Free Lunch

George Svetlichny*

October 24, 2018

Abstract

The Free Lunch Principle: Nature thrives on freebies. She chooses nothing, and no one helps Her. She must use canonical mathematical structures as there is no one to tell Her otherwise. With this I show where variational principles are superfluous and Noether’s theorem is trivial. This paper is based on a talk I gave at the Harvey Brown Festschrift in Oxford, July 2015.

1 Introduction

There ain’t no such thing as a free lunch. This is an oft quoted fact of life but it just ain’t so in mathematics. By a free lunch I mean a mathematical structure that exists without external specification. These are among what mathematicians call canonical structures and these arise from the mere construction of other structures. I suggest that it is these structures that Nature uses. For simplicity, my examples will be taken from classical particle mechanics. In what follows I use standard notation and constructs from manifold theory, essentially vector fields, differential forms, and various bundles. Readers unfamiliar with these should consult subsection 4.1 of the Appendix for a quick tour of the needed material. Free lunches are just ain’t visible without this perspective.

We shall deal with a manifold $M$ with local coordinates $q^i$, $i = 1, . . . , n$ representing the configuration space (positions of the particles). Two immediate objects of interest are the tangent bundle $T(M)$ (where Lagrangians live) and the cotangent bundle $T^*(M)$, known as phase space (where Hamiltonians live).

An important example of a free lunch is the canonical 1-form on phase space $\theta = p_i dq^i$ (summation convention in force). Usually to have a one-form on a manifold you have to specify it externally for your own obscure reasons. On the cotangent bundle it’s just there. It arises from the very definition of the cotangent bundle. The canonical 1-form then defines the symplectic form $d\theta = dp_i \wedge dq^i$ along with the family of Hamiltonian dynamical systems. This may account for the prevalence of Hamiltonian systems in nature. It’s a class of

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*Departamento de Matemática, Pontifícia Universidade Católica, Rio de Janeiro, Brazil
svetlich@mat.puc-rio.br

http://www.mat.puc-rio.br/~svetlich

1There may be no global system of coordinates.
dynamical systems free for the taking. This family is a quintessential free lunch. More on this later.

Why is $\theta$ a free lunch? A 1-form on $M$ is written as $\alpha_jdq^j$. The pairs $(q^i, \alpha_j)$ are coordinate functions on $T^*(M)$. Physicists usually write the coordinates as $(q^i, p_j)$ and we shall do so from now on. Note that this pair represents a point $q$ in $M$ and a 1-form $p_jdq^j$ on $M$ at the given point $q$. Shortly we shall use the expression $p_jdq^j$ to mean the 1-form $\theta$ on $T^*(M)$ at the point $(q^i, p_j)$. Such double meanings of expressions is common among free lunches. To define $\theta$ at the point $(q^i, p_j)$ we have to say how it acts (contracts) on a vector at the same point. Let $X = \eta^i \frac{\partial}{\partial q^i} + \sigma_j \frac{\partial}{\partial p_j}$ be such a vector. Define $\langle \theta, X \rangle = \eta^i$. Notice that in this definition one just uses structures already present and no further objects need be introduced by the mathematician, or physicist, or anybody else. See subsection 12 of the Appendix for a geometric depiction of this argument.

There are many other free lunches such as the canonical map $j : V \rightarrow V''$ between a vector space $V$ and its double dual $V''$ given by $\langle j(v), \phi \rangle = \langle \phi, x \rangle$. Note the double meaning of $\phi$ in this expression, as a point in a vector space and as a linear form.

Another free lunch is the Lie-Jordan algebra associated to an associative algebra $A$. There is the Lie product: $[a, b] = ab - ba$ and the Jordan product: $a \ast b = ab + ba$. Every associative algebra is a Lie-Jordan algebra, extra structure free to use. Is it a surprise that quantum mechanics uses them?

Akin to a free lunch is the central limit theorem of probability. Many independent stochastic influences lead to the universal existence of the Gaussian distribution. This may have something to do with physical laws being beautiful and at most second order. As the universe is made up of very many parts, all ugliness and higher order contributions just get squeezed out by something like central limit theorem. Taking this a bit further, if any behavior of constituent parts, when there are very many of them, leads to a universal pattern one could take the attitude that such a universal pattern can exist for no reason at all. It’s just a pattern there for the taking, and Nature takes it. If anything leads to a pattern then just nothing itself leads to it. Maybe there are no fundamental building blocks to the world. Nature could just be manifesting the universal patterns that are just there.

Free lunches need no reason to be used. There is no deity or some “fundamental principle” or “law” needed to put them into practice. Just take and enjoy. So if Nature is consistently doing something we can’t quite understand, the question to ask is “where’s the free lunch?” If there is some central kitchen cooking up free lunches, then, if we can find it, we shall have a theory of everything as Nature would be feasting there.

I won’t try to define free lunch with any rigor, but just paraphrase the American Supreme Court Justice Potter Stewart: “I can’t define free lunch, but I know it when I see it!”

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2For the curious, the Honorable Justice Stewart was referring to pornography.
2 Where variational principles are not needed

I was always bothered by variational principles in physics. Why should Nature care about extremizing an action integral

\[ S(\phi) = \int L(x, \phi(x), \partial_\mu \phi(x)) \, d^n x \]

I of course don’t question this when one is really trying to minimize length or area or energy or something like that, but why care about the action integral for the Yang-Mills gauge theory coupled to fermions? What is Nature looking for? I decided She doesn’t care about the integral. Besides, the integral may not exist for the fields one is interested in. Most of the variational calculations done by theoretical physicists are formal in any case, so they too don’t care about the integral as such. What is going on?

If the action integral is a decoy, there should be a way of understanding all the benefits of “variation” without recourse to it. Where’s the free lunch? For a starter, consider just systems with a finite number of degrees of freedom and first order Lagrangians \( L(q^i, v^j) \) where \( v \) is velocity4. Variational calculus supplies us with a second-order ordinary differential equation starting with \( L \) and it’s first and second order derivatives. We must imagine how this can proceed in a coordinate free fashion. Taking first order derivatives means essentially computing \( dL \) which thus lives (is a section of) on \( T^*(TM) \). There must yet be another step to get to second order derivatives, which we won’t need here. Where do second-order ordinary differential equations live? They are special sections of \( TT(M) \), to be explained below. So how does one get from a section of \( T^*T(M) \) and wherever the second order derivatives live to a special section of \( TT(M) \) in a coordinate free fashion and universally for any \( L \) that you may choose? This is only possible if there are canonical relations that are just there, among the various bundles.

The intrinsic mathematical description of variation calculus involves iterated tangent and cotangent bundles. The four \( TT(M), TT^*(M), T^*T(M), T^*T^*(M) \) cover most of the situations for first order Lagrangians. These four double bundles offer a modest free banquet.

My previous attempt at getting Euler’s equations without the variational principle can be found in [1] where equivariance was the guiding principle. I did not get Euler’s equation uniquely from equivariance, but a narrow class of possible equations. Below I show a much quicker way based only on dimensional analysis and free snacks in the iterated bundle.

First some elementary mechanics.

Consider the kinetic energy:

\[ KE = \frac{1}{2} mv^2 = \frac{1}{2m} p^2. \]

4As a bothersome aside, the action integral seems to acquire importance in quantization using Feynman’s integrals, but that’s another kettle of fish. Where’s the free lunch?! Here I’ll be classical for now.

4What velocity really means we’ll see later.
One can’t square a vector or co-vector\(^5\) but \(\frac{1}{2}mv^2 = \frac{1}{2}(mv)v = \frac{1}{2}pv\). This is a perfectly legitimate contraction, a vector with a co-vector.

I will state Newton’s Laws as:

\[
v = \frac{dq}{dt}, \quad f = \frac{dp}{dt}
\]

(1)

Newton did not consider the first equation as a physical law but as an expression of velocity. As one of the creators of calculus, he would not object to the equation. I consider this equation as a physical law. Velocity is an attribute that a particle (or system) has instantaneously in contrast to Zeno’s idea that it would be instantaneously at rest\(^6\) Thus the first equation is a physical law on par with the second.

We need to consider \(p\) and \(v\) together. Now \(v\) lives in \(T(M)\), \(p\) in \(T^*(M)\).\(^7\)

To see them together try \(T^*T(M)\) which has elements

\[
(q^i, v^i; A_i dq^i + B_i dv^i).
\]

(2)

Denote these by \((q, v; A, B)\). As was pointed out above, \(v\) here is considered as a physical entity (Zeno’s oversight). Mathematicians don’t think like this, but physicists should.

Under change of coordinates \(q \mapsto \tilde{q}(q)\) one has:

\[
\tilde{A}_i = A_j \frac{\partial q^j}{\partial \tilde{q}^i} + B_j \tilde{v}^k \frac{\partial^2 q^j}{\partial q^k \partial \tilde{q}^i}, \quad \tilde{B}_i = B_j \frac{\partial q^j}{\partial \tilde{q}^i}
\]

\(B\) is a co-vector identified with momentum, (writing it as \(p\) would be more appropriate), but what is \(A\)?

It turns out that \(A\) is force. There are various ways to see this, we present two:

1. The free lunch way: \(\theta = p^i dq^i\) is free lunch, take time derivative: \(\dot{\theta} = f^i dq^i + p^i dv^i\). Compare with (2).

2. A more physical way: \(p_i v^i\) is a canonical scalar in \(T^*T(M)\) so is its time derivative. Thus \(\dot{f}_i \tilde{v}^i + \dot{p}_i \tilde{a}^i = f_i v^i + p_i a^i\). All transformations other then for \(f\) are known, and we deduce \(\dot{f}_i = f_{,a} \frac{\partial q^a}{\partial \tilde{q}^i} + p_j \frac{\partial^2 q^a}{\partial \tilde{q}^i \partial \tilde{q}^j} \tilde{v}^b\), same as for \(A\).

We can now give a variationless derivation of Euler’s equation. We have identified the physical dimensions of the components of \(T^*T(M)\):

\[
(q, v; A, B) = (\text{position, velocity; force, momentum})
\]

\(^5\)One needs a metric, but here is no free metric on \(T(M)\) or \(T^*(M)\).

\(^6\)Zeno was an earthling (as far as I know) and if he had known general relativity then he would have realized that with the earth present, the space time metric around a flying arrow is essentially different from that around an arrow at rest. Thus space-time knows the difference and Zeno should have. The first equation is an approximation to the true physical situation.

\(^7\)That we can write \(\text{KE} = \frac{1}{2}pv\) shows that mass establishes a relation between \(T(M)\) and \(T^*(M)\), a fact not yet explored in the literature.
Consider now a Lagrangian $L(q,v)$ and take its differential:

$$dL = \frac{\partial L}{\partial q^i} dq^i + \frac{\partial L}{\partial v^i} dv^i$$

So we now conclude: $\frac{\partial L}{\partial q_i}$ is force and $\frac{\partial L}{\partial v_i}$ is momentum.

By Newton’s laws we conclude:

$$v^i = \frac{dq^i}{dt}, \quad \frac{\partial L}{\partial q^i} = \frac{d}{dt} \frac{\partial L}{\partial v^i} \quad (3)$$

The second equation is Euler’s equations, showing it as being a free lunch. No need for a variational principle.

Other iterated bundles are also of interest. We have:

$$TT^*(M) : \left( q^i, p_i, S^i \frac{\partial}{\partial q^i} + T_j \frac{\partial}{\partial p_j} \right) = (q, p; S, T)$$

$$T^*T^*(M) : \left( q^i, p_i, Q_i dq^i + R_j dp_j \right) = (q, p; Q, R)$$

$$TT(M) : \left( q^i, v^i, U_i \frac{\partial}{\partial q^i} + V^j \frac{\partial}{\partial v^j} \right) = (q, v; U, V)$$

One has the following canonical isomorphisms (free lunches):

$$T^*T(M) \rightarrow TT^*(M) : (q, v; A, B) \mapsto (q, B; v, A) \quad (4)$$

$$T^*T^*(M) \rightarrow TT^*(M) : (q, p; Q, R) \mapsto (q, p; R, -Q) \quad (5)$$

Thus the physical dimensions are:

In $T^*T(M)$: (position, velocity; force, momentum)

In $TT^*(M)$: (position, momentum; velocity, force)

In $T^*T^*(M)$: (position, momentum; -force, velocity)

The bundle $TT(M)$ apparently has no canonical relation to the other three bundles. It’s physical role is very different and not at all clear. The physical dimensions of its components are (position, velocity; velocity, acceleration). There are two velocities, the first one, the above referred to physical attribute, the second one, the time derivative of position. The acceleration should also be considered as a physical entity being the time derivative of the physical velocity. This bundle carries some indication that the first of Newton’s laws is a true physical law and not a definition. I’ll come back to this bundle in the next section.

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8 “Canonical” here means the isomorphisms commute with coordinate changes on $M$.

9 Because of (4) and (5) any two iterated bundles such as $T^*TT^* \cdots T^*TM$ having the same number of iterations and at least one $T^*$ functor applied, are canonically isomorphic. The bundle $TTTT^* \cdots TT^*M$ with purely the $T$ functor applied, stands apart.
Consider now \( H(q, p) \), the hamiltonian function in \( T^*(M) \). One has the differential
\[
dH = \frac{\partial H}{\partial q} dq^i + \frac{\partial H}{\partial p_i} dp_i.
\]
The physical dimensions in \( T^*T^*(M) \) are: (position, momentum; -force, velocity), thus:
\[
\frac{\partial H}{\partial q^i} \text{ is -force, and } \frac{\partial H}{\partial p_i} \text{ is velocity.}
\]
By Newton’s laws (1):
\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}
\]
which are precisely Hamilton’s equations.

3 Hamilton is cleaner

It is known that the two schemes, Euler and Hamilton, are equivalent and there is a geometric way of seeing this.

The graph of \( dH \) as a subset of \( T^*T^*(M) \) is the following set..
\[
\left\{ \left( q^i, p_i, \frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i} \right) \mid (q, p) \in T^*(M) \right\}.
\]
We now use the canonical equivalence \( T^*T^*(M) \simeq T^*T(M) \) to bring this sets to \( T^*T(M) \).
\[
\left\{ \left( q^i, \frac{\partial H}{\partial p_i}; -\frac{\partial H}{\partial q^i}, p_i \right) \mid (q, p) \in T^*(M) \right\}.
\]
Since differentials of functions in \( T(M) \) live in \( T^*T(M) \), this should be the graph of \( dL \) for some function \( L \) which physicists call the Lagrangian. This means that
\[
p_i = \frac{\partial L}{\partial v^i} \left( q, \frac{\partial H}{\partial p} \right), \quad \frac{\partial H}{\partial q^i} = \frac{\partial L}{\partial q^i} \left( q, \frac{\partial H}{\partial p} \right)
\]
These are well know relations from which \( L \) can be reconstructed from \( H \) as
\[
L \left( q, \frac{\partial H}{\partial p} \right) = p_i \frac{\partial H}{\partial p_i} - H
\]
To express this in \( (q, v) \) coordinates one must solve \( v^i = \frac{\partial H}{\partial p_i} \) for \( p \) as a function of \( v \) and this can be done if \( (q, p) \mapsto \left( q, \frac{\partial H}{\partial p} \right) \) is invertible and we assume this from now on. Equivalently the map \( (q, v) \mapsto \left( q, \frac{\partial L}{\partial v} \right) \) is also invertible and it is under these conditions that Euler and Hamilton are equivalent.

\(^{10}\)Under a condition to be stated below
A geometric way of seeing this is to note that we have switched two “axes” (second and fourth) which depict the graph of the \( p \) derivative of \( H \). A sign was changed also.

Switching axes on graphs of derivatives is a well known procedure known as the Legendre transform.

\[
\begin{aligned}
\text{Axes} & \quad \xrightarrow{\text{switch}} \\
 f'(x) & \quad \quad g'(x)
\end{aligned}
\]

Now \( g \), the Legendre transform of \( f \), is defined by: \( g(x) = x(f')^{-1}(x) - f((f')^{-1}(x)) \). A possibly more familiar form is \( g(f'(x)) = xf'(x) - f(x) \). This can be easily shown to correspond to switching axes on graphs of derivatives. As before we are ignoring additive constants.

The equivalence \( TT^*(M)\simeq T^*T^*(M) \) exchanges vectors and 1-forms. The vector corresponding to \( \theta = p_i \, dq^i \) is \( \Theta = p_i \frac{\partial}{\partial p_i} \).

Now on any vector space \( V \), with linear coordinates \( x_i \), \( \Theta_V = x_i \frac{\partial}{\partial x^i} \) is a canonical vector field (free lunch).

The Legendre transforms \( L \mapsto \Theta_{TM} L - L = H \) and \( H \mapsto \Theta_{T^*M} H - H = L \) makes use of this free lunch.\(^{11}\)

A first order ordinary differential equation \( \dot{q}^i = \xi^i(q) \) is a vector field \( \mathcal{X} \) on \( M \) given by \( \mathcal{X} = \xi^i \frac{\partial}{\partial q^i} \). A second order differential equation should by all rights be a vector field on \( T(M) \), but it’s not any vector field that defines such an equation. Consider the equation \( \ddot{q}^i = \xi^i(q, \dot{q}) \). To associate a vector field to it we turn it into a first order system:

\[
\begin{aligned}
\dot{q}^i &= v^i \\
\dot{v}^i &= \xi^i(q, v)
\end{aligned}
\]

which corresponds to the vector field which at a point \( (q, v) \) is \( v^i \frac{\partial}{\partial q^i} + \xi^i \frac{\partial}{\partial v^i} \).

Note that this is on that part of the bundle where \textit{velocity} = velocity, that is \(^{11}\)In the first equation \( H \) is in \( (q, v) \) coordinates and in the second equation \( L \) is in \( (q, p) \) coordinates. In fairness, one could add constants to these transforms which does not change anything essential. If we want to include gravity, then the absolute scale of energy is important and constants need to be considered.
where the first of Newton’s laws (1) holds. The vector field \( \mathcal{X} \) on \( T(M) \) that correspond to Euler’s equation (second equation of (3)) is easily shown to be determined by:

\[
\mathcal{X} d\theta_T + dH_T = 0,
\]

(9)

where \( H_T = \Theta_{TM} L - L \) is the Hamiltonian function in \( (q,v) \) coordinates in \( T(M) \) and \( \theta_T = \frac{\partial L}{\partial v^i} dq^i \), one of the equivariant 1-forms in \( TM \) and the pullback of the canonical (free lunch) \( \theta \) in \( T^*(M) \) by \( (q,v) \mapsto (q, \frac{\partial L}{\partial v}) \).

Thus to define the 2nd order differential equation that is Euler’s, indirect references have to be made to structures in \( T^*(M) \) (phase space), a different bundle.

Compare (9) with the equation defining the Hamiltonian field in \( T^*(M) \):

\[
\mathcal{X} d\theta + dH = 0.
\]

(10)

In contrast to (9), this equation uses the free-lunch canonical \( \theta \) and no indirect references to another bundle. One might say that Euler’s equation is Hamilton’s equation seen through a glass darkly. It also hints that on the Hamiltonian side, in \( T^*(M) \), many constructs could be clearer and more natural, that is, free lunches, as we shall see below.

We now show the equivalence of Hamilton’s and Euler’s solutions.

Let \( (q(t), p(t)) \) be a path in \( T^*(M) \) satisfying Hamilton’s equations. Lifted to \( TT^*(M) \) it becomes

\[
(q(t), p(t); \dot{q}(t), \dot{p}(t))
\]

(11)

The Hamiltonian vector field (section of \( TT^*(M) \)) has the form:

\[
(q, p; \dot{q}, \dot{p} = \frac{\partial H}{\partial q}, -\frac{\partial H}{\partial p})
\]

(12)

As before, using a canonical isomorphism we transfer (11) and (12) to \( T^*T(M) \) (where \( dL \) lives):

\[
(q(t), \dot{q}(t); \dot{p}(t), p(t))
\]

(13)

\[
(q, \dot{q}, \dot{p} = \frac{\partial H}{\partial q}, -\frac{\partial H}{\partial p}, p)
\]

(14)

The latter is the graph of \( dL \), that is:

\[
\left( q, \dot{q}, \dot{p} = \frac{\partial L}{\partial q}, \frac{\partial H}{\partial q}, \frac{\partial L}{\partial v} \left( q, \dot{q}, \dot{p} \right) \right)
\]

But \( \frac{\partial H}{\partial p} = \dot{q} \) so on the path this is:

\[
\left( q(t), \dot{q}(t), \frac{\partial L}{\partial q} (q(t), \dot{q}(t)), \frac{\partial L}{\partial v} (q(t), \dot{q}(t)) \right)
\]

(15)
Comparing entries in (13) and (15) one has on the given path:
\[ v = \frac{dq}{dt}, \quad \frac{\partial L}{\partial q} = \frac{d}{dt} \frac{\partial L}{\partial v} \]
That is, the path satisfies Euler’s equation.

The argument can be run backwards to go from a solution of Euler’s equation to that of Hamilton’s.

Free lunch may not really explain why Lagrangian theories are so successful, but it does shed a light. So we all shove stuff into the physicist’s machine and get stuff back. We say, “Wow! We can get General Relativity, we can get the Standard Model, we can Tame Ferocious Tigers, aren’t we clever!” Nature doesn’t care we do this, we are her naughty kids; besides the factory is free to use by anyone. Exhilarating stuff, but let’s look around and ask (1) What are Nature’s machines made of; how do they work? (2) What is She banging together in Her hangar. Every time I have ever asked the first question the answer has been: Nature builds from parts that are just there and are free for the taking. What else could She use? There are no suppliers. I wish I knew the answer to the second question.

4 Where Noether’s theorem is trivial

First I review the usual variational argument that leads to Noether’s theorem. Let \( L(q, v) \) be a Lagrangian and consider the integral:
\[ \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) \, dt. \quad (16) \]
Consider now a “variation” of the \( q \) coordinates \( \tilde{q}^i = q^i + \epsilon \xi^i \) where \( \epsilon \) is considered infinitesimal and \( \xi^i \) are functions of \( q \). The assumption that is now made is the vanishing of the “variation” of (16) meaning the vanishing of the linear term in \( \epsilon \) in a Taylor expansion of:
\[ \int_{t_1}^{t_2} L(\tilde{q}(t), \tilde{\dot{q}}(t)) \, dt. \]
A simple chain rule calculation reveals that the “variation” is:
\[ \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} (q(t), \dot{q}(t)) \xi^i(q(t)) + \frac{\partial L}{\partial v^i} (q(t), \dot{q}(t)) \frac{\partial \xi^i}{\partial q^j} (q(t)) \dot{q}^j(t) \right) \, dt. \quad (17) \]
Now the hypothesis is that this must vanish identically for all paths \( q(t) \). Thus the integrand must vanish on all paths but the physics literature does not take

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12 Why would you want to do this?
13 The physics literature often writes \( \epsilon \xi^i \) as \( \delta q^i \) but this practice obscures even further what is really going on.
14 We are dealing here with point symmetries. More complicated “variations” can be considered, but these already make the main point.
this step and using integration by parts rewrites the integral as:

\[
\int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial v^i} \right) \xi^i + \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \xi^i \right) dt.
\]

The identical vanishing of the integrand is written thus:

\[
\left( \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial v^i} \right) \xi^i + \frac{d}{dt} \left( \frac{\partial L}{\partial v^i} \xi^i \right) = 0.
\]

Now comes the leap of the quantum cat: on a path that satisfies Euler’s equations the first term vanishes, so on solutions of Euler’s equations the second term vanishes, and the expression \( \frac{\partial L}{\partial v^i} \xi^i \) is a constant. Under vanishing of the “variation” of solutions of Euler’s equation have an associated constant of motion. This is Noether’s theorem which is obviously an important insight but the above deduction is strangely mysterious. Physics literature just presents the steps without any justification or insight beyond it just working.

Now to the simplification. The “variation” of the coordinates is just an obscure way of talking about a vector field: \( \mathcal{X} = \xi^i \frac{\partial}{\partial q^i} \). Lagrangians live on \( T(M) \) and the vector field \( \mathcal{X} \) has a canonical lifting \( \mathcal{X}_T \) to \( T(M) \) given by

\[
\mathcal{X}_T = \xi^k \frac{\partial}{\partial q^k} + v^j \frac{\partial}{\partial q^i} \frac{\partial q^j}{\partial v^k}. \]

The identical vanishing of the integrand in (17) for all paths \( q(t) \) is now simply seen as the statement:

\[
\mathcal{X}_T(L) = 0, \tag{18}
\]

which is a direct statement of invariance, the Lagrangian is constant on the integral paths of the vector field. One could just take this as the initial assumption of symmetry without invoking the integral and going through all those manipulations. One can rewrite (18) to get Noether’s theorem just as before but the situation is even simpler if we consider Noether’s theorem in \( T^*(M) \) and I do so now.

The vector field \( \mathcal{X} \) in \( M \) also lifts canonically to \( T^*(M) \) as:

\[
\mathcal{X}_{T^*} = \xi^i \frac{\partial}{\partial q^i} - p_j \frac{\partial \xi^j}{\partial q^i} \frac{\partial}{\partial p_i}.
\]

Invariance of the Hamiltonian, i.e. symmetry of dynamics is given by \( \mathcal{X}_{T^*}(H) = 0 \). Due to the equivalence \( T^*T^*(M) \simeq TT^*(M) \), the associated 1-form to \( \mathcal{X}_{T^*} \)

15I’m leaving out the arguments \( q(t) \) and \( \dot{q}(t) \) of the functions involved

16See subsection 4.1.1 of the Appendix.

17There is a canonical way of doing this but this would take us too far afield, and besides, the free lunch here is not very tasty.

18See subsection 4.1.1 of the Appendix.
is: \( p_i \frac{\partial \xi^j}{\partial q^i} dq^i + \xi^i dp_i \). This form is exact, equal to \( dK_X = d(p_i \xi^i) \). Thus the symmetry flow is Hamiltonian, and \( \mathcal{X}_T(H) = 0 \) now becomes: \( \{ K_X, H \} = 0 \). This is Noether’s Theorem! Read one way – deformation of \( H \) generated by \( K_X \) is zero, this is a statement of symmetry. Read the other way – time derivative of \( K_X \) under time evolution is zero, this is a statement of conservation.

The conserved quantity is \( p_i \xi^i \) which in \((q, v)\) coordinates is \( \xi^i \frac{\partial L}{\partial v^i} \) just as in the variational case.

Rather than “a profound connection between symmetries and conservation” as is constantly stated in physics books, Noether’s theorem here is just reading a vanishing Poisson bracket in two separate ways. Getting the theorem through variational calculus gives an impression of profundity, but that’s an illusion. The only bit of mathematics here is recognizing that the lifted field to \( T^*(M) \) is Hamiltonian, which is rather immediate.

The above view was presaged by a question posed by Dwight E. Neuen-schwarder in American Journal of Physics, 63, 489 (1995). Paraphrasing: “Is there a Noether’s theorem for discrete symmetries?” Two opposite answers were given in American Journal of Physics, 64, 849 (1996)). The “no” answer was given by Benito Hernández-Bermejo, and the “yes” answer for quantum theory by Robert Mills who wrote:

“The observable \( A \) is invariant under the transformation generated by \( \hat{B} \) if and only if the observable \( B \) is invariant under the transformation generated by \( \hat{A} \).” This is reading \([\hat{A}, \hat{B}] = 0\) in two ways.

Leaving the question of discrete symmetries aside, this is precisely the argument I gave above for classical theories.

So we have Noether’s theorem in \( T(M) \) and \( T^*(M) \). How do they compare? One has the following theorem:

\[ \mathcal{X}_{T^*}(H) = 0 \iff \mathcal{X}_T(L) = 0 \]

Thus Lagrangian and Hamiltonian point symmetries are the same. The proof is straightforward but tedious. See the subsection \( 4.3 \) in the Appendix.

Appendix

4.1 Quick tour of relevant manifold formalism

The configuration space (positions of the particles) will be a manifold \( M \) with local coordinates \( q^i, i = 1, \ldots, n \). The particles would have velocities \( v^i \) and joining both coordinates as \((q, v)\) creates a new manifold called the tangent bundle of \( M \), denoted by \( T(M) \). I will use the mathematician’s designation for vectors: \( v = v^i \frac{\partial}{\partial q^i} \), where I have adopted the summation convention in that

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19 See subsection 4.1.2 of the Appendix.
20 There may not be a global system of coordinates.
a repeated upper and lower index is summed over. A vector is a differential operator and given a vector \( \mathcal{X} = \xi^i \frac{\partial}{\partial q^i} \) and a function \( f \) of the variables \( q \) one has \( \mathcal{X} f = \xi^i \frac{\partial f}{\partial q^i} \).

A differential 1-form is a linear form on vectors (also known as a co-vector, or covariant vector) and is usually written with subscript indices: \( \alpha_i \). The form \( \alpha \) applied to the vector \( v \) gives the number \( \langle \alpha, v \rangle = \alpha_i v^i \) known as the contraction of \( \alpha \) with \( v \). The mathematician’s designation for a one-form is \( \alpha = dq^i \). The form \( dq^i \) is a special case of converting a function \( f \) of the variables \( q \) to a one form known as the differential of \( f \) by defining \( df = \frac{\partial f}{\partial q^i} dq^i \). In physics, momenta are 1-forms. Joining the coordinate \( q^i \) with the coefficients of a 1-forms \( \alpha \) as \( (q, \alpha) \) creates a new manifold called the cotangent bundle of \( M \), denoted by \( T^*(M) \). Physicists know \( T^*(M) \) as phase space.

There are higher order forms, a \( k \)-form is a totally antisymmetric covariant \( k \)-tensor \( \omega_{i_1 i_2 \cdots i_k} \). Mathematicians write this as

\[
\omega = \omega_{i_1 i_2 \cdots i_k} dq^{i_1} \wedge dq^{i_2} \wedge \cdots \wedge dq^{i_k}.
\]

The number \( k \) is known as the order of the form and we write \( k = |\omega| \). A 0-form is just a function, and on a manifold of dimension \( n \) there are no \( k \)-forms for \( k > n \). The product \( \wedge \) is bilinear, distributes over sums, and satisfies \( \alpha \wedge \beta = (-1)^{[\alpha][\beta]} \beta \wedge \alpha \). There is a differential operator \( d \) called exterior derivative defined by \( d \omega = d\omega_{i_1 i_2 \cdots i_k} dq^{i_1} \wedge dq^{i_2} \wedge \cdots \wedge dq^{i_k} \). This is a \( (k + 1) \)-form. One has the Leibnitz rule \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{[\alpha]} \alpha \wedge d\beta \), furthermore \( d^2 = 0 \). A \( k \)-form \( \alpha \) is exact if \( \alpha = d\beta \) for some \( \beta \) and is closed if \( d\alpha = 0 \). Locally closed forms are exact, but globally may not be.

A vector field \( \mathcal{X} \) and a \( k \)-form \( \omega \) can be contracted to produce a \( (k - 1) \)-form designated by mathematicians as \( \mathcal{X}[\omega] \) and given by

\[
\mathcal{X}[\omega] = v^a \omega_{a i_2 \cdots i_k} dq^{i_2} \wedge dq^{i_3} \wedge \cdots \wedge dq^{i_k}
\]

Both \( T(M) \) and \( T^*(M) \) are manifolds in their own right and so one can apply the functors \( T \) and \( T^* \) to these to get higher order iterated bundles such as \( TTT^*T(M) \) or \( T^*T^*(M) \) etc. Four of these are of fundamental importance and will be introduced at the appropriate time.

Let \( \phi : M \to N \) be a map between two manifolds. Given a vector \( \mathcal{X} \) at \( q \in M \) we can push it to a vector \( \phi_*(\mathcal{X}) \) (also denoted by \( d\phi(\mathcal{X}) \)) at the point \( \phi(q) \in N \) as follows: Choose a path \( q(t) \in M \) such that \( q(0) = q \) and \( q'(0) = \mathcal{X} \), then \( \phi_*(\mathcal{X}) = (\phi(q(t)))' \) at \( t = 0 \). One can also pull back a 1-form \( \omega \) at a point

\(\text{[21]}\) Without this convention a mathematician would write \( v = \sum_i v^i \frac{\partial}{\partial q^i} \).

\(\text{[22]}\) In practically all elementary physics books there is no distinction between vectors and 1-forms as the metric in 3-space can be used to convert one to the other. This completely obscures their truly different physical nature. Physicists, and especially their students, should know this.
r ∈ N to a 1-form φ∗(ω) to any point q ∈ M such that φ(q) = r. To define φ∗(ω) it’s enough to state what ⟨φ∗(ω), X⟩ is for any vector X at q, and we define ⟨φ∗(ω), X⟩ = ⟨ω, φ∗(X)⟩.

4.1.1 Lifting of vector fields

Given a vector field X = ξi∂/∂qi on M there are canonical lifting of it to T(M) and T∗(M). Think of the field as defining an infinitesimal coordinate transformation as ˜q^i = q^i + tξ^i with t infinitesimal. A vector transforms as ˜v^i = v^j∂q^j = v^i + tv^j∂ξ^i∂q^j. This means the lifted field in T(M) is X_{T,M} = ξ^i∂/∂q^i + v^j∂ξ^i∂/∂v^j.

For T∗(M) a 1-form transforms as ˜ω^i = ω^j∂q^j = ω^i − tω^j∂ξ^j∂q^i. This means the lifted field in T∗(M) is X_{T∗,M} = ξ^i∂/∂q^i − p^j∂ξ^j∂/∂p^i.

4.1.2 Poisson structure

The canonical equivalence T∗T∗(M) ≃ TT∗(M) exchanges 1-forms and vector fields on T∗(M). The vector field corresponding to df = ∂f/∂q^i dq^i + ∂f/∂p^i dp^i is X_f = −∂f/∂p^i ∂/∂q^i + ∂f/∂q^i ∂/∂p^i. Physicists would call such a function f a hamiltonian and X_f the hamiltonian vector field. Given two functions f and g we define the Poisson bracket by:

\{f, g\} = X_f(g).

A simple calculation reveals the usual formula:

\{f, g\} = ∂f/∂q^i ∂g/∂p^i − ∂f/∂p^i ∂g/∂q^i.

The Poisson bracket is anti-symmetric, which the first of the two equations above doesn’t show. Another canonical way of defining it is

\{f, g\} = X_f(X_g)dθ

where the antisymmetry is clear.
4.2 Geometry of $\theta$

In this picture $p$ stands for two things, a 1-form at the point $q$ in $M$, explicitly $p_i \, dq^i$, of which the $p_i$ are its components, and also the last $n$ coordinates of the point $(q, p) \in T^*(M)$. The 1-form $\theta$ at that point is also written as $p_i \, dq^i$ where the $p_i$ are the mentioned coordinates of the point $(q, p)$ used as components of $\theta$. Of course numerically the $p_i$ are the same in both usages. In this picture 1-forms are depicted as a series of level curves, actually lines, that is, if $\alpha$ is a 1-form, then the level surfaces $\langle \alpha, v \rangle = \text{const.}$, on the tangent space of the corresponding point, are a set of lines, and is a graphic way of depicting the 1-form.

The expression $\langle p, \pi_*(V) \rangle$ is precisely the expression $p_i \eta^i$ given in the introduction.

One sees that this picture does nothing more than display geometrical structures of the cotangent bundle, nothing extraneous to this bundle is brought in. It’s remarkable that these structures already pick out a class of dynamical systems, the Hamiltonian ones, which Nature actually employs. This is free lunch.
4.3 Equivalence of Euler and Hamilton symmetries

Let $X$ be a vector field on $M$ and $X_T = \xi^k \frac{\partial}{\partial q^k} + v^j \frac{\partial \xi^k}{\partial q^j} \frac{\partial}{\partial v^k}$ be its lift to $T(M)$ and $X_{T^*} = \xi^i \frac{\partial}{\partial q^i} - p_a \frac{\partial \xi^a}{\partial q^k} \frac{\partial}{\partial p_k}$ its lift to $T^*(M)$. Let $H$ be a Hamiltonian.

For any function $f(q,p)$, denote by $f^-$ the function $f(q,p(v))$ where $p(q,v) = p^-$ is the inverse function to $\frac{\partial H}{\partial p}$, that is $(\frac{\partial H}{\partial p})^- = v$.

One has by the chain rule
\[
\frac{\partial f^-}{\partial q^k} = \left( \frac{\partial f}{\partial q^k} \right)^- + \left( \frac{\partial f}{\partial p_a} \right)^- \frac{\partial p_a^-}{\partial q^k} \tag{19}
\]

Suppose $X_{T^*}(H) = 0$, we then have:
\[
\xi^i \left( \frac{\partial H}{\partial q^k} \right)^- = p_a^- \frac{\partial \xi^a}{\partial q^k} \left( \frac{\partial H}{\partial p_k} \right)^- = p_a^- \frac{\partial \xi^a}{\partial q^k} v^k \tag{20}
\]

One has
\[
X_T(L) = \left( \xi^k \frac{\partial}{\partial q^k} + v^m \frac{\partial \xi^k}{\partial q^m} \frac{\partial}{\partial v^m} \right) \left( p_a^- v^a - H^- \right) \tag{21}
\]

Expanding this, lexicographically, we have four contributions
\[
\xi^k \frac{\partial p_a^-}{\partial q^k} v^a \tag{22}
\]
\[
-\xi^k \frac{\partial H^-}{\partial q^k} = -\xi^k \left( \frac{\partial H}{\partial q^k} \right)^- - \xi^k v^a \frac{\partial p_a^-}{\partial q^k} \tag{23}
\]
\[
v^m \frac{\partial \xi^k}{\partial q^m} \frac{\partial p_a^-}{\partial v^a} + v^k \frac{\partial \xi^a}{\partial q^k} p_a \tag{24}
\]
\[
-v^k \frac{\partial \xi^a}{\partial q^k} \frac{\partial H^-}{\partial v^a} = -v^k \frac{\partial \xi^a}{\partial q^k} \left( \frac{\partial H}{\partial v^a} \right)^- - \frac{\partial p_a^-}{\partial v^a} = -v^k \frac{\partial \xi^a}{\partial q^k} v^m \frac{\partial p_m^-}{\partial v^a} \tag{25}
\]

Using (20) in (23) one sees that all contributions cancel out and we conclude $X_T(L) = 0$. A similar calculation in the other direction proves the converse.

Acknowledgements

My thanks to Harvey Brown for his wealth of ideas and his wisdom.

References

[1] George Svetlichny, “Equivariance, Variational Principles, and The Feynman Integral,” SIGMA, 4, 032 (2008), Doi: 10.3842/SIGMA.2008.032; arXiv:0711.4550