Generalised Inverse-Cowling Approximation for Polar $w$-mode Oscillations of Neutron Stars

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ABSTRACT

Adopting the Lindblom-Detweiler formalism for polar oscillations of neutron stars, we study the $w$-mode oscillation and find that the Lagrangian change in pressure, measured by the physical quantity $X$, is negligibly small. Based on this observation, we develop the generalised inverse-Cowling approximation (GICA) with the approximation $X = X' = 0$, where $X'$ is the derivative of $X$ with respect to the circumferential radius, for $w$-mode oscillations of neutron stars. Under GICA, $w$-mode oscillations are described by a second-order differential system, which can yield accurate frequencies and damping rates of quasi-normal modes.

Key words: gravitational waves - quasi-normal modes of compact stars - stars: neutron - stars: oscillations - relativity

1 INTRODUCTION

The search for gravitational waves emitted from stellar collapse has been the major goal of many relativists and astrophysicists since the early seventies of last century. At the beginning of a new millennium, several ground-based interferometric detectors (e.g. LIGO, VIRGO, GEO 600, TAMA 300) are currently endeavouring to detect gravitational waves from extraterrestrial objects (see, e.g. Hughes 2003, Mason 2004, and references therein). With the upgrade of these detectors and the availability of a space-based interferometric detector, LISA, which is expected to be launched at the beginning of the next decade, it is widely believed that gravitational wave detection and hence astronomy in the gravitational wave window will become achievable in the near future (Hughes 2003, Mason 2004). In fact, there have been four science runs on LIGO and GEO 600 since 2002. Despite that so far no positive evidence for the existence of gravitational waves has been found, relevant observational data were used to place limits on the ellipticity and the gravitational wave strain of a number of known pulsars (Abbott et al. 2005, 2007).

One major kind of gravitational wave emitting astrophysical processes is certainly stellar gravitational collapse, often resulting in remnants of neutron stars or black holes, which are also possible gravitational wave emitters (for a review on gravitational waves emitted in gravitational collapse see, e.g. Frayer & New 2003, and references therein). For example, gravitational waves emitted in the formation of neutron stars in our galaxy via stellar core collapses have been shown to be detectable for sufficiently fierce asymmetric collapse (Frayer et al. 2002, Lindblom et al. 1998). Since such gravitational waves are likely to carry the information about the internal structure of the neutron star involved, such as its mass, radius, density distribution and in turn its equation of state (EOS), the characteristics of gravitational wave signals emitted from undulating neutron stars have become a subject of much study (see, e.g., Andersson & Kokkotas 1999, 1998, Benhar et al. 1999, Kokkotas et al. 2001, Benhar et al. 2004).

The spectrum of gravitational waves from compact stellar objects are commonly described in terms of quasi-normal modes (QNMs). Each of these QNMs is characterized by a complex eigenfrequency $\omega = \omega_r + i\omega_i$ and has a time dependence $\exp(i\omega t)$ (Press 1971, Leaver 1986, Ching et al. 1996, Kokkotas & Schmidt 1999). The QNM frequencies of gravitational waves emitted from a neutron star are generally model-sensitive and are expected to depend significantly on the EOS used in the stellar model. In spite of this fact, some universal behaviours in the frequency $\omega_i$ and the damping time $\tau \equiv 1/\omega_i$ of the $f$-mode and
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the \( w \)-mode oscillations of non-rotating neutron stars can still be identified (Andersson & Kokkotas 1998, Benhar et al. 1999), which have recently been explained by Tsui & Leung (2005a). Based on such universal behaviours, it has been shown that the radius and the mass of a neutron star can be inferred from the pulsation frequencies of its fundamental fluid \( f \)-mode and the first \( w \)-mode (Andersson & Kokkotas 1998, Benhar et al. 1999). Most interestingly, Tsui & Leung (2005a) and Tsui et al. (2006) showed that the EOS of a neutron star can be inferred from the eigenfrequencies of a few (say, e.g., 3) \( w \)-mode QNMs. These discoveries all underscore the physical significance of QNMs of neutron stars. On the other hand, it is important to note that detection of gravitational waves from neutron stars with sufficient accuracy to infer their internal structure is a mission impossible in the short (or even medium) term. As estimated by Andersson & Comet (2001) and Tsui et al. (2006), realisation of such schemes has to await the availability of more advanced gravitational-wave observatory like EURO.

Non-radial oscillations of relativistic neutron stars, first studied by Thorne & Campolattaro (1967), can be analysed by decomposing the perturbed Einstein equations into spherical harmonics. Such oscillations are categorized into axial (toroidal) and polar (spherical) modes according to the parity of the harmonics. In axial-mode oscillations, which can be described by a single second-order wave equation (Chandrasekhar & Ferrari 1991), fluid elements of a star are merely spectators in the sense that they are not affected by gravitational waves generated (Thorne & Campolattaro 1967). On the other hand, in polar-mode oscillations, fluid motion and gravitational wave are coupled together. Due to the complicated interplay between the matter and the metric, polar-mode oscillations are usually governed by a system of ordinary differential equations (ODEs) involving several physical variables. In the pioneering work of Thorne & Campolattaro (1967), polar oscillations are described by a fifth-order system of ODEs. Later, Lindblom & Detweiler (1983) explicitly reduced the equation set into a fourth-order one, hereafter referred to as the LD formalism, where the oscillation of a star is described by four independent physical quantities \( H_1, K, W, \) and \( X \). While \( H_1 \) and \( K \) are measures of the metric, \( W \) and \( X \) are respectively proportional to the Lagrangian displacement in the radial direction and the pressure change of the fluid constituting the star (see Sect. 2 for more details). Besides the scheme proposed by Lindblom & Detweiler (1983), there are other schemes describing polar oscillations with two coupled second-order ODEs (Ipser & Price 1991, Price & Ipser 1991, Kojima 1992, Allen et al. 1998). They are in fact equivalent to fourth-order systems.

As in the Newtonian theory of stellar pulsations, polar oscillations can be classified into the fundamental \( (f) \) mode, the pressure \( (p) \) mode, and the gravity \( (g) \) mode. In addition to these, there is an extra kind of mode in the relativistic theory, namely the spacetime \( (w) \) mode, representing mainly perturbations in the metric (Kojima 1988, Kojimas & Schutz 1992, Andersson et al. 1995, Andersson & Kokkotas 1999).

Soon after the discovery of the polar \( w \)-mode, it has been conjectured that the fluid in a relativistic star will not be affected much by \( w \)-mode pulsations (Kokkotas & Schutz 1992, Andersson et al. 1995). Aiming at understanding qualitatively the physical nature of polar \( w \)-mode pulsations, Andersson et al. (1996) suggested the Inverse-Cowling Approximation (ICA) for polar \( w \)-mode oscillations. In ICA the fluid motion is completely neglected from the outset (details of ICA will be discussed in Sect. 2 of the present paper) and the metric variables are governed by a second-order ODE system. The approximation can successfully reproduce qualitative features of polar \( w \)-mode oscillations and clarify the nature of the oscillations (Andersson et al. 1996). However, as shown in Fig. 1, it fails to reproduce accurate values of the QNM frequencies. This is a sort of expected because the main concern of ICA was qualitative discussion on polar \( w \)-mode oscillations.

The major objective of the present paper is to establish a quantitatively correct approximation for the polar \( w \)-mode, referred to as Generalised Inverse-Cowling Approximation (GICA) in the present paper, which is capable of locating polar \( w \)-mode with high accuracy (see Fig. 1). In our analysis we find that the fluid motion of a star is in fact non-negligible. However, \( X \) is so small that the eigenfrequency of polar \( w \)-mode can be accurately determined with the approximation \( X = X' = 0 \) (hereafter a prime indicates differentiation with respect to circumferential radius \( r \)) by solving two coupled first-order ODEs. Thus, under GICA polar \( w \)-mode oscillations are describable with one single second-order ODE system.

The organization of our paper is as follows. In Sect. 2, we briefly review the LD formalism. The ICA developed by Andersson et al. (1996) is discussed in Sect. 3 and we propose GICA in Sect. 4. We then conclude our paper in Sect. 5 with a short conclusion and discussion. Unless otherwise stated, geometrized units in which \( G = c = 1 \) are adopted in the following discussion.

2 LD FORMALISM

In spherical coordinates, \( (t, r, \theta, \varphi) \), the geometry of spacetime around a non-rotating unperturbed neutron star is given by the line element:

\[
ds^2 = -e^{\lambda(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \tag{1}
\]

The metric coefficient \( e^{\lambda(r)} \) is determined by the mass distribution function \( m(r) \), the mass inside circumferential radius \( r \),

\[
e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}. \tag{2}
\]
The other metric coefficient $e^{\nu(r)}$ and the mass distribution function $m(r)$ can be obtained from the solution to the Tolman-Oppenheimer-Volff (TOV) equations [Tolman 1939, Oppenheimer & Volkoff 1939]:

\[
\begin{align*}
\frac{d\nu}{dr} &= \frac{2m + 8\pi r^3 p}{r(r - 2m)}, \\
\frac{dm}{dr} &= 4\pi r^2 \rho, \\
\frac{dp}{dr} &= -\frac{1}{2}(\rho + p)\frac{d\nu}{dr},
\end{align*}
\]  

(3) \hspace{1cm} (4) \hspace{1cm} (5)

where $\rho$ and $p$ are the mass density and pressure, respectively.

On the other hand, in the LD formalism for a pulsating relativistic star with radius $R$ [Lindblom & Detweiler 1983], the perturbation in the metric, $h_{\mu\nu}$, can be expressed in terms of three physical quantities $H_0, H_1$ and $K$ and displayed in a matrix form as follows:

\[
h_{\mu\nu} = \begin{pmatrix}
-e^{\nu} \rho^l H_0 & -i\omega m l H_1 & 0 & 0 \\
-i\omega m^l H_1 & -i\nu^l \rho H_0 & 0 & 0 \\
0 & 0 & -\nu^l r^{2} K & 0 \\
0 & 0 & 0 & -\nu^l r^{2} \sin^2 \theta K
\end{pmatrix} e^{i\omega t} Y_{lm}(\theta, \phi),
\]

(6)

where $\omega$ is eigenfrequency of the polar mode oscillation, $l$ is the angular momentum quantum number, and $\mu$ is

\[
\mu = \begin{cases}
\frac{r}{R} & r < R; \\
1 & r \geq R.
\end{cases}
\]

(7)

Correspondingly, the Lagrangian displacements of the fluid are expressed by:

\[
\begin{align*}
\xi^r &= \mu l r^{1} e^{-\lambda/2} W Y_{lm}(\theta, \phi) e^{i\omega t}, \\
\xi^\theta &= -\mu l r^{1} V \partial_\theta Y_{lm}(\theta, \phi) e^{i\omega t}, \\
\xi^\phi &= -\mu l r^{1} (r \sin \theta) \partial_\phi Y_{lm}(\theta, \phi) e^{i\omega t}.
\end{align*}
\]

(8) \hspace{1cm} (9) \hspace{1cm} (10)

In other words, $W$ and $V$ together measure the spatial displacements of the fluid.

Putting all these perturbations into the Einstein equations, [Lindblom & Detweiler 1983] showed that these physical variables are related by four first-order ODEs:

\[
\begin{align*}
H_1' &= \frac{1}{r} \left[ l + \frac{1}{2} \frac{me^\lambda}{r} + 4\pi r^2 e^\lambda (p - \rho) \right] H_1 + \frac{1}{r} e^\lambda H_0 + K - 16\pi (\rho p + p V), \\
K' &= \frac{1}{r} H_0 + \frac{1}{2r} (l + 1) H_1 - \frac{1}{2} \frac{l + 1}{r} - \frac{1}{2} \frac{l + 1}{r^2} K - 8\pi (\rho p + p V)\frac{e^{\nu/2}}{r} W, \\
W' &= \frac{1}{r} \left[ l + 1 \right] W + re^{\lambda/2} \left[ \frac{e^{nu/2}}{\gamma p} X - \frac{8(r + 1)}{r^2} V + \frac{1}{2} H_0 + K \right], \\
X' &= -\frac{1}{r} \left[ X + (p + p) e^{\nu/2} \right] \left\{ \frac{1}{2} \left( \frac{1}{r} - \frac{1}{2} \frac{l + 1}{r} \right) H_0 + \frac{1}{2} \left[ r \omega^2 e^{-\nu} + \frac{1}{2} (l + 1) \right] H_1 + \frac{1}{2} \left( \frac{3}{2} \nu - \frac{1}{r} \right) K - \frac{1}{2} \frac{l + 1}{r^2} \nu V \right. \\
&\quad \left. - \frac{1}{r} \left[ 4\pi (p + p) e^{\lambda/2} + \omega^2 e^{\lambda/2 - \nu} - \frac{1}{2} \frac{1}{r^2} \frac{d}{dr} \left( \frac{e^{\lambda/2} \nu}{r^2} \right) \right] W \right\},
\end{align*}
\]

(11) \hspace{1cm} (12) \hspace{1cm} (13) \hspace{1cm} (14)

where $\rho(r)$, $p(r)$ are respectively the density and the pressure at a radius $r$, $\gamma$ is the adiabatic index, defined by

\[
\gamma = \frac{\rho + p}{p} \frac{dp}{dr},
\]

(15)

and

\[
X = \omega^2 (p + p) e^{-\nu/2} V - \frac{1}{r} \nu' e^{(\nu - \lambda)/2} W + \frac{1}{2} (p + p) e^{\nu/2} H_0.
\]

(16)

The physical significance of $X$ is that it is a measure of $\Delta p$, the Lagrangian change in the fluid pressure. In fact, it can be readily shown from the theory developed by [Thorne & Campolattaro 1967] and [Lindblom & Detweiler 1983] that:

\[
\Delta p = -e^{-\nu/2} \mu \cdot X Y_{lm}(\theta, \phi) e^{i\omega t}.
\]

(17)

Besides, the metric perturbation $H_0$ is expressed in terms of $H_1, K$ and $X$:

\[
\begin{align*}
\left[ 3m + \frac{1}{2} (l + 2)(l - 1) r + 4\pi r^3 \right] H_0 &= 8\pi r^3 e^{-\nu/2} X - \left[ \frac{1}{2} (l + 1) (m + 4\pi r^3) - \omega^2 r^3 e^{-(\nu + \lambda)} \right] H_1 \\
&\quad + \left[ \frac{1}{2} (l + 2)(l - 1) r - \omega^2 r^3 e^{-\nu} - \frac{1}{r} e^\lambda (m + 4\pi r^3) (3m - 4\pi r^3) \right] K.
\end{align*}
\]

(18)
3 INVERSE-COWLING APPROXIMATION

Working in LD formalism and motivated by the conjecture that the fluid in a relativistic star is hardly excited by \( w \)-mode pulsations (Kokkotas & Schutz 1992; Andersson et al. 1993, Andersson et al. 1996) attempted to study the physical nature of polar \( w \)-mode oscillations by proposing the ICA with \( V = W = 0 \). Under such assumption, the motion of the fluid is completely ignored. To gauge the accuracy of the assumption \( V = W = 0 \), we show in Fig. 1 the \( l = 2 \) (always assumed for all numerical results in this paper) polar \( w \)-mode QNM frequencies of (a) a polytropic star with \( p = \kappa \rho^2 \) and \( \kappa = 100 \), which was used in the paper by Andersson et al. (1996), and (b) an APR1 star (Akmal et al. 1998), which typifies other realistic neutron stars, and compare the numerical results obtained respectively from exact LD formalism, ICA with \( V = W = 0 \) and GICA (to be developed in Sect. 4 of the present paper). It is obvious that in both cases the eigenfrequency obtained from the assumption \( V = W = 0 \) deviates markedly from the exact one and the agreement between the approximate and exact values is only qualitative. As noted by Andersson et al. (1996), the imaginary parts of QNM frequencies obtained from ICA are typically 20-30 percent smaller than the exact values and the relative error in the frequency spacing is about 10 percent. This is not really surprising because the original motivation of ICA was merely to verify the spacetime nature of polar \( w \)-mode oscillations.

The inadequacy of the approximation \( V = W = 0 \) can be clearly shown by comparing the contributions of various terms in (17), whose physical meaning will be discussed at the end of this paper. In Fig. 2, the absolute values of \( \omega^2 (p + p) e^{-\nu^2 /2} V \), \( p' e^{(\nu^2 - \lambda^2) /2} W / r \), \( (p + p) e^{\nu^2 /2} H_0 /2 \) and \( X \) are compared for the a typical polar \( w \)-mode of the APR1 neutron star considered in Fig. 1b. We note the term \( \omega^2 (p + p) e^{-\nu^2 /2} V \), a measure of the fluid motion in the tangential direction, is comparable to the term \( (p + p) e^{\nu^2 /2} H_0 /2 \), a perturbation in the metric. Besides, the term \( p' e^{(\nu^2 - \lambda^2) /2} W / r \), measuring the motion in the radial direction, is not as small as expected. Judging from the results shown in Figs. 1 and 2, we deem that the approximation \( V = W = 0 \) is unjustified. Instead, as suggested by the comparison shown in Fig. 2, the approximation \( X = X' = 0 \) seems to be more appropriate. Similar behaviour also prevails in stars constructed with other realistic EOSs. We note that this point has been discussed and exploited by (Tsui & Leung 2005a) to explain the universality observed in the polar \( w \)-mode. In the following section we will base on GICA with \( X = X' = 0 \) to establish a second-order ODE system that is able to locate \( w \)-mode QNMs accurately.

4 GENERALISED INVERSE-COWLING APPROXIMATION

Under GICA where \( X = X' = 0 \), (14) and (16) yield respectively

\[
0 = \frac{1}{2} \left( \frac{1}{2} - \frac{\nu'}{2} \right) H_0 + \frac{1}{2} \left[ r \omega^2 e^{-\nu} + \frac{1}{2} (l + 1) \right] H_1 + \frac{1}{2} \left( \frac{3}{2} \frac{\nu'}{r} - \frac{1}{r} \right) K - \frac{1}{2} \frac{l(l+1)}{r^2} \nu' V \\
- \frac{1}{r} \left[ 4 \pi (p + p) e^{\nu^2 /2} + \omega^2 e^{\nu^2 /2 - \nu} - \frac{1}{2} r^2 \frac{d}{dr} \left( \frac{e^{-\nu^2 /2} \nu'}{r^2} \right) \right] W; \tag{19}
\]

and

\[
V = \frac{-\nu'}{2 \omega^2} H_0 - \frac{e^\nu}{r^2 (-2m + r) \omega^2} W. \tag{20}
\]

Solving these two equations, we can express \( W \) in terms of the metric coefficients \( H_0, H_1 \) and \( K \):

\[
W = \frac{1}{D} \left\{ e^{\nu^2 /2} \left[ \frac{-5m + m - 12 \pi r^3}{2(r - 2m)} K + \frac{1}{4} \left[ (l + 1) + 2 \pi r^2 \frac{\omega^2}{2} \right] H_1 \right. \right.
\]

\[
+ \left. \left[ e^{\nu^2 (l+1)} (m + 4 \pi r^3) \right] + r^2 \omega^2 (3m - r + 4 \pi r^3) \right\}, \tag{21}
\]

where

\[
D = -4 \pi (p + p) + \frac{7m^2 - 4 \pi r^4 (p - r + 4 \pi r^2)}{r^2 (r - 2m)^2} - 4m (r + 2 \pi r^3). \tag{22}
\]
Moreover, following directly from (13) and \( X = 0 \), \( H_0 \) becomes:

\[
H_0 = \frac{1}{3m + \frac{2}{3}(l+1)l + 4\pi r^2} \left\{ \frac{-l(l+1)}{2} \left( m + 4\pi r^3 \right) + e^{-(\lambda + \nu)} e^{r^2} \right\} H_1 + \left[ \frac{1}{2} (l - 1) (l + 2) r - \frac{e^\lambda}{r} \left( m + 4\pi r^3 \right) \left( 3m - r + 4\pi r^3 \right) - r^3 \omega^2 e^{-\nu} \right] K, \tag{23}
\]

As a result, \( V, W \) and \( H_0 \) are all expressible in terms of \( H_1 \) and \( K \), which are governed by two first-order ODEs (11) and (12).

To solve this second-order ODE system, we first discuss the boundary conditions at the center \( r = 0 \). Straightforward expansion about the center shows that Eqs. (11) and (12) lead to:

\[
0 = -\frac{l+1}{r} H_1(0) + \frac{1}{r} \left[ H_0(0) + K(0) - 16\pi (p_0 + \rho_0) V(0) \right], \tag{24}
\]

\[
0 = \frac{1}{r} H_0(0) + \frac{1}{2r} l(l+1) H_1(0) - \frac{l+1}{r} K(0) - \frac{8\pi (p_0 + \rho_0)}{r} W(0), \tag{25}
\]

Comparing (24) and (25), we see that:

\[
W(0) = -l V(0). \tag{26}
\]

In addition, (23) gives

\[
K(0) = H_0(0), \tag{27}
\]

at the center. Putting (20) and (24) back into (20), we have:

\[
K(0) = H_0(0) = \left[ -\frac{8\pi}{3} (3p + \rho) + \frac{2\omega^2 e^{-\nu}}{l} \right] W(0). \tag{28}
\]

Finally, from (24) or (25),

\[
H_1(0) = \frac{1}{l(l+1)} \left[ 2l K(0) + 16\pi (p + \rho) W(0) \right]. \tag{29}
\]

Therefore, for a given \( W(0) \), which merely provides a scale of the solution, the values of \( H_1(0), K(0), H_0(0) \) and \( V(0) \) are all determined. Compared with the boundary conditions at the center, the boundary condition at the stellar surface \( r = R \) is much simpler. There are two independent solutions near the surface with arbitrarily assigned values of \( H_1(R) \) and \( K(R) \). Integrating (11) and (12) outward from \( r = 0 \) and inward from \( r = R \), and matching the values of \( H_1, K \) at some intermediate value of \( r \), say \( r = R/2 \), can numerically determine the solution of the second-order system up to a multiplicative constant.

The QNM frequencies obtained from GICA are shown in Fig. 1 and Table 1, which agree nicely with the exact values. In particular, both the percentage errors in the frequency and the damping rate, \( E_\nu \) and \( E_\eta \), decrease with increasing frequency, reflecting the decrease in the Lagrangian change in pressure in high-frequency polar \( w \)-modes. Meanwhile, it is also worthy to note that even the \( w_{11} \) mode, characterized by large damping rates, can be reproduced accurately. Hence, GICA proposed here, which assumes \( X = 0 \), clearly outperforms the ICA with the approximation \( V = W = 0 \).

## 5 CONCLUSION AND DISCUSSION

We have developed in the present paper an accurate scheme, GICA, to describe \( w \)-mode oscillations of neutron stars. Instead of completely ignoring the displacement of the fluid element of a star under consideration, we have shown that the term \( X \), which is proportional to the Lagrangian change in the pressure, is negligible. Under GICA, \( w \)-mode oscillations of neutron stars are governed by two first-order ODE in the metric coefficients \( H_1 \) and \( K \). The displacements of matter, measured by \( V \) and \( W \), are properly considered and found to be expressible in terms of \( H_1 \) and \( K \). Numerical results obtained directly from the approximation \( X = X' = 0 \) clearly demonstrate the validity and accuracy of GICA (see Fig. 1 and Table 1).

In addition to yielding reliable numerical values of QNM frequencies for \( w \)-mode oscillations, GICA proposed here also provides a proper physical picture for such oscillations. A star undergoing polar \( w \)-mode oscillations has non-negligible displacements in the tangential and radial directions. However, as shown in Fig. 2, such displacements are driven by the perturbed spacetime, instead of the excess pressure due to the motion of matter.

It is well known that axial \( w \)-mode oscillations of compact stars, which are merely oscillations in the metric variables, are governed by a second-order Regge-Wheeler wave equation (Chandrasekhar & Ferrari 1991). Our finding that polar \( w \)-mode oscillations are adequately described by one single second-order ODE system thus puts polar and axial \( w \)-modes on an equal footing in the sense that both of them are governed by second-order ODE systems. As shown in the review by
Kokkotas & Schmidt (1999). QNMs of these two kinds of \( w \)-mode lie asymptotically on a common smooth curve in the complex frequency plane. In addition, Kojima et al. (1995) and Andersson et al. (1996) also noted that for incompressible ultra compact stars the oscillation frequencies of these two \( w \)-modes agree to within one percent. GICA proposed here may be the first step in the long process of seeking quantitative interpretation of such interesting asymptotic behaviour and the similarity between two kinds of \( w \)-mode.

We have adopted LD formalism to describe polar oscillations in the present paper. In fact, there are other alternative formalisms for polar oscillations of relativistic stars (see, e.g., Ipser & Price 1991, Kojima 1992, Allen et al. 1998, Nagar et al. 2004). In particular, in the formalism proposed by Allen et al. (1998), and later used by Nagar et al. (2004) to study accretion-driven gravitational waves from compact stars, such oscillations are described by two coupled second-order wave equations in the metric variables plus a Hamiltonian constraint yielding the Eulerian change in the fluid pressure. The approach is also equivalent to a fourth-order differential equation system with the tortoise ordinate being the independent variable. The formalism is evidently more amenable to decoupling metric and matter variables. It is worthy to develop quantitatively accurate approximation scheme for polar \( w \)-mode oscillations based on such formalism. Currently we are working along this direction.

Lastly, it is worthy to note the physical meaning of (16), which can be rewritten as follows:

\[
\mu l \omega (\rho + p) Y_{lm}(\theta, \phi)e^{i\omega t} = -e^{\nu} \delta p - \frac{\mu l (\rho + p) e^{\nu} H_0}{2} Y_{lm}(\theta, \phi)e^{i\omega t},
\]

where

\[
\delta p \equiv \Delta p - p' \xi = \Delta p - \frac{\mu l p' e^{-\lambda/2}W}{r} Y_{lm}(\theta, \phi)e^{i\omega t}
\]

is the Eulerian change in pressure. Each term in (30) has an obvious analogue in Newtonian theory of stellar pulsation (see, e.g. Eq. (17.21b) in Cox 1980). The term \( \mu l \omega (\rho + p) Y_{lm}(\theta, \phi)e^{i\omega t} \) in the LHS is a measure of inertia \( \times \) acceleration, whereas the two terms in the RHS, \( -e^{\nu} \delta p \) and \( -\mu l (\rho + p) e^{\nu} H_0 Y_{lm}(\theta, \phi)e^{i\omega t}/2 \), correspond to the pressure and gravity forces, respectively. Thus, (30) can be understood as the extension of application of Newton’s second law to general relativistic theory of stellar pulsation. Following directly from Fig. 2 and (30), the net driving forces for \( w \)-mode pulsation is the sum of gravity and the Eulerian change in pressure.

To compare \( w \)-mode with \( f \) and \( p \) modes, we show in Fig. 3 and Fig. 4 the corresponding physical quantities appearing in (16) for the \( f \)-mode and the first \( p \)-mode of the same star. It is interesting to note that for \( f \)-mode oscillations the major driving force is the Eulerian change in pressure \( \delta p \) and the gravity in fact partially cancels the effect of \( \delta p \). On the other hand, the major driving force for \( p \)-mode oscillations is evidently due to the Lagrangian change in pressure \( \Delta p \). We expect that these findings can cast light on eigen-mode analysis for pulsations of relativistic stars and assist in developing appropriate approximation schemes under different circumstances. Relevant study is now underway and will be reported elsewhere in due course.

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Table 1. Polar $v$-mode QNM frequencies of the two typical stars considered in Fig. 1 are obtained from GICA and exact LD formalism, denoted respectively by $\tilde{\omega}$ and $\omega$, and tabulated here. The percentage errors in the frequency and damping rate, $E_r = |\tilde{\omega}_r / \omega_r - 1| \times 100\%$ and $E_i = |\tilde{\omega}_i / \omega_i - 1| \times 100\%$, are also included.

| Polytropic star | $M\tilde{\omega}_1$ | $M\tilde{\omega}_1$ | $M\omega_r$ | $M\omega_i$ | $E_r$ | $E_i$ |
|-----------------|---------------------|---------------------|------------|------------|------|------|
| Polytrope 0.0360 | 0.7045              | 0.0385              | 0.7064     | 6.5        | 0.27 | |
| Polytrope 0.3377 | 0.3804              | 0.3427              | 0.3781     | 1.5        | 0.6  | |
| Polytrope 0.5027 | 0.2593              | 0.5029              | 0.2645     | 0.022      | 2.0  | |
| Polytrope 0.8741 | 0.3637              | 0.8752              | 0.3694     | 0.12       | 1.5  | |
| Polytrope 1.2235 | 0.4193              | 1.2241              | 0.4250     | 0.05       | 1.3  | |
| Polytrope 1.5683 | 0.4596              | 1.5687              | 0.4652     | 0.026      | 1.2  | |
| Polytrope 1.9111 | 0.4915              | 1.9114              | 0.4971     | 0.016      | 1.1  | |
| Polytrope 2.2528 | 0.5180              | 2.2531              | 0.5235     | 0.011      | 1.1  | |
| Polytrope 2.5938 | 0.5407              | 2.5940              | 0.5462     | 0.0077     | 1.0  | |
| Polytrope 2.9343 | 0.5606              | 2.9344              | 0.5660     | 0.0059     | 0.97 | |
| Polytrope 3.2743 | 0.5783              | 3.2745              | 0.5838     | 0.0051     | 0.94 | |
| Polytrope 3.6141 | 0.5943              | 3.6143              | 0.5998     | 0.0046     | 0.91 | |
| Polytrope 3.9536 | 0.6090              | 3.9538              | 0.6144     | 0.0041     | 0.88 | |

| APR1 EOS | $M\tilde{\omega}_1$ | $M\tilde{\omega}_1$ | $M\omega_r$ | $M\omega_i$ | $E_r$ | $E_i$ |
|----------|---------------------|---------------------|------------|------------|------|------|
| APR1 0.2963 | 0.3551             | 0.2983              | 0.3533     | 0.69       | 0.51 | |
| APR1 0.5057 | 0.2924             | 0.5073              | 0.2963     | 0.3        | 1.3  | |
| APR1 0.9247 | 0.3777             | 0.9266              | 0.3818     | 0.21       | 1.1  | |
| APR1 1.3138 | 0.4307             | 1.3155              | 0.4353     | 0.13       | 1.1  | |
| APR1 1.6958 | 0.4700             | 1.6973              | 0.4749     | 0.086      | 1.0  | |
| APR1 2.0754 | 0.5001             | 2.0767              | 0.5051     | 0.063      | 1.0  | |
| APR1 2.4551 | 0.5238             | 2.4564              | 0.5287     | 0.054      | 0.93 | |
| APR1 2.8361 | 0.5431             | 2.8376              | 0.5478     | 0.053      | 0.85 | |
| APR1 3.2188 | 0.5594             | 3.2205              | 0.5637     | 0.055      | 0.77 | |
| APR1 3.6028 | 0.5733             | 3.6049              | 0.5771     | 0.058      | 0.66 | |
Figure 1. Polar \(w\)-mode QNMs of (a) a polytropic star with \(\gamma = 2\) and compactness = 0.211; and (b) an APR1 star with compactness = 0.2 are obtained from exact LD formalism, ICA proposed previously and GICA developed here, respectively. Here \(M\) is the mass of the star.

Figure 2. The absolute values of various terms in \(\omega^2(\rho + p)e^{-\nu/2}V, p'e^{(\nu-\lambda)/2}W/r, (\rho + p)e^{\nu/2}H_0/2\) and \(X\), are compared for the least damped polar \(w\)-mode \((l = 2)\) of the APR1 neutron star considered in Fig. 1b.
Figure 3. The absolute values of various terms in Eq. (16), \( \omega^2 (\rho + p)e^{-\nu/2V} \), \( p'e^{(\nu-\lambda)/2W\nu} \), \( (\rho + p)e^{\nu/2H_0} \), and \( X \), are compared for the \( f \)-mode \((l = 2)\) of the APR1 neutron star considered in Fig. 1b.
Figure 4. The absolute values of various terms in \(\omega^2(\rho + p)e^{-\nu/2}V, p'e^{(\nu - \lambda)/2}W/r, (\rho + p)e^{\nu/2}H_0/2\) and \(X\), are compared for the first \(p\)-mode \((l = 2)\) of the APR1 neutron star considered in Fig. 1b.