EQUILIBRIUM FLUCTUATION OF
THE ATLAS MODEL

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Abstract. We study the fluctuation of the Atlas model, where a unit drift is assigned to the lowest ranked particle among a semi-infinite ($\mathbb{Z}_+$-indexed) system of otherwise independent Brownian particles, initiated according to a Poisson point process on $\mathbb{R}_+$. In this context, we show that the joint law of ranked particles, after being centered and scaled by $t^{-1/4}$, converges as $t \to \infty$ to the Gaussian field corresponding to the solution of the Additive Stochastic Heat Equation (ASHE) on $\mathbb{R}_+$ with Neumann boundary condition at zero. This allows us to express the asymptotic fluctuation of the lowest ranked particle in terms of a $\frac{1}{4}$-Fractional Brownian Motion ($\frac{1}{4}$-fBm). In particular, we prove a conjecture of Pal and Pitman [17] about the asymptotic Gaussian fluctuation of the ranked particles.

1. Introduction

In this paper we study the infinite particles Atlas model. That is, we consider the $\mathbb{R}^{\mathbb{Z}_+}$-valued process $\{X_i(t)\}_{i \in \mathbb{Z}_+}$, each coordinate performing an independent Brownian motion except for the lowest ranked particle receiving a drift of strength $\gamma > 0$. For suitable initial conditions, this process is given by the unique weak solution of

$$dX_i(t) = \gamma 1_{\{X_i(t) = X_{(0)}(t)\}} dt + dB_i(t), \quad i \in \mathbb{Z}_+. \quad (1.1)$$

Hereafter $B_i(t), i \in \mathbb{Z}_+$, denote independent standard Brownian motions and $X_{(i)}(t), i \in \mathbb{Z}_+$, denote the ranked particles, i.e. $X_{(0)}(t) \leq X_{(1)}(t) \leq \ldots$. More precisely, recall that $(x_i) \in \mathbb{R}^{\mathbb{Z}_+}$ is rankable if there exists a bijection $\pi : \mathbb{Z}_+ \to \mathbb{Z}_+$ (i.e. permutation) such that $x_{\pi(i)} \leq x_{\pi(j)}$ for all $i \leq j \in \mathbb{Z}_+$. Such ranking permutation is unique up to ties, which we break in lexicographic order. The equation (1.1) is then well-defined if $(X_i(t))_{i \in \mathbb{Z}_+}$ is rankable at all $t \geq 0$ with a measurable ranking permutation.

The Atlas model (1.1) is a special case of diffusions with rank dependent drifts. In finite dimensions, such systems are studied in [1], motivated by questions in filtering theory, and in [8, 14], in the context of stochastic portfolio theory. See also [4, 5, 10, 11, 12], for their ergodicity and sample path properties, and [6, 18] for their large deviations properties as the dimension tends to infinity. The Atlas model is a simple special case

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(where the drift vector is specialized to \((\gamma, 0, \ldots, 0)\)) that allows more detailed analysis. In particular, Pal and Pitman \cite{20} consider the infinite dimensional Atlas model \((1.1)\), establishing well-posedness and the existence of an explicit invariant measure, see also \cite{15, 16}.

In this paper we study the long-time behavior of the ranked particles, in particular the lowest ranked particle. This amounts to understanding competition between the drift \(\gamma\) and the push-back from the bulk of particles (due to ranking). These two effects act against each other, and balance exactly at the critical density \(2\gamma\). More precisely, recall from \cite{20} that, starting from \(\{X(i)(0)\} \sim \text{PPP}_+(2\gamma)\), the Poisson Point Process with density \(2\gamma\) on \(\mathbb{R}_+ := [0, \infty)\), \((1.1)\) admits a unique weak solution (which is rankable) such that \(\{X(i)(t) - X(0)(0)\}_{i \in \mathbb{Z}_+}\) retains the \(\text{PPP}_+(2\gamma)\) law for all \(t \geq 0\). At this critical density, we show that, for large \(t\) and for all \(i\), \(X(i)(t)\) fluctuates at \(O(t^{1/4})\), and the joint law of the fluctuations of the particles scales to a Gaussian field characterized by \(\text{ASHE}\).

Hereafter we fix \(\{X(i)(t)\}_{i \in \mathbb{Z}_+}\) to be the unique weak solution of \((1.1)\) starting from \(\text{PPP}_+(2\gamma)\). With \(Y_i(t) := X(i)(t) - X(0)(t)\) denoting the \(i\)-th gap, such initial condition are equivalent to \(X(0)(0) = 0\) and \(\{Y_i(0)\}_{i \in \mathbb{Z}_+} \sim \bigotimes_{i \in \mathbb{Z}_+} \text{Exp}(2\gamma)\). We consider the process

\[
\mathcal{X}^\varepsilon_i(t) := \varepsilon^{1/4} \left[ i_x(x) - 2\gamma X(i_x(x)) (\varepsilon^{-1} t) \right], \quad i_x(x) := \left[ (2\gamma^{1/2})^{-1} x \right]. \tag{1.2}
\]

Recall that the the relevant solution of the \(\text{ASHE}\), \((1.5)\), is invariant under the scaling \(\mathcal{X}_i(x) \mapsto a^{1/4} \mathcal{X}_i/(a^{1/2})\), which suggests the scaling of \((1.2)\). Alternatively, this scaling can be understood as choosing the diffusive scaling of \((t, x)\) to respect \(B_i(\cdot)\), and choosing the \(\varepsilon^{1/4}\) factor to capture the Gaussian fluctuation of \(\text{PPP}_+(2\gamma)\).

Let \(p(x) = \Phi'(x) = (2\pi)^{-1/2} e^{-x^2/2}\) be the standard Gaussian density, with \(p_t(x) := p(xt^{-1/2})\) the heat kernel and \(\Phi_t(x) = \Phi(xt^{-1/2})\) the scaled error function. We use \(p_t^N(y, x) := p_t(y - x) + p_t(y + x)\) for the Neumann heat kernel and

\[
\Psi_t(y, x) := 2 - \Phi_t(y - x) - \Phi_t(y + x). \tag{1.3}
\]

Hereafter we endow the space of right-continuous-left-limit functions on \(\mathbb{R}_+\), the topology of uniform convergence on compact sets, and use \(\Rightarrow\) to denote weak convergence of probability measures. Our main result is as follows.

**Theorem 1.1.** Let \(\mathcal{X}_i(\cdot)\) denote the \(C(\mathbb{R}_+^2, \mathbb{R})\)-valued centered Gaussian process with covariance

\[
E(\mathcal{X}_i(x) \mathcal{X}_i(x')) = 2\gamma \left( \int_0^\infty \Psi_t(y, x) \Psi_t(y, x') dy + \int_{t \wedge t'} \int_0^\infty p_t^N(y, x) p_{t'}^N(y, x') dy ds \right). \tag{1.4}
\]

Then, \(\mathcal{X}^\varepsilon_i(\cdot) \Rightarrow \mathcal{X}_i(\cdot)\), as \(\varepsilon \to 0\).

**Remark 1.2.** The limiting process \(\mathcal{X}_i(\cdot)\) can be equivalently characterized by the solution of the \(\text{ASHE}\) on \(\mathbb{R}_+\),

\[
\left( \partial_t - \frac{1}{2} \partial_{xx} \right) \mathcal{X}_i(x) = (2\gamma)^{1/2} \mathcal{Y}_i, \quad t, x > 0, \tag{1.5}
\]
with the initial condition $X_0(x) = (2\gamma)^{1/2}B(x)$ and a suitable boundary condition. Here $B(x)$ denotes a standard Brownian motion and $\mathcal{W}(t,x)$ denotes a 2-dimensional white noise, independent of $B(\cdot)$. In the course of proving Theorem 1.1, extracting the boundary condition requires a special choice of the test function (see (1.12)). From this, we end up with the Neumann boundary condition. That is, we declare the semi-group of (1.5) to be $p^N_t(y, x)$, whereby obtaining $X_t = W_t(x) + M_t(x)$, for

$$W_t(x) := \int_0^\infty p^N_t(y, x)X_0(y)dy = (2\gamma)^{1/2}\int_0^\infty \Psi_t(y, x)dB(y), \quad (1.6)$$

$$M_t(x) := (2\gamma)^{1/2}\int_0^t \int_0^\infty p^N_{t-s}(y, x)d\mathcal{W}(y, s). \quad (1.7)$$

The former and latter, measurable with respect to $B(\cdot)$ and $\mathcal{W}(\cdot, \cdot)$, respectively, are independent. From (1.6) and (1.7), one then concludes the covariance as given in (1.4).

In retrospect, the Neumann boundary condition represents the conservation of particles at $x = 0$. It is shown in [3] that at the equilibrium density we consider here, $\sup_{s \in [0, t]} \{\varepsilon^{1/2}|X_0(\varepsilon^{-1}t)|\} \to 0$ almost surely. That is, at the scale $\varepsilon^{-1/2}$ of space, the lowest rank particle stays very close to $x = 0$. Consequently, the flux at $x = 0$ should be zero, which amounts to the Neumann boundary condition.

**Remark 1.3.** If starting (1.1) at deterministic equi-distant particle positions, i.e. $X_{(i)}(0) = 2\gamma i$, one should naturally expect to end up with the limiting process $X_t(x) := M_t(x)$ (corresponding to $X_0(x) = 0$). However, our proof of Theorem 1.1 relies on the stationarity of $\{X_{(i)}(\cdot) - X_0(\cdot)\}$ to simplify a-priori estimates, and hence does not apply to this deterministic initial condition.

An important consequence of Theorem 1.1 is:

**Corollary 1.4.**

(a) Let $B^{(H)}(\cdot)$ denote the fractional Brownian motion with Hurst parameter $H$. As $\varepsilon \to 0$, $\varepsilon^{-1/4}X_0(\varepsilon^{-1}\cdot)$, the scaled fluctuation of the lowest ranked particle, weakly converges to $(2\pi)^{1/3}\gamma^{-1/2}B^{(H)}(\cdot)$.

(b) As $t \to \infty$, $(X_{(i \varepsilon(x))}(t) - X_{(i \varepsilon(x))}(0))t^{-1/4}$ weakly converges to a centered Gaussian with variance $\sigma^2(x)$, satisfying $\sigma(0) = (2/\pi)^{1/4}\gamma^{-1/2}$ and $\lim_{x \to \infty} \sigma(x) = (2\pi)^{-1/4}\gamma^{-1/2}$.

Indeed, it is not difficult to deduce from (1.4) the covariance of the center Gaussian process $K_4(x) := (2\gamma)^{-1}(X_4(x) - X_0(x))$ for the special case of $x = 0$ and $x \to \infty$, and to arrive at

$$E(K_t(0)K_{t'}(0)) = \gamma^{-1}(2\pi)^{-1/2}(t^{1/2} + (t')^{1/2} - |t - t'|^{1/2}), \quad (1.8)$$

$$\lim_{x \to \infty} E(K_t(x)K_{t'}(x)) = \gamma^{-1}(8\pi)^{-1/2}(t^{1/2} + (t')^{1/2} - |t - t'|^{1/2}). \quad (1.9)$$

From (1.8)–(1.9) Corollary 1.4 immediately follows.

Theorem 1.1 is the first result of asymptotic fluctuations of (1.1), with Corollary 1.4(b) resolving the conjecture of Pal and Pitman [17, Conjecture 3]. Further, Theorem 1.1 establishes the previously undiscovered connection of (1.1) to ASHE.
Remark 1.5. In [3], the hydrodynamic limits of the Atlas model (1.1) is studied. For out-of-equilibrium initial conditions, it is shown that \( \varepsilon^{1/2}X_{(0)}(\varepsilon^{-1}x) \) converges to a deterministic limit described by the one-sided Stefan’s problem. For the symmetric simple exclusion process on \( \mathbb{Z} \), [15] shows that the hydrodynamic limit of a tagged particle is described by the two-sided Stefan’s problem. For the same model, [16] shows that the fluctuation scales to a generalized Ornstein–Uhlenbeck process related to ASHE.

Remark 1.6. Harris [9] introduces a closely related model of i.i.d. \( \mathbb{Z} \)-indexed Brownian particles \( B_j(t) \), which can be regarded as the bulk version of (1.1). Using an explicit formula for the law of \( B(0)(t) \), he shows that at equilibrium with density \( 2\gamma \), \( \lim_{t \to \infty} t^{-1/4}(B(0)(t) - B(0)(0)) \Rightarrow (2\pi)^{-1/4}\gamma^{-1/2}B(1) \). This result is further extended by [7] to the functional convergence \( \varepsilon^{1/4}(B(0)(\varepsilon^{-1}x) - B(0)(0)) \Rightarrow (2\pi)^{-1/4}\gamma^{-1/2}B(1/4)(\cdot) \).

Intuitively, we expect the Atlas model to behavior similarly to the Harris model once we match the equilibrium density. This is indeed confirmed in (1.9). That is, at the bulk \( x \to \infty \) the asymptotic fluctuation of the two systems are approximately equal, to \( (2\pi)^{-1/4}\gamma^{-1/2}B(1/4)(\cdot) \). Somewhat unexpectedly, as shown in Corollary 1.4(a), the \( \frac{1}{4} \)-FBM fluctuation also appears at \( x = 0 \), but with a different prefactor.

Remark 1.7. Applying our technique to the Harris model, one may rederive the results of [7, 9]. This provides an explanation of the scaling and the \( \frac{1}{4} \)-FBM limit as the fluctuation of ASHE at \( x = 0 \). Specifically, the scaling limit of the Harris model should be ASHE on \( \mathbb{R} \) with no boundary condition. Since no drift presents in the Harris model, the latter scaling limit could be deduced directly from the time evolution equation.

Our strategy of proving Theorem 1.1 is to focus on the empirical measure. While this strategy has been widely used for interacting particle systems, in the context of Atlas model, or more generally diffusions with rank-dependent drifts, analyzing empirical measure is a new approach that has only been used here and in [3]. It completely bypasses the need of local times, which is a major a challenge when analyzing diffusions with rank-depended drifts.

To define the empirical measure, we consider \( w(y) := e^{-y} \wedge 1, \| \phi \|_\omega := \sup_{y \in \mathbb{R}} |\phi(y)|/w(y) \), and \( \mathcal{Q} := \{ \phi \in L^\infty(\mathbb{R}) : |\phi(y)|_\omega < \infty \} \). Let \( X_i^\varepsilon(t) := \varepsilon^{1/2}X_i(\varepsilon^{-1}t) \), \( X_i(t) := \varepsilon^{1/2}X_i(\varepsilon^{-1}t) \) and, for any \( \phi \in \mathcal{Q} \), let

\[
\langle Q_i^\varepsilon, \phi \rangle := \sum_{i=0}^{\infty} \phi(X_i^\varepsilon(t)),
\]

\[
\langle \tilde{Q}_i^\varepsilon, \phi \rangle := \varepsilon^{1/4} \left( \langle Q_i^\varepsilon, \phi \rangle - 2\gamma \varepsilon^{-1/2} \int_0^\infty \phi(y)dy \right),
\]

which are well-defined (see Lemma 3.1). As we are at equilibrium, \( Q_i^\varepsilon \) is a PPP\((2\gamma\varepsilon^{-1/2})\) translated by \( X(0)(t) \), so \( \tilde{Q}_i^\varepsilon \) captures the Gaussian fluctuation of PPP\((2\gamma\varepsilon^{-1/2})\) around \( 2\gamma\varepsilon^{-1/2}1_{\mathbb{R}_+}(y)dy \).

Under this framework, the main challenge of proving Theorem 1.1 is to choose the test function \( \mathcal{F}_{i,\delta}^\varepsilon(x) \) that \( i \) identifies the relevant boundary condition; and \( ii \) relates itself
to the process $X_\varepsilon(x)$. With
\[ F^\varepsilon_\delta(x) := (\hat{Q}_\varepsilon, \Psi_\delta(\cdot, x)), \tag{1.12} \]
establishing (ii) amounts to approximating the displacement of a ranked particle by the net flux of particles, which we achieve by using stationarity. In Sections 4 and 5 we prove Propositions 1.8 and 1.9, respectively, from which Theorem 1.1 follows immediately.

**Proposition 1.8.** Fix any $b \in (0, 1/4)$. As $(\varepsilon, \delta) \to (0, 0)$, $F^\varepsilon_\delta(\cdot + \varepsilon b) \Rightarrow X_\varepsilon(\cdot)$, for $X_\varepsilon(x)$ given as in Theorem 1.1.

**Proposition 1.9.** Fix any $a \in (1/2, \infty)$ and $b \in (0, 1/4)$. As $\varepsilon \to 0$, $F^\varepsilon_x \rightarrow (\cdot + \varepsilon b) - X^\varepsilon_\varepsilon(\cdot) \Rightarrow 0$.

2. **Outline of the Proof of Propositions 1.8 and 1.9**

Without lost of generality, we scale the drift $\gamma > 0$ to unity by $X_i(t) \mapsto \gamma X_i(\gamma^{-2}t)$. Hereafter, we fix $\gamma := 1$ and use $C(a,b,\ldots)$ to denote generic positive finite (deterministic) constant that depends only on the designated variables.

We proceed to describe the time evolution of $\hat{Q}_\varepsilon^\varepsilon$. To this end, let
\[ \mathcal{D}_T := \left\{ \psi, (\cdot) \in C^2(\mathbb{R} \times [0, T]) : |\psi|_{\mathcal{D}_T} < \infty \right\}, \]
\[ |\psi|_{\mathcal{D}_T} := \sup_{t \in [0,T]} (|\partial_t \psi_t|_\mathcal{D} + |\partial_x \psi_t|_\mathcal{D} + |\partial_{xx} \psi_t|_\mathcal{D} + |\psi_t|_\mathcal{D}). \]
We decompose $\hat{Q}_\varepsilon^\varepsilon = A_\varepsilon^\varepsilon + W_\varepsilon^\varepsilon$, where
\[ \langle A_\varepsilon^\varepsilon, \phi \rangle := -2\varepsilon^{-1/4} \int_0^{X_\varepsilon^\varepsilon(t)} \phi(y) dy \tag{2.1} \]
records the fluctuation of the lowest ranked particle, and
\[ \langle W_\varepsilon^\varepsilon, \phi \rangle := \varepsilon^{1/4} \left( \langle Q_\varepsilon^\varepsilon, \phi \rangle - 2\varepsilon^{-1/2} \int_{X_\varepsilon^\varepsilon(t)}^\infty \phi(y)dy \right) \tag{2.2} \]
accounts for the fluctuations of the bulk of particles. For any $\psi \in \mathcal{D}_T$ and $t_0 \in [0, T]$, let
\[ M^{\varepsilon}_{t_0, t}(\psi, k) := \sum_{i=0}^{k} \int_{t_0}^t \partial_q \psi_i(X^\varepsilon_i(s)) dB^\varepsilon_i(s), \tag{2.3} \]
which is a $C([t_0, T], \mathbb{R})$-valued martingale in $t$.

**Proposition 2.1.** For any $T \in \mathbb{R}_+$, $t_0 \in [0, T]$ and $\psi \in \mathcal{D}_T$, there exists a $C([t_0, T], \mathbb{R})$-valued martingale $M^{\varepsilon}_{t_0, \cdot}(\psi, \infty)$ such that, for all $q \geq 1$,
\[ \left\| \sup_{t \in [t_0, T]} |M^{\varepsilon}_{t_0, t}(\psi, k) - M^{\varepsilon}_{t_0, t}(\psi, \infty)| \right\|_q \to 0. \tag{2.4} \]
Furthermore, almost surely
\[
\langle \tilde{Q}_t^\varepsilon, \psi_t \rangle - \langle \tilde{Q}_0^\varepsilon, \psi_0 \rangle = \int_0^t \left( W_s^\varepsilon \left( \partial_s + \frac{1}{2} \partial_{yy} \right) \psi_s \right) ds + \int_0^t \langle A_s^\varepsilon, \partial_s \psi_s \rangle ds + M_{0,t}^\varepsilon(\psi, \infty),
\]
for all \( t \in [0, T] \).

**Remark 2.2.** Proposition 2.1 is established in Section 3, where we derive (2.5) via Ito calculus. In this derivation, the driving Brownian motions \( B_i(t), i \in \mathbb{Z}_+ \), collectively contribute
\[
\left( \varepsilon^{1/4} \langle Q_t^\varepsilon, (\partial_t + 2^{-1} \partial_{yy}) \psi_t \rangle - 2\varepsilon^{-1/4} \int_0^\infty \partial_s \psi_s(y) dy \right) dt + dM_{0,t}^\varepsilon(\psi, \infty)
\]
whereas the drift \( \gamma = 1 \) at the lowest ranked particle contributes
\[
\varepsilon^{-1/4} \partial_y \psi_s(X_{(0)}^\varepsilon(t)) dt = \left( -\varepsilon^{-1/4} \int_{X_{(0)}^\varepsilon(t)}^\infty \partial_{yy} \psi_s(y) dy \right) dt.
\]
These, when combined together, give the expression (2.5).

Based on Proposition 2.1, in Section 3 we establish the following a-priori estimate of \( X_{(0)}^\varepsilon(\cdot) \).

**Proposition 2.3.** Fixing any \( q > 1, b \in [0, 1/4] \) and \( T \in \mathbb{R}_+ \), we let \( \tau_b^\varepsilon := \inf \{ t \geq 0 : |X_{(0)}^\varepsilon(t)| \geq \varepsilon^b \} \). There exists \( C = C(T, b, q) < \infty \) such that, for all \( \varepsilon \in (0, (2q)^{-2}] \),
\[
P \left( \tau_b^\varepsilon \leq T \right) \leq C e^{(1/4-b)q-1}.
\]

**Remark 2.4.** Proposition 2.3 implies, for any \( T \in \mathbb{R}_+ \) and \( b \in (0, 1/4) \), we have \( P(\sup_{t \in [0,T]} |X_{(0)}^\varepsilon(t)| \leq \varepsilon^b) \rightarrow 1 \). This is almost optimal, since we know a-posteriori from Theorem 1.1 that \( \varepsilon^{-1/4} X_{(0)}^\varepsilon(t) = \mathcal{X}_t^\varepsilon(x) \) converges weakly.

Turning to the proof of Proposition 1.8, for each \( t, \delta, \eta > 0, x \in \mathbb{R}_+ \), we apply Proposition 2.1 for \( \psi_s(y) := \Psi_{t+\delta-s}(y, x + \eta) \in \mathcal{D}_t \). With \( \psi_s(y) \) solving the backward heat equation \( (\partial_s + 2^{-1} \partial_{yy}) \psi_s = 0 \), one easily obtains that
\[
\mathcal{F}_{t}^{\varepsilon, \delta}(x + \eta) = \mathcal{W}_{t}^{\varepsilon}(x) + \mathcal{M}_{t}^{\varepsilon}(x) + \mathcal{A}_{t}^{\varepsilon}(x),
\]
where \( \varepsilon := (\varepsilon, \delta, \eta) \),
\[
\Psi_{t}^{\varepsilon}(y, x) := \Psi_{t+\delta}(y, x + \eta), \quad p_{t}^{N, \varepsilon}(y, x) := p_{t+\delta}^{N}(y, x + \eta),
\]
\[
\mathcal{W}_{t}^{\varepsilon}(x) := \langle \tilde{Q}_0^{\varepsilon}, \Psi_{t}^{\varepsilon}(\cdot, x) \rangle,
\]
\[
\mathcal{M}_{t}^{\varepsilon}(x) := M_{0,t}^{\varepsilon}(\Psi_{t}^{\varepsilon}(\cdot, x), \infty) = \varepsilon^{1/2} \sum_{i=0}^{\infty} \int_{0}^{t} p_{t-s}^{N, \varepsilon}(X_{t}^{\varepsilon}(s), x) dB_{s}^{i}(s),
\]
\[
\mathcal{A}_{t}^{\varepsilon}(x) := \int_{0}^{t} \langle A_{s}^{\varepsilon}, \partial_s \Psi_{t-s}^{\varepsilon}(\cdot, x) \rangle ds.
\]
Since $W_t^\varepsilon(x)$ and $M_t^\varepsilon(x)$, consisting respectively of the contribution of $\{X_t^\varepsilon(0)\}$ and $\{B_t^\varepsilon(\cdot)\}$, are independent, Proposition 1.8 is an immediate consequence of:

**Proposition 2.5.**

(a) Fix any $b \in (0, 1/4)$. As $\varepsilon, \delta \to (0, 0)$, $A_{\varepsilon,\delta}(\cdot) \Rightarrow 0$.

(b) As $\varepsilon \to 0$, $W_t^\varepsilon(\cdot) \Rightarrow W_\varepsilon(\cdot)$, where $W_\varepsilon(\cdot)$ is a centered Gaussian process with

$$E(W_t^\varepsilon(W_{t'}^\varepsilon)) = 2 \int_0^\infty \Psi_t(y, x) \Psi_{t'}(y, x') dy. \quad (2.11)$$

(c) As $\varepsilon \to 0$, $M_t^\varepsilon(\cdot) \Rightarrow M_\varepsilon(\cdot)$, where $M_\varepsilon(\cdot)$ is a centered Gaussian process with

$$E(M_t^\varepsilon M_{t'}^\varepsilon) = 2 \int_0^{t \wedge t'} \int_0^\infty p_{t-s}^N(y, x) p_{t'-s}^N(y, x') dy ds. \quad (2.12)$$

**Remark 2.6.** Our special choice of $\psi_s(y)$ is what makes Proposition 2.5(a) valid. To see this, note that $X_{(0)}^\varepsilon(t) = O(\varepsilon^{1/4})$ for all $b \in (0, 1/4)$ (by Proposition 2.3) and that $A_t^\varepsilon(x) = \int_0^t \langle A_s, \xi_s \rangle ds$ for $\xi_s(y) = \partial_s \Psi_{t+\delta-s}(y, x)$. With $\xi_s(0) = 0$, by (2.1) we can approximate $\langle A_s, \xi_s \rangle$ by $\varepsilon^{-1/4}O((X_{(0)}^\varepsilon(s))^2)$, which indeed tends to zero. Further, we expect Proposition 2.5(b) and (c) to hold by comparing (1.6) with (2.8), and (1.7) with (2.9), since $\tilde{Q}_0^\varepsilon$ approximates $2dB_0(\cdot)$, and $\varepsilon^{1/2}Q_t^\varepsilon$ approximates $21_{\mathbb{R}_+}(x)dx$, respectively.

For the proof of Proposition 1.9, we require the following notations:

$$G_t^\varepsilon(x) := \langle \tilde{Q}_t^\varepsilon, 1_{(-\infty, x]} \rangle = \varepsilon^{1/4} \langle \tilde{Q}_t^\varepsilon, 1_{(-\infty, x]} \rangle - 2\varepsilon^{-1/4}x, \quad (2.13)$$

$$I_t^\varepsilon(x) := \inf \{ i \in \mathbb{Z}_+ : X_{(i)}^\varepsilon(t) > x \} = \langle \tilde{Q}_t^\varepsilon, 1_{(-\infty, x]} \rangle, \quad (2.14)$$

$$\tilde{X}_t^\varepsilon(x) := \varepsilon^{1/4}(I_t^\varepsilon(x) - 2X_{(I_t^\varepsilon(x)))}(\varepsilon^{-1}t)). \quad (2.15)$$

Up to a centering and scaling, $G_t^\varepsilon(x)$ counts the total number of particles to the left of $x$, and $\tilde{X}_t^\varepsilon(x)$ records the trajectory of $X_{(I_t^\varepsilon(x)))}(\cdot)$, where $X_{(I_t^\varepsilon(x)))}(0)$ the first particle to the right of $x$ at time $0$. Proposition 1.9 is then an immediate consequence of:

**Proposition 2.7.** Let $a \in (1/2, \infty)$ and $b \in (0, 1/4)$.

(a) As $\varepsilon \to 0$, $\mathcal{F}_t^{\varepsilon+a}(\cdot + \varepsilon^b) - G_t^\varepsilon(\cdot + \varepsilon^b) \Rightarrow 0$.

(b) As $\varepsilon \to 0$, $G_t^\varepsilon(\cdot + \varepsilon^b) - \tilde{X}_t^\varepsilon(\cdot + \varepsilon^b) \Rightarrow 0$.

(c) As $\varepsilon \to 0$, $\tilde{X}_t^\varepsilon(\cdot + \varepsilon^b) - X_t^\varepsilon(\cdot) \Rightarrow 0$.

Letting

$$\rho_t^\varepsilon(x) := X_{(I_t^\varepsilon(x)))}(\varepsilon^{-1}t) - \varepsilon^{-1/2}x = \varepsilon^{-1/2}[X_{(I_t^\varepsilon(x)))}(t) - x], \quad (2.16)$$

$$\mathcal{D}^\varepsilon(j, j', t) := j - j' - 2(X_j(t)(\varepsilon^{-1}t) - X_{j'}(t)(\varepsilon^{-1}t)) \quad (2.17)$$

$$= \text{sign}(j - j') \sum_{i \in [j', j) \cup [j, j')} (1 - 2Y_i(\varepsilon^{-1}t)), \quad (2.18)$$

in Section 5, we establish Proposition 2.7 relying on the following exact relations

$$\rho_t^\varepsilon(x) \in (0, Y_{I_t(x)-1}(\varepsilon^{-1}t)), \quad \text{for all } x \text{ such that } x \geq X_{(0)}^\varepsilon(t), \quad (2.19)$$
\[ \mathcal{G}^\varepsilon_t(x) - \tilde{\mathcal{X}}^\varepsilon_t(x) = \varepsilon^{1/4}D^\varepsilon(I^\varepsilon_t(x), I^\varepsilon_0(x), t) + 2\varepsilon^{1/4}\rho^\varepsilon_t(x), \]  
\[ \tilde{\mathcal{X}}^\varepsilon_t(x + \varepsilon^b) - \mathcal{X}_t^\varepsilon(x) = \varepsilon^{1/4}D^\varepsilon(I^\varepsilon_0(x + \varepsilon^b), \iota_t(x), t). \]

Indeed, (2.19) holds since \( \rho^\varepsilon_t(x) \) represents the gap between \( \varepsilon^{-1/2}x \) and the first particle to its right, (2.20) follows by combining (2.13)–(2.14) and (2.16), and (2.21) follows by combining (2.15) and (1.2).

The starting point of proving Proposition 2.7 is as follows. We establish part (a) based on using \( \Psi_\delta(y, x + \varepsilon^b) = 1_{(-\infty, -x-\varepsilon^b)}(y) + 1_{(-\infty, x+\varepsilon^b)}(y) \), for \( b \in (0, 1/4) \) to ensure that \( \langle \hat{Q}_t^\delta, 1_{(-\infty, -x-\varepsilon^b)} \rangle \approx 0 \). As for parts (b) and (c), by shifting each \( x \) by \( \varepsilon^b \), we use (2.19) to ensure that \( \varepsilon^{1/4}\rho^\varepsilon_t(x + \varepsilon^b) \approx 0 \), and by using stationarity, we have \( D^\varepsilon(j, j', t) = O(|j - j'|^{1/2}) \). Consequently, we reduce showing parts (b) and (c) to showing

\[ \varepsilon^{1/4}|I^\varepsilon_t(x) - I^\varepsilon_0(x)|^{1/2} \approx 0, \quad \varepsilon^{1/4}|I^\varepsilon_0(x + \varepsilon^b) - \iota_t(x)|^{1/2} \approx 0. \]

The former should hold since, by (2.13)–(2.14), we have \( I^\varepsilon_t(x) - I^\varepsilon_0(x) = \varepsilon^{-1/4}(\mathcal{G}^\varepsilon_t(x) - \mathcal{G}^\varepsilon_0(x)) = O(\varepsilon^{-1/4}) \), and we expect the latter to be true since \( I^\varepsilon_0(x + \varepsilon^b) \approx \text{Pois}(2\varepsilon^{-1/2}(x + \varepsilon^b)) \) and \( \iota_t(x) = 2\varepsilon^{-1/2}x + O(1) = 2\varepsilon^{-1/2}(x + \varepsilon^b) + O(\varepsilon^{-1/2+b}) \).

The rest of this paper is organized as follows. Section 3 is primarily devoted to the proof of Propositions 2.1 and 2.3. In Sections 4 and 5, we prove Propositions 2.5 and 2.7, respectively.

3. A-priori estimates: Proof of Propositions 2.1 and 2.3

Let \( X_i^{\varepsilon, 1}(t) := X_i(0) + B_i^\varepsilon(t) \), \( X_i^{\varepsilon, r}(t) := X_i^{\varepsilon, 1}(t) + \varepsilon^{-1/2} t \), \( X_{(i)}^{\varepsilon, 1}(t) \) and \( X_{(i)}^{\varepsilon, r}(t) \) be the corresponding ranked processes. We then have from (1.1) (for \( \gamma = 1 \)) that, almost surely, for all \( i \in \mathbb{Z}_+ \) and \( t \geq 0 \),

\[ X_i^{\varepsilon, 1}(t) \leq X_i(t) \leq X_i^{\varepsilon, r}(t), \]

from which it easily follows that

\[ X_{(i)}^{\varepsilon, 1}(t) \leq X_{(i)}(t) \leq X_{(i)}^{\varepsilon, r}(t). \]

Based on (3.1)–(3.2), we now establish bounds on the mass of the empirical measure on intervals of the form \( (-\infty, x] \).

**Lemma 3.1.** Fix any \( a > 0 \), \( q \in [1, \infty) \), \( t \in \mathbb{R}_+ \) and \( j \in \mathbb{Z}_+ \). There exists \( C = C(a, q, t) < \infty \) such that, for all \( \varepsilon \in (0, (aq)^{-2}] \),

\[ \sum_{i = j}^\infty \left\| \sup_{s \in [0, t]} \exp(-aX_i^\varepsilon(s)) \right\|_q \leq C\varepsilon^{-1/2}e^{-j\varepsilon^{1/2}a/4}, \]

\[ \left\| \sum_{i = j}^\infty \sup_{s \in [0, t]} \exp(-aX_{(i)}^\varepsilon(s)) \right\|_q \leq C\varepsilon^{-1/2}e^{-j\varepsilon^{1/2}a/4}. \]

**Proof.** Fix \( t \in \mathbb{R}_+ \), \( q \in [1, \infty) \), \( a > 0 \) and \( j_* \in \mathbb{Z}_+ \). Let \( X_i^{\varepsilon, 1,*}(s) := X_{i+j_*}^{\varepsilon, 1}(s) \) be the \( i \)-th (unranked) particle among \( \{X_j^{\varepsilon, 1}\}_{j \geq j_*} \). Let \( F_i^\varepsilon := \sup_{s \in [0, t]} \exp(-aX_i^\varepsilon(s)) \), \( F_{(i)}^\varepsilon := \)
sup \varepsilon \in [0, q] \exp(-aX^\varepsilon_{(i)}(s))

By (3.1), \( F^\varepsilon_i \leq F^\varepsilon_1 \), hence \( \sum_{i=j}^\infty \| F^\varepsilon_i \|_q \leq \sum_{i=j}^\infty \| F^\varepsilon_1 \|_q \). Let \( r := 2^{-1}aq \varepsilon^{1/2} \) and \( \overline{B^\varepsilon_i(t)} := sup_{s \in [0, t]} |B^\varepsilon_i(s)| \). With \( F^\varepsilon_i(t) \) defined as in the preceding, we have

\[
E \left( F^\varepsilon_i \right)^q \leq (E e^{-2rY_0(0)})^i E \left( e^{aq\overline{B^\varepsilon_i(t)}} \right) = (1 + r)^{-i} E \left( e^{aq\overline{B^\varepsilon_i(t)}} \right).
\] (3.5)

Further, by the reflection principle, \( E[\exp(-aq\overline{B^\varepsilon_i(t)})] \leq 2E[\exp(aqB^\varepsilon_i(t))] = C(a, q, t) \). Consequently,

\[
\sum_{i=j}^\infty \left\| F^\varepsilon_i \right\|_q \leq \frac{(1 + r)^{-(j-1)/q}}{(1 + r)^{1/q} - 1} C.
\]

With \( r \in (0, 1] \), further using the elementary inequalities \((1 + r)^{1/q} \geq 1 + r/q \) and \((1 + r)^{-j/q} \leq \exp(-j/r(2q)) \), we conclude (3.3).

We next show (3.4). Since, by definition, \( X^\varepsilon_{(i)}(s) \) is the \( i \)-th smallest particles among \( \{X^\varepsilon_{(j)}(s)\}_{j \geq j_0} \), we have that \( X^\varepsilon_{(i)}(s) \leq X^\varepsilon_{(i+j_0)}(s) \leq X^\varepsilon_{(i+j_0)}(s) \) and, therefore, \( F^\varepsilon_{(i+j_0)} \leq F^\varepsilon_{(i+j_0)} \). Summing both sides over \( i \), we further obtain \( \sum_{i=0}^\infty F^\varepsilon_{(i+j_0)} \leq \sum_{i=0}^\infty F^\varepsilon_{(i)} = \sum_{i=0}^\infty F^\varepsilon_{i+j_0} = \sum_{i=0}^\infty F^\varepsilon_{i+j_0} \). From this and (3.3) we conclude (3.4).

Based on (3.1), we now establish the continuity of the process \( X^\varepsilon_{(j)}(\cdot) \).

**Lemma 3.2.** There exists \( C < \infty \) such that for any \( [t_1, t_2] \subset [0, \infty) \), \( j \in \mathbb{Z}_+ \) and \( \varepsilon \in (0, 1] \),

\[
P \left( \sup_{t \in [t_1, t_2]} |X^\varepsilon_{(j)}(t) - X^\varepsilon_{(j)}(t_1)| \geq \alpha \right) \leq C \exp\left(-\frac{\alpha}{2} - \varepsilon^{-1}(t_2 - t_1)\right). \] (3.6)

**Proof.** It clearly suffices to show that

\[
E \left[ \exp\left(\varepsilon^{-1/2} \sup_{t \in [t_1, t_2]} |X^\varepsilon_{(j)}(t) - X^\varepsilon_{(j)}(t_1)|\right) \right] \leq C \exp\left(\varepsilon^{-1}(t_2 - t_1)\right), \] (3.7)

Since \( (Y_t(\cdot))_t \in \mathbb{Z}_+ \) is at equilibrium, we have

\[
(X^\varepsilon_{(i)}(\cdot + t_1) - X^\varepsilon_{(j)}(t_1))_{i \in \mathbb{Z}_+} \overset{\text{distr.}}{=} (X^\varepsilon_{(i)}(\cdot) - X^\varepsilon_{(i)}(0))_{i \in \mathbb{Z}_+},
\]

so without loss of generality we assume that \( t_1 = 0 \). Let

\[
U^\varepsilon_{i+j}(t, i, j) := \sup_{s \in [0, t]} \left\{ \exp\left[\varepsilon^{-1/2} \left(X^\varepsilon_{i+j}(s) - X^\varepsilon_{(j)}(0)\right)\right] \right\}, \] (3.8)

\[
U^\varepsilon_{1}(t, i, j) := \sup_{s \in [0, t]} \left\{ \exp\left[-\varepsilon^{-1/2} \left(X^\varepsilon_{(i)}(t) - X^\varepsilon_{(i)}(0)\right)\right] \right\}. \] (3.9)

Similar to (3.5), we have

\[
E \left( U^\varepsilon_{i+j}(t, i, j) \right) \leq (E(e^{-Y_0(0)}))^{j-1} E \left( e^{\varepsilon^{-1/2} \overline{B^\varepsilon}(t) + \varepsilon^{-1}t} \right) \leq (2/3)^{i-j} C e^{2\varepsilon^{-1}t}, \forall i \leq j, \] (3.10)

\[
E \left( U^\varepsilon_{1}(t, i, j) \right) \leq (E(e^{-Y_0(0)}))^{i-j} E \left( e^{\varepsilon^{-1/2} \overline{B^\varepsilon}(t)} \right) \leq (2/3)^{i-j} C e^{\varepsilon^{-1}t}, \forall i \geq j. \] (3.11)
Based on Lemma 3.1, we now establish the following a-priori estimate of the empirical measure.

**Lemma 3.3.** Fix $T \in \mathbb{R}_+$, $q \in [1, \infty)$ and $a \in (0, \infty)$. Let $J^\varepsilon_j := \lfloor \varepsilon^{-1/2} j, \varepsilon^{-1/2} (j + 1) \rfloor \cap \mathbb{Z}$ and $f_i, i \in \mathbb{Z}_+$, be $\mathbb{R}_+$-valued random variables. There exists $C = C(T, q, a) < \infty$ such that for all $t \in [0, T]$ and $\varepsilon \in (0, (aq)^{-2})$,

$$
\left\| \sum_{i=0}^{\infty} e^{-aX^\varepsilon_i(t)} \right\|_q \leq C \varepsilon^{-1/4} \sum_{j=0}^{\infty} e^{-ja/4} \left( \sum_{i \in J^\varepsilon_j} \| f_i \|^2_{2q} \right)^{1/2}.
$$

**Proof.** For each $j \in \mathbb{Z}_+$, by the Cauchy–Schwarz inequality we have

$$
\left\| \sum_{i \in J^\varepsilon_j} e^{-aX^\varepsilon_i(t)} \right\|_q \leq \left\| \sum_{i \in J^\varepsilon_j} e^{-2aX^\varepsilon_i(t)} \right\|^{1/2} \left\| \sum_{i \in J^\varepsilon_j} (f_i)^2 \right\|^{1/2}.
$$

On the r.h.s., replacing $\left\| \sum_{i \in J^\varepsilon_j} (f_i)^2 \right\|_q$ with $\sum_{i \in J^\varepsilon_j} \| f_i \|^2_{2q}$, and replacing $\left\| \sum_{i \in J^\varepsilon_j} e^{-2aX^\varepsilon_i(t)} \right\|_q$ with $\sum_{i \geq \varepsilon^{-1/2} j} e^{-2aX^\varepsilon_i(t)}$, which, by (3.4), is bounded by $C \varepsilon^{-1/2} \exp(-ja/2)$, we conclude (3.12).}$

Now we establish a decomposition of $W^\varepsilon_t$ into $W^\varepsilon_{\star t}$ and $R^\varepsilon_t$ as follows. As we show latter in (3.16), $R^\varepsilon_t$ becomes negligible as $\varepsilon \to 0$, so $W^\varepsilon_t \approx W^\varepsilon_{\star t}$.

**Lemma 3.4.** Fix $t \in \mathbb{R}_+$, $\varepsilon \in (0, 1]$ and $\phi \in \mathcal{D}$ such that $\frac{d\phi}{dy} \in \mathcal{D}$, and let

$$
\langle W^\varepsilon_{\star t}, \phi \rangle := \varepsilon^{1/4} \sum_{i=0}^{\infty} \phi \left( X^\varepsilon_{(i)}(t) \right) \left( 1 - 2Y_i(\varepsilon^{-1} t) \right),
$$

$$
\langle R^\varepsilon_t, \phi \rangle := \varepsilon^{-1/4} \sum_{i=0}^{\infty} \int_{X^\varepsilon_{(i)}(t)}^{X^\varepsilon_{(i+1)}(t)} (X^\varepsilon_{(i+1)}(t) - y) \phi(y) dy.
$$

Then,

$$
\langle W^\varepsilon_t, \phi \rangle = \langle W^\varepsilon_{\star t}, \phi \rangle - 2 \left\langle R^\varepsilon_t, \frac{d\phi}{dy} \right\rangle.
$$

**Proof.** Since the gaps are at equilibrium, $X^\varepsilon_{(i)}(t) - X^\varepsilon_{(0)}(t)$ is the sum of the i.i.d. Exp(2$\varepsilon^{-1/2}$) random variables, so by the Law of Large Numbers we have $\lim_{k \to \infty} X^\varepsilon_{(k)}(t) = \infty$, hence

$$
\langle W^\varepsilon_t, \phi \rangle = \varepsilon^{1/4} \sum_{i=0}^{\infty} \left( \phi \left( X^\varepsilon_{(i)}(t) \right) - 2\varepsilon^{-1/2} \int_{X^\varepsilon_{(i)}}^{X^\varepsilon_{(i+1)}} \phi(y) dy \right).
$$

With $\int_{x_1}^{x_2} \phi(y) dy = (x_2 - x_1) \phi(x_1) + \int_{x_1}^{x_2} (x_2 - y) \phi'(y) dy$, we obtain the desired decomposition.
Based on Lemma 3.3, we next establish bounds on $\langle R^\varepsilon_t, \phi \rangle$ and $\langle W^i_{t, \varepsilon^*, \phi} \rangle$. We note here that, while these bounds fall short of proving Proposition 2.5, they suffice for justifying the use of Itô calculus in Proposition 2.1.

Hereafter, when the context is clear, we sometimes use $\phi^\varepsilon_i$, $Y^\varepsilon_i$ and $X^\varepsilon_i$, respectively, to denote $\phi(X^\varepsilon_i(t))$, $Y_i(\varepsilon^{-1}t)$ and $X^\varepsilon_i(t)$.

**Lemma 3.5.** Fix $T \in \mathbb{R}_+$, $q \in [1, \infty)$ and $\phi \in \mathcal{D}$ such that $\frac{d \phi}{dy} \in \mathcal{D}$. There exists $C = C(T, q) < \infty$ such that for all $t \in [0, T]$ and $\varepsilon \in (0, (2q)^{-2}]$,

$$
\|\langle R^\varepsilon_t, \phi \rangle \|_q \leq C \varepsilon^{1/4} \|\phi\|_{\mathcal{D}}, \quad (3.16)
$$

$$
\|\langle W^i_{t, \varepsilon^*, \phi} \rangle \|_q \leq C \left| \frac{d \phi}{dy} \right| \mathcal{D}. \quad (3.17)
$$

**Proof.** Fixing $T \in \mathbb{R}_+$, $t \in [0, T]$, $q \in [1, \infty)$, $\varepsilon \in (0, (2q)^{-2}]$ and $\psi \in \mathcal{D}$, we let $C = C(T, q) < \infty$. To show (3.16), in (3.14), we use $X^\varepsilon_{i+1} - y \leq \varepsilon^{1/2} Y_i$ and

$$
\sup_{y \in [X^\varepsilon_i, X^\varepsilon_{i+1}]} |\phi(y)| \leq |\phi\|_{\mathcal{D}} \exp(-X^\varepsilon_i)
$$

to obtain $|\langle R^\varepsilon_t, \phi \rangle | \leq \varepsilon^{3/4} |\phi\|_{\mathcal{D}} \sum_{i=0}^{\infty} (Y_i)^2 \exp(-X^\varepsilon_i)$. Combining this with (3.12) for $i = (Y_i)^2$, we arrive at

$$
\|\langle R^\varepsilon_t, \phi \rangle \|_q \leq C \varepsilon^{1/2} |\phi\|_{\mathcal{D}} \sum_{j=0}^{\infty} \exp(-j/4) \left( \|Y_i\|_{2q} \right)^{1/2}. \quad (3.19)
$$

Further using $\|Y_i\|_{2q} = C$ and $|J^*_j| \leq \varepsilon^{-1/2} + 1$, we conclude (3.16) upon summing $j$.

Turning to showing (3.17), we assume without lost of generality $q \in \mathbb{Z}_+ \cap [1, \infty)$. Letting $Z_k := \sum_{i=0}^{k} (1 - 2Y_i)$, with $\phi \in \mathcal{D}$, using summation by parts in (3.13), we obtain

$$
\langle W^i_{t, \varepsilon^*, \phi} \rangle := \varepsilon^{1/4} \sum_{i=0}^{\infty} (\phi^\varepsilon_i - \phi^\varepsilon_{i+1}) Z_i. \quad (3.18)
$$

To bound this expression, we combine

$$
|\phi_{i+1}^\varepsilon - \phi_i^\varepsilon| \leq \left| \frac{d \phi}{dy} \right|_{\mathcal{D}} \int_{X^\varepsilon_i}^{X^\varepsilon_{i+1}} e^{-y} dy \leq \left| \frac{d \phi}{dy} \right|_{\mathcal{D}} \varepsilon^{1/2} Y^\varepsilon_i \exp(-X^\varepsilon_i),
$$

(where the second inequality is obtained by using $e^y \leq e^{-X^\varepsilon_i}$) and (3.12) for $i = Y_i Z_i$ to obtain

$$
\|\langle W^i_{t, \varepsilon^*, \phi} \rangle \|_q \leq C \varepsilon^{1/2} \left| \frac{d \phi}{dy} \right|_{\mathcal{D}} \sum_{j=0}^{\infty} e^{-j/2} \left( \sum_{i \in J^*_j} \|Z_i Y_i\|_{2q} \right)^{1/2}. \quad (3.19)
$$

With $\|Y_i\|_{4q} = C$ and $\|Z_i\|_{4q} \leq (i + 1)^{1/2} C$, we have $\|Y_i Z_i\|_{2q} \leq (i + 1) C$. Plugging this into (3.19), we further obtain

$$
\|\langle W^i_{t, \varepsilon^*, \phi} \rangle \|_q \leq C \varepsilon^{1/2} \left| \frac{d \phi}{dy} \right|_{\mathcal{D}} \sum_{j=0}^{\infty} \left[ |J^*_j| \varepsilon^{-1/2} (j + 1) \right]^{1/2} e^{-j/4}.
$$

With $|J^*_j| \leq \varepsilon^{-1/2} + 1$, upon summing over $j$ we conclude (3.17). \qed
Based on Lemma 3.3, we now establish a bound on $M^\varepsilon_{t_0,t}(\psi, j)$, as defined in (2.3). Hereafter we adopt the convention that $M^\varepsilon_{t_0,t}(\psi, -1) := 0$.

**Lemma 3.6.** Let $\sigma \in [0, \infty]$ be arbitrary stopping time (with respect to the underlying sigma algebra). Fix $T \in \mathbb{R}_+$ and $q \in (1, \infty)$. There exists $C = C(T, q) < \infty$ such that, for all $\psi \in \mathcal{D}_T$, $t_0 \in [0, T]$, $j, j' \geq -1$ and $\varepsilon \in (0, 1]$,

$$
\left\| \sup_{t \in [t_0, T]} \left| M^\varepsilon_{t_0,t\wedge\sigma}(\psi, j) - M^\varepsilon_{t_0,t\wedge\sigma}(\psi, j') \right| \right\|^2_q \leq C \frac{1}{T} \exp\left( -(j \wedge j') \varepsilon^{1/2}/2 \right).
$$

(3.20)

**Proof.** Fixing such $T$, $q$, $t_0$, $j, j'$, $\varepsilon$, $\psi$ and $\sigma$, we let $C = C(T, q) < \infty$. We assume without lost of generality $j > j'$. Applying Doob's $L^q$-inequality and the Burkholder–Davis–Gundy (BDG) inequality (e.g. [19, Theorem II.1.7 and Theorem IV.4.1]) to the $C([t_0, T], \mathbb{R})$-valued martingale $M^\varepsilon_{t_0,\cdot \wedge \sigma}(\psi, j) - M^\varepsilon_{t_0,\cdot \wedge \sigma}(\psi, j')$, we obtain

$$
\left\| \sup_{t \in [t_0, T]} \left| M^\varepsilon_{t_0,t\wedge\sigma}(\psi, j) - M^\varepsilon_{t_0,t\wedge\sigma}(\psi, j') \right| \right\|^2_q \leq C \frac{1}{T} \int_{t_0}^T \left( \sum_{i=j'+1}^j (\partial_y \psi_s(X^\varepsilon_s)) \right)^2 ds.
$$

(3.21)

In the last expression, replacing $(\partial_y \psi_s(y))^2$ with $|\psi|^2_{\mathcal{F}_T} e^{-2y}$ and replacing $j$ with $\infty$, and then applying (3.3) for $a = 2$, we further obtain the bound $C \frac{1}{T} \exp\left( -j \varepsilon^{1/2}/2 \right)$, thereby concluding (3.20). \qed

**Proof of Proposition 2.1.** Fix $\psi \in \mathcal{D}_T$. The bound (3.20) implies that $\{M^\varepsilon_{t_0,\cdot}(\psi, j)\}_j$ is Cauchy in the complete space $L^q(C([t_0, T], \mathbb{R}, \mathcal{B}, \mathbb{P})$, whereby we conclude (2.4). Further, for all $q > 1$,

$$
\left\| \sup_{t \in [t_0, T]} \left| M^\varepsilon_{t_0,t}(\psi, \infty) \right| \right\|_q \leq \lim_{j \to \infty} \left\| \sup_{t \in [t_0, T]} \left| M^\varepsilon_{t_0,t}(\psi, j) \right| \right\|_q \leq C(T, q) \left| \psi \right|_{\mathcal{F}_T},
$$

(3.22)

where the last inequality follows by (3.20) for $j' = -1$.

To derive (2.5), we apply Ito’s formula to

$$
\left\langle \tilde{Q}_{k,s}^\varepsilon, \psi_s \right\rangle := \varepsilon^{1/4} \left( \sum_{i=0}^k \psi_t(X^\varepsilon_t(s)) - 2\varepsilon^{-1/2} \int_0^\infty \psi_s(y) dy \right)
$$

to obtain

$$
\left\langle \tilde{Q}_{k,s}^\varepsilon, \psi_s \right\rangle \bigg|_{s=0}^{s=t} = \int_0^t \left\langle \varepsilon^{1/4} Q_{k,s}^\varepsilon, \left( \partial_y + \frac{1}{2} \partial_{yy} \right) \psi_s \right\rangle ds - 2\varepsilon^{-1/4} \int_0^t \int_0^\infty \partial_y \psi_s(y) dy ds
$$

$$
+ M^\varepsilon_{t_0,t}(\psi, k) + \varepsilon^{-1/4} \int_0^t \left( \partial_y \psi_s \right) (X^\varepsilon_{0}(s)) \sum_{i=0}^k 1_{\{X(s) = X^\varepsilon_{0}(s)\}} ds.
$$
Clearly, almost surely for all $s \in [0, T]$, \( \langle \hat{Q}_{k,s}^\varepsilon, \phi \rangle \to \langle \hat{Q}_s^\varepsilon, \phi \rangle \) and \( \sum_{i=0}^k 1_{\{X_{(i)}(s) = X_{(i)}(s)\}} \to 1 \) as \( k \to \infty \). As for \( M_{0,t}^\varepsilon(\psi, k) \), from (2.4) (for large enough \( q \)) we deduce that, almost surely for all \( t \in [0, T] \), \( M_{0,t}^\varepsilon(\psi, k) \to M_{0,t}^\varepsilon(\psi, \infty) \). Hence letting \( k \to \infty \) we arrive at

\[
\langle \hat{Q}_s^\varepsilon, \psi_s \rangle |_{s=0}^{s=t} = \int_0^t \langle \varepsilon^{1/4}Q_s^\varepsilon, \left( \partial_s + \frac{1}{2} \partial_{yy} \right) \psi_s \rangle ds - 2\varepsilon^{-1/4} \int_0^t \int_0^\infty \partial_s \psi_s(y) dy ds + 2\varepsilon^{-1/4} \int_0^t \int_0^\infty \partial_y \psi_s(y) dy ds - 2\varepsilon^{-1/4} \int_0^t \int_0^\infty \partial_y \psi_s(y) dy ds. \tag{3.23}
\]

With \( A_t^\varepsilon \) and \( W_t^\varepsilon \) defined as in (2.1)–(2.2), the r.h.s. of (3.23) equals

\[
\int_0^t \langle W_t^\varepsilon, (\partial_s + 2^{-1} \partial_{yy}) \psi_s \rangle ds + \int_0^t \langle A_t^\varepsilon, \partial_s \psi_s \rangle ds + \varepsilon^{-1/4} \int_0^t \int_0^\infty \partial_{yy} \psi_s ds dy ds. \tag{3.24}
\]

The last term in (3.25) cancels the first term in (3.24), so (2.5) follows. \( \Box \)

**Corollary 3.7.** For any \( T \in \mathbb{R}_+ \) and \( q \in (1, \infty) \), there exists \( C = C(T, q) < \infty \) such that for all \( q > 1, \varepsilon \in (0, (2q)^{-2}] \) and \( t \in [0, T] \),

\[
\left\| \int_0^{X_{(0)}(t)} \operatorname{sech}(y) dy \right\|_q \leq C \varepsilon^{1/4}. \tag{3.26}
\]

**Proof.** Applying Proposition 2.1 for \( \psi(y) := \operatorname{sech}(y) \in \mathcal{O}_T \), we obtain

\[
\langle A_s^\varepsilon + W_s^\varepsilon, \operatorname{sech} \rangle |_{s=0}^{s=t} = 2^{-1} \int_0^t \left\langle W_s^\varepsilon, \frac{\psi_s}{\psi_s^2} \operatorname{sech} \right\rangle ds + M_{0,t}^\varepsilon(\operatorname{sech}, \infty),
\]

or equivalently

\[
\langle A_t^\varepsilon, \operatorname{sech} \rangle = \langle W_0^\varepsilon - W_t^\varepsilon, \operatorname{sech} \rangle + 2^{-1} \int_0^t \left\langle W_s^\varepsilon, \frac{\psi_s}{\psi_s^2} \operatorname{sech} \right\rangle ds + M_{0,t}^\varepsilon(\operatorname{sech}, \infty).
\]

Recall from (3.15) we have \( \langle W_s^\varepsilon, \phi \rangle = \langle W_s^\varepsilon, \phi \rangle - 2 \langle R_s^\varepsilon, \frac{d\phi}{dy} \rangle \). As \( \psi \in C^\infty(\mathbb{R}) \) and \( \frac{d\psi}{dy} \) \( \operatorname{sech} \) \( \mathcal{O}_T \) for all \( k \in \mathbb{Z}_+ \), further applying (3.16)–(3.17) and (3.22), we conclude (3.26). \( \Box \)

**Proof of Proposition 2.3.** Fix \( T \in \mathbb{R}_+, b \in [0, 1/4) \) and \( q > 1 \). Applying Chebyshev’s inequality in (3.26), we obtain that, for all \( t \in [0, T], q > 1 \) and \( \varepsilon \in (0, (2q)^{-2}] \),

\[
\mathbb{P} \left( \left| X_{(0)}(t) \right| \geq \lambda \right) \leq \varepsilon^{1/4} C(T, q) \left( \int_0^\lambda \operatorname{sech}(y) dy \right)^{-q}. \tag{3.27}
\]

Indeed, letting \( t_k^\varepsilon := \varepsilon \varepsilon_k \) we have

\[
\left\{ \varepsilon_k \leq T \right\} \subset \bigcup_{k \leq \varepsilon^{-1} T} \left\{ \left| X_{(0)}(t_k^\varepsilon) \right| \geq \frac{\varepsilon b}{2} \right\} \cup \left\{ \sup_{t \in \left[ t_k^\varepsilon, t_{k+1}^\varepsilon \right]} \left| X_{(0)}(t) - X_{(0)}(t_k^\varepsilon) \right| \geq \frac{\varepsilon b}{2} \right\}. \tag{3.28}
\]

From (3.27) and (3.6) we deduce

\[
\mathbb{P} \left( \left| X_{(0)}(t_k^\varepsilon) \right| \geq \varepsilon b/2 \right) \leq C \varepsilon^{(1/4 - b)q}. \tag{3.29}
\]
Proof. With \(\langle Q_t^ε,\phi \rangle := \langle Q_t^ε,\phi(\cdot + X_0^ε(t)) \rangle\), we let
\[
S_0^ε(t) := 1_{\{\sup_{s\in[0,t]}|X_0^ε(s)| \leq ε^b\}}.
\]

**Lemma 3.8.** Fix \(s,t \in (0,\infty), \ x, y' \in \mathbb{R}, \ q \in [1,\infty), \ b \in [0,1/4). \) There exists \(C = C(q) < \infty\) such that, for all \(ε \in (0,1]\),
\[
\|S_0^ε(t) \left\langle \widetilde{Q}_t^ε, p_s^N(\cdot - y', x) \right\rangle \|_q \leq (|\log s| + 1)C,
\]
\[
\|\left\langle \widetilde{Q}_0^ε, p_s^N(\cdot, x) \right\rangle \|_q \leq C.
\]

**Proof.** With \(p_s^N(y, x) := p_s(y - x) + p_s(y + x)\) and \(S_0^ε(t)\) decreasing in \(b\), it clearly suffices to prove, for any fixed \(x' \in \mathbb{R}\),
\[
\|S_0^ε(t) \left\langle \widetilde{Q}_t^ε, p_s(\cdot - x') \right\rangle \|_q \leq (|\log s| + 1)C,
\]
\[
\|\left\langle \widetilde{Q}_0^ε, p_s(\cdot - x') \right\rangle \|_q \leq C.
\]

Since \(p(z)\) decreases in \(|z|\), we have \(p_s(z) \leq s^{-1/2} \sum_{j=0}^{\infty} p(j) 1_{[j,j+1)}(|z|s^{-1/2}).\) Using this, we obtain
\[
S_0^ε(t)\left\langle \widetilde{Q}_t^ε, p_s(\cdot - x') \right\rangle = S_0^ε(t)ε^{1/2}\langle Q_t^ε, p_s(\cdot - x') \rangle \leq \sum_{j=0}^{\infty} S_0^ε(t) F_j^ε(t,s)p(j),
\]
\[
\langle \widetilde{Q}_0^ε, p_s(\cdot - x') \rangle = ε^{1/2}\langle Q_0^ε, p_s(\cdot - x') \rangle \leq \sum_{j=0}^{\infty} G_j^ε(s)p(j),
\]
where
\[
F_j^ε(t,s) := s^{-1/2}ε^{1/2}\left\langle Q_t^ε, 1_{[j,j+1)}(|\cdot - x'|s^{-1/2}) \right\rangle,
\]
\[
G_j^ε(s) := s^{-1/2}ε^{1/2}\left\langle Q_0^ε, 1_{[j,j+1)}(|\cdot - x'|s^{-1/2}) \right\rangle.
\]

With \(Q_0^ε \sim \text{PPP}_+(2ε^{-1/2}),\) we have that \(\|G_j^ε\|_q \leq C(q).\) Combining this with (3.38), using \(\sum_{j=0}^{\infty} p(j) < \infty,\) we conclude (3.36). As for (3.37), letting
\[
H_j^ε(t,s) := \sup_{|x''-x'| \leq 1} \left\{s^{-1/2}\langle Q_t^ε(0), 1_{[j,j+1)}(|\cdot - x'|s^{-1/2}) \rangle \right\},
\]
since \(Q_t^ε\) and \(Q_t^ε(0)\) differ only by the shift of \(X_0^ε(s),\) with \(S_0^ε(t)\) as in (3.32), we have \(S_0^ε(t)F_j^ε(t,s) \leq H_j^ε(t,s).\) With \(Q_t^ε(0) \sim \text{PPP}_+(2ε^{-1/2}),\) (3.35) now follows in a way similar to (3.36). The only difference is the maximum over \(\{x'' : |x'' - x'| \leq 1\},\) which results in the extra \(|\log s|\) factor. \(\square\)
4. Proof of Proposition 2.5

4.1. Proof of part (a). Fixing \( b \in (0, 1/4), b' \in (1/8, 1/4) \cap [b, \infty) \) and \( T \in \mathbb{R}_+ \), we show

\[
\lim_{(\varepsilon, \delta) \to (0, 0)} S_{b'}^\varepsilon(T) \left( \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}_+} \left| A_t^{\varepsilon, \delta, \varepsilon b}(x) \right| \right) = 0. \tag{4.1}
\]

The desired result \( A_t^{\varepsilon, \delta, \varepsilon b}(\cdot) \Rightarrow 0 \) then follows since \( S_{b'}^\varepsilon(T) \to_p 1 \) (by Proposition 2.3).

Turning to proving (4.1), fixing \( t \in [0, T] \), by (2.1) and (2.10) we have

\[
S_{b'}^\varepsilon(T) \left| A_t^{\varepsilon, \delta, \varepsilon b}(x) \right| \leq 2\varepsilon^{-1/4} S_{b'}^\varepsilon(T) \int_0^t \int_0^s \left| \partial_s \Psi_{t+\delta, \varepsilon b}(y, x + \varepsilon b) \right| dy ds.
\]

Since here \( \sup_{s \in [0, T]} \{ |X_0^\varepsilon(s)| \} \leq \varepsilon' \), we may integrate over \( \int_{-\delta}^{T+1} \int_{-\varepsilon'}^{\varepsilon'} \) instead. After exchanging the order of integrations, we integrate over \( s \in (-\delta, T+1) \) using the readily verified identity \( |\partial_s \Psi_s(y, x + \varepsilon b)| = -\text{sign}(y) \partial_s \Psi(y, x + \varepsilon b) \) to obtain

\[
S_{b'}^\varepsilon(T) \left| A_t^{\varepsilon, \delta, \varepsilon b}(x) \right| \leq 2\varepsilon^{-1/4} \int_{-\varepsilon'}^{\varepsilon'} \left| \Psi_{T+1+\delta}(y, x + \varepsilon b) - \Psi_0(y, x + \varepsilon b) \right| dy. \tag{4.2}
\]

Let \( f(y) := \Psi_{T+1+\delta}(y, x + \varepsilon b) - 1 \). Since \( \Psi_0(y, x + \varepsilon b) = 1 \), for all \( x \geq 0 \) and \( |y| \leq \varepsilon' \leq \varepsilon_b \), we have \( |\Psi_{T+1+\delta}(y, x + \varepsilon b) - \Psi_0(y, x + \varepsilon b)| = |f(y)| \). Further, since \( f(0) = 0 \) and \( f'(y) = -p_{T+1+\delta}(y, x + \eta) \), we further deduce \( |f(y)| \leq C|y|(T+1+\delta)^{-1/2} \leq C|y| \). Plugging this into (4.2), we obtain \( S_{b'}^\varepsilon(T) |A_t^{\varepsilon, \delta, \varepsilon b}(x)| \leq C\varepsilon^{-1/4+2\varepsilon} \), thereby, with \( b' > 1/8 \), concluding (4.1).

4.2. Proof of part (b). Recall \( \Psi_t^\varepsilon(y, x) \) and \( p_t^{N, \varepsilon}(y, x) \) are defined as in (2.7). By Lemma 3.4, we have \( \mathcal{W}_t^\varepsilon(x) = \mathcal{W}_t^{\varepsilon, x}(x) - 2\mathcal{R}_t^\varepsilon(x) \), for

\[
\mathcal{W}_t^{\varepsilon, x}(x) := \varepsilon^{-1/4} \sum_{i=0}^\infty (1 - 2Y_i(0))\Psi_t^\varepsilon(X_{i+1}^\varepsilon(0), x), \tag{4.3}
\]

\[
\mathcal{R}_t^\varepsilon(x) := \varepsilon^{-1/4} \sum_{i=0}^\infty \int_{X_{i+1}^\varepsilon(0)} X_{i+1}^\varepsilon(0) - y) p_t^{N, \varepsilon}(y, x) dy. \tag{4.4}
\]

We first show that \( \mathcal{R}_t^\varepsilon(\cdot) \Rightarrow 0 \), or more explicitly,

\[
\mathbb{E} \left( \sup_{t \in [0, T]} \sup_{x \in [0, L]} \left| \mathcal{R}_t^\varepsilon(x) \right| \right) \leq C\varepsilon^{-1/4} \log \varepsilon, \tag{4.5}
\]

for some \( C = C(T, L) < \infty \) and for all \( \varepsilon \in (0, 1/4] \) and \( \delta, \eta \in (0, 1] \).

**Proof of (4.5).** Fixing \( T, L \geq 0 \), we let \( C = C(T, L) \). To bound \( \mathcal{R}_t^\varepsilon(x) \), in (4.4) we replace \( (X_{i+1}^\varepsilon(0) - y) \) with \( \varepsilon^{1/2} Y_i(0) \), and then divide the sum into the sums over \( i \leq \varepsilon^{-1} \) and over \( i > \varepsilon^{-1} \). For the former replacing each \( Y_i(0) \) (with \( i \leq \varepsilon^{-1} \)) by \( \overline{Y} := \sup_{i \leq \varepsilon^{-1}} Y_i(0) \), we obtain

\[
\sup_{t \in [0, T]} \sup_{x \in [0, L]} \left| \mathcal{R}_t^\varepsilon(x) \right| \leq R_1 + R_2, \tag{4.6}
\]
\[ R_1^\varepsilon := \varepsilon^{1/4} Y^\varepsilon \int_{X_{(i-1)}(0)}^{X_i(0)} p_t^{N,\varepsilon}(y, x) dy \leq 2\varepsilon^{1/4} Y^\varepsilon, \]

\[ R_2^\varepsilon := \varepsilon^{1/4} \sum_{i > \varepsilon^{-1}} Y_i \sup_{t \in [0, T]} \sup_{x \in [0, L]} \int_{X_{(i)}(0)}^{\infty} p_t^{N,\varepsilon}(y, x) dy. \]  \hspace{1cm} (4.7)

With \( \{ Y_i(0) \} \sim \bigotimes_{i \in \mathbb{Z}_+} \text{Exp}(2) \), we have \( \mathbb{E}(R_1^\varepsilon) \leq C \varepsilon^{1/4} |\log \varepsilon| \). As for \( R_2^\varepsilon \), from (1.3) we have

\[ 0 \leq \Psi^\varepsilon_t(x, y) \leq C(T, L)(e^{-y} \wedge 1), \quad \forall t \in [0, T], \; x \in [0, L], \; y \in \mathbb{R}_+. \]  \hspace{1cm} (4.8)

Plugging this into (4.7), we obtain \( R_2^\varepsilon \leq C \varepsilon^{1/4} \sum_{i > \varepsilon^{-1}} Y_i \exp(-X^\varepsilon_{(i)}(0)) \). Further applying (3.12) for \( f_i = Y_i \), we conclude

\[ \mathbb{E}(R_2) \leq C \sum_{j=0}^{\infty} e^{-j/4} \left( \sum_{i \in I_j^\varepsilon} \| Y_i \|_{2}^2 \right)^{1/2} \leq \varepsilon^{-1/4} C \exp\left(-\varepsilon^{-1/2}/4\right). \]

Combining the preceding bounds on \( \mathbb{E}(R_1) \) and \( \mathbb{E}(R_2) \) with (4.6), we conclude (4.5). \( \square \)

With (4.1), it then suffices to show:

**Lemma 4.1.** We have that \( \{ \mathcal{W}^{\varepsilon, \bullet}_t(\cdot) \}_\varepsilon \subset C(\mathbb{R}^2_+, \mathbb{R}) \) and the processes are tight in \( C(\mathbb{R}^2_+, \mathbb{R}) \).

**Lemma 4.2.** As \( \varepsilon \to 0 \), \( \{ \mathcal{W}^{\varepsilon, \bullet}_t(\cdot) \}_\varepsilon \) converges in finite dimensional distribution to a centered Gaussian process \( \mathcal{W}_t(\cdot) \) with the covariance (2.11).

We prove Lemma 4.1 (as well as Lemma 4.6) by applying the following special form of the Kolmogorov–Chentsov criterion of tightness (see [13, Corollary 14.9]).

**Lemma 4.3** (Kolmogorov–Chentsov). A given collection of \( C(\mathbb{R}^2_+, \mathbb{R}) \)-valued processes \( \{ K^t(\cdot) \}_\varepsilon \) is tight if, for some \( \alpha \in (0, 1] \), and for all \( q \in (1, \infty) \), \( T, L \in \mathbb{R}_+ \), there exists \( C = C(T, L, \alpha, q) \geq 0 \) such that

\[ \| K^t_0(0) \|_q \leq C, \]  \hspace{1cm} (4.9)

\[ \| K^t_0(x) - K^t_{t'}(x') \|_q \leq |x - x'|^{\alpha/2} C, \]  \hspace{1cm} (4.10)

\[ \| K^t_0(x) - K^t_{t'}(x) \|_q \leq |t - t'|^{\alpha/4} C, \]  \hspace{1cm} (4.11)

for all \( t, t' \in [0, T] \), \( x, x' \in [0, L] \), \( \varepsilon, \delta \) and \( \eta \) sufficiently small.

**Proof of Lemma 4.1.** For each \( i \in \mathbb{Z}_+ \), \( (t, x) \mapsto (1 - 2Y_i(0))\Psi^\varepsilon_t(X^\varepsilon_{(i)}(0), x) \) is continuous. The series (4.3) defining \( \mathcal{W}^{\varepsilon, \bullet}_t(\cdot) \) converges absolutely, hence \( \mathcal{W}^{\varepsilon, \bullet}_t(\cdot) \in C(\mathbb{R}^2_+, \mathbb{R}) \).

Fixing \( T, L \in \mathbb{R}_+, \; q \in (1, \infty), \; x, x' \in [0, L] \) and \( t < t' \in [0, T] \) letting \( C = C(T, L, q) < \infty \), we next show (4.9)–(4.11) for \( K^t_0(x) = \mathcal{W}^{\varepsilon, \bullet}_t(\cdot) \) and \( \alpha = 1 \). Consider the discrete time martingale

\[ k \mapsto m^\varepsilon_k(t, x) := \varepsilon^{1/4} \sum_{i=0}^{k} (1 - 2Y_i(0))\Psi^\varepsilon_t(X^\varepsilon_{(i)}(0), x). \]  \hspace{1cm} (4.12)
With $W_t^{\epsilon,*}(x) = m^\epsilon_t(t,x)$, showing (4.9)–(4.11) amounts to bounding the quadratic variation of $m^\epsilon_t(t,x)$, which we do by using $Q_0^e \sim \text{PPP}_+(2\varepsilon^{-1/2})$.

Let $\langle \tilde{Q}_0^e : f \rangle := \varepsilon^{1/2} \sum_{i=0}^k f(X_{(i)}^\epsilon(0))$ be the $k$-th approximation of $\tilde{Q}_0^e$. The martingale $m^\epsilon_k(t,x)$ has quadratic variation $\langle \tilde{Q}_0^e , \Psi^\epsilon(\cdot,x) \rangle^2$. Consequently, by the BDG inequality and Fatou’s lemma, letting $k \to \infty$ we have

\[ \| W_0^{\epsilon,*}(0) \|^2 \leq C \left\| \langle \tilde{Q}_0^e, (\Psi_0^\epsilon(\cdot,0))^2 \rangle \right\|_{q/2}, \]  \[ (4.13) \]

\[ \| W_t^{\epsilon,*}(x) - W_t^{\epsilon,*}(x') \|^2 \leq C \left\| \langle \tilde{Q}_0^e, (\Psi_t^\epsilon(\cdot, x) - \Psi_t^\epsilon(\cdot, x'))^2 \rangle \right\|_{q/2}, \]  \[ (4.14) \]

\[ \| W_t^{\epsilon,*}(x) - W_t^{\epsilon,*}(x') \|^2 \leq C \left\| \langle \tilde{Q}_0^e, (\Psi_t^\epsilon(\cdot, x) - \Psi_t^\epsilon(\cdot, x'))^2 \rangle \right\|_{q/2}. \]  \[ (4.15) \]

The estimate (4.9) follows by applying $\Psi_0^\epsilon(y,0) \leq Ce^{-y}$ (by (4.8)) to (4.13) and then using $\| \langle \tilde{Q}_0^e, \exp(-2\cdot) \rangle \|_{q/2} \leq C$ (by (3.4) for $j = 0$). To show (4.10), since $0 \leq \Psi_t^\epsilon(y, x) \leq 2$, we have

\[ (\Psi_{t+s}(y, x) - \Psi_t^\epsilon(y, x'))^2 \leq 2 \int_x^{x'} |\partial_x \Psi_t^\epsilon(z, x)| dz = 2 \int_x^{x'} p_t^{N,\epsilon}(y, z) dz. \]  \[ (4.16) \]

Using this in (4.14), we bound the r.h.s. of (4.14) by $C \int_x^{x'} \| \langle \tilde{Q}_0^e, \tilde{p}_{t+s}^{N,\epsilon}(\cdot, z) \rangle \|_{q/2} dz$. This, by (3.34), is bounded by $C|x - x'|$, whereby we conclude (4.10). Turning to showing (4.11), letting $\tilde{\Psi}_{t, t'}^e(y) := \Psi_t^\epsilon(y, x) - \Psi_{t, t'}^\epsilon(y, x)$, similar to (4.16) we have

\[ \left( \tilde{\Psi}_{t, t'}^\epsilon(y) \right)^2 \leq 2 \int_t^{t'} |\partial_x \Psi_{t+s}^\epsilon(y, x)| ds \]

\[ = \int_t^{t'} s^{-1} |y + x + \eta| p_s^{\epsilon}(y + x) + (y - x - \eta)|p_s^{\epsilon}(y - x)| ds. \]  \[ (4.17) \]

However, due to the $s^{-1}$ singularity, the argument for proving (4.10) does not apply. To circumvent this problem, letting $g(y) := \mathbf{1}_{\{|x+y| \leq 1/2\}} \vee \mathbf{1}_{\{|x+y| \leq 1/2\}}$, we bound $F_1^\epsilon := \langle \tilde{Q}_0^e, (1-g)(\tilde{\Psi}_{t, t'}^e)^2 \rangle$ and $F_2^\epsilon := \langle \tilde{Q}_0^e, g(\tilde{\Psi}_{t, t'}^e)^2 \rangle$ separately. For $F_1^\epsilon$, in (4.17) using $|s^{-1} z p_s(z)| \leq C |z|^{-1} p_{2s}(z)$ and $|x + \eta \pm y| \geq |t' - t|^{1/2}$, we obtain

\[ \left( \tilde{\Psi}_{t, t'}^\epsilon(y) \right)^2 (1 - g(y)) \leq (t - t')^{1/2} C \int_t^{t'} p_{2s}^{N,\epsilon}(y, x) ds. \]

Given this inequality, we now conclude $\| F_1^\epsilon \|_{q/2} \leq C|t - t'|$ by the same argument following (4.16). As for $F_2^\epsilon$, using $|\tilde{\Psi}_{t, t'}^\epsilon(y)| \leq 2$, we obtain

\[ F_2^\epsilon \leq 4 \varepsilon^{1/2} \left\langle \tilde{Q}_0^e, 1_{\{|x+y| \leq 1/2\}} \vee 1_{\{|x+y| \leq 1/2\}} + 1_{\{|x-y| \leq 1/2\}} \right\rangle. \]

Combining this with $\tilde{Q}_0^e \sim \text{PPP}_+(2\varepsilon^{-1/2})$, we conclude $\| F_2^\epsilon \|_{q/2} \leq C|t' - t|^{1/2}$. \[ \Box \]

Next we prove Lemma 4.2 using the martingale Central Limit Theorem of [2, Theorem 2], which we state here in the form convenient for our purpose.
Lemma 4.4 (martingale Central Limit Theorem). Suppose that for any fixed $\epsilon \in (0, 1]^3$, $(N_\epsilon^i, \mathcal{F}_\epsilon^i), i = -1, 0, 1, \ldots, n_\epsilon$, is a discrete time $L^2$-martingale, starting at $N_\epsilon^i = 0$, with the corresponding martingale differences $D_\epsilon^i := N_\epsilon^{i+1} - N_\epsilon^i$ and predictable compensator $\langle N_\epsilon^i \rangle := \sum_{i=0}^{n_\epsilon} \mathbb{E}[|D_\epsilon^i|^3]$. If, for some $\sigma_* \in \mathbb{R}_+$, as $\epsilon \to 0$,

\[
\sum_{i=0}^{n_\epsilon} \mathbb{E}[|D_\epsilon^i|^3] \to 0, \tag{4.18}
\]

\[
\langle N_\epsilon^i \rangle \to \sigma_*^2, \tag{4.19}
\]

then $N_{n_\epsilon}^\epsilon \to \mathcal{N}(0, \sigma_*)$, the mean zero Gaussian with variance $\sigma_*^2$.

Remark 4.5. Although the proof of [2, Theorem 2] is for a single martingale, the same proof applies for a collection of of martingales $\{(N_\epsilon^i, \mathcal{F}_\epsilon^i)\}_\epsilon$ as we consider here. In particular, the truncation argument of [2] applies equally well here, by letting $\tau_{\epsilon,L} := \inf \{i : \langle N_\epsilon^i \rangle > L\}$, whereby $\mathbb{P}(N_\epsilon^i = N_\epsilon^i_{\tau_{\epsilon,L}}, \forall i) \to 1$ as $L \to \infty$, uniformly in $\epsilon$.

Proof of Lemma 4.2. Let $\Gamma_{t,t'}(x, x') := 2 \int_0^\infty \Psi_t(y, x)\Psi_{t'}(y, x')dx$. Fixing arbitrary $t_1, \ldots, t_l$ and $x_1, \ldots, x_l \in \mathbb{R}_+$, we let $C = C(t_1, \ldots, t_l, x_1, \ldots, x_l) < \infty$ and

\[
W_{\epsilon,*} := (W_{t_1\epsilon,*}(x_1), \ldots, W_{t_l\epsilon,*}(x_l)) \in \mathbb{R}^l.
\]

Our goal is to show $W_{\epsilon,*} \Rightarrow \mathcal{N}(0, \Sigma)$, where $\Sigma := (\Gamma_{t_j,t_{j'}}(x_j, x_{j'}))_{j,j' = 1}^l$. Equivalently, fixing arbitrary $v = (v_i) \in \mathbb{R}^l$ and letting $\sigma_* := |\sum_{j,j'=1}^l v_j v_{j'} \Gamma_{t_j,t_{j'}}(x_j, x_{j'})|^{1/2}$, we show

\[
v \cdot W_{\epsilon,*} = \sum_{j=1}^l v_j m_\epsilon^j(t_j, x_j) \Rightarrow \mathcal{N}(0, \sigma_*),
\]

where $m_\epsilon^j(t, x)$ is defined as in (4.12). To this end, letting $n_\epsilon := \lceil \epsilon^{-1} \rceil$, we consider the martingale

\[
N_\epsilon^i := \sum_{j=1}^l v_j m_\epsilon^j(t_j, x_j), \quad k_i := i \mathbf{1}_{\{i < n_\epsilon\}} + \infty \mathbf{1}_{\{i = n_\epsilon\}}. \tag{4.20}
\]

It then suffices to verify i) (4.18); and ii) (4.19).

(i) Let $F_\epsilon^i := \sum_{j=1}^l v_j \psi_{t_j}(X^\epsilon_{(i)}(0), x_j)$. With $N_\epsilon^i$ defined as in (4.20), we have

\[
\sum_{i=0}^{n_\epsilon} \mathbb{E}[|D_\epsilon^i|^3] \leq \varepsilon^{3/4} \left[ \sum_{i \leq \epsilon^{-1}} \mathbb{E}[|1 - 2 Y_i(0)|^3 (F_\epsilon^i)^3] + \mathbb{E}\left( \sum_{i \geq \epsilon^{-1}} (1 - Y_i(0)) (F_\epsilon^i)^3 \right) \right]. \tag{4.21}
\]

We now show that the r.h.s. tends to zero based on the a-priori estimates (3.4) and (3.12). From (4.8) we obtain $|F_\epsilon^i| \leq C \exp(-X_{(i)}^\epsilon(0))$. Using this in (4.21), we bound the r.h.s. by $g_1^\epsilon + g_2^\epsilon$, where

\[
g_1^\epsilon := \varepsilon^{3/4} \mathbb{E}\left( \sum_{i \leq \epsilon^{-1}} |1 - 2 Y_i(0)|^3 e^{-3 X_{(i)}^\epsilon(0)} \right), \tag{4.22}
\]

\[
g_2^\epsilon := \varepsilon^{3/4} \left\| \sum_{i \geq \epsilon^{-1}} (1 - Y_i(0)) e^{-X_{(i)}^\epsilon(0)} \right\|_3^2.
\]
In (4.22), replacing each $|1 - 2Y_i(0)|^3$ with $\overline{Y}_\varepsilon^{i,:} := \sup_{t \leq -1} \{1 - 2Y_i(0)|^3\}$, we obtain

$$g_1^\varepsilon \leq \varepsilon^{3/4} \left\| \overline{Y}_\varepsilon^{i,:} \right\|_2 \left\| \sum_{i=0}^{\infty} e^{-3X_0(i)} \right\|_2.$$ 

With $\{Y_i(0)\} \sim \otimes \text{Exp}(2)$, we have $\left\| \overline{Y}_\varepsilon^{i,:} \right\|_2 \leq C(\log \varepsilon + 1)$, and by (3.4) for $j = 0$, we have $\left\| \sum_{i=0}^{\infty} e^{-3X_0(i)} \right\|_2 \leq C\varepsilon^{-1/2}$, whereby we conclude $g_1^\varepsilon \rightarrow 0$. As for $g_2^\varepsilon$, applying (3.12) for $f_i = |1 - Y_i(0)|$, we obtain $g_2^\varepsilon \leq \varepsilon^{3/4}\varepsilon\varepsilon^{-1/2} \exp(-\varepsilon^{-1/2}/C)^3 \rightarrow 0$.

(ii) With $N_\varepsilon^\varepsilon$ defined as in (4.20), we have $\langle N_\varepsilon^\varepsilon \rangle_{\varepsilon} = \sum_{j,j'} v_j v_{j'} \Gamma_{t,t'}^\varepsilon(x_j, x_{j'})$, where

$$\Gamma_{t,t'}^\varepsilon(x, x') = \varepsilon^{1/2} \sum_{i=0}^{\infty} \Psi_t^\varepsilon(X_{(i)}^t(0), x) \Psi_{t'}^\varepsilon(X_{(i)}^t(0), x').$$ (4.23)

In (4.23), if we replace each $X_{(i)}^t(0)$ by $E(X_{(i)}^t(0)) = \varepsilon^{1/2} 2^{-1-i} := x_i^\varepsilon$, we obtain the expression

$$\Gamma_{t,t'}^\varepsilon(x, x') := \varepsilon^{1/2} \sum_{i=0}^{\infty} \Psi_t^\varepsilon(x_i^\varepsilon, x) \Psi_{t'}^\varepsilon(x_i^\varepsilon, x').$$ (4.24)

This, with $x_{i+1}^\varepsilon - x_i^\varepsilon = 2^{-1}\varepsilon^{1/2}$, is a Riemann sum approximation of $\Gamma_{t,t'}^\varepsilon(x, x')$. In particular, by using the continuity of $(y, \varepsilon) \mapsto \Psi_t^\varepsilon(y, x)$, it is not hard to show that $\Gamma_{t,t'}^\varepsilon(x, x') \rightarrow \Gamma_{t,t'}(x, x')$. Consequently, showing (4.19) is reduced to showing $\Gamma_{t,t'}^\varepsilon(x, x') - \Gamma_{t,t'}(x, x') \rightarrow 0$, which is in turn implied by

$$\varepsilon^{1/2} \sum_{i=0}^{\infty} E \left| \Psi_t^\varepsilon(X_{(i)}^t(0), x) \Psi_{t'}^\varepsilon(X_{(i)}^t(0), x') - \Psi_t^\varepsilon(x_i^\varepsilon, x) \Psi_{t'}^\varepsilon(x_i^\varepsilon, x') \right| \rightarrow 0.$$ (4.25)

We now prove (4.25) by using the continuity of $y \mapsto \Psi_t^\varepsilon(y, x)$ and the control on $|X_{(i)}^t(0) - x_i^\varepsilon| = |X_{(i)}^t(0) - E(X_{(i)}^t(0))|$. For any $L > 0$, we divide the expression in (4.25) into $G_1^\varepsilon L + G_2^\varepsilon L$, for

$$G_1^\varepsilon L := \varepsilon^{1/2} \sum_{i > L \varepsilon^{-1/2}} \left| \Psi_t^\varepsilon(X_{(i)}^t(0), x) \Psi_{t'}^\varepsilon(X_{(i)}^t(0), x') - \Psi_t^\varepsilon(x_i^\varepsilon, x) \Psi_{t'}^\varepsilon(x_i^\varepsilon, x') \right|,$$ (4.26)

$$G_2^\varepsilon L := \varepsilon^{1/2} \sum_{i \leq L \varepsilon^{-1/2}} \left| \Psi_t^\varepsilon(X_{(i)}^t(0), x) \Psi_{t'}^\varepsilon(X_{(i)}^t(0), x') - \Psi_t^\varepsilon(x_i^\varepsilon, x) \Psi_{t'}^\varepsilon(x_i^\varepsilon, x') \right|.$$ (4.27)

By (4.8) and (3.4), for the tail term $G_1^\varepsilon L$, we have $E(G_1^\varepsilon L) \leq C \exp(-L/C)$. With this, it then suffices to show

$$\lim_{\varepsilon \rightarrow 0} E(G_2^\varepsilon L) = 0,$$ (4.28)

(since we can then further take $L \rightarrow \infty$ after taking $\varepsilon \rightarrow 0$). To this end, fixing arbitrary $L > 0$, we let $d^\varepsilon := \sup_{i \leq -1/2 L} |X_{(i)}^t(0) - x_i^\varepsilon|$. With $\{X_{(i)}^t(0)\} \sim \text{PPP}_+(2\varepsilon^{-1/2})$ we have $P(\{|d^\varepsilon| > \varepsilon^{1/8}\}) \rightarrow 0$. By telescoping, in (4.27), for each $i$, we bound the corresponding term by $|\Psi_t^\varepsilon(X_{(i)}^t(0), x) - \Psi_t^\varepsilon(x_i^\varepsilon, x)| \Psi_{t'}^\varepsilon(x_i^\varepsilon, x') + \Psi_t^\varepsilon(x_i^\varepsilon, x) |\Psi_{t'}^\varepsilon(X_{(i)}^t(0), x') - \Psi_t^\varepsilon(x_i^\varepsilon, x')|$. 

Further using \(|\Psi_\epsilon^t(x, y)| \leq 2\) and \(|\Psi_\epsilon^t(a, x) - \Psi_\epsilon^t(b, x)| = \int_a^b p_{\eta, t}^{N, \epsilon}(z, x) dz\), we obtain

\[
G_2^{\epsilon, L} \leq C \int_{-d^\epsilon}^{d^\epsilon} \left( \tilde{Q}_{0, t}^\epsilon \cdot \tilde{p}_{t}^{N, \epsilon}(\cdot, x) + p_{t}^{N, \epsilon}(\cdot, z, x') \right) dz. \tag{4.29}
\]

Now consider the cases \(d^\epsilon \leq \epsilon^{1/8}\) and \(d^\epsilon > \epsilon^{1/8}\) separately. For the former combining (4.29) and (3.36), we obtain \(E(G_2^{\epsilon} 1_{\{|d^\epsilon| \leq \epsilon^{1/8}\}}) \leq C \epsilon^{1/8} \to 0\). For the latter using \(G_2^{\epsilon, L} \leq C(L)\) (since \(|\Psi_\epsilon^t(\cdot)| \leq 2\), we conclude \(E(G_2^{\epsilon} 1_{\{|d^\epsilon| > \epsilon^{1/8}\}}) \leq C(L) P(d^\epsilon > \epsilon^{1/8}) \to 0\). Therefore (4.28) follows.

\[
\square
\]

4.3. **Proof of part (c).** Recall \(S_0^\epsilon(\cdot)\) is defined as in (3.32). Letting \(\hat{t}_\epsilon := t \wedge \tau_{1/8}^\epsilon\) and

\[
N_{t, t'}^\epsilon(x) := M_{t, t'}^\epsilon \left( p_{t, \hat{t}_\epsilon}^{N, \epsilon}(\cdot, x), \infty \right)
\]

\[
= \epsilon^{1/4} \sum_{i=0}^\infty \int_0^{t'} S_{1/8}^\epsilon(s) p_{t + \delta - s}^{N, \epsilon}(X_i^\epsilon(s), x) dB_i^\epsilon(s),
\]

we recall from Proposition 2.3, that for any \(T \in \mathbb{R}_+\), \(\lim_{\epsilon \to 0} P(M_{0, t}^\epsilon(x) = N_{0, t}^\epsilon(x), \forall t \in [0, T], x \in \mathbb{R}_+) = 1\), so without lost of generality we replace \(M_{t}^\epsilon(x)\) with \(N_{t}^\epsilon(x) := N_{0, t}^\epsilon(x)\).

**Lemma 4.6.** The collection of processes \(\{N_{t}^\epsilon(\cdot)\} \in C(\mathbb{R}_+^2, \mathbb{R})\) is tight in \(C(\mathbb{R}_+^2, \mathbb{R})\).

**Proof.** The process \((t, x) \mapsto N_{t}^\epsilon(x)\), as the uniform limit (as \(k \to \infty\)) of the continuous martingale \(M_{0, t}^\epsilon(\psi, x)\) for \(\psi(y) := p_{0, t}^{N, \epsilon}(y, x)\), is continuous.

Fixing \(T, L \in \mathbb{R}_+, q \in (1, \infty)\) and \(\alpha \in (0, 1)\), hereafter we let \(C = C(T, L, q, \alpha) < \infty\). For this fixed \(\alpha\), we next verify the conditions (4.9)–(4.11) for \(K_{t}^\epsilon(x) = N_{t}^\epsilon(x)\). The first condition (4.9) follows trivially since \(N_{0}^\epsilon(0) = 0\). As for (4.10)–(4.11), fixing \(t < t' \in [0, T]\), \(x < x' \in [0, L]\), our goal is to bound the moments of \(N_{t}^\epsilon := N_{t}^\epsilon(x') - N_{t}^\epsilon(x)\) and \(N_{t}^\epsilon(x) - N_{t}^\epsilon(x) = N_{t}^\epsilon + N_{3}^\epsilon\), where \(N_{2}^\epsilon := M_{t}^\epsilon(p_{t, t'}^{N, \epsilon}(\cdot, x)) - M_{t}^\epsilon(p_{t, t'}^{N, \epsilon}(\cdot, x))\) and \(N_{3}^\epsilon := M_{t}^\epsilon(p_{t, t'}^{N, \epsilon}(\cdot, x))\). To this end, we control the quadratic variation

\[
V_1^\epsilon := \int_0^t S_{1/8}^\epsilon(s) \left( \tilde{Q}_s^\epsilon(p_{t}^{N, \epsilon}(\cdot, x') - p_{t}^{N, \epsilon}(\cdot, x))^2 \right) ds, \tag{4.31}
\]

\[
V_2^\epsilon := \int_0^t S_{1/8}^\epsilon(s) \left( \tilde{Q}_s^\epsilon(p_{t}^{N, \epsilon}(\cdot, x) - p_{t}^{N, \epsilon}(\cdot, x))^2 \right) ds, \tag{4.32}
\]

\[
V_3^\epsilon := \int_t^{t'} S_{1/8}^\epsilon(s) \left( \tilde{Q}_s^\epsilon(p_{t}^{N, \epsilon}(\cdot, x'))^2 \right) ds, \tag{4.33}
\]

of the martingales \(N_{j}^\epsilon, j = 1, 2, 3\), respectively. By using

\[
|p_{t}^{N, \epsilon}(y, x) - p_{t}^{N, \epsilon}(y', x')| \leq C t^{-(\alpha+1)/2}(|x - x'|^\alpha + |y - y'|^\alpha), \tag{4.34}
\]

\[
|p_{t}^{N, \epsilon}(y, x) - p_{t}^{N, \epsilon}(y, x)| \leq C t^{-(\alpha+1)/2} (t' - t)^{\alpha/2}, \tag{4.35}
\]

\[
|p_{t}^{N, \epsilon}(y, x)| \leq C (s')^{-1/2}
\]
(where (4.34) and (4.35) follow from the \( \alpha \)-Hölder continuity of \( \exp(-z^2/2) \) and \( z \exp(z^2/2) \), respectively), we obtain
\[
\left( p_{t-s}^{N,\epsilon}(\cdot, x') - p_{t-s}^{N,\epsilon}(\cdot, x) \right)^2 \leq C|x - x'|^\alpha \left( p_{t-s}^{N,\epsilon}(\cdot, x) + p_{t-s}^{N,\epsilon}(\cdot, x') \right),
\]
\[
\left( p_{t-s}^{N,\epsilon}(\cdot, x) - p_{t-s}^{N,\epsilon}(\cdot, x) \right)^2 \leq C|t - t'|^{\alpha/2} \left( p_{t-s}^{N,\epsilon}(\cdot, x') + p_{t-s}^{N,\epsilon}(\cdot, x') \right),
\]
\[
\left( p_{t-s}^{N,\epsilon}(\cdot, x') \right)^2 \leq C(t' - s)^{-1/2} p_{t-s}^{N,\epsilon}(\cdot, x').
\]
Plugging this in (4.31)–(4.33), and using (3.34), we further obtain
\[
\|V_1^\epsilon\|_q \leq C|x - x'|\alpha \int_0^t (t' - s)^{-(1+\alpha)/2}(|\log(t - s)| + 1)ds,
\] (4.36)
\[
\|V_2^\epsilon\|_q \leq C|t' - t'|^{\alpha/2} \int_0^t (t' - s)^{-(1+\alpha)/2}(|\log(t' - s)| + 1)ds,
\] (4.37)
\[
\|V_3^\epsilon\|_q \leq C \int_t^{t'} (t' - s)^{-1/2}(|\log(t' - s)| + 1)ds,
\] (4.38)
respectively. By the Burkholder–Davis–Gundy inequality, we have \( \|N_j^\epsilon\|_q \leq C(\|V_j^\epsilon\|_q/2)^{1/2} \). Combining this with (4.36)–(4.38), we thus conclude (4.10)–(4.11) for \( K_t^\epsilon(x) = N_t^\epsilon(x) \). 

**Lemma 4.7.** As \( \epsilon \to 0 \), \( \{N_t^\epsilon(\cdot)\}_\epsilon \) converges in finite dimensional distribution to a centered Gaussian process \( M_\cdot(\cdot) \) with the covariance (2.12).

**Proof.** Fixing arbitrary \( t_1, \ldots, t_m, x_1, \ldots, x_m \in \mathbb{R}_+ \), we let \( C = C(t_1, \ldots, t_m, x_1, \ldots, x_m) < \infty \), \( v := (v_1, \ldots, v_m) \in \mathbb{R}^m \) and \( \iota := \sqrt{-1} \). Consider the characteristic functions \( \varphi_\epsilon(v) := E[\exp(i \sum_{j=1}^m v_j N_j^\epsilon(x_j))] \) and \( \varphi(v) := E[\exp(i \sum_{j=1}^m v_j M_j(x_j))] \) of \( (N_j^\epsilon(x_j))_{j=1}^m \) and \( (M_j(x_j))_{j=1}^m \), respectively. By Lévy’s continuity theorem, it suffices to show \( \varphi_\epsilon \to \varphi \).

Letting \( q_{s,t,t'}^{N}(y, x, x') := p_{t-s}^{N}(y, x)p_{t'-t}^{N}(x, x') \) and \( \mathcal{V}_{t,t'}(x, x') := \int_{0}^{t \land t'} \int_{s}^{s \land t'} q_{s,t,t'}^{N}(y, x, x')dyds \), we recall that
\[
\varphi(v) = \exp \left( -2^{-1} \sum_{j, j'=1}^m v_j v_{j'} \mathcal{V}_{t, t'}(x_j, x_{j'}) \right).
\]
As for \( \varphi_\epsilon \), since \( N_t^\epsilon(x) \) is a continuous martingale of quadratic variation
\[
\mathcal{V}_{t,t'}^\epsilon(x, x') := \int_{0}^{t \land t'} \mathcal{S}_{1/8}^\epsilon(s) \left( \mathcal{Q}_{s}^\epsilon, q_{s,t,t'}^{N,\epsilon}(\cdot, x, x') \right)ds
\]
\[
= \int_{0}^{t \land t'} \mathcal{S}_{1/8}^\epsilon(s) \left( \varepsilon^{1/2} \sum_{i=0}^{\infty} q_{s,t,t'}^{N,\epsilon}(X_{(i)}^\epsilon)(s), x, x' \right)ds,
\] (4.39)
where \( q_{s,t,t'}^{N,\epsilon}(y, x, x') := p_{t-s}^{N,\epsilon}(y, x)p_{t'-t}^{N,\epsilon}(x, x') \), we have
\[
\varphi_\epsilon(v) = E \left[ \exp \left( -2^{-1} \sum_{j, j'=1}^m v_j v_{j'} \mathcal{V}_{t, t'}^\epsilon(x_j, x_{j'}) \right) \right].
\]
Given these expressions of $\varphi$ and $\varphi_\varepsilon$, by the bounded convergence theorem, it suffices to show that

$$\mathcal{V}_{t,t'}^\varepsilon(x, x') \to \mathcal{V}_{t,t'}(x, x'),$$

(4.40) for all $t, t', x, x' \geq 0$. Similar to (4.23)–(4.24), in (4.39), if we replace $X^\varepsilon_{(i)}(s)$ with $x^\varepsilon_i = E(X^\varepsilon_{(i)}(0))$, the resulting expression $\mathcal{V}_{t,t'}^\varepsilon(x, x')$ represents a Riemann sum approximation of $\mathcal{V}_{t,t'}(x, x')$. The only difference here is the extra factor of $S_{1/8}(s)$, which satisfies $P(S_{1/8}(s) = 1, \forall s \in [0, T]) \to 1$ (by Proposition 2.3). Hence, in the same way $\Gamma_{t,t'}^\varepsilon(x, x') \to \Gamma_{t,t'}(x, x')$, we have $\mathcal{V}_{t,t'}^\varepsilon(x, x') \to \mathcal{V}_{t,t'}(x, x')$, thereby reducing showing (4.40) to showing

$$\int_0^{t \wedge t'} \mathbf{E}\left(S_{1/8}^\varepsilon(s) \varepsilon^\varepsilon(s)\right) ds \longrightarrow 0,$$

(4.41) for $r^\varepsilon(s) := \varepsilon^{1/2} \sum_{i=0}^{\infty} |q_{i,s,t,t'}^{N,\varepsilon}(X^\varepsilon_{(i)}(s), x, x') - q_{i,s,t,t'}^{N,\varepsilon}(x^\varepsilon_i, x, x')|$. We now prove (4.41) by using the continuity of $y \mapsto q_{s,t,t'}^{N,\varepsilon}(y, x, x')$ and the control on $S_{1/8}^\varepsilon(s)|X^\varepsilon_{(i)}(s) - x^\varepsilon_i|$, similar to the proof of (4.25). Expressing $X^\varepsilon_{(i)}(s)$ as $X^\varepsilon_{(i)}(s) - X^\varepsilon_{(0)}(s) + X^\varepsilon_{(0)}(s)$, with $D^\varepsilon(j, j', t)$ defined as in (2.17), we have

$$S_{1/8}^\varepsilon(s)|X^\varepsilon_{(i)}(s) - x^\varepsilon_i| \leq 2^{-1} \varepsilon^{1/2} D^\varepsilon(0, i, s) + \varepsilon^{1/8} := d_i^\varepsilon(s).$$

(4.42)

Since the gaps are at equilibrium, from (2.18) we deduce

$$\|D^\varepsilon(j, j', t)\|_n \leq C(n)|j - j'|^{1/2}.\leqno(4.43)$$

Using this for $(j, j') = (0, i)$, with $i = 2\varepsilon^{-1/2}x^\varepsilon_i$, we obtain

$$\|d_i^\varepsilon(s)\|_n \leq C(n)\varepsilon^{1/8} [1 + (x^\varepsilon_i)^{1/2}], \ \forall n \in \mathbb{Z}.\leqno(4.44)$$

Next, by telescoping, we bound $r^\varepsilon(s)$ by $F_1^\varepsilon(s) + F_2^\varepsilon(s)$, where

$$F_1^\varepsilon(s) := \varepsilon^{1/2} \sum_{i=0}^{\infty} \left| P_{i-s}^{N,\varepsilon}(X^\varepsilon_{(i)}(s), x) - P_{i-s}^{N,\varepsilon}(x^\varepsilon_i, x) \right| P_{i-s}^{N,\varepsilon}(X^\varepsilon_{(i)}(s), x'),$$

(4.45)

$$F_2^\varepsilon(s) := \varepsilon^{1/2} \sum_{i=0}^{\infty} P_{i-s}^{N,\varepsilon}(x^\varepsilon_i, x) \left| P_{i-s}^{N,\varepsilon}(X^\varepsilon_{(i)}(s), x') - P_{i-s}^{N,\varepsilon}(x^\varepsilon_i, x') \right|.$$  

(4.46)

In (4.45)–(4.46), using (4.34) for $\alpha = 1/2$ and using (4.42), we obtain

$$S_{1/8}^\varepsilon(s)F_1^\varepsilon(s) \leq C(t - s)^{-3/4} \varepsilon^{1/2} \sum_{i=0}^{\infty} (d_i^\varepsilon(s))^{1/2} S_{1/8}^\varepsilon(s) P_{i-s}^{N,\varepsilon}(X^\varepsilon_{(i)}(s), x'),$$

(4.47)

$$S_{1/8}^\varepsilon(s)F_2^\varepsilon(s) \leq C(t' - s)^{-3/4} \varepsilon^{1/2} \sum_{i=0}^{\infty} (d_i^\varepsilon(s))^{1/2} P_{i-s}^{N,\varepsilon}(x^\varepsilon_i, x).$$

(4.48)

Plugging (4.44) in (4.48), we obtain

$$\mathbf{E}\left(S_{1/8}^\varepsilon(s)F_2^\varepsilon(s)\right) \leq C(t' - s)^{-3/4} \varepsilon^{1/16}.\leqno(4.49)$$
As for $F_1^\varepsilon(s)$, fixing $q \in (1, 2)$, in (4.47), for each $i$, multiplying and dividing by the factor $\exp(-X^{\varepsilon}_{(i)}(s))$, we apply Hölder’s inequality (with respect to $E[\sum_i(\cdot)]$) to obtain $E(F_1^\varepsilon(s)) \leq (t-s)^{-3/4}f_{11}(s)f_{12}(s)$, where

$$f_{11}(s) := \left[ E \left( \varepsilon^{1/2} \sum_{i=0}^{\infty} (d_{i}^\varepsilon(s))^{q'/2} e^{-q' X^{\varepsilon}_{(i)}(s)} \right) \right]^{1/q'},$$

$$f_{12}(s) := \left[ E \left( S_b^\varepsilon(s) \langle \tilde{Q}^\varepsilon \exp(q\cdot) \langle \tilde{p}_{t-s}^N(\cdot) \rangle^q \rangle \right) \right]^{1/q},$$

and $1/q' + 1/q = 1$. Combining (4.44) and (3.12) for $f_i = (d_{i}^\varepsilon(s))^{q'}$, we obtain $f_{11}(s) \leq C\varepsilon^{1/16}$. As for $f_{12}(s)$, with $e^{qy}p_d(y-z) = C(\sigma, z)p_d(y - q\sigma - z)$, we have

$$e^{qy} \left( p_{t-s}^N(y, x') \right)^q = \left[ p_{t-s}^N(y, x') \right]^{-q-1} e^{qy} p_{t-s}^N(y, x')$$

$$\leq C(L, T, q)(t' - s)^{-(q-1)/2} p_{t-s}^N(y - q(t' + \delta - s), x').$$

Using this and (3.33) for $y' = q(t + \delta - s)$, we obtain $f_{12} \leq C(t-s)^{-(q-1)/2q}(|\log(t' - s)| + 1)^{1/q}$. Consequently,

$$E(F_1^\varepsilon(s)) \leq C\varepsilon^{1/16}(t-s)^{-3/4-(q-1)/2q}(|\log(t' - s)| + 1). \tag{4.50}$$

With $(q-1)/2q < 1/4$, from (4.49)–(4.50) we conclude (4.41).

5. Proof of Proposition 2.7

Recall $t^\varepsilon_k := \varepsilon^{-1}k$. We first establish the following estimates on the continuity in $t$ of $G_\varepsilon^\varepsilon(x)$, $\tilde{X}_\varepsilon^\varepsilon(x)$ and $X_i^\varepsilon(x)$.

**Lemma 5.1.** For any fixed $T, L \in \mathbb{R}_+$,

$$F_{X^\varepsilon}(T, L) := \sup \left\{ |X^\varepsilon_i(x) - X^\varepsilon_i(x)| : k \leq T\varepsilon^{-1}, t \in [t_k, t_{k+1}], x \in [0, L] \right\} \rightarrow 0, \tag{5.1}$$

$$F_{\tilde{X}^\varepsilon}(T, L) := \sup \left\{ |\tilde{X}^\varepsilon_i(x) - \tilde{X}^\varepsilon_i(x)| : k \leq T\varepsilon^{-1}, t \in [t_k, t_{k+1}], x \in [0, L] \right\} \rightarrow 0, \tag{5.2}$$

$$F_{G^\varepsilon}(T, L) := \sup \left\{ |G^\varepsilon_i(x) - G^\varepsilon_i(x)| : k \leq T\varepsilon^{-1}, t \in [t_k, t_{k+1}], x \in [0, L] \right\} \rightarrow 0. \tag{5.3}$$

**Proof.** We say that events $\{A^\varepsilon\}$ happen at Super-Polynomially Rate (SPR) if, for each $q \geq 1$, $P((A^\varepsilon)^c)\varepsilon^{-q}$ is uniformly bounded. By (1.2), $X^\varepsilon_i(x) - X^\varepsilon_i(x) = 2 \varepsilon^{1/4}(X(i\varepsilon(x))(\varepsilon^{-1}t) - X(i\varepsilon(x))(\varepsilon^{-1}t_k))$. Fixing arbitrary $a > 0$, from (3.6) we deduce that

$$\sup_{t \in [t_k, t_{k+1}]} \left\{ \varepsilon^{1/4}|X(i\varepsilon(x))(\varepsilon^{-1}t) - X(i\varepsilon(x))(\varepsilon^{-1}t_k)| \right\} \leq a, \quad \text{at SPR.}$$

By taking the union bound over $k \leq T\varepsilon^{-1}$ and over $i\varepsilon(x) \in \mathbb{Z} \cap [0, L\varepsilon^{-1/2} + 1]$, which is a union size $C\varepsilon^{-3/2}$, we conclude that $F_{X^\varepsilon}(T, L) \leq a$ at SPR. As $a > 0$ is arbitrary, we obtain (5.1).

As for (5.1), by (2.15) we have $\tilde{X}^\varepsilon_i(x) - \tilde{X}^\varepsilon_i(x) = 2 \varepsilon^{1/4}(X(i\varepsilon(x))(\varepsilon^{-1}t) - X(i\varepsilon(x))(\varepsilon^{-1}t_k))$. Further, with $\{X^\varepsilon(i)(0)\} \sim PPP_+(2\varepsilon^{-1/2})$, we have

$$I_0^\varepsilon(x) \leq (4L + 1)\varepsilon^{-1/2} \quad \text{as SPR,} \tag{5.4}$$

The proof of Proposition 2.7 is complete.
so (5.2) follows by the same argument for (5.1).

Letting \( F_{G^\varepsilon}(k,L) := \sup\{ |G^\varepsilon_t(x) - G^\varepsilon_0(x)| : t \in [t_k,t_{k+1}], x \in [0,L] \} \), we next show (5.3) by showing, for each \( k \) and fixed \( a > 0 \), \( F_{G^\varepsilon}(k,L) \leq a \) at spr. By stationarity, \( |G^\varepsilon_t(x) - G^\varepsilon_0(x)| \overset{\text{distr.}}{=} |G^\varepsilon_{t+tk}(x + X^\varepsilon_{(0)}(tk)) - G^\varepsilon_{tk}(x + X^\varepsilon_{(0)}(tk))| \) and by Proposition 2.3 \( |X^\varepsilon_{(0)}(tk)| \leq 1 \) at spr. Hence it suffices to show \( F_{G^\varepsilon}(L) \to_\text{p} 0 \), where \( F_{G^\varepsilon}(L) := \sup\{ |G^\varepsilon_t(x) - G^\varepsilon_0(x)| : t \in [0,\varepsilon], x \in [-1,L + 1] \} \to_\text{p} 0 \). With \( G^\varepsilon_t(x) \) defined as in (2.13), we have that \( G^\varepsilon_t(x) - G^\varepsilon_0(x) = \varepsilon^{1/4}G^\varepsilon(t,x) \), where

\[
G^\varepsilon(t,x) = \langle Q^\varepsilon_t, 1_{(-\infty,x]} \rangle - \langle Q^\varepsilon_0, 1_{(-\infty,x]} \rangle
\]

is the net flux of particles across \( x \) within \([0,t] \). Let

\[
H^\varepsilon(j) := \sum_{i \geq j} H^\varepsilon_1(i,j) + \sum_{i < j} H^\varepsilon_x(i,j),
\]

(5.6)

\[
H^\varepsilon_1(i,j) := 1\left\{ \inf_{t \in [0,\varepsilon]} X^\varepsilon_{(t)} \leq X^\varepsilon_{(0)}(0) \right\}, \quad H^\varepsilon_x(i,j) := 1\left\{ \sup_{t \in [0,\varepsilon]} X^\varepsilon_x(t) > X^\varepsilon_{(j)}(0) \right\}.
\]

In (5.5), using (3.1), \( X^\varepsilon_{(I^*_0(x)-1)}(0) > x \) and \( X^\varepsilon_{(I^*_0(x))}(0) \leq x \), we then obtain that

\[
\sup_{t \in [0,\varepsilon]} \{ |G^\varepsilon(t,x)| \} \leq \varepsilon^{1/4}H^\varepsilon(I^*_0(L)).
\]

Combining this with (5.4), we now arrive at

\[
F_{G^\varepsilon}(L) \leq \sup_{j \in \mathcal{K}(L)} \{ \varepsilon^{1/4}H^\varepsilon(j) \} \quad \text{at spr},
\]

(5.7)

where \( \mathcal{K}(L) := \{ j : |j| \leq 4(L + 1)\varepsilon^{-1/2} \} \). Recall \( U^\varepsilon_x(t,i,j) \) and \( U^\varepsilon_1(t,i,j) \) are defined as in (3.8)–(3.9). Fixing any \( q \geq 1 \), in (5.6) taking the \( q \)-th norm of both sides, we obtain

\[
\|H^\varepsilon(j)\|_q \leq \sum_{i \geq j} \|H^\varepsilon_1(i,j)\|_q + \sum_{i < j} \|H^\varepsilon_x(i,j)\|_q
\]

\[
\leq \sum_{i \geq j} \left( \mathbb{E} U^\varepsilon_1(\varepsilon,i,j) \right)^{1/q} + \sum_{i < j} \left( \mathbb{E} U^\varepsilon_x(\varepsilon,i,j) \right)^{1/q}.
\]

Further using (3.10)–(3.11) in the last expression, we conclude \( \|H^\varepsilon(j)\|_q \leq C(q) \). With \( q \geq 1 \) being arbitrary and \( |\mathcal{K}(L)| \leq C\varepsilon^{-1/2} \), we have \( \sup_{j \in \mathcal{K}(L)} \{ \varepsilon^{1/4}H^\varepsilon(j) \} \leq a \) at spr, for arbitrary fixed \( a > 0 \). Combining this with (5.7), we thus complete the proof.

Recall \( D^\varepsilon(j,j',t) \) is defined as in (2.17). Let

\[
\mathcal{J}(T,L) := \{ (j,j',k) \in \mathbb{Z}^3_+ : j,j' \leq 4(L + 1)\varepsilon^{-1/2}, |j - j'| \leq \varepsilon^{-\mu'}, k \leq T\varepsilon^{-1} \}.
\]

(5.8)

Lemma 5.2. For each fixed \( T, L \in \mathbb{R}_+ \) and \( \mu' \in (0,1/2) \),

\[
\sup_{(j,j',k) \in \mathcal{J}(T,L)} \varepsilon^{1/4}D^\varepsilon(j,j',t_k) \to_\text{p} 0.
\]

(5.9)
Proof. Fix such \(L, T, \mu'\) and any \(a > 0\). By (4.43) we have \(P(\varepsilon^{1/4}|D^\varepsilon(j, j', t_k)| \geq a) \leq C(n, a)(|j - j'|\varepsilon^{1/2})^n\). Form this, fixing \(n > 2/(1/2 - \mu')\), using the union bound, we obtain
\[
P\left( \bigcup_{(j, j', k) \in \mathcal{F}_{\mu'}(T, L)} \{\varepsilon^{1/4}|D^\varepsilon(j, j', t_k)| \geq a\} \right) \leq C(n, T, L, a)\varepsilon^{n(1/2 - \mu')}|\mathcal{F}_{\mu'}(T, L)|.
\]
With \(|\mathcal{F}_{\mu'}(T, L)| \leq C(T, L)\varepsilon^{-2}\) and \(n(1/2 - \mu') - 2 > 0\), the r.h.s. tends to zero as \(\varepsilon \to 0\). Since \(a > 0\) is arbitrary, we conclude (5.9).

Hereafter we say events \(\{\mathcal{A}\}_\varepsilon\) occur with with an Overwhelming Probability (OP) if \(P(\mathcal{A}\varepsilon) \to 1\) as \(\varepsilon \to 0\).

Proof of Proposition 2.7(a). Fixing any \(L, T \in \mathbb{R}_+\), \(a \in (1/2, \infty)\) and \(b \in (0, 1/4)\), our goal is to show
\[
\sup_{x \in [e^b, L]} \sup_{t \in [0, T]} \left| F^\varepsilon_t(x) - G^\varepsilon_t(x) \right| \to 0.
\]
Let
\[
f^\varepsilon(y, x) := \Psi^\varepsilon(y, x) - 1_{(-\infty, x]}(y) = 1 - \Phi^\varepsilon(y + x) + 1_{(x, \infty)}(y) - \Phi^\varepsilon(y - x).
\]
Recall from (1.12) and (2.13) that \(F^\varepsilon_t(x) - G^\varepsilon_t(x) = \langle \hat{Q}^\varepsilon_t, f^\varepsilon(\cdot, x) \rangle = F^\varepsilon_t(t, x) + f_2^\varepsilon(x)\), where
\[
F^\varepsilon_t(t, x) := \varepsilon^{1/4}\langle Q^\varepsilon_t, f^\varepsilon(\cdot, x) \rangle,
\]
\[
f_2^\varepsilon(x) := -2\varepsilon^{-1/4} \int_0^\infty f^\varepsilon(y, x)dy.
\]
From (5.11) we deduce that, for \((x, y) \in \mathbb{R}_+^2\),
\[
|f^\varepsilon(y, x)| \leq \Phi^\varepsilon(-y - x) + \Phi^\varepsilon(-y - x).\]

In particular, \(\int_0^\infty |f^\varepsilon(y, x)|dy \leq C\int_0^\infty \Phi^\varepsilon\varepsilon(z)dz = C\varepsilon^{a/2}\). Using this in (5.13), with \(a > 1/2\), we conclude that \(\sup_{x \geq 0} \{|F^\varepsilon_1(t, x)|\} \to 0\). It then suffices to show
\[
\sup \{|F^\varepsilon_1(t, x)|: t \in [0, T], x \in [e^b, L]\} \to 0.
\]

We now show (5.15) by using (5.14) and \(Q^\varepsilon_t(0) \sim \text{PPP}_+(2\varepsilon^{-1/2})\). Recall on \((y, x) \in \mathbb{R}_+^2\) we have \(\Phi^\varepsilon(-y - x) \leq C\exp[-2(x + y)/\varepsilon^{a/2}]\) and \(\Phi^\varepsilon(-|y - x|) \leq C\exp[-2|y - x|/\varepsilon^{a/2}]\). Combining this with (5.14), we obtain
\[
|f^\varepsilon(y, x)| \cdot 1_{\{x \geq e^b\}} \leq Ce^{-2b-a/2 - 2y e^{-a/2}} + Ce^{2|y - x| e^{-a/2}}.
\]
This decays fast expect when \(|y - x|\) is small, so fixing \(a' \in (1/2, a)\), we further deduce
\[
|f^\varepsilon(y, x)| \cdot 1_{\{x \geq e^b\}} \leq C\left(e^{-2b-a/2} e^{-y} + e^{-a' a/2} e^{-(y - x)}\right) + C \cdot 1_{\{|y - x| < e^{a'/2}\}}.
\]
Plugging this in (5.12), we arrive at $F_1^\varepsilon(t, x) \leq CF_{11}^\varepsilon(t, x) + CF_{12}^\varepsilon(t, x)$, where
\[
F_{11}^\varepsilon(t) := \varepsilon^{1/4} \left( e^{-\varepsilon b - \alpha G} + e^{-\varepsilon (a' - \alpha)/2} \right) \langle Q_t^\varepsilon \exp (- \cdot) \rangle,
\]
\[
F_{12}^\varepsilon(t, x) := \varepsilon^{1/4} \langle Q_t^\varepsilon, 1_{(x - \varepsilon a'/2, x + \varepsilon a'/2)}(\cdot) \rangle.
\]

For $F_{11}^\varepsilon(t)$, with $a > a' \lor (2b)$, applying (3.4) we conclude that $\operatorname{sup}_{t \in [0, T]} \{F_{11}^\varepsilon(t)\} \to 0$ in $L^1$ and hence in probability. Turning to bounding $F_{12}^\varepsilon(t, x)$, we let $N(t, x) := \varepsilon^{1/4} \langle Q_t^\varepsilon(0), 1_{(x - \varepsilon a'/2, x + \varepsilon a'/2)} \rangle \sim \text{Pois}(4\varepsilon (a' - 1)/2)$. Since $Q_t^\varepsilon$ and $Q_t^\varepsilon(0)$ differ only by the shift of $X_{(0)}^\varepsilon(t)$, which by Proposition 2.3 is at most 1 with an OP, we have
\[
\sup_{x \in [0, L]} F_{12}^\varepsilon(t, x) \leq \sup_{x \in [-1, L+1]} \left\{ \varepsilon^{1/4} N(t, x) \right\} \leq 2 \sup_{|i| \leq (L+2)e^{-a'/2}} \left\{ \varepsilon^{1/4} N(t, 2ie^{a'/2}) \right\},
\]
for all $t \in [0, T]$ with an OP. Further, with $a' > 1/2$, from the large deviations bound of $\text{Pois}(4\varepsilon (a' - 1)/2)$, we deduce that
\[
\sup_{k \leq T e^{-1}} \sup_{|i| \leq (L+2)e^{-a'/2}} \left\{ \varepsilon^{1/4} N(t_k, 2ie^{a'/2}) \right\} \to 0,
\]
thereby concluding
\[
\sup_{k \leq T e^{-1}} \sup_{x \in [0, L]} \{F_{12}^\varepsilon(t_k, x)\} \to_p 0.
\]

Now, since (by (2.13)) $F_{12}^\varepsilon(t, x) = G_t^\varepsilon(x + \varepsilon a'/2) - G_t^\varepsilon(x - \varepsilon a'/2) - 2\varepsilon a'/2 - 1/4$, combining (5.16) and (5.3), we conclude $\operatorname{sup}_{t \in [0, T]} \sup_{x \in [0, L]} \{F_{12}^\varepsilon(t, x)\} \to_p 0$. \hfill \square

**Proof of Proposition 2.7(b).** Fixing $L, T \geq 0$ and $b \in (0, 1/4)$, by (5.2)–(5.3) and (2.20), it suffices to show
\[
\sup_{k \leq T e^{-1}} \sup_{x \in [e^b, L]} \left| \varepsilon^{1/4} D^\varepsilon (I_{t_k}^\varepsilon(x), I_{0}^\varepsilon(x), t_k) - 2\varepsilon^{1/4} \rho_{t_k}^\varepsilon(x) \right| \to_p 0,
\]
as $\varepsilon \to 0$. Letting
\[
G_1^\varepsilon := \sup_{k \leq T e^{-1}} \sup_{x \in [e^b, L]} \left\{ \varepsilon^{1/4} \rho_{t_k}^\varepsilon(x) \right\}, \quad G_2^\varepsilon := \sup_{k \leq T e^{-1}} \sup_{x \in [e^b, L]} \left\{ \varepsilon^{1/4} D^\varepsilon (I_{t_k}^\varepsilon(x), I_{0}^\varepsilon(x), t_k) \right\},
\]
we next show

1. $G_1^\varepsilon \to_p 0$; and
2. $G_2^\varepsilon \to_p 0$.

(i) From (2.14) and (3.31), we have $I_{t_k}^\varepsilon(x) = \langle Q_{t_k}^\varepsilon, 1_{(-\infty, x)} \rangle = \langle Q_{t_k}^\varepsilon(0), 1_{(-\infty, x - X_{(0)}^\varepsilon(t_k))} \rangle$. Further, by Proposition 2.3, with an OP we have $\sup_{t \in [0, T]} \{\| X_{(0)}^\varepsilon(t) \| \} \leq 1$, so, with an OP, $I_{t_k}^\varepsilon(L) \leq \langle Q_{t_k}^\varepsilon(0), 1_{(-\infty, L+1)} \rangle$ for all $k \leq T e^{-1}$. Using this and the large deviations bound of $\langle Q_{t_k}^\varepsilon(0), 1_{(-\infty, L+1)} \rangle \sim \text{Pois}(2\varepsilon^{-1/2}(L + 1))$, we then conclude that
\[
\left\{ I_{t_k}^\varepsilon(L) \leq 4(L + 1)\varepsilon^{-1/2}, \forall k \leq T e^{-1} \right\} \text{ holds with an OP.}
\]

Next, By Proposition 2.3, with an OP, for all $x \in [e^b, L]$ and $t \in [0, T]$, we have $X_{(0)}^\varepsilon(t) \leq \varepsilon^b \leq x$. Consequently, by (2.19), $|\rho_{t_k}^\varepsilon(x)| \leq Y_{t_k}^\varepsilon(x) (\varepsilon^{-1} t)$, with an OP. Combining this with (5.17), we then conclude that, with an OP, $G_1^\varepsilon \leq \sup_{k \leq T e^{-1}} \sup_{|i| \leq 4(L + 1)\varepsilon^{-1/2}} \left\{ \varepsilon^{1/4} Y_{t_k}^\varepsilon \right\}$, which clearly converges to zero in probability.
(ii) By Proposition 1.8 and Proposition 2.7(a), the process \((t, x) \mapsto (\mathcal{G}^e_t(x) - \mathcal{G}^e_0(x)) \mathbf{1}_{[e^b, \infty)}(x)\) converges weakly. The latter, by (2.13)–(2.14), is equal to \(\varepsilon^{1/4}(I^e_t(x) - I^e_0(x)) \mathbf{1}_{[e^b, \infty)}(x)\). From this, we conclude that, for any \(\mu' > 1/4\),

\[
\lim_{\alpha \to \infty} \lim_{\varepsilon \to 0} \mathbf{P}\left( \sup_{t \in [0,T]} \sup_{x \in [e^b, L]} |I^e_t(x) - I^e_0(x)| \leq \varepsilon^{-\mu'} \right) = 1. \tag{5.18}
\]

Fix arbitrary \(\mu' \in (1/4, 1/2)\). With \(\mathcal{I}_{\mu}(T, L)\) defined as in (5.8), combining (5.17) and (5.18), we arrive at

\[
\lim_{\varepsilon \to 0} \mathbf{P}\left( G^e_{\mu} \leq \sup_{(j,j',k) \in \mathcal{I}_{\mu}(T, L)} \{ \varepsilon^{1/4}D_{(j,j',k)} \} \right) = 1.
\]

From this and Lemma 5.2, we conclude \(G^e_{\mu} \to \mathbb{P} 0\). \(\square\)

**Proof of Proposition 2.7(c).** Fixing \(L, T \geq 0\) and \(b \in (0,1/4)\), by (5.1)–(5.2) and (2.21), it suffices to show

\[
\sup_{k \leq T / \varepsilon^b} \sup_{x \in [0,L]} |\varepsilon^{1/4}D_{(x^b)}(I^e_0(x + \varepsilon^b) - i_\varepsilon(x), t_k)| \to 0, \tag{5.19}
\]

as \(\varepsilon \to 0\). As shown in the proof of Proposition 2.7(b), this amounts to showing, for some \(\mu' \in (0,1/2)\), \(\lim_{\varepsilon \to 0} \mathbf{P}\left( \sup_{x \in [0,L]} |I^e_0(x + \varepsilon^b) - i_\varepsilon(x)| \leq \varepsilon^{-\mu'}, \forall x \in [0,L] \right) = 1\). By Markov’s inequality, with \(b < 1/4\), this in turn follows from

\[
\mathbf{E}\left( \sup_{x \in [0,L]} \varepsilon^{1/2-b} |I^e_0(x + \varepsilon^b) - i_\varepsilon(x)| \right)^2 \leq C. \tag{5.20}
\]

With \(I^e_0(x') = \langle Q^e_0, \mathbf{1}_{[0,x]} \rangle\) and \(i_\varepsilon(x) := \lfloor 2\varepsilon^{-1} x \rfloor\), we have \(\varepsilon^{1/2-b} |I^e_0(x + \varepsilon^b) - i_\varepsilon(x)| = \varepsilon^{1/2-b} m^e(x + \varepsilon^b) - 2 + \varepsilon^{1/2-b} r_{\varepsilon^b}^e\), for some \(|r^e| \leq 1\) and for \(m^e(x') := \langle Q^e_0, \mathbf{1}_{[0,x']} \rangle - 2\varepsilon^{-1/2} x'\). The process \(m^e(\cdot)\) is a martingale since \(Q^e_0 \sim \mathbb{P} + (2\varepsilon^{-1/2})\). With \(b \in (0,1/4)\), applying Doob’s \(L^2\) maximal inequality to \(m^e(\cdot)\), we obtain \(\mathbf{E}(\sup_{x \in [0,L+1]} \{ \varepsilon^{1/4} m^e(x) \})^2 \leq C\), thereby concluding (5.20). \(\square\)

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