Time-periodic quantum states of weakly interacting bosons in a harmonic trap

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We consider identical quantum bosons with weak contact interactions in a two-dimensional isotropic harmonic trap, and focus on states at the Lowest Landau Level (LLL). At linear order in the coupling parameter $g$, we exploit the rich algebraic structure of the problem to give an explicit construction of a large family of quantum states with energies of the form $E_0 + g E_1 / 4 + O(g^2)$, where $E_0$ and $E_1$ are integers. As a result, any superposition of these states evolves periodically with a period of at most $8 \pi / g$ until, at much longer time scales of order $1 / g^2$, corrections to the energies of order $g^2$ become important and may upset this perfectly periodic behavior. We further construct coherent-like combinations of these states that naturally connect to classical dynamics in an appropriate regime, and explain how our findings relate to the known time-periodic features of the corresponding weakly nonlinear classical theory. We briefly comment on possible generalizations of our analysis to other numbers of spatial dimensions and other analogous physical systems.

I. INTRODUCTION

Systems of identical interacting bosons in a harmonic trap are a standard topic in the physics of cold atomic gases [4]. While much of the effort in this domain has been given to the emergence of Bose-Einstein condensation, the more basic and general question of the structure of quantum energy levels in the presence of interactions has likewise attracted a considerable amount of attention [2–11].

In our previous article [12], we systematically addressed the structure of fine splitting of the energy levels in a two-dimensional isotropic harmonic trap due to weak contact interactions at linear order in the coupling parameter [13]. Our goal has been to explore the operation of the symmetries of the system with respect to the diagonalization problems defining the level splitting, to describe optimizations in the computation of the spectra due to the symmetry structure, to establish bounds on the ranges of the spectra, and to investigate, in the spirit of the quantum chaos theory [14], the energy level statistics within the interaction-induced fine level splitting, in order to identify, or rule out, additional symmetry structures.

In this article, we shall present a more refined treatment of the energy levels within the Lowest Landau Level (LLL), identifying large families of explicit energy eigenstates. Particles in LLL states have the maximal amount of angular momentum for a given energy, and LLL restrictions have often been studied in relation to rapidly rotating Bose-Einstein condensates [15–17]. With respect to the fine structure splitting, the analysis of the LLL levels completely separates from all other levels, as explained in [12], and can be treated independently.

Our specific attention to the LLL levels is motivated by the known peculiarities of the corresponding classical problem. Indeed, the splitting of the energy levels at linear order in the coupling parameter is governed by the resonant LLL Hamiltonian [12]. The classical dynamics of this Hamiltonian has been investigated in a few works [18–20], driven by rather different motivations. Explicit classical solutions are known for the LLL Hamiltonian, and these known solutions have mode amplitude spectra exactly periodic in time [19]. One can then ask what features of the quantum energy eigenstates underlie these simple, explicitly known classical behaviors.

What we find is much more than what we ask for. While in principle, the time-periodic properties of the classical orbits could emerge from asymptotic properties of the quantum energy eigenstates at high energies, without any corresponding exact periodicities in the quantum theory, what we find is that there are ladders of equispaced energy levels at weak coupling, with distances between such ladder state energies given by a fixed number times the coupling constant. The wavefunctions of the ladder states can be constructed explicitly, and while these state form a small subset of all energy eigenstates, their number grows without bound as the number of particles increases. These states underlie the classical periodic behaviors, to which they can be explicitly connected by forming coherent-like combinations, and in fact provide a much richer array of periodic behaviors than what is known to exist in the classical theory.

While the derivations in our paper will be given for nonrelativistic bosons common in the physics of cold atomic gases, closely related structures arise in other branches of physics. Relativistic analogs of the systems we study exist in Anti-de Sitter spacetimes, displaying classical features closely reminiscent of the LLL Hamiltonian [21–23]. These features can be distilled into a specific class of partially solvable resonant Hamiltonians [24], and are rooted in the existence of breathing modes [25], similar to the Pitaevskii-Rosch breathing mode [2] or to the exactly periodic center-of-mass motion [26] in our present system. Quantum energy level cor-
rections due to interactions in Anti-de Sitter spacetimes, which are directly parallel to the nonrelativistic topics of our current paper, have attracted attention within high-energy theory [27]. Generalizations of our current findings to these relativistic cases (which we shall not immediately pursue here), will provide new results in that domain of study.

We shall now proceed with presenting the concrete results of our investigations. We will first review, in the next section, the basics of weakly nonlinear energy level splitting for bosons with contact interactions in a two-dimensional harmonic trap. In section III, we shall present an explicit construction of families of eigenstates with equidistant energies, so that arbitrary superpositions of such states evolve periodically in time. In section IV, we shall present coherent-like combinations of these states that have an explicit connection to time-periodic solutions of the classical weakly nonlinear theory. We shall conclude with a discussion, highlighting possible generalizations, and the way our quantum considerations elucidate, in a nontrivial way, the behavior of the corresponding classical theory.

II. WEAKLY INTERACTING BOSONS AT THE LOWEST LANDAU LEVEL

We shall give a very brief review of the energy level splitting for bosons in a harmonic trap due to weak contact interactions. Further details can be found in [12].

A. General energy levels

Our main object of study is the following second-order Hamiltonian:

\[ H = H_0 + g H_{\text{int}}, \]

\[ H_0 = \frac{1}{2} \int (\nabla \Psi^\dagger \cdot \nabla \Psi + (x^2 + y^2) \Psi^\dagger \Psi) \, dx \, dy, \]

\[ H_{\text{int}} = \pi \int \Psi^{12} \Psi^2 \, dx \, dy, \]

where the interaction strength \( g \) is taken to be small, \( 0 < g \ll 1 \). (The restriction to positive \( g \) is completely inessential, but simply adopted to simplify the wording in our considerations.) The field operators \( \Psi(x, y) \) and \( \Psi^\dagger(x, y) \) satisfy the standard commutation relations

\[ [\Psi^\dagger(x, y), \Psi(x', y')] = -\delta(x - x') \delta(y - y'). \]  

(2)

(There are known subtleties in defining contact interac-
tions discussed, for instance, in [3], which are, in particular, crucial in the context of [13], but they play no role in our treatment as we restrict our attention to the linear order in \( g \) [12].)

In a noninteracting theory with \( g = 0 \), the bosons independently occupy the energy levels in the harmonic potential, and the total energy is the sum of the individual (integer) energies. The field operator \( \Psi \) can be decomposed in terms of the harmonic oscillator eigenfunctions \( \psi_{nm} \) carrying \( n + 1 \) units of energy and \( m \) units of angular momentum (one must have \( m \in \{-n, -n + 2, \ldots, n - 2, n\} \)):

\[ \Psi(x, y) = \sum_{n,m} a_{nm} \psi_{nm}(x, y). \]  

(3)

The creation-annihilation operators \( a^\dagger_{nm} \) and \( a_{nm} \) satisfy

\[ [a^\dagger_{nm}, a_{n'm'}] = -\delta_{nn'}\delta_{mm'}. \]  

(4)

The Hamiltonian \( H_0 \) is diagonal in the Fock basis generated by these operators. To obtain a state with occupation numbers \( \{\eta_{nm}\} \) from the vacuum \( |0\rangle \), one writes

\[ |\{\eta_{nm}\}\rangle = \prod_{nm} \left( \frac{(a^\dagger_{nm})^{\eta_{nm}}}{\sqrt{\eta_{nm}!}} \right) |0\rangle. \]  

(5)

The corresponding energies (minus the vacuum energy) are then simply

\[ E(\eta) = \sum_{nm} n \eta_{nm}. \]  

(6)

The degeneracies of these energy levels are very high, given by the number of ways to partition a given integer \( N \) into different modes with particles. The degeneracies grow without bound at higher values of \( E \).

Turning on weak interactions, one can write

\[ H_{\text{int}} = \frac{1}{2} \sum_{n_1, n_2, n_3, n_4 \geq 0} \sum_{m_1 + m_2 = m_3 + m_4} a^\dagger_{n_1 m_1} a^\dagger_{n_2 m_2} a_{n_3 m_3} a_{n_4 m_4}. \]  

(7)

The condition \( m_1 + m_2 = m_3 + m_4 \) is dictated by the angular momentum conservation. We have furthermore introduced the interaction coefficients \( C_{n_1 n_2 n_3 n_4} \), which are a set of numbers computed from integrals of products of \( \psi_{nm} \) [12]. To compute the energy shifts at linear order in \( g \), one simply has to evaluate all matrix elements

\[ \langle \{\eta\}|H_{\text{int}}|\{\eta'\}\rangle \]  

(8)

between Fock states \( |\{\eta\}\rangle \) and \( |\{\eta'\}\rangle \) with the same energy \( E \), same number of particles \( N = \sum_{nm} \eta_{nm} \) and the same angular momentum \( M = \sum_{nm} m \eta_{nm} \). For any given \( N \), \( E \) and \( M \), this is a finite-sized matrix whose eigenvalues \( \varepsilon_\eta \) give the energy shifts in the fine structure of the original energy level \( E \) at order \( g \):

\[ \tilde{E}_\eta = E + g \varepsilon_\eta. \]  

(9)

One may notice that only those terms in (7) that satisfy \( n_1 + n_2 = n_3 + n_4 \) may contribute to the
matrix elements (8), since $|\eta\rangle$ and $|\eta'\rangle$ carry the same energy. One can then replace $H_{\text{int}}$ in (8) by

$$H_{\text{res}} = \frac{1}{2} \sum_{n_1, n_2, n_3, n_4} C_{n_1 n_2 n_3 n_4} a_{n_1 m_1}^\dagger a_{n_2 m_2} a_{n_3 m_3} a_{n_4 m_4}.$$  \hspace{1cm} (10)

The classical version of this resonant Hamiltonian is frequently encountered in studies of weakly nonlinear long-term dynamics of resonant PDEs, see, for instance, [18] for applications to the nonlinear Schrödinger equation in a harmonic potential, the classical version of (1).

Besides conserving $N$, $M$, and $E$, the resonant Hamiltonian (10) possesses extra conservation laws whose explicit form can be found in [12], originating from the breathing modes [25] of (1). These extra conservation laws impose relations between energy shifts in unperturbed energy levels with different values of $N$, $M$ and $E$. A detailed description of the patterns in the fine structure spectra induced by these symmetries can be found in [12], together with the ways the symmetries can be used for optimizing computations of the spectra. We shall give below an explicit description of these algebraic structures for the LLL levels, the case of interest for us in this article, where the mathematical details simplify and can be stated compactly.

**B. The LLL truncation**

The classical dynamics corresponding to (10) can be consistently truncated to the set of modes with the maximal amount of rotation for a given energy, namely, the modes in (3) satisfying $n = m$. Such a truncation has appeared in the literature under the name of the Lowest Landau Level (LLL) equation [18–20]. One does not in general expect that consistent classical truncations have direct implications in the quantum theory, since quantum variables cannot be simply set to zero. It turns out, however, that there is a precise quantum counterpart of the classical LLL truncation. Namely, one can convince oneself [12] that the only way to have $E = M$ in an unperturbed energy level (5) is if the only nonzero occupation numbers $\eta_{nm}$ are for modes with $n = m$. Thus, states of this type have vanishing matrix elements (8) with any other states, while their nonvanishing matrix elements among themselves are completely governed by the part of the resonant Hamiltonian (10) that only depends on the creation-annihilation operators with $n = m$. This part, the LLL Hamiltonian, can be expressed simply as

$$H_{\text{LLL}} = \frac{1}{2} \sum_{n_1, n_2, n_3, n_4} C_{n_1 n_2 n_3 n_4} a_{n_1 m_1}^\dagger a_{n_2 m_2} a_{n_3 m_3} a_{n_4 m_4}.$$ \hspace{1cm} (11)

with

$$C_{n_1 n_2 n_3 n_4} = \frac{(n_1 + n_2 + n_3 + n_4)/2)!}{2^{n_1 + n_2} \sqrt{n_1! n_2! n_3! n_4!}}.$$ \hspace{1cm} (12)

We have renamed $a_{n_i m_i}$ with $n_i = m_i$ to $a_{n_i}$.

The LLL Hamiltonian (11-12) commutes with the following operators, which reflect a subset of the conserved quantities of (10) that we have already briefly mentioned:

$$N = \sum_{k=0}^{\infty} a_{k}^\dagger a_{k}, \quad M = \sum_{k=1}^{\infty} k a_{k}^\dagger a_{k},$$

$$Z = \sum_{k=0}^{\infty} \sqrt{k + 1} a_{k+1}^\dagger a_{k}.$$ \hspace{1cm} (13)

Physically, $Z$ is a raising operator for the center-of-mass motion (which is an independent harmonic oscillator decoupled from the other degrees of freedom [26]). The algebra of these conserved quantities is given by

$$[M, Z] = Z, \quad [M, Z^\dagger] = -Z^\dagger,$$

$$[Z, Z^\dagger] = -N,$$ \hspace{1cm} (14)

with the remaining commutators vanishing.

The generalities of finding the spectra for systems of the form (11), with arbitrary interaction coefficients $C$, have been explained in [29], and are a simplified version of the brief presentation in the previous section (because the set of modes is simpler). We shall now review the construction of the spectra of (11-12) in more detail because this material forms essential backgrounds for the new derivations we shall present in this article.

The LLL Fock basis is given by

$$|\eta_0, \eta_1, \ldots\rangle = \prod_{k=0}^{\infty} \frac{(a_{k}^\dagger)^{\eta_{k}}}{\sqrt{\eta_{k}!}} |0, 0, 0, \ldots\rangle,$$ \hspace{1cm} (15)

such that, for any $k$,

$$a_{k}^\dagger a_{k} |\eta_0, \eta_1, \ldots\rangle = \eta_{k} |\eta_0, \eta_1, \ldots\rangle,$$ \hspace{1cm} (16)

where $\eta_{k}$ are nonnegative integers. The conserved quantities $N$ and $M$ are diagonal in this basis, and the corresponding eigenvalues are

$$N = \sum_{k=0}^{\infty} \eta_{k}, \quad M = \sum_{k=1}^{\infty} k \eta_{k}. $$ \hspace{1cm} (17)

Since $N$ and $M$ commute with the Hamiltonian, the diagonalization is performed independently at each value of $N$ and $M$. The number of states with a given value of $N$ and $M$ is the number of integer partitions of $M$ into at most $N$ parts [29], a well-known number-theoretic function denoted as $p_N(M)$. One thus has to diagonalize an explicit $p_N(M) \times p_N(M)$ numerical matrix to find the energy shifts (9) at each value of $N$ and $M$. 
The above description is true [29] for general values of the interaction coefficients $C$ in (11). For the specific values given by (12), there is an extra conserved quantity $Z$, which adds a new twist to the story. $Z$ commutes with $\mathcal{H}_{LLL}$, but it acts as a raising operator for $M$ according to (14). As a result, if $|\Psi\rangle$ is an eigenstate of $\mathcal{H}_{LLL}$ at level $(N, M)$ giving an energy shift $\varepsilon_I$, then $Z|\Psi\rangle$ is an eigenstate of $\mathcal{H}_{LLL}$ at level $(N, M + 1)$ with the same eigenvalue $\varepsilon_I$. The action of $Z$ thus copies the energy shifts from levels with lower $M$ to levels with higher $M$. In this way, one starts with the vector $|N, 0, 0, \ldots\rangle$ at level $(N, 0)$ satisfying

$$\mathcal{H}_{LLL}|N, 0, 0, \ldots\rangle = \frac{N(N - 1)}{2}|N, 0, 0, \ldots\rangle. \quad (18)$$

The corresponding energy shift $N(N - 1)/2$ will evidently be copied by the action of $Z$ to all other blocks with the same value of $N$. This shift, given by $N(N - 1)/2$, remains the largest one at all higher levels; it is also true that all eigenvalues of $\mathcal{H}_{LLL}$ are nonnegative [12]. One can show that, the energy shifts at each level $(N, M)$ are fully exhausted by the energy shifts copied through the action of $Z$ from levels with the same $N$ and lower $M$, and the new shifts that always correspond to eigenvalues in the kernel of $Z^\dagger$, i.e., states satisfying

$$Z^\dagger|\Psi\rangle = 0. \quad (19)$$

It is thus in principle enough to study the kernel (19) at each $(N, M)$-level to reconstruct all possible shifts at all values of $N$ and $M$, utilizing the action of $Z$ described above.

III. LADDER STATES

Having spelled out the systematics of diagonalizing the LLL Hamiltonian (11-12) and its connection to the energy shifts (9) at order $g$ for harmonically trapped bosons with contact interactions described by (1), we proceed with the main result of our paper, namely, the construction of an explicit family of eigenstates of (11-12), which we call the ladder states. In each $Z^\dagger$-kernel (19) of an $(N, M)$-level (with $2 \leq M \leq N$), there is precisely one such state with the eigenvalue

$$\varepsilon^{(N, M)} = \frac{N(N - 1)}{2} - \frac{NM}{4}. \quad (20)$$

Of course, the action of $Z$ copies such energies from lower levels to higher levels producing, within a given level, a ladder of states with evenly spaced energies of the form

$$\varepsilon^{(N, M)}_m = \frac{N(N - 1)}{2} - \frac{Nm}{4}, \quad (21)$$

where $m$ is any integer number satisfying $2 \leq m \leq \min(N, M)$. By our discussion above, among such levels, the ones with $m = M$ belong to the kernel (19).

Evidently, when the eigenvalues (21) are introduced as energy shifts in (9), one obtains sets of energy levels of (1) separated by gaps integer in units of $g/4$, neglecting the higher corrections of order $g^2$. This would mean that any superposition of the corresponding energy eigenstates will evolve periodically with a period of at most $8\pi/g$, until at very late times of order $1/g^2$ this periodic behavior may be upset by higher-order corrections to the energies.

In the rest of this section, our goal will be to establish the existence of states with energies of the form (20), and give explicit expressions for their wavefunctions. To this end, we start by examining the following operator:

$$B = \mathcal{H}_{LLL} - \frac{N(N - 1)}{2} + \frac{1}{4}(NM - ZZ^\dagger). \quad (22)$$

By (14), $B$ commutes with both $Z$ and $Z^\dagger$. Using (11-12) and (13), $B$ is itself of the form (11) with modified interaction coefficients $C$:

$$B = \mathcal{H}_{LLL} + \frac{1}{4} \sum_{n,m=0}^{\infty} (n - 2) a_n^\dagger a_m^\dagger a_n a_m - \frac{1}{4} \sum_{n,m=0}^{\infty} \sqrt{n + 1} \sqrt{m + 1} a_{n+1}^\dagger a_n a_{m+1}. \quad (22)$$

One can re-express this as

$$B = \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=0}^{j} A_{jk}^\dagger \left[ \frac{j!}{2^j} \sum_{l=0}^{j} A_{jl} + \frac{k!(j-k)!}{4} \right] \times \left( (j-4) A_{jk} - (k+1) A_{j,k+1} - (j-k+1) A_{j,k-1} \right)$$

$$\equiv \sum_{j=0}^{\infty} \sum_{k=0}^{j} B_{kl}^{(j)} A_{jk}^\dagger A_{jl}, \quad (23)$$

where we have defined $A_{jk} = \frac{a_k a_j}{\sqrt{k!(j-k)!}}$, $(A_{j,-1}$ and $A_{j,j+1}$, which formally occur at the boundaries of the $k$-summation in (23), must be understood as 0.) Since $A_{jk}$ is invariant under $k \to j - k$, to understand the features of (23), it suffices to study, for each $j$, the quadratic forms $v_{\ell}^{(j)} B_{kl}^{(j)} v_{l}^{(j)}$ within the subspaces $v_{k}^{(j)} = v_{j-k}^{(j)}$. The following properties of the matrices $(B_{kl}^{(j)})_{kl}$ within these subspaces deserve to be mentioned:

1. $(B_{kl}^{(j)})_{kl}$ vanishes for $j \leq 3$.

2. There are two zero eigenvalues corresponding to the eigenvectors

$$v_{k}^{(j)} = \frac{j!}{k!(j-k)!}, \quad (24)$$

and

$$v_{k}^{(j)} = \frac{(j-2)!}{(k-1)!(j-k-1)!}.$$
3. All other eigenvalues are positive. (We do not have an analytic proof for this last statement, but we have verified it for a number of different values of \(j\) by explicit diagonalization.)

Taken together, these properties imply that \(B\) of (22) is a nonnegative operator:

\[
\forall |\Psi\rangle : \langle \Psi | B |\Psi\rangle \geq 0. \tag{25}
\]

If this bound is saturated,

\[
B|\Psi\rangle = 0. \tag{26}
\]

If, in addition, (19) is satisfied, (20) is ensured, which opens a way to construct the ladder states. We note that the bound (25) guarantees that, if a state with the energy (20) exists within a given \((N, M)\)-level, it is the lowest energy state within that \((N, M)\)-level. As mentioned above, we do not have a complete proof of this bound, because property 3 in the list has not been proved analytically. This is, however, completely irrelevant for our construction of the ladder states below, since they are explicitly defined as states satisfying (26) and (19). (The conjectured bound (25) ascertains that the energies of the ladder states we find explicitly are the lowest ones within their respective \((N, M)\)-levels, which is indeed seen in numerical diagonalization of the LLL Hamiltonian within concrete \((N, M)\)-levels, but it is not necessary to construct the ladder states wavefunctions.)

We now proceed with an explicit construction of states with energies given by (20), for which one needs to solve (26) and (19) simultaneously. Property 1 listed above implies that \(B\) annihilates the following states

\[
|N - M, M, 0, 0, 0, \ldots\rangle, \tag{27}
\]

\[
|N - M + 1, M - 2, 1, 0, 0, 0, \ldots\rangle.
\]

(Indeed, all terms in \(B\) contain either at least one \(a_k\) with \(k \geq 3\), or \((a_j)^2\), all of which annihilate the above states.) These states, however, do not satisfy (19). One can remedy for that, remembering that \(Z\) commutes with \(B\), and therefore leaves (26) intact, while changing \(M\) to \(M + 1\). One can therefore take (27) at a lower value of \(M\), and transport them to the current \((N, M)\)-level by repeated action of \(Z\). This gives a large set of vectors satisfying (26) within the current \((N, M)\)-level, and one might hope to form a linear combination of these vectors satisfying both (26) and (19), yielding the desired ladder state with the energy (20).

An efficient way to implement in practice the general idea described above is as follows. Consider the set of vectors

\[
|\phi_{m}^{NM}\rangle = Z^{m} Z^{|m|} |N - M, M, 0, \ldots\rangle, \tag{28}
\]

for \(m = 0 \ldots M \leq N\). This is equivalent to considering the states \(Z^{m} |N - M + m, M - m, 0, \ldots\rangle\) since

\[
Z^{|m|} |N - M, M, 0, \ldots\rangle = \sqrt{\frac{M!(N - M + m)!}{(N - M)!(M - m)!}} |N - M + m, M - m, 0, \ldots\rangle. \tag{29}
\]

As \(B\) commutes with \(Z\) and \(Z^\dagger\) and annihilates \(|N - M, M, 0, \ldots\rangle\), all of these vectors solve (26). Note also that this set of vectors is linearly independent. This can be understood by noting that \(|\phi_{m}^{NM}\rangle\) contains a term with \(a_{m+1}^\dagger\), while all of the \(|\phi_{m}^{NM}\rangle\) with \(n < m\) have zero occupation number for this mode and only contain lower modes. Using (14) and (29), we observe that

\[
Z^\dagger |\phi_{m}^{NM}\rangle = Z^{m} Z^{|m|} Z^{|m|} |N - M, M, 0, \ldots\rangle
\]

\[
+ mN Z^{m-1} Z^{|m-1|} Z^{|m-1|} |N - M, M, 0, \ldots\rangle
\]

\[
= \sqrt{\frac{N - M + 1}{M}} (|\phi_{m}^{NM-1}\rangle + mN |\phi_{m-1}^{NM-1}\rangle).
\]

We now consider a linear combination of the vectors (28) at fixed \(M\) and \(N\):

\[
|\psi_{M}^{N}\rangle = \sum_{m=0}^{M} b_{m} |\phi_{m}^{NM}\rangle, \tag{30}
\]

and determine the coefficients for which this linear combination satisfies (19), whose left-hand side is now given by

\[
Z^\dagger |\psi_{M}^{N}\rangle \sim \sum_{m=1}^{M-1} (b_{m} |\phi_{m}^{NM-1}\rangle + mN b_{m} |\phi_{m-1}^{NM-1}\rangle)
\]

\[
+ b_{0} |\phi_{0}^{NM-1}\rangle + mN b_{0} |\phi_{0}^{NM-1}\rangle.
\]

Because the vectors (28) are linearly independent, imposing (19) gives the following recursion relation

\[
b_{m} = \frac{(-1)^{m}}{m! N^{m}} b_{0}, \tag{31}
\]

yielding an explicit (unnormalized) ladder state in the \(Z^\dagger\)-kernel of level \((N, M)\) in the form

\[
|\tilde{\psi}_{M}^{N}\rangle = \sum_{m=0}^{M} \frac{(-1)^{m}}{m! N^{m}} Z^{m} Z^{|m|} |N - M, M, 0, \ldots\rangle. \tag{32}
\]

By construction, this state satisfies (26) and (19), and hence it is an eigenstate of (11-12) with the eigenvalue (20). Another representation of such energy eigenstates can be found using the commutation relations of \(Z\) and \(Z^\dagger\):

\[
|\tilde{\psi}_{M}^{N}\rangle = \sum_{m=0}^{M} \frac{(-1)^{m}}{m! N^{m}} \prod_{k=1}^{m} (Z Z^\dagger - kN) |N - M, M, 0, \ldots\rangle.
\]

To compute the norm squared of (32), we keep in mind that \(|\tilde{\psi}_{M}^{N}\rangle\) satisfies (19), which yields

\[
\langle \tilde{\psi}_{M}^{N} | \tilde{\psi}_{M}^{N}\rangle = b_{0} \langle N - M, M, 0, \ldots | \tilde{\psi}_{M}^{N}\rangle.
\]

Using (29) and remembering that \(b_{0} = 1\) in (32), this can be written as

\[
\langle \tilde{\psi}_{M}^{N} | \tilde{\psi}_{M}^{N}\rangle = \sum_{m=0}^{M} \frac{(-1)^{m}}{m! N^{m}} \frac{M!}{(N - (M - m))! (M - m)!}.
\]
The right-hand side can be summed up in a closed form in terms of a generalized Laguerre polynomial:

\[
\langle \tilde{\psi}_M^N | \tilde{\psi}_M^N \rangle = \frac{M!}{N! M!} I^{-1}_M(-N). \tag{33}
\]

We then define the normalized ladder state in the \(Z^1\)-kernel of level \((N, M)\) with \(2 \leq M \leq N\):

\[
|\psi_M^N \rangle = \frac{\tilde{\psi}_M^N}{\sqrt{\langle \tilde{\psi}_M^N | \tilde{\psi}_M^N \rangle}}. \tag{34}
\]

It is instructive to compute the occupation numbers in the ladder states (34). We shall assume large \(M\) and \(N\) and a fixed ratio \(d^2 \equiv M/N < 1\). In this limit, as shown in Appendix A,

\[
\langle \psi_M^N | a_n^\dagger a_n | \psi_M^N \rangle \approx \frac{N}{n!} e^{d^2-d} (n-1-d^2)^2 (d-d^2)^n. \tag{35}
\]

This is exactly the same, under the identification \(|p| = \sqrt{d-d^2}\) as the amplitude spectrum \(|\alpha_n|^2\) of the classical stationary solutions of the LLL Hamiltonian given by (21) of [19] under the constraint that the classical expression for Z, which is (29) of [19], vanishes. Of course, while the amplitudes of the classical stationary solution are reproduced by (34), the classical evolution of the phases is not, since (34) is an eigenstate of the quantum Hamiltonian. One should think of (34) as an analog of energy eigenstates of a simple harmonic oscillator, whereas what one needs to fully relate to the classical evolution are analogs of coherent states. We shall construct such states explicitly in the next section, developing a thorough connection to the classical considerations of [19].

IV. COHERENT-LIKE STATES AND CONNECTION TO CLASSICAL THEORY

The ladder states given by (32) and (34) discovered in the previous section have the property that any superposition of such states evolves periodically with a period proportional to \(1/g\). This is very reminiscent of the periodicity properties of an explicit family of classical solutions of the LLL Hamiltonian constructed in [19]. It is natural to look for specific superpositions of the ladder state that connect to the classical dynamics of [19] as closely as possible. To this end, we define the following coherent-like states:

\[
|\alpha, \beta \rangle = e^{-|\alpha|^2/2} \sum_{N=0}^{\infty} \frac{\alpha^N}{\sqrt{N!}} (1 + |\beta|^2)^{-N/2} \beta^N.
\]

(36)

defined above become semiclassical in the sense that the standard deviation of \(a_n\) becomes small compared to the expectation value. This will be made manifest in our subsequent computations. (In all of our subsequent computations, the mode number \(n\) is kept fixed as \(|\alpha|\) is taken large.)

The first thing we would like to analyze in relation to (36) is the expectation value \(|\langle \alpha, \beta | a_n \rangle|\). The details of the computation are given in Appendix B. In a nutshell, the only nonvanishing matrix elements of the form \(\langle \psi_M^{N'} | a_n | \psi_M^N \rangle\) are for \(N' = N - 1\) and \(M' = M - n\). The sum over \(M\) is then evaluated using the fact that the binomial distribution is sharply peaked around its maximum at large \(N\), while the sum over \(N\) reduces to

\[
\sum_{N=1}^{\infty} \frac{N \xi_N}{N!} = \xi e^\xi. \tag{37}
\]

The end result is

\[
|\langle \alpha, \beta | a_n \rangle| \approx -a e^{-\frac{|\alpha|^2}{2}} \left( \frac{n-\frac{1}{2} |\beta|^2 + 1}{1 - \frac{|\beta|^2}{|\beta|^2 + 1}} \right) \frac{p^n}{\sqrt{n!}} \tag{38}
\]

where \(a\) and \(b\) can be read off by comparing the last line to the previous line, and

\[
p = -\frac{|\beta|}{\beta^*} \sqrt{\frac{1}{1 + |\beta|^2} - \frac{|\beta|^2}{1 + |\beta|^2}}. \tag{39}
\]

This expectation value thus agrees with the classical ansatz (21) in [19], supporting our view that the ladder states are a quantum counterpart of the time-periodical classical solutions of [19].

One can similarly evaluate the expectation values of the quantum operators \(N, M\) and \(Z\). Indeed, we give an explicit analysis in Appendix B of the expectation values of \(a_n^\dagger a_n\) in (36). At large \(|\alpha|\), the result is

\[
|\langle \alpha, \beta | a_n^\dagger a_n \rangle| \approx \frac{|\alpha|^2 e^{-|\alpha|^2}}{n!} \frac{\left( n - \frac{1}{2} |\beta|^2 + 1 \right)^2}{1 - \sqrt{\frac{|\beta|^2}{|\beta|^2 + 1}}} \tag{40}
\]

where \(|p|^2 = \sqrt{\frac{|\beta|^2}{|\beta|^2 + 1}} - \frac{|\beta|^2}{|\beta|^2 + 1}\). One then has the following expressions for the expectation values of \(N\) and \(M\):

\[
\sum_{n=0}^{\infty} \langle \alpha, \beta | a_n^\dagger a_n \rangle \langle \alpha, \beta \rangle = |\alpha|^2 \tag{41}
\]

\[
\sum_{n=0}^{\infty} \langle \alpha, \beta | n a_n^\dagger a_n \rangle \langle \alpha, \beta \rangle = |\alpha|^2 \frac{|\beta|^2}{1 + |\beta|^2}. \tag{42}
\]
Furthermore, \( \langle \alpha, \beta | Z | \alpha, \beta \rangle = 0 \) since (36) satisfy (19). Note that, if one extracts \( a \) and \( b \) from (38) and substitutes them to the classical expressions for the conserved quantities, given by (27-29) of [19], one obtains the same values: \( N = |\alpha|^2, M = |\alpha|^2 \frac{|\beta|^2}{1 + |\beta|^2} \), \( Z = 0 \). We shall explain how to obtain more general states with nonzero values of \( Z \) at the end of this section.

We now explore the time dependence of the states (36) under the evolution defined by the LLL Hamiltonian (11-12). In the language of the original system (1), this corresponds to viewing the system in a reference frame rotating with the harmonic trap frequency, and in terms of the slow time \( \tau = gt \). (In the formulas below giving time dependences, we shall simply use \( t \) to refer to this slow time.)

We start with computing

\[
\langle a_n \rangle_t = \langle \alpha, \beta | e^{i H_{LLL} t} a_n e^{-i H_{LLL} t} | \alpha, \beta \rangle.
\]

The computation is essentially identical to (38), which is described in Appendix B. The only nonvanishing matrix elements are still \( \langle \psi_{N-1}^\alpha | a_n | \psi_N^\alpha \rangle \), except that now they are multiplied with the phase factors \( e^{i \omega_{MN} t} \), where \( \omega_{MN} \) is expressed through the energies (20):

\[
\omega_{MN} = \varepsilon^{(N-1,M-n)} - \varepsilon^{(N,M)} = \frac{(n - 4)(N - 1) + M}{4}.
\]

One then has to repeat the computations of Appendix B taking account these phase factors, which is straightforward as the extra phases are linear in \( N, M, \) and \( t \). The sum over \( M \) is evaluated as in Appendix B. Thereafter, the sum over \( N \) is evaluated using (37) with \( \xi = |\alpha|^2 e^{-\frac{1}{4} \left( |\beta|^2 + n - 4 \right)} \), which gives

\[
\langle a_n \rangle_t = \langle a_n \rangle_0 e^{\frac{1}{4} \left( |\beta|^2 - 1 \right)} e^{-\frac{1}{8} \left( |\alpha|^2 - 1 \right)} e^{-\frac{1}{8} \left( |\alpha|^2 - 1 \right)} e^{-\frac{1}{8} \left( |\alpha|^2 - 1 \right)},
\]

(45)

where \( \langle a_n \rangle_0 \) is the same as in (38).

In order to study the classical limit, one needs to restore the factors of \( \hbar \). In the classical limit \( \hbar \to 0 \), the canonical coordinates \( x_n \sim \sqrt{\hbar}(a_n + a_n^\dagger) \) should stay finite, together with their conjugate canonical momenta, and the LLL Hamiltonian (11) must stay finite in terms of these canonical coordinates and momenta. This means that the LLL Hamiltonian (11) must be prefaced with \( \hbar^2 \). Together with the standard \( \hbar \) in front of the time derivative in the Schrödinger equation, this means that reinserting \( \hbar \) effectively amounts to replacing \( t \) by \( \hbar t \) in our expressions. The classical limit is then obtained by taking \( \hbar \to 0 \) while keeping \( \langle a_n \rangle_0^2 \hbar \) fixed. The quantum dynamics is expected to reproduce classical features as long as the square of the expectation value of \( a_n \) in the coherent state is small with respect to the modulus squared of its expectation value. By computations closely retracing the steps of Appendix B, we find

\[
\langle a_n \rangle_t^2 - \langle a_n \rangle _{t=0}^2 \approx \frac{\alpha^2 \hbar^2 t^2}{\sqrt{n!}}. 
\]

(47)

At \( t \to 1 \), \( \langle a_n \rangle_0^2 \sim 1/\hbar \) and \( \hbar \to 0 \), this expression vanishes, defining a classical state. One can similarly reinstate \( \hbar \) into (45) and take the \( \hbar \to 0 \) limit to recover the classical LLL solutions of [19].

Classical states of the form (21) in [19] define a three-dimensional invariant manifold. The coherent states (36) are constructed with two free parameters \( \alpha \) and \( \beta \) which tune \( N \) and \( M \), but are constrained to satisfy \( Z^t | \alpha, \beta > = 0 \) as they are explicitly built out of \( Z^t \)-kernel eigenstates. The third free parameter can be reinstated by applying the unitary operator \( e^{q^* Z - q Z^t} \) to the coherent states (36):

\[
| \alpha, \beta, q > \equiv e^{q^* Z - q Z^t} | \alpha, \beta >.
\]

(48)

This is in parallel to the classical story of [19], where the \( Z \)-transformation can be applied to solutions with \( Z = 0 \), to obtain other time-periodic solutions. After some algebra, one can show (see Appendix C) that these states reproduce the classical transformations given by (35) of [19] for \( a, b \) and \( p \),

\[
\langle \alpha, \beta, q | a_n | \alpha, \beta, q \rangle = \frac{e^{-pq - \frac{1}{2} q^2}}{\sqrt{n!}} \left( \frac{an}{p + q^* + b - aq} \right) (p + q^*)^n.
\]

V. DISCUSSION

Motivated by time-periodic behaviors with periods of order \( 1/g \) in the classical theory of Bose-Einstein condensates at the lowest Landau level [19], we have analyzed the fine splitting of the energy levels of the corresponding quantum problem at order \( g \). In the fine structure emanating from an integer unperturbed LLL level with \( N \) particles, \( M \) units of energy and \( M \) units of angular momentum, we have discovered ladders of equispaced energy eigenstates (with energy distances proportional to \( g \)). The energy shifts of these states (relatively to the unperturbed level of noninteracting bosons from which they split
of, in units of $g$, are given by (21), while their explicit (unnormalized) wavefunctions are
\[ |\psi_{m}^{(N,M)}\rangle = Z^{M-m} |\psi_{m}^{N}\rangle, \]  
(50)
with $|\psi_{m}^{N}\rangle$ given by (32) and $Z$ given by (13). Coherent-like combinations of these states can be constructed in the form (36) and (48). In an appropriate semiclassical regime, these states closely approximate the classical time-periodic dynamics discussed in [19]. We should emphasize, however, that the amount of time-periodicity one discovers in the quantum theory is much greater than what the classical theory would naively suggest. Indeed, the entire family of special time-periodic classical solutions in [19] is parametrized by three complex numbers. One the other hand, as manifest from our discussion, the corresponding time periodicities in the quantum theory are present for arbitrary superpositions of the giant family of states we construct.

While our main motivations were in exploring the structures of the spectra in the quantum theory, our considerations shed light on the corresponding classical dynamics, which is unusual, as one normally thinks of the quantum theory as being much more complicated than its classical counterpart. Indeed, since the bound (25) is established by analyzing the stucture of (23), a similar statement should hold in the classical theory as an inequality
\[ \mathcal{H}_{LLL} \geq \frac{N^2}{2} - \frac{NM - |Z|^2}{4}. \]  
(51)
It turns out that the time-periodic classical solutions of [19] precisely saturate this inequality, in an immediate relation to the eigenvectors (24). They thus lie at the bottom of a valley in phase space, which gives a natural explanation for the consistency of the corresponding ansatz, a puzzle since it was established in [19]. We remark that it would be very difficult to guess the inequality (51) from purely classical reasoning, while its analog in the corresponding quantum theory is strongly suggested by the patterns in the energy spectra visible from straightforward numerics in the spirit of [12, 29], which is how we have arrived at it in practice. Quantization, in this case, brings in surprising benefits.

There is a number of ways our considerations may generalize. The LLL Hamiltonian (11-12) is a particularly simple representative of a large class of resonant Hamiltonians constructed in [24], admitting special time-periodic classical solutions. Such Hamiltonians naturally arise in weakly nonlinear classical theory for Bose-Einstein condensates [30], for Hartree-type equations [23] and for relativistic analogs of these problems [21, 22], which are of interest in gravitational and high-energy physics. The generalities of how such solvable weakly nonlinear dynamics may arise as a controlled approximation to PDEs have been spelled out in [25]. This, in particular, implies that such analytic structures are to be expected for the Gross-Pitaevskii equation with a harmonic potential in one dimension, and, within the maximally rotating sector, in three dimensions, generalizing the two-dimensional considerations of our paper. It is legitimate to expect that the corresponding quantum systems will display features similar to what we have described here, and can be analyzed by similar means. The solvable time-periodic features often arise from truncating the corresponding theories to subsets of modes (spherically symmetric or maximally rotating, for example) in the weakly nonlinear regime. Some of such truncations (say, the maximally rotating sectors) will be inherited by the corresponding quantum theory, as happened to the LLL states in our present treatment. In other cases, the truncation may no longer be strictly speaking valid in the quantum theory, but it should still be approximately valid for states of high energy, where classical dynamics is approached. In such situations, one expects that ladders in the fine structure splitting, analogous to what we have seen for the LLL Hamiltonian, will emerge asymptotically in the fine structure of sufficiently high unperturbed energy levels. We intend to investigate some of these questions elsewhere [31].

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Appendix A: Expectation values in the ladder states

We start with computing the expectation value \( \langle \psi_M^N | a_n^\dagger a_n | \psi_M^N \rangle \) in the ladder states (34). In general, commuting \( a_n \) past \( Z^m \) contained in \( | \psi_M^N \rangle \) as per (32) gives

\[
a_n Z^m = \sum_{k=\max(0,n-m)}^n Z^{k+m-n} a_k \sqrt{\frac{n!}{(n-k)!(m-n+k)!}} Z^{m-k} Z^{m-k-1},
\]

(\text{A1})

In our case, any \( a_k \) with \( k \geq 2 \) annihilates \( Z^m | N-M, M, 0... \rangle \) contained in \( | \psi_M^N \rangle \), as seen from (29), effectively reducing (A1) to two terms at most. One can similarly commute \( a_k^\dagger \) to the left past \( Z^m \) contained in \( | \psi_M^N \rangle \), utilizing the same structure. As a result, three types of terms are left, according to which \( a_k^\dagger \) and \( a_k \) operate on the bra and the ket in this procedure (both with \( k = 0 \), one with 0 and one with 1, both with 1). The action of \( a_0 \) or \( a_1 \) on \( Z^m | N-M, M, 0... \rangle \) is straightforward, obtained by acting with these operators on (29). After that, evaluating the expectation value amounts to applying the identity [32]

\[
Z^m Z^{m'} = \sum_{k=0}^{\min(m,m')} \frac{N^k m! m'!}{k!(m-k)!(m'-k)!} Z^{m'-k} Z^{m-k},
\]

(A2)

followed again by (29).

One thus obtains the following expression, containing three terms, in accord with the three different ways \( a_n \) may produce nonvanishing contributions in the expectation value mentioned above:

\[
\langle \psi_M^N | a_n^\dagger a_n | \psi_M^N \rangle = A_1 \frac{n^2}{n!} + A_2 \frac{2n}{n!} + A_3 \frac{1}{n!},
\]

(A3)

with

\[
A_1 = \sum_{m,m'=n}^{M} \sum_{k=\max(0,-M+m+m'+2-n)}^{\min(m,n+1,m'-n+1)} C_{mm'k} \frac{(N+1-M+m+m'-n-k)!}{(m-n-k+1)!(m'-n-k+1)! (M-m-m'-2+n)!}
\]

\[
A_2 = \sum_{m,n=1}^{M} \sum_{k=\max(0,-M+m+m'-1-n)}^{\min(m,n+1,m'-n)} C_{mm'k} (N-M-m') \frac{(N-M+m+m'-n-k)!}{(m-n-k+1)!(m'-n-k+1)! (M-m-m'-1+n+k)!}
\]

\[
A_3 = \sum_{m,n=1}^{M} \sum_{k=\max(0,-M+m+m'-n)}^{\min(m,n,m'-n)} C_{mm'k} (N-M-m') (N-M-m) \frac{(N-1-M+m+m'-n-k)!}{(m-n-k)!(m'-n-k)! (M-m-m'+n+k)!}
\]

\[
C_{mm'k} \frac{(-1)^m m' (N-1)^k}{N^m m'} \frac{M!}{(N-M)!}
\]

Those three coefficients can be given a more uniform appearance in the following way,

\[
A_1 = \sum_{u,u'=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')}^{\min(u,u')} C_{uu'k} N^{-2n+2} \frac{(N-1-(M-n)+u+u'-k)!}{(u-k)!(u'-k)! (M-n-u-u'+k)!}
\]

\[
A_2 = - \sum_{u,u'=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')}^{\min(u,u')} C_{uu'k} N^{-2n+1} \frac{(N-(M-n)-(u')!(M-n-u-u'+k)!)}{(u-k)!(u'-k)! (M-n-u-u'+k)!}
\]

\[
A_3 = \sum_{u,u'=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')}^{\min(u,u')} \frac{C_{uu'k} N^{-2n} (N-(M-n)-(u')!(M-n-u-u'+k)!)}{(u-k)!(u'-k)!} \times \frac{(N-1-(M-n)+u+u'-k)!}{(M-n-u-u'+k)!}
\]

We now assume that \( N \) is large and apply the following simplification to the sums:

\[
\sum_{u=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')} (-1)^u (N-1)^k \frac{(N-1-(M-n)+u+u'-k)!}{(M-n-u-u'+k)!k!(u-k)!(u'-k)!} \]

\[
\sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(M-n-u-u'+k)!} \]

\[
\sum_{u=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')} \frac{(-1)^u (N-1)^k}{(M-n-u-u'+k)!k!(u-k)!(u'-k)!}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(M-n-u-u'+k)!} \]

\[
\sum_{u=0}^{M-n} \sum_{k=\max(0,-(M-n)+u+u')} \frac{(-1)^u (N-1)^k}{(M-n-u-u'+k)!k!(u-k)!(u'-k)!}
\]

\[
\sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{(M-n-u-u'+k)!} \]
where we used that $N^n \approx (N - 1)^n$ for large $N$. The last sum vanishes when $u \neq 0$, reducing the three sums to only one over $u$, with $u'$ and $k$ set to 0.

The case of $A_3$ needs one more step because of its additional $u'$-dependent factor $(N - (M - n - u))$. First, one notes that this factor can be split in two terms,

\[ (N - (M - n - u)) = (N - (M - n - u + k)) + k. \]

The first term can be approximated as before

\[
\sum_{u=0}^{M-n} \sum_{k=max(0,-(M-n)+u+u')}^{\min(u,u')} \frac{(-1)^u(N-1)^k}{N^u} \frac{(N-1-(M-n)+u+u'-k)!}{(M-n-u-u'+k)! k! (u-k)! (u'-k)!} (N-(M-n-u+k)) 
\]

\[
\approx \sum_{i=0}^{M-n-u'} \frac{(-1)^i(N-1-(M-n)+u'+l)!}{(N-1)! l!(M-n-u-l)!} (N-(M-n-l)) \sum_{k'=0}^{u'} \frac{(-1)^{k'} u'}{u! k'} \frac{1}{k'!}, \tag{A5}
\]

while for the second term, we have

\[
\sum_{u=0}^{M-n} \sum_{k=max(0,-(M-n)+u+u')}^{\min(u,u')} \frac{(-1)^u(N-1)^k}{N^u} \frac{(N-1-(M-n)+u+u'-k)!}{(M-n-u-u'+k)! k! (u-k)! (u'-k)!} (N-(M-n-u+k)) 
\]

\[
\approx \sum_{i=0}^{M-n-u'} \frac{(-1)^i(N-1-(M-n)+u'+l)!}{(N-1)! l!(M-n-u-l)!} \sum_{k'=0}^{u'} \frac{(-1)^{k'} u'}{u! k'} \frac{1}{k'!}. \tag{A6}
\]

This last sum over $k'$ is only nonzero for $u' = 1$. Such that

\[
A_3 \approx \frac{N^{-2n} M!}{(N-M)!} \sum_{i=0}^{M-n} \frac{(-1)^i}{(N-1)! l!(M-n-l)!} \frac{(N-1-(M-n)+l)!}{(N-(M-n))(N-(M-n+l))} 
\]

\[
- \frac{N^{-2n} M!}{(N-M)! N} \sum_{i=0}^{M-n-1} \frac{(-1)^i}{(N-1)! l!(M-n-l-1)!} \frac{(N-1-(M-n)+l+1)!}{(N-(M-n-l-1))(N-(M-n-l))(N-(M-n))}. 
\]

This can further be simplified by noting that

\[
\sum_{i=0}^{M-n} \frac{(-1)^i}{(N-1)! l!(M-n-l)!} = -1 \sum_{i=0}^{M-n-1} \frac{(-1)^i}{(N-1)! l!(M-n-l-1)!}. 
\]

In the limit of large $N$, $A_3$ therefore reduces to

\[
A_3 \approx \frac{N^{-2n} M!}{(N-M)!} \sum_{i=0}^{M-n} \frac{(-1)^i}{(N-1)! l!(M-n-l)!} (N-(M-n))^2, \tag{A7}
\]

which allows us to write (A3) as a complete square

\[
\frac{\langle \tilde{\psi}_M^N | a_{M n} | \tilde{\psi}_M^N \rangle}{\langle \psi_M^N | \tilde{\psi}_M^N \rangle} = \frac{A N^{-2n}}{B n!} (n(N-1) - (N-M))^2, \tag{A8}
\]

with

\[
A = \sum_{i=0}^{M-n} \frac{(-1)^i}{(N-1)! l!(M-n-l)!}, \tag{A9}
\]

\[
B = \sum_{i=0}^{M} \frac{(-1)^i}{i!(M-i)!}. \tag{A10}
\]
The ratio $A/B$ can be written as a function of generalized Laguerre polynomials as follows,

$$\frac{A}{B} = \frac{(N-1)^{-M+n}(N-1-(M-n))!L_{M-n}^N(-N+1)}{N^{-M}(N-M)!L_{M-n}^{-N-1}(-N)}.$$ 

For large $M, N$ with $M/N \equiv d^2$ fixed, we find

$$\frac{A}{B} = e^{d^2}N2^n \frac{(1-d^2)^n L_{M-n}^N(-N+1)}{N-M} L_{M-n}^{-N-1}(-N).$$

Using the limiting behavior of Laguerre polynomials of the form $L_n^{\alpha_n}(z_n)$ with $\lim_{n \to \infty} -\alpha_n/n > 1$, derived in [33, 34], one can show that for large $M$ and $N$, with fixed $M/N = d^2$,

$$\frac{L_{M-n}^N(-N+1)}{L_{M-n}^{-N-1}(-N)} \approx (1+d)e^{-d} \left(\frac{1}{1+1/d}\right)^n.$$  \hspace{1cm} (A11)

In particular, (7.5) of [33] describes the behavior of generalized Laguerre polynomials with a large parameter in the region of the complex plane of interest for us, while an explicit representation of the various functions appearing in this expression are given in (3.1) and (3.2) of [33] and (4.14) and (4.15) of [34].

Combining all of this, we find

$$\langle \psi_M^{N'}|a^\dagger_n a_n|\psi_M^N\rangle \approx \frac{N}{n!} \frac{e^{d^2-d}}{(1-d)^2} (n-(1-d^2))^2 (d-d^2)^n.$$  \hspace{1cm} (A12)

As a cross-check, this result in fact agrees well with numerical computations of such expectation values. It also has the right normalization of the amplitude spectrum:

$$\langle \psi_M^N|\sum_{n=0}^M a^\dagger_n a_n |\psi_M^N\rangle = N$$

$$\langle \psi_M^N|\sum_{n=0}^M n a^\dagger_n a_n |\psi_M^N\rangle = M,$$

for $M, N \gg 1$.

We now turn to matrix elements of a single annihilation operator $\langle \psi_M^{N'}|a_n|\psi_M^N\rangle$, which are only nonvanishing for $N' = N - 1$ and $M' = M - n$ and thus involve ladder states originating from different $(N, M)$-levels. Similarly to the previous computation, after having made use of (A1), only terms with $k + m - n = 0$ with $k = 1$ or $k = 0$ will contribute to the expectation value because $\langle \psi_M^N|Z = 0$. The expectation value of an annihilation operator can therefore be shown to be

$$\langle \psi_M^{N-1}|a_n|\psi_M^N\rangle = \frac{(-1)^n}{N^n \sqrt{n!}} \frac{M!(M-n)!}{(N-M)!(N-1-(M-n))!} \frac{(N-M-n(N-1))A}{\sqrt{\langle \psi_M^N|\psi_M^N\rangle \langle \psi_{M-n}^{N-1}|\psi_{M-n}^{N-1}\rangle}}$$  \hspace{1cm} (A13)

with

$$A = \sum_{u=0}^{M-n} \frac{(-1)^u}{u!(N-1)^u} \frac{(N-1-(M-n-u))!}{(M-n-u)!} = \frac{(N-1-(M-n))!}{(N-1)^{M-n}} L_{M-n}^{-N}(1-N).$$

Note that, as opposed to the computation of the expectation value of the product $a^\dagger_n a_n$, no approximations were made yet to obtain (A13). Using the representation of the norms as generalized Laguerre polynomials (33)

$$\langle \psi_M^N|\psi_M^N\rangle = \frac{M!}{N^M} L_M^{-N-1}(-N)$$

$$\langle \psi_{M-n}^{N-1}|\psi_{M-n}^{N-1}\rangle = \frac{(M-n)!}{(N-1)^{M-n}} L_{M-n}^{-N}(-N+1)$$
one obtains

$$\langle \psi_{M-n}^{(N-1)} | a_n | \psi_M^{N} \rangle = \frac{(-1)^n N^{M/2}}{N^n (N-1)^{(M-n)/2}} \sqrt{(N-1 - (M-n)!) (N-M-n(!)} \sqrt{N-M-n}(1-N) \sqrt{L_{M-n}^{-N}} (1-N).$$

In this expression, we recognize the fraction of Laguerre polynomials which was computed in (A11). Therefore, in the large \(N\) and \(M\) limit and fixed ratio \(d^2 \equiv M/N\) one finds

$$\langle \psi_{M-n}^{(N-1)} | a_n | \psi_M^{N} \rangle \approx \frac{(-1)^n \sqrt{N} (1 - d^2 - n) e^{d_0 - d_1/2}}{(1 - d)^{1/2} \sqrt{n!}} (d - d^2)^{n/2}.$$  \(\text{(A14)}\)

**Appendix B: Expectation values in the coherent states**

Given the expectation values in ladder states (A12) and (A14), one can evaluate similar expectation values in the coherent-like states (B2). We first compute the expectation value of \(a_n\) in \(|\alpha, \beta\rangle\):

$$\langle \alpha, \beta | a_n | \alpha, \beta \rangle = e^{-|\alpha|^2} \sum_{N=1}^\infty \frac{|\alpha|^{2N} \sqrt{N} / \alpha^*}{N!} (1 + |\beta|^2)^{-N+1/2} \times \sum_{M=n}^N \frac{|\beta|^{2M}}{\beta^{2n}} \sqrt{\frac{N!}{M!(N-M)!}} \sqrt{\frac{(N-1)!}{(M-n)!(N-1-(M-n))!}} \langle \psi_M^{(N-1)} | a_n | \psi_M^{N} \rangle.$$  \(\text{(B1)}\)

(Note that the second sum has \(N-1\) as its upper bound in the case \(n = 0\).) For \(|\alpha| \gg 1\) the first sum will be dominated by terms with \(N \sim |\alpha|^2\). Because of the presence of the binomial coefficient in the second sum, for large \(N\), the leading terms will be centered around \(M \sim N|\beta|^2/(1 + |\beta|^2)\), which means that we are in the regime where (A14) is a good approximation. Keeping in mind that

$$\sqrt{\frac{N!}{M!(N-M)!}} \sqrt{\frac{(N-1)!}{(M-n)!(N-1-(M-n))!}} \approx \frac{N!}{M!(N-M)!} \sqrt{\frac{(N-M)}{N}} \left(\frac{M}{N-M}\right)^{n/2},$$

the sum over \(M\) becomes schematically of the form \(\sum_M x^M(N/M) f(M/N)\), where \(f\) is a smooth function. The distribution \(x^M(N/M)\) has mean \(Nx/(1 + x)\) and a standard deviation of order \(\sqrt{N}\). Within the standard deviation, \(f\) essentially does not vary, provided that \(N\) is large, and can be taken outside the sum as \(f(x/(1 + x))\). Although this approximation is only valid in the large \(N\) limit, if one takes \(|\alpha|\) to be large, only terms with \(N\) large will contribute in the \(N\)-sum, so the approximation can be legitimately employed. In conclusion, one obtains

$$\langle \alpha, \beta | a_n | \alpha, \beta \rangle = e^{-|\alpha|^2} \sum_{N=1}^\infty \frac{|\alpha|^{2N} N e^{\beta^2/2} \sqrt{N} / \alpha^*}{N! \alpha^*} \sqrt{1 - \beta^2 / |\beta|^2 + 1} \beta^{2n} (-1)^n \left(\frac{|\beta|^2}{1 + |\beta|^2} + 1 \right)^{n/2} \left(\frac{1}{|\beta|^2 + 1} - n \right)^{n/2}.$$  \(\text{(B2)}\)

with

$$p = \frac{\sqrt{1 + |\beta|^2} - \sqrt{|\beta|^2}}{1 + |\beta|^2}. $$
Repeating these steps for the expectation value of $a_n^\dagger a_n$ in $|\alpha,\beta\rangle$

$$(\alpha, \beta | a_n^\dagger a_n | \alpha, \beta) = e^{-|\alpha|^2} \sum_{N=0}^{\infty} \frac{|\alpha|^{2N}}{N!} (1 + |\beta|^2)^{-N} \sum_{M=n}^{N} \frac{|\beta|^{2M}}{M!(N-M)!} \langle \psi_M^N | a_n^\dagger a_n | \psi_M^N \rangle,$$

and taking into account (A12), one finds:

$$(\alpha, \beta | a_n^\dagger a_n | \alpha, \beta) \approx \frac{e^{-|\alpha|^2}}{n!} \sum_{N=1}^{\infty} \frac{|\alpha|^{2N}}{N!} \frac{e^{-|\beta|^2} \left( n - \frac{1}{|\beta|^2+1} \right)^2 |\beta|^{2n}}{1 - \sqrt{\frac{|\beta|^2}{|\beta|^2+1}}},$$

with $|\beta|^2 = \sqrt{\frac{|\beta|^2}{|\beta|^2+1}} - \frac{|\beta|^2}{|\beta|^2+1}$.

**Appendix C: Construction of coherent-like states with nonzero $Z$**

We examine the expectation value of $a_n$ in the states (48). From the Baker-Campbell-Hausdorff identity,

$$e^{q^*Z - qZ^\dagger} = e^{q^*Z} e^{-qZ^\dagger} e^{-\frac{1}{2}|q|^2 N},$$

and (A1), we get

$$a_n e^{q^*Z} = e^{q^*Z} \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} a_m.$$  

(C2)

Moreover, on states $|\psi\rangle$ in the kernel of $Z^\dagger$, one finds

$$Z^\dagger a_m |\psi\rangle = (-1)^l \sqrt{\frac{(m+l)!}{m!}} a_{m+l} |\psi\rangle,$$

which implies

$$e^{qZ^\dagger} a_m |\psi\rangle = \sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \sqrt{\frac{(m+l)!}{m!}} a_{m+l} |\psi\rangle.$$  

(C4)

Therefore,

$$(\alpha, \beta, q | a_n | \alpha, \beta, q) = \langle \alpha, \beta | e^{-q^*Z} e^{qZ^\dagger} e^{-\frac{1}{2}|q|^2 N} a_n e^{q^*Z} e^{-qZ^\dagger} e^{-\frac{1}{2}|q|^2 N} |\alpha, \beta\rangle$$

$$= \langle \alpha, \beta | e^{qZ^\dagger} e^{-\frac{1}{2}|q|^2 N} a_n e^{q^*Z} e^{-\frac{1}{2}|q|^2 N} |\alpha, \beta\rangle$$

$$= \langle \alpha, \beta | e^{qZ^\dagger} e^{-\frac{1}{2}|q|^2 N} a_n e^{q^*Z} a_m |\alpha, \beta\rangle$$

$$= \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} \langle \alpha, \beta | e^{qZ^\dagger} e^{-\frac{1}{2}|q|^2 N} e^{q^*Z} a_m |\alpha, \beta\rangle$$

$$= \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} e^{-\frac{1}{2}|q|^2} \langle \alpha, \beta | e^{qZ^\dagger} e^{q^*Z} a_m |\alpha, \beta\rangle$$

$$= \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} e^{-\frac{1}{2}|q|^2} \langle \alpha, \beta | e^{qZ^\dagger} a_m |\alpha, \beta\rangle$$

$$= \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} e^{-\frac{1}{2}|q|^2} \langle \alpha, \beta | a_m |\alpha, \beta\rangle$$

$$= \sum_{m=0}^{\infty} \frac{(q^*)^{n-m}}{(n-m)!} \sqrt{\frac{n!}{m!}} e^{-\frac{1}{2}|q|^2} \langle \alpha, \beta | a_m |\alpha, \beta\rangle.$$
\[
\sum_{l=0}^{\infty} \frac{(-q)^l}{l!} \sum_{m=0}^{n} (q^*)^{n-m} \frac{n!}{m!(n-m)!} \left( \frac{a(m + l)}{p} + b \right) p^{m+l}
\]

where we used

\[
a_n e^{-\frac{4\pi^2}{3} N} = e^{-\frac{4\pi^2}{3}(N+1)} a_n
\]

in going from the second to the third line, and the Baker-Campbell-Hausdorff identity in going from the fourth to the fifth line.

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