Subtraction method of computing QCD jet cross sections at NNLO accuracy

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We present a general subtraction method for computing radiative corrections to QCD jet cross sections at next-to-next-to-leading order accuracy. The steps needed to set up this subtraction scheme are the same as those used in next-to-leading order computations. However, all steps need non-trivial modifications, which we implement such that that those can be defined at any order in perturbation theory. We give a status report of the implementation of the method to computing jet cross sections in electron-positron annihilation at the next-to-next-to-leading order accuracy.

1. Introduction

Accurate predictions of QCD jet cross sections require the computation of radiative corrections at least at next-to-leading order (NLO) accuracy, but in some cases also at higher order. The physical cases when computations at the next-to-next-to-leading order (NNLO) are important have been discussed extensively in the literature [1]. Although perturbation theory is expected to be a rather systematic procedure, this has not been reflected for many years in the computation of radiative corrections to QCD jet cross sections. The main reason for this is that the higher order corrections are sums of several contributions which are separately divergent in \(d = 4\) spacetime dimensions, only their sum is finite. Furthermore, these contributions have different numbers of particles in the final state therefore, their combination is not straightforward.

2. Subtraction methods at NLO accuracy

The perturbative expansion of any jet cross section can formally be written as \(\sigma = \sigma^{\text{LO}} + \sigma^{\text{NLO}} + \sigma^{\text{NNLO}} + \ldots\). Let us consider \(e^+e^- \rightarrow m\) jet production, when \(\sigma^{\text{LO}}\) is the integral of the fully exclusive Born cross section over the available phase space defined by the jet function \(J_m\),

\[
\sigma^{\text{LO}} = \int_m d\sigma_m^B J_m \equiv \int d\phi_m |M_m^{(0)}|^2 J_m .
\]  

(1)

The NLO correction is the sum of two contributions. We have to consider the fully exclusive cross section \(d\sigma^R\) for producing \(m+1\) partons and the one-loop correction \(d\sigma^V\) to the production of \(m\) partons,

\[
\sigma^{\text{NLO}} = \int_{m+1} d\sigma_{m+1}^R J_{m+1} + \int_m d\sigma_{m}^V J_m
\]

\[
= \int d\phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}
\]

\[
+ \int d\phi_m 2\text{Re}(M_m^{(1)} \cdot M_m^{(0)}) J_m .
\]  

(2)

These two contributions are separately divergent in \(d = 4\) dimensions although their sum is finite.

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for infrared safe observables. We assume that ultraviolet renormalization has been carried out, so the divergences are purely of infrared origin and are regularized by defining the integrals in $d = 4 - 2\varepsilon$ dimensions.

There are several general methods of computing the finite NLO correction. Most of these rely on the same principles, namely one defines an approximate cross section $d\sigma^A$ that regularizes the real correction in $d$ dimensions in all its infrared singular limits, so the cross section

$$\sigma_{m+1}^{\text{NLO}} = \int_{m+1} \left[ (d\sigma^R)_{\varepsilon=0} J_{m+1} - (d\sigma^A)_{\varepsilon=0} J_m \right]$$

(3)

is finite. The subtraction term in this equation is symbolic in the sense that it is actually a sum of different terms and the jet function depends on different momenta in each of these terms.

The first step in defining the approximate cross section is to derive the universal factorization properties of QCD matrix elements when one external momentum becomes soft, or collinear to another one (referred to as unresolved). These are well-known both at NLO and at NNLO. In writing the factorization formulae we use the colour-state notation \[2\], and also some notation introducing the factorization formulae we use the colourization wecolor.

The difficulty of using the multiple infrared factorization formulae for constructing the approximate cross sections is amply demonstrated by the slow progress in setting up a (general) subtraction scheme.

The second step is to write the IR factorization formulae in such a way that intersecting limits can be identified and disentangled so that multiple subtraction is avoided. At the NLO accuracy, the only such intersection occurs in the regions of phase space where one parton is simultaneously soft and also collinear to a second (hard) parton and the overlap of the soft and collinear limits can easily be identified to be the collinear limit of the soft factorization formulae \[3\],

$$C_{ir} |M_{m+1}^{(0)}(p_i, p_r, \ldots)|^2 \propto \frac{2}{s_{ir}} \frac{z_1}{1 - z_1} T^2 |M_m^{(0)}(p_i, \ldots)|^2.$$  

(6)

Thus the candidate subtraction

$$A_{1} |M_{m+1}^{(0)}|^2 = \sum_r \left[ \sum_{i\neq r} \frac{1}{2} C_{ir} + \left( S_r - \sum_{i\neq r} C_{ir} S_r \right) \right] |M_{m+1}^{(0)}|^2$$

(7)

has the same singular limit as the real correction itself and is free of double subtractions. However, disentangling the unresolved limits at higher orders, when multiple soft, collinear and soft-collinear limits overlap in a complicated way, is far more cumbersome \[3\]. This calls for a simple and systematic procedure.

In a physical gauge the collinear singularities are due to the collinear splitting of an external parton \[13,19\]. The overall colour structure of
the event does not change, the splitting is entirely described by the Altarelli–Parisi functions which are products of colour factors and kinematical functions describing the dynamics of the collinear splitting. The emission of soft gluons is just the opposite; it does not affect the kinematics of the radiating partons, but it does affect their colour, because it always carries away some colour charge. If we want to identify the collinear contributions in the soft factorization formulae to any order in perturbation theory, we can use the following simple procedure: (i) employ the soft insertion rules \[20,11\] to obtain the usual expression

\[ S_r |M_{m+1}^{(0)}(p_r,\ldots)|^2 \propto \sum_{i=1}^{m} \sum_{k=1}^{m} \varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r) \times \frac{2p_\mu^I p_\nu^J}{s_{ir} s_{kr}} (M_{m}^{(0)}(\ldots)|T_i \cdot T_k| M_{m}^{(0)}(\ldots)) , \]

(8)

with

\[ \sum_{\text{hel.}} \varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r) = -g^{\mu\nu} + \frac{p_\mu^I n^J + p_\nu^I n^J}{p_r \cdot n} ; \]

(9)

(ii) fix the gauge vector to \( n^\mu = Q^\mu - p_\mu^Q Q^2 / s_{rQ} , s_{rQ} = 2p_r \cdot Q \) (\( Q^\mu \) is the total incoming momentum) to identify the collinear contribution in the colour-diagonal terms

\[ S_r |M_{m+1}^{(0)}(p_r,\ldots)|^2 \propto \sum_{i=1}^{m} \left[ \frac{1}{2} \sum_{k \neq i} \left( \frac{2s_{ik}}{s_{ir} s_{rk}} - \frac{2s_{iQ}}{s_{rQ} s_{ir}} - \frac{2s_{kQ}}{s_{rQ} s_{kr}} \right) \right. \]

\[ \left. \times \langle M_m^{(0)}(\ldots)|T_i \cdot T_k| M_m^{(0)}(\ldots) \rangle \right] - T_i^2 \frac{2s_{iQ}}{s_{ir} s_{rQ}} |M_m^{(0)}(\ldots)|^2 ; \]

(10)

(iii) define momentum fractions in the Sudakov parametrization of momenta \( p_i^\mu \) and \( p_k^\mu \) being collinear as \( z_i = \frac{s_{iQ}}{s_{Q} + s_{Q}} \), so that the colour-diagonal terms become equal to the collinear limit of the soft factorization formula. Then the pure soft contributions are given by

\[ S_{r}^{\text{pure}} |M_{m+1}^{(0)}(p_r,\ldots)|^2 \propto \]

\[ \sum_{i=1}^{m} \left[ \frac{1}{2} \sum_{k \neq i} \left( \frac{2s_{ik}}{s_{ir} s_{rk}} - \frac{2s_{iQ}}{s_{rQ} s_{ir}} - \frac{2s_{kQ}}{s_{rQ} s_{kr}} \right) \right. \]

\[ \left. \times \langle M_m^{(0)}(\ldots)|T_i \cdot T_k| M_m^{(0)}(\ldots) \rangle \right] . \]

(11)

We checked explicitly that this procedure leads to non-overlapping factorization formulae that describe the analytic behaviour of the squared matrix elements in any IR limit at the NNLO accuracy \[21\].

The third step is the definition of the subtraction terms. The factorization formulae in Eqs. \[11\] and \[9\] (and similar ones at NNLO) are valid in the strict limits only and have to be extended over the whole phase space. This extension requires momentum mappings \( \{ p \}_{m+1} \rightarrow \{ \tilde{p} \}_m \) that

- implement exact momentum conservation,
- lead to exact phase-space factorization,
- can be generalized to any number of unresolved partons and
- respect the (delicate) structure of cancellations among the subtraction terms.

For any such mapping we can write the exact factorization of the phase space in the following symbolic form

\[ d\phi_{m+1}(p_1,\ldots;Q) = d\phi_{m}(\tilde{p}_1,\ldots;Q)[dp_r] . \]

(12)

Then the singular integral over the momentum of the unresolved parton \( r \) can be computed independently of the jet function and the rest of the phase-space integration, leading to

\[ \int d\sigma^A = d\sigma^B \otimes I(\epsilon) , \]

(13)

where \( I(\epsilon) \) is an operator in colour space with universal pole part,

\[ I(\epsilon) \propto \frac{\alpha_s}{2\pi} \sum_i \left[ \frac{1}{\epsilon} - \frac{2}{\epsilon^2} \sum_{k \neq i} T_i \cdot T_k \left( \frac{4\pi\mu^2}{s_{ik}} \right) \right] \]

\[ + O(\epsilon^0) , \]

(14)
with $\gamma_1$ being the flavour constants defined in for instance, Ref. [2].

The fourth step is to identify the universal IR pole structure of one-loop QCD matrix elements,

$$|\mathcal{M}_m^{(1)}(\{p\})| = -\frac{1}{2} I(\epsilon)|\mathcal{M}_m^{(0)}(\{p\})| + O(\epsilon^0).$$

We observe that the poles in the insertion operator $I(\epsilon)$ are equal, but opposite in sign to the poles of the virtual correction, so that the $m$-parton integral

$$\sigma_{m}^{\text{NLO}} = \int_m \left[ d\sigma^V + \int_1 d\sigma^A \right] \epsilon = 0$$

is finite and we can take the physical limit $\epsilon \rightarrow 0$. Therefore, the sum of the two finite contributions $\sigma_{m}^{\text{NLO}}$ and $\sigma_{m+1}^{\text{NLO}}$ can be computed numerically in four dimensions and is equal to $\sigma^{\text{NNLO}}$.

### 3. A subtraction scheme at NNLO accuracy

The physical motivation for higher accuracy and the success of the subtraction schemes at NLO lead one to consider the extension of the subtraction method to NNLO, when three terms contribute: the double-real, the real-virtual and the double-virtual cross sections,

$$\sigma_{m}^{\text{NNLO}} = \sigma_{m+2}^{\text{RR}} + \sigma_{m+1}^{\text{RV}} + \sigma_{m}^{VV}$$

$$\equiv \int_{m+2} d\sigma_{m+2}^{\text{RR}} J_{m+2} + \int_{m+1} d\sigma_{m+1}^{\text{RV}} J_{m+1}$$

$$+ \int_m d\sigma_{m}^{VV} J_m.$$

The reorganization of the NNLO contributions into three finite cross sections,

$$\sigma_{m+2}^{\text{NNLO}} = \sigma_{m+2}^{\text{NNLO}} + \sigma_{m+1}^{\text{NNLO}} + \sigma_{m}^{\text{NNLO}},$$

is governed by the jet function as follows:

$$\sigma_{m+2}^{\text{NNLO}} = \int_{m+2} \left\{ d\sigma_{m+2}^{\text{RR}} J_{m+2} - d\sigma_{m+2}^{\text{RR}, A2} J_m - \left( d\sigma_{m+2}^{\text{RR}, A1} J_{m+1} - d\sigma_{m+2}^{\text{RR}, A2} J_m \right) \right\},$$

$$\sigma_{m+1}^{\text{NNLO}} = \int_{m+1} \left\{ \left( d\sigma_{m+1}^{\text{RV}} + \int_1 d\sigma_{m+2}^{\text{RR}, A1} \right) J_{m+1} - \left[ d\sigma_{m+1}^{\text{RV}, A1} + \left( \int_1 d\sigma_{m+2}^{\text{RR}, A1} \right) A_1 \right] J_m \right\},$$

$$\sigma_{m}^{\text{NNLO}} = \int_m \left\{ d\sigma_{m}^{VV} + \int_2 \left( d\sigma_{m+2}^{\text{RR}, A2} - d\sigma_{m+2}^{\text{RR}, A2} \right) \right\}$$

$$+ \int_1 \left\{ d\sigma_{m+1}^{\text{RV}, A1} + \left( \int_1 d\sigma_{m+2}^{\text{RR}, A1} \right) A_1 \right\} J_m.$$

(21)

Here we see that at NNLO accuracy one has to disentangle the overlapping singularities also among the singly- and doubly-unresolved limits. The purpose of the approximate cross section $d\sigma_{m+2}^{\text{RR}, A2}$ is to regularize the singly-unresolved limits of $d\sigma_{m+2}^{\text{RR}, A2}$ and the doubly-unresolved limits of $d\sigma_{m+2}^{\text{RR}, A2}$ simultaneously. Similarly, the approximate cross section $\left( \int_1 d\sigma_{m+2}^{\text{RR}, A1} \right) A_1$ regularizes the singly-unresolved limits of $\int_1 d\sigma_{m+2}^{\text{RR}, A1}$ and the $\epsilon$ poles of $d\sigma_{m+2}^{\text{RV}, A1}$, respectively. This puts severe constraints on the phase-space mappings needed for the definition of the subtractions.

In Ref. [23] we introduced new types of momentum mappings, one for collinear- and another for soft-type subtractions, that can easily be generalized to any order in perturbation theory. The key feature of these mappings is that in the factorized $m$-particle phase space all momenta take away the recoil instead of a single one as in the case of the dipole (or antennae) subtractions. In this way the factorization of the phase space can be done in a way that respects the (delicate) structure of cancellations among the various subtraction terms.

The complete subtraction scheme at NNLO, based on these new, fully local approximate cross sections is defined in Refs. [22,23,24]. We employed this subtraction scheme for computing the finite cross sections $\sigma_{m+2}^{\text{NNLO}}$ and $\sigma_{m+1}^{\text{NNLO}}$ of the C-parameter and thrust distributions in electron-positron annihilation. Fig. 1 shows the distributions normalized to the total cross section at $O(\alpha_s^2)$ accuracy. The computer time needed for obtaining these distributions is fairly little. The plots shown here can be obtained on a desktop computer in about 50 hours. The still missing three-parton contribution is a smooth function as compared to the four- and five-parton contributions, therefore its numerical integration does not raise any serious stability issues.
Figure 1. The five- and four-parton contributions (lower panels) to the C-parameter (left) and thrust (right) distributions. The upper panels show the distributions at the LO, NLO and incomplete (without the three-parton contribution) NNLO accuracy.

In order to have the complete physical prediction we also have to add $\sigma^{\text{NNLO}}$, which requires the integration of the subtraction terms over the singly- and doubly-unresolved factorized phase spaces.

The necessary one-particle integrals have been computed in Refs. [25,26,27]. In Ref. [25] we used standard techniques of partial fractioning, iterated sector decomposition [28,29] and residuum subtraction to find the Laurent expansion of the one-particle integrals in

$$\int \left[ d\sigma_{m+1}^{\text{RV},A_1} + \left( \int d\sigma_{m+2}^{\text{RR},A_1} \right)^{A_1} \right].$$  \hspace{1cm} (22)$$

With this technique the expansion coefficients are given as finite integrals. In order to find explicit analytic expressions, one can use the integration-by-parts technique to reduce the integrations to master integrals to be evaluated by solving differential equations [26]. An alternative solution is to employ the Mellin-Barnes technique to evaluate the expansion coefficients in terms of harmonic sums [27]. Our study shows that the latter technique is in general more efficient. We expect that the same techniques can also be employed for the computation of the coefficients in the $\epsilon$-expansion of the two-particle integral

$$\int_2 \left( d\sigma_{m+2}^{\text{RR},A_2} - d\sigma_{m+2}^{\text{RR},A_1} \right).$$  \hspace{1cm} (23)$$

This work is in progress.

In this contribution we have discussed how the same steps of setting up a subtraction scheme for computing QCD jet cross sections at NLO accuracy can be used, but need to be modified when generalizing to NNLO accuracy. We introduced these modifications such that a systematic perturbative expansion can in principle be given at any order of perturbation theory. The explicit computations of course become rather cumbersome already at NNLO.

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