Existence and uniqueness of fixed point for ordered contraction type operator in Banach Space

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Abstract. In this paper, we investigate the existence and uniqueness of fixed point for partially ordered contraction type operators in Banach Space. We also present applications to integral and differential equations.

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1. Introduction

Existence of fixed points for contraction type maps in partially ordered metric space has been considered recently in [6]-[11], where some applications to matrix equation, ordinary differential equations and integral equations are presented, see [12]-[18]. The following generalization of Banach’s contraction principle is due to Geraghty [21].

Let ζ denotes the class of those functions $\beta : [0, \infty) \rightarrow [0, 1)$ which satisfy the condition

$$\beta(t_n) \rightarrow 1 \Rightarrow t_n \rightarrow 0. \quad (1.1)$$

Theorem 1.1. Let $(M, d)$ be a complete metric space and let $f : M \rightarrow M$ be a map. Suppose there exists $\beta \in \zeta$ such that for each $x, y \in M$,

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y). \quad (1.2)$$

Then $f$ has a unique fixed point $z \in M$, and $\{f^n(x)\}$ converges to $z$, for each $x \in M$. 

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In [19], J. Harjani and K. Sadarangani studied fixed point theorems for weakly contractive mappings in partially ordered sets. Very recently, Amini-Harandi and Emami [20] proved the following existence theorem which is a version of Theorem 1.1 in the context of partially ordered complete metric spaces and a generalization of results in [19]:

**Theorem 1.2.** Let \((M, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) in \(M\) such that \((M, d)\) is a complete metric space. Let \(f : M \to M\) be an increasing map such that there exists an element \(x_0 \in M\) with \(x_0 \preceq f(x_0)\). Suppose that there exists \(\beta \in \zeta\) such that

\[
d(f(x), f(y)) \leq \beta(d(x,y))d(x,y), \quad x, y \in M, y \preceq x. \tag{1.3}
\]

Assume that either \(f\) is continuous or \(M\) is such that if an increasing sequence \(\{x_n\} \to x\) in \(M\), then \(x_n \preceq x, \forall n\).

Besides, if

for each \(x, y \in M\), there exists \(z \in M\), which is comparable to \(x\) and \(y\).

Then \(f\) has a unique fixed point.

In this paper, we generalize Theorem 1.2 from three aspects. Firstly, the contraction condition (2.1) is merely about partial order, while in (1.3) the contraction is about metric and \(\beta : [0, \infty) \to [0, 1)\). The major difficult brought by (2.1) is that in (2.6) the contraction constant \(Nf(\| u - v \|)\) may bigger than 1, as the normal constant \(N\) of a cone is bigger than 1, see [22] in Lemma 2.1. Secondly, we do not need continuity or the equivalent condition of the operator as in [8]-[20]. Thirdly, we don’t need any upper or lower solution as in [8]-[20]. Our methods are different from that in [20]. In Section 3, an application to an integral equation is given.

Let us recall some preliminaries first.

**Definition 1.3 ([1]).** Let \(E\) be a real Banach space. A nonempty convex closed set \(P \subset E\) is called a cone if

(i) \(x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P\);

(ii) \(x \in P, -x \in P \Rightarrow x = \theta, \theta\) is the zero element in \(E\).

In the case that \(P\) is a given cone in a real Banach space \((E, \| \cdot \|)\), a partial order ”\(\leq\)” can be induced on \(E\) by \(x \leq y \iff y - x \in P\). The cone \(P\) is called normal if there exists a constant \(N > 0\) such that for all \(x, y \in E, \theta \leq x \leq y\) implies that \(\| x \| \leq N \| y \|\). The minimal such number \(N\) is called the normal constant of \(P\). Details about cones and fixed point of operators can be found in [1]-[5].

**Lemma 1.4 ([1]).** A cone \(P\) is normal if and only if there exists a norm \(\| \cdot \|_1\) in \(E\) which is equivalent to \(\| \cdot \|\) such that for any \(\theta \leq x \leq y, \| x \|_1 \leq \| y \|_1\), i.e., \(\| \cdot \|_1\) is monotone. The equivalence of \(\| \cdot \|\) and \(\| \cdot \|_1\) means that there exist \(M > m > 0\) such that \(m \| \cdot \|_1 \leq \| \cdot \| \leq M \| \cdot \|_1\).
Lemma 1.5 ([1]). Let \( P \) be a normal cone in a real Banach space \( E \). Suppose that \( \{x_n\} \) is a monotone sequence which has a subsequence \( \{x_{n_i}\} \) converging to \( x^* \), then \( \{x_n\} \) also converges to \( x^* \). Moreover, if \( \{x_n\} \) is an increasing sequence, then \( x^* \leq \{x_n\}(n = 1, 2, 3, ...) \); if \( \{x_n\} \) is a decreasing sequence, then \( x^* \leq \{x_n\}(n = 1, 2, 3, ...) \).

2. Main results

We suppose that \( E \) is a partially ordered Banach space. \( P \) is a normal cone and the normal constant is \( N \). The partial order "\( \leq \)" on \( E \) is induced by the cone \( P \).

Theorem 2.1 (Main Theorem). Suppose that \( A : E \to E \) is a decreasing operator and satisfies the following ordered contraction type condition:

\[
(H) \quad \text{There exists an increasing function } f : (0, +\infty) \to (0, 1) \text{ such that } Au - Av \leq f(\|v - u\|)(v - u), \quad \forall u, v \in E, \ u \leq v. \tag{2.1}
\]

Besides, if

\[
\text{for each } x, y \in E, \text{there exist both } \inf\{x, y\} \text{ and } \sup\{x, y\}. \tag{2.2}
\]

Then \( A \) has unique fixed point in \( E \).

Proof. Let \( u_0 \in E \), we have \( Au_0 \in E \). So we have the following two cases.

**Case I:** When \( u_0 \) is comparable to \( Au_0 \). Firstly, without loss of generality, we suppose that

\[
u_0 \leq Au_0. \tag{2.3}\]

If \( u_0 = Au_0 \), then the proof is finished. Suppose that \( u_0 < Au_0 \). Since \( A \) is decreasing we obtain \( Au_0 \geq A^2u_0 \) and it is easy to prove that \( A^2 \) is increasing. Using the contractive condition (2.1), we have

\[
Au_0 - A^2u_0 \leq f(\|Au_0 - u_0\|)(Au_0 - u_0) \leq Au_0 - u_0. \tag{2.4}
\]

So \( A^2u_0 \geq u_0 \), that is

\[
u_0 \leq A^2u_0. \tag{2.5}\]

From (2.1) and the normality of cone \( P \), we have

\[
\|Au - Av\| \leq Nf(\|v - u\|)\|v - u\|, \quad \forall u, v \in E, \ u \leq v, \tag{2.6}
\]

\[
A^2v - A^2u \leq f(\|Au - Av\|)(Au - Av) \leq f(Nf(\|u - v\|)\|u - v\|)(v - u) \leq f(Nf(\|u - v\|)\|u - v\|)(v - u). \tag{2.7}
\]

Let \( A^2 = B \). From (2.5) and the above inequalities we have the following two conclusions:

(a) There exists a nondecreasing function \( f : (0, +\infty) \to (0, 1) \) such that for \( u, v \in E \) with \( u \leq v \)

\[
Bv - Bu \leq f(Nf(\|u - v\|)\|u - v\|)(v - u), \tag{2.7}
\]

(b) There exists \( u_0 \in E \) such that \( u_0 \leq Bu_0 \).
We assert that the operator \( B \) has unique fixed point in \( E \). In fact, we can use the method of iteration to construct the fixed point of \( B \). Consider the iterative sequence

\[
x_{n+1} = Bx_n, \quad n = 0, 1, 2, \cdots. \tag{2.8}
\]

Since \( x_0 \leq Bx_0 \) and the operator \( B \) is increasing, we have

\[
x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \tag{2.9}
\]

This means that \( \{x_n\} \) is an increasing sequence. So

\[
\theta \leq x_{n+1} - x_n = Bx_n - Bx_{n-1} \leq f(Nf(\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|)(x_n - x_{n-1}).
\]

Since \( P \) is normal, from Lemma 1.5 we have

\[
\|x_{n+1} - x_n\| \leq f(Nf(M\|x_n - x_{n-1}\|)M\|x_n - x_{n-1}\|)f(M\|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|1.
\]

Since \( f(t) \in (0, 1) \) for all \( t \geq 0 \), so \( \|x_{n+1} - x_n\| \leq \|x_n - x_{n-1}\|1 \), i.e., \( \{\|x_n - x_{n-1}\|1\}(n = 1, 2, \cdots) \) is a nonnegative decreasing sequence. From \( f \) is increasing we know

\[
f(M\|x_n - x_{n-1}\|1) \leq f(M\|x_1 - x_0\|1) < 1.
\]

So

\[
\|x_{n+1} - x_n\| \leq f(Nf(M\|x_1 - x_0\|1)M\|x_1 - x_0\|1)f(M\|x_1 - x_0\|1)\|x_n - x_{n-1}\|1.
\]

Let \( \lambda = f(Nf(M\|x_1 - x_0\|1)M\|x_1 - x_0\|1)f(M\|x_1 - x_0\|1) \), then \( \lambda \in (0, 1) \).

So

\[
\|x_{n+1} - x_n\| \leq \lambda\|x_n - x_{n-1}\|1 \leq \cdots \leq \lambda^n\|x_1 - x_0\|1.
\]

We can assert that \( \{x_n\} \) is a Cauchy sequence in \( (E, \| \cdot \|) \). In fact, for any positive integer \( n, m \),

\[
\|x_{n+m} - x_n\|1 \leq \|x_{n+m} - x_{n+m-1}\|1 + \cdots + \|x_{n+1} - x_n\|1 \leq (\lambda^{n+m-1} + \cdots + \lambda^n)\|x_1 - x_0\|1 \leq \frac{\lambda^n}{1 - \lambda}\|x_1 - x_0\|1.
\]

It follows in a standard way that \( \{x_n\} \) is a Cauchy sequence in \( E \). Since \( E \) is complete, we can suppose that \( x_n \to x_1 \in E \). (2.9) together with Lemma 1.5 implies that

\[
x_n \leq x_1. \tag{2.10}
\]

(2.10), together with (2.7) and the equivalence of \( \| \cdot \|_1 \) and \( \| \cdot \| \) implies that

\[
\|Bx_1 - Bx_n\|1 \leq f(Nf(M\|x_1 - x_n\|1)M\|x_1 - x_n\|1)f(M\|x_1 - x_n\|1)\|x_1 - x_n\|1.
\]
So
\[
\| x^*_n - Bx^*_n \|_1 \leq \| x^*_n - x^* \|_1 + \| Bx^*_n - x^* \|_1 = \| x^*_n - x^* \|_1 + \| Bx^*_n - Bx^*_1 \|_1 \\
\leq \| x^*_n - x^* \|_1 + f(Nf(M\|x^*_n - x^*\|_1)M\|x^*_n - x^*\|_1)f(M\|x^*_n - x^*\|_1)\|x^*_n - x^*\|_1.
\]
Let \( n \to \infty \), we obtain \( \| x^*_n - Bx^*_n \|_1 = 0 \). So \( x^*_n = Bx^*_n \), i.e., \( x^*_n \) is a fixed point of \( B \) in \( E \) and \( \lim_{n \to \infty} Bx^*_n = \lim_{n \to \infty} B^n x_0 = Bx^*_n = x^*_n \).

Then we will prove the uniqueness of the fixed point. On the contrary, if \( \overline{x} \) is another fixed point of \( B \), we will get \( \overline{x} = x^*_n \). In fact, the first case, when \( \overline{x} \) is comparable with \( x_0 \). Without loss of generality, we suppose that \( \overline{x} \leq x_0 \). Since \( B \) is increasing, \( B^n \overline{x} \leq B^n x_0 \). Similar to the proof of the monotonicity of the sequence \( \{\|x_n - x_{n-1}\|_1\}(n = 1, 2, 3, \cdots) \), we can obtain \( \{\|B^n \overline{x} - B^n x_0\|_1\}(n = 0, 1, 2, \cdots) \) is also an increasing sequence and
\[
\|B^n \overline{x} - B^n x_0\|_1 \leq \lambda_1 \|B^{n-1} \overline{x} - B^{n-1} x_0\|_1 \leq \cdots \leq \lambda^n_1 \| \overline{x} - x_0 \|_1,
\]
in which \( \lambda_1 = f(Nf(M\| \overline{x} - x_0\|_1)M\| \overline{x} - x_0\|_1)f(M\| \overline{x} - x_0\|_1) \), then \( \lambda_1 \in (0, 1) \). Let \( n \to \infty \), we have
\[
\overline{x} = \lim_{n \to \infty} B^n \overline{x} = \lim_{n \to \infty} B^n x_0 = x^*_n.
\]
(2.11)

The second case, when \( \overline{x} \) can not compare with \( x_0 \). From (2.1), we obtain \( x_1 = \inf\{\overline{x}, x_0\}, x_2 = \sup\{\overline{x}, x_0\} \in E \) satisfying
\[
x_1 \leq \overline{x} \leq x_2, x_1 \leq x_0 \leq x_2
\]
i.e., \( \overline{x} \) is comparable with \( x_1, x_2 \) and \( x_0 \) is comparable with \( x_1, x_2 \). Since \( B \) is increasing, we know
\[
Bx_1 \leq B\overline{x} \leq Bx_2, Bx_1 \leq Bx_0 \leq Bx_2,
\]
and for any natural number \( n \)
\[
B^n x_1 \leq B^n \overline{x} \leq B^n x_2, B^n x_1 \leq B^n x_0 \leq B^n x_2.
\]
So we know \( B^n \overline{x} \) can compare with \( B^n x_1 \) and \( B^n x_2 \). Similarly we can prove that \( \{\|B^n x_i - B^n x_0\|_1\}(n = 0, 1, 2, \cdots) \) and \( \{\|B^n x_i - B^n \overline{x}\|_1\}(n = 0, 1, 2, \cdots) \) are also nonnegative and decreasing sequences(in which \( B^0 x = x \)) and
\[
\|B^n x_i - B^n x_0\|_1 \leq \lambda_2 i \|B^{n-1} x_i - B^{n-1} x_0\|_1 \leq \cdots \leq \lambda^n_2 \|x_i - x_0\|, \tag{2.12}
\]
\[
\|B^n x_i - B^n \overline{x}\|_1 \leq \lambda_3 i \|B^{n-1} x_i - B^{n-1} \overline{x}\|_1 \leq \cdots \leq \lambda^n_3 \|x_i - \overline{x}\|, \tag{2.13}
\]
in which
\[
\lambda_2 i = f(Nf(M\|x_i - x_0\|_1)M\|x_i - x_0\|_1)f(M\|x_i - x_0\|_1),
\]
\[
\lambda_3 i = f(Nf(M\| \overline{x} - x_i\|_1)M\| \overline{x} - x_i\|_1)f(M\| \overline{x} - x_i\|_1),
\]
\( \lambda_2 i, \lambda_3 i \in (0, 1) \). Let \( n \to \infty \) in (2.12) and (2.13), we have
\[
\lim_{n \to \infty} B^n x_i = \lim_{n \to \infty} B^n x_0 = x^*_n,
\]
\[
\lim_{n \to \infty} B^n x_i = \lim_{n \to \infty} B^n \overline{x} = \overline{x}.
\]
So
\[ x^* = \overline{x}. \quad (2.14) \]
(2.11) together with (2.14) implies that \( x^* \) is unique fixed point of \( B \).
Next we will prove that the unique fixed point of \( B \) is also the unique fixed point of \( A \).
Since
\[ A^2 x^* = B x^* = x^*, \]
and
\[ A^2 (Ax^*) = A(A^2 x^*) = Ax^*, \]
i.e., \( B(Ax^*) = Ax^* \). From the uniqueness of the fixed point of \( B \) we know
\[ Ax^* = x^*. \quad (2.15) \]
So \( x^* \) is the unique fixed point of \( A \) in \( E \).

**Case II**: Another case, when \( u_0 \) is not comparable to \( Au_0 \). From the assumption (H), we know there exists \( v_0 \in E \) such that \( \inf\{Au_0, u_0\} = v_0 \).
That is \( v_0 \leq Au_0, v_0 \leq u_0 \). Since \( A \) is a decreasing operator, we have
\[ A^2 u_0 \leq Av_0, Au_0 \leq Av_0. \]
This shows that
\[ v_0 \leq Av_0. \quad (2.16) \]
Similarly as the proof of Case I, we can get that \( A \) has unique fixed point in \( E \). □

### 3. Applications

In this section, we present two examples where our Theorem can be applied.

**Example 1.** We consider the self-feedback stability of a signal outlet function in nonlinear suppressed interference channel. When outlet signals are fed back to the input process, we want to know whether the final signals are stable. We suppose that the signal period is 1. We only consider situations in a period. The signal space is \( C[0, 1] \) and the signal output function is (only in the case of real number)
\[ Au(t) = \frac{1}{2\pi + u(t)} - \frac{\pi^2}{16} \int_0^1 (s^2 + t^2) \frac{1 + u(t)s^2}{2\pi M} ds, \]
\( M \) is a positive integer. Let \( P = \{u(t)|u(t) \geq 0, \ t \in [0, 1]\} \), then \( P \) is a normal cone in \( C[0, 1] \). The partial order \( \leq \) induced by \( P \) is: \( u \leq v \Leftrightarrow u(t) \leq v(t) \) for all \( t \in [0, 1] \). \( E = C[0, 1] \) is a partially ordered Banach space. Evidently,
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A is a decreasing operator. For all \( u(t), v(t) \in E \) satisfying \( u(t) \leq v(t) \), we obtain that

\[
Au(t) - Av(t) = \frac{1}{2\pi + u(t)} - \frac{\pi^2}{16} \int_0^1 (s^2 + t^2) \frac{1 + u(t)s^2}{2\pi M} ds
- \frac{1}{2\pi + v(t)} + \frac{\pi^2}{16} \int_0^1 (s^2 + t^2) \frac{1 + v(t)s^2}{2\pi M} ds
= \frac{v(t) - u(t)}{(2\pi + u(t))(2\pi + v(t))} + \frac{\pi^2}{16} \int_0^1 (s^2 + t^2) (v(t) - u(t))s^2 \frac{1}{2\pi M} ds
\leq \frac{v(t) - u(t)}{4\pi^2} + \frac{\pi}{60M} (v(t) - u(t))
\leq \frac{3}{20} (v(t) - u(t)).
\]

When take \( f(t) = \frac{3}{20} \) in Theorem 2.1, it is easy to know that the conclusion of Theorem 2.1 holds, i.e., there is unique \( u^* \in E \) such that \( Au^* = u^* \). This means that the signal outlet function has self-feedback stability.

**Example 2.** Now, we study the existence of solution for the following first-order periodic problem

\[
\begin{align*}
    u'(t) &= F(t, u(t)), t \in [0, 1] \\
    u(0) &= u(1).
\end{align*}
\]

(3.1)

where \( F : [0, 1] \times R \to R \) is a continuous function.

We consider the space \( C(I)(I = [0, 1]) \) of continuous functions defined on \([0, 1] \). Obviously, this space with the metric given by

\[
d(x, y) = \sup |x(t) - y(t)|, t \in I, x, y \in C(I)
\]

ia a Banach space. \( C(I) \) can be equipped with a partial order induced by a cone

\[
P = \{ y - x : y(t) - x(t) \geq 0, t \in I \}.
\]

Obviously, \( P \) is a normal cone and assume that its normal constant is \( N \). And the order relation in \( C(I) \) induced by \( P \) is:

\[
x, y \in C(I), x \leq y \iff x(t) \leq y(t), t \in I.
\]

**Theorem 3.1.** Consider problem (3.1) with \( F : I \times R \to R \) continuous and suppose that there exists \( \lambda > 0 \) and \( 0 < \alpha \leq \lambda N \) such that for \( x, y \in R \) with \( x \geq y \),

\[
0 \leq F(t, y) + \lambda y - (F(t, x) + \lambda x) \leq \alpha (x - y) \ln[(1 + \frac{1}{x - y})^{x-y}].
\]

Then there exists unique solution for problem (3.1).

**Proof.** Problem (3.1) can be written as

\[
u(t) = \int_0^1 G(t, s)[F(s, u) + \lambda u(s)]ds,
\]
where $G(t, s)$ is the Green function given by

$$G(t, s) = \begin{cases} 
\frac{e^{\lambda(1+s-t)}}{e^{\lambda} - 1}, & 0 \leq s < t \leq 1; \\
\frac{e^{\lambda(s-t)}}{e^{\lambda} - 1}, & 0 \leq t < s \leq 1.
\end{cases}$$

Define $T : C(I) \to C(I)$ by

$$(Tu)(t) = \int_0^1 G(t, s)[F(s, u(s)) + \lambda u(s)]ds.$$ 

Note that if $u \in C(I)$ is a fixed point of $T$ then $u \in C^1(I)$ is a solution of (3.1).

In what follows, we check that hypotheses of Theorem 2.1 are satisfied. Clearly, $(C(I), \leq)$ satisfies condition (2.2), since for $x, y \in C(I)$ the functions $\max\{x, y\}, \min\{x, y\}$ are least upper and greatest lower bounds of $x$ and $y$, respectively.

The operator $T$ is decreasing, since for $u \geq v$, and using our assumption, we can obtain

$$F(t, u) + \lambda u \leq F(t, v) + \lambda v,$$

which implies, since $G(t, s) > 0$, that for $t \in I$,

$$(Tu)(t) = \int_0^1 G(t, s)[F(s, u(s)) + \lambda u(s)]ds$$

$$\leq \int_0^1 G(t, s)[F(s, v(s)) + \lambda v(s)]ds = (Tv)(t),$$

Besides, for $u \geq v$, we have

$$\|Tu - Tv\| = \sup_{t \in I} |(Tu)(t) - (Tv)(t)|$$

$$\leq \sup \int_0^1 G(t, s) |F(s, u(s)) + \lambda u(s) - F(s, v(s)) - \lambda v(s)| ds$$

$$\leq \sup \int_0^1 \alpha G(t, s)(u - v) \ln[(1 + \frac{1}{u-v})^{u-v}] ds$$

$$\leq \alpha \|u - v\| f(\|u - v\|) \sup \int_0^1 G(t, s) ds$$

$$= \alpha \|u - v\| f(\|u - v\|) \sup_{t \in I} \frac{1}{e^{\lambda} - 1} \left[ e^{\lambda(1+s-t)}|t|_0 + \frac{1}{\lambda} e^{\lambda(s-t)}|t|_1 \right]$$

$$= \alpha \|u - v\| f(\|u - v\|) \frac{1}{\lambda e^{\lambda} - 1} (e^\lambda - 1)$$

$$= \alpha \|u - v\| f(\|u - v\|) \frac{1}{\lambda}$$

$$\leq N f(\|u - v\|) \|u - v\|.$$

This implies that $T$ satisfies condition (2.6) which can be used to prove the uniqueness of solution. And (2.6) is deduced by (2.1).

In the above inequalities we choose $f(t) = t \ln(1 + \frac{1}{t})$. It is easy to prove that $f(t)$ is increasing and $f(t) : (0, +\infty) \to (0, 1)$.
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Finally, Theorem 2.1 gives that $T$ has an unique fixed point. □

Competing Interests
The authors declare that they have no competing interests.

Authors’ Contributions
J. Mao proved main Theorems. Z. Zhao gave the application of the manuscript. All authors read and approved the manuscript.

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