Backward Uniqueness of Kolmogorov Operators

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Abstract

The backward uniqueness of the Kolmogorov operator
\[ L = \sum_{i,k=1}^{n} \partial_{x_i}(a_{i,k}(x,t)\partial_{x_k}) + \sum_{l=1}^{m} x_l\partial_{y_l} - \partial_t, \]
was proved in this paper. We obtained a weak Carleman inequality via Littlewood-Paley decomposition for the global backward uniqueness. Moreover, a monotonicity inequality was also proved for the Kolmogorov equation.

keywords: Carleman inequality, Kolmogorov operator, backward uniqueness, Littlewood-Paley decomposition

1 Introduction

The Kolmogorov equation has many applications in various models (see [PP], [WZ]), for example, Prandtl's boundary layer equation in the Crocco variables and Boltzmann-Landau equation. One of the simplest form of the Kolmogorov operator is given in the following equation
\[ \partial_{xx}u + x\partial_y u - \partial_t u = 0. \]

In this paper, we consider the following more general backward Kolmogorov operator:
\[ Lu = (\sum_{i,k=1}^{n} \partial_{x_i}(a_{i,k}(x,t)\partial_{x_k}) + \sum_{l=1}^{m} x_l\partial_{y_l} - \partial_t)u, \]
where \( m \leq n, \ x = (x_1, \cdots , x_n), \ y = (y_1, \cdots , y_m), \) and \( (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0, T). \) Assume that the coefficients are symmetric and uniformly elliptic:
\[ a_{ik}(x, t) = a_{ki}(x, t), \ 1 \leq i, k \leq n; \ \lambda^{-1} |\xi|^2 \leq \sum_{i,k=1}^{n} \xi_i a_{ik}(x, t) \xi_k \leq \lambda |\xi|^2, \quad (1.1) \]

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for any \((x,t) \in \mathbb{R}^n \times (0,T)\) and \(\xi \in \mathbb{R}^n\). Here \(\lambda > 1\) is a constant.

Our interest in the backward uniqueness of Kolmogorov operator arises from the study of regularity of Kolmogorov operators and recent progress in the backward uniqueness of the parabolic equations where some important applications have been found. In fact, the backward uniqueness of parabolic operator in half-space is crucial for the proof of smoothness of solutions of Navier-Stokes equations in \(L^3,\infty\) (see \[ESS2\]). Some new techniques are developed in the proof of their result, as well as in the sequel papers (for example \[EKPV\]).

Their main results state as follows:

Suppose that \(u\) and the generalized derivatives \(\partial_t u\) and \(\nabla^2 u\) are square integrable over any bounded domain of \(\mathbb{R}^N_+ \times (0,T)\)

\[
|\Delta u + \partial_t u| \leq M(|u| + |\nabla u|), \quad |u| \leq e^{M|x|^2} \quad \text{in} \quad \mathbb{R}^N_+ \times (0,T)
\]

and \(u(x,0) = 0\) in \(\mathbb{R}^N_+\). Then \(u(x,t) \equiv 0\), in \(\mathbb{R}^N_+ \times (0,T)\). (see \[ESS\])

There is a long history of this type of backward uniqueness of parabolic equations. In the papers like \[LPR\] and \[LA\], some type of Carleman inequalities are obtained under the \(C^2\) smoothness assumptions of the coefficients. On the other hand, the well known example of Miller \[MI\] (where an operator having coefficients which are Hölder-continuous of order 1/6 with respect to \(t\) and \(C^\infty\) with respect to \(x\) does not have the uniqueness property) shows that a certain amount of regularity assumptions on the \(a_{ik}\)'s are necessary for the uniqueness.

The main idea in the proof of the backward uniqueness is to obtain certain type of Carleman inequality which implies the backward uniqueness property. For the backward uniqueness of Kolmogorov operator, an uniform Carleman inequality is necessary. However, it is difficult to obtain such inequalities because of degeneracy of the Kolmogorov operator. By combining the Littlewood-Paley decomposition and the approach of Carleman-type inequality in Escauriaza, Seregin, and Šverák \[ESS\], we obtained a weak version of Carleman inequality, which is also useful for unique continuation (see \[LIN\], \[CHE\], \[ESC\], \[EV\], \[EF\], \[FER\], \[ESS\], \[SP\] and so on). On the other hand, monotonicity inequality of frequency functions could also be used to prove unique continuation, for example, see \[PO\] and \[EKPV\].

For the backward uniqueness of Kolmogorov operator, an uniform Carleman inequality is necessary. However, it is difficult to obtain such inequalities because of degeneracy of the Kolmogorov operator. By combining the Littlewood-Paley decomposition and the approach of Carleman-type inequality in Escauriaza, Seregin, and Šverák \[ESS\], we obtained a weak version of Carleman inequality which implies the backward uniqueness property.

Our main idea is first to establish the Carleman-type inequality for the low frequency part in the degenerated direction. Under the assumption of coefficients which are independent of \(y\), we make use of the Littlewood-Paley decomposition of the solution \(u\) in \(y\) direction to prove that the \(L^2\) norm of \(\triangle_j \partial_y u\) can be controlled by the \(L^2\) norm of \(\triangle_j u\). Hence the vanishing property of \(\triangle_j u\) for any \(j\) implies that \(u\) vanishes.

We assume that \(u\) satisfies

\[
\begin{cases}
Lu = (\sum_{i,k=1}^n \partial_{xi}(a_{ik}(x,t)\partial_{xi}) + \sum_{i=1}^m x_i \partial_y - \partial_i)u = c(x,t)u + \sum_{i=1}^n d(x,t)\partial_{xi}u, \\
u(x,y,0) = 0 \quad (x,y) \in \mathbb{R}^{n+m}, \\
\nabla_x u, (\sum_{i=1}^m x_i \partial_y + \partial_t)u \in L^2(\mathbb{R}^{n+m} \times (0,T)),
\end{cases}
\]

(1.2)

where the coefficients are some suitable regular functions.

Let \(g = \chi(t)u\), where \(\chi(t)\) is a \(C^\infty\) smooth function for \(0 < t_1 < t_2 < T\) and

\[
\chi(t) = \begin{cases}
1, & t \leq t_1, \\
0, & t > t_2.
\end{cases}
\]
For fixed $\alpha_0 > 1$, we choose $j_0 = \max\{j \in \mathbb{Z}; 4^{j+1} \leq \frac{\alpha_0 - 1}{4t}, t \in (0, t_2]\}$. When $\alpha > \alpha_0$, we then choose a small $t_2$ and obtain the following Carleman-type inequality

$$
\int_{\mathbb{R}^{n+m} \times (0,T)} t^{-2\alpha+1} |L\Delta_j g|^2 dx dy \geq \int_{\mathbb{R}^{n+m} \times (0,T)} \left( \frac{1}{4\lambda} t^{-2\alpha} |\partial_x \Delta_j g|^2 + \frac{\Omega - 1}{4} t^{-2\alpha-1} |\Delta_j g|^2 \right) dx dy,
$$

where $\Delta_j$ is Littlewood-Paley decomposition operator, $j \in \mathbb{Z}$, $j \leq j_0$. (more details see Section 2.)

The backward uniqueness is proved by applying the above Carleman-type inequality.

**Theorem 1.1** Suppose that $u$ satisfies (1.2) and (1.1). We assume that $a(x,t) \in C^{0,1}$ and $c(x,t), d(x,t)$ are bounded functions. Then $u \equiv 0$ in $\mathbb{R}^{n+m} \times (0,T)$.

**Remark 1.1** Here $a(x,t) \in C^{0,1}$ means that $\nabla_x u$ and $\partial_t u$ are bounded. Under the condition of (1.2), the operator $L$ satisfies the well-known Hörmander finite rank condition. The Kolmogorov operator, although degenerated in some sense, still retains most of the properties of the parabolic operator. For example, the interior regularity of the Kolmogorov operator is similar to that of the parabolic operator (see [PP], [Zhang] and [WZ]), as well as the backward uniqueness at least under some additional assumptions.

**Remark 1.2** For the general Kolmogorov operator, for example, $L_1 = \partial_{xx} + x\partial_y + y\partial_z + \partial_t$, we still don't know if it has the backward uniqueness property, which is interesting and need new idea.

We also give an alternative proof for the above results by frequency functions as the parabolic case given by [PO] and [EKPV].

## 2 Proof of the Main Theorem

We first introduce some of the notations which are used throughout this paper.

Set $g = \chi(t)u$, and $\chi(t)$ is a $C^\infty$ smooth function satisfying

$$
\chi(t) = \begin{cases} 
1, & t \leq t_1, \\
0, & t > t_2.
\end{cases}
$$

where $0 < t_1 < t_2 < T$, to be chosen.

Let $\phi(t) = (t + b)^{-\alpha}$ and $f = \phi g$, where $\alpha > 0$ and $b$ is a constant satisfying $0 < b \leq t_2$.

We introduce the Littlewood-Paley decomposition on $\mathbb{R}^m$. Let $\varphi(\xi)$ be a smooth cut-off function such that

$$
\varphi(\xi) = \begin{cases} 
1, & |\xi| \leq \frac{1}{2}, \\
0, & |\xi| > 1.
\end{cases}
$$

Let $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. For any integer $j$, as usual, we denote $\Delta_j$ and $S_j$,

$$
\Delta_j h(x) = \mathcal{F}^{-1}(\psi(\frac{\xi}{2^j})\mathcal{F}(h)(\xi)),
$$

$$
S_j h(x) = \mathcal{F}^{-1}(\varphi(\frac{\xi}{2^j})\mathcal{F}(h)(\xi)),
$$

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where \( h(x) \in \mathcal{S}'(\mathbb{R}^m) \) and \( \mathcal{F}^{-1} \) is the inverse of Fourier transformation. For \( h \in L^2(\mathbb{R}^m) \), denote \( \mathcal{F}(h) = \hat{h} \), and we have

\[
\int_{\mathbb{R}^m} |h|^2 \, dy = \int_{\mathbb{R}^m} \hat{h}^2 \, dy = \int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \Delta_j h|^2 \, dy.
\]

And it is easy to see that there exists \( K > 0 \) only dependant on the dimension, such that

\[
K^{-1} \int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \Delta_j h|^2 \, dy \leq \int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \Delta_j h|^2 \, dy \leq K \int_{\mathbb{R}^m} \sum_{j \in \mathbb{Z}} \Delta_j h|^2 \, dy
\]

(2.1)

Let \( \Omega_T = \mathbb{R}^{m+n} \times (0, T) \). By our assumption, we may assume that for some positive constant \( \lambda > 1 \), for all \( (x, t) \in \mathbb{R}^n \times (0, T) \) and \( \xi \in \mathbb{R}^n \)

\[
\lambda^{-1} |\xi|^2 \leq \xi a_{ij} \xi_j \leq \lambda |\xi|^2
\]

and

\[
|\nabla_x a_{ij}(x, t)|, |\partial_t a_{ij}(x, t)|, |c(x, t)|, |d(x, t)| \leq \lambda.
\]

We make the Littlewood-Paley decomposition in the \( y \) direction. For convenience we set

\[ f_j = \phi \Delta_j g. \]

We are going to prove a Carleman-type inequality for the function \( \Delta_j g \) which enable us to overcome the difficulty of the degeneracy in the \( y \) direction.

**Lemma 2.1** Under the assumptions of Theorem 1.1, the function \( \Delta_j g \) satisfies the following Carleman inequality

\[
\int_{\Omega_T} (t + b)^{-2\alpha + 1} |L \Delta_j g|^2 \geq \int_{\Omega_T} \frac{1}{4\lambda} (t + b)^{-2\alpha} |\partial_x \Delta_j g|^2 + \frac{\alpha - 1}{4} (t + b)^{-2\alpha - 1} |\Delta_j g|^2.
\]

**Remark 2.1** The function \( \Delta_j g \) is the Littlewood-Paley decomposition in the \( y \) direction of the solution \( u \) of problem (1.2). One can easily check that

\[
\Delta_j g = \chi(t) \mathcal{F}^{-1}(\psi(\frac{\xi}{2t}) \mathcal{F}(u)(\xi)).
\]

**Proof:** We need to estimate the integral of the function \( \Delta_j g \) and its derivative in the \( x \) direction in terms of

\[
I \equiv \int_{\Omega_T} (t + b)|\phi L \Delta_j g|^2 \, dx \, dy \, dt.
\]

Recall the notation \( \phi(t) = (t + b)^{-\alpha} \), \( g = \chi(t)u \) and let \( f_j = \phi \Delta_j g \). By the equation of (1.2), on the one hand, we have

\[
\int_{\Omega_T} (t + b)|\phi L \Delta_j g|^2 \, dx \, dy \, dt
\]

(2.2)

\[
= \int_{\Omega_T} (t + b)\left( \sum_{i,k=1}^n \partial_x(a_{ik}(x,t) \partial_x k) + \sum_{l=1}^m \partial_t y_l + \phi \frac{\phi'}{\phi}(\phi \Delta_j g) \right)^2 \, dx \, dy \, dt
\]

\[
\geq \int_{\Omega_T} 2(t + b)\left( \sum_{i,k=1}^n \partial_x(a_{ik}(x,t) \partial_x k) - \phi \frac{\phi'}{\phi}(\phi \Delta_j g) \right) f_j \left( \sum_{l=1}^m \partial_t y_l + \phi \right) f_j \, dx \, dy \, dt
\]

\[
= - \int_{\Omega_T} 2(t + b) \sum_{i,k=1}^n a_{ik} \partial_x f_j \partial_y k f_j \, dx \, dy \, dt + \int_{\Omega_T} \sum_{i,k=1}^n (a_{ik} + (t + b) \partial t a_{ik}) \partial_x f_j \partial_x f_j \, dx \, dy \, dt
\]
Hence when following Carleman-type inequality

\[ I = \int_{\Omega_T} \left| \left( \sum_{i, k=1}^{n} \partial_{x_i}(a_{i k}(x, t) \partial_{x_k}) + \sum_{l=1}^{m} x_l \partial_{y_l} + \partial_t - \frac{\partial_j}{\phi} (\tilde{\phi} \Delta_j g) \right)^2 dxdydt \right| \]

where in the last equality we used the fact \( \tilde{\phi} = (t + b)^{\frac{1}{2}} \tilde{\phi} = (t + b)^{\frac{1}{2}} - \alpha \), and \( \tilde{f}_j = \tilde{\phi} \Delta_j g = (t + b)^{\frac{1}{2}} f_j \). Then we have

\[ I \geq \int_{\Omega_T} \left( \sum_{i, k=1}^{n} a_{i k} \partial_{x_i} \tilde{f}_j \partial_{y_k} \tilde{f}_j + \sum_{i, k=1}^{n} \partial_t a_{i k} \partial_{x_i} \tilde{f}_j \partial_{x_k} \tilde{f}_j + \left( \frac{\partial_j}{\phi} \right)^2 (\tilde{f}_j)^2 \right) dxdydt \]

\[ = - \int_{\Omega_T} (t + b) \sum_{i, k=1}^{n} a_{i k} \partial_{x_i} \partial_{y_k} \tilde{f}_j \tilde{f}_j dxdydt + \int_{\Omega_T} \sum_{i, k=1}^{n} (t + b) \partial_t a_{i k} \partial_{x_i} \tilde{f}_j \partial_{x_k} \tilde{f}_j dxdydt \]

\[ + \int_{\Omega_T} \frac{\alpha - 1}{(t + b)} |f_j|^2 dxdydt \quad (2.3) \]

where in the last equality we used the fact \( \tilde{f}_j = (t + b)^{\frac{1}{2}} f_j \).

Now we choose \( t_2 \leq (16\lambda^2)^{-1} \) and then fixed from now on in the definition of function \( \chi(t) \). Combining the two inequalities \((2.2)\) with \((2.3)\), a simple calculation yields that

\[ I \geq - \int_{\Omega_T} 2(t + b) \sum_{i, k=1}^{n} a_{i k} \partial_{x_i} \tilde{f}_j \partial_{y_k} \tilde{f}_j dxdydt \]

\[ + \int_{\Omega_T} \sum_{i, k=1}^{n} \left( \frac{1}{2} a_{i k} + (t + b) \partial_t a_{i k} \right) \partial_{x_i} \tilde{f}_j \partial_{x_k} \tilde{f}_j dxdydt + \int_{\Omega_T} \frac{\alpha - 1}{2(t + b)} |f_j|^2 dxdydt \]

\[ \geq \int_{\Omega_T} \frac{1}{4\lambda} |\nabla x f_j|^2 - \lambda (t + b) |\nabla y f_j|^2 + \frac{\alpha - 1}{2(t + b)} |f_j|^2. \quad (2.4) \]

Note that

\[ \int_{\mathbb{R}^m} |\partial_y f_j|^2 = \int_{\mathbb{R}^m} \eta^2 |\psi(\eta/2^j) \tilde{f}(\eta)|^2 \leq 4^{j+1} \int_{\mathbb{R}^m} \eta^2 |\psi(\eta/2^j) \tilde{f}(\eta)|^2. \quad (2.5) \]

Let \( \alpha_0 > 1 \), and we may assume that the parameter \( \alpha \) in the function \( \phi(t) = (t + b)^{-\alpha} \) satisfies

\[ \alpha > \alpha_0 > 0. \]

Moreover, we choose

\[ j_0 = \max\{j \in \mathbb{Z}; 4^{j+1} \leq \frac{\alpha_0 - 1}{4(t + b)}, t \in (0, t_2)\}. \]

Hence when \( \alpha > \alpha_0 \) and \( t_2 \leq (16\lambda^2)^{-1} \), for \( j \leq j_0 \), from \((2.4)\) and \((2.5)\), we obtain the following Carleman-type inequality

\[ \int_{\Omega_T} (t + b)^2 |\phi L \Delta_j g|^2 \geq \int_{\Omega_T} \frac{1}{4\lambda} |\nabla x f_j|^2 + \frac{\alpha - 1}{4(t + b)} |f_j|^2 \]

\[ = \int_{\Omega_T} \frac{1}{4\lambda} (t + b)^{-2\alpha} |\nabla x \Delta_j g|^2 + \frac{\alpha - 1}{4} (t + b)^{-2\alpha - 1} |\Delta_j g|^2. \quad (2.6) \]
Remark 2.2
The assumption that the coefficients \(a_{ik}, c, d\) are independent of \(y\) seems to be only a technique assumption. However, we do not know how to remove it in general. The main difficulty for the Kolmogorov operator in our case is the lose of derivative estimates in \(y\) direction. On the other hand, the recent regularity result (see [WZ]) shows that one can recover the regularity even in \(y\) direction.

Here we give another proof by using the frequency function method (see [PO], [EKPV]). We consider the differential inequality

\[
|\left(\sum_{i,k=1}^n \partial_{x_i}(a_{ik}(x,t)\partial_{x_k})u + \sum_{l=1}^m x_l\partial_{y_l}u + \partial_t u\right)| \leq \lambda(|u| + |\nabla_x u|).
\]

(2.10)

Let

\[
es(t) = \int_{\mathbb{R}^{n+m}} u^2 dx dy,
\]

Then we finished the proof of Lemma 2.1. □

**Proof of Theorem 1.1.** Since \(g = \chi(t)u\), then

\[
L\Delta_j g = \Delta_j (\chi' u + \chi c(x,t) u + \chi d(x,t) \cdot \nabla_x u).
\]

By the above Carleman inequality (2.6), we deduce

\[
\int_{\Omega_T} (t + b)^{-2\alpha + 1}|(\chi' \Delta_j u + \chi c(x,t) \Delta_j u + \chi d(x,t) \cdot \nabla_x \Delta_j u)|^2 \geq \int_{\Omega_T} \frac{1}{4\lambda} (t + b)^{-2\alpha}|\chi \nabla_x \Delta_j u|^2 + \frac{\alpha - 1}{8}(t + b)^{-2\alpha - 1}|\chi \Delta_j u|^2,
\]

(2.7)

where we used the assumption \(|c(x,t)|, |d(x,t)| \leq \lambda\) and the choice of \(t_2\) which satisfies \(t_2 \leq (16\lambda^2)^{-1}\). Consequently

\[
\int_{\Omega_T} (t + b)^{-2\alpha + 1}|(\chi' \Delta_j u)|^2 \geq \int_{\Omega_T} \frac{\alpha - 1}{8}(t + b)^{-2\alpha - 1}|\chi \Delta_j u|^2.
\]

Summing for all \(j \leq j_0\), by the inequality (2.1), we obtain

\[
\int_{\Omega_T} (t + b)^{-2\alpha + 1}|(\chi' u)|^2 \geq C(K) \int_{\Omega_T} \frac{\alpha - 1}{8}(t + b)^{-2\alpha - 1}|\chi \sum_{j=0}^{j_0-1} u|^2.
\]

Then

\[
\int_0^{t_1} \int_{\mathbb{R}^{n+m}} |\sum_{j=0}^{j_0-1} u|^2 \leq \frac{16}{C(K)(\alpha - 1)(t_2 - t_1)^2} \int_{t_1}^{t_2} \int_{\mathbb{R}^{n+m}} \frac{(t_1 + b)^{2\alpha + 1}}{(t + b)^{2\alpha - 1}} |u|^2.
\]

(2.9)

Now we let \(\alpha \to \infty\) in (2.9), then we obtain

\[
\sum_{j=0}^{j_0-1} u \equiv 0.
\]

And then \(u \equiv 0\) in \(R^2 \times (0, t_1)\) by the choice of \(j_0\) and \(\alpha_0 \to \infty\). Again we obtain that \(u \equiv 0\) in \(R^2 \times (0, t_2)\) since \(t_1\) can approach \(t_2\). Finally, we have \(u \equiv 0\) in \(R^2 \times (0, T)\) after the iteration, since \(t_2\) only depends on \(L\). Hence we have completed the proof of Theorem 1.1. □

**Remark 2.2** The assumption that the coefficients \(a_{ik}, c, d\) are independent of \(y\) seems to be only a technique assumption. However, we do not know how to remove it in general. The main difficulty for the Kolmogorov operator in our case is the lose of derivative estimates in \(y\) direction. On the other hand, the recent regularity result (see [WZ]) shows that one can recover the regularity even in \(y\) direction.
\[ d(t) = \int_{\mathbb{R}^{n+m}} \left( \sum_{i,k=1}^{n} a_{ik}(x,t) \partial_{x_i} u \partial_{x_k} u \right) dx dy, \]

and

\[ h(t) = \frac{d(t)}{e(t)}. \]

Then we have the following monotonicity inequality lemma.

**Lemma 2.2** Suppose that \( u \) satisfies (2.10). In addition to the condition of Theorem 1.1, we assume that for some constant \( M \)

\[ \int_{\mathbb{R}^{n+m}} |\nabla_y u|^2 \leq M \int_{\mathbb{R}^{n+m}} |u|^2. \tag{2.11} \]

Then there exists a constant \( C = C(\lambda, M) \) such that

\[ \dot{h}(t) \geq -C(\lambda, M)[h(t) + 1]. \tag{2.12} \]

**Remark 2.3** This is the corresponding monotonicity inequality for the Kolmogorov operator. There is an additional assumption (2.11) which seems necessary in our approach. And in application, we again need to make use of the Littlewood-Paley decomposition in \( y \) direction.

**Proof:** By our assumption, one can calculate directly

\[ \dot{e}(t) = 2 \int_{\mathbb{R}^{n+m}} uu_t = \int_{\mathbb{R}^{n+m}} u Lu + 2 \int_{\mathbb{R}^{n+m}} u (\partial_t u + \sum_{l=1}^{m} x_l \partial_{y_l} u - \frac{1}{2} Lu), \]

and

\[ d(t) = \int_{\mathbb{R}^{n+m}} u (\partial_t u + \sum_{l=1}^{m} x_l \partial_{y_l} u - \frac{1}{2} Lu) - \frac{1}{2} \int_{\mathbb{R}^{n+m}} u Lu. \]

Hence

\[ \dot{e}(t)d(t) = 2 \left[ \int_{\mathbb{R}^{n+m}} u (\partial_t u + \sum_{l=1}^{m} x_l \partial_{y_l} u - \frac{1}{2} Lu) \right]^2 - \frac{1}{2} \left[ \int_{\mathbb{R}^{n+m}} u Lu \right]^2. \tag{2.13} \]
By our assumption on the coefficient \( a_{ik}(x, t) \), we have
\[
\dot{d}(t) = \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} \partial_{i} a_{ik}(x, t) \partial_{x_i} u \partial_{x_k} u + 2 \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} a_{ik}(x, t) \partial_{x_i} u \partial_{x_k} \partial_{t} u
\]
\[
= \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} \partial_{i} a_{ik}(x, t) \partial_{x_i} u \partial_{x_k} u - 2 \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} \partial_{x_k} (a_{ik}(x, t) \partial_{x_i} u) \partial_{t} u
\]
\[
= \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} \partial_{i} a_{ik}(x, t) \partial_{x_i} u \partial_{x_k} u + 2 \int_{\mathbb{R}^{n+m}} (\partial_{i} u + \sum_{l=1}^{m} x_l \partial_{y_l} u) (\partial_{t} u + \sum_{l=1}^{m} x_l \partial_{y_l} - Lu) dx dy
\]
\[
- 2 \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} a_{ik}(x, t) \partial_{x_i} u \partial_{y_k} u
\]
\[
= \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} \partial_{i} a_{ik}(x, t) \partial_{x_i} u \partial_{x_k} u + 2 \int_{\mathbb{R}^{n+m}} (\partial_{i} u + \sum_{l=1}^{m} x_l \partial_{y_l} u - \frac{1}{2} Lu)^2
\]
\[
- \frac{1}{2} \int_{\mathbb{R}^{n+m}} |Lu|^2 - 2 \int_{\mathbb{R}^{n+m}} \sum_{i,k=1}^{n} a_{ik}(x, t) \partial_{x_i} u \partial_{y_k} u
\]
\[
\geq 2 \int_{\mathbb{R}^{n+m}} (\partial_{i} u + \sum_{l=1}^{m} x_l \partial_{y_l} u - \frac{1}{2} Lu)^2 - \frac{1}{2} \int_{\mathbb{R}^{n+m}} |Lu|^2 - C(\lambda, M)[d(t) + e(t)].
\]
Hence together with (2.13) we obtain
\[
e(t)d(t) - d(t)e(t) \geq -C(\lambda, M)[d(t) + e(t)] - \frac{1}{2} \int_{\mathbb{R}^{n+m}} |Lu|^2. \tag{2.14}
\]
Consequently, we have the following monotonicity inequality
\[
\dot{h}(t) = \frac{\dot{d}(t)e(t) - \dot{e}(t)d(t)}{e(t)^2} \geq -C(\lambda, M)[h(t) + 1] - \frac{\int_{\mathbb{R}^{n+m}} |Lu|^2}{2e(t)}.
\]
Together with (2.10), then (2.12) follows easily. Then we proved Lemma 2.2. \(\square\)

**Proof of Theorem 1.1 by the monotonicity inequality:**
We assume that \( e(t) \equiv 0 \) for \( 0 \leq t \leq t_1 \), and \( e(t) > 0 \) as \( t_1 < t \leq T \). For \( t_1 < t < t_2 \leq T \), we integrate the above inequality from \( t \) to \( t_2 \)
\[
\log \frac{h(t_2) + 1}{h(t) + 1} \geq -C(\lambda, M)T,
\]
which yields
\[
h(t) \leq C(\lambda, M, T, h(t_2)). \tag{2.15}
\]
Since
\[
\frac{\dot{e}(t)}{e(t)} \leq 3h(t) + C(\lambda) \leq C(\lambda, M, T, h(t_2)),
\]
integrating from \( t \) to \( t_2 \), we have
\[
e(t_2) \leq e(t)C(\lambda, M, T, h(t_2)). \tag{2.16}
\]
Let $t \to t_1$, we get $e(t_2) = 0$ which is a contradiction. Using the same Littlewood-Paley decomposition as Theorem 1.1, we could replace $u$ with $\Delta_j u$, since $\Delta_j u$ satisfies the inequality (2.11). Then the remaining arguments are similar to that of the previous proof of Theorem 1.1. We complete the proof of our theorem by the frequency function method. □

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