Topology measurement within the histories approach

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Abstract

An idealised experiment estimating the spacetime topology is considered in both classical and quantum frameworks. The latter is described in terms of histories approach to quantum theory. A procedure creating combinatorial models of topology is suggested. The correspondence between these models and discretised spacetime models is established.
1 Introduction

Within the conventional account of the relativity theory the structure of spacetime as differentiable manifold is supposed to be given and it is the metric structure that is subject to measurement and changes. So, the topology of spacetime is not *an observable*.

Nowadays there is no fully fledged theory in which the spacetime topology would be a variable, nor even in a sense perceivable entity. However, even if such theory does not exist, we may try to consider idealized experiments which would let us know the spacetime topology. That means, we should assume the spacetime to be a manifold, and we only wish to *determine* its topological structure. In accordance with it, any observer should believe that the topology of the area of his observation (that is, appropriate coordinate neighborhood) is that of a ball. So, in order to recover the entire spacetime topology we have to find out how the balls do overlap. However, any realistic experiment (having at most finite number of outcomes) can not let us know it. We are only able to know if the regions have common points (section 2).

Such experimental scheme inevitably needs *several* observers, but the problem of event identification arises: two observers registering an event should be made sure they really see the same. We emphasize that this is a matter of *convention*: two observers should have a way to identify remote events. This leads to the concept of organized observation (section 3).

The obtained results of observations then ought to be somehow interpreted. We may do that in classical either quantum way. In the classical approach this leads to the Sorkin discretization scheme (section 5).

The attempt to put a scheme of topology estimation into the framework of quantum mechanics requires the cooperative nature of the observations to be explicitly captured in the theory. It is the notion of *homogeneous history* in the histories approach to quantum theory [7] that can be used for this purpose. Within the histories approach we introduce the notion of the ‘team’ (organized set of observers). To carry out the mathematical description of the team we had to impose an additional mathematical structure. It turned out that this structure can be represented by that of associative algebra (section 3). It is worthy to mention that such structures can be introduced in different ways reflecting different ways of organization of the team of observers.

\[
\begin{pmatrix}
\text{topology} \\
\text{measurement}
\end{pmatrix}
= \begin{pmatrix}
\text{homogeneous} \\
\text{history}
\end{pmatrix}
+ \begin{pmatrix}
\text{organization} \\
\text{of observers}
\end{pmatrix}
\]

There is no spacetime points at all within the histories approach, and the goal of the introduced additional structure is to 'manufacture' them. We suggest an algebraic machinery building topological spaces (namely, the Rota topologies on primitive spectra of appropriate algebras) and call it *spatialization procedure* (section 4).
In order to make sure of the viability of our quantum construction we should take care of correspondence principle: we should be able to carry out quasiclassical measurements. That means to possess such an organization scheme for the team that the result of the spatialization procedure would be the same as in classical approach. It reduces to a purely mathematical problem of existence of appropriate algebraic structure. We suggest a constructive solution of this problem using so-called incidence algebras (section 3).

2 Empirical topology

Let us consider, following [2], an idealized experimental scheme for determining the topological structure of spacetime. Consider a team $\Lambda$ of observers. Each of them assumes himself to be in the center of an area $O_\lambda$ ($\lambda \in \Lambda$) homeomorphic to an open ball. We require it to satisfy the correspondence principle: in fact, looking around we do not see holes or borders in the sky. These areas $\{O_\lambda\}$ will form an atlas for the spacetime manifold in which they are. Then the problem of learning the structure of the entire manifold arises. It was solved by Alexandrov [1] by introducing the notion of nerve of the covering, namely the result is encoded in the structure of mutual intersections of the elements of the covering.

Within the proposed scheme, the problem is to experimentally verify which areas $O_\lambda$ do overlap. This is done by exchanging information between observers about the events they observe. The results of the observations could be put into the following table (Tab. 1) whose rows correspond to events and columns correspond to the observers.

| Event label | $O_1$ | $O_2$ | ... | $O_\lambda$ | ... |
|-------------|-------|-------|-----|-------------|-----|
| 1           | +     | −     | ... | +           | ... |
| 2           | +     | +     | ... | +           | ... |
| ...         | ...   | ...   | ... | ...         | ... |
| $n$         | −     | +     | ... | +           | ... |
| ...         | ...   | ...   | ... | ...         | ... |

Table 1: The results of observations: if an observer $\lambda$ registers the event $i$ we put ”+” into the appropriate cell of the table, otherwise ”−” is put.

The consequences we make out of the experiments necessarily have the statistical nature. In particular, the statement ”the areas of two observers do overlap” is merely a statistical hypothesis. To verify it the following criterion is suggested:
If it occurs that the observers $O_1$ and $O_2$ have registered the same event, then the areas of their observations do overlap.

Note that this criterion is statistical rather than logical. We emphasize that after the observations were carried out we only accept or reject the appropriate hypothesis.

When such a hypothesis is accepted, it gives us the complete information about the nerve of the covering. One might think that now we are able to recover the global topology by gluing the balls together. But this is an illusion: the obstacle is that we have nothing to glue! Moreover the geometrical realization by nerve may be a source of artifacts: for instance we can cover an interval $(0, 1)$ (having dimension 1) in such a way

$$O_1 = (0, 0.6)$$
$$O_2 = (0.4, 1)$$
$$O_3 = (0.2, 0.8)$$

that the appropriate nerve is realized by a triangle (having dimension 2). Supposed we could exhaust all the points of spacetime, the ”real ultimate” structure of spacetime manifold would be recovered. However what we can really carry out is to realize a ”homogeneous history” whose outcome is recorded in the table like Tab. 1.

3 Entanglement in histories approach

In this section we introduce topology measurements into the histories approach to quantum theory. It will be based on the algebraization scheme of the histories approach suggested by C.Isham. The key issues of this scheme are

- to consider propositions about histories rather than histories themselves
- to span a linear space on elementary propositions about histories
- to endow the propositions themselves by the additional structure of orthoalgebra

thus organizing them in a way similar to conventional quantum mechanics.

In this paper a similar idea is realized. We consider

* propositions about topologies rather than the topologies themselves
* a linear space spanned on the elementary propositions about topologies
the structure of associative algebra on this linear space

Let us specify what do we mean by propositions about topologies. There are at least three ways to introduce a topology on a set $M$. First two of them are in a sense exhaustive: to define (to list out) all open sets either to define the operation of closure on all subsets of $M$. The third way is more 'economic': to declare which sequences do converge. It will be suitable for us to replace topology by convergencies for both technical and operationalistic reasons (section 4). In fact, any realistic experiment can yield us at most a finite sequence of results. The associative algebras related with propositions about topology will be built in section 6.

Let us figure out how the notion of organized team of observers can be incorporated into the histories approach. Let

$$A_1^2 U_{i_1} A_2^2 \ldots U_{i_{n-1}} A_n^2 \psi_0$$

be a homogeneous history. The operators $A_i$ are assumed to act in a Hilbert space $\mathcal{H}$. It was suggested by Isham to describe the history (2) by an element of the tensor product $\otimes_{i=1}^n \mathcal{H}$. Then we assume that there is an 'organizer' of the history whose status is a priori the same as that of every member of the team. That means that he has the same state space $\mathcal{H}$. Thus each history, that is, a vector from $\otimes_{i=1}^n \mathcal{H}$, should be associated with a vector in the state space $\mathcal{H}$ of the organizer. The suggested correspondence should meet the following requirements:

(i) Neither the number of observers nor their particular choice of what to measure should influence the form of this organization

(ii) If we have an experiment which is a refinement of different coarser experiments, their results should not contradict

(iii) This correspondence should be linear in order to support the superposition principle

Mathematically this correspondence is introduced by defining a family of linear mappings $O_n$ ($n = 1, \ldots, n$):

$$O_n : \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H} \rightarrow \mathcal{H}$$

whose form is specified by a particular organization of the topology measurement. The requirement (iii) is expressed in the linearity of $O_n$. To meet the requirement (i) we are dealing with the family $\{O_n\}$ rather than with a single mapping.

Now the requirement (ii) can be formulated as a relation between the mappings $O_n$. First,
\[ O_1(x) = x \]

and

\[ O_{p+q}(x_1 \otimes \ldots \otimes x_{p+q}) = O_2(O_p(x_1 \otimes \ldots \otimes x_p) \otimes O_q(x_1 \otimes \ldots \otimes x_q)) \]

In particular

\[ O_2(O_2(x \otimes y) \otimes z) = O_2(x \otimes O_2(y \otimes z)) \quad (4) \]

therefore all the mappings \( O_n \) can be inductively expressed through \( O_2 \).

\[ O_n(x_1 \otimes \ldots \otimes x_{n-1} \otimes x_n) = O_2(O_{n-1}(x_1 \otimes \ldots \otimes x_{n-1}) \otimes x_n) \]

Being a linear mapping, \( O_2 : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \) generates a bilinear mapping \( \mathcal{H} \times \mathcal{H} \to \mathcal{H} \) whose action on the pair \((x, y)\) we denote simply by \( x \cdot y \). Then the relation (4) reads:

\[ (x \cdot y) \cdot z = x \cdot (y \cdot z) \]

So, the organization of a topology measurement is mathematically expressed by defining an associative product in \( \mathcal{H} \). Then the 'organizing operator' (3) takes the form:

\[ O_n(x \otimes y \otimes \ldots \otimes z) = x \cdot y \cdot \ldots \cdot z \]

Suppose a history realizing a topology measurement 'took place', that is, the team of observers had carried out a number of yes-no experiments (Table 1), or, in other words, the operators \( A^i_{t_i} \) are projectors in \( \mathcal{H} \):

\[ P^1_{t_1} U_{t_1 t_2} P^2_{t_2} \ldots U_{t_{n-1} t_n} P^n_{t_n} \psi \quad (5) \]

The outcome of each of these yes-no measurements is the selection of a subspace of \( \mathcal{H} \) (associated with appropriate projector). The organizer has a collection of subspaces of \( \mathcal{H} \) in his disposal. Now let us return to requirement (i): what invariant object may he construe out of them having only these subspaces and the product in \( \mathcal{H} \)? This is the algebra \( \mathcal{A} \) spanned on these subspaces. So, all the available information about the spacetime topology is encoded in this subalgebra of \( \mathcal{H} \). A way to extract it is to apply the spatialization procedure described in the next section.
4 Spatialization procedure and Rota topology

Let us consider what sort of spaces can be extracted from algebras. Begin with the discussion of what points ought to be. Suppose for a moment that the obtained algebra $\mathcal{A}$ is commutative. In this case it can be canonically represented by a functional algebra on an appropriate topological space. This may be obtained using Gel’fand representation. In this case the points of this space can be thought of as characters. The characters are, in turn, one-dimensional irreducible representations whose kernels are maximal ideals. There are several ways to impose a topology on the set of points.

In general, when the algebra $\mathcal{A}$ may be non-commutative, the scheme of geometrization remains in principle unchanged: we only pass from characters to classes of irreducible representations, and, respectively, from maximal ideals to primitive ones. For a more detailed analysis of the relevance of primitive ideals the reader is referred to [10]. So

$$X = \text{Prim} \mathcal{A}$$

that is, the points are the elements of the primitive spectrum of $\mathcal{A}$ (equivalence classes of IRRs). Note that at this point we have $X$ as a set not yet endowed by any structure. The straightforward way to ‘topologize’ $X$ could be to use the Jacobson topology. Unfortunately, in the finitary context we are (section 2) this topology (as well as the other standard ones) reduces to the trivial case of discrete one. So, let us seek for a weaker structure which could produce us a reasonable topology on $X$.

It is the notion of convergence space [8] which is the closest to topological structure. It is formed by declaring a relation $(x_n) \rightarrow y$ of convergence between sequences and points:

$$x_1, x_2, \ldots, x_n, \ldots \rightarrow y$$

which always gives rise to the following relation on the points of $X$:

$$x \rightarrow y \text{ if and only if } x, x, \ldots, x, \ldots \rightarrow y \quad (6)$$

Having any relation $\rightarrow$ on $X$, we are always in a position to define a topology on $X$ as the strongest topology in which (6) holds. So, we shall introduce a topology on $X = \text{Prim} \mathcal{A}$ according to the following scheme:

$$(\text{relation on } X) \rightarrow (\text{topology on } X)$$

Recall that the elements of $X$ are the primitive ideals of $\mathcal{A}$, which are, in turn, subsets of $\mathcal{A}$. Having two such ideals $X, Y$ we can form both their
intersection $X \cap Y$ and their product $X \cdot Y$ as the ideal spanned on all products $x \cdot y$ with $x \in X$, $y \in Y$. Note that in general $X \cdot Y \neq Y \cdot X$. However both $X \cdot Y$ and $Y \cdot X$ always lie in (but may not coincide with) $X \cap Y$. Relations between primitive ideals were investigated. G.-C. Rota [12] introduced the following relation in the context of enumerative combinatorics:

$$X \rho Y \text{ if and only if } X \cdot Y \subseteq X \cap Y$$  \hspace{1cm} (7)

We shall call the topology generated by this relation ”\(\rho\)” the ROTA TOPOLOGY on the set of primitive ideals.

We see that in order to judge on the measured topology it suffices to build an ’organizing’ algebra. A particular form of this algebra should be produced using the table of observations (like Tab. 1). There is no \textit{a priori} preferred way to build such an algebra: different models of ’data processing’ may give different spatializations. However, there exists a ’classical spatialization’ using no quantum models — this is the Sorkin discretization scheme (section 5).

The problem of correspondence then arises is it possible to suggest such an organizing algebra based on the table of results that the appropriate topological spaces (Rota and Sorkin topologies) would coincide. This problem will be solved in section 6.

5  Finitary substitutes

The Sorkin spatialization procedure imposes the topology on the set $N$ of events whose prebase is formed by the subsets of events observed by each observer. Consider this construction in more detail following the account suggested in [2, 10].

Associate with any event $i$ the set $\Lambda_i \subseteq \Lambda$ of observers which registered it:

$$\Lambda_i = \{ \lambda \subseteq \Lambda \mid \text{the event } i \text{ was registered by } \lambda \}$$  \hspace{1cm} (8)

and consider the relation $\rightarrow$ on the set of events:

$$i \rightarrow j \text{ if and only if } \forall \lambda j \in N_\lambda \Rightarrow i \in N_\lambda$$  \hspace{1cm} (9)

Note that the relation $\rightarrow$ is evidently reflexive ($i \rightarrow i$) and transitive ($i \rightarrow j, j \rightarrow k \Rightarrow i \rightarrow k$). Such relations are called QUASIORDEARS. Consider the equivalence relation $\leftrightarrow$ on the set of events $N$:

$$i \leftrightarrow j \text{ if and only if } i \rightarrow j \text{ and } j \rightarrow i$$  \hspace{1cm} (10)

and consider the quotient set
called finitary spacetime substitute [13] or pattern space [14]. For \( x, y \in X \) introduce the relation \( x \rightarrow y \):

\[
x \rightarrow y \quad \text{if and only if} \quad \forall i \in x, \forall j \in y \quad i \rightarrow j
\]

(note that the expressions like \( i \in x \) make sense since the elements of \( X \) are subsets of \( N \)). The relation \( \rightarrow \) on \( X \) is:

(i) reflexive: \( x \rightarrow x \)

(ii) transitive: \( x \rightarrow y, y \rightarrow z \Rightarrow x \rightarrow z \)

(iii) antisymmetric: \( x \rightarrow y, y \rightarrow x \Rightarrow x = y \)

The relations having these three properties are called partial orders. It is known (see, e.g. [13]) that there is 1–1 correspondence between partial orders and topologies on finite sets, and that the topology of the manifold can be recovered when the number of events and observers grows to infinity [13].

To conclude this section consider an example of finitary substitute from [2]. Suppose there are four observers \( O_1, \ldots, O_4 \) living on the circle \( e^{i\varphi} \) whose areas of observations are:

\[
\begin{align*}
O_1 & : \{-2\pi/3 < \varphi < 2\pi/3\} \\
O_2 & : \{\pi/3 < \varphi < 5\pi/3\} \\
O_3 & : \{-3\pi/4 < \varphi < 2/3\pi\} \\
O_4 & : \{\pi/4 < \varphi < 3\pi/4\}
\end{align*}
\]

Then the table of outcomes takes the form:

| Event label | \( O_1 \) | \( O_2 \) | \( O_3 \) | \( O_4 \) |
|-------------|-----------|-----------|-----------|-----------|
| 0           | +         | -         | -         | -         |
| \( \pi/2 \) | +         | +         | -         | +         |
| \( \pi \)   | -         | +         | -         | -         |
| 3\( \pi/2 \) | +         | +         | +         | -         |

Then the relation ”\( \rightarrow \)” (9) is already partial order:

\[
\begin{align*}
\pi/2 & \rightarrow 0 \quad ; \quad \pi/2 \rightarrow \pi \\
3\pi/2 & \rightarrow 0 \quad ; \quad 3\pi/2 \rightarrow \pi
\end{align*}
\]

and the equivalence relation (10) turns out to be the equality. Hence the finitary substitute \( X \) is the whole set of events:
Figure 1: The finitary substitute of the circle.

\[ x = \{\pi/2\} , \; y = \{3\pi/2\} , \; z = \{0\} , \; w = \{\pi\} \]

and the partial order on \( X \) is:

Remark. As we have already mentioned, there is an equivalent way to define topology in terms of converging sequences. It worthy to mention that we use the symbol \( \rightarrow \) for the partial order (9) due to the following fact:

\[ x \rightarrow y \quad \text{if and only if} \quad \lim \{x, x, \ldots, x, \ldots\} = y \]

6 Incidence algebras

There is no direct evidence of the compatibility of Sorkin and histories approaches to empirical spacetime topologies. In this section we solve this problem. We explicitly suggest the construction which starting from the table of observations produces an algebra whose space of primitive ideals endowed with the Rota topology is homeomorphic to the Sorkin finitary substitute obtained from the same table.

As it was studied in section \( \text{3} \) in order to build a model of organized spacetime observations we need to introduce an algebra, that is the two following objects:

- A linear space \( H \)
- A product operation on the space \( H \)
somehow generated by the table of observations Tab. 4, where $H$ will stand
for a model of $\mathcal{H}$ and the product will capture the organization.

As we promised, we shall deal with a linear space $H$ spanned on the
elementary propositions about topology. What could be the form of such
propositions? Each of them should involve at least two points, since the
matter of topology is just to study the mutual positions of events. We shall
choose the simplest model, namely, that of two-point statements (a higher
order situation was considered in [16]). Such elementary statements were
already formulated (1).

The form of the algebra we suggest will be similar to that introduced in
[4]. Let $p, q$ be two events, denote by the symbol (sic!) $|p><q|$ the propo-
sition associated with this pair. Form the linear span of all such symbo-
ls: $\text{span}\{|p><q|\}$ and define the product on it:

$$
|p><q| \cdot |r><s| = <q|r> \cdot |p><s| \quad (14)
$$

where $<q|r> = \delta_{qr}$. Note the obtained product is associative but not com-
mutative in general.

In order to take into account the results of the measurement (Table 1)
we form the linear space

$$
H = \text{span}\{|p><q| \text{ such that } p \rightarrow q\}
$$

where $p \rightarrow q$ (3) means that $p$ was registered by a greater set of observers than
$q$. To assure that the obtained algebra can really describe an organization
(in the sense of section 3, we have to check that $H$ is an algebra.

**Proposition 1.** The linear space $H$ with the product (14) is associative
algebra.

*Proof.* Let $|p><q|$ and $|r><s|$ are in $H$, that means $p \rightarrow q$ and $r \rightarrow s$. 
If $q \neq r$ then their product is zero. If $q = r$ then, according to (14) their
product is $|p><q| \cdot |q><s| = |p><s|$, which is the element of $H$ since the
relation $\rightarrow$ is transitive: $p \rightarrow q, q \rightarrow s$ implies $p \rightarrow s$.

**Remark.** The algebras of such sort (called incidence algebras) were intro-
duced by Rota in [12] in a slightly different way.

Now let us apply the spatialization procedure described in section 4. The
primitive spectrum of the algebra $H$ was calculated in [5]; it consists of all
the ideals of the form:

$$
X_s = \text{span}\{|p><q| : |p><q| \neq |s><s|\} \quad (15)
$$
where \( s \) ranges over all equivalence classes with respect to the relation ”\( \leftrightarrow \)” on \( \mathbb{N} \), that is, events. So, at the first stage of the spatialization procedure we already have a canonical bijection between the elements of the primitive spectrum of the algebra \( H \) and the events in the Sorkin’s discretization scheme (section 5). In order to show the compatibility of the two schemes we have to show that the Rota topology on the set \( X_s \) is the same as that of Sorkin.

Let us figure out the form of the relation \( \rho \) for the suggested algebra. By the way we shall see that the relation \( \rho \) can be thought of as a sort of ’proximity’ between events. So, let \( r, s \) be two events.

**Proposition 2.** Let \( X_r, X_s \) be two primitive ideals. Then \( X_r \rho X_s \) if and only if \( r \leftrightarrow s \) and there is no \( t \neq r, s \) such that \( r \to t \to s \).

**Proof** will be carried out exhaustively: we shall consider all possible cases.

- **Case 1.** \( r = s \). Consider \( X_r \cdot X_r \). To prove that this product coincides with \( X_r \) recall its definition (15). Let \( |a><b| \in X_r \) (that is \( a \neq r \) or \( b \neq r \)) and prove that it can be represented as the product of two elements from \( X_r \). If \( a \neq r \) then \( |a><b| = |a><a| \cdot |a><b| \). If \( b \neq r \) then \( |a><b| = |a><b| \cdot |b><b| \). Therefore \( X_r \cdot X_r = X_r \), and \( X_r \rho X_r \).

- **Case 2.** \( r \not\leftrightarrow s \). Consider an element \( |p><q| \) from the intersection of the ideals:

\[
X_r \cap X_s = \text{span}\{|p><q| : |p><q| \notin |r><r| \text{ and } |p><q| \notin |s><s|\}
\]

and show that it belongs to the product

\[
X_r \cdot X_s = \text{span}\{|p><q||a><b| : |p><q| \notin |r><r| \text{ and } |a><b| \notin |s><s|\}
\]

If \( p = q \neq r, s \) then \( |p><p| = |p><p| \cdot |p><p| \). If \( p \neq q \) then \( p \neq r \) or \( q \neq s \) (since \( r \not\leftrightarrow s \)). Then \( |p><q| = |p><p| \cdot |p><q| \text{ or } |p><p| \cdot |q><q| \), respectively. Therefore \( X_r \rho X_s \).

- **Case 3.** \( r \to s \) and there is \( t \neq r, s \) such that \( r \to t \to s \). Consider an element \( |p><q| \) from the intersection of the ideals:

\[
X_r \cap X_s = \text{span}\{|p><q| : |p><q| \notin |r><r| \text{ and } |p><q| \notin |s><s|\}
\]

and show that it belongs to the product

\[
X_r \cdot X_s = \text{span}\{|p><q||a><b| : |p><q| \notin |r><r| \text{ and } |a><b| \notin |s><s|\}
\]
If \( p = q \neq r, s \) then \( \mathcal{P} > < \mathcal{P} = \mathcal{P} > \mathcal{Q} \cdot \mathcal{P} > \mathcal{P} \). Let \( p \neq q \) and \( p \neq r \) or \( q \neq s \), then \( \mathcal{P} > \mathcal{Q} = \mathcal{P} > \mathcal{P} \cdot \mathcal{P} > \mathcal{Q} \) (if \( p \neq r \)) or \( \mathcal{P} > \mathcal{Q} = \mathcal{P} > \mathcal{Q} \cdot \mathcal{Q} > \mathcal{Q} \) (if \( q \neq s \)). Finally let \( \mathcal{P} > \mathcal{Q} = \mathcal{R} > \mathcal{S} \), then \( \mathcal{R} > \mathcal{S} = \mathcal{R} > \mathcal{T} \cdot \mathcal{T} > \mathcal{S} \), and we again have \( \mathcal{X} \mathcal{P} \).

- **Case 4.** \( r \rightarrow s \) and there is no \( t \neq r, s \) such that \( r \rightarrow t \rightarrow s \). Let us show that the element \( \mathcal{R} > \mathcal{S} \) from the intersection of the ideals:

\[
\mathcal{X}_r \cap \mathcal{X}_s = \text{span}\{ \mathcal{P} > \mathcal{Q} : \mathcal{P} > \mathcal{Q} \neq \mathcal{R} > \mathcal{R} \text{ and } \mathcal{P} > \mathcal{Q} \neq \mathcal{S} > \mathcal{S} \}\]

is not an element of the product

\[
\mathcal{X}_r \cdot \mathcal{X}_s = \text{span}\{ \mathcal{P} > \mathcal{Q} | a > b : \mathcal{P} > \mathcal{Q} \neq \mathcal{R} > \mathcal{R} \text{ and } a > b \neq \mathcal{S} > \mathcal{S} \}\]

Suppose this is not the case, then \( \mathcal{R} > \mathcal{S} = \sum C_{abc} \mathcal{A} > \mathcal{T} \cdot \mathcal{T} > \mathcal{C} \).

Multiply this equality (in \( H \)) by \( \mathcal{R} > \mathcal{R} \) from the left and by \( \mathcal{S} > \mathcal{S} \) from the right. Then \( \mathcal{R} > \mathcal{S} = \sum C_{rst} \mathcal{R} > \mathcal{T} \cdot \mathcal{T} > \mathcal{S} \). However there is no \( t \) such that \( r \rightarrow t \rightarrow s \), therefore this sum is zero, while \( \mathcal{R} > \mathcal{S} \neq 0 \).

So we see that two primitive ideals are binded by the relation \( \rho \) if and only if they are as close as possible on the Hasse diagram of the partial order associated with the Sorkin topology (Case 4).

The results of this section can be summarized in the following theorem.

**Theorem.** *The Sorkin topology of a finitary substitute coincides with the Rota topology of its incidence algebra.*

**Concluding remarks**

Initially, in the histories approach to quantum mechanics the existence of the spacetime as a fixed manifold was presupposed \(^9\). An algebraic version \(^9\) of this approach did not give up this presupposition, however rendered it rudimentary. In this paper we do the next step, and the spacetime becomes an observable up to its combinatorial approximation.

The core of the suggested quantum scheme is the spatialization procedure. We have realized it as close as possible to the standard spatialization due to Gel’fand. The peculiarity of our approach is that we impose a new topology, namely, that of Rota (section 4). The reason for us to do it was that in finite dimensional situations (which we considered as realistic ones) the Gel’fand topology reduced to the trivial discrete one. The suggested machinery is a bridge between the algebraic version of the histories approach \(^4\) and combinatorial models such as lattice-like discretization schemes \(^13\).
From the other side, beyond the histories approach an algebraic construction merging finitary substitutes into the quantum-like environment was already carried out by the ‘poseteers group’ (the term introduced by F. Lizzi) in [2, 10]. The comparison of the two constructions is in Table 2.

| Poseteers’ approach | Incidence algebras |
|---------------------|--------------------|
| **The algebras**    |                    |
| $C^*$-algebras of infinite dimensions | finite dimensional algebras with no involution |

| **The points**       |                    |
| kernels of irreducible $*$-representations | kernels of irreducible representations |

| **The topology**     |                    |
| Jacobson topology    | Rota topology      |

Table 2: The comparison of the two algebraic schemes.

Note that the incidence algebras are not semisimple. At first sight this seems to be an essential drawback of the theory bringing it far from the existing quantum constructions. However, in the case of finite dimension the Jacobson topology will always be discrete. From the other side, the lack of semisimplicity makes it possible to develop differential calculi on finitary models, which may be considered as a link between the poseteers’ approach and the formalism of discrete differential manifolds [4].

In [4] finite dimensional semisimple commutative algebras are studied and a differential structure is built in terms of moduli of differential forms (being the conjugate to the module of derivations in the classical case), while there is no nonzero derivations in the algebras themselves. Contrarily, the incidence algebras possess derivations which makes it possible to introduce tensor calculi based on the notion of duality [4]. Moreover, since all their derivatives are inner, they already contain vectors (for details the reader is referred to [15]).

Finally, we dwell on the algebra of symbols $|p><q|$ we introduced in section 6. Irrespective to the particular form of the organizing algebra (like the incidence algebra in our case) we can always write down expressions like
where the operation "$\circ$" is a multiplication generating a particular organization of the team (section 3), and $q_i, q_{i+1}$ are neighbor (in the sense of Rota topology) events. So, this expression can be thought of as the sum over trajectories.

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