ON DEGREES OF BIRATIONAL MAPPINGS

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ABSTRACT. We prove that the degrees of the iterates \( \deg(f^n) \) of a birational map satisfy \( \liminf(\deg(f^n)) < +\infty \) if and only if the sequence \( \deg(f^n) \) is bounded, and that the growth of \( \deg(f^n) \) cannot be arbitrarily slow, unless \( \deg(f^n) \) is bounded.

1. DEGREE SEQUENCES

Let \( k \) be a field. Consider a projective variety \( X \), a polarization \( H \) of \( X \) (given by hyperplane sections of \( X \) in some embedding \( X \subset \mathbb{P}^N \)), and a birational transformation \( f \) of \( X \), all defined over the field \( k \). Let \( k \) be the dimension of \( X \). The degree of \( f \) with respect to the polarization \( H \) is the integer

\[
\deg_H(f) = (f^*H) \cdot H^{k-1}
\]

where \( f^*H \) is the total transform of \( H \), and \( (f^*H) \cdot H^{k-1} \) is the intersection product of \( f^*H \) with \( k-1 \) copies of \( H \). The degree is a positive integer, which we shall simply denote by \( \deg(f) \), even if it depends on \( H \). When \( f \) is a birational transformation of the projective space \( \mathbb{P}^k \) and the polarization is given by \( O_{\mathbb{P}^k}(1) \) (i.e. by hyperplanes \( H \subset \mathbb{P}^k \)), then \( \deg(f) \) is the degree of the homogeneous polynomial formulas defining \( f \) in homogeneous coordinates.

The degrees are submultiplicative, in the following sense:

\[
\deg(f \circ g) \leq c_{X,H} \deg(f) \deg(g)
\]

for some positive constant \( c_{X,H} \) and for every pair of birational transformations. Also, if the polarization \( H \) is changed into another polarization \( H' \), there is a positive constant \( c \) which depends on \( X, H \) and \( H' \) but not on \( f \), such that

\[
\deg_H(f) \leq c \deg_{H'}(f)
\]

We refer to [11, 16, 18] for these fundamental properties.

The degree sequence of \( f \) is the sequence \( (\deg(f^n))_{n \geq 0} \); it plays an important role in the study of the dynamics and the geometry of \( f \). There are
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infinitely, but only countably many degree sequences (see [4, 19]); unfortunately, not much is known on these sequences when \( \dim(X) \geq 3 \) (see [3, 10] for \( \dim(X) = 2 \)). In this article, we obtain the following basic results.

- The sequence \( (\deg(f^n))_{n \geq 0} \) is bounded if and only if it is bounded along an infinite subsequence (see Theorems A and B in § 2 and § 3).
- If the sequence \( (\deg(f^n))_{n \geq 0} \) is unbounded, then its growth cannot be arbitrarily slow; for instance, \( \max_{0 \leq j \leq n} \deg(f^j) \) is asymptotically bounded from below by the inverse of the diagonal Ackermann function when \( X = \mathbb{P}^k \) (see Theorem C in § 4 for a better result).

We focus on birational transformations because a rational dominant transformation which is not birational has a topological degree \( \delta > 1 \), and this forces an exponential growth of the degrees: \( 1 < \delta^{1/k} \leq \lim_{n} (\deg(f^n)^{1/n}) \) where \( k = \dim(X) \) (see [11] and [6], pages 120–126).

2. AUTOMORPHISMS OF THE AFFINE SPACE

We start with the simpler case of automorphisms of the affine space; the goal of this section is to introduce a \( p \)-adic method to study degree sequences.

**Theorem A (Urech).**— Let \( f \) be an automorphism of the affine space \( \mathbb{A}^k \). If \( \deg(f^n) \) is bounded along an infinite subsequence, then it is bounded.

2.1. Urech’s proof. In [19], Urech proves a stronger result. Writing his proof in an intrinsic way, we extend it to affine varieties:

**Theorem 2.1.** Let \( X = \text{Spec} \, A \) be an irreducible affine variety of dimension \( k \) over the field \( k \). Let \( f : X \to X \) be an automorphism. If \( (\deg(f^n)) \) is unbounded there exists \( \alpha > 0 \) such that \( \# \{ n \geq 0 \mid \deg(f^n) \leq d \} \leq \alpha d^k \); in particular, \( \max_{0 \leq j \leq n} \deg(f^j) \) is bounded from below by \( (n/\alpha)^{1/k} \).

Here, the degree of \( f^n \), depends on the choice of a projective compactification \( Y \) of \( X \) and an ample line bundle \( L \) on \( Y \). However, by Equation (1.3), the statement of Theorem 2.1 does not depend on the choice of \( (Y, L) \). Since automorphisms of \( X \) always lift to its normalization, we may assume that \( X \) is normal. To prove this theorem, we shall introduce another equivalent notion of degree.

2.1.1. Degrees on affine varieties. Consider \( X \) as a subvariety \( X \subseteq \mathbb{A}^N \subseteq \mathbb{P}^N \). Let \( \bar{X} \) be the Zariski closure of \( X \) in \( \mathbb{P}^N \) and \( H_1 := \mathbb{P}^N \setminus \mathbb{A}^N \) be the hyperplane at infinity. Let \( \pi : Y \to \bar{X} \) be its normalization: \( Y \) is a normal projective
compactification of $X$. Since $\pi: Y \to \tilde{X}$ is finite, there exists $m \geq 1$ such (i) $H := \pi^*(mH|_{\tilde{X}})$ is very ample on $Y$ and (ii) $H$ is projectively normal on $Y$ i.e. for every $n \geq 0$, the morphism $(H^0(Y, H))^\otimes n \to H^0(Y, nH)$ is surjective.

If $P \in A$ is a regular function on $Y$, we denote by $(P) = (P)_0 - (P)_\infty$ the divisor defined by $P$ on $Y$, and we define

$$\Delta(P) = \min\{d \geq 0 \mid (P) + dH \geq 0 \text{ on } Y\},$$

$$A_d = \{P \in A \mid \Delta(P) \leq d\}, \quad (\forall d \geq 0).$$

Then $A = \bigcup_{d \geq 0} A_d$. Since $Y \setminus X$ is the support of $H$, we get an isomorphism $i_n : H^0(Y, nH) \to A_n \subseteq A$ for every $n \geq 0$. Thus, $A_1$ generates $A$ and the morphism $A_1^\otimes n \to A_n$ is surjective. Now we define

$$\deg^H(f) = \min\{m \geq 0 \mid \Delta(f^*P) \leq m \text{ for every } P \in A_1\}. \quad (2.3)$$

For every $P \in A_n$, we can write $P = \sum_{i=1}^l g_{i,j} \cdots g_{1,n}$ for some $g_{i,j} \in A_1$. We get $f^*P = \sum_{i=1}^l f^*g_{i,j} \cdots f^*g_{1,n} \in A_{\deg^H(f)n}$ and

$$\Delta(f^*P) \leq \deg^H(f)\Delta(P). \quad (2.4)$$

Since $A$ is generated by $A_1$, we get an embedding

$$\text{End}(A) \subseteq \text{Hom}_k(A_1, A) = \bigcup_{d \geq 1} \text{Hom}_k(A_1, A_d). \quad (2.5)$$

Set $\text{End}(A)_d = \text{End}(A) \cap \text{Hom}_k(A_1, A_d)$. For any automorphism $f: X \to X$, $\deg^H(f) \leq d$ if and only if $f \in \text{End}(A)_d$. By Riemann-Roch theorem, there exists $\gamma > 0$ such that $\dim A_n \leq \gamma d^k$, and this gives the upper bound

$$\dim \text{End}(A)_d \leq \text{Hom}_k(A_1, A_d) \leq (\gamma d^k) \dim A_1. \quad (2.6)$$

The following proposition, proved in the Appendix, shows that this new degree $\deg^H(f)$ is equivalent to the degree $\deg_H(f)$ introduced in Section 1.

**Proposition 2.2.** For every automorphism $f \in \text{Aut}(X)$ we have

$$\frac{1}{k} \deg^H(f) \leq \frac{1}{(H^k)} \deg_H(f) \leq \deg^H(f).$$

**2.1.2. Proof of Theorem 2.1.** By Proposition 2.2, the initial notion of degree can be replaced by $\deg^H$. Let $\gamma$ be as in Equation (2.2). Set $\ell = (\gamma d^k) \dim A_1 + 1$, and assume that $\deg^H(f^{n_i}) \leq d$ for some sequence of positive integers $n_1 < n_2 < \ldots < n_\ell$. Each $(f^*)^{n_i}$ is in $\text{End}(A)_d$ and, because $\ell > \dim \text{End}(A)_d$, there is a non-trivial linear relation between the $(f^*)^{n_i}$ in the vector space $\text{End}(A)_d$:

$$(f^*)^n = \sum_{m=1}^{n-1} a_m (f^*)^m \quad (2.7)$$
for some integer \( n \leq n_\ell \) and some coefficients \( a_m \in k \). Then, the subalgebra \( k[f^\ast] \subseteq \text{End}(A) \) is of finite dimension and \( k[f^\ast] \subseteq E_B \) for some \( B \geq 0 \). This shows that the sequence \( \langle \deg^H(f^n) \rangle_{N \geq 0} \) is bounded.

Thus, if we set \( \alpha = \gamma \dim A_1 \), and if the sequence \( \langle \deg^H(f^n) \rangle \) is not bounded, we obtain \#\{ \( n \geq 0 \) \: \( \deg^H(f^n) \leq d \} \leq \alpha d^k \). This proves the first assertion of the theorem; the second follows easily.

2.2. The \( p \)-adic argument. Let us give another proof of Theorem A when \( \text{char}(k) = 0 \), which will be generalized in § 3 for birational transformations.

2.2.1. Tate diffeomorphisms. Let \( p \) be a prime number. Let \( K \) be a field of characteristic 0 which is complete with respect to an absolute value \(| \cdot |\) satisfying \(|p| = 1/p\); such an absolute value is automatically ultrametric (see [13], Ex. 2 and 3, Chap. I.2). Let \( R = \{ x \in K ; |x| \leq 1 \} \) be the valuation ring of \( K \); in the vector space \( K^k \), the unit polydisk is the subset \( U = R^k \).

Fix a positive integer \( k \), and consider the ring \( R[x] = R[x_1, \ldots, x_k] \) of polynomial functions in \( k \) variables with coefficients in \( R \). For \( f \) in \( R[x] \), define the norm \( \| f \| \) to be the supremum of the absolute values of the coefficients of \( f \):

\[
\| f \| = \sup_{I} |a_I|
\]

where \( f = \sum_{I=(i_1, \ldots, i_k)} a_I x^I \). By definition, the Tate algebra \( R(x) \) is the completion of \( R[x] \) with respect to this norm. It coincides with the set of formal power series \( f = \sum_{I} a_I x^I \) converging (absolutely) on the closed polydisk \( R^k \). Moreover, the absolute convergence is equivalent to \( |a_I| \to 0 \) as \( \text{length}(I) \to \infty \).

Every element \( g \) in \( R(x)^k \) determines a Tate analytic map \( g : U \to U \).

For \( f \) and \( g \) in \( R(x) \) and \( c \) in \( R_+ \), the notation \( f \in p^c R(x) \) means \( \| f \| \leq |p|^c \) and the notation \( f \equiv g \mod(p^c) \) means \( \| f - g \| \leq |p|^c \); we then extend such notations component-wise to \( (R(x))^m \) for all \( m \geq 1 \).

For indeterminates \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_m) \), the composition \( R(y) \times R(x)^m \to R(x) \) is well defined, and coordinatewise we obtain

\[
R(y)^n \times R(x)^m \to R(x)^n.
\]

When \( m = n = k \), we get a semigroup \( R(x)^k \). The group of (Tate) analytic diffeomorphisms of \( U \) is the group of invertible elements in this semigroup; we denote it by \( \text{Diff}^\text{an}(U) \). Elements of \( \text{Diff}^\text{an}(U) \) are bijective transformations \( f : U \to U \) given by \( f(x) = (f_1, \ldots, f_k)(x) \) where each \( f_i \) is in \( R(x) \) with an inverse \( f^{-1} : U \to U \) that is also defined by power series in the Tate algebra.

The following result is due to Jason Bell and Bjorn Poonen (see [1, 17]).
Theorem 2.3. Let \( f \) be an element of \( R[x]^k \) with \( f \equiv \text{id} \mod (p^c) \) for some real number \( c > 1/(p-1) \). Then \( f \) is a Tate diffeomorphism of \( U = R^k \) and there exists a unique Tate analytic map \( \Phi: R \times U \to U \) such that

1. \( \Phi(n, x) = f^n(x) \) for all \( n \in \mathbb{Z} \);
2. \( \Phi(s+t, x) = \Phi(s, \Phi(t, x)) \) for all \( t, s \in R \).

2.2.2. Second proof of Theorem A. Denote by \( S \) the finite set of all the coefficients that appear in the polynomial formulas defining \( f \) and \( f^{-1} \). Let \( R_S \subseteq k \) be the ring generated by \( S \) over \( \mathbb{Z} \), and let \( K_S \) be its fraction field:

\[
\mathbb{Z} \subset R_S \subset K_S \subset k. \tag{2.10}
\]

Since \( \text{char}(k) = 0 \), there exists a prime \( p > 2 \) such that \( R_S \) embeds into \( \mathbb{Z}_p \) (see [15], §4 and 5, and [1], Lemma 3.1). We apply this embedding to the coefficients of \( f \) and get an automorphism of \( \mathbb{A}^k_p \) which is defined by polynomial formulas in \( \mathbb{Z}_p[x_1, \ldots, x_k] \); for simplicity, we keep the same notation \( f \) for this automorphism (embedding \( R_S \) in \( \mathbb{Z}_p \) does not change the value of the degrees \( \deg(f^n) \)). Since \( f \) and \( f^{-1} \) are polynomial automorphisms with coefficients in \( \mathbb{Z}_p \), they determine elements of \( \text{Diff}^{an}(U) \), the group of analytic diffeomorphisms of the polydisk \( U = \mathbb{Z}_p^k \).

Reducing the coefficients of \( f \) and \( f^{-1} \) modulo \( p^2 \mathbb{Z}_p \), one gets two permutations of the finite set \( \mathbb{A}^k_p(\mathbb{Z}_p/p^2\mathbb{Z}) \) (equivalently, \( f \) and \( f^{-1} \) permute the balls of \( U = \mathbb{Z}_p^k \) of radius \( p^{-2} \), and these balls are parametrized by \( \mathbb{A}^k_p(\mathbb{Z}_p/p^2\mathbb{Z}) \); see [7]). Thus, there exists a positive integer \( m \) such that \( f^m(0) \equiv 0 \mod (p^2) \). Taking some further iterate, we may also assume that the differential \( D_{f^m} \) satisfies \( D_{f^m} \equiv \text{id} \mod (p) \). We fix such an integer \( m \) and replace \( f \) by \( f^m \). The following lemma follows from the submultiplicativity of degrees (see Equation (1.2) in Section 1). It shows that replacing \( f \) by \( f^m \) is harmless if one wants to bound the degrees of the iterates of \( f \).

Lemma 2.4. If the sequence \( \deg(f^{mn}) \) is bounded for some \( m > 0 \), then the sequence \( \deg(f^n) \) is bounded too.

Denote by \( x = (x_1, \ldots, x_k) \) the coordinate system of \( \mathbb{A}^k \), and by \( m_p \) the multiplication by \( p \): \( m_p(x) = px \). Change \( f \) into \( g := m_p^{-1} \circ f \circ m_p \); then \( g \equiv \text{id} \mod (p) \) in the sense of Section 2.2.1. Since \( p \geq 3 \), Theorem 2.3 gives a Tate analytic flow \( \Phi: \mathbb{Z}_p \times \mathbb{A}^k_p(\mathbb{Z}_p) \to \mathbb{A}^k_p(\mathbb{Z}_p) \) which extends the action of \( g \): \( \Phi(n, x) = g^n(x) \) for every integer \( n \in \mathbb{Z} \). Since \( \Phi \) is analytic, one can write

\[
\Phi(t, x) = \sum_j A_j(t)x^j \tag{2.11}
\]
where $J$ runs over all multi-indices $(j_1, \ldots, j_k) \in (\mathbb{Z}_{\geq 0})^k$ and each $A_J$ defines a $p$-adic analytic curve $Z_p \rightarrow \mathbb{A}^k(\mathbb{Q}_p)$. By submultiplicativity of the degrees, there is a constant $C > 0$ such that $\deg(g^n) \leq CB^m$. Thus, we obtain $A_J(n_i) = 0$ for all indices $i$ and all multi-indices $J$ of length $|J| > CB^m$. The $A_J$ being analytic functions of $t \in Z_p$, the principle of isolated zeros implies that $A_J = 0$ in $Z_p \langle t \rangle$, $\forall J$ with $|J| > CB^m$. (2.12)

Thus, $\Phi(t, x)$ is a polynomial automorphism of degree $\leq CB^m$ for all $t \in Z_p$, and $g^n(x) = \Phi(n, x)$ has degree at most $CB^m$ for all $n$. By Lemma 2.4, this proves that $\deg(f^n)$ is a bounded sequence.

3. **BIRATIONAL TRANSFORMATIONS**

**Theorem B.**— Let $k$ be a field of characteristic 0. Let $X$ be a projective variety and $f: X \dashrightarrow X$ be a birational transformation of $X$, both defined over $k$. If the sequence $(\deg(f^n))_{n \geq 0}$ is not bounded, then it goes to $+\infty$ with $n$: 

$$\liminf_{n \rightarrow +\infty} \deg(f^n) = +\infty.$$ 

This extends Theorem A to birational transformations. With a theorem of Weil, we get: if $f$ is a birational transformation of the projective variety $X$, over an algebraically closed field of characteristic 0, and if the degrees of its iterates are bounded along an infinite subsequence $f^{n_i}$, then there exist a birational map $\psi: Y \dashrightarrow X$ and an integer $m > 0$ such that $f_Y := \psi^{-1} \circ f \circ \psi$ is in $\text{Aut}(Y)$, and $f_Y^m$ is in the connected component $\text{Aut}(Y)^0$ (see [5] and references therein).

Urech’s argument does not apply to this context; the basic obstruction is that rational transformations of $\mathbb{A}^k_k$ of degree $\leq B$ generate an infinite dimensional $k$-vector space for every $B \geq 1$ (the maps $z \in \mathbb{A}^1_k \mapsto (z - a)^{-1}$ with $a \in k$ are linearly independent); looking back at the proof in Section 2.1.2, the problem is that the field of rational functions on an affine variety $X$ is not finitely generated as a $k$-algebra. We shall adapt the $p$-adic method described in Section 2.2.2. In what follows, $f$ and $X$ are as in Theorem B; we assume, without loss of generality, that $k = \mathbb{C}$ and $X$ is smooth. We suppose that there is an infinite sequence of integers $n_1 < \ldots < n_j < \ldots$ and a number $B$ such that $\deg(f^{n_j}) \leq B$ for all $j$. We fix a finite subset $S \subset \mathbb{C}$ such that $X$, $f$ and $f^{-1}$ are defined by equations and formulas with coefficients in $S$, and we embed the ring $R_S \subset \mathbb{C}$ generated by $S$ in some $Z_p$, for some prime number $p > 2$. According to [7, Section 3], we may assume that $X$ and $f$ have good reduction modulo $p$. 
3.1. **The Hrushovski’s theorem and \( p \)-adic polydisks.** According to a theorem of Hrushovski (see [12]), there is a periodic point \( z_0 \) of \( f \) in \( X(\mathbf{F}) \) for some finite field extension \( \mathbf{F} \) of the residue field \( \mathbf{F}_p \), the orbit of which does not intersect the indeterminacy points of \( f \) and \( f^{-1} \). If \( \ell \) is the period of \( z_0 \), then \( f^\ell(z_0) = z_0 \) and \( Df^\ell_{z_0} \) is an element of the finite group \( \text{GL}((TX_{\mathbf{F}_q})_{z_0}) \simeq \text{GL}(k, \mathbf{F}_q) \). Thus, there is an integer \( m > 0 \) such that \( f^m(z_0) = z_0 \) and \( Df^m_{z_0} = \text{Id} \).

Replace \( f \) by its iterate \( g = f^m \). Then, \( g \) fixes \( z_0 \) in \( X(\mathbf{F}) \), \( g \) is an isomorphism in a neighborhood of \( z_0 \), and \( Dg_{z_0} = \text{Id} \). According to [2] and [7, Section 3], this implies that there is

- a finite extension \( K \) of \( \mathbf{Q}_p \), with valuation ring \( R \subset K \);

- a point \( z \) in \( X(K) \) and a polydisk \( V_z \simeq R^k \subset X(K) \) which is \( g \)-invariant and such that \( g|_{V_z} \equiv \text{Id} \mod (p) \) (in the coordinate system \((x_1, \ldots, x_k)\) of the polydisk).

When the point \( z_0 \) is in \( X(\mathbf{F}_p) \) and is the reduction of a point \( z \in X(\mathbf{Z}_p) \), the polydisk \( V_z \) is the set of points \( w \in X(\mathbf{Z}_p) \) with \( |z - w| < 1 \); one identifies this polydisk to \( U = (\mathbf{Z}_p)^k \) via some \( p \)-adic analytic diffeomorphism \( \varphi: U \to V_z \); changing \( \varphi \) into \( \varphi \circ m_p \) if necessary, we obtain \( g|_{V_z} \equiv \text{Id} \mod (p) \) (see Section 2.2.2 and [7, Section 3.2.1]). In full generality, a finite extension \( K \) of \( \mathbf{Q}_p \) is needed because \( z_0 \) is a point in \( X(\mathbf{F}) \) for some extension \( \mathbf{F} \) of \( \mathbf{F}_p \).

3.2. **Controlling the degrees.** As in Section 2.2.1, denote by \( U \) the polydisk \( R^k \simeq V_z \); thus, \( U \) is viewed as the polydisk \( R^k \) and also as a subset of \( X(K) \). Applying Theorem 2.3 to \( g \), we obtain a \( p \)-adic analytic flow

\[
\Phi: R \times U \to U, \quad (t, x) \mapsto \Phi(t, x)
\]

(3.1)

such that \( \Phi(n, x) = g^n(x) \) for every integer \( n \). In other words, the action of \( g \) on \( U \) extends to an analytic action of the additive compact group \((R, +)\).

Let \( \pi_1: X \times X \to X \) denote the projection onto the first factor. Denote by \( \text{Bir}_D(X) \) the set of birational transformations of \( X \) of degree \( D \); once birational transformations are identified to their graphs, this set becomes naturally a finite union of irreducible, locally closed algebraic subsets in the Hilbert scheme of \( X \times X \) (see [5], Section 2.2, and references therein). Taking a subsequence, there is a positive integer \( D \), an irreducible component \( B_D \) of \( \text{Bir}_D(X) \), and a strictly increasing, infinite sequence of integers \((n_j)\) such that

\[
g^{n_j} \in B_D
\]

(3.2)

for all \( j \). Denote by \( \overline{B_D} \) the Zariski closure of \( B_D \) in the Hilbert scheme of \( X \times X \). To every element \( h \in \overline{B_D} \) corresponds a unique algebraic subset \( G_h \) of
Given any subset $t \subseteq h$ hence, Lemma 3.1. There is a finite subset $E \subset U \subset X(K)$ with the following property. Given any subset $E$ of $U \times U$ with $\pi_1(E) = E$, there is at most one element $h \in \overline{B_D}$ such that $\bar{E} \subset G_h$.

Fix such a set $E$, and order it to get a finite list $E = (x_1, \ldots, x_{\ell_0})$ of elements of $U$. Let $E' = (x_1, \ldots, x_{\ell_0}, x_{\ell_0+1}, \ldots, x_\ell)$ be any list of elements of $U$ which extends $E$. For every element $h$ in $\overline{B_D}$, the variety $G_h$ determines a correspondence $G_h \subset X \times X$. The subset of elements $(h, (x_i, y_i)_{1 \leq i \leq \ell})$ in $\overline{B_D} \times (X \times X)^{\ell}$ defined by the incidence relation

$$(3.3) \quad (x_i, y_i) \in G_h$$

for every $1 \leq i \leq \ell$ is an algebraic subset of $\overline{B_D} \times (X \times X)^{\ell}$. Add one constraint, namely that the first projection $(x_i)_{1 \leq i \leq \ell}$ coincides with $E'$, and project the resulting subset on $(X \times X)^{\ell}$: we get a subset $G(E')$ of $(X \times X)^{\ell}$. Then, define a $p$-adic analytic curve $\Lambda: R \to (X \times X)^{\ell}$ by

$$(3.4) \quad \Lambda(t) = (x_i, \Phi(t, x_i))_{1 \leq i \leq \ell}.$$ 

If $t = n_j$, $g^{n_j}$ is an element of $B_D$ and $\Lambda(n_j)$ is contained in the graph of $g^{n_j}$; hence, $\Lambda(n_j)$ is an element of $G(E')$. By the principle of isolated zeros, the analytic curve $t \mapsto \Lambda(t) \subset (X \times X)^{\ell}$ is contained in $G(E')$ for all $t \in R$. Thus, for every $t$ there is an element $h_t \in \overline{B_D}$ such that $\Lambda(t)$ is contained in the subset $G_{h_t}^{\ell}$ of $(X \times X)^{\ell}$. From the choice of $E$ and the inclusion $E \subset E'$, we know that $h_t$ does not depend on $E'$. Thus, the graph of $\Phi(t, \cdot)$ coincides with the intersection of $G_{h_t}$ with $U \times U$. This implies that the graph of $g^n(\cdot) = \Phi(n, \cdot)$ coincides with $G_{h_n}$, and that the degree of $g^n$ is at most $D$ for all values of $n$.

4. Lower bounds on degree growth

We now prove that the growth of $(\deg(f^n))$ can not be arbitrarily slow unless $(\deg(f^n))$ is bounded. For simplicity, we focus on birational transformations of the projective space; there is no restriction on the characteristic of $k$.

4.1. A family of integer sequences. Fix two positive integers $k$ and $d$; $k$ will be the dimension of $\mathbb{P}_k$, and $d$ will be the degree of $f: \mathbb{P}^k \to \mathbb{P}^k$. Set

$$m = (d-1)(k+1).$$

(4.1)
Then, consider an auxiliary integer $D \geq 1$, which will play the role of the degree of an effective divisor in the next paragraphs, and define

$$q = (dD + 1)^m.$$ (4.2)

Thus, $q$ depends on $k$, $d$ and $D$ because $m$ depends on $k$ and $d$. Then, set

$$a_0 = \binom{k + D}{k} - 1, \quad b_0 = 1, \quad c_0 = D + 1.$$ (4.3)

Starting from the triple $(a_0, b_0, c_0)$, we define a sequence $((a_j, b_j, c_j))_{j \geq 0}$ inductively by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j, b_j - 1, q c_j^2)$$ (4.4)

if $b_j \geq 2$, and by

$$(a_{j+1}, b_{j+1}, c_{j+1}) = (a_j - 1, q c_j^2, q c_j^2) = (a_j - 1, c_{j+1}, c_{j+1})$$ (4.5)

if $b_j = 1$. By construction, $(a_1, b_1, c_1) = (a_0 - 1, q c_0^2, q c_0^2)$.

Define $\Phi : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$\Phi(c) = q c^2.$$ (4.6)

**Lemma 4.1.** Define the sequence of integers $(F_i)_{i \geq 1}$ recursively by $F_1 = q(D + 1)^2$ and $F_{i+1} = \Phi^{F_i}(F_i)$ for $i \geq 1$ (where $\Phi^{F_i}$ is the $F_i$-iterate of $\Phi$). Then

$$(a_1 + F_1 + \cdots + F_i, b_1 + F_1 + \cdots + F_i, c_1 + F_1 + \cdots + F_i) = (a_0 - i - 1, F_{i+1}, F_{i+1}).$$

The proof is straightforward. Now, define $S : \mathbb{Z}^+ \to \mathbb{Z}^+$ as the sum

$$S(j) = 1 + F_1 + F_2 + \cdots + F_j$$ (4.7)

for all $j \geq 1$; it is increasing and goes to $+\infty$ extremely fast with $j$. Then, set

$$\chi_{d,k}(n) = \max \left\{ D \geq 0 \mid S\left( \binom{k + D}{k} - 2 \right) < n \right\}. $$ (4.8)

**Lemma 4.2.** The function $\chi_{d,k} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is non-decreasing and goes to $+\infty$ with $n$.

**Remark 4.3.** The function $S$ is primitive recursive (see [9], Chapters 3 and 13). In other words, $S$ is obtained from the basic functions (the zero function, the successor $s(x) = x + 1$, and the projections $(x_i)_{1 \leq i \leq m} \to x_i$) by a finite sequence of compositions and recursions. Equivalently, there is a program computing $S$, all of whose instructions are limited to (1) the zero initialization $V \leftarrow 0$, (2) the increment $V \leftarrow V + 1$, (3) the assignment $V \leftarrow V'$, and (4) loops of definite length. Writing such a program is an easy exercise. Now, consider the diagonal Ackermann function $A(n)$ (see [9], Section 13.3). It grows asymptotically
faster than any primitive recursive function; hence, the inverse of the Ackermann diagonal function $\alpha(n) = \max\{D \geq 0 \mid \text{Ack}(D) \leq n\}$ is, asymptotically, a lower bound for $\chi_{d,k}(n)$. Showing that $\chi_{d,k}$ is in the $L_6$ hierarchy of [9], Chapter 13, one gets an asymptotic lower bound by the inverse of the function $f_7$ of [9], independent of the values of $d$ and $k$.

4.2. Statement of the lower bound. We can now state the result that will be proved in the next paragraphs.

Theorem C.– Let $f$ be a birational transformation of the complex projective space $\mathbb{P}_k^d$ of degree $d$. If the sequence $(\max_{0 \leq j \leq n}(\deg(f^j)))_{n \geq 0}$ is unbounded, then it is bounded from below by the sequence of integers $(\chi_{d,k}(n))_{n \geq 0}$.

Remark 4.4. There are infinitely, but only countably many sequences of degrees $(\deg(f^n))_{n \geq 0}$ (see [4, 19]). Consider the countably many sequences

$$\left(\max_{0 \leq j \leq n}(\deg(f^j))\right)_{n \geq 0} \quad (4.9)$$

restricted to the family of birational maps for which $(\deg(f^n))$ is unbounded. We get a countable family of non-decreasing, unbounded sequences of integers. Let $(u_i)_{i \in \mathbb{Z}_{\geq 0}}$ be any countable family of such sequences of integers $(u_i(n))$. Define $w(n)$ as follows. First, set $v_j = \min\{u_0, u_1, \ldots, u_j\}$; this defines a new family of sequences, with the same limit $+\infty$, but now $v_j(n) \geq v_{j+1}(n)$ for every pair $(j, n)$. Then, set $m_0 = 0$, and define $m_{n+1}$ recursively to be the first positive integer such that $v_{n+1}(m_{n+1}) \geq v_n(m_n) + 1$. We have $m_{n+1} \geq m_n + 1$ for all $n \in \mathbb{Z}_{\geq 0}$. Set $w(n) := v_{r_n}(m_{r_n})$ where $r_n$ is the unique non-negative integer satisfying $m_{r_n} \leq n \leq m_{r_n} + 1$. By construction, $w(n)$ goes to $+\infty$ with $n$ and $u_i(n)$ is asymptotically bounded from below by $w(n)$.

In Theorem C, the result is more explicit. Firstly, the lower bound is explicitly given by the sequence $(\chi_{d,k}(n))_{n \geq 0}$. Secondly, the lower bound is not asymptotic: it works for every value of $n$. In particular, if $\deg(f^j) < \chi_{d,k}(n)$ for $0 \leq j \leq n$ and $\deg(f) = d$, then the sequence $(\deg(f^n))$ is bounded.

4.3. Divisors and strict transforms. To prove Theorem C, we consider the action of $f$ by strict transform on effective divisors. As above, $d = \deg(f)$ and $m = (d-1)(k+1)$ (see Section 4.1).
4.3.1. **Exceptional locus.** Let \( X \) be a smooth projective variety and \( \pi_1, \pi_2 : X \to \mathbb{P}^k \) be two birational morphisms such that \( f = \pi_2 \circ \pi_1^{-1} \); then, consider the exceptional locus \( \text{Exc}(\pi_2) \subset X \), project it by \( \pi_1 \) into \( \mathbb{P}^k \), and list its irreducible components of codimension 1: we obtain a finite number

\[
E_1, \ldots, E_{m(f)}
\]

of irreducible hypersurfaces, contained in the zero locus of the jacobian determinant of \( f \). Since this critical locus has degree \( m \), we obtain:

\[
m(f) \leq m, \quad \text{and} \quad \deg(E_i) \leq m \quad (\forall i \geq 1).
\]

4.3.2. **Effective divisors.** Denote by \( M \) the semigroup of effective divisors of \( \mathbb{P}^k \). There is a partial ordering \( \leq \) on \( M \), which is defined by \( E \leq E' \) if and only if the divisor \( E' - E \) is effective.

We denote by \( \text{deg} : M \to \mathbb{Z}_{\geq 0} \) the degree function. For every degree \( D \geq 0 \), we denote by \( M_D \) the set \( \mathbb{P}(H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(D))) \) of effective divisors of degree \( D \); thus, \( M \) is the disjoint union of all the \( M_D \), and each of these components will be endowed with the Zariski topology of \( \mathbb{P}(H^0(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(D))) \). The dimension of \( M_D \) is equal to the integer \( a_0 = a_0(D, k) \) from Section 4.1:

\[
\dim(M_D) = \binom{k + D}{k} - 1.
\]

Let \( G \subset M \) be the semigroup generated by the \( E_i \):

\[
G = \bigoplus_{i=1}^{m(f)} \mathbb{Z}_{\geq 0} E_i.
\]

The elements of \( G \) are the effective divisors which are supported by the exceptional locus of \( f \). For every \( E \in G \), there is a translation operator \( T_E : M \to M \), defined by \( T_E : E' \mapsto E + E' \); it restricts to a linear projective embedding of the projective space \( M_D \) into the projective space \( M_{D + \text{deg}(E)} \). We define

\[
M_D^\circ = M_D \setminus \bigcup_{E \in G \setminus \{0\}, \text{deg}(E) \leq D} T_E(M_{D - \text{deg}(E)}).
\]

Thus, \( M_D^\circ \) is the complement in \( M_D \) of finitely many proper linear projective subspaces. Also, \( M_D^\circ_0 = M_D^\circ \) is a point and \( M_D^\circ_1 \) is obtained from \( M_D^\circ_0 = (\mathbb{P}^k)^\vee \) by removing finitely many points, corresponding to the \( E_i \) of degree 1 (the hyperplanes contracted by \( f \)). Set \( M^\circ = \bigcup_{D \geq 0} M_D^\circ \). This is the set of effective divisors without any component in the exceptional locus of \( f \). The inclusion of \( M^\circ \) in \( M \) will be denoted by \( \iota : M^\circ \to M \). There is a natural projection \( \pi_G : M \to G \); namely, \( \pi_G(E) \) is the maximal element such that \( E - \pi_G(E) \) is effective.
We denote by $\pi: M \to M^o$ the projection $\pi = \text{id} - \pi_G$; this homomorphism removes the part of an effective divisor $E$ which is supported on the exceptional locus of $f$.

**Remark 4.5.** The restriction of the map $\pi$ to the projective space $M_D$ is piecewise linear, in the following sense. Consider the subsets $U_{E,D}$ of $M_D$ which are defined for every $E \in G$ with $\deg(E) \leq D$ by

$$U_{E,D} = T_E(M_D - \deg(E)) \setminus \bigcup_{E' > E, E' \in G, \deg(E') \leq D} T_{E'}(M_D - \deg(E')).$$

They define a stratification of $M_D$ by (open subsets of) linear subspaces, and $\pi$ coincides with the linear map inverse of $T_E$ on each $U_{E,D}$. Moreover, $\pi_\circ Z$ is closed for any closed subset $Z \subseteq M_D$.

We say that a scheme theoretic point $x \in M$ (resp. $M^o$) is irreducible if the divisor of $P_k$ corresponding to $x$ is irreducible. In other words, $x$ is irreducible, if a general closed point $y \in \{x\} \subseteq M$ is irreducible.

**4.3.3. Strict transform.** First, we consider the total transform $f^*: M \to M$, which is defined by $f^*(E) = (\pi_1)_1\pi_2^*(E)$ for every divisor $E \in M$. This is a homomorphism of semigroups; it is injective on non-closed irreducible points. Let $[x_0, \ldots, x_k]$ be homogeneous coordinates on $\mathbb{P}^k$. If $E$ is defined by the homogeneous equation $P = 0$, then $f^*(E)$ is defined by $P \circ f = 0$; thus, $f^*$ induces a linear projective embedding of $M_D$ into $M_{D'}$ for every $D$.

Then, we denote by $f^\circ: M^o \to M^o$ the strict transform. It is defined by

$$f^\circ(E) = (\pi_\circ \circ f^* \circ \iota)(E).$$

This is a homomorphism of semigroups. If $x \in M$ is an irreducible point, its total transform $f^*(x)$ is not necessarily irreducible, but $f^\circ(x)$ is irreducible.

In general, $(f^\circ)^n \neq (f^n)^\circ$, but for non-closed irreducible point $x \in M$, we have $(f^\circ)^n(x) = (f^n)^\circ(x)$ for $n \geq 0$. Indeed, a non-closed irreducible point $x \in M$ can be viewed as an irreducible hypersurface on $X$ which is defined over some transcendental extension of $k$, but not over $k$. Then $f^\circ(x)$ is the unique irreducible component $E$ of $f^*(x)$, on which $f|_E$ is birational to its image. (Note that when $k$ is uncountable, one can also work with very general points of $M_D$ for every $D \geq 1$, instead of irreducible, non-closed points).

**4.4. Proof of Theorem C.** Let $\eta$ be the generic point of $M_1^\circ$ ($\eta$ corresponds to a generic hyperplane of $\mathbb{P}^k_\eta$). Note that $\eta$ is non-closed and irreducible. The
degree of $f^*(\eta)$ is equal to the degree of $f$, and since $\eta$ is generic, $f^*(\eta)$ coincides with $f^\circ(\eta)$. Thus, $\deg(f) = \deg(f^\circ(\eta))$ and more generally
\[
\deg(f^n) = \deg((f^\circ)^n\eta) \quad (\forall n \geq 1).
\] (4.16)

Fix an integer $D \geq 0$. Write $M^\circ_{\leq D}$ for the disjoint union of the $M^\circ_{D'}$ with $D' \leq D$, and define recursively $Z_D(0) = M^\circ_{\leq D}$ and
\[
Z_D(i + 1) = \{ E \in Z_D(i) \mid f^\circ(E) \in Z_D(i) \}
\] (4.17)
for $i \geq 0$. A divisor $E \in M^\circ_{\leq D}$ is in $Z_D(i)$ if its strict transform $f^\circ(E)$ is of degree $\leq D$, and $f^\circ(f^\circ(E))$ is also of degree $\leq D$, up to $(f^\circ)^i(E)$ which is also of degree at most $D$.

Let us describe $Z_D(i + 1)$ more precisely. For each $i$, and each $E \in G$ of degree $\deg(E) \leq dD$ consider the subset $T_E(\{Z_D(i)\}) \cap M_{dD}$; this is a subset of $M_{dD}$ which is made of divisors $W$ such that $\pi_0(W)$ is contained in $Z_D(i)$, and the union of all these subsets when $E$ varies is exactly the set of points $W$ in $M_{dD}$ with a projection $\pi_0(W)$ in $Z_D(i)$. Thus, we consider
\[
(f^*)^{-1}(T_E(\{Z_D(i)\})) = \{ V \in M_{\leq D} \mid f^*(V) \in T_E(\{Z_D(i)\}) \}.
\] (4.18)
These sets are closed subsets of $M_{\leq D}$, and
\[
Z_D(i + 1) = Z_D(i) \bigcap \bigcup_{E \in G, \deg(E) \leq dD} \pi_0\left((f^*)^{-1}(T_E(\{Z_D(i)\}))\right).
\] (4.19)
Since $Z_D(0)$ is closed in $M^\circ_{\leq D}$ and $\pi_0$ is closed on $M_{\leq D}$, by induction, $Z_D(i)$ is closed for all $i \geq 0$. The subsets $Z_D(i)$ form a decreasing sequence of Zariski closed subsets (in the disjoint union $M^\circ_{\leq D}$ of the $M^\circ_{D'}$, $D' \leq D$). The strict transform $f^\circ$ maps $Z_D(i + 1)$ into $Z_D(i)$. By Noetherianity, there exists a minimal integer $\ell(D) \geq 0$ such that
\[
Z_D(\ell(D)) = \bigcap_{i \geq 0} Z_D(i);
\] (4.20)
we denote this subset by $Z_D(\infty) = Z_D(\ell(D))$. By construction, $Z_D(\infty)$ is stable under the operator $f^\circ$; more precisely, $f^\circ(Z_D(\infty)) = Z_D(\infty) = (f^\circ)^{-1}(Z_D(\infty))$.

Let $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a lower bound for the inverse function of $\ell$:
\[
\ell(\tau(n)) \leq n \quad (\forall n \geq 0).
\] (4.21)
Assume that $\max\{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)$ for some $n_0 \geq 1$. Then $\deg((f^\circ)^i(\eta)) \leq \tau(n_0)$ for every integer $i$ between 0 and $n_0$; this implies that $\eta$ is in the set $Z_{\tau(n_0)}(\ell(\tau(n_0))) = Z_{\tau(n_0)}(\infty)$, so that the degree of $(f^\circ)^m(\eta)$ is
bounded from above by \( \tau(n_0) \) for all \( m \geq 0 \). From Equation (4.16) we deduce that the sequence \((\deg(f^n))_{m \geq 0}\) is bounded. This proves the following lemma.

**Lemma 4.6.** Let \( \tau \) be a lower bound for the inverse function of \( \ell \). If
\[
\max \{\deg(f^m) \mid 0 \leq m \leq n_0\} \leq \tau(n_0)
\]
for some \( n_0 \geq 1 \), then the sequence \((\deg(f^n))_{n \geq 0}\) is bounded by \( \tau(n_0) \).

So, to conclude, we need to compare \( \ell : \mathbb{Z}_{\geq 0} \to \mathbb{Z}^+ \) to the function \( S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}^+ \) of paragraph 4.1 (recall that \( S \) depends on the parameters \( k = \dim(\mathbb{P}^k) \) and \( d = \deg(f) \) and that \( \ell \) depends on \( f \)). Now, write \( Z'_D(i) = Z_D(i) \setminus Z_D(\infty) \), and note that it is a strictly decreasing sequence of open subsets of \( Z_D(i) \) with \( Z'_D(j) = \emptyset \) for all \( j \geq \ell(D) \). We shall say that a closed subset of \( M_{\leq D} \setminus Z_D(\infty) \) for the Zariski topology is \textbf{piecewise linear} if all its irreducible components are equal to the intersection of \( M_{\leq D} \setminus Z_D(\infty) \) with a linear projective subspace of some \( M_D \), \( D' \leq D \). We note that the intersection of two irreducible linear projective subspaces is still an irreducible linear projective subspace.

Let \( \text{Lin}(a,b,c) \) be the family of closed piecewise linear subsets of \( M_{\leq D} \setminus Z_D(\infty) \) of dimension \( a \), with at most \( c \) irreducible components, and at most \( b \) irreducible components of maximal dimension \( a \). Then,
\begin{enumerate}
  \item \( Z'_D(i+1) = \{ F \in Z'_D(i) \mid f^*(F) \in Z'_D(i) \} = \pi_c(f^*Z'_D(i) \cap \bigcup E T_E(Z'_D(i))) \), where \( E \) runs over the elements of \( G \) of degree \( \deg(E) \leq dD \);
  \item in this union, each irreducible component of \( T_E(Z'_D(i)) \) is piecewise linear.
\end{enumerate}
Recall that \( q = (dD + 1)^m \) (see Section 4.1). If \( Z \) is any closed piecewise linear subset of \( M_{\leq D} \setminus Z_D(\infty) \) that contains exactly \( c \) irreducible components, the set
\[
\pi_c(f^*Z \bigcap \bigcup_{E \in G, \deg(E) \leq dD} T_E(Z)) = \bigcup_{E \in G, \deg(E) \leq dD} \pi_c(f^*Z \bigcap T_E(Z)) = \bigcup_{E \in G, \deg(E) \leq dD} T_E^{-1}(T_E(f^*Z \bigcap T_E(Z))
\]
has at most \( qc^2 = (dD + 1)^m c^2 \) irreducible components (this is a crude estimate: \( f^*Z \bigcap T_E(Z) \) has at most \( c^2 \) irreducible components, \( T_E^{-1}(T_E(f^*Z \bigcap T_E(Z)) \) is injective and the factor \((dD + 1)^m \) comes from the fact that \( G \) contains at most \((dD + 1)^m \) elements of degree \( \leq dD \). Let us now use that the sequence \( Z'_D(i) \) decreases strictly as \( i \) varies from 0 to \( \ell(D) \), with \( Z'_D(\ell(D)) = \emptyset \). If \( 0 \leq i \leq \ell(D) - 1 \), and if \( Z'_D(i) \) is contained in \( \text{Lin}(a,b,c) \), we obtain
\begin{enumerate}
  \item if \( b \geq 2 \), then \( Z'_D(i+1) \) is contained in \( \text{Lin}(a,b-1,qc^2) \);
(2) if \( b = 1 \), then \( Z^2_f(i + 1) \) is contained in \( \text{Lin}(a - 1, qc^2, qc^2) \).

This shows that

\[
\ell(D) \leq S\left( \frac{k + D}{k} \right) - 2 + 1
\]

(4.22)

where \( S \) is the function introduced in the Equation (4.7) of Section 4.1. Since \( \chi_{d,k} \) satisfies \( \ell(\chi_{d,k}(n)) \leq n \) for every \( n \geq 1 \), the conclusion follows.

5. Appendix: Proof of Proposition 2.2

We keep the notation from Section 2.1.1. Let \( f \) be an automorphism of \( X \). There exist a normal projective irreducible variety \( Z \) and two birational morphisms \( \pi_1: Z \to Y \) and \( \pi_2: Z \to Y \) such that \( \pi_1 \) and \( \pi_2 \) are isomorphisms over \( X \), and \( f = \pi_2 \circ \pi_1^{-1} \).

**Lemma 5.1.** We have \( \Delta(f^*P) \leq k(H^k)^{-1}\Delta(P)\deg_H(f) \) for every \( P \in A \).

**Proof of Lemma 5.1.** By Siu’s inequality (see [14] Theorem 2.2.15, and [8] Theorem 1), we get

\[
\pi_2^*H \leq \frac{k(\pi_2^*H \cdot (\pi_2^*H)^{k-1})}{((\pi_2^*H)^k)} \pi_1^*H = \frac{k\deg_H(f)}{(H^k)} \pi_1^*H. \tag{5.1}
\]

Since \( (P) + \Delta(P)H \geq 0 \) we have \( (\pi_2^*P) + \Delta(P)\pi_2^*H \geq 0 \). It follows that

\[
(\pi_2^*P) + \frac{\Delta(P)k\deg_H(f)}{(H^k)} \pi_1^*H \geq 0. \tag{5.2}
\]

Since \( (\pi_1)_* \circ (\pi_1)^* = \text{Id} \) we obtain \( (f^*P) + (k\Delta(P)(H^k)^{-1}\deg_H(f))H \geq 0 \). This implies \( \Delta(f^*P) \leq k(H^k)^{-1}\Delta(P)\deg_H(f) \). \( \square \)

Lemma 5.1 shows that \( \deg_H(f) \leq k(H^k)^{-1}\deg_H(f) \). We now prove the reverse direction: \( \deg_H(f) \leq (H^k)^{-1}\deg_H(f) \).

Since \( H \) is very ample, Bertini’s theorem gives an irreducible divisor \( D \in |H| \) such that \( \pi_2^*(E) \nsubseteq D \) for every prime divisor \( E \) of \( Z \). \( \pi_2^*D \) is equal to the strict transform \( \pi_2^*D \). By definition, \( D = (P) + H \) for some \( P \in A \). Thus, \( (\pi_1)_* \pi_2^*D \) is linearly equivalent to \( (\pi_1)_* \pi_2^*D = (\pi_1)_* \pi_2^*D \), and this irreducible divisor \( (\pi_1)_* \pi_2^*D \) is the closure \( D_{f,P} \) of \( \{ f^*P = 0 \} \subseteq X \) in \( Y \). Writing \( (f^*P) = D_{f,P} - F \) where \( F \) is supported on \( Y \setminus X \) we also get that \( (\pi_1)_* \pi_2^*H \) is linearly equivalent to \( F \). Since \( \Delta(f^*P) \leq \deg_H(f)\Delta(P) = \deg_H(f) \), the definition of \( \Delta \) gives

\[
D_{f,P} - F + \deg_H(f)H = (f^*P) + \deg_H(f)H \geq 0. \tag{5.3}
\]

Thus, \( F \leq \deg_H(f)H \) because \( D_{f,P} \) is irreducible and is not supported on \( Y \setminus X \). Altogether, this gives \( \deg_H(f) = ((\pi_1)_* \pi_2^*H \cdot H^{k-1}) = (F \cdot H^{k-1}) \leq \deg_H(f)(H^k) \).
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