DISTINGUISHING SOME GENUS ONE KNOTS USING FINITE QUOTIENTS

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Abstract. We give a criterion for distinguishing a prime knot $K$ in $S^3$ from every other knot in $S^3$ using the finite quotients of $\pi_1(S^3 \setminus K)$. Using recent work of Baldwin-Sivek, we apply this criterion to the hyperbolic knots $5_2$, $15n_{43522}$, and the three-strand pretzel knots $P(-3,3,2n+1)$ for every integer $n$.

1. Introduction

Finite quotients of the fundamental group are useful for distinguishing 3-manifolds; in particular, for a 3-manifold $M$, a finite quotient of $\pi_1(M)$ corresponds to the deck group of a finite-sheeted regular cover of $M$. If the fundamental groups of two 3-manifolds $M$ and $N$ have different finite quotients, then $\pi_1(M) \not\cong \pi_1(N)$ and the 3-manifolds $M$ and $N$ are not homeomorphic. When $M$ is a compact 3-manifold, its fundamental group $\pi_1(M)$ is residually finite [Hem16], and so the set $C(\pi_1(M))$ of finite quotients of $\pi_1(M)$ is non-empty and infinite.

One of the consequences of the residual finiteness of the fundamental groups of compact 3-manifolds is that the unknot is the only knot in $S^3$ whose knot complement has a fundamental group with only finite cyclic quotients. Boileau [Boi18] has conjectured that every prime knot $K \subset S^3$ is completely determined by the set of finite quotients of $\pi_1(S^3 \setminus K)$, that is, if for two prime knots $J$ and $K$, $\pi_1(S^3 \setminus J)$ and $\pi_1(S^3 \setminus K)$ have the same set of finite quotients, then $J$ and $K$ are isotopic. By work of Boileau-Friedl [BF20] and Bridson-Reid [BR20], it is known that the figure-eight knot $4_1$ and the trefoil knot $3_1$ are completely determined by the finite quotients of their knot groups even amongst the fundamental groups of compact 3-manifolds. Furthermore, Wilkes [Wil19] has shown that knots in $S^3$ whose complements are graph manifolds are distinguished by the finite quotients of their knot groups.

The purpose of this note is to show:

Theorem 1.1. Let $K$ be the knot $5_2$ (shown in Figure 1A), one of the hyperbolic pretzel knots $P(-3,3,2n+1)$ ($n \in \mathbb{Z}$) (Figure 1B), or the knot $15n_{43522}$ (shown in Figure 1C), then $K$ is distinguished from every other knot in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus K)$.

Theorem 1.1 will follow from our next theorem, using recent work of Baldwin-Sivek [BS22, BS22b], and the work of Wilkes [Wil18] (as described in Section 3). To state the theorem, we recall the definition of a characterizing slope $\alpha \in \mathbb{Q}$ for a knot $K$. For a knot $K \subset S^3$, let $S_\alpha(K)$ be the 3-manifold obtained by $\alpha -$Dehn surgery on $K$. 


Figure 1. Hyperbolic knots in the statement of Theorem 1.1

Definition 1.2. A slope $\alpha$ is a characterizing slope for a knot $K \subset S^3$ if for any knot $J \subset S^3$, $S^3_\alpha(J) \cong S^3_\alpha(K)$ if and only if $J$ is isotopic to $K$.

Theorem 1.3. Let $K$ be a hyperbolic knot in $S^3$ for which

1. $0$ is a characterizing slope for $K$,  
2. $S^3_0(K)$ is distinguished from every other compact, irreducible 3-manifold by the finite quotients of $\pi_1(S^3_0(K))$.

then $K$ is distinguished from other knots in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus K)$.

We point out that every knot in $S^3$ has infinitely many characterizing slopes [Lac19]. On the other hand, it is known that some knots have infinitely many non-characterizing (integral) slopes [BM18]. Our note relies on recent work of Baldwin-Sivek [BS22, BS22b] showing that $0$ is a characterizing slope for a family of genus 1 knots. Prior to this, the only knots previously known to have $0$ as a characterizing slope were the unknot (Property R), the trefoil, and the figure-eight knot [Gab87].

Theorem 1.3 also gives a different proof that the figure-eight knot is distinguished from every other knot in $S^3$, recovering a result in [BF20] and [BR20] (see Section 3).

Corollary 1.4. The knot $4_1$ is distinguished from every other knot in $S^3$ by the finite quotients of $\pi_1(S^3 \setminus 4_1)$.

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2. Profinite Completions

For a finitely generated, residually finite group $G$, we can organize the set of finite quotients of $G$ into an inverse system whose inverse limit $\hat{G}$ is a finitely generated, profinite group, the \emph{profinite completion} of $G$. There is a bijective correspondence between the finite index subgroups of $G$ and the open subgroups of $\hat{G}$. There is a canonical inclusion $G \hookrightarrow \hat{G}$ that has dense image because $G$ is residually finite.

Moreover, the assignment of finitely generated groups to their profinite completions is functorial, and an epimorphism of groups $G \twoheadrightarrow H$ will induce a continuous epimorphism of the profinite completions $\hat{G} \twoheadrightarrow \hat{H}$. The profinite completion completely captures the data of finite quotients of a group. In particular, two residually finite groups $G$ and $H$ will have isomorphic profinite completions if and only if they have the same set of finite quotients $C(G) = C(H)$ ([RZ00], Corollary 3.2.8).

We say that a finitely generated group is \emph{profinetely rigid} if it is completely determined by its profinite completion among all finitely generated, residually finite groups. We say a compact, orientable 3-manifold $M$ is \emph{relatively profinetely rigid} if $\pi_1(M)$ is completely determined by its profinite completion among the fundamental groups of compact, orientable 3-manifolds.

\textbf{Notation.} For a subgroup $K < G$, we write the closure of $K$ in $\hat{G}$ as $\bar{K}$. If $K$ is closed in the profinite topology on $G$, $\bar{K} \cong \hat{K}$ and we say that $K$ is a \emph{separable} subgroup of $G$.

3. Proofs

We first prove Theorem 1.1 and Corollary 1.4 assuming Theorem 1.3. As noted in the introduction, this relies on recent work of Baldwin-Sivek [BS22] [BS22b]. Using their classification of \emph{nearly fibered} genus-1 knots, they prove:

\textbf{Theorem 3.1 (Theorem 1.1. [BS22], Theorem 1.1 [BS22b]).} Let $K$ be any of the knots $5_2, 15n_{43522}, Wh^-(T_{2,3}, 2), Wh^+(T_{2,3}, 2), P(-3, 3, 2n + 1) \ (n \in \mathbb{Z})$ or their mirrors. Then 0 is a characterizing slope for $K$.

\textbf{Proof.} (Theorem 1.1) Let $K$ be the knot $5_2$ or the knot $P(-3, 3, 2n + 1)$ for some $n \in \mathbb{Z}$, as in the statement of Theorem 1.1. By Theorem 3.1 0 is a characterizing slope for $K$. Let $T$ be an incompressible torus in $S^3_0(K)$ obtained by capping off a genus 1 Seifert surface for $K$. Since $K$ is a Montesinos knot that is not the trefoil, we can apply the proof of Lemma 3.1 [BS22b] to conclude that $S^3_0(K)$ is not Seifert fibered, and in addition, when we cut $S^3_0(K)$ along $T$, we obtain the complement of the $(2, 4)$--torus link $T_{2,4} \subset S^3$, which is Seifert fibered over the annulus [CC93]. Thus, $S^3_0(K)$ has a non-trivial JSJ decomposition with a JSJ graph having a single vertex and cycle, which in particular is not bipartite.
When \( K = 15n_{43522} \), by Theorem 3.1, 0 is a characterizing slope for \( K \). Using Regina [BBP+99], one can check that \( S_0^3(K) = \{ A : (2, 1) \}/(\frac{3}{2}, \frac{1}{1}) \); the result of gluing the two torus boundary components of a Seifert-fibered space with base orbifold an annulus with one cone point by the homeomorphism \( (\frac{3}{2}, \frac{1}{1}) \). Thus, \( S_0^3(K) \) has a non-trivial JSJ decomposition with JSJ graph having a single vertex and cycle and therefore the JSJ graph is not bipartite.

Thus, by Theorem A [Wil18], for all \( K \) in the statement of Theorem 1.1, \( S_0^3(K) \) is (relatively) profinitely rigid. Hence, these knots all satisfy the hypothesis of Theorem 1.3, and the result follows.

Proof. (Corollary 1.4) By [Gab87], 0 is a characterizing slope for the figure-eight knot \( 4_1 \). It follows from [Fun14] that \( S_0^3(4_1) \) is relatively profinitely rigid. Briefly, the manifold \( S_0^3(4_1) \) is a torus bundle with SOLV geometry and monodromy \( (\frac{2}{1}, \frac{1}{1}) \). The eigenvalues of this monodromy are \( \lambda = (\frac{3\pm \sqrt{5}}{2} \), and so generate \( \mathbb{Q}(\sqrt{5}) \) which has class number 1. By Corollary 1.2 in [Fun14], the number of compact 3-manifolds whose fundamental groups have the same finite quotients as \( S_0^3(4_1) \) is bounded above by the class number, 1. Hence \( S_0^3(4_1) \) is (relatively) profinitely rigid as claimed, and so \( 4_1 \) satisfies the hypotheses of Theorem 1.3 which proves the corollary.

To establish Theorem 1.3, we will use the following lemma:

Lemma 3.2. Let \( J \) and \( K \) be hyperbolic knots in \( S^3 \) with \( \lambda_J, \lambda_K \) the homological longitudes of \( J \) and \( K \) respectively. If \( \hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K) \) then, upon identification, \( \langle \lambda_J \rangle \) and \( \langle \lambda_K \rangle \) have conjugate closures in \( \hat{\pi}_1(S^3 \setminus K) \).

Proof. Let \( \mu_J, \mu_K \) be the meridians of \( J \) and \( K \) respectively, and \( P_J = \langle \mu_J, \lambda_J \rangle \) and \( P_K = \langle \mu_K, \lambda_K \rangle \) in \( \pi_1(S^3 \setminus J) \) and \( \pi_1(S^3 \setminus K) \) respectively. By the main theorem of [Ham01], abelian subgroups of 3-manifold groups are separable and so \( \overline{\langle \lambda_J \rangle} \cong \overline{\langle \lambda_J \rangle}, \overline{P_J} \cong \overline{P_J} < \overline{\pi_1(S^3 \setminus J)} \) and \( \overline{\langle \lambda_K \rangle} \cong \overline{\langle \lambda_K \rangle}, \overline{P_K} \cong \overline{P_K} < \overline{\pi_1(S^3 \setminus K)} \).

Let \( \phi : \hat{\pi}_1(S^3 \setminus J) \to \hat{\pi}_1(S^3 \setminus K) \) be an isomorphism. By Theorem 9.3 of [WZ17], it follows that \( \phi(\overline{P_J}) \) and \( \overline{P_K} \) are conjugate in \( \hat{\pi}_1(S^3 \setminus K) \); so there exists \( g \in \hat{\pi}_1(S^3 \setminus K) \) such that \( g\phi(\overline{P_J})g^{-1} = \overline{\phi(\overline{P_K})} \). Since \( \overline{\langle \lambda_K \rangle} \) is the intersection of \( P_K \) with the kernel of the unique epimorphism \( \pi_1(S^3 \setminus K) \to \mathbb{Z} \), it follows that \( \overline{\langle \lambda_K \rangle} \) is the intersection of \( \overline{P_K} \) with the kernel of the induced epimorphism \( \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}} \). To see this, we observe that any element of \( \overline{P_K} \) is a \( \hat{\mathbb{Z}} \)-linear combination of \( \mu_K, \lambda_K \), and so if any element of \( \overline{P_K} \), say \( \alpha = c_1\mu_K + c_2\lambda_K \) with \( c_1, c_2 \in \hat{\mathbb{Z}} \), is in the kernel of the epimorphism \( \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}} \) induced by the unique epimorphism \( \pi_1(S^3 \setminus K) \to \mathbb{Z}, c_1 = 0 \) because the image of \( \mu_K \) in \( \hat{\mathbb{Z}} \) is non-trivial while the image of \( \lambda_K \) is trivial. As \( c_1 = 0 \), \( \alpha \in \overline{\langle \lambda_K \rangle} \).

Next, observe that since the epimorphism \( \pi_1(S^3 \setminus J) \to \mathbb{Z} \) is unique, it can also be described as the homomorphism obtained by the following composition of maps

\[
\pi_1(S^3 \setminus J) \hookrightarrow \hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}} \quad (\clubsuit)
\]
where the first map is the canonical inclusion of $\pi_1(S^3 \setminus J)$ into its profinite completion, and the last map $\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}$ is the epimorphism induced by the unique epimorphism $\pi_1(S^3 \setminus K) \to \mathbb{Z}$. Furthermore, $(\lambda_J)$ is the intersection of $P_J \cong \hat{P}_J$ with the kernel of the map $\hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}$ because $(\lambda_J)$ is the intersection of $P_J$ with the kernel of the unique epimorphism $\pi_1(S^3 \setminus J) \to \mathbb{Z}$ which we have noted is the same map as the composition of maps $\circledast$. Since $g^{-1}\lambda_Kg$ is in the intersection of $\phi(\hat{P}_J)$ with the kernel of the epimorphism $\hat{\pi}_1(S^3 \setminus K) \to \hat{\mathbb{Z}}$, $g^{-1}\lambda_Kg \in (\phi(\lambda_J))$. By interchanging $J$ and $K$, we can apply the foregoing argument to show that $g\phi(\lambda_J)g^{-1} \in (\lambda_K)$ and so the closures of $\langle \phi(\lambda_J) \rangle$ and $\langle \lambda_K \rangle$ are conjugate in $\hat{\pi}_1(S^3 \setminus K)$.

With this lemma in hand, we can prove Theorem 1.3. We recall the hypotheses; for a knot $K$, the slope 0 is characterizing for $K$ and the 0-Dehn surgery on $K$ is (relatively) profinitely rigid.

**Proof.** (Theorem 1.3) Let $J$ be a knot in $S^3$ with $\hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K)$. By Theorem A of [WZ17], $J$ is a hyperbolic knot. Assuming that $Q$ is a finite quotient of $\pi_1(S^3_0(K))$, we can precompose with the Dehn-filling epimorphism $\pi_1(S^3 \setminus K) \to \pi_1(S^3_0(K))$ to obtain an epimorphism from $\pi_1(S^3 \setminus K)$ to $Q$ under which $\lambda_K$ maps trivially. As in the proof of Lemma 3.2, we can choose some identification $\phi : \hat{\pi}_1(S^3 \setminus J) \cong \hat{\pi}_1(S^3 \setminus K)$, and thereby obtain an epimorphism $\pi_1(S^3 \setminus J) \to Q$ by the following composition:

$$\pi_1(S^3 \setminus J) \hookrightarrow \hat{\pi}_1(S^3 \setminus J) \xrightarrow{\phi} \hat{\pi}_1(S^3 \setminus K) \to Q$$

By Lemma 3.2, since $\langle \phi(\lambda_J) \rangle$ and $\langle \lambda_K \rangle$ have conjugate closures and $\lambda_K$ maps trivially, $\lambda_J$ will also map trivially. It follows that $Q$ is a finite quotient of $\pi_1(S^3_0(J))$.

Applying this argument with the roles of $J$ and $K$ reversed shows that every finite quotient of $\pi_1(S^3_0(J))$ is also a finite quotient of $\pi_1(S^3_0(K))$, and we conclude that $\hat{\pi}_1(S^3_0(J)) \cong \hat{\pi}_1(S^3_0(K))$.

However by the second hypothesis, $S^3_0(K)$ is (relatively) profinitely rigid and so $S^3_0(J)$ is homeomorphic to $S^3_0(K)$. By the first hypothesis, 0 is a characterizing slope for $K$ and so $J$ and $K$ are isotopic.

**Remark 3.3.** Note that the knots $Wh^+(T_{2,3}, 2)$ and $Wh^-(T_{2,3}, 2)$ are satellite knots and thus the proof of Theorem 1.3 does not apply.

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