Classical geometry from the quantum Liouville theory

Leszek Hadasz
M. Smoluchowski Institute of Physics,
Jagellonian University, Reymonta 4, 30-059 Kraków, Poland

Zbigniew Jaskólski
Marcin Piątek
Institute of Theoretical Physics
University of Wrocław
pl. M. Borna, 950-204 Wrocław Poland

Abstract

Zamolodchikov’s recursion relations are used to analyze the existence and approximations to the classical conformal block in the case of four parabolic weights. Strong numerical evidence is found that the saddle point momenta arising in the classical limit of the DOZZ quantum Liouville theory are simply related to the geodesic length functions of the hyperbolic geometry on the 4-punctured Riemann sphere. Such relation provides new powerful methods for both numerical and analytical calculations of these functions. The consistency conditions for the factorization of the 4-point classical Liouville action in different channels are numerically verified. The factorization yields efficient numerical methods to calculate the 4-point classical action and, by the Polyakov conjecture, the accessory parameters of the Fuchsian uniformization of the 4-punctured sphere.

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1e-mail: hadasz@th.if.uj.edu.pl
2e-mail: jask@ift.uni.wroc.pl
3e-mail: piatek@ift.uni.wroc.pl
1 Introduction

A few years ago a considerable progress in the Liouville theory has been achieved [1]. The solution to the quantum theory based on the structure constants proposed by Otto and Dorn [2] and by A. and Al. Zamolodchikov [3] was completed by Ponsot and Teschner [4–6] and by Teschner [1, 7]. Along with the techniques of calculating conformal blocks developed by Al. Zamolodchikov [8–10] the DOZZ theory provides explicit formulae for quantum correlators.

On the other hand there exists so called geometric approach originally proposed by Polyakov [11] and further developed by Takhtajan [12–15] (see also [16–19]). In contrast to the operator formulation of the DOZZ theory the correlators of primary fields with elliptic and parabolic weights are expressed in terms path integral over conformal class of Riemannian metrics with prescribed singularities at the punctures. The underlying structure of this formulation is the classical hyperbolic geometry of the Riemann surface.

Although the relation between these formulations is not yet completely understood [20–22] it is commonly believed that the quasiclassical limit of the DOZZ theory exists and is correctly described by the classical Liouville action of the geometric approach. This is for instance justified by explicit calculation of the classical limit of the DOZZ structure constants and the classical Liouville action for the Riemann sphere with three punctures [3, 23].

Some of the predictions derived from the path integral representation of the geometric approach can be rigorously proved and lead to deep geometrical results. This can be seen as an additional support for the correctness of the picture the geometric formulation provides for the semiclassical limit of the DOZZ theory. One of the results of this type is the so called Polyakov conjecture obtained as a classical limit of the Ward identity [24–29]. It states that the classical Liouville action is a generating function for the accessory parameters of the Fuchsian uniformization of the punctured sphere yielding an essentially new insight into this classical long standing problem. Its usefulness for solving the uniformization is however restricted by our ability to calculate the classical Liouville action for more than three singularities.

The existence of the semiclassical limit of the Liouville correlation function with the projection on one intermediate conformal family implies a semiclassical limit of the BPZ quantum conformal block [30] with heavy weights $\Delta = Q^2 \delta$, $\Delta_i = Q^2 \delta_i$, with $\delta, \delta_i = O(1)$ in the following form:

$$F_{1+6Q^2, \Delta \left[ \begin{array}{cc} \Delta_3 & \Delta_2 \\ \Delta_4 & \Delta_1 \end{array} \right]} (x) \sim \exp \left\{ Q^2 f_\delta \left[ \begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array} \right] (x) \right\}. \quad (1.1)$$

The function $f_\delta \left[ \begin{array}{cc} \delta_3 & \delta_2 \\ \delta_4 & \delta_1 \end{array} \right] (x)$ is called the classical conformal block [3] or (a bit confusing) the "classical action" [9, 10].

The existence of the semiclassical limit (1.1) was first postulated in [9, 10] where it was pointed out that the classical block is related to a certain monodromy problem of a null vector decoupling equation in a similar way the classical Liouville action is related to the Fuchsian
uniformization. This relation was further used to derive the $\Delta \to \infty$ limit of the conformal block and its expansion in powers of the $q$ variable.

The 4-point function of the DOZZ theory can be defined as an integral of $s$-channel conformal blocks and the DOZZ couplings over the continuous spectrum of the theory. In the semiclassical limit the integrand can be expressed in terms of the 3-point classical Liouville action and the classical block, and the integral itself is dominated by the saddle point $\Delta_s = Q^2 \delta_s(x)$. One thus gets the factorization [3]

$$ S^{(cl)}(\delta_4, \delta_3, \delta_2, \delta_1; x) = S^{(cl)}(\delta_4, \delta_3, \delta_s(x)) + S^{(cl)}(\delta_s(x), \delta_2, \delta_1) - f_{\delta_s(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (x) - \bar{f}_{\delta_s(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (\bar{x}), $$

where $S^{(cl)}(\delta_4, \delta_3, \delta_2, \delta_1; x)$ is the classical action for the weights $\delta_1, \delta_2, \delta_3, \delta_4$ located at $0, x, 1$ and $\infty$, respectively, and $S^{(cl)}(\delta_3, \delta_2, \delta_1)$ is the classical Liouville action for the weights $\delta_1, \delta_2, \delta_3$ at the locations $0, 1, \infty$. Since the semiclassical limit should be independent of the choice of the channel in the representation of the DOZZ 4-point function one gets the consistency conditions

$$ S^{(cl)}(\delta_4, \delta_3, \delta_s(x)) + S^{(cl)}(\delta_s(z), \delta_2, \delta_1) - f_{\delta_s(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (x) - \bar{f}_{\delta_s(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (\bar{x}) $$

$$ = S^{(cl)}(\delta_4, \delta_1, \delta_t(x)) + S^{(cl)}(\delta_t(x), \delta_2, \delta_3) - f_{\delta_t(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (1 - x) - \bar{f}_{\delta_t(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] (1 - \bar{x}) $$

$$ = 2\delta_2 \log xx + S^{(cl)}(\delta_1, \delta_3, \delta_u(x)) + S^{(cl)}(\delta_u(x), \delta_2, \delta_4) - f_{\delta_u(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] \left( \frac{1}{x} \right) - \bar{f}_{\delta_u(x)} \left[ \delta_3 \delta_2 \delta_1 \delta_4 \right] \left( \frac{1}{\bar{x}} \right), $$

where the saddle weights $\delta_t(x), \delta_u(x)$ in the $t$- and $u$-channel are simply related to the $s$-channel saddle point:

$$ \delta_t(x) = \delta_s(1 - x), \quad \delta_u(x) = \delta_s \left( \frac{1}{x} \right). $$

As the conditions (1.3) follow from the semiclassical limit of the bootstrap equations of the quantum DOZZ theory we shall call them the classical bootstrap equations.

The question arises what is the geometric interpretation of the saddle point conformal weight $\delta_s(x)$. Let us recall that the classical solution describes a unique hyperbolic geometry with singularities at the locations of conformal weights. For elliptic, parabolic and hyperbolic weights one gets conical singularities, punctures and holes with geodesic boundaries, respectively [23, 29, 31]. In the latter case the (classical) conformal weight $\delta$ is related to the length $\ell$ of the corresponding geodesic by

$$ \delta = \frac{1}{4} + \frac{\mu}{4} \left( \frac{\ell}{2\pi} \right)^2 $$

(1.4)

where the scale of the classical configuration is set by the condition $R = -\frac{\mu}{2}$ imposed on the constant scalar curvature $R$. 

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Let us note that the relation (1.4) appears also in the context of the quantization of the Teichmüller space ([32], [33] and references therein). With this relation assumed one can show that the Hilbert space arising in the quantization of the Teichmüller space of a Riemann surface and the space of conformal blocks of the Liouville theory on this surface are isomorphic as representations of the mapping class group.

In the case of 4 singularities at the standard locations 0, x, 1, ∞ there are three closed geodesics Γ_s, Γ_t, Γ_u separating the singular points into pairs (x, 0|1, ∞), (x, 1|0, ∞) and (x, ∞|0, 1) respectively. Since the spectrum of DOZZ theory is hyperbolic the singularities corresponding to the saddle point weights δ_i(x) are geodesic holes. One may expect that these weights are related to the lengths ℓ_i of the closed geodesics Γ_i in corresponding channels:

\[ δ_i(x) = \frac{1}{4} + \frac{μ}{4} \left( \frac{ℓ_i(x)}{2π} \right)^2, \quad i = s, t, u. \]  

If the formula above proved true it would provide a new powerful tool for calculating the geodesic length functions which play an important role in analyzing the structure of the moduli space of Riemann surfaces ([34], and references therein). It would also pave a way for an explicit uniformization of at least 4-punctured sphere.

Our aim in the present paper is a numerical verification of the three conjectures mentioned above: the asymptotic (1.1), the classical bootstrap equations (1.3), and the relation (1.5). Up to our knowledge no rigorous proof of any of these relations is known. The basic difficulty is the conformal block itself which except of some special cases is only known as a formal power series in the x variable. Since the coefficients are defined in terms of inverses of the Gram matrices of a Verma module a direct analysis is prohibitively difficult.

A more efficient method based on a recurrence relation for the coefficients was developed by Al. B. Zamolodchikov [8]. The assumption crucial for this method is that the formal power series defining the conformal block converges. It is actually believed (and well justified by all special cases where the conformal block can be calculated explicitly) that the radius of convergence is 1. Another commonly accepted hypothesis, still waiting for its rigorous proof, states that the only singularities of the conformal block with respect to the z variable are branching points (in general of a transcendental kind) at 0, 1, and ∞ [36]. This implies that the conformal block is a single-valued analytic function on the universal covering of a 3-punctured Riemann sphere and can be expressed by a power series convergent in the entire domain of its analyticity [10]. A recurrence relation for calculating coefficients of this so called q-expansion [10] provides an extremely efficient method for numerical analysis of conformal block and was applied for testing the conformal bootstrap equations [3, 37]. Let us note that both the x- and q-expansion recurrence methods involve formulae which successfully passed numerical tests but a formal proof of them is still lacking.

The organization of the paper is as follows. In Sect. 2 we briefly present the classical Liouville action and its relation to the monodromy problem of Fuchsian uniformization. In Sect. 3 the classical conformal block is introduced and its relation to a certain monodromy
problem of the null-vector decoupling equation is clarified. These sections contain a material which is basically known and were added mainly for completeness and to set the notation.

In Sect. 4 the Zomolodchikov’s recurrence method is applied to calculate the $x$- and $q$-expansion of the classical conformal block in the case of parabolic external weights up to 7th and 16th power, respectively. It is also checked by symbolic calculation that up to these orders the asymptotic (1.1) is correct.

In Sect. 5 the saddle point weights and momenta are defined and numerically calculated in same special cases in which the lengths of geodesics in each channel are exactly known. An excellent agreement is obtained using both the $x$- and $q$-expansions of classical conformal block. It is also verified that the geodesic length function numerically calculated from the formula (1.5) satisfies the upper and lower bounds known to mathematicians [35]. The calculations of this section provide a strong evidence that the formula (1.5) is correct and along with the $q$-expansion for classical conformal block give an extremely efficient method for numerical calculations of the geodesic length functions on the 4-punctured sphere.

In Sect. 6 some numerical tests of the classical bootstrap are presented. It is shown that it is satisfied with high precision, improving with the number of terms taken into account in the $q$-expansion of the conformal block. This allows for calculating the classical Liouville action for 4-puncture sphere and, by the Polyakov conjecture, the accessory parameters of the Fuchsian uniformization. Finally, Sect. 7 contains conclusions and discussion of possible extensions of the present work.

## 2 Classical Liouville action and the Polyakov conjecture

A conformal factor of the hyperbolic metric on the surface $X$ parametrized with a complex coordinate $z$ is a solution of the Liouville equation

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = \frac{\mu}{2} e^{\phi(z, \bar{z})}$$

Given a genus $g$ of $X$ and a set of points $z_1, \ldots, z_n$ removed from $X$ this metric is determined by the singular behavior of $\phi$ at $z_j-s$. In the present paper we shall consider the case of $X$ being a punctured sphere ($g = 0$) and choose a complex coordinates on $X$ in such a way that $z_n = \infty$. The existence and the uniqueness of the solution of the equation (2.1) on the sphere with the elliptic singularities,

$$\phi(z, \bar{z}) = \begin{cases} 
-2 (1 - \xi_j) \log |z - z_j| + O(1) & \text{as } z \to z_j, \quad j = 1, \ldots, n - 1, \\
-2 (1 + \xi_n) \log |z| + O(1) & \text{as } z \to \infty,
\end{cases}$$

was proved by Picard [38, 39] (see also [40] for a modern proof). The solution can be interpreted as a conformal factor of the complete, hyperbolic metric on $X = \mathbb{C} \setminus \{z_1, \ldots, z_{n-1}\}$ with the conical singularities of the opening angles $0 < 2 \pi \xi_j < 2 \pi$ at the punctures $z_j$. The solution is known to exist also in the case of parabolic singularities (corresponding to $\xi_j \to 0$),
with the asymptotic behavior of the Liouville field of the form
\[
\phi(z, \bar{z}) = \begin{cases} 
-2 \log |z - z_j| - 2 \log |\log |z - z_j|| + O(1) & \text{as } z \to z_j, \\
-2 \log |z| - 2 \log |\log |z|| + O(1) & \text{as } z \to \infty.
\end{cases}
\tag{2.3}
\]

The central objects in the geometric approach to the quantum Liouville theory are the partition functions on \(X\):
\[
\langle X \rangle = \int_\mathcal{M} D\phi \ e^{-Q^2 S_L[\phi]},
\tag{2.4}
\]
where \(\mathcal{M}\) is the space of conformal factors on \(X\) with the asymptotics (2.2) or (2.3), and the correlation functions of the energy–momentum tensor
\[
\langle T(u_1) \ldots T(u_k) \bar{T}(\bar{w}_1) \ldots \bar{T}(\bar{w}_l) X \rangle = \int_\mathcal{M} D\phi \ e^{-Q^2 S_L[\phi]} \ T(u_1) \ldots T(u_k) \bar{T}(\bar{w}_1) \ldots \bar{T}(\bar{w}_l),
\tag{2.5}
\]
with
\[
T(u) = \frac{Q^2}{2} \left[ -\frac{1}{2} (\partial_u \phi(u, \bar{u}))^2 + \partial_u^2 \phi(u, \bar{u}) \right].
\tag{2.6}
\]

The singular nature of the Liouville field at the punctures requires regularizing terms in the Liouville action:
\[
S_L[\phi] = \frac{1}{4\pi} \lim_{\epsilon \to 0} S_L^\epsilon[\phi],
\]
\[
S_L^\epsilon[\phi] = \int_{X_\epsilon} d^2 z \left[ |\partial \phi|^2 + \mu e^\phi \right] + \sum_{j=1}^{n-1} (1 - \xi_j) \int_{|z - z_j| = \epsilon} |dz| \ \kappa_z \phi + (1 + \xi_n) \int_{|z| = 1/\epsilon} |dz| \ \kappa_z \phi
\tag{2.7}
\]
\[
-2\pi \sum_{j=1}^{n-1} (1 - \xi_j)^2 \log \epsilon - 2\pi (1 + \xi_n)^2 \log \epsilon,
\]
where \(X_\epsilon = \mathbb{C} \setminus \left\{ \bigcup_{j=1}^n \{|z - z_j| < \epsilon\} \cup \{|z| > \frac{1}{\epsilon}\} \right\}\). The form of (2.7) is valid for parabolic singularities (with corresponding \(\xi_j = 0\) as well.

One can check by perturbative calculations of the correlators (2.5) [12] that the central charge reads
\[
c = 1 + 6Q^2.
\tag{2.8}
\]

The transformation properties of (2.4) with respect to the global conformal transformations show [12] that the punctures behave like primary fields with the dimensions
\[
\Delta_j = \bar{\Delta}_j = \frac{Q^2}{4} \ (1 - \xi_j^2).
\tag{2.9}
\]

As for fixed \(\xi_j\) the dimensions scale like \(Q^2\), the punctures correspond to heavy fields of the operator approach [3].

In the classical limit \(Q^2 \to \infty\) with all
\[
\delta_i \overset{\text{def}}{=} \frac{\Delta_i}{Q^2} = \frac{1 - \xi_j^2}{4}
\tag{2.10}
\]
kept fixed we expect the path integral to be dominated by the classical action:

$$\langle X \rangle \sim e^{-Q^2 S^{(\text{cl})}(\delta_i; z_i)}, \quad (2.11)$$

where the classical action $S^{(\text{cl})}(\delta_i; z_i)$ is the functional $S_L[\cdot]$ (2.7) evaluated at the classical solution $\varphi$ of (2.1) with the asymptotics (2.2) or (2.3). Similarly:

$$\langle T(z)X \rangle \sim T^{\text{cl}}(z) e^{-Q^2 S^{(\text{cl})}(\delta_i; z_i)}, \quad (2.12)$$

where $T^{\text{cl}}(z)$ is the classical energy–momentum tensor.

From (2.6) and (2.2) or (2.3) it follows that

$$T^{\text{cl}}(z) \sim \frac{\Delta_j}{(z - z_j)^2} \quad \text{for } z \to z_j,$$

$$T^{\text{cl}}(z) \sim \frac{\Delta_j}{z^2} \quad \text{for } z \to \infty, \quad (2.13)$$

and consequently

$$T^{\text{cl}}(z) = Q^2 \sum_{j=1}^{n-1} \left[ \frac{\delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right]. \quad (2.14)$$

Combining now (2.11), (2.12) and (2.14) with the conformal Ward identity [30]

$$\langle T(z)X \rangle = \sum_{j=1}^{n-1} \left[ \frac{\Delta_j}{(z - z_j)^2} + \frac{1}{z - z_j} \frac{\partial}{\partial z_j} \right] \langle X \rangle,$$  

we get the relation

$$c_j = -\frac{\partial S^{(\text{cl})}(\delta_i; z_i)}{\partial z_j} \quad (2.16)$$

known as the Polyakov conjecture.

It is amazing that this relation obtained by general heuristic path integral arguments turned out to provide an exact solution to a long standing problem of the so called accessory parameters of the Fuchsian uniformization of the punctured sphere. Indeed, it was rigorously proved [24–29] that the formula (2.16) yields the accessory parameters $c_j$ for which the Fuchsian equation

$$\partial^2 \psi(z) + \frac{1}{Q^2 T^{\text{cl}}(z)} \psi(z) = 0 \quad (2.17)$$

admits a fundamental system of solutions with $SU(1, 1)$ monodromies around all singularities. Note that if $\{\chi_1(z), \chi_2(z)\}$ is such a system then the function $\varphi(z, \bar{z})$ determined by the relation

$$e^{\varphi(z, \bar{z})} = \frac{4 |w'|^2}{\mu(1 - |w|^2)^2}, \quad w(z) = \frac{\chi_1(z)}{\chi_2(z)}, \quad (2.18)$$

satisfies (2.1) and (2.2) (or (2.3)). The $SU(1, 1)$ monodromy condition is then equivalent to the existence of the well defined hyperbolic metric on $X$. 

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From (2.13) it follows that \( c_j \) satisfy the relations
\[
\sum_{j=1}^{n-1} c_j = 0, \quad \sum_{j=1}^{n-1} (\delta_j + z_j c_j) = \delta_n.
\] (2.19)

In the case of three singularities the only two accessory parameters are completely determined by (2.19). Hence one can solve the equation (2.17), find the classical Liouville field and calculate the classical action. The result is [3]:

\[
S^{(cl)}(\delta_3, \delta_2, \delta_1) \equiv S^{(cl)}(\delta_3, \delta_2, \delta_1; \infty, 1, 0) =
\frac{1}{2}(1 - \xi_1 - \xi_2 - \xi_3) \log \mu + \sum_{\sigma_2, \sigma_3 = \pm} F\left(\frac{1 - \xi_1}{2} + \sigma_2 \frac{\xi_2}{2} + \sigma_3 \frac{\xi_3}{2}\right) - \sum_{j=1}^{3} F(\xi_j) + \text{const},
\]

(2.20)

where

\[
F(x) = \int_{1/2}^{x} dy \log \frac{\Gamma(y)}{\Gamma(1-y)}.
\]

For the elliptic and parabolic singularities one has \( \delta \leq \frac{1}{4} \). As we shall see below, it is useful to know the classical Liouville action also for \( \delta > \frac{1}{4} \), what corresponds to the hyperbolic singularities. The relevant construction of the metric and the classical action was given in [23]. If we write

\[
\delta_3 = \frac{1}{4} + p^2, \quad p \in \mathbb{R},
\]

(2.21)

then in the case of two elliptic/parabolic and one hyperbolic singularity

\[
S^{(cl)}(\delta_3, \delta_2, \delta_1) = \frac{1}{2}(1 - \xi_1 - \xi_2) \log \mu + \sum_{\sigma_2, \sigma_3 = \pm} F\left(\frac{1 - \xi_1}{2} + \sigma_2 \frac{\xi_2}{2} + i\sigma_3 p\right)
\]

\[
- \sum_{j=1}^{2} F(\xi_j) + H(2ip) + \pi|p| + \text{const},
\]

(2.22)

with

\[
H(x) = \int_{0}^{x} dy \log \frac{\Gamma(-y)}{\Gamma(y)}.
\]

### 3 Classical conformal block

The partition function (2.4) corresponds in the operator formulation to the correlation function of the primary fields \( V_{\alpha_j}(z_j, \bar{z}_j) \),

\[
\langle X \rangle = \langle V_{\alpha_n}(\infty, \infty) \ldots V_{\alpha_1}(z_1, \bar{z}_1) \rangle,
\]

(3.1)

where

\[
\Delta_j = \alpha_j(Q - \alpha_j), \quad \alpha_j = \frac{Q}{2} (1 + \xi_j).
\]
The DOZZ 4-point correlation function with the standard locations \( z = 0, x, 1, \infty \) is expressed as an integral over the continuous spectrum

\[
\left\langle V_{4,i}(\infty, \infty)V_{3}(1, 1)V_{2}(x, \bar{x})V_{1}(0, 0) \right\rangle = \int_{\mathbb{R}^+} d\alpha C(\alpha_4, \alpha_3, \alpha)C(Q - \alpha, \alpha_2, \alpha_1) \left| \mathcal{F}_{1+6Q^2, \Delta} \left[ \frac{\Delta_3 \Delta_2}{\Delta_1} \right](x) \right|^2.
\]

(3.2)

Let

\[
1_{\Delta, \Delta} = \sum_I (|\xi_{\Delta, I}| \otimes |\xi_{\Delta, I}|)\langle \xi_{\Delta, I} \otimes \xi_{\Delta, I} \rangle
\]

be an operator that projects onto the space spanned by the states form the conformal family with the highest weight \( \Delta \). The correlation function with the \( 1_{\Delta, \Delta} \) insertion factorizes into the product of the holomorphic and anti-holomorphic factors,

\[
\left\langle V_{4}(\infty, \infty)V_{3}(1, 1)1_{\Delta, \Delta}V_{2}(x, \bar{x})V_{1}(0, 0) \right\rangle = \]

\[
C(\alpha_4, \alpha_3, \alpha)C(Q - \alpha, \alpha_2, \alpha_1) \mathcal{F}_{1+6Q^2, \Delta} \left[ \frac{\Delta_3 \Delta_2}{\Delta_1} \right](x) \mathcal{F}_{1+6Q^2, \Delta} \left[ \frac{\Delta_3 \Delta_2}{\Delta_1} \right](\bar{x}).
\]

(3.3)

Assuming a path integral representation of the l.h.s. one should expect in the limit \( Q \to \infty \), with all the weights being heavy \( \Delta, \Delta_i \sim Q^2 \), the following asymptotic behavior

\[
\left\langle V_{4}(\infty, \infty)V_{3}(1, 1)1_{\Delta, \Delta}V_{2}(x, \bar{x})V_{1}(0, 0) \right\rangle \sim e^{-Q^2S^{(cl)}(\delta_i, x; \delta)}.
\]

(3.4)

On the other hand one can calculate this limit for the DOZZ coupling constants \([3, 23]\) obtaining

\[
C(\alpha_4, \alpha_3, \alpha)C(Q - \alpha, \alpha_2, \alpha_1) \sim e^{-Q^2\left(S^{(cl)}(\delta_4, \delta_3, \delta) + S^{(cl)}(\delta, \delta_2, \delta_1)\right)}.
\]

(3.5)

It follows that the conformal block should have the \( Q \to \infty \) asymptotic (1.1) so that

\[
S^{(cl)}(\delta_i, x; \delta) = S^{(cl)}(\delta_4, \delta_3, \delta) + S^{(cl)}(\delta, \delta_2, \delta_1) - f_\delta \left[ \frac{\delta_3 \delta_2}{\delta_4 \delta_1} \right](x) - \bar{f}_\delta \left[ \frac{\delta_3 \delta_2}{\delta_4 \delta_1} \right](\bar{x}).
\]

(3.6)

It should be stressed that the asymptotic behavior (1.1) is a nontrivial statement on the (quantum) conformal block. Although there is no proof of this property yet it seems to be well justified by sample numerical calculations, as well as by its consequences. We shall briefly describe two of them.

The first one is the classical bootstrap mentioned in the introduction. In the semiclassical limit the l.h.s of the formula (3.2) takes the form \( e^{-Q^2S^{(cl)}(\delta_4, \delta_3, \delta_1; x)} \), where

\[
S^{(cl)}(\delta_4, \delta_3, \delta_2, \delta_1; \infty, 1, x, 0).
\]

The r.h.s. of (3.2) is in this limit determined by the saddle point approximation

\[
e^{-Q^2S^{(cl)}(\delta_i, x; \delta)} \approx \int_0^\infty dp e^{-Q^2S^{(cl)}(\delta_i, x; \delta)}
\]
where $\delta_s = \frac{1}{4} + p_s^2$ and the saddle point Liouville momentum $p_s$ is determined by

$$\frac{\partial}{\partial p} s^{(cl)}(\delta, x; \frac{1}{4} + p_s^2)|_{p=p_s} = 0. \quad (3.7)$$

One thus gets the relation (1.2) first obtained in [3] and the classical bootstrap (1.3) as its consistency condition.

The second implication of the asymptotic (1.1) is the relation of the classical block to certain monodromy problem, which in fact proved to be essential for developing a very effective recursive method for calculating the conformal block itself (and therefore also its classical asymptotic) [8, 10, 36]. Consider the null vectors on the second level of the Verma module, given by

$$|\chi_{\pm}\rangle = \left( L_{-2} - \frac{3}{2(2\Delta_{\pm} + 1)} L_{-1}^2 \right) |\Delta_{\pm}\rangle, \quad (3.8)$$

where $|\Delta_{\pm}\rangle$ are the highest weight states with (for $c \geq 25$)

$$\Delta_{\pm} = \frac{1}{16} \left( 5 - c \pm \sqrt{(c-1)(c-25)} \right).$$

This expression simplifies if we use convenient parametrization of the central charge

$$c = 1 + 6Q^2, \quad Q = \frac{1}{b} + b, \quad (3.9)$$

so that

$$\Delta_+ = -\frac{1}{2} - \frac{3}{4}b^2, \quad \Delta_- = -\frac{1}{2} - \frac{3}{4b^2}. \quad (3.10)$$

It follows from (3.8) that the correlators

$$\langle \hat{\chi}_{\pm}(z)X\rangle_{\Delta} \overset{\text{def}}{=} \left\langle V_4(\infty, \infty)V_3(1, 1)1_{\Delta, \Delta} \hat{\chi}_{\pm}(z)V_2(x, \bar{x})V_1(0, 0)\right\rangle, \quad (3.11)$$

where $\hat{\chi}_{\pm}(z)$ are the null fields corresponding to $|\chi_{\pm}\rangle$, satisfy the null-vector decoupling equation:

$$\left[ \frac{\partial^2}{\partial z^2} + \gamma_{\pm} \left( \frac{1}{z} - \frac{1}{1-z} \right) \frac{\partial}{\partial z} \right] \langle \hat{\chi}_{\pm}(z)X\rangle_{\Delta} =$$

$$\gamma_{\pm} \left[ \frac{\Delta_1}{z^2} + \frac{\Delta_2}{(z-x)^2} + \frac{\Delta_3}{(1-z)^2} + \frac{\Lambda_{\pm}}{z(z-1)} + \frac{x(1-x)}{z(x)(1-x)} \frac{\partial}{\partial x} \right] \langle \hat{\chi}_{\pm}(z)X\rangle_{\Delta},$$

with

$$\Lambda_{\pm} = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_\pm - \Delta_4$$

and

$$\gamma_{\pm} = \frac{2}{3}(2\Delta_{\pm} + 1) \quad \Rightarrow \quad \gamma_+ = -b^2, \quad \gamma_- = -\frac{1}{b^2}.$$ 

For $Q \to \infty$ we have either $b \to 0$ or $b \to \infty$. To fix the notation we shall concentrate on the first possibility. For $b \to 0$ the operator with the weight $\Delta_+$ remains “light” ($\Delta_+ = O(1)$)
and its presence in the correlation function has no influence on the classical solution of the field equations\(^4\). Consequently, for \(b \to 0\)

\[
\langle \hat{\chi}_+(z) X \rangle_\Delta \sim \chi^{cl}(z) e^{\frac{-1}{2b} \left( S^{(cl)}(\delta_4, \delta_3, \delta) + S^{(cl)}(\delta, \delta_2, \delta_1) - f_\delta [\delta_4, \delta_3, \delta_1] (x) - f_\delta [\delta_4, \delta_3, \delta_1] (\bar{x}) \right)}
\]

(3.12)

and we get from (3.11):

\[
\partial_z^2 \chi^{cl}(z) + \left[ \frac{\delta_1}{z^2} + \frac{\delta_2}{(z - x)^2} + \frac{\delta_3}{(1 - z)^2} + \frac{\delta_1 + \delta_2 + \delta_3 - \delta_4}{z(1 - z)} + \frac{x(1 - x) C(x)}{(z - x)(1 - z)} \right] \chi^{cl}(z) = 0,
\]

(3.13)

where

\[
C(x) = \frac{d}{dx} f_\delta [\delta_4, \delta_3, \delta_1] (x).
\]

(3.14)

The accessory parameter \(C(x)\) can be determined from the following monodromy problem. Consider the correlation function

\[
G(z) \overset{\text{def}}{=} \langle V_4(\infty, \infty) V_3(1, 1) \hat{\chi}_+(z) V_\Delta(0, 0) \rangle
\]

(3.15)

where \(V_\Delta\) is the primary field corresponding to the highest weight state \(\xi_\Delta\). The null vector decoupling equation for this correlator reads

\[
\left( \frac{1}{b^2} \frac{d^2}{dz^2} + \left( \frac{1}{1 - z} - \frac{1}{z} \right) \frac{d}{dz} + \frac{\Delta_3}{z^2} + \frac{\Delta + \Delta_+ + \Delta_3 - \Delta_4}{z(1 - z)} \right) G(z) = 0.
\]

(3.16)

Substituting into this equation the most singular term in the OPE

\[
\hat{\chi}_+(z) V_\Delta(0, 0) \sim z^\kappa V_\Delta'(0, 0)
\]

we get

\[
\kappa(\kappa - 1) + b^2 (\Delta - \kappa) = 0.
\]

(3.17)

In the limit \(b \to 0\):

\[
\Delta = \frac{Q^2}{4}(1 - \xi^2) = \frac{1}{4b^2}(1 - \xi^2) + O(1)
\]

and (3.17) transforms to

\[
\kappa(\kappa - 1) + \frac{1}{4}(1 - \xi^2) = 0 \quad \Rightarrow \quad \kappa = \frac{1}{2}(1 \pm \xi).
\]

For \(z \to e^{2\pi i} z\):

\[
\begin{pmatrix} z^{1 - \xi} \\ z^{1 + \xi} \end{pmatrix} \to \begin{pmatrix} e^{-i\pi \xi} & 0 \\ 0 & e^{i\pi \xi} \end{pmatrix} \begin{pmatrix} z^{1 - \xi} \\ z^{1 + \xi} \end{pmatrix}
\]

and (minus) the trace of the monodromy matrix (an invariant with respect to the choice of the basis in the space of solutions of (3.16)) is equal to

\[2 \cos \pi \xi.\]

\(^4\)This is reflected by the fact that limit \(b \to 0\) of the equation (3.11) exists only in the “+” case.
Note that we get the same monodromy invariant if we replace in (3.15) the primary field $V_{\Delta}$ with any of its descendants. Note also that (in the classical limit) the monodromy invariant of the two independent solutions of (3.16) for a curve encircling 0 is by construction equal to the monodromy invariant for a basis in the space of solutions of (3.13) along a curve encircling both 0 and $x$ (all point on this curve can be taken "to the left" of the operator $1_\Delta$). This condition fixes the accessory parameter $C(x)$ that appears on the r.h.s. of (3.13).

4 Zamolodchikov’s recursion methods

The BPZ 4-point conformal block is defined [30] as a formal power series

$$F_{c,\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right](x) = x^{\Delta - \Delta_{2} - \Delta_{1}} \left( 1 + \sum_{n=1}^{\infty} x^{n} F_{c,\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right] \right). \tag{4.1}$$

Studying the analytic structure of its coefficients Al. Zamolodchikov derived the recursion relation [8]:

$$F_{c,\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right] = g_{\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right] + \sum_{r \geq 2} \sum_{s \geq 1} \sum_{n \geq r s \geq 2} \tilde{R}_{rs}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right] F_{c,\Delta+rs}^{n-rs} \left[ \Delta_{4} \Delta_{1} / \Delta_{4} \Delta_{1} \right] \tag{4.2}$$

where

$$c_{rs}(\Delta) = 13 - 6 \left( T_{rs}(\Delta) + \frac{1}{T_{rs}(\Delta)} \right),$$

$$T_{rs}(\Delta) = \frac{r s - 1 + 2 \Delta + \sqrt{(r-s)^2 + 4(r s - 1) \Delta + 4 \Delta^2}}{r^2 - 1},$$

and $g_{\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right]$ are coefficients of the expansion of the hypergeometric function,

$$2F_{1}(\Delta + \Delta_{2} - \Delta_{1}, \Delta + \Delta_{3} - \Delta_{4}, 2\Delta, x) = \sum_{n=0}^{\infty} g_{\Delta}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right] x^n.$$

An exact form of the coefficients $\tilde{R}_{rs}^{n} \left[ \Delta_{3} \Delta_{2} / \Delta_{4} \Delta_{1} \right]$ (see Appendix) was partially derived and partially guessed in [8]. Although no proof of this form exists it is well justified by numerical calculations.

Once the expansion (4.1) is known one can calculate the coefficients of the power expansion of the classical conformal block

$$f_{\delta}^{\left[ \delta_{3} \delta_{2} / \delta_{1} \delta_{1} \right]}(x) = (\delta - \delta_{1} - \delta_{2}) \log x + \sum_{n=1}^{\infty} x^{n} f_{\delta}^{n} \left[ \delta_{3} \delta_{2} / \delta_{4} \delta_{1} \right] \tag{4.3}$$

It is believed that the series (4.1) converges on the unit disc but up to our knowledge there is no rigorous proof of this fact yet.
directly from the asymptotic (1.1)
\[
\sum_{n=1}^{\infty} x^n f_n^{\beta}[\delta_1, \delta_2, \delta_3, \delta_4] = \lim_{Q^2 \to \infty} \frac{1}{Q^2} \log \left( 1 + \sum_{n=1}^{\infty} x^n \mathcal{F}_{c,\Delta}^{n} \left[ \frac{\Delta_1}{\Delta_4}, \frac{\Delta_2}{\Delta_1} \right] \right),
\]
where on the r.h.s. one first expand the logarithm into a power series and then the limit is taken for each term separately.

In the present work we were interested in a special case of all parabolic external weights \(\Delta_i = \frac{Q^2}{4}\), the central charge \(c = 1 + 6Q^2\), and the intermediate weight \(\Delta = Q^2 \left( \frac{1}{4} + p^2 \right)\) parameterized by the Liouville momenta \(p\). We have checked by symbolic calculations that up to \(m = 7\) the limits in (4.4) exist yielding explicit formulae for the first seven coefficients of the expansion (4.3). Up to the first five terms:

\[
f(p, x) = f_{\frac{1}{4} + p^2} \left[ \frac{\frac{1}{4}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}} \right](x)
= (p^2 - \frac{1}{4}) \log x + \left( \frac{9}{128} + \frac{13 p^2}{64} + \frac{1}{1024 (1 + p^2)} \right) x^2
+ \left( \frac{19}{384} + \frac{23 p^2}{192} + \frac{1}{1024 (1 + p^2)} \right) x^3
+ \left( \frac{1257}{32768} + \frac{2701 p^2}{32768} - \frac{1}{2097152 (1 + p^2)^2}ight) x^4
+ \frac{1}{8388608 (1 + p^2)^2} + \frac{7439}{8388608 (1 + p^2)} + \frac{81}{8388608 (4 + p^2)} \right) x^5 + \ldots
\]

The limitation of the formulae (4.1) and (4.3) is that the power series involved are supposed to converge only for \(|x| < 1\).

A more convenient representation of the conformal block was developed by Al. Zamolodchikov [10] who proposed to regard it as a function of the variable

\[
q(x) = e^{-\pi \frac{K(1-x)}{K(x)}}, \quad K(x) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - xt^2)}}.
\]

The map \(\mathbb{C} \setminus \{0, 1, \infty\} \ni x \to q(x) \in \mathbb{D}\) yields a uniformization of the 3-punctured sphere by the Poincaré disc \(\mathbb{D}\). If the points 0, 1, \(\infty\) are the only singular points of the conformal block, then the block is a single valued function on \(\mathbb{D}\) and the series in \(q\) variable converges uniformly on each subset \(\{q : |q| < e^{-\epsilon} < 1\}\). It was shown in [10] that the conformal block can be expressed as

\[
\mathcal{F}_{c,\Delta}^{\frac{\Delta_1}{\Delta_4}, \frac{\Delta_2}{\Delta_1}}(x) = x^{\frac{\Delta_1}{\Delta_4} - \Delta_1 - \Delta_2(1 - x)}^{\frac{\Delta_2}{\Delta_1} - \Delta_1 - \Delta_4} \times \theta_3(q)^{\frac{c-1}{2} - 4(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)}
\]
where for the central charge parameterized as $c$

asymptotic (1.1) up to the terms $q$

Using this formula in the special case of all parabolic external weights we have checked the

where

The coefficients in (4.10) can be calculated step by step from the asymptotic (1.1)

where

The representation (4.6) and the asymptotic (1.1) imply the following representation for

The coefficients in (4.7) are uniquely determined by the recursion relation:

where for the central charge parameterized as $c = 1 + 6 \left( b + \frac{1}{b} \right)^2$:

and $R_{c}^{rs} \left[ \Delta_3 \Delta_2 \right]_{\Delta_4 \Delta_1}$ are related to the coefficients $\tilde{R}_{c}^{rs} \left[ \Delta_3 \Delta_2 \right]_{\Delta_4 \Delta_1}$ and their explicit form is known [8] (see Appendix). There are some important advantages of the formulae above. First of all one gets a series which is supposed to converge on the whole domain of analyticity of the conformal block. Secondly if we choose the parametrization (3.9) of the central charge the formulae does not contain square roots which simplifies symbolic calculations a lot.

The representation (4.6) and the asymptotic (1.1) imply the following representation for the classical conformal block:

\[
\begin{align*}
    f_{\delta}^{\left[ \delta_3 \Delta_2 \delta_1 \right]}(x) &= \left( \frac{3}{4} - \delta_1 - \delta_2 \right) \log x + \left( \frac{1}{4} - \delta_1 - \delta_3 \right) \log (1 - x) \\
    &+ \left( \frac{3}{4} - 2(\delta_1 + \delta_2 + \delta_3 + \delta_4) \right) \log \left( \frac{2}{\pi} K(x) \right) \\
    &+ (\delta - \frac{1}{4}) \log 16 - (\delta - \frac{1}{4}) \pi \frac{K(1 - x)}{K(x)} + h_{\delta}^{\left[ \delta_3 \Delta_2 \delta_1 \right]}(q),
\end{align*}
\]

where

\[
h_{\delta}^{\left[ \delta_3 \Delta_2 \delta_1 \right]}(q) = \sum_{n=1}^{\infty} (16q)^n h_{\delta}^{n} \left[ \delta_3 \Delta_2 \delta_1 \right]_{\Delta_4 \Delta_1}.
\]

The coefficients in (4.10) can be calculated step by step from the asymptotic (1.1)

\[
\sum_{n=1}^{\infty} (16q)^n h_{\delta}^{n} \left[ \delta_3 \Delta_2 \delta_1 \right]_{\Delta_4 \Delta_1} = \lim_{Q^2 \to \infty} \frac{1}{Q^2} \log \left( 1 + \sum_{n=1}^{\infty} (16q)^n H_{c, \Delta}^{n} \left[ \Delta_3 \Delta_2 \Delta_1 \right] \right).
\]

Using this formula in the special case of all parabolic external weights we have checked the asymptotic (1.1) up to the terms $q^{16}$. We have also calculated the coefficients of (4.10) up to
this order. The first few terms of the series (4.10) read:

\[
\begin{align*}
    h(p, q) &\equiv h_{\frac{1}{4}+p^2 } \left[ \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \right] (q) = 1 + \frac{1}{4(1+p^2)} q^2 \\
    &+ \left( \frac{-1}{32(1+p^2)^3} + \frac{3}{128(1+p^2)^2} + \frac{15}{128(1+p^2)} + \frac{81}{128(4+p^2)} \right) q^4 \\
    &+ \left( \frac{1}{96(1+p^2)^4} - \frac{5}{384(1+p^2)^3} + \frac{9}{256(1+p^2)^2} + \frac{2048(1+p^2)}{16384} \right) q^6 \\
    &+ \left( \frac{-5}{1024(1+p^2)^5} + \frac{35}{4096(1+p^2)^4} - \frac{8192(1+p^2)}{1653} + \frac{332239}{1048576} \right) q^8 + \ldots.
\end{align*}
\]

5 Geodesic length function

According to the uniformization theorem the punctured sphere \( X \) is conformally equivalent to \( \mathbb{H}/G \), where \( \mathbb{H} \) is the upper half plane endowed with the Poincare hyperbolic metric and \( G \) is the Fuchsian group uniquely (up to conjugation in \( \text{PSL}(2, \mathbb{R}) \)) determined by \( X \) and isomorphic to its fundamental group.

The group \( G \) is generated by \( T_i \in \text{PSL}(2, \mathbb{R}) \), \( i = 1, \ldots, n \). It is possible to chose them in such a way that

\[ T_1 T_2 \ldots T_n = I. \]

If all the punctures correspond to parabolic singularities, then

\[ |\text{Tr} T_i| = 2. \]

For each pair of punctures, say \( z_i \) and \( z_j \), there exists a unique closed geodesics of the hyperbolic metrics on \( X \), separating \( z_i \) and \( z_j \) from the remaining singularities. Its length \( \ell(\gamma_{ij}) \) can be determined from the relation

\[ 2 \cosh \frac{\ell(\gamma_{ij})}{2} = |\text{Tr} T_i T_j|. \]  

Setting for the four punctured sphere (by an appropriate global conformal transformation) \( z_1 = 0 \), \( z_2 = x \), \( z_3 = 1 \) and \( z_4 = \infty \) we have in the notation from the previous sections \( \ell(\gamma_{12}) \equiv \ell(\gamma_s) \), \( \ell(\gamma_{23}) \equiv \ell(\gamma_t) \), \( \ell(\gamma_{24}) \equiv \ell(\gamma_u) \).
In the case of locations \( x = \frac{1}{9}, \frac{1}{2}, e^{-\frac{\pi i}{3}} \) the group \( G \) is explicitly known \[35\]

\[
\begin{array}{c|ccc|c}
 x & T_1 & T_2 & T_3 & T_4 \\
\hline
\frac{1}{9} & \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix} & \begin{pmatrix} 2 & -9 \\ 1 & -4 \end{pmatrix} & \begin{pmatrix} -1 & -6 \\ 0 & -1 \end{pmatrix} \\
\hline
\frac{1}{2} & \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} & \begin{pmatrix} 3 & -4 \\ 4 & -5 \end{pmatrix} & \begin{pmatrix} 3 & -8 \\ 2 & -5 \end{pmatrix} & \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix} \\
\hline
e^{-\frac{\pi i}{3}} & \begin{pmatrix} -1 & 0 \\ \frac{3}{2} & -1 \end{pmatrix} & \begin{pmatrix} 5 & -6 \\ 6 & -7 \end{pmatrix} & \begin{pmatrix} 2 & -6 \\ \frac{5}{2} & -4 \end{pmatrix} & \begin{pmatrix} -1 & -8 \\ 0 & -1 \end{pmatrix}
\end{array}
\]

Taking into account the locations obtained from \( x = \frac{1}{9}, \frac{1}{2}, e^{-\frac{\pi i}{3}} \) by \( SL(2, \mathbb{C}) \) transformations preserving the set \( \{0, 1, \infty\} \) one gets

\[
\begin{array}{c|cccc|c cc}
 x & \frac{1}{9} & -\frac{1}{9} & \frac{8}{9} & \frac{9}{9} & -8 & 9 & \frac{1}{2} & -1 & 2 & e^{-\frac{\pi i}{3}} & e^{+\frac{\pi i}{3}} \\
\hline
\cosh \frac{\ell(\gamma_s)}{2} & 2 & 2 & 5 & 8 & 5 & 8 & 3 & 3 & 7 & \frac{7}{2} & \frac{7}{2}
\end{array}
\]

(5.2)

All the cases listed above concern the metrics on \( \mathbb{H}/G \), induced by the hyperbolic Poincaré metric on \( \mathbb{H} \) with the scalar curvature \(-2\). This corresponds to \( \mu = 4 \) in the relation (1.5) which in terms of the \( s \)-channel saddle point Liouville momentum (3.7) reads

\[
\ell(\gamma_s(x)) = 4\pi p_s(x).
\]

(5.3)

Taking into account the explicit formula (2.22) for the 3-point classical action one can write the saddle point equation (3.7) in the form:

\[
-\pi + 2i \log \frac{\Gamma(1 - 2ip)\Gamma^2(\frac{\ell}{2} + ip)}{\Gamma(1 + 2ip)\Gamma^2(\frac{\ell}{2} - ip)} = \mathfrak{Re} \frac{\partial}{\partial p} f(p, x).
\]

(5.4)

The precision of a numerical solution to this equation is in practice determined only by the precision of an approximation to the classical conformal block \( f(p, x) \) at hand.

For the locations inside the unit disc \( x = \frac{1}{9}, -\frac{1}{9}, \frac{1}{2}, \frac{8}{9} \) one can consider approximation given by the first \( n \) terms of the \( x \)-expansion (4.5). We have done the numerical calculations for \( n = 0, 1, \ldots, 7 \). The deviations from the exact values measured by the differences

\[
\Delta(x) = \cosh \left[ \frac{\ell(\gamma_s(x))}{2} \right] - \cosh [2\pi p_s(x)]
\]

are presented in Tab.1. They provide a very good confirmation of the formula (5.3).

A much better precision can be achieved if we use the \( q \)-expansion of \( f(p, x) \) (4.12). The results of numerical calculations with the approximations of \( f(x, p) \) up to the terms \( q^{2k}, k = 0, 1, \ldots, 8 \) are presented in Tab.2. The agreement with the conjectured exact formula (5.3) is perfect.
The analytic bounds for the geodesic length function for the locations set to be slightly less than 16 decimal digits (Tab.3).

\[
\text{deviations from the exact values (5.2) closed to the precision of our numerical calculations}
\]

The advantage of the \(q\)-expansion is that it converges rapidly for all locations \(x\) on the complex plane except tiny neighborhoods of \(x\) for all the cases listed in (5.2). Using the \(q\)-expansion up to the terms \(q^{16}\) we get the deviations from the exact values (5.2) closed to the precision of our numerical calculations set to be slightly less than 16 decimal digits (Tab.3).

As another verification of the formula (5.3) one can compare the numerical results with the analytic bounds for the geodesic length function for the locations \(x\) such that the geodesic \(\gamma_s(x)\) is contained in the unit disc [35]:

\[
\frac{2\pi^2}{\log(256|x|^{-1} + 136)} < \ell_s(x) < \frac{2\pi^2}{\log\left(\frac{16\sqrt{1 - |x|}}{|x|}\right) - \frac{\pi^2}{4\log\left(\frac{16\sqrt{1 - |x|}}{|x|}\right)}}
\]  

(5.5)

We present sample numerical calculations of the geodesic length function along the rays

\[
x(r) = re^{i\frac{\pi k}{4}}, \quad k = 0, 1, 2, 3, 4.
\]  

(5.6)
| $x$  | $\Delta(x)$     | $x$  | $\Delta(x)$     |
|------|------------------|------|------------------|
| $\frac{1}{2}$ | $-1.3 \times 10^{-15}$ | $\frac{1}{2}$ | $1.3 \times 10^{-15}$ |
| $-\frac{1}{2}$ | $1.8 \times 10^{-15}$ | $-1$ | $4.4 \times 10^{-15}$ |
| $\frac{3}{4}$  | $1.5 \times 10^{-14}$ | $2$  | $-7.1 \times 10^{-15}$ |
| $\frac{9}{4}$  | $1.4 \times 10^{-13}$ | $e^{-\frac{9}{4}}$ | $2.7 \times 10^{-15}$ |
| $-8$          | $-8.9 \times 10^{-15}$ | $e^{\frac{9}{4}}$ | $2.7 \times 10^{-15}$ |
| $9$           | $1.4 \times 10^{-13}$ |

Tab. 3

The results for the $q$-expansion up to the term $q^{16}$ are plotted on Fig. 1 for two different ranges of $r$.

Let us close this section with the remark that it is straightforward to work out an analytic approximation for the saddle point momentum $p_s$ and, consequently, for the geodesic length $\ell(\gamma_s)$ for small $x$. Indeed, with the help of the Lagrange duplication formula

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{2})$$

eq (\Omega_{p_x})^2 \frac{\partial}{\partial p} f(p, x).

Performing a series expansion of the l.h.s. according to the formula

$$\frac{1}{\Gamma(1+z)} = \exp \left\{ \gamma z - \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k \right\},$$

where $\gamma$ is the Euler–Marcheroni constant, we can rewrite (5.7) in the form

$$-\pi + 16p \log 2 + 2i \log \frac{\Gamma(1 + 2ip)}{\Gamma(1 - 2ip)} \Gamma^2(1 + ip) = \Re \frac{\partial}{\partial p} f(p, x).$$

Using the explicit form of $f(p, x)$ one can express a solution of (5.8) in the form of an expansion in $\Re x$ and $\frac{1}{\log x}$. For instance, keeping in (5.8) terms up to $p^3$ and $x^3$ we get

$$p_s = \log x + 16 \log 2 - \Re x - \frac{207}{512} \Re x^2 - \frac{205}{1536} \Re x^3 + \frac{8\zeta(3)}{\pi} \left( \Re x^2 + \Re x^3 \right)$$

$$= \log x + 16 \log 2 - \Re x - \frac{207}{512} \Re x^2 - \frac{205}{1536} \Re x^3 + \frac{8\pi^3\zeta(3)}{\pi} \left( \Re x^2 + \Re x^3 \right) + \mathcal{O} \left( \frac{1}{\log x} \right)^4.$$
6 Classical bootstrap

The $q$-expansion yields an extremely efficient method of numerical calculation of the classical conformal block and the saddle point momenta for all locations $x \neq 0, 1, \infty$. The precision of the $s$-channel calculations certainly worsens near $x = 1, x = \infty$ singularities. Still the range of rapid convergence of the $q$-expansion is huge enough to provide a reasonable testing ground for the classical bootstrap equations (1.3). For instance in the region $|x| < 4.6 \times 10^5$ with the small disc $|1 - x| < 0.0003$ around $x = 1$ removed one has $|q(x)| < \frac{1}{2}$.

In the present paper we are interested in checking the classical bootstrap equations in the simplest case of four parabolic singularities. The $s$-channel factorization of the 4-point classical action (1.2) yields the expression

$$S_4(x) \equiv S^{(cl)} \left( \frac{1}{T}, \frac{1}{T}, \frac{1}{T}, \frac{1}{T}; x \right) = 2S_3(p_s(x)) - 2\Re f(p_s(x), x)$$
where \( f(p, x) \) is the classical conformal block (4.5), \( p_s(x) \) is the saddle point momentum in the \( s \)-channel given by (5.4), and \( S_3(p) \) is the 3-point classical action (2.22) for two parabolic and one hyperbolic weight\(^6\):

\[
S_3(p) \equiv S^{(cl)}(\frac{1}{4}, \frac{1}{4}, \frac{1}{4} + p^2) = 4F \left( \frac{1}{2} + ip \right) + H(2ip) + \pi p,
\]

The efficiency of numerical methods provides a considerable freedom in testing the classical bootstrap equations. As an example we have chosen the calculation of the relative deviation from these equations measured by the functions:

\[
\Delta_{st}(x) = \frac{S_4(1 - x) - S_4(x)}{S_4(x)}
\]
\[
\Delta_{su}(x) = \frac{S_4(\frac{1}{x}) - \log |x| - S_4(x)}{S_4(x)}.
\]

The largest deviation should be expected around the locations \( x = 0, 1, \infty \). The behavior of the functions \( \Delta_{st}(x), \Delta_{su}(x) \) in these regions is well represented by their values along the real axis. We present the results for two approximations to the classical conformal block: up to the \( q^{12} \) terms and up to the \( q^{16} \) terms in the expansion (4.12). The results are shown on Fig.2 and Fig.3. They provide an excellent numerical verification of the classical bootstrap.

![Graph showing relative deviation from the classical bootstrap equations along the real axis near x = 0, 1 singularites.](image)

7 Conclusions

The numerical tests presented in this paper provide a convincing evidence that

- the classical limit of the conformal block exists yielding a consistent definition of the classical conformal block;

\(^6\)In order to simplify the formula an appropriate constant was chosen in (2.22).
Fig.3: Relative deviation from the classical bootstrap equations along the real axis near $x = \infty$ singularity.

- the geodesic length functions of the hyperbolic geometry on the 4-punctured Riemann sphere can be calculated by the saddle point equation involving the classical conformal block and the 3-point classical Liouville action;

- the classical Liouville action on the 4-punctured Riemann sphere can be calculated in terms of the geodesic length function, the classical conformal block and the 3-point classical Liouville action.

The statements above were derived by heuristic field theoretical arguments within the path integral representation of the quantum Liouville theory and should be regarded as well motivated conjectures. It might be surprising that they are so well supported by numerical calculations. Still a challenging problem is to provide rigorous mathematical proofs for them. Any attempt in this direction seem to require a better understanding of the classical conformal block itself and in particular a direct way to calculate it. The main motivation for this line of research is the long standing problem of the uniformization of the 4-punctured Riemann sphere. Indeed if the factorization of the classical Liouville action holds and the classical conformal block and the geodesic length functions are available one can calculate the 4-point Liouville action and, by the Polyakov conjecture, the accessory parameter of the appropriate Fuchsian equation.

The efficiency of numerical calculations based on the $q$-expansion used in this paper are completely satisfactory. As an example we present on Fig.4 the contour plots of the geodesic lengths in different channels in the case of the hyperbolic metric with the constant scalar curvature $R = -1$, i.e. the lines of constant value of the length Frenkel–Nielsen coordinate on the moduli space of the 4-puncture sphere as a function of the Koba–Nielsen coordinate $x$. Let us note that also the twist Frenkel–Nielsen coordinate can be calculated. Indeed it was proved in [24, 25] that the second derivative of the classical Liouville action with respect to the location of the punctures, $\frac{\partial S_{\text{cl}}}{\partial z_i \partial \bar{z}_j}$, gives the Weil-Petersson metric on the moduli space. With the 4-point classical Liouville action at hand one can calculate the Weil-Petersson metric and the twist coordinate along each line of constant length.
Fig. 4: Geodesic length functions in different channels for the hyperbolic metric with the scalar curvature $R = -1$ (the Gaussian curvature $-\frac{1}{2}$).
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Appendix

The coefficients \( \tilde{R}_\Delta^{rs}\left[\frac{\Delta_3}{\Delta_1}\Delta_2\right] \) of the \( x \)-expansion can be written as

\[
\tilde{R}_\Delta^{rs}\left[\frac{\Delta_3}{\Delta_1}\Delta_2\right] = \tilde{A}_\Delta^{rs}\tilde{P}_\Delta^{rs}\left[\frac{\Delta_3}{\Delta_1}\right] \tilde{P}_\Delta^{rs}\left[\Delta_2\right],
\]

\[
\tilde{A}_\Delta^{rs} = -\frac{\partial c_{rs}(\Delta)}{\partial \Delta} A_{rs}(T_{rs}(\Delta)) = \frac{24(T_{rs}(\Delta))^2 - 1}{1 - s^2 + (r^2 - 1)T_{rs}(\Delta)^2} A_{rs}(T_{rs}(\Delta)),
\]

\[
\tilde{P}_\Delta^{rs}\left[\Delta_2\right] = P_{rs}(T_{rs}(\Delta), \Delta_1 + \Delta_2, \Delta_1 - \Delta_2).
\]

For arbitrary positive integers \( m, n \) the function \( A_{mn}(\alpha^2) \) is defined by the relations

\[
A_{mn}(\alpha^2) = -\frac{1}{2} \left( \prod_{k=1}^{m} \prod_{l=1}^{n} \frac{1}{k\alpha - \frac{l}{\alpha}} \right), \tag{A.1}
\]

where the prime on the symbol of products means that the factors \((k, l) = (0, 0)\) and \((k, l) = (m, n)\) must be omitted. \( A_{mn}(\alpha^2) \) can be rewritten in the form

\[
A_{mn}(\alpha^2) = \frac{1}{2mn} \prod_{k=1}^{m-1} \prod_{l=1}^{n-1} \frac{\alpha^4}{(l^2 - k^2\alpha^4)^2} \prod_{k=1}^{m-1} \frac{1}{k^2 \alpha^4 - n^2} \prod_{l=1}^{n-1} \frac{\alpha^4}{l^2 \alpha^4 - m^2 \alpha^4}.
\]

For arbitrary positive integers \( m, n \) the function \( P_{mn}(\alpha^2, \Delta, \delta) \) is defined by the relations

\[
P_{mn}(\alpha^2, \Delta_1 + \Delta_2, \Delta_1 - \Delta_2) = \prod_{p=1}^{m-1} \prod_{q=1}^{n-1} \left( \frac{\alpha_1 + \alpha_2 - pa + \frac{q}{\alpha}}{2} \right) \left( \frac{\alpha_1 - \alpha_2 - pa + \frac{q}{\alpha}}{2} \right) \tag{A.2}
\]

where the variables \( \alpha_i \) are related to \( \Delta_i \) via \( \Delta_i = -\frac{1}{4} \left( \alpha - \frac{1}{\alpha} \right)^2 + \frac{\alpha^2}{4} \).

\( P_{mn}(\alpha^2, \Delta, \delta) \) can be expressed in the form

\[
P_{mn}(\alpha, \Delta, \delta) = \prod_{k=1}^{4} P_{mn}^{k}(\alpha, \Delta, \delta)
\]

where

\[
P_{mn}^{1}(\alpha, \Delta, \delta) = \prod_{m-1 \geq p > 0} \prod_{n-1 \geq q > 1-n} \prod_{p+m=1 \text{ mod } 2} \prod_{q+n=1 \text{ mod } 2} Q_{p,q}(\alpha, \Delta, \delta)Q^{-p,q}(\alpha, \Delta, \delta)
\]
\[
P_{mn}(\alpha, \Delta, \delta) = \begin{cases} 
\prod_{q+n=1 \text{ mod } 2} q > 0 \\
 Q_{0,q}(\alpha, \Delta, \delta) & \text{if } m \text{ is odd} \\
 1 & \text{otherwise}
\end{cases}
\]

\[
P_{mn}^3(\alpha, \Delta, \delta) = \begin{cases} 
\prod_{p+m=1 \text{ mod } 2} p > 0 \\
 Q_{p,0}(\alpha, \Delta, \delta) & \text{if } n \text{ is odd} \\
 1 & \text{otherwise}
\end{cases}
\]

\[
P_{mn}^4(\alpha, \Delta, \delta) = \begin{cases} 
\delta & \text{if } m \text{ and } n \text{ are odd} \\
 1 & \text{otherwise}
\end{cases}
\]

and

\[
Q^{\mu, \nu}(\alpha^2, \Delta, \delta) = \left[ \frac{1}{16} \left( \frac{q^2}{\alpha^2} + p^2 \alpha^2 - 2pq \right) \left( \frac{q^2 - 4}{\alpha^2} + (p^2 - 4) \alpha^2 + 2(4 - pq) - 8\Delta \right) + \delta^2 \right].
\]

The coefficients \(R_{c}^{rs} \left[ \frac{\Delta_1 \Delta_2}{\Delta_1 \Delta_1} \right] \) that appear in the \(q\)-expansion of the conformal block can be written as

\[
R_{c}^{rs} \left[ \frac{\Delta_1 \Delta_2}{\Delta_1 \Delta_1} \right] = A_{c}^{rs} P_{c}^{rs} \left[ \frac{\Delta_1}{\Delta_1} \right] P_{c}^{rs} \left[ \frac{\Delta_2}{\Delta_1} \right],
\]

\[
A_{c}^{rs} = A_{rs}(-b^2) = A_{rs}(b^2),
\]

\[
P_{c}^{rs} \left[ \frac{\Delta_2}{\Delta_1} \right] = P_{rs}(-b^2, \Delta_1 + \Delta_2, \Delta_1 - \Delta_2),
\]

where the functions \(A_{rs}\), and \(P_{rs}\) are defined by (A.1) and (A.2) respectively.

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