Embedded Flexible Spherical Cross-Polytopes with Nonconstant Volumes

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Abstract—We construct examples of embedded flexible cross-polytopes in the spheres of all dimensions. These examples are interesting from two points of view. First, in dimensions 4 and higher, they are the first examples of embedded flexible polyhedra. Notice that, in contrast to the spheres, in the Euclidean and Lobachevsky spaces of dimensions 4 and higher still no example of an embedded flexible polyhedron is known. Second, we show that the volumes of the constructed flexible cross-polytopes are nonconstant during the flexion. Hence these cross-polytopes give counterexamples to the Bellows Conjecture for spherical polyhedra. Earlier a counterexample to this conjecture was constructed only in dimension 3 (V.A. Alexandrov, 1997), and it was not embedded. For flexible polyhedra in spheres we suggest a weakening of the Bellows Conjecture, which we call the Modified Bellows Conjecture. We show that this conjecture holds for all flexible cross-polytopes of the simplest type, which includes our counterexamples to the ordinary Bellows Conjecture. Simultaneously, we obtain several geometric results on flexible cross-polytopes of the simplest type. In particular, we write down relations for the volumes of their faces of codimensions 1 and 2.

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1. INTRODUCTION

Let $X^n$ be one of the three $n$-dimensional spaces of constant curvature, that is, the Euclidean space $E^n$, the sphere $S^n$, or the Lobachevsky space $\Lambda^n$. For convenience, we will always normalize metrics on the sphere $S^n$ and Lobachevsky space $\Lambda^n$ so that their curvatures are equal to 1 and $-1$, respectively. For consistency of terminology, great spheres in $S^n$ will often be called planes. Spheres in $S^n$ that are not great ones will be called small spheres.

A flexible polyhedron in $X^n$ is a closed connected $(n-1)$-dimensional polyhedral surface $P$ in $X^n$ that admits a continuous deformation $P_u$ such that every face of $P_u$ remains isometric to itself during the deformation, but the deformation $P_u$ is not induced by a rotation of the ambient space $X^n$. The surface $P_u$ is allowed to be self-intersecting. However, non-self-intersecting (or embedded) polyhedra are of a special interest. A precise definition will be given in Section 2.

First flexible polyhedra, namely, flexible octahedra in $E^3$, were constructed by Bricard [6]. Moreover, Bricard classified all flexible octahedra in $E^3$. In particular, he proved that all of them are self-intersecting. The first example of an embedded flexible polyhedron in $E^3$ was constructed by Connelly [7]. Particular examples of flexible polyhedra in the spaces $E^1$, $S^3$, and $\Lambda^3$ were constructed by Walz and Stachel (see [21, 22]). In a recent paper [13], the author managed to generalize Bricard’s results to all spaces of constant curvature of arbitrary dimensions, that is, to construct and classify flexible cross-polytopes in all spaces $E^n$, $S^n$, and $\Lambda^n$. Here and below, by an $n$-dimensional

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cross-polytope we mean an arbitrary polyhedron of the combinatorial type of a regular cross-polytope, i.e., of a regular polytope dual to the $n$-dimensional cube. In particular, a two-dimensional cross-polytope is a quadrangle, and a three-dimensional cross-polytope is an octahedron.

Notice that there exists a very simple method that allows one to construct a flexible polyhedron in $\mathbb{S}^n$ from every flexible polyhedron in $\mathbb{S}^{n-1}$: to do this, one can just take the bipyramid (the suspension) with vertices at the poles of $\mathbb{S}^n$ over the given flexible polyhedron lying in the equatorial great sphere $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. Since any nondegenerate polygon in $\mathbb{S}^2$ with at least four sides is flexible, we can easily obtain many examples of flexible polyhedra in $\mathbb{S}^n$ (including embedded ones) by iterating the above construction. Examples of such kind are not interesting. Therefore, it seems appropriate to study flexible polyhedra in the open hemisphere $\mathbb{S}^n_+ \subset \mathbb{S}^n$.

Until now, no example of an embedded flexible polyhedron in $\mathbb{E}^n$, $\mathbb{S}^n_+$, or $\Lambda^n$ was known for $n \geq 4$. In the present paper we will show that there exist embedded flexible cross-polytopes in the open hemispheres $\mathbb{S}^n_+$ of all dimensions. Moreover, we will prove the following theorem.

**Theorem 1.1.** For every $n \geq 2$, in the sphere $\mathbb{S}^n$ there exists a flexible cross-polytope $P_u$, $u \in \mathbb{R} = \mathbb{R} \cup \{\infty\}$, possessing the following properties:

1. there is a $\delta > 0$ such that the cross-polytope $P_u$ is embedded whenever $u \in (-\delta, \delta)$;
2. $P_0$ is the equatorial great sphere $\mathbb{S}^{n-1} \subset \mathbb{S}^n$ with a simplicial decomposition combinatorially equivalent to the boundary of the $n$-dimensional cross-polytope;
3. $P_u$ is contained in the upper hemisphere $\mathbb{S}^n_+$ for $u > 0$ and is contained in the lower hemisphere $\mathbb{S}^n_-$ for $u < 0$;
4. $P_{\infty}$ is contained in the equatorial great sphere $\mathbb{S}^{n-1}$ but is not embedded.

This theorem will be proved in Section 5.

As mentioned above, the classification of all flexible cross-polytopes in spaces of constant curvature was obtained by the author in [13]. However, this classification was given in algebraic terms. So the question of which of the constructed flexible cross-polytopes are self-intersecting and which are embedded is nontrivial and was not considered in [13]. The proof of Theorem 1.1 reduces to showing that some of the flexible cross-polytopes constructed in [13] for an appropriate choice of parameters possess the required properties (1)--(4). We will show that such cross-polytopes can be found in the class of flexible cross-polytopes of the simplest type, which were constructed in [13, Sect. 5] as multidimensional generalizations of Bricard’s flexible octahedra of the third type (also known as skew flexible octahedra). The simplest type of flexible cross-polytopes occupies a special place because the flexions of cross-polytopes of this type admit a rational parametrization, while all other types of flexible cross-polytopes have an elliptic parametrization that degenerates to a rational parametrization only for some special values of edge lengths.

One of important problems in the theory of flexible polyhedra is concerned with their volumes and is related to the so-called *Bellows Conjecture*. This conjecture, which was suggested by Connelly [8] and was proved by Sabitov [17–19] in 1996, claims that the volume of any flexible polyhedron in the Euclidean three-space is constant during the flexion. Since by a polyhedron we mean a polyhedral surface, we need to specify that by the volume of a polyhedron we mean the volume of the region bounded by this polyhedral surface. If a polyhedron is self-intersecting, the volume of an arbitrary flexible polyhedron in $\mathbb{E}^n$, $n \geq 4$, is constant, was proved by the author [11, 12]. The question naturally arises as to whether the analog of the Bellows Conjecture holds in the spheres $\mathbb{S}^n$ and Lobachevsky spaces $\Lambda^n$, $n \geq 3$. More precisely, we should again replace the
spheres $S^n$ with the open hemispheres $S^+_n$, because in $S^n$ the question is trivial. Indeed, flexible polyhedra in $S^n$ with nonconstant volumes can be obtained by taking iterated bipyramids over flexible spherical polygons with nonconstant areas, as was described above. In 1997 Alexandrov [3] constructed an example of a flexible self-intersecting polyhedron with nonconstant volume in the open hemisphere $S^3_+$. The combinatorial type of this polyhedron is the bipyramid over the hexagon. Theorem 1.1 easily yields the following result.

**Corollary 1.2.** The volume of the flexible cross-polytope $P_u$ in Theorem 1.1 is nonconstant on any arbitrarily small interval of parameters $(0, u_0)$. Thus, for every $n \geq 2$, there exists an embedded flexible cross-polytope with nonconstant volume in $S^+_n$.

**Proof.** For $u$ small enough, the surface $P_u$ divides the sphere $S^n$ into two parts. To speak of the volume of the cross-polytope $P_u$, we should decide the volume of which of these two parts we consider. The sum of these two volumes is constant and is equal to the volume $\sigma_n$ of the sphere $S^n$. Since we are interested only in the question of nonconstancy of the volume, it is completely irrelevant for us which of the two volumes to consider. To be specific, we will assume that the volume $V(P_u)$ of the polyhedron $P_u$ is the volume of the part that contains the north pole. Then $V(P_0) = \sigma_n/2$ and $V(P_u) < \sigma_n/2$ whenever $0 < u < \delta$, since the polyhedron $P_0$ is embedded and is contained in $S^+_n$. Hence the function $V(P_u)$ is nonconstant in any neighborhood of zero. $\square$

Quite recently the author has proved the Bellows Conjecture for flexible polyhedra in the odd-dimensional Lobachevsky spaces. This result was announced in [14]; its proof will be presented in a separate paper. The question of whether the Bellows Conjecture is true for flexible polyhedra in the even-dimensional Lobachevsky spaces remains open. It is well known that the volumes of polyhedra in the sphere $S^n$ and in the Lobachevsky space $\Lambda^n$ are closely related to each other. For instance, this manifests itself in the fact that the functions expressing the volumes of simplices in $S^n$ and $\Lambda^n$ in terms of their dihedral angles are obtained from each other (up to a multiplicative constant) by an appropriate analytic continuation [10, 4] (see also [1]). Hence the fact that the Bellows Conjecture is true in the odd-dimensional Lobachevsky spaces makes it rather plausible to assume that a certain proper analog of the Bellows Conjecture in spheres should still be true. To formulate this conjecture, which will be referred to as the Modified Bellows Conjecture, we will need a special operation on polyhedra in $S^n$.

First of all, we note that when studying flexible polyhedra, we can restrict ourselves to studying only simplicial flexible polyhedra, i.e., polyhedra all of whose faces are simplices. Indeed, for an arbitrary flexible polyhedron, we can decompose its faces into simplices, possibly adding new vertices. Then the obtained polyhedron will again be flexible. Notice that for simplicial polyhedra, a deformation preserving the combinatorial type is a flexion if and only if all edge lengths are constant during this deformation. Let $P_u$ be an arbitrary simplicial flexible polyhedron in $S^n$, and let $a(u)$ be a vertex of it. Denote by $-a(u)$ the point of $S^n$ antipodal to the point $a(u)$. Consider a new flexible polyhedron $\tilde{P}_u$ of the same combinatorial type as $P_u$ such that all vertices of $P_u$ except $a(u)$ and all faces of $P_u$ not containing $a(u)$ remain vertices and faces of $\tilde{P}_u$, respectively, the vertex $a(u)$ of $P_u$ is replaced by the vertex $-a(u)$ of $\tilde{P}_u$, and every face $[a(u)b_1(u)\ldots b_k(u)]$ of $P_u$ is replaced by the face $[(-a(u))b_1(u)\ldots b_k(u)]$ of $\tilde{P}_u$. We will say that the flexible polyhedron $\tilde{P}_u$ is obtained from the flexible polyhedron $P_u$ by replacing the vertex $a(u)$ with its antipode. It is easy to see that the replacements of two different vertices of $P_u$ with their antipodes commute.

**Conjecture 1.3** (Modified Bellows Conjecture). Let $P_u$ be an arbitrary flexible simplicial polyhedron in $S^n$. Then we can replace some vertices of $P_u$ with their antipodes so that the generalized volume of the obtained flexible polyhedron will remain constant during the flexion.

To support this conjecture, we will show that it is true for all flexible cross-polytopes of the simplest type, which, by Corollary 1.2, contains counterexamples to the ordinary Bellows Conjecture. The proof will be based on an explicit calculation of the volumes of flexible cross-polytopes.
of the simplest type (Theorem 7.8). During the calculation, we will obtain a series of results on the geometry of flexible cross-polytopes of the simplest type in \( \mathbb{X}^n = \mathbb{S}^n, \mathbb{E}^n, \mathbb{S}^n \). We believe that these results are of independent interest. In Section 4 we derive formulas for the dihedral angles of flexible cross-polytopes of the simplest type. Each flexible cross-polytope \( P_u \) of the simplest type is flat (i.e., is contained in a hyperplane) and satisfies geometric properties. Namely, the \((n - 1)\)-dimensional flexible cross-polytopes for the two values \( u = 0 \) and \( u = \infty \) of the parameter. In Section 6 we prove that flat cross-polytopes \( P_0 \) and \( P_\infty \) have certain surprising geometric properties. Namely, the \((n - 1)\)-dimensional flexible cross-polytopes obtained from \( P_0 \) (or \( P_\infty \)) by deleting different pairs of opposite vertices are either all circumscribed about concentric spheres or satisfy certain other similar properties (Theorem 6.1). For flexible octahedra in the three-dimensional Euclidean space, this result was obtained by Bennett [5]. In Section 7 we derive linear relations for the volumes of \((n - 1)\)- and \((n - 2)\)-dimensional faces of flexible cross-polytopes of the simplest type. In the case of the three-dimensional Euclidean space, these relations turn into relations for the areas of faces and the lengths of edges of skew flexible octahedra, which were known to Bricard [6].

2. DEFINITION OF FLEXIBLE POLYHEDRA

**Definition 2.1.** A finite simplicial complex \( K \) is called a \( k \)-dimensional pseudomanifold if

1. every simplex of \( K \) is contained in a \( k \)-dimensional simplex of \( K \);
2. every \((k - 1)\)-dimensional simplex of \( K \) is contained in exactly two \( k \)-dimensional simplices of \( K \);
3. \( K \) is strongly connected, i.e., any two \( k \)-dimensional simplices of \( K \) can be connected by a finite sequence of \( k \)-dimensional simplices such that any two consecutive simplices in this sequence have a common \((k - 1)\)-dimensional face.

We say that a pseudomanifold \( K \) is oriented if all its \( k \)-dimensional simplices are endowed with orientations such that, for any \((k - 1)\)-simplex \( \tau \) of \( K \), the orientations induced on \( \tau \) by the chosen orientations of the two \( k \)-dimensional simplices containing \( \tau \) are opposite to each other.

**Definition 2.2.** Let \( K \) be an oriented \((n - 1)\)-dimensional pseudomanifold. A nondegenerate polyhedron (or polyhedral surface) of combinatorial type \( K \) in \( \mathbb{X}^n \) is a mapping \( P : K \to \mathbb{X}^n \) such that

1. for each simplex \([v_0 \ldots v_l]\) of \( K \) the points \( P(v_0), \ldots, P(v_l) \) are independent, i.e., do not lie in an \((l - 1)\)-dimensional plane in \( \mathbb{X}^n \), and the restriction of \( P \) to the simplex \([v_0 \ldots v_l]\) is a homeomorphism onto the convex hull of the points \( P(v_0), \ldots, P(v_l) \);
2. \( K \) cannot be decomposed into a union of two subcomplexes \( K_1 \) and \( K_2 \) such that \( \text{dim } K_1 = \text{dim } K_2 = n - 1 \) and the set \( P(K_1 \cap K_2) \) is contained in an \((n - 2)\)-dimensional plane in \( \mathbb{X}^n \).

A polyhedron \( P : K \to \mathbb{X}^n \) is called embedded if \( P \) is an embedding, and is called self-intersecting otherwise. The number \( n \) is called the dimension of the polyhedron \( P \). The images of simplices of \( K \) under the mapping \( P \) are called faces of the polyhedron. Faces of codimension 1, i.e., of dimension \( n - 1 \), are called facets. The whole polyhedron is not thought of as a face of itself.

It is completely irrelevant what homeomorphism is used to map a simplex \([v_0 \ldots v_l]\) onto the convex hull of the points \( P(v_0), \ldots, P(v_l) \). This means that we do not distinguish between polyhedra \( P \) and \( P' \) of the same combinatorial type \( K \) such that \( P(v) = P'(v) \) for all vertices \( v \) of \( K \).

**Definition 2.3.** A continuous family of mappings \( P_u : K \to \mathbb{X}^n \) is called a nondegenerate flexible polyhedron if

1. for all but finitely many \( u \), \( P_u \) is a nondegenerate polyhedron of combinatorial type \( K \);
2. the lengths of all edges \([P_u(v_1)P_u(v_2)]\) of the polyhedron \( P_u \) are constant as \( u \) varies;
(3) for any two \( u_1 \neq u_2 \) sufficiently close to each other, the polyhedra \( P_{u_1} \) and \( P_{u_2} \) are not congruent to each other.

Let us give an example showing why we need condition (2) in Definition 2.2. Consider the two-dimensional pseudomanifold \( K \) with seven vertices \( a_1, a_2, b_1, b_2, c_1, c_2, \) and \( d \) and ten two-dimensional simplices \( [a_i a_2 b_j], [a_i b_j c_j], \) and \( [a_i c_j d] \), \( i, j \in \{1, 2\} \). Consider the polyhedron \( P: K \rightarrow \mathbb{E}^3 \) with the vertices
\[
P(a_i) = ((-1)^i, 0, 0), \quad P(b_i) = (0, (-1)^i, 0), \quad P(c_i) = (0, 0, (-1)^i), \quad P(d) = (0, 0, 0).
\]

Geometrically this polyhedron is two tetrahedra with a common edge. Naturally, it admits flexions consisting in rotations of these tetrahedra around their common edge. Condition (2) is introduced to exclude such examples. Notice that, for an embedded polyhedron, this condition always holds automatically. In what follows, we always mean that all flexible polyhedra under consideration are nondegenerate without mentioning this explicitly.

It is easy to check that, for flexible cross-polytopes, the nondegeneracy condition is equivalent to the requirement that each dihedral angle is neither identically 0 nor identically \( \pi \) during the flexion. This requirement was imposed in [13] when we classified flexible cross-polytopes, and it was called essentiality.

Consider a face \( G \) of dimension \( k < n - 1 \) of a polyhedron \( P: K \rightarrow \mathbb{X}^n \). Take a point \( x \) in the relative interior of \( G \). (As usual, we will conveniently assume that \( G \) is a vertex of \( K \).) Denote by \( S^{n-k-1}_x \) the sphere of unit vectors orthogonal to \( G \) in the tangent space \( T_x \mathbb{X}^n \). For each face \( F \supset G \) of \( P \), the cone of tangent vectors to \( F \) at \( x \) cuts out a spherical simplex of dimension \( F - k - 1 \) in the sphere \( S^{n-k-1}_x \). These simplices for all faces \( F \supset G \) constitute an \( (n - k - 1) \)-dimensional spherical polyhedron in \( S^{n-k-1}_x \), which will be denoted by \( L(G, P) \) and called the link of \( G \) in \( P \). Up to an isometry, the link of \( G \) is independent of the choice of the point \( x \). If \( P_u \) is a flexible polyhedron in \( \mathbb{X}^n \), then all faces of \( P_u \) remain congruent to themselves during the flexion. Hence, for each pair of faces \( F \supset G \) the spherical simplex corresponding to it also remains congruent to itself. Therefore, \( L(G, P_u) \) is a flexible polyhedron in \( S^{n-k-1}_x \).

For an embedded polyhedron in the Euclidean or Lobachevsky space, one can naturally define its interior dihedral angles at faces of codimension 2. For an embedded polyhedron in the sphere this definition is ambiguous, since we need to specify which of the two components of the complement of the polyhedral surface is called the interior. For a self-intersecting polyhedron, the concept of an interior dihedral angle does not make sense. Hence, for arbitrary polyhedra, the proper object is oriented dihedral angles that can be defined in the following way. First of all, we need to fix an orientation of the pseudomanifold \( K \) and an orientation of the space \( \mathbb{X}^n \). For a facet \( \Delta_i \) of a nondegenerate polyhedron \( P: K \rightarrow \mathbb{X}^n \), the unit exterior normal vector to it at a point \( x \) of \( P \) is, by definition, the unit vector \( m_i \in T_x \Delta_i \) orthogonal to \( \Delta_i \) such that the product of the direction of \( m_i \) by the orientation of \( \Delta_i \) induced by the given orientation of \( K \) yields the positive orientation of \( \mathbb{X}^n \). Let \( F \) be an \( (n - 2) \)-dimensional face of \( P \), and let \( \Delta_1 \) and \( \Delta_2 \) be the two facets containing \( F \). Choose an arbitrary point \( x \in F \). Let \( m_1 \) and \( m_2 \) be the unit exterior normal vectors to the facets \( \Delta_1 \) and \( \Delta_2 \), respectively, at the point \( x \). For \( i = 1, 2 \), we denote by \( n_i \) the unit interior normal vector to the face \( F \) of the simplex \( \Delta_i \) at the point \( x \), that is, the unit vector in \( T_x \Delta_i \) orthogonal to the simplex \( F \) and pointing inside \( \Delta_i \). Choose a positive rotation around direction around the face \( F \) such that the vector \( n_1 \) is obtained from the vector \( m_1 \) by the rotation through an angle of \( \pi/2 \) in the positive direction. Now, we denote by \( \psi_F \) the rotation angle from the vector \( n_1 \) to the vector \( n_2 \) in this positive direction. This angle is well defined up to \( 2\pi q, q \in \mathbb{Z} \). We will regard this angle as an element of the group \( \mathbb{R}/(2\pi\mathbb{Z}) \). It is easy to show that the angle \( \psi_F \) does not depend on the choice of the point \( x \), nor does it depend on which of the two facets containing \( F \) is denoted by \( \Delta_1 \). This angle will be called the oriented dihedral angle of the polyhedron \( P \) at the face \( F \).
3. FLEXIBLE CROSS-POLYTOPES OF THE SIMPLEST TYPE

In this section we present our construction of flexible cross-polytopes of the simplest type from [13, Sect. 5]. We will always identify the Euclidean space \( \mathbb{E}^n \) with the Euclidean vector space \( \mathbb{R}^n \), the sphere \( S^n \) with the unit sphere in the Euclidean vector space \( \mathbb{R}^{n+1} \), and the Lobachevsky space \( \Lambda^n \) with a half of the two-sheeted hyperboloid \( \langle x, x \rangle = -1 \), \( x_0 > 0 \), in the pseudo-Euclidean space \( \mathbb{R}^{n,1} \) with the scalar product

\[
\langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \ldots + x_n y_n.
\]

To unify the notation, we will denote by \( \mathbb{V} \) the spaces \( \mathbb{R}^n, \mathbb{R}^{n+1} \), and \( \mathbb{R}^{n,1} \) in the cases \( \mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n, \) and \( \Lambda^n \), respectively.

We consider the simplicial complex \( K_n \) with \( 2n \) vertices \( a_1, a_2, \ldots, a_n, b_1, \ldots, b_n \), with edges connecting all pairs of vertices except the pairs \( (a_i, b_i) \), \( i = 1, \ldots, n \), and with simplices spanned by all sets of vertices that are pairwise joined by edges. The simplicial complex \( K_n \) is isomorphic to the boundary of the regular \( n \)-dimensional cross-polytope. Any polyhedron of combinatorial type \( K_n \) will be called a cross-polytope. With some abuse of notation, we will denote the vertices of a cross-polytope \( P \) that are the images of the vertices \( a_i \) and \( b_i \) of \( K_n \) again by \( a_i \) and \( b_i \), respectively.

We choose the orientation of the simplicial complex \( K_n \) such that the simplex \( [a_1 \ldots a_n] \) is positively oriented. Denote the set \( \{1, \ldots, n\} \) by \([n]\). For any disjoint subsets \( I, J \subset [n] \), we denote by \( \Delta_{I,J} \) the face of \( P \) spanned by all vertices \( a_i, \ i \in I \), and all vertices \( b_j, \ j \in J \). Obviously, \( \dim \Delta_{I,J} = |I| + |J| - 1 \). The fact that the subsets \( I \) and \( J \) in notation like \( \Delta_{I,J} \) are disjoint is always implied and is not stated explicitly.

A flexible cross-polytope \( P_u \) of the simplest type in \( \mathbb{X}^n \) corresponds to a pair \( (G, \lambda) \), where

1. \( G = (g_{ij}) \) is a symmetric \( n \times n \) matrix with units on the diagonal such that all its principal minors of sizes \( 2 \times 2, \ldots, (n-1) \times (n-1) \) are strictly positive and \( \det G > 0 \), \( \det G = 0 \), and \( \det G < 0 \) in the cases \( \mathbb{X}^n = \mathbb{S}^n, \mathbb{X}^n = \mathbb{E}^n, \) and \( \mathbb{X}^n = \Lambda^n \), respectively;
2. \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is a row of nonzero real numbers such that \( \lambda_i \neq \pm \lambda_j \) for \( i \neq j \).

If \( \mathbb{X}^n = \mathbb{S}^n \) or \( \mathbb{X}^n = \Lambda^n \), then the flexible cross-polytope \( P_u \) is built in the following way:

1. Choose vectors \( n_1, \ldots, n_n \in \mathbb{V} \) with the Gram matrix \( G \) and a vector \( m \in \mathbb{V} \) orthogonal to them such that \( \langle m, m \rangle = 1 \). This can be done in a unique way up to a (pseudo)orthogonal transformation of \( \mathbb{V} \).
2. Determine elements of an \( n \times n \) matrix \( H = (h_{ij}) \) by

\[
h_{ij} = \frac{2\lambda_i (\lambda_j g_{ij} - \lambda_j)}{\lambda_i^2 - \lambda_j^2}
\]

whenever \( i \neq j \), and put \( h_{ii} = 1 \) for all \( i \).
3. In the subspace \( \text{span}(n_1, \ldots, n_n) \), take the basis \( c_1, \ldots, c_n \) dual to the basis \( n_1, \ldots, n_n \), and put

\[
d_i(u) = \sum_{j=1}^{n} h_{ij} c_j - \frac{2\lambda_i^2 u^2}{\lambda_i^2 u^2 + 1} n_i + \frac{2\lambda_i u}{\lambda_i^2 u^2 + 1} m. \tag{3.1}
\]
4. Then the parametrization of the flexion of the cross-polytope is given by

\[
a_i(u) = \frac{s_i c_i}{|c_i|}, \quad b_i(u) = \frac{s'_i d_i(u)}{|d_i(u)|}. \tag{3.2}
\]

where the signs \( s_i, s'_i = \pm 1 \) are chosen arbitrarily in the case \( \mathbb{X}^n = \mathbb{S}^n \) and are chosen so that the points \( a_i(u) \) and \( b_i(u) \) belong to the connected component \( \Lambda^n \) of the hyperboloid \( \langle x, x \rangle = -1 \) in
the case \( X^n = \Lambda^n \). If \( X^n = \Lambda^n \), we use the notation \( |x| = \sqrt{-\langle x, x \rangle} \) for time-like vectors \( x \in \mathbb{R}^{n,1} \). (In fact, a straightforward computation shows that the length of the vector \( d_i(u) \) is independent of \( u \)).

If \( X^n = E^n \), then steps 1 and 2 are the same as above, while steps 3 and 4 are as follows:

3. Take a hyperplane \( \Pi \subset E^n \) orthogonal to \( m \), and take a simplex \([a_1 \ldots a_n]\) in \( \Pi \) such that the vectors \( n_1, \ldots, n_n \) are orthogonal to the facets of \([a_1 \ldots a_n]\) opposite to the vertices \( a_1, \ldots, a_n \), respectively. It is easy to see that such a simplex is unique up to homothety and parallel translation. For each \( i \), take the length of the altitude of the simplex \([a_1 \ldots a_n]\) drawn from the vertex \( a_i \), and multiply it by the sign \( s_i \) that is equal to \(+1\) whenever \( n_i \) is the interior normal vector to the corresponding facet of \([a_1 \ldots a_n]\) and is equal to \(-1\) whenever \( n_i \) is the exterior normal vector to the corresponding facet of \([a_1 \ldots a_n]\). Denote the obtained number by \( a_i \).

4. Then the parametrization of the flexion of the cross-polytope is given by

\[
\begin{align*}
\mathbf{a}_i(u) &= a_i, \\
\mathbf{b}_i(u) &= b_i\left(\sum_{j=1}^{n} \frac{h_{ij}a_j}{a_j} - \frac{2\lambda^2u^2}{\lambda^2u^2+1}n_i + \frac{2\lambda_iu}{\lambda_i^2u^2+1}m\right), \\
&= \left(\sum_{j=1}^{n} \frac{h_{ij}}{a_j}\right)^{-1},
\end{align*}
\]

In this case, we denote by \( s'_i \) the sign of the number \( b_i \), \( i = 1, \ldots, n \).

**Remark 3.1.** In the Euclidean and spherical cases, the above construction yields a flexible cross-polytope if and only if none of the denominators in the formulas written above vanishes, hence, it yields a flexible cross-polytope for all pairs \((G, \lambda)\) satisfying the above conditions (1) and (2) outside some subset of positive codimension. For the Lobachevsky space the situation is somewhat more difficult. Namely, the cross-polytope is well defined only if the vectors \( d_i(u) \) computed by (3.1) turn out to be time-like.

**Remark 3.2.** The fact that the formulas written above actually yield flexible polyhedra, i.e., that the lengths of all edges actually remain constant during the obtained deformations, was proved in [13]. However, this fact can actually be checked directly by a simple calculation without using the results of [13].

**Remark 3.3.** In [13] the author has also shown that, for flexible cross-polytopes of the simplest type described above, the dihedral angles adjacent to the facet \([a_1 \ldots a_n]\) vary during the flexion so that the tangents of the halves of any two of them are either directly or inversely proportional to each other. Moreover, in the same paper it has been shown that for \( X^n \neq E^2 \) this property is characteristic of flexible polyhedra of the simplest type. Namely, if the dihedral angles adjacent to a facet of a nondegenerate flexible cross-polytope vary so that the tangents of the halves of any two of them are either directly or inversely proportional to each other, then the flexion of this cross-polytope can be parametrized as indicated above. In the Euclidean plane \( E^2 \), besides flexible cross-polytopes (quadrangles) of the simplest type, this characteristic property is also fulfilled for flexible parallelograms, which cannot be obtained by the construction described above (see [13, Lemma 4.8, Remark 4.9]).

It is easy to see that, for each \( i \), changing simultaneously the signs of the numbers \( \lambda_i, s_i, \) and \( s'_i \), of all matrix elements \( g_{ij} = g_{ji} \) such that \( j \neq i \), and of the vectors \( m \) and \( n_i \) does not change the cross-polytope \( P_u \). Hence, without loss of generality, we may assume that all coefficients \( \lambda_i \) are positive. In addition, renumbering the vertices of the cross-polytope, we can make it so that \( 0 < \lambda_1 < \ldots < \lambda_n \). In what follows, we will always assume that these inequalities are satisfied.

In the rest of this paper, \( P_u \) is always a flexible cross-polytope of the simplest type, \((G, \lambda)\) are the corresponding data, and \( s = (s_1, \ldots, s_n, s'_1, \ldots, s'_n) \) is the corresponding row of signs.
4. DIHEDRAL ANGLES

For the oriented dihedral angles of the cross-polytope $P_u$ to be well defined, we need to choose the orientation of the space $\mathbb{X}^n$. We will conveniently choose this orientation so that the vector $m$ is the interior normal vector to the simplex $[a_1 \ldots a_n]$ if $s_1 \ldots s_n = 1$ and is the exterior normal vector to the simplex $[a_1 \ldots a_n]$ if $s_1 \ldots s_n = -1$. We introduced the sign $s_1 \ldots s_n$ in order that in the spherical case the orientation of the sphere $\mathbb{S}^n$ does not change if we replace some vertices of $P_u$ with their antipodes.

Each $(n - 2)$-dimensional face of $P_u$ has the form $\Delta_{I,J}$, where $|I| + |J| = n - 1$. We put $\psi_{I,J}(u) = \psi_{\Delta_{I,J}}(u)$.

For each $k \in [n]$, we consider the set

$$X_k = \{i \in [n] \mid ((i < k) \cap (s_is'_i = 1)) \lor ((i > k) \cap (s_is'_i = -1))\},$$

where $\land$ and $\lor$ denote the logical “and” and “or,” respectively.

**Lemma 4.1.** If either $n \geq 3$ or $n = 2$ and $\mathbb{X}^2 = \mathbb{S}^2$, then the oriented dihedral angles of the cross-polytope $P_u$ are given by

$$\psi_{I,J}(u) = (-1)^{|J \cap X_k|} s_k \varphi_k(u) + \begin{cases} 0 & \text{if } s_k s'_k = 1, \\ \pi & \text{if } s_k s'_k = -1, \end{cases}$$

where $k$ is a unique element of the set $[n] \setminus (I \cup J)$ and $\varphi_k(u) = 2 \arctan(\lambda_k u)$.

**Proof.** First of all, we consider the dihedral angles $\psi_k(u) = \psi_{\Delta_{\emptyset,k}}(u)$ adjacent to the facet $\Delta_{[n],\emptyset} = [a_1 \ldots a_n]$. For them, formula (4.1) was essentially obtained in the author's paper [13]. Indeed, in that paper, when deriving the parametrization of the cross-polytopes of the simplest type, we used the variables $t_k$ that were originally defined as the tangents of the halves of the dihedral angles adjacent to the facet $[a_1 \ldots a_n]$. Then, to simplify the formulas, we applied several special algebraic transformations, which were called *elementary reversions*. As a result of these transformations, each of the variables $t_k$ was replaced with one of the values $\pm t_k^{-1}$. Afterwards, the new variables $t_k$ were parametrized by $t_k = \lambda_k u$. Thus, the results of [13] immediately imply that $\tan(\psi_k(u)/2) = \pm (\lambda_k u)^{\pm 1}$, i.e., that $\psi_k(u)$ is one of the angles $\pm \varphi_k(u)$ or $\pm \varphi_k(u) + \pi$. (Recall that the angle $\psi_k(u)$ is defined modulo $2\pi \mathbb{Z}$.) It remains to show that the sign $\pm$ at $\varphi_k(u)$ is equal to $s_k$ and the summand $\pi$ is present if and only if $s_k s'_k = -1$. The sign of the tangent of one half of the angle $\psi_k(u)$ is equal to the sign of the sine of the angle $\psi_k(u)$ and, hence, to the sign of the scalar product $(b_k(u), m)$, which is equal to $s'_k \text{sgn}(\lambda_k u)$. Therefore, the dihedral angle $\psi_k(u)$ equals either $s'_k \varphi_k(u)$ or $\pi - s'_k \varphi_k(u)$. Further, from formulas (3.1)–(3.4), which provide the parametrization for the flexion of the cross-polytope $P_u$, one can easily deduce that $\text{dist}_{\mathbb{X}^n}(a_k, b(0)) < \text{dist}_{\mathbb{X}^n}(a_k, b(\infty))$ whenever $s_k s'_k = 1$ and $\text{dist}_{\mathbb{X}^n}(a_k, b(0)) > \text{dist}_{\mathbb{X}^n}(a_k, b(\infty))$ whenever $s_k s'_k = -1$. Thus, if $s_k s'_k = 1$, then $\psi_k(0) = 0$, $\psi_k(\infty) = \pi$, and hence $\psi_k(u) = s_k \varphi_k(u)$, while if $s_k s'_k = -1$, then $\psi_k(0) = \pi$, $\psi_k(\infty) = 0$, and hence $\psi_k(u) = s_k \varphi_k(u) + \pi$.

Now, let $U, W \subset [n]$ be subsets such that $U \cap W = \emptyset$ and $|U| + |W| = n - 2$, and let $k$ and $l$ be the two distinct elements of the set $[n] \setminus (U \cup W)$. Consider the formula

$$\psi_{U \cup W \cup \{l\}} = \begin{cases} -\psi_{U \cup \{l\},W} & \text{if } l \in X_k, \\ -\psi_{U \cup \{l\},W} & \text{if } l \notin X_k. \end{cases}$$

We will prove formulas (4.1) and (4.2) by the simultaneous induction on $|J|$ and $|W|$. Formula (4.1) has already been proved for $|J| = 0$. First, we will show that formula (4.1) for all pairs $(I, J)$ such that $|J| = p$ implies formula (4.2) for all pairs $(U, W)$ such that $|W| = p$. Second, we will show that formula (4.2) for all pairs $(U, W)$ such that $|W| = p$ and formula (4.1) for all
pairs \((I, J)\) such that \(|J| = p\) imply formula (4.1) for all pairs \((I, J)\) such that \(|J| = p + 1\). As a result we will complete the proofs of formulas (4.1) and (4.2).

1. Suppose that formula (4.1) holds true for all pairs \((I, J)\) such that \(|J| = p\). Consider an arbitrary pair \((U, W)\) such that \(|W| = p\). If \(n \geq 3\), then the link \(L_{U \cup W}(u) = L(\Delta_{U \cup W}, P_u)\) is a spherical quadrangle. We denote by \(A, B, C,\) and \(D\) the vertices of this quadrangle corresponding to the \((n - 2)\)-dimensional faces \(\Delta_{U \cup \{l\}, W}, \Delta_{U \cup \{k\}, W}, \Delta_{U, W \cup \{l\}},\) and \(\Delta_{U, W \cup \{k\}}\), respectively. Then \(A, B, C,\) and \(D\) are consecutive vertices of the quadrangle \(L_{U \cup W}(u)\) (in the cyclic order).

In the exceptional case of \(n = 2\) and \(X^2 = S^2\), the condition \(|U| + |W| = n - 2 = 0\) immediately implies that \(U = W = \emptyset\). Hence the face \(\Delta_{U, W} = \Delta_{\emptyset, \emptyset}\) does not exist. Nevertheless, we will conveniently adhere to the convention that the cross-polytope \(P_u\) has an additional empty face \(\Delta_{\emptyset, \emptyset}\), which is formally assigned the dimension \(-1\). The link of this face is, by definition, the cross-polytope \(P_u\) itself. Thus, in the case of \(n = 2\) and \(X^2 = S^2\), the link \(L_{U \cup W}(u) = L_{\emptyset, \emptyset}(u)\) is again a spherical quadrangle with the vertices \(A = \Delta_{\{l\}, \emptyset} = a_l, B = \Delta_{\{k\}, \emptyset} = a_k, C = \Delta_{\emptyset, \{l\}} = b_l,\) and \(D = \Delta_{\emptyset, \{k\}} = b_k\), where \((k, l) = (1, 2)\) or \((2, 1)\).

The oriented angles \(\psi_A(u), \psi_B(u), \psi_C(u),\) and \(\psi_D(u)\) of the spherical quadrangle \(L_{U \cup W}(u)\) at the vertices \(A, B, C,\) and \(D\), respectively, are equal to \(\psi_{U \cup \{l\}, W}(u), \psi_{U \cup \{k\}, W}(u), \psi_{U, W \cup \{l\}}(u),\) and \(\psi_{U, W \cup \{k\}}(u)\), respectively. By the induction hypothesis, formula (4.1) holds true for the pairs \((U \cup \{l\}, W)\) and \((U \cup \{k\}, W)\). Hence, during the flexion of the quadrangle \(L_{U \cup W}(u)\), its angles at the vertices \(A\) and \(B\) vary in such a way that the tangents of their halves are either directly or inversely proportional to each other. Flexible spherical quadrangles with this property were completely described by Bricard in his study [6, §II] of flexible octahedra in \(E^3\). (Bricard considered tetrahedral angles in \(E^3\) rather than spherical quadrangles, but these two objects are obviously equivalent.) By the result of Bricard, for each such flexible spherical quadrilateral, either its opposite sides are pairwise equal to each other or the sum of the lengths of each pair of its opposite sides is equal to \(\pi\). There exist three types of such quadrangles; they are shown in Fig. 1. The vertices of these quadrangles are denoted by \(P, Q, R,\) and \(S\) instead of \(A, B, C,\) and \(D\), since they can be identified with \(A, B, C,\) and \(D\) in several ways that differ from each other by cyclic permutations. For quadrangles of the first and second types shown in Figs. 1a and 1b, respectively, we have \(PQ = RS\) and \(QR = SP\), and the tangents of the halves of neighboring angles are inversely proportional to each other during the flexion. Hence these cases occur when \(s_k s'_k = -s_l s'_l\). For quadrangles of the third type shown in Fig. 1c, we have \(PQ + RS = QR + SP = \pi\), and the tangents of the halves of neighboring angles are directly proportional to each other during the flexion. Hence this case occurs when \(s_k s'_k = s_l s'_l\).

Assume that the quadrangle \(L_{U \cup W}(u)\) is of the first type. Then

\[
\psi_{U \cup \{l\}, W}(u) = \psi_{C, W}(u) = \psi_{A, U}(u) = \psi_{U \cup \{l\}, W}(u).
\]

(4.3)

Let us show that, in this case, \(l \notin X_k\). Take \(u > 0\). If \(s_k s'_k = 1\) and \(s_l s'_l = -1\), then formula (4.1) for the pairs \((U \cup \{l\}, W)\) and \((U \cup \{k\}, W)\) implies that \(\psi_A(u) = \pm \varphi_k(u)\) and \(\psi_B(u) = \pm \varphi_l(u) + \pi\).
Hence, the unoriented interior angles of the quadrangle $L_{U,W}(u)$ are equal to $\varphi_k(u)$ and $\pi - \varphi_l(u)$. Since the sum of the angles of a spherical quadrangle is greater than $2\pi$, this implies that $\varphi_k(u) > \varphi_l(u)$ and, therefore, $l < k$. Thus, $l \not\in X_k$. If $s_k s'_k = -1$ and $s_l s'_l = 1$, then the interior angles of the quadrangle $L_{U,W}(u)$ are equal to $\pi - \varphi_k(u)$ and $\varphi_l(u)$, so $l > k$, and again $l \not\in X_k$.

If the quadrangle $L_{U,W}(u)$ is of the second type, then
\[
\psi_{U,W\cup\{l\}}(u) = \psi_C(u) = -\psi_A(u) = -\psi_{U\cup\{l\},W}(u). \tag{4.4}
\]

Similarly to the previous case, we easily deduce that $l \in X_k$ from the fact that the sum of the angles indicated by one and two arcs in Fig. 1b is less than $\pi$.

Assume that the quadrangle $L_{U,W}(u)$ is of the third type. If the vertex $A$ is identified either with $P$ or with $R$, then equality (4.4) holds true, and if $A$ is identified either with $Q$ or with $S$, then equality (4.3) holds true. Similarly to the previous cases, since the angle indicated by one arc is greater than the angle indicated by two arcs in Fig. 1c, it follows easily that $l \in X_k$ whenever $A$ is either $P$ or $R$, while $l \not\in X_k$ whenever $A$ is either $Q$ or $S$.

Thus, we see that equality (4.2) holds true in all cases.

2. Suppose that formula (4.2) holds true for all pairs $(U,W)$ such that $|W| \leq p$ and formula (4.1) holds true for all pairs $(I,J)$ such that $|J| = p + 1$. Then formula (4.1) for the pair $(I,J)$ follows immediately from formula (4.1) for the pair $(I \cup \{l\}, J \setminus \{l\})$ and formula (4.2) for the pair $(I,J \setminus \{l\})$, where $l$ is an arbitrary element of $J$. □

For each $(n - 2)$-dimensional face $F = \Delta_{I,J}$, we will conveniently introduce the notation
\[
\lambda_F = (-1)^{|J\setminus X_k|} s_k \lambda_k,
\]
where $k$ is a unique element of the set $[n] \setminus (I \cup J)$. Then $\psi_F(u) = 2 \arctan(\lambda_F u)$ if $s_k s'_k = 1$ and $\psi_F(u) = 2 \arctan(\lambda_F u) + \pi$ if $s_k s'_k = -1$.

**Remark 4.2.** For flexible cross-polytopes (quadrangles) of the first type in $\mathbb{E}^2$ or $\Lambda^2$, the above proof does not work. Indeed, in these cases the quadrangle $L_{\partial,\partial}(u) = P_u$ is either Euclidean or hyperbolic rather than spherical, while the fact that it is spherical plays an important role in the proof (for instance, in the claim that the sum of its angles is greater than $2\pi$ in the case shown in Fig. 1a). In fact, one can show that formulas (4.1) still hold true for flexible quadrangles of the simplest type in $\mathbb{E}^2$ but do not hold true for some flexible quadrangles of the simplest type in $\Lambda^2$. We will not do this in the present paper, because the case of quadrangles of the simplest type in $\mathbb{E}^2$ and $\Lambda^2$ is not of serious interest in itself, and it will not be used in the sequel.

5. EMBEDDED SPHERICAL FLEXIBLE CROSS-POLYTOPES

**Theorem 5.1.** Let $P_u : K_n \to S^n$ be a spherical flexible cross-polytope of the simplest type corresponding to a triple $(G, \lambda, s)$ such that $0 < \lambda_1 < \ldots < \lambda_n$ and $s_i s'_i = -1$ for all $i$. Then the mapping $P_0$ is the homeomorphism of $K_n$ onto the equatorial sphere $S^{n-1} \subset S^n$. Hence, the mapping $P_u$ is an embedding for all sufficiently small $u$.

**Proof.** We will prove the assertion of the theorem by induction on the dimension $n$.

**Base of induction $n = 2$**. By (4.1), all angles of the spherical quadrangle $P_0$ are equal to $\pi$. Hence this quadrangle is a decomposition of the great circle $S^1 \subset S^2$ into four arcs. Thus, the assertion of the theorem is true. Notice that in this reasoning it is important that the length of every side of the quadrangle $P_0$ is strictly less than $\pi$. This implies that the perimeter of $P_0$ is less than $4\pi$, and so $P_0$ cannot be “wound” around the great circle $S^1$ two or more times.
Inductive step. Suppose that \( n \geq 3 \). Assume that the assertion of the theorem is true for \( m \)-dimensional flexible cross-polytopes of the simplest type for all \( m < n \). We will prove the assertion of the theorem for an \( n \)-dimensional flexible cross-polytope of the simplest type \( P_u \). Recall that a continuous mapping \( f : X \to Y \) of topological spaces is called a local homeomorphism at a point \( x \in X \) if it maps homeomorphically a neighborhood of \( x \) in \( X \) onto a neighborhood of \( f(x) \) in \( Y \). Let us prove that the mapping \( P_0 : K_n \to S^{n-1} \) is a local homeomorphism at all points of \( K_n \). For points in the interiors of \((n - 1)\)-dimensional simplices, this follows immediately, since all faces of \( P_0 \) are nondegenerate. For points in the relative interiors of \((n - 2)\)-dimensional simplices, this also follows immediately, since, by (4.1), the dihedral angles at all \((n - 2)\)-dimensional faces of \( P_0 \) are equal to \( \pi \). Consider an \((n - k - 1)\)-dimensional simplex \( \Delta \) of \( K_n \), where \( k > 1 \), and a point \( x \) in its relative interior. With some abuse of notation, we denote the face of \( P_u \) that is the image of the simplex \( \Delta \) again by \( \Delta \). The oriented dihedral angles of the link \( L(\Delta, P_u) \) at its \((k - 2)\)-dimensional faces are equal to the oriented dihedral angles of \( P_u \) at the corresponding \((n - 2)\)-dimensional faces of it. Therefore, the link \( L(\Delta, P_u) \) is itself a flexible spherical cross-polytope of the simplest type, and all its dihedral angles become equal to \( \pi \) for \( u = 0 \). Hence, renumbering its vertices, we can ensure that this cross-polytope corresponds to a triple \((G, \lambda, \mathbf{s})\), where \( 0 < \lambda_1 < \ldots < \lambda_k \). It follows from (4.1) that \( \bar{s}_i s'_i = -1 \), \( i = 1, \ldots, k \). Consequently, by the induction hypothesis, we conclude that the surface \( L(\Delta, P_u) \) is embedded and coincides with the \((k - 1)\)-dimensional great sphere \( S_{\Delta}^{k-1} \) cut in \( S_{\Delta}^n \) by the tangent space to the great sphere \( S^{n-1} \) containing \( P_0 \). Therefore, the mapping \( P_0 : K_n \to S^{n-1} \) is a local homeomorphism at \( x \).

Thus, the mapping \( P_0 : K_n \to S^{n-1} \) is a local homeomorphism at all points of \( K_n \). Since the simplicial complex \( K_n \) is compact, it follows that this mapping is a finite-sheeted nonramified covering (see, for instance, [15, Ch. V, Sect. 2]). However, \( n \geq 3 \); hence the sphere \( S^{n-1} \) is simply connected; therefore, this covering is a homeomorphism (see [15, Ch. V, Sect. 6; 16, Ch. 4, Sect. 2]). Obviously, a polyhedron sufficiently close to an embedded polyhedron is embedded too. Therefore, since the cross-polytope \( P_u \) is embedded for \( u = 0 \), we conclude that it is embedded for all sufficiently small \( u \).

Remark 5.2. Similarly, it can be proved that the mapping \( P_\infty \) is a homeomorphism onto the equatorial sphere \( S^{n-1} \) whenever \( s_is'_i = 1 \) for all \( i \).

Proof of Theorem 1.1. Consider a flexible cross-polytope \( P_u \) of the simplest type in \( S^n \) that corresponds to a triple \((G, \lambda, \mathbf{s})\), where \( G \) is a positive definite symmetric matrix with units on the diagonal, \( 0 < \lambda_1 < \ldots < \lambda_n \), and \( s_i = -1, s'_i = 1 \) for \( i = 1, \ldots, n \). This cross-polytope is well defined for almost all pairs \((G, \lambda)\) (see Remark 3.1). By Theorem 5.1, \( P_u \) is embedded for sufficiently small \( u \), that is, satisfies property (1) in Theorem 1.1. Formulas (3.1), (3.2), which give a parametrization of the flexion of \( P_u \), immediately imply that \( P_u \) also satisfies properties (2) and (4) in Theorem 1.1 and “almost satisfies” property (3). Namely, for \( u > 0 \) the cross-polytope \( P_u \) is contained in the closed positive hemisphere \( S_+^n \subset S^n \) consisting of all unit vectors that have nonnegative scalar products with \( \mathbf{m} \), while for \( u < 0 \) the cross-polytope \( P_u \) is contained in the corresponding closed negative hemisphere \( \overline{S}^n_+ \).

A flexible cross-polytope that, in addition, satisfies property (3) can be obtained by rotating the whole sphere \( S^n \). In the space \( \mathbb{R}^{n+1} \) containing the sphere \( S^n \), we choose a vector \( \mathbf{k} \) orthogonal to \( \mathbf{m} \) and making acute angles with the vectors \( \mathbf{c}_1, \ldots, \mathbf{c}_n \). For instance, we can take \( \mathbf{k} = \mathbf{n}_1 + \ldots + \mathbf{n}_n \). Consider the \((n - 1)\)-dimensional vector subspace \( U \subset \mathbb{R}^{n+1} \) orthogonal to \( \mathbf{m} \) and \( \mathbf{k} \). Denote by \( R_\alpha \) the rotation around \( U \) through the angle \( \alpha \), where the direction of the rotation is chosen so that, for small positive \( \alpha \), the vectors \( R_\alpha(\mathbf{c}_i) \) form obtuse angles with \( \mathbf{m} \).

For each \( u \), we denote by \( \rho(u) \) the smallest of the distances (in the metric of \( S^n \)) from the vertices \( \mathbf{b}_1(u), \ldots, \mathbf{b}_n(u) \) of \( P_u \) to the equatorial great sphere \( S^{n-1} \) orthogonal to \( \mathbf{m} \). Then \( \rho : \mathbb{R} \to \mathbb{R} \) is a continuous function such that \( \rho(0) = \rho(\infty) = 0 \) and \( \rho(u) > 0 \) for \( u \neq 0, \infty \).
Consider the flexible cross-polytope
\[ \tilde{P}_u = R_{\alpha(u)}(P_u), \quad \alpha(u) = \frac{1}{2} \text{sgn}(u)\rho(u). \]

Let us show that it satisfies properties (1)–(4) in Theorem 1.1. Since the cross-polytope \( \tilde{P}_u \) is obtained from \( P_u \) by a rotation and, in addition, \( \tilde{P}_0 = P_0 \) and \( \tilde{P}_{\infty} = P_{\infty} \), properties (1), (2), and (4) for \( \tilde{P}_u \) follow from the same properties for \( P_u \). Let us prove property (3). The definition of the rotations \( R_{\alpha} \) immediately implies that the vertices \( \tilde{a}_i(u) = R_{\alpha(u)}(a_i) \) lie in the open hemisphere \( S_u^+ \) whenever \( u > 0 \) and in the open hemisphere \( S_u^- \) whenever \( u < 0 \). The vertices \( b_i(u) \) of the initial cross-polytope \( P_u \) also lie in \( S_u^+ \) whenever \( u > 0 \) and in \( S_u^- \) whenever \( u < 0 \). Under the rotation through \( \rho(u)/2 \), these vertices are shifted by distances that are not greater than \( \rho(u)/2 \); hence, they cannot leave these hemispheres. Therefore, \( \tilde{P}_u \) is contained in \( S_u^+ \) for \( u > 0 \) and in \( S_u^- \) for \( u < 0 \). □

**Remark 5.3.** In some particular cases, the proof of Theorem 5.1 can be obtained without using topological facts and without applying formulas (4.1) for the oriented dihedral angles. Consider a special case of the cross-polytopes of the simplest type corresponding to the unit matrix \( \varepsilon \). We will say that the cross-polytope \( P \) is tangent to the equidistant hypersurface \( \Omega \) if \( \Omega \) is circumscribed yields a decomposition of the facets of \( P \) into two classes in the following way. Choose some orientations of the hypersurface \( \Omega \) and of the polyhedral surface \( P \). To the tangent point of \( \Omega \) and the hyperplane of a facet \( F \) of \( P \) we assign the sign “+” if their orientations coincide at the tangent point and the sign “−” otherwise. All facets of \( P \) are decomposed into two classes \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) depending on these signs. If we reverse either the orientation of \( \Omega \) or the orientation of \( P \), then the classes \( \mathcal{F}_+ \) and \( \mathcal{F}_- \) will be interchanged. The obtained decomposition without specifying
which of the two classes is positive and which is negative will be called the decomposition given by $\Omega$.

Now, assume that the hyperplanes of all facets of a cross-polytope $P$ intersect a hyperplane $H \subset \mathbb{X}^m$ at the same angle $\alpha \in (0, \pi/2)$. Since $\alpha < \pi/2$, we see that, at the intersection point of the hyperplane of the facet $F$ and the hyperplane $H$, the orthogonal projection provides a correspondence between the orientations of these two hyperplanes. Thus, we again obtain a decomposition of the facets of $P$ into two classes. This decomposition will be called the decomposition given by $H$. Exactly in the same way, one can obtain a decomposition of the facets of $P$ into two classes if $\mathbb{X}^m = \Lambda^m$ and the hyperplanes of all facets of $P$ are parallel to the same hyperplane $H$. (Recall that two hyperplanes in $\Lambda^m$ are called parallel if they have no common points in $\Lambda^m$ and have exactly one common point on the absolute.)

**Theorem 6.1.** For each flexible cross-polytope $P_u$ of the simplest type in $\mathbb{X}^n$, $n \geq 3$, we have one of the following two possibilities:

1. The cross-polytopes $P_{(1)}, \ldots, P_{(n)}$ are circumscribed about concentric $(n-2)$-dimensional spheres or orispheres $\Omega_1, \ldots, \Omega_n$ in $\mathbb{X}^{n-1}$; moreover, the spheres $\Omega_1, \ldots, \Omega_n$ are not great spheres if $\mathbb{X}^{n-1} = \mathbb{S}^{n-1}$.

2. There is a hyperplane $H \subset \mathbb{X}^{n-1}$ such that, for each of the cross-polytopes $P_{(1)}, \ldots, P_{(n)}$, one of the following three conditions is fulfilled:

   (a) $P_{(k)}$ is circumscribed about an equidistant hypersurface $\Omega_k$ with base $H$;

   (b) (only in the case $\mathbb{X}^{n-1} = \Lambda^{n-1}$) the hyperplanes of all facets of $P_{(k)}$ are parallel to $H$;

   (c) the hyperplanes of all facets of $P_{(k)}$ intersect $H$ at the same angle $\alpha_k \in (0, \pi/2)$.

In addition, for each $k$, the decomposition of facets of $P_{(k)}$ into two classes given by $\Omega_k$ in cases 1 and 2(a) and by $H$ in cases 2(b) and 2(c) is as follows: One of the classes consists of all facets $\Delta_{i,j}$ such that the set $J \setminus X_k$ has even cardinality, and the other class consists of all facets $\Delta_{i,j}$ such that $J \setminus X_k$ has odd cardinality.

**Remark 6.2.** If $\mathbb{X}^n = \mathbb{S}^n$, then possibilities 1 and 2 coincide. Indeed, if the cross-polytope $P_{(k)}$ is circumscribed about an $(n-2)$-dimensional small sphere $\Omega_k$, then all facets of $P_{(k)}$ intersect the great sphere $H$ concentric to $\Omega_k$ at the same angle $\alpha_k \in (0, \pi/2)$, and vice versa.

**Remark 6.3.** The assertion of Theorem 6.1 remains literally true if we replace the flat position $P_0$ of $P_u$ with the other flat position $P_\infty$ of it. This is not surprising, since it is clear that these two cases are completely similar to each other. However, it is interesting that in both cases the resulting decompositions of facets of the cross-polytopes $P_{(k)}$ into two classes are governed by the evenness of the cardinalities of the same sets $J \setminus X_k$. This can be explained in the following way. Let $(G, \bar{\mathbf{x}}, \mathbf{s})$ be the set of data corresponding to the flexible cross-polytope $P_u$. Consider the flexible cross-polytope $\tilde{P}_u$ corresponding to the set of data $(G, \bar{\mathbf{x}}, \mathbf{s})$, where $\bar{\lambda}_i = 1/\lambda_i$, $\bar{s}_i = s_i$, and $\bar{s}'_i = -s'_i$ for all $i$. It can be immediately checked that $P_u = \tilde{P}_u \cup \tilde{P}_u$ for all $u$; in particular, $P_\infty = \tilde{P}_0$. Hence, to reduce the assertion of the theorem for $P_\infty$ to the assertion of the theorem for $\tilde{P}_0$, it is enough to show that the sets $\tilde{X}_k$ for the flexible cross-polytopes $\tilde{P}_u$ and $\tilde{P}_u$ are identical to each other. At first sight, this is not the case, since we have reversed all signs $s'_i$. Nevertheless, the coefficients $\tilde{\lambda}_i$ decrease rather than increase. Therefore, all the results described above, including Theorem 6.1, will become true for $\tilde{P}_u$ only after renumbering its vertices in the opposite order or, equivalently, reversing the order on the set $[n]$. Now, we only need to notice that, simultaneously reversing the order on the set $[n]$ and all signs $s'_i$, we do not change the sets $\tilde{X}_k$.

Let $G = \Delta_{U,W}$ be an arbitrary $(n-3)$-dimensional face of $P_{(k)}$, and let $l$ be a unique element of the set $[n] \setminus (U \cup W \cup \{k\})$. Let $F$ and $F'$ be the two facets of $P_{(k)}$ that contain the face $G$. Consider the dihedral angle between the $(n-2)$-dimensional simplices $F$ and $F'$ in $\mathbb{X}^{n-1}$. If $l \in X_k$,
then we denote by $B_{k,G}$ the interior bisecting hyperplane of the dihedral angle between $F$ and $F'$, while if $l \notin X_k$, then we denote by $B_{k,G}$ the exterior bisecting hyperplane of the dihedral angle between $F$ and $F'$.

In the case of the Euclidean space $\mathbb{E}^{n-1}$, we consider its projectivization $\mathbb{RP}^{n-1}$ and denote by $\hat{B}_{k,G}$ the projectivization of the hyperplane $B_{k,G}$. In the case of the Lobachevsky space $\Lambda^{n-1}$, we consider its Beltrami–Klein model in which it is identified with a disk in the projective space $\mathbb{RP}^{n-1}$, and denote by $\hat{B}_{k,G}$ the projective hyperplane in $\mathbb{RP}^{n-1}$ containing the hyperplane $B_{k,G}$.

In the spherical case, we denote by $\hat{B}_{k,G}$ the image of the great sphere $B_{k,G}$ under the natural two-sheeted projection $S^{n-1} \to \mathbb{RP}^{n-1}$.

Lemma 6.4. The projective hyperplanes $\hat{B}_{k,G}$ corresponding to all pairs $(k,G)$ such that $G$ is an $(n-3)$-dimensional face of $P(k)$ intersect exactly at one point.

Lemma 6.5. Any dihedral angle of any of the cross-polytopes $P(k)$ is equal neither to 0 nor to $\pi$.

We will show that Theorem 6.1 follows from Lemmas 6.4 and 6.5, and then we will prove these lemmas.

Proof of Theorem 6.1. Let $O$ be the intersection point of the projective hyperplanes $\hat{B}_{k,G}$, which exists and is unique by Lemma 6.4. Consider the following cases:

1. Suppose that $O$ lies in $X^{n-1} = \mathbb{E}^{n-1}$ or $\Lambda^{n-1}$ or corresponds to a pair of antipodal points of $X^{n-1} = S^{n-1}$. In the latter case, we denote by $O$ one of the two antipodal points of $S^{n-1}$ projecting to $O \in \mathbb{RP}^{n-1}$. Then, for each pair $(k,G)$, the bisecting hyperplane $B_{k,G}$ of the dihedral angle between the two $(n-2)$-dimensional faces $F$ and $F'$ of $P(k)$ containing $G$ passes through $O$. Hence, each cross-polytope $P(k)$ is circumscribed about a sphere $\Omega_k$ centered at $O$.

2. Suppose that $X^{n-1} = \Lambda^{n-1}$ and $O$ is a point on the absolute. Similarly, we find that each cross-polytope $P(k)$ is circumscribed about an orisphere $\Omega_k$ centered at $O$.

3. Suppose that $X^{n-1} = \mathbb{E}^{n-1}$ and $O$ is a point at infinity. Consider an arbitrary hyperplane $H \subset \mathbb{E}^{n-1}$ orthogonal to lines passing through $O$. Then, for each pair $(k,G)$, the bisecting hyperplane $B_{k,G}$ of the dihedral angle between the facets $F$ and $F'$ of $P(k)$ containing $G$ is perpendicular to $H$. Therefore, all facets of each cross-polytope $P(k)$ form the same angle with $H$.

4. Suppose that $X^{n-1} = \Lambda^{n-1}$ and $O \in \mathbb{RP}^{n-1}$ is a point outside the absolute. We denote by $H \subset \Lambda^{n-1}$ the hyperplane that is the polar of $O$ with respect to the absolute. Then, for each pair $(k,G)$, the bisecting hyperplane $B_{k,G}$ of the dihedral angle between the facets $F$ and $F'$ of $P(k)$ containing $G$ intersects the hyperplane $H$ and is perpendicular to it. Hence either the hyperplanes of both facets $F$ and $F'$ are divergent with $H$ and are at the same distance from it, or the hyperplanes of both facets $F$ and $F'$ are parallel to $H$, or the hyperplanes of both facets $F$ and $F'$ intersect $H$ at the same angle. Hence, for each of the cross-polytopes $P(k)$, one of assertions 2(a)–2(c) in Theorem 6.1 holds.

It follows from Lemma 6.5 that, in case 1, none of the spheres $\Omega_k$ is a great sphere in $S^{n-1}$, and in cases 3 and 4, the facets of $P(k)$ are neither contained in $H$ nor perpendicular to $H$.

In each of the cases considered, the facets $F$ and $F'$ of $P(k)$ belong to the same class in the decomposition corresponding either to the hypersurface $\Omega_k$ or to the hyperplane $H$ if and only if $B_{k,G}$ is the interior bisecting hyperplane of the dihedral angle between $F$ and $F'$, i.e., if and only if $l \in X_k$, where $l$ is a unique element of the set $\{n\} \setminus (U \cup W \cup \{k\})$ and $G = \Delta_{U,W}$. This easily implies that one of the two classes consists of all facets $\Delta_{l,J}$ of $P(k)$ with even cardinality $|J \setminus X_k|$, and the other class consists of all facets $\Delta_{l,J}$ of $P(k)$ with odd cardinality $|J \setminus X_k|$. □

Let $F$ and $F'$ be two $(n-2)$-dimensional simplices in $X^{n-1}$ with a common $(n-3)$-dimensional face $G$, and let $H \subset X^{n-1}$ be an arbitrary hyperplane passing through $G$. Choose a co-orientation of $G$ in $X^{n-1}$, that is, a direction of the positive circuit around it, and choose one of the two half-planes $H_+ \subset H$ bounded by the plane of $G$. Define the oriented angle $\angle(F,H_+) \in \mathbb{R}/(2\pi\mathbb{Z})$ to be
the angle of the rotation of $F$ in the positive direction around $G$ to the half-plane $H_+$, and put $\angle(H_+, F_l) = -\angle(F, H_+)$, and similarly for $F'$. Then the ratio
\[
 r(F, H, F') = \frac{\sin \angle(F, H_+)}{\sin \angle(H_+, F')}
\]
is determined solely by the simplices $F$ and $F'$ and the hyperplane $H$ and is independent of the choice of the co-orientation of $G$ and of the choice of the half-plane $H_+$. In addition,
\[
 r(F, H, F') = r(F', H, F)^{-1}.
\]

Now, we take for $G$ an arbitrary $(n - 3)$-dimensional face $\Delta_{U,W}$ of the cross-polytope $P_0$. Let \( \{k, l\} = [n] \setminus (U \cup W) \). Then $G$ is a face of the cross-polytopes $P_{(k)}$ and $P_{(l)}$. Let $F_1$ and $F'_1$ be the two facets of $P_{(k)}$ containing $G$, and let $F_2$ and $F'_2$ be the two facets of $P_{(l)}$ containing $G$. (Obviously, $F_1$ and $F'_1$ are exactly the faces $\Delta_{U \cup \{l\},W}$ and $\Delta_{U,W \cup \{l\}}$, but we do not want to specify which of them is $F_1$ and which of them is $F'_1$ to avoid considering several cases in the sequel; similarly for $F_2$ and $F'_2$.)

**Lemma 6.6.** The hyperplanes $\mathcal{B}_{k,G}$ and $\mathcal{B}_{l,G}$ coincide, and
\[
 r(F_1, B_G, F_2) = \frac{\lambda_{F_2}}{\lambda_{F_1}},
\]
where $\mathcal{B}_G = \mathcal{B}_{k,G} = \mathcal{B}_{l,G}$.

**Proof.** Suppose that $k < l$. The link $L_G(u) = L(G, P_u)$ is the flexible spherical quadrangle with the vertices $A$, $B$, $C$, and $D$ cut by the tangent cones to the faces $F_1$, $F_2$, $F'_1$, and $F'_2$, respectively, in the sphere $S_x^2$ for an interior point $x$ of $G$. By Lemma 4.1, the tangents of the halves of the oriented dihedral angles $\psi_A(u) = \psi_{F_1}(u)$ and $\psi_B(u) = \psi_{F_2}(u)$ are either directly or inversely proportional to each other. Hence the flexible quadrangle $L_G(u)$ is of one of the three types shown in Fig. 1. Denote by $\alpha$ and $\beta$ the lengths of the sides $AB$ and $AD$, respectively. Then the lengths of the sides $CD$ and $BC$ are equal to $\alpha$ and $\beta$, respectively, for the quadrangles in Figs. 1a and 1b, and are equal to $\pi - \alpha$ and $\pi - \beta$, respectively, for the quadrangle in Fig. 1c. In addition, $\alpha \neq \beta$ and $\alpha + \beta \neq \pi$.

Indeed, it is easy to check that if one of the equalities $\alpha = \beta$ or $\alpha + \beta = \pi$ held true, then one of the angles of $L_G(u)$, i.e., one of the dihedral angles of $P_u$, would be either identically 0 or identically $\pi$ during the flexion, which is impossible, since $\lambda_k \neq 0$ and $\lambda_l \neq 0$. Bricard [6, § II] showed that, for each of the quadrangles in Figs. 1a and 1b, we have one of the two equalities
\[
 \tan \frac{\psi_A(x)}{2} \tan \frac{\psi_B(x)}{2} = \frac{\cos \alpha - \beta}{\cos \alpha + \beta}, \quad \tan \frac{\psi_A(x)}{2} \tan \frac{\psi_B(x)}{2} = \frac{\sin \beta - \alpha}{\sin \beta + \alpha}, \tag{6.1}
\]
and for the quadrangle in Fig. 1c we have one of the two equalities
\[
 \tan \frac{\psi_A(x)}{2} \tan \frac{\psi_B(x)}{2} = \frac{\sin \alpha - \beta}{\sin \alpha + \beta}, \quad \tan \frac{\psi_A(x)}{2} \tan \frac{\psi_B(x)}{2} = \frac{-\cos \alpha - \beta}{\cos \alpha + \beta}.
\]

Consider the flat position $L_G(0)$ of $L_G(u)$ lying in the great circle $S_x^1 \subset S_x^2$ cut by the tangent space $T_x \mathbb{X}^{n-1}$. We will study in detail the case of the quadrangle $L_G(u)$ shown in Fig. 1a; the two other cases are completely similar. In the case in Fig. 1a, we have $s_k s'_k = -s_l s'_l$. Consider two subcases:

1. Suppose that $s_k s'_k = 1$ and $s_l s'_l = -1$. Then $k \in X_l$ and $l \in X_k$. Hence, $B_{k,G}$ and $B_{l,G}$ are interior bisecting hyperplanes of the dihedral angles between $F_1$ and $F'_1$ and between $F_2$ and $F'_2$, respectively. Therefore, the tangent space $T_x B_{k,G}$ intersects the circle $S_x^1$ at the midpoints of the two arcs with endpoints $A$ and $C$, and the tangent space $T_x B_{l,G}$ intersects $S_x^1$ at the midpoints of

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Since \( \alpha, \beta \) are parallel to each other, hence, the same diameter of the arcs with endpoints \( B \) and \( D \). The flat quadrangle \( L_G(0) \) in the circle \( S^1_x \) has the form shown in Fig. 2a if \( \alpha > \beta \) and the form shown in Fig. 2b if \( \alpha < \beta \). In both cases the midpoints \( E \) and \( E' \) of the arcs with endpoints \( A \) and \( C \) coincide with the midpoints of the arcs with endpoints \( B \) and \( D \) (see Figs. 2a and 2b). Hence, \( B_{k,G} = B_{l,G} \). The ratio \( r(F_1, B_G, F_2) \) is equal to the ratio of the sines of the oriented lengths of the arcs \( AE \) and \( EB \). Therefore,

\[
r(F_1, B_G, F_2) = \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}}.
\]

Since \( s_k s'_k = 1 \) and \( s_l s'_l = -1 \), we have \( \tan \frac{\varphi_A(x)}{2} = \lambda_{F_1} x \) and \( \tan \frac{\varphi_B(x)}{2} = -\lambda_{F_2}^{-1} x^{-1} \). Hence it follows from (6.1) that the ratio \( \lambda_{F_2}/\lambda_{F_1} \) is equal either to \( -(\cos \frac{\alpha + \beta}{2})/(\cos \frac{\alpha - \beta}{2}) \) or to \( (\sin \frac{\alpha + \beta}{2})/(\sin \frac{\alpha - \beta}{2}) \). Since \( \alpha, \beta \in (0, \pi) \), we easily obtain

\[
\left| \frac{\cos \frac{\alpha + \beta}{2}}{\cos \frac{\alpha - \beta}{2}} \right| < 1, \quad \left| \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}} \right| > 1.
\]

However, \( |\lambda_{F_1}| < |\lambda_{F_2}| \) since \( k < l \). Thus,

\[
\frac{\lambda_{F_2}}{\lambda_{F_1}} = \frac{\sin \frac{\alpha + \beta}{2}}{\sin \frac{\alpha - \beta}{2}} = r(F_1, B_G, F_2).
\]

2. Suppose that \( s_k s'_k = -1 \) and \( s_l s'_l = 1 \). Then \( k \notin X_l \) and \( l \notin X_k \). Hence, \( B_{k,G} \) and \( B_{l,G} \) are the exterior bisecting hyperplanes of the dihedral angles between \( F_1 \) and \( F_1' \) and between \( F_2 \) and \( F_2' \), respectively. Therefore, the tangent space \( T_X B_{k,G} \) intersects \( S^1_X \) at the endpoints of the diameter parallel to the chord \( AC \), and the tangent space \( T_X B_{l,G} \) intersects \( S^1_X \) at the endpoints of the diameter parallel to the chord \( BD \). The flat quadrangle \( L_G(0) \) has the form shown in Fig. 2c if \( \alpha > \beta \) and has the form shown in Fig. 2d if \( \alpha < \beta \). In both cases, the chords \( AC \) and \( BD \) are parallel to each other; hence, the same diameter \( EE' \) is parallel to both of them. Therefore, \( B_{k,G} = B_{l,G} \) and

\[
r(F_1, B_G, F_2) = -\frac{\sin \frac{\pi + \alpha - \beta}{2}}{\sin \frac{\pi - \alpha + \beta}{2}} = -\frac{\cos \frac{\alpha - \beta}{2}}{\cos \frac{\alpha + \beta}{2}} = \frac{\lambda_{F_2}}{\lambda_{F_1}}.
\]

The latter equality follows from the formulas \( \tan \frac{\varphi_A(x)}{2} = -\lambda_{F_1}^{-1} x^{-1} \) and \( \tan \frac{\varphi_B(x)}{2} = \lambda_{F_2} x \), formulas (6.1), and the inequality \( |\lambda_{F_1}| < |\lambda_{F_2}| \). \( \square \)

**Proof of Lemma 6.5.** As mentioned in the proof of the previous lemma, the link \( L_G(u) \) of every \((n-3)\)-dimensional face \( G \) of \( P_u \) has the form of one of the quadrangles shown in Fig. 1, and, in addition, the lengths \( \alpha \) and \( \beta \) of its sides \( AB \) and \( AD \) are not equal to each other and their sum is not equal to \( \pi \). Hence, in the flat position \( L_G(0) \) of this quadrangle, any two of its vertices neither

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coincide nor are antipodal to each other. Therefore, the dihedral angles of the cross-polytopes $P_{(k)}$ and $P_{(i)}$ at $G$ are neither zero nor straight. $\square$

**Lemma 6.7.** Let $F_1$, $F_2$, and $F_3$ be three pairwise distinct $(n-2)$-dimensional faces of the cross-polytope $P_0$ that lie in the same $(n-1)$-dimensional face of $P_0$. Then

$$\hat{B}_{F_1\cap F_2} \cap \hat{B}_{F_2\cap F_3} = \hat{B}_{F_2\cap F_3} \cap \hat{B}_{F_3\cap F_1} = \hat{B}_{F_3\cap F_1} \cap \hat{B}_{F_1\cap F_2}. \quad (6.2)$$

**Proof.** Put $G_{ij} = F_i \cap F_j$ and $Q = F_1 \cap F_2 \cap F_3$. Then $\dim G_{ij} = n-3$ and $\dim Q = n-4$. Suppose that $n \geq 4$. Let $x$ be a point in the relative interior of $Q$. In the space $T_x X^{n-1}$ we consider the sphere $S^2_x$ consisting of all unit vectors orthogonal to $Q$. The tangent cones to the faces $F_1$, $F_2$, and $F_3$ intersect the sphere $S^2_x$ along arcs of great circles, which we denote by $f_1$, $f_2$, and $f_3$, respectively, and the tangent spaces to the hyperplanes $B_{G_{12}}$, $B_{G_{23}}$, and $B_{G_{31}}$ intersect $S^2_x$ along great circles, which we denote by $b_{12}$, $b_{23}$, and $b_{31}$, respectively. Each great circle $b_{ij}$ passes through the common endpoint of the arcs $f_i$ and $f_j$. By Lemma 6.6, we have

$$r(f_i, b_{ij}, f_j) = r(F_i, B_{G_{ij}}, F_j) = \frac{\lambda_{F_i}}{\lambda_{F_j}}.$$ 

Hence,

$$r(f_1, b_{12}, f_2) r(f_2, b_{23}, f_3) r(f_3, b_{31}, f_1) = 1.$$

By spherical Ceva's theorem in the trigonometric form, the great circles $b_{12}$, $b_{23}$, and $b_{31}$ intersect at a pair of antipodal points of $S^2_x$. In addition, since every of the ratios $r(f_i, b_{ij}, f_j)$ is neither zero nor infinity, no two of the great circles $b_{12}$, $b_{23}$, and $b_{31}$ coincide. This immediately implies (6.2). If $n = 3$, then equality (6.2) follows in the same way from the trigonometric form of Ceva's theorem in $X^2$ applied to the lines $B_{G_{12}}$, $B_{G_{23}}$, and $B_{G_{31}}$. $\square$

**Proof of Lemma 6.4.** Let $F$ be an $(n-2)$-dimensional face of the cross-polytope $P_0$. We denote by $O_F$ the intersection of the $n-1$ projective hyperplanes $\hat{B}_G \subset \mathbb{RP}^{n-1}$, where $G$ runs over all $(n-3)$-dimensional faces of $F$. Then $O_F$ is nonempty. Let us show that $O_F$ is a point. If this were not correct, then the intersection of the projective plane $O_F$ with the projective hyperplane of the face $F$ would also be nonempty. This is impossible, since the intersection of the projective hyperplane of the face $F$ with $\hat{B}_G$ is exactly the projective plane of the face $G$, and the intersection of the projective planes of all $(n-3)$-dimensional faces $G \subset F$ is empty. (No hyperplane $\hat{B}_G$ can coincide with the hyperplane of $F$, since the coefficient $\lambda_F$ is neither zero nor infinity.) To prove the lemma, it remains to show that all points $O_F$ coincide. To this end, it is sufficient to show that $O_{F_1} = O_{F_2}$ whenever $F_1$ and $F_2$ are two $(n-2)$-dimensional faces of $P_0$ that are contained in an $(n-1)$-dimensional face $\Delta$ of $P_0$. Put $G_0 = F_1 \cap F_2$. For each $(n-2)$-dimensional face $F \subset \Delta$ such that $F \neq F_1, F_2$, Lemma 6.7 implies that $\hat{B}_{F_1\cap F} \cap \hat{B}_{G_0} = \hat{B}_{F_2\cap F} \cap \hat{B}_{G_0}$. As $F$ runs over all $(n-2)$-dimensional faces $F \subset \Delta$ such that $F \neq F_1, F_2$, the intersection $F_1 \cap F$ runs over all $(n-3)$-dimensional faces of $F_1$ different from $G_0$, and the intersection $F_2 \cap F$ runs over all $(n-3)$-dimensional faces of $F_2$ different from $G_0$. Therefore, the intersection of all hyperplanes $\hat{B}_G$ for $G$ running over all $(n-3)$-dimensional faces of $F_1$ coincides with the intersection of all hyperplanes $\hat{B}_G$ for $G$ running over all $(n-3)$-dimensional faces of $F_2$, i.e., $O_{F_1} = O_{F_2}$. $\square$

7. VOLUMES

Recall that the $n$-dimensional volume of the unit $n$-dimensional sphere $S^n$ is equal to

$$\sigma_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$ 

We will denote the $k$-dimensional volume of the $k$-dimensional simplex $\Delta$ by $V_k(\Delta)$. For each face $\Delta_{I,J}$ of the cross-polytope $P_n$, we denote its $(|I| + |J| - 1)$-dimensional volume by $V_{I,J}$. 

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7.1. Relations for the volumes of \((n - 1)\)-dimensional faces. For the flat positions \(P_0\) and \(P_\infty\) of a flexible cross-polytope \(P_n\) of the simplest type in \(\mathbb{S}^n\), we can consider the degrees \(\text{deg} P_0\) and \(\text{deg} P_\infty\) of the mappings \(P_0, P_\infty: K_n \to \mathbb{S}^{n-1}\), respectively. Recall that, by definition, the \textit{degree} is the algebraic number of pre-images of an arbitrary regular value of the mapping under consideration, with each point in the pre-image assigned the plus sign if the differential of the mapping at this point preserves the orientation and the minus sign otherwise. All necessary facts on the degrees of mappings can be found, for instance, in [16, Ch. 3, Sect. 2.7, Ch. 5, Sect. 4.1].

To fix the signs of the degrees \(\text{deg} P_0\) and \(\text{deg} P_\infty\), we need to choose the orientation of the great sphere \(\mathbb{S}^{n-1}\). We will choose this orientation in such a way that the restrictions of the mappings \(P_0\) and \(P_\infty\) to the simplex \([a_1 \ldots a_n]\) (these restrictions coincide) preserve the orientation.

**Lemma 7.1.** The degree \(\text{deg} P_0\) is equal to 1 if \(s_is_i' = -1\) for all \(i\) and is equal to 0 if at least one of the numbers \(s_is_i' = 1\) for all \(i\) and is equal to 0 if at least one of the numbers \(s_is_i' = 1\) is equal to \(-1\).

**Proof.** We will prove the first assertion of the lemma. The proof of the second assertion is completely similar. It follows easily from Theorem 5.1 that \(\text{deg} P_0 = 1\) if \(s_is_i' = -1\) for all \(i\).

Assume that \(s_is_i' = 1\) for some \(k\). Consider the flexible cross-polytope \(\bar{P}_n\) obtained from \(P_n\) by replacing all its vertices \(b_i(u)\) such that \(s_is_i' = 1\) with the antipodal points. Then the signs corresponding to the flexible cross-polytope \(\bar{P}_n\) are \(\tilde{s}_i = s_i\) and \(\tilde{s}_i' = -s_i\) for all \(i\). Hence, by Theorem 5.1, the cross-polytope \(\bar{P}_0\) is embedded. We will denote the vertices \(a_i(0)\) and \(b_i(0)\) of the flat cross-polytope \(P_0\) simply by \(a_i\) and \(b_i\), respectively. The vertices of the cross-polytope \(\bar{P}_0\) are the points \(a_i\) and \(b_i = -s_is_i'\bar{b}_i\). Let us show that the point \(\bar{b}_k = -b_k\) does not lie in the image of the mapping \(P_0\), that is, in the union of faces of the cross-polytope \(P_0\). Assume the contrary. Then \(\bar{b}_k \in \Delta_{A,B}\) for some facet \(\Delta_{A,B}\) of \(P_0\), \(A \cup B = [n]\). Therefore,

\[
\bar{b}_k = \sum_{i \in A} \mu_i a_i + \sum_{i \in B} \mu_i b_i
\]

for some nonnegative coefficients \(\mu_i\). Denote by \(B_+\) and \(B_-\) the subsets of \(B\) consisting of all \(i\) such that \(s_is_i' = 1\) and \(s_is_i' = -1\), respectively. Then

\[
\bar{b}_k + \sum_{i \in B_+} \mu_i \bar{b}_i = \sum_{i \in A} \mu_i a_i + \sum_{i \in B_-} \mu_i \bar{b}_i.
\]

Since the vectors \(b_1, \ldots, b_n\) are linearly independent, the vector \(v\) standing on both sides of this equality is nonzero. Then the point \(v/|v|\) lies in the faces \(\Delta_{\emptyset, B_+ \cup \{k\}}\) and \(\Delta_{\emptyset, B_-}\) of \(\bar{P}_0\), which is impossible, since these faces do not have common vertices and the cross-polytope \(\bar{P}_0\) is embedded. (The number \(k\) may or may not belong to \(B_+\) but it certainly does not belong to \(B_-\).) The obtained contradiction shows that the image of the mapping \(P_0: K_n \to \mathbb{S}^{n-1}\) does not coincide with the whole sphere \(\mathbb{S}^{n-1}\), which immediately implies that \(\text{deg} P_0 = 0\). \(\square\)

We denote by \(Y_+\) (respectively, by \(Y_-\)) the subset of \([n]\) consisting of all \(i\) such that \(s_is_i' = 1\) (respectively, \(s_is_i' = -1\)) for all \(i\) such that \(s_is_i' = 1\); then \(Y_+ \cup Y_- = [n]\).

**Theorem 7.2.** Let \(P_u\) be a flexible cross-polytope of the simplest type in \(\mathbb{X}^n\), and let \(Y\) be one of the two subsets \(Y_+\) and \(Y_-\) corresponding to it. Then

\[
\sum_{A \cup B = [n]} (-1)^{|B \cap Y|} V_{A,B} = \begin{cases} 0 & \text{if } \mathbb{X}^n = \mathbb{E}^n \text{ or } \Lambda^n, \\ 0 & \text{if } \mathbb{X}^n = \mathbb{S}^n \text{ and } Y \neq \emptyset, \\ \sigma_{n-1} & \text{if } \mathbb{X}^n = \mathbb{S}^n \text{ and } Y = \emptyset. \end{cases}
\]  

(7.1)

**Proof.** We will prove relation (7.1) for \(Y = Y_+\) by studying the flat position \(P_0\) of the cross-polytope. Relation (7.1) for \(Y = Y_-\) is obtained in the same way by studying the flat position \(P_\infty\).
If $Y_+ = \emptyset$, then it follows from Theorem 5.1 that the sum of the volumes of the facets of $P_0$ is equal to the volume $\sigma_{n-1}$ of $S^{n-1}$, which yields (7.1).

Assume that either $X^n = E^n$, or $X^n = \Lambda^n$, or $X^n = S^n$ and $Y_+ \neq \emptyset$. Then the degree of the mapping $P_0: K_n \to X^{n-1}$ is zero. For $X^n = S^n$, this follows from Lemma 7.1. For $X^n = E^n$ and $\Lambda^n$, this is also true, since the image $P_0(K_n)$ is compact and hence the mapping $P_0$ is not surjective. Therefore, a generic point in $X^{n-1}$ is covered by the facets of $P_0$ embedded in $X^{n-1}$ in an orientation-preserving way as many times as by the facets of $P_0$ embedded in $X^{n-1}$ in an orientation-reversing way. Hence the sum of the volumes of the facets of $P_0$ embedded in $X^{n-1}$ in an orientation-preserving way is equal to the sum of the volumes of the facets of $P_0$ embedded in $X^{n-1}$ in an orientation-reversing way. The embedding of a facet $\Delta_{A,B}$ into $X^{n-1}$ preserves the orientation if and only if it is possible to travel from $\Delta_{[n],\emptyset}$ to $\Delta_{A,B}$ in $K_n$ while crossing finitely many times the $(n-2)$-dimensional faces of $P_0$ in such a way that the $(n-2)$-dimensional faces of $P_0$ at which the dihedral angles are equal to $0$ are crossed an even number of times. It follows from formula (4.1) that this is the case if and only if the number $|B \cap Y_+|$ is even, which yields formula (7.1).

Corollary 7.3. If $X^n = E^n$ or $\Lambda^n$, then not all of the numbers $s_is_i'$ are the same.

7.2. A property of circumscribed cross-polytopes. Let $P$ be an $m$-dimensional cross-polytope in $X^m$, $m \geq 2$, such that either $P$ is circumscribed about a hypersurface $\Omega$ which is a sphere, an orisphere, or an equidistant hypersurface, or the hyperplanes of all facets of $P$ form the same angle $\alpha \in (0, \pi/2)$ with some hyperplane $H$, or $X^m = \Lambda^m$ and the hyperplanes of all facets of $P$ are parallel to the same hyperplane $H$. Let $F = F_+ \cap F_-$ be the decomposition of the set of facets of $P$ into two classes given by the hypersurface $\Omega$ or by the hyperplane $H$. Consider the chess coloring of facets of $P$ and denote by $F_h$ and $F_w$ the sets of black and white facets, respectively.

It is well known that for a quadrangle circumscribed about a circle the sums of the lengths of its opposite sides are equal to each other. The following lemma is a direct generalization of this fact.

Lemma 7.4. If $X^m = E^m$ or $\Lambda^m$, then

$$\sum_{F \in F_+ \cap F_h} V_{m-1}(F) - \sum_{F \in F_- \cap F_h} V_{m-1}(F) - \sum_{F \in F_+ \cap F_w} V_{m-1}(F) + \sum_{F \in F_- \cap F_w} V_{m-1}(F) = 0. \quad (7.2)$$

Proof. The proof is the same as in the two-dimensional case. For each facet $F$, we denote by $A_F$ either the tangent point of the hyperplane of $F$ and the hypersurface $\Omega$ or the intersection point of the hyperplane of $F$ and the hyperplane $H$. (If $X^m = \Lambda^m$ and the hyperplane of $F$ is parallel to $H$, then the point $A_F$ lies on the absolute.) Then the volume of the simplex $F$ can be decomposed into the algebraic sum of the volumes of the simplices $[A_F G]$ spanned by the point $A_F$ and facets $G$ of the simplex $F$:

$$V_{m-1}(F) = \sum_{G \subseteq F, \dim G = m-2} \pm V_{m-1}([A_F G]). \quad (7.3)$$

It is not hard to check that the signs in formulas (7.2) and (7.3) are consistent so that the volumes of the simplices $[A_{F_1} G]$ and $[A_{F_2} G]$ enter the left-hand side of (7.2) with opposite signs for any two facets $F_1$ and $F_2$ with a common $(m-2)$-dimensional face $G$. However, the simplices $[A_{F_1} G]$ and $[A_{F_2} G]$ are congruent to each other. Hence their volumes are equal, which implies equality (7.2).

If $X^m = S^m$, then formula (7.2) is in general not correct. The matter is that formula (7.3) is not correct. Namely, the left- and right-hand sides of (7.3) can differ by the volume $\sigma_{m-1}$ of $S^{n-1}$. Nevertheless, there is an important special case in which formula (7.2) is valid.
Lemma 7.5. Suppose that $X^m = S^m$. Let $S^m_+ \subset S^m$ be a closed hemisphere bounded by the $(m - 1)$-dimensional great sphere concentric with the small sphere $\Omega$ about which the cross-polytope $P$ is circumscribed. Assume that $P$ is contained in $S^m_+$. Then formula (7.4) holds true for $P$.

Proof. It is sufficient to note that the point $A_F$ is the projection of the center of the sphere $\Omega$ onto the $(m - 1)$-dimensional great sphere $H_F$ containing $F$. Hence, the point $A_F$ lies in the open $(m - 1)$-dimensional hemisphere $H_{F'} = H_F \cap S^m_0$, whose closure contains $F$. Therefore, formula (7.3) holds true in this case, and the proof of Lemma 7.4 can be repeated literally. $\square$

7.3. Relations for the volumes of $(n - 2)$-dimensional faces.

Theorem 7.6. Suppose that $n \geq 3$. Then, for a flexible cross-polytope $P_u$ of the simplest type either in $E^n$ or in $\Lambda^n$, we have

$$\sum_{I \cup J = [n] \setminus \{k\}} (-1)^{|J \cap \lambda_k|} V_{I,J} = 0, \quad k = 1, \ldots, n, \tag{7.4}$$

and, for a flexible cross-polytope $P_u$ of the simplest type in $S^n$, we have

$$\sum_{I \cup J = [n] \setminus \{k\}} (-1)^{|J \cap \lambda_k|} V_{I,J} = 0 \quad \text{if} \quad X_k \neq \emptyset, \tag{7.5}$$

$$\sum_{I \cup J = [n] \setminus \{k\}} V_{I,J} = \sigma_{n-2} \quad \text{if} \quad X_k = \emptyset. \tag{7.6}$$

Proof. Consider the flat position $P_0$ of $P_u$ and apply Lemmas 7.4 and 7.5 to the $(n - 1)$-dimensional cross-polytopes $P_k$ defined in Section 6. The facets of $P_k$ are $(n - 2)$-dimensional faces $\Delta_{I,J}$ of $P_0$ such that $I \cup J = [n] \setminus k$. By Theorem 6.1, either the cross-polytope $P_k$ is circumscribed about a hypersurface that is a sphere, an orisphere, or an equidistant hypersurface, or all facets of $P_k$ are parallel to some hyperplane, or all facets of $P_k$ intersect some hyperplane at the same angle. In addition, the class to which a facet $\Delta_{I,J}$ belongs is determined by the evenness of the cardinality of the set $J \setminus X_k$. On the other hand, the color of the facet $\Delta_{I,J}$ is determined by the evenness of the cardinality of the set $J$. Hence, for flexible cross-polytopes in $E^n$ and $\Lambda^n$, Lemma 7.4 immediately yields formula (7.4).

Now, let $P_u$ be a flexible cross-polytope of the simplest type in $S^n$. Consider the $2^n$ spherical flexible cross-polytopes $P_u^s$ obtained from $P_u$ by replacing some of its vertices with the antipodal points, i.e., corresponding to the same pair $(G, \lambda)$ and all rows of signs $s = (s_1, \ldots, s_n, s'_1, \ldots, s'_n)$. For each row of signs $s$, we put

$$\varepsilon_k(s) = s_1 \ldots \hat{s}_k \ldots s_n = s_k(s_1 \ldots s_n).$$

The sets $X_k$, the faces $\Delta_{I,J}$, and the volumes $V_{I,J}$ depend on $s$. We will denote them by $X_k(s)$, $\Delta_{I,J}(s)$, and $V_{I,J}(s)$, respectively. We denote by $S_k(s)$ the left-hand side of (7.5) or (7.6) for the flexible cross-polytope $P_u^s$.

At least one of the cross-polytopes $P_u^s$ is contained in the closed $(n - 1)$-dimensional hemisphere bounded by the $(n - 2)$-dimensional great sphere concentric with the small spheres $\Omega_1, \ldots, \Omega_n$. Therefore, completely in the same way as we have deduced formula (7.4) from Lemma 7.4, we can deduce from Lemma 7.5 that there exists a row of signs $s^*$ such that $S_k(s^*) = 0$ for all $k = 1, \ldots, n$. Notice that all sets $X_k(s^*)$ are nonempty, $k = 1, \ldots, n$. Indeed, if a set $X_k(s^*)$ were empty, then the corresponding sum $S_k(s^*)$ would be a sum of positive terms and, therefore, would be nonzero.

Let us prove formulas (7.5) and (7.6) by induction on the dimension $n$. Obviously, for $n = 2$ these formulas hold true, since the volume of any zero-dimensional face is equal to 1 and $\sigma_0 = 2$. 

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We assume that formulas (7.5) and (7.6) hold true for all \((n - 1)\)-dimensional spherical flexible cross-polytopes of the simplest type and prove them for an \(n\)-dimensional spherical flexible cross-polytope \(P_u\) of the simplest type. Let us study how the value \(S_k(s)\) changes when we change the row of signs \(s\). We put

\[
S_k(s) = \begin{cases} 
\varepsilon_k(s)S_k(s) & \text{if } X_k(s) \neq \emptyset, \\
\varepsilon_k(s)(S_k(s) - \sigma_{n-2}) & \text{if } X_k(s) = \emptyset.
\end{cases}
\]

**Lemma 7.7.** Take any \(k \in [n]\). Then the numbers \(S_k(s)\) are the same for all rows of signs \(s\).

**Proof.** It is sufficient to prove that \(S_k(s^{(1)}) = S_k(s^{(2)})\) for any two rows of signs \(s^{(1)}\) and \(s^{(2)}\) that differ by one coordinate only. Let this coordinate be either \(s_i\) or \(s_j\).

We put \(P_u^{(i)} = P_u^{(i)}(s), \ X_k^{(i)} = X_k(s^{(i)}), \ \Delta_{I,J}^{(i)} = \Delta_{I,J}(s^{(i)}), \ S_k^{(i)} = S_k(s^{(i)}), \ etc., \ i = 1, 2.\) The vertices of \(P_u^{(1)}\) will be denoted by \(a_j\) and \(b_j\) instead of \(a_j^{(1)}\) and \(b_j^{(1)}\), respectively. The cross-polytope \(P_u^{(2)}\) is obtained from the cross-polytope \(P_u^{(1)}\) by replacing one of its vertices \(v\) with the antipodal point of the sphere. We have \(v = a_i\) if the rows \(s^{(1)}\) and \(s^{(2)}\) differ by the coordinate \(s_i\), and \(v = b_i\) if the rows \(s^{(1)}\) and \(s^{(2)}\) differ by the coordinate \(s_j\).

If \(l = k\), then \(P_u^{(1)} = P_u^{(2)}\), hence, \(S_k^{(1)} = S_k^{(2)}\). In addition, \(X_k^{(1)} = X_k^{(2)}\) and \(\varepsilon_k^{(1)} = \varepsilon_k^{(2)}\). Therefore, equality \(S_k^{(1)} = S_k^{(2)}\) holds true.

Suppose that \(l \neq k\). It is easy to see that the sets \(X_k^{(1)}\) and \(X_k^{(2)}\) differ by the occurrence of the element \(l\) only, that is, either \(l \notin X_k^{(1)}\) and \(X_k^{(2)} = X_k^{(1)} \cup \{l\}\) or \(l \notin X_k^{(2)}\) and \(X_k^{(1)} = X_k^{(2)} \cup \{l\}\). We assume that \(l \notin X_k^{(1)}\) and \(X_k^{(2)} = X_k^{(1)} \cup \{l\}\), since the second case is completely similar.

Let \(\tilde{P}_u = L(v, P_u^{(1)})\) be the link of the vertex \(v\) in the flexible cross-polytope \(P_u^{(1)}\). As it was shown in the Introduction, the link of a vertex of a flexible polyhedron is itself a spherical flexible polyhedron. Moreover, the dihedral angles of the link are equal to the corresponding dihedral angles of the initial polyhedron. Since \(P_u^{(1)}\) is a flexible cross-polytope of the simplest type, its dihedral angles vary so that the tangents of their halves are either directly or inversely proportional to each other. Hence the same property holds true for the dihedral angles of the spherical flexible cross-polytope \(P_u\), which implies that \(P_u\) is also a flexible cross-polytope of the simplest type (see Remark 3.3). We denote by \(\tilde{a}_j\) and \(\tilde{b}_j\) the vertices of \(\tilde{P}_u\) that are the unit tangent vectors to the edges \([va_j]\) and \([vb_j]\), respectively, \(j \neq l\). For the cross-polytope \(\tilde{P}_u\) all objects introduced above will be marked by a tilde, for example, \(\tilde{G}, \tilde{A}, \tilde{s}, \tilde{\Delta}_{I,J}, \tilde{S}_J, \) etc. Notice that pairs of opposite vertices of the cross-polytope \(\tilde{P}_u\) are indexed by elements of the set \([n]\) \(\setminus\) \{\(l\)\} rather than \([n - 1]\). This is completely inessential. The only difference is that the set \(\tilde{X}_1\) and the sets \(I\) and \(J\) for faces \(\tilde{\Delta}_{I,J}\) are also subsets of \([n]\) \(\setminus\) \{\(l\)\}. It is easy to see that \(\tilde{A}\) is the row \(A\) with the coordinate \(\lambda_l\) deleted. Hence, the entries of the row \(\tilde{A}\) are positive and increasing. Therefore, by the induction hypothesis, formulas (7.5) and (7.6) hold true for the cross-polytope \(\tilde{P}_u\).

For \(u = 0\), all dihedral angles of the cross-polytope \(P_u^{(1)}\) at facets \(\Delta_{I,J}^{(1)}\) such that \(I \cup J = [n] \setminus \{j\}\) degenerate to the zero angles if \(s_j^{(1)}s_j^{(1)} = 1\) and to the straight angles if \(s_j^{(1)}s_j^{(1)} = -1\). The same is true for the dihedral angles of the \((n - 1)\)-dimensional cross-polytope \(\tilde{P}_u^{(1)}\). In addition, the dihedral angle of \(P_u\) at the \((n - 3)\)-dimensional face \(\tilde{F}\) cut by the tangent cone to the \((n - 2)\)-dimensional face \(F\) of \(P_u^{(1)}\) is equal to the dihedral angle of \(P_u^{(1)}\) at \(F\). Therefore, \(s_j\tilde{s}_j = s_j^{(1)}s_j^{(1)}\) for all \(j \neq l\). Hence, \(\tilde{X}_k = X_k^{(1)}\).

If an \((n - 2)\)-dimensional face \(\Delta_{I,J}^{(1)}\) does not contain the vertex \(v\), then \(\Delta_{I,J}^{(1)} = \Delta_{I,J}^{(2)}\), hence, \(V_{I,J}^{(1)} = V_{I,J}^{(2)}\). If an \((n - 2)\)-dimensional face \(\Delta_{I,J}^{(1)}\) contains the vertex \(v\), then the faces \(\Delta_{I,J}^{(1)}\)
and $\Delta_{i,j}^{(2)}$ constitute together an $(n-2)$-dimensional spherical $(n-2)$-hedron with the two antipodal vertices $v$ and $-v$, and the intersection of this $(n-2)$-hedron with the equatorial great sphere with respect to the poles $v$ and $-v$ is isometric to the simplex $\tilde{\Delta}_{i,j}$ (see Fig. 3). Hence,

$$V_{i,j}^{(1)} + V_{i,j}^{(2)} = \frac{\sigma_{n-2}}{\sigma_{n-3}} \tilde{V}_{i,j}.$$ 

If $v = a_l$, then the signs $(-1)^{|J\cap X_k^{(1)}|}$ and $(-1)^{|J\cap X_k^{(2)}|}$ in the sums $S_k^{(1)}$ and $S_k^{(2)}$, respectively, coincide whenever $v \in \Delta_{i,j}$ and are opposite to each other whenever $v \notin \Delta_{i,j}$. If $v = b_l$, then, conversely, the signs $(-1)^{|J\cap X_k^{(1)}|}$ and $(-1)^{|J\cap X_k^{(2)}|}$ are opposite to each other whenever $v \in \Delta_{i,j}$ and coincide whenever $v \notin \Delta_{i,j}$. Hence,

$$S_k^{(1)} = \pm S_k^{(2)} + \frac{\sigma_{n-2}}{\sigma_{n-3}} \tilde{S}_k,$$

where we choose the sign “$-$” if $v = a_l$ and the sign “$+$” if $v = b_l$. By the induction hypothesis, $\tilde{S}_k = \sigma_{n-3}$ if $\tilde{X}_k = \emptyset$ and $\tilde{S}_k = 0$ if $\tilde{X}_k \neq \emptyset$. Since $\tilde{X}_k = X_k^{(1)}$ and the set $X_k^{(2)}$ is nonempty, this immediately implies the required equality $S_k^{(1)} = S_k^{(2)}$. □

Now, we are ready to complete the proof of Theorem 7.6. Since $S_k(s^*) = 0$ and $X_k(s^*) \neq \emptyset$ for $k = 1, \ldots, n$, Lemma 7.7 implies that $S_k(s) = 0$ for all $k$ and $s$. Hence, $S_k(s) = 0$ whenever $X_k(s) \neq \emptyset$, and $S_k(s) = \sigma_{n-2}$ whenever $X_k(s) = \emptyset$. Therefore, the assertion of Theorem 7.6 is true for all cross-polytopes $P_u^n$, including the initial cross-polytope $P_u^n$. □

7.4. Volumes of flexible cross-polytopes of the simplest type. First of all, let us give a rigorous definition of the volume of a not necessarily embedded polyhedron $P : K \rightarrow \mathbb{R}^n$.

Suppose that $\mathbb{X}^n = \mathbb{E}^n$ or $\Lambda^n$. For each point $x \in \mathbb{X}^n \setminus P(K)$, we denote by $\kappa(x)$ the algebraic intersection index of a continuous curve $\gamma$ from $x$ to the infinity with the oriented cycle $P(K)$. Obviously, this intersection index is independent of the choice of the curve $\gamma$. Then $\kappa$ is an almost everywhere defined integer-valued piecewise constant function on $\mathbb{X}^n$ with compact support. By definition, the generalized oriented volume of the polyhedron $P$ is the number

$$\nu(P) = \int_{\mathbb{X}^n} \kappa(x) dV(x),$$  \hspace{1cm} (7.7)

where $dV$ is the standard volume form on $\mathbb{X}^n$.

The case $\mathbb{X}^n = \mathbb{S}^n$ is somewhat more complicated, since the sphere $\mathbb{S}^n$ does not contain the infinity, which implies that there is no canonical choice of the function $\kappa$. Consider an arbitrary

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**Fig. 3.** The union of the faces $\Delta_{i,j}^{(1)}$ and $\Delta_{i,j}^{(2)}$. 

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Theorem 7.8. The generalized volume of any flexible cross-polytope of the simplest type in the spaces $\mathbb{E}^n$ and $\Lambda^n$, $n \geq 3$, is identically equal to zero during the flexion. The generalized volume of any flexible cross-polytope of the simplest type in the sphere $\mathbb{S}^n$, $n \geq 2$, is identically equal to zero during the flexion, except for the following cases:

- If $s_1s_1' = \ldots = s_ns_n' = 1$, then
  \[ \mathcal{V}(P_u) = \frac{s_1\sigma_n}{\pi} \arctan(\lambda_1u). \]  
  \[ \text{(7.8)} \]

- If $s_1s_1' = \ldots = s_ns_n' = -1$, then
  \[ \mathcal{V}(P_u) = \frac{\sigma_n}{2} + \frac{s_n\sigma_n}{\pi} \arctan(\lambda_nu). \]  
  \[ \text{(7.9)} \]

- If $s_1s_1' = \ldots = s_ks_k' = -1$ and $s_{k+1}s_{k+1}' = \ldots = s_ns_n' = 1$ for some $k$, $1 \leq k < n$, then
  \[ \mathcal{V}(P_u) = \frac{\sigma_n}{\pi} (s_k \arctan(\lambda_ku) + s_{k+1} \arctan(\lambda_{k+1}u)). \]  
  \[ \text{(7.10)} \]

Proof. In the Euclidean space $\mathbb{E}^n$, the volume of any flexible polyhedron is constant during the flexion (see [17–19, 9] for $n = 3$, [11] for $n = 4$, and [12] for $n \geq 5$). Hence, $\mathcal{V}(P_u) = \mathcal{V}(P_0) = 0$. The latter equality is true, since the cross-polytope $P_0$ is flat, i.e., is contained in the hyperplane $\mathbb{E}^{n-1}$.

In the two-dimensional sphere $\mathbb{S}^2$, flexible cross-polytopes of the simplest type are exactly flexible quadrangles shown in Fig. 1. For each of them the assertion of the theorem can be checked directly.

Suppose that $\mathbb{X}^n = \Lambda^n$ or $\mathbb{S}^n$, $n \geq 3$. We introduce a parameter $\varepsilon$ equal to $1$ for $\mathbb{S}^n$ and to $-1$ for $\Lambda^n$. We apply Schl"afli's classical formula for the differential of the oriented volume of an arbitrary polyhedron in $\mathbb{X}^n$ that is deformed with preservation of its combinatorial type:

\[ \mathrm{d}\mathcal{V} = \frac{\varepsilon}{n-1} \sum_F V_{n-2}(F) \mathrm{d}\psi_F, \]

where the sum is taken over all $(n-2)$-dimensional faces $F$, and $\psi_F$ is the oriented dihedral angle of the polyhedron at the face $F$. In our case, this formula takes the form

\[ \mathrm{d}\mathcal{V}(P_u) = \frac{\varepsilon}{n-1} \sum_{(I,J)} V_{I,J} \mathrm{d}\psi_{I,J}(u), \]

where the sum is taken over all pairs of nonintersecting subsets $I, J \subset [n]$ such that $|I| + |J| = n - 1$.

Remark 7.9. Schl"afli’s formula is usually written for convex polytopes (see, for instance, [1]). Schl"afli’s formula for arbitrary nondegenerate polyhedra in the sense of our definition in Section 2 follows from Schl"afli’s formula for simplices, since the indicator function $\chi_P(x)$ of any $n$-dimensional polyhedron $P: K \to \mathbb{X}^n$ can be represented as an algebraic sum of the indicator functions of simplices with vertices at the vertices of $P$. Indeed, if $v_0$ is an arbitrary vertex of the pseudomanifold $K$.
and \([v_1^{(j)} \ldots v_n^{(j)}], j = 1, \ldots, N,\) are all positively oriented \((n - 1)\)-dimensional simplices of \(K\) that do not contain the vertex \(v_0\), then

\[
\varepsilon_P(x) = \sum_{j=1}^{N} \eta_j \varepsilon[P(v_0)P(v_1^{(j)}) \ldots P(v_n^{(j)})](x),
\]

where \(\eta_j\) is the sign of the orientation of the simplex \([P(v_0)P(v_1^{(j)}) \ldots P(v_n^{(j)})]\).

If \(X^n = \Lambda^n\), then, using formulas (4.1) and (7.4), we obtain \(dV(P_u) = 0\). Hence, \(V(P_u) = V(P_0) = 0\).

Similarly, if \(X^n = S^n\), then, using formulas (4.1), (7.5), and (7.6), we obtain

\[
dV(P_u) = \frac{2\sigma_{n-2}}{n-1} \sum_{k: X_k = \emptyset} s_k \operatorname{arctan}(\lambda_ku).
\]

Since \(\sigma_n = 2\pi\sigma_{n-2}/(n - 1)\), this can be rewritten in the form

\[
V(P_u) = V(P_0) + \frac{\sigma_n}{\pi} \sum_{k: X_k = \emptyset} s_k \operatorname{arctan}(\lambda_ku).
\]

The boundary of the cross-polytope \(P_0\) is contained in the great sphere \(S^{n-1}\). Hence, the oriented volume \(V(P_0)\) is equal to 0 if \(\deg P_0 = 0\) and is equal to \(\sigma_n/2\) if \(\deg P_0 = 1\). (In the latter case, the sign is irrelevant, since the volume is defined up to an integer multiple of \(\sigma_n\).) By Lemma 7.1, we obtain \(V(P_0) = \sigma_n/2\) if \(s_i s'_i = -1\) for all \(i\), and \(V(P_0) = 0\) in all other cases.

It can be directly checked that all sets \(X_k\) are always nonempty except for the three special cases listed in Theorem 7.8. Hence the oriented volume of \(P_u\) is always identically equal to zero except for these three cases. If \(s_1 s'_1 = \ldots = s_n s'_n = 1\), then \(X_1\) is the only empty set among the sets \(X_k\), and we obtain (7.8). If \(s_1 s'_1 = \ldots = s_n s'_n = -1\), then \(X_n\) is the only empty set among the sets \(X_k\), and we obtain (7.9). If \(s_1 s'_1 = \ldots = s_k s'_k = -1\) and \(s_{k+1} s'_{k+1} = \ldots = s_n s'_n = 1\) for a \(k\) such that \(1 \leq k < n\), then \(X_k = X_{k+1} = \emptyset\) and all other sets \(X_j\) are nonempty. Hence, we obtain formula (7.10). \(\square\)

**Remark 7.10.** In the Euclidean case, Schl"affi’s formula does not allow one to obtain any expression for the volume of a polyhedron, since the volume does not enter this formula. Instead, in the Euclidean case, Schl"affi’s formula implies that the total mean curvature of a flexible polyhedron is constant during the flexion (see [2]).

**Corollary 7.11.** The Modified Bellows Conjecture (Conjecture 1.3) is true for all spherical flexible cross-polytopes of the simplest type.

**Proof.** For a spherical flexible cross-polytope \(P_u\) of the simplest type, we can always replace several vertices with their antipodes to ensure that the signs \(s_j\) and \(s'_j\) corresponding to the obtained cross-polytope satisfy the conditions \(s_1 s'_1 = 1\) and \(s_2 s'_2 = -1\). Then, by Theorem 7.8, the generalized oriented volume of the obtained flexible cross-polytope is identically equal to zero during the flexion. \(\square\)

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