‘Spectrum generating algebras’ of classical systems: the Kepler-Coulomb potential

Ş. Kuru¹ and J. Negro²

¹ Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey.
² Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, 47011 Valladolid, Spain.
E-mail: kuru@science.ankara.edu.tr
E-mail: jnegro@fta.uva.es

Abstract. We construct the classical ‘spectrum generating algebra’ for the Kepler-Coulomb potential and find a type of constants of motion depending explicitly on time. Such constants give rise to the motion of this system in an algebraic way.

1. Introduction

The supersymmetric quantum mechanics (or susy-qm) methods are often used to deal with exactly solvable problems as well as the quasi-exactly solvable ones. As we know the application of these factorization methods allow us to find the spectrum generating algebras and potential algebras in quantum mechanics [1, 2, 3, 4]. But in classical mechanics also it can be defined the ‘spectrum generating algebra’ closing with Poisson brackets [5, 6, 7, 8, 9, 10]. In the case of classical systems this algebra allows us to find trajectories algebraically.

Our aim is here to construct the algebraic structure of the classical one-dimensional Kepler-Coulomb problem which is analogue of the corresponding quantum system that can be solved by means of the factorization method. This algebraic structure is related with the so called spectrum generating algebras in quantum mechanics, as can be seen replacing Poisson brackets by commutators: \{·,·\} → −i[·,·]. We also obtain two explicit time-dependent constants of motion that determine the trajectories algebraically. Finally, we have characterized the ‘spectrum generating’ algebra for Kepler-Coulomb problem, and in this way have determined the motion [11].

2. ‘Factorization’ of Kepler-Coulomb Potential

The Hamiltonian function for Kepler-Coulomb problem is

\[ H = \frac{p^2}{2m} + \frac{\ell^2}{2mr^2} - \frac{k}{r} \] (1)

where \(r,p\) are canonical coordinates and \(\{r,p\} = 1\). For the sake of simplicity, \(1/2m = 1\) and \(k = 1\) will be taken. As the Hamiltonian does not depend explicitly on time, so it is a constant...
of motion and its value is the energy of the system: \(H = E\). Following the quantum mechanical case, we multiply the Hamiltonian by \(r^2\) [11, 12]. Then, we arrive to the following factorization

\[
r^2 p^2 - r - r^2 H = A^+ A^- + \gamma(H) = -\ell^2.
\]

(2)

For this Hamiltonian, the functions \(A^\pm\) have the form:

\[
A^\pm = (\mp i r p + a\sqrt{-H} r + \frac{b}{\sqrt{-H}}) e^{\pm f(r,p)}
\]

(3)

where the function \(f(r,p)\) is arbitrary. \(H\) and \(A^\pm\) are functions of the canonical coordinates \((r,p)\) and we want they satisfy the following algebra with the Poisson brackets [7]

\[
\{H, A^\pm\} = \mp i \alpha(H) A^\pm
\]

(4)

\[
\{A^+, A^-\} = i \beta(H).
\]

(5)

Here \(\alpha(H)\) and \(\beta(H)\) are functions of \(H\) to be determined. This kind of algebra is called a classical 'spectrum generating algebra'. Substituting \(A^\pm\) given by (3) in (2), \(a, b\) and \(\gamma(H)\) are obtained:

\[
a = 1, \quad b = -\frac{1}{2}, \quad \gamma(H) = \frac{1}{4H}.
\]

(6)

Next, \(A^\pm\) will satisfy the conditions (4) and (5), if we take \(f(r,p) = -2 i r p \sqrt{-H}\). Finally, the factor functions are

\[
A^\pm = (\mp i r p + \sqrt{-H} r - \frac{1}{2\sqrt{-H}}) e^{\pm 2 i r p \sqrt{-H}}
\]

(7)

together with \(\alpha(H) = -4 H \sqrt{-H}\) and \(\beta(H) = 1/\sqrt{-H}\). Then, the commutators (4) and (5) take on the form

\[
\{H, A^\pm\} = \pm 4 i H \sqrt{-H} A^\pm
\]

(8)

\[
\{A^+, A^-\} = \frac{i}{\sqrt{-H}}.
\]

(9)

Defining \(A_0 = 1/2 \sqrt{-H}\), we get the \(su(1,1)\) algebra

\[
\{A_0, A^\pm\} = \mp i A^\pm
\]

(10)

\[
\{A^+, A^-\} = 2 i A_0.
\]

(11)

Now we can define time-dependent constants of motion in terms of the factor functions as

\[
Q^\pm = A^\pm e^{\mp i \alpha(H)t}.
\]

(12)

It is easy to check that these are constants of motion, that is,

\[
\frac{dQ^\pm}{dt} = \{Q^\pm, H\} + \frac{\partial Q^\pm}{\partial t} = 0.
\]

(13)

The values of these constants of motion are complex numbers, \(Q^\pm = q_0 e^{\pm i \theta_0}\). Because of the factorization (2), \(q_0\) is a function of the energy, \(q_0(E)\). We know from [7] that we can determine the motion of the particle using these time-dependent constants of motion. For this potential the motion has two different characters depending on the energy: bounded \((E < 0)\) or unbounded \((E > 0)\). Here we consider both of them separately.
• Bounded motion ($E < 0$): $Q^\pm$ are written as follows

$$Q^\pm = \left( \mp i r p + \sqrt{-E} r - \frac{1}{2 \sqrt{-E}} \right) e^{\mp (2 i r p \sqrt{-E} - 4 i E \sqrt{-E} t)} = q_0 e^{\pm i \theta_0}. \quad (14)$$

or explicitly,

$$-i r p + \sqrt{-E} r - \frac{1}{2 \sqrt{-E}} = q_0 e^{i (2 r p \sqrt{-E} - 4 E \sqrt{-E} t + \theta_0)}$$

$$i r p + \sqrt{-E} r - \frac{1}{2 \sqrt{-E}} = q_0 e^{-i (2 r p \sqrt{-E} - 4 E \sqrt{-E} t + \theta_0)} \quad (15)$$

where $q_0$ has the form: $q_0 = \sqrt{-1 - \frac{1}{4E} - l^2}$, since $A^+ A^- = -\frac{1}{4E} - l^2$. From these equations we obtain the following implicit relations for $r$ and $rp$,

$$2 \sqrt{-E} r - \frac{1}{2 \sqrt{-E}} = 2 q_0 \cos (2 r p \sqrt{-E} - 4 E \sqrt{-E} t + \theta_0)$$

$$r p = -q_0 \sin (2 r p \sqrt{-E} - 4 E \sqrt{-E} t + \theta_0). \quad (16)$$

Although from them we can not find explicitly $r$ and $p$ as a function of $t$, we can find $t$ in terms of $r$ (or in terms of $rp$)

$$t = -\frac{1}{4 E \sqrt{-E}} \left[ \arccos \left( \frac{1}{2 q_0 \sqrt{-E}} \left( 1 + 2 \sqrt{-E} r \right) \right) - 2 \sqrt{-E} ( E r^2 - l^2 + r )^{1/2} - \theta_0 \right]. \quad (17)$$

Let us make the following definitions

$$e = -2 q_0 \sqrt{-E}, \quad a = -\frac{1}{2 E} \quad (18)$$

where $e$ and $a$ correspond eccentricity and semimajor axis, respectively [13]. Then, we obtain $t$ in the form

$$t = -\frac{1}{4 E \sqrt{-E}} \left[ \arccos \left( \frac{1}{e} \left( 1 - \frac{r}{a} \right) \right) - (e^2 - (1 - \frac{r}{a})^2)^{1/2} - \theta_0 \right]. \quad (19)$$

Equation (19) is the same equation as in [13] with the definitions, $k = 1, m = 1/2$. For this case the angular frequency is given in the form $\omega = -4 E \sqrt{-E} = \alpha(H)$. This means that the frequency of motion is given by the underlying algebra. We know that in bounded motion the orbits are closed and the motion is periodic in time as can be seen in Figs. (1) and (2).

• Unbounded motion ($E > 0$):

In this case, since the character of the factor functions changes, the expressions given above must adapt to this situation: replacing the root $\sqrt{-H}$ with $i \sqrt{H}$. Then, $\alpha(H) = -4 i H \sqrt{-H}$ and $\beta(H) = -i \sqrt{H}$. The factor functions for this case are pure imaginary and the time-dependent constants of motion (also pure imaginary) take the form

$$Q^\pm (r, p, t) = i \left( \mp r p + \sqrt{H} r + \frac{1}{2 \sqrt{H}} \right) e^{\pm (2 r p \sqrt{H} - 4 H \sqrt{H} t)} \quad (20)$$

The values of these constants are $q^\pm = i q_0 e^{\pm \theta_0}$ where $q_0 = (\frac{1}{4E} + l^2)^{1/2}$. Then, we have the following equations

$$2 \sqrt{E} r + \frac{1}{\sqrt{E}} = 2 q_0 \cosh (2 r p \sqrt{E} - 4 E \sqrt{E} t + \theta_0) \quad (21)$$
These equations are implicit for \( r \) and \( p \), and we can not find explicit expressions for them as a function of time. But we can express for instance \( t \) in terms of \( r \)

\[
t = \frac{1}{4E\sqrt{E}} \left[ \text{arccosh} \left( \frac{1}{e} (1 + 2Er) \right) - \left( (1 + 2Er)^2 - e^2 \right)^{1/2} - \theta_0 \right]
\]

(23)

where \( e = 2q_0 \sqrt{E} \). As we know, for unbounded motion the trajectories are open and the motion is not periodic which is seen in Figs. (1) and (2). It is easy also to check that the classical ‘spectrum generating algebra’ has the same form when we make following definition, \( A_0 = \frac{-i}{2\sqrt{H}} \).

Figure 1. Plot of \( r - t \) for \( E < 0 \) (the left one) and for \( E > 0 \) (the right one).

Figure 2. Orbits in phase plane.

3. Conclusions
In this work we have shown how it is constructed a classical ‘spectrum generating algebra’ for Kepler-Coulomb potential systems. Using this algebra we have presented two explicitly time-dependent constants of motion. Then, we have determined the motion algebraically. We have seen that our result is the same as those well known in text books [13], but the method we have used here is very practical and easy to follow. We can also say that the classical ‘spectrum generating algebras’ are similar as in the corresponding quantum mechanical systems.

It is interesting that there is a connection between this algebra and the action-angle variables. The angle variable has a frequency given by the coefficient \( \alpha(E) \) and the action is essentially determined by \( \beta(E) \) in the Poisson brackets of the classical ‘spectrum generating algebra’, (8)
and (10) [11]. This connection is specially attractive in the construction of coherent states [14, 15].

This algebraic approach has already been extended to 2-dimensional cases by Hussin et al. [8] and to complex potentials by Sinha et al. [9]. This type of algebras is also important in the search of integrability properties of classical systems [10].

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