Lattices and hypergraphs associated to square-free monomial ideals

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ABSTRACT
Given a square-free monomial ideal \( I \) in a polynomial ring \( R \) over a field \( K \), one can associate it with its LCM-lattice and its hypergraph. In this short note, we establish the connection between the LCM-lattice and the hypergraph, and in doing so we provide a sufficient condition for removing higher dimension edges of the hypergraph without impacting the projective dimension of the square-free monomial ideal. We also offer algorithms to compute the projective dimension of a class of square-free monomial ideals built using the new result and previous results of Lin-Mantero.

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1. Introduction
Finding the projective dimension or the Castelnuovo-Mumford regularity of a homogeneous ideal, \( I \), in a graded polynomial ring \( R = K[x_1, \ldots, x_n] \) over a field \( K \) has been an active research field over the last decades. See, for example, the survey papers [5] and [15]. These two invariants give important information about the ideal, and they measure the complexity of the ideal. Moreover, they play important roles in algebraic geometry, commutative algebra, and combinatorial algebra. In general, one finds the graded minimal free resolution of an ideal to obtain those invariants, but the computation can be difficult and computationally expensive. Alternatively one can try finding bounds for these invariants using properties of the ideal. Studying monomial ideals and specifically square-free monomial ideals is important in this strategy. In particular, it is well known that the regularity of a given ideal is bounded by the regularity of its initial ideal (see for example Theorem 22.9 [16].) The polarization of a monomial ideal does not change its projective dimension (see for example Theorem 21.10 [16]), hence one may use square-free monomial ideals to understand projective dimensions of monomial ideals in general. Finally, when \( I \) is a square-free monomial ideal, there is a dual relation between the projective dimension and the regularity with respect to the Alexander dual [19]. Thus, finding the projective dimension of a square-free monomial ideal is a central problem in this field, see for instance [3]. Additionally, one can use the projective dimension to decide whether an ideal is Cohen-Macaulay.

This paper focuses on using two combinatorial objects associated to a square-free monomial ideal in place of the minimal free resolution: the dual hypergraph and the LCM-lattice. Kimura, Terai, and Yoshida define the dual hypergraph of a square-free monomial ideal in order to

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compute its arithmetical rank [10]. Since then, there are papers using this combinatorial object to study various properties, for example, [6] and [13]. In particular, Lin and Mantero use it to show that ideals with the same dual hypergraph have the same total Betti numbers and projective dimension [11], which has found use in other papers, such as in [8].

For the second combinatorial object, Gasharov, Peeva, and Welker define the LCM-lattice of a monomial ideal. They show that if there is a map between two LCM-lattices which is a bijection on the atoms and preserves joins, then a resolution of an ideal in the domain is the resolution of the other with respect to the map, that is, when the map is an isomorphism those two ideals have the same total Betti numbers and projective dimension [4]. Phan and Mapes show that every finite atomic lattice is the LCM-lattice of a monomial ideal via a special construction, [14] and [17]. It is natural to inquire if there is a connection between the dual hypergraph and the LCM-lattice of a given square-free monomial ideal. The positive answer is one of the first results in this paper (Theorem 4.2). Specifically, one can construct the LCM-lattice of a monomial ideal via its dual hypergraph and vice versa as shown in Section 4.

The results in [11] and [12] focus mostly on determining the projective dimension when the dual hypergraph of an ideal consists only of vertices and edges with cardinality 2 (i.e. the dual hypergraph is “1-dimensional”). Moreover, the work of Kimura, Rinaldo, and Terai shows that the projective dimension of a monomial ideal depends on the 1-skeleton structure of the dual hypergraph [9]. It is clear that sometimes a higher dimensional edge of a dual hypergraph can be removed without impacting the projective dimension of a monomial ideal. This paper focuses mostly on the question: Under what conditions can one remove higher dimensional edges without changing the projective dimension of a hypergraph? The work by Lin and Mantero answers part of this question [12] with some restrictions on the 1-skeleton of the hypergraph. In this paper, using the connection to LCM-lattices, we show a sufficient condition for when removing the higher dimensional edge has no impact on total Betti numbers and hence the projective dimension (Corollary 4.4). We explain why the work of Kimura, Rinaldo, and Terai shows that the projective dimension of a monomial ideal depends on the 1-skeleton structure of the dual hypergraph [9] and explain the result in Lin and Mantero in a combinatorial construction (Remark 4.8). We then proceed with our results concerning higher dimensional edges on bushes in Section 5. In the end, we provide algorithms for computing the projective dimension of certain square-free monomial ideals using hypergraphs without the computation of the minimal free resolution of the square-free monomials. Throughout this paper, ideals are square-free monomial ideals in a polynomial ring \( R \) over the field \( \mathbb{K} \).

### 2. Lattices and LCM-lattices

A lattice is a set \((P, <)\) with an order relation \(<\), which is transitive and antisymmetric satisfying the following properties:

1. \( P \) has a maximum element denoted by \( \hat{1} \).
2. \( P \) has a minimum element denoted by \( \hat{0} \).
3. Every pair of elements \( a \) and \( b \) in \( P \) has a join \( a \lor b \), which is the least upper bound of the two elements.
4. Every pair of elements \( a \) and \( b \) in \( P \) has a meet \( a \land b \), which is the greatest lower bound of the two elements.

We define an atom of a lattice \( P \) to be an element \( x \in P \) such that \( x \) covers \( \hat{0} \) (i.e. \( x > \hat{0} \) and there is no element \( a \) such that \( x > a > \hat{0} \)). We will denote the set of atoms as \( \text{atoms}(P) \).

**Definition 2.1.** If \( P \) is a lattice and every element in \( P - \{ \hat{0} \} \) is the join of atoms, then \( P \) is an atomic lattice. Further, if \( P \) is finite, then it is a finite atomic lattice.
Given a lattice \( P \), an element \( x \in P \) is *meet-irreducible* if \( x \neq a \land b \) for any \( a > x, b > x \). The set of meet-irreducible elements in \( P \) is denoted by \( \text{mi}(P) \). Given an element \( x \in P \), the *filter* of \( x \) is \( [x] = \{ a \in P | x \leq a \} \).

**Remark 2.2.** Lemma 2.3 in [14] guarantees that if \( P \) is a finite atomic lattice, then every element \( p \) in \( P - \{ 1 \} \) is the meet of all the meet-irreducible elements greater than \( p \).

For the purposes of this paper, it will often be convenient to consider finite atomic lattices as sets of sets in the following way. Let \( S \) be a set of subsets of \( \{ 1, ..., n \} \), closed under intersections, and containing the entire set, the empty set, and the sets \( \{ i \} \) for all \( 1 \leq i \leq n \). Then it is easy to see \( S \) is a finite atomic lattice by ordering the sets in \( S \) by inclusion. This set obviously has a minimal element, a maximal element, and \( n \) atoms. By [18, Proposition 3.3.1], we need to show that it is a meet-semilattice. Here the meet of two elements would be defined to be their intersection. Since \( S \) is closed under intersections, this is a meet-semilattice. Conversely, it is clear that all finite atomic lattices can be expressed in this way, simply by letting

\[
S_P = \{ \sigma \mid \sigma = \text{supp}(p), p \in P \},
\]

where \( \text{supp}(p) = \{ a_i \mid a_i \leq p, a_i \in \text{atoms}(P) \} \).

The only poset that we are interested in this paper is the LCM-lattice of a monomial ideal. As a poset, the LCM-lattice of \( I \), typically denoted as \( L_I \), is the set of all least common multiples of subsets of monomial generators of \( I \) partially ordered by divisibility. It is easy to show that the LCM-lattice is in fact a finite atomic lattice.

### 2.1. Coordinatizations of LCM-lattices

LCM-lattices became important in the study of resolutions of monomial ideals in the paper by Gasharov, Peeva, and Welker [4]. In Theorem 3.3, they show that given monomial ideals \( I \) and \( I' \) in polynomial rings \( R \) and \( R' \) with LCM-lattices \( L \) and \( L' \), respectively, if there is a join-preserving map \( f : L \to L' \) which is a bijection on atoms then a minimal resolution of \( R/I \) can be relabeled to be a resolution of \( R'/I' \). In Theorem 2.1, they observe that if \( f \) is an isomorphism then the relabeled resolution is a minimal resolution of \( R'/I' \). Those two primary results will be important to this work.

Continuing this study, one of the main results (Theorem 5.1) of [17] showed that every finite atomic lattice is in fact the LCM-lattice of a monomial ideal. This result was generalized by a modified construction in [14], which also showed that all monomial ideals can be realized this way. We include a brief description of this work here for the convenience of the reader.

Define a *labeling* of a finite atomic lattice \( P \) as any assignment of non-trivial monomials \( M = \{ m_{p_1}, ..., m_{p_t} \} \) to some elements \( p_i \in P \). It will be convenient to think of unlabeled elements as having the label 1. Define the monomial ideal \( M_M \) to be the ideal generated by monomials

\[
x(a) = \prod_{p \in [a]^c} m_p
\]

for each \( a \in \text{atoms}(P) \) where \( [a]^c \) means take the complement of \( [a] \) in \( P \). We say that the labeling \( M \) is a *coordinatization* if the LCM-lattice of \( M_M \) is isomorphic to \( P \).

The following theorem, which is Theorem 3.2 in [14], gives a criterion for when a labeling is a coordinatization.

**Theorem 2.3.** Any labeling \( M \) of elements in a finite atomic lattice \( P \) by monomials satisfying the following two conditions will yield a coordinatization of \( P \).

(C1) If \( p \in \text{mi}(P) \), then \( m_p \neq 1 \). (i.e. all meet-irreducibles are labeled.)
(C2) If \( \gcd(m_p, m_q) \neq 1 \) for some \( p, q \in P \), then \( p \) and \( q \) must be comparable. (i.e. each variable only appears in monomials along one chain in \( P \).)

Example 2.4. In Figure 1 we see an example of a poset \( P \) with a labeling on the vertices. We can see that this labeling satisfies both conditions of Theorem 2.3 and so one can check that the corresponding monomial ideal \( I_1 = (bce, abc, acd, abd) \) has \( P \) as its LCM-lattice. Note that this ideal is square-free and only the meet irreducible elements are labeled, this is an example of a “minimal monomial ideal associated to \( P \)” as defined in [17].

Using the more general construction stated here though we can also obtain other monomial ideals with the same lcm-lattice. The labeling given in Figure 2 would produce the monomial ideal \( I_2 = (bce, abc, a^2c, a^2b) \), and the labeling in Figure 3 would yield the ideal \( I_3 = (bce, abc, acd, abd) \). In what follows we will be interested in the relationship between the ideal produced in Figure 1 and the one in Figure 3.

3. Hypergraph of a square-free monomial ideal

Kimura, Terai, and Yoshida associate a square-free monomial ideal with a hypergraph in [10]; see Definition 3.1. Note that this construction is different from the construction, associating ideals with hypergraphs, which is extended from the study of edge ideals. In particular relative to edge ideals, the hypergraph of Kimura, Terai, and Yoshida might be more aptly named the “dual hypergraph”. The construction of dual hypergraphs is first introduced by Berge in [1]. In the edge ideal case, one associates a square-free monomial with a hypergraph by setting variables as vertices and each monomial corresponds to an edge of the hypergraph (see for example [5]). In the following definition, we actually associate variables with edges of the hypergraph and vertices with the monomial generators of the ideal, and in practice, this is the dual hypergraph of the hypergraph in the edge ideal construction.

**Definition 3.1.** Let \( I \) be a square-free monomial ideal in a polynomial ring with \( n \) variables with minimal monomial generating set \( \{m_1, ..., m_\mu\} \). Let \( V \) be the set \( \{1, ..., \mu\} \). We define \( \mathcal{H}(I) \) (or \( \mathcal{H} \) when \( I \) is understood) to be the hypergraph associated to \( I \) which is defined as \( \{\{j \in V : x_i \mid m_j\} : i = 1, 2, ..., n\} \). We call the sets \( \{j \in V : x_i \mid m_j\} \) the edges of the hypergraph.

Note that if you start with a hypergraph you can create a monomial ideal by assigning a variable to each edge, then each vertex (or element in \( V \)) would be assigned the monomial product of the variables corresponding to the edges using that vertex. The issue however that doing this will not always produce a minimal generating set. To obtain a minimal generating set the hypergraph needs to be separated, where \( \mathcal{H} \) is separated if in addition for every \( 1 \leq j_1 < j_2 \leq \mu, \) there

![Figure 1](image-url) The LCM-lattice of \( I_1 \) in Example 2.4.
exist edges $F_1$ and $F_2$ in $\mathcal{H}$ so that $j_1 \in F_1 \cap (V - F_2)$ and $j_2 \in F_2 \cap (V - F_1)$. All hypergraphs in this paper will be separated unless otherwise stated.

**Example 3.2.** Let

$I = (abo, bcp, cdepq, efgr, ghr, hijoq, jk, kl, lmo, mn) = (m_1, m_2, ..., m_{11})$

Figure 4 is the hypergraph, $\mathcal{H}(I)$, associated to $I$ via Definition 3.1 where

$$\mathcal{H}(I) = \{(1) = F_a, \{1, 2\} = F_b, \{2, 3\} = F_c, \{3\} = F_d, ..., \{10, 11\} = F_m, \{11\} = n,$$

$$\{1, 4, 7, 10\} = F_o, \{2, 3, 9\} = F_p, \{3, 7\} = F_q, \{4, 5, 6\} = F_r\}.$$

We notice that the edge corresponding to the variable $q$ or the variable $r$ is a union of 2 or more distinct edges.

Some important terminology regarding these hypergraphs is the following. We say a vertex $i \in V$ of $\mathcal{H}$ is an open vertex if $\{i\}$ is not in $\mathcal{H}$, and otherwise, $i$ is closed. In Figure 4, we can see that the vertices labeled by $a, d, i, n$ are all closed, and the rest are open. Let $\mathcal{H}^i = \{F \in H : |F| \leq i + 1\}$ denote the $i$-dimensional subhypergraph of $\mathcal{H}$ where $|F|$ is the cardinality of the $F$. We call $\mathcal{H}^1$, the 1-skeleton of $\mathcal{H}$. We say $\mathcal{H}$ has a spanning 1-skeleton if $V(\mathcal{H}) = V(\mathcal{H})$. Figure 4 has a spanning 1-skeleton.

Let $R = \mathbb{K}[x_1, ..., x_n]$ be a polynomial ring over a field $\mathbb{K}$. The minimal free resolution of $R/I$ for a graded ideal $I \subset R$ is an exact sequence of the form

$$0 \rightarrow \bigoplus_j S(-j)^{\beta_{i,j}(R/I)} \rightarrow \cdots \rightarrow \bigoplus_j S(-j)^{\beta_{i,j}(R/I)} \rightarrow R \rightarrow R/I \rightarrow 0$$

The exponents $\beta_{i,j}(R/I)$ are invariants of $R/I$, called the graded Betti numbers of $R/I$. In general, finding Betti numbers is still a wide-open question. This project focuses on studying the projective dimension of $R/I$, denoted $pd(R/I)$, which is defined as follows.
Recently there have been a number of results concerning determining the projective dimension of square-free monomial ideals from the associated hypergraph. The proposition below allows us to talk about the projective dimension of a hypergraph rather than an ideal.

**Proposition 3.3 ([11, Proposition 2.2]).** If $I_1$ and $I_2$ are square-free monomial ideals associated to the same separated hypergraph $\mathcal{H}$, then the total Betti numbers of the two ideals coincide.

From now on, we will use $\text{pd}(R/I)$ in the place of $\text{pd}(\mathcal{H}(I))$ throughout the paper. If a hypergraph $\mathcal{H} = \mathcal{H}(I)$ is an union of two disconnected hypergraphs $G_1 = \mathcal{H}(I_1)$ and $G_2 = \mathcal{H}(I_2)$, we have $\text{pd}(\mathcal{H}) = \text{pd}(G_1) + \text{pd}(G_2)$ as one can construct the minimal resolution of $R/I$ using the tensor of minimal resolutions of $R/I_1$ and $R/I_2$.

### 4. Connection between dual hypergraphs and LCM-lattices

In this section, we want to show that one can re-build the LCM-lattice $L_I$ of the monomial ideal $I$ from $\mathcal{H}(I)$. Moreover, in order to do so, we will need to prove an important result that will allow us to detect meet-irreducible elements of the LCM-lattice from the hypergraph itself.

First let us define a finite atomic lattice from a given hypergraph $\mathcal{H}$. Thinking of a finite atomic lattice as a set of sets, define $L_{\mathcal{H}}$ to be the meet-closure (or intersection-closure) of the set $\{F^c \mid F \in \mathcal{H}\}$, where $F^c$ means taking the complement of each edge of $\mathcal{H}$ in the set of vertices of $\mathcal{H}$. This meet-closure will be a meet-semilattice (partially ordered by inclusion). In order to make it a lattice, we add the set of all the vertices of $\mathcal{H}$ (i.e. a maximal element).

Our claim is that $L_{\mathcal{H}(I)} = L_I$, and we will prove this by constructing a coordinatization of $L_{\mathcal{H}(I)}$ that will produce the same monomial ideal $I$. First though, we need to identify which elements of $L_{\mathcal{H}(I)}$ are meet-irreducible.

Recall that a meet-irreducible of a finite atomic lattice $L$ is an element that is not the meet of any 2 elements. $\hat{1} \neq \sigma \in L$ is meet-irreducible if it is not the intersection of 2 (or more) subsets $\tau_1, ..., \tau_I$ of $L$ where none of these $\tau_i$ are $\sigma$. The way that we construct $L_{\mathcal{H}(I)}$ leads to the following description of the meet-irreducible elements as edges of $\mathcal{H}(I)$.

**Proposition 4.1.** If $F \in \mathcal{H}(I)$ is the union of 2 or more distinct edges of $\mathcal{H}(I)$, then the edge $F$ corresponds to an element which is a meet in $L_{\mathcal{H}(I)}$.

**Proof.** Suppose $F = \bigcup_{i=1}^t G_i$ where $G_i$ is also an edge of $\mathcal{H}(I)$. Then by De Morgan’s Laws, the corresponding elements in $L_I$ are $F^c = \bigcap_{i=1}^t G_i^c$ where the notation $F^c$ means complement in $\{1, ..., l\}$. In terms of thinking of $L_{\mathcal{H}(I)}$ as sets of sets closed under intersections, this says precisely that $F^c$ is the meet of $G_{i_1}^c, ..., G_{i_t}^c$. \(\square\)

Now we are ready to show that $L_{\mathcal{H}(I)}$ and $L_I$ are actually the same.
Theorem 4.2. If $I$ is a square-free monomial ideal, then $L_{H(I)} = L_I$.

Proof. We begin by constructing a coordinatization for $L_{H(I)}$, which will produce the ideal $I$ as follows. By Equation (2.1), we can see that if $F_i$ is an edge of $H(I)$ corresponding to the variable $x_i$, then in $L_{H(I)}$ we label the element \[
\bigvee_{j \in [n] \setminus j \not\in F_i} a_j = F_i^c
\]
(where $a_j$’s are the atoms of $L_{H(I)}$) with the variable $x_i$. Note that here the equality is a bit of an abuse of notation where on one side we are thinking of elements as joins of atoms and on the other side we are thinking of them as subsets of the vertex set.

Note that this labeling by definition will satisfy condition (C2) since each variable only gets used once, so it remains to consider what condition (C1) means in this case. Now, consider the fact that condition (C1) requires that all meet-irreducible elements of $L_{H(I)}$ are labeled. By Proposition 4.1, we have a precise description of the meet-irreducible elements as being a subset of the edges of $H(I)$. As we are labeling all elements of $H(I)$, condition (C1) is satisfied and so the labeling we have given is in fact a coordinatization.

Now if we can show that the coordinatization we produced gives the ideal $I$, then we will know the lattice we coordinated is the LCM-lattice of $I$, thus proving the theorem. By construction, the monomial associated to the atom $a_i$ will be the product of the variables corresponding to the edges that contain $i$, which is precisely the ideal $I$.

This relationship between the LCM-lattice and the hypergraph is best seen in the following example.

Example 4.3. This example gives the relationship between the LCM-lattice and the hypergraph.
Let $I = (ab, bcg, cdg, de, efg) = (m_1, m_2, m_3, m_4, m_5)$. Figure 5 is the hypergraph of the $I$ such that $H(I) = \{\{1\} = F_a, \{1, 2\} = F_b, \{2, 3\} = F_c, \{3, 4\} = F_d, \{4, 5\} = F_e, \{5\} = F_f, \{2, 3, 5\} = F_g\}$. We take the compliments of individual edges of $H(I)$, and obtain $\{\{2, 3, 4, 5\}, \{3, 4, 5\}, \{1, 4, 5\}, \{1, 2, 5\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 4\}\}$. We then take the closure under intersections of all edges to obtain $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 2\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{1, 2, 3\} = [5] \setminus \{F_e\}, \{1, 2, 5\} = [5] \setminus \{F_d\}, \{1, 4, 5\} = [5] \setminus \{F_c\}, \{2, 3, 4\}, \{3, 4, 5\} = [5] \setminus \{F_b\}, \{1, 2, 3, 4\} = [5] \setminus \{F_a\}, \{1, 2, 3, 4, 5\}\}$.

Figure 6 shows the LCM-lattice of $I$ and the connection.

Note that there can be numerous cases where $H(I) \neq H(I')$ but $L_I = L_{I'}$. In these cases the difference here between $H(I)$ and $H(I')$ has to be in the edges that do not correspond to meet-irreducibles. Proposition 4.1 determines which edges in $H(I)$ correspond to elements which are
not meet-irreducible in the corresponding $L_I$. We have the following result that is an extension of Proposition 3.3.

**Corollary 4.4.** Let $I_1$ and $I_2$ be square-free monomial ideals such that $\mathcal{H}(I_1) = \mathcal{H}(I_2) \cup F$ where $F \subseteq \mathcal{H}(I_1)$ is the union of 2 or more distinct edges of $\mathcal{H}(I_1)$, then the total Betti numbers of the two ideals coincide.

**Proof.** Combining Proposition 4.1 and Theorem 4.2 with the work of [4] and [14], we can see that removing edges $F$ which are the union of other edges preserves the LCM-lattice and thus preserves all the total Betti numbers. \qed

**Example 4.5.** Going back to Example 2.4 in Figures 1 and 3, we could see from the coordinizations that the variable $f$ was not needed to produce a monomial ideal with the given lcm-lattice. On the hypergraph side, the hypergraph of the ideal $I_1 = (bce, abc, adc, abd)$ is shown in Figure 7 and the hypergraph of the ideal $I_2 = (bcef, abc, acdf, abdf)$ is shown in Figure 8 with $F_f = \{1, 3, 4\}$ as the brick wall. We see that $F_f = \{1, 3, 4\} = \{1\} \cup \{3, 4\} = F_f \cup F_{g}$, and hence by Corollary 4.4, we can remove the edge $F_f$ when considering the projective dimension of the ideal.

**Example 4.6.** In Example 4.3, the edge corresponding to the variable $g$ is $\{2, 3, 5\}$ and it is a union of $\{2, 3\}$ and $\{5\}$. Hence by Proposition 4.1 and Theorem 4.2, the LCM-lattice of the ideal is the same after we remove the edge corresponding to the variable $g$. In other words by Corollary 4.4, the projective dimension of the hypergraph is the same as the projective dimension of the hypergraph after we remove the edge $\{2, 3, 5\}$. We can see from Figure 6 that $g$ does not correspond to a meet-irreducible of $L_I$.

**Example 4.7.** In Figure 4, the edge corresponding to the variable $q$ or the variable $r$ is a union of 2 or more distinct edges. Hence the projective dimension of the hypergraph is the same as the projective dimension of the hypergraph after we remove those two edges.

**Remark 4.8.** Using Corollary 4.4, we can see that in order to further extend the previous results on computing the projective dimension by using combinatorial formulas on $\mathcal{H}(I)$, we need only consider certain classes of hypergraphs that do not have edges which would be deemed irrelevant by Proposition 4.1. This class of ideals would include hypergraphs that are “union-free families” as defined in [2] but this definition only considers when one edge is the union of two other edges. A more general notion of “union-free” families also exists in the literature [7] but this notion excludes edges that are contained in unions of other edges and is not appropriate in our setting.

**Figure 6.** The LCM-lattice of Example 4.3.
This also explains the results of [9] and [11] where the authors only focus on 1-skeleton of the hypergraph, or more precisely, they focus on the induced subgraph on the open vertices of the 1-skeleton. In the work of [11], the authors show that any edges that have closed vertices can be removed without impact on the projective dimension (cf. Theorem 2.9 (d) [12]). This is because such an edge is a union of other edges, that is, closed vertices. In [9, Theorem 4.3], the authors show that if the open vertices of the 1-skeleton of $H(I)$ contain a complete bipartite graph as a spanning subgraph then its projective dimension is exactly the number of vertices minus 1. With Proposition 4.1 we can now see that this result follows because the complete bipartite graph as a spanning subgraph forces higher dimensional edges to be a union of 2 or more distinct edges if their vertices are not closed.

5. Bushes with higher dimensional edges

Throughout this section, $\mathcal{H}$ is a hypergraph, and $I = I(\mathcal{H})$ is the standard square-free monomial ideal associated to it in the polynomial ring $R$. Let $v$ be a vertex in $\mathcal{H}$ and $m_v \in I$ be the monomial generator associated to it. Let $\mathcal{G}(I) = \{m_1, ..., m_r\}$ be the minimal generating set of $I. \mathcal{H}_v = \mathcal{H}(I_v)$ is the hypergraph associated to the ideal $I_v = (\mathcal{G}(I) \backslash \{m_v\})$. Similarly, $W \subseteq V = [\mu]$, $\mathcal{H}_W = \mathcal{H}(I_W)$ is the hypergraph associated to the ideal $I_W = (\mathcal{G}(I) \backslash \{m_w\} | w \in W)$. A hypergraph $\mathcal{H}$ with $V = [\mu]$ is a string if $\{i, i+1\} \in \mathcal{H}$ for all $i = 1, ..., \mu - 1$ and the only edges containing the vertex $i$ are $\{i - 1, i\}, \{i, i+1\}$ and, possibly, $\{i\}$. We say that a string is an open string if all vertices other than 1 and $\mu$ are open (Note that for $\mathcal{H}$ to be separated 1 and $\mu$ must be closed).

Definition 5.1. The degree of a vertex is the number of faces containing the vertex. We say a vertex is a joint on a hypergraph if its degree is at least 3. Let $v$ be a vertex in a hypergraph $\mathcal{H}$, and let $\mathcal{H}_1, ..., \mathcal{H}_r$ be the connected components of $\mathcal{H}_v$; if one of them, say $\mathcal{H}_1$, is a string hypergraph, we call $\mathcal{H}_1$ a branch of $\mathcal{H}$ (from $v$). We say a hypergraph is a bush, if its spanning 1-skeleton has branches of length at most 2.

The smallest case of a bush is a 2-star where there is exactly one joint and every branch has a length less than or equal 2. Furthermore, by the definition of the branch or the string, the end vertices of a 2-star must be closed.

In this section we focus on the projective dimension of hypergraphs which are bushes. More precisely, we want to see the impact of the higher dimensional edges on the projective dimension.
One technique that is used in [11] and [12] which we will need here, is using the short exact sequences obtained by looking at colon ideals. Specifically, there is one type of colon ideal that we are interested in, and we explain below what the operation looks like on the associated hypergraphs.

**Definition 5.2.** Let $F$ be an edge in $\mathcal{H}$ and let $x_F \in R$ be the variable associated to $F$.

- The hypergraph $\mathcal{H} : F$, obtained by removing $F$ in $\mathcal{H}$, is the hypergraph associated to the ideal $I : x_F$.
- The hypergraph $(\mathcal{H}, x_F)$, obtained by adding a vertex corresponding to the variable $x_F$ in $\mathcal{H}$, is the hypergraph associated to the ideal $(I, x_F)$.

The following results appearing in references [11] and [12] will be very useful to us in this section. We put them here for the self-containment of this work and for the reader’s convenience.

**Theorem 5.3.**

1. [11, Corollary 3.8] An open string hypergraph with $\mu > 1$ vertices has projective dimension $\mu - \lfloor \frac{\mu}{2} \rfloor$.
2. [12, Theorem 2.9 (c)] If $\mathcal{H}' \subseteq \mathcal{H}$ are hypergraphs with $\mu(\mathcal{H}') = \mu(\mathcal{H})$, then $\text{pd}(\mathcal{H}') \leq \text{pd}(\mathcal{H})$ where $\mu(\ast)$ denotes the number of vertices of $\ast$.
3. [12, Proposition 4.7 and 4.9] Let $\mathcal{H}$ be a 1-dimensional hypergraph, $w$ be a joint in $\mathcal{H}$, and $S$ be a branch departing from $w$ with $v_1, \ldots, v_n$ vertices. Suppose vertices of $S$, $v_1, \ldots, v_{n-1}$, are open and the end vertex $v_n$, that is, the leaf of $S$, is the only closed vertex of $S$. Let $E$ be the unique edge connecting $w$ to $v_1$. Then $\text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}')$, where $\mathcal{H}'$ is the following hypergraph: (a) if $n \equiv 1$ mod 3, then $\mathcal{H}' = \mathcal{H} : E$; (b) if $n \equiv 2$ mod 3, then $\mathcal{H}' = \mathcal{H}_w$.

The following lemma extends the result of [12], because of the new connection of LCM-lattice and hypergraphs. It is a special case of Theorem 4.3 in [9]. We provide a different proof here.

**Lemma 5.4.** Let $\mathcal{H}$ be a hypergraph such that its spanning 1-skeleton is a 2-star with at least one open vertex. Then $\text{pd}(\mathcal{H}) = |V(\mathcal{H})| - 1$.

**Proof.** We use induction on the number of edges on $\mathcal{H}$ such that their dimensions are at least 2. Suppose there are no higher dimensional edges on $\mathcal{H}$ then the lemma follows from Proposition 4.16 of [12]. Assume there is an edge $F$ on $\mathcal{H}$ such that it is at least 2-dimensional. We assume that $F$ is not a union of some other edges, otherwise, we are done because of Corollary 4.4.

Notice that the number of vertices of $F$ must be at least 3. Therefore the number of vertices of $\mathcal{H}_{V(F)}$, the induced hypergraph after removing vertices of $F$, is at most $|V(\mathcal{H})| - 3$, and the projective dimension of $\mathcal{H}_{V(F)}$ is at most $|V(\mathcal{H})| - 3$. We consider the short exact sequence

$$0 \rightarrow (\mathcal{H}, x_F) \rightarrow \mathcal{H} \rightarrow (\mathcal{H} : x_F) \rightarrow 0$$

where $x_F$ is the variable corresponding to the edge $F$. The induction hypothesis gives $\text{pd}(\mathcal{H} : F) = |V(\mathcal{H} : F)| - 1 = |V(\mathcal{H})| - 1$ as $V(\mathcal{H} : F) = V(\mathcal{H}) = V(\mathcal{H}^2)$ and $\mathcal{H} : F$ has one fewer higher dimensional edge than $\mathcal{H}$.

Moreover, by Theorem 5.3(2), we have $\text{pd}(\mathcal{H} : F) \leq \text{pd}(\mathcal{H})$. We observe that the hypergraph $(\mathcal{H}, x_F)$ is the union of $\mathcal{H}_{V(F)}$ and an isolated vertex corresponding to $x_F$, hence its projective dimension is at most $|V(\mathcal{H})| - 3 + 1 = |V(\mathcal{H})| - 2$. Using the short exact sequence on the projective dimension, we have

$$\text{pd}(\mathcal{H}) \leq \max\{\text{pd}(\mathcal{H} : F), \text{pd}(\mathcal{H}, x_F)\} = \text{pd}(\mathcal{H} : F) \leq \text{pd}(\mathcal{H}).$$

The conclusion follows.
The following lemma is a rewrite of the result in [12] where the authors did not see the broader implication.

**Lemma 5.5.** Let \( v \) be a joint on a hypergraph \( H \) such that there is no higher dimensional edge on the branches of \( v \) and there is a branch on \( v \) with length 2. Then \( \text{pd}(H) = \text{pd}(H_v) \).

**Proof.** The proof follows similarly to the proof of Proposition 4.9 of [12] or Theorem 5.3(3). The only assumption that is needed in the proof of Proposition 4.9 of [12] is that \( v \) has no higher dimensional edge on the branches of \( v \). Let \( v_1 \), and \( v_2 \) be two vertices on the branch of length 2 such that \( v_1 \) is adjacent to \( v \) and let \( E \) be the edge connecting \( v \) and \( v_1 \). The hypergraphs, \( H, H : E, H_{v_1}, (H : E)_v, H_{v_2}, H_{v_2} : E \), and \( Q_{v_2} = Q_{v_1} : v_2 \) have the exact same relationships described in the proof of Proposition 4.9 of [12] as long as there is no higher dimensional edge on the branches of \( v \), hence the relationships of projective dimensions are the same.

We conclude this section with the following proposition which is the extension of the results of [12]. Notice that the connected closed vertices assumption is harmless via Corollary 4.4. This also shows that higher dimensional edges can be removed or disregarded with the new connection we built from previous sections.

**Proposition 5.6.** Let \( H \) be a bush that does not have any connected closed vertices. Let \( W \) be the subset of \( V(H) \) such that each vertex is a joint of \( H \) having at least one branch of length 2. If all the higher dimensional edges of \( H \) have vertices on the branches of the same joint, then \( \text{pd}(H) = \text{pd}(H_W) \).

**Proof.** We use induction on the number of joints and the number of higher dimensional edges on the joints. Suppose \( H \) has only one joint and this joint has branches of length 1, then nothing is to be proven. Suppose that \( H \) has a unique joint with at least one branch of length 2. Then by Lemma 5.4, we are done.

Suppose \( H \) has at least two joints, and we assume that \( v \) is a joint having at least one branch of length 2. Suppose there are no higher dimensional edges on branches of \( v \) then by Lemma 5.5, \( \text{pd}(H) = \text{pd}(H_v) \) where \( H_v \) is the hypergraph obtained from \( H \) by removing the joint \( v \). Notice that \( H_v \) is a union of branches of \( v \) and another hypergraph which satisfies the assumptions of the theorem. Moreover, the vertices of branches of \( v \) all become closed in \( H_v \), because the lengths of the branches are at most 2. Let \( V(S_v) \) be the vertex set of all branches of \( v \) including \( v \) and \( H_{V(S_v)} \) be the hypergraph obtained from \( H \) by removing all the vertices of \( V(S_v) \) and \( (H_{V(S_v)})_W \) be the hypergraph obtained from \( H_{V(S_v)} \) by removing all of the joints of \( H_{V(S_v)} \) having at least one branch of length 2. Then \( \text{pd}(H_v) = |V(S_v)| - 1 + \text{pd}(H_{V(S_v)}) = |V(S_v)| - 1 + \text{pd}(H_{V(S_v)})_W \) by induction. Since \( H_W \) is a union of branches of \( v \) without \( v \) and \( (H_{V(S_v)})_W \), we have \( \text{pd}(H_W) = \text{pd}(H_v) = \text{pd}(H) \).

Suppose the branches of \( v \) have at least one higher dimensional edge. Let \( F \) be a higher dimensional edge and \( x_F \) be the variable corresponding to the edge. We consider the same short exact sequence:

\[
0 \rightarrow (H, x_F) \rightarrow H \rightarrow H : F \rightarrow 0.
\]

Notice that \( (H : F) \) is a hypergraph obtained from \( H \) with the edge \( F \) removed. By the induction hypothesis, \( \text{pd}(H : F) = \text{pd}(H : F)_W \). Since \( v \) is a joint with at least one branch of length 2, \( v \) will be removed in \( H_W \) and \( (H : F)_W = H_W : F \). Moreover, all the vertices of branches of \( v \) will become closed because the branches have length at most 2, hence the vertices of \( F \) are closed in \( H_W \) and \( (H : F)_W \). By Corollary 4.4 again, we have \( \text{pd}(H : F)_W = \text{pd}(H_W : F) = \text{pd}(H_W) \). We are left to show \( \text{pd}(H : F) = \text{pd}(H) \). With Equation (5.1), it is sufficient to show that \( \text{pd}(H : F) > \text{pd}(H, x_F) \).
By induction on the number of higher dimensional edges on the branches of \( v \), we have \( \text{pd}(\mathcal{H}, x_F) = \text{pd}(\langle \mathcal{H}, x_F \rangle_W) \) because \( \mathcal{H}, x_F \) has no edge \( F \) on the branches of \( v \). Once we show \( \text{pd}(\langle \mathcal{H}, x_F \rangle_W) > \text{pd}(\langle \mathcal{H}, x_F \rangle_W) \) then we are done. The hypergraphs \( \langle \mathcal{H}, F \rangle_W \) and \( \langle \mathcal{H}, x_F \rangle_W \) are unions of branches of \( v \), and \( \langle \mathcal{H}, F \rangle_{V(S_i)} \cup \{x_F\} = \langle \mathcal{H}, x_F \rangle_{V(S_i)} \). We now just need to compare the structure of branches of \( v \) on \( \langle \mathcal{H}, F \rangle_W \) and \( \langle \mathcal{H}, x_F \rangle_W \). The vertices of branches of \( v \) on \( \langle \mathcal{H}, F \rangle_W \) and \( \langle \mathcal{H}, x_F \rangle_W \) are closed because \( v \) is a joint with branches of length at most 2. Hence the projective dimension of the branches of \( v \) is the number of vertices. The number of vertices on the branches of \( v \) in \( \langle \mathcal{H}, F \rangle_W \) is \( |V(S_i)| - 1 \) but the number of vertices on the branches of \( v \) with \( \{x_F\} \) in \( \langle \mathcal{H}, x_F \rangle_W \) is \( |V(S_i)| - |V(F)| + 1 < |V(S_i)| - 1 \). Hence we have \( \text{pd}(\mathcal{H}, F) = \text{pd}(\langle \mathcal{H}, F \rangle_W) > \text{pd}(\langle \mathcal{H}, x_F \rangle_W) = \text{pd}(\mathcal{H}, x_F). \)

Proposition 5.6 provides us an inductive process to obtain the projective dimension of a hypergraph or a square-free monomial ideal. One just needs to remove joints one by one until all of the branches of length 2 are separated. This actually covers a large class of ideals. Specifically, the class of ideals has most of the variables only appearing in one or two generators. When a variable appears in multiple generators, we are able to create a new ideal where the variable appears in fewer generators after removing the joints or if some of its vertices become closed, hence we can apply Corollary 4.4. In the next section, we provide a process to see the efficient reduction. Note that this reduction process does not work well with the case where most of the variables appear in many generators, that is, the 1-skeleton of the hypergraph involves very few generators. One classical ideal is the initial ideal of the determinantal ideal.

**Appendix: algorithmic procedures and one example**

Recall a hypergraph \( \mathcal{H} \) with \( V = [\mu] \) is a string if \( \{i, i + 1\} \) is in \( \mathcal{H} \) for all \( i = 1, \ldots, \mu - 1 \), and the only edges containing \( i \) are \( \{i - 1, i\}, \{i, i + 1\} \) and possibly \( \{i\} \). Also, \( \mathcal{H} \) is a \( \mu \)-cycle if \( \mathcal{H} = \mathcal{H} \cup \{\mu, 1\} \) where \( \mathcal{H} \) is a string. Let \( \mathcal{H} \) be a hypergraph that satisfies the assumptions of Proposition 5.6. Furthermore, if \( \mathcal{H}_W \) is a union of bushes, 2-star strings, and cycles of 2-stars, then one can obtain \( \text{pd}(\mathcal{H}) \) by first removing all the joints having branches of length 2, and then one can apply Proposition 4.18 in [12] to obtain the projective dimension of \( \mathcal{H} \). This is because all the higher dimensional edges on the branches of joints of length 2 can be removed in \( \mathcal{H}_W \) by Corollary 4.4 and the fact that all the vertices on the branches of joints of length 2 become closed in \( \mathcal{H}_W \). In this section, we present algorithmic procedures to compute the projective dimension of a bush hypergraph. We use Algorithm A.1 in [12] to decide if a vertex is a joint or an endpoint. We write \( d(i) \) as the degree of any given vertex \( i \) in \( \mathcal{H} \). We just need to know if \( d(i) = 0, 1, 2 \), or greater than 2 (a joint) for the purpose of computation.

The following result provides an algorithm to obtain the hypergraph \( \mathcal{H}_W \) in the statement of Proposition 5.6. We use the variable \( i \) to detect the vertices with degree one (if any). The variable \( j \) runs through the other vertices looking for neighbors of \( i \), and \( k \) looks for the other neighbor of \( j \) (if any).

**Algorithm 6.1.** Input: A connected hypergraph \( \mathcal{H} \) that is a bush and all the higher dimensional edges of \( \mathcal{H} \) have vertices on the branches of the same joint. Let the vertex set be \( V(\mathcal{H}) = \{1, 2, \ldots, \mu\} \). The output is a hypergraph \( \mathcal{H}_W \) such that \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}_W) \).

Step 0: Set \( i = 1 \).

Step 1: If \( \mathcal{H} = \emptyset \), stop the process and give \( \mathcal{H} \) as output.

If \( \mathcal{H} \neq \emptyset \) set \( j = k = 1 \), and do the following: if \( i \leq \mu \), then go to Step 2, if \( i = \mu + 1 \), then stop the process and give \( \mathcal{H} \) as output.

Step 2: If \( i \notin V(\mathcal{H}) \), then set \( i = i + 1 \) and start Step 2 again.

If \( i \in V(\mathcal{H}) \), compute \( d(i) \) using Algorithm A.1 in [12] if \( d(i) = 0, 1, 2 \), then set \( i = i + 1 \) and start Step 2 again; if \( d(i) > 2 \) then go to Step 3.

Step 3: If \( j > \mu \) then set \( i = i + 1 \) and go to Step 2. If \( j = i \) or if \( \{i, j\} \notin \mathcal{H} \) then set \( j = j + 1 \) and start again Step 3. If \( \{i, j\} \in \mathcal{H} \) then go to Step 4.

Step 4: Use Algorithm A.1 in [12] to compute \( d(j) \). If \( d(j) = 2 \), then go to Step 5; otherwise set \( j = j + 1 \) and go to Step 3.
Step 5: If \( k = i \), or if \( k = j \), or if \( \{j, k\} \not\in \mathcal{H} \), then set \( k = k + 1 \) and start again Step 5. If \( \{j, k\} \in \mathcal{H} \) then set \( \mathcal{H} = \mathcal{H}_i \) and \( i = i + 1 \), go to Step 1. (this procedure stops because \( d(j) = 2 \))

The following result provides an algorithm to apply Corollary 4.4: after removing higher dimensional edges of a hypergraph, the projective dimension stays the same.

**Algorithm 6.2.** Input: A connected hypergraph \( \mathcal{H} = \bigcup \{F_i\}_{i=1}^p \) such that all edges that have cardinality greater than 2 are the union of 2 or more distinct edges of \( \mathcal{H} \). The output is a hypergraph \( \mathcal{H}' \) such that \( \mathcal{H}' \) has no higher dimensional edges and \( \text{pd}(\mathcal{H}) = \text{pd}(\mathcal{H}') \).

Step 0: Set \( i = 1 \).
Step 1: If \( \mathcal{H} = \emptyset \), stop the process and give \( \mathcal{H} \) as output.
   If \( \mathcal{H} \neq \emptyset \) then do the following: if \( i \leq p \), then go to Step 2, if \( i = p + 1 \), then stop the process and give \( \mathcal{H} \) as output.
Step 2: If \( |F_i| < 3 \) then set \( i = i + 1 \) and start Step 1 again. If \( |F_i| \geq 3 \) then set \( \mathcal{H} = \mathcal{H} \setminus \{F_i\} \) and \( i = i + 1 \), and go to Step 1.

**Remark 6.3.** We can combine Algorithm 5.6 in [11], Algorithms A.1 and A.2 in [12], and Algorithms 6.1 and 6.2 to compute the projective dimension of a hypergraph that is a bush and its higher dimensional edges are on the same joints. The example below illustrates the process.

**Example 6.4.** Let \( \mathcal{H} \) be a hypergraph as in Figure 9. It presents a square-free monomial ideal minimally generated by 43 generators with 70 variables. By Proposition 5.6 and Algorithm 6.1, we remove the red vertices that are the joints of \( \mathcal{H} \) having at least one branch of length 2 to obtain the hypergraph \( \mathcal{H}_1 \) as in Figure 10. This step removes 5 generators and 7 variables.
By Corollary 4.4 and Algorithm 6.2, we can remove green and blue edges as they are union of other lower dimensional edges. We obtain the hypergraph $H_2$ as in Figure 11. This step removes 4 variables. Finally, we remove edges using Corollary 4.4, Theorem 5.3(3), and Algorithms A.2 in [12] to obtain the hypergraph $H_3$ as in Figure 12. In this step, we remove edges that are either unions of lower dimensional edges or are branches of length one, and we remove 19 variables.

Figure 12 presents a square-free monomial ideal minimally generated by 38 generators with 40 variables. We cut down 5 generators from original ideal and remove 30 variables. Then by Theorem 5.3(1) and Algorithm 5.1 in [11], we have the projective dimension of $H$ is equal to $31 + 7 − 2 = 36$ which comes from 31 isolated vertices, and one string of 7 vertices with two open substrings of length 3 and 1. Notice that the maximum possible projective dimension for a square-free monomial ideal generated by 43 elements is 43 by [13, Proposition 4.1]. This example highlights the effectiveness of the algorithm.

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