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Igor E. Shparlinski, Wolfgang Steiner

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On digit patterns in expansions of rational numbers with prime denominator

IGOR E. SHPARLINSKI
Department of Computing, Macquarie University
Sydney, NSW 2109, Australia
igor.shparlinski@mq.edu.au

WOLFGANG STEINER
LIAFA, CNRS, Université Paris Diderot – Paris 7
Case 7014, 75205 Paris Cedex 13, France
steiner@liafa.univ-paris-diderot.fr

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Abstract
We show that, for any fixed $\varepsilon > 0$ and almost all primes $p$, the $g$-ary expansion of any fraction $m/p$ with gcd$(m, p) = 1$ contains almost all $g$-ary strings of length $k < (5/24 - \varepsilon) \log_g p$. This complements a result of J. Bourgain, S. V. Konyagin, and I. E. Shparlinski that asserts that, for almost all primes, all $g$-ary strings of length $k < (41/504 - \varepsilon) \log_g p$ occur in the $g$-ary expansion of $m/p$.

1 Introduction
Let us fix some integer $g \geq 2$. It is well-known that if gcd$(n, gm) = 1$ then the $g$-ary expansion of the rational fractions $m/n$ is purely periodic with period $t_n$, which is independent of $m$ and equals the multiplicative order of $g$ modulo $n$, see [9]. In the series of works [3, 8, 9], the distribution of digit patterns in such expansions has been studied. In particular, for positive integers $k$ and $m < n$ with gcd$(n, gm) = 1$, we denote by $T_{m,n}(k)$ the number
of distinct $g$-ary strings $(d_1, \ldots, d_k) \in \{0, 1, \ldots, g - 1\}^k$ that occur among
the first $t_n$ strings $(\delta_r, \ldots, \delta_{r+k-1})$, $r = 1, \ldots, t_n$, from the $g$-ary expansion
\[
m = \sum_{r=1}^{\infty} \delta_r g^{-r}, \quad \delta_r \in \{0, 1, \ldots, g - 1\}.
\] (1)

Motivated by applications to pseudorandom number generators, see [1], we are interested in describing the conditions under which $T_{m,n}(k)$ is close to its trivial upper bound
\[
T_{m,n}(k) \leq \min\{t_n, g^k\}.
\]
Since $t_n \leq n$, it is clear that only values $k \leq \lceil \log_g n \rceil$ are of interest. It has been shown in [8, Theorem 11.1] that, for any fixed $\varepsilon > 0$ and for almost all primes $p$ (that is, for all but $o(x/\log x)$ primes $p \leq x$), we have $T_{m,p}(k) = g^k$, provided that $k \leq (3/37 - \varepsilon) \log_g p$. The coefficient $3/37$ has been increased up to $41/504$ in [3, Corollary 8]. Here we show that, for almost all primes $p$, we have $T_{m,p}(k) = (1 + o(1))g^k$ for much larger string lengths $k$.

**Theorem 1.** For any fixed $\varepsilon > 0$, for almost all primes $p$, we have
\[
T_{m,p}(k) = (1 + o(1))g^k
\]
as $p \to \infty$, provided that $k \leq (5/24 - \varepsilon) \log_g p$.

Our arguments depend on the reduction of the problem to the study of intersections of intervals and multiplicative groups modulo $p$ generated by $g$, that has been established in [8]. In turn, the question about the intersections of intervals and subgroups in residue rings has been studied in a number of works [3, 4, 8]. In particular, the results of [3, Corollary 8] and [8, Theorem 11.1] are based on estimates of the length of the longest interval that is not hit by a subgroup of the multiplicative group $\mathbb{F}_p^*$ of the field $\mathbb{F}_p$ of $p$ elements. To prove Theorem 1, we use the results and ideas of [3] to estimate the total number of intervals of a given length that do not intersect a given subgroup of $\mathbb{F}_p^*$.

## 2 Multiplicative Orders

We recall the following well-known implication of the classical result of [5].

**Lemma 2.** For almost all primes $p$, the multiplicative order $t$ of $g$ modulo $p$ satisfies $t > p^{1/2}$.
3 Bounds of Some Exponential Sums

Let $p$ be prime and let $G \subseteq \mathbb{F}_p^*$ be a subgroup of order $t$, where $\mathbb{F}_p$ is a finite field of $p$ elements.

We denote
$$e_p(z) = \exp(2\pi i z/p)$$
and define exponential sums
$$S_\lambda(p; G) = \sum_{v \in G} e_p(\lambda v).$$

Using [6, Lemma 3] (see also [8, Lemma 3.3]) if $t < p^{2/3}$, and the well known bounds
$$|S_\lambda(p; G)| \leq p^{1/2} \quad \text{and} \quad \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; G)|^2 \leq pt$$
(see [8, Equations (3.4) and (3.15)]) if $t \geq p^{2/3}$, we derive:

**Lemma 3.** For any prime $p$ and a subgroup $G \subseteq \mathbb{F}_p^*$ of order $t$, we have
$$\sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; G)|^4 \ll pt^{5/2}.$$

4 Intervals Avoiding Subgroups

As before, let $p$ be prime and let $G \subseteq \mathbb{F}_p^*$ be a subgroup of order $t$.

Let $U(p; G, H)$ be the set of $u \in \mathbb{F}_p$ such the congruence
$$v \equiv u + x \pmod{p}, \quad v \in G, \ 0 \leq x < H,$$
has no solution.

**Lemma 4.** Assume that $G$ is of order $t > p^{1/2}$. Then, for any fixed integer $\nu \geq 1$, we have
$$\#U(p; G, H) \leq p^{2-1/4(\nu+1)+o(1)} H^{-1/2} t^{-5/4+2\nu+1}/4\nu(\nu+1)$$
$$+ p^{5/2-1/2\nu+o(1)} H^{-1} t^{-5/4+1/2\nu}.$$
Proof. Let us fix some \( \varepsilon > 0 \). We put

\[
s = \left\lceil \frac{3}{2} (1 + \varepsilon^{-1}) \right\rceil, \quad h = \left\lceil \frac{p^{1+\varepsilon}}{H} \right\rceil, \quad Z = \lceil H/s \rceil.
\]

We can assume that \( h < p/2 \), as otherwise the bound is trivial (for example, it follows immediately from the bound of Heath-Brown and Konyagin [6, Theorem 1]). Obviously

\[
\mathcal{U}(p; \mathcal{G}, H) \subseteq \mathcal{W}_s(p; \mathcal{G}, Z),
\]

where \( \mathcal{W}_s(p; \mathcal{G}, Z) \) is the set of \( u \in \mathbb{F}_p \) such the congruence

\[
v \equiv u + x_1 + \ldots + x_s \pmod{p}, \quad v \in \mathcal{G}, \quad 0 \leq x_1, \ldots, x_s < Z,
\]

has no solution.

For the number \( Q_s(p; \mathcal{G}, Z, u) \) of solutions to the congruence (3), exactly as in the proof of [8, Lemma 7.1], we obtain

\[
Q_s(p; \mathcal{G}, Z, u) = \frac{1}{p} \sum_{|a| < p/2} e_p(-au) \left( \sum_{0 \leq x < Z} e_p(ax) \right)^s S_a(p; \mathcal{G}).
\]

where the sums \( S_a(p; \mathcal{G}) \) are defined in Section 3.

Separating the term \( tZ^s p^{-1} \) corresponding to \( a = 0 \) and summing over all \( u \in \mathcal{W}_s(p; \mathcal{G}, Z) \) yields

\[
0 = \sum_{u \in \mathcal{W}_s(p; \mathcal{G}, Z)} Q_s(p; \mathcal{G}, Z, u) \geq \frac{tWZ^s}{p} - \frac{\sigma}{p},
\]

where

\[
W = \#\mathcal{W}_s(p; \mathcal{G}, Z)
\]

and

\[
\sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{u \in \mathcal{W}_s(p; \mathcal{G}, Z)} e_p(au) \right| \left| \sum_{0 \leq x < Z} e_p(ax) \right|^s |S_a(p; \mathcal{G})|.
\]
Using the Cauchy inequality, and then the orthogonality relation for exponential functions, we obtain

\[
\sigma^2 \leq \sum_{1 \leq |a| < p/2} \left| \sum_{u \in W_s(p; G; Z)} e_p(au) \right|^2 \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} e_p(ax) \right|^{2s} |S_a(p; G)|^2
\]

\[\leq pW \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} e_p(ax) \right|^{2s} |S_a(p; G)|^2.
\]

Hence

\[W \leq \frac{p}{t^2 Z^{2s}} \Sigma,
\]

where

\[\Sigma = \sum_{1 \leq |a| < p/2} \left| \sum_{0 \leq x < Z} e_p(ax) \right|^{2s} |S_a(p; G)|^2.
\]

Following the idea of the proof of [8, Lemma 7.1], we write

\[\Sigma = \Sigma_1 + \Sigma_2,
\]

where

\[\Sigma_1 = \sum_{1 \leq |a| \leq h} \left| \sum_{0 \leq x < Z} e_p(ax) \right|^{2s} |S_a(p; G)|^2,
\]

\[\Sigma_2 = \sum_{h < |a| < p/2} \left| \sum_{0 \leq x < Z} e_p(ax) \right|^{2s} |S_a(p; G)|^2.
\]

For \(1 \leq |a| \leq h\), we use the trivial estimate

\[\left| \sum_{0 \leq x < Z} e_p(ax) \right| \leq Z
\]

and derive

\[\Sigma_1 \leq Z^{2s} \sum_{1 \leq |a| \leq h} |S_a(p; G)|^2 = \frac{Z^{2s}}{t} \sum_{1 \leq |a| \leq h} \sum_{w \in G} |S_{aw}(p; G)|^2
\]

\[= \frac{Z^{2s}}{t} \sum_{\lambda \in \mathbb{F}_p^*} M_{\lambda}(p; G, h) |S_\lambda(p; G)|^2,
\]

5
where \( M_\lambda(p; \mathcal{G}, h) \) denotes the number of solutions to the congruence
\[
\lambda \equiv aw \pmod{p}, \quad 1 \leq |a| \leq h, \quad w \in \mathcal{G}.
\]

Hence, by the Cauchy inequality
\[
\Sigma_1 \leq \frac{Z^{2s}}{t} \left( \sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \right)^{1/2} \left( \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2}.
\]

As in [3, Section 3.3], we have
\[
\sum_{\lambda \in \mathbb{F}_p^*} M_\lambda(p; \mathcal{G}, h)^2 \leq tN(p; \mathcal{G}, h),
\]
where \( N(p; \mathcal{G}, h) \) is the number of solutions of the congruence
\[
ux \equiv y \pmod{p}, \quad 0 < |x|, |y| \leq h, \quad u \in \mathcal{G}.
\]

Therefore,
\[
\Sigma_1 \leq \frac{Z^{2s}}{t^{1/2}} N(p; \mathcal{G}, h)^{1/2} \left( \sum_{\lambda \in \mathbb{F}_p^*} |S_\lambda(p; \mathcal{G})|^4 \right)^{1/2}.
\] (6)

It is shown in [3, Theorem 1] that if \( t \geq p^{1/2} \) then for any fixed integer \( \nu \) and any positive number \( h \), we have
\[
N(p; \mathcal{G}, h) \leq ht^{(2\nu+1)/2}(\nu+1) p^{-1/2(\nu+1)+o(1)} + h^2 t^{1/\nu} p^{-1/2+o(1)}.
\] (7)

Therefore, using Lemma 3 and the bound (7) we derive from (6) that
\[
\Sigma_1 \leq p^{1/2} t^{3/4} Z^{2s} \left( h^{1/2} t^{(2\nu+1)/4(\nu+1)} p^{-1/4(\nu+1)+o(1)} + ht^{1/2\nu} p^{-1/2+o(1)} \right). \] (8)

If \( h < |a| < p/2 \), then we use the bound
\[
\sum_{0 \leq x < Z} e_p(ax) \ll \frac{p}{|a|},
\]
see [7, Bound (8.6)]. From the trivial bound
\[
|S_a(p; \mathcal{G})| \leq t,
\]
recalling the choice of $h$, we obtain

$$
\Sigma_2 \ll \sum_{h < |a| < p/2} \left( \frac{p}{|a|} \right)^{2s} t^2 \ll t^2 \frac{p^{2s}}{h^{2s-1}} \ll \frac{t^2 Z^{2s} h}{p^{2s \varepsilon}} \ll \frac{Z^{2s} p^3}{p^{2s \varepsilon}} \ll Z^{2s},
$$

as $2s \varepsilon > 3$ for our choice of $s$. Thus the bound on $\Sigma_2$ is dominated by the bound (8) on $\Sigma_1$. Using (4) and (5), we obtain

$$
W \leq p^{3/2} t^{-5/4} \left( h^{1/2} t^{(2\nu+1)/4\nu(\nu+1)} p^{-1/4(\nu+1)+o(1)} + h t^{1/2\nu} p^{-1/2\nu+o(1)} \right).
$$

Recalling (2), the choice of $h$ and that $\varepsilon$ is arbitrary, after simple calculations, we obtain the result.

**Corollary 5.** Assume that $G$ is of order $t > p^{1/2}$. Then for any $\varepsilon > 0$ and

$$
H \geq p^{19/24 + \varepsilon}
$$

we have

$$
\#U(p; G, H) = o(p).
$$

**Proof.** Since $t > p^{1/2}$, we have, for any fixed integer $\nu \geq 1$,

$$
\#U(p; G, H) \leq p^{11/8+1/8\nu(\nu+1)+o(1)} H^{-1/2} + p^{15/8-1/4\nu+o(1)} H^{-1}.
$$

Taking $\nu = 2$ or $\nu = 3$, we conclude the proof. \qed

### 5 Proof of Theorem 1

By Lemma 2 it is enough to consider prime $p$ for which the multiplicative order $t$ of $g$ modulo $p$ satisfies $t > p^{1/2}$.

We now take a positive integer $k \leq (5/24 - \varepsilon) \log_g p$ and consider the intervals $[D, D + 1)$ and $[D/k, D/k + 1)$. As in the proof of [8, Theorem 11.1], we observe that, for any integer $\ell \geq 0$ and any $g$-ary string $(d_1, \ldots, d_k)$, we have $\delta_{\ell+i} = d_i$, $i = 1, \ldots, k$, if and only if

$$
\frac{mg^\ell}{p} - \left[ \frac{mg^\ell}{p} \right] \in \left( \frac{D}{g^k}, \frac{D + 1}{g^k} \right),
$$

with $m = \frac{ht^{1/2\nu} p^{1/2\nu+o(1)}}{Z} 

\frac{p^{1/2}}{Z}. $
where \( D = d_1 g^{k-1} + d_2 g^{k-2} + \cdots + d_k \) and the \( \delta_r, r = 1, 2, \ldots, \) are defined by (1) with \( n = p \). Thus, if a string \((d_1, \ldots, d_k)\) is not present in the \( g \)-ary expansion of \( m/p \), then each interval \([u, u + H]\) with

\[
u = \left\lfloor \frac{D}{g^k p} \right\rfloor, \ldots, \left\lfloor \frac{D + 1/2}{g^k p} \right\rfloor \quad \text{and} \quad H = \left\lfloor \frac{1}{2g^k p} \right\rfloor
\]

contains no element of the conjugacy class \( mG_p \) of the group \( G_p \) generated by \( g \) modulo \( p \). Clearly, different strings \((d_1, \ldots, d_k)\) correspond to different intervals of the values of \( u \), and each of them contains

\[
\left\lfloor \frac{D + 1/2}{g^k p} \right\rfloor - \left\lfloor \frac{D}{g^k p} \right\rfloor \gg \frac{p}{g^k}
\]

values of \( u \). Therefore, the number of missing strings \((d_1, \ldots, d_k)\) satisfies

\[
g^k - T_{m,p}(k) \ll \frac{g^k}{p} \#U(p; G_p, H).
\]

Since \( g^k \leq p^{5/24 - \varepsilon} \), we infer from Corollary 5 that \( \#U(p; G_p, H) = o(p) \), which proves Theorem 1.

6 Composite Denominators

It is quite likely that one can also study \( T_{m,n}(k) \) for almost all composite \( n \) by supplementing the ideas of this work with those of [2] (to get an analogue of Lemma 3) and also using the result of [10] that gives an analogue of Lemma 2.

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