Szegedy quantum walks with memory on regular graphs

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Abstract
Quantum walks with memory (QWM) are types of modified quantum walks that record the walker’s latest path. The general model of coined QWM is presented in Li et al. (Phys Rev A 93:042323, 2016). In this paper, we present the general Szegedy QWM model and we describe its relationship with the coined QWM model. A coined QWM can be transformed into a Szegedy QWM, while a Szegedy QWM can be transformed into a coined QWM with any partition. These results may help in the analysis of the coined QWM. By transforming a coined QWM into a Szegedy QWM, the essential structure of the coined QWM is revealed. We give an example and we prove that two known QWMs are equal when they have a proper position-dependent coin operator.

Keywords Quantum walks · Quantum walks with memory · Szegedy quantum walks with memory · Line digraph

1 Introduction
Quantum walk is an essential model to realize quantum computation. Quantum walk provides a method to explore all possible paths in a parallel way due to constructive quantum interference along the paths. Many kinds of quantum walks have been proposed, such as single-particle quantum walks [1–4], two-particle quantum walks [5–7], three-state quantum walks [8,9], controlled interacting quantum walks [10,11], indistinguishable particle quantum walks [12,13], and disordered quantum walk [14,15]. Each type of quantum walk has its own special features and advantages. Therefore,
algorithms based on quantum walks have been established as a dominant technique in quantum computation, ranging from element distinctness [16] to database searching [17–20], from constructing quantum hash functions [10,11] to graph isomorphism testing [21,22].

Most of the quantum walks that have been studied are quantum walks without memory (QWoM) on regular graphs. Quantum walks with memory (QWM) have been studied in Refs. [23–27], while classical walks with memory have been used in research on the behavior of hunting, searching, and building human memory search model. The standard QWM is a kind of modified quantum walk that has many extra coins to record the walker’s latest path. Rohde et al. presented a kind of coined QWM provided by recycled coins and a memory of the coin-flip history [23]. Mc Gettrick presented another kind of coined QWM whose coin state decides whether the shift as ‘Reflect’ or ‘Transmit’ [24,25]. Konno and Machida provided limit theorems for Mc Gettrick’s QWM [26]. Li et al. [27] presented the generic model of coined quantum walks with memory by introducing the relation of QWM on a regular graph and QWoM on the line digraph of the regular graph. With this model, it becomes possible to build any required coined QWM on regular graphs and to study properties of different kinds of QWM.

Apart from the coined quantum walk, the Szegedy quantum walk is also a very important version of quantum walks. Szegedy quantum walk was introduced by Szegedy by means of quantizing Markov chains described by some transition matrix [28]. Later, Portugal [29] showed the connection between the coined quantum walk and Szegedy’s quantum walk. Konno et al. [30] introduced the notion of a partition-based quantum walk, and he proved that the two-step coined quantum walks, an extension of the Szegedy quantum walks for multigraphs, and the two-tessellate staggered quantum walks are unitary equivalent. Liu et al. [31] constructed Szegedy quantum walks on regular uniform hypergraphs.

It is proven that Szegedy quantum walk is used in a variety of different applications, such as verifying matrix products [32], searching triangles [33], testing group commutativity [34], approximating the effective resistance in electrical networks [35], and quantum Pagerank algorithm [36] for determining the relative importance of nodes in a graph.

In this paper, the generic model of the Szegedy QWM is presented in Sect. 2. Then, the relation between the Szegedy QWM and the coined QWM is revealed in Sect. 3. By transforming coined QWM to Szegedy QWM, some facts are shown in Sect. 4. These facts change the existing understanding of coined QWM. Finally, a conclusion is given in Sect. 5.

## 2 Generic model of Szegedy QWM

In this section, we present the general model of Szegedy QWM.

Firstly, we define some notation. Let $G = (V, E)$ be a digraph with vertex set $V(G)$ and arc set $E(G)$. With fixed labeling of vertices, the adjacency matrix of a digraph $G$ with $N$ vertices, denoted by $M(G)$, is the $N \times N$ $(0,1)$-matrix with $ij$-th element defined by $M_{i,j}(G) = 1$ if $(x_i, x_j) \in E(G)$ and $M_{i,j}(G) = 0$, otherwise.
The line digraph of a digraph $G$, denoted by $\overrightarrow{L}G$, is defined as follows: the vertex set of $\overrightarrow{L}G$ is $E(G)$; for $x_a, x_b, x_c, x_d \in V(G)$, $((x_a, x_b), (x_c, x_d)) \in E(\overrightarrow{L}G)$ if and only if $(x_a, x_b)$ and $(x_c, x_d)$ are both in $E(G)$ and $x_b = x_c$. The line digraph of $\overrightarrow{L}G$ is denoted by $\overrightarrow{L}^2G$. Similarly, there are $\overrightarrow{L}^dG$s with $d \in \mathbb{N}^*$. For simplicity, all of them are called line digraph of $G$.

According to Ref. [27], coined quantum walks with $d$ memory on a regular graph $G$ can be transformed to coined QWoM on $\overrightarrow{L}^dG$. Similarly, Szegedy quantum walks with $d$ memory on a regular graph $G$ could be seen as a Szegedy QWoM on $\overrightarrow{L}^dG$. The general model of the Szegedy QWM is given as follows.

**Definition 1** Let $G$ be an $m$-regular graph. Define $\pi$ to be an edge dicycle partition of $\overrightarrow{L}^dG$ such that

$$\pi : \overrightarrow{L}^dG \to \{C_1, C_2, \ldots\},$$

where $\{C_k|k = 1, \ldots\}$ satisfy that $\bigcap_k E(C_k) = \emptyset, \bigcup_k E(C_k) = E(\overrightarrow{L}^dG)$ and each $C_k$ is a Eulerian digraph. We denote the set of edge dicycle partitions of $\overrightarrow{L}^dG$ by $\Pi_{\overrightarrow{L}^dG}$.

**Definition 2** For $\pi \in \Pi_{\overrightarrow{L}^dG}$ with $\overrightarrow{L}^dG \xrightarrow{\pi} \{C_1, C_2, \ldots\}$, define $f$ such that for any $(v_1, v_2) \in E(\overrightarrow{L}^dG),

$$f : (v_1, v_2) \to (v_2, v_3)$$

where $(v_1, v_2)$ and $(v_2, v_3)$ belong to same $E(C_k)$.

$\pi$ is an edge dicycle partition, hence $f$ is a bijection. $f$ requires the walker to walk along a subgraph $C_k$.

**Definition 3** For a Szegedy QWoM on the line digraph of $G$ denoted by $\overrightarrow{L}^dG$, i.e., Szegedy QWM on $G$, the evolution is decomposed into two steps, $U = SR$, defined as

$$R = 2 \sum_v |\psi_v\rangle\langle\psi_v| - I$$

$$S = \sum_{(v, w) \in E(\overrightarrow{L}^dG)} |f(v, w)\rangle\langle v, w|,$$

where

$$|\psi_v\rangle = |v\rangle \otimes \sum_w \sqrt{q_{vw}} |w\rangle.$$ 

$q_{vw}$ is the probability of leaping from $v$ to $w$.

Definition 3 presents the Szegedy QW with $d$ memory on $G$. It is not only the new form of QWM, but also a useful tool in analyzing coined QWM, which is shown in Sect. 4.
3 Coined QWM and its relation with Szegedy QWM

3.1 Coined QWM

To uncover the relation of Szegedy QWM and coined QWM, the model of coined QWM is introduced in Definitions 4, 5, and 6. For more details, please refer to Ref. [27].

Definition 4 Let $G$ be an $m$-regular graph. Define $\pi'$ to be a partition of $\vec{L}^d G$ such that

$$\pi' : \vec{L}^d G \rightarrow \{C_1', C_2', \ldots, C_m'\}, \tag{6}$$

where $\{C_k'|k = 1, \ldots, m\}$ satisfy that $V(C_k') = V(\vec{L}^d G)$, $\bigcup_k E(C_k') = E(\vec{L}^d G)$ and for every vertex $v \in V(C_k')$, the outdegree is 1. Dicycle partition is a kind of partition which satisfies that for every vertex $v \in V(C_k)$, the outdegree and indegree are 1. The set of partitions of $\vec{L}^d G$ are denoted by $\Pi'_{\vec{L}^d G}$.

Definition 5 For $\pi' \in \Pi'_{\vec{L}^d G}$ with $\vec{L}^d G \xrightarrow{\pi'} \{C_1', C_2', \ldots, C_m'\}$, define

$$f'_{C_k'} : V(\vec{L}^d G) \rightarrow V(\vec{L}^d G) \tag{7}$$

such that for any $v \in V(\vec{L}^d G)$,

$$(v, f'_{C_k'}(v)) \in E(C_k') \tag{8}$$

Definition 6 For a coined QWoM on the line digraph of $G$ denoted by $\vec{L}^d G$, i.e., coined QWM on $G$, the evolution is decomposed into two steps, $U' = DC$, defined by

$$C : |v, c) \longrightarrow \sum_j A_{c,c_j}|v, c_j); \tag{9}$$

$$D : |v, c_j) \longrightarrow |f'_{C_j}(v), gc(v, c_j)) \tag{10}$$

The coin shift function $gc$ has to satisfy

$$\{gc(v_i, c_k), \ldots) = \{c_1, c_2, \ldots, c_m\} with v = f'_{C_k}(v_i). \tag{11}$$

For a coined QWM, whether the partition is a dicycle partition affects the choice of coin shift function. Through the analysis in Sect. 3.2, the gap between dicycle partition and partition is bridged by Szegedy QWM.
3.2 The relation of coined QWM and Szegedy QWM

Szegedy quantum walk is a kind of quantum walk that the walker wander on edges of the graph, while coined quantum walk is a kind of quantum walk that the walker wander on vertices of the graph. Szegedy QWM lives in the Hilbert space $H_E$ spanned by $|v,w⟩$, where $(v,w) ∈ E(\overrightarrow{Ld}G)$. Coined QWM lives in the Hilbert space $H_{V,C}$ spanned by $|v,c⟩$, where $v ∈ V(\overrightarrow{Ld}G)$, $c$ denotes the state of coin. The movement of Szegedy QWM is controlled by the adjacent matrix, an edge dicycle partition of the graph, while the movement of coined QWM is controlled by a coin operator, a partition of the graph and a coin shift function.

From the above mentioned, Szegedy QWM and coined QWM do not look the same. However, the relation of Szegedy QWM and coined QWM can be revealed by following analysis.

A brief summary of correspondence between Szegedy QWM and coined QWM is given in Table 1.

1. For an $m$-regular graph $G$, $H_E$ and $H_{V,C}$ are the same size, $\text{size}(V(\overrightarrow{Ld}G)) \cdot m$.
2. Let $v$ in $|v,w⟩$ denote the current position of the walker. Then, $|v,w⟩$ denotes a directed line from $v$ to $w$. At the same time, $|v,c⟩$ also denotes a directed line from $v$, and the target point of this directed line is based on the partition of $\overrightarrow{Ld}G$. Therefore, $|v,w⟩$ corresponds to $|v,c⟩$. And this fact builds a bridge between Szegedy QWM and coined QWM.

3. A coined QWM with the partition $\pi'$ and coin shift function $g_c$ can be transformed to a Szegedy QWM. $|v,c_j⟩$ denotes a directed line from $v$ to $w$, where $(v,w) ∈ E(C'_j)$. Equation 11 transforms $|v,c_j⟩$ to $|f'_{C'_j}(v),g_c(v,c_j)⟩$. Depending on Definition 5, $f'_{C'_j}(v) = w$. $|f'_{C'_j}(v),g_c(v,c_j)⟩$ denotes a directed line from $w$ to $u$, where $(w,u) ∈ E(C'_k)$, $c_k = g_c(v,c_j)$. Let $f(v,w) = (w,u)$, then $f$ and the corresponding edge partition $\pi$ for a Szegedy QWM are fixed. The operator $S$ in Eq. 4 and $D$ in Eq. 11 have the same effect: $|v,w⟩ \rightarrow |w,u⟩$. The requirement of $g_c$ in Eq. 11 is to make sure the operator $D$ is unitary. Therefore, the edge partition $\pi$, fixed by $\pi'$ and $g_c$, is an edge dicycle partition.

On the other hand, a Szegedy QWM with an edge dicycle partition $\pi$ can be transformed to a coined QWM with any partition $\pi'$. The operator $S$ in Eq. 4 has the effect $|v,w⟩ \rightarrow |w,u⟩$. Let $(v,w) ∈ E(C'_j)$, then $|v,c_j⟩$ denotes a same directed line from $v$ to $w$. Hence, $f'_{C'_j}(v) = w$. Let $(w,u) ∈ E(C'_k)$, $g_c(v,c_j) = c_k$. Then, the correspondence coined QWM with the partition $\pi'$ and coin shift function $g_c$ is fixed.
4. The operator $R$ in Eq. 3 has the following effect:

$$R : |v, w\rangle \rightarrow |v\rangle \otimes [(2q_{vw} - 1)|w\rangle + \sum_{w' \neq w} 2\sqrt{q_{vw}q_{vw'}}|w'\rangle]$$

(12)

With the relation of $|v, w\rangle$ and $|v, c\rangle$, $R$ can be transformed to

$$R : |v, c\rangle \rightarrow \sum_j A_{c,j}|v, c_j\rangle.$$  

(13)

$A$ is in the form of

$$A = \begin{bmatrix} 2q_{vw} - 1 & 2\sqrt{q_{vw}q_{vw'}} \\ 2\sqrt{q_{vw}q_{vw'}} & 2q_{vw} - 1 \end{bmatrix}.$$  

(14)

or

$$A = \begin{bmatrix} 2q_{vw'} - 1 & 2\sqrt{q_{vw}q_{vw'}} \\ 2\sqrt{q_{vw}q_{vw'}} & 2q_{vw} - 1 \end{bmatrix}.$$  

(15)

for two partitions, which are different at the position $v$, respectively. Therefore, by choosing proper coin operator, $R$ corresponds to $C$ in Eq. 9 with $A$ as a real operator. By extending $|\psi_v\rangle = |v\rangle \otimes \sum_w \sqrt{q_{vw}}|w\rangle$ in Eq. 5 to $|\psi_v\rangle = |v\rangle \otimes \sum_w \alpha_{vw}|w\rangle$, the coin operator $A$ in Ref. 14 can be extend to

$$A = \begin{bmatrix} 2|\alpha_{vw}|^2 - 1 & 2\alpha_{vw}\alpha_{vw'} \\ 2\alpha_{vw'}^*\alpha_{vw} & 2|\alpha_{vw'}|^2 - 1 \end{bmatrix}.$$  

(16)

which is a complex operator.

5. Till now, the way of relating the evolution operators of Szegedy QWM and coined QWM has been uncovered. The choice of initial state is also an element which affects the probability distribution. According to the relation of $|v, w\rangle$ and $|v, c\rangle$, given the initial state for Szegedy QWM, the initial state for coined QWM can be constructed to produce the same probability distribution. It is vice versa.

In conclusion, there is correspondence between Szegedy QWM and coined QWM. However, the correspondence is not one-to-one. A coined QWM can be transformed to a Szegedy QWM, while a Szegedy QWM can be transformed to a coined QWM with any partition. In the next section, we will show that transforming the evolution of coined QWM to Szegedy QWM can help us to analyze coined QWM.

## 4 Analysis of coined QWM

In Ref. [27], coined QWM is controlled by a coin operator, a partition $\pi'$, and a proper coin shift function $g_c$. Different partitions will lead to different choices of coin shift
function and different QWMs. However, by transforming coined QWM to Szegedy QWM, we find different coined QWMs with different partition may be the same QWM in essence. With this fact, we should take a new look at coined QWM from another angle.

### 4.1 Essence of coined QWM

Section 3.2 shows the relation of coined QWM and Szegedy QWM. A coined QWM can be transformed to a Szegedy QWM, while a Szegedy QWM can be transformed to a coined QWM with any partition. That means all coined QWM can be transformed to a coined QWM with a same partition $\pi'$. Partition $\pi 2'$ in Fig. 2 is a dicycle partition, which has some good properties for analysis. Therefore, we suggest researchers to choose the partition $\pi 2'$. Then, all coined QWM can be transformed to QWM with the dicycle partition $\pi 2'$ and a coin shift function with the constraint:

$$\{g_c(v_i, c_k), \ldots\} = \{c_1, c_2, \ldots, c_m\} \text{ with } v = f'_{C_k}(v_i).$$ (17)

Then, the elements which affect the evolution of coined QWM are the coin operator and the coin shift function $g_c$. At the same time, the evolution of Szegedy QWM is affected by the adjacent matrix and the edge dicycle partition. Each form of QWM has its advantages. Coined QWM has concise form for experiment and designing algorithm. Szegedy QWM shows the essential evolution of QWM. Furthermore, except coined QWM with partition $\pi 2'$, other forms of coined QWM have their special meanings. The QWM in Ref. [23] was presented by considering recycled coins. The QWM in Ref. [24,25] was presented by using the coin state to decide the shift is ‘Reflect’ or ‘Transmit.’ Our results do not mean that other forms of coined QWM do not have value to study. Our results show the essential structure of coined QWM.

### 4.2 QWM1 and QMW2

Szegedy QWM can help us to analyze coined QWM. An example is shown as follows. There are two kinds of coined QWM on the line in Ref. [23] and Ref. [25,26], respectively. The two coined QWMs were the only two QWMs before the generic model of coined QWM was presented in Ref. [27]. However, through the analysis in Sect. 3.2, these two quantum walks with 1 memory are in fact the same one when they have a proper position-dependent coin operator.

QWM1 is a coined QW with 1 memory with partition $\pi 1'$ in Fig. 1a and $g_{c_1}$, which is as follows.

$$g_{c_1}(v_{x,x+1}, 1) = 1, \quad g_{c_1}(v_{x,x-1}, 1) = -1,$$

$$g_{c_1}(v_{x,x+1}, -1) = 1, \quad g_{c_1}(v_{x,x-1}, -1) = -1.$$ (18)
Fig. 1 Partition $\pi_1'$ of the line digraph. Black lines show the movement of QWM1

(a) (-1, 0) (0, 1) (1, 2)
(0, -1) (1, 0) (2, 1)

(b) (-1, 0) (0, 1) (1, 2)
(0, -1) (1, 0) (2, 1)

(c) (-1, 0) (0, 1) (1, 2)
(0, -1) (1, 0) (2, 1)

(d) (-1, 0) (0, 1) (1, 2)
(0, -1) (1, 0) (2, 1)
Fig. 2 Partition $\pi 2'$ of the line digraph. Black lines show the movement of QWM2.
Let the coin operator be

\[
H_1 = \begin{cases} 
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} 
\end{cases}
\]

at positions \((x,x+1)\); \((x+1,x)\).

(19)

QWM2 is a coined QW with 1 memory with a dicycle partition \(\pi \mathcal{P}_2^\prime\) in Fig. 2a and \(gc_2(v, c_j) = c_j\). Let the coin operator be

\[
H_2 = \begin{bmatrix} 
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix}.
\]

(20)

In Fig. 1a, the partition \(\pi \mathcal{P}_1^\prime\) of \(\mathcal{L} G\) is shown. According to the operator \(D\) in Eq. 9, three kinds of movement of QWM1 are shown by black arrows in Fig. 1b–d, while three kinds of movement of QWM2 are shown by black arrows in Fig. 2b–d. The movements of QWM1 and QWM2 are exactly the same. However, \(\pi \mathcal{P}_1^\prime\) and \(\pi \mathcal{P}_2^\prime\) are different at lower positions of \(\mathcal{L} G\), such as positions \((0, -1), (1, 0), (2, 1)\). Therefore, by choosing two different operators at these positions in the form of Eqs. 14 and 15, QWM1 and QWM2 have the same evolution.

Then, the initial states \(\alpha|\bar{x} - 1, x, 1\rangle + \beta|x - 1, x, -1\rangle + \alpha'|x + 1, x, 1\rangle + \beta'|x + 1, x, -1\rangle\) for QWM1 and \(\alpha|\bar{x} - 1, x, 1\rangle + \beta|x - 1, x, -1\rangle + \beta'|x + 1, x, 1\rangle + \alpha'|x + 1, x, -1\rangle\) for QWM1 can generate same probability distribution. It is also backed up by simulation results.

5 Summary

QWM are a type of modified quantum walks that have many extra coins to record the walker’s latest path. In this paper, we present the general Szegedy QWM model and we describe its relationship with the coined QWM model. Through the analysis, we expose the fact that a coined QWM can be transformed into a Szegedy QWM, while a Szegedy QWM can be transformed into a coined QWM with any partition. That means any coined QWM can be transformed into a coined QWM with a same partition. Therefore, transforming a coined QWM into a Szegedy QWM provides a new angle for analyzing the coined QWM. With the above results, we get the essential structure of the coined QWM that the coined QWM is only affected by the coin operator and coin shift function. We give an example. By transforming QWM1 and QWM2 into Szegedy QWM, we prove that these two quantum walks with 1 memory are equal when they have a proper position-dependent coin operator.

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