Heat Trace and Functional Determinant in One Dimension

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We study the spectral properties of the Laplace type operator on the circle. We discuss various approximations for the heat trace, the zeta function and the zeta-regularized determinant. We obtain a differential equation for the heat kernel diagonal and a recursive system for the diagonal heat kernel coefficients, which enables us to find closed approximate formulas for the heat trace and the functional determinant which become exact in the limit of infinite radius. The relation to the generalized KdV hierarchy is discussed as well.
1 Introduction

The heat kernel of elliptic partial differential operators is one of the most powerful tools in mathematical physics (see, for example, [12, 13, 4, 5, 17, 22] and further references therein). Of special importance are the spectral functions such as the heat trace, the zeta function and the functional determinant that enable one to study the spectral properties of the corresponding operator.

The one-dimensional case is a very special one which exhibits an underlying symmetry that has deep relations to such diverse areas as integrable systems, infinite-dimensional Hamiltonian systems, isospectrality etc (see [1, 20, 21, 9, 6, 10, 7, 14, 15, 16], for example). Moreover, it has been shown that one can obtain closed formulas which express the functional determinant in one dimension in terms of a solution to a particular initial value problem; see, e.g., [19, 11, 18]. Although the goals of our paper and the previous paper are similar our approach is completely different. Our results are also formulated in a completely different way. We try to obtain direct formulas for spectral invariants in terms of the potential terms and some new operators rather than solutions of some initial value problem.

We study the heat kernel of a Laplace type partial differential operator on the circle $M = S^1$ of radius $a$. Let $V$ be a $N$-dimensional vector bundle over $S^1$ and $Q$ be a smooth Hermitian endomorphism of the bundle $V$. Let $L : C^\infty(V) \rightarrow C^\infty(V)$ be a second-order differential operator defined by

$$L = -D^2 + Q,$$

where $D = \partial_x$ denotes the derivative with respect to the local coordinate $x$ on $S^1$, with $0 \leq x \leq 2\pi a$.

The heat kernel $U(t; x, x')$ of the operator $L$ is the fundamental solution of the heat equation

$$(\partial_t + L)U(t; x, x') = 0$$

for $t \geq 0$ with the initial condition

$$U(0; x, x') = \delta(x - x')$$

It is well known that the operator $L$ is essentially self-adjoint in $L^2(V)$ and has a discrete real spectrum bounded from below. Moreover, each eigenvalue has a finite multiplicity and the corresponding eigenvectors are smooth sections that can be chosen to form an orthonormal basis in $L^2(V)$. Let us denote the eigenvalues
and the eigenfunctions of the operator $L$ by $(\lambda_n, \varphi_n)_{n=1}^{\infty}$ where each eigenvalue is taken with multiplicity. Then the heat kernel has the form

$$U(t; x, x') = \sum_{n=1}^{\infty} \exp(-t\lambda_n)\varphi_n(x)\varphi_n^*(x').$$

(1.4)

We note that the heat kernel diagonal $U(t; x, x)$ is a smooth self-adjoint endomorphism.

In this paper we report on various approximations for the heat trace and functional determinant and discuss its relation to the Korteweg-de Vries hierarchy. Although it is heavily based upon our previous work there are many new original ideas and results obtained in this paper.

This paper is organized as follows. In Sec. 2 we introduce the spectral invariants such as the heat trace, the zeta function and a new very powerful invariant which is defined in terms of the Mellin transform of the heat trace. In particular, it immediately gives the functional determinant in one dimension. In Sec. 3 we develop a perturbation theory in the potential term $Q$ and compute the linear and quadratic terms in the heat trace. In Sec. 4 we describe a scheme for the asymptotic expansion of the heat kernel in powers of $t$ and in the Taylor series in space coordinates. In Sec. 5 we compute the leading derivatives terms in the diagonal values of the heat kernel coefficients and use this to compute the terms linear and quadratic in the potential term in the heat trace and functional determinant. In Sec. 6 we prove an algebraic lemma for the heat semigroup of the sum of two self-adjoint operators and apply this lemma to obtain a differential equation directly for the heat kernel diagonal. In Sec. 7 we use that equation to obtain a new recursive system for the diagonal heat kernel coefficients and obtain a closed formula for the whole sequence of all diagonal heat kernel coefficients. We then use this formula to obtain some closed formulas for the heat kernel diagonal and the functional determinant. Even though these formulas are not exact on the circle they become exact in the limit of infinite radius. Of course, the heat trace and the functional determinant diverge on a noncompact space, such as the real line. That is why, we write our formulas in terms of the circle. In Sec. 8 we describe the bi-Hamiltonian systems and define an abstract generalized KdV hierarchy. Then we apply this formalism to our differential operator in one dimension and obtain the standard KdV hierarchy, whose integrals of motion are exactly the global heat kernel coefficients.
2 Spectral Invariants

We will be interested in the spectral invariants of the operator $L$. One of them, called the heat trace, is the trace of the heat kernel and reads [12]

$$\Theta(t) = \text{Tr } \exp(-tL) = \int_{S^1} dx \text{tr } U(t; x, x) = \sum_{n=1}^{\infty} \exp(-t\lambda_n). \quad (2.1)$$

Another important spectral invariant is the zeta function defined by [17, 5]

$$\zeta(s, \lambda) = \text{Tr } (L - \lambda)^{-s} = \sum_{n=1}^{\infty} (\lambda_n - \lambda)^{-s}, \quad (2.2)$$

where $\lambda$ is a sufficiently large negative parameter so that the operator $L - \lambda$ is positive and $s$ is a complex parameter with sufficiently large positive real part. The zeta function can be expressed in terms of the Mellin transform of the heat trace

$$\zeta(s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty dt \; t^{s-1} e^{t\lambda} \Theta(t). \quad (2.3)$$

The zeta function enables one to define the functional determinant as follows [17]

$$\log \text{Det } (L - \lambda) = -\zeta'(0, \lambda), \quad (2.4)$$

where $\zeta'(s, \lambda) = \partial_s \zeta(s, \lambda)$.

Next, we define a function $\Omega(t)$ by

$$\Theta(t) = (4\pi t)^{-1/2} \Omega(t), \quad (2.5)$$

and a new function $B_q(\lambda)$ of a complex variable $q$ as the modified Mellin transform of this function

$$B_q(\lambda) = \frac{1}{\Gamma(-q)} \int_0^\infty dt \; t^{-q-1} e^{t\lambda} \Omega(t). \quad (2.6)$$

As was shown in [4], the integral (2.6) converges for Re $q < 0$, and, therefore, by integrating by parts it can be analytically continued to an entire function of $q$, that is, for Re $q < N$,

$$B_q(\lambda) = (-1)^N \frac{1}{\Gamma(-q + N)} \int_0^\infty dt \; t^{-q-1+N} \partial_t^N \left[ e^{t\lambda} \Omega(t) \right]. \quad (2.7)$$
It is also easy to see that the function $B_q(\lambda)$ is an analytic function of $\lambda$ for sufficiently large negative real part of $\lambda$, that is, for $\text{Re}\lambda << 0$. Moreover, the values of the function $B_q(\lambda)$ at non-negative integer values of $q$, that is, $q = k = 0, 1, 2, \ldots$, are equal to the Taylor coefficients of the function $\exp(t\lambda)\Omega(t)$ at $t = 0$.

$$B_k(\lambda) = (-\partial_t)^k \left[ e^{t\lambda} \Omega(t) \right] \bigg|_{t=0},$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^j A_{k-j},$$

where

$$A_k = (-\partial_t)^k \Omega(t) \bigg|_{t=0}.$$  

(2.9)

Notice that for integer $q$ the functions $B_k(\lambda)$ are polynomials in $\lambda$; obviously, $B_k(0) = A_k$. However, for non-integer $q$ the functions $B_q(\lambda)$ might be singular at $\lambda = 0$.

This function enables one to express the zeta function in the form

$$\zeta(s, \lambda) = (4\pi)^{-1/2} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s)} B_{\frac{1}{2} - s}(\lambda).$$

(2.10)

Now, by using the fact that $\Gamma(s)$ has a pole at $s = 0$ with residue 1, we obtain

$$\zeta(0, \lambda) = 0,$$

(2.11)

and a very simple formula for the determinant in one dimension

$$\log \det (L - \lambda) = B_{1/2}(\lambda) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t}} \partial_t \left[ e^{t\lambda} \Omega(t) \right].$$

(2.12)

We will expand the potential $Q$ in the Fourier series

$$Q(x) = \sum_{n \in \mathbb{Z}} q_n e^{inx/a},$$

(2.13)

where

$$q_n = \frac{1}{2\pi a} \int_{S^1} dx \, e^{-inx/a} Q(x)$$

(2.14)

and

$$q_n^* = q_{-n}.$$  

(2.15)
3 Perturbation Theory

We introduce a formal small parameter $\varepsilon$ and consider the perturbation theory for the heat trace of the operator $L = -D^2 + \varepsilon Q$ which can be obtained as a perturbation series in powers of $\varepsilon$; we set $\varepsilon = 1$ at the end.

By using the Duhamel series for the heat semigroup

$$U(t) = U_0(t) - \int_0^t dv U_0(t - v)QU_0(v)$$

$$+ \int_0^t dv_2 \int_0^{v_2} dv_1 U_0(t - v_2)QU_0(v_2 - v_1)QU_0(v_1) + O(\varepsilon^3),$$

we get the trace

$$\text{Tr } U(t) = \text{Tr } U_0(t) - t\text{Tr } QU_0(t)$$

$$+ \int_0^t dv_2 \int_0^{v_2} dv_1 \text{Tr } QU_0(v_2 - v_1)QU_0(t - v_2 + v_1) + O(\varepsilon^3).$$

Now, by using the formula

$$\int_0^t dv_2 \int_0^{v_2} dv_1 f(v_2 - v_1) = \int_0^t dv (t - v)f(v),$$

we obtain

$$\text{Tr } U(t) = \text{Tr } U_0(t) - t\text{Tr } QU_0(t)$$

$$+ \int_0^t dv (t - v)\text{Tr } QU_0(v)QU_0(t - v) + O(\varepsilon^3),$$

Finally, by changing the variable

$$v = \frac{t}{2}(1 + \xi)$$
and using the symmetry of the integrand we get

$$\text{Tr } U(t) = \text{Tr } U_0(t) - t \text{Tr } QU_0(t)$$
$$+ \frac{t^2}{2} \int_0^1 d\xi \text{Tr } QU_0 \left( t \frac{1 + \xi}{2} \right) QU_0 \left( t \frac{1 - \xi}{2} \right) + O(\varepsilon^3). \quad (3.7)$$

The heat kernel of the operator $L_0 = -D^2$ is well known and has the form

$$U_0(t; x, x') = \frac{1}{2\pi a} \sum_{n \in \mathbb{Z}} \exp \left( -\frac{t}{a^2}n^2 + in \frac{2\pi}{a}(x - x') \right)$$
$$= (4\pi t)^{-1/2} \sum_{n \in \mathbb{Z}} \exp \left( -\frac{1}{4t}(x - x' + 2\pi an)^2 \right). \quad (3.8)$$

where the second form is obtained by the Poisson duality. The heat trace then is

$$\text{Tr } U_0(t) = 2\pi a N (4\pi t)^{-1/2} \theta \left( \frac{t}{a^2} \right), \quad (3.9)$$

where

$$\theta(t) = \sum_{n \in \mathbb{Z}} \exp \left( -\frac{1}{t} n^2 \right) = \frac{t^{1/2}}{\sqrt{\pi}} \sum_{n \in \mathbb{Z}} \exp \left( -tn^2 \right). \quad (3.10)$$

This function can be expressed in terms of the Jacobi $\theta$-function,

$$\theta(t) = \theta_3(0, e^{-\pi^2/t}). \quad (3.11)$$

By using this equation we easily obtain

$$\text{Tr } U(t) = (4\pi t)^{-1/2} \theta \left( \frac{t}{a^2} \right) \left( 2\pi a N - t \int_{S^1} dx \text{tr } Q \right)$$
$$+ \frac{t^2}{2} \int_{S^1 \times S^1} dx dx' \text{tr } Q(x) F(t; x, x') Q(x') + O(\varepsilon^3), \quad (3.12)$$

where

$$F(t; x, x') = \int_0^1 d\xi U_0 \left( t \frac{1 - \xi}{2}; x, x' \right) U_0 \left( t \frac{1 + \xi}{2}; x', x \right). \quad (3.13)$$
Now, by using the explicit form of the heat kernel and the Poisson duality formula in one of the heat kernels we compute

\[ F(t; x, x') = (4\pi t)^{-1/2} \sum_{n \in \mathbb{Z}} \exp \left( -\frac{a^2}{4t} \pi^2 n^2 \right) F_n(t; x, x'), \quad (3.14) \]

where

\[ F_n(t; x, x') = \frac{1}{2\pi a} \sum_{k \in \mathbb{Z}} \exp \left( ikx - x' \right) \int_0^1 d\xi \exp \left( -\frac{t}{a^2} \frac{1 - \xi^2}{4} k^2 - ikn(1 + \xi)\pi \right). \quad (3.15) \]

Note that this function is the integral kernel of the operator

\[ F_n(t) = \int_0^1 d\xi \exp \left( \frac{1 - \xi^2}{4t} D^2 - n(1 + \xi)\pi \right), \quad (3.16) \]

and the function \( F(t; x, x') \) is the kernel of the operator

\[ F(t) = (4\pi t)^{-1/2} \sum_{n \in \mathbb{Z}} \exp \left( -\frac{a^2}{4t} \pi^2 n^2 \right) F_n(t). \quad (3.17) \]

Further, we can rewrite this equation in the spectral form

\[ \Omega(t) = \theta \left( \frac{t}{a^2} \right) 2\pi a \left( N - t\text{tr} q_0 \right) + \pi a^2 \sum_{k \in \mathbb{Z}} |q_k|^2 \beta_k \left( \frac{t}{a^2} \right) + O(\varepsilon^3), \quad (3.18) \]

where \( |q_k|^2 = \text{tr} q_k q_k^* \) and

\[ \beta_k(t) = \int_0^1 d\xi \sum_{n \in \mathbb{Z}} \exp \left( -t \frac{1 - \xi^2}{4} k^2 - \frac{1}{t} \pi^2 n^2 - ikn(1 + \xi)\pi \right) \]
\[ = \frac{t^{1/2}}{\sqrt{\pi}} \int_0^1 d\xi \sum_{n \in \mathbb{Z}} \exp \left\{ -t \left[ n^2 + \frac{(1 + \xi)}{2} (k^2 - 2nk) \right] \right\}. \quad (3.19) \]

These formulas enable one to compute the zeta function and the determinant with the same accuracy, that is, up to cubic terms in the potential \( Q \). However, we will not do it in general. Rather we will be interested in the limit of large
radius, \(a \to \infty\). We will do this in another section below by a completely different method.

Let us just note that asymptotically as \(a \to \infty\)
\[
\begin{align*}
\theta(t) & \sim 1, \\
F(t) & \sim (4\pi t)^{-1/2} \alpha \left(-tD^2\right), \\
\beta_k(t) & \sim \alpha \left(tk^2\right),
\end{align*}
\]
where \(\alpha(z)\) is a function defined by
\[
\alpha(z) = \int_0^1 d\xi \exp \left(-\frac{1 - \xi^2}{4}z\right). \tag{3.23}
\]
This is an entire function of \(z\). By using the well known integral
\[
\int_0^1 d\xi \left(\frac{1 - \xi^2}{4}\right)^q = \frac{\Gamma(q + 1)\Gamma(q + 1)}{\Gamma(2q + 2)} \tag{3.24}
\]
one can obtain the power series representation of this function
\[
\alpha(z) = \sum_{k=0}^{\infty} \frac{k!}{(2k + 1)!} (-z)^k. \tag{3.25}
\]
By using either this series or by the integration by parts one can show that this function satisfies the differential equation
\[
\left(4\partial_t + 1 + \frac{2}{t}\right)\alpha(t) = \frac{2}{t}. \tag{3.26}
\]

4 Heat Kernel Asymptotic Expansion

It is useful to introduce various scales parametrized by dimensionless parameters \(\tau, \varepsilon\) and \(\delta\) as follows. The parameter \(\tau\) measures the relative radius of the circle,
\[
\tau = \frac{t}{a^2}. \tag{4.1}
\]
The parameter \(\varepsilon\) measures the relative amplitude of the potential, that is,
\[
tQ = O(\varepsilon), \tag{4.2}
\]
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while the parameter $\delta$ measures the derivatives of the potential, that is,

$$ r^{k+1} \delta^{2k} Q = O(\delta^k \epsilon). \quad (4.3) $$

We assume now that $t$ is smaller than all other parameters of the same dimension, that is,

$$ \tau \ll 1, \quad \epsilon \ll 1, \quad \delta \ll 1. \quad (4.4) $$

Also, we consider the neighborhood of the diagonal $x = x'$, that is, we assume that

$$ x - x' = o(t^{1/2}). \quad (4.5) $$

It is well known that there is an asymptotic expansion as $\tau, \epsilon, \delta \to +0$ near the diagonal of the form [12]

$$ U(t; x, x') \sim (4\pi t)^{-1/2} \exp \left\{ -\frac{1}{4t} (x - x')^2 \right\} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k(x, x'), \quad (4.6) $$

where $a_k(x, x')$ are the so-called heat kernel coefficients. We will denote by square brackets the diagonal values of two-point functions, i.e.,

$$ [f] = \lim_{x \to x'} f(x, x'). \quad (4.7) $$

Then the asymptotic expansion of the heat kernel diagonal as $t \to 0$ is

$$ [U(t)] \sim (4\pi t)^{-1/2} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} [a_k], \quad (4.8) $$

and, therefore, there is the corresponding asymptotics of the heat trace function $\Omega(t)$

$$ \Omega(t) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A_k, \quad (4.9) $$

where

$$ A_k = \int_{S^1} dx \text{tr} [a_k] \quad (4.10) $$

are the spectral invariants of the operator $L$ called global heat kernel coefficients or simply heat invariants.
The first heat kernel coefficient $a_0$ is determined from the initial condition (1.3) and is equal to

$$a_0 = 1.$$  \hspace{1cm} (4.11)

The higher-order heat kernel coefficients $a_k, k \geq 1$, satisfy the following recurrence relations [4, 3, 5]

$$\left(1 + \frac{1}{k}(x - x')\partial_x\right)a_k = La_{k-1}, \quad k \geq 1. \hspace{1cm} (4.12)$$

A powerful method for calculation of the heat kernel coefficients was developed in [4, 3, 5]. In the one-dimensional case it takes a very simple form [7]. First of all, we fix the point $x'$. We introduce the following notation. For every non-negative integer $n$ we define the functions

$$|n\rangle = \frac{1}{n!}(x - x')^n; \hspace{1cm} (4.13)$$

we also let, by definition, $|n\rangle = 0$ for $n < 0$. Then

$$D|n\rangle = |n - 1\rangle. \hspace{1cm} (4.14)$$

We also define the operator $D^{-1}$ by

$$D^{-1}f(x) = \int_{x'}^x dy f(y); \hspace{1cm} (4.15)$$

then for any non-negative $n$

$$D^{-1}|n\rangle = |n + 1\rangle. \hspace{1cm} (4.16)$$

Next, for every non-negative integer $m$ we define the operators

$$\langle m|f\rangle = [\partial^m_x f], \hspace{1cm} (4.17)$$

and the matrix elements of a differential operator $L$ by

$$\langle m|L|n\rangle = \frac{1}{n!}[\partial^m_x L(x - x')^n]. \hspace{1cm} (4.18)$$

Then the matrix elements of the identity operator are obviously

$$\langle m|n\rangle = \delta_{mn}. \hspace{1cm} (4.19)$$
where $\delta_{mn}$ is the usual Kronecker symbol, therefore, the matrix elements of the first and the second derivative have the form

\[
\langle m | D | n \rangle = \langle m + 1 | n \rangle = \delta_{n,m+1}, \quad (4.20)
\]
\[
\langle m | D^2 | n \rangle = \langle m + 2 | n \rangle = \delta_{n,m+2}. \quad (4.21)
\]

Also, for a function $Q$ for $m \geq n$ we have rather

\[
\langle m | Q | n \rangle = \frac{1}{n!} \left[ \partial_x^n \{ Q(x)(x - x')^n \} \right] = \left( \begin{array}{c} m \\ n \end{array} \right) Q^{(m-n)}, \quad m \geq n, \quad (4.22)
\]

where

\[
Q^{(n)} = \partial_x^n Q. \quad (4.23)
\]

For $m < n$ these matrix elements obviously vanish

\[
\langle m | Q | n \rangle = 0, \quad m \leq n - 1, \quad (4.24)
\]

In general, the matrix elements of a differential operator $L$ of order $p$ vanish for $m \leq n - p - 1$,

\[
\langle m | L | n \rangle = 0 \quad \text{for } m \leq n - p - 1. \quad (4.25)
\]

For a pseudo-differential (nonlocal) operator it is not so—all matrix elements are, in general, non-zero. For example,

\[
\langle m | D^{-1} | n \rangle = \langle m - 1 | n \rangle = \langle m | n + 1 \rangle = \delta_{n,m-1}. \quad (4.26)
\]

The matrix representation of the operators is very convenient in so far that the products, the powers and the commutators of the operators are given by the product, the powers and the commutators of the infinite matrices. For example, two commuting operators must have commuting matrices etc.

By using the above equations we obtain the matrix elements of the Schrödinger operator $(1.1)$

\[
\langle m | L | n \rangle = -\langle m | D^2 | n \rangle + \langle m | Q | n \rangle = -\delta_{n,m+2} + \left( \begin{array}{c} m \\ n \end{array} \right) Q^{(m-n)}.
\]

These matrix elements form an infinite matrix

\[
\langle \langle m | L | n \rangle \rangle = \begin{pmatrix}
Q & 0 & -\mathbb{I} & 0 & 0 & \ldots \\
Q^{(1)} & Q & 0 & -\mathbb{I} & 0 & \ldots \\
Q^{(2)} & \left( \begin{array}{c} 1 \\ 2 \end{array} \right) Q^{(1)} & Q & 0 & -\mathbb{I} & \ldots \\
Q^{(3)} & \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) Q^{(2)} & \left( \begin{array}{c} 1 \\ 2 \end{array} \right) Q^{(1)} & Q & 0 & \ldots \\
Q^{(4)} & \left( \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \right) Q^{(3)} & \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) Q^{(2)} & \left( \begin{array}{c} 1 \\ 2 \end{array} \right) Q^{(1)} & Q & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(4.28)
Now, by using the technique developed in [4] one can express the coefficients $a_k(x, x')$ in terms of the Taylor series

$$a_k(x, x') = \sum_{n=0}^{\infty} \frac{1}{n!} (x - x')^n \langle n|a_k \rangle,$$  \hspace{2cm} (4.29)

where

$$\langle n|a_k \rangle = \left[ \partial_x^n a_k \right] = \sum_{n_1, \cdots, n_{k-1} \geq 0} \frac{k}{(k + n)} \cdot \frac{(k - 1)}{(k - 1 + n_{k-1})} \cdots \frac{1}{(1 + n_1)}$$

$$\times \langle n|L|n_{k-1} \rangle \langle n_{k-1}|L|n_{k-2} \rangle \cdots \langle n_1|L|0 \rangle,$$  \hspace{2cm} (4.30)

These coefficients are differential polynomials of the potential $Q$ evaluated at the point $x'$. It is important to note that

$$\langle m|L|m + 1 \rangle = 0,$$  \hspace{2cm} (4.31)

and

$$\langle m|L|m + 2 + k \rangle = 0 \quad \text{for} \quad k \geq 1,$$  \hspace{2cm} (4.32)

and, therefore, the summation over $n_i$ in (4.30) is limited from above and ranges over

$$0 \leq n_1 \leq n_2 + 2 \leq \cdots \leq n_{k-1} + 2(k - 1) \leq n + 2(k - 1).$$  \hspace{2cm} (4.33)

By using this technique it is easy to obtain the diagonal values of some low-order heat kernel coefficients [4]

$$[a_1] = Q,$$  \hspace{2cm} (4.34)

$$[a_2] = Q^2 - \frac{1}{3} Q''.$$

$$[a_3] = Q^3 - \frac{1}{2} (QQ'' + Q''Q + Q'Q') + \frac{1}{10} Q^{(4)}.$$  \hspace{2cm} (4.36)

The general formula for an arbitrary coefficient $[a_k]$ is presented in [7].

5 Leading Derivatives in Heat Kernel Coefficients

The technique described above can be used to analyse the general structure of the heat kernel coefficients, in particular, to compute the leading derivative terms in
all heat kernel coefficients \([a_k]\). This has been done in [2, 4, 5] for general Laplace type operators. The leading derivatives in the heat kernel coefficients for \(k \geq 2\) have the following form

\[
[a_k] = \frac{k!(k-1)!}{(2k-1)!} \left\{(-D^2)^{k-1} Q + (2k-1)Q(-D^2)^{k-2} Q \right\} + O(\partial(QQ)) + O(\varepsilon^3).
\] (5.1)

Here total derivatives (and commutators) of quadratic terms denoted by \(O(\partial(QQ))\) and the terms of higher order in \(Q\) denoted by \(O(\varepsilon^3)\) are omitted.

Now, by using the integral (3.24) one can sum up the asymptotic expansion of the heat kernel diagonal to get the asymptotic expansion as \(\tau, \varepsilon \to 0\)

\[
[U(t)] \sim (4\pi t)^{-1/2} \left\{1 - t\alpha\left(-tD^2\right)Q + \frac{t^2}{2}Q\alpha\left(-tD^2\right)Q + O(\partial(QQ)) + O(\varepsilon^3) \right\},
\] (5.2)

where \(\alpha(z)\) is exactly the same function defined by (3.23). This is an asymptotic expansion as \(\tau, \varepsilon \to 0\) but the parameter \(\delta\) does not have to be small, \(\delta \sim 1\).

After integrating the heat kernel diagonal all total derivatives vanish and we obtain then the asymptotic expansion as \(\tau, \varepsilon \to 0\) of the heat trace function \(\Omega(t)\)

\[
\Omega(t) \sim 2\pi a N + \omega(t) + O(\varepsilon^3),
\] (5.3)

where

\[
\omega(t) = -t \int_{S^1} dx \text{tr} \left(\frac{1}{2} q \left(-\lambda\right)^q + b_q(\lambda) + O(\varepsilon^3)\right)
\] (5.4)

This formula should be compared with the results of Sec. 3. It can be obtained by taking the limit of large radius \(a \to \infty\) in the equation (3.18).

Next, by using the equation (2.6) we compute the asymptotic expansion of the function \(B_q(\lambda)\) as \(a\lambda \to -\infty\) and \(\varepsilon \to 0\)

\[
B_q(\lambda) \sim 2\pi a N(-\lambda)^q + b_q(\lambda) + O(\varepsilon^3),
\] (5.5)

where

\[
b_q(\lambda) = q(-\lambda)^{q-1} \int_{S^1} dx \text{tr} \left(\frac{1}{2} q(1) \left(-\lambda\right)^{q-2} \int \frac{D^2}{\lambda} Q\right).\] (5.6)
and

\[ f_q(z) = \int_0^1 d\xi \left(1 + \frac{\xi^2}{4}z\right)^q. \] (5.7)

It is easy to see that for positive values of \(z\) the function \(f_q(z)\) is an entire function of \(q\); by using eq. (3.24) it can be represented as a power series

\[ f_q(z) = \sum_{j=0}^{\infty} \frac{\Gamma(q+1)\Gamma(j+1)}{\Gamma(q-j+1)\Gamma(2j+2)} z^j. \] (5.8)

Notice that for non-negative integer values \(q = 0, 1, 2, \ldots\) this series terminates and is, in fact, a polynomial of \(z\) of order \(q\). One can also compute the asymptotics as \(z \to \infty\)

\[ f_q(z) = \frac{\Gamma(q+1)\Gamma(q+1)}{\Gamma(2q+2)} z^q + O(z^{q-1}). \] (5.9)

Finally, by using this result the functional determinant, (2.12), takes the form (within the same accuracy, that is, as an asymptotic series as \(a\lambda \to -\infty\) and \(\varepsilon \to 0\))

\[ \log \mathrm{Det} \left(L - \lambda\right) = 2\pi a N(-\lambda)^{1/2} + \gamma(\lambda) + O(\varepsilon^3), \] (5.10)

where

\[ \gamma(\lambda) = \frac{1}{2(-\lambda)^{1/2}} \int_{S^1} dx \mathrm{tr} Q - \frac{1}{8(-\lambda)^{3/2}} \int_{S^1} dx \mathrm{tr} Q f_{-3/2} \left(\frac{D^2}{\lambda}\right) Q. \] (5.11)

The function \(f_{-3/2}\) can be easily computed from (5.7); it has a very simple form

\[ f_{-3/2}(z) = \frac{4}{z+4}. \] (5.12)

Therefore,

\[ \gamma(\lambda) = \frac{1}{2(-\lambda)^{1/2}} \int_{S^1} dx \mathrm{tr} Q - \frac{1}{4(-\lambda)^{1/2}} \int_{S^1} dx \mathrm{tr} Q \left(-D^2 - 4\lambda\right)^{-1} Q \] (5.13)

The functions \(\omega(t)\), \(b_q(\lambda)\) and \(\gamma(\lambda)\) can be written in the spectral form as

\[ \omega(t) = -2\pi at \mathrm{tr} q_0 + 2\pi a t^2 \sum_{n=1}^{\infty} |q_n|^2 \alpha \left(\frac{t}{a^2 n^2}\right), \] (5.14)

\[ b_q(\lambda) = 2\pi a q(-\lambda)^{q-1} \mathrm{tr} q_0 + q(q-1)2\pi a(-\lambda)^{q-2} \sum_{n=1}^{\infty} |q_n|^2 f_{q-2} \left(-\frac{n^2}{\lambda a^2}\right) \] (5.15)

\[ \gamma(\lambda) = \frac{a^3}{(-\lambda)^{1/2}} \mathrm{tr} q_0 - \pi \frac{a^3}{(-\lambda)^{1/2}} \sum_{n=1}^{\infty} \frac{|q_n|^2}{n^2 - 4\lambda a^2}, \] (5.16)
where \(|q_n|^2 = \text{tr} q_n^* q_n\). We stress here once again that all these results are valid only in the limit as \(a \lambda \to -\infty\) and \(\varepsilon \to 0\). As was noted in Sec. 2 the function \(B_q(\lambda)\) and the functional determinant may be singular at \(\lambda \to 0\). Even though the limit \(\lambda \to 0\) is not well defined we quote the result

\[
\gamma(\lambda) = \frac{\pi a}{(-\lambda)^{1/2}} \left\{ \text{tr} q_0 - a^2 \sum_{n=1}^{\infty} |q_n|^2 \frac{1}{n^2} + o(\lambda) \right\}, \tag{5.17}
\]

which is indeed singular as \(\lambda \to 0\).

6 Equation for Heat Kernel Diagonal

Now, following [6], we derive another recursion system for the heat kernel coefficients, which gives directly the diagonal heat kernel coefficients \([a_k]\). It is based on the following purely algebraic lemma.

Let \(L(H)\) be an algebra of operators on some Hilbert space \(H\). Every operator \(Y : H \to H\) from the algebra defines the standard action on the algebra itself, \(Y : L(H) \to L(H)\), by left multiplication \(X \to YX\) for any \(X \in L(H)\); we will denote this action by the same symbol \(Y\) which should not cause any confusion.

There is also another operator \(\text{Ad}_Y : L(H) \to L(H)\) defined by the commutator

\[
\text{Ad}_Y X = [Y, X] \tag{6.1}
\]

for any \(X \in L(H)\).

**Lemma 1** Let \(D, Q \in L(H)\) be two operators from the algebra \(L(H)\) and \(L \in L(H)\) be an operator defined by

\[
L = -D^2 + Q; \tag{6.2}
\]

let \(U(t) = \exp(-tL)\) be its heat semigroup.

Suppose that the operator \(\text{Ad}_D\) is an injection and that the image of the operator \(\text{Ad}_Q\) is a subset of the image of the operator \(\text{Ad}_D\), that is, \(\text{Ad}_Q(L(H)) \subseteq \text{Ad}_D(L(H))\); then the operator \(\hat{E} : L(H) \to L(H)\) defined by

\[
\hat{E} = \text{Ad}_D^3 - 2Q\text{Ad}_D - 2\text{Ad}_D Q + \text{Ad}_Q\text{Ad}_D + \text{Ad}_D\text{Ad}_Q + \text{Ad}_Q\text{Ad}_D^{-1}\text{Ad}_Q \tag{6.3}
\]

is well defined.

Then:
1. For any \( t \geq 0 \),
\[
\left( 4\partial_t \text{Ad}_D - \hat{E} \right) U(t) = 0,
\] (6.4)

2. and for any non-negative integer \( k \geq 0 \),
\[
-4\text{Ad}_D L^{k+1} = \hat{E} L^k.
\] (6.5)

**Proof.** The semigroup satisfies obviously the equations
\[
\partial_t U = -LU = -UL.
\] (6.6)

Therefore, we have
\[
4\partial_t \text{Ad}_D U = -4DLU + 4ULD
\]
\[
= -DLU + ULD - 3DUL + 3LUD
\]
\[
= D^3U - DQU - UD^3 + QUD
\]
\[
+3DUD^2 - 3DUQ - 3D^2UD + 3QUD
\] (6.7)

Now, from the commutativity of the operators \( L \) and \( U(t) \),
\[
[(D^2 + Q), U] = 0,
\] (6.8)
we have also
\[
[Q, U] = [D^2, U].
\] (6.9)

On the other hand,
\[
[D^2, U] = [D, (DU + UD)],
\] (6.10)
and, therefore,
\[
[Q, U] = [D, (DU + UD)].
\] (6.11)

This equation can be written as
\[
\text{Ad}_Q U = \text{Ad}_D(DU + UD),
\] (6.12)
and, hence,
\[
\text{Ad}_D^{-1}\text{Ad}_Q U = DU + UD.
\] (6.13)

Therefore,
\[
\text{Ad}_Q\text{Ad}_D^{-1}\text{Ad}_Q U = QDU + QUD - DUQ - UDQ.
\] (6.14)
Next, we compute directly
\[
\begin{align*}
\text{Ad}_D^3 U &= D^3 U - 3D^2 U D + 3 D U D^2 - UD^3, \\
Q \text{Ad}_D U &= QDU - QUD, \\
\text{Ad}_D Q U &= DQU - QUD, \\
\text{Ad}_Q \text{Ad}_D U &= QDU - QUD - DUQ + UDQ, \\
\text{Ad}_D \text{Ad}_Q U &= DQU - DUQ - QUD + UQD.
\end{align*}
\]

By using these results it is easy to show that
\[
\begin{align*}
\hat{E} U &= D^3 U - DQU - UD^3 + UQD \\
&+ 3DUD^2 - 3DUQ - 3D^2 UD + 3QUD,
\end{align*}
\]
and, therefore, the heat semigroup satisfies the equation
\[
\left(4 \partial_t \text{Ad}_D - \hat{E}\right) U(t) = 0.
\]

By expanding this equation in power series in \( t \) we also obtain immediately the commutators of the operator \( D \) with the powers of the operator \( L \)
\[
-4 \text{Ad}_D L^{k+1} = \hat{E} L^k.
\]
This is a very important purely algebraical equation that can be proved also by mathematical induction.

Now, we apply this lemma to a particular case when when \( D = \partial_x \) is the derivative operator, \( Q \) is the operator of multiplication by a matrix-valued function and
\[
U(t; x, x') = U(t) \delta(x - x'),
\]
is the heat kernel. Our goal is now to take the equation in the kernel form, i.e. to apply it to the delta-function \( \delta(x - x') \), and then to compute its diagonal value \( [U(t)] = U(t; x, x) \).

**Corollary 1** The heat kernel diagonal \([U(t)]\) of the operator \( L = -D^2 + Q \) satisfies the equation
\[
(4 \partial_t D - E) [U(t)] = 0,
\]
where
\[
E = D^3 - 2QD - 2DQ + \text{Ad}_Q D + D \text{Ad}_Q + \text{Ad}_Q D^{-1} \text{Ad}_Q.
\]

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Remark. It should be understood that this operator acts on functions and not on operators as the operator (6.3). In the scalar case all the commutators vanish and \( E \) becomes a differential operator

\[
E = D^3 - 2QD - 2DQ. \tag{6.26}
\]

The equation (6.24) was obtained in [6] by a completely different method. Similar equations were obtained in [20, 9].

Proof. First, notice that for every smooth two-point function \( f(x, x') \) there holds

\[
(\partial_x + \partial_{x'}) f(x, x') \bigg|_{x = x'} = \partial[f], \tag{6.27}
\]

where, as usual \([f] = f(x, x)\) denotes the diagonal value. Also, for any function \( f \) considered as an operator of multiplication by this function we have

\[
\text{Ad}_D f = [D, f] = Df, \tag{6.28}
\]

and, therefore,

\[
\text{Ad}_D U(t) \delta(x - x') \bigg|_{x = x'} = D[U(t)], \tag{6.29}
\]

\[
\text{Ad}_Q^3 U(t) \delta(x - x') \bigg|_{x = x'} = D^3[U(t)], \tag{6.30}
\]

\[
\text{Ad}_Q U(t) \delta(x - x') \bigg|_{x = x'} = \text{Ad}_Q [U(t)], \tag{6.31}
\]

\[
\text{Ad}_D^{-1} \text{Ad}_Q U(t) \delta(x - x') \bigg|_{x = x'} = D^{-1} \text{Ad}_Q [U(t)]. \tag{6.32}
\]

By using these equations into eq. (6.4) we obtain finally the equation (6.24).

One should point out that the equation (6.24) for the heat kernel diagonal is a new nontrivial equation that expresses deep underlying symmetry of the one-dimensional spectral problem. It is this equation that leads to the existence of an infinite-dimensional completely integrable Hamiltonian system (Korteweg-De Vries hierarchy).

It is worth noting the following fact. First, we compute

\[
(DU(t) + U(t)D) \delta(x - x') \bigg|_{x = x'} = W(t), \tag{6.33}
\]

where

\[
W(t) = (\partial_x - \partial_{x'}) U(t; x, x') \bigg|_{x = x'}. \tag{6.34}
\]
Next, we also have
\[ \text{Ad}_Q \delta(x - x') \bigg|_{x=x'} = \text{Ad}_Q[U(t)]. \] (6.35)

Therefore, by using eq. (6.12) we obtain
\[ \text{DW}(t) = \text{Ad}_Q[U(t)]. \] (6.36)

By using the heat kernel expansion (4.6) we see that the function \( W \) has the asymptotic expansion as \( t \to 0 \)
\[ W(t) \sim (4\pi t)^{-1/2} \sum_{k=1}^{\infty} \frac{(-t)^k}{k!} W_k, \] (6.37)
where
\[ W_k = (\partial_x - \partial_{x'})a_k(x, x') \bigg|_{x=x'}. \] (6.38)

By comparing this expansion with the eq. (4.8) we see that the commutators of the diagonal heat kernel coefficients \([a_k]\) with the potential \( Q \) are also given by the derivative of some differential polynomials \( W_k \), i.e.
\[ DW_k = \text{Ad}_Q[a_k] \] (6.39)

Note that in the scalar case all commutators with \( Q \) vanish and, therefore,
\[ W(t) = W_k = 0. \] (6.40)

### 7 Closed Formulas for Spectral Invariants

Substituting the asymptotic expansion of the heat kernel diagonal (4.8) into the eq. (6.24) we find a direct recursion system for the diagonal heat kernel coefficients \([a_k]\)
\[ D[a_k] = -\frac{k}{2(2k - 1)} E[a_{k-1}]. \] (7.1)
or, which is equivalent,
\[ [a_k] = \frac{k}{2(2k - 1)} A[a_{k-1}], \] (7.2)
where \( A \) is an operator defined by
\[ A = -D^{-1} E. \] (7.3)
A similar formula has been found in [6,20,9]. Note, that the operator $A$ is not a differential operator but a nonlocal pseudo-differential one. In the scalar case the operator $A$ has a simple form

$$A = -D^2 + 2Q + 2D^{-1}QD = -D^2 + 4Q - 2D^{-1}Q'.$$  \hfill (7.4)

The recursion system (7.2) can be formally solved: for $k \geq 1$,

$$[a_k] = \frac{k!(k-1)!}{(2k-1)!} A^{k-1} Q.$$  \hfill (7.5)

Thus, all diagonal heat kernel coefficients $[a_k]$ can be obtained by acting with the powers of the operator $A$ on $Q$.

Now, by using this solution for the heat kernel coefficients the asymptotic expansion of the heat kernel diagonal can be summed formally. Indeed, by using eqs. (4.9) and (3.24) we obtain

$$[U(t)] \sim (4\pi t)^{-1/2} \left\{ 1 - t\alpha(tA)Q \right\},$$  \hfill (7.6)

where $\alpha(z)$ is the function defined by (3.23). Indeed, by using the eq. (3.26) it is easy to see that the heat kernel diagonal satisfies the eq. (6.24). It is also instructive to compare this result with eq. (5.2) obtained by summing the leading derivatives. The result (7.6) goes much further in the sense that it also sums all powers of the potential $Q$. That is, this equation sums all powers of the parameter $\delta$ and $\varepsilon$. However, it is only valid in the asymptotic limit $\tau \to 0$ (that is, in the limit of the infinite radius of the circle, $a \to \infty$). That is why we do not use the equality sign here. Furthermore, all integrals below over the circle $S^1$ (of infinite radius) can be replaced by the integral over the real line $\mathbb{R}$. We do not do it since strictly speaking the potential $Q$ is defined on the circle and we do not assume anything about its behavior at infinity.

This gives the spectral function

$$\Omega(t) \sim 2\pi aN - t \int_{S^1} dx \operatorname{tr} \alpha(tA)Q,$$  \hfill (7.7)

The closed form (7.7) gives then the trace of the heat kernel, the zeta-function and all other spectral functions. In particular, we have for any complex $q$

$$B_q(\lambda) \sim 2\pi N a(-\lambda)^q + q(-\lambda)^{q-1} \int_{S^1} dx \operatorname{tr} f_{q-1} \left( -\frac{A}{\lambda} \right) Q.$$  \hfill (7.8)
where \( f_q(z) \) is the function defined by (5.7). Thus, the functional determinant (2.12) takes the form

\[
\log \text{Det} (L - \lambda) \sim 2\pi Na(-\lambda)^{1/2} + \frac{1}{2(-\lambda)^{1/2}} \int_{S^1} dx \text{tr} f_{-1/2}(-\frac{A}{\lambda})Q. \tag{7.9}
\]

The function \( f_{-1/2} \) can be easily computed from the definition (5.7); we get

\[
f_{-1/2}(z) = \frac{2}{\sqrt{z}} \sin^{-1}\left(1 + \frac{4}{z}\right)^{-1/2}. \tag{7.10}
\]

These formal expressions are very useful and provide, for example, a good framework to obtain the asymptotic expansion of the functional determinant as \( \lambda \to -\infty \). Although the limit \( \lambda \to 0 \) is not well defined, we write the formal formulas in this case too

\[
B_q(0) \sim \frac{\Gamma(q + 1)\Gamma(q)}{\Gamma(2q)} \int_{S^1} dx \text{tr} A^{q-1}Q, \tag{7.11}
\]

\[
\log \text{Det} L \sim \frac{\pi}{2} \int_{S^1} dx \text{tr} A^{-1/2}Q. \tag{7.12}
\]

8 Korteweg-de Vries Hierarchy

We describe briefly the formalism of an infinitely-dimensional bi-Hamiltonian system \([10]\). Let \( Q = Q(s) \) be a one-parameter family of self-adjoint operators acting on a Hilbert space \( \mathcal{H} \). Let \( H = H(Q) \) be a functional of \( Q \). Then we define another self-adjoint operator \( \delta H/\delta Q \) on \( \mathcal{H} \) called the variational derivative of \( H \) with respect to \( Q \) as follows

\[
\partial_s H = \text{Tr} \left( \frac{\delta H}{\delta Q} \partial_s Q \right). \tag{8.1}
\]

Let \( D \) be an anti-self-adjoint operator on the Hilbert space \( \mathcal{H} \). We define a Poisson bracket on the space of all functionals of \( Q \) by

\[
\{F, G\}_D = \text{Tr} \frac{\delta F}{\delta Q} \text{Ad}_D \frac{\delta G}{\delta Q}. \tag{8.2}
\]
Obviously, it is antisymmetric and satisfies the Jacobi identity, that is,

\[ \{F, G\}_D = -\{G, F\}_D, \quad \{H, \{F, G\}_D\}_D + \{F, \{G, H\}_D\}_D + \{G, \{H, F\}_D\}_D = 0. \]  

(8.3)

Further, we define a second Poisson bracket by the operator \( \hat{E} \)

\[ \{H, G\}_E = \text{Tr} \frac{\delta H}{\delta Q} \frac{\delta G}{\delta Q}. \]  

(8.4)

One can show that this form is indeed antisymmetric and also satisfies the Jacobi identity.

Now, let \( L = L(s) \) be another one-parameter family of self-adjoint operators on the Hilbert space defined by \( L = -D^2 + Q \). Let \( U(t) = \exp(-tL) \) be its semigroup and

\[ H(t) = -\frac{1}{t} \text{Tr} \exp(-tL) \]  

(8.5)

be its heat trace; we also define a sequence of functionals

\[ H_k(t) = \partial_t^k H(t). \]  

(8.6)

Then it is easy to show, first, that

\[ \frac{\delta H(t)}{\delta Q} = U(t). \]  

(8.7)

Next, as we know from \( (6.21) \), the heat semigroup satisfies the equation

\[ 4 \text{Ad}_D \partial_t U(t) = \hat{E} U(t), \]  

(8.8)

where \( \hat{E} \) is the operator defined by \( (6.3) \). By multiplying this equation by \( U(\tau) \) and taking the trace we obtain

\[ 4 \text{Tr} U(\tau) \text{Ad}_D \partial_t U(t) = \text{Tr} U(\tau) \hat{E} U(t), \]  

(8.9)

which can be written as

\[ \{H(\tau), 4\partial_t H(t)\}_D = \{H(\tau), H(t)\}_E. \]  

(8.10)

The right hand side of this equation is equal to

\[ \{H(\tau), H(t)\}_E = -\{H(t), H(\tau)\}_E = \{H(t), 4\partial_t H(\tau)\}_D = \{4\partial_t H(\tau), H(t)\}_D. \]  

(8.11)
Therefore,
\[ \{H(\tau), \partial_\tau H(t)\}_D = \{\partial_\tau H(\tau), H(t)\}_D. \quad (8.13) \]

Let us define the matrix
\[ M_{kn} = \{H_k, H_n\}_D, \quad (8.14) \]
with \( n, k \geq 0 \), where the functionals \( H_k \) are defined by \((8.7)\); it is obviously, anti-symmetric
\[ M_{kn} = -M_{nk}. \quad (8.15) \]
Now, by differentiating the eq. \((8.13)\) and setting \( \tau = t \) we see that this matrix satisfies the equation
\[ M_{nk} = M_{n+1,k-1}. \quad (8.16) \]
Therefore, the matrix \( M \) vanishes on the main diagonal and on the next to the main diagonal
\[ M_{nn} = M_{n,n+1} = 0. \quad (8.17) \]
Now, we show by induction that it vanishes on all diagonals; we have for any \( n, k \geq 0 \)
\[ M_{n,n+2k} = M_{n+1,n+2k-1} = \cdots = M_{n+k,n+k} = 0, \quad (8.18) \]
and
\[ M_{n,n+2k+1} = M_{n+1,n+2k} = \cdots = M_{n+k,n+k+1} = 0. \quad (8.19) \]
This proves that this matrix is equal to zero, \( M_{kn} = 0 \). That is, the derivatives of the function \( H(t) \) are all in involution
\[ \{H_k, H_n\}_D = 0 \quad (8.20) \]
for any \( k, n \geq 0 \).

Next, we define an hierarchy of Hamiltonian systems (that we call a generalized KdV hierarchy)
\[ \partial_s Q = \text{Ad}_D H_k, \quad (8.21) \]
with the parameter \( t \) in \( H_k(t) \) being fixed here. Then for any functional \( \Phi \) of the operator \( Q \) we have
\[ \partial_s \Phi = \text{Tr} \frac{\delta \Phi}{\delta Q} \partial_s Q = \text{Tr} \frac{\delta \Phi}{\delta Q} \text{Ad}_D \frac{\delta H_k}{\delta Q} = \{\Phi, H_k\}_D. \quad (8.22) \]
Therefore, a functional \( \Phi \) is an integral of motion of the Hamiltonian system if and only if its Possion bracket with the Hamiltonian \( H_k \) vanishes (that is, it is in
involution with the Hamiltonian). Thus, all functionals $H_k$ are integrals of motion of the whole hierarchy of Hamiltonian systems, that is, for any $n,k$,

$$\partial_s H_n = 0. \quad (8.23)$$

A special motivation for the study of the one-dimensional heat kernel is its relation to the Korteweg-de Vries (KdV) hierarchy. We consider a second-order differential operator of the form $L = -D^2 + Q$; to be specific, we assume that the potential $Q$ is a real symmetric matrix.

We will need to study the deformation of spectral invariants under the variation of the potential $Q$. More specifically we consider a one parameter family of operators $L(s) = -D^2 + Q(s)$, where $s$ is a real parameter. Then we have

$$\partial_s \Theta(t) = \partial_s \text{Tr} \exp(-tL) = -t \text{Tr} [\partial_s Q \exp(-tL)]. \quad (8.24)$$

This means that

$$\frac{\delta \Theta(t)}{\delta Q} = -t[U(t)]. \quad (8.25)$$

Expanding both sides of this equation in the asymptotic series as $t \to 0$, we see that

$$\partial_s A_k = k \int_{S^1} dx \text{ tr} \partial_s Q[a_{k-1}], \quad (8.26)$$

where $A_k$ are the global heat kernel coefficients (4.10) of the operator $L$ and $[a_k]$ are the diagonal local heat kernel coefficients introduced in the Sec. 4. Therefore,

$$\frac{\delta A_k}{\delta Q} = k[a_{k-1}]. \quad (8.27)$$

Now, we rescale the sequence of global heat invariants $A_k$ and define a new sequence $I_k$ by

$$I_k = (-1)^{k} \frac{(2k)!}{k!(k+1)!} A_{k+1}. \quad (8.28)$$

Then by using eqs. (7.2) and (7.3)

$$\frac{\delta I_k}{\delta Q} = -2(D^{-1}E)^{k-1} Q, \quad (8.29)$$

and, therefore,

$$D \frac{\delta I_k}{\delta Q} = E \frac{\delta I_{k-1}}{\delta Q}, \quad (8.30)$$

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These functionals define the KdV hierarchy
\[
\frac{\partial Q}{\partial s} = D \frac{\delta I_k}{\delta Q}, \quad k = 1, 2, \ldots
\] (8.31)

This system is an infinitely-dimensional bi-Hamiltonian system. We define two Poisson brackets
\[
\{H, G\}_D = \int_{S^1} dx \text{tr} \frac{\delta H}{\delta Q} D \frac{\delta G}{\delta Q},
\] (8.32)
\[
\{H, G\}_E = \int_{S^1} dx \text{tr} \frac{\delta H}{\delta Q} E \frac{\delta G}{\delta Q},
\] (8.33)

where \(E\) is the operator defined by (4-4.48). Now by using (6.25) one can show that this form is indeed antisymmetric in spite of the fact, that the operator \(E\) is not anti-self-adjoint, in general. In the scalar case the operator \(E\) given by (6.26) is anti-self-adjoint and the corresponding form \(\{F, G\}_E\) (8.33) is antisymmetric automatically. This means that the Poisson brackets are related by
\[
\{I_n, I_k\}_D = \{I_n, I_{k-1}\}_E.
\] (8.34)

Now, exactly as above, this enables one to show that

i) all functionals \(I_k\) are in involution, that is, for any \(n, k,\)
\[
\{I_n, I_k\}_D = \{I_n, I_k\}_E = 0,
\] (8.35)

ii) and, therefore, are integrals of motion, that is, for any \(n,\)
\[
\partial_s I_n = 0.
\] (8.36)

The generalization of this scheme further (to partial differential operators on manifolds, pseudo-differential operators, discrete operators, etc) is an interesting and intriguing problem related to the whole area of spectral geometry and isospectral deformations. What one has to do is to find two anti-self-adjoint operators \(D\) and \(E,\) such that the heat kernel diagonal satisfies the equation
\[
(4D\partial_t - E)[U(t)] = 0.
\] (8.37)

If such operators are found and \(D\) satisfies additionally the Jacobi identity, then the whole construction can be carried out to obtain a completely integrable infinitely dimensional Hamiltonian system.
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