Algebraic surfaces with canonical map of degree 13
15 17 18 21 22

NGUYEN BIN

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Abstract

In this note, we construct some minimal smooth surfaces of general type with canonical map of degree 13, 15, 17, 18, 21, 22. These surfaces are constructed as $\mathbb{Z}_2^3$-covers of a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$.

1 Introduction

The existence of surfaces of general type with canonical map of high degree has been studied by many authors in the last decades. This problem is motivated by the work of A. Beauville [1]. For recent account of the subject we refer the reader to the preprint by M. Mendes Lopes and R. Pardini [7]. Let $X$ be a minimal smooth complex surface of general type and denote by $\phi_X : X \dashrightarrow \mathbb{P}^g(X)^{-1}$ the canonical map of $X$, where $K_X$ is the canonical divisor of $X$ and $p_g(X) = \dim H^0(X, K_X)$ is the geometric genus. It is of interest to know which positive integers $d$ occur as the degree of such canonical maps for surfaces of general type. By the Bogomolov-Miyaoka-Yau inequality, for surfaces of general type, the degree $d$ of the canonical map is at most 36 [13, Proposition 5.7]. Some surfaces with $d = 9, 10, 11, 12, 14, 16, 20, 24, 27, 32, 36$ were constructed by U. Persson [13], S. L. Tan [18], C. Rito [14, 15, 16, 17], C. Gleissner, R. Pignatelli and C. Rito [3], Ching-Jui Lai and Sai-Kee Yeung [11], F. Fallucca and C. Gleissner [2], and the author [8, 9, 11]. We are interested in finding examples of surfaces with canonical map of new degree. The aim of this note is to construct surfaces with $d = 13, 15, 17, 18, 21, 22$.

Theorem 1. There exist minimal surfaces of general type $X$ satisfying the following

| $d$ | $q(X)$ | $p_g(X)$ | $K_X^2$ |
|-----|--------|--------|--------|
| 22  | 0      | 3      | 28     |
| 21  | 0      | 3      | 28     |
| 18  | 0      | 3      | 23     |
| 17  | 0      | 3      | 21     |
| 15  | 0      | 3      | 23     |
| 13  | 0      | 3      | 21     |

In the above theorem, $q(X) = \dim H^1(X, K_X)$ is the irregularity of $X$. The surfaces are constructed as $\mathbb{Z}_2^3$-covers of a blow-up $Y$ of $\mathbb{P}^1 \times \mathbb{P}^1$. The building data $\{L_\chi, D_\sigma\}_{\chi, \sigma}$ of $\mathbb{Z}_2^3$-covers (see Section 2) are chosen such that there are three characters $\chi_1, \chi_2, \chi_3$ of $\mathbb{Z}_2^3$ with $h^0(L_{\chi_1} + K_Y) + h^0(L_{\chi_2} + K_Y) + h^0(L_{\chi_3} + K_Y) = 3$ and that $h^0(L_X + K_Y)$ vanishes for all other characters $\chi$ of $\mathbb{Z}_2^3$. From the decomposition of the space of 2-forms of the surfaces (see Proposition 3)

$$H^0(X, K_X) = H^0(Y, K_Y) \oplus \bigoplus_{\chi \neq 00} H^0(Y, K_Y + L_{\chi}),$$

such a choice of the building data allows to describe three generators of the canonical linear system $|K_X|$ (see Proposition 4). From the three generators, we determine the moving part $|M|$ and the base points of $|K_X|$. Furthermore, the total branch locus of $\mathbb{Z}_2^3$-covers are chosen in such a way that the surfaces $X$ are smooth. This leads to the fact that $d = M^2 - n$, where $n$ is the number of base points of $|K_X|$. 

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Throughout this note all surfaces are projective algebraic over the complex numbers. The linear equivalence of divisors is denoted by \( \equiv \). We call a surface \( X \) no non-trivial 3-torsion if the only 3-torsion in \( \text{Pic}(X) \) is \( \mathcal{O}_X \). A character \( \chi \) of the group \( G \) is a homomorphism from \( G \) to \( \mathbb{C}^* \), the multiplicative group of the non-zero complex numbers.

2 \( \mathbb{Z}_3^2 \)-coverings

The construction of abelian covers was studied by R. Pardini in [12]. For details about the building data of abelian covers and their notations, we refer the reader to Section 1 and Section 2 of R. Pardini’s work ([12]). For the sake of completeness, we recall some facts on \( \mathbb{Z}_3^2 \)-covers in [10], in a form which is convenient for our later constructions. From [12, Theorem 2.1 and Proposition 2.1] we have

**Proposition 1.** Given \( Y \) a smooth projective surface, let \( L_\chi \) be divisors of \( Y \) such that \( L_\chi \neq \mathcal{O}_Y \) for all non-trivial characters \( \chi \) of \( \mathbb{Z}_3^2 \) and let \( D_\sigma \) be effective divisors of \( Y \) for all \( \sigma \in \mathbb{Z}_3^2 \setminus \{(0,0)\} \) such that the total branch divisor \( B := \sum_{\sigma \neq (0,0)} D_\sigma \) is reduced. If \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \) is the building data of a \( \mathbb{Z}_3^2 \)-cover

\[
f : X \longrightarrow Y,
\]

then

\[
\begin{align*}
3L_{10} & \equiv \equiv \quad D_{10} + 2D_{20} + \quad 2D_{21}, \\
3L_{01} & \equiv \quad D_{01} + 2D_{02} + \quad 2D_{11} + 2D_{12} + \quad 2D_{21}, \\
3L_{20} & \equiv \quad D_{01} + D_{02} + \quad D_{11} + 2D_{22} + \quad D_{12} + 2D_{21}, \\
3L_{02} & \equiv \quad 2D_{01} + D_{02} + \quad D_{11} + 2D_{22} + \quad D_{12} + 2D_{21}, \\
3L_{11} & \equiv \quad 2D_{01} + D_{02} + \quad D_{11} + 2D_{22} + \quad D_{12} + 2D_{21}, \\
3L_{22} & \equiv \quad 2D_{01} + D_{02} + \quad 2D_{10} + D_{11} + 2D_{22} + \quad D_{12} + 2D_{21}, \\
3L_{12} & \equiv \quad 2D_{01} + D_{02} + \quad 2D_{10} + D_{11} + 2D_{22} + \quad D_{12} + 2D_{21}, \\
3L_{21} & \equiv \quad D_{01} + 2D_{02} + \quad 2D_{10} + D_{11} + 2D_{22} + \quad 2D_{12} + D_{21}.
\end{align*}
\]

Conversely, if \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \) satisfies the above conditions and the surface \( Y \) has no non-trivial 3-torsion, then we can associate a \( \mathbb{Z}_3^2 \)-cover \( f : X \longrightarrow Y \) in a natural way.

In the above proposition, the last statement follows from the fact that if the surface \( Y \) has no non-trivial 3-torsion, then \( 3L \equiv D \) uniquely defines \( L \).

From [12, Proposition 3.1] we have:

**Proposition 2.** Let \( Y \) be a smooth surface and let \( f : X \longrightarrow Y \) be a \( \mathbb{Z}_3^2 \)-cover with the building data \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \) such that \( X \) is normal. Then the surface \( X \) is smooth above a point \( y \in Y \) if and only if one of the following conditions holds:

1. the point \( y \) is not a branch point of \( f \).
2. the point \( y \) belongs only to one component \( \Delta \) of \( \cup_{\sigma} D_\sigma \) and \( y \) is a smooth point of \( \cup_{\sigma} D_\sigma \).
3. the point \( y \) belongs only to \( D_\sigma \) and \( D_{\sigma'} \) for some \( \sigma, \sigma' \in \mathbb{Z}_3^2 \setminus \{(0,0)\} \) and:

   (a) \( D_\sigma \) and \( D_{\sigma'} \) are smooth at \( y \) and they meet transversally at \( y \);
   (b) the inertia groups of \( D_\sigma \) and \( D_{\sigma'} \) are different.

Also from [12, Lemma 4.2, Proposition 4.2] we have:

**Proposition 3.** Let \( f : X \longrightarrow Y \) be a smooth \( \mathbb{Z}_3^2 \)-cover with the building data \( \{L_\chi, D_\sigma\}_{\chi,\sigma} \). The surface \( X \) satisfies the following:

\[
3K_X \equiv f^* \left( 3K_Y + \sum_{\sigma \neq (0,0)} 2D_\sigma \right);
\]

\[
f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \bigoplus_{\chi \neq \chi_0} L_{\chi}^{-1}.
\]

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This implies that
\[
H^0(X, K_X) = H^0(Y, K_Y) \oplus \bigoplus_{\chi \neq \chi_{00}} H^0(Y, K_Y + L_\chi)
\]
\[
K^3_X = \left(3K_Y + \sum_{\sigma \neq (0,0)} 2D_\sigma\right)
\]
\[
p_g(X) = p_g(Y) + \sum_{\chi \neq \chi_{00}} h^0(L_\chi + K_Y)
\]
\[
\chi(\mathcal{O}_X) = 9\chi(\mathcal{O}_Y) + \sum_{\chi \neq \chi_{00}} \frac{1}{2}(L_\chi + K_Y)L_\chi.
\]

Let \( Y \) be a smooth surface and \( f : X \rightarrow Y \) be a smooth \( \mathbb{Z}^3 \)-cover with building data \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \). We get from [5, Section 3.4] that
\[
f_* \omega_X = \left(\prod_{(H, \psi)} x_{(H, \psi)}^2\right) \omega_Y \oplus \bigoplus_{\chi \neq 1} \left(\prod_{(H, \psi)} x_{(H, \psi)}^{2-b_{\chi, H, \psi}}\right) \omega_Y(L_\chi),
\]
where \( x_{(H, \psi)} \) is the local equation for \( R_{(H, \psi)} \), \( R_{(H, \psi)} \) is the reduced divisor supported on \( f^*(D_{(H, \psi)}) \), and the integer numbers \( b_{\chi, H, \psi} \) are the coefficients in the following formula
\[
3L_\chi \equiv \sum_{(H, \psi)} b_{\chi, H, \psi}D_{(H, \psi)}.
\]

Moreover, by the Hurwitz formula, we have that
\[
K_X \equiv f^*(K_Y) + \sum_{(H, \psi)} 2R_{(H, \psi)}.
\]

Combining (1) and (2), the generators of the canonical linear system \( |K_X| \) are obtained by the following:

**Proposition 4.** If \( Y \) is a smooth surface and \( f : X \rightarrow Y \) is a smooth \( \mathbb{Z}^3 \)-cover with building data \( \{L_\chi, D_\sigma\}_{\chi, \sigma} \), the canonical linear system \( |K_X| \) is generated by
\[
f^*((K_Y + L_\chi)) + \sum_{(H, \psi)} (2 - b_{\chi, H, \psi})R_{(H, \psi)}, \forall \chi \in J
\]
where \( J := \left\{ \chi' : |K_Y + L_{\chi'}| \neq \emptyset \right\} \), \( R_{(H, \psi)} \) is the reduced divisor supported on \( f^*(D_{(H, \psi)}) \), and the integer numbers \( b_{\chi, H, \psi} \) are the coefficients in the following formula
\[
3L_\chi \equiv \sum_{(H, \psi)} b_{\chi, H, \psi}D_{(H, \psi)}.
\]

For the proof of a general version of Proposition 4, we refer the reader to (c.f. [2, Proposition 3.15]).

### 3 Constructions

In this section, we construct surfaces described in Theorem 1.

#### 3.1 Construction of the surface with \( d = 22 \).

Let \( Y \) be the blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at four general points \( P_1, P_2, P_3, P_4 \). We denote by

- \( F, G \): the pull-back of general fibers \( f, g \), respectively of \( \mathbb{P}^1 \times \mathbb{P}^1 \),
- \( E_{P_1}, E_{P_2}, E_{P_3}, E_{P_4} \): the exceptional divisors corresponding to \( P_1, P_2, P_3, P_4 \), respectively,
- \( F_{P_1}, F_{P_2}, F_{P_3}, F_{P_4} \): the strict transform of the fibers \( f \) through \( P_1, P_2, P_3, P_4 \), respectively,
- \( G_{P_1}, G_{P_2}, G_{P_3}, G_{P_4} \): the strict transform of the fibers \( f \) through \( P_1, P_2, P_3, P_4 \), respectively.
The canonical class $K_Y \equiv -2F - 2G + EP_1 + EP_2 + EP_3 + EP_4$. We consider the following smooth divisors of $Y$:

$$D_{02} := G_{P_1} + G_{P_2}, D_{10} := G_{P_3} + G_{P_4}, D_{20} := E_1 + E_2,$$

$$D_{22} := F_{P_1} + F_{P_2} + F_1, D_{21} := C + F_{P_1} + F_{P_2}.$$

and $D_{ij} = 0$ for the other $(i, j)$, where $F_1 \in |F|, C \in |F + G - EP_1 - EP_2 - EP_3 - EP_4|$ are distinct divisors such that no more than two of these divisors go through the same point. We consider the following non-trivial divisors:

$$L_{10} := 2F + G - EP_3 - EP_4,$$

$$L_{01} := 3F + 2G - 2EP_1 - 2EP_2 - EP_3 - EP_4,$$

$$L_{20} := 4F + 2G - EP_1 - EP_2 - 2EP_3 - 2EP_4,$$

$$L_{02} := 3F + G - EP_1 - EP_2 - EP_3 - EP_4,$$

$$L_{11} := 2F + 2G - EP_3 - EP_4,$$

$$L_{22} := F + 2G - EP_3 - EP_4,$$

$$L_{12} := 2F + 2G - EP_1 - EP_2 - EP_3 - EP_4,$$

$$L_{21} := F + 3G - EP_1 - EP_2 - EP_3 - EP_4.$$

These divisors $D_\sigma$, $L_\chi$ satisfy the following relations:

$$3L_{10} \equiv D_{01} + 2D_{02}, 2D_{20} + D_{22} + D_{21} \equiv 6F + 3G - 3EP_3 - 3EP_4,$$

$$3L_{20} \equiv 2D_{01} + D_{02}, 2D_{20} + 2D_{22} + D_{21} \equiv 9F + 6G - 6EP_1 - 6EP_2 - 6EP_3 - 6EP_4,$$

$$3L_{02} \equiv 2D_{01} + D_{02}, 2D_{20} + D_{22} + D_{21} \equiv 9F + 3G - 3EP_1 - 3EP_2 - 3EP_3 - 3EP_4,$$

$$3L_{11} \equiv D_{01} + 2D_{02} + 2D_{20} + 2D_{22} + 2D_{21} \equiv 6F + 6G - 3EP_1 - 3EP_2 - 3EP_3 - 3EP_4,$$

$$3L_{22} \equiv 2D_{01} + D_{02} + D_{20} + D_{21} \equiv 3F + 6G - 3EP_1 - 3EP_2 - 3EP_3 - 3EP_4,$$

$$3L_{12} \equiv 2D_{01} + D_{02} + 2D_{20} + 2D_{21} \equiv 6F + 9G - 3EP_1 - 3EP_2 - 3EP_3 - 3EP_4,$$

Thus by Proposition 1, the divisors $D_\sigma$, $L_\chi$ define a $\mathbb{Z}_2^2$-cover $\varphi : X \longrightarrow Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^* (6F + 4G - EP_1 - EP_2 - 3EP_3 - 3EP_4).$$

We notice that a surface is of general type and minimal if the canonical divisor is big and nef (see e.g. [6, Section 2]). The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal and of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = (6F + 4G - EP_1 - EP_2 - 3EP_3 - 3EP_4)^2 = 28,$$

$$p_g (X) = 3, \chi (OX) = 4, q (X) = 0.$$

By Proposition (4), the linear system $|K_X|$ is generated by the three following divisors:

$$K_{20} := 2R_{02} + R_{20} + \varphi^* (FP_3) + \varphi^* (FP_4),$$

$$K_{11} := R_{10} + 2R_{21} + \varphi^* (EP_1) + \varphi^* (EP_2),$$

$$K_{12} := R_{02} + R_{10} + 2R_{22},$$

where $R_\sigma$ are the reduced divisors supported $\varphi^* (D_\sigma)$, for all $\sigma$. We have that

$$K_{20} = 2\overline{EP}_1 + 2\overline{EP}_2,$$

$$K_{11} = \overline{EP}_1 + \overline{EP}_2 + \overline{EP}_3 + \overline{EP}_4,$$

$$K_{12} = \overline{EP}_1 + \overline{EP}_2 + \overline{EP}_3 + \overline{EP}_4,$$

where $\overline{EP}_1, \overline{EP}_2, \overline{EP}_3, \overline{EP}_4, \overline{EP}_5, \overline{FP}_1, \overline{FP}_2, \overline{FP}_3, \overline{FP}_4, \overline{FP}_5, \overline{G}_P_1, \overline{G}_P_2, \overline{G}_P_3, \overline{G}_P_4$ are the reduced divisors supported $\varphi^* (C), \varphi^* (EP_3), \varphi^* (EP_4), \varphi^* (EP_1), \varphi^* (EP_2), \varphi^* (FP_1), \varphi^* (FP_2), \varphi^* (FP_3), \varphi^* (FP_4), \varphi^* (G_P_1), \varphi^* (G_P_2), \varphi^* (G_P_3), \varphi^* (G_P_4)$, respectively.

The intersections $\overline{EP}_1 \cap \overline{EP}_2, \overline{EP}_3 \cap \overline{EP}_4, \overline{EP}_5 \cap \overline{EP}_1, \overline{EP}_1 \cap \overline{EP}_2, \overline{FP}_1 \cap \overline{FP}_2, \overline{FP}_3 \cap \overline{FP}_4, \overline{FP}_5 \cap \overline{FP}_1, \overline{FP}_1 \cap \overline{FP}_2$ are the all base points of $|K_X|$. So we obtain that $d = K_X^2 - 6 = 22$. 

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3.2 Construction of the surface with $d = 21$.

Let $Y'$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at three general points $P_1, P_2, P_3$. We denote by

$$F, G;$$
the pull-back of general fibers $f, g$, respectively in $\mathbb{P}^1 \times \mathbb{P}^1$,

$$E_P, E_P', E_{P_1}, E_{P_2}, E_{P_3};$$
the exceptional divisors corresponding to $P_1, P_2, P_3$, respectively,

$$F_{P_1}, F_{P_2}, F_{P_3};$$
the strict transform of the fibers $f$ through $P_1, P_2, P_3$, respectively,

$$G_{P_1}, G_{P_2}, G_{P_3};$$
the strict transform of the fibers $g$ through $P_1, P_2, P_3$, respectively.

The canonical class $K_{Y'} \equiv -2F - 2G + E_{P_1} + E_{P_2} + E_{P_3}$. Let $Y$ be the blow-up of $Y'$ at the intersection $P'_3 := G_{P_3} \cap E_{P_3}$. By the abuse of notation, we denote by

$$F, G, E_{P_1}, E_{P_2}, E_{P_3}, F_{P_1}, F_{P_2}, F_{P_3}, G_{P_1}, G_{P_2}, G_{P_3};$$
the pull-back of $F, G, E_{P_1}, E_{P_2}, E_{P_3}, F_{P_1}, F_{P_2}, F_{P_3}, G_{P_1}, G_{P_2}, G_{P_3}$, respectively,

$$E_{P_1}', E_{P_2}', E_{P_3}';$$
the exceptional divisor,

$$E_{P_1}'';$$
the strict transform of $E_{P_1}$, respectively.

The canonical class $K_Y \equiv -2F - 2G + E_{P_1} + E_{P_2} + E_{P_3} + 2E_{P_3}'$. We consider the following smooth divisors of $Y$:

$$D_{01} := G_{P_3}''', D_{02} := G_1 + E_{P_3}'', D_{20} := C + G_{P_1} + G_{P_2}, D_{11} := E_{P_2},$$

$$D_{22} := F_{P_1} + F_{P_2}, D_{12} := E_{P_1}, D_{21} := F_1 + F_{P_2},$$

and $D_{10} = 0$, where $F_1 \in |F|$, $G_1 \in |G|$, $C \in |F + G - E_{P_1} - E_{P_2} - E_{P_3} - E_{P_3}'|$. Let $D$ be a divisor such that no more than two of these divisors go through the same point. We consider the following non-trivial divisors:

$$L_{10} := 2F + 2G - E_{P_1} - E_{P_2} - E_{P_3} - E_{P_3}'' - E_{P_3},$$

$$L_{01} := 2F + G - E_{P_1} - E_{P_2} - E_{P_3}'' - E_{P_3},$$

$$L_{20} := 3F + G - E_{P_1} - E_{P_2} - E_{P_3}'' - E_{P_3},$$

$$L_{02} := 2F + G - E_{P_1} - E_{P_2} - E_{P_3}'' - 2E_{P_3},$$

$$L_{11} := 2F + 3G - 2E_{P_1} - E_{P_2} - E_{P_3}'' - 2E_{P_3},$$

$$L_{22} := 2F + 3G - E_{P_1} - E_{P_2} - E_{P_3}'' - 2E_{P_3},$$

$$L_{12} := 3F + 2G - 2E_{P_1} - E_{P_2} - E_{P_3}'' - 2E_{P_3},$$

$$L_{21} := F + 2G - E_{P_2} - E_{P_3}'' - E_{P_3}. $$

These divisors $D_\sigma, L_\chi$ satisfy the following relations:

$$3L_{10} \equiv 2D_{20} + D_{22} + D_{21} \equiv 6F + 6G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{01} \equiv D_{01} + 2D_{02} + D_{22} + 2D_{21} \equiv 6F + G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{20} \equiv 2D_{01} + D_{02} + 2D_{22} + 2D_{21} \equiv 6F + G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{02} \equiv D_{01} + 2D_{02} + D_{22} + 2D_{21} \equiv 6F + 6G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{11} \equiv D_{01} + 2D_{02} + 2D_{20} + 2D_{22} \equiv 6F + 9G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{22} \equiv 2D_{01} + D_{02} + D_{20} + D_{22} \equiv 3F + 6G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{12} \equiv D_{01} + 2D_{02} + 2D_{20} + D_{22} \equiv 6F + 9G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

$$3L_{21} \equiv D_{01} + 2D_{02} + 2D_{20} + 2D_{22} \equiv 3F + 6G - 3E_{P_1} - 3E_{P_1} - 3E_{P_3} - 3E_{P_3}$$

Thus by Proposition 1, the divisors $D_\sigma, L_\chi$ define a $\mathbb{Z}_3^2$-cover $\varphi : X \to Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^* \left( 4F + 4G - E_{P_1} - E_{P_1} - E_{P_3} - 2E_{P_3}' \right).$$

The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = \left( 4F + 4G - E_{P_1} - E_{P_1} - E_{P_3} - 2E_{P_3}' \right)^2 = 28,$$

$$p_g(X) = 3, \chi(O_X) = 4, q(X) = 0.$$
where $R_\sigma$ are the reduced divisors supported $\varphi^*(D_\sigma)$, for all $\sigma$. We have that

\[
\begin{align*}
K_{10} &= \overline{G}_{P_3 P_2} + 2\overline{E}_1 + 2\overline{L}_P + \overline{P}_3 + \overline{F}_1 + 2\overline{P}_2 + 3\overline{F}_1 + 3\overline{G}_P, \\
K_{11} &= \overline{G}_{P_2 P_1} + 2\overline{E}_1 + 2\overline{L}_P + \overline{P}_3 + 2\overline{F}_1 + 2\overline{P}_2 + 3\overline{G}_P, \\
K_{12} &= \overline{G}_{P_1 P_1} + 2\overline{E}_1 + 2\overline{L}_P + \overline{P}_3 + \overline{F}_1 + 2\overline{P}_2 + 3\overline{G}_P,
\end{align*}
\]

where $\overline{G}_{P_i P_j}$, $\overline{E}_P$, $\overline{P}_P$, $\overline{P}_P$, $\overline{E}_P$, $\overline{F}_P$, $\overline{G}_P$, $\overline{G}_P$ are the reduced divisors supported $\varphi^*(G_{P_i P_j})$, $\varphi^*(G_1)$, $\varphi^*(E_{P_3 P_2})$, $\varphi^*(E_{P_3 P_2})$, $\varphi^*(E_{P_1 P_2})$, $\varphi^*(E_{P_1 P_2})$, $\varphi^*(E_{P_1 P_2})$, $\varphi^*(E_{P_1 P_2})$, $\varphi^*(E_{P_1 P_2})$, respectively.

The intersections $\overline{G}_{P_i P_j} \cap \overline{P}_P$, $\overline{E}_P \cap \overline{P}_P$, $\overline{P}_P \cap \overline{P}_P$, $\overline{E}_P \cap \overline{P}_P$, $\overline{P}_P \cap \overline{P}_P$, $\overline{F}_P \cap \overline{P}_P$, $\overline{P}_P \cap \overline{P}_P$ are the all base points of $|K_X|$. So we obtain that $d = K_X^2 = 7$.

### 3.3 Construction of the surface with $d = 18$.

Let $Y$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point $P$. We denote by

- $F, G$: the pull-back of general fibers $f, g$, respectively of $\mathbb{P}^1 \times \mathbb{P}^1$,
- $E_P$: the exceptional divisor,
- $F_P, G_P$: the strict transform of the fibers $f, g$ through $P$, respectively.

The canonical class $K_Y \equiv -2F - 2G + E_P$. We consider the following smooth divisors of $Y$:

$$D_{01} := G_1 + G_2, D_{02} := G_3, D_{20} := F_1 + F_P, D_{22} := C_1 + E_P, D_{21} := C_2 + G_P$$

and $D_{ij} = 0$ for the other $(i, j)$, where $F_1 \in |F|, G_1, G_2, G_3 \in |G|, C_1 \in |F + G|, C_2 \in |F + G - E_P|$ are distinct divisors such that no more than two of these divisors go through the same point. We consider the following non-trivial divisors:

\[
\begin{align*}
L_{10} &= 2F + 2G - E_P, \\
L_{01} &= F + 3G - E_P, \\
L_{20} &= 2F + 2G - E_P, \\
L_{02} &= F + 3G, \\
L_{11} &= 2F + 2G, \\
L_{22} &= F + 2G, \\
L_{12} &= 2F + 3G - 2E_P, \\
L_{21} &= F + 2G - E_P.
\end{align*}
\]

These divisors $D_\sigma$, $L_X$ satisfy the following relations:

\[
\begin{align*}
3L_{10} &= 2D_{20} + D_{22} + D_{21} = 6F + 3G - 3E_P, \\
3L_{01} &= D_{01} + 2D_{02} + D_{22} + 2D_{21} = 3F + 9G - 3E_P, \\
3L_{20} &= D_{20} + 2D_{22} + 2D_{21} = 6F + 6G - 3E_P, \\
3L_{02} &= 2D_{01} + D_{02} + 2D_{22} + D_{21} = 3F + 9G, \\
3L_{11} &= D_{01} + 2D_{02} + 2D_{20} + 2D_{22} = 6F + 6G, \\
3L_{22} &= 2D_{01} + D_{02} + D_{20} + 2D_{22} = 3F + 6G, \\
3L_{12} &= 2D_{01} + D_{02} + 2D_{20} + 2D_{21} = 6F + 9G - 6E_P, \\
3L_{21} &= D_{01} + 2D_{02} + 2D_{20} + D_{21} = 3F + 6G - 3E_P.
\end{align*}
\]

Thus by Proposition 1, the divisors $D_\sigma$, $L_X$ define a $\mathbb{Z}_3^2$-cover $\varphi : X \longrightarrow Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^*(2F + 6G - E_P).$$

The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal and of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = (2F + 6G - E_P)^2 = 23,$$

$$p_g(X) = 3, \chi(O_X) = 4, q(X) = 0.$$
By Proposition (4), the linear system \(|K_X|\) is generated by the three following divisors:

\[
\begin{align*}
K_{20} &= 2R_0 + 2R_2 + R_{20}, \\
K_{11} &= R_0 + 2R_2 + \varphi^*(E_P), \\
K_{12} &= R_2 + 2R_2 + \varphi^*(G_P),
\end{align*}
\]

where \(R_{\sigma}\) are the reduced divisors supported \(\varphi^*(D_{\sigma})\), for all \(\sigma\). We have that

\[
\begin{align*}
K_{20} &= 2G_1 + 2G_2 + 2G_3 + F_1 + F_P, \\
K_{11} &= G_1 + G_2 + G_3 + 3E_P + 2E_2 + 2G_P, \\
K_{12} &= G_3 + 2G_1 + 2G_P + 3G_P,
\end{align*}
\]

where \(G_1, G_2, G_3, F_1, F_P, G_1, E_P, G_2, G_P\) are the reduced divisors supported \(\varphi^*(G_1), \varphi^*(G_2), \varphi^*(G_3), \varphi^*(F_1), \varphi^*(F_P), \varphi^*(C_1), \varphi^*(E_P), \varphi^*(C_2), \varphi^*(G_P)\), respectively.

The intersections \(G_1 \cap G_2, G_2 \cap G_1, G_3 \cap G_2, E_P \cap F_P, G_P \cap F_P\) are all base points of \(|K_X|\). So we obtain that \(d = K_X^2 = 18\).

### 3.4 Construction of the surface with \(d = 17\).

Let \(Y^\prime\) be the blow-up of \(\mathbb{P}^1 \times \mathbb{P}^1\) at two distinct points \(P_1, P_2\). We denote by

\[
\begin{align*}
F, G: & \quad \text{the pull-back of general fibers } f, g, \text{ respectively in } \mathbb{P}^1 \times \mathbb{P}^1, \\
E_{P_1}, E_{P_2}: & \quad \text{the exceptional divisors corresponding to } P_1, P_2, \text{ respectively,} \\
F_{P_1}, F_{P_2}: & \quad \text{the strict transform of the fibers } f \text{ through } P_1, P_2, \text{ respectively,} \\
G_{P_1}, G_{P_2}: & \quad \text{the strict transform of the fibers } g \text{ through } P_2, \text{ respectively.}
\end{align*}
\]

The canonical class \(K_{Y^\prime} = -2F - 2G + E_{P_1} + E_{P_2}\). Let \(Y\) be the blow-up of \(Y^\prime\) at the intersection \(P^\prime_2 := G_{P_2} \cap E_{P_2}\). By the abuse of notation, we denote by

\[
\begin{align*}
F, G, E_{P_1}, F_{P_1}, E_{P_2}, F_{P_2}, E_{P_3}, G_{P_1}, G_{P_2}, G_{P_3}: & \quad \text{the pull-back of } F, G, E_{P_1}, F_{P_1}, F_{P_2}, G_{P_1}, \text{ respectively,} \\
E_{P_2}, E_{P_2}': & \quad \text{the exceptional divisor,} \\
E_{P_2}, E_{P_2}': & \quad \text{the strict transform of } E_{P_2}, G_{P_2}, G_{P_2}', \text{ respectively.}
\end{align*}
\]

The canonical class \(K_Y = -2F - 2G + E_{P_1} + E_{P_2} + 2E_{P_2}\). We consider the following smooth divisors of \(Y\):

\[
\begin{align*}
D_{01} := G_1 + G_{P_2}P_2, \\
D_{02} := G_2 + E_{P_3}P_2, \\
D_{20} := F_{P_2} + F_{P_1}, \\
D_{22} := C_1 + E_{P_1}, \\
D_{21} := C_2 + G_{P_1},
\end{align*}
\]

and \(D_{ij} = 0\) for the other \((i, j)\), where \(G_1, G_2 \in |G|, C_1 \in |F + G - E_{P_3}P_2 - E_{P_2}'|, C_2 \in |F + G - E_{P_1}|\) are distinct divisors such that no more than two of these divisors go through the same point.

We consider the following non-trivial divisors:

\[
\begin{align*}
L_{10} := & \quad 2F + G - E_{P_1} - E_{P_2} - E_{P_2}' \\
L_{01} := & \quad F + 3G - E_{P_1} \\
L_{20} := & \quad 2F + 2G - E_{P_1} - E_{P_2} - E_{P_2}' \\
L_{02} := & \quad F + 3G - E_{P_2} - 2E_{P_2}' \\
L_{11} := & \quad 2F + 2G - E_{P_1} - E_{P_2} - 2E_{P_2}' \\
L_{22} := & \quad F + 2G - E_{P_1} - 2E_{P_1} - 2E_{P_2} \\
L_{12} := & \quad 2F + 3G - 2E_{P_2} - 2E_{P_2}' - 2E_{P_2} \\
L_{21} := & \quad F + 2G - E_{P_1} - E_{P_2} - 2E_{P_2}'.
\end{align*}
\]

These divisors \(D_{\sigma}, L_{\chi}\) satisfy the following relations:

\[
\begin{align*}
3L_{10} & \equiv 2D_{20} + D_{22} + D_{21} \equiv 6F + 3G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2}' \\
3L_{01} & \equiv D_{01} + 2D_{02} + D_{22} + 2D_{21} \equiv 3F + 9G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2}' \\
3L_{20} & \equiv D_{20} + 2D_{22} + 2D_{21} \equiv 6F + 6G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2}' \\
3L_{02} & \equiv 2D_{01} + D_{02} + D_{21} \equiv 3F + 9G - 3E_{P_1} - 3E_{P_2} - 6E_{P_2}' \\
3L_{11} & \equiv D_{01} + 2D_{02} + 2D_{20} + 2D_{22} \equiv 6F + 9G - 6E_{P_1} - 3E_{P_2} - 6E_{P_2}' \\
3L_{22} & \equiv 2D_{01} + D_{02} + D_{20} + D_{22} \equiv 3F + 6G - 3E_{P_2} - 6E_{P_2}' \\
3L_{12} & \equiv 2D_{01} + D_{02} + 2D_{20} + 2D_{22} + 2D_{21} \equiv 6F + 9G - 6E_{P_1} - 3E_{P_2} - 6E_{P_2}' \\
3L_{21} & \equiv D_{01} + 2D_{02} + D_{20} + D_{21} \equiv 3F + 6G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2}'.
\end{align*}
\]
Thus by Proposition 1, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2$-cover $\varphi : X \dashrightarrow Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^* \left( 2F + 6G - E_{P_1} - E_{P_2} - 2E_{P'_2} \right).$$

The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal and of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = (2F + 6G - E_{P_1} - E_{P_2} - 2E_{P'_2})^2 = 21,$$

$p_g(X) = 3, \chi(O_X) = 4, q(X) = 0$.  

By Proposition (4), the linear system $|K_X|$ is generated by the three following divisors:

$$K_{20} := 2R_{01} + 2R_{02} + R_{20} + \varphi^*(E_{P'_2}),$$
$$K_{11} := R_{01} + 2R_{21} + \varphi^*(E_{P_1}),$$
$$K_{12} := R_{02} + 2R_{22} + \varphi^*(G_{P_1}),$$

where $R_{0\sigma}$ are the reduced divisors supported $\varphi^*(D_\sigma)$, for all $\sigma$. We have that

$$K_{20} = \begin{cases} 2\sigma_1 + 2\sigma_{P_2}, & \varphi^*(E_{P'_2}) \\ G_{1} + 2\sigma_{P_1} & + 2E_{P_2}, \varphi^*(E_{P'_1}) \\ \sigma_2 + 2E_{P_2}, \varphi^*(E_{P'_1}) \\ \sigma_2 + 2E_{P_2}, \varphi^*(E_{P'_1}) \end{cases}$$

where $\overline{M}$ is denoted the reduced divisors supported $\psi^*(M)$ for divisor $M$ in $Y_{11}$. 

The intersections $\overline{C}_1 \cap \overline{C}_2, \overline{C}_2 \cap \overline{C}_1, E_{P_1} \cap F_{P_1}, C_{P_2} \cap F_{P_2}$ are the all base points of $|K_X|$. So we obtain that $d = K_X^2 = 4 = 17$. 

### 3.5 Construction of the surface with $d = 15$. 

Let $Y$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point $P$. We denote by

- $F, G$: the pull-back of general fibers $f, g$, respectively in $\mathbb{P}^1 \times \mathbb{P}^1$,
- $E_P$: the exceptional divisor,
- $F_P, G_P$: the strict transform of the fibers $f, g$ through $P$, respectively.

The canonical class $K_Y \equiv -2F - 2G + E_P$. We consider the following smooth divisors of $Y$:

$$D_{01} := G_1 + G_2, D_{02} := G_3 + E_P, D_{20} := F_1 + F_2, D_{22} := C_1, D_{21} := C_2 + G_P$$

and $D_{ij} = 0$ for the other $(i, j)$, where $F_1, F_2 \in |F|$, $G_1, G_2, G_3 \in |G|, C_1, C_2 \in |F + G - E_P|$ are distinct divisors such that no more than two of these divisors go through the same point. We consider the following non-trivial divisors:

$$L_{10} := 2F + G - E_P, \quad L_{01} := F + 3G - E_P, \quad L_{20} := 2F + 2G - E_P, \quad L_{02} := F + 3G - E_P, \quad L_{11} := 2F + 2G, \quad L_{22} := F + 2G, \quad L_{12} := 2F + 3G - E_P, \quad L_{21} := F + 2G.$$ 

These divisors $D_\sigma, L_X$ satisfy the following relations:

$$
\begin{align*}
3L_{10} & \equiv \quad 3L_{01} + 2D_{02} + 2D_{20} + D_{22} + D_{21} & \equiv 6F + 3G - 3E_P \\
3L_{20} & \equiv \quad 3L_{02} + D_{02} + D_{20} + D_{22} + D_{21} & \equiv 3F + 9G - 3E_P \\
3L_{02} & \equiv \quad 2D_{01} + D_{02} + 2D_{02} + 2D_{20} + 2D_{22} & \equiv 6F + 6G \\
3L_{22} & \equiv \quad 2D_{01} + D_{02} + 2D_{02} + 2D_{20} + 2D_{22} & \equiv 3F + 6G \\
3L_{12} & \equiv \quad 2D_{01} + D_{02} + 2D_{02} + 2D_{20} + 2D_{22} & \equiv 6F + 9G - 3E_P \\
3L_{21} & \equiv \quad 2D_{01} + D_{02} + 2D_{02} + 2D_{20} + 2D_{22} & \equiv 3F + 6G.
\end{align*}
$$

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Thus by Proposition 1, the divisors $D_2, L_X$ define a $\mathbb{Z}_2$-cover $\varphi : X \to Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^* (2F + 6G - E_P).$$

The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal and of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = (2F + 6G - E_P)^2 = 23,$$

$$p_g (X) = 3, \chi (\mathcal{O}_X) = 4, q (X) = 0.$$

By (4), the linear system $|K_X|$ is generated by the three following divisors:

$$K_{11} := R_{01} + 2R_{21} + \varphi^* (E_P),$$

$$K_{12} := R_{02} + 2R_{22} + \varphi^* (G_1),$$

$$K'_{12} := R_{02} + 2R_{22} + \varphi^* (G_5),$$

where $R_\sigma$ are the reduced divisors supported $\varphi^* (D_\sigma)$, for all $\sigma$ and $G_4, G_5 \in |G|$ which are different from $G_1, G_2, G_3$. We have that

$$K_{11} = \overline{G}_1 + \overline{G}_2 + 3\overline{E}_P + 2\overline{C}_2 + 2\overline{G}_P + \varphi^* (G_4)$$

$$K_{12} = \overline{C}_3 + \overline{E}_P + 2\overline{C}_1 + \varphi^* (G_4)$$

$$K'_{12} = \overline{C}_3 + \overline{E}_P + 2\overline{C}_1 + \varphi^* (G_5)$$

where $\overline{G}_1, \overline{G}_2, \overline{G}_3, \overline{E}_P, \overline{C}_1, \overline{C}_2, \overline{C}_3$ are the reduced divisors supported $\varphi^* (G_1), \varphi^* (G_2), \varphi^* (G_3), \varphi^* (E_P), \varphi^* (C_1), \varphi^* (C_2), \varphi^* (G_5)$, respectively. Since the divisor $\overline{E}_P$ is the common component of the three generators of the linear system $|K_X|$, the divisor $\overline{E}_P$ is the fixed part of $|K_X|$. Thus the moving part $|M|$ of $|K_X|$ is generated by the following divisors:

$$M_{11} := \overline{G}_1 + \overline{G}_2 + 2\overline{E}_P + 2\overline{C}_2 + 2\overline{G}_P$$

$$M_{12} := \overline{G}_3 + 2\overline{C}_1 + 2\overline{C}_2 + 2\overline{C}_1$$

$$M'_{12} := \overline{G}_3 + 2\overline{C}_1 + 2\overline{C}_2 + 2\overline{G}_P$$

We have that

$$M^2 = M^2_{11} = M^2_{12} = M^2_{12}' = 20.$$
and $D_{10} = 0$, where $F_i \in |F|$, $G_1, G_2, G_3 \in |G|$, $C \in |F + G - E_{P_2} - E_{P_2'}|$ are distinct divisors such that no more than two of these divisors go through the same point. We consider the following non-trivial divisors:

\[
\begin{align*}
L_{10} &:= 2F + G - E_{P_1} - E_{P_2} - E_{P_2'} \\
L_{01} &:= F + 2G - E_{P_1} - E_{P_2'} \\
L_{20} &:= 2F + G - E_{P_1} - E_{P_2} - E_{P_2'} \\
L_{02} &:= F + 4G - E_{P_1} - E_{P_2'} - 2E_{P_2'} \\
L_{11} &:= 2F + 2G - E_{P_2} - 2E_{P_2'} \\
L_{22} &:= F + 3G - E_{P_1} - E_{P_2} - 2E_{P_2'} \\
L_{12} &:= 2F + 3G - E_{P_1} - E_{P_2} - 2E_{P_2'} \\
L_{21} &:= F + 2G - E_{P_1} - E_{P_2}.
\end{align*}
\]

These divisors $D_\sigma$, $L_X$ satisfy the following relations:

\[
\begin{align*}
3L_{10} &\equiv 2D_{20} + D_{22} + D_{21} \equiv 6F + 3G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{01} &\equiv D_{01} + 2D_{02} \equiv 3F + 6G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{20} &\equiv D_{20} + 2D_{22} + 2D_{21} \equiv 6F + 3G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{02} &\equiv 2D_{02} + D_{02} + 2D_{22} + D_{21} \equiv 3F + 12G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{11} &\equiv D_{01} + 2D_{02} + 2D_{20} + 2D_{22} \equiv 6F + 6G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{22} &\equiv D_{01} + 2D_{02} + D_{20} + D_{22} \equiv 3F + 9G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{12} &\equiv 2D_{01} + D_{02} + 2D_{20} + 2D_{22} \equiv 6F + 9G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'} \\
3L_{21} &\equiv D_{01} + 2D_{02} + D_{20} + D_{21} \equiv 3F + 6G - 3E_{P_1} - 3E_{P_2} - 3E_{P_2'}.
\end{align*}
\]

Thus by Proposition 1, the divisors $D_\sigma, L_X$ define a $\mathbb{Z}_2^3$-cover $\varphi : X \to Y$. Moreover, by Propositions 2 and 3, the surface $X$ is smooth and satisfies the following:

$$3K_X \equiv \varphi^* (2F + 6G - E_{P_1} - E_{P_2} - E_{P_2'}) .$$

The canonical divisor $K_X$ is big and nef since the divisor $3K_X$ is the pull-back of a big and nef divisor. Thus, the surface $X$ is minimal and of general type. Furthermore, by Proposition 3, the surface $X$ possesses the following invariants:

$$K_X^2 = (2F + 6G - E_{P_1} - E_{P_2} - E_{P_2'})^2 = 21,$$

$$p_g(X) = 3, \chi(O_X) = 4, q(X) = 0.$$

By (4), the linear system $|K_X|$ is generated by the three following divisors:

$$K_{11} := R_{01} + 2R_{12} + R_{21} + \varphi^* (E_{P_1}) ,$$
$$K_{12} := R_{02} + 2R_{22} + R_{12} + \varphi^* (G_4),$$
$$K_{12} := R_{02} + 2R_{22} + R_{12} + \varphi^* (G_5),$$

where $R_\sigma$ are the reduced divisors supported $\varphi^* (D_\sigma)$, for all $\sigma$ and $G_4, G_5 \in |G|$ which are different from $G_1, G_2, G_3$. We have that

\[
\begin{align*}
K_{11} &= G_1 + G_2 + G_3 + \overline{G}_{P_2}, \\
K_{12} &= E_{P_2} P_1 + 2C + \overline{G}_{P_1}, \\
K_{12} &= \overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \overline{P}_4 + \overline{P}_5 + \varphi^* (E_{P_1}) + \varphi^* (G_4) + \varphi^* (G_5),
\end{align*}
\]

where $G_1, G_2, G_3, G_{P_2}, \overline{G}_{P_2}, \overline{C}, \overline{G}_{P_1}, \overline{P}_1$ are the reduced divisors supported $\varphi^* (G_1)$, $\varphi^* (G_2)$, $\varphi^* (G_3)$, $\varphi^* (G_{P_2})$, $\varphi^* (E_{P_2})$, $\varphi^* (C)$, $\varphi^* (G_4)$, $\varphi^* (G_5)$. Since the divisor $\overline{P}_1$ is the common component of the three generators of the linear system $|K_X|$, the divisor $\overline{P}_1$ is the fixed part of $|K_X|$. Thus the moving part $|M|$ of $|K_X|$ is generated by the following divisors:

\[
\begin{align*}
M_{11} &= \overline{G}_1 + \overline{G}_2 + \overline{G}_3 + \overline{G}_{P_2}, \\
M_{12} &= \overline{E}_{P_2} + \overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \varphi^* (E_{P_1}) + \varphi^* (G_4) + \varphi^* (G_5),
\end{align*}
\]

We have that

$$M^2 = M_{11}^2 = M_{12}^2 = 18.$$
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Mathematics Division,
National Center for Theoretical Sciences,
National Taiwan University,
Taiwan.
E-mail address: nguyenbin@ncts.ntu.edu.tw

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