WEAK KAM THEORY FOR SUBRIEMANNIAN LAGRANGIANS

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ABSTRACT. We extend weak KAM theory to Lagrangians that are defined only on the horizontal distribution of a sub-Riemannian manifold. The main tool is Tonelli’s theorem which allows dispending on a Lagrangian dynamics.

1. INTRODUCTION

Lagrangians that are not defined on the whole tangent bundle of a manifold arise naturally in mechanics with non-holonomic constraints. We approach the study of such kind of systems based on variational principles, more specifically the study of global minimizing curves and the related weak KAM solutions.

A basic fact that should be established to start, is a Tonelli type theorem. We accomplish that in section 2. In section 3 we consider the infinite horizon discounted problem and obtain some bounds for the discounted value function, uniform respect to the discount. Using those bounds we prove in section 4 the existence of weak KAM solutions. Then we introduce the Lax-Oleinik semigroup, the action potential, the Peierls barrier and the Aubry set, and establish usual properties of these objects. We also prove the long time convergence of the Lax-Oleinik semigroup under a suitable growth condition for the Lagrangian. In section 5 we consider viscosity solutions of the Hamilton-Jacobi equation and prove that weak KAM solutions are viscosity solutions. We don’t know if the converse is true. In section 6 we consider minimizing measures and define the effective Lagrangian and Hamiltonian. We don’t know if the support of any minimizing measure is contained in the Aubry set. In section 7 we observe that the normalized discounted value function converges, as in [DFTZ], to a particular weak KAM solution. In section 8 we consider homogenization of the Hamilton Jacobi following the approach in [CIS].

Several authors have studied some aspects of Calculus of Variations and Hamilton-Jacobi equation in our setting.

Gomes [G] has studied the existence and properties of minimizing measures as well as the solutions of the dual min-max problem.

Manfredi and Stroffolini [MS] gave a Hopf-Lax formula in the Heisenberg group, and Balogh et al [BCP] did it for Carnot groups.

Birindelli and Wigniolli [BW] studied homogenization of the Hamilton-Jacobi equation in the Heisenberg group and Stroffolini [S] did it for Carnot groups.
2. Tonelli’s theorem

2.1. Preliminaries. Let \((M, \mathcal{D}, \langle \cdot, \cdot \rangle)\) be a sub-Riemannian manifold such that \(\mathcal{D}\) is bracket generating. Denote by \(\pi : TM \to M\) the natural projection.

**Definition 1.** We say that

- A continuous curve \(\gamma : [a, b] \to M\) is **absolutely continuous** iff \(\varphi \circ \gamma\) is absolutely continuous for any smooth and compactly supported \(\varphi : M \to \mathbb{R}\).
- An absolutely continuous curve \(\gamma : [a, b] \to M\) is **horizontal** if \(\dot{\gamma}(t) \in \mathcal{D}\) for a.e. \(t \in [a, b]\).
- We denote by \(W^{1,1}_\mathcal{D}([a, b])\) the set of horizontal absolutely continuous curves defined on the interval \([a, b]\).

**Proposition 1.** A continuous curve \(\gamma : [a, b] \to M\) is absolutely continuous if and only if there is a partition \(a = t_0 < t_1 < \cdots < t_p = b\) and charts \((U_i, \varphi_i)\) of \(M\) for \(1 \leq i \leq p\) such that \(\gamma([t_{i-1}, t_i]) \subset U_i, \) and \(\varphi_i \circ \gamma|_{[t_{i-1}, t_i]}\) is absolutely continuous.

For \(\gamma \in W^{1,1}_\mathcal{D}([a, b])\) we define its sub-Riemannian length by

\[ \ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt. \]

From that we define the *Carnot-Caratheodory* distance \(d\) on \(M\) which in turn defines a topology on \(M\) that coincides with the original topology of \(M\) as a manifold.

**Definition 2.**

- A family \(F\) of curves \(\gamma : [a, b] \to M\) is **d-absolutely equicontinuous** if \(\forall \varepsilon > 0, \exists \delta > 0\) s.t. \(\forall \gamma \in F\) and any disjoint subintervals \([a_1, b_1], \ldots, [a_N, b_N]\) of \([a, b]\):

\[ \sum_{i=1}^N b_i - a_i < \delta \implies \sum_{i=1}^N d(\gamma(b_i), \gamma(a_i)) < \varepsilon \]

- A family \(G \subset L^1([a, b], m)\), \(m\) the Lebesgue measure on \([a, b]\), is uniformly integrable if given \(\varepsilon > 0\) there is \(\delta > 0\) such that

\[ f \in G, E \subset [a, b] \text{ measurable, } m(E) < \delta \implies \int_E |f| < \varepsilon. \]

**Remark 1.**

1. A family \(F \subset W^{1,1}_\mathcal{D}([a, b])\) such that \(\{\|\dot{\gamma}\| : \gamma \in F\}\) is uniformly integrable, is d-absolutely equicontinuous.
2. A d-absolutely equicontinuous family is d-equicontinuous.
3. An uniform limit of d-absolutely equicontinuous functions is d-absolutely continuous.

**Theorem 1.** Let \((\gamma_n)_n\) be a sequence in \(W^{1,1}_\mathcal{D}([a, b])\) that converges d-uniformly to \(\gamma\), and such that \((\|\dot{\gamma}_n\|)_n\) is uniformly integrable. Then \(\gamma \in W^{1,1}_\mathcal{D}([a, b])\).
Thus $V$ is continuous. We observe that according to exercise 3.52 in \textit{ABB}, $V(\eta) = \ell(\eta)$.

Let $[c, d] \subset [a, b]$, and $c = t_0 < t_1 < \cdots < t_m = d$, then

$$
\sum_{i=1}^{m} d(\gamma(t_{i-1}), \gamma(t_i)) = \lim_{n \to \infty} \sum_{i=1}^{m} d(\gamma_n(t_{i-1}), \gamma_n(t_i)) \leq \liminf_{n \to \infty} V(\gamma_n|[c, d])
$$

Thus $V(\gamma|[c, d]) \leq \liminf_{n \to \infty} V(\gamma_n|[c, d])$.

Given $\varepsilon > 0$ there is $\delta > 0$ such $n \in \mathbb{N}, m(E) < \delta \implies \int_E \|\dot{\gamma}_n\| < \varepsilon$. Let $]c_1, d_1[, \cdots , ]c_k, d_k[$ be disjoint subintervals of $[a, b]$, with $\sum_k (d_i - c_i) < \delta$, then

$$
0 \leq \sum_{i=1}^{k} v(d_i) - v(c_i) = \sum_{i=1}^{k} V(\gamma|[c_i, d_i]) \leq \sum_{i=1}^{k} \liminf_{n \to \infty} V(\gamma_n|[c_i, d_i])
$$

$$
\leq \liminf_{n \to \infty} \sum_{i=1}^{k} V(\gamma_n|[c_i, d_i]) = \liminf_{n \to \infty} \sum_{i=1}^{k} \int_{c_i}^{d_i} \|\dot{\gamma}_n\| \leq \varepsilon
$$

Step 2. There is reparametrization $\bar{\gamma} : [0, L] \to M$ of $\gamma$, $L = V(\gamma)$, such that $d(\bar{\gamma}(r), \bar{\gamma}(s)) \leq |r - s|$ for all $r, s \in [0, L]$, and so $\gamma \in W^{1,1}_D([a, b])$.

Since $v$ is nondecreasing and absolutely continuous, the function $\phi : [0, L] \to [a, b]$ defined by

$$
\phi(s) = \inf\{t : v(t) \geq s\}
$$

satisfies $v(\phi(s)) = s$. $\phi$ is strictly increasing with a denumerable set of jump discontinuities

$$
\lim_{s \to r^-} \phi(s) = a_n < b_n = \lim_{s \to r^+} \phi(s),
$$

which correspond to the intervals $[a_n, b_n]$ where $v$ is constant. $v$ is a bijection from $[0, 1] - \bigcup[a_n, b_n]$ onto $[0, L]$.

For a Riemannian metric $g$ on $M$ defining a distance $d_g$ there is $C > 0$ such that

$$
C d_g(\gamma(t), \gamma(s)) \leq d(\gamma(t), \gamma(s)) \leq V(\gamma|[t, s]) = v(s) - v(t),
$$

thus, if $v'(t) = 0$ then $\dot{\gamma}(t) = 0$. If $t \in [a, b]$ is a number where $v$ and $\gamma$ are differentiable and $\dot{\gamma}(t) \neq 0$, then $t \notin \bigcup[a_n, b_n]$, and thus $t = \phi(s)$ for a number $s \in [0, L]$ where $\phi$ is continuous. Then

$$
\lim_{r \to s} \frac{\phi(r) - \phi(s)}{r - s} = \lim_{r \to s} \frac{1}{v(\phi(r)) - v(\phi(s))} = \frac{1}{v'(t)},
$$

so that $\phi$ is differentiable at $s$.

Define $\bar{\gamma} : [0, L] \to M$ by $\bar{\gamma}(s) = \gamma(\phi(s))$, for $r < s$ we have

$$
d(\bar{\gamma}(r), \bar{\gamma}(s)) = d(\gamma(\phi(r), \gamma(\phi(s)))) \leq V(\gamma|[\phi(r), \phi(s)]) = v(\phi(s)) - v(\phi(r)) = s - r
$$
By Proposition 3.50 in [ABB] we have that \( \tilde{\gamma} \in W^{1,1}_D([0, L]) \). If \( t \in [a, b] \) is a number where \( v \) and \( \gamma \) are differentiable and \( \dot{\gamma}(t) \neq 0 \), then \( t = \phi(s) \) for a number \( s \in [0, L] \) where \( \phi \) is differentiable, thus so is \( \tilde{\gamma} \) and \( \dot{\gamma}(s) = \gamma(t)/v'(t) \). Thus \( \gamma \in W^{1,1}_D([a, b]) \).

### 2.2. Tonelli’s Theorem

We assume that \( L : D \to \mathbb{R} \) is a \( C^2 \) function satisfying the following properties

**Uniform superlinearity:** For all \( K \geq 0 \) there is \( C(K) \in \mathbb{R} \) such that
\[
L(v) \geq K\|v\| + C(K) \quad \text{for all} \quad v \in D.
\]

**Uniform boundedness:** For all \( R \geq 0 \), we have
\[
A(R) = \sup\{L(v) : \|v\| \leq R\} < +\infty.
\]

**Strict convexity:** there is \( a > 0 \) such that \( \partial_{vv}^2 L(v)(w, w) \geq a \) for all \( v, w \in D \) with \( \|w\| = 1 \), where \( \partial_v L \) denotes the derivative along the fibers.

We define the Energy function \( E : D \to \mathbb{R} \) by \( E(v) = \partial_v L(v) \cdot v - L(v) \). Notice that \( \frac{dE(sv)}{ds} = s\partial_{vv}^2 L(sv)(v, v) \), and therefore
\[
E(v) = E(0) + \int_0^1 \frac{dE(sv)}{ds} ds = -L(0) + \int_0^1 s\partial_{vv}^2 L(sv)(v, v) \geq -A(0) + a\|v\|^2.
\]

For \( \gamma \in W^{1,1}_D([a, b]) \), \( \lambda \geq 0 \) we define the discounted action
\[
A_{L, \lambda}(\gamma) = \int_a^b e^{\lambda t} L(\dot{\gamma}(t)) \, dt.
\]

**Lemma 1.** For \( c \in \mathbb{R} \) let \( \mathcal{F}(c, r) = \{ \gamma \in W^{1,1}_D([a, b]) : A_{L, \lambda}(\gamma) \leq c, \lambda \in [0, r] \} \). Then the families \( \{e^{\lambda t}\|\dot{\gamma}(t)\| : \gamma \in \mathcal{F}(c, r), \lambda \in [0, r]\} \) \( \{\|\dot{\gamma}(t)\| : \gamma \in \mathcal{F}(c, r)\} \) are uniformly integrable.

**Proof.** By the uniform superlinearity, \( L(v) \geq C(0) \). Given \( \varepsilon > 0 \) take \( K > 0 \) such that
\[
\max_{\lambda \in [0, r]} \left( c - C(0) \int_a^b e^{\lambda t} dt \right) < K\varepsilon.
\]

By the uniform superlinearity,
\[
L(v) \geq K\|v\| + C(K).
\]

Let \( \gamma \in \mathcal{F}(c) \) and \( E \subset [a, b] \) be measurable, then
\[
C(K) \int_E e^{\lambda t} dt + K \int_E e^{\lambda t}\|\dot{\gamma}(t)\| \, dt \leq \int_E e^{\lambda t} L(\dot{\gamma}(t)) \, dt
\]
and
\[
C(0) \int_{[a, b] \setminus E} e^{\lambda t} dt \leq \int_{[a, b] \setminus E} e^{\lambda t} L(\dot{\gamma}(t)) \, dt
\]
Adding the inequalities we get
\[
(C(K) - C(0)) \int_E e^{\lambda t} dt + C(0) \int_a^b e^{\lambda t} dt + K \int_E e^{\lambda t} \|\dot{\gamma}(t)\| dt \leq A_{L,\lambda}(\gamma) \leq c
\]
which gives
\[
\int_E e^{\lambda t} \|\dot{\gamma}(t)\| dt \leq \frac{c - C(0)}{K} \int_a^b e^{\lambda t} dt + \frac{C(0) - C(K)}{K} \int_E e^{\lambda t} dt \leq c + \frac{C(0) - C(K)}{K} \int_E e^{\lambda t} dt.
\]
This gives the uniform integrability of \(\{e^{\lambda t} \|\dot{\gamma}(t)\| : \gamma \in \mathcal{F}(c, r), \lambda \in [0, r]\}\). The uniform integrability of \(\{|\ddot{\gamma}(t)| : \gamma \in \mathcal{F}(c, r)\}\) follows from
\[
\int_E e^{\lambda t} \|\dot{\gamma}(t)\| dt \geq e^{\lambda a} \int_E \|\dot{\gamma}\| \geq \min_{\lambda \in [0, r]} e^{\lambda a} \int_E \|\dot{\gamma}\|
\]
\[\square\]

**Theorem 2.** Let \((\gamma_n)_n\) be a sequence in \(W^{1,1}_D([a, b])\) converging d-uniformly to \(\gamma\), \(\lambda_n \geq 0\) converging to \(\lambda\) and
\[
\liminf_{n \to \infty} A_{L,\lambda_n}(\gamma_n) < +\infty.
\]
Then \(\gamma \in W^{1,1}_D([a, b])\) and
\[(1) \quad A_{L,\lambda}(\gamma) \leq \liminf_{n \to \infty} A_{L,\lambda_n}(\gamma_n).
\]

**Proof.** Let \(l = \liminf_{n \to \infty} A_{L,\lambda_n}(\gamma_n)\), extracting subsequence still denoted \(\gamma_n, \lambda_n\) we have \(l = \lim_{n \to \infty} A_{L,\lambda_n}(\gamma_n)\), and forgetting some of the first curves \(\gamma_n\), we can suppose that \(\gamma_n \in \mathcal{F}(l + 1, \lambda + 1)\) for all \(n\). Lemma \([\text{I}]\) implies that \((\|\dot{\gamma}_n\|)_n\) and \((e^{\lambda_n t} \|\dot{\gamma}_n(t)\|)_n\) are uniformly integrable, and Theorem \([\text{I}]\) that \(\gamma \in W^{1,1}_D([a, b])\).

Let us show how we can reduce the proof of \([\text{I}]\) to the case where \(M\) is an open subset of \(\mathbb{R}^d\), \(d = \dim M\) and the horizontal distribution is trivial. The lagrangian \(L\) is bounded below by \(C(0)\). If \([a', b'] \subset [a, b]\), for all \(n\) we have
\[
A_{L,\lambda_n}(\gamma_n|[a', b']) \leq A_{L,\lambda_n}(\gamma_n) - C(0) \int_{[a, b]\setminus[a', b']} e^{\lambda_n t} dt
\]
so that
\[
\liminf_{n \to \infty} A_{L,\lambda_n}(\gamma_n|[a', b']) < +\infty.
\]
Let now the partition \(a_0 = 0 < a_1 < \ldots < a_p = 1\) and \(U_1, \ldots, U_p\) be as in the proof of Theorem \([\text{I}]\) so that the horizontal distribution is trivial on \(U_i\) and \(\gamma([a_{i-1}, a_i]) \subset U_i, i = 1, \ldots, p\). It is enough to prove that
\[
A_{L,\lambda_n}(\gamma|[a_{i-1}, a_i]) \leq \liminf_{n \to \infty} A_{L,\lambda_n}(\gamma_n|[a_{i-1}, a_i])
\]
because that implies

\[ A_{L,\lambda}(\gamma) = \sum_{i=1}^{p} A_{L,\lambda}(\gamma[a_{i-1}, a_i]) \leq \sum_{i=1}^{p} \liminf_{n \to \infty} A_{L,\lambda_n}(\gamma[a_{i-1}, a_i]) \]

\[ \leq \liminf_{n \to \infty} \sum_{i=1}^{p} A_{L,\lambda_n}(\gamma[a_{i-1}, a_i]) = \liminf_{n \to \infty} A_{L,\lambda}(\gamma_n) \]

In the sequel we suppose that \( M = U \) is an open set of \( \mathbb{R}^d \), the horizontal distribution is \( U \times \mathbb{R}^m \) with \( \langle , \rangle \) the euclidean inner product in \( \mathbb{R}^m \), and that \( \gamma([a, b]) \) and all \( \gamma_n([a, b]) \) are contained \( U \). We will write \( \gamma(t) = (\gamma(t), \zeta(t)) \), \( \gamma_n(t) = (\gamma_n(t), \zeta_n(t)) \).

**Lemma 2.** Let \( U \subset \mathbb{R}^d \) be open, \( L \in C^2(U \times \mathbb{R}^m) \) be strictly convex and uniformly superlinear on the second variable with \( \sup\{ L(x, v) : \|v\| \leq R \} < +\infty \) for each \( R > 0 \). Given \( K \subset U \) compact, \( C > 0 \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( x \in K \), \( |x - y| \leq \delta \), \( \|v\| \leq C \) and \( w \in \mathbb{R}^m \), then

\[ L(y, w) \geq L(x, v) + \partial_v L(x, v) \cdot (w - v) - \varepsilon. \]

For a fixed constant \( C \) let

\[ E_C = \{ t \in [a, b] : \|\zeta(t)\| \leq C \} \]

Given \( \varepsilon > 0 \), the compact set \( K = \gamma([a, b]) \cup \bigcup_{n \in \mathbb{N}} \gamma_n[a, b] \) and the constant \( C \) fixed above, we apply Lemma 2 to get \( \delta > 0 \) satisfying its conclusion. By the compactness of \([a, b]\) and the continuity of \( \gamma \), there is \( \eta > 0 \) such that \( t \in [a, b] \), \( d(x, \gamma(t)) < \eta \) imply \( |x - \gamma(t)| < \delta \). Since \( \gamma_n \) converges \( d\)-uniformly to \( \gamma \), there exists an integer \( n_0 \) such that, for each \( n \geq n_0 \) we have \( d(\gamma_n(t), \gamma(t)) < \eta \) for each \( t \in [a, b] \). Hence, for each \( n \geq n_0 \) and almost all \( t \in E_C \), we have

\[ L(\gamma_n(t), \zeta_n(t)) \geq L(\gamma(t), \zeta(t)) + \partial_v L(\gamma(t), \zeta(t)) \cdot (\zeta_n(t) - \zeta(t)) - \varepsilon, \]

and from the uniform superlinearity, \( L(\gamma_n(t), \zeta_n(t)) \geq C(0) \) a.e.where. Thus

\[ A_{L,\lambda_n}(\gamma_n) \geq \int_{E_C} e^{\lambda t} L(\gamma(t), \zeta(t)) \, dt + C(0) \int_{[a, b] \setminus E_C} e^{\lambda t} \, dt \]

\[ + \int_{E_C} e^{\lambda t} \partial_v L(\gamma(t), \zeta(t)) \cdot (\zeta_n(t) - \zeta(t)) \, dt - \varepsilon e^{\lambda b} m(E_C) \]

Since \( \{ \|e^{\lambda t} \zeta_n(t)\| \} \) is uniformly integrable, \( e^{\lambda t} \zeta_n(t) \) converges to \( e^{\lambda t} \zeta(t) \) in the weak topology \( \sigma(L^1, L^\infty) \). Since \( \|\zeta(t)\| \leq C \) for \( t \in E_C \), the function \( \chi_{E_C}(t) \partial_v L(\gamma(t), \zeta(t)) \) is bounded. Thus

\[ \int_{E_C} \partial_v L(\gamma(t), \zeta(t)) \cdot (e^{\lambda t} \zeta_n(t) - e^{\lambda t} \zeta(t)) \, dt, \int_{E_C} \partial_v L(\gamma(t), \zeta(t)) \cdot (e^{\lambda t} - e^{\lambda t}) \zeta(t) \, dt \to 0, \]

as \( n \to \infty \). Taking limit in \( 2 \) we have

\[ l = \lim_{n \to \infty} A_{L,\lambda_n}(\gamma_n) \geq \int_{E_C} e^{\lambda t} L(\gamma(t), \zeta(t)) \, dt + C(0) \int_{[a, b] \setminus E_C} e^{\lambda t} \, dt - \varepsilon e^{\lambda b} m(E_C). \]
Letting $\varepsilon \to 0$ we have

$$l = \lim_{n \to \infty} A_{L,\lambda_n}(\gamma_n) \geq \int_{E_C} e^{\lambda t} L(\gamma(t), \zeta(t)) \, dt + C(0) \int_{[a,b] \setminus E_C} e^{\lambda t} \, dt.$$  

Since $\zeta(t)$ is defined and finite for almost all $t \in [a,b]$ we have that $E_C \nearrow E_\infty$ as $C \nearrow +\infty$ with $m([a,b] \setminus E_\infty) = 0$. Since $L(\gamma(t), \zeta(t))$ is bounded below by $C(0)$, the monotone convergence theorem gives

$$\int_{E_C} e^{\lambda t} L(\gamma(t), \zeta(t)) \, dt \to \int_{a}^{b} e^{\lambda t} L(\gamma(t), \zeta(t)) \, dt \text{ as } C \to +\infty.$$  

Letting $C \nearrow +\infty$ in (3) we finally obtain

$$l = \lim_{n \to \infty} A_{L,\lambda_n}(\gamma_n) \geq A_{L,\lambda}(\gamma).$$

\[ \square \]

**Corollary 1.** The action $A_{L,\lambda} : W_D^{1,1}([a,b]) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous for the topology of $d$-uniform convergence in $W_D^{1,1}([a,b])$.

**Proof.** Let $\gamma_n$ be a sequence in $W_D^{1,1}([a,b])$ that converges $d$-uniformly to $\gamma \in W_D^{1,1}([a,b])$. We must show that

$$\liminf_{n \to \infty} A_{L,\lambda}(\gamma_n) \geq A_{L,\lambda}(\gamma).$$

If $\liminf_{n \to \infty} A_{L,\lambda}(\gamma_n) = +\infty$ there is nothing to prove. In other case, the result follows from Theorem 2 \[ \square \]

**Corollary 2** (Tonelli’s Theorem). If $K \subset M$ is compact, $c \in \mathbb{R}$ then the set

$$\mathcal{F}(K,c) = \{ \gamma \in W_D^{1,1}([a,b]) : [a,b] \cap K \neq \emptyset, A_{L,\lambda}(\gamma) \leq c \}$$

is a compact subset of $W_D^{1,1}([a,b])$ for the topology of $d$-uniform convergence.

**Proof.** By the compactness of $K$ and Theorem 2 the subset $\mathcal{F}(K,c)$ is closed in the space of continuous curves $C([a,b],M)$. By the uniform superlinearity, $L(v) \geq \|v\| + C(1)$ for all $v \in D$. Thus for any $\gamma \in W_D^{1,1}([a,b])$ and any $t,s \in [a,b]$ with $t \leq s$, we have

$$C(1) \int_{t}^{s} e^{\lambda r} \, dr + e^{\lambda a} \int_{t}^{s} \|\dot{\gamma}\| \leq A_{L,\lambda}(\gamma).$$

If $\gamma \in \mathcal{F}(c,K)$ we have

$$d(\gamma(s), \gamma(t) \leq e^{-\lambda a} \left( c + |C(1)| \int_{a}^{b} e^{\lambda r} \, dr \right),$$

and thus, letting $r = e^{-\lambda a} (c + |C(1)| \int_{a}^{b} e^{\lambda r} \, dr)$, we get

$$\gamma([a,b]) \subset \{ y \in M : d(y,K) \leq r \}.$$
Since $\mathcal{F}(c)$ is $d$-absolutely equicontinuous by Theorem 1 and Remark (1), the Arzelá-Ascoli Theorem implies that $\mathcal{F}(K, c)$ is a compact subset of $W^{1,1}_D([a, b])$ for the topology of $d$-uniform convergence.

Denote $C_{a,b}(x, y) := \{ \gamma \in W^{1,1}_D([a, b]) : \gamma(a) = x, \gamma(b) = y \}$.

**Corollary 3** (Tonelli minimizers). For each $x, y \in M$ and each $a, b \in \mathbb{R}$, $a < b$, there exists $\gamma \in C_{a,b}(x, y)$, $A_{L,\lambda}(\gamma) \leq A_{L,\lambda}(\alpha)$ for any $\alpha \in C_{a,b}(x, y)$.

**Proof.** Set $C = \inf \{ A_{L,\lambda}(\gamma) : \gamma \in C_{a,b}(x, y) \}$. By Corollary 2 the set

$$\{ \gamma \in C_{a,b}(x, y) : A_{L,\lambda}(\gamma) \leq \bar{C} + 1 \}$$

is a compact subset of $W^{1,1}_D([a, b])$ for the topology of $d$-uniform convergence. Choose $\gamma_n \in C_{a,b}(x, y)$ such that $A_{L,\lambda}(\gamma_n) < \bar{C} + \frac{1}{n}$. Then $\gamma_n$ has a subsequence that converges $d$-uniformly to some $\gamma \in C_{a,b}(x, y)$. By Theorem 2, $A_{L,\lambda}(\gamma) = \bar{C}$.

## 3. The discounted value function

For $\lambda > 0$ we define the discounted value function

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda t} L(\dot{\gamma}(t)) \, dt$$

where the infimum is taken over the curves $\gamma \in W^{1,1}_D([ - \infty, 0])$, with $\gamma(0) = x$.

**Proposition 2.** The discounted value function has the following properties

1. $\min L \leq \lambda u_\lambda \leq A(0)$.
2. For any $\alpha \in W^{1,1}_D([a, b])$

$$u_\lambda(\alpha(b)) e^{\lambda b} - u_\lambda(\alpha(a)) e^{\lambda a} \leq A_{L,\lambda}(\alpha)$$

**Proof.** 1) Taking $\gamma(t) \equiv x$ we have

$$u_\lambda(x) = \int_{-\infty}^{0} e^{\lambda t} A(0) \, dt = \frac{A(0)}{\lambda}$$

Given $\varepsilon > 0$ take $\gamma \in W^{1,1}_D([ - \infty, 0])$ with $\gamma(0) = x$ such that

$$\frac{\min L}{\lambda} \leq \int_{-\infty}^{0} e^{\lambda t} L(\dot{\gamma}(t)) \, dt < u_\lambda(x) + \varepsilon.$$

2) For $\alpha \in W^{1,1}_D([a, b])$ define $\alpha_b(t) = \alpha(t + b)$, then (5) is equivalent to

$$u_\lambda(\alpha_b(0)) - u_\lambda(\alpha_b(a - b)) e^{\lambda(a-b)} \leq A_{L,\lambda}(\alpha_b),$$

and so we can assume that $b = 0$. Given $\gamma \in W^{1,1}_D([ - \infty, 0])$ with $\gamma(0) = \alpha(a)$ define $\gamma_a(s) = \gamma(s - a)$. From the definition of $u_\lambda$ we have

$$u_\lambda(\alpha(0)) \leq \int_{-\infty}^{a} e^{\lambda s} L(\dot{\gamma}_a(s)) \, ds + A_{L,\lambda}(\alpha) = e^{\lambda a} \int_{-\infty}^{0} e^{\lambda t} L(\dot{\gamma}(t)) \, dt + A_{L,\lambda}(\alpha)$$
Taking the infimum over the curves $\gamma \in W^{1,1}_D([-\infty, 0])$, with $\gamma(0) = \alpha(a)$ we get

$$u_\lambda(\alpha(0)) \leq u_\lambda(\alpha(a))e^{\lambda a} + A_{L,\lambda}(\alpha)$$

\[\square\]

**Theorem 3** (Dynamic Programming Principle). The function $u_\lambda$ satisfies

$$(6) \quad u_\lambda(x) = \inf\{u_\lambda(\gamma(-T))e^{-\lambda T} + A_{L,\lambda}(\gamma) : \gamma \in W^{1,1}_D([-T, 0]), \gamma(0) = x\}.$$ 

Moreover the infimum is attained

**Proof.** The inequality $\leq$ in (6) follows from (5). Given $\varepsilon > 0$ take $\gamma \in W^{1,1}_D([-\infty, 0])$ with $\gamma(0) = x$ such that

$$\int_{-\infty}^0 e^{\lambda t}L(\dot{\gamma}(t))\,dt < u_\lambda(x) + \varepsilon.$$ 

$$u_\lambda(\gamma(-T))e^{-\lambda T} \leq \int_{-\infty}^{-T} e^{\lambda t}L(\dot{\gamma}(t))\,dt = \int_{-\infty}^0 e^{\lambda t}L(\dot{\gamma}(t))\,dt - A_{L,\lambda}(\gamma|_{[-T,0]})$$ 

$$< u_\lambda(x) + \varepsilon - A_{L,\lambda}(\gamma|_{[-T,0]}),$$

giving the inequality $\geq$ in (6). To prove that the infimum is attained we consider a minimizing sequence $\gamma_n \in W^{1,1}_D([-T, 0])$ with $\gamma_n(0) = x$. For $n$ large enough

$$A_{L,\lambda}(\gamma_n) \leq 1 + u_\lambda(x) - u_\lambda(\gamma_n(-T))e^{-\lambda T} \leq 1 + 2\|u_\lambda\|_{\infty}.$$ 

By Corollary 2 up to subsequences, $\gamma_n$ converges $d$-uniformly to $\gamma \in W^{1,1}_D([-T, 0])$ with $\gamma(0) = x$ and by Theorem 2

$$A_{L,\lambda}(\gamma) \leq \liminf_{n \to \infty} A_{L,\lambda}(\gamma_n) = u_\lambda(x) - u_\lambda(\gamma(-T))e^{-\lambda T}$$

\[\square\]

**Proposition 3.** $u_\lambda$ is Lipschitz for the metric $d$, uniformly in $\lambda > 0$

**Proof.** Let $x, y \in M$, $d = d(x, y)$ and take a minimizing geodesic $\gamma \in W^{1,1}_D([-d, 0])$ with $\gamma(-d) = y$, $\gamma(0) = x$ and $\|\dot{\gamma}\| = 1$ a.e.

$$u_\lambda(x) \leq u_\lambda(y)e^{-\lambda d} + \int_{-d}^0 e^{\lambda s}L(\dot{\gamma}(s))\,ds \leq u_\lambda(y)e^{-\lambda d} + \frac{1 - e^{-\lambda d}}{\lambda}(A(1))$$

$$u_\lambda(x) - u_\lambda(y) \leq \frac{1 - e^{-\lambda d}}{\lambda}(-\lambda u_\lambda(y) + A(1)) \leq \frac{1 - e^{-\lambda d}}{\lambda}(A(1) - \min L) \leq d(A(1) - \min L)$$

\[\square\]

**Proposition 4.** Given $\lambda > 0$, $x \in M$ there exists $\gamma_{\lambda,x} \in W^{1,1}_D([-\infty, 0])$ such that $\gamma_{\lambda,x}(0) = x$ and for any $t \geq 0$

$$(7) \quad u_\lambda(x) = u_\lambda(\gamma_{\lambda,x}(-t))e^{-\lambda t} + A_{L,\lambda}(\gamma_{\lambda,x}|_{[-t,0]}).$$
Moreover $\|\dot{\gamma}_{\lambda,x}\|_\infty$ is bounded uniformly in $\lambda > 0, x \in M$ and

$$u_\lambda(x) = \int_{-\infty}^{0} e^{\lambda t}L(\dot{\gamma}_{\lambda,x}(t)) \, dt.$$  \hfill (8)

**Proof.** By Theorem 3, for each $T > 0$ there is $\gamma^T \in W^{1,1}_D([-T,0])$ with $\gamma^T(0) = x$ and

$$u_\lambda(x) = u_\lambda(\gamma^T(-T))e^{-\lambda T} + A_{L,\lambda}(\gamma).$$

Since $u$ satisfies (5), for any $t \in [0, T]$ we have

$$u_\lambda(x) - e^{-\lambda t}u_\lambda(\gamma^T(-t)) = A_{L,\lambda}(\gamma^T_{-t,0}).$$

As in the proof Theorem 3

$$A_{L,\lambda}(\gamma^T_{-t,0}) \leq 2\|u_\lambda\|_\infty$$

By Tonelli’s Theorem there is a sequence $T_j \to \infty$ such that $\gamma^{T_j}_{-t,0}$ converges uniformly with the metric $d$. Applying this argument to a sequence $t_k \to \infty$ and using a diagonal trick one gets a sequence $\gamma_{\lambda,x} \in W^{1,1}_D([-\infty,0])$ such that $\gamma_{\sigma_n}[-t,0]$ converges $d$-uniformly to $\gamma_{\lambda,x}[-t,0]$ for any $t > 0$. By the continuity of $u$, for any $t > 0$, $u_\lambda(\gamma_{\lambda,x}(-t)) = \lim_{n \to \infty} u_\lambda(\gamma_{\sigma_n}(-t))$. By Theorem 2

$$u_\lambda(x) = u_\lambda(\gamma_{\lambda,x}(-t))e^{-\lambda t} + A_{L}(\gamma_{\lambda,x}[-t,0]).$$

By Proposition 3 there is a uniform Lipschitz constant $K$ for $u_\lambda$. By the superlinearity, $L(v) = (K + 1)\|v\| + C(K + 1)$. For $a < b < 0$

$$u_\lambda(\gamma_{\lambda,x}(b))e^{\lambda b} - u_\lambda(\gamma_{\lambda,x}(a))e^{\lambda a} = A_{L,\lambda}(\gamma_{\lambda,x}|_{[a,b]})$$

$$\geq e^{\lambda a}(K + 1) \int_a^b \|\dot{\gamma}_{\lambda,x}\| + C(K + 1) \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}. $$

$$u_\lambda(\gamma_{\lambda,x}(b))e^{\lambda b} - u_\lambda(\gamma_{\lambda,x}(a))e^{\lambda a} \leq u_\lambda(\gamma_{\lambda,x}(b))(e^{\lambda b} - e^{\lambda a}) + Ke^{\lambda a}d(\gamma_{\lambda,x}(b), \gamma_{\lambda,x}(a))$$

$$\leq A(0)e^{\lambda b} - e^{\lambda a} + Ke^{\lambda a} \int_a^b \|\dot{\gamma}_{\lambda,x}\|$$

Thus

$$\frac{1}{b - a} \int_a^b \|\dot{\gamma}_{\lambda,x}\| \leq (A(0) - C(K + 1)) \frac{e^{\lambda(b-a)} - 1}{\lambda(b-a)}$$

implying that $\|\dot{\gamma}_{\lambda,x}\| \leq (A(0) - C(K + 1)$ a.e. Letting $t \to +\infty$ in (7) we get (8) by the dominated convergence theorem. \hfill \Box

4. **Weak KAM theory**

In this section we extend weak KAM theory to our setting. The difference with the standard case is the lack of a Lagrangian dynamics and Tonelli’s theorem takes care of this difficulty. Denote $A_{L,0}$ by $A_L$
4.1. The weak KAM theorem.

**Definition 3.** Let $c \in \mathbb{R}$, we say that

- $u : M \to \mathbb{R}$ is $L + c$ dominated if for any $\gamma \in W^{1,1}_D([a, b])$ we have \[ u(\gamma(b)) - u(\gamma(a)) \leq A_{L+c}(\gamma) \]
- $\gamma \in W^{1,1}_D([a, b])$ is calibrated by an $L + c$ dominated function $u : M \to \mathbb{R}$ if \[ u(\gamma(b)) - u(\gamma(a)) = A_{L+c}(\gamma) \]
- $u : M \to \mathbb{R}$ is a backward (forward) $c$-weak KAM solution if it is $L + c$ dominated and for any $x \in M$ there is $\gamma \in W^{1,1}_D([\gamma_0, \gamma_1]) (\gamma \in W^{1,1}_D([0, \infty)))$ calibrated by $u$ such that $\gamma(0) = x$.

Suppose $M$ is compact. Consider the value function $u_\lambda$, $\lambda > 0$. Since $\lambda u_\lambda$ is uniformly bounded, $u_\lambda$ is uniformly Lipschitz there is a sequence $\lambda_n$ converging to zero such that $\lambda_n u_{\lambda_n}$ converges to a constant $-c$. Thus $u_\lambda - \max u_\lambda$ is uniformly bounded and Lipschitz, and then, through some sequence converges to a Lipschitz function $v$.

**Theorem 4** (weak KAM theorem). Suppose $M$ is compact. Let $v_\lambda = u_\lambda - \min u_\lambda$. Suppose that $v = \lim_{k \to \infty} v_{\lambda_k}$ and $c = - \lim_{k \to \infty} \lambda_k \min u_{\lambda_k}$ for a sequence $\lambda_k \to 0$. Then $v$ is a backward $c$-weak KAM solution.

**Proof.** Let $\gamma \in W^{1,1}_D([a, b])$, By (3), for any $\lambda > 0$

\[ v_\lambda(\gamma(b))e^{\lambda a} - v_\lambda(\gamma(b))e^{\lambda b} \leq (e^{\lambda a} - e^{\lambda b}) \min u_\lambda + A_{L,\lambda}(\gamma). \]

Taking limits $\lambda_k \to 0$,

\[ v(\gamma(b)) - v(\gamma(a)) \leq (b - a)c + A_L(\gamma), \]

giving that $v$ is $L + c$ dominated.

Let $x \in M$ and $\gamma_\lambda \in W^{1,1}_D([\gamma_0, \gamma_1])$ be a curve such that $\gamma_\lambda(0) = x$ for any $t \geq 0$

\[ u_\lambda(x) = u_\lambda(\gamma(-t))e^{\lambda t} + A_{L,\lambda}(\gamma_\lambda), \]

so that

\[ v_\lambda(x) = v_\lambda(\gamma(-t))e^{\lambda t} + (e^{\lambda t} - 1) \min u_\lambda + A_{L,\lambda}(\gamma_\lambda). \]

Then, as in the proof of Proposition 4, there is a subsequence $(\lambda^n)_n$ of $(\lambda_k)_k$ and a curve $\gamma \in W^{1,1}_D([\gamma_0, \gamma_1])$ such for any $t > 0$, $\gamma_\lambda^n([-t, 0])$ converges $d$-uniformly to $\gamma|_{[-t, 0]}$. Taking limits $\lambda^n \to 0$, by Theorem 2 we have

\[ v(x) = u(\gamma(-t)) + ct + A_L(\gamma) \]

showing that $\gamma$ is calibrated by $v$. Thus $v$ is a $c$-weak KAM solution.  \qed
Suppose there is a function $u : M \to \mathbb{R}$ that is $L + c$ dominated and let $x \in M$. Taking $\gamma : [a, b] \to M$ the constant curve $\gamma(t) = x$ we have $L(x, 0) + c \geq 0$. Thus $c \geq -\min_{x \in M} L(x, 0)$. Define $c(L) = \inf\{c \in \mathbb{R} : \exists u : M \to \mathbb{R} \text{ $L + c$ dominated}\}$.

If $k \geq -\min L$, then $L + k \geq 0$ and so any constant function is $L + k$ dominated. Thus $c(L) \leq -\min L$.

**Example 1.** Let $M$ be compact, $U : M \to \mathbb{R}$ be smooth and $L(x, v) = \|v\|^2 - U(x)$. Then $\min_{x \in M} L(x, 0) = -\max U$. Thus $c(L) = \max U$.

**Proposition 5.**

1. There exists $u : M \to \mathbb{R}$ that is $L + c(L)$ dominated.
2. If $M$ is compact and there is a backward $c$-weak KAM solution $u : M \to \mathbb{R}$, then $c = c(L)$.

The proof of (1) is as the one of Theorem 4.2.7 in [F]. The proof of (2) is as the one of Corollary 4.3.7 in [F].

**Proposition 6.** Let $u : M \to \mathbb{R}$ be $L + c$ dominated then

1. $|u(y) - u(x)| \leq (A(1) + c)d(x, y)$
2. If $\gamma \in W^{1,1}_2(I)$ then $u \circ \gamma$ is absolutely continuous and $(u \circ \gamma)'(t) \leq L(\gamma(t)) + c$ at any $t \in I$ where $u \circ \gamma$ and $\gamma$ are differentiable.

**Proof.** (1) Taking a minimizing geodesic $\alpha \in C_d(x, y)$ with $\|\dot{\alpha}\| = 1$ a.e. we have

$$u(y) - u(x) \leq A_{L+c}(\alpha) \leq (A(1) + c) d(x, y).$$

(2) Since $u$ is Lipschitz with constant $K = A(1) + c$, given $\varepsilon > 0$ there is $\delta > 0$ such that for any disjoint $I_{a_1, b_1}, \ldots, I_{a_k, b_k} \subseteq I$,

$$\sum_{i=1}^k b_i - a_i < \delta \implies \sum_{i=1}^k |u \circ \gamma(b_i) - u \circ \gamma(a_i)| \leq K \sum_{i=1}^k d(\gamma(a_i), \gamma(b_i)) \leq K \sum_{i=1}^k \int_{a_i}^{b_i} \|\dot{\gamma}\| < \varepsilon.$$

Let $t \in I$ be a point of differentiability of $u \circ \gamma$ and $\gamma$, dividing $u(\gamma(t + h)) - u(\gamma(t)) \leq A_{L+c}(\gamma|_{t, t+h})$ by $h$ and letting $h \to 0$ we get $(u \circ \gamma)'(t) \leq L(\dot{\gamma}(t)) + c$. \qed

4.2. **The Lax-Oleinik semigroup.** Set $C_t(x, y) = C_{0,t}(x, y)$ and $C(x, y) = \bigcup_{t > 0} C_t(x, y)$.

**Definition 4.** We define the minimal action to go from $x$ to $y$ in time $t > 0$ as the function $h : M \times M \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$h(x, y, t) := h_t(x, y) := \inf\{A_L(\gamma) : \gamma \in C_t(x, y)\}$$
Proposition 7. The minimal action has the following properties

(1) \( h_t(x, y) \geq t \inf_{M} L \)
(2) \( h_{t+s}(x, z) = \inf_{y \in M} h_t(x, y) + h_s(y, z) \)
(3) \( \gamma \in W^{1,1}_{D}([a, b]), a < b \) is minimizing if and only if \( h_{b-a}(\gamma(a), \gamma(b)) = A_L(\gamma) \).
(4) For each \( x, y \in M, t > 0 \), there exists \( \gamma \in C_t(x, y) \) such that \( h_t(x, y) = A_L(\gamma) \).

Proposition 8. Assume that there are \( C_1, C_2 \geq 1 \) such that

\[ \partial_t L \cdot v \leq C_1 L + C_2 \]

and let \( \Delta_\delta = \{ x, y \in M : d(x, y) \leq 1/\delta \} \) for \( \delta > 0 \). Then \( h \) is Lipschitz on \( \Delta_\delta \times [\delta, 1/\delta] \). In particular, \( h \) is continuous.

**Proof.** First observe that \( h \) is bounded on \( \Delta_\delta \times [\delta, 1/\delta] \). Indeed, let \( d = d(x, y) \) and take a minimizing geodesic \( \sigma \in C_d(x, y) \) with \( \| \tilde{\sigma} \| = 1 \) a.e. and \( \gamma(s) = \sigma(s) / t \), then \( \| \gamma \| = d(x, y) / t \leq \delta^{-2} \) for \( (x, y) \in \Delta_\delta, t \geq \delta \). Thus \( h(x, y, t) \leq A(\delta^{-2}) / \delta \).

Next observe that the assumption (10) implies that

\[ L(sv) \leq sC_1 L(v) + \frac{C_1}{C_2}(sC_1 - 1), \quad s \geq 1. \]

Let \( x, y \in M, t > 0 \). For \( z \in M, d = d(y, z) \), take a minimizing geodesic \( \sigma \in C_d(y, z) \) with \( \| \tilde{\sigma} \| = 1 \) a.e. For \( \gamma \in C_t(x, y) \) define \( \tilde{\gamma} \in C_t(x, z) \) by \( \tilde{\gamma}[0, t] = \gamma, \tilde{\gamma}(s) = \sigma(s-t) \) for \( s \in [t, t+\delta] \). Thus \( h(x, z, t+d) \leq A_L(\tilde{\gamma}) \leq A_L(\gamma) + A(1)d \) and then

\[ h(x, z, t+d(y, z)) \leq h(x, y, t) + A(1)d(y, z). \]

Similarly, defining \( \tilde{\tau} \in C_t(y, z), \tau > t \) by \( \tilde{\gamma}[0, t] = \gamma, \tilde{\gamma}(s) = y \) for \( s \in [t, \tau] \) we get

\[ h(x, y, \tau) \leq h(x, y, t) + A(0)(\tau - t). \]

Let \( r > 0 \) an \( \gamma \in C_{t+r}(x, y) \) and define \( \tilde{\gamma} \in C_t(x, y) \) by \( \tilde{\gamma}(s) = \gamma((t+r)s/t) \) we have

\[ h(x, y, t) \leq A_L(\tilde{\gamma}) = \frac{t+r}{t} \int_{0}^{t+r} L\left( \frac{t+r}{t} \right) \]

\[ \leq \left( \frac{t+r}{t} \right)^{C_1-1} A_L(\gamma) + \frac{C_1}{C_2} \left( \left( \frac{t+r}{t} \right)^{C_1} - 1 \right), \]

and then

\[ h(x, y, t) \leq \left( \frac{t+r}{t} \right)^{C_1-1} h(x, y, t+r) + \frac{C_1}{C_2} \left( \left( \frac{t+r}{t} \right)^{C_1} - 1 \right). \]

Thus

\[ h(x, y, t) - h(x, y, t+r) \leq \left( \left( \frac{t+r}{t} \right)^{C_1-1} - 1 \right) h(x, y, t+r) + \frac{C_1}{C_2} \left( \left( \frac{t+r}{t} \right)^{C_1} - 1 \right) \]

\[ \leq C_1 r(1 + \delta^{-2})^{C_1-1} \left( \frac{A(\delta^{-2})}{\delta} + \frac{C_1}{C_2} \right). \]
Letting \( d = d(y, z) \), from (11), (13) we get
\[
    h(x, z, t) - h(x, y, t) \leq \left( \left( \frac{t + d}{t} \right)^{C_1} - 1 \right) h(x, z, t + d) + \frac{C_1}{C_2} \left( \left( \frac{t + d}{t} \right)^{C_1} - 1 \right) + h(x, z, t + d) - h(x, y, t)
\]
and similarly for \( h(z, y, t) - h(x, y, t) \).

Corollary 4. Assume \( M \) is compact and there are \( C_1, C_2 > 0 \) such that (10) holds. For any \( \delta > 0 \) there is \( K_\delta > 0 \) such that \( h_t \) is \( K_\delta \)-Lipschitz for any \( t \geq \delta \).

Proof. Since \( h \) is continuous and \( M \) is compact, for \( x, y, t, s > 0 \) there is \( z \in M \) such that \( h_{t+s}(x, y) = h_t(x, z) + h_s(z, y) \).

Let \( \delta > 0 \), by Proposition 8 there is \( K_\delta > 0 \) such that \( h \) is \( K_\delta \) Lipschitz on \( M \times M \times [\delta, 1/\delta] \). If \( t > 1/\delta \) choose \( t_0 < t_1 < \cdots < t_k = t \) such that \( t_i - t_{i-1} \in [\delta, 1/\delta] \). Let \( x_1, x_2, y_1, y_2 \in M \) and suppose \( h_t(x_1, y_1) \geq h_t(x_2, y_2) \). There are \( z_0 = x_2, z_1, \ldots, z_l = y_2 \in M \) such that \( h_t(x_2, y_2) = \sum_{i=1}^{l} h_{t_i-t_{i-1}}(z_{i-1}, z_i) \).

Since
\[
    h_t(x_1, y_1) \leq h_{t_1}(x_1, z_1) + \sum_{i=2}^{l} h_{t_i-t_{i-1}}(z_{i-1}, z_i) + h_{t_l-t_{l-1}}(z_{l-1}, y_1),
\]
we have
\[
    h_t(x_1, y_1) - h_t(x_2, y_2) \leq h_{t_1}(x_1, z_1) - h_{t_1}(x_2, z_1) + h_{t_l-t_{l-1}}(z_{l-1}, y_1) - h_{t_l-t_{l-1}}(z_{l-1}, y_2) \leq K_\delta (d(x_1, x_2) + d(y_1, y_2))
\]

Definition 5 (Lax-Oleinik semi-group). For \( u : M \to [\infty, +\infty] \) and \( t > 0 \) we define \( \mathcal{L}_t u : M \to [\infty, +\infty] \) by
\[
    \mathcal{L}_t u(x) = \inf_{y \in M} \{ u(\gamma(0)) + A_L(\gamma) : \gamma \in W^{1,1}_{D}([0, t]), \gamma(t) = x \}
\]

Notice that
\[
    \mathcal{L}_t u(x) = \inf_{y \in M} u(y) + h_t(y, x)
\]

Proposition 9. The family of maps \( \{ \mathcal{L}_t \}_{t \geq 0} \) has the following properties
\begin{enumerate}
    \item \( u : M \to \mathbb{R} \) is \( L + c \) dominated if and only if \( u \leq \mathcal{L}_t u + ct \) for all \( t \geq 0 \).
    \item For any \( s, t > 0 \), \( \mathcal{L}_{t+s} = \mathcal{L}_t \circ \mathcal{L}_s \).
    \item If \( u : M \to \mathbb{R} \) is Lipschitz for the metric \( d \), then for each \( x \in M \) and \( t > 0 \) there is \( \gamma \in W^{1,1}_{D}([0, t]) \) such that \( \gamma(t) = x \) and
\[
    \mathcal{L}_t u(x) = u(\gamma(0)) + A_L(\gamma) = u(\gamma(0)) + h_t(\gamma(0), x).
\]
(4) If $M$ is compact, $u \in C(M)$ and that there are $C_1, C_2 > 0$ such that \[ \| u \|_{C_b(M)} \leq C \] holds, then for each $x \in M$, $t > 0$ there is $y \in M$ such that $\mathcal{L}_t u(x) = u(y) + h_t(y, x)$.

Proof. (3) Using the constant curve with value $x$ we have $\mathcal{L}_t u(x) \leq u(x) + t A(0)$. Thus

$$\mathcal{L}_t u(x) = \inf \{ u(\gamma(0)) + A_L(\gamma) : \gamma \in \mathcal{F}(u, x, t) \}$$

where

$$\mathcal{F}(u, x, t) = \{ \gamma \in W^{1,1}_D([0, t]) : \gamma(t) = x, u(\gamma(0)) + A_L(\gamma) \leq u(x) + t A(0) \}.$$ 

If $u$ is $K$-Lipschitz, $u(x) - u(\gamma(0)) \leq K d(x, \gamma(0))$.

By the superlinearity of $L$ we have

$$C(K + 1)t + (K + 1)c(\gamma) \leq A_L(\gamma),$$

thus, if $\gamma \in \mathcal{F}(u, x, t)$ then $d(x, \gamma(0)) \leq c(\gamma) \leq t(A(0) - C(K + 1))$. Therefore

$$\mathcal{L}_t u(x) = \inf \{ u(\gamma(0)) + A_L(\gamma) : \gamma \in \mathcal{F}(x, t, K) \}$$

where

$$\mathcal{F}'(x, t, K) = \{ \gamma \in W^{1,1}_D([0, t]) : \gamma(t) = x, A_L(\gamma) \leq t(A(0) + K(A(0) - C(K + 1))) \}.$$ 

By Tonelli’s Theorem the set $\mathcal{F}'(x, t, K)$ is compact for the topology of uniform convergence with the metric $d$. Choose $\gamma_n \in \mathcal{F}'(x, t, K)$ such that $u(\gamma_n(0)) + A_L(\gamma_n) \leq \mathcal{L}_t u(x) + \frac{1}{n}$. Then $\gamma_n$ has a subsequence $\gamma_{n_j}$ that converges uniformly with the metric $d$ to some $\gamma \in \mathcal{F}'(x, t, K)$. By continuity of $u$, $u(\gamma(0)) = \lim u(\gamma_{n_j}(0))$. By Theorem 2 $u(x) = \mathcal{L}_t u(x) = u(\gamma(0)) + A_L(\gamma)$.

Proposition 10. Let $c \in \mathbb{R}$. A function $u : M \to \mathbb{R}$ satisfies $u = \mathcal{L}_t u + ct$ for all $t \geq 0$ if and only it is a weak KAM solution.

Proof. It is clear that a weak KAM solution $u$ satisfies $u = \mathcal{L}_t u + ct$ for all $t \geq 0$. Assuming that $u = \mathcal{L}_t u + ct$ for all $t \geq 0$, by item (1) in Proposition 9 we have that $u$ is $L + c$ dominated.

Let $x \in M$ by item (3) of Proposition 9 for each $s > 0$ there is $\gamma_s \in W^{1,1}_D([-s, 0])$ with $\gamma(0) = x$ and $\mathcal{L}_s u(x) = u(\gamma_s(-s)) + A_L(\gamma)$; then $u(x) = u(\gamma_s(-s)) + A_{L+c}(\gamma)$. Since $u$ is $L + c$ it is dominated, it is $K$-Lipschitz for $K = c + A(1)$ we have that $u(x) = u(\gamma_s(-t)) + A_{L+c}(\gamma_s[-t, 0])$ for any $t \in [0, s]$. From the proof of item (3) of Proposition 9 for $s > t$ we have

$$\int_t^s L(\gamma_s) \leq t(A(0) + K(A(0) + C(K + 1))).$$

By Tonelli’s Theorem there is a sequence $s_j \to \infty$ such that $\gamma_{s_j}[-t, 0]$ converges uniformly with the metric $d$ Aplying this argument to a sequence $t_k \to \infty$ and using a diagonal trick one gets a sequence $\sigma_n \to \infty$ and a curve $\gamma \in W^{1,1}_D([-\infty, 0])$ such that $\gamma_{\sigma_n}[-t, 0]$ converges uniformly with the metric $d$ to $\gamma[-t, 0]$ for any $t > 0$. 

By the continuity of $u$, for any $t > 0$, $u(\gamma(-t)) = \lim_{n \to \infty} u(\gamma_{\alpha_n}(-t))$. By Theorem 2
\[ \mathcal{L}_t u(x) = u(\gamma(-t)) + A_L(\gamma|[-t,0]). \]

**Corollary 5.** Let $M$ be compact. For each $\delta > 0$ there is $C_\delta > 0$ such that $|h_t(x,y) + c(L)t| \leq C_\delta$ for $x, y \in M$, $t \geq \delta$

**Proof.** For $\gamma \in C_t + \delta(x,y)$ we have
\[ A_L(\gamma) = A_L(\gamma|[0,1]) + A_L(\gamma|[\delta/2, t + \delta/2]) + A_L(\gamma|[t + \delta/2, t + \delta]) \geq 2 \inf h_{\frac{\delta}{2}} + \inf h_t, \]
thus
\[ h_{t+\delta}(x,y) \geq 2 \inf h_{\frac{\delta}{2}} + \inf h_t. \]

For any $z, w \in M$
\[ h_{t+\delta}(x,y) \leq h_{\frac{\delta}{2}}(x,z) + h_t(z,w) + h_{\frac{\delta}{2}}(w,y) \leq 2 \sup h_{\frac{\delta}{2}} + h_t(z,w), \]
thus
\[ h_{t+\delta}(x,y) \leq 2 \sup h_{\frac{\delta}{2}} + \inf h_t. \]

Therefore, for $t \geq 2$ we have that
\[ h_t - \inf h_t \leq 2(\sup h_{\frac{\delta}{2}} - \inf h_{\frac{\delta}{2}}) := a \]
Let $u : M \to \mathbb{R}$ be such that $u = \mathcal{L}_t u + c(L)t$ for all $t \geq 0$. For $t \geq \delta$ we have
\[ -2\|u\|_{\infty} \leq u(y) - u(x) \leq h_t(x,y) + c(L)t \leq a + \inf h_t + c(L)t \]
\[ = a + \inf h_t + u - \mathcal{L}_t u \leq a + 2\|u\|_{\infty} \]
and then
\[ |h_t(x,y) + c(L)t| \leq 2(\sup h_{\frac{\delta}{2}} - \inf h_{\frac{\delta}{2}} + \|u\|_{\infty}). \]

4.3. **Static curves and the Aubry set.** We assume that $M$ is compact and set $c = c(L)$.

**Definition 6.** We define
- The **Peierls barrier** $h : M \times M \to \mathbb{R}$ by
  \[ h(x,y) = \lim_{t \to \infty} \inf h_t(x,y) + ct, \]
- The **Mañé potential** $\Phi : M \times M \to \mathbb{R}$
  \[ \Phi(x,y) = \inf_{t>0} h_t(x,y) + ct. \]

It is clear that $\Phi \leq h$

**Proposition 11.** Functions $h$ and $\Phi$ have the following properties
1. $u$ is $L + c$ dominated $\iff u(y) - u(x) \leq \Phi(x,y)$.
2. For each $x \in M$, $\Phi(x,x) = 0$. 
We refer to Example 1 for a more precise definition of semi-static and static curves.

The following Proposition says that semi-static curves have energy $c(L)$.

Example 2. Let $M$, $U$ and $L$ be as in example 1. If $\gamma : [0, t] \to U^1([-\infty, \max U - \varepsilon])$ then $A_{L + c(L)}(\gamma) \geq \varepsilon t$, therefore if $\gamma_n \in C_{t_n}(x, y)$ is a sequence such that $t_n \to \infty$ and $A_{L + c(L)}(\gamma_n)$ converges to $h(x, y)$ then for any $\varepsilon > 0$ there is $N$ such that $\gamma_n([0, t_n]) \cap U^1([-\max U - \varepsilon, +\infty]) \neq \emptyset$ for $n \geq N$.

If $U(x) = \max U$ and $\gamma : [0, t] \to M$ is the constant curve $x$, then $A_{L + c(L)}(\gamma) = 0$ and thus $h(x, x) = 0$.

If $U(x) < \max U$ there are $\delta, \varepsilon > 0$ such that $U(y) \leq \max U - \varepsilon$ for $d(x, y) \leq \delta$. It is not hard to see that for any $\gamma \in C_t(x, x)$ such that $\gamma([0, t]) \cap U^1([-\max U - \varepsilon, \max U]) \neq \emptyset$ we have $A_{L + c(L)}(\gamma) \geq 2\sqrt{\delta \varepsilon}$ and then $h(x, x) \geq 2\sqrt{\delta \varepsilon}$.

Thus $A = U^{-1}(\max U)$.

The following Proposition says that semi-static curves have energy $c(L)$.
Proposition 12. Let \( \eta \in W^{1,1}_D(J) \) be semi-static. For a. e. \( t \in J \)
\[
E(\dot{\eta}(t)) = L_v(\dot{\eta}(t))\dot{\eta}(t) - L(\dot{\eta}(t)) = c
\]
Proof. For \( \lambda > 0 \), let \( \eta_{\lambda}(t) := \eta(\lambda t) \) so that \( \dot{\eta}_{\lambda}(t) = \lambda \dot{\eta}(\lambda t) \) a. e.
For \( r, s \in J \) let
\[
A_{rs}(\lambda) := \int_{r/\lambda}^{s/\lambda} \left[ L(\dot{\eta}_{\lambda}(t)) + c \right] dt = \int_{r}^{s} \left[ \frac{L(\lambda \dot{\eta}(t)) + c}{\lambda} \right] dt.
\]
Since \( \eta \) is a free-time minimizer, differentiating \( A_{rs}(\lambda) \) at \( \lambda = 1 \), we have that
\[
0 = A'_r(1) = \int_{r}^{s} [L(\dot{\eta})\dot{\eta} - L(\dot{\eta}) - c].
\]
Since this holds for any \( r, s \in J \) we have
\[
E(\dot{\eta}(t)) = L_v(\dot{\eta}(t))\dot{\eta}(t) - L(\dot{\eta}(t)) = c
\]
for a. e. \( t \in J \). \( \square \)

Example 3. Let \( M, U : M \to \mathbb{R} \) and \( L \) be as in example 1. If \( \gamma \in W^{1,1}_D([a, b]) \) is semi-static, then \( \frac{1}{2} \| \dot{\gamma}(t) \|^2 + U(\gamma(t)) = \max U \). Thus
\[
A_{L+c}(\gamma) = \int_{a}^{b} \| \dot{\gamma} \|^2 = \int_{a}^{b} \sqrt{2(\max U - U(\gamma))} \| \dot{\gamma} \| = \int_{0}^{b} \sqrt{2(\max U - U(\gamma))} ds
\]
where \( s(t) = \int_{a}^{t} \| \dot{\gamma} \| \) is the arc length function, \( s(b) = l \).

We have that \( \Phi(x, y) \) is the distance from \( x \) to \( y \) for the Maupertuis type sub-riemannian metric \( g_x(v, w) = 2(\max U - U(x))\langle v, w \rangle \).

The following Propositions prove the existence of static curves and that \( A \neq \emptyset \).

Proposition 13. Let \( u \) be \( L + c \) dominated, \( \gamma \in W^{1,1}_D([-\infty, 0]) \) be calibrated by \( u \). Take a sequence \( t_n \to \infty \) such that the sequence \( \gamma(-t_n) \) converges. Then there is a subsequence \( s_k = t_{n_k} \) and \( \eta \in W^{1,1}_D(\mathbb{R}) \) such that
(i) For each \( l > 0 \), \( \gamma(-s_k + \cdot)[-l, l] \) converges uniformly to \( \eta([-l, l]) \).
(ii) \( \eta \) is static.

Proof. (i) Set \( c = c(L) \) and let \( r < l \), then
\[
\int_{-t_n + r}^{-t_n + l} L(\dot{\gamma}) + c(l - r) = u(\gamma(-t_n + l)) - u(\gamma(-t_n + r)) \leq 2\| u \|.
\]
As we had before, Tonelli’s Theorem implies that there is a sequence \( n_k \to \infty \) and a curve \( \eta \in W^{1,1}_D(\mathbb{R}) \) satisfying (i).
(ii) For $l > 0$

$$0 \leq A_{L+c}(\eta|_{[-L,l]}) + \Phi(\eta(l),\eta(-l))$$

$$\leq \lim_k \left\{ A_{L+c}(\gamma|_{[-s_k-l,-s_k+l]}) + \lim_m A_{L+c}(\gamma|_{[-s_k+l,-s_m-l]}) \right\}$$

$$= \lim_k \lim_m A_{L+c}(\gamma|_{[-s_k-l,-s_m-l]})$$

$$= \lim_k \Phi(-s_k-l,\gamma(-s_m-l))$$

$$= \Phi(\eta(-l),\eta(-l)) = 0.$$  

proving that $\eta$ is static. \hfill $\square$

**Proposition 14.** If $\gamma \in W^{1,1}_D([0,\infty)) \cup W^{1,1}_D([-\infty,0])$ is static, then $p = \gamma(0) \in A$.

**Proof.** For $\gamma \in W^{1,1}_D([0,\infty))$ take a sequence $r_n \to \infty$ such that the sequence $\gamma(r_n)$ converges to $y \in M$. Then

$$0 \leq h(p,p) \leq h(p,y) + \Phi(y,p)$$

$$\leq \lim_n A_{L+c}(\gamma|_{[0,r_n]}) + \Phi(y,p)$$

$$= \lim_n -\Phi(\gamma(r_n),p) + \Phi(y,p) = 0.$$  

Analogously for $\gamma \in W^{1,1}_D([-\infty,0])$. \hfill $\square$

**Proposition 15.** For any $x \in M$, $h(x,\cdot)$ is a backward weak KAM solution and $-h(\cdot,x)$ is a forward weak KAM solution.

**Proof.** By items (1), (4) of Proposition 11, $h(x,\cdot)$ is $L + c$ dominated. Let $\gamma_n : [-t_n,0] \to M$ be a sequence of minimizing curves connecting $x$ to $y$ such that

$$h(x,y) = \lim_{n \to \infty} A_{L+c}(\gamma_n).$$

Let $l > 0$, for $n$ big enough $t_n > l$ and $a(n,l) = A_{L+c}(\gamma_n[-l,0])$ is bounded from above, because if there were a sequence $n_k \to \infty$ such that $a(n_k,l) \to +\infty$, passing to a subsequence $\gamma_{n_k}(-l)$ would converge to $z \in M$, and then we would have that

$$h(x,z) \leq \liminf_{k \to \infty} A_{L+c}(\gamma_{n_k}[-t_{n_k},-l]).$$

As we had before, Tonelli’s Theorem implies that there is a sequence $n_k \to \infty$ such $\gamma_{n_k}[-l,0]$ converges uniformly with the metric $d$ to $\gamma[-l,0]$ for any $t > 0$. For $s < 0$ define $\gamma(s) = \lim_{k \to \infty} \gamma_{n_k}(s)$. Fix $t < 0$, for $k$ large $t + t_{n_k} \geq 0$ and

$$A_{L+c}(\gamma_{n_k}) = \int_{-t_{n_k}}^t L(\dot{\gamma}_{n_k}) + c(t + t_{n_k}) + \int_t^0 L(\dot{\gamma}_{n_k}) - ct.$$
Since $\gamma_{n_k}$ converges to $\gamma$ uniformly on $[t, 0]$, we have
\[
\liminf_{k \to \infty} \int_t^0 L(\gamma_{n_k}, \dot{\gamma}_{n_k}) \geq \int_t^0 L(\gamma, \dot{\gamma}).
\]
From item (6) of Proposition 11 we have
\[
h(x, \gamma(t)) \leq \liminf_{k \to \infty} \int_t^{-t_{n_k}} L(\gamma_{n_k}, \dot{\gamma}_{n_k}) + c(t + t_{n_k}).
\]
Taking $\liminf_{k \to \infty}$ in (14) we get
\[
h(x, y) \geq h(x, \gamma(t)) + \int_t^0 L(\gamma, \dot{\gamma}) - ct.
\]
which implies that $\gamma$ is calibrated by $h(x, \cdot)$.

**Corollary 7.** If $x \in \mathcal{A}$ there exists a curve $\gamma \in W^{1,1}_D(\mathbb{R})$ such that $\gamma(0) = x$ and for all $t \geq 0$
\[
\Phi(\gamma(t), x) = h(\gamma(t), x) = -\int_0^t L(\dot{\gamma}) - ct \\
\Phi(x, \gamma(-t) = h(x, \gamma(-t)) = -\int_{-t}^0 L(\dot{\gamma}) - ct.
\]
In particular the curve $\gamma$ is static.

Denote by $\mathcal{K}$ the family of static curves $\eta : \mathbb{R} \to M$, and for $y \in \mathcal{A}$ denote by $\mathcal{K}(y)$ the set of curves $\eta \in \mathcal{K}$ with $\eta(0) = y$. On $\mathcal{K}$ we have a dynamics given by time translation along a static curve.

**Proposition 16.** $\mathcal{K}$ is a compact metric space with respect to the uniform convergence on compact intervals.

**Proof.** Let $\{\eta_n\}$ be a sequence in $\mathcal{K}$. By Proposition 12 for a. e. $t \in \mathbb{R}$, $E(\eta_n(t)) = c$, and then $\{A_L(\eta_n)\}$ is bounded. As in Proposition 10 we obtain a sequence $n_k \to \infty$ such that $\eta_{n_k}$ converges to $\eta : \mathbb{R} \to M$ uniformly on each $[a, b]$ and then $\eta$ is static.

**Proposition 17.** Two dominated functions that coincide on $\mathcal{M} = \bigcup_{\eta \in \mathcal{K}} \omega(\eta)$ also coincide on $\mathcal{A}$.

**Proof.** Let $\varphi_1, \varphi_2$ be two dominated functions coinciding on $\mathcal{M}$. Let $y \in \mathcal{A}$ and $\eta \in \mathcal{K}(y)$. Let $(t_n)_n$ be a diverging sequence such that $\lim_n \eta(t_n) = x \in \mathcal{M}$. By (5) in Corollary 6
\[
\varphi_i(y) = \varphi_i(\eta(0)) - \Phi(y, \eta(0)) = \varphi_i(\eta(t_n)) - \Phi(y, \eta(t_n))
\]
for every \( n \in \mathbb{N}, i = 1, 2 \). Sending \( n \to \infty \), we get
\[
\varphi_1(y) = \lim_{n \to \infty} \varphi_1(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_1(x) - \Phi(y, x) = \varphi_2(x) - \Phi(y, x)
\]
\[
= \lim_{n \to \infty} \varphi_2(\eta(t_n)) - \Phi(y, \eta(t_n)) = \varphi_2(y).
\]

**Proposition 18.** Let \( \eta \in \mathcal{K}, \psi \in C(M) \) and \( \varphi \) be a dominated function. Then the function \( t \mapsto (L_t \psi)(\eta(t)) - \varphi(\eta(t)) \) is nonincreasing on \( \mathbb{R}_+ \).

**Proof.** From (5) in Corollary 6, for \( t < s \) we have
\[
(L_s \psi)(\eta(s)) - (L_t \psi)(\eta(t)) \leq \int_t^s L(\eta(\tau), \dot{\eta}(\tau)) d\tau = \varphi(\eta(s)) - \varphi(\eta(t)).
\]

**Lemma 3.** There is \( R > 0 \) such that, if \( \eta \) is any curve in \( \mathcal{K} \) and \( \lambda \) is sufficiently close to 1, we have
\[
\int_{t_1}^{t_2} L(\eta_\lambda(t), \dot{\eta}_\lambda(t)) dt \leq \Phi(\eta_\lambda(t_1), \eta_\lambda(t_2)) + R(t_2 - t_1)(\lambda - 1)^2
\]
for any \( t_2 > t_1 \), where \( \eta_\lambda(t) = \eta(\lambda t) \).

**Proof.** Let \( K = \max\{\|v\| : E(v) = c\} \), \( a = \sup\{\|L_{vv}(v)\| : \|v\| \leq 2K\} \). For \( \lambda \in (1 - \delta, 1 + \delta) \) fixed, using Proposition 12
\[
\int_{t_1}^{t_2} L(\eta_\lambda(t), \dot{\eta}_\lambda(t)) dt = \int_{t_1}^{t_2} \left[ \frac{1}{2} L(v(\eta_\lambda(t)), \dot{v}(\eta_\lambda(t))) \dot{\eta}_\lambda(t) dt \right.
\]
\[
+ \frac{1}{2} (\lambda - 1)^2 L_{vv}(v(\eta_\lambda(t)), \dot{v}(\eta_\lambda(t))) \dot{\eta}_\lambda(t)^2 dt
\]
\[
\leq \lambda \int_{t_1}^{t_2} L(\eta(\lambda t), \dot{\eta}(\lambda t)) dt + \frac{1}{2} (t_2 - t_1) a^2 K^2 (\lambda - 1)^2
\]
\[
= \Phi(\eta(\lambda t_1), \eta(\lambda t_2)) + \frac{1}{2} (t_2 - t_1) a^2 K^2 (\lambda - 1)^2
\]

**Proposition 19.** Let \( \eta \in \mathcal{K}, \psi \in C(M) \) and \( \varphi \) be a dominated function. Assume that \( D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\} \neq \emptyset \) where \( D^+ \) denote the super-differential. Then for all \( t > 0 \) we have
\[
(L_t \psi)(\eta(t)) - \varphi(\eta(t)) < \psi(\eta(0)) - \varphi(\eta(0))
\]
Proof. Fix $t > 0$. By (5) in Corollary 6 it is enough to prove (16) for $\varphi = -\Phi(\cdot, \eta(t))$. Since $L_t(\psi + a) = L_t\psi + a$ we can assume that $\psi(\eta(0)) = \varphi(\eta(0))$.

\[
(L_t\psi)(\eta(t)) - \varphi(\eta(t)) = (L_t\psi)(\eta(t)) \leq \int_{(1/\lambda) t}^{t/\lambda} L(\eta_\lambda, \dot{\eta}_\lambda) + \psi((1 - \lambda)t)),
\]

thus, by Lemma 3

\[
(L_t\psi)(\eta(t)) - \varphi(\eta(t)) \leq \psi((1 - \lambda)t) + o((1 - \lambda)t)) + Mt(\lambda - 1)^2.
\]

If $m \in D^+((\psi - \varphi) \circ \eta)(0) \setminus \{0\}$, we have

\[
(L_t\psi)(\eta(t)) - \varphi(\eta(t)) \leq m((1 - \lambda)t) + o((1 - \lambda)t)) + Mt(\lambda - 1)^2,
\]

where $\lim_{\lambda \to 1} o((1 - \lambda)t)/1 - \lambda = 0$. Choosing appropriately $\lambda$ close to 1, we get

\[
(L_t\psi)(\eta(t)) - \varphi(\eta(t)) < 0.
\]

□

Proposition 20. Let $u : M \to \mathbb{R}$ be a backward weak KAM solution, then

\[
u(x) = \min_{p \in \mathcal{A}} u(p) + h(p, x).
\]

$\nu(x, y) = \min_{q \in \mathcal{A}} h(x, q) + h(q, y) = \min_{q \in \mathcal{A}} \Phi(x, q) + \Phi(q, x)$

Corollary 8. Let $C \subset M$ and $w_0 : C \to \mathbb{R}$ be bounded from below. Let

\[
w(x) = \inf_{z \in C} w_0(z) + \Phi(z, x)
\]

(1) $w$ is the maximal dominated function not exceeding $w_0$ on $C$.

(2) If $C \subset \mathcal{A}$, $w$ is a backward weak KAM solution.

(3) If for all $x, y \in C$

\[
w_0(y) - w_0(x) \leq \Phi(x, y),
\]

then $w$ coincides with $w_0$ on $C$.

4.4. Convergence of the Lax-Oleinik semigroup. We assume again that $M$ is compact. Adding $c(L)$ to $L$ we can assume that $c(L) = 0$. The main result of this section is Theorem 6.

Proposition 21. Let $u \in C(M)$

(a) Suppose $\psi = \lim_{n \to \infty} L_{t_n} u$ for some $t_n \to \infty$, then

\[
\psi \geq v(x) := \min_{z \in M} u(z) + h(z, x).
\]

(b) Suppose $L_t u$ converges as $t \to \infty$, then the limit is function $v$ defined in (17).
Proof. (a) For \( x \in M \), \( n \in \mathbb{N} \) let \( \gamma_n \in W^{1,1}_D([0,t_n]) \) be such that \( \gamma(t_n) = x \) and
\[
\mathcal{L}_{t_n} u(x) = u(\gamma_n(0)) + A_L(\gamma_n).
\]
Passing to a subsequence if necessary we may assume that \( \gamma_n(0) \) converges to \( y \in M \). Taking \( \lim \inf \) in (18), we have from item (5) of Proposition 11
\[
\psi(x) = u(y) + \lim \inf_{n \to \infty} A_L(\gamma_n) \geq u(y) + h(y, x).
\]

(b) For \( x \in M \) let \( z \in M \) be such that \( v(z) = u(z) + h(z, x) \). Since \( \mathcal{L}_t u(x) \leq u(z) + h_t(z, x) \), we have
\[
\lim_{t \to \infty} \mathcal{L}_t u(x) \leq \lim \inf_{t \to \infty} u(z) + h_t(z, x) = v(z)
\]
which together with item (a) gives \( \lim_{t \to \infty} \mathcal{L}_t u = v \). \( \square \)

Using Proposition 20 we can write (17) as
\[
v(x) = \min_{y \in \mathcal{A}} \Phi(y, x) + w(y)
\]
(19)
\[
w(y) := \inf_{z \in M} u(z) + \Phi(z, y)
\]
(20)

Item (1) of Corollary 8 states that \( w \) is the maximal dominated function not exceeding \( u \). Items (2), (3) of Corollary 8 imply that \( v \) is the unique backward weak KAM solution that coincides with \( w \) on \( \mathcal{A} \).

**Proposition 22.** Suppose that \( u \) is dominated, then \( \mathcal{L}_t u \) converges uniformly as \( t \to \infty \) to the function \( v \) given by (17).

**Proof.** Since \( u \) is dominated, the function \( t \mapsto \mathcal{L}_t u \) is nondecreasing. As well, in this case, \( w \) given by (20) coincides with \( u \). Items (1) and (3) of Corollary 8 imply that \( v \) is the maximal dominated function that coincides with \( u \) on \( \mathcal{A} \) and then \( u \leq v \) on \( M \).

Since the semigroup \( \mathcal{L}_t \) is monotone and \( v \) is a backward weak KAM solution
\[
\mathcal{L}_t u \leq \mathcal{L}_t v = v \text{ for any } t > 0.
\]
Thus the uniform limit \( \lim_{t \to \infty} u \) exists. \( \square \)

Henceforth we assume that there are \( C_1, C_2 > 0 \) such that (10) holds.

**Proposition 23.** Let \( \delta > 0 \) and suppose \( u : M \to \mathbb{R} \) is bounded, then the family \( \{ \mathcal{L}_t u : t \geq \delta \} \) is uniformly bounded and equicontinuous.

**Proof.** By Proposition 4 there is \( K_\delta > 0 \) such that \( \{ h_t : t \geq \delta \} \) is uniformly \( K_\delta \)-Lipschitz.

Let \( x, y \in M, t \geq \delta \). For \( \varepsilon > 0 \) there is \( z_t \in M \) such that
\[
\mathcal{L}_t u(y) > u(z_t) + h_t(z_t, y) - \varepsilon
\]
(21)
Thus

\[ \mathcal{L}_t u(x) - \mathcal{L}_t u(y) \leq u(z_t) + h_t(z_t, x) - u(z_t) - h_t(z_t, y) + \varepsilon \leq K_\delta d(x, y) + \varepsilon, \]

implying that \( \{ \mathcal{L}_t u : t \geq \delta \} \) is equicontinuous.

Let \( C_\delta > 0 \) be given by Corollary 5. From (21) we get

\[ -\|u\|_\infty - C_\delta - \varepsilon \leq u(z_t) + h_t(z_t, y) - \varepsilon \]

\[ < \mathcal{L}_t u(y) \leq u(z_t) + h_t(z_t, y) \leq \|u\|_\infty + C_\delta \]

proving that \( \{ \mathcal{L}_t u : t \geq \delta \} \) is uniformly bounded. \( \square \)

To prove the convergence of \( \mathcal{L}_t \) we will follow the lines in .

For \( u \in C(M) \) let

\[ \omega_L(u) := \{ \psi \in C(M) : \exists t_n \to \infty \text{ such that } \psi = \lim_{n \to \infty} \mathcal{L}_{t_n} u \}. \]

(22) \[ u(x) := \sup_{\omega_L(u)} \{ \psi(x) : \psi \in \omega_L(u) \} \]

(23) \[ \underline{u}(x) := \inf_{\omega_L(u)} \{ \psi(x) : \psi \in \omega_L(u) \} \]

**Corollary 9.** Let \( u \in C(M) \), \( v \) be the function given by (17), \( \underline{u}, \bar{u} \) defined in (22) and (23). Then

(24) \[ v \leq \underline{u} \leq u \]

**Proposition 24.** For \( u \in C(M) \), function \( u \) given by (22) is dominated.

**Proof.** Let \( x, y \in M \). Given \( \varepsilon > 0 \) there is \( \psi = \lim_{n \to \infty} \mathcal{L}_{t_n} u \) such that \( u(x) - \varepsilon < \psi(x) \).

For \( n > N(\varepsilon) \) and \( a > 0 \)

\[ u(x) - 2\varepsilon < \psi(x) - \varepsilon \leq \mathcal{L}_{t_n} u(x) = \mathcal{L}_a(\mathcal{L}_{t_n-a} u)(x) \leq \mathcal{L}_{t_n-a} u(y) + h_a(y, x). \]

Choose a divergent sequence \( n_j \) such that \( (\mathcal{L}_{t_n-a} u) \) converges uniformly. For \( j > N(\varepsilon) \), \( \mathcal{L}_{t_n-a} u(y) < u(y) + \varepsilon \), and then

\[ u(x) - 3\varepsilon < \mathcal{L}_{t_n-a} u(y) + h_a(y, x) - \varepsilon < u(y) + h_a(y, x). \]

\( \square \)

**Proposition 25.** Suppose \( \varphi \) is dominated and \( \psi \in \omega_L(u) \). For any \( y \in M \) there exists \( \gamma \in K(y) \) such that the function \( t \mapsto \psi(\gamma(t)) - \varphi(\gamma(t)) \) is constant.

**Proof.** Let \( (s_k)_k \) and \( (t_k)_k \) be diverging sequences, \( \eta \) be a curve in \( K \) such that \( y = \lim_k \eta(s_k) \), and \( \psi \) is the uniform limit of \( \mathcal{L}_{t_k} u \). As in Proposition 15 we can assume that the sequence of functions \( t \mapsto \eta(s_k + t) \) converges uniformly on compact intervals to \( \gamma : \mathbb{R} \to M \), and so \( \gamma \in K \). We may assume moreover that \( t_k - s_k \to \infty \), as \( k \to \infty \), and that \( \mathcal{L}_{t_k-s_k} u \) converges uniformly to \( \psi_1 \in \omega_L(u) \). By (2) and (5) in Proposition 9

\[ \| \mathcal{L}_{t_k} u - \mathcal{L}_{s_k} \psi_1 \|_\infty \leq \| \mathcal{L}_{t_k-s_k} u - \psi_1 \|_\infty \]
which implies that $\mathcal{L}_{s_k} \psi_1$ converges uniformly to $\psi$. From Proposition 18 we have that for any $\tau \in \mathbb{R}$ $s \mapsto (\mathcal{L}_{s} \psi_1)(\eta(\tau + s)) - \varphi(\eta(\tau + s))$ is a nonincreasing function in $\mathbb{R}^+$, and hence it has a limit $l(\tau)$ as $s \to \infty$, which is finite since $l(\tau) \geq -\|\mathbf{\tau} - \varphi\|_\infty$.

Given $t > 0$, we have

$$l(\tau) = \lim_{k \to \infty} \left( \mathcal{L}_{s_k + t} \psi_1)(\eta(s_k + \tau + t)) - \varphi(\eta(s_k + \tau + t)) \right) = (\mathcal{L}_t \psi)(\gamma(\tau + t)) - \varphi(\gamma(\tau + t))$$

The function $t \mapsto (\mathcal{L}_t \psi)(\gamma(\tau + t)) - \varphi(\gamma(\tau + t))$ is therefore constant on $\mathbb{R}^+$. Applying Proposition 19 to the curve $\gamma(\tau + \cdot) \in \mathcal{K}$, we have $D^+((\psi - \varphi) \circ \gamma)(\tau) \setminus \{0\} = \emptyset$ for any $\tau \in \mathbb{R}$. This implies that $\psi - \varphi$ is constant on $\gamma$.

**Proposition 26.** Let $\eta \in \mathcal{K}$, $\psi \in \omega_{\mathcal{L}}(u)$ and $v$ be defined by (17). For any $\varepsilon > 0$ there exists $\tau \in \mathbb{R}$ such that

$$\psi(\eta(\tau)) - v(\eta(\tau)) < \varepsilon.$$

**Proof.** Since the curve $\eta$ is contained in $\mathcal{A}$, we have

$$v(\eta(0)) = \min_{z \in M} u(z) + \Phi(z, \eta(0)),$$

and hence $v(\eta(0)) = u(z_0) + \Phi(z_0, \eta(0))$, for some $z_0 \in M$. Take a curve $\gamma : [0, T] \to M$ such that

$$v(\eta(0)) + \varepsilon = u(z_0) + \Phi(z_0, \eta(0)) + \varepsilon > u(z_0) + \int_0^T L(\gamma, \dot{\gamma}) \geq \mathcal{L}_T u(\eta(0)).$$

Choosing a divergent sequence $(t_n)_n$ such that $\mathcal{L}_{t_n} u$ converges uniformly to $\psi$ we have for $n$ sufficiently large

$$\|\mathcal{L}_{t_n} u - \psi\|_\infty < \frac{\varepsilon}{2}, \quad t_n - T > 0.$$

Take $\tau = t_n - T$

$$\psi(\eta(\tau)) - \frac{\varepsilon}{2} < \mathcal{L}_{t_n} u(\eta(\tau)) = \mathcal{L}_t \mathcal{L}_T u(\eta(\tau)) = \mathcal{L}_T u(\eta(0)) + \int_0^T L(\eta, \dot{\eta})$$

$$< \frac{\varepsilon}{2} + v(\eta(0)) + \int_0^T L(\eta, \dot{\eta}) = \frac{\varepsilon}{2} + v(\eta(\tau))$$

□

From Propositions 25 and 26 we obtain

**Theorem 5.** Let $\psi \in \omega_{\mathcal{L}}(u)$ and $v$ be defined by (17). Then $\psi = v$ on $\mathcal{M}$.

**Theorem 6.** Assume that there are $C_1, C_2 > 0$ such that (10) holds. Let $u \in C(M)$, then $\mathcal{L}_t u$ converges uniformly as $t \to \infty$ to $v$ given by (17).

**Proof.** The function $u$ is dominated and coincides with $v$ on $\mathcal{M}$ by Theorem 5. Proposition 17 implies that $u$ coincide with $v$ on $\mathcal{A}$ and so does with $w$. By item (1) of Corollary 8 we have $u \leq v$. □
5. HAMILTON-JACOBI EQUATION

5.1. Viscosity solutions.

**Definition 8.** Let $N$ be a manifold, $F : T^*N \to \mathbb{R}$ be continuous and consider the Hamilton-Jacobi equation

\[(25)\quad F(u(z), du(z)) = 0\]

We say that $u \in C(N)$ is a *viscosity subsolution* of (25) if $\forall \varphi \in C^1(N)$ s.t. $z_0 \in N$ is a point of local maximum of $u - \varphi$ we have

\[F(u(x_0), d\varphi(x_0)) \leq 0.\]

We say that $u \in C(N)$ is a *viscosity supersolution* of (25) if $\forall \varphi \in C^1(N)$ s.t. $z_0 \in N$ is a point of local minimum of $u - \varphi$ we have

\[F(u(x_0), d\varphi(z_0)) \geq 0.\]

We say that $u$ is a *viscosity solution* if it is both, a subsolution and a supersolution.

Let $\pi^* : T^*M \to M$ be the natural projection and define the sub-Riemannian Hamiltonian $H : T^*M \to \mathbb{R}$ by

\[H(p) = \max\{p(v) - L(v) : v \in \mathcal{D}, \pi(v) = \pi^*(p)\}.\]

For $\omega \in \Omega^1(M)$, the Hamiltonian associated to $L - \omega$ is $H(p + \omega(\pi^*(p)))$

**Proposition 27.** Let $u \in C(M)$, then $w : M \times [0, \infty[ \to \mathbb{R}$ given by $w(x, t) = L_t u(x)$, is a viscosity solution of

\[
\begin{align*}
&w_t(x, t) + H(d_x w(x, t)) = 0, \quad x \in M, t > 0 \\
&w(x, 0) = u(x)
\end{align*}
\]

**Proposition 28.** Let $\lambda \geq 0$, then $u \in C(M)$ is a viscosity subsolution of

\[(26)\quad \lambda u(x) + H(du(x)) - c = 0,
\]

if and only if

\[(27)\quad u(\gamma(b))e^{\lambda b} - u(\gamma(a))e^{\lambda a} \leq A_{L+c,\lambda}(\gamma) \text{ for any } \gamma \in W^1_\mathcal{D}([a, b])
\]

**Corollary 10.**

1. A viscosity subsolution of (26) is Lipschitz for the metric $d$.
2. Considering the case $\lambda = 0$

\[(28)\quad H(du(x)) - c = 0,
\]

\[c(L) = \inf\{c \in \mathbb{R} : (28) \text{ has a viscosity subsolution } u \in C(M)\}\]

**Proposition 29.**

1. For $\lambda > 0$, let $u_\lambda$ be the value function, then $u_\lambda + c/\lambda$ is a viscosity solution of (26).
(2) If \( u = \mathcal{L}_t u + ct \) for any \( t > 0 \), then \( u \) is a viscosity solution of (28).

Item (2) of Proposition 29 can be considered a special case of Proposition 27 since \( w(x, t) = \mathcal{L}_t u(x) = u(x) - ct \) and then \( d_x w(x, t) = du(x), w_t = -c \).

Under some mild assumption that can be fulfilled for a sub-Riemannian Hamiltonian, the comparison principle for (26) holds for \( \lambda > 0 \). In particular (26) has a unique viscosity solution. An instance of such an assumption is

\[
\left| \frac{\partial H}{\partial x} (x, p) \right| \leq (1 + |p|)\Phi(H(x, p))
\]

for a continuous function \( \Phi : \mathbb{R} \to \mathbb{R}^+ \), see page 35 in [Ba]. This fact implies the following.

**Corollary 11.** Assume that (29) holds, then there is only one constant \( c \) such that (28) has viscosity solutions.

**Proof.** Suppose that there are \( c_1 < c_2 \) such that (28) admits viscosity solutions \( u_1, u_2 \) respectively. Adding a constant to \( u_1 \) we may suppose \( u_1 > u_2 \). Let \( c_1 < c < c_2 \). For \( \lambda > 0 \) small enough \( u_1 \) and \( u_2 \) are a subsolution and a supersolution of (26). By comparison, \( u_1 \leq u_2 \) which is a contradiction. \( \square \)

### 6. Aubry-Mather theory

In this section we extend Aubry-Mather theory to our setting. We take Mañé’s approach using closed measures. We assume again that \( M \) is compact.

Let \( g \) be a Riemannian metric on \( M \) that tames the sub-Riemannian metric. The values of \( g \) outside \( \mathcal{D} \) will not play any role. Let \( C^0_\mathcal{E} \) be the set of continuous functions \( f : TM \to \mathbb{R} \) having linear growth, i.e.

\[
\|f\|_\mathcal{E} := \sup_{v \in \mathcal{D}} \frac{|f(v)|}{1 + \|v\|_g} < +\infty.
\]

Let \( \mathcal{P}_\mathcal{E} \) be the set of Borel probability measures on \( TM \) such that

\[
\int_{TM} \|v\|_g \, d\mu < +\infty,
\]

endowed with the topology such that \( \lim_n \mu_n = \mu \) if and only if

\[
\lim_n \int_{TM} f \, d\mu_n = \int_{TM} f \, d\mu
\]

for all \( f \in C^0_\mathcal{E} \).

Let \( (C^0_\mathcal{E})' \) be the dual of \( C^0_\mathcal{E} \). Then \( \mathcal{P}_\mathcal{E} \) is naturally embedded in \( (C^0_\mathcal{E})' \) and its topology coincides with that induced by the weak* topology on \( (C^0_\mathcal{E})' \). This topology is metrizable, indeed, let \( \{f_n\} \) be a sequence of functions with compact support on
\(C^0_\ell\) which is dense on \(C^0_\ell\) in the topology of uniform convergence on compact sets of \(TM\). The metric \(d_\ell\) on \(\mathcal{P}_\ell\) defined by

\[
d_\ell(\mu_1, \mu_2) = \left| \int_{TM} \|v\|_g \, d\mu_1 - \int_{TM} \|v\|_g \, d\mu_2 \right| + \sum_n \frac{1}{2^n \|f_n\|_\infty} \left| \int_{TM} f_n \, d\mu_1 - \int_{TM} f_n \, d\mu_2 \right|
\]

gives the topology of \(\mathcal{P}_\ell\).

Let \(\mathcal{P}_D\) be the set of measures in \(\mathcal{P}_\ell\) that are supported in \(D\).

**Proposition 30.** The function \(A_L : \mathcal{P}_D \to \mathbb{R} \cup \{+\infty\}\), defined as

\[
A_L(\mu) = \int_{TM} \psi(\pi(v)) v \, d\mu(v)
\]

for all \(\psi \in C^1(M)\). We denote by \(V_D\) the convex closed set of vakonomic measures.

**Definition 9.** A measure \(\mu\) in \(\mathcal{P}_D\) is called closed or vakonomic (see [G]) if

\[
(30) \quad \int_{TM} D\varphi(\pi(v)) v \, d\mu(v) = 0
\]

for all \(\varphi \in C(M)\). We denote by \(V_D\) the convex closed set of vakonomic measures.

For \(\mu \in V_D\) we define its rotation vector \(\rho(\mu) \in H_1(M, \mathbb{R})\) by

\[
(31) \quad \langle [\omega], \rho(\mu) \rangle = \int_{TM} \omega \, d\mu, \quad \omega \in \Omega^1(M), d\omega = 0
\]

For \(\gamma \in W^{1,1}_D([a,b])\) we define the measure \(\mu_\gamma\) by

\[
(32) \quad \int_{TM} f \, d\mu_\gamma = \frac{1}{b-a} \int_a^b f(\gamma), \quad f \in C^0_\ell,
\]

when \(\gamma(a) = \gamma(b), \mu_\gamma \in V_D\) and it is called holonomic. We denote by \(H_D\) the set of holonomic measures.

More generally, if \(\gamma_n \in W^{1,1}_D([a_n, b_n])\), \(b_n - a_n \to \infty\) and \(\mu = \lim_n \mu_{\gamma_n}\) exists then \(\mu \in V_D\). Indeed, we observe again that if \(\alpha \in W^{1,1}_D\) is a minimizing geodesic with \(\|\dot{\alpha}\| = 1\) a.e. we have that \(-C(0)\ell(\alpha) \leq A_L(\alpha) \leq A(1)\ell(\alpha)\). Fix \(x_0 \in M\), take unit speed geodesics \(\alpha_n\) from \(x_0\) to \(\gamma_n(a_n)\) and \(\beta_n\) from \(\gamma_n(b_n)\) to \(x_0\), and the concatenation \(\delta_n\) of \(\alpha_n, \gamma_n\) and \(\beta_n\). Then \(\delta_n\) is a closed curve and for any \(f \in C^0_\ell\) we have

\[
\lim_n \int_{TM} f \, d\mu_{\delta_n} = \lim_n \int_{TM} f \, d\mu_{\gamma_n} = \int_{TM} f \, d\mu.
\]

Thus \(\mu \in V_D\) and \(\rho(\mu) = \lim \rho(\mu_{\delta_n})\).

Set \(\mathbb{G} := \{ h \in H_1(M, \mathbb{Z}) : kh \neq 0 \text{ for all } k \in \mathbb{N} \}\) and consider the abelian cover \(\Pi : \bar{M} \to M\) whose group of deck transformations is \(\mathbb{G}\). The distribution \(\mathcal{D}\) lifts to a bracket generating distribution \(\bar{\mathcal{D}} \subset T\bar{M}\).
Fix a basis \( c_1, \ldots, c_k \) of \( H^1(M, \mathbb{R}) \) and fix closed 1-forms \( \omega_1, \ldots, \omega_k \) in \( M \) such that \( \omega_i \) has cohomology class \( c_i \). Let \( G : \bar{M} \rightarrow H_1(M, \mathbb{R}) \) be given by

\[
\langle \sum_i a_ic_i, G(x) \rangle = \int_{\bar{x}_0}^x \sum_i a_i\bar{\omega}_i,
\]

where \( \bar{x}_0 \) is a base point in \( \bar{M} \) and \( \bar{\omega}_i \) is the lift of \( \omega_i \) to \( \bar{M} \). Since \( \bar{\omega}_i \) is exact, the integral does not depend on the choice of the path from \( \bar{x}_0 \) to \( x \). Notice that the function \( G \) depends on the choice of \( \bar{x}_0 \), on the choice of the basis \( \{c_i\} \) and of representatives \( \omega_i \) in \( c_i \). In general, if \( x, y \) belong to the same fiber, then \( G(x) - G(y) \) is the transformation in \( \mathbb{G} \) carrying \( y \) to \( x \), condition which uniquely identifies it, conversely, any deck transformation admits a representation of this type.

Let \( \bar{\gamma} \in W^{1,1}_D([a, b]) \) and \( \gamma = \Pi \circ \bar{\gamma} \). If \( \omega \in \Omega^1(M) \), \( d\omega = 0 \), there is \( f \in C^1(M) \) such that

\[
\int_{T \bar{M}} \omega \, d\mu_{\gamma} = \frac{1}{b-a}(\langle [\omega], G(\gamma(b)) - G(\gamma(a)) \rangle + f((\gamma(b)) - f(\gamma(a))].
\]

If \( T \in \mathbb{G}, a < b \), take \( \bar{\gamma} \in W^{1,1}_D([a, b]) \) such that \( \bar{\gamma}(b) = T \bar{\gamma}(a) \), then \( T = G(\bar{\gamma}(b)) - G(\bar{\gamma}(a)) \). If \( \gamma = \Pi \circ \bar{\gamma} \) then \( \gamma(\bar{\gamma}(a)) = \gamma(b) \) and \( \rho(\mu_{\gamma}) = T/(b-a) \).

If \( \bar{\gamma}_n \in W^{1,1}_D([a_n, b_n]) \), \( \gamma_n = \Pi \circ \bar{\gamma}_n \), \( b_n - a_n \rightarrow \infty \) and \( \mu = \lim_n \mu_{\gamma_n} \) exists then

\[
h = \lim_{n \rightarrow \infty} \frac{G(\bar{\gamma}(b_n)) - G(\bar{\gamma}(a_n))}{b_n - a_n}
\]

exists and \( \rho(\mu) = h \).

**Lemma 4.** The map \( \rho : \mathcal{V}_D \rightarrow H_1(M, \mathbb{R}) \) is surjective.

**Proof.** Since \( \mathcal{V}_D \) is convex and the map \( \rho \) is affine, we have that \( \rho(\mathcal{V}_D) \) is convex, and we have proved that \( \rho(\mathcal{V}_D) \) contains \( \mathbb{G} \).

**Lemma 5.** If \( M \) is connected the set \( \overline{\mathcal{H}_D} \) is convex.

**Proof.** Let \( \eta_1, \eta_2 \in \overline{\mathcal{H}_D} \), and let \( \lambda_1 \) and \( \lambda_2 \) in \([0, 1]\) be such that \( \lambda_1 + \lambda_2 = 1 \). There are sequences \( \gamma_n^i \in \mathcal{C}_{S_i}(x_n^i, x_n^i), i = 1, 2 \) such that \( \mu_{\gamma_n} \) converges to \( \eta_i \). There are \( m_1, m_2 \in \mathbb{N} \) such that \( T_n^i = m_n^iS_n^i \rightarrow +\infty \) and \( T_n^i \overline{T_n^2} \) converges to \( \frac{\lambda_1}{\lambda_2} \), then \( \frac{T_n^i}{\overline{T_n^2}} \) converges to \( \lambda_i \).

Taking a minimizing geodesic \( \alpha_n \in \mathcal{C}_1(x_n^1, x_n^2) \), define the curve \( \gamma_n \in \mathcal{C}_{T_n^1 + T_n^2 + 2}(x_n^1, x_n^2) \) by

\[
\gamma_n(t) = \begin{cases} 
\gamma_n^1(t) & t \in [0, T_n^1] \\
\alpha_n(t - T_n^1) & t \in [T_n^1, T_n^1 + 1] \\
\gamma_n^2(t - T_n^1 - 1) & t \in [T_n^1 + 1, T_n^1 + T_n^2 + 1] \\
\alpha_n(T_n^1 + T_n^2 + 2 - t) & t \in [T_n^1 + T_n^2 + 1, T_n^1 + 2].
\end{cases}
\]

Then \( \mu_{\gamma_n} \) converges to \( \lambda_1 \eta_1 + \lambda_2 \eta_2 \).
Theorem 7. $\mathcal{V}_D = \overline{H}_D$

Proof. As in [B], by Lemma 5 and Proposition 31.

Proposition 31. Let $C$ be a closed convex subset of $\mathcal{P}_\ell$ and let

$$C^+ = \{ f \in C^0_\ell : \int_{TM} f d\mu \geq 0 \ \forall \mu \in C \}.$$ 

Then

$$C = \{ \mu \in \mathcal{P}_\ell : \int_{TM} f d\mu \geq 0 \ \forall f \in C^+ \},$$

it is enough to prove that $\overline{H}_D^+ \subset \mathcal{V}_D^+$. The proof of this fact is the same as the proof of a similar fact in [B]. □

Proposition 32. If $\mu \in \mathcal{V}_D$ then there is a sequence $\mu_{\eta_n} \in H_D$ converging to $\mu$ such that $A_L(\mu_{\eta_n})$ converges to $A_L(\mu)$

Proof. The proof is similar to the one of Proposition 2-3.3 in [CI]. □

Proposition 33. 

$$\min \{ A_L(\mu) : \mu \in \mathcal{V}_D \} = -c(L).$$

Proof. Set $c = c(L)$ and let $x \in M$, then there is a sequence $x_n \in C_{t_n}(x, x)$ with $t_n \to \infty$ such that

$$h(x, x) = \lim_{n \to \infty} A_L(\gamma_n) + c t_n.$$ 

Let $\mu_0 \in \mathcal{V}_D$ be the limit of a subsequence of $\mu_{\gamma_n}$, then

$$0 = \lim_{n \to \infty} \frac{1}{t_n} A_L(\gamma_n) + c = \lim_{n \to \infty} A_L(\mu_{\gamma_n}) + c \geq A_L(\mu_0) + c \geq \inf \{ A_L(\mu) : \mu \in \mathcal{V}_D \} + c.$$ 

Let $\mu \in \mathcal{V}_D$, by Proposition 32 there is a sequence $\gamma_n \in C_{t_n}(x_n, x_n)$ such that

$$\frac{1}{t_n} A_L(\gamma_n) = A_L(\mu_{\gamma_n})$$

converges to $-c(L)$. Since $A_L(\gamma_n) + c t_n \geq \Phi(x_n, x_n) = 0$,

$$A_L(\mu) + c = \lim_{n \to \infty} \frac{1}{t_n} A_L(\gamma_n) + c \geq 0.$$ 

Thus $\inf \{ A_L(\mu) : \mu \in \mathcal{V}_D \} + c \geq 0$ and then

$$A_L(\mu_0) = \inf \{ A_L(\mu) : \mu \in \mathcal{V}_D \} = -c$$

□

We define the set of Mather measures as $\mathfrak{M}(L) = \{ \mu \in \mathcal{V}_D : A_L(\mu) = -c(L) \}$, and the set of projected Mather measures $\mathfrak{M}_0(L) = \{ \pi_\# \mu : \mu \in \mathfrak{M}(L) \}$.
Example 4. Let $M, U : M \to \mathbb{R}$ and $L$ be as in example 1. Since $\min L = -\max U = -c(L)$, the support of any $\mu \in \mathcal{M}(L)$ is contained in $\{0\} \times U^{-1}(\max U) = \{0\} \times A$.

If $\mu$ is a probability measure in $TM$ supported in $\{0\} \times A$, then $\mu \in \mathcal{V}_A$ and $\int L d\mu = -\max U$ and thus $\mu \in \mathcal{M}(L)$. Therefore $\mathcal{M}_0(L)$ is the set of probability measures in $M$ supported in $U^{-1}(\max U) = A$.

Definition 10. We define the effective Lagrangian $L : H^1(M, \mathbb{R}) \to \mathbb{R}$, or Mather $\beta$ function, by

\[(33)\quad L(h) = \inf \{ A_L(\mu) : \rho(\mu) = h \}\]

It is clear that $L$ is convex and its Legendre transform, called effective Hamiltonian, $H : H^1(M, \mathbb{R}) \to \mathbb{R}$ is given by

\[(34)\quad H([\omega]) = -\inf \{ A_{L-\omega}(\mu) : \mu \in \mathcal{V}_A \} = c(L - \omega)\]

7. CONVERGENCE OF THE DISCOUNTED VALUE FUNCTION

Suppose $M$ is compact and let $c = c(L)$. For $\lambda > 0$ let $u_\lambda$ be the discounted value function.

Proposition 34. For every $\mu \in \mathcal{M}_0(L)$ we have $\int_M u_\lambda \ d\mu \leq -\frac{c}{\lambda}$.

Proof. Let $\gamma_k \in C_{t_k}(x_k, x_k)$. By item (2) of Proposition 2

\[u_\lambda(\gamma_k(t)) \leq e^{-\lambda t}(u_\lambda(x_k) + \int_0^t e^{\lambda s} L(\dot{\gamma}_k(s)) \ ds)\]

Integration by parts gives

\[\int_0^{t_k} u_\lambda(\gamma_k(t)) \ dt \leq \frac{1 - e^{-\lambda t_k}}{\lambda} u_\lambda(x_k) + \int_0^{t_k} e^{-\lambda t} \int_0^t e^{\lambda s} L(\dot{\gamma}_k(s)) \ ds \ dt\]

\[= \frac{1 - e^{-\lambda t_k}}{\lambda} u_\lambda(x_k) - e^{-\lambda t_k} \int_0^{t_k} e^{\lambda s} L(\dot{\gamma}_k(s)) \ ds + \frac{1}{\lambda} \int_0^{t_k} L(\dot{\gamma}_k)\]

\[\leq \frac{A_L(\gamma_k)}{\lambda}\]

If the sequence $\mu_{\gamma_k} \in \mathcal{H}_A$ converges to $\nu \in \mathcal{V}_A$ we have

\[\int_M u_\lambda \ d\pi_#\nu \leq \frac{A_L(\nu)}{\lambda}\]

In particular, for $\nu \in \mathcal{M}(L)$ choose a sequence $\mu_{\gamma_k} \in \mathcal{H}_A$ converging to $\nu$, and then

\[\int_M u_\lambda \ d\pi_#\nu \leq -\frac{c}{\lambda}\]

\[\square\]

Proposition 35. $u_\lambda + \frac{c}{\lambda}$ is uniformly bounded.
Proof. Fix $z \in A$ and let $v(x) = h(z, x)$. Let $x \in M$.

By Proposition 4 there is $\gamma_{\lambda, x} \in W^{1,1}_D([-\infty, 0])$ such that $\gamma_{\lambda, x}(0) = x$ and

$$u_\lambda(x) = \int_{-\infty}^{0} e^{\lambda t} L(\gamma_{\lambda, x}(t)) \, dt \geq \int_{-\infty}^{0} e^{\lambda t} ((v \circ \gamma_{\lambda, x})' - c) \, dt$$

$$\geq v(x) - \int_{-\infty}^{0} e^{\lambda t} (\lambda v \circ \gamma_{\lambda, x} - c) \, dt \geq v(x) - \max v - \frac{c}{\lambda}.$$ 

There is a sequence $\gamma_n \in C_{t_n}(x, x)$ with $t_n \to \infty$ such that $h(x, x) = \lim_{n \to \infty} A_{L}(\gamma_n) + c t_n$.

By (27), $u_\lambda(x)(e^{\lambda t_n} - 1) \leq A_{L, \lambda}(\gamma_n)$. Considering the function $a_n(t) = A_{L}(\gamma_n |_{[0, t]})$ and integrating by parts

$$u_\lambda(x)(e^{\lambda t_n} - 1) \leq A_{L}(\gamma_n) e^{\lambda t_n} - \int_{0}^{t_n} e^{\lambda t} a_n(t) \, dt$$

$$\leq A_{L}(\gamma_n) e^{\lambda t_n} + \int_{0}^{t_n} e^{\lambda t} (v(x) - v(\gamma_n(t)) + c) \, dt$$

$$\leq A_{L}(\gamma_n) e^{\lambda t_n} + (e^{\lambda t_n} - 1)(v(x) - \min v) + c(e^{\lambda t_n} - \frac{e^{\lambda t_n} - 1}{\lambda})$$

$$= (e^{\lambda t_n} - 1)(A_{L}(\gamma_n) + c t_n + v(x) - \min v - \frac{c}{\lambda}) + A_{L}(\gamma_n) + c t_n$$

dividing by $e^{\lambda t_n} - 1$ and letting $n \to \infty$,

$$u_\lambda(x) + \frac{c}{\lambda} \leq h(x, x) + v(x) - \min v \leq h(x, z) + 2v(x) - \min v.$$ 

□

Corollary 12. If $u_{\lambda_k} + \frac{c}{\lambda_k}$ converges uniformly to $u$ for $\lambda_k \to 0$, then $u$ is a weak KAM solution and $\int_M u \, d\mu \leq 0$

Proof. The first assertion follows as in Theorem 4. We already proved that $\lambda_k \int_M u_{\lambda_k} \, d\mu \leq -c$ for $\mu \in M_0$. Thus $\int_M (u_{\lambda_k} + \frac{c}{\lambda_k}) \, d\mu \leq 0$ and letting $k \to \infty$ we get $\int_M u \, d\mu \leq 0$. □

Theorem 8. Let $u_\lambda$ be the discounted value function, then $u_\lambda + \frac{c}{\lambda}$ converges uniformly as $\lambda \to 0$ to the weak KAM solution $u_0$ given by either of the following ways

(i) it is the largest viscosity subsolution of (28) such that $\int_M u \, d\mu \leq 0$ for any $\mu \in M_0(L)$.

(ii) $u_0(x) = \inf_M \{ \int_M h(y, x) \, d\mu(y) : \mu \in M_0(L) \}$. 


Now the proof of Theorem 8 goes as in [DFIZ]

8. Homogenization

As in section 6, let $M$ be a compact manifold. We follow the approach of [CIS] and refer to that paper for the definition of the convergence of a family of metric spaces $(M_n, d_n)$ to a metric space $(M, d)$ through continuous maps $F_n : M_n \to M$. For the Carnot Caratheodory metric $d$ on $\bar{M}$ for the distribution $\bar{D}$, we have as in [CIS] Proposition 36.

$$\lim_{\varepsilon \to 0} (\bar{M}, \varepsilon d, \varepsilon G) = H_1(M, \mathbb{R}).$$

Let $\bar{L} : \bar{D} \to \mathbb{R}$ be the lift of $L$ and

$$\bar{H}(p) = \max\{p(v) - \bar{L}(v) : \bar{\pi}(v) = \bar{\pi}^*(p)\}.$$

We are interested in the limit as $\varepsilon \to 0$ of the Cauchy problem

$$(35) \quad \begin{cases} w_t + \bar{H}(dw/\varepsilon) = 0, & x \in \bar{M}, t > 0; \\
 w(x, 0) = \tilde{f}_\varepsilon(x). 
\end{cases}$$

for $f_\varepsilon : \bar{M} \to \mathbb{R}$ continuous, uniformly $\varepsilon G$-converging to some $f : H_1(M, \mathbb{R}) \to \mathbb{R}$. Since

$$\bar{H}(p/\varepsilon) = \max\{p(v) - \bar{L}(\varepsilon v) : \bar{\pi}(v) = \bar{\pi}^*(p)\},$$

letting $L^\varepsilon_t$ be the Lax-Oleinik semi-group for the Lagrangian $\bar{L}(\varepsilon v)$, by Proposition 27 we have that $u_\varepsilon$ given by $u_\varepsilon(x, t) = L^\varepsilon_t f_\varepsilon(x)$ is a viscosity solution to (35). Under assumption (29) the viscosity solution is unique. We have

$$(36) \quad u_\varepsilon(x, t) = \inf\{f_\varepsilon(\tilde{\gamma}(0)) + \int_0^t \bar{L}(\varepsilon \dot{\tilde{\gamma}}) : \tilde{\gamma} \in W^{1,1}_D([0, t])\}$$

$$= \inf\{f_\varepsilon(\alpha(0)) + \varepsilon A_L(\bar{\alpha}) : \bar{\alpha} \in W^{1,1}_D([0, t/\varepsilon])\}$$

where $\bar{h}_t : \bar{M} \times \bar{M} \to \mathbb{R}$ is the minimal action for the Lagrangian $\bar{L}$.

Theorem 9. Let $f_\varepsilon : \bar{M} \to \mathbb{R}$, $f : H_1(M, \mathbb{R}) \to \mathbb{R}$ be continuous, with $f$ having at most linear growth, such that $f_\varepsilon$ uniformly $\varepsilon G$-converges to $f$. Then the family of functions $u_\varepsilon : \bar{M} \times [0, +\infty[ \to \mathbb{R}$ locally uniformly $\varepsilon G$-converges in $\bar{M} \times [0, +\infty[$ to the viscosity solution $u : H_1(M, \mathbb{R}) \times [0, +\infty[ \to \mathbb{R}$ of the problem

$$(37) \quad \begin{cases} u_t + \mathbb{H}(du) = 0, & h \in H_1(M, \mathbb{R}), t > 0; \\
 u(h, 0) = f(h). 
\end{cases}$$

We need the following Lemma
Lemma 6 ([CIS]). Given a compact subset $K$ in $H^1(M, \mathbb{R})$, and a compact interval $I \subset [0, +\infty]$, there is a constant $C > 0$ such that $\varepsilon d(x, y) \leq C$ for $\varepsilon > 0$ suitably small $\varepsilon G(x) \in K$, $t \in I$ and $y \in \bar{M}$ realizing the minimum in the (36).

Proof of Theorem 9. The solution for the limit problem (37) is
\[
\liminf_{n \to \infty} u_{\varepsilon_n}(x_n, t_n) = \inf_{\mu \in H_1(M, \mathbb{R})} \{ f(q) + tL(\frac{z - q}{t}) : q \in H_1(M, \mathbb{R}) \}.
\]

Let $h \in H_1(M, \mathbb{R})$, $t > 0$. Let $\{\varepsilon_n\}$ be a sequence converging to zero and let $\{x_n\}, \{t_n\}$ be subsequences in $\bar{M}$, $(0, +\infty)$ such that $\varepsilon_n G(x_n)$ converges to $z$ and $t_n$ converges to $t$. Our goal is to prove that $u_{\varepsilon_n}(x_n, t_n)$ converges to $u(z, t)$.

Let $y_n \in \bar{M}$, and $\gamma_n \in \mathcal{C}_{t_n/\varepsilon_n}(y_n, x_n)$ be such that
\[
u_{\varepsilon_n}(x_n, t_n) = f_{\varepsilon_n}(y_n) + \varepsilon_n \bar{h}_{t_n/\varepsilon_n}(y_n, x_n) = f_{\varepsilon_n}(y_n) + \varepsilon_n A_L(\gamma_n).
\]

By Lemma 6, the sequence $\varepsilon_n G(y_n)$ is bounded, so it converges up to subsequences, to $q \in H_1(M, \mathbb{R})$ and then
\[
\lim_n \varepsilon_n(G(x_n) - G(y_n)) = z - q.
\]

Consider the measure $\mu_{\gamma_n}$ for $\gamma_n = \Pi \circ \gamma_n$
\[
\bar{h}_{t_n/\varepsilon_n}(x_n, t_n) = A_L(\gamma_n) = t_n A_L(\mu_{\gamma_n})/\varepsilon_n.
\]

Passing to a further subsequence we can assume that $\mu_{\gamma_n}$ converges to $\mu$, then we have
\[
\rho(\mu) = \frac{z - q}{t} \quad \text{and} \quad \liminf_n A_L(\mu_{\gamma_n}) \geq A_L(\mu).
\]

Thus
\[
\liminf_n \varepsilon_n \bar{h}_{t_n/\varepsilon_n}(x_n, x_n) = \liminf_n t_n A_L(\mu_{\gamma_n}) \geq t A_L(\mu) \geq tL(\frac{z - q}{t}).
\]

By the the uniform $\varepsilon_n G$-convergence of $f_{\varepsilon_n}$ to $f$, we obtain
\[
\liminf_n u_{\varepsilon_n}(x_n, t_n) = \liminf_n f_{\varepsilon_n}(y_n) + \varepsilon \bar{h}_{t_n/\varepsilon_n}(y_n, x_n) \geq f(q) + tL(\frac{z - q}{t}) \geq u(z, t)
\]

Let $l = \limsup_n u_{\varepsilon_n}(x_n, t_n)$ and take a subsequence of $\{\varepsilon_n\}$, still denoted $\{\varepsilon_n\}$, such that $l = \lim_{n \to \infty} u_{\varepsilon_n}(x_n, t_n)$. Choose $q \in H_1(M, \mathbb{R})$ such that $u(z, t) = f(q) + tL(\frac{z - q}{t})$. Let $\mu \in \mathcal{V}_D$ be such that $\rho(\mu) = \frac{z - q}{t}$ and $A_L(\mu) = L(\frac{z - q}{t})$. There is a sequence $\mu_{\eta_k} \in \mathcal{H}_D$ converging to $\mu$ such that $\{A_L(\mu_{\eta_k})\}$ converges to $A_L(\mu)$.

We have that $\eta_k \in \mathcal{C}_{T_k}(p_k, p_k)$ and we can assume that $T_k \geq 1$. Take a subsequence $\{\varepsilon_{n_k}\}$ such that $\varepsilon_{n_k} T_k$ converges to $0$.

Choose a lift $y_k$ of $p_{n_k}$ such that $d_k^* = d(y_k, x_{n_k}) \leq \text{diam } M$. Let $m_k \in \mathbb{N}$ be such that
\[
(1 + m_k)T_k \leq t_{n_k}/\varepsilon_{n_k} < (2 + m_k)T_k,
\]

then $\varepsilon_{n_k} m_k T_k$ converges to $t$.
Let $\beta_k \in W^{1,1}_D([0, m_k T_n k])$ be the lift of the curve $\eta_k$ travelled $m_k$ times such that $\beta_k(m_k T_k) = y_k$, then $\frac{G(y_k) - G(\beta_k(0))}{m_k T_k} = \rho(\mu_{\eta_k})$. Thus $\varepsilon_{n_k}(G(y_k) - G(\beta_k(0))) = \varepsilon_{n_k} m_k T_k \rho(\mu_{\eta_k})$ converges to $z - q$ and since $\varepsilon_{n_k}(G(x_{n_k}) - G(y_k))$ converges to 0, we have that $\varepsilon_{n_k} G(\beta_k(0))$ converges to $q$.

Take a minimizing geodesic $\beta_k \in C_{0,d^*}(y_k, x_{n_k})$ with $\|\dot{\beta}_k\| = 1$ a.e. Letting $\alpha_k(s) = \beta_k(d^*_k s / (t_{n_k} - m_k T_k))$ we have that $|\dot{\alpha}_k| \leq d^*_k$ and then $A_L(\alpha_k) \leq 2A(diam M) T_k$.

\[ l = \lim_k u_{\varepsilon_{n_k}}(x_{n_k}, t_{n_k}) \leq \limsup_k f_{\varepsilon_{n_k}}(\beta_k(0)) + \varepsilon_{n_k}(A_{L}(\beta_k) + A_{L}(\alpha_k)) \]

\[ \leq \lim_k f_{\varepsilon_{n_k}}(\beta_k(0)) + \varepsilon_{n_k} m_k T_k A_{L}(\mu_{\eta_k}) + 2\varepsilon_{n_k} T_k A(diam M) = f(q) + tL \left( \frac{z - q}{t} \right). \]

9. Open questions

- Is any viscosity solution of (28) a weak KAM solution?
- Is the support of any $\mu \in \mathcal{M}_0(L)$ contained in $\mathcal{A}$?

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