MIYAOKA-YAU INEQUALITY FOR COMPACT KÄHLER MANIFOLDS WITH SEMI-POSITIVE CANONICAL BUNDLE

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Abstract. In this paper, we prove the Miyaoka-Yau inequality for compact Kähler manifolds with semi-positive canonical bundle. The key point of the proof is the estimate for the $L^2$-norm of the scalar curvature along the Kähler-Ricci flow.

1. Introduction

Let $X$ be a complex manifold of dimension $n$ and $K_X$ be the canonical bundle of $X$. The Miyaoka-Yau inequality is the following inequality for Chern classes of $X$ which holds under suitable positivity condition on $K_X$:

$$(MY) \quad (2(n+1)c_2(X) - nc_1(X)^2) \cdot (-c_1(X))^{n-2} \geq 0.$$  

In 1977, Yau [Yau77] showed (MY) under the ampleness of $K_X$ and Miyaoka [Miy77] under the bigness of $K_X$ and $n = 2$. After that, Y. Zhang [ZhaY09] obtained (MY) for smooth minimal projective varieties of general type and Guenancia-Taji [GT16] for minimal projective varieties. For more detailed historical account and related results, we refer to [GT16, Section 1]. We remark that all most all results for (MY) are proved under the projectivity assumption. Our motivation here is to extend it to compact Kähler manifolds. The main theorem is stated as follows:

Theorem 1.1. All compact Kähler manifolds with semi-positive canonical bundle satisfy (MY).

Here, semi-positive means that there exists a smooth Hermitian metric on $K_X$ whose Chern curvature is semi-positive. It is natural to expect that (MY) holds even when compact Kähler manifolds with nef canonical bundle. However, in our argument, we only prove when $K_X$ is semi-positive.

Before we outline the proof of Theorem 1.1, we fix some notations. Let $X$ be a compact Kähler manifold of dimension $n$. For a Kähler form $\omega$ on $X$, we denote $\text{Rm}(\omega)$ the Riemann curvature tensor of $\omega$, $\text{Ric}(\omega) \in 2\pi c_1(X)$ the Ricci curvature of $\omega$, and $R(\omega)$ the scalar curvature of $\omega$. The Kähler-Ricci flow starting from a Kähler form $\omega_0$ is the smooth family $\{\omega_t\}_{t \geq 0}$ of Kähler forms satisfying

$$\begin{cases} \frac{\partial}{\partial t} \omega_t = -\text{Ric}(\omega_t) - \omega_t, \\
\omega_t|_{t=0} = \omega_0, \end{cases}$$

which we simply write $\omega_t$.

For the proof of Theorem 1.1, we first note that since the case when $K_X$ is nef and big (this condition is equivalent to $K_X$ is semi-positive and big) is shown by [ZhaY09], we
only need to show (MY) if $K_X$ is semi-positive and not big. We follow the argument in [ZhaY09]. The essential point of his proof is to reduce (MY) to the uniform boundedness of the scalar curvature along the Kähler-Ricci flow which is shown by [Zha09] when $K_X$ is nef and big. In our setting, we need the following new scalar curvature estimate.

**Theorem 1.2** (=Theorem 2.6). Let $\omega_0$ be a Kähler form satisfying $[\omega_0] - 2\pi c_1(K_X) > 0$ and $\omega_t$ be the Kähler-Ricci flow starting from $\omega_0$. Assume that $K_X$ is semi-positive and not big. Then the following estimate holds:

$$\int_0^\infty dt \int_X R(\omega_t)^2 \omega_t^n < \infty.$$  

For the proof, we consider the function $E(t)$ defined by

$$E(t) := \int_X \sqrt{-1} \partial f_t \wedge \bar{\partial} f_t \wedge \omega_t^{n-1},$$

where $f_t := \log(\omega_t^n/\Omega)$ and $\Omega$ is a volume form on $X$ such that $-\text{Ric}(\Omega) \geq 0$. The function $E(t)$ is the Dirichlet norm of $f_t$ and is also similar to the 1st derivative of the Mabuchi’s energy. The key observation is that the time derivative of $E(t)$ can be used to estimate the $L^2$-norm of the scalar curvature of $\omega_t$.

We remark that Song-Tian [ST16] showed that if $K_X$ is semi-ample, then the scalar curvature is uniformly bounded along the Kähler-Ricci flow. However, we cannot apply this result to our setting.

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2. Proof of the Theorems

We first recall the argument of [ZhaY09]. The following formula for Chern classes is well-known.

**Proposition 2.1** (see [Kob87, Chapter 4]). For any Kähler form $\omega$, the following estimate holds.

\[
(2(n + 1)c_2(X) - nc_1(X)^2) \cdot [\omega]^{n-2} \\
= \frac{1}{4\pi^2(n+1)} \int_X \left( (n+1)|\text{Rm}^\omega(\omega)|^2 - (n+2)|\text{Ric}^\omega(\omega)|^2 \right) \omega^n \\
\geq \frac{1}{4\pi^2(n+1)} \int_X \left( (n+1)|\text{Rm}^\omega(\omega)|^2 - (n+2) |\text{Ric}(\omega) + \omega|_\omega^2 \right) \omega^n.
\]

Here we set

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

$$\text{Rm}(\omega)_{\bar{7}\bar{8}l} := \text{Rm}(\omega)_{\bar{7}\bar{8}l} - \frac{R(\omega)}{n(n+1)}(g_{\bar{7}\bar{8}}g_{l\bar{9}} + g_{l\bar{9}}g_{\bar{7}\bar{8}}),$$

$$\text{Ric}(\omega) := \text{Ric}(\omega) - \frac{R(\omega)}{n}\omega.$$
Thanks to this proposition, in order to prove (MY), we only need to find Kähler forms \( \{\omega_i\}_{i=1}^{\infty} \) satisfying
\[
\lim_{i \to \infty} [\omega_i] = -2\pi c_1(X) = 2\pi c_1(K_X),
\]
(2.2)

\[
\lim_{i \to \infty} \int_X |\text{Ric}(\omega_i) + \omega_i|_{\omega_i}^n \omega_i^n = 0.
\]
(2.3)

In the following argument, we will prove that the Kähler-Ricci flow \( \omega_t \) satisfies these two conditions.

We first recall the long time existence theorem for the Kähler-Ricci flow.

**Theorem 2.4** ([TZ06]). For any Kähler form \( \omega_0 \), the Kähler-Ricci flow \( \omega_t \) starting from \( \omega_0 \) exists for \( t \in [0, \infty) \) if and only if the canonical bundle \( K_X \) is nef. Furthermore, in this setting, the cohomology class \( \alpha_t \) of \( \omega_t \) satisfies
\[
\alpha_t = e^{-t} [\omega_0] + (1 - e^{-t}) 2\pi c_1(K_X) \to 2\pi c_1(K_X) \text{ as } t \to \infty.
\]
In particular, the Kähler-Ricci flow \( \omega_t \) satisfies (2.2) if \( K_X \) is nef.

We now focus on (2.3). In general, it is a hard problem to estimate the Ricci curvature along the Kähler-Ricci flow. However, we can reduce (2.3) to the estimate for the scalar curvature. More precisely, we have the following proposition:

**Proposition 2.5.** If the canonical bundle \( K_X \) is nef, then there exists a constant \( C > 0 \) such that for any \( T > 0 \), we have the following estimate:
\[
\int_0^T dt \int_X |\text{Ric}(\omega_t) + \omega_t|_{\omega_t}^2 \omega_t^n \leq \int_0^T dt \int_X R(\omega_t)^2 \omega_t^n + C,
\]

Here \( C > 0 \) is a constant which depends only on the cohomology classes \([\omega_0]\) and \( c_1(K_X)\).

**Proof.** Recall that the scalar curvature and the volume form evolves as
\[
\frac{\partial}{\partial t} R(\omega_t) = \Delta_{\omega_t} R(\omega_t) + |\text{Ric}(\omega_t) + \omega_t|_{\omega_t}^2 - (R(\omega_t) + n),
\]
\[
\frac{\partial}{\partial t} \omega_t^n = -(R(\omega_t) + n)\omega_t^n,
\]
(for instance [BEG13, (3.56)]). Then, we get
\[
\int_X |\text{Ric}(\omega_t) + \omega_t|_{\omega_t}^2 \omega_t^n = \int_X \left( \frac{\partial}{\partial t} R(\omega_t) \right) \omega_t^n + \int_X (R(\omega_t) + n)\omega_t^n
\]
\[
= \left( \frac{d}{dt} \int_X R(\omega_t)\omega_t^n + \int_X R(\omega_t)(R(\omega_t) + n)\omega_t^n \right) - \frac{d}{dt} \int_X \omega_t^n
\]
\[
= \frac{d}{dt} \left( -n \left( 2\pi c_1(K_X) \cdot \alpha_t^{n-1} \right) + \int_X R(\omega_t)^2 \omega_t^n - n^2 \left( 2\pi c_1(K_X) \cdot \alpha_t^{n-1} \right) - \frac{d}{dt} \left( \alpha_t^n \right) \right)
\]
\[
\leq \frac{d}{dt} \left( -n \left( 2\pi c_1(K_X) \cdot \alpha_t^{n-1} \right) + \int_X R(\omega_t)^2 \omega_t^n - \frac{d}{dt} \left( \alpha_t^n \right). \right.
\]
We remark that since \( K_X \) is nef and \( \alpha_t \) is Kähler, \( (2\pi c_1(K_X) \cdot \alpha_t^{n-1}) \geq 0 \) holds for any \( t \geq 0 \). Therefore, by integrating with respect to \( t \), we obtain the conclusion. \( \square \)

The following new estimate gives the desired bound for the scalar curvature.
Theorem 2.6. Assume that the canonical bundle $K_X$ is semi-positive and not big. Let $\omega_0$ be a Kähler form on $X$ satisfying $[\omega_0] - 2\pi c_1(K_X) > 0$ and $\omega_t$ be the Kähler-Ricci flow starting from $\omega_0$. Let $\Omega$ be a smooth volume form on $X$ such that $-\text{Ric}(\Omega) \geq 0$. We set $f_t$ and $E(t)$ by

$$f_t := \log \frac{\omega_t^n}{\Omega}, \quad E(t) := \int_X \sqrt{-1} \partial f_t \wedge \bar{\partial} f_t \wedge \omega_t^{n-1}.$$ 

Then the following estimate holds:

$$\int_0^\infty dt \int_X R(\omega_t)^2 \omega_t^n \leq \frac{n}{2} E(0) + C,$$

where $C > 0$ is a constant depends only on $\omega_0$ and $c_1(K_X)$.

Proof. We first note that $f_t$ satisfies

$$\frac{\partial}{\partial t} f_t = \text{tr}_{\omega_t} \left( \frac{\partial}{\partial t} \omega_t \right) = -(R(\omega_t) + n), \quad \sqrt{-1} \bar{\partial} f_t = -\text{Ric}(\omega_t) + \text{Ric}(\Omega).$$

Then, we get the following:

$$\begin{align*}
\frac{d}{dt} E(t) &= -2 \int_X \partial_t f_t \sqrt{-1} \bar{\partial} f_t \wedge \omega_t^{n-1} \\
&\quad - \int_X f_t \sqrt{-1} \bar{\partial} f_t \wedge (n-1) \omega_t^{n-2} \wedge \frac{\partial}{\partial t} \omega_t \\
&= -2 \int_X -(R(\omega_t) + n)(-\text{Ric}(\omega_t) + \text{Ric}(\Omega)) \wedge \omega_t^{n-1} \\
&\quad + (n-1) \int_X \sqrt{-1} \partial f_t \wedge \bar{\partial} f_t \wedge \omega_t^{n-2} \wedge (-\text{Ric}(\omega_t) - \omega_t) \\
&\quad (2.7) = \frac{-2}{n} \int_X R(\omega_t)^2 \omega_t^n - 2 \int_X R(\omega_t)(-\text{Ric}(\Omega)) \wedge \omega_t^{n-1} \\
&\quad + (n-1) \int_X \sqrt{-1} \partial f_t \wedge \bar{\partial} f_t \wedge \omega_t^{n-2} \wedge (-\text{Ric}(\omega_t) - \omega_t).
\end{align*}$$

The second term of (2.7) is estimated as follows: Let $C > 0$ be a constant satisfying $R(\omega_t) \geq -C$ for $t \in [0, \infty)$ which always exists by a maximum principle argument and only depends on $\omega_0$ (see [BEG13, Theorem 3.2.2]). Since the volume form $\Omega$ satisfies $-\text{Ric}(\Omega) \geq 0$, the second term is estimated as

$$\begin{align*}
\int_X R(\omega_t)(-\text{Ric}(\Omega)) \wedge \omega_t^{n-1} &= \int_X (R(\omega_t) + C)(-\text{Ric}(\Omega)) \wedge \omega_t^{n-1} - C \int_X (-\text{Ric}(\Omega)) \wedge \omega_t^{n-1} \\
&= \int_X (R(\omega_t) + C)(-\text{Ric}(\Omega)) \wedge \omega_t^{n-1} - C(2\pi c_1(K_X) \cdot \alpha_t^{n-1}) \\
&\geq -C(2\pi c_1(K_X) \cdot \alpha_t^{n-1}) \\
&\geq -C' e^{-(n-\nu)t},
\end{align*}$$

where $\nu$ is the numerical dimension of $K_X$, i.e. $\nu := \max\{k = 0, \ldots, n \mid (c_1(K_X)^k \cdot [\omega_0]^{n-k}) \neq 0\}$. Since $K_X$ is not big, we have $\nu < n$.

The third term of (2.7) is less than or equal to zero since $[-\text{Ric}(\omega_t) - \omega_t] = -e^{-t}([\omega_0] - 2\pi c_1(K_X)) < 0$. 


Therefore, we obtain
\[
\frac{d}{dt} E(t) \leq -\frac{2}{n} \int_X R(\omega_t) \omega_t^n + C' e^{-(n-\nu)t}.
\]
By integrating \( t \) from 0 to \( \infty \), we get the conclusion since \( E(t) \geq 0 \).
\[\square\]

**Proof of Theorem 1.1.** We assume that \( K_X \) is semi-positive and not big. Let \( \omega_t \) be the Kähler-Ricci flow as in Theorem 2.6. We now prove that \( \omega_t \) satisfies (2.2), (2.3). Proposition 2.4 and the semi-positivity of \( K_X \) imply (2.2). By Proposition 2.5 and Theorem 2.6 we have
\[
\int_0^\infty dt \int_X |\text{Ric}(\omega_t) + \omega_t|_{\omega_t}^2 \omega_t^n < \infty.
\]
Then, we can find a sequence \( \{t_i\}_{i=1}^\infty \subset \mathbb{R} \) such that \( t_i \to \infty \) as \( i \to \infty \) and \( \{\omega_{t_i}\}_{i=1}^\infty \) satisfies (2.3). Therefore, by Proposition 2.1, we obtain (MY).
\[\square\]

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