Higher Nash blowup on normal toric varieties

Daniel Duarte*

Abstract

The higher Nash blowup of an algebraic variety replaces singular points with limits of certain spaces carrying higher order data associated to the variety at non-singular points. In the case of normal toric varieties we give a combinatorial description of the higher Nash blowup in terms of a Gröbner fan. This description will allow us to prove the analogue of Nobile’s theorem on the usual Nash blowup in this context. More precisely, we prove that for a normal toric variety, the higher Nash blowup is an isomorphism if and only if the variety is non-singular.

Introduction

The classical Nash blowup is a natural modification of an algebraic variety that replaces singular points by limits of tangent spaces at non-singular points. Recently, Takehiko Yasuda in [Y] has generalized this construction by considering not only first-order data, as with the tangent space, but also higher-order one. In his construction, instead of tangent spaces, the author considers nth infinitesimal neighborhoods of non-singular points. Then one replaces singular points by limits of these infinitesimal neighborhoods at non-singular points. The resulting variety is called higher Nash blowup of order n and is denoted by $\text{Nash}_n(X)$. Yasuda then conjectures that for $n \gg 0$, $\text{Nash}_n(X)$ is non-singular ([Y], Conjecture 0.2). If the conjecture is true, this process would give resolution of singularities in one step.

In this paper we consider Yasuda’s higher Nash blowup in the case of normal toric varieties. Let $\sigma \subset \mathbb{R}^d$ be a strictly convex rational polyhedral cone, $X$ the associated normal toric variety, and $\overline{\text{Nash}_n(X)}$ the normalization of $\text{Nash}_n(X)$. To begin with, we will see that $\overline{\text{Nash}_n(X)}$ has a natural structure of toric variety and so it is defined by some fan. Our first result shows that this fan can be identified with the Gröbner fan of the ideal $J_n = (x^a_1 - 1, \ldots, x^a_s - 1)^{n+1} \subset k[x^{a_1}, \ldots, x^{a_s}] = k[\hat{\sigma} \cap \mathbb{Z}^d]$ (Theorem 2.5). This will be done essentially by comparing the action of the torus on the distinguished point of the dense orbit of $\overline{\text{Nash}_n(X)}$ and the induced action on

*Research supported by CONACYT (México)
the ideal $J_n$. By taking suitable limits, the same action will give us the distinguished points of orbits in $\overline{\text{Nash}_n(X)}$ and initial ideals of $J_n$.

The idea of comparing the fan defining $\overline{\text{Nash}_n(X)}$ with a Gröbner fan is inspired by a similar idea that appears in another paper of Yasuda in which the author defines a variant of $\text{Nash}_n(X)$ in positive characteristic. In the case of toric varieties, the author proves, using similar arguments, that this variant is determined by a Gröbner fan ([YT], Proposition 3.5). We also mention that a much more explicit combinatorial description of the usual Nash blowup of toric varieties has been recently given by P. González and B. Teissier in [GT] and by D. Grigoriev and P. Milman in [GM].

Later in the paper, we will study an analogue of the following well-known theorem of A. Nobile ([No]): In characteristic zero, the Nash blowup of a variety $X$ is an isomorphism if and only if $X$ is non-singular. One can naturally ask if this theorem also holds for the higher Nash blowup. We answer this question affirmatively when $X$ is a normal toric variety (Corollary 3.8). Using the description of $\overline{\text{Nash}_n(X)}$ in terms of a Gröbner fan, the problem can be reduced to showing that this fan is a non-trivial subdivision of the cone, say $\sigma$, defining $X$. By general results on the Gröbner fan, this is equivalent to showing that there exists an element of some reduced Gröbner basis with the property that its initial part with respect to some $w \in \sigma$ changes as we vary $w$ in $\sigma$.

The paper is organized as follows. In the first section we recall the basics of Gröbner bases but in a slightly more general setting: instead of a polynomial ring we will consider a monomial subalgebra of the polynomial ring. We give the definition of a Gröbner basis and Gröbner fan in this context and prove some of their basic properties. Then, in the second and third section we prove, respectively, the description of $\overline{\text{Nash}_n(X)}$ for normal toric varieties in terms of a Gröbner fan and the analogue of Nobile’s theorem for normal toric varieties.

Finally, I want to thank Mark Spivakovsky for his encouragement, his guidance, and his comments regarding this paper. He read previous versions of this manuscript and detected several mistakes. I also want to thank Takehiko Yasuda for having kindly answered several questions I had on the higher Nash blowup. Some ideas presented here started from discussions with him.

1 Gröbner fan of ideals in monomial subalgebras

In this section we want to consider an intrinsic theory of Gröbner bases of ideals in monomial subalgebras of the polynomial ring. It can be verified that the basic theory of Gröbner bases (up to the existence and uniqueness of a reduced Gröbner
Finally, for all monomial order. Then every non-zero ideal $I \supseteq A$ is contained in an ideal $\mathfrak{a}$ in the usual way. For instance, any monomial order on $k[x_1, \ldots, x_d]$ restricts to a monomial order on $k[x_1^a, \ldots, x_d^a]$. However, the converse is not true.

**Example 1.1.** Consider the subalgebra $k[x, y] \subset k[x, y]$. Let $w = (\sqrt{3}, -1)$. Define a monomial order $\succ$ on the monomials of $k[x, y]$ as follows:

$$x^a y^b \succ x^c y^d \iff w \cdot (a, b) > w \cdot (c, d).$$

Suppose $\succ$ extends to a monomial order $\succ'$ on $k[x, y]$. Since, by definition, every monomial $x^a y^b \in k[x, y]$ must satisfy $x^a y^b \succ' 1$, then, in particular, we must have $y \succ' 1$. But then $x \cdot y \succ' 1 \cdot x = x$. Since $xy$ and $x$ are monomials on $k[x, xy]$ we should have $xy \succ x$, which is clearly not true. Therefore, the monomial order $\succ$ cannot be extended to $k[x, y]$.

Let $\succ$ be a monomial order on $k[x^a, \ldots, x^a]$, $f = \sum_{i=1}^{r} \lambda_i x^{\beta_i}$ be a nonzero polynomial in $k[x^a, \ldots, x^a]$, where $\beta_1 > \beta_2 > \cdots > \beta_r$. Define $\text{lc}(f) := \lambda_{\beta_1}$, the leading coefficient of $f$; $\text{lt}(f) := \lambda_{\beta_1} x^{\beta_1}$, the initial form or leading term of $f$. In addition, we define $\text{lm}(0) = \text{lc}(0) = \text{lt}(0) = 0$. Finally, for $S \subset k[x^a, \ldots, x^a]$, we define the initial ideal of $S$, denoted $\text{in}_>(S)$, to be the ideal generated (in $k[x^a, \ldots, x^a]$) by the leading terms of elements of $S$ with respect to $\succ$.

**Definition 1.2.** Fix a monomial order. A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ contained in an ideal $I \subset k[x^a, \ldots, x^a]$, is called a Gröbner basis for $I$ if for each $f \in I \setminus \{0\}$, there exists $i \in \{1, \ldots, t\}$ such that $\text{lm}(g_i)$ divides $\text{lm}(f)$ in $k[x^a, \ldots, x^a]$.

**Definition 1.3.** A Gröbner basis $G = \{g_1, \ldots, g_t\}$ is called reduced if $\text{lc}(g_i) = 1$ for all $i$, and no non-zero monomial of $g_i$ is divisible by any $\text{lt}(g_j)$ for $j \neq i$.

**Theorem 1.4.** Fix a monomial order. Then every non-zero ideal $I$ has a unique reduced Gröbner basis with respect to this monomial order.

**Proof.** As we said before, this can be proved in exactly the same way as in the polynomial ring case (cf. [AL], Chapter 1).
1.2 Gröbner fan

The Gröbner fan of an ideal in $k[x_1, \ldots, x_d]$ is a subdivision of $\mathbb{R}^d_{\geq 0}$ (see [M], Ch. 2, Def. 2.4.10). Since we want to deal with monomial subalgebras, we will need to consider subdivisions of a little more general cone in $\mathbb{R}^d$. In this section we give the analogous definition of Gröbner fan of an ideal in $k[x^{a_1}, \ldots, x^{a_s}]$. As before, to prove that this Gröbner fan is indeed a fan, we can reproduce, word by word, the proof of the polynomial ring case.

Let $\bar{\sigma} := \mathbb{R}_{\geq 0}(a_1, \ldots, a_s) \subset \mathbb{R}^d_{\geq 0}$ be the cone generated by $a_1, \ldots, a_s$, and let $\sigma \subset \mathbb{R}^d_{\geq 0}$ be its dual cone. Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[x^{a_1}, \ldots, x^{a_s}]$. Define the initial form $\text{in}_w(f)$ as the sum of terms $c_u x^u$ in $f$ with $w \cdot u$ maximized. The initial ideal of $I$ with respect to $w$ is defined as $\text{in}_w(I) := \langle \text{in}_w(f) | f \in I \rangle$.

**Proposition 1.5.** Let $I$ be an ideal in $k[x^{a_1}, \ldots, x^{a_s}]$, let $w \in \sigma$ and consider

$$C[w] := \{ w' \in \sigma | \text{in}_w(I) = \text{in}_{w'}(I) \}.$$  

Then $C[w]$ is the relative interior of a polyhedral cone inside $\sigma$.

**Proof.** As in the polynomial ring case, it can be checked that

$$C[w] = \{ w' \in \sigma | \text{in}_{w'}(g) = \text{in}_w(g), \text{ for all } g \in G \}, \quad (1)$$

where $G = \{ g_1, \ldots, g_r \}$ is the reduced Gröbner basis of $I$ with respect to $>_w$, where $>$ is any monomial order on $k[x^{a_1}, \ldots, x^{a_s}]$. For $g_i \in G$, write $g_i = \sum_j c_{ij} x^{a_{ij}} + \sum_j c'_{ij} x^{b_{ij}}$, where $\text{in}_w(g_i) = \sum_j c_{ij} x^{a_{ij}}$. The proposition then follows because the right-hand side set of (1) equals

$$\{ w' \in \sigma | w' \cdot a_{ij} = w' \cdot a_{ik}, w' \cdot a_{ij} > w' \cdot b_{ik} \text{ for } i = 1, \ldots, r, \text{ and all } j, k \}.$$  

This is the relative interior of a polyhedral cone by definition. See [St], Ch. 2, Prop. 2.3, or [M], Ch. 2, Prop. 2.4.6 for details.

**Proposition 1.6.** Let $C[w]$ be the closure of $C[w]$ in $\mathbb{R}^d$. Then the set $\text{GF}(I) := \{ C[w] | w \in \sigma \}$ forms a polyhedral fan.

**Proof.** See [St], Ch. 2, Prop. 2.4, or [M], Ch. 2, Prop. 2.4.9.

**Definition 1.7.** The set $\text{GF}(I)$ is called the Gröbner fan of $I$.

In order to compute examples of Gröbner bases of ideals in $k[x^{a_1}, \ldots, x^{a_s}]$, we use the so-called extrinsic algorithm for computing intrinsic Gröbner bases (see [St], Algorithm 11.24).
Example 1.8. Let \( J = \langle xy + x, x^3y^3 + x^2y^3 \rangle \subset k[x, xy, x^2y^3] \). Let \( > \) be the lexicographic order. Let \( w = (1, 1) \). Implementing the extrinsic algorithm in SINGULAR 3-1-6, we obtain the following reduced Gröbner basis with respect to \( >_w \) (the leading terms are listed first): \( \{ xy + x, x^4 + x^3, x^2y^3 - x^3 \} \). Therefore (see prop. 1.5), \( C[(1,1)] = \{ (p,q) \in \sigma | q > 0, p > 0, 3q > p \} \). Similarly,
\[
C[(4,1)] = \{ (p,q) \in \sigma | q > 0, p > 0, p > 3q, 2p + 3q > 0 \},
\]
\[
C[(2,-1)] = \{ (p,q) \in \sigma | 0 > q, 2p + 3q > 0, p > 0 \}.
\]
The resulting fan is shown in figure 1.

![Gröbner fan of J.](image)

2 Higher Nash blowup of toric varieties

In this section we introduce the notion of higher Nash blowup, defined by Takehiko Yasuda in [Y]. We state some basic properties and related results. Then we prove that for normal toric varieties the higher Nash blowup is determined by the Gröbner fan of a certain ideal.

The definition of the usual Nash blowup goes as follows (see [No]):

**Definition 2.1.** Let \( X \subset \mathbb{C}^m \) be an algebraic variety of pure dimension \( d \). Consider the Gauss map:
\[
G : X \setminus \text{Sing}(X) \to G(d,m) \quad x \mapsto T_xX,
\]
where $G(d, m)$ is the Grassmanian parameterizing the $d$-dimensional vector spaces in $\mathbb{C}^m$, and $T_x X$ is the direction of the tangent space to $X$ at $x$. Denote by $X^*$ the Zariski closure of the graph of $G$. Call $\nu$ the restriction to $X^*$ of the projection of $X \times G(d, m)$ to $X$. The pair $(X^*, \nu)$ is called the Nash blowup of $X$.

We can directly generalize this definition as follows (see \cite{OZ}, Section 1). Let us consider an irreducible algebraic variety $X \subset \mathbb{C}^m$. Let $R$ be the ring of regular functions of $X$. Consider the ideal $I = \ker(R \otimes R \to R)$, where $r \otimes r' \mapsto rr'$. We see $I$ as an $R$–module via the map $R \to R \otimes R$, $r \mapsto r \otimes 1$. For any $x \in X$, let $(R_x, \mathfrak{m}_x)$ be the localization of $R$ in $x$. Consider the following $\mathbb{C} \cong R_x/\mathfrak{m}_x$–vector space:

$$T^n_x X := (I_x/I_x^{n+1} \otimes \mathbb{C})^\vee.$$ 

This is a vector space of dimension $N = \binom{d+n}{d} - 1$ whenever $x$ is a non-singular point. Since $X \subset \mathbb{C}^m$ we have that $T^n_x X \subset T^n_x \mathbb{C}^m \cong \mathbb{C}^M$ where $M = \binom{m+n}{m} - 1$, that is, we can see $T^n_x X$ as an element of the grassmanian $G(N, M)$. Now consider the Gauss map:

$$G_n : X \setminus \text{Sing}(X) \to G(N, M) \quad x \mapsto T^n_x X.$$ 

Denote by $X_n$ the Zariski closure of the graph of $G_n$. Call $\nu$ the restriction to $X_n$ of the projection of $X \times G(N, M)$ to $X$. The pair $(X_n, \nu)$ is called the Nash blowup of $X$ relative to $I/I^{n+1}$ (this is a special case of a more general construction appearing in \cite{OZ}). Viewed like this, it is clear that for $n = 1$ this is exactly the usual Nash blowup of $X$ (in this case, $T^1_x X = T_x X$, according to \cite{H}, Proposition 8.7).

This notion of Nash blowup of $X$ relative to $I/I^{n+1}$ is equivalent to the definition of higher Nash blowup given by Yasuda (\cite{Y}, Proposition 1.8). The main difference between these constructions is that Yasuda replaces the Grassmanian by a different parameter space of the variety: the Hilbert scheme of points.

### 2.1 Higher Nash blowup

Let $X := \text{Spec } R$, where $R = k[y_1, \ldots, y_s]/I$, $I$ is a prime ideal, and $k$ is an algebraically closed field of characteristic zero. Consider $x \in X$ a $k$–point and let $\mathfrak{m}$ be its corresponding maximal ideal in $R$. Let $d = \dim X$. Let $x^{(n)} := \text{Spec } (R/\mathfrak{m}^{n+1})$ be the $n$th infinitesimal neighborhood of $x$. If $X$ is smooth at $x$, then $x^{(n)}$ is a closed subscheme of $X$ of length $N = \binom{d+n}{d}$ (i.e., $R/\mathfrak{m}^{n+1}$ has length $N$ as an $R$–module). Therefore, it corresponds to a point

$$[x^{(n)}] \in \text{Hilb}_N(X),$$

where $\text{Hilb}_N(X)$ is the Hilbert scheme of $N$ points of $X$ (see \cite{Na}, Definition 1.2). If $X_{sm}$ denotes the smooth locus of $X$, then we have a map

$$\delta_n : X_{sm} \to \text{Hilb}_N(X), \quad x \mapsto [x^{(n)}].$$
Definition 2.2. ([Y], Definition 1.2) We define the $n$th Nash blowup of $X$, denoted by $\text{Nash}_n(X)$, to be the closure of the graph of $\delta_n$ with reduced scheme structure in $X \times_k \text{Hilb}_N(X)$. By restricting the projection $X \times_k \text{Hilb}_N(X) \to X$ we obtain a map

$$\pi_n : \text{Nash}_n(X) \to X.$$ 

This map is projective, birational, and it is an isomorphism over $X_{\text{sm}}$. In addition, $\text{Nash}_1(X)$ is canonically isomorphic to the classical Nash blowup of $X$ (see [Y], Section 1).

In [Y], the author conjectures that for $n$ big enough, the $n$th Nash blowup of $X$ is non-singular (see [Y], Conjecture 0.2). If the conjecture is true, this construction would give a one-step resolution of singularities. In the same paper, the author proves that the conjecture is true for curves:

Theorem 2.3. ([Y], Corollary 3.7) Let $X$ be a variety of dimension 1. Then for $n$ big enough, $\text{Nash}_n(X)$ is non-singular.

For varieties of higher dimension the answer remains unknown, even though Yasuda has stated that the $A_3$-singularity is probably a counterexample to his conjecture (see [Y1], Remark 1.5).

2.2 Normalization of the higher Nash blowup of a normal toric variety

Let $\sigma \subset \mathbb{R}^d$ be a strictly convex rational polyhedral cone of dimension $d$. Let $k[A] := k[\sigma \cap \mathbb{Z}^d] = k[x^{a_1}, \ldots, x^{a_s}]$. After suitable change of coordinates, we can assume that $k[x^{a_1}, \ldots, x^{a_s}] \subset k[x_1, \ldots, x_d]$. Let $X := \text{Spec } k[A]$ be the corresponding $d$-dimensional normal toric variety with torus $T \subset X$. Since the torus $T$ is dense in $X$ we first remark that

$$\text{Nash}_n(X) = \{(x, \delta_n(x))| x \in X_{\text{sm}}\} = \{(x, \delta_n(x))| x \in T\}.$$ 

In addition, $T \cong \pi_n^{-1}(T) = \{(x, \delta_n(x))| x \in T\} = \{(x, [x^{(n)}])| x \in T\}$, i.e., $\text{Nash}_n(X)$ contains an open set isomorphic to the torus $T$. The action of $T$ on $X$ induces the following action of $T$ on $\text{Nash}_n(X)$:

$$T \times \text{Nash}_n(X) \to \text{Nash}_n(X), \quad (t, (x, [Z])) \mapsto (t \cdot x, [t \cdot Z]).$$

Over points $x \in T$, i.e., $(x, [x^{(n)}]) \in \pi_n^{-1}(T)$, this action looks like:

$$T \times \pi_n^{-1}(T) \to \pi_n^{-1}(T), \quad (t, (x, [x^{(n)}])) \mapsto (t \cdot x, [(t \cdot x)^{(n)}]).$$

Since this action extends the action of the torus $T \cong \pi_n^{-1}(T)$ over itself, we obtain:
Proposition 2.4. Let $X$ be an affine normal toric variety. Then, for all $n \in \mathbb{N}$, $\text{Nash}_n(X)$ is a toric variety with dense torus $\pi_n^{-1}(\mathbb{T}) \cong \mathbb{T}$.

This proposition is our starting point. Now we want to give a description of $\text{Nash}_n(X)$ in terms of fans and cones. Since $\text{Nash}_n(X)$ may not be a normal variety, we do not have such a description automatically (see [GT], Section 7). Therefore we consider its normalization:

$$\eta : \overline{\text{Nash}_n(X)} \to \text{Nash}_n(X).$$

Let $U := \eta^{-1}(\mathbb{T})$, which is dense since $\overline{\text{Nash}_n(X)}$ is irreducible. Moreover, since $\mathbb{T}$ is contained in the normal locus of $\text{Nash}_n(X)$, we have that $U$ is isomorphic to $\mathbb{T}$. The action of $\mathbb{T}$ on $\text{Nash}_n(X)$ induces the following action of $\mathbb{T}$ on $U$:

$$\mathbb{T} \times U \to U, \quad (t, \eta^{-1}(x, [x^{(n)}])) \mapsto \eta^{-1}(t \cdot x, [(t \cdot x)^{(n)}]).$$

Since this action commutes with the normalization map restricted to $U$ then, by [Se], Lemma 6.1, there is a unique action of $\mathbb{T}$ on $\overline{\text{Nash}_n(X)}$ extending the action on $U$ and such that it commutes with $\eta$. This implies that $\overline{\text{Nash}_n(X)}$ is a (normal) toric variety with torus $U \cong \mathbb{T}$.

Now, since $\overline{\text{Nash}_n(X)}$ is a normal toric variety, there exists a fan $\Sigma \subset N_{\mathbb{R}}$, where $N$ is a lattice of rank $d$, such that its associated normal toric variety is isomorphic to $\overline{\text{Nash}_n(X)}$. The composition $\pi_n \circ \eta : \overline{\text{Nash}_n(X)} \to X$ is a morphism of toric varieties that sends the torus $U \subset \overline{\text{Nash}_n(X)}$ to the torus $\mathbb{T} \subset X$ in such a way that this restriction is a homomorphism of groups. Thus it is a toric morphism. By [O], Theorem 1.13, there exists a morphism of lattices $\phi : N \to \mathbb{Z}^d$ compatible with $\Sigma$ and $\sigma$, and such that the induced morphism on the toric varieties is $\pi_n \circ \eta$. On the other hand, since the normalization map is proper and birational we have that the composition $\pi_n \circ \eta$ is a proper birational map of normal toric varieties. This implies that $\phi$ is an isomorphism and $\sigma = \bigcup_{\tau \in \Sigma} \phi_{\mathbb{R}}(\tau)$, where $\phi_{\mathbb{R}} : N \otimes \mathbb{R} \to \mathbb{Z}^d \otimes \mathbb{R}$ is the tensor of $\phi$ and $\mathbb{R}$ (see [O], Chapter 1, Corollary 1.17). Because of this, we can assume that $N = \mathbb{Z}^d$, $\phi$ is the identity, and $\Sigma$ is a refinement of $\sigma$.

Let $1 = (1, \ldots, 1)$ be the distinguished point of the dense torus $\mathbb{T} \hookrightarrow X$ (see [CLS], Chapter 3, Section 2, for the definition of distinguished point and its basic properties). Since $\mathbb{T} \cong \pi_n^{-1}(\mathbb{T}) \cong \eta^{-1}(\pi_n^{-1}(\mathbb{T}))$, and since the action of $\mathbb{T}$ on $\overline{\text{Nash}_n(X)}$ is induced by that of $\mathbb{T}$ on $X$, we have that $\eta^{-1}(1, 1^{(n)})$ is the distinguished point of the dense torus $\eta^{-1}(\pi_n^{-1}(\mathbb{T})) \subset \overline{\text{Nash}_n(X)}$.

Consider $w \in \sigma$, and $f = \sum c_u x^u \in k[A]$. Let $d(f) := \text{max}\{w \cdot u|c_u \neq 0\}$. Define

$$f_t := t^{d(f)} f(t^{-w_1} x_1, \ldots, t^{-w_d} x_d) = t^{d(f)} f(t^{-w_1} x^{a_1}, \ldots, t^{-w_d} x^{a_d}).$$

8
Then we have \( f_t = \text{in}_w(f) + t \cdot f' \), for some \( f' \in k[A][t] \). Let \( I_t := \langle f_t | f \in I \rangle \) be the ideal in \( k[A][t] \) generated by the \( f_t \).

Let \( w \in \sigma \) and consider the one-parameter subgroup \( \lambda_w : k^* \to (k^*)^d \), \( t \mapsto t^w = (t^{w_1}, \ldots, t^{w_d}) \). Then, for any \( t \in k^* \),

\[
\lambda_w(t) \cdot (1, 1^{(n)}) = (\lambda_w(t) \cdot 1, (\lambda_w(t) \cdot 1)^{(n)}) \\
= \left( (t^{w_{a_1}}, \ldots, t^{w_{a_s}}), \text{Spec} \frac{k[A]}{\langle x^{a_1} - t^{w_{a_1}}, \ldots, x^{a_s} - t^{w_{a_s}} \rangle_{n+1}} \right) \\
= \left( (t^{w_{a_1}}, \ldots, t^{w_{a_s}}), \text{Spec} \frac{k[A]}{\langle t^{-w_{a_1}}x^{a_1} - 1, \ldots, t^{-w_{a_s}}x^{a_s} - 1 \rangle_{n+1}} \right)
\]

It can be proved by induction on \( n \) that in the ring \( k[A][t, t^{-1}] \) we have:

\[
(t^{-w_{a_1}}x^{a_1} - 1, \ldots, t^{-w_{a_s}}x^{a_s} - 1)^{n+1} = (\langle x^{a_1} - 1, \ldots, x^{a_s} - 1 \rangle_{n+1})_t.
\]

In particular,

\[
\lambda_w(t) \cdot (1, 1^{(n)}) = \left( (t^{w_{a_1}}, \ldots, t^{w_{a_s}}), \text{Spec} \frac{k[A]}{\langle x^{a_1} - 1, \ldots, x^{a_s} - 1 \rangle_{n+1}} \right).
\]

Let \( J_n := \langle x^{a_1} - 1, \ldots, x^{a_s} - 1 \rangle_{n+1} \). According to [E], Theorem 15.17 (this theorem is stated for the polynomial ring but the same proof works if we replace polynomial ring by monomial subalgebra), we obtain:

\[
\lim_{t \to 0}(\lambda_w(t) \cdot (1, 1^{(n)})) = \left( \lim_{t \to 0}(\lambda_w(t) \cdot 1), \text{Spec} \frac{k[A]}{\langle x^{a_1} - 1, \ldots, x^{a_s} - 1 \rangle_{n+1}} \right).
\]

**Remark 2.5.** The notation we use for the limits of one-parameter subgroups is not standard. Usually the limit is denoted just as \( \lim_{t \to 0} \lambda_w(t) \). Since we will be taking these limits at different levels \((X, \text{Nash}_n(X), \text{Nash}_0(X))\) we need to modify the standard notation in order to distinguish in which toric variety we are working on.

**Proposition 2.6.** Let \( X = \text{Spec} k[A] \) be the normal toric variety associated to the cone \( \sigma \). Let \( \Sigma \) be the fan associated to the normalization of \( \text{Nash}_n(X) \) and let \( GF(J_n) \) be the Gröbner fan of \( J_n \). Then \( \Sigma \) is a refinement of \( GF(J_n) \). In particular, there exists a surjective morphism of normal toric varieties

\[
\text{Nash}_n(X) \xrightarrow{\phi} X_{GF(J_n)}.
\]

**Proof.** To begin with, recall that the support of both \( \Sigma \) and \( GF(J_n) \) is \( \sigma \). Let \( w \) be in the relative interior of \( \sigma_1 \), where \( \sigma_1 \) is a cone of \( \Sigma \) different from \{0\}. Then there exists a unique cone \( \sigma_2 \) of \( GF(J_n) \) such that \( w \) belongs to the relative interior of \( \sigma_2 \).
Denote by $\gamma_{\sigma_1}$ the distinguished point of $\sigma_1$ in $\overline{\text{Nash}_n(X)}$. Now let $w' \neq w$ be in the relative interior of $\sigma_1$. By [CLS], Proposition 3.2.2, we have

$$\lim_{t \to 0}(\lambda_w(t) \cdot \eta^{-1}((1, 1^{(n)}))) = \gamma_{\sigma_1} = \lim_{t \to 0}(\lambda_{w'}(t) \cdot \eta^{-1}((1, 1^{(n)}))) = (3)$$

By definition, $\lambda_w(t) \cdot \eta^{-1}((1, 1^{(n)})) = \eta^{-1}((\lambda_w(t) \cdot 1, (\lambda_w(t) \cdot 1)^{(n)}))$. But now, since $\eta$ is a continuous map,

$$\eta(\lim_{t \to 0}(\lambda_w(t) \cdot \eta^{-1}((1, 1^{(n)})))) = \lim_{t \to 0}(\eta(\lambda_w(t) \cdot \eta^{-1}((1, 1^{(n)}))))$$

$$= \lim_{t \to 0}(\eta(\eta^{-1}((\lambda_w(t) \cdot 1, (\lambda_w(t) \cdot 1)^{(n)}))))$$

$$= \lim_{t \to 0}(\lambda_w(t) \cdot 1, (\lambda_w(t) \cdot 1)^{(n)})$$

Similarly, $\eta(\lim_{t \to 0}(\lambda_{w'}(t) \cdot \eta^{-1}((1, 1^{(n)})))) = \lim_{t \to 0}(\lambda_{w'}(t) \cdot (1, 1^{(n)}))$. Thus, by (1) and (2), Spec $k[A]/in_w(J_n) = \text{Spec } k[A]/in_{w'}(J_n)$. This is an equality of closed subschemes of Spec $k[A]$ according to [E], Theorem 15.17. This implies that $in_w(J_n) = in_{w'}(J_n)$, i.e., $w'$ belongs to the relative interior of $\sigma_2$. Therefore $\sigma_1 \subset \sigma_2$. Since $\Sigma$ and $GF(J_n)$ have the same support, we conclude that $\Sigma$ is a refinement of $GF(J_n)$.

\begin{remark}
Notice that in the previous proof we cannot give a similar argument to show that $GF(J_n)$ is a refinement of $\Sigma$ since the normalization map may fail to be $1$-$1$ over the non-normal locus. More precisely, let $\{r_i\}, \{s_i\}$ be two sequences in $\pi_n^{-1}(\mathbb{T}) \subset \text{Nash}_n(X)$ such that $\lim r_i = l = \lim s_i$, where $l \in \text{Nash}_n(X) \setminus \pi_n^{-1}(\mathbb{T})$. Then it may happen that $\lim \eta^{-1}(r_i) \neq \lim \eta^{-1}(s_i)$.
\end{remark}

Now we have the following two morphisms:

$$\begin{array}{ccc}
\text{Nash}_n(X) & \xrightarrow{\phi} & X_{GF(J_n)} \\
\downarrow & & \downarrow \eta \\
\text{Nash}_n(X) & & \\
\end{array}$$

The normalization map is a finite morphism. If we could give a morphism $\psi : X_{GF(J_n)} \to \text{Nash}_n(X)$ such that $\eta = \psi \circ \phi$, then, since $\phi$ is surjective, both morphisms $\psi$ and $\phi$ must also be finite. Since $X_{GF(J_n)}$ and $\text{Nash}_n(X)$ are normal varieties this would imply $X_{GF(J_n)} \cong \overline{\text{Nash}_n(X)}$ (the normalization of any variety is unique). In what follows, we will try to define such a morphism $\psi$ by giving a map of sets $X_{GF(J_n)} \to \text{Nash}_n(X)$ extending the existing birational map between them (which is given by the torus). Since $X_{GF(J_n)}$ is normal, this map of sets is
actually a morphism of varieties (this is a consequence of one version of Zariski’s Main Theorem, see [H], Theorem 5.2).

To begin with, let us recall the construction of the map \( \phi \), which is obtained as the induced morphism of the identity on the lattice \( \mathbb{Z}^d \) (the identity is compatible with the fans \( \Sigma \) and \( GF(J_n) \)). For any \( \sigma \in \Sigma \), there is a cone \( \sigma' \in GF(J_n) \) containing \( \sigma \), and so there is a toric morphism \( \phi_i : X_\sigma \to X_{\sigma'} \), where \( X_\sigma \) and \( X_{\sigma'} \) are the affine toric varieties corresponding to \( \sigma \) and \( \sigma' \). These maps glue together to give the morphism \( \phi \) (see [CLS], Theorem 3.3.4). Moreover, for any cone \( \sigma' \in GF(J_n) \) that is not subdivided in \( \Sigma \) (i.e., a cone that appears in both fans), the corresponding morphism \( \phi_i : X_{\sigma'} \to X_{\sigma'} \) is an isomorphism.

We want to define a map \( \psi : X_{GF(J_n)} \to Nash_n(X) \) making the following diagram commutative:

\[
\begin{array}{ccc}
Nash_n(X) & \xrightarrow{\phi} & X_{GF(J_n)} \\
\downarrow{\eta} & & \downarrow{\psi} \\
Nash_n(X) & &
\end{array}
\]

We already have a morphism from the torus \( T \subset X_{GF(J_n)} \) to the torus \( T \subset Nash_n(X) \) (over these sets, \( \eta \) and \( \phi \) are isomorphisms). Since \( X_{GF(J_n)} \) is a normal toric variety and since \( \phi \) and \( \eta \) are equivariant, it suffices to define the map for the distinguished points of every cone in \( GF(J_n) \). Let \( \tau \in GF(J_n) \) be a cone that is subdivided in \( \Sigma \). Let us denote by \( \sigma_1, \ldots, \sigma_r \) the cones of \( \Sigma \) such that \( \bigcup_{i=1}^r \sigma_i = \tau \).

Moreover, the relative interior of every \( \sigma_i \) is contained in the relative interior of \( \tau \). Let \( \gamma_\tau \) denote the distinguished point of \( \tau \) in \( X_{GF(J_n)} \). According to [CLS], Lemma 3.3.21, the following is true:

(a) \( \phi(\gamma_\sigma) = \gamma_\tau \), for all \( i = 1, \ldots, r \);

(b) \( \phi(O(\sigma_i)) \subseteq O(\tau) \), where \( O(\cdot) \) denotes the orbit corresponding to a cone.

Let us define \( \psi(\gamma_\tau) := \eta(\gamma_{\sigma_1}) \) (the choice of \( \sigma_1 \) is arbitrary, we could use any of the \( \sigma_i \)). To see that this definition makes the diagram above commutative, we need to verify that \( \eta(\gamma_{\sigma_i}) = \eta(\gamma_{\sigma_1}) \) for any \( \gamma_{\sigma_i} \). For this we are going to use once again the characterization of distinguished points as limits of one-parameter subgroups. Let
$w_i$ be in the relative interior of $\sigma_i$. By [CLS], Proposition 3.2.2, we have:

$$\lim_{t \to 0} \lambda_{w_i}(t) \cdot \eta^{-1}((1, 1^{(n)})) = \gamma_{\sigma_i}.$$ 

Now, since $\eta$ is a continuous map,

$$\eta(\gamma_{\sigma_i}) = \eta(\lim_{t \to 0} (\lambda_{w_i}(t) \cdot \eta^{-1}((1, 1^{(n)}))))
= \lim_{t \to 0} (\eta(\lambda_{w_i}(t) \cdot \eta^{-1}((1, 1^{(n)}))))
= \lim_{t \to 0} (\eta(\eta^{-1}((\lambda_{w_i}(t) \cdot 1, (\lambda_{w_i}(t) \cdot 1)^{(n)}))))
= \lim_{t \to 0} (\lambda_{w_i}(t) \cdot 1, (\lambda_{w_i}(t) \cdot 1)^{(n)})
= \left(\lim_{t \to 0} (\lambda_{w_i}(t) \cdot 1), \Spec \frac{k[A]}{in_{w_i}(J_n)}\right).$$

On the other hand, every $w_i$ is also contained in the relative interior of $\tau$. This implies, by definition of Gröbner fan, that $\text{in}_{w_i}(J_n) = \text{in}_{w_j}(J_n)$ for all $1 \leq i, j \leq r$. Consequently, $\eta(\gamma_{\sigma_i}) = \eta(\gamma_{\sigma_1})$ for all $i$.

Therefore, by defining $\psi(t \cdot \gamma_\tau) := t \cdot \eta(\gamma_{\sigma_1})$, and using facts (a) and (b) above, we obtain a morphism $\psi : X_{GF(\sigma)} \to \text{Nash}_n(X)$ making the diagram above commutative. With this, as we said before, the morphism $\phi$ must be an isomorphism. We have proved the following theorem:

**Theorem 2.8.** Let $X = \Spec k[A]$ be the normal toric variety associated to the cone $\sigma$. Let $\Sigma$ be the fan associated to the normalization of $\text{Nash}_n(X)$ and let $GF(\sigma)$ be the Gröbner fan of $J_n$. Then $\Sigma = GF(\sigma)$.

Let us illustrate the previous theorem by an example. Let $\sigma$ be the cone generated by $(0, 1)$ and $(4, -3)$. Then $\sigma \cap \mathbb{Z}^2$ is generated by $\{(1, 0), (1, 1), (3, 4)\}$. Let $X = \Spec \mathbb{C}[\sigma \cap \mathbb{Z}^2] = \Spec \mathbb{C}[x, y, z]/(xy - z^4)$. It is well known that $\text{Nash}_1(X)$ is the blowup of the ideal $\langle x, y, z^3 \rangle$ (see [No], Theorem 1). According to [GM], Section 4.3, the fan corresponding to $\text{Nash}_1(X)$ is the fan in figure 2.

Let us compare this fan with the Gröbner fan of the ideal $J_1 = \langle u - 1, uv^3 - 1, uv - 1 \rangle^2 \subset \mathbb{C}[u, v, w]$. Let us consider the following vectors: $w_1 = (1, 0)$, $w_2 = (3, -2)$. Implementing the so-called extrinsic algorithm for computing intrinsic Gröbner bases (see [ST], Algorithm 11.24) in **SINGULAR** 3-1-6, we find that the reduced Gröbner bases of $J_1$ with respect to $w_1$ and $w_2$ are, respectively,

$$\{u^2 v^2 - 2uv + 1, u^2 v - u - uv + 1, u^2 - 2u + 1, u^3 v^4 + u - 4uv + 2\},$$
$$\{u^2 v^2 - 2uv + 1, u^4 v^5 - u^3 v^4 - uv + 1, u^6 v^8 - 2u^3 v^4 + 1, u + u^3 v^4 - 4uv + 2\}.$$
As in the proof of proposition 1.5 we obtain the following open cones:

\[ C[w_1] = \{(a, b) \in \sigma | a + b > 0, a > 0, a + 2b > 0, 2a + 3b > 0, 3a + 4b > 0\}, \]

\[ C[w_2] = \{(a, b) \in \sigma | a + b > 0, 3a + 4b > 0, -a - 2b > 0, -b > 0, a > 0\}. \]

The closures of these cones give precisely the fan in figure 2.

![Figure 2: Fan for Nash\(_1(X)\).](image)

3 An analogue of Nobile’s theorem

In this final section we study an analogue of the following well-known theorem of A. Nobile ([No], Theorem 2):

**Theorem 3.1.** Let \(X\) be an algebraic variety of pure dimension \(d\) over an algebraically closed field of characteristic zero. Let \((X^*, \nu)\) be the Nash blowup of \(X\). Then, \(\nu\) is an isomorphism if and only if \(X\) is non-singular.

We will prove the analogue of this theorem in our particular context, that is, we consider only normal toric varieties and Nash blowup is replaced by normalized higher Nash blowup. In view of the results of the previous sections, we will be able to give a combinatorial proof using the theory of Gröbner bases. Once this is done, it is an immediate consequence that the analogue of Nobile’s theorem for higher Nash blowup without normalization is also true for normal toric varieties (see Corollary 3.8).

Let \(X\) be a normal toric variety. Let \((\overline{\text{Nash}_n(X)}, \pi_n \circ \eta)\) be the \(n\)th normalized higher Nash blowup of \(X\). One direction of the analogue of Nobile’s theorem is clear; namely, if \(X\) is non-singular then \(\pi_n\) is an isomorphism (\(\pi_n\) only modifies singular points) and so is \(\eta\). Therefore, if \(X\) is non-singular, \(\pi_n \circ \eta\) is an isomorphism.
Let us suppose now that $X$ is singular. We want to prove that $\pi_n \circ \eta$ is not an isomorphism. Let $\sigma \subset \mathbb{R}^d$ be a strictly convex polyhedral cone such that $X$ is the associated normal toric variety. By theorem 2.3 the fan corresponding to $\overline{Nash}_n(X)$ is given by the Gröbner fan of the ideal $J_n = \langle x^{a_1} - 1, \ldots, x^{a_s} - 1 \rangle^{n+1} \subset k[x^{a_1}, \ldots, x^{a_s}] = k[\sigma \cap \mathbb{Z}^d]$. To prove that $\pi_n \circ \eta$ is not an isomorphism it suffices to prove that the Gröbner fan of $J_n$ truly subdivides $\sigma$. Indeed, suppose that $GF(J_n)$ is a non-trivial subdivision of $\sigma$ and consider two cones $\sigma_1 \neq \sigma_2$ in $GF(J_n)$, whose relative interiors are contained in the relative interior of $\sigma$. Denote by $\gamma_{\sigma_1}$, $\gamma_{\sigma_2}$, and $\gamma_{\sigma}$ the respective distinguished points. Then by [CLS], Lemma 3.3.21, we have:

$$(\pi_n \circ \eta)(\gamma_{\sigma_1}) = \gamma_{\sigma} = (\pi_n \circ \eta)(\gamma_{\sigma_2}).$$

Since $\gamma_{\sigma_1} \neq \gamma_{\sigma_2}$, we see that $\pi_n \circ \eta$ is not injective, so it is not an isomorphism.

Therefore, by definition of Gröbner fan, we need to find $w, w' \in \sigma$ such that $in_w(J_n) \neq in_{w'}(J_n)$. As we saw in previous sections (see prop. 1.5), this is equivalent to the following fact. Fix some $w$ in the interior of $\sigma$ and let $>$ be any monomial order on $k[x^{a_1}, \ldots, x^{a_s}]$. Define a new order $>_w$ for which $x^u >_w x^v$ if $u \cdot w > v \cdot w$ or $u \cdot w = v \cdot w$ and $u > v$. Let $G$ be the reduced Gröbner basis of $J_n$ with respect to $>_w$. Then $in_w(J_n) \neq in_{w'}(J_n)$ for some $w' \in \sigma$ if and only if $in_w(g) \neq in_{w'}(g)$ for some $g \in G$.

**Remark 3.2.** We could formulate a similar question for ideals other than $J_n$, for $n \geq 1$. Is it true that the fact that the Gröbner fan of some ideal in $k[x^{a_1}, \ldots, x^{a_s}]$ does not subdivide $\sigma$ implies that $\sigma$ is regular? The answer is no in general. Take for instance any monomial ideal. Any minimal monomial basis is already the reduced Gröbner basis with respect to any $w \in \sigma$. The initial parts of these monomials are trivially preserved when varying $w \in \sigma$. However, this does not imply regularity of $\sigma$. But even for non-monomial ideals, something similar happens. Consider the ideal $J_0$. Here the generators $\{x^{a_1} - 1, \ldots, x^{a_s} - 1\}$ form the reduced Gröbner basis of $J_0$ with respect to any $w \in \sigma$ and they also trivially satisfy the conditions on the initial parts but this does not imply regularity of the cone $\sigma$.

The strategy for the proof of the analogue of Nobile’s theorem is to find an element of the reduced Gröbner basis whose initial part changes as we vary $w \in \sigma$. To illustrate the method we consider the following family of normal toric surfaces.

**Proposition 3.3.** Let us consider the $A_m$-singularity, and let $k[x, x^m y^{m+1}, xy]$ be its ring of regular functions. Let $J_n = \langle x - 1, x^m y^{m+1} - 1, xy - 1 \rangle^{n+1}$. Then $GF(J_n)$ has no trivial subdivisions.

**Proof.** Let $\sigma \subset \mathbb{R}^2$ be the cone generated by $(0, 1)$ and $(m, -m - 1)$. Denote by $R_1$ and $R_2$ the rays generated by $(0, 1)$ and $(m, -m - 1)$, respectively. Fix some $w_0$ in the relative interior of $\sigma$ and sufficiently close to $R_2$. Let $>$ be any monomial order.
on \( k[x, x^m y^{n+1}, xy] \) and let \( G \) be the reduced Gröbner basis of \( J_n \) with respect to \( >_{w_0} \). We are going to show that there exists some \( g \in G \) such that its initial part changes as we vary \( w \in \sigma \).

Since \((x-1)^{n+1} \in J_n\), there exists \( g \in G \) such that \( lt_{>w_0}(g)|x^{n+1} \), i.e., \( lt_{>w_0}(g) = x^p \), \( p \leq n+1 \). We consider two cases:

1. First suppose there is another monomial in \( g \) different from a power of \( x \). Since there are only a finite number of monomials in \( g \), then if \( w_0 \) is sufficiently close to \( R_2 \) we have that \( in_{w_0}(g) = x^p \). But now by taking \( w \) sufficiently close to \( R_1 \), we have \( in_w(g) \neq x^p \). This implies that \( GF(J_n) \) has non-trivial subdivisions.

2. Now suppose that \( g = x^p + \alpha_1 x^{p-1} + \cdots + \alpha_p x + \alpha_p \). Applying the division algorithm to \((x-1)^{n+1}\) and \( g \) we obtain \((x-1)^{n+1} = g \cdot q + r\), where \( r = 0 \) or \( r \neq 0 \) and \( \deg_x r < \deg_x g \). If \( r \neq 0 \) then there is some \( g' \in G \), \( g' \neq g \) such that \( lt_{>w_0}(g')|lt_{>w_0}(r) \) which implies that \( lt_{>w_0}(g')|lt_{>w_0}(g) \), contradicting the fact that \( G \) is reduced. Therefore \( r = 0 \) and so \( g = (x-1)^{p} \). Once again, we consider two cases:

2.1. Suppose \( p < n+1 \). In particular, \( g = (x-1)^{p} \in J_n \) but this is impossible by lemma 3.6 proved below.

2.2. Suppose \( p = n+1 \). We are going to show that there is an element \( h \in J_n \) such that \( lt_{>w_0}(h) = x^n \), which again contradicts the fact that \( G \) is reduced. We proceed by induction on \( n \). First we show that there is an element \( h_1 \in J_1 \) such that \( lt_{>w_0}(h_1) = x \). Assume for the moment that such an element exists. Let \( h_i := (x-1) \cdot h_{i-1} \in J_i \), \( i \geq 2 \). Then, by induction, \( lt_{>w_0}(h_i) = x^{i} \). Now we prove that such an \( h_1 \) exists. Let \( n = 1 \) and consider the following telescopic sums:

\[
x^{m+1} y^{m+1} + 1 = (xy - 1) \cdot \left[ \sum_{j=0}^{m} (xy)^{m-j} \right] + 2
\]

\[
= (xy - 1) \cdot \left[ (xy - 1) \cdot \left( \sum_{j=1}^{m} j \cdot (xy)^{m-j} \right) + (m + 1) \right] + 2
\]

\[
= (xy - 1)^2 \cdot \left( \sum_{j=1}^{m} j \cdot (xy)^{m-j} \right) + (xy - 1) \cdot (m + 1) + 2.
\]

This implies:

\[
x^{m+1} y^{m+1} - x^m y^{m+1} - x + 1 = (xy - 1)^2 \cdot \left( \sum_{j=1}^{m} j \cdot (xy)^{m-j} \right)
\]

\[
- x^m y^{m+1} - x + (m + 1) \cdot xy - (m + 1) + 2.
\]
The term on the left equals \((x - 1) \cdot (x^m y^{m+1} - 1) \in J_1\). Since \((xy - 1)^2\) is also in \(J_1\) we have that \(h_1 := x^m y^{m+1} + x - (m + 1) \cdot xy + (m + 1) - 2 \in J_1\). If \(w_0\) is sufficiently close to \(R_2\), then \(m_{\omega_0}(h_1) = x\) and so \(l_{\geq w_0}(h_1) = x\), as desired.

Therefore, by (2.1) and (2.2), case (2) is impossible. By case (1) we are done. \(\square\)

**Remark 3.4.** Notice that the previous proof is also valid for any normal toric surface, since, according to [O], Proposition 1.21, there is an identical relation to that of \(x, xy, x^m y^{m+1}\), among any three consecutive generators in the minimal generating set of the semigroup associated to the toric surface.

Now we move into the general case. As before, let \(\sigma \subset \mathbb{R}^d\) be a strictly convex rational polyhedral cone of dimension \(d\) and such that \(\tilde{\sigma} \subset \mathbb{R}_{\geq 0}\). Let \(\{a_1, \ldots, a_s\} \subset \mathbb{Z}_{\geq 0}^d\) be the minimal set of generators of \(\tilde{\sigma} \cap \mathbb{Z}^d\). We need two preliminary lemmas.

According to [CLS], Proposition 1.2.23, the set \(\{a_1, \ldots, a_s\}\), contains the ray generators of the edges of \(\tilde{\sigma}\) which we denote, after renumbering if necessary, by \(\{a_1, \ldots, a_r\}\), as well as possibly some points in the relative interior of \(\{\sum_{i=1}^r \lambda_i a_i | 0 \leq \lambda_i \leq 1\}\). Since \(\tilde{\sigma}\) has dimension \(d\), we must have \(r \geq d\). Let us assume that \(\sigma\) is not a regular cone.

**Lemma 3.5.** There exist \(h \in J_n\) and some \(w\) in the relative interior of \(\sigma\) such that \(l_{\geq w}(h) = (x^{a_i})^n\), for some \(i \in \{1, \ldots, r\}\).

**Proof.** We proceed by induction on \(n\). We are going to show that there exist \(h_1 \in J_1\) and some \(w \in \sigma\) such that \(l_{\geq w}(h_1) = x^{a_i}\) for some \(i \in \{1, \ldots, r\}\). Assume for the moment that such \(h_1\) and \(w\) exist. Let \(h_l := (x^{a_i} - 1) \cdot h_{l-1} \in J_l\), \(l \geq 2\). Then, by induction, \(l_{\geq w}(h_l) = (x^{a_i})^l\). Now we prove that such \(h_1\) and \(w\) exist. Let \(n = 1\) and consider the following map of \(k\)-algebras:

\[
\phi : k[y_1, \ldots, y_s] \to k[x^{a_1}, \ldots, x^{a_r}], \quad y_i \mapsto x^{a_i}.
\]

Let \(J_1 := \langle y_1 - 1, \ldots, y_s - 1 \rangle^2 + \ker \phi\). Since \(\sigma\) is not a regular cone, we must have \(s > d\). Consider a subset of \(\{a_1, \ldots, a_r\}\) consisting of \(d\) linearly independent elements (such a subset exists since \(\tilde{\sigma}\) has dimension \(d\)). After renumbering, if necessary, we may assume that this subset is \(\{a_1, \ldots, a_d\}\). Let \(A\) be the transpose of the matrix whose rows are \(a_1, \ldots, a_d\), in this order. Let \(\lambda' := (\lambda'_{1}, \ldots, \lambda'_{d})\) be the solution of the equation \(A z = a_{d+1}\), i.e., \(\lambda' = A^{-1} a_{d+1}\). The entries of \(A\) are all integers as well as those of \(a_{d+1}\), whence \(\lambda' \in \mathbb{Q}^d\). By multiplying by suitable integers and after renumbering, if necessary, we obtain the following relation:

\[
\lambda_1 a_1 + \cdots + \lambda_t a_t = \lambda_{t+1} a_{t+1} + \cdots + \lambda_{d+1} a_{d+1},
\]

where \(\lambda_i \in \mathbb{Z}_{\geq 0}\) for all \(i\), and for some \(t \in \{1, \ldots, d\}\). This implies that \(y_1^{\lambda_1} \cdots y_t^{\lambda_t} - y_{t+1}^{\lambda_{t+1}} \cdots y_{d+1}^{\lambda_{d+1}} \in \ker \phi\).
Consider the change of coordinates $y_i \mapsto y'_i + 1$. Then
\[(y'_1 + 1)^{\lambda_1} \cdots (y'_r + 1)^{\lambda_r} - (y'_{r+1} + 1)^{\lambda_{r+1}} \cdots (y'_{d+1} + 1)^{\lambda_{d+1}}\]
belongs to $K$, where $K$ is the image of ker $\phi$ under the change of coordinates, and consequently, it also belongs to $(y'_1, \ldots, y'_s)^2 + K$. Since $(y'_1, \ldots, y'_s)^2$ contains all monomials of degree two in the variables $y'_i$, the polynomial $\delta_1 y'_1 + \cdots + \delta_{d+1} y'_{d+1}$ is also in $(y'_1, \ldots, y'_s)^2 + K$, for some non-zero coefficients $\delta_i$. Undoing the change of coordinates, we obtain $h := \delta_1 y_1 + \cdots + \delta_{d+1} y_{d+1} + c \in T$, where $c$ is a constant. Hence $h_1 := \phi(h) = (\delta_1 x^{a_1} + \cdots + \delta_{d+1} x^{a_{d+1}} + c) \in J_1$. Now consider two cases (recall that $r$ denotes the number of edges of $\tilde{\sigma}$):

(1) If $r > d$ then $a_{d+1} \in \{a_1, \ldots, a_r\}$. Consequently, $lt_{>\omega}(h) = x^{a_i}$, for some $i \in \{1, \ldots, r\}$ and any $w \in \sigma$, as desired.

(2) Suppose that $r = d$ and recall that $\{a_1, \ldots, a_s\}$ is the minimal set of generators of $\tilde{\sigma} \cap \mathbb{Z}^d$. In particular, $a_{d+1} = \sum_{i=1}^d \lambda_i a_i$, where $0 \leq \lambda_i < 1$. Denote by $H$ the hyperplane generated by $\{a_1, \ldots, a_{d-1}\}$. Then $H \cap \tilde{\sigma}$ is a facet of $\tilde{\sigma}$, i.e., there exists $w \in \sigma$ such that $w^\perp = H$. In particular, $w \cdot a_i = 0$ for $i = 1, \ldots, d-1$, and $w \cdot a_d > 0$. If $a_{d+1} \in H$ then $lt_{>\omega}(h) = x^{a_d}$, as desired. Otherwise, $w \cdot a_{d+1} > 0$. Now we choose $w'$ sufficiently close to $w$ in the relative interior of $\sigma$ and such that $0 < w' \cdot a_i < w' \cdot a_d$ and $0 < w' \cdot a_i < w' \cdot a_{d+1}$ for all $i = 1, \ldots, d-1$. We know that $a_{d+1} = \sum_{i=1}^d \lambda_i a_i$, where, in particular, $0 < \lambda_d < 1$. This fact allow us to choose $w'$ satisfying also $w' \cdot a_{d+1} < w' \cdot a_d$. Therefore, $lt_{>\omega}(h) = x^{a_d}$. This concludes the proof of the lemma.

\[\Box\]

**Lemma 3.6.** If $p \leq n + 1$, then $(x^{a_i} - 1)^p \notin J_n$, for every $i$.

**Proof.** For convenience of notation, we take $i = 1$ and we assume that $a_{11} > 0$. Let $f_t := (x^{a_1} - 1)^{t_1} \cdot (x^{a_2} - 1)^{t_2} \cdots (x^{a_s} - 1)^{t_s}$, where $\sum_j t_j = n + 1$. Suppose that
\[(x^{a_1} - 1)^p = \sum h_tf_t, \tag{4}\]
for some $h_t \in k[x^{a_1}, \ldots, x^{a_s}]$. We will get a contradiction by taking derivatives with respect to $x_1$. When we take the first derivative with respect to $x_1$ of $\sum h_tf_t$, every summand $h_tf_t$ produces two summands, according to Leibniz’ rule of derivation. Each of these new summands contains a factor $(x^{a_1} - 1)^{r_1} \cdot (x^{a_2} - 1)^{r_2} \cdots (x^{a_s} - 1)^{r_s}$, where $n \leq \sum_j r_j \leq n + 1$. Continuing this way, after differentiating $p$ times with respect to $x_1$, every summand in the resulting sum contains a factor $(x^{a_1} - 1)^{r_1} \cdot (x^{a_2} - 1)^{r_2} \cdots (x^{a_s} - 1)^{r_s}$, where $0 < n + 1 - p \leq \sum_j r_j \leq n + 1$. 

17
On the other hand, the first derivative with respect to \( x_1 \) of \((x^{a_1} - 1)^p\) is \((x^{a_1} - 1)^{p-1} \cdot m\), where \(m\) is some monomial. The second derivative will produce two summands, each one being a product of \((x^{a_1} - 1)^r\) where \(p - 1 \leq r \leq p\), and some monomial. Continuing this way, after \(p - 1\) derivations, the resulting sum consists of summands of the form \((x^{a_1} - 1)^r \cdot m\), \(1 \leq r \leq p\), and where there is exactly one summand such that \(r = 1\). The next derivation produces a non-zero monomial plus summands of the form \((x^{a_1} - 1)^r \cdot m\), where \(1 \leq r \leq p\).

Therefore, after derivating each side of equation (1) \(p\) times, and evaluating the resulting polynomials in \((1, 1, \ldots, 1)\) we obtain zero on the right hand and something different from zero on the left hand. This is a contradiction.

\[\] 

**Proof.** Let \(\pi(w)\) be as in lemma 3.5. Let \(G\) be the reduced Gröbner basis of \(J_n\) with respect to \(>w\), where \(>\) is any monomial order on \(k[x^{a_1}, \ldots, x^{a_r}]\). By definition, \((x^{a_i} - 1)^{n+1} \in J_n\). For each \(i\), there exists \(g_i \in G\) such that \(lt_{>w}(g_i)\) is not an isomorphism. Consequently, \(C[w] \neq C[w''']\) and so the Gröbner fan of \(J_n\) is not trivial.

Now we are ready to prove the analogue of Nobile’s theorem in our context.

**Theorem 3.7.** Let \(X\) be the normal toric variety defined by \(\sigma\). Let \(\pi_n \circ \eta : Nash_n(X) \to X\) be the normalized higher Nash blowup of \(X\). Then if \(X\) is singular, \(\pi_n \circ \eta\) is not an isomorphism.

**Proof.** Let \(w \in \sigma\) be as in lemma 3.5. Let \(G\) be the reduced Gröbner basis of \(J_n\) with respect to \(>w\), where \(>\) is any monomial order on \(k[x^{a_1}, \ldots, x^{a_r}]\). By definition, \((x^{a_i} - 1)^{n+1} \in J_n\). For each \(i\), there exists \(g_i \in G\) such that \(lt_{>w}(g_i)\) is not an isomorphism. Consequently, \(C[w] \neq C[w''']\) and so the Gröbner fan of \(J_n\) is not trivial. Here \(C[w]\) denotes the equivalence class of \(w\) in the Gröbner fan of \(J_n\).

(1) Suppose there is some \(i \in \{1, \ldots, r\}\) such that \(g_i\) contains some monomial \(x^\delta\) that is not a power of \(x^{a_i}\). By definition of \(>w\), \((x^{a_i})^{p_i}\) is a monomial of \(in_w(g_i)\). On the other hand, since \(p_ia_i\) is in the ray generated by \(a_i\) (which is a ray of the cone \(\sigma\)), there exists \(w' \in \sigma\) such that \(w'.(p_ia_i) = 0\) and \(w'.\delta > 0\).

Now we choose \(w''\) sufficiently close to \(w'\) in the relative interior of \(\sigma\) and such that \(0 < w''.(p_ia_i) < w''.\delta\). This implies that \((x^{a_i})^{p_i}\) is not a monomial of \(in_{w''}(g_i)\). Consequently, \(C[w] \neq C[w''']\) and so the Gröbner fan of \(J_n\) is not trivial.

(2) Suppose that \(g_i = (x^{a_i})^{p_i} + \alpha_{i,1}(x^{a_i})^{p_i-1} + \cdots + \alpha_{i,p_i-1}(x^{a_i}) + \alpha_{i,p_i}\), where \(i \in \{1, \ldots, r\}\). Applying the division algorithm in one variable we obtain:

\[(x^{a_i} - 1)^{n+1} = g_i \cdot q_i + r_i,\]

where \(r_i = 0\) or \(r_i \neq 0\) and \(\deg_{x^{a_i}}(r_i) < \deg_{x^{a_i}}(g_i)\). If \(r_i \neq 0\) for some \(i\), the previous equality implies \(r_i \in J_n\), and so there exists \(g \in G\) such that \(lt_{>w}(g)\) divides \(lt_{>w}(r_i)\). This implies that \(lt_{>w}(g)\) divides \(lt_{>w}(g_i)\), which contradicts the fact that \(G\) is reduced. Therefore \(r_i = 0\) for all \(i\), implying \(g_i = (x^{a_i} - 1)^{p_i}\), where \(p_i \leq n + 1\). By lemma 3.6 \(p_i\) cannot be smaller than \(n + 1\), i.e., \(p_i = n + 1\) for all \(i\). According to lemma 3.5 there exists \(h \in J_n\) such that \(lt_{>w}(h) = (x^{a_i})^n\). Once again, this gives a contradiction.
By (1) and (2), the Gröbner fan of $J_n$ has non-trivial subdivisions and so $\pi_n \circ \eta$ is not an isomorphism.

As an immediate consequence, the analogue of Nobile’s theorem for higher Nash blowup without normalization is also true for normal toric varieties.

**Corollary 3.8.** Let $X$ be a normal toric variety and let $(\text{Nash}_n(X), \pi_n)$ be its $n$th higher Nash blowup. Then $\pi_n$ is an isomorphism if and only if $X$ is non-singular.

**Proof.** Suppose $\pi_n$ is an isomorphism. In particular, $\text{Nash}_n(X)$ is normal whence $\text{Nash}_n(X) \cong \text{Nash}_n(X)$. By the previous theorem, this implies that $X$ is non-singular.

**References**

[AL] Adams, W., Loustaunau, P.; *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics, 3, AMS, Providence, 1994.

[CLO] Cox, D., Little, J., O’Shea, D.; *Ideals, Varieties, and Algorithms*, Third Edition, Undergraduate Texts in Mathematics, Springer, New York, 2007.

[CLS] Cox, D., Little, J., Schenck, H.; *Toric varieties*, Graduate Studies in Mathematics, Volume 124, AMS, 2011.

[DGPS] Decker, W.; Greuel, G.-M.; Pfister, G.; Schönemann, H.; *Singular 3-1-6 — A computer algebra system for polynomial computations*. http://www.singular.uni-kl.de (2012).

[E] Eisenbud, D.; *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 150, Springer-Verlag, New York, 1995.

[GT] Gonzalez, P., Teissier, B.; *Toric Geometry and the Semple-Nash modification*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales, Serie A Matemáticas, DOI 10.1007/s13398-012-0096-0, (2012).

[GM] Grigoriev, D., Milman, P.; *Nash desingularization for binomial varieties as Euclidean division, a priori termination bound, polynomial complexity in dim 2*, Adv. Math. 231 (2012), no. 6, 3389-3428.

[H] Hartshorne, R.; *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, 1977.

[M] Maclagan, D., Thomas, R.; *Computational Algebra and Combinatorics of Toric Ideals*, Proceedings of the International Conferences organized by
Bhaskaracharya Pratishthana, Pune and Harish-Chandra Research Institute, Allahabad, during 8 - 13 December, 2003, Ramanujan Mathematical Society Lecture Notes Series, Vol. 4.

[Na] Nakajima, H.; *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, Vol. 18, American Mathematical Society, Providence, 1991.

[No] Nobile, A.; *Some properties of the Nash blowing-up*, Pacific Journal of Mathematics, 60, (1975), 297-305.

[O] Oda, T.; *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 15, Springer-Verlag, Berlin, 1988.

[OZ] Oneto, A., Zatini, E.; *Remarks on Nash blowing-up*, Rend. Sem. Mat. Univ. Politec. Torino 49 (1991), no. 1, 71-82, Commutative algebra and algebraic geometry, II (Italian) (Turin 1990).

[Se] Seshadri, C. S.; *Quotient Spaces Modulo Reductive Algebraic Groups*, Annals of Mathematics, Vol. 95, No. 3, (1972).

[St] Sturmfels, B.; *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. 8, American Mathematical Society, Providence, 1996.

[Y] Yasuda, T.; *Higher Nash blowups*, Compositio Math. 143 (2007), no. 6, 1493-1510.

[Y1] Yasuda, T.; *Universal flattening of Frobenius*, American Journal of Mathematics, Volume 134, No. 2, 2012, p. 349-378.