WHITEHEAD MODULES OVER LARGE PRINCIPAL IDEAL DOMAINS

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Abstract. We consider the Whitehead problem for principal ideal domains of large size. It is proved, in ZFC, that some p.i.d.'s of size $\geq \aleph_2$ have non-free Whitehead modules even though they are not complete discrete valuation rings.

A module $M$ is a Whitehead module if $\text{Ext}^1_R(M, R) = 0$. The second author proved that the problem of whether every Whitehead $\mathbb{Z}$-module is free is independent of ZFC + GCH (cf. [5], [6], [7]). This was extended in [1] to modules over principal ideal domains of cardinality at most $\aleph_1$. Here we consider the Whitehead problem for modules over principal ideal domains (p.i.d.'s) of cardinality $> \aleph_1$.

If $R$ is any p.i.d. which is not a complete discrete valuation ring, then an $R$-module of countable rank is Whitehead if and only if it is free (cf. [3]). On the other hand, if $R$ is a complete discrete valuation ring, then it is cotorsion and hence every torsion-free $R$-module is a Whitehead module (cf. [2, XII.1.17]).

It will be convenient to decree that a field is not a p.i.d. and to use the term “slender” to designate a p.i.d. which is not a complete discrete valuation ring, or equivalently, is not cotorsion (cf. [3, III.2.9]). We will say that a module is $\kappa$-generated if it is generated by a subset of size $\leq \kappa$ and that it is $\kappa$-free if every submodule generated by $< \kappa$ elements is free. (Note that, by Pontryagin’s Criterion and induction on $\kappa$, every $\aleph_1$-free module which has rank $\leq \kappa$ is $\kappa$-generated.)

An argument due to the second author (cf. [3] or [4]) shows that it is consistent with ZFC + GCH that for any p.i.d. $R$ (of arbitrary size), there are Whitehead $R$-modules of rank $\geq |R|$ which are not free.

If the p.i.d. $R$ is slender and has cardinality at most $\aleph_1$, the Axiom of Constructibility ($V = L$) implies that every Whitehead $R$-module is free (cf. [2]). Our main result is that the story is different for p.i.d.'s of larger size. We will prove the following theorems in ZFC.

**Theorem 1.** There is a slender p.i.d. $R$ of cardinality $2^{\aleph_1}$ such that every $\aleph_1$-free $\aleph_1$-generated $R$-module is a Whitehead module. Hence there are non-free Whitehead $R$-modules which are $\aleph_1$-generated.

**Theorem 2.** There is a p.i.d. $R$ of cardinality $\aleph_2$ such that an $\aleph_1$-generated $R$-module is Whitehead only if it is free.

Assuming $V = L$ and using the existing theory (cf. [6]) one easily obtains the following:

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Corollary 3. (V = L) There are principal ideal domains $R_1$ and $R_2$ each of cardinality $\aleph_2$ and non-slender such that:

1. an $R_1$-module $M$ (of arbitrary cardinality) is Whitehead if and only if $M$ is the union of a continuous chain, $M = \bigcup_{\alpha<\lambda} M_\alpha$ for some $\lambda$, such that for all $\alpha<\lambda$, $M_{\alpha+1}/M_\alpha$ is $R_1$-free and $R_1$-generated;

2. an $R_2$-module $M$ (of arbitrary cardinality) is Whitehead if and only if $M$ is free.

The theorems can be generalized to other cardinals: see Theorems 8 and 9 at the end of the sections.

1. Proof of Theorem 3

The ring $R$ in Theorem 3 will be constructed by a transfinite induction so that for every module $F/K$ ($F$ free) which is $\mathfrak{m}$-free and $\mathfrak{m}$-generated, Ext$(F/K,R) = 0$, i.e., every homomorphism from $K$ to $R$ extends to a homomorphism from $F$ to $R$. The following proposition provides the inductive step.

Proposition 4. Let $R$ be a local slender p.i.d. with maximal ideal $pR$, and let $K \subseteq F$ be free $R$-modules of rank $\aleph_1$ such that $F/K$ is $\aleph_1$-free. Let $\psi : K \to R$ be an $R$-homomorphism. Then there is a local slender p.i.d. $R^+$ containing $R$ as subring, with maximal ideal $pR^+$ and of cardinality $|R| + \aleph_1$ such that the $R^+$-homomorphism $1_{R^+} \otimes_R \psi : R^+ \otimes_R K \to R^+ \otimes_R R$ extends to an $R^+$-homomorphism $\varphi : R^+ \otimes_R F \to R^+ \otimes_R R$.

Proof. Write $F = \bigcup_{\alpha<\omega_1} F_\alpha$ as a continuous union of submodules of countable rank with $F_0 = 0$. For each $\alpha < \omega_1$, $F_\alpha + K/K$ is free; let $\{b^i_\alpha : i \in I_\alpha\}$ be a linearly independent subset of $F_\alpha$ such that $\{b^i_\alpha + K : i \in I_\alpha\}$ is a basis of $F_\alpha + K/K$.

(1) $\varphi(b^i_\alpha + K) = x^i_\alpha$. Let $x^i_\alpha < \omega_1$.

We claim that there is a local slender p.i.d. $R^+$ of cardinality $|R| + \aleph_1$ containing $R$ as subring and with maximal ideal $pR^+$ and elements $x^i_\alpha \in R^+$ ($\alpha < \omega_1$, $i \in I_\alpha$) such that $x^i_\alpha = \sum_{j \in I_\beta} r^{i,j}_\alpha x^j_\beta + s^{i,j}_\alpha$ for all $\alpha < \beta < \omega_1$ and $i \in I_\alpha$.

Supposing this for the moment, let us finish the proof. Clearly $\{b^i_\alpha : \alpha < \omega_1, i \in I_\alpha\}$ generates $R^+ \otimes_R F$ as $R^+$-module. Define $\varphi$ extending $1_{R^+} \otimes_R \psi$ by $\varphi(1 \otimes b^i_\alpha) = x^i_\alpha \otimes 1$. We must check that this is well-defined. For this it suffices to prove that $\varphi(1 \otimes (1 \otimes b^i_\alpha)) = \sum_{j \in I_\beta} r^{i,j}_\alpha x^j_\beta + s^{i,j}_\alpha \varphi(1 \otimes b^j_\beta) + (1 \otimes \psi)(1 \otimes k^{i,j}_\alpha)$ for all $\alpha < \beta < \omega_1$ and $i \in I_\alpha$. But this is implied by the assumption that $x^i_\alpha = \sum_{j \in I_\beta} r^{i,j}_\alpha x^j_\beta + s^{i,j}_\alpha$.

So it remains to define $R^+$. Let $R^0 = R$ and for $0 < \alpha < \omega_1$, let $R^\alpha = R[\{x^i_\alpha : i \in I_\alpha\}]$, the polynomial ring over $R$ in the commuting indeterminates $x^i_\alpha$, $i \in I_\alpha$. Let $\pi^i_\alpha : R^\alpha \to R^\beta$ be the ring homomorphism which is the identity on $R$ and takes $x^i_\alpha$ to $\sum_{j \in I_\beta} r^{i,j}_\alpha x^j_\beta + s^{i,j}_\alpha$. It is easy to check, using the fact that the $\{b^i_\alpha : i \in I_\alpha\}$ are linearly independent, that $\pi^i_\delta \circ \pi^i_\beta = \pi^i_\gamma$ whenever $\alpha < \beta < \gamma < \omega_1$.

Let $R'$ with maps $\pi^i_\alpha : R^\alpha \to R'$ be the direct limit of this $\aleph_1$-directed system of homomorphisms. Clearly each $R^\alpha$ is a unique factorization domain such that $p$ is prime in $R^\alpha$. Since the system is directed, $R'$ is an integral domain and $p$ is prime in $R'$. Moreover, since the system is $\aleph_1$-directed, $\bigcap_{n \in \omega} R^p R' = 0$ since the same is true in each $R^\alpha$. If $\{a_n : n \in \omega\}$ is a Cauchy sequence in $R$ which does not have a
limit (in the $p$-adic topology), then \( \{ \pi^0(a_n) : n \in \omega \} \) does not have a limit in the $p$-adic topology on \( R^\alpha \) for all $t \in R^\alpha - pR^\alpha$. Hence, by the $\aleph_1$-directedness, the same holds for \( \{ \pi^0(a_n) : n \in \omega \} \) in \( R' \).

Finally, let $R^+$ be the localization of $R'$ at the prime $p$. We appeal to the following elementary Lemma to finish.

**Lemma 5.** Suppose $R'$ is an integral domain with a prime $p$ such that $\bigcap_{n \in \omega} p^n R' = 0$. Then the localization $R'_*(p)$ of $R'$ at $p$ is a p.i.d.

**Proof.** Given a non-zero proper ideal $I$ of $R'_*(p)$, let $I' = I \cap R'$ ($= \{ r \in R' : \frac{r}{p} \in I \}$). Let $m$ be minimal such that $I' \cap (p^m R' - p^{m+1} R') \neq \emptyset$. Clearly $m$ exists, by hypothesis and since $I'$ is non-zero. We claim that $I = p^m R'_*(p)$. Let $a \in I' \cap (p^m R' - p^{m+1} R')$; then $a = p^m r$ for some $r \in R'$ and $r \notin pR'$; so $r$ is a unit in $R'_*(p)$ and thus $p^m \in I$. Now for any non-zero $\frac{b}{p} \in I$, $b \in I' - \{ 0 \}$ so $b \in I' \cap (p^n R' - p^{n+1} R')$ for some $n \geq m$. Thus $b = p^m c$ for some $c \in R'$ and $n \geq m$. But then $\frac{b}{p} = p^{m-n} \frac{c}{p} \in p^m R'_*(p)$. Therefore $I = p^m R'_*(p)$.

**Proof of Theorem 6.** Let $\lambda = 2^{\aleph_1}$. We define a ring $R$ on the set $\lambda$ which is the union of a continuous chain of rings $R_\nu$ ($\nu < \lambda$) such that for each $\nu < \lambda$, $R_{\nu+1}$ is of the form $(R_\nu)^+$ for some quadruple $(R_\nu, K_\nu, F_\nu, \psi_\nu)$ satisfying the hypotheses of the Proposition. We begin, for example, with $R_0 = \mathbb{Z}_p$. It is easy to see that $R$ is a local p.i.d. with prime $p$. Moreover, the proof of the Proposition shows that a witnessing Cauchy sequence to the incompleteness of $R_0$ is preserved at each stage and therefore also in $R$ since $\omega_1$ has cofinality $> \omega$. Because $\lambda^{\aleph_1} = \lambda$, we can choose the enumeration of quadruples $(R_\nu, K_\nu, F_\nu, \psi_\nu)$ such that for every $\aleph_1$-generated $\aleph_1$-free $R$-module $F/K$ (where $K \subseteq F$ are free $R$-modules) and every $R$-homomorphism $\psi : K \to R$, there is a $\nu < \lambda$ such that $R \otimes_{R_\nu} F_\nu$ is isomorphic to $F$ under an isomorphism which takes $R \otimes_{R_\nu} F_\nu$ to $K$ and identifies $1_R \otimes_{R_\nu} \psi_\nu$ with $\psi$ under the natural isomorphism of $R \otimes_{R_\nu} R_\nu$ with $R$. (Note that $K \subseteq F$ and $\psi$ can each be completely described by a sequence of $\aleph_1$ elements of $R = \lambda$.)

By using a direct system indexed by the countable rank submodules of $F/K$ in the proof of the Proposition, we can prove the following more general version of the theorem. Part (1) of Corollary 3 can be correspondingly generalized.

**Theorem 6.** For any cardinal $\kappa \geq \aleph_1$, there is a local slender p.i.d. $R$ of cardinality $2^\kappa$ such that every $\aleph_1$-free $\kappa$-generated $R$-module is a Whitehead module.

2. Proof of Theorem 6

Let $R$ be the polynomial ring $F[X]$ where $F = \mathbb{Q}((t_\nu : \nu < \omega_2))$ and $\{ t_\nu : \nu < \omega_2 \}$ is an algebraically independent set.

Let $A$ be an $\aleph_1$-generated $\aleph_1$-free $R$-module which is not free and let $A = \bigcup_{\alpha < \omega_1} A_\alpha$ be an $\aleph_1$-filtration of $A$. Then there is a stationary set $S$ of limit ordinals such that for $\gamma \in S$, $A_{\gamma+1}/A_\gamma$ is not free. Without loss of generality we can assume that there is a $d \in \omega$ such that for all $\gamma \in S$, $A_{\gamma+1}/A_\gamma$ is of rank $d+1$ and not free but every submodule of rank $\leq d$ is free. (Note that we allow $A_\alpha/A_\gamma$ to be non-free for $\alpha \notin S$.) Thus $A_{\gamma+1}/A_\gamma$ is isomorphic to $F'/K'$, where $F'_\gamma$ is free on \( \{ y_{\gamma,n} : n \in \omega \} \cup \{ x_{\gamma,\ell} : \ell < d \} \) and $K'_\gamma$ has a basis $\{ w'_{\gamma,n} : n \in \omega \}$ where

$$w'_{\gamma,n} = p_{\gamma,n} y_{\gamma,n+1} - y_{\gamma,n} - \sum_{\ell < d} s_{\gamma,n,\ell} x_{\gamma,\ell}$$
for some $p_{\gamma,n}, s_{\gamma,n,\ell} \in R$ where the $p_{\gamma,n}$ are non-units of $R$ (not necessarily prime).

(Compare, for example, Observation 3.1.)

Let $F = \bigoplus_{\beta < \omega} F^\beta$ and $K = \bigoplus_{\beta < \omega} K^\beta$ be as in Lemma XII.1.4; that is, for all $\alpha < \omega_1, \bigoplus_{\beta < \alpha} F^\beta / \bigoplus_{\beta < \alpha} K^\beta \cong A_\alpha$ and $\bigoplus_{\beta < \alpha} F^\beta / (\bigoplus_{\beta < \alpha} F^\beta + K^\alpha) \cong A_{\alpha+1}/A_\alpha$. Moreover, by the proof of Lemma XII.1.4, we can assume that for $\gamma \in S$, $F^\gamma$ is a summand of $F_\gamma$ and $K_\gamma$ has a basis which includes $\{w_{\gamma,n} : n \in \omega\}$ where

$$w_{\gamma,n} = w^\gamma_{\gamma,n} - a_{\gamma,n},$$

for some $a_{\gamma,n} \in \bigoplus_{\beta < \gamma} F^\beta$ (and $\psi_\gamma(w^\gamma_{\gamma,n}) = \varphi_\gamma(a_{\gamma,n}) \in A_\gamma$). Fix a basis $B$ of $F$ which is the union of a basis $B^\beta$ for each $F^\beta$ and which includes $\bigcup_{y, n \in \omega} \{y_{\gamma,n} : n \in \omega\} \cup \{x_{\gamma,\ell} : \ell < d\}$. Also fix a basis of $K$ which includes $\bigcup_{w \in S} \{w_{\gamma,n} : n \in \omega\}$. Given an element $r$ of $R$, we will say $\mu \in \omega_2$ occurs in $r$ if $r$ does not belong to $Q(\{t_\nu : \nu \in \omega_2 - \{\mu\}\})[X]$. Given an element $z$ of $F$ we will say that $\mu$ occurs in $z$ if it occurs in some coefficient of the unique linear combination of elements of $B$ which equals $z$. There is a subset $I$ of $\omega_2$ of cardinality $\aleph_1$ such that all of the $p_{\gamma,n}$ and $s_{\gamma,n,\ell}$ ($\gamma \in S, n \in \omega, \ell < d$) belong to $Q(\{t_i : i \in I\})[X]$. Moreover, we can choose $I$ such that it contains every $\mu$ which occurs in some coefficient of a linear combination of elements of $B$ which equals some $a_{\gamma,n} (\gamma \in S, n \in \omega)$. Without loss of generality (by renumbering the $t_\nu$), $I = \omega_1$.

Now we define $\psi : K \rightarrow R$ by defining

$$\psi(w_{\gamma,n}) = t_{\omega_1 + \omega_\gamma + n}$$

and letting $\psi$ be arbitrary on the other basis elements of $K$. We will show that $\text{Ext}(A, R) \neq 0$ by showing that $\psi$ cannot be extended to a homomorphism from $F$ into $R$. Suppose to the contrary that there is a homomorphism $\varphi : F \rightarrow R$ extending $\psi$. For each $\alpha < \omega_1$, let $T^\alpha$ be the set of all $\mu \in \omega_2$ which occur in $\varphi(b)$ for some $b \in \bigcup\{B^\beta : \beta < \alpha\}$. Then the $T^\alpha$ ($\alpha \in \omega_1$) form a continuous chain of countable subsets of $\omega_2$ and there is $\delta \in S$ such that $T^\delta \cap \{\omega_1 + \beta : \beta < \omega_1\} \subseteq \{\omega_1 + \beta : \beta < \delta\}$. There is a finite subset $Y$ of $\omega_2$ such that every $\mu$ which occurs in $\varphi(y_{\delta,0})$ or in $\varphi(x_{\delta,\ell})$ for some $\ell < d$ belongs to $Z$. Let $R^* = Q(\{t_\nu : \nu \in \omega_1 \cup T^\delta \cup Z\})[X]$, a subring of $R = F[X]$. Now for all $n \in \omega$ we have $\varphi(w_{\delta,n}) = \psi(w^\delta_{\delta,n}) = t_{\omega_1 + \omega_\delta + n} - p_{\delta,n} \varphi(y_{\delta,n+1}) - \varphi(y_{\delta,n}) - \sum_{\ell < d} s_{\delta,n,\ell} \varphi(x_{\delta,\ell}) - \varphi(a_{\delta,n})$.

If we can show that this implies that $t_{\omega_1 + \omega_\delta + n}$ belongs to $R^*$ for all $n \in \omega$, we will have a contradiction of the choice of $T^\delta$ and the fact that $Z$ is finite. We will show this by induction on $n$ along with simultaneously proving that $\varphi(y_{\delta,n+1}) \in R^*$. We begin with $n = -1$: $\varphi(y_{\delta,0})$ belongs to $R^*$ by definition of $Z$. Now suppose the inductive hypothesis is true for $n - 1$ and we prove it for $n$. By the last displayed formula, the inductive hypothesis and the choice of $R^*$, there is an element $r_n \in R^*$ such that $p_{\delta,n} \varphi(y_{\delta,n+1}) = r_n - t_{\omega_1 + \omega_\delta + n}$. If $t_{\omega_1 + \omega_\delta + n} \notin R^*$, there is an automorphism $\Theta$ of $R$ which fixes $R^*$ and takes $t_{\omega_1 + \omega_\delta + n}$ to $t_\tau$ for some $\tau \notin T^\delta$. Then $p_{\delta,n} \Theta(\varphi(y_{\delta,n+1})) = r_n - t_\tau$. (Remember that $p_{\delta,n} \in R^*$.) Therefore, subtracting, $p_{\delta,n}$ divides $t_{\omega_1 + \omega_\delta + n} - t_\tau$, which is impossible since $p_{\delta,n}$ is a non-unit. Thus $t_{\omega_1 + \omega_\delta + n}$ and hence $p_{\delta,n} \varphi(y_{\delta,n+1})$ belong to $R^*$. But then since $p_{\delta,n} \in R^*$ we can prove by induction on $m$ that the coefficient of $X^m$ in $\varphi(y_{\delta,n+1}) \in F[X]$, belongs to $Q(\{t_\nu : \nu \in \omega_1 \cup T^\delta \cup Z\})$, and hence that $\varphi(y_{\delta,n+1})$ belongs to $R^*$. 

We can even find a principal ideal domain of cardinality $\aleph_1$ which satisfies the conclusion of Theorem 2. Namely, let $R = F_1[X]$ where $F_1 = Q(\{t_\nu : \nu < \omega_1\})$. 

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Define $\psi(w_{\delta,n})$ to be $t_{\omega\delta+\sigma_{\delta}+n}$ where $\omega\delta+\sigma_{\delta}$ is larger than any $\mu$ which occurs in any $p_{\delta,k}$ or $s_{\delta,k,\ell}$ for $k \in \omega$, $\ell < d$. Define $T_{\delta}$ as before and choose $\delta \in S$ such that $T_{\delta} \cap \omega_{1} \subseteq \omega_{\delta}$. Let $R^{*} = \mathbb{Q}(\{t_{\nu} : \nu \in \omega_{\delta} + \sigma_{\delta} \cup T_{\delta} \cup Z\})[X]$.

We can also localize without affecting the property of the ring that we desire. More generally, we have:

**Theorem 7.** For any $\kappa \geq \aleph_1$ there is a local p.i.d. $R$ of cardinality $\kappa$ such that an $R$-module of cardinality $\leq \kappa$ is Whitehead only if it is free.

References

[1] T. Becker, L. Fuchs and S. Shelah, *Whitehead modules over domains*. Forum Math. 1 (1989), 53–68.
[2] P. C. Eklof and A. H. Mekler, *Almost Free Modules*, North-Holland (1990).
[3] O. Gerstner, L. Kaup and H. G. Weidner, *Whitehead-Modul abzählbare Ranges über Hauptidealringen*, Arch. Math. (Basel) 20 (1969), 503–514.
[4] R. Göbel and S. Shelah, *Cotorsion theories and splitters*, Trans. Amer. Math. Soc, to appear.
[5] S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math 18 (1974), 243–256.
[6] S. Shelah, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel J. Math. 21 (1975), 319–349.
[7] S. Shelah, *Whitehead groups may not be free even assuming CH, II*, Israel J. Math 35 (1980), 257–285.
[8] J. Trlifaj, *Non-perfect rings and a theorem of Eklof and Shelah*, Comment. Math. Univ. Carolinae 32 (1991), 27–32.

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