Baryon Asymmetry of the Universe without Boltzmann or Kadanoff-Baym

Jean-Sébastien Gagnon
École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland and
Technische Universität Darmstadt, Darmstadt, Germany

Mikhail Shaposhnikov
École Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

(Dated: December 7, 2010)

Abstract

We present a formalism that allows the computation of the baryon asymmetry of the universe from first principles of statistical physics and quantum field theory that is applicable to certain types of beyond the Standard Model physics (such as the neutrino Minimal Standard Model – νMSM) and does not require the solution of Boltzmann or Kadanoff-Baym equations. The formalism works if a thermal bath of Standard Model particles is very weakly coupled to a new sector (sterile neutrinos in the νMSM case) that is out-of-equilibrium. The key point that allows a computation without kinetic equations is that the number of sterile neutrinos produced during the relevant cosmological period remains small. In such a case, it is possible to expand the formal solution of the von Neumann equation perturbatively and obtain a master formula for the lepton asymmetry expressed in terms of non-equilibrium Wightman functions. The master formula neatly separates CP-violating contributions from finite temperature correlation functions and satisfies all three Sakharov conditions. These correlation functions can then be evaluated perturbatively; the validity of the perturbative expansion depends on the parameters of the model considered. Here we choose a toy model (containing only two active and two sterile neutrinos) to illustrate the use of the formalism, but it could be applied to other models.

*Electronic address: jean-sebastien.gagnon@physik.tu-darmstadt.de
†Electronic address: mikhail.shaposhnikov@epfl.ch
The baryon asymmetry of the universe (BAU) is a very important quantity in cosmology. It is quantified by the (dimensionless) baryon-to-photon ratio $\eta$. This ratio has been measured very precisely by the WMAP collaboration \[1\] and is given by $\eta = (n_b - n_{\bar{b}})/n_\gamma = (6.1 \pm 0.2) \times 10^{-10}$. Here $n_b$ ($n_{\bar{b}}$) is the number density of baryons (anti-baryons) and $n_\gamma$ is the number density of photons. If this excess of baryons over anti-baryons is not simply an initial condition of our universe, then it is the goal of any particle physics models to explain this asymmetry.

In principle, the Standard Model possesses all the necessary ingredients to produce an asymmetry (i.e. it fulfills the three Sakharov conditions \[2\]): baryon number violating processes are mediated by sphalerons, $CP$-violation is hidden in the Cabibbo Kobayashi Maskawa (CKM) matrix and a first order phase transition would provide the necessary out-of-equilibrium condition. On the other hand, it has been shown that in the Standard Model there is no first order electroweak phase transition for Higgs masses above 80 GeV (it is a smooth crossover) \[3\]. Since the lower bound on the Higgs mass from LEP is 114 GeV, the out-of-equilibrium condition necessary for baryogenesis is not satisfied and no asymmetry is produced. In extensions of the Standard Model, this conclusion may change; thus new physics beyond the Standard Model is necessary to explain the BAU.

There exists many mechanisms/models that could explain the BAU. The most well-known are Grand Unified Theories (GUT) baryogenesis, electroweak baryogenesis in extensions of the Standard Model, leptogenesis and the Affleck-Dine mechanism. See Refs. \[4-10\] for reviews. A common feature of these models is that they require heavy degrees of freedom that are hard to detect with present accelerators (the LHC may discover some of these particles in the near future).

An alternative model is the neutrino Minimal Standard Model ($\nu$MSM) \[11-19\]; it is a minimal extension of the Standard Model with three sterile right-handed neutrinos with masses below the electroweak scale. This model has interesting features. For instance, it could explain simultaneously three shortcomings of the Standard Model (namely neutrino oscillations, dark matter and BAU). Also, since it contains no very heavy degree of freedom, it could in principle be tested experimentally with present facilities \[20, 21\].

For baryogenesis to occur some degrees of freedom must be out of thermal equilibrium.
in order to fulfill the third Sakharov condition. The usual approach to baryon excess computations is to use Boltzmann equations. Various assumptions are used in the derivation of Boltzmann equations, one of them being that the coherence length of the processes involved must be much smaller than the mean free path of the particles. As long as these assumptions are satisfied, Boltzmann equations can describe systems that are arbitrarily out-of-equilibrium. But it is shown in Refs. [22, 23] that coherence effects are important in baryogenesis, and thus a more refined quantum mechanical treatment is needed.

Out-of-equilibrium quantum field theory is a notoriously difficult subject. Significant progress has been made recently in formal aspects (e.g. [24–26]) but applications to realistic baryogenesis computations are still lacking (although see the recent progress in Refs. [27–32]).

Our ultimate goal is to compute the BAU in the phenomenologically interesting $\nu$MSM. Previous calculations of the BAU in the $\nu$MSM [12] show that the produced asymmetry is in the right range. The production of a baryon asymmetry in the $\nu$MSM happens via coherent active-sterile neutrino oscillations and requires appropriate kinetic equations for its treatment. These calculations are performed using kinetic equations of the form [12, 17, 33, 34]:

$$i \frac{d \rho}{dt} = [H, \rho] - \frac{i}{2} \{\Gamma, \rho\} + \frac{i}{2} \{\Gamma_p, 1 - \rho\},$$  \hspace{1cm} (1)

where $\rho$ is the complete neutrino density matrix, $H$ is the Hamiltonian and $\Gamma_p$ and $\Gamma$ are production and destruction rates, respectively. The approach of Ref. [12] based on Eq. (1) has several weak points. First, Eq. (1) relies on the usual assumptions of kinetic theory (with the additional assumption that the duration of a collision is small compared to the various oscillation times of $\rho$). Second, the calculations in Ref. [12] are done in the relaxation time approximation and it is assumed that the integrals in the collision term are dominated by $O(T)$ momenta. This assumption might not be warranted; since this is basically an oscillation problem with many particles, there are many timescales involved and all components might not relax in the same way. Another weak point of Eq. (1) is that it is not systematically derived from first principles and thus there is no real control over the error. This has important consequences for phenomenology since “factors of a few” in the determination of the allowed parameter range of the $\nu$MSM are crucial for the experimental searches of its particles. Thus in order to better constrain the model and study its phenomenological
implications, we need a first principles calculation based on quantum field theory. The first principles formalism presented here and the treatment based on Eq. (1) could also be compared in a region of parameter space where the two approaches should match in order to estimate the accuracy of kinetic theory approaches.

In this paper, we present the first steps toward a computation of the baryon asymmetry from first principles of statistical physics and quantum field theory. We focus on a particular class of models similar to the νMSM, namely a minimal extension of the Standard Model with an arbitrary number of sterile neutrinos. Our approach is sufficiently general that it could be applied to other models. The baryogenesis scenario studied here is similar in spirit to leptogenesis (with crucial differences). The key idea is that in some region of parameter space where the Yukawa couplings of the sterile neutrinos are small, it is possible to use conventional perturbation theory without having recourse to more sophisticated non-equilibrium tools (such as Kadanoff-Baym equations). For a similar treatment used in the context of sterile neutrino production, see Ref. [13].

The rest of the paper is organized as follows. In Sect. II we present the model Lagrangian used in this study and outline our baryogenesis scenario. The core of the paper is Sect. III where we derive a perturbative formula that expresses the (lepton) asymmetry in terms of Wightman functions and discuss its range of validity. In Sect. IV we test the formula derived in Sect. III by applying it to a toy model that can be solved both perturbatively and exactly. This computation also serves as an illustration of the inner workings of the perturbative formula. Some elements of this paper have already been published (in a different form) in the proceedings of Strong and Electroweak Matter 2008 [35].

II. THEORETICAL BACKGROUND

A. Model Lagrangian

A generic Lagrangian containing A active neutrinos and B sterile (right handed) neutrinos can be written in the chiral basis as (α runs from 1 to A and I from 1 to B):

\[ L_{AB} = \bar{L}_\alpha i \rho L_\alpha + \bar{N}_I i \rho N_I - \frac{M_{IJ}}{2} \bar{N}_I^c N_J - F_{\alpha I} \bar{L}_\alpha N_I \Phi + h.c., \]  

(2)

where \( L \) is the active lepton doublet, \( N \) the sterile neutrino singlet, \( N^c \) the charge conjugated \( N \), \( \Phi \) the Higgs doublet, \( \rho \) the Higgs field, \( M_{IJ} \) are Majorana masses and \( F_{\alpha I} \) are Yukawa...
Table I:

| A | B | \(M_{IJ}\) | Yuk. + Mix. | Phases | Total |
|---|---|---|---|---|---|
| 3 | 3 | 3 | 9 | 6 | 18 |
| 3 | 2 | 2 | 6 | 3 | 11 |
| 3 | 1 | 1 | 3 | 0 | 4 |
| 2 | 2 | 2 | 4 | 2 | 8 |
| 2 | 1 | 1 | 2 | 0 | 3 |
| 1 | 2 | 2 | 2 | 1 | 5 |

couplings. Here sterile means singlet under the Standard Model gauge group. Note that for the purpose of deriving a master formula for the asymmetry, only active-sterile transitions are necessary and other Standard Model particles are left out here. The case \(A = 3\) and \(B = 3\) (and including Standard Model particles) corresponds to the \(\nu\)MSM. Note that in the phenomenologically relevant region of the \(\nu\)MSM parameter space where the right abundance of dark matter is obtained, the “dark matter sterile neutrino” has a tiny Yukawa coupling and essentially decouples \([11, 12]\). Thus the \(\nu\)MSM has effectively three active and two sterile neutrinos.

We can check that the Lagrangian (2) satisfies all three Sakarov’s conditions for baryogenesis. First, the presence of the Majorana mass term breaks lepton number conservation (and thus baryon number conservation via sphaleron processes). Second, if \(F_{\alpha I}\) is complex, then there can be \(CP\) violation in the system (similar to the case of the CKM matrix). Table (I) shows the counting of physical parameters (masses, Yukawa couplings, mixing angles and complex phase) in (2) for various numbers of active and sterile neutrinos. Note that complex phases can only be present for \(B \geq 2\). \(CP\) violation is thus possible in the \(\nu\)MSM. Third, the out-of-equilibrium condition is provided here by the expansion of the universe. If the rate of sterile neutrino production \(\Gamma_{\text{sterile}}\) is less than the rate of expansion of the universe \(H\), then sterile neutrinos do not interact enough to preserve equilibrium. This ratio can be estimated as follows:

\[
\frac{\Gamma_{\text{sterile}}}{H} \sim \frac{f^2 T}{T^2/M_{\text{Pl}}} \sim 0.1,
\]

where we used \(T = T_{\text{sph}} \sim 100\) GeV (temperature at which sphalerons become inefficient) and \(f = 10^{-9}\) (a possible Yukawa coupling in the \(\nu\)MSM). Since the ratio is inversely propor-
tional to the temperature, the sterile neutrinos are out-of-equilibrium for all temperatures of interest. Note that this ratio depends on $f$ and that the Yukawa couplings in the $\nu$MSM are constrained by observations \cite{17}. The value used in the above estimate is for a typical choice of parameters; there is a choice of coupling when sterile neutrinos equilibrate at $T > 100$ GeV. In the latter case our formalism does not work and kinetic equations (or more sophisticated non-equilibrium quantum field theory tools) must be used.

B. Outline of the Baryogenesis Scenario

The scenario for baryon asymmetry generation in sterile neutrino oscillations was proposed in Ref. \cite{34} and developed in Refs \cite{12, 17}. Just after reheating ($t_i = 0$), we have a thermal distribution of Standard Model particles and we assume that there is no sterile neutrino initially. This is true if there is no source of sterile neutrino during inflation, such as in recent models where the inflaton is played by the Higgs \cite{15, 36}. There are no experimental data that could constrain the initial conditions at the moment. The final result for the asymmetry of course depends on the initial conditions. Note that the formalism presented in Sect. \ref{sec:formalism} is valid for any initial distribution of sterile neutrinos.

During the following cosmological evolution, the three Sakharov conditions are satisfied and a lepton asymmetry is produced via coherent and resonant oscillations of active-sterile neutrinos. The necessary resonance condition for a sufficient asymmetry generation is that two of the sterile neutrinos should be nearly degenerate in mass. This lepton asymmetry is converted into a baryon asymmetry via sphaleron processes. Since sphalerons become inefficient at a temperature around $T_{\text{sph}} \sim 100$ GeV, the conversion process stops at around $t_f = t_{\text{sph}} = 10^{13}$ GeV$^{-1}$.

This scenario is similar in spirit to “usual” or thermal leptogenesis, although the physics is different. In the case of thermal leptogenesis, the parameters of the model are different (large Yukawa couplings and large masses for the sterile neutrinos) and the sterile neutrinos are initially in thermal equilibrium. It is after this initial period of thermal equilibrium that they go out-of-equilibrium and decay, producing a lepton asymmetry. This period of out-of-equilibrium is relatively short. In the case of the $\nu$MSM, sterile neutrinos are not in equilibrium initially and typically stay out-of-equilibrium (due to their very small couplings) until sphaleron processes become inefficient. The exact time at which sterile neutrinos reach
thermal equilibrium depends on the parameters of the model (see Sect. III B).

III. PERTURBATIVE FORMULA FOR THE (LEPTON) ASYMMETRY

A. Derivation of the Master Formula

The total lepton asymmetry per unit volume is given by the difference between the average number density of leptons minus the average number density of anti-leptons. The asymmetry at time $t$ for a particular flavor $\alpha$ of active neutrino $\nu_\alpha$ is:

$$\Delta_\alpha(t, \vec{x}) = \text{Tr} \left[ \hat{\rho}(t) \nu_\alpha^\dagger(t, \vec{x}) \nu_\alpha(t, \vec{x}) \right]. \quad (4)$$

All fields in Eq. (4) are in the interaction picture. Here $\hat{\rho}(t)$ is the appropriate density operator. The initial condition for the density matrix is expressed as:

$$\hat{\rho}(0) = \hat{\rho}_S \otimes \hat{\rho}_{SM}, \quad (5)$$

where $\hat{\rho}_S$ is the out-of-equilibrium density matrix for the sterile neutrinos and $\hat{\rho}_{SM} = e^{-\hat{H}_{SM}/T}$ is the usual equilibrium density operator for Standard Model particles. The precise form of $\hat{\rho}_S$ is irrelevant for the derivation of the master formula, but the final value of the asymmetry is dependent on it. For the case of the baryogenesis scenario outlined in Sect. III B, the density matrix for sterile neutrinos is $\hat{\rho}_S = |0\rangle \langle 0|$ at $t = 0$, where $|0\rangle$ is a sterile neutrino vacuum state.

Equations (4) and (5) constitutes the system that we want to solve (for any particular flavor). In equilibrium, the density operator is an exponential and is time-independent; the machinery to solve such problems is very well developed. In the present case, Eq. (3) shows that the system is out-of-equilibrium, meaning that the density operator is time-dependent and that usual equilibrium techniques fail. A simple way to see why such problems are hard is that propagators at finite temperature depend on distribution functions. The building blocks of perturbation theory are propagators and vertices. In out-of-equilibrium situations, distribution functions are time-dependent, implying that propagators change with time in a non-trivial way. Thus usual perturbation theory fails and resummation techniques must be used \cite{26,37,38}, unless changes in the propagators are small on timescales relevant for the problem at hand. In the following, we treat Eqs. (4)-(5) perturbatively and show in the next section under what condition perturbation theory is valid.
The first step is to find the time evolution of the density operator. It is given by the von Neumann equation:

$$i \frac{d\hat{\rho}(t)}{dt} = [\hat{H}_I(t), \hat{\rho}(t)] ,$$

(6)

where $\hat{H}(t) = \hat{H}_0 + \hat{H}_I(t)$ is the interaction Hamiltonian corresponding to the Lagrangian \[2\]. All operators in Eq. (6) are in the interaction picture. Note that we work in flat spacetime here; effects due to an expanding background can be incorporated using the usual procedure of expressing the equations in conformal time and going back to physical time at the end of the computation. The rest of the derivation does not depend on the background.

We can find an iterative solution to Eq. (6):

$$\hat{\rho}(t) = \hat{\rho}(0) - i \int_0^t dt' \left[ \hat{H}_I(t'), \hat{\rho}(0) \right] - \int_0^t dt' \int_0^{t'} dt'' \left[ \hat{H}_I(t'), \left[ \hat{H}_I(t''), \hat{\rho}(t'') \right] \right] + O(\hat{H}_I^3).$$

(7)

Note that the expansion parameter in Eq. (7) is $\hat{H}_I t$. Thus the iterative solution (7) is also a perturbative solution if the criterion $\hat{H}_I t \ll 1$ is satisfied. We use quotation marks here to indicate that this is not a precise criterion since $\hat{H}_I$ is an operator and $t$ is a number; we present a more precise criterion in the next section.

Substituting the iterative solution (7) back into Eq. (4), we schematically get:

$$\Delta_{\alpha}(t', \vec{x}) = \text{Tr} \left[ \left( \hat{\rho}(0) + \hat{\rho}^{(1)}(t') + \hat{\rho}^{(2)}(t') + \ldots \right) \nu_{\alpha}^\dagger(t', \vec{x}) \nu_{\alpha}(t', \vec{x}) \right].$$

(8)

The first contribution is just an irrelevant infinite constant independent of the temperature; it is an artefact of using the lepton current to compute the asymmetry and we discard it in the following. The second contribution (and all contributions $\hat{\rho}^{(n)}$ with $n$ odd) are automatically zero because they contain an odd number of creation/annihilation operators. The first non-trivial contribution is thus (we only keep terms up to $O(\hat{H}_I^2)$ in the following):

$$\Delta_{\alpha}(t', \vec{x}) = \int_0^{t'} dt_1 \text{Tr} \left[ \left( -[\hat{H}_I(t), [\hat{H}_I(t_1), \hat{\rho}(0)]] \right) \nu_{\alpha}^\dagger(t', \vec{x}) \nu_{\alpha}(t', \vec{x}) \right].$$

(9)

To make progress, the interaction Hamiltonian need to be specified. A convenient form for our purposes is to decompose it as:

$$\hat{H}_I(t) = \int d^3 \vec{x} \left( \bar{\nu}_{\beta}(t, \vec{x}) J_{\beta}(t, \vec{x}) + \bar{J}_{\beta}(t, \vec{x}) \nu_{\beta}(t, \vec{x}) \right),$$

(10)

where $J$ contains all the information about the interaction except the field associated to the desired asymmetry. The exact form of $J$ is not important for the rest of the derivation.
Note that $J$ contains the out-of-equilibrium sterile neutrino field; in the case of the $\nu$MSM, it also contains the Higgs field and the Yukawa couplings.

Inserting this form for the interaction Hamiltonian into Eq. (9) and using Wick’s theorem, we obtain:

$$\Delta_\alpha(t', \bar{x}) = -\int_0^{t'} dt \int_0^t dt_1 \int d^3 y \int d^3 y_1 \times 2 \text{Re} \left[ \langle J_\beta \bar{J}_\gamma \rangle \langle \nu_\alpha^\dagger \nu_\gamma \rangle - \langle J_\gamma \bar{J}_\beta \rangle \langle \nu_\alpha^\dagger \nu_\beta \rangle \langle \nu_\alpha \bar{\nu}_\gamma \rangle \right. $$

$$\left. - \langle J_\beta \bar{J}_\gamma \rangle \langle \nu_\gamma \nu_\alpha^\dagger \rangle \langle \nu_\lambda \bar{\nu}_\beta \rangle + \langle J_\gamma \bar{J}_\beta \rangle \langle \nu_\gamma \nu_\alpha^\dagger \rangle \langle \nu_\alpha \bar{\nu}_\beta \rangle \right] + O(\hat{H}_4^4),$$

(11)

where we have used a condensed notation $J_\beta \equiv J_{\beta a}(t, \vec{y})$ with the first and second indices being flavor and spinor indices, respectively; $\alpha$ always refers to the external flavor index and spacetime coordinates $(t', \bar{x})$, while $\beta, \gamma, ...$ correspond to internal flavor/spinor indices and to spacetime coordinates $(t, \vec{y}), (t_1, \vec{y}_1), ...$. The equilibrium correlators $\langle \nu \bar{\nu} \rangle$ are the usual active neutrino propagators. The correlators $\langle J \bar{J} \rangle$ contain the out-of-equilibrium sterile neutrino fields $N$ and potentially other fields $\Phi$ that are in equilibrium; they can be decomposed as $\langle J \bar{J} \rangle = \text{Tr} \left[ \hat{\rho}_S \otimes \hat{\rho}_SM \bar{J} \hat{J} \right] \sim \text{Tr} \left[ \hat{\rho}_S N \bar{N} \right] \otimes \text{Tr} \left[ \hat{\rho}_SM \bar{\Phi} \Phi \right]$. Note that for the scenario outlined in Sect. II B we have $\hat{\rho}_S = |0\rangle\langle 0|$ and thus zero temperature propagators can be used to describe the evolution of the sterile neutrinos. This replacement is allowed if perturbation theory is valid; we come back to this point in the next section.

It is possible to simplify Eq. (11) further by making an assumption. First note that for any propagator we have $\langle \psi(x) \bar{\psi}(y) \rangle \sim \langle \bar{\psi}(x) \psi(y) \rangle$, where the $\sim$ means everything is equal except for their Dirac structures which are complex conjugate of each other. Looking at Eq. (11), we see that the first and second terms and the third and fourth terms are the same except that their coupling constants and their Dirac structures are complex conjugate of each other. In the following, we assume that the Dirac structure computation for each term gives a real scalar; thus for all practical purposes, the Dirac structure is irrelevant when comparing these pairs of terms. This assumption may be true in general, but its validity must be checked explicitly for each interaction Hamiltonian.

Taking into account the previous assumption and defining $J_\beta \equiv F_{\beta I} \bar{J}_I$, the asymmetry at second order in the interaction Hamiltonian finally becomes:

$$\Delta_\alpha(t', \bar{x}) = -\int_0^{t'} dt \int_0^t dt_1 \int d^3 y \int d^3 y_1 \text{Re} \left[ 2i \text{Im}(F_{\beta I} F^{*}_{\gamma J}) \langle \bar{J}_I \bar{J}_J \rangle \langle \nu_\alpha^\dagger \nu_\gamma \rangle \langle \nu_\alpha \bar{\nu}_\beta \rangle \right.$$ 

$$ \left. -2i \text{Im}(F^{*}_{\gamma J} F_{\beta I}) \langle \bar{J}_J \bar{J}_I \rangle \langle \nu_\gamma \nu_\alpha^\dagger \rangle \langle \nu_\alpha \bar{\nu}_\beta \rangle \right]$$

9
\[
= \int_0^{t'} dt \int_0^t dt_1 \int d^3 y \int d^3 y_1 4 \left[ \text{Im}(F_{\beta I} F^{*}_{\gamma J}) \text{Im}(\langle \bar{J}_I \tilde{J}_J \rangle \langle \nu^\dagger_\alpha \nu_\gamma \rangle \langle \nu_\alpha \bar{\nu}_\beta \rangle) - \text{Im}(F^{*}_{\gamma J} F_{\beta I}) \text{Im}(\langle \tilde{J}_J \bar{J}_I \rangle \langle \nu_\gamma \nu^\dagger_\alpha \rangle \langle \nu_\alpha \bar{\nu}_\beta \rangle) \right]. \tag{12}
\]

This last equation, or more specifically its \(O(H_I^4)\) version (see the discussion below), are the main results of this paper. It expresses the lepton asymmetry for a particular flavor in terms of Wightman correlators. If perturbation theory is valid, then these correlators can be computed using conventional tools. If the result is zero, then higher order contributions in \(\tilde{H}_I\) must be computed.

Formula (12) is quite general, applicable in principle to leptogenesis-type models. The validity of perturbation theory is the only assumption that enters into its derivation (in addition to the minor assumption about the Dirac structure).

The last formula satisfies the third Sakharov condition. As mentioned previously, the \(J\) operator contains the sterile neutrino operator, and the sterile neutrinos are not in thermal equilibrium. Assuming that the sterile neutrinos are in equilibrium, then the \(J\) operator would obey the Kubo-Martin-Schwinger condition \[39, 40\]
\[
\langle J_a(t_1) \bar{J}_b(t_2) \rangle = \langle \bar{J}_b(t_2) J_a(t_1 + i\beta) \rangle \quad (\beta = 1/T \text{ is the inverse temperature}).
\]
Using the Kubo-Martin-Schwinger condition and time translation invariance, it can be shown that the asymmetry is automatically zero in equilibrium.

Formula (12) also neatly separates \(CP\) violating effects (contained in the imaginary part of the couplings) and dynamical effects. It is thus easy to see that it satisfies the second Sakharov condition: if there is no \(CP\) violating phase, then the Yukawa couplings \(F\)'s are real and the formula gives automatically zero. Furthermore, using the fact that propagators are diagonal in flavor space (\(\langle \nu_\alpha \bar{\nu}_\beta \rangle \propto \delta_{\alpha\beta}\) and \(\langle \bar{J}_I \tilde{J}_J \rangle \propto \delta_{IJ}\)), we get that \(\text{Im} \left( F_{\beta I} F^{*}_{\gamma J} \right) = 0\) and that the formula for the asymmetry is automatically zero at \(O(H_I^2)\). This is another way of saying that a lepton asymmetry is a quantum effect (generally coming from the interference of a tree level diagram and a loop diagram).

The first non-trivial order in the expansion of the density operator is thus \(O(H_I^4)\). Expanding the iterative solution (7) up to \(O(H_I^4)\) and repeating the same procedure, we finally obtain:

\[
\Delta_\alpha(t', \vec{x}) = - \int_0^{t'} dt \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int d^3 y \int d^3 y_1 \int d^3 y_2 \int d^3 y_3 \times 4 \left[ \text{Im}(F_{\beta I} F_{\gamma J} F^{*}_{\delta K} F^{*}_{\epsilon L}) \text{Im}(\langle \bar{J}_I \tilde{J}_J \bar{J}_K \tilde{J}_L \rangle) \right]
\]
Thus the only inputs in Eq. (13) are the $F_{\alpha\gamma}$ and $\bar{\nu}\nu$ propagators. The formula allows the computation of the BAU up to $O(f^4)$ and is exact in all other couplings of the model. Since perturbation theory is assumed to be valid, it is possible to resolve the 4-point functions $\langle \tilde{J}_I \tilde{J}_J \tilde{J}_K \tilde{J}_L \rangle$ into products of 2-point functions using Wick's theorem. Thus the only inputs in Eq. (13) are the $F_{\alpha\gamma}$'s and the various propagators of the model. The resulting terms can be interpreted as Feynman diagrams.
FIG. 1: Typical Feynman diagrams appearing in the perturbative formula (13). The solid lines correspond to active $\nu$ and sterile $N$ neutrino propagators and the dashed lines correspond to Higgs $H$ propagators.

The creation of the asymmetry takes place in the early universe where a thermal bath of Standard Model particles is present. It is well known that resummed propagators (with thermal masses and dampings) must be used in order to obtain correct results. For instance, interactions of particles with the medium may open new decay channels that are otherwise kinematically forbidden (see for example [41]). The inclusion of damping is also important to deal with so-called secular terms or “pinch” singularities (see Sect. III B). Hard thermal loop resummations are moreover necessary to obtain gauge invariant results in some cases [42].

To end this section, we discuss qualitatively the case of the $\nu$MSM in order to illustrate the use of the perturbative formula (13) (for a similar discussion in a kinetic theory setup, see [12, 17]). We restrict ourselves to the symmetric phase here. The typical Feynman diagrams appearing in the perturbative formula are given in Fig. (1). In order to compute the asymmetry, we need the (Wightman) active/sterile neutrino and Higgs propagators. The exact form of these propagators depend on the self-energies that are resummed into them. For sterile neutrinos, no resummation is necessary since the propagators are at zero temperature and the self-energy corrections are small (i.e. $O(f^2)$). For active neutrinos, it is necessary to include $W, Z$ boson and charged lepton loops. The absence of charged leptons would imply a new symmetry for the $\nu$MSM, allowing the removal of $CP$ phases in $F_{\alpha I}$ by rotating active neutrino fields and leading to a vanishing asymmetry [51]. For Higgs bosons, the dominant contribution to the self-energy comes from top quark loops of size $O(m_t^2T^2/v^2)$; this is parametrically large compared to $W, Z$ boson loops of size $O(m_{W,Z}^4/v^2)$ and Higgs loops of size $O(m_H^4/v^2)$ (all the estimates are for large temperatures). The self-energies that need to be resummed in the propagators are illustrated in Fig. (2).

As can be seen from the above qualitative discussion, the computation of the BAU in the
νMSM is not a simple task. For this reason and as a warm up, we analyze a simpler toy model involving only Yukawa interactions with sterile neutrinos and no other interaction. The full computation of the BAU in the νMSM will be the subject of a separate publication.

B. Validity of Perturbation Theory

For the iterative solution $⟨H_1 t \ll 1⟩$ to be also a perturbative solution, the criterion “$\hat{H}_1 t \ll 1$” needs to be satisfied. Said differently, it means that perturbation theory is bound to break down for sufficiently long times; these are the infamous secular terms that plague non-equilibrium quantum field theory (e.g. [26]). These secular terms also appear in a different form in quantum field theory computations using the real-time formalism of finite temperature field theory [43]. In the real-time formalism (where time is taken to go from minus infinity to plus infinity and back), ill-defined products of delta functions (or “pinch” singularities) arise naturally [44, 45] and blow up with the time-volume [46]. Fortunately these pinch singularities cancel in equilibrium due to the Kubo-Martin-Schwinger condition [47]. Since the Kubo-Martin-Schwinger condition is not valid out-of-equilibrium, the pinch singularity problem remains for non-thermal systems and more sophisticated methods must be used [26, 37, 38].

In the present paper, we use the von Neumann equation and time is finite: if the final time is taken to be small, then secular terms should also stay small (see [48] for a similar argument phrased in terms of pinch singularities). We can estimate the size of the secular terms in the following way. Roughly speaking, “$\hat{H}_1 t$” gives the number of sterile neutrinos produced per unit time ($\hat{H}_1$) times the time ($t$). The total number of sterile neutrinos
produced is given by:

\[ n_{\text{sterile}} \sim \int_0^{t_{\text{sph}}} \Gamma_{\text{sterile}}(t') dt' \approx \Gamma_{\text{sterile}} \left( \frac{t_{\text{sph}}}{H} \right) \],

where in the last step we use the fact that both the rates of sterile neutrino production and of the universe’s expansion are power laws. Thus the size of secular terms (or equivalently the total number of sterile neutrinos produced) is given by the out-of-equilibrium criterion (3) evaluated at the time when sphalerons become inefficient. For \( T = T_{\text{sph}} = 100 \text{ GeV} \) and \( f = 10^{-9} \), we get \( n_{\text{sterile}} \approx 0.1 \ll 1 \) and thus perturbation theory is justified in this part of parameter space. It also shows that the distribution function for the sterile neutrinos does not evolve much during the relevant period for baryogenesis and justifies our use of zero temperature propagators to describe their evolution (in the case of the baryogenesis scenario outlined in Sect. II B).

IV. APPLICATION TO A TOY MODEL

Before using the formula for the asymmetry (13) on a realistic but more complicated model (such as the \( \nu \text{MSM} \)), we would like to test it on a simpler (but unrealistic) model and verify the range of validity of perturbation theory. We thus make the following simplifications to the model Lagrangian (2). First, we neglect interactions (damping) with Standard Model particles. This implies that the total active lepton asymmetry is zero (see the discussion below Eq. (13)) and only individual flavor asymmetries are non-zero. Second, we assume that the Higgs is non-dynamical, making the Lagrangian quadratic in the fields. This simplified model thus describes a bunch of non-interacting harmonic oscillators and no thermalization is possible. Third, we reduce the number of free parameters by reducing the number of active/sterile neutrinos. Looking at Table (I), the minimal model containing only one \( C P \) violating phase has one active and two sterile neutrinos. For reasons that will become clear in the next section, the case with one active neutrino and two sterile neutrinos trivially gives a vanishing asymmetry. We thus opt for the next-to-minimal model (i.e. two active and two sterile neutrinos) to test our formula. The Lagrangian for such a toy model is:

\[
L_{\text{toy}} = \psi_{1R}^\dagger i\sigma^\mu \partial_\mu \psi_{1R} + \psi_{2R}^\dagger i\sigma^\mu \partial_\mu \psi_{2R} + \psi_{3L}^\dagger i\tilde{\sigma}^\mu \partial_\mu \psi_{3L} + \psi_{4L}^\dagger i\tilde{\sigma}^\mu \partial_\mu \psi_{4L} - M \psi_{1R}^T i\sigma^2 \psi_{2R} - \frac{\Delta M}{2} (\psi_{1R}^T i\sigma^2 \psi_{1R} + \psi_{2R}^T i\sigma^2 \psi_{2R}) - f v \psi_{3L}^\dagger \psi_{1R} - f v \psi_{3L}^\dagger \psi_{2R} - \delta f v \psi_{4L}^\dagger \psi_{2R} + h.c.,
\]

(15)
where we take the Higgs field to be a constant $v$, $\psi_{1,2R}$ are the sterile neutrino right-handed Weyl fields, $\psi_{3,4L}$ are the active neutrino left-handed Weyl fields, $M$ is the common mass of the sterile neutrinos, $\Delta M$ is the sterile neutrino mass difference, $f$ and $\epsilon$ and $\delta$ are Yukawa couplings and $\eta$ is a $CP$ violating phase.

This toy model has five real parameters and one $CP$ violating phase [52] (compare this to the 8 real parameters and 3 phases of the $\nu$MSM including constraints from dark matter abundance). The physics of leptogenesis in this toy model is thus simpler. For instance, if $\eta$ or $\epsilon$ or $\Delta M$ are sent to zero, then the Lagrangian does not contain any $CP$ or baryon number violating terms anymore and the asymmetry vanishes (if $\delta$ is zero then we come back to the one active neutrino case and the asymmetry is also zero). These features should also appear in the solution.

We can solve this model in two ways. Since the Higgs takes its expectation value, the Lagrangian is quadratic in the fields and it is possible to solve the system exactly. If the Yukawa coupling $f$ is small, then the system can also be solved perturbatively using the master formula for the lepton asymmetry presented in Sect. III A. The exact and perturbative results can be compared and should match in some time interval. This is what we present in the next sections.

A. Exact Solution

As in the perturbative case, the lepton asymmetry (for the active flavor $a = 3, 4$) is given by the average lepton current density:

$$\Delta_\alpha(t, \bar{x}) = \text{Tr} \left[ \rho(0) \psi^\dagger_{\alpha L}(t, \bar{x}) \psi_{\alpha L}(t, \bar{x}) \right],$$

where all fields are in the Heisenberg picture. The calculation of the asymmetry is now purely quantum mechanical and very similar to neutrino oscillation computations (e.g. [49]). For times $t < 0$, there is no interaction and there is no sterile neutrino. At $t = 0$, the interaction is adiabatically switched on and sterile neutrino fields are initially in flavor eigenstates (in which the thermal averages are defined). This last point is very important because otherwise the problem is trivial. Indeed, since the Lagrangian (15) can be diagonalized such that all four masses are real (see below), then it means that all $CP$ violating phases can be absorbed and there is no asymmetry if the thermal averages are defined in the mass basis. In the
case where the thermal averages are defined in the initial flavor basis, the asymmetry is in principle non-zero.

The goal is to compute the asymmetry at some time \( t > 0 \), hence the fields \( \eta_{3L}(t, \vec{x}) \) must be time evolved from 0 to some time \( t \). Time evolution in quantum mechanics is most conveniently done using energy eigenstates. In order to obtain the energy eigenstates, we need the exact masses. The mass matrix \( \mathcal{M} \) in (15) is:

\[
\mathcal{M} = \begin{pmatrix}
\Delta M & M & f v & 0 \\
M & \Delta M & efve^{-i\eta} & \delta f v \\
f v & efve^{-i\eta} & 0 & 0 \\
0 & \delta f v & 0 & 0
\end{pmatrix}.
\] (17)

The above symmetric complex mass matrix can be diagonalized using Takagi’s factorization \( \mathcal{M} = \mathcal{U} \mathcal{D} \mathcal{U}^T \) where \( \mathcal{U} \) is unitary (e.g. [50]). The results for the masses are:

\[
\tilde{m}_1 = M + \frac{f^2 v^2}{2M} (1 + \epsilon^2 + \delta^2 + 2\rho^2),
\]

\[
\tilde{m}_2 = M + \frac{f^2 v^2}{2M} (1 + \epsilon^2 + \delta^2 - 2\rho^2),
\]

\[
\tilde{m}_3 = \frac{f^2 v^2}{M} (\epsilon + \sqrt{\epsilon^2 + \delta^2}),
\]

\[
\tilde{m}_4 = \tilde{m}_3 \left( \frac{\delta^2}{2\epsilon^2 + \delta^2 + 2\epsilon\sqrt{\epsilon^2 + \delta^2}} \right),
\] (18)

where we used the parametrization \( \Delta M \equiv \left( \frac{f^2v^2(\epsilon+\delta)}{M} \right) \kappa \) and defined \( \rho e^{i\theta} \equiv \sqrt{\epsilon e^{i\eta} + (\epsilon + \delta)\kappa} \).

Flavor and energy eigenstates are related as \( \psi_R = \mathcal{U} \tilde{\psi}_R \) and \( \psi_L = \mathcal{U}^* \tilde{\psi}_L \) where \( \psi_R \equiv (\psi_{1R} \ \psi_{2R} \ i\sigma^2 \psi_{3L}^* \ i\sigma^2 \psi_{4L}^*)^T \) and \( \psi_L \equiv (-i\sigma^2 \psi_{1R}^* \ -i\sigma^2 \psi_{2R}^* \ \psi_{3L} \ \psi_{3L})^T \). In the basis where the mass matrix is diagonal, the neutrino fields are given by:

\[
\tilde{\psi}_{iL}(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{h \pm 1} \left[ \tilde{a}_{p,h,i} \sqrt{E_p} - |\vec{p}| h \chi(h)(\vec{p}) e^{-ipx} + h \tilde{a}_{p,h,i}^\dagger \sqrt{E_p} + |\vec{p}| h \chi(-h)(\vec{p}) e^{ipx} \right],
\] (19)

where \( h \) is the helicity and the \( \chi(h)(\vec{p}) \)'s are two-component helicity eigenstate spinors. The exact energies are given by \( \tilde{E}_p = \sqrt{|\vec{p}|^2 + \tilde{m}_i^2} \) and the exact masses \( \tilde{m}_i \)'s are given in Eq. (18).

After \( t = 0 \), the initial flavor eigenstates become linear combinations of energy eigenstates and start to oscillate, with each energy eigenstate oscillating with a different frequency depending on the exact masses. At time \( t \), we have:

\[
\Delta_\alpha(t, \vec{x}) = \text{Tr} \left[ \rho(0)[\tilde{\psi}_L^\dagger(t, \vec{x}) \mathcal{U}^T]_\alpha [\mathcal{U}^* \tilde{\psi}_L(t, \vec{x})]_\alpha \right].
\] (20)
In the last expression, the neutrino fields are expressed in terms of creation/annihilation operators $\tilde{a}_{p,h,i}$ that create/annihilate quanta corresponding to states with a certain mass. To do the thermal averages, it is necessary to re-express these “mass” creation/annihilation operators as linear combinations of “flavor” creation/annihilation operators $a_{p,h,i}$. This is done using Bogoliubov-type transformations. The appropriate transformations are:

$$
\tilde{a}_{p,h,i} = \frac{\sqrt{E_i} + |\vec{p}|h}{\sqrt{2E_i}} \sum_{j=1,2} (U^\dagger V)_{ij} \frac{1}{\sqrt{2E_j}} \left( a_{p,h,j} \sqrt{E_j + |\vec{p}|h} - h\phi(\vec{p}, h)a^\dagger_{-p,h,j} \sqrt{E_j - |\vec{p}|h} \right) + \sum_{k=3,4} \sum_{j=1,2} (U^T V^*)_{ik} \left( b_{p,1,k} \delta_{1,h} + \phi(\vec{p}, -1)a^\dagger_{-p,1,k} \delta_{-1,h} \right) + \frac{\sqrt{E_i} - |\vec{p}|h}{\sqrt{2E_i}} \sum_{j=1,2} (U^T V^*)_{ij} \frac{1}{\sqrt{2E_j}} \left( a_{p,h,j} \sqrt{E_j - |\vec{p}|h} + h\phi(\vec{p}, h)a^\dagger_{-p,h,j} \sqrt{E_j + |\vec{p}|h} \right) + \sum_{k=3,4} \sum_{j=1,2} (U^T V^*)_{ik} \left( a_{-1,k} \delta_{-1,h} + \phi(\vec{p}, 1)b^\dagger_{-p,1,k} \delta_{1,h} \right),

(21)
$$

where $V$ is the matrix that diagonalizes the free part (i.e. $f = 0$) of the Lagrangian \[15\], $E_i = \sqrt{|\vec{p}|^2 + m_i^2}$ are the eigenenergies corresponding to the masses of the free Lagrangian and $\phi(\vec{p}, h)$ is some phase that satisfies $\phi(-\vec{p}, -h) = \phi^*(\vec{p}, h)$ and $\phi(\vec{p}, -h) = -\phi^*(\vec{p}, h)$ \[49\]. One can verify that these transformations preserve the canonical anti-commutation relations.

Inserting the neutrino field operator \[19\] into Eq. \[20\] and using the relations between the two sets of creation/annihilation operators \[21\], the asymmetry becomes (after some algebra):

$$
\Delta_\alpha(t, \vec{x}) = - \sum_{i,j=1}^{2+4} \sum_{\kappa=3}^{2+4} \frac{d^3p}{(2\pi)^3} \left[ \frac{2|\vec{p}|}{E_i E_j} n(E_\kappa) \text{Im} \left( U_{ai} U^*_{\alpha j} U^*_{\alpha i} U_{\kappa j} \right) \left( (E_i + \bar{E}_j) \sin (E_i - \bar{E}_j)t - (\bar{E}_i - E_j) \sin (E_i + \bar{E}_j)t \right) \right]
$$

(22)

This is the final result for the exact lepton asymmetry. The above formula is true for two sterile neutrinos and any number of active neutrino flavors $A$. The solution contains a $CP$ structure part and a dynamical part consisting of two oscillating functions (one with a large amplitude and small frequency and one with a small amplitude and large frequency) for each value of $i, j$. We note that the $CP$ structure part $\text{Im} \left( U_{ai} U^*_{\alpha j} U^*_{\alpha i} U_{\kappa j} \right)$ is similar to the one obtained in Ref. \[12\]. We also immediately see that the total active lepton number is zero, i.e. $\sum_\alpha \Delta_\alpha(t) \propto \sum_\alpha \sum_{i<j=1}^{2+4} \sum_{\kappa=3}^{2+4} \text{Im} \left( U_{ai} U^*_{\alpha j} U^*_{\alpha i} U_{\kappa j} \right) = 0$. In particular, we get $\Delta_\alpha(t) = 0$ for
only one flavor of active neutrinos; this explains our study of the two active and two sterile neutrino case.

B. Perturbative Solution

If the Yukawa coupling \( f \) is small, then the active-sterile neutrino interactions are small and can be treated as a perturbation. Diagonalizing the quadratic part of the Lagrangian \((13)\) with the transformations \( \psi_{1R} \rightarrow (\eta_{1R} + i\eta_{2R})/\sqrt{2} \) and \( \psi_{2R} \rightarrow (\eta_{1R} - i\eta_{2R})/\sqrt{2} \) (the active neutrino fields stay the same \( \psi_{3AL} \rightarrow \eta_{3AL} \)), we obtain:

\[
\mathcal{L}_{\text{toy}} = \frac{\eta_{1R}^\dagger i \sigma^\mu \partial_\mu \eta_{1R}}{2} + \frac{\eta_{2R}^\dagger i \sigma^\mu \partial_\mu \eta_{2R}}{2} + \frac{\eta_{3L}^\dagger i \sigma^\mu \partial_\mu \eta_{3L}}{2} + \frac{\eta_{4L}^\dagger i \sigma^\mu \partial_\mu \eta_{4L}}{2}
\]

\[
-\frac{fv}{\sqrt{2}} \eta_{3L}^\dagger (\eta_{1R} + i\eta_{2R}) - \frac{\epsilon f ve_i n}{\sqrt{2}} \eta_{3L}(\eta_{1R} - i\eta_{2R}) - \frac{\delta f v}{\sqrt{2}} \eta_{4L}(\eta_{1R} - i\eta_{2R}) + h.c., (23)
\]

The perturbative masses are \( m_{1,2} = M \pm \Delta M \) for the sterile neutrinos and zero for the active neutrinos. The interaction Hamiltonian can be obtained from the above Lagrangian and zero:

\[
H_I = \eta_{3L}^\dagger J_3 + \eta_{3L}^\dagger J_3 + \eta_{4L}^\dagger J_4 + \eta_{4L}^\dagger J_4,
\]

where the \( J \) operators are given by:

\[
J_3 = \frac{fv}{\sqrt{2}} (1 + e^{i\eta}) \eta_{1R} + \frac{ifv}{\sqrt{2}} (1 - e^{i\eta}) \eta_{2R} \equiv F_{31} \eta_{1R} + F_{32} \eta_{2R},
\]

\[
J_4 = \frac{\delta fv}{\sqrt{2}} \eta_{1R} - \frac{i\delta fv}{\sqrt{2}} \eta_{2R} \equiv F_{41} \eta_{1R} + F_{42} \eta_{2R}.
\]

Substituting the operators \((25)\) in Eq. \((13)\), it is a straightforward but tedious exercise to compute the asymmetry. In the following we present the computation of (a part of) the first term in Eq. \((13)\); the others are done in a similar way. Note that the assumption about the Dirac structure is verified here (see the discussion following Eq. \((11)\)). The first term of Eq. \((13)\) is:

\[
\Delta_\alpha(t', \bar{x})_1 = -\int_0^{t'} dt \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int d^3 y \int d^3 y_1 \int d^3 y_2 \int d^3 y_3 \left[ -4 \text{Im} \left( F_{\alpha I} F_{\gamma J} F_{\alpha K}^* F_{\gamma N}^* \right) \text{ Im} \left( \langle 0 | \bar{J}_{\alpha I}(y) \bar{J}_{\beta J}(y_1) \bar{J}_{\gamma K}(y_2) \bar{J}_{\delta N}(y_3) | 0 \rangle \right) \right.
\]

\[
\left. \langle \eta_{\alpha La}^\dagger(x) \eta_{\alpha La}(y_2) \rangle \langle \eta_{\gamma La}^\dagger(y_1) \eta_{\gamma La}(y_3) \rangle \left( \langle \eta_{\alpha La}(y) \eta_{\alpha La}(x) \rangle + \langle \eta_{\alpha La}(x) \eta_{\alpha La}(y) \rangle \right) \right] + 4 \text{Im} \left( F_{\alpha I} F_{\gamma J} F_{\alpha K}^* F_{\alpha N}^* \right) \text{ Im} \left( \langle 0 | \bar{J}_{\alpha I}(y) \bar{J}_{\beta J}(y_1) \bar{J}_{\gamma K}(y_2) \bar{J}_{\delta N}(y_3) | 0 \rangle \right)
\]

\[
\langle \eta_{\alpha La}^\dagger(x) \eta_{\alpha La}(y_3) \rangle \langle \eta_{\gamma La}^\dagger(y_1) \eta_{\gamma La}(y_2) \rangle \left( \langle \eta_{\alpha La}(y) \eta_{\alpha La}(x) \rangle + \langle \eta_{\alpha La}(x) \eta_{\alpha La}(y) \rangle \right) \right),
\]

(26)
where \(a, b, ..., e\) are spinor indices, \(\alpha, \gamma\) refer to active neutrino flavors and \(I, J, K, N\) refer to sterile neutrino flavors. The case where \(\alpha = \gamma\) corresponds to one active neutrino flavor only and gives zero. For definiteness we thus consider \(\alpha = 3\) and \(\gamma = 4\) in the following. Keeping only non-zero contributions coming from the imaginary part of the coupling constants, we obtain:

\[
\Delta_\alpha(t', \vec{x})_1 = -\int_0^{t'} dt \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int d^3y \int d^3y_1 \int d^3y_2 \int d^3y_3 \\
\quad \left[ -4\text{Im} \left( F_{31} F_{41} F_{32}^* F_{42}^* \right) \right] \text{Im} \left( \langle 0 | \eta_{1Ra}(y) \eta_{1Rb}(y_1) \eta_{2Ra}^\dagger(y_2) \eta_{2Rd}(y_3) | 0 \rangle \right) \\
\quad \langle \eta_{3La}^\dagger(x) \eta_{3Le}(y_3) \rangle \langle \eta_{4La}(y_1) \eta_{4Ld}(y_3) \rangle \left( \langle \eta_{3La}(y) \eta_{3Le}(x) \rangle + \langle \eta_{3Le}(x) \eta_{3La}(y) \rangle \right) \\
\quad -4\text{Im} \left( F_{31} F_{42} F_{32}^* F_{41}^* \right) \text{Im} \left( \langle 0 | \eta_{1Ra}(y) \eta_{2Rb}(y_1) \eta_{2Ra}^\dagger(y_2) \eta_{1Rd}(y_3) | 0 \rangle \right) \\
\quad \langle \eta_{3La}^\dagger(x) \eta_{3Le}(y_2) \rangle \langle \eta_{4La}(y_1) \eta_{4Ld}(y_3) \rangle \left( \langle \eta_{3La}(y) \eta_{3Le}(x) \rangle + \langle \eta_{3Le}(x) \eta_{3La}(y) \rangle \right) \\
\quad +4\text{Im} \left( F_{31} F_{41} F_{32}^* F_{42}^* \right) \text{Im} \left( \langle 0 | \eta_{1Ra}(y) \eta_{1Rb}(y_1) \eta_{2Ra}^\dagger(y_2) \eta_{2Rd}(y_3) | 0 \rangle \right) \\
\quad \langle \eta_{3La}^\dagger(x) \eta_{3Le}(y_3) \rangle \langle \eta_{4La}(y_1) \eta_{4Ld}(y_3) \rangle \left( \langle \eta_{3La}(x) \eta_{3Le}(y) \rangle + \langle \eta_{3Le}(y) \eta_{3La}(x) \rangle \right) \\
\quad +4\text{Im} \left( F_{31} F_{42} F_{32}^* F_{41}^* \right) \text{Im} \left( \langle 0 | \eta_{1Ra}(y) \eta_{2Rb}(y_1) \eta_{2Ra}^\dagger(y_2) \eta_{1Rd}(y_3) | 0 \rangle \right) \\
\quad \langle \eta_{3La}^\dagger(x) \eta_{3Le}(y_3) \rangle \langle \eta_{4La}(y_1) \eta_{4Ld}(y_3) \rangle \left( \langle \eta_{3La}(x) \eta_{3Le}(y) \rangle + \langle \eta_{3Le}(y) \eta_{3La}(x) \rangle \right) \\
\quad + \text{(same terms with } 1R \leftrightarrow 2R \text{ and } F_{a1} \leftrightarrow F_{a2} \text{)} .
\] (27)

Assuming the validity of perturbation theory, all the neutrino fields are free fields and we can use Wick’s theorem to decompose the above 4-point functions into products of 2-point functions. Since the system is translationally invariant, we can do the Fourier transform over space. Concentrating on the first of the eight terms in Eq. (27), we have:

\[
\Delta_\alpha(t', \vec{x})_{1,1} = \int_0^{t'} dt \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int \frac{d^3p}{(2\pi)^3} 4\text{Im} \left( F_{31} F_{41} F_{32}^* F_{42}^* \right) \text{Im} \left[ \langle \eta_{3La}^\dagger(x) (\vec{p}) \rangle \langle 0 | \eta_{2Ra}^\dagger \eta_{2Rd}^\dagger | 0 \rangle (\vec{p}) \langle \eta_{4La} \eta_{4Ld} (\vec{p}) \rangle (-\vec{p}) \right] \\
\quad \langle 0 | \eta_{1Ra} \eta_{1Rb} | 0 \rangle (-\vec{p}) \langle \eta_{3La} \eta_{3Le} (\vec{p}) \rangle \left( \langle \eta_{3Le} \eta_{3La} (\vec{p}) \rangle + \langle \eta_{3La} \eta_{3Le}^\dagger (\vec{p}) \rangle \right) .
\] (28)

To make progress, we need the form of the propagators for massless left-handed active neutrinos and for massive right-handed sterile neutrinos. These are obtained from the expansion of the fields in terms of creation/annihilation operators (cf. Eq. (19)). The propagators are:

\[
\langle \eta_{L}(t_1) \eta_{L}^\dagger(t_2) | (\vec{p}) \rangle = (1 - n(E_p)) \chi(-1)(\vec{p}) \chi_{-1}(\vec{p}) e^{-iE_p(t_1-t_2)} \\
\quad + n(E_p) \chi(-1)(-\vec{p}) \chi_{-1}(\vec{p}) e^{iE_p(t_1-t_2)}.\]
Inserting these propagators in Eq. (28), we obtain:

\[
\langle \eta_R(t_1)\eta_R(t_2) \rangle \langle \bar{p} \rangle = -\frac{1}{2E_p} \sum_h \left[ (1 - n(E_p))h m \chi_h(\bar{p}) \hat{\chi}_{-h}(\bar{p}) e^{-iE_p(t_1 - t_2)} + n(E_p) h M \chi_{-h}(\bar{p}) \hat{\chi}_{h}(\bar{p}) e^{iE_p(t_1 - t_2)} \right], \\
\langle \eta_R(t_1) \eta_R(t_2) \rangle \langle \bar{p} \rangle = -\frac{1}{2E_p} \sum_h \left[ (1 - n(E_p))h m \chi^\dagger_h(\bar{p}) \hat{\chi}^\dagger_{-h}(\bar{p}) e^{-iE_p(t_1 - t_2)} + n(E_p) h M \chi^\dagger_{-h}(\bar{p}) \hat{\chi}^\dagger_{h}(\bar{p}) e^{iE_p(t_1 - t_2)} \right].
\] (29)

The other terms in Eq. (13) can be done in a similar way and they all have similar structures (with different arrangements of energies in the sines and denominators). There are 12 terms similar to Eq. (31) in the final result for the perturbative lepton asymmetry. Because of its size (there are no obvious simplifications) and since it is not very instructive, we do not write the full expression of the perturbative lepton asymmetry here.

Even at the level of individual terms we see that Eq. (31) has the features expected from the first two Sakharov conditions. First it is clear that Eq. (31) is zero when \( \eta = 0, \epsilon = 0 \) or \( \delta = 0 \) (i.e. no \( CP \) violation). It also vanishes when the sterile neutrino masses are degenerate (i.e. no lepton number violation). To see that, note that each terms in Eq. (31) is paired up with a similar term with \( 1R \leftrightarrow 2R \) and \( F_{\alpha 1} \leftrightarrow F_{\alpha 2} \) (cf. Eq. (27)). The asymmetry is thus proportional to \( E_1 - E_2 \) and vanishes when the two energies are equal (i.e. when \( \Delta M = 0 \)).
C. Comparison Between Exact and Perturbative Solutions

The exact and perturbative computations of the asymmetry start with the same initial condition and the dynamics is dictated by the same Lagrangian. They should therefore give the same result up to some time where secular terms become important. To estimate this time, it is not possible to use the considerations of Sect. [III B] because there is no interaction (thus no thermalization) and spacetime is not expanding. We use instead the following argument.

The toy model considered in Sect. [IV] is quadratic in the field, thus the asymmetry production should be oscillatory. Looking at the exact (22) and perturbative (31) solutions, we note that both solutions are sums of oscillatory functions with different frequencies. The “exact” and “perturbative” frequencies are given by \( \tilde{\omega}_{ij,\pm} = (\tilde{E}_i \pm \tilde{E}_j) \) and \( \omega_{ij,\pm} = (E_i \pm E_j) \). Since the masses are different in the exact and perturbative cases, the frequencies are also different. This implies that the two solutions develop a phase difference over time. This phase difference is secular. Thus even in a non-interacting theory, secular terms are present because of the building up of phase difference between solutions.

The exact and perturbative solutions should agree when this phase difference is small. We estimate this phase difference in the following. The frequencies can be approximated as \( \omega_{ij,\pm} \approx |\vec{p}| (1 \pm 1) + (m_i^2 \pm m_j^2)/2|\vec{p}| \) (we assume that \( m < |\vec{p}| \) here). The criterion for the smallness of secular terms is thus:

\[
|\omega_{ij,\pm} - \tilde{\omega}_{ij,\pm}| t \ll 1 \Rightarrow t \ll \frac{2|\vec{p}|}{\left| (m_i^2 \pm m_j^2) - (\tilde{m}_i^2 \pm \tilde{m}_j^2) \right|}.
\]  (32)

Thus the set of masses that produces the largest frequency difference gives the lowest time at which the two solutions should differ from each other.

In the following we compare the exact result (22) and the complete perturbative result (c.f. Eq. (31)) numerically. We plot both results for representative time intervals. For simplicity we plot the asymmetry per unit phase space volume and take a typical momentum \( p \sim T \). The values of the parameters used are \( f = 3 \times 10^{-4}, v = 174 \text{ GeV}, \epsilon = 0.8, \delta = 0.3, \eta = 0.6, \kappa = 1.1, M = 1.5 \text{ GeV}, T = 100 \text{ GeV} \) and \( f = 10^{-7} \). The results are shown in Fig. (3).

We can estimate the time \( t_s \) at which secular terms become important using the criterion (32) and the expression for the masses (18). For the parameters chosen here, the largest
FIG. 3: Plots of active lepton asymmetry production (for one flavor) per unit of phase space as a function of time. The red and blue lines correspond to the exact and perturbative results, respectively. Each plot covers a different time interval (time is expressed in GeV$^{-1}$). We note that the exact and perturbative results agree very well up to $t \approx 6000 \approx t_s/3$, where we start to see small discrepancies that grow larger with time.

frequency difference is produced by $|\omega_{12,\pm} - \tilde{\omega}_{12,\pm}|$:

$$t_s \ll \frac{2|\vec{p}|}{|(m_1^2 \pm m_2^2) - (\tilde{m}_1^2 \pm \tilde{m}_2^2)|} \sim \frac{|\vec{p}|}{M\Delta M \left(\frac{1+\epsilon^2+\delta^2}{(\epsilon+\delta)\kappa}\right)},$$

(33)

which gives $t_s \sim 21000$ GeV$^{-1}$. Note that all estimates are roughly given by $t_s \sim |\vec{p}|/M\Delta M$.

As expected the two solutions follow each other nicely until $t \sim 6000 \sim t_s/3$ where
discrepancies appear. Those discrepancies are relatively small (roughly 10%), but they steadily grow with time; see Fig. 4 for plots of the absolute error between the two results. At $t \sim t_s \sim 21000$ the absolute error becomes roughly as large as the absolute value of both asymmetries. We thus conclude that the exact and perturbative methods agree with each other from small times to a time $t_s$ at which secular terms are large and perturbation theory breaks down.

V. CONCLUSION

First principles computations of the lepton (baryon) asymmetry are difficult because some degrees of freedom need to be out-of-equilibrium in order to get a non-zero result, and the treatment of those degrees of freedom using quantum field theory is unwieldy. In this paper we have derived from first principles of quantum field theory and statistical mechanics a (simple) formula that could be used as a starting point for a perturbative computation of the baryon asymmetry of the universe. Our formalism is quite general and can be applied to other models. The only assumption entering into our derivation is that perturbation theory must be valid; physically this translates into the condition that the total number of out-of-equilibrium degrees of freedom (sterile neutrinos in our case) must remain small. This last condition depends on the parameters of the model under study.

We have also tested this formula for the asymmetry on an exactly solvable toy model. We have confidence that the method works and that it can be applied to a more complicated model involving damping. The application of this formalism to the \nu\text{MSM} and the study of
its phenomenology is work in progress.

Acknowledgments

The authors would like to thank T. Asaka, T. Hambye and J. Louis for useful comments and discussions. This work was supported in part by the Swiss National Science Foundation.

[1] E. Komatsu et al. (WMAP), Astrophys. J. Suppl. 180, 330 (2009), 0803.0547.
[2] A. D. Sakharov, Pisma Zh. Eksp. Teor. Fiz. 5, 32 (1967).
[3] K. Kajantie, M. Laine, K. Rummukainen, and M. E. Shaposhnikov, Phys. Rev. Lett. 77, 2887 (1996), hep-ph/9605288.
[4] M. Trodden, Rev. Mod. Phys. 71, 1463 (1999), hep-ph/9803479.
[5] A. Riotto and M. Trodden, Ann. Rev. Nucl. Part. Sci. 49, 35 (1999), hep-ph/9901362.
[6] M. Dine and A. Kusenko, Rev. Mod. Phys. 76, 1 (2004), hep-ph/0303065.
[7] W. Buchmuller (2007), 0710.5857.
[8] S. Davidson, E. Nardi, and Y. Nir, Phys. Rept. 466, 105 (2008), 0802.2962.
[9] A. Pilaftsis (2009), 0904.1182.
[10] M. Shaposhnikov, J. Phys. Conf. Ser. 171, 012005 (2009).
[11] T. Asaka, S. Blanchet, and M. Shaposhnikov, Phys. Lett. B631, 151 (2005), hep-ph/0503065.
[12] T. Asaka and M. Shaposhnikov, Phys. Lett. B620, 17 (2005), hep-ph/0505013.
[13] T. Asaka, M. Laine, and M. Shaposhnikov, JHEP 06, 053 (2006), hep-ph/0605209.
[14] T. Asaka, M. Laine, and M. Shaposhnikov, JHEP 01, 091 (2007), hep-ph/0612182.
[15] F. L. Bezrukov and M. Shaposhnikov, Phys. Lett. B659, 703 (2008), 0710.3755.
[16] M. Laine and M. Shaposhnikov, JCAP 0806, 031 (2008), 0804.4543.
[17] M. Shaposhnikov, JHEP 08, 008 (2008), 0804.4542.
[18] A. Roy and M. Shaposhnikov, Phys. Rev. D82, 056014 (2010), 1006.4008.
[19] L. Canetti and M. Shaposhnikov, JCAP 1009, 001 (2010), 1006.0133.
[20] D. Gorbunov and M. Shaposhnikov, JHEP 10, 015 (2007), 0705.1729.
[21] M. Shaposhnikov, J. Phys. Conf. Ser. 136, 022045 (2008), 0809.2028.
[22] W. Buchmuller and S. Fredenhagen, Phys. Lett. B483, 217 (2000), hep-ph/0004145.
[23] M. Lindner and M. M. Muller, Phys. Rev. D73, 125002 (2006), hep-ph/0512147.
[24] J. Berges, Nucl. Phys. A699, 847 (2002), hep-ph/0105311.
[25] G. Aarts, D. Ahrensmeier, R. Baier, J. Berges, and J. Serreau, Phys. Rev. D66, 045008 (2002), hep-ph/0201308.
[26] J. Berges, AIP Conf. Proc. 739, 3 (2005), hep-ph/0409233.
[27] A. De Simone and A. Riotto, JCAP 0708, 002 (2007), hep-ph/0703175.
[28] A. Anisimov, W. Buchmuller, M. Drewes, and S. Mendizabal, Annals Phys. 324, 1234 (2009), 0812.1934.
[29] V. Cirigliano, C. Lee, M. J. Ramsey-Musolf, and S. Tulin, Phys. Rev. D81, 103503 (2010), 0912.3523.
[30] M. Garny, A. Hohenegger, A. Kartavtsev, and M. Lindner, Phys. Rev. D80, 125027 (2009), 0909.1559.
[31] M. Garny, A. Hohenegger, A. Kartavtsev, and M. Lindner, Phys. Rev. D81, 085027 (2010), 0911.4122.
[32] M. Beneke, B. Garbrecht, C. Fidler, M. Herranen, and P. Schwaller (2010), 1007.4783.
[33] G. Sigl and G. Raffelt, Nucl. Phys. B406, 423 (1993).
[34] E. K. Akhmedov, V. A. Rubakov, and A. Y. Smirnov, Phys. Rev. Lett. 81, 1359 (1998), hep-ph/9803255.
[35] J. S. Gagnon, Nucl. Phys. A820, 199c (2009).
[36] F. Bezrukov, D. Gorbunov, and M. Shaposhnikov, JCAP 0906, 029 (2009), 0812.3622.
[37] E. Calzetta and B. L. Hu, Phys. Rev. D37, 2878 (1988).
[38] D. Boyanovsky and H. J. de Vega, Ann. Phys. 307, 335 (2003), hep-ph/0302055.
[39] R. Kubo, J. Phys. Soc. Jap. 12, 570 (1957).
[40] P. C. Martin and J. S. Schwinger, Phys. Rev. 115, 1342 (1959).
[41] V. S. Rychkov and A. Strumia, Phys. Rev. D75, 075011 (2007), hep-ph/0701104.
[42] E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569 (1990).
[43] D. Boyanovsky, H. J. de Vega, and S.-Y. Wang, Phys. Rev. D61, 065006 (2000), hep-ph/9909369.
[44] T. Altherr and D. Seibert, Phys. Lett. B333, 149 (1994), hep-ph/9405396.
[45] T. Altherr, Phys. Lett. B341, 325 (1995), hep-ph/9407249.
[46] C. Greiner and S. Leupold, Eur. Phys. J. C8, 517 (1999), hep-ph/9804239.
In the case of 3 active and 3 sterile neutrinos, we have 6 CP violating phases (see Table I). In the absence of interactions between charged leptons and $W$ bosons, the neutrino fields are not constrained anymore and it is possible to make a unitary transformation on them $\nu' = U\nu$ (where $U$ contains 3 real parameters and 6 phases). This additional freedom in $U$ can be used to remove the 6 CP phases in the Yukawa matrix $F_{\alpha I}$ and make the asymmetry vanish.

Note that the toy model has 5+1 parameters instead of the 6+2 parameters indicated in Table II. This reduction in the number of parameters is obtained by taking linear combinations of the neutrino fields and rearranging the Lagrangian. It is only possible in the absence of damping.