The Spectrum of Mapping Ideals of Type Variable Exponent Function Space of Complex Variables with Some Applications

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Abstract

The topological and geometric behaviors of the variable exponent formal power series space, as well as the prequasi-ideal construction by \( s \)-numbers and this function space of complex variables, are investigated in this article. Upper bounds for \( s \)-numbers of infinite series of the weighted \( n \)th power forward and backward shift operator on this function space are being investigated, with applications to some entire functions.

1. Introduction

Operator ideal theory has various applications in the geometry of Banach spaces, fixed point theory, spectral theory, and other areas of mathematics, among other areas of knowledge. Throughout the article, we will adhere to the etymological conventions listed below. If any other sources are used, we will make a note of them.

The topological and geometric behaviors of the variable exponent formal power series space, as well as the prequasi-ideal construction by \( s \)-numbers and this function space of complex variables, are investigated in this article. Upper bounds for \( s \)-numbers of infinite series of the weighted \( n \)th power forward and backward shift operator on this function space are being investigated, with applications to some entire functions.

1.1. Conventions

\[ \mathbb{N} = \{0, 1, 2, \ldots\}, \mathbb{C}: \text{complex number space} \]
\[ \mathbb{R}^\mathbb{N}: \text{the space of all real sequences} \]
\[ \ell_{00}^\mathbb{N}: \text{the space of bounded real sequences} \]
\[ \ell^r: \text{the space of } r \text{-absolutely summable real sequences} \]
\[ c_0: \text{the space of null real sequences} \]
\[ c_0 = \{0, 0, \ldots, 1, 0, 0, \ldots\}, \text{as } 1 \text{ lies at the } \ell \text{th coordinate, for all } \ell \in \mathbb{N} \]
\[ \mathcal{S}: \text{the space of each sequence with finite nonzero coordinates} \]
\[ \text{card} \{\emptyset\}: \text{the number of elements of the set } \emptyset \]
\[ m_i>: \text{the space of all monotonic increasing sequences of positive reals} \]
\[ L: \text{the ideal of all bounded linear operators between any arbitrary Banach spaces} \]
\[ F: \text{the ideal of finite rank operators between any arbitrary Banach spaces} \]
\[ \Lambda: \text{the ideal of approximable operators between any arbitrary Banach spaces} \]
\[ L_c: \text{the ideal of compact operators between any arbitrary Banach spaces} \]
\[ L(\mathcal{X}, \mathcal{Y}): \text{the space of all bounded linear operators from a Banach space } \mathcal{X} \text{ into a Banach space } \mathcal{Y} \]
\[ L(\mathcal{X}): \text{the space of all bounded linear operators from a Banach space } \mathcal{X} \text{ into itself} \]
\[ F(\mathcal{X}, \mathcal{Y}): \text{the space of finite rank operators from a Banach space } \mathcal{X} \text{ into a Banach space } \mathcal{Y} \]
\[ F(\mathcal{X}): \text{the space of finite rank operators from a Banach space } \mathcal{X} \text{ into itself} \]
\[ \Lambda(\mathcal{X}, \mathcal{Y}): \text{the space of approximable operators from a Banach space } \mathcal{X} \text{ into a Banach space } \mathcal{Y} \]
\[ \Lambda(\mathcal{X}): \text{the space of approximable operators from a Banach space } \mathcal{X} \text{ into itself} \]
\[ L_c(\mathcal{X}, \mathcal{Y}): \text{the space of compact operators from a Banach space } \mathcal{X} \text{ into a Banach space } \mathcal{Y} \]
\[ L_c(\mathcal{X}): \text{the space of compact operators from a Banach space } \mathcal{X} \text{ into itself} \]
\[ (s_n(G))_{n\in\mathbb{N}}: \text{the sequence of } s \text{-numbers of the bounded linear operator } G \]
looked into the quasi-ideals $S$, showed that for any Banach spaces $X$.

### 1.2. Notations 1.2 (see [1]). $S_{\omega} = \{ S_{\omega}(X, Y) ; X$ and $Y$ are Banach Spaces $\}$, where

\[
S_{\omega}(X, Y) = \left\{ P \in L(X, Y) : f_\omega \in \mathcal{H}, \text{ where } f_\omega(z) = \sum_{n=0}^{\infty} s_{\omega}(P)z^n \text{ converges for any } z \in \mathbb{C} \right\}
\]

$S^\omega_{\omega}$: the operator ideals formed by the sequence of approximation numbers in any sequence space $V$.

$S^{Kol}_{\omega}$: the operator ideals formed by the sequence of Kolmogorov numbers in any sequence space $V$.

Several operator ideals in the class of Banach or Hilbert spaces are defined by sequences of real numbers. Let, for example, is produced by $(d_n(G))_{n \in \mathbb{N}}$ and $c_0$. Pietsch [2] looked into the quasi-ideals $S^\omega_{\omega}$, for $0 < t < \infty$. He demonstrated how $t^2$ and $t^4$ yield the ideals of Hilbert Schmidt operators and nuclear operators between Hilbert spaces, respectively. In addition, he proved that $F = S_{\omega}$, for $1 \leq t < \infty$, and $S_{\omega}$ is a simple Banach space. Pietsch [3] explained that $S_{\omega}$, where $0 < t < \infty$, is small. Makarov and Faried [4] showed that for any Banach spaces $X$ and $Y$ with dim $(X) = \dim (Y) = \infty$, then for every $r > t > 0$, one has $S^\omega_{\omega}(X, Y) \subset S^\omega_{\omega}(X, Y) \subset L(X, Y)$. The concept of prequasi-ideal was developed by Faried and Bakery [5], who elaborated on the concept of quasi-ideal. They investigated some geometric and topological properties of the spaces $S_{\omega}(t)$ and $S_{\omega}$. According to the spectral decomposition theorem [2], for $A \in L_c(H)$, where $H$ is a Hilbert space, one has $A(y) = \sum_{n=0}^{\infty} a_n \varphi_n$, where $\varphi_n$ and $\{ \gamma_n \}$ are orthonormal families in $H$. Suppose $(\gamma_n)_{n \in \mathbb{N}}$ be decreasing and $D : (\gamma_n)_{n \in \mathbb{N}} \rightarrow (t \eta_n)$ be the diagonal operator on $\ell^p$ with $p \geq 1$. Therefore, $s_{\omega}(D) = t$, Shields [6] investigated an indication to the weighted shift operators as formal power series in unilateral shifts and formal Laurent series in bilateral shifts. Hedaya
tian [7] offered the space of formal power series with power $r$, $\mathcal{R}^r((ba))$, where $(ba)$ is a sequence of positive numbers with $b_0 = 1$ and $r > 0$. By the space $\mathcal{R}^r((ba))$, he meant that the set of all formal power series $\sum_{n=0}^{\infty} f_n \zeta^n$ with $\sum_{n=0}^{\infty}$
functions.

However, Emamirad and Heshmati [8] explored the idea of functions evident on the Bargmann space by $f(z) = \sum_{n=0}^{\infty} c_n (z^n/\sqrt{a})$ with $\|f\| = \sum_{n=0}^{\infty} |c_n|^2 < \infty$, where $\{z^n/\sqrt{a} : a \in \mathbb{N}\}$ is an orthonormal basis. Faried et al. [9] introduced the upper bounds for $s$-numbers of infinite series of the weighted $n$th power forward shift operator on $\mathcal{H}''((bu))$, for $1 \leq r < \infty$, with some applications to some entire functions.

The paper is arranged as follows. In Section 3, we offer the definition of the space $\mathcal{H}''(a)$ with definite function $\rho$. We introduce the sufficient conditions on $\mathcal{H}''(a)$ to generate premodular special space of formal power series. This gives that $\mathcal{H}''(a)$ is a prequasinormed space. In Section 4, firstly, we give the sufficient conditions on $\mathcal{H}''(a)$ such that the class $S(\mathcal{H}''(a))$ generates an operator ideal. Secondly, we explain enough settings (not necessary) on $\mathcal{H}''(a)$ so that $F = S(\mathcal{H}''(a))$. This shows the nonlinearity of $s$-type ($\mathcal{H}''(a)$) spaces which gives an answer of Rhoades [10] open problem. Thirdly, we investigate the conditions on $\mathcal{H}''(a)$ such that the prequasi-ideal $S(\mathcal{H}''(a))$, are Banach and closed. Fourthly, we examine the sufficient conditions on $\mathcal{H}''(a)$ such that $S(\mathcal{H}''(a))$ is strictly contained for different powers. We show the smallness of $S(\mathcal{H}''(a))$. Fifthly, we investigate the simplicity of $S(\mathcal{H}''(a))$. Sixthly, we present the enough setup on $\mathcal{H}''(a)$ such that the class $L$ with its sequence of eigenvalues in $\mathcal{H}''(a)$ equals $S(\mathcal{H}''(a))$. In Section 5, we estimate the upper bounds for $s$-numbers of infinite series of the weighted $n$th power forward and backward shift operator on $\mathcal{H}''(a)$ with approaches to some entire functions.

2. Definitions and Preliminaries

Definition 1 (see [11]). A function $s : L(\mathcal{X}, \mathcal{Y}) \to [0, \infty)^\mathbb{N}$ is called an $s$-number, if the sequence $(s_n(B))_{n=0}^\infty$, for all $B \in L(\mathcal{X}, \mathcal{Y})$, shows the following settings:

(a) If $B \in L(\mathcal{X}, \mathcal{Y})$, then $\|B\| = s_0(B) \geq s_1(B) \geq s_2(B) \geq \cdots \geq 0$
(b) $s_{k+n-1}(B_1 + B_2) \leq s_k(B_1) + s_n(B_2)$, for every $B_1, B_2 \in L(\mathcal{X}, \mathcal{Y}), b, a \in \mathbb{N}$
(c) The inequality $s_k(ABD) \leq \|A\|s_k(B)\|D\|$ holds, if $D \in L(\mathcal{X}_0, \mathcal{X}), B \in L(\mathcal{X}, \mathcal{Y})$, and $A \in L(\mathcal{Y}, \mathcal{Y}_0)$, where $\mathcal{X}_0$ and $\mathcal{Y}_0$ are arbitrary Banach spaces
(d) Suppose $A \in L(\mathcal{X}_0, \mathcal{X})$ and $\lambda \in \mathbb{R}$, then $s_\lambda(\lambda A) = |\lambda| s_\lambda(A)$
(e) Let $\text{rank}(A) \leq b$ then $s_\lambda(A) = 0$, whenever $A \in L(\mathcal{X}_0, \mathcal{X})$

(f) Assume $I_\lambda$ indicates the identity operator on the $\lambda$-dimensional Hilbert space $\ell^2_\lambda$, then $s_{\lambda n}(I_\lambda) = 0$ or $s_{\lambda n}(I_\lambda) = 1$

Consider the following examples of $s$-numbers:

(i) The $b$th approximation number, $a_b(A)$, where

$$a_b(A) = \inf \{\|A - B\| : B \in L(X, Y) \text{ and rank } (B) \leq b\}$$

(ii) The $b$th Kolmogorov number, $d_b(A)$, where

$$d_b(A) = \inf_{\text{dim } Y \leq \|A\|} \sup_{v \in Y} \inf_{\|u\| = 1} \|Au - v\|.$$
Theorem 8 (see [5]). Suppose \( g \) is a quasinorm on the ideal \( U \), then \( g \) is a prequasinorm on the ideal \( U \).

Theorem 9 (see [12]). Assume \( s \)-type \( \mathcal{V} \nu := \{ f = (sr(T)) \in \mathbb{R}^N : T \in L(\mathcal{X}, \mathcal{Y}) \text{and} \nu(f) < \infty \} \). If \( \mathcal{S} \nu \) is an operator ideal, then we have

1. \( \mathcal{S} \subset s \)-type \( \mathcal{V} \nu \)

2. Assume \( (s_r(T))_{r \in \mathbb{N}} \in s \)-type \( \mathcal{V} \nu \) and \( (s_r(T)_{r \in \mathbb{N}}) \in s \)-type \( \mathcal{V} \nu \), then \( (s_r(T + T))_{r \in \mathbb{N}} \in s \)-type \( \mathcal{V} \nu \)

3. Suppose \( \lambda \in \mathbb{R} \) and \( (s_r(T))_{r \in \mathbb{N}} \in s \)-type \( \mathcal{V} \nu \), then \( |\lambda| \)

4. The sequence space \( \mathcal{V} \nu \) is solid, i.e., when \( (s_r(G))_{r \in \mathbb{N}} \in s \)-type \( \mathcal{V} \nu \) and \( s_r(T) \leq s_r(G) \), for every \( r \in \mathbb{N} \) and \( T \), \( G \in L(\mathcal{X}, \mathcal{Y}) \), then \( (s_r(T))_{r \in \mathbb{N}} \in s \)-type \( \mathcal{V} \nu \).

Lemma 10 (see [13]). If \( \{ \xi_i \}_{i \in \mathbb{N}} \) is a bounded family of \( \mathbb{R} \). We have

\[
\sup_{\text{card } (G) = \infty} \inf_{i \in G} \xi_i = \inf_{\text{card } (G) = \infty} \sup_{i \in G} \xi_i. \tag{8}
\]

Lemma 11 (see [14]). If \( (r_n), (a_n) \in \mathbb{R}_+^\infty \) and \( (a_n) \in (0, \infty)_N \), with \( \mathcal{K} = \max \{ 1, 2^{n+1} \} \) and \( \mathcal{K} = \max \{ 1, 1, 2^{n+1} \} \), then

\[
|a_n| \leq K(|r_n| + |t_n|). \tag{9}
\]

Definition 12 (see [1]). The linear space of formal power series

\[
\mathcal{H} = \left\{ f : f(z) = \sum_{m=0}^{\infty} f_n z^n \text{ converges for any } z \in \mathcal{C} \right\}, \tag{10}
\]

is called a special space of formal power series (or in short (ssfps)), if it shows the following settings:

1. \( e^{(m)} \in \mathcal{H} \), for all \( m \in \mathbb{N} \), where \( e^{(m)}(z) = \sum_{n=0}^{\infty} e^{\infty}_n z^n = z^n \)

2. If \( g \in \mathcal{H} \) and \( \sum_{n=0}^{\infty} \theta_n |z|^n \), for all \( n \in \mathbb{N} \), then \( f \in \mathcal{H} \)

3. Suppose \( f \in \mathcal{H} \), then \( f_{[\lambda]} \in \mathcal{H} \), where \( f_{[\lambda]}(z) = \sum_{m=0}^{\infty} f_{\lambda} z^\lambda \) and \( \lfloor b \rfloor \) marks the integral part of \( b/2 \)

Theorem 13 (see [1]). If \( \mathcal{H} \) is a (ssfps), then \( S_\mathcal{H} \) is an operator ideal.

By \( \overline{S} \), we explain the space of finite formal power series, i.e. for \( f \in \overline{S} \), one has \( l \in \mathbb{N} \) with \( f(z) = \sum_{m=0}^{l} f_n z^n \).

Definition 14 (see [1]). A subspace \( \mathcal{H} \rho \) of the (ssfps) is called a premodular (ssfps), if there is a function \( \rho : \mathcal{H} \to [0, \infty) \) verifies the next conditions:

(i) For \( f \in \mathcal{H} \), we have \( \rho(f) \geq 0 \) and \( f = 0 \iff \rho(f) = 0 \),

(ii) Suppose \( f \in \mathcal{H} \) and \( \lambda \in \mathbb{R} \), then there is \( l \geq 1 \) with

\[
\rho(\lambda f) = |\lambda| \rho(f)
\]

(iii) Let \( f, g \in \mathcal{H} \), then there is \( K \geq 1 \) such that \( \rho(f + g) \leq K(\rho(f) + \rho(g)) \)

(iv) Suppose \( |f| \leq |g| \), for every \( b \in \mathbb{N} \), then \( \rho(f) \leq \rho(g) \)

(v) There is \( K_0 \geq 1 \) so that \( \rho(f) \leq K_0 \rho(\xi) \).

(vi) \( \overline{\mathcal{H}} = \mathcal{H} \rho \)

(vii) one has \( \xi > 0 \) with \( \rho(\lambda e^{(0)}) \geq \xi |\lambda| \rho(e^{(0)}) \), where \( \lambda \in \mathbb{R} \)

Note that the continuity of \( \rho(f) \) at \( 0 \) comes from condition (ii). Condition (1) in Definition 12 and condition (vii) in Definition 14 investigate that \( (e^{(m)})_{m=0}^{\infty} \in \mathcal{H} \rho \) is a Schauder basis of \( \mathcal{H} \rho \).

The (ssfps) \( \mathcal{H} \rho \) is called a prequasinormed (ssfps) if \( \rho \) shows the conditions (i)–(iii) of Definition 14, and if the space \( H \) is complete under \( \rho \), then \( \mathcal{H} \rho \) is called a prequasi-Banach (ssfps).

Theorem 15 (see [1]). Every premodular (ssfps) \( \mathcal{H} \rho \) is a prequasinormed (ssfps).

Definition 16 (see [1]). Assume \( \mathcal{H} \rho \) is a prequasinormed (ssfps). An operator \( V_z : \mathcal{H} \rho \to \mathcal{H} \rho \) is called forward shift, if \( V_z f = z f \), for all \( f \in \mathcal{H} \rho \), where \( V_z f(z) = \sum_{n=0}^{\infty} f_{n+1} z^n \) converges for every \( z \in \mathcal{C} \) and \( \rho(V_z f) < \infty \).

Definition 17 (see [1]). Suppose \( \mathcal{H} \rho \) is a prequasinormed (ssfps). An operator \( B_z : \mathcal{H} \rho \to \mathcal{H} \rho \) is called backward shift, if \( B_z f(z) = (f(z) - f(0))/z \), for all \( f \in \mathcal{H} \rho \), where \( B_z f(z) = \sum_{n=0}^{\infty} f_{n+1} z^n \) converges for every \( z \in \mathcal{C} \) and \( \rho(B_z f) < \infty \).

Definition 18 (see [9]). By using the power series of an entire function \( g(z) = \sum_{m=0}^{\infty} a_m z^m \), the shift operator \( V_{g(z)} \) is defined as

\[
V_{g(z)}(f(z)) = \left( \sum_{m=0}^{\infty} a_m V_z^m \right)(f(z)). \tag{11}
\]

Definition 19 (see [9]). By using the power series of an entire function \( g(z) = \sum_{m=0}^{\infty} a_m z^m \), the shift operator \( B_{g(z)} \) is defined as

\[
B_{g(z)}(f(z)) = \left( \sum_{m=0}^{\infty} a_m B_z^m \right)(f(z)). \tag{12}
\]
3. Main Results

3.1. The Space of Functions ($\mathcal{H}_{p(\cdot)}$). We define in this section the space ($\mathcal{H}_{p(\cdot)}$) under the function $p$ and give enough conditions on it to create pre-modular (ssfps) which implies that is a prequasi-Banach (ssfps).

If $p = (p_v)_{v \in \mathbb{N}}$, we define the new space of functions:

$$\mathcal{H}_{p(\cdot)} = \left\{ f : f(z) = \sum_{v=0}^{\infty} f_v z^v \text{ converges for any } z \in \mathbb{C} \text{ and } \rho(f) < \infty, \text{for some } \zeta > 0 \right\},$$

(13)

where

$$\rho(f) = \sum_{v=0}^{\infty} \frac{1}{p_v} |f_v|^p_v.$$

(14)

$$\mathcal{H}_{p(\cdot)} = \left\{ f : f(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \text{ converges for any } z \in \mathbb{C} \text{ and } \rho(\hat{f}z^\zeta) < \infty, \text{for some } \zeta > 0 \right\}$$

(15)

If $(p_v) \in \ell_\infty$, one has

**Theorem 20.** Consider $(p_v) \in \ell_\infty \cap \ell_\infty$ with $p_0 > 0$, one has ($\mathcal{H}_{p(\cdot)}$) is a premodular Banach (ssfps).

Proof (1-i). Let $f, g \in \mathcal{H}_{p(\cdot)}$. Therefore, $f(z) = \sum_{v=0}^{\infty} f_v z^v$ and $g(z) = \sum_{v=0}^{\infty} g_v z^v$ converge for any $z \in \mathbb{C}$. Then, $(f + g)(z) = \sum_{v=0}^{\infty} (f_v + g_v) z^v$ converges for any $z \in \mathbb{C}$. From $(p_v) \in \ell_\infty$, we have

$$\sum_{v=0}^{\infty} (1/p_v) |f_v + g_v|^p_v \leq K(\sum_{v=0}^{\infty} (1/p_v) |f_v|^p_v + \sum_{v=0}^{\infty} (1/p_v) |g_v|^p_v) < \infty,$$

so $f + g \in \mathcal{H}_{p(\cdot)}$.

(1-ii) Let $\lambda \in \mathbb{R}$ and $f \in \mathcal{H}_{p(\cdot)}$. Therefore, $f(z) = \sum_{v=0}^{\infty} f_v z^v$ converges for any $z \in \mathbb{C}$. Then, $(\lambda f)(z) = \sum_{v=0}^{\infty} \lambda f_v z^v$ converges for any $z \in \mathbb{C}$. From $(p_v) \in \ell_\infty$, we have

$$\sum_{v=0}^{\infty} (1/p_v) |\lambda f_v|^p_v \leq |\lambda|^p \sum_{v=0}^{\infty} (1/p_v) |f_v|^p_v < \infty.$$

So $\lambda f \in \mathcal{H}_{p(\cdot)}$. Therefore, from conditions (1-i) and (1-ii), the space $\mathcal{H}_{p(\cdot)}$ is linear. To prove $e^{(m)} \in \mathcal{H}_{p(\cdot)}$,

for all $m \in \mathbb{N}$, where $e^{(m)}(z) = \sum_{v=0}^{\infty} e^{(m)} f_v z^v = z^m$ and $\sum_{v=0}^{\infty} (1/p_v) |e^{(m)} f_v|^p_v = 1/p_m$.

(2) Assume $|f_v| \leq |g_v|$, for all $v \in \mathbb{N}$ and $g \in \mathcal{H}_{p(\cdot)}$.

Then, converges for any $z \in \mathbb{C}$

One has

$$\sum_{v=0}^{\infty} \frac{1}{p_v} |f_v|^p_v \leq \sum_{v=0}^{\infty} \frac{1}{p_v} |g_v|^p_v < \infty.$$
So, \( f(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \) and \( z \in \mathbb{C} \) and \( \rho(f) < \infty \). Hence, \( f \in \mathcal{H}_p(\cdot) \).

(3) Let \( f \in \mathcal{H}_p(\cdot) \) and \((p_v) \in \ell_\infty \cap \ell_\infty \) with \( p_0 > 0 \). Then, \( f(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \) converges for any \( z \in \mathbb{C} \).

and \( \rho(f) < \infty \). One has

\[
\rho (f^{(i)}) = \frac{\sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v}{\sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v} = \frac{\sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v}{\sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v} \leq 2 \sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v = 2\rho(f)
\]

(17)

Hence, \( f^{(i)}(z) = \sum_{v=0}^{\infty} \hat{f}_v z^v \) converges for any \( z \in \mathbb{C} \) and \( \rho(f^{(i)}) < \infty \). Then \( f^{(i)} \in \mathcal{H}_p(\cdot) \).

(i) Obviously, if \( f \in \mathcal{H}_p(\cdot) \), one gets \( \rho(f) \geq 0 \) and \( \rho(f) = 0 \Leftrightarrow f = \theta \).

(ii) There is \( l = \max \{1, \sup \eta^p \} \geq 1 \), for all \( \eta \in \mathbb{R} \setminus \{0\} \) and \( l \geq 1 \), for \( \eta = 0 \) so that

\[
\rho(\eta f) = \sum_{v=0}^{\infty} 1 |\eta f[v+2, v]|^p_v \leq \sup \eta^p \sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v \leq l \sup \rho(f),
\]

(18)

for all \( f \in \mathcal{H}_p(\cdot) \).

(iii) There is \( K = \max \{1, 2 \} \\sup \eta^p \) \geq 1 \) so that

\[
\rho(\eta f) = \sum_{v=0}^{\infty} 1 |\eta f[v+2, v]|^p_v \leq \sup \eta^p \sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v \leq l \eta \rho(f),
\]

(19)

for every \( f, g \in \mathcal{H}_p(\cdot) \).

(iv) Obviously from the proof part (2).

(v) From the proof part (3), one has \( K_0 = 2 \geq 1 \).

(vi) Clearly, \( \hat{\mathcal{H}} = \mathcal{H}_p(\cdot) \).

(vii) One has \( \zeta = 0 < \zeta \leq \eta \\sup \eta^p \) with \( \rho(\eta^p e^{(i)}) \geq \zeta |\eta| \rho(e^{(i)}) \), for each \( \eta \neq 0 \) and \( \zeta > 0 \), when \( \eta = 0 \). Therefore, the space \( (\mathcal{H}_p(\cdot))_\rho \) is a prequasi-Banach (ssfps). To show that \( (\mathcal{H}_p(\cdot))_\rho \) is a premodular Banach (ssfps), we suppose \( f^{(i)} \) to be a Cauchy sequence in \( (\mathcal{H}_p(\cdot))_\rho \), then for every \( \epsilon \in (0, 1) \), there is \( i_0 \in \mathbb{N} \) such that for all \( i, j \geq i_0 \), one gets

\[
\rho \left( f^{(i)} - f^{(j)} \right) = \sum_{v=0}^{\infty} 1 |f[v+2, v]|^p_v < \epsilon^p \epsilon
\]

(20)

For \( i, j \geq i_0 \) and \( v \in \mathbb{N} \), we have

\[
\left| f^{(i)} - f^{(j)} \right| < \epsilon.
\]

(21)

So, \( f^{(i)} = \sum_{v=0}^{\infty} \hat{f}_v z^v \) is a Cauchy sequence in \( \mathbb{R} \), for fixed \( v \in \mathbb{N} \), hence \( \lim_{i, j \to \infty} f^{(i)} = f^{(0)} \), for fixed \( v \in \mathbb{N} \).

Therefore, \( \rho(f^{(i)} - f^{(0)}) \leq \epsilon \), for every \( i \geq i_0 \). Finally, to show that \( f^{(0)} \in \mathcal{H}_p(\cdot) \), we have

\[
\rho(f^{(0)}) = \rho(f^{(i)} - f^{(0)} + f^{(0)}) \leq K \left( \rho(f^{(i)} - f^{(0)}) + \rho(f^{(0)}) \right) < \infty.
\]

(22)

Hence, \( f^{(0)} \in \mathcal{H}_p(\cdot) \). Then, the space \( (\mathcal{H}_p(\cdot))_\rho \) is a premodule Banach (ssfps).

In view of Theorems 15 and 20, we conclude the following theorem.

\( \square \)

Theorem 21. If \( (p_v) \in \ell_\infty \cap \ell_\infty \) with \( p_0 > 0 \), then the space \( (\mathcal{H}_p(\cdot))_\rho \) is a prequasi-Banach (ssfps), where

\[
\rho(f) = \sum_{v=0}^{\infty} \left( |f[v+2, v]|^p_v \right) \text{for all } f \in \mathcal{H}_p(\cdot).
\]

Theorem 22. Suppose \( (p_v) \in \ell_\infty \cap \ell_\infty \) with \( p_0 > 0 \), one has \( (\mathcal{H}_p(\cdot))_\rho \) is a prequasiclosed (ssfps), where

\[
\rho(f) = \sum_{v=0}^{\infty} \left( |f[v+2, v]|^p_v \right) \text{for all } f \in \mathcal{H}_p(\cdot).
\]

(23)

Proof. According to Theorem 21, the space \( (\mathcal{H}_p(\cdot))_\rho \) is a prequasinormed (ssfps). To explain that \( (\mathcal{H}_p(\cdot))_\rho \) is a prequasiclosed (ssfps), let \( \{f^{(i)}\}_{i=0}^{\infty} \in (\mathcal{H}_p(\cdot))_\rho \) and \( \lim_{i \to \infty} \rho(f^{(i)} - f^{(0)}) = 0 \), we have for all \( \epsilon \in (0, 1) \), there is \( i_0 \in \mathbb{N} \) such that for all \( i \geq i_0 \), one gets

\[
\epsilon > \rho \left( f^{(i)} - f^{(0)} \right) = \left[ \sum_{v=0}^{\infty} \left( |f[v+2, v]|^p_v \right) \right]^{1/\rho}\epsilon
\]

(24)

So, for \( i \geq i_0 \) and \( a \in \mathbb{N} \), we have \( |\hat{f}_{a}^{(i)} - \hat{f}_{a}^{(0)}| < \epsilon \). Therefore, \( (\hat{f}_{a}^{(i)}) \) is a convergent sequence in \( \mathbb{R} \), for fixed \( a \in \mathbb{N} \).

Then, \( \lim_{i \to \infty} \hat{f}_{a}^{(i)} = \hat{f}_{a}^{(0)} \) for fixed \( a \in \mathbb{N} \). Finally to prove
that $f^{(0)} \in (\mathcal{H}_{\rho})_p$, we have
\[
p\left(f^{(0)}\right) = p\left(f^{(0)} - f^{(i)} + f^{(i)}\right) \leq p\left(f^{(i)} - f^{(0)}\right) + p\left(f^{(i)}\right) < \infty,
\]
(25)
this gives $f^{(0)} \in (\mathcal{H}_{\rho})_p$ which shows that $(\mathcal{H}_{\rho})_p$ is a prequasiclosed (ssfps).

4. Properties of Operator Ideal

Throughout this section, some geometric and topological properties of the prequasi-ideals formed by $s$-numbers and $(\mathcal{H}_{\rho})_p$ are presented.

4.1. Ideal of Finite Rank Operators. In this part, enough settings (not necessary) on $(\mathcal{H}_{\rho})_p$ so that $\mathcal{F} = S(\mathcal{H}_{\rho})_p$ are given. This explains the nonlinearity of the $s$-type $(\mathcal{H}_{\rho})_p$ spaces (Rhoades open problem [10]).

In view of Theorems 13 and 20, we conclude the next theorem.

Theorem 23. Consider $(p_s) \in m \cap \ell_{\infty}$ with $p_s > 0$, then $S(\mathcal{H}_{\rho})_p$ is an operator ideal.

Theorem 24. If $(p_m) \in m \cap \ell_{\infty}$ with $p_m > 0$, then $\mathcal{F} = S(\mathcal{H}_{\rho})_p$, where
\[
\rho(f) = \sum_{v=0}^{\infty} \frac{1}{v!} \left| f^{(v)} \right|, \text{ for every } f \in (\mathcal{H}_{\rho})_p.
\]

Proof. Clearly, $\overline{F(\mathcal{X}, \mathcal{Y})} \subseteq S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$, since the space $S(\mathcal{H}_{\rho})_p$ is an operator ideal. Therefore, we have to show that $S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y}) \subseteq \overline{F(\mathcal{X}, \mathcal{Y})}$. By letting $T \in S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$, then, $f_s \in (\mathcal{H}_{\rho})_p$ with $f_s(z) = \sum_{v=0}^{\infty} \frac{1}{v!} s_v(T) z^v$ converges for any $z \in \mathcal{C}$. So, $\rho(f_s) < \infty$, fix $\in (0, 1)$, we have $m \in \mathbb{N} - \{0\}$ with $\rho(f_s - \sum_{v=0}^{m-1} s_v(T) v^v) < \varepsilon/4$. As $(s_v(T))_{v \in \mathbb{N}}$ is decreasing, we have
\[
\sum_{v=m+1}^{2m} \frac{1}{v} \left(\sum_{v=m+1}^{2m} \frac{1}{v} \left| f_s(v)(T) \right| \right) \leq \sum_{v=m+1}^{2m} \frac{1}{v} \left| f_s(T) \right| \leq \sum_{v=m+1}^{2m} \frac{1}{v} \left| f_s(T) \right| ^v < \varepsilon/4.
\]
(27)
Therefore, we have $A \in F_{2m}(\mathcal{X}, \mathcal{Y})$, rank $A \leq 2m$ and
\[
\sum_{v=2m+1}^{3m} \frac{1}{v} \left| T - A \right| ^v \leq \sum_{v=2m+1}^{3m} \frac{1}{v} \left| T - A \right| ^v < \varepsilon/4.
\]
(28)
As $(p_s) \in \ell_{\infty}$, then
\[
\sum_{v=0}^{m} \frac{1}{v!} \left| T - A \right| ^v < \varepsilon/4.
\]
(29)
Since $T - A \in S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$, then $h_z \in (\mathcal{H}_{\rho})_p$, where $h_z(z) = \sum_{v=0}^{\infty} \frac{1}{v!} (T - A) z^v$ converges for any $z \in \mathcal{C}$. Because $(p_z)$ is increasing and from the inequalities (27)–(29), we get
\[
d(T, A) = p(h_z) = \sum_{v=0}^{3m-1} \frac{1}{v!} (s_v(T - A)) ^v + \sum_{v=3m+1}^{\infty} \frac{1}{v!} (s_v(T - A)) ^v
\]
\[
\leq \sum_{v=0}^{3m-1} \frac{1}{v!} \left| T - A \right| ^v + \sum_{v=3m+1}^{\infty} \frac{1}{v!} (s_v(T - A)) ^v < \varepsilon.
\]
(30)
Since $I_x S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$ but the condition $(p_s) \in m \cap \ell_{\infty}$ is not verified which explain a negative example of the converse statement. This finishes the proof.

We can reformulate Theorem 24 as follows: if $(p_m) \in m \cap \ell_{\infty}$ with $p_m > 0$, then every compact operators can be approximated by finite rank operators and the converse is not always true.

4.2. Banach and Closed Prequasi-Ideal. In this part, enough settings on $(\mathcal{H}_{\rho})_p$ so that the prequasioperator ideal $S(\mathcal{H}_{\rho})_p$ is Banach and closed are investigated.

Theorem 25. Assume $(p_m) \in m \cap \ell_{\infty}$ with $p_m > 0$, then the function $g(P) = \rho(f_s)$ is a prequasinorm on $(\mathcal{H}_{\rho})_p$, where $f_s(z) = \sum_{v=0}^{\infty} s_v(T) z^v$ converges for any $z \in \mathcal{C}$ and
\[
\rho(f_s) = \sum_{v=0}^{\infty} \frac{1}{v!} s_v(T) ^v, \text{ for every } f_s \in (\mathcal{H}_{\rho})_p.
\]
(31)

Proof. One has $g$ verifies the next setups:

(1) Let $P \in S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$, $g(P) = \rho(f_s) \geq 0$ and $g(P) = \rho(f_s) = 0 \Rightarrow s_v(P) = 0$, for all $v \in \mathbb{N} \Rightarrow P = 0$.

(2) There is $l \geq 1$ with $g(P \lambda) = \rho(f_s \lambda) = l \lambda \rho(f_s) = l \lambda \left| g(P) \right|$, for every $P \in S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$ and $\lambda \in \mathbb{R}$.

(3) One has $K_0 \geq 1$, for $P_1, P_2 \in S(\mathcal{H}_{\rho})_p(\mathcal{X}, \mathcal{Y})$, then $f_1(z) = \sum_{v=0}^{\infty} s_v(P_1) z^v$ and $f_2(z) = \sum_{v=0}^{\infty} s_v(P_2) z^v$ converge for any $z \in \mathcal{C}$. Therefore, for $h_z(z) = \sum_{v=0}^{\infty} s_v(P_1 + P_2) z^v$, one has
\[
g(P_1 + P_2) = \rho(h_z) \leq \rho\left(f_1|_{\ell_{\infty}} + f_2|_{\ell_{\infty}}\right) \leq K \left| g\left(f_1|_{\ell_{\infty}}\right)\right| + p\left(f_2|_{\ell_{\infty}}\right) \leq KK_0 (g(P_1) + g(P_2))
\]
(32)
(4) We have \( C \geq 1 \), let \( A \in L(\mathfrak{X}, \mathfrak{X}), B \in S(\mathfrak{X}, \mathfrak{Y}) \) and \( D \in L(\mathfrak{Y}, \mathfrak{Y}) \). Then, \( f_s(z) = \sum_{m=0}^{\infty} s_m(B)z^m \) converges for all \( z \in \mathbb{C} \). Therefore, for \( h_s(z) = \sum_{m=0}^{\infty} s_m(D)z^m \) converges for any \( z \in \mathbb{C} \) and

\[
g(DBA) = \rho(h_s) \leq \rho(||A|| ||D|| f_s) \leq C ||A|| \rho(B) ||D||
\]

(33)

\[\square\]

**Theorem 26.** Assume \( \mathfrak{X} \) and \( \mathfrak{Y} \) are Banach spaces, and \((p_\nu) \in m_i \cap \ell_\infty \) with \( p_\nu > 0 \), then \((S(\mathfrak{X}, \mathfrak{Y}), \| g \|)\) is a quasi-Banach operator ideal, where \( g(P) = \rho(f_s), h_s(z) = \sum_{m=0}^{\infty} s_m(P)z^m \) converges for any \( z \in \mathbb{C} \) and

\[
p(f_s) = \sum_{m=0}^{\infty} s_m(P)z^m, \text{ for every } f_s \in \mathfrak{H}(p) \tag{34}
\]

**Proof.** As \((p_\nu) \in m_i \cap \ell_\infty \) with \( p_\nu > 0 \), one has the function \( g(P) = \rho(f_s) \) is a quasi-norm on \( S(\mathfrak{X}, \mathfrak{Y}) \). Let \((P_m)\) be a Cauchy sequence in \( S(\mathfrak{X}, \mathfrak{Y}) \). Therefore, \( f_s^{(m)} \in (\mathfrak{H}(p), \rho) \) and \( f_s^{(m)}(z) = \sum_{m=0}^{\infty} s_m(P_m)z^m \) converges for any \( z \in \mathbb{C} \). Suppose \( h_s(z) = \sum_{m=0}^{\infty} s_m(P_i - P_j)z^m \), then from parts (iv) and (vii) of Definition 14 and since \( L(\mathfrak{X}, \mathfrak{Y}) \supseteq S(\mathfrak{X}, \mathfrak{Y}) \), we have

\[
g(P_i - P_j) = \rho(h_s) \geq \rho\left( s_0(P_i - P_j)e^{(0)} \right) = \rho\left( ||P_i - P_j||e^{(0)} \right)
\]

\[
\geq \xi ||P_i - P_j|| \rho\left( e^{(0)} \right),
\]

then \((P_m)_{m \in \mathbb{N}}\) is a Cauchy sequence in \( L(\mathfrak{X}, \mathfrak{Y}) \). Since the space \( L(\mathfrak{X}, \mathfrak{Y}) \) is a Banach space, there is \( P \in L(\mathfrak{X}, \mathfrak{Y}) \) with \( \lim_{m \to \infty} ||P_i - P_j|| = 0 \) and as \( f_s^{(m)} \in (\mathfrak{H}(p), \rho) \), for every \( m \in \mathbb{N} \). Hence, by using Theorem 25 and the continuity of \( \rho \) at \( \theta \), we have

\[
g(P) = g(P - P_m + P_m) \leq KK_0(g(P_m - P) + g(P_m))
\]

\[
= KK_0\rho\left( ||P_m - P|| \sum_{m=0}^{\infty} e^{(m)} \right) + KK_0 \rho\left( f_s^{(m)} \right) < \varepsilon,
\]

(35)

so \( f_s \in (\mathfrak{H}(p), \rho) \), which implies \( P \in S(\mathfrak{X}, \mathfrak{Y}) \). \( \square \)

**Theorem 27.** Suppose \( \mathfrak{X} \) and \( \mathfrak{Y} \) are Banach spaces, and \((p_\nu) \in m_i \cap \ell_\infty \) with \( p_\nu > 0 \), then \((S(\mathfrak{X}, \mathfrak{Y}), \| g \|)\) is a quasi-Banach operator ideal, where \( g(P) = \rho(f_s), h_s(z) = \sum_{m=0}^{\infty} s_m(P)z^m \) converges for any \( z \in \mathbb{C} \) and

\[
\rho(f_s) = \sum_{m=0}^{\infty} s_m(P)z^m, \text{ for every } f_s \in \mathfrak{H}(p) \tag{37}
\]

**Proof.** As \((p_\nu) \in m_i \cap \ell_\infty \) with \( p_\nu > 0 \), so the function \( g(P) = \rho(f_s) \) is a quasi-norm on \( S(\mathfrak{X}, \mathfrak{Y}) \). Let \( P_m \in S(\mathfrak{X}, \mathfrak{Y}) \), with \( m \in \mathbb{N} \) and \( \lim_{m \to \infty} g(P_m - P) = 0 \). Then, \( f_s^{(m)} \in (\mathfrak{H}(p), \rho) \) and \( f_s^{(m)}(z) = \sum_{m=0}^{\infty} s_m(P_m)z^m \) converges for any \( z \in \mathbb{C} \). Suppose \( h_s(z) = \sum_{m=0}^{\infty} s_m(P_i - P_j)z^m \), then from parts (iv) and (vii) of Definition 14 and since \( L(\mathfrak{X}, \mathfrak{Y}) \supseteq S(\mathfrak{X}, \mathfrak{Y}) \), one obtains

\[
g(P - P_j) = \rho(h_s) \geq \rho\left( s_0(P - P_j)e^{(0)} \right) = \rho\left( ||P - P_j||e^{(0)} \right)
\]

\[
\geq \xi ||P - P_j|| \rho\left( e^{(0)} \right),
\]

(38)

then \((P_m)_{m \in \mathbb{N}}\) is a convergent sequence in \( L(\mathfrak{X}, \mathfrak{Y}) \). Since the space \( L(\mathfrak{X}, \mathfrak{Y}) \) is a Banach space, then there is \( P \in L(\mathfrak{X}, \mathfrak{Y}) \) with \( \lim_{m \to \infty} ||P_m - P|| = 0 \) and as \( f_s^{(m)} \in (\mathfrak{H}(p), \rho) \), for every \( m \in \mathbb{N} \), by using Theorem 25 and the continuity of \( \rho \) at \( \theta \), one has

\[
g(P) = g(P - P_m + P_m) \leq KK_0(g(P_m - P) + g(P_m))
\]

\[
= KK_0\rho\left( ||P_m - P|| \sum_{m=0}^{\infty} e^{(m)} \right) + KK_0 \rho\left( f_s^{(m)} \right) < \varepsilon,
\]

(39)

hence, \( f_s \in (\mathfrak{H}(p), \rho) \), which gives \( P \in S(\mathfrak{X}, \mathfrak{Y}) \). \( \square \)

According to Theorem 9, we introduce the following properties of the \( s \)-type \( (\mathfrak{H}(p), \rho) \).

**Theorem 28.** For \( s \)-type \( (\mathfrak{H}(p), \rho) := \{ (s(T)) \in \mathbb{R}^N : T \in S(\mathfrak{X}, \mathfrak{Y}) \} \). The next settings are verified.

(1) We have \( s \)-type \( (\mathfrak{H}(p), \rho) \supset \mathfrak{F} \)

(2) Suppose \( (s(T_1))_{r=0}^{\infty} \in s \)-type \( (\mathfrak{H}(p), \rho) \) and \( (s(T_2))_{r=0}^{\infty} \in s \)-type \( (\mathfrak{H}(p), \rho) \), then \( (s(T_1 + T_2))_{r=0}^{\infty} \in s \)-type \( (\mathfrak{H}(p), \rho) \)

(3) One has \( \lambda \in \mathbb{R} \) and \( (s(T))_{r=0}^{\infty} \in s \)-type \( (\mathfrak{H}(p), \rho) \), then \( |\lambda|(s(T))_{r=0}^{\infty} \in s \)-type \( (\mathfrak{H}(p), \rho) \)

(4) The \( s \)-type \( (\mathfrak{H}(p), \rho) \) is solid
4.3. Small Prequasi-Banach Ideal. We introduce here some inclusion relations concerning the space $S_{(\mathcal{F}, \rho)}$ for different $(p, \rho)$.

**Theorem 29.** Let $X$ and $Y$ be Banach spaces with $\dim(X) = \dim(Y) = \infty$, and $(p, \rho), (q, \tau) \in \mathcal{I}_{\infty}$ with $p, q > 0$ and $p < q$, for all $v \in \mathbb{N}$, we have

$$S_{(\mathcal{F}, \rho)}(X, Y) \subset S_{(\mathcal{F}, \rho)}(X, Y) \subset L(X, Y).$$

(40)

**Proof.** Assume $T \in S_{(\mathcal{F}, \rho)}(X, Y)$. Therefore, $f_{\rho} \in (\mathcal{F}, \rho)$ and $f_{\rho}(z) = \sum_{v=0}^{\infty} s_v(T)z^v$ converges for any $z \in \mathbb{C}$. Then,

$$\sum_{v=0}^{\infty} \frac{1}{p_v} (s_v(T))^{p_v} < \sum_{v=0}^{\infty} \frac{1}{q_v} (s_v(T))^{q_v} < \infty,$$

(41)

hence, $T \in S_{(\mathcal{F}, \rho)}(X, Y)$. Next, by taking $T$ with $s_v(T) = (p_v/(v+1))^{1/p_v}$, one has $T \notin S_{(\mathcal{F}, \rho)}(X, Y)$ and $T \notin S_{(\mathcal{F}, \rho)}(X, Y)$. Clearly, $S_{(\mathcal{F}, \rho)}(X, Y) \subset L(X, Y)$. Again, by choosing $s_v(T) = (q_v/(v+1))^{1/q_v}$, one has $T \notin S_{(\mathcal{F}, \rho)}(X, Y)$ and $T \notin L(X, Y)$. This finishes the proof.

In this part, we examine the sufficient setting for which $S_{(\mathcal{F}, \rho)}$ is small.

**Theorem 30.** Let $X$ and $Y$ be Banach spaces with $\dim(X) = \dim(Y) = \infty$. Assume $(p, \rho) \in \mathcal{I}_{\infty}$ with $p > 0$, then $S_{(\mathcal{F}, \rho)}$ is small.

**Proof.** Obviously, the space $(S_{(\mathcal{F}, \rho)} \cap \mathcal{G})$ generates a prequasi-Banach operator ideal, with $g(T) = \sum_{v=0}^{\infty} (1/p_v) (s_v(T))^{p_v}$. Let $S_{(\mathcal{F}, \rho)}(X, Y) = \mathcal{L}(X, Y)$. Hence, there is $C > 0$ with $g(T) \leq C ||T||$, for all $T \in L(X, Y)$. According to Dvoretzky’s theorem [15] with $r \in \mathbb{N}$, there are quotients spaces $X/\lambda_r$ and subspaces $\eta_r$ of $X/\lambda_r$ that are isomorphic to $C^1/\lambda_r$, $B_r$, with $||D^1||/||D^1|| \leq 2$ and $||B_r||/||B_r|| \leq 2$. Suppose $I_r$ be the identity operator on $C^1/\lambda_r$ and $f_r$ be the quotient operator from $X$ onto $X/\lambda_r$ and $f_r$ be the natural embedding operator from $f_r$ into $Y$. Let $h_\alpha$ be the Bernstein numbers [16], we have

$$1 = h_{\alpha}(I_r) = h_{\alpha}(B_rI_r^1D_rI_r^1) \leq ||B_r|| h_{\alpha}(B_rI_r^1D_rI_r^1) ||D_r^1|| $$

$$= ||B_r|| h_{\alpha}(B_rI_r^1D_rI_r^1) ||D_r^1|| \leq ||B_r|| d_{\alpha}(B_rI_r^1D_rI_r^1) ||D_r^1|| $$

$$= ||B_r|| d_{\alpha}(B_rI_r^1D_rI_r^1) ||D_r^1|| \leq ||B_r|| \alpha_{\alpha}(B_rI_r^1D_rI_r^1) ||D_r^1||,$$

(42)

for $0 \leq j \leq r$. Then for $l \geq 1$, one has

$$\frac{1}{P_j} \leq \frac{\left(\|B_r\|\|D_r^1\|^p\right)}{P_j} \left(\alpha_{\alpha}(I_rB_r^1D_rI_r^1)\right)^p \Rightarrow \frac{1}{P_j}$$

$$\leq l\|B_r\|\left(\alpha_{\alpha}(I_rB_r^1D_rI_r^1)\right)^p \|D_r^1\| \Rightarrow \sum_{j=0}^{l-1} \frac{1}{P_j}$$

$$\leq lC\|B_r\|\|D_r^1\| \left(\alpha_{\alpha}(I_rB_r^1D_rI_r^1)\right)^p \Rightarrow \sum_{j=0}^{l-1} \frac{1}{P_j}$$

$$\leq lC\|B_r\|\|D_r^1\| \left(\alpha_{\alpha}(I_rB_r^1D_rI_r^1)\right)^p \Rightarrow \sum_{j=0}^{l-1} \frac{1}{P_j}$$

$$\leq lC\|B_r\|\|D_r^1\| \left(\alpha_{\alpha}(I_rB_r^1D_rI_r^1)\right)^p \Rightarrow \sum_{j=0}^{l-1} \frac{1}{P_j} \leq 4lC.$$
Corollary 33. Let \((p_v), (q_v) \in \mathfrak{m} \cap \ell_\infty\) with \(1 \leq p_v < q_v\) for each \(v \in \mathbb{N}\), then
\[
L\left(\mathcal{S}(\mathcal{P}_{p_v}), \mathcal{S}(\mathcal{P}_{q_v})\right) = L_C\left(\mathcal{S}(\mathcal{P}_{p_v}), \mathcal{S}(\mathcal{P}_{q_v})\right),
\] (46)

Proof. Clearly, as \(A \subseteq L_c\). □

Theorem 34. Assume \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 \geq 1\), then \(\mathcal{S}(\mathcal{P}_{p_v})\) is simple.

Proof. Suppose \(T \in L_C(\mathcal{S}(\mathcal{P}_{p_v}))\) and \(T \notin \mathcal{A}(\mathcal{S}(\mathcal{P}_{p_v}))\). In view of Lemma 3, we have \(G, B \in L(S(\mathcal{P}_{p_v}))\) so as to \(BTG_k = I_k\).

One gets \(I_S(\mathcal{P}_{p_v}) \in L_C(S(\mathcal{P}_{p_v}))\). Therefore, \(L(S(\mathcal{P}_{p_v})) = L_C(S(\mathcal{P}_{p_v}))\). This implies one and only one nontrivial closed ideal \(\Lambda(S(\mathcal{P}_{p_v}))\) in \(L(S(\mathcal{P}_{p_v}))\). □

4.5. Spectrum of Prequasi-Ideal. In this part, we introduce enough settings on \((\mathcal{H}_{p_v})\), so that the class \(L\) with sequence of eigenvalues in \((\mathcal{H}_{p_v})\) equals \(\mathcal{S}(\mathcal{P}_{p_v})\).

Theorem 35. If \(X\) and \(Y\) are Banach spaces with \(\dim(X) = \dim(Y) = \infty\). Suppose \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 > 0\), we have
\[
\left(\mathcal{S}(\mathcal{P}_{p_v})\right)^A(X, Y) = \mathcal{S}(\mathcal{P}_{p_v})(X, Y).
\] (47)

Proof. Let \(T \in (\mathcal{S}(\mathcal{P}_{p_v}))^A(X, Y)\), then \(f_A \in ((\mathcal{H}(\mathcal{P}_{p_v}))\), where \(f_A(z) = \sum_{v=0}^{\infty} \lambda_v(T)z^v\) converges for all \(z \in C\) with \(\rho(f_A) = \sum_{v=0}^{\infty} (1/p_v)\lambda_v(T)|p_v^v < \infty\), and \(\|T - \lambda_v(T)I\| = 0\) for all \(v \in \mathbb{N}\). We have \(T = \lambda_v(T)I\), with \(v \in \mathbb{N}\), hence \(\lambda_v(T) = \lambda_v(T)\), with \(v \in \mathbb{N}\). As a result, \(f_A \in (\mathcal{H}(\mathcal{P}_{p_v}))\), then \(T \in \mathcal{S}(\mathcal{P}_{p_v})(X, Y)\). Secondly, assume \(T \in \mathcal{S}(\mathcal{P}_{p_v})(X, Y)\).

Hence, \(f_A \in (\mathcal{H}(\mathcal{P}_{p_v}))\), where \(f_A(z) = \sum_{v=0}^{\infty} \lambda_v(T)z^v\) converges for all \(z \in C\) with \(\rho(f_A) = \sum_{v=0}^{\infty} (1/p_v)\lambda_v(T)|p_v^v < \infty\). One has
\[
\sum_{v=0}^{\infty} \frac{1}{p_v} |\lambda_v(T)|^{p_v} \geq \frac{1}{\sup_{p_v \in \mathbb{N}} \sum_{v=0}^{\infty} |\lambda_v(T)|^{p_v}}.
\] (48)

Therefore, \(\lim_{v \to \infty} s_v(T) = 0\). Let \(\|T - s_v(T)I\|^{-1}\) exists, for all \(v \in \mathbb{N}\). Hence, \(\|T - s_v(T)I\|^{-1}\) exists and bounded, for all \(v \in \mathbb{N}\). Therefore, \(\lim_{v \to \infty} \|T - s_v(T)I\|^{-1} = \|T\|^{-1}\) exists and bounded. By using the prequasi-operator ideal of \((\mathcal{S}(\mathcal{P}_{p_v}), \mathcal{G}\), one has
\[
I = TT^{-1} \in \mathcal{S}(\mathcal{P}_{p_v})(X, Y) \Rightarrow (s_v(I))_{v=0}^{\infty} \in (\mathcal{H}(\mathcal{P}_{p_v})) \Rightarrow \lim_{v \to \infty} s_v(I) = 0.
\] (49)

Since \(\lim_{v \to \infty} s_v(I) = 1\). Hence, \(\|T - s_v(T)I\| = 0\), for all \(v \in \mathbb{N}\). This gives \(T \in \mathcal{S}(\mathcal{P}_{p_v})(X, Y)\).

This shows the proof. □

5. Weighted Shift Operators on \((\mathcal{H}_{p_v})\)

In this section, we present the upper bounds of s-numbers for infinite series of the weighted nth power forward and backward shift operator on \(\mathcal{H}(\mathcal{P}_{p_v})\) with applications to some entire functions.

Theorem 36. Assume \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 > 0\), then \(V_z \in L((\mathcal{H}(\mathcal{P}_{p_v}))\) with
\[
\|V_z\| = \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p},
\] (50)

where \(\rho(f) = [\sum_{r=0}^{\infty} |1/p_r| |f, A|^{p_r}]^{1/\rho_p}\), for all \(f \in (\mathcal{H}(\mathcal{P}_{p_v}))\).

Proof. Suppose the setups are verified. For \(f \in (\mathcal{H}(\mathcal{P}_{p_v}))\). Since \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 > 0\), then \(V_z f = \rho(f) = [\sum_{r=0}^{\infty} (1/p_{r+1}) |f, A|^{p_{r+1}}]^{1/\rho_p} \leq \sum_{r \neq 0} (1/p_r) |f, A|^{p_r}^{1/\rho_p} \leq \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p} |f, A|^{p_r}.\)

Therefore, \(V_z \in L((\mathcal{H}(\mathcal{P}_{p_v}))\) with \(\|V_z\| \leq \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p} |f, A|^{p_r}.\) Since \(V_z \in L((\mathcal{H}(\mathcal{P}_{p_v}))\) . Then, there is \(A > 0\) with \(\rho(V_z f) \leq Ap(f)\), for all \(f \in (\mathcal{H}(\mathcal{P}_{p_v}))\). Hence, \(\rho(V_z f) \leq Ap(f)\), one gets \(\sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p} \leq \|V_z\|\).

This completes the proof. □

Theorem 37. Consider \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 > 0\), then \(B_z \in L((\mathcal{H}(\mathcal{P}_{p_v}))\) with
\[
\|B_z\| = \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p},
\] (51)

where \(\rho(f) = [\sum_{r=0}^{\infty} |1/p_r| |f, A|^{p_r}]^{1/\rho_p}\), for every \(f \in (\mathcal{H}(\mathcal{P}_{p_v}))\).

Proof. Let the given settings hold for every \(f \in (\mathcal{H}(\mathcal{P}_{p_v}))\). Since \((p_v) \in \mathfrak{m} \cap \ell_\infty\) with \(p_0 > 0\), then
\[
\rho(B_z f) = \left[\sum_{r=0}^{\infty} \left(1/p_r\right)^{1/\rho_p}\right]^{1/\rho_p} \leq \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p} \left[\sum_{r=0}^{\infty} \left(1/p_r\right)^{1/\rho_p}\right] \leq \sup_{r \neq 1} \left(\frac{p_r}{p_{r+1}}\right)^{1/\rho_p} \rho(f).
\] (52)
Theorem 38. Let $(p_0) \in \mathbb{N} \cap \ell_\infty$ with $p_0 \geq 1$. Suppose \( \limsup_{v \to \infty} (1/\sqrt{p_v}) = 1 \), then every function in \( \mathcal{H}(\rho_\pi) \) is analytic on the open unit disc \( D \). Moreover, the convergence in \( \mathcal{H}(\rho_\pi) \) implies the uniform convergence on compact subsets of \( D \), where \( \rho(f) = \sum_{v=0}^{\infty} (1/p_v) |f(A)|^{1/p_v} \), for any \( f \in \mathcal{H}(\rho_\pi) \).

**Proof.** Suppose \( \limsup_{v \to \infty} (1/\sqrt{p_v}) = 1 \), and \( f \in \mathcal{H}(\rho_\pi) \). Therefore, \( f(z) = \sum_{v=0}^{\infty} f_v z^v \) converges for every \( z \in C \) and \( \rho(f) = \sum_{v=0}^{\infty} (1/p_v) |f(A)|^{1/p_v} < \infty \). Hence, \( \limsup_{v \to \infty} (1/\sqrt{p_v}) |f(A)|^{1/p_v} < 1 \). We have

\[
\limsup_{v \to \infty} \sqrt{|f(A)|^{1/p_v}} < \frac{1}{\limsup_{v \to \infty} (1/\sqrt{p_v})} = 1. \quad (53)
\]

Since \( (p_v) \in \mathbb{N} \cap \ell_\infty \) with \( p_0 \geq 1 \), we obtain \( \limsup_{v \to \infty} (1/\sqrt{p_v}) |f(A)|^{1/p_v} < \limsup_{v \to \infty} (1/\sqrt{p_v}) < 1 \), for all \( z \in D \). Hence, \( f(z) = \sum_{v=0}^{\infty} f_v z^v \) converges for every complex value of \( z \in D \). Assume \( A \) is a compact subset of \( D \) and \( f(z) \in A \), for all \( k \in N \). Let \( f_k \) converges to \( f \in \mathcal{H}(\rho_\pi) \), we have

\[
\left| f_k(z) - f(z) \right| = \sum_{v=0}^{\infty} \left| f_k^v - f_v \right| z^v \leq \sum_{v=0}^{\infty} (\rho(p_v) |z|^v)^{1/p_v} \leq \sum_{v=0}^{\infty} (\rho(p_v) |z|^v)^{1/p_v} \leq \sum_{v=0}^{\infty} (\rho(p_v) |z|^v)^{1/p_v} = \sum_{v=0}^{\infty} (\rho(p_v) |z|^v)^{1/p_v} \rho(f_k - f). \quad (54)
\]

where \((q_v) \in \mathbb{N} \cap \ell_\infty \) with \( q_0 \geq 1 \) and \( (1/p_v) + (1/q_v) = 1 \), for all \( v \in N \). Clearly, \( \limsup_{v \to \infty} (1/\sqrt{p_v}) |z|^v < 1 \), then \( \sum_{v=0}^{\infty} (\rho(p_v) |z|^v)^{1/p_v} < \infty \). So \lim_{v \to \infty} f_k(z) = f(z) \in A. \]

Theorem 39. Assume \( V_z \) is the forward shift operator on \( \mathcal{H}(\rho_\pi) \), with \( \rho(f) = \sum_{v=0}^{\infty} (1/p_v) |f(A)|^{1/p_v} \), for all \( f \in \mathcal{H}(\rho_\pi) \). Then

\[
\sup_{\text{card } \xi = r+1} \left( \frac{P_k}{P_{k+n}} \right)^{1/p_v} \leq s_\xi(V_z) \leq \sup_{\text{card } \xi = r+1} \left( \frac{P_k}{P_{k+n}} \right)^{1/p_v}, \quad (55)
\]

where \( A_n = \left( \sum_{k=0}^{\infty} (1/p_k) |f(A)|^{1/p_v} \right)^{1/p_v} \).
numbers, we have

\[ s_r(V^n_\ell) = \alpha_r(V^n_\ell) \leq \Vert V^n_\ell - R^n_\ell \Vert \leq \sup_{f(z) \neq 0} \frac{(V^n_\ell - R^n_\ell) f(z)}{|f(z)|} \]

\[ \leq \sup_{f(z) \neq 0} \left( \frac{1}{\rho_k} \right)^{1/\alpha_p} \sup_{f(z) \neq 0} \left( \sum_{k \in \mathbb{C}} |f(z)| \right) \]

\[ \leq \sup_{k \in \mathbb{C}} \left( \frac{1}{\rho_k} \right)^{1/\alpha_p}. \]  

(61)

This inequality is verified for every card \( \xi = r \) and by using Lemma 10, one has

\[ \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} \leq s_r(V^n_\ell) \leq \inf_{\text{card } \xi = r \leq k} \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} \]

\[ = \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p}. \]  

(62)

This completes the proof.

\[ \square \]

**Theorem 40.** If \( B_\ell \) is the backward shift operator on \((\mathbb{H}(\mathbb{P}(\rho)))_\rho^\ast\)
with \( \rho(f) = \left[ \sum_{k=0}^{\infty} \left( 1/p_k \right) |f_k|^\rho \right]^{1/\rho} \), for all \( f \in (\mathbb{H}(\mathbb{P}(\rho)))_\rho^\ast \), then

\[ \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} \leq s_r(B^n_\ell) \leq \sup_{\text{card } \xi = r \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} \]

(63)

where \( G_n = \left[ \sum_{k=0}^{\infty} \left( 1/p_k \right) |f_k|^\rho \right]^{1/\rho} \).

**Proof.** Assume card \( \xi = r + 1 \), and let \( \alpha_r(B^n_\ell) \) be for every \( f \in (\mathbb{H}(\mathbb{P}(\rho)))_\rho \), where \( f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k \) converges for any \( z \in \mathbb{C} \) and \( \rho(f) = \left[ \sum_{k=0}^{\infty} (1/p_k) |f_k|^\rho \right]^{1/\rho} < \infty \). Therefore, \( B^n_\ell f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k \) and \( \rho(B^n_\ell f(z)) = \left[ \sum_{k=0}^{\infty} (1/p_k) |f_k|^\rho \right]^{1/\rho} < \infty \).

Suppose \( P_k \) is an operator on \((\mathbb{H}(\mathbb{P}(\rho)))_\rho \) with rank \( P_k = r = 1 \) and

\[ \left( \frac{P_k}{P_{k+n}} \right) (z) = P_k \left( \sum_{k=0}^{\infty} \hat{f}_k z^k \right) = \sum_{k \in \mathbb{C}} \hat{f}_k z^k. \]

(64)

As \( \rho(P \xi g) = \left[ \sum_{k \in \mathbb{C}} (1/p_k) |f_k|^\rho \right]^{1/\rho} = \rho(g) \), this implies \( \| P_k \| \leq 1 \).

Let define an operator \( S^n_\ell \) by \( (S^n_\ell h)(z) = S^n_\ell \left( \sum_{k \in \mathbb{C}} \hat{f}_k z^k \right) = \sum_{k=0}^{\infty} \hat{f}_k z^k \), one gets

\[ \rho(S^n_\ell h) = \left[ \sum_{k \in \mathbb{C}} (1/p_k) \left| \hat{f}_k \right|^\rho \right]^{1/\rho} \leq U_n \left[ \sum_{k \in \mathbb{C}} \left| \hat{f}_k \right|^\rho \right]^{1/\alpha_p} = U_n \rho(h). \]

(65)

Therefore, \( \| S^n_\ell \| \leq U_n \), where \( 1 \leq U_n = \left[ \sum_{k \in \mathbb{C}} \left( 1/p_k \right) |f_k|^\rho \right]^{1/\rho} \). Hence, the identity operator will be \( I_{n+1} = P_k B^n_\ell S^n_\ell \), in view of the definition of \( s \)-numbers, one has

\[ s_r(I_{n+1}) = 1 \leq \| S^n_\ell \| s_r(B^n_\ell) \| S^n_\ell \| \leq s_r(B^n_\ell) \| S^n_\ell \| \Rightarrow \]

(66)

This inequality is confirmed for all card \( \xi = r + 1 \), and we have

\[ s_r(B^n_\ell) \geq \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} 1 \]

(67)

On the other hand, suppose \( \xi \) is a subset of \( \mathbb{N} \) with card \( \xi = r \). Define the finite rank operator \( R^n_\ell \) by \( (R^n_\ell v)(z) = R^n_\ell (\sum_{k=0}^{\infty} \hat{f}_k z^k) \). Define the approximation numbers, one gets

\[ s_r(B^n_\ell) \leq \alpha_r(B^n_\ell) \leq \| B^n_\ell - R^n_\ell \| \leq \sup_{\| f(z) \| \neq 0} \frac{|B^n_\ell - R^n_\ell f(z)|}{|f(z)|} \]

\[ = \sup_{\| f(z) \| \neq 0} \left( \frac{\sum_{k \in \mathbb{C}} \hat{f}_k z^k}{|f(z)|} \right) \leq \sup_{\| f(z) \| \neq 0} \left( \sum_{k \in \mathbb{C}} \hat{f}_k z^k \right) \]

\[ \leq \sup_{k \in \mathbb{C}} \left( \frac{1}{p_k} \right)^{1/\alpha_p}. \]

(68)

This inequality is satisfied by any card \( \xi = r \) and from Lemma 10, we have

\[ \sup_{\text{card } \xi = r + 1 \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} 1 \]

\[ \leq s_r(B^n_\ell) \leq \sup_{\text{card } \xi = r \leq k} \left( \frac{P_k}{P_{k+n}} \right)^{1/\alpha_p} \]

(69)

This finishes the proof.

\[ \square \]
Next, the upper and lower bounds of norm $\sum_{m=0}^{\infty} \| \mathcal{e}_m V^m_z \|$ on the space $(\mathcal{H}_p)_\rho$ have been explained.

**Theorem 41.** The effect of $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z$ on the space $(\mathcal{H}_p)_\rho$, where $\rho(f) = \| \sum_{r=0}^{\infty} 1/p_r f[p_r] \|_1$, for all $f \in (\mathcal{H}_p)_\rho$, $(\mathcal{e}_m)_{m=0}^{\infty} \in \ell(p_m) \rho$, and $(p_r) \in \mathfrak{m}_r \cap \ell_{\cos}$ with $p_0 \geq 1$, we have

$$\sup_{m<k} \left( \frac{\rho_1}{\rho_2} \right) \leq \sup_{m<k} \left( \frac{\rho_1}{\rho_2} \right).$$

**Proof.** Assume $f \in (\mathcal{H}_p)_\rho$, we have $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z)$. Then,

$$\left\| \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right\| = \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right) = \left\| \sum_{m=0}^{\infty} (1/p_{km}) \mathcal{e}_m V^m_z \right\|^{1/\rho} \leq \sup_{m<k} \left( \frac{\rho_1}{\rho_2} \right).$$

Since $\rho$ satisfies the triangle inequality, we get

$$\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z = \sup_{f \in \mathcal{H}_p} \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z) \right) \rho(f).$$

**Theorem 42.** The effect of $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z$ on the space $(\mathcal{H}_p)_\rho$, where $\rho(f) = \| \sum_{r=0}^{\infty} 1/p_r f[p_r] \|_1$, for all $f \in (\mathcal{H}_p)_\rho$, $(\mathcal{e}_m)_{m=0}^{\infty} \in \ell(p_m) \rho$, and $(p_r) \in \mathfrak{m}_r \cap \ell_{\cos}$ with $p_0 \geq 1$, we have

$$\sup_{m<k} \left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right)^{1/\rho} \leq \sup_{m<k} \left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right)^{1/\rho}.$$

**Proof.** Suppose $f \in (\mathcal{H}_p)_\rho$, one has $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z)$. We have

$$\left\| \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right\| = \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right) = \left\| \sum_{m=0}^{\infty} (1/p_{km}) \mathcal{e}_m V^m_z \right\|^{1/\rho} \leq \sup_{m<k} \left( \frac{\rho_1}{\rho_2} \right).$$

As $\rho$ verifies the triangle inequality, one can see

$$\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \leq \sup_{f \in \mathcal{H}_p} \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z) \right) \rho(f).$$

The following theorem indicates an upper estimation to the $s$-numbers of $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z$ on the space $(\mathcal{H}_p)_\rho$.

**Theorem 43.** The effect of $\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z$ on the space $(\mathcal{H}_p)_\rho$, where $\rho(f) = \| \sum_{r=0}^{\infty} 1/p_r f[p_r] \|_1$, for all $f \in (\mathcal{H}_p)_\rho$, the $s$-numbers of this operator are presented by

$$s_r \left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right) \leq \sup_{m} \inf_{\mathcal{E}_{r+1} \xi} \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right) \rho(f) \leq \sup_{m} \inf_{\mathcal{E}_{r+1} \xi} \rho\left( \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z \right) \rho(f).$$

for all $(\mathcal{e}_m)_{m=0}^{\infty} \in \ell(p_m) \rho$ and $(p_r) \in \mathfrak{m}_r \cap \ell_{\cos}$ with $p_0 \geq 1$.

**Proof.** Let $\xi$ be a subset of $\mathbb{N}$ and $\text{card} \xi = r$. By using the definition of $s$-numbers. Define the finite rank operator $R$ by $R f(z) = \rho(\sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z)) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{e}_m V^m_z f(z)$. In view of the definition of approximation numbers and since $\rho$ satisfies...
the triangle inequality, we have

\[
 s_\tau \left( \sum_{m=0}^\infty c_m V^m_z \right) \leq \alpha_\tau \left( \sum_{m=0}^\infty c_m V^m_z \right) \leq \left\| \sum_{m=0}^\infty c_m V^m_z - R \right\| \\
 \leq \sup_{\rho(f) \neq 0} \frac{\rho \left( \sum_{m=0}^\infty c_m V^m_z f - R f \right)}{\rho(f)} \\
 \leq \sum_{m=0}^\infty \left[ \sum_{k \in \mathbb{L}} \left( 1/p_{m+1} \right) \left( |c_m| \left| f_{k+1} \right| \right) \right]^{1/\alpha_p} \\
 \leq \sup_{k \in \mathbb{L}} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p}. 
\] (77)

This inequality is verified for every card \( \xi = r \), and one has

\[
 s_\tau \left( \sum_{m=0}^\infty c_m V^m_z \right) \leq \inf_{\text{card } \xi = r} \left( \sum_{m=0}^\infty \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p} \right) \\
 = \sup_{\text{card } \xi + 1} \inf_{k \in \mathbb{L}} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p}. 
\] (78)

This implies the proof. \( \square \)

The next theorem investigates an upper estimation to the \( s \)-numbers of \( \sum_{m=0}^\infty c_m B^m_z \) on the space \(( \mathbb{H}_{p(\cdot)} )_\rho^\ast \).

**Theorem 44.** Acting \( \sum_{m=0}^\infty c_m B^m_z \) on the space \(( \mathbb{H}_{p(\cdot)} )_\rho^\ast \), where \( \rho(f) = \left[ \sum_{m=0}^\infty \left( 1/p_{m+1} \right) \left| f_{m+1} \right| \right]^{1/\alpha_p} \), for every \( f \in ( \mathbb{H}_{p(\cdot)} )_\rho^\ast \), the \( s \)-numbers of this operator satisfy

\[
 s_\tau \left( \sum_{m=0}^\infty c_m B^m_z \right) \leq \sup_{\text{card } \xi + 1} \inf_{k \in \mathbb{L}} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p}, 
\] (79)

for all \((c_m)_{m=0}^\infty \in \ell^{(1/p_{m+1}/\alpha_p)} \) and \((p_y) \in m_i \cap \ell_{\infty} \) with \( p_0 \geq 1 \).

**Proof.** Assume \( \xi \) is a subset of \( \mathbb{N} \) and card \( \xi = r \). From the definition of \( s \)-numbers. Define the finite rank operator \( R f_k = \sum_{m=0}^k c_m f_{m+k} \). From the definition of approximation numbers and as \( \rho \) verifies the triangle inequality, one has

\[
 s_\tau \left( \sum_{m=0}^\infty c_m B^m_z \right) \leq \alpha_\tau \left( \sum_{m=0}^\infty c_m B^m_z \right) \leq \left\| \sum_{m=0}^\infty c_m B^m_z - R \right\| \\
 \leq \sup_{\rho(f) \neq 0} \frac{\rho \left( \sum_{m=0}^\infty c_m B^m_z f - R f \right)}{\rho(f)} \\
 \leq \sum_{m=0}^\infty \left[ \sum_{k \in \mathbb{L}} \left( 1/p_{m+1} \right) \left| f_{k+1} \right| \right]^{1/\alpha_p} \\
 \leq \sup_{k \in \mathbb{L}} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p}. 
\] (80)

This inequality is satisfied for all card \( \xi = r \), and we have

\[
 s_\tau \left( \sum_{m=0}^\infty c_m B^m_z \right) \leq \inf_{\text{card } \xi = r} \sup_{k \in \mathbb{L}} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p} \\
 = \sup_{\text{card } \xi + 1} \inf_{m} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty |c_m|^{1/p_{m+1}/\alpha_p}. 
\] (81)

This completes the proof. \( \square \)

The following theorems are direct consequences of Theorem 43 and Definition 18, for some entire functions, for example, the exponential and the sine functions.

**Theorem 45.** Let \((p_y) \in m_i \cap \ell_{\infty} \) with \( p_0 \geq 1 \). Assume \( B_{e^z} \) is a shift operator on \( ( \mathbb{H}_{p(\cdot)} )_\rho^\ast \), where for \( \rho(f) = \left[ \sum_{m=0}^\infty \left( 1/p_{m+1} \right) \left| f_{m+1} \right| \right]^{1/\alpha_p} \), for all \( f \in ( \mathbb{H}_{p(\cdot)} )_\rho^\ast \) and \( e^z = \sum_{m=0}^\infty e^{z m}/m! \).

The upper estimation of the \( s \)-number of \( V_{e^z} \) is given by

\[
 s_\tau \left( V_{e^z} \right) \leq \sup_{\text{card } \xi + 1} \inf_{m} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty \left( \frac{1}{m!} \right)^{1/p_{m+1}/\alpha_p}. 
\] (82)

**Theorem 46.** Let \((p_y) \in m_i \cap \ell_{\infty} \) with \( p_0 \geq 1 \). Suppose \( B_{\sin(z)} \) is a shift operator on \( ( \mathbb{H}_{p(\cdot)} )_\rho^\ast \), where \( \rho(f) = \left[ \sum_{m=0}^\infty \left( 1/p_{m+1} \right) \left| f_{m+1} \right| \right]^{1/\alpha_p} \), for all \( f \in ( \mathbb{H}_{p(\cdot)} )_\rho^\ast \) and \( \sin(z) = \sum_{m=0}^\infty (-1)^m (z^{2m+1}/(2m+1)!). \)

The upper estimation of the \( s \)-number of \( V_{\sin(z)} \) is presented by

\[
 s_\tau \left( V_{\sin(z)} \right) \leq \sup_{\text{card } \xi + 1} \inf_{m} \left( \frac{p_k}{p_{k+1}} \right)^{1/\alpha_p} \sum_{m=0}^\infty \left( \frac{1}{(2m+1)!} \right)^{1/p_{m+1}/\alpha_p}. 
\] (83)

The following theorems are direct consequences of Theorem 44 and Definition 19, for some entire functions, for example, the exponential and the sine functions.
Theorem 47. Assume \((p_0) \in \ell_\infty \cap \ell_{\infty}^\infty \) with \(p_0 \geq 1\). Suppose \(B\) is a shift operator on \((\mathcal{H}_{\rho})\), where \(\rho(f) = \sum_{r=0}^{\infty} \|f\|_r p_r^{-1/q_r}\), for every \(f \in (\mathcal{H}_{\rho})\) and \(C = \sum_{m=0}^{\infty} e^{m+1} m!\). The upper estimation of the \(s\)-numbers of \(B\) is pretended by
\[
s_{i}(B) \leq \sup_{m} \inf_{\ell} \sup_{k} \left( \frac{p_{k+m}}{p_{k}} \right)^{i/q_{r}} \sum_{m=0}^{\infty} \left( \frac{1}{m!} \right) p_{n}^{j/q_{r}}.
\]

Theorem 48. Suppose \((p_0) \in \ell_\infty \cap \ell_{\infty}^\infty \) with \(p_0 \geq 1\). Assume \(B\) is a shift operator on \((\mathcal{H}_{\rho})\), where \(\rho(f) = \sum_{r=0}^{\infty} \|f\|_r p_r^{-1/q_r}\), for every \(f \in (\mathcal{H}_{\rho})\) and \(s_{\infty} = \sum_{m=0}^{\infty} (-1)^{m}(2m+1)!m!\). The upper estimation of the \(s_{\infty}\) of \(B\) is presented by
\[
s_{i}(B) \leq \sup_{m} \inf_{\ell} \sup_{k} \left( \frac{p_{k+m}}{p_{k}} \right)^{i/q_{r}} \sum_{m=0}^{\infty} \left( \frac{1}{(2m+1)!} \right) p_{n}^{j/q_{r}}.
\]

Data Availability
No data were used.

Conflicts of Interest
The authors declare that they have no competing interests.

Authors’ Contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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