}
\int_T^{T+H} (\Delta(x+U) - \Delta(x))^2 dx \\
= \frac{1}{4\pi^2} \sum_{n \leq T^{3/2}} \frac{d^2(n)}{n^{3/2}} \int_T^{T+H} x^{1/2} \left| \exp\left(2\pi i(n/x)^{1/2}U\right) - 1 \right|^2 dx \\
+ O(T^{1+\varepsilon} + H U^{1/2} T^\varepsilon),
\end{equation}

which implies that the estimate

\begin{equation}
\int_T^{T+H} (\Delta(x+U) - \Delta(x))^2 dx \approx H U \log^3(T^{1/2}/U)
\end{equation}
holds for $HU \gg T^{1+\varepsilon}$ and $T^{\varepsilon} \ll U \leq T^{1/2}$.

For $E(t)$ Jutila proved that the asymptotic formula

\begin{equation}
\int_{T}^{T+H} (E(t+U) - E(t))^2 \, dt = \frac{1}{\sqrt{2\pi}} \sum_{n \leq T/2U} \frac{d^2(n)}{n^{3/2}} \int_{T}^{T+H} t^{1/2} |\exp(i(2\pi n/t)^{1/2}U) - 1|^2 \, dt \\
+ O(T^{1+\varepsilon} + HU^{1/2}T^{\varepsilon}),
\end{equation}

holds for $1 \leq U \ll T^{1/2} \ll H \ll T$.

Jutila [8] also raised the problem of extending (1.8) and (1.10) to higher power moments. Especially, he conjectured that the estimates

\begin{equation}
\int_{2}^{T} (\Delta(x+U) - \Delta(x))^4 \, dx \ll T^{1+\varepsilon}U^2
\end{equation}

and

\begin{equation}
\int_{2}^{T} (E(t+U) - E(t))^4 \, dx \ll T^{1+\varepsilon}U^2
\end{equation}

are true for $1 \ll U \ll T^{1/2}$. He also pointed out that if (1.12) were true, then the important bound

$$\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 \, dt \ll T^{1+\varepsilon}$$

would follow.

When $H = T$, Recently Ivić[6] obtained substantial improvements on this problem. He proved a much more explicit asymptotic formula for the integral $\int_{T}^{2T} (\Delta(x+U) - \Delta(x))^2 \, dx$. He also proved that the estimate (1.11) holds for $T^{3/8} \ll U \ll T^{1/2}$. Similar results were also established for $E(t)$. We remark that the range $T^{3/8} \ll U \ll T^{1/2}$ seems to be the limit of the present methods.

Kiuchi and Tanigawa studied the mean square for short intervals of Jutila’s type for other arithmetical functions [9, 10].

**Notation.** $\varepsilon$ is a sufficiently small positive constant. $f \ll g$ means $|f| \leq Cg$ for some positive constant $C$. $n \asymp N$ means $N \ll n \ll N$, $n \sim N$ means $N < n \leq 2N$. $\mu(n)$ is the Möbius function. $SC(\Sigma)$ denotes the summation conditions of the sum $\Sigma$ when it is complicated.
2 Main results

Jutila and Ivić established asymptotic formulas for the mean square of \( \Delta(x + U) - \Delta(x) \). However for the fourth moment, only an upper bound result was proved, which seems very weak comparison to the mean square case. So it is natural to ask if one can find an asymptotic formula for the fourth power moment of \( \Delta(x + U) - \Delta(x) \) in some range of \( U \). The aim of this paper is to solve this problem. We shall establish an asymptotic formula for the fourth power moment of \( \Delta(x + U) - \Delta(x) \), which can be viewed as an analogue of (1.8).

**Theorem 1.** Suppose \( T, H, U \) are large real numbers such that

\[
T^{3/7} \ll U \ll T^{1/2}, \quad H \leq T, \quad H^8 U^{21} \gg T^{17},
\]

then for a small constant \( c \) we have

\[
\int_T^{T+H} (\Delta(x + U) - \Delta(x))^4 \, dx = \frac{3}{2\pi^4} \sum_{n_j \leq e(T/U)^{1/4}} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \int_T^{T+H} x \prod_{j=1}^4 \sin \frac{\pi U \sqrt{n_j}}{x^{1/2}} \, dx + O \left( T^{17/8 + \varepsilon} U^{-5/8} + HT^{3/16 + \varepsilon} U^{25/16} + H^{1/4} T^{15/16 + \varepsilon} U^{21/16} \right).
\]

and

\[
\int_T^{T+H} (\Delta(x + U) - \Delta(x))^4 \, dx = \frac{3}{\pi^4} \int_T^{T+H} x \left( \sum_{n \leq e(T/U)^{1/4}} \frac{d^2(n)}{n^{3/2}} \sin \frac{\pi U \sqrt{n}}{x^{1/2}} \right)^2 \, dx + O \left( HU^2 (T/U^2)^{-1/2} T^{\varepsilon} + HT^{3/16 + \varepsilon} U^{25/16} \right) + O \left( T^{17/8 + \varepsilon} U^{-5/8} + H^{1/4} T^{15/16 + \varepsilon} U^{21/16} \right).
\]

**Remark 1.** Our result holds only in the range \( T^{3/7} \ll U \ll T^{1/2} \), which is narrower than Ivić’s range \( T^{3/8} \ll U \ll T^{1/2} \). However, our theorem gives an asymptotic formula for the fourth power moment of \( \Delta(x + U) - \Delta(x) \). We note that the range \( T^{3/7} \ll U \ll T^{1/2} \) seems to be the limit we can establish asymptotic result for the fourth power moment by the present methods.
Remark 2. If the condition (2.1) is replaced by
\[ T^{3/7} \ll U \ll T^{1/2}, \quad T^{5/6} \ll H \leq T, \quad H^{8} U^{21} \gg T^{17}, \]
then the term \( H^{1/4} T^{15/16+\epsilon} U^{21/16} \) in (2.2) and (2.3) can be removed. If (2.1) is replaced by
\[ T^{3/7} \ll U \ll T^{1/2}, \quad H \leq T, \quad H^{16} U^{36} \gg T^{31}, \]
then both \( H^{1/4} T^{15/16+\epsilon} U^{21/16} \) and \( T^{17/8+\epsilon} U^{-5/8} \) in (2.2) and (2.3) can be removed.

Corollary 1. Suppose
\[ T^{3/7} \ll U \ll T^{1/2-\epsilon}, \quad H \leq T, \quad H^{8} U^{21} \gg T^{17+\epsilon}. \]
Then we have
\[ \int_{T}^{T+H} (\Delta(x + U) - \Delta(x))^4 dx \asymp H U^{2} \log^{6} \frac{T}{U^2}. \]

Theorem 2. Suppose \( T, H, U \) are large real numbers such that
\[ T^{3/7} \ll U \ll T^{1/2}, \quad T^{205/227} \ll H \leq T, \quad H^{8} U^{21} \gg T^{17}, \]
c is a small positive constant, then we have
\[ \int_{T}^{T+H} (E(t + U) - E(t))^4 dt = \frac{12}{\pi} \sum_{\sqrt{\pi n} \leq c(T/U)^{1/4}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1 n_2 n_3 n_4)^{3/4}} \]
\[ \times \int_{T}^{T+H} t \prod_{j=1}^{4} \sin \left( \frac{\sqrt{n_1} \pi j}{2t} \right)^{1/2} dt + O \left( H T^{3/16+\epsilon} U^{25/16} + T^{17/8+\epsilon} U^{-5/8} \right) \]
and
\[ \int_{T}^{T+H} (E(t + U) - E(x))^4 dt \]
\[ = \frac{24}{\pi} \sum_{n \leq c(T/U)^{1/4}} \frac{d^2(n)}{n^{3/2}} \sum_{m \leq c(T/U)^{1/4}} \frac{d^2(m)}{m^{3/2}} \int_{T}^{T+H} t \sin^{2} \left( \frac{\sqrt{n} \pi}{2t} \right)^{1/2} \sin^{2} \left( \frac{\sqrt{m} \pi}{2t} \right)^{1/2} dt \]
\[ + O \left( H U^{2} \left( \frac{T}{U^2} \right)^{-1/2} T^{\epsilon} + H T^{3/16+\epsilon} U^{25/16} + T^{17/8+\epsilon} U^{-5/8} \right). \]
Corollary 2. Suppose
\[ T^{3/7} \ll U \ll T^{1/2-\varepsilon}, \quad T^{205/227} \ll H \leq T, \quad H^{8} U^{21} \gg T^{17+\varepsilon}, \]
then
\[
\int_{T}^{T+H} (E(x+U) - E(x))^4 \, dx \asymp H U^2 \log^6 \frac{T}{U^2}.
\]

3 Some preliminary lemmas

Lemma 3.1. If \( 1 \ll N \ll x \), then
\[
\Delta(x) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n} x - \frac{\pi}{4} \right) + O(x^{1/2+\varepsilon} N^{-1/2}).
\]

Proof. This is the well-known truncated Voronoi’s formula (see for example, (3.17) of Ivić [5]). □

Lemma 3.2. If \( \alpha^* = \sqrt{n_1} + \sqrt{n_2} \pm \sqrt{n_3} - \sqrt{n_4} \neq 0 \), then
\[
|\alpha^*| \gg \max(n_1, n_2, n_3, n_4)^{-3/2}(n_1 n_2 n_3 n_4)^{-1/2}.
\]

Proof. This is a variant of Lemma 2 of Ivić and Sargos [7]. □

Lemma 3.3. Let \( N \geq 2, \Delta > 0 \) and let \( \mathcal{A}(N; \Delta) \) denote the number of solutions of the inequality
\[
|n_1^{1/2} + n_2^{1/2} - n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N \ (j = 1, 2, 3, 4),
\]
then
\[
\mathcal{A}(N; \Delta) \ll (\Delta N^{7/2} + N^2) N^\varepsilon.
\]

Proof. This is a special case of Theorem 2 of Robert and Sargos [12]. □

Lemma 3.4. Let \( N_j \geq 2, \Delta > 0 \) and let \( \mathcal{A}_\pm(N_1, N_2, N_3, N_4; \Delta) \) denote the number of solutions of the inequality
\[
|n_1^{1/2} + n_2^{1/2} \pm n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N \ (j = 1, 2, 3, 4),
\]
then
\[
\mathcal{A}_\pm(N_1, N_2, N_3, N_4; \Delta) \ll \prod_{j=1}^{4} (\Delta^{1/4} N_j^{7/8} + N_j^{1/2}) N_j^\varepsilon.
\]
Proof. This is Lemma 3 of Zhai [17].

**Lemma 3.5.** Suppose $f_j(t)(1 \leq j \leq k)$ and $g(t)$ are continuous, monotonic real-valued functions on $[a, b]$ and let $g(t)$ have a continuous, monotonic derivative on $[a, b]$. If $|f_j(t)| \leq A_j(1 \leq j \leq k), |g'(t)| \gg \Delta$ for any $t \in [a, b]$, then

$$\int_a^b f_1(t) \cdots f_k(t) e(g(t)) dt \ll A_1 \cdots A_k \Delta^{-1}.$$ 

Proof. This is Lemma 15.3 of Ivić [5]. We note that Lemma 3.5 is still true if the function $e(t)$ is replaced by $\cos t$ and $\sin t$.  

**Lemma 3.6.** The square roots of different square-free numbers are linearly independent over the integers.

Proof. This is a classical result of Besicovitch [1].

### 4 Estimates of some special sums

In this section we shall prove the estimates of several special sums used in our proof.

**Lemma 4.1.** Let $z \geq 10$. Then

$$\sum_{n \leq z} d^2(n)n^{-1/2} \asymp z^{1/2} \log^3 z,$$

and

$$\sum_{n > z} d^2(n)n^{-3/2} \asymp z^{-1/2} \log^3 z.$$ 

Proof. These two estimates follow from the partial summation formula and the well-known estimate

$$\sum_{n \leq z} d^2(n) \asymp z \log^3 z.$$ 


Lemma 4.2. Let \( z \geq 10 \). Define
\[
c_1(z) := \sum_{n_1, n_2, n_3, n_4} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}n_4^{1/4}}.
\]
Then \( c_1(z) \ll 1 \).

Proof. Suppose \( \sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} = \sqrt{n_4} \). By Lemma 3.6, we have
\[
n_j = lm_j^2, m_1 + m_2 + m_3 = m_4, \mu(l) \neq 0.
\]
So we have (noting that \( d(ab) \leq d(a)d(b) \))
\[
c_1(z) \ll \sum_{m_1 + m_2 + m_3 = m_4} \frac{d(lm_j^2) d(lm_j^2) d(lm_j^2) d(lm_j^2)}{l^{5/2}(m_1m_2m_3)^{3/2}m_4^{1/2}}
\ll \sum_{l \leq z} \frac{d^4(l)}{l^{5/2}} \sum_{m_1 + m_2 + m_3 = m_4} \frac{d(m_j^2) d(m_j^2) d(m_j^2) d(m_j^2)}{(m_1m_2m_3)^{3/2}m_4^{1/2}}
\ll \sum_{l \leq z} \frac{d^4(l)}{l^{5/2}} \sum_{m_1, m_2, m_3 \leq (z/l)^{1/2}} \frac{d(m_j^2) d(m_j^2) d(m_j^2) d((m_1 + m_2 + m_3)^2)}{(m_1m_2m_3)^{3/2}(m_1 + m_2 + m_3)^{1/2}}
\ll 1.
\]

Lemma 4.3. Let \( z \geq 10 \). Define
\[
c_2(z) := \sum_{n_1, n_2, n_3, n_4} \frac{d(n_1)d(n_2)d(n_3)d(n_4)(\sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4})}{(n_1n_2n_3n_4)^{3/4}}.
\]
Then \( c_2(z) \ll \log^4 z \).

Proof. We have \( c_2(z) \ll c_2'(z) \), where
\[
c_2'(z) := \sum_{n_1, n_2, n_3, n_4} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}n_4^{1/4}}.
\]
It is easily seen that the equation \( \sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4} \) is equivalent to the following two cases:

(1) \( z \leq n \leq 2z \)
(2) \( 2z < n \leq 4z \)
Thus we may write
\[ c_2'(z) = 2c_{21}(z) + c_{22}(z) \]

with
\[
c_{21}(z) := \sum_{n \leq z} \frac{d^2(n)}{n^{3/2}} \sum_{m \leq z} \frac{d^2(m)}{m},
\]
\[
c_{22}(z) := \sum_{\sqrt{n_1 + n_2} = \sqrt{n_3} + \sqrt{n_4}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}n_4^{1/4}},
\]

where \( \sum^* \) means the condition \( n_1 \neq n_3, n_1 \neq n_4 \).

Obviously, \( c_{21}(z) \ll \log^4 z \). Now suppose \( \sqrt{n_1 + \sqrt{n_2}} = \sqrt{n_3} + \sqrt{n_4} \) such that \( n_1 \neq n_3, n_1 \neq n_4 \). By Lemma 3.6 we have
\[
(4.1) \quad n_j = l m_j^2, \quad m_1 + m_2 = m_3 + m_4, \quad \mu(l) \neq 0.
\]

Thus
\[
c_{22}(z) \ll \sum_{l \leq z} \frac{d^4(l)}{l^{10/4}} \sum_{m_1 + m_2 = m_3 + m_4} \frac{d(m_1^2)d(m_2^2)d(m_3^2)d(m_4^2)}{(m_1m_2m_3m_4)^{3/2}m_4^{1/2}},
\]
\[
\ll \sum_{l \leq z} \frac{d^4(l)}{l^{10/4}} \sum_{n \leq (z/l)^{1/2}} r(n) R(n),
\]

where
\[
r(n) := \sum_{n = m_1 + m_2} \frac{d(m_1^2)d(m_2^2)}{(m_1m_2)^{3/2}} \quad \text{and} \quad R(n) = \sum_{n = m_3 + m_4} \frac{d(m_3^2)d(m_4^2)}{m_3^{3/2}m_4^{1/2}}.
\]

Obviously
\[
r(n) \ll n^{-3/2+\varepsilon} \quad \text{and} \quad R(n) \ll n^{-1/2+\varepsilon}.
\]

So we have \( c_{22}(z) \ll 1 \). Now Lemma 4.3 follows from the above estimates.

\[ \square \]

**Lemma 4.4.** Suppose \( u > z \geq 10 \). Define
\[
c(z, u) = \sum_{l \leq z} \frac{1}{(n_1n_2n_3n_4)^{3/4}} \prod_{j=1}^{4} \min \left( \frac{\sqrt{n_j}}{\sqrt{z}}, 1 \right),
\]
\[
SC(\Sigma_1) : \sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}, \quad n_1 \neq n_3, \quad n_1 \neq n_4, \quad n_j \leq u \quad (j = 1, 2, 3, 4).
\]
Then we have \( c(z, u) \ll z^{-3/2} \).

Proof. Let

\[
g(n_1, n_2, n_3, n_4) = \frac{1}{(n_1 n_2 n_3 n_4)^{3/4}} \prod_{j=1}^{4} \min \left( \frac{\sqrt{n_j}}{\sqrt{z}}, 1 \right).
\]

Obviously we have

\[
c(z, u) \ll \sum_2 g(n_1, n_2, n_3, n_4),
\]

\begin{align*}
SC(\Sigma_2) : \sqrt{n_1} + \sqrt{n_2} & = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4, n_j \leq u \ (j = 1, 2, 3, 4), \\
n_1 \leq n_2, n_3 \leq n_4, n_2 < n_4.
\end{align*}

We write

\[
\sum_2 g(n_1, n_2, n_3, n_4) = (\sum_3 + \sum_4) g(n_1, n_2, n_3, n_4),
\]

\begin{align*}
SC(\Sigma_3) : \sqrt{n_1} + \sqrt{n_2} & = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4, \\
n_1 \leq n_2, n_3 \leq n_4, n_2 < n_4, n_4 \leq z,
\end{align*}

\begin{align*}
SC(\Sigma_4) : \sqrt{n_1} + \sqrt{n_2} & = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4, n_j \leq u \ (j = 1, 2, 3, 4), \\
n_1 \leq n_2, n_3 \leq n_4, n_2 \leq n_4, n_4 > z.
\end{align*}

We estimate \( \Sigma_3 \) first. From (4.1) we get

\[
\sum_3 g(n_1, n_2, n_3, n_4) \ll z^{-2} \sum_2 (n_1 n_2 n_3 n_4)^{-1/4}
\]

\[
\ll z^{-2} \sum_{l < z} l^{-1} \sum_{m_1 + m_2 = m_3 + m_4} (m_1 m_2 m_3 m_4)^{-1/2}
\ll z^{-2} \sum_{l < z} l^{-1} \sum_{n \leq 2(z/l)^{1/2}} f^2(n) \ll z^{-2} \sum_{l < z} l^{-1} \sum_{n \leq 2(z/l)^{1/2}} 1
\ll z^{-2} \sum_{l < z} l^{-1} (z/l)^{1/2} \ll z^{-3/2},
\]

where we used the estimate

\[
f(n) = \sum_{n = m_1 + m_2} (m_1 m_2)^{-1/2} \ll 1.
\]
Now we estimate $\Sigma_4$. From the condition $SC(\Sigma_4)$ we get $n_2 \geq z/2$. If $n_3 \geq z$, then $n_1 \geq z$. So we can write

$$\sum_4 g(n_1, n_2, n_3, n_4) = \left( \sum_{41} + \sum_{42} + \sum_{43} \right) g(n_1, n_2, n_3, n_4),$$

$SC(\Sigma_{41}) : \sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4,$

$n_4 > z, n_2 > z/2, n_3 < z, n_1 < z,$

$SC(\Sigma_{42}) : \sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4,$

$n_4 > z, n_2 > z, n_3 < z, n_1 > z,$

$SC(\Sigma_{43}) : \sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}, n_1 \neq n_3, n_1 \neq n_4,$

$n_4 > z, n_2 > z, n_3 \geq z, n_1 \geq z.$

By (4.1) we get

$$\sum_{41} g(n_1, n_2, n_3, n_4) \ll z^{-1} \sum_{41} n_1^{-1/4} n_2^{-3/4} n_3^{-1/4} n_4^{-3/4}$$

$$\ll z^{-1} \sum_l l^{-2} \sum_{411} m_1^{-1/2} m_2^{-3/2} m_3^{-1/2} m_4^{-3/2}$$

$$= z^{-1} \sum_l l^{-2} \sum_{n>(z/l)^{1/2}} f_1(n) g_1(n),$$

where

$SC(\Sigma_{411}) : m_1 + m_2 = m_3 + m_4, m_1 \leq (z/l)^{1/2}, m_3 \leq (z/l)^{1/2},$

$m_2 > (z/2l)^{1/2}, m_4 > (z/l)^{1/2};$

$f_1(n) := \sum_{412} m_1^{-1/2} m_2^{-3/2},$ $SC(\Sigma_{412}) : m_1 \leq (z/l)^{1/2}, m_2 > (z/2l)^{1/2}, m_1 \leq m_2;$

$g_1(n) := \sum_{413} m_3^{-1/2} m_4^{-3/2},$ $SC(\Sigma_{413}) : m_3 \leq (z/l)^{1/2}, m_4 > (z/l)^{1/2}.$

It is easy to see that

$$f_1(n) \ll n^{-3/2} (z/l)^{1/4}, \quad g_1(n) \ll n^{-3/2} (z/l)^{1/4}.$$

Thus we have

$$\sum_{41} g(n_1, n_2, n_3, n_4) \ll z^{-1} \sum_l l^{-2} (z/l)^{1/2} \sum_{n>(z/l)^{1/2}} n^{-3} \ll z^{-3/2}.$$
Similarly we can prove
\[ \sum_{42} g(n_1, n_2, n_3, n_4) \ll z^{-3/2}, \]
\[ \sum_{43} g(n_1, n_2, n_3, n_4) \ll z^{-3/2}. \]

Now Lemma 4.4 follows from the above estimates. \( \square \)

## 5 Identities involving the functions \( \sin u \) and \( \cos u \)

We need some formulas about the functions \( \sin u \) and \( \cos u \). Let \( k \geq 1 \) be a fixed integer, \( \alpha_1, \ldots, \alpha_k \) be real numbers. Define

\[
S_k(\alpha_1, \cdots, \alpha_k) := \sum_{j_1=0}^{1} \cdots \sum_{j_k=0}^{1} (-1)^{\sum_{i=1}^{k} j_i} \sin(j_1 \alpha_1 + \cdots + j_k \alpha_k),
\]
\[
C_k(\alpha_1, \cdots, \alpha_k) := \sum_{j_1=0}^{1} \cdots \sum_{j_k=0}^{1} (-1)^{\sum_{i=1}^{k} j_i} \cos(j_1 \alpha_1 + \cdots + j_k \alpha_k).
\]

**Lemma 5.1.** Suppose \( k = m + l, m \geq 1, l \geq 1 \), then we have

\[
S_k(\alpha_1, \cdots, \alpha_k) = S_m(\alpha_1, \cdots, \alpha_m) C_l(\alpha_{m+1}, \cdots, \alpha_k) + C_m(\alpha_1, \cdots, \alpha_m) S_l(\alpha_{m+1}, \cdots, \alpha_k), \tag{5.1}
\]
\[
C_k(\alpha_1, \cdots, \alpha_k) = C_m(\alpha_1, \cdots, \alpha_m) C_l(\alpha_{m+1}, \cdots, \alpha_k) - S_m(\alpha_1, \cdots, \alpha_m) S_l(\alpha_{m+1}, \cdots, \alpha_k). \tag{5.2}
\]

**Proof.** These two identities follow from the well-known formulas

\[
\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta
\]

and

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,
\]

respectively. \( \square \)

**Lemma 5.2.** We have

\[
S_k(\alpha_1, \cdots, \alpha_k) = \begin{cases} 
2^k (-1)^{r+1} \left( \prod_{j=1}^{k} \sin \frac{\alpha_j}{2} \right) \cos \frac{\alpha_1 + \cdots + \alpha_k}{2}, & \text{if } k \equiv r \pmod{4}, \ r = 1, 3, \\
2^k (-1)^{r} \left( \prod_{j=1}^{k} \sin \frac{\alpha_j}{2} \right) \sin \frac{\alpha_1 + \cdots + \alpha_k}{2}, & \text{if } k \equiv r \pmod{4}, \ r = 2, 4,
\end{cases}
\]
\[ \mathcal{C}O_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 2^k(-1)^r \left( \prod_{j=1}^k \sin \frac{\alpha_j}{2} \right) \sin \frac{\alpha_1 + \cdots + \alpha_k}{2}, & \text{if } k \equiv r \pmod{4}, r = 1, 3, \\ 2^k(-1)^{r+1} \left( \prod_{j=1}^k \sin \frac{\alpha_j}{2} \right) \cos \frac{\alpha_1 + \cdots + \alpha_k}{2}, & \text{if } k \equiv r \pmod{4}, r = 2, 4. \end{cases} \]

**Proof.** Trivially for \( k = 1 \) we have

\[ \mathcal{S}I_1(\alpha_1) = -\sin \alpha_1 = -2 \sin \frac{\alpha_1}{2} \cos \frac{\alpha_1}{2}, \]

(5.3) \[ \mathcal{C}O_1(\alpha_1) = 1 - \cos \alpha_1 = 2 \sin^2 \frac{\alpha_1}{2}. \]

(5.4) The cases \( k = 2, 3, 4 \) follow easily from Lemma 5.1 and (5.3).

Now suppose \( m \geq 1 \) be a fixed integer such that Lemma 5.2 is true for \( k \leq 4m \). We shall show that Lemma 5.2 is true for \( k = 4m + r, r = 1, 2, 3, 4 \). We only prove the result for \( \mathcal{S}I_k(\alpha_1, \ldots, \alpha_k) \) with \( r = 1, 3 \). The other cases are similar. By Lemma 5.1 we have

\[ \mathcal{S}I_k(\alpha_1, \ldots, \alpha_k) = \mathcal{S}I_{4m}(\alpha_1, \ldots, \alpha_{4m}) \mathcal{C}O_r(\alpha_{4m+1}, \ldots, \alpha_k) + \mathcal{C}O_{4m}(\alpha_1, \ldots, \alpha_{4m}) \mathcal{S}I_r(\alpha_{4m+1}, \ldots, \alpha_k) \]

\[ = (-1)^r 2^k \prod_{j=1}^k \sin \frac{\alpha_j}{2} \left\{ \sin \frac{\alpha_1 + \cdots + \alpha_{4m}}{2} \sin \frac{\alpha_{4m+1} + \cdots + \alpha_k}{2} \right\} \]

\[ + (-1)^{r+1} 2^k \prod_{j=1}^k \sin \frac{\alpha_j}{2} \left\{ \cos \frac{\alpha_1 + \cdots + \alpha_{4m}}{2} \cos \frac{\alpha_{4m+1} + \cdots + \alpha_k}{2} \right\} \]

\[ = (-1)^{r+1} 2^k \prod_{j=1}^k \sin \frac{\alpha_j}{2} \left\{ \cos \frac{\alpha_1 + \cdots + \alpha_{4m}}{2} \cos \frac{\alpha_{4m+1} + \cdots + \alpha_k}{2} \right\} \]

\[ - \sin \frac{\alpha_1 + \cdots + \alpha_{4m}}{2} \sin \frac{\alpha_{4m+1} + \cdots + \alpha_k}{2} \right\} \]

\[ = (-1)^{r+1} 2^k \prod_{j=1}^k \sin \frac{\alpha_j}{2} \cos \frac{\alpha_1 + \cdots + \alpha_k}{2}. \]

\( \square \)
6 Proof of Theorem 1

Suppose the condition (2.1) holds. Let \( y := c(T/U)^{1/4} \), where \( c \) is a small positive constant. For any \( T \leq x \ll T \), define

\[
R_1(x) := \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{xn} - \frac{\pi}{4}),
\]

\[
R_2(x) := \Delta(x) - R_1(x).
\]

By Lemma 3.1, we can write

\[
\Delta(x + U) - \Delta(x) = S_1(x) + S_2(x),
\]

where

\[
S_1(x) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \left\{ \cos \left( 4\pi \sqrt{n(x + U)} - \frac{\pi}{4} \right) - \cos \left( 4\pi \sqrt{nx} - \frac{\pi}{4} \right) \right\},
\]

\[
S_2(x) = R_2(x + U) - R_2(x) + M(x),
\]

and

\[
M(x) = \frac{(x + U)^{1/4} - x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n(x + U)} - \frac{\pi}{4} \right).
\]

\[
= \frac{(x + U)^{1/4} - x^{1/4}}{(x + U)^{1/4}} R_1(x + U).
\]

It is easy to see that

\[
M(x) \ll U x^{-1} |R_1(x + U)|.
\]

6.1 Evaluation of the integral \( \int_T^{T+H} S_1^4(x) dx \)

First we evaluate the integral \( \int_T^{T+H} S_1^4(x) dx \). \( S_1(x) \) can be written as

\[
S_1(x) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{j=0}^{1} (-1)^{j+1} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n(x + jU)} - \frac{\pi}{4} \right).
\]

In order to write the 4-th power in a simple way, we introduce here the following notations. Let \( I = \{0, 1\} \) and let \( \mathbf{j} = (j_1, j_2, j_3, j_4) \) and \( \mathbf{i} = (i_1, i_2, i_3) \) be elements in \( I^4 \) and \( I^3 \), respectively, and put \( |\mathbf{j}| = j_1 + \cdots + j_4 \).
Using the elementary formula
\[ \cos a_1 \cos a_2 \cos a_3 \cos a_4 = \frac{1}{2} \sum_{i \in I^3} \cos(a_1 + (-1)^i a_2 + (-1)^i a_3 + (-1)^i a_4) \]
we have

\[ S_4^4(x) = \frac{x}{(\pi \sqrt{2})^4} \sum_{j \in I^4} (-1)^{\mid j \mid + 4} \sum_{n_1 \leq y} \ldots \sum_{n_4 \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \]
\[ \times \prod_{l=1}^{4} \cos \left( 4\pi \sqrt{n_l(x + j_l U)} - \frac{\pi}{4} \right) \]
\[ = \frac{x}{2^5 \pi^4} \sum_{j \in I^4} (-1)^{\mid j \mid} \sum_{i \in I^3} \sum_{n_i \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \cos \left( 4\pi \alpha(x) - \frac{\pi \beta}{4} \right), \]
where
\[ \alpha(x) = \sqrt{n_1(x + j_1 U)} + (-1)^{i_1} \sqrt{n_2(x + j_2 U)} + (-1)^{i_2} \sqrt{n_3(x + j_3 U)} \]
\[ + (-1)^{i_3} \sqrt{n_4(x + j_4 U)}, \]
\[ \beta = 1 + (-1)^{i_1} + (-1)^{i_2} + (-1)^{i_3}. \]

Let
\[ \alpha^* = \sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + (-1)^{i_3} \sqrt{n_4}, \]
and
\[ \alpha_* = j_1 \sqrt{n_1} + (-1)^{i_1} j_2 \sqrt{n_2} + (-1)^{i_2} j_3 \sqrt{n_3} + (-1)^{i_3} j_4 \sqrt{n_4}. \]

We divide the sum \((6.2)\) into two parts according to \(\alpha^* = 0\) or \(\alpha^* \neq 0\), respectively. Thus we get

\[ S_4^4(x) = \frac{1}{2^5 \pi^4} (S_{11}(x) + S_{12}(x)), \]

where
\[ S_{11}(x) = x \sum_{j \in I^4} (-1)^{\mid j \mid} \sum_{i \in I^3} \sum_{n_i \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \cos \left( 4\pi \alpha(x) - \frac{\pi \beta}{4} \right), \]
and
\[ S_{12}(x) = x \sum_{j \in I^4} (-1)^{\mid j \mid} \sum_{i \in I^3} \sum_{n_i \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \cos \left( 4\pi \alpha(x) - \frac{\pi \beta}{4} \right). \]
It is easy to see that
\[ 4\pi\alpha(x) = 4\pi\alpha^* \sqrt{x} + \frac{2\pi U \alpha_s}{x^{1/2}} + O \left( \frac{U^2(\sqrt{n_1} + \cdots + \sqrt{n_4})}{x^{3/2}} \right). \]

Hence, if \( \alpha^* = 0 \), we have
\[
\cos \left( 4\pi\alpha(x) - \frac{\pi\beta}{4} \right) = \cos \left( \frac{2\pi U \alpha_s}{x^{1/2}} - \frac{\pi\beta}{4} + O \left( \frac{U^2(\sqrt{n_1} + \cdots + \sqrt{n_4})}{x^{3/2}} \right) \right).
\]

Therefore
\[
(6.6) \quad \int_T^{T+H} S_{11}(x) dx = \sum_{j \in I^4} (-1)^{|j|} \sum_{i \in I^3 \atop n_i \leq y} \sum_{\alpha^* = 0} d(n_1) \cdots d(n_4) \frac{\alpha^*}{(n_1 \cdots n_4)^{3/4}}
\]
\[
\times \int_T^{T+H} x \cos \left( \frac{2\pi U \alpha_s}{x^{1/2}} - \frac{\pi\beta}{4} \right) dx
\]
\[
+ O \left( HT^{-1/2} U^2 \sum_{\alpha^* = 0} d(n_1) \cdots d(n_4)(\sqrt{n_1} + \cdots + \sqrt{n_4}) \right).
\]

For \( i = (i_1, i_2, i_3) \in I^3, i \neq (0, 0, 0) \), we let
\[
I(i) = \sum_{j \in I^4} (-1)^{|j|} \sum_{\substack{n_i \leq y \\ \alpha^* = 0}} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}}
\]
\[
\times \int_T^{T+H} x \cos \left( \frac{2\pi U \alpha_s}{x^{1/2}} - \frac{\pi\beta}{4} \right) dx,
\]

where \( \alpha^* \) and \( \alpha_s \) are defined by (6.3) and (6.4). It is easily seen that
\[
I(0, 0, 1) = I(0, 1, 0) = I(1, 0, 0) = I(1, 1, 1),
\]
\[
I(0, 1, 1) = I(1, 1, 0) = I(1, 0, 1).
\]

Concerning the sum in \( O \)-term in (6.6), Lemma 4.2 and 4.3 imply that
\[
\sum_{\substack{n_i \leq y \\ \alpha^* = 0}} \frac{d(n_1) \cdots d(n_4)(\sqrt{n_1} + \cdots + \sqrt{n_4})}{(n_1 \cdots n_4)^{3/4}} \ll \log^4 y.
\]
Therefore
\[
\int_{T}^{T+H} S_{11}(x)dx = 4 \sum_{n_1+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}} d(n_1)d(n_2)d(n_3)d(n_4) 
\]
\[
\times \int_{T}^{T+H} xST_4 \left( \frac{2\pi U\sqrt{n_1}}{x^{1/2}}, \frac{2\pi U\sqrt{n_2}}{x^{1/2}}, \frac{2\pi U\sqrt{n_3}}{x^{1/2}}, \frac{2\pi U\sqrt{n_4}}{x^{1/2}} \right) dx 
\]
\[
+ 3 \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}+\sqrt{n_4}} d(n_1)d(n_2)d(n_3)d(n_4) 
\]
\[
\times \int_{T}^{T+H} xCO_4 \left( \frac{2\pi U\sqrt{n_1}}{x^{1/2}}, \frac{2\pi U\sqrt{n_2}}{x^{1/2}}, \frac{2\pi U\sqrt{n_3}}{x^{1/2}}, \frac{2\pi U\sqrt{n_4}}{x^{1/2}} \right) dx 
\]
\[+ O(H^2 U^2 T^{-1/2} \log^4 y). \]

By Lemma 5.2, we have
\[
ST_4 \left( \frac{2\pi U\sqrt{n_1}}{x^{1/2}}, \frac{2\pi U\sqrt{n_2}}{x^{1/2}}, \frac{2\pi U\sqrt{n_3}}{x^{1/2}}, \frac{2\pi U\sqrt{n_4}}{x^{1/2}} \right) = 0 
\]
for $\sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} = \sqrt{n_4}$, and
\[
CO_4 \left( \frac{2\pi U\sqrt{n_1}}{x^{1/2}}, \frac{2\pi U\sqrt{n_2}}{x^{1/2}}, \frac{2\pi U\sqrt{n_3}}{x^{1/2}}, \frac{2\pi U\sqrt{n_4}}{x^{1/2}} \right) = 16 \prod_{j=1}^{4} \frac{\sin \pi U^n_j}{x^{1/2}} 
\]
for $\sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}$. Hence
\[
(6.7) \quad \int_{T}^{T+H} S_{11}(x)dx = 48 \sum_{\sqrt{n_1}+\sqrt{n_2}=\sqrt{n_3}+\sqrt{n_4}} \frac{d(n_1)\cdots d(n_4)}{(n_1\cdots n_4)^{3/4}} 
\]
\[
\times \int_{T}^{T+H} x \prod_{j=1}^{4} \sin \frac{\pi U\sqrt{n_j}}{x^{1/2}} dx + O(H^2 U^2 T^{-1/2} \log^4 y). 
\]

Now consider the integral $\int_{T}^{T+H} S_{12}(x)dx$. Note that $S_{12}(x)$ corresponds to the sum under the condition $\alpha^* \neq 0$. From the definition of $\alpha(x)$, we have
\[
\alpha'(x) = \frac{\alpha^*}{2\sqrt{x}} + O \left( \frac{U(\sqrt{n_1} + \cdots + \sqrt{n_4})}{x^{3/2}} \right). 
\]
By Lemma 3.2 we get \(|\alpha^*| \gg \max(n_1, n_2, n_3, n_4)^{-7/2}\), which combining with
\(y = c(T/U)^{1/4}\) with a small positive constant \(c\) implies that
\[
\alpha'(x) \gg |\alpha^*|T^{-1/2}.
\]

By Lemma 3.5 we get
\[
\int_T^{T+H} x \cos(4\pi \alpha(x) - \pi \beta/4) \, dx \ll \frac{T^{3/2}}{|\alpha^*|}.
\]

Hence
\[
\int_T^{T+H} S_{12}(x) \, dx \ll T^{3/2} \sum_{j \in I^4} \sum_{i \in I^3} \sum_{n_i \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \frac{1}{|\alpha^*|}
\]
\[
\ll G_1 + G_2,
\]
where
\[
G_1 = T^{3/2} \sum_{n_i, n_j^y \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \frac{1}{|\sqrt{n_1} + \sqrt{n_2} - \sqrt{n_3} - \sqrt{n_4}|},
\]
\[
G_2 = T^{3/2} \sum_{n_i, n_j^y \leq y} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \frac{1}{|\sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} - \sqrt{n_4}|}.
\]

We only estimate the sum \(G_1\). The estimate for \(G_2\) is the same.

Let \(\eta = \sqrt{n_1} + \sqrt{n_2} - \sqrt{n_3} - \sqrt{n_4}\). By a splitting argument we get
\[
G_1 \ll G_1(N_1, N_2, N_3, N_4) \log^4 T
\]
for some \(N_1, N_2, N_3, N_4, N_j \ll y\) (\(j = 1, 2, 3, 4\)), where
\[
G_1(N_1, N_2, N_3, N_4) = T^{3/2} \sum_{n_i \sim N_i} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \frac{1}{|\eta|}.
\]

Without loss of generality, we may assume that \(N_1 \leq N_2, N_3 \leq N_4\) and \(N_2 \leq N_4\). Then by Lemma 3.2 we have
\[
\eta_0 := N_4^{-3/2}(N_1N_2N_3N_4)^{-1/2} \ll |\eta| \ll N_4^{1/2}
\]

\[\text{Page 18}\]
By a splitting argument again we get that 

\[ G_1(N_1, N_2, N_3, N_4) \ll \frac{T^{3/2+\varepsilon}}{(N_1N_2N_3N_4)^{3/4}} \sum_{\delta <|\eta| \leq 2\delta} 1. \]

By Lemma 3.4 we have 

\[ G_1(N_1, N_2, N_3, N_4) \ll \frac{T^{3/2+\varepsilon}}{(N_1N_2N_3N_4)^{3/4}} \prod_{j=1}^{4} N_j^{1/2} (\delta^{1/4} N_j^{3/8} + 1) \]

\[ \ll \frac{T^{3/2+\varepsilon}}{(N_1N_2N_3N_4)^{1/4}} \left( \delta^{3/4} (N_1N_2N_3)^{3/8} + \delta^{1/2} N_4^{3/4} + \delta^{1/4} N_4^{3/8} + 1 \right) \]

\[ \ll \frac{T^{3/2+\varepsilon}}{(N_1N_2N_3N_4)^{1/4}} \left( N_4^{3/8} + \delta^{-1/4} \right) + \frac{T^{3/2+\varepsilon}}{(N_1N_2N_3N_4)^{1/4}} \left( \delta^{-1/4} N_4^{9/8} + \delta^{-1} \right) \]

\[ \ll T^{3/2+\varepsilon} N_4^{5/2} \]

where in the last step we used (6.8). Thus we have \( G_1 \ll T^{3/2+\varepsilon} y^{5/2} \). Similarly we have \( G_2 \ll T^{3/2+\varepsilon} y^{5/2} \). Combining the above estimates we get

\[ \int_T^{T+H} S_{12}(x) dx \ll T^{3/2+\varepsilon} y^{5/2}. \]

From (6.5), (6.7) and (6.9), we get

\[ \int_T^{T+H} S_1^4(x) dx = \frac{3}{2\pi^4} \sum_{\eta_j \leq y} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}} \]

\[ \times \int_T^{T+H} x \prod_{j=1}^{4} \sin \frac{\pi U \sqrt{n_j}}{x^{1/2}} dx + O(T^{3/2+\varepsilon} y^{5/2} + HU^2 T^{-1/2} \log^4 T). \]

It is easy to see that the sum in the right hand side of (6.10) can be
written as $\sum_6 + \sum_7$, where

(6.11)
\[
\sum_6 = \frac{3}{\pi^3} \sum_{n \leq y^{3/2}} \frac{d^2(n)}{n^{3/2}} \sum_{m \leq y^{3/2}} \frac{d^2(m)}{m^{3/2}} \int_T^{x} x \sin^2 \frac{\pi U \sqrt{n}}{x^{1/2}} \sin^2 \frac{\pi U \sqrt{m}}{x^{1/2}} \, dx
\]
\[
= \frac{3}{\pi^3} \int_T^{x} x \left( \sum_{n \leq y^{3/2}} \frac{d^2(n)}{n^{3/2}} \sin^2 \frac{\pi U \sqrt{n}}{x^{1/2}} \right)^2 \, dx,
\]
and

(6.12)
\[
\sum_7 = \frac{3}{2 \pi^4} \sum_{\sqrt{n_1^2 + \sqrt{n_2^2 - \sqrt{n_3^2 + \sqrt{n_4^2}}} \neq n_3, n_1 \neq n_4} \frac{d(n_1) \cdots d(n_4)}{(n_1 \cdots n_4)^{3/4}} \int_T^{x} x \prod_{j=1}^{4} \sin \frac{\pi U \sqrt{n_j}}{x^{1/2}} \, dx.
\]

By Lemma 4.1 we have

(6.13)
\[
\sum_6 \ll H U^2 \log^6 T,
\]
while by Lemma 4.4 we have

(6.14)
\[
\sum_7 \ll T^{1+\varepsilon} H (T/U^2)^{-3/2} \ll H U^2 T^{\varepsilon} (T/U^2)^{-1/2}.
\]

From (6.10)–(6.14) we get

(6.15)
\[
\int_T^{T+H} S_4^4(x) \, dx \ll H U^2 T^{\varepsilon} + T^{3/2+\varepsilon} y^{5/2} \ll H U^2 T^{\varepsilon}
\]
if we note the condition $H^8 U^{21} \gg T^{17}$.

### 6.2 On the integral $\int_T^{T+H} S_2^4(x) \, dx$

In this subsection we estimate the integral $\int_T^{T+H} S_2^4(x) \, dx$. Since $S_2(x) = R_2(x+U) - R_2(x) + M(x)$, it is sufficient to consider the integrals of $R_2^4(x)$ and $M^4(x)$.

Recall that $(T \leq x \leq T + H)$
\[
R_2(x) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{y < n \leq x} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{nx} - \pi/4) + O(T^\varepsilon).
\]

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Let $J$ be a positive integer such that $2^{J+1} \times H^2 y^{-3} T^{-1}$ and let $z = 2^{J+1} y$, then $y < z < T$. We divide $R_2(x)$ into two parts:

$$R_2(x) = R_{21}(x) + R_{22}(x),$$

where

$$R_{21}(x) = \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{y < n \leq z} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{nx} - \frac{\pi}{4}\right),$$

$$R_{22}(x) = \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{z < n \leq T} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{nx} - \frac{\pi}{4}\right) + O(T^\varepsilon).$$

Obviously we have

$$R_{22}(x) \ll x^{1/2+\varepsilon} z^{-1/2}.$$

For the mean square estimate for $R_{22}(x)$, we have

$$\int_T^{T+H} R_{22}^2(x) dx \ll T^{1/2+\varepsilon} H z^{-1/2} + T^{1+\varepsilon}$$

(see e.g. (2.3) of Zhai [18]). Hence

$$(6.16) \quad \int_T^{T+H} R_{22}^2(x) dx \ll \max_{T \leq x \leq T+H} R_{22}^2(x) \int_T^{T+H} R_{22}^2(x) dx \ll T^{1+\varepsilon} z^{-1} (T^{1/2} H z^{-1/2} + T).$$

For $R_{21}(x)$ we can write

$$R_{21}(x) = \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{y < n \leq z} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{nx} - \frac{\pi}{4}\right)$$

$$= \sum_{0 \leq j \leq J} \frac{x^{1/4}}{\sqrt{2\pi}} \sum_{2^j y < n \leq 2^{j+1} y} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{nx} - \frac{\pi}{4}\right).$$

By Hölder’s inequality we get

$$|R_{21}(x)|^4 \ll T J^3 \sum_{0 \leq j \leq J} \left| \sum_{2^j y < n \leq 2^{j+1} y} \frac{d(n)}{n^{3/4}} \cos\left(4\pi \sqrt{nx} - \frac{\pi}{4}\right) \right|^4.$$
Correspondingly we have

\[ \int_T^{T+H} |R_{21}(x)|^4 dx \]

\[ \ll TJ^3 \sum_{0 \leq j \leq J} \int_T^{T+H} \left| \sum_{2^j y < n \leq 2^{j+1} y} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{n}x - \pi/4) \right|^4 dx \]

\[ \ll TJ^4 \int_T^{T+H} \left| \sum_{2^j y < n \leq 2^{j+1} y} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{n}x - \pi/4) \right|^4 dx \]

for some \( 0 \leq j \leq J \). Let \( N = 2^j y \). Thus by Lemma 3.5 we get (note that \( J \ll \log T \))

\[ \int_T^{T+H} |R_{21}(x)|^4 dx \ll T^{1+\epsilon} \int_T^{T+H} \left| \sum_{n \sim N} \frac{d(n)}{n^{3/4}} e(2\sqrt{n}x) \right|^4 dx \]

\[ = T^{1+\epsilon} \sum_{n,m,k,l \sim N} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \int_T^{T+H} e(2\eta x^{1/2}) dx \]

\[ \ll \frac{T^{1+\epsilon}}{N^3} \sum_{n,m,k,l \sim N} \min \left( H, \frac{T^{1/2}}{\eta} \right), \]

where we put \( \eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l} \). By Lemma 3.3 the contribution of \( H \) is (note that in this case \( |\eta| \leq \frac{T^{1/2}}{H} \))

\[ \ll HT^{1+\epsilon} N^{-3} (T^{1/2} H^{-1} N^{7/2} + N^2) \]

\[ \ll T^{3/2+\epsilon} N^{1/2} + HT^{1+\epsilon} N^{-1} \]

\[ \ll T^{3/2+\epsilon} z^{1/2} + HT^{1+\epsilon} y^{-1}. \]

By Lemma 3.3 again we see that the contribution of \( T^{1/2} |\eta|^{-1} \) (in this case \( |\eta| > T^{1/2} H^{-1} \)) is

\[ \ll T^{3/2+\epsilon} N^{-3} \max_{\delta \gg T^{1/2} H^{-1}} \frac{1}{\delta} \sum_{n,m,k,l \sim N} \frac{1}{\delta < |\eta| \leq 2\delta} \]

\[ \ll \max_{\delta \gg T^{1/2} H^{-1}} (T^{3/2+\epsilon} N^{1/2} + T^{3/2+\epsilon} N^{-1} \delta^{-1}) \]

\[ \ll T^{3/2+\epsilon} z^{1/2} + HT^{1+\epsilon} N^{-1} \]

\[ \ll T^{3/2+\epsilon} z^{1/2} + HT^{1+\epsilon} y^{-1}. \]
Combining (6.16)–(6.19) and noting $z \approx H^2y^{-2}T^{-1}$ we get that
\begin{equation}
\int_T^{T+H} |R_2(x)|^4 dx \ll HT^{1+\varepsilon}y^{-1} + T^{3/2+\varepsilon}z^{1/2} + T^{3/2+\varepsilon}H^{-3/2} + T^{2+\varepsilon}z^{-1} \ll HT^{1+\varepsilon}y^{-1} + T^{3+\varepsilon}y^3H^{-2}.
\end{equation}

On the other hand, since $M(x) \ll \frac{U}{4}|R_1(x+U)|$ and
\begin{equation}
\int_T^{T+H} |R_1(x)|^4 dx \ll \int_T^{2T} |R_1(x)|^4 dx \ll T^2
\end{equation}
we have
\begin{equation}
\int_T^{T+H} M^4(x) dx \ll U^4T^{-4} \int_T^{T+H} |R_1(x+U)|^4 dx \ll U^4T^{-4}T^2 \ll 1.
\end{equation}

From (6.20) and (6.21)
\begin{equation}
\int_T^{T+H} S_2^4(x) dx \ll T^{1+\varepsilon}Hy^{-1} + T^{3+\varepsilon}y^3H^{-2}.
\end{equation}

We also remark that by (2.1) the right hand side of (6.22) is bounded above by $T^{\varepsilon}HU^2$.

6.3 Proof of Theorem 1

By the elementary estimate $(a + b)^4 - a^4 \ll |a^3b| + |b|^4$ we can write
\begin{equation}
(\Delta(x + U) - \Delta(x))^4 = S_1^4(x) + O(|S_1^3(x)S_2(x)| + |S_2(x)|^4).
\end{equation}

By (6.15), (6.22) and the Hölder’s inequality we get
\begin{equation}
\int_T^{T+H} |S_1^3(x)S_2(x)| dx \ll \left( \int_T^{T+H} S_1^4(x) dx \right)^{3/4} \left( \int_T^{T+H} S_2^4(x) dx \right)^{1/4} \ll T^{\varepsilon}(HU^2)^{3/4}(HTy^{-1})^{1/4} + T^{\varepsilon}(HU^2)^{3/4}(T^3y^3H^{-2})^{1/4} \ll HT^{3/16+\varepsilon}U^{25/16} + H^{1/4}T^{15/16+\varepsilon}U^{21/16}.
\end{equation}
Now Theorem 1 follows from (6.10)-(6.14) and (6.22)-(6.24).

Finally we prove Corollary 1. Recall the definition $\Sigma_6$ in Section 6.1. Since $U \ll T^{1/2-\epsilon}$, by Lemma 4.1 we see that

\begin{equation}
\Sigma_6 \asymp HU^2 \log^6 \frac{T}{U^2}.
\end{equation}

which combining the formula (2.3) of Theorem 1 gives Corollary 1.

7 Proof of Theorem 2

In this section we prove Theorem 2. We begin with the well-known Atkin-
son’s formula (see Ivic [5, Chapter 15])

\begin{equation}
E(t) = \Sigma_1(t) + \Sigma_2(t) + O(\log^2 t),
\end{equation}

where

\begin{align*}
\Sigma_1(t) &:= \frac{1}{\sqrt{2}} \sum_{n \leq N} h(t, n) \cos(f(t, n)), \\
\Sigma_2(t) &:= -2 \sum_{n \leq N'} d(n)n^{-1/2} \left(\log \frac{t}{2\pi n}\right)^{-1/4} \cos \left( t \log \frac{t}{2\pi n} - t + \frac{\pi}{4} \right), \\
h(t, n) &:= (-1)^n d(n)n^{-1/2} \left(\frac{t}{2\pi n} + \frac{1}{4}\right)^{-1/4} (g(t, n))^{-1}, \\
g(t, n) &:= \operatorname{arsinh}\left(\frac{\pi n}{2t}\right)^{1/2}, \\
f(t, n) &:= 2t g(t, n) + (2\pi nt + \pi^2 n^2)^{1/2} - \pi/4, \\
At \leq N \leq A't, N' := t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2},
\end{align*}

where $0 < A < A'$ are any fixed constants.

We also need the error term $\Delta^*(t)$, defined by

\begin{equation}
\Delta^*(t) := \frac{1}{2} \sum_{n \leq 4t} (-1)^n d(n) - t(\log t + 2\gamma - 1).
\end{equation}

$\Delta^*(t)$ also has the truncated Voronoi’s formula (see Ivic [5] (15.68)):

\begin{align*}
\Delta^*(t) &= t^{1/4} \sum_{n \leq N} \frac{(-1)^n d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{nt} - \frac{\pi}{4} \right) + O(t^{1/2+\epsilon} N^{-1/2}),
\end{align*}
for $1 \ll N \ll t$

Let $y := c(T/U)^{1/4}$, where $c$ is a small positive constant. For $T \ll t \ll T$, define

$$E_1(t) := \frac{1}{\sqrt{2}} \sum_{n \leq y} h(t, n) \cos(f(t, n)),$$

$$E_2(t) := E(t) - E_1(t),$$

$$R_1^*(t) := \frac{t^{1/4}}{\sqrt{2\pi}} \sum_{n \leq y} \frac{(-1)^n d(n)}{n^{3/4}} \cos \left(4\pi \sqrt{nt} - \frac{\pi}{4}\right),$$

$$R_2^*(t) := \Delta^*(t) - R_1^*(t).$$

Step 1. The upper bound of $\int_T^{T+H} \mathcal{E}_2^4(t) dt$

In this subsection we shall show that

$$\int_T^{T+H} \mathcal{E}_2^4(t) dt \ll HT^{1+\varepsilon}y^{-1}. \quad (7.4)$$

Let $z := (T/y)^{1/2}$ and define

$$E_{21}(t) := \frac{1}{\sqrt{2}} \sum_{y < n \leq z} h(t, n) \cos(f(t, n)),$$

$$E_{22}(t) := E_2(t) - E_{21}(t).$$

Let $1/4 < \theta < 1/3$ be a constant such that $E(t) \ll t^\theta$. Following Ivić [5], the second author proved in [18] that

$$\int_T^{T+H} |\Delta(t)|^A dt \ll HT^{A/4+\varepsilon}$$

holds for $H \geq T^{1+\theta(A-2)-A/4}$ if $2 < A < 2\theta/(\theta-1/4)$. By the same argument we can show that the estimate

$$\int_T^{T+H} |E(t)|^A dt \ll HT^{A/4+\varepsilon}$$

holds for $H \geq T^{1+\theta(A-2)-A/4}$ if $2 < A < 2\theta/(\theta-1/4)$. A slight modification shows that if $H \geq T^{1+\theta(A-2)-A/4}$ and $2 < A < 2\theta/(\theta-1/4)$, then

$$\int_T^{T+H} |E_{22}(t)|^A dt \ll HT^{A/4+\varepsilon}. \quad (7.5)$$

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Similar to the formula (2.8) of Zhai [18], we can easily show that

\[(7.6) \quad \int_T^{T+H} |E_{22}(t)|^2 dt \ll HT^{1/2}z^{-1/2} \log^3 T.\]

We omit the proofs of the above formulas. From (7.5), (7.6) and Hölder’s inequality we get

\[(7.7) \quad \int_T^{T+H} |E_{22}(t)|^4 dt \ll HT^{1+\varepsilon}z^{-1/3} \ll HT^{1+\varepsilon}(T/y)^{-1/6} \ll HT^{1+\varepsilon}y^{-1}\]

holds in the range \(T^{205/227} \ll H \leq T\) if noting that \(T^{3/7} \ll U \ll T^{1/2}\).

Now we estimate the integral \(\int_T^{T+H} |E_{21}(t)|^4 dt\). Similar to (6.17) we have

\[(7.8) \quad \log^{-4} T \int_T^{2T} |E_{21}(t)|^4 dt \ll \int_T^{2T} \left| \sum_{n \sim N} h(t, n)e(f(t, n)) \right|^4 dt = \sum_{N < n, m, k, l \leq 2N} \int_T^{2T} H(t; n, m, k, l)e(F(t; n, m, k, l)) dt\]

holds for some \(y < N < z\), where

\[H(t; n, m, k, l) = h(t, n)h(t, m)h(t, k)h(t, l),\]

\[F(t; n, m, k, l) = f(t, n) + f(t, m) - f(t, k) - f(t, l).\]

From (7.2) it is easy to check that

\[(7.9) \quad h(t, n) = \frac{2^{3/4}}{\pi^{1/4}} \frac{(-1)^n d(n)}{n^{3/4}} \frac{1}{t^{1/4}} (1 + O(n/t)),\]

\[f(t, n) = 2^{3/2}(\pi nt)^{1/2} - \pi/4 + O(n^{3/2}t^{-1/2}),\]

\[f'(t, n) = 2^{1/2}(\pi n)^{1/2}t^{-1/2} + O(n^{3/2}t^{-3/2}).\]

Let \(\eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}\). From the third formula of (7.9) we get

\[F'(t; n, m, k, l) = (2\pi)^{1/2}\eta t^{-1/2} + O(z^{3/2}t^{-3/2}).\]
Let $C > 0$ be a large real constant such that if $|\eta| \geq Cz^{3/2}T^{-1}$ then $|F'(t; n, m, k, l)| \gg |\eta|T^{-1/2}$.

If $|\eta| \leq Cz^{3/2}T^{-1}$, then by the trivial estimate and Lemma 3.3 we get

\begin{equation}
\int_{T}^{T+H} \sum_{N < n, m, k, l \leq 2N} \frac{H(t; n, m, k, l)e(F(t; n, m, k, l))}{|\eta| \leq Cz^{3/2}T^{-1}} dt \ll \frac{HT^{1+\epsilon}}{N^3} \sum_{N < n, m, k, l \leq 2N} 1 \ll \frac{HT^{1+\epsilon}}{N^3} (Cz^{3/2}T^{-1}N^{7/2} + N^2)N^\epsilon \ll HT^{1+\epsilon}y^{-1} + Hz^{2}T^\epsilon \ll HT^{1+\epsilon}y^{-1}.
\end{equation}

Now suppose $|\eta| > Cz^{3/2}T^{-1}$. By Lemma 3.5 and 3.3 we get

\begin{equation}
\begin{split}
\int_{T}^{T+H} \sum_{N < n, m, k, l \leq 2N} H(t; n, m, k, l)e(F(t; n, m, k, l))dt &\ll \frac{T^{3/2+\epsilon}}{N^3} \sum_{N < n, m, k, l \leq 2N} \frac{1}{|\eta|} \ll \frac{T^{3/2+\epsilon}}{N^3 \delta} \sum_{N < n, m, k, l \leq 2N} 1 \\
&\ll \frac{T^{3/2+\epsilon}}{N^3 \delta}(\delta N^{7/2} + N^2)N^\epsilon \ll \frac{HT^{1+\epsilon}}{y} + \frac{T^{3/2+\epsilon}}{N\delta} \ll \frac{HT^{1+\epsilon}}{y} + \frac{T^{5/2+\epsilon}}{Nz^{3/2}} \ll \frac{HT^{1+\epsilon}}{y} + \frac{T^{7/4+\epsilon}}{y^{1/4}} \ll \frac{HT^{1+\epsilon}}{y}.
\end{split}
\end{equation}

From (7.8), (7.10) and (7.11) we get

\begin{equation}
\int_{T}^{T+H} |\mathcal{E}_{21}(t)|^4 dt \ll HT^{1+\epsilon}y^{-1} + T^{3/2+\epsilon}z^{1/2},
\end{equation}

which combining (7.7) gives (7.4).

**Step 2.** The upper bound of $\int_{T}^{T+H} (\mathcal{E}_1(t) - 2\pi \mathbb{R}_1^{*}(t/2\pi))^4 dt$
It is easy to see that
\[
2\pi R_1^* \left( \frac{t}{2\pi} \right) = \frac{(2t)^{1/4}}{\pi^{1/4}} \sum_{n \leq y} \frac{(-1)^n d(n)}{n^{3/4}} \cos \left( \frac{2^{3/2}(\pi nt)^{1/2} - \pi}{4} \right).
\]

Thus we can write
\[
E_1(t) - 2\pi R_1^* \left( \frac{t}{2\pi} \right) = S_3(t) + S_4(t),
\]
\[
S_3(t) = \sum_{n \leq y} h_1(t, n) \cos(f(t, n)),
\]
\[
h_1(t, n) = \frac{h(t, n)}{\sqrt{2}} - \frac{(2t)^{1/4} (-1)^n d(n)}{\pi^{1/4} n^{3/4}},
\]
\[
S_4(t) = \sum_{n \leq y} \frac{(2t)^{1/4} (-1)^n d(n)}{\pi^{1/4} n^{3/4}} \left( \cos(f(t, n)) - \cos \left( \frac{2^{3/2}(\pi nt)^{1/2} - \pi}{4} \right) \right).
\]

From (7.9) we have \( h_1(t, n) \ll d(n) n^{1/4} t^{-3/4} \) and \( S_3(t) \ll y^{5/4} t^{-3/4} \log y \). Thus

\[
\int_T^{T+H} |S_3(t)|^4 dt \ll y^5 H^{-3} \log^4 y \ll 1.
\] (7.13)

By the simple relation
\[
\cos(u - v) - \cos(u + v) = 2 \sin u \sin v
\]
we can write
\[
S_4(t) = \sum_{n \leq y} h_2(t, n) \sin(f_1(t, n)),
\]
\[
h_2(t, n) = \frac{(2t)^{1/4} (-1)^n d(n)}{\pi^{1/4} n^{3/4}} \sin \frac{f(t, n) - 2^{3/2}(\pi nt)^{1/4} + \pi/4}{2},
\]
\[
f_1(t, n) = \frac{f(t, n) + 2^{3/2}(\pi nt)^{1/4} - \pi/4}{2}.
\]

It is easy to check that
\[
\frac{f(t, n) - 2^{3/2}(\pi nt)^{1/4} + \pi/4}{2} = \beta_3 n^{3/2} t^{-1/2} + \beta_5 n^{5/2} t^{-3/2} + \ldots,
\]
where $\beta_3, \beta_5, \cdots$ are real constants. So for each $n \leq y$, $h_2(t, n)$ is a monotonic function of $t$ and $h_2(t, n) \ll d(n)n^{3/4}t^{-1/4}$. By (7.14) again we can write

$$|S_4(t)|^2 = \sum_{n,m \leq y} h_2(t, n)h_2(t, m) \sin(f_1(t, n)) \sin(f_2(t, m))$$

$$= S_5(t) + S_6(t) + S_7(t),$$

$$S_5(t) = \sum_{n \leq y} h_2^2(t, n) \sin^2(f_1(t, n)),$$

$$S_6(t) = \frac{1}{2} \sum_{n,m \leq y \atop n \neq m} h_2(t, n)h_2(t, m) \cos(f_1(t, n) - f_1(t, m)),$$

$$S_7(t) = \frac{1}{2} \sum_{n,m \leq y \atop n \neq m} h_2(t, n)h_2(t, m) \cos(f_1(t, n) + f_1(t, m)).$$

Trivially $S_5(t) \ll y^{5/2}T^{1/2} \log^3 y$, which implies

(7.15) $\int_T^{T+H} S_5(t)dt \ll y^{5/2}HT^{-1/2} \log^3 y.$

From the third formula of (7.9) it is easy to check that

$$|f_1'(t, n) - f_1'(t, m)| \gg |\sqrt{n} - \sqrt{m}|T^{-1/2}.$$  

By Lemma 3.5 we have

(7.16) $\int_T^{T+H} S_6(t)dt \ll \sum_{n \neq m} \frac{d(n)d(m)(nm)^{3/4}}{|\sqrt{n} - \sqrt{m}|}$

$$\ll \sum_{|\sqrt{n} - \sqrt{m}| \geq (nm)^{1/4}/100} \frac{d(n)d(m)(nm)^{3/4}}{|\sqrt{n} - \sqrt{m}|}$$

$$+ \sum_{|\sqrt{n} - \sqrt{m}| < (nm)^{1/4}/100} \frac{d(n)d(m)(nm)^{3/4}}{|\sqrt{n} - \sqrt{m}|}$$

$$\ll \sum_{n,m \leq y} d(n)d(m)(nm)^{1/2} + \sum_{n \times m \leq y} \frac{d(n)d(m)(nm)^{3/4}}{n^{1/2}|n - m|}$$

$$\ll y^3 \log^2 y + \sum_{n \times m \leq y} \frac{d^2(n)n^2}{|n - m|} \ll y^3 \log^4 y.$$  

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Similarly
\[ \int_T^{T+H} S_7(t) dt \ll \sum_{n \neq m} \frac{d(n)d(m)(nm)^{3/4}}{\sqrt{n} + \sqrt{m}} \ll y^3 \log^2 y. \] (7.17)

From (7.15)-(7.17) we get
\[ \int_T^{T+H} |S_4(t)|^2 dt \ll y^{5/2}HT^{-1/2} \log^3 y + y^3 \log^4 y \ll y^{5/2}HT^{-1/2} \log^3 y, \]
which combining the trivial estimate \( S_4(t) \ll y^{7/4}t^{-1/4} \log y \) gives
\[ \int_T^{T+H} |S_4(t)|^4 dt \ll y^6 HT^{-1} \log^3 y \ll HT \log y. \] (7.18)

So from (7.13) and (7.18) we get
\[ \int_T^{T+H} |E_1(t) - 2\pi R_1^*(t/2\pi)|^4 dt \ll HT \log y. \] (7.19)

**Step 3. Proof of Theorem 2**

For \( T \leq t \leq T + H \), we can write
\[ E(t + U) - E(t) = E_1(T + U) - E_1(t) + E_2(T + U) - E_2(t) \]
\[ = 2\pi R_1^*(\frac{t + U}{2\pi}) - 2\pi R_1^*(\frac{t}{2\pi}) + E_1(T + U) - 2\pi R_1^*(\frac{t + U}{2\pi}) \]
\[ - E_1(t) + 2\pi R_1^*(\frac{t}{2\pi}) + E_2(T + U) - E_2(t) \]
\[ = S_8(t) + S_9(t) \]
say, where
\[ S_8(t) : = 2^{1/4} \pi^{-1/4} t^{1/4} \sum_{n \leq y} \frac{(-1)^n d(n)}{n^{3/4}} \cos \left( 2^{3/2} \pi^{1/2} \sqrt{n t} - \frac{\pi}{4} \right), \]
\[ S_9(t) : = E_1(T + U) - 2\pi R_1^*(\frac{t + U}{2\pi}) - E_1(t) + 2\pi R_1^*(\frac{t}{2\pi}) \]
\[ + E_2(T + U) - E_2(t) + M^*(t), \]
\[ M^*(t) : = 2^{1/4} \pi^{-1/4} \left( (t + U)^{1/4} - t^{1/4} \right) \]
\[ \times \sum_{n \leq y} \frac{(-1)^n d(n)}{n^{3/4}} \cos \left( 2^{3/2} \pi^{1/2} \sqrt{n (t + U)} - \frac{\pi}{4} \right) \]
\[ \ll UT^{-1} |R_1^*(t + U)|. \]
Similar to (6.21), we have

$$\int_T^{T+H} M^*(t) dt \ll 1,$$

which combining (7.4) and (7.19) gives

(7.21) $$\int_T^{T+H} S^4(t) dt \ll HT^{1+\varepsilon} y^{-1}.$$ 

For $S_8(t)$, similar to (6.10) under the condition (2.1) we have the asymptotic formula

(7.22) $$\int_T^{T+H} S_8^4(t) dt = \frac{12}{\pi} \sum_{n \leq y} \frac{d(n_1)d(n_2)d(n_3)d(n_4)}{(n_1n_2n_3n_4)^{3/4}}$$

$$\times \int_T^{T+H} t \prod_{j=1}^{4} \sin \frac{U \sqrt{\pi n_j}}{(2t)^{1/2}} dx + O(HU^2T^{-1/2} \log^4 T)$$

$$+ O(HT^{3/16+\varepsilon} U^{25/16}).$$

Now Theorem 2 follows from (7.20)–(7.22).

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