EXTENSION OF THE TOTAL MASS OF LOG-CONCAVE FUNCTIONS AND RELATED INEQUALITIES

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Abstract. The aim of this paper is to deal with the log-concave functions in $\mathbb{R}^n$, endowed with a suitable algebraic structure corresponds to the structure of convex bodies in $\mathbb{R}^n$, when restricted to the subclass of characteristic functions. In this paper, the functional Quermassintegrals of log-concave functions in $\mathbb{R}^n$ are discussed, we express the functional mixed Quermassintegrals as the integral of the support function of $f$ on some measures. We obtain a functional counterpart of the mixed Quermassintegrals inequality for convex bodies. Moreover, as a special case a weak log Quermassintegral inequality is obtained.

1. INTRODUCTION

The fundamental Brunn-Minkowski inequality for convex bodies (compact convex subsets with nonempty interiors) states that for convex bodies $K$ and $L$ in Euclidean $n$-space, $\mathbb{R}^n$, the volume of the bodies and of their Minkowski sum $K + L = \{x + y : x \in K, \text{ and } y \in L\}$ are given by

$$V(K + L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}. \quad (1.1)$$

with equality if and only if $K$ and $L$ are homothetic, namely they agree up to a translation and a dilation. As the first milestone of the Brunn-Minkowski theory, the Brunn-Minkowski inequality is a far-reaching generalization of the isoperimetric inequality, affine isoperimetric inequality and the uniqueness issue in the solution of the Minkowski’s problem. The Brunn-Minkowski inequality exposes the crucial logarithmic concavity of the volume in $\mathbb{R}^n$, because it has an equivalent formulation as: for all real $t \in [0,1]$,

$$V((1-t)K + tL) \geq V(K)^{1-t}V(L)^t, \quad (1.2)$$

and for $t \in (0,1)$. There is equality if and only if $K$ and $L$ are translates, see for example [27,28,54] for more about the Brunn-Minkowski inequality.

Another important geometric inequality related to the convex bodies $K$ and $L$ is the mixed volume inequality, which also is called as the Minkowski’s first
inequality

\[ V_1(K, L) := \frac{1}{n} \lim_{t \to 0^+} \frac{V(K + tL) - V(K)}{t} \geq V(K) \frac{n-1}{n} V(L) \frac{1}{n}, \]  

for \( K, L \in \mathbb{K}^n \), the set of convex bodies in \( \mathbb{R}^n \). Inequality (1.3) can be easily obtained from (1.1), and in fact they are equivalent to each other. Specially, when choose \( L \) to be a unit ball, up to a factor, \( V_1(K, L) \) is exactly the perimeter of \( K \), and inequality (1.3) turns out to be the isoperimetric inequality in the class of convex bodies. Moreover, the mixed volume \( V_1(K, L) \) admits a simple integral representation (see [40, 41])

\[ V_1(K, L) = \frac{1}{n} \int_{S_{n-1}} h_L dS_K, \]  

where \( h_L \) is the support function of \( L \), \( S_K \) is the area measure of \( K \).

Let \( K \in \mathbb{R}^n \), the Quermassintegrals \( W_i(K) \) \( (i = 0, 1, \cdots, n) \) of \( K \), which are defined by letting \( W_0(K) = V_n(K) \), the volume of \( K \); \( W_n(K) = \omega_n \), the volume of the unit ball \( B_2^n \) in \( \mathbb{R}^n \); and for general \( i = 1, 2, \cdots, n-1 \),

\[ W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G_{i,n}} \text{vol}_i(K|L_i) dL_i, \]  

where the \( G_{i,n} \) is the Grassmannian manifold of \( i \)-dimensional linear subspaces of \( \mathbb{R}^n \), \( dL_i \) is the normalized Haar measure on \( G_{i,n} \), \( K|L_i \) denotes the orthogonal projection of \( K \) onto the \( i \)-dimensional subspaces \( L_i \), and \( \text{vol}_i \) is the \( i \)-dimensional volume on space \( L_i \).

In the 1930s, Aleksandrov, Fenchel and Jessen (see [1, 23]) proved that for a convex body \( K \) in \( \mathbb{R}^n \), there exists a regular Borel measure \( S_{n-1-i}(K) \) \( (i = 0, 1, \cdots, n-1) \) on \( S_{n-1} \), the unit sphere in \( \mathbb{R}^n \), such that for any convex bodies \( K \) and \( L \), the following representations hold

\[ W_i(K, L) = \frac{1}{n-i} \lim_{\epsilon \to 0^+} \frac{W_i(K + \epsilon L) - W_i(K)}{\epsilon} \]

\[ = \frac{1}{n} \int_{S_{n-1-i}} h_L(u) dS_{n-1-i}(K, u). \]  

Where \( K + tL = \{ x + ty : x \in K, y \in L \} \), the quantity \( W_i(K, L) \) is called the \( i \)-th mixed Quermassintegral of \( K \) and \( L \).

In the 1960s, the Minkowski addition was extended to the \( L^p \)-Minkowski sum \( K +_p t \cdot L \), that is (see [24])

\[ h_{K+_p t \cdot L}^p = h_K^p + th_L^p. \]  

The extension of the mixed Quermassintegrals to the mixed \( L^p \)-Quermassintegrals due to Lutwak [40]. In his paper, he establishes the mixed \( L^p \)-Quermassintegral inequalities and solves the \( L^p \)-Minkowski problem. See [31, 41–46, 55–57] for more about the \( L^p \)-Minkowski theory and \( L^p \)-Minkowski inequalities. The first variation of the mixed \( L^p \)-Quermassintegrals are defined by

\[ W_{p,i}(K, L) := \frac{p}{n-i} \lim_{t \to 0^+} \frac{W_i(K +_p t \cdot L) - W_i(L)}{t}, \]  

(1.8)
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for $i = 0, 1, \cdots, n - 1$. In particular, for $p = 1$, the mixed $L^p$-Quermassintegrals $W_{p,i}(K, L)$ are just $W_i(K, L)$ defined by (1.6). $W_{p,0}(K, L)$ is also denoted by $V_i(K, L)$, which is called the $L_p$-mixed volume of $K$ and $L$. Similarly, the mixed $L^p$-Quermassintegral has the following integral representation (see [40]):

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S_{n-1}} h_L^p(u) dS_{p,i}(K, u),$$

for all $L \in \mathcal{K}_0^n$. The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h_K(\cdot)^{1-p}.$$

Specially, the case $p = 1$ of the representation (1.9) is just the representation (1.6).

In most recently, the interest in the log-concave functions has been considerably increasing, motivated by the analogy properties between the log-concave functions and the volume convex bodies in $\mathbb{R}^n$. The classical Prékopa-Leindler inequality (see [15, 38, 47–49]) firstly shows the connections of the volume of convex bodies and log-concave functions. The functional form of the Blaschke-Santaló inequality for even case is established by Ball in [8, 9]. The general case is proved by Artstein-Avidan, Klartag and Milman [4], other proofs are given by Fradelizi, Meyer [26] and Lehec [36, 37]. More about the functional Blaschke-Santaló inequality, such as the inverse form, the stability and others see [10, 25, 32, 53]. The functional version of the mean width for log-concave function has been introduced by Klartag, Milman and Rotem [34, 51, 52]. The functional affine isoperimetric inequality for log-concave functions are proved by Artstein-Avidan, Klartag, Schütt and Werner [7]. The John ellipsoid for log-concave functions have been establish by Gutiérrez, Merino Jiménez and Villa [2]. The LYZ ellipsoid for log-concave functions are established by Fang and Zhou [21]. See [3, 6, 11, 16–18, 39] for more about the pertinent results. To establish the functional versions of inequalities and problems from the points of convex geometric analysis is a new research fields.

We consider the following log-concave functions of $\mathbb{R}^n$:

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f = e^{-u},$$

where $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. The log-concave function is closely related to the convex geometry of $\mathbb{R}^n$. An example of a log-concave function is the characteristic function $\chi_K$ of a convex body $K$ in $\mathbb{R}^n$, which is defined by

$$\chi_K(x) = e^{-I_K(x)} = \begin{cases} 1, & \text{if } x \in K; \\ 0, & \text{if } x \notin K, \end{cases}$$

where $I_K$ is a lower semi-continuous convex function, and the indicator function of $K$ is,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K; \\ \infty, & \text{if } x \notin K. \end{cases}$$
Maybe, the above function is the most nature way to embed the set of convex bodies in that of log-concave functions. There are many analogies between the theory of convex bodies and that of log-concave functions. The breakthrough in the discovery of parallel behaviours of convex bodies and log-concave functions was the Prékopa-Leindler inequality. It states that, for any given nonnegative functions \( f, g, h \in \mathbb{R}^n \), for \( t \in (0, 1) \), satisfying
\[
h((1 - t)x + ty) \geq f(x)^{1 - t}g(x)^t \quad \forall x, y \in \mathbb{R}^n,
\]
then
\[
\int_{\mathbb{R}^n} h(x)dx \geq \left( \int_{\mathbb{R}^n} f(x)dx \right)^{1 - t} \left( \int_{\mathbb{R}^n} g(x)dx \right)^t.
\] (1.13)

Equality holds if and only if the functions \( f \) and \( g \) are log-concave functions, and \( f(x) = g(x - x_0) \) for some \( x_0 \in \mathbb{R}^n \). Beyond the Prékopa-Leindler inequality, it is recognized as the functional version of the classical Brunn-Minkowski inequality (see \([14, 19, 27, 47–49]\)).

The fundamental issues on the class of log-concave functions is studying the algebraic structure, which are called “sum” and “scalar multiplication”. How to define a suitable algebraic structure on the class of log-concave functions and, it should inherit analogous structure of the class of convex bodies, is very important. Actually, the Fenchel conjugate and the infimal convolution give the answers to the problem.

Let \( f = e^{-u}, g = e^{-v} \) be log-concave functions, \( \alpha, \beta > 0 \), the “sum” and “scalar multiplication” of log-concave functions are defined as,
\[
\alpha \cdot e^{-u} \oplus \beta \cdot e^{-v} := e^{-w}, \quad \text{where } w^* = \alpha u^* + \beta v^*.
\] (1.14)

Here \( w^* \) denotes as usual the Fenchel conjugate of the convex function \( \omega \). The total mass integral \( J(f) \) is defined by, \( J(f) = \int_{\mathbb{R}^n} fdx \). In paper of Colesanti and Fragała \([20]\), the quantity \( \delta J(f, g) \), which is called as the first variation of \( J \) at \( f \) along \( g \),
\[
\delta J(f, g) = \lim_{t \to 0^+} \frac{J(f \oplus t \cdot g) - J(f)}{t},
\]
is discussed. It has been shown that \( \delta J(f, g) \) is finite and is given by
\[
\delta J(f, g) = \int_{\mathbb{R}^n} v^*d\mu(f),
\]
where \( \mu(f) \) is the measure of \( f \) on \( \mathbb{R}^n \). See \([20]\) for more about the discussion of the \( \delta J(f, g) \).

Inspired by the paper of Colesanti and Fragała \([20]\), in this paper, we define the functional \( i \)-th Quermassintegrals \( W_i(f) \) as the \( i \)-dimensional average total mass of \( f \),
\[
W_i(f) := \frac{\omega_n}{\omega_{n-i}} \int_{\mathcal{G}_{n-i,n}} J_{n-i}(f) dL_{n-i}.
\]

Where \( J_i(f) \) denotes the \( i \)-dimensional total mass of \( f \), \( \mathcal{G}_{i,n} \) is the Grassmannian manifold of \( \mathbb{R}^n \). We show that the \( W_i(f) \) is \( GL \) invariant and translation invariant.
Moreover, we define the first variation of $W_i$ at $f$ along $g$, which is

$$W_i(f, g) = \lim_{t \to 0^+} \frac{W_i(f \oplus t \cdot g) - W_i(f)}{t}.$$  \hfill (1.15)

We also call it as the functional $i$-th mixed Quermassintegral, say that it is natural extension of the Quermassintegrals of convex bodies in $\mathbb{R}^n$. In fact, if one takes $f = \chi_K$, and $\text{dom}(f) = K \in \mathbb{R}^n$, then the Quermassintegrals $W_i(f)$ turn out to be $W_i(K)$, and $W_i(\chi_K, \chi_L)$ equals to the $W_i(K, L)$.

In the section 4, we focus on how can we represent the functional $i$-th mixed Quermassintegrals $W_i(f, g)$ similar as $W_i(K, L)$, which can represent as the integrals of the support function $h_L$ with some measure $S_i(K)$, here $S_i(K)$ is some surface measure. Owing to the Blaschke-Petkantschin formula and the similar definition of the support function of $f$, in Theorem 4.7 we obtain the integral represent of the functional $i$-th mixed Quermassintegrals $W_i(f, g)$. After Theorem 4.7 been proved, in section 5, we turn our attentions to the functional inequalities involving $W_i(f, g)$, we proved the functional form of Quermassintegral Minkowski inequality, that is our Theorem 5.1. Specially, the weak log-Quermassintegral inequality for convex bodies is obtained as a Corollary. In section 3, the projection of the log-concave functions is defined, the Fenchel conjugate and the infimal convolution of convex functions are discussed.

2. preliminaries

In this paper, we work in $n$-dimensional Euclidean space, $\mathbb{R}^n$, endowed with the usual scalar product $\langle x, y \rangle$ and norm $\|x\|$. Let $B^n_2 = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ denote the standard unit ball in $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ denote the unit sphere in $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$, and $\mathcal{K}^n_0$ be the subclass of convex bodies $K$ whose relative interior $\text{int}(K)$ is nonempty. For $i \leq n$, let $\mathcal{H}^i$ be the $i$-dimensional Hausdorff measure, we indicate by $V(K) = \mathcal{H}^n(K)$ the $n$-dimensional volume.

Let $h_K(\cdot) : \mathbb{R}^n \to \mathbb{R}$ be the support function of $K$; i.e., for $x \in \mathbb{R}^n$,

$$h_K(x) = \max \{ \langle x, y \rangle : y \in K \},$$

where $\langle x, y \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Let $n_K(x)$ be the unit outer normal at $x \in \partial K$, then

$$h_K(n_K(x)) = \langle n_K(x), x \rangle.$$ \hfill (2.1)

It is shown that the sublinear support function characterizes a convex body and, conversely, every sublinear function on $\mathbb{R}^n$ is the support function of a nonempty compact convex set. Two convex bodies $K$, $L$ satisfy $K \subseteq L$ if and only if $h_K(\cdot) \leq h_L(\cdot)$. By the definition of the support function, it follows immediately that the support function of the image $gK := \{gy : y \in K\}$ is given by

$$h_{gK}(x) = h_K(g^Tx)$$

for $g \in \text{GL}(n)$. Here $g^T$ denotes the transpose of $g$. 
Let $K \in K_0^n$ be a convex body that contains the origin in its interior, the polar body $K^\circ$ is defined by

$$K^\circ = \{ y \in \mathbb{R}^n : \langle y, x \rangle \leq 1, \text{ for all } x \in K \}.$$ 

For convex body $K$, the gauge function $\| \cdot \|_K$ is defined by

$$\| x \|_K = \min \{ a \geq 0 : x \in \alpha K \} = \max_{y \in K^\circ} \langle x, y \rangle = h_{K^\circ}(x).$$

It is clear that $\| x \|_K = 1$ whenever $x \in \partial K$.

We denote by $I_K$ and $\chi_K$ the indicatrix function and characteristic function of $K$, defined respectively by formula (1.12) and (1.11).

In the following, we discuss in the functional setting in $\mathbb{R}^n$. Let $u : \mathbb{R}^n \to (-\infty, +\infty]$ be a convex function, that is $u((1-t)x + ty) \leq (1-t)u(x) + tu(y)$ for $t \in [0, 1]$. We set $\text{dom}(u) = \{ x \in \mathbb{R}^n : u(x) \in \mathbb{R} \}$. By the convexity of $u$, $\text{dom}(u)$ is a convex set in $\mathbb{R}^n$. We say that $u$ is proper if $\text{dom}(u) \neq \emptyset$, and $u$ is of class $C^2_+$ if it is twice differentiable on $\text{int(\text{dom}(u))}$, with a positive definite Hessian matrix. In the following we define the subclass of $f$,

$$\mathcal{L} = \{ u : \mathbb{R}^n \to (-\infty, +\infty] : u \text{ is convex, and low semicontinuous, and } \lim_{\|x\| \to +\infty} u(x) = +\infty \};$$

Recall that the Fenchel conjugate of $u$ is the convex function defined by

$$u^*(y) = \sup_{x \in \mathbb{R}^n} \{ \langle x, y \rangle - u(x) \}. \quad (2.2)$$

It is obvious that $u(x) + u^*(y) \geq \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$, and there is an equality if and only if $x \in \text{dom}(u)$ and $y$ is in the subdifferential of $u$ at $x$, that means

$$u^*(\nabla u(x)) + u(x) = \langle x, \nabla u(x) \rangle. \quad (2.3)$$

Moreover, if $u$ is a lower semi-continuous convex function, then also $u^*$ is a lower semi-continuous convex function, and $u^{**} = u$.

Specially, the Fenchel conjugate of the indicatrix $I_K$ of a convex body is precisely its support function $h_K$, one has

$$\alpha \cdot \chi_K \oplus \beta \cdot \chi_L = \chi_{\alpha K + \beta L}.$$ 

Therefore we can say that it is a natural extension on the convex bodies.

The infimal convolution of functions $u$ and $v$ from $\mathbb{R}^n$ to $(-\infty, +\infty]$ defined by

$$u \Box v(x) = \inf_{y \in \mathbb{R}^n} \{ u(x - y) + v(y) \}. \quad (2.4)$$

The right scalar multiplication by a nonnegative real number $\alpha$:

$$(u\alpha)(x) := \begin{cases} \alpha u\left(\frac{x}{\alpha}\right), & \text{if } \alpha > 0; \\ I_{\{0\}}, & \text{if } \alpha = 0. \end{cases} \quad (2.5)$$

The following proposition below gathers some elementary properties of the Fenchel conjugate and the infimal convolution of $u$ and $v$, which can be found in [20, 50].
Proposition 2.1. Let \( u : \mathbb{R}^n \to (-\infty, +\infty) \) be a convex function. Then:

1. \((u\square v)^* = u^* + v^*;\)
2. \((u\alpha)^* = \alpha u^*, \ \alpha > 0;\)
3. \(\text{dom}(u\square v) = \text{dom}(u) + \text{dom}(v);\)
4. it holds \(u^*(0) = -\inf(u),\) in particular if \(u\) is proper, then \(u^*(y) > -\infty;\)
   \(\inf(u) > -\infty\) implies \(u^*\) is proper.

For quick reference we recall some basic definition and notations in convex geometry that is required for our results. Good references are Gardner [28], Gruber [29], Schneider [54].

Now we introduce the Legendre conjugate of the pair \((C, u)\), see [20, 50] for more about it. Given a differentiable real valued function \(u\) on an open subset \(C\) of \(\text{dom}(u)\), the Legendra conjugate of the pair \((C, u)\) is defined to be the pair \((D, v)\), where \(D\) is the image of \(C\) through the gradient mapping \(\nabla u\), and

\[
v(y) = (\nabla u^{-1}(y), y) - u(\nabla u^{-1}(y)),
\]

where \(\nabla u^{-1}(y) := \{x : \nabla u(x) = y\}\). The above definition of \(v\) is well posed whenever for any \(y \in D\), the value of \((x, y) - u(x)\) turns out to be independent from the choice of the point \(x \in \nabla u^{-1}(y)\). We see that a pair \((C, u)\) is a convex function of Legendre type if:

1. \(C\) is a nonempty open convex set;
2. \(u\) is differentiable and strictly convex on \(C\);
3. \(\lim \|\nabla u(x_i)\| \to +\infty\) whenever \(\{x_i\} \subset C\) is a sequence converging to some \(x \in \partial C\).

The following position about the Fenchel and Legendre conjugates are obtained in [50].

Proposition 2.2. Let \( u : \mathbb{R}^n \to (-\infty, +\infty) \) be a closed convex function, and set \(C := \text{int}(\text{dom}(u)), \ \mathcal{C}^* := \text{int}(\text{dom}(u^*)).\) Then \((C, u)\) is a convex function of Legendre type if and only if \(\mathcal{C}^*, u^*\) is. In this case \((\mathcal{C}^*, u^*)\) is the Legendre conjugate of \((C, u)\) (and conversely). Moreover, \(\nabla u := C \to \mathcal{C}^*\) is a continuous bijection, and the inverse map of \(\nabla u\) is precisely \(\nabla u^*\).

Let us introduce the classes of functions we deal with in this paper. A function \(f : \mathbb{R}^n \to (-\infty, +\infty]\) is called log-concave if for all \(x, y \in \mathbb{R}^n\) and \(0 < t < 1\), we have

\[
f((1-t)x + ty) \geq f^{1-t}(x)f^t(y).
\]

If \(f\) is a strictly positive log-concave function on \(\mathbb{R}^n\), then there exist a convex function \(u : \mathbb{R}^n \to (-\infty, +\infty]\) such that \(f = e^{-u}\).

Let \(f = e^{-u} : \mathbb{R}^n \to (-\infty, +\infty]\) be log-concave functions, we define the subclass of \(f\) by

\[
\mathcal{A} = \{f : \mathbb{R}^n \to (0, +\infty] : f = e^{-u}, u \in \mathcal{L}\}.
\]

In the following, we will give some examples and basis properties of functions in \(\mathcal{L}\), the class of log-concave functions \(\mathcal{A}\) can be endowed with an algebraic structure which extends in a natural way the usual Minkowski’s structure on \(\mathcal{K}^n\). For examples, for any \(K \in \mathcal{K}^n\), the function \(u = I_K\) belongs to \(\mathcal{L}\). Notice that
\( \alpha \cdot f \boxplus \beta \cdot g = e^{-[(\alpha u) \Box (\beta v)]} \).

(2.8)

That means

\[
(\alpha \cdot f \boxplus \beta \cdot g)(x) = \sup_{y \in \mathbb{R}^n} f \left( \frac{x - y}{\alpha} \right)^{\alpha} g \left( \frac{y}{\beta} \right)^{\beta}.
\]

(2.9)

In particular, when \( \alpha = 0 \) and \( \beta > 0 \), we have \((\alpha \cdot f \boxplus \beta \cdot g)(x) = g \left( \frac{x}{\beta} \right)^{\beta}\); when \( \alpha > 0 \) and \( \beta = 0 \), then \((\alpha \cdot f \boxplus \beta \cdot g)(x) = f \left( \frac{x}{\alpha} \right)^{\alpha}\); finally, when \( \alpha = \beta = 0 \), we have \((\alpha \cdot f \boxplus \beta \cdot g)(x) = I_{\{0\}}\).

The following Lemma is obtained in [20].

**Lemma 2.3.** Let \( u \in \mathcal{L} \), then there exist constants \( a \) and \( b \), with \( a > 0 \), such that, for \( \forall x \in \mathbb{R}^n \)

\[
u(x) \geq a \|x\| + b.
\]

(2.10)

Moreover \( u^* \) is proper, and satisfies \( u^*(y) > -\infty, \forall y \in \mathbb{R}^n \).

The Lemma 2.3 grants that \( \mathcal{L} \) is closed under the operations of infimal convolution and right scalar multiplication defined in (2.4) and (2.5) are closed (see [20]).

**Proposition 2.4.** Let \( u \) and \( v \) belong both to the same class \( \mathcal{L}' \), and \( \alpha, \beta \geq 0 \). Then \((\alpha u \Box v \beta)\) belongs to the same class as \( u \) and \( v \).

Let \( f \in \mathcal{A} \) be a log-concave, according to a series of papers by Artstein-Avidan and Milman [5], Rotem [51], the support function of \( f = e^{-u} \) is defined as,

\[
h_f(x) = (-\log f(x))^* = u^*(x).
\]

(2.11)

Here the \( u^* \) is the Legendre transform. The definition of \( h_f \) is a proper generalization of the support function \( h_K \), in fact, one can easily checks \( h_{\chi_K} = h_K \). Obviously, the support function \( h_f \) share the most of the important properties of support functions \( h_K \). Specifically, it is easy to check that the function \( h : \mathcal{A} \to \mathcal{L} \) has the following properties [52]:

1. \( h \) is a bijective map from \( \mathcal{A} \to \mathcal{L} \).
2. \( h \) is order preserving: \( f \leq g \) if and only if \( h_f \leq h_g \).
3. \( h \) is additive: for every \( f, g \in \mathcal{A} \) we have \( h_{f \boxplus g} = h_f + h_g \).

The polar function is defined by

\[
f^\circ = e^{-u^*}.
\]

(2.12)

Specifically,

\[
f^\circ(y) = \inf_{x \in \mathbb{R}^n} \left\{ \frac{e^{-\langle x, y \rangle}}{f(x)} \right\}.
\]
hence, we obtain that $f^\circ$ is also a log-concave function.

The following proposition shows that $h_f$ is $GL(n)$ covariant which is proved in [21].

**Proposition 2.5.** Let $f \in \mathcal{A}$. For $T \in GL(n)$ and $x \in \mathbb{R}^n$, then

$$h_{f \circ T}(x) = h_f(T^{-t}x).$$

(2.13)

Moreover, for the polar function of $f$,

$$(f \circ T)^\circ = f^\circ \circ T^{-t}.$$  

(2.14)

Let $u, v \in \mathcal{L}$, denote by $u_t = u \square vt \ (t > 0)$, and $f_t = e^{-ut}$. The following Lemmas describe the monotonous and convergence of $u_t$ and $f_t$, respectively, see Colesanti and Fragalà [20].

**Lemma 2.6.** Let $f = e^{-u}, \ g = g^{-v} \in \mathcal{A}$. For $t > 0$, set $u_t = u \square (vt)$ and $f_t = e^{-ut}$. Assume that $v(0) = 0$, then for every fixed $x \in \mathbb{R}^n$, $u_t(x)$ and $f_t(x)$ are respectively pointwise decreasing and increasing with respect to $t$; in particular it holds

$$u_1(x) \leq u_t(x) \leq u(x) \quad \text{and} \quad f(x) \leq f_t(x) \leq f_1(x) \quad \forall x \in \mathbb{R}^n \ \forall t \in [0, 1].$$

(2.15)

**Lemma 2.7.** Let $u$ and $v$ belong both to the same class $\mathcal{L}$ and, for any $t > 0$, set

$$u_t := u \square (vt).$$

Assume that $v(0) = 0$, then

$(1) \ \forall x \in \text{dom}(u), \lim_{t \to 0^+} u_t(x) = u(x)$;

$(2) \ \forall E \subset \subset \text{dom}(u), \lim_{t \to 0^+} \nabla u_t(x) = \nabla u$ uniformly on $E$.

**Lemma 2.8.** Let $u$ and $v$ belong both to the same class $\mathcal{L}$ and for any $t > 0$, let $u_t := u \square (vt)$. Then $\forall x \in \text{int}(\text{dom}(u_t))$, and $\forall t > 0$,

$$\frac{d}{dt} u_t(x) = -\psi(\nabla u_t(x)),$$

(2.16)

where $\psi := v^\ast$.

### 3. Projection of functions onto linear subspace

Let $G_{i,n}$ ($0 \leq i \leq n$) be the Grassmanian manifold of $i$-dimensional linear subspace of $\mathbb{R}^n$. The elements of $G_{i,n}$ will usually be denoted by $L$, and $L_i^\perp$ stands for the orthogonal complement of $L_i$ which is a $(n - i)$-dimensional subspace of $\mathbb{R}^n$. Note that $G_{i,n}$ can be regarded as a $i(n - i)$-dimensional smooth compact submanifold of a Euclidean space (see [22]) and we can equip it with the Hausdorff measure $\mathcal{H}^{i(n - i)}$. The total mass of the measure of $G_{i,n}$ is given by,

$$\mathcal{H}^{i(n - i)}(G_{i,n}) = \int_{G_{i,n}} dL_i = c_{n,i}.$$  

(3.1)

here $c_{n,i} = \frac{O_iO_{i-1} \cdots O_1}{O_iO_{i-1} \cdots O_1}$, and $O_k = \mathcal{H}^{k-1}(S^{k-1}) = \frac{2\pi^{k/2}}{\Gamma(k/2)}$.

A function $f \in \mathcal{A}$ is non-degenerate and integrable if and only if

$$\lim_{\|x\| \to +\infty} \frac{u(x)}{\|x\|} = +\infty.$$
Then, let
\[ L' = \{ u \in \mathcal{L} : \text{dom}(u) = \mathbb{R}^n, u \in \mathcal{C}^2_{+}(\mathbb{R}^n), \lim_{\|x\| \to +\infty} \frac{u(x)}{\|x\|} = +\infty \}, \]
and
\[ \mathcal{A}' = \{ f : \mathbb{R}^n \to (0, +\infty) : f = e^{-u}, u \in L' \}. \]

Let \( L_i \in \mathcal{G}_{i,n} \) and \( f : \mathbb{R}^n \to \mathbb{R} \). The projection of \( f \) onto \( L_i \) is defined by (see [30, 34])
\[ f|_{L_i}(x) := \max\{ f(y) : y \in x + L_i^\perp \}, \quad \forall x \in \text{dom}(u)|_{L_i}. \tag{3.2} \]
where \( L_i^\perp \) is the orthogonal complement of \( L_i \) in \( \mathbb{R}^n \), \( \text{dom}(u)|_{L_i} \) is the set of \( \text{dom}(u) \) project onto \( L_i \). By the definition of the log-concave function \( f = e^{-u} \), for every \( x \in \text{dom}(u)|_{L_i} \), one can rewrite (3.2) as
\[ f|_{L_i}(x) = \exp \left\{ \max\{ -u(y) : y \in x + L_i^\perp \} \right\} = e^{-u|_{L_i}(x)}. \tag{3.3} \]
Regards the the “sum” and the “multiplication” of \( f \), we say that the projection keeps the structure on \( \mathbb{R}^n \). In other words, we have the following Proposition.

**Proposition 3.1.** Let \( f, g \in \mathcal{A}', L_i \in \mathcal{G}_{i,n} \) and \( \alpha > 0 \). Then

1. \( (\alpha \cdot f)|_{L_i} = \alpha \cdot f|_{L_i} \),
2. \( (f \oplus g)|_{L_i} = f|_{L_i} \oplus g|_{L_i} \).

**Proof.** Let \( f, g \in \mathcal{A}' \), since \( u(x) = -\log f(x) \), by the definition of the right multiplication of \( u \), it is easy to say that, \((u\alpha)|_{L_i} = u|_{L_i}\alpha \) holds for \( \forall x \in \text{dom}(u)|_{L_i} \).
\[ (\alpha \cdot f)|_{L_i}(x) = \max \{ \exp\{ -u\alpha(y) \} : y \in x + L_i^\perp \} \]
\[ = \exp \{ \max\{ -u(y) : y \in x + L_i^\perp \} \alpha \} \]
\[ = \alpha \cdot (f|_{L_i})(x). \]

So we get the proof of the first result.

Similarly, for any \( x \in \text{dom}(u)\]|_{L_i} \), by the sum of \( f \) and \( g \) we have
\[ \max \left\{ - (u \Box v)(y) : y \in x + L_i^\perp \right\} \]
\[ = \max \left\{ - \inf_{\mathcal{F} \in x + E_i^+} \{ u(y - \mathcal{F}) + v(\mathcal{F}) : y, \mathcal{F} \in x + L_i^\perp \} \right\} \]
\[ = \inf_{\mathcal{F} \in x + E_i^+} \left\{ - \max\{ u(y - \mathcal{F}) + v(\mathcal{F}) : y, \mathcal{F} \in x + L_i^\perp \} \right\} \]
\[ = \inf_{\mathcal{F} \in x + E_i^+} \left\{ - \max\{ u(y - \mathcal{F}) \} - \max\{ v(\mathcal{F}) \} : y, \mathcal{F} \in x + L_i^\perp \right\} \]
\[ = -u|_{L_i}(x) \Box v|_{L_i}(x). \]
So we have
\[ (f \oplus g)|_{L_i}(x) = \exp \left\{ \max_{y} \left\{ - (u \Box v)(y) : y \in x + L_i^\perp \right\} \right\} \]
\[ = \exp \left\{ - \left\{ u|_{L_i}(x) \Box v|_{L_i}(x) \right\} \right\} \]
\[ = f|_{L_i}(x) \oplus g|_{L_i}(x). \]
Then we obtain \((f \oplus g)|_{L_i} = f|_{L_i} \oplus g|_{L_i}\).

The following Proposition grants the monotonicity of the projection of functions.

**Proposition 3.2.** Let \(L_i \in \mathcal{G}_{i,n}\), \(f, g\) are functions on \(\mathbb{R}^n\), such that \(f(x) \leq g(x)\) holds for \(x \in \mathbb{R}^n\). Then
\[
f|_{L_i} \leq g|_{L_i},
\]
holds for any \(x \in L_i\).

*Proof.* Since for \(x \in \mathbb{R}^n\), \(f(x) \leq g(x)\), and
\[
\max\{f(y) : y \in x + L_i^\perp\} \leq \max\{g(y) : y \in x + L_i^\perp\}
\]
By the definition of the projection, we complete the proof. \(\square\)

For the convergence of the functions \(f\) we have the following.

**Proposition 3.3.** Let \(\{f_i\}\) are functions such that \(\lim_{n \to \infty} f_n = f_0\), \(L_i \in \mathcal{G}_{i,n}\), then
\[
\lim_{n \to \infty} (f_n|_{L_i}) = f_0|_{L_i}.
\]

*Proof.* Since \(\lim_{n \to \infty} f_n = f_0\). It means for \(\forall \epsilon > 0\), there exist \(N_0\), \(\forall n > N_0\), we have \(f_0 - \epsilon \leq f_n \leq f_0 + \epsilon\), by the definition of the projection, we have \(f_0|_{L_i} - \epsilon \leq f_n|_{L_i} \leq f_0|_{L_i} + \epsilon\). Hence each \(\{f_n|_{L_i}\}\) has a convergent subsequence, we also denoted also by \(\{f_n|_{L_i}\}\), converging to some \(f_0'|_{L_i}\). Then for \(x \in L_i\), we have
\[
f_0|_{L_i}(x) - \epsilon \leq f_0'|_{L_i}(x) = \lim_{n \to \infty} (f_n|_{L_i})(x) \leq f_0|_{L_i}(x) + \epsilon.
\]
By the arbitrariness of \(\epsilon\) we have \(f_0'|_{L_i} = f_0|_{L_i}\), so we complete the proof. \(\square\)

Combine with Proposition 3.3 and Proposition 2.7 it is easy to obtain the following Proposition.

**Proposition 3.4.** Let \(u\) and \(v\) belong both to the same class \(\mathcal{L}'\), for any \(t > 0\), set \(u_t = u \Box (vt)\). Assume that \(v(0) = 0\) and \(L_i \in \mathcal{G}_{i,n}\), then
1. \(\forall x \in \text{dom}(u)|_{L_i}, \lim_{t \to 0^+} u_t|_{L_i}(x) = u|_{L_i}(x)\),
2. \(\forall x \in \text{int}(\text{dom}(u)|_{L_i}), \lim_{t \to 0^+} \nabla u_t|_{L_i} = \nabla u|_{L_i}\).

*Proof.* Let \(x \in \text{dom}(u)|_{L_i}\) be fixed, assume that \(v(0) = 0\), we know that \(u_t(x) \leq u(x)\) for every \(t \geq 0\), by Proposition 3.2 we have \(u_t|_{L_i} \leq u|_{L_i}\), then we obtain
\[
\lim_{t \to 0^+} \sup u_t|_{L_i}(x) \leq u|_{L_i}(x).
\]
On the other hand, assume \(u, v \in \mathcal{L}'\), since \((u_t|_{L_i})^*(y) = \sup_{x \in L_i} \{\langle x, y \rangle - (u_t|_{L_i})(x)\}\), then we obtain
\[
u_t|_{L_i}(x) = \sup_{y \in L_i} \{\langle x, y \rangle - (u_t|_{L_i})^*(y)\}
= \sup_{y \in L_i} \{\langle x, y \rangle - (u|_{L_i})^*(y) - t(v|_{L_i})^*(y)\}.
\]
Note that whenever \( y = \nabla u_t|_{L_i}(x) \), the supremum holds, then we choose \( r = \|\nabla u_t|_{L_i}(x)\| \) and \( B_i^r \) be the ball in \( L_i \) with radius \( r \), by the boundness of \( v^* \) we set \( c := \sup_{B_r} v^* \), then
\[
\begin{align*}
  u_t|_{L_i}(x) &\geq \sup_{y \in B_i^r} \{ \langle x, y \rangle - (u|_{L_i})^*(y) \} - tc. \\
  &= \langle x, \nabla u_t|_{L_i}(x) \rangle - (u|_{L_i})^*(\nabla u_t|_{L_i}(x)) - tc = u|_{L_i}(x) - tc,
\end{align*}
\]
when \( t \to 0^+ \), the we have
\[
\lim_{t \to 0^+} \inf u_t|_{L_i}(x) \geq u|_{L_i}(x).
\]

The continuous of \( u \) and by passing to the inferior limit as \( t \to 0^+ \), we complete the proof of (1).

The convexity of \( u_t \), implies the convexity of \( u_t|_{L_i} \). In fact, \( \forall x, y \in L_i, \lambda, \mu \in (0, 1) \) and satisfy \( \lambda + \mu = 1 \).
\[
\begin{align*}
  u_t|_{L_i}(\lambda x_1 + \mu x_2) &= \max\{ u_t(y) : y \in (\lambda x_1 + \mu x_2) + E_i^\perp \}. \\
  \text{Then there exist } y_1, y_2 \in L_i^+, \text{ such that } &\lambda x_1 + y_1 + \mu x_2 + y_2 \in (\lambda x_1 + \mu x_2) + E_i^\perp, \\
  u_t|_{L_i}(\lambda x_1 + \mu x_2) &= u_t(\lambda x_1 + y_1 + \mu x_2 + y_2) \\
  &\leq \lambda u_t(x_1 + y_1') + \mu u_t(x_2 + y_2') \\
  &\leq \lambda \max\{ u_t(\overline{y}_1) : \overline{y}_1 \in x_1 + L_i^+ \} \\
  &\quad + \mu \max\{ u_t(\overline{y}_2) : \overline{y}_2 \in x_2 + L_i^+ \} \\
  &= \lambda u_t|_{L_i}(x_1) + \mu u_t|_{L_i}(x_2).
\end{align*}
\]
Combing with the differentiability of their pointwise limit \( u|_{L_i} \) in the interior of its domain, we have the result. \( \square \)

Now let us introduce some fact about the functions \( u_t = u\Box(v t) \) with respect to the parameter \( t \), more about see [20].

**Lemma 3.5.** Let \( u \) and \( v \) belong both to the same class \( \mathcal{L}' \), \( u_t := u\Box(v t) \) \((t > 0)\).

Let \( L_i \) be \( i \)-dimensional linear subspace of \( \mathbb{R}^n \).

\[
\frac{d}{dt}(u_t|_{L_i})(x) = -\psi \left( \nabla (u_t|_{L_i})(x) \right),
\]
where \( \psi := v^*|_{L_i}, x \in \text{dom}(u_t|_{L_i}), \) and \( t > 0 \).

**Proof.** Set \( D_t := \text{dom}(u_t|_{L_i}) \subset L_i \), for every fixed \( x \in \text{int}(D_t) \), the map \( t \to \nabla (u_t|_{L_i})(x) \) is differentiable on \((0, +\infty)\). Indeed, by the definition of Fenchel conjugate and the definition of projection \( u \), it is easy to see that \( (u|_{L_i})^* = u^*|_{L_i} \) and \( (u\Box u t)|_{L_i} = u|_{L_i}\Box u t|_{L_i} \) holds. The Lemma 2.4 and the property of the projection grants the differentiability. Set \( \varphi := u|_{L_i} \) and \( \psi := v^*|_{L_i} \), and \( \varphi_t = \varphi + t\psi \), then \( \varphi_t \) belongs to the class \( \mathcal{C}^2 \) on \( L_i \). Then \( \nabla^2 \varphi_t = \nabla^2 \varphi + t\nabla^2 \psi \) is nonsingular on \( L_i \). So the equation
\[
\nabla \varphi(y) + t\nabla \psi(y) - x = 0,
\]
locally defines a map \( y = y(x, t) \) which is of class \( \mathcal{C}^1 \). By Proposition 2.2, we have \( \nabla (u_t|_{L_i}) \) is the inverse map of \( \nabla \varphi_t \), that is \( \nabla \varphi_t(\nabla (u_t|_{L_i}(x)) = x \), which means
that for every $x \in \text{int}(D_t)$ and every $t > 0$, $t \to \nabla(u_t|_{L_i})$ is differentiable. Using the equation (2.3) again, we have

$$u_t|_{L_i}(x) = \langle x, \nabla(u_t|_{L_i})(x) \rangle - \varphi_t(\nabla(u_t|_{L_i})(x)), \quad \forall x \in \text{int}(D_t). \quad (3.7)$$

Moreover, note that $\varphi_t = \varphi + t\psi$ we have

$$u_t|_{L_i}(x) = \langle x, \nabla(u_t|_{L_i})(x) \rangle - \varphi(\nabla(u_t|_{L_i})(x)) - t\psi(\nabla(u_t|_{L_i})(x))$$

$$= u_t|_{L_i}(\nabla(u_t|_{L_i})(x)) - t\psi(\nabla(u_t|_{L_i})(x)).$$

Differential the above formal we obtain,

$$\frac{d}{dt}(u_t|_{L_i})(x) = -\psi(\nabla(u_t|_{L_i})(x)).$$

Then we complete the proof of the result. \qed

4. Differentiability of the Functional Quermassintegrals of Log-concave Function

In this section, we discussed the functional $i$-th Quermassintegrals $W_i(f)$, we obtain the integral representation of the functional $i$-th mixed Quermassintegrals.

**Definition 4.1.** Let $f \in \mathcal{A}'$ be a integrable log-concave function on $\mathbb{R}^n$, $L_i \in \mathcal{G}_{i,n}$ ($i = 1, 2, \cdots, n$). Let $x \in \text{dom}(u)|_{L_i}$, the $i$-th total mass of $f$ is defined as

$$J_i(f) := \int_{L_i} f|_{L_i}(x) dx, \quad (4.1)$$

where $f|_{L_i}$ is the projection of $f$ onto $L_i$ defined by (3.2), $dx$ is the $i$-dimensional volume element in $L_i$.

**Remark 4.1.** (1) The definition of the $J_i(f)$ follows the $i$-dimensional volume of the projection a convex body. If $i = 0$, we defined $J_0(f) := \omega_n$, the volume of the unit ball in $\mathbb{R}^n$, for the completeness.

(2) When take $f = \chi_K$, the characteristic function of a convex body $K$, one has $J_i(f) = V_i(K)$, the $i$-dimensional volume in $L_i$.

**Definition 4.2.** Let $f \in \mathcal{A}'$ be a integrable log-concave function in $\mathbb{R}^n$. Set $L_i \in \mathcal{G}_{i,n}$ be $i$-dimensional linear subspace and, for any $x \in \text{dom}(u)|_{L_i}$, the functional $i$-th Quermassintegrals of $f$ (or the $i$-dimensional mean projection mass of $f$) are defined as

$$W_{n-i}(f) := \frac{\omega_n}{\omega_i} \int_{\mathcal{G}_{i,n}} J_i(f) dL_i, \quad (4.2)$$

where $J_i(f)$ is the $i$-th total mass of $f$ defined by (4.1), $dL_i$ is the normalized Haar measure on $\mathcal{G}_{i,n}$.

**Remark 4.2.** (1) The definition of the $W_i(f)$ follows the definition of the $i$-th Quermassintegral $W_i(K)$, that is, the $i$-th mean total mass of $f$ on $\mathcal{G}_{i,n}$. Also in the recently paper of Bobkov, Colesanti and Fragala [12], the authors give the same definition by defining the Quermassintegral of the support set for the quasi-concave functions.
(2) When \( i \) equals to \( n \) in (4.2), we have \( W_0(f) = \int_{\mathbb{R}^n} f(x)dx = J(f) \), the total mass function of \( f \) defined by Colesanti and Fragalà [20]. Then we can say that our definition of the \( W_i(f) \) is a nature extension of the total mass function of \( J(f) \).

(3) Form the definition of the Quermassintegrals \( W_i(f) \), the following properties are obtained (see also [12]).

- Positivity. \( 0 \leq W_i(f) \leq +\infty \).
- Monotonicity. \( W_i(f) \leq W_i(g) \), if \( f \leq g \).
- Generally speaking, the \( W_i(f) \) has no homogeneity under dilations. That is \( W_i(\lambda \cdot f) = \lambda^{n-i} W_i(f^\lambda) \), where \( \lambda \cdot f(x) = \lambda f(x/\lambda) , \lambda > 0 \).

Inspired by the definition of the mixed Quermassintegral and paper of Colesanti and Fragalà, we give the following definition.

**Definition 4.3.** Let \( f, g \in \mathcal{A}' \) are integrable functions of \( \mathbb{R}^n \). \( \oplus \) and \( \cdot \) denote the operations of “sum” and “scalar multiplication” in \( \mathcal{A}' \), \( W_i(f) \) and \( W_i(g) \) are, respectively, the Quermassintegrals of \( f \) and \( g \). Whenever the following limit exists

\[
W_i(f,g) = \frac{1}{(n-i)} \lim_{t \to 0^+} \frac{W_i(f \oplus t \cdot g) - W_i(f)}{t}, \tag{4.3}
\]

we denote it by \( W_i(f,g) \), and call it as the first variation of \( W_i \) at \( f \) along \( g \), or the functional \( i \)-th mixed Quermassintegrals of \( f \) and \( g \).

**Remark 4.3.** Let \( f = \chi_K \) and \( g = \chi_L \), with \( K, L \in \mathcal{K}^n \). In this case \( W_i(f \oplus t \cdot g) = W_i(K + tL) \), then \( W_i(f,g) = W_i(K,L) \). In general, \( W_i(f,g) \) has no analog properties of \( W_i(K,L) \), for example, \( W_i(f,g) \) is not always nonnegative and finite.

The following is devote to prove that \( W_i(f,g) \) exist under the fairly weak hypothesis. First, we prove that the first \( i \)-dimensional total mass of \( f \) is translation invariant.

**Lemma 4.4.** Let \( L_i \in \mathcal{G}_{i,n} \), \( f = e^{-u} \), and \( g = e^{-v} \) are integrable log-concave functions in \( \mathcal{A}' \). Let \( c = \inf u|_{L_i} =: u(0) \), \( d = \inf v|_{L_i} := v(0) \), and set \( \bar{u}_i(x) = u|_{L_i}(x) - c \), \( \bar{v}_i(x) = v|_{L_i}(x) - d \), \( \bar{\varphi}_i(y) = (\bar{u}_i)^*(y), \bar{\psi}_i(y) = (\bar{v}_i)^*(y) \), and \( \bar{f}_i = e^{-\bar{u}_i}, \bar{g}_i = e^{-\bar{v}_i}, \bar{f}_i|_{L_i} = \bar{f} \oplus t \cdot \bar{g} \). Then if

\[
\lim_{t \to 0^+} \frac{J_i(\bar{f}_i) - J_i(\bar{f})}{t} = \int_{L_i} \bar{\psi}_i d\mu_i(\bar{f}),
\]

holds, then we have

\[
\lim_{t \to 0^+} \frac{J_i(f_i) - J_i(f)}{t} = \int_{L_i} \psi_i d\mu_i(f).
\]

**Proof.** By the construction, we have \( \tilde{u}_i(0) = 0 \), \( \tilde{v}_i(0) = 0 \), and \( \tilde{v}_i \geq 0, \tilde{\varphi}_i \geq 0, \tilde{\psi}_i \geq 0 \). Further, we have \( \tilde{\psi}_i(y) = \psi_i(y) + d \), and \( \bar{f}_i = e^c \tilde{f}_i \). Then we have

\[
\lim_{t \to 0^+} \frac{J_i(\bar{f}_i) - J_i(\bar{f})}{t} = \int_{L_i} \bar{\psi}_i d\mu_i(\bar{f}) = e^c \int_{L_i} \psi_i d\mu_i(f) + dc \int_{L_i} d\mu_i(f). \tag{4.4}
\]
On the other hand, since
\[ f_i \oplus t \cdot g_i = e^{-(c+dt)} (f_i \oplus t \cdot g_i), \]
we have,
\[ J_i(f \oplus t \cdot g) = e^{-(c+dt)} J_i(f_i \oplus t \cdot g_i). \tag{4.5} \]
Derivative both sides of the above formula, we obtain
\[
\lim_{t \to 0^+} \frac{J_i(f \oplus t \cdot g) - J_i(f)}{t} \\
= -de^{-c} \lim_{t \to 0^+} J_i(f_i \oplus t \cdot g_i) dx + e^{-c} \lim_{t \to 0^+} \left[ \frac{J_i(f_i \oplus t \cdot g_i) - J_i(f_i)}{t} \right] \\
= -de^{-c} J_i(f_i) + \int_{L_i} \psi_i d\mu_i(f) + d \int_{L_i} d\mu_i(f) \\
= \int_{L_i} \psi_i d\mu_i(f).
\]
So we complete the proof. \hfill \Box

**Theorem 4.5.** Let \( f, g \in \mathcal{A}' \), and satisfy \(-\infty \leq \inf \log g \leq +\infty \) and \( W_i(f) > 0 \). Then \( W_j(f, g) \) is differentiable at \( f \) along \( g \), and it holds
\[
W_j(f, g) \in [-k, +\infty],
\tag{4.6}
\]
where \( k = \max \{ d, 0 \} W_i(f) \).

**Proof.** Since \( f|_{L_i} = e^{-u|_{L_i}} \), for every \( L_i \in \mathcal{G}_{i,n} \),
\[
u|_{L_i} := -\log(f|_{L_i}) = -(\log f)|_{L_i} \quad \text{and} \quad v|_{L_i} := -\log(g|_{L_i}) = -(\log g)|_{L_i}.
\]
By the definition of \( f_i \) and the Proposition 3.1 we obtain,
\[
f_i|_{L_i} = (f \oplus t \cdot g)|_{L_i} = f|_{L_i} \oplus t \cdot g|_{L_i}.
\]
Notice that \( v|_{L_i}(0) = v(0) \), set
\[
d := v(0), \quad \tilde{v}|_{L_i}(x) := v|_{L_i}(x) - d \\
\tilde{g}|_{L_i}(x) := e^{-\tilde{v}|_{L_i}(x)}, \quad \tilde{f}|_{L_i} := f|_{L_i} \oplus t \cdot \tilde{g}|_{L_i}.
\]
Up to a translation of coordinates, without loss of generality, we may assume \( \inf(v) = v(0) \). The Lemma 2.6 says that for every \( x \in L_i \),
\[
f|_{L_i} \leq \tilde{f}|_{L_i} \leq \tilde{f}|_{L_i}, \quad \forall x \in \mathbb{R}^n, \forall t \in [0, 1]
\]
Then there exists \( \tilde{f}|_{L_i}(x) := \lim_{t \to 0^+} \tilde{f}|_{L_i}(x) \), moreover, it holds \( \tilde{f}|_{L_i}(x) \geq f|_{L_i}(x) \) and \( \tilde{f}|_{L_i} \) is pointwise decreasing as \( t \to 0^+ \). Moreover, by Lemma 2.3 and Proposition 2.4, it shows that
\[
f|_{L_i} \oplus t \cdot \tilde{g}|_{L_i} \in \mathcal{A}', \quad \forall t \in [0, 1],
\]
Then \( J_i(f) \leq J_i(\tilde{f}_i) \leq J_i(\tilde{f}_i) \), moreover, \( -\infty \leq J_i(f), J_i(\tilde{f}_i) < \infty \). Hence, by the monotone and convergence, we have
\[
\lim_{t \to 0^+} W_i(\tilde{f}_i) = W_i(\tilde{f}).
\]
In fact, by definition we have $\tilde{f}_t|_{L_i}(x) = e^{-\inf\{u|_{L_i}(x-y)+tv|_{L_i}(y)\}}$, and
$$-\inf\{u|_{L_i}(x-y)+tv|_{L_i}(y)\} \leq -\inf u|_{L_i}(x-y) - t \inf v|_{L_i}(y),$$
Note that $-\infty \leq \inf(v|_{L_i}) \leq +\infty$, then $-\inf u|_{L_i}(x-y) - t \inf v|_{L_i}(y)$ is continuous function of variable $t$, then
$$\tilde{f}|_{L_i}(x) := \lim_{t \to 0^+} \tilde{f}_t|_{L_i}(x) = f|_{L_i}(x). \tag{4.7}$$
Moreover, $W_i(\tilde{f}_t)$ is continuous function of $t$ ($t \in [0, 1]$), then
$$\lim_{t \to 0^+} W_i(\tilde{f}_t) = W_i(f).$$
Since $f_t|_{L_i} = e^{-dt} \tilde{f}_t|_{L_i}(x)$, we have
$$\frac{W_i(f_t) - W_i(f)}{t} = W_i(f)e^{-dt} - 1 + e^{-dt}\frac{W_i(\tilde{f}_t) - W_i(f)}{t}, \tag{4.8}$$
Since $\tilde{f}_t|_{L_i} \geq f_{L_i}$, we have the following two cases, that is:
$$\exists t_0 > 0 : W_i(\tilde{f}_{t_0}) = W_i(f) \quad \text{or} \quad W_i(\tilde{f}_t) = W_i(f) \quad \forall t > 0.$$
For the first case, since $W_i(\tilde{f}_t)$ is a monotone increasing function of $t$, it must holds $W_i(\tilde{f}_t) = W_i(f)$ for every $t \in [0, t_0]$. Hence we have
$$\lim_{t \to 0^+} \frac{W_i(f_t) - W_i(f)}{t} = -dW_i(f),$$
the statement of the theorem holds true.
In the latter case, since $\tilde{f}_t|_{L_i}$ is increasing non-negative function, then it means that $\log(W_i(\tilde{f}_t))$ is an increasing concave function of $t$. Then
$$\exists \frac{\log(W_i(\tilde{f}_t)) - \log(W_i(f))}{t} \in [0, +\infty].$$
On the other hand, since
$$\log' \left( W_i(\tilde{f}_t) \right) \bigg|_{t=0} = \frac{1}{W_i(f)} = \lim_{t \to 0^+} \frac{\log(W_i(\tilde{f}_t)) - \log(W_i(f))}{W_i(\tilde{f}_t) - W_i(f)}.$$
Then
$$\lim_{t \to 0^+} \frac{W_i(\tilde{f}_t) - W_i(f)}{\log(W_i(\tilde{f}_t)) - \log(W_i(f))} = W_i(f) > 0. \tag{4.9}$$
From above we infer that
$$\exists \lim_{t \to 0^+} \frac{W_i(\tilde{f}_t) - W_i(f)}{t} \in [0, +\infty]. \tag{4.10}$$
Combining the above formula we obtain
$$\lim_{t \to 0^+} \frac{W_i(f_t) - W_i(f)}{t} \in [-\max\{d, 0\}W_i(f), +\infty].$$
In view of the example of the mixed Quermassintegral, it is natural to ask whether in general, \( W_i(f, g) \) has some kind of integral representation. The following theorem establishes the integral representation of \( W_i(f, g) \).

Let us begin by introducing the measures which intervene in the representation formulae for \( W_i(f, g) \).

**Definition 4.4.** Let \( L_i \in G_{i,n} \) and \( f = e^{-u} \in A' \) be integrable function of \( \mathbb{R}^n \). Consider the gradient map \( \nabla u : \mathbb{R}^n \to \mathbb{R}^n \), the Borel measure \( \mu_i(f) \) on \( L_i \) is defined by

\[
\mu_i(f) := \frac{\langle \nabla u_{|L_i} \rangle}{\|x\|^{n-1}}(f_{|L_i}),
\]

(4.11)

When \( \text{dom}(u) =: K \in \mathcal{K}^i \), we also set \( \sigma_i(f) \) the Borel measure on \( S^{i-1} \) defined by

\[
\sigma_i(f) := (\nu_K)_x(f\mathcal{H}^{i-1} \setminus \partial K),
\]

(4.12)

here \( \mathcal{H}^i \) is the \( i \)-dimensional Hausdorff measure measure.

Recall that the following Blaschke-Petkantschin formula is useful (see [33]).

**Proposition 4.6.** Let \( L_i \in G_{i,n} \) \((i = 1, 2, \cdots, n)\) be linear subspace of \( \mathbb{R}^n \), \( f \) be a non-negative bounded Borel function on \( \mathbb{R}^n \), then

\[
\int_{\mathbb{R}^n} f(x)dx = \frac{\omega_n}{\omega_i} \int_{G_{i,n}} \int_{L_i} f(x)\|x\|^{n-i}dxdL_i.
\]

(4.13)

**Theorem 4.7.** Let \( f, g \in A' \) be integrable functions on \( \mathbb{R}^n \). Let \( \mu_i(f) \) be the \( i \)-dimensional measure of \( f \), and \( W_i(f) \) \((i = 0, 1, \cdots, n - 1)\) are the Quermassintegrals of \( f \). Then

\[
W_i(f, g) = \frac{1}{n-i} \int_{\mathbb{R}^n} h_g d\mu_i(f),
\]

(4.14)

where \( h_g \) is the support function of \( g \).

**Proof.** By the definition of the \( i \)-th Quermassintegral of \( f \), we have

\[
\frac{W_i(f_t) - W_i(f)}{t} = \int_{G_{n-i,n}} J_{n-i}(f_t) - J_{n-i}(f) dx dL_{n-i}.
\]

Let \( t > 0 \) be fixed, take \( C \subset \subset \text{dom}(u)|_{L_{n-i}} \), and by reduction \( 0 \in \text{int}(\text{dom}(v)|_{L_{n-i}}) \), we have \( C \subset \subset \text{dom}(u)|_{L_{n-i}} \), by the Lemma 3.5, we obtain

\[
\lim_{h \to 0} \frac{J_{n-i}(f_{t+h})(x) - J_{n-i}(f_{t})(x)}{h} = \lim_{h \to 0} \int_{L_{n-i}} \frac{f_{t+h}L_{n-i}(x) - f_tL_{n-i}(x)}{h} dx,
\]

\[
= \int_{L_{n-i}} \psi(\nabla u_{|L_{n-i}}(x)) f_tL_{n-i}(x) dx.
\]
Remark 4.8. From the integral representation (4.14) it is easy seen that the functional $i$-th mixed Quermassintegral is linear in its second argument, with the sum in $A'$, for $f, g, h \in A'$

$$W_i(f, g \oplus h) = W_i(f, g) + W_i(f, h).$$

(4.15)

Specially, when $f = g$, and we show that $W_i(f, f)$ admits a nice representation in terms of the entropy of $f$. Let $\mu$ be a Probability measure, for every non-negative measurable function $f$, the entropy (see Ledoux [35], or Colesanti and Frugalà [20]) is defined by,

$$\text{Ent}(f) = \int_{\mathbb{R}^n} f \log f \, dx - \left( \int_{\mathbb{R}^n} f \, dx \right) \left( \log \int_{\mathbb{R}^n} f \, dx \right).$$

(4.16)

Here we define the entropy of the $i$-dimensional average total mass of $f$.

Definition 4.5. For every $f \in A'$ be a integrable function, the entropy of the $i$-dimensional average total mass of $f$ is defined by

$$\text{Ent}_i(f) = \frac{1}{n - i} \left[ \frac{\omega_n}{\omega_i} \int_{\mathbb{R}^n} f \log f \, dx - W_i(f) \log W_i(f) \right],$$

(4.17)

here $W_i(f)$ is the Quermassintegral of $f$. 

Where $h_g = \psi = \nu^*|_{L_{n-i}}$. Moreover, we have

$$\lim_{h \to 0} \frac{W_i(f_{t+h}) - W_i(f_t)}{h} = \frac{\omega_n}{\omega_{n-i}} \int_{\mathbb{R}^n} \int_{L_{n-i}} \frac{\psi(\nabla u_t|_{L_{n-i}}(x)) f_t|_{L_{n-i}}(x)}{\|x\|^{n-i}} \|x\|^{n-i} \, dx \, dL_{n-i},$$

$$= \int_{\mathbb{R}^n} \psi(\nabla u_t|_{L_{n-i}}(x)) f_t|_{L_{n-i}}(x) \, dx,$$

$$= \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f_t).$$

So we have

$$W_i(f_{t+h}) - W_i(f_t) = \int_0^t \left\{ \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f_s) \right\} ds.$$

The continuous of $\psi$ implies

$$\lim_{s \to 0^+} \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f_s) ds = \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f) ds.$$

Therefore,

$$\lim_{t \to 0^+} \frac{W_i(f_t) - W_i(f)}{t} = \frac{d}{dt} W_i(f_t)|_{t=0^+} = \lim_{s \to 0^+} \frac{d}{dt} W_i(f_t)|_{t=s}$$

$$= \lim_{s \to 0^+} \frac{d}{dt} \int_0^t \left\{ \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f_s) \right\} ds$$

$$= \int_{\mathbb{R}^n} \psi \, d\mu_{n-i}(f).$$

Since $h_g = \psi$, so we have $W_i(f, g) = \frac{1}{n-i} \lim_{t \to 0^+} \frac{W_i(f_t) - W_i(f)}{t} = \frac{1}{n-i} \int_{\mathbb{R}^n} h_g \, d\mu_i(f).$ 

□
Corollary 4.9. Let \( f \in \mathcal{A}' \) be integrable function on \( \mathbb{R}^n \), then \( \text{Ent}_i(f) \in (-\infty, +\infty) \) and

\[
W_i(f, f) = W_i(f) \left[ 1 + \frac{1}{n - i} \log W_i(f) \right] + \text{Ent}_i(f). \tag{4.18}
\]

Proof. Since \( f \in \mathcal{A}' \) then be integrable function, then \( f|_{L_i} \in \mathcal{A}' \). So by the Proposition 2.3, we have \( \text{Ent}_i(f) \in (-\infty, +\infty) \). Also we have \( u \sqcup u t = u(1 + t) \), then we get \( u \sqcup u t|_{L_i} = (1 + t) u|_{L_i} \). Then

\[
\frac{J_i(f \oplus t f) - J_i(f)}{t} = \frac{1}{t} \left[ (1 + t)^i \int_{L_i} e^{-(1+t)u|_{L_i}} dx - \int_{L_i} e^{-u|_{L_i}} dx \right]
= \frac{[(1 + t)^i - 1]}{t} \int_{L_i} e^{-(1+t)u|_{L_i}} dx + \int_{L_i} e^{-u|_{L_i}} \left( e^{-tu|_{L_i}} - 1 \right) dx.
\]

Now take limits when \( t \to 0^+ \), then we obtain

\[
\lim_{t \to 0^+} \frac{J_i(f \oplus t f) - J_i(f)}{t} = iJ_i(f) + \int_{L_i} f|_{L_i} \log f|_{L_i} dx. \tag{4.19}
\]

Then we have

\[
\lim_{t \to 0^+} \frac{W_i(f \oplus t f) - W_i(f)}{t} = \frac{\omega_n}{\omega_{n-i}} \int_{G_{n-i,n}} \lim_{t \to 0^+} \frac{J_{n-i}(f \oplus t f) - J_{n-i}(f)}{t} dL_{n-i}
= \frac{\omega_n}{\omega_{n-i}} \int_{G_{n-i,n}} \left[ (n - i)J_{n-i}(f) + \int_{L_{n-i}} f|_{L_{n-i}} \log f|_{L_{n-i}} dx \right] dL_{n-i}
= (n - i)W_i(f) + \frac{\omega_n}{\omega_{n-i}} \int_{G_{n-i,n}} \int_{L_{n-i}} f|_{L_{n-i}} \log f|_{L_{n-i}} dxdL_{n-i}.
\]

By the definition we obtain

\[
W_i(f, f) = \frac{1}{n - i} \lim_{t \to 0^+} \frac{W_i(f \oplus t f) - W_i(f)}{t}
= W_i(f) + \frac{\omega_n}{(n - i)\omega_{n-i}} \int_{G_{n-i,n}} \int_{L_{n-i}} f|_{L_{n-i}} \log f|_{L_{n-i}} dxdL_{n-i}
= W_i(f) \left[ 1 + \frac{1}{n - i} \log W_i(f) \right] + \text{Ent}_i(f).
\]

Then we complete the proof. \( \square \)

5. Mixed Quermassintegral Inequality for Log-Concave Function

Now we will discuss the functional form of Minkowski’s first inequality for Quermassintegrals.

Theorem 5.1. Let \( f \) and \( g \) be log-convex functions of \( \mathcal{A}' \), then we have

\[
W_i(f, g) \geq W_i(f) \left[ 1 + \frac{1}{n - i} \log W_i(g) \right] + \text{Ent}_i(f). \tag{5.1}
\]

With equality hold if and only if there exists \( x_0 \in \mathbb{R}^n \) such that \( g(x) = f(x - x_0) \), for all \( x \in \mathbb{R}^n \).
The inequality (5.1) is called the functional Brunn-Minkowski’s first inequality for Quermassintegrals or functional mixed Quermassintegral inequality. In the following we will give some special case of (5.1).

In fact, if we take $f = \chi_K$ and $g = \chi_L$, with $K, L \in K^n$. In this case $\chi_K \oplus t \cdot \chi_L = \chi_{K+tl}$, $J_i(\chi_K) = V_i(K)$, here $V_i$ denotes the $i$-dimensional volume in $L_i$, $W_i(\chi_K) = W_i(K)$, and $W_i(\chi_K, \chi_L) = W_i(K, L)$. In this case

$$Ent_i(\chi_K) = -\frac{1}{n-i} W_i(K) \log W_i(K).$$

Then (5.1) turn out to be

$$W_i(K, L) \geq W_i(K) \left[ 1 + \frac{1}{n-i} \log W_i(L) \right] + \frac{1}{n-i} W_i(K) \log \frac{W_i(L)}{W_i(K)}. \quad (5.2)$$

We can rewrite the above formula (5.2) equivalent to the following

$$\frac{W_i(K, L) - W_i(K)}{W_i(K)} \geq \frac{1}{n-i} \log \frac{W_i(L)}{W_i(K)}. \quad (5.3)$$

We define the $i$-cone volume probability measure $\overline{V}_{iK}$ similar with the $\overline{V}_K$ defined by Böröczky [13],

$$d\overline{V}_{iK} = \frac{1}{n} h_K dS_{iK},$$

where the $dS_{iK}$ is the $i$-th Borel measure on $S^{n-1}$. The normalized $i$-cone volume probability measure $\overline{V}_{Ki}$ is defined as

$$d\overline{V}_{Ki} = \frac{1}{W_i(K)} d\overline{V}_{iK}. \quad (5.4)$$

Then the normalized $i$-mixed Quermassintegrals $\overline{W}_i(K, L)$ is,

$$\overline{W}_i(K, L) = \frac{W_i(K, L)}{W_i(K)} = \int_{S^{n-1}} \frac{h_L}{h_K} d\overline{V}_{iK}. \quad (5.5)$$

Moreover by the integral representation of $W_i(K)$, we have

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K dS_{iK} = \int_{S^{n-1}} d\overline{V}_i(K)$$

Then formula (5.3) reads

$$\int_{S^{n-1}} \left( \frac{h_L}{h_K} - 1 \right) d\overline{V}_{iK} \geq \frac{1}{n-i} \log \frac{W_i(L)}{W_i(K)}. \quad (5.6)$$

We call (5.6) the weak $i$-the log Quermassintegral inequality. In fact, since

$$\frac{h_L}{h_K} - 1 \geq \log \frac{h_L}{h_K}, \quad (5.7)$$
for all \( u \in S^{n-1} \), and the equality holds if and only if \( \frac{h_L}{h_K} = 1 \), that is, \( K = L \).

For \( i = 0 \) and \( n = 2 \), since \( d\nu_{0K} = d\nu_K \), the cone volume probability measure of \( K \), then by (5.7) and (5.6) we obtain

\[
\int_{S^1} \left( \frac{h_L}{h_K} - 1 \right) d\nu_K \geq \int_{S^1} \log \frac{h_L}{h_K} d\nu_K \geq \frac{1}{2} \log \frac{V(L)}{V(K)}. \tag{5.8}
\]

So we have the following Corollary.

**Corollary 5.2.** Let \( K, L \in \mathcal{K}^n \), \( W_i(K) \) denotes the \( i \)-th Quermassintegrals, \( \nabla_{iK} \) be the normalized \( i \)-cone volume probability measure, then

\[
\int_{S^{n-1}} \left( \frac{h_L}{h_K} - 1 \right) d\nabla_{iK} \geq \frac{1}{n - i} \log \frac{W_i(L)}{W_i(K)}. \tag{5.9}
\]

When \( h_K = h_L \) then equality holds.

**Lemma 5.3.** Let \( f, g \in \mathcal{A}' \) are integrable functions, then

\[
\lim_{t \to 0^+} \frac{W_i((1 - t) \cdot f \oplus t \cdot g) - W_i(f)}{t} = (n - i) \left[ W_i(f, g) - W_i(f, f) \right]. \tag{5.10}
\]

**Proof.** First by Lemma 4.4, without lose of generality, we may assume that the function \( v = -\log g \) satisfies the condition \( v(0) = 0 \). For \( t \in (0, 1) \), let \( s(t) = \frac{t}{1-t} \), by (2.9) we obtain

\[
(1 - t) \cdot f \oplus t \cdot g = (1 - t) \cdot \left( f \oplus s(t) \cdot g \right).
\]

Let \( f_{s(t)} = f \oplus s(t) \cdot g \), then we have

\[
\frac{W_i((1 - t) \cdot f \oplus t \cdot g) - W_i(f)}{t} = \frac{W_i((1 - t) \cdot f_{s(t)}) - W_i(f_{s(t)})}{t} + \frac{W_i(f_{s(t)}) - W_i(f)}{t}. \tag{5.11}
\]

Concerning the first term of the right hand side (5.11), by Lemma 2.6 we know that the function \( f_{s(t)}(x) \) converge decreasingly to some pointwise limit \( f(x) \) as \( t \to 0^+ \), since \( s(t) \to 0^+ \) as \( t \to 0^+ \). In fact, we have

\[
\lim_{t \to 0^+} f_{s(t)}(x) = \lim_{t' \to 0^+} f_{t'}(x) = f(x).
\]

Then we obtain that

\[
\lim_{t \to 0^+} \frac{W_i((1 - t) \cdot f_{s(t)}) - W_i(f_{s(t)})}{t} = \lim_{t \to 0^+} \frac{W_i((1 - t) \cdot f) - W_i(f)}{t} = -(n - i)W_i(f, f). \tag{5.12}
\]

Concerning the second term, we have

\[
\lim_{t \to 0^+} \frac{W_i(f_{s(t)}) - W_i(f)}{t} = \lim_{t \to 0^+} \frac{W_i(f \oplus s(t) \cdot g) - W_i(f)}{t} = \lim_{t \to 0^+} \frac{W_i(f \oplus s(t) \cdot g) - W_i(f)}{s(t)} \cdot \frac{s(t)}{t} = (n - i)W_i(f, g). \tag{5.13}
\]
Then, one can show the conclude by combining the (5.12) and (5.13). □

Now we are in the position of the proof the functional mixed Quermassintegrals inequalities.

Proof of Theorem 5.1. First, we construct a function

$$\Psi(t) = \log \left( W_i((1 - t) \cdot f \oplus t \cdot g) \right).$$  \hspace{1cm} (5.14)

In fact, for every \(f, g, h \in A'\) and for every \(t \in [0, 1]\), since

$$h|_{L_i}(z) = \left( (1 - t) \cdot f|_{L_i} \oplus t \cdot g|_{L_i} \right)(z)
= \sup \left\{ f|_{L_i}^{1-t} g|_{L_i}^t : (1 - t)x + ty = z \right\}$$ \hspace{1cm} (5.15)

By the Prékopa-Leindler inequality, we have

$$\int_{L_i} h|_{L_i} dz \geq \left( \int_{L_i} f|_{L_i} dx \right)^{1-t} \left( \int_{L_i} g|_{L_i} dy \right)^t.$$

That means,

$$J_i(h) \geq J_i(f)^{1-t} J_i(g)^t.$$  \hspace{1cm} (5.16)

Integral both sides (5.16) on \(G_{i,n}\) with measure \(L_i\), by the Prékopa-Leindler inequality once again, we obtain

$$W_i((1 - t) \cdot f \oplus t \cdot g) \geq W_i(f)^{1-t} W_i(g)^t.$$  \hspace{1cm} (5.17)

Since \(\Psi(t) := \log \left( W_i((1 - t) \cdot f \oplus t \cdot g) \right)\), we conclude that, \(\Psi(t)\) is a concave on \([0, 1]\). Then, it holds

$$\frac{\Psi(t) - \Psi(0)}{t} \geq \Psi(1) - \Psi(0), \quad \forall t \in [0, 1].$$  \hspace{1cm} (5.18)

It means that \(\Psi(t)|_{t=0} \geq \Psi(1) - \Psi(0)\).

By Lemma (5.3), we have

$$\Psi(t)|_{t=0} = \frac{W_i((1 - t) \cdot f \oplus t \cdot g)}{W_i((1 - t) \cdot f \oplus t \cdot g)^t} \bigg|_{t=0} = \frac{(n - i) \left[ W_i(f,g) - W_i(f,f) \right]}{W_i(f)}.$$  

On the other hand, note that

$$\Psi(1) - \Psi(0) = \log \left( W_i(g) \right) - \log \left( W_i(f) \right).$$

Therefore, we obtain

$$\frac{(n - i) \left[ W_i(f,g) - W_i(f,f) \right]}{W_i(f)} \geq \log \left( W_i(g) \right) - \log \left( W_i(f) \right).$$
Then, combining with formula (4.18), we obtain

\[
W_i(f, g) \geq \frac{1}{n-i} W_i(f) \left[ \log(W_i(g)) - \log W_i(f) \right] + W_i(f, f) = W_i(f) \left[ 1 + \frac{1}{n-i} \log W_i(g) \right] + \text{Ent}_i(f)
\]

Concerning the equality case, first, assume that \( g(x) = f(x - x_0) \), by (4.18) and the invariance of the integral by translation of coordinates, we know that (5.1) hold with equality. On the other hand, if (5.1) holds with equality sign, by inspection of the above proof, one may see that the inequalities (5.16), (5.17) and (5.18) must hold as equalities. Moreover, whenever inequalities (5.16) and (5.17) hold with equality sign, then (5.18) automatic hold with equality. This entails that the Prékopa-Leindler inequality holds as an equality, therefore \( f \) and \( g \) must agree up to a translation.

\[\square\]

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References

[1] A.D. Aleksandrov, On the theory of mixed volumes. I: Extensions of certain concepts in the theory of convex bodies, Mat. Sb. 2 (1937), 947–972.
[2] D. Alonso-Gutiérrez, B. Merino, C. Jiménez, and R. Villa, John’s Ellipsoid and the Integral Ratio of a Log-Concave Function, J. Geom. Anal. 28 (2018), 1182–1201.
[3] D. Alonso-Gutiérrez, B.G. Merino, C.H. Jiménez, and R. Villa, Rogers-Shephard inequality for log-concave function, J. Funct. Anal. 271 (2016), 3269–3299.
[4] S. Artstein-Avidan, B. Klartag, and V.D. Milman, The Santaló point of a function, and a functional form of the Santaló inequality, Mathematika 51 (2004), 33–48.
[5] S. Artstein-Avidan and V.D. Milman, A characterization of the support map, Adv. Math. 223(1) (2010), 379–391.
[6] S. Artstein-Avidan and B.A. Slomka, A note on Santaló inequality for the polarity transform and its reverse, Proc. Amer. Math. Soc. 143(4) (2015), 1693–1904.
[7] S. Avidan, B. Klartag, C. Schütt, and E. Werner, Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality, J. Funct. Anal. 262 (2012), 4181–4204.
[8] K. Ball, Isometric Problems in l_p and Section of Convex Sets, Ph.D dissertation, Cambridge, 1986.
[9] _____, Logarithmically concave functions and sections of convex sets in \( \mathbb{R}^n \), Studia Math. 88 (1989), 69–84.
[10] F. Barchiesia, K.J. Böröczky, and M. Fradelizi, Stability of the functional forms of the Blaschke-Santaló inequality, Monatsh. Math. 173 (2014), 135–159.
[11] M. Barchiesia, G.M. Capranaib, N. Fusco, and G. Pisante, Stability of Pólya-Szego inequality for log concave functions, J. Funct. Anal. 267 (2014), 2264–2297.
[12] S. Bobkov, A. Colesanti, and I. Fragalà, Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities, Manuscripta Math. 143 (2014), 131–169.
[13] K.J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, The log-Brunn-Minkowski inequality, Adv. Math. 231 (2012), 1974–1997.
[14] H. Brascamp and E. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to diffusion equation*, J. Funct. Anal. 22 (1976), 366–389.

[15] H.J. Brascamp. and L. Leindler, *On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22(4) (1976), 366–389.

[16] U. Caglar and E. Werner, *Mixed f-divergence and inequalities for log concave functions*, Proc. Lond. Math. Soc. 110(2) (2015), 271–290.

[17] U. Caglar and E. Werner, *Divergence for s-concave and log concave functions*, Adv. Math. 257 (2014), 219–247.

[18] U. Caglar and D. Ye, *Affine isoperimetric inequalities in the functional Orlicz-Brunn-Minkowski theory*, Adv. Appl. Math. 81 (2016), 78–114.

[19] M.A. Hernández Cifre and J. Yepes Nicolás, *Brunn-Minkowski and Prékopa-Leindler’s inequalities under projection assumptions*, J. Math. Anal. Appl. 455 (2017), 1257–1271.

[20] A. Colesanti and I. Fragalà, *The first variation of the total mass of log-concave functions and related inequalities*, Adv. Math. 244 (2013), 708–749.

[21] N. Fang and J. Zhou, *LYZ ellipsoid and Petty projection body for log-concave functions*, Adv. Math. 340 (2018), 914–959.

[22] H. Federer, *Geometric Measure Theory*, Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, Berlin, 1969.

[23] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe Körper*, Danske Vid 16 (1938), 1–31.

[24] W.J. Firey, *p-means of convex bodies*, Math. Scand 10 (1962), 17–24.

[25] M. Fradelizi, Y. Gordon, M. Meyer, and S. Reisner, *The case of equality for an inverse Santaló inequality*, Adv. Geom. 10 (2010), 621–630.

[26] M. Fradelizi and M. Meyer, *Some functional forms of Blaschke-Santaló inequality*, Math. Z. 256 (2007), 379–395.

[27] R.J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. 39 (2002), 355–405.

[28] ______, *Geometric Tomography*, 2nd edition, Encyclopedia Math. Appl., vol. 58, Cambridge University Press, Cambridge, 2006.

[29] P.M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin Heidelberg, 2007.

[30] D.A. Gutiérrez, S.A. Avidan, B.G. Merino, C.H. Jiménez, and R. Villa, *Rogers-Shephard and local Loomis-Whitney type inequalities*, Mathe. Annal. (2019).

[31] C. Haberl and F. Schuster, *Asymmetric affine Lp Sobolev inequalities*, J. Funct. Anal. 257 (2009), 641–648.

[32] J. Haddad, C.H. Jiménez, and M. Montenegro, *Asymmetric Blaschke-Santaló functional inequalities*, J. Funct. Anal. (2019), https://doi.org/10.1016/j.jfa.2019.108319.

[33] E.B.V. Jensen, *Local Stereology*, World Scientific, New York, 1998.

[34] B. Klartag and V.D. Milman, *Geometry of log-concave functions and measures*, Geom. Dedicata 112 (2005), 169–182.

[35] M. Ledoux, *The concentration of Measure Phenomenon*, in: Mathematical Surveys and Monographs, vol. 89, American Mathematical Societey, 2001.

[36] J. Lehec, *A direct proof of the functional Santaló inequality*, A. R. Acad. Sci. Paris, Ser. I 347 (2009), 55–58.

[37] ______, *Partitions and functional Santaló inequality*, Arch. Math. (Basel) 92 (2009), 89–94.

[38] L. Leindler, *On a certain converse of Hölder inequality II*, Acta Sci. Math. 33 (1972), 217–223.

[39] Y. Lin, *Affine Orlicz Pólya-Szegő for log-concave functions*, J. Funct. Anal. 273 (2017), 3295–3326.

[40] E. Lutwak, *The Brunn-Minkowski-Firey Theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom. 38 (1993), 131–150.
[41] , The Brunn-Minkowski-Firey Theory II: Affine and geominimal surface area, Adv. Math. 118 (1996), 224–194.
[42] E. Lutwak, D. Yang, and G. Zhang, $L_p$ affine isoperimetric inequalities, J. Differential Geom. 56 (2000), 111–132.
[43] , Sharp affine $L_p$ sobolev inequalities, J. Differential Geom. 62 (2002), 17–38.
[44] , On the $L_p$-Minkowski problem, Trans. Amer. Math. Soc. 356(11) (2004), 4359–4370.
[45] , Volume inequalities for subspaces of $L_p$, J. Differential Geom. 68 (2004), 159–184.
[46] , Optimal Sobolev norms and the $L_p$ Minkowski problem, Int. Math. Res. Not. (2006), 1–21, IMRN 62987.
[47] A. Prekopa, Logarithmic concave measures with applications to stochastic programming, Acta Sci. Math. (Szeged) 32 (1971), 335–343.
[48] , On logarithmic concave measures with applications to stochastic programming, Acta Sci. Math. (Szeged) 34 (1973), 335–343.
[49] , New proof for the basic theorem of logconcave measures, Alkalmaz. Mat. Lapok 1 (1975), 385–389.
[50] T. Rockafellar, Convex analysis, Princeton Press, Princeton, 1970.
[51] L. Rotem, On the mean width of log-concave functions, in: Geometric Aspects of Functional Analysis, in: Geometric Aspects of Functional Analysis, in: Lecture Notes in Math., vol. 2050, Springer, Berlin, 2012.
[52] , Support functions and mean width for $\alpha$-concave functions, Adv. Math. 243 (2013), 168–186.
[53] , A sharp Blaschke-Santaló inequality for $\alpha$-concave functions, Geom. Dedicata 172 (2014), 217–228.
[54] R. Schneider, Convex Bodies: The Brunn-Minkowski Theory, Encyclopedia Math. Appl., vol. 58, Cambridge University Press, Cambridge, 1993.
[55] E. Werner, On $L_p$ affine surface areas, Indiana Univ. Math. 56 (2007), 2305–2324.
[56] E. Werner and D. Ye, New $L_p$ affine isoperimetric inequalities, Adv. Math. 218 (2008), 762–780.
[57] , Inequalities for mixed $p$-affine surface area, Math. Ann. 347 (2010), 703–737.

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