Accelerating Optimization and Reinforcement Learning
with Quasi-Stochastic Approximation

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Abstract

The ODE (ordinary differential equation) method has been a workhorse for algorithm design and analysis since the introduction of the stochastic approximation technique of Robbins and Monro in the early 1950s. It is now understood that convergence theory amounts to establishing robustness of Euler approximations for ODEs, while theory of rates of convergence requires finer probabilistic analysis. This paper sets out to extend this theory to quasi-stochastic approximation (QSA), based on algorithms in which the “noise” or “exploration” is based on deterministic signals, much like quasi-Monte Carlo. The main results are obtained under minimal assumptions: the usual Lipschitz conditions for ODE vector fields, and for rate results it is assumed that there is a well defined linearization near the optimal parameter \( \theta^* \), with Hurwitz linearization matrix \( A^* \). Algorithm design is performed in continuous time, in anticipation of discrete-time implementation based on Euler approximations, or high-fidelity alternatives.

The main contributions are summarized as follows:

(i) If the algorithm gain is chosen as \( a_t = g/(1 + t)^\rho \) with \( g > 0 \) and \( \rho \in (0, 1) \), then the rate of convergence of the algorithm is \( 1/t^\rho \). There is also a well defined “finite-\( t \)” approximation:

\[
a_t^{-1} (\Theta_t - \theta^*) = \bar{Y} + \Xi_t + o(1)
\]

where \( \bar{Y} \in \mathbb{R}^d \) is a vector identified in the paper, and \( \{\Xi_t\} \) is bounded with zero temporal mean.

(ii) With gain \( a_t = g/(1 + t) \) the results are not as sharp: the rate of convergence \( 1/t \) holds only if \( I + gA^* \) is Hurwitz. Hence we obtain the optimal rate of convergence only if \( g > 0 \) is chosen sufficiently large.

(iii) Based on the Ruppert-Polyak averaging technique of stochastic approximation, one would expect that a convergence rate of \( 1/t \) can be obtained by averaging:

\[
\Theta_t^{RP} = \frac{1}{T} \int_0^T \Theta_t \, dt
\]

where the estimates \( \{\Theta_t\} \) are obtained using the gain in (i). The preceding sharp bounds imply that averaging results in \( 1/t \) convergence rate if and only if \( \bar{Y} = 0 \). This condition holds if the noise is additive, but appears to fail in general.

(iv) The theory is illustrated with applications to gradient-free optimization, and policy gradient algorithms for reinforcement learning.

Note: This pre-print is written in a tutorial style so it is accessible to new-comers. It will be a part of a handout for upcoming short courses on RL. A more compact version suitable for journal submission is in preparation.

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1 Introduction

The ODE method was coined by Ljung in his 1977 survey of stochastic approximation (SA) techniques for analysis of recursive algorithms [25]. The theory of SA was born more than 25 years earlier with the publication of the work of Robbins and Monro [33], and research in this area has barely slowed in the 70 years since its publication. The goal of SA is a simple root finding problem: compute or approximate the vector $\theta^* \in \mathbb{R}^d$ solving $f(\theta^*) = 0$, in which $f: \mathbb{R}^d \to \mathbb{R}^d$ is defined by an expectation:

$$f(\theta) \equiv \mathbb{E}[f(\theta, \Phi)], \quad \theta \in \mathbb{R}^d,$$

where $f: \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, and $\Phi$ is a random vector taking values in a set $\Omega$ (assumed in this paper to be a subset of Euclidean space).

Given the evolution of this theory, it is useful to reconsider the meaning of the method. Rather than merely a method for analyzing an algorithm, the ODE method is an approach to algorithm design, broadly described in two steps:

**Step 1:** Design $f$ (and hence $f$) so that the following ODE is globally asymptotically stable:

$$\frac{d}{dt} \theta_t = \dot{f}(\theta_t).$$

**Step 2:** Obtain a discrete-time algorithm via a “noisy” Euler approximation:

$$\theta_{n+1} = \theta_n + \alpha_{n+1} [f(\theta_n) + \Xi_n], \quad n \geq 0,$$

in which $\{\alpha_n\}$ is the non-negative “gain” or “step-size” sequence, and the sequence $\{\Xi_n\}$ has mean that vanishes as $n \to \infty$.

Each step may require significant ingenuity. Step 1 may be regarded as a control problem: design dynamics to reach a desirable equilibrium from each initial condition. There are algorithm design questions in Step 2, but in the original formulation of SA and nearly all algorithms that follow, recursion (3) takes the form

$$\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, \Phi_{n+1}), \quad n \geq 0,$$

in which $\Phi_{n+1}$ has the same distribution as $\Phi$, or the distribution of $\Phi_{n+1}$ converges to that of $\Phi$ as $n \to \infty$. A useful approximation requires assumptions on $f$, the “noise” $\Phi_{n+1}$, and the step-size sequence $\alpha$. The required assumptions, and the mode of analysis, are not very different than what is required to successfully apply a deterministic Euler approximation [10].

The motivation for the abstract formulation of Step 2 is to reinforce the idea that we are not bound to the traditional recursion. For example, control variate techniques offer alternatives: an example is the introduction of the “advantage function” in policy gradient techniques for reinforcement learning (RL) [41, 40].

In much of the SA literature, and especially in the applications considered in this paper, it is assumed that $\Phi$ is independent and identically distributed (i.i.d.), or more generally, it is a Markov chain. Just as quasi Monte-Carlo algorithms are motivated by fast approximation of integrals, the quasi-stochastic approximation (QSA) algorithms considered in this paper are designed to speed convergence for root finding problems. It is useful to pose the algorithm in continuous time:

$$\frac{d}{dt} \Theta_t = a_t f(\Theta_t, \xi_t).$$

where in the main results we restrict to gains of the form

$$a_t = g/(1 + t)^\rho, \quad \text{with } 0 < \rho \leq 1 \text{ and } g > 0$$

The probing signal $\xi$ is generated from a deterministic (possibly oscillatory) signal rather than a stochastic process. Two canonical choices are the $m$-dimensional mixtures of periodic functions:

$$\xi_t = \sum_{i=1}^K v^i [\phi_i + \omega_i t + 1] (\text{mod } 1)$$

$$\xi_t = \sum_{i=1}^K v^i \sin(2\pi [\phi_i + \omega_i t])$$
for fixed vectors \( \{ v^i \} \subset \mathbb{R}^m \), phases \( \{ \phi_i \} \), and frequencies \( \{ \omega_i \} \). Under mild conditions on \( f \) we can be assured of the existence of this limit defining the mean vector field:

\[
\bar{f}(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\theta, \xi_t) \, dt, \quad \text{for all } \theta \in \mathbb{R}^d.
\]  

(8)

Our aim is to obtain tight bounds between solutions of (5) and

\[
d \frac{d}{dt} \Theta_t = a_t \bar{f}(\Theta_t), \quad t \geq t_0, \quad \Theta_{t_0} = \Theta_0
\]

(9)

where the choice of \( t_0 \) depends on the stability properties of the associated ODE (2) with constant gain.

**Contributions** It is assumed throughout the paper that (2) is globally asymptotically stable, with unique equilibrium denoted \( \theta^* \in \mathbb{R}^d \). Convergence of (5) is also assumed: sufficient conditions based on the existence of a Lyapunov function can be found in [6, 7]. This prior work is reviewed in the Appendix, along with a new sufficient condition extending the main result of [11].

We say that the rate of convergence of (5) is \( 1/t^{\varrho_0} \) if

\[
\limsup_{t \to \infty} t^{\varrho} \| \tilde{\Theta}_t \| = \begin{cases} 
\infty & \varrho > \varrho_0 \\
0 & \varrho < \varrho_0
\end{cases}
\]

(10)

where \( \tilde{\Theta}_t \triangleq \Theta_t - \theta^* \) is the estimation error. By careful design we can achieve \( \varrho_0 = 1 \), which is optimal in most cases (such as for Monte-Carlo – see Section 2.1). Solutions to (9) and (2) are related by a time-transformation (see Lemma C.1—a common transformation in the SA literature), so that \( \tilde{\Theta}_t \to \theta^* \). Conditions are imposed so the rate of convergence is faster than \( 1/t \), which justifies the following scaling:

\[
Z_t = \frac{1}{a_t} \left( \Theta_t - \tilde{\Theta}_t \right), \quad t \geq t_0
\]

(11)

We obtain general conditions under which \( Z \) is bounded and non-vanishing using the gain (6), which then implies the rate of convergence of (5) is \( 1/t^{\varrho} \). A key assumption is that the ODE is smooth near the equilibrium \( \theta^* \), so that there is a well defined linearization matrix \( A^* = \partial \bar{f}(\theta^*) \).

Given this background, we are ready to summarize the main results:

1. The following dichotomy is established:

   (i) With \( \varrho < 1 \) we obtain \( 1/t^{\varrho} \) rate of convergence, provided \( A^* \) is Hurwitz, regardless of the value of \( g > 0 \) appearing in (6).

   (ii) With \( \varrho = 1 \) we obtain the optimal rate of convergence \( 1/t \) provided \( I + gA^* \) is Hurwitz.

In either case, an exact approximation of the scaled error is obtained:

\[
Z_t = \tilde{Y} + \Xi_t + o(1)
\]

(12)

where \( \tilde{Y} \in \mathbb{R}^d \) is a vector identified in (45), and

\[
\Xi_t = \int_0^t f(\theta^*, \xi_r) \, dr
\]

This is assumed bounded in \( t \) (justified for natural classes of probing signals in Appendix B).

2. For those well-versed in stochastic approximation theory, the preceding conclusions would motivate the use of averaging:

\[
\Theta_T^{RP} = \frac{1}{T} \int_0^T \Theta_t \, dt, \quad T > 0
\]

where \( \{ \Theta_t \} \) are obtained using the gain (6) with \( \varrho < 1 \). The superscript refers to the averaging technique for stochastic approximation introduced independently by Ruppert and Polyak [34, 30].
In general, this approach fails: the rate of convergence of \( \{ \Theta^p_T \} \) to \( \theta^* \) is 1\( /T \) if and only if \( \bar{Y} = 0 \). A sufficient condition for this is additive noise, for which (5) becomes

\[
\frac{d}{dt} \Theta_t = a_t \{ \bar{f}(\Theta_t) + D(\bar{\xi}_t) \}
\]

(13)

In this case

\[
\Xi^t_1 = \int_0^t D(\bar{\xi}_r) \, dr
\]

3. These theoretical results motivate application to gradient-free optimization, and policy-gradient techniques for RL. Theory and examples are presented in Sections 4 and 5.

Literature review This paper spans three areas of active research:

1. Stochastic approximation Section 2 and some of the convergence theory in the appendix is adapted from [6, 7], which was inspired by the prior results in [27, 35]; [16] contains applications to gradient-free optimization with constraints. The first appearance of QSA methods appears to have originated in the domain of quasi-Monte Carlo methods applied to finance; see [22, 23].

The contributions here are a significant extension of [6, 7], which considered exclusively the special case in which the function \( f \) is linear, with \( f(\theta, \xi) = A\theta + B\xi \) for matrices \( A, B \). The noise is thus additive: (13) holds with \( D(\xi_t) = B\xi_t \). Using the gain (6) with \( \rho = 1 \), the optimal rate of convergence was obtained under the assumption that \( I + gA \) is Hurwitz. The assumptions on \( g \) using \( \rho = 1 \) are stronger than what is imposed in stochastic approximation, which requires that \( \frac{1}{2}I + gA \) is Hurwitz. On the other hand, the conclusions for stochastic approximation algorithms are weaker: from (12) we obtain

\[
t^2\|\Theta_t - \theta^*\|^2 = \|\bar{Y} + \Xi^t_1 + o(1)\|^2
\]

In the theory of stochastic approximation we must introduce an expectation, and settle for a much slower rate:

\[
\lim_{t \to \infty} tE[\|\Theta_t - \theta^*\|^2] = \text{trace} (\Sigma_\theta)
\]

That is, the rate is \( 1/\sqrt{t} \) rather than \( 1/t \): see [20, 14] for refinements of this result and history.

The “ODE@\( \infty \)” (73) was introduced in [11] for stability verification in stochastic approximation. Thm. A.1 is an extension of the Borkar-Meyn Theorem [11, 10], which has been refined considerably in recent years [31, 32].

2. Gradient free optimization The goal is to minimize a loss function \( L(\theta) \) over \( \theta \in \mathbb{R}^d \). It is possible to observe the loss function at any desired value, but no gradient information is available.

Gradient-free optimization has been studied in two, seemingly disconnected lines of work: techniques intended to directly approximate gradient descent through perturbation techniques, known as Simultaneous Perturbations Stochastic Approximation (SPSA), and Extremum-Seeking Control (ESC) which is formulated in a purely deterministic setting. The contributions of the present paper are motivated by both points of view, but leaning more on the former. See [42, 24, 2] for the nearly century-old history of ESC.

Theory for SPSA began with the algorithm of Keifer-Wolfowitz [21], which requires at each iteration access to two perturbations per dimension to obtain a stochastic gradient estimate. This computational barrier was addressed in the work of Spall [36, 39, 37]. Most valuable for applications in RL is the one-measurement form of SPSA introduced in [37]: this can be expressed in the form (4), in which

\[
f(\theta_n, \Phi_{n+1}) = L(\theta_n + \varepsilon \Phi_{n+1})\Phi_{n+1}
\]

(15)

where \( \Phi \) is a zero-mean and i.i.d. vector-valued sequence. The qSGD (quasi-Stochastic Gradient Descent) algorithm (56) is a continuous time analog of this approach.

The introduction of [29] suggests that there is an older history of improvements to SPSA in the Russian literature: see eqn. (2) of that paper and surrounding discussion. Beyond history, the contributions of [29] include rates of convergence results for standard and new SPSA algorithms. Information theoretic lower bounds for optimization methods that have access to noisy observations of the true function was derived in [19]. This class of algorithms also has some bandits history [1, 12].

In all of the SPSA literature surveyed above, a gradient approximation is obtained through the introduction of an i.i.d. probing signal. For this reason, the best possible rate is of order \( 1/\sqrt{n} \). The algorithms
introduced in the present work are designed to achieve $O(1/n)$ convergence rate for optimization and root-finding problems.

More closely related to the present work is [8] which treats SPSA using a specially designed class of deterministic probing sequences. There are no comparable contributions, but this previous work motivates further research on rates of convergence for the algorithms proposed.

3. Policy gradient techniques

These may be regarded as a special case of gradient-free optimization, in which the goal is to minimize average cost for an MDP based solely on input-output measurements. The standard dynamic programming formulation for optimal control is replaced with the following architecture: given a parameterized family of (possibly randomized) state-feedback polices $\{\phi^\theta : \theta \in \mathbb{R}^d\}$, the goal is to minimize the associated average cost $L(\theta) = \mathbb{E}_\theta[c(X_k, U_k)]$, where the expectation is in steady-state, subject to $U_k = \phi^\theta(X_k)$ for all $k$ (the state description must be extended if the policy is randomized). Williams’ REINFORCE [43] is an early example, while the most popular algorithms today are of the “actor-critic” category in which the approximation of the gradient of $L$ is based on an estimate of a value function. In the widely cited recent paper [26] it is shown that SPSA algorithms such as a modified version of (15) can sometimes outperform actor-critic methods.

Organization

Section 2 contains some simple examples, introduced mainly to clarify notation and motivation, much of which is adapted from [6, 7]. The main results are summarized in Section 3, along with sketches of the proofs. Applications to gradient-free optimization are summarized in Section 4, and to policy gradient RL in Section 5. Section 6 contains conclusions and directions for future research. The appendix contains three sections: Appendix A concerns convergence theory of QSA based on [6, 7]. The main challenge is to establish boundedness of trajectories of the algorithm. A new sufficient condition is established, based on a generalization of the Borkar-Meyn Theorem. Ergodic theory for dynamical systems is required in the main assumptions: justification for a broad class of “probing signals” is contained in Appendix B. Appendix C contains the details of the proofs of the main results.

2 Simple Examples

The following subsections contain examples to illustrate theory of QSA, and also a glimpse at applications. Sections 2.1 and 2.3 are adapted from [6, 7].

2.1 Quasi Monte-Carlo

Consider the problem of obtaining the integral over the interval $[0, 1]$ of a function $y: \mathbb{R} \to \mathbb{R}$. In a standard Monte-Carlo approach we would draw independent random variables $\{\Phi_{n+1}\}$, with distribution uniform on the interval $[0, 1]$, and then average:

$$\theta_n = \frac{1}{n} \sum_{k=0}^{n-1} y(\Phi_k)$$

A QSA analog is described as follows: the probing signal is the one-dimensional sawtooth function, $\xi_t \equiv t$ (modulo 1) and consider the analogous average

$$\Theta_t = \frac{1}{t} \int_0^t y(\xi_r) \, dr$$

Alternatively, we can adapt the QSA model (5) to this example, with

$$f(\theta, \xi) \equiv y(\xi) - \theta.$$  

The averaged function is then given by

$$\bar{f}(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\theta, \xi_t) \, dt = \int_0^1 y(\xi_t) \, dt - \theta$$
so that \( \theta^* = \int_0^1 y(\xi_t) \, dt \) is the unique root of \( \tilde{f} \). Algorithm (5) is given by:

\[
\frac{d}{dt} \Theta_t = a_t [y(\xi_t) - \Theta_t].
\] (19)

This Monte-Carlo approach (17) can be transformed into something resembling (19). Taking derivatives of each side of (17), we obtain using the product rule of differentiation, and the fundamental theorem of calculus,

\[
\frac{d}{dt} \Theta_t = -\frac{1}{t^2} \int_0^t y(\xi_r) \, dr + \frac{1}{t} y(\xi_t) = \frac{1}{t} [y(\xi_t) - \Theta_t]
\]

This is precisely (19) with \( a_t = 1/t \) (not a great choice for an ODE design, since it is not bounded as \( t \downarrow 0 \)).

The numerical results that follow are based on \( y(\theta) = e^{4\theta} \sin(100\theta) \), whose mean is \( \theta^* \approx -0.5 \). The differential equation (19) was approximated using a standard Euler scheme with sampling interval \( 10^{-3} \). Several variations were simulated, differentiated by the gain \( a_t = g/(1 + t) \). Fig. 1 shows typical sample paths of the resulting estimates for a range of gains, and common initialization \( \Theta_0 = 10 \). In each case, the estimates converge to the true mean \( \theta^* \approx -0.5 \), but convergence is very slow for \( g > 0 \) significantly less than one. Recall that the case \( g = 1 \) is very similar to what was obtained from the Monte-Carlo approach (17).

Independent trials were conducted to obtain variance estimates. In each of \( 10^4 \) independent runs, the common initial condition was drawn from \( N(0, 10) \), and the estimate was collected at time \( T = 100 \). Fig. 2 shows three histograms of estimates for standard Monte-Carlo (16), and QSA using gains \( g = 1 \) and 2. An alert reader must wonder: why is the variance reduced by 4 orders of magnitude when the gain is increased from 1 to 2? The relative success of the high-gain algorithm is explained in Thm. 3.1.

\[
\begin{align*}
\mu &= -0.47, \\
\sigma^2 &= 2 \times 10^{-3} ; \\
\text{Monte Carlo} &; g = 1
\end{align*}
\]

\[
\begin{align*}
\mu &= -0.48, \\
\sigma^2 &= 1 \times 10^{-3} ; \\
\text{QSA} &; g = 1
\end{align*}
\]

\[
\begin{align*}
\mu &= -0.48, \\
\sigma^2 &= 1 \times 10^{-3} ; \\
\text{QSA} &; g = 2
\end{align*}
\]

Figure 2: Histograms of Monte-Carlo and Quasi Monte-Carlo estimates after \( 10^4 \) independent runs. The optimal parameter is \( \theta^* \approx -0.4841 \).

\subsection*{2.2 Constant gain algorithm}

In later sections we will consider the linear approximation:

\[
f(\theta, \xi_t) = A(\theta - \theta^*) + B \xi_t
\] (20)

This provides insight, and sometimes we can show strong coupling between the linear and nonlinear QSA ODEs. We briefly consider here this linear model in which \( a_t = \alpha \) is constant. Then QSA is a time-invariant linear system:

\[
\frac{d}{dt} \Theta_t = \alpha [A \Theta_t + B \xi_t], \quad \Theta_0 = \dot{\theta}_0
\]

where \( \tilde{\Theta}_t \equiv \Theta_t - \theta^* \) is the error at time \( t \). For this simple model we can solve the ODE when the probing signal is the mixture of sinusoids (7b).
A linear system satisfies the principle of superposition. To put this to work, consider the probing signal (7b), and for each \(i\), consider the ODE

\[
\frac{d}{dt} \tilde{\Theta}_t = \alpha [A \tilde{\Theta}_t + B v^i \sin(2\pi [\phi_i + \omega_i t])] , \quad \tilde{\Theta}_0 = 0
\]

The principle states that the solution to the ODE is the sum:

\[
\tilde{\Theta}_t = e^{\alpha A t} \tilde{\theta}_0 + B K \sum_{i=1}^{K} \tilde{\Theta}_t^i
\]  

We see that the response to the initial error \(\tilde{\theta}_0 = \theta_0 - \theta^*\) decays to zero exponentially quickly. Moreover, to understand the steady-state behavior of the algorithm it suffices to fix a single value of \(i\).

Let’s keep things simple, and stick to sinusoids. And it is much easier to work with complex exponentials:

\[
\frac{d}{dt} \tilde{\Theta}_t = \alpha [A \tilde{\Theta}_t + B v \exp(j \omega t)] , \quad \tilde{\Theta}_0 = 0
\]

with \(\omega \in \mathbb{R}\) and \(v \in \mathbb{R}^d\) (dropping the scaling \(2\pi\) for simplicity, and the phase \(\phi\) is easily returned by a time-shift). We can express the solution as a convolution:

\[
\tilde{\Theta}_t = \alpha \int_0^t \exp(\alpha A r) B v \exp(j \omega (t-r)) \, dr
\]

Writing \(D = [\alpha A - j \omega I]\), the integral of the matrix exponential is expressed,

\[
\int_0^t e^{D r} \, dr = D^{-1} [e^{Dt} - I]
\]

Using linearity once more, and the fact that the imaginary part of \(e^{j \omega t}\) is \(\sin(\omega t)\), we arrive at a complete representation for (21):

**Proposition 2.1.** Consider the linear model with \(A\) Hurwitz, and probing signal (7b), for which the constant-gain QSA algorithm has the solution (21). Then

\[
\tilde{\Theta}_t^i = \alpha \Gamma^i v^i \quad \text{for each } i \text{ and } t , \quad \text{with}
\]

\[
\Gamma^i = \text{Im} \left( [\alpha A - j \omega I]^{-1} \left[ \exp(\alpha A t) - \exp(2\pi j [\phi_i + \omega_i t]) I \right] \right)
\]

Prop. 2.1 illustrates a challenge with fixed gain algorithms: if we want small steady-state error, then we require small \(\alpha\) (or large \(\omega\), but this brings other difficulties for computer implementation—never forget Euler!). However, if \(\alpha > 0\) is very small, then the impact of the initial condition in (21) will persist for a long time.

The Ruppert-Polyak averaging technique can be used to improve the steady-state behavior—more on this can be found in Section 3 for vanishing-gain algorithms. It is easy to illustrate the value for the special case considered here. One form of the technique is to simply average some fraction of the estimates:

\[
\Theta^\text{RP}_T \equiv \frac{1}{T - T_0} \int_{T_0}^T \Theta_t \, dt
\]

For example, \(T_0 = T - T/5\) means that we average the final 20%.

**Corollary 2.2.** Suppose that the assumptions of Prop. 2.1 hold, so in particular \(f\) is linear. Consider the averaged estimates (23) in which \(T_0 = T - T/K\) for fixed \(K > 1\). Then,

\[
\Theta^\text{RP}_T = \theta^* + M_T \theta_0 + B \sum_{i=1}^{K} \Theta^\text{RP}_T^i
\]
Consider the nonlinear state space model in continuous time,

\[ \frac{dx}{dt} = f(x, u), \quad t \geq 0 \]

with \( x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m \). Given a cost function \( c: \mathbb{R}^{n+m} \rightarrow \mathbb{R} \), our goal is to approximate the optimal value function

\[ J^*(x) = \min_u \int_0^\infty c(x_t, u_t) \, dt, \quad x = x_0 \]

and approximate the optimal policy. For this we first explain how policy iteration extends to the continuous time setting.

For any feedback law \( u_t = \phi(x_t) \), denote the associated value function by

\[ J^\phi(x) = \int_0^\infty c(x_t, \phi(x_t)) \, dt, \quad x = x_0. \]

This solves a dynamic programming equation:

\[ 0 = c(x, \phi(x)) + \nabla J^\phi(x) \cdot f(x, \phi(x)) \]

The policy improvement step in this continuous time setting defines the new policy as the minimizer:

\[ \phi^+(x) \in \arg \min_u \{c(x, u) + \nabla J^\phi(x) \cdot f(x, u)\} \]

Consequently, approximating the term in brackets is key to approximating PIA.

An RL algorithm is constructed through the following steps. First, add \( J^\phi \) to each side of the fixed-policy dynamic programming equation:

\[ J^\phi(x) = J^\phi(x) + c(x, \phi(x)) + \nabla J^\phi(x) \cdot f(x, \phi(x)) \]

The right-hand side motivates the following definition of the fixed-policy Q-function:

\[ Q^\phi(x, u) = J^\phi(x) + c(x, u) + f(x, u) \cdot \nabla J^\phi(x). \]

The policy update can be equivalently expressed \( \phi^+(x) = \arg \min_u Q^\phi(x, u) \), and this Q-function solves the fixed point equation

\[ Q^\phi(x, u) = Q^\phi(x) + c(x, u) + f(x, u) \cdot \nabla Q^\phi(x) \quad (24) \]

where \( H^\phi(x) = H(x, \phi(x)) \) for any function \( H \) (note that this is a substitution, rather than the minimization appearing in Q-learning).

Consider now a family of functions \( Q^\theta \) parameterized by \( \theta \), and define the Bellman error for a given parameter as

\[ E^\theta(x, u) = -Q^\theta(x, u) + Q^\phi(x) + c(x, u) + f(x, u) \cdot \nabla Q^\theta(x) \quad (25) \]
A model-free representation is obtained, on recognizing that for any state-input pair \((x_t, u_t)\),
\[
\mathcal{E}^\theta(x_t, u_t) = -Q^\theta(x_t, u_t) + Q^\theta(x_t) + c(x_t, u_t) + \frac{d}{dt}Q^\theta(x_t)
\]  
(26)

The error \(\mathcal{E}^\theta(x_t, u_t)\) can be observed without knowledge of the dynamics \(f\) or even the cost function \(c\). The goal is to find \(\theta^*\) that minimizes the mean square error:
\[
||\mathcal{E}^\theta||^2 \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T [\mathcal{E}^\theta(x_t, u_t)]^2 dt.
\]  
(27)

We choose a feedback law with “exploration”:
\[
u_t = \bar{\phi}(x_t, \xi_t)
\]  
(28)
chosen so that the resulting state trajectories are bounded for each initial condition, and that the joint process \((x, u, \xi)\) admits an ergodic steady state.

Whatever means we use to obtain the minimizer, this approximation technique defines an approximate version of PIA: given a policy \(\phi\) and approximation \(Q^\theta\), the policy is updated:
\[
\phi^+(x) = \arg \min_u Q^\theta(x, u)
\]
(29)

This procedure is repeated to obtain a recursive algorithm.

**Least squares solution**  Consider for fixed \(T\) the loss function
\[
L_T(\theta) = \frac{1}{T} \int_0^T [\mathcal{E}^\theta(x_t, u_t)]^2 dt
\]
If the function approximation architecture is linear,
\[
Q^\theta(x, u) = d(x, u) + \theta^T \psi(x, u), \quad \theta \in \mathbb{R}^d.
\]  
(30)
in which \(d: X \times U \to \mathbb{R}\). Then \(L_T\) is a quadratic function of \(\theta\):
\[
L_T(\theta) = \theta^T M \theta - 2b^T \theta + L_T(0) = (\theta - \theta^*)^T M (\theta - \theta^*) + L_T(0)
\]
We leave it to the reader to find expressions for \(M, b\), and \(L_T(0)\).

In this special case we do not need gradient descent techniques: the matrices \(M\) and \(b\) can be represented as Monte-Carlo, as surveyed in Section 2.1, and then \(\theta^* = M^{-1}b\).

**Gradient descent**  The first-order condition for optimality is expressed as a root-finding problem: \(\nabla_{\theta}||\mathcal{E}^\theta||^2 = 0\), and the standard gradient descent algorithm in ODE form is
\[
\frac{d}{dt} \theta_t = -\frac{1}{2} \theta_t \nabla_\theta ||\mathcal{E}^\theta||^2 = -a \nabla_\theta \nabla_\theta \mathcal{E}^\theta
\]
with \(a > 0\). This is an ODE of the form (2), whose QSA counterpart (5) is the QSA steepest descent algorithm,
\[
\frac{d}{dt} \Theta_t = -a_t \nabla_\Theta \mathcal{E}^\theta(x_t, u_t) \Theta_t
\]
\[
\zeta^\theta_t \overset{\text{def}}{=} \nabla_\theta \mathcal{E}^\theta(x_t, u_t)
\]  
(31)
Where, based on (26) we can typically swap derivative with respect to time and derivative with respect to \(\theta\) to obtain
\[
\nabla_\theta \mathcal{E}^\theta(x_t, u_t) = -\nabla_\theta Q^\theta(x_t, u_t) + \left\{ \nabla_\theta Q^\theta(x_t, \phi(x_t)) + \frac{d}{dt} \nabla_\theta Q^\theta(x_t, \phi(x_t)) \right\}
\]

The QSA gradient descent algorithm (31) is best motivated by a nonlinear function approximation, but it is instructive to see how the ODE simplifies for the the linearly parameterized family (30). We have in this case
\[
\zeta_t = -\psi(x_t, u_t) + \psi(x_t, \phi(x_t)) + \frac{d}{dt} \psi(x_t, \phi(x_t))
\]
and $\mathcal{E}^\theta(x_t, u_t) = b_t + \zeta_t^\top \theta$ using

$$b_t = c(x_t, u_t) - d(x_t, u_t) + d(x_t, \Phi(x_t)) + \frac{d}{dt}d(x_t, \Phi(x_t))$$

so that (31) becomes

$$\frac{d}{dt} \Theta_t = -a_t \left[ \zeta_t \zeta_t^\top \Theta_t + b_t \zeta_t \right]$$ (32)

The convergence of (32) may be very slow if the matrix

$$G \equiv \lim_{t \to \infty} \frac{1}{t} \int_0^t \zeta_t \zeta_t^\top dt$$ (33)

has eigenvalues close to zero (see Prop. 3.3 for a simple explanation). This can be resolved through the introduction of a larger gain $a$, or a matrix gain. One approach is to estimate $G$ from data and invert:

$$\hat{G}_t = \frac{1}{t} \int_0^t \zeta_t \zeta_t^\top dt, \quad 0 \leq t \leq T$$ (34a)

$$\frac{d}{dt} \Theta_t = -a_t \hat{G}_t^{-1} \left[ \zeta_t \zeta_t^\top \Theta_t + b_t \zeta_t \right], \quad t \geq T$$ (34b)

This might be motivated by the ODE approximation

$$\frac{d}{dt} \theta = -a \{ \theta - \theta^* \}$$

This idea is the motivation for Zap SA and Zap Q-learning [15, 17].

![Figure 3: Comparison of QSA and Stochastic Approximation (SA) for policy evaluation.](image)

**Numerical example** Consider the LQR problem in which $f(x, u) = Ax + Bu$, and $c(x, u) = x^\top M x + u^\top Ru$, with $M \geq 0$ and $R > 0$. Given the known structure of the problem, we know that the function $Q^\phi$ associated with any linear policy $\phi(x) = K x$, takes the form

$$Q^\phi = \begin{bmatrix} x \end{bmatrix}^\top \begin{bmatrix} M & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} A^\top P + PA + P & PB \\ B^\top P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

where $P$ solves the Lyapunov equation

$$A^\top P + PA + K^\top RK + Q = 0$$

This motivates a quadratic basis, which for the special case $n = 2$ and $m = 1$ becomes

$$\{ \psi_1, \ldots, \psi_6 \} = \{ x_1^2, x_2^2, x_1 x_2, x_1 u, x_2 u, u^2 \}$$

and there is no harm in setting $d(x, u) \equiv 0$.

In order to implement the algorithm (34b) we begin with selecting an input of the form

$$u_t = K_0 x_t + \xi_t$$ (35)

where $K_0$ is a stabilizing controller and $\xi_t = \sum_{j=1}^q v_j \sin(\omega_j t + \phi_j)$. Note that $K_0$ need not be the same $K$ whose value function we are trying to evaluate.
The numerical results that follow are based on a double integrator with friction:
\[
\dot{y} = -0.1 \dot{y} + u
\]
which can be expressed in state space form using \(x = (y, \dot{y})^T\):
\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]  
(36)

We took a relatively large cost on the input:
\[
M = I \quad R = 10
\]

and gain \(a_t = 1/(1 + t)\).

Fig. 3 shows the evolution of the QSA algorithm for the evaluation of the policy \(K = [-1, 0]\) using the stabilizing controller \(K_0 = [-1, -2]\) and \(\xi\) in (35) as the sum of 24 sinusoids with random phase shifts and whose frequency was sampled uniformly between 0 and 50 rad/s. The QSA algorithm is compared with the related SA algorithm in which \(\xi\) is “white noise” instead of a deterministic signal\(^1\)

Fig. 4 shows the weighted error for the feedback gains obtained using the approximate policy improvement algorithm (29) and the optimal controller \(K^*\) (which can be easily computed for an LQR problem). Each policy evaluation was performed by the model-free algorithm (34). The PIA algorithm indeed converges to the optimal control gain \(K^*\).

3 Main Results

3.1 Assumptions

The following assumptions are imposed throughout the remainder of the paper:

**(A1)** The process \(a\) is non-negative, monotonically decreasing, and
\[
\lim_{t \to \infty} a_t = 0, \quad \int_0^\infty a_r \, dr = \infty. \quad (37)
\]

**(A2)** The functions \(\bar{f}\) and \(f\) are Lipschitz continuous: for a constant \(\ell_f < \infty\),
\[
\begin{align*}
\|\bar{f}(\theta') - \bar{f}(\theta)\| &\leq \ell_f \|\theta' - \theta\|, \\
\|f(\theta', \xi) - f(\theta, \xi)\| &\leq \ell_f \|\theta' - \theta\|, \quad \theta', \theta \in \mathbb{R}^d, \quad \xi \in \Omega
\end{align*}
\]

There exists a constant \(b_0 < \infty\), such that for all \(\theta \in \mathbb{R}^d\), \(T > 0\),
\[
\left\| \frac{1}{T} \int_0^T f(\theta, \xi_t) \, dt - \bar{f}(\theta) \right\| \leq b_0 \frac{1}{T} (1 + \|\theta\|) \quad (38)
\]

**(A3)** The ODE (2) has a globally asymptotically stable equilibrium \(\theta^*\).

The process \(a\) in (A1) is a continuous time counterpart of the standard step size schedules in stochastic approximation. The Lipschitz condition in (A2) is standard, and (38) is only slightly stronger than ergodicity of \(\xi\) as given by (8). General sufficient conditions on both \(f\) and \(\xi\) for the ergodic bound (38) are given in Lemma B.1. Assumptions (A1) and (A2), along with a strengthening of (A3), imply convergence of (5) to \(\theta^*\): see Appendix A.

We henceforth assume that (5) is convergent, and turn to identification of the convergence rate. This requires a slight strengthening of (A2):

\(^1\)For implementation, both (34) and the linear system (36) were approximated using Euler’s method, with time-step of 0.01s.
(A4) The vector field $\bar{f}$ is differentiable, with derivative denoted

$$A(\theta) = \partial_{\theta} \bar{f}(\theta)$$  \hspace{1cm} (39)

That is, $A(\theta)$ is a $d \times d$ matrix for each $\theta \in \mathbb{R}^d$, with $A_{i,j}(\theta) = \frac{\partial}{\partial \theta_j} \bar{f}_i(\theta)$.

Moreover, the derivative $A$ is Lipschitz continuous, and $A^* = A(\theta^*)$ is Hurwitz.

The matrix-valued function $A$ is uniformly bounded over $\mathbb{R}^d$, subject to the global Lipschitz assumption on $\bar{f}$ imposed in (A2). The Hurwitz assumption implies that the ODE (2) is exponentially asymptotically stable.

The final assumption is a substantial strengthening of the ergodic limit (38). For this it is simplest to adopt a “Markovian” setting in which the probing signal is itself the state process for a dynamical system:

$$\frac{d}{dt} \xi = H(\xi)$$  \hspace{1cm} (40)

where $H: \Omega \rightarrow \Omega$ is continuous, with $\Omega$ a compact subset of Euclidean space. A canonical choice is the $K$-dimensional torus: $\Omega = \{ x \in \mathbb{C}^K : |x_i| = 1, \ 1 \leq i \leq K \}$, and $\xi$ defined to allow modeling of excitation as a mixture of sinusoids:

$$\xi_\ell = [ \exp(j\omega_1 t), \ldots, \exp(j\omega_K t) ]^T$$  \hspace{1cm} (41)

with distinct frequencies, ordered for convenience: $0 < \omega_1 < \omega_2 < \cdots < \omega_K$. The dynamical system (40) is linear in this special case. It is ergodic, in a sense made precise in Lemma B.1.

Lemma B.1 also provides justification for the following assumptions for the special case (41), and under mild smoothness conditions on $f$. We do not assume (41) in our main results.

(A5) The probing signal is the solution to (40), with $\Omega$ a compact subset of Euclidean space. It has a unique invariant measure $\pi$ on $\Omega$, and satisfies the following ergodic theorems for the functions of interest, for each initial condition $\xi_0 \in \Omega$:

(i) For each $\theta$ there exists a solution to Poisson’s equation $\hat{f}(\theta, \cdot)$ with forcing function $\hat{f}(\theta, \cdot) = f(\theta, \cdot) - \bar{f}(\theta)$. That is,

$$\hat{f}(\theta, \xi_{t_0}) = \int_{t_0}^{t_1} [ f(\theta, \xi_t) - \bar{f}(\theta) ] \, dt + \hat{f}(\theta, \xi_{t_1}), \quad 0 \leq t_0 \leq t_1$$  \hspace{1cm} (42)

with

$$\hat{f}(\theta) = \int_{\Omega} f(\theta, x) \pi(dx) \quad \text{and} \quad 0 = \int_{\Omega} \hat{f}(\theta, x) \pi(dx)$$

(ii) The function $\hat{f}$, and derivatives $\partial_\theta \hat{f}$ and $\partial_\theta f$ are Lipschitz continuous in $\theta$. In particular, $\hat{f}$ admits a derivative $\hat{A}$ satisfying

$$\hat{A}(\theta, \xi_{t_0}) = \int_{t_0}^{t_1} [ A(\theta, \xi_t) - A(\theta) ] \, dt + \hat{A}(\theta, \xi_{t_1}), \quad 0 \leq t_0 \leq t_1$$

where $A(\theta, \xi) = \partial_\theta f(\theta, \xi)$ and $A(\theta) = \partial_\theta \bar{f}(\theta)$ was defined in (39). Lipschitz continuity is assumed uniform with respect to the exploration process: for $\ell_f < \infty$,

$$\| \hat{f}(\theta', \xi) - \hat{f}(\theta, \xi) \| \leq \ell_f \| \theta' - \theta \|$$

$$\| A(\theta', \xi) - A(\theta, \xi) \| \leq \ell_f \| \theta' - \theta \|$$

$$\| \hat{A}(\theta', \xi) - \hat{A}(\theta, \xi) \| \leq \ell_f \| \theta' - \theta \|, \quad \theta', \ \theta \in \mathbb{R}^d, \ \xi \in \Omega$$

(iii) Denote $\gamma_t = [ \hat{A}(\theta^*, \xi_0) - \hat{A}(\theta^*, \xi_t) ] f(\theta^*, \xi_t)$. The following limit exists:

$$\bar{\gamma} \overset{\text{def}}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T \gamma_t \, dt = \int_{\Omega} \hat{A}(\theta^*, x) f(\theta^*, x) \pi(dx)$$

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and the partial integrals are bounded:

$$\sup_{T \geq 0} \left| \int_0^T [\mathcal{Y}_t - \bar{\mathcal{Y}}] \, dt \right| < \infty$$

Assumption (A5) (iii) is imposed because the vector $\bar{\mathcal{Y}}$ arises in an approximation for the scaled error $\mathcal{Z}_t$. This assumption is not much stronger than the others. In particular, the partial integrals will be bounded if there is a bounded solution to Poisson’s equation:

$$\bar{\mathcal{Y}}_{t_0} = \int_{t_0}^{t_1} [\mathcal{Y}_t - \bar{\mathcal{Y}}] \, dt + \bar{\mathcal{Y}}_{t_1}$$

The notation $\hat{f}$ in (42) is used to emphasize the parallels with Markov process and stochastic approximation theory: this is precisely the solution to Poisson’s equation (with forcing function $\hat{f}(\cdot) = \hat{f}(\theta, \cdot) - \hat{f}(\cdot)$) that appears in theory of simulation of Markov processes, average-cost optimal control, and stochastic approximation [18, 3, 28, 5]. For the one-dimensional probing signal defined by the sawtooth function $\xi_t \overset{\text{def}}{=} t \mod 1$, $t \geq 1$, a solution to Poisson’s equation has a simple form:

$$\hat{g}(z) = -\int_0^z [g(x) - \bar{g}] \, dx + \hat{g}(0), \quad z \in [0, 1)$$

where $\bar{g} = \int_0^1 g(x) \, dx$

It will be useful to make the change of notation: for $\theta \in \mathbb{R}^d$ and $T \geq 0$,

$$\Xi_T^1(\theta) = \int_0^T [f(\theta, \xi_t) - \hat{f}(\theta)] \, dt = \hat{f}(\theta, \xi_0) - \hat{f}(\theta, \xi_T)$$

(43)

where the second equality follows from (42). The special case $\theta = \theta^*$ deserves special notation:

$$\Xi_T^1 = \Xi_T^1(\theta^*) = \int_0^T f(\theta^*, \xi_t) \, dt$$

3.2 Main results

For simplicity we restrict to the vanishing gain algorithm, using $a_t = 1/(1 + t)^\rho$, with $\rho \in (0, 1]$.

**Theorem 3.1.** Suppose that (A1)–(A5) hold, and the gain is $a_t = 1/(1 + t)^\rho$. Then,

(i) $\rho < 1$. Then the following hold:

$$Z_t = \bar{\mathcal{Y}} + \Xi_t^1 + o(1)$$

$$\Theta_t = \theta^* + a_t[\bar{\mathcal{Y}} + \Xi_t^1] + o(a_t)$$

(44)

where

$$\bar{\mathcal{Y}} = (A^*)^{-1} \bar{\mathcal{Y}}$$

(45)

(ii) $\rho = 1$. If $I + A^*$ is Hurwitz, then the convergence rate is $1/t$:

$$Z_t = \bar{\mathcal{Y}} + \Xi_t^1 + o(1)$$

$$\Theta_t = \theta^* + a_t[\bar{\mathcal{Y}} + \Xi_t^1] + o(a_t)$$

(46)
The averaging technique of Ruppert and Polyak is often presented as a two time-scale algorithm, for which a continuous time formulation is defined as follows:

\begin{align}
\frac{d}{dt} \Theta_t &= a_t f(\Theta_t, \xi_t), \\
\frac{d}{dt} \Theta^{RP}_t &= \frac{1}{1 + t} [\Theta_t - \Theta^{RP}_t]
\end{align}

(47a) (47b)

What is crucial in this estimation technique is that the first gain is relatively large: \( \lim_{t \to \infty} (1 + t) a_t = \infty \).

Regardless of \( a \), the estimates can be expressed as an approximate average:

\[ \Theta_T^{RP} = \frac{1}{1 + T} \Theta_0^{RP} + \frac{1}{1 + T} \int_0^T \Theta_t \, dt, \quad T \geq 0 \]

It may not make sense to average over the entire history, since you may be including wild transients. Instead, choose a constant \( K > 1 \), and define

\[ \Theta_T^{RP} = \frac{1}{T - T_0} \int_{T_0}^T \Theta_t \, dt, \quad T \geq 0, \quad T_0 = T - T/K \]

(47c)

The choice of \( T_0 \) is made so that \( 1/(T - T_0) = K/T \). The set of equations (47) will be called "Ruppert-Polyak" averaging.

**Theorem 3.2.** Suppose that the assumptions of Thm. 3.1 hold, so that in particular \( a_t = g/(1 + t)^\rho \) using any \( g > 0 \), and with \( \rho \in (\frac{1}{2}, 1) \). Then, with \( \Theta_T^{RP} \) defined by either (47b) or (47c), we have

\[ \Theta_T^{RP} = \theta^* + a_T (1 - \rho)^{-1} \bar{\gamma} + \Phi_T/T + o(1/T) \]

where \( \{\Phi_t\} \) is a bounded function of time. Consequently, the convergence rate is \( 1/T \) if and only if \( \bar{\gamma} = 0 \). \( \Box \)

### 3.3 Proof outline

The first step is to explain why we can replace \( \theta^* \) with \( \Theta_t \) in the definition of the scaled error \( Z_t \). Prop. 3.3 provides justification, and makes clear the enormous difference between the choice of \( \rho = 1 \) or \( \rho < 1 \) when using the gain \( a_t = 1/(1 + t)\rho \):

**Proposition 3.3.** Suppose that \( (A3) \) holds, and that \( \bar{f} \) is \( C^1 \) with \( A^* \) Hurwitz. Fix \( q_0 > 0 \) satisfying \( \text{Re} \lambda(X) < -q_0 \) for every eigenvalue \( \lambda \) for \( A^* \). Then, there exists \( b_0 > 0 \), \( B < \infty \) such that whenever \( ||\theta_{t_0} - \theta^*|| \leq b_0 \), the solution \( \{\Theta_t : t \geq t_0\} \) of the ODE (2) satisfies

\[ ||\theta_t - \theta^*|| \leq B ||\theta_{t_0} - \theta^*|| \exp(-q_0 [t - t_0]), \quad t \geq t_0 \]

Consequently, the following hold for the solution to the ODE (9) using the gain \( a_t = 1/(1 + t)\rho \), with \( 0 \leq \rho \), and \( ||\Theta_{t_0} - \theta^*|| \leq b_0 \):

(i) If \( \rho = 1 \) then \( ||\Theta_t - \theta^*|| \leq B ||\theta_{t_0} - \theta^*|| [t_0/t]^{q_0} \) for \( t \geq t_0 \).

(ii) Using any \( 0 < \rho < 1 \),

\[ ||\Theta_t - \theta^*|| \leq B t_0 ||\theta_{t_0} - \theta^*|| \exp(-q_0 (1 - \rho)^{-1} (1 + t) 1^{1 - \rho}), \quad t \geq t_0 \]

(48)

where \( B t_0 = B \exp(q_0 (1 - \rho)^{-1} (1 + t_0) 1^{1 - \rho}) \). \( \Box \)

A significant conclusion is that when \( \rho < 1 \), so that eq. (48) holds, then

\[ Z_t = \frac{1}{a_t} (\Theta_t - \theta^*) + \varepsilon_t^z \]

with \( \varepsilon_t^z = [\theta^* - \Theta_t]/a_t \) vanishing quickly as \( t \to \infty \).

Prop. 3.4 shows how the nonlinear ODE is naturally “linearized”, provided it is convergent.
Proposition 3.4. Suppose that (A1)–(A4) hold, and that solutions to (5) converge to \( \theta^* \) for each initial condition. Then, the scaled error admits the representation
\[
\frac{d}{dt} Z_t = [r_t I + a_t A(\Theta_t)] Z_t + a_t \Delta_t + \Xi_t, \quad Z_{t_0} = 0
\]
where \( r_t = -\frac{1}{\Theta_t} \log(a_t) \), \( \Xi_t = f(\Theta_t, \xi_t) - f(\Theta_t) \), and \( \|\Delta_t\| = o(\|Z_t\|) \) as \( t \to \infty \).

In particular, setting \( A^* = A(\theta^*) \),

(i) With \( a_t = g/(1 + t) \),
\[
\frac{d}{dt} Z_t = a_t [g^{-1} I + A^*] Z_t + a_t \Delta_t + \Xi_t
\]
(ii) For any \( \rho \in (0, 1) \), using the gain \( a_t = g/(1 + t)^\rho \) gives
\[
\frac{d}{dt} Z_t = a_t A^* Z_t + a_t \Delta_t + \Xi_t
\]
where the definition of \( \Delta_t \) is different in each appearance, but in each case satisfies \( \|\Delta_t\| = o(\|Z_t\|) \).

Note that \( r_t = -\frac{1}{\Theta_t} \log(a_t) \) is always non-negative under (A1).

The challenge in applying Prop. 3.4 is that the “noise” process \( \Xi_t \) appearing in (49) is non-vanishing, and is not scaled by a vanishing term. This is resolved through the change of variables: denote for \( t \geq 0 \),
\[
Y_t \doteq Z_t - \Xi^t(\Theta_t)
\]
where \( \Xi^t(\Theta_t) \) is defined in (43). It is shown in Prop. C.2 that \( Y_t \) solves the differential equation
\[
\frac{d}{dt} Y_t = a_t \left[ A^* Y_t + \Delta^Y_t - Y_t + A^* \Xi^t \right] + r_t [Y_t + \Xi^t]
\]
where \( \Xi^t = \Xi^t(\theta^*) \), and \( \|\Delta^Y_t\| = o(1 + \|Y_t\|) \) as \( t \to \infty \), and from this we obtain convergence: \( \lim_{t \to \infty} Y_t = \bar{Y} \).

This leads easily to Thm. 3.1, and Thm. 3.2 then follows—the details can be found in the Appendix, along with proofs of the other results.

4 Gradient-Free Optimization

Consider the unconstrained minimization problem
\[
\min_{\theta \in \mathbb{R}^d} L(\theta).
\]

It is assumed that \( L : \mathbb{R}^d \to \mathbb{R} \) has a unique minimizer, denoted as \( \theta^* \). The goal here is to estimate \( \theta^* \) based on observations of \( L(\Psi_t) \), where \( \Psi \) is chosen by design.

The algorithms described below are each based on the following architecture: construct an ODE of the form
\[
\frac{d}{dt} \Theta_t = -a_t \tilde{V}_L(t)
\]
where \( a \) is a non-negative, time-varying gain, and \( \tilde{V}_L(t) \) is designed to approximate the gradient (in an average sense).

The algorithms are similar to both SPSA and ESC (see introduction). We opt for quasi Stochastic Gradient Descent (qSGD) here, as algorithms of the form (54) are cousins of SGD algorithms that are also designed to approximate gradient descent. Each algorithm is defined based on a \( d \)-dimensional probing signal. For simplicity, we impose the following normalization conditions, unless stated otherwise:

\[
\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \xi_t \, dt = 0 \quad (55a)
\]
\[
\lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} \xi_t^2 \, dt = I \quad (55b)
\]

This is easily arranged using a mixture of sinusoids.

The first example is a straightforward translation of (15):
qSGD #1
For a given $d \times d$ positive definite matrix $G$, and $\Theta_0 \in \mathbb{R}^d$,
\[
\frac{d}{dt} \Theta_t = -a_t \frac{1}{\varepsilon} G \Psi_t L(\Psi_t)
\]
\[
\Psi_t = \Theta_t + \varepsilon \xi_t
\]
(56a)
\[
(56b)
\]

The second example falls into the class of ESC algorithms:

qSGD #2
For a given $d \times d$ positive definite matrix $G$, and initial condition $\Theta_0$,
\[
\frac{d}{dt} \Theta_t = -a_t \frac{1}{\varepsilon} G \xi_t \frac{d}{dt} L(\Psi_t)
\]
\[
\Psi_t = \Theta_t + \varepsilon \xi_t
\]
(57a)
\[
(57b)
\]
where $\xi_t = \frac{d}{dt} \xi_t$.

The algorithm qSGD #1 takes the form (5), with
\[
f(\Theta_t, \xi_t) = -\frac{1}{\varepsilon} G \xi_t L(\Theta_t + \varepsilon \xi_t)
\]
(58)

Under (55), a second order Taylor series expansion gives
\[
\bar{f}_\varepsilon(\theta) \overset{df}{=} \lim_{T \to \infty} \frac{1}{T} \int_{t=0}^{T} f(\theta, \xi_t) dt = -G \nabla L(\theta) + o(\varepsilon)
\]
(59)

where $\|o(\varepsilon)\| = O(\varepsilon^2)$ if $L$ is a $C^2$ function. A similar approximation holds for qSGD #2.

The representation (59) suggests that the algorithm will approximate gradient descent, provided the right hand side of (59) and (58) is Lipschitz continuous (cf. Assumption (A2)). However, in many problems we know that $\nabla L$ is globally Lipschitz continuous, but $L$ is not. Consider for example a quadratic function. In this case $f$ is not Lipschitz continuous, contradicting Assumption (A2) of the QSA theory. We can only guarantee global convergence of (56) if we employ projection of estimates onto a compact region.

A slightly more complex algorithm resolves this issue. This is again a simple extension of one of Spall’s SPSA algorithms with two function evaluations.

qSGD #3
For a given $d \times d$ positive definite matrix $G$, and initial condition $\Theta_0$,
\[
\frac{d}{dt} \Theta_t = -a_t \frac{1}{2\varepsilon} G \xi_t \left\{ L(\Theta_t + \varepsilon \xi_t) - L(\Theta_t - \varepsilon \xi_t) \right\}
\]
(60)

Denoting by $a_t f(\Theta_t, \xi_t)$ the right hand side of (60), we can show that $f(\theta, \xi) = -G \xi T \nabla L(\theta) + O(\varepsilon^2)$; it is Lipschitz in its first variable whenever this is true for $\nabla L$.

The vector field $f_\varepsilon$ for (60) admits the same approximation (59) under slightly milder conditions on the probing signal with the zero-mean assumption (55a) dropped. Moreover, the following global consistency result can be established:

Proposition 4.1. Suppose that the following hold for function and algorithm parameters in qSGD #3:
Figure 5: Minimizing a loss function with multiple local minima: While $L(\Theta_t)$ is highly oscillatory, the estimate $\Theta^{RP}_t$ is nearly optimal.

(i) Assumption (A1) holds.

(ii) The probing signal satisfies (55b).

(iii) $\nabla L$ is globally Lipschitz continuous, and $L$ is strongly convex, with unique minimizer $\theta^* \in \mathbb{R}^d$.

Then for each $\varepsilon > 0$, there is a unique root $\theta^*_\varepsilon$ of $f_\varepsilon$, satisfying $\|\theta^*_\varepsilon - \theta^*\| \leq O(\varepsilon^2)$. And convergence holds from each initial condition: $\lim_{t \to \infty} \Theta_t = \theta^*_\varepsilon$.

Proof: The hypotheses of the proposition imply that Assumptions (A1)-(A2) of the QSA theory hold for $f(\theta, \xi) = -G \xi L(\theta) + O(\varepsilon^2)$ and $f_\varepsilon$ defined in (59). Since $L$ is strongly convex, it holds that there is a unique solution to $\nabla L(\theta) = O(\varepsilon^2)$ for any $\varepsilon > 0$, so that Assumption (A3) holds as well. Therefore, for each $\varepsilon > 0$, $\Theta_t$ converges to the unique root $\theta^*_\varepsilon$ of $f_\varepsilon$ satisfying $\nabla L(\theta^*_\varepsilon) = O(\varepsilon^2)$. Due to strong convexity, we have:

$$L(\theta^*) \geq L(\theta^*_\varepsilon) + (\nabla L(\theta^*_\varepsilon))^T(\theta^* - \theta^*_\varepsilon) + \frac{\eta}{2}\|\theta^*_\varepsilon - \theta^*\|^2$$

for some $\eta > 0$. Therefore

$$\frac{\eta}{2}\|\theta^*_\varepsilon - \theta^*\|^2 \leq L(\theta^*) - L(\theta^*_\varepsilon) + (\nabla L(\theta^*_\varepsilon))^T(\theta^*_\varepsilon - \theta^*)$$

$$\leq \|\nabla L(\theta^*_\varepsilon)\|\|\theta^*_\varepsilon - \theta^*\|$$

implying that $\|\theta^*_\varepsilon - \theta^*\| \leq O(\varepsilon^2)$.

\[\square\]

qSGD and simulated annealing The primary motivation for SPSA, ESC, and qSGD algorithms is that they can be run based purely on observations of the loss function. A secondary benefit is that the probing can be designed to emulate a simulated annealing algorithm. The example described below was designed to illustrate this point.

The first plot on the left in Fig. 5 is a highly non-convex function $L$, defined as the “soft min” of convex quadratic functions:

$$L(\theta) = -\log \left( \sum_{k=1}^{4} \exp \left( -\{z_i + \|\theta - \theta^i\|^2 \}/\sigma^2 \right) \right)$$

with $\sigma = 1/10$, and

$$\{\theta^i, z_i\} = \{[(-1), -1], [(-1), -2], [(1), -2], [(1), -3]\}$$

The minimizer is $\theta^* \approx (1)$, with $L(\theta^*) \approx -300$.

The gain process $a_t = \min\{\pi(1 + t)^{-\rho}\}$ was used with $\rho = 0.9$ and $\pi = 10^{-3}$. The probing signal was chosen to satisfy (55):

$$\xi_t = \sqrt{2}[\sin(t\omega_1), \sin(t\omega_2)]^T$$

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with \( \omega_1 = 1/4 \) and \( \omega_2 = 1/e^2 \) chosen to obtain attractive plots—higher frequencies lead to faster convergence. The value \( \varepsilon = 0.15 \) was chosen for the scaling (recall (56b)). These meta-parameters were obtained by trial and error: if \( \sigma \) or \( \varepsilon \) is too small, then we are trapped in a local minima.

The ODE was approximated using a standard Euler scheme with 1 sec sampling interval: the crude ODE model.

The two plots on the right hand side in Fig. 5 show the evolution of \( \Theta_t \) in \( \mathbb{R}^2 \) for \( 0 \leq t \leq 5 \times 10^4 \), with \( \Theta_0 = (-2, -2)^\top \). Plots (b) and (d) indicate that the estimates exhibit significant variation throughout the run, but in plot (d) it is clear that they are trapped within the region of attraction of the global minimum.

What’s more, averaging is highly successful: the estimate \( \Theta_T^{\text{av}} \) was obtained as the average of \( \Theta_t \) over the final 20% of the run. It is found that \( L(\Theta_T^{\text{av}}) \) is only a small fraction of one percent greater than \( L(\theta^*) \).

\section{5 Quasi Policy Gradient Algorithms}

A simple example is illustrated in Fig. 6, in which the two dimensional state space is position and velocity:

\[
x^k \in X = [z^\text{min}, z^\text{goal}] \times [-\mathbb{R}, \mathbb{R}]
\]

where \( z^\text{min} \) is a lower limit for position, and the target position is \( z^\text{goal} \). The velocity is bounded in magnitude by \( \mathbb{R}^+ \). The input \( u \) is the throttle position (which is negative when the car is in reverse). The special case adopted in [40, Ch. 10] is modeled in discrete time:

\[
\begin{align*}
z_{k+1} &= [x_1 + x_2]_1 \\
v_{k+1} &= [v_k + 10^{-3}u_k - 2.5 \times 10^{-3}\cos(3z_k)]_2
\end{align*}
\]

(61)

with \( z^\text{min} = -1.2 \), \( z^\text{goal} = 0.5 \), and \( \mathbb{R}^+ = 7 \times 10^{-2} \). The brackets are projecting the values of \( z_{k+1} \) to the interval \( [z^\text{min}, z^\text{goal}] = [-1.2, 0.5] \), and \( v_{k+1} \) to the interval \( [-\mathbb{R}^+, \mathbb{R}^+] \).

Due to state and input constraints, a feasible policy will sometimes put the car in reverse, and travel at maximal speed away from the goal to reach a higher elevation to the left. Several cycles back and forth may be required to reach the goal.

The control objective is to reach the goal in minimal time, but this can also be cast as a total cost optimal control problem. Let \( x^c = (z^\text{goal}, 0)^\top \), and reduce the state space so that \( x^c \) is the only state \( x = (z, u)^\top \in X \) satisfying \( z = z^\text{goal} \). Let \( c(x, u) = 1 \) for all \( x, u \) with \( x \neq x^c \), and \( c(x^c, u) \equiv 0 \).

For \( \theta \in \mathbb{R} \), consider the policy

\[
u_k = \phi_\theta(x^k) = \begin{cases} 1 & \text{if } z_k + v_k \leq \theta \\ \text{sign}(v_k) & \text{else} \end{cases}
\]

(62)

The policy accelerates the car towards the goal whenever the estimate \( \hat{z}_{k+1} = z_k + v_k \) is at or below the threshold \( \theta \).

Fig. 7 shows trajectories of position as a function of time from three initial conditions, and with two instances of this policy: \( \theta = -0.8 \), and \( \theta = -0.2 \). The former is a much better choice from initial condition \( z(0) = -0.6 \): we see that the time to reach the goal is nearly twice as long when using \( \theta = -0.2 \) as compared to \( \theta = -0.8 \).

Application of qSGD to estimate \( \theta^* \) is a form of policy gradient (PG) reinforcement learning.

The total cost in this example coincides with the time to reach the goal. For fixed initial condition \( x^0 \in \mathbb{R}^d \), we might apply qSGD to estimate the minimizer of the corresponding total cost \( J_\theta(x^0) \) over \( \theta \). To avoid infinite values, we consider the loss function \( L(\theta) = \min\{ J^\text{max}, J_\theta(x^0) \} \); the value \( J^\text{max} = 5 \times 10^3 \) was used in these experiments.

A discrete-time counterpart of qSGD #1 is

\[
\begin{align*}
\Theta_{n+1} &= \Theta_n + \alpha_n + \frac{1}{\varepsilon} G\xi_{n+1}L(\Psi_{n+1}) \\
\Psi_{n+1} &= \Theta_n + \varepsilon \xi_{n+1}
\end{align*}
\]

(63a)

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Figure 7: Trajectories for the Mountain Car for two policies (differentiated by $\theta$), and three initial positions.

The algorithm is episodic in the sense that the observation of $L(\Psi_{n+1})$ is obtained only when the car reaches the goal, or the time limit $J_{\text{max}}$ is reached.

It may be more valuable to introduce randomization in the initial condition. In this case we introduce a second probing signal $\{\xi_n^z\}$. We proceed as above, but define

$$L(\Psi_{n+1}) = J_\theta(x_{n+1})$$

with

$$x_{n+1} = [x^0 + \varepsilon^x \xi_{n+1}^x]$$

(64)

where the brackets again indicate projection onto the state space $X$. The goal then is to minimize the average cost:

$$L(\theta) = \mathbb{E}[J_\theta(X)] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \min\{J_{\text{max}}, J_\theta(\xi_n^x)\}$$

(65)

The signal $\{\xi_n^x = (\xi_n^z, \xi_n^r)^T\}$ was chosen to cover the state space uniformly: introduce two signals that are quasi-uniform and independent on $[0,1]$:

$$W_n^u = \text{frac}(nr_u), \quad W_n^z = \text{frac}(nr_z),$$

where “frac” denotes the fractional part of a real number, $r_u, r_z$ are irrational, and their ratio is also irrational. Then define

$$\xi_n^z = \pi(2W_n^u - 1)$$

$$\xi_n^r = z_{\text{min}} + [z_{\text{goal}} - z_{\text{min}}]W_n^z$$

(66)

The values $r_\pi = \pi$ and $r_e = e$ were chosen in these experiments. Also, $x^0 = 0$ and $\varepsilon^x = 0$ in (64), giving $x_{n+1} = \xi_{n+1}^x$.

Figure 8: qSGD #1 for the Mountain Car using the gradient-free optimization algorithm (63) using a large constant step-size.

A run using the episodic algorithm (63, 64) is shown in Fig. 8, with constant step-size $\alpha_n = 0.1$, $\varepsilon = 0.05$, and $\xi_n = \sin(n)$. The large step-size was chosen simply to illustrate the exotic nonlinear dynamics that emerge from this algorithm. It would seem that the algorithm has failed, since the estimates oscillate between $-1.2$ and $-0.3$ in steady-state, while the actual optimizer is $\theta^* \approx -0.8$. The dashed line shows the average of $\{\Theta_n\}$ over the final 20% of estimates. This average is very nearly optimal, since the objective function is nearly flat for $\theta$ near the optimizer.

Fig. 9 (a) shows results from $10^3$ independent runs, each with horizon length $T = 10^4$. In each case, the parameter estimates evolve according to (63), to obtain estimates $\{\Theta_n: 1 \leq n \leq T, \ 1 \leq i \leq 10^3\}$. The two columns are distinguished by the probing signals $\xi_n$ and $\xi_n^x$. For QSA the probing signal $\xi_n$ was a sinusoid, with phase $\phi$ selected independently in the interval $[0,1)$, in each of the $10^3$ runs. The probing sequence $\xi_n^x$ was fixed as (66).

The results displayed in the second column of Fig. 9 (a) used an independent sequence for the probing signals, each uniform on their respective ranges (in particular, the distribution of $\xi_n$ was chosen uniform on
the interval \([-1, 1]\) for each \(n\). The label 1SPSA refers to the algorithm of Spall based on i.i.d. exploration \([37, 38]\) (recall (15)).

It was found that qSGD #3 results in far more reliable estimates: see comparison in Fig. 9 (b). However, this method will not be reliable in the presence of system disturbances, since it requires on two observations (recall (15)).

Algorithm qSGD #2 is also easily adapted to this application:

\[
\Theta_{n+1} = \Theta_n + \frac{1}{\varepsilon} G \xi'_{n+1} L(\Psi_{n+1})' \\
\Psi_{n+1} = \Theta_n + \varepsilon \xi_{n+1}
\]

(67a)

(67b)

where the primes denote approximations of the derivatives appearing in (57a):

\[
\xi'_{n+1} = \frac{1}{\delta} (\xi_{n+1} - \xi_n), \quad L(\Psi_{n+1})' = \frac{1}{\delta} (L(\Psi_{n+1}) - L(\Psi_n))
\]

with \(\delta > 0\) the sampling interval.

A histogram and sample path of parameter estimates are shown in Fig. 9 (c), based on algorithm (67) with \(\delta^2 = 0.5\), and all of the same choices for parameters, except that the step-size was reduced to avoid large initial transients: \(\alpha_n = \min(1/n^{0.75}, 0.05)\). This results in \(\alpha_n = 1/n^{0.75}\) for \(n \geq 55\).

Based on the histogram, the performance appears slightly worse than observed for method #1 in Fig. 9 (a), but these outcomes are a product of particular choices for algorithm parameters.

6 Conclusions

It is evident that QSA is worthy of far greater attention in terms of both theory and applications. The case for applications to optimization and optimal control was made in Sections 4 and 5, where the most obvious examples are deterministic control systems, such as in robotics. Theoretically speaking, we have shown under general conditions that the rate of convergence is much faster for QSA as opposed to SA. If there is one take-home message from these results, it is this: do not inject randomness in your algorithm.

There are several open questions for future research:

- Theory in the quasi Monte-Carlo literature may shed doubt on the \(1/t\) convergence rate established in Thms. 3.1 and 3.2, since it is well known that the optimal rate is of the form \([\log(t)]^d/t\), with \(d\) the dimension (Section 3 of Chapter 9 of [3] contains an accessible survey). It may be that sharper results obtained in the present paper are a result of the smoothness conditions imposed on the functions involved. Another possibility is that the theory of QMC is posed in discrete time, and that an Euler approximation will destroy the \(1/t\) rate. If so, this motivates a more accurate ODE approximation.
- The convergence-rate obtained in Thm. 3.2 for Ruppert-Polyak averaging is currently a mystery, because the vector \(\tilde{Y}\) defined in (A5) is not well understood. A current topic of research concerns conditions under which this is zero, so we obtain the optimal \(1/t\) convergence rate.
- More exciting questions concern the creation of more efficient schemes to optimize the convergence rate. The crisp approximation obtained in Thm. 3.1 will likely lead to new approaches.
- What about high dimensions? The qSGD algorithms are easy to code, and quickly converge to an approximately optimal parameter for the examples considered in this paper. In high dimensions we can’t expect to
blindly apply any of these algorithms. If \( \theta \in \mathbb{R}^d \) with large \( d \), then the complexity of the probing signal must grow as well. For example, consider the choice of probing signal (7), where \( i \) ranges from 1 to \( d = K = 1000 \). If the frequencies \( \{\omega_i\} \) are chosen in a narrow range, then the limit (55b) will converge very slowly. The rate will be faster if the frequencies are widely separated, but we then need a much higher resolution Euler approximation to implement an algorithm.

This challenge is well understood in the optimization literature. One approach to create a reliable algorithm is to employ block coordinate descent to effectively reduce the dimension of the optimization problem. This requires two ingredients:

(i) A sequence of timepoints \( T_0 = 0 < T_1 < T_2 < \cdots \)

(ii) A sequence of “parameter blocks” \( B_k \subset \{1, \ldots, d\} \) for each \( k \geq 0 \), where the number of elements \( d_B \) in \( B_k \) is far smaller than \( d \).

The qSGD ODE (54) is modified so that \( \Theta_t(i) \) is held constant on the interval \([T_k, T_{k+1}]\) for \( i \notin B_k \), and

\[
\frac{d}{dt} \Theta_t(i) = -a_i[\tilde{\nabla}_L(t)]_i, \quad i \in B_k, \quad t \in [T_k, T_{k+1})
\]

The alternating direction method of multipliers (ADMM) employs a similar scheme.

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Appendices

This Appendix is organized into three sections: some of the convergence theory in Appendix A is taken from [6, 7] (duplicated here for convenience). The material supporting the proof of Thm. A.1 is all new. Appendix B contains theory to justify Assumption (A5), and Appendix C contains proofs of the main results related to rates of convergence for QSA.

A Convergence of QSA

Analysis of QSA is based on consideration of the associated ODE (2) in which the “averaged” vector field is introduced in (8):

\[
\bar{f}(\theta) = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\theta, \xi_t) \, dt, \quad \text{for all } \theta \in \mathbb{R}^d.
\]  

(68)

The first step in the theory is to find assumptions to ensure that the limit exists. Following this, stability conditions are found for the averaged ODE (2) that ensure that ODE (2) and the QSA ODE (5) converge to the same limit.

The starting point in an ODE approximation is a temporal transformation: substitute in (5) the new time variable given by

\[
\tau = g_t \defeq \int_0^t a_r \, dr, \quad t \geq 0.
\] 

(69)

The time-scaled process is then defined by

\[
\hat{\Theta}_\tau \defeq \Theta_{g^{-1}(\tau)} = \Theta|_{t=g^{-1}(\tau)}
\] 

(70)

For example, if \( a_r = (1 + r)^{-1} \), then

\[
\tau = \log(1 + t) \quad \text{and} \quad \xi(g^{-1}(\tau)) = \xi(e^\tau - 1).
\] 

(71)

The chain rule of differentiation gives

\[
\frac{d}{d\tau} \Theta(g^{-1}(\tau)) = f(\Theta(g^{-1}(\tau)), \xi(g^{-1}(\tau)).
\] 

That is, the time-scaled process solves the ODE,

\[
\frac{d}{d\tau} \hat{\Theta}_\tau = f(\hat{\Theta}_\tau, \xi(g^{-1}(\tau)).
\] 

(72)

The two processes \( \Theta \) and \( \hat{\Theta} \) differ only in time scale, and hence, proving convergence of one proves that of the other. For the remainder of this section we will deal exclusively with \( \hat{\Theta} \); it is on the ‘right’ time scale for comparison with \( \Theta \), the solution of (2).

The first step in establishing convergence of QSA is to show that the solutions are bounded in time. Two approaches can be borrowed from the literature: Lyapunov function techniques, or the ODE at \( \infty \) introduced in [11, 10].

When applying the techniques of [11, 10] we require the vector field at \( \infty \):

\[
\bar{f}_\infty(\theta) \defeq \lim_{r \to \infty} r^{-1} \bar{f}(r\theta), \quad \theta \in \mathbb{R}^d
\] 

(73)

See [13] for a proof of the following:

**Theorem A.1.** Suppose that Assumptions (A1)–(A3) hold, along with the following two conditions:

(i) The limit (73) exists for all \( \theta \) to define a continuous function \( \bar{f}_\infty : \mathbb{R}^d \to \mathbb{R}^d \).
(ii) The origin is globally asymptotically stable for the ODE at \( \infty \):

\[
\frac{d}{dt} \theta_t^\infty = \vec{f}_\infty(\theta_t^\infty), \quad \theta \in \mathbb{R}^d
\]

(74)

Then the solution to (5) converges to \( \theta^* \) for each initial condition.

When applying Lyapunov function techniques we impose the following:

(QSV1) There exists a continuous function \( V : \mathbb{R}^d \to \mathbb{R}_+ \) and constants \( c_0 > 0, \delta_0 > 0 \) such that, for any initial condition \( \theta_0 \) of (2), and any \( 0 \leq T \leq 1 \), the following bounds hold whenever \( \| \theta_s \| > c_0 \),

\[
V(\theta_{s+T}) - V(\theta_s) \leq -\delta_0 \int_0^T \| \theta_t \| \, dt
\]

The Lyapunov function is Lipschitz continuous: there exists a constant \( \ell_V < \infty \) such that

\[
\| V(\theta') - V(\theta) \| \leq \ell_V \| \theta' - \theta \| \quad \text{for all } \theta, \theta'.
\]

Assumption (QSV1) ensures that there is a Lyapunov function \( V \) with a strictly negative drift whenever \( \theta \) escapes a ball of radius \( c_0 \). This is used to establish boundedness of \( \Theta \). If \( V \) is differentiable then we have

\[
\frac{d}{dt} V(\theta_t) \leq -\| \theta_t \|^2, \quad \text{whenever } \| \theta_t \| > c_0
\]

The integral form is chosen since sometimes it is easier to establish a bound in this form. In particular, the proof of Thm. A.1 below is based on the construction of a solution to (QSV1).

Verifying (QSV1) for a linear system. Consider the ODE (2) in which \( \vec{f}(x) = Ax \) with \( A \) a Hurwitz \( d \times d \) matrix. There is a quadratic function \( V_2(x) = x^T P x \) with \( P \in \mathbb{R}^{d \times d} \) satisfying the Lyapunov equation \( PA + A^T P = -I \), with \( P > 0 \). Consequently, solutions to (2) satisfy

\[
\frac{d}{dt} V_2(\theta_t) = -\| \theta_t \|^2
\]

Choose \( V = \sqrt{V_2} \), so that by the chain rule

\[
\frac{d}{dt} V(\theta_t) = -\frac{1}{2} \frac{1}{\sqrt{V_2(\theta_t)}} \| \theta_t \|^2
\]

This \( V \) is a Lipschitz solution to (QSV1), for any \( c_0 > 0 \).

Theorem A.2. Under Assumptions (A1)–(A3) and (QSV1), the solution to (5) converges to \( \theta^* \) for each initial condition.

A.1 ODE Solidarity

Here we establish Thms. A.1 and A.2 under the assumption that solutions to (5) are ultimately bounded in the following sense: there exists \( b < \infty \) such that for each \( \theta \in \mathbb{R}^d \), \( z \in \Omega \), there is a \( T_\theta \) such that

\[
\| \hat{\Theta}_\tau \| \leq b \quad \text{for all } \tau \geq T_\theta, \quad \text{when } \hat{\Theta}_0 = \theta, \hat{\xi}_0 = z
\]

(75)

Verification of this stability condition is contained in [6], under the assumptions of Thm. A.2. A complete proof under the assumptions of Thm. A.1 is contained in Appendix A.2.

Define \( \theta^*(w), w \geq \tau, \) to be the unique solution to (2) ‘starting’ at \( \hat{\Theta}_\tau \):

\[
\frac{d}{dw} \theta^*(w) = f(\theta^*(w)), \quad w \geq \tau, \quad \theta^*_\tau = \hat{\Theta}_\tau
\]

(76)

The following result is required in the proof of either stability result. The proof of Lemma A.3 can be found in [6], and is similar to results in the SA literature (e.g., Lemma 1 in Chapter 2 of [9]).
Lemma A.3. Assume that $\hat{\Theta}$ is bounded. Then, for any $T > 0$,
\[
\lim_{\tau \to \infty} \sup_{v \in [0, T]} \left\| \int_{\tau}^{\tau + v} [f(\hat{\Theta}_w, \xi(g^{-1}(w)) - \hat{f}(\hat{\Theta}_w)] \, dw \right\| = 0
\]
\[
\lim_{\tau \to \infty} \sup_{v \in [0, T]} \| \hat{\Theta}_{\tau + v} - \hat{\Theta}(\tau + v) \| = 0.
\]

Proposition A.4. (Boundedness Implies Convergence) Suppose that (A1)–(A3) hold, and that the ultimate boundedness assumption (75) holds. Then, the solution to (5) converges to $\theta^*$ for each initial condition.

Proof. Under the assumptions of the proposition, there exists $b < \infty$ such that for any $\theta \in \mathbb{R}^d$, there is a time $T_{\theta}$ such that $\|\hat{\Theta}_{\tau}\| = \|\hat{\Theta}_\tau\| \leq b$, for $\tau \geq T_{\theta}$.

By the definition of global asymptotic stability, for every $\varepsilon > 0$, there exists a $T_{\varepsilon} > 0$, independent of the value $\hat{\Theta}_\tau$, such that $\|\hat{\Theta}_\tau(\tau + v) - \theta^*\| < \varepsilon$ for all $v \geq T_{\varepsilon}$, whenever $\|\hat{\Theta}_\tau\| \leq b$.

Lemma A.3 gives,
\[
\limsup_{\tau \to \infty} \| \hat{\Theta}_{\tau + T_{\varepsilon}} - \theta^* \| \leq \limsup_{\tau \to \infty} \| \hat{\Theta}_{\tau + T_{\varepsilon}} - \hat{\Theta}_\tau^T(\tau + T_{\varepsilon}) \| + \limsup_{\tau \to \infty} \| \hat{\Theta}_\tau^T(\tau + T_{\varepsilon}) - \theta^* \| \leq \varepsilon.
\]
Since $\varepsilon$ is arbitrary, we have the desired limit. $\square$

A.2 ODE@\infty

The remainder of this section contains technical results concerning the proof of Thm. A.1. It may not be surprising that the following result makes several appearances in the proofs.

Proposition A.5. (Grnwall Inequality) Let $\alpha$, $\beta$ and $z$ be real-valued functions defined on an interval $[0, T]$, with $T > 0$. Assume that $\beta$ and $z$ are continuous.

(i) If $\beta$ is non-negative and if $z$ satisfies the integral inequality
\[
z_t \leq \alpha_t + \int_0^t \beta_s z_s \, ds \tag{77}
\]
Then Grnwall Inequality holds:
\[
z_t \leq \alpha_t + \int_0^t \alpha_s \beta_s \exp\left(\int_s^t \beta_r \, dr \right) ds, \quad 0 \leq t \leq T. \tag{78}
\]

(ii) If, in addition, the function $\alpha$ is non-decreasing, then
\[
z_t \leq \alpha_t \exp\left(\int_0^t \beta_s \, ds \right), \quad 0 \leq t \leq T. \tag{79}
\]

$\square$

Grnwall’s Inequality implies a crude bound that is needed in approximations:

Proposition A.6. Consider the ODE (2), subject to the Lipschitz condition in (A2). Then,

(i) There is a constant $B_f$ depending only on $f$ such that
\[
\|\hat{\theta}_t\| \leq (B_f + \|\theta_0\|) e^{B_f t} - B_f, \quad t \geq 0
\]
(ii) If there is an equilibrium $\theta^*$, then for each initial condition,
\[ \|\theta_t - \theta^*\| \leq \|\theta_0 - \theta^*\| e^{\ell t}, \quad t \geq 0 \]
\[ \Box \]

The main step in the proof of Thm. A.1 is to show that the assumptions of the theorem imply that the ODE (2) is ultimately bounded. We first need to better understand the special properties of the solution to (74):

Lemma A.7. Suppose that (A2) holds, and the limit (73) exists for all $\theta$ to define a continuous function $\bar{f}_{\infty} : \mathbb{R}^d \to \mathbb{R}^d$. Suppose moreover that the origin is asymptotically stable for (74). Then the following hold:

(i) For each $\theta \in \mathbb{R}^d$ and $s \geq 0$,
\[ \bar{f}_{\infty}(s\theta) = s \bar{f}_{\infty}(\theta) \]

(ii) If $\{\theta_t^\infty : t \geq 0\}$ is any solution to the ODE (74), and $s > 0$, then $\{y_t = s\theta_t^\infty : t \geq 0\}$ is also a solution, starting from $y_0 = s\theta_0^\infty \in \mathbb{R}^d$.

(iii) The origin is globally asymptotically stable for (74), and convergence to the origin is exponentially fast: for some $R < \infty$ and $\rho > 0$,
\[ \|\theta_t^\infty\| \leq Re^{-\rho t}\|\theta_0^\infty\|, \quad \theta_0^\infty \in \mathbb{R}^d \]

Proof. Consider first the scaling result in part (i): from the definition (73), with $s > 0$,
\[ \bar{f}_{\infty}(s\theta) = s \lim_{r \to \infty} (sr)^{-1} \bar{f}(sr\theta) = s \bar{f}_{\infty}(\theta) \]

The case $s = 0$ is trivial, since it is clear that $\bar{f}_{\infty}(0) = 0$. This establishes (i).

Next, write
\[ \theta_t^\infty = \theta_0^\infty + \int_0^t \bar{f}_{\infty}(\theta_\tau^\infty) \, d\tau \]

Multiplying both sides by $s$ and applying (i) gives (ii).

Under the assumption that the origin is asymptotically stable, there exists $\varepsilon > 0$ such that $\lim_{t \to \infty} \theta_t = 0$, whenever $\|\theta_0\| \leq \varepsilon$. Moreover, the convergence is uniform: there exists $T_0 > 0$ such that
\[ \|\theta_{T_0}\| \leq \frac{1}{2} \varepsilon \quad \text{whenever } \|\theta_0\| \leq \varepsilon \]

Next, apply scaling: for any initial condition $\theta_0$, consider $y_t = s\theta_t^\infty$ using $s = \varepsilon / \|\theta_0\|$, chosen so that $\|y_0\| = \varepsilon$. Then $\|y_{T_0}\| \leq \frac{1}{2} \varepsilon = \frac{1}{2} \|y_0\|$, implying
\[ \|\theta_{T_0}\| \leq \frac{1}{2} \|\theta_0\| \quad \theta_0 \in \mathbb{R}^d \]

This easily implies (iii) by iteration, as follows: for any $t$ we can write $t = nT_0 + t_0$, with $0 \leq t_0 < T_0$, so that
\[ \|\theta_t\| \leq \frac{1}{2} \|\theta_{(n-1)T_0+t_0}\| \leq 2^{-n} \|\theta_{T_0}\| \]

Grönwall’s Inequality gives $\|\theta_{n0}\| \leq e^{\ell t_0} \|\theta_0\|$, so that
\[ \|\theta_t\| \leq 2e^{\ell t_0} 2^{-(n+1)} \|\theta_0\| \]

where the right hand side has been arranged to make use of the bound $t \leq (n + 1)T_0$, giving $2^{-(n+1)} \leq \exp(-\log(2)t/T_0)$. We arrive at the bound in (iii) with $R = 2e^{\ell T_0}$ and $\rho = \log(2)/T_0$. □
Lemma A.8. Under the assumptions of Lemma A.7, for each $T < \infty$ and $\varepsilon \in (0, 1]$, there exists $K_T < \infty$ independent of $\varepsilon$, and $B_T(\varepsilon) < \infty$ such that for all solutions to eqs. (2) and (74) from common initial condition $\theta_0$,

$$\|\theta_t - \theta_t^\infty\| \leq B_T(\varepsilon) + K_T[1 + \|\theta_0\|] \varepsilon$$

Proof. Denote

$$E(\theta) = \|\bar{f}(\theta) - \bar{f}_\infty(\theta)\|$$

so that by Lemma A.7, with $s = \|\theta\|$,

$$s^{-1}E(\theta) = \|\bar{f}_s(\theta/s) - \bar{f}_\infty(\theta/s)\|$$

Because the functions $\{\bar{f}_s : s \geq 1\}$ are uniformly Lipschitz continuous, the right hand side converges to zero uniformly in $\theta \neq 0$. Consequently,

$$E(\theta) = o(\|\theta\|)$$

Let’s think about what this means: for any $\varepsilon > 0$, there exists $N(\varepsilon) < \infty$, such that $E(\theta) \leq \varepsilon\|\theta\|$ whenever $\|\theta\| \geq N(\varepsilon)$. From this we get the simpler looking bound:

$$E(\theta) \leq B_\varepsilon + \varepsilon\|\theta\|, \quad \text{where } B_\varepsilon = \max\{E(\theta) : \|\theta\| \leq N(\varepsilon)\}$$

(80)

For any initial condition $\theta_0$ we compare the two solutions:

$$\theta_t = \theta_0 + \int_0^t \bar{f}(\theta_\tau) d\tau$$

$$\theta_t^\infty = \theta_0 + \int_0^t \bar{f}_\infty(\theta_\tau^\infty) d\tau$$

Write $z_t = \|\theta_t - \theta_t^\infty\|$ and use the preceding definition to obtain,

$$z_t \leq \int_0^t \|\bar{f}_\infty(\theta_\tau) - \bar{f}_\infty(\theta_\tau^\infty)\| d\tau + \int_0^t E(\theta_\tau) d\tau$$

$$\leq \ell f \int_0^t \|\theta_\tau - \theta_\tau^\infty\| d\tau + \int_0^t E(\theta_\tau) d\tau$$

Grnwall’s Inequality in its second form (79) holds, with $\beta_t \equiv \ell f$, and $\alpha_t$ the second integral, giving

$$z_t \leq e^{\ell f t} \int_0^t E(\theta_\tau) d\tau \leq e^{\ell f t} \int_0^t \left\{B_\varepsilon + \varepsilon\|\theta_\tau\|\right\} d\tau$$

where the second inequality uses (80), with $\varepsilon > 0$ to be chosen. Prop. A.6 gives $\|\theta_\tau\| \leq \{B_f + \|\theta_0\|\} e^{\ell f \tau}$, so that

$$z_t = \|\theta_t - \theta_t^\infty\| \leq \tau e^{\ell f t} B_\varepsilon + \varepsilon e^{\ell f t} \left\{B_f + \|\theta_0\|\right\} \left\{e^{\ell f t} - 1\right\}$$

Prop. A.9 combined with Thm. A.2 imply Thm. A.1.

Proposition A.9. Under the assumptions of Thm. A.1, there is a Lipschitz continuous function $V$ that satisfies (QSV1).

Proof. Choose $T > 0$ so that $\|\theta_t^\infty\| \leq \frac{1}{2}\|\theta\|$ when $t \geq T$, for any solution to eq. (74), from any initial condition $\theta_0^\infty = \theta$. We then define

$$V^\infty(\theta) = \int_0^T \|\theta_t^\infty\| dt, \quad \theta_0^\infty = \theta$$

$$V(\theta) = \int_0^T V(\theta_t) dt, \quad \theta_0 = \theta$$

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The Gronwall Inequality implies that each is a Lipschitz continuous function of \( \theta \). Moreover, applying Lemma A.7 it follows that the first is radially homogeneous: \( V^\infty(s\theta) = sV^\infty(\theta) \) for each \( \theta \) and \( s > 0 \), and satisfies the lower bound for some \( \delta > 0 \):

\[
V^\infty(\theta) \geq \delta \|\theta\|
\]

Consequently, this is a Lyapunov function for the ODE for each initial condition \( \theta^0 = \theta \),

\[
V^\infty(\theta^\infty(\theta)) = \int_0^T \|\theta^\infty(\theta)\| \, dt \leq \frac{1}{2} V^\infty(\theta) \leq V^\infty(\theta) - \frac{1}{2} \delta \|\theta\|
\]

The next step is to show that a similar bound holds with \( \theta^\infty \) replaced by \( \vartheta^\infty \). Let \( \ell_V \) denote the Lipschitz constant for \( V^\infty \). The bound above combined with Lemma A.8 gives

\[
V^\infty(\vartheta^\infty) \leq V^\infty(\theta) - \frac{1}{2} \delta \|\theta\| + \ell_V \left( B_T(\vartheta) + K_T[1 + \|\theta\|] \varepsilon \right)
\]

Fix \( \varepsilon \in (0, 1) \) so that

\[
\ell_V K_T \varepsilon \leq \delta/4
\]

giving

\[
V^\infty(\vartheta^\infty) \leq V^\infty(\theta) - \frac{1}{4} \delta \|\theta\| + K^\infty_V \]

with \( K^\infty_V = \ell_V (B_T(\varepsilon) + K_T) \).

To complete the proof, write

\[
V(\vartheta_s) = \int_s^T V^\infty(\vartheta_t) \, dt + \int_0^s V^\infty(\vartheta_{T+t}) \, dt
\]

The preceding bound gives

\[
V^\infty(\vartheta_{T+t}) \leq V^\infty(\vartheta_t) - \frac{1}{2} \delta \|\vartheta_t\| + K^\infty_V
\]

so that

\[
V(\vartheta_s) \leq V(\vartheta_0) - \frac{1}{4} \delta \int_0^s \|\vartheta_t\| \, dt + sK^\infty_V
\]

This bound is essentially equivalent to (QSV1).

\[\square\]

### B Deterministic Markovian model

Here we explain how to verify assumption (A5), and obtain representations for the solution to Poisson’s equation. To simplify notation we fix \( \theta \) and \( 1 \leq i \leq d \), denote \( g(\xi_t) \equiv f_i(\theta, \xi_t) \), and consider the associated Poisson equation (42) in the new notation:

\[
\hat{g}(\xi_{t_0}) = \int_{t_0}^{t_1} \left[ g(\xi_t) - \bar{g} \right] \, dt + \hat{g}(\xi_{t_1}), \quad 0 \leq t_0 \leq t_1
\]

Recall that \( g \) is known as the forcing function, \( \bar{g} \) its steady-state mean, and \( \hat{g} \) is the solution (known as the relative value function in some applications).

Assumption (A5) is analogous to common assumptions in the study of simulation or stochastic approximation algorithms when \( \xi \) is a Markov process [18, 4]. Conditions for a well behaved solution to Poisson’s equation are available, subject to conditions on the Markov process and the function. In particular, for stochastic differential equations (SDEs), a non-degeneracy condition known as hypoellipticity is a first step, and then a solution to Poisson’s equation exists subject to a Lyapunov function drift condition [18].

While the process \( \xi \) defined by (40) is Markovian, it is purely degenerate in the sense that Poisson’s equation (81) in differential form is a first order PDE:

\[
g(z) + \partial \hat{g}(z) \cdot H(z) = \bar{g}, \quad z \in \Omega
\]

To the best of our knowledge, a general theory for solutions is not presently available. Lemma B.1 concerns the special case (41) for which \( H \) is a linear function of \( z \in \mathbb{C}^K \).
Lemma B.1.  Suppose that $g : \mathbb{C}^K \rightarrow \mathbb{R}$ admits the Taylor series representation,

$$g(z) = \sum_{n_1, \ldots, n_K} a_{n_1, \ldots, n_K} z_1^{n_1} \cdots z_K^{n_K}, \quad z \in \Omega,$$

where the sum is over all $K$-length sequences in $\mathbb{Z}_+^K$, and the coefficients $\{a_{n_1, \ldots, n_K}\} \subset \mathbb{C}^K$ are absolutely summable:

$$\sum_{n_1, \ldots, n_K} |a_{n_1, \ldots, n_K}| < \infty \quad (83)$$

Then, with $\xi$ defined in (41),

(i) The ergodic limit holds:

$$\hat{g} = \lim_{T \to \infty} \frac{1}{T} \int_0^T g(\xi_t) \, dt = \int_0^1 \cdots \int_0^1 g(e^{2\pi i t_1}, \ldots, e^{2\pi i K}) \, dt_1 \cdots dt_K$$

where $\hat{g} = a_0$ (the coefficient when $n_i = 0$ for each $i$).

(ii) There exists a solution $\hat{g} : \mathbb{C}^K \rightarrow \mathbb{R}$ to (81). It is of the form (82):

$$\hat{g}(x) = \sum_{n_1, \ldots, n_K} \hat{a}_{n_1, \ldots, n_K} x_1^{n_1} \cdots x_K^{n_K} \quad (84)$$

in which $|\hat{a}_{n_1, \ldots, n_K}| \leq |a_{n_1, \ldots, n_K}|/\omega_1$ for each coefficient.

Proof. Complex exponentials and the Fourier representation are used to obtain the simple formula:

$$g(\xi_t) = \sum_{n_1, \ldots, n_K} a_{n_1, \ldots, n_K} \exp\{(n_1 \omega_1 + \cdots + n_K \omega_K)jt\}$$

The absolute-summability assumption (83) justifies Fubini’s Theorem:

$$\int_{t_0}^{t_1} [g(\xi_t) - \hat{g}] \, dt = \sum_{n_1, \ldots, n_K} a_{n_1, \ldots, n_K} \int_{t_0}^{t_1} \exp\{(n_1 \omega_1 + \cdots + n_K \omega_K)jt\} \, dt = g(\xi_{t_1}) - \hat{g}(\xi_{t_1})$$

where $\hat{g}$ is given by (84) with $\hat{a}_0 = 0$ (that is, $n_k = 0$ for each $k$), and for all other coefficients

$$\hat{a}_{n_1, \ldots, n_K} = a_{n_1, \ldots, n_K} \{n_1 \omega_1 + \cdots + n_K \omega_K\}^{-1}$$

The lemma then justifies (A5) provided $f_i(\theta, \cdot)$ satisfies the Taylor series bound for each $i$ and $\theta$, along with the derivatives $\frac{\partial}{\partial \theta_j} f_i(\theta, \cdot)$, for each $i, j$. While an explicit formula for $\hat{f}$ is not required in any algorithm, bounds may be valuable in finer convergence rate analysis of QSA algorithms. In particular, the approximation of the scaled error $Z_t \equiv \frac{1}{a_t} (\Theta_t - \hat{\Theta}_t)$ obtained in Thm. 3.1 depends on $\hat{f}(\theta^*, \xi_t)$.

C Technical Proofs

The solution to (2) is related to the solution to (9) through a temporal transformation: the proof of Lemma C.1 is an application of the chain rule.

Lemma C.1.  Let $\{\Theta_t : t \geq \tau_0\}$ denote the solution to (2) with $\Theta_{\tau_0} = \Theta_0$, and $\tau_0 = g_{\tau_0}$ (with time-change defined in (69)). The solution to (9) is then given by

$$\hat{\Theta}_t = \Theta_{\tau}, \quad t \geq t_0, \quad \text{with} \quad \tau = g_t = \int_0^t a_r \, dr$$

\[\square\]
Proof of Prop. 3.4. Taking derivatives of each side of (11) gives, by the product rule,

\[
\frac{d}{dt} Z_t = \frac{d}{dt} \left( \frac{1}{a_t} (\Theta_t - \bar{\Theta}_t) \right) \\
= \left( -\frac{1}{a_t} \frac{d}{dt} a_t \right) (\Theta_t - \bar{\Theta}_t) + f(\Theta_t, \xi_t) - \bar{f}(\bar{\Theta}_t) \\
= r_t Z_t + f(\Theta_t, \xi_t) - \bar{f}(\bar{\Theta}_t)
\]

where in the final equation we used the chain rule for the derivative of a logarithm (recall that \( r_t = -\frac{d}{dt} \log(a_t) \)), and the definition of \( Z_t \).

On adding and subtracting \( \bar{f}(\bar{\Theta}_t) \), we arrive at a suggestive decomposition:

\[
\frac{d}{dt} Z_t = r_t Z_t + \left[ \bar{f}(\bar{\Theta}_t) - \bar{f}(\bar{\Theta}_t) \right] + \left[ f(\Theta_t, \xi_t) - \bar{f}(\bar{\Theta}_t) \right]
\]

That is, under the assumptions of Prop. 3.4,

\[
R_t = A(\bar{\Theta}_t)[\Theta_t - \bar{\Theta}_t] + \epsilon^1_t
\]

where, under the Lipschitz condition on \( A \),

\[
\| \epsilon^1_t \| = o(\| \Theta_t - \bar{\Theta}_t \|) = o(a_t \| Z_t \|)
\]

This completes the proof of (49), with \( \Delta_t = \epsilon^1_t / a_t \).

If \( a_t = g/(1 + t) \) we obtain \( r_t = 1/(1 + t) = g^{-1} a_t \). Equation (49) thus implies the approximation (50), where the definition of \( \| \Delta_t \| \) is modified to include the error from replacing \( A(\bar{\Theta}_t) \) with its limit \( A^* = A(\Theta^*) \).

Consider next the “larger gain” \( a_t = g/(1 + t)^{\rho} \), with \( \rho \in (0, 1) \), so that \( r_t = \rho/(1 + t) \). The simpler approximation (51) follows, in which \( \| \Delta_t \| \) has two additional terms: once again, we replace \( A(\bar{\Theta}_t) \) with its limit \( A^* \), and also use the approximation \( r_t = o(a_t) \). \( \square \)

Recall the change of variables: \( Y_t \overset{\text{def}}{=} Z_t - \Xi^1_t(\Theta_t) \) was introduced as a means to remove the non-vanishing noise \( \Xi_t \) in (49). Prop. C.2 establishes a differential equation for \( Y \), similar to the quasi stochastic approximation algorithm (5).

Proposition C.2. Under the assumptions of Thm. 3.1, suppose that \( r_t \leq b a_t \) for a constant \( b \), and all \( t \geq 0 \). Then, the vector-valued process \( Y \) satisfies the differential equation,

\[
\frac{d}{dt} Y_t = a_t \left[ A^* Y_t + \Delta^Y_t - \gamma_t + A^* \Xi^1_t \right] + r_t [Y_t + \Xi^1_t]
\]

(85)

where \( \Xi^1_t = \Xi^1_t(\Theta^*) \), and \( \| \Delta^Y_t \| = o(1 + \| Y_t \|) \) as \( t \to \infty \). That is, for scalars \( \{ \epsilon^Y_t \} \),

\[
\| \Delta_t \| \leq \epsilon^Y_t \{1 + \| Y_t \|\}, \quad t \geq t_0
\]

with \( \epsilon^Y_t \to 0 \) as \( t \to \infty \).

Proof. Using the chain rule, we have

\[
\frac{d}{dt} \{ \Xi^1_t(\Theta_t) \} = \{ f(\Theta_t, \xi_t) - \bar{f}(\bar{\Theta}_t) \} + \partial_0 \Xi^1_t(\Theta_t) \cdot \{ \frac{d}{dt} \Theta_t \}
\]

where the second equation follows from the definition \( \Xi_t \overset{\text{def}}{=} f(\Theta_t, \xi_t) - \bar{f}(\bar{\Theta}_t) \), and the dynamics (5). Rearranging terms we obtain

\[
\Xi_t = \frac{d}{dt} \{ \Xi^1_t(\Theta_t) \} - a_t Y_t(\Theta_t)
\]

(86)

where \( Y_t(\Theta_t) = \partial_0 \Xi^1_t(\Theta_t) \cdot f(\Theta_t, \xi_t) \).
The following is then obtained on substitution into (49):

\[
\frac{d}{dt} Y_t = a_t \left[ A(\bar{\Theta}_t) Y_t + \Delta_t - Y_t(\Theta_t) + A(\bar{\Theta}_t) \Xi^t(\Theta_t) \right] + r_t(Y_t - \Xi^t(\Theta_t))
\]

To go from this ODE to (85) we must bound the error:

\[
\Delta^Y_t \equiv \Delta^a_t + \Delta^b_t
\]

where \( \Delta^a_t = \Delta_t + \left[ A(\bar{\Theta}_t) - A^* \right] (Y_t + \Xi^t) \)

\[
\Delta^b_t = A(\bar{\Theta}_t) (\Xi^t(\Theta_t) - \Xi^t) - (Y_t(\Theta_t) - Y_t) + \frac{r_t}{a_t} (\Xi^t(\Theta_t) - \Xi^t)
\]

We have \( \|\Delta_t\| = o(1+\|Y_t\|) \) because of the prior assertion that \( \|\Delta_t\| = o(\|Z_t\|) \) as \( t \to \infty \), and the assumption that \( \Xi^t(\Theta_t) \) is bounded in \( t \) (recall (42)). Prop. 3.3 combined with Lipschitz continuity of \( A \) then implies that \( \|\Delta^a_t\| = o(1+\|Y_t\|) \).

To complete the proof, we must bound the error in replacing \( \Theta_t \) with \( \theta^* \) in each appearance in \( \Delta^b_t \). The representation (43) combined with (A5) implies that for a Lipschitz constant \( \ell \),

\[
\|A(\bar{\Theta}_t) - A^*\| \leq \ell \|\Theta_t - \theta^*\| \quad \|\Xi^t(\Theta_t) - \Xi^t\| \leq \ell \|\Theta_t - \theta^*\|
\]

and hence both error terms are vanishing, and also \( \|Y_t(\Theta_t) - Y_t\| = o(1) \) by Lipschitz continuity of \( \partial_\theta \Xi^t(\theta) \):

from (43) and (A5):

\[
\partial_\theta \Xi^t(\theta) = \hat{A}(\theta, \bar{\xi}_d) - \hat{A}(\theta, \xi_d)
\]

These bounds show that \( \|\Delta^b_t\| = o(1) \).

\[\square\]

**Proof of Thm. 3.1.** First rewrite (85) as

\[
\frac{d}{dt} Y_t = a_t \left[ (A^* + \frac{r_t}{a_t} I) Y_t + \Delta^Y_t - Y_t + (A^* + \frac{r_t}{a_t} I) \Xi^t \right]
\]

where \( r_t/a_t = o(1) \) for \( \rho < 1 \), and \( r_t/a_t \equiv 1 \) if \( \rho = 1 \). The above can be regarded as a linear QSA ODE with vanishing disturbance. Let \( r(\rho) = 1 \{ \rho = 1 \} \) (equal to zero for \( \rho < 1 \), and \( r(1) = 1 \)). Under the condition that \( A^* + r(\rho)I \) is Hurwitz, the proof of Thm. A.2 can be used with no significant changes to establish convergence:

\[
\hat{Y} = \lim_{t \to \infty} Y_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \Xi^t + [A^* + r(\rho)I]^{-1} \partial_\theta \Xi^t \cdot f(\theta^*, \bar{\xi}_d) \right] dt = [A^* + r(\rho)I]^{-1} \hat{Y}
\]

where the second equality holds because \( \lim_{T \to \infty} \frac{1}{T} \int_0^T \Xi^t dt = 0 \) under (43), and from the definition of \( \hat{Y} \) in (A5). This gives the coupling result \( Z_t = \hat{Y} + \Xi^t + o(1) \).

The second approximation in (44) follows from the first: applying the definition (11) gives

\[
\Theta_t = \bar{\Theta}_t + a_t [\hat{Y} + \Xi^t] + o(a_t)
\]

For \( \rho < 1 \) we have \( \bar{\Theta}_t = \theta^* + o(a_t) \) since \( \bar{\Theta}_t \) converges to \( \theta^* \) faster than \( t^{-N} \) for any \( N \).

For \( \rho = 1 \), Prop. 3.3 (i) implies that \( \bar{\Theta}_t = \theta^* + O(t^{-\infty}) \) where \( \text{Real}(\lambda) < -\rho_0 \) for every eigenvalue \( \lambda \) for \( A^* \). Therefore, \( \bar{\Theta}_t = \theta^* + o(t^{-1}) \) if \( I + A^* \) is Hurwitz.

\[\square\]

**Proof of Thm. 3.2.** Let \( \Theta^{\text{RP}}_t \) be defined by (47). By Prop. 3.3 (ii), the convergence rate of \( \Theta_t \) is exponential in \( t \) and we obtain the approximation

\[
\Theta^{\text{RP}}_t - \theta^* \equiv \frac{1}{T - T_0} \int_{T_0}^T [\Theta_t - \theta^*] dt = \frac{1}{T - T_0} \int_{T_0}^T [\Theta_t - \bar{\Theta}_t] dt + o(1)
\]

Combining the definition \( a_t Z_t = \Theta_t - \bar{\Theta}_t \) and

\[
\frac{d}{dt} Z_t = a_t A^* Z_t + a_t \Delta_t + \Xi_t
\]
(see (51)), gives by the Fundamental Theorem of Calculus:

\[ Z_T - Z_{T_0} = A^* \int_{T_0}^T [\Theta_t - \bar{\Theta}_t] \, dt + \int_{T_0}^T [a_t \Delta_t + \bar{\zeta}_t] \, dt \]

Under (A4), we have \( \|a_t \Delta_t\| = a_t O(\|\Theta_t - \bar{\Theta}_t\|) \). It then follows from Thm. 3.1 (i) that \( \int_{T_0}^T a_t \Delta_t \, dt = o(1) \) for \( \rho \in (\frac{1}{2}, 1) \). Therefore,

\[ \int_{T_0}^T [\Theta_t - \bar{\Theta}_t] \, dt = [A^*]^{-1} \left\{ Z_T - Z_{T_0} - \int_{T_0}^T \bar{\zeta}_t \, dt + o(1) \right\} \quad (88) \]

Next recall (86) and (43), which gives

\[ \int_{T_0}^T \bar{\zeta}_t \, dt = [\Xi_T(\Theta_T) - \Xi_{T_0}(\Theta_{T_0})] - \int_{T_0}^T a_t Y_t(\Theta_t) \, dt \]

\[ = [\Xi_T(\Theta_T) - \Xi_{T_0}(\Theta_{T_0})] - \int_{T_0}^T a_t Y_t \, dt + \int_{T_0}^T a_t O(\|\Theta_t - \bar{\Theta}_t\|) \, dt \]

where \( \int_{T_0}^T a_t O(\|\Theta_t - \bar{\Theta}_t\|) \, dt = o(1) \). Recalling the definition \( Y_t \triangleq Z_t - \Xi_t(\Theta_t) \) in (52) gives,

\[ Z_T - Z_{T_0} - \int_{T_0}^T \bar{\zeta}_t \, dt = Y_T - Y_{T_0} + \int_{T_0}^T a_t Y_t \, dt + o(1) \quad (89) \]

Combining (88) and (89) gives

\[ \frac{1}{T - T_0} \int_{T_0}^T [\Theta_t - \bar{\Theta}_t] \, dt = \frac{1}{T - T_0} [A^*]^{-1} \left\{ Y_T - Y_{T_0} + \int_{T_0}^T a_t Y_t \, dt + o(1) \right\} \]

The last integral is the crucial term. Write

\[ \bar{\chi}_t = Y_t - \bar{Y} \quad \text{and} \quad \bar{\chi}_t' = \int_0^t [Y_r - \bar{Y}] \, dr \]

where \( \bar{Y} \) is the ergodic mean introduced in (A5), and both terms are bounded in \( t \) by assumption. We then obtain

\[ \int_{T_0}^T a_t Y_t \, dt = \bar{Y} \int_{T_0}^T a_t \, dt + \int_{T_0}^T a_t \, d\bar{\chi}_t' \]

Then, using integration by parts, the second term above is bounded:

\[ \int_{T_0}^T a_t \, d\bar{\chi}_t' = a_T \bar{\chi}_T \bigg|_{t=T_0}^T - \int_{T_0}^T \frac{d}{dt} a_t \, \bar{\chi}_t' \, dt \]

\[ = [a_T \bar{\chi}_T - a_{T_0} \bar{\chi}_{T_0}'] + \rho \int_{T_0}^T \frac{1}{(1 + t)^{1+\rho}} \, \bar{\chi}_t' \, dt \]

This establishes the conclusion: if \( \bar{Y} \) is non-zero, then averaging* does not achieve \( 1/T \) convergence rate, since \( \int_{T_0}^T a_t \, dt \) is \( O(T^{1-\rho}) \). And, if \( \bar{Y} \) is zero, the \( 1/T \) convergence rate holds for \( \Theta_T^{RP} - \theta^* \). \( \Box \)