Symmetric ordering effect on Casimir energy in $\kappa$–Minkowski spacetime

Hyeong-Chan Kim\textsuperscript{1} \textsuperscript{*}, Chaiho Rim\textsuperscript{2} \textsuperscript{†} and Jae Hyung Yee\textsuperscript{1} \textsuperscript{‡}

\textsuperscript{1} Department of Physics, Yonsei University, Seoul 120-749, Republic of Korea and
\textsuperscript{2} Department of Physics and Research Institute of Physics and Chemistry, Chonbuk National University, Jeonju 561-756, Korea.

We present the Casimir energy of spherical shell, for the symmetrically deformed scalar field in $\kappa$-Minkowski space-time, satisfying Dirichlet boundary condition. The Casimir energy shows the particle anti-particle symmetry contrary to the asymmetrically deformed case. In addition, the deformation effect starts from $O(1/\kappa)$ term unlike in the parallel plates.

PACS numbers: 11.10.Nx, 11.30.Cp, 02.40.Gh
Keywords: Casimir energy, non-commutative field theory, $\kappa$-Minkowski spacetime, $\kappa$-deformed Poincaré symmetry

I. INTRODUCTION

The Casimir energy is dependent on the geometry as shown in \cite{1, 2}: The Casimir force is attractive between parallel plates but repulsive in a sphere. When the spacetime is $\kappa$-deformed \cite{3}, the Poincaré algebra is deformed \cite{4} so that the energy-momentum relation from the Einstein special relativity is deformed. In this case, the Casimir energy provides a useful information about the vacuum structure of the theory as shown in \cite{5}. It turns out that the $\kappa$-deformed Poincaré algebra has many different versions, which originate from the ordering of the space and time coordinates. In our previous paper in \cite{5}, the Casimir energy for the so-called asymmetric ordering case is investigated, where the vacuum is shown to break the particle and anti-particle symmetry. Thus, one needs to investigate if there exists a case where the vacuum respects the particle and anti-particle symmetry. One obvious choice is the symmetric ordering case, where particle and antiparticle dispersion relation is symmetric. In this brief report, we evaluate the Casimir energy for the particle and anti-particle contribution for the symmetric ordering case. In section II a brief summary is given how to evaluate the Casimir energy. In section III the Casimir energy is given with an appropriate measure, and summary and discussion is given in section IV.

II. CASIMIR ENERGY OF A SPHERICAL SHELL

The Casimir energy is the zero point vacuum energy of massless scalar fields, which in momentum space is given by $E_c^\pm = \frac{1}{2} \int \hbar \omega_p$ where $\int_\mathbf{p}$ denotes the momentum integration with $\kappa$-Poincaré invariant measure $\int \frac{d^3p}{(2\pi)^3} e^{i\omega_p/\kappa}$ with $\alpha = 3/2$, and $\omega_p$ is the positive mode (particle) dispersion relation for the symmetric ordering case

$$\omega_p = 2\kappa \ln \left( \frac{|p|}{2\kappa} + \sqrt{1 + \frac{p^2}{4\kappa^2}} \right) = -2\kappa \ln \left( -\frac{|p|}{2\kappa} + \sqrt{1 + \frac{p^2}{4\kappa^2}} \right).$$

The negative mode (anti-particle) contribution $E_c^-$ is obtained using the same $\omega_p$ but with the measure changed $\int \frac{d^3p}{(2\pi)^3} e^{-i\omega_p/\kappa}$. We note that $E_c^-$ can be formally obtained if one changes the sign of $\kappa$ to $-\kappa$. Following the prescription given in Ref. \cite{5}, we can put the Casimir energy in the form

$$E_c(a) = \sum_{n=0}^{\infty} \left( E_n(a) - E_n(\eta R) \right)$$

\textsuperscript{*}Electronic address: hckim@phya.yonsei.ac.kr
\textsuperscript{†}Electronic address: rim@chonbuk.ac.kr
\textsuperscript{‡}Electronic address: jhyee@yonsei.ac.kr
where \( a \) represents the radius of the sphere, \( \eta R \) is radius of a large sphere introduced to regularize the mode with \( 0 < \eta < 1 \) and

\[
\mathcal{E}_n(r) = -\frac{1}{r} \sum_l \frac{\kappa}{\nu_l^{2n-2}} \int_0^\infty dy \left( e^{i\nu y e^{-i\phi}} g(r, i\nu y e^{-i\phi}) \right) \frac{d}{dy} q_n(y e^{-i\phi})
\]

(3)

where the limit \( R \to \infty, \sigma \to 0, \phi \to 0 \) is assumed at the end. \( g(r, z) \equiv -e^{i\omega(z,r)/\kappa \omega(z, r)/\kappa} \) and \( q_n(y) \) corresponds to the series expansion of the large order Bessel function whose explicit form for \( n = 0, 1, 2 \) is given in [3]. The merit of this decomposition is that \( \mathcal{E}_0(a) - \mathcal{E}_0(\eta R) = 0 \), and all other components are finite, and allow the systematic expansion in \( 1/(\kappa a) \). \( \mathcal{E}_n(r) \) is conveniently put as, after the integration by part

\[
\mathcal{E}_n(r) = \frac{1}{r} \sum_l \frac{B_n(\nu, r)}{\nu_l^{2n-2}} ; \quad B_n(\nu, r) = \frac{1}{\pi} \int_0^\infty dy \, q_n(y) \, G \left( \frac{y}{2\kappa r} \right)
\]

(4)

where \( G(x) \) is an even function of \( x 

\[
G(x) = \begin{cases} \frac{\theta(1-x^2)}{\sqrt{1-x^2}} & \text{for } \alpha = 0 \\ \frac{\theta(1-x^2)}{\sqrt{1-x^2}} \left( 1 - 4x^2 + \frac{3(x^2-3)}{\sqrt{1-x^2}} \sin^{-1} x \right) + \theta(x^2-1) \left( \frac{1 - 3\log \left( \frac{\sqrt{1-x^2} + 1}{\sqrt{1-x^2} + 1} \right)}{\sqrt{1-x^2}} \right) & \text{for } \alpha = 3/2 \end{cases}
\]

(5)

The \( \kappa \to \infty \) limit of the Casimir energy is given as \( \mathcal{E}^{(0)}_n(a) = 0.0002819/a \), the commutative result [6] since \( G(0) = 1 \). Its correction up to \( O(1/\kappa^2) \) is given [3] in terms of \( \Delta \mathcal{E}_1(r), \Delta \mathcal{E}_2(r) \) and

\[
\sum_{n \geq 3} \Delta \mathcal{E}_n(a) = \frac{G_1}{\kappa a} \sum_{n \geq 3} \sum_l \frac{1}{\nu_l^{2n-2}} \int_0^{1/b} dy q_n(y)(by)^2 + O(1/\kappa^3)
\]

(6)

where \( \Delta \mathcal{E}_n(r) = \mathcal{E}_n(r) - \mathcal{E}^{(0)}_n(r) \) with \( \mathcal{E}^{(0)}_n(r) = \lim_{\kappa \to \infty} \mathcal{E}_n(r) \), \( b = \frac{\nu}{2\kappa a} \) and \( G_1 = \frac{1}{2} \frac{d^2}{dx^2} G(x) \bigg|_{x=0} \).

### III. Casimir Energy When \( \alpha = 0 \) And \( \alpha = 3/2 \)

When one neglects the measure factor (\( \alpha = 0 \)), one has \( B_1(\nu, a) \equiv B_1(b) = -(1 + 16b^2)/(128(1 + b^2)^{5/2}) \) and \( B_2(\nu, a) \equiv B_2(b) = (35 + 174b^2 - 780b^4 + 23552b^6 - 4094b^8)/(32768(1 + b^2)^{11/2}) \). \( \mathcal{E}_1(r) \) can be evaluated as a series expansion in \( 1/(\kappa a) \) using a formula in [3]

\[
\mathcal{E}_1(r) = \frac{1}{r} \sum_{l=0}^{\infty} B_l(\nu, r) = \frac{1}{r} \left( 2\kappa r \int_0^{\infty} db \, B_1(b) + O(\kappa r)^{-3} \right) = \frac{3\kappa}{32} + \frac{O(\kappa r)^{-3}}{r}.
\]

(7)

Thus \( \mathcal{E}_1(a) - \mathcal{E}_1(\eta R) \) is the order of \( O(\kappa a)^{-3} \). To evaluate \( \mathcal{E}_2(r) \), one needs to take care of the fictitious singularity at \( b = 0 \) using [3]: One may put, \( B_2(b) = B_2^{\text{ref}}(b) + (B_2(b) - B_2^{\text{ref}}(b)) \) with \( B_2^{\text{ref}}(b) = \frac{1}{32\kappa a} \left( \frac{35}{1+b^2} \right) \) \( B_2^{\text{ref}}(0) = B_2(0) \) and \( B_2^{\text{ref}}(b \to \infty) = 0 \) to sum exactly \( B_2^{\text{ref}}(b) \), and use [3] for the rest of the sum:

\[
\mathcal{E}_2(r) = \frac{35\pi}{65536 \kappa a} + \frac{35\pi}{131072 \kappa a^2} + \frac{O(\kappa a)^{-3}}{r}.
\]

(8)

(One can check the result does not depend on the explicit choice of \( B_2^{\text{ref}}(b) \)). The \( \kappa \) independent term is contained in \( E^{(0)}_c(a) \) and the rest gives the deformed correction \( \Delta \mathcal{E}_2(r) \). Finally, \( G_1 = 1/2 \) in [6] yields

\[
\sum_{n \geq 3} \Delta \mathcal{E}_n(a) = \frac{J_1}{8\pi a (\kappa a)^2} + O(1/\kappa^3)
\]

(9)

where \( J_1 \approx 0.001713 \) is evaluated in Ref. [5]. Combining Eqs. (7), (8), and (9), one has the Casimir energy,

\[
E_c = \frac{1}{a} \left( 0.0002819 + \frac{35\pi}{131072 \kappa a} + \frac{0.00006816}{(\kappa a)^2} + O \left( \frac{1}{\kappa a} \right)^3 \right).
\]

(10)
When the measure factor is included ($\alpha = 3/2$), one may conveniently arrange $B_n(\nu, r)$ into 3 pieces $B_n(\nu, r) \equiv (I_n(b) + K_n(b) + R_n(b))/\pi$ where

$$I_n(b) \equiv \int_0^{1/b} dy (1 - 4b^2y^2) q_n(y), \quad R_n(b) \equiv \int_{1/b}^{\infty} dy q_n(y)G(by)$$

$$K_n(b) \equiv \int_0^{1/b} dy \left(G(by) - (1 - 4b^2y^2)\right) q_n(y).$$

Explicitly, $I_1(b) = -\frac{1}{16} \left(b(108b^4 + 179b^2 + 41)/b^2 + 1^2 + (1 - 108b^2)\cot^{-1}(b)\right)$ is $O(b^{-3})$ for large $b$ and is finite for $b \to 0$. Using (13) one has $\sum_{l=0}^{\infty} I_1(b) = \kappa a/16 - 5/(384\kappa a) + O(1/(\kappa a)^3)$. $K_1(b)$ is easy to calculate if the sum over $l$ is done first. Note that $\sum_{l=0}^{\infty} q_n(b)/b = \frac{1}{2}(-4s^2 S_2(s) + 3s^4 S_3(s))$ where

$$S_n(s) \equiv \sum_{l=0}^{\infty} \frac{b}{(b^2 + s^2)^n} = \frac{\kappa a}{(n - 1)s^{2(n-1)}} + \frac{1}{48\kappa a s^{2n}} + O\left(\frac{1}{\kappa a}\right)^3.$$

Thus, summing $K_1(b)$ over $l$ (change of the integration variable $y \to y/b$ is used) gives $\sum_{l=0}^{\infty} K_1(b) = -\kappa a/16 - (3C - 2)/(64\kappa a) + O(1/(\kappa a)^3)$ where $C \simeq 0.915966$ is the Catalan’s constant. The sum of $R_1$ over $l$ is done in the same way: $\sum_{l=0}^{\infty} R_1(b) = -(7 - 18C + 3\pi)/384\kappa a + O\left(\frac{1}{\kappa a}\right)^3$. Combining all the contributions one has

$$\Delta(E_1(a) - E_1(\eta R)) = -\frac{1}{a}\left(\frac{1}{128 \kappa a} + O\left(\frac{1}{(\kappa a)^3}\right)\right).$$

Similarly, the rest of the terms are given as

$$\Delta(E_2(a) - E_2(\eta R)) = \frac{1}{a}\left(\frac{-191 + 108\pi}{9216\pi(\kappa a)} + O\left(\frac{1}{(\kappa a)^3}\right)\right)$$

$$\sum_{n \geq 3} \Delta E_n(a) = \frac{1}{a}\left(\frac{G_1J_1}{4\pi(\kappa a)^2} + O\left(\frac{1}{(\kappa a)^3}\right)\right)$$

with $G_1 = -13$ and $J_1 \simeq 0.001713$. Thus, the Casimir energy is given as

$$E_c = \frac{1}{a}\left(0.002819 - \frac{0.002691}{\kappa a} - \frac{0.001772}{(\kappa a)^2} + O\left(\frac{1}{(\kappa a)^3}\right)\right).$$

IV. SUMMARY AND DISCUSSIONS

The Casimir energy of massless scalar field in $\kappa$-Minkowski space-time is presented in (14) when the measure factor is neglected, and in (16) when the measure factor is taken care of. The negative mode (anti-particle) contribution is formally given if $\kappa \to -\kappa$ in (14) and (16). However, the formula obtained in (14) and (16) is valid only for $\kappa > 0$. One can extend the result to $\kappa < 0$ using the even property of $G(x)$ under the sign change of $\kappa$ to $-\kappa$:

$$E_c^{\alpha=0}(a) = \frac{1}{a}\left(0.002819 + \frac{0.0005389}{\kappa a} + \frac{0.0006816}{(\kappa a)^2} + O\left(\frac{1}{(\kappa a)^3}\right)\right),$$

$$E_c^{\alpha=3/2}(a) = \frac{1}{a}\left(0.002819 - \frac{0.002691}{\kappa a} - \frac{0.001772}{(\kappa a)^2} + O\left(\frac{1}{(\kappa a)^3}\right)\right).$$

This demonstrates that the particle contribution and the anti-particle contribution are equal. It should be emphasized that the symmetric result (16) originates from the analytic property of the dispersion relation (11) (the integration range in (11) is from 0 to $\infty$). This is contrasted with the asymmetric ordering case in (14) where the presence of a branch-cut spoils the particle and anti-particle symmetry.

The dispersion relation (11) is obtained from the Casimir invariant $M_2^a(p) = (2\kappa \sinh p_0/(2\kappa))^2 - p^2 = 0$. When $|p| \geq 2\kappa$, however, there appears another real mode, so called the high momentum mode, from the relation $M_2^a(p) = (2\kappa \sinh p_0/(2\kappa))^2 - p^2 = -4\kappa^2$. Considering the result of black body radiation [2], where the high momentum mode spoils the $\kappa \to \infty$ limit, one needs to exclude the high momentum mode on the mass-shell condition.

Finally, it is worth to mention that the $\kappa$ correction to the Casimir energy (14) is of $O(1/\kappa)$. This is contrasted with results shown in the parallel plate case. According to [8] the Casimir energy of electromagnetic field (electric and magnetic modes) per unit area is of the $O(1/\kappa^2)$. 
Acknowledgments

This work was supported in part by the Korea Research Foundation Grant funded by Korea Government (MOEHRD, Basic Research Promotion Fund, (KRF-2005-075-C00009; H.-C.K.) and(KRF-2007-2-313-C00153;R) and by the the Center for Quantum Spacetime (CQUeST) of Sogang University with grant (R11-2005-021). It is also acknowledged by R that this report has been completed during his visit to Korea Institute for Advanced Study (KIAS).

APPENDIX A: APPROXIMATION OF SUMMATION BY USING INTEGRATION

Let us consider the following summation: \( \sum_{l=0}^{\infty} f(\frac{l+1/2}{L}) \) where \( \lim_{x \to \infty} f(x) = 0 \). For \( L \to \infty \), this becomes the integral \( \int_0^{\infty} f(x)dx \) for non-singular continuous function \( f(x) \). However, for large but finite \( L \), this is not the case, and we are presenting a reasonable approximation of this sum as an integral. Using the Taylor series at \( x_c = (l+1/2)/L \), \( f(x) = \sum_{n=0}^{\infty} f^{(n)}(x_c)(x-x_c)^n/n! \) one can put

\[
\int_{x_c-1/(2L)}^{x_c+1/(2L)} f(x)dx = \frac{f(x_c)}{L} + \sum_{k=1}^{\infty} f^{(2k)}(x_c) \frac{2}{(2L)^{2k+1}(2k+1)!}.
\]

(A1)

Successive application of this formula gives

\[
f(x_c) = L \int_{x_c-1/(2L)}^{x_c+1/(2L)} \left[ f(x) - \frac{1}{24L^2} f^{(2)}(x) \right] + \sum_{k=2}^{\infty} \left( \frac{2k(2k+1)}{6} - 1 \right) \frac{f^{(2k)}(x_c)}{(2L)^{2k}(2k+1)!}.
\]

(A2)

Noting that the last term is of \( O(L^{-4}) \), one has

\[
\sum_{l=0}^{\infty} f(\frac{l+1/2}{L}) = L \int_0^{\infty} dx f(x) + \frac{f'(0)}{24L} + O(L^{-3}).
\]

(A3)

For the special case when \( f(x) \) is an even function, the summation over \( f(x) \) is approximated by its integration only since the right-hand side of (A3) does not contain any even number of derivatives to all orders in \( 1/L \). For example, if \( f(x) = (1 + x^2)^{-1} \), (A3) gives \( L \int_0^{\infty} dx f(x) = L\pi/2 \) to all orders in \( 1/L \). On the other hand, one may perform exact summation to have \( (L\pi/2) \tan(L\pi) \). The difference of the two is of \( O(e^{-L\pi}) \), which is exponentially small for large \( L \). This is just an example of the fact that the exponentially small correction is not well represented by a Taylor series.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).
[2] T. H. Boyer, Phys. Rev. 174, 1764 (1968); B. Davis, J. Math. Phys. 13, 1324 (1972); R. Balian and B. Duplantier, Ann. Phys. (N.Y.) 112, 165 (1978); K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (N.Y.) 115, 388 (1978).
[3] S. Majid and H. Ruegg, Phys. Lett. B334, 348 (1994) [arXiv:hep-th/9405107]; S. Zakrzewski, J. Phys A27, 2075 (1994).
[4] J. Lukierski, A. Nowicki, H. Ruegg, and V. N. Tolstoy, Phys. Lett. B264, 331 (1991); J. Lukierski and H. Ruegg, Phys. Lett. B329, 189 (1994) [arXiv:hep-th/9310117].
[5] H.-C. Kim, C. Rim, J. H. Yee, [arXiv:0710.5633].
[6] C. M. Bender and K. A. Milton, Phys. Rev. D 50, 6547 (1994) [hep-th/9406045]; A. Romeo, Phys. Rev. D 52, 7308 (1995).
[7] H.-C. Kim, J. H. Yee, and C. Rim, Phys. Rev. D 76, 105012 (2007) [arXiv:0705.4628].
[8] J. P. Bowes and P. D. Jarvis, [arXiv:gr-qc/9602016]; M. V. Cougo-Pinto, C. Farina, and J. F. M. Mendes, Phys. Lett. B529, 256 (2002).