Newtonian limit of fully nonlinear cosmological perturbations in Einstein’s gravity

Jai-chan Hwang  
Department of Astronomy and Atmospheric Sciences, Kyungpook National University, Daegu 702-701, Republic of Korea  
E-mail: jchan@knu.ac.kr

Hyerim Noh  
Korea Astronomy and Space Science Institute, Daejeon 305-348, Republic of Korea  
E-mail: hr@kasi.re.kr

Abstract.  
We prove that in the infinite speed-of-light limit (i.e., non-relativistic and subhorizon limits), the relativistic fully nonlinear cosmological perturbation equations in two gauge conditions, the zero-shear gauge and the uniform-expansion gauge, exactly reproduce the Newtonian hydrodynamic perturbation equations in the cosmological background; as a consequence, in the same two gauge conditions, the Newtonian hydrodynamic equations are exactly recovered in the Minkowsky background.

1. Introduction  
Newton’s gravity and Einstein’s gravity are two competing and complementary theoretical frames where the current cosmological research is practically based. Only with the advent of Einstein’s gravity the history of modern physical cosmology has begun [1]. Equations describing the cosmological world model were first derived based on Einstein’s gravity by Friedmann in 1922 [2], and the Newtonian study followed later by Milne and McCrea in 1934 [3]. The two results coincided in the zero-pressure limit. Similarly, the linear perturbation equations were first derived based on Einstein’s gravity by Lifshitz in 1946 [4], and the Newtonian study followed later by Bonnor in 1957 [5]. For density perturbation, the two results again coincided in the zero-pressure limit [6].

The Newtonian limit of Einstein’s gravity in cosmology, however, is not a settled subject. The problem is rather serious in the level of background world model where the well known Newtonian cosmology [3] is known to be a special one motivated and guided by the results in Einstein’s gravity; without additional symmetry in the matter distribution or the boundary condition at infinity the Newtonian dynamics and gravitation could lead to the world models different from the Friedmann’s one based on Einstein’s gravity, see [7]. Even in the post-Newtonian approach in cosmology we
have to *subtract* the background equations based on Einstein’s gravity to get proper Newtonian equations for the perturbation [8].

Despite such a trouble in the background level, curiously we often have quite successful relativistic/Newtonian correspondence in the perturbation level. Exact Newtonian perturbation equations are recovered in the Newtonian limit of the post-Newtonian approximation independently of the gauge condition [8]; exact Newtonian equations are recovered in Minkowsky background [9]. In the perturbation theory, however, the analysis and results crucially depend on the gauge choice [10, 11, 12].

The exact correspondence of all three Newtonian perturbation variables (perturbed density, velocity, and gravitational potential) is not available in a single gauge condition even to the linear order [11, 12]. Such a correspondence of all variables is available only in the subhorizon limit in the zero-shear gauge and the uniform-expansion gauge [12], which is shown to be valid even to the second order in perturbation [13].

However, whether the relativistic/Newtonian correspondence in the subhorizon limit is valid to the fully nonlinear order was not known in the literature. Here we provide a proof of the correspondence in the two fundamental gauge conditions. For the correspondence we will show that we need weak gravity and slow-motion limits, in addition to the subhorizon limit as well as negligible pressure and internal energy density compared with the energy density; all these limits can be summarized as the infinite speed-of-light limit. Such a proof is now available mainly due to a recent advent of the exact and fully nonlinear cosmological perturbation theory in Einstein’s gravity [14]; for the concrete nature of our formulation with a certain limit caused by *ignoring* the transverse-tracefree part of the metric tensor, see section 2.

Section 2 is a summary of the fully nonlinear cosmological perturbation theory presented in [14]. Section 3 presents our main result, proving the Newtonian limit. In sections 4 and 5 we clarify a couple of issues. Section 6 is a discussion.

### 2. Fully nonlinear perturbations

We consider the scalar- and vector-type perturbations in a *flat* background with the metric convention [15, 14]

\[
ds^2 = -a^2 (1 + 2\alpha) \, dx^0 dx^0 - 2a\chi_i dx^0 dx^i + a^2 (1 + 2\varphi) \delta_{ij} dx^i dx^j,\]

(1)

where \(a(x^0)\) is the cosmic scale factor, and \(\alpha, \varphi\) and \(\chi_i\) are functions of spacetime with *arbitrary* amplitudes; index of \(\chi_i\) is raised and lowered by \(\delta_{ij}\) as the metric; notice that \(x^0 = \eta\) with \(ad\eta \equiv cdt\). The spatial part of the metric is simple because we already have taken the spatial gauge condition without losing any generality to fully nonlinear order [15, 14].

The arbitrary perturbation variables \(\alpha, \varphi\) and \(\chi_i\) give only five independent degrees of freedom whereas we should have six independent physical degrees of freedom for the most general perturbations. This is because we have *ignored* the transverse-tracefree part corresponding to two degrees of freedom associated with the gravitational waves
Newtonian limit of fully nonlinear cosmological perturbations in Einstein's gravity

(to the linear order) and other transverse-tracefree distortion of the three-dimensional hypersurface to the nonlinear order. In addition, although we have fixed the spatial gauge condition with three gauge (congruence) degrees of freedom, we have not fixed the temporal gauge (slicing) degree of freedom yet, thus enabling the basic set of equations presented in a sort of gauge-ready form. Thus, our perturbation variables contain four physical degrees of freedom (two for scalar-type perturbation and the other two for the vector-type perturbation) and one temporal gauge degree of freedom. We have freedom to impose the slicing condition depending on the mathematical simplification and/or physical interpretation. Our equations are designed so that after imposing any of the fundamental slicing condition, the gauge degrees of freedom are completely removed, and consequently each variable in those gauge can be equivalently regarded as a unique gauge-invariant variable, to nonlinear perturbation order, see equation (19) and below.

In order to have the fully nonlinear and exact equations it is essentially important to take the spatial gauge condition as we do and ignore the transverse-tracefree part of the metric. By taking the first condition we do not lose any generality especially because the spatial gauge condition we take is the unique one which (together with any temporal gauge condition in the pool of our suggested fundamental slicing conditions) leaves the remaining variables spatially gauge invariant. Whereas neglecting the transverse-tracefree part of the metric should be regarded as an important shortcoming of our formulation. In this sense our formulation can be regarded as not exact. At the moment we cannot formulate the fully nonlinear and exact equations in the presence of the transverse-tracefree part of the metric, and these should be handled perturbatively only. We still call our formulation fully nonlinear and exact despite this important shortcoming because we have not imposed any condition on our metric and energy-momentum perturbation variables and formulated equations in exact forms, see equations (10)-(17).

We consider a fluid without anisotropic stress. The energy momentum tensor is given as

\[ \tilde{T}_{ab} = \tilde{\rho} c^2 \tilde{u}_a \tilde{u}_b + \tilde{p} (\tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b), \]

(2)

where tildes indicate covariant quantities; \( \tilde{u}_a \) is the normalized fluid four-vector; \( \tilde{\rho} \) and \( \tilde{p} \) are the mass density and pressure, respectively. We decompose the fluid quantities to the background and perturbation as

\[ \tilde{\rho} = \rho + \delta \rho, \quad \tilde{p} = p + \delta p, \quad \tilde{u}_i \equiv a \frac{v_i}{c}, \]

(3)

where the index of \( v_i \) is raised and lowered by \( \delta_{ij} \) as the metric. In the explicit presence of the internal energy \( \tilde{\rho} \tilde{\Pi} \), \( \tilde{\rho} \) should be replaced by \( \tilde{\rho} (1 + \tilde{\Pi}/c^2) \) [9]; in the latter expression \( \tilde{\rho} \) is the material density.

In [14] we have introduced several different definitions of the fluid three-velocity (see the Appendix D in [14]). Although mathematically equivalent, in this work we will use the following definition. The fluid three-velocity measured by the Eulerian observer
Newtonian limit of fully nonlinear cosmological perturbations in Einstein’s gravity

with the normal four-vector $\bar{n}^{c}$ is introduced as

$$\hat{V}^{i} \equiv \frac{\bar{h}^{i}h^{c}a^{c}}{-\bar{n}^{c}a^{c}} = \frac{1}{N} \left( \bar{n}^{i} \hat{u}^{0} + N^{i} \right), \tag{4}$$

where $\bar{h}^{i}h^{c}a^{c}$ is the projection tensor normal to $\bar{n}^{c}$, and the index of $\hat{V}^{i}$ is raised and lowered by the ADM three-space metric $h_{ij}$; $N$ and $N^{i}$ are the lapse and shift vector, respectively, in the ADM formulation. In order to use the perturbation notation, we introduce

$$\hat{V}_{i} \equiv a^{i} \tilde{v}_{c}, \tag{5}$$

where the index of $\tilde{v}_{i}$ is raised and lowered by $\delta_{ij}$. Compared with $v_{i}$ we have

$$v_{i} \equiv \hat{\gamma} \tilde{v}_{i}, \tag{6}$$

where

$$\hat{\gamma} = \sqrt{1 + \frac{v^{k}v_{k}}{c^{2}(1 + 2\varphi)}} = \frac{1}{\sqrt{1 - \frac{\varphi^{2}v_{k}v^{k}}{c^{2}(1 + 2\varphi)}}}, \tag{7}$$

is the Lorentz factor. In terms of $\tilde{v}_{i}$ the Lorentz factor becomes a well known form.

We can decompose $\chi_{i}$, $v_{i}$ and $\hat{v}_{i}$ to scalar- and vector-type perturbations even to the nonlinear order as

$$\chi_{i} = c\chi_{i} + a\Psi_{i}^{(v)}, \quad v_{i} \equiv -v_{i} + v_{i}^{(v)}, \quad \hat{v}_{i} \equiv -\tilde{v}_{i} + \tilde{v}_{i}^{(v)}, \tag{8}$$

with $\Psi_{i}^{(v)} \equiv 0$ and $v_{i}^{(v)} \equiv 0 \equiv \tilde{v}_{i}^{(v)}$. Due to the nonlinear relation between $v_{i}$ and $\hat{v}_{i}$ the scalar- and vector-decompositions for $v_{i}$ and $\hat{v}_{i}$ do not coincide with each other to the nonlinear order. To the nonlinear order our scalar- and vector-type perturbations are coupled in the equation level.

We assign dimensions as

$$[a] = [\bar{u}^{ab}] = [\bar{u}] = [\alpha] = [\varphi] = [\chi^{i}] = [v^{i}/c] = [\tilde{\varphi}^{i}/c] = 1, \quad [v/c] = L, \quad [x^{i}] = L, \quad [\chi] = T, \quad [\kappa] = T^{-1}, \quad [\bar{T}_{ab}] = [\bar{\varphi}^{2}c] = [\bar{p}], \quad [G\bar{\varphi}] = T^{-2}, \tag{9}$$

where $\kappa$, a perturbed part of the trace of extrinsic curvature (equivalently, a perturbed part of the expansion scalar of the normal-frame vector with a minus sign) will be introduced in equation (10).

Now, using $\tilde{v}_{i}$ as the fluid three-velocity, a complete set of fully nonlinear perturbation equations without taking the temporal gauge are the following [14].

Definition of $\kappa$:

$$\kappa \equiv 3\frac{\dot{a}}{a} \left( 1 - \frac{1}{N} \right) - \frac{1}{N(1 + 2\varphi)} \left[ 3\dot{\varphi} + \frac{c}{a^{2}} \left( \chi_{,k}^{k} + \frac{\chi^{k}\varphi_{,k}}{1 + 2\varphi} \right) \right]. \tag{10}$$

ADM energy constraint:

$$-\frac{3}{2} \left( \frac{\dot{a}^{2}}{a^{2}} - \frac{8\pi G}{3} \bar{\rho} - \frac{\Lambda c^{2}}{3} \right) + \frac{\dot{a}}{a} \kappa + \frac{c^{2}\Delta \varphi}{a^{2}(1 + 2\varphi)^{2}}
= \frac{1}{6} \kappa^{2} - 4\pi G \left( \bar{\rho} + \bar{p} \right) \left( \dot{\gamma}^{2} - 1 \right) + \frac{3}{2} \frac{c^{2}\varphi^{,i}\varphi_{,i}}{a^{2}(1 + 2\varphi)^{3}} - \frac{c^{2}}{4} K_{j}^{i}K^{j}_{i} \tag{11}$$
Covariant energy conservation:

\[ \frac{2}{3} \rho_N + \frac{c}{2a^2 N(1 + 2\phi)} \left( \Delta \chi_i + \frac{1}{3} \chi_{i,k} \right) + 8\pi G \left( \tilde{\theta} + \tilde{\rho} \right) a^2 \tilde{v}_i \]

\[ = \frac{c}{a^2 N(1 + 2\phi)} \left\{ \left( \frac{N_j}{N} - \frac{\varphi_j}{1 + 2\phi} \right) \left[ \frac{1}{2} \left( \chi^j,i + \chi_i^j \right) - \frac{1}{3} \delta^j_i \chi^k,k \right] \right. \]

\[- \left. \frac{\varphi^j}{(1 + 2\phi)^2} \left( \chi_i \varphi_j + \frac{1}{3} \chi_j \varphi_i \right) + \frac{N_j}{1 + 2\phi} \nabla_j \left[ \frac{1}{N} \left( \chi^i \varphi,i + \chi^j \varphi^j - \frac{2}{3} \delta^j_i \chi^k,k \right) \right] \right\}. \] (12)

Trace of ADM propagation:

\[- \frac{3}{N} \left( \frac{\ddot{a}}{a} \right) - 3 \frac{\dot{a}^2}{a^2} - 4\pi G \left( \tilde{\theta} + 3 \frac{\tilde{\rho}}{c^2} \right) + \Lambda \dot{c}^2 + \frac{1}{N} \dot{\kappa} + 2 \frac{\dot{\kappa}}{\kappa} + \frac{c^2 \Delta N}{a^2 N(1 + 2\phi)} \]

\[ = \frac{3}{\kappa^2} + 8\pi G \left( \tilde{\theta} + \frac{\tilde{\rho}}{c^2} \right) \left( \tilde{\gamma}^2 - 1 \right) - \frac{c}{a^2 N(1 + 2\phi)} \left( \chi^i \kappa,i + \chi^j \varphi^j - \frac{2}{3} \delta^j_i \chi^k,k \right) + c^2 \nabla^j \nabla^i. \] (13)

Tracefree ADM propagation:

\[ \left( \frac{1}{N} \frac{\partial}{\partial t} + \frac{3}{a} \frac{\dot{a}}{a} - \kappa + \frac{c \chi^k}{a^2 N(1 + 2\phi)} \nabla_k \right) \left\{ \frac{c}{a^2 N(1 + 2\phi)} \right. \]

\[ \times \left[ \frac{1}{2} \left( \chi^i,j + \chi_j^i \right) - \frac{1}{3} \delta^i_j \chi^k,k - \frac{1}{1 + 2\phi} \left( \chi^i \varphi,j + \chi_j \varphi^i - \frac{2}{3} \delta^j_i \chi^k,k \right) \right] \}

\[ - \frac{c^2}{a^2(1 + 2\phi)} \left[ \frac{1}{1 + 2\phi} \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) \varphi + \frac{1}{N} \left( \nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta \right) N \right] \]

\[ = 8\pi G \left( \tilde{\theta} + \frac{\tilde{\rho}}{c^2} \right) \left[ \frac{\tilde{\gamma}^2 \tilde{\varphi} - \tilde{\varphi}}{c^2(1 + 2\phi)^2} - \frac{3}{3} \delta^i_j \varphi^j \right] + \frac{c^2}{a^4 N^2(1 + 2\phi)^2} \]

\[ \times \left[ \frac{1}{2} \left( \chi^i,j \chi_{j,k} - \chi_{k,j} \chi^i,k \right) + \frac{1}{1 + 2\phi} \left( \chi^i \kappa,j \chi_{j,k} - \chi^j \kappa,i \chi_{j,k} \right) \right] - \frac{c^2}{a^2(1 + 2\phi)^2} \]

\[ \times \left[ \frac{3}{1 + 2\phi} \left( \varphi^i \varphi,j - \frac{1}{3} \delta^i_j \varphi^j \right) \right] + \frac{N}{N} \left( \varphi^i \varphi,j + \varphi,j \varphi^i - \frac{2}{3} \delta^j_i \varphi^k \right). \] (14)

Covariant energy conservation:

\[ \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\phi)} \left( N \dot{\nabla}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \tilde{\theta} + \tilde{\rho} \left( \dot{\theta} + \frac{\dot{\rho}}{c^2} \right) \left\{ N \left( \frac{3}{a} \kappa - \kappa \right) \right. \]

\[ + \frac{(N \dot{\nabla})_{,k}}{a(1 + 2\phi)} + \frac{N \dot{\nabla} \varphi_{,k}}{a(1 + 2\phi)^2} + \frac{1}{c^2} \dot{\gamma} \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\phi)} \left( N \dot{\nabla}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \tilde{\gamma} \right\} = 0. \] (15)

Covariant momentum conservation:

\[ \frac{1}{a} \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\phi)} \left( N \dot{\nabla}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \left( a \hat{\gamma} \hat{v}_i \right) + \hat{\nabla}^k \hat{v}_i \left( \frac{c \chi_k}{a^2(1 + 2\phi)} \right) \]

\[ + \frac{c^2}{a} \dot{N}_i = \left( 1 - \frac{1}{\dot{\gamma}^2} \right) \frac{c^2 \varphi_{,i}}{a(1 + 2\phi)} \]

\[ + \frac{1}{\tilde{\theta} + \frac{\tilde{\rho}}{c^2}} \left\{ \frac{N}{a \dot{\gamma}^2} \hat{\nabla}^i + \hat{\nabla}_i \left[ \frac{\partial}{\partial t} + \frac{1}{a(1 + 2\phi)} \left( N \dot{\nabla}^k + \frac{c}{a} \chi^k \right) \nabla_k \right] \tilde{\rho} \right\} = 0, \] (16)
Newtonian limit of fully nonlinear cosmological perturbations in Einstein’s gravity

where

\[ N \equiv \sqrt{1 + 2\alpha + \frac{\chi^k\chi_k}{a^2(1 + 2\varphi)}} \]

\[ K^i_j K^j_i = \frac{1}{a^4 N^2 (1 + 2\varphi)^2} \left\{ \frac{1}{2} \chi^i_j (\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i_i \chi^j_j \right\} \]

\[ = \frac{4}{1 + 2\varphi} \left[ \frac{1}{2} \chi^i_j \varphi^j (\chi_{i,j} + \chi_{j,i}) - \frac{1}{3} \chi^i_i \chi^j_j \varphi_j \right] + \frac{2}{(1 + 2\varphi)^2} \left( \chi^i_i \varphi^j \varphi_j + \frac{1}{3} \chi^i_j \varphi_i \varphi_j \right) \]

(17)

with \( N \) introduced as \( N \equiv aN; \) \( N \) is the ADM lapse function with \( N \equiv 1/\sqrt{-g^{00}} \).

These equations were derived in Section 3 of [14]; here we express the equations using \( \hat{v} \) instead of \( v \) used in [14], set the energy density as \( \bar{\mu} \equiv \bar{c}^2 \), and recover \( c \); in our notation \( \bar{c} \) includes the internal energy.

To the background order, equations (11), (13) and (15) give

\[ \frac{\dot{a}}{a^2} = \frac{8\pi G}{3} \varrho + \frac{\Lambda c^2}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \varrho + \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3}, \quad \dot{\varrho} + 3\frac{\dot{a}}{a} \left( \varrho + \frac{p}{c^2} \right) = 0. \]

(18)

As mentioned, contrary to common beliefs, even with \( p = 0 = \Lambda \), these equations are pure relativistic in their origin; for a clarification, see [7].

Equations (10)-(16) are presented without taking the temporal gauge (hypersurface or slicing) condition. As the temporal gauge condition we can impose any one of the following conditions

- **comoving gauge**: \( \hat{v} \equiv 0 \),
- **zero-shear gauge**: \( \chi \equiv 0 \),
- **uniform-curvature gauge**: \( \varphi \equiv 0 \),
- **uniform-expansion gauge**: \( \kappa \equiv 0 \),
- **uniform-density gauge**: \( \delta \equiv 0 \).

(19)

or combinations of these to each perturbation order; we may call these the fundamental gauge conditions. With the imposition of any of these temporal gauge conditions the remaining perturbation variables are free from the remnant (spatial and temporal) gauge mode, and have unique gauge-invariant combinations. Thus, we can regard each perturbation variable in these gauges as the gauge-invariant one to the nonlinear order [15, 14]. The temporal synchronous gauge condition sets \( \alpha = 0 \); however, as this condition fails to fix the gauge degree of freedom completely, and leaves the remnant gauge mode even from the linear order, we do not consider it as the fundamental gauge condition in the nonlinear perturbation analysis. As the comoving gauge condition we suggested \( \hat{v} \equiv 0 \) which is the same as \( v \equiv 0 \) for vanishing vector-type perturbation; in the presence of vector-type perturbation these two conditions differ from each other from the third-order perturbation.
3. Newtonian correspondence

3.1. Newtonian equations

The Newtonian hydrodynamic equations in the cosmological background are

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} \rho = - \frac{1}{a} \nabla \cdot (\bar{\rho} \mathbf{v}), \quad (20)
\]

\[
\dot{\mathbf{v}} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{a} \nabla U - \frac{1}{a \bar{\rho}} \nabla \bar{p}, \quad (21)
\]

\[
\frac{\Delta}{a^2} U = -4\pi G (\bar{\rho} - \rho), \quad (22)
\]

where \(\bar{\rho}\) is the material density. These equations properly reduce to the Newtonian hydrodynamic equations in the Minkowsky background where the background order quantities become \(a \equiv 1\) and \(\rho \equiv 0\) with no \(\Lambda\). From equations (20) and (22) we have

\[
\dot{U} + \frac{\dot{a}}{a} U = 4\pi G a \Delta^{-1} \nabla \cdot (\bar{\rho} \mathbf{v}). \quad (23)
\]

Our aim in this work is to show that equations (10)-(16) properly reduce to equations (20)-(23) in the infinite speed-of-light limit in the zero-shear gauge and in the uniform-expansion gauge to the fully nonlinear order.

Equation (20) can be decomposed to the background order part (the third one of equation (18) with vanishing pressure term) and the perturbed part

\[
\dot{\delta} = - \frac{1}{a} \nabla \cdot [(1 + \delta) \mathbf{v}], \quad (24)
\]

where \(\delta \equiv \delta \rho / \rho\). Newtonian derivation of equations (21), (22) and (24) can be found in sections 7-9 of [16].

3.2. Infinite speed-of-light limit

The relativistic perturbation variables \(\alpha, \varphi, \bar{v}^i, \bar{\rho}\) and \(\bar{p}\) in this paragraph are valid in both the zero-shear gauge and the uniform-expansion gauge. As the non-relativistic limit we consider

\[
\alpha \ll 1, \quad \varphi \ll 1, \quad \bar{v}^k \bar{v}_k / c^2 \ll 1, \quad \bar{p} \ll \bar{\rho} c^2, \quad \bar{\Pi} / c^2 \ll 1. \quad (25)
\]

We identify

\[
\alpha = - \frac{1}{c^2} U, \quad \varphi = \frac{1}{c^2} V, \quad v^i = \bar{v}^i = \mathbf{v}, \quad (26)
\]

where \(\mathbf{v}\) is the perturbed Newtonian velocity; \(U\) and \(V\) correspond to the Newtonian and the post-Newtonian perturbed gravitational potentials, respectively [9, 8]; later we will show \(\varphi = -\alpha\), thus \(V = U\). As the subhorizon limit, we take the dimensionless quantity

\[
\frac{k^2 c^2}{a^2 H^2} \gg 1, \quad (27)
\]

where \(k\) the comoving wave-number with \(\Delta = -k^2\) in the Fourier space notation; \(H \equiv \dot{a} / a\); in the presence of \(\Lambda\) we consider \(H^2 \sim 8\pi G \bar{\rho}\). The first two conditions in
equation (23) are the weak gravity (general relativistic) limits, the third one is the slow-motion (special relativistic) limit, and the last two conditions imply that we ignore the pressure and internal energy density compared with the energy density. Both the non-relativistic limits in equation (23) and the subhorizon limit in equation (27) correspond to taking $c \rightarrow \infty$ limit.

To the linear order, the vector-type part of equation (12) gives
\[ c^2 \Delta \Psi^{(v)} = -16\pi G \tilde{\rho} \bar{v}_{i}^{(v)} c, \tag{28} \]
This is “the initial value equation for the frame-dragging potential $\Psi_i^{(v)}$”, see equation (4.12) in [10]; $\Psi_i^{(v)}$ is supported by $\bar{v}_i^{(v)}$ which is related to the fluid vorticity [10]. Thus, we notice that in the subhorizon limit $\Psi_i^{(v)}$ is suppressed by a factor in equation (27) compared with the $\bar{v}_i^{(v)}/c$. Therefore, we can ignore $\Psi_i^{(v)}$ in the non-relativistic limit while keeping $\bar{v}_i^{(v)}$. Thus, we have
\[ \chi_i = c \chi_{i}, \tag{29} \]

3.3. Proof in the Zero-shear gauge

The zero-shear gauge sets $\chi \equiv 0$ [17]. As we have $\chi_i = c \chi_{i}$, due to the reason presented above equation (29), we have $\chi_i = 0$. The condition $\chi_i = 0$ leads to vanishing shear of the normal-frame vector to the fully nonlinear order [10, 14], thus we term it as the zero-shear gauge; in the literature it is often known as the conformal Newtonian gauge, the longitudinal gauge, or the Poisson’s gauge, with less clear reasons.

In the subhorizon limit together with the non-relativistic limits, equations (12) and (14), respectively, give
\[ \kappa = -\frac{12\pi G \tilde{\rho}}{c^2} \Delta^{-1} \nabla \cdot (\bar{v} \bar{\nabla}), \tag{30} \]
\[ \varphi = -\alpha. \tag{31} \]
Using equations (30) and (31) and using the Newtonian identifications in equation (26) we can show the following. Equation (15) gives equation (20) Equation (16) gives equation (21). Both equations (11) and (13) give equation (22). Finally, equation (10) gives equation (23). Therefore, we proved the exact Newtonian correspondence in the zero-shear gauge.

Equation (15) is the energy conservation equation. Later in section 4 we will show that to the leading ($c^0$, thus Newtonian) order we have equation (20) which is the mass conservation, whereas, to the next ($c^{-2}$, thus first post-Newtonian) order we have the proper energy conservation equation.

3.4. Proof in the Uniform-expansion gauge

The uniform-expansion gauge sets $\kappa \equiv 0$; this was introduced as the uniform-Hubble gauge in [10].
In the subhorizon limit together with the non-relativistic limits, equations (12) and (14), respectively, give
\[ c\chi = -\frac{12\pi Ga^3}{c^2} \Delta^{-1} \nabla \cdot (\bar{\rho}\mathbf{v}), \quad (32) \]
\[ \varphi = -\alpha. \quad (33) \]
Using equations (32) and (33) and using the Newtonian identifications in equation (20) we can show the following. Equation (15) gives equation (20) which is the mass conservation equation; for the energy conservation part, see section 4. Equation (16) gives equation (21). Both equations (11) and (13) give equation (22). Finally, equation (10) gives equation (23). Therefore, we proved the exact Newtonian correspondence in the uniform-expansion gauge.

4. Energy conservation equation as the post-Newtonian order

The referee has suggested us to show the energy conservation equation in Newton’s gravity, and here we present it. The energy conservation equation follows from equation (15) by keeping the internal energy and the pressure terms to first post-Newtonian (1PN) order. In both the zero-shear gauge and the uniform-expansion gauge equation (15) gives
\[ \left( 1 + \frac{1}{c^2} \bar{\Pi} \right) \left[ \bar{\rho} + 3 \frac{\dot{a}}{a} \bar{\rho} + \frac{1}{a} \nabla \cdot (\bar{\rho}\mathbf{v}) \right] + \frac{1}{c^2} \bar{\rho} \left[ \bar{\Pi} + \frac{1}{a} \mathbf{v} \cdot \nabla \bar{\Pi} + \left( 3 \frac{\dot{a}}{a} + \frac{1}{a} \nabla \cdot \mathbf{v} \right) \frac{\bar{p}}{\bar{\rho}} \right] = 0. \quad (34) \]
The first part gives the mass conservation in equation (20) and the second part gives the energy conservation equation
\[ \dot{\bar{\Pi}} + \frac{1}{a} \mathbf{v} \cdot \nabla \bar{\Pi} + \left( 3 \frac{\dot{a}}{a} + \frac{1}{a} \nabla \cdot \mathbf{v} \right) \frac{\bar{p}}{\bar{\rho}} = 0. \quad (35) \]
Apparently the pure energy conservation part comes from the 1PN order. The proper 1PN expansion gives the pure energy conservation part as the 1PN correction terms in the mass conservation equation; see [9] in the Minkowsky background, and equations (57)-(63) in [8] which include the anisotropic stress and flux.

In the covariant notation, the energy-conservation equation is
\[ 0 = \bar{u}^a \bar{T}_{ab} = \bar{u}^a \left\{ \bar{\rho} \left( c^2 + \bar{\Pi} \right) + \bar{p} \right\} \bar{u}_a + \bar{p} \delta_{ab}, \]
\[ = - \left( c^2 + \bar{\Pi} \right) (\bar{\rho}\bar{u}^c)_{;c} - \bar{\rho} \left( \bar{\Pi}, \bar{u}^c + \bar{\theta} \bar{p}/\bar{\rho} \right), \quad (36) \]
where \( \bar{\theta} \equiv \bar{u}^c_{;c} \) is the expansion scalar. The mass conservation demands \( (\bar{\rho}\bar{u}^c)_{;c} = 0 \) and the remaining part of equation (36) gives the energy conservation equation. Equation (36) leads to equation (34).

5. Newtonian limit in Einstein’s gravity

Our proof of the Newtonian limit in the two fundamental gauge conditions shows that, in order to have the Newtonian hydrodynamic equations we may set \( \chi_i = 0 \) but should
Newtonian limit of fully nonlinear cosmological perturbations in Einstein’s gravity

have \( \varphi = -\alpha \), thus the metric should be \( \tilde{g}_{00} \)

\[
\begin{align*}
\text{ds}^2 & = -\left(1 - \frac{1}{c^2} 2U\right) c^2 dt^2 + a^2 \left(1 + \frac{1}{c^2} 2U\right) \delta_{ij} dx^i dx^j.
\end{align*}
\] (37)

This may look strange because it is well known that the Newtonian equations can be recovered from the Newtonian gravitational potential contained in \( \tilde{g}_{00} \) only. Whereas, our derivation apparently demanded the presence of nontrivial \( \tilde{g}_{ij} \) as well; the \( U \) term in \( \tilde{g}_{ij} \) is the well-known post-Newtonian correction important in affecting the light deflection, see equation (149) of [8]. Here we would like to make a comment on this subtle issue.

In order to have the conservation equations and the Poisson’s equation in (20)-(22) from equations (15), (10) and (13), we only need the \( U \) term in \( \tilde{g}_{00} \) part. Whereas, to get equations (22) and (23) from equations (11) and (10), respectively, we require \( \varphi = -\alpha \) which is given by equation (14), thus demanding the presence of \( U \) term in \( \tilde{g}_{ij} \) part which is ordinarily the first post-Newtonian contribution.

Therefore, the situation can be summarized as the following. The complete Newtonian equations indeed follow from parts of Einstein’s equations based on \( \tilde{g}_{00} \) part only. Meanwhile, the validity of the rest of Einstein’s equations demands the presence of nontrivial \( \tilde{g}_{ij} \) for the self consistency. We can also interpret the situation as that the \( \tilde{g}_{00} \) part gives the Newtonian limit, and \( \tilde{g}_{ij} \) part is simply determined by remaining parts of the Einstein’s equation for self consistency with the Newtonian limit; yet in another words, parts of Einstein’s equation mix the Newtonian and the post-Newtonian orders [9, 18, 19].

6. Discussion

In this work we have proved that, in the subhorizon limit together with the non-relativistic (weak gravity, slow-motion, and negligible pressure and internal energy density compared with the energy density) limits, Newtonian nonlinear perturbation equations are exactly recovered from the general relativistic ones in the zero-shear gauge and in the uniform-expansion gauge. As a consequence, we also have shown that the Newtonian hydrodynamic equations are exactly recovered in the Minkowsky background.

We point out that in our previous proof, based on the growing mode solutions in the zero-pressure medium, of the correspondence to the second order we only have assumed the subhorizon limit, thus the correspondence is valid for relativistic (strong gravity and fast motion) case [13], whereas in our present general proof valid to fully nonlinear order we additionally have assumed the weak gravity as well as slow-motion limits.

Our proof of the relativistic/Newtonian correspondence to the fully nonlinear order in the two gauges is not necessarily a trivial one. For example, the correspondence is not available in other fundamental gauge conditions, like the comoving gauge (\( v \equiv 0 \)), the uniform-curvature gauge (\( \varphi \equiv 0 \)), uniform-density gauge (\( \delta \equiv 0 \)), and the synchronous gauge (\( \alpha \equiv 0 \)), even to the linear order in the subhorizon limit [11, 12, 13]; notice
that each one of these other fundamental gauges sets the fundamental fluid (density or velocity) or potential perturbation variables in Newtonian theory equal to zero. Our result that the correspondence is possible only in the two particular gauge conditions is consistent with the fact that the post-Newtonian approach is consistent only with the zero-shear gauge and the uniform-expansion gauge, see [8, 20].

For the correspondences we have demanded the non-relativistic as well as the sub-horizon limits. As we approach the horizon scale and as soon as the (both special and general) relativistic effects are not negligible, we have pure Einstein’s gravity correction terms appearing. In order to properly handle the relativistic effects we may have three different but complimentary approaches: the relativistic (nonlinear) perturbation theory, the cosmological post-Newtonian approach, and the full-blown general relativistic numerical relativity.

The relativistic/Newtonian correspondence in all cosmological scales was studied up to the second-order perturbation in [12, 13]. The relativistic/post-Newtonian correspondence was studied based on the linear perturbation theory and the first post-Newtonian approximation in [20]. The results show that, for the correspondence, different gauges are suitable for different variables: e.g., the comoving gauge shows exact relativistic/Newtonian correspondences for the density and the velocity perturbations up to the second order, whereas the zero-shear gauge shows exact relativistic/Newtonian correspondence for the gravitational potential perturbation to the linear order only [13].

In this work we have considered the situation where the pressure term is negligible compared with the energy density. Thus we ignored any gravitating role of the pressure. The situation is applicable in the matter dominated era in the ΛCDM cosmology; Λ is the cosmological constant and CDM indicates the cold dark matter. However, as we consider other than Λ as the dark energy or other than CDM as the dominant dark matter, some form of pressure term will become important, thus demanding situations deviating from the Newtonian theory. The radiation pressure term could be important in the early matter dominated era just after the radiation-dominated era or the recombination. In these situations where the pressure term plays the gravitating roles it is necessary that we should go back to Einstein’s gravity displayed in equations (10)-(17). Whether the gravitating role of pressure can be handled by a minimal extension of Newtonian equations is left for a future study.

Acknowledgments

We wish to thank the referee for useful and important suggestions. H.N. was supported by grant No. 2012 R1A1A2038497 from NRF. J.H. was supported by KRF Grant funded by the Korean Government (KRF-2008-341-C00022).

References

[1] A. Einstein, Königlich Preussische Akademie der Wissenschaften (1917); translated in J. Bernstein and G. Feinberg, eds, *Cosmological Constants: Papers in Modern Cosmology* (Columbia Univ.
Newtonian limit of fully nonlinear cosmological perturbations in Einstein’s gravity

Press, 1989).

[2] A.A. Friedmann, Zeitschrift fur Physik 10, 377 (1922); translated in Gen. Rel. Grav. 31, 1991 (1999).

[3] E.A. Milne, Quart. J. Math. 5, 64 (1934); W.H. McCrea and E.A. Milne, Quart. J. Math. 5, 73 (1934).

[4] E.M. Lifshitz, J. Phys. (USSR) 10, 116 (1946).

[5] W.B. Bonnor, Mon. Not. R. Astron. Soc. 117, 104 (1957).

[6] H. Nariai, Prog. Theor. Phys., 41, 686 (1969).

[7] D. Layzer, Astron. J. 59, 268 (1954); D.S. Lemons, Am. J. Phys. 56, 502 (1988).

[8] J. Hwang, H. Noh and D. Puetzfeld, JCAP 03, 010 (2008).

[9] S. Chandrasekhar, Astrophys. J. 142, 1488 (1965).

[10] J.M. Bardeen, Phys. Rev. D 22, 1882 (1980).

[11] J. Hwang, Astrophys. J. 427, 533 (1994).

[12] J. Hwang and H. Noh, Gen. Rel. Grav. 31, 1131 (1999).

[13] J. Hwang, H. Noh and J. Gong, Astrophys. J. 752, 50 (2012).

[14] J. Hwang and H. Noh, Mon. Not. R. Astron. Soc. submitted, arXiv:1207.0264v2 (2013).

[15] J.M. Bardeen, Particle Physics and Cosmology, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988) 1; J. Hwang, Astrophys. J. 375, 443 (1991).

[16] P.J.E. Peebles, The large-scale structure of the universe, (Princeton Univ. Press, Princeton, 1980).

[17] E.R. Harrison, Rev. Mod. Phys. 39, 862 (1967).

[18] L. Kofman and D. Pogosyan, Astrophys. J. 442, 30 (1995).

[19] S. Matarrese and D. Terranova, Mon. Not. R. Astron. Soc. 283, 400 (1996).

[20] H. Noh and J. Hwang, Astrophys. J. 757, 145 (2012).