Abstract: This note is a continuation of the author’s paper [9]. We prove that if the metric $g$ of a 4-manifold has bounded Ricci curvature and the curvature has no local concentration everywhere, then it can be smoothed to a metric with bounded sectional curvature. Here we don’t assume the bound for local Sobolev constant of $g$ and hence this smoothing result can be applied to the collapsing case.

1 Introduction

It is well known that if a Riemannian manifold has bounded sectional curvature, then at least its local structure is better understood than that with weaker curvature bounds. A natural question is how to generalize the results which hold for spaces with bounded curvatures to those with bounded Ricci curvatures. Therefore, it is important to discover whether we can deform or smooth a metric with bounded Ricci curvature to a metric with bounded curvature. In particular, if one wants to study the compactness theory for a sequence of metrics with uniform bounded Ricci curvatures, the first step is to construct “good” coordinates of definite size, say harmonic coordinates, at the place where the metric has no local curvature concentration such that the space will not develop singularities. However, if, in addition, we don’t assume the noncollapsing condition, such
coordinates will not exist. A standard argument dealing with the collapsing situation is
to lift the metric to the tangent space and construct “good” coordinates there, but, we
cannot immediately do so under the assumption of bounded Ricci curvature. Therefore,
it is necessary to smooth the metric to the one with bounded curvature.

In this note we study the problem whether we can smooth a metric with bounded Ricci
curvature to that with bounded curvature in dimension 4. Due to the reason mentioned
above, our result does not depend on the noncollapsing condition.

Let \( X = S^2 \times S^2 / \sigma \), where \( \sigma \) denotes the Cartan involution that rotates \( \pi \) on each sphere.
We can resolve the four resulting singularities to obtain \( M = \mathbb{C}P^2 \# 5 \mathbb{C}P^2 \) which admits
a sequence of Kähler-Einstein metrics converging to \( X \). With regard to this example,
in general, we cannot smooth a metric with bounded Ricci curvature to the one with
bounded curvature. Consequently, additional geometric conditions must be needed.

To the author’s knowledge, two methods are known in the literature to do the smoothing.
One is the heat flow method. If we assume that the initial metric has bounded curvature,
then we can show the short-time existence of Ricci flow and obtain the covariant derivative
bounds for the curvature tensors along the Ricci flow, for example, \([4]\), \([3]\) and \([13]\). In the
case that the initial metric has only bounded Ricci curvature, some additional condition is
needed for the smoothing procedure. In \([8]\), Dai, Wei and Ye studied how to do smoothing
on a compact manifold with bounded Ricci curvature and conjugate radius bound. Here
the conjugate radius bound ensures that one can lift the metric to the tangent space
and hence obtain certain initial conditions to run the flow. Thus to smooth the metrics
one needs to find out the suitable conditions on the geometry of the space to rule out
the possible singularities developing, which is not convenient in general. So one may
consider the smoothing procedure locally, which can be easily performed in practice. In
\([16]\), Yang introduced the local Ricci flow and proved its short-time existence under
the assumptions that the initial metric has some suitable integral curvature bounds and
satisfies noncollapsing condition, and as a consequence, the metric can be smoothed in a
given geodesic ball with a definite volume. Since Yang’s result is purely local, we don’t
need the global geometric assumptions in the space.

The other way is the embedding method, for example, \([6]\), \([1]\) and \([10]\). The basic idea
of the embedding method is that if the injectivity radius of \( M \) has a positive lower bound,
we can embed \( M \) into \( L^2(M) \) by a map \( I \) involving the distance function such that a
sufficiently smooth submanifold can be found in a neighbor of \( I(M) \). We can perform
this construction in each coordinate chart of \( M \) and glue them together to get the desired
smooth metric. If we don’t have the assumption on the injectivity radius, then each small neighborhood of \( M \) is a quotient of a Riemannian manifold satisfying injectivity radius lower bound by an action of a pseudofundamental group. However, embedding method may not be as convenient as the heat flow method if the metric has no bounded sectional curvature.

Recently, in [15], using local Ricci flow developed by Yang [16], Xu proved the short-time existence of Ricci flow under the assumptions that the initial metric satisfies the volume doubling property, local Sobolev constant bound and some integral bounds on curvature such that the curvature has local concentration nowhere and thus the space will not develop singularities. The purpose of this note is two-fold. Firstly, if the curvature has no local concentration everywhere, one can prove the short-time existence of Ricci flow directly without using the local Ricci flow. As a matter of face, we can employ a covering argument as in [8] to obtain the corresponding energy estimates needed in the short-time existence of Ricci flow and thus avoid using the local Ricci flow. This can then simplify the proof given in [15]. Secondly, based on the recent breakthrough by Cheeger and Tian in the work of collapsing Einstein 4-manifolds, [7], if the initial metric has bounded Ricci curvature, we can remove the assumption on the local Sobolev constant in dimension 4, or equivalently, the lower volume bound of each geodesic ball involved, and thus the result can be applied to the collapsing case in dimension 4. Since the proof in [7] depends on Gauss-Bonnet-Chern formula, it is not known if one can generalize the result to higher dimensions.

For convenience, we will give a brief description of removing the noncollapsing assumption in dimension 4. In [7], Cheeger and Tian obtained the following technical result: Given a 4-dimensional Einstein manifold with Einstein constant \( |\lambda| \leq 3 \), if there exists some \( \varepsilon > 0 \) such that \( \int_{B_r(x)} |\text{Rm}|^2 < \varepsilon \), then for some constant \( c > 0 \),

\[
\frac{\text{Vol}(B_{cr}(\underline{x})}{\text{Vol}(B_{cr}(x))} \int_{B_{cr}(x)} |\text{Rm}|^2 < \varepsilon,
\]

where \( \underline{x} \) is a point in the simply connected space of constant curvature \(-1\). In [9], the author generalized it to the 4-dimensional bounded Ricci curvature case. First we need to smooth the metric in some suitable local scale to obtain the curvature bound for some nearby metric. Here the estimates are not uniform and depend on the chosen local scale. Then by using a local equivariant version of good chopping and an iteration technique
one can show the following key estimate
\[
\frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(B_r(x))} \int_{B_r(x)} |\mathrm{Rm}|^2 \leq C,
\]
where \( C \) is a definite constant. This result is nontrivial in the collapsing case, since the volume of a geodesic ball is arbitrarily small. If we can lift the metric to the tangent space, the corresponding \( L^2 \)-norm of curvature may not be small and even worse it may be unbounded. However the above estimate indicates that although the \( L^2 \)-norm of curvature may not be small, it is still bounded and thus rule out the worse situation. Finally, one can show that the quantity
\[
\frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(B_r(x))} \int_{B_r(x)} |\mathrm{Rm}|^2
\]
can be sufficiently small if one shrinks the ball to a smaller concentric one, whose radius is comparable to \( r \). The proof used the Gauss-Bonnet-Chern formula, an estimate on the transgression form in terms of volume growth rate, and a controlled and smooth approximation of the distance function.

According to [5], [14], [11], [12] and [2], etc., there exists a constant \( C \) depending only on the bounds of Ricci curvature and the dimension of \( M \) such that the Sobolev constant for the geodesic ball \( B_r(x) \) can be controlled as follows: for \( r \leq 1 \),
\[
C_s(B_r(x)) \leq C \left( \frac{r^4}{\operatorname{Vol}(B_r(x))} \right)^{\frac{1}{2}}.
\]
This implies that the \( L^2 \)-norm of curvature actually can be made arbitrarily small against local Sobolev constant.

Our main result is the following

**Theorem 1.1.** Let \( (M,g_0) \) be a complete noncompact Riemannian 4-manifold. There exist constant \( \varepsilon \) and \( C_1 \) such that for \( r \leq 1 \), if
\[
\int_{B_r(x)} |\mathrm{Rm}(g_0)|^2 \, dV_{g_0} \leq \varepsilon, \quad \text{for any } x \in M,
\]
and
\[
|\mathrm{Ric}(g_0)| \leq K,
\]
then the Ricci flow
\[
\begin{cases}
\frac{\partial g}{\partial t} = -2 \mathrm{Ric}(g), \\
g(0) = g_0.
\end{cases}
\]
has a smooth solution for \( t \in [0, T) \), where

\[
T \geq C_1 \cdot \min \left( r^2, K^{-1} \right).
\]

Moreover, for \( t \in (0, T) \), the Riemannian curvature tensor satisfies the following bound,

\[
\| Rm \|_\infty \leq C_2 t^{-1}.
\]

Here \( \varepsilon, C_1 \) and \( C_2 \) only depend on the dimension of \( M \).

This note is organized as follows. In Section 2, we show the Moser’s iteration for linear heat equations. In Section 3, we will study the short-time existence of nonlinear equaiton and apply the result to Ricci flow.

Acknowledgement The author would like to thank Professors Jeff Cheeger and Gang Tian for many helpful discussions on their work [7] and their support. The author is also grateful to Professor Laurent Saloff-Coste for discussion on the literature of the local Sobolev constant bounds.

2 Moser’s Iteration for Linear Heat Equations

In this section we study Moser’s weak maximum principle for linear equations. We mainly follow the lines in [16]. The difference is that the equation discussed here is not local.

Fix a geodesic ball \( B_r(x) \subset M^4 \) and a smooth compactly supported function \( \phi \in C_0^\infty(B_r(x)) \).

Let \( g(t), 0 \leq t \leq T \), be a 1-parameter family of smooth Riemannian metrics. Let \( \nabla \) denote the covariant differentiation with respect to the metric \( g(t) \) and \( -\Delta \) be the corresponding Laplace-Beltrami operator. Let \( A > 0 \) be a constant that satisfies the standard Sobolev inequality

\[
\left( \int_{B_r(x)} f^4 \, dV_g \right)^{\frac{1}{2}} \leq A \int_{B_r(x)} |\nabla f|^2 \, dV_g, \quad f \in C_0^\infty(B_r(x)),
\]

with respect to each metric \( g(t), 0 \leq t \leq T \).

Assume that for each \( t \in [0, T] \),

\[
\frac{1}{2}g_{ij}(0) \leq g_{ij}(t) \leq 2g_{ij}(0) \quad \text{on} \quad B_r(x).
\]
All geodesic balls in this section are defined with respect to the metric $g(0)$, and therefore, are fixed open subsets of $M$, and independent of $t$.

We want to study the heat equation:

$$\frac{\partial f}{\partial t} \leq \nabla f + uf, \quad 0 \leq t \leq T,$$

where $f$ and $u$ are nonnegative functions on $B_r(x) \times [0, T]$, such that

$$\frac{\partial}{\partial t} dV_g \leq c \cdot u dV_g$$

and

$$(\int_{B_r(x)} u^2)^{\frac{1}{2}} \leq \mu t^{\frac{1}{4}}.$$

The following two lemmas are the consequences of direct computations.

**Lemma 2.1.** Given $p > 1$, $\phi \in C_0^\infty(B_r(x))$, $f \in C^\infty(M)$, $f \geq 0$,

$$\int_{B_r(x)} |\nabla (\phi f^{\frac{1}{2}})|^2 \leq \frac{p^2}{2(p-1)} \int_{B_r(x)} \phi^2 f^{p-1}(-\nabla f) dV_g + \left(1 + \frac{1}{(p-1)^2}\right) \int_{B_r(x)} |\nabla \phi|^2 f^p dV_g.$$

**Proof:** Using integration by parts, we have

$$\int |\nabla (\phi f^{\frac{1}{2}})|^2 = -\int \phi f^{\frac{1}{2}} \nabla f$$

$$= \frac{p}{2} \int \phi^2 f^{p-1}(-\nabla f) + \int f^p |\nabla \phi|^2 - \frac{p(p-2)}{4} \int \phi^2 f^{p-2} |\nabla f|^2$$

$$= \frac{p^2}{2(p-1)} \int \phi^2 f^{p-1}(-\nabla f) + \frac{p}{2(p-1)} \int \phi^2 f^{p-1} \nabla f$$

$$+ \int f^p |\nabla \phi|^2 - \frac{p(p-2)}{4} \int \phi^2 f^{p-2} |\nabla f|^2.$$

On the other hand, by Cauchy inequality,

$$\frac{p}{2(p-1)} \int \phi^2 f^{p-1} \nabla f = -\frac{p}{2(p-1)} \int \nabla (\phi^2 f^{p-1}) \nabla f$$

$$= -\frac{p}{p-1} \int \phi f^{p-1} \nabla \phi \nabla f - \frac{p}{2} \int \phi^2 f^{p-2} |\nabla f|^2$$

$$\leq \frac{1}{(p-1)^2} \int f^p |\nabla \phi|^2 + \frac{p^2}{4} \int \phi^2 f^{p-2} |\nabla f|^2 - \frac{p}{2} \int \phi^2 f^{p-2} |\nabla f|^2$$

$$= \frac{1}{(p-1)^2} \int f^p |\nabla \phi|^2 + \frac{p(p-2)}{4} \int \phi^2 f^{p-2} |\nabla f|^2.$$
This proves the lemma. □

**Lemma 2.2.** Suppose that \( f \) and \( u \) are nonnegative functions on \( B_r(x) \times [0, T) \) which satisfy (2.2), (2.3) and (2.4). For \( p > 1 \), we have

\[
\frac{\partial}{\partial t} \int \phi^2 f^p + \frac{p-1}{p} \int |\nabla (\phi f^p)|^2 \leq C_p \int |\nabla \phi|^2 f^p + C_p \mu^3 A^2 t^{-1} \int \phi^2 f^p. \tag{2.5}
\]

**Proof:** By Lemma 2.1, we have

\[
\frac{\partial}{\partial t} \int \phi^2 f^p + 2 \left( 1 - \frac{1}{p} \right) \int |\nabla (\phi f^p)|^2 \leq C_p \int |\nabla \phi|^2 f^p + (p+c) \int u \phi^2 f^p.
\]

Using Hölder, Sobolev, Cauchy inequalities, and (2.4), we see that

\[
\int u \phi^2 f^p \leq \left( \int u^3 \right)^{\frac{1}{3}} \left( \int (\phi^2 f^p) \right)^{\frac{1}{3}} \left( \int \phi^4 f^{2p} \right)^{\frac{1}{3}}
\]

\[
\leq \mu t^{-\frac{1}{3}} \left( \int (\phi^2 f^p) \right)^{\frac{1}{3}} \cdot A^2 \left( |\nabla (\phi f^p)|^2 \right)^{\frac{1}{3}}
\]

\[
\leq (\mu t^{-\frac{1}{3}})^3 \epsilon^{-\frac{1}{3}} \int \phi^2 f^p + \epsilon^\frac{4}{3} A^2 \int \nabla (\phi f^p)^2.
\]

Choosing \( \epsilon \) so that \( \epsilon^\frac{4}{3} A^2 = \frac{p-1}{p} \), we complete the proof of lemma 2.2 □

Now given \( 0 < \tau < \tau' < T \), let

\[
\psi(t) = \begin{cases} 0, & 0 \leq t \leq \tau, \\ \frac{t-\tau}{\tau'-\tau}, & \tau \leq t \leq \tau', \\ 1, & \tau' \leq t \leq T. \end{cases}
\]

Multiplying (2.5) by \( \psi \), we have

\[
\frac{\partial}{\partial t} \left( \psi \int \phi^2 f^p \right) + \psi \int |\nabla (\phi f^p)|^2 \leq C_p \psi \int |\nabla \phi|^2 f^p + \left( \hat{C}(t) \psi + |\psi'| \right) \int \phi^2 f^p,
\]

where \( \hat{C}(t) = C_p \mu^3 A^2 t^{-1} \). Integrating this with respect to \( t \) from \( \tau \) to \( T \) and throwing away \( \int_\tau^{\tau'} \psi |\nabla (\phi f^p)|^2 \) on the left-hand side, we obtain

**Lemma 2.3.** For \( \tau' \leq t \leq T \), we have

\[
\int_t^{\tau'} \phi^2 f^p + \int_t^{\tau'} \int |\nabla (\phi f^p)|^2 \leq C_p \int_\tau^T \int |\nabla \phi|^2 f^p + \left( \hat{C}(\tau') + \frac{1}{\tau'-\tau} \right) \int_\tau^T \int \phi^2 f^p.
\]
Now given $p > 1$, $0 \leq \tau < T$, denote

$$H(p, \tau, r) = \int_{\tau}^{T} \int_{B_r(x)} f^p.$$

**Lemma 2.4.** Given $p \geq p_0$, $0 \leq \tau < \tau' < T$ and $r' < r$

$$H\left(\frac{3}{2}p, \tau', r'\right) \leq A \left(\hat{C}(\tau') + \frac{1}{\tau' - \tau} + \frac{C_1}{(r - r')^2}\right)^{\frac{3}{2}} H(p, \tau, r)^{\frac{3}{2}}.$$

**Proof:** Choosing a suitable cut-off function $\phi$ and noticing $|\nabla \phi|_t \leq 2|\nabla \phi|_0$, we have

$$H\left(\frac{3}{2}p, \tau', r'\right) \leq \int_{\tau'}^{T} \int_{B_r(x)} (\phi^2 f^p)^{\frac{3}{2}}$$

$$\leq \int_{\tau'}^{T} \left(\int_{B_r(x)} \phi^2 f^p\right)^{\frac{1}{2}} \left(\int_{B_r(x)} \phi^4 f^{2p}\right)^{\frac{1}{2}} dt$$

$$\leq \left(\sup_{\tau' \leq t \leq T} \int_{B_r(x)} \phi^2 f^p\right)^{\frac{1}{2}} A \int_{\tau'}^{T} \int_{B_r(x)} |\nabla (\phi f^{\frac{p}{2}})|^2 dt$$

$$\leq A \left(4 \int_{\tau}^{T} |\nabla \phi|^2 f^p + \left(\hat{C}(\tau') + \frac{1}{\tau' - \tau}\right) \int_{\tau}^{T} \int_{B_r(x)} \phi^2 f^p\right)^{\frac{3}{2}}$$

$$\leq A \left(\hat{C}(\tau') + \frac{1}{\tau' - \tau} + \frac{C_1}{(r - r')^2}\right) \left(\int_{\tau}^{T} \int_{B_r(x)} \phi^2 f^p\right)^{\frac{3}{2}}$$

This proves the lemma. □

The following theorem is the consequence of Moser’s iteration.

**Theorem 2.5.** Let $f$ and $u$ be non-negative functions on $B_r(x) \times [0, T)$ satisfying (2.3), (2.3) and (2.4). Then for $t \in [0, T)$ and $p_0 > 2$, 

$$|f(x, t)| \leq CA^{\frac{3}{2}} \left(1 + A^2 \mu^3 t^{-1} + \frac{1}{r^2}\right)^{\frac{3}{p_0}} \left(\int_{0}^{T} \int_{B_r(x)} f^{p_0}\right)^{\frac{1}{p_0}},$$

where $C$ depends on the dimension of $M$, $p_0$.

**Proof:** Denote $\nu = \frac{3}{2}$. Fix $0 < t < T$, and set

$$p_k = p_0 \nu^k,$$

$$\tau_k = t(1 - \nu^{-k-1}),$$

$$r_k = \frac{r}{2} (1 + \nu^{-k/2}),$$

$$\Phi_k = H(p_k, \tau_k, r_k)^{\frac{1}{p_k}}.$$
Applying Lemma 2.4,

\[ H(p_{k+1}, \tau_{k+1}, r_{k+1}) \leq AC \left( (1 + A^2 \mu^3) t^{-1} + \frac{1}{r^2} \right)^\nu \nu^{\nu} H(p_{k}, \tau_{k}, r_{k})^\nu. \]

Therefore,

\[ \Phi_{k+1} \leq (AC)^{\frac{1}{p_{k+1}}} \left( (1 + A^2 \mu^3) t^{-1} + \frac{1}{r^2} \right)^{\frac{1}{\nu}} \nu^{\nu} \Phi_{k}. \]

Hence,

\[ \Phi_{k+1} \leq (AC)^{\frac{\sigma_{k+1}}{p_0}} \left( (1 + A^2 \mu^3) t^{-1} + \frac{1}{r^2} \right)^{\frac{\sigma_k}{p_0}} \nu^{\frac{\sigma_k}{p_0}} H(p_0, 0, r)^{\frac{1}{p_0}}, \]

where \( \sigma_k = \sum_{i=0}^{k} \nu^{-i}, \sigma_k' = \sum_{i=0}^{k} i \nu^{-i}. \) Letting \( k \to \infty, \) we obtain

\[ |f(x, t)| \leq C A^{\frac{\sigma_k}{p_0}} \left( (1 + A^2 \mu^3) t^{-1} + \frac{1}{r^2} \right)^{\frac{3}{p_0}} \left( \int_0^T \int_{B_r(x)} f^{p_0} \right)^{\frac{1}{p_0}}. \]

Now let \( T \to t. \) This proves the theorem. \( \square \)

### 3 Short-time Existence for Ricci Flow

In this section we study the short-time existence of Ricci flow and hence obtain the smoothing result. We will follow the lines in [16], together with the covering argument in [8] to get the desired energy estimates in Ricci flow.

Let \( M \) be a complete noncompact manifold with Riemannian metric \( g_0. \) Consider the following evolution equation

\[
\begin{align*}
\frac{\partial g}{\partial t} &= -2 \text{Ric}(g), \\
g(0) &= g_0.
\end{align*}
\]  

(3.1)

It is easy to check that the curvature tensor \( Rm \) and Ricci tensor \( \text{Ric} \) satisfy the following equations respectively,

\[
\frac{\partial Rm}{\partial t} = \triangle Rm + Q_1(Rm, Rm),
\]

(3.2)

and

\[
\frac{\partial \text{Ric}}{\partial t} = \triangle \text{Ric} + Q_2(Rm, \text{Ric}),
\]

(3.3)

where \( Q_i \) are multi-linear functions of their arguments, \( i = 1, 2. \) Their definitions depend only on the dimension of \( M. \)
**Theorem 3.1.** There exist constant $C_1$ and $C_2$ such that if for all $x \in M$

\[
\left( \int_{B_r(x)} |Rm(g_0)|^2 dV_{g_0} \right)^{\frac{1}{2}} \leq [C_1Cs(B_r(x))]^{-1}
\]

and

\[|Ric(g_0)| \leq K,
\]

then the equation (3.1) has a smooth solution for $t \in [0, T)$, where

\[T \geq C_2 \cdot \min \left( r^2, K^{-1} \right).
\]

Moreover, for $t \in (0, T)$, the Riemannian curvature tensor satisfies the following bound,

\[
\|Rm\|_{\infty} \leq \frac{C_3}{t}.
\]

(3.4)

Here $C_1$, $C_2$ and $C_3$ only depend on the dimension of $M$.

**Remark 3.2.** By the same argument as in the proof, we can show that Theorem 3.1 holds for $n \geq 3$, where the $L^2$-norm of the curvature should be replaced by $L^{\frac{n}{2}}$-norm.

**Remark 3.3.** The assumption of the Ricci curvature is only used to guarantee that we have a standard covering property on $M$. So if we assume that $(M, g_0)$ has such covering property, then we can weaken the assumption of Ricci curvature by $\int_{B_r(x)} |Ric(g_0)|^p \leq K$, for each $x \in M$, where $p > \frac{n}{2}$. Note that Corollary 3.3 will give the required estimate on Ricci curvature along the Ricci flow.

To prove Theorem 3.1, we first show a result for the scalar function. In the following we always assume that $(M, g_0)$ is a Riemannian manifold with bounded Ricci curvature and that for each $t \in [0, T]$ and $x \in M$,

\[
\frac{1}{2} g_0 \leq g(t) \leq 2g_0,
\]

\[
\left( \int_{B_r(x)} f^4 dV_g \right)^{\frac{1}{2}} \leq A \int_{B_r(x)} |\nabla f|^2 dV_g, \quad f \in C_0^\infty(B_r(x)).
\]
Theorem 3.4. Let \( f \geq 0 \) solve
\[
\frac{\partial f}{\partial t} \leq \Delta f + C_0 f^2, \quad 0 \leq t \leq T,
\]
on \( M \times [0,T) \). Assume that
\[
\frac{\partial}{\partial t} dV_{g(t)} \leq c f \, dV_{g(t)}
\]
and that
\[
\left( \int_{B_r(x)} f_0^2 \right)^{\frac{1}{2}} \leq (6 C_0 A)^{-1}, \quad \text{for any } x \in M
\]
where \( f_0(x) = f(x,0) \). Then
\[
|f(x,t)| \leq C_1 t^{-1},
\]
where \( 0 < t < \min(T, C_2 r^2) \), \( C_1 \) and \( C_2 \) depend on the dimension of \( M \) and \( C_0 \).

Proof: Let \([0,T'] \subset [0,T)\) be the maximal interval such that
\[
e_0 = \sup_{x \in M, \, 0 \leq t \leq T'} \left( \int_{B_r(x)} f^2 \right)^{\frac{1}{2}} \leq (3 C_0 A)^{-1}.
\]
By a direct calculation, we have, for \( 0 \leq t \leq T' \),
\[
\frac{\partial}{\partial t} \int \phi^{q+2} f^p + \int |\nabla (\phi f^{p/2})|^2 \leq C_{p,q} \| \nabla \phi \|_\infty^2 \int \phi^q f^p.
\]
(3.6)
Since \( g_0 \) has bounded Ricci curvature and all the metrics \( g(t) \) are equivalent, by a standard covering theorem there exists a definite constant \( N \) such that
\[
B_{2r}(x) \subset \bigcup_{i=1}^N B_r(y_i), \quad y_i \in B_{2r}(x).
\]
Then we can choose \( \phi \) such that the following holds
\[
\frac{\partial}{\partial t} \int_{B_r(x)} f^2 \leq 2 N \| \nabla \phi \|_\infty^2 e_0^2.
\]
Integrating this, we obtain
\[
\int_{B_r(x)} f^2 \, dV_g \leq \int_{B_r(x)} f_0^2 \, dV_{g(0)} + 2 N \| \nabla \phi \|_\infty^2 e_0^2 t.
\]
Therefore,
\[
(1 - 2 N \| \nabla \phi \|_\infty^2 t) e_0^2 \leq (6 C_0 A)^{-1}
\]
Since $x$ is chosen arbitrary, for $T' < \frac{3}{8N} \| \nabla \phi \|_\infty^{-2}$, we have

$$e_0 < (3C_0A)^{-1}.$$ 

This contradicts the assumed maximality of $[0, T']$. We can therefore assume that $T' \geq \min(C_{2r^2}, T)$.

Multiplying (3.6) by $t$ and then integrating with respect to $t$, we have

$$\int \phi^{q+2} f^p + \int_0^t \int |\nabla (\phi f^{p/2})|^2 \leq Ct^{-1} \int_0^t \int \phi^q f^p$$

Then the covering argument yields that

$$\int_{B_r(x)} f^3 \leq \sum_{i=1}^N \int_{B_r(y_i)} \phi_i^4 f^3 \leq Ct^{-1} \int_0^t \left( \int_{B_r(y_i)} f^2 \right)^{\frac{1}{2}} \left( \int_{B_r(y_i)} (\phi_i f)^4 \right)^{\frac{1}{2}} \leq Cc_0 At^{-2} \int_0^t \int_{B_r(y_i)} f^2 \leq CAe_0^3 t^{-1},$$

where $\phi_i$ is a cutoff function in $B_r(y_i)$.

Setting $\mu^3 = CNAe_0^3$, we can apply Theorem 2.5 which still holds, as $p_0 \to 2$.

On the other hand, for $t \in [0, T']$

$$A^2 \mu^3 \leq CNA^3 e_0^3 \leq CNA^3 (3C_0A)^{-3} \leq CNC_0^{-3}.$$ 

We then obtain the desired estimate. □

The argument also implies the following

**Corollary 3.5.** Let $f$ satisfy the assumptions of Theorem 3.4. Then given $u \geq 0$ such that

$$\frac{\partial u}{\partial t} \leq \phi^2 (\Delta u + c_0 fu),$$

the following estimate holds for $0 \leq t < \min(T, C_{2r^2})$,

$$|u(x, t)| \leq C_1 A^\frac{2}{3} t^{-\frac{2}{3}} \left( \int_{B_{3r}(x)} u_0^3 \right)^{\frac{1}{3}},$$

where $u_0(x) = u(x, 0)$. 

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Proof: Using the covering argument and the proof of Lemma 2.2, we obtain
\[
\frac{\partial}{\partial t} \int \phi^2 u^3 + \frac{2}{3} \int |\nabla (\phi u^3)|^2 \leq C_3 \int u^3 + C_4 A \left( \int f^2 \right)^{\frac{1}{2}} \int |\nabla (\phi u^3)|^2.
\]
This implies that
\[
\int_{B_r(x)} u^3 \leq C \int_{B_{3r}(x)} u^3_0.
\]
By the above theorem, we have
\[
\int_{B_r(x)} f^3 \leq \mu^3 t^{-1}.
\]
Thus the result follows from Theorem 2.5 with \( p_0 = 3 \).

Proof of Theorem 3.1: It is well known that the equation (3.1) has a smooth solution on a sufficiently small time interval starting at \( t = 0 \). Let \([0, T_{\text{max}})\) be a maximal time interval on which (3.1) has a smooth solution and such that the following hold for each metric \( g(t) \) and \( x \in M \)
\[
\|f\|_2^2 \leq 4A_0 \|\nabla f\|_2^2, \quad f \in C_0^\infty (B_r(x)), \tag{3.7}
\]
\[
\frac{1}{2} g_0 \leq g(t) \leq 2 g_0, \tag{3.8}
\]
\[
\left( \int_{B_r(x)} |\text{Rm}| \right)^{\frac{1}{2}} \leq 2 (C_1 A_0)^{-1}, \tag{3.9}
\]
Suppose that \( T_{\text{max}} < T_0 = C_2 \cdot \min (r^2, K^{-1}) \). We will show that this leads to a contradiction.

First, notice that the curvature tensor \( \text{Rm} \) satisfies (3.2), then we have
\[
\frac{\partial}{\partial t} |\text{Rm}|^2 = \phi^2 \Delta |\text{Rm}|^2 - 2 |\nabla \text{Rm}|^2 + \text{Rm} \ast \text{Rm} \ast \text{Rm}.
\]
According to the proof of Theorem 3.4, we obtain
\[
\sup_{x \in M} \|\text{Rm}(g(t))\|_{2;B_r(x)} < 2 \sup_{x \in M} \|\text{Rm}(g_0)\|_{2;B_r(x)} \leq 2 [C_1 A_0]^{-1},
\]
which implies a strict inequality for (3.9).
Next, since the Ricci curvature satisfies (3.3), then Corollary 3.5 and the covering argument imply that

\[ |\text{Ric}(g(t))| \leq C A_0^2 t^{-\frac{2}{3}} \left( \int_{B_{3r}(x)} |\text{Ric}(g_0)|^3 \right)^{\frac{1}{3}} \]
\[ \leq C t^{-\frac{2}{3}} \left( K A_0^2 \int_{B_{3r}(x)} |\text{Ric}(g_0)|^2 \right)^{\frac{1}{3}} \]
\[ \leq C t^{-\frac{2}{3}} \]

where \( \|\nabla \phi\|_\infty \) can be evaluated at \( g(0) \), since the metrics \( g(t) \) are equivalent within the maximal time \( T_{\text{max}} \).

Applying the bound on Ric to the following
\[ \left| \frac{d}{dt} \int f^p \, dV_g \right| \leq 2 \| \text{Ric} \|_\infty \int f^p \, dV_g, \]
we have
\[ -2 C t^{-\frac{4}{3}} \, dt \leq \, d \log \int f^p \, dV_g \leq 2 C t^{-\frac{4}{3}} \, dt, \]
which implies that for some suitably chosen constants,
\[ \left| \log \frac{\|f\|_p(t)}{\|f\|_p(0)} \right| < \log 2. \]

The differential inequality
\[ \left| \frac{d}{dt} \int |\nabla f|^2 \, dV_g \right| \leq 2 \| \text{Ric} \|_\infty \int |\nabla f|^2 \, dV_g \]
leads to a similar estimate. Therefore, it follows that for any \( t \leq T_0 \),
\[ \|f\|_2^2(t) < 2 \|f\|_2^2(0) \leq 2 A_0 \|\nabla f\|_2^2(0) < 4 A_0 \|\nabla f\|_2^2(t), \]
that is to say (3.7) holds with strict inequality.

To show that (3.8) holds with strict inequality, we use Hamilton’s trick. Simply fix a tangent vector \( v \) with respect to \( g(t) \), then
\[ \frac{d}{dt} \|v\|_g^2(t) = \frac{d}{dt} (g_{ij}(t)v^i v^j) = g'_{ij}(t)v^i v^j \]
implies
\[ \left| \frac{d}{dt} \log \|v\|_g^2(t) \right| \leq |g'_{ij}(t)| \leq 2 |\text{Ric}|. \]
So for $0 \leq t \leq T_2 < T_0$, 
\[
\log \frac{|v|_{g(t)}^2}{|v|_{g(0)}^2} \leq \int_0^{T_2} |g'_{ij}(t)| \, dt \leq 2 \| \text{Ric} \|_\infty T_2 < \log 2,
\]
which implies 
\[
\frac{1}{2} |v|_{g(0)}^2 < |v|_{g(t)}^2 < 2 |v|_{g(0)}^2,
\]
for $t < T_0$.

Now we can show that $g(t)$ has a smooth limit as $t \to T_{\max}$. If $T_{\max} < T_0$, we would be able to extend the solution to (3.1) smoothly beyond $T_{\max}$ with (3.7), (3.8) and (3.9) still holding. This contradicts the assumed maximality of $T_{\max}$. Hence, we conclude that $T_{\max} \geq T_0$.

The estimate (3.4) follows from Theorem 3.4. □

Therefore, the previous theorem can be restated as the following.

**Theorem 3.6.** There exist constant $\varepsilon$ and $C_1$ such that for $r \leq 1$, if 
\[
\frac{r^4}{\text{Vol}(B_r(x))} \int_{B_r(x)} |\text{Rm}(g_0)|^2 \, dV_{g_0} \leq \varepsilon
\]
and 
\[
|\text{Ric}(g_0)| \leq K,
\]
then the equation (3.1) has a smooth solution for $t \in [0, T)$, where 
\[
T \geq C_1 \cdot \min \left( r^2, K^{-1} \right).
\]
Moreover, for $t \in (0, T)$, the Riemannian curvature tensor satisfies the following bound, 
\[
\| \text{Rm} \|_\infty \leq C_2 t^{-1}.
\]
Here $\varepsilon$, $C_1$ and $C_2$ only depend on the dimension of $M$.

**Proof:** When using the covering argument, we shall use the following result. For any $y_1, y_2 \in B_{2r}(x)$, there exists a definite constant $c_1 > 0$ such that 
\[
\frac{1}{c_1} \text{Vol}(B_r(y_1)) \leq \text{Vol}(B_r(y_2)) \leq c_1 \text{Vol}(B_r(y_1)).
\]
This follows from the volume comparison theorem, the facts that within the maximal time all metrics are equivalent and that at $t = 0 \ B_r(y_i) \cap B_r(x) \neq \emptyset$, $i = 1, 2$. 

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Then we can proceed as in the proof of the previous theorem. □

Let $M$ be a 4-dimensional Riemannian manifold with bounded Ricci curvature. Based on the result in [7], we can show that if there exists some $\varepsilon > 0$ such that

$$\int_{B_r(x)} |\text{Rm}|^2 < \varepsilon,$$

then for some constant $c < 1$, we have

$$\frac{r^4}{\text{Vol}(B_{cr}(x))} \int_{B_{cr}(x)} |\text{Rm}|^2 < \varepsilon.$$

Therefore we obtain the main theorem as required.

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