Riemann surfaces and AF-algebras

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Abstract

For a generic set in the Teichmüller space, we construct a covariant functor with the range in a category of the AF-algebras. The functor maps isomorphic Riemann surfaces to the stably isomorphic AF-algebras. Our construction is based on a Hodge theory for measured foliations, elaborated by Hubbard, Masur and Thurston. As a special case, one obtains a categorical correspondence between the complex and noncommutative tori.

Key words and phrases: Riemann surfaces, AF-algebras

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1 Introduction

1. Setting of problems.

A. Let \( \mathbb{C} \) be the complex plane and \( \tau \in \mathbb{C} \) a point, such that \( \text{Im} \tau > 0 \). Denote by \( \mathcal{B} \) a category whose objects, \( \text{Ob}(\mathcal{B}) \), are the complex tori \( \mathcal{B}_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}) \) and morphisms, \( \text{Mor}(\mathcal{B}) \), are the biholomorphic maps modulo the ones, which are isotopic to the identity map with respect to a fixed topological marking. Likewise, let \( \mathcal{A} \) be a category, whose objects \( \text{Ob} (\mathcal{A}) \) are the AF-algebras \( \mathcal{A}_\theta \) (the Effros-Shen algebras [6]), given by the following Bratteli diagram:

\[
\begin{array}{cccccc}
\ \ & a_0 & a_1 & \cdots & \\
\ \ & \ \ & \ \ & \ \ & \\
\ \ & \ \ & \ \ & \ \ & \\
\end{array}
\]

\[
\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}
\]

Figure 1: The Effros-Shen algebra \( \mathcal{A}_\theta \).

and morphisms \( \text{Mor} (\mathcal{A}) \) are the stable isomorphisms of the AF-algebras modulo the inner automorphisms. (Here \( \theta \) is an irrational number, \( a_0 \in \mathbb{N} \cup \{0\} \) and

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\( \alpha_i \in \mathbb{N} \) are the multiplicities of edges of the graph.) The following phenomenon is well known.

**Observation.** For a generic \( \mathcal{A}_\theta \in \mathcal{A} \) and \( \mathcal{B}_\tau \in \mathcal{B} \), it holds \( \text{Mor} (\mathcal{A}_\theta) \cong \text{Mor} (\mathcal{B}_\tau) \cong SL_2(\mathbb{Z})/\pm I \).

(Proof. Indeed, \( \mathcal{B}_\tau, B_\tau \) are isomorphic, if and only if, \( \tau' = \frac{a\tau + b}{c\tau + d} \), where \( ad - bc = 1 \) and \( a, b, c, d \in \mathbb{Z} \). Generically, the group of the inner automorphisms of \( \mathcal{B}_\tau \) is \( \pm I \), where \( I \) is the identity matrix. Similarly, the \( AF \)-algebras \( \mathcal{A}_\theta, A_\theta \) are stably isomorphic, i.e. \( \mathcal{A}_\theta \otimes K \cong \mathcal{A}_\theta' \otimes K \), where \( K \) is the \( C^* \)-algebra of compact operators, if and only if, \( \theta' = \frac{a\theta + b}{c\theta + d} \), where \( ad - bc = 1 \) and \( a, b, c, d \in \mathbb{Z} \). Again, for a generic \( \mathcal{A}_\theta \) the group of stable inner automorphisms is \( \pm I \). □)

**B.** Our paper is about this phenomenon and its higher genus manifestations. Our goal is a general correspondence between the Riemann surfaces and the \( AF \)-algebras. It is assumed at the outset, that the observed phenomenon is part of a categorical correspondence. Namely, recall that covariant functor is a pair of functions \( F_1 : \text{Ob} (\mathcal{B}) \to \text{Ob} (\mathcal{A}) \) and \( F_2 : \text{Mor} (\mathcal{B}) \to \text{Mor} (\mathcal{A}) \), denoted by the same symbol \( F \), such that \( F(\varphi_1 \varphi_2) = F(\varphi_1)F(\varphi_2) \) for any \( \varphi_1, \varphi_2 \in \text{Mor} (\mathcal{B}) \).

Our objective can be expressed as follows.

**Main problem.** Construct a covariant functor \( F : \mathcal{B} \to \mathcal{A} \) (if any), which maps isomorphic complex tori to the stably isomorphic Effros-Shen algebras.

One can generalize the main problem as follows. Let \( \mathcal{S} \) be a category whose objects, \( \text{Ob} (\mathcal{S}) \), are the Riemann surfaces \( \mathcal{S} \) and morphisms, \( \text{Mor} (\mathcal{S}) \), are isomorphisms of the Riemann surfaces. Likewise, let \( \mathcal{A}^* \) be a category whose objects, \( \text{Ob} (\mathcal{A}^*) \), are certain \( AF \)-algebras, \( \mathcal{A} \), which we shall call toric \( AF \)-algebras\(^1\), and morphisms, \( \text{Mor} (\mathcal{A}^*) \), are stable isomorphisms of the \( AF \)-algebras.

**Extended main problem.** Construct a covariant functor \( F^* : \mathcal{S} \to \mathcal{A}^* \) (if any), which maps isomorphic Riemann surfaces to the stably isomorphic toric \( AF \)-algebras.

2. **Background.**

The functor \( F : \mathcal{B} \to \mathcal{A} \) first emerged in the context of quantum algebras. According to a remarkable insight of Manin \([12]\), the Effros-Shen algebra \( \mathcal{A}_\theta \), where \( \theta \) is the real root of an irreducible quadratic polynomial over \( \mathbb{Z} \), can be used to explicitly construct the abelian extensions of the real quadratic number fields (the real multiplication problem). Note that the Effros-Shen algebra is very close to a \( C^* \)-algebra called a noncommutative torus. (In fact, \( \mathcal{A}_\theta \) embeds into a noncommutative torus, whose dimension group is isomorphic to such of \( \mathcal{A}_\theta \).) In a series of papers by Polishchuk \([15], [18]\], starting with the work of Polishchuk and Schwarz \([19]\), the noncommutative tori have been treated as noncommutative algebraic varieties, endowed with the holomorphic vector

\(^1\)The name has been suggested to us by Yu. I. Manin.
bundles. A different set up has been suggested by Soibelman and Vologodsky [21], where the category $\mathcal{A}$ is viewed as quantum deformation of category $\mathcal{B}$. The principle of quantum deformation has been further elaborated by Kontsevich and Soibelman in [11] and Soibelman in [20].

3. Objectives.
The extended functor $F^* : S \to \mathcal{A}^*$ attracted less attention so far. The objective of present paper is detailed construction of such a functor. We build a functor $F^*$, such that the restriction of $F^*$ to the Riemann surfaces of genus one coincides with functor $F : \mathcal{B} \to \mathcal{A}$. Thus, we solve the main and the extended main problems. However, a difference between the two cases exists. Unless $g = 1$, functor $F^*$ is defined only for a generic set in the Teichmüller space $T(g)$. This effect is due to a (non-generic) divergence of the Jacobi-Perron continued fractions. Let us note, that functor $F^*$ can be extended to the non-generic cases as well. However, group $\text{Mor} (\mathfrak{A}, \mathfrak{A} \in \mathcal{A}^*)$ is no longer isomorphic to the matrix group $\text{GL}_{6g-6}(\mathbb{Z})$.

4. Construction of the mapping $F^*$.
A. Let us outline the construction of $F^*$. It is based on a Hodge theory for the measured foliations developed by Hubbard-Masur [10] and inspired by Thurston [23]. The measured foliation, $\mathcal{F}$, can be defined as a local-global structure on the surface $X$, consisting of a finite collection of the singular points $x_i$ and regular leaves (1-dimensional submanifolds). Each $x_i$ has a local chart $V$, such that $\mathcal{F} = \mathcal{F}(x_i, \phi_i)$ is given by the trajectories of a closed 1-form $\phi_i = Im (z^{k_i}dz)$, where $z = u + iv \in V$ and $k_i \in \mathbb{N}$. Everywhere else, $\mathcal{F}$ looks as a family of the parallel lines. It was observed by Jenkins and Strebel, that every measured foliation occurs uniquely as foliation $Im q = 0$ given by a holomorphic quadratic form $q$ on the Riemann surface $S$. This fact has been established for the foliations, whose regular leaves are homeomorphic to $S^1$ (the Jenkins-Strebel foliations, or $JS$-foliations), see e.g. [22]. The equivalence classes of the $JS$-foliations is a non-empty set of the real dimension $6g - 6$. The uniqueness of $q$ implies (via a standard argument, see below), that the $JS$-foliations parametrize the Teichmüller space of surface $X$. The parametrization has been rapidly adopted by Harer, Kontsevich and Mumford to represent the Teichmüller space via the metric ribbon graphs, see e.g. §6 of [9] for an account.

B. Unlike the $JS$-foliation, each regular leaf of a minimal foliation is dense (by definition) in $X$. Hubbard and Masur extended Jenkins-Strebel’s results to the minimal foliations, which we shall call the Hubbard-Masur, or $HM$-foliations. A key step in Hubbard-Masur’s proof involves the uniqueness of the $JS$-foliations, since they are dense in the space of all measured foliations. It is helpful to think of the $HM$-foliations as of the set of irrational numbers, while the $JS$-foliations form a set of the rational numbers of the real line. These two families of measured foliations are complementary to each other: the $JS$-foliations are topologically trivial, but rich in measure, while the $HM$-foliations are poor in measure, yet topologically rich. Let us give details of the Hubbard-Masur construction.
C. Denote by $T(g)$ the Teichmüller space of genus $g \geq 1$ and let $S \in T(g)$. Let $q \in H^0(S, \Omega^{0,2})$ be a holomorphic quadratic differential on the Riemann surface $S$, such that all zeroes of $q$ are simple. By $\tilde{S}$ we understand a double cover of $S$. Note that there is an involution on the homology groups $H_*(\tilde{S})$ induced by the covering map $\tilde{S} \to S$. Let $H_1^{\text{odd}}(\tilde{S})$ be the odd part of the first (integral) homology of $\tilde{S}$ with respect to this involution, relatively the zeroes of $q$. By the formulas for the relative homology:

$$H_1^{\text{odd}}(\tilde{S}) \cong \mathbb{Z}^n,$$

where $n = \begin{cases} 6g - 6, & \text{if } g \geq 2 \\ 2, & \text{if } g = 1. \end{cases}$

D. Let $\mathcal{F}$ be a measured foliation with the simple zeroes (i.e. $k_i = 1$), $Q$ the vector bundle of all quadratic differentials over the Teichmüller space $T(g)$ and $0$ the zero section of $Q$. Consider those quadratic differentials, $E_\mathcal{F} \subset Q - \{0\}$, whose horizontal trajectories induce $\mathcal{F}$. Note that, if $\mathcal{F}$ is a measured foliation with the simple zeroes (a generic case), then $E_\mathcal{F} \cong \mathbb{R}^n - 0$, while $T(g) \cong \mathbb{R}^n$.

**Hubbard-Masur homeomorphism ([10]).** There exists a homeomorphism (an embedding) $h : E_\mathcal{F} \to T(g)$.

The Hubbard-Masur result implies that the measured foliations parametrize the space $T(g) - \{pt\}$, where $pt = h(0)$. Indeed, denote by $\mathcal{F}'$ a vertical trajectory structure $Re q = 0$ of $q$. Since $\mathcal{F}$ and $\mathcal{F}'$ define $q$, and $\mathcal{F} = \text{Const}$ for all $q \in E_\mathcal{F}$, one gets a homeomorphism between $T(g) - \{pt\}$ and $\Phi_X$, where $\Phi_X \cong \mathbb{R}^n - 0$ is the space of equivalence classes of the measured foliations $\mathcal{F}'$ on $X$.

E. On the face of it, the parametrization of $T(g) - \{pt\}$ by measured foliations is not unique in view of the fact that it depends on the arbitrary choice of foliation $\mathcal{F}$. However, this is not the case, as the following (spectral) construction shows. Let $Sp(S)$ be the length spectrum of the Riemann surface $S \in T(g)$, i.e. the set of lengths of simple closed geodesics of $S$. Similarly, let $Sp(\mathcal{F}')$ be the infimum of the integrals $\int_\gamma |\phi|$, where $\phi$ is a closed 1-form, whose trajectories give $\mathcal{F}'$ and $\gamma_i$ runs over all simple closed curves on $X$. The Hubbard-Masur homeomorphism $h_{E_\mathcal{F}} : \Phi_X \to T(g) - \{pt\}$ is canonical, if there exist a foliation $\mathcal{F}$, such that $Sp(\mathcal{F}') = Sp(h_{E_\mathcal{F}}(\mathcal{F}'))$ for $\forall \mathcal{F}' \in \Phi_X$.

F. Recall that $T(g) \cong \mathbb{R}^n$, where $n = \dim H_1^{\text{odd}}(\tilde{S})$. In view of the Hubbard-Masur homeomorphism, any $n$-parameter family of the measured foliations can be used to parametrize the space $T(g) - \{pt\}$. As explained, there are two notable families of such foliations: the $JS$ and $HM$-foliations. Note that the equivalence classes of the $JS$-foliations form a dense countable subset in the space of all measured foliations. For the case of $g = 1$, the $JS$-foliations are given by the straight lines of a rational slope, parametrized by $\mathbb{Q}$. To better understand a difference between the $JS$ and $HM$-foliations, let us denote them by $\mathcal{F}_{JS}$ and $\mathcal{F}_{HM}$, respectively. A leaf space of the foliation $\mathcal{F}$ on surface $X$ is the factor space $X/\mathcal{F}$, which identifies the points of $X$ belonging to the same leaf of $\mathcal{F}$. It is well known that generically (i.e. when all the zeroes of the foliation are
simple), the space $X/\mathcal{F}_{JS} \cong \Gamma_n$, where $\Gamma_n$ is a 3-valent metric graph embedded in $X$ (a metric ribbon graph), see e.g. [9]. The edges of the graph correspond to the heights of the cylinders carved by the periodic trajectories of the $JS$-foliations, while the vertices are bijective with the zeroes of the foliation. An $n$-parameter family of the $JS$-foliations is obtained by a variation of the lengths of the edges of $\Gamma_n$. The graphs $\Gamma_n$ can be used to calculate homology of the mapping class group (Harer) and to settle the Witten conjecture (Kontsevich).

G. Unlike the $JS$-foliations, the leaf space $X/\mathcal{F}_{HM}$ is no longer a 1-dimensional CW-complex (a graph), but a non-Hausdorff topological space. Due in part to this fact, the $HM$-foliations have been reluctantly used as a model for the Teichmüller spaces. As we shall see, the $AF$-algebras appear to be a proper category in this sense. Indeed, let

$$\lambda_i = \int_{\gamma_i} \phi,$$

where $\phi = \text{Re} \, q'$ is a closed 1-form and $\gamma_i$ are elements of a basis in $H^0_{\text{odd}}(\tilde{S})$. (The periods $\lambda_i$ are well defined in the sense that a $\mathbb{Z}$-module $\Lambda_n = \sum_{i=1}^n \lambda_i \mathbb{Z}$ depends neither on $\gamma_i$ nor the equivalence class of the 1-form $\phi$, see the proof of lemma 1.) It was shown by Douady and Hubbard in [9], that $\lambda_i$ are coordinates in the space of all measured foliations. Every $n$-parameter family of the $HM$-foliations is obtained by variation of the reals $\lambda_i$ in a generic set of $\mathbb{R}^n$. Thus, $\lambda_i$ form a coordinate system in a generic part of $T(g)$.

H. Let $\text{Mod} (X) = \text{Diff}^+(X) / \text{Diff}^0(X)$ be the mapping class group of the surface $X$. (For the case $g = 1$, $\text{Mod} (X) \cong SL_2(\mathbb{Z})$ is the familiar modular group, hence the notation.) The following $\mathbb{Z}$-module, attached to an $HM$-foliation, will be critical. Assuming that $\lambda_i$ are linearly independent over $\mathbb{Q}$, consider a $\mathbb{Z}$-module:

$$\Lambda_n = \lambda_1 \mathbb{Z} + \ldots + \lambda_n \mathbb{Z} \subset \mathbb{R},$$

thought of as a dense subset of the real line. One can ask what happens to $\Lambda_n = \Lambda_n(S)$, when one passes to an isomorphic Riemann surface $S'$, i.e. $S' = \varphi(S)$ for an $\varphi \in \text{Mod} (X)$? It is not hard to prove, that $\Lambda_n(S) = \Lambda_n(\varphi(S))$ for $\forall \varphi \in \text{Mod} (X)$, see lemma 1 (The action of $\varphi$ corresponds to a change of basis in the module $\Lambda_n$.) In other words, the modules $\Lambda_n$ make a category, which is covariant to the category $S$. Roughly speaking, the rest of the note is a phrasing of this property of $\Lambda_n$ in terms of the toric $AF$-algebras. Note that it is in a sharp contrast to the ribbon graph picture, where $\Gamma_n(S) \neq \Gamma_n(\varphi(S))$ for almost all $\varphi \in \text{Mod} (X)$.

I. To finalize construction of the mapping $F^*: \mathcal{S} \to A^*$, let us denote by $\theta = (\theta_1, \ldots, \theta_{n-1})$ a vector with the coordinates $\theta_i = \lambda_i / \lambda_1$, whenever $\lambda_1 \neq 0$. (Note that $\lambda_1 = 0$ is not a generic condition.) Consider the Jacobi-Perron continued fraction:

$$\left( \begin{array}{c} 1 \\ \theta \end{array} \right) = \lim_{k \to \infty} \left( \begin{array}{cc} 0 & 1 \\ 1 & b_1 \end{array} \right) \ldots \left( \begin{array}{cc} 0 & 1 \\ 1 & b_k \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$

where $b_i = (b_1^{(i)}, \ldots, b_n^{(i)})^T$ is a vector of the non-negative integers, $I$ the unit
matrix and $\mathbb{I} = (0, \ldots, 0, 1)^T$. Denote by $\mathbb{A} := \mathbb{A}_{\theta_1, \ldots, \theta_{n-1}}$ an $AF$-algebra:

$$
\begin{align*}
\mathbb{Z}^n & \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix}} \mathbb{Z}^n \\
& \xrightarrow{\begin{pmatrix} 0 & 1 \\ I & b_2 \end{pmatrix}} \mathbb{Z}^n \\
& \xrightarrow{\mathbb{Z}^n \ldots,}
\end{align*}
$$

which is defined as the inductive limit of the positive isomorphisms given by the unimodular integral matrices $\begin{pmatrix} 0 & 1 \\ I & b_i \end{pmatrix}$ for $i = 1, \ldots, \infty$. The map $F^*$ is given by the formula $S(\lambda_1, \ldots, \lambda_n) \mapsto \mathbb{A}_{\theta_1, \ldots, \theta_{n-1}}$, where $S \in \mathcal{S}$ and $A \in \mathcal{A}$. Note that in the case $g = 1$, the $AF$-algebra $\mathbb{A}$ coincides with an Effros-Shen algebra $\mathbb{A}_{\theta_1}$. We shall call the $AF$-algebra $\mathbb{A}$ a toric $AF$-algebra, whenever $g \geq 2$.

5. Main results.
A summary of our results is contained in the following statement.

**Theorem.** Let $V = \text{Ob} (\mathcal{S})$, $W = \text{Ob} (\mathcal{A}^*)$ and $F^*: V \to W$ be as before. The mapping $F^*$ has the following properties:

(i) $V \cong W \times (0, \infty)$ is a trivial bundle, whose projection map $\pi: V \to W$ coincides with $F^*$;

(ii) $F^*$ is a covariant functor, which maps isomorphic Riemann surfaces $\mathcal{S}, \mathcal{S}' \in \mathcal{S}$ to the stably isomorphic toric $AF$-algebras $\mathbb{A}, \mathbb{A}' \in \mathcal{A}^*$.

6. Structure of the paper.
The article is organized as follows. In section 2 we set up a background for the main construction. The definition of a toric $AF$-algebra is deferred to section 3. The definition of $F^*$ and a main result (the functoriality of $F^*$) appears in section 4. In section 5, we prove the main result. The special case $g = 1$, which corresponds to a functor between the complex tori and Effros-Shen algebras, is considered in section 6. Finally, in section 7, an invariant of the conformal equivalence is introduced.

2 Preliminaries
Present section contains a minimal background information necessary to prove our main results. Measured foliations are expounded in [8], [10] and [23]. A full exposition of the remarkable connection between measured foliations and the Teichmüller spaces (Hubbard-Masur theory) can be found in [10]. For the $C^*$-algebra part, especially $AF$-algebras, we refer the reader to [4]. Finally, a comprehensive account of the Jacobi-Perron continued fractions can be found in [2].

2.1 Measured foliations

1. Definition
A measured foliation, $\mathcal{F}$, on a surface $X$ is a partition of $X$ into the singular points $x_1, \ldots, x_n$ of order (multiplicity) $k_1, \ldots, k_n$ and regular leaves (1-dimensional submanifolds). On each open cover $U_i$ of $X - \{x_1, \ldots, x_n\}$ there exists a non-vanishing real-valued closed 1-form $\phi_i$ such that

(i) $\phi_i = \pm \phi_j$ on $U_i \cap U_j$;

(ii) at each $x_i$ there exists a local chart $(u, v) : V \to \mathbb{R}^2$ such that for $z = u + iv$, it holds $\phi_i = \text{Im}(z^{\frac{k_i}{2}} dz)$ on $V \cap U_i$ for some branch of $z^{\frac{k_i}{2}}$.

The pair $(U_i, \phi_i)$ is called an atlas for measured foliation $\mathcal{F}$. Finally, a measure $\mu$ is assigned to each segment $(t_0, t) \in U_i$, which is transverse to the leaves of $\mathcal{F}$, via the integral $\mu(t_0, t) = \int_{t_0}^{t} \phi_i$. The measure is invariant along the leaves of $\mathcal{F}$, hence the name.

2. Singular points

The configuration of leaves near singular points of order $k$ is shown in Fig.2. The singular point of order 0 is referred to as a fake saddle. The singular point of even order is called oriented. When all singular points of a measured foliation are oriented, $\mathcal{F}$ is called a flow. Any flow on a compact surface is defined by the trajectories of a closed 1-form $\phi$. If $\mathcal{F}$ is measured foliation, the index theorem implies

$$\sum_{i=1}^{n} \frac{k_i}{2} = 2g - 2.$$ 

In particular, there exists only a finite number of the singular points (which are not the fake saddles) for any measured foliation $\mathcal{F}$. Via a double cover construction (to be considered later on), each measured foliation is covered by a flow on an appropriate surface.

![Figure 2: Singular points of measured foliations.](image)

3. Method of zippered rectangles

Note that a fake saddle is not a singular point, since $\phi = \text{Im}(dz)$ does not vanish in any local chart of such a saddle. The fake saddle is a useful formalism, which allows a uniform exposition, e.g. of the method of zippered rectangles (see below).
A. There exists a remarkable construction, which allows to produce a flow from the given set of positive reals \((\lambda_1, \ldots, \lambda_n)\). Let \(\pi\) be a permutation of \(n\) symbols. Consider a rectangle with the base \(\lambda_1 + \ldots + \lambda_n\) and the top \(\lambda_{\pi(1)} + \ldots + \lambda_{\pi(n)}\). We shall identify the open interval \(\lambda_i\) in the base with the open interval \(\lambda_{\pi(i)}\) at the top for all \(i = 1, \ldots, n\), as it is shown in the Fig.3. The resulting object will be a \(k\)-holed topological surface, \(X\), of genus \(g = \frac{1}{2}(n - N(\pi) + 1)\), where \(N(\pi)\) is the number of cyclic permutations in the prime decomposition of \(\pi\) [24]. A flow \(F\) on \(X\) is defined by the vertical lines given by the closed 1-form \(\phi = dx\). The order of singular points of \(F\) depends on the length of the elementary cyclic permutations and the total number of the singular points equals to \(k = N(\pi)\). The singular points are located at the holes of the surface \(X\).

![Figure 3: Zippering of the rectangle.](image)

B. To recover \(\lambda_i\) from the 1-form \(\phi\), notice that

\[ n = 2g + N(\pi) - 1 = \dim H_1(X, \text{Sing} F; \mathbb{Z}), \]

where the last symbol stays for the relative homology of \(X\) with respect to the set of singular points of the flow \(F\). Since \(\phi = dx\), one arrives at the elementary, but an important formula:

\[ \lambda_i = \int_{\gamma_i} \phi, \]

where \(\gamma_i\) are elements of a basis in \(H_1(X, \text{Sing} F; \mathbb{Z})\). It will be eventually shown, that \(\lambda_i\) are the coordinates in the space of measured foliations.

4. Example: zippered torus

We shall consider the case \(n = 2\) in more detail. Since each permutation of two intervals is cyclic, \(N(\pi) = 1\) and therefore \(g = \frac{1}{2}(2 + 1 - 1) = 1\). In other words, \(X\) is a torus with \(k = N(\pi) = 1\) hole. As explained, the flow \(F\) is given by the trajectories of the 1-form \(\phi = dx\). To determine the order of the singular point \(x_i\) situated at the hole, consider the index equation \(\sum \frac{k_i}{2} = 2g - 2 = 0\). Therefore, \(k_1 = 0\) and the singular point is a fake saddle, shown in Fig.4.

5. The representation of measured foliation by a flow
A. Unless all singular points of a measured foliation are of an even order, the foliation $\mathcal{F}$ cannot be given by a closed 1-form on the surface $X$. Fortunately, the situation can be corrected on a surface $\tilde{X}$, which is a double cover of $X$ ramified over the singular points of $\mathcal{F}$ of the odd order. Namely, denote by $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ the set of singular points of $\mathcal{F}$ of the even and odd order, respectively. Note that in view of the index formula, $m$ is always an even integer. Let $\tilde{X}$ be a surface, which covers surface $X$ with a ramification of index 2 over the points $y_1, \ldots, y_m$. The genus of a covering surface (the Riemann-Hurwitz formula) writes as:

$$\tilde{g} = 2g + \frac{m}{2} - 1,$$

where $\tilde{g}$ and $g$ are the genera of the surfaces $\tilde{X}$ and $X$, respectively.

B. Denote by $\pi : \tilde{X} \to X$ a covering projection and let $\mathcal{F}$ be a measured foliation on $X$. We shall denote by $\tilde{\mathcal{F}} = \pi^{-1}(\mathcal{F})$ the resulting measured foliation on the surface $\tilde{X}$. The set $\text{Sing}(\tilde{\mathcal{F}}) = \{2x_1, \ldots, 2x_n; \tilde{y}_1, \ldots, \tilde{y}_m\}$, where $2x_i$ denote two copies of singular point $x_i$ and $\tilde{y}_i$ a singular point of order $2(k_i + 1)$, whenever $k_i$ is the order of $y_i$. Thus, $\tilde{\mathcal{F}}$ has no singular points of an odd order and therefore $\tilde{\mathcal{F}}$ is a flow. We call the $\tilde{\mathcal{F}}$ a covering flow of the measured foliation $\mathcal{F}$.

6. Measured foliations and quadratic differentials

Let $\mathcal{S}$ be a Riemann surface, and $q \in H^0(\mathcal{S}, \Omega^{\otimes 2})$ a holomorphic quadratic differential on $\mathcal{S}$. The lines $\text{Re } q = 0$ and $\text{Im } q = 0$ define a pair of measured foliations on $\mathcal{S}$, which are transversal to each other outside the set of singular points. The set of singular points is common to the both foliations and coincides with the zeroes of $q$. The above measured foliations are said to represent the vertical and horizontal trajectory structure of $q$, respectively.

2.2 Hubbard-Masur theory

1. Hubbard-Masur homeomorphism
Let $T(g)$ be the Teichmüller space of the topological surface $X$ of genus $g \geq 1$, i.e. the space of the complex (conformal) structures on $X$. Consider the vector bundle $p : Q \to T(g)$ over $T(g)$, whose fiber above the point $S \in T(g)$ is a vector space $H^0(S, \Omega^\otimes 2)$. Given a non-zero $q \in Q$ above $S$, we can consider a horizontal measured foliation $F_q \in \Phi_X$, where $\Phi_X$ denotes the space of the equivalence classes of measured foliations on $X$. If $\{0\}$ is the zero section of $Q$, the above construction defines a map $Q - \{0\} \to \Phi_X$. In other words, $E_F$ is a subspace of the holomorphic quadratic forms, whose horizontal trajectory structure coincides with the measured foliation $F$.

**Theorem (Hubbard-Masur [10])** The restriction $E_F \to T(g)$ of $p$ to $E_F$ is a homeomorphism.

2. Parametrization of the Teichmüller space by measured foliations

One obtains an important corollary from the Hubbard-Masur theorem. We shall assume that $F \in \Phi_X$ is a measured foliation, whose set of singular points consists of simple zeroes (i.e. $k_1 = 1$ for all the singular points). Denote by $F'$ a measured foliation defined by the vertical trajectories $Re q = 0$, where $q \in E_F$. Since $F = \text{Const}$ for all $q \in E_F$, we conclude that $F' \in \Phi_X$ parametrize the space $E_F$. The $F'$ and $F$ have a common set of the singular points, and therefore all zeroes of $F'$ are simple. Thus, $F'$ varies in a subset of $\Phi_X$ of dimension $n = 6g - 6$ ($n = 2$) if $g \geq 2$ ($g = 1$). Recall that $\Phi_X \cong \mathbb{R}^n - 0$, while $T(g) \cong \mathbb{R}^n$. We let $pt$ be the image of the zero quadratic differential under the Hubbard-Masur homeomorphism.

**Corollary** The mapping $h_{E_F} : \Phi_X \to T(g) - \{pt\}$ is a homeomorphism.

3. The canonical homeomorphism $h : \Phi_X \to T(g) - \{pt\}$.

It follows from the Hubbard-Masur construction, that the homeomorphism $h_{E_F}$ depends on an initial choice of the measured foliation $F$. Let us show, however, that there exists a canonical map $h : \Phi_X \to T(g) - \{pt\}$. Let $S$ be the set of homotopy equivalence classes of simple closed curves on the surface $X$. Define a map $i : \Phi_X \to \mathbb{R}^S$ by the formula

$$\phi \mapsto \inf_{\gamma_i} \int |\phi(t)|\gamma_i'(t)dt, \quad \phi \in \Phi_X, \quad \gamma_i \in S,$$

where the infimum is taken over all simple closed curves in the homotopy class $\gamma_i$. Recall that a length spectrum of a Riemann surface $S \in T(g)$ is an element of $\mathbb{R}^S$ consisting of the lengths of the shortest simple closed geodesic representatives in each of the homotopy classes of $S$. We shall denote the latter function by $j : T(g) \to \mathbb{R}^S$. The homeomorphism $h : \Phi_X \to T(g) - \{pt\}$ will be called **canonical** if the following diagram is commutative:
Further, by a *Hubbard-Masur homeomorphism* we shall understand the canonical homeomorphism $h$.

4. *Douady-Hubbard coordinates in the space of measured foliations*

**A.** Recall that $\Phi_X$ is the space of equivalence classes of measured foliations on the topological surface $X$. Following Douady and Hubbard [3], we consider a coordinate system on $\Phi_X$, suitable for the proof of theorem [1]. For clarity, let us make a generic assumption that $q \in H^0(S, \Omega \otimes^2)$ is a non-trivial holomorphic quadratic differential with only simple zeroes. We wish to construct a Riemann surface of $\sqrt{q}$, which is a double cover of $S$ with ramification over the zeroes of $q$. Such a surface, denoted by $\tilde{S}$, is unique and has an advantage of carrying a holomorphic differential $\omega$, such that $\omega^2 = q$. We further denote by $\pi: \tilde{S} \to S$ the covering projection. The vector space $H^0(\tilde{S}, \Omega)$ splits into the direct sum $H^0_{even}(\tilde{S}, \Omega) \oplus H^0_{odd}(\tilde{S}, \Omega)$ in view of the involution $\pi^{-1}$ of $\tilde{S}$, and the vector space:

$$H^0(S, \Omega \otimes^2) \cong H^0_{odd}(\tilde{S}, \Omega).$$

Let $H^0_{odd}(\tilde{S})$ be the odd part of the homology of $\tilde{S}$ relatively the zeroes of $q$. Consider the pairing $H^0_{odd}(\tilde{S}) \times H^0(S, \Omega \otimes^2) \to \mathbb{C}$, defined by the integration $(\gamma, q) \mapsto \int_{\gamma} \omega$. We shall take the associated map $\psi_q: H^0(S, \Omega \otimes^2) \to \text{Hom}(H^0_{odd}(\tilde{S}); \mathbb{C})$ and let $h_q = \text{Re} \psi_q$.

**Lemma (Douady-Hubbard [3])** The map

$$h_q: H^0(S, \Omega \otimes^2) \to \text{Hom}(H^0_{odd}(\tilde{S}); \mathbb{R}),$$

is an $\mathbb{R}$-isomorphism.

**B.** Since each $F \in \Phi_X$ is the vertical foliation $\text{Re} \ q = 0$ for a $q \in H^0(S, \Omega \otimes^2)$, the Douady-Hubbard lemma implies that $\Phi_X \cong H^0(\text{Hom}(H^0_{odd}(\tilde{S}); \mathbb{R})$. By formulas for the relative homology, one finds that $H^0_{odd}(\tilde{S}) \cong \mathbb{Z}^n$, where

$$n = \begin{cases} 
6g - 6, & \text{if } g \geq 2 \\
2, & \text{if } g = 1.
\end{cases}$$

Finally, each $h \in \text{Hom}(\mathbb{Z}^n; \mathbb{R})$ is given by the reals $\lambda_1 = h(e_1), \ldots, \lambda_n = h(e_n)$, where $(e_1, \ldots, e_n)$ is a basis in $\mathbb{Z}^n$. Thus, the reals $(\lambda_1, \ldots, \lambda_n)$ make a coordinate system in the linear space $\Phi_X$. 

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2.3 AF-algebras

1. $C^*$-algebras

A. Rings of bounded operators on Hilbert space have been introduced in the 1930s by Murray and von Neumann. These rings, called von Neumann algebras, have a plethora of remarkable properties, with deep and unexpected ties to geometry, topology and representation theory.

B. By the $C^*$-algebra one understands a noncommutative Banach algebra with an involution. Namely, a $C^*$-algebra $A$ is an algebra over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$, $a \in A$, such that $A$ is complete with respect to the norm, and such that $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$ for every $a,b \in A$. Every $C^*$-algebra is isomorphic to a subalgebra of algebra of bounded linear operators on a Hilbert space.

C. If $A$ is commutative, then the Gelfand theorem says that $A$ is isomorphic to the $C^*$-algebra $C_0(X)$ of continuous complex-valued functions on a locally compact Hausdorff space $X$. For otherwise, $A$ represents a noncommutative topological space $X$.

2. Dimension groups

Let $A$ be a unital $C^*$-algebra and $V(A)$ be the union (over $n$) of projections in the $n \times n$ matrix $C^*$-algebra with entries in $A$. Projections $p, q \in V(A)$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$. The equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p \oplus q]$. The Grothendieck completion of this semigroup to an abelian group is called a $K_0$-group of algebra $A$. The functor $A \to K_0(A)$ maps a category of unital $C^*$-algebras into the category of abelian groups so that projections in algebra $A$ correspond to a positive cone $K_0^+ \subseteq K_0(A)$ and the unit element $1 \in A$ corresponds to an order unit $u \in K_0(A)$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a dimension group.

3. Isomorphism of dimension groups

We will need to define an isomorphism for the dimension groups. Recall that by an ordered group one understands an abelian group $G$ together with a subset $P = G^+$ such that $P + P \subseteq P, P \cap (-P) = \{0\}$, and $P - P = G$. We call $P$ the positive cone on $G$. We write $a \leq b$ (or $a < b$) if $b - a \in P$ (or $b - a \in P \setminus \{0\}$). Given ordered groups $G$ and $H$, we say that a homomorphism $\varphi : G \to H$ is positive if $\varphi(G^+) \subseteq H^+$, and that $\varphi : G \to H$ an order-isomorphism if $\varphi(G^+) = H^+$. An isomorphism $\cong$ between dimension groups $(K_0(A), K_0^+(A), u)$ and $(K_0(A'), K_0^+(A'), u')$ is an order-isomorphism such that $\varphi(u) = u'$.

4. Dimension groups as dense subgroups of $\mathbb{R}$

A positive homomorphism $s : G \to \mathbb{R}$ is called a state if $s(u) = 1$, where $u \in G^+$ is an order unit of $G$. We let $S(G)$ be the state space of $G$, i.e. the set of states
on $G$ endowed with natural topology. $S(G)$ is a compact convex subset of vector space $\text{Hom}(G; \mathbb{R})$. By the Krein-Milman theorem, $S(G)$ is the closed convex hull of its extreme points. When $S(G)$ consists of one point, the abelian group $G$ is said to be \textit{totally ordered}. The totally ordered abelian groups of rank $n \geq 2$ will be of main interest for us, and the standard way to produce such groups is this. Let $G \subset \mathbb{R}$ be a dense abelian subgroup of rank $n$. The positive cone of $G$ is defined to be the set $G^+ = \{ g \in G : g > 0 \}$ and the order unit $u = 1$. Any totally ordered abelian group appears in this way \cite{4}, Corollary 4.7.

5. The $AF$-algebras, Bratteli diagrams and Elliott invariants

A. One of the major classes of the $C^*$-algebras whose structure and invariants are well-understood, are the $AF$-algebras. Recall that every finite-dimensional $C^*$-algebra is isomorphic to $A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_r}(\mathbb{C})$, where $n_1, \ldots, n_r$ are positive integers and $M_{n_i}(\mathbb{C})$ are the $C^*$-algebras of square matrices with the complex entries. If $A, B$ are finite-dimesional $C^*$-algebras, then $\text{Hom}(A, B)$ are described by systems of the matrix units in $A, B$. Namely, the matrix unit $E_{ij}^{n_k}$ in $A$ is a matrix whose entries are zero except $e_{ij} = 1$. For very obvious reasons, the matrix units of $A$ is a linear basis of $A$ over $\mathbb{C}$. If $h \in Hom (A, B)$, then $h$ is defined by the image, $h(E_{ij}^{n_k})$, in the set of matrix units of $B$. If there are other matrix units of $A$ which map to $h(E_{ij}^{n_k})$, we denote by $\mu$ their total number. It easy to see that numbers $\mu$ are all one needs to describe $h$. Indeed, since $A$ and $B$ are direct sums of their matrix units, one can rearrange the summands in $A$ and $B$, so that $h$ acts by the formula $\oplus_{j=1}^r \mu_{ij} a_i \mapsto b_j$, see \cite{4} for details of the construction. The $AF$-algebra, $\mathbb{A}$, is the inductive limit of finite-dimensional $C^*$-algebras

$$A_1 \xrightarrow{\mu_1}, A_2 \xrightarrow{\mu_2}, \ldots$$

induced by the homomorphisms $\mu_k : A_k \rightarrow A_{k+1}$. Here $\mu_k = (\mu_{ij}^k)$ are the multiplicity matrices specified above.

B. The $\infty$-partite graph whose incidence matrix on the $k$-th step is $\mu_k$, is called a \textit{Bratteli diagram} of $\mathbb{A}$. For an example of such a diagram, see Fig.1. Note that $\mathbb{A}$ is no longer a finite-dimensional $C^*$-algebra, hence the name $AF$ (the approximately finite-dimensional $C^*$-algebra).

C. Denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the integers and non-negative integers, respectively. We shall restrict to the $AF$-algebras, such that $\mu_i \in GL_n(\mathbb{Z})$. Since $K_0(A_i) \cong \mathbb{Z}^{n_i}$, we conclude that $K_0(\mathbb{A}) \cong \mathbb{Z}^{n}$. Similarly, $K_0^+(\mathbb{A}) = \lim_{i \rightarrow -\infty} \mu_1 \ldots \mu_i(\mathbb{Z}_+^n)$. Thus, at least formally, we know the dimension group of $\mathbb{A}$. The following beautiful result describes the isomorphism and stable isomorphism classes of the $AF$-algebras.

Lemma (Elliott \cite{7}) Let $\mathbb{A}, \mathbb{A}'$ be the $AF$-algebras. Then:

(i) $\mathbb{A} \cong \mathbb{A}'$ if and only if $(K_0(\mathbb{A}), K_0^+(\mathbb{A}), u) \cong (K_0(\mathbb{A}'), K_0^+(\mathbb{A}'), u')$;
(ii) $\mathbb{A} \otimes \mathbb{K} \cong \mathbb{A}' \otimes \mathbb{K}$ if and only if $(K_0(\mathbb{A}), K_0^+(\mathbb{A})) \cong (K_0(\mathbb{A}'), K_0^+(\mathbb{A}'))$.\]
2.4 The Jacobi-Perron continued fractions

1. Regular continued fractions

Let \( a_1, a_2 \in \mathbb{N} \) such that \( a_2 \leq a_1 \). Recall that the greatest common divisor of \( a_1, a_2, GCD(a_1, a_2) \), can be determined from the Euclidean algorithm:

\[
\begin{align*}
   a_1 &= a_2b_1 + r_3 \\
   a_2 &= r_3b_2 + r_4 \\
   r_3 &= r_4b_3 + r_5 \\
   & \vdots \\
   r_{k-3} &= r_{k-2}b_{k-1} + r_{k-1} \\
   r_{k-2} &= r_{k-1}b_k,
\end{align*}
\]

where \( b_i \in \mathbb{N} \) and \( GCD(a_1, a_2) = r_{k-1} \). The Euclidean algorithm can be written as the regular continued fraction

\[
\theta = \frac{a_1}{a_2} = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots + \frac{1}{b_k}}} = [b_1, \ldots, b_k].
\]

If \( a_1, a_2 \) are non-commensurable (i.e. \( \theta \in \mathbb{R} - \mathbb{Q} \)), then the Euclidean algorithm never stops and \( \theta = [b_1, b_2, \ldots] \). Note that the regular continued fraction can be written in a (projective) matrix form:

\[
\begin{pmatrix} 1 \\ \theta \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & b_k \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

2. The Jacobi-Perron continued fractions

The Jacobi-Perron algorithm and connected (multidimensional) continued fraction generalizes the Euclidean algorithm to the case \( GCD(a_1, \ldots, a_n) \) when \( n \geq 2 \). Namely, let \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \( \lambda_i \in \mathbb{R} - \mathbb{Q} \) and \( \theta_{i-1} = \frac{\lambda_i}{\lambda_1} \), where \( 1 \leq i \leq n \) and \( \lambda_1 \neq 0 \). Consider a projective equivalence class \((1, \theta_1, \ldots, \theta_{n-1})\) of the vector \( \lambda \).

**Definition** The continued fraction

\[
\begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_{n-1} \end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & b_1^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1}^{(1)} \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & b_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{n-1}^{(k)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]

where \( b_i^{(j)} \in \mathbb{N} \cup \{0\} \), is called the Jacobi-Perron fraction of the vector \( \lambda \in \mathbb{R}^n \).
To recover the integers $b_i^{(k)}$ from the vector $(\theta_1, \ldots, \theta_{n-1})$, one has to repeatedly solve the following system of equations:

\[
\begin{align*}
\theta_1 &= b_1^{(1)} + \frac{1}{\theta_{n-1}^{(1)}}, \\
\theta_2 &= b_2^{(1)} + \frac{1}{\theta_{n-1}^{(1)}}, \\
&\vdots \\
\theta_{n-1} &= b_{n-1}^{(1)} + \frac{1}{\theta_{n-1}^{(1)}},
\end{align*}
\]

where $(\theta_1', \ldots, \theta_{n-1}')$ is the next input vector. Thus, each vector $(\theta_1, \ldots, \theta_{n-1})$ gives rise to a formal Jacobi-Perron continued fraction. Whether the fraction is convergent or not, is yet to be determined.

3. Convergent Jacobi-Perron continued fractions

Let us introduce some notation, see p. 13 of [2]. Define $A_{i}^{(j)} = \delta_{ij}$ to be the Kronecker delta, where $i, j = 0, \ldots, n - 1$. Suppose that the integers $A_1^{(\nu)}, \ldots, A_{n-1}^{(\nu)}$ are known. One can define by the induction:

\[
A_i^{(\nu+n)} = \sum_{j=0}^{n-1} b_i^{(\nu)} A_i^{(\nu+j)},
\]

where $b_i^{(0)} = 1$ for $i = 0, \ldots, n - 1$ and $\nu = 0, 1, \ldots, \infty$.

**Definition**  The Jacobi-Perron continued fraction is said to be convergent, if $\theta_i = \lim_{k \to \infty} A_i^{(k)} / A_0^{(k)}$ for all $i = 1, \ldots, n - 1$.

Unless $n = 2$, the convergence of the Jacobi-Perron fractions is a delicate question. To the best of our knowledge, there exists no intrinsic necessary and sufficient conditions for such a convergence. However, the Bauer criterion and Masur-Veech theorem (see below) imply that the Jacobi-Perron fractions converge for a generic set of vectors $\lambda \in \mathbb{R}^n$. We shall give in §2.4.5 an intrinsic sufficient condition, which we shall use to construct the toric $AF$-algebras.

4. Bauer’s criterion

Convergence of the Jacobi-Perron continued fractions can be characterized in terms of measured foliations. Let $\mathcal{F} \in \Phi_X$ be measured foliation on a surface $X$ of genus $g \geq 1$. Recall that $\mathcal{F}$ is called uniquely ergodic if every invariant measure of $\mathcal{F}$ is a multiple of the Lebesgue measure. By the Masur-Veech theorem, there exists a generic subset $V \subset \Phi_X$ such that each $\mathcal{F} \in V$ is uniquely ergodic [13, 24]. We let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector with coordinates $\lambda_i = \int_{\gamma_i} \phi$, where $\gamma_i \in H_1^{odd}(\tilde{X})$ and $\phi$ is the covering 1-form (a flow) of the foliation $\mathcal{F}$ on the covering surface $\tilde{X}$, see §2.1.5. By an abuse of notation, we shall say that $\lambda \in V$. Note that the covering flow $\phi$ can be obtained from $\lambda$ by the
method of zippered rectangles; in this case an interval exchange map \((\lambda, \pi)\) is defined. The last observation allows us to obtain the following characterization of convergence of the Jacobi-Peron fractions.

**Lemma (Bauer [1])** The Jacobi-Perron continued fraction of \(\lambda\) converges if and only if \(\lambda \in V \subset \mathbb{R}^n\).

5. **Sufficient conditions of convergence**

The following sufficient condition is due to O. Perron.

**Lemma (Perron [14], Satz II)** The Jacobi-Perron continued fraction

\[
\lim_{k \to \infty} \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)} \\
\end{pmatrix} \cdots \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(k)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(k)} \\
\end{pmatrix} \begin{pmatrix}
0 \\
\vdots \\
\end{pmatrix}
\]

is convergent, if for all \(i = 1, \ldots, n - 1\) and \(k = 1, \ldots, \infty\), it holds:

\[
0 < \frac{1}{b_{n-1}^{(k)}} \leq C \quad \text{and} \quad 0 \leq \frac{b_i^{(k)}}{b_{n-1}^{(k)}} < C,
\]

where \(C\) is a real constant independent of \(k\).

Perron’s lemma indicates when the continued fractions diverge. In particular, the divergence is caused by a fast growth of the entry \(b_i^{(k)}\) with respect to \(b_{n-1}^{(k)}\) as \(k \to \infty\). It is easy to see, that the regular continued (i.e. \(n = 2\)) and periodic Jacobi-Perron fractions are always convergent.

6. **Uniqueness of representation of vectors by the Jacobi-Perron continued fractions**

It is known that the positive irrational numbers are bijective with the regular continued fractions. Up to a multiple, the same result is valid for the convergent Jacobi-Perron continued fractions. Namely, the following is true.

**Lemma (Perron [14], Satz IV)** Let \(\tilde{\lambda}, \lambda \in \mathbb{R}^n\) be represented by the convergent Jacobi-Perron continued fractions \(b_i^{(j)}\) and \(\tilde{b}_i^{(j)}\), respectively. If \(\tilde{\lambda} = \mu \lambda\) for a \(\mu > 0\), then \(b_i^{(j)} = \tilde{b}_i^{(j)}\) for \(i = 1, \ldots, n\) and \(j = 1, \ldots, \infty\).

3 **The toric AF-algebras**

In the present section we introduce an object, which we shall call a toric AF-algebra. This is a special type of the AF-algebra, which can be regarded as a generalization of the Effros-Shen algebra to the case \(g \geq 2\). We start with the noncommutative torus (an irrational rotation algebra), whose dimension group
is order-isomorphic to such of an Effros-Shen algebra. We remind a classic theorem of Effros-Shen, Elliott, Pimsner-Voiculescu and Rieffel, which classifies the noncommutative tori up to a stable equivalence. Finally, a wider class of the $AF$-algebras is introduced and the corresponding examples are discussed.

3.1 The noncommutative torus

Let $\theta \in \mathbb{R} - \mathbb{Q}$. The noncommutative torus, $A_\theta$, is a norm-closed $C^*$-algebra generated by the unitary operators in the Hilbert space $L^2(S^1)$:

$$Uf(t) = z(t)f(t), \quad Vf(t) = f(t - \theta),$$

which are the multiplication by a unimodular function $z(t)$ and the rotation operators. It can be easily verified that $UV = e^{2\pi i \theta} VU$. Therefore, $A_\theta$ can be viewed as a universal $C^*$-algebra generated by unitaries $U,V$ subject to the commutation relation $UV = e^{2\pi i \theta} VU$.

3.2 The Effros-Shen algebra

Note that $K_1(A_\theta) = \mathbb{Z}^2$ and thus $A_\theta$ is not an $AF$-algebra. (Recall that every $AF$-algebra $\mathfrak{A}$ must have $K_1(\mathfrak{A}) = 0$.) However, there exists an important $AF$-algebra, closely related to $A_\theta$. We shall need the following classical result (due to Effros-Shen, Elliott, Pimsner-Voiculescu, Rieffel).

**Theorem** Let $A_\theta$ be a noncommutative torus and $[a_0, a_1, \ldots]$ be the continued fraction of $\theta$. Denote by $K_\theta$ an $AF$-algebra shown in Fig. 1 of introduction. Then $A_\theta$ can be embedded (as a $C^*$-subalgebra) into the $AF$-algebra $K_\theta$ and the dimension groups of $A_\theta$ and $K_\theta$ are order-isomorphic. Moreover, if $\alpha = [a_0, a_1, \ldots]$ and $\beta = [b_0, b_1, \ldots]$ then $K_\alpha$ and $K_\beta$ are stably isomorphic if and only if $a_{m+k} = b_m$ for an integer number $k \in \mathbb{Z}$. In other words, the irrational numbers $\alpha$ and $\beta$ are modular equivalent: $\beta = (a\alpha + b)/(c\alpha + d)$, $ad - bc = 1$, where $a, b, c, d \in \mathbb{Z}$.

In view of the theorem, $K_\theta$ can be seen as a “linear part” of the noncommutative torus $A_\theta$. We shall refer to the $AF$-algebra $K_\theta$ as an Effros-Shen algebra [6].

3.3 The toric $AF$-algebras

We conclude by a generalization of the Effros-Shen algebras to the case of genus $g \geq 2$. Following a suggestion of Yu. I. Manin (private communication), we shall call the resulting object a toric $AF$-algebra.

1. Definition
Let
\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix}
= \lim_{k \to \infty} \prod_{i=1}^{k}
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(i)} \\
0 & 1 & \ldots & 1 & b_2^{(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(i)}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix},
\]

where
\[
n = \begin{cases}
6g - 6, & \text{if } g \geq 2 \\
2, & \text{if } g = 1,
\end{cases}
\]
be a convergent Jacobi-Perron continued fraction. Let \( \mathbb{A} := \mathbb{A}_{\theta_1, \ldots, \theta_{n-1}} \) be an \( AF \)-algebra given by the following Bratteli diagram (shown for the case \( g = 2 \)):}

![Bratteli diagram](image)

**Figure 5:** The toric \( AF \)-algebra of genus \( g = 2 \).

where \( b_j^{(i)} \) is the multiplicity of edges of the graph. The \( AF \)-algebra \( \mathbb{A} \) will be called a toric \( AF \)-algebra of genus \( g \geq 2 \).

2. Equivalent definition

In view of Bauer’s criterion (§2.4.4), one can define a toric \( AF \)-algebra as an \( AF \)-algebra with a unique trace. Namely, the following lemma is true.

**Lemma** The \( AF \)-algebra \( \mathbb{A} \) is a toric \( AF \)-algebra if and only if it has a unique trace.

**Proof.** Let \( \mathbb{A} \) be a toric \( AF \)-algebra. Then the convergence of the corresponding Jacobi-Perron continued fraction implies the unique trace, see an argument found in the proof of Theorem 4 of Bauer’s paper [1].
The converse, that the unique trace implies convergence of the Jacobi-Perron fraction, is rather direct consequence of the definitions, or one may argue as in the proof of Theorem 3.2 of Effros-Shen [5]. □

3. Examples
We conclude with two examples, which illustrate a difference between the toric $AF$-algebras and general $AF$-algebras. One example is based on the Perron sufficient conditions for a convergence of the Jacobi-Perron fractions, another (a counterexample) on the Effros-Shen calculations made in [5].

**Example 1.** Let $A = A_{\theta_1, \ldots, \theta_n - 1}$ be an $AF$-algebra, whose Jacobi-Perron continued fraction satisfies the conditions $0 < \frac{1}{b_{n-1}} \leq C$ and $0 \leq \frac{1}{b_{n-1}} \leq C$, where $C$ is a real constant independent of $k$. Then, by Perron’s sufficient conditions (§2.4.6), the continued fraction is convergent. Thus, the $AF$-algebra $A$ is a toric $AF$-algebra.

**Example 2.** Let $n = 3$ and the $AF$-algebra $A := A_{\theta_1, \theta_2}$ has the Jacobi-Perron continued fraction of the form:

$$
\lim_{k \to \infty} \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & \beta_1 \\
0 & 1 & 0
\end{pmatrix} \cdots \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & \beta_k \\
0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix},
$$

where $\sum_{k=1}^{\infty} \frac{1}{\beta_k} < 1$. It has been shown by Effros and Shen [5] (Proposition 4.1), that the above Jacobi-Perron continued fraction is divergent. Thus, $A$ is the $AF$-algebra, which is not a toric $AF$-algebra.

4 Main results
The present section contains a precise definition of the category $S$ of generic Riemann surfaces and the category $A^*$ of the toric $AF$-algebras, as hinted in the introduction. We construct a map $F^*$ between the sets $\text{Obj} (S)$ and $\text{Obj} (A^*)$. Our main result says that map $F^*$ so constructed, is actually a functor.

4.1 The map $F^* : Ob (S) \to Ob (A^*)$

1. Category $S$.
Let $T(g)$ be the Teichmüller space of genus $g \geq 1$. By the Hubbard-Masur theory (§2.2), $T(g) - \{pt\} \cong \Phi_X$, where $\Phi_X$ is the space of measured foliations on a surface of genus $g$. Recall that the $HM$-foliation is a measured foliation, whose regular leaves are dense in $X$. By $V \subset T(g)$ we shall understand a preimage under homomorphism $h : T(g) - \{pt\} \to \Phi_X$ of the set of $HM$-foliations such that for any $F \in V$: 
Sing $\mathcal{F}$ consists of singular points, which are:
   
   (a) the singular points with $k = 1$ (tripods), if $g \geq 2$, or
   
   (b) the singular points with $k = 0$ (fake saddles), if $g = 1$;

(ii) $\mathcal{F}$ is a uniquely ergodic measured foliation.

Lemma The $V$ is a generic subset of $T(g)$.

Proof. Let $g \geq 2$ and $h(V^{i})$ a subset of $\Phi_X$ satisfying condition (i). Singular points of degree 1 correspond to simple zeroes of holomorphic quadratic forms on $X$, endowed with a complex structure. Holomorphic forms with such zeroes are stable with respect to small perturbation in the space of holomorphic quadratic forms and, in view of Morse theory, make a generic set therein. Since $h$ is continuous, $V^{i}$ is a generic set of $T(g)$. In case $g = 1$, observe that fake saddle is a removable singularity, i.e. the corresponding holomorphic form has no singular points. In this case $h(V^{i})$ coincides with the space of all $HM$-foliations, which are generic on torus for obvious reasons.

Let $h(V^{ii})$ be a subset of $\Phi_X$ satisfying condition (ii). It has been shown by Masur [13] and Veech [24] that the measured foliations with a uniquely ergodic invariant measure are generic in $\Phi_X$. Therefore, by continuity of $h$, $V^{ii}$ is generic in $T(g)$. It has been shown in the above cited works, that $h(V^{ii})$ is generic in $h(V^{i})$. Let $V = V^{i} \cap V^{ii}$. Since a generic subset of generic set is generic in $T(g)$, lemma follows. □

Definition 1 Let $\text{Ob} (\mathcal{S}) = \{ \mathcal{S} \in \mathcal{V} \mid \varphi(\mathcal{S}) \in \mathcal{V}, \varphi \in \text{Mod} (X) \}$ be a maximal subset of $\mathcal{V}$ closed under the isomorphisms of Riemann surfaces. The category $\mathcal{S} = \text{Ob} (\mathcal{S}) \cup \text{Mor} (\mathcal{S})$, where $\text{Mor} (\mathcal{S})$ are isomorphisms between the Riemann surfaces $\mathcal{S} \in \text{Ob} (\mathcal{S})$.

2. Category $\mathcal{A}^{*}$.

Denote by $NCS(g)$ a set of the toric $AF$-algebras of genus $g$, where $g \geq 1$ and

$$n = \begin{cases} 
6g - 6, & \text{if } g \geq 2 \\
2, & \text{if } g = 1.
\end{cases}$$

Definition 2 Let $\text{Ob} (\mathcal{A}^{*}) = \{ k_{\theta_1, \ldots, \theta_{n-1}} \in NCS(g) \mid \text{if } k_{\theta'_1, \ldots, \theta'_{n-1}} \cong k_{\theta_1, \ldots, \theta_{n-1}}, \\
\text{then } k_{\theta'_1, \ldots, \theta'_{n-1}} \in NCS(g) \}$ be maximal subset of $NCS(g)$ closed under the stable isomorphisms between the $AF$-algebras. The category $\mathcal{A}^{*} = \text{Ob} (\mathcal{A}^{*}) \cup \text{Mor} (\mathcal{A}^{*})$, where $\text{Mor} (\mathcal{A}^{*})$ are stable isomorphisms between the $AF$-algebras $k_{\theta_1, \ldots, \theta_{n-1}} \in \text{Ob} (\mathcal{A}^{*})$.

Note that $\mathcal{A}^{*} = \mathcal{A}$ whenever $g = 1$.

3. Mapping $F^{*} : \text{Ob} (\mathcal{S}) \to \text{Ob} (\mathcal{A}^{*})$.  

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Let $S \in \text{Ob}(\mathcal{S})$ be a Riemann surface of genus $g \geq 1$. The assignment $S \mapsto \mathcal{A}_{\theta_1, \ldots, \theta_{n-1}}$, where $\mathcal{A}_{\theta_1, \ldots, \theta_{n-1}} \in \mathcal{A}^*$, is as follows. Let $h : T(g) - \{pt\} \to \Phi_X$ be the Hubbard-Masur homeomorphism. Then $h(S) = \mathcal{F}(\lambda_1, \ldots, \lambda_n)$, where $(\lambda_1, \ldots, \lambda_n)$ are the Douady-Hubbard coordinates in $\Phi_X$. Without loss of generality, we assume $\lambda_1 \neq 0$ and let $\theta_1 = \frac{\lambda_1}{\lambda_1}, \ldots, \theta_{n-1} = \frac{\lambda_{n-1}}{\lambda_1}$.

**Lemma** $\mathcal{A}_{\theta_1, \ldots, \theta_{n-1}}$ is a toric $\mathcal{A}F$-algebra.

**Proof.** Indeed, by item (i) of definition of the category $\mathcal{S}$, all singular points of the foliation $\mathcal{F}$ are the tripods (simple zeroes). In this case, the argument of [3] shows that $\mathcal{F}$ depends on $n = 6g - 6$ ($n = 2$) real parameters if $g \geq 2$ ($g = 1$).

On the other hand, by item (ii) of the definition of the category $\mathcal{S}$, $\mathcal{F}$ is a uniquely ergodic $HM$-foliation. By Bauer’s criterion, the Jacobi-Perron continued fraction for the vector $(\theta_1, \ldots, \theta_{n-1})$ is convergent. Thus, $\mathcal{A}_{\theta_1, \ldots, \theta_{n-1}}$ is a toric $\mathcal{A}F$-algebra. □

**Definition 3** By $F^* : \text{Ob}(\mathcal{S}) \to \text{Ob}(\mathcal{A}^*)$, we understand a map given by the formula $S \mapsto \mathcal{A}_{\theta_1, \ldots, \theta_{n-1}}$.

### 4.2 Results

The main result of present paper is a functoriality of the map $F^*$. The latter means that $F^*$ maps each pair of the isomorphic Riemann surfaces to a pair of the stably isomorphic toric $\mathcal{A}F$-algebras. It should be noted, that $F^*$ is not an injective functor, and therefore $\mathcal{S}$ and $\mathcal{A}^*$ are not the equivalent categories. (This is by no means a devaluing factor – compare with the classical homology functor, which maps topological spaces to the abelian groups – but is not an injective functor.) A summary of our results can be expressed as follows.

**Theorem 1** Let $V = \text{Ob}(\mathcal{S})$ and $W = \text{Ob}(\mathcal{A}^*)$. The map $F^*$ satisfies the following properties:

(i) $V \cong W \times (0, \infty)$ is a trivial fiber bundle, such that the fiber map $\pi : V \to W$ coincides with $F^*$;

(ii) $F^*$ is a covariant functor from the category $\mathcal{S}$ to the category $\mathcal{A}^*$, which maps isomorphic Riemann surfaces $S, S' \in \mathcal{S}$ to the stably isomorphic toric $\mathcal{A}F$-algebras $\mathcal{A}, \mathcal{A}' \in \mathcal{A}^*$.

### 5 Proofs

#### 5.1 Proof of Theorem 1

Let us outline the proof. We shall consider the following sets of objects:

(i) generic Riemann surfaces $V$;

(ii) pseudo-lattices $\mathcal{PL}$;

We owe our terminology to Yu. I. Manin, who introduced pseudo-lattices in the context of the real multiplication problem [12].
The proof takes the following steps:

(a) to show that $V \cong \mathcal{P}\mathcal{L}$ are equivalent categories such that isomorphic Riemann surfaces $S, S' \in V$ map to isomorphic pseudo-lattices $PL, PL' \in \mathcal{P}\mathcal{L}$;

(b) a non-injective functor $F: \mathcal{P}\mathcal{L} \to \mathcal{P}\mathcal{P}\mathcal{L}$ is constructed. The $F$ maps isomorphic pseudo-lattices to isomorphic projective pseudo-lattices and $\text{Ker } F \cong (0, \infty)$;

(c) to show that a subcategory $U \subseteq \mathcal{P}\mathcal{P}\mathcal{L}$ and $W$ are the equivalent categories.

In other words, we have the following diagram:

\[
\begin{array}{ccc}
V & \overset{\alpha}{\longrightarrow} & \mathcal{P}\mathcal{L} & \overset{F}{\longrightarrow} & U & \overset{\beta}{\longrightarrow} & W,
\end{array}
\]

where $\alpha$ is an injective map, $\beta$ is a bijection and $\text{Ker } F \cong (0, \infty)$.

Category $V$. A Riemann surface is a triple $(X, S, j)$, where $X$ is a topological surface of genus $g \geq 1$, $j: X \to S$ is a complex (conformal) parametrization of $X$ and $S$ is a Riemann surface. A morphism of Riemann surfaces $(X, S, j) \to (X, S', j')$ is a biholomorphic map modulo the ones, which are isotopic to the identity map with respect to a fixed topological marking of $X$. A category of generic Riemann surfaces $S$, consists of $\text{Ob } (S)$ which are Riemann surfaces $S \in V \subset T(\mathbb{S})$ and morphisms $H(S, S')$ between $S, S' \in \text{Ob } (V)$ which coincide with the morphisms specified above. For any $S, S', S'' \in \text{Ob } (S)$ and any morphisms $\varphi': S \to S'$, $\varphi'': S' \to S''$ a morphism $\phi: S \to S''$ is the composite of $\varphi'$ and $\varphi''$, which we write as $\phi = \varphi'' \varphi'$. The identity morphism, $1_S$, is a morphism $H(S, S)$.

Category $\mathcal{P}\mathcal{L}$. A pseudo-lattice (of rank $n$) is a triple $(\Lambda, R, j)$, where $\Lambda \cong \mathbb{Z}^n$ and $j: \Lambda \to R$ is a homomorphism. A morphism of pseudo-lattices $(\Lambda, R, j) \to (\Lambda, R, j')$ is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z}^n & \overset{j}{\longrightarrow} & \mathbb{R} \\
\varphi \downarrow & & \psi \\
\mathbb{Z}^n & \overset{j'}{\longrightarrow} & \mathbb{R}
\end{array}
\]

where $\varphi$ is a group homomorphism and $\psi$ is an inclusion map, i.e. $j'(\Lambda') \subseteq j(\Lambda)$. Any isomorphism class of a pseudo-lattice contains a representative given by
$j : \mathbb{Z}^n \to \mathbb{R}$ such that

\[
j(1,0,\ldots,0) = \lambda_1, \quad j(0,1,\ldots,0) = \lambda_2, \quad \ldots, \quad j(0,0,\ldots,1) = \lambda_n,
\]

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are positive reals. The pseudo-lattices of rank $n$ make up a category, which we denote by $\mathcal{PL}_n$.

The following lemma implies a functoriality of the map $F^*$: Roughly, the lemma says that the $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ is an invariant of the isomorphism class of the Riemann surface $\mathcal{S}$ (a modulus of $\mathcal{S}$). The action of $\text{Mod}(X)$ on such a module corresponds to a transformation of the basis of the module. The proof of lemma is based on the explicit formulas for $\lambda_i$.

**Lemma 1 (Basic lemma)** Let $g \geq 2$ ($g = 1$) and $n = 6g - 6$ ($n = 2$). There exists an injective covariant functor $\alpha : V \to \mathcal{PL}_n$ which maps isomorphic Riemann surfaces $\mathcal{S}, \mathcal{S}' \in V$ to the isomorphic pseudo-lattices $\mathcal{PL}, \mathcal{PL}' \in \mathcal{PL}_n$.

**Proof.** Let $\alpha : T(g) - \{pt\} \to \text{Hom}(H_1^{\text{odd}}(\mathcal{S}); \mathbb{R}) - 0$ be a Hubbard-Masur map, see §2.2. Since $\alpha$ is a homeomorphism between the respective spaces, we conclude that $\alpha$ is an injective map. The first claim of lemma is proved.

Let us show that $\alpha$ sends morphisms of $\mathcal{S}$ to morphisms of $\mathcal{PL}$. Let $\varphi \in \text{Mod}(X)$ be a diffeomorphism of $X$. Suppose that all the zeroes of measured foliations are generic (simple) and let $p : \tilde{X} \to X$ be the double cover of $X$ as explained in §2.2.3. (Note that the case of torus does not require a double cover, and thus one can assert $p = \text{Id}$ in the argument below.) Denote by $\tilde{\varphi}$ a diffeomorphism of $\tilde{X}$, which makes the following diagram commutative:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{\varphi} & X
\end{array}
\]

One can consider the effect of $\varphi, \tilde{\varphi}$ and $p$ on the respective (relative) integral homology groups:

\[
H_1^{\text{odd}}(\tilde{X}) \oplus H_1^{\text{even}}(\tilde{X}) \xrightarrow{\tilde{\varphi}^*} H_1^{\text{odd}}(\tilde{X}) \oplus H_1^{\text{even}}(\tilde{X})
\]

\[
\begin{array}{ccc}
H_1(X, \text{Sing } \mathcal{F}) & \xrightarrow{\varphi_*} & H_1(X, \text{Sing } \mathcal{F}) \\
\downarrow p_* & & \downarrow p_*
\end{array}
\]
where $\text{Ker } p_\ast \cong H_1^{\text{even}}(\bar{X})$. Since $p_\ast : H_1^{\text{odd}}(\bar{X}) \to H_1(X, \text{Sing } \mathcal{F})$ is an isomorphism, we conclude that $\bar{\varphi}_\ast \in GL_n(\mathbb{Z})$, where $n = \dim H_1^{\text{odd}}(\bar{X})$. It is easy to see, that $\bar{\varphi}_\ast$ acts on a pseudo-lattice by a transformation of its basis, and therefore, $\bar{\varphi}_\ast \in \text{Mor } (\mathcal{PL})$.

Let us show that $\alpha$ is a functor. Indeed, let $\mathcal{S}, \mathcal{S}' \in V$ be isomorphic Riemann surfaces, such that $\mathcal{S}' = \varphi(\mathcal{S})$ for a $\varphi \in \text{Mod } (X)$. Let $a_{ij}$ be the elements of matrix $\bar{\varphi}_\ast \in GL_n(\mathbb{Z})$. Recall that:

$$\lambda_i = \int_{\gamma_i} \phi$$

for a closed 1-form $\phi = \text{Re } \omega$ and $\gamma_i \in H_1^{\text{odd}}(\bar{X})$. Then

$$\gamma_j = \sum_{i=1}^n a_{ij} \gamma_i, \quad j = 1, \ldots, n,$$

are the elements of a new basis in $H_1^{\text{odd}}(\bar{X})$. By the integration rules,

$$\lambda'_j = \int_{\gamma'_j} \phi = \int_{\sum a_{ij} \gamma_i} \phi = \sum_{i=1}^n a_{ij} \lambda_i,$$

Finally, let $j(\Lambda) = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $j'(\Lambda) = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Since $\lambda'_j = \sum_{i=1}^n a_{ij} \lambda_i$ and $(a_{ij}) \in GL_n(\mathbb{Z})$, we conclude that:

$$j(\Lambda) = j'(\Lambda).$$

In other words, the $\mathbb{Z}$-module $\mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ is an invariant of $\text{Mod } (X)$. In particular, the pseudo-lattices $(\Lambda, \mathbb{R}, j)$ and $(\Lambda, \mathbb{R}, j')$ are isomorphic. Hence, $\alpha : V \to \mathcal{PL}$ maps isomorphic Riemann surfaces to the isomorphic pseudo-lattices, i.e. $\alpha$ is a functor.

Finally, let us show that $\alpha$ is a covariant functor. Indeed, let $\varphi_1, \varphi_2 \in \text{Mor}(\mathcal{S})$. Then $\alpha(\varphi_1 \varphi_2) = (\varphi_1 \varphi_2)_\ast = (\bar{\varphi}_1)_\ast (\bar{\varphi}_2)_\ast = \alpha(\varphi_1) \alpha(\varphi_2)$. Lemma 1 follows. □

Category $\mathcal{PPL}$. A projective pseudo-lattice (of rank $n$) is a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^n$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of projective pseudo-lattices $(\Lambda, \mathbb{C}, j) \to (\Lambda, \mathbb{R}, j')$ is a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z}^n & \xrightarrow{j} & \mathbb{R} \\
\downarrow \varphi & & \downarrow \psi \\
\mathbb{Z}^n & \xrightarrow{j'} & \mathbb{R}
\end{array}
\]
where \( \varphi \) is a group homomorphism and \( \psi \) is an \( \mathbb{R} \)-linear map. (Notice that unlike the case of pseudo-lattices, \( \psi \) is a scaling map as opposite to an inclusion map. This allows the two pseudo-lattices to be projectively equivalent, while being distinct in the category \( \mathcal{P}L_n \).) It is not hard to see that any isomorphism class of a projective pseudo-lattice contains a representative given by \( j : \mathbb{Z}^n \to \mathbb{R} \) such that

\[
\begin{align*}
    j(1,0,\ldots,0) &= 1, \\
    j(0,1,\ldots,0) &= \theta_1, \\
    j(0,0,\ldots,1) &= \theta_{n-1},
\end{align*}
\]

where \( \theta_i \) are positive reals. The projective pseudo-lattices of rank \( n \) make up a category, which we denote by \( \mathcal{P}PL_n \).

Category \( W \). Finally, the toric \( AF \)-algebras \( A = A_{\theta_1,\ldots,\theta_{n-1}} \), modulo the stable isomorphism between them, make up a category, which we shall denote by \( W_n \).

The following equivalence of categories is immediate.

**Lemma 2** Let \( U_n \subseteq \mathcal{P}PL_n \) be a subcategory consisting of the projective pseudo-lattices \( \mathcal{P}PL = \mathcal{P}PL(1,\theta_1,\ldots,\theta_{n-1}) \) for which the Jacobi-Perron fraction of the vector \( (1,\theta_1,\ldots,\theta_{n-1}) \) converges to the vector. Define a map \( \beta : U_n \to W_n \) by the formula

\[
\mathcal{P}PL(1,\theta_1,\ldots,\theta_{n-1}) \mapsto A_{\theta_1,\ldots,\theta_{n-1}}.
\]

Then \( \beta \) is a bijective functor, which maps isomorphic projective pseudo-lattices to the stably isomorphic toric \( AF \)-algebras.

**Proof.** It is evident that \( \beta \) is injective and surjective. Let us show that \( \beta \) is a functor. Indeed, according to \( \S 2.3.4 \), every totally ordered abelian group of rank \( n \) has form \( \mathbb{Z} + \theta_1 \mathbb{Z} + \ldots + \theta_{n-1} \mathbb{Z} \). The latter is a projective pseudo-lattice \( \mathcal{P}PL \) from the category \( U_n \). On the other hand, by the Elliott theorem (\( \S 2.3.5.C \)), the \( \mathcal{P}PL \) defines a stable isomorphism class of the \( AF \)-algebra \( A_{\theta_1,\ldots,\theta_{n-1}} \in W_n \). Therefore, \( \beta \) maps isomorphic projective pseudo-lattices (from the set \( U_n \)) to the stably isomorphic toric \( AF \)-algebras, and vice versa. Lemma 2 follows. \( \square \)

Let \( PL(\lambda_1,\lambda_2,\ldots,\lambda_n) \in \mathcal{P}L_n \) and \( \mathcal{P}PL(1,\theta_1,\ldots,\theta_{n-1}) \in \mathcal{P}PL_n \). To finish the proof of theorem 1 it remains to show the following.

**Lemma 3** Let \( F : \mathcal{P}L_n \to \mathcal{P}PL_n \) be a map given by formula

\[
PL(\lambda_1,\lambda_2,\ldots,\lambda_n) \mapsto \mathcal{P}PL\left(1,\frac{\lambda_2}{\lambda_1},\ldots,\frac{\lambda_n}{\lambda_1}\right).
\]

Then \( \text{Ker } F = (0,\infty) \) and \( F \) is a functor which maps isomorphic pseudo-lattices to isomorphic projective pseudo-lattices.

**Proof.** Indeed, \( F \) can be thought as a map from \( \mathbb{R}^n \) to \( \mathbb{R}P^{n-1} \). Hence \( \text{Ker } F = \{ \lambda_1 : \lambda_1 > 0 \} \cong (0,\infty) \). The second part of lemma is evident. \( \square \)

Assuming \( n = 6g - 6 \) (\( n = 2 \) for \( g \geq 2 \) (\( g = 1 \)), one gets theorem 1 from lemmas 1, 2, 3. \( \square \)
6 The complex tori and the Effros-Shen algebras

Recall that our study has been motivated by a correspondence between the complex tori and the Effros-Shen algebras. In view of theorem 1, we have the following corollary.

**Corollary 1** There exists a covariant (non-injective) functor $F : \mathcal{B} \to \mathcal{A}$, which maps isomorphic complex tori to the stably Effros-Shen algebras.

*Proof.* Let $g = 1$ and $n = 2$ in §5.1, and repeat the argument. $\square$

Given that the case $g = 1$ is of special beauty, we wish to independently prove corollary 1. As the reader will see, the exposition simplifies and some formulas become explicit. Our goal is the moduli space of the Effros-Shen algebras. One starts with the definition of the complex tori and the related moduli space. We analyze the Hubbard-Masur homeomorphism and measured foliations in the context of the complex tori. The explicit formulas for the functor $F$ are given. Finally, we take a look at the “moduli space” for the Effros-Shen algebras.

1. Complex tori

**A.** Let $\omega_1$ and $\omega_2$ be non-zero complex numbers which are linearly independent over $\mathbb{R}$. Recall that the quotient space $\mathbb{B}(\omega_1, \omega_2) = \mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ is called a complex torus. The conformal transformation $z \mapsto \pm \frac{\omega_2}{\omega_1} z$ brings $\mathbb{B}(\omega_1, \omega_2)$ to a normal form with $\omega_1 = \tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and $\omega_2 = 1$ and we let $\mathbb{B}_\tau = \mathbb{B}(\tau, 1)$.

**B.** Let $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ be a lattice in the complex plane $\mathbb{C}$. Recall that $\Lambda$ defines an elliptic curve $E(\mathbb{C}) : y^2 = 4x^3 - g_2x - g_3$ via the complex analytic map $\mathbb{C}/\Lambda \to E(\mathbb{C})$ given by formula $z \mapsto (\wp(z, \Lambda), \wp'(z, \Lambda))$, where $g_2 = 60 \sum_{\omega \in \Lambda^\times} \omega^{-4}$, $g_3 = 140 \sum_{\omega \in \Lambda^\times} \omega^{-6}$, $\Lambda^\times = \Lambda - \{0\}$ and

$$\wp(z, \Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^\times} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

is the Weierstrass $\wp$ function. We identify elliptic curves $E(\mathbb{C})$ with complex tori $\mathbb{C}/\Lambda$. Taking elliptic curves with coefficients in different subfields of $\mathbb{C}$, one gets an important connection of complex tori to number theory.

**C.** Let $\mathbb{B}(\omega_1, \omega_2)$ be complex torus, corresponding to the elliptic curve $E(\mathbb{C}) : y^2 = 4x^3 - g_2x - g_3$. The holomorphic differential:

$$\omega_N = \frac{dx}{2y} = \frac{dy}{12x^2 - g_2}$$

is invariant under translations in $\mathbb{C}$. We refer to $\omega_N$ as a Néron differential. It is connected to the periods $\omega_i$ by simple formulas:

$$\omega_1 = \int_{\gamma_1} \omega_N, \quad \omega_2 = \int_{\gamma_2} \omega_N.$$
where \( \{ \gamma_1, \gamma_2 \} \) is a homology basis of complex torus.

D. Let us find, when complex tori \( B(\omega_1, \omega_2) \) and \( B(\omega'_1, \omega'_2) \) are isomorphic. If \( \Lambda = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z} \) is a lattice with basis \( (\omega_1, \omega_2) \), then any new basis of \( \Lambda \) has form \( \omega'_1 = a\omega_1 + b\omega_2, \omega'_2 = c\omega_1 + d\omega_2 \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \). Let \( \Lambda' = \omega'_1 \mathbb{Z} + \omega'_2 \mathbb{Z} \). Since \( B(\omega_1, \omega_2) \) and \( B(\omega'_1, \omega'_2) \) are isomorphic if and only if \( \Lambda' = \alpha \Lambda \) for an \( \alpha \in \mathbb{C} - 0 \), we conclude that \( \tau' = \frac{\omega'_1}{\omega'_2} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\tau + b}{c\tau + d} \). Thus, \( B_\tau, B_{\tau'} \) are isomorphic whenever \( \tau' \equiv \tau \mod SL_2(\mathbb{Z}) \).

2. Moduli of complex tori

A. The moduli, \( \mathcal{M} \), is a space of isomorphism classes of complex tori (Riemann surfaces). By formulas above, \( \mathcal{M} = \mathbb{H}/SL_2(\mathbb{Z}) \). To study action of \( SL_2(\mathbb{Z}) \) on the open upper half-plane \( \mathbb{H} \), recall that \( A \in SL_2(\mathbb{Z}) \) is called hyperbolic (parabolic; elliptic) whenever \( |Tr A| > 2 \) (\( |Tr A| = 2 \); \( |Tr A| < 2 \)). Every hyperbolic (parabolic; elliptic) transformation has two real (one real; two complex conjugate) fixed points. Accordingly, the action of hyperbolic (parabolic) elements on \( \mathbb{H} \) is free, while elliptic must have fixed points.

B. Let us specify the fundamental region (orbit space) of \( SL_2(\mathbb{Z}) \). Recall that open half-plane \( \mathbb{H} = \{ x + iy \in \mathbb{C} | y > 0 \} \) admits a hyperbolic metric \( ds = |dz|/y \) such that \( SL(2,\mathbb{Z}) \) acts on \( \mathbb{H} \) by isometries (linear-fractional transformations). The tessellation of \( \mathbb{H} \) by fundamental regions is shown on Fig.6.

![Figure 6: \( \mathbb{H} \) and fundamental region \( \Omega \).](image)

C. Let \( A \) be a fixed point of transformation \( A \). Recall that the order of \( z \) is the smallest positive integer \( n \), such that \( A^n = Id \). To determine fixed points and their order, denote by \( U(z) = z + 1 \) a parabolic and by \( T(z) = -\frac{1}{2}z \) an elliptic transformation. Then the fixed points at the boundary of \( \Omega \) are as follows:

(i) \( z = \infty \) is a fixed point of \( U \) of infinite order;
(ii) \( z = i \) is a fixed point of \( T \) of order 2;
(iii) \( z = \pm \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \) are fixed points of \( S = TU^{-1} \) of order 3.

3. The Hubbard-Masur theory

A. Let us formulate the Hubbard-Masur theory (§2.2) in the case \( g = 1 \). The main simplification is that holomorphic quadratic differentials on complex tori are squares of abelian (degree 1) differentials, i.e. \( q = \omega^2 \), where \( \omega \) is a holomorphic differential. The last fact is an implication of index formula, telling that quadratic differentials cannot have zeroes on a complex torus. (Indeed, if \( q(p) = 0 \), then \( p \) is a singular point of negative index, which contradicts zero Euler characteristic of tori.) Thus, the covering surface \( \tilde{X} = T^2 \cup T^2 \) (a double copy of the torus).

B. Let \( \phi = \Re \omega \) be a 1-form defined by \( \omega \). Since \( \omega \) is holomorphic, \( \phi \) is a closed form on \( T^2 \). The \( \mathbb{R} \)-isomorphism \( h_q : H^0(S, \Omega) \to \text{Hom}(H_1(T^2); \mathbb{R}) \), as explained, is given by formulas:

\[
\begin{align*}
\lambda_1 &= \int_{\gamma_1} \phi \\
\lambda_2 &= \int_{\gamma_2} \phi,
\end{align*}
\]

where \( \{\gamma_1, \gamma_2\} \) is a homology basis of \( T^2 \). With no restriction, one can assume that \( \lambda_1, \lambda_2 \) are positive.

4.Measured foliations on tori

Denote by \( \Phi_{T^2} \) the space of measured foliations on \( T^2 \). Each \( F \in \Phi_{T^2} \) is measure equivalent to foliation by a family of parallel lines of slope \( \theta \) and an invariant transverse measure \( \mu \) (Fig.7).

![Figure 7: Measured foliation \( F \) on \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \).](image)

For brevity, we shall use notation \( F^\theta_\mu \) for such a foliation. There exists a simple relationship between \( (\lambda_1, \lambda_2) \) and \( (\theta, \mu) \). Indeed, \( \phi = \text{Const} \) defines a measured foliation, \( F^\theta_\mu \), such that:

\[
\begin{align*}
\lambda_1 &= \int_{\gamma_1} \phi = \int_0^1 \mu dx \\
\lambda_2 &= \int_{\gamma_2} \phi = \int_0^1 \mu dy, \text{ where } \frac{dy}{dx} = \theta.
\end{align*}
\]

By an integration, we get

\[
\begin{align*}
\lambda_1 &= \int_0^1 \mu dx = \mu \\
\lambda_2 &= \int_0^1 \mu \theta dx = \mu \theta.
\end{align*}
\]
Thus,
\[ \mu = \lambda_1 \quad \text{and} \quad \theta = \frac{\lambda_2}{\lambda_1}. \]

5. The functor \( F : B \to A \)

Let us establish explicit formulas for the functor \( F \). Recall (§6.1.C) that \( \omega_1 = \int_{\gamma_1} \omega_N \) and \( \omega_2 = \int_{\gamma_2} \omega_N \), where \( \omega_N \) is the Néron differential and \( \{\gamma_1, \gamma_2\} \) is a homology basis of complex torus. The functor \( F : B \to A \) is given by composition of \( \pi \) and \( h \), presented by the diagram:

\[
\tau = \frac{\int_{\gamma_2} \omega_N}{\int_{\gamma_1} \omega_N} \rightarrow h \rightarrow F \left( \frac{\int_{\gamma_1} \phi}{\int_{\gamma_2} \phi} \right) \rightarrow \frac{\int_{\gamma_2} \phi}{\int_{\gamma_1} \phi} \rightarrow A \left( \frac{\int_{\gamma_2} \phi}{\int_{\gamma_1} \phi} \right),
\]

where \( h : T(1) - \{pt\} \to \Phi_{T^2} \) is the Hubbard-Masur homeomorphism. We shall use the above formula for a calculation of the moduli of the Effros-Shen algebras.

6. Moduli of the Effros-Shen algebras (fixed points)

A. The aim of present section is to use the explicit formulas for the functor \( F \) to elucidate a moduli problem for the Effros-Shen algebras. As the first step, we wish to find the fixed points of the parameter space \((\theta, \mu)\) under the equivalence transformations, as it was done for the complex tori in §6.2.C. It will be proved, that unlike the complex tori, only parabolic transformations have the fixed points in the plane \((\theta, \mu)\).

B. Let \( P = \{ (\theta, \mu) \mid \mu > 0, \theta \in \mathbb{R} \} \) be a parameter space of the measured foliations. To obtain the explicit formulas for the transformations of \( P \), induced by the modular transformations of \( H \), let \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \). Since

\[
\begin{cases}
\lambda_1 = \int_{\gamma_1} \phi \\
\lambda_2 = \int_{\gamma_2} \phi
\end{cases}
\quad \text{and} \quad
\begin{cases}
\lambda'_1 = \int_{\gamma'_1} \phi \\
\lambda'_2 = \int_{\gamma'_2} \phi
\end{cases}
\]

we conclude (by the integration rules), that

\[
\begin{cases}
\gamma'_1 = a\gamma_1 + b\gamma_2 \\
\gamma'_2 = c\gamma_1 + d\gamma_2,
\end{cases}
\]

Since \( \mu = \lambda_1 \) and \( \theta = \frac{\lambda_2}{\lambda_1} \), one gets that the modular group \( SL_2(\mathbb{Z}) \) acts on the points of plane \( P \) by the formulas:

\[
\begin{cases}
\theta \mapsto \frac{e + d\theta}{a + b\theta} \\
\mu \mapsto \mu(a + b\theta).
\end{cases}
\]

For brevity, let us call the above transformation a \( P \)-transformation.
C. Let $r : G \times X \to X$ be action of a group $G$ on the set $X$. Recall that a stabilizer of the fixed set $Z \subseteq X$ of $r$ is a subgroup $G'$, such that $g(x) = x$, where $g \in G'$ and $x \in Z$. The following lemma gives a description of the set of fixed points of $P$-transformations and their stabilizers.

**Lemma** Let $Z = \{(\theta, \mu) \in P \mid \theta = \frac{p}{q}, p, q \in \mathbb{Z}, \mu > 0\}$. The set of fixed points of $P$-transformations coincides with $Z$ and the stabilizer of every line $\theta = \frac{p}{q}$ is a parabolic element of $SL_2(\mathbb{Z})$.

**Proof.** To find the fixed points and their stabilizers, one has to solve the following system of equations:

\[
\begin{aligned}
\theta &= \frac{c+\mu d}{a+b\theta} \\
\mu &= \mu(a+b\theta).
\end{aligned}
\]

Note that the above equations do not depend on $\mu$ (since $\mu \neq 0$) and are equivalent to:

\[
\begin{aligned}
\frac{c+\mu d}{a+b\theta} &= \frac{c+(d-1)\theta}{(a-1)+b\theta} \\
1 &= a+b\theta \\
\end{aligned}
\]

On order the last system of equations to be solvable (for $\theta$), the following determinant must vanish:

\[
\begin{vmatrix}
c & d-1 \\
a-1 & b
\end{vmatrix} = 0.
\]

Therefore, $bc - (a-1)(d-1) = bc - ad - 1 + a + d = 0$ and, since $ad - bc = 1$, one arrives at $a + d = 2$. Thus, a solution $\theta = \frac{1}{2} \in \mathbb{Q}$ exists, if and only if, the $P$-transformation is parabolic. Note that because our equations are independent of $\mu$, the vertical line $\theta = \frac{p}{q}$ in the plane $P$ is a fixed set of the parabolic transformation.

Let us show that every rational point $\theta \in \mathbb{R}$ is the fixed point of a parabolic $P$-transformation. Indeed, it is known that any $x, y \in \mathbb{Q}$ are equivalent modulo $SL_2(\mathbb{Z})$, i.e. there exists $T \in SL_2(\mathbb{Z})$, such that $y = T(x)$. Let $A(x) = x$ be a parabolic transformation with the fixed point $x$. Then $TAT^{-1}(y) = TA(x) = T(x) = y$ and therefore $y \in \mathbb{Q}$ is the fixed point for the parabolic transformation $TAT^{-1}$. □

7 Projective curvature

The functor $F$ allows to introduce new invariants of the Riemann surfaces. We shall discuss one such invariant (a projective curvature) for the complex tori and the Riemann surfaces of higher genus. It has been conjectured by Yu. I. Manin [12] (real multiplication program), that the projective curvature of an elliptic curve with the complex multiplication plays a prominent rôle in the explicit class field theory (the abelian extensions of the real quadratic number fields).
7.1 Projective curvature of a complex torus

Let $F : \mathcal{B} \to \mathcal{A}$ be a functor from the category of complex tori to the category of Effros-Shen algebras. By a projective curvature of the complex torus $B \tau \in \mathcal{B}$, one understands a real number $\theta$, such that $F(B \tau) = A \theta$.

Example. Let $\tau \in \mathbb{Q}(\sqrt{-d})$ be an algebraic number belonging to an imaginary quadratic number field. Then the projective curvature of $B \tau$ is an irrational number belonging to a real quadratic number field. (This is a starting point of Manin’s real multiplication program, see [12].)

Although it is possible to evaluate projective curvature for certain complex tori, an analytic formula for the function $\theta = f(\tau)$ seems to be out of reach so far. It is likely that the function is of a class $C^0$.

Problem 1 Find a formula for the function $f$.

7.2 Projective curvature of a Riemann surface

A. Let $\mathbb{A}_01,\ldots,\mathbb{A}_n-1$ be a toric $AF$-algebra, whose Jacobi-Perron continued fraction has the form:

\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(1)} \\
\vdots  & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)}
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_1^{(2)} \\
\vdots  & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(2)}
\end{pmatrix}
\begin{pmatrix}
0 \\
\vdots \\
\vdots \\
1
\end{pmatrix}.
\]

Consider a real number $\theta$, given by the regular continued fraction:

\[\theta = [b_1^{(1)} + 1, \ldots, b_{n-1}^{(1)} + 1, b_1^{(2)} + 1, \ldots, b_{n-1}^{(2)} + 1, \ldots].\]

In view of Perron’s uniqueness lemma (§2.4.6), the mapping $\nu : \mathcal{A}^* \to \mathbb{R}$ defined by the formula $\mathbb{A}_01,\ldots,\mathbb{A}_n-1 \mapsto \theta$ is injective. By a projective curvature of a Riemann surface $S \in \mathcal{S}$, one understands the real number $\theta = \nu(\mathbb{A}_01,\ldots,\mathbb{A}_n-1)$, where $\mathbb{A}_01,\ldots,\mathbb{A}_n-1 = F^*(S)$.

B. There exists a geometric interpretation of the projective curvature. Let $F$ be a measured foliation the torus. Then $F$ is equivalent to a foliation by the parallel lines of a slope $\theta$. The notion of “slope” extends to the surfaces of higher genera. Although there exists no globally defined parallel lines on the surfaces of genus $g \geq 2$ (due to the existence of the non-trivial singular points), one can define a slope as an average inclination of the leaves of measured foliation with respect to the generators of the first homology group. From this point of view, the projective curvature is a measure of the slope.

C. We conclude by an open question regarding the projective curvature. Intuitively, the projective curvature is responsible for an “asymptotic geometry” of the Riemann surface. The following question seems to be interesting.
Problem 2 To classify the Riemann surfaces in terms of their projective curvature. In particular, what are the values of $\theta$ on a generic Riemann surface belonging to the category $\mathcal{A}^*$?

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