ABSTRACT YOUNG PAIRS FOR SIGNED PERMUTATION GROUPS

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Abstract. The notion of an Abstract Young (briefly: AY) representation is a natural generalization of the classical Young orthogonal form. The AY representations of the symmetric group are characterized in [U2]. In this paper we present several types of minimal AY representation of $D_n$ associated with standard D-Young tableaux which are a natural generalization of usual standard Young tableaux. We give an explicit combinatorial view (the representation space is spanned by certain standard tableaux while the action is a generalized Young orthogonal form) of representations which are induced into $D_n$ from minimal AY representations of one of the natural embeddings of $S_n$ into $D_n$. Then we show that these induced representations are isomorphic to the direct sum of two or three minimal AY representations of $D_n$ also associated with standard D-Young tableaux. It is done by constructing a continuous path between representation matrices where one end of the path is the mentioned direct sum; another end is the classical form of induced representation. In the last section we briefly explain how the similar results may be obtained for the group $B_n$ instead of $D_n$.

1. INTRODUCTION

One of the important problems in combinatorial representation theory is to find a unified combinatorial construction of Weyl groups representations. A most important breakthrough in this area was the introduction of Kazhdan-Lustig cell representations [KL]. Other results were achieved by Vershik [V], Vershik and Okounkov [OV], Cherednik [Ch] and Ram [Ra].

A unified axiomatic approach, in steps of the above works of Vershik and Ram, to the representation theory of Coxeter groups and their Hecke algebras was presented in [U1]. This was carried out by a natural assumption on the representation matrices, avoiding a priori use of external concepts (such as Young tableaux).

Let $(W, S)$ be a Coxeter system, and let $K$ be a finite subset of $W$. Let $F$ be a suitable field of characteristic zero and let $\rho$ be a representation of $W$ on the vector space $V_K := \text{span}_F \{C_w | w \in K\}$, with basis vectors indexed by elements of $K$. Adin, Brenti and Roichman in [U1] and [U2] study the sets $K$ and representations $\rho$ which satisfy the following axiom:

(A) For any generator $s \in S$ and any element $w \in K$ there exist scalars $a_s(w), b_s(w) \in F$ such that

$$\rho_s(C_w) = a_s(w)C_w + b_s(w)C_{ws}.$$ 

If $w \in K$ but $ws \notin K$ we assume $b_s(w) = 0$.

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A pair \((\rho, K)\) satisfying Axiom (A) is called an abstract Young (AY) pair; \(\rho\) is an AY representation, and \(K\) is an AY cell. If \(K \neq \emptyset\) and has no proper subset \(\emptyset \subset K' \subset K\) such that \(V_{K'}\) is \(\rho\)-invariant, then \((\rho, K)\) is called a minimal AY pair. (This is much weaker than assuming \(\rho\) to be irreducible.)

In [U1] it was shown that an AY representation of a simply laced Coxeter group is determined by a linear functional on the root space. Thus it may be obtained by restriction of Ram’s calibrated representations of affine Hecke algebras (see [Ra]) to the corresponding Weyl groups. In [U2] it is shown that, furthermore, the values of the linear functional on the “boundary” of the AY cell determine the representation.

In [U2] this result is used to characterize AY cells in the symmetric group. This characterization is then applied to show that every irreducible representation of \(S_n\) may be realized as a minimal abstract Young representation. AY representations of Weyl groups of type \(B\) are not determined by a linear functional. However, it is shown in [U2] that irreducible representations of \(B_n\), similarly to irreducible representations of \(S_n\), may be realized as minimal AY representations.

In this work we present several types of minimal AY representation of \(D_n\) which arise from D-Young tableaux introduced in Section 3.1. These D-Young tableaux are a special case of Ram’s negative rotationally symmetric tableaux [Ra]. It is shown in [U2] Theorem 3.9 that the representation of a Coxeter group \(W\) which is induced from a minimal AY representation of its parabolic subgroup \(P\) is a minimal AY representation of \(W\). In section 5.1 of this work we give an explicit combinatorial view (the representation space is spanned by certain standard tableaux while the action is a generalized Young orthogonal form) of representations which are induced into \(D_n\) from minimal AY representations of one of the natural embeddings of \(S_n\) into \(D_n\). In section 5.2 we show that these induced representations are isomorphic to the direct sum of two or three minimal AY representations of \(D_n\) associated with standard D-Young tableaux. It is done by constructing a continuous path between representation matrices where one end of the path is the mentioned direct sum; another end is the classical form of induced representation (see [U2] Remark 3.10). In section 6 we briefly explain how the similar results may be obtained for the group \(B_n\) instead of \(D_n\).

2. Preliminaries and notations

A Coxeter system is a pair \((W, S)\) consisting of a group \(W\) and a set \(S\) of generators for \(W\), subject only to relations of the form

\[(st)^{m(s,t)} = 1,\]

where \(m(s,s) = 1\) and \(m(s,t) = m(t,s) \geq 2\) for \(s \neq t\) in \(S\). In case no relation occurs for a pair \((s,t)\), we make the convention that \(m(s,t) = \infty\). \(W\) is called a Coxeter group. If \(m(s,t) \leq 3\) for all \(s \neq t\) then \((W, S)\) is called simply laced.

Let \((W, S)\) be a Coxeter system, and let \(X = X(W, S)\) be the corresponding Cayley graph (with generators acting on the right): its vertices are the elements of \(W\), and \(x, y \in W\) are connected by an edge if and only if \(x^{-1}y \in S\). \(X\) is a connected undirected graph. A subset \(K \subseteq W\) is called convex if, for any \(x, y \in K\), all the geodesics (paths of shortest length) connecting \(x\) to \(y\) in \(X\) have all their vertices in \(K\).

Let \(P\) be a poset, and let \(X\) be its undirected Hasse diagram. Thus \(X\) is an undirected graph which has \(P\) as a vertex set, with an edge \(\{x, y\}\) whenever \(x\) either
covers or is covered by \( y \). A subset \( \mathcal{K} \) of \( P \) is called *convex* if, for any \( x, y \in \mathcal{K} \), all geodesics (shortest paths) connecting \( x \) to \( y \) in \( X \) have all their vertices in \( \mathcal{K} \).

Below, \( P \) will be a Coxeter group \( W \) with the right weak Bruhat order, namely the transitive closure of the relation

\[ w < ws \iff w \in W, s \in S \text{ and } \ell(w) < \ell(ws). \]

Clearly, \( \mathcal{K} \subseteq W \) is convex in the right Cayley graph \( X(W,S) \) if and only if it is convex in the right weak Bruhat poset \( P \).

Let \( V \) be the root space of a Coxeter system \((W,S)\). Thus \( V \) is a vector space over \( \mathbb{R} \) with a basis \( \{\alpha_s \mid s \in S\} \) indexed by the group generators. A symmetric bilinear form \( B \) is defined on \( V \) by

\[ B(\alpha_s, \alpha_t) := -\cos \frac{\pi}{m(s,t)} \quad (\forall s,t \in S) \]

(interpreted to be \(-1\) in case \( m(s,t) = \infty \)). For each generator \( s \in S \) define a linear map \( \sigma_s : V \rightarrow V \) by

\[ \sigma_s(v) := v - 2B(v, \alpha_s)\alpha_s \quad (\forall v \in V). \]

This yields a faithful \( B \)-preserving action \( \sigma \) of \( W \) on \( V \) (see, e.g., [H]), and defines the corresponding root system

\[ \Phi := \{\sigma_w(\alpha) \mid w \in W, s \in S\} \subseteq V. \]

2.1. **The Coxeter groups of type B.** The Coxeter group of type \( B \), \( B_n \), is the group of all signed permutations. Let \( S_{2n} \) be the group of all permutations of the numbers \( \pm 1, \pm 2, \ldots, \pm n \). Then

\[ B_n = \{\pi \in S_{2n} : \pi(-i) = -\pi(i) \text{ for } i = 1, 2, \ldots, n\} \]

The group \( B_n \) is generated by the Coxeter generators \( \{s_0, s_1, \ldots, s_{n-1}\} \) defined by:

\[ s_0 = (1, 1) \]

and

\[ s_i = (i, i + 1)(-i, -i - 1), \quad 1 \leq i \leq n - 1. \]

2.2. **The Coxeter groups of type D.** The Coxeter group of type \( D \), \( D_n \), can be defined as the normal subgroup of \( B_n \) consisting of all signed permutations \( \pi \) satisfying: \( |\{i \in [n] \mid \pi(i) < 0\}| \) is even. We embed \( D_n \) in \( S_{2n} \) in the natural way. The group \( D_n \) is generated by the Coxeter generators \( \{s_0, s_1, \ldots, s_{n-1}\} \) defined by:

\[ s_0 = (1, -2)(2, -1) \]

and

\[ s_i = (i, i + 1)(-i, -i - 1), \quad 1 \leq i \leq n - 1. \]

Notice that \( s_0 \) of \( B_n \) differs from \( s_0 \) of \( D_n \) while other Coxeter generators are the same. An element of \( D_n \) is called a *reflection* if it is conjugate to a Coxeter generator. The reflections of \( D_n \) are \( (i, j) = (i, j)(-i, -j) \in S_{2n} \). Notice that in this notation \( (i, j) = (-i, -j) \).

Let \( V = \mathbb{R}^n \) be the root space of \( D_n \) with \( \{e_1, \ldots, e_n\} \) as its standard basis. The simple roots are \(-e_1 - e_2, e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n \). Denote

\[ e_{-i} = -e_i \quad \text{for} \quad 1 \leq i \leq n \]
The positive root $\alpha_{ij} \in \mathbb{R}^n$ corresponding to the reflection $(i, j) \in D_n$ is
$$\alpha_{ij} = e_i - e_j \quad \text{for} \quad i, j \in [\pm n], \; i \neq -j, \; i < j$$
Notice that $\alpha_{ij} = \alpha_{-j,-i} = -\alpha_{ji}$. We have the following identity:
$$\alpha_{ir} + \alpha_{rj} = e_i - e_j = e_i - e_j = \alpha_{ij}$$

**Definition 2.1.** For $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ denote $v_{-i} = -v_i$ and define the derived vector
$$\Delta v = (v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1})$$
$$= (v_1 + v_2, v_2 - v_1, v_3 - v_2, \ldots, v_n - v_{n-1}) \in \mathbb{R}^n.$$

2.3. **The Young Orthogonal Form.** Let $\lambda$ be a partition of $n$, and let $\rho^\lambda$ be the corresponding irreducible representation of $S_n$ (on the Specht module $S^\lambda$). Let $\{v_Q \mid Q \text{ standard Young tableau of shape } \lambda\}$ be the basis of $S^\lambda$ obtained from the basis of polytabloids by the Gram-Schmidt process (see [J, §25]). Let $Q$ be a standard Young tableau. If $k \in \{1, \ldots, n\}$ is in box $(i, j)$ of $Q$, then the content of $k$ in $Q$ is $c_k := j - i$.

The $k$th hook-distance is defined as
$$h(k) := c_{k+1} - c_k \quad (1 \leq k \leq n - 1).$$
Finally, let $Q^{si}$ be the tableau obtained from $Q$ by interchanging $i$ and $i + 1$.

**Theorem 2.2.** (Classical Young Orthogonal Form) [J, §25.4] In the above notations,
$$\rho^\lambda(s_i)(v_Q) = \frac{1}{h(i)} v_Q + \sqrt{1 - \frac{1}{h(i)^2}} v_{Q^{si}}.$$  

This setting is generalized naturally to skew shapes $\lambda/\mu$. Standard Young tableaux of shape $\lambda/\mu$, hook distances on these tableaux and skew Specht modules $S^{\lambda/\mu}$ are defined analogously. In particular,

**Theorem 2.3.** (Classical Young Orthogonal Form for Skew Specht Modules) Let $\{v_Q \mid Q \text{ standard Young tableau of shape } \lambda/\mu\}$ be the basis of the skew Specht module $S^{\lambda/\mu}$ obtained by the Gram-Schmidt process from the polytabloid basis. Then
$$\rho^{\lambda/\mu}(s_i)(v_Q) = \frac{1}{h(i)} v_Q + \sqrt{1 - \frac{1}{h(i)^2}} v_{Q^{si}}.$$  

2.4. **Abstract Young Cells and Representations.** This section surveys results from [U1] which will be used in this paper. Recall the definition of AY cells and representations from the introduction.

**Observation 2.4.** [U1, Observation 3.3] Every nonempty AY cell is a left translate of an AY cell containing the identity element of $W$.

**Proposition 2.5.** [U1, Corollary 4.4] Every minimal AY cell is convex (in the right Cayley graph $X(W, S)$ or, equivalently, under right weak Bruhat order). In particular, $b_s(w) \neq 0$ whenever $s \in S$ and $w, ws \in K$.  

Important examples of AY cells are (standard) descent classes.

Surprisingly, Axiom (A) leads to very concrete matrices, whose entries are essentially inverse linear.

In [U1] it is shown that, under mild conditions, Axiom (A) is equivalent to the following more specific version. Here $T$ is the set of all reflections in $W$.

(B) For any reflection $t \in T$ there exist scalars $\hat{a}_t, \hat{b}_t, \check{a}_t, \check{b}_t \in \mathbb{F}$ such that, for all $s \in S$ and $w \in K$:

$$
\rho_s(C_w) = \begin{cases} 
\hat{a}_{wsw^{-1}}C_w + \hat{b}_{wsw^{-1}}C_{ws}, & \text{if } \ell(w) < \ell(ws); \\
\check{a}_{wsw^{-1}}C_w + \check{b}_{wsw^{-1}}C_{ws}, & \text{if } \ell(w) > \ell(ws).
\end{cases}
$$

If $w \in K$ and $ws \not\in K$ we assume that $\check{b}_{wsw^{-1}} = 0$ (if $\ell(w) < \ell(ws)$) or $\hat{b}_{wsw^{-1}} = 0$ (if $\ell(w) > \ell(ws)$).

**Theorem 2.6.** [U1, Theorem 5.2] Let $(\rho, K)$ be a minimal AY pair for the Iwahori-Hecke algebra of $(W, S)$. If $a_s(w) = a_s(w') \implies b_s(w) = b_s(w')$ $(\forall s, s' \in S, w, w' \in K)$, then $\rho$ satisfies Axiom (B).

**Theorem 2.7.** [U1, Theorem 11.1] The coefficients $\hat{a}_t$ $(t \in T)$ determine all the character values of $\rho$.

The assumption regarding the coefficients $b_s(w)$ in Theorem 2.6 is merely a normalization condition. Theorem 2.6 shows that the coefficients $a_s(w)$ and $b_s(w)$ in Axiom (A) depend only on the reflection $wsw^{-1} \in T$ and on the relation between $w$ and $ws$ in the right weak Bruhat order. It turns out that for simply laced Coxeter groups the coefficients $\hat{a}_t$ are given by a linear functional.

**Definition 2.8.** For a convex subset $K \subseteq W$ define:

$$
T_K := \{wsw^{-1} | s \in S, w \in K, ws \in K\}, \\
T_{\partial K} := \{wsw^{-1} | s \in S, w \in K, ws \not\in K\}.
$$

**Definition 2.9.** (K-genericity)

Let $K$ be a convex subset of $W$ containing the identity element. A vector $f$ in the root space $V$ is $K$-generic if:

(i) For all $t \in T_K$, 
$$
\langle f, \alpha_t \rangle \not\in \{0, 1, -1\}.
$$

(ii) For all $t \in T_{\partial K}$, 
$$
\langle f, \alpha_t \rangle \in \{1, -1\}.
$$

(iii) If $w \in K$, $s, t \in S$, $m(s, t) = 3$ and $ws, wt \not\in K$, then 
$$
\langle f, \alpha_{wsw^{-1}} \rangle = \langle f, \alpha_{wtw^{-1}} \rangle (= \pm 1).
$$

By Observation 2.4, we may assume that $id \in K$.

**Theorem 2.10.** [U1, Theorem 7.4] Let $(W, S)$ be an irreducible simply laced Coxeter system, and let $K$ be a convex subset of $W$ containing the identity element. If $f \in V^*$ is $K$-generic, then 
$$
\hat{a}_t := \frac{1}{\langle f, \alpha_t \rangle} \quad (\forall t \in T_K \cup T_{\partial K}),
$$
together with \( \dot{a}_t, \dot{b}_t \) and \( \ddot{b}_t \) satisfying
\[
\dot{a}_t + \ddot{a}_t = 0 \\
\dot{b}_t \cdot \ddot{b}_t = (1 - \ddot{a}_t)(1 - \ddot{a}_t)
\]
define a representation \( \rho \) such that \((\rho, K)\) is a minimal AY pair satisfying Axiom (B).

Remark 2.11. Various normalizations for \( \dot{b}_t \) and \( \ddot{b}_t \) are possible: symmetric (\( \ddot{b}_t = \dot{b}_t \)), seminormal (\( \ddot{b}_t = 1 \)), row stochastic (\( \dot{a}_t + \dot{b}_t = \ddot{a}_t + \ddot{b}_t = 1 \)), etc. [U1, Subsection 5.2]. By Theorem 2.7, all these normalizations give isomorphic representations.

The following theorem is complementary.

Theorem 2.12. [U1, Theorem 7.5] Let \((W, S)\) be an irreducible simply laced Coxeter system and let \( K \) be a subset of \( W \) containing the identity element. If \((\rho, K)\) is a minimal AY pair satisfying Axiom (B) and \( \dot{a}_t \neq 0 \) (\( \forall t \in T_K \)), then there exists a \( K \)-generic \( f \in V \) such that
\[
\dot{a}_t = \frac{1}{\langle f, \alpha_t \rangle} \quad (\forall \ t \in T_K \cup T_{\partial K}).
\]

2.5. Boundary Conditions. In this section it is shown that the action of the group \( W \) on the boundary of a minimal AY cell determines the representation up to isomorphism.

Definition 2.13. Let \( f \in V \) be an arbitrary vector on the root space \( V \) of \( W \).

(1) Define
\[
K^f := \{ w \in W \mid \forall t \in A_f, \ell(tw) > \ell(w) \}
\]
where
\[
A_f = \{ t \in T \mid \langle f, \alpha_t \rangle \in \{ \pm 1 \} \}.
\]
(2) If \( f \) is \( K^f \)-generic (as in Definition 2.9), then the corresponding AY representation of \( W \) (as in Theorem 2.10), with the symmetric normalization \( \dot{b}_t = \ddot{b}_t \) (\( \forall t \in T_K \)), will be denoted \( \rho^f \).

3. D-Young Tableaux and Minimal Cells in \( D_n \)

In this section we show that standard Young tableaux of skew shape lead to minimal AY cells.

3.1. Cells and Skew Shapes. In this subsection we study minimal AY cells \( K \subseteq D_n \). By Observation 2.4, every minimal AY cell is a translate of a minimal AY cell containing the identity element; thus we may assume that \( \text{id} \in K \).

We identify the root space of \( D_n \) with \( V = \mathbb{R}^n \) and for a vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) and recall the notation from Definition 2.1
\[
\Delta v = (v_1 + v_2, v_2 - v_1, \ldots, v_n - v_{n-1}) \in \mathbb{R}^n.
\]

For a (skew) tableau \( T \) denote
\[
c_k := j - i
\]
where \( k \) is the entry in row \( i \) and column \( j \) of \( T \). Call \( \text{cont}(T) := (c_1, \ldots, c_n) \) the content vector of \( T \), and call \( \Delta \text{cont}(T) \) the derived content vector of \( T \). Below we sometimes shall denote for brevity \( \text{cont}(T) \) as \( c(T) \).
Definition 3.1. Let $\lambda$ be a diagram of a skew shape. Define a $D$-Young tableau of shape $\lambda$ to be a filling of $\lambda$ by the $2n$ numbers $\pm 1, \pm 2, \ldots, \pm n$ in such a way that $c_{i-1} = -c_i$ for $1 \leq i \leq n$. A $D$-Young tableau is called standard if the numbers are increasing in rows and in columns. If $c_i = 0$, then we allow the numbers $\pm i$ to occupy the same box.

Remark 3.2. Our standard $D$-Young tableau doesn’t change if we multiply all its entries by $-1$ and rotate it by $180^\circ$. Indeed standard $D$-Young tableaux (except of the case when $\pm i$ occupy the same box) are “negative rotationally symmetric standard tableaux” considered by Ram in [Ra] just with additional requirement that the all assigned boxes form a skew shape.

Remark 3.3. (1) The contents $c_i$ are integer numbers $...,-2,-1,0,1,2,...$ We emphasize it here because in section 6 we shall consider noninteger contents.

(2) The need of considering the negative entries and the negative contents, as well as the need to allow two numbers in the same box will be discussed in subsection 5.2.5.

Here are three simple examples:

\[
\begin{array}{cccccc}
-9 & -8 & -7 & -6 & -5 \\
-4 & -3 & -2 & -1 \\
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 & 9
\end{array}
\quad
\begin{array}{cccc}
-4 & 1 \\
-3 & \pm 2 & 3 \\
-1 & 4
\end{array}
\quad
\begin{array}{cccc}
-4 & -2 & 1 & 4 \\
-3 & -2 & 1 & 4 \\
-4 & -1 & 2 & 3
\end{array}
\]

Recall the notations $K^f$ and $\rho^f$ from Definition 2.13. Also recall that for each $v \in \mathbb{R}^n$, $\langle v, \alpha_{ij} \rangle = v_i - v_j$.

The following theorem generalizes the part of sufficiency of Theorem 5.1 from [12].

Theorem 3.4. Let $T$ be a standard $D$-Young tableau with $2n$ boxes, let $c = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ be its content vector. Consider the sets $A_c = \{t \in T \mid \langle c, \alpha_t \rangle \in \{1,-1\}\}$ and $K^c = \{w \in D_n \mid \ell(tw) > \ell(w), \ \forall t \in A_c\}$. Then $c$ is $K^c$-generic and therefore gives rise to a minimal $AY$ pair $(\rho^c, K^c)$.

See [Cr, Theorem 2.3.4] for the proof of Theorem 3.4.

3.2. DAY Cells: definition and structure.

Definition 3.5. Let $T$ be a standard $D$-Young tableau with $2n$ boxes. It follows from Theorem 3.4 that $T$ gives rise to a minimal $AY$ pair $(K^c, \rho^c)$ where $c = \text{cont}(T)$. We call such a cell $K^c$ a DAY cell and such a representation $\rho^c$ a DAY representation.

The following theorem describes a DAY cell by a certain set of standard $D$-Young tableaux. Its statement and proof are similar to the statement and the proof of Theorem 5.5 from [12]. The proof of Theorem 3.6 may be found in [Cr Theorem 2.3.12].

Theorem 3.6. Let $T$ be a standard $D$-Young tableau and let $c = \text{cont}(T)$. Then for any $\pi \in D_n$,

$$\pi \in K^c \iff \text{The tableau } T^{\pi^{-1}} \text{ is a standard } D\text{-Young tableau.}$$
4. Standard D-Young tableaux: construction and enumeration.

We deal here only with tableaux which have at most two boxes on the zero content diagonal. The reason for it is that tableaux without boxes with zero content lead to representations induced to $D_n$ from $S_n$, while tableaux with one or two boxes of zero content give rise to minimal AY representations which are subrepresentations of these induced representations.

4.1. Tableaux without zero-content boxes.

Definition 4.1. Let $m \in \mathbb{N}$. Denote by $T[\lambda, m,+]$ the set which consists of all standard D-Young tableaux whose smallest nonnegative content is equal to $m$, whose boxes with positive contents form the shape $\lambda$ and the number of negative entries in the boxes with positive contents is even. The set $T[\lambda, m,-]$ is defined similarly to $T[\lambda, m,+]$, just the number of negative entries in the boxes with positive contents is odd.

Further, when we consider both $T[\lambda, m,+]$ and $T[\lambda, m,-]$ we sometimes write $T[\lambda, m,\pm]$ for brevity. Obviously from the definition, tableaux from sets $T[\lambda, m,\pm]$ have no zero-content boxes.

Recall that if the number of negative entries in the boxes of $T$ with positive contents is even, then it is also even for $T^\pi$ with any $\pi \in D_n$. For example, if we start with $T = \begin{array}{ccc} -3 & -2 & 1 \\ 2 & 3 & \end{array}$, $\text{cont}(T) = (2, 3, 4)$ we obtain the 4-element set $T[(3), 2,+]$.

Proposition 4.2. Let $\lambda$ be a shape (straight or skew) with $n$ boxes and $m \in \mathbb{N}$. Then

$$\#T[\lambda, m,+] = \#T[\lambda, m,-] = 2^{n-1} f^\lambda$$

Proof. We are dealing now with tableaux that have no box on zero-content diagonal. This means that the standardness (i.e. increasing of the entries in rows and columns) of the sub tableau with positive contents is not affected at all by the part with negative contents. In other words, when there is no box with content zero, then one can build a standard D-Young tableau just filling the shape $\lambda$ (which is the shape of $n$ boxes with positive contents) in the standard way (i.e. increasing in rows and columns) by the numbers $\{x_1, ..., x_n\}$ such that $\{|x_1|, \cdots, |x_n|\} = \{1, 2, \cdots, n\}$. For each choice of such numbers $\{x_1, ..., x_n\}$ there are $f^\lambda$ ways to arrange them in the shape $\lambda$ increasing in rows and columns. For the tableaux of the set $T[\lambda, m,+]$ the number of negative entries in the shape $\lambda$ has to be even i.e. it must be 0,2,4,... and for $T[\lambda, m,-]$ this number must be 1,3,5,... Therefore

$$\#T[\lambda, m,+] = \left( \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots \right) f^\lambda = 2^{n-1} f^\lambda$$

and

$$\#T[\lambda, m,-] = \left( \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots \right) f^\lambda = 2^{n-1} f^\lambda.$$
4.2. Tableaux having one or two boxes with zero content. Now consider tableaux with at most two boxes on the diagonal of zero content. Such tableau can be divided (sometimes not uniquely) into two sub tableaux: one contains \( n \) boxes with nonnegative contents and another (which is a reflection of the first one because \( c_{-i} = -c_i \)) contains the boxes with non positive contents.

4.2.1. Tableaux having one box with zero content. First consider the case in which the tableaux have one box with zero content. The condition \( c_{-i} = -c_i \) implies that this unique zero-content box must be occupied by two numbers \( \pm i \). For example,

\[
\begin{array}{c}
-3 \\
+2 \\
-1 \\
3
\end{array}
\]

Here the shape (but not a sub tableau!) of nonnegative contents can be defined uniquely and is

\[\lambda = \begin{array}{c}
\end{array} = (2, 1)\]

**Definition 4.3.** We define the set \( T[\lambda, \mathbb{X}] \) to be the set of all standard D-Young tableaux which have one box with zero content (always occupied by \( \pm i \) for some \( i \)) and shape of the boxes with nonnegative contents is \( \lambda \).

For example, the set \( T[(2, 1), \mathbb{X}] \) is

\[
\begin{array}{ccccccc}
-3 & -2 & -3 & 1 & -3 & -1 & -3 & 2 & -2 & 3 \\
+1 & +2 & +1 & +2 & +1 & +1 & +1 & +1 & +1 & +1 \\
2 & 3 & -1 & 3 & 1 & 3 & -2 & 3 & -2 & 3 \\
\end{array}
\]

while the set \( T[(4), \mathbb{X}] \) consists of only one tableau

\[
\begin{array}{ccccccc}
-4 & -3 & -2 & +1 & 2 & 3 & 4 \\
\end{array}
\]

4.2.2. Tableaux having two boxes with zero content. Now suppose we have a tableau with two different boxes on the zero content diagonal.

**Definition 4.4.** Denote by \( T[\lambda, \mathbb{X}, |, +] \ (T[\lambda, \mathbb{X}, |, -]) \) the set of standard D-Young tableaux which have exactly two boxes on the zero content diagonal and which can be divided by the vertical (horizontal) straight line into two parts (with \( n \) boxes in each part)– one consists of the boxes with nonnegative contents and another of non positive contents – and \( \lambda \) is the shape (straight or skew) of boxes with nonnegative contents after this separation and the number of negative entries in the boxes of \( \lambda \) is even. When the number of negative entries in the boxes of \( \lambda \) is odd we have \( T[\lambda, \mathbb{X}, |, -] \ (T[\lambda, \mathbb{X}, |, +]) \).

Consider, for example, the following tableau:

\[
\begin{array}{c}
-3 \\
+2 \\
-2 \\
1 \\
3
\end{array}
\]

The diagonal of zero contents passes through the number 3 and \(-3\) and this tableau can be divided as required above by the vertical line:

\[
\begin{array}{c}
-3 \\
+2 \\
-2 \\
3
\end{array} \quad \rightarrow \quad \begin{array}{c}
-3 \\
+2 \\
-2 \\
3
\end{array}
\]
Here \( \lambda = \begin{array}{c}
\hline
2
\hline
\end{array} + (2,1) \) and the number of negative entries in \( \lambda \) is equal to zero, thus even. The complete set \( T[(2,1),\cdot|\cdot,+] \) is

\[
\begin{array}{ccc}
-3 & 1 & 2 \\
-2 & 1 & 3 \\
\end{array} ;
\begin{array}{ccc}
-3 & 1 & 2 \\
1 & 2 & 3 \\
\end{array} ;
\begin{array}{ccc}
-3 & 2 & 1 \\
-3 & 1 & 2 \\
\end{array} 
\]

while the set \( T[(2,1),\cdot|\cdot,-] \) is

\[
\begin{array}{ccc}
-3 & 1 & 2 \\
-2 & 1 & 3 \\
\end{array} ;
\begin{array}{ccc}
-3 & 1 & 2 \\
1 & 2 & 3 \\
\end{array} ;
\begin{array}{ccc}
-3 & 2 & 1 \\
-3 & 1 & 2 \\
\end{array} 
\]

Not every standard D-young tableau can be divided by a vertical line into two halves of nonnegative and nonpositive contents. For example, if we divide by the vertical line the following tableau

\[
\begin{array}{ccc}
-4 & -3 \\
-2 & 1 \\
-1 & 2 \\
3 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
-4 & -3 \\
-2 & 1 \\
-1 & 2 \\
3 & 4 \\
\end{array} 
\]

each half has positive and negative contents together.

The following tableau can be divided as required above by the horizontal line:

\[
\begin{array}{ccc}
-3 \\
-2 & -1 \\
1 & 2 \\
3 \\
\end{array} \rightarrow \begin{array}{ccc}
-3 \\
-2 & -1 \\
1 & 2 \\
3 \\
\end{array} 
\]

Here the skew shape \( \lambda = \begin{array}{c}
\hline
2
\hline
\end{array} + (2,2)/(1) \) and the complete set is

\[
T[(2,2)/(1),\div,+] = \left\{ \begin{array}{ccc}
\begin{array}{ccc}
-3 \\
-2 & -1 \\
1 & 2 \\
3 \\
\end{array} ;
\begin{array}{ccc}
-3 & 1 \\
-2 & 1 \\
1 & 3 \\
-2 & 3 \\
\end{array} ;
\begin{array}{ccc}
-3 & -1 \\
-2 & 1 \\
1 & 3 \\
2 & -1 \\
\end{array} \right\}
\]

The structure of the D-tableaux with two boxes on the zero diagonal which belong to both \( T[\lambda,\div,+] \) and \( T[\lambda,\cdot|\cdot,-] \) is described in the following observation (this observation will not be used later):

**Observation 4.5.** Let \( \lambda = \mu/\nu \) for some partitions \( \mu = (\mu_1, \ldots, \mu_h) \vdash n + k \) where \( h = \mu'_1 \) and \( \nu = (\nu_1, \ldots, \nu_\ell) \vdash k \) where \( \ell = \nu'_1 \), and \( h > \ell \) and suppose that \( h - \ell = 1 \) and \( \mu_h \geq 2 \). Then

\[
T[\lambda,\div,+] = T[\lambda,\cdot|\cdot,-] \quad \text{and} \quad T[\lambda,\div,-] = T[\lambda,\cdot|\cdot,+]
\]

where \( \lambda = \mu/\nu \) for \( \mu = (\mu_1, \ldots, \mu_h, \mu_h+1) \) with \( \mu_1 = \mu_1 - 1 \), \( \mu_2 = \mu_2 - 1 \), ..., \( \mu_h = \mu_h - 1 \), \( \mu_h+1 = 1 \) and \( \nu = (\nu_1, \ldots, \nu_\ell) \) with \( \nu_1 = \nu_1 - 1 \), \( \nu_2 = \nu_2 - 1 \), ...,
Moreover, if $\nu T \lambda$ diagonal, there exists a shape $\lambda$ such that this tableau belongs to $T[\lambda, \cdot, \pm]$ or $T[\lambda, \div, \pm]$.

Remark 4.6. For each standard D-Young with exactly two boxes on the zero diagonal, there exists a shape $\lambda$ such that this tableau belongs to $T[\lambda, \cdot, \pm]$ or $T[\lambda, \div, \pm]$.

4.2.3. Constructing D-Young tableaux from a shape with n boxes. We continue to study sets $T[\lambda, \cdot, \pm]$, $T[\lambda, \div, \pm]$, $T[\lambda, m, \pm]$ and $T[\lambda, m, \div]$ in more detail. Lemmas 4.10 and 4.11 which are proved in this subsection are needed for the proof of the main result – Theorem 5.11.

Suppose we are given a shape (straight or skew) $\lambda$ with $n$ boxes. We shall address the following question: what sets of standard D-Young tableaux may be constructed by putting the lower left box of $\lambda$ on the diagonal of zero content?

Obviously it is always (i.e. for any $\lambda$) possible to construct the set $T[\lambda, \emptyset]$. The conditions which provide the existence of $T[\lambda, \cdot, \pm]$ and $T[\lambda, \div, \pm]$ are in given in the following simple remark.

Remark 4.7. If $h - \ell \geq 2$ and $\mu_h = 1$, then the sets $T[\lambda, \cdot, \pm]$ are defined but the sets $T[\lambda, \div, \pm]$ are not defined and we assume $T[\lambda, \div, \pm] = \emptyset$. If $h - \ell = 1$ and $\mu_h \geq 2$, then the sets $T[\lambda, \div, \pm]$ are defined but the sets $T[\lambda, \cdot, \pm]$ don’t exist and we assume $T[\lambda, \cdot, \pm] = \emptyset$. If $h - \ell \geq 2$ and $\mu_h \geq 2$, then both $T[\lambda, \div, \pm]$ and $T[\lambda, \cdot, \pm]$ are available. If $\mu_h = 1$ and $h - \ell = 1$, then the only thing we can construct putting the lower left box of $\lambda$ on the diagonal of zero content is the set $T[\lambda, \emptyset]$ while the sets $T[\lambda, \cdot, \pm]$ and $T[\lambda, \div, \pm]$ are empty.

Definition 4.8. Denote

$$T[\lambda, 0, +] := T[\lambda, \cdot, +] \cup T[\lambda, \div, +] \cup T[\lambda, \emptyset]$$

and

$$T[\lambda, 0, -] := T[\lambda, \cdot, -] \cup T[\lambda, \div, -] \cup T[\lambda, \emptyset].$$

The following theorem describes a bijection between $T[\lambda, 0, \pm]$ and $T[\lambda, m, \pm]$ with $m = 1, 2, 3, ...$

Theorem 4.9. Let $\lambda$ be a skew or straight shape with $n$ boxes. Then for any $m \in \mathbb{N}$ there exists a natural bijection:

$$T[\lambda, 0, +] = T[\lambda, \emptyset] \cup T[\lambda, \div, +] \cup T[\lambda, \cdot, +] \leftrightarrow T[\lambda, m, +].$$

and

$$T[\lambda, 0, -] = T[\lambda, \emptyset] \cup T[\lambda, \div, -] \cup T[\lambda, \cdot, -] \leftrightarrow T[\lambda, m, -].$$

Theorem 4.9 will be proved using Lemmas 4.10 and 4.11. It should be noted that Lemma 4.10 will be used later, when analyzing the associated representations.

Lemma 4.10. Let $\lambda$ be a straight or a skew shape with $n$ boxes, i.e. $\lambda = \mu/\nu$ for some partitions $\mu = (\mu_1, \ldots, \mu_k) \vdash n + k$ where $h = \mu_1'$ and $\nu = (\nu_1, \ldots, \nu_k) \vdash k$ where $\ell = \nu_1'$, and $h > \ell$, and let $T$ be a standard Young tableau of shape $\lambda$ filled by entries $x_1, x_2, \ldots, x_n$ such that $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, n\}$. If the number of negative entries in $T$ is even (odd), then there exists a unique standard D-Young
tableau $[T]_0 \in T[\lambda, 0, +]$ ($[T]_0 \in T[\lambda, 0, -]$, respectively) such that $T$ is its half of shape $\lambda$ with nonnegative contents.

Proof. (1) Let $h - \ell \geq 2$ and $\mu_\ell \geq 2$. Then the lower left corner of $T$ has at least two boxes in both directions:

$$
\begin{array}{|c|c|}
\hline
\ast & \ast \\
\hline
a & \ast \\
\hline
b & c \\
\hline
\end{array}
$$

where $a < b < c$.

Denote

$$
X =
\begin{array}{|c|c|c|}
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\end{array},
Y =
\begin{array}{|c|c|c|}
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\end{array},
Z =
\begin{array}{|c|c|c|}
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\ast & \ast & \ast \\
\hline
\end{array}.
$$

If $|a| < |b| < |c|$, then $X$ is a standard D-Young tableau while $Y$ and $Z$ are obviously non standard and we put $[T]_0 = X \in T[\lambda, \cdot|\cdot, +]$.

If $|a| > |b| > |c|$, then $Y$ is standard while $X$ and $Z$ aren’t and so we put $[T]_0 = Y \in T[\lambda, \div, +]$.

Finally, if $|a| > |b| < |c|$, then $Z$ is standard while $X$ and $Y$ aren’t and so we put $[T]_0 = Z \in T[\lambda, \otimes]$.

The case $|a| < |b| > |c|$ is impossible because $a < b < c$.

Consider the examples. Take

$$
T =
\begin{array}{|c|c|}
\hline
-2 & -1 \\
\hline
3 & 4 \\
\hline
\end{array}
$$

with its reflection

$$
\begin{array}{|c|c|}
\hline
-4 & -3 \\
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array},
$$

Then $[T]_0 =
\begin{array}{|c|c|c|c|}
\hline
-4 & -3 & -2 & -1 \\
\hline
1 & 2 & 3 & 4 \\
\hline
\end{array} \in T[(2, 2), \cdot|\cdot, +]$ while the tableaux

$$
\begin{array}{|c|c|}
\hline
-2 & -1 \\
\hline
3 & 4 \\
\hline
-4 & -3 \\
\hline
1 & 2 \\
\hline
\end{array}
$$

and

$$
\begin{array}{|c|c|}
\hline
-2 & -1 \\
\hline
-4 & +3 \\
\hline
1 & 2 \\
\hline
\end{array}
$$

are obviously not standard.

Two more examples:

$$
T =
\begin{array}{|c|c|}
\hline
-4 & -3 \\
\hline
-2 & -1 \\
\hline
\end{array}
\Rightarrow
[T]_0 =
\begin{array}{|c|c|}
\hline
-4 & -3 \\
\hline
-2 & -1 \\
\hline
1 & 2 \\
\hline
3 & 4 \\
\hline
\end{array} \in T[(2^2), \div, +]
$$
and

\[ T = \begin{array}{cc} -4 & 2 \\ -1 & 3 \end{array} \quad \Rightarrow \quad [T]_0 = \begin{array}{cc} -4 & 2 \\ -3 & 1 \\ -2 & 4 \end{array} \in T[(2^2), \mathbb{R}] . \]

(2) If \( h - \ell = 1 \) and \( \mu_h \geq 2 \), then the box \( a \) is absent in \( T \) and the tableau \( X \) doesn’t exist, \( T[\lambda, \cdot, +] = \emptyset \). If \( h - \ell \geq 2 \) and \( \mu_h = 1 \), then the box \( c \) is absent in \( T \) and the tableau \( Y \) doesn’t exist, \( T[\lambda, \cdot, +] = \emptyset \). The above item (1) of our proof can be repeated with slight changes: we have to erase \( a \) or \( c \), respectively, from all the places where they appear. If \( h - \ell = 1 \) and \( \mu_h = 1 \), then both \( a \) and \( c \) are absent, both of the sets \( T[\lambda, \cdot, +] \) and \( T[\lambda, \cdot, +] \) are empty, the lower left corner of \( T \) consists of a single box and \([T]_0 \in T[\lambda, \emptyset] \).

Thus, for any standard Young tableau \( T \) of an arbitrary shape \( \lambda \) with \( n \) boxes filled by entries \( x_1, x_2, \ldots, x_n \) such that \( \{ |x_1|, |x_2|, \ldots, |x_n| \} = \{ 1, 2, \ldots, n \} \) and the number of minuses in \( T \) is even we have uniquely defined a standard D-Young tableau \([T]_0 \in T[\lambda, 0, +]\) such that \( T \) is its half of shape \( \lambda \) with nonnegative contents, as claimed. When the number of negative entries in \( T \) is odd, we can repeat with appropriate slight changes all the arguments of the above proof to define \([T]_0 \in T[\lambda, 0, -] \).

\[ \square \]

**Lemma 4.11.** Let \( \lambda \) be a straight or a skew shape with \( n \) boxes and let \( T \) be a standard Young tableau of shape \( \lambda \) filled by entries \( x_1, x_2, \ldots, x_n \) such that \( \{ |x_1|, |x_2|, \ldots, |x_n| \} = \{ 1, 2, \ldots, n \} \). If the number of negative entries in \( T \) is even (odd), then for any \( m \in \mathbb{N} \) there exists a unique standard D-Young tableau \([T]_m \in T[\lambda, m, +] \) ((\([T]_m \in T[\lambda, m, -] \), respectively) such that \( T \) is its half of shape \( \lambda \) with positive contents.

**Proof.** Take \( T \) and put it in such a way that the box at its lower left corner has the content \( m \). Then put \( T \)'s reflection in such a way that its upper right corner box has the content \(-m\) and the whole picture is a skew shape, i.e. \( T \) is strictly to the northeast of its reflection. Clearly we obtain a D-Young tableau. Since there is no box on the zero content diagonal, the halves of positive and negative contents don’t touch each other and therefore the obtained D-Young tableau is standard and we denote it as \([T]_m \). Example:

\[ \lambda = (2) \vdash 2, \quad T = \begin{array}{c} -1 \end{array} \begin{array}{c} 2 \end{array}, \quad m = 1 \implies [T]_1 = \begin{array}{c} -1 \end{array} \begin{array}{c} 2 \end{array} \in T[(2), 1, -] \]

\[ \square \]

From Lemmas 4.10 and 4.11 we can easily conclude the following

**Proof of Theorem 4.9** For any standard D-Young tableau \( X \in T[\lambda, 0, \pm] \) or \( T[\lambda, m, \pm] \), we have \( X = [T]_0 \) or \( [T]_m \) where \( T \) is the half of \( n \) boxes with non-negative (positive) contents of shape \( \lambda \) in \( X \), and our claim now follows. \[ \square \]

5. Several types of DAY representations.

Given a standard D-Young tableau \( T \) with its content vector

\[ c(T) = (c_1(T), c_2(T), \ldots, c_n(T)) \]
As discussed in previous sections, $c(T)$ is a generic vector (see Theorem 3.4) and thus it gives rise to a minimal AY cell in $D_n$ (see Theorem 2.10). Denote

\[ f(T) = (f_0(T), f_1(T), ..., f_{n-1}(T)) \]
\[ = \Delta c(T) = (c_1 + c_2, c_2 - c_1, c_3 - c_2, ..., c_n - c_{n-1}) . \]

The corresponding minimal AY representation $\rho$, which we called a DAY representation (see Definition 3.5) acts on the space spanned by all the standard $D$-Young tableaux obtained from $T$ by the natural action of $D_n$. The action of representation matrices of generators is defined by

\[ \rho_{s_i}(T) = \frac{1}{f_i(T)} T + \sqrt{1 - \frac{1}{f_i(T)^2}} T^{s_i} \text{ for } i = 0, 1, 2, ..., n - 1 . \]

Notice (see Theorem 3.6) that $T^{s_i}$ is not standard if and only if

\[ \sqrt{1 - \frac{1}{f_i(T)^2}} = 0 \iff f_i = \pm 1 . \]

**Definition 5.1.** For $m \in \{0\} \cup \mathbb{N}$ we denote by $\rho^{\lambda,m,+}$, $\rho^{\lambda,m,-}$, $\rho^{\lambda,\cdot,+}$, $\rho^{\lambda,\cdot,-}$, $\rho^{\lambda,\cdot,\cdot,\cdot}$ DAY representations defined above by $(\ast)$ on spaces spanned by sets $T[\lambda, m,+], T[\lambda, m,-], T[\lambda, \cdot,+], T[\lambda, \cdot,-], T[\lambda, \cdot,\cdot], T[\lambda, \cdot,\cdot,\cdot], T[\lambda, \cdot,\cdot,\cdot,\cdot]$, respectively.

**Remark 5.2.** The sets $T[\lambda, \cdot,+], T[\lambda, \cdot,+]$, $T[\lambda, \cdot]$, $T[\lambda, \cdot,-], T[\lambda, \cdot,\cdot], T[\lambda, \cdot,\cdot,-], T[\lambda, \cdot,\cdot,\cdot]$ are pairwise disjoint because they consist of tableaux with different shapes and therefore

\[ \rho^{\lambda,0,+} = \rho^{\lambda,\cdot,+} \oplus \rho^{\lambda,\cdot,\cdot} \]

and similarly

\[ \rho^{\lambda,0,-} = \rho^{\lambda,\cdot,-} \oplus \rho^{\lambda,\cdot,\cdot} . \]

When we write $\rho^{\lambda,m,\cdot}$, $\rho^{\lambda,\cdot,\cdot} \cdot \cdot$ or $\rho^{\lambda,\cdot,\cdot}$ we mean “representations $\rho^{\lambda,m,+}$ and $\rho^{\lambda,m,-}$” or “representations $\rho^{\lambda,\cdot,+}$ and $\rho^{\lambda,\cdot,-}$” or “representations $\rho^{\lambda,\cdot,+}$ and $\rho^{\lambda,\cdot,-}$”, respectively.

5.1. **Representations $\rho^{\lambda,m,\cdot}$ are induced from $S_n$ to $D_n$.** There are two relevant embeddings of $S_n$ in $D_n$. Denote them

\[ S^1_n = \langle s_1, s_2, \ldots, s_{n-1} \rangle \]

and

\[ S^0_n = \langle s_0, s_2, \ldots, s_{n-1} \rangle . \]

When $n$ is odd $S^1_n$ and $S^0_n$ are conjugate subgroups in $D_n$. When $n$ is even they are not conjugate.

There is the following known fact:

**Fact 5.3.** The cosets are $D_n / S^1_n = \{ \sigma_1 S^1_n, \sigma_2 S^1_n, \ldots, \sigma_{n-1} S^1_n \}$ where

\[ \sigma_1 = e = (1 2 3 4 \ldots n) , \]
\[ \sigma_2 = s_0 = (-2 - 1 3 4 \ldots n) , \]
\[ \sigma_3 = s_2 s_0 = (-3 - 1 2 4 \ldots n) , \]
\[ \sigma_4 = s_1 s_2 s_0 = (-3 - 2 1 4 \ldots n) , \]
\[ \sigma_5 = s_3 s_2 s_0 = (-4 - 1 2 3 \ldots n) , \]
\[ \sigma_6 = s_1 s_3 s_2 s_0 = s_3 s_1 s_2 s_0 = (-4 - 2 1 3 \ldots n) , \]
\[\sigma_7 = s_2s_1s_3s_{2s_0} = (-4 - 3 1 2 \ldots n),\]
\[\sigma_8 = s_0s_2s_1s_3s_{2s_0} = (-4 - 3 - 2 - 1 \ldots n),\]
\[\sigma_{2n-1} = (-n - (n - 1) - (n - 2) \ldots - 2 - 1) if n is even and\]
\[\sigma_{2n-1} = (-n - (n - 1) - (n - 2) \ldots - 2 1) if n is odd.\]

These \(\sigma_i\) are coset representatives of minimal length.

**Fact 5.4.** To obtain the minimal length representatives of cosets of \(D_n / S_n^0\) which we shall denote as \(\tilde{\sigma}_i\) we just have to change the roles of \(s_1\) and \(s_0\) in each \(\sigma_i\). It is convenient to index them by the increasing sequences with odd number of minuses which are obtained by the action of each \(\tilde{\sigma}_i\) on the sequence \((-1 2 3 \ldots n)\):

\[\tilde{\sigma}_1 = e \longleftrightarrow (-1 2 3 4 \ldots n),\]
\[\tilde{\sigma}_2 = s_1 \longleftrightarrow (-2 1 3 4 \ldots n),\]
\[\tilde{\sigma}_3 = s_2s_1 \longleftrightarrow (-3 1 2 4 \ldots n),\]
\[\tilde{\sigma}_4 = s_0s_2s_1 \longleftrightarrow (-3 - 2 - 1 4 \ldots n),\]
\[\tilde{\sigma}_5 = s_3s_2s_1 \longleftrightarrow (-4 1 2 3 \ldots n),\]

\[\tilde{\sigma}_6 = s_4s_3s_2s_1 \longleftrightarrow (-5 1 \ldots n).\]

Now we prove that for any \(m \in \mathbb{N}\) our minimal AY representations \(\rho^{\lambda,m,+}\) are induced into \(D_n\) from \(S_n^1\) while \(\rho^{\lambda,m,-}\) are induced into \(D_n\) from \(S_n^0\).

Let \(\lambda\) be a straight or skew shape with \(n\) boxes and let \(S^\lambda\) denote the representation of \(S_n\) associated with \(\lambda\) via classical Young orthogonal form (see Theorems 2.2 and 2.3). (If \(\lambda\) is a straight shape, then \(S^\lambda\) is an irreducible Specht module and if \(\lambda\) is a skew shape, then \(S^\lambda\) is a skew Specht module.)

**Theorem 5.5.** For any natural \(m = 1, 2, 3, \ldots\)

1. \(\rho^{\lambda,m,+} \cong S^\lambda \uparrow_{S_n^1}^{D_n}\)
2. \(\rho^{\lambda,m,-} \cong S^\lambda \uparrow_{S_n^0}^{D_n}\)

**Proof.** We give the proof of (1) in detail and, then briefly explain how the proof of (2) follows.

Recall that the representation space of \(\rho^{\lambda,m,+}\) is spanned by the set of standard D-Young tableaux \(T[\lambda, m, +]\). As we already know (see Lemma 4.1) each standard tableau \(T\) which is a filling of \(\lambda\) (which is the shape of boxes with positive contents) by \(n\) numbers from the set \(\{\pm 1, \pm 2, \ldots, \pm n\}\) in such a way that from each pair \(\pm i\) only one number (i.e. or \(i\) or \(-i\)) appears in the tableau and the number of negatives in the tableau is even leads to the unique standard D-Young tableau \([T]_m \in T[\lambda, m, +]\) for which \(T\) is its sub tableau of shape \(\lambda\) of boxes with positive contents. So we can work with standard tableaux \(T\) of shape \(\lambda\) disregarding the parts with negative contents and from now on in this subsection we identify a standard D-Young tableau \([T]_m \in T[\lambda, m, +]\) with \(T\).

From now on in this subsection we denote for brevity \(\rho = \rho^{\lambda,m,+}\).

Fix some standard Young tableau \(T\) of shape \(\lambda\) filled with numbers \(1, 2, 3, \ldots, n\). Notice that for any other standard Young tableau \(\tilde{T}\) of shape \(\lambda\) filled with numbers \(1, 2, 3, \ldots, n\) there exists \(\pi \in S_n^1\) such that \(T = T^\pi\) (see proof of Theorem 3.6 for more details). Obviously the subspace of \(\text{Span}\{T[\lambda, m, +]\}\) spanned by all (standard D-Young tableaux \([T]_m\) whose halves of positive contents are) standard Young tableaux of shape \(\lambda\) filled with numbers \(1, 2, 3, \ldots, n\) is invariant under the action of \(\rho_s\) for \(0 < i = 1, 2, \ldots, n - 1\) (i.e. is invariant under the action of \(S_n^1\)) and the corresponding representation is exactly the representation \(S^\lambda\) of \(S_n^1\).
The following obvious observation is important:

**Observation 5.6.** Let $SYT(\lambda) = \{T_1, T_2, \ldots, T_{f_\lambda}\}$ be the set of all standard Young tableaux of shape $\lambda$ filled with numbers $1, 2, 3, \ldots, n$. Then all the tableaux $T_j^{\sigma_i}$ for $j = 1, 2, \ldots, f_\lambda$ and $i = 1, 2, \ldots, 2^n - 1$ with $\sigma_i$ defined in Fact 5.3 are also standard, i.e. their entries are increasing in rows and columns, and $T_j^{\sigma_i} \neq T_j^{\sigma_{i'}}$ for $j \neq j'$. Moreover, it is easy to see by induction on $\ell(\sigma_i)$ (the length of $\sigma_i$) that there exist signed permutations $\omega \in D_n$ and coefficients $a_{j,i,\omega}^{m_i} \neq 0$, $a_{j,i,\omega}^{m_i} \neq 0$ such that tableaux $T_j^{\sigma_i}$ are standard and

$$
\rho_{\sigma_i}(T_j) = \sum a_{j,i,\omega}^{m_i} T_j^{\omega} + \sum a_{j,i,\sigma_i}^{m_i} T_j^{\sigma_i} \quad \text{with} \quad \ell(\sigma_i) > \ell(\omega) \quad \text{for any} \quad \omega.
$$

In order to prove the first statement of Theorem 5.5 we must show that

$$
\text{Span}\{T[\lambda, m, +]\} = \bigoplus_{i=1}^{2^n-1} \rho_{\sigma_i}(\text{Span}\{SYT(\lambda)\})
$$

where

$$
\rho_{\sigma_i}(\text{Span}\{SYT(\lambda)\}) = \text{Span}\{\rho_{\sigma_i}(T_1) , \rho_{\sigma_i}(T_2) , \rho_{\sigma_i}(T_3) , \ldots , \rho_{\sigma_i}(T_{f_\lambda})\}.
$$

The number $m$ affects only on the certain values of coefficients $a_{j,i,\omega}^{m_i}$ but all the arguments of the proof remain the same for any $m \in \mathbb{N}$.

Now we explain why the subspaces $\rho_{\sigma_i}(\text{Span}\{SYT(\lambda)\})$ and

$$
\rho_{\sigma_i}(\text{Span}\{SYT(\lambda)\}) = \text{Span}\{\rho_{\sigma_i}(T_1) , \rho_{\sigma_i}(T_2) , \rho_{\sigma_i}(T_3) , \ldots , \rho_{\sigma_i}(T_{f_\lambda})\}
$$

have trivial intersection:

**Lemma 5.7.** In current notations

$$
\rho_{\sigma_i}(\text{Span}\{SYT(\lambda)\}) \cap \rho_{\sigma_{i'}}(\text{Span}\{SYT(\lambda)\}) = \{0\} \quad \text{for} \quad i \neq i'.
$$

The proof of this lemma will follow from two claims.

**Claim 5.8.** $T_j^{\sigma_i}$ cannot appear in the decomposition of $\rho_{\sigma_{i'}}(T_{j'})$ for $i' < i$ and any $j'$.

**Proof of Claim 5.8** Let the coset representatives $\sigma_i$ be arranged by increasing of length, i.e. $\ell(\sigma_i) \geq \ell(\sigma_{i'})$ when $i > i'$. Recall that $T_j, T_{j'} \in SYT(\lambda)$ and hence $T_{j'} = (T_j)^{\sigma_i}$ for some $\pi \in S_n$ (if $j' = j$, then $\pi = e$) and assume to the contrary that $T_j^{\sigma_i}$ appears in the decomposition of $\rho_{\sigma_{i'}}(T_{j'})$ for some $i' < i$. Then by Observation 5.6, we must have $T_j^{\sigma_i} = T_j^{\sigma_{i'}}$ or $T_j^{\sigma_i} = T_{j'}^{\sigma_i}$ for some certain $\omega \in D_n$ with $\ell(\sigma_i) \geq \ell(\sigma_{i'}) > \ell(\omega)$. The equality $T_j^{\sigma_i} = T_j^{\sigma_{i'}}$ is impossible because $T_j^{\sigma_i} = T_j^{\sigma_{i'}} = ((T_j)^{\pi})^{\sigma_{i'}} = (T_j)^{\sigma_{i'}}$ for some $\pi \in S_n$ means that $\sigma_i$ and $\sigma_{i'}$ belong to the same coset which is a contradiction. The equality $T_j^{\sigma_i} = T_{j'}^{\sigma_i}$ with $\pi \in S_n$, $\omega \in D_n$ and $\ell(\sigma_i) > \ell(\omega)$ is also impossible because it means that $\sigma_i = \omega \pi$ and hence $\sigma_i$ is not a coset representative of minimal length: $\omega = \sigma_i \pi^{-1} \in \sigma_i S_n$ and has shorter length than $\sigma_i$. This contradiction completes the proofs that $T_j^{\sigma_i}$ can not appear in the decomposition of $\rho_{\sigma_{i'}}(T_{j'})$ for $i' < i$ and any $j'$ and thus Claim 5.8 is proved. \qed
Claim 5.9. Let \( \{u_1, u_2, \ldots, u_k\} \) and \( \{v_1, v_2, \ldots, v_k\} \) be two linearly independent subsets of a vector space \( U \) such that \( \text{Span}\{u_i\}_{i=1}^k \cap \text{Span}\{v_i\}_{i=1}^k = \{0\} \). Then

\[
\text{Span}\{u_1, u_2, \ldots, u_k\} \cap \text{Span}\{v_1 + u_1, u_2 + v_2, \ldots, u_k + v_k\} = \{0\}.
\]

The proof of Claim 5.9 is a simple exercise in linear algebra. □

Proof of Lemma 5.7 Combine Observation 5.6 with Claims 5.8 and 5.9.

The subspace \( \rho_{\sigma_i}(\text{Span}\{\text{SYT}(\lambda)\}) = \text{Span}\{\text{SYT}(\lambda)\} \) is invariant under the action of \( \rho_{s_1}, \rho_{s_2}, \ldots, \rho_{s_{n-1}} \) (it is an \( S_n^1 \)-representation \( S^\lambda \)). By the construction, the \( \rho \)-action of \( D_n \) on \( \text{Span}\{T[\lambda, m, +]\} \) permutes the subspaces \( \rho_{\sigma_i}(\text{Span}\{\text{SYT}(\lambda)\}) \) and because these subspaces have trivial intersection their sum is direct. Also clear that for each \( i \) we have \( \dim \rho_{\sigma_i}(\text{Span}\{\text{SYT}(\lambda)\}) = f^\lambda \) and so

\[
\bigoplus_{i=1}^{n-1} \rho_{\sigma_i}(\text{Span}\{\text{SYT}(\lambda)\}) = \text{Span}\{T[\lambda, m, +]\}
\]

All what we said above means that \( \rho^\lambda, m, + \cong S^\lambda \uparrow_{S_n^1}^{D_n} \) by the definition of induced representation and thus the proof of the statement (1) of Theorem 5.5 is completed.

The proof of the statement (2) of Theorem 5.5 is very similar to above. Instead of \( \text{SYT}(\lambda) \) filled by numbers \( 1, 2, \ldots, n \) take the set \( \text{SYT}(\lambda) \) of standard tableaux of shape \( \lambda \) filled by numbers \( -1, 2, \ldots, n \). Easy to see that the subspace \( \text{Span}\{\text{SYT}(\lambda)\} \) is invariant under the action of \( \rho_{s_0}, \rho_{s_2}, \rho_{s_3}, \ldots, \rho_{s_{n-1}} \) and gives an \( S_n^1 \)-representation \( S^\lambda \). We list the coset representatives in Fact 5.4 and the proof goes similar to above with appropriate changes.

□

Now we illustrate the above proof with the example.

Example 5.10. Consider the set \( T[(2, 1), 1, +] \) and show explicitly that \( \rho^{(2, 1),1,+} \cong S^{(2,1)} \uparrow_{S_3^1}^{D_3} \). The minimal length coset representatives are

\[
\begin{align*}
\sigma_1 &= e = (1 \ 2 \ 3) , \\
\sigma_2 &= s_0 = (-2 \ -1 \ 3) , \\
\sigma_3 &= s_2s_0 = (-3 \ -1 \ 2) , \\
\sigma_4 &= s_1s_2s_0 = (-3 \ -2 \ 1) .
\end{align*}
\]

The set \( \text{SYT}(\lambda) = \{T_1, T_2\} \) is

\[
T_1 = \begin{array}{c}
1 \\
3 \\
2
\end{array} \quad \text{and} \quad T_2 = T_1^{s_2} = \begin{array}{c}
1 \\
2 \\
3
\end{array} .
\]

We disregard the part with negative contents and the content of the lower left box in \( T_1 \) and \( T_2 \) is 1.

Then

\[
\begin{align*}
T_1^{s_0} &= \begin{array}{c}
-2 \\
3 \\
-1
\end{array} , \\
T_2^{s_0} &= \begin{array}{c}
-2 \\
-1 \\
3
\end{array} , \\
T_1^{s_2s_0} &= \begin{array}{c}
-3 \\
-1 \\
2
\end{array} , \\
T_2^{s_2s_0} &= \begin{array}{c}
-3 \\
-1 \\
2
\end{array} .
\end{align*}
\]
\[
T_1^{s_1 s_2 s_0} = \begin{pmatrix} -3 & -2 \\ 1 \\ -2 \end{pmatrix}, \quad T_2^{s_1 s_2 s_0} = \begin{pmatrix} -3 & 1 \\ -2 \end{pmatrix}.
\]

So
\[
\rho_{\sigma_2}(T_1) = \frac{1}{10} T_1 + \frac{\sqrt{24}}{5} T_0^{s_0}, \quad \rho_{\sigma_2}(T_2) = \frac{1}{3} T_2 + \frac{\sqrt{8}}{3} T_0^{s_0},
\]
\[
\rho_{\sigma_3}(T_1) = \frac{1}{10} T_1 + \frac{\sqrt{24}}{10} T_0^{s_0} + \frac{\sqrt{24}}{15} T_0^{s_0} + \frac{192}{15} T_0^{s_0},
\]
\[
\rho_{\sigma_3}(T_2) = -\frac{1}{6} T_2 + \frac{\sqrt{24}}{6} T_0^{s_0} - \frac{\sqrt{8}}{15} T_0^{s_0} + \frac{192}{15} T_0^{s_0},
\]
\[
\rho_{\sigma_4}(T_1) = \frac{1}{10} T_1 - \frac{\sqrt{3}}{10} T_0^{s_2} + \frac{\sqrt{24}}{15} T_0^{s_0} + \frac{2\sqrt{3}}{15} T_0^{s_2} + \frac{2}{\sqrt{5}} T_0^{s_2 s_0},
\]
\[
\rho_{\sigma_4}(T_2) = \frac{1}{6} T_2 + \frac{\sqrt{24}}{6} T_0^{s_2} + \frac{\sqrt{8}}{15} T_0^{s_0} + \frac{2\sqrt{3}}{15} T_0^{s_2} + \frac{2}{\sqrt{5}} T_0^{s_2 s_0}.
\]

Here one can easily see that each \( \rho_{\sigma_i}(T_j) \) is linearly independent of all other \( \rho_{\sigma_j}(T_{j'}) \) and therefore
\[
\bigoplus_{i=1}^{4} \text{Span}\{\rho_{\sigma_i}(T_1), \rho_{\sigma_i}(T_2)\} = \text{Span}\{T[(2,1),1,+]\}
\]
which together with the fact that the \( \rho \)-action of \( D_3 \) by the construction permutes the subspaces \( \text{Span}\{\rho_{\sigma_i}(T_1), \rho_{\sigma_i}(T_2)\} \), \( 1 \leq i \leq 4 \), implies that \( \rho = \rho^{(2,1),1,+} \cong S^{(2,1)} / S_3 \).

5.2. **Decomposition of induced representation into minimal AY representations.** In this section we prove the main result

5.2.1. **Decomposition Rule.**

**Theorem 5.11.** Let \( \lambda \) be a straight or skew shape with \( n \) boxes and \( m \in \mathbb{N} \). Then

1. \( \rho^{\lambda, \Box} \otimes \rho^{\lambda, \cdot, +} \otimes \rho^{\lambda, \cdot, +} \cong \rho^{\lambda, m, +} \)
2. \( \rho^{\lambda, \Box} \otimes \rho^{\lambda, \cdot, -} \otimes \rho^{\lambda, \cdot, -} \cong \rho^{\lambda, m, -} \)

Note that, when the set \( T[\lambda, \cdot, \pm] \) or \( T[\lambda, \cdot, \pm] \) is empty, then the representation \( \rho^{\lambda, \cdot, \pm} \) or \( \rho^{\lambda, \cdot, \pm} \), respectively, is the zero module.

Notice also that in Theorem 5.11 it is enough to deal with \( m = 1 \) because it follows from Theorem 5.5 that representations \( \rho^{\lambda, m_1, +} \) and \( \rho^{\lambda, m_2, +} \) for any \( m_1, m_2 \in \mathbb{N} \) are isomorphic.

5.2.2. **Proof of Theorem 5.11.** First we will prove the following lemma:

**Lemma 5.12.** There exist matrix functions

\[
g_i : \mathbb{C} \to M_d(\mathbb{C})
\]
for \( i = 0, 1, 2, ..., n - 1 \) and \( d = 2^{n-1} \ell(\lambda) \) such that

\[
g_i(m) = \rho^{\lambda, m, +} \quad m = 0, 1, 2, ...
\]

The entries of \( g_i(x) \) are analytic single valued functions on \( \{ x \in \mathbb{C} : Re x > -1 \} \) and are continuous from the right at \( x = -1 \) as functions of real variable. The same holds for \( \rho^{\lambda, m, -} \).
The following obvious observation is crucial for the proof of Lemma 5.12.

**Observation 5.13.** If \((c_1, c_2, ..., c_n)\) is the content vector of a D-Young tableau \(T\), then the content vector of \(T^\pi\) is \((c_{\pi^{-1}(1)}, c_{\pi^{-1}(2)}, ..., c_{\pi^{-1}(n)})\).

**Proof of Lemma 5.12.** Take some standard D-Young tableau \(Q = [T]_m\) (for the definition of \([T]_m\) see Lemmas 4.10 and 4.11) and consider its half of shape \(\lambda\) with nonnegative contents denoted by \(T\). If for a certain \(\pi \in D_n\), \(T^\pi\) is standard, then \(Q^\pi = ([T]_m)^\pi = [T^\pi]_m\) is also standard for \(m = 1, 2, 3, \ldots\); however, for \(m = 0\), we can have a situation when \(T^\pi\) is standard while \(Q^\pi = ([T]_0)^\pi\) is not standard. In such a case, \(T^\pi\) is a half of shape \(\lambda\) with nonnegative contents of the standard D-Young tableau \([T^\pi]_0\) from some other subset of \([T][\lambda, 0, \pm]\), not from that which contains \(Q\). For example,

\[
Q = \begin{bmatrix}
-3 & 1 & 2 \\
-2 & -1 & 3
\end{bmatrix} \in T[(2, 1), [1], [+] \quad T = \begin{bmatrix}
1 & 2 \\
3
\end{bmatrix}
\]

and take \(\pi = \begin{bmatrix}
-3 & -2 & -1 & 1 & 2 & 3 \\
-2 & 1 & 3 & -3 & -1 & 2
\end{bmatrix}\) \(\pi \in D_3\). Then \(T^\pi = \begin{bmatrix}
-3 & -1 \\
2
\end{bmatrix}\) is standard while \(Q^\pi = ([T]_0)^\pi = \begin{bmatrix}
-2 & -3 & -1 \\
1 & 3 & 2
\end{bmatrix} \notin T[(2, 1), [1], [+]\) because is not standard.

Here \(T^\pi\) is the half of shape \(\lambda = (2, 1)\) of nonnegative contents for the standard D-Young tableau \([T^\pi]_0 = \begin{bmatrix}
-3 & -1 \\
1 & 3
\end{bmatrix} \in T[(2, 1), \mathbb{X}]\).

If we want to prove Theorem 5.11(1) we take a standard Young tableau \(T\) with \(n\) boxes of shape \(\lambda\) filled by numbers \(1, 2, 3, ..., n\) (the number of negative entries is zero, thus even). Let

\[
c = \text{cont} ([T]_0) = (c_1, c_2, ..., c_n)
\]

and define for \(x \in \{-1\} \cup \{x \in \mathbb{C} : \operatorname{Re} x > -1\}\) the vector function

\[
[c](x) = \left((|c|(x))_1, (|c|(x))_2, ..., (|c|(x))_n\right) = (c_1 + x + 1, c_2 + x + 1, ..., c_n + x + 1).
\]

For \(\pi \in D_n\) define also

\[
[c]^\pi(x) = \left((|c|(x))_{\pi^{-1}(1)}, (|c|(x))_{\pi^{-1}(2)}, ..., (|c|(x))_{\pi^{-1}(n)}\right).
\]

Note that \([e](x) = [c]^e(x)\) where \(e \in D_n\) is the identity element. Also note that when \(\pi^{-1}(j) = -i\) for some \(1 \leq i, j \leq n\), then we have

\[
(|c|(x))_{\pi^{-1}(j)} = (|c|(x))_{-i} = -(|c|(x))_i = -(c_i + x + 1).
\]

Clearly, by the construction of \([T]_m\) presented in proofs of Lemmas 4.10 and 4.11 each entry of \(c([T]_m)\) is greater by 1 than the corresponding entry of \(c([T]_{m-1})\) i.e.

\[
c([T]_m) = c([T]_{m-1}) + (1, 1, ..., 1).
\]

Note that this holds for \(m = 1\) too.

These functions \([c](x)\) are defined in such a way that substitutions \(x = -1, 0, 1, 2, ...\) in \([c](x)\) lead to content vectors of our tableaux \([T]_m\) with \(m \in \{0\} \cup \mathbb{N}\), i.e.,

\[
[c](m - 1) = c([T]_m);
\]
and therefore by definition of $[c]^{\pi}(x)$ and Observation [5.13] we have

$$[c]^{\pi}(m - 1) = c([T^{\pi}]_m).$$

Now define

$$[f](x) = \Delta [c](x), \quad [f]^{\pi}(x) = \Delta [c]^{\pi}(x).$$

The following is a most important observation of this proof: for integer $x = m - 1$ where $m = 0, 1, 2, \ldots$, we have

$$[f](m - 1) = \Delta [c](m - 1) = \Delta c([T]_m) = f([T]_m),$$

$$[f]^{\pi}(m - 1) = \Delta [c]^{\pi}(m - 1) = \Delta c([T^{\pi}]_m) = f([T^{\pi}]_m).$$

Now consider the space spanned by the set $R$ of standard tableaux of shape $\lambda$ with even number of negative entries

$$R = \{P : P \text{ is standard and } P = T^{\pi} \text{ for some } \pi \in D_n \}$$

and define for each $i = 0, 1, 2, \ldots, n - 1$ a linear operator on $\text{Span}(R)$ by its action on the basis set $R$:

$$g_i(x + 1)(T^{\pi}) = \frac{1}{([f]^{\pi}(x))_i} T^{\pi} + \sqrt{1 - \frac{1}{([f]^{\pi}(x))_i^2}} T^{s_i \pi},$$

where $([f]^{\pi}(x))_i$ denotes the $i$-th entry of the vector $[f]^{\pi}(x)$ and the branch of square root is $\sqrt{1} = 1$. Observe that when $T^{s_i \pi}$ is not standard, then $([f]^{\pi}(x))_i = \pm 1$, so we shall never exit $\text{Span}(R)$. The matrices of these operators with respect to the basis $R$ are matrix functions $g_i(x + 1)$. By the construction, these are exactly the representation matrices of the generators $\rho^{\lambda,m,+}_i$ for $x = m - 1$. This holds since the generator matrices of the representation $\rho^{\lambda,m,+}$ are completely determined by entries of derived content vectors of tableaux of the set $T^{[\lambda,m,+]} = \{[T^{\pi}]_m : T^{\pi} \text{ is standard}\}$.

In order to complete the proof of Lemma [5.12] we have to prove the following

**Claim 5.14.** For $i = 0, 1, 2, \ldots, n - 1$, mappings $g_i(x + 1)$ are well-defined for $x \in \mathbb{C}$ with $Re \, x > -1$. The coefficients $\frac{1}{([f]^{\pi}(x))_i}$ and $\sqrt{1 - \frac{1}{([f]^{\pi}(x))_i^2}}$ are single valued and analytic.

**Proof.** We must show that $|([f]^{\pi}(x))_i| > 1$ for $x \in \mathbb{C}$ with $Re \, x > -1$. By definition,

$$([f]^{\pi}(x))_0 = ([c]^{\pi}(x))_1 + ([c]^{\pi}(x))_2 = ([c](x))_{\pi^{-1}(1)} + ([c](x))_{\pi^{-1}(2)},$$

$$([f]^{\pi}(x))_i = ([c]^{\pi}(x))_{i+1} - ([c]^{\pi}(x))_i = ([c](x))_{\pi^{-1}(i+1)} - ([c](x))_{\pi^{-1}(i)}$$

for $i = 1, 2, \ldots, n - 1$. Obviously, if $\pi^{-1}(1)$ and $\pi^{-1}(2)$ have opposite signs, then $([f]^{\pi}(x))_0$ is constant, i.e. does not depend on $x$, and if $\pi^{-1}(i+1)\pi^{-1}(i) > 0$, then $([f]^{\pi}(x))_i$ does not depend on $x$. If $\pi^{-1}(1)\pi^{-1}(2) > 0$ or $\pi^{-1}(i+1)\pi^{-1}(i) < 0$, then for $i = 0, 1, 2, \ldots, n - 1$

$$|([f]^{\pi}(x))_i| = |c_{i_1} + c_{i_2} + 2x + 2|,$$

where $i_1 = |\pi^{-1}(i+1)|$ and $i_2 = |\pi^{-1}(i)|$ for $i \geq 1$, or $i_1 = |\pi^{-1}(1)|$ and $i_2 = |\pi^{-1}(2)|$ for $i = 0$. So,

$$|([f]^{\pi}(x))_i| = \sqrt{c_{i_1} + c_{i_2} + 2Re \, x + 2} + \sqrt{(2Im \, x)^2} \geq |c_{i_1} + c_{i_2} + 2Re \, x + 2|.$$

Numbers $c_{i_1}, c_{i_2}$ are nonnegative because they entries of the content vector of $[T]_0 \in T^{[\lambda,0,+]}$ and at most one of them may be equal to zero. Hence $c_{i_1} + c_{i_2} \geq 1$. 

By the assumption of our claim, $\Re x > -1$, and therefore $2\Re x + 2 > 0$. Thus, $|([f]^T(x))_i| \geq c_{i_1} + c_{i_2} + 2\Re x + 2 > 1$, and we are done. \hfill \square

The proof of Lemma 5.12 is completed. \hfill \square

For the further arguments we need the following Carlson’s theorem:

**Theorem 5.15. Carlson’s theorem.** If $f(z)$ is regular and of exponential type in the half plane $x \geq 0$ (i.e. if $f$ is analytic in the open half plane $x > 0$ and continuous in $x \geq 0$) and $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) < 2\pi$, then $f(z) \equiv 0$ if $f(n) = 0$, $n \in \mathbb{Z}$, where $h(\theta)$ is the indicator function of $f$:

$$h(\theta) = \limsup_{r \to +\infty} \frac{\log |f(re^{i\theta})|}{r}$$

(See [B] for the details about the indicator function $h(\theta)$ and the proof of the Carlson’s theorem.)

**Proof of Theorem 5.11.** Now we give two rather similar arguments such that each of them completes the proof of Theorem 5.11.

**Argument (1)** Take some $w = s_{i_1} \cdots s_{i_2} s_{i_1} \in D_n$ and denote

$$\chi_w(x) = \text{Trace}(g_{i_1}(x+1) \cdots g_{i_2}(x+1) g_{i_1}(x+1)).$$

If we fix $w$, then $\chi_w$ is a function of $x$ and it is a polynomial in several expressions of the form

$$\pm \frac{1}{c_i + c_j + 2x + 2}, \quad \sqrt{1 + \frac{1}{(c_i + c_j + 2x + 2)^2}},$$

where $c_i, c_j$ are some entries of the content vector of $|T|_0$, as we have seen in the proof of Claim 5.14. Obviously, there exist real positive constants $R$ and $d$ such that

$$\forall x = re^{i\theta} \in \mathbb{C} : |x| = r \geq R \implies |\chi_w(x)| = |\chi_w(re^{i\theta})| \leq r^d.$$

Notice that $\chi_w(m-1)$ (with $m = 1, 2, 3, \ldots$) is the value of character of the representation $\rho^{\lambda,m-1}$. We already know by Theorem 5.9 that all $\rho^{\lambda,m-1}$ for $m = 1, 2, 3, \ldots$ are isomorphic and therefore $\chi_w(m-1) = \text{const}$ for $m = 1, 2, 3, \ldots$. Considering $\chi_w(x)$ as a function of $x$ and choosing for square roots the branch with $\sqrt{T} = 1$ (the branching points of our square roots are at the left of zero (indeed at the left of $-1$) as we showed in Claim 5.14 and so we can take the branches which are analytic in the right half plane), we see that $\chi_w(x)$ satisfies conditions of Carlson’s theorem (see Theorem 5.15):

$$\forall \theta : h(\theta) = \limsup_{r \to +\infty} \frac{\log |\chi_w(re^{i\theta})|}{r} \leq \limsup_{r \to +\infty} \frac{\log (r^d)}{r} = 0 \implies h\left(\frac{\pi}{2}\right) + h\left(-\frac{\pi}{2}\right) = 0 < 2\pi.$$

Hence, by Carlson’s theorem, $\chi_w(x)$ must be constant for any number with positive real part, in particular for real $x > 0$. Moreover, by the uniqueness of analytic extension, it must be constant also for the numbers with real part between $-1$ and $0$. So $\chi_w(x) = \text{const}$ for $\{x \in \mathbb{C} : \Re x > -1\}$. Because at the point $x = -1$ the function $\chi_w(x)$ is continuous from the right function of real variable, we have $\chi_w(-1) = \chi_w(x) = \text{const}$. But at $x = -1$, the value $\chi_w(-1)$ is the value of character of the representation $\rho^{\lambda,0,+}$ calculated at $w$. The element $w$ is an
arbitrary element of our group $D_n$, so the character of $\rho^{\lambda,0,+}$ is the same as the character of $\rho^{\lambda,m,+}$ which implies that

$$\rho^{\lambda,0,+} = \rho^{\lambda,0,+} \oplus \rho^{\lambda,1,+} \oplus \rho^{\lambda,2,+} \cong \rho^{\lambda,m,+},$$

and the first statement of Theorem 5.11 is proved.

Here is another argument which also proves the first statement of Theorem 5.11.

**Argument (2)** The group $D_n$ has $(n^2 + n)/2$ Coxeter relations:

$$(s_is_j)^{m_{ij}} = 1, \quad 0 \leq i < j \leq n - 1,$$

where $m_{ii} = 1$ and $m_{ij} = 2$ or $m_{ij} = 3$ for $i < j$.

According to these relations, we introduce $(n^2 + n)/2$ matrix functions of $x$ for $0 \leq i \leq j \leq n - 1$:

$$A_{ij}(x) = (g_i(x + 1)g_j(x + 1))^{m_{ij}}.$$

For $x = -1, 0, 1, 2, 3, ...$, the matrices $g_i(x + 1)$ (for $i = 0, 1, 2, ..., n - 1$) are generator matrices of representations $\rho^{\lambda,0,+}$, $\rho^{\lambda,1,+}$, $\rho^{\lambda,2,+}$, ... and therefore the generator matrices $\rho^{\lambda,0,+}$, $\rho^{\lambda,1,+}$, $\rho^{\lambda,2,+}$, ... must satisfy the defining relations of the group which implies

$$A_{ij}(x) = I \quad \text{for } 0 \leq i \leq j \leq n - 1 \quad \text{and} \quad x = -1, 0, 1, 2, 3, ...,$$

where $I$ is the identity matrix. But the entries of $A_{ij}(x)$ are polynomials in several expressions of the form

$$\pm \frac{1}{c_i + c_j + 2x + 2}, \quad \sqrt{1 - \frac{1}{(c_i + c_j + 2x + 2)^2}},$$

where $c_i$, $c_j$ are some entries of the content vector of $[T]_0$, as we have seen in the proof of Claim 5.14. By the same argument (which involves the Carlson’s theorem) as above (in Argument (1)), we conclude that

$$A_{ij}(x) = I \quad \text{for } 0 \leq i \leq j \leq n - 1 \quad \text{and for any real } x \geq -1,$$

not only for integer $x = -1, 0, 1, 2, 3, ...$ as before. This means that the matrices $g_i(x + 1)$ are generator matrices of representations not only for integer $x = -1, 0, 1, 2, 3, ...$ but for any real $x \geq -1$. Denote this representation $\rho^{(x+1)}$. As above, take some fixed $w \in D_n$ and denote by $\chi_w(x)$ the character of the representation $\rho^{(x+1)}$ evaluated at $w$. The character is a polynomial in the entries of representation matrices, so in our case $\chi_w(x)$ is a polynomial in several expressions of the form

$$\pm \frac{1}{c_i + c_j + 2x + 2}, \quad \sqrt{1 - \frac{1}{(c_i + c_j + 2x + 2)^2}}.$$

and therefore $\chi_w(x)$ is a continuous function of $x \geq -1$. By discreteness of the character values and continuity $\chi_w(x)$, we have $\chi_w(x) = \text{const}$ which implies that all the representations $\rho^{(x+1)}$ for $x \geq -1$ are isomorphic. In particular, substituting $x = -1, 0, 1, 2, 3, ...$ we get

$$\rho^{\lambda,0,+} \oplus \rho^{\lambda,1,+} \oplus \rho^{\lambda,2,+} \cong \rho^{\lambda,m,+},$$

and the first statement of Theorem 5.11 is proved.

In order to prove the second statement of Theorem 5.11 we have to start with standard tableau $T$ filled by numbers $-1, 2, 3, ..., n$. Everything is similar to the above with very slight changes. The proof of Theorem 5.11 is completed. □
Remark 5.16. In the argument (2) we did not use Theorem 5.5. Moreover, in argument (2) we proved that all the representations \( \rho^{(x)} \) are isomorphic and denoting
\[
\rho_{s_i}^{(\infty)} = \lim_{x \to +\infty} \rho_{s_i}^{(x)} \quad \text{for } i = 0, 1, 2, \ldots, n - 1
\]
we obtain the representation \( \rho^{(\infty)} \) which is exactly the classical form of induced representation as given, for example, in [U1], proof of Theorem 9.3, page 32. This observation itself can be used to give another proof of Theorem 5.5 because, as we have seen above in argument (2), \( \chi_w(x) = \text{const} \) and has the same value when \( x \to \infty \) which implies that representations \( \rho^{(x)} \) (and in particular \( \rho^{\lambda,m,+} \) for \( m \in \mathbb{N} \)) are isomorphic to the representation \( \rho^{(\infty)} \) obtained when \( x \) tends to infinity.

5.2.3. An Example. Now we give an example to illustrate the above proof of Theorem 5.11.

Claim 5.17. For \( m \in \mathbb{N} \)
\[
\rho^{(3)}(\mathbb{E}) \oplus \rho^{(3)}(\div , +) \cong \rho^{(3),m,+}
\]
In this case the set \( T[(3), \div , +] \) is empty. Hence, Claim 5.17 is equivalent to
\[
\rho^{(3),\mathbb{E}} \oplus \rho^{(3),\div,+} \cong \rho^{(3),m,+}.
\]
We can “formulate” this claim graphically:

Here zero shows the cells with zero content and each diagram describes the corresponding representation:

\[
\rho^{(3),\mathbb{E}} \rightarrow \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \rho^{(3),\div,+} \rightarrow \begin{bmatrix} 0 & 0 \end{bmatrix}
\]
and \( \rho^{(3),1,+} \rightarrow \begin{bmatrix} \end{bmatrix} \).

Of course, this claim can be easily verified by direct calculations because the group \( D_3 \) is small (occasionally it is isomorphic to \( S_4 \)) and its character table is well known, but we shall follow the above general proof.

Consider four following tableaux
\[
T_1 = T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad T_2 = T^{s_0} = \begin{bmatrix} -2 & -1 & 3 \end{bmatrix},
\]
\[
T_3 = T^{s_0 s_2} = \begin{bmatrix} -3 & -1 & 2 \end{bmatrix}, \quad T_4 = T^{s_1 s_2 s_0} = \begin{bmatrix} -3 & -2 & 1 \end{bmatrix}
\]
and denote
\[
R = \{ T_1, T_2, T_3, T_4 \} = \{ \pi^\pi : \pi \in \{ e, s_0, s_2 s_0, s_1 s_2 s_0 \} \}.
\]

The half with nonnegative contents of shape \( \lambda = (3) \) of standard \( D \)-Young tableau from the set \( T[(3), m, +] \) for \( m \in \{ 0 \} \cup \mathbb{N} \) must be one of the four tableaux from the set \( R \).

For each \( m \in \{ 0 \} \cup \mathbb{N} \) the set \( T[(3), m, +] = \{ [T_1]_m, [T_2]_m, [T_3]_m, [T_4]_m \} \) for each \( m \in \{ 0 \} \cup \mathbb{N} \), and we have \( T[(3), 0, +] = T[(3), \mathbb{E}] \cup T[(3), \div , +] \).
Consider $T[(3), \Xi]$ which consists of only one tableau $[T_1]_0$ and write down its content vector $c([T_1]_0)$ and its derived content vector $f([T_1]_0)$:

$$[T_1]_0 = \begin{bmatrix} -3 & -2 & 1 & 2 & 3 \end{bmatrix}$$

$$c([T_1]_0) = (0, 1, 2)$$

$$f([T_1]_0) = \Delta c([T_1]_0) = (c_1 + c_2, c_2 - c_1, c_3 - c_2) =$$

$$= (0 + 1, 1 - 0, 2 - 1) = (1, 1, 1).$$

Obviously the corresponding representation $\rho^{(3),\Xi}$ is the trivial representation of $D_3$ because $\rho_{s_0}^{(3),\Xi} = \rho_{s_1}^{(3),\Xi} = \rho_{s_2}^{(3),\Xi} = 1$.

Now consider the set $T[(3), \hat{+}, +] = \{ [T_2]_0, [T_3]_0, [T_4]_0 \}$ which is:

$$[T_2]_0 = \begin{bmatrix} -2 & -1 & 3 \\ -3 & 1 & 2 \end{bmatrix}$$

$$c([T_2]_0) = (-1, 0, 2)$$

$$f([T_2]_0) = \Delta c([T_2]_0) = (c_1 + c_2, c_2 - c_1, c_3 - c_2) =$$

$$= (-1 + 0, 0 - (-1), 2 - 0) = (-1, 1, 2),$$

$$[T_3]_0 = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

$$f([T_3]_0) = (1, 3, -2),$$

$$[T_4]_0 = \begin{bmatrix} -3 & -2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$f([T_4]_0) = (1, -3, 1).$$

Now we denote for brevity $\rho^{(0)} = \rho^{(3),0,+} = \rho^{(3),\Xi} \oplus \rho^{(3),\hat{+},+}$ and write down the representation matrices of generators for the representation $\rho^{(0)}$. The upper left $1 \times 1$ block in each generator matrix corresponds to the 1-dimensional trivial representation $\rho^{(0)}_{s_0}$ and the lower right $3 \times 3$ block corresponds to the 3-dimensional representation $\rho^{(3),\hat{+},+}$:

$$\rho^{(0)}_{s_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho^{(0)}_{s_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{-1}{3} & 0 \end{pmatrix},$$

$$\rho^{(0)}_{s_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{-1}{3} & \frac{-\sqrt{3}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now consider the minimal AY representation $\rho^{(1)} = \rho^{(3),1,+}$ which is induced from the trivial representation of $S_3$ to $D_3$ as it follows from Theorem 5.5. The set $T[(3), 1, +]$ is:

$$[T_1]_1 = \begin{bmatrix} 1 & 2 & 3 \\ -3 & -2 & 1 \end{bmatrix}$$

$$c([T_1]_1) = (1, 2, 3)$$

$$f([T_1]_1) = (3, 1, 1),$$

$$[T_2]_1 = \begin{bmatrix} -2 & -1 & 3 \\ -3 & 1 & 2 \end{bmatrix}$$

$$f([T_2]_1) = (-3, 1, 4),$$

$$[T_3]_1 = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

$$f([T_3]_1) = (1, 5, -4).$$
The generator matrices in this case are

\[ [T_4]_1 = \begin{pmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \end{pmatrix} \quad f([T_4]_1) = (1, -5, 1). \]

The representation \( \rho \) of the group as in the natural representation of \( \rho \) is a permutation representation, moreover its generator matrices are isomorphic.

Define matrix functions:

\[ g_0(x + 1) = \begin{pmatrix} \frac{1}{2x+3} & \frac{\sqrt{(2x+3)^2-1}}{2x+3} & 0 & 0 \\ \frac{\sqrt{(2x+3)^2-1}}{2x+3} & \frac{1}{2x+3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ g_1(x + 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2x+5} & \frac{\sqrt{(2x+5)^2-1}}{2x+5} \\ 0 & 0 & \frac{\sqrt{(2x+5)^2-1}}{2x+5} & \frac{1}{2x+5} \end{pmatrix}, \]

\[ g_2(x + 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2x+4} & \frac{\sqrt{(2x+4)^2-1}}{2x+4} \\ 0 & 0 & \frac{\sqrt{(2x+4)^2-1}}{2x+4} & \frac{1}{2x+4} \end{pmatrix}. \]

Easy to see that substituting \( x = -1, 0 \) we get matrices \( \rho_i^{(0)}, \rho_i^{(1)} \) for \( i = 0, 1, 2 \), respectively. The entries and the traces of the products of these matrix functions are polynomials in \( \frac{1}{2x+3}, \frac{\sqrt{(2x+3)^2-1}}{2x+3}, \frac{1}{2x+5}, \frac{\sqrt{(2x+5)^2-1}}{2x+5}, \frac{1}{2x+4}, \frac{\sqrt{(2x+4)^2-1}}{2x+4} \). Therefore, the Carlson’s theorem is applicable, as we explained in argument (1) of the proof of Theorem \[5.11\] and both of arguments (1) and (2) work here properly to show that representations \( \rho^{(0)} \) and \( \rho^{(1)} \) are isomorphic.

The matrices of generators in the representation \( \rho^{(\infty)} \) which was introduced in Remark \[5.10\] are

\[ \rho_{s_0}^{(\infty)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{s_1}^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{s_2}^{(\infty)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The representation \( \rho^{(\infty)} \) is induced into \( D_3 \) from the trivial representation of \( S_3 = \langle s_1, s_2 \rangle \) and is the classical form of induced representation as given, for example, in [51], proof of Theorem 9.3, page 32. We see that in our case the representation \( \rho^{(\infty)} \) is a permutation representation, moreover its generator matrices are exactly as in the natural representation of \( S_4 \). It is not a surprise because \( D_3 \cong S_4 \) and
the representation which is induced from the trivial representation is a permutation representation obtained from the action of $D_3$ on the set $D_3/S_1^3$ which consists of four cosets. It is a well known fact that the natural representation of the symmetric group $S_n$ is isomorphic to the direct sum the trivial representation $S_n$ and the irreducible representation of $S_n$ associated with the partition $(n-1,1)$ ⊢ $n$. In our example we have a special case of this observation for $n = 4$: the representation $\rho^{(3)}_{\lambda,\pm}$ is the trivial representation of $D_3 \cong S_4$ while the representation $\rho^{(3),\pm,\pm}$ is isomorphic to the irreducible representation of $S_4 \cong D_3$ associated with the partition $(3,1) \vdash 4$.

In the previous example the decomposition had only two factors (we had there $\lambda = (3)$, the set $T[\lambda,\cdot,\pm] = \emptyset$ and the representation $\rho^{\lambda,\cdot,\pm}$ was the zero representation). If we take $\lambda = (2,2)$ then all three factors $\rho^{\lambda,\cdot,\pm}$, $\rho^{\lambda,\pm,\pm}$ and $\rho^{\lambda,\cdot,\cdot}$ are nonzero representations. This can be expressed graphically

Here zero shows the boxes with zero content and each diagram describes the corresponding representation:

Concerning this example we notice also the following: Theorem 5.5 implies that the representation $\rho^{2,\cdot,\cdot,\cdot}$ and $\rho^{\cdot,\cdot,\cdot,\cdot}$ are isomorphic to the representations induced to $D_4$ from the irreducible two dimensional representation of $S_1^4$ and $S_0^4$ respectively, associated with the Young diagram $2^2 = (2,2)$.

5.2.4. Remarks. We finish this section with several remarks.

Remark 5.18. It is possible to avoid using Carlson’s theorem (which is a highly nontrivial result from the theory of analytic functions) in arguments (1) and (2) of the proof of Theorem 5.11 if we take a stochastic normalization instead of symmetric (see Remark 2.11), then the characters will be rational functions of $x$ (without square roots) and a rational function which has infinitely many zeros is zero because a polynomial can’t have infinitely many zeros.

There is a natural question: when the $D_n$-representations $\rho^{\lambda,m,\pm}$ and $\rho^{\lambda,m,-}$ are isomorphic? By Theorem 5.5 we have $\rho^{\lambda,m,\pm} \cong S^\lambda \uparrow_{S_n^m}$ and $\rho^{\lambda,m,-} \cong S^\lambda \uparrow_{S_n^m}$. The
subgroups $S_n^1$ and $S_n^0$ are always conjugate in $B_n$. If $n$ is odd, then $S_n^1$ and $S_n^0$ are also conjugate as subgroups in $D_n$ (the element $\pi$ such that $\pi(1) = 1$ and $\pi(i) = -i$ for $i = 2, 3, \ldots n$ belongs to $D_n$ when $n$ is odd and $\pi^{-1}s_1\pi = s_0$, $\pi^{-1}s_i\pi = s_i$ for $i = 2, 3, \ldots n - 1$) which easily implies that the induced representations are isomorphic. When $n$ is even the representations $\rho^{\lambda,m,+}$ and $\rho^{\lambda,m,-}$ can be non isomorphic. The irreducible representations of $B_n$ are indexed by ordered pairs of partitions $(\alpha, \beta)$ such that $|\alpha| + |\beta| = n$. Restricting irreducible representations $(\alpha, \beta)$ and $(\beta, \alpha)$ with $\alpha \neq \beta$ from $B_n$ to $D_n$ we get the same irreducible representation of $D_n$ and therefore irreducible representations of $D_n$ are indexed by unordered pairs of partitions $\{\alpha, \beta\}$ such that $|\alpha| + |\beta| = n$. For even $n$ there are two non isomorphic irreducible representations of $D_n$ $\{\alpha, \alpha\}^+$ and $\{\alpha, \alpha\}^-$ where $\alpha + \frac{n}{2}$ which split from the irreducible representation of $B_n$ associated with $(\alpha, \alpha)$. It can be easily shown that

$$1 \uparrow_{S_n^0}^{B_n} \cong 1 \uparrow_{S_n^1}^{B_n} \cong \bigoplus_{k=0}^{n} ((n-k), (k))$$

and therefore

$$\rho^{(n),m,+} \cong 1 \uparrow_{S_n^0}^{D_n} \cong \bigoplus_{k=0}^{\frac{n}{2}-1} \{(n-k), (k)\} \oplus \left\{ \left(\frac{n}{2}\right), \left(\frac{n}{2}\right) \right\}^+$$

and

$$\rho^{(n),m,-} \cong 1 \uparrow_{S_n^0}^{D_n} \cong \bigoplus_{k=0}^{\frac{n}{2}-1} \{(n-k), (k)\} \oplus \left\{ \left(\frac{n}{2}\right), \left(\frac{n}{2}\right) \right\}^-$$

and therefore $\rho^{(n),m,+} \ncong \rho^{(n),m,-}$.

**Remark 5.19.** Using the character table of $D_4$ one can see that the representation $\rho^{(2^2),\cdot,-}$ whose representation space is spanned by

$$
\begin{pmatrix}
-4 & 3 & 2 & 1 \\
-2 & 3 & 4 & 1
\end{pmatrix},
\begin{pmatrix}
-4 & -3 & -2 & 1 \\
2 & 1 & 3 & 4
\end{pmatrix},
\begin{pmatrix}
-4 & -2 & -1 & 3 \\
-3 & 2 & 4 & 1
\end{pmatrix}
$$

and whose generator matrices are

$$\rho_{s_0}^{(2^2),\cdot,-} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\rho_{s_1}^{(2^2),\cdot,-} = 
\begin{pmatrix}
1 & \frac{\sqrt{3}}{2} & 0 \\
\frac{1}{2} & 0 & -1 \\
0 & 0 & 1
\end{pmatrix},
\rho_{s_2}^{(2^2),\cdot,-} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & \frac{\sqrt{3}}{2} & 1
\end{pmatrix},
\rho_{s_3}^{(2^2),\cdot,-} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}$$

is isomorphic to the irreducible three-dimensional split representation $\{\{2\}, \{2\}\}^-$ while the representation $\rho^{(2^2),\cdot,+}$ whose representation space is spanned by

$$
\begin{pmatrix}
-4 & -3 & 1 & 2 \\
-2 & -1 & 3 & 4
\end{pmatrix},
\begin{pmatrix}
-4 & -2 & 1 & 3 \\
-3 & 1 & 2 & 4
\end{pmatrix},
\begin{pmatrix}
-4 & -3 & -2 & -1 \\
1 & 2 & 3 & 4
\end{pmatrix}
$$

is isomorphic to another irreducible three-dimensional split representation $\{\{2\}, \{2\}\}^+$. Of course, $\rho^{(2^2),\cdot,+} \ncong \rho^{(2^2),\cdot,-}$ because $\{\{2\}, \{2\}\}^+ \ncong \{\{2\}, \{2\}\}^-$. Similarly, one can verify directly that $\rho^{(2^2),\cdot,-} \cong \{\{1^2\}, \{1^2\}\}^-$ and $\rho^{(2^2),\cdot,+} \cong \{\{1^2\}, \{1^2\}\}^+$. 
The following conjecture seems to be true:

**Conjecture 5.20.** Let \( n \in \mathbb{N} \) be even. Then

\[
\rho \left( \left( \frac{n}{2} \right)^2 \cdot \frac{1}{2} \right) \cong \left\{ \left( \frac{n}{2} \right), \left( \frac{n}{2} \right) \right\}^-, \rho \left( \left( \frac{n}{2} \right)^2 \cdot \frac{1}{2} \right) \cong \left\{ \left( \frac{n}{2} \right), \left( \frac{n}{2} \right) \right\}^+,
\]

\[
\rho \left( \frac{2n}{2} \cdot \frac{1}{2} \right) \cong \left\{ \left( \frac{1}{2} \right), \left( \frac{1}{2} \right) \right\}^- , \rho \left( \frac{2n}{2} \cdot \frac{1}{2} \right) \cong \left\{ \left( \frac{1}{2} \right), \left( \frac{1}{2} \right) \right\}^+.
\]

**5.2.5. Why we need the negative contents?** The representations \( \rho^{\lambda, m, \pm} \) (with \( m \in \mathbb{N} \)), \( \rho^{\lambda, \cdot \mid \cdot, \pm} \), \( \rho^{\lambda, \cdot \div \cdot, \pm} \), \( \rho^{\lambda, \cdot \wedge \cdot, \pm} \) are completely determined by the derived content vector \( f = \Delta c \) which itself is completely determined by the half tableau of shape \( \lambda \) which consists of boxes with nonnegative contents (when the shape is placed; i.e. \( m \) is given or the conditions \( \cdot \mid \cdot \) or \( \cdot \div \cdot \) hold). The natural question is: why do we need the half with non positive contents? The answer is: we want the minimal AY cell \( K^c \) which corresponds to the minimal AY representation \( \rho^c \) to be described bijectively by the set of standard D-Young tableaux and to provide this we need

\[ f_i(T) = \pm 1 \iff T^{s_1} \text{ is not standard.} \]

Consider, for example, \( T = \begin{array}{c|c|c}
1 & 2 & 3 \\
\end{array} \) where the box \( 1 \) has the zero content. Then the derived content vector

\[ f(T) = (f_0, f_1, f_2) = (c_1 + c_2, c_2 - c_1, c_3 - c_2) = (0 + 1, 1 - 0, 2 - 1) = (1, 1, 1). \]

We have \( f_0 = 1 \) and therefore the tableaux \( T^{s_0} \) must be not standard, but \( T^{s_0} = \begin{array}{c|c|c|c}
-2 & -1 & 3 \\
-3 & 2 & 1 & 3 \\
\end{array} \) is standard.

However, if we add to \( T \) the part with non positive contents and let \( T = \begin{array}{c|c|c|c}
-3 & -2 & 1 & 3 \\
2 & 1 & 3 \\
\end{array} \) we have \( T^{s_0} = \begin{array}{c|c|c}
-3 & 1 & 2 \\
-2 & -1 & 3 \\
\end{array} \) which is indeed not standard as we wish.

When the tableaux has no boxes of zero content (i.e. in the case of representation \( \rho^{\lambda, m, \pm} \) with \( m \in \mathbb{N} \)), then indeed the part with negative contents does not play any role and we introduced it only in order to have a unified definition of a D-Young tableau.

Ram in [Ra] requires for standardness increasing of entries in the same diagonal (from northwest to southeast) without allowing two entries to occupy the same box. In the above example it works properly:

\[ T = \begin{array}{c|c|c|c}
-3 & -2 & -1 \\
1 & 2 & 3 \\
\end{array} \]

is standard and

\[ T^{s_0} = \begin{array}{c|c|c}
-3 & 1 & 2 \\
-2 & -1 & 3 \\
\end{array} \]

has 2 to the northwest of \( -2 \) in the same diagonal and hence is not standard. However for

\[ T = \begin{array}{c|c|c|c}
-1 & 2 & 3 \\
1 & 2 & 3 \\
\end{array} \]

with \( f_0(T) = c_1(T) + c_2(T) = 0 + 3 = 3 \neq \pm 1 \),

\[ T = \begin{array}{c|c|c|c}
-3 & -2 \\
-1 & 1 \\
\end{array} \]
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we have

\[ T^{s_0} = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 1 \\ -2 \end{bmatrix} \]

with 2 to the northwest of −2 in the same diagonal, i.e. \( T^{s_0} \) is not “standard” while \( f_0(T) \neq \pm 1 \). If we allow the numbers ±i to occupy the same box the (derived) content vector does not change; however, we avoid here the mentioned problem:

\[ f_0(T) \neq \pm 1 \text{ and } f_0(T) = \pm 1 \]

are both standard.

6. Analogous constructions for \( B_n \)

In this section we show how to obtain representations induced from \( S_n \) to \( B_n \) by the construction which is similar to presented above construction for \( D_n \).

The tableaux in this case will be with semi integer contents:

**Definition 6.1.** For a (straight or a skew) tableau \( T \) denote

\[ c_k := j - i + \frac{1}{2} \]

where \( k \) is the entry in row \( i \) and column \( j \) of \( T \). Call \( \text{cont}(T) := (c_1, \ldots, c_n) \) the content vector of \( T \), and call \( \Delta \text{cont}(T) \) the derived content vector of \( T \). Below we sometimes shall denote for brevity \( \text{cont}(T) \) as \( c(T) \).

**Definition 6.2.** Let λ be a diagram of skew shape. Define a B-Young tableau of shape \( \lambda \) to be a filling of \( \lambda \) by the \( 2n \) numbers \( \pm 1, \pm 2, \ldots, \pm n \) in such a way that \( c_{-i} = -c_i \) for \( 1 \leq i \leq n \) where \( c_i \) are semi integer contents defined above in Definition 6.1. A B-Young tableau is called standard if the numbers are increasing in rows and in columns.

The additional generator \( s_0 = (-1, 1) \) of \( B_n \) differs from \( s_0 \) of \( D_n \) (see sections 2.1 and 2.2) and its simple root is \( e_1 \).

**Definition 6.3.** For the B-Young tableau \( T \) with its content vector \( c(T) \) with semi integer entries we define the derived content vector

\[ f(T) = (f_0(T), f_1(T), f_2(T), \ldots, f_{n-1}(T)) = \Delta c(T) = (2c_1, c_2 - c_1, c_3 - c_2, \ldots, c_n - c_{n-1}) \]
6.1.1. **Standard B-Young tableaux without boxes of content** $\pm \frac{1}{2}$.

**Definition 6.4.** Let $\lambda$ be a straight or skew shape with $n$ boxes and let $m \in \mathbb{N}$. Then denote by $T[\lambda, m]$ the set of all standard B-Young tableaux whose shape of boxes with positive contents is $\lambda$ and the smallest positive content is $m + \frac{1}{2}$.

For example, the set $T[1^3, 1] = \{T_1, T_2, ..., T_8\}$ consists of the following tableaux:

$$
T_1 = \begin{array}{ccc}
1 & 2 & 3 \\
-3 & -2 & -1 \\
\end{array} ;
T_2 = \begin{array}{ccc}
-2 & 3 & 2 \\
-3 & 1 & 3 \\
\end{array} ;
T_3 = \begin{array}{ccc}
-3 & 1 & 2 \\
-2 & 3 & 3 \\
\end{array} ;
T_4 = \begin{array}{ccc}
2 & 1 & 3 \\
-1 & -2 & -3 \\
\end{array} .
$$

$$
T_5 = \begin{array}{ccc}
-3 & -2 & 1 \\
-1 & 2 & 3 \\
\end{array} ;
T_6 = \begin{array}{ccc}
-2 & -3 & 2 \\
1 & 3 & 1 \\
\end{array} ;
T_7 = \begin{array}{ccc}
-3 & 1 & 2 \\
3 & 2 & -1 \\
\end{array} ;
T_8 = \begin{array}{ccc}
1 & -2 & 2 \\
-1 & 3 & 1 \\
\end{array} .
$$

**Proposition 6.5.** Let $\lambda$ be a straight or skew shape with $n$ boxes and let $m \in \mathbb{N}$. Then

$$
\#T[\lambda, m] = 2^n f^\lambda .
$$

*Proof.* Similar to the proof of Proposition 4.2. □

Notice that there exists an obvious natural bijection

$$
T[\lambda, m] \leftrightarrow T[\lambda, m, +] \bigcup T[\lambda, m, -] \text{ for } m \in \mathbb{N} .
$$

6.1.2. **Standard B-Young tableaux with a single box on $\frac{1}{2}$-content diagonal.**

**Definition 6.6.** Denote by $T[\lambda, \cdot]$ the set of standard B-Young tableaux which have a single box on $\frac{1}{2}$-content diagonal (and also a single box with content $-\frac{1}{2}$) and which can be divided by the **vertical** straight line into two parts (with $n$ boxes in each part) – one consists of the boxes with negative contents and another of positive contents – and $\lambda$ is the shape (straight or skew) of boxes with positive contents after this separation.

**Definition 6.7.** Denote by $T[\lambda, \div]$ the set of standard B-Young tableaux which have a single box on $\frac{1}{2}$-content diagonal and which can be divided by the **horizontal** straight line into two parts (with $n$ boxes in each part) – one consists of the boxes with negative contents and another of positive contents – and $\lambda$ is the shape (straight or skew) of boxes with positive contents after this separation.

For example, $T[(2, 1), \cdot]$ consists of following eleven tableaux:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
-3 & 3 & -2 \\
-2 & -1 & -3 \\
\end{array} ;
\begin{array}{ccc}
1 & 3 & 2 \\
-2 & 2 & -3 \\
-1 & -1 & -3 \\
\end{array} ;
\begin{array}{ccc}
-1 & 2 & 3 \\
-3 & 3 & -2 \\
-2 & 1 & -3 \\
\end{array} .
$$
The set $T[(2, 1), \cdot|\cdot]$ consists of following five tableaux:

\[
\begin{array}{cccc}
-2 & 1 & -2 & 3 \\
-3 & 3 & -1 & 1 \\
-1 & 2 & -3 & 2 \\
-2 & 3 & -3 & 1 \\
-3 & 2 & -1 & 1 \\
\end{array}
\]

Notice that unlike the case of $D_n$, here the sets $T[\lambda, \cdot|\cdot]$ and $T[\lambda, \div]$ never are empty.

In the above example we see that

$$
\# T[(2, 1), \cdot|\cdot] + \# T[(2, 1), \div] = 11 + 5 = 2 \cdot 2 = \# T[(2, 1), m] , \ m = 1, 2, ...
$$

It does not surprise in view of the following proposition:

**Proposition 6.8.** Let $\lambda$ be a straight or skew shape with $n$ boxes and let $m \in \mathbb{N}$. There exists a natural bijection

$$
T[\lambda, \cdot|\cdot] \bigcup T[\lambda, \div] \leftrightarrow T[\lambda, m]
$$

*Proof.* Take some tableau $T \in T[\lambda, m]$ and denote by $T_{pos}$ its half of shape $\lambda$ with positive contents and by $T_{neg}$ its half with negative contents. If the lower left box of $T_{pos}$ is occupied by a positive entry, then we can stick $T_{pos}$ and $T_{neg}$ by a common vertical edge to obtain a tableau which belongs to $T[\lambda, \cdot|\cdot]$. For example, if $\lambda = (2, 1)$ and $T_{pos} = \begin{pmatrix} -3 & -1 \\ 2 \end{pmatrix}$, $T_{neg} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, then we have a tableau

$$
\begin{array}{cc}
-3 & -1 \\
-2 & 2 \\
1 & 3 \\
\end{array}
\in T[\lambda, \cdot|\cdot]
$$

However sticking in the second way we obtain a tableau

$$
\begin{array}{cc}
-3 & -1 \\
-2 & 2 \\
1 & 3 \\
\end{array}
$$

which is not standard.

If the lower left box of $T_{pos}$ is occupied by a negative entry, then we can stick $T_{pos}$ and $T_{neg}$ by a common horizontal edge to obtain a tableau which belongs to $T[\lambda, \div]$. For example, if $T_{pos} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$, then we stick it to $T_{neg} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
from above to obtain a standard B-Young tableau $$\begin{array}{c}
-3 & 2 \\
-1 & 1 \\
-2 & 3 \\
\end{array} \in T[(2, 1), \div]$$ while sticking in the other way leads to a non standard tableau $$\begin{array}{c}
-3 & 2 \\
1 & 1 \\
-2 & 3 \\
\end{array}.$$ For the inverse mapping we take a tableau from $$T[\lambda, m] \cup T[\lambda, \div]$$ and separate it by a vertical or horizontal straight line, respectively, into two halves $$T_{pos}$$ and $$T_{neg}$$ with positive and negative contents and then put these halves in such a way that the lower left box of $$T_{pos}$$ will have content $$m + \frac{1}{2}$$ (and of course the upper right box of $$T_{neg}$$ will occupy the box with content $$-(m + \frac{1}{2})$$). This gives us a standard B-Young tableau from the set $$T[\lambda, m].$$ □

**Definition 6.9.** Denote $$T[\lambda, 0] := T[\lambda, \cdot | \cdot] \cup T[\lambda, \div].$$

Reformulating Proposition 6.8 we can say that there exists a natural bijection between $$T[\lambda, 0]$$ and $$T[\lambda, m]$$ with $$m = 1, 2, 3, \ldots$$

6.2. **Representations arising from standard B-Young tableaux.** As above, for a B-Young tableau $$T$$ and $$\pi \in B_n$$ we denote by $$T^\pi$$ the B-Young tableau where each entry $$i$$ is replaced by $$\pi(i).$$

Take a set of standard B-Young tableaux $$T[\lambda, m]$$ or $$T[\lambda, \cdot | \cdot]$$ or $$T[\lambda, \div]$$ and define a representation $$\rho^{\lambda,m}$$ or $$\rho^{\lambda,\cdot|\cdot}$$ or $$\rho^{\lambda,\div}$$ by Young orthogonal form

$$\rho_s(T) = \frac{1}{f_i(T)} T + \sqrt{1 - \frac{1}{f_i(T)^2}} T^{s_i} \quad \text{for } i = 0, 1, 2, \ldots, n - 1.$$ Here $$s_0$$ is $$s_0$$ of $$B_n$$ and the numbers $$f_i(T)$$ are the entries of the derived content vector of B-Young tableaux $$T$$ introduced in Definition 6.3.

(It is a routine but not so difficult work to verify that this is indeed a representation i.e. that matrices $$\rho_s$$ satisfy the group relations of $$B_n.$$ We already know that for $$i = 1, 2, \ldots, n - 1$$ the relations are satisfied from the classical Young orthogonal form for $$S_n.$$ Easy to see that

$$\rho_{s_0}^2(T) = \rho_{s_0} (\rho_{s_0}(T)) = \rho_{s_0} \left( \frac{1}{2c_1} T + \sqrt{1 - \frac{1}{(2c_1)^2}} T^{s_0} \right) =$$

$$= \frac{1}{2c_1} \left( \frac{1}{2c_1} T + \sqrt{1 - \frac{1}{(2c_1)^2}} T^{s_0} \right) +$$

$$+ \sqrt{1 - \frac{1}{(2c_1)^2}} \left( -\frac{1}{2c_1} T^{s_0} + \sqrt{1 - \frac{1}{(2c_1)^2}} T \right) = T.$$ Also it is rather easy to show that $$\rho_{s_0}$$ commutes with $$\rho_s$$ for $$i = 2, 3, \ldots, n - 1.$$ The most long is the calculation which verifies that $$(\rho_{s_0}\rho_{s_1})^4 = id.$$ Now we can explain why for $$B_n$$ we use semi integer contents. As we mentioned above, to obtain a representation from the derived content vector its entries must
repeat the form of simple roots of the group, therefore the first entry of the derived content vector of B-Young tableau must be $c_1$ (or its multiple) because the root which correspond to $s_0$ in $B$ is $c_1$. In the definition of the representation $\rho$ the entry $f_0(T) = 2c_1(T)$ appears in the denominator and therefore to avoid division by zero we introduce semi integer contents $\ldots, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$ We want the representation space to be spanned by standard tableaux and so we need $T^{s_0}$ to be not standard if and only if $\sqrt{1 - \frac{1}{\rho(T)^2}} = 0$ which occurs if and only if $f_1 = \pm 1$. This is the reason for taking $2c_1$ but not $c_1$.

The analogues of Theorems 5.9 and 5.11 here are as follows:

**Theorem 6.10.** Let $\lambda$ be a straight or skew shape with $n$ boxes and let $S^\lambda$ denote the representation of $S_n = S_n^\lambda = \langle s_1, s_2, \ldots, s_{n-1} \rangle$ associated with $\lambda$ via classical Young orthogonal form (see Theorems 2.2 and 2.3). (If $\lambda$ is a straight shape, then $S^\lambda$ is an irreducible Specht module and if $\lambda$ is a skew shape, then $S^\lambda$ is a skew Specht module.) Then for any natural $m = 1, 2, 3, \ldots$

$$\rho^{\lambda, m} \cong S^\lambda \uparrow_{B_n}^{S_n}$$

**Theorem 6.11.** In the conditions of the above Theorem 6.10

$$\rho^{\lambda, \uparrow} \oplus \rho^{\lambda, \downarrow} \cong \rho^{\lambda, m}$$

The proofs of Theorems 6.10 and 6.11 are very similar to the proofs of Theorems 5.9 and 5.11 respectively.

A natural question is why we don’t consider tableaux with semi integer contents to obtain AY representations of $D_n$? We explain the reason by the following example:

$$T = \begin{array}{ccc|cc}
-3 & -2 & -1 & 1 & 2 & 3
\end{array} \\
c(T) = \left(\begin{array}{c}
\frac{1}{2} & 3 & \frac{5}{2}
\end{array}\right)$$

For $B_3$ we have $f(T) = (2c_1, c_2 - c_1, c_3 - c_2) = (1, 1, 1)$ (this is the trivial representation of $B_3$) and indeed for $s_0 = (-1, 1)$ the first entry of $f(T)$ is $f_0 = 2c_1 = 2 \cdot \frac{1}{2} = 1$ and $T^{s_0} = \begin{array}{ccc|cc}
-3 & -2 & 1 & -1 & 2 & 3
\end{array}$ is not standard. However for $D_3$ with $s_0 = (-1, 2)(-2, 1)$ the tableau $T^{s_0} = \begin{array}{ccc|cc}
-3 & 1 & 2 & -2 & -1 & 3
\end{array}$ is still not standard while the entry $f_0 = c_1 + c_2 = \frac{1}{2} + \frac{3}{2} = 2 \neq \pm 1$. We avoid this situation because, as we already mentioned in the section 5.2.5, we want the minimal AY cell $K^f$ which corresponds to the minimal AY representation $\rho^f$ to be described bijectively by the set of standard D-Young tableaux and to provide this we need $f_1(T) = \pm 1 \iff T^{s_0}$ is not standard.

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