The Origin of the Planck’s Constant

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Abstract
In this paper, we discuss an equation which does not contain the Planck’s constant, but it will turn out the Planck’s constant when we apply the equation to the problems of particle diffraction.

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1 Introduction
In 1900, M. Planck assumed that the energy of a harmonic oscillator can take on only discrete values which are integral multiples of $\hbar \nu$, where $\nu$ is the vibration frequency and $\hbar$ is a fundamental constant, now either $\hbar$ or $\bar{\hbar} = \hbar / 2\pi$ is called as Planck’s constant. The Planck’s constant next made its appearance in 1905, when Einstein used it to explain the photoelectric effect, he assumed that the energy in an electromagnetic wave of frequency $\omega$ is in the form of discrete quanta (photons) each of which has an energy $\bar{\hbar} \omega$ in accordance with Planck’s assumption. From then, it has been recognized that the Planck’s constant plays a key role in the quantum mechanics.

In this paper, we discuss an equation which does not contain the Planck’s constant, but it will turn out the Planck’s constant when we apply the equation to the problems of particle diffraction.

Consider a particle of mass $m$ and charge $q$ moving in an electromagnetic field in a Minkowski’s space ($x_1, x_2, x_3, x_4 =ict$), the 4-vector velocity of the particle is denoted by $u_\mu$, the 4-vector potential of the electromagnetic field is denoted by $A_\mu$, where and below we use Greek letters for subscripts that range from 1 to 4.

We write a theorem to specify our argument.

Theorem: No mater how to move or when to move in the Minkowski’s space, the motion of the particle is governed by a potential function $\Phi$ as

\[ mu_\mu + qA_\mu = \partial_\mu \Phi \] (1)

For applying Eq. (1) to specific applications, we set $\Phi = -i\kappa \psi$, then Eq. (1) is rewritten as

\[ (mu_\mu + qA_\mu)\psi = -i\kappa \partial_\mu \psi \] (2)

the coefficient $\kappa$ is subject to the interpretation of $\psi$.

Eq. (1) was obtained in the author’s previous paper [1], here we shall not discuss its deduction, conversely, shall discuss how to use it and reveal its relation with the Planck’s constant.

There are three mathematical properties of $\psi$ worth recording here. First, if there is a path $\gamma_i$ joining initial point $x_0$ to final point $x$, then

\[ \psi_i = e^{\frac{i}{\hbar} \int_{x_0}^{x} (mu_\mu + qA_\mu)dx_\mu} \] (3)

Second, the integral of Eq. (3) is independent of the choice of path. Third, the superposition principle is valid for $\psi_i$, i.e., if there are $N$ paths from $x_0$ to $x$, then

\[ \psi = \sum_{i}^{N} \psi_i \] (4)

\[ m\bar{u}_\mu = \sum_{i}^{N} mu_\mu \psi_i / \sum_{i}^{N} \psi_i \] (5)

\[ (m\bar{u}_\mu + qA_\mu)\psi = -i\kappa \partial_\mu \psi \] (6)

where $m\bar{u}_\mu$ is average momentum.

To gain further insight into physical meanings of this theorem, we shall discuss four applications.

2 Two slit experiment
As shown in Fig.1, suppose that the electron gun emits a burst of electrons at $x_0$ at time $t = 0$, the electrons arrive at the point $x$ on the screen at time $t$. There are two paths for the electron to go to the destination, according to our above theorem, $\psi$ is given by
The Aharonov-Bohm effect

Let us consider the modification of the two-slit experiment, as shown in Fig. 2. Between the two slits there is located a tiny solenoid S, designed so that a magnetic field perpendicular to the plane of the figure can be produced in its interior. No magnetic field is allowed outside the solenoid, and the walls of the solenoid are such that no electron can penetrate to the interior. Like Eq. (3), the amplitude $\psi$ is given by

$$\psi = e^{\frac{\imath}{\hbar} \int_{x_0(l_1)}^{x} (m_\mu + qA_\mu) dx_\mu} + e^{\frac{-\imath}{\hbar} \int_{x_0(l_2)}^{x} (m_\mu + qA_\mu) dx_\mu}$$

and the probability is given by

$$W = \psi(x)\psi^*(x)$$

$$= 2 + e^{\frac{\imath}{\hbar} \int_{x_0(l_1)}^{x} (m_\mu + qA_\mu) dx_\mu} - e^{\frac{-\imath}{\hbar} \int_{x_0(l_2)}^{x} (m_\mu + qA_\mu) dx_\mu}$$

$$+ e^{\frac{-\imath}{\hbar} \int_{x_0(l_1)}^{x} (m_\mu + qA_\mu) dx_\mu} - e^{\frac{\imath}{\hbar} \int_{x_0(l_2)}^{x} (m_\mu + qA_\mu) dx_\mu}$$

$$= 2 + 2 \cos \left[ \frac{p}{\kappa} (l_1 - l_2) \right]$$

where $p$ is the momentum of the electron. We find a typical interference pattern with constructive interference when $l_1 - l_2$ is an integral multiple of $\kappa/p$, and destructive interference when it is a half integral multiple. This kind of experiment has been done a long age, no matter what kind of particle, the comparision of the experiment to Eq. (8) leads to two consequences: (1) the complex function $\psi$ is found to be probability amplitude, i.e., $\psi(x)\psi^*(x)$ expresses the probability of finding a particle at location $x$ in the Minkowski’s space. (2) $\kappa$ is the Planck’s constant.

3 The Aharonov-Bohm effect

Now, constructive (or destructive) interference occurs when

$$\frac{p}{\kappa} (l_1 - l_2) + \frac{q\phi}{\kappa} = 2\pi n \quad (or \quad n + \frac{1}{2})$$

(11)
where \( n \) is an integer. When \( \kappa \) takes the value of the Planck’s constant, we know that this effect is just the Aharonov-Bohm effect which was shown experimentally in 1960.

4 The hydrogen atom

The hydrogen atom is one of the few physically significant quantum-mechanical systems for which an exact solution can be found and the theoretical predictions compared with experiment.

Rutherford’s model of a hydrogen atom consists of a nucleus made up of a single proton and of a single electron outside the nucleus, the electron moves in an orbit about the nucleus. Here we consider two points denoted by \( x_0 \) and \( x \) in the orbit, and two paths \( l_1 \) and \( l_2 \) from \( x_0 \) to \( x \) along different directions, as shown in Fig.3. Then, according to our above theorem, the probability amplitude \( \psi \) is given by

\[
\psi = e^{i \int_{x_0(l_1)} (mu + qA) dx} + e^{i \int_{x_0(l_2)} (mu + qA) dx} \tag{12}
\]

and the probability is given by

\[
W = \psi(x)\psi^*(x) = 2 + 2 \cos \left[ \frac{1}{\kappa} \int_{l_1}^{l_2} (mu + qA) dx \right] \tag{13}
\]

where \( k = 1, 2, 3, p_k = mu_k \). For the stationary states, the integral about time will be automatically eliminated because the probability should be stable. The probability of the electron at every point in the orbit should be the same because these points in the orbit are equivalent, this leads to

\[
\oint_{\text{orbit}} p_k dx_k = 2\pi \kappa n \tag{14}
\]

When \( \kappa = \hbar \), Eq.(14) is just the Bohr-Sommerfeld quantization rule for the hydrogen atom.

The probability of the electron outside the orbit should vanish, in where the momentum of the electron should become imaginary.

5 The motion of particle in a potential well

Let us now restrict ourselves to one dimensional well. We choose point \( x_0 \) to locate at the left turning point and \( x \) at arbitrary point in the well, as shown in Fig.4, likewise, there are two paths \( l_1 \) and \( l_2 \) from \( x_0 \) to \( x \) to correspond to "coming" (\( l_1 \)) and "back" (\( l_2 \)) for the particle motion, like Eq.(12) and (13), we obtain the probability as

\[
W = \psi(x)\psi^*(x) = 2 + 2 \cos \left[ \frac{1}{\kappa} \int_{l_1}^{l_2} (mu + qA) dx \right] \tag{15}
\]

The integral about time vanishes for the stationary state. The probability has a distribution in the well, but it will vanish at the right turning point for satisfying boundary condition, this leads to

\[
\int_{l_1}^{l_2} pdx = 2\pi \kappa (n + \frac{1}{2}) \tag{16}
\]

where the integral is evaluated over one whole period of classical motion, from the left turning point to the right and back. We again meet the Bohr-Sommerfeld quantization rule for the old quantum theory when we
take $\kappa = \hbar$, although it was originally written in the form of Eq.(14) in 1915 due to A. Sommerfeld and W. Wilson.

6 Discussion

The above formulation based on the theorem of Eq.(1) is successful to the quantum mechanics, but we emphasize that Eq.(1) is essentially different from the Schrodinger’s equation. In the author’s previous paper we have proved that we can derive the Schrodinger’s equation from our Eq.(1), inversely we can not obtain Eq.(1) from the Schrodinger’s equation.

We always assume that the path integral about time vanishes for stationary state, because we always investigate stable experimental phenomena. If we can be equipped to investigate dynamic processes, the path integral about time will display its effects.

7 Conclusion

The Planck’s constant is an fundamental constant which can be well defined in the theorem of Eq.(1).

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