THE C = 1 MATRIX MODEL FORMULATION
OF TWO DIMENSIONAL YANG-MILLS THEORIES

Stefano Panzeri

Istituto Nazionale di Fisica Nucleare, Sezione di Torino
Dipartimento di Fisica Teorica dell’Università di Torino
via P.Giuria 1, I-10125 Torino, Italy

Abstract

We find the exact matrix model description of two dimensional Yang-Mills theories on a cylinder or on a torus and with an arbitrary compact gauge group. This matrix model is the singlet sector of a $c = 1$ matrix model where the matrix field is in the fundamental representation of the gauge group. We also prove that the basic constituents of the theory are Sutherland fermions in the zero coupling limit, and this leads to an interesting connection between two dimensional gauge theories and one dimensional integrable systems. In particular we derive for all the classical groups the exact grand canonical partition function of the free fermion system corresponding to a two dimensional gauge theory on a torus.
1 Introduction

Due to its property of exact solvability \( [1] \), two dimensional QCD (QCD2) may be a starting point for the analytical investigation of the properties of the confined phase of four dimensional QCD. Recently, some progress in this direction was done by Gross and Taylor \( [2] \). In fact they showed that QCD2 can be interpreted, in the large \( N \) limit, as a closed string theory, by proving that all the coefficients of the expansion of the partition function in powers of \( 1/N \) have a geometrical interpretation in terms of maps of an orientable string worldsheet onto a two dimensional target space.

Later, the nature of the string theory underlying QCD2 was further clarified by finding the explicit matrix model description of QCD2, at least on the cylinder and on the torus \( [3, 4, 5] \). In \( [3, 4] \) it was indeed shown that the hamiltonian for this theory in the large \( N \) limit essentially coincides with the one found by Das and Jevicki \( [6] \) for the \( c = 1 \) matrix model, the only difference being that the corresponding matrix field is now a unitary matrix field rather than an Hermitian one, while in \( [5] \) it was found that QCD2 on the torus and on the cylinder is, even for finite \( N \), exactly equivalent to a one dimensional matrix model of type proposed in \( [7] \) by Kazakov and Migdal (KM), where now the eigenvalues of the matter fields live on a circle rather than on a line.

It is very interesting to understand whether two dimensional Yang-Mills theories with an arbitrary compact gauge group admit a similar closed string theory formulation. In \( [8] \) the large \( N \) expansion of the partition function of YM2 with gauge group \( SO(N) \) or \( Sp(2N) \) was studied, and a string theory description similar to that of \( [2] \) was obtained, the main difference being that the worldsheet of the string may be nonorientable. In this letter we proceed in a different direction, by proving that YM2 with an arbitrary compact gauge group on a cylinder and on a torus are exactly the singlet sector of a \( c = 1 \) matrix model whose matrix field is in the fundamental representation of the gauge group and is precisely the path-ordered integral of the gauge field around the compactified dimension. This result is obtained both in the continuum and in the lattice formulation.

Further, it is discussed the free fermion content of the theory, by showing that YM2 can be viewed as the zero coupling limit of the Sutherland integrable systems \( [9] \). This free fermion interpretation is the physical background for a Das-Jevicki matrix model realization. Finally, we remark that we obtain for the partition function on the torus (which corresponds to finite temperature Sutherland systems) a new interesting expression in terms of Jacobi theta functions, whose behaviour under modular transformations is well known. The significance of the modular inversion in the context of Sutherland systems is also clarified.

All the results are presented for simple compact Lie groups. The generalization to an arbitrary compact group is easily done by taking semidirect products.

The paper is organized as follows: in section 2 we derive the matrix model action for YM2 and clarify its connection with the modular inversion for the kernel on the cylinder; in section 3 we fully exploit the fermion content of the theory and...
obtain new expressions for the partition functions on the torus; in section 4 we present our conclusions.

2 The matrix model description of YM2

It is well known by now that two dimensional Yang-Mills theories defined on a manifold $M$ of genus $p$ and with a metric $g_{\mu\nu}$ are exactly solvable. The partition function is given by

$$Z_M(A) = \int D\!A e^{-\frac{1}{g^2} \int_M d^2 x \sqrt{g} F_{\mu\nu} F^{\mu\nu}} = \sum_R d_R^2 - 2p - n d_R e^{-\frac{1}{2} A \tilde{g}^2 C_2(R)},$$

(1)

where the sum is over all equivalence classes of irreducible representations $R$ of the gauge group $G$, $d_R$ is their dimension and $C_2(R)$ is the quadratic Casimir in the representation $R$. Similarly, the heat kernel defined by a surface of genus $p$ and $n$ boundaries is given by

$$K_{p,n}(g_1, \ldots, g_n; N, A) = \sum_R d_R^2 - 2p - n d_R \chi_R(g_1) \cdots \chi_R(g_n) e^{-\frac{1}{2} A \tilde{g}^2 C_2(R)},$$

(2)

where $g_i$ are the Wilson loops evaluated along the boundaries, and $\chi_R$ denotes the Weyl character of the representation $R$. For dimensional reasons, and because of the invariance of the action under area preserving diffeomorphisms, eqs. (1) and (2) depend only on the variable $\tilde{g}^2 A$. We will henceforth denote this variable by $t$.

Since in this paper we are dealing only with Kernels defined on $p = 0$ surfaces, from now on we will denote simply $K_{0,n}$ by $K_n$. The heat kernels in (2) are expressed in terms of exponentials of $g^2$, but in order to formulate two dimensional YM theories on a general simple Lie group as a matrix model, we need to construct, as pointed out in [5], a similar representation for the kernel defined by the cylinder ($K_{0,2}$) in terms of exponentials of $1/g^2$. This representation was first derived by Altschuler and Itzykson (AI) [10] in the context of the study of multimatrix models by using only the algebraic properties of characters. It may be very useful to give a brief account of this proof, as follows.

First we have to fix some notations. Let $G$ be a compact simple Lie group (of rank $r$) and $\mathcal{G}$ its Lie algebra. Let $\mathcal{G}^C$ be the complexification of $\mathcal{G}$. Let $\langle , \rangle$ be an invariant form on $\mathcal{G}^C$ which is positive definite on $i\mathcal{G}$. Choose a Cartan subalgebra $\mathcal{H}^C$ of $\mathcal{G}^C$, let $\mathcal{H} = \mathcal{H}^C \cap \mathcal{G}$, and choose a set of positive roots $\Sigma_+ \subset i\mathcal{H}$. We identify $\mathcal{H}^C$ with its dual by means of $\langle , \rangle$. To each positive root $\alpha$ we associate the corresponding coroot $\hat{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$. The coroot lattice $\hat{Q}$ is the lattice generated by the coroots. Let $P$ be the weight lattice, which is the dual of $\hat{Q}$.

The analog of the Vandermonde determinant for the Hermitian matrices, for $h \in i\mathcal{H}^C$ is the polynomial

$$\Delta(h) = \prod_{\alpha \in \Sigma_+} \langle \alpha, h \rangle$$

(3)
It is the infinitesimal version of the Weyl's denominator,

\[ \sigma(h) = e^{i\langle \rho, h \rangle} \prod_{\alpha \in \Sigma_+} \left( 1 - e^{-i\langle \alpha, h \rangle} \right) = \prod_{\alpha \in \Sigma_+} 2i \sin \left( \frac{\langle \alpha, h \rangle}{2} \right) \]  

(4)

where \( \rho \) denotes

\[ \rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha \]  

(5)

The set of the highest weights of all the irreducible unitary representations of \( G \) (irreps in the following) is

\[ P_+ = P \cap \{ x \in iH | \langle x, \alpha \rangle \geq 0, \alpha \in \Sigma_+ \} \]  

(6)

Denote by \( \chi_\lambda(g) \) and \( d_\lambda \) the character and the dimension of the irrep corresponding to \( \lambda \in P_+ \). Then the heat kernel on the disk is given by

\[ K_1(g, t) = \sum_{\lambda \in P_+} d_\lambda \chi_\lambda(g) e^{-C_2(\lambda)t/2} \]  

(7)

where the quadratic Casimir \( C_2(\lambda) \) has the following expression:

\[ C_2(\lambda) = |\lambda + \rho|^2 - |\rho|^2 \]  

(8)

At this point we have to compute the kernel on the cylinder:

\[ K_2(g_1, g_2^{-1}, t) = \int_G d^G K_1(g_1gg_2^{-1}g_2^{-1}, t) = \sum_{\lambda \in P_+} \chi_\lambda(g_1) \chi_\lambda(g_2^{-1}) e^{-C_2(\lambda)t/2} \]  

(9)

where \( d^G \) denotes the normalized Haar measure on \( G \) and the latter expression is a consequence of the relation:

\[ \int d^G \chi_\lambda(xgyg^{-1}) = d_\lambda^{-1} \chi_\lambda(x) \chi_\lambda(y) \]  

(10)

The definition of the Weyl character \( \chi_\lambda \) is the following:

\[ \chi_\lambda(g) = \frac{\nu_{\lambda+\rho}(\phi)}{\sigma(\phi)} \]  

(11)

where \( \phi_i \) (\( i = 1, \ldots, r \)) are the invariant angles of \( g \) (\( g = e^{i\phi}; \ i\phi \in H \)) and

\[ \nu_\lambda(\phi) = \sum_{w \in W} \epsilon(w) e^{i\langle w(\lambda), \phi \rangle} \]  

(12)

where \( W \) is the Weyl group. In order to express (4) in terms of theta functions, we have to rewrite the sum over the irreps in the form of an unconstrained sum over the whole weight lattice. This is easily done by substituting in (3) the Weyl’s
formula (11) and by noticing that \( \nu_\lambda(\phi) = 0 \) if the stabilizer of \( \lambda \) in \( W \) is non-trivial. The result is the following:

\[
\sum_{\lambda \in \mathcal{P}^+} e^{-C_2(\lambda) t / 2 \nu_{\lambda+\rho}(\phi) \nu_{\lambda+\rho}(\theta)} = e^{i |\theta|^2 / 2} \sum_{w \in W} \epsilon(w) \sum_{\lambda \in \mathcal{P}} e^{i (\lambda, \phi - w(\theta))} e^{-|\lambda|^2 t / 2}
\]

(13)

where \( \epsilon(w) = (-1)^{l(w)} \), \( l(w) \) = length of \( w \) expressed as a product of Weyl’s reflections and we denote by \( \phi \) [resp. \( \theta \)] the invariant angles of \( g_1 \) [resp. \( g_2 \)]. By means of the Poisson summation formula we arrive finally at (apart from some irrelevant multiplicative factor):

\[
K_2(g_1, g_2^{-1}, t) \equiv K_2(\phi, \theta, t) = e^{|\rho|^2 t / 2} \left( 2\pi / t \right)^{\sigma(\phi) / 2} \delta(W(0), g_1) \delta(W(2\pi), g_2^{-1}) \psi(g_1) \psi(g_2^{-1}).
\]

(14)

Recently, the formula (14) was derived, for the \( SU(N) \) and \( U(N) \) groups, in the context of the study of QCD2 [5]. The main advantage of the derivation [5] is that it allows to prove that QCD2 on a cylinder and on a torus is described by a matrix model which is exactly a one dimensional KM model. The formula (14) then arises by diagonalizing the matrix model and by fixing the boundary conditions. So, in order to find this \( c = 1 \) matrix model description also for the general case of a compact simple Lie group, it is needed to extend the procedure worked out in [5] to the general case.

Recovering the matrix model action describing YM2 on a cylinder, is a trivial extension of the calculation done in [5] for QCD2: therefore we will outline only the main ideas of the procedure [5]. By working in the first order formalism, it is convenient to fix the gauge \( \partial_0 A_0 = 0 \); then all the non static modes of the Fourier expansion (in the time coordinate) of the fields can be integrated away and one obtains the following expression for \( K_c \):

\[
K_c(g_1, g_2^{-1}, t) = \int \mathcal{D}B \mathcal{D}A e^{-\frac{i}{2} \text{Tr} \int_0^{2\pi} dx [\partial B - i[A,B]]^2} \times \\
\times \delta(W(0), g_1) \delta(W(2\pi), g_2^{-1}) \psi(g_1) \psi(g_2^{-1}).
\]

(15)

where \( B(x) \) and \( A(x) \) (matrix fields on the algebra) denote the static modes of the \( A_0(x, \tau) \) and \( A_1(x, \tau) \) gauge fields respectively, \( W(x) \) denotes the path-ordered integral of the gauge field around the compactified direction:

\[
W(x) = \mathcal{P} e^{i \int_0^{2\pi} d\tau A_0(x, \tau)},
\]

(16)

1The coordinates on the cylinder are denoted by \( x, \tau \). Notice that we choose the time direction (with coordinate \( \tau \)) as the compactified one. This notation is opposite to that of [3, 4], but is more convenient for the study of finite temperature lattice models considered below.
and, in order to fix (resp. to $g_1$ and $g_2^{-1}$) the values of $W$ at the two boundaries of the cylinder in a gauge invariant manner, we have introduced delta function acting on the space of conjugation invariant functions (class functions), denoted by $\hat{\delta}(g, h)$ and defined by

$$\hat{\delta}(g, h) = \int dU \delta(UgU^{-1}h).$$

(17)

The factors $\psi(g_1)$ and $\psi(g_2)$ are just normalization factors; they depend only on the eigenvalues of $g_1$ and $g_2$ and they will be chosen so that one obtains the partition function on a torus by identifying the two boundaries of the cylinder and by a group integration over the boundary conditions. It is also important to remark that, since the matrix field is $W(x)$, the compactified time direction disappears from this description and we have an exact dimensional reduction.

A less trivial step is the diagonalization of the model (15) and the recovering of the AI formulas (13,14). In order to do this, one has to write, for a general compact simple group, the conjugation invariant delta functions in (15) in terms of periodic delta functions of the invariant angles of the boundary conditions. In order to obtain the latter expression, we note that by substituting in (17) the character expansion for the delta functions and by using the identity (10) one obtains

$$\int dg \delta(g^ie^{i\phi}g^{-1}e^{-i\theta}) = \sum_{\lambda \in P_+} \chi_\lambda(e^{i\phi})\chi_\lambda(e^{-i\theta})$$

(18)

By writing explicitly the Weyl’s character formula (11) and by using the formula (13) evaluated in $t = 0$, one finds finally the following expression:

$$\int dg \delta(g^ie^{i\phi}g^{-1}e^{-i\theta}) = \sum_{\lambda \in P_+} \chi_\lambda(e^{i\phi})\chi_\lambda(e^{-i\theta})$$

(18)

(19)

where the latter relation is valid up to terms which are vanishing when one performs the integral over $\phi, \theta$. At this point, by means of techniques which are by now standard in matrix models, the matrix $B(x)$ can be diagonalized, the functional integral over its eigenvalues performed and, with the help of the expression (19), one obtains the Altschuler-Itzykson formula (14). In retrieving (14) the normalization factor $\psi$ has also been determined. It is given by $\psi(g) = \sigma(g)/\Delta(g)$.

Let’s now clarify the string theory description arising from our results: the action in (15) is a KM model in one continuous dimension (the spatial dimension of the cylinder), where the role of KM gauge field [resp. KM matter field] is played by $A(x)$ [resp. $B(x)$]. The main difference between (15) and a KM model of the ordinary type is that the boundary conditions are now depending on $e^{2\pi i B}$ rather than $B$. Therefore the string theory interpretation of (15) is straightforward,
although slightly different if we pass from the cylinder to the torus. Let us consider first YM2 on a torus: here the target space of the matrix model \( [13] \) is a circle and thus, exactly as the KM model on a circle describes the singlet (vortex free) sector of the \( c = 1 \) compactified hermitian matrix model \([11, 12]\), YM2 on a torus turns out to be the singlet sector of a matrix model on a circle whose matrix field is defined on the group rather than on the algebra. If YM2 lives on a cylinder, the target space of the corresponding matrix model is a line. In this case the KM gauge field can be gauged away and we have an ordinary \( G \)-matrix model on a line (also in the singlet sector since the boundary conditions are imposed in a gauge invariant manner at the boundaries of this target space).

This is in complete agreement with the results obtained for \( U(N) \) and \( SU(N) \) by Minahan and Polychronakos \([3]\): they found, by choosing the gauge \( A_1(x, \tau) = 0 \), that QCD2 on a cylinder is equivalent to the singlet sector of a \( c = 1 \) unitary matrix model, where the unitary matrix field is just \( W = e^{2\pi i B(x)} \). However, their derivation is not valid for the case of the torus, where it is not possible to fix the above mentioned gauge.

Finally, we want to show that the matrix-model description of YM2 simply arises also in the lattice formulation. To see this, first consider a \( d \)-dimensional \( G \)-matrix model with continuum action

\[
S = -\beta \text{ Tr} \int d^d x DU(x)D^{-1}(x)
\]  

(20)

where \( U \) is a matrix field in the fundamental representation of \( G \). The \( D \) symbol denotes a covariant derivative operator and thus this model is an extension of the KM one, where now the local degrees of freedom are on the group rather than on the algebra. For \( d = 1 \), as discussed before, (20) is the matrix model formulation of YM2. In order to recover the same result on the lattice, let’s write the discretization of (20):

\[
S = -\beta_0 \text{ Tr} \sum_x \sum_{\mu=1}^d \left( U_x - V_{x,\mu} U_{x+\mu} V_{x,\mu}^{-1} \right) \left( U_x^{-1} - V_{x,\mu} U_{x+\mu}^{-1} V_{x,\mu}^{-1} \right)
\]

\[
= \beta_0 \sum_{x,\mu} \text{ Re} \text{ Tr} \left( U_x V_{x,\mu} U_{x+\mu} V_{x,\mu}^{-1} - 2 \right)
\]  

(21)

where \( \beta_0 = \beta/a^{3-d} \) and \( a \) is the lattice spacing. The KM matter fields on the sites \( x \) of a regular hypercubic lattice are now \( U_x \), while the angular variables on the links are \( V_{x,\mu} \). We shall call this lattice field theory the KM G-matrix model.

It is interesting to note that, by thinking \( U_x \) as a link matrix along an extra compactified timelike direction, (21) is just the Wilson action for the \( d + 1 \)-dimensional YM theory at finite temperature, where we are keeping only the timelike plaquettes. In the timelike direction we have actually only one site, but, at least in the large \( \beta \) limit, this is essentially irrelevant \([13]\). As further discussed in \([13]\), this model should capture most of the features of the infinite temperature limit of
Yang-Mills theories. In this context, for \( d > 1 \), \( \beta \) plays the role of the coupling for timelike plaquettes, and is proportional to the temperature.

At this point the diagonalization procedure is very similar to that of the KM model, and one remains only with the invariant angles \( \phi_x \) of the \( U(x) \) matrices on the sites. The analog of the Itzykson-Zuber formula for the integration over angular link variables is

\[
I(\phi_x, \phi_{x+\mu}) = \int_G D V_{x,\mu} \exp \left[ \beta_0 \text{Re} \text{Tr} U_{x,\mu} V_{x+\mu} V_{x+\mu}^{-1} \right] = \sum_R \lambda_R(\beta_0) \chi_R(e^{i\phi_x}) \chi_R(e^{-i\phi_{x+\mu}})
\]

(22)

where we used the character expansion of Wilson action. The coefficients of this expansion are:

\[
\lambda_R(\beta_0) \equiv d_R^{-1} \int_G dU \chi_R(U) e^{\beta_0 \text{Tr}(U + U^{-1})}
\]

(23)

and have the following asymptotic behaviour as \( \beta_0 \to \infty \):

\[
\lambda_R(\beta_0) \sim 1 - C_2(R)/2\beta_0 + O(1/\beta_0^2)
\]

(24)

Here we are interested only to the \( d = 1 \) model, since we want to show that it is equivalent to YM2 on a cylinder. In the \( d = 1 \) case, no approximation is needed in keeping only the timelike plaquettes in the Wilson action for YM theories (there is no spacelike plaquette contribution) and the continuous limit (\( \beta_0 \to \infty \)) is well defined. So, let’s consider in detail the discretization of YM2 with Wilson action on a cylinder, and denote with \( N_0 \) the number of sites in the compactified time direction and with \( N_1 \) the number of sites in the space direction. The boundary conditions are assigned by fixing in a conjugation invariant manner the value of the Polyakov loop which winds around the compactified direction. We can actually, by using the orthogonality properties of characters, reduce ourself to the \( N_0 = 1 \) case (This leads, in the continuous limit, only to a rescaling of the coupling constant \( \beta \to \beta/N_0 \)). At this point we remain with a KM G-matrix model on a one dimensional lattice with \( N_1 \) sites, which is thus equivalent in the continuous limit to YM2 on the cylinder. In the \( \beta_0 \to \infty \) limit, it is also easy to obtain from (22, 23, 24) the usual continuum expression for the kernel on the cylinder.

Notice that the G-matrices on the sites are the lattice counterparts of the path-ordered integrals of the gauge field around the time direction. The extension to the case of the torus is straightforward, the only difference being that the KM G-matrix model is defined on a circle.

### 3 The free fermion content of YM2

In this section we show that, as expected from the matrix model interpretation \[14\], YM2 can be viewed as a quantum theory of free fermions living on a circle. More precisely, it is the zero coupling limit of the Sutherland integrable systems \[9\].

\footnote{This correspondence was first noticed, in a different context, in \[13\].}
One can start by noticing, in analogy with the well known analysis for Hermitian $c = 1$ matrix model [17], that the quantization of (15) is equivalent to the singlet sector of G-matrix quantum mechanics. Thus the hamiltonian written in terms of the invariant angles on G is:

$$H = -\Delta_G \equiv -\sigma^{-1}(\phi) \left[ \sum_i \frac{\partial^2}{\partial \phi_i^2} + |\rho|^2 \right] \sigma(\phi) \tag{25}$$

where $\Delta_G$ is the “radial” part of the Laplace operator on the group manifold [18] and $\rho$ is defined in (5). This is a consequence of the fact that the kernels (2) of YM2, for example the kernel on the disk $K_1$, are solutions of the heat equation:

$$\left( \frac{\partial}{\partial t} + \frac{1}{2} \Delta_G \right) K_1(\phi, t) = 0 \tag{26}$$

After redefining the kernel by $K_1 \rightarrow \sigma(\phi)K_1(\phi, t) \equiv \hat{K}_1(\phi, t)$, the eq. (26) becomes the (euclidean) free Schrödinger equation for $r$ fermions on a circle. It is important to notice that the boundary conditions are determined by this redefinition, and are just the same boundary conditions for the fermions in the zero coupling limit of the Sutherland integrable model related to the Lie algebra G [19].

As a consequence one can rewrite all the quantities of YM2 in the language of the Sutherland model. In fact, the invariant angles $\phi_i$ on the gauge group G are the coordinates of the fermions; the eigenvalues of the hamiltonian are $C_2(\lambda)$ (the spectrum is discrete because of the compactness of the configuration space); the eigenfunctions corresponding to $C_2(\lambda)$ are the class functions $\psi(\phi) = \psi_0(\phi)\chi_\lambda(\phi)$, where $\psi_0(\phi) \equiv \sigma(\phi)$ is the ground state eigenfunction. The transition amplitude between the configuration $\phi_i = 0$ at $t = 0$ and the configuration $\phi_i$ at time $t$ is proportional to the kernel on the disk $K_1$, and it is given by $\hat{K}_1$. Similarly the propagator from the configuration $\phi_i$ at $t = 0$ to the configuration $\theta_i$ at time $t$ is essentially the kernel on the cylinder:

$$\hat{K}_2(\phi, \theta, t) = \sigma(\phi)\sigma(-\theta)K_2(\phi, \theta, t) \tag{27}$$

It is now clear the meaning, in the Sutherland system language, of the modular inversion leading to the expression (14) for $K_2$: the usual character expansion (13) corresponds to the momentum representation for the Sutherland propagator (in fact the elements of the weight lattice labels the momentum eigenvalues), while the inverted expression (14) is a gaussian in the invariant angles and thus corresponds to the coordinate representation for the fermion propagator (the integers numbers which label the coroot lattice in (14) have to be interpreted as winding numbers). In [16] it is shown that the Sutherland model can be related to the Laplace operator $\Delta_G$ also for a set of nonzero values of the coupling constant. The formula (14) provides, similarly as the noninteracting case, a modular inversion for the one and two point functions also in this cases.
Even more interesting is the interpretation of the YM2 partition function on a \( p = 1 \) surface, obtained by simply sewing together the two ends of the cylinder, according to (28):

\[
Z_{p=1}(t) = \int dg K_2(g, g^{-1}, t) = \int_{0}^{2\pi} \prod_{i=1}^{r} d\phi_i \sum_{w \in W} c(w) \sum_{\lambda \in P} e^{i(\lambda, \phi - w(\phi))} e^{-|\lambda|^2 t/2}
\]

where with \( \phi_i \) we denote the invariant angles of \( g \). Since we are now fixing periodic boundary conditions in the euclidean time direction, the partition function on the torus (28) is just the finite temperature version of the corresponding Sutherland model, the inverse temperature being proportional to \( t = g^2 A \). Here we want to show that, as suggested in [5] for the \( SU(N) \) group, it is possible to find for the partition function on the torus an alternative expression, different from the usual character expansion. The interest of this result is twofold: on one hand we have an expression for \( Z_{p=1} \) in terms of theta functions \( \theta_2 \) and \( \theta_3 \), whose behaviour under modular transformations is well known; on the other hand this allows us to write in a new way the energy levels of the zero coupling limit (at finite temperature) of the Sutherland systems related to a given Lie algebra. Since this expressions are very simple in the grand canonical formalism, we restrict ourself to the classical simple groups [4].

In the case of the classical simple groups, the Weyl group \( W \) has the permutation group of the invariant angles as a subgroup. This suggests us to calculate the above integral by decomposing, as done in [11] for the one-dimensional KM model, each permutation belonging to \( W \) into its cycles. At this point we have to consider separately the four series of classical simple groups: \( Sp(2N) \), \( SO(2N+1) \), \( SO(2N) \), and \( SU(N) \) (corresponding to the four series of Lie algebras \( C_N, B_N, D_N \) and \( A_N \)).

### 3.1 The partition function for \( Sp(2N) \)

The \( Sp(2N) \) group has rank \( N \) and the Weyl group is the group of the permutations and sign changes of \( N \) elements. By writing explicitly the structure of the weight lattice \( P \) and the structure of \( W \) (as a sum over permutations \( S \) and sign changes \( \epsilon_i \)), the partition function (28) has the form:

\[
Z_{p=1}(N, t) = \int_{0}^{2\pi} \prod_{i=1}^{r} d\phi_i \sum_{\epsilon_i = \pm 1} \sum_{S} (-1)^S \prod_{i} \epsilon_i \sum_{\{t_i\}_{i=-\infty}^{+\infty}} \times
\]

\[
\times \exp \left( i \sum_{k} l_k(\phi_k - \phi_{S(k)})\epsilon_{S(k)} - \sum_{k} t_k^2 t/2 \right)
\]

\(^3\)Notice that in (28) we have ignored, for simplicity, the overall factor \( \exp[||\rho||^2 t/2] \) and this corresponds just to a shift of the zero point energy [13].

\(^4\)For the exceptional groups, it is obviously not possible to use the grand-canonical formalism; however one can still write the canonical partition function as a sum of a finite number of theta functions.
The Weyl group is larger than the permutation group; if one first performs the sum over sign changes in (29), then as one decomposes each permutation $S$ into product of cycles, each integral at the r.h.s. of (29) decomposes into the product of integrals corresponding to the cycles in the decomposition of $S$. In this way (see [11] for the details of cycle decomposition) one can show that

$$Z_{p=1}(N, t) = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp \left\{ -iN\theta - \sum_{j=1}^{\infty} \frac{(-1)^j}{j} F_j e^{i\theta j} \right\}$$

(30)

With $F_j$ we denote a contribution from a cycle of a length $j$; it is given by:

$$F_j = \sum_{\epsilon_i = \pm 1} \prod_{i=1}^j \epsilon_i \int_0^{2\pi} d\phi_i \sum_{\{t_i\} = -\infty} \epsilon^i \sum_k t_k (\phi_k - \phi_{k+1}) e^{-\sum_k t_k^2/2}$$

$$= \left(2\pi\right)^j 2^{j-1} \left[ \theta_3(0, \tau_j) - 1 \right]$$

(31)

where from now on $\tau = \frac{\pi}{2\pi}$. This result can be expressed in a rather elegant and interesting form if one consider the grand-canonical partition function for the $C_N$ series:

$$Z_C(q, t) = \sum_N Z_{p=1}(N, t) q^N$$

(32)

In this case one obtains:

$$Z_C(q, t) = \exp \left\{ - \sum_{j=1}^{\infty} \frac{(-4\pi q)^j}{2j} [\theta_3(0, \tau_j) - 1] \right\}$$

(33)

The sum over $j$ in the exponents can be performed if one replaces the theta functions with their expressions as infinite sums, finally leading to the result

$$Z_C(q, t) = \prod_{n=1}^{\infty} \left(1 + 4\pi q e^{-\frac{t}{n^2}}\right)$$

(34)

In this infinite product we easily recognize the grand-canonical partition function of a gas of free fermions at finite temperature and in a compactified one-dimensional space.

### 3.2 The partition function for $SO(2N+1)$

The $SO(2N+1)$ group has rank $N$, the Weyl group is isomorphic to that of $SP(2N)$ and the sum over the weight lattice is given by two independent sums: the first sum corresponds to the vectorial representations (labeled by integers numbers), while the second sum corresponds to the spinorial ones (labeled by half-integers numbers). Therefore the partition function is conveniently written as the sum of two integrals:

$$Z_{p=1}(N, t) = Z_{p=1}^{(1)}(N, t) + Z_{p=1}^{(2)}(N, t)$$

(35)
where

\[ Z_{p=1}^{(1)}(N,t) = \int_0^{2\pi} \prod_{i=1}^r d\phi_i \sum_{\epsilon_i=\pm 1} \sum_S (-1)^S \prod_i \epsilon_i \sum_{\{l_i\}=-\infty}^{+\infty} \times \]
\[ \times \exp \left\{ i \sum_k l_k (\phi_k - \phi_{S(k)} \epsilon_{S(k)}) - \sum_k l_k^2 t/2 \right\} \]

\[ Z_{p=1}^{(2)}(N,t) = \int_0^{2\pi} \prod_{i=1}^r d\phi_i \sum_{\epsilon_i=\pm 1} \sum_S (-1)^S \prod_i \epsilon_i \sum_{\{m_i\}=-\infty}^{+\infty} \times \]
\[ \times \exp \left\{ i \sum_k \left( m_k + \frac{1}{2} \right) (\phi_k - \phi_{S(k)} \epsilon_{S(k)}) - \sum_k \left( m_k + \frac{1}{2} \right)^2 t/2 \right\} \]

(36)

are the contribution to the partition function on the torus arising respectively from vectorial and spinorial representations. Each of the two terms in (35) can be computed as before; the contributions of a cycle of length \( j \) to the cyclic decomposition of \( S \) are then:

\[ F_j^{(1)} = (2\pi)^j 2^{j-1} \left[ \theta_3(0,\tau_j) - 1 \right] \]
\[ F_j^{(2)} = (2\pi)^j 2^{j-1} \theta_2(0,\tau_j) \]

(37)

Thus the generating function \( Z_B \) for the \( SO(2N+1) \) partition functions on a torus has the form:

\[ Z_B(q,t) = \exp \left\{ - \sum_{j=1}^{\infty} \frac{(-4\pi q)^j}{2j} \left[ \theta_3(0,\tau_j) - 1 \right] \right\} \]
\[ + \exp \left\{ - \sum_{j=1}^{\infty} \frac{(-4\pi q)^j}{2j} \theta_2(0,\tau_j) \right\} \]

(38)

or equivalently

\[ Z_B(q,t) = \prod_{n=1}^{\infty} \left( 1 + 4\pi q e^{-n^2 t/2} \right) + \prod_{n=0}^{\infty} \left( 1 + 4\pi q e^{-(n+\frac{1}{2})^2 t/2} \right) \]

(39)

### 3.3 The partition function for \( SO(2N) \)

By looking at the \( SO(2N) \) group, we note that the weight lattice has the same structure as that of \( SO(2N+1) \), but the Weyl group is different: it is the group of the permutations and sign changes of \( N \) objects, but with the constraint that the total number of minus signs must be even. Therefore the partition function has a form similar to (35, 36), the only difference being that the sum over the sign changes is not the sum over the \( N \) independent \( \epsilon_i \), but they do satisfy the relation \( \prod_i \epsilon_i = 1 \).
By writing, as in eq. (35), \( Z_p = 1 \) as a sum of two pieces and by decomposing each term into cycles one obtains:

\[
Z_p^{(i)}(N, t) = \sum_{m=0}^{\infty} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-2im\varphi} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-iN\theta} \times \exp \left\{ - \sum_{j=1}^{\infty} \frac{(-1)^j}{j} F_j^{(i)} e^{i\theta_j} \right\} \tag{40}
\]

where \( i = 1, 2 \) and

\[
F_j^{(1)} = (2\pi)^j 2^{j-1} \left[ \theta_3(0, \tau j) - e^{i\varphi} \right]
\]

\[
F_j^{(2)} = (2\pi)^j 2^{j-1} \theta_2(0, \tau j) \tag{41}
\]

The sum over the positive integers \( m \) and the integral over the Lagrange multiplier \( \varphi \) has been inserted in (40) in order to constraint the sign changes variables to satisfy the above mentioned relation. Now we can write the grand-canonical partition function:

\[
Z_D(q, t) = (1 + 2\pi q) \prod_{n=1}^{\infty} \left( 1 + 4\pi q e^{-n^2t/2} \right) + \prod_{n=0}^{\infty} \left( 1 + 4\pi q e^{-(n+\frac{1}{2})^2t/2} \right) \tag{42}
\]

It is interesting to remark that, for the orthogonal groups, due to the contribution of spinorial representations, the winding numbers (i.e. the integers numbers labeling the coroot lattice) in the Sutherland propagator (14) are only even, as usual when we are dealing with fermionic representations.

### 3.4 The partition function for SU(N)

Let us conclude this section by remembering that for \( SU(N) \) this procedure was worked out in ref. [5]. For the sake of completeness, we report here the expression of \( Z_A \) that one finds in this case:

\[
Z_A(q, t) = \left( \frac{t}{4\pi} \right)^{1/2} \int_0^{2\pi} d\beta \prod_{n=-\infty}^{+\infty} \left( 1 + q e^{-\frac{1}{2} \left( n - \frac{\beta}{2\pi} \right)^2} \right). \tag{43}
\]

The \( SU(N) \) group has actually rank \( N - 1 \), although its root lattice is most conveniently embedded in a \( N \) dimensional space. This fact, for the system of fermions, means that the wave function of the center of mass of the \( N \) fermions is completely localized, and therefore the corresponding momentum undetermined. This is the reason of the integration over \( \beta \) in (43).

Finally we emphasize that the grand canonical partition functions has been defined keeping \( t \) independent of \( N \), so all these grand canonical expressions cannot be used as such to calculate the large \( N \) limit of \( Z_{p=1}(N, t) \).
4 Conclusions

We have shown in this paper that YM2 on a cylinder or on a torus, with gauge group an arbitrary compact Lie group $G$ is exactly a one dimensional matrix model of the type proposed by Kazakov and Migdal, where the KM matter field is now in the fundamental representation of the gauge group. The string interpretation of this model requires expanding the group matrices as exponentials of the matrices on the algebra; therefore in our formulation it is quite direct to retrieve the result found in [2, 3] that the worldsheet is orientable for the $SU(N)$ gauge group, while it may be both orientable and nonorientable for YM2 with gauge group $SO(N)$ or $Sp(2N)$. However, we are not able to extend these results to YM2 on a higher genus surface.

We also prove that the fundamental constituents of the theory are free fermions excitations, most conveniently described as the zero coupling limit of the Sutherland systems and it is very intriguing, for future developments, to note that the large $N$ limit of these integrable systems, as shown in [20] for the Sutherland systems related to $SU(N)$, is described by a two dimensional conformal field theory.

I am very indebted with M. Caselle and A. D’Adda for daily discussions, for many suggestions and for a careful reading of the manuscript. I would like also to thank M. Billò, F. Gliozzi and L. Magnea for useful discussions.

References

[1] B.Ye. Rusakov, *Mod. Phys. Lett.* A5 (1990) 693; E. Witten, *Commun. Math. Phys.* 141 (1991) 153; M. Blau and G. Thompson, *Int. Jour. Mod. Phys.* A16 (1992) 3781

[2] D. Gross, Princeton preprint PUIT-1356, hep-th/9212149; D. Gross and W. Taylor, preprints PUIT-1376 hep-th/9301068 and PUIT-1382 hep-th/9303046

[3] J. A. Minahan and A. P. Polychronakos, *Equivalence of Two Dimensional QCD and the c = 1 Matrix Model*, preprint CERN-TH-6843/93 UVA-HET-93-02 (March 1993)

[4] M. R. Douglas, *Conformal Field Theory Techniques for Large N Group Theory*, preprint RU-93-13 NSF-ITP-93-39 (March 1993)

[5] M. Caselle, A. D’Adda, L. Magnea and S. Panzeri, *Two dimensional QCD is a one dimensional Kazakov-Migdal model*, preprint DFTT 15/93, April 1993

[6] S. R. Das and A. Jevicki, *Mod. Phys. Lett.* A5 (1990) 1639

[7] V.A. Kazakov and A.A. Migdal, *Nucl. Phys.* B397 (1993) 214
[8] S. G. Naculich, H. A. Riggs, H. J. Schnitzer, *Two-dimensional Yang-Mills Theories Are String Theories*, preprint BRX-TH-346 JHU-TIPAC-930015; S. Ramgoolam, *Comment on Two Dimensional O(N) and Sp(N) Yang-Mills Theories as String Theories*, preprint YCTP-P16-93

[9] B. Sutherland, *J. Math. Phys.* **12** (1971) 246

[10] D. Altschuler and C. Itzykson, *Ann. Inst. Henri Poincaré Phys. Theor.* **54** (1991) 1

[11] M. Caselle, A. D’Adda and S. Panzeri, *Phys. Lett.* **B293** (1992) 161

[12] D. V. Boulatov and V. A. Kazakov, *Int. Jour. Mod. Phys.* **A8** (1993) 809

[13] M. Caselle, A. D’Adda and S. Panzeri, *Phys. Lett.* **B302** (1993) 80

[14] A. P. Polychronakos *Phys. Lett.* **B266** (1991) 29

[15] A. Gorsky and N. Nekrasov, *Hamiltonian Systems of Calogero Type and Two Dimensional Yang-Mills Theory*, preprint UUITP-6/93 ITEP-20/93, March 1993

[16] M. A. Olshanetsky and A. M. Perelomov, *Phys. Rep.* **94** (1983) 313

[17] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35

[18] J. S. Dowker, *Ann. of Phys.* **62** (1971) 361; P. Menotti and E. Onofri, *Nucl. Phys.* **B190** (1981) 288

[19] J. E. Hetrick, *Canonical Quantization of two dimensional Gauge Fields*, preprint UVA-ITFA 93-15 (May 1993)

[20] N. Kawakami and S.K. Yang, *Phys. Rev. Lett.* **67** (1991) 2493