Stability for small data: the drift model of the conformal method

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Abstract
The conformal method in general relativity aims at successfully parametrising the set of all initial data associated with globally hyperbolic space-times. One such mapping was suggested by Maxwell D (2014 Initial data in general relativity described by expansion, conformal deformation and drift (arXiv:1407.1467)). For closed manifolds, I verify that the solutions of the corresponding conformal system are stable, in the sense that they present a priori bounds under perturbations of the system’s coefficients. This result holds in dimensions $3 \leq n \leq 5$, when the metric is conformally flat, the drift is small. A scalar field with suitably high potential is considered in this case.

Keywords: general relativity, constraint equations, mathematical physics

(Some figures may appear in colour only in the online journal)

1. Introduction

A spacetime is defined as the equivalence class, up to an isometry, of Lorentzian manifolds $(\tilde{M}, \tilde{g})$ of dimension $n + 1$, which satisfy the Einstein field equations

$$Ric_{\alpha\beta}(\tilde{g}) = \frac{1}{2}R(\tilde{g})\tilde{g}_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$$\alpha, \beta = 1, n + 1.$$ We have used the following notation: $R(\tilde{g})$ is the scalar curvature of $\tilde{g}$, $Ric$ the Ricci curvature and $T_{\alpha\beta}$ the stress–energy tensor. If $T_{\alpha\beta} = 0$, we describe the vacuum. If

$$T_{\alpha\beta} = \tilde{\nabla}_\alpha \tilde{\psi} \tilde{\nabla}_\beta \tilde{\psi} - \left(\frac{1}{2} |\tilde{\nabla} \tilde{\psi}|_\tilde{g}^2 + V(\tilde{\psi})\right) \tilde{g}_{\alpha\beta},$$

the model corresponds to the existence of a scalar field $\tilde{\psi} \in C^\infty(M)$ having potential
$V \in C^\infty(\mathbb{R})$. By correctly choosing $\psi$ and $V$, we can describe the vacuum with cosmological constant and the Einstein–Klein–Gordon setting.

A globally hyperbolic spacetime admits initial data $(M, g, K, \psi, \pi)$, where

- $(M, g)$ is an $n$-dimensional Riemannian manifold,
- $K$ is a symmetric two-tensor corresponding to the second fundamental form,
- $\psi$ represents the scalar field in $M$, and
- $\pi$ is its time derivative.

The associated spacetime development takes the form $(M \times \mathbb{R}, \tilde{g}, \tilde{\psi})$, where $\tilde{g}$ is a Lorentzian metric that verifies $\tilde{g}|_M = \hat{g}$, the Lie derivative on $M$ in the time direction $\partial_t = T$ is equal to $L_T \tilde{g} = 2K$, $\tilde{\psi}$ is a scalar field such that $\tilde{\psi}|_M = \hat{\psi}$ and $\partial_t \tilde{\psi}|_M = \hat{\pi}$.

Through the work of Choquet-Bruhat and Geroch, having the initial data verify the constraint equations is proved to be not only a necessary, but also sufficient condition for the development of a maximal, globally hyperbolic space-time [FB52, CBG69]:

$$
R(\tilde{g}) + (\text{tr}_\tilde{g} \tilde{K})^2 - |\tilde{K}|^2_\tilde{g} + |\nabla \tilde{\psi}|^2_\tilde{g} + 2V(\tilde{\psi})
$$

The above system is clearly under-determined, which allows for a good amount of freedom in choosing $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$.

The conformal method is based on the work of André Lichnerowicz from 1944, who provided a blueprint to construct solutions of null mean curvature [Lic44]. It was later extended by James W York Jr to tackle constant mean curvature (CMC) solutions [Yor73]. The method was further developed together with Niall Ó Murchadha to include non constant mean curvature (non-CMC) solution [ÓMY74]. Two decades later, York introduced the conformal ‘thin-sandwich’ (CTS) model as an alternative for treating the non-CMC case, followed by a variant in a joint paper with Harald P Pfeiffer, called the Hamiltonian conformal ‘thin-sandwich’ (CTS-H) method [PY03, Yor99]. The non-CMC methods were later all shown to be equivalent in a paper by David Maxwell, which also contains a detailed overview of the conformal method [Max14a].

The conformal method allows for the constraint equations to be transformed into a determined system of equations by fixing well-chosen quantities (see Choquet-Bruhat, Isenberg and Pollack [CBIP07]). Essentially, the technique maps a space of parameters to the space of solutions.

Given an initial data set $(\hat{g}, \hat{K}, \hat{\psi}, \hat{\pi})$, the classical choice of parameters is $(g, U, \tau, \psi, \pi; \alpha)$: in this case, the conformal class $g$ is represented by a Riemannian metric $g$, the smooth function $\tau = \hat{g}^{ab} K_{ab}$ is the mean curvature a two-tensor $U_{ab}$ that is both trace-free and divergence-free with respect to $g$ (a transverse-traceless tensor). We sometimes prefer to indicate the volume gauge by the densitized lapse

$$
\tilde{N}_{\hat{g},\alpha} := \frac{\alpha}{\tilde{N}_{\hat{g}}}.
$$

In order to have a better understanding of the drift method, we recall a basic fact of differential geometry: any metric is uniquely identified by its conformal class together with its volume form. In fact,

$$
\mathcal{M} = \mathcal{C} \times \mathcal{V},
$$

where $\mathcal{C}$ and $\mathcal{V}$ are the spaces of conformal classes and volume forms, respectively.
where $\mathcal{M}$ is the space of metrics, $\mathcal{V}$ is the space of volume forms and $\mathcal{C}$ is the space of conformal classes. In the context of the Einstein equations, it makes sense to consider $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{V}$ modulo diffeomorphisms $D_0$, with $D_0$ the connected component of the identity in the diffeomorphism group.

In his papers, Maxwell describes in great detail how the spaces $\mathcal{M}$, $\mathcal{C}$ and $\mathcal{V}$, together with their tangent, cotangent and quotient spaces, are represented within the choice of parameters [Max14a, Max14b]. The TT-tensor $U$, for example, is shown to correspond to an element of $T_g (\mathcal{C}/D_0)$, which can be interpreted as a conformal method. By this interpretation, it becomes clear that $\mathcal{C}$ is prioritized over $\mathcal{V}$ when it comes to choices of parameters.

In [Max14b], Maxwell introduces a variant to the standard conformal method. Very succinctly, the drift model differs from its predecessor in that it replaces the mean curvature $\tau$ with two new conformal data, a volumetric momentum $\tau^*$ and a drift $V$. These new quantities are defined by the volumetric equivalent to the York splitting [Max15]:

$$\tau = \tau^* + \frac{1}{N_{g,\omega}} \text{div}(V + Q) \quad (1.6)$$

where $\tau^* \in \mathbb{R}$, $V$ is a smooth vector field and $Q$ is a conformal Killing field. The volumetric momentum $\tau^*$ as measured by $\omega$ is uniquely determined and can be rewritten as

$$\tau^* = \frac{\int_M N_{g,\omega} \tau \, dV_g}{\int_M N_{g,\omega} \, dV_g}. \quad (1.7)$$

The vector field $V$ is uniquely determined up to a $g$ divergence-free vector field.

According to Maxwell [Max09, Max15], $\tau^* = 0$ seems to be a common property of the known non-CMC cases of singularities in the parametrisation proposed by the conformal method, or more explicitly of an infinity of solutions corresponding to the same data set. The drawback of the classical conformal method is that the value of $\tau^*$ cannot be calculated a priori from a choice of representatives. One needs to first solve the corresponding system, as

$$\tau^* = \tau^*(g, u) \quad (1.8)$$

Coming back to ($Q_1$), this is an argument against the classical conformal model.

A drift $[V]_g$ at $g$ is the equivalence class of $V$ modulo $\text{Ker} \, L_g + \text{Ker} \, d\omega$. The space of drifts at $g$ is denoted as $\text{Drift}_g$. David Maxwell introduces the concept of drift as an infinitesimal motion in the space of metrics, modulo diffeomorphisms, that preserves conformal class, up to a diffeomorphism, and the volume form, also up to a diffeomorphism.

Assuming that $g$ admits non-trivial conformal Killing field and therefore that $Q \equiv 0$, one can obtain the initial data $(\tilde{g}_{ab}, \tilde{K}_{ab})$ from a conformal data set, given a gauge $\omega$, as follows.

(a) Choose an arbitrary representative $g_{ab} \in \mathfrak{g}$.
(b) Choose the unique densitized lapse $N_{g,\omega}$.
(c) Choose the unique TT-tensor $U_{ab}$.
(d) Choose a vector field $V$, unique up to a conformal Killing field. We use the tilde to differentiate the drift from the potential, while still staying true to Maxwell’s initial notation.

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Both $u$ and $W$ are unknown. We write

$$\hat{g}_{ab} = u^{q-2}g_{ab},$$

$$\hat{K}_{ab} = u^{-2} \left[ \frac{1}{2N_{g,\omega}}(\mathcal{L}_g W)_{ab} + U_{ab} \right] + \frac{1}{n} u^{q-2}g_{ab} \left( \tau^* + \frac{1}{N_{g,\omega}} \text{div}(V) \right).$$  \hspace{1cm} (1.9)

We plug these quantities into the constraint equations to obtain

$$\Delta_g u + \frac{n - 2}{4(n - 1)} (R(g) + |\nabla \psi|^2) u = \frac{(n - 2)|U + \mathcal{L}_g W|^2 + \pi^2}{4(n - 1)u^{q+1}} + \frac{n - 2}{4(n - 1)} \times \left[ 2V(\psi) - \frac{n - 1}{n} \left( \tau^* + \frac{\text{div}_g(u^q \nabla V)}{N_{g,\omega}u^{q+1}} \right) \right] u^{q+1}$$

$$\text{div}_g \left( \frac{1}{2N_{g,\omega}} \mathcal{L}_g W \right) = \frac{n - 1}{n} u^{q+1} \left( \frac{\text{div}_g(u^q \nabla V)}{2N_{g,\omega}u^{q+1}} \right) + \pi \nabla \psi = 0.$$ \hspace{1cm} (1.10)

The following table regroups for $n = 3$ the conformal data and their dimensions (columns 2 and 3), the expressions of physical data as functions of representatives of conformal data (column 1) and the dimensions of the remaining unknowns (column 4).

| Physical data                                      | Parameters | Degrees of freedom | Unknowns |
|---------------------------------------------------|------------|--------------------|----------|
| $\hat{g} = u^{q-2}g$                              | $g$        | 5                  | 1        |
| $\hat{K}_{ab} = u^{-2}\left[ \frac{1}{2N_{g,\omega}}(\mathcal{L}_g W)_{ab} + U_{ab} \right] + \frac{1}{n} u^{q-2}g_{ab} \left( \tau^* + \frac{1}{N_{g,\omega}} \text{div}(V) \right)$ | $U, \tau^*, N, V$ | 2 + 1 + 3 | 3        |
| $\hat{\psi} = \psi$                              | $\psi$     | 1                  | 0        |
| $\hat{\pi} = u^{-9}\pi$                          | $\pi$      | 1                  | 0        |

(1.11)

1.1. The viability of conformal method models

The conformal method essentially provides a mapping from the set of conformal data representatives to the set of initial data,

$$\text{Conformal data representatives} \rightarrow \text{Initial data}.$$  \hspace{1cm} (1.12)

More precisely, in the case of the classical conformal method, given a volume gauge $\omega$, the mapping presents as

$$(g_{ab}, U_{ab}, \tau; N) \xrightarrow{\text{solv}(u,W)} (\hat{g}, \hat{K}).$$  \hspace{1cm} (1.13)
By the nature of the conformal method, the mapping is onto: from any set of initial data, one can calculate a set of corresponding conformal data representatives. We list a number of criteria by which the strength of a conformal method may be judged.

1) Is the mapping a bijection?

An ideal parametrisation would establish a one-to-one correspondence between the sets of conformal data and of the initial data, which would imply that any initial data is well characterized by the conformal method. So far, none of the known conformal models have been proved to verify this condition.

In lieu of such a strong result, one may ask:
- Where is the mapping well-defined (in the sense that there exists $\hat{\mathbf{g}}, \hat{\mathbf{K}}$ corresponding to a fixed set of conformal data representatives)? As long as we properly identify the problem sets, we can simply remove them from the domain.
- Where is the mapping one-to-one? If we obtain multiple solutions, where does this happen?

2) Is the mapping continuous?

This question tests that the mapping is, in some sense, physically relevant.

1.2. The main result

Let $(M, g)$ be a closed locally conformally flat manifold of dimension $n$, which can be 3, 4 and 5. Let

$$\Delta_{\hat{g}} = -\text{div}_{\hat{g}}\nabla$$

be the Laplace–Beltrami operator with non-negative eigenvalues. Similarly, let

$$\tilde{\Delta}_{\hat{g}} W_i = -\text{div}_{\hat{g}}(\mathcal{L}_x W)_i = - (\mathcal{L}_x W)_{i,j}$$

be the corresponding Lamé operator. The volumetric drift model proposed by Maxwell leads to the reworking of the Einstein constraint equations as

$$\Delta_{\hat{g}} u + \frac{n-2}{4(n-1)} (R(g) + |\nabla \psi|^2_{\hat{g}}) u = \frac{(n-2)\hat{U} + \hat{\mathcal{L}}_x W^2 + \pi^2}{4(n-1)u^{n+1}} + \frac{n-2}{4(n-1)}$$

$$\times \left(2V(\psi) - \frac{n-1}{n} \tau^* + \frac{n-1}{n} \frac{\tilde{\text{div}}_x (u^3 \tilde{\mathbf{V}})^2}{u^{n+1}} \right) u^{n-1}$$

$$\text{div}_{\hat{g}} \left(\frac{\tilde{\mathbf{N}}}{2} \mathcal{L}_x W \right) = \frac{n-1}{n} u^{3} d \left(\frac{\tilde{\text{div}}_x (u^3 \tilde{\mathbf{V}})}{2u^{n+1}} \right) + \pi \nabla \psi.$$  

The existence of solutions to this system was treated in [HMM18] in the non-focusing case, and in [Vâl19] for the focusing case. The classical conformal method, also in the focusing regime, is treated in [Pre14]. See [DH09] for the precursor of the asymptotic techniques used in the existence proofs.

The second equation may be rewritten as:
\[ \nabla \ln \tilde{N}, \mathcal{L}_g W \] + 2 \frac{n-1}{n-2} \left( \frac{3n-2}{n-2} \left( \frac{\nabla u, \nabla \tilde{V}}{u^2} - \frac{\nabla^2 u, \nabla \tilde{V}}{u} \right) \right) \\
+ 2 \frac{n-1}{n-2} \left( - \frac{\nabla u, \tilde{V}}{u} \nabla \ln \tilde{N} + \nabla \nabla \tilde{V} \frac{\nabla u}{u} - \frac{\nabla \nabla \tilde{V}, \nabla u}{u} \right) \\
- \frac{n-1}{n} \left( \nabla \nabla \tilde{V} \ln \tilde{N} + \nabla \nabla \tilde{V} \right) - 2 \tilde{N}^{-1} \pi \nabla \psi. \tag{1.17} \]

In the present paper, it sometimes proves useful to work with the more general equation

\[ \Delta_g u + hu = f u^{q-1} + \rho_1 + |\Psi + \rho_2 \mathcal{L}_g W|_{g}^2 \]
\[ - \frac{b}{u^\alpha} - c_{\alpha} (\nabla u, Y) \left( \frac{\partial}{\partial u^\alpha} + \frac{1}{u^{\alpha+2}} \right) - \frac{\nabla u, Y, \nabla u}{u^{\alpha+2}} \tag{1.18} \]

where, in the drift method, we have:

\[ h = \frac{n-2}{4(n-1)} \left( \mathcal{R}_g - |\nabla \psi|_{g}^2 \right), \quad f = \frac{n-2}{4(n-1)} \left( 2V(\psi) - \frac{n-1}{n} \tau^2 \right) \]
\[ \rho_1 = \frac{n-2}{4(n-1)} \left( \psi - \frac{n-1}{n} \bar{N}^2 \nabla \tilde{V} \right), \quad \rho_2 = \sqrt{\frac{n-2}{n-1}} \tilde{N}, \quad \Psi = \sqrt{\frac{n-2}{n-1}} \frac{U}{\tilde{N}}. \tag{1.19} \]
\[ b = \frac{n-2}{2n} \tau^* \bar{N} \nabla \tilde{V}, \quad c = \frac{n-2}{n}, \quad d = \tau^*, \quad Y = \sqrt{\frac{n}{n-2}} \bar{N}. \]

Consider \((u_{\alpha}, W_{\alpha})_{\alpha \in \mathbb{N}}\) a sequence of smooth solutions of perturbations of the system (1.18),

\[ \Delta_g u_{\alpha} + h_{\alpha} u_{\alpha} = f_{\alpha} u^{q-1} + \rho_{1,\alpha} + |\Psi_{\alpha} + \rho_{2,\alpha} \mathcal{L}_g W_{\alpha}|_{g}^2 \]
\[ - \frac{b_{\alpha}}{u_{\alpha}^\alpha} - c_{\alpha} (\nabla u_{\alpha}, Y_{\alpha}) \left( \frac{\partial}{\partial u_{\alpha}^\alpha} + \frac{1}{u_{\alpha}^{\alpha+2}} \right) - \frac{\nabla u_{\alpha}, Y_{\alpha}}{u_{\alpha}^{\alpha+2}} \tag{1.20} \]

Here, we ask that the perturbed coefficients converge towards the initial ones in a sufficiently regular way, e.g. in \(C^{2,0}\) norm. The scalar solutions \(u_{\alpha}\) are uniformly bounded from below by a positive constant as long as the same holds true for the sequence \(\rho_{1,\alpha}\). To prove this, let \(m_{\alpha} = \min_{x \in M} u_{\alpha}(x) = u_{\alpha}(x_0) > 0\) and let

\[ a_{\alpha} = \rho_{1,\alpha} + |\Psi_{\alpha} + \rho_{2,\alpha} \mathcal{L}_g W_{\alpha}|_{g}^2. \tag{1.21} \]

Since \(\nabla u_{\alpha}(x_0) = 0\) and since \(\Delta_g u_{\alpha}(x_0) \leq 0\), we have

\[ h_{\alpha}(x_0) m_{\alpha} - f_{\alpha}(x_0) m_{\alpha}^{q-1} - \frac{a_{\alpha}(x_0)}{m_{\alpha}^{\alpha+1}} + \frac{b_{\alpha}(x_0)}{m_{\alpha}} \geq 0. \]

Since \(a_{\alpha} \to a\) in \(C^0(M)\) as \(\alpha \to +\infty\) and \(a > 0\) in \(M\), there exists \(\varepsilon > 0\) such that \(m_{\alpha} \geq \varepsilon\), meaning that

\[ u_{\alpha} \geq \varepsilon > 0 \quad \text{for all } x \in M \text{ and all } \alpha. \tag{1.22} \]
We would like to prove the \textit{a priori} estimate

\[ \|u_\alpha\|_{C^{2,\eta}} + \|W_\alpha\|_{C^{1,\theta}} \leq C, \]  

(1.23)

where the constant \( C \) can depend on \( M \), \( g \) and the constants \( \theta \) and \( T \) defined in the statement of the theorem below, but is independent of \( \alpha \).

If this is true, then by standard elliptic theory there exists, up to a subsequence, a \( C^{2,\eta} \) limit of \((u_\alpha, W_\alpha)\) solving the limiting system (1.18). In effect, since the system (1.18) is invariant by the addition of conformal Killing fields as it only depends on \( L_\theta W_\alpha \) and \( \Delta_\theta W_\alpha \), and thus the size of \( W_\alpha \) can be controlled by the addition of conformal Killing fields, it suffices to show that

\[ \|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|L_\theta W_\alpha\|_{L^\infty} \leq C. \]  

(1.24)

The proof follows by contradiction. We assume instead that there exists a sequence of solutions \((u_\alpha, W_\alpha)\) of the perturbed system such that

\[ \|u_\alpha\|_{L^\infty} + \|\nabla u_\alpha\|_{L^\infty} + \|\nabla^2 u_\alpha\|_{L^\infty} + \|L_\theta W_\alpha\|_{L^\infty} \to \infty \quad \text{as} \quad \alpha \to \infty. \]  

(1.25)

The main theorem in this paper is the following.

\textbf{Theorem 1.} Let \((M, g)\) be a closed Riemannian manifold of dimension \( n = 3, 4, 5 \), where \( g \) is locally conformally flat. Let \( \frac{1}{4} < \eta < 1 \) and \( 0 < \alpha < 1 \). Let \( a, b, c, d, f, h, \rho_1, \rho_2, \psi, \pi, \tilde{N} \) be smooth functions on \( M \), let \( \tilde{V} \) and \( Y \) be smooth vector field on \( M \). For any \( 0 < \theta < T \), there exists \( S_{\theta,T} \) and \( \vartheta_{\theta,T} \) such that, given any parameters within

\[ \mathcal{E}_{\theta,T} \coloneqq \left\{ (f, a, b, c, d, h, \rho_1, \rho_2, Y) \times (\tilde{N}, \tilde{V}, \psi, \pi), \quad f \geq \theta, \quad a \geq \theta, \quad \tilde{N} \geq \theta, \quad \|f\|_{C^{1,\eta}} \leq T, \quad \|a\|_{C^{1,\eta}}, \|b\|_{C^{1,\eta}}, \|c\|_{C^{1,\eta}}, \|d\|_{C^{1,\eta}}, \right. \]

\[ \left. \|\rho_1\|_{C^{1,\eta}}, \|\rho_2\|_{C^{1,\eta}}, \|h\|_{C^{1,\eta}}, \|Y\|_{C^{1,\eta}} \leq T, \quad \text{and} \quad \|\tilde{N}\|_{C^{2,\eta}}, \|\tilde{V}\|_{C^{2,\eta}} \leq T \right\}, \]

(1.26)

with

\[ \|Y\|_{C^{1,\eta}}, \|\tilde{V}\|_{C^{2,\eta}} \leq \vartheta_{\theta,T}, \]  

(1.27)

then any smooth solution \((u, W)\) (1.16), with \( u > 0 \), satisfies

\[ \|u\|_{C^{2,\eta}} + \|L_{\theta} W\|_{C^{1,\eta}} \leq S_{\theta,T}. \]  

(1.28)

A few remarks are in order at this point. We have taken the decision to write the theorem using the physical coordinates for the second equation, and the general coefficients for the first. The same is true for the ensuing proof. This forcibly leads to some redundancies. We recall that \( Y = \sqrt{\frac{n-2}{n-4}} \tilde{N} \), so asking for bounds on \( \tilde{N} \) and \( \tilde{V} \) imply bounds on \( Y \). The reasons why we still choose this writing are as follows:

(a) The general notation of the first equation is the same as the ones used in the paper proving the existence of solutions to the system, and are more readable than the physical coordinates one [Vâl19]. Moreover, they more accurately capture the nature of the scalar equation and make it easier to handle, since one can follow each of the different non-linear terms separately.
(b) Writing the second equation in more general terms can prove counterproductive. For one, introducing new coefficients would actually burden the notation in this particular case.

(c) Most importantly, one hopes that there is a better way to treat potential blow-ups caused by $L \circ W^\alpha$. This could follow from a more detailed analysis of the second equation, where even the exact size of each of the dimensional constants can potentially play a role, given the coupling of the system.

In the proof, we use the smallness of $Y$ (and thus, $\tilde{V}$, since $\tilde{N} \geq \theta$) as sparsely as possible, and we take care to emphasize it each time. We do this out of the desire to provide what we hope is a useful insight into current technical difficulties. By looking at similar systems, such as the Yamabe problem, one can hope that by advancing the necessary techniques, one can successfully remove the smallness hypothesis altogether. The hope for future advancement is to obtain a theorem of stability that does not require the conformally flat hypothesis on $g$ or the smallness condition (1.27).

For now, in the argument by contradiction, we are working with

$$Y_\alpha \to 0 \quad \text{in} \quad C^{1,\alpha} \quad \text{and} \quad \tilde{V}_\alpha \to 0 \quad \text{in} \quad C^{2,\alpha}.$$  \hspace{1cm} (1.29)

The fact that $g$ is locally conformally flat is a condition we impose to get the improved estimates on $L \circ W^\alpha$ that we need. We briefly explain the reasoning. The Green representation formula is applied on balls of diminishing radius $B_{x_\alpha}(\delta_\alpha)$, $\delta_\alpha \to 0$, where $x_\alpha$ is a concentration point. Moreover, we impose Neumann boundary conditions, so that there is no dependency on $W^\alpha$, but just $L \circ W^\alpha$. The bounds need to be uniform with respect to $\alpha$, which is why we need the kernel of $\Delta_{g^\alpha}$ to have the same dimension as that of $\Delta_{\xi}$, with $g^\alpha = \exp_{x_\alpha}^* (\delta_\cdot)$.

The fact that we are working on compact manifolds is essential, given the techniques we use in the present paper. Extending these results to the non-compact setting is highly non-trivial, not only because new kinds of blow-up behaviours need to be considered, including bubbles travelling towards infinity.

The condition that $3 \leq n \leq 5$ comes from the $a$ priori estimates: it ensures that the asymptotic behaviour of bubbles clustering together is sufficiently constraining to lead to a contradiction. In fact, it is very unlikely that we can extend the stability to $n \geq 6$, given that, in the classical model, Bruno Premoselli was able to find explicit examples of instability [Pre16]. He also obtains similar stability results for the classical model in $3 \leq n \leq 6$.

**Outline of the paper.** The proof is structured as follows. In section 2, we conformally change $(u_\alpha, W^\alpha)$ on $(M, g)$ to $(v_\alpha, Z^\alpha)$ defined in a Euclidean domain. In section 3, we begin by obtaining pointwise estimates on both $v_\alpha$, $\nabla v_\alpha$, $\nabla^2 v_\alpha$ and $L \circ Z^\alpha$. Section 4 begins with an immediate consequence of the aforementioned bounds: they yield a Harnack inequality on $v_\alpha$ (4.4). Green’s representation theory, applied to the elliptic operators of both the first and second equation, plays a central role in both obtaining and improving the weak bounds on $L \circ Z^\alpha$ we obtained previously. The next step consists of using the techniques of asymptotic analysis to describe potential blow-up behaviour, and their interactions. All this leads to a contradiction.

### 2. Conformal changes of coordinates

Since $(M, g)$ is assumed to be locally conformally flat, for any sequence $x_\alpha \in M$ with $x_\alpha \to x$ as $\alpha \to +\infty$ and for any $\delta > 0$ small enough, there exist smooth diffeomorphisms

$$\Phi_\alpha : \tilde{\Omega} \subset M \mapsto B_0(\delta) \subset \mathbb{R}^n$$  \hspace{1cm} (2.1)
and \( \varphi_\alpha \in \mathcal{C}^\infty(\mathcal{B}_0(\delta)) \) where \( \tilde{\Omega}_\alpha \) is some neighbourhood of \( x_\alpha \) in \( M \) such that

\[
(\Phi_\alpha)_*g = \varphi_\alpha^{\rho - 2} \xi
\]

(2.2)

where \( \xi \) is the Euclidean metric. Moreover we can choose the diffeomorphisms \( \Phi_\alpha \) and the functions \( \varphi_\alpha \) to be uniformly bounded in any \( \mathcal{C}^k \) for \( k \leq m \), \( m \) fixed as we want. Note that we can also choose \( \varphi_\alpha(0) = 1 \) and \( \nabla \varphi_\alpha(0) = 0 \). For \( x \in \mathcal{B}_0(\delta) \), consider the change of functions

\[
v_\alpha(x) = \varphi_\alpha(x) u_\alpha \circ \Phi_\alpha^{-1}(x) \quad \text{and} \quad Z_\alpha(x) = \varphi_\alpha(x)^2 \Phi_\alpha^{-1} W_\alpha(x).
\]

(2.3)

This change of functions will be used repeatedly in the sequel. First of all, note that, by (1.22), there exists \( \varepsilon' > 0 \) such that

\[
v_\alpha \geq \varepsilon'.
\]

(2.4)

Then it is convenient to recall the following formulas. We see that the Laplace–Beltrami operator becomes

\[
\Delta_\xi v_\alpha = \varphi_\alpha^{\rho - 1}(x) \left( \Delta g u_\alpha + \frac{n - 2}{4(n - 1)} R(g) u_\alpha \right) (\Phi_\alpha^{-1})
\]

and that

\[
\varphi_\alpha^{\rho - 2} \xi Z_\alpha = (\Phi_\alpha)_* \left( \mathcal{L}_g W_\alpha \right).
\]

(2.5)

(2.6)

At last, the Lamé type operator transforms as

\[
\overrightarrow{\Delta}_\xi \left( \varphi_\alpha^{2 - q}(\Phi_\alpha)_* W_\alpha \right) - q \xi^d \partial_k (\ln \varphi_\alpha) \mathcal{L}_\xi \left( \varphi_\alpha^{2 - q}(\Phi_\alpha)_* W_\alpha \right) \right) = (\Phi_\alpha)_* \left( \overrightarrow{\Delta}_g W_\alpha \right);
\]

(2.7)

so

\[
\left( \overrightarrow{\Delta}_\xi Z_\alpha \right) - q (\nabla \ln \varphi_\alpha, \mathcal{L}_\xi Z_\alpha)_i = (\Phi_\alpha)_* \left( \overrightarrow{\Delta}_g W_\alpha \right)_i
\]

(2.8)

Simple but tedious computations lead then to the transformation of the system (1.20) into

\[
\Delta \xi v_\alpha(x) + \tilde{h}_\alpha(x) v_\alpha(x) = \tilde{f}_\alpha(x) v_\alpha^{\rho - 1}(x) + \tilde{a}_\alpha(x) \frac{\tilde{b}_\alpha(x)}{v_\alpha^{\rho + 1}(x)} v_\alpha(x)
\]

\[
- (\nabla v_\alpha(x), \tilde{Y}_\alpha(x)) \left( \frac{\tilde{c}_\alpha(x)}{v_\alpha^{\rho + 2}(x)} + \frac{\tilde{d}_\alpha(x)}{v_\alpha^{\rho + 3}(x)} \right) = \frac{(\nabla v_\alpha(x), \tilde{Y}_\alpha(x))^2}{v_\alpha^{\rho + 1}(x)}
\]

\[
\left( \overrightarrow{\Delta}_\xi Z_\alpha \right)_i = q (\nabla \ln \varphi_\alpha, \mathcal{L}_\xi Z_\alpha)_i + \tilde{R}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha, \mathcal{L}_\xi Z_\alpha)_i
\]

(2.9)
where

\[ \tilde{Y}_\alpha = \varphi_\alpha^2(\Phi_\alpha)Y_\alpha, \quad \tilde{f}_\alpha = f_\alpha \circ \Phi_\alpha^{-1}, \]

\[ \tilde{h}_\alpha = \varphi_\alpha^{-2}\left(h_\alpha - \frac{n - 2}{4(n - 1)}R(g)\right) \circ \Phi_\alpha^{-1}, \]

\[ \tilde{b}_\alpha = \varphi_\alpha^q b_\alpha \circ \Phi_\alpha^{-1} - \varphi_\alpha(\nabla \varphi_\alpha, (\Phi_\alpha)_*, Y_\alpha)c_\alpha \circ \Phi_\alpha^{-1}, \]

\[ \tilde{c}_\alpha = c_\alpha \circ \Phi_\alpha^{-1}, \quad \tilde{d}_\alpha = 2\varphi_\alpha(\nabla \varphi_\alpha, (\Phi_\alpha)_*, Y_\alpha) + \varphi_\alpha^q d_\alpha \circ \Phi_\alpha^{-1}, \]

\[ \tilde{\alpha}_a = \tilde{\rho}_{1,a} + |\tilde{\Psi}_a + \tilde{\rho}_{2,a}\mathcal{L}_\xi Z_\alpha|_2 \]

\[ \tilde{\alpha}_{1,a} = \varphi_\alpha^{2q} \tilde{\rho}_{1,a} \circ \Phi_\alpha^{-1} + \varphi_\alpha^{q + 1}(\nabla \varphi_\alpha, (\Phi_\alpha)_*, Y_\alpha)d_\alpha \circ \Phi_\alpha^{-1} - \varphi_\alpha^2(\nabla \varphi_\alpha, (\Phi_\alpha)_*, Y_\alpha)^2, \]

\[ \tilde{\alpha}_{2,a} = \varphi_\alpha^q \tilde{\rho}_{2,a} \circ \Phi_\alpha^{-1}, \quad \tilde{\Psi}_a = \varphi_\alpha^q (\Phi_\alpha)_a \Psi_\alpha \]

(2.10)

and

\[ \tilde{R}_\alpha(v_\alpha, \nabla v_\alpha, \nabla^2 v_\alpha, \mathcal{L}_\xi Z_\alpha) = (\Phi_\alpha)_* R_\alpha(u_\alpha, \nabla u_\alpha, \nabla^2 u_\alpha, \mathcal{L}_g W_\alpha) \]

\[ = \langle (\Phi_\alpha)_* \nabla \ln \tilde{N}, \mathcal{L}_\xi Z_\alpha \rangle + 2 \frac{n - 1}{n - 2} \varphi_\alpha^{2-q} \]

\[ \times \left( \frac{3n - 2}{n - 2} \frac{(\nabla v_\alpha, (\Phi_\alpha)_* V_\alpha) \nabla v_\alpha}{v_\alpha^2} - \frac{(\nabla^2 v_\alpha, (\Phi_\alpha)_* V_\alpha)}{v_\alpha} \right) \]

\[ + \tilde{\mathcal{T}}_\alpha(v_\alpha, \nabla v_\alpha). \]

(2.11)

Here, \( \tilde{\mathcal{T}}_\alpha \) denotes the lower order terms of the second equation (2.11). It is clear that we have

\[ |\tilde{\mathcal{T}}_\alpha(v_\alpha, \nabla v_\alpha)| \leq C \left( 1 + ||(\Phi_\alpha)_* V_\alpha||_C \right) \left| \frac{\nabla v_\alpha}{v_\alpha} \right|. \]

(2.12)

### 3. Weak pointwise estimates

The following result describes a pointwise estimate that holds everywhere on \( M \). It provides a way to identify a set of points \( S_\alpha \) where \( u_\alpha \) or \( \mathcal{L}_g W_\alpha \) can potentially explode.

**Lemma 1.** Let \( (u_\alpha, W_\alpha) \) be a sequence of solutions of the perturbed system (1.20), verifying the non-compactness hypothesis (1.25). There exists an integer \( N_\alpha \in \mathbb{N}^* \) and a set of critical points \( S_\alpha = (x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha}) \) of \( u_\alpha \) such that

\[ d_\alpha(x_{i,\alpha}, x_{j,\alpha})^p u_\alpha(x_{i,\alpha})^p \geq 1, \]

for all \( 1 \leq i, j \leq N_\alpha, i \neq j \), and

\[ \left( \min_{1 \leq i \leq N_\alpha} d_\alpha(x_{i,\alpha}, x) \right)^{n} u_\alpha(x) \leq 1 \]

(3.1)
for any $x$ critical point of $u$, in $M$, and
\[
\left( \min_{1 \leq i \leq N_\alpha} d_\alpha(x_{i,\alpha}, x) \right)^n \left( u_{\alpha}^n(x) + \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right)^n + \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \frac{2}{2} + |L_p W_\alpha|_p(x) \right) \leq C, \tag{3.3}
\]
where $C$ is a constant that is independent of $\alpha$.

**Proof.** Step 1: setting up the proof by contradiction. For every $\alpha \in \mathbb{N}^*$, we may define the integer $N_\alpha \in \mathbb{N}^*$ and the set of critical points
\[
S_\alpha = \{ x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha} \}
\]
of $u$, by the following lemma, which holds very generally for any sufficiently regular function.

**Lemma 2.** Let $u$ be a positive real-valued $C^2$ function defined in a compact manifold $M$. Then there exists $N \in \mathbb{N}^*$ and $(x_1, x_2, \ldots, x_N)$ a set of critical points of $u$ such that
\[
d_{\alpha}(x_i, x_j) \geq 1 \tag{3.4}
\]
for all $i, j \in \{1, \ldots, N\}$, $i \neq j$, and
\[
\left( \min_{i=1,\ldots,N} d_{\alpha}(x_i, x) \right)^{\frac{n}{n-2}} u(x) \leq 1 \tag{3.5}
\]
for all critical points $x$ of $u$.

The lemma and its proof may be found in Druet and Hebey’s paper [DH09]. We define the operator
\[
\Psi_\alpha(x) = \left( \min_{1 \leq i \leq N_\alpha} d_{\alpha}(x_{i,\alpha}, x) \right)^n \times \left( u_{\alpha}^n(x) + \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right)^n + \frac{\nabla^2 u_\alpha(x)}{u_\alpha(x)} \frac{2}{2} + |L_p W_\alpha|_p(x) \tag{3.6}
\]
for $x \in M$. Assume that there exists $(x_\alpha)_{\alpha} \in M$ such that
\[
\Psi_\alpha(x_\alpha) = \sup_M \Psi_\alpha \to \infty \tag{3.7}
\]
as $\alpha \to \infty$.

**Step 2: rescaling.** We denote the injectivity radius of $M$ by $i_\delta(M)$. Let
\[
0 < \delta_\alpha < \frac{1}{2} i_\delta(M) \tag{3.8}
\]
be radii around $x_\alpha$. Consider the following rescalings of the conformal factors:
\[
\tilde{u}_\alpha(x) = \mu_{\alpha} \frac{u_{\alpha}^n}{\varphi_\alpha(\mu_{\alpha}) u_\alpha \circ \Phi_{\alpha}^{-1}(\mu_{\alpha} x)}
\]
\[
\hat{Z}_\alpha(x) = \mu_{\alpha} \varphi_\alpha(\mu_{\alpha} x)^{-1-q} (\Phi_{\alpha}), W_\alpha(\mu_{\alpha} x), \tag{3.9}
\]
where $x \in \Omega_\alpha$, with $\Omega_\alpha := B_{\alpha_\alpha} \left( \frac{x}{\alpha_\alpha} \right)$ and
\[
\mu_{\alpha}^{-1} := u_\alpha(x_\alpha)^n + \left( \frac{\nabla u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right)^n + \left( \frac{\nabla^2 u_\alpha(x_\alpha)}{u_\alpha(x_\alpha)} \right)^{\frac{2}{2}} + |L_p W_\alpha|_p(x_\alpha). \tag{3.10}
\]
Moreover, because \( M \) is compact, and by (3.6) and (3.7),
\[
\frac{d\eta(x, t\cdot)}{\mu_\alpha} \to \infty \quad \text{and} \quad \mu_\alpha \to 0.
\] (3.11)

We consider the rescaled perturbed system corresponding to (3.9),
\[
\Delta \dot{\tilde{v}}_\alpha = -\mu_\alpha^2 \tilde{h}_\alpha \dot{\tilde{v}}_\alpha + \dot{\tilde{f}}_\alpha \tilde{v}_\alpha^{-1} + \frac{\tilde{a}_\alpha}{\tilde{v}_\alpha^{q+1}} - \mu_\alpha^2 \frac{\tilde{\dot{h}}_\alpha}{\tilde{v}_\alpha} - \mu_\alpha^2 \frac{\langle \nabla \tilde{v}_\alpha, \tilde{Y}_\alpha \rangle}{\tilde{v}_\alpha}
\times \left( \frac{\tilde{\dot{d}}_\alpha}{\tilde{v}_\alpha^{q+1}(\mu_\alpha)} + \frac{\tilde{\dot{\tilde{c}}}_\alpha}{\tilde{v}_\alpha^{q+1}(\mu_\alpha)} \right) - \mu_\alpha^2 \frac{(\nabla \tilde{v}_\alpha, \tilde{Y}_\alpha)^2}{\tilde{v}_\alpha^{q+1}(\mu_\alpha)}
\]
\[
\Delta \tilde{\tilde{Z}}_\alpha = q \mu_\alpha \xi^k \partial_k (\nabla \dot{\tilde{v}}_\alpha) \left( \nabla \tilde{Z}_\alpha \right) + \tilde{R}_\alpha (\tilde{v}_\alpha, \nabla \tilde{v}_\alpha, \nabla^2 \tilde{v}_\alpha, \tilde{Z}_\alpha),
\] (3.12)

where
\[
\begin{align*}
\tilde{h}_\alpha(x) &= \tilde{h}_\alpha(\mu_\alpha x), \\
\tilde{f}_\alpha(x) &= \tilde{f}_\alpha(\mu_\alpha x), \\
\tilde{\rho}_\alpha^1(x) &= \tilde{\rho}_\alpha^1(\mu_\alpha x), \\
\tilde{\rho}_\alpha^2(x) &= \tilde{\rho}_\alpha^2(\mu_\alpha x), \\
\tilde{\Psi}_\alpha(x) &= \tilde{\Psi}_\alpha(\mu_\alpha x), \\
\tilde{Y}_\alpha(x) &= \tilde{Y}_\alpha(\mu_\alpha x), \\
\tilde{\alpha}_\alpha(x) &= \tilde{\alpha}_\alpha(\mu_\alpha x), \\
\tilde{\dot{c}}_\alpha(x) &= \tilde{c}_\alpha(\mu_\alpha x), \\
\tilde{\dot{\tilde{c}}}_\alpha(x) &= \tilde{\tilde{c}}_\alpha(\mu_\alpha x), \\
\tilde{\tilde{d}}_\alpha(x) &= \tilde{d}_\alpha(\mu_\alpha x).
\end{align*}
\] (3.13)

and
\[
\tilde{R}_\alpha (\tilde{v}_\alpha, \nabla \tilde{v}_\alpha, \nabla^2 \tilde{v}_\alpha, \tilde{Z}_\alpha) \leq C_R \left( \mu_\alpha^{n+1} + \mu_\alpha^n \left| \frac{\nabla \tilde{v}_\alpha}{\tilde{v}_\alpha} \right|^2 + \mu_\alpha^{n-1} \left| \frac{\nabla^2 \tilde{v}_\alpha}{\tilde{v}_\alpha} \right| + \mu_\alpha \left| \nabla \tilde{Z}_\alpha \right| \right).
\] (3.14)

where \( C'_R \) is a constant.

By the definition (3.10),
\[
\dot{\tilde{v}}_\alpha(0) + \left| \frac{\nabla \tilde{v}_\alpha(0)}{\tilde{v}_\alpha(0)} \right|^n + \left| \frac{\nabla^2 \tilde{v}_\alpha(0)}{\tilde{v}_\alpha(0)} \right|^2 + \left| \tilde{Z}_\alpha(0) \right| = 1
\] (3.15)

and for any \( R > 0 \),
\[
\sup_{x \in \partial_0 R} \left( \left| \frac{\nabla \tilde{v}_\alpha(x)}{\tilde{v}_\alpha(x)} \right|^n + \left| \frac{\nabla^2 \tilde{v}_\alpha(x)}{\tilde{v}_\alpha(x)} \right|^2 + \left| \tilde{Z}_\alpha(x) \right| \right) \leq 1 + o(1)
\] (3.16)

and thereby
\[
\sup_{x \in \partial_0 R} |\nabla \ln \tilde{v}_\alpha(x)| \leq 1 + o(1)
\] (3.17)

for all \( R > 0 \). As a consequence,
\[
\tilde{v}_\alpha(0)e^{-2|x|} \leq \tilde{v}_\alpha(x) \leq \tilde{v}_\alpha(0)e^{2|x|}.
\] (3.18)
Step 3: $|L_\xi \tilde{Z}_\alpha|$ converges to zero. By Green’s representation formula applied to the first equation of (3.12) on $B_\varepsilon(3R)$, we get

$$
\hat{v}_\alpha(x) \geq \int_{B_\varepsilon(3R)} G_{3R}(x, y) \left[ \frac{\hat{a}_\alpha(y)}{v^{\alpha+1}_\alpha(y)} - \mu_\alpha^2 \tilde{h}_\alpha(y) \hat{v}_\alpha(y) - \frac{\mu_\alpha^2}{v_\alpha^\prime(\mu_\alpha y)} \hat{b}_\alpha(y) \right. \\
- \mu_\alpha^2 \left( \frac{\nabla \hat{v}_\alpha(y), \hat{Y}_\alpha(y)}{\hat{v}_\alpha(y)} + \frac{\hat{c}_\alpha(y)}{v^\alpha_\alpha(\mu_\alpha y)} \right) \\
\left. - \frac{\mu_\alpha^2}{v^{\alpha+1}_\alpha(\mu_\alpha y)} \right] dy.
$$

(3.19)

Here, $G_{3R}(x, y) := \frac{1}{(n-2s_\alpha-1)} \left| (x - y)^{2-n} - (3R)^{2-n} \right|$. By taking $\alpha$ large, we get the bulk integral estimate

$$
\int_{B_\varepsilon(2R)} |x - y|^{2-n} |L_\xi \tilde{Z}_\alpha|^2(y) dy \leq C,
$$

(3.20)

where $C$ is a positive constant independent of $R$ or $\alpha$. Therefore, we may find $s_\alpha \in (\frac{1}{2}R, 2R)$ such that the boundary estimate

$$
\int_{\partial B_\varepsilon(s_\alpha)} |L_\xi \tilde{Z}_\alpha|^2(y) d\sigma(y) \leq CR^{n-3}
$$

(3.21)

holds. Moreover,

$$
\left| q^{\mu_\alpha} \xi^\beta (\ln \varphi_\alpha)(\mu_\alpha y) \left( L_\xi \tilde{Z}_\alpha \right)_y \right| \leq Cq^{\mu_\alpha^2} |y| \left| L_\xi \tilde{Z}_\alpha \right|_x.
$$

(3.22)

Turning to the second equation of (3.12), we use the Green representation formula for the Lamé type operator $\Delta_\xi$ in $B_\varepsilon(2R)$. This yields

$$
\left| L_\xi \tilde{Z}_\alpha \right|_x \leq C \int_{B_\varepsilon(s_\alpha)} |x - y|^{1-n} \left| \Delta_\xi \tilde{Z}_\alpha \right| dy + C \int_{\partial B_\varepsilon(s_\alpha)} |x - y|^{1-n} \left| L_\xi \tilde{Z}_\alpha \right|(y) d\sigma(y)
$$

$$
\leq CR_{\mu_\alpha} + \frac{C'}{R}.
$$

(3.23)

for positive constants $C$ and $C'$. We therefore get an improvement on the pointwise estimate of the rescaled $\tilde{W}_\alpha$ from (3.16):

$$
\left| L_\xi \tilde{Z}_\alpha \right|_x \rightarrow 0
$$

(3.24)

and

$$
\tilde{a}_\alpha \rightarrow 0
$$

(3.25)

in $C_{L^\infty}(\mathbb{R}^n)$ as $\alpha \rightarrow \infty$.

Step 4: the study of potential blow-up profiles. We turn to the study of the remaining terms of (3.10). From (3.15) and (3.24), we deduce that

$$
\lim_{\alpha \rightarrow \infty} \left( \hat{v}_\alpha^\beta(0) + \left| \frac{\nabla \hat{v}_\alpha(0)}{\tilde{v}_\alpha(0)} \right|^\alpha + \left| \frac{\nabla^2 \hat{v}_\alpha(0)}{\tilde{v}_\alpha(0)} \right|^{\frac{\beta}{2}} \right) = 1.
$$

(3.26)
Let us set
\[
  w_\alpha(x) := \frac{\hat{v}_\alpha(x)}{v_\alpha(0)} = \frac{u_\alpha(\exp_{x_\alpha}(\mu_\alpha x))}{u_\alpha(x_\alpha)}.
\] (3.27)

It follows from that
\[
  w_\alpha(0) = 1 \quad \text{and} \quad \lim_{\alpha \to \infty} \left( \left| \frac{\nabla w_\alpha(0)}{w_\alpha(0)} \right|^n + \left| \frac{\nabla^2 w_\alpha(0)}{w_\alpha(0)} \right|^\frac{n}{2} \right) \leq 1
\] (3.28)
and therefore from (3.17) that
\[
  e^{-2|x|} \leq w_\alpha(x) \leq e^{2|x|}.
\] (3.29)

We divide the first equation of system (3.12) by \( \bar{u}_\alpha(0) \) and obtain
\[
  \Delta_\xi w_\alpha = -\mu_\alpha^2 \bar{h}_\alpha w_\alpha + \int_\alpha w_\alpha^{q-1} \varepsilon^{q-2}(0) + \frac{\mu_\alpha^2}{\bar{e}_{\alpha}^{q+1}(x)} \hat{h}_\alpha
\]
\[
  + \frac{1}{\bar{e}_{\alpha}^{q+1}(x)} \bigg( \frac{\Psi_\alpha(x) + \hat{\rho}_{2\alpha}(x) \mathcal{L}_\xi \hat{Z}_\alpha(x)}{v_\alpha(x)} \bigg)^2 - \mu_\alpha \mathcal{L}_\xi \hat{w}_\alpha
\]
\[
  \times \bigg( \frac{\bar{d}_\alpha}{\bar{e}_{\alpha}^{q+1}(x)} + \frac{\hat{e}_\alpha}{\bar{e}_{\alpha}^{q+1}(\exp_{x_\alpha}(\mu_\alpha x))} \bigg)
\].

Up to a subsequence, we denote
\[
  \hat{\omega}_\alpha = \hat{v}_\alpha(0), \quad \text{with} \quad \lim_{\alpha \to \infty} \hat{\omega}_\alpha = : \hat{\omega} \in [0, 1],
\]
\[
  l_\alpha = v_\alpha^{-1}(x_\alpha), \quad \text{with} \quad \lim_{\alpha \to \infty} l_\alpha = : l \in [0, e^{-1}]
\] (3.30)
which follows from (3.9) and (3.26) in the case of the first limit, and from (1.22) for the second. Furthermore, (3.10) implies that
\[
  \hat{\omega} = \lim_{\alpha \to \infty} \frac{\hat{\omega}_\alpha}{\hat{\omega}_\alpha^2} = 0.
\] (3.31)

**Remark 1.** It is here that we use the hypothesis \( V_\alpha(x_\alpha) \to 0 \).

We denote
\[
  \frac{\mathcal{L}_\xi \hat{Z}_\alpha(x)}{\hat{\omega}_\alpha^{q+2}} \to \mathcal{L}_\xi Z.
\] (3.32)
By standard elliptic theory, we find that there exists $w := \lim_{\alpha \to \infty} w_\alpha$ in $C^1_{\text{loc}}(\mathbb{R}^n)$, and by dividing the first equation by $\dot{l}$, we obtain

$$
\Delta w = \frac{n - 2}{4(n - 1)} \left( 2V(\psi(x_0)) - \frac{n - 1}{n} \tau^2 \right) w^{q-1} \dot{l}^{q-2} + \frac{n - 2}{16(n - 1)} \tilde{N}^2(x_0) |\mathcal{L}_\zeta Z|_{\xi}^2 \tag{3.33}
$$

From this, we can easily tackle the slightly more general equation

$$
\Delta w = f(x_0) w^{q-1} \dot{l}^{q-2}.
$$

In fact, we can easily tackle the slightly more general equation

$$
\Delta w = f(x_0) w^{q-1} \dot{l}^{q-2} - \frac{\langle \nabla w, Y(x_0) \rangle^2}{w^2} \dot{l}^{q+2}, \quad x \in \mathbb{R}^n, \tag{3.34}
$$

even if $Y(x_0) \neq 0$. Based on the observation (3.31), we consider three separate cases.

**First case.** Let

$$
l = 0 \quad \text{and} \quad \dot{l} \neq 0. \tag{3.35}
$$

Then by passing to the limit in the first equation of (3.12), we obtain

$$
\Delta \tilde{B} = f(x_0) \tilde{B}^{q-1} \tag{3.36}
$$
in $\mathbb{R}^n$. The exact form of the solutions of this equation is known, thanks to the work of Caffarelli, Gidas and Spruck [CGS89]:

$$
\tilde{B}(x) = \left( 1 + \frac{f(x_0)|x - y_0|^2}{n(n - 2)} \right)^{1 - \frac{q}{2}} \text{ or } \tilde{B} \equiv 0. \tag{3.37}
$$

If $U$ is non-trivial, with $y_0 \in \mathbb{R}^n$ the unique maximum point, there exist $(y_\alpha)_\alpha$ local maxima of $(u_\alpha)_\alpha$ approaching $y_0$ such that

$$
d_\alpha(x_\alpha, y_\alpha) = O(\mu_\alpha) \tag{3.38}
$$

and

$$
\mu_\alpha^{-\frac{q-2}{2}} v_\alpha(y_\alpha) \to 1 \quad \text{as} \quad \alpha \to \infty. \tag{3.39}
$$

Since $(y_\alpha)_\alpha$ are critical points, the hypothesis (3.5) implies that

$$
d_\alpha(S_\alpha, y_\alpha) \mu_\alpha^{-\frac{q-2}{2}} v_\alpha(y_\alpha) \leq 1
$$

for all $\alpha \in \mathbb{N}$, so by (3.39), $d_\alpha(S_\alpha, y_\alpha) = O(\mu_\alpha)$; together with (3.38), the triangle inequality implies $d_\alpha(S_\alpha, x_\alpha) = O(\mu_\alpha)$, which contradicts (3.11).
If \( U \equiv 0 \), then
\[
\lim_{\alpha \to \infty} \hat{v}_\alpha(0) = 0,
\]
which contradicts (3.35).

**Second case.** Let
\[
l \neq 0 \quad \text{and} \quad \hat{l} = 0.
\]
Since \( l \neq 0 \), thanks to (1.22) and (3.27), \( w \) is bounded from below by a constant,
\[
w \geq \varepsilon l.
\]
Note also that (3.34) implies that \( w \) is subharmonic and that
\[
\Delta w^{-\alpha} \leq \alpha \left[ \frac{\|\nabla w\|^2}{w^{\alpha+1}} \left[ \frac{\|Y(x_0)\|^2}{\varepsilon^{\alpha+1}} - (\alpha + 1) \right] \right],
\]
so \( w^{-\alpha} \) is subharmonic for \( \alpha \) large. By applying lemma 9 (see the appendix A), we deduce that \( w \) is constant, in contradiction with (3.28).

**Third case.** Let
\[
l = 0 \quad \text{and} \quad \hat{l} = 0,
\]
then \( w \) is a non-negative harmonic function on \( \mathbb{R}^n \). Thus, \( w \) is constant and furthermore, by (3.28), \( w \equiv 1 \). But \( \hat{l} = 0 \) implies that
\[
\left| \frac{\nabla w(0)}{w(0)} \right|^\beta + \left| \frac{\nabla^2 w(0)}{w(0)} \right|^{\frac{\beta}{2}} = 1.
\]
which leads to a contradiction. \( \square \)

### 4. Asymptotic analysis

In this section, we assume that \((u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}\) is an \( L^\infty \) blow-up sequence, i.e. we ask that there exist a sequence \((x_\alpha)_{\alpha \in \mathbb{N}}\) of critical points of \((u_\alpha)_{\alpha \in \mathbb{N}}\) and a series of positive real numbers \((\rho_\alpha)_{\alpha \in \mathbb{N}}\), where
\[
0 < \rho_\alpha < \frac{1}{16} i_4(M),
\]
such that
\[
\rho_\alpha^n \sup_{B_{\rho_\alpha}(0)} u_{\alpha}^n(\rho_\alpha x) \to \infty \quad \text{as} \ \alpha \to \infty,
\]
and moreover we ask that
\[
d_g(x_\alpha, x)\left( u_{\alpha}^n(x) + \left| \frac{\nabla u_{\alpha}(x)}{u_{\alpha}(x)} \right|^\beta + \left| \frac{\nabla^2 u_{\alpha}(x)}{u_{\alpha}(x)} \right|^{\frac{\beta}{2}} + |L_g W_{\alpha}|_{x}(x) \right) \leq C,
\]
for \( x \in B_{\rho_\alpha}(8\rho_\alpha) \).

In the reminder of this section, we assume that \((u_\alpha, W_\alpha)_{\alpha \in \mathbb{N}}\) is a blow-up sequence, and we look at the kind of asymptotic profiles we can potentially obtain. At the very end, we rule all
of them out, and thus obtain our compactness result. Note that, if we were to assume that \((4.2)\) holds for a sequence \(\{x_\alpha\}\) in \(S_\alpha\), with \(\rho_\alpha\) smaller than the distance of \(x_\alpha\) to any other point in \(S_\alpha\), then \((4.3)\) holds as well.

### 4.1. Harnack inequality

The following is a Harnack-type inequality. It is a direct consequence of the weak estimate and it plays a key role in ruling out clusters of bubbles where some are much larger than others.

**Lemma 3.** Let \((u_\alpha, \rho_\alpha)\), be a blow-up sequence such that \((4.2)\) and \((4.3)\) hold. Then there exists a constant \(C_3 > 1\) such that for any sequence \(0 < s_\alpha \leq \rho_\alpha\), we get

\[
 s_\alpha^2 \|\nabla^2 u_\alpha\|_{L^\infty(\Omega_\alpha)} + s_\alpha \|\nabla u_\alpha\|_{L^\infty(\Omega_\alpha)} \leq C_3 \sup_{\Omega_\alpha} u_\alpha \leq C_3^2 \inf_{\Omega_\alpha} u_\alpha, \tag{4.4}
\]

where \(\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)\).

**Remark 2.** When considering a rescaling of the type

\[
 \bar{u}(x) = s_\alpha^{-2} u_\alpha(\exp_{x_\alpha}(s_\alpha x)), \tag{4.5}
\]

and \(\bar{\Omega}_\alpha = B_0(6) \setminus B_0(\frac{1}{6})\), then the above lemma gives

\[
 \|\nabla^2 \bar{u}_\alpha\|_{L^\infty(\bar{\Omega}_\alpha)} + \|\nabla \bar{u}_\alpha\|_{L^\infty(\bar{\Omega}_\alpha)} \leq C_3 \sup_{\bar{\Omega}_\alpha} \bar{u}_\alpha \leq C_3^2 \inf_{\bar{\Omega}_\alpha} \bar{u}_\alpha. \tag{4.6}
\]

**Proof of lemma 3.** Estimate \((3.10)\) implies that

\[
 \left| \frac{\nabla u_\alpha(x)}{u_\alpha(x)} \right| \leq C_2 d(x_\alpha, x)^{-1} \text{ in } \Omega_\alpha, \tag{4.7}
\]

and therefore

\[
 s_\alpha |\nabla \ln u_\alpha(x)| \leq 6C_2 \text{ in } \Omega_\alpha. \tag{4.8}
\]

Similarly, it holds true that

\[
 s_\alpha^2 |\nabla^2 \ln u_\alpha(x)| \leq 6C_2 \text{ in } \Omega_\alpha. \tag{4.9}
\]

Taking \(C_3 \geq 6C_2\), we get the first inequality from \((4.6)\). Then, from \((4.8)\) and from the fact that the domain is an annulus \(\Omega_\alpha = B_{x_\alpha}(6s_\alpha) \setminus B_{x_\alpha}(\frac{1}{6}s_\alpha)\), we estimate that

\[
 \sup_{\Omega_\alpha} \ln u_\alpha - \inf_{\Omega_\alpha} \ln u_\alpha \lesssim \ell_s(\Omega_\alpha) \|\nabla \ln u_\alpha\|_{L^\infty(\Omega_\alpha)} \lesssim 42C_2,
\]

where \(\ell_s(\Omega_\alpha)\) is the infimum of the length of a curve in \(\Omega_\alpha\) drawn between a maximum and a minimum of \(u_\alpha\). Equivalently

\[
 \sup_{\Omega_\alpha} u_\alpha \leq e^{42C_2} \inf_{\Omega_\alpha} u_\alpha,
\]

so it suffices to take \(C_3 = e^{42C_2}\). \(\square\)

Let \((B_{x_\alpha}(16), \Phi_\alpha)\) be a conformal chart around \(x_\alpha\). We study the blow-up sequence in a Euclidean framework through these charts. By the properties we have imposed on \(\varphi_\alpha\),
\(|q^k_\xi \partial_k (\ln \varphi_\alpha) (\mathcal{L}_\xi Z_\alpha)_x| \leq C|y||\mathcal{L}_\xi Z_\alpha|_x, \quad \text{on } B_0(8\rho_\alpha). \tag{4.10}\)

By the definition of a blow-up sequence, we also get that
\[
|\xi^{\alpha} (v^{\alpha}_q(x) + \frac{\nabla v^{\alpha}_q(x)}{v^{\alpha}_q(x)}| + \frac{\nabla^2 v^{\alpha}_q(x)}{v^{\alpha}_q(x)} | \leq C, \quad \text{on } B_0(8\rho_\alpha). \tag{4.11}\]

4.2. Strong estimate on \(v^{\alpha}_q\) in \(B_0(\mu_\alpha)\)

The following result is a strong estimate on the size of a blow-up sequence in a very small ball \(B_0(\mu_\alpha)\).

**Lemma 4.** Let \((u^{\alpha}_q, W^{\alpha}_q), \alpha \in \mathbb{N}\) be a blow-up sequence verifying (4.1)–(4.3). Let
\[
\mu^{1-\frac{\alpha}{2}} := u^{\alpha}_q(x_\alpha) = v^{\alpha}_q(0). \tag{4.12}\]

Up to a subsequence, we have
\[
\mu_\alpha \to 0 \quad \text{and} \quad \frac{\rho_\alpha}{\mu_\alpha} \to \infty. \tag{4.13}\]

Moreover, we see that
\[
m^{2-\frac{\alpha}{n}}_\alpha v^{\alpha}_q(\mu_\alpha x) \to \tilde{B}(x) \quad \text{in } \mathcal{C}^{2,0}_\text{loc}(\mathbb{R}^n) \quad \text{as } \alpha \to \infty \tag{4.14}\]

and
\[
\mu^{\alpha}_\alpha |\mathcal{L}_\xi Z_\alpha|_x(\mu_\alpha x) \to 0 \quad \text{and} \quad \mathcal{C}^{0}_\text{loc}(\mathbb{R}^n) \quad \text{as } \alpha \to \infty. \tag{4.15}\]

We have denoted
\[
x_0 = \lim_{\alpha \to \infty} x_\alpha \tag{4.16}\]

and
\[
\tilde{B}(x) = \left(1 + \frac{f_0(x_\alpha)}{n(n - 2)}|x|^2 \right)^{1-\frac{\alpha}{2}}. \tag{4.17}\]

**Proof.** The proof involves arguments similar to the ones used for lemma 1. Let \(y_\alpha \in B_{\mu_\alpha}(8\rho_\alpha)\) be such that
\[
u^{\alpha}_\alpha := u^{\alpha}_q(y_\alpha) + \frac{\nabla u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} | + \frac{\nabla^2 u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} | \frac{\nabla u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} + |\mathcal{L}_\xi W^{\alpha}_q(y_\alpha)| \tag{4.18}\]

and let
\[
u^{\alpha-\frac{\alpha}{n}} := u^{\alpha}_q(y_\alpha) + \frac{\nabla u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} | + \frac{\nabla^2 u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} | \frac{\nabla u^{\alpha}_q(y_\alpha)}{u^{\alpha}_q(y_\alpha)} + |\mathcal{L}_\xi W^{\alpha}_q(y_\alpha)|. \tag{4.19}\]
Conditions (4.2) and (4.3) imply that
\[ \frac{\rho_\alpha}{\nu_\alpha} \to \infty \quad \text{and} \quad \nu_\alpha \to 0 \quad \text{as} \quad \alpha \to \infty. \] (4.20)

Moreover,
\[ d_\varepsilon (x_\alpha, y_\alpha) \leq C_{\frac{1}{2}} \nu_\alpha, \] (4.21)

which implies that the coordinates of \( y_\alpha \) in the exponential chart around \( x_\alpha \), defined as \( \tilde{y}_0 = \nu_\alpha^{-1} \exp^{-1}_{x_\alpha} (y_\alpha) \), are bounded by \( C_{\frac{1}{2}} \). Up to a subsequence, we may choose a finite limit \( \tilde{y}_0 = \lim_{\alpha \to \infty} \tilde{y}_\alpha \). We denote
\[ \hat{v}_\alpha (x) = \nu_\alpha^{-1} u_\alpha \left( \exp_{x_\alpha} (\nu_\alpha x) \right) \quad \text{and} \quad \hat{Z}_\alpha (x) = \nu_\alpha^{n-2} Z_\alpha \left( \exp_{x_\alpha} (\nu_\alpha x) \right) \] (4.22)

for \( x \in \Omega_\alpha := B_0 \left( \frac{8 \rho_\alpha}{\nu_\alpha} \right) \). As before,
\[ \hat{v}_\alpha (x) = O(1), \quad \left| \nabla \hat{v}_\alpha (x) \right| = O(1), \quad \left| \nabla^2 \hat{v}_\alpha (x) \right| = O(1) \] (4.23)

and
\[ |\mathcal{L}_\xi \hat{Z}_\alpha (x)|_\xi \to 0. \] (4.24)

This implies that
\[ \hat{v}_\alpha (\tilde{y}_\alpha) + \left|\frac{\nabla \hat{v}_\alpha (\tilde{y}_\alpha)}{\hat{v}_\alpha (\tilde{y}_\alpha)}\right|^n + \left|\frac{\nabla^2 \hat{v}_\alpha (\tilde{y}_\alpha)}{\hat{v}_\alpha (\tilde{y}_\alpha)}\right|^\frac{\xi^2}{\nu_\alpha} = 1. \] (4.25)

By applying the same analysis as in the proof of lemma 1, we get that, up to passing to a subsequence, there exists \( \tilde{B}_\lambda := \lim_{\alpha \to \infty} v_\alpha \) in \( C^2_{\text{loc}} (\mathbb{R}^n) \), with \( x_0 := \lim_{\alpha \to \infty} x_\alpha \), such that
\[ \Delta \tilde{B}_\lambda = f(x_0) \tilde{B}_\lambda^{-1}, \] (4.26)

Since \( \nabla \tilde{B}_\lambda (0) = 0 \), it holds that
\[ \tilde{B}_\lambda (x) = \lambda^{\frac{\xi^2}{\nu_\alpha}} \left( 1 + \frac{f(x_0) \lambda^2 |x|^2}{n(n-2)} \right)^{1-\frac{\xi^2}{\nu_\alpha}} \] (4.27)

for some \( \lambda > 0 \), where
\[ \frac{\nu_\alpha}{\rho_\alpha} \to \lambda. \] (4.28)

This yields (4.13)–(4.15), thanks to (4.20), (4.27) and (4.24).

\[ \square \]

4.3. The sphere of dominance around a blow-up point

We denote
\[ B_\alpha (x) = \mu_\alpha^{\frac{\xi^2}{\nu_\alpha}} \left( \mu_\alpha^2 + \frac{f_\alpha (x_\alpha) \lambda^2 |x|^2}{n(n-2)} \right)^{1-\frac{\xi^2}{\nu_\alpha}} \] (4.29)
and
\[ \theta_\alpha(x) = \sqrt{\mu_\alpha^2 + |x|^2}. \]  
(4.30)

Our next goal is to extend the estimates from a ball of size $\mu_\alpha$ to one of size $\rho_\alpha$. We define the radius on which the estimates continue to hold as
\[ r_\alpha = \sup R_\alpha \]  
(4.31)

where
\[ R_\alpha = \{ 0 < r \leq \rho_\alpha, \quad v_\alpha \leq (1 + \varepsilon)B_\alpha, \quad |\nabla(v_\alpha - B_\alpha)|_\xi \leq \varepsilon|\nabla B_\alpha|_\xi, \]  
and $B_0(r) \setminus B_0(2R_\alpha \mu_\alpha)$}  
(4.32)

and
\[ R_\alpha^2 = \frac{n(n - 2)}{f_\alpha(x_\alpha)}. \]  
(4.33)

The following two properties hold for $r_\alpha$:
\[ r_\alpha = O(\sqrt{\mu_\alpha}) \]  
(4.34)

and
\[ r_\alpha \gg \mu_\alpha. \]  
(4.35)

By the previous lemma, we know that the $C^{2,2}$ limit holds on balls of radius $\rho_\alpha$ and by definition also of radius $r_\alpha$, which is to say that the two are comparable. As a result, (4.13) implies (4.34).

In order to get the second estimate, it suffices to note that, by the definition of $r_\alpha$ and by (1.22),
\[ \varepsilon \leq C \mu_\alpha^{\frac{n+1}{2}} r_\alpha^{-n}, \]  
(4.36)

which directly implies (4.35).

### 4.3.1 First order estimates of $v_\alpha$ on $B_0(8\delta_\alpha)$.

**Lemma 5.** Let $(\delta_\alpha)_{\alpha}$, $0 < \delta_\alpha \leq r_\alpha$ be a sequence of radii. Then for any $z_\alpha \in B_0(8\delta_\alpha)$ there holds:
\[ v_\alpha(z_\alpha) + |\nabla v_\alpha(z_\alpha)||z_\alpha| + |\nabla^2 v_\alpha(z_\alpha)||z_\alpha|^2 \leq CB_\alpha(z_\alpha). \]  
(4.37)

Moreover, there exists a sequence of positive numbers $(\kappa_\alpha)_{\alpha \in \mathbb{N}}$ such that
\[ (1 - \kappa_\alpha)B_\alpha(z_\alpha) \leq v_\alpha(z_\alpha). \]  
(4.38)

If $\delta_\alpha \to 0$, then $\kappa_\alpha \to 0$.

**Proof.** For $x \in B_0(8)$:
\[ \bar{v}_\alpha(x) = r_\alpha^{\frac{n-2}{2}} v_\alpha(r_\alpha x). \]  
(4.39)

Then $\bar{v}_\alpha$ satisfies
\[\Delta \bar{v}_\alpha(x) + r^2 \Delta \bar{h}_\alpha(x) = \bar{f}_\alpha(x) \bar{v}_{\alpha}^{q-1}(x) + r^{2\alpha} \bar{a}_\alpha(x) \bar{v}_{\alpha}^{q+1}(x) - r^2 \bar{b}_\alpha(x)\]

where

\[
\bar{a}_\alpha(x) = \bar{a}_\alpha(r_\alpha x), \quad \bar{h}_\alpha(x) = \bar{h}_\alpha(r_\alpha x), \quad \bar{f}_\alpha(x) = \bar{f}_\alpha(r_\alpha x), \\
\bar{c}_\alpha(x) = \bar{c}_\alpha(r_\alpha x), \quad \bar{d}_\alpha(x) = \bar{d}_\alpha(r_\alpha x), \quad \bar{Y}_\alpha(x) = \bar{Y}_\alpha(r_\alpha x).
\]

By the definition of \(r_\alpha\), we know that

\[
\bar{v}_\alpha(x) \leq C \left( \frac{\mu_\alpha}{r_\alpha} \right)^{\frac{n-2}{2}}
\]

in \(B_0(1) \setminus B_0 \left( \frac{1}{4} \right)\). By the weak estimate we know that

\[
\bar{v}_\alpha \leq C \\
\frac{\nabla \bar{v}_\alpha(x)}{\bar{v}_\alpha(x)} \leq C \\
\frac{\nabla^2 \bar{v}_\alpha(x)}{\bar{v}_\alpha(x)} \leq C \\
r^{2\alpha} \bar{a}_\alpha \leq C
\]

in \(B_0(8) \setminus B_0 \left( \frac{1}{4} \right)\). We conclude the proof by lemma 3.

We consider \(G_\alpha\), the Green function of \(\Delta \bar{v} + \bar{h}_\alpha\) in \(M\). For any sequence \((z_\alpha)\) of points in \(B_0(8r_\alpha)\):

\[
v_\alpha(z_\alpha) \geq \varphi_\alpha(z_\alpha) \int_{B_0(r_\alpha)} \varphi_\alpha(y) G_\alpha \left( \Phi_{\alpha}^{-1}(z_\alpha), \Phi_{\alpha}^{-1}(y) \right) f_\alpha(y) v_\alpha^{q-1}(y) \, dy.
\]

In particular,

\[
\frac{\bar{v}_\alpha(z_\alpha)}{B_\alpha(z_\alpha)} \geq \frac{\varphi_\alpha(z_\alpha)}{B_\alpha(z_\alpha)} \int_{B_0 \left( \frac{1}{4} r_\alpha \right)} \varphi_\alpha(\mu_\alpha y) \bar{f}_\alpha(\mu_\alpha y) \left( \frac{\mu_\alpha}{r_\alpha} \right)^{\frac{n-2}{2}} v_\alpha(\mu_\alpha y) \, dy \\
\times G_\alpha \left( \Phi_{\alpha}^{-1}(z_\alpha), \Phi_{\alpha}^{-1}(\mu_\alpha y) \right) d_\delta \left( \Phi_{\alpha}^{-1}(z_\alpha), \Phi_{\alpha}^{-1}(\mu_\alpha y) \right)^{n-2} \\
\times \left( \frac{\mu_\alpha^2 + \frac{\mu_\alpha c_\alpha}{\mu_\alpha^2} |c_\alpha|^2}{d_\delta \left( \Phi_{\alpha}^{-1}(z_\alpha), \Phi_{\alpha}^{-1}(\mu_\alpha y) \right)^2} \right)^{\frac{n-2}{2}} \, dy.
\]
4.3.2. Improved weak estimate of $\mathcal{L}_{\xi}Z_{\alpha}$ on $B_0(7\delta_{\alpha})$.

**Lemma 6.** Let $(\delta_{\alpha})_\beta$ be a sequence of positive numbers such that $\delta_{\alpha} \gg \mu_{\alpha}$ and $\delta_{\alpha} \leq \sqrt{\mu_{\alpha}}$. We get for any $x \in B_0(7\delta_{\alpha})$,

$$\int_{B_0(6\delta_{\alpha})} |x-y|^{2-n} |\mathcal{L}_{\xi}Z_{\alpha}(y)|^2 \nu_{\alpha}^{n-1}(y) \, dy \leq C(B_{\alpha}(x) + O(1)) \quad (4.46)$$

and as a consequence

$$\int_{B_0(6\delta_{\alpha})} |\mathcal{L}_{\xi}Z_{\alpha}|^2 \, dy \leq C \left( \mu_{\alpha}^{2n-2} \delta_{\alpha}^{2-3n} + \frac{\delta_{\alpha}}{\mu_{\alpha}} \delta_{\alpha}^{-1} \delta_{\alpha}^{2n-1} \right). \quad (4.47)$$

Further, there exists a sequence $s_{\alpha} \in (5\delta_{\alpha}, 6\delta_{\alpha})$ such that

$$\int_{\partial B(s_{\alpha})} |\mathcal{L}_{\xi}Z_{\alpha}|^2 \, d\sigma \leq C \left( \mu_{\alpha}^{2n-2} \delta_{\alpha}^{2-3n} + \frac{\delta_{\alpha}}{\mu_{\alpha}} \delta_{\alpha}^{-1} \delta_{\alpha}^{2n-1} \right). \quad (4.48)$$

**Proof.** We use the Green’s representation theorem for $\Delta_{\xi} + \tilde{h}_{\alpha}$ in $B_0(7\delta_{\alpha})$ in the first equation, and obtain

$$\int_{B_0(6\delta_{\alpha})} |x-y|^{2-n} \frac{|\mathcal{L}_{\xi}Z_{\alpha}(y)|^2}{\nu_{\alpha}^{n+1}(y)} \, dy \leq C (B_{\alpha}(x) + H_1 + H_2 + H_3), \quad (4.49)$$

where

$$H_1 = \int_{B_0(6\delta_{\alpha})} \frac{\tilde{b}_{\alpha}(y)}{\nu_{\alpha}(y)} |x-y|^{2-n} \, dy,$$

$$H_2 = \int_{B_0(6\delta_{\alpha})} \left( \nabla \nu_{\alpha}(y), \tilde{V}(y) \right) \left( \frac{\tilde{d}_{\alpha}(y) - \tilde{c}_{\alpha}(y)}{\nu_{\alpha}^{n+2}(y)} \right) |x-y|^{2-n} \, dy, \quad (4.50)$$

$$H_3 = \int_{B_0(6\delta_{\alpha})} \left( \nabla \nu_{\alpha}(y), \tilde{V}(y) \right)^2 \left( \frac{\tilde{d}_{\alpha}(y)}{\nu_{\alpha}^{n+2}(y)} \right)^2 |x-y|^{2-n} \, dy.$$

Lemma 5 yields the following estimates:

$$H_1 \leq C \int_{B_0(6\delta_{\alpha})} \frac{\tilde{b}_{\alpha}}{\mu_{\alpha}} \theta_{\alpha}^{n-2} |x-y|^{2-n} \, dy \leq C \mu_{\alpha} \left( \frac{\delta_{\alpha}^2}{\mu_{\alpha}} \right)^{\frac{3}{2}}, \quad (4.51)$$

$$H_2 \leq C \int_{B_0(6\delta_{\alpha})} \theta_{\alpha}^{-2}(y) \left( \frac{1}{\mu_{\alpha}} \theta_{\alpha}^{3n-2} + \frac{\delta_{\alpha}}{\mu_{\alpha}} \theta_{\alpha}^{n-2} \right) |x-y|^{2-n} \, dx,$$

$$\leq C \left( \frac{\delta_{\alpha}^{2}}{\mu_{\alpha}} \right)^{\frac{3}{2}} + C \left( \frac{\delta_{\alpha}^{2}}{\mu_{\alpha}} \right)^{\frac{3}{2}} \quad (4.52)$$

and

$$H_3 \leq C \int_{B_0(6\delta_{\alpha})} \frac{\tilde{b}_{\alpha}}{\mu_{\alpha}} \theta_{\alpha}^{n-2} |x-y|^{2-n} \, dx \leq C \left( \frac{\delta_{\alpha}^{2}}{\mu_{\alpha}} \right)^{\frac{3n-2}{2}}. \quad (4.53)$$
As a consequence,
\[
\int_{B_\delta(0)} |x - y|^{2-n} \frac{|\mathcal{L}_\xi Z_\alpha(y)|^2}{\epsilon_n^{2-n}(y)} \, dy \leq C \left( \frac{\nu_{\alpha}^2}{\mu_{\alpha}^2} \theta_{\alpha}^{-n}(x) + 1 \right) .
\] (4.54)

In particular, we get (4.47) and (4.48).

4.3.3. First order estimate of \( \mathcal{L}_\xi Z_\alpha \) on \( B_\delta(3\delta_\alpha) \). We use the previous improved weak estimate in order to get a first order estimate of \( \mathcal{L}_\xi Z_\alpha \). For \( x \neq 0 \), let
\[
\mathcal{G}_\alpha(x) = -\frac{1}{4(n-1)\omega_{n-1}} |x|^{2-n} \left( (3n-2)\delta_j + (n-2)\frac{y_jy_j}{|x|^4} \right) \]
be the fundamental solution of \( \overrightarrow{\Delta}_\xi \) in \( \mathbb{R}^n \). We define on \( \mathbb{R}^n \) the vector field
\[
V_\alpha(x) = -\frac{n^2}{2(n-2)} \ln \left( 1 + \frac{|x|^2}{\mu_{\alpha}^2} \right) \tilde{V}_\alpha(0) + \frac{n}{\mu_{\alpha}^2 + |x|^2} \left( x, \tilde{V}_\alpha(0) \right) x_i \]
and a vector field \( R_\alpha \) such that
\[
\overrightarrow{\Delta}_\xi (V_\alpha + R_\alpha)(x) = 2n - 1 \left( \frac{\nabla^2 B_\alpha(x), \tilde{V}_\alpha(0)}{B_\alpha(x)} + \frac{3n}{n-2} \frac{\nabla B_\alpha(x), \tilde{V}_\alpha(0)}{B_\alpha^2(x)} \nabla B_\alpha(x) \right) .
\] (4.57)

Note, in particular, that
\[
|\overrightarrow{\Delta}_\xi R_\alpha(x)| \leq C |\tilde{V}_\alpha(0)| \mu_{\alpha}^2 \theta_{\alpha}^{-3}(x)^{-4} .
\] (4.58)

Thus,
\[
C |\tilde{V}_\alpha(0)| \mu_{\alpha}^2 \theta_{\alpha}^{-3}(x) \quad n = 5
\]
\[
|\mathcal{L}_\xi R_\alpha(x)| \leq C |\tilde{V}_\alpha(0)| \mu_{\alpha}^2 \theta_{\alpha}^{-3}(x) \ln \left( 1 + \frac{\theta_{\alpha}(x)}{\mu_{\alpha}} \right) n = 4. \quad (4.59)
\]
\[
C |\tilde{V}_\alpha(0)| \mu_{\alpha}^2 \theta_{\alpha}^{-2}(x) \quad n = 3.
\]

By direct calculation, we see that
\[
\left( \mathcal{L}_\xi V_\alpha \right)_{ij}(x) = -\frac{2n}{n-2} \frac{\tilde{V}_\alpha(0)}{\mu_{\alpha}^2 + |x|^2} \left( x_j \tilde{V}_\alpha(0) \right)_{j} + \frac{x_i}{|x|^2} \tilde{V}_\alpha(0) - \frac{2n}{n-2} \left( \frac{x}{|x|^2}, \tilde{V}_\alpha(0) \right).
\]

Note that
\[
|\mathcal{L}_\xi V_\alpha(x)| \leq C |\tilde{V}_\alpha(0)| \theta_{\alpha}^{-1}(x) .
\] (4.61)
Lemma 7. Let \((\delta_\alpha)\) be a sequence of positive numbers such that
\[
\frac{\mu_\alpha}{\delta_\alpha} \rightarrow 0 \quad \text{and} \quad \delta_\alpha \leq \min(\rho, \sqrt{\mu_\alpha}).
\] (4.62)

For any \(x \in B_0(3\delta_\alpha)\), we get the following estimate:
\[
|\mathcal{L}_\xi Z_\alpha(x)| \leq \theta^{-1}_\alpha(x) + \mu_\alpha^{-1} \delta_\alpha^{1-2\alpha}. \quad (4.63)
\]

Proof. Without mentioning of the conformal change factor \(\varphi_\alpha\), we apply the Green representation theorem on the 2nd equation. Let \(G_{\alpha,j}\) be the Green one-form for \(\Delta_\xi\) with Neumann boundary conditions on \(B_0(s_\alpha)\), \(s_\alpha \leq 4\delta_\alpha\). Similarly, let
\[
\mathcal{H}_{ij}(x, y)_p = \partial_j G_{\alpha,j}(x, y)_p + \partial_j G_{\alpha,j}(x, y)_p - \frac{2}{n} \xi_j \sum_{k=1}^n \partial_k G_{\alpha,k}(x, y)_p. \quad (4.64)
\]

There holds that
\[
\mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)_j(x) = \int_{\partial B_0(s_\alpha)} \mathcal{H}_{ij}(x, y)_p \Delta_\xi(Z_\alpha - V_\alpha - R_\alpha)_j(y) \, dy
+ \int_{\partial B_0(s_\alpha)} \mathcal{H}_{ij}(Z_\alpha, y)_p \mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)_j(y) \, ds. \quad (4.65)
\]

Keeping in mind that \(R_\alpha\) is negligible compared to \(V_\alpha\), we obtain the estimate
\[
|\mathcal{L}_\xi(Z_\alpha - V_\alpha - R_\alpha)_j(x)| \leq C (I_1 + I_2 + I_3 + I_4 + J_1 + J_2), \quad (4.66)
\]

where the bulk terms are
\[
I_1 = \int_{B_0(6\delta_\alpha)} |x - y|^{1-n} |\mathcal{L}_\xi Z_\alpha(y)| \, dy
\]
\[
I_2 = \left| \frac{n+1}{n-2} \int_{B_0(6\delta_\alpha)} \left( \frac{\nabla v_\alpha(y) \nabla \tilde{v}_\alpha(y)}{v_\alpha(y)} - \frac{\nabla B_\alpha(y) \nabla \tilde{B}_\alpha(y)}{B_\alpha(y)} \right) \right| |x - y|^{1-n} \, dy
\]
\[
I_3 = \left| \int_{B_0(6\delta_\alpha)} |x - y|^{1-n} \left( \frac{\nabla^2 v_\alpha(y) \nabla \tilde{v}_\alpha(y)}{v_\alpha(y)} - \frac{\nabla^2 B_\alpha(y) \nabla \tilde{B}_\alpha(y)}{B_\alpha(y)} \right) \right| \nabla \ln \tilde{N}_\alpha(y)
+ \text{div} \tilde{V}_\alpha(y) \nabla \frac{v_\alpha(y)}{v_\alpha(y)} - \frac{\nabla \tilde{V}_\alpha(y) \nabla v_\alpha(y)}{v_\alpha(y)} \right| \, dy
\]
\[
I_4 = \int_{B_0(6\delta_\alpha)} |x - y|^{1-n} \, dy
\] (4.67)
and the boundary terms
\[ J_1 = \int_{\partial B_0} |x - y|^{1-n} |L_{\xi} V_\alpha(x)| \, d\sigma, \]
\[ J_2 = \int_{\partial B_0} |x - y|^{1-n} |L_{\xi} Z_\alpha(x)| \, d\sigma. \] (4.68)

Then, by (4.61),
\[ J_1 \leq C \delta_\alpha^{-1}, \] (4.69)
and by (4.48),
\[ J_2 \leq \mu^{n-1} \delta_\alpha^{1-2n}. \] (4.70)

Next, we see that
\[ |I_2| \leq C \int_{B_0(6\theta_\alpha)} |x - y|^{1-n} \theta_\alpha^{-2}(y) \, dy \]
\[ \leq C \theta_\alpha^{-1}(x) \int_{B_0(6\theta_\alpha)} \left| \frac{x}{\theta_\alpha(x)} - \frac{1}{\left( \frac{\mu_\alpha}{\theta_\alpha(x)} \right)^2 + |z|^2} \right| \, dz, \] (4.71)
so that
\[ |I_2| \leq \theta_\alpha^{-1}(x). \] (4.72)

The term \( I_3 \) is in fact negligible when compared to \( I_2 \) and so
\[ |I_3| \leq \theta_\alpha^{-1}(x) \] (4.73)
also. It is also clear that
\[ I_4 \leq C \delta_\alpha. \] (4.74)

Coming back to (4.66) with all these estimates, we thus obtain that
\[ |L_{\xi} Z_\alpha(x)| \leq C \left( \theta_\alpha(x)^{-1} + \mu^{n-1} \delta_\alpha^{1-2n} \right) + I_1. \] (4.75)

It remains to estimate \( I_1 \). We shall use an iterative argument to do it. Assume that
\[ |L_{\xi} Z_\alpha(x)| \leq C \left( \theta_\alpha(x)^{-\beta} + \mu^{n-1} \delta_\alpha^{1-2n} \right) \] (4.76)
for some \( 1 < \beta \leq n \). Note that, thanks to the weak estimate (4.11) on \( L_{\xi} Z_\alpha \), it holds for \( \beta = n \). If (4.76) holds, we can write that
\[ I_1 \leq C \mu^{n-1} \delta_\alpha^{1-2n} \int_{B_0(6\theta_\alpha)} |x - y|^{1-n} \, dy + C \int_{B_0(6\theta_\alpha)} |x - y|^{1-n} \theta_\alpha(y)^{-\beta} \, dy \]
\[ \leq C \mu^{n-1} \delta_\alpha^{2-2n} + \begin{cases} \theta_\alpha(x)^{1-\beta} & \text{if } \beta < n \\ \theta_\alpha(x)^{1-n} \ln \left( 1 + \frac{\theta_\alpha(x)}{\mu_\alpha} \right) & \text{if } \beta = n \end{cases} \] (4.77)
Remember here that $\beta > 1$. Coming back to (4.75), we obtain that, if (4.76) holds for some $1 < \beta \leq n$, it necessarily also holds when $\beta$ is replaced by $\beta - \frac{1}{2}$. Since, as already said, it holds for $\beta = n$, we obtain by induction that it holds for all $\beta = n - \frac{k}{2}$ as long as $n - \frac{k}{2} > 1$. Thus, it holds for $\beta = 1$. But this is exactly the estimate (4.63).

\[ \square \]

Remark 3. For $\delta_\alpha = r_\alpha$, we get that

\[ |\mathcal{L}_\xi Z_\alpha| \leq \theta_\alpha^{-1} + \mu_\alpha^{-1} \delta_\alpha^{-2n} \]  

(4.78)

implies

\[ |\mathcal{L}_\xi Z_\alpha| \leq \left( \frac{\mu_\alpha}{r_\alpha} \right)^{n-1} \theta_\alpha^{-n}. \]  

(4.79)

4.3.4. Asymptotic profile on $B_0(\beta)$.

Lemma 8. Up to a subsequence, it holds that

\[ v_\alpha(0) r_\alpha^{-2n} v_\alpha(r_\alpha x) \rightarrow \frac{R_0^{\beta - 2}}{|x|^n} + H(x) \quad \text{in } C^2_{\text{loc}}(B_0(2) \setminus \{0\}), \]  

(4.80)

where $H$ is a non-negative superharmonic function in $B_0(2)$. We recall that, by (4.12),

\[ v_\alpha(0) = \mu_\alpha^{-\frac{q}{4}}. \]  

(4.81)

Proof. Step 1: let

\[ \hat{v}_\alpha(x) = \mu_\alpha^{-\frac{q}{4}} r_\alpha^{-2n} v_\alpha(r_\alpha x), \quad x \in B_0(2), \]  

(4.82)

where $\mu_\alpha$ is defined in (4.12). Then

\[ \Delta_\xi \hat{v}_\alpha = \hat{F}_\alpha, \]  

(4.83)

with

\[
\hat{F}_\alpha = -r_\alpha^2 \hat{b}_\alpha(x) \hat{v}_\alpha(x) + \mu_\alpha^2 r_\alpha^{-2} \hat{\mathcal{L}}_\alpha(x) \hat{v}_\alpha^{-1}(x) \\
+ \mu_\alpha^{2-2n} r_\alpha^{2n-3} \hat{\mathcal{L}}_\alpha(x) + |\hat{\mathcal{F}}_\alpha(x) + \hat{\mathcal{F}}_\alpha(x) \hat{\mathcal{L}}_\alpha(x)|^2 \hat{v}_\alpha^{-1}(x) \\
- \mu_\alpha^{2-2n} r_\alpha^{2n-3} \hat{b}_\alpha(x) \hat{v}_\alpha(x) - \frac{\partial}{\partial \alpha} r_\alpha^{\beta-2} \left( \frac{\langle \nabla \hat{v}_\alpha(x), \hat{Y}_\alpha(x) \rangle^2}{\hat{v}_\alpha^2(x) \alpha^{n-1}} \right) \\
- \mu_\alpha^{2-2n} r_\alpha^{2n-3} \left( \frac{\langle \nabla \hat{v}_\alpha(x), \hat{Y}_\alpha(x) \rangle}{\hat{v}_\alpha(x)} \left( \frac{\hat{d}_\alpha(x)}{\hat{v}_\alpha(x)} + \frac{\hat{\mathcal{L}}_\alpha(x)}{\hat{v}_\alpha(x)} \right) \right). 
\]  

(4.84)

This implies that

\[
\Delta_\xi \hat{v}_\alpha = \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 \hat{F}_\alpha \hat{v}_\alpha^{-1} + \left( \frac{r_\alpha^2 \mu_\alpha}{\mu_\alpha} \right)^2 \left( \frac{\mu_\alpha}{r_\alpha} \right)^{2n} \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 |\hat{\mathcal{L}}_\alpha(x) \hat{v}_\alpha(x)|^2 \\
- \left( \frac{r_\alpha^2 \mu_\alpha}{\mu_\alpha} \right)^{2n-2} \left( \frac{\langle \nabla \hat{v}_\alpha(x), \hat{Y}_\alpha(x) \rangle}{\hat{v}_\alpha^{n+3}} \right)^2 + o(1); 
\]  

(4.85)
where

\[ \tilde{f}_\alpha(x) = \tilde{f}_\alpha(r_\alpha x), \quad \rho_{2,\alpha} = \tilde{\rho}_{2,\alpha}(r_\alpha x) \quad \text{and} \quad \tilde{Y}_\alpha(x) = \tilde{\tilde{Y}}_\alpha(r_\alpha x). \]  \tag{4.86}

By the definition (4.32), there holds for some positive \( C \)

\[ \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{1/2} \leq \tilde{\tilde{e}}_\alpha(x) \leq \left( \frac{f_\alpha(x_\alpha)}{n(n-2)} \right)^{1/2}. \]  \tag{4.87}

Similarly,

\[ |\nabla \tilde{\tilde{e}}_\alpha(x)| \leq \left( \frac{f_\alpha(x_\alpha)}{n(n-2)} \right)^{-1/2}. \]  \tag{4.88}

Moreover, for any \( x \in B_0(2) \):

\[ \frac{\tilde{a}_\alpha(x)}{\tilde{\tilde{\alpha}}^{\alpha^n}(x)} \leq C \left( \frac{\mu_\alpha}{r_\alpha} \right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{1/2} \in L^\infty(B_0(2) \setminus \{0\}). \]  \tag{4.89}

We recall that we have assumed \( \tilde{Y}_\alpha \to 0 \) in \( C^{0,0}_\text{loc}(\cdot) \). By standard elliptic theory, we see that

\[ \tilde{\tilde{e}}_\alpha \to \tilde{\tilde{v}} \quad \text{in } C^{1,1}_\text{loc}(B_0(2) \setminus \{0\}) \quad \text{as } \alpha \to \infty. \]  \tag{4.90}

For \( x \neq 0 \),

\[ \tilde{\tilde{v}}(x) = \frac{\lambda_0}{|x|^n-2} + H(x), \]  \tag{4.91}

where \( H \) is a superharmonic function in \( B_0(2) \) and \( \lambda_0 = \left( \frac{\text{area} - 2}{\text{volume}} \right) \). Moreover, \( H \geq 0 \) in \( B_0(2) \). If \( r_\alpha < \rho_\alpha \), then \( H(0) > 0 \). Indeed, by the definition (4.32), there exists \( y_\alpha \in B_0(r_\alpha) \) such that at least one of the following conditions hold:

(a) \( v_\alpha(y_\alpha) = (1 + \varepsilon)B_\alpha(y_\alpha) \),

(b) \( |\nabla v_\alpha(y_\alpha)| \geq (1 + \varepsilon)|\nabla B_\alpha(y_\alpha)| \).

Letting \( \bar{y}_\alpha = \frac{r_\alpha}{\rho_\alpha} y_\alpha \), we see that either \( H(\bar{y}_\alpha) \) or \( \nabla H(\bar{y}_\alpha) \) are non-zero, and since \( H \) is a non-negative superharmonic function, then \( H(0) > 0 \). Independently, we show that \( H(0) \leq 0 \). The Pohozaev identity writes as

\[
\int_{B_0(\delta r_\alpha)} \left( x^k \partial_k v_\alpha(x) + \frac{n-2}{2} v_\alpha(x) \right) \Delta v_\alpha(x) \, dx = \int_{\partial B_0(\delta r_\alpha)} \left( \frac{1}{2} \delta r_\alpha |\nabla v_\alpha(x)|^2 - \frac{n-2}{2} v_\alpha(x) \partial_n v_\alpha(x) - \delta r_\alpha (\partial_n v_\alpha(x))^2 \right) \, d\sigma. \tag{4.92}
\]

Thanks to (4.80), we can estimate the boundary terms as

\[
\int_{\partial B_0(\delta r_\alpha)} \left( \frac{1}{2} \delta r_\alpha |\nabla v_\alpha(x)|^2 - \frac{n-2}{2} v_\alpha(x) \partial_n v_\alpha(x) - \delta r_\alpha (\partial_n v_\alpha(x))^2 \right) \, d\sigma
\]

\[ = \left( \frac{\mu_\alpha}{r_\alpha} \right)^{n-2} \left( \int_{\partial B_0(\delta)} \left( \frac{1}{2} |\nabla \Psi|^2 - \frac{n-2}{2} \Psi \partial_n \Psi - \delta (\partial_n \Psi)^2 \right) \, d\sigma + o(1) \right) \tag{4.93}
\]
where $\Psi(x) = R_0^{n-2}|x|^{2-n} + H(x)$. Simple computations lead then to  

$$
\int_{\partial B_{0}(\delta r_{n})} \left( \frac{1}{2} \delta r_{n} |\nabla v_{n}(x)|^{2} - \frac{n-2}{2} v_{n}(x) \partial_{r} v_{n}(x) - \delta r_{n} (\partial_{r} v_{n}(x))^{2} \right) \, \sigma \, dx \\
= \left( \frac{\mu_{n}}{r_{n}} \right)^{n-2} \left( \frac{\omega_{n-1} R_{0}^{n-2} H(0) + O(\delta) \right). \tag{4.94}
$$

On the other hand, the LHS reads as  

$$
\int_{B_{0}(\delta r_{n})} \left( x^{k} \partial_{k} v_{n}(x) + \frac{n-2}{2} v_{n}(x) \right) \Delta_{\xi} v_{n}(x) \, dx = J_1 + J_2 + J_3 + J_4, \tag{4.95}
$$

where

$$
J_1 = - \int_{B_{0}(\delta r_{n})} \left( x^{k} \partial_{k} v_{n}(x) + \frac{n-2}{2} v_{n}(x) \right) \times \left( \tilde{h}_{n}(x) v_{n}(x) + \tilde{b}_{n}(x) \frac{\tilde{v}_{n}(x)}{v_{n}(x)} + \langle \nabla v_{n}(x), \tilde{Y}_{n}(x) \rangle \frac{\tilde{d}_{n}(x)}{v_{n}^{2}(x)} + \tilde{\alpha}_{n}(x) \right) \, dx
$$

$$
J_2 = \int_{B_{0}(\delta r_{n})} \left( x^{k} \partial_{k} v_{n}(x) + \frac{n-2}{2} v_{n}(x) \right) \tilde{f}_{\alpha}(x) v_{n}^{\alpha-1}(x) \, dx
$$

$$
J_3 = \int_{B_{0}(\delta r_{n})} \left( x^{k} \partial_{k} v_{n}(x) + \frac{n-2}{2} v_{n}(x) \right) \frac{\tilde{\mu}_{\alpha}(x) + |\tilde{\Psi}_{\alpha}(x) + \tilde{\rho}_{\alpha}(x) \tilde{Z}_{\alpha}(x)|^{2}}{v_{n}^{\alpha+1}(x)} \, dx
$$

$$
J_4 = - \int_{B_{0}(\delta r_{n})} \left( x^{k} \partial_{k} v_{n}(x) + \frac{n-2}{2} v_{n}(x) \right) \left( \frac{\nabla v_{n}(x), \tilde{Y}_{n}(x)}{v_{n}^{\alpha+1}(x)} \right)^{2} \, dx
$$

(4.96)

We find estimates for each quantity in turn. In the case of $J_1$, we notice that  

$$
\left| \int_{B_{0}(\delta r_{n})} \tilde{h}_{n}(x) B_{n}(x) \, dx \right| \leq C \begin{cases} \mu_{n}^{2} \delta r_{n} & \text{if } n = 5 \\
\mu_{n}^{2} \ln \left( \frac{r_{n}}{\mu_{n}} \right) \delta r_{n} \mu_{n} & \text{if } n = 4 \\
\delta r_{n} \mu_{n} & \text{if } n = 3 \end{cases} \tag{4.97}
$$

Then we have that

$$
\left| \int_{B_{0}(\delta r_{n})} \tilde{b}_{n}(x) \, dx \right| \leq C (\delta r_{n})^{n}, \tag{4.98}
$$

and

$$
\left| \int_{B_{0}(\delta r_{n})} \left( \frac{\nabla B_{n}(x), \tilde{Y}_{n}(x)}{B_{n}(x)} \right) \left( \tilde{d}_{n}(x) + \tilde{c}_{n}(x) \right) \, dx \right| \leq C \int_{B_{0}(\delta r_{n})} \theta_{n}^{\alpha-1}(x) \, dx \leq C (\delta r_{n})^{n-1}. \tag{4.99}
$$
For $J_3$, we obtain
\[
\left| \int_{B_0(\delta_\alpha)} \frac{|C_\alpha Z_\alpha(x)|^2}{B_0'(x)} \, dx \right| \leq \int_{B_0(\delta_\alpha)} \left( \frac{\mu_\alpha}{r_\alpha} \right)^{2n-2} \mu_\alpha^{-n} \theta_\alpha^n(x) \, dx \\
\leq C \left( \frac{\mu_\alpha}{r_\alpha} \right)^{n-2} r_\alpha \delta^{2n},
\] (4.100)
while for $J_4$, we get
\[
\left| \int_{B_0(\delta_\alpha)} \frac{\langle \nabla B_\alpha(x), \bar{Y}_\alpha(x) \rangle^2}{B_0'(x)} \, dx \right| \leq \int_{B_0(\delta_\alpha)} \theta_\alpha^{2n-2}(x) \mu_\alpha^{-n} \, dx \\
\leq C \left( \frac{\mu_\alpha}{r_\alpha} \right)^n \left( \frac{r_\alpha^2}{\mu_\alpha} \right)^n r_\alpha^{-2} \delta^{3n-2}.
\] (4.101)

For $J_2$, lengthy, yet straightforward computations as those seen in [Vâl19] (see the proof of lemma 9) lead to
\[
J_2 = o \left( \frac{\mu_\alpha}{r_\alpha} \right)^{n-2}.
\] (4.102)

We conclude that
\[
H(0) = o \left( \frac{\mu_\alpha}{r_\alpha} \right)^{n-2} (1 + O(\delta)), \quad \forall \alpha, \forall \delta > 0,
\] (4.103)
and thus $H(0) = 0$.

### 4.4. Proof of the stability theorem

Consider the sets $S_\alpha$ and let
\[
16\delta_\alpha := \min_{1 \leq i < j \leq N_\alpha} |x_{i,\alpha} - x_{j,\alpha}|.
\] (4.104)

We first prove that $\delta_\alpha \to 0$ as $\alpha \to +\infty$. Assuming that the contrary holds, we can apply the results of lemma 4 with $x_0 = x_{i,\alpha}$ and $\rho_\alpha = \delta$ for some $\delta > 0$ fixed. This contradicts (4.32).

We reorder the elements of the sets $S_\alpha$ in order of distance, so that
\[
16\delta_\alpha = |x_{1,\alpha} - x_{2,\alpha}|.
\] (4.105)

For $R > 1$, let $1 \leq M_{R,\alpha}$ be such that
\[
|x_{1,\alpha} - x_{i_0,\alpha}| \leq R\delta_\alpha \quad \text{for} \quad i_0 \in \{1, \ldots, M_{R,\alpha}\}, \\
|x_{1,\alpha} - x_{i_0,\alpha}| > R\delta_\alpha \quad \text{for} \quad i_0 \in \{M_{R,\alpha} + 1, \ldots, N_\alpha\}.
\] (4.106)

For $x \in B_0(8\delta_\alpha)$, we define the rescaled quantities
\[
\tilde{v}_\alpha(x) := \delta_\alpha^{-q} \phi_\alpha(\delta_\alpha x) u_\alpha \circ \Phi^{-1}_\alpha(\delta_\alpha x)
\] (4.107)
and
\[
\tilde{Z}_\alpha(x) = \delta_\alpha^{n-1} \phi_\alpha^{-q+2}(\delta_\alpha x) (\Phi_\alpha)_\alpha W_\alpha(\delta_\alpha x).
\] (4.108)
In the exponential chart, the elements of $\mathcal{S}_\alpha$ become
\[ \tilde{x}_{i,\alpha} := \delta_{\alpha}^{-1} \exp_{\tilde{x}_{i,\alpha}}^{\mathbb{Z}} (x_{i,\alpha}) , \tag{4.109} \]
where $1 \leq i \leq N_l$. Note that $B_{\alpha_{\lambda}} (8\delta_{\alpha})$ and $B_{\alpha_{\lambda}} (8\delta_{\alpha})$ are disjoint. We define two types of concentration points: the first type of bubbles verify
\[ \sup_{B_{\alpha_{\lambda}} (8)} \left( \tilde{v}_\alpha (x)^q + \frac{\nabla \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + \frac{\nabla^2 \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + |\nabla \tilde{Z}_\alpha (x)| \right) = O(1) \tag{4.110} \]
and the second type of bubbles behave such that
\[ \sup_{B_{\alpha_{\lambda}} (8)} \left( \tilde{v}_\alpha (x)^q + \frac{\nabla \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + \frac{\nabla^2 \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + |\nabla \tilde{Z}_\alpha (x)| \right) \to \infty. \tag{4.111} \]

**1. We consider a cluster with only the first type of points, i.e. where all bubbles are of a comparable size.** Assume $\tilde{x}_{i,\alpha}$ corresponds to the first type. Since for all $j \leq M_{R,\alpha}$,
\[ |\tilde{x}_{i,\alpha} - \tilde{x}_{j,\alpha}| \geq 2, \tag{4.112} \]
then
\[ \tilde{v} (\tilde{x}_{i,\alpha}) \geq 2C(R). \tag{4.113} \]
Since $\tilde{v}_\alpha$ is uniformly bounded in $C^2$, there exists $r_1 > 0$ such that
\[ \inf_{B_{\alpha_{\lambda}} (r_1)} \tilde{v}_\alpha \geq C(R). \tag{4.114} \]
By following the arguments of lemmas 1 and 4, there exists a $C^2(B_0(R))$ limit,
\[ \tilde{v} = \lim_{\alpha \to \infty} \tilde{v}_\alpha \tag{4.115} \]
such that
\[ \Delta \tilde{v} = f (0) \tilde{v}^{q-1}; \tag{4.116} \]
since $\tilde{v}$ has at least two maxima, this leads to a contradiction.

**2. We consider a cluster with both type of points, i.e. where there exists at least one pair of bubbles such that one is much greater than the other.** Around the second type of concentration point, we consider two cases: either
\[ \sup_{B_{\alpha_{\lambda}} (8)} \tilde{v}_\alpha (x) \leq M \quad \text{and} \quad \sup_{B_{\alpha_{\lambda}} (8)} \left( \frac{\nabla \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + \frac{\nabla^2 \tilde{v}_\alpha (x)^k}{\tilde{v}_\alpha (x)} + |\nabla \tilde{Z}_\alpha (x)| \right) \to \infty \tag{4.117} \]
or
\[ \sup_{B_{\alpha_{\lambda}} (8)} \tilde{v}_\alpha (x) \to \infty. \tag{4.118} \]
From lemma 8, we know that
\[ |\tilde{v}_\alpha - \tilde{B}_\alpha| = o(\delta_{\alpha}^{- \frac{1}{2}}). \tag{4.119} \]
where

\[ \tilde{B}_\alpha(x) = \tilde{\mu}_\alpha \left( \frac{\tilde{\mu}_\alpha}{\mu} - \frac{\tilde{f}(\tilde{x}_\alpha)}{n(n - 2)} |x|^2 \right) \]  

(4.120)

with

\[ \tilde{\mu}_\alpha = \frac{\mu_\alpha}{\delta_\alpha} = \bar{u}_\alpha(\tilde{x}_\alpha)^{-q + 2}. \]  

(4.121)

Up to a subsequence,

\[ \bar{u}_\alpha(\tilde{x},j,\alpha) \bar{u}_\alpha(x) \to \frac{\lambda_i}{|x - \tilde{x}|^{n - 2}} + H_j(x) \]  

(4.122)

in \( B_{\tilde{\mu}} (\frac{1}{4} \{ \tilde{x}_i \}) \), with \( \lambda_i > 0 \), where \( H_j \) is superharmonic in \( B_{\tilde{\mu}} (\frac{1}{4} \{ \tilde{x}_i \}) \) with \( H(\tilde{x}_i) = 0 \). This means that \( \tilde{u}_\alpha \to 0 \) in \( C^0 \left( B_{\tilde{\mu}} (\frac{1}{2} \{ B_{h_\alpha} \left( \frac{1}{4} \right) \} ) \right) \) By the Harnack type result, lemma 3, we get a contradiction.

3. We consider a cluster with only the second type of points. Let \( \tilde{G}_\alpha(x, \cdot) \) be the Green function of the operator \( \Delta + \frac{\delta^2}{\delta^2} h_\alpha \) in \( B_\epsilon(3R) \). It converges to the Green function of \( \Delta + h_0 \) is coercive, for any \( y \in B_\mu(2R) \), and since \( Y_\alpha \to 0 \),

\[ \bar{u}_\alpha(x) \geq \int_{B_\mu \left( \frac{1}{2} \right)} \tilde{G}_\alpha(x, y) \tilde{f}_\alpha(y) \tilde{v}^{q - 1} \alpha(y) \, dy + \int_{B_{2\mu \left( 2 \right)}} \tilde{G}_\alpha(x, y) \tilde{f}_\alpha(y) \tilde{v}^{q - 1} \alpha(y) \, dy. \]  

(4.123)

This yields

\[ \bar{u}_\alpha(x) \geq (1 + \alpha(1))(\tilde{B}_1\alpha(x) + \tilde{B}_2\alpha(x)) - \frac{C}{R^{n - 2}} \left( \mu_1\alpha + \mu_2\alpha \right) \]  

(4.124)

For \( |x| \leq \frac{1}{4}, x \neq 0 \), we approximate the RHS with \( \tilde{B}_1\alpha \) to get

\[ \left( \frac{\mu_1\alpha}{\mu_2\alpha} \right)^{\frac{1}{q - 2}} \frac{1}{|x|^{n - 2}} \left( |x - \tilde{x}_2|^{2 - n} - CR^{2 - n} + o(1) \right) \leq o(1) + \frac{C}{R^{n - 2}} |x|^{2 - n}. \]  

(4.125)

We divide the previous equation by \( |x| \) and take \( x \to 0 \) to get, for \( R \) large,

\[ \limsup_{\alpha \to \infty} \left( \frac{\mu_1\alpha}{\mu_2\alpha} \right)^{\frac{1}{q - 2}} \leq C \frac{16^{n - 2}}{R^{n - 2} - C16^{n - 2}}. \]  

(4.126)

By switching the roles of \( \tilde{x}_1\alpha \) and \( \tilde{x}_2\alpha \), we obtain

\[ \limsup_{\alpha \to \infty} \left( \frac{\mu_2\alpha}{\mu_1\alpha} \right)^{\frac{1}{q - 2}} \leq C \frac{16^{n - 2}}{R^{n - 2} - C16^{n - 2}}. \]  

(4.127)

This is a contradiction.  \[ \square \]
5. Discussion: is the drift model a better alternative?

We recall that not much is known about far-from-CMC solutions. The classical conformal method seems to display a number of singularities in the parametrisation, and these singularities are sometimes difficult to find \textit{a priori} without first solving the corresponding conformal system \cite{Max11, Max15}.

As we have discussed in the introduction, an advantage of the drift model is that the singularities identified by Maxwell can be found \textit{a priori} known conformal data sets—i.e. when the volumetric momentum is null.

Apart from being more natural from a physical and geometrical point of view, another feature of Maxwell’s model is that it prescribes more than 10 parameters. At first glance, it ‘over-describes’ the initial data. An important idea underlying the works presented in the sequel is the hope to use these four additional parameters to ‘tilt’ the coordinate system (the other ten parameters) in the neighbourhood of a singularity. Another way to think about this is that the manifold of initial data cannot accurately be covered by only one chart; by changing the additional drift parameters whenever we approach a singularity, we essentially switch to a different chart. In this way, we might be able to prove that the set of solutions to the constraint equations does not possess any real singularity, but only spurious ones due to the choice of coordinates. Naively, one might think of a curve having a vertical tangent which is not well parametrized by its $x$-axis. The price we pay is that the drift system is analytically much more complicated than the classical one.

The goal is to find a viable alternative to the conformal method that gives insight into the structure of the set of solutions of the constraint equations. The drift method proposed by Maxwell provides a promising way forward. The following steps may begin to provide a way forward, in order to achieve this goal:

a. \textit{Existence for small data}. Verify that Maxwell’s system is reasonable, in the sense that it can be solved even in the case of focusing non lineairities. An immediate consequence is that the set of solutions is non-empty. For the non-focusing case, existence is proved in \cite{HMM18}, whereas for the focusing case, we cite \cite{Val19}.

b. \textit{Stability}. Check that, given a perturbation of the coefficients, the set of solutions to the perturbed system is bounded. One might always extract a sequence that converges to a solution of the limiting system. This is the purpose of chapter 3.

c. \textit{The study of bifurcations}. This is where the extra parameters of Maxwell’s method might come into play, by allowing us the freedom to continuously change our mapping as needed.
Indeed, as proved by Premoselli [Pre15], there is no hope that a single choice of $N$ and $V$ lead to a nice smooth parametrisation of the set of solutions. Bifurcations must occur. Even in the defocusing case, such bifurcations can occur, as shown by James Dilts, Michael Holst and David Maxwell [DHKM17]. Thus, tilting the coordinates (the parameters) in a neighbourhood of these bifurcations is a way to understand them and the extra parameters give an opportunity to do so.

We summarize this programme with the help of the figure 1. Point a. allows us to start the process of proving that solutions exist for small parameters. Point b. roughly says that the only problem could come from bifurcations corresponding to folding (at least for the parameters for which stability holds). We rule out vertically asymptotic branches. Part c. consists intuitively in tilting the coordinates with the four added parameters, as shown in the figure. These three steps should permit to obtain a nice smooth description of the set of solutions.

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Appendix A

A.1. Standard elliptic theory for the Lamé operator

If $X$ is a one-form in $M$, the Lamé operator is written in coordinate form as:

$$\nabla^2 g X_i = \nabla^j \nabla^i X_j + \nabla^i \nabla^j X^j - \frac{2}{n} \nabla^i (\text{div}_g X). \tag{6.1}$$

The operator $\nabla^2 g$ is uniformly elliptic on $M$. It satisfies the strong ellipticity condition (also known as the Legendre–Hadamard condition): for any $x \in M$ and any $\eta \in T^*_x M$:

$$\left(\nabla^2 g(x, \xi) \eta, \eta\right) = \left|\xi\right|^2 g_{\eta} + \left(1 - \frac{2}{n}\right) \left|\langle \xi, \eta \rangle\right|^2 g_{\eta} \geq \left|\xi\right|^2 g_{\eta}. \tag{6.2}$$

The Lamé operator is self-adjoint on $H^1(M)$ on any closed manifold $M$, since by integration by parts one gets, for any 1 forms $X$ and $Y$,

$$\int_M \langle \nabla^2 g X, Y \rangle_g \ d v_g = \frac{1}{2} \int_M \langle \mathcal{L}_g X, \mathcal{L}_g Y \rangle_g \ d v_g. \tag{6.3}$$

This implies that for any one-form $X$ on $M$,

$$\nabla^2 g X = 0 \iff \mathcal{L}_g X = 0. \tag{6.4}$$

The standard elliptic theory for (self-adjoint) strongly elliptic operators acting on vector bundles on a compact manifold apply (see theorem 5.20 in Giaquinta–Martinazzi [GM05]):

**Proposition 1.** For any $p \in (1, \infty)$, there exists constants $C_1 = C_1(g, p)$ and $C_2 = C_2(g, p)$ such that for any one-form $X$ in $M$:

$$\|X\|_{W^{2, p}(M)} \leq C_1 \|\nabla^2 g X\|_{L^p(M)} + C_2 \|X\|_{L^1(M)}. \tag{6.5}$$

In addition, if $X$ satisfies
\[
\int_M \langle X, K \rangle \, dv_g = 0
\]  
(6.6)

for all conformal Killing one-forms \( K \), then we can choose \( C_2 = 0 \).

We now turn to the case of \( \mathbb{R}^n \). For any \( 1 \leq i \leq n \), we define the one-form \( \mathbb{R}^n \backslash \{ 0 \} \) by:

\[
\mathcal{G}_i(y) = -\frac{1}{4(n-1)\omega_{n-1}} |y|^{2-n} \left( (3n-2)\delta_{ij} + \frac{y_j y_i}{|y|^2} \right)
\]  
(6.7)

for any \( y \neq 0 \). Note that the matrices \( (\mathcal{G}_i(y))_{ij} \) thus defined are symmetric: for any \( y \neq 0 \),

\[
\mathcal{G}_i(y) = \mathcal{G}_j(y).
\]  
(6.8)

Let \( X \) be a field of one-form in \( \mathbb{R}^n \). For any \( R > 0 \) and for any \( x \in B_0(R) \) there holds:

\[
X_i(x) = \int_{B_0(R)} \mathcal{G}_i(x - y) \Delta \xi X(y) \, dy + \int_{\partial B_0(R)} \mathcal{L}_\xi X(y) \nu(y) \mathcal{G}_i(x - y) \, d\sigma
\]

\[
- \int_{\partial B_0(R)} \mathcal{L}_\xi (\mathcal{G}_i(x - \cdot))(y) \nu(y) X(y) \, d\sigma.
\]  
(6.9)

If \( Y \) is a smooth one-form in \( L^1(\mathbb{R}^n) \), then

\[
W_i(x) = \int_{\mathbb{R}^n} \mathcal{G}_i(x - y) Y(y) \, dy = (\mathcal{G}*Y)_i(x)
\]  
(6.10)

satisfies

\[
\Delta \xi W_i(x) = Y_i(x).
\]  
(6.11)

The system (1.18) is invariant up to adding a conformal Killing one-form in \( M \) to \( W_\alpha \). Let

\[
\mathcal{K}_R = \{ X \in H^1(M) | B_0(R) \} \text{, } \mathcal{L}_\xi X = 0
\]  
(6.12)

is the subspace of one-forms associated to the kernel to the Neumann problem for \( \Delta \xi \) in \( B_0(R) \).

The \( H^1 \) orthogonal space is defined as the space of one-forms \( Y \in H^1(B_0(R)) \) such that for any \( X \in \mathcal{K}_R \):

\[
\int_{B_0(R)} \langle Y, K \rangle \xi \, dx = 0.
\]  
(6.13)

For any one-form \( X \in B_0(B_0(R)) \), we define the orthogonal projection on \( \mathcal{K}_R \) by

\[
\pi_R(X) = \frac{1}{m} \left( \int_{B_0(R)} \langle K_j, X \rangle \, dx \right) K_j.
\]  
(6.14)

The existence of Green one-forms satisfying Neumann boundary conditions is described in the following proposition:

**Proposition 2.** For any \( 1 \leq i \leq n \) and any \( R > 0 \), there exists a unique \( \mathcal{G}_{i,R} \) defined in \( B_0(R) \times B_0(R) \) \( \text{\{x,x\}} \), \( x \in B_0(R) \) so that:



\[ (X - \pi_R(X))(x) = \int_{R_0(R)} G_{i,R}(x,y) \Delta \xi X(y) \, dx \]
\[ + \int_{\partial B_0(R)} L_{i,k} X(y) \nu_k(y) G_{i,R}(x,y) \, d\sigma. \quad (6.15) \]

Moreover, $G_{i,R}$ is analytic in $B_0(R) \times B_0(R) \setminus D$. Furthermore, if $K$ denotes any compact set in $B_0(R)$, there holds for any $x, y \in M$

\[ |x - y||\nabla G_{i,R}(x,y)| + |G_{i,R}(x,y)| \leq C(\delta)|x - y|^{2-n}, \quad (6.16) \]
where \[ \delta = \frac{1}{R} d(K, \partial B_0(R)) > 0. \quad (6.17) \]

**A.2. Limiting equation**

The following lemma has been proved in [Val19].

**Lemma 9.** Let $u \in C^2$ be a bounded subharmonic function defined on $\mathbb{R}^n$. If there exists $0 < \varepsilon \leq u$ which bounds $u$ from below and $\alpha > 0$ such that $u^{-\alpha}$ is a subharmonic function, then $u$ is a constant.

**Proof.** Let us denote

\[ \bar{u}_i(R) := \frac{1}{\omega_{n-1} R^{n-1}} \int_{\partial B_i(R)} u(y) \, dy \]

the average of a smooth function $u$ over the sphere $\partial B_i(R)$. We will sometimes use the simplified notation $\bar{u}(R)$. Recall that, given any subharmonic function $u, x \in \mathbb{R}^n$ and for any two radii $R \leq \bar{R}$, then

\[ \bar{u}_i(R) \leq \bar{u}_i(\bar{R}). \quad (6.18) \]

This follows from

\[ r^{n-1} \bar{u}'(r) = \frac{1}{\omega_{n-1}} \int_{\partial B_1(r)} \partial_x u(y) \, dy = -\frac{1}{\omega_{n-1}} \int_{B_1(r)} \Delta u(y) \, dy \geq 0 \]

where $r > 0$ and $\nu$ is the exterior normal.

Note that $u^{-\alpha} \leq e^{-\alpha}$ implies that the average of $u^{-\alpha}$ on arbitrary subsets is uniformly bounded. Let us fix $x \in \mathbb{R}^n$. Since $u^{-\alpha} \leq \varepsilon^{-\alpha}$ is bounded, there exists a constant $M > 0$ and a sequence of radii $R_i \to \infty$ as $i \to \infty$ such that

\[ M^{-\alpha} := \lim_{i \to \infty} \bar{u}^{-\alpha}_{i}(R_i). \quad (6.19) \]

In fact, because the averages are increasing (6.18) and bounded, any sequence $\bar{R} \to \infty$ around any point in $\mathbb{R}^n$ leads to the same limit $M$, since one may always find a subsequence of $R_i$ such that $B_i(R_i)$ includes the new sequence.

As $u^{-\alpha}$ is subharmonic,

\[ u^{-\alpha}(x) \leq \bar{u}^{-\alpha}_{i}(R) \]

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and therefore $u^{-\alpha}(x) \leq M^{-\alpha}$, or equivalently
\[ M \leq u(x). \] \hspace{1cm} (6.20)

For $z \in \mathbb{R}^n$, let $R := |z - x|$ and $\tilde{R} > R$. By Green’s representation theorem (see for example proposition 1.22 and corollary 1.23 of [HL11]), we get
\[
u(z) \leq \int_{\partial B_x(\tilde{R})} \frac{u(y) - \tilde{R}^2 - R^2}{\omega_{n-1}|z - y|^2} \, dy \]
\[ \leq \frac{(\tilde{R} + R)^{n-2}}{\omega_n} \frac{\tilde{R} - R}{\tilde{R} - R^2} \omega_n - 1 \quad \tilde{R}_n(\tilde{R}). \] \hspace{1cm} (6.21)

For $\delta > 0$, we denote
\[ \Omega_{\delta,R} := \{ z \in \partial B_x(R), u(z) \geq M + \delta \} \]
a subset of $\partial B_x(R)$ and let
\[ \theta_{\delta,R} := \frac{|\Omega_{\delta,R}|}{|\partial B_x(R)|} \in [0,1] \]
be the corresponding relative size of its volume. Note that $\theta_{\delta,R} \to 0$ as $R \to \infty$. Otherwise, if there exists $\varepsilon \in (0,1]$ such that
\[ \limsup_{R \to \infty} \frac{|\{ z \in \partial B_x(R), u(z) \geq M + \delta \}|}{|\partial B_x(R)|} = \varepsilon \]
then
\[ \limsup_{R \to \infty} \sup_{\lambda_{\delta,i}} \frac{u^{-\alpha}(x)}{\lambda_{\delta,i}^{1 - \delta^i}} \leq \varepsilon (M + \delta)^{-\alpha} + (1 - \varepsilon)M^{-\alpha} < M^{-\alpha} \]
which contradicts our definition (6.19) of $M$.

By choosing $R$ large, $\theta_{\delta,R} \leq \delta$. Let
\[ \lambda_{\delta,i} := u_i(2^i R) \]
Note that, by (6.21), $\lambda_{\delta,i} \leq 3 \times 2^{n-2}\lambda_{\delta,i+1}$. Since
\[ u(x) \leq \lambda_{\delta,i} \leq (M + \delta)(1 - \theta_{\delta,2^i R}) + \lambda_{\delta,i+1} \times \theta_{\delta,2^i R} \]
then, by induction,
\[ u(x) \leq (M + \delta) \frac{1 - \delta^i}{1 - \delta} + \lambda_{\delta,i} \]
for all $i \in \mathbb{N}$. As we take $i \to \infty$,
\[ u(x) \leq (M + \delta) \frac{1}{1 - \delta} \]
for any $\delta > 0$, and therefore $u(x) \leq M$. By (6.20), $u(x) \equiv M$.

We may apply the same argument to any other $\tilde{x} \in \mathbb{R}^n$ and obtain the same value $u(\tilde{x}) = M$. Indeed, assuming that
\[ \tilde{M}^{-\alpha} := \lim_{\tilde{R} \to \infty} \sup_{\lambda_{\tilde{\delta},i}} \frac{u^{-\alpha}(\tilde{x})}{\lambda_{\tilde{\delta},i}^{1 - \tilde{\delta}^i}} \]

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so that $\tilde{M}^{-\alpha} \geq M^{-\alpha}$, then for $\tilde{R}$ large, $u^{-\alpha}(\tilde{R}) \geq M^{-\alpha}$. But, at the same time, given any fixed $\tilde{R}$, then for $R$ sufficiently large, by (6.21), $u^{-\alpha}(\tilde{R}) \leq u^{-\alpha}(R)$. Thus we obtain that $u \equiv M$ in $\mathbb{R}^n$. □

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