Computation of Miura surfaces for general Dirichlet boundary conditions

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Abstract

The Miura ori is a very classical origami pattern used in numerous applications in Engineering. The study of the shapes that surfaces using this pattern can assume is still lacking. A nonlinear partial differential equation (PDE) that models the possible shapes that a periodic Miura tessellation can take in the homogenization limit has been established recently and solved only in specific cases. In this paper, the existence and uniqueness of a solution to the PDE is proved for general Dirichlet boundary conditions. Then a $H^2$-conforming discretization is introduced to approximate the solution of the PDE and a fixed point algorithm is proposed to solve the associated discrete problem. A convergence proof for the method is given as well as a convergence rate. Finally, numerical experiments show the robustness of the method and that non trivial shapes can be achieved using periodic Miura tessellations.

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1 Introduction

Origami inspired structures are used for multiple engineering applications. A classical example is solar panels for satellites. Indeed, the panels can be folded along the crease lines in very compact structures easily stored in a rocket and then unfolded in very wide panels when in space. More recently, origami inspired structures have gained attention as a mean to produce materials with a negative Poisson ratio and metamaterials. The science behind origami is also used to fold airbags for optimal deployment.

The Miura ori or Miura tessellation, first introduced in [12], is a well-known type of origami-inspired tessellation that has drawn a lot of attention over the years. It has found applications in solar panels and biology. Miura tessellations have often been deemed useful when unfolded in a flat plane. But, they can also achieve many non-planar shapes when partially unfolded which has recently allowed new applications. However, a lot remains unknown regarding the modeling of the shapes that Miura tessellations can take. Indeed, simulating their exact shape through mechanical modeling is computationally involved due to the large number of degrees of freedom (dofs) considered. The foldability of origami gives rise to notoriously difficult computational problems. Only a few periodic cells can be simulated as for instance in [19]. In [15, 18], the authors managed to proved that the in-plane and out-of-plane Poisson ratios of Miura tessellations are equal in norm but of opposite signs which indicates that Miura surfaces are generally saddle shaped. But that does not provide a way to compute Miura tessellations.

To remedy that issue, [13, 10] introduced a homogenization process leading to a set of equations that describe the shapes that Miura tessellations can fit in the limit $r/R \to 0$, where $r$ is the size of the pattern and $R$ is the global size of the structure. The resulting equations describe parametric
surfaces that are no longer discrete but continuous. The main advantage of the homogenization approach is that it greatly reduces the computational cost of simulating Miura surfaces as one does not need to take into account all the dofs stemming from each individual polygon any longer. Using this approach, the authors managed to determine all axisymmetric Miura surfaces. They also built an algorithm to produce some non-planar Miura tessellations but it fails in certain situations. The homogenization process produces a nonlinear elliptic PDE, as described in [10], that has remained unsolved. We believe that solving and providing a systematic and robust method to compute Miura surfaces will allow the exploration of the possible shapes that can be created with Miura tessellations. It might also help shed some light on determining what surfaces can Miura tessellations fit, which, to the best of our knowledge, remains unknown.

In Section 2, under regularity assumptions on the boundary conditions and the domain of the parametric surfaces, existence and uniqueness of solutions of the equation are proved. \( C^{2,\alpha} \) regularity of the parametric surfaces is also proved in the process. In Section 3, a \( H^2 \)-conforming finite element method (FEM) coupled to a fixed point method is introduced to approximate the solution of the elliptic equation. Subsequently, a first order convergence rate in \( H^2 \)-norm is proved for the FEM approximation. In Section 4, the convergence rate is verified on an analytical solution and then several non-analytical surfaces are computed for various Dirichlet boundary conditions so as to demonstrate the versatility and robustness of the proposed method.

2 Continuous equations

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex polygon that can be perfectly fitted by triangular meshes. Note that, due to the convexity hypothesis, the boundary \( \partial \Omega \) is Lipschitz [7] and verifies an exterior sphere condition. We impose strongly the Dirichlet boundary conditions \( \varphi = \varphi_D \) on \( \partial \Omega \) where \( \varphi_D \in (C(\partial \Omega))^3 \). \( \varphi_D \) is actually assumed to be more regular, as it should verify a bounded slope condition, as described p. 309 of [6], with constants \( K_i > 0 \), for each component \( \varphi_D^i \), where \( i \in \{1, 2, 3\} \).

2.1 Strong form equations

Let \( \varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \) be a parametrization of the homogenized surface constructed from a Miura tessellation. The coordinates of \( \varphi \) are written as \( \varphi^i \) for \( i \in \{1, 2, 3\} \). As proved in [10], \( \varphi \) is a solution of the following strong form equation:

\[
p(\varphi)\varphi_{xx} + q(\varphi)\varphi_{yy} = 0 \in \mathbb{R}^3,
\]

where

\[
p(\varphi) = \frac{4}{4 - |\varphi_x|^2}, \quad q(\varphi) = \frac{4}{|\varphi_y|^2},
\]

and the subscripts \( x \) and \( y \) stand respectively for \( \partial_x \) and \( \partial_y \).

Remark 1. Note that, because (1) is derived from zero energy deformation modes, it is not variational in the sense that it does not derive from an energy that could be interpreted as the elastic energy of the system. This has implications in the proof of existence of solutions as variational techniques cannot be used.

2.2 Main result

We introduce the Hilbert space \( V := (H^2(\Omega))^3 \cap (W^{1,\infty}(\Omega))^3 \). We consider the convex subset \( V_D := \{ \varphi \in V \mid \varphi = \varphi_D \text{ on } \partial \Omega \} \) as our solution space and the corresponding homogeneous space
is $V_0 := \{ \varphi \in V \mid \varphi = 0 \text{ on } \partial \Omega \}$. $V$ is equipped with the usual $(H^2(\Omega))^3$ Sobolev norm. Note that due to Rellich–Kondrachov theorem, Theorem 9.16 of [4], $V \subset \left(C^0(\Omega)\right)^3$. Let $A : V \to \mathbb{R}^3$ be the operator defined for $\varphi \in V$ and $\psi \in V$ as

$$A(\varphi)\psi := p(\varphi)\psi_{xx} + q(\varphi)\psi_{yy} \in \mathbb{R}^3.$$  \hfill (2)

Note that because the operator $A$ has no cross derivative terms, the maximum principle can be applied to each individual component. Solving Eq. (1) thus consists in finding $\varphi \in V_D$ such that

$$A(\varphi)\varphi = 0 \in \mathbb{R}^3.$$  \hfill (3)

To ensure ellipticity, we need $p(\varphi) > 0$. This is achieved through the following assumptions.

**Hypothesis 2** (Bounded slope condition). Let $K := (K_1, K_2, K_3) \in \mathbb{R}^3$, from the bounded slope condition. It is assumed thereafter that $\partial \Omega$ and $\varphi_D$ are such that,

$$|K|^2 \leq 4.$$  \hfill (4)

We make the following final assumption, so as to also make (1) a uniformly elliptic equation.

**Hypothesis 3** (Non-singular surface). It is assumed that there exists $\eta > 0$, the solutions of (1) verify $\eta \leq |\varphi_y|^2$ a.e. in $\Omega$, if they exist.

This assumption is reasonable since we are not interested in computed singular surfaces. Hypotheses 2 and 3 are assumed throughout the entire paper.

**Remark 4.** Hypothesis 3 is necessary to prove existence of solutions to the equation with the method used in the following. Indeed, using regularization techniques for non-uniformly elliptic equations, as in Theorem 12.7 of [6], would require $A(\varphi)$ to be defined over all of $\mathbb{R}^{3\times 2}$. Also, the uniform ellipticity allows to prove the Cordès condition below.

The main result of this section is the following

**Theorem 5** (Existence of a regular solution). There exists a unique solution $\varphi \in V_D$ of (3). $\varphi$ has the following extra regularity: there exists $\alpha \in (0, 1)$, $\varphi \in (C^{2,\alpha}(\Omega))^3$.

The proof will follow a similar path to the proof of Theorem 12.5 of [6]. It consists in getting regularity from the linear equation obtained by freezing the coefficients of $A(\varphi)$ and then using a fixed point argument.

### 2.3 Existence proof

In this entire section, we assume $\varphi \in V_D \cap C^{1,\alpha}(\Omega)^3$ is defined such that $\|\nabla \varphi\|_{C^0(\Omega)} \leq |K|$. The operator $A(\varphi)$ is then strictly elliptic as, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$p(\varphi)\xi_1^2 + q(\varphi)\xi_2^2 \geq |\xi|^2.$$  

We first focus on solving a linear problem related to the nonlinear problem (3).

**Proposition 6.** The equation search for $\psi \in V_D$ such that

$$A(\varphi)\psi = 0,$$  \hfill (5)

admits a unique strong solution and there exists $\alpha \in (0, 1)$ such that $\psi \in (C^{2,\alpha}(\Omega))^3$. The solution $\psi$ also verifies the following gradient estimates,

$$\sup_\Omega |\nabla \psi^i| \leq K_i, \quad \forall i \in \{1, 2, 3\},$$  \hfill (6)

where $K_i > 0$ are the constants from the bounded slope condition.
Proof. \( \Omega \) being a bounded convex domain, it satisfies an exterior sphere cone condition at every boundary point. Also, \( p(\varphi) \) and \( q(\varphi) \) are Hölder continuous and \( \varphi_D \in (C^0(\partial \Omega))^3 \). We can thus apply the classical result, Theorem 6.13 of [6] for strongly elliptic linear equations to obtain the existence of \( \psi \in V_D \), solution of (5). The \( C^{2,\alpha} \) regularity follows from the same theorem. The fact that \( A(\varphi) \) is diagonal is fundamental as the cited result is proved using the maximum principle which is not true in general for systems. Lemma 12.6 of [6] can finally be applied to obtain (6) because \( \partial \Omega \) and \( \varphi_D \) are assumed to verify a bounded slope condition with constants \( (K_i)_i \).

**Lemma 7** (Uniform ellipticity). The solution \( \psi \in V_D \) of Eq. (5) verifies \( p(\psi) > 0 \) and \( A(\psi) \) is uniformly elliptic.

**Proof.** Using (6) and Hypothesis 2, one has \( |\psi_x|^2 + |\psi_y|^2 \leq |K|^2 \leq 4 \) and thus, using Hypothesis 3,

\[
|\psi_x|^2 \leq 4 - \eta < 4.
\]

Thus \( p(\psi) > 0 \). Similarly, one has

\[
1 \leq p(\psi) \leq \frac{4}{\eta} \quad \text{and} \quad 1 \leq q(\psi) \leq \frac{4}{\eta}.
\]

Letting \( \lambda(\psi) \) and \( \Lambda(\psi) \) be respectively the smallest and largest eigenvalues of \( A(\psi) \), one has

\[
\frac{\Lambda(\psi)}{\lambda(\psi)} \leq \max\left(4, \frac{4}{\eta}\right) := \gamma.
\]

and thus \( A(\psi) \) is uniformly elliptic. \( \square \)

Let us now focus on the fixed point argument. Let \( T : V_D \ni \varphi \mapsto \psi(\varphi) \in V_D \) be the map that, given a \( \varphi \in V_D \), associates the solution to (5).

**Proposition 8.** The map \( T \) admits a fixed point \( \varphi^* \in V_D \) which is a solution of Eq. and \( \varphi^* \) benefits from the regularity results from Proposition 6.

**Proof.** We use the Schauder fixed point theorem, see Corollary 11.2 of [6]. The proof consists of three steps.

**Stability under \( T \)** We are going to prove that \( B \subset (C^{1,\alpha}(\Omega))^3 \) is stable under \( T \): \( T(B) \subset B \). First, let us notice that using the maximum principle, see Theorem 3.7 of [6], one has

\[
\|\psi\|_{C^0(\Omega)} = \max_{\Omega} |\psi| \leq \max_{\partial \Omega} |\varphi_D| = \|\varphi_D\|_{C^0(\partial \Omega)}.
\]

We have a \( C^1 \) bound on \( \psi \) because of (6). We now give a Hölder estimate of \( \nabla \psi \). Let \( |\cdot|_{\alpha}^* \) for \( \alpha \in (0,1) \) be a semi-norm such that

\[
|\nabla \psi|_{\alpha}^* := \sup_{z, z' \in \Omega} \min_{z \neq z'} (\text{dist}(z, \partial \Omega), \text{dist}(z', \partial \Omega))^{1+\alpha} \frac{|\nabla \psi(z) - \nabla \psi(z')|}{|z - z'|^{\alpha}}.
\]

Using Theorem 12.4 of [6], one has the following bound:

\[
|\nabla \psi|_{\alpha}^* \leq C\|\psi\|_{C^0(\Omega)} \leq C\|\varphi_D\|_{C^0(\partial \Omega)},
\]

(9)
where $C > 0$ only depends on $\gamma$ and not on $\varphi$. We define,

$$
B = \left\{ \tilde{\psi} \in \left( C^{1,\alpha}(\Omega) \right)^3 : \|\tilde{\psi}\|_{C^0} \leq \|\varphi_D\|_{C^0(\partial\Omega)}, \sup_{\Omega} |\nabla \tilde{\psi}|^2 \leq |K|^2, \right. \\
\left. |\nabla \tilde{\psi}|^*_\alpha \leq C\|\varphi_D\|_{C^0(\partial\Omega)} \right\},
$$

which is a closed convex subset of the Banach space $(C^1(\Omega))^3$ associated to the semi-norm:

$$
|\nabla \tilde{\psi}|^*_1 := \sup_{z \in \Omega} \text{dist}(z, \partial\Omega)|\nabla \tilde{\psi}(z)|.
$$

Using (8), (6) and (9), one notices that $T(B) \subset B$. Let $\| \cdot \|_1^*$ denote the norm associated with the semi-norm $| \cdot |^*_1$ applied to the gradient.

**Precompactness of $T(B)$**  The proof of this result is based on Lemma 6.33 of [6] which is similar to the Ascoli–Arzelà theorem, see Theorem 4.25 of [4]. By definition, the functions of $B$ are equicontinuous at every point in $\Omega$. Let us now prove a similar result on $\partial\Omega$. Let $z \in \Omega$ and $z_0 \in \partial\Omega$. Using Remark 3, p. 105 of [6], there exists a barrier function $w$, which only depends on $\gamma$, such that for $\epsilon > 0$,

$$
|\psi(z) - \varphi_D(z_0)| \leq \epsilon + k_\epsilon w(z),
$$

where $k_\epsilon$ is independent of $\varphi \in V_D$. Using $w(z) \to 0$, when $z \to z_0$, one proves the equicontinuity of $T(B)$ at $z_0 \in \partial\Omega$. Thus the functions of $T(B)$ are equicontinuous over $\tilde{\Omega}$ and since $T(B)$ is a bounded equicontinuous subset of $(C^{1,\alpha}(\Omega))^3$, using Lemma 6.33 of [6], $T(B)$ is precompact in $(C^1(\Omega))^3$.

**Continuity of $T$**  We prove that $T$ is continuous over $(C^1(\Omega))^3$ for the norm $\| \cdot \|_1^*$. Let $(\varphi_n)_n$ be a sequence of $B$ such that $\varphi_n \stackrel{n \to +\infty}{\longrightarrow} \varphi \in B$ for the norm $\| \cdot \|_1^*$. Let $\psi := T\varphi$ and $\psi_n := T\varphi_n$, for $n \in \mathbb{N}$. We want to prove that $\psi_n \stackrel{n \to +\infty}{\longrightarrow} \psi$ for $\| \cdot \|_1^*$. An immediate consequence of (6) is the equicontinuity of $(\nabla \psi_n)_n$, on compact subdomains of $\Omega$. Using Corollary 6.3, p. 93 of [6], one has in particular,

$$
d^2\|\nabla^2 \psi_n\|_{C^0(\Omega')} + d^{2+\alpha}\|\nabla^2 \psi_n|_{\alpha,\Omega'}^*_1 \leq C\|\psi_n\|_{C^0(\Omega)} \leq C\|\varphi_D\|_{C^0(\partial\Omega)},
$$

where $d \leq \text{dist}(\Omega', \partial\Omega)$ and $\Omega' \subset \tilde{\Omega}$. The constant $C > 0$ above does not depend on $\varphi_n$ because it depends on the $C^\alpha$ norm of $p(\varphi_n)$ and $q(\varphi_n)$ with $(\varphi_n)_n$ bounded in the $C^\alpha$ norm and $p$ and $q$ are $C^1$. Therefore, $(\nabla^2 \psi_n)_n$ is equicontinuous on compact subdomains of $\Omega$. Therefore, a subsequence $(\psi_{u(n)})_n$, converges uniformly for all $\Omega' \subset \tilde{\Omega}$, in $C^{2,\alpha}(\Omega')$ towards a function $\tilde{\psi} \in C^{2,\alpha}(\Omega)$ verifying $A(\varphi)\tilde{\psi} = 0$ in $\Omega$. Let us now look at the boundary conditions verified by $\tilde{\psi}$. Let $z \in \Omega$ and $z_0 \in \partial\Omega$. Using again an argument similar to Remark 3, p. 105 of [6], one has

$$
|\psi_{u(n)}(z) - \varphi_D(z_0)| \leq \epsilon + k_\epsilon w(z),
$$

where $w$ is independent of $n$, and thus gets $\tilde{\psi}(z) \to \varphi_D(z_0)$, when $n \to \infty$ and $z \to z_0$. Thus, by uniqueness of the solutions of (5), one has $\tilde{\psi} = \psi$. Therefore $T\varphi_{u(n)} \to \psi = T\varphi$ in $\tilde{\Omega}$, when $n \to \infty$. As $T(B)$ is precompact in $(C^1(\Omega))^3$, $T\varphi_{u(n)} \to \psi$, up to a subsequence, for $\| \cdot \|_1^*$. The expected result has been proved only for a subsequence. The reasoning above applies to any subsequence of $(T\psi_n)_n$ and thus $(T\psi_n)_n$ has $\psi$ as its only accumulation point. As the sequence $(T\psi_n)_n$ is bounded in $(C^1(\Omega))^3$ for $\| \cdot \|_1^*$ and has a unique accumulation point, the entire sequence $(T\psi_n)_n$ converges to $\psi$ for $\| \cdot \|_1^*$. Therefore, $T$ is continuous.
Conclusion Using a Schauder fixed point theorem, see Corollary 11.2 of [6], one has the existence of a solution $\varphi^*$ to (3) and using Proposition 6, $\varphi^* \in (C^{2,\alpha}(\Omega))^3$ for $0 < \alpha < 1$.  

2.4 Weak form equation

Let $W_D := T(B)$. The previous section proved that there exists strong solutions to (1) in $W_D$. However, working on the strong form equation is not well suited for finite element methods. We thus resort to a weak form equation that will still be solved by the solutions of (1). We consider a test function $\tilde{\psi} \in V_0$ and $\varphi \in W_D$ and define the form

$$a(\varphi, \tilde{\psi}) := \int_{\Omega} \Gamma(\varphi)A(\varphi) \varphi \cdot \Delta \tilde{\psi}, \quad (10)$$

where $\Delta : \mathbb{R}^3 \to \mathbb{R}^3$ is the Hodge Laplacian, which computes the Laplacian of each of the coordinates and

$$\Gamma(\varphi) := \frac{p(\varphi) + q(\varphi)}{p(\varphi)^2 + q(\varphi)^2}.$$

Lemma 9. There exists $0 < \Gamma_0 \leq \Gamma_1$, for all $\varphi \in W_D$,

$$0 < \Gamma_0 \leq \Gamma(\varphi) \leq \Gamma_1.$$ 

Proof. One has

$$1 \leq p(\varphi) \leq \frac{4}{\eta}, \quad 1 \leq q(\varphi) \leq \frac{4}{\eta}.$$ 

Thus,

$$0 < \frac{\eta^2}{32} \leq \Gamma(\varphi) \leq \frac{4}{\eta}.$$ 

Lemma 10. $a : W_D \times V \to \mathbb{R}$ is well defined.

Proof. Let $\varphi \in V_D$ and $\tilde{\psi} \in V_0$.

$$|a(\varphi, \tilde{\psi})| \leq \frac{4}{\eta} \int_{\Omega} (p(\varphi)|\varphi_{xx}| + q(\varphi)|\varphi_{yy}|) |\Delta \tilde{\psi}|,$$

$$\leq \frac{4\Gamma_1}{\eta} \int_{\Omega} (|\varphi_{xx}| + |\varphi_{yy}|) (|\tilde{\psi}_{xx}| + |\tilde{\psi}_{yy}|),$$

$$\leq \frac{16\Gamma_1}{\eta} \|\varphi\|_{H^2(\Omega)} \|\tilde{\psi}\|_{H^2(\Omega)}.$$ 

Equation (3) is thus reformulated into search for $\varphi \in W_D$ such that

$$a(\varphi, \tilde{\psi}) = 0, \quad \forall \tilde{\psi} \in V_0. \quad (11)$$

Lemma 11. Eq. (3) and (11) are equivalent.

Proof. Proving that solutions of (3) verify (11) is trivial. Let us consider $\varphi \in W_D$ solution of (11). Let $\tilde{\psi} \in V_0$. Classically, there exists a unique $\Psi \in (H^2(\Omega))^3$ such that $\Delta \Psi = \tilde{\psi}$ in $(L^2(\Omega))^3$ and $\Psi = 0$ on $\partial \Omega$. Thus

$$\int_{\Omega} \Gamma(\varphi)A(\varphi) \varphi \cdot \Psi = 0, \quad \forall \Psi \in (L^2(\Omega))^3.$$ 

Therefore, as $\Gamma(\varphi) \geq \Gamma_0 > 0$, $\varphi$ solves (3).
2.5 Uniqueness proof

To prove the uniqueness of solutions to (3), we define for $\varphi \in W_D$, the auxiliary bilinear form such that, for $\psi \in V_D$ and $\tilde{\psi} \in V_0$,

$$a(\varphi; \psi, \tilde{\psi}) := \int_\Omega \Gamma(\varphi)A(\varphi)\psi \cdot \Delta \tilde{\psi}. \tag{12}$$

Using $a$, we are going to prove that the map $T$ of Proposition 8 is Lipschitz which provides uniqueness of a solution of (3). First, we need to prove the coercivity of $a$. We follow [16] and prove the following lemma.

**Lemma 12 (Coercivity).** For $\varphi \in W_D$, the bilinear form $a(\varphi)$ is coercive over $V_0 \times V_0$. There exists $C > 0$, independent of $\varphi$, for all $\Psi \in V_0$,

$$C\|\Psi\|_{H^2(\Omega)}^2 \leq a(\varphi; \Psi, \Psi).$$

**Proof.** Following [16], we use the Miranda–Talenti theorem which states, there exists $C > 0$, for all $\Psi \in (H^2(\Omega))^3 \cap (H^1_0(\Omega))^3$,

$$|\Psi|_{H^2(\Omega)} \leq \|\Delta \Psi\|_{L^2(\Omega)},$$

$$\|\Psi\|_{H^2(\Omega)} \leq C\|\Delta \Psi\|_{L^2(\Omega)},$$

where $C$ is a constant depending only on the diameter of $\Omega$. Following [17], we use the fact that uniform ellipticity implies the Cordès condition: there exits $\varepsilon \in (0, 1]$, $p(\varphi)^2 + q(\varphi)^2 \leq \frac{1}{1 + \varepsilon}$.

The $\varepsilon$ does not depend on $\varphi$, because the bounds on $p(\varphi)$ and $q(\varphi)$ are independent from $\varphi$. Therefore

$$\|\Gamma(\varphi)A(\varphi) - I_2\|^2 = \Gamma(\varphi)^2(p(\varphi)^2 + q(\varphi)^2) - 2\Gamma(\varphi)(p(\varphi) + q(\varphi)) + 2$$

$$= 2 - \frac{(p(\varphi) + q(\varphi))^2}{p(\varphi)^2 + q(\varphi)^2} \leq 1 - \varepsilon,$$

where $I_2$ is the identity matrix. Let us now prove the coercivity of $a(\varphi)$,

$$a(\varphi; \Psi, \Psi) = \|\Delta \Psi\|_{L^2} - \int_\Omega (\Delta - \Gamma(\varphi)A(\varphi))\Psi \cdot \Delta \Psi$$

$$\geq \|\Delta \Psi\|_{L^2} - \sqrt{1 - \varepsilon}\|\Psi\|_{H^2(\Omega)}\|\Delta \Psi\|_{L^2}$$

$$\geq \frac{1 - \sqrt{1 - \varepsilon}}{C^2}\|\Psi\|_{H^2(\Omega)},$$

where $C > 0$ is the constant from the Miranda–Talenti theorem. \(\square\)

**Proposition 13 (Uniqueness).** Eq. (3) admits a unique solution.

**Proof.** We prove that $T$ is Lipschitz. Let $\varphi, \hat{\varphi} \in W_D$. Thus, there exists $\psi, \hat{\psi} \in W_D$,

$$a(\varphi; \psi, \tilde{\psi}) = a(\hat{\varphi}; \hat{\psi}, \tilde{\psi}), \quad \forall \tilde{\psi} \in V_0.$$

Let $\tilde{\psi} \in V_0$, one thus has

$$\int_\Omega (\Gamma(\varphi)A(\varphi) - \Gamma(\hat{\varphi})A(\hat{\varphi}))\psi \cdot \Delta \tilde{\psi} = -a(\hat{\varphi}; \psi - \hat{\psi}, \tilde{\psi}).$$
Note that $\bar{H}_F \subseteq C^2$, which are Lipschitz over $W_D$ and thus $\Gamma A$ is a Lipschitz operator. Thus, $|a(\bar{\varphi}; \bar{\psi}, \bar{\varphi})| \leq C\|\bar{\varphi} - \hat{\varphi}\|_{W^1,\infty(\Omega)}\|\Delta \bar{\psi}\|_{L^2(\Omega)}\|\Delta \hat{\psi}\|_{L^2(\Omega)}$. $\psi \in C^2(\Omega) = H^2$ over $\Omega$ which is bounded and is thus bounded in $H^2$-norm. Applying Lemma 12 with $\hat{\psi} := \psi - \hat{\psi}$, one has $\|\psi - \hat{\psi}\|_{H^2} \leq C\|\varphi - \hat{\varphi}\|_{W^1,\infty(\Omega)}$, where $C > 0$ is a generic constant independent of $\varphi$ and $\hat{\varphi}$.

3 Numerical scheme

Approximate solutions to (11) are computed using $H^2$-conformal finite elements and a fixed point method.

3.1 Discrete Setting

Let $(\mathcal{T}_h)_h$ be a family of quasi-uniform and shape regular triangulations [5], perfectly fitting $\Omega$. For a cell $c \in \mathcal{T}_h$, let $h_c := \text{diam}(c)$ be the diameter of $c$. Then, we define $h := \max_{c \in \mathcal{T}_h} h_c$ as the mesh parameter for a given triangulation $\mathcal{T}_h$ and $\mathcal{E}_h$ as the set of its edges. The set $\mathcal{E}_h$ is partitioned as $\mathcal{E}_h^I \cup \mathcal{E}_h^b$, where for all $e \in \mathcal{E}_h^b$, $e \subset \partial \Omega$ and $\cup_{e \in \mathcal{E}_h^b} e = \partial \Omega$.

As Eq. (11) is written in a subset of $(H^2(\Omega))^3$, we resort to discretizing it using vector Bell FEM [2], which are $H^2$-conformal. Let

$$V_h := \left\{ \varphi_h \in \mathbb{P}_3(\mathcal{T}_h)^3 | \forall e \in \mathcal{E}_h, \frac{\partial \varphi_h}{\partial n_e} \in \mathbb{P}_3(e)^{3 \times 2} \right\},$$

where $n_e$ is the normal to an edge $e \in \mathcal{E}_h$. Let $V_{hD} := \{ \varphi_h \in V_h | \varphi_h = \mathbb{I}_h \varphi_D \text{ on } \partial \Omega \}$, where $\mathbb{I}_h$ is the Bell interpolant [3] and $V_{h0}$ is the corresponding homogeneous space. To have a discrete equivalent to Hypothesis 3 and Eq. (6), we define for $\varphi \in V_D \cap C^1(\Omega)^3$,

$$\bar{p}(\varphi) := \left\{ \begin{array}{ll} \frac{4}{\eta} & \text{if } |\varphi_x|^2 \geq 4 - \eta \\ p(\varphi) & \text{otherwise} \end{array} \right., \quad \bar{q}(\varphi) := \left\{ \begin{array}{ll} 1 & \text{if } |\varphi_y|^2 \geq 4 \\ \frac{4}{\eta} & \text{if } |\varphi_y|^2 \leq \eta \\ q(\varphi) & \text{otherwise} \end{array} \right..$$

Note that $\bar{p}$ and $\bar{q}$ are bounded and Lipschitz. We define for $\varphi, \psi \in V_D \cap C^1(\Omega)^3$, the following operator

$$\bar{A}(\varphi)\psi := \bar{p}(\varphi)\psi_{xx} + \bar{q}(\varphi)\psi_{yy}.$$  

Note that for $\varphi_D \in W_D$ solution of (3), $\bar{A}(\varphi)\varphi = A(\varphi)\varphi = 0$. Let $\bar{\Gamma}$ be define as $\Gamma$ but using $\bar{p}$ and $\bar{q}$. We write the discrete problem as: search for $\varphi_h \in V_{hD}$, such that,

$$\bar{a}(\varphi_h, \psi_h) := \int_{\Omega} \bar{\Gamma}(\varphi_h)\bar{A}(\varphi_h)\varphi_h \cdot \Delta \psi_h = 0, \quad \forall \psi_h \in V_{h0}. \quad (15)$$

To compute a solution to (15), we resort to fixed point iterations. We thus define for $\varphi_h, \psi_h \in V_{hD}$, the following auxiliary bilinear form, such that for all $\psi_h \in V_{h0}$,

$$\bar{a}(\varphi_h; \psi_h, \psi_h) := \int_{\Omega} \bar{\Gamma}(\varphi_h)\bar{A}(\varphi_h)\psi_h \cdot \Delta \psi_h.$$  

(16)
Lemma 14. Given $\varphi_h \in V_{hD}$, the equation search for $\psi_h \in V_{hD}$ such that
\[ \bar{a}(\varphi_h; \psi_h, \tilde{\psi}_h) = 0, \quad \forall \tilde{\psi}_h \in V_{h0}, \] (17)
admits a unique solution.

Proof. Let $\varphi_h \in V_{hD}$. The coercivity of $\bar{a}(\varphi_h)$ over $V_{h0} \times V_{h0}$ is proved similarly as Lemma 12 by replacing $A$, $p$ and $q$ with $\bar{A}$, $\bar{p}$ and $\bar{q}$. As $\bar{a}(\varphi_h)$ is coercive on $V_{h0} \times V_{h0}$, we resort to defining $\Psi_h := \psi_h - \mathcal{I}_h \phi$, where $\phi \in (H^2(\Omega))^3$ is such that $\phi = \varphi_D$ on $\partial \Omega$. The solution of $\Delta \phi = 0$ with a similar Dirichlet boundary condition, is an example of such a function. We are now interested in searching for $\Psi_h \in V_{h0}$,
\[ \bar{a}(\varphi_h; \Psi_h, \tilde{\psi}_h) = -\bar{a}(\varphi_h; \mathcal{I}_h \phi, \tilde{\psi}_h), \quad \forall \tilde{\psi}_h \in V_{h0}. \]
This equation has a unique solution as $\bar{a}(\varphi_h)$ is coercive. \hfill \Box

3.2 Fixed point method

Proposition 15. The map $T_h : V_{hD} \ni \varphi_h \mapsto \psi_h(\varphi_h) \in V_{hD}$, where $\psi_h$ is the unique solution of (17), admits a fixed point $\varphi^*_h \in V_{hD}$ which verifies
\[ \bar{a}(\varphi^*_h, \tilde{\psi}_h) = \bar{a}(\varphi^*_h; \varphi^*_h, \tilde{\psi}_h) = 0, \quad \forall \tilde{\psi}_h \in V_{h0}, \] (18)
and there exists $C > 0$, independent of $h$,
\[ \|\varphi^*_h\|_{H^2(\Omega)} \leq C. \] (19)

Proof. Let us first find a stable convex domain $B \subset (H^2(\Omega))^3$. Let $\Psi_h$ be the solution of
\[ \bar{a}(\varphi_h; \Psi_h, \tilde{\psi}_h) = -\bar{a}(\varphi_h; \mathcal{I}_h \phi, \tilde{\psi}_h), \quad \forall \tilde{\psi}_h \in V_{h0}. \]
Therefore, using the coercivity of $\bar{a}(\varphi_h)$,
\[ \|\Psi_h\|_{H^2(\Omega)} \leq \frac{C^2}{1 - \sqrt{1 - \varepsilon}} \bar{a}(\varphi_h; \Psi_h, \Psi_h) = \frac{C^2}{1 - \sqrt{1 - \varepsilon}} \bar{a}(\varphi_h; \mathcal{I}_h \phi, \Psi_h) \leq \frac{C^2 \Gamma_1}{1 - \sqrt{1 - \varepsilon}} \|\mathcal{I}_h \phi\|_{H^2(\Omega)} \|\Psi_h\|_{H^2(\Omega)}. \]
We define $\psi_h := \Psi_h - \mathcal{I}_h \phi$. Thus
\[ \bar{a}(\varphi_h; \psi_h, \tilde{\psi}_h) = 0, \quad \tilde{\psi}_h \in V_{h0}. \]
Therefore, one has
\[ \|\psi_h\|_{H^2(\Omega)} \leq \|\mathcal{I}_h \phi\|_{H^2(\Omega)} + \|\Psi_h\|_{H^2(\Omega)} \leq \left(1 + \frac{C^2 \Gamma_1}{1 - \sqrt{1 - \varepsilon}} \frac{16}{\eta}\right) \|\phi\|_{H^2(\Omega)}. \]
Let $B$ be the ball of $V_h \subset (H^2(\Omega))^3$ centred in 0 with radius given in the previous equation. Thus $T_h(B) \subset B$.

Let us now prove the continuity of $T_h$. We follow [3] and actually prove that $T_h$ is Lipschitz. Let $\varphi_h, \tilde{\varphi}_h \in V_{hD}$ and $\tilde{\psi}_h, \tilde{\psi}_h \in V_{h0}$, one thus has
\[ \int_{\Omega} \left(\bar{A}(\varphi_h) - \bar{A}(\tilde{\phi}_h)\right) T_h \varphi_h \cdot \Delta \tilde{\psi}_h = -\bar{a}(\tilde{\phi}_h; T_h \varphi_h - T_h \tilde{\varphi}_h, \tilde{\psi}_h). \]
Because $\bar{\Gamma}A$ is Lipschitz, one has
\[
|\bar{a}(\bar{\varphi}_h; T_h\bar{\varphi}_h - T_h\hat{\varphi}_h, \tilde{v}_h)| \leq C\|\nabla \varphi_h - \nabla \hat{\varphi}_h\|_{L^2(\Omega)}\|\Delta \tilde{v}_h\|_{L^2(\Omega)},
\]
\[
\leq C\|\nabla \varphi_h - \nabla \hat{\varphi}_h\|_{L^2(\Omega)}\|\psi_h\|_{H^2(\Omega)},
\]
\[
\leq C\|\nabla \varphi_h - \nabla \hat{\varphi}_h\|_{L^2(\Omega)},
\]
where $C > 0$ is a generic constant, independent of $h$. Using $\tilde{v}_h := T_h\varphi_h - T_h\hat{\varphi}_h$ and the coercivity of $\bar{a}$, one has
\[
\|T_h\varphi_h - T_h\hat{\varphi}_h\|_{H^2} \leq C\|\varphi_h - \hat{\varphi}_h\|_{H^2(\Omega)}.
\]
As a consequence of the Brouwer fixed point theorem \cite{4}, there exists $\varphi^*_h \in V_{hD}$ solution to (15). Also, the solution of (15) is unique because $T_h$ is Lipschitz.

A fixed point method requires an initial guess. A good initial guess is to consider the solution of $\bar{\varphi}$ on the second equation, one has $\varphi^*_h \in \psi \in C^{p,q}$ as a consequence of the Brouwer fixed point theorem \cite{4}, there exists $\varphi^*_h \in V_{hD}$ solution to (15). Let $\varepsilon > 0$ be a tolerance. Note

\section{Numerical Scheme}

Algorithm 1 Fixed point

1: Compute $\psi_{h,0} \in V_{hD}$ solution of $\int_{\Omega} \nabla \psi_{h,0} \cdot \nabla \tilde{v}_h = 0$, for all $\tilde{v}_h \in V_{h0}$.
2: For $n \in \mathbb{N}$,
   1: Let $\varphi_{h,n+1} := \psi_{h,n}$.
   2: Compute $\psi_{h,n+1} \in V_{hD}$ solution of $\bar{a}(\varphi_{h,n+1}; \psi_{h,n+1}, \tilde{v}_h)$, for all $\tilde{v}_h \in V_{h0}$.
   3: Check if $\|\varphi_{h,n+1} - \psi_{h,n+1}\|_{H^2(\Omega)} \leq \varepsilon$. If true, end the computation and $\psi_{h,n+1}$ is the result. If false, start again at step 2.

that, as expected, a linear rate of convergence with respect to the number of iterations $n$, has been observed with Algorithm 1.

\subsection{Convergence rate}

To be able to give a convergence rate, we make the stronger regularity assumption $\varphi_D \in \left(H^2_0(\Omega)\right)^3$. Before giving the convergence rate, we prove the following lemma.

\begin{lemma}
The solution $\varphi \in W_D$ of (3) is such that $\varphi \in H^3(\Omega)^3$.
\end{lemma}

\begin{proof}
Let $p := p(\varphi)$, $q := q(\varphi)$, $\psi := \varphi_x$ and $\hat{\psi} := \varphi_y$. Note that, with the previous results, $p, q \in C^\infty(\Omega)$. We derive in the sense of distributions (3) with respect to $x$ and $y$ and get
\[
\begin{cases}
p\psi_{xx} + q\psi_{yy} = -p_x \varphi_{xx} - q_x \varphi_{yy} =: f \in L^2(\Omega)^3, \\
p\hat{\psi}_{xx} + q\hat{\psi}_{yy} = -p_y \varphi_{xx} - q_y \varphi_{yy} =: g \in L^2(\Omega)^3.
\end{cases}
\]
As in the proof of Lemma 12, we consider $\Psi := \psi - \phi$, where $\phi \in \left(H^2(\Omega)\right)^3$ is equal to the trace of $\psi$ on $\partial \Omega$. Let $\hat{f} := -p\phi_{xx} - q\phi_{yy} \in L^2(\Omega)^3$. Then one has
\[
p\Psi_{xx} + q\Psi_{yy} = f + \hat{f}.
\]
Using Lemma 12, one has
\[
C\|\Delta \Psi\|_{L^2(\Omega)} \leq \Gamma_1\|f + \hat{f}\|_{L^2(\Omega)}\|\Delta \Psi\|_{L^2(\Omega)},
\]
where $C > 0$. Using the Miranda–Talenti lemma, one has $\Psi \in \left(H^2(\Omega)\right)^3$. Using a similar treatment on the second equation, one has $\varphi \in \left(H^3(\Omega)\right)^3$.
\end{proof}
Theorem 17. The sequence \((\varphi_h)_h \in V_{hD}^N\) of solutions of (11) converges towards \(\varphi \in V_D\), solution of (3), with the following convergence estimate,
\[
\|\varphi - \varphi_h\|_{H^2(\Omega)} \leq C(\varphi) h, \tag{20}
\]
where \(C(\varphi) > 0\) is a constant depending on \(\varphi\).

Proof. As \(\varphi \in (H^3(\Omega))^3\), one has only the classical interpolation error [3],
\[
\|I_h\varphi - \varphi\|_{H^2(\Omega)} \leq Ch||\varphi||_{H^3(\Omega)},
\]
where \(C > 0\) is independent of \(\varphi\).

Let us define the following scalar product, for \(\psi, \tilde{\psi} \in V\),
\[
(\psi, \tilde{\psi})_h := \frac{1}{K} \int_\Omega \tilde{\Gamma}(\varphi_h)\tilde{A}(\varphi_h)\psi \cdot \Delta \tilde{\psi},
\]
where \(K > 0\) whose value is chosen below. The associated norm over \(V_0\) is written \(\|\psi\|^2_h := \sqrt{(\psi, \tilde{\psi})_h}\).

Lemma 10, shows that there exists \(C > 0\), independent of \(\varphi_h\),
\[
\|\psi\|_h \leq C\|\psi\|_{H^2(\Omega)}.
\]

Lemma 12, shows that \(| \cdot |_h\) is nonnegative and combined with the previous statement that it is equivalent to the \(H^2\)-norm over \(V_0\). The nonlinear discrete equation is reformulated into,
\[
(\varphi_h, \tilde{\psi}_h)_h = 0, \quad \forall \tilde{\psi}_h \in V_{h0},
\]
while for the continuous nonlinear equation, one has
\[
(\varphi, \tilde{\psi}_h)_h = \int_\Omega (\tilde{\Gamma}(\varphi_h)\tilde{A}(\varphi_h) - \tilde{\Gamma}(\varphi)\tilde{A}(\varphi))\varphi \cdot \Delta \tilde{\psi}_h, \quad \forall \tilde{\psi}_h \in V_{h0}.
\]

Letting \(\tilde{\psi}_h \in V_{h0}\), one thus has
\[
(I_h\varphi, \tilde{\psi}_h)_h = (I_h\varphi - \varphi, \tilde{\psi}_h)_h + \int_\Omega (\tilde{\Gamma}(\varphi_h)\tilde{A}(\varphi_h) - \tilde{\Gamma}(\varphi)\tilde{A}(\varphi))\varphi \cdot \Delta (I_h\varphi - \varphi_h).
\]

Taking \(\tilde{\psi}_h = I_h\varphi - \varphi_h\), one has
\[
\|I_h\varphi - \varphi_h\|^2_h = (I_h\varphi - \varphi, I_h\varphi - \varphi_h)_h + \int_\Omega (\tilde{\Gamma}(\varphi_h)\tilde{A}(\varphi_h) - \tilde{\Gamma}(\varphi)\tilde{A}(\varphi))\varphi \cdot \Delta (I_h\varphi - \varphi_h).
\]

The first term in the right-hand side of the previous equation, written \(T_1\), is bounded as such,
\[
T_1 \leq \|I_h\varphi - \varphi\|\|I_h\varphi - \varphi_h\|_h.
\]

Regarding the second, written \(T_2\), one has
\[
T_2 \leq \frac{C}{K}\|\varphi_h - \varphi\|_{H^2(\Omega)}\|\Delta (I_h\varphi - \varphi)\|_{L^2(\Omega)} \leq \frac{C}{K}\left(\|I_h\varphi - \varphi_h\|^2_{L^2(\Omega)} + \|I_h\varphi - \varphi\|_{L^2(\Omega)}\|I_h\varphi - \varphi_h\|_{H^2(\Omega)}\right),
\]
where the constant \(K > 0\) is chosen so that \(C < K\). One thus has,
\[
\|I_h\varphi - \varphi_h\|_h \leq C\|I_h\varphi - \varphi\|_{H^2(\Omega)},
\]
and thus
\[
\|\varphi - \varphi_h\|_h \leq C\|I_h\varphi - \varphi\|_{H^2(\Omega)} \leq C\|\varphi\|_{H^3(\Omega)}.
\]

To finish, we invoke the equivalence of \(| \cdot |_h\) with the \(H^2\)-norm over \(V_0\).
4 Numerical tests

The method is implemented in the FEM software Firedrake [14]. Because it is not possible to impose strongly Dirichlet boundary conditions in Firedrake for the Bell finite element at the moment, we resort to a least-square penalty. Let us define the bilinear form

$$b_h(\psi_h, \tilde{\psi}_h) := \bar{\eta} \sum_{e \in \mathcal{E}_h} h_e^{-4} \int_e \psi_h \cdot \tilde{\psi}_h,$$

(21)

where $\bar{\eta}$ is a user defined penalty coefficient and $h_e := \text{diam}(e)$ for $e \in \mathcal{E}_h$. The corresponding right-hand side is

$$l_h(\tilde{\psi}_h) := \bar{\eta} \sum_{e \in \mathcal{E}_h} h_e^{-4} \int_e \varphi_D \cdot \tilde{\psi}_h.$$

(22)

We thus solve at each iteration,

$$\bar{a}(\varphi_h; \psi_h, \tilde{\psi}_h) + b_h(\psi_h, \tilde{\psi}_h) = l_h(\tilde{\psi}_h), \quad \forall \tilde{\psi}_h \in V_h.$$

(23)

For all numerical tests, the $\varepsilon$ in Algorithm 1 is set to $\varepsilon := 10^{-5}$ and the penalty parameter $\bar{\eta}$ is set to $\bar{\eta} := 10$. 

4.1 Minimal surface

The domain is a rectangle $\Omega = [0, L] \times [0, H]$, where $L = 2$ and $H = 1$. The boundary of the domain is folded by an angle $\alpha$ along the segment $[AC]$ as sketched in Figure 1. Therefore, on the lines of equation $x = 0$ and $y = 0$, the imposed boundary conditions is $\varphi_{D1}(x, y) = (x, y, 0)^T$, whereas on the lines of equation $x = L$ and $x = H$, the imposed boundary condition is $\varphi_{D2}(x, y) = (1 - \frac{x}{L})B'C + (1 - \frac{y}{H})B'A + OB'$, where

$$\overrightarrow{DB'} = BD \sin(\alpha)(0, 0, 1)^T + \cos(\alpha)\overrightarrow{BB'}.$$

Figure 2 shows the surface computed with a structured mesh of size $h = 5.00 \cdot 10^{-2}$ and containing 15,138 dofs. We note that the computation stops after computing the first iteration. That suggests that the initial guess, for which $\Delta \varphi_h = 0$, is actually the solution and thus the solution is a minimal surface. That is confirmed by the fact that $\max_\Omega |\Delta \varphi_h| \sim 10^{-9}$. A closer inspection shows that actually $|\varphi_{h,xx}| \sim 10^{-9}$ and $|\varphi_{h,yy}| \sim 10^{-9}$. Therefore the surface is actually a ruled surface, which confirms the prediction made in [10], that Miura surfaces can be ruled surfaces. One can even deduce in that case that the analytical solution is

$$\varphi(x, y) = \frac{1}{LH} \left( \overrightarrow{OB'} - (L, 0, 0)^T - (0, H, 0)^T \right) xy + (1, 0, 0)^T x + (0, 1, 0)^T y.$$

As $\varphi \in V_h$, $\varphi_h = \varphi$ on any mesh.
4 NUMERICAL TESTS

4.2 Axisymmetric surface

This test case comes from [10]. The reference solution is

\[ \phi(x, y) = (\rho(x) \cos(\alpha y), \rho(x) \sin(\alpha y), z(x))^T, \]

where

\[ \begin{cases} 
\rho(x) = \sqrt{4c_0^2 x^2 + 1}, \\
z(x) = 2s_0 x, 
\end{cases} \]

\[ \alpha = (1 - s_0^2)^{-1/2}, \ c_0 = \cos\left(\frac{\theta}{2}\right), \ s_0 = \sin\left(\frac{\theta}{2}\right) \text{ and } \theta \in (0, \frac{2\pi}{\alpha}). \]

The domain is \( \Omega = [-s_0^*, s_0^*] \times [0, \frac{2\pi}{\alpha}], \) where \( s_0^* = \sin\left(\frac{1}{2} \cos^{-1}\left(\frac{1}{2 \cos\left(\frac{\theta}{2}\right)}\right)\right). \)

A convergence test is performed to show that the convergence rate proved in Theorem 17 is correct. Let \( \theta = \frac{\pi}{2}. \) The reference solution \( \phi \) is used as Dirichlet boundary condition on \( \partial \Omega. \) Table 1 contains the errors and estimated convergence rate.

| \( h \)    | nb dofs | \( H^2 \)-error | convergence rate | nb iterations |
|------------------|----------|------------------|------------------|---------------|
| 0.140            | 9,450    | 1.916e-04        | -                | 6             |
| 0.0702           | 35,838   | 2.377e-05        | 3.13             | 6             |
| 0.0351           | 139,482  | 3.005e-06        | 3.04             | 6             |
| 0.0175           | 550,242  | 3.817e-07        | 3.01             | 6             |

Table 1: Axisymmetric surface: estimated convergence rate and number of fixed point iterations.

The convergence rate is estimated using the formula

\[ \log \left( \frac{e_1}{e_2} \right) \log \left( \frac{\text{card}(T_{h_1})}{\text{card}(T_{h_2})} \right)^{-1}, \]

where \( e_1 \) and \( e_2 \) are the errors in \( H^2 \) semi-norm. The convergence rates presented in Table 1 are well above the first order rate proved in (20). This is due to the fact that \( \phi \) is far more regular than in the general case. Indeed, \( \phi \in C^\infty(\Omega)^3. \) In that case, the classical interpolation result [3] is \( \|I_h \phi - \phi\|_{H^2(\Omega)} \leq C h^3 \|\phi\|_{H^3(\Omega)} \) and we recover a convergence order of 3 as estimated in Table 1.

Further computations are performed with more realistic boundary conditions. Mirror boundary conditions are imposed on the lines of equation \( y = 0 \) and \( y = \frac{2\pi}{\alpha} \) of \( \Omega, \) which translates into the
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fact that the dofs on the two planes are one and the same and not doubled, but still unknown. The Dirichlet boundary condition imposed on the lines of equations $x = -s_0^*$ and $x = s_0^*$ are then only a circle centered around the $z$ axis, of radius $\sqrt{4c_0^2(s_0^*)^2 + 1}$ and contained in the planes of equations $z = \pm 2s_0s_0^*$. Figure 3 shows the computed surface for $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{4}$. We recover the two expected hyperboloids.

4.3 Non axisymmetric surface

The boundary conditions imposed will be those of a half cone. The initial domain is $\Omega = [0, \pi] \times [0, 1]$ and the imposed Dirichlet boundary condition is

$$
\varphi_D(x, y) = (x \cos(y), x \sin(y), x)^T.
$$

We use a structured triangular mesh of size $h = 0.041$ with 59,058 dofs. The resulting surface is presented in Figure 4 and required 17 fixed point iterations. Even though the left picture in Figure 4 looks very much like a cone, the right picture does not. Indeed towards the top of the surface, it tends to flatten. Therefore, if we glue a reflexion of the surface, it will be continuous but not $C^1$. As we proved that the surface is at least $C^1$, we cannot have a solution for a domain $\Omega$ larger in the $y$ component. This is confirmed by the numerical method that stops converging for such domains. This result confirms the results from [10] which classified all axisymmetric surfaces and cones are not one of them.

4.4 Deformed hyperboloid

This numerical test consists in deforming the hyperboloid of Section 4.2. The lower part of the cylinder stays unchanged whereas the upper part is slightly modified. The domain is $\Omega = [0, L] \times [0, H]$, where $L = 0.765$, $H = \frac{2\pi}{\alpha}$ and $\alpha = 1.41$. Periodicity is imposed on the lines of equations
$y = 0$ and $y = H$. Therefore, Dirichlet boundary conditions are imposed only on the lines of equation $x = 0$ and $x = L$ as

$$
\begin{align*}
\varphi_{D_1}(y) &= (R\cos(\alpha y), R\sin(\alpha y), -l)^T \text{ on } x = 0, \\
\varphi_{D_2}(y) &= (R\cos(\alpha y), R\sin(\alpha y), R\sin(\beta)\cos(\alpha y))^T \text{ on } x = L,
\end{align*}
$$

where $l = 1.08$, $R = 1.14$ and $\beta = \frac{\pi}{4}$. Figure 5 shows the computed surface for a structured mesh of size $h = 0.222$ and containing 14,760 dofs. The computation requires 53 fixed point iterations before reaching a satisfactory error. Such a non-trivial Miura surface could not be computed with previous methods not relying on solving Eq. (1).

5 Conclusion

In this paper, it was proved that assuming a specific bounded slope condition on the Dirichlet boundary condition, the existence of a unique solution to (1) can be asserted. Subsequently, a numerical method based on conforming finite elements and fixed point iterations was presented and proved to converge at order one towards the solution of (1). Finally, the convergence rate was validated on a numerical example and the method was used to compute a few non-trivial and non-analytical surfaces.

Further investigations into the homogenization process that produced (1) could prove valuable. This work raises the question of what kind of surfaces can be approximated by a Miura surface, which is left for future work. Investigations into how to build a Miura tessellation of a given size $r > 0$ from a given Miura surface seem to be a natural next step.

Competing interests

The author declares no competing interests.

Code availability

The code is available at https://github.com/marazzaf/Miura.git
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