Dark–Bright Solitons in Inhomogeneous Bose-Einstein Condensates

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We investigate dark–bright vector solitary wave solutions to the coupled non–linear Schrödinger equations which describe an inhomogeneous two-species Bose-Einstein condensate. While these structures are well known in non–linear fiber optics, we show that spatial inhomogeneity strongly affects their motion, stability, and interaction, and that current technology suffices for their creation and control in ultracold trapped gases. The effects of controllably different interparticle scattering lengths, and stability against three-dimensional deformations, are also examined.

Among the many surprising features of non–linear equations, the emergence of solitons is one of the most prominent. For the non–linear Schrödinger equation (NLSE), which governs both non–linear modes in fibers and dilute Bose–Einstein condensates (BECs), two different kinds of scalar solitons, bright and dark, are known \cite{1}. In optics, bright and dark solitons arise in media with anomalous dispersion and normal dispersion, respectively, and for BECs the s–wave scattering interaction is the determining factor (attractive for bright solitons, repulsive for dark). Whereas in gaseous Bose–condensates dark solitons only, and only recently, have been observed \cite{2,3}, optical solitons are well investigated and on the verge of industrial application \cite{4}. In addition to the bright and dark scalar solitons there are also various multicomponent (vector) solitons known, which arise as solutions to systems of coupled NLSEs. An elegant example is the so–called dark–bright soliton, where a bright optical solitary wave exists in a system with normal dispersion because it is trapped within a copropagating dark soliton \cite{5,6,7}. In this Letter we investigate the behavior of dark–bright solitons (and solitary waves) in repulsively interacting two-component Bose-Einstein condensates. We examine the effects of spatial inhomogeneity, three dimensional geometry, and dissipation, which are all important features of BEC experiments.

In the context of cold atomic gases the two vector components evolving under the Gross-Pitaevskii NLSE are the macroscopic wave functions of Bose-condensed atoms in two different internal states, which we will denote as $|D\rangle$ and $|B\rangle$. The non–linear interactions are due to elastic s–wave scattering among the atoms, and are effectively repulsive (positive scattering length) for both systems ($^{23}$Na and $^{87}$Rb) in which multicomponent condensates have been realized. In both cases the three 1D interaction strengths ($g_{DD}, g_{BB}, g_{DB}$) may easily be made equal to within a few percent; and quasi-one dimensional traps that are longitudinally very flat are under active experimental development \cite{8}. So we will begin with the two-component NLSE with all interaction strengths equal, in one dimension, with no potential, and then add more realistic features in succession. In the standard natural units, the general equations read

\begin{equation}
\begin{align*}
\dot{\psi}_B &= -\frac{1}{2}\psi''_B + [V_B + |\psi_B|^2 + G_D|\psi_D|^2 - \mu - \Delta]|\psi_B|, \\
\dot{\psi}_D &= -\frac{1}{2}\psi''_D + [V_D + |\psi_D|^2 + G_B|\psi_B|^2 - \mu]|\psi_D|,
\end{align*}
\end{equation}

where the chemical potentials $\mu_D = \mu$ and $\mu_B \equiv \mu + \Delta$ have been introduced in the standard way. For the present we set the coupling strength ratios $G_{B,D} = 1$ and assume no external potentials, $V_{D,B} = 0$. The dark–bright soliton solution to eqs. (1) is then given by

\begin{equation}
\begin{align*}
\psi_B &= \sqrt{\frac{N_B}{2}}e^{i\phi}e^{i\Omega_Bt}e^{ix\kappa}\text{sech}\left(\kappa|x-q(t)|\right), \\
\psi_D &= i\sqrt{\mu}\sin\alpha + \sqrt{\mu}\cos\alpha\tanh\left(\kappa|x-q(t)|\right),
\end{align*}
\end{equation}

where $N_B \equiv \int dx |\psi_B|^2$ is the rescaled number of particles in state $|B\rangle$, the soliton inverse length is $\kappa = \sqrt{\mu \cos^2\alpha + (\Omega_B/4)^2} - N_B/4$, the bright component frequency shift is $\Omega_B \equiv \kappa^2(2-\tan^2\alpha)/2-\Delta$, and the soliton position is $q(t) = q(0) + \kappa t \tan\alpha$. The ‘binding energy’ of the bright component in the well formed by the $\psi_D$ mean field, in the co-moving frame, is clearly $\kappa^2/2$; the bright component phase shift $\phi$ is only of significance if there are two or more solitons. Readers familiar with the scalar solitons of the one-component NLSE will recognize $\psi_D$ as a dark (or ‘grey’) soliton of velocity-angle $\alpha$, and $\psi_B$ as a bright soliton, which can only be found in single-component condensates if they have negative scattering.
length (and so are prone to collapse) (see Fig. 1(a)). The Thomas-Fermi-like expansion with \( N_B \) of the trapped bright component makes the soliton size \( \kappa^{-1} \) longer than for a single-component dark soliton at the same \( \mu \) (see Fig. 1(b)).

The integrable system of eqs. (3) with \( G_{D,B} = 1 \) and \( V_{D,B} = 0 \), also known as the Manakov equation, conserves the free energy

\[
G = \frac{1}{2} \int dx \left[ |\psi_D'|^2 + |\psi_B'|^2 + (|\psi_D|^2 + |\psi_B|^2 - \mu)^2 + 2\Delta|\psi_B|^2 \right] = \frac{4}{3} \kappa^3 + \frac{1}{2} N_B \kappa^2 (1 + \tan^2 \alpha) + N_B \Delta.
\]

Since \( G \) decreases with increasing soliton velocity, the soliton is formally unstable (to acceleration!). But one implication of integrability is that perturbations of eqs. (3) due to interactions with other waves (solitary or ordinary) will not cause dissipation. If an inhomogeneous potential is added, however, by allowing non-zero \( V_{D,B}(x) \) in (3), then the system is no longer integrable, and the soliton can interact non-trivially with the surrounding condensate. Nevertheless, if \( V \) varies slowly on the soliton scale \( \kappa \), then excitations of the background that have high enough temporal frequency to accept energy resonantly from the soliton must also have short enough spatial wavelength to ‘see’ \( V_{D,B}(x) \) as approximately constant, and hence tend to decouple from the soliton as they do in the truly integrable case. The result is that \( G(q,\dot{q}) \) (given by replacing \( \mu \to \mu - V_D(q), \Delta \to \Delta - V_B(q) + V_D(q) \) in eq. (3) and inverting \( \dot{q} = \kappa(q, \cos \alpha) \tan \alpha \) is approximately conserved, and this determines the motion of the soliton in the potential. (A more sophisticated multiple time scale boundary layer analysis supports this simple argument.)

The motion this implies simplifies in the limit of velocities much smaller than the speed of sound, because

\[
G = \frac{4}{3} \left[ \mu + \frac{N_B^2}{16} - V_D(q) \right]^{3/2} + N_B \left[ V_B(q) - \frac{V_D(q)}{2} \right] - 2\dot{q}^2 \left[ \mu + \frac{N_B^2}{16} - V_D(q) \right] + O(\dot{q}^4),
\]

dropping a term which is constant if we assume that none of the conserved state \(|B\rangle\) atoms escape from the soliton. This implies the low-velocity equation of motion

\[
\ddot{q} = -\frac{V_D'(q)}{2} - \frac{N_B[V_B'(q) - 2V_D'(q)]}{8\sqrt{\mu + N_B^2/16} - V_D(q)},
\]

which, together with its numerical confirmation shown in Fig. 2, is the primary result of this paper. In the limit \( N_B \to 0 \) we recover the equation of motion of the dark soliton, and as \( N_B \) increases we find that the soliton is more and more insulated from the effect of \( V_D \), and more sensitive to \( V_B - V_D \). In the limit \( N_B \gg \sqrt{\mu} \), where the soliton is expanded by the large bright component to many healing lengths in size, we have

\[
\ddot{q} = \left( 1 - 8\frac{\mu - V_D}{N_B^2} \right) [V_B'(q) - V_D'(q)] - 4\frac{\mu - V_D}{N_B^2} V_D'(q),
\]

so that a small differential force on the bright component will predominate. Our assumption that the whole soliton is small compared to the trap scale, however, means that the dark component retains its dramatic effect of giving the soliton an effectively negative mass: the soliton accelerates in the opposite direction to a force exerted through \( V_B \). If \( V_B \) and \( V_D \) are kept equal, on the other hand, a highly expanded dark–bright soliton with \( N_B \gg \sqrt{\mu} \) moves in the potential as if it had a very large positive mass (because as one can see from eq. (4) the soliton’s potential energy is also \( \sim -V_D \)). Numerical integration of the coupled NLSEs shows excellent agreement with eq. (5) (see Fig. 2). Note that for harmonic \( V_B > V_D \) eq. (5) implies that there is a critical \( N_B \) above which the soliton will escape from the trap instead of oscillating. While the precise transition point between very slow oscillation and very slow escape is difficult to check numerically, our numerical results confirm that escape does occur in this case at larger \( N_B \).

A trapping potential also modifies the interactions between solitons. While generally solitons that are more than a few soliton lengths apart are essentially unaffected by each other, at closer ranges they can distort each other significantly. Although the respective bright component numbers \( \propto N_B \) of two solitons are simply conserved dur-
the trap confining species $j = D, B$. Since in our standard natural units we have absorbed in $\psi_{D,B}$ the self-interaction strengths $g_{DD}$ and $g_{BB}$, in this more general case we must allow the co-efficients $G_B \equiv g_{DB}/g_{BB}$ and $G_D \equiv g_{DD}/g_{DD}$ to differ from unity in the NLSE system \[ \text{(6)} \]. It is straightforward to show that

$$G_D = \frac{a_{DB}}{a_{DD}} \left( 1 + \frac{A_D - A_B}{A_D + A_B} \right),$$

$$G_B = \frac{a_{DB}}{a_{BB}} \left( 1 - \frac{A_D - A_B}{A_D + A_B} \right),$$

so that varying the relative tightness of radial confinement for the two species yields one free control parameter, which can allow significant retuning of $G_{D,B}$ even without the measure of modifying the scattering lengths themselves.

If $G_{D,B} \neq 1$, solutions that are distorted versions of the dark–bright soliton certainly exist, although they may only be given in closed form for special cases. (For instance, in the limit of small $N_B$ one may discard the $|\psi_B|^2$ terms in the NLSE, and find dark soliton solutions for $\psi_D$, with $\psi_B \propto \text{sech}^n[\kappa(x - q)]$ for $n(\nu + 1) = 2G_D$. For $G_D > 1$, there are also one or more excited bound states of $\psi_B$.) Such solutions are often referred to as solitary waves, rather than solitons, to indicate that they may interact nontrivially with other solitary or ordinary waves. This means, for example, that a collision between dark–bright solitary waves may effect a net transfer of bright component from one solitary wave to the other; see Fig. (6). It also implies that unlike true solitons, which are transparent to all quasiparticle modes, dark–bright solitary waves will suffer from dissipation due to collisions with thermal particles and phonons, even when the one-dimensional approximation is excellent. It is a problem beyond the scope of this Letter to compute scattering rates with $G_{D,B}$ significantly different from unity. For $G_{D,B}$ close to unity, however, simple estimates for the anti-damping rate due this effect show it to be negligible, at attainably low temperatures, because the cross sections for dissipative collisions are proportional to the squares of the scattering length differences. In current experiments, however, soliton lifetimes are limited not by one-dimensional dissipation, but by the breakdown of the one-dimensional approximation. In more than one dimension, single-component dark solitons suffer from the well-known ‘snake mode’ dynamical instability, in which bending of the plane of the density minimum grows exponentially even without dissipation. This is the presumed cause of decay of solitons that have been produced in BEC experiments to date. As shown by Muryshev et al. \[ \text{(12)} \], a single-component dark soliton is only unstable to snake modes of wavelength greater than the soliton size, so that radial confinement to within a healing length should stabilize dark solitons. By an extension of their method of analysis, one can show that the dark–bright

**FIG. 3:** Symmetric collision of two dark–bright solitons in a harmonic trap ($\gamma = 1$). Degree of brightness indicates $|\psi_B|^2$ as a function of $x$ and $t$ (both in trap units), for repulsive ($\Delta \phi = 0$) and attractive ($\Delta \phi = \pi$) interaction.

**FIG. 4:** Collision of two dark–bright solitons in a harmonic trap ($\gamma = 1$). Shown is $|\psi_B|^2$ as a function of $x$ and $t$, both in trap units. Initially one soliton is at rest (i.e. $q = 0$); after the collision both solitons are oscillating.
soliton is also stable against snake modes of wavelength less than its size \( \kappa^{-1} \). Since for \( N_B \gg \sqrt{\mu} \) this can be much larger than the healing length, stabilization of dark–bright solitons against snakeing should easily be possible in current traps. (The proof is simple in the case of a motionless dark-bright soliton in three-dimensional bulk. We take eqs. (2) for \( \alpha = 0 \), and expand the free energy \( G \) to second order in \( \delta \psi = \Phi_j(x) \cos k(\cos \theta + z \sin \theta - \beta) \), where \( R_j \) and \( S_j \) are real, and \( j = B, D \). The result for this perturbation of wave number \( k \) is
\[
\delta G_k = \int dx \Gamma_k(x) \int dy dz \cos^2 k(\cos \theta + z \sin \theta - \beta)
\]
\[
\Gamma_k(x) = 2 \left[ \sqrt{\pi} \Re[\Phi_D(x)] \tanh(\kappa x) + \sqrt{\frac{N_B \kappa}{2}} \Re[\Phi_B(x)] \sech(k x) \right]^2
\]
\[
\hat{H}_k = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{k^2}{2} - k^2 \sech^2(k x)
\]
Hence, for the \( k \) sector to have positive definite free energy, it is obviously a sufficient condition that \( \hat{H}_k \) have a positive spectrum. Considering the possibility of purely imaginary \( \Phi_D \) shows that this is also a necessary condition. Considering the eigenfunction \( \sech \kappa x \) (which is clearly the ground state because it is real and nodeless) shows that the spectrum of this particular \( \hat{H}_k \) is positive for \( |k| > \kappa \).

Even more conveniently, we note that a robust method for the controlled creation of dark–bright solitons has already been presented and analysed in detail (without being explicitly recognized as such) [3]. Dum et al. have shown that dark solitons may be created in one condensate component by adiabatic transfer of population from a condensate in another internal state, and have already noted that in the late stages of this procedure the second component appears as a stretched dark soliton around the remaining population of the first, whose wave function approaches a hyperbolic secant. Stopping short of complete adiabatic transfer will in fact produce a dark-bright soliton of arbitrary \( N_B \). Engineering by masked Rabi transfer with phase imprinting may also be possible, and the smoother total density profile and larger size of a highly-expanded dark–bright soliton should simplify the creation of slow and stable solitons by this method.

Our conclusion therefore is that dark–bright solitons move in trapped condensates much as dark solitons (though more slowly, if \( V_D = V_B \)), and have strong advantages in stability and controllability. Since \( |\psi_B|^2 \) can be imaged separately and non-destructively, they also offer a realization of Reinhardt’s and Clark’s proposal to track solitons by trapping distinguishable atoms inside them [4]. And in addition to advancing soliton studies into the inhomogeneous regime of BECs, production of dark-bright solitons in BECs would be the development of dark-bright solitons. (though more slowly, if \( V_D = V_B \)), and have strong advantages in stability and controllability. Since \( |\psi_B|^2 \) can be imaged separately and non-destructively, they also offer a realization of Reinhardt’s and Clark’s proposal to track solitons by trapping distinguishable atoms inside them [4]. And in addition to advancing soliton studies into the inhomogeneous regime of BECs, production of dark-bright solitons in BECs would be the development of atom optical tweezers with potentially sub-micron precision: the trapping and manipulation of ultracold atoms by ultracold atoms.

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