UNITARY EQUIVALENCE OF AUTOMORPHISMS
OF SEPARABLE C*-ALGEBRAS

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ABSTRACT. We prove that the automorphisms of any separable C*-algebra that does not have continuous trace are not classifiable by countable structures up to unitary equivalence. This implies a dichotomy for the Borel complexity of the relation of unitary equivalence of automorphisms of a separable unital C*-algebra: Such relation is either smooth or not even classifiable by countable structures.

1. Introduction

If $A$ is a separable C*-algebra, the group $\operatorname{Aut}(A)$ of automorphisms of $A$ is a Polish group with respect to the topology of pointwise norm convergence. An automorphism of $A$ is called (multiplier) inner if it is induced by the action by conjugation of a unitary element of the multiplier algebra $M(A)$ of $A$. Inner automorphisms form a Borel normal subgroup $\operatorname{Inn}(A)$ of the group of automorphisms of $A$. The relation of unitary equivalence of automorphisms of $A$ is the coset equivalence relation on $\operatorname{Aut}(A)$ determined by $\operatorname{Inn}(A)$. (The reader can find more background on C*-algebras in Section 2.) The main result presented here asserts that if $A$ does not have continuous trace, then it is not possible to effectively classify the automorphisms of $A$ up to unitary equivalence using countable structures as invariants; in particular this rules out classification by K-theoretic invariants. In the course of the proof of the main result we will show that the existence of an outer derivation on a C*-algebra $A$ is equivalent to a seemingly stronger statement, that we will refer to as Property AEP (see Definition 4.4), implying in particular the existence of an outer derivable automorphism of $A$.

The notion of effective classification can be made precise by means of Borel reductions in the framework of descriptive set theory (the monographs [20] and [15] are standard references for this subject). If $E$ and $E'$ are equivalence relations on standard Borel spaces $X$ and $X'$ respectively, then a Borel reduction from $E$ to $F$ is a Borel function $f : X \to X'$ such that for every $x, y \in X$, $x E y$ if and only if $f(x) E' f(y)$. The Borel function $f$ witnesses an effective classification of the objects of $X$ up to $E$, with $E'$-equivalence classes of objects of $X'$ as invariants. This framework captures the vast majority of concrete classification results in mathematics. (In [11] and [12] the computation of most of the invariants in the theory of C*-algebras is shown to be Borel.)
If $E$ and $F$ are, as before, equivalence relations on standard Borel spaces, then $E$ is defined to be **Borel reducible** to $F$ if there is a Borel reduction from $E$ to $F$. This can be interpreted as a notion that allows one to compare the complexity of different equivalence relations. Some distinguished equivalence relations are used as benchmarks of complexity. Among these are the relation $=_Y$ of equality for elements of a Polish space $Y$, and the relation $\simeq_C$ of isomorphism within some class of countable structures $C$. If $E$ is an equivalence relation on a standard Borel space $X$, we say that:

- $E$ is **smooth** (or the elements of $X$ are concretely classifiable up to $E$) if $E$ is Borel reducible to $=_Y$ for some Polish space $Y$;
- $E$ is **classifiable by countable structures** (or the elements of $X$ are classifiable by countable structures up to $E$) if $E$ is Borel reducible to $\simeq_C$ for some class $C$ of countable structures.

A nontrivial example of smooth equivalence relation is the relation of unitary equivalence of irreducible representations of a Type I C*-algebra (see [4] Definition IV.1.1.1). Since all uncountable Polish spaces are Borel isomorphic to $\mathbb{R}$, the class of smooth equivalence relations includes only the equivalence relations that are effectively classifiable using real numbers as invariants. The class of equivalence relations that are classifiable by countable structures is much wider. In fact most classification results in mathematics involve some class of countable structures as invariants. Elliott’s seminal classification of AF algebras by the ordered $K_0$ group in [9] is of this sort, as well as the K-theoretical classification of purely infinite simple nuclear C*-algebras in the UCT class obtained by Kirchberg and Phillips in [22] and [37]. Nonetheless, in the last decade a number of natural equivalence relations arising in different areas of mathematics have been shown to be not classifiable by countable structures. A key role in this development has been played by the theory of turbulence, developed by Greg Hjorth in the second half of the 1990s.

Turbulence is a dynamic condition on a continuous action of a Polish group on a Polish space, implying that the associated orbit equivalence relation is not classifiable by countable structures. Many nonclassifiability results were established directly or indirectly using this criterion. For instance Hjorth showed in [17] (Section 4.3) that the orbit equivalence relation of a turbulent Polish group action is Borel reducible to the relation of homeomorphism of compact spaces, which in turn is reducible to the relation of isomorphism of separable simple nuclear unital C*-algebras by a result of Farah-Toms-Törnquist (Corollary 5.2 of [11]). As a consequence these equivalence relations are not classifiable by countable structures.

In this paper, we use Hjorth’s theory of turbulence to prove the following theorem:

**Theorem 1.1.** If $A$ is a separable C*-algebra that does not have continuous trace, then the automorphisms of $A$ are not classifiable by countable structures up to unitary equivalence.

Theorem 1.1 strengthens Theorem 3.1 from [34], where the automorphisms of $A$ are shown to be not concretely classifiable under the same assumptions on the C*-algebra $A$. We will in fact show that the same conclusion holds even if one only considers the subgroup consisting of approximately inner automorphisms of $A$, i.e. pointwise limits of inner automorphisms. A C*-algebra has continuous trace (see Definition IV.1.4.12 and Proposition IV.1.4.19 of [4]) if it has Hausdorff spectrum
and it is generated by its abelian elements. The class of C*-algebras that do not have continuous trace is fairly large, and in particular includes C*-algebras that are not Type I. More information about C*-algebras with continuous trace can be found in the monograph [39].

A particular implication of Theorem 1.1 is that it is not possible to classify the automorphisms of any separable C*-algebra that does not have continuous trace up to unitary equivalence by K-theoretic invariants. This should be compared with the classification results of (sufficiently outer) automorphisms up to other natural equivalence relations, such as outer conjugacy (see [31] Section 3). Nakamura showed in [31] (Theorem 9) that aperiodic automorphisms of Kirchberg algebras are classified by their KK-classes up to outer conjugacy. Theorem 1.4 of [24] asserts that there is only one outer conjugacy class of uniformly aperiodic automorphisms of UHF algebras. These results were more recently generalized and expanded to classification of actions of $\mathbb{Z}^2$ and $\mathbb{Z}^n$ up to outer conjugacy or cocycle conjugacy (see [28], [27], [19], and [29]).

Phillips and Raeburn obtained in [36] a cohomological classification of automorphisms of a C*-algebra with continuous trace up to unitary equivalence. Such classification implies that if $A$ has continuous trace and the spectrum of $A$ is homotopy equivalent to a compact space, then the normal subgroup $\text{Inn}(A)$ of inner automorphisms is closed in $\text{Aut}(A)$ (see Theorem 0.8 of [38]). In particular (cf. Corollary II.6.5.7 in [4]) this conclusion holds when $A$ is unital and has continuous trace. It follows from a standard result in descriptive set theory (see Exercise 6.4.4 of [15]) that the automorphisms of $A$ are concretely classifiable up to unitary equivalence if and only if $\text{Inn}(A)$ is a closed subgroup of $\text{Aut}(A)$. Theorem 0.8 of [38] and Theorem 1.1 therefore imply the following dichotomy result:

**Theorem 1.2.** If $A$ is a separable unital C*-algebra, then the following statements are equivalent:

1. the automorphisms of $A$ are concretely classifiable up to unitary equivalence;
2. the automorphisms of $A$ are classifiable by countable structures up to unitary equivalence;
3. $A$ has continuous trace.

More generally the same result holds if $A$ is a separable C*-algebra with (not necessarily Hausdorff) compact spectrum. Without this hypothesis the implication $3 \Rightarrow 1$ of Theorem 1.2 does not hold, as pointed out in Remark 0.9 of [38]. We do not know if the implication $3 \Rightarrow 2$ holds for a not necessarily unital C*-algebra $A$. This is commented on more extensively in Section 7.

In particular Theorem 1.2 offers another characterization of unital C*-algebras that have continuous trace, in addition to the classical Fell-Dixmier spectral condition (see [14], [8]) or the reformulation in terms of central sequences by Akemann and Pedersen (see [2] Theorem 2.4).

The dichotomy in the Borel complexity of the relation of unitary equivalence of automorphisms of a unital C*-algebra expressed by Theorem 1.2 should be compared with the analogous phenomenon concerning the relation of unitary equivalence of irreducible representations of a C*-algebra $A$. It is a classical result of Glimm from [16] that such a relation is smooth if and only if $A$ is Type I. It was proved in [21] and, independently, in [13] that the irreducible representations of a C*-algebra that is not Type I are in fact not classifiable by countable structures up to unitary equivalence.
The strategy of the proof of Theorem 1.1, summarized in Figure 1, is the following: We first introduce properties AEP and AEP⁺, named after Akemann, Elliott, and Pedersen since they can be found in nuce in their works [2] and [10]. We then show that Property AEP⁺ is stronger than Property AEP; moreover Property AEP is equivalent to the existence of an outer derivation, and it implies that the conclusion of Theorem 1.1 holds. This concludes the proof under the assumption that the C*-algebra \( A \) has an outer derivation. We then assume that \( A \) does not have continuous trace and has only inner derivations. Using a characterization of C*-algebras with only inner derivations due to Elliott (the main theorem in [10]) and a characterization of continuous trace C*-algebras due to Akemann-Pedersen (Theorem 2.4 in [2]), we infer that in this case \( A \) has a simple nonelementary direct summand. We then deduce that \( A \) contains a central sequence that is not hypercentral (a similar result was proved by Phillips in the unital case, cf. Theorem 3.6 of [35]). The proof is finished by proving that the existence of a central sequence that is not hypercentral implies that the conclusion of Theorem 1.1 holds.

This paper is organized as follows: Section 2 contains some background on C*-algebras and introduces the notations used in the rest of the paper; Section 3 infers from Hjorth’s theory of turbulence a criterion of nonclassifiability by countable structures (Criterion 3.3) to be applied in the proof of Theorem 1.1; Section 4 establishes Theorem 1.1 in the case of C*-algebras with outer derivations, while Section 5 deals with the case of C*-algebras with only inner derivations; Section
2. Background on C*-algebras and notation

A C*-algebra is a norm-closed self-adjoint subalgebra of the Banach *-algebra $B(H)$ of bounded linear operators on some Hilbert space $H$. (The reader can consult [30] as a reference for the basic theory of C*-algebras.) The group $\text{Aut}(A)$ of automorphisms of $A$ is a Polish group with respect to the topology of pointwise convergence (see [38] page 4). A C*-algebra is called unital if it contains a multiplicative identity, usually denoted by 1. If $A$ is unital and $u$ is a unitary element of $A$ (i.e. such that $uu^* = u^*u = 1$), then

$$\text{Ad}(u) (x) = uxu^*$$

defines an automorphism $\text{Ad}(u)$ of $A$. When $A$ is not unital one can consider unitary elements of the multiplier algebra of $A$. The multiplier algebra $M(A)$ of $A$ is the largest unital C*-algebra containing $A$ as an essential ideal (see [4] II.7.3). It can be regarded as the noncommutative analog of the Stone-Cech compactification of a locally compact Hausdorff space. The multiplier algebra of a separable C*-algebra $A$ is not norm separable (unless $A$ is unital, in which case $M(A)$ coincides with $A$). Nonetheless the strict topology (see [4] II.7.3.11) of $M(A)$ is Polish and induces a Polish group structure on the group $U(A)$ of unitary elements of $M(A)$. If $u$ is a unitary multiplier of $A$, i.e. an element of $U(A)$, then one can define as before the automorphism $\text{Ad}(u)$ of $A$. An automorphism of $A$ is called inner if it is of the form $\text{Ad}(u)$ for some unitary multiplier $u$, and outer otherwise. Inner automorphisms of a separable C*-algebra $A$ form a Borel normal subgroup of $\text{Aut}(A)$ (see [32] Proposition 2.4). Two automorphisms $\alpha$ and $\beta$ of $A$ are called unitarily equivalent if $\alpha \circ \beta^{-1}$ is inner or, equivalently,

$$\alpha(x) = \beta(uxu^*)$$

for some unitary multiplier $u$ and every $x \in A$. This defines a Borel equivalence relation on $\text{Aut}(A)$.

In the rest of the paper, we assume all C*-algebras to be norm separable, apart from multiplier algebras and enveloping von Neumann algebras. The enveloping von Neumann algebra or second dual $A^{**}$ of a C*-algebra $A$ (see [33] 3.7.6 and 3.7.8) is a von Neumann algebra isometrically isomorphic to the second dual of $A$. The σ-weak topology on $A^{**}$ is the weak* topology of $A^{**}$ regarded as the dual Banach space of $A^*$. The algebra $A$ can be identified with a σ-weakly dense subalgebra of $A^{**}$. Moreover (see [33] 3.12.3) we can identify the multiplier algebra $M(A)$ of $A$ with the idealizer of $A$ inside $A^{**}$, i.e. the algebra of elements $x$ such that $xa \in A$ and $ax \in A$ for every $a \in A$. Analogously, the unitization $\hat{A}$ of $A$ (see [4] II.1.2) is identified with the subalgebra of $M(A)$ generated by $A$ and 1. If $x$ is a normal element of $A$, i.e. commuting with its adjoint, and $f$ is a complex-valued continuous function defined on the spectrum of $x$, $f(x)$ denotes the element of $\hat{A}$ obtained from $x$ and $f$ using functional calculus (II.2 of [4] is a complete reference for the basic notions of spectral theory and continuous functional calculus in operator algebras). If $x, y$ are element of a C*-algebra, then $[x, y]$ denotes their commutator $xy - yx$; moreover if $S$ is a subset of a C*-algebra $A$, then $S' \cap A$ denotes the relative commutant of $S$ in $A$ (see [4] I.2.5.3). The set $\mathbb{N}$ of natural numbers is supposed not to contain 0. Boldface letters $t$ and $s$ indicate sequences
of real numbers whose $n$-th terms are $t_n$ and $s_n$ respectively. Analogously $x$ stands for the sequence $(x_n)_{n \in \mathbb{N}}$ of elements of a C*-algebra $A$.

3. Nonclassifiability criteria

Recall that a subset $A$ of a Polish space $X$ has the Baire property (Definition 8.21 of [20]) if its symmetric difference with some open set is meager. A function between Polish spaces is Baire measurable (Definition 8.37 of [20]) if the inverse image of any open set has the Baire property. Observe that, in particular, any Borel function is Baire measurable. Suppose that $E$ and $R$ are equivalence relations on Polish spaces $X$ and $Y$ respectively. We say that $E$ is generically $R$-ergodic if, for every Baire measurable function $f : X \to Y$ such that $f(x)Rf(y)$ whenever $xEy$, there is a comeager subset $C$ of $X$ such that $f(x)Rf(y)$ for every $x, y \in C$ (cf. [15] Definition 10.1.4). Observe that if $E$ is generically $R$-ergodic and no equivalence class of $E$ is comeager then, in particular, $E$ is not Borel reducible to $R$.

One of the main tools in the study of Borel complexity of equivalence relations is Hjorth’s theory of turbulence. A standard reference for this subject is [17]. Turbulence is a dynamical property of a continuous group action of a Polish group $G$ on a Polish space $X$ (see [17] Definition 3.13). The main result about turbulent actions is the following result of Hjorth (Theorem 3.21 in [17]):

The orbit equivalence relation $E^G_X$ associated with a turbulent action $G \curvearrowright X$ of a Polish group $G$ on a Polish space $X$ is generically $\simeq_C$-ergodic for every class $C$ of countable structures, where $\simeq_C$ denotes the relation of isomorphism for elements of $C$. Moreover $E^G_X$ has meager equivalence classes, and hence it is not classifiable by countable structures.

This result is valuable not only on its own, but also because it allows one to prove nonclassification results via the following two lemmas:

**Lemma 3.1.** Suppose that $E$, $F$, and $R$ are equivalence relations on Polish spaces $X$, $Y$, and $Z$ respectively, and that $F$ is generically $R$-ergodic. If there is a comeager subset $\tilde{C}$ of $Y$ and a Baire measurable function $f : \tilde{C} \to X$ such that:

- $f(x)Ef(y)$ for any $x, y \in \tilde{C}$ such that $xFy$;
- $f[C]$ is comeager in $X$ for every comeager subset $C$ of $\tilde{C}$;

then the relation $E$ is generically $R$-ergodic as well.

**Proof.** Suppose that $g : X \to Z$ is a Baire measurable function such that $g(x)Rg(x')$ for any $x, x' \in X$ such that $xEx'$. The composition $g \circ f$ is a Baire measurable function from $\tilde{C}$ to $Z$ such that $(f \circ g)(y)R(f \circ g)(y')$ for any $y, y' \in \tilde{C}$ such that $yFy'$. Since $\tilde{C}$ is comeager in $Y$, and $F$ is generically $R$-ergodic, there is a comeager subset $C$ of $\tilde{C}$ such that $(g \circ f)(y)R(g \circ f)(y')$ for every $y, y' \in C$. Therefore, $f[C]$ is a comeager subset of $X$ such that $g(x)Rg(x')$ for every $x, x' \in f[C]$. \hfill \Box

Observe that if $f$ is continuous, open, and onto, then it will automatically satisfy the second condition of Lemma 3.1.

**Lemma 3.2.** Suppose that $E$ and $F$ are equivalence relations on Polish spaces $X$ and $Y$ respectively, and $F$ is generically $\simeq_C$-ergodic for every class $C$ of countable structures. If there is a Baire measurable function $f : Y \to X$ such that:

- $f(x)Ef(y)$ whenever $xFy$;
• for every comeager subset $C$ of $Y$ there are $x, y \in C$ such that $f(x) \not\equiv_f(y)$; then the relation $E$ is not classifiable by countable structures.

Proof. Suppose by contradiction that there is a class $C$ of countable structures and a Borel reduction $g : X \to C$ of $E$ to $\simeq_C$. The composition $g \circ f : Y \to C$ is a Baire measurable function from $Y$ to $C$ such that $(g \circ f)(y) \simeq_C (g \circ f)(y')$ for any $y, y' \in Y$ such that $yFy'$. Since $F$ is generically $\simeq_C$-ergodic, there is a comeager subset $C$ of $Y$ such that $(g \circ f)(y) \simeq_C (g \circ f)(y')$ for every $y, y' \in C$. Therefore, being $g$ a reduction of $E$ to $\simeq_C$, $f(y) \not\equiv_f(y')$ for every $y, y' \in C$. This contradicts our assumptions. \qed

Consider $\mathbb{R}^\mathbb{N}$ as a Polish space with the product topology and $\ell^1$ as a Polish group with its Banach space topology. The fact that the action of $\ell^1$ on $\mathbb{R}^\mathbb{N}$ by translation is turbulent is a particular case of Proposition 3.25 in [17]. It then follows by Hjorth’s turbulence theorem that the associated orbit equivalence relation $E^\ell_\mathbb{N}$ is generically $\simeq_C$-ergodic for every class $C$ of countable structures. It is not difficult to see that the function $f : (\mathbb{R}\setminus\{0\})^\mathbb{N} \to (0,1)^\mathbb{N}$ defined by

$$f(t) = \left( \frac{|t_n|}{|t_n| + 1} \right)_{n \in \mathbb{N}}$$

satisfies both the first (being continuous, open, and onto) and the second condition of Lemma 3.1 where:

• $F$ is the relation $E^\ell_\mathbb{N}$ of equivalence modulo $\ell^1$ of sequences of real numbers;
• $E$ is the relation $E^\ell_{(0,1)^\mathbb{N}}$ of equivalence modulo $\ell^1$ of sequences of real numbers between 0 and 1.

It follows that the latter relation is generically $\simeq_C$-ergodic for every class $C$ of countable structures. Considering the particular case of Lemma 3.2 when $F$ is the relation $E^\ell_{(0,1)^\mathbb{N}}$ one obtains the following nonclassifiability criterion:

**Criterion 3.3.** If $E$ is an equivalence relation on a Polish space $X$ and there is a Baire measurable function $f : (0,1)^\mathbb{N} \to X$ such that:

• $f(x) \not\equiv_f(y)$ for any $x, y \in (0,1)^\mathbb{N}$ such that $x \not\equiv y \in \ell^1$;
• any comeager subset of $(0,1)^\mathbb{N}$ contains elements $x, y$ such that $f(x) \not\equiv_f(y)$;
then the relation $E$ is not classifiable by countable structures.

In order to apply Criterion 3.3 we will need the following fact about nonmeager subsets of $(0,1)^\mathbb{N}$:

**Lemma 3.4.** If $X$ is a nonmeager subset of $(0,1)^\mathbb{N}$, then there is an uncountable $Y \subseteq X$ such that, for every pair of distinct points $s, t$ of $Y$, $\|s - t\|_\infty \geq \frac{1}{4}$, where $\|s - t\|_\infty = \sup_{n \in \mathbb{N}} |t_n - s_n|$.

Proof. Define for every $s \in (0,1)$,

$$K_s = \left\{ t \in (0,1)^\mathbb{N} \mid \|t - s\| \leq \frac{1}{4} \right\}.$$ 

Observe that $K_s$ is a closed nowhere dense subset of $(0,1)^\mathbb{N}$. Consider the class $\mathcal{A}$ of subsets $Y$ of $X$ with the property that, for every $s, t$ in $Y$ distinct, $\|s - t\| \geq \frac{1}{4}$. If
A is partially ordered by inclusion, then it has some maximal element \( Y \) by Zorn’s lemma. By maximality,

\[
X \subset \bigcup_{t \in Y} \left\{ s \in (0, 1)^{\mathbb{N}} \mid \| t - s \|_{\infty} \leq \frac{1}{4} \right\}.
\]

Being \( X \) nonmeager, \( Y \) is uncountable. \( \square \)

### 4. The case of algebras with outer derivations

The aim of this section is to show that if a C*-algebra \( A \) has an outer derivation, then the relation of unitary equivalence of approximately inner automorphisms of \( A \) is not classifiable by countable structures. In proving this fact we will also show that any such C*-algebra satisfies a seemingly stronger property, that we will refer to as Property AEP (see Definition 4.4).

A derivation of a C*-algebra \( A \) is a linear function

\[
\delta : A \to A
\]
satisfying the derivation identity:

\[
\delta(xy) = \delta(x)y + x\delta(y)
\]
for \( x, y \in A \). The derivation identity implies that \( \delta \) is a bounded linear operator on \( A \) (see [33] Proposition 8.6.3). The set \( \Delta(A) \) of derivations of \( A \) is a closed subspace of the Banach space \( B(A) \) of bounded linear operators on \( A \). A derivation is called a *-derivation if it is a positive linear operator, i.e. it sends positive elements to positive elements. Any element \( a \) of the multiplier algebra of \( A \) defines a derivation \( \text{ad}(ia) \) of \( A \), by

\[
\text{ad}(ia)(x) = [ia, x].
\]
This is a *-derivation if and only if \( a \) is self-adjoint. A derivation of this form is called inner, and outer otherwise. More generally, if \( a \) is an element of the enveloping von Neumann algebra of \( A \) that derives \( A \), i.e. \( ax - xa \in A \) for any \( x \in A \), then one can define the (not necessarily inner) derivation \( \text{ad}(ia) \) of \( A \). Since any derivation is linear combination of *-derivations (see [33] 8.6.2), the existence of an outer derivation is equivalent to the existence of an outer *-derivation. The set \( \Delta_0(A) \) of inner derivations of \( A \) is a Borel (not necessarily closed) subspace of \( \Delta(A) \). The norm on \( \Delta_0(A) \) defined by

\[
\| \text{ad}(ia) \|_{\Delta_0(A)} = \inf \{ \| a - z \| \mid z \in A' \cap A \}
\]
makes \( \Delta_0(A) \) a separable Banach space isometrically isomorphic to the quotient of \( A \) by its center \( A' \cap A \). The inclusion of \( \overline{\Delta_0(A)} \) in \( \Delta(A) \) is continuous, and the closure \( \overline{\Delta_0(A)} \) of \( \Delta_0(A) \) in \( \Delta(A) \) is a closed separable subspace of \( \Delta(A) \). If \( \delta \) is a *-derivation then the exponential \( \exp(\delta) \) of \( \delta \), regarded as an element of the Banach algebra \( B(A) \) of bounded linear operators of \( A \), is an automorphism of \( A \). Automorphisms of this form are called derivable. If \( \delta = \text{ad}(ia) \) is inner then

\[
\exp(\delta) = \text{Ad}(\exp(ia))
\]
is inner as well. Lemma 4.1 provides a partial converse to this statement. (The converse is in fact false in general.) For more information on derivations and derivable automorphisms, the reader is referred to [33] Section 8.6.

Recall that an irreducible representation of a C*-algebra \( A \) is a *-homomorphism \( \pi : A \to B(H) \), where \( H \) is a (necessarily separable) Hilbert space, such that no
nontrivial proper subspace of $H$ is invariant for $\pi[A]$ (see [4] Definition II.6.1.1 and Proposition II.6.1.8). Two irreducible representations $\pi_0, \pi_1$ of $A$ are unitarily equivalent if there is a unitary element $u$ of $B(H)$ such that $\pi_1(x) = u\pi_0(x)u^*$ for every $x \in A$. An ideal of a C*-algebra $A$ is called primitive if it is the kernel of an irreducible representation of $A$. A C*-algebra $A$ is called primitive if $\{0\}$ is a primitive ideal in $A$, i.e. $A$ has a faithful irreducible representation. The primitive spectrum $\hat{A}$ of $A$ is the space of primitive ideals of $A$ endowed with the hull-kernel topology described in [33] 4.1.3. The spectrum $\hat{A}$ of $A$ is the space of unitary equivalence classes of irreducible representations of $A$ endowed with the Jacobson topology defined in [33] 4.1.12.

**Lemma 4.1.** Suppose that $A$ is a primitive C*-algebra. If $\delta$ is a *-derivation of $A$ with operator norm strictly smaller than $2\pi$ such that $\exp(\delta)$ is inner, then $\delta$ is inner.

The lemma is proved in [18] (Theorem 4.6 and Remark 4.7) under the additional assumption that $A$ is unital. It is not difficult to check that the same proof works without change in the nonunital case.

**Definition 4.2.** Suppose that $A$ is a C*-algebra, $(a_n)_{n \in \mathbb{N}}$ is a dense sequence in the unit ball of $A$, and $x = (x_n)_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal positive contractions of $A$ such that for every $n \in \mathbb{N}$ and $i \leq n$,

$$||[x_n, a_i]|| \leq 2^{-n}.$$  

Since the $x_n$’s are pairwise orthogonal, if $t$ is a sequence of real numbers of absolute value at most 1, then the series

$$\sum_{n \in \mathbb{N}} t_n x_n$$

converges to a self-adjoint element of $A^{**}$. Moreover, the sequence of inner automorphisms

$$\left( \text{Ad} \left( \exp \left( i \sum_{k \leq n} t_k x_k \right) \right) \right)_{n \in \mathbb{N}}$$

of $A$ converges to the approximately inner automorphism

$$\alpha_t := \text{Ad} \left( \exp \left( i \sum_{n \in \mathbb{N}} t_n x_n \right) \right).$$

The equivalence relation $E_x$ on $(0,1)^N$ is defined by

$$s E_x t \iff \alpha_s \text{ and } \alpha_t \text{ are unitarily equivalent.}$$

Observe that this equivalence relation is finer than the relation of $\ell^1$-equivalence introduced in Section 3. In fact if $s, t \in (0,1)^N$ and $s - t \in \ell^1$, then the series

$$\sum_{n \in \mathbb{N}} (t_n - s_n) x_n$$

converges in $A$. It is then easily verified that

$$u := \exp \left( i \sum_{n \in \mathbb{N}} (t_n - s_n) x_n \right)$$
is a unitary multiplier of $A$ such that

$$\text{Ad}(u) \circ \alpha_s = \alpha_t.$$ 

Therefore, if the equivalence classes of $E_x$ are meager, the continuous function

$$(0,1)^N \rightarrow \text{Aut}(A)$$

$$t \mapsto \alpha_t$$

satisfies the hypothesis of Criterion 3.3. This concludes the proof of the following lemma:

**Lemma 4.3.** Suppose that $A$ is a C*-algebra. If for some sequence $x$ of pairwise orthogonal positive contractions of $A$ the equivalence relation $E_x$ has meager equivalence classes, then the approximately inner automorphisms of $A$ are not classifiable by countable structures.

Lemma 4.3 motivates the following definition.

**Definition 4.4.** A C*-algebra $A$ has Property AEP if for every dense sequence $(a_n)_{n \in \mathbb{N}}$ in the unit ball of $A$ there is a sequence $x = (x_n)_{n \in \mathbb{N}}$ of pairwise orthogonal positive contractions of $A$ such that:

1. $\| [x_n, a_i] \| < 2^{-n}$ for $i \in \{1, 2, \ldots, n\}$;
2. the relation $E_x$ as in Definition 1.2 has meager conjugacy classes.

It is clear that if a C*-algebra $A$ has Property AEP, then $A$ has an outer $*$-derivation. In fact, if $s, t \in (0,1)^N$ are such that $s \not\in E_x t$, then the self-adjoint element

$$a = \sum_{n \in \mathbb{N}} (t_n - s_n) x_n$$

of $A^{**}$ derives $A$. The automorphism $\text{Ad}(\exp(ia))$ is outer, and hence such is the $*$-derivation $\text{ad}(ia)$. The rest of this section is devoted to prove that, conversely, if $A$ has an outer $*$-derivation, then $A$ has Property AEP.

The following lemma shows that primitive nonsimple C*-algebras have Property AEP. The main ingredients of the proof are borrowed from Lemma 6 of [10] and Lemma 3.2 of [2].

**Lemma 4.5.** If $A$ is a primitive nonsimple C*-algebra, then it has Property AEP.

**Proof.** Fix a faithful irreducible representation $\pi : A \rightarrow B(H)$. By Theorem 3.7.7 of [33] $\pi$ extends to a $\sigma$-weakly continuous representation $\pi^{**} : A^{**} \rightarrow B(H)$. Fix a dense sequence $(a_n)_{n \in \mathbb{N}}$ in the unit ball of $A$ and a strictly positive contraction $b_0$ of $A$ (see Proposition II.4.2.1 of [4] for a characterization of strictly positive elements). As in the proof of Lemma 3.2 in [2], one can define a sequence $x = (x_n)_{n \in \mathbb{N}}$ of pairwise orthogonal projections such that for some $\varepsilon > 0$ and every $k, n \in \mathbb{N}$ such that $k \leq n$,

- $\| x_n b_0 \| > \varepsilon$;
- $\| [x_n, a_k] \| < 2^{-n}$.

Now suppose by contradiction that the equivalence relation $E_x$ has a nonmeager equivalence class $X$. Thus for every $t, s \in X$ the automorphism

$$\alpha_{t,s} = \text{Ad}\left( \exp\left( i \sum_{n \in \mathbb{N}} (t_n - s_n) x_n \right) \right)$$
is inner. Fix \( s, t \in X \). Observe that \( \alpha_{t,s} \) is the exponential of the *-derivation

\[
\delta_{t,s} = \text{ad} \left( i \sum_{n \in \mathbb{N}} (t_n - s_n)x_n \right).
\]

By Lemma 4.1 the *-derivation \( \delta_{t,s} \) is inner. Thus, there is an element \( z_{t,s} \) of the center of the enveloping von Neumann algebra of \( A \) such that

\[
\sum_{n \in \mathbb{N}} (t_n - s_n)x_n + z_{t,s} \in M(A).
\]

The image of a central element of \( A^{**} \) under \( \pi \) belongs to the relative commutant of \( \pi[A] \) in \( B(H) \), which consists only of scalar multiples of the identity by II.6.1.8 of [4]. Thus,

\[
\pi \left( \sum_{n \in \mathbb{N}} (t_n - s_n)x_n \right) \in \pi^{**} [M(A)].
\]

Hence

\[
\pi \left( b_0 \sum_{n \in \mathbb{N}} (t_n - s_n)x_n \right) \in \pi[A].
\]

By Lemma 4.1 one can find an uncountable subset \( Y \) of \( X \) such that any pair of distinct elements of \( Y \) has uniform distance at least \( \frac{1}{4} \). Fix \( s \in Y \). For all \( t, t' \in Y \), there is \( m \in \mathbb{N} \) such that

\[
|t_m - t'_m| \geq \frac{1}{4}.
\]

Henceforth,

\[
\| \pi \left( b_0 \left( \sum_{n \in \mathbb{N}} (t_n - s_n)x_n \right) \right) - \pi \left( b_0 \left( \sum_{n \in \mathbb{N}} (t'_n - s_n)x_n \right) \right) \| = \| \pi \left( b_0 \sum_{n \in \mathbb{N}} (t_n - t'_n)x_n \right) \| = \| b_0 \sum_{n \in \mathbb{N}} (t_n - t'_n)x_n \| \geq \| b_0 \sum_{n \in \mathbb{N}} (t_n - t'_n)x_n x_m a_0 \| \geq |t_m - t'_m| \|(x_m b_0)^*(x_m b_0)\| \geq \frac{\varepsilon^2}{4}.
\]

Since \( Y \) is uncountable this contradicts the separability of \( \pi [A] \). \( \square \)

In order to prove Property AEP for all C*-algebra with outer *-derivations we need the fact that Property AEP is liftable. This means that if a *-homomorphic image of a C*-algebra \( A \) has Property AEP, then \( A \) has Property AEP. (For an exhaustive introduction to liftable properties the reader is referred to Chapter 8 of [26].)

**Lemma 4.6.** If \( \pi : A \to B \) is a surjective *-homomorphism and \( B \) has Property AEP, then \( A \) has Property AEP.
Proof. Suppose that \((a_n)_{n \in \mathbb{N}}\) is a dense sequence in \(A\). Thus, \((\pi(a_n))_{n \in \mathbb{N}}\) is a dense sequence in \(B\). Pick a sequence \((y_n)_{n \in \mathbb{N}}\) in \(B\) obtained from \((\pi(a_n))_{n \in \mathbb{N}}\) as in the definition of Property AEP. By Lemma 10.1.12 of [26], there is a sequence \((z_n)_{n \in \mathbb{N}}\) of pairwise orthogonal positive contractions of \(A\) such that \(\pi(z_n) = y_n\) for every \(n \in \mathbb{N}\). Fix an increasing quasicentral approximate unit of \(A\) of \(B\) of pairwise orthogonal positive contractions of \(A\) such that \(\pi(z_n) = y_n\) for every \(n \in \mathbb{N}\). Denote by \(\pi\) of Theorem 4.2 of [1].

Thus, \(\pi\) is outer. Suppose that \(\pi\) is outer. Thus, there is \(k_n \in \mathbb{N}\) such that

\[
x_n = \frac{1}{z_n} (1 - u_{k_n}) z_n,
\]

then

\[
\|x_n a_i - a_i x_n\| < 2^{-n}
\]

for every \(i \leq n\). Observe that \((x_n)_{n \in \mathbb{N}}\) is a sequence of pairwise orthogonal positive contractions of \(A\). Moreover, if \(E \subset (0, 1)^{\mathbb{N}}\) is nonmeager, consider \(s, t \in E\) such that the automorphism

\[
\text{Ad} \left( \exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) y_n \right) \right)
\]

of \(B\) is outer. We claim that the automorphism

\[
\text{Ad} \left( \exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) x_n \right) \right)
\]

of \(A\) is outer. Suppose that this is not the case. Thus, there is \(z\) in the center of the enveloping von Neumann algebra of \(A\) such that

\[
\exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) x_n \right) + z \in U(A).
\]

Denoting by \(\pi^* : A^{**} \to B^{**}\) the normal extension of \(\pi\) (see III.5.2.10 of [4]), one has that

\[
\exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) y_n \right) + \pi^* (z) = \pi^* \left( \exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) x_n \right) + z \right) \in U(B)
\]

by Theorem 4.2 of [1]. Since \(\pi^* (z)\) belongs to the center of the enveloping von Neumann algebra of \(B\),

\[
\exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) y_n \right) + \pi^* (z)
\]

is a unitary multiplier of \(B\) that implements

\[
\text{Ad} \left( \exp \left( \sum_{n \in \mathbb{N}} (t_n - s_n) y_n \right) \right).
\]
Hence, the latter automorphism of $B$ is inner, contradicting the assumption. □

Liftability of Property AEP allows one to easily bootstrap Property AEP from primitive nonsimple C*-algebras to C*-algebra whose primitive spectrum is not $T_1$.

**Lemma 4.7.** If $A$ is a C*-algebra whose primitive spectrum $\hat{A}$ is not $T_1$, then $A$ has Property AEP.

**Proof.** Since $\hat{A}$ is not $T_1$, by 4.1.4 of [33] there is an irreducible representation $\pi$ of $A$ whose kernel is not a maximal ideal. This implies that the image of $A$ under $\pi$ is a nonsimple primitive C*-algebra. By Lemma 4.5 the latter C*-algebra has Property AEP. Therefore, being Property AEP liftable by Lemma 4.6, $A$ has Property AEP. □

In order to show that a C*-algebra $A$ has Property AEP, it is sometimes easier to show that it has a stronger property that we will refer to as Property AEP$^+$. Property AEP$^+$ appears, without being explicitly defined, in the proofs of Lemma 17, Theorem 18, and the main Theorem of [10], as well as in the proofs of Lemma 3.5 and Lemma 3.6 of [2].

Recall that a bounded sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $A$ is called central if for every $a \in A$,

$$\lim_{n \to +\infty} \|[x_n, a]\| = 0.$$  

The beginning of Section 5 contains a discussion about the notion of central sequence, the related notion of hypercentral sequence, and their basic properties.

**Definition 4.8.** A C*-algebra $A$ has Property AEP$^+$ if there is a sequence $(\pi_n)_{n \in \mathbb{N}}$ of irreducible representations of $A$ such that, for some positive contraction $b_0$ of $A$ and a central sequence $(x_n)_{n \in \mathbb{N}}$ of pairwise orthogonal positive contractions of $A$:

- the sequence
  $$(\pi_n((x_n - \lambda)b_0))_{n \in \mathbb{N}}$$
  does not converge to 0 for any $\lambda \in \mathbb{C}$;
- $x_m \in \operatorname{Ker}(\pi_n)$ for every pair of distinct natural numbers $n, m$.

To prove that Property AEP$^+$ is stronger than Property AEP we will need the following lemma:

**Lemma 4.9.** Fix a strictly positive real number $\eta$. For every $\epsilon > 0$ there is $\delta > 0$ such that for every C*-algebra $A$ and every pair of positive contractions $x, b$ of $A$ such that $\|a\| \geq \eta$, if

$$\|(\exp(it) - \mu)b\| \leq \delta$$

for some $\mu \in \mathbb{C}$ then

$$\|(x - \lambda)b\| \leq \epsilon$$

for some $\lambda \in \mathbb{C}$.

**Proof.** Fix $\epsilon > 0$. Pick a polynomial

$$p(Z) = \rho_0 + \rho_1 Z + \ldots + \rho_n Z^n$$

such that

$$|p(\exp(it)) - t| \leq \frac{\epsilon}{2}$$
for every \( t \in [0,1] \). If \( \mu \in \mathbb{C} \) is such that \( |\mu| \leq \frac{2}{\eta} \), define \( p_\mu(Z) \) to be the polynomial in \( Z \) obtained by \( p(Z) \) replacing the indeterminate \( Z \) by \( Z + \mu \). Observe that the \( j \)-th coefficient of \( p_\mu(Z) \) is
\[
\rho_j^\mu = \sum_{i=j}^{n} \rho_i \binom{i}{j} \mu^{j-i}
\]
for \( 0 \leq j \leq n \). Finally define
\[
C = \sum_{1 \leq j \leq i \leq n} |\rho_i| \binom{i}{j} \left( \frac{3}{\eta} \right)^{j-i} \left( \frac{2}{\eta} \right)^{j-i}
\]
and
\[
\delta = \min \left\{ \frac{\varepsilon}{2C}, 1 \right\}.
\]
Suppose that \( A \) is a C*-algebra and \( x, b \in A \) are positive contractions such that \( \|a\| \geq \eta \) and, for some \( \mu \in \mathbb{C} \),
\[
\|(\exp(ix) - \mu)b\| \leq \delta.
\]
Thus,
\[
|\mu| \leq \frac{2}{\eta}.
\]
Moreover
\[
\|(x - \rho_0^\mu)b\| = \|p(\exp(ix)) - \rho_0^\mu b\| + \frac{\varepsilon}{2}
\]
\[
= \left\| \left( \sum_{j=1}^{n} \rho_j^\mu (\exp(ix) - \mu)^j \right) b \right\| + \frac{\varepsilon}{2}
\]
\[
\leq \sum_{j=1}^{n} |\rho_j^\mu| \|\exp(ix) - \mu\|^{j-1} \delta + \frac{\varepsilon}{2}
\]
\[
\leq \sum_{j=1}^{n} \sum_{i=j}^{n} |\rho_i| \binom{i}{j} \left( \frac{2}{\eta} \right)^{j-i} \left( \frac{3}{\eta} \right)^{j-1} \delta + \frac{\varepsilon}{2}
\]
\[
\leq C \delta + \frac{\varepsilon}{2} \leq \varepsilon.
\]
This concludes the proof. \( \square \)

We can now prove that Property AEP\(^+\) is stronger than property AEP.

**Proposition 4.10.** If a C*-algebra \( A \) has Property AEP\(^+\), then it has property AEP.

**Proof.** Suppose that \( (\pi_n)_{n \in \mathbb{N}} \) is a sequence of irreducible representations of \( A \), \( b_0 \) is a positive contraction of \( A \) of norm 1, and \( (x_n)_{n \in \mathbb{N}} \) is a sequence of orthogonal positive elements of \( A \) as in the definition of Property AEP\(^+\). Fix a dense sequence \( (a_n)_{n \in \mathbb{N}} \) in the unit ball of \( A \). After passing to a subsequence of the sequence \( (x_n)_{n \in \mathbb{N}} \), we can assume that for some \( \delta > 0 \), for every \( \lambda \in \mathbb{C} \) and every \( n \in \mathbb{N} \),
\[
\|\pi_n((x_n - \lambda)b_0)\| \geq \delta
\]
and
\[
\|[x_n, a_1]\| < 2^{-n}
\]
for $i \leq n$. Thus, for every $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, and $t \in \left(\frac{1}{4}, 1\right)$,
\[
\|\pi_n((tx_n - \lambda)b_0)\| \geq \frac{\delta}{4}.
\]
Observe that, in particular,
\[
\|\pi_n(b_0)\| \geq \delta
\]
for every $n \in \mathbb{N}$. By Lemma 4.9, this implies that for some $\varepsilon > 0$, for every $t \in \left(\frac{1}{4}, 1\right)$, $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$,
\[
\|\pi_n((\exp(itx_n) - \mu)b_0)\| \geq \varepsilon.
\]
Assume by contradiction that there is a nonmeager subset $X$ of $(0, 1)^\mathbb{N}$ such that for every $s, t \in X$, the automorphism
\[
\text{Ad} \left( \exp \left( i \sum_{n \in \mathbb{N}} (t_n - s_n)x_n \right) \right)
\]
of $A$ is inner. If $s, t \in X$, then there is an element $z_{t,s}$ in the center of the enveloping von Neumann algebra of $A$ such that
\[
\exp \left( i \sum_{n \in \mathbb{N}} (t_n - s_n)x_n + z_{t,s} \right)
\]
multiplies $A$. Hence
\[
y_{t,s} = \exp \left( i \sum_{n \in \mathbb{N}} (t_n - s_n)x_n + z_{t,s} \right) b_0
\]
is an element of $A$. By Lemma 3.4 one can find an uncountable subset $Y$ of $X$ such that, for any $t, s \in Y$, there is $m \in \mathbb{N}$ such that
\[
|t_m - s_m| \geq \frac{1}{4}.
\]
Fix $s \in Y$ and observe that, for $t, t' \in Y$,
\[
\pi_{n_0}(\exp(z_{t',s} - z_{t,s})) = \mu 1
\]
is a scalar multiple of the identity. Therefore
\[
\|y_{t,s} - y_{t',s}\| = \left\| \left( \exp \left( i \sum_{n \in \mathbb{N}} (t_n - t'_n)x_n \right) - \exp(z_{t',s} - z_{t,s}) \right) a_0 \right\| \geq \|\pi_{n_0}((\exp((t_{n_0} - t'_{n_0})x_n) - \mu)a_0)\| \geq \varepsilon.
\]
This contradicts the separability of $A$. \(\square\)

The proofs of Lemma 4.11 and Lemma 4.12 are contained, respectively, in the proofs of Lemmas 3.6 and 3.7 of [2] and in the proof of the implication $(i) \Rightarrow (ii)$ at page 139 of [10].

Recall that a point $x$ of a topological space $X$ is called separated if, given any point $y$ of $X$ distinct from $x$, there are disjoint open neighborhoods of $x$ and $y$. 

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Lemma 4.11. Suppose that $A$ is a C*-algebra whose primitive spectrum $\hat{A}$ is $T_1$. Consider a sequence $(\xi_n)_{n \in \mathbb{N}}$ of separated points in $\hat{A}$. Define $F$ to be the set of limit points of the sequence $(\xi_n)_{n \in \mathbb{N}}$ and $I$ to be the closed self-adjoint ideal of $A$ corresponding to $F$. If either the quotient $A/I$ does not have continuous trace, or the multiplier algebra of $A/I$ has nontrivial center, then $A$ has Property AEP$^+$.

Lemma 4.12. If $A$ is a C*-algebra whose spectrum $\hat{A}$ is homeomorphic to the one-point compactification of a countable discrete space, then $A$ has Property AEP$^+$.

We can now prove the main result of this section that Property AEP as defined in [4.3] is equivalent to having an outer *-derivation.

Theorem 4.13. If $A$ is a C*-algebra, the following statements are equivalent:

1. $A$ has an outer derivation;
2. $A$ has Property AEP.

Proof. We have already pointed out that Property AEP implies the existence of an outer *-derivation. It remains only to show the converse. Suppose that $A$ has an outer derivation. By Lemma 16 of [10], either there is a quotient $B$ of $A$ whose spectrum $\hat{B}$ is homeomorphic to the one point compactification of a countable discrete space, or the primitive spectrum $\hat{A}$ of $A$ is not Hausdorff. In the first case, $A$ has Property AEP by virtue of Lemma 4.12 and Lemma 4.6. Suppose that, instead, the primitive spectrum $\hat{A}$ of $A$ is not Hausdorff. If $\hat{A}$ is not even $T_1$, the conclusion follows from Lemma 4.7. Suppose now that $\hat{A}$ is $T_1$. Since $\hat{A}$ is not Hausdorff, there are two points $\rho_0, \rho_1$ of $\hat{A}$ that do not admit any disjoint open neighbourhoods. By separability, and since separated points are dense in $\hat{A}$ by Proposition 1 of [7], one can find a sequence $(\xi_n)_{n \in \mathbb{N}}$ of separated points of $\hat{A}$ whose set $F$ of limit points contains both $\rho_0$ and $\rho_1$. Define $I$ to be the closed self-adjoint ideal $I$ of $A$ corresponding to the closed subset $F$ of $\hat{A}$. By Lemma 3.1 of [2], either $A/I$ does not have continuous trace or the multiplier algebra of $A/I$ has nontrivial center. In either cases, it follows that $A$ has Property AEP$^+$ and, in particular, the weaker Property AEP by Lemma 4.11. $\square$

5. The case of algebras with only inner derivations

In this section we will prove that, if a C*-algebra $A$ with only inner derivations does not have continuous trace, then the relation of unitary equivalence of approximately inner automorphisms of $A$ is not classifiable by countable structures. In proving this fact we will also show that any such C*-algebra contains a central sequence that is not hypercentral.

If $A$ is a C*-algebra, denote by $A^\infty$ the quotient of the direct product $\prod_{n \in \mathbb{N}} A$ by the direct sum $\bigoplus_{n \in \mathbb{N}} A$ (defined as in [14] II.8.1.2.). Identifying as it is customary $A$ with the algebra of elements of $A^\infty$ admitting constant representative sequence, denote by $A_\infty$ the relative commutant

$$A' \cap A^\infty = \{ x \in A^\infty \mid \forall y \in A, \ [x, y] = 0 \}.$$ 

Finally define

$$\text{Ann}(A, A_\infty) = \{ x \in A_\infty \mid \forall y \in A, \ xy = 0 \}$$

to be the annihilator ideal of $A$ in $A_\infty$. Observe that, if $A$ is unital, then $\text{Ann}(A, A_\infty)$ is the trivial ideal.
A central sequence in a C*-algebra $A$ is a bounded sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $A$ that asymptotically commute with any element of $A$. This means that for any $a \in A$,

$$\lim_{n \to +\infty} \|[x_n, a]\| = 0.$$ 

Equivalently the image of $(x_n)_{n \in \mathbb{N}}$ in the quotient of $\prod_{n \in \mathbb{N}} A$ by $\bigoplus_{n \in \mathbb{N}} A$ belongs to $A_\infty$. From the last characterization it is clear that if $(x_n)_{n \in \mathbb{N}}$ is a central sequence of elements $A$ with spectra contained in some subset $D$ of $\mathbb{C}$ and $f : D \to \mathbb{C}$ is a continuous function such that $f(0) = 0$, then the sequence $(f(x_n))_{n \in \mathbb{N}}$ is central. It is not difficult to verify that, if $(x_n)_{n \in \mathbb{N}}$ is a central sequence and $b \in M(A)$, then the sequence $(\langle b, x_n \rangle)_{n \in \mathbb{N}}$ converges strictly to 0.

If $A$ is unital, a central sequence $(x_n)_{n \in \mathbb{N}}$ is called hypercentral (see [35] Section 1) if it asymptotically commutes with any other central sequence. This amounts to say that for any other central sequence $(y_n)_{n \in \mathbb{N}}$

$$\lim_{n \to +\infty} \|[x_n, y_n]\| = 0.$$ 

Equivalently the image of $(x_n)_{n \in \mathbb{N}}$ in the quotient of $\prod_{n \in \mathbb{N}} A$ by $\bigoplus_{n \in \mathbb{N}} A$ belongs to the center of $A_\infty$. In order to generalize the notion of hypercentral sequence to the nonunital setting it is convenient for our purposes to consider the strict topology rather than the norm topology. Henceforth we give the following definition:

**Definition 5.1.** If $A$ is a (not necessarily unital) C*-algebra, a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $A$ is called hypercentral if it is central and, for any other central sequence $(y_n)_{n \in \mathbb{N}}$, the sequence

$$(\langle x_n, y_n \rangle)_{n \in \mathbb{N}}$$

converges to 0 in the strict topology.

Observe that a central sequence $(x_n)_{n \in \mathbb{N}}$ is hypercentral if and only if the image of the element of $A_\infty$ having $(x_n)_{n \in \mathbb{N}}$ as representative sequence in the quotient $A_\infty / \text{Ann}(A, A_\infty)$ belongs to the center of $A_\infty / \text{Ann}(A, A_\infty)$. It is clear from this characterization that, if $(x_n)_{n \in \mathbb{N}}$ is a hypercentral sequence of elements of $A$ with spectra contained in some subset $D$ of $\mathbb{C}$ and $f : D \to \mathbb{C}$ is a complex-valued continuous function such that $f(0) = 0$, then the sequence $(f(x_n))_{n \in \mathbb{N}}$ is hypercentral. When $A$ is unital the ideal $\text{Ann}(A, A_\infty)$ is trivial, and hence this definition agrees with the usual definition of hypercentral sequence.

The fact that a unital simple C*-algebra contains a central sequence that is not hypercentral is a particular case of Theorem 3.6 of [35]. We will show here how one can generalize this fact to all simple nonelementary C*-algebras. The proof deeply relies on ideas from [35].

**Lemma 5.2.** If $(x_n)_{n \in \mathbb{N}}$ is a hypercentral sequence in $A$ and $\alpha$ is an approximately inner automorphism of $A$, then $(\alpha(x_n) - x_n)_{n \in \mathbb{N}}$ converges strictly to 0.

**Proof.** Suppose that $\varepsilon > 0$ and $\alpha$ is an element of $A$. Since $(x_n)_{n \in \mathbb{N}}$ is a hypercentral sequence, by strict density of the unit ball of $A$ in the unit ball of $M(A)$ (see [4] II.7.3.11 and [25] Proposition 1.4) there is a finite subset $F$ of the unit ball of $A$, a positive real number $\delta$, and a natural number $n_0$ such that, for every $n \geq n_0$ and every $y$ in the unit ball $M(A)$ such that $\|[y, z]\| < \delta$ for every $z \in F$,

$$\max \{|\alpha(x_n y - y x_n)|, |(x_n y - y x_n)\alpha|\} \leq \varepsilon.$$
Consider the open neighbourhood
\[ U = \{ \alpha \in \text{Aut}(A) \mid \| \alpha(x) - x \| < \delta \text{ for every } x \in F \} \]
of $id_A$ in Aut($A$). Observe that if $\beta \in U$ is inner, then for every $n \geq n_0$
\[ \| (\beta(x_n) - x_n)a \| \leq \varepsilon \]
and
\[ \| a(\beta(x_n) - x_n) \| \leq \varepsilon. \]
Approximating with inner automorphisms, one can see that the same is true if $\beta \in U$ is just approximately inner. Since $\alpha$ is approximately inner, there is a unitary multiplier $u$ of $A$ and an approximately inner automorphism $\beta$ of $A$ in $U$ such that
\[ \alpha = \beta \circ \text{Ad}(u). \]
Consider a natural number $n_1 \geq n_0$ such that, for $n \geq n_1$,
\[ \| \beta^{-1}(a)[x_n, u] \| \leq \varepsilon \]
and
\[ \| [x_n, u^*]\beta^{-1}(a) \| \leq \varepsilon. \]
It follows that, if $n \geq n_1$,
\[ \| a(\alpha(x_n) - x_n) \| \leq \| a\beta(\text{Ad}(u)(x_n) - x_n) \| + \| \beta(x_n) - x_n \| \]
\[ \leq \| \beta^{-1}(a)(ux_nu^* - x_n) \| + \varepsilon \]
\[ = \| \beta^{-1}(a)[x_n, u] \| + \varepsilon \]
\[ \leq 2\varepsilon \]
and, analogously,
\[ \| (\alpha(x_n) - x_n)a \| \leq 2\varepsilon. \]
Since $\varepsilon$ was arbitrary, this concludes the proof of the fact that
\[ (a(x_n) - x_n)_{n \in \mathbb{N}} \]
converges strictly to 0.

If $\alpha$ is an automorphism of a $C^*$-algebra $A$, then $\alpha^{**}$ denotes the unique extension of $\alpha$ to a $\sigma$-weakly continuous automorphism of the enveloping von Neumann algebra $A^{**}$ of $A$ (defined as in Proposition III.5.2.10 of [4]).

**Lemma 5.3.** Suppose that $A$ is a $C^*$-algebra such that every central sequence in $A$ is hypercentral. If $\alpha$ is an approximately inner automorphism of $A$, then $\alpha^{**}$ fixes pointwise the center of $A^{**}$, i.e. $\alpha^{**}(z) = z$ for every central element of $A^{**}$.

**Proof.** Observe that $z$ derives $A$, since
\[ za - az = 0 \in A \]
for every $a \in A$. Thus, by Lemma 1.1 of [2], there is a bounded net $(z_\lambda)$ in $A$ converging strongly to $z$ such that, for every $a \in A$,
\[ \lim_{\lambda} \| [z_\lambda - z, a] \| = 0. \]
Recall that strong and $\sigma$-strong topology agree on bounded sets, and that the $\sigma$-strong topology is stronger than the $\sigma$-weak topology (see Definition I.3.1.1 of [4]). Thus the net $(z_\lambda)$ converges $a$ fortiori $\sigma$-weakly to $z$. Since the $\sigma$-weak topology on $A^{**}$ is the weak* topology on $A^{**}$ regarded as the dual space of $A^*$, the unit
ball of $A^{**}$ is $\sigma$-weakly compact by Alaoglu’s theorem (Theorem 2.5.2 in [32]). Moreover by Kaplanski’s Density Theorem (Theorem 2.3.3 in [33]) the unit ball of $A$ is $\sigma$-weakly dense in the unit ball of $A^{**}$. As a consequence the unit ball of $A^{**}$ is $\sigma$-weakly metrizable, and the same holds for balls of arbitrary radius. Thus we can find a bounded sequence $(z_n)_{n\in\mathbb{N}}$ in $A$ converging $\sigma$-weakly to $z$ such that, for every $a \in A$,

$$\lim_{n\to+\infty} \| [z_n - z, a] \| = 0.$$ 

Since $[z_n - z, a] = [z_n, a]$ for every $n \in \mathbb{N}$, $(z_n)_{n\in\mathbb{N}}$ is central and hence hypercentral (every central sequence of $A$ is hypercentral by assumption). Being $\alpha^{**}$ a $\sigma$-weakly continuous automorphism of $A^{**}$ extending $\alpha$, $(\alpha(z_n))_{n\in\mathbb{N}}$ converges $\sigma$-weakly to $\alpha^{**}(z)$. It follows from Lemma 5.3 and from the facts that $\alpha$ is approximately inner and the sequence $(z_n)_{n\in\mathbb{N}}$ is hypercentral that the bounded sequence $(z_n - \alpha(z_n))_{n\in\mathbb{N}}$ converges strictly to 0. By Lemma 1.3.1 of [26] and since weak and $\sigma$-weak topology agree on bounded sets, the sequence $(z_n - \alpha(z_n))_{n\in\mathbb{N}}$ converges $\sigma$-weakly to 0. Therefore $z = \alpha^{**}(z)$. □

A $C^*$-algebra is called elementary if it is *-isomorphic to the algebra of compact operators on some Hilbert space (see Definition IV.1.2.1 in [4]). By Corollary 1 of Theorem 1.4.2 in [3] any elementary $C^*$-algebra is simple. Moreover by Corollary 3 of Theorem 1.4.4 in [3] any automorphism of an elementary $C^*$-algebra is inner; in particular the group $\text{Inn}(A)$ of inner automorphisms of an elementary $C^*$-algebra $A$ is closed inside the group $\text{Aut}(A)$ of all automorphisms. Conversely if the group of inner automorphisms of a simple $C^*$-algebra $A$ is closed, then $A$ is elementary by Theorem 3.1 of [34] together with Corollary IV.1.2.6 and Proposition IV.1.4.19 of [4].

Recall that in this paper all $C^*$-algebras (apart from multiplier algebras and enveloping von Neumann algebras) are assumed to be norm separable. In particular separability of $A$ is assumed in Proposition 5.4; however we do not know if the separability assumption is necessary there.

**Proposition 5.4.** If $A$ is a simple $C^*$-algebra such that every central sequence in $A$ is hypercentral, then $A$ is elementary.

**Proof.** It is enough to show that $\text{Inn}(A)$ is closed in $\text{Aut}(A)$ or, equivalently, that no outer automorphism is approximately inner. Fix an outer automorphism $\alpha$ of $A$. Since $A$ is simple, by [28] Corollary 2.3, there is an irreducible representation $\pi$ such that $\pi$ and $\pi \circ \alpha$ are not equivalent. If $z$ is the central cover of $\pi$ in $A^{**}$ (defined as in [33] 3.8.1), then $\alpha^{**}(z)$ is the central cover of $\pi \circ \alpha$; moreover, being $\pi$ and $\pi \circ \alpha$ not equivalent, $\alpha^{**}(z)$ is different from $z$ by Theorem 3.8.2 of [33]. Thus $\alpha^{**}$ does not fixes pointwise the center of $A^{**}$ and, by Lemma 5.3 $\alpha$ is not approximately inner. □

Proposition 5.4 shows that any simple nonelementary $C^*$-algebra contains a central sequence that is not hypercentral. It is clear that the same conclusion holds for any $C^*$-algebra containing a simple nonelementary $C^*$-algebra as a direct summand. By Theorem 3.9 of [2], this class of $C^*$-algebras coincides with the class of $C^*$-algebras that have only inner derivations and do not have continuous trace. This concludes the proof of the following proposition:
Proposition 5.5. If $A$ is a C*-algebra that does not have continuous trace and has only inner derivations, then $A$ contains a central sequence that is not hypercentral.

In view of this result, in order to finish the proof of Theorem 1.1 it is enough to show that its conclusion holds for a C*-algebra $A$ containing a central sequence that is not hypercentral.

**Proposition 5.6.** If $A$ is a C*-algebra containing a central sequence that is not hypercentral, then the approximately inner automorphisms of $A$ are not classifiable by countable structures up to unitary equivalence.

**Proof.** Fix a dense sequence $(a_n)_{n \in \mathbb{N}}$ in the unit ball of $A$. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a central sequence in $A$ that is not hypercentral. Thus there is a central sequence $(y_n)_{n \in \mathbb{N}}$ in $A$ such that the sequence $(\langle x_n, y_n \rangle)_{n \in \mathbb{N}}$

does not converge strictly to 0. This implies that, for some positive contraction $b$ in $A$, then the sequence $(\langle b x_n, y_n \rangle)_{n \in \mathbb{N}}$

does not converge to 0 is norm. Without loss of generality we can assume that, for every $n \in \mathbb{N}$, $x_n$ and $y_n$ are positive contractions. Observe that the sequence $(\exp(itx_n) - 1)_{n \in \mathbb{N}}$ is not hypercentral for any $t \in (0, 1)$. After passing to subsequences, we can assume that for some strictly positive real number $\varepsilon$, for every $t \in (0, 1)$, every $s \in (\frac{1}{t}, 1)$, and every $n, k \in \mathbb{N}$ such that $k \leq n$:

- $\| [a_k, \exp(itx_n)] \| < 2^{-n}$;
- $\| b x_n, y_n \| \geq \varepsilon$;
- $\| b \exp(isx_n), y_n \| \geq \varepsilon$.

Define $\eta = \frac{\varepsilon}{2^n}$. After passing to a further subsequence, we can assume that, for every $t \in (0, 1)$ and every $n, k \in \mathbb{N}$ such that $k \leq n$:

- $\| \exp(itx_k), y_n \| < 2^{-n} \eta$;
- $\| [y_k, \exp(itx_n)] \| < 2^{-n} \eta$;
- $\| [\exp(itx_k), \exp(isx_n)] \| < 2^{-n} \eta$.

It is not difficult to verify that, if $t \in (0, 1)^\mathbb{N}$, then the sequence $(\Ad(\exp(itx_n)))_{n \in \mathbb{N}}$

is Cauchy in $\text{Aut}(A)$. Denoting by $f(t)$ its limit, one obtains a function

$f : (0, 1)^\mathbb{N} \to \text{Im}(A)$.

In the rest of the proof we will show that $f$ satisfies the hypothesis of Criterion 3.3. Suppose that $M$ is a Lipschitz constant for the function $t \mapsto \exp(it)$ on $[0, 1]$. If $t, s \in (0, 1)^\mathbb{N}$ and $n \in \mathbb{N}$ are such that $|t_k - s_k| < \varepsilon$ for $k \in \{1, 2, \ldots, n\}$, then it is easy to see that

$\| f(t)(a_k) - f(s)(a_k) \| \leq 2^{-n+1} + \varepsilon M$

for $k \leq n$. This shows that the function $f$ is continuous, particularly, Baire measurable. Moreover, if $t, s \in (0, 1)^\mathbb{N}$ are such that $s - t \in \ell^1$, then the sequence

$(\exp(it_1x_1) \cdots \exp(it_nx_n)\exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n \in \mathbb{N}}$

is Cauchy in $U(A)$, and hence has a limit $u \in U(A)$. It is now readily verified that

$f(t) = \text{Ad}(u) \circ f(s)$.
and hence \( f(t) \) and \( f(s) \) are unitarily equivalent. Finally, suppose that \( C \) is a comeager subset of \( (0, 1)^\mathbb{N} \). Thus, there are \( t, s \in C \) such that \( |t_n - s_n| \in (\frac{1}{2}, 1) \) for infinitely many \( n \in \mathbb{N} \). We claim that \( f(t) \) and \( f(s) \) are not unitarily equivalent. Suppose by contradiction that this is not the case. Thus there is \( u \in U(A) \) such that

\[
f(t) = Ad(u) \circ f(s).
\]

This implies that the sequence

\[
(u \exp(it_1x_1) \cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n \in \mathbb{N}}
\]

in \( U(A) \) is central, i.e. asymptotically commutes (in norm) with any element of \( A \). Fix now any \( n_0 \in \mathbb{N} \) such that \( |t_{n_0} - s_{n_0}| \in (\frac{1}{2}, 1) \) and

\[
\|b[y_n, u]\| < \eta
\]

for \( n \geq n_0 \). Suppose that \( n > n_0 \). Using the fact that the elements \( \exp(it_nx_m) \) and \( \exp(it_kx_k) \) commute up to \( \eta 2^{-m} \) for \( k, m \in \mathbb{N} \), one can show that

\[
b(y_n_0)u \exp(it_1x_1) \cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(-is_1x_1)
\]

is at distance at most \( 5\eta \) from

\[
b(y_n_0) \exp(i(t_{n_0} - s_{n_0})x_{n_0}) \exp(it_1x_1) \cdots \exp(it_{n_0}x_{n_0})
\]

\[
\cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(is_{n_0}x_{n_0}) \cdots \exp(-is_1x_1),
\]

where \( \exp(it_{n_0}x_{n_0}) \) and \( \exp(is_{n_0}x_{n_0}) \) indicate omitted terms in the product. Similarly

\[
b \exp(it_1x_1) \cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(-is_1x_1)y_{n_0}
\]

is at distance at most \( 5\eta \) from

\[
b \exp(i(t_{n_0} - s_{n_0})x_{n_0}) y_{n_0} \exp(it_1x_1) \cdots \exp(it_{n_0}x_{n_0})
\]

\[
\cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(is_{n_0}x_{n_0}) \cdots \exp(-is_1x_1).
\]

Thus, the norm of the commutator of

\[
u \exp(it_1x_1) \cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(-is_1x_1)
\]

and \( y_0 \) is at least

\[
\|b[\exp(i(t_{n_0} - s_{n_0})x_{n_0}), y_{n_0}]\| - 10\eta \geq \varepsilon - 10\eta \geq \frac{\varepsilon}{2}.
\]

This contradicts the fact that the sequence

\[
(u \exp(it_1x_1) \cdots \exp(it_nx_n) \exp(-is_nx_n) \cdots \exp(-is_1x_1))_{n \in \mathbb{N}}
\]

is central and concludes the proof. \( \square \)

6. A dichotomy for derivations

If \( A \) is a C*-algebra, then we denote as in Section 4 by \( \Delta_0(A) \) the separable Banach space of inner derivations of \( A \) endowed with the norm \( \|\cdot\|_{\Delta_0(A)} \) and by \( \overline{\Delta_0(A)} \) the closure of \( \Delta_0(A) \) inside the space \( \Delta(A) \) of derivations of \( A \) endowed with the operator norm. Suppose that \( E_{\Delta(A)} \) is the Borel equivalence relation on \( \overline{\Delta_0(A)} \) defined by

\[
\delta_0 E_{\Delta(A)} \delta_1 \iff \delta_0 - \delta_1 \in \Delta_0(A).
\]
Observe that $E_{\Delta(A)}$ is the orbit equivalence relation associated with the continuous action of the additive group of $\Delta_0(A)$ on $\Delta_0(A)$ by translation.

**Theorem 6.1.** If $A$ is a unital C*-algebra, then the following statements are equivalent:

1. $\Delta_0(A)$ is closed in $\Delta(A)$;
2. $E_{\Delta(A)}$ is smooth;
3. $E_{\Delta(A)}$ is classifiable by countable structures;
4. $A$ has continuous trace.

The equivalence of 1 and 4 follows from Theorem 5.3 of [18] together with the equivalence of 1 and 3 in Theorem 1.2. The implication 1 $\Rightarrow$ 2 follows from Exercise 4.4 of [15]. Trivially 2 is stronger than 3. Finally observe that $\Delta_0(A)$ and $\Delta_0(A)$ satisfy the hypothesis of Lemma 2.1 of [40]. In fact, as discussed at the beginning of Section 5, $\Delta_0(A)$ endowed with the norm

$$\|\text{ad}(ia)\|_{\Delta(A)} = \inf \{\|a - z\| | z \in A' \cap A\}$$

is a separable Banach space. Moreover the inclusion map of $\Delta_0(A)$ in $\Delta_0(A) \subset \Delta(A)$ is bounded with norm at most 2. Thus, if $\Delta_0(A)$ is not closed in $\Delta(A)$, then the continuous action of the additive group $\Delta_0(A)$ on $\Delta_0(A)$ by translation is turbulent. Hjorth’s turbulence theorem recalled at the beginning of Section 3 concludes the proof of the implication 3 $\Rightarrow$ 1.

7. Questions

As pointed out in Section 1, the implication 3 $\Rightarrow$ 1 of Theorem 1.1 does not hold in general. Remark 0.9 of [38] provides an example of a C*-algebra $A$ that has continuous trace such that the group Inn($A$) of inner automorphisms of $A$ is not closed inside Aut($A$). This implies that the automorphisms of $A$ are not concretely classifiable up to unitary equivalence. It would be interesting to know if the automorphisms of $A$ are at least classifiable by countable structures up to unitary equivalence. More generally we would like to suggest the following question:

**Question 7.1.** Is there a C*-algebra $A$ such that the automorphisms of $A$ are classifiable by countable structures but not concretely classifiable?

By Theorem 1.1 and the discussion preceding Theorem 1.2 such C*-algebra would necessarily have continuous trace and spectrum not homotopically equivalent to a compact space. It is clear that Question 7.1 has negative answer if and only the dichotomy expressed by the equivalence of 1 and 2 in Theorem 1.2 holds for any (not necessarily unital) C*-algebra.

It would also be interesting to study the Borel complexity of the equivalence relation of conjugacy inside the automorphism group Aut($A$) of a C*-algebra $A$. Recall that two automorphisms $\alpha, \beta$ of $A$ are conjugate if there is a third automorphism $\gamma$ of $A$ such that $\alpha = \gamma \circ \beta \circ \gamma^{-1}$. Observe that this is the orbit equivalence relation associated with the action of Aut($A$) on itself by conjugation.

It is worth noting that a dichotomy result as in Theorem 1.2 does not hold for the equivalence relation of conjugacy even for unital commutative C*-algebras. If $X$ is a compact metrizable space, denote by $C(X)$ the unital commutative C*-algebra of complex-valued continuous functions on $X$ (a classic result of Gelfand and Naimark asserts that any unital commutative C*-algebra is of this form, see
Theorem II.2.2.4 of \[4\]. Observe that by II.2.2.5 of \[4\] the group Aut(C(X)) of automorphisms of C(X) is isomorphic as a Polish group to the group Homeo(X) of homeomorphisms of X endowed with the topology of pointwise convergence. Theorem 4.9 and Corollary 4.11 of \[17\] assert that the equivalence relation of conjugacy inside Homeo([0,1]) is Borel complete (see Definition 13.1.1 \[15\]); in particular it is classifiable by countable structures, but it is not smooth and not Borel. As a consequence the same is true for the equivalence relation of conjugacy inside the automorphism group of C([0,1]). An analogous result holds for the automorphism group of C(2N) by Theorem 5 of \[6\]. On the other hand the equivalence relation of conjugacy inside the automorphism group of C([0,1]^2) is not classifiable by countable structures by Theorem 4.17 of \[17\].

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