Anomalous real spectra of non-Hermitian quantum graphs in a strong-coupling regime

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Abstract
A family of one-dimensional quantum systems described by certain weakly non-Hermitian Hamiltonians with real spectra is studied. The presence of a microscopic spatial defect (i.e. of a fundamental length scale \( \theta > 0 \)) is mimicked by the replacement of the real line of coordinates \( \mathbb{R} \) by a non-tree graph. By construction, this graph only differs from \( \mathbb{R} \) on a small interval of size \( \mathcal{O}(\theta) \). An unexpected return of the reality of the whole spectrum in an anomalous strong-coupling regime (i.e. the emergence of an isolated island of stability of the system) is then revealed, proved and tentatively attributed to the topological nontriviality of the graph.

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1. Introduction

1.1. Schrödinger equations on linear lattices

For any concrete Hamiltonian with a real spectrum and with the property \( H \neq H^\dagger \) valid in an auxiliary, unphysical Hilbert space \( \mathcal{H}^{(F)} \), one must introduce another ‘standard’ physical Hilbert space \( \mathcal{H}^{(S)} \) in a way explained, say, in [1]. Most often this is achieved via the physics-determining metric operator \( \Theta = \Theta^\dagger \neq I \) which must be compatible with the ‘cryptohermiticity’ condition

\[
H = \Theta^{-1} H^\dagger \Theta := H^\dagger,
\]

guaranteeing the necessary Hermiticity of \( H \) in the correct Hilbert space \( \mathcal{H}^{(S)} \) [2].

Perturbation-expansion constructions of the required ad hoc Hilbert-space metrics \( \Theta = \Theta(H) \) are difficult (cf their characteristic sample in [3]). In what follows the simpler approach will be employed in which one approximates the (real) interval of coordinates by a
discrete lattice composed, say, of \( N = 2K \) grid points:

\[
\xi_{-K+1}, \xi_{-K+2}, \ldots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \ldots, \xi_{K-1}, \xi_K.
\]  

(2)

This converts the solution of equation (1) to the straightforward application of linear algebra.

Without additional interactions the free motion confined to lattice (2) with \( K < \infty \) may be controlled by the Runge–Kutta recipe leading to the \( 2K \)-dimensional matrix Schrödinger equation

\[
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots \\
0 & -1 & 2 & -1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & -1 & 2 & -1 \\
0 & \cdots & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix}
\psi(\xi_{-K+1}) \\
\psi(\xi_{-K+2}) \\
\psi(\xi_{-K+3}) \\
\vdots \\
\psi(\xi_{K-1}) \\
\psi(\xi_K)
\end{bmatrix}
= \begin{bmatrix}
\psi(\xi_{-K+1}) \\
\psi(\xi_{-K+2}) \\
\psi(\xi_{-K+3}) \\
\vdots \\
\psi(\xi_{K-1}) \\
\psi(\xi_K)
\end{bmatrix}.
\]  

(3)

A systematic amendment of precision may be mediated by an ad hoc increase of \( K \). For more details the readers may consult our recent paper [4] on bound states (to be abbreviated as BI in what follows). Lattice (2) with finite \( K < \infty \) has been interpreted there as an approximate representation of the straight real line. Equally well one can turn attention to the unbounded equidistant grid with \( K = \infty \) (cf., e.g., our papers [5] on scattering).

In paper BI we tried to enhance the phenomenological appeal of the model and introduced a specific \( \mathcal{PT} \)-symmetric interaction in it. In a way inspired by [5] the purely kinetic Hamiltonian of equation (3) has been replaced there by the manifestly non-Hermitian tridiagonal and \( 2K \)-dimensional matrix \( H^{(2K)}(\nu) \). From the resulting sequence of finite-dimensional toy models

\[
H^{(2)}(\nu) = \begin{bmatrix}
2 & -1 & 0 \\
-1 + \nu & 2
\end{bmatrix},
\]

\[
H^{(4)}(\nu) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 + \nu & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix},
\]

\[
H^{(6)}(\nu) = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 + \nu & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix},
\]

we extracted the respective spectra of energies \( E = E^{(2K)}_n(\nu), n = 1, 2, \ldots, N, N = 2K \). Empirically, we revealed that these spectra remain strictly real inside a \( K \)-independent weak-coupling interval of \( \nu \in (-1, 1) \).

The key problem addressed in paper BI was the construction of the complete family of metrics \( \Theta \) or, equivalently, of all the eligible inner products

\[
\langle\langle \psi | \phi \rangle = \sum_{m,n=-K+1}^{K} \psi^*(\xi_m) \Theta_{m,n} \phi(\xi_n)
\]

(4)

between elements \( \psi \) and \( \phi \) of the ‘standard’ Hilbert space \( \mathcal{H}^{(S)} \). We simplified this construction by working with the non-complex, real Hamiltonian matrices and with their real left and right
Figure 1. The simplest topologically nontrivial localized modification of the real interval of coordinates.

... eigenvectors. Then, the resulting metrics (which can be, in general, complex valued) were also real.

Even then, the selection of the appropriate (i.e. symmetric and positive-definite) matrix \( \Theta = \Theta(H) \) in equation (4) was not unique [1]. Naturally, the ambiguity of \( \Theta \) represents one of the main weaknesses of the theory. Among its suppressions available in the literature, one may mention not only the most common (often called Dirac’s) unit-matrix choice of \( \Theta^{(\text{Dirac})} = I \) but also the increasingly popular specification of the metric \( \Theta^{(\text{CP})} = CP \), where \( P \) is the parity, while \( C \) denotes a charge ([6] may be recommended as a source of more details). In this context, the main physical message delivered by our paper BI may be read as the conjecture of the classification of the eligible physical metrics \( \Theta \) in terms of a certain microscopic fundamental length scale \( \theta \in \mathbb{R}^+ \). Within such a scheme one can treat \( \Theta^{(\text{Dirac})} \) and \( \Theta^{(\text{CP})} \) as extreme special cases characterized by \( \theta = 0 \) and \( \theta = \infty \), respectively.

1.2. Nonlinear lattices and graphs simulating the finite-size microscopic defects

One of the first applications of the freedom of the choice of any finite \( \theta \in (0, \infty) \) was described in our paper BII [7]. We tried to move there beyond the one-dimensional quantum systems and, tentatively, we replaced the real line of coordinates \( \mathbb{R} \) by the \( q \)-pointed star-shaped graph \( G(q) \) embedded, presumably, in a more-dimensional space \( \mathbb{R}^d \).

Fortunately, the discretization method proved applicable at all the not too large integers \( q > 2 \). At the same time, the scaling invariance of the corresponding tree-shaped quantum graphs \( G(q) \) created a subtle inconsistency in the new theory. Indeed, on a pragmatic phenomenological level the scaling invariance of graphs \( G(q) \) contradicts the existence of a finite (though, presumably, very small) ‘smearing length’ \( \theta \lesssim \infty \) removing the ambiguity of the physical metric \( \Theta \).

From another perspective, the presence of a non-vanishing ‘smearing length’ \( \theta \) implies the admissibility of topologically nontrivial anomalies in the graphs themselves. In what follows we intend to develop this idea in more detail. In one of the simplest realizations of such a self-consistently non-local quantum-graph project we shall replace the real line of paper BI by a suitable short-range graph-supported modification of the real line.

For the sake of simplicity, we shall only contemplate the simplest non-tree graph sampled in figure 1, with the admissible size of the defect \( G - \mathbb{R} \) restricted just to the order of magnitude of \( \theta \).

2. Non-Hermitian interactions near vertices

2.1. Single-loop model

There exist many topologically nontrivial (i.e. non-tree, loops-containing) generalizations of the trivial real-line graph (i.e. of \( \mathbb{R} \equiv G^{(2)} \) in our present notation) which might be discretized along the lines discussed in papers BI and BII. Here, in a way guided by figure 1 we shall
select the first nontrivial, non-tree discrete-graph lattice in the form

\[
\begin{array}{c}
\hspace{1cm} \vdash x_{-K} \hspace{1cm} \cdots \hspace{1cm} \vdash x_{-2} \hspace{1cm} \vdash x_{-1} \hspace{1cm} \vdash x_{1} \hspace{1cm} \vdash x_{2} \hspace{1cm} \cdots \hspace{1cm} \vdash x_{K} \\
\vdash x_{0+} \\
\vdash x_{0-}
\end{array}
\]  \quad (5)

possessing the four vertices \(x_{-K}, x_{-1}, x_{1}, x_{K}\) and four wedges. For the sake of simplicity of our present constructive considerations, just the external wedges will be assumed of variable length, \(K = 1, 2, \ldots\).

In the next step following the current practice \cite{8}, we shall start from the use of the most common discrete Laplacean \(\Delta\) on this lattice, with

\[
\Delta \psi(\xi_k) \sim -\frac{\psi(\xi_{k+1}) - u\psi(\xi_k) + \psi(\xi_{k-1})}{h^2}, \quad k \neq \pm 1,
\]

where we choose \(u = 2\), and with

\[
\Delta \psi(\xi_k) \sim -\frac{\psi(\xi_j) - u\psi(\xi_k) + \psi(\xi_{0^+}) + \psi(\xi_{0^-})}{h^2}, \quad j = 2k, \quad k = \pm 1,
\]

where we choose \(u = 3\).

In the final step, the natural generalization of the model will be obtained when we append elementary Hermiticity-violating nearest-neighbor interaction terms to the two outmost vertices \(x_{-K}\) and \(x_{K}\) and, independently, also to the remaining two inner vertices \(x_{-1}\) and \(x_{1}\). For illustration purposes, we may take graph (5) and indicate the position of the interaction terms by symbols ♠ (representing the elements proportional to a real coupling \(g\)), ♣ (representing the coupling \(h\)) and ♥ (representing the coupling \(z\)). At any \(K\) this will define our present three-parametric family of toy Hamiltonians \(H = H^{(K)}(g, h; z)\). At an illustrative choice of \(K = 3\) this means that we may pick up the one-loop discrete graph

\[
\begin{array}{c}
\hspace{1cm} \vdash x_{-3} \hspace{1cm} \vdots \hspace{1cm} \vdash x_{-2} \hspace{1cm} \vdash x_{-1} \hspace{1cm} \vdash x_{1} \hspace{1cm} \vdash x_{2} \hspace{1cm} \vdash x_{3} \\
\vdash x_{0+} \\
\vdash x_{0-}
\end{array}
\]  \quad (6)

as leading to the eight-dimensional sparse-matrix Hamiltonian

\[
H^{(3)}(g, h; z) = \begin{pmatrix}
2 & -1 & -1 - z \\
1 + z & 2 & -1 \\
-1 & 3 & -1 - g & -1 - h \\
-1 + g & 2 & -1 + h \\
-1 + h & 2 & -1 + g \\
-1 - h & -1 - g & 3 & -1 \\
-1 & 2 & -1 + z \\
-1 - z & 2
\end{pmatrix}.
\]

2.2. Factorizable secular equation

From the purely computational point of view a remarkable property of our \(K = 3\) model is that its secular equation gets factorized. Indeed, once we put \(H = H^{(3)}(\gamma + \delta, \gamma - \delta; z)\), we reveal that the energies may be identified with the roots of one of the following pair of the two algebraic (and, in principle, exactly solvable) polynomial secular equations

\[
E^4 - 9E^3 + P_+ E^2 + Q_+ E + R_+ = 0
\]  \quad (8)
with the respective coefficients

\[ P_+ = P_+(z, \gamma) = z^2 + 24 + 4\gamma^2, \quad Q_+ = Q_+(z, \gamma) = -5z^2 - 19 - 16\gamma^2, \]
\[ R_+ = R_+(z, \gamma) = 2z^2 + 4\gamma^2z^2 + 12\gamma^2 + 2 \]

(9)

which do not depend on \( \delta \) and

\[ P_- = P_-(z, \delta) = 28 + z^2 + 4\delta^2, \quad Q_- = Q_-(z, \delta) = -35 - 5z^2 - 16\delta^2, \]
\[ R_- = R_-(z, \delta) = 14 + 6z^2 + 12\delta^2 + 4\delta^2z^2 \]

(10)

(which do not depend on \( \gamma \)). This means that at each ‘outer’ coupling \( z \) the spectrum is composed of the two one-parametric quadruplets of levels which might be expressed in closed form, in principle at least.

A numerical illustration of the parametric dependence of the spectrum is provided in figure 2. We see there that the whole octuplet of the \( z \)-dependent energies remains real if and only if we stay in the weak-coupling regime with \( |z| \leq 1 \). The first merger occurs at \( |z| = 1 \) involving the second and the third root of the ±-subscripted secular polynomial (8). In the strong-coupling domain, i.e. at \( |z| > 1 \), these two energies form a complex-conjugate pair with the growing common function \( |\text{Im}E_\pm| \).

The separation of the ‘physical’ weak-coupling regime from its ‘unphysical’ strong-coupling complement is ‘generic’ within a fairly broad class of models. This has been demonstrated in our purely numerical study [9] where the ‘generic’ spectrum has been found composed of a part called ‘robust’ (i.e. real at all the coupling strengths) and a complement called ‘fragile’ (sampled by the two ellipses in figure 2). The same or similar pattern has also been detected via the exactly solvable models [10] (offering also the explicit forms of the imaginary parts of the energies) as well as via their simplest, semi-numerically tractable generalizations [11].

In such a setting our present introduction of topologically nontrivial quantum graphs may be characterized as opening new perspectives and moving beyond the classification offered in [9].

After the change of notation with \( g = \gamma + \delta \) and \( h = \gamma - \delta \) we may keep \( \delta = 0 \) and let the second parameter grow, \( \gamma = g = h > 0 \). Empirically, we reveal (and, subsequently,
easily prove by elementary means) that the overall pattern represented in figure 2 gets only inessentially deformed in the whole interval of \( \gamma \in (0, 1) \). For illustration, figure 3 displays the shape (i.e. the smoothly deformed form) of the spectrum at \( g = h = 0.98 \) (i.e. near the weak-coupling physical horizon).

3. Anomalous domains of stability

The inspection of coefficients (9) and (10) in secular polynomials (8) reveals that the contribution of the growth of the positive quantity \( \gamma^2 \) or \( \delta^2 \) shares the direction with the contribution of \( z^2 \). Hence, these pairs of parameters play a partially interchangeable dynamical role. No particularly surprising changes of the overall form of the spectrum can be expected at small \( \gamma \), therefore. Moreover, from the purely qualitative point of view one of the three parameters can be considered redundant so we shall simplify the discussion by setting \( \delta = 0 \) in what follows.

3.1. Numerical experiments with the Hamiltonian \( H^{(3)}(\gamma; z) \) in the strong-coupling regime

In the specific \( K = 3 \) model with vanishing \( \delta = g - h = 0 \) and subcritical positive \( \gamma < 1 \), our numerical experiments indicated that all the spectrum may become doubly degenerate in the limit \( \gamma \to 1 \). Due to the relative simplicity of the model this expectation can be rigorously confirmed by the explicit evaluation of the secular eigenvalue condition at \( \gamma = 1 \),

\[
(E - 2)^2 [E^3 - 7E^2 + (14 + z^2)E - 7 - 3z^2]^2 = 0.
\]

One may now ask what happens when one moves beyond this ‘natural’ boundary of the weak-coupling cryptohermiticity domain, i.e. in the language of mathematics, beyond the boundary of the well-established domain of the guaranteed reality of the spectrum, i.e. from the point of view of quantum physics, beyond the boundary of the domain of the safely stable and safely unitary time evolution of the system.

A deeper change of the pattern can be expected to emerge in the upper vicinity of the critical parameters, i.e. say, at \( \gamma \gtrsim 1 \). We performed a number of numerical experiments
in such a new domain, with a typical result obtained at $\gamma = 1.035$ and sampled in figure 4. In full accord with expectations one reveals, first of all, that the spectrum ceases to be real in the half-strong-/half-weak-coupling regime with $|z| < 1$. Indeed, the numerical spectrum becomes unphysical there since a pair of the energies becomes complex (this 'missing pair' is absent in the picture of course). In parallel, one can note that during the growth of $\gamma \gtrsim 1$ the two lowest energy levels still remain very close to each other forming an almost degenerate doublet while the constant root $E = 2$ ceases to be degenerate.

The comparison of figures 3 and 4 at $|z| > 1$ (i.e. in the genuine strong-coupling regime) reveals that while the maximal (i.e. the eighth) energy remains safely real, the seventh and sixth level merge and complexify beyond certain fairly large pair of 'exceptional-point' values of $z = \pm |z(\text{EP})(\gamma)|$ (numerically one finds that $z(\text{EP})(\gamma) \approx \pm 3$ at $\gamma = 1.035$).

For these reasons (easily supported also by an elementary algebra) there still exists a comparatively smaller domain of strong couplings where the situation remains ambiguous. A magnified detail of figure 4 displayed in figure 5 reveals the presence of an extremely anomalous behavior of the numerical energy spectrum at $\gamma > 1$ and $|z| > 1$. Indeed, while the two largest energies (lying out of the frame of this picture) remain safely real we discover that all the remaining sextuplet of the energies also remains real inside the interval of $z \in (1, 1+a) \approx (1.000, 1.022)$.

Such an anomalous return of the complete stability and observability of the system in the strong-coupling regime looks surprising and unexpected. Certainly, the confirmation of its existence must be performed by non-numerical means. Fortunately, such a proof is rendered possible by the factorizability (8) of the underlying secular equation.

### 3.2. The domain of reality of the first, $\gamma$-independent quadruplet of energies

The study of the spectrum of the Hamiltonian $H^{(3)}(\gamma, \gamma; z)$ (where $\delta = 0$) gets perceivably facilitated by the $\gamma$-independence of the minus-subscripted coefficients (10) in one of our quartic polynomial secular equations (8).

The quadruplet of the related (here, minus-superscripted) energies is easy to find because one of them is constant, $E^{(-)}_0 = 2$, while the other three roots will only vary with $z$ in a way

dictated by their definition

\[-7 - 3z^2 + 14E + Ez^2 - 7E^2 + E^3 = 0.\]

At \(|z| = 1\), this equation may be easily factorized so that the triplet of energies \(E_j^{(-)}(z)\), \(j = 1, 2, 3\), will coincide with the explicitly known roots of the polynomial

\[Q(E) = -10 + 15E - 7E^2 + E^3 = (E - 2) \left( E - \frac{5 + \sqrt{3}}{2} \right) \left( E - \frac{5 - \sqrt{3}}{2} \right).\]

Once we consider positive \(z^2 - 1 = \lambda\), the graphical representation of our secular equation in the form of the sum \(Q(E) + (E - 3)\lambda = 0\) indicates that with the (unlimited) growth of \(\lambda \in (0, \infty)\), the rightmost root \(E_3^{(-)}(z)\) will remain real. It will merely decrease from its maximum \(E_3^{(-)}(\pm 1) = 3.618\) to its minimum \(E_3^{(-)}(\pm \infty) = 3\). Hence, we may restrict our discussion to the specification of the domain of the survival of the reality of the two smaller roots \(E_1^{(-)}(z)\) and \(E_2^{(-)}(z)\).

There certainly exists an interval of \(\lambda \in (0, \lambda_{\text{max}})\) in which the latter roots remain real while moving toward each other. This motion starts at their respective initial values \(E_1^{(-)}(\pm 1) = 1.382\) and \(E_2^{(-)}(\pm 1) = 2\), and terminates at the common degenerate value of \(E_1^{(-)}(\pm \sqrt{1 + \lambda_{\text{max}}}) = E_2^{(-)}(\pm \sqrt{1 + \lambda_{\text{max}}}) = y\).

The latter point is the position of an extreme (in fact, of the maximum) of our secular polynomial so that its value must satisfy the conditions \(Q'(y) + \lambda = 0\) and \(Q''(y) < 0\). The latter one is safely satisfied since we surely have \(y < 7/3\).

The former condition looks tractable as a quadratic equation with the two roots \(y\) of which we need the smaller one,

\[y = \frac{7 - \sqrt{4 - 3\lambda}}{3}.\]

Nevertheless, for our present purposes we should rather treat this condition differently, namely as the linear definition of \(\lambda = \lambda(y)\). The insertion of this quantity in secular equation finally yields the single cubic equation for \(y\),

\[-35 + 42y - 16y^2 + 2y^3 = 0.\]
This equation gives the unique real value $y \sim 1.702843492$ of the maximal energy which agrees with the prediction displayed in figure 5. The exact analytic formula also exists for the unique maximal admissible value of $\lambda_{\text{max}} \sim 0.14078102$. Its knowledge provides the first rigorous necessary condition of validity of our numerically supported stability-pattern predictions.

### 3.3. The domain of reality of the second, $\gamma$-dependent part of the spectrum

A deeper inspection of figure 5 indicates that the plus-subscripted secular equation (8) + (9) should be expected more restrictive. Still, its analysis may proceed along similar lines, using positive $\lambda = z^2 - 1$ and $\mu = y^2 - 1$ in our second secular equation

$$E^4 - 9E^3 + (29 + 4\lambda + 4\mu)E^2 + (-40 - 16\mu - 5\lambda)E + 16 + 2\lambda + 4(1 + \mu)(1 + \lambda) + 12\mu = 0.$$  

At the boundary $\lambda = \mu = 0$ the left-hand-side polynomial gets factorized,

$$\hat{Q}(E) = E^4 - 9E^3 + 29E^2 - 40E + 20 = (E^2 - 5E + 5)(E - 2)^2.$$  

In the strong-coupling regime with $\lambda > 0$ and $\mu > 0$ the graphical interpretation of the plus-subscripted secular equation may again be based on the split of the secular polynomial into the easily factorizable polynomial $\hat{Q}(E)$ and the correction term

$$\hat{R}(E) = E^2\lambda + 4E^2\mu - 16\lambda\mu - 5E\lambda + 6\lambda + 16\mu + 4\mu\lambda$$

exhibiting the three-component form

$$\hat{R}(E) = 4\lambda\mu + \lambda(E - 2)(E - 3) + 4\mu(E - 2)^2.$$  

Obviously, the polynomial $\hat{Q}(E)$ vanishes at its four roots $\hat{E}_1^{(\pi)} \sim 1.382$, $\hat{E}_2^{(\pi)} = \hat{E}_3^{(\pi)} = 2$ and $\hat{E}_4^{(\pi)} \sim 3.618$ so that the spectrum is real at $\lambda = \mu = 0$. Now, our task is to prove that this quadruplet of energies remains real in a non-empty domain $\mathcal{D}$ of nontrivial $\lambda > 0$ and $\mu > 0$.

After the reparametrization of our variables

$$E \equiv \frac{x + \hat{\mu}}{2}, \quad \hat{\mu} = 16\mu, \quad \hat{\lambda} = 4\lambda,$$

we get the simpler form of the secular equation

$$S(\hat{\lambda}, \hat{\mu}, x) = (x^2 + \hat{\mu} - 5)(x + 1)^2 + \hat{\lambda}(x^2 + \hat{\mu} - 1) = 0,$$  

(12)

which may be further rewritten in the non-polynomial form

$$1 + \left(\frac{\hat{\lambda}}{(x + 1)^2}\right) = \frac{4}{x^2 + \hat{\mu} - 1},$$  

(13)

comparing the left-hand-side expression $F_{\text{LHS}}[(x + 1)^2, \hat{\lambda}]$ (which is solely controlled by the parameter $\hat{\lambda}$ and exhibits the reflection symmetry with respect to the off-central point $x = -1$) with the right-hand-side symmetric function $F_{\text{RHS}}(x^2, \hat{\mu})$ of $x$ (which is independent of $\hat{\lambda}$).

Obviously, the latter equation cannot have any real roots for $\hat{\mu} \in (5, \infty)$ because in this interval we have $F_{\text{LHS}}[(x + 1)^2, \hat{\lambda}] > 1$ while $F_{\text{RHS}}(x^2, \hat{\mu}) < 1$. With the decrease of $\hat{\mu} < 5$ there emerges, on the purely geometric grounds, the first degenerate and positive pair of real roots $x_{1,4}(\hat{\lambda}, \hat{\mu})$ which lies on the right branch of the spike $F_{\text{LHS}}[(x + 1)^2, \hat{\lambda}]$. With the further decrease of $\hat{\mu}$ (and at any $\hat{\lambda} > 0$), the rightmost root $x_4(\hat{\lambda}, \hat{\mu})$ can only move to the right so that its reality remains granted. Similarly, its neighbor $x_3(\hat{\lambda}, \hat{\mu})$ moves to the left and also remains real and larger than $-1$.

This means that the reality of the spectrum is controlled by the reality of the pair of the smaller roots $x_{1,2}(\hat{\lambda}, \hat{\mu})$ which have to lie on the left branch of the spike $F_{\text{LHS}}[(x + 1)^2, \hat{\lambda}]$. The
existence of the latter two intersections with the right-hand-side curve $F_{\text{RHS}}(x^2, \mu)$ requires that $\mu < 4$. As long as the domain of stability $D$ must lie in the strip of not too large $\mu \in (0, 4)$ with bounded $\hat{\lambda} \in (0, \hat{\lambda}_{\text{max}}(\mu))$, let us now return to the polynomial version of equation (12), paying attention just to its relevant $x < -1$ part.

In this interval the reality of the two leftmost roots $x_{1,2}(\hat{\lambda}, \mu)$ of the polynomial $S(\hat{\lambda}, \hat{\mu}, x)$ becomes more easily interpreted after we separate this polynomial into two components, namely $S(\hat{\lambda}, \hat{\mu}, x) = Q(\hat{\mu}, x) + \hat{\lambda} R(\hat{\mu}, x)$, where $Q(\hat{\mu}, x) = (x^2 + \hat{\mu} - 5)(x + 1)^2$. The reason is that $R(\hat{\mu}, x) = x^2 + \hat{\mu} - 1$ is safely positive in all the intervals of $x < -1$. On this background the reality of the roots $x_{1,2}(\hat{\lambda}, \hat{\mu})$ may be deduced from the facts that

- the auxiliary $\hat{\lambda} = 0$ curve $Q(\hat{\mu}, x)$ remains negative (and has a minimum) in a non-empty interval of $x \in (-b, -1)$ where $b = -x_1(0, \hat{\mu}) = \sqrt{5 - \hat{\mu}}$;
- secular polynomial $S(\hat{\lambda}, \hat{\mu}, x)$ remains negative (and has a minimum at some point $y$) inside a smaller interval of $x \in (-b + \Delta_1, -1 - \Delta_2)$;
- both the shifts $\Delta_{1,2}$ vanish at $\hat{\lambda} = 0$ and grow with growing $\hat{\lambda} \leq \hat{\lambda}_{\text{max}}(\hat{\mu})$.

The auxiliary quantity $y = y(\hat{\mu})$ and the function $\hat{\lambda} = \hat{\lambda}_{\text{max}}(\hat{\mu})$ which determines the boundary of the domain $D$ are both defined by the pair of polynomial equations

$$S(\hat{\lambda}_{\text{max}}(\hat{\mu}), \hat{\mu}, y) = 0, \quad S'(\hat{\lambda}_{\text{max}}(\hat{\mu}), \hat{\mu}, y) = 0.$$  

The easy elimination of $\hat{\lambda}_{\text{max}}(\hat{\mu})$ yields the final implicit exact definition of the unknown auxiliary parameter $y = y(\hat{\mu})$,

$$y^4 + 2\mu y^2 - 2 y^2 + 4 y + \hat{\mu}^2 - 6 \hat{\mu} + 5 = 0.$$  

We may re-read this relation as the explicit definition of the doublet of the eligible inverse functions

$$\hat{\mu} = \hat{\mu}_\pm(y) = -y^2 + 3 \pm 2\sqrt{-y^2 + 1 - y}. \quad (14)$$  

They remain real for $y \in (-c, -1)$ with $c = (1 + \sqrt{5})/2 \sim 1.618$. Using the Taylor series it is easy to prove that also the smaller function $\hat{\mu}_-(y)$ remains positive in the right vicinity of $y = -c$. The inspection of figure 6 indicates, moreover, that the lower bound of $\hat{\mu}_-(y) > -0.4$...
Figure 7. The pair of bracketing functions $\hat{\lambda}_{\text{max}}(y)$.

is still not too negative so that with $g = h = \sqrt{1 + \hat{\mu}(y)/16} > 0.987$ we may keep speaking about the strong coupling regime.

By the backward insertion, we obtain the explicit form of the doublet of the functions of our main interest displayed in figure 7 and given by formulae

$$
\hat{\lambda}_{\text{max}}[\hat{\mu}(y)] = \hat{\lambda}_{\text{max}}[\hat{\mu}(y)] = \frac{(y + 1)(y^2 + y - 2 \mp 2\sqrt{y^2 + 1 - y})}{(-y)}, \quad (15)
$$

Equations (14) and (15) form our final result. They provide the parametric definition of the boundary of the domain $D$ where the anomalous, strong-coupling stability of the quantum model with the Hamiltonian $H^{3\text{D}}(\gamma, \gamma; z)$ takes place. Thus, any parameter $y \in (-c, -1)$ bounded by $c = (1+\sqrt{5})/2 \sim 1.618$ specifies the coupling constants $g = h = \sqrt{1 + \hat{\mu}_{\pm}(y)/16}$ via equation (14). The spectrum of energies will be then real for all the couplings $z \in (1, z_{\text{max}})$ restricted by the upper bound $z_{\text{max}} = \sqrt{1 + \hat{\lambda}_{\text{max}}(y)/4}$ which is determined by formula (15).

4. Discussion

As long as our secular polynomial of the eighth degree gets factorized into two polynomials (8) of the fourth degree, one could, in principle, employ the discriminant techniques for quartic polynomials and try to extract the reality conditions for the spectrum. In fact, our present analytic approach to the description of the stability islands in parameter space represents just a formally equivalent, purely algebraic implementation of such an idea.

In doing so we heavily profited from additional simplifications offered by our particular example in a way recommended and used in a similar study of a certain specific $\mathcal{PT}$-symmetric chain model family in [12]. In this manner, the unbelievably clumsy explicit discriminants (or, alternatively, their merely perturbative versions) were replaced by a perceivably more efficient equivalent specification of the exceptional points.

This being said, it is necessary to add that the main weak point of such an algebraic approach to the problem of horizons of the physical domains lies in its insensitivity to perturbations of various kinds. In particular, also our present determination of the anomalous behavior of the energies as sampled in figure 5 may prove sensitive to perturbations. The
simplest possible form of such a perturbation may be mimicked by a trivial scalar shift of the secular determinant making it either slightly positive or slightly negative. Such a type of a purely numerical experiment reveals the occurrence of the two alternative scenarios where the complexification of the energies caused by the decrease of \( z > 1 \) involves *either* the mere two central energy levels (cf figure 8) or the two non-central neighboring energy-level pairs (cf figure 9), respectively.

A more systematic classification of the similar alternatives would already move us beyond the scope of our present paper. Interested readers could find a complementary study of this problem in [13] where a purely combinatorial full classification of all of the possible confluences of the energy levels (i.e. of a complete menu of eligible quantum catastrophes) has been found, in full generality, due to the extreme simplicity of the underlying toy model.
In an opposite extreme, [14] could be consulted for a realistic physical example where the authors were able to identify certain helical turbulence function and its derivative as effective parameters responsible for the different kinds of the unfoldings of eigenvalues in a complicated magnetohydrodynamic system.

For a deeper formal insight in the theory, the Kato monograph [15] could be consulted. Supplementary comments on the practical aspects of the explicit determination of the boundaries $\partial D$ (carrying, usually, the nickname of ‘exceptional points’ [16]) may be sought not only in magnetohydrodynamics [17] but also in the context of classical mechanics [18] or experimental optics [19].

Recently, the very real correspondence between the exceptional points and stability of quantum systems attracted attention to the study of concrete models. In many of them the necessary assumption of Hermiticity of observables takes place ‘in disguise’. For illustration of details one can recollect, e.g., the studies of the interacting boson models in nuclear physics [1] as well as of perturbation expansions in field theory [20]. Many new applications as reviewed, e.g., by Bender [6] range from the supersymmetric systems [21] to integrable models [22] and from the first-quantized relativistic systems [23] to the model-building in quantum cosmology [24]. Important innovations also occurred in the description of the time-dependent quantum systems [25] or of the theory of scattering [5, 26].

In many of these applications an important role is played by the proofs of the reality of the spectrum. Often, it can only be mediated by perturbation theory [27] so that the difference between our present ‘normal’, weak-coupling real spectrum and its ‘anomalous’ strong-coupling version might be perceived as purely conventional. Similarly, the role of the ‘smallness of size’ of our present anomalous domain of parameters guaranteeing the stability of the system may be also declared just supportive. This being said, we still believe that the ultimate source of the emergence of an isolated island of stability in our present model should be sought in the implicit continuous-limit connection of this model with the topologically anomalous, scaling-non-invariant graph of figure 1.

5. Summary

In a toy model a loss of locality on microscopic distances has been simulated by the brute-force replacement of the real line of coordinates by a topologically nontrivial loop-containing graph. We also recalled our recent papers BI and BII and complemented such a ‘kinematical’ form of the loss of locality by its parallel, dynamically generated form in which the interaction ceases to be naively Hermitian.

Both our toy-model kinematic and dynamical nonlocalities were assumed restricted to a small-size spatial domain. On this background we performed certain numerical experiments which revealed certain spectral anomalies involving, among other things, the puzzling disappearance of the instabilities in the strong-coupling dynamical regime. The existence of this phenomenon has been confirmed by its thorough non-numerical description and analytic explanation.

The resulting anomalous parametric dependence of the spectra certainly reflects the presence of the hypothetical microscopic deviation of the space in which we live from its standard topologically trivial picture. We believe that the similar quantum phenomena indicating possible deeper connections between kinematical topology and dynamical analysis will deserve an intensive study in the future. Our present encouraging technical message is that the use of the quantum graphs and of nontrivial discrete lattices seems to offer one of the feasible mathematical ways toward the similar phenomenological analyses.
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