SOLVING THE STRONGLY COUPLED 2D GRAVITY:
2. FRACTIONAL-SPIN OPERATORS, AND
TOPOLOGICAL THREE-POINT FUNCTIONS.

Jean-Loup GERVAIS
Jean-François ROUSSEL
Laboratoire de Physique Théorique de l’École Normale Supérieure,
24 rue Lhomond, 75231 Paris CEDEX 05, France.

Abstract
We report progress along the line of a previous article — nb. 1 of the series — by one of us (J.-L. G.). One main point is to include chiral operators with fractional quantum group spins (fourth or sixth of integers) which are needed to achieve the necessary correspondence between the set of conformal weights of primaries and the physical spectrum of Virasoro highest weights. This is possible by extending the study of the chiral bootstrap (recently completed by E. Cremmer, and the present authors) to the case of semi-infinite quantum-group representations which correspond to positive integral screening numbers. In particular, we prove the Biedenharn-Elliot and Racah identities for q-deformed 6-j symbols generalized to continuous spins. The decoupling of the family of physical chiral operators (with real conformal weights) at the special values $C_{\text{Liouville}} = 7, 13, \text{and } 19$, is shown to provide a full solution of Moore and Seiberg’s equations, only involving operators with real conformal weights. Moreover, our study confirms the existence of the strongly coupled topological models put forward earlier. The three-point functions are determined. They are given by a product of leg factors similar to the ones of the weakly coupled models. However, contrary to this latter case, the equality between the quantum group spins of the holomorphic and antiholomorphic components is not preserved by the local vertex operator. Thus the “c=1” barrier appears as connected with a deconfinement of chirality.
1 INTRODUCTION

At the present time, the only way\cite{1,2,3} to go through the “$c = 1$ barrier” is to use the operator approach to 2D gravity. The basic reason seems to be that one should treat the two screening charges symmetrically in the strong coupling regime, since they are complex conjugate. This is in sharp contrast with what is currently done in the weak coupling regime, say using matrix models. In the operator approach, the quantum group structure was shown\cite{4,5} to be of the type $U_q(sl(2)) \odot \hat{U}_q(sl(2))$, with $q = \exp i\hbar$, $\hat{q} = \exp i\hat{\hbar}$, and

$$h = \frac{\pi}{12} \left( C - 13 - \sqrt{(C - 25)(C - 1)} \right),$$

$$\hat{h} = \frac{\pi}{12} \left( C - 13 + \sqrt{(C - 25)(C - 1)} \right),$$

(1.1)

where $C$ is the central charge of Liouville theory. Each quantum group parameter is associated with a screening charge by the relations $h = \pi(\alpha_-)/2$, $\hat{h} = \pi(\alpha_+)^2/2$. In the strong coupling regime $1 \leq C \leq 25$, $h$ and $\hat{h}$ are complex conjugate. Thus, treating them symmetrically, as was done in refs.\cite{1,2,3}, is the key. A major progress was made in ref.\cite{3} by proving a unitary truncation theorem, that holds for special values of $C$. The point of this theorem is as follows. The basic family of $(r,s)$ chiral operators in 2D gravity may be labelled by two quantum group spins $J$ and $\hat{J}$, with $r = 2\hat{J} + 1$, $s = 2J + 1$, so that the spectrum of Virasoro weights is given by

$$\Delta_{J\hat{J}} = \frac{C - 1}{24} - \frac{1}{24} \left( (J + \hat{J} + 1)\sqrt{C - 1} - (J - \hat{J})\sqrt{C - 25} \right)^2,$$

(1.2)

in agreement with Kac’s formula. The weak coupling regime corresponds to $C > 25$ where this formula is automatically real, for real $J$ and $\hat{J}$. In this paper, we deal with the strong coupling regime $1 < C < 25$. Then, since $\sqrt{C - 1}$ is real, and $\sqrt{C - 25}$ pure imaginary, the formula just written gives complex results in general. However—in a way that is reminiscent of the truncations that give the minimal unitary models—for $C = 7, 13, and 19$, there is a consistent truncation of the above general family down to an operator algebra involving operators with real Virasoro conformal weights only. These are of two types. The first has spins $\hat{J} = J$, and Virasoro weights that are negative; the second has $\hat{J} = -J - 1$, and Virasoro weights that are positive.

In this article, we improve with respect to the previous situation of ref.\cite{3} in several respects. First, ref.\cite{3} only dealt with the fusion algebra at the level of primaries, making use of the quantum group structure unravelled in refs.\cite{4,5}. The recent articles \cite{6,7} which combine the operator approach with the Moore and Seiberg scheme \cite{8}, will allow us to deal with arbitrary descendants. Second, ref.\cite{3} did not fully clarify the role of the coupling constants — that are not governed solely by the quantum group symmetry. This may now be done using the general formulae of refs.\cite{6,7}. Third, the spectrum of zero-modes of the physical Hilbert space of the strongly coupled theories is such that the corresponding set of operators...
should involve in some cases chiral primary fields with quantum group spins that are rational numbers instead of halves of integers. These operators were not considered in ref. [3], since the structure of the chiral operator algebra was not yet known well enough to generalize to non-half-integer spins. This is now possible thanks to ref. [6,7]. Moreover, a recent study [9,10] based on an operator Coulomb-gas realization makes it possible to derive the braiding of chiral vertex operators with arbitrary spins provided the screening numbers remain integer. Finally, the Liouville string associated with the strongly coupled theories under consideration, remarkable as they may be [11], remain complicated. It was proposed earlier [3] to consider strongly coupled topological models, obtained by considering together two copies of Liouville, so that the total central charge is 26. Our study confirms the consistency of these models. Using our determination of the coupling constants, we may calculate their three-point functions, it is found to be a product of leg factors much like the result of matrix models for $c \leq 1$ as recovered from the continuous Liouville theory in the present framework [12]. However, a strikingly novel feature appears. In the strong coupling regime, the Liouville exponentials cannot be used, since they do not preserve the physical Hilbert space. They are replaced [3] by other local vertex operators which shift the zero modes of the left- and right-movers in uncorrelated ways, so that it is not consistent to assume that their quantum group spins $J$ and $\tilde{J}$ are equal. This is in sharp contrast with the Liouville field. It shows that the strong coupling regime may be characterized by a deconfinement of chirality.

2 GETTING STARTED

In this section we go through some background material as a preparation for the main body of the paper. All notations are the same as in the previous articles using the operator method. We shall not fully re-explain the conventions. One outcome of ref. [6] was the fusion and braiding of the general chiral operators $V_{m}^{J,j}$, also denoted $V_{m}^{(J,j)}$, where underlined symbols denote double indices $\underline{J} \equiv (J, \tilde{J})$, $\underline{m} \equiv (m, \tilde{m})$, which were all taken to be half-integers:

$$\mathcal{P}_{1} V_{\underline{L}_{12} \underline{L}_{23}}^{(\underline{d}_{1})} V_{\underline{L}_{23 \underline{L}_{12}}}^{(\underline{d}_{2})} = \sum_{\underline{L}_{12}} g_{\underline{L}_{12}}^{\underline{L}_{23}} g_{\underline{L}_{23}}^{\underline{L}_{12}} \{ \underline{d}_{1} \underline{d}_{2} \underline{d}_{12} \underline{d}_{23} \} \times$$

$$\mathcal{P}_{1} \sum_{\{ \nu \}} V_{\underline{L}_{12} \underline{L}_{23}}^{(\underline{d}_{1})} (\nu) < \omega_{\underline{L}_{12}}, \{ \nu \} \vert V_{\underline{L}_{23 \underline{L}_{12}}}^{(\underline{d}_{2})} \vert \omega_{\underline{L}_{23}} >,$$  \hspace{1cm} (2.1)

$$\mathcal{P}_{1} V_{\underline{L}_{12} \underline{L}_{23}}^{(\underline{d}_{1})} V_{\underline{L}_{23 \underline{L}_{12}}}^{(\underline{d}_{2})} = \sum_{\underline{L}_{12}} e^{\pm i \pi (\Delta_{\underline{L}_{12}} + \Delta_{\underline{L}_{23}} - \Delta_{\underline{L}_{12}} - \Delta_{\underline{L}_{23}})} \times$$

$$g_{\underline{L}_{12}}^{\underline{L}_{13}} g_{\underline{L}_{23}}^{\underline{L}_{12}} \{ \underline{d}_{1} \underline{d}_{2} \underline{d}_{12} \underline{d}_{23} \} \mathcal{P}_{1} V_{\underline{L}_{23 \underline{L}_{12}}}^{(\underline{d}_{2})} V_{\underline{L}_{12 \underline{L}_{23}}}^{(\underline{d}_{1})}.$$  \hspace{1cm} (2.2)

In these formulae, world-sheet variables are omitted, and $\omega$ is the rescaled zero-mode of the underlying Bäcklund free field that characterizes the Verma modules $\mathcal{H}(\omega)$.
spanned by states noted $|\varpi, \{\nu\}>$, where $\{\nu\}$ is a multi-index. The symbol $\varpi_\perp$ stands for $\varpi_0 + 2J + 2J_\pi/h$ where $\varpi_0 = 1 + \pi/h$ corresponds to the $sl(2)$-invariant vacuum; $\mathcal{P}_\perp$ is the projector on $\mathcal{H}(\varpi_\perp)$. The above formulae contain the recoupling coefficients for the quantum group structure $U_q(sl(2)) \otimes U_q'(sl(2))$, which are defined by

$$\{ \hat{J}_1, \hat{J}_2, \hat{J}_{12} \} = (-1)^{f_V(J_1, J_2, J_3, J_4, J_5, J_6)} \{ J_1, J_2, J_{12} \} \{ \hat{J}_1, \hat{J}_2, \hat{J}_{12} \}$$

(2.3)

where $f_V(J_1, J_2, J_3, J_4, J_5, J_6)$ is an integer to which we shall come back. The symbol $\{ \hat{J}_1, \hat{J}_2, \hat{J}_{12} \}$ is the 6-j coefficient associated with $U_q(sl(2))$, while $\{ J_1, J_2, J_{12} \}$ stands for the 6-j associated with $U_q'(sl(2))$. In addition to these group theoretic features there appear the coupling constants $g_{n,r}^{J_1 J_2}$ whose general expression was derived earlier.

On the other hand, the following notions were introduced in the previous work on strongly coupled Liouville theory [1, 3].

a) The physical Hilbert space. It is given by

$$\mathcal{H}_{s, \text{phys}} \equiv \bigoplus_{r=0}^{\infty} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_s(\varpi_{r,n}),$$

(2.4)

$$\varpi_{r,n} \equiv \left( \frac{r}{2} - s \right) (1 - \frac{\pi}{h}).$$

(2.5)

The integer $s$ is such that the special values correspond to

$$C = 1 + 6(s + 2), \quad s = 0, \pm 1, \quad h + \tilde{h} = s\pi.$$  

(2.6)

The weight $\Delta(\varpi_{r,n}) \equiv (1 + \pi/h)^2 h/4\pi - h \varpi_{r,n}^2/4\pi$ is positive and in $\mathcal{H}_{s, \text{phys}}$ the representation of the Virasoro algebra is unitary. In Eq.2.4 we added a subscript $s$ to indicate that the Hilbert spaces depend upon the central charge, so that they are explicit functions of $s$. This will be useful in section 7. The partition function corresponds to compactification on a circle with radius $R = \sqrt{2(2 - s)}$ (see refs. [1, 2]).

b) The restricted set of conformal weights. The truncated family only involves operators of the type $(2J + 1, 2J + 1)$ noted $\chi^{(J)}_-$ and $(-2J - 1, 2J + 1)$ noted $\chi^{(J)}_+$. Their Virasoro conformal weights [1, 2, 3] which are respectively given by

$$\Delta^-(J, C) = -\frac{C - 1}{6} J(J + 1), \quad \Delta^+(J, C) = 1 + \frac{25 - C}{6} J(J + 1),$$

(2.7)

are real. $\Delta^-(J)$ is negative for all $J$ (except for $J = -1/2$ where it becomes equal to $\Delta^-(-1/2) = (s + 2)/4$). $\Delta^+(J)$ is always positive, and is larger than one if $J \neq -1/2$.

c) The truncated families: $\mathcal{A}^\pm_{\text{phys}}$ is the set of operators noted $\chi^{(J)}_\pm$, introduced in [1, 2, 3], whose conformal weights are given by Eq.2.7. In ref. [3], the case of integer $2J$ was completely solved at the level of primaries. They were expressed as specific linear combinations of $V_{M, -M}^{(J, J)}$ (resp $V_{M, -M}^{(-J, -1, J)}$), so that the following holds.
THE UNITARY TRUNCATION THEOREM:

For $C = 1 + 6(s + 2)$, $s = 0, \pm 1$, and when it acts on $\mathcal{H}_{s,\text{phys}}$, the set $\mathcal{A}_{\text{phys}}^+$ (resp. $\mathcal{A}_{\text{phys}}^-$) of operators $\chi_+^{(J)}$ (resp. $\chi_-^{(J)}$) is closed by fusion and braiding, and only gives states that belong to $\mathcal{H}_{\text{phys}}$.

With the formulae just recalled we may explain the main point of our paper. The fusion and braiding relations Eqs. 2.1 and 2.2 are operator relations computed between states $\mathcal{H}(\varpi_r)$ on the left and $\mathcal{H}(\varpi_{J^3_r})$ on the right. Yet all spins are treated on the same footing, following the basic scheme of Moore and Seiberg, where there is a one-to-one correspondence between the spectra of highest weights and conformal weights of primary fields. This will force us to an extensive generalization of the work of ref.[6, 7]. First, in Eqs. 2.1, 2.2, the spins $2J$, $2\hat{J}$, $2J_3$, and $2\hat{J}_3$ were assumed to be integers. On the other hand, we may write Eq. 2.5 as

$$\varpi_{r,n} = \varpi_{J^3_r,n}, \quad J_{r,n} = \frac{r}{2(2-s)} + \frac{n-1}{2}, \quad \hat{J}_{r,n} = -\frac{r}{2(2-s)} - \frac{n+1}{2}. \quad (2.8)$$

One sees that, for $s = -1$, and $s = 0$, this introduces spins that are rational numbers, but not halves of integers. Once we go away from half-integer spins, we may as well consider continuous spins as is done in refs.[9, 10]. Second, the treatment of ref.[3] had another basic difference between the spectrum of highest-weight states and of conformal weights of primary operators: the physical spectrum of the latter involves two types of fields ($\mathcal{A}_{\text{phys}}^+$), with $\hat{J} = J$, and $\hat{J} = -J - 1$, respectively, while the physical Hilbert space only involved spins satisfying the latter type condition, as is clear from Eq. 2.8. The reason is that the physical Hilbert space was taken to have positive highest-weights so that the Virasoro representation be unitary. From the viewpoint of the operator algebra we are led to enlarge the Hilbert space and consider

$$\mathcal{H}_{s,\text{phys}}^\pm \equiv \bigoplus_{r=0}^{\frac{1+s}{2}} \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_s(\varpi_{r,n}^\pm), \quad (2.9)$$

$$\varpi_{r,n}^\pm \equiv \left(\frac{r}{2 + s} + n\right)(1 \mp \frac{\pi}{h}). \quad (2.10)$$

Now Eq. 2.14 is generalized to

$$\varpi_{r,n}^\pm = \varpi_{J^3_{r,n}}^\pm, \quad J_{r,n}^\pm = \frac{r}{2(2+s)} + \frac{n-1}{2}, \quad \hat{J}_{r,n}^\pm = \mp \frac{r}{2(2+s)} \pm \frac{n-1}{2}. \quad (2.11)$$

One sees that for $s = 0$, the $J$’s may be fourths of integers, while, for $s = 1$ (resp. $s = -1$) the $J^-$’s (resp. the $J^+$’s) may be sixths of integers. A parenthetical remark is in order at this point. In general, a highest-weight state $|\varpi>$ is eigenstate of $L_0$ with an eigenvalue $\left(1 + \frac{\pi}{h}\right)^2 h/4 + h \varpi^2/4\pi$ that is invariant under $\varpi \to -\varpi$. Since the corresponding Verma module $\mathcal{H}(\varpi)$ may be deduced group theoretically once this eigenvalue is fixed, it follows that $\mathcal{H}(\varpi)$ and $\mathcal{H}(-\varpi)$ are the same Hilbert space. Changing $\varpi_{r,n} \to -\varpi_{r,n}$ in Eq. 2.10 is equivalent to changing $J_{r,n}^\pm \to -J_{r,n}^\pm - 1$, and $\hat{J}_{r,n}^\pm \to -\hat{J}_{r,n}^\pm - 1$. It is thus related to the symmetry put forward in ref.[3]. We shall make use of this freedom below. Returning to our main line, let us note that,
in $\mathcal{H}_{\text{phys}}$, the highest weights are real but negative. What is its physical meaning?

This brings in the proposal of ref. [3]. There it was remarked that the spectrum of
physical conformal weights Eq.2.7 is such that

$$\Delta^\pm(J, C) + \Delta^\mp(J, 26 - C) = 1.$$  \hspace{1cm} (2.12)

Moreover, if $C$ takes the special values Eq.2.6 one has

$$26 - C = 1 + 6(s' + 2), \quad s' = -s.$$  \hspace{1cm} (2.13)

Thus we may construct a consistent string model with two copies of strongly coupled
Liouville theories, one with $s$ playing the role of gravity, and the other with $s'$ being
considered as the matter. Following the usual counting where the BRST cohomology
removes two degrees of freedom, this theory has essentially no degree of freedom,
meaning that it is topological. This is why the Hilbert space $\mathcal{H}_{\text{phys}}^-$ may be used:
the excitations that would be negative-normed states decouple. This situation is of
course similar to the one of the $c \leq 1$ topological models. Thus we shall follow
closely the Liouville derivation[12] of the three-point function for the $c \leq 1$ models.

The plan of the article is as follows. In section 3 we first prove some mathematical
properties concerning the extension of the 6-j symbols to non-integer spins. These
are necessary to generalize the operator algebra to non-integer spins, as done in
section 4. This extension to non-integer spins will be shown to consistently include
representations in a generalized sense that are semi-infinite. In the language of
the Coulomb-gas approach to conformal theories, they correspond to chiral fields
with numbers of screening operators that are positive integers. This is actually how
they are explicitly constructed in refs.[9, 13]. As such, they are not what is needed
to define the operators of the family $\mathcal{A}^+\text{phys}$, since these involve negative spins and
negative screening numbers. In section 5 we thus continue our results to negative
“screening numbers” thanks to the continuation $J_i \rightarrow -J_i - 1$, put forward in
ref.[3], which we apply to the generalized 6-j symbols. This involves a non trivial
identity on q-deformed hypergeometric functions $_4F_3$. We are then ready to build
the physical consistent algebra (in section 6), and show that it is fully closed by
fusion and braiding to all order in the secondaries. In other words, at the special
values we have a complete truncation of the chiral bootstrap equations down to the
one describing the OPA of a set of chiral fields with real conformal weights. Finally,
in section 7, we apply the formulae just derived to compute the three-point function
of the topological models mentioned above.

Our discussion will follow the line of ref.[3], by establishing relations between the
quantum group symbols of $U_q(sl(2))$, and $\hat{U}_q(sl(2))$, that hold at the special values,
and ensure that the truncation theorem holds. However, we will be led to change the
definition of the $\chi$ fields. Let us comment about this now. In ref.[3], the following
expression for the $\chi^{(J)}_{-G}$ fields was used\textsuperscript{2}

$$\chi^{(J)}_{-G} = \sum_{m=-J}^J C^{(J)}_m(\tau) e^{im(\bar{\omega} - \omega + (h - \hat{h})/2)} \psi^{(J, J)}_{m, -m}$$  \hspace{1cm} (2.14)

\textsuperscript{2}The central charge of matter is noted $c$.

\textsuperscript{3}It is distinguished by the subscript “G”.

\begin{itemize}
\item \textsuperscript{2}The central charge of matter is noted $c$.
\item \textsuperscript{3}It is distinguished by the subscript “G”.
\end{itemize}
C_m^{(J)}(\varpi) \propto (-1)^{J-m} \left( \frac{2J}{J-m} \right)^{m/2} \prod_{t=0}^{2J} \frac{\varpi - J + m - t}{\varpi + 2m}, \quad (2.15)

where \( \propto \) means that we work up to factors that only depend upon \( J \). The chiral fields \( \psi_{m,-m}^{(J,J)} \) differ from \( V_{m,-m}^{(J,J)} \) by normalization factors, noted \( E_{m,-m}^{(J,J)} \) in the recent papers. In ref.[3], the method used was as follows. One starts from the ansatz for the simplest case
\[
\chi^{-1/2} = a(\varpi)\psi^{-1/2,1/2} + b(\varpi)\psi^{1/2,1/2},
\]
Closure by braiding requires
\[
a(\varpi_{rn})b(\varpi_{rn+1}) = \lambda \sin(h\varpi_{rn}) \sin(h\varpi_{rn+1}) \quad (2.16)
\]
where \( \lambda \) is an arbitrary constant. The following solution of these equations was used in refs.[2, 3]:
\[
a = e^{ih\varpi} e^{ih/2} \sin(h\varpi), \quad b = -e^{ih\varpi} e^{ih/2} \sin(h\varpi),
\]
which, using leading-order fusion, leads to Eq.2.14. This particular choice was made in order to arrive at simple expressions in terms of the quantum-group covariant chiral fields \( \xi_{M,-M}^{(J,J)} \). At that time, the complete braiding matrix was only known for the \( \xi \) fields, so that the discussion made an extensive use of them. The present viewpoint is somewhat different. We shall use the \( V \) fields instead of the \( \psi \)'s, or the \( \xi \)'s. Moreover, we consider fusion to all orders in the descendants. The definitions which appear natural from the present standpoint look rather different from the ones of ref.[3]. It is thus useful to sketch how Eq.2.14 is related with the general expression we will derive in section 6. This is done in an appendix.

3 GENERALIZATION OF 6-J SYMBOLS

In this section we propose a generalization of the 6-j symbols to non-half-integer spins, and prove the corresponding generalized polynomial equations\[^4\] in particular the pentagonal relation. Although purely mathematical, this generalization stems from the needs of physics, and we shall therefore follow this guide, in two steps.

The standard three-leg vertex operators intertwine three standard representations of \( U_q(sl(2)) \) labelled by positive\[^5\] half-integers. Their algebra was completely elucidated in ref.[3], using Moore and Seiberg formalism. However, since the early eighties Gervais and Neveu have introduced operators of positive half-integer spins, but acting on a Hilbert space described by a continuous zero-mode \( \varpi \) or equivalently, by a continuous spin. This will lead to the first step of generalization. The discrete values \( \varpi = \varpi_J = \varpi_0 + 2J \), with \( 2J \in \mathbb{Z}_+ \), give back the standard case.

\[^4\] This is an abusive use of the name “polynomial equations” which usually refers to consistency equations for fusion and braiding coefficients. But the 6-j coefficients, which are solutions of them, satisfy parallel equations, namely orthogonality, Racah identity, Biedenharn-Elliot identity..., that we generically call polynomial equations as well.

\[^5\] All along this section, we use positive for non-negative, including zero.
The second step will come from operators with three continuous spins. Their braiding has already been elucidated in refs.\[9, 10\] using a Coulomb-gas approach. It is essentially given by the generalized 6-j introduced by Askey and Wilson\[11\] who proved the orthogonality relation. Our more generalized 6-j coincide with theirs, and moreover we prove the other polynomial equations.

These two generalizations could seem to be of a very different kind, but they actually follow one another very naturally. The right language is not the one of half-integer positive or continuous spins but a question of number of restrictions on the spins, the former being only a consequence of the latter. Let us be more explicit. We use dotted vertices, a dot on one leg meaning that the sum of the spins of the two other legs minus the spin of this leg is constrained to be a positive integer. Since there are three legs at a vertex we may have one, two or three dots. We call them type TI1, TI2, TI3, where the letters T and I stand for triangular inequalities. We illustrate the meaning of the dots:

\[
\begin{align*}
\text{j}_{12} & | j_1 \quad j_2 \quad \rightarrow \quad j_1 + j_2 - j_{12} \text{ positive integer} \\
\text{j}_{12} & | j_1 \quad j_2 \quad \rightarrow \quad j_2 + j_{12} - j_1 \text{ positive integer}^6
\end{align*}
\]

Adding dots to a vertex adds other restrictions. For the type TI3, we have

\[
\begin{align*}
\text{J}_{12} & | J_1 \quad J_2 \quad \rightarrow \quad \left\{ \begin{array}{c}
J_1 + J_2 - J_{12} \quad \text{positive integer} \\
J_{12} + J_2 - J_1 \quad \text{positive integer} \\
J_1 + J_{12} - J_2 \quad \text{positive integer}
\end{array} \right\} \Rightarrow \quad 2J_1, 2J_2, 2J_{12} \quad \text{positive integers.} \quad (3.1)
\end{align*}
\]

These are but the usual (full) triangular inequalities, or the branching rules for \(sl(2)\), with half-integer spins. So, the standard operators are of the TI3 type.

The first step of generalization is to relax one restriction, which gives the TI2 type:

\[
\begin{align*}
\text{j}_{12} & | J_1 \quad J_2 \quad \rightarrow \quad \left\{ \begin{array}{c}
J_1 + j_2 - j_{12} \quad \text{positive integer} \\
J_1 + j_{12} - j_2 \quad \text{positive integer}
\end{array} \right\} \Rightarrow \quad 2J_1 \quad \text{positive integer} \quad (3.2)
\end{align*}
\]

\(j_{12}\) and \(j_2\) being arbitrary\[^7\]. In this TI2 case, \(J_1\) is a positive half-integer, and \(j_2 - j_{12}\) is a half-integer (positive or negative). This is fixed by the number of restrictions. The second step is of course to keep only one restriction (type TI1):

\[
\begin{align*}
\text{j}_{12} & | J_1 \quad J_2 \quad \rightarrow \quad J_1 + j_2 - j_{12} \quad \text{positive integer} \quad (3.3)
\end{align*}
\]

\[^6\] This vertex does not exist by itself as the only ones allowed are 3.1, 3.2, 3.3. It is only represented in order to illustrate the dot convention.

\[^7\] For the time being, we denote half-integer positive spins by capital letters and continuous ones by small letters, but this is only a consequence of the type of vertex and in no case an a priori assumption.
and none of the spins are half-integers. Then the fusion or braiding of such generalized vertices lead to 6-j symbols generalized in a very specific way, and the “miracle” is that it is mathematically consistent. In the first step of generalization, fusion and braiding lead to two different generalized 6-j, whereas they lead to only one kind of 6-j in the second step.

3.1 The first step of generalization

First, we try to define a 6-j coefficient for the following fusion operation:

\[
\begin{array}{c}
\begin{array}{c}
J_1 \\
\downarrow \scriptstyle j
\end{array}
\quad \begin{array}{c}
J_2 \\
\downarrow \scriptstyle j + m_1
\end{array}
\quad \rightarrow
\begin{array}{c}
J_{12} \\
\downarrow \scriptstyle j + m_{12}
\end{array}
\end{array}
\end{array}
\]

It follows from the general rules for these three TI2 and one TI3 vertices that \( J_1, J_2, J_{12} \) are positive half-integers, that \( J_i \pm m_i \), for \( i = 1, 2, 12 \) are positive integers (\( m_{12} \equiv m_1 + m_2 \)), and that \( J_1, J_2, J_{12} \) satisfy the full triangular inequality.

The standard\(^{[19, 20]}\) (i.e. four TI3) definition of the q-deformed 6-j symbols that is to be generalized, is:

\[
\begin{align*}
\{ \begin{array}{c}
J_1 \\
\downarrow \scriptstyle j
\end{array} | \begin{array}{c}
J_2 \\
\downarrow \scriptstyle j + m_1
\end{array} \} &= \sqrt{2J_{12} + 1}[2j + 2m_1 + 1]e^{i\pi(\frac{1}{2}J_1 + J_2 - J_{12} - m_{12} - 2j)} \times \\
\Delta(J_1, J_2, J_{12})\Delta(J_1, j, j + m_1)\Delta(J_{12}, j, j + m_1)\Delta(J_2, j + m_1, j + m_{12}) \times \\
\sum_z \frac{e^{i\pi z} [z + 1]}{[z - J_1 - J_2 - J_{12}] [z - 2j - J_1 - m_1] [z - 2j - J_2 - m_{12}] [z - 2j - J_{12} - m_{12}]} \times \\
\frac{1}{[J_1 + J_2 + m_{12} + 2j - z] [J_1 + J_2 + m_1 + m_{12} + 2j - z] [J_1 + J_2 + m_1 + 2j - z] [J_2 + J_{12} + m_1 + 2j - z]}
\end{align*}
\]

with

\[
\Delta(l, j, k) = \sqrt{\frac{[-l + j + k] [l - j + k] [l + j - k]}{[l + j + k + 1]}}
\]

and

\[
|n|! \equiv \prod_{r=1}^{n} |r|, \quad |x| \equiv \frac{\sin(hx)}{\sin h}.
\]

These 6-j coefficients involve square roots with sign ambiguities. In the building of strong coupling regime operators, the cancellation of signs is of a great importance (see section \[5\]), and we must therefore treat them consistently. Following ref.\([12]\), and since the basic mathematical tool is the relationship between sine and gamma functions, we define square roots of sine functions for non-integer argument from the relation \( \sqrt{\sin(\pi z)} \equiv \sqrt{\pi}/(\sqrt{\Gamma(z)}\sqrt{\Gamma(1 - z)}) \). The choice of the sheet for \( \sqrt{\Gamma(z)} \) is determined so that the relation \( \sqrt{\Gamma(z + 1)} = \sqrt{\pi}\sqrt{\Gamma(z)} \) holds, where \( \sqrt{\pi} \) is defined
as usual with a cut on the negative real axis. This specifies the relation between 
\( \sqrt{\sin(\pi(z + 1))} \) and \( \sqrt{\sin(\pi(z))} \). In the present case, one has typically \( z \sim (J_i + \ldots)h/\pi \). If \( h \) is complex, as it is the case in the strong coupling regime, \( z \) is far from the cut. For the weak coupling regime, on the contrary, \( h \) is real, and one should temporarily give a small imaginary part to \( J \) or to \( h \) to specify the sheet.

For simplicity we lump the square roots of 6-j symbols into coefficients noted \( \Xi \). Those \( \Xi \) factors are chosen so that they can be seen as normalization factors of the vertices of Eq.3.4 and will cancel (or factorize) out of the polynomial equations when applying successive fusions (or braidings). So, the polynomial equations are fundamentally rational equations (without square roots). We could take \( \Xi = \Delta \), but \( \Delta \) involves factorials with arguments which will not remain integers in the forthcoming generalization to non-half-integer spins. This is why we define \((p_{1,2} \equiv j_1 + j_2 - j_{12})\)

\[
\Xi_{j_1, j_2}^{j_{12}} = \prod_{k=1}^{p_{1,2}} \sqrt{\frac{[2j_1 - k + 1][2j_2 - k + 1][2j_{12} + k + 1]}{[k]}}
\]

\[
= \sqrt{\frac{[j_1 - j_2 + j_{12} + 1]_{p_{1,2}}[j_1 + j_2 + j_{12} + 1]_{p_{1,2}}[2j_{12} + 2]_{p_{1,2}}}{[p_{1,2}]!}}
\]

which we wrote, by anticipation, in a way that will be valid for continuous spins since \( p_{1,2} \) will remain an integer. The symbol \([x]_n\) stands in general for \( \prod_{k=1}^{n} [x + k - 1] \). This gives for half-integer spins

\[
\Xi_{j_1, j_2}^{j_{12}} = \sqrt{\frac{[2J_1]![2J_2]![2J_{12} + 2]}{[2J_{12} + 1]![\Delta(J_1, J_2, J_{12})]}}
\]

Using this, we compute

\[
\left\{ \begin{array}{l}
J_1, J_2 \mid j_{12}^{j+m_1} \\
J_{j+m_1}^{j+m_1} \end{array} \right\} = \Xi_{j_1, j_2}^{j_{12}} e^{i\pi(2J_1 + 2J_2 - 2m_1 - 2J_{12})} \left[ 2j + 2m_1 + 1 \right]
\]

\[
(\Delta(J_2, j + m_1, j + m_1) \Delta(J_1, j, j + m_1))^2
\]

\[
\sum_{z} \frac{e^{i\pi[z + 1]}}{[z - J_1 - J_2 - J_{12}]![z - 2J_1 - J_2 - J_{12}]![z - 2J_2 - J_1 - m_1]![z - 2J_1 - J_2 - m_1]![z - 2J_2 - J_1 - m_2]![z - 2J_1 - J_2 - m_2]!}
\]

\[
\left[ 2j + 2m_1 + 2j - z \right]! \left[ 2j + 2m_1 + 2j - z \right]! \left[ 2j + 2m_1 + 2j - z \right]!
\]

\[
= 1
\]

\[
\sum_{z} e^{i\pi[z + 1]}
\]

\[
\sum_{z} \frac{1}{[2j + 2m_1 + 2j - z]!}
\]

The sum is for integer \( z \) such that the factorials have positive arguments that is,
for
\[
\begin{align*}
  z - 2j &\leq J_1 + J_2 + m_{12} \quad \text{(sup1)} \\
  z - 2j &\leq J_1 + J_{12} + m_1 + m_{12} \quad \text{(sup2)} \\
  z - 2j &\leq J_2 + J_{12} + m_1 \quad \text{(sup3)} \\
  z - 2j &\geq J_1 + m_1 \quad \text{(inf1)} \\
  z - 2j &\geq J_{12} + m_{12} \quad \text{(inf2)} \\
  z - 2j &\geq J_2 + m_1 + m_{12} \quad \text{(inf3)} \\
  z &\geq J_1 + J_2 + J_{12} \quad \text{(inf4)}
\end{align*}
\]
(3.11)

The TI3 conditions can be seen to come from the conditions of existence of a non-empty range for \( z \): the three upper bounds must individually be greater than the four lower bounds, which yields twelve inequalities that can be rearranged in four TI3’s.

This suggests the following generalization to non-half-integer \( j \): in this case, if \( z \) were kept integer, the first six bounds would not remain integer and should therefore be removed as they would no longer be given by (one over) poles of gamma functions. It seems much more sensible to choose \( z = 2j + \) integer, or equivalently to sum over \( y \equiv z - 2j = \) integer. With this choice, it is the bound (inf4), on the contrary, that is no longer integer (for \( y \)) and has to be removed. We are left with the conditions
\[
\begin{align*}
  J_1 + m_1 \\
  J_{12} + m_{12} \\
  J_2 + m_1 + m_2
\end{align*}
\]
\[
\begin{align*}
  \leq y = z - 2j \leq \begin{cases} 
  J_1 + J_2 + m_{12} \\
  J_1 + J_{12} + m_1 + m_{12} \\
  J_2 + J_{12} + m_{12}
\end{cases}.
\end{align*}
\]
(3.12)

It is remarkable that the three remaining upper and lower bounds give nine inequalities that can precisely be combined in the three TI2’s and one TI3 of the fusion-diagram.

The mechanism may be summarized as follows. Choose some particular TI3, TI2 or TI1 for the four vertices. It follows that certain (sum of) spins must be integers. The generalization of the 6-j’s is obtained by only keeping the integer bounds among the seven of Eq.3.11, with a “good” choice for \( z \pmod{1} \). Then, the existence of a non-empty range of summation turns out to be equivalent to the positivity conditions of the TI3, TI2 and TI1 type initially chosen. The same mechanism will work in the case of the braiding with four TI2’s and braiding/fusing with four TI1’s (the second step of generalization).

Then, the remaining factorials with non-integer arguments must be combined two by two, in order to avoid q-deformed gamma functions, in the sum as well as in the \( \Delta \) functions. It is already done in the \( \Xi \) factors. We perform the following substitutions
\[
\frac{[z+1]}{[z-J_1-J_2-J_{12}]} \rightarrow [z-J_1-J_2-J_{12}+1]_{J_1+J_2+J_{12}+1}
\]
for two factorials in the sum, and
\[
\frac{(\Delta(j,j+m))^2}{[J+m][J-m][2j+m-J+1]}_{2j+1}
\]
for the others which only appear in the \( \Delta \) functions. This, together with the use of the new summation variable \( y \equiv z - 2j \), shows that these newly defined 6-j symbols
are in fact rational fractions in the variable $e^{ihj}$ (or $j$ in the non-deformed case). It can be checked that this generalized definition gives back the standard one when applied to half-integer spins such that the four TI3’s are satisfied, as the lower bound of summation (inf4) is restored by the poles of the corresponding gamma function.

This is only a (natural) definition, we have now to prove the polynomial relations on 6-j symbols. Let us concentrate on the pentagonal relations which are the basic identities for fusion. It is convenient to introduce the following function of $e^{ihj}$, where the $J_i$’s and $m_i$’s are fixed parameters, and $m_{ij} = m_i + m_j$,

$$P(e^{ihj}) \equiv \{\frac{J_1}{j+m_{12}} j \left| \frac{J_2}{j+m_{1}} j \right. \} \{\frac{J_3}{j+m_{12}} j \left| \frac{J_4}{j+m_{1}} j \right. \} \{\frac{J_5}{j+m_{12}} j \left| \frac{J_6}{j+m_{1}} j \right. \}.$$  

(3.13)

The range of summation over $J_{23}$ is fixed by the individual conditions for the 6-j:

$$|J_2 - J_3|, |J_1 - J_{123}|, |m_{23}| \leq J_{23} \leq J_2 + J_3, J_1 + J_{123}.  \tag{3.14}$$

$P$ is a sum of rational fractions of $e^{ihj}$, with a range of summation which is independent from $j$. It is thus a rational fraction of $e^{ihj}$. If $2j$ is chosen to be an integer large enough, all the TI3’s are fulfilled. Then, the 6-j’s reduce to the standard ones, the usual pentagonal equation shows that $P(e^{ihj}) = 0$ (there is an extra upper bound $2j + 2m_1 + m_2 + m_3$ for $J_{23}$ in the standard pentagonal equation, but it is irrelevant for $j$ large enough). Hence, $P$ is a rational fraction with an infinite number of distinct zeros $e^{ihn/2}$ ($h/\pi$ is not rational). This shows that $P$ is identically 0 and, therefore, the pentagonal relation holds for any $j$.

We repeat that the square roots of the $\Xi$ factors cause no trouble as the $\Xi$ factors with one entry equal to $J_{23}$ cancel out and the others can be factorized.

The braiding

$$\begin{array}{cccc}
J_1 & J_2 & J_3 \\
\downarrow & \downarrow & \downarrow \\
j & j + m_1 & j + m_{12} \\
\end{array} \quad \rightarrow \quad \begin{array}{cccc}
J_2 & J_1 \\
\downarrow & \downarrow \\
j & j + m_2' & j + m_{12} \\
\end{array}$$

leads to the definition of an other generalized 6-j coefficient. As before, the choice of a continuous $j$ leads to choose $z - 2j = y = \text{integer}$. This time the upper bound (sup3) is the one naturally released, leaving only the four desired TI2. We only write down the result:

$$\frac{\{J_1}{j+m_{12}} j \left| \frac{J_2}{j+m_{2}} j \right. \} \{\frac{J_3}{j+m_{12}} j \left| \frac{J_4}{j+m_{2}} j \right. \} e^{i\pi(J_1 - J_2 + m_{12} - 2m_2')} [2j + 2m_1 + 1] \Delta(J_1, j, j + m_1)^2$$

(3.15)

The relation always holds even though for $j$ half-integer not large enough some 6-j can have poles cancelled by zeros of others.
We define now (twice-generalized) 6-j coefficients for the fusion and braiding of operators with only TI1 conditions (figure 3.3). The fusion is then

\[
(\Delta(J_2, j + m_1, j + m_{12}))^2 \sum_y \frac{(-1)^y [2j + m_{12} + m_1 + m'_2 - y + 1]_{2y-m_{12}-m_1-m'_2} \times 1}{[y - J_1 - m_1][y - J_1 - m_{12} - m'_2][y - J_2 - m'_1][y - J_2 - m_{12}][J_1 + J_2 + m_1 + m'_2 - y]}
\]

for

\[
\begin{align*}
J_1 + m_1 & \\
J_1 + m_{12} + m'_2 & \\
J_2 + m'_2 & \\
J_2 + m_{12} + m_1
\end{align*}
\]

\[
\leq y \leq \begin{cases} J_1 + J_2 + m_{12} \\ J_1 + J_2 + m_1 + m'_2 \end{cases}.
\]

It is straightforward to prove the orthogonality relation for generalized 6-j of this type, using the same method as for the pentagonal relation.

### 3.2 The second step of generalization

We define now (twice-generalized) 6-j coefficients for the fusion and braiding of operators with only TI1 conditions (figure 3.3). The fusion is then

\[
\begin{array}{cccc}
& j_1 & \rightarrow & j_{12} \\
& j & \rightarrow & j + m_1 \\
& j + m_1 & \rightarrow & j + m_{12} \\
\end{array}
\]

\[
\begin{array}{cccc}
& j_2 & \rightarrow & j \\\n& j + m_{12} & \rightarrow & j \\
\end{array}
\]

It now follows from the general rules, that all spins are continuous and that the only conditions are that \(j_i + m_i, i = 1, 2, 12\) and \(j_1 + j_2 - j_{12}\) be positive integers.

Now that the principles of generalization have been written, the twice-generalized 6-j will be easily deduced. Among the bounds \((3.11)\), only (inf1), (inf2), (sup1) and (sup2) are integers in the case of four TI1’s, so that two more conditions (inf3) and (sup3) have to be released. It is then easy to see that, here again, the conditions of existence of a non-empty range for \(y\), with these bounds (selected only according to their integer character), precisely says that \(j_i + m_i, i = 1, 2, 12\) and \(j_1 + j_2 - j_{12}\) must be positive integers.

It is possible to write the result without any gamma function in the following rational form:

\[
\begin{aligned}
\{j_1 \begin{array}{c|c|c}
  j & j_1 & j_2 \\
  \hline
  j + m_1 & j_1 + m_1 & j + m_{12} \\
  \hline
  j + m_{12} & j_1 + m_1 & j_1 + m_{12} \\
\end{array}\} &= \frac{\Xi_j^{j+m_1} \Xi_j^{j+m_{12}} \Xi_j^{j_1+m_1} \Xi_j^{j_1+m_{12}}}{\Xi_j^{j_2} \Xi_j^{j_1+j+m_{12}}}[2j + m_1 - j_1 + 1][2j + 2m_1 + 2][2j + m_{12} + 2] \\
& \times \sum_y \frac{(-1)^y [2j + j_1 + m_1 + 2]_{y-j_1+m_1} [2j - j_1 - j_2 + y + 1]_{(j_1+m_1)+(j_1+j_2-j_{12})-y}}{[y - (j_{12} + m_{12})][y - (j_1 + m_1)]}
\end{aligned}
\]

\(^{10}\text{We shall actually only use the case where they are rational. At this level of the discussion this makes no difference (more about this below, however).} \)
\[
\frac{|j_2 + j_{12} + m_1 - y + 1|}{[(j_1 + m_1) + (j_2 + m_2) - y!][(j_1 + m_1) + (j_12 + m_12) - y!]} (3.19)
\]

with
\[
\begin{align*}
&j_1 + m_1, \\
&j_{12} + m_{12}
\end{align*}
\]

This definition is similar up to normalization to the definition of ref.\[14\]. There, the generalized 6-j coefficient is expressed with a \(_4F_3\) hypergeometric function. We give such an expression in section 5, but here we prefer keeping an explicit sum in order to see more clearly what are the bounds of summation, as it is a key point of this generalization procedure.

This 6-j can be considered as a function of the three continuous variables \(e^{ih_j}, e^{ih_j_1}, e^{ih_j_2}\), and of three independent discrete variables chosen among \(j_1 + m_1, j_2 + m_2, j_{12} + m_{12}\) and \(j_1 + j_2 - j_{12}\) (the latter must be positive integers but are clearly not independent). For fixed values of these discrete variables, it is immediate to see that the 6-j is a rational fraction of the three continuous variables.

Going along a line similar to the previous one, it is straightforward to check that, if \(j_1\) and \(j_2\) are chosen to be half-integers large enough for the three TI2 and one TI3 of the first step of generalization to be fulfilled, the range of summation is reduced by the gamma functions so that we get back the once-generalized 6-j of last subsection.

Let us now prove the twice generalized pentagonal equation. We define
\[
Q(e^{ih_{j_1}}, e^{ih_{j_2}}, e^{ih_{j_3}}) = \sum_{j_{23}} \frac{\{j_{12}\}_{j+m_{12}} j_{12} \{j_{23}\}_{j+m_{23}} j_{23}}{\{j_{1}\}_{j+m_{1}} j_{1} \{j_{3}\}_{j+m_{3}} j_{3}}
\]

We have to prove that \(Q = 0\). \(Q\) is a function of the fixed discrete variables \(j_i + m_i, j_i + j_k - j_{ik}\) and of \(e^{ih_j}\), that we keep fixed.

Here again the range of summation for \(j_{23}\) naturally comes from the individual conditions for the 6-j, and can be written: \(j_{23} = j_2 + j_3 - n\) for \(n\) integer such that
\[
0 \leq n \leq (J_2 + m_2) + (J_3 + m_3), (j_1 + j_2 - j_{12}), (j_3 + j_{12} - j_{123})
\]

which shows that the range of summation only depends on the fixed discrete variables. Hence, \(Q\) which is a sum on a fixed interval of rational fractions of \(e^{ih_{j_1}}, e^{ih_{j_2}}\) and \(e^{ih_{j_3}}\), is a rational fraction as well.

Let us see that, for \(j_1, j_2, j_3\) half-integers taken large enough, \(Q\) is equal to \(P\) (of last subsection) and hence to zero. With such \(j_1, j_2, j_3\), the twice-generalized 6-j’s become once-generalized 6-j’s, as noted above. So, we only have to prove that the range of summation over \(j_{23}\) is identical to the one \((3.14)\) of \(P\). In terms of the better suited variable \(n = j_2 + j_3 - j_{23}\), the interval \((3.14)\) is
\[
0, j_2 + j_3 - j_{1} - j_{123} \leq n \leq j_2 + j_3 - |j_2 - j_3|, j_2 + j_3 - |j_2 - j_3|, j_2 + j_3 - |m_{23}|.
\]

One lower bound and two out of the six upper bounds of \((3.23)\) (a bound with an absolute value being decomposed in two bounds) are already verified thanks to
Eq. 3.22. And the other bounds of 3.23 are not relevant for $j_1$, $j_2$, $j_3$ large enough: noting that for fixed discrete variables ($j_i + m_i$, ... = constants) $j_{123}$ is equal to $j_1 + j_2 + j_3 + \text{constant}$, and $m_{23}$ to $-j_2 - j_3 + \text{constant}$, we see that the extra bounds of 3.23 are less restrictive than the bounds 3.22. For example $j_2 + j_3 - j_1 - j_{123} = -2j_1 + \text{constant}$ and is lower than the lower bound of 3.22 for $j_1$ large enough. It can easily be seen that for $j_1$, $j_2$, $j_3$ larger than $(j_1 + m_1) + (j_2 + m_2) + (j_3 + m_3)$ for example (this is much larger than needed) the bounds 3.14 are equivalent to 3.22. This proves that in this infinite number of cases $Q = P = 0$. Then, as $Q$ is a rational fraction, it is zero for any $j_1$, $j_2$, $j_3$, which proves the pentagonal relation.

The twice-generalized braiding 6-j can be deduced from the once-generalized ones, but it gives the same twice-generalized 6-j as fusion, not surprisingly as both come from four TII’s vertices as shows the braiding of TI1 operators:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array{...
\[
\sum_{p_{1,2}, p_{12,3} \leq y \leq p_{1,2} + p_{12,3} + p_{1,2}} (-1)^y [j_1 + j_{23} + j_{123} + 2]_{y-p_{1,2}} [y + 2j_{123} - j_1 - j_2 - j_{12} + 1]_{p_{12,3} + p_{1,2} - y}
\]

\[
[y + 2j_{123} - j_2 - j_3 - j_{23} + 1]_{p_{12,3} + p_{1,2} - y} [j_2 + j_{12} + j_{23} - j_{123} - y + 1]_{y-p_{1,2}} [p_{12,3} + p_{1,2} - y]
\]

where the \( p \)'s are such that

\[
P_{1,2} \equiv j_1 + j_2 - j_{12}, \quad P_{2,3} \equiv j_2 + j_3 - j_{23}, \quad P_{12,3} \equiv j_{12} + j_{23} - j_{123}, \quad P_{1,23} \equiv j_{1} + j_{23} - j_{123}, \quad \} \rightarrow P_{k,l} \in Z_+, \quad P_{1,2} + P_{12,3} = P_{1,23} + P_{2,3}.
\]

We note that we only have the residual symmetry

\[
\{j_1 j_2 j_{123} \mid j_{12} \} = \{j_1 j_2 j_{123} \mid j_{12} \}.
\]

The other symmetries are lost due to the particular choice of the quantities \( p_{k,l} \) to be positive integers.

We give the orthogonality relation:

\[
\sum_{j_{123} - j_1 \leq j_{23} \leq j_1 + j_2} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_3 j_{123} \\ j_{123} \end{array} \right\} = \delta_{j_{12} j'_{12}}
\]

with \( j_{123} - j_3 \leq j_{12} \leq j_1 + j_2 \) and \( j_{123} - j_3 \leq j'_{12} \leq j_1 + j_2 \).

Then we study the properties of these 6-j with some spins shifted by \( \alpha \pi / h \). Since they are trigonometric functions of the spins multiplied by \( h \), and if the set of shifted spins is well chosen, this only results in an overall sign for half-integer \( \alpha \) — or in nothing for integer \( \alpha \). These properties will be needed in next section. We represent the shifts \( s_i \) for every \( j_i \) in an array recalling the 6-j symbols for readability: \( \left( \begin{array}{ccc} s_1 & s_2 & s_{12} \\ s_3 & s_{123} & s_{23} \end{array} \right) \). First of all, if we want to have a simple relation between the original and the shifted 6-j, we have to keep the \( p_{k,l} \) unchanged, as they control the range of summation. This condition gives four linear equations on the six shifts \( s_i \), leaving three as independent. The latter may be chosen in different ways, for instance, \( (0 \, 1 \, 1), (1 \, -1 \, 0), \text{ and } (1 \, 0 \, -1) \). An equivalent choice is \( (0 \, 0 \, 0), (1 \, 0 \, 1), \text{ and } (0 \, 1 \, 0) \).

These shifts however affect the linear combinations of spins that cannot be written by means of \( p_{k,l} \) and will only give a relation between both 6-j if they are taken proportional to \( \alpha \pi / h \) with \( 2 \alpha \) an integer, thanks to \( [x + n \pi / h] = (-1)^n [x] \) and the rules for square roots given above. The actual calculation shows that the 6-j shifted by the above (array of) shifts times \( \pi / h \) times an integer are unchanged. When the 6-j arguments are shifted by the above shifts times \( \pi / h \) times half an integer, they only get an extra sign, at most. We give it in three cases that will prove useful in next section:

\[
\left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} + \alpha \frac{\pi}{h} \left( \begin{array}{c} 0 \, 0 \, 0 \\ 1 \, 1 \, 1 \end{array} \right) = \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\}
\]

\[
\left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} + \alpha \frac{\pi}{h} \left( \begin{array}{c} 1 \, 0 \, 1 \\ 0 \, 1 \, 0 \end{array} \right) = \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\}
\]

\[
\left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} + \alpha \frac{\pi}{h} \left( \begin{array}{c} 0 \, 0 \, 0 \\ 0 \, 1 \, 1 \end{array} \right) = (-1)^{2\alpha} (j_{12} + j_{23} - j_2 - j_{123}) \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\} \left\{ \begin{array}{c} j_1 j_2 j_{123} \\ j_{12} \end{array} \right\}
\]

for \( 2 \alpha \in Z \).
4 ALGEBRA OF GENERALIZED OPERATORS

In this section, we determine the algebra of generalized vertex operators using the quantum-group invariant operator basis, in two steps.

4.1 The case of $U_q(sl(2))$.

The fusion and braiding matrices of the standard operators, — i.e. of operators with only half-integers spins verifying the full triangular inequalities (TI3), and represented on figure 3.1 — were computed recently in ref. [6] (see Eq.2.1, with all hatted spins set equal zero). However, the operators considered by Gervais et al from the beginning were not of this kind. They were $V_{m}^{(j)}$ operators with $J + m \in \mathbb{Z}_+$ and $J - m \in \mathbb{Z}_+$, but matrix elements with arbitrary zero-modes were considered. This situation corresponds to the case of only two conditions (TI2) represented on figure 3.2. Before ref. [6], however, one could only elucidate the full algebra in the basis of the quantum group covariant operators $\xi_m^{(j)}$ where the dependence in the zero-mode disappears [4, 5]. Hence the first step of generalization of the algebra merely amounts to finishing the work on well known operators, proving that the braiding and fusing of these traditional $V_{m}^{(j)}$ operators is essentially given by generalized 6-j of the first step (Eqs.3.10, 3.12 for fusion and Eqs.3.16, 3.17 for braiding). Whereas it was useful in last section to introduce the general method, the first step of generalization is straightforward here, and we shall skip it for brevity sake. The interested reader may deduce it by restriction\(^{11}\) of the second step to which we are going directly. In the general case, the operators are of a different kind, since the quantum group spins are no more half-integers. The Coulomb-gas picture provides a convenient way to build the operators [9]. One has

$$V_{m}^{(j)} \propto V_{-j}^{(j)} S_{j + m} \quad (4.1)$$

where $V_{-j}^{(j)}$ is the exponential of the free Bäcklund field, and $S$ is the screening operator. This makes sense for arbitrary $j$, provided $J + m \in \mathbb{Z}_+$, i.e. if the number of screening operators is a positive integer. Remarkably, these are our twice-generalized operators TI1, the ones with only one inequality represented on figure 3.3. This correspondence is fully clarified in refs. [9, 10], where the braiding of these operators is computed. Our next point is the fusion matrix which seems difficult to compute, using the approach of refs. [9, 10]. Of course, the fusion can be obtained from the braiding by the three-leg symmetry following MS [8]. We derive it nevertheless, for completeness and in order to show how far one can go by using the null-vector equations. As we shall see, all the spins can be taken continuous except one (say $J_1$) which has to be kept half-integer, as there must be at least one degenerate conformal primary (of the kind $(1, 2J_1 + 1)$) so that the decoupling of the null vector yields the differential equation which is the starting point. We follow closely the recursive proof of ref. [9], generalizing it to non-half-integer spins, this is why we will be sometimes sketchy and refer the interested reader to the details

\(^{11}\) keeping in mind the caveats given at the end of section 6,
given in ref. [3]. We want to prove that these generalized operators fit in Moore and Seiberg scheme [8]. Thus their fusion must be of the form

\[ P_j V_{m_1}^{(j_1)} V_{m_2}^{(j_2)} = P_j \sum_{j_{12}=-m_1-m_2}^{j_1+j_2} F_{j+m_1,j_{12}}^{j_1,j_2} | \omega_{j_{12}} \rangle \langle \omega_{j_2-j_{12}} |, \]

(4.2)

where \( j_1 + m_i \) and \( j_1 + (j_2 - j_{12}) \) are positive integers (these conditions determine the summation range).

As said above, we have to keep one spin half-integer. We choose \( J_1 = 1/2 \) to begin the recursion. It gives a degenerate family of the BPZ type (1,2) and the differential equation coming from the decoupling of the null vector at level two allows us to compute the four-point function

\[ < \omega_{123} | V_{1/2}^{(j_1)} | \omega_{j_2} \rangle = z_2^{\Delta_{j_2} - \Delta_{j_3}} z_1^{\Delta_{1/2} + \Delta_{j_{12}}} \times \]

\[ \left( \frac{z_2}{z_1} \right)^{\Delta_{j_{123}}^{1/2}} \left( 1 - \frac{z_2}{z_1} \right)^{-h_{j_2}/\pi} \mathbf{2}F_1 \left( a_{\pm}, b_{\pm}; c_{\pm}; \frac{z_2}{z_1} \right); \]

(4.3)

\[ a_{\pm} = \frac{1}{2} + \frac{h}{2\pi} \left[ -\omega_2 \mp (\omega_{123} - \omega_3) \right]; \quad b_{\pm} = \frac{1}{2} + \frac{h}{2\pi} \left[ -\omega_2 \mp (\omega_{123} + \omega_3) \right]; \]

\[ c_{\pm} = 1 \mp \frac{h \omega_{123}}{\pi}; \quad \omega_i \equiv \omega_{ji} = \omega_0 + 2j_i \]

(4.4)

where \( j_2, j_3, j_{123} \) can indeed be taken continuous. The operator \( V^{(j_2)} \) with continuous \( j_2 \) was not explicitly built in the Gervais et al approach yet (hence the usefulness of the Coulomb-gas approach). However, without this explicit construction, if one assumes that such an operator exists, the differential equation allows to compute this four-point function.

The transformation properties of the hypergeometric function \( 2F_1 \) in Eq. (4.3) allows us to show that the fusion of \( V^{(1/2)} \) and \( V^{(j)} \) are of the MS form recalled above with

\[ F_{j_{123} + \epsilon_1/2, j_2 + \epsilon_2/2}^{j_{123} + \epsilon_1/2} \left( \frac{1/2}{j_{123} + \epsilon_1} \right) = \frac{\Gamma(1 - \epsilon_1 \omega_{123} h/\pi)}{\Gamma(1/2 + (-\epsilon_1 \omega_{123} + \epsilon_2 \omega_2 + \omega_3) h/2\pi)} \frac{\Gamma(\epsilon_2 \omega_2 h/\pi)}{\Gamma(1/2 + (-\epsilon_1 \omega_{123} + \epsilon_2 \omega_2 - \omega_3) h/2\pi)}. \]

(4.5)

Still following ref. [3], we try the ansatz

\[ F_{j_{23}, j_{12}}^{j_{12}} \left( j_{12} \right) = g_{j_{12}}^{j_{12}} g_{j_{12}}^{j_{12}} \mathbf{2}F_1 \left( j_1 j_2; j_{12}; j_{12}; j_{12} \right). \]

(4.6)

It is possible to see that the fusion coefficient computed above can be cast under the form:

\[ F_{j_{123} + \epsilon_1/2, j_2 + \epsilon_2/2}^{j_{123} + \epsilon_1/2} \left( \frac{1/2}{j_{123} + \epsilon_1} \right) = \frac{g_{j_{12}}^{j_{12}} g_{j_{12}}^{j_{12}}}{g_{j_{12}}^{j_{12}} g_{j_{12}}^{j_{12}}} \mathbf{2}F_1 \left( j_{123}; j_{123}; j_{123}; j_{123}; j_{123}; j_{123}; j_{123} + \epsilon_1/2 \right). \]

(4.7)
with (letting as usual $F(z) \equiv \Gamma(z)/\Gamma(1-z)$)

$$g_{j_1j_2}^{j_{12}} = (g_0)^{j_1+j_2-j_{12}} \prod_{k=1}^{j_1+j_2-j_{12}} \sqrt[4]{F((\omega_1-k)h/\pi)F((\omega_2-k)h/\pi)F((-\omega_{12}-k)h/\pi)/F(1+kh/\pi)}.$$  

(4.8)

Then, we use the pentagonal equation for generalized 6-j symbols that was proven in last section. It would allow to make a recursion on the integer quantities $j_i + j_k - j_{ik}$, and compute the most general fusion coefficients. However we are short of a starting point general enough for this recursion, as our starting point (4.7), the fusion of spin $1/2$ and a continuous spin $j_2$, does not involve two continuous spins. Hence, in order to be able to begin the recursion, we have to only use pentagonal equations involving fusion operations of the following type:

$$J_1 \ j_2 \ j_3 \ j_{12} \ j_{23} \ j_{123} \ j_{234} \ j_{34} \ j_4$$

which can be seen as a restricted case of the twice-generalized one, due to the half-integer character of $J_1$ (remember the dot convention of last section). All the results of the second step of generalization apply, with the extra condition that $J_1 + j_{123} - j_{23}$ and $J_1 + j_{12} - j_2$ be positive. Our starting point, the fusion coefficient (4.7), corresponds to the fusion (4.9) with $J_1 = 1/2$. So, we make use of the following pentagonal relation:

$$F_{j_{123}j_{234}j_{34}j_4}^{j_{12}j_2j_3j_4} = g_{j_{234}}^{j_{12}j_2j_3j_4} g_{j_{34}}^{j_{12}j_2j_3j_4} \{j_1 j_2 j_3 j_{12} j_{23} \}$$

(4.10)

with the restrictions (in comparison with the most general case) that $J_2$ be half-integer and that $J_2 + j_{234} - j_{34}$ and $J_2 + j_{23} - j_3$ be positive. The final step to the ansatz (4.6) is then a conjecture. It seems however very reasonable as it is only
the result of the symmetrization of Eq. (4.10) to 6-j symbols that satisfy the required polynomial equations. Moreover it is confirmed by the result already proven for braiding in ref. [11].

4.2 The complete algebra

We have done half of the work only, since there are two different screening charges, or equivalently two deformation parameters such that \( h/\pi = (\alpha_-)^2/2 \) and \( \tilde{h}/\pi = (\alpha_+)^2/2 \). In the half-integer case they give the BPZ degenerate families \((1, 2J + 1)\) and \((2J + 1, 1)\) respectively, and all the families \((2J + 1, 2J + 1)\) by fusion. This is the double \( U_q(sl(2)) \odot U_q(sl(2)) \) structure recalled in the introduction. The half-integer spin operators of the full algebra \( V^{(J, \tilde{J})}_{m, \tilde{m}} \) were described by the four discrete quantum numbers \( J, \tilde{J}, m, \tilde{m} \). The question is to know what will be the good quantum numbers in the continuous spin case. Let us first stick to the half-integer spin case for a short while. The Coulomb gas realization of ref. [9, 10] with both screening charges has a trivial braiding and fusion matrices — simple sign factors — with any operator that we call effective spin, and not of \( J \) so that \( V^e_{J, \tilde{J}} \) may be regarded as a function of the combination \( J^e \equiv J + \tilde{J}\pi/h \), that we call effective spin, and not of \( J \) and \( \tilde{J} \) separately. Concerning the states, a similar phenomenon occurs. For a state associated with half-integer spins \( J \) and \( \tilde{J} \), the corresponding zero-mode is \( \varpi = \varpi_{J, \tilde{J}} \equiv \varpi_0 + 2J + 2\tilde{J}\pi/h \). It is only a function of \( J^e \) and we have \( \varpi_{J, \tilde{J}} = \varpi_0 + 2J^e \), also denoted \( \varpi_J \). One may verify that the fusion and braiding matrices may be written in terms of these effective spins (more about this below). Of course, if \( h \) is irrational, using \( J, \tilde{J} \) or \( J^e \) is immaterial. In this half-integer spin case, there is a remarkable fact, which is related: by using the properties of the gamma functions under integer shifts and of the sinus functions under shifts by \((\pi \times \text{integer})\), one can show that any operators of the \( U_q(sl(2)) \) family has a trivial braiding and fusion matrices — simple sign factors — with any operator of the \( U_q(sl(2)) \) type (see refs. [3, 4, 5] and references therein). This is the meaning of the \( \odot \) in \( U_q(sl(2)) \odot U_q(sl(2)) \), it is a sort of graded tensorial product. These sign factors simply come out when one returns from the effective spins to the half-integer ones.

For continuous case, the operators are \( V^{(J^e)}_{J, \tilde{J}} S^{J+m} \tilde{S}^{\tilde{J}+\tilde{m}} \). They are specified by \( J^e \) and the two screening numbers \( J + m \), and \( \tilde{J} + \tilde{m} \) that are positive integers. The hatted and unhatted quantum numbers \( J \) and \( \tilde{J} \) can no longer be separated, and the effective spins \( J^e \) turn out to be the only possible good variables. As a result the relative fusion and braiding of the two families with a single screening charge, become non trivial, as shown in ref. [10], and as we shall see. So, we review what
the TI3 and TI1 vertices become in the case of the full algebra, in terms of effective spins. Treating the three legs symmetrically (instead of using the shift quantities \( m \)), the TI3 conditions for the full algebra may be written as

\[
\begin{align*}
\mathcal{J}^e_{23} \rightarrow \left\{ \begin{array}{l}
J^e_1 + J^e_2 - J^e_{12} \in Z_+ + (\pi/h)Z_+ \\
J^e_{12} + J^e_2 - J^e_1 \in Z_+ + (\pi/h)Z_+ \\
\end{array} \right. & \Rightarrow 2J^e_1, 2J^e_2, 2J^e_{12} \in Z_+ + \frac{\pi}{h}Z_+.
\end{align*}
\]

These conditions are clearly equivalent to the conditions [3.14] applied to unhatted and hatted spins. Now that the good variables have been chosen, the generalization to continuous spins works for \( U_q(sl(2)) \odot \mathcal{U}_q^2(sl(2)) \) just like in the case of the simple algebra \( U_q(sl(2)) \). The generalized vertex is

\[
\begin{align*}
\mathcal{J}^e_{12} & \rightarrow J^e_1 + J^e_2 - J^e_{12} \in Z_+ + \frac{\pi}{h}Z_+.
\end{align*}
\]

More generally, we extend the notation of Eq.3.26 to the full algebra, and for every vertex \( \mathcal{J}^e_{ki} \), we introduce the quantities \( p_{k,l} \) and \( \hat{p}_{k,l} \) defined by \( j_k + j_l - j_{kl} \equiv p_{k,l} + \hat{p}_{k,l}(\pi/h) \) which are constrained to be positive integers\(^{[3]}\). The vertex 4.14 represents the free field raised at a continuous power \( J^e_i \) screened by integer numbers \( p_{1,2} \) and \( \hat{p}_{1,2} \) of the two different screening operators.

It leads us to the following ansatz:

\[
F_{\mathcal{J}^e_{23},\mathcal{J}^e_{12},\mathcal{J}^e_1,\mathcal{J}^e_2,\mathcal{J}^e_3} = \mathcal{F}_{\mathcal{J}^e_{23},\mathcal{J}^e_{12},\mathcal{J}^e_1,\mathcal{J}^e_2,\mathcal{J}^e_3} \left\{ \left\{ \mathcal{J}^e_{12}, \mathcal{J}^e_2, \mathcal{J}^e_1 \right\} \right\} ,
\]

whose precise definition follows next. The \( \hat{J}^e_i \)'s are rescaled from the \( J^e_i \)'s by \( \hat{J}^e_i = J^e_i h/\pi \) as usual. The fusion matrix \( F \) and the \( g \) coefficients, although noted as before, are a generalization of the previous ones with the condition Eq.4.14 for the full algebra vertices. These \( g \)'s have to be computed. The 6-j symbols in Eq.4.15 are noted with double braces, which means that they are (slightly) different from the ones of last section. This is of course necessary as the \( J^e_i \)'s verify the conditions 4.14 of the full algebra type whereas the 6-j-symbol arguments must verify standard conditions of the type 3.26. We shall call them effective 6-j coefficients. One may verify that we will satisfy these standard conditions if we shift the effective \( J^e_i \)'s by suitable linear combinations either of \( p_{k,l} \)'s or of the \( \hat{p}_{k,l} \)'s. In fact, for 6-j coefficients computed with one deformation parameter — i.e. one of the last two factors on the r.h.s. of Eq.4.13, say the first one — it is convenient to shift the \( J^e_i \)'s by linear combinations of the \( \hat{p}_{k,l} \)'s, so that conditions 4.13 reduces to conditions Eqs.3.26 that only involve the \( p_{k,l} \)'s. This leads to the possible definition

\[
\left\{ \left\{ \mathcal{J}^e_{12}, \mathcal{J}^e_2, \mathcal{J}^e_1 \right\} \right\} \equiv \left\{ \mathcal{J}^e_{12} + \mathcal{J}^e_{23}, \mathcal{J}^e_2 + \mathcal{J}^e_{23}, \mathcal{J}^e_1 + \mathcal{J}^e_{23} \right\} \equiv \left\{ J^e_{12} + \hat{p}_{1,2,3} \mathcal{J}^e_{23}, \mathcal{J}^e_2 + \hat{p}_{1,2,3} \mathcal{J}^e_{23}, \mathcal{J}^e_1 + \hat{p}_{1,2,3} \mathcal{J}^e_{23} \right\} ,
\]

\(^{[12]} Z_+ + (\pi/h)Z_+ \equiv \{ a + b\pi/h : a, b \text{ positive integers} \}

\(^{[13]} \)This notation assumes a specific choice of labels for the six \( j \) parameters, but it has the advantage of simplicity.
The difficulty is that when replacing the effective 6-j’s by one of their definitions, in these equations, the conditions allows as well to show that the ansatz Eq. 4.15 verifies the pentagonal equation. This proves the equality of the corresponding 6-j coefficients thanks to Eqs. 3.29.



\[
\{ \{ J^1_1 J^2_2 | J^1_2 \} = \{ J^1_1 J^2_2 \} \mid J^1_1 \}
\]

since one can easily see that the spins of the 6-j symbol of the r.h.s. satisfy the conditions 3.26 of the half-algebra. A potential trouble is that there are many possible choices, none of them seemingly better than the other. The solution of the puzzle is that all the choices give the same value:

\[
\left\{ J^1_1 J^2_2 \mid J^1_2 \right\} = \left\{ J^1_1 J^2_2 \mid J^1_2 + \hat{p}_{1,2} \pi / h \right\}
\]

where we wrote various possibilities. These equalities are easy consequences of the properties proven at the end of section 3. This result is not surprising as the differences between the different choices are precisely shifts of the spins by (integer $x \pi / h$), such that the $p_{k,l}$ are unchanged, which are precisely the kind of shifts that were proven not to change the value of the 6-j coefficients in section 3.3. For instance, we compare the choice of Eq. 4.16 and the first one of Eq. 4.17 (using the notations of subsection 3.3 for the addition of sixtuples):

\[
\begin{align*}
&\frac{\left( J^1_1 J^2_2 \right)}{\left( J^1_1 J^2_2 \right)} = \left( J^1_1 J^2_2 \right) + \left( \hat{p}_{1,2} \pi / h \right) \\
&\left( J^1_1, J^2_2 \right) = \left( \hat{p}_{1,2} \pi / h \right)
\end{align*}
\]

This invariance of the 6-j coefficients under some particular shifts of their arguments allows as well to show that the ansatz Eq. 4.15 verifies the pentagonal equation. In these equations, the $g$ coefficients cancel out and the sum to be proven factorises in a sum over $p_{2,3}$ of effective 6-j’s and a sum over $\hat{p}_{2,3}$ of hatted effective 6-j’s. The difficulty is that when replacing the effective 6-j’s by one of their definitions in terms of (normal) 6-j’s, all the 6-j’s do not have the same $J^1_i$’s, they are shifted by some $\hat{p}_{k,l} \pi / h$, and the pentagonal equation 3.21 cannot be applied directly. It is however easy to see that thanks to some shifts of the previous kind, it reduces to the pentagonal equation 3.21.

So, now that this ansatz has been proven to be consistently defined, let us demonstrate it. Let us get rid of the value of the $g$ coefficients first. The fusion of a spin-one-half operator and of a generalized one belonging to the the $U_q(sl(2)) \otimes U_q(sl(2))$ algebra, involves two $g$ coefficients of the most general type. There is no difficulty in computing the corresponding four-point functions and to obtain this fusion coefficient by transformation of the gamma function. It yields consistent necessary conditions for $g$ which turns out to be the most natural generalization \[14\] of the integer spin case of ref. 3:

\[
g_{J^1_1 J^2_2} = (-1)^{\hat{p}}(i/2)^{p+\hat{q}}H_{pp}(\omega J^1_1)H_{pp}(\omega J^2_2)H_{pp}(\omega J^1_2)
\]

\[\text{H} \frac{H_{pp}(\omega J^1_1)H_{pp}(\omega J^2_2)H_{pp}(\omega J^1_2)}{H_{pp}(\omega p/2, \hat{p}/2)} \] (4.18)}

\[14\] The sign $(-1)^{\hat{p}}$ was added in comparison with ref. 3 (more about signs below).
Eventually, the pentagonal equation applied to verifying conjecture that such is the case for $J$. There, the 6-j coefficients had entries verifying $j$. This does not agree with the one of ref.\[6\]. The latter was derived with a treatment of signs where the $(h)$ is a half-integer spin operator built with deformation parameter $h$. This does not matter for the polynomial equations, since the difference may be not complete, so that it is not really correct.

A parenthetical remark is in order at this point. The expression just given does not agree with the one of ref.\[6\]. The latter was derived with a treatment of signs of square roots that is not completely consistent, so that it is not really correct. This does not matter for the polynomial equations, since the difference may be re-absorbed by a change of coupling constants

$$g_{\ell_1 \ell_2} \rightarrow (-1)^{2\ell_1 \ell_2 + 2\ell_1 \ell_2} g_{\ell_1 \ell_2},$$

but Eq.\[1.21\] is the one which is completely consistent.

For brevity sake, we use in this paragraph the following short notation: $V(J)$ (resp. $V(\hat{J})$) is a half-integer spin operator built with deformation parameter $h$ (resp. $\hat{h}$), $V(J)$ (resp. $V(\hat{J})$) is a continuous spin operator of the half-algebra with deformation parameter $h$ (resp. $\hat{h}$), hence verifying $J_i + J_{right} - J_{left} \in Z_+$, and $V(J)$ is a continuous spin operator of the full algebra, hence verifying $J_i^c + J_{right}^c - J_{left}^c \in Z_+ + (\pi/h)Z_+.$
We have to check that our result for continuous spins reduces to this (corrected)
old one when applied to spins \( J_1^e = J_i + J_1 \pi / h \). The \( g \) coefficients are trivially the
restriction of ours. The non-trivial work is to prove

\[
\begin{align*}
\{ \{ J_1 + J_2 \pi / h, J_2 + J_3 \pi / h \} & \ ( J_{12} + J_{23} \pi / h) \} \\
\{ \{ J_3 + J_4 \pi / h, J_{12}, J_4 \pi / h \} & = \\
\{ \{ J_3 + J_4 \pi / h, J_{12}, J_4 \pi / h \} & \ ( J_{12} + J_{23} \pi / h) \} \\
( -1)^f g( \{ \{ J_1, J_2, J_3, J_4 \} & J_1 J_2 J_3 J_4 \} J_{12} J_{23}) \{ J_1 J_2 J_3 J_4 \} \{ \{ J_1 J_2 J_3 J_4 \} \} = \\
( -1)^f & \sum_{\nu_{12}} \gamma^{(J_1 J_2)} \gamma^{(J_3 J_4)} < \nu_{12}, \nu_{12} \} | V^{(J_1^e J_2^e)} | \nu_{12} > .
\end{align*}
\]

By applying the definition 4.10, for example, to the first effective 6-j of Eq.4.22, and
noticing that

\[
\begin{align*}
( J_1 + J_2 \pi / h, J_2 + J_3 \pi / h, J_{12} + (J_1 + J_3) \pi / h) & \\
& = ( J_1 J_2 J_3 J_4 ) \left( J_{12}, J_{23} \right) + \gamma_1 \left( J_{12}, J_{23} \right) \frac{\gamma_3 \gamma_1}{h} \left( J_{12}, J_{23} \right) + \gamma_3 \left( J_{12}, J_{23} \right) \frac{\gamma_3 \gamma_1}{h} \left( J_{12}, J_{23} \right),
\end{align*}
\]

one can see that this 6-j \( (J_i + J_1 \pi / h) \) reduces to the half-integer spin 6-j \( (J_i) \) up to a sign thanks to Eqs.[3.29]. Doing the same for the hatted 6-j and collecting all the
signs, one easily completes the proof.

For completeness, we write the full fusion-equation

\[
\begin{align*}
\mathcal{P}^{(J_1^e J_2^e)} \mathcal{P}^{(J_3^e J_4^e)} \mathcal{P}^{(J_5^e J_6^e)} = & \sum_{(J_{12})} \frac{g_{J_{12} J_3} g_{J_3 J_4}}{g_{J_3 J_4} g_{J_1 J_2}} \left\{ \left\{ J_{12} J_3 J_4 \right\} \left\{ J_1 J_2 J_3 \right\} \right\} (4.22)
\end{align*}
\]

The sum over \( J_{12}^e \) is a sum over all the values of \( J_{12} \) allowed by the four full-algebra vertex-conditions of the type Eq.4.14. It can be viewed as a double sum on \( p_{12} \) and \( \hat{p}_{12} \). The value of the \( g \) coefficients for the full algebra is given in Eq.4.18. The definition of the effective 6-j coefficients (double brace) is given in Eqs.4.10, 4.17.

We give the braiding equation as well. It was computed in ref.[9] for one half of the algebra and in ref.[10] for the full algebra. It can be deduced from the fusion by the three leg symmetry of the vertex-operators\[8, 11]:

\[
< \nu_{12} | V^{(J_1^e J_2^e)} | \nu_{21} > = e^{i \pi (\Delta(J_1^e) + \Delta(J_2^e) - \Delta(J_{12}^e))} < \nu_{12} | V^{(J_2^e)} | \nu_{12} > .
\]

This gives

\[
\begin{align*}
\mathcal{P}^{(J_1^e J_2^e)} \mathcal{P}^{(J_3^e J_4^e)} \mathcal{P}^{(J_5^e J_6^e)} = & \sum_{J_{12}} \frac{g_{J_{12} J_3} g_{J_3 J_4}}{g_{J_3 J_4} g_{J_1 J_2}} \left\{ \left\{ J_{12} J_3 J_4 \right\} \left\{ J_1 J_2 J_3 \right\} \right\} \left\{ \left\{ J_{12} J_3 J_4 \right\} \left\{ J_1 J_2 J_3 \right\} \right\} (4.25)
\end{align*}
\]

where again the sum over \( J_{12}^e \) is to be understood as a double sum.

These new expressions have the virtue that, besides the extension to continuous
spins, they give expressions of the fusing and braiding coefficients which are analytic
in the spins. It did not seem easy to derive formulae of this type from the expression
for half-integer spins due to the sign \((-1)^f\). The analytic expression for the \( g \)
coefficients is of great importance as well, as we shall see in section 7: it directly
leads to the three-point function.
5 THE SYMMETRY $j \to -j - 1$

In this section, we study the behaviour of the 6-j coefficients under the transformation $j_i \to -j_i - 1$ applied to all the spins. We prove that there exists a suitable continuation for which the 6-j coefficients are unchanged. This part essentially follows the line of ref. [3], but deals with higher quantum-group symbols.

The operators with transformed spins $-j_i - 1$ are needed in next section to build real-positive-weight operators in the strong coupling regime [3]. They are needed in the weak coupling regime [12], as well. The basic feature in this transformation is that the screening numbers $p_{k,l}$ of the four vertex operators involved are transformed into $-p_{k,l} - 1$, and so, this symmetry defines the fusion coefficient of vertex operators with negative screening numbers.

The transformation $j_i \to -j_i - 1$ will be performed on an expression of the 6-j coefficients in terms of $4 \, F_3$ hypergeometric function. This is why we shall first prove the following useful identity on q-deformed $4 \, F_3$ hypergeometric functions, following a method explained for instance in refs. [15, 16]:

$$4 \, F_3 \left( x, y, z, -N; a, b, c; 1 \right) = \binom{b - z}{N} \binom{c - z}{N} \binom{b}{N} \binom{c}{N} 4 \, F_3 \left( a - x, a - y, z; a - z - b + 1, z - N - c + 1; 1 \right), \quad (5.1)$$

with the conditions that $N$ be a positive integer (i.e. that the series be terminating) and that $x + y + z - N + 1 = a + b + c$ (i.e. that the series be Saalschützian). Our conventions for the hypergeometric functions are the following:

$$4 \, F_3 \left( \alpha, \beta, \gamma, \delta; z \right) \equiv \sum_{\nu=0}^{\infty} \left[ \binom{\alpha}{\nu} \binom{\beta}{\nu} \binom{\gamma}{\nu} \binom{\delta}{\nu} \right] z^\nu. \quad (5.2)$$

Eq. (5.1) is given in ref. [16] in the non-q-deformed case. Although some q-deformed hypergeometric function properties are given in this reference, this one is not computed explicitly. This is why we rederive it rapidly, verifying that the proof of the classical case extends to the q-deformed case without problem.

The basic idea is to write a finite double sum in two different ways by inverting the summations:

$$\sum_{n=0}^{\infty} \sum_{p=0}^{n} \beta_n \alpha_p u_{n-p} = \sum_{p=0}^{\infty} \sum_{n=p}^{\infty} \beta_n \alpha_p u_{n-p}. \quad (5.3)$$

We formally let summations go to infinity for simplicity, but they are all finite as $\beta_n$ is zero for $n > N$ (it can be checked on the explicit values given below in Eq. (5.6)).

We define

$$S_n = \sum_{p=0}^{n} \alpha_p u_{n-p}, \quad T_p = \sum_{n=p}^{\infty} \beta_n u_{n-p}, \quad (5.4)$$

so that Eq. (5.3) reads

$$\sum_{n=0}^{\infty} \beta_n S_n = \sum_{p=0}^{\infty} \alpha_p T_p. \quad (5.5)$$

\footnotetext{[16]They are borrowed from ref. [3]. The appendix B of this reference shows the connection with the usual mathematical notation. For the case considered, the two agree.}
Let us choose

\[
\alpha_p \equiv \frac{|x|_p}{|a|_p} \frac{|y|_p}{|b|_p} , \quad \beta_n \equiv \frac{|z|_n}{|c|_n} , \quad \gamma_k \equiv \frac{(a - x - y)_k}{(k)!} ,
\]

so that the sums \( S_n \) and \( T_p \) can be computed. We use identities of the type

\[
[a]_n = 1/[a+n]_n , \quad [a]_n/[a]_m = [a+m]_{n-m} , \quad [a]_n = (-1)^n [-a-n+1]_n ,
\]

to which a meaning can be given for any signs of the integers \( n, m \) — thanks to the following usual continuation for the products

\[
\prod_{x=a}^{b} f(x) = 1/\prod_{x=b+1}^{a-1} f(x) .
\]

One can transform \( S_n \) into the following \( _3F_2 \) hypergeometric function

\[
S_n = \frac{(a-x)_n (a-y)_n}{[n]!!} \frac{\Gamma(a,x+y-a-n+1)}{\Gamma(a,x)} .
\]

It is a terminating \((-n \text{ is a negative integer})\) Saalschützian \((x + y + (-n) + 1 = a + (x + y - a - n + 1))\) \( _3F_2 \) series, and can therefore be summed, thanks to the \( q \)-deformed Saalschütz theorem given in ref.\[16]\:

\[
3F_2 \left( \frac{x,y,-n}{a,x+y-a-n+1};1 \right) = \frac{(a-x)_n (a-y)_n}{[n]!!} \frac{\Gamma(a,x+y-a-n+1)}{\Gamma(a,x)} .
\]

The same happens to \( T_p \) so that Eq.5.3 reduces to one sum on each side. These sums can both be put under the form of a \( _4F_3 \) hypergeometric series, which finally gives Eq.5.1. Q.E.D.

The generalized 6-j coefficient of Eq.\[3.25\] can easily be written with a \( _4F_3 \) series:

\[
\{ j_{123} \}_{j_{123}} = \frac{x_{j_{23}}^{j_{123}} x_{j_{123}}^{j_{23}}}{x_{j_{123}}^{j_{123}}} (-1)^{p_{1.2}} [2j_{23} + 1] [p_{1.2}] (j_{23} + j_{12} + 1)_{p_{1.2}}
\]

\[
\frac{[j_{123} + j_{1} - j_{2} - j_{3} + 1]}{[j_{1} - j_{2} - j_{123}]} \frac{4F_3 \left( \frac{[j_{1} + j_{23} + j_{123} + 2j_{1} - j_{2} - j_{12} - p_{1.2}]}{[p_{1.2} - p_{1.2} + j_{123} + j_{23} - j_{12} + 1]} \right)}{[j_{1} - j_{2} - j_{123}]_{p_{1.2} - p_{1.2}}},
\]

where we wrote the \( _4F_3 \) summation index \( \nu = y - p_{1.2} \) (\( \nu \) of Eq.\[5.2\] and \( y \) of Eq.\[3.25\]). The factorial \([p_{1.2} - p_{1.2}]\) causes no problem when \( p_{1.2} - p_{1.2} < 0 \), since its vanishing just compensates the poles of the hypergeometric function — which arise for \( \nu \geq p_{1.2} - p_{1.2} \) — and cancels the other terms (\( \nu < p_{1.2} - p_{1.2} \)) of the summation, thereby giving the correct formula.

It is not possible to continue this \( _4F_3 \) function when transforming all the spins by \( j_i \rightarrow -j_i - 1 \). The reason is that all the \( p_{k,l} \) would also be transformed into \(-p_{k,l} - 1\) and the transformed \( _4F_3 \) would not have any negative-integer upper argument. It
would not terminate and, possibly, diverge. This is why we first transform it using Eq. 5.1.

This hypergeometric series verifies the conditions of Eq. 5.1: it is terminating and Saalschützian. We can therefore transform it thanks to Eq. 5.1 with \( N = -p_{23}, z = j_1 - j_2 - j_{12}, a = p_{123} - p_{12,3} + 1 \) and write it in the strictly equivalent form

\[
\begin{align*}
\{ j_1, j_2 \} & \quad \{ j_{12}, j_{23} \} = \sum_{j_{123}} \sum_{j_{123}} \sum_{j_{123}} (-1)^{p_{23}} \frac{[2j_{12} + 1]}{[p_{12,3}]} \frac{[j_{12} + j_{123} - j_3 + 1]}{p_{2,3}} \\
& \quad \frac{[j_3 + j_{12} - j_2 + 1]}{[p_{12} - p_{23}]} \mathcal{F}_3 \left( -j_{12} - j_3 - j_{123}; j_1 - j_2 - j_{12} = p_{2,3} + 1; \right. \\
& \left. -j_{123}, p_{12} - j_{23}, j_{12} + j_{123} - j_3 + 1 \right) \pmod{1}. (5.12)
\end{align*}
\]

Although equivalent to the definition Eq. 5.11 for all \( p_{kl} \) positive integers, this expression Eq. 5.12 leads to a different continuation when \( j_i \)'s are mapped into \( j_i - 1 \): the transformed \( \mathcal{F}_3 \) is a well defined series as it is terminating thanks to its upper parameter \( -p_{123} \). So, the 6-j symbol with spins transformed by \( j_i \rightarrow j_i - 1 \) has a consistent definition. However, the transformed expression does not have the same arguments as the defining formula 5.11. So, in order to compare them, we look for an other expression of the 6-j coefficients by transforming again the definition Eq. 5.11 by Eq. 5.1. This time, we take \( N = -p_{12,3}, z = j_1 + j_{23} - j_{123} + 1 \) and \( a = p_{123} - p_{12,3} + 1 \). Then we make the simple change of variables \( j_1 \leftrightarrow j_3, j_{12} \leftrightarrow j_{23} \) and get (using the symmetry 3.27)

\[
\begin{align*}
\{ j_1, j_2 \} & \quad \{ j_{123} \} = \sum_{j_{123}} \sum_{j_{123}} \sum_{j_{123}} (-1)^{p_{23}} \frac{[2j_{12} + 1]}{[p_{12,3}]} \frac{[j_2 + j_3 + j_{123} - j_1 + 2]}{p_{123}} \\
& \quad \frac{[j_3 + j_{23} - j_2 + 1]}{[p_{12} - p_{23}]} \mathcal{F}_3 \left( -j_{12} - j_3 - j_{123} + 2p_{2,3} + 1; j_2 + j_{123} - j_1 + 1; \right. \\
& \left. -j_{123}, p_{12} - j_{23}, j_2 + j_{123} - j_1 + 2 \right) \pmod{1}. (5.13)
\end{align*}
\]

One already sees that Eq. 5.12 transformed by \( j_i \rightarrow j_i - 1 \) gives the same \( \mathcal{F}_3 \) function as Eq. 5.13. A little work is required to see that the prefactors are indeed the same, in particular, thanks to Eqs. 5.7, 5.8, the factorials transform like

\[
[p_{23}] / [p_{12,3}] \rightarrow [-p_{23} - 1] / [-p_{12,3} - 1] = (-1)^{p_{12,3} - p_{23}} [p_{12,3}] / [p_{23}] \pmod{1} (5.14)
\]

and consequently

\[
\begin{align*}
\sum_{j_{123}} \sum_{j_{123}} \sum_{j_{123}} \sum_{j_{123}} (-1)^{j_{123}} \frac{[2j_{12} + 1]}{p_{12,3}} \frac{[j_2 + j_3 + j_{123} - j_1 + 2]}{p_{2,3}} \\
& \quad \frac{[j_3 + j_{23} - j_2 + 1]}{p_{12} - p_{23}} \mathcal{F}_3 \left( -j_{12} - j_3 - j_{123}; j_1 - j_2 - j_{12} = p_{2,3} + 1; \right. \\
& \left. -j_{123}, p_{12} - j_{23}, j_2 + j_{123} - j_1 + 2 \right) \pmod{1}
\end{align*}
\]

After simplifications, this proves that

\[
\{ j_{123} \} = \{ j_1, j_2, j_{123} \} \pmod{1} (5.15)
\]

Finally, we write this transformation for effective 6-j coefficients, as they are the ones of interest in fusion or braiding. We apply the transformation 5.13 to their definition 4.16 and get

\[
\left\{ \begin{array}{c} j_1 \ j_2 \\ j_{123} \ j_{23} \end{array} \right\} = \left\{ \begin{array}{c} j_1 \ j_2 \\ j_{123} \ j_{23} \end{array} \right\} \pmod{1} (5.16)
\]
This is the definition of an effective 6-j with entries $-J_i^e - 1$. But this transformation does not treat $J_i^e$ and $\hat{J}_i^e$ symmetrically and will not be convenient for later use. So, we shift the previous 6-j entries by

$$-\frac{\pi}{\hbar} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} - \frac{\pi}{\hbar} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\pi}{\hbar} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(5.17)

which does not change its value thanks to Eqs.3.29. This gives the more symmetric transformation

$$\begin{array}{c}
\begin{array}{c}
\{J_i^e J_j^e J_k^e J_l^e J_m^e J_n^e \mid J_{12}^e, J_{23}^e \} = \\
\{\{J_i^e, J_l^e, J_m^e \mid -J_i^e - (1 + \pi/\hbar), -J_l^e - (1 + \pi/\hbar) \} - J_{12}^e - (1 + \pi/\hbar), \}
\end{array}
\end{array}$$

(5.18)

which we use later on.

### 6 STRONG COUPLING REGIME OPERATORS

#### 6.1 Relation between 6-j symbols

In this section we restrict ourselves to the central charges of interest for the strong coupling regime: $C = 1 + 6(2 + s)$. As explained in the preliminary section (2), the Hilbert spaces to be considered are $\mathcal{H}_{\text{phys}}^\pm$ defined by Eqs.2.9, 2.10. The effective spins, corresponding to Eq.2.10, are

$$\{J_i^e = J + J\pi/\hbar, 2J = n + r/(2 + s), n \in \mathbb{Z}, r = 0...1 + s\},$$

(6.1)

with real negative weights, for $\mathcal{H}_{\text{phys}}^-$, and

$$\Delta(J^e) = \Delta(J, J) = -(2 + s)J(J + 1),$$

(6.2)

or

$$\{J_i^e = -J - 1 + J\pi/\hbar, 2J = n + r/(2 - s), n \in \mathbb{Z}, r = 0...1 - s\},$$

(6.3)

with real positive weights, for $\mathcal{H}_{\text{phys}}^+$, and

$$\Delta(J^e) = \Delta(-J - 1, J) = 1 + (2 - s)J(J + 1).$$

(6.4)

As before we define hatted quantities by $\hat{J}^e = J^e \pm h/\pi$, so we have

$$\hat{J}^e = J + J\pi/\hbar, \quad \hat{J}^e = J + (-J - 1)h/\pi.$$  

(6.5)

The aim of this section is a truncation theorem of the type displayed in ref.[3]. Indeed, we will see that, for these particular values of $C$, these two subsets of operators are individually closed for braiding and fusion.

The main tool to prove the closure of the algebra of the physical operators is that the hatted 6-j coefficients are equal to unhatted 6-j coefficients up to a sign, provided $C = 1 + 6(s + 2)$, $s = -1, 0, 1$ if the spins considered are of the form given in Eq.6.3 or Eq.6.1. This type of reasoning was already presented in ref.[3], albeit for
different quantum-group symbols — 3-j’s, and universal R matrix — and for half-integer spins only. Before coming to this result, we have to prove some preliminary properties.

Using the fact that
\[
h + \hat{h} = (C - 13)\pi/6 = s\pi
\]
(6.6)
it is straightforward to prove that for \( A \) of the type
\[
A = n + r/(2 + s) \quad , \quad n \in \mathbb{Z} , \quad r = 0...1 + s
\]
(6.7)
one has \((k \in \mathbb{Z})\)
\[
\hat{A}(1 + \pi/\hat{h} + k) = (-1)^{(2+s)(A+k+1)} [A(1 + \pi/h + k)]
\]
(6.8)
and consequently that \((p \in \mathbb{Z}_+)\)
\[
\hat{A}(1 + \pi/\hat{h} + k) = (-1)^{(2+s)p(A+k+(p+1)/2)} [A(1 + \pi/h + k)] \quad , \quad \hat{A}(p) = (-1)^{sp(p-1)/2} [p] \quad .
\]
(6.9)
(6.10)
Eqs.6.3, 6.10 allow to prove the following identity about hypergeometric functions:
\[
k+1\hat{F}_k\left((-1, \pi/\hat{h})_{i=1...k+1} ; z\right) = k+1F_k\left((-1, \pi/h)_{i=1...k+1} ; z\right) \quad , \quad \left(-1\right)^{\left(\sum (n_i+r_i) - \sum (n'_i+r'_i) + 1\right)}
\]
(6.11)
with the \( k + 1 \) upper entries and \( k \) lower entries obtained from \( A_i \) and \( A'_i \) of the type
\[
A_i = n_i + r_i/(2 + s) , \quad n_i \in \mathbb{Z} , \quad r_i = 0...1 + s ; \quad A'_i = n'_i + r'_i/(2 + s) , \quad n'_i \in \mathbb{Z} , \quad r'_i = 0...1 + s.
\]

There is a similar property for hypergeometric function with arguments including no \( \pi/h \) part”, but they are then restricted to be integers\(^3\). So, it is clear that if we want to be able to relate hatted and unhatted 6-j coefficients for fractionnal spins we have to consider effective spins introduced in 6.1 (and 6.3) only. We shall generically note \( J'_{i^-} = J_i + J_i\pi/h \) and \( J'_{i^+} = -J_i - 1 + J_i\pi/h \) the spins of the type
\[
6.1 , \quad 6.3
\]
respectively.

Consider first the hatted part of the effective 6-j coefficients, using their definition
\[
4.16:
\]
\[
\langle \{ \hat{J}_1, \hat{J}_2, \hat{J}_3 \} J_{123} | \hat{J}_{12} \rangle = \langle \{ J_1, J_2, J_3 \} J_{123} | J_{12} \rangle
\]
(6.12)
We insert the expression 5.11 of the 6-j in terms of 4\( F_3 \) function and see that its arguments are precisely of the type \( A_i(1 + \pi/h) \). So, using Eq.6.11 for the 4\( F_3 \) function and Eqs.6.9, 6.10 for the prefactors, we notice that Eq.6.11 yields no extra sign for the argument \( z \) in this case, as 6-j symbols involve Saalschützian 4\( F_3 \) functions. Thus we get
\[
\langle \{ \hat{J}_1, \hat{J}_2, \hat{J}_3 \} J_{123} | \hat{J}_{12} \rangle
\]
with

\[ \phi(J_1, J_2, J_3, J_{12}, J_{23}, J_{123}) = p_{1,2}(2J_2 + 2J_3) + p_{2,3}2J_3 + p_{1,23}2J_{23} + p_{1,2,3}2J_3 \]

\[ + p_{1,2}(p_{1,2} + 1)/2 + p_{2,3}(p_{2,3} + 1)/2 + p_{1,23}(p_{1,23} + 1)/2 + p_{1,2,3}(p_{1,2,3} + 1)/2 \]

(6.14)

and

\[ p_{k, l} \equiv \hat{j}_k + \hat{j}_l - J \]

as usual. In terms of effective 6-j coefficients, it reads

\[ \{\{J_1^e - J_3^e / J_{123}^e\} \} = (-1)^{2s} \phi(J_1, J_2, J_3, J_{12}, J_{23}, J_{123}) \{\{J_1^e - J_2^e / J_{123}^e\} \} \]

(6.15)

In the argument just given, the hatted spins are obtained from the unhatted ones by exchanging \( h \) and \( \hat{h} \). This is not true for the other case to which we come next. According to Eq. 6.5, exchanging \( h \) and \( \hat{h} \) in \( J^e \) does not give \( J^{e+} \), but rather

\[ -(J + 1) + Jh/\pi \equiv -J^{e+} - 1 - \pi/h \]

which is not the same in general. We shall see that the transformations of the type \( 6.11 \) will still correspond to exchanging \( h \) and \( \hat{h} \) and not \( J^e \) and \( J^{e+} \) (one could not distinguish between them in the \( J^e \) case). It is one novelty of the positive-weight spin case to which we are coming now. Another is that we will have to use the results of last section, as the \( J_i^{e+} \)'s involve negative spins and more fundamentally, since the unhatted 6-j coefficients with spins \( J_i^{e+} \) involve negative screening-numbers \( p_{k, l} \). The final result will be a relation, where \( J_i^{e+} \) is transformed into \( J_i^{e+} \), similar to the previous case. The counterpart of Eqs. 6.7–6.11 is now that, for

\[ B = n + r/(2 - s) \text{ , } n \in Z, r = 0...1 - s \]

(6.16)

the useful identities are

\[ \hat{B}(1 - \pi/\hat{h}) + \hat{k} \]

\[ = (-1)^{(2s - (B + k + 1)/2)} \hat{B}(B - \pi/h + k) \]

(6.17)

and

\[ k + h F_k \left( \{B(1 - \pi/\hat{h})\}_{i=1\ldots k + 1}; z \right) = k + h F_k \left( \{B(1 - \pi/h)\}_{i=1\ldots k}; z \right) \]

(6.18)

(6.19)

with

\[ B_i = n_i + r_i/(2 - s) \text{ , } n_i \in Z, r_i = 0...1 - s \text{ ; } B'_i = n'_i + r'_i/(2 - s) \text{ , } n'_i \in Z, r'_i = 0...1 - s. \]

As said above, it turns out that the transformation \( 6.13 \) allows to relate unhatted 6-j coefficients with hatted 6-j coefficients with symmetrized spins. By definition

\[ \{\{J_1^{e+} J_2^{e+} J_{12}^{e+} / J_{123}^{e+}\} \} \equiv \{\{J_1 J_2 J_{12} / J_{123}\} \} \]

(6.20)

and, like in the negative weight case, using Eqs. 6.10, 6.18, 6.19 it can be transformed into

\[ (-1)^{(2s - (1 - J_1 J_2 J_{12} J_{123} J_{123}))/2} \hat{J} = \hat{J}_{12} - J_{12} + (J_1 + J_2)\pi/h \]

(6.21)

or

\[ \{\{J_1^{e+} J_2^{e+} J_{12}^{e+} / J_{123}^{e+}\} \} \equiv \{\{J_1 J_2 J_{12} / J_{123}\} \} \]

(6.22)

and

\[ (-1)^{(2s - (1 - J_1 J_2 J_{12} J_{123} J_{123}))/2} \hat{J} = J_{12} - 1 + (J_1 + J_2)\pi/h \]

(6.23)

or

\[ \{\{J_1^{e+} J_2^{e+} J_{12}^{e+} / J_{123}^{e+}\} \} \equiv \{\{J_1 J_2 J_{12} / J_{123}\} \} \]

(6.24)

and

\[ (-1)^{(2s - (1 - J_1 J_2 J_{12} J_{123} J_{123}))/2} \hat{J} = J_{12} - 1 - (J_1 + J_2)\pi/h \]

(6.25)

As a consequence, any kind of 6-j coefficient with unhatted entries can be transformed into an equivalent one with hatted entries and symmetrized spins.
or, in terms of effective spins and effective $6$-j's

\[
\left\{ \left\{ j_1^e, j_2^e, j_3^e \mid j_{12}^e, j_{13}^e, j_{23}^e \right\} \right\} = (-1)^{(2-s)} \phi(J_1, J_2, J_3, J_{12}, J_{13}, J_{23}) \left\{ \left\{ \hat{j}_1^e, \hat{j}_2^e, \hat{j}_3^e \mid \hat{j}_{12}^e, \hat{j}_{13}^e, \hat{j}_{23}^e \right\} \right\}
\]

following the remark above that the symmetrized spins $-J_i - 1 + J_i \pi / \hbar$ can be written $-\hat{j}_i^e - 1 - \pi / \hbar$. Then, the result of last section, Eq.5.18, makes the link with positive screening number $6$-j symbols and gives

\[
\left\{ \left\{ j_1^e, j_2^e, j_3^e \mid j_{12}, j_{13}, j_{23} \right\} \right\} = (-1)^{(2-s)} \phi(J_1, J_2, J_3, J_{12}, J_{13}, J_{23}) \left\{ \left\{ \hat{j}_1^e, \hat{j}_2^e, \hat{j}_3^e \mid \hat{j}_{12}, \hat{j}_{13}, \hat{j}_{23} \right\} \right\}.
\]

In this case of the positive weight spins, we transformed the unhatted effective spins and $6$-j’s into their hatted counterparts, whereas in the negative weight case we made the contrary. It is because the hatted $6$-j symbols are the ones with positive screening numbers.

We would like to emphasize that the compact notation should not hide the very different natures of the transformations 6.21 and 5.18 which are necessary to obtain Eq.6.22. The transformation 5.18 relates $6$-j coefficients with positive number of screening charges with $6$-j coefficients with negative number of screening charges, defining the latter by a suitably chosen continuation of $_4F_3$ function. It is valid for generic central charge and spins. The transformation 6.21 relates hatted and unhatted $6$-j coefficients with the same type (positive or negative) of screening charge number. It involves no continuation but is only valid for $C = 1 + 6(2 + s)$ and spins $J_i^e$ of the type 6.3.

### 6.2 The physical fields with negative conformal weights

The necessary tools are ready now and we build the physical operators. The sign $(2 \pm s) \phi(J_i)$ that shows up in Eqs.6.13, 6.22 can be grouped in four identical terms for the four vertices of fusion and an extra fifth term. It is therefore natural to include the four terms in the definition of the operators in order to get rid of them and be able to use the orthogonality of the $6$-j coefficients to prove the closure of the algebra. So, we define

\[
\mathcal{P}_{J_{12}^-} \chi^{(J_1)}_{-} = \sum_{J_{2,1},p_{2,1} \in \mathbb{Z}_+} (-1)^{(2+s)}(2J_{2p_{1,2},1} + \frac{p_{1,2}(p_{1,2}+1)}{2}) g_{J_{12}^-}^{J_{12}^e} \mathcal{P}_{J_{12}^-} V^{(J_1^-)} \mathcal{P}_{J_{12}^-} (6.23)
\]

with $p_{1,2} = J_1 + J_2 - J_{12}$. It can as well be summed over $J_{12}^e$ and written, using the closure relation in the Hilbert space for the l.h.s.,

\[
\chi^{(J_1)}_{-} = \sum_{J_{12},J_{2,1} \in \mathbb{Z}_+} (-1)^{(2+s)}(2J_{2p_{1,2},1} + \frac{p_{1,2}(p_{1,2}+1)}{2}) g_{J_{12}^-}^{J_{12}^e} \mathcal{P}_{J_{12}^-} V^{(J_1^-)} \mathcal{P}_{J_{12}^-} (6.24)
\]

The fusion of such operators is obtained from Eq.4.23:

\[
\chi^{(J_1)}_{-} \chi^{(J_2)}_{-} = \sum_{J_{123},J_{23} \in \mathbb{Z}_+} (-1)^{(2+s)}(2J_{23p_{2,3}} + 2J_{3p_{2,3}} + \frac{p_{2,3}(p_{2,3}+1)}{2} + \frac{p_{1,2}(p_{1,2}+1)}{2}) \mathcal{P}_{J_{123}^-} V^{(J_1^-)} \mathcal{P}_{J_{123}^-}.
\]
\[
\chi^e_{\{J_1,\nu_1\}} \chi^e_{\{J_2,\nu_2\}} \mathcal{P}^e_{J_3} = \sum_{J_{12}, J_{3}} \left(-1\right)^{(2+s)2} g^e_{J_1,J_2} g^e_{J_3} P^e_{J_{123}} \mathcal{P}_{J_3}^e V(J_1^e) \mathcal{P}^e_{J_2} V(J_2^e) \mathcal{P}^e_{J_3} \]
\[
\left\{ \left\{ J_1^e, J_2^e, J_{12}^e \right\} \left\{ J_3^e, J_{13}^e, J_{23}^e \right\} \sum_{\nu_1, \nu_2} \mathcal{P}_{J_123}^e V(J_{12}^e, \nu_1) \mathcal{P}^e_{J_3} < J_{12}^e, \{ \nu_1 \} \left| V(J_1^e) \right| J_2^e > .
\right. 
\]
equivalently sum over $J_3$ and use the operator $\varpi$ of eigenvalues $\varpi e^- = \varpi_0 + 2J e^-$ (which does not commute with the vertex operators):

$$\chi_-(J_1)\chi_-(J_2) = \sum_{J_{12} < J_1 + J_2, \{\nu_{12}\}} \chi_{(J_{12}, \{\nu_{12}\})}^{(J_1, J_2)} (-1)^{(1+\hbar/\pi)(\varpi-\varpi_0)(J_{1}+J_{2}-J_{12})} < J_{12}^-; \{\nu_{12}\}|\chi^{(J_1)}|J_2^- \rangle.$$  \hspace{1cm} (6.29)

The closure of the braiding works in the same way. The only differences are in the signs, in particular the extra phase of the braiding Eq. 4.25. The braiding of the two $\chi$ fields of Eq. 6.25 involves the 6-j coefficients

$$\{ \begin{array}{ccc} \gamma_{1a}^- & \gamma_{1a}^- & \gamma_{1a}^- \\ \gamma_{2a}^- & \gamma_{2a}^- & \gamma_{2a}^- \\ \gamma_{13}^- & \gamma_{13}^- & \gamma_{13}^- \end{array} \} \{ \begin{array}{ccc} \delta_{1a}^- & \delta_{1a}^- & \delta_{1a}^- \\ \delta_{2a}^- & \delta_{2a}^- & \delta_{2a}^- \\ \delta_{13}^- & \delta_{13}^- & \delta_{13}^- \end{array} \}$$

which yield the sign $(-1)^{(s+2)\phi(J_1, J_3, J_2, J_{13}, J_{23}, J_{123})}$. The signs combine correctly thanks to the following identity

$$(s + 2) \left[ p_{1,3}(2J_3 + 2J_2) + p_{2,3}2J_2 + p_{1,23}2J_{23} + p_{13,2}2J_2 \right]$$

$$+ \epsilon \left[ \Delta(J_{13}^e) + \Delta(J_{23}^e) - \Delta(J_3^e) - \Delta(J_{123}^e) \right] =$$

$$(s + 2) \left[ p_{1,23}2J_{23} + p_{2,3}2J_3 + p_{2,13}2J_{13} + p_{1,3}2J_3 - 2\epsilon J_1 J_2 \right] \pmod{2}. \hspace{1cm} (6.30)$$

Using this we prove

$$\chi_{(J_1)}(J_2) \chi_{-}^{(J_1)} = e^{-2\pi \epsilon(2+s)J_1 J_2} \chi_{-}^{(J_2)} \chi_{-}^{(J_1)}. \hspace{1cm} (6.31)$$

It is easy to check that the $\chi$ operators verify the polynomial equations. For example, the orthogonality is satisfied thanks to the $\epsilon$ sign in the braiding Eq. 6.31. The pentagonal identity expressing the associativity of

$$\chi_{(J_1)}(J_2) \chi_{-}^{(J_1)} = \sum_{J_{1234}, J_{1234}} \mathcal{P}_{J_{1234}}^{J_{1234}} \chi_{-}^{(J_1)} \mathcal{P}_{J_{1234}}^{J_{1234}} \chi_{-}^{(J_2)} \mathcal{P}_{J_{1234}}^{J_{1234}} \chi_{-}^{(J_3)} \mathcal{P}_{J_{1234}}^{J_{1234}}$$

works thanks to

$$(2 + s) \left[ (J_2 + J_3 - J_{23})2J_4 + (J_1 + J_{23} - J_{123})2J_4 + (J_1 + J_2 - J_{12})2J_3 \right]$$

$$= (s + 2) \left[ (J_1 + J_2 - J_{12})2J_{34} + (J_{12} + J_3 - J_{123})2J_4 \right] \pmod{2}.$$

\section{6.3 The physical fields with positive conformal weights}

The operators with positive weights are built similarly. They involve negative screening numbers and thus required some more work in last section, but now that the formula 6.22, similar to 6.13, relating hatted and unhatted 6-$j$’s has been worked out, they can be built along the same line. We define

$$\chi_{+}^{(J_1)} \equiv \sum_{J_{12}, J_{12}, J_{12}, J_{12} \in Z_{+}} (-1)^{(2-s)(2J_{2p_{1,2}} + \frac{L_{12}(p_{1,2}+1)}{2})} g_{J_{12}^+ J_{12}^+}^{J_{12}^+ J_{12}^+} \mathcal{P}_{J_{12}}^{J_{12}} \mathcal{V}(J_{12}^+)^{J_{12}^+} \mathcal{P}_{J_{12}^+}^{J_{12}^+}. \hspace{1cm} (6.32)$$
The closure of fusion works similarly and gives
\[
\chi_+^{(J_1)} \chi_+^{(J_2)} \mathcal{P}_{J_3}^{e^-} = \sum_{J_{12} < J_1 + J_2} (-1)^{2-s} 2 J_3 (J_1 + J_2 - J_{12})
\]
\[
\sum_{\nu_{12}} \chi_+^{(J_{12}, \nu_{12})} \mathcal{P}_{J_3}^{e^-} < J^e_+, \{\nu_{12}\} |\chi^{(J_1)}| J^e_2 > .
\]
(6.33)

There is a slight change concerning the braiding, as the weights are not the same.
The positive weights of Eq.6.4 can be obtained from the negative ones of Eq.6.2 by a change of sign and a change of \(s\) into \(-s\) (up to an extra 1). Thus we get the right formula from Eq.6.30 by changing \(s\) and \(\epsilon\) in their opposite, and finally we have for braiding
\[
\chi_+^{(J_1)} \chi_+^{(J_2)} = e^{2i\pi(2-s)J_1 J_2} \chi_+^{(J_2)} \chi_+^{(J_1)} .
\]
(6.34)

Finally, we note that we could have chosen
\[
\{ J^{e+} = J + (-J - 1)\pi/\hbar, 2J = n + r/(2-s), n \in Z, r = 0...1 - s \},
\]
(6.35)

instead of Eq.6.3. This amounts to exchanging \(\hbar\) and \(\hat{\hbar}\) everywhere in the above formulae, and the discussion is the same. This possibility will be important in the coming section. This other possibility may be deduced very simply from the above, by noting that one only needs to change \(J\) into \(-J - 1\) everywhere. This leads to the replacement of Eq.6.32 by
\[
\chi_+^{(J_1)} \equiv \sum_{J_{12}, J_{2,12} \in Z} (-1)^{2-s} 2 J_{12} (p_{1,2} + 1) g_{J_1^{e+}, J_2^{e+}}^{J_{12}^{e+}, J_{12}^{e+}} \mathcal{P}_{J_1^{e+}} \mathcal{P}_{J_2^{e+}} .
\]
(6.36)

### 6.4 More about the half-integer case

As we saw in section 3 for the 6-j coefficients and in section 4 for the operators, it is possible to extend these algebras to continuous spins provided the full triangular inequalities (TI3) are replaced by TI1 (one inequality per vertex). These TI1’s are the “selection rules” we used for our vertex operators to build the physical \(\chi\) operators. But then, a question arises: what happens when we use TI1 and that some spins happen “accidentally” to be half-integers?

When deriving the general algebra in sections 3 and 4 we could think that this case was of probability zero and we did not ask this question. But in the case of our operators of the strong coupling regime with spins \(J_i = (n+r/(2 \pm s))/2\), this is very likely to happen. For example by fusing the spins \(J_1 = 5/6\) and \(J_2 = 1/6\) we get \(J_{12} = 1, 0, -1,...\). The questions are then: Are there singularities arising? Are we entitled to go to negative \(J_{12}\) in such a case as is allowed by TI1? Or, on the contrary, should we consider TI3 (or simply TI2) in such a case?....

The answer can be found in the fact that the polynomial equations, being rational, always guarantee the consistency of the algebra. There are three different algebras with TI3, TI2 or TI1. Once one of these three possibilities is chosen the algebra is consistent, but, if one of the spins that was not foreseen to be half-integer
when making this choice happens “accidentally” to be half-integer, there are zeros and singularities arising. They cancel however in the polynomial equations due to their rational character. In particular, this is the case in the traditional construction of Gervais et al where the zero-mode of the incoming and outgoing states is continuous, so that one is in the case TI2. It can easily be checked, — even for basic $J = 1/2$ operators, the braiding and fusion of which are directly deduced from the differential equation — that there are zeros and singularities arising if the zero-mode $\omega$ ever happens to be equal to $\omega_0 + 2J$, with $2J$ integer. At the level of 6-j’s, one can either keep TI2 and have singularities for some discrete values of the zero-mode, or use TI3 and have perfectly finite results, as was done in ref.\cite{[6]}.

In the strong coupling case, however, we must consider spins of the form $J_i = (n + r/(2 \pm s))/2$ in order to complete modular invariance\cite{[2]}. So we have to use TI1. And half-integer spins arise naturally from non-half-integer ones. Hence, we need such vertex operators like

$$
\begin{array}{c|c|c|c|c|c}
1/2 & 0 & 3/2 & -5/6 & 1/2 & 100/3
\end{array}
$$

(6.37)

Although surprising when one is familiar with TI3, they can indeed be constructed. The first one is simply a screening operator and the second one has got many extra screening operators. They are not considered usually as they are singular. What saves us in this case of strong coupling regime is that the singularities of the operators that we must consider cancel each other thanks to the particular choice of $J_i$ and $\hat{J}_i$. More precisely, it means that although the 6-j coefficients considered all along this section, and in particular in Eq.6.27, may be singular or zero, nevertheless, the orthogonality relation Eq.3.28 holds, as it is an equation between rational functions, and it allows to go from Eq.6.27 to Eq.6.28, cancelling the singularities\cite{[3]}. This is one more miracle of this construction for the strong coupling regime: although for the basic $V$ operators the choice of spins \[6.1\] or \[6.3\] leads to singularities coming from the 6-j coefficients (until Eq.6.27), these singularities disappear for the physical $\chi$ operators in Eqs.6.28, 6.31, 6.33, 6.34.

We want to point out that it has some astonishing implications, like the following: the orthogonality Eq.3.28 can be used with spins $j_{12}$ and $j_{23}$ half-integers such that it is possible to find spins $j_{12}$ and $j_{23}$ such that 4 TI3’s are satisfied. However, we sum over $j_{23}$ verifying TI1 and thanks to the cancellation of zeros and singularities the result is still the identity matrix for a range of $j_{12}$ and $j'_{12}$ determined by TI1: $j_{12}, j'_{12} \in [j_{123} - j_3, j_{1} + j_2]$. It seems curious as for the same spins there is as well a standard orthogonality relation with 4 TI3’s. There is no incompatibility between both of them: if instead, we chose to sum over $j_{23}$ with TI3 i.e. $|j_{123} - j_1|, |j_2 - j_3| \leq j_{23} \leq j_2 + j_3, j_1 + j_2$, the result is now (even for $j_{12}$ and $j'_{12}$ verifying TI1, hence in $[j_{123} - j_3, j_1 + j_2]$) the projector on the space of spins $j_{12}$ and $j'_{12}$ verifying TI3: $j_{12}, j'_{12} \in [\max(|j_{123} - j_3|, |j_1 - j_2|), \min(j_2 + j_3, j_{123} + j_3)]$. This is the standard orthogonality relation.

\[\text{\footnote{There is however a little subtelity in the fact that the equations \[6.1\], \[6.3\] relating hatted and unhatted 6-j coefficients are not identities at the level of rational fractions but for the particular values \[6.1\] and \[6.3\] of the spins.}}\]
orthogonality relation. So, if the spins are half-integers, we can consistently choose either to stay in the subspace of vertices verifying TI3, or to go to the larger space of vertices verifying TI1 (and have singularities in the case of $V$ operators). Once this choice is made on the initial vertex operators, the vertex operators obtained by braidings and fusions remain of the same kind. The same is true with the intermediate case of TI2. All this was checked numerically on many examples.

Since TI1 must be the selection rules in every case here, the fusion Eqs.6.28, 6.29, 6.33 involve an infinite sum as $J_{12}$ is not bounded from below. This is not different from the braiding Eqs.6.31, 6.34, where the intermediate states between $\chi^{(J_2)}$ and $\chi^{(J_1)}$ are in infinite number. However, if all the four external spins are specified (i.e. if the left and right states are specified as well), these infinite sums disappear.

The same question must be asked concerning the $g$ coefficients. The answer is the same: they are finite in our case in contrast with the general case. More precisely, one can see that in the general case, if some spins ever turn out to be half-integer whereas the full triangular inequalities are not fulfilled, there are zeros and poles arising. They do not cancel with the ones of the 6-j coefficients in general (see e.g. the fusion coefficient for spin 1/2 Eq.4.5). However, a careful analysis, that we do not detail here for brevity sake, shows that the $g$'s of interest here, i.e. with all spins such that $\tilde{J} = J$ or $\tilde{J} = -J - 1$, never bring poles (to see this, use the expression of $g$ in terms of arbitrary path Eq.4.20, note that there are only zeros or poles when changing of quarter of plane, and that in both of our cases, the beginning and ending points of the paths are on the same diagonal, which involves two changes of quarter plane that cancel one another). They are never zero except when one of the three spins involved in a $g$ coefficient is $J = -1/2$, $\tilde{J} = -J - 1 = -1/2$. This agrees with the leg-factors of the final result Eq.7.26 which are zero in such a case only. This is not really surprising as the representation of spin -1/2 is very peculiar.

### 7 TOPOLOGICAL MODELS

#### 7.1 The vertex

As already indicated, we may consider two copies of the strongly coupled models under consideration with central charges

$$C = 1 + 6(s + 2), \quad c = 1 + 6(-s + 2). \quad (7.1)$$

The second plays the role of matter, while the first may be considered as gravity. The latter gives a proper dressing, since obviously,

$$C + c = 26. \quad (7.2)$$

In this section, we determine the corresponding three-point functions. Let us first establish the framework. Since it is very close to the one already put forward for the weak coupling regime (see ref.[12]), we shall be rather brief.

First, we shall be discussing closed surfaces, so that we should include both left and right movers. As explained in ref.[12], the discussion carried out so far only
applies to the holomorphic components, which are functions of \( z = \tau + i\sigma \) (\( \tau \) is the time coordinate on the cylinder). For the antiholomorphic components, one should change \( i \) into \(-i\) everywhere. Thus if we call \( \chi^{(J)}_{\pm} \), the corresponding physical fields, the braiding relations Eqs.6.31, 6.34 are changed into
\[
\chi^{(J_1)}_{\pm}(\tau - i\sigma_1)\chi^{(J_2)}_{\pm}(\tau - i\sigma_2) = e^{2\pi i(2 + s)J_1 J_2(\sigma_1 - \sigma_2)}\chi^{(J_1)}_{\pm}(\tau - i\sigma_2)\chi^{(J_2)}_{\pm}(\tau - i\sigma_1). \tag{7.3}
\]
Since the braiding of the holomorphic components is a simple phase, we construct local fields simply from products of the form \( \chi^{(J)}_{\pm}(\tau + i\sigma)\chi^{(J)}_{\pm}(\tau - i\sigma) \) with suitable \( J_1, J_2 \). We shall use them to describe the coupling of strongly coupled gravity. They are the generalizations of the exponentials of the Liouville field, to the present situation. Second the matter (with central charge \( c \)) is treated much like gravity. Since by construction, \( c \) belongs to the set of special values, there exist physical fields noted \( \chi^{(J)}_{\pm}(\tau + i\sigma) \), and \( \chi^{(J)}_{\pm}(\tau - i\sigma) \) similar to the above. The quantum group structure has parameters \( h' \), and \( \hat{h}' \) (we systematically distinguish symbols related with matter by a prime), such that
\[
h + \hat{h}' = \hat{h} + h' = 0. \tag{7.4}
\]
We consider only the simplest solution of the BRST cohomology, and take vertex operators of the form
\[
V_{J', \mathcal{J}} = \chi^{(J')}_{+}\chi^{(\mathcal{J})}_{+} \chi^{(J')}_{-}\chi^{(\mathcal{J})}_{-}. \tag{7.5}
\]
It follows from Eq.2.7 that \( V_{J', \mathcal{J}} \) has conformal weights \((1, 1)\) as required. At this point let us note a basic difference with the weak coupling regime. There the Liouville field only involves left and right movers with equal quantum group spins. Thus, only three-point functions with \( J' = \mathcal{J} \) are considered. Here, the quantum spins of the two holomorphic components are not related so far, and will not be set equal (more about this). The values of the quantum group spins which appear in the last equation may be seen on Eqs.6.24 and 6.32. Concerning the latter, we have to make an important point. As noted at the end of the previous section, there are two ways to define the fields \( \chi_+ \), which correspond to the existence of the two screening charges. For physical reasons, which we shall spell out below, we shall take one choice (Eq.6.3) for the holomorphic components, and the other (Eq.6.35) for the antiholomorphic one. Thus we let
\[
J^e \equiv J^{e+} = (-J' - 1) + J'\pi/h, \quad J^c \equiv J^{c+} = J' + J'\pi/h' = J'(1 - h/\pi),
\]
\[
\mathcal{J} \equiv \mathcal{J}^{e+} = J' + (1 - \mathcal{J})\pi/h, \quad \mathcal{J}^c \equiv \mathcal{J}^{c+} = J' + \mathcal{J}\pi/h' = J'(1 - h/\pi). \tag{7.6}
\]
As just noted, the upper indices \pm are omitted from now on. Concerning the Hilbert spaces, our previous discussion shows that the fields \( \chi_+ \) (resp. \( \chi_- \)) have a consistent restriction to the physical Hilbert space \( \mathcal{H}_{s, \text{phys}}^+ \) (resp. \( \mathcal{H}_{s, \text{phys}}^- \)). Thus we shall work in the Hilbert space \( \mathcal{H}_{s, \text{phys}}^+ \otimes \mathcal{H}_{s, \text{phys}}^- \otimes \mathcal{H}_{-s, \text{phys}}^+ \otimes \mathcal{H}_{-s, \text{phys}}^- \) (this notation should be self explanatory). The BRST cohomology selects the states such that \( L_0 + L_0' \),
\[\text{As usual we take the two holomorphic components to commute.}\]
and \(L_0 + L_0'\) have eigenvalues equal to one, which correspond to the vertex Eq. 7.3. In the same way, as in the weak coupling regime, this is satisfied if \(\varpi^2 = -\varpi'\), and \(\varpi' = \varpi^2\). We shall make choices which are consistent with Eq. 7.6 by letting \(\varpi = -\varpi'\) (or equivalently \(\varpi = \varpi')\), and \(\varpi = \varpi'\) (or equivalently \(\varpi = -\varpi'\)). Thus the spectrum of \(\varpi\) is given by

\[
\varpi_{r,n} = -\left(\frac{r}{2-s} + n\right)(1 - \frac{\pi}{h}), \quad \varpi'_{r,n} = \left(\frac{r}{2-s} + n\right)(1 - \frac{\pi}{h});
\]

\[
\varpi_{r,\bar{n}} = \left(\frac{\bar{r}}{2-s} + \bar{n}\right)(1 - \frac{\pi}{h}), \quad \varpi'_{r,\bar{n}} = \left(\frac{\bar{r}}{2-s} + \bar{n}\right)(1 - \frac{\pi}{h}). \tag{7.7}
\]

The corresponding spins are

\[
J_{r,n} = -\frac{r}{2(2-s)} - \frac{n+1}{2}, \quad \tilde{J}_{r,n} = J'_{r,n} = \tilde{J}'_{r,n} = \frac{r}{2(2-s)} + \frac{n-1}{2},
\]

\[
\tilde{J}_{r,\bar{n}} = -\frac{\bar{r}}{2(2-s)} - \frac{\bar{n}+1}{2}, \quad \tilde{J}_{r,\bar{n}} = J'_{r,\bar{n}} = \tilde{J}'_{r,\bar{n}} = \frac{\bar{r}}{2(2-s)} + \frac{\bar{n}-1}{2}. \tag{7.8}
\]

Clearly, one has \(J_{r,n} = -\tilde{J}_{r,n} - 1\), \(J'_{r,n} = \tilde{J}'_{r,n}\), and so on, so that the spectrum of highest-weights states (Eq. 7.7), and of vertex operators (Eq. 7.6), are identical.

Our next point is the choice of \(J\) and \(\mathcal{T}\) in the vertex operator Eq. 7.5. The basic requirement is that these operators must commute at equal \(\tau\). It follows from Eqs. 6.31 and 6.34 that

\[
V_{j_1',\tau_1}(\sigma_1, \tau)V_{j_2',\tau_2}(\sigma_2, \tau) = e^{i\pi(2-s)(2J_1'\nu_2 + 2J_2'\nu_1)}V_{j_2',\tau_2}(\sigma_2, \tau)V_{j_1',\tau_1}(\sigma_1, \tau). \tag{7.9}
\]

It is easy to see that the exponential factor becomes equal to one if we impose

\[
J_1' - j_1 = \nu_i, \quad \nu_i \in Z, \tag{7.10}
\]

Indeed it becomes equal to \(\exp\{-i\pi(2-s)(2J_1'\nu_2 + 2J_2'\nu_1)\}\). It follows from Eq. 7.8 that \(2(2-s)J_1' \in Z\), which completes the derivation. Thus we shall consider that \(J'\) and \(\mathcal{T}\) in Eq. 7.5 are constrained to satisfy a condition of the type Eq. 7.10. Next let us show that this is consistent with the requirement that the properties of vertex operators be symmetric between the three legs. Indeed, it follows from the definitions Eqs. 6.23 and 6.32 that, in general, \(<\varpi_{L'}|\chi_{\pm}^{(J')}|\varpi_{K'}>\neq 0\) only for \(J' + K' - L' \in Z_+\). For the vertex Eq. 7.3, where \(\mathcal{T} - J' \in Z\), and if we consider matrix elements of the type just written, with \(\mathcal{K} - K' \in Z\), we will also have \(\mathcal{T} - L' \in Z\). Thus it is consistent to assume that conditions of the type Eq. 7.10 hold for the three legs of the vertex operators. Note that the stronger condition \(\mathcal{K} - K' = 0\) is not preserved since in Eq. 7.3 the summations over left and right movers are independent. Thus it would be inconsistent to consider vertex operators with \(J' = \mathcal{T}\) only. Since we take \(J' \neq \mathcal{T}\), we have to discuss the monodromy properties of our operators. In general, for \(V_{\nu_1,\nu_2}^{(J)}\) they are given by

\[
V_{\nu_1,\nu_2}^{(J)}(\tau + i\sigma + 2i\pi) = e^{-ih\varpi^2/2}V_{\nu_1,\nu_2}^{(J)}(\tau + i\sigma)e^{ih\varpi^2/2}. \tag{7.11}
\]
It thus follows from Eqs. 6.23 and 6.32 that
\[
< \varpi_{j_{12}^\pm} | \chi_J^{(J)}(\tau, \sigma + 2\pi)| \varpi_{j_2^\pm} > = 
\]
\[
e^{\pi i 2(2s)(J_{12} - J_2)(J_{12} + J_2 + 1)} < \varpi_{j_{12}^\pm} | \chi_J^{(J)}(\tau, \sigma)| \varpi_{j_2^\pm} > .
\]
(7.12)
Consider the vertex for gravity, that is $\chi_J^{(J - \overline{J})}$. One gets the factor
\[
e^{-i\pi 2(2 - s)(J_{12} - J_2)(J_{12} + J_2 + 1) - (J_2 - J_{12})(J_{12} + J_2 + 1)}.
\]
It follows from the conditions $J_1 + J_2 - J_{12} \in Z_+$, $\overline{J}_1 + \overline{J}_2 - \overline{J}_{12} \in Z_+$ of Eq. 6.32, and from Eq. 7.10, that $J_2 - J_{12} - (\overline{J}_2 - \overline{J}_{12}) \in Z$. Since the values of $J$ are such that $2(2 - s)J$ is an integer, one easily deduces that the last exponent is equal to $i\pi$ multiplied by an integer. Thus the gravity part is at most multiplied by a minus sign when $\sigma \rightarrow \sigma + 2\pi$. On the other hand, using the special choice of $J$'s and $\overline{J}$'s (Eqs. 7.6), one easily sees that the matter part gives the same factor, so that $V_\sigma$ is invariant under this $2\pi$ rotation as expected physically.

In the weak coupling regime, the Liouville field, and thus the vertex operators preserve\(^{12}\) the condition $J' - \overline{J} = 0$. This is not true for our vertex operator in the strong coupling regime. Thus the transition through the $c = 1$ barrier may be considered as related with a deconfinement of this quantum number.

### 7.2 The three-point function

Next, our aim is to compute the three-point functions of the form
\[
\langle \prod_{\ell=1}^3 V_{J_\ell, \overline{J}_\ell}(\sigma_\ell, \tau_\ell) \rangle = C_{1,2,3}/\left( \prod_{k<\ell} |z_k - z_\ell|^{2} \right), \quad z_\ell = e^{\tau_\ell + i\sigma_\ell}
\]
(7.13)
where $C_{1,2,3}$ is the coupling constant. Applying the same reasoning\(^{13}\) as in ref.\([12]\), one sees that
\[
C_{1,2,3} = \langle \varpi_0, -\varpi_0, -\varpi_0'| V_{J_2, \overline{J}_2} \rangle = \langle -\varpi_3, -\varpi_3, -\varpi_3'| V_{J_2, \overline{J}_2} \varpi_1, \varpi_1| \varpi_1' \rangle \times \langle \varpi_1, \varpi_1, \varpi_1'| V_{J_1, \overline{J}_1} \rangle
\]
(7.14)
where the $\varpi$'s are associated with the spin $J_\ell$, and of the type Eq. 7.6, Eq. 7.7. By definition, the highest-weight matrix elements of the $V$ fields are equal to one or zero. It thus follows, from Eqs. 6.24, 6.32 and 6.36 that
\[
< \varpi_k, \varpi_k'| V_{J_k, \overline{J}_k} \varpi_\ell, \varpi_\ell'| \varpi_\ell' > = (-1)^{(2 - s)[2J_k(J_k + J_k - J_\ell) + 2J_\ell(J_\ell + J_\ell - J_k) + 2J_k(J_k + J_k - J_\ell) + 2J_\ell(J_\ell + J_\ell - J_k)]} \times
\]
\[
(1)^{(2 - s)[2J_k(J_k + J_k - J_\ell) + 2J_\ell(J_\ell + J_\ell - J_k)]} g^{J_k}_{J_k', J_k''} g'^{J_\ell}_{J_\ell', J_\ell''} \varpi_k J_k' \varpi_k J_k'' \varpi_\ell J_\ell' \varpi_\ell J_\ell''
\]
(7.15)
\(^{19}\)The left- and right-most operators require some special treatment as in the weak coupling regime. Since it is completely similar, we do not discuss this point here again.
where the spins are the appropriate effective ones, provided the appropriate differences between spins are integers (recall Eq. 3.3). Next, it is easy to see that the first and the last matrix elements in Eq. 7.14 involve coupling constants with one vanishing spin (of the form $g_{S,0}$ or $g_{-\pi/h, S-1/\pi/h}$) which are equal to one. The middle one, gives

$$C_{1,2,3} = g_{J^e_{2,1}}^{-1-\pi/h} g_{J^e_{2,1}^e, J^e_{1}}^{-1-\pi/h} \frac{\hat{T}_3^{-1-\pi/h}}{\hat{T}_2} \frac{\hat{T}_3^{-1-\pi/h}}{\hat{T}_1}$$  \hspace{1cm} (7.16)

where the effective spins take the form displayed on Eqs. 7.6. These coupling constants are given\textsuperscript{20} by formulae of the type Eq. 4.18, with

$$p \equiv J_1 + J_2 + J_3 + 1, \quad \tilde{p} \equiv \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + 1,$$

$$p' \equiv J'_1 + J'_2 + J'_3 + 1, \quad \tilde{p}' \equiv \tilde{J}'_1 + \tilde{J}'_2 + \tilde{J}'_3 + 1,$$

$$\hat{p} \equiv \hat{T}_1 + \hat{T}_2 + \hat{T}_3 + 1, \quad \hat{p}' \equiv \hat{T}'_1 + \hat{T}'_2 + \hat{T}'_3 + 1.$$  \hspace{1cm} (7.17)

Applying a reasoning similar to the one already given\textsuperscript{12} for the weak coupling regime, one easily sees that the three sets of $J$’s satisfy the same relations:

$$J_i = -\tilde{J}_i - 1, \quad \tilde{J}_i = J'_i, \quad J_i = -\hat{J}_i - 1, \quad J'_i = \hat{J}_i \hspace{1cm} (7.18)$$

Next we determine the product of the first two terms of Eq. 7.16 — the last two are similar. According to Eq. 4.18, we have

$$g_{J^e_{2,1}}^{-1-\pi/h} g_{J^e_{2,1}^e, J^e_{1}}^{-1-\pi/h'} = \left( \frac{i}{2} \right)^{p+\tilde{p}+p'+\tilde{p}'} \frac{\prod_{k=1}^{3} H_{pp'}(\omega_k H'_{\tilde{p}'\tilde{p}'}(\omega'_k))}{H_{pp'}(\omega_{p/2,\tilde{p}/2}) H'_{\tilde{p}'\tilde{p}'}(\omega'_{p/2,\tilde{p}/2})}$$  \hspace{1cm} (7.19)

The $p$-parameters are related by

$$p' = \tilde{p}', \quad p = -\tilde{p} - 1, \quad p = -\tilde{p}' - 1, \quad \tilde{p} = p'$$  \hspace{1cm} (7.20)

the first two relations are specific to the strong coupling regime, while the last two are not. The calculation we are going to perform will actually not make use of the former so that it also applies to the weakly coupled case. First consider the numerator in the r.h.s. of Eq. 7.19. Each term is of the same type. So we compute

$$H_{pp'}(\omega) H'_{\tilde{p}'\tilde{p}'}(\omega') = \prod_{r=1}^{p} \sqrt{F(\omega - rh/\pi)} \prod_{\tilde{r}=1}^{\tilde{p}} \sqrt{F(\omega' - \tilde{r}'\pi/h')} \times$$

$$\frac{\prod_{r=1}^{\tilde{p}} \sqrt{F(\omega - \tilde{r}'\pi/h')} \prod_{\tilde{r}=1}^{p'} \sqrt{F(\omega' - r'h'/\pi)}}{\prod_{r=1}^{p} \prod_{\tilde{r}=1}^{\tilde{p}} \left( (\omega - r)\sqrt{h/\pi} - \tilde{r}'\sqrt{\pi/h'} \right) \prod_{r=1}^{p'} \prod_{\tilde{r}=1}^{\tilde{p}} \left( (\omega' - r')\sqrt{h'/\pi} - \tilde{r}'\sqrt{\pi/h'} \right)}.$$  \hspace{1cm} (7.21)
Each pair of factors is seen to simplify tremendously, using the fact that \( \varpi = -\varpi' \), and \( \hat{\varpi} = \varpi' \). One finds
\[
\prod_{r=1}^{p} \sqrt{F(\varpi - rh/\pi)} \prod_{\hat{r}=1}^{\hat{p}} \sqrt{F(\varpi' - \hat{r}h/\pi)} = \frac{\prod_{r=1}^{p} \sqrt{F(\varpi - rh/\pi)}}{\prod_{\hat{r}=1}^{\hat{p}+1} \sqrt{F(\varpi - (r - 1)h/\pi)}} = \frac{1}{\sqrt{F(\varpi)}},
\]
where we use the general substitution rule Eq.5.8 that follows from the symmetry \( J \to -J - 1 \). On the other hand,
\[
\prod_{\hat{r}=1}^{\hat{p}} \sqrt{F(\varpi - \hat{r}\pi/h)} \prod_{r'=1}^{p'} \sqrt{F(\varpi' - r'h/\pi)} = \prod_{\hat{r}=1}^{\hat{p}} \sqrt{F(\varpi - \hat{r}\pi/h)F(-\varpi + \hat{r}\pi/h)}
\]
\[
= \prod_{\hat{r}=1}^{\hat{p}} i/(\varpi - \hat{r}\pi/h).
\]
Combining with the denominator, we finally derive
\[
H_{p\hat{p}}(\varpi)H'_{p'\hat{p}'}(\varpi') = (\pi/h)^{\hat{p}/2} e^{i\pi\hat{p}(p+1)/2} e^{i\pi\hat{p}/2}/\sqrt{F(\varpi)}. \tag{7.22}
\]

The calculation of the denominator of Eq.7.14 is not quite the same since the relation between \( \varpi_{p/2,\hat{p}/2} \) and \( \varpi'_{p'/2,\hat{p}'/2} \) differs from the one between \( \varpi_k \) and \( \varpi'_k \). Applying the same method, one finds for the two pairs of factors
\[
\prod_{r=1}^{p} \sqrt{F(\hat{p} + 1 + (p - r + 1)h/\pi)} \prod_{\hat{r}=1}^{\hat{p}} \sqrt{F(\hat{p} + 1 - (p + \hat{r})\pi/h)} = \frac{1}{\sqrt{F(\hat{p} + 1)}},
\]
\[
\prod_{\hat{r}=1}^{\hat{p}} \sqrt{F(\hat{p} + 1 + (\hat{p} - \hat{r} + 1)\pi/h)} \prod_{r'=1}^{p'} \sqrt{F(-p - (p' - r' + 1)\pi/h)} = 1.
\]

One easily sees that the denominator is equal to \( \exp[i\pi\hat{p}(p+1)/2](h/\pi)^{\hat{p}/2}/\hat{p}! \). Using the fact that \( F(\hat{p} + 1) = \hat{p}!^2 \sin(\pi(\hat{p} + 1))/\pi \), one finally finds
\[
H_{p\hat{p}}(\varpi_{p/2,\hat{p}/2})H'_{p'\hat{p}'}(\varpi'_{p'/2,\hat{p}'/2}) = e^{-i\pi(\hat{p}+1)/2} e^{-i\pi\hat{p}(p+1)/2} \left( \frac{h}{\pi} \right)^{-\hat{p}/2} \sqrt{\pi}. \tag{7.23}
\]

It immediately follows from Eq.7.17 that any term of the form \( \beta \hat{p} \) may be written as \( \beta \prod_{k=1}^{n}(\beta) \hat{J}_k \). Using this remark, to simplify Eq.7.22 and 7.23, we finally derive the following expression of the l.h.s. of Eq.7.19
\[
g_{J_2}^{J_0} = \frac{g_{J_2}^{J_0}}{g_{J_2}^{J_0}} = \frac{a(-1)^{p\hat{p}} \prod_{k=1}^{3} (b^{\hat{J}_k})}{\sqrt{F(2\hat{J}_k + 1 + (2J_k + 1)h/\pi)}} \tag{7.24}
\]
where \( a \) and \( b \) are independent from the \( J \)'s.

Concerning the second factor in Eq.7.13, the calculation is essentially the same. The only difference is the choice of \( \hat{\varpi} \) and \( \hat{\varpi}' \). One may see that it amounts to
exchanging the rôles of unhatted and hatted quantities. Combining everything together one gets

\[ C_{1,2,3} = a^2 (-1)^p (-1)^{(2-s)}2(J_1 p + \bar{J}_1 \bar{\rho} + J'_1 \rho' + (\bar{J}_1 + 1)(\bar{\rho} + 1)) \times \]

\[ \prod_{k=1}^{3} \frac{(b)_{J_k} (\bar{b})_{\bar{J}_k}}{\sqrt{F} \left( 2J_k + 1 + (2J_k + 1)h/\pi \right) \sqrt{F} \left( 2\bar{J}_k + 1 + (2\bar{J}_k + 1)\pi/h \right)}. \] (7.25)

The –1 factor may be simplified. Using Eqs. 7.18, 7.20 one easily sees that

\[ (-1)^{(2-s)}2[J_1 p + \bar{J}_1 \bar{\rho} + J'_1 \rho' + (\bar{J}_1 + 1)(\bar{\rho} + 1)] = (-1)^{(2-s)}(J_1 + \bar{J}_1) = 1. \]

The final result is

\[ C_{1,2,3} = a^2 \prod_{k=1}^{3} \frac{(b)_{J'_k} (\bar{b})_{\bar{J}'_k}}{\sqrt{F} \left( 2J'_k + 1 (1 - \frac{h}{\pi}) \right) \sqrt{F} \left( 2\bar{J}'_k + 1 (1 - \frac{\pi}{h}) \right)}. \] (7.26)

Since \( h \) is not real, this three-point coupling is complex. However, taking complex conjugate simply exchanges \( J'_k \) with \( \bar{J}'_k \). Thus left and right movers are exchanged, which makes sense physically.

8 OUTLOOK

There are several interesting physical points to make about our results. In particular, the vertex \( V_{0,0} \) seems to define a cosmological term which differs drastically from the one (the Liouville exponential) which is relevant for the weak-coupling regime. Its study should throw light on the nature of the “c=1” barrier. The associated string susceptibility seems computable. The result is real, while the weak-coupling formula is complex in the strong coupling case. From the viewpoint of conformal theories, the cosmological term is the marginal operator that takes us away from free-field theory. Thus at \( c = 1 \) a new marginal operator replaces the standard one, and this is why the theory for \( c > 1 \) looks so different. We have seen that the barrier seems to be related to a deconfinement of chirality. This is also clear on the expression of \( V_{0,0} \): it corresponds to a metric tensor which is a simple product of one analytic function by its anti-analytic counterpart. In a way, the surface becomes degenerate. We shall return to these points in a separate article. Another remark is that the present new topological models may be simple enough so that their n-point functions are derivable in closed form.

The quantum group technology we have developed, is clearly interesting in itself. It should be helpful, to make progress for the Liouville string theories in full-fledged space-times. In the weak coupling regime, going to continuous \( J \) also seems a key step. For instance, it allows to define the Liouville field itself\[10\].
The present method should be extendable to the $N = 1$ super-Liouville theory, making use of the quantum group structure exhibited in ref. [17]. This will be useful to study the Liouville superstrings [11], which are very interesting physically. In particular the five-dimensional model seems related with the supersymmetric string solutions recently put forward in ref. [18].

There are clearly many more relevant comments. We leave them for future publications.

**Acknowledgements**

We are indebted to Jens Schnittger for repeated discussions that were very useful. This work was supported in part by the E.U. network “Capital Humain et Mobilité”, contract # CHRXCT920035.

### A Appendix: Relation with the earlier definition

In this appendix, we compare our general expressions for the $\chi$ fields with the earlier one of ref. [3]. For this, it is necessary to use the fact that braiding and fusing equations are invariant under the following, which we call gauge transformations:

$$V^{(J,\tilde{J})}_{m,\tilde{m}} \rightarrow S(\overline{\omega})V^{(J,\tilde{J})}_{m,\tilde{m}} S^{-1}(\overline{\omega}), \quad g_{L_1 L_2}^{J_1 J_2} \rightarrow \frac{S(\overline{\omega_{L_1} L_2})}{S(\overline{\omega_L})} g_{L_1 L_2}^{J_1 J_2},$$

where $S$ is an arbitrary function of $\overline{\omega}$, independent from the other mode of the underlying Bäcklund free field. One may show, using Eq.3.28 of ref. [6], that the relation between $\psi$ and $V$ fields may be cast under the form (we do not specify the indices $K, L, M$ of $g$ for brevity)

$$\psi^{(J,\tilde{J})}_{m,-m} \propto \rho(\overline{\omega}) \frac{1}{\sqrt{C_m^{(J)}(\overline{\omega}) C_{-m}^{(J)}(\overline{\omega})}} g_{K L}^{M} V^{(J,\tilde{J})}_{m,-m} \rho^{-1}(\overline{\omega}).$$

where

$$\rho(\overline{\omega}) = \frac{1}{\sqrt{\Gamma[\overline{\omega h}/\pi] \Gamma[\overline{\omega h}/\pi] \sqrt{\overline{\omega} \overline{\omega}}}.$$ (A.3)

Substituting into Eq.2.14, one finds

$$\chi^{(J)}_{-G} \propto \rho(\overline{\omega}) \left( \sum_{m=-J}^{J} e^{im[\tilde{h} \overline{\omega} - h \overline{\omega} + (h - \tilde{h})/2]} \sqrt{\frac{C_m^{(J)}(\overline{\omega})}{C_{-m}^{(J)}(\overline{\omega})}} g_{K L}^{M} V^{(J,\tilde{J})}_{m,-m} \right) \rho^{-1}(\overline{\omega}).$$ (A.4)

On the other hand, one may check that

$$\sqrt{\frac{C_m^{(J)}(\overline{\omega})}{C_{-m}^{(J)}(\overline{\omega})}} = \propto e^{iJ[\tilde{h} \overline{\omega} - h \overline{\omega}] + i\pi s [Jm + m^2/2]}.$$ (A.5)

21This idea was already used in ref. [10] to show that the different operator quantizations of Liouville theory are actually related by transformations of this type.

22For this it is convenient to choose $g_0 = h/\pi$, instead of $g_0 = 2\pi$ as was done in refs. [3], [4].
We dropped factors that only depend upon \( J \), and factors of the form exp\((2\alpha m)\), \( \alpha \) independent from \( J \) and \( \varpi \). The latter factors are pure gauges in the sense of Eq.A.1. Indeed one has
\[
e^{2\alpha m}V^{(J,J)}_{m,-m} = e^{-\alpha \varpi/(1-\pi/h)}V^{(J,J)}_{m,-m}e^{\alpha \varpi/(1-\pi/h)}
\]
as is easily verified. The terms that only depend upon \( J \) only change the overall normalization which could not be seen in ref.[3] where the coupling constants were left undetermined. On the other hand, we have
\[
e^{im(h-\hbar)/2}V^{(J,J)}_{m,-m} = e^{-ih(1+\pi/h)\varpi/4}V^{(J,J)}_{m,-m}e^{ih(1+\pi/h)\varpi/4}
\]
so that the factor exp\( im(h-\hbar)/2 \) in Eq.A.4 may be absorbed by a change of \( \rho(\varpi) \). Collecting all the factors, one finally finds
\[
\chi^{(J)}_{-G} \propto \tilde{\rho}(\varpi) \left( \sum_{m=-J}^J e^{i\pi s[2J_{12}(J+m)-(J+m)(J+m+1)/2]}\sum_{K,L} P^{(J,J)}_{K,L} V^{(J,J)}_{m,-m} \right) P^{12}_{J,12} \tilde{\rho}^{-1}(\varpi). \tag{A.6}
\]
The symbol \( \tilde{\rho} \) stands for the modified gauge transformation operator. Up to this gauge transformation, this expression coincides with our general formula Eq.6.23 for half-integer \( J \)’s. Thus the normalizations of the present article are deduced from the previous one by a gauge transformation of the type Eq.A.1. Note that \( \chi^{(J)}_{-G} \) is not really equivalent to the one we have introduced, since it was defined in \( H^{s}_{phys} \) instead of \( H^{-s}_{phys} \). The expression just given is unsensitive to this modification.

References

[1] J.-L. Gervais and A. Neveu,  Phys. Lett. B151 (1985) 271.
[2] J.-L. Gervais, B. Rostand,  Nucl. Phys. B346 (1990) 473.
[3] J.-L. Gervais,  Comm. Math. Phys. 138 (1991) 301.
[4] O. Babelon,  Phys. Lett. B215 (1988) 523.
[5] J.-L. Gervais,  Comm. Math. Phys. 130 (1990) 257.
[6] E. Cremmer, J.-L. Gervais, J.-F. Roussel, “The genus-zero bootstrap of chiral vertex operators in Liouville theory”, preprint LPTENS-93/29, Nucl. Phys. B to be published.
[7] E. Cremmer, J.-L. Gervais, J.-F. Roussel, “The quantum group structure of 2D gravity and minimal models II: The genus-zero chiral bootstrap”, preprint LPTENS-93/19, hep-th/9305043, Comm. Math. Phys. to be published.
[8] G. Moore, N. Seiberg,  Comm. Math. Phys. 123 (1989) 77.
[9] J.-L. Gervais, J. Schnittger,  Phys. Lett. B315 (1993) 258.
[10] J.-L. Gervais, J. Schnittger, “Continuous spins in 2D gravity: chiral vertex operators and construction of the local Liouville field”, preprint LPTENS-93/40.

[11] A. Bilal, J.-L. Gervais, Nucl. Phys. B284 (1987) 397, Phys. Lett. B187 (1987) 39, Nucl. Phys. B293 (1987) 1, Nucl. Phys. B295 [FS21] (1988) 277.

[12] J.-L. Gervais, “Quantum group derivation of 2D gravity-matter coupling” Invited talk at the Stony Brook meeting String and Symmetry 1991. “Gravity-Matter couplings from Liouville Theory”, preprint LPTENS 91-22, hep-th/9205034, Nucl. Phys. B391 (1993) 287.

[13] J.-L. Gervais, J. Schnittger, “The many faces of the quantum Liouville exponentials” preprint LPTENS-93/30, hep-th/9308134, Nucl. Phys. B to be published.

[14] R. Askey, J. Wilson, “A set of orthogonal polynomials that generalize the Racah coefficients or 6-j symbols” SIAM J. Math. Anal. 10 (1979) 1008.

[15] G. Andrews “q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra”, Conference board of the Mathematical Sciences, Regional Conference in Mathematics, # 66, A.M.S. ed.

[16] L.C. Slater, “Generalized hypergeometric functions” Cambridge University press 1966.

[17] J.-L. Gervais, B. Rostand, Comm. Math. Phys. 143 (1992) 175.

[18] I.Antoniadis, S. Ferrara, C. Kounnas, “Exact supersymmetric string background solutions in curved gravitational backgrounds” preprint hep-th/9402073.

[19] A. Kirilov, N. Reshetikhin, Infinite Dimensional Lie Algebras and Groups, Advanced Study in Mathematical Physics vol. 7, Proceedings of the 1988 Marseille Conference, V. Kac editor, p. 285, World scientific.

[20] Bo-yu Hou, Bo-yuan Hou, Zhong-qi Ma, Comm. Theor. Phys., 13, 181, (1990); Comm. Theor. Phys., 13, 1990, 341.
