Every smooth Jordan curve has an inscribed rectangle with aspect ratio equal to $\sqrt{3}$

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Abstract

We use Batson’s lower bound on the nonorientable slice genus of $(2n, 2n-1)$-torus knots to prove that for any $n \geq 2$, every smooth Jordan curve has an inscribed rectangle of of aspect ratio $\tan(\frac{\pi}{2n})$ for some $k \in \{1, \ldots, n-1\}$. Setting $n = 3$, we have that every smooth Jordan curve has an inscribed rectangle of aspect ratio $\sqrt{3}$.

1 Introduction

Every Jordan curve has an inscribed rectangle. Vaughan proved this fact by parameterizing a Möbius strip above the plane which bounds the curve, and for which self-intersections correspond to inscribed rectangles. [5] If we fix a positive real number $r$, we may ask if every Jordan curve has an inscribed rectangle of aspect ratio $r$. For all $r \neq 1$, this problem is open, even for smooth or polygonal Jordan curves. The case of $r = 1$ is the famous inscribed square problem, which has been resolved for a large class of curves. The known partial results include [4], in which it is shown that curves which are suitably close to being convex must have inscribed rectangles of aspect ratio $\sqrt{3}$, and [1], in which it is shown that every convex curve has an inscribed rectangle of every aspect ratio.

We present a 4-dimensional generalization of Vaughan’s proof which lets us get some control on the aspect ratio of the inscribed rectangles. In particular, we will resolve the case of $r = \sqrt{3}$ for all smooth Jordan curves.

In regards to notation, $M$ will denote the Möbius strip $\text{Sym}_2(S^1) = (S^1 \times S^1)/(\mathbb{Z}/2\mathbb{Z})$, where the $\mathbb{Z}/2\mathbb{Z}$ action consists of swapping the elements of the ordered pair. We write $\{x, y\}$ to denote the equivalence class represented by the pair $(x, y)$. As a notational convenience, we abide by the convention that $S^1$ denotes the unit complex numbers.

2 The Proof

Theorem 1. Let $\gamma : S^1 \to \mathbb{C}$ be a $C^\infty$ injective function with nowhere vanishing derivative. Then for all integers $n \geq 2$, there exists an integer $k \in \{1, \ldots, n-1\}$ so that $\gamma$ has an inscribed rectangle of aspect ratio $\tan(\frac{\pi k}{2n})$.

We first prove a lemma.

Lemma 1. Let $K_n$ denote the knot in $\mathbb{C} \times S^1$ parameterized by $g \mapsto (g, g^{2n})$ for $g \in S^1$. Then if $n \geq 3$, there is no smooth embedding of the Möbius strip $M \hookrightarrow \mathbb{C} \times S^1 \times \mathbb{R}_{\geq 0}$ such that $\partial M$ maps to $K_n \times \{0\}$.

Proof. The manifold $\mathbb{C} \times S^1$ is diffeomorphic to the interior of the solid torus, so given any embedding of the solid torus into $S^3$, the image of $K_n$ under this embedding yields a knot in $S^3$ which must have non-orientable 4-genus at most that of $K_n$. By embedding the solid torus with a single axial twist, we can make the image of $K_n$ be the torus knot $T_{2n, 2n-1}$. In [2], Batson uses Heegaard-Floer homology to show that the non-orientable 4-genus of $T_{2n, 2n-1}$ is at least $n - 1$. This means that for $n \geq 3$, the knot $K_n$ cannot be bounded by a Möbius strip in the 4-manifold $\mathbb{C} \times S^1 \times \mathbb{R}_{\geq 0}$.

Proof of Theorem 1. Suppose $\gamma : S^1 \to \mathbb{C}$ is a $C^\infty$ injective function with nowhere vanishing derivative, and no inscribed rectangles of aspect ratio $\tan(\frac{\pi k}{2n})$ for any $k \in \{1, \ldots, n-1\}$. By the smooth inscribed
square theorem, we can assume \( n \geq 3 \). We define \( \mu : M \to \mathbb{C}^2 \) by the formula

\[
\mu(x, y) = \left( \frac{\gamma(x) + \gamma(y)}{2}, (\gamma(y) - \gamma(x))^{2n} \right)
\]

We see that if \( \mu(x, y) = \mu(w, z) \), then the pairs share a midpoint \( m \), and the angle at \( m \) between \( x \) and \( w \) must be a multiple of \( \pi/n \). Therefore, either \( \{x, y\} = \{w, z\} \) or \( \{x, w, y, z\} \) form the vertices of a rectangle of aspect ratio \( \tan(\frac{\pi k}{2n}) \) for some \( k \in \{1, ..., n-1\} \). Furthermore, we see that the differential of \( \mu \) is non-degenerate. Therefore, \( \mu \) is a smooth embedding of \( M \) into \( \mathbb{C}^2 \). If we take a small tubular neighborhood \( N \) around \( C \times \{0\} \), the Möbius strip \( \mu(M) \) intersects \( \partial N \) at a knot which is isotopy equivalent to the knot \( K_n \) described in our lemma. We have therefore constructed a smooth Möbius strip bounding \( K_n \), which contradicts Lemma 1.

**Corollary 1.** Every smooth Jordan curve has an inscribed rectangle of aspect ratio \( \sqrt{3} \).

**Proof.** Apply Theorem 1 to \( n = 3 \). We have a rectangle of aspect ratio \( \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} \) or \( \tan\left(\frac{\pi}{2}\right) = \sqrt{3} \). Aspect ratio is only defined up to reciprocals, so either way, we have a rectangle of the desired aspect ratio.

**References**

[1] Arseniy Akopyan and Sergey Avvakumov. Any cyclic quadrilateral can be inscribed in any closed convex smooth curve. *arXiv:1712.10205v1 [math.MG]*, 2017.

[2] Joshua Batson. Nonorientable four-ball genus can be arbitrarily large. *Mathematical Research Letters*, 21(3), 2014.

[3] H.B. Griffiths. The topology of square pegs in round holes. *Proceedings of the London Mathematical Society*, s3-62(3):647–672, 1991.

[4] Benjamin Matschke. *Equivariant topology methods in discrete geometry*. PhD thesis, Freie Universität Berlin, 2011.

[5] Benjamin Matschke. A survey on the square peg problem. *Notices of the AMS*, 61(4), 2014.

[6] M. J. Nielsen and S. E. Wright. Rectangles inscribed in symmetric continua. *Geometriae Dedicata*, 56(3):285–297, 1995.

[7] Terence Tao. An integration approach to the toeplitz square peg problem. *Forum of Mathematics, Sigma*, (5), 2017.