CONTRIBUTION ON L-WEAKLY COMPACT SETS
AND OPERATORS

HASSAN KHABAOUI, JAWAD H’MICHAINE, AND KAMAL EL FAHRI

Abstract. In this paper we introduce and study a new class of operators related to norm bounded sets on Banach Lattice and which brings together several classical classes of operators (as o-weakly compact operators, b-weakly compact operators, M-weakly compact operators, L-weakly compact operators, almost Dunford Pettis operators). As consequences, we give some new lattice approximation properties of these classes of operators.

1. Introduction

Along this paper $E, F$ mention Banach lattices, $X, Y$ are Banach spaces. The positive cone of $E$ will be denoted by $E^+$. Recall that a net $(x_\alpha) \subset E$ is unbounded absolutely weakly convergent (abb, uaw-convergent) to $x$ if $(|x_\alpha - x| \land u)$ converges weakly to zero for every $u \in E^+$, we write $x_\alpha \xrightarrow{uaw} x$. We note that every disjoint sequence of a Banach lattice is uaw-null [7, Lemma 2]. A net $(x'_\alpha)$ is unbounded absolutely weak* convergent (abb, uaw*-convergent) to $x'$ if $(|x'_\alpha - x'| \land u')$ converges weak* to zero for every $0 \leq u' \in E'$, we write $x'_\alpha \xrightarrow{uaw*} x$. Recall from [6] that a norm bounded subset $A$ of a Banach lattice $E$ is L-weakly compact if $\lim_{n \to +\infty} \|x_n\| = 0$ for every disjoint sequence $(x_n)$ contained in $\text{sol}(A)$, where $\text{sol}(A) := \{x \in E : \exists y \in A \text{ with } |x| \leq |y|\}$ is the solid hull of the set $A$. Alternatively, $A$ is L-weakly compact if and only if $\|x_n\| \to 0$ for every norm bounded uaw-nul sequence $(x_n)$ of $\text{sol}(A)$ [4, Proposition 3.3].

In this paper, we introduce and study a new class of operators attached on a norm bounded subset of the starting space (Definition 3.1) and which groups together several classes of operators, as M-weakly compact operators (Corollary 3.2), order weakly compact operators (Theorem 3.4), b-weakly compact operators (Theorem 3.5), almost...
Dunford-Pettis operators (Proposition 3.6) and L-weakly compact operators (Corollary 3.3). As consequences, we obtain new characterizations of L-weakly compact sets (Corollary 3.5), of order continuous Banach lattice (Corollary 3.6), of KB-space (Corollary 3.7) and of positive Schur property (Corollary 3.9).

2. Preliminaries and notations

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space \((E, \| \cdot \|)\) such that \(E\) is a vector lattice and its norm satisfies the following property: for each \(x, y \in E\) such that \(|x| \leq |y|\), we have \(\|x\| \leq \|y\|\). \(E\) is order continuous if for each net \((x_\alpha)\) such that \(x_\alpha \downarrow 0\) in \(E\), the net \((x_\alpha)\) converges to 0 for the norm \(\| \cdot \|\), where the notation \(x_\alpha \downarrow 0\) means that the net \((x_\alpha)\) is decreasing, its infimum exists and \(\inf(x_\alpha) = 0\).

We will use the term operator \(T : E \rightarrow F\) from \(E\) to \(F\) to mean a bounded linear mapping. \(T'\) will be the adjoint operator of \(T : E \rightarrow F\) defined from \(F'\) into \(E'\) by \(T'(f)(x) = f(T(x))\) for each \(f \in F'\) and each \(x \in E\). An operator \(T : E \rightarrow F\) is positive if \(T(x) \in F^+\) whenever \(x \in E^+\). For more information on positive operators see the book of Aliprantis-Burkinshaw [1].

We need to recall definitions of the following operators:

1. An operator \(T : X \rightarrow F\) is said to be L-weakly compact, if \(T(B_X)\) is an L-weakly compact subset of \(F\).
2. An operator \(T : E \rightarrow X\) is said to be order weakly compact, if \(T([0, x])\) is a relatively weakly compact subset of \(X\) for every \(x \in E^+\).
3. An operator \(T : E \rightarrow X\) is said to be b-weakly compact, if \(T([0, x] \cap E)\) is a relatively weakly compact subset of \(X\) for every \(x \in (E^\prime)^+\).
4. An operator \(T : E \rightarrow X\) is said to be M-weakly compact, if \(T(x_n) \rightharpoonup 0\), for every norm bounded disjoint sequence \((x_n)\) in \(E\).
5. An operator \(T : E \rightarrow X\) is said to be almost Dunford-Pettis, if \(T(x_n) \rightharpoonup 0\) for every disjoint weakly null sequence \((x_n)\) in \(E\).

3. Main results

We start this section by the following definition.
Definition 3.1. Let $S$ be a norm bounded subset of $E$. An operator $T : E \to Y$ is said to be $S$-L-weakly compact (abbr, $S$-Lwc) if for every uaw-null sequence $(x_n) \subset \text{sol}(S)$, we have $T(x_n) \overset{||\cdot||}{\longrightarrow} 0$.

Observing that for a norm bounded subset $S$ of $E$, $Id_E$ is $S$-Lwc if and only if $S$ is an L-weakly compact subset of $E$ ([4, Proposition 3.3]) and that $T$ is M-weakly compact if and only if $T$ is a $B_E$-Lwc [4, Corollary 3.1], where $B_E$ denotes the closed unit ball of $E$.

For a norm bounded subset $S$ of $E$, we note by $LWC_S(E,Y)$ the space of all $S$-Lwc operators from $E$ into $Y$. It is a norm closed vector subspace of $L(E,Y)$, the space of all operators from $E$ into $Y$, and it is a left ideal in $L(E,Y)$. In particular, if $S$ is an L-weakly compact subset of $E$, then every operator $T$ defined from $E$ to $Y$ is $S$-Lwc. On the other hand, note that if $A$, $B$ are two norm bounded subsets of $E$ such that $A \subset B$ and $T$ is an operator from $E$ into $Y$, then $T$ is $A$-Lwc whenever $T$ is $B$-Lwc. On the other hand, $T$ is $B$-Lwc if and only if $T$ is $\text{sol}(B)$-Lwc.

Proposition 3.1. Let $T : E \to Y$ be an operator and $S$ be a norm bounded subset of $E$. If $T$ is $S$-Lwc, then for every $\lambda \in \mathbb{R}$ the operator $T$ is $\lambda S$-Lwc.

Proof. It follows from the fact that for every $\lambda \in \mathbb{R}$, $\text{sol}(\lambda S) = \lambda \text{sol}(S)$.

Proposition 3.2. Let $T : E \to Y$ be an operator and $A$, $B$ are norm bounded subsets of $E$. If $T$ is $A$-Lwc and $B$-Lwc, then $T$ is $(A + B)$-Lwc.

Proof. Let $(x_n)$ be a uaw-null sequence of $\text{sol}(A + B)$, then there exist two sequences $(a_n) \subset A$ and $(b_n) \subset B$ such that $x_n^+ \leq |a_n + b_n|$. Therefore, by the Riesz decomposition property [1, Theorem 1.13] there exist two positive elements $a_n^+$ and $b_n^+$ satisfying $x_n^+ = a_n^+ + b_n^+$, $|a_n^+| \leq |a_n|$ and $|b_n^+| \leq |b_n|$ for each $n$. So, $(a_n^+)$ is a uaw-null sequence of $\text{sol}(A)$ and $(b_n^+)$ is a uaw-null sequence of $\text{sol}(B)$. As $T$ is $A$-Lwc and $B$-Lwc, then $(Ta_n^+) \overset{||\cdot||}{\longrightarrow} 0$ and $T(b_n^+) \overset{||\cdot||}{\longrightarrow} 0$ and hence $T(x_n^+) \overset{||\cdot||}{\longrightarrow} 0$. By the same reason, we found $T(x_n^-) \overset{||\cdot||}{\longrightarrow} 0$. Therefore, $T(x_n) \overset{||\cdot||}{\longrightarrow} 0$. That is, $T$ is $(A + B)$-Lwc, as claimed.

As immediate consequences of the preceding result, we have the following results.

Corollary 3.1. Let $T : E \to Y$ be an operator and $A$, $B$ are norm bounded subsets of $E$. We have the following statements:
Let \( x = (\lambda A + \mu B) - Lwc \).

(2) If \( T \) is \( A - Lwc \) and \( B - Lwc \), then \( T \) is \((A \cup B) - Lwc \).

(3) If \( T \) is \( A - Lwc \) or \( B - Lwc \), then \( T \) is \((A \cap B) - Lwc \).

**Proposition 3.3.** Let \( T \) be an operator from \( E \) to \( Y \), \( S \) be a norm bounded subset of \( E \) and \( I \) be the ideal generated by \( S \). If \( T \) is \( S - Lwc \), then for each \( x \in I \) we have \( T - Lwc \).

**Proof.** Let \( S \) be a norm bounded subset of \( E \), \( I \) be the ideal generated by \( S \) and \( x \in I \), then there exist \( \alpha > 0 \) and some vectors \( x_1, \ldots, x_n \in S \) with \( |x| \leq \alpha \sum_{i=1}^{n} |x_i| \). By the Riesz decomposition property [1, Theorem 1.13] we have

\[
[-|x|, |x|] \subset \alpha [-|x_1|, |x_1|] + \cdots + \alpha [-|x_n|, |x_n|].
\]

We observe that, for each \( i = 1, \ldots, n \) we have \( [-|x_i|, |x_i|] \subset sol(S) \), then

\[
[-|x|, |x|] \subset \alpha sol(S) + \cdots + \alpha sol(S).
\]

Since \( T \) is \( S - Lwc \), then \( T \) is \( sol(S) - Lwc \) and so by Corollary 3.1, we infer that \( T \) is \( \alpha sol(S) + \cdots + \alpha sol(S) - Lwc \). Therefore, \( T \) is \( [-|x|, |x|] - Lwc \).

**Theorem 3.1.** Let \( T : E \rightarrow Y \) be an operator and \( u \in E^+ \). Then, the following statements are equivalent:

(1) \( T \) is \([u, u] - Lwc \).

(2) For each \( \varepsilon > 0 \), there exists some \( g \in (E')^+ \) such that

\[
\forall x \in [u, u], \quad \|T(x)\| \leq g(|x|) + \varepsilon.
\]

**Proof.** 1) \( \Rightarrow \) 2) Let \( \varepsilon > 0 \) and \( u \in E^+ \). As \( T \) is \([u, u] - Lwc \), we see that every disjoint sequence of \([0, u]\) converges uniformly to zero on \( T'(B_Y^+) \), then by [1, Theorem 4.40] there exists some \( g \in (E')^+ \) such that

\[
(T'(f) - g)^+(u) < \varepsilon \quad \text{holds for all } f \in B_Y^+.
\]

Let \( x \in [u, u] \), then for each \( f \in B_Y \), we have

\[
\|f(T(x))\| \leq |T'(f)(|x|)| \leq g(|x|) + \varepsilon,
\]

so

\[
\|T(x)\| \leq g(|x|) + \varepsilon
\]

2) \( \Rightarrow \) 1) Let \( \varepsilon > 0 \) and \( (x_n) \) be a uaw-null sequence of \([u, u] \). We have to show that \( T x_n \overset{\|\cdot\|}{\longrightarrow} 0 \). By our hypothesis, there exists some
$\mathcal{S}$-weakly compact operators

$g \in (E')^+$ such that

$$||T(x_n)|| \leq g(|x_n|) + \frac{\varepsilon}{2} \text{ for every } n \in \mathbb{N}.$$ 

As $x_n \xrightarrow{uaw} 0$ in $E$, then $g(|x_n|) = g(|x_n| \wedge u) \to 0$ and hence there exists some integer $m$ such that $g(|x_n|) \leq \frac{\varepsilon}{2}$ for every $n \geq m$. So, for every $n \geq m$ we have $||T(x_n)|| \leq \varepsilon$ which implies that $T(x_n) \xrightarrow{||.||} 0$. Therefore, $T$ is $[-u, u]$-Lwc. \hfill $\square$

In the following result, we present some characterizations of $\mathcal{S}$-Lwc operators.

**Theorem 3.2.** For an operator $T : E \to Y$ and a norm bounded subset of $\mathcal{S} \subset E$, the following statements are equivalent:

1. $T$ is $\mathcal{S}$-Lwc.
2. For each $\varepsilon > 0$, there exist some $g \in (E')^+$ and $u \in E^+$ such that

$$||T(x)|| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in \text{sol}(\mathcal{S}).$$

3. For every uaw-null net $(x_\alpha)$ of $\text{sol}(\mathcal{S})$, we have $T(x_\alpha) \xrightarrow{||.||} 0$.

**Proof.** $1) \Rightarrow 2)$ Let $\varepsilon > 0$; since $T$ is $\mathcal{S}$-Lwc, then by [1, Theorem 4.36] there exists some $u \in E^+$ lying in the ideal generated by $\text{sol}(\mathcal{S})$ such that

$$||T(|x|) - u||) \leq \frac{\varepsilon}{4} \text{ for all } x \in \text{sol}(\mathcal{S}).$$

This implies that,

$$||T(|x|)|| \leq ||T(|x| \wedge u)|| + \frac{\varepsilon}{4} \text{ for all } x \in \text{sol}(\mathcal{S}).$$

Hence, for every $x \in \text{sol}(\mathcal{S})$, we have

$$||T(x^+)|| \leq ||T(x^+ \wedge u)|| + \frac{\varepsilon}{4}$$

and

$$||T(x^-)|| \leq ||T(x^- \wedge u)|| + \frac{\varepsilon}{4}.$$

This implies that for every $x \in \text{sol}(\mathcal{S})$, we have

$$||T(x)|| \leq ||T(x^+ \wedge u)|| + ||T(x^- \wedge u)|| + \frac{\varepsilon}{2} \text{ (\ast).}$$

On the other hand, by Proposition 3.3 $T$ is $[-u, u]$-Lwc, and so by Theorem 3.1, there exists some $g \in (E')^+$ such that for every $x \in \text{sol}(\mathcal{S})$, we have

$$||T(x^+ \wedge u)|| \leq g(x^+ \wedge u) + \frac{\varepsilon}{4}$$
and
\[ \|T(x^{-} \wedge u)\| \leq g(x^{-} \wedge u) + \frac{\varepsilon}{4}. \]

On the other hand, using the fact that \(|x| \wedge u = x^{+} \wedge u + x^{-} \wedge u\), we see that for every \(x \in \text{sol}(S)\) we have
\[ \|T(x^{+} \wedge u)\| + \|T(x^{-} \wedge u)\| \leq g(|x| \wedge u) + \frac{\varepsilon}{2} \quad (**) \]

Therefore, by combining (*) and (**), we have
\[ \|T(x)\| \leq g(|x| \wedge u) + \varepsilon \quad \text{for every} \quad x \in \text{sol}(S). \]

2) \(\Rightarrow\) 3) Let \(\varepsilon > 0\) and \((x_{\alpha})\) a net of \(\text{sol}(S)\). By our hypothesis, there exist some \(g \in (E')^{+}\) and \(u \in E^{+}\) such that,
\[ \|T(x_{\alpha})\| \leq g(|x_{\alpha}| \wedge u) + \frac{\varepsilon}{2} \quad \text{for all} \quad \alpha. \]

Since \(x_{\alpha} \xrightarrow{uaw} 0\), then there exists some \(\alpha_0\) such that \(g(|x_{\alpha}| \wedge u) < \frac{\varepsilon}{2}\) for every \(\alpha \geq \alpha_0\). Hence, for every \(\alpha \geq \alpha_0\) we have \(\|T(x_{\alpha})\| \leq \varepsilon\), which implies that \(T(x_{\alpha}) \xrightarrow{||\|} 0\).

3) \(\Rightarrow\) 1) Obvious. \(\square\)

As a consequence of the preceding theorem, we have the following characterizations of M-weakly compact operators which is exactly the [4, Corollary 3.1].

**Corollary 3.2.** For an operator \(T : E \rightarrow F\) the following statements are equivalent:

1) \(T\) is M-weakly compact.
2) For each \(\varepsilon > 0\), there exist some \(g \in (E')^{+}\) and \(u \in E^{+}\) such that
\[ \|T(x)\| \leq g(|x| \wedge u) + \varepsilon \quad \text{for all} \quad x \in B_{E}. \]
3) For every net \((x_{\alpha}) \subset B_{E}\) such that \(x_{\alpha} \xrightarrow{uaw} 0\), we have \(T(x_{\alpha}) \xrightarrow{||\|} 0\).
4) For every sequence \((x_{n}) \subset B_{E}\) such that \(x_{n} \xrightarrow{uaw} 0\), we have \(T(x_{n}) \xrightarrow{||\|} 0\).

In a similar way, we may prove the following result which present a dual version of the Theorem 3.2. The proof of this theorem is similar to that of the Theorem 3.2, so we decided to delete it.

**Theorem 3.3.** Let \(T : E \rightarrow F\) be an operator and \(S\) a norm bounded subset of \(F'\). Then, the following statements are equivalent:

1) \(T'\) is \(S\)-Lwc.
For each \( \varepsilon > 0 \) there exist some \( g \in (F')^+ \) and \( u \in F^+ \) such that
\[
\|T'(f)\| \leq (|f| \wedge g)(u) + \varepsilon \quad \text{for all } f \in \text{sol}(S).
\]

(3) For every uaw\(^*\)-null net \((f_\alpha) \subset \text{sol}(S)\), we have \( T'(f_\alpha) \rightharpoonup 0 \).

(4) For every uaw\(^*\)-null sequence \((f_n) \subset \text{sol}(S)\), we have \( T'(f_n) \rightharpoonup 0 \).

As consequence of the Theorem 3.3, we obtain new characterizations of L-weakly compact operators.

**Corollary 3.3.** Let \( T : E \longrightarrow F \) be an operator. Then, the following statements are equivalent:

1. \( T \) is L-weakly compact.
2. For each \( \varepsilon > 0 \) there exist some \( g \in (F')^+ \) and \( u \in F^+ \) such that
\[
\|T'(f)\| \leq (|f| \wedge g)(u) + \varepsilon \quad \text{for all } f \in B_{F'}.
\]

(3) For every net \((f_\alpha) \subset B_{F'} \) such that \( f_\alpha \rightharpoonup^* 0 \), we have \( T'(f_\alpha) \rightharpoonup 0 \).

(4) For every sequence \((f_n) \subset B_{F'} \) such that \( f_n \rightharpoonup^* 0 \), we have \( T'(f_n) \rightharpoonup 0 \).

Recall that a bounded subset \( A \) in Banach lattice \( E \) is said uaw-compact whenever every net \((x_\alpha)\) in \( E \) has a subnet, which is uaw-convergent. Note that the standard basis \((e_n)\) of \( \ell_1 \) is uaw-null in \( \ell_1 \), then the set \( A = \{e_n : n \in \mathbb{N}\} \) is relatively uaw-compact; but \( A \) is not relatively compact.

**Proposition 3.4.** Let \( S \) be a norm bounded subset of \( E \). For an \( S \)-Lwc operator \( T : E \longrightarrow Y \), we have

1. \( T(A) \) is compact for every norm bounded uaw-compact subset \( A \subset \text{sol}(S) \).
2. \( T(A) \) is relatively compact for every norm bounded relatively uaw-compact subset \( A \subset \text{sol}(S) \).

**Proof.**
1) Let \( A \) be a uaw-compact subset of \( \text{sol}(S) \) and let \((x_\alpha)\) be a net of \( A \), then there exist a subnet of \((x_\alpha)\) which we denoted by \((x_\alpha)\) and \( x \in A \) such that \( x_\alpha \rightharpoonup^* x \). As \( T \) is \( S \)-Lwc, then by Corollary 3.1 and Theorem 3.2, \( T(x_\alpha) \rightharpoonup T(x) \). Therefore, \( T(A) \) is a compact subset of \( Y \).

2) Let \( A \) be a relatively uaw-compact subset of \( \text{sol}(S) \) and let \((x_\alpha)\) be a net of \( A \), then \((x_\alpha)\) has a uaw-convergent subnet which we denoted
by \((x_\alpha)\). Since the subnet \((x_\alpha)\) is norm bounded and uaw-Cauchy, then the double net \((x_\alpha - x_\beta)_{(\alpha,\beta)}\) is uaw-null. As \(T\) is \(S\)-Lwc, then by Theorem 3.2 and Corollary 3.1 the net \((T(x_\alpha))\) is norm Cauchy and hence \((T(x_\alpha))\) is norm convergent. Therefore, \(T(A)\) is a relatively compact subset of \(Y\).

In the following result we give characterizations of \(S\)-Lwc operators which are uaw-continuous.

**Proposition 3.5.** Let \(S\) be a norm bounded subset of \(E\). For a uaw-continuous operator \(T : E \rightarrow F\) the following statements are equivalent:

1. \(T\) is \(S\)-Lwc.
2. \(T(A)\) is relatively compact for every relatively uaw-compact subset \(A \subset \text{sol}(S)\).
3. \(T(A)\) is compact for every uaw-compact subset \(A \subset \text{sol}(S)\).

*Proof.* 1) \(\Rightarrow\) 2) Follows from the Proposition 3.4.

2) \(\Rightarrow\) 3) Obvious.

3) \(\Rightarrow\) 1) Let \((x_n)\) be a uaw-null sequence of \(\text{sol}(S)\), then the set \(A = \{x_n : n \in \mathbb{N}\} \cup \{0\}\) is uaw-compact, and hence it follows from our hypothesis that \(T(A)\) is relatively compact. So, there exist a subsequence \((x_{\phi(n)})\) of \((x_n)\) and \(y \in T(A)\) such that \(T(x_{\phi(n)}) \longrightarrow y\). Since the sequence \((x_n)\) is uaw-null, then \(x_{\phi(n)} \xrightarrow{\text{uaw}} 0\) and since \(T\) is uaw-continuous then \(y = 0\). Therefore, \(T(x_n) \longrightarrow 0\) which implies that \(T\) is \(S\)-Lwc.

An immediate consequence of the preceding result is the following.

**Corollary 3.4.** For a uaw-continuous operator \(T : E \rightarrow F\), the following statements are equivalent:

1. \(T\) is M-weakly compact.
2. \(T(A)\) is compact for every uaw-compact subset \(A \subset B_E\).
3. \(T(A)\) is relatively compact for every relatively uaw-compact subset \(A \subset B_E\).

As another consequence of Theorem 3.2 and Proposition 3.5, we have the following characterizations of L-weakly compact sets.

**Corollary 3.5.** For a norm bounded subset \(S\) of \(E\), the following statements are equivalent:

1. \(S\) is L-weakly compact.
(2) For each $\varepsilon > 0$, there exist some $g \in (E')^+$ and $u \in E^+$ such that
$$||x|| \leq g(|x| \wedge u) + \varepsilon \text{ for every } x \in \text{sol}(S).$$

(3) For every net $(x_\alpha) \subset \text{sol}(S)$ such that $x_\alpha \xrightarrow{\text{uaw}} 0$, we have $x_\alpha \xrightarrow{||||} 0$.

(4) Every norm bounded uaw-compact subset $A \subset \text{sol}(S)$ is compact.

(5) Every norm bounded relatively uaw-compact subset $A \subset \text{sol}(S)$ is relatively compact.

Note that from [4, Proposition 3.4], it is easy to see that an operator $T : E \rightarrow Y$ is order weakly compact if and only if for every $v \in E^+$, the operator $T$ is $\{v\}$-Lwc.

With the help of Theorem 3.2 and Proposition 3.4, we are now in a position to present new characterizations of the order weakly compact operators.

**Theorem 3.4.** For an operator $T : E \rightarrow Y$, the following statements are equivalent:

1. $T$ is order weakly compact.
2. For every relatively compact subset $S$ of $E$, the operator $T$ is $S$-Lwc.
3. For every $v \in E^+$, the operator $T$ is $\{v\}$-Lwc.
4. For every $\varepsilon > 0$ and $v \in E^+$, there exist some $g \in (E')^+$ and $u \in E^+$ such that
$$||T(x)|| \leq g(|x| \wedge u) + \varepsilon \text{ for every } x \in [-v,v].$$
5. For every $v \in E^+$ and for every net $(x_\alpha) \subset [-v,v]$ such that $x_\alpha \xrightarrow{\text{uaw}} 0$, we have $T(x_\alpha) \xrightarrow{||||} 0$.
6. For every $v \in E^+$ and for every sequence $(x_n) \subset [-v,v]$ such that $x_n \xrightarrow{\text{uaw}} 0$, we have $T(x_n) \xrightarrow{||||} 0$.
7. For every order bounded uaw-compact subset $A$ of $E$, $T(A)$ is compact.
8. For every order bounded relatively uaw-compact subset $A$ of $E$, $T(A)$ is relatively compact.

**Proof.** It remains to show 1) $\Rightarrow$ 2) and 8) $\Rightarrow$ 1), the other implications are already seen before in the preceding results.

1) $\Rightarrow$ 2) Let $T$ be an order weakly compact operator, $S$ a relatively compact set of $E$ and $(x_n)$ a uaw-nul sequence of $\text{sol}(S)$ and let $\varepsilon > 0$. Since $(x_n) \subset \text{sol}(S)$ and $S$ a relatively compact set of $E$, then there
exist \( u \in E^+ \) and a sequence \( (y_n) \subset S \) such that
\[
||| (x_n^+ - u)^+ ||| \leq |||(|y_n| - u)^+ ||| \leq \frac{\varepsilon}{4 ||T||} 
\]
for all \( n \), which implies that for all \( n \), we have
\[
||| T(x_n^+) ||| \leq ||| T(x_n^+ \wedge u) ||| + \frac{\varepsilon}{4}.
\]
Since \( T \) is order weakly compact then, \( T \) is \([-u, u]\)-Lwc operator, thus by Theorem 3.1, there exists \( g \in (E')^+ \) such that
\[
||| T(x_n^+ \wedge u) ||| \leq g(x_n^+ \wedge u) + \frac{\varepsilon}{4} \text{ for all } n,
\]
So for every \( n \) we have
\[
||| T(x_n^+) ||| \leq g(x_n^+ \wedge u) + \frac{\varepsilon}{2} \text{ for all } n.
\]
Since \( x_n \uarrow 0 \), then there exists \( n_0 \) such that \( g(x_n^+ \wedge u) < \frac{\varepsilon}{2} \) for all \( n \geq n_0 \). Thus, for each \( n \geq n_0 \) we have \( ||| T(x_n^+) ||| \leq \varepsilon \), which implies that \( T(x_n^+) \xrightarrow{|||} 0 \). By the same reason, we found \( T(x_n^-) \xrightarrow{|||} 0 \). Hence, \( T(x_n) \xrightarrow{|||} 0 \), as desired.

8) \( \Rightarrow \) 1) Let \( v \in E^+ \) and \( (x_n) \subset [-v, v] \) be a uaw-null sequence. The set \( A = \{x_n : n \in \mathbb{N}\} \) is relatively uaw compact, implies that \( (T(x_n)) \) is relatively compact. By observing that the sequence \( (x_n) \) is weakly null we infer that \( T(x_n) \xrightarrow{|||} 0 \).

\[ \square \]

As a consequence of Theorem 3.4, we obtain the following characterizations of order continuous Banach lattices.

**Corollary 3.6.** The following statements are equivalent:

1. \( E \) is order continuous.
2. Every relatively compact subset \( S \) of \( E \) is \( L \)-weakly compact.
3. For every \( x \in E^+ \), \( Id_E \) is \( \{x\}\)-Lw.
4. For every \( \varepsilon > 0 \) and \( v \in E^+ \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
\[
||x|| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in [-v, v].
\]
5. For every \( v \in E^+ \) and for every net \( (x_\alpha) \subset [-v, v] \) such that \( x_\alpha \uarrow 0 \), we have \( x_\alpha \xrightarrow{|||} 0 \).
6. For every \( v \in E^+ \) and for every sequence \( (x_n) \subset [-v, v] \) such that \( x_n \uarrow 0 \), we have \( x_n \xrightarrow{|||} 0 \).
7. Every order bounded uaw-compact subset \( A \subset E \) is compact.
(8) Every order bounded relatively uaw-compact subset \( A \subset E \) is relatively compact.

Recall from [5] that a subset \( A \) of a Banach lattice \( E \) is said to be b-semi compact if it is almost order bounded as a subset of \( E'' \), that is, for every \( \varepsilon > 0 \) there exists \( u \in E'' \) such that \( A \subset [-u, u] + \varepsilon B_{E''} \).

By repeating the proof of Theorem 3.4, we can prove a similar result for b-weakly compact operators as follows.

**Theorem 3.5.** For an operator \( T : E \rightarrow Y \), the following statements are equivalent:

1. \( T \) is b-weakly compact.
2. For every b-semi compact subset \( S \) of \( E \), the operator \( T \) is \( S \)-Lwc.
3. For every \( v \in (E'')^+ \), \( T \) is \( (-v, v] \cap E \)-Lwc.
4. For each \( \varepsilon > 0 \) and \( v \in (E'')^+ \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
   \[
   \|T(x)\| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in [-v, v] \cap E.
   \]
5. For every \( v \in (E'')^+ \) and for every net \( (x_\alpha) \subset [-v, v] \cap E \) such that \( x_\alpha \xrightarrow{\text{uaw}} 0 \), we have \( T(x_\alpha) \xrightarrow{\|\|} 0 \).
6. For every \( v \in (E'')^+ \) and for every sequence \( (x_n) \subset [-v, v] \cap E \) such that \( x_n \xrightarrow{\text{uaw}} 0 \), we have \( T(x_n) \xrightarrow{\|\|} 0 \).
7. For every b-order bounded uaw-compact subset \( A \subset E \), \( T(A) \) is compact.
8. For every b-order bounded relatively uaw-compact subset \( A \subset E \), \( T(A) \) is relatively compact.

As a consequence of Theorem 3.5, we have the following characterizations of KB-spaces.

**Corollary 3.7.** The following statements are equivalent:

1. \( E \) is a KB space.
2. Every b-semi compact subset \( S \) of \( E \) is L-weakly compact.
3. For every \( v \in (E'')^+ \), \( \text{Id}_E \) is \( (-v, v] \cap E \)-Lwc.
4. For each \( \varepsilon > 0 \) and \( v \in (E'')^+ \), there exist some \( g \in (E')^+ \) and \( u \in E^+ \) such that
   \[
   \|x\| \leq g(|x| \wedge u) + \varepsilon \text{ for all } x \in [-v, v] \cap E.
   \]
5. For every \( v \in (E'')^+ \) and for every net \( (x_\alpha) \subset [-v, v] \cap E \) such that \( x_\alpha \xrightarrow{\text{uaw}} 0 \), we have \( x_\alpha \xrightarrow{\|\|} 0 \).
6. For every \( v \in (E'')^+ \) and for every sequence \( (x_n) \subset [-v, v] \cap E \) such that \( x_n \xrightarrow{\text{uaw}} 0 \), we have \( x_n \xrightarrow{\|\|} 0 \).
(7) Every b-order bounded uaw-compact subset \( A \subset E \) is compact.

(8) Every b-order bounded relatively uaw-compact subset \( A \subset E \) is relatively compact.

In terms of relatively weakly compact sets and \( S \)-Lwc operators the almost Dunford-Pettis operators are characterized as follows.

**Proposition 3.6.** For an operator \( T : E \rightarrow Y \), the following statements are equivalent:

1. \( T \) is almost Dunford-Pettis.
2. For every relatively weakly compact subset \( S \) of \( E \), the operator \( T \) is \( S \)-Lwc.
3. For every relatively weakly compact subset \( S \) of \( E \) and for every net \( (x_\alpha) \subset \text{sol}(S) \) such that \( x_\alpha \xrightarrow{\text{uaw}} 0 \), we have \( T(x_\alpha) \xrightarrow{||}\ 0 \).
4. For every relatively weakly compact subset \( S \) of \( E \) and for every sequence \( (x_n) \subset \text{sol}(S) \) such that \( x_n \xrightarrow{\text{uaw}} 0 \), we have \( T(x_n) \xrightarrow{||}\ 0 \).

**Proof.**

1) \( \Rightarrow \) 2) Let \( S \) be a relatively weakly compact subset of \( E \) and \( (x_n) \) be a uaw-null sequence of \( \text{sol}(S) \), then by [4, Theorem 3.1], we have \( x_n^+ \xrightarrow{w} 0 \) and \( x_n^- \xrightarrow{w} 0 \). On the other hand, since \( T \) is almost Dunford-Pettis, it follows from [3, Theorem 2.2] that \( T(x_n^+) \xrightarrow{||}\ 0 \) and \( T(x_n^-) \xrightarrow{||}\ 0 \), and so \( T(x_n) \xrightarrow{||}\ 0 \). Therefore, \( T \) is \( S \)-Lwc.

2) \( \Rightarrow \) 3) Obvious.

3) \( \Rightarrow \) 4) Obvious.

4) \( \Rightarrow \) 1) Let \( (x_n) \) be a disjoint weakly null sequence of \( E \). Put \( K = \{x_n : n \in \mathbb{N}\} \), we note that \( K \) is relatively weakly compact and the sequence \( (x_n) \) is uaw-null, hence by our hypothesis \( T(x_n) \xrightarrow{||}\ 0 \), as desired.

\( \Box \)

As consequences of Proposition 3.5 and Proposition 3.6, we have the following results which present new characterizations of almost Dunford-Pettis operators which are uaw-continuous.

**Corollary 3.8.** For a uaw-continuous operator \( T : E \rightarrow F \), the following statements are equivalent:

1. \( T \) is almost Dunford-Pettis.
2. For every relatively weakly compact subset \( S \) of \( E \) and for every uaw-compact subset \( A \) of \( \text{sol}(S) \), we have \( T(A) \) is compact.
3. For every relatively weakly compact subset \( S \) of \( E \) and for every relatively uaw-compact subset \( A \) of \( \text{sol}(S) \), we have \( T(A) \) is relatively compact.
New characterizations of the positive Schur property are obtained as a consequence of Proposition 3.6 and Corollary 3.8.

**Corollary 3.9.** The following statements are equivalent:

1. $E$ has the positive Schur property.
2. Every relatively weakly compact subset of $E$ is L-weakly compact.
3. For every relatively weakly compact subset $S$ of $E$ and for every net $(x_\alpha) \subset \text{sol}(S)$ such that $x_\alpha \overset{uaw}{\longrightarrow} 0$, we have $x_\alpha \overset{||}{\longrightarrow} 0$.
4. For every relatively weakly compact subset $S$ of $E$ and for every sequence $(x_n) \subset \text{sol}(S)$ such that $x_n \overset{uaw}{\longrightarrow} 0$, we have $x_n \overset{||}{\longrightarrow} 0$.
5. For every relatively weakly compact subset $S$ of $E$ and for every uaw-compact subset $A$ of $\text{sol}(S)$, we have $A$ is compact.
6. For every relatively weakly compact subset $S$ of $E$ and for every relatively uaw-compact subset $A$ of $\text{sol}(S)$, we have $A$ is relatively compact.

The last result of this section presents a set characterization of Banach lattices with weakly sequentially continuous lattice operations.

**Proposition 3.7.** The following statements are equivalent:

1. The lattice operations of $E$ are weakly sequentially continuous.
2. Every relatively weakly compact subset of $E$ is relatively uaw-compact.

**Proof.**
1) $\Rightarrow$ 2) It follows from [4, Corollary 3.5].

2) $\Rightarrow$ 1) Let $T : E \rightarrow \ell^\infty$ be an almost Dunford-Pettis operator and let $S$ be a relatively weakly compact subset of $E$, by our hypothesis we have $S$ is a relatively uaw-compact set of $E$. Now by Proposition 3.6 we see that $T$ is $S$-Lwc, therefore by Proposition 3.4 the set $T(S)$ is relatively compact. That is, $T$ is a Dunford-Pettis operator, and so by [2, Corollary 2.4] the lattice operations of $E$ are weakly sequentially continuous, and the proof of the theorem is finished. \qed

**References**

[1] C.D. Aliprantis and O. Burkinshaw, Positive operators. Reprint of the 1985 original. Springer, Dordrecht, 2006.
[2] B. Aqzzouz and K. Bouras, Weak and almost Dunford-Pettis operators on Banach lattices. Demonstratio Mathematica, vol. 46, no. 1, 2013, pp. 165-179. https://doi.org/10.1515/dema-2013-0431.
[3] B. Aqzzouz, A.Elbour, Some characterizations of almost Dunford–Pettis operators and applications. Positivity 15, 369–380 (2011). https://doi.org/10.1007/s11117-010-0083-7.
[4] K. El Fahri, H. Khabaoui and J. H’michane, Some characterizations of L-weakly compact sets using the unbounded absolute weak convergence and applications. Positivity 26, 42 (2022). https://doi.org/10.1007/s11117-022-00912-2.
[5] N. Machrafi, K. El Fahri and M. Moussa, A note on b-semicompact sets and operators. Rend. Circ. Mat. Palermo 65, 47–53 (2016). https://doi.org/10.1007/s12215-015-0217-7
[6] P. Meyer-Nieberg, Banach Lattices, Universitext, Springer-Verlag, Berlin, 1991.
[7] O. Zabeti, Unbounded absolute weak convergence in Banach lattices. Positivity 22, 501–505 (2018). https://doi.org/10.1007/s11117-017-0524-7

HASSAN Khabaoui, Moulay Ismail University, Faculty of sciences, Department of Mathematics, B.P. 11201 Zitoune, Meknès, Morocco.  
Email address: khabaoui.hassan2@gmail.com

ENGINEERING SCIENCES LAB. ENSA, B.P 241, Ibn Tofail University, Kenitra, Morocco. 
Email address: hm1982jad@gmail.com

KAMAL EL FAHRI, Ibn Zohr University, Faculty of sciences, Department of Mathematics, Laboratory of Mathematics and Application, Functional analysis team, Morocco. 
Email address: kamalelfahri@gmail.com