A MULTIPLICATIVE MEASURE ON
THE POSITIVE REAL AXIS

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Abstract. We construct a measure $\mu$ and a $\sigma$-algebra $\mathcal{M}$ of subsets of the positive real axis, $\mathbb{R}_{>0}$, with the following multiplicative property:

$$\mu \left( \bigcup_j E_j \right) = \prod_j \mu(E_j)$$

for every countable collection $\{E_j\}$ of pairwise disjoint subsets of $\mathcal{M}$. For them, we apply the Carathéodory’s procedure to the triplet $(\mathbb{R}_{>0}, \cdot, \tau)$, where $\cdot$ is the product of $\mathbb{R}$ and $\tau$ is the usual topology on $\mathbb{R}_{>0}$.

We conclude this note describing the connection between this multiplicative measure $\mu$ and the Lebesgue measure.

1. Introduction

Roughly speaking, the Carathéodory’s procedure in constructing the Lebesgue measure in $\mathbb{R}$ is the following: first, it is built a outer measure on $\mathbb{R}$ from the length of the closed intervals of $\mathbb{R}$. Once the outer measure is constructed, the next step is to define the Lebesgue measurable sets. The collection of measurable subsets of $\mathbb{R}$ turns out to be a $\sigma$-algebra. Then the Lebesgue measure is defined as the outer measure restricted on the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Such a measure, say $m$, satisfies

$$m \left( \bigcup_j E_j \right) = \sum_j m(E_j)$$

for every countable collection $\{E_j\}$ of pairwise disjoint measurable sets. This property of $m$ is called countable additivity (see, e.g. [3]).

The purpose of this note is to construct a multiplicative measure on the positive real axis $\mathbb{R}_{>0} = (0, +\infty)$. For them, we adapt the Carathéodory’s procedure to our new setting. We consider the positive real axis $\mathbb{R}_{>0}$ with the usual topology and define the ”length” of each closed interval $I = [a, b]$ in $\mathbb{R}_{>0}$ by

$$\ell(I) = b \cdot a^{-1}.$$ Then, to apply the Carathéodory’s idea with these elements, we will obtain a measure $\mu$ and a $\sigma$-algebra $\mathcal{M}$ in $\mathbb{R}_{>0}$ with the following properties:

$$\mu(\emptyset) = 1, \quad \mu((0, 1)) = +\infty, \quad \mu(\mathbb{R}_{>0}) = +\infty,$$

$$\mu \left( \bigcup_j E_j \right) = \prod_j \mu(E_j)$$

for every countable collection $\{E_j\}$ of pairwise disjoint sets of $\mathcal{M}$. We call to this property: countable multiplicativity. By this reason, we say that $\mu$ is a multiplicative measure.
Once constructed the measure $\mu$ and the $\sigma$-algebra $\mathcal{M}$, we will show that $\mathcal{M}$ coincides with the $\sigma$-algebra of the Lebesgue measurable subsets of $\mathbb{R}_{>0}$ and $\mu(E) = \lambda(E)$ for each $E \in \mathcal{M}$, where

$$\lambda(E) = \exp \left( \int_E \frac{1}{x} \, dx \right).$$

In Section 2 we give some preliminaries about single and double infinite products of positive real numbers which are necessary to our construction. The multiplicative measure $\mu$ is constructed in Section 3. Finally, in Section 4, we show that $\mu(E) = \lambda(E)$ for each $E \in \mathcal{M}$.

**Notation.** The positive real axis will be denoted by $\mathbb{R}_{>0} = (0, +\infty)$ and $\mathbb{R}_{\geq 1} = [1, +\infty)$.

## 2. Preliminaries

In this section we present some basic facts on certain single and double infinite products of real numbers (see, e.g. [1]).

### 2.1. Single infinite products.
Since we will consider only infinite products with positive factors, the following simplified definition will be adopted.

Given a sequence $\{a_j\}$ of positive real numbers, let

$$p_N = \prod_{j=1}^N a_j = a_1 \cdot a_2 \cdots a_N.$$

The number $p_N$ is called the $N$th partial product of the sequence $\{a_j\}$.

**Definition 1.** We say that an infinite product of positive real numbers $\prod_{j=1}^{+\infty} a_j$ converges if there exists a number $p > 0$ such that

$$\lim_{N \to +\infty} p_N = p.$$

In this case, $p$ is called the value of the infinite product and we write $p = \prod_{j=1}^{+\infty} a_j$.

**Remark 2.** If $p = 0$ or $p = +\infty$ or the limit does not exist we say that the product diverges. A necessary condition for convergence of an infinite product is

$$\lim_{j \to +\infty} a_j = 1,$$

but this is not sufficient.

**Remark 3.** If $\{a_j\}_{j=1}^{+\infty}$ is a sequence of real numbers greater than or equal to 1, then $1 \leq p_N \leq p_{N+1}$ for all $N \geq 1$. So, $\prod_{j=1}^{+\infty} a_j = \sup\{p_N : N \in \mathbb{N}\} \in [1, +\infty]$. This is, $\prod_{j=1}^{+\infty} a_j$ converges in $\mathbb{R}_{\geq 1}$ or $\prod_{j=1}^{+\infty} a_j = +\infty$.

The following theorem on rearrangements is an immediate consequence of Theorem 3.3 in [2] and from remark 3.

**Theorem 4.** If $\{a_j\}_{j=1}^{+\infty}$ is a sequence of real numbers greater than or equal to 1, then

$$\prod_{k=1}^{+\infty} a_{\sigma(k)} = \prod_{j=1}^{+\infty} a_j$$

for each bijection $\sigma : \mathbb{N} \to \mathbb{N}$.
The proof of Theorem 4 is simpler. Indeed, for $a_j \geq 1$, we have that

$$\prod_j a_j = \exp \left( \sum_j \log(a_j) \right) = \exp \left( \sum_k \log(a_{\sigma(k)}) \right) = \prod_j a_{\sigma(k)}.$$ 

2.2. Double infinite products. Formally, a double infinite product of positive real numbers is of the form

$$\prod_{i,j=1}^{+\infty} a_{ij}$$

where the $a_{ij}$'s are indexed by pair of natural numbers $i, j \in \mathbb{N}$.

A sequence of real numbers of the form $\{a_{ij}\}$ is called a double sequence.

Our main interest is in giving a definition of a double infinite product of real numbers greater than or equal to 1 that does not depend on the order of its factors.

If $F \subset \mathbb{N} \times \mathbb{N}$ is a finite subset of pairs of natural numbers, then we denote by

$$\prod_F a_{ij} = \prod_{(i,j) \in F} a_{ij}$$

the partial product of all $a_{ij}$'s whose indices $(i, j)$ belong to $F$.

**Definition 5.** The unordered infinite product of real numbers $a_{ij} \geq 1$ is

$$\prod_{\mathbb{N} \times \mathbb{N}} a_{ij} = \sup_{F \in \mathcal{F}} \left\{ \prod_F a_{ij} \right\}$$

where the supremum is taken over the collection $\mathcal{F}$ of all finite subsets $F \subset \mathbb{N} \times \mathbb{N}$.

Such an unordered infinite product converges if the supremum is finite and diverges to $+\infty$ if the supremum is $+\infty$. Note that this supremum exists in $\mathbb{R} \geq 1$ if and only if the finite partial products are bounded from above.

Next, we define rearrangements of a double infinite product into single infinite product and show that every rearrangement of a convergent unordered double infinite product converges to the same product.

**Definition 6.** A rearrangement of a double infinite product of positive real numbers

$$\prod_{i,j=1}^{+\infty} a_{ij}$$

is a single infinite product of the form

$$\prod_{k=1}^{+\infty} b_k, \quad b_k = a_{\sigma(k)}$$

where $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a one-to-one and onto map.

The following result is an adaptation of Theorem 8.42 in [1], pp. 201, to the context of double products.

**Theorem 7.** If the unordered product of a double sequence of real numbers greater than or equal to 1 converges, then every rearrangement of the double infinite product into a single infinite product converges to the unordered product.
Proof. Suppose that the unordered product \( \prod a_{ij}, \ a_{ij} \geq 1 \), converges with \( \prod_{N \times N} a_{ij} = s \in \mathbb{R}_{\geq 1} \), and let
\[
\prod_{k=1}^{+\infty} b_k
\]
be a rearrangement of the double product corresponding to a map \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \). For \( N \in \mathbb{N} \), let
\[
F_N = \{ \sigma(k) \in \mathbb{N} \times \mathbb{N} : 1 \leq k \leq N \},
\]
so that
\[
\prod_{k=1}^{N} b_k = \prod_{F_N} a_{ij}.
\]
Given \( \epsilon > 0 \), let \( F \subset \mathbb{N} \times \mathbb{N} \) be a finite set such that
\[
s - \epsilon < \prod_{F} a_{ij} \leq s,
\]
and let \( K \in \mathbb{N} \) be defined by \( K = \max\{ \sigma^{-1}(i, j) : (i, j) \in F \} \).
If \( N \geq K \), then \( F_N \supseteq F \) and since \( a_{ij} \geq 1 \), we obtain
\[
s - \epsilon < \prod_{F} a_{ij} \leq \prod_{F_N} a_{ij} \leq s.
\]
This implies that
\[
\left| \prod_{k=1}^{N} b_k - s \right| \leq \epsilon, \ \forall N \geq K.
\]
Thus,
\[
\prod_{k=1}^{+\infty} b_k = \prod_{N \times N} a_{ij}.
\]
\( \square \)

The rearrangement of a double infinite product into a single infinite product is one natural way to interpret a double infinite product in terms of single infinite products. Another way is to use iterated products of single infinite product.

Given a double product \( \prod a_{ij} \), one can define two iterated products, obtained by multiplying first over one index followed by the other:
\[
\prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right) = \lim_{N \rightarrow +\infty} \prod_{i=1}^{N} \left( \lim_{M \rightarrow +\infty} \prod_{j=1}^{M} a_{ij} \right),
\]
\[
\prod_{j=1}^{+\infty} \left( \prod_{i=1}^{+\infty} a_{ij} \right) = \lim_{M \rightarrow +\infty} \prod_{j=1}^{M} \left( \lim_{N \rightarrow +\infty} \prod_{i=1}^{N} a_{ij} \right).
\]

**Theorem 8.** A unordered product of real numbers \( a_{ij} \geq 1 \) converges if and only if either one of the iterated products
\[
\prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right), \ \prod_{j=1}^{+\infty} \left( \prod_{i=1}^{+\infty} a_{ij} \right)
\]
converges. Moreover, both iterated products converge to the unordered product:
\[ \prod_{N \times N} a_{ij} = \prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right) = \prod_{j=1}^{+\infty} \left( \prod_{i=1}^{+\infty} a_{ij} \right) \]

**Proof.** Suppose that the unordered product converges. Since \( a_{ij} \geq 1 \) and \( \sup_I (\sup_J c_{ij}) = \sup_{I \times J} c_{ij} \) for each double sequence of reals, we obtain
\[ \prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right) = \sup_N \left\{ \prod_{i=1}^{N} \left( \sup_{M} \prod_{j=1}^{M} a_{ij} \right) \right\} = \sup_{(N,M) \in \mathbb{N} \times \mathbb{N}} \left\{ \prod_{i=1}^{N} \prod_{j=1}^{M} a_{ij} \right\} \leq \prod_{N \times N} a_{ij}. \]

Conversely, suppose that one of the iterated products exists. Without loss of generality, we suppose that
\[ \prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right) < +\infty. \]

Let \( F \subset \mathbb{N} \times \mathbb{N} \) be a finite subset. Then, there exist two natural numbers \( N \) and \( M \) such that
\[ F \subset \{1, 2, ..., N\} \times \{1, 2, ..., M\} =: R. \]

So that,
\[ \prod_{F} a_{ij} \leq \prod_{R} a_{ij} = \prod_{i=1}^{N} \left( \prod_{j=1}^{M} a_{ij} \right) \leq \prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right). \]

Therefore, the unordered product converges and
\[ \prod_{N \times N} a_{ij} \leq \prod_{i=1}^{+\infty} \left( \prod_{j=1}^{+\infty} a_{ij} \right). \]

\[ \Box \]

### 3. Main Results

To construct our multiplicative measure \( \mu \), we consider the positive real axis \( \mathbb{R}_{>0} \) with the usual topology and for each closed interval \( I = [a, b] \) in \( \mathbb{R}_{>0} \), we define its "length", \( \ell(I) \), by
\[ \ell(I) = b \cdot a^{-1}, \]
where \( \cdot \) is the product of \( \mathbb{R} \). It is easy to check that

1. \( \ell(I) \geq 1 \) for each closed interval \( I = [a, b] \) in \( \mathbb{R}_{>0} \).
2. If \( I, J_1 \) and \( J_2 \) are three closed intervals in \( \mathbb{R}_{>0} \) such that \( I \subseteq J_1 \cup J_2 \), then \( \ell(I) \leq \ell(J_1) \cdot \ell(J_2) \).

In the sequel, a cover of a set \( E \subseteq \mathbb{R}_{>0} \) is a countable collection \( S \) of intervals \( I = [a, b] \), with \( 0 < a < b < +\infty \), such that \( E \subset \bigcup_{I \in S} I \).

Now we shall construct the exterior measure of an arbitrary subset \( E \) of \( \mathbb{R}_{>0} \). Given a set \( E \subseteq \mathbb{R}_{>0} \), we cover \( E \) by a countable collection of intervals \( S = \{I_j\} \) in \( \mathbb{R}_{>0} \), and let
\[ \nu(S) = \prod_{I_j \in S} \ell(I_j). \]
We point out that the value $\nu(S)$ is independent of the rearrangement of the cover $S = \{I_j\}$ of $E$ (see Theorem 3). The exterior measure of $E$, denoted $\mu_e(E)$, is defined by

$$\mu_e(E) = \inf_S \nu(S),$$

where the infimum is taken over all such covers $S$ of $E$. Thus, $1 \leq \mu_e(E) \leq +\infty$. Moreover, if $E_1 \subset E_2 \subset \mathbb{R}_{>0}$, then $\mu_e(E_1) \leq \mu_e(E_2)$.

**Theorem 9.** Let $I = [a, b] \subset \mathbb{R}_{>0}$. Then $\mu_e(I) = \ell(I)$.

**Proof.** That $\mu_e(I) \leq \ell(I)$ is obvious. To show the opposite inequality we choose a countable cover $S = \{I_j\}$ of $I$. Let $0 < \epsilon < 1$ be fixed. For each $j \in \mathbb{N}$, let $I_j^\ast$ be an interval of $\mathbb{R}_{>0}$ such that its interior $(I_j^\ast)^\circ \supset I_j$ and $\ell(I_j^\ast) \leq \frac{1+\epsilon}{1+\epsilon^N+1} \cdot \ell(I_j)$. Then $I \subset \bigcup_j(I_j^\ast)^\circ$. Since $I$ is closed and bounded, by the Heine-Borel Theorem, there exists $N \in \mathbb{N}$ such that $I \subset \bigcup_{j=1}^N I_j^\ast$. Now, the property 2. above implies that $\ell(I) \leq \prod_{j=1}^N \ell(I_j^\ast)$. Therefore,

$$\ell(I) \leq \frac{1+\epsilon}{1+\epsilon^N+1} \prod_{j=1}^N \ell(I_j) \leq (1 + \epsilon) \cdot \nu(S).$$

Then letting $\epsilon \to 0^+$, we obtain $\ell(I) \leq \nu(S)$ and hence $\ell(I) \leq \mu_e(I)$.

**Theorem 10.** If $E = \bigcup_j E_j$ is a countable union of subsets of $\mathbb{R}_{>0}$, then

$$\mu_e(E) \leq \prod_j \mu_e(E_j).$$

**Proof.** Without loss of generality, we may assume that $\prod_j \mu_e(E_j) < +\infty$. Now, we fix $0 < \epsilon < 1$. Given $j \in \mathbb{N}$, we choose intervals $I_{ij}$ of $\mathbb{R}_{>0}$ such that $E_j \subset \bigcup_i I_{ij}$ and

$$\prod_i \ell(I_{ij}) \leq \frac{1+\epsilon}{1+\epsilon^N+1} \mu_e(E_j).$$

Let $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection, then the collection $\{J_k\}$ defined by $J_k = I_{\sigma(k)}$ is a cover of $E$. So that,

$$\mu_e(E) \leq \prod_k \ell(J_k) = \prod_{N \times \mathbb{N}} \ell(I_{ij}) = \prod_j \left( \prod_i \ell(I_{ij}) \right) \leq (1 + \epsilon) \prod_j \mu_e(E_j),$$

where the first equality follows from Theorem 7, the second from Theorem 8, and the last inequality from 11 and

$$\prod_{j=1}^N \left( \frac{1+\epsilon}{1+\epsilon^N+1} \mu_e(E_j) \right) = \frac{1+\epsilon}{1+\epsilon^N+1} \frac{\prod_{j=1}^N \mu_e(E_j)}{\prod_{j=1}^{+\infty} \mu_e(E_j)} \to (1 + \epsilon) \prod_{j=1}^{+\infty} \mu_e(E_j).$$

Finally, the result follows by letting $\epsilon \to 0$ in 12.

The next theorem relates the exterior measure of an arbitrary set of $\mathbb{R}_{>0}$ with the exterior measure of open sets and $G_\delta$ sets of $\mathbb{R}_{>0}$. This theorem is an analogous of the theorems 3.6 and 3.8 in [3], pp. 44.

**Theorem 11.** Let $E \subseteq \mathbb{R}_{>0}$, then

(i) given $0 < \epsilon < 1$, there exists and open set $G$ such that $E \subset G$ and $\mu_e(G) \leq (1 + \epsilon) \cdot \mu_e(E)$;

(ii) there exists a set $H$ of type $G_\delta$ in $\mathbb{R}_{>0}$ such that $E \subset H$ and $\mu_e(E) = \mu_e(H)$. 

\[
\begin{align*}
\text{(i)} & \quad \text{given } 0 < \epsilon < 1, \text{ there exists an open set } G \text{ such that } E \subset G \text{ and } \mu_e(G) \leq (1 + \epsilon) \cdot \mu_e(E); \\
\text{(ii)} & \quad \text{there exists a set } H \text{ of type } G_\delta \text{ in } \mathbb{R}_{>0} \text{ such that } E \subset H \text{ and } \mu_e(E) = \mu_e(H).
\end{align*}
\]
Indeed, given \( 0 < \epsilon < E \), we have \( \mu \{ a \} = 0 \) and choose a cover \( \{ I_j \} \) of \( E \) in \( \mathbb{R}_{>0} \) such that \( \prod_j \ell(I_j) \leq \sqrt{(1 + \epsilon)} \cdot \mu_e(E) \). For each \( j \in \mathbb{N} \), let \( I^*_j \) be an interval of \( \mathbb{R}_{>0} \) such that its interior \( (I^*_j)^\circ \supset I_j \) and \( \ell(I^*_j) \leq \frac{\sqrt{(1 + \epsilon)}}{\sqrt{(1 + \epsilon^2 + 1)}} \cdot \ell(I_j) \). If \( G = \bigcup_j (I^*_j)^\circ \), then \( G \) is an open set such that \( E \subset G \) and

\[
\mu_e(G) \leq \prod_j \ell(I^*_j) \leq \sqrt{(1 + \epsilon)} \cdot \prod_j \ell(I_j) \leq (1 + \epsilon) \cdot \mu_e(E).
\]

\( \square \)

The following corollary is an immediate consequence of Theorem \( \textbf{11} \) (i).

**Corollary 12.** Let \( E \subset \mathbb{R}_{>0} \). Then

\[
\mu_e(E) = \inf \mu_e(G),
\]

where the infimum is taken over all open sets \( G \) of \( \mathbb{R}_{>0} \) containing \( E \).

A set \( E \subset \mathbb{R}_{>0} \) is said to be measurable if given \( \epsilon > 0 \), there exists an open set \( G \) of \( \mathbb{R}_{>0} \) such that \( E \subset G \) and

\[
\mu_e(G \setminus E) < 1 + \epsilon.
\]

If \( E \subset \mathbb{R}_{>0} \) is measurable, its exterior measure is called its measure, which we denote by \( \mu(E) \), i.e.:

\[
\mu(E) = \mu_e(E), \quad \text{for measurable } E.
\]

We say that a set \( E \) is \( \mu \)-measurable if satisfies the above definition.

The next list of examples and properties of measurable sets of \( \mathbb{R}_{>0} \) follows from the previous results and the definition of measurability.

1) Every open set of \( \mathbb{R}_{>0} \) is \( \mu \)-measurable. For instance, the empty set and the sets \( \mathbb{R}_{>0} \) and \( (0, 1) \) are \( \mu \)-measurable with \( \mu(\emptyset) = 1 \), \( \mu(\mathbb{R}_{>0}) = +\infty \) and \( \mu((0, 1)) = +\infty \).

2) Every set of exterior measure \( 1 \) is \( \mu \)-measurable. For instance, every point \( \{ a \} \) of \( \mathbb{R}_{>0} \) is \( \mu \)-measurable with \( \mu(\{ a \}) = 1 \).

The claim 2) follows from Theorem \( \textbf{11} \) (i).

3) If \( E = \bigcup_j E_j \) is a countable union of \( \mu \)-measurable sets of \( \mathbb{R}_{>0} \), then \( E \) is \( \mu \)-measurable and

\[
\mu(E) \leq \prod_j \mu(E_j).
\]

Indeed, given \( 0 < \epsilon < 1 \), for each \( j = 1, 2, \ldots \), we take an open set \( G_j \) such that \( E_j \subset G_j \) and \( \mu_e(G_j \setminus E_j) < \frac{1 + \epsilon}{1 + \epsilon^2 + 1} \). Then \( G = \bigcup_j G_j \) is an open such that \( E \subset G \), and since \( G \setminus E \subset \bigcup_j (G_j \setminus E_j) \), we have

\[
\mu_e(G \setminus E) \leq \mu_e \left( \bigcup_j (G_j \setminus E_j) \right) \leq \prod_j \mu_e((G_j \setminus E_j)) \leq (1 + \epsilon),
\]

where the second inequality follows from Theorem \( \textbf{10} \). Thus \( E \) is \( \mu \)-measurable and \( \mu(E) = \mu_e(E) \leq \prod_j \mu_e(E_j) = \prod_j \mu(E_j) \).

4) Every interval \( I \subset \mathbb{R}_{>0} \) is \( \mu \)-measurable. If \( I \) is bounded and \( \inf I > 0 \), then \( \mu(I) = \ell(I) \); opposite case \( \mu(I) = +\infty \).
The claim 4) follows from Theorem [9] and from 1) and 2) above.

5) If \( I_j \) is a finite collection of nonoverlapping intervals of \( \mathbb{R}_{>0} \), then \( \bigcup_j I_j \) is \( \mu \)-measurable and \( \mu \left( \bigcup_j I_j \right) = \prod_j \mu(I_j) \).

The claim 5) is an extension of Theorem [9].

6) If \( E_1 \) and \( E_2 \) are two subsets of \( \mathbb{R}_{>0} \) such that
\[
d(E_1, E_2) := \inf \{|x-y| : x \in E_1, y \in E_2\} > 0,
\]
then \( \mu_e(E_1 \cup E_2) = \mu_e(E_1) \cdot \mu_e(E_2) \).

The proof of the claim 6) is similar to that of Lemma 3.16, in [3], but reformulated to our setting.

7) Every closed set \( F \) of \( \mathbb{R}_{>0} \) is \( \mu \)-measurable.

To see 7), we assume first that \( F \) is a compact set in \( \mathbb{R}_{>0} \), so inf \( F > 0 \) and hence \( \mu_e(F) < +\infty \). Given \( 0 < \epsilon < 1 \), by Theorem [11] statement (i), there exists an open set \( G \) such that \( F \subset G \) and \( \mu_e(G) \leq (1 + \epsilon) \cdot \mu_e(F) \). Since \( G \setminus F \) is open in \( \mathbb{R}_{>0} \), Theorem 1.11 in [3], implies there exists a collection of nonoverlapping closed intervals \( \{I_j\} \), where \( I_j = [a_j, b_j] \) with \( 0 < a_j < b_j \), such that \( G \setminus F = \bigcup_j I_j \). Thus, \( \mu_e(G \setminus F) \leq \prod_j \mu_e(I_j) \). Then, to prove the \( \mu \)-measurability of \( F \) it suffices to show that \( \prod_j \mu_e(I_j) \leq 1 + \epsilon \). We have \( G = F \cup (\bigcup_j I_j) \supset F \cup (\bigcup_{j=1}^N I_j) \) for each \( N \). Since \( F \) and \( \bigcup_{j=1}^N I_j \) are disjoint and compact, by 6), we obtain
\[
\mu_e(G) \geq \mu_e \left( F \cup \left( \bigcup_{j=1}^N I_j \right) \right) = \mu_e(F) \cdot \mu_e \left( \bigcup_{j=1}^N I_j \right).
\]
By 5), we have \( \prod_{j=1}^N \mu_e(I_j) = \mu_e \left( \bigcup_{j=1}^N I_j \right) \leq \frac{\mu_e(G)}{\mu_e(F)} \leq 1 + \epsilon \). This proves the result in the case when \( F \) is compact.

To end, let \( F \) be any closed subset of \( \mathbb{R}_{>0} \) and write \( F = \bigcup_j F_j \), where \( F_j = F \cap \{x > 0 : j^{-1} \leq x \leq j\} \), \( j = 2, 3, \ldots \). Since, each \( F_j \) is compact and, thus, \( \mu \)-measurable; then the \( \mu \)-measurability of \( F \) follows from 3).

8) The complement of a \( \mu \)-measurable set of \( \mathbb{R}_{>0} \) is \( \mu \)-measurable.

The proof of 8) is similar to that of Theorem 3.17 in [3].

9) If \( \{E_j\} \) is a countable collection of \( \mu \)-measurable sets of \( \mathbb{R}_{>0} \), then \( \bigcap_j E_j \) is \( \mu \)-measurable.

The proof of 9) is similar to that of Theorem 3.18 in [3].

10) A set \( E \) of \( \mathbb{R}_{>0} \) is \( \mu \)-measurable if and only if given \( \epsilon > 0 \), there exists a closed set \( F \subset E \) such that \( \mu_e(E \setminus F) < 1 + \epsilon \).

The proof of 10) is similar to that of Lemma 3.22 in [3].

The following result follows from 3), 8) and 9).
Theorem 13. Let $\mathcal{M}$ be the collection of $\mu$-measurable subsets of $\mathbb{R}_{>0}$. Then, $\mathcal{M}$ is a $\sigma$-algebra.

From 1) and this theorem we obtain the following corollary.

Corollary 14. Every Borel set of $\mathbb{R}_{>0}$ is $\mu$-measurable.

We are now in a position to prove the following theorem.

Theorem 15. If $\{E_j\}$ is a countable collection of disjoint $\mu$-measurable subsets of $\mathbb{R}_{>0}$, then

$$\mu \left( \bigcup_j E_j \right) = \prod_j \mu(E_j).$$

Proof. First, we assume that each $E_j$ is bounded with $\inf E_j > 0$. Given $0 < \epsilon < 1$ and $j = 1, 2, \ldots$, by 10), there exists a closed $F_j \subset E_j$ such that $\mu(E_j \setminus F_j) < \frac{\epsilon j!}{(1+\epsilon)^j}$. Then, $\mu(E_j) \leq 1 + \frac{\epsilon j!}{1+\epsilon} \cdot \mu(F_j)$. Since the $E_j$ are bounded with $\inf E_j > 0$ and disjoint, the $F_j$ are compact and disjoint. Then, by 6), we have $\mu \left( \bigcup_{j=1}^N F_j \right) = \prod_{j=1}^N \mu(F_j)$, for each $N$.

So,

$$\mu \left( \bigcup_j E_j \right) \geq \prod_{j=1}^N \mu(F_j) \geq \frac{1 + \epsilon^N + 1}{1 + \epsilon} \prod_{j=1}^N \mu(E_j) \geq \frac{1}{1 + \epsilon} \prod_{j=1}^N \mu(E_j),$$

for each $N$. By letting $N \to +\infty$ and since $0 < \epsilon < 1$ is arbitrary, we obtain $\mu \left( \bigcup_j E_j \right) \geq \prod_{j=1}^{+\infty} \mu(E_j)$. The opposite inequality follows from Theorem 33. This proves the theorem in the case when the $E_j$ are bounded with $\inf E_j > 0$.

For the general case, let $I_k = [k^{-1}, k], \ k = 1, 2, \ldots$, and we define $J_1 = I_1$ and $J_k = I_k \setminus I_{k-1}$ for $k \geq 2$. Then the sets $A_{jk} = E_j \cap J_k$, with $j, k = 1, 2, \ldots$, are bounded, disjoint, $\mu$-measurable and with $\inf A_{jk} > 0$. Since $E_j = \bigcup_k A_{jk}$ and $\bigcup_j E_j = \bigcup_{j,k} A_{jk}$, by the case already established and proceeding, with the double product, as in the proof of Theorem 33, we have

$$\mu \left( \bigcup_j E_j \right) = \mu \left( \bigcup_{j,k} A_{jk} \right) = \prod_{N \times N} \mu(A_{jk}) = \prod_{j=1}^{+\infty} \left( \prod_{k=1}^{+\infty} \mu(A_{jk}) \right) = \prod_{j=1}^{+\infty} \mu(E_j).$$

This completes the proof. \qed

We also have the Carathéodory’s characterization for $\mu$-measurable sets of $\mathbb{R}_{>0}$.

The proof of the following theorem is similar to that of Theorem 3.30, in [3], but reformulated to our setting.

Theorem 16. A set $E$ of $\mathbb{R}_{>0}$ is $\mu$-measurable if and only if for every set $A$ of $\mathbb{R}_{>0}$

$$\mu_e(A) = \mu_e(A \cap E) \cdot \mu_e(A \setminus E).$$

We summarize our main result in the following theorem.

Theorem 17. Let $\mathbb{R}_{>0}$ with the usual topology. Then there exist a $\sigma$-algebra $\mathcal{M}$ containing every Borel set of $\mathbb{R}_{>0}$ and a measure $\mu : \mathcal{M} \to [1, +\infty]$ such that

(i) $\mu(\emptyset) = 1$.

(ii) For each interval $I$ of $\mathbb{R}_{>0}$, we have that $\mu(I) = \ell(I)$ if $I$ is bounded and $\inf I > 0$; opposite case $\mu(I) = +\infty$.
(iii) \[ \mu \left( \bigcup_j E_j \right) = \prod_j \mu(E_j) \]

for every countable collection \( \{E_j\} \) of pairwise disjoint sets of \( \mathcal{M} \). We call this property: countable multiplicativity.

By this Theorem, we say that \( \mu \) is a multiplicative measure. So, we have constructed a multiplicative measure on \( \mathbb{R}_{>0} \) using only the topology usual on \( \mathbb{R}_{>0} \) and its multiplicative structure.

In the next section, we shall describe the connection between our measure \( \mu \) and the Lebesgue measure.

4. The measure \( \lambda \)

Let \( \lambda \) be now the (multiplicative) measure defined on the \( \sigma \)-algebra of Lebesgue measurable subsets of \( \mathbb{R}_{>0} \) by

\[ \lambda(E) = \exp \left( \int_E \frac{1}{x} \, dx \right). \]

It is clear that for each interval \( I = [a, b] \subset \mathbb{R}_{>0} \), \( \mu(I) = \lambda(I) \). So, \( \mu(G) = \lambda(G) \) for each open set \( G \subset \mathbb{R}_{>0} \). If we show that the \( \sigma \)-algebra \( \mathcal{M} \) coincides with the \( \sigma \)-algebra of the Lebesgue measurable sets of \( \mathbb{R}_{>0} \), then it will follow that \( \mu(E) = \lambda(E) \) for all \( E \in \mathcal{M} \).

From the definition of the \( \mu \)-measurability follows that \( E \in \mathcal{M} \) (or \( E \) is \( \mu \)-measurable) if and only if \( E = H \setminus U \) where \( H \) is of type \( G_\delta \) and \( \mu(U) = 1 \). Thus, to prove that \( E \) is \( \mu \)-measurable if and only if \( E \) is Lebesgue measurable it suffices to show that

\[ \mu(U) = 1 \text{ if and only if } |U|_e = 0, \]

here \( | \cdot |_e \) denote the outer Lebesgue measure. Suppose \( \mu(U) = 1 \). By Theorem 1.1, in [3], claim (i), given \( \epsilon > 0 \), there exists an open set \( G \), such that \( U \subset G \) and

\[ \mu(G) \leq 1 + \epsilon. \]

(3)

Theorem 1.1, in [3], implies there exists a collection of nonoverlapping closed interval \( \{I_j = [a_j, b_j]\} \) such that \( G = \bigcup_j I_j \). Then, \( \log(G) \subset \bigcup_j [\log(a_j), \log(b_j)] \). Since \( \lambda(G) = \mu(G) \), we obtain, by [3], that

\[ |\log(G)|_e \leq \sum_j [\log(b_j) - \log(a_j)] = \log(\lambda(G)) = \log(\mu(G)) \leq \log(1 + \epsilon). \]

So, \( |\log(G)|_e = 0 \) and, hence, \( |\log(U)|_e = 0 \). Then, to apply the exponential map on the region \( \log(U) \), we have \( |U|_e = 0 \).

Similarly, it is proved that \( |U|_e = 0 \) implies \( \mu(U) = 1 \). Thus, the \( \sigma \)-algebra \( \mathcal{M} \) coincides with the \( \sigma \)-algebra of the Lebesgue measurable sets of \( \mathbb{R}_{>0} \), and \( \mu \equiv \lambda \).
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