Anomalous propagations of electromagnetic waves in anisotropic media with a unique dispersion relation

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Abstract

We investigate the reflection and refraction behaviors of electromagnetic waves at the interface between an isotropic material and the anisotropic medium with a unique dispersion relation. We show that the refraction angle of whether phase or energy flow for $E$-polarized waves is opposite to that for $H$-polarized waves, though the dispersion relations for $E$- and $H$-polarized waves are the same in such anisotropic media. For a certain polarized wave the refraction behaviors of wave vector and energy flow are also significantly different. It is found that waves exhibit different propagation behaviors in anisotropic media with different sign combinations of the permittivity and permeability tensors. Some interesting properties of propagation are also found in the special anisotropic media, such as amphoteric refraction, oblique total transmission for both $E$- and $H$-polarized waves, and the inversion of critical angle. The special anisotropic media can be realized by metamaterials and lead to potential applications, such as to fabricate polarization splitters with higher efficiencies than conventional counterparts.

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I. INTRODUCTION

Over thirty years ago, Veselago pioneered the concept of left-handed material (LHM) with simultaneously negative permittivity and negative permeability \[1\]. In this kind of material, \( k \), \( E \) and \( H \) form a left-handed set of vectors, and the material is thus called left-handed material. It leads to many exotic electromagnetic properties among which the most well-known is negative refraction and can be used to fabricate a perfect lens \[2\]. Since the negative refraction was experimentally observed in a structured metamaterial formed by split ring resonators and copper strips \[3\], the LHM has attracted much attention and interest. By now, more experiments \[4, 5\] and numerical simulations \[6, 7, 8, 9\] have added new proofs to the negative refraction. Moreover, it has been found that photonic crystals \[10, 11, 12, 13, 14\], optical crystals \[15\] and anisotropic media \[16, 17, 18, 19\] can exhibit negative refraction except for the isotropic material, so the concept of LHM should be extended.

Recently, an increasing amount of effort has been devoted to the study on negative refraction in anisotropic media. Lindell et al firstly studied the negative refraction in uniaxially anisotropic media in which the permittivity tensor \( \varepsilon \) and the permeability tensor \( \mu \) are not necessarily negative \[16\]. In Refs. \[18, 19, 20\], the general behavior of wave propagations in uniaxial media is investigated in detail. These researches are mainly concentrated on isotropic and uniaxial media, but less work has been done on negative refraction in generally anisotropic media. In addition, only one kind of polarized waves, e.g., \( E \)-polarized waves, is considered in the previous work on negative refraction in generally anisotropic media \[21, 22, 23\]. Then, one enquires naturally: How on earth are \( H \)-polarized waves refracted in anisotropic media? Whether \( H \)-polarized waves are refracted anomalously or regularly when \( E \)-polarized waves exhibit negative refraction?

In this paper, we investigate the propagation of electromagnetic waves in the anisotropic media with a uniform dispersion relation for any polarized waves. We demonstrate that the refraction behaviors of both phase and energy flow for \( E \)-polarized waves are different from those for \( H \)-polarized waves, though the dispersion relations for \( E \) - and \( H \)-polarized waves are the same. Moreover, for a certain polarized wave the refraction behaviors of wave vector and energy flow are significantly different. We also find other interesting characteristics of wave propagation, including oblique total transmission for both \( E \) - and \( H \)-polarized waves and the inversion of critical angle. The special anisotropic media can be realized by metamaterial...
terials and can lead to potential applications, such as to fabricate polarization splitters with a higher efficiency than conventional counterparts. Our results show that it is necessary to study the propagation of phase and energy flow for both $E$- and $H$-polarized waves in order to obtain a complete knowledge on characteristics of wave propagation in anisotropic media.

This paper is organized in the following way. Section II gives the unique dispersion of any polarized waves in the special anisotropic media. We discuss the reflection and refraction at the interface between an isotropic regular material and the special anisotropic medium in Sec. III. Section IV shows in detail that the reflection and refraction characteristics of waves are different in the anisotropic media with different sign combinations of $\varepsilon$ and $\mu$, gives corresponding numerical results, and discuss how to realize the media. A summary is given in Sec. V.

II. DISPERSION RELATIONS

In this section we present the dispersion relation of electromagnetic wave propagation in the special anisotropic media.

For simplicity, we assume the permittivity and permeability tensors of the anisotropic media are simultaneously diagonal in the principal coordinate system,

$$
\varepsilon = \begin{pmatrix}
\varepsilon_x & 0 & 0 \\
0 & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{pmatrix}, \quad \mu = \begin{pmatrix}
\mu_x & 0 & 0 \\
0 & \mu_y & 0 \\
0 & 0 & \mu_z
\end{pmatrix}.
$$

Note that the anisotropic medium can be realized by metamaterials composed of periodic arrays of split-ring resonators and conducting wires [23, 24, 25], or of periodic inductor-capacitor loaded transmission line circuits [26, 27, 28, 29, 30, 31, 32]. In order to disclose the basic characteristics of the propagation of waves, we do not consider losses here as in Refs. [16, 18, 21], though no practical material is ideally lossless. In fact, the metamaterial of low losses has been realized [32]. To simplify the analyses, we assume the medium in a Cartesian coordinates $(O, x, y, z)$ coincident with the principal coordinate system and $e_x, e_y, e_z$ are unit vectors along the axes.

Let us consider a plane wave of angular frequency $\omega$ propagating from an isotropic regular material into the special anisotropic medium. We assume that the electric field is $E = E_0 e^{i k r - i \omega t}$ and that the magnetic field is $H = H_0 e^{i k r - i \omega t}$. The dispersion relation of plane
FIG. 1: The surface of wave vectors at a fixed frequency determined by the dispersive relation Eq. (4) with \( C < 0 \) is always a single-sheet hyperboloid. Here, it has been chosen that \( \varepsilon_z \mu_y > 0, \varepsilon_z \mu_x > 0, \varepsilon_y \mu_x < 0, \) and \( \omega/c \) is the unit of coordinate axes.

A wave in the isotropic regular medium is

\[
k^2_x + k^2_y + k^2_z = \varepsilon_I \mu_I \frac{\omega^2}{c^2},
\]

where \( k_x, k_y, k_z \) are the components of the \( k \) vector, and \( \varepsilon_I \) and \( \mu_I \) are the permittivity and the permeability. For the anisotropic media, if the condition \( \varepsilon_x \mu_x = \varepsilon_y \mu_y = \varepsilon_z \mu_z = C \) is satisfied, the dispersion relations of any polarized plane waves are the same

\[
\frac{q^2_x}{\varepsilon_z \mu_y} + \frac{q^2_y}{\varepsilon_z \mu_x} + \frac{q^2_z}{\varepsilon_y \mu_x} = \frac{\omega^2}{c^2},
\]

where \( C \) is a constant, and \( q_j \) denotes the component of the wave vector \( q \) in the \( j \) direction \( (j = x, y, z) \). When \( C > 0 \), the anisotropic medium is regarded quasiisotropic because \( E- \) and \( H- \) polarized waves in it exhibits the same propagation behaviors, just as in isotropic medium \([20, 33, 34]\). In the present paper we are interested in the anisotropic medium with \( C < 0 \). We shall show that \( E- \) and \( H- \) polarized waves exhibits different propagation behaviors. That is why such anisotropic media can not be regarded as quasiisotropic. The surface of wave vectors decided by the dispersion relation with \( C < 0 \) in wave-vector space is always a single-sheet hyperboloid as shown in Fig. 1.

For simplicity we assume in the rest of the paper that the incident, reflected and refracted waves are all in the \( x-z \) plane, the boundary is at \( z = 0 \) and the \( z \) axis is directed into the
anisotropic medium. Then we can write the incident angle as

$$\theta_I = \tan^{-1} \left[ \frac{k_x}{k_z} \right].$$

(5)

The refractive angle of the transmitted wave vector or of the phase is

$$\beta_P = \tan^{-1} \left[ \frac{q_x}{q_z} \right].$$

(6)

The refraction of phase is regarded regular if \( k_z \cdot \mathbf{q} > 0 \) and anomalous if \( k_z \cdot \mathbf{q} < 0 \). In general the occurrence of refraction requires that the \( z \) component of the refracted wave vector must be real. According to Eqs. (2) and (4) we obtain

$$\varepsilon_y \mu_x \left( \frac{k^2}{\varepsilon_I \mu_I} - \frac{q^2_z}{\varepsilon_z \mu_y} \right) \geq 0,$$

(7)

where \( k^2 = k_x^2 + k_z^2 \). At the same time, we have

$$q_z = \sigma \sqrt{\varepsilon_y \mu_x \left( \frac{\omega^2}{c^2} - \frac{q^2_x}{\varepsilon_z \mu_y} \right)},$$

(8)

where \( \sigma = +1 \) or \( \sigma = -1 \). The choice of the sign should ensure that the power of electromagnetic waves propagates away from the surface to the \( +z \) direction.

### III. REFLECTION, REFRACTION AND BREWSTER ANGLES

In this section we discuss the reflection and refraction between an isotropic regular material and the special anisotropic medium. We show that the refraction angles of both phase and energy flow for \( E \)-polarized waves are opposite to the corresponding parts for \( H \)-polarized, though the dispersion relations for \( E \)- and \( H \)-polarized waves are the same in the special anisotropic medium.

For \( E \)-polarized plane wave the electric fields can be expressed as

$$\mathbf{E}_I = E_0 \mathbf{e}_y e^{ik_x x + ik_z z - i\omega t},$$

(9)

$$\mathbf{E}_R = R_E E_0 \mathbf{e}_y e^{ik_x x - ik_z z - i\omega t},$$

(10)

$$\mathbf{E}_T = T_E E_0 \mathbf{e}_y e^{iq_x^{(E)} x + iq_z^{(E)} z - i\omega t},$$

(11)

for incident, reflected and refracted waves, respectively. Here \( R_E \) and \( T_E \) are the reflection and transmission coefficients, and the components of refracted wave vector \( (q_x^{(E)}, q_z^{(E)}) \) are
decided by the dispersion relation Eq. (4). By boundary conditions we find that \( q_x = k_x \) and the reflection and transmission coefficients are

\[
R_E = \frac{\mu_x k_z - \mu_I q_z^{(E)}}{\mu_x k_z + \mu_I q_z^{(E)}}, \quad T_E = \frac{2\mu_x k_z}{\mu_x k_z + \mu_I q_z^{(E)}}. \tag{12}
\]

The time-average Poynting vector is defined as \( \mathbf{S} = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*) \). For the refracted wave we have

\[
\mathbf{S}_T^{(E)} = \text{Re} \left[ \frac{T_E^2 E_0^2 k_x}{2\omega \mu_z} \mathbf{e}_x + \frac{T_E^2 E_0^2 q_z^{(E)}}{2\omega \mu_x} \mathbf{e}_z \right]. \tag{13}
\]

We should note that the sign of \( q_x \cdot \mathbf{S}_T^{(E)} \) cannot be used to demonstrate whether the refraction of energy flow is regular or anomalous as in isotropic or uniaxial anisotropic material [16, 18]. Instead, we consider the inner product of the tangential component of \( q^{(E)} \) and \( \mathbf{S}_T^{(E)} \)

\[
q_x^{(E)} \cdot \mathbf{S}_T^{(E)} = \text{Re} \left[ \frac{T_E^2 E_0^2 k_x^2}{2\omega \mu_z} \right]. \tag{14}
\]

The refraction of energy flow is regular if \( q_x^{(E)} \cdot \mathbf{S}_T^{(E)} > 0 \) and anomalous if \( q_x^{(E)} \cdot \mathbf{S}_T^{(E)} < 0 \). Hence, the sign of \( \mu_z \) indicates whether the refraction is regular or anomalous.

Following similar way, for \( H \)-polarized incident waves we obtain the reflection and transmission coefficients

\[
R_H = \frac{\varepsilon_x k_z - \varepsilon_I q_z^{(H)}}{\varepsilon_x k_z + \varepsilon_I q_z^{(H)}}, \quad T_H = \frac{2\varepsilon_x k_z}{\varepsilon_x k_z + \varepsilon_I q_z^{(H)}}, \tag{15}
\]

the Poynting vector is calculated to be

\[
\mathbf{S}_T^{(H)} = \text{Re} \left[ \frac{T_H^2 H_0^2 k_x}{2\omega \varepsilon_z} \mathbf{e}_x + \frac{T_H^2 H_0^2 q_z^{(H)}}{2\omega \varepsilon_x} \mathbf{e}_z \right], \tag{16}
\]

and the inner product of the tangential component of wave vector \( q^{(H)} \) and \( \mathbf{S}_T^{(H)} \) is

\[
q_x^{(H)} \cdot \mathbf{S}_T^{(H)} = \text{Re} \left[ \frac{T_H^2 H_0^2 k_x^2}{2\omega \varepsilon_z} \right]. \tag{17}
\]

Likewise, whether the refraction of energy flow for \( H \)-polarized waves is regular is decided by the sign of \( \varepsilon_z \). Evidently, if \( \mu_x \neq \mu_z \) or \( \varepsilon_x \neq \varepsilon_z \) in Eqs. (13) or (16), then a bending angle always exists between \( \mathbf{k} \) and \( \mathbf{S} \), and therefore \( \mathbf{k}, \mathbf{E} \) and \( \mathbf{H} \) cannot form a strictly left-handed system. Further, \( \mathbf{k} \) is not coincident with \( \mathbf{S} \) judging by Eqs. (14) and (17). This point can be seen more clearly in the next section. The refraction angle of the energy flow is defined to be

\[
\beta_S^{(E)} = \tan^{-1} \left[ \frac{\mu_x k_x}{\mu_z q_z^{(E)}} \right], \quad \beta_S^{(H)} = \tan^{-1} \left[ \frac{\varepsilon_x k_x}{\varepsilon_z q_z^{(H)}} \right], \tag{18}
\]
for \( E \)- and \( H \)-polarized waves, respectively.

Let us note that the refraction direction should obey two rules: The tangential components of the incident, reflected, refracted wave vectors on the interface are continuous; The Poynting vector of the refracted wave points away from the interface. Embodied in the problem we are concerned with, the two rules are: \( q_x = k_x \) and \( S_T z > 0 \). Then, making use of Eqs. (5) and (7), we find there exists a critical value

\[
\theta_C = \sin^{-1} \left( \frac{\varepsilon_\mu_y}{\varepsilon_\mu_I} \right) \tag{19}
\]

for the incident angle if \( 0 < \varepsilon_\mu_y < \varepsilon_\mu_I \). At the same time, from Eqs. (13) and (16), we conclude that \( q_{z}^{(E)} \) must have the same sign as \( \mu_x \) for \( E \)-polarized waves and \( q_{z}^{(H)} \) must have the same sign as \( \varepsilon_x \) for \( H \)-polarized waves. Therefore, the sign of \( \mu_x \) suggests whether the refraction of phase in Eq. (6) is regular or anomalous for \( E \)-polarized waves, and the sign of \( \varepsilon_x \) does the same for \( H \)-polarized waves. Considering Eq. (8) and \( \varepsilon_x/\mu_x < 0 \), we come to the conclusion

\[
\beta_S^{(E)} = -\beta_S^{(H)}, \tag{20}
\]

That is to say, the refraction angle of the phase for \( E \)-polarized wave is always opposite to that for \( H \)-polarized wave. Then, together with Eqs. (8), (18) and (20), we find

\[
\beta_S^{(E)} = -\beta_S^{(H)}, \tag{21}
\]

which means that the refraction angle of energy flow for \( E \)-polarized wave is always opposite to that for \( H \)-polarized wave.

We next explore the problem of oblique total transmission. The reflectivity and the transmissivity are defined to be

\[
r = -\frac{e_z \cdot S_R}{e_z \cdot S_I}, \quad t = \frac{e_z \cdot S_T}{e_z \cdot S_I}, \tag{22}
\]

respectively, where \( S_I \) and \( S_R \) are the respective Poynting vectors of incident and reflected waves. Then we have for \( E \) and \( H \)-polarized waves

\[
r_E = |R_E|^2, \quad t_E = \frac{\mu_x q_z^{(E)}}{\mu_x k_x} |T_E|^2; \tag{23}
\]

\[
r_H = |R_H|^2, \quad t_H = \frac{\varepsilon_x q_z^{(H)}}{\varepsilon_x k_x} |T_H|^2. \tag{24}
\]

It is easy to show that \( r_E + t_E = 1 \) and \( r_H + t_H = 1 \), which indicates the power conservation on the boundary. When \( r_E = 0 \) or \( r_H = 0 \), the incident angle is known as the Brewster
angle \[24\]. Substituting Eq. (12) into Eq. (23), we find that when the condition

\[
0 < \frac{\mu_z (\varepsilon_y \mu_I - \varepsilon_I \mu_x)}{\varepsilon_I (\mu_I^2 - \mu_x \mu_z)} < 1 \tag{25}
\]

is satisfied, the Brewster angle for \(E\)-polarized wave is

\[
\theta_B^{(E)} = \sin^{-1} \left[ \sqrt{\frac{\mu_z (\varepsilon_y \mu_I - \varepsilon_I \mu_x)}{\varepsilon_I (\mu_I^2 - \mu_x \mu_z)}} \right]. \tag{26}
\]

Similarly, inserting Eq. (15) into Eq. (24) we can see that under the condition

\[
0 < \frac{\varepsilon_z (\mu_y \varepsilon_I - \mu_I \varepsilon_x)}{\mu_I (\varepsilon_I^2 - \varepsilon_x \varepsilon_z)} < 1 \tag{27}
\]

there exists a Brewster angle for \(H\)-polarized wave

\[
\theta_B^{(H)} = \sin^{-1} \left[ \sqrt{\frac{\varepsilon_z (\mu_y \varepsilon_I - \mu_I \varepsilon_x)}{\mu_I (\varepsilon_I^2 - \varepsilon_x \varepsilon_z)}} \right]. \tag{28}
\]

At the Brewster angle, the reflectivity is zero and the oblique total transmission occurs. It is worthy of mentioning that such oblique total transmissions for \(E\)- and \(H\)-polarized waves can not occur simultaneously in a conventional isotropic material. Therefore the oblique total transmission is due to the anisotropy of the material \[19\].

### IV. DETAILED ANALYSES AND NUMERICAL RESULTS

In the following we shall apply the conclusions in the above section to exploring in detail the reflection and refraction characteristics at the boundary between an isotropic material and the special anisotropic medium. Aggregately there are six kinds of sign combinations for \(\varepsilon\) and \(\mu\) of the anisotropic medium. We show that with different sign combinations of \(\varepsilon\) and \(\mu\), the behavior of propagation is different. According to the curve form of refracted wave vector at a fixed frequency (isofrequency curve) in the wave vector plane \[21\], we discuss in three cases.

#### A. the isofrequency curve is a hyperbola with foci on the \(k_x\) axis

When the signs of the permittivity and permeability tensors are chosen as \(\varepsilon = (-, -, +)\) and \(\mu = (+, +, -)\), the isofrequency curve of the refracted wave vector is a hyperbola with the foci in the \(k_x\) axis, as shown in Fig. 2 (a). Note that this case corresponds to the
FIG. 2: The isofrequency curves to illustrate the refraction. The circle and the hyperbola represent the surfaces of wave vectors in an isotropic regular medium and the anisotropic media, respectively. In (a), \( \beta_P^{(E)} = -\beta_P^{(H)} > 0, \beta_S^{(E)} = -\beta_S^{(H)} < 0 \), while \( \beta_P^{(H)} = -\beta_P^{(E)} > 0, \beta_S^{(H)} = -\beta_S^{(E)} < 0 \) in (b).

For the \( E \)-polarized wave the wave vector is refracted regularly, but the Poynting vector is refracted anomalously. For the \( H \)-polarized wave the situation is just opposite. More evidently,

\[
\beta_P^{(E)} = -\beta_P^{(H)} > 0, \quad \beta_S^{(E)} = -\beta_S^{(H)} < 0.
\]

If \( 0 < \varepsilon_x \mu_y - \varepsilon_I \mu_I \), the refraction can occur only if the incident angle is in the branch

\[
\theta_C < |\theta_I| < \pi/2,
\]

where \( \theta_C \) is the critical angle defined by Eq. (19), or else \( 0 < |\theta_I| < \pi/2 \). That is to say, the incident angle can be larger than the critical angle, different from the regular situation where the incident angle is smaller than the critical angle. At the same time, using Eqs. (19), (26) and (28) one can easily show that the Brewster angle is larger than the critical angle

\[
\theta_B^{(E)} > \theta_C, \quad \theta_B^{(H)} > \theta_C.
\]

This phenomenon is called the inversion of critical angle [19, 24]. From Eqs. (23) and (24) one can show that a bit shift of \( \theta_I \) from \( \theta_C \) can lead to a big change in reflectivity, as can be seen in Fig. 3.
Incident angle $\theta$ (degree)

Reflectivity $r$

| Incident angle $\theta_i$ (degree) | $r_E$ | $r_H$ |
|-----------------------------------|-------|-------|
| 30                                | 0.2   | 0.2   |
| 45                                | 0.4   | 0.4   |
| 60                                | 0.6   | 0.6   |
| 75                                | 0.8   | 0.8   |
| 90                                | 1.0   | 1.0   |

FIG. 3: Diagram of reflectivity as a function of incident angle. For the isotropic material $\varepsilon_I = 1$ and $\mu_I = 1$, while $\varepsilon = (-0.5, -0.4, 1)$ and $\mu = (1, 0.8, -2)$ for the anisotropic medium. There exist Brewster angles for both $E$- and $H$-polarized waves, i.e., $\theta_B^{(E)}$ and $\theta_B^{(H)}$. The $\theta_C$ is the critical angle.

If the signs of the permittivity and permeability are chosen as $\varepsilon = (+, +, -)$ and $\mu = (-, -, +)$, the iso-frequency curve of the refracted wave vector is shown in Fig. 2(b). We can see that the refraction behaviors of $E$- and $H$-polarized waves are just opposite to the counterparts in Fig. 2(a), though the dispersion relation is invariant. This is due to the sign inversions of $\varepsilon$ and $\mu$ in Fig. 2(b) compared with those in Fig. 2(a).

For the purpose of illustration, a numerical example is given in Fig. 3. We find that there exists a Brewster angle for both $E$- and $H$-polarized waves and that the Brewster angles are larger than the critical angle. This phenomenon is different from the situation in conventional isotropic nonmagnetic materials where the Brewster angle only exists for $H$-polarized waves. In a regular nonmagnetic material, the physical mechanism of Brewster angle is that the component of energies radiated by electric dipoles under the effect of transmitted electric fields is zero along the direction perpendicular to the reflected wave. However, the appearance of Brewster angles here relies on parameters of $\varepsilon$ and $\mu$.

First, let us consider an example: For a light incident from a regular magnetic material with $\varepsilon = 1$ and $\mu = 2$ into another regular material with $\varepsilon' = 1$ and $\mu' = 1$, there exists a Brewster angle for $E$-polarized waves. Evidently, the cause is not the response of electric dipoles, but the response of magnetic dipoles. Therefore, the existence of Brewster angles for $E$- and $H$-polarized waves here is due to the compound operations of the electric and magnetic
FIG. 4: The isofrequency curves to illustrate the refraction. The circle and the ellipse represent the surfaces of wave vectors in an isotropic regular material and the anisotropic medium. In (a), $\beta_P^{(E)} = -\beta_P^{(H)} > 0$, $\beta_S^{(E)} = -\beta_S^{(H)} > 0$. The refraction situation is reversed in (b), i.e., $\beta_P^{(H)} = -\beta_P^{(E)} > 0$, $\beta_S^{(H)} = -\beta_S^{(E)} > 0$.

responses in the anisotropic media.

B. the isofrequency curve is an ellipse

If the signs of the permittivity and permeability tensors have the form of $\varepsilon = (-, +, -)$ and $\mu = (+, -, +)$, the isofrequency curve of the refracted wave vector is an ellipse as shown in Fig. 4 (a). This case corresponds to the cutoff indefinite medium in Ref. [23]. For this case we find

$$\mu_x > 0, \quad q_z^{(E)} > 0; \quad \mu_z > 0, \quad q_x^{(E)} \cdot S_T^{(E)} > 0. \quad (34)$$

$$\varepsilon_x < 0, \quad q_z^{(H)} < 0; \quad \varepsilon_z < 0, \quad q_x^{(H)} \cdot S_T^{(H)} < 0. \quad (35)$$

The wave vector and the Poynting vector are both refracted regularly for the $E$-polarized wave, while anomalously for the $H$-polarized wave. By virtue of the analyses in the previous section, we obtain

$$\beta_P^{(E)} = -\beta_P^{(H)} > 0, \quad \beta_S^{(E)} = -\beta_S^{(H)} > 0. \quad (36)$$

From the figure we can also see that, when $0 < \varepsilon_z \mu_y < \varepsilon_I \mu_I$, the refraction can occur only if the incident angle is in the branch

$$-\theta_C < \theta_I < \theta_C, \quad (37)$$
FIG. 5: Diagram of reflectivity as a function of incident angle. Here \( \varepsilon_I = 1, \mu_I = 1, \varepsilon = (-0.5, 0.4, -1) \) and \( \mu = (1, -0.8, 2) \). The \( \theta_B^{(H)} \) is the Brewster angle for \( H \)-polarized waves and \( \theta_C \) is the critical angle.

where \( \theta_C \) is the critical angle, or else \(-\pi/2 < \theta_I < \pi/2\). In the case of \( \varepsilon = (+, -, +) \) and \( \mu = (-, +, -) \) the isofrequency curve of the refracted wave vector is shown in Fig. 4 (b). We can see that the refraction behaviors of the \( E \)- and \( H \)-polarized waves are just the results after exchanging the \( E \)- and \( H \)-polarized waves in Fig. 4 (a). This result manifests again that \( E \)- and \( H \)-polarized waves do not necessarily exhibit the same propagation, even if their dispersion relations are the same. As an example, we examine the case of \( \varepsilon = (-0.5, 0.4, -1) \) and \( \mu = (1, -0.8, 2) \), with the results illustrated in Fig. 5. We find that there exists a Brewster angle for \( H \)-polarized waves. Actually, we can choose appropriate material parameters of \( \varepsilon \) and \( \mu \) for \( E \)-polarized waves to exhibit a Brewster angle.

C. the isofrequency curve is a hyperbola with foci in the \( k_z \) axis

If the signs of the permittivity and permeability are of the form \( \varepsilon = (-, +, +) \) and \( \mu = (+, -, -) \), the isofrequency curve of the refracted wave vector is a hyperbola with the foci in the \( z \) axis, as shown in Fig. 6 (a). Let us note that this case corresponds to the never cutoff indefinite medium in Ref. [23]. In view of the discussions in the previous section, we have

\[
\mu_x > 0, \quad q_z^{(E)} > 0; \quad \mu_z < 0, \quad q_x^{(E)} \cdot S_T^{(E)} < 0. \tag{38}
\]

\[
\varepsilon_x < 0, \quad q_z^{(H)} < 0; \quad \varepsilon_z > 0, \quad q_x^{(H)} \cdot S_T^{(H)} > 0. \tag{39}
\]
FIG. 6: The isofrequency curves to illustrate the refraction. The circle and the hyperbola represent the surfaces of wave vectors in an isotropic regular material and the anisotropic medium. In (a), \( \beta_p^{(E)} = -\beta_p^{(H)} > 0, \quad \beta_S^{(E)} = -\beta_S^{(H)} < 0 \). In (b), \( \beta_p^{(H)} = -\beta_p^{(E)} > 0, \quad \beta_S^{(H)} = -\beta_S^{(E)} < 0 \). There does not exist any critical angle.

For the \( E \)-polarized wave the wave vector is refracted regularly, but the Poynting vector is refracted anomalously. For the \( H \)-polarized wave the situation is just opposite. It is easy to show that

\[
\beta_p^{(E)} = -\beta_p^{(H)} > 0, \quad \beta_S^{(E)} = -\beta_S^{(H)} < 0. \tag{40}
\]

Different from the case shown in Fig. 2 (a), there does not exist any critical value for the incident angle, namely

\[
0 < |\theta_I| < \pi/2. \tag{41}
\]

As for the case of \( \varepsilon = (+, -, -) \) and \( \mu = (-, +, +) \), the isofrequency curve of the refracted wave vector is shown in Fig. 6 (b). We can see that the refraction behaviors of \( E \)- and \( H \)-polarized waves are just opposite to those in Fig. 6 (a). We give a numerical result of \( \varepsilon = (-0.5, 0.4, 1) \) and \( \mu = (1, -0.8, -2) \) in Fig. 7. We find that there exists a Brewster angle for \( E \)-polarized waves.

In the above discussions we find that \( E \)-polarized waves propagate remarkably distinct from \( H \)-polarized waves in the special anisotropic media. The two polarized waves are always refracted into different directions. Even if the propagation direction is along one principal axis, the two eigenmodes for one propagation direction are always in two different directions, \( i.e., q_x^{(E)} = -q_x^{(H)} \) for one \( q_x \). Then, the refractive indices \( n \) are always different for the two polarizations and are relevant to the propagation directions. Meanwhile, optic
Incident angle $\theta$ (degree)

Reflectivity $r$

$0$ $0.2$ $0.4$ $0.6$ $0.8$ $1$

$\varepsilon_{I} = 1$, $\mu_{I} = 1$, $\varepsilon = (-0.5, 0.4, 1)$ and $\mu = (1, -0.8, -2)$. The $\theta_B^{(E)}$ is the Brewster angle for $E$-polarized waves.

Axes are the directions for which the two values of the phase velocity $v_p = c/n$ are equal. Accordingly, there does not exist any optic axis in the anisotropic medium, and the concept of ordinary waves or extraordinary waves in uniaxial materials is not valid here. In this sense, waves in the anisotropic medium all can be regarded as extraordinary waves. The property that the two eigenmodes for one propagation direction are always refracted into two different directions indicates that the anisotropic media are polarization-sensitive and can be put into potential applications, such as dividing a beam into two pure polarizations. The polarization beam splitter made of such media can divide beams with larger splitting angles and splitting distances than conventional ones. In the above discussions we have assumed that the principal axes are coincident with the coordinate axes in which the media lie. If not so, new propagation phenomena will occur except for those in the above and one can investigate this issue by methods in Refs. [17, 22, 35].

Finally, we discuss how to realize the anisotropic media of a unique dispersion. Several recent developments make the special anisotropic media available. Firstly, metamaterials composed of periodic arrays of split-ring resonators and conducting wires have been demonstrated to be able to construct anisotropic media of the unique dispersion relation. A more promising choice is the metamaterial composed of periodic inductor-capacitor loaded transmission line circuits because it has lower loss and wider bandwidth. Such metamaterials can also exhibit the above dispersion relations. In addition, certain designs of
photonic crystals have been shown to be able to model the dispersion relation of anisotropic materials [40, 41, 42]. Therefore, there is no physical or technical obstacles to make the special anisotropic media.

V. CONCLUSION

In summary, we have investigated the wave propagation in anisotropic media for which the dispersion relation of any polarized waves is the same. We have analysed the reflection and refraction behaviors of electromagnetic waves at the interface between an isotropic regular medium and the anisotropic media. We show that in the anisotropic media, the refraction angles of both phase and energy flow for $E$-polarized waves are opposite to the counterparts for $H$-polarized waves, that is, $\beta_{P}^{(E)} = -\beta_{P}^{(H)}$ and $\beta_{S}^{(E)} = -\beta_{S}^{(H)}$, though the dispersion relation for $E$-polarized waves is the same as that for $H$-polarized waves. The refraction behaviors of wave vector and energy flow are significantly different for a certain polarized wave. In addition, we find many interesting characteristics of wave propagation in the medium. Firstly, the wave propagation exhibits different behaviors with different sign combinations of the permittivity and permeability tensors. Secondly, the reflection coefficient becomes zero at certain angles not only for $H$-polarized wave but also for $E$-polarized wave. Therefore it is reasonable to extend the concept of Brewster angle from $H$-polarized wave to $E$-polarized wave. Lastly, the Brewster angle can be larger than the critical angle, which is called the inversion of critical angle. Our results indicate that it is necessary to study the propagation of the phase and the energy for both $E$- and $H$-polarized waves in order to obtain a complete knowledge on the wave propagation in anisotropic media.

Due to the unique dispersion relation, characteristics of electromagnetic wave propagation in the special anisotropic media are remarkably distinct from those in isotropic or uniaxial LHM. The most attracting one is that the anisotropic media are polarization-sensitive, which can lead to many applications, such as to fabricate polarization splitters with higher efficiencies than conventional counterparts. Before we end this paper, we stress that the special anisotropic media can be realized by anisotropic metamaterials having been constructed in laboratory [24, 25, 32, 39]. Then the properties of wave propagation in them can be studied experimentally and favorable applications can be realized by the anisotropic media.
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