Results for Wieferich Primes

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Abstract: Let \( v \geq 2 \) be a fixed integer, and let \( x \geq 1 \) and \( z \geq x \) be large numbers. The exact asymptotic formula for the number of Wieferich primes \( p \), defined by \( v^{p-1} \equiv 1 \mod p^2 \), in the short interval \([x, x + z]\) is proposed in this note. The search conducted on the last 100 years have produced two primes \( p < x = 10^{15} \) such that \( 2^{p-1} \equiv 1 \mod p^2 \). The probabilistic and theoretical information within predicts the existence of another base \( v = 2 \) prime on the interval \([10^{15}, 10^{40}]\). Furthermore, a result for the upper bound on the number of Wieferich primes is used to demonstrate that the subset of nonWieferich primes has density 1.

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1 Introduction

Let $p \geq 3$ denotes a prime, and let $v \geq 2$ be a fixed integer base. The set of Wieferich primes is defined by the congruence $v^{p-1} \equiv 1 \mod p^2$, see [48, p. 333]. These numbers are of interest in Diophantine equations, see [56], [39, Theorem 1], algebraic number theory, the theory of primitive roots, see [44], additive number theory, see [20], and many other topics in mathematics.

In terms of the order of the element $v \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ in the finite ring, this set has the equivalent description
\[
W_v = \{ p : \text{ord}_{p^2}(v) | p-1 \}.
\] (1.1)

For a large number $x \geq 1$, the corresponding counting function for the number of Wieferich primes up to $x$ is defined by
\[
W_v(x) = \# \{ p \leq x : \text{ord}_{p^2}(v) | p-1 \}.
\] (1.2)

The heuristic argument in [48, p. 413], [8], et alii, claims that
\[
W_v(x) \approx \sum_{p \leq x} \frac{1}{p} \ll \log \log x.
\] (1.3)

The basic idea in this heuristic was considerably improved and generalized in [26, Section 2]. The conditional analysis is based on some of the statistical properties of the Fermat quotient. Specifically, this is a map
\[
\begin{align*}
(Z/p^2Z)^\times &\to \mathbb{F}_p, \\
v &\to q_v(p),
\end{align*}
\] (1.4)
defined by
\[
q_v(p) \equiv v^{p-1} - 1 \mod p^2.
\]

Each integer $v \in \mathbb{N}$ is mapped into an infinite sequence
\[
\{ x_p(v) : p \geq 2 \} = (q_v(2), q_v(3), q_v(5), \ldots) \in \mathcal{P}(v).
\] (1.5)

The product space $\mathcal{P}(v) = \prod_{p \mid v} \mathbb{F}_p$ is the set of sequences $\{ x_p(v) = q_v(p) : p \nmid v \}$. Evidently, the subset of sequences $\{ x_p(v) = q_v(p) = 0 : p \nmid v \}$ is equivalent to $W_v$, and the counting function is
\[
W_v(x) = \# \{ p \leq x : p \nmid v, x_p(v) \equiv 0 \mod p \}.
\] (1.6)

1.1 Summary of Heuristics

A synthesis of some of the previous works, such as Artin heuristic for primitive roots, and the heuristics arguments in [48, 8, 26], et alii, are spliced together here.

**Conjecture 1.1.** For any integer $v \geq 2$, the subset of elements $\{ x_p(v) \} \in \mathcal{P}(v)$ for which $x_p(v) = 0$ has the asymptotic formula
\[
\# \{ p \leq x : p \nmid v, x_p(v) \equiv 0 \mod p \} = c_v \log \log x + o((\log \log x)^{1+\epsilon/2}),
\] (1.7)
where $\epsilon > 0$ is a small number, and the correction factor is defined by
\[
c_v = \sum_{n \geq 1} \sum_{d \mid n} \frac{\mu(n) \gcd(dn,k)}{dn\phi(dn)},
\]
where $v = ab^k$ with $a \geq 1$ squarefree; as $x \to \infty$. 

The correction factor $c_v \geq 0$ accounts for the dependencies among the primes. According to the analysis in [26], for any random integer $v \geq 2$, the correction factor $c_v = 1$ with probability one. Thus, $c_v \neq 1$ on a subset of integers $v \geq 2$ of zero density. For example, at the odd prime powers $v \equiv 1 \mod 4$, see Theorem 10.1 for more details.

### 1.2 Results In Short Intervals

The purpose of this note is to continue the investigation of the asymptotic counting functions for Wieferich primes in (18.1). A deterministic analysis demonstrates that the number of Wieferich primes is very sparse, and it is infinite.

**Theorem 1.1.** Let $v \geq 2$ be a base, and let $x \geq 1$ and $z \geq x$ be large numbers. Then, the number of Wieferich primes in the short interval $[x, x + z]$ has the asymptotic formula

$$W_v(x + z) - W_v(x) = c_v \left( \log \log(x + z) - \log \log(x) \right) + E_v(x, z),$$

where $c_v \geq 0$ is the correction factor, and $E_v(x, z)$ is an error term.

This is equivalent to the asymptotic form

$$W_v(x) \sim \log \log(x),$$

and demonstrates that the subset of primes $W_v$ is infinite.

**Theorem 1.2.** Let $v \geq 2$ be a base, and let $x \geq 1$ be a large number. Then, the number of Wieferich primes on the interval $[1, x]$ has the upper bound

$$W_v(x) \leq 4v \log \log x.$$  

Sequences of integers divisible by high prime powers fall on the realm of the abc conjecture. Thus, it is not surprising to discover that the sequence of integers $s_p = 2^{p-1} - 1$, prime $p \geq 2$, has finitely many terms divisible by $p^3$.

**Theorem 1.3.** Let $v \geq 2$ be a fixed integer base. Then, the subset of primes

$$A_v = \{ p : \text{ord}_{p^3}(v) \mid p - 1 \}$$

is finite. In particular, the congruence $v^{p-1} - 1 \equiv 0 \mod p^3$ has at most finitely many primes $p \geq 2$ solutions.

### 1.3 Average Order

The average orders of random subsets $W_v$ has a simpler to determined asymptotic formula, and the average correction factor $c_v = 1$.

**Theorem 1.4.** Let $x \geq 1$ and $z \geq x$ be large numbers, and let $v \geq 2$ random integer. Then, the average order of the number of Wieferich primes has the asymptotic formula

$$W_v(x + z) - W_v(x) = \log \log(x + z) - \log \log(x) + E_v(x, z)$$

where $E_v(x, z)$ is an error term.

The asymptotic form

$$W_v(x) \sim \log \log(x)$$

is evident in this result.
1.4 Guide

Sections 2 to 8 provide the basic foundation, and additional related results, but not required. The proofs of Theorems 1.1, 1.2, and 1.3 appear in Section 9, and the proof of Theorem 1.4 appears in Section 11. Section 10 is concerned with the derivation of the correction factor $c_v \geq 0$. Estimates for the next Wieferiech primes to bases $v = 2$ and $v = 5$ are given in Section 17. A proof for the density of non-Wieferiech primes appears in Theorem 16.1.

These ideas and Conjecture 1.1 take the more general forms

$$W_{n,v}(x) = \# \left\{ p \leq x : v^{p^{n-1}(p-1)} - 1 \equiv 0 \mod p^{n+1} \right\}$$

for $n \geq 1$, and the subset $A_{n,v} = \{ p: \text{ord}_{p^{n+2}}(v) | p-1 \}$ is finite.

In synopsis, the sequence of integers investigated here have the followings divisibility properties.

1. The sequence of integers $2^{p-1} - 1$ is divisible by every prime $p \geq 2$, for example, $2^{p-1} - 1 \equiv 0 \mod p$. The corresponding subset of primes has density 1, this follows from Fermat little theorem.

2. The sequence of integers $2^{p-1} - 1$ is divisible by infinitely many prime powers $p^2 \geq 4$, for example, $2^{p-1} - 1 \equiv 0 \mod p^2$. The corresponding subset of primes has zero density, this follows from Theorem 1.1.

3. The sequence of integers $2^{p-1} - 1$ is divisible by finitely many prime powers $p^3 \geq 8$, for example, $2^{p-1} - 1 \equiv 0 \mod p^3$. The corresponding subset of primes has zero density, this follows from Theorem 1.3.

Other examples of interesting sequences of integers that might have similar divisibility properties are the followings.

4. For fixed pair $a > b > 0$, as $n \to \infty$, the sequence of integers $a^n - b^n$ is divisible by infinitely many primes $p \geq 2$, for example, $a^n - b^n \equiv 0 \mod p$. The corresponding subset of primes has nonzero density, this follows from Zsigmondy theorem, see also [32], [7], et cetera.

5. For fixed pair $a > b > 0$, as $n \to \infty$, the sequence of integers $a^n - b^n$ is probably divisible by infinitely many prime powers $p^2$, for example, $a^n - b^n \equiv 0 \mod p^2$. This is expected to be a subset of primes of zero density, but there is no proof in the literature.

6. For a fixed pair $a > b > 0$, as $n \to \infty$, the sequence of integers $a^n - b^n$ is probably divisible by finitely many prime powers $p^3$, for example, $a^n - b^n \equiv 0 \mod p^3$. This is expected to be a a subset of primes of zero density, but there is no proof in the literature.

More general and deeper information on the prime divisors of sequences and prime power divisors of sequences appears in [17] Chapter 6, and the references within.
2 Basic Analytic Results

A few analytic concepts and results are discussed in this Section. The logarithm integral is defined by

$$\text{li}(x) = \int_2^x \frac{1}{\log t} dt = \frac{x}{\log x} + a_2 \frac{x}{\log^2 x} + a_3 \frac{x}{\log^3 x} + a_4 \frac{x}{\log^4 x} + O\left(\frac{1}{\log^5 x}\right).$$  \hspace{1cm} (2.1)

A refined version over the complex numbers appears in [37, p. 20]. The corresponding prime counting function has the Legendre form

$$\pi(x) = \frac{x}{\log x} + a_2 \frac{x}{\log^2 x} + a_3 \frac{x}{\log^3 x} + a_4 \frac{x}{\log^4 x} + O\left(\frac{1}{\log^5 x}\right),$$  \hspace{1cm} (2.2)

2.1 Sums And Products Over The Primes

Lemma 2.1. (Mertens) Let $x \geq 1$ be a large number, then the followings hold.

(i) The prime harmonic sum satisfies

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + b_0 + \frac{b_1}{\log x} + \frac{b_2}{\log^2 x} + \frac{b_3}{\log^3 x} + O\left(\frac{1}{\log^5 x}\right)$$  \hspace{1cm} (2.3)

where $b_0 > 0, b_1, b_2,$ and $b_3$ are constants, as $x \to \infty$.

(ii) The prime harmonic product satisfies

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{1}{e^\gamma \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right),$$  \hspace{1cm} (2.4)

where $\gamma = 0.5772\ldots$ is a constant.

where $a_2 > 0, a_3,$ and $a_4$ are constants.

Proof. (i) Use the Legendre form of the prime number theorem in (2.2).

More general versions for primes over arithmetic progressions appear in [35], and a general version of the product appears in the literature.

Explicit estimates can be handled with the formulas

$$\left|\sum_{p \leq x} \frac{1}{p} - \log \log x - b_0\right| < \frac{1}{10 \log^2 x} + \frac{4}{15 \log^3 x}$$  \hspace{1cm} (2.5)

for $x \geq 10400$ and $b_0$ is a constant; and

$$\frac{1}{e^\gamma \log x} \left(1 - \frac{2}{\log^2 x}\right) < \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{1}{e^\gamma \log x} \left(1 + \frac{2}{\log^2 x}\right),$$  \hspace{1cm} (2.6)

for $x \geq 2$, see [49], [15].
2.2 Totients Functions

Let \( n = p_1^{e_1}p_2^{e_2} \cdots p_t^{e_t} \) be an arbitrary integer. The Euler totient function over the finite ring \( \mathbb{Z}/n\mathbb{Z} \) is defined by \( \varphi(n) = \prod_{p|n} (1 - 1/p) \). While the more general Carmichael totient function over the finite ring \( \mathbb{Z}/n\mathbb{Z} \) is defined by

\[
\lambda(n) = \begin{cases} 
\varphi(2^e), & n = 2^e, e = 0, 1, \text{ or } 2, \\
\varphi(2^e)/2, & n = 2^e, e \geq 3, \\
\varphi(p^e), & n = p^e \text{ or } 2p^e \text{ and } e \geq 1,
\end{cases}
\]  

(2.7)

where \( p \geq 3 \) is prime, and

\[
\lambda(n) = \text{lcm}(\lambda(p_1^{e_1})\lambda(p_2^{e_2}), \ldots, \lambda(p_t^{e_t})).
\]

(2.8)

The two functions coincide, that is, \( \varphi(n) = \lambda(n) \) if \( n = 2, 4, p^m, \) or \( 2p^m, m \geq 1 \). And \( \varphi(2^m) = 2\lambda(2^m) \). In a few other cases, there are some simple relationships between \( \varphi(n) \) and \( \lambda(n) \).

2.3 Sums Of Totients Functions Over The Integers

The Carmichael totient function has a more complex structure than the Euler totient function, however, many inherited properties such as

\[
\lambda(n) \mid \varphi(n), \quad \varphi(n) = \xi(n)\lambda(n),
\]

can be used to derive information about the Carmichael totient function.

**Theorem 2.1.** For all \( x \geq 16 \), then the followings hold.

(i) The average order has the asymptotic formula

\[
\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).
\]

(ii) The normalized function has the asymptotic formula

\[
\sum_{n \leq x} \frac{\varphi(n)}{n} = 6 \frac{\pi^2}{x^2} + O(\log x).
\]

**Theorem 2.2.** ([16, Theorem 3]) For all \( x \geq 16 \), then the followings hold.

(i) The average order has the asymptotic formula

\[
\frac{1}{x} \sum_{n \leq x} \lambda(n) = \frac{x}{\log x} e^{\frac{B \log \log x \log \log \log x}{\log \log x} (1+o(1))},
\]

where the constants are \( \gamma = 0.5772 \ldots \) and

\[
B = e^{-\gamma} \prod_{p \geq 2} \left( 1 - \frac{1}{(p-1)^2(p+1)} \right) = 0.34537 \ldots.
\]

(ii) The normalized function has the asymptotic formula

\[
\sum_{n \leq x} \frac{\lambda(n)}{n} = \frac{x}{\log x} e^{\frac{B \log \log x \log \log \log x}{\log \log x} (1+o(1))}.
\]
Proof. (ii) Set \( U(x) = \sum_{n \leq x} \lambda(n) \). Summation by part yields
\[
\sum_{n \leq x} \frac{\lambda(n)}{n} = \int_1^x \frac{1}{t} dU(t) = \frac{U(x)}{x} - \frac{U(1)}{1} + \int_1^x \frac{U(t)}{t^2} dt.
\]
(2.9)
Invoke case (i) to complete the claim. \( \blacksquare \)

Lemma 2.2. Let \( x \geq 16 \), and \( \lambda(n) \) be the Carmichael totient function. Then
\[
\sum_{n \leq x} \frac{\varphi(\lambda(n))}{\lambda(n)} \geq k_1 \frac{x}{\log \log x} (1 + o(1)),
\]
(2.10)
where \( k_1 > 0 \) is a constant.

Proof. Begin with the expression
\[
\frac{\varphi(\lambda(n))}{\lambda(n)} = \prod_{p|\lambda(n)} \left(1 - \frac{1}{p}\right) \geq \frac{1}{e^{\gamma} \log \log \lambda(n)} \left(1 + O\left(\frac{1}{\log \log \lambda(n)}\right)\right)
\]
holds for all integers \( n \geq 1 \), with equality on a subset of integers of zero density. In light of the lower bound and upper bound
\[
\frac{n}{(\log n)^a \log \log n} \leq \lambda(n) \leq \frac{n}{(\log n)^b \log \log n}
\]
(2.12)
for some constants \( a > 0 \) and \( b > 0 \) and large \( n \geq 1 \), see [16, Theorem 1], proceed to determine a lower bound for the sum:
\[
\sum_{n \leq x} \frac{\varphi(\lambda(n))}{\lambda(n)} \geq k_1 \int_2^x \frac{1}{\log \log t} dt \geq k_1 \frac{x}{\log \log x} + k_2 \int_2^x \frac{1}{t(\log t)(\log \log t)^2} dt.
\]
(2.13)
where \( k_1 > 0 \) and \( k_2 \) are constants. This is sufficient to complete the claim. \( \blacksquare \)

More advanced techniques for the composition of arithmetic functions are studied in [40], [6], et cetera.

The ratio \( \varphi(\varphi(n))/\varphi(n) = \varphi(\lambda(n))/\lambda(n) \) for all integers \( n \geq 1 \). But, the composition only satisfy \( \varphi(\varphi(n)) \geq \varphi(\lambda(n)) \), and agree on a subset of integers of zero density.

Lemma 2.3. ([42, Theorem 2]) Let \( x \geq 1 \) be a large number. Then
\[
\#\{n : \varphi(\varphi(n)) \geq \varphi(\lambda(n))\} \ll \frac{x}{(\log \log x)^c}
\]
(2.14)
for any constant \( c > 0 \).
2.4 Sums Of Totients Functions Over The Primes

The ratio ϕ(p−1)/(p−1) for all primes p ≥ 2 has an established literature. But, the ratio λ(p−1)/(p−1) does not have any meaningful literature, its average order is estimated here in Lemma 2.5, and a more precise version is stated in the exercises.

**Lemma 2.4. ([53 Lemma 1])** Let x ≥ 1 be a large number, and let ϕ(n) be the Euler totient function. Then

\[
\sum_{p \leq x} \frac{\varphi(p-1)}{p-1} = a_0 \text{li}(x) + O\left(\frac{x}{\log^B x}\right),
\]

(2.15)

where the constant

\[
a_0 = \prod_{p \geq 2} \left(1 - \frac{1}{p(p-1)}\right) = .37399581\ldots,
\]

(2.16)

and li(x) is the logarithm integral, and B > 1 is an arbitrary constant, as x → ∞.

More general versions of Lemma 2.4 are proved in [54], and [21].

**Lemma 2.5.** Let x ≥ 1 be a large number, and let λ(n) be the Carmichael totient function. Then

\[
\sum_{p \leq x} \frac{\lambda(p-1)}{p-1} \gg \frac{x}{(\log x)(\log \log x)},
\]

(2.17)

**Proof.** Begin with the expression

\[
\frac{\lambda(p-1)}{p-1} = \prod_{q | \lambda(p-1)} \left(1 - \frac{1}{q}\right) \gg \frac{1}{\log \lambda(p-1)}
\]

(2.18)

holds for all integers p − 1 ≥ 1, with equality on a subset of integers of zero density. In light of the lower bound and upper bound

\[
\frac{n}{(\log n)^a \log \log n} \leq \lambda(n) \leq \frac{n}{(\log n)^b \log \log n}
\]

(2.19)

for some constants a > 0 and b > 0 and large n ≥ 1, see [16] Theorem 1, proceed to determine a lower bound for the sum:

\[
\sum_{p \leq x} \frac{\lambda(p-1)}{p-1} \gg k_1 \int_2^x \frac{1}{\log \log t} d\pi(t) \gg \frac{x}{(\log x)(\log \log x)}.
\]

(2.20)

Confer the exercises for similar information.

2.5 Sums Of Totients Functions Over Subsets Of Integers

The asymptotic formulas for the normalized summatory totient function ϕ(n)/n and ϕ(n) over the subset of integers \(A = \{n \geq 1 : \gcd(\varphi(n), q) = 1\}\) are computed here. These results are based on the counting function \(A(x) = \{n \leq x : n \in A\}\).
**Theorem 2.3.** (EE Theorem 2]) For a prime power $q \geq 2$, and a large number $x \geq 1$, the counting function $A(x)$ has the asymptotic formula

$$\sum_{n \leq x} \frac{1}{\gcd(\varphi(n), q)} = c_q \frac{x}{\log^{1/(q-1)} x} \left(1 + O_q \left(\frac{\log \log x}{\log x}\right)\right),$$

(2.21)

where $c_q > 0$ is a constant.

**Theorem 2.4.** For large number $x \geq 1$, the average order for the normalized Euler totient function $\varphi(n)/n$ over the subset $A$ has the asymptotic formula

$$\sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \frac{\varphi(n)}{n} = 6 c_q \frac{x}{\pi^2 \log^{1/(q-1)} x} \left(1 + O_q \left(\frac{\log \log x}{\log x}\right)\right),$$

(2.22)

where $c_q > 0$ is a constant.

**Proof.** Let $A = \{n \geq 1 : \gcd(\varphi(n), q) = 1\}$ and let $A(x) = \{n \leq x : n \in A\}$ be the corresponding the counting function. Using the standard indentity $\varphi(n) = n \sum_{d|n} \mu(n)/d$ the average order is expressed as

$$\sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \frac{\varphi(n)}{n} = \sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \sum_{d|n} \frac{\mu(n)}{d} = \sum_{d \leq x} \frac{\mu(n)}{d} \sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \frac{1}{d},$$

(2.23)

where $\mu(n) \in \{-1, 0, 1\}$ is the Mobius function. Applying Theorem 2.3 leads to

$$\sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \frac{\varphi(n)}{n} = \sum_{d \leq x} \frac{\mu(n)}{d} \left(c_q \frac{x/d}{\log^{1/(q-1)} x/d} \left(1 + O_q \left(\frac{\log \log x/d}{\log x/d}\right)\right)\right) \sum_{d \leq x} \frac{\mu(d)}{d^2},$$

(2.24)

where the implied constant absorbs a negligible dependence on $d$. Now, use Lemma ?? to approximate the finite sum as

$$\sum_{n \leq x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} + O \left(\frac{1}{x \log^2 x}\right),$$

(2.25)

and to complete the proof.

**Theorem 2.5.** For large number $x \geq 1$, the average order of the Euler totient function $\varphi(n)$ over the subset $A = \{n \geq 1 : \gcd(\varphi(n), q) = 1\}$ has the asymptotic formula

$$\sum_{\substack{n \leq x \\ \gcd(\varphi(n), q) = 1}} \varphi(n) = 3 c_q \frac{x^2}{\pi^2 \log^{1/(q-1)} x} \left(1 + O_q \left(\frac{\log \log x}{\log x}\right)\right),$$

(2.26)

where $c_q > 0$ is a constant.
Proof. By Theorem 2.4, the appropriate measure is $W(x) = \sum_{n \leq x, \gcd(\varphi(n), q) = 1} \varphi(n)/n$, and summation by part yields

$$\sum_{\substack{n \leq x \\text{gcd}(\varphi(n), q) = 1}} \varphi(n) = \sum_{\substack{n \leq x \\text{gcd}(\varphi(n), q) = 1}} n \cdot \frac{\varphi(n)}{n} = \int_1^x t \, dW(t) = xW(x) + O(1) - \int_1^x W(t)dt = x \left( \frac{6c_q}{\pi^2} \frac{x}{\log^{1/(q-1)} x} \left( 1 + O_q \left( \frac{\log \log x}{\log x} \right) \right) \right) - \int_1^x W(t)dt = \frac{3c_q}{\pi^2} \frac{x}{\log^{1/(q-1)} x} \left( 1 + O_q \left( \frac{\log \log x}{\log x} \right) \right).$$

(2.27)

$\blacksquare$

2.6 Problems

1. Prove that the average order of the Euler totient function over the shifted primes satisfies

$$\sum_{p \leq x} \frac{\varphi(p - a)}{p - a} = a_1 \text{li}(x) + o(\text{li}(x)),$$

where $\text{li}(x)$ is the logarithm function, and $a_1 > 0$ is a constant depending on $a \geq 1$.

2. Prove that the average order of the Carmichael totient function over the shifted primes satisfies

$$\sum_{p \leq x} \frac{\lambda(p - a)}{p - a} = b_1 \frac{\text{li}(x)}{\log \log x} + o \left( \frac{\text{li}(x)}{\log \log x} \right).$$

where $b_1 > 0$ is a constant depending on $a \geq 1$.

3. Estimate the normal order of the totient ratio $\xi(n) = \varphi(n)/\lambda(n)$: For any number $\varepsilon > 0$, there exists a function $f(n)$ such that

$$f(n) - \varepsilon \leq \xi(n) \leq f(n) + \varepsilon.$$

4. Estimate the average order of the totient ratio $\xi(n) = \varphi(n)/\lambda(n)$ over the integers and over the shifted primes:

$$\sum_{n \leq x} \xi(n) = \sum_{n \leq x} \frac{\varphi(n)}{\lambda(n)} \quad \text{and} \quad \sum_{p \leq x} \xi(p - 1) = \sum_{p \leq x} \frac{\varphi(p - 1)}{\lambda(p - 1)}.$$

5. Estimate the average order of the inverse totient function over the integers and over the shifted primes:

$$\sum_{n \leq x} \frac{1}{\varphi(n)} \quad \text{and} \quad \sum_{p \leq x} \frac{1}{\varphi(p - 1)}.$$
6. Estimate the average order of the inverse lambda function over the integers and over the shifted primes:

\[
\sum_{n \leq x} \frac{1}{\lambda(n)} \quad \text{and} \quad \sum_{p \leq x} \frac{1}{\lambda(p - 1)}.
\]
3 Finite Cyclic Groups

Let \( n = p_1^{v_1} p_2^{v_2} \cdots p_t^{v_t} \) be an arbitrary integer, and let \( \mathbb{Z}/n\mathbb{Z} \) be a finite ring. Some properties of the group of units of the finite ring

\[
(\mathbb{Z}/n\mathbb{Z})^\times = U(p^{v_1}) \times U(p^{v_2}) \times \cdots \times U(p^{v_t}),
\]

where \( U(n) \) is a cyclic group of order \( \#U(n) = \varphi(n) \) are investigated here.

3.1 Multiplicative Orders

**Definition 3.1.** The order of an element \( v \in G \) in a cyclic group \( G \) is defined by \( \text{ord}_G(v) = \min\{n : v^n \equiv 1 \mod G\} \), and the index is defined by \( \text{ind}_G(v) = \frac{\#G}{\text{ord}_G(v)} \).

**Definition 3.2.** A subset of integers \( B \subset \mathbb{Z} \) with respect to a fixed base \( v \geq 2 \) if the order and the index are nearly equal: \( \text{ord}_n(v) \approx \text{ind}_n(v) \approx \sqrt{n} \) for each \( n \in B \).

**Lemma 3.1.** The order \( \text{ord} : G \rightarrow \mathbb{N} \) is multiplicative function on a multiplicative subgroup \( G \) of cardinality \( \#G = \lambda(n) \) in \( \mathbb{Z}/n\mathbb{Z} \), and it has the followings properties.

(i) \( \text{ord}_n(u \cdot v) = \text{ord}_n(u) \text{ord}_n(v) \), \( \text{if gcd}(\text{ord}_n(u), \text{ord}_n(v)) = 1 \).

(ii) \( \text{ord}_n(u^k) = \frac{\text{ord}_n(u)}{\gcd(k, n)} \), \( \text{for any pair of integers } k, n \geq 1 \).

The Carmichael function specifies the maximal order of a cyclic subgroup \( G \) of the finite ring \( \mathbb{Z}/n\mathbb{Z} \), and the maximal order \( \lambda(n) = \max\{m \geq 1 : v^m \equiv 1 \mod n\} \) of the elements in a finite cyclic group \( G \).

**Definition 3.3.** An integer \( u \in \mathbb{Z} \) is called a primitive root \( \mod n \) if the least exponent \( \min\{m \in \mathbb{N} : u^m \equiv 1 \mod n\} = \lambda(n) \).

In synopsis, primitive elements in a cyclic group have the maximal orders \( \text{ord}_G(v) = \#G \), and minimal indices \( \text{ind}_G(v) = 1 \).

**Lemma 3.2.** (Primitive root test) Let \( p \geq 3 \) be a prime, and let \( a \geq 2 \) be an integer such \( \gcd(a, p) = 1 \). Then,
the integer \( a \) is a primitive root if and only if

\[
a^{(p-1)/q} - 1 \not\equiv 0 \mod p
\]

for every prime divisor \( q | p - 1 \).

**Proof.** This is a restricted version of the Pocklington primality test, see [1] p. 175.

**Lemma 3.3.** For any integer \( n \geq 1 \), and the group \( \mathbb{Z}/n\mathbb{Z} \), the followings hold.

(i) The group of units \( (\mathbb{Z}/n\mathbb{Z})^\times \) has \( \varphi(n) \) units.

(ii) The number of primitive root is given by

\[
\varphi(\varphi(n)) = \varphi(n) \prod_{p | \varphi(n)} \left(1 - \frac{1}{p^{m(p)}}\right),
\]

where \( m(p) \geq 1 \) is the number of invariant factor associate with \( p \).
The Euler totient function and the more general Carmichael totient function over the finite ring $\mathbb{Z}/n\mathbb{Z}$ are seamlessly linked by the Fermat-Euler Theorem.

**Lemma 3.4.** (Fermat-Euler) If $a \in \mathbb{Z}$ is an integer such that $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \mod n$.

The improvement provides the least exponent $\lambda(n) \mid \varphi(n)$ such that $a^{\lambda(n)} \equiv (1 \mod n)$.

**Lemma 3.5.** ([12]) Let $n \in \mathbb{N}$ be any given integer. Then

(i) The congruence $a^{\lambda(n)} \equiv 1 \mod n$ is satisfied by every integer $a \geq 1$ relatively prime to $n$, that is $\gcd(a, n) = 1$.

(ii) In every congruence $x^{\lambda(n)} \equiv 1 \mod n$, a solution $x = u$ exists which is a primitive root mod $n$, and for any such solution $u$, there are $\varphi(\lambda(n))$ primitive roots congruent to powers of $u$.

**Proof.** (i) The number $\lambda(n)$ is a multiple of every $\lambda(p^v) = \varphi(p^v)$ such that $p^v \mid n$.

Ergo, for any relatively prime integer $a \geq 2$, the system of congruences

$$a^{\lambda(n)} \equiv 1 \mod p_1^{v_1}, \quad a^{\lambda(n)} \equiv 1 \mod p_2^{v_2}, \quad \ldots, \quad a^{\lambda(n)} \equiv 1 \mod p_t^{v_t}, \quad (3.4)$$

where $t = \omega(n)$ is the number of prime divisors in $n$, is valid.

### 3.2 Maximal Cyclic Subgroups

The multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ has $\xi(n) = \varphi(n)/\lambda(n)$ maximal cyclic subgroups

$$G_1 \cup G_2 \cup \cdots \cup G_t = (\mathbb{Z}/n\mathbb{Z})^\times \quad (3.5)$$

of order $\#G_i = \lambda(n)$, and $G_i \cap G_j = \{1\}$ for $i \neq j$ with $1 \leq i, j \leq t = \xi(n)$. Each maximal subgroup $G_i$ has a unique subset of $\varphi(\lambda(n))$ primitive roots. The optimal case $G_1 = (\mathbb{Z}/n\mathbb{Z})^\times$ for $\xi(n) = 1$ occurs on a subset of integers of zero density, the next lemma is the best known result, see also Lemma 2.4.

**Lemma 3.6.** (Gauss) Let $p \geq 3$ be a prime, and let $n \geq 1$ be an integer. Then, the multiplicative groups has the following properties.

(i) $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is cyclic of order $\varphi(p^n)$, and there exists a primitive root of the same order.

(ii) $(\mathbb{Z}/2p^n\mathbb{Z})^\times$ is cyclic of order $\varphi(2p^n)$, and there exists a primitive root of the same order.

**Proof.** The proof and additional information appear in [3, Theorem 10.7], and [16, Theorem 2.6].

Extensive details on this topic appear in [10].
4 Characteristic Functions

The indicator function or characteristic function $\Psi : G \rightarrow \{0,1\}$ of some distinguished subsets of elements are not difficult to construct. Many equivalent representations of the characteristic function $\Psi$ of the elements are possible.

4.1 Characteristic Functions Modulo Prime Powers

The standard method for constructing characteristic function for primitive elements are discussed in [24, Corollary 3.5], [33, p. 258], and some characteristic functions for finite rings are discussed in [46]. These type of characteristic functions detect the orders of the elements $v \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ by means of the divisors of $\varphi(p^2) \neq \emptyset$. A new method for constructing characteristic functions for certain elements in cyclic groups is developed here. These type of characteristic functions detect the orders of the elements $v \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ by means of the solutions of the equation $\tau^{p^m} - v \equiv 0 \pmod{p^2}$, where $v, \tau$ are constants, and $n$ is a variable such that $1 \leq n < p - 1$, and $gcd(n,p - 1) = 1$. The formula $\varphi(n) = \prod_{p|n}(1 - 1/p)$ denotes the Euler totient function.

**Lemma 4.1.** Let $p \geq 3$ be a prime, and let $\tau$ be a primitive root mod $p^2$. Let $v \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ be a nonzero element. Then

$$\Psi_v(p^2) = \sum_{1 \leq n < p^2 \atop \gcd(n,p-1)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi(v^{p^m} - v)/\varphi(p^2)} = \left\{ \begin{array}{ll} 1 & \text{if } \ord_{p^2}(v) = p - 1, \\ 0 & \text{if } \ord_{p^2}(v) \neq p - 1. \end{array} \right. \quad (4.1)$$

**Proof.** Let $\tau \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ be a fixed primitive root of order $p(p - 1) = \varphi(p^2)$. As the index $n \geq 1$ ranges over the integers relatively prime to $p - 1$, the element $\tau^{p^m} \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ ranges over the elements of order $\ord_{p^2}(\tau^{p^m}) = p - 1$. Hence, the equation

$$\tau^{p^m} - v = 0 \quad (4.2)$$

has a solution if and only if the fixed element $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is an elements of order $\ord_{p^2}(v) = p - 1$. Setting $w = e^{i2\pi(\tau^{p^m} - v)/\varphi(p^2)}$ and summing the inner sum yield

$$\sum_{\gcd(n,p-1)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} w^m = \left\{ \begin{array}{ll} 1 & \text{if } \ord_{p^2}(v) = p - 1, \\ 0 & \text{if } \ord_{p^2}(v) \neq p - 1. \end{array} \right. \quad (4.3)$$

This follows from the geometric series identity $\sum_{0 \leq m \leq x-1} w^m = (w^x - 1)/(w - 1), w \neq 1$ applied to the inner sum. \hfill \blacksquare

The characteristic function for any element $v \geq 2$ of order $\ord_{p^2}(v) = d \mid p - 1$ in the cyclic group $(\mathbb{Z}/p^2\mathbb{Z})^\times$ is a sum of characteristic functions.

**Lemma 4.2.** Let $v \geq 2$ be a fixed base, let $p \geq 3$ be a prime, and let $\tau$ be a primitive root mod $p^2$. The indicator function for the subset of primes such that $v^{p^2-1} - 1 \equiv 0 \pmod{p^2}$ is given by

$$\Psi_0(p^2) = \sum_{d \mid p-1} \sum_{1 \leq n < p-1 \atop \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi(v^{d^m} - v)/\varphi(p^2)} = \left\{ \begin{array}{ll} 1 & \text{if } \ord_{p^2}(v) \mid p - 1, \\ 0 & \text{if } \ord_{p^2}(v) \nmid p - 1. \end{array} \right. \quad (4.4)$$
Proof. Suppose that $\text{ord}_p(v) = p - 1$. Then, there is a unique pair $d$ such that $\tau_{dpn} - v \equiv 0 \mod p^2$. Otherwise, $\tau_{dpn} - v \not\equiv 0 \mod p^2$ for all pairs $d$ and $\gcd(n, (p-1)/d) = 1$. Proceed as in the proof of Lemma 4.1. $\blacksquare$

Lemma 4.3. Let $p \geq 3$ be a prime, and let $\tau$ be a primitive root $\mod p^k$. Let $v \in (\mathbb{Z}/p^k\mathbb{Z})^\times$ be a nonzero element. Then

$$\Psi_v(p^k) = \sum_{1 \leq n < p^k} \frac{1}{\varphi(p^k)} \sum_{0 \leq m < \varphi(p^k)} e^{i2\pi \frac{\tau^{m-1}n-v}{\varphi(p^k)}} \left\{ \begin{array}{ll} 1 & \text{if } \text{ord}_{p^k}(v) = p-1, \\ 0 & \text{if } \text{ord}_{p^k}(v) \neq p-1. \end{array} \right. \quad (4.5)$$

Proof. Modify the proof of Lemma 4.1 to fit the finite ring $\mathbb{Z}/p^k\mathbb{Z}$. $\blacksquare$

4.2 Characteristic Functions Modulo $n$

The indicator function for primitive root in a maximal cyclic group $G \subset \mathbb{Z}/n\mathbb{Z}$ is simpler than the indicator function for primitive root in $\mathbb{Z}/n\mathbb{Z}$, which is a sum of indicator functions for its maximal cyclic groups $G_1, G_2, \ldots, G_e$, with $e \geq 1$.

Lemma 4.4. Let $n \geq 3$ be an integer, and let $\tau \in G$ be a primitive root $\mod n$ in a maximal cyclic subgroup $G \subset (\mathbb{Z}/n\mathbb{Z})^\times$. If $v \neq \pm u^2$ is an integer, then

$$\Psi_v(G) = \sum_{1 \leq \tau < \varphi(n)} \frac{1}{\varphi(n)} \sum_{0 \leq r < \varphi(n)} e^{i2\pi \frac{\tau^m-n-v}{\varphi(n)}} \left\{ \begin{array}{ll} 1 & \text{if } \text{ord}_{n}(v) = \lambda(n), \\ 0 & \text{if } \text{ord}_{n}(v) \neq \lambda(n). \end{array} \right. \quad (4.6)$$

Proof. Let $\tau \in G$ be a fixed primitive root of order $\lambda(n)$, see Lemma 3.6. As the index $m \geq 1$ ranges over the integers relatively prime to $\lambda(n)$, the element $\tau^m \in G$ ranges over the elements of order $\text{ord}_{n}(\tau^m) = \lambda(n)$. Hence, the equation

$$\tau^m - v = 0 \quad (4.7)$$

has a solution if and only if the fixed element $v \in G$ is an element of order $\text{ord}_{n}(v) = \lambda(n)$. Next, let $w = e^{i2\pi (\tau^m-n)/\varphi(n)}$. Summing the inner sum yields

$$\sum_{\gcd(m,\lambda(n))=1} \frac{1}{\varphi(n)} \sum_{0 \leq r < \varphi(n)} e^{i2\pi \frac{\tau^m-n-v}{\varphi(n)}} \left\{ \begin{array}{ll} 1 & \text{if } \text{ord}_{n}(v) = \lambda(n), \\ 0 & \text{if } \text{ord}_{n}(v) \neq \lambda(n). \end{array} \right. \quad (4.8)$$

This follows from the geometric series identity $\sum_{0 \leq r \leq x-1} w^r = (w^x - 1)/(w - 1), w \neq 1$ applied to the inner sum. $\blacksquare$

Lemma 4.5. Let $n \geq 3$ be an integer, and let $\xi(n) = \varphi(n)/\lambda(n)$. Let $\tau_i \in G_i$ be a primitive root $\mod n$ in a maximal cyclic subgroup $G_i \subset (\mathbb{Z}/n\mathbb{Z})^\times$. If $v \neq \pm u^2$ is an integer, then

$$\Psi_1(n) = \sum_{1 \leq \tau \leq \xi(n)} \frac{1}{\varphi(n)} \sum_{1 \leq \tau < \varphi(n)} e^{i2\pi \frac{\tau^m-n-v}{\varphi(n)}} \left\{ \begin{array}{ll} 1 & \text{if } \text{ord}_{n}(v) = \lambda(n), \\ 0 & \text{if } \text{ord}_{n}(v) \neq \lambda(n). \end{array} \right. \quad (4.9)$$

Proof. This is a sum of $\xi(n) \geq 1$ copies of the indicator function proved in Lemma 4.1. $\blacksquare$
The last one considered is the indicator function for elements of order \( \text{ord}_{n^2}(v) | \lambda(n) \) in \( \mathbb{Z}/n^2\mathbb{Z} \). This amounts to a double sum of indicator functions for its maximal cyclic groups \( G_1, G_2, \ldots, G_e \), with \( e \geq 1 \).

**Lemma 4.6.** Let \( n \geq 3 \) be an integer, and let \( \xi(n) = \varphi(n)/\lambda(n) \). Let \( \tau_i \in G_i \) be a primitive root mod \( n \) in a maximal cyclic subgroup \( G_i \subset \left( \mathbb{Z}/n^2\mathbb{Z} \right) \). If \( n \neq \pm u^2 \) is an integer, then

\[
\Psi_0(n^2) = \sum_{1 \leq i \leq \xi(n)} \sum_{d | \lambda(n)} \sum_{1 \leq m < \lambda(n) \atop \gcd(m,\lambda(n)/d)=1} \frac{1}{\varphi(n^2)} \sum_{0 \leq r < \varphi(n^2)} e^{i2\pi \left( \frac{m}{\varphi(p^2)} \right) r} = \begin{cases} 1 & \text{if } \text{ord}_{n^2}(v) | \lambda(n), \\ 0 & \text{if } \text{ord}_{n^2}(v) \nmid \lambda(n). \end{cases}
\]

(4.10)

**Proof.** For each divisor \( d | \lambda(n) \), this is a sum of \( \xi(n) \geq 1 \) copies of the indicator function proved in Lemma 4.3.

\[ \square \]

### 4.3 Problems

1. Let \( p \geq 3 \) be a prime. Show that the characteristic function of quadratic nonresidue in the finite ring \( \left( \mathbb{Z}/p^2\mathbb{Z} \right) \) is

\[
\Psi_{q^2}(p^2) = \sum_{1 \leq n < \varphi(p^2) \atop \gcd(2,n)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi \left( \frac{m}{\varphi(p^2)} \right) n} = \begin{cases} 1 & \text{if } \text{ord}_{p^2}(v) = 2, \\ 0 & \text{if } \text{ord}_{p^2}(v) \neq 2. \end{cases}
\]

2. Let \( p = 3a + 1 \geq 7 \) be a prime. Show that the characteristic function of cubic nonresidue in the finite ring \( \left( \mathbb{Z}/p^2\mathbb{Z} \right) \) is

\[
\Psi_{q^3}(p^2) = \sum_{1 \leq n < \varphi(p^2) \atop \gcd(3,n)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi \left( \frac{m}{\varphi(p^2)} \right) n} = \begin{cases} 1 & \text{if } \text{ord}_{p^2}(v) = 3, \\ 0 & \text{if } \text{ord}_{p^2}(v) \neq 3. \end{cases}
\]

### 5 Equivalent Exponential Sums

For any fixed \( 0 \neq s \in \mathbb{Z}/p^2\mathbb{Z} \), an asymptotic relation for the exponential sums

\[
\sum_{\gcd(n,\varphi(p^2))=1} e^{i2\pi s n / \varphi(p^2)} \quad \text{and} \quad \sum_{\gcd(n,\varphi(p^2))=1} e^{i2\pi n / \varphi(p^2)},
\]

(5.1)
is provided in Lemma 5.1. This result expresses the first exponential sum in (5.1) as a sum of simpler exponential sum and an error term. The proof is based on Lagrange resolvent in the finite ring \( \mathbb{Z}/p^2\mathbb{Z} \). Specifically,

\[
(\omega^t, \zeta^s) = \zeta^s + \omega^{t} \zeta^{s^1} + \omega^{-2t} \zeta^{s^2} + \cdots + \omega^{-(p-2)t} \zeta^{s^{(p-2)^{-1}}},
\]

(5.2)

where \( \omega = e^{i2\pi / p}, \quad \zeta = e^{i2\pi / \varphi(p^2)}, \) and the variables \( 0 \neq s \in \mathbb{Z}/p^2\mathbb{Z}, \) and \( 0 \neq t \in \mathbb{Z}/p\mathbb{Z} \).

**Lemma 5.1.** Let \( p \geq 2 \) be a large prime. If \( \tau \) be a primitive root modulo \( p^2 \), then

\[
\sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi s n d \tau / \varphi(p^2)} = \sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi \tau n d p / \varphi(p^2)} + O(p^{1/2} \log^3 p),
\]

(5.3)

for any fixed \( d \mid p - 1, \) and \( 0 \neq s \in \mathbb{Z}/p^2\mathbb{Z} \).
Proof. Summing (5.2) times $\omega^t n$ over the variable $t \in \mathbb{Z}/p\mathbb{Z}$ yields, (all the nontrivial complex $p$th root of unity),

$$p \cdot e^{i 2\pi \tau n d p / \varphi(p^2)} = \sum_{0 \leq t \leq p-1} (\omega^t, \zeta^{s \tau n d p}) \omega^t n. \quad (5.4)$$

Summing (5.3) over the variable $n \geq 1$, for which $\gcd(n, (p-1)/d) = 1$, yields

$$p \cdot \sum_{\gcd(n,(p-1)/d)=1} e^{i 2\pi \tau n d p / \varphi(p^2)} = \sum_{\gcd(n,(p-1)/d)=1} \sum_{0 \leq t \leq p-1} (\omega^t, \zeta^{s \tau n d p}) \omega^t n \quad (5.5)$$

$$= \sum_{1 \leq t \leq p-1} (\omega^t, \zeta^{s \tau n d p}) \sum_{\gcd(n,(p-1)/d)=1} \omega^t n - p.$$ 

The first index $t = 0$ contributes $p$, see [36, Equation (5)] for similar calculations. Likewise, the basic exponential sum for $s = 1$ can be written as

$$p \cdot \sum_{\gcd(n,(p-1)/d)=1} e^{i 2\pi \tau n d p / \varphi(p^2)} = \sum_{1 \leq t \leq p-1} (\omega^t, \zeta^{s \tau n d p}) \sum_{\gcd(n,(p-1)/d)=1} \omega^t n - p, \quad (5.6)$$

Differencing (5.5) and (5.6) produces

$$S_1 = \quad p \cdot \left( \sum_{\gcd(n,(p-1)/d)=1} e^{i 2\pi \tau n d p / \varphi(p^2)} - \sum_{\gcd(n,(p-1)/d)=1} e^{i 2\pi \tau n d p / \varphi(p^2)} \right)$$

$$= \sum_{1 \leq t \leq p-1} \left( (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right) \sum_{\gcd(n,(p-1)/d)=1} \omega^t n. \quad (5.7)$$

The right side sum $S_1$ can be rewritten as

$$S_1 = \sum_{1 \leq t \leq p-1} \left( (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right) \sum_{\gcd(n,(p-1)/d)=1} \omega^t n \quad (5.8)$$

$$= \sum_{1 \leq t \leq p-1} \left( (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right) \sum_{e \leq (p-1)/d} \mu(e) \frac{\omega^{et} - \omega^{et(\frac{p-1}{d}+1)}}{1 - \omega^{et}}$$

$$= \sum_{1 \leq t \leq p-1} \sum_{e \leq (p-1)/d} \left( (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right) \mu(e) \frac{\omega^{et} - \omega^{et(\frac{p-1}{d}+1)}}{1 - \omega^{et}}. \quad (5.9)$$

The second line follows from Lemma 5.2i. The upper bound

$$|S_1| \leq \sum_{1 \leq t \leq p-1} \sum_{e \leq (p-1)/d} \left| (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right| \mu(e) \frac{\omega^{et} - \omega^{et(\frac{p-1}{d}+1)}}{1 - \omega^{et}}$$

$$\leq \sum_{1 \leq t \leq p-1} \sum_{e \leq (p-1)/d} \left| (\omega^t, \zeta^{s \tau n d p}) - (\omega^t, \zeta^{s \tau n d p}) \right| \mu(e) \frac{\omega^{et} - \omega^{et(\frac{p-1}{d}+1)}}{1 - \omega^{et}}$$

$$\leq \sum_{1 \leq t \leq p-1} \sum_{e \leq (p-1)/d} \left( 2p^{1/2} \log p \right) \left( 2p \log p \right) \frac{\omega^{et} - \omega^{et(\frac{p-1}{d}+1)}}{1 - \omega^{et}}$$

$$\leq \sum_{1 \leq t \leq p-1} \left( 2p^{1/2} \log p \right) \left( \frac{2p \log p}{\pi t} \right)$$

$$\leq \left( 4p^{3/2} \log^2 p \right) \sum_{1 \leq t \leq p-1} \frac{1}{t} \quad (5.9)$$

$$\leq 8p^{3/2} \log^3 p.$$
The third line follows the upper bound for Lagrange resolvents, and the fourth line follows from Lemma 5.2.ii. Here, the difference of two Lagrange resolvents, (Gauss sums), has the upper bound

\[
\left| \left( \omega^t, \zeta^{t \sigma_{dp}} \right) - \left( \omega^t, \zeta^{t \sigma_{dp}} \right) \right| \leq 2 \left| \sum_{1 \leq t \leq p-1} \chi(t) e^{i2\pi t/p} \right| \leq 2p^{1/2} \log p, \tag{5.10}
\]

where \( |\chi(t)| = 1 \) is a root of unity. Taking absolute value in (5.7) and using (5.9) and (??) return

\[
\frac{\sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi n/p} - \sum_{\gcd(n,(p-1)/d)=1} e^{i2\pi n/d}}{p} \leq |S_1| \leq 8p^{3/2} \log^3 p. \tag{5.11}
\]

The last inequality implies the claim.

**Lemma 5.2.** Let \( p \geq 2 \) be a large prime, and let \( \omega = e^{i2\pi/p} \) be a \( p \)-th root of unity. Then,

(i)

\[
\sum_{\gcd(n,(p-1)/d)=1} \omega^{tn} = \sum_{e \leq (p-1)/d} \mu(e) \frac{\omega^{et} - \omega^{e\left(\frac{t}{d} + 1\right)}}{1 - \omega^{et}},
\]

(ii)

\[
\left| \sum_{\gcd(n,(p-1)/d)=1} \omega^{tn} \right| \leq \frac{2p \log p}{\pi t},
\]

where \( \mu(k) \) is the Mobius function, for any fixed pair \( d | p-1 \) and \( t \in [1, p-1] \).

**Proof.** (i) Use the inclusion-exclusion principle to rewrite the exponential sum as

\[
\sum_{\gcd(n,(p-1)/d)=1} \omega^{tn} = \sum_{n \leq (p-1)/d} \omega^{tn} \sum_{e \mid n} \mu(e)
\]

\[
= \sum_{e \leq (p-1)/d} \mu(e) \sum_{n \leq (p-1)/d} \omega^{tn}
\]

\[
= \sum_{e \leq (p-1)/d} \mu(e) \sum_{m \leq (p-1)/de} \omega^{tm}
\]

\[
= \sum_{e \leq (p-1)/d} \mu(e) \frac{\omega^{et} - \omega^{e\left(\frac{t}{d} + 1\right)}}{1 - \omega^{et}}. \tag{5.12}
\]

(ii) Observe that the parameters \( \omega = e^{i2\pi/p} \), the integers \( t \in [1, p-1] \), and \( e \leq (p-1)/d \) imply that \( \pi et/p \neq k\pi \) with \( k \in \mathbb{Z} \), so the sine function \( \sin(\pi et/p) \neq 0 \) is well defined. Using standard manipulations, and \( z/2 \leq \sin(z) < z \) for \( 0 < |z| < \pi/2 \), the last expression becomes

\[
\left| \frac{\omega^{et} - \omega^{e\left(\frac{t}{d} + 1\right)}}{1 - \omega^{et}} \right| \leq \left| \frac{2}{\sin(\pi et/p)} \right| \leq \frac{2p}{\pi et}. \tag{5.13}
\]
for $1 \leq d \leq p - 1$. Finally, the upper bound is

$$\sum_{e \leq (p-1)/d} \mu(e) \frac{\omega^e - \omega^{e \frac{(p-1)}{d}+1}}{1 - \omega^e} \leq \frac{2p}{\pi t} \sum_{e \leq (p-1)/d} \frac{1}{e},$$

(5.14)

$$\leq \frac{2p \log p}{\pi t}. \quad \blacksquare$$

6 Upper Bound For The Main Term

An estimate for the finite sum occurring in the evaluation of the main term is considered in this section.

**Lemma 6.1.** Let $x \geq 1$ be a large number, and let $\varphi(n)$ be the Euler totient function. Then

$$\sum_{p \leq x} \frac{1}{\varphi(p)} \sum_{d \mid p-1, \gcd(n,(p-1)/d) = 1} 1 \leq 2 \log \log x. \quad (6.1)$$

**Proof.** Use the identity $\sum_{d \mid n} \varphi(d) = n$ to eliminate the inner double sum in the following way:

$$\sum_{d \mid p-1, \gcd(n,(p-1)/d) = 1} 1 = \sum_{d \mid p-1} \varphi((p-1)/d) = p - 1. \quad (6.2)$$

Substituting this returns

$$\sum_{p \leq x} \frac{1}{\varphi(p)} \sum_{d \mid p-1, \gcd(n,(p-1)/d) = 1} 1 = \sum_{p \leq x} \frac{1}{\varphi(p)} : (p-1) = \sum_{p \leq x} \frac{1}{p}. \quad (6.3)$$

Lastly, apply Mertens theorem to the prime harmonic sum. \quad \blacksquare
7 Evaluations Of The Main Terms

Various types of finite sums occurring in the evaluations of the main terms of various results are considered in this section.

7.1 Sums Over The Primes

Lemma 7.1. Let $x \geq 1$ be a large number, and let $\varphi(n)$ be the Euler totient function. Then

(i) $\sum_{p \leq x} \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} \frac{\varphi(p^2)}{d} = \log \log x + b_0 + O \left( \frac{1}{\log x} \right)$,

where $b_0 > 0$ is a constant.

(ii) $\sum_{p \leq x} \varphi(p^2) \sum_{\gcd(n,p-1)=1} 1 = a_0 \log \log x + a_1 + O \left( \frac{1}{\log x} \right)$,

where $a_0 > 0$ and $a_1$ are constants.

Proof. (i) Use the identity $\sum_{d \mid n} \varphi(d) = n$ to eliminate the inner double sum in the following way:

\[ \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} \varphi((p-1)/d) = p - 1. \]  

(7.1)

Substituting this returns

\[ \sum_{p \leq x} \varphi(p^2) \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} 1 = \sum_{p \leq x} \frac{1}{p} = \frac{1}{p} (x - \{x\}). \]  

(7.2)

Lastly, take the identity $x = x - \{x\}$, where $\{x\}$ is the fractional function, to complete the proof. (ii) The proof of this case is similar.

7.2 Sums Over The Bases

The other form of the main term deals with the summation over the bases $v \geq 2$.

Lemma 7.2. Let $x \geq 1$ be a large number, and let $\varphi(n)$ be the Euler totient function. Then

(i) $\sum_{v \leq x} \frac{1}{\varphi(v^2)} \sum_{d \mid v, \gcd(n,(v-1)/d)=1} 1 = \frac{1}{p} x + O \left( \frac{1}{p} \right)$.

(ii) $\sum_{v \leq x} \frac{1}{\varphi(v^2)} \sum_{\gcd(n,v-1)=1} 1 = \frac{\varphi(p-1)}{\varphi(p^2)} x + O \left( \frac{1}{p} \right)$.

Proof. (i) Use the identity $\sum_{d \mid n} \varphi(d) = n$ to eliminate the inner double sum:

\[ \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} \varphi((p-1)/d) = p - 1. \]  

(7.3)

Substituting this returns

\[ \sum_{v \leq x} \frac{1}{\varphi(v^2)} \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} 1 = \sum_{v \leq x} \frac{1}{p} = \frac{1}{p} (x - \{x\}). \]  

(7.4)

Lastly, take the identity $[x] = x - \{x\}$, where $\{x\}$ is the fractional function, to complete the proof. (ii) The proof of this case is similar.
7.3 Sums Over The Bases And Primes

Lemma 7.3. Let \( x \geq 1 \) be a large number, and let \( \varphi(n) \) be the Euler totient function. Then

(i) \[ \frac{1}{x} \sum_{n \leq x} \sum_{p \leq x} \frac{1}{d_{p-1}} \sum_{\gcd(n,(p-1)/d) = 1} \varphi(p^2) \sum_{\gcd(n,p-1) = 1} 1 = \log \log x + b_0 + O \left( \frac{1}{\log x} \right), \]

(ii) \[ \frac{1}{x} \sum_{n \leq x} \sum_{p \leq x} \frac{1}{d_{p-1}} \sum_{\gcd(n,(p-1)/d) = 1} \varphi(p^2) \sum_{\gcd(n,p-1) = 1} 1 = \log \log x + a_1 + O \left( \frac{1}{\log x} \right), \]

where \( a_0 > 0, a_1 \) and \( b_0 > 0 \) are constants.

Proof. (i) Using the identity \( \sum_{d|p-1} \varphi(d) = p - 1 \) is used to eliminate the inner double sum yield

\[
\frac{1}{x} \sum_{n \leq x} \sum_{p \leq x} \frac{1}{d_{p-1}} \sum_{\gcd(n,(p-1)/d) = 1} \varphi(p^2) \sum_{\gcd(n,p-1) = 1} 1 = \frac{1}{x} \sum_{p \leq x} \sum_{v \leq x} \frac{1}{d_{p-1}} \sum_{\gcd(n,(p-1)/d) = 1} \varphi(p^2) \sum_{\gcd(n,p-1) = 1} 1
\]

\[
= \frac{1}{x} \sum_{p \leq x} \sum_{v \leq x} \frac{p-1}{d_{p-1}} \varphi(p^2)
\]

\[
= \frac{1}{x} \sum_{p \leq x} \left( \frac{x}{p} + O \left( \frac{1}{p} \right) \right). \tag{7.5}
\]

Applying Lemma 2.1 yields

\[
\sum_{p \leq x} \frac{1}{p} + O \left( \frac{1}{x} \sum_{p \leq x} \frac{1}{p} \right) = \log \log x + b_0 + O \left( \frac{1}{\log x} \right), \tag{7.6}
\]

where \( b_0 > 0 \) is a constant. (ii) The proof of this case is similar, but it uses Lemma 2.4. □

7.4 Sums Over The Integers

Lemma 7.4. Let \( x \geq 1 \) be a large number, and let \( \varphi(n) \) be the Euler totient function. Then

(i) \[ \sum_{n \leq x} \sum_{1 \leq i \leq \xi(n)} \frac{1}{\varphi(n^2)} \sum_{d|\lambda(n), \gcd(n,\lambda(n)/d) = 1} \sum_{\gcd(n,\lambda(n)/d) = 1} 1 = \log x + \gamma + O \left( \frac{1}{x} \right), \]

where \( \gamma > 0 \) is Euler constant.

(ii) \[ \sum_{n \leq x} \sum_{1 \leq i \leq \xi(n)} \frac{1}{\varphi(n^2)} \sum_{\gcd(n,\lambda(n)) = 1} 1 \gg \frac{\log x}{\log \log x}. \]

Proof. (i) Use the identity \( \sum_{d|n} \varphi(d) = n \) to eliminate the inner double sum in the following way:

\[
\sum_{d|\lambda(n), \gcd(n,\lambda(n)/d) = 1} \sum_{\gcd(n,\lambda(n)/d) = 1} \varphi(\lambda(n)/d) = \lambda(n). \tag{7.7}
\]

Substituting this, and using the identities \( \varphi(n^2) = n \varphi(n) \), and \( \varphi(n) = \xi(n) \lambda(n) \) return

\[
\sum_{n \leq x} \sum_{1 \leq i \leq \xi(n)} \frac{1}{\varphi(n^2)} \varphi(n) \lambda(n) = \sum_{n \leq x} \frac{1}{\varphi(n^2)} \xi(n) \lambda(n) = \sum_{n \leq x} \frac{1}{n}. \tag{7.8}
\]
Lastly, apply the usual formula to the harmonic sum. (ii) The proof of this case is similar:

\[
\sum_{n \leq x} \sum_{1 \leq i \leq \xi(n)} \frac{1}{\varphi(n^2)} \sum_{\gcd(n, \lambda(n)) = 1} \frac{1}{\varphi(n^2)} \cdot \varphi(\lambda(n)) = \sum_{n \leq x} \sum_{1 \leq i \leq \xi(n)} \frac{1}{n \varphi(n)} \cdot \varphi(\lambda(n)) = \sum_{n \leq x} \frac{1}{n} \cdot \frac{\varphi(\lambda(n))}{\lambda(n)} \cdot \frac{1}{\varphi(\lambda(n))} \cdot \frac{1}{\varphi(\lambda(n))} = \sum_{n \leq x} \frac{1}{n} \prod_{p \mid \lambda(n)} \left(1 - \frac{1}{p}\right)
\]

\[
\gg \sum_{n \leq x} \frac{1}{n \log \log n} \gg \frac{\log x}{\log \log x},
\]

but it has no simple exact form. ■

7.5 Problems

1. Determine an exact asymptotic formula for

\[
\sum_{n \leq x, 1 \leq i \leq \xi(n)} \frac{1}{\varphi(n^2)} \sum_{\gcd(n, \lambda(n)) = 1} 1 = c_0 \frac{\log x}{\log \log x} \left(1 + O\left(\frac{1}{(\log \log x)^2}\right)\right),
\]

where \(c_0 > 0\) is a constant.
8 Estimates For The Error Terms

Upper bounds for the error terms occurring in the proofs of several results as Theorem 1.1 to Theorem 1.4 are determined here.

8.1 Error Terms In Long Intervals

The estimates for the long interval \([1, x]\) computed here are weak, but sufficient in many applications.

**Lemma 8.1.** Let \(x \geq 1\) be large number. Let \(p \geq 2\) be a large prime, and let \(\tau \in (\mathbb{Z}/p^2\mathbb{Z})^\times\) be a primitive root mod \(p^2\). If the element \(v \geq 2\) and \(\gcd(v, \phi(p^2)) = w\), then,

\[
\begin{align*}
(i) \sum_{p \leq x} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} & \sum_{1 \leq m < \varphi(p^2)} e^{i2\pi (x/p^2 - v/m)} \\
& \leq 2v \log \log x,
\end{align*}
\]

\[
\begin{align*}
(ii) \sum_{p \leq x} \sum_{\gcd(n,p-1)=1} \frac{1}{\varphi(p^2)} & \sum_{1 \leq m < \varphi(p^2)} e^{i2\pi (x/p^2 - v/m)} \\
& \leq 2v \log \log x,
\end{align*}
\]

where \(w \leq v\), for all sufficiently large numbers \(x \geq 1\).

**Proof.** (i) Rearrange the inner triple finite sum in the form

\[
E(x) = \sum_{p \leq x} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i2\pi (x/p^2 - v/m)}
\]

\[
= \sum_{p \leq x} \frac{1}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} e^{i2\pi \frac{dvm}{\varphi(p^2)}}
\]

\[
= \sum_{p \leq x} \frac{1}{\varphi(p^2)} \left( \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} e^{i2\pi \frac{dvm}{\varphi(p^2)}} \right)
\]

\[
+ O \left( \sum_{p \leq x} \frac{1}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \sum_{d|p-1} p^{1/2} \log^3 p \right)
\]

\[
= T_1 + T_2.
\]

The third line in (8.1) follows from Lemma 8.1. To complete the estimate, apply Lemma 8.2 and Lemma 8.3 to the terms \(T_1\) and \(T_2\) respectively. The proof of statement (ii) is similar.

**Lemma 8.2.** For any fixed integer \(v \geq 2\), and a large number \(x \geq 1\),

\[
\left| \sum_{p \leq x} \frac{1}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} e^{i2\pi \frac{dvm}{\varphi(p^2)}} \right| \leq v \log \log x.
\]
Proof. Trivially \( \left| e^{i2\pi \frac{dpn}{\varphi(p^2)}} \right| = 1 \), so the double inner inner sum reduces to
\[
\sum_{d|p-1, \gcd(n,(p-1)/d)=1} 1 = p - 1.
\]
(8.4)

Plugging this trivial value returns
\[
\left| T_1 \right| \leq \left| \sum_{p \leq x} \frac{p-1}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \right|
= w \sum_{p \leq x} \frac{1}{p} = w \log \log x,
\]
where where \( \varphi(p^2) = p(p-1) \), the parameter \( \gcd(v, p(p-1)) = w \leq v \), and
\[
\left| \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \right| = w
\]
is an exact evaluation. ■

Lemma 8.3. For any small number \( \varepsilon > 0 \), a fixed integer \( v \geq 2 \), and a large number \( x \geq 1 \),
\[
\left| \sum_{p \leq x} \frac{1}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \sum_{d|p-1} p^{1/2} \log^3 p \right| \leq .
\]
(8.7)

Proof. The asymptotic estimate \( \sum_{d|p-1} = O(p^k) \) is the maximal number of divisors. Thus,
\[
\left| T_2 \right| \leq \left| \sum_{p \leq x} \frac{p^{1/2+\varepsilon} \log^3 p}{\varphi(p^2)} \sum_{0 < m < \varphi(p^2)} e^{-i2\pi \frac{vm}{\varphi(p^2)}} \right|
\leq \sum_{p \leq x} \frac{p^{1/2+\varepsilon} \log^3 p}{\varphi(p^2)}
\leq \sum_{p \leq x} \frac{1}{p}
= \log \log x,
\]
where \( \varphi(p^2) = p(p-1) \), and \( \sum_{0 < m < \varphi(p^2)} e^{-i2\pi pm/\varphi(p^2)} = -1 \). ■

Lemma 8.4. Let \( x \geq 1 \) be large number. Let \( p \geq 2 \) be a large prime, and let \( \tau \in (\mathbb{Z}/p^k\mathbb{Z})^\times \) be a primitive root mod \( p^2 \). If the element \( v \geq 2 \) and \( \gcd(v, \varphi(p^k)) = w \), then,
\[
(i) \sum_{p \leq x} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^k)} \sum_{1 \leq m < \varphi(p^k)} e^{i2\pi \frac{1}{\varphi(p^k)} \frac{v \tau^{k-1} dm - v m}{\varphi(p^k)} } = O(\log \log x),
\]
(ii) and \[ \sum_{p \leq x, \gcd(n,p-1)=1} \sum_{1 \leq m < \varphi(p^k)} e^{i \frac{2 \pi x^m}{\varphi(p^k)}} = O \left( \log \log x \right), \]
for all sufficiently large numbers \( x \geq 1 \).

\[ \text{Proof.} \] Generalize the proof of Lemma 8.1 to fit the finite ring \( \mathbb{Z}/p^k \mathbb{Z} \).

\section{Error Terms In Short Intervals}

This calculations show that the error terms for short intervals \([x, x + z]\), with \( z = O(x) \) are nontrivials, and easy to determine using elementary method. But fail for very large intervals \([x, x^D]\), with \( D > 1 \).

\begin{lemma}
Let \( x \geq 1 \) and \( z \geq 1 \) be large numbers. Let \( p \geq 2 \) be a large prime, and let \( \tau \in (\mathbb{Z}/p^2 \mathbb{Z})^\times \) be a primitive root mod \( p^2 \). If the element \( v \geq 2 \) and \( \gcd(v, \varphi(p^2)) = w \), then,

(i) \[ \sum_{x \leq p \leq x + z, d \mid p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i \frac{2 \pi x^m}{\varphi(p^2)}} = O \left( \frac{z^{1/2}}{x^{1/2} \log x} \right), \]

(ii) \[ \sum_{x \leq p \leq x + z, d \mid p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i \frac{2 \pi x^m}{\varphi(p^2)}} = O \left( \frac{z^{1/2}}{x^{1/2} \log x} \right), \]

for all sufficiently large numbers \( x \geq 1 \).

\[ \text{Proof.} \] (i) Use the value \( \varphi(p^2) = p(p-1) \) to rearrange the triple finite sum in the form

\[ \sum_{x \leq p \leq x + z, d \mid p-1, \gcd(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i \frac{2 \pi x^m}{\varphi(p^2)}} = \sum_{x \leq p \leq x + z, \gcd(n,(p-1)/d)=1} \left( \frac{1}{p} \sum_{0 \leq m < \varphi(p^2)} e^{-i \frac{2 \pi x^m}{\varphi(p^2)}} \right) \left( \frac{1}{p-1} \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} \sum_{1 \leq m < \varphi(p^2)} e^{i \frac{2 \pi \tau^m d}{\varphi(p^2)}} \right). \]

Let

\[ A_p = \frac{1}{p} \sum_{0 \leq m < \varphi(p^2)} e^{-i \frac{2 \pi x^m}{\varphi(p^2)}} \]

and

\[ B_p = \frac{1}{p-1} \sum_{d \mid p-1, \gcd(n,(p-1)/d)=1} \sum_{1 \leq m < \varphi(p^2)} e^{i \frac{2 \pi \tau^m d}{\varphi(p^2)}}. \]

Utilize the prime number theorem \( \pi(x) = x/\log x + O(x/\log^2 x) \) for \( x \geq 1 \), and the exact value of finite sum \( \sum_{0 \leq m < \varphi(p^2)} e^{-i \frac{2 \pi \tau^m d}{\varphi(p^2)}} = -w \) whenever \( \gcd(v,p(p-1)) = w \), to estimate the first sum as

\[ \sum_{x \leq p \leq x + z, \gcd(n,(p-1)/d)=1} |A_p|^2 = \sum_{x \leq p \leq x + z} \left| \frac{1}{p} \sum_{0 \leq m < \varphi(p^2)} e^{-i \frac{2 \pi \tau^m d}{\varphi(p^2)}} \right|^2 = \sum_{x \leq p \leq x + z} \frac{w}{p^2} = w \int_{x}^{x+z} \frac{1}{t^2} \log(t) dt = O \left( \frac{1}{x \log x} \right), \]
where $w \leq v$ is a fixed number. Similarly, use the upper bound $\pi(x + z) - \pi(x) \leq 2z/\log x$ for $x \geq 1$, and $z \geq x^{3/4}$; and $\sum_{d | p-1} \varphi(p-1) = p - 1$ for $p \geq 3$, to obtain a trivial estimate for the second sum as

$$\sum_{x \leq p \leq x + z} |B_p|^2 = \sum_{x \leq p \leq x + z} \left| \frac{1}{p - 1} \sum_{d | p-1, \gcd((p-1)/d) = 1} e^{i2\pi nt/dp} / \varphi(d^2) \right|^2$$

$$\leq \sum_{x \leq p \leq x + z} \left| \frac{1}{p - 1} \sum_{d | p-1} \varphi((p-1)/d) \right|^2$$

$$= \sum_{x \leq p \leq x + z} 1$$

$$\leq \frac{2z}{\log x}.$$  

(8.13)

Now apply the Cauchy inequality

$$\left| \sum_{x \leq p \leq x + z} A_p \cdot B_p \right| \leq \left( \sum_{x \leq p \leq x + z} |A_p|^2 \right)^{1/2} \cdot \left( \sum_{x \leq p \leq x + z} |B_p|^2 \right)^{1/2}$$

$$\ll \left( \frac{1}{x \log x} \right)^{1/2} \cdot \left( \frac{2z}{\log x} \right)^{1/2}$$

$$\ll \frac{z^{1/2}}{x^{1/2} \log x}.$$  

(8.14)

(ii) The proof of this case is similar. ■
9 Counting Function For The Wieferich Primes

The subset of primes $\mathcal{W}_2 = \{ p : \text{ord}_p(2) \mid p - 1 \} = \{ 1093, 3511, \ldots \}$ associated with the base $v = 2$ is the best known case. But, many other bases have been computed too, see [13], [28].

9.1 Proof Of Theorem 1.1

**Proof.** (Theorem 1.1) Let $x \geq 1$ be a large number, and fix an integer $v \geq 2$. Consider the sum of the characteristic function for the fixed element $v$ of order $\text{ord}_p(v) \mid p - 1$ over the primes in the short interval $[x, x + z]$. Then

$$W_v(x + z) - W_v(x) = \sum_{x \leq p \leq x + z} \Psi_v(p^2).$$

Replacing the characteristic function, see Lemma 4.2, and expanding the difference equation (9.1) yield

$$\sum_{x \leq p \leq x + z} \Psi_v(p^2) = \sum_{x \leq p \leq x + z} \sum_{d \mid p - 1, \gcd(n,(p-1)/d)=1} 1 \sum_{1 \leq m < \varphi(p^2)} e^{i2\pi(v^n - v)m / \varphi(p^2)}.$$

The main term $M_v(x, z)$ is determined by the index $m = 0$, and the error term $E_v(x, z)$ is determined by the range $1 \leq m < \varphi(p^2)$. Applying Lemma 7.1 to the main term and applying Lemma 8.5 to the error term yield

$$M_v(x, z) + E_v(x, z) = c_v \left( \log \log(x+z) - \log \log(x) \right) + O \left( \frac{1}{\log x} \right) + O \left( \frac{z^{1/2}}{x^{1/2} \log x} \right),$$

Next, assuming that $z = O(x)$ it reduces to

$$W_v(x + z) - W_v(x) = c_v \left( \log \log(x+z) - \log \log(x) \right) + O \left( \frac{1}{\log x} \right),$$

where $c_v \geq 0$ is the density constant.

The specific constant $c_v \geq 0$ for a given fixed base $v \geq 2$ is a problem in algebraic number theory, see Theorem 10.1 for some details.

9.2 Proof Of Theorem 1.2

**Proof.** (Theorem 1.2) Let $x \geq 1$ be a large number, and fix an integer $v \geq 2$. The sum of the characteristic function for the fixed element $v$ of order $\text{ord}_p(v) \mid p - 1$ over the primes
in the interval \([1, x]\) is written as
\[
W_v(x) = \sum_{p \leq x} \Psi_0(p^2). \tag{9.5}
\]
Replacing the characteristic function, see Lemma 4.2, and expanding yield
\[
\sum_{p \leq x} \Psi_0(p^2) = \sum_{p \leq x} \sum_{d | p-1, \gcd(n, (p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i 2\pi (x^{p^2}-v)m/\varphi(p^2)}
\]
\[
= \sum_{p \leq x} \frac{1}{\varphi(p^2)} \sum_{d | p-1, \gcd(n, (p-1)/d)=1} 1
\]
\[
+ \sum_{x \leq p \leq x + z, d | p-1, \gcd(n, (p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i 2\pi (x^{p^2}-v)m/\varphi(p^2)}
\]
\[
= M_v(x) + E_v(x). \tag{9.6}
\]
The main term \(M_v(x)\) is determined by the index \(m = 0\), and the error term \(E_v(x)\) is determined by the range \(1 \leq m < \varphi(p^2)\). Applying Lemma 7.1 to the main term and applying Lemma 8.1 to the error term yield
\[
W_v(x) = M_v(x) + E_v(x)
\]
\[
\leq 2 \log \log(x) + 2v \log \log x
\]
\[
\leq 4v \log \log x.
\]
This verifies the upper bound. \(\blacksquare\)

### 9.3 Wieferich Constants

An appropriate upper bound of a primes counting problem immediately provides information on the convergence of the infinite series \(\sum_{p \geq 2} 1/p\). The best known case is the Brun constant
\[
B = \sum_{\text{twin primes } p, p+2} \frac{1}{p} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \approx 1.902160583104 \ldots, \tag{9.7}
\]
see [50], p. 9. A conjecture claims that \(1.90216054 < B < 1.90216063\), see [25], p. 215]. However, the arithmetic properties, such as rationality and irrationality, of \(B\) remains hopelessly unresolved. Similar problems arise for the sequences of Wieferich primes.

**Corollary 9.1.** Let \(v \geq 2\) be a small fixed integer. Then,
\[
\sum_{p \geq 2} \frac{1}{p} < \infty. \tag{9.8}
\]
In particular,
\[
\sum_{p \geq 2} \frac{1}{p} = 1/1093 + 1/3511 + 8c \log \log(10^{15})/10^{15} < 0.00119974,
\]
for some small constant \(c > 0\).
\[
\sum_{p \geq 2} \frac{1}{p} = \frac{1}{11} + \frac{1}{1006003} + \frac{12c \log \log(10^{14})}{10^{14}} < 0.0909102,
\]
for some small constant \(c > 0\).

**Proof.** (i) For base \(v = 2\), and using the numerical data in [13], the series is written a sum of two simpler subsums, and each subsums is evaluated or estimated:

\[
\sum_{p \geq 2} \frac{1}{p^{2p-1-1 \equiv 0 \mod p^2}} = \sum_{p \leq 10^{15}} \frac{1}{p} + \sum_{p > 10^{15}} \frac{1}{p} = \frac{1}{1093} + \frac{1}{3511} + \int_{10^{15}}^\infty \frac{1}{t} dW_2(t),
\]

where \(W_2(t) \leq 8 \log \log t\).

\[\text{(ii)}\ \sum_{p \geq 2} \frac{1}{p^{2p-1-1 \equiv 0 \mod p^2}} \leq 2^{10^3} + 1 + \sum_{p > 10^{15}} \frac{1}{p} \equiv 0 \mod p^2
\]

\[\text{for some small constant } c > 0.\]

9.4 **Proof Of Theorem 1.3**

Some numerical data has been compiled for this case in [11] and [28]. For examples, \(42^{22} - 1 \equiv 0 \mod 23^3\), and \(68^{112} - 1 \equiv 0 \mod 113^3\), for the ranges \(v < 100\), and \(p < 2^{32}\). But, these are very rare.

**Proof.** (Theorem 1.3) Let \(x \geq 1\) be a large number, and fix a small integer \(v \geq 2\). Let \(k = 3\) in Lemmas 4.3 and 4.2, to obtain a characteristic function to fit the finite ring \(\mathbb{Z}/p^3\mathbb{Z}\). Summing the characteristic function for the fixed element \(v\) of order \(\text{ord}_p(v)\) over the primes in the interval \([1, x]\) yields

\[
\sum_{p \leq x} \Psi_0(p^3) = \sum_{p \leq x} \sum_{d|p-1, \gcd(n, (p-1)/d) = 1} \frac{1}{\varphi(p^3)} \sum_{0 \leq m < \varphi(p^3)} e^{2\pi i \frac{d^2 n \varphi^2}{\varphi(p^3)} m}. \tag{9.10}
\]

Break the quadruple sum into a main term and an error term. The main term \(M_v(x)\) is determined by the index \(m = 0\), and the error term is determined by the range \(1 \leq m < \varphi(p^3)\), that is

\[
\sum_{p \leq x} \sum_{d|p-1, \gcd(n, (p-1)/d) = 1} \frac{1}{\varphi(p^3)} \sum_{0 \leq m < \varphi(p^3)} e^{2\pi i \frac{d^2 n \varphi^2}{\varphi(p^3)} m} = \sum_{p \leq x} \frac{1}{\varphi(p^3)} \sum_{d|p-1, \gcd(n, (p-1)/d) = 1} 1 \tag{9.11}
\]

\[+ \sum_{p \leq x} \frac{1}{\varphi(p^3)} \sum_{d|p-1, \gcd(n, (p-1)/d) = 1} \sum_{1 \leq m < \varphi(p^3)} e^{2\pi i \frac{d^2 n \varphi^2}{\varphi(p^3)} m}.\]
Use the identity $\sum_{d|n} \varphi(d) = n$ and $\varphi(p^2) = p(p-1)$ to verify that the main term is finite:

$$
\sum_{p \leq x} \frac{1}{\varphi(p^3)} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} 1 = \sum_{p \leq x} \frac{1}{p^2} \cdot (p-1) = \sum_{p \leq x} \frac{1}{p^2} \cdot (p-1) = O(1). \tag{9.12}
$$

Similarly, to verify that the error term is finite, use the value $\sum_{1 \leq m < \varphi(p^3)} e^{i2\pi vm/\varphi(p^3)} = -w$ whenever $\gcd(v, p^2(p-1)) = w$, and the trivial estimate for the double inner sum:

$$
\sum_{p \leq x} \frac{1}{\varphi(p^3)} \sum_{d|p-1, \gcd(n,(p-1)/d)=1} 1 \sum_{1 \leq m < \varphi(p^3)} e^{i2\pi vm/\varphi(p^3)} \leq \sum_{p \leq x} \frac{1}{\varphi(p^3)} \cdot w \cdot (p-1) \leq 2w \sum_{p \leq x} \frac{1}{p^2} = O(1), \tag{9.13}
$$

where $w \leq v$ is a small fixed number. Therefore

$$
\sum_{p \leq x} \Psi_0(p^3) = O(1), \tag{9.14}
$$

and this implies that the number of primes such that $v^{p-1} - 1 \equiv 0 \pmod{p^3}$ is finite. ■

### 9.5 Problems

1. Modify the characteristic function in Lemma 4.3 suitable for $\mathbb{Z}/p^n+3\mathbb{Z}$, to prove that the subset of primes $A(v) = \{p : v^{p^n(p-1)} - 1 \equiv 0 \pmod{p^{n+3}}\}$ is finite, $n \geq 0$.

2. A Wieferich prime pair $p$ and $q$ satisfies the reciprocity condition

$$
p^{p-1} - 1 \equiv 0 \pmod{q^2} \quad \text{and} \quad q^{p-1} - 1 \equiv 0 \pmod{q^2}.
$$

Many pairs of these primes are known, for example,

$$(p, q) = (83, 4871; (2903, 18787); (911, 318917) \leq 10^6$$

are known, see [39], and [28 p. 935]. Prove that there are infinitely many, and give an estimate of its counting function.

3. Let $v \geq 2$ be a small fixed integer. Apply the abc conjecture to show that the subset of primes $\{p : v^{p-1} - 1 \equiv 0 \pmod{p^3}\}$ is finite.
4. Let $p > 3$ be a prime. Use the Wilson result $(p - 1)! \equiv -1 \mod p$ to prove the Wolstenhome lemma, [47, p. 94]:

$$1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{p - 1} \equiv 0 \mod p^2.$$

5. Let $n > 9$ be an integer. Use Gauss generalization $K(n) = \prod_{\gcd(k,n) = 1} k \equiv \pm 1 \mod n$ of the Wilson result $(p - 1)! \equiv -1 \mod p$ to prove:

$$1 + \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{\phi(n)}} \equiv 0 \mod n^2,$$

where $\gcd(a_k, n) = 1$, if and only if $K(n) = -1$.

6. Let $B_k \in \mathbb{Q}$ be the $k$-th Bernoulli number, and let $p \geq 3$ be a prime. Generalize the power sum congruence

$$\sum_{1 \leq m \leq p} m^k \equiv pB_k \mod p,$$

see [31], to composite integers $p = n \geq 1$. 

RESULTS FOR WIEFERICH PRIMES
10 Correction Factors

About a quarter century ago the Lehmers discovered a discrepancy in the theory of primitive roots. This discovery led to the concept of correction factors, see [52] for the historical and technical details. This topic has evolved into a full fledged area of algebraic number theory, see [34], [45, Chapter 2], [4], [5], etc.

The correction factors $c_v \geq 0$ accounts for the dependencies among the primes in Conjecture 1.1 and Theorem 1.1. According to the analysis in [26, Section 2], for any random integer $v \geq 2$, the corresponding correction factor associated to a random subset $W_v$ of Wieferich primes has the value $c_v = 1$ with probability one. Thus, the nonunity correction factors $c_v \neq 1$ occur on a subset of integers $v \geq 2$ zero density. For example, at the odd prime powers $v = q^k \equiv 1$ mod $4$. A more precise expression for the value of the correction factors is given below.

**Theorem 10.1.** Let $v = ab^k$ with $a \geq 2$ squarefree. Then

$$c_v = \sum_{n \geq 1} \sum_{d|n} \frac{\mu(n) \gcd(dn, k)}{dn \phi(dn)},$$

(10.1)

where $K_r = \mathbb{Q}(\zeta_r, v^{1/r})$ is a number field extension, $\zeta_r$ is a primitive $r$th root of unity.

**Proof.** Let $r = nd$ with $d \mid n - 1$, and let $|K_r : \mathbb{Q}|$ be the index of the finite extension. The splitting field of the pure equation $x^{p(p-1)/d} - v = 0$ is the $r$th-cyclotomic numbers field $K_r$. By the Frobenius density theorem, [23, p. 134], (or the Chebotarev density theorem), the proportion of primes that split completely is $1/|K_{nd} : \mathbb{Q}|$. Therefore, the proportion of primes that do not split completely is

$$1 - \frac{1}{|K_{nd} : \mathbb{Q}|}.$$  

(10.2)

In light of this information, the inclusion-exclusion principle leads to

$$c_v = \sum_{n \geq 1} \sum_{d|n} \frac{\mu(n)}{|K_{nd} : \mathbb{Q}|}.$$  

(10.3)

Furthermore, for $v = ab^k$ with $a \geq 2$ squarefree, the index has the following form

$$|K_r : \mathbb{Q}| = \begin{cases} \frac{\varphi(r)}{\varphi(k, r)} & \text{if } r \mid 2a, \text{ and } a \equiv 1 \text{ mod } 4, \\ \frac{\varphi(r)}{\gcd(k, r)} & \text{otherwise}, \end{cases}$$  

(10.4)

see [22, p. 214], [2, p. 5].

This technique explicates the fluctuations in the numbers of primes with respect to the bases $v \geq 2$, see [13, Section 4.1] for a discussion. The best known case is $v = 5$. In this case, the dependency between two primes 2 and 5 is captured in the index calculation for $r = 10$. Here, the number field extension index fails to be multiplicative:

$$20 = |K_{10} : \mathbb{Q}| \neq |K_2 : \mathbb{Q}| \cdot |\mathbb{Q}(\zeta_5) : \mathbb{Q}| = 10.$$  

(10.5)

In this case $\pm \sqrt{5} = \zeta_5 + \zeta_5^{-1} - \zeta_5^2 - \zeta_5^{-2}$ so $\pm \sqrt{5} \in \mathbb{Q}(\zeta_5)$, see [35, p. 13], [2, Chapter 2], [52, Section 3],
10.1 Problems

1. Show that the ring of integers $\mathcal{O}_K$ of the numbers field $\mathbb{Q}(\sqrt{2})$ is not $\mathbb{Z}[\sqrt{2}]$ for $n = 1093$ and 3511. Hint: show that it contains half integers $(1 + \sqrt{2})/2$. 
11 Average Order Of A Random Subset Of Wieferich Primes

Quite often, the calculation of average density of an intractable primes distribution problem is the first line approach to solving the individual primes distribution problem, see [53], [4], [30], et cetera.

Proof. (Theorem 1.4) Let \( x \geq 1 \) be a large number, and let \( v \geq 2 \) be a random integer. The average number of Wieferich primes in base \( v \) over the interval \( [x, x+z] \) is given by

\[
\frac{1}{x} \sum_{v \leq x} (W_v(x + z) - W_v(x)) = \frac{1}{x} \sum_{v \leq x, x \leq p \leq x + z} \Psi_0(v). \tag{11.1}
\]

Replacing the characteristic function, Lemma 4.2, and expanding the difference equation (11.1) yield

\[
\frac{1}{x} \sum_{v \leq x, x \leq p \leq x + z} \Psi_0(v) = \frac{1}{x} \sum_{v \leq x, x \leq p \leq x + z} \frac{1}{\varphi(p^2)} \sum_{d | p-1, \text{gcd}(n,(p-1)/d)=1} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi \frac{m n}{\varphi(p^2)}}
\]

\[
= \frac{1}{x} \sum_{v \leq x, x \leq p \leq x + z} \frac{1}{\varphi(p^2)} \sum_{d | p-1, \text{gcd}(n,(p-1)/d)=1} 1
\]

\[
+ \frac{1}{x} \sum_{v \leq x, x \leq p \leq x + z} \sum_{d | p-1, \text{gcd}(n,(p-1)/d)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i2\pi \frac{m n}{\varphi(p^2)}}
\]

\[
= M_0(x, z) + E_0(x, z).
\]

The main term \( M_0(x, z) \) is determined by the index \( m = 0 \), and the error term \( E_0(x, z) \) is determined by the range \( 1 \leq m < \varphi(p^2) \). Applying Lemma 7.1 to the main term and applying Lemma 8.5 to the error term yield

\[
\frac{1}{x} \sum_{v \leq x} (W_v(x + z) - W_v(x)) = M_0(x, z) + E_0(x, z) \tag{11.3}
\]

\[
= \log \log(x + z) - \log \log(x) + O \left( \frac{1}{\log x} \right)
\]

\[
+ O \left( \frac{z^{1/2}}{x^{1/2} \log x} \right).
\]

Next, assuming that \( z = O(x) \) it reduces to

\[
\frac{1}{x} \sum_{v \leq x} (W_v(x + z) - W_v(x)) = \log \log(x + z) - \log \log(x) + O \left( \frac{1}{\log x} \right). \tag{11.4}
\]
12 Balanced Subsets

The balanced index \( \text{ind}_{p^2}(v) = p \) occurs whenever the base \( v \geq 2 \) has balanced order \( \text{ord}_{p^2}(v) = p - 1 \).

The balanced subset is
\[
B_v = \{ p \leq x : \text{ord}_{p^2}(v) = p - 1 \}.
\]

For a large number \( x \geq 1 \), the corresponding counting function for the number of Wieferich primes up to \( x \) with respect to a base of balanced order is defined by
\[
B_v(x) = \# \{ p \leq x : \text{ord}_{p^2}(v) = p - 1 \}.
\]

**Theorem 12.1.** Let \( v \geq 2 \) be a base, and let \( x \geq 1 \) and \( z \geq x \) be large numbers. Then, the number of Wieferich primes \( p \) such that \( \text{ord}_{p^2}(v) = p - 1 \) in the short interval \( [x, x+z] \) has the asymptotic formula
\[
B_v(x+z) - B_v(x) = c_v \left( \log \log (x+z) - \log \log (x) \right) + E_v(x, z),
\]
where \( c_v \geq 0 \) is the correction factor, and \( E_v(x) \) is an error term.

**Proof.** Let \( x \geq 1 \) be a large number, and fix an integer \( v \geq 2 \). Consider the sum of the characteristic function for the fixed element \( v \) of order \( \text{ord}_{p^2}(v) = p - 1 \) over the primes in the short interval \( [x, x+z] \). Then
\[
B_v(x+z) - B_v(x) = \sum_{x \leq p \leq x+z} \Psi(v).
\]

Replacing the characteristic function, Lemma 4.1, and expanding the existence equation (12.2) yield
\[
\sum_{x \leq p \leq x+z} \Psi(v) = \sum_{x \leq p \leq x+z} \sum_{\gcd(n,p-1)=1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i 2 \pi (x^{\varphi(p^2)} - v)^m} \varphi(p^2)
\]
\[
= \sum_{x \leq p \leq x+z} \frac{1}{\varphi(p^2)} \sum_{\gcd(n,p-1)=1} 1
\]
\[
+ \sum_{x \leq p \leq x+z} \sum_{\gcd(n,p-1)=1} \frac{1}{\varphi(p^2)} \sum_{1 \leq m < \varphi(p^2)} e^{i 2 \pi (x^{\varphi(p^2)} - v)^m} \varphi(p^2)
\]
\[
= M_v(x, z) + E_v(x, z).
\]

The main term \( M_v(x, z) \) is determined by the index \( m = 0 \), and the error term \( E_v(x, z) \) is determined by the range \( 1 \leq m < \varphi(p^2) \). Applying Lemma 7.2 to the main term and applying Lemma 8.1 to the error term yield
\[
\sum_{x \leq p \leq x+z} \Psi(v) = M_v(x, z) + E_v(x, z)
\]
\[
= c_v \left( \log \log (x+z) - \log \log (x) \right) + O \left( \frac{1}{\log x} \right) + O \left( \frac{z}{x \log x} \right).
\]

Next, assuming that \( z = O(x) \) it reduces to
\[
B_v(x+z) - B_v(x) = c_v \left( \log \log (x+z) - \log \log (x) \right) + O \left( \frac{1}{\log x} \right),
\]
where \( c_v \geq 0 \) is the density constant. ■
The average constant is $a_0 = 0.37399581\ldots$, see (2.16). The specific constant $c_v \geq 0$ for a given fixed base $v \geq 2$ is a problem in algebraic number theory explicated in the next section.
13 Data For Next Primes

The numerical data demonstrates that the number of primes \( p \leq x = 4 \times 10^{15} \) with respect to a fixed base \( v \geq 2 \) is a small quantity and vary from 0 to about 6. Some estimates for the intervals that contain the next Wieferich primes with respect to several fixed bases are sketched in this section.

13.1 Calculations For The Subset \( \mathcal{W}_2 \)

It have taken about a century to determine the number of base \( v = 2 \) primes up to \( x = 10^{15} \). The numerical data demonstrate that the subset is just

\[
\mathcal{W}_2 = \{ p : \text{ord}_p(2) \mid p - 1 \} = \{1093, 3511, \ldots, \}
\]

(13.1)

see [13]. To estimate the size of an interval \([10^{15}, 10^D]\) with \( D > 15 \), that contains the next Wieferich prime, assume that the correction factor \( c_2 > 0 \) of the set of Wieferich primes \( \mathcal{W}_2 \) is the same as the average density \( c_0 = 1 \) of a random set of Wieferich primes

\[
\mathcal{W}_0 = \{ p : \text{ord}_{p^2}(v) = p - 1 \}.
\]

(13.2)

Specifically, \( c_2 = c_0 = 1 \). Let \( x = 10^{15} \) and let \( x + z = 10^{15w} \). By assumption, and Theorem 1.2 it follows that

\[
1 \leq c_2 (\log \log(x + z) - \log \log x) + E(x, z)
= c_0 (\log \log(x + z) - \log \log x) + E(x, z)
= \log \frac{\log 10^{15w}}{\log 10^{15}} + E(x, z),
\]

(13.3)

where \( E(x, z) \) is an error term. Therefore, it is expected that the next Wieferich prime \( p > 10^{15} \) is in the interval \([x = 10^{15}, x + z = 10^{40}]\).

13.2 Calculations For The Subset \( \mathcal{W}_5 \)

The numerical data for the number of primes up to \( x = 10^{15} \) with respect to base \( v = 5 \) demonstrates that the subset

\[
\mathcal{W}_5 = \{2, 20771, 40487, 53471161, 1645333507, 6692367337, 188748146801, \ldots, \}
\]

(13.4)

where each prime satisfies \( 5^{p-1} - 1 \equiv 0 \mod p^2 \), see [13], contains about twice as many primes as the numerical data for many other subsets \( \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4, \mathcal{W}_6, \mathcal{W}_7 \), et cetera. A larger than average correction factor \( c_5 > 1 \), and many other bases \( v \) explicates this disparity, see Theorem 10.1.

To estimate the size of an interval \([10^{15}, 10^D]\) with \( D > 15 \), that contains the next base \( v = 5 \) Wieferich prime, assume that the density constant \( c_5 > 1 \) of the set of Wieferich primes

\[
\mathcal{W}_5 = \{ p : \text{ord}_{p^2}(5) \mid p - 1 \}
\]

(13.5)

is the same as the average density \( c_0 = 1 \) of a random set of Wieferich primes

\[
\mathcal{W}_0 = \{ p : \text{ord}_{p^2}(v) = p - 1 \}.
\]

(13.6)
Specifically, \( c_5 > c_0 = 1 \). Let \( x = 10^{15} \) and let \( x + z = 10^{15w} \). By assumption, and Theorem [12], it follows that

\[
1 \leq c_5 (\log \log (x + z) - \log \log (x)) + O \left( \frac{1}{\log x} \right)
\]

\[
= c_5 (\log \log (x + z) - \log \log (x)) + O \left( \frac{1}{\log x} \right)
\]

\[
= c_5 \log \frac{10^{15w}}{\log 10^{15}} + O \left( \frac{1}{\log x} \right).
\]

Therefore, it is expected that the next Wieferich prime \( p > 10^{15} \) is in the interval \([x = 10^{15}, x + z = 10^{40}]\).

### 13.3 Calculations For The Balanced Subset

The previous estimate assume that the order of the base \( \text{ord}_{p^2}(2) | p - 1 \) can vary as the prime \( p \) varies. In contrast, if the order of the base remains exactly \( \text{ord}_{p^2}(2) = p - 1 \) as the prime \( p \) varies, then the estimated interval is significantly larger as demonstrated below.

The subset of balanced primes in base \( v = 2 \) is

\[
\mathcal{B}_2 = \{ p : \text{ord}_{p^2}(2) = p - 1 \}
\]

and its density is the same as the average density \( a_0 = .37399581 \ldots \) of a random set of Wieferich primes

\[
\mathcal{W}_v = \{ p : \text{ord}_{p^2}(v) | p - 1 \}.
\]

Specifically, \( c_2 = a_0 \). Let \( x = 10^{15} \) and let \( x + z = x^{15w} \). By assumption, and Theorem [12], it follows that

\[
1 \leq c_2 (\log \log (x + z) - \log \log (x)) + O \left( \frac{1}{\log x} \right)
\]

\[
= a_0 (\log \log (x + z) - \log \log (x)) + O \left( \frac{1}{\log x} \right)
\]

\[
= .37399581 \log \frac{10^{15w}}{\log 10^{15}} + O \left( \frac{1}{\log x} \right).
\]

Therefore, it is expected that the next Wieferich prime \( p > 10^{15} \), for which the order \( \text{ord}_{p^2}(2) = p - 1 \) and its index is \( \text{ind}_{p^2}(2) = p \), is in the interval \([x = 10^{15}, x + z = 10^{218}]\).
14 Least Primitive Root In Finite Rings

The primitive roots \( v \geq 2 \) in residues finite rings modulo \( p \) rarely fail to be a primitive root in residues finite rings modulo \( p^2 \).

**Definition 14.1.** A primitive root \( v = v(p) \) modulo \( p \) is called nilpotent if the congruence \( v^{p-1} - 1 \equiv 0 \mod p^2 \) holds. The subset of primes such that \( v \) is nilpotent is denoted by \( \mathcal{N}_v = \{ p : \text{ord}_p(v) = p - 1 \text{ and } \text{ord}_{p^2}(v) = p - 1 \} \).

**Definition 14.2.** The least primitive root modulo a prime \( p \geq 3 \) is denoted by \( g = g(p) \geq 2 \) and least primitive root modulo a prime \( p^2 \) is denoted by \( h = h(p) \geq 2 \).

The occurrence of nilpotent primitive roots \( v = v(p) \) are very common. A sample is provided in the table.

| \( v(p) \) | \( p \) | \( v(p)\) | \( p \) |
|---|---|---|---|
| 3 | 1006003 | 7 | 5 |
| 5 | 40487 | 10 | 487 |
| 6 | 66161 | 11 | 71 |

However, the occurrence of nilpotent and least primitive roots simultaneously, are rarer. A complete list of the known cases is provided in the table.

| Nilpotent \( v(p) \) | Least \( g(p) \) | Least \( h(p^2) \) | Modulo \( p \) or \( p^2 \) |
|---|---|---|---|
| 5 | 5 | 7 | 40487 |
| 5 | 5 | 7 | 6692367337 |

The computational data for all the primes \( p \leq 10^{15} \) are compiled in [28, p. 929] and [44], [13], et alii.

**Lemma 14.1.** If \( g \geq 2 \) is a primitive root modulo a prime \( p \geq 3 \). Then, either \( g \) or its inverse \( \overline{g} \) is a primitive root modulo \( p^k \) for every \( k \geq 1 \).

The existence of nilpotent primitive roots, which causes the sporadic divisibility of the integers \( g^{p-1} - 1 \) by prime powers \( p^2 \), has an interesting effect on the distribution of primitive roots in the cyclic groups \( (\mathbb{Z}/p^k\mathbb{Z})^\times \) with \( k \geq 1 \), and the noncyclic groups \( (\mathbb{Z}/n\mathbb{Z})^\times \) with \( n \geq 1 \) is an arbitrary integer. The precise criterion for cyclic groups is specified below.

**Lemma 14.2.** A primitive root \( g \geq 2 \) modulo \( p \geq 3 \) is primitive root \( g \geq 2 \) modulo \( p^k \) for all \( k \geq 1 \) if and only if \( g^{p-1} - 1 \not\equiv 0 \mod p^2 \).

**Theorem 14.1.** Let \( x \geq 1 \) be a large number. Then

(i) The number of primes such that \( g(p) \geq 2 \) is a primitive root modulo \( p \), but \( g(p^2) \geq g(p) + 1 \) is infinite.

(ii) The counting function has the asymptotic formula

\[
\# \{ p \leq x : g(p) \text{ and } g(p^2) \geq g(p) + 1 \} \sim a_0 \log \log x, \quad (14.1)
\]

where \( a_0 = .37399581 \ldots \).
Proof. Without loss in generality, let \( v = 2 \), and let \( \tau \) be a primitive root modulo \( p^2 \). Suppose that the integer \( v = 2 \) is a primitive root modulo \( p \), but not modulo \( p^2 \). Then, the equation
\[
\tau^{pn} - 2 = 0
\]  
(14.2)
has a solution in prime \( p \geq 2 \), and \( n \geq 1 \) such that \( \gcd(n, p - 1) = 1 \) if and only if \( 2^{p-1} - 1 \not\equiv 0 \mod p \) and \( 2^{p-1} - 1 \equiv 0 \mod p^2 \). Constructing a indicator function, see Lemma 4.1 and sum it over the primes lead to
\[
\sum_{p \leq x} \Psi_v(p^2).
\]  
(14.3)
Expanding the indicator function in (14.3) yield
\[
\sum_{p \leq x} \Psi_v(p^2) = \sum_{p \leq x, \gcd(n, p - 1) = 1} \frac{1}{\varphi(p^2)} \sum_{0 \leq m < \varphi(p^2)} e^{i2\pi \frac{(\tau^{pn} - v)m}{\varphi(p^2)}}
\]  
(14.4)
\[
= M_v(x) + E_v(x).
\]
The main term \( M_v(x) \) is determined by the index \( m = 0 \), and the error term \( E_v(x) \) is determined by the range \( 1 \leq m < \varphi(p^2) \). Applying Lemma 5.1 to the main term and applying Lemma 8.1 to the error term yield
\[
M_v(x) + E_v(x) = c_v \log \log x + O \left( \frac{1}{\log x} \right) + O \left( (\log \log x)^{1-\varepsilon} \right)
\]  
(14.5)
where \( \varepsilon > 0 \) is a small number, and \( c_v \geq 0 \) is the density constant.

Lemma 14.3. If \( g \) is a primitive root modulo \( p \), then \( g + mp \) is a primitive root modulo \( p^2 \) for all \( m \in [0, p-1] \) but one exceptional value.

Proof. [46] Theorem 2.5

15 The Order Series \( \sum_{n \geq 2} 1/n \ord_n(v) \)

The Romanoff problem is concerned with the evaluation of the series \( \sum_{n \geq 2} 1/(n \ord(2)) \). This series occurs in the calculation of the density of the binary additive problem \( n = p + 2^k \). Much more general versions of this series are used in similar additive problems.

15.1 Order Series Over The Integers

Theorem 15.1. (43) Let \( f_v(n) = \ord_n(v) \). Then

(i) If \( \varepsilon > 0 \) is an arbitrary small number, then there is an absolute constant \( c_2 \) for which,
\[
\sum_{n \geq 2} \frac{1}{nf_v(n)\varepsilon} \leq e^\gamma (\log \log n + \varepsilon^{-1} + c_2).
\]  
(15.1)
(ii) If $\varepsilon > 0$ is an arbitrary small number, let $x \geq 2$, and let $v = 1 + \text{lcm}[1,2,\ldots, x]$, then

$$\sum_{n \geq 2} \frac{1}{nf_v(n)^\varepsilon} \geq e^\varepsilon \log v + O (\log \log \log v).$$

(15.2)

15.2 Order Series Over Subsets Of Integers

Let $q \geq 2$ be a prime power, and let $v \geq 1$ be a fixed integer. The asymptotic formulas for the restrictions to relatively primes subsets of integers $\mathcal{A} = \{n \geq 1 : \gcd(\text{ord}_n(v), q) = 1\}$ are considered in this section. These results are based on the counting function $A(x) = \{n \leq x : n \in \mathcal{A}\}$.

**Theorem 15.2.** ([38, Theorem 4]) For a prime power $q \geq 2$, and a large number $x \geq 1$, the counting function $A(x)$ has the asymptotic formula

$$\sum_{n \leq x} 1 = a(q,v) \frac{x}{\log c(q,v)} \left(1 + O_q \left(\frac{(\log \log x)^5}{(\log x)^c(q,v)+1}\right)\right),$$

(15.3)

where $a(q,v) > 0$ and $c(q,v) > 0$ are constants.

**Theorem 15.3.** For any prime power $q \geq 2$, and fixed integer let $v \geq 1$, the order series converges:

$$\sum_{n \leq x} \frac{1}{n \text{ord}_n(v)} < \infty.$$  

(15.4)

**Proof.** Let $\mathcal{A} = \{n \geq 1 : \gcd(\text{ord}_n(v), q) = 1\}$ and let $A(x) = \{n \leq x : n \in \mathcal{A}\}$ be the corresponding the counting function. The series has an integral representation as

$$\sum_{n \leq x} \frac{1}{n \text{ord}_n(v)} = \int_1^\infty \frac{1}{t \text{ord}_t(v)} dA(t).$$

(15.5)

Use the bounds of the order function $1/t < 1/\text{ord}_t(v) < 1/\log t$ and its derivatives

$$-\frac{1}{t^2} < \frac{d}{dt} \frac{1}{\text{ord}_t(v)} < -\frac{1}{t},$$

(15.6)

and Theorem [16.1] which gives $A(t) \ll x \log^{-c(q,v)} x$, to estimate the integral:

$$\int_1^\infty \frac{1}{t \text{ord}_t(v)} dA(t) = \frac{A(t)}{t \text{ord}_t(v)} \int_1^\infty \left(\frac{-1}{t^2 \text{ord}_t(v)} + \frac{1}{t \text{ord}_t(v)} \right) A(t) dt$$

$$\ll O \left(\frac{1}{(\log x)^{c(q,v)+1}}\right) + \int_1^\infty \left(\frac{1}{t^2 \log t} + \frac{1}{t \log t}\right) t \log c(q,v) dt$$

(15.7)

$$= O \left(\frac{1}{(\log x)^{c(q,v)}}\right),$$

where $c(q,v) > 1$ is a constant.  

$\blacksquare$
15.3 An Estimate For The Series $\sum_p \omega(p)$

The new result in Theorem 1.2 is used to sharpen the numerical evaluation of the series

$$\sum_{p \geq 2} \frac{1}{\omega(p)} \leq 0.9091 \ldots, \quad (15.8)$$

where $\omega(p) = \text{ord}_p(2)$. The above estimate was computed in [20]. Similar routines are used here too.

Lemma 15.1. Let $\omega(p) = \text{ord}_p(2)$. Then

$$\sum_{p \geq 2} \frac{1}{\omega(p)} \leq 0.811049529055567378261719 \ldots. \quad (15.9)$$

Proof. Start substituting the data

(i) $\text{ord}_p(2) = \text{ord}_p(2)$ if $2^{p-1} - 1 \equiv 0 \mod p^2$, and

(ii) $\text{ord}_p(2) = p \text{ord}_p(2)$ if $2^{p-1} - 1 \not\equiv 0 \mod p^2$,

into the series:

$$\sum_{p \geq 2} \frac{1}{\omega(p)} = \sum_{p \geq 2} \frac{1}{\omega(p)} + \sum_{p \geq 2} \frac{1}{\omega(p)}$$

$$= \sum_{p \geq 2} \frac{1}{\text{ord}_p(2)} + \sum_{p \geq 2} \frac{1}{p \text{ord}_p(2)}.$$  

Using $\text{ord}_p(2) \geq \log p/\log 2$, the upper bound $W_2(x) \leq 8 \log \log x$, see Theorem 1.2 and the numerical data in [20] and [13], set $x = 7 \times 10^{15}$, the first subseries reduces to

$$\sum_{p \geq 2} \frac{1}{\text{ord}_p(2)} = \sum_{p \leq 10^{15}} \frac{1}{\text{ord}_p(2)} + \sum_{p > 10^{15}} \frac{1}{p \text{ord}_p(2)}$$

$$\leq \frac{1}{\text{ord}_{10932}(2)} + \frac{1}{\text{ord}_{35121}(2)} + \sum_{p > 10^{15}} \frac{\log 2}{\log p}$$

$$\leq \frac{1}{364} + \frac{1}{1755} + \int_{10^{15}}^{\infty} \frac{\log 2}{\log t} dW_2(t)$$

$$\leq \frac{1}{364} + \frac{1}{1755} + \frac{8c \log 2 \log \log 10^{15}}{10^{15}}$$

$$\leq 0.2766564971799087434188077,$$  

where $0 < c \leq 10$ is a small constant. Fix a number $x = 10^4$, then the second subseries reduces to

$$\sum_{p \geq 2} \frac{1}{p \text{ord}_p(2)} = \sum_{p \leq x} \frac{1}{p \text{ord}_p(2)} + \sum_{p > x} \frac{1}{p \text{ord}_p(2)}$$

$$\leq 0.5343930318756586348429114 \ldots.$$  

(15.11)
where the lower tail is computed by a computer algebra system:

\[ \sum_{p \leq x} \frac{1}{p \ord_p(2)} = 0.3172457909240327210173469 \ldots, \quad (15.13) \]

and the upper tail is estimated by an integral approximation:

\[ \sum_{p > x} \frac{\log 2}{p \log p} \leq \frac{2}{\log x} = 0.2171472409516259138255645 \ldots, \quad (15.14) \]

For very large \( x \geq 1 \) the series is approximately

\[ \sum_{p \geq 2} \frac{1}{\omega(p)} \leq .5939022881039414644436155 + \frac{2}{\log x}. \quad (15.15) \]

Thus, the numerical value can be reduced to \( \sum_{p \geq 2} 1/\omega(p) \leq .624 \) by increasing \( x > 10^{50} \).

### 15.4 Problems

1. Given an arbitrary small number \( \varepsilon > 0 \), use the upper bound \( \ord_n(v) < n \) of the order modulo \( n \) to show that

\[ \sum_{n \geq \ell} n \ord_n(v)^\varepsilon \geq \zeta(1+) \prod_{p \mid v} \left( 1 - \frac{1}{p^{1+\varepsilon}} \right). \]

2. Evaluate the squarefree oprdr series

\[ \sum_{n \leq 2} \frac{\mu(n)^2}{n \ord_n(v)} \geq \frac{6e^\gamma}{\pi^2} \log \log v + O(1). \]

3. Evaluate the limit

\[ \lim_{m \to \infty} \sum_{n \geq 2} \frac{1}{n} = 0. \]

4. Evaluate the finite sum

\[ \sum_{m \leq x} \sum_{n \geq 2} \frac{1}{n} = a_v \log x + o(\log x), \]

where \( a_v \) is a constant.
16 NonWieferich Primes

Let \( \mathbb{P} = \{2, 3, 5, 7, \ldots\} \) be the set of prime numbers, and let \( v \geq 2 \) be a fixed integer. The subsets of Wieferich primes and nonWieferich primes are defined by

\[
\mathcal{W}_v = \{ p \in \mathbb{P} : v^{p-1} - 1 \equiv 0 \mod p^2 \}
\]

and

\[
\overline{\mathcal{W}}_v = \{ p \in \mathbb{P} : v^{p-1} - 1 \not\equiv 0 \mod p^2 \}
\]

respectively. The set of primes \( \mathbb{P} = \mathcal{W}_v \cup \overline{\mathcal{W}}_v \) is a disjoint union of these subsets.

The subsets \( \mathcal{W}_v \) and \( \overline{\mathcal{W}}_v \) have other descriptions by means of the orders \( \text{ord}_p(v) = d | p - 1 \) and \( \text{ord}_{p^2}(v) = pd \neq p \) of the base \( v \) in the cyclic group \( (\mathbb{Z}/p\mathbb{Z})^\times \) and \( (\mathbb{Z}/p^2\mathbb{Z})^\times \) respectively. The order is defined by \( \text{ord}_{n}(v) = \min\{m \geq 1 : v^m - 1 \equiv 0 \mod n\} \).

Specifically,

\[
\mathcal{W}_v = \{ p : \text{ord}_{p^2}(v) | p - 1 \}
\]

and

\[
\overline{\mathcal{W}}_v = \{ p : \text{ord}_{p^2}(v) \not| p - 1 \}.
\]

For a large number \( x \geq 1 \), the corresponding counting functions for the number of such primes up to \( x \) are defined by

\[
\mathcal{W}_v(x) = \# \{ p \leq x : \text{ord}_{p^2}(v) | p - 1 \}
\]

and

\[
\overline{\mathcal{W}}_v(x) = \pi(x) - \mathcal{W}_v(x),
\]

where \( \pi(x) = \# \{ p \leq x \} \) is the primes counting function, respectively.

Assuming the \( abc \) conjecture, several authors have proved that there are infinitely many nonWieferich primes, see [51], [19], et alii. These results have lower bounds of the form

\[
\overline{\mathcal{W}}_v(x) \gg \frac{\log x}{\log \log x}
\]

or slightly better. In addition, assuming the Erdos binary additive conjecture, there is a proof that the subset of nonWieferich primes has nonzero density. More precisely,

\[
\overline{\mathcal{W}}_v(x) \geq c \frac{x}{\log x}
\]

where \( c > 0 \) is a constant, see [20] Theorem 1] for the details.

16.1 Result For Nonzero Density

Here, it is shown that the subset of nonWieferich primes has density 1 in the set of primes unconditionally.

**Theorem 16.1.** Let \( v \geq 2 \) be a small base, and let \( x \geq 1 \) be a large number. Then, the number of nonWieferich primes has the asymptotic formula

\[
\overline{\mathcal{W}}_v(x) = \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right).
\]
Proof. The upper bound $W_v(x) \leq 4v \log \log x$, confer Theorem 1.2, is used below to derive a lower bound for the counting function $\overline{W}_v(x)$. This is as follows:

\[
\overline{W}_v(x) = \# \{p \leq x : \text{ord}_p(v) \nmid p - 1\} = \pi(x) - W_v(x) \geq \pi(x) - 4v \log \log x = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),
\]

where $\pi(x) = \#\{p \leq x\} = x/\log x + O(x/\log^2 x)$. ■

Corollary 16.1. Almost every odd integer is a sum of a squarefree number and a power of two.

Proof. Same as Theorem 1 in [20], but use the upper bound $W_v(x) \leq 4v \log \log x$. ■
17  Generalizations

The concept of Wieferich primes extends in many different directions, see [55], [26].

**Definition 17.1.** An integer \( n \geq 1 \) is a pseudoprime to base \( v \geq 2 \) if the congruence \( v^{n-1} \equiv 1 \mod n \) holds.

**Definition 17.2.** An integer \( n \geq 1 \) is a Carmichael number if the congruence \( v^{n-1} \equiv 1 \mod n \) holds for every \( v \) such that \( \gcd(v, n) = 1 \).

**Lemma 17.1.** Every Carmichael number is squarefree and satisfies the followings properties.

(i) \( n = q_1q_2\cdots q_t \) with \( q_1, q_2 < \cdots < q_t \) primes in increasing order.

(ii) \( q_i \leq \sqrt{n} \)

(iii) \( q - 1 \mid n - 1 \) for every prime divisor \( q \mid n \)

**Lemma 17.2.** (Cipolla) Let \( n \geq 3 \) be an integer, and let \( v \geq 2 \) be a fixed integer. Then, the congruence \( v^{n-1} \equiv 1 \mod n \) has infinitely many solutions \( n \geq 3 \).

A proof appears in [47, p. 125].

**Definition 17.3.** An integer \( n \geq 1 \) is a Wieferich pseudoprime to base \( v \geq 2 \) if the congruence \( v^{n-1} \equiv 1 \mod n^2 \) holds.

Some information on the calculation of the constants \( c_v \) for pseudoprimes is available in [57].
18 Counting Function For The Abel Numbers

The subset of integers $\mathcal{A}_v = \{ n : \text{ord}_{v^2}(n) \mid \lambda(n) \}$ associated with the base $v \geq 2$ congruence $v^{\lambda(n)} - 1 \equiv 0 \mod n^2$. These numbers will be referred to as Abel numbers to commemorate the earliest research on this topic, see [28], [48, p. 413]. The corresponding counting function is defined by

$$A_v(x) = \# \{ n \leq x : \text{ord}_{v^2}(n) \mid \varphi(n) \}.$$ (18.1)

The heuristic argument in [48, p. 413] is not conclusive, but claims something as

$$A_v(x) \approx \sum_{p \leq x} \frac{1}{p} \ll \log x.$$ (18.2)

18.1 Proof Of Theorem 18.1

Theorem 18.1. Let $v \geq 2$ be a base, and let $x \geq 1$ and $z \geq x$ be large numbers. Then, the number of Abel numbers in the short interval $[x, x + z]$ has the asymptotic formula

$$A_v(x + z) - A_v(x) = c_v (\log(x + z) - \log(x)) + E_v(x, z),$$ (18.3)

where $c_v \geq 0$ is the correction factor, and $E_v(x, z)$ is an error term.

Proof. Let $x \geq 1$ be a large number, and fix an integer $v \geq 2$. Consider the sum of the characteristic function for the fixed element $v$ of order $\text{ord}_{v^2}(n) \mid \varphi(n)$ over the integers in the short interval $[x, x + z]$. Then

$$A_v(x + z) - A_v(x) = \sum_{x \leq n \leq x + z} \Psi_0(n^2).$$ (18.4)

Replacing the characteristic function, see Lemma 4.6 and expanding the difference equation (18.4) yield

$$\sum_{x \leq n \leq x + z} \Psi_0(n^2) = \sum_{x \leq n \leq x + z} \sum_{d \mid \lambda(n), \text{gcd}(r, \lambda(n)/d) = 1} \frac{1}{\varphi(n^2)} \sum_{0 \leq m < \varphi(n^2)} e^{\frac{2\pi i (x^{v^2} - v^2) n}{\varphi(n^2)}}$$

$$= \sum_{x \leq n \leq x + z} \sum_{d \mid \lambda(n), \text{gcd}(r, \lambda(n)/d) = 1} \frac{1}{\varphi(n^2)} \sum_{1 \leq m < \varphi(n^2)} e^{\frac{2\pi i (x^{v^2} - v^2) n}{\varphi(n^2)}}$$

$$= M_v(x, z) + E_v(x, z).$$ (18.5)

The main term $M_v(x, z)$ is determined by the index $m = 0$, and the error term $E_v(x, z)$ is determined by the range $1 \leq m < \varphi(n^2)$. Applying Lemma 5.5 to the main term and applying Lemma 8.5 to the error term yield

$$M_v(x, z) + E_v(x, z)$$

$$= a_v (\log(x + z) - \log(x)) + O \left( \frac{1}{\log x} \right) + O \left( \frac{z^{1/2}}{x^{1/2} \log x} \right).$$ (18.6)
Next, assuming that \( z = O(x) \) it reduces to

\[
A_v(x + z) - A_v(x) = a_v \left( \log(x + z) - \log(x) \right) + O \left( \frac{1}{\log x} \right), 
\]

(18.7)

where \( a_v \geq 0 \) is the density constant.

The specific constant \( a_v \geq 0 \) for a given fixed base \( v \geq 2 \) is a problem in algebraic number theory, see Theorem 10.1 for some details.
References

[1] Agoh, Takashi; Dilcher, Karl; Skula, Ladislav Wilson quotients for composite moduli. Math. Comp. 67 (1998), no. 222, 843-861.

[2] Ambrose, Christopher Daniel. On Artin’s primitive root conjecture. Doctoral thesis, 2014.

[3] Apostol, Tom M. Introduction to analytic number theory. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[4] Balog, Antal; Cojocaru, Alina-Carmen; David, Chantal. Average twin prime conjecture for elliptic curves. Amer. J. Math. 133 (2011), no. 5, 1179-1229.

[5] Julio Brau, Character sums for elliptic curve densities. arXiv:1703.04154.

[6] W.D. Banks, F. Luca, F. Saidak and P. Stanica. Compositions with the Euler and Carmichael Functions, Abh. Math. Sem. Univ. Hamburg., 75 (2005), 215-244.

[7] Bilu, Yu.; Hanrot, G.; Voutier, P. M. Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte. J. Reine Angew. Math. 539 (2001), 75-122.

[8] Crandall, Richard; Dilcher, Karl; Pomerance, Carl. A search for Wieferich and Wilson primes. Math. Comp. 66 (1997), no. 217, 433-449.

[9] Crandall, Richard; Pomerance, Carl. Prime numbers. A computational perspective. Second edition. Springer, New York, 2005.

[10] Peter J. Cameron and D. A. Preece, Notes on primitive lambda-roots, http://www.maths.qmul.ac.uk/pjccsgnoteslambda.pdf.

[11] Carmichael, R. D.; On Euler’s $\phi$-function. Bull. Amer. Math. Soc. 13 (1907), no. 5, 241-243.

[12] Carmichael, R. D. Note on a new number theory function. Bull. Amer. Math. Soc. 16 (1910), no. 5, 232-238.

[13] Dorais, Francois G.; Klyve, Dominic. A Wieferich prime search up to $6.7 \times 10^{15}$. J. Integer Seq. 14 (2011), no. 9, Article 11.9.2, 14 pp.

[14] DeKoninck, J.-M.; Doyon, N. On the set of Wieferich primes and of its complement. Ann. Univ. Sci. Budapest. Sect. Comput. 27 (2007), 3-13.

[15] Dusart, Pierre Inegalites explicites pour $\psi(x)$, $\theta(x)$, and $\pi(x)$ et les nombres premiers. C. R. Math. Acad. Sci. Soc. R. Can. 21 (1999), no. 2, 53-59.

[16] Erdos, Paul; Pomerance, Carl; Schmutz, Eric. Carmichael’s lambda function. Acta Arith. 58 (1991), no. 4, 363-385.

[17] Everest, Graham; van der Poorten, Alf; Shparlinski, Igor; Ward, Thomas. Recurrence sequences. Mathematical Surveys and Monographs, 104. American Mathematical Society, Providence, RI, 2003.
[18] Gauss, Carl Friedrich. *Disquisitiones arithmeticae*. Translated by Arthur A. Clarke. Revised by William C. Waterhouse, Cornelius Greither and A. W. Grootendorst. Springer-Verlag, New York, 1986.

[19] H. Graves, M. Ram Murty, The abc conjecture and non-Wieferich primes in arithmetic progressions, J. Number Theory 133 (2013) 1809-1813.

[20] Granville, Andrew; Soundararajan, K. A binary additive problem of Erdos and the order of 2 mod $p^2$. Ramanujan J. 2 (1998), no. 1-2, 283-298.

[21] Hinz, Jurgen G. Character sums and primitive roots in algebraic number fields. Monatsch. Math. 95 (1986), no. 4, 275-286.

[22] C. Hooley, On Artins conjecture, J. Reine Angew. Math. 225, 209-220, 1967.

[23] Janusz, Gerald J. *Algebraic number fields*. Pure and Applied Mathematics, Vol. 55. Academic Press, New York-London, 1973.

[24] Johnsen, John. On the distribution of powers in finite fields. J. Reine Angew. Math. 251 1971 10-19.

[25] Klyve, Dominic Explicit bounds on twin primes and Brun’s Constant. Thesis (Ph.D.)-Dartmouth College. 2007.

[26] Katz, Nicholas M. Wieferich past and future. Topics in finite fields, 253-270, Contemp. Math., 632, Amer. Math. Soc., Providence, RI, 2015.

[27] Srinivas Kotyada, Subramani Muthukrishnan, Non-Wieferich primes in number fields and ABC conjecture, [arXiv:1610.00488](https://arxiv.org/abs/1610.00488). Computers in Mathematical Research, North-Holland, 1968, pp. 84-88.

[28] Wilfrid Keller, Jorg Richstein, Solutions of the congruence $a^{p-1} \equiv 1 \mod p^r$, Math. Comp. 74 (2005), 927-936.

[29] Joshua Knauer, and Jorg Richstein, The continuing search for Wieferich primes, Math. Comp. 74 (2005), 1559-1563.

[30] James, Kevin; Smith, Ethan. Average Frobenius distribution for the degree two primes of a number field. Math. Proc. Cambridge Philos. Soc. 154 (2013), no. 3, 499-525.

[31] Lehmer, Emma On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson. Ann. of Math. (2) 39 (1938), no. 2, 350-360.

[32] Lagarias, J. C. Errata to: The set of primes dividing the Lucas numbers has density 2/3, Pacific J. Math. 118 (1985), no. 2, 449-461; MR0789184. Pacific J. Math. 162 (1994), no. 2, 393-396.

[33] Lidl, Rudolf; Niederreiter, Harald. *Finite fields*. With a foreword by P. M. Cohn. Second edition. Encyclopedia of Mathematics and its Applications, 20. Cambridge University Press, Cambridge, 1997.

[34] Lenstra, H. W., Jr.; Stevenhagen, P.; Moree, P. Character sums for primitive root densities. Math. Proc. Cambridge Philos. Soc. 157 (2014), no. 3, 489-511.
[35] Languasco, Alessandro; Zaccagnini, Alessandro. Computing the Mertens and Meissel-Mertens constants for sums over arithmetic progressions. Experiment. Math. 19 (2010), no. 3, 279-284.

[36] Mordell, L. J. On the exponential sum $\sum_{1 \leq x \leq X} \exp\left(2\pi i (ax + bg^x)/p\right)$. Mathematika 19 (1972), 84-87.

[37] Moreno, Carlos Julio. Advanced analytic number theory: $L$-functions. Mathematical Surveys and Monographs, 115. American Mathematical Society, Providence, RI, 2005.

[38] Muller, Helmut On the distribution of the orders of $2$ (mod $u$) for odd $u$. Arch. Math. (Basel) 84 (2005), no. 5, 412-420.

[39] Mihailescu, Preda. A class number free criterion for Catalan’s conjecture. J. Number Theory 99 (2003), no. 2, 225-231.

[40] Greg Martin, Carl Pomerance. The iterated Carmichael $\lambda$-function and the number of cycles of the power generator. arXiv:math/0406335.

[41] Montgomery, Peter L. New solutions of $a^{p-1} \equiv 1 \pmod{p^2}$. Math. Comp. 61 (1993), no. 203, 361-363.

[42] Muller, Thomas W.; Schlage-Puchta, Jan-Christoph On the number of primitive $\lambda$-roots. Acta Arith. 115 (2004), no. 3, 217-223.

[43] Murty, M. Ram; Rosen, Michael; Silverman, Joseph H. Variations on a theme of Romanoff. Internat. J. Math. 7 (1996), no. 3, 373-391.

[44] Paszkiewicz, A. A new prime $p$ for which the least primitive root mod $p$ and the least primitive root mod $p^2$ are not equal. Math. Comp. 78 (2009), no. 266, 1193-1195.

[45] Palenstijn, Willem Jan. Radicals in Arithmetic, Leiden University dissertation, 2014.

[46] Redmond, Don. Number theory. An introduction. Monographs and Textbooks in Pure and Applied Mathematics, 201. Marcel Dekker, Inc., New York, 1996.

[47] Ribenboim, Paulo, The new book of prime number records, Berlin, New York: Springer-Verlag, 1996.

[48] Rosser, J. Barkley; Schoenfeld, Lowell Approximate formulas for some functions of prime numbers. Illinois J. Math. 6 1962 64-94.

[49] Sebah, Pascal; Gourdon, Xavier. "Introduction to twin primes and Brun’s constant computation", Preprint 2002.

[50] J. H. Silverman, Wieferich criterion and the abc-conjecture, J. Number Theory 30 (1988) no. 2, 226-237.

[51] Stevenhagen, Peter. The correction factor in Artin’s primitive root conjecture. Les XXII èmes Journees Arithmetiques (Lille, 2001). J. Theor. Nombres Bordeaux 15 (2003), no. 1, 383-391.
[53] Stephens, P. J. An average result for Artin conjecture. Mathematika 16, (1969), 178-188.

[54] Vaughan, R. C. Some applications of Montgomery’s sieve. J. Number Theory 5 (1973), 64-79.

[55] Voloch, Jose Felipe. Elliptic Wieferich primes. J. Number Theory 81 (2000), no. 2, 205-209.

[56] A. Wieferich, Zum letzten Fermatschen Theorem, J. Reine Angew. Math. 136 (1909), 293-302.

[57] Wagstaff, Samuel S., Jr. Pseudoprimes and a generalization of Artin’s conjecture. Acta Arith. 41 (1982), no. 2, 141-150.