Rank-one convexity implies polyconvexity for isotropic, objective and isochoric elastic energies in the two-dimensional case

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We show that in the two-dimensional case, every objective, isotropic and isochoric energy function which is rank-one convex on $GL^+(2)$ is already polyconvex on $GL^+(2)$. Thus we negatively answer Morrey’s conjecture in the subclass of isochoric nonlinear energies, since polyconvexity implies quasiconvexity. Our methods are based on different representation formulae for objective and isotropic functions in general as well as for isochoric functions in particular. We also state criteria for these convexity conditions in terms of the deviatoric part of the logarithmic strain tensor.

1 Morrey’s conjecture

Different notions of convexity play a fundamental role in nonlinear elasticity theory \cite{1}, especially the concepts of polyconvexity (i.e. convexity in terms of minors) of an energy function $\varphi$ to the weak lower semicontinuity of the energy functional $\varphi$ (which is tantamount to the weak lower semicontinuity of the energy functional $\varphi \mapsto \int W(\nabla \varphi) \, dx$ on appropriate Sobolev spaces) and rank-one convexity (or Legendre-Hadamard ellipticity). Polyconvexity, in particular, has been an important notion in continuum mechanics and a cornerstone of the direct methods of the calculus of variations since its introduction to elasticity theory in John Ball’s seminal paper \cite{2,3}. It is well known that the implications

\[ \text{polyconvexity } \Rightarrow \text{quasiconvexity } \Rightarrow \text{rank-one convexity} \]

hold for arbitrary dimension $n$. However, it is also known that rank-one convexity does not imply polyconvexity in general, and that for $n = 2$ rank-one convexity does not imply quasiconvexity. The question whether rank-one convexity implies quasiconvexity in the two-dimensional case is considered to be one of the major open problems in the calculus of variations. Charles B. Morrey conjectured in 1952 that the two are not equivalent \cite{4}, i.e. that there exists a function $W : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ which is rank-one convex but not quasiconvex.

2 Isochoric energy functions

In \cite{5}, we present a condition under which rank-one convexity implies polyconvexity (and thus quasiconvexity), thereby further complicating the search for a counterexample: any function $W : GL^+(2) \to \mathbb{R}$ which is isotropic and objective (i.e. bi-SO(2)-invariant) as well as isochoric is rank-one convex if and only if it is polyconvex. A function $W : GL^+(2) \to \mathbb{R}$ is called isochoric if

\[ W(\alpha F) = W(F) \quad \text{for all } \alpha \in \mathbb{R}_+ := (0, \infty). \]

Such energy functions play an important role in nonlinear elasticity theory \cite{6,7}, where an additive volumetric-isochoric split

\[ W(F) = W_{\text{iso}}(F) + W_{\text{vol}}(\det F) = W_{\text{iso}} \left( \frac{F}{(\det F)^{1/n}} \right) + W_{\text{vol}}(\det F) \]

of the elastic energy potential $W$ into an isochoric part $W_{\text{iso}}$ and a volumetric part $W_{\text{vol}}$ is oftentimes assumed. This constitutive requirement is equivalent \cite{8} to the existence of a function $p : \mathbb{R}_+ \to \mathbb{R}$ with

\[ \frac{1}{n} \text{tr}(\sigma(V)) = p(\det V) = p(\det F), \]

where $\sigma(V)$ denotes the Cauchy-stress tensor corresponding to the left Biot stretch tensor $V = \sqrt{FF^T}$.

Our approach is based on certain representation formulae for isochoric energy functions: it is well known that any objective, isotropic function can be expressed in terms of the singular values of $F$, i.e. that there exists a uniquely determined function

\[ (*) \quad \sigma(V) = \sqrt{FF^T}. \]

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for all $F \in \text{GL}^+(2)$ with singular values $\lambda_1, \lambda_2 \in \mathbb{R}_+$. Our main result in [5] is the following:

**Theorem 2.1** Let $W: \text{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ be an objective, isotropic and isochoric function, and let $h: \mathbb{R}_+ \to \mathbb{R}$, $g: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ denote the functions uniquely determined by (1). Then the following are equivalent:

i) $W$ is polyconvex,

ii) $W$ is rank-one convex,

iii) $g$ is separately convex,

iv) $h$ is convex on $\mathbb{R}_+$,

v) $h$ is convex and non-decreasing on $[1, \infty)$.

In particular, rank-one convexity (and thus quasiconvexity) implies polyconvexity for such functions. The implication $\vDash \forall i$ follows from an earlier observation [9] that the function $Z: \text{GL}^+(2) \to [1, \infty)$ with $Z(F) = \frac{1}{\lambda_2}$ for $F \in \text{GL}^+(2)$ with singular values $\lambda_1 \geq \lambda_2$ is polyconvex on $\text{GL}^+(2)$.

Any isochoric, isotropic and objective function $W$ on $\text{GL}^+(2)$ can also be expressed in terms of the deviatoric quadratic Hencky strain energy $\|\text{dev} \log U\|^2 = \left\| \log \frac{U}{(\det U)^{1/2}} \right\|^2$, i.e. there exists a unique function $f: [0, \infty) \to \mathbb{R}$ such that

$$W(F) = f(\|\text{dev} \log U\|^2).$$

**Proposition 2.2** Let $W: \text{GL}^+(2) \to \mathbb{R}$, $F \mapsto W(F)$ be an objective, isotropic and isochoric function and let $f: [0, \infty) \to \mathbb{R}$ denote the functions uniquely determined by (2). If $f \in C^2([0, \infty))$, then the following are equivalent:

i) $W$ is polyconvex (i.e. $W(F) = P(F, \det F)$ with $P: \mathbb{R}^{2 \times 2} \to \mathbb{R}$ convex),

ii) $W$ is rank-one convex (i.e. $D^2W(F).(\xi \otimes \eta, \xi \otimes \eta) \geq 0$),

iii) $2 \eta f''(\eta) + (1 - \sqrt{2\eta}) f'(\eta) \geq 0$ for all $\eta \in (0, \infty)$.

### 3 Applications

By integrating the polyconvexity criterion given in Proposition 2.2, we can obtain an exponential growth condition for the function $f$ which is necessarily satisfied if $W$ is rank-one convex (i.e. polyconvex).

**Corollary 3.1** Let $W: \text{GL}^+(2) \to \mathbb{R}$ with $W(F) = \tilde{f}(\|\text{dev} \log U\|^2) = f\left(\log^2 \frac{\lambda_2}{\lambda_1}\right)$ be a polyconvex energy function with $f \in C^2([0, \infty))$. If $f'(\theta) \neq 0$ for all $\theta > 0$, then the function $f$ satisfies the inequality

$$f(\theta) \geq (e^{\sqrt{\theta}} - 1) \frac{\sqrt{\theta}}{e^{\sqrt{\theta}}} f'(\varepsilon) + f(0) \quad \text{for all } \theta, \varepsilon > 0.$$  

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**References**

[1] J. Schröder and P. Neff, Poly-, quasi-and rank-one convexity in applied mechanics (Springer Science & Business Media, 2010).

[2] J. Ball, Archive for Rational Mechanics and Analysis 63, 337–403 (1977).

[3] B. Dacorogna, Direct Methods in the Calculus of Variations., 2. edition, Applied Mathematical Sciences, Vol. 78 (Springer, Berlin, 2008).

[4] C. Morrey, Pacific J. Math 2(1), 25–53 (1952).

[5] R.J. Martin, I.D. Ghiba, and P. Neff, to appear in Proceedings of the Royal Society Edinburgh A (2015), available at arXiv:1507.00266.

[6] P. Neff, B. Eidel, and R. J. Martin, to appear in Archive for Rational Mechanics and Analysis (2016).

[7] P. Neff, I. D. Ghiba, and J. Lankeit, Journal of Elasticity 121(2), 143–234 (2015).

[8] H. Richter, Z. Angew. Math. Mech. 28(7–8), 205–209 (1948).

[9] I. Ghiba, P. Neff, and M. Šilhavý, Int. J. Non-Linear Mech. 71, 48–51 (2015).