Efimov universality with Coulomb interaction

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(Dated: January 14, 2019)

Abstract

The universal properties of charged particles are modified by the presence of a long-range Coulomb interaction. We investigate the modification of Efimov universality as a function of the Coulomb strength using the Gaussian expansion method. The resonant short-range interaction is described by Gaussian potentials to which a Coulomb potential is added. We calculate binding energies and root mean square radii for the three- and four-body systems of charged particles and present our results in a generalised Efimov plot. We find that universal features can still be discerned for weak Coulomb interaction, but break down for strong Coulomb interaction. The root-mean-square radius plateaus at increasingly smaller values for strong Coulomb interaction and the probability distributions of the states become more concentrated inside the Coulomb barrier. As an example, we apply our universal model to nuclei with an $\alpha$-cluster substructure. Our results point to strong non-universal contributions in that sector.
I. INTRODUCTION

Few-body systems of strongly-interacting particles show universal properties independent of the details of their interaction at short distances [1,2]. The simplest example is given by identical bosons with large S-wave scattering length $a$ and mass $m$. If $a$ is positive and much larger than the range of the interaction $r_0$, the system has a shallow two-body bound state with binding energy

$$B_2 = \frac{\hbar^2}{ma^2} (1 + \mathcal{O}(r_0/a)),$$

and mean-square separation $a^2/2$.

In systems of three and more bosons, a three-body parameter $\kappa_*$ is required to characterize the system. For fixed scattering length, this implies universal correlations between different few-body observables such as the Phillips [3] and Tjon lines [4] if the two-body interaction has the same on-shell properties. In the three-body system, the Efimov effect [1] generates a universal spectrum of three-body bound states with binding energy

$$B_3(1 + \mathcal{O}(r_0/a)) = -\frac{\hbar^2}{ma^2} + \left[e^{-2\pi n f(\xi)}\right]^{1/s_0} \frac{\kappa_*^2}{m},$$

where $s_0 = 1.00624\ldots$ is a transcendental number, the angle $\xi$ is defined by $\tan \xi = -(mB_3)^{1/2}a/\hbar$, and $f(\xi)$ is a universal function with $f(-\pi/2) = 1$ (see Ref. [5] for more details). This spectrum is represented in the Efimov plot shown in Fig. 1. It consists of the two-dimensional plane spanned by the energy variable $K = \text{sgn}(E)\sqrt{m|E|}$ shown as a function of the inverse scattering length $1/a$. The solid lines indicate the Efimov states while the hashed areas give the scattering thresholds. The dashed vertical line illustrates an exemplary physical system with fixed scattering length.

![Efimov plot](image)

FIG. 1. Efimov plot of the three-body bound state spectrum. The energy variable $K = \text{sgn}(E)\sqrt{m|E|}$ shown as a function of the inverse scattering length $1/a$. The solid lines indicate the Efimov states while the hashed areas give the scattering thresholds. The dashed vertical line illustrates an exemplary physical system with fixed scattering length.

by the energy variable $K = \text{sgn}(E)\sqrt{m|E|}$ and the inverse scattering length $1/a$. The three-body parameter $\kappa_*$ is not shown explicitly, as it only sets the overall scale of the plot. The shaded area of radius $1/r_0$ shows the region where universality is expected to hold, while the solid lines indicate the Efimov states and the hashed areas give the scattering thresholds. The dashed vertical line illustrates an exemplary system with fixed scattering length. The spectrum displays invariance under discrete scaling transformations with the scaling factor $\exp(\pi/s_0) \approx 22.7$: the knowledge of the scattering length dependence of one state determines the scattering length dependence of all other states.
In the unitary limit $1/a = r_0 = 0$, the six-dimensional three-body problem reduces to a one-dimensional Schrödinger equation in the hyperadius with an attractive inverse square potential. The Efimov spectrum then becomes geometric:

$$B_3 = \left(e^{-2\pi/s_0}\right)^{n-n_*} \frac{\kappa_*^2}{m},$$

(3)

where $\kappa_*$ is identified as the binding momentum of the state with label $n_*$, and the binding momenta of neighbouring states differ by the scaling factor $\exp(\pi/s_0) \approx 22.7$. Two universal four-body states with binding energies

$$B^0_4 = 4.610(1) B_3 \quad \text{and} \quad B^1_4 = 1.00227(1) B_3$$

(4)

are attached to each Efimov state [6–10]. This pattern is expected to hold for higher-body states as well [11–14]. However, at some point their size becomes so small that they leave the universal region. The properties of these systems are generically referred to as *Efimov physics* [2,15]. In ultracold atoms, signatures of these universal states have been observed for up to five particles [16,17].

The Efimov effect is also relevant for the binding of weakly-bound states in nuclear and particle physics, such as hadronic molecules and halo nuclei. For reviews of the status of Efimov physics in these systems, see Refs. [18–20]. In systems with two or more charged constituents, however, the short-range strong interaction is accompanied by long-range Coulomb interactions. The effect of Coulomb interactions on Efimov physics and discrete scale invariance is therefore an important issue. Qualitatively, one expects shallow states with sizes of the order of the Bohr radius or larger to be Coulombic while deeper states with size smaller than the Bohr radius are Efimovian in character [21]. In the simpler case of a two-body system with inverse square interaction, these expectations were confirmed by explicit calculations [22,23]. An analysis of the gross properties of three-body halo nuclei with Coulomb interaction was carried out in Ref. [24]. Moreover, various studies have focused on the treatment of Coulomb interactions in short-range effective field theories of the three-nucleon system (see, e.g., Refs. [25–28]). In particular, König et al. [29] have considered the universal properties of three- and four-nucleon systems in a strict expansion around the unitary limit, where Coulomb forces are subleading. Their counting is based on the assumption that light, and possibly heavier, nuclei are bound weakly enough to be insensitive to the details of the interactions but strongly enough to be insensitive to the exact size of the two-nucleon system. By construction, their counting is not applicable at very low energies where the finite scattering lengths strongly influence the spectrum and the Coulomb interaction cannot be treated in perturbation theory.

Here, we attempt to fill a gap in previous calculations and focus explicitly on the modification of Efimov universality and discrete scale invariance in the presence of Coulomb forces. We follow the standard Efimov scenario and include the scattering lengths at leading order. Our results are summarized in generalized Efimov plots. The paper is organised as follows. First we will introduce the method that is used in our calculations, the Gaussian expansion method, in Sec. [II A] and the Coulomb-modified scattering length in Sec. [II B]. We will then present our Hamiltonian in Sec. [II C] and show our results in natural units in Sec. [III] and Sec. [IV]. In the last part (Sec. [V]), we apply our calculations to the $N\alpha$ system and discuss the difficulties arising there. We then summarize our findings and indicate possible future projects in the outlook (Sec. [VI]).

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1 The hyperadius $\rho_{hyp}$ is the mean separation of the three particles, i.e. $\rho_{hyp}^2 = (r_{12}^2 + r_{13}^2 + r_{23}^2)/3$ for equal masses.
II. METHOD

A. Gaussian Expansion Method

We use the Gaussian Expansion Method (GEM) \cite{30} to calculate binding energies and wave functions of the systems of $N$ identical charged bosons under investigation.

Successful applications of this method include various hypernuclear systems up to $N = 5$ \cite{31,36} and systems of atomic $^4\text{He}$ \cite{37,39}.

The GEM utilizes products of Gaussian basis functions to describe the system in terms of Jacobi coordinates. The Jacobi coordinates are illustrated in Fig. 2. There is only one configuration (“channel”) in the three-body system and two in the four-body system, because we treat identical particles.

We then solve the Schrödinger equation

$$(H - E)\Psi = 0.$$  \hfill (5)$$

The details of the Hamiltonian $H$ can be found in Sec. II C. The total wave function $\Psi$ is comprised of Gaussian functions as explained in the following.

The basis functions for each Jacobi coordinate take the form of

$$\phi_{nlm}(r) = \phi_{nl}^G(r) \ Y_{lm}(\hat{r}),$$
$$\phi_{nl}(r) = N_{nl} \ r^l \ e^{-(r/\rho_n)^2},$$

$$\psi_{NLM}(R) = \psi_{NL}^G(R) \ Y_{LM}(\hat{R}),$$
$$\psi_{NL}(R) = N_{NL} \ R^L \ e^{-(R/R_n)^2},$$

$$\xi_{\nu\lambda\mu}(\rho) = \xi_{\nu\lambda\mu}^G(\rho) \ Y_{\lambda\mu}(\hat{\rho}),$$
$$\xi_{\nu\lambda}(\rho) = N_{\nu\lambda} \ \rho^\lambda \ e^{-(\rho/\rho_\nu)^2},$$

where $N_{nl}, N_{NL}, N_{\nu\lambda}$ denote the normalization constants. The range parameters of the Gaussians are chosen to form a geometric progression:

$$r_n = r_{\min} \ a^{n-1},$$
$$R_n = R_{\min} \ A^{N-1},$$
$$\rho_n = \rho_{\min} \ \alpha^{\nu-1},$$

$$a = \left( \frac{r_{\max}}{r_{\min}} \right)^{\frac{1}{n_{\max} - 1}} \ (n = 1 \sim n_{\max}),$$
$$A = \left( \frac{R_{\max}}{R_{\min}} \right)^{\frac{1}{N_{\max} - 1}} \ (N = 1 \sim N_{\max}),$$
$$\alpha = \left( \frac{\rho_{\max}}{\rho_{\min}} \right)^{\frac{1}{\nu_{\max} - 1}} \ (\nu = 1 \sim \nu_{\max}).$$

This ensures that short-range correlations and long-range contributions can both be described very well and makes it especially well-suited to the problem discussed in this study.
The total wave function is then

\[
\Psi_{JM}^{\text{trimer}} = \sum_c \sum_{n_c,N_c} \sum_{\ell_c,L_c} \sum_{b \in p(c)} C_n^{(c)} \sum_{p(c)} \left[ \phi_{n_c \ell_c}^{(c)}(r_b) \psi_{N_c L_c}^{(c)}(R_b) \right]_{JM}
\]

\[
\Psi_{JM}^{\text{tetramer}} = \sum_c \sum_{n_c,N_c} \sum_{\ell_c,L_c} \sum_{b \in p(c)} \sum_{c' \in p(c')} C_n^{(c)} \sum_{p(c)} \sum_{p(c')} \left[ \phi_{n_c \ell_c}^{(c)}(r_b) \psi_{N_c L_c}^{(c)}(R_b) \right]_{JM},
\]

where \( c \) is the configuration channel as illustrated in Fig. 2 and \( p(c) \) are all possible permutations of particles in a given channel. \( C_n^{(c)} \) and \( C_n^{(c')} \) are coefficients.

Since we are interested in the leading order results we take all angular momentum quantum numbers (\( \ell, L, \lambda \)) to be zero. This also takes care of the symmetrization of our identical bosons. We are also only calculating states with total angular momentum zero.

With this the Schrödinger equation for the three-body system is

\[
\sum_c \sum_{n_c,N_c} \sum_{\ell_c,L_c} \sum_{b \in p(c)} \sum_{b' \in p(c')} C_n^{(c)} \times \left[ \phi_{n_c \ell_c}^{(c)}(r_b) \psi_{N_c L_c}^{(c')}^{(c)}(R_{b'}) \right]_{JM} \left( H - E \right) \left[ \phi_{n_c \ell_c}^{(c)}(r_b) \psi_{N_c L_c}^{(c)}(R_b) \right]_{JM} = 0,
\]

which due to the non-orthogonality of the basis states corresponds to a general eigenvalue problem:

\[
\sum_{\tilde{n}} (H_{\tilde{n}'\tilde{n}} - E N_{\tilde{n}'\tilde{n}}) C_{\tilde{n}} = 0,
\]

where \( N_{\tilde{n}'\tilde{n}} \) is a normalization matrix and \( \tilde{n} \) comprises all indices that are summed over. The matrix elements can be calculated analytically for all potentials used here and we solve the equation using standard linear algebra methods. The four-body system can be treated analogously.

### B. Coulomb-modified Scattering Length

The scattering length is an important parameter in the discussion of the Efimov effect. However, with the long-range Coulomb interaction present, the “usual” scattering length \( a \) does not exist and one has to define a Coulomb-modified scattering length \( a_C \). It is related to the phase shift \( \delta_0(p) \) between ingoing and outgoing Coulomb waves, much like the scattering length \( a \) is related to the phase shift between ingoing and outgoing free waves.

The Coulomb-modified effective range expansion for angular momentum \( \ell = 0 \) is

\[
C_{\eta,0}^{2} p \cot \delta_0(p) + \gamma h(\eta) = -\frac{1}{a_C} + \frac{1}{2} r_C p^2 + \ldots,
\]

with \( C_{\eta,0}^{2} = 2 \pi \eta/(e^{2 \pi \eta} - 1) \), \( h(\eta) = \text{Re} \left[ \Gamma'(i \eta)/\Gamma(i \eta) \right] - \log(\eta) \), \( \gamma = 2 \mu \alpha \alpha Z_1 Z_2 \), and \( \eta = \gamma/(2p) \). Here, \( p \) and \( \gamma \) are wave numbers and have the unit \( \text{fm}^{-1} \). More details can be found in Refs. [40]–[42].

This definition introduces some complications in practical calculators. For example, it is not possible to numerically calculate the Coulomb-modified scattering length \( a_C \) directly at zero energy. This means that we have to extrapolate to zero to obtain the scattering length. The extrapolation, however, is much more accurate than the accuracy of the results of our main calculation, so uncertainties stemming from this are negligible.
C. Interaction

In this subsection, we specify our interaction Hamiltonian. We consider equal mass particles of mass $m$. The two-body Hamiltonian contains a Gaussian potential to model the short-range interaction and a long-range Coulomb potential,

$$H_2 = -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial r^2} + V_0 e^{-\frac{r^2}{2r_0^2}} + \hbar c Z_1 Z_2 \frac{\alpha}{r},$$

(16)

where $\mu = m/2$ is the reduced mass while $V_0$ and $r_0$ determine the strength and the range of the Gaussian potential, respectively. The short-range part has been used in many works to investigate the Efimov effect [43–45]. It is well suited to the investigation of universal physics because it can be regarded as a contact term with Gaussian smearing. If one considers only energies that are small compared to the natural scale $E_s = \hbar^2/(2\mu r_0^2)$, it corresponds to the leading order of an effective theory for large scattering length. As we are only interested in the universal behaviour and not in the short-range details of any particular interaction, we will use this Hamiltonian in our analysis. Adding the Coulomb interaction to this Hamiltonian is straightforward. $Z_1$ and $Z_2$ are the charge numbers of particles 1 and 2, respectively, and $\alpha \approx 1/137$ is the fine structure constant.

The generalization of the Hamiltonian to systems of three and four particles is also straightforward:

$$H_3 = -\frac{\hbar^2}{2\mu_r} \nabla_r^2 - \frac{\hbar^2}{2\mu_R} \nabla_R^2 + V_0 \sum_{i<j} e^{-\frac{r_{ij}^2}{2r_0^2}} + \hbar c \sum_{i<j} Z_i Z_j \frac{\alpha}{r_{ij}},$$

(17)

and

$$H_4 = -\frac{\hbar^2}{2\mu_r} \nabla_r^2 - \frac{\hbar^2}{2\mu_R} \nabla_R^2 - \frac{\hbar^2}{2\mu_\rho} \nabla_\rho^2 + V_0 \sum_{i<j} e^{-\frac{r_{ij}^2}{2r_0^2}} + \hbar c \sum_{i<j} Z_i Z_j \frac{\alpha}{r_{ij}},$$

(18)

with $r_{ij}$ the distance between particle $i$ and $j$. In the kinetic part, we have used the Jacobi coordinates $r, R, \rho$ with the corresponding reduced masses $\mu_r, \mu_R, \mu_\rho$.

It is useful to express our Hamiltonian in natural units by dividing out the natural energy scale $E_s$. Using the reduced distance $\tilde{r} = r/r_0$, this gives us for the two-body Hamiltonian

$$\tilde{H}_2 = -\nabla_{\tilde{r}}^2 + \tilde{V}_0 e^{-\frac{\tilde{r}^2}{2}} + \frac{\tilde{c}}{\tilde{r}},$$

(19)

with the definitions $\tilde{V}_0 = V_0/E_s$ and $\tilde{c} = c_c/c_s$ with $c_c = \hbar c Z_i Z_j \alpha$ and $c_s = \hbar^2/(mr_0)$. In analogy to $E_s$, $c_s$ can be regarded as natural scale of the Coulomb coupling strength.

It is now obvious that the same result for different $r_0$ can be obtained, as long as $\tilde{c}$ stays the same. Note, however, that expressing the charge in terms of $\tilde{c}$: $Z_i Z_j = \tilde{c} h/(c_0 \alpha n r_0)$. Thus the same value of $\tilde{c}$ corresponds to different physical charges for different $r_0$ and a value for $r_0$ has to be chosen when comparing to physical systems. We will discuss this issue in more details in Sec. V and present our approach to choosing $r_0$. First, however, we will concentrate on observables in natural units. We will present results for fixed values of $\tilde{c}$, which can be viewed as representatives of different universality classes.

III. GENERALISED EFIMOV PLOT

We start by summarizing our results for different potential strengths $V_0$ in the generalized Efimov plot of Fig. 3. Plotted in natural units, the values for different ranges of the Gaussian potential
FIG. 3. Comparison of the Generalised Efimov Plot for different Coulomb interaction strength parameters in terms of $V_0$. The blue lines are the dimers, the red lines the trimers and the green lines the tetramers. Excited states are not shown to avoid confusion. The solid lines are without Coulomb interaction, the dashed lines for $\tilde{c} = 0.007$, the dotted lines for $\tilde{c} = 0.07$ and the dash-dotted lines for $\tilde{c} = 0.7$. $r_0$ lie on the same line, which is clear from Eq. (19). The Efimov plot is almost unchanged for very small values of the Coulomb interaction strength parameter $\tilde{c}$. Figure 3 also shows that states do become shallower with increasing repulsive Coulomb interaction as expected. In the following, we will show the energies as a function of the inverse Coulomb-modified scattering length $1/a_C$, which is an observable quantity and thus better suited to discuss universal properties.

The case of weak Coulomb interaction is shown in Fig. 4 a). Near the continuum threshold, the dimer shows slight deviations from the universal $1/a^2$ behaviour. Since we calculated the identical boson case, the scaling factor is expected to be $\lambda = 22.7$. Due to this high scaling factor, we could only resolve the lowest two trimers in our calculation.

Comparing the scaling factors of the no-Coulomb case and the weak Coulomb case, however, reveals already some drastic change. First, we note that the scaling factors calculated at the three-body threshold and in the unitary limit are the same in the universal limit. In our calculations, however, there are some non-universal effects due to the fact that we calculate the ground state without a repulsive three-body force tuned to push the ground state into the universal region. We estimate these effects by calculating the scaling factors in both limits. In the no-Coulomb case with $c_c = 0$ the scaling factor is $18 \pm 1$ when comparing the intersections of the states with the three-body threshold ($a^-$) and $23 \pm 1$ when comparing the square roots of the binding energies in the unitary limit, where only the numerical errors are quoted. For the $c_c/c_s = 0.007$ case calculated with the same Hamiltonian, the corresponding scaling factors become $55 \pm 5$ and $32 \pm 3$, respectively. As discussed above, some slight asymmetry (as in the $c_c = 0$ case) is expected since the ground state for this Hamiltonian is not fully universal, but the large discrepancy seen in the weak-Coulomb case cannot be explained by non-universal behaviour of the ground state alone. So, even though the Efimov plot does not look very different for the two cases at first sight, a significant breaking of the discrete scaling symmetry can already be observed.

Comparing the ratio between binding energy at unitarity $B_3$ and inverse scattering length at the three-body threshold $a^0$ is also instructive. The most precise universal value from the literature is $a^0 \sqrt{mB_3}/\hbar = 1.5077(1)$ [46]. Without Coulomb interaction, we obtain 2.1(1) for the trimer ground state and 1.6(1) for the excited state. For the case of weak Coulomb interaction
(\(c_c/c_s = 0.007\)), we have 2.3(1) for the ground state and 4(1) for the excited trimer. This shows that the strong deviation we observe for the scaling factor mainly stems from the deformation of the excited state by the Coulomb interaction while the more compact ground state remains relatively unaffected. This is in agreement with the simple scale argument from Sec. [I] For stronger Coulomb interaction, however, the ground state trimer also starts to become deformed and we obtain the factors 4 for \(\tilde{c} = 0.07\) and 40 for \(\tilde{c} = 0.7\).

If the Coulomb interaction is made stronger by tuning up the strength parameter \(\tilde{c}\), the Generalised Efimov plot changes significantly as seen in panels b) and c) of Fig. 4. The excited states move towards positive scattering length and larger binding energies. They eventually vanish from the spectrum as can be seen in Fig. 4(b) and c). When considering the fact that near the threshold, states typically are more dilute this behaviour is to be expected. The larger the average distance between particles, the more influence the long-range Coulomb force gains over the short-range attraction. Therefore, one expects the Coulomb interaction to have the strongest effect for the shallow states.

Since there is no excited state anymore for \(\tilde{c} = 0.07\) in the unitary limit, the scaling factor between two consecutive trimers is not well-defined anymore. We can, however, investigate the evolution of the scaling factor that connects the tetramer to the trimer. In the case without Coulomb, we obtain 2.3 ± 0.1 for the scaling of the intersection of the ground state of the tetramer...
and the ground state of the trimer with the \( N \)-body threshold and \( 2.4 \pm 0.1 \) for the scaling of the square root of the binding energies for the same states at unitarity. This is reasonably close to the universal value \( B_4^0/B_3 = 4.610(1) \) from Eq. (4). Our result without Coulomb is \( 5.8 \pm 0.5 \). The difference from the universal value again gives an estimate of the size of non-universal effects in our calculation. For the ratio of the scattering lengths corresponding to the intersections of the states with the \( N \)-body threshold, Deltuva obtained \( 0.4254(2) \) \[46\]. This should be compared to the inverse of our number, i.e. \( 0.43 \pm 0.01 \). This agrees much better, because it is extracted from threshold properties, where non-universal effects are less severe.

For weak Coulomb interaction (\( \tilde{c} = 0.007 \)), the scaling factors become slightly smaller, \( 2.2 \pm 0.1 \) for the intersection with the \( N \)-body threshold and \( 2.3 \pm 0.1 \) for the square root of the energies. But compared to the large modification of the scaling factor between the ground and excited state of the trimer, the change is negligible. This might be due to the fact that here we compare two ground states that have similar distances between the particles (see Sec. IV) and are therefore affected in similar ways by the Coulomb interaction. If we go to \( \tilde{c} = 0.07 \), the scaling factor becomes even smaller, \( 2 \pm 0.1 \) and \( 2.1 \pm 0.1 \), respectively. A caveat is in order for the second of these values, because the ground state of the tetramer is not strictly inside the range of our theory at unitarity. It is still evident, however, that although the Generalised Efimov plot shown in Fig. 4 c) does not much resemble the standard Efimov plot any more, the scaling between the ground state tetramer and trimer remains qualitatively the same.

In the case of a strong Coulomb interaction, illustrated in Fig. 4 c), the excited states have vanished completely from the region where \( E < E_s \). The scaling factor between the trimer and tetramer ground states is \( 1.7 \pm 0.1 \) if read off from the intersections of the states with the \( N \)-body threshold. If one were to compare the energies as well, the resulting factor would be \( 1.9 \pm 0.1 \), but the tetramer energy is quite far outside the range of \( E/E_s < 1 \), so it should be disregarded. However, just from the first value we can observe that the scaling factor continues to decrease, but is still not as drastically influenced as other aspects of the Efimov plot.

IV. STRUCTURE

To better understand the effect of the Coulomb interaction on the Efimov spectrum, we analyse the structure of the states as well. There are two parts to the analysis. In the first part, we investigate the root mean square distance between two particles, \( \sqrt{\langle r^2_{ij} \rangle} \). Since we have identical particles and thus no way to single out a specific particle pair, this value will be averaged over all particle pairs. This will give us an idea about the general size of the bound state. In the second part we will then supplement this analysis with 2D visualisations of the particle density distribution, in order to capture different topologies.

A. Rms Distance Analysis

To calculate the rms pair distance we calculated the matrix elements for \( (\sum_{i,j} r^2_{ij})/N_{\text{pairs}} \) and multiplied this with the wavefunctions for the respective states that we obtained from our calculations. Then we took the square root of the result to obtain \( \sqrt{\langle r^2_{ij} \rangle} \).

The results for the dimer are shown in the left panel of Fig. 5 for different strengths of the Coulomb interaction. An interesting observation that can be made is that while for the no-Coulomb case the dimer seems to become arbitrarily large for smaller and smaller binding energies, the
FIG. 5. Value of the root mean square radius $r_{\text{rms}}$ of the dimer (left panel), trimer (middle panel) and tetramer (right panel) in natural units against the binding energy in natural units for different strengths of the Coulomb potential $\tilde{c}$. The value of $\tilde{c}$ is indicated in the legend. The solid lines represent the ground state and the dashed lines the excited state.

The presence of the Coulomb interaction leads to a plateau in the maximal size. For larger binding energies, the dimer seems almost unaffected even in the strong-Coulomb case. This explains the feature observed in the generalised Efimov plots, where the dimer follows almost a straight line which corresponds to the universal dimer $\sim 1/\alpha^2$ until it becomes too shallow and curves up to the break-up threshold very steeply. There seems to be a maximal separation distance up to which the short-range potential can support a bound state against the Coulomb repulsion. When this maximal separation is reached, the state vanishes into the threshold very quickly.

This behaviour is similar for the first excited trimer (middle panel of Fig. 5) which becomes very large close to threshold without the Coulomb interaction, but reaches its maximal size much more quickly with even a very weak Coulomb interaction present. For even stronger Coulomb interaction, the excited state vanishes into the dimer+atom threshold, so it is not possible to further trace the behavior. Shortly before vanishing, the rms radius becomes larger, which is probably due to one of the atoms separating from the other two. The ground state stays generally much smaller than the excited state, because it is further away from unitarity, i.e. at smaller negative values of the scattering length, for the same binding energy. This qualitative behavior is in agreement with the results of Fedorov et al. for three-body halos with Coulomb [24].

An often used criterion for Efimov states is that the size of the state $R$ should be much larger than the range of the interaction (cf. [9]). If we take $r_0$ as the range of the short-range interaction, we can see that the Coulomb barrier forces the states to become smaller with increasing Coulomb interaction strength. This pushes the states out of the universal region and away from the Efimov criterion $r_{\text{rms}} \gg r_0$. For weak Coulomb interaction the criterion is still well satisfied, but for stronger Coulomb repulsion the rms radius is of the order of $r_0$, which also explains the large modifications of the spectrum shown in the previous section.

For the tetramer, a similar picture emerges, as is evidenced in the right panel of Fig. 5. The rms distance of the ground state again becomes smaller when the Coulomb interaction becomes stronger. The excited tetramer is larger than the ground states for all strengths of the Coulomb interaction, but vanishes into the trimer+atom threshold if the Coulomb interaction becomes too strong.
FIG. 6. Contour plot of the ground state for different $\tilde{c}$, at the point where the binding energy is $0.05E_s$. From left to right $\tilde{c} = 0, 0.007, 0.07, 0.7$. The red bar shows $r_{\text{rms}}/r_0$ for the corresponding state. The $x$ and $y$ axes show distance in units of $r_0$. The plot shows sampled distributions of particles with their center of mass in the origin in the principal axis frame. They were rotated so that the principal axis with the smallest moment of inertia is on the $y$ axis, and if necessary mirrored on $x$ or $y$ axis to ensure that the left upper quadrant is empty. The colors correspond to the summed probabilities of finding a particle in any given bin after this procedure with white being zero probability and dark green being the highest probability in a given plot.

B. Contour Plot Analysis

In this section, we show the particle distribution of the trimers as a 2D heatmap. This is achieved by aligning the sampled particles such that the principal axis with the smallest moment of inertia lies on the $y$ axis, and then mirroring along the $x$ and $y$ axis such that there is one particle in the right upper quadrant and two particles in the lower half of the plot. This helps to visualise the structure of the sampled state.

In Fig. 6, we show the contour plots of the ground state trimer for different strengths of the Coulomb potential at a binding energy of $0.05E_s$, which was chosen because it represents a rather shallow state. An interesting observation that can be made from the plots is that for increasing Coulomb interaction strength the particle clouds become less fanned out, while the positions with the highest probability density stay almost the same. So the differences in rms radii stem in a large part from the removal of dilute but comparatively improbable configurations and less from a reduction in size of the highest-probability configuration.
FIG. 7. Excited states without Coulomb interaction (left) and with weak Coulomb interaction ($\tilde{c} = 0.007$) (right), at the point where the binding energy is $0.05E_s$. For more information on the type of plot, refer to Fig. 6.

V. REALISATION IN SYSTEMS OF $\alpha$ PARTICLES

As mentioned in section II C when comparing our results to physical systems, we have to fix $r_0$ in some way. Because the charges are fixed by nature, fixing $r_0$ gives us the $\tilde{c}$ universality class of the system via $\tilde{c} = Z_iZ_jmr_0\alpha\frac{\hbar\epsilon}{\hbar^2}$.

One obvious way to determine $r_0$ is to choose a value for $r_0$ such that the Coulomb-modified effective range ($r_{\text{eff}}^C$) is reproduced. This ansatz has already been used in [48] to describe $^{17}\text{F}(\frac{1}{2}^+)$ as a dimer of $^{16}\text{O}$ and a proton, leading to consistent results for the dimer system. To investigate the applicability of our results in bigger nuclear systems, we will use systems of three and four $\alpha$ particles as an example. When treating clusters of $\alpha$ particles, our present approach is of limited use because most clusters of $\alpha$ particles are resonances and not bound states with respect to breakup into $\alpha$ particles. This is, e.g., true for $^{8}\text{Be}$ and the Hoyle state in $^{12}\text{C}$. However, there is a $0^+$ state of $^{16}\text{O}$ at $14.032\text{MeV}$ [49] (commonly referred to as $^{16}\text{O}(0^+_5)$) that could be described as an $\alpha$ cluster state. In order to investigate whether this state can be interpreted as the remnant of a universal tetramer connected to an Efimov state, we calculate the generalised Efimov plot for the three- and four-$\alpha$ system.

As for the $^{16}\text{O}+\text{p}$ system treated in [48], we first determine the $r_0$ that reproduces the correct Coulomb modified effective range for the $\alpha\alpha$ system. This is necessary because the Coulomb interaction introduces an additional scale which means that results for different ranges of the Gauss potential $r_0$ are different. We choose $r_0$ such that $r_{\text{eff}}^C$ has the correct physical value. The connection between $r_0$ and $r_{\text{eff}}^C$ for the $\alpha\alpha$ system is shown in Fig. 8. The values for the scattering length and the effective range are taken from [50]. For Fig. 8 we select points where the scattering length is within the specified errors from [50]:

$$a_C \approx (1920 \pm 90)\text{fm}. \quad (20)$$
This leads to the spread of effective ranges $r_{\text{eff}}^C$ in Fig. 8. The effective range band drawn in the insert of Fig. 8 gives the value from [50], $(1.098 \pm 0.005)$ fm, including errors. Thus choosing $r_0$ to be 2.3 fm reproduces the physical values of $r_{\text{eff}}^C$ and $a_C$.

We now turn to the three- and four-\(\alpha\) system. It turns out that with this value for $r_0$, it is impossible to describe the highest excited state of $^{16}$O below the four-\(\alpha\) breakup threshold without at the same time having a three-\(\alpha\) bound state above it. However, such a state does not exist in nature. This is illustrated in Fig. 9 where the energies of the two-, three-, and four-\(\alpha\) systems are given as function of $1/a_C$ for physical $r_{\text{eff}}^C$. The vertical solid line gives the physical value for $1/a_C$ in the $\alpha\alpha$ system. Thus we conclude that the highest $0^+$ excited state of $^{16}$O cannot be connected to a remnant of an Efimov state, at least not with our strategy of fixing $r_0$ and omitting higher-order interactions. This could point to significant contributions of short-range non-\(\alpha\)-cluster configurations to this state.

Moreover, when analysing the rms distance values, we find that $r_{\text{rms}}$ for the tetramer without three-body force at the physical scattering length (corresponding to the left panel of Fig. 9) is only 3.1 fm, which is only barely larger than $r_0 = 2.3$ fm. Adding a three-body force increases the rms distance between two $\alpha$ particles insubstantially. For $w_0 = 0.775 E_s$, which is the value of the three-body potential strength parameter that reproduces the $^{16}$O ($0^+_1$) binding energy (middle panel of Fig. 9), $r_{\text{rms}} = 3.4$ fm. This again suggests that this state is heavily influenced by short-range effects and cannot be accurately described in a universal theory of $\alpha$ clusters.

From the differences between our universal model for $\alpha$ clusters and the $0^+$ bound states in $^{12}$C and $^{16}$O, we can conclude that significant non-universal contributions play a role for the bound states of three and four clusters.
FIG. 9. Left panel: Nα system with binding energies and widths of 0+ states from Refs. [49, 51, 52]. The dotted horizontal line represents the 8Be ground state, the dashed horizontal lines are the ground state of 12C and the Hoyle state from bottom to top, and the dash-dotted lines are 16O states. The vertical solid line is at the αα scattering length $a_{αα} \approx 1800$ fm. In addition results for the dimer (blue solid line), trimer (red solid line) and tetramer (green solid line) without three-body force are shown. The dashed horizontal black line marks the natural energy scale $E_s$, which is a measure for the range of validity of our theory. Middle panel: Detail of the left plot with the three-body force fitted to reproduce the shallowest bound tetramer state. Right panel: Same as the middle panel with the three-body force fitted such that the trimer is unbound at the physical αα scattering length.

VI. SUMMARY AND OUTLOOK

We have investigated the universal properties of systems with a short-range attractive interaction (modeled by Gaussian interactions with a range $r_0$) and a repulsive long-range Coulomb interaction. Such systems can be realized in nature in α cluster nuclei and few-body systems of ions. A particular focus was placed on connecting such systems to Efimov states and their higher-body analogs in the case without Coulomb interaction and studying their evolution as a function of the Coulomb interaction strength.

Our results are summarized in generalised Efimov plots where the scattering length is replaced by the Coulomb-modified scattering length. Our plots can be used to identify universal states of charged particles. In natural units, the spectra are fully universal, i.e. independent of the Gaussian range $r_0$. Varying the dimensionless Coulomb strength $\tilde{c}$, we find that although the structure of the Efimov plot remains qualitatively the same, the discrete scaling symmetry is broken strongly already for very weak Coulomb interaction. The long-range Coulomb interaction has a strong effect close to threshold but leaves the deeper states almost unchanged, which clearly breaks the scaling symmetry. Moreover, the scaling factors connecting the three- and four-body ground states are also only weakly affected by the Coulomb interaction. A study of the structure of these states reveals that their constituents become more localised for stronger Coulomb force, while the highest-probability configuration is almost unaffected. This can be interpreted as an effect of the rising Coulomb barrier [24].

Since the Coulomb interaction introduces an additional scale, the results for different ranges of the Gaussian potential $r_0$ lead to different physical scenarios. Thus, we have fixed the range $r_0$ to reproduce the Coulomb-modified effective range $r_{eff}$ to its physical value. However, other
strategies to deal with this problem are possible and will be investigated in future work.

As an example, we have applied our model to three- and four-body systems of $\alpha$ particles. With our strategy for fixing $r_0$ we find that the $0^+$ bound state spectrum of $^{12}\text{C}$ and $^{16}\text{O}$ cannot be described by our universal model, at least without including higher-order interactions. From our investigations of the structure of these states, we conclude that this is related to the small size of these states which implies significant non-universal contributions.

This leads to the interesting question of whether the universality discussed in this work could be observed in the resonance sector of $^{12}\text{C}$ and $^{16}\text{O}$, i.e. above the breakup threshold into $\alpha$ particles. In addition, the universality may be relevant for other systems where the Coulomb interaction is weaker in comparison to the short-range force such as systems of cold ions [54, 55]. Work in both directions is in progress.

ACKNOWLEDGMENTS

We thank Artem Volosniev for discussions and Dörte Blume for providing details on the contour plots in Refs. [56]. This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 279384907 - SFB 1245 and the Federal Ministry of Education and Research (BMBF) under contract 05P18RDFN1.

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