Moduli space of noncommutative flat connections and finite-fold noncommutative coverings

Petr Ivankov
Researcher, Dinamika, Moscow State University, Moscow
E-mail: monster.ivankov@gmail.com

Abstract. If we regard classical differential geometry then the moduli space of flat connections depends on the fundamental group. Here we consider the noncommutative generalization of this result. This work states the correspondence between noncommutative coverings and flat connections.

1. Introduction
In the classical commutative geometry the moduli space of flat connections is given by the following theorem.

Theorem 1. Let $X$ be any smooth manifold, and let $\mathcal{M}_X$ be the space of flat connections. Choose a basepoint $x_i$ in each component of $X$. Then the holonomy provides a natural identification

$$\mathcal{M}_X = \prod_i \text{Hom}(\pi_1(X,x_i), G) / G$$

which is independent of the basepoints, where

- $G$ is the gauge group,
- $G$ acts on $\text{Hom}(\pi_1(X,x_i), G)$ by conjugations.

Here we would like to generalize the Theorem 1. To do it one needs to give noncommutative generalizations of geometric notions of the Theorem 1. Following table contains the mapping between geometric and algebraic notions.
Table 1. The mapping between geometric and algebraic notions

| Geometry                                      | Algebra                                      |
|-----------------------------------------------|----------------------------------------------|
| Riemannian manifold $M$                       | Spectral triple $(C^\infty(M), L^2(M,S), D)$ |
| Topological covering $\tilde{M} \to M$        | Noncommutative covering, $(C(M), C(\tilde{M}), G(\tilde{M} | M))$ |
| Natural structure of Riemannian manifold on the covering space $\tilde{M}$ | Triple $(C^\infty(\tilde{M}), L^2(\tilde{M},S), \tilde{D})$ |
| Group homomorphism $G(\tilde{M} | M) \to GL(n,C)$ | Action $G(\tilde{M} | M) \times \mathbb{C}^n \to \mathbb{C}^n$ |
| Trivial bundle $\tilde{M} \times \mathbb{C}^n$ | Free module $C^\infty(\tilde{M}) \otimes \mathbb{C}^n$ |
| Canonical flat connection on $\tilde{M} \times \mathbb{C}^n$ | Trivial flat connection $C^\infty(\tilde{M}) \otimes \mathbb{C}^n$ |
| Action of $G(\tilde{M} | M)$ on $\tilde{M} \times \mathbb{C}^n$ | Action of $G(\tilde{M} | M)$ on $C^\infty(\tilde{M}) \otimes \mathbb{C}^n$ |
| Quotient space $P = (\tilde{M} \times \mathbb{C}^n)/G(\tilde{M} | M)$ | Invariant module $\mathcal{E} = (C^\infty(\tilde{M}) \otimes \mathbb{C}^n)^{G(\tilde{M} | M)}$ |
| Geometric flat connection on $P$ | Algebraic flat connection on $\mathcal{E}$ |

The following text contains detail discussion about ingredients of the Table 1.

2. Motivation. Preliminaries

2.1. Coverings

Definition 1. [7] Let $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}$ be a continuous map. An open subset $\mathcal{U} \subset \mathcal{X}$ is said to be evenly covered by $\tilde{\pi}$ if $\tilde{\pi}^{-1}(\mathcal{U})$ is the disjoint union of open subsets of $\tilde{\mathcal{X}}$ each of which is mapped homeomorphically onto $\mathcal{U}$ by $\tilde{\pi}$. A continuous map $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}$ is called a covering projection if each point $x \in \mathcal{X}$ has an open neighborhood evenly covered by $\tilde{\pi}$. $\tilde{\mathcal{X}}$ is called the covering space and $\mathcal{X}$ the base space of the covering.
Definition 2. [7] Let \( p : \tilde{X} \rightarrow X \) be a covering. A self-equivalence is a homeomorphism \( f : \tilde{X} \rightarrow \tilde{X} \) such that \( p \circ f = p \). This group of such homeomorphisms is said to be the \emph{group of covering transformations} of \( p \) or the \emph{covering group}. Denote by \( G (\tilde{X} \mid X) \) this group.

Remark 3. Above results are copied from [7]. Below the \emph{covering projection} word is replaced with covering.

2.2. Flat connections in the differential geometry

Here I follow to [5]. Let \( M \) be a manifold and \( G \) a Lie group. A \emph{(differentiable) principal bundle over} \( M \) \emph{with group} \( G \) consists of a manifold \( P \) and an action of \( G \) on \( P \) satisfying the following conditions:

(a) \( G \) acts freely on \( P \) on the right: \((u,a) \in P \times G \mapsto ua = R_a u \in P\);
(b) \( M \) is the quotient space of \( P \) by the equivalence relation induced by \( G \), i.e., \( M = P/G \), and the canonical projection \( \pi : P \rightarrow M \) is differentiable;
(c) \( P \) is locally trivial, that is, every point \( x \) of \( M \) has an open neighborhood \( U \) such that \( \pi^{-1} (U) \) is isomorphic to \( U \times G \) in the sense that there is a diffeomorphism \( \psi : \pi^{-1} (U) \rightarrow U \times G \) such that \( \psi (ua) = (\pi (u), \varphi (a)) \) where \( \varphi \) is a mapping of \( \pi^{-1} (U) \) into \( G \) satisfying \( \psi (ua) = (\varphi (a)) \) for all \( u \in \pi^{-1} (U) \) and \( a \in G \).

A principal fibre bundle will be denoted by \( P (M, G, \pi) \), \( P (M, G) \) or simply \( P \).

Let \( P (M, G) \) be a principal fibre bundle over a manifold with group \( G \). For each \( u \in P \) let \( T_u (P) \) be a tangent space of \( P \) at \( u \) and \( G_u \) the subspace of \( T_u (P) \) consisting of vectors tangent to the fibre through \( u \). A \emph{connection} \( \Gamma \) in \( P \) is an assignment of a subspace \( Q_u \) of \( T_u (P) \) to each \( u \in P \) such that

(a) \( T_u (P) = G_u \oplus Q_u \) \emph{(direct sum)};
(b) \( Q_{ua} = (R_a)_* Q_u \) for every \( u \in P \) and \( a \in G \), where \( R_a \) is a transformation of \( P \) induced by \( a \in G \), \( R_a u = u a \).

Let \( P = M \times G \) be a trivial principal bundle. For each \( a \in G \), the set \( M \times \{a\} \) is a submanifold of \( P \). The canonical flat connection in \( P \) is defined by taking the tangent space to \( M \times \{a\} \) at \( u = (x, a) \) as the horizontal tangent subspace at \( u \). A connection in any principal bundle is called \emph{flat} if every point has a neighborhood such that the induced connection in \( P | U = \pi^{-1} (U) \) is isomorphic with the canonical flat connection.

Corollary 1. \emph{(Corollary II 9.2 [5])} Let \( \Gamma \) be a connection in \( P (M, G) \) such that the curvature vanishes identically. If \( M \) is paracompact and simply connected, then \( P \) is isomorphic to the trivial bundle and \( \Gamma \) is isomorphic to the canonical flat connection in \( M \times G \).

If \( \tilde{\pi} : \tilde{M} \rightarrow M \) is a covering then the \( \tilde{\pi} \)-lift of \( P \) is a principal \( \tilde{P} (\tilde{M}, G) \) bundle, given by

\[
\tilde{P} = \{ (u, \tilde{x}) \in P \times \tilde{M} \mid \pi (u) = \tilde{\pi} (\tilde{x}) \}.
\]

If \( \Gamma \) is a connection on \( P (M, G) \) and \( \tilde{M} \rightarrow M \) is a covering then is a canonical connection \( \tilde{\Gamma} \) on \( \tilde{P} (\tilde{M}, G) \) which is the lift of \( \Gamma \), that is, for any \( \tilde{u} \in \tilde{P} \) the horizontal space \( \tilde{Q}_{\tilde{u}} \) is isomorphically mapped onto the horizontal space \( Q_{\tilde{\pi} (\tilde{u})} \) associated with the connection \( \Gamma \). If \( \Gamma \) is flat then from the Proposition (II 9.3 [5]) it turns out that there is a covering \( \tilde{M} \rightarrow M \) such that \( \tilde{P} (\tilde{M}, G) \)
(which is the lift of $P(M,G)$) is a trivial bundle, so the lift $\tilde{\Gamma}$ of $\Gamma$ is a canonical flat connection (cf. Corollary 1). From the the Proposition (II 9.3 [5]) it follows that for any flat connection $\Gamma$ on $P(M,G)$ there is a group homomorphism $\varphi : G \left( \tilde{M} \mid M \right) \to G$ such that

(a) There is an action $G \left( \tilde{M} \mid M \right) \times \tilde{P} \to \tilde{P} \approx \tilde{M} \times G$ given by

$$g(\tilde{x}, a) = (g\tilde{x}, \varphi(g)a); \forall \tilde{x} \in \tilde{M}, a \in G,$$

(b) There is the canonical diffeomorphism $P = \tilde{P}/G \left( \tilde{M} \mid M \right)$.

(c) The lift $\tilde{\Gamma}$ of $\Gamma$ is a canonical flat connection.

**Definition 4.** In the above situation we say that the flat connection $\Gamma$ is induced by the covering $\tilde{M} \to M$ and the homomorphism $G \left( \tilde{M} \mid M \right) \to G$, or we say that $\Gamma$ comes from $G \left( \tilde{M} \mid M \right) \to G$.

**Remark 5.** The Proposition (II 9.3 [5]) assumes that $\tilde{M} \to M$ is the universal covering however it is not always necessary requirement.

**Remark 6.** If $\pi_1(M,x_0)$ is the fundamental group [7] then there is the canonical surjective homomorphism $\pi_1(M,x_0) \to G \left( \tilde{M} \mid M \right)$. So there exist the composition $\pi_1(M,x_0) \to G \left( \tilde{M} \mid M \right) \to G$. It follows that any flat connection comes from the homomorphisms $\pi_1(M,x_0) \to G$.

Suppose that there is the right action of $G$ on $P$ and suppose that $F$ is a manifold with the left action of $G$. There is an action of $G$ on $P \times F$ given by $a(u,\xi) = (ua, a^{-1}\xi)$ for any $a \in G$ and $(u,\xi) \in P \times F$. The quotient space $P \times_G F = (P \times F)/G$ has the natural structure of a manifold and if $E = P \times_G F$ then $E(M,F,G,P)$ is said to be the fibre bundle over the base $M$, with (standard) fibre $F$, and (structure) group $G$ which is associated with the principal bundle $P$ (cf. [5]). If $P = M \times G$ is the trivial bundle then $E$ is also trivial, that is, $E = M \times F$. If $F = \mathbb{C}^n$ is a vector space and the action of $G$ on $\mathbb{C}^n$ is a linear representation of the group then $E$ is the linear bundle. Denote by $T(M)$ (resp. $T^*(M)$) the tangent (resp. contangent) bundle, and denote by $\Gamma(E)$, $\Gamma(T(M))$, $\Gamma(T^*(M))$ the spaces of sections of $E$, $T(M)$, $T^*(M)$ respectively. Any connection $\Gamma$ on $P$ gives a covariant derivative on $E$, that is, for any section $X \in \Gamma(T(M))$ and any section $\xi \in \Gamma(E)$ there is the derivative given by

$$\nabla_X(\xi) \in \Gamma(E).$$

If $E = M \times \mathbb{C}^n$, $\Gamma$ is the canonical flat connection and $\xi$ is a trivial section, that is, $\xi = M \times \{x\}$ then

$$\nabla_X \xi = 0, \ \forall X \in T(M). \quad (1)$$

For any connection there is the unique map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes T^*(M)) \quad (2)$$

such that

$$\nabla_X \xi = (\nabla \xi, X)$$

where the pairing $\langle \cdot, \cdot \rangle : \Gamma(E \otimes T^*(M)) \times \Gamma(T(M)) \to \Gamma(E)$ is induced by the pairing $\Gamma(T^*(M)) \times \Gamma(T(M)) \to C^\infty(M)$. 



2.3. Noncommutative generalization of connections

The noncommutative analog of manifold is a spectral triple and there is the noncommutative analog of connections.

2.3.1. Connection and curvature

Definition 7. [2]

(a) A cycle of dimension \( n \) is a triple \((\Omega, d, \int)\) where \( \Omega = \bigoplus_{j=0}^{n} \Omega^j \) is a graded algebra over \( \mathbb{C} \), 
\( d \) is a graded derivation of degree 1 such that \( d^2 = 0 \), and \( f : \Omega^n \to \mathbb{C} \) is a closed graded trace on \( \Omega \), 
(b) Let \( \mathcal{A} \) be an algebra over \( \mathbb{C} \). Then a cycle over \( \mathcal{A} \) is given by a cycle \((\Omega, d, \int)\) and a homomorphism \( \mathcal{A} \to \Omega^0 \).

Definition 8. [2] Let \( \mathcal{A} \xrightarrow{\rho} \Omega \) be a cycle over \( \mathcal{A} \), and \( \mathcal{E} \) a finite projective module over \( \mathcal{A} \). Then a connection \( \nabla \) on \( \mathcal{E} \) is a linear map \( \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \) such that

\[
\nabla (\xi x) = \nabla (\xi) x = \xi \otimes d\rho (x); \forall \xi \in \mathcal{E}, \forall x \in \mathcal{A}.
\] (3)

Here \( \mathcal{E} \) is a right module over \( \mathcal{A} \) and \( \Omega^1 \) is considered as a bimodule over \( \mathcal{A} \).

Remark 9. The map \( \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \) is an algebraic analog of the map \( \nabla : \Gamma (\mathcal{E}) \to \Gamma (\mathcal{E} \otimes T^* (\mathcal{M})) \) given by (2).

Proposition 10. [2] Following conditions hold:

(a) Let \( e \in \text{End}_{\mathcal{A}} (\mathcal{E}) \) be an idempotent and \( \nabla \) is a connection on \( \mathcal{E} \); then

\[
\xi \mapsto (e \otimes 1) \nabla \xi
\] (4)

is a connection on \( e\mathcal{E} \),

(b) Any finite projective module \( \mathcal{E} \) admits a connection,

(c) The space of connections is an affine space over the vector space \( \mathcal{H}_{\text{Hom}_{\mathcal{A}}} (\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1) \),

(d) Any connection \( \nabla \) extends uniquely up to a linear map of \( \tilde{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{A}} \Omega \) into itself such that

\[
\nabla (\xi \otimes \omega) = \nabla (\xi) \omega + \xi \otimes d\omega; \forall \xi \in \mathcal{E}, \omega \in \Omega.
\] (5)

A curvature of a connection \( \nabla \) is a (right \( \mathcal{A} \)-linear) map

\[
F_{\nabla} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2
\] (6)

defined as a restriction of \( \nabla \circ \nabla \) to \( \mathcal{E} \), that is, \( F_{\nabla} = \nabla \circ \nabla |_{\mathcal{E}} \). A connection is said to be flat if its curvature is identically equal to 0 (cf. [1]).

Remark 11. Above algebraic notions of curvature and flat connection are generalizations of corresponding geometrical notions explained in [5] and the Section 2.2.

For any projective \( \mathcal{A} \) module \( \mathcal{E} \) there is a trivial connection

\[
\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega,
\]

\[
\nabla = \text{Id}_{\mathcal{E}} \otimes d.
\]

From \( d^2 = d \circ d = 0 \) it follows that \( (\text{Id}_{\mathcal{E}} \otimes d) \circ (\text{Id}_{\mathcal{E}} \otimes d) = 0 \), i.e. any trivial connection is flat.
Lemma 1. If $\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1$ is a trivial connection and $e \in \text{End}_A(\mathcal{E})$ is an idempotent then the given by (4)

$$\xi \mapsto (e \otimes 1) \nabla \xi$$

connection $\nabla_e : e\mathcal{E} \to e\mathcal{E} \otimes \Omega^1$ on $e\mathcal{E}$ is flat.

Proof. From

$$(e \otimes 1)(\text{Id}_E \otimes d) \circ (e \otimes 1)(\text{Id}_E \otimes d) = e \otimes d^2 = 0$$

it turns out that $\nabla_e \circ \nabla_e = 0$, i.e. $\nabla_e$ is flat.

Remark 12. The notion of the trivial connection is an algebraic version of geometrical canonical connection explained in the Section 2.2.

2.3.2. Spectral triples

This section contains citations of [8].

Definition of spectral triples

Definition 13. [8] A (unital) spectral triple $(A, H, D)$ consists of:

- a pre-$C^*$-algebra $A$ with an involution $a \mapsto a^*$, equipped with a faithful representation on:
- a Hilbert space $H$; and also
- a selfadjoint operator $D$ on $H$, with dense domain $\text{Dom}D \subset H$, such that $a(\text{Dom}D) \subset \text{Dom}D$ for all $a \in A$.

There is a set of axioms for spectral triples described in [8].

Noncommutative differential forms

Any spectral triple naturally defines a cycle $\rho : A \to \Omega_D$ (cf. Definition 8). In particular for any spectral triple there is an $A$-module $\Omega^1_D \subset B(H)$ of order-one differential forms which is a linear span of operators given by

$$a \left[ D, b \right] : a, b \in A.$$ (7)

There is the differential map

$$d : A \to \Omega^1_D, \quad a \mapsto \left[ D, a \right].$$ (8)

3. Noncommutative finite-fold coverings

3.1. Coverings of $C^*$-algebras

Definition 14. If $A$ is a $C^*$-algebra then an action of a group $G$ is said to be involutive if $ga^* = (ga)^*$ for any $a \in A$ and $g \in G$. The action is said to be non-degenerated if for any nontrivial $g \in G$ there is $a \in A$ such that $ga \neq a$.

Definition 15. Let $A \to \tilde{A}$ be an injective $\ast$-homomorphism of unital $C^*$-algebras. Suppose that there is a non-degenerated involutive action $G \times \tilde{A} \to \tilde{A}$ of a finite group $G$, such that $A = \tilde{A}^G \overset{\text{def}}{=} \left\{ a \in \tilde{A} \mid a = ga; \forall g \in G \right\}$. There is an $A$-valued product on $\tilde{A}$ given by

$$\langle a, b \rangle_{\tilde{A}} = \sum_{g \in G} g(a^*b)$$ (9)
and $\tilde{A}$ is an $A$-Hilbert module. We say that a triple $\left(\tilde{A}, \tilde{A}, G\right)$ is an unital noncommutative finite-fold covering if $\tilde{A}$ is a finitely generated projective $A$-Hilbert module.

**Remark 16.** Above definition is motivated by the Theorem 2.

**Theorem 2.** [6]. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p: \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

$$(f\xi)(y) = f(y)\xi(p(y)), \; f \in C(\mathcal{Y}), \; \xi \in C(\mathcal{X}),$$

then $p$ is a finite-fold covering.

**3.2. Coverings of spectral triples**

**Definition 17.** Let $(A, \mathcal{H}, D)$ be a spectral triple, and let $A$ be the $C^*$-norm completion of $A$. Let $\left(\tilde{A}, \tilde{\mathcal{H}}, \tilde{D}\right)$ be an unital noncommutative finite-fold covering such that there is the dense inclusion $A \to \tilde{A}$. Let $\tilde{\mathcal{H}} \overset{\text{def}}{=} \tilde{A} \otimes_A \mathcal{H}$ is a Hilbert space such that the Hilbert product $(\cdot, \cdot)_{\tilde{\mathcal{H}}}$ is given by

$$(a \otimes \xi, b \otimes \eta)_{\tilde{\mathcal{H}}} = \frac{1}{|G|} \left(\xi, \sum_{g \in G} g \left(\tilde{a} \cdot \tilde{b}\right)\right)_{\mathcal{H}}; \; \forall \tilde{a}, \tilde{b} \in \tilde{A}, \; \xi, \eta \in \mathcal{H}$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the Hilbert product on $\mathcal{H}$. There is the natural representation $\tilde{A} \to B(\tilde{\mathcal{H}})$. A spectral triple $\left(\tilde{A}, \tilde{\mathcal{H}}, \tilde{D}\right)$ is said to be a $(A, \tilde{A}, G)$-lift of $(A, \mathcal{H}, D)$ if following conditions hold:

(a) $\tilde{A}$ is a $C^*$-norm completion of $A$,

(b) $\tilde{D} \left(1_{\tilde{A}} \otimes_A \xi\right) = 1_{\tilde{A}} \otimes_A D\xi; \; \forall \xi \in \text{Dom}D$,

(c) $\tilde{D} \left(g \tilde{\xi}\right) = g \left(\tilde{D}\xi\right)$ for any $g \in G$.

**Remark 18.** It is proven in [3] that for any spectral triple $(A, \mathcal{H}, D)$ and any unital noncommutative finite-fold covering $(A, \tilde{A}, G)$ there is the unique $(A, \tilde{A}, G)$-lift $(\tilde{A}, \tilde{\mathcal{H}}, \tilde{D})$ of $(A, \mathcal{H}, D)$.

**Remark 19.** It is known that if $M$ is a Riemannian manifold and $\tilde{M} \to M$ is a covering, then $\tilde{M}$ has the natural structure of Riemannian manifold (cf. [5]). The existence of lifts of spectral triples is a noncommutative generalization of this fact (cf. [3])

**4. Construction of noncommutative flat connections**

Let $(A, \mathcal{H}, D)$ be a spectral triple, let $\left(\tilde{A}, \tilde{\mathcal{H}}, \tilde{D}\right)$ is the $(A, \tilde{A}, G)$-lift of $(A, \mathcal{H}, D)$. Let $V = \mathbb{C}^n$ and with left action of $G$, i.e. there is a linear representation $\rho: G \to GL(\mathbb{C}, n)$. Let $\tilde{\mathcal{E}} = \tilde{A} \otimes \mathbb{C}^n \approx \tilde{A}^n$ be a free module over $\tilde{A}$, so $\tilde{\mathcal{E}}$ is a projective finitely generated $A$-module (because $\tilde{A}$ is a finitely generated projective $A$-module). Let $\nabla: \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \Omega_D^1$ be the trivial flat connection. In [3] it is proven that $\Omega_D^1 = \tilde{A} \otimes_A \Omega_D^1$ it follows that the connection $\nabla: \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \Omega_D^1$ can be regarded as a map $\nabla': \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \tilde{A} \otimes_A \Omega_D^1$ i.e. one has a connection

$$\nabla': \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \Omega_D^1.$$

From $\nabla \circ \nabla|_{\tilde{\mathcal{E}}} = 0$ it turns out that $\nabla' \circ \nabla'|_{\tilde{\mathcal{E}}} = 0$, i.e. $\nabla'$ is flat. There is the action of $G$ on $\tilde{\mathcal{E}} = \tilde{A} \otimes \mathbb{C}^n$ given by

$$g (\tilde{a} \otimes x) = g\tilde{a} \otimes gx; \; \forall g \in G, \; \tilde{a} \in \tilde{A}, \; x \in \mathbb{C}^n.$$  \hspace{1cm} (10)
Denote by
\[ \mathcal{E} = G = \{ \xi \in \tilde{\mathcal{E}} \mid G\tilde{\xi} = \tilde{\xi} \} \]  
(11)

Clearly \( \mathcal{E} \) is an \( A-A \)-bimodule. For any \( \tilde{\xi} \in \tilde{\mathcal{E}} \) there is the unique decomposition
\[ \tilde{\xi} = \xi + \xi_\perp, \]
\[ \xi = \frac{1}{|G|} \sum_{g \in G} g\tilde{\xi}, \]
\[ \xi_\perp = \tilde{\xi} - \xi. \]
(12)

From the above decomposition it turns out the direct sum \( \tilde{\mathcal{E}} = \tilde{\mathcal{E}}^G \oplus \mathcal{E}_\perp \) of \( A \)-modules. So \( \mathcal{E} = \tilde{\mathcal{E}}^G \) is a projective finitely generated \( A \)-module, it follows that there is an idempotent \( e \in \text{End}_A \tilde{\mathcal{E}} \) such that \( \mathcal{E} = e\tilde{\mathcal{E}} \). The Proposition 10 gives the canonical connection
\[ \nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1_D \]  
(13)

which is defined by the connection \( \nabla' : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes_A \Omega^1_D \) and the idempotent \( e \). From the Lemma 1 it turns out that \( \nabla \) is flat.

**Definition 20.** We say that \( \nabla \) is a **flat connection induced by noncommutative covering** \( (A, \tilde{A}, G) \) and the linear representation \( \rho : G \to GL(\mathbb{C}, n) \), or we say the \( \nabla \) **comes from the representation** \( \rho : G \to GL(\mathbb{C}, n) \).

It is proven in [4] that any flat connection given by the Definition 20 with commutative \( A \) and \( \tilde{A} \) corresponds to a classical flat connection described in the Section 2.2. This fact enables us to construct the mapping given by the Table 1 and the noncommutative analog of the Theorem 1.

5. Acknowledgments

Author would like to acknowledge members of the Moscow State University seminars "Noncommutative geometry and topology" and "Algebras in analysis" led by professors A. S. Mishchenko and A. Ya. Helemskii for a discussion of this work.

References

[1] Brzeziński T 2006 Flat connections and (co)modules Preprint arXiv:math/0608170
[2] Connes A 1994 *Noncommutative Geometry*, Academic Press, San Diego, CA, 661 p., ISBN 0-12-185860-X
[3] Ivanov P 2017 Coverings of Spectral Triples Preprint math.OA/1705.08651
[4] Ivanov P 2018 Finite Noncommutative Coverings and Flat Connections Preprint arXiv:1801.09587
[5] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry: Volume 1* (Interscience publishers a division of John Willey & Sons: New York - London)
[6] Pavlov A and Troitskii E 2011 *Russian J. of Math. Phys.* 18 338
[7] Spanier E 1966 *Algebraic Topology* (New York McGraw-Hill)
[8] Várilly J 2006 *An Introduction to Noncommutative Geometry* (European Mathematical Society).