LIMIT THEOREMS FOR SKEW PRODUCTS WITH MIXING BASE MAPS AND EXPANDING ON THE AVERAGE FIBERS WITHOUT FIBERWISE CENTERING

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Abstract. In this paper we show how to apply classical probabilistic tools for globally centered partial sums \( \sum_{j=0}^{n-1} \varphi \circ \tau^j \) generated by skew product \( \tau \), built over a sufficiently well mixing base map and a random expanding dynamical system. Under certain regularity assumptions on the observable \( \varphi \), we obtain a central limit theorem (CLT) with rates, a functional CLT, an almost sure invariance principle (ASIP), a moderate deviations principle, several exponential concentration inequalities and Rosenthal type moment estimates for skew products with \( \alpha, \phi \) or \( \psi \) mixing base maps and expanding on the average random fiber maps. All of the results are new even in the uniformly expanding case. The main novelty here is that the random maps are not independent (contrary to [3]) and that the underlying observable is not fiberwise centered (contrary to [29]).

1. Introduction

1.1. Quenched limit theorems for random random dynamical systems. Let \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) be an invertible ergodic probability preserving system. Let \(T_\omega: X \to X, \omega \in \Omega\) be a family of maps so that the corresponding skew product \(\tau\) given by \(\tau(\omega, x) = (\sigma \omega, T_\omega x)\) is measurable. A random dynamical system is formed by the sequence of compositions

\[ T^n_\omega x, n \geq 0 \quad \text{where} \quad T^n_\omega = T^{n-1}_\omega \circ \cdots \circ T_\omega \circ T_\omega \]

taken along the orbit of a “random” point \(\omega\). Random dynamical systems are key tools to model many natural phenomena, such as the transport in complex environments such as in the ocean or the atmosphere [6]. The system \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) is often referred to as the driving system, and the map \(\sigma\) is often referred to as the base map.

Let \(\varphi: \Omega \times X \to \mathbb{R}\) be a measurable function ("an observable") and let \(\mu\) be a \(\tau\)-invariant probability measure on \(\Omega \times X\). Then \(\mu\) can be decomposed as \(\mu = \int \mu_\omega d\mathbb{P}(\omega)\), where \(\mu_\omega\) is a family of probability measures on \(X\) so that \((T_\omega)_* \mu_\omega = \mu_{\sigma \omega}\) for \(\mathbb{P}\)-a.e. \(\omega\). Set \(S_n \varphi = \sum_{j=0}^{n-1} \varphi \circ \tau^j\). Then

\[ S_n \varphi(\omega, x) := S_n^\omega \varphi(x) = \sum_{j=0}^{n-1} \varphi_{\sigma^j \omega} \circ T^j_\omega, \]

where \(\varphi_{\omega}(\cdot) = \varphi(\omega, \cdot)\). For \(\mathbb{P}\) almost every \(\omega\) we can consider the sequence of functions \(S_n^\omega \varphi(\cdot)\) on the probability space \((X, \mathcal{B}, \mu_\omega)\) as random variables. Limit theorems for such sequences are called quenched limit theorems. Probably the first papers dealing with quenched limit theorems for random dynamical systems are [36, 37], and since then limit theorems for several classes of random dynamical systems were vastly studied. We refer to [8, 10, 15, 16, 17, 21, 22, 24, 18, 28, 30, 31, 51, 52].
for a partial list of relatively recent results of this kind. We note that in many of the examples these results are obtained for the unique measure \( \mu \) such that \( \mu_\omega \) is absolutely continuous with respect to \( m \). However, some results hold true even for maps \( T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\tau \omega} \subset X \) which are defined on random subsets of \( X \) (see [38]), where in this case the most notable choice of \( \mu_\omega \) is the, so called, random Gibbs measure (see [28, 43]).

1.1.1. Random vs deterministic centering. In general, quenched limit theorems hold under the fiberwise centering condition \( \mu_\omega(\varphi_\omega) = 0 \), namely for \( S_n^\omega \varphi = \mu_\omega(S_n^\omega \varphi) \). We refer to [2], in which the question of fiberwise limit theorems with deterministic centering is studied (see also [31, 45]).

It is shown that when \( \varphi(\omega, x) = \varphi(x) \) does not depend on \( \omega \), then a central limit theorem with deterministic centering can only hold in general when \( \mu_\omega \) does not depend on \( \omega \) (as in [31, 45, 3]).

When \( \varphi \) depends also on \( \omega \) it is less likely that such a result will hold in general, even when \( \mu_\omega = \nu \) does not depend on \( \omega \), since then \( S_n^\omega \varphi \) can be decomposed as \( S_n^\omega \varphi = S_n^\omega \bar{\varphi} + \sum_{j=0}^{n-1} \nu(\varphi_{\tau^j \omega}) \), where \( \bar{\varphi}(\omega, x) = \varphi_\omega - \nu(\varphi_\omega) \). Under appropriate mixing conditions the sum on the right hand also satisfies several limit theorems, but, in general, the terms on the right hand seem to be correlated. The relevance of this discussion to the present paper will be clear after reading the beginning of the next section.

1.2. Annealed limit theorems. Limit theorems for the sums \( S_n \varphi = \sum_{j=0}^{n-1} \varphi \circ \tau^j \), considered as random variables on the probability space \( (\Omega \times X, \mathcal{F} \times \mathcal{B}, \mu) \), are called annealed limit theorems. We would like to stress that, in contrast to the quenched case, the summands \( \varphi \circ \tau^j \) form a stationary sequence, however, without some kind of mixing assumptions on the base map \( \sigma \) we do not expect to get any limit theorem in a general framework. We would like to emphasize that when quenched distributional limit theorems hold with a deterministic centering then the annealed counterparts follow. However, as discussed in Section 1.1.1 this could happen only in special circumstances. Moreover, stronger results like annealed almost sure approximation by sums of Gaussian random variables (i.e. almost sure invariance principle) cannot be derived from the corresponding quenched ones, even when \( \mu_\omega \) and \( \varphi_\omega \) do not depend on \( \omega \), since the approximating Gaussians depend also on \( \omega \). Therefore, in general, annealed limit theorems differ from the quenched ones.

For deterministic dynamical systems, i.e. when \( T_\omega = T \) and \( \varphi(\omega, x) = \varphi(x) \) do not depend on \( \omega \), many of the limit theorems follow from spectral properties of the transfer operator \( L_T \) corresponding to \( T \) (namely, the dual of the Koopman operator \( g \to g \circ T \) with respect to the underlying invariant measure \( \mu \), or from a sufficiently fast convergence of \( L_T^n \) towards a one dimensional projection. While several quenched limit theorems are based on an appropriate random counterpart of such spectral properties (see [15, 16, 17, 21, 22, 20, 18, 28, 30, 14]), such annealed “spectral” techniques are not fully developed as the quenched ones. A very notable exception is the case of iid maps, discussed in the next section.

1.2.1. iid maps. Suppose that \( \Omega = \mathbb{Y}^\mathbb{Z} \) is a product space and the coordinates \( \omega_j \) of \( \omega = (\omega_j) \) are independents (with \( \sigma \) being the left shift) and that \( T_\omega = T_{\omega_0} \) depends only on the 0-th coordinate. Then the statistical behavior of the skew product \( \tau \) can be investigated using the, so called, averaged transfer operator, given by (see [3, 7, 34]),

\[
\mathcal{A} g(x) = \int L_{\omega} g(x) d\mathbb{P}(\omega)
\]

where \( L_{\omega} \) is the transfer operator corresponding to \( T_\omega \) and the underlying reference measure \( m \). In [3] the authors showed that for several classes of random expanding maps, the operator \( \mathcal{A} \) is quasi compact. Using that, they obtained a variety of limit theorems for random variables of the form \( \sum_{j=0}^{n-1} \varphi(T_{\omega_{j-1}} \circ \cdots \circ T_{\omega_0} x) \) where \( (\omega, x) \) are distributed according to a \( \tau \)-invariant measure \( \mu \) of the form \( \mathbb{P} \times (h dm) \) for some continuous function \( h \), which satisfies \( \mathcal{A} h = h \). The point is that once quasi compactness is achieved the classical Negaev-Guivarch method method [30, 37, 49, 24, 52] can
be applied. Independence here is crucial, since it yields that the iterates on the averaged operator can be written as

\[ A^n g = \int L_{\omega}^n g \, d\mathbb{P}(\omega), \]

where \( L_{\omega}^n = L_{\sigma^{n-1} \omega} \circ \cdots \circ L_{\sigma \omega} \circ L_{\omega} \), which is the transfer operator of \( T_{\omega}^n \). Hence, the statistical behavior of the iterates \( \tau^n \) of the skew product can be described by the iterates of \( A \). We would like to stress that even in the iid case this approach works only when \( \varphi(\omega, x) = \varphi(x) \) does not depend on \( \omega \) since it requires substituting \( \varphi \) (and appropriate functions of \( \varphi \)) into the averaged operator.

1.2.2. The motivation behind the present paper: non iid maps, random functions and deterministic centering. The starting point of this paper is the observation that when the coordinates \( (\omega_j) \) are not independent there is no apparent relation between the dynamics of iterates of \( \tau \) and the iterates of the averaged operator \( A \) defined above. Thus, a very natural question arising from \[3\] is whether limit theorems hold for mixing base maps with non-independent coordinates, and functions \( \varphi \) which depend on \( \omega \). We believe that this question is especially important since “asymptotic” independence is a more realistic assumption than independence.

This was also one of the main motivations in \[29\], where an annealed local central limit theorem and an annealed renewal theorem were obtained for several classes of mixing base maps such as Markov shifts and non-uniform Young towers, together with uniformly expanding random maps. These results were (also) obtained by a certain type of integration argument, however the method of \[29\] does not involve the iterates of the averaged operator, and instead we studied directly integrals of the form \( \int L_{\sigma}^n g_{\omega} \, d\mathbb{P}(\omega) \), and their complex perturbations (relying on the fiberwise “spectral” properties and the periodic point approach developed in \[28\]). While \[29\] was the first paper to discuss annealed limit theorem for non independent maps and random potentials, all the results there were obtained for fiberwise centered observables \( \varphi \) (so that \( \mu_{\omega}(\varphi_{\omega}) = 0 \)). Moreover, the maps \( T_{\omega} \) in \[29\] were uniformly expanding, the base map had a periodic point and the random transfer operators satisfied certain regularity assumptions as functions of \( \omega \) on the finite periodic orbit. Furthermore, we did not consider in \[29\] any of the limit theorems obtained in the present paper, which will be discussed in the following sections.

1.3. Our new results and the method of the proofs. The goal in this paper is to obtain limit theorems with deterministic centering conditions for skew products \( \tau \) built over mixing base maps and non-uniformly expanding maps \( T_{\omega} \). More precisely, we still consider a product space \( \Omega = Y^\mathbb{Z} \), but with “weakly-dependent” coordinates \( \omega_j \) instead of independent ones. We consider a family on non-uniformly expanding map \( T_{\omega} = T_{\omega_0} \) and observables of the form \( \varphi(\omega, x) = \varphi_{\omega_0}(x) \) and prove limit theorems for sequences of the form \( Z_n = S_n \varphi - n \int \varphi \, d\mu \), where

\[ S_n \varphi(\omega, x) = \sum_{j=0}^{n-1} \varphi_{\omega_j}(T_{\omega_{j-1}} \circ \cdots \circ T_{\omega_0} x) = \sum_{j=0}^{n-1} \varphi_{\sigma^j \omega}(T_{\omega}^j(x)) \]

considered as a random variables on the probability space \( (\Omega \times X, \mathcal{F} \times \mathcal{B}, \mu) \), where \( \mu \) is the unique \( \tau \)-invariant measure with \( \mu_{\omega} \) being absolutely continuous with respect to \( \mu \) (or a random Gibbs measure). These results are obtained for a certain type of observables \( \varphi \) so that \( \varphi_{\omega}(\cdot) \) have bounded variation, uniformly in \( \omega \). When the maps \( T_{\omega} \) are only expanding on the average we will also have a certain scaling assumption\[4\], which was shown in \[19\] to be necessary for quenched limit theorems, and which is similarly necessary for annealed ones. In what follows we will always assume that \( \int \varphi \, d\mu = 0 \), which is not really a restriction since we can always replace \( \varphi \) with \( \varphi - \int \varphi \, d\mu \).

We obtain our results using two different methods, as described below.\[3\]

\[1\] That is \( \text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi_{\omega}\|_{BV}) < \infty \) for some tempered random variable \( K \).

\[2\] Section 1.3.1 appears before Section 1.3.2 only because \( \alpha \) mixing is a weaker notion than \( \phi \) or \( \psi \) mixing.
1.3.1. Limit theorems via the method of cumulants for \(\alpha\)-mixing driving systems via the method of cumulants.

We assume here that the coordinates \(\omega_n\) are \(\alpha\)-mixing, with the \(n\)-th \(\alpha\) mixing coefficient \(\alpha_n\) (defined in [2,3]) satisfy \(\alpha_n = O(e^{-cn^\eta})\) for some \(c, \eta > 0\) (i.e. it is stretched exponential). The first step towards limit theorems is standard for stationary processes: we show that under the weaker condition \(\sum_n n\alpha_n < \infty\), the limit
\[
 s^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var}_\mu(S_n), \ S_n = S_n \varphi
\]
eexists and that it vanishes if and only if \(\varphi\) admits a certain co-boundary representation. When \(s^2 > 0\) we show that \(n^{-1/2}S_n\) converges in distribution towards a centered normal random variable with variance \(s^2\). More precisely, we have the convergence rate
\[
 \sup_{t \in \mathbb{R}} |\mu(S_n \leq ts\sqrt{n}) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2} dx| \leq Cn^{-\frac{\eta}{4}}, \gamma = 1/\eta.
\]

We also obtain certain type of large deviations results, which is often referred to as a moderate deviations principle (see [13]). These results yield, for instance, that for every closed interval \([a,b]\) we have
\[
 \lim_{n \to \infty} \frac{1}{a_n^2} \ln \mu \left\{ (\omega, x) : \frac{S_n(\omega, x)}{a_n}\sqrt{n}/2 \in [a, b] \right\} = -\frac{1}{2} \inf_{x \in [a,b]} x^2
\]
where \(a_n\) is a sequence so that \(a_n \to \infty\) and \(a_n = o(n^{1/4})\). We also obtain several types of “stretched” exponential concentration inequalities \((2.7), (2.8)\) and Gaussian moment estimates of Rosenthal type \((2.9)\). These results are obtained using the method of cumulants. More precisely, we first obtain a certain type of multiple correlation estimates in Proposition 3.3, which using a general result yields that the \(k\)-th cumulant of the sum \(S_n\) is at most of order \(n(k!)^{1+\gamma}(c_0)^{k-2}\) for \(k \geq 3\), where \(c_0\) is some constant (see Theorem 3.1). Then we can apply the method of cumulants \([33, 14]\).

The above multiple correlation estimates together with the method of cumulants and the Rosenthal type inequalities also yield a functional CLT. Let us consider the random function \(S_n(t) = n^{-1/2}S_{[nt]}\) on \([0, 1]\). Then we show that it converges in distribution in the Skorokhod space \(D[0, 1]\) to \(sW\), where \(W\) is a standard Brownian motion and \(s^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var}_\mu(S_n)\).

1.3.2. Limit theorems via a martingale coboundary representation for \(\phi\) or \(\psi\) mixing driving systems via martingale methods.

For several classes of expanding or hyperbolic deterministic dynamical systems \((X, B, T)\) and a centered observable \(\varphi\), so that \(\chi = \sum_{n \geq 1} L^n \varphi\) converges (say in \(L^\infty\)), where \(L\) is the transfer operator with respect to the underlying invariant measure, we can set \(u = \varphi + \chi \circ T - \chi\). Then (see [32]), \((u \circ T^n)\) is a reverse martingale difference with respect to the reverse filtration \(\{T^{-n}B\}\). For several classes of maps \(T\), this, together with additional rates of convergence for the iterates of the transfer operator, yields (see [12]) an almost sure invariance principle for the sums \(W_n = \sum_{j=0}^{n-1} \varphi \circ T^j\), which means that there is a coupling with a sequence of iid centered normal random variables \(Z_j\) with variance \(s^2 = \lim_{n \to \infty} \frac{1}{n} \text{Var}(W_n)\) so that
\[
 \sup_{1 \leq k \leq n} \left| W_k - \sum_{j=1}^{k} Z_j \right| = O(n^{1/4} \log n)^{1/2} (\log \log n)^{1/4}, \text{ almost surely.}
\]

While the results described above were essentially generalized to the quenched case \([19]\), in the annealed case the situation is more complicated since the global transfer operator has the form \(L_\varphi(\omega, x) = (L_{\sigma^{-1} \varphi} g(\sigma^{-1} \omega, \cdot)) (x)\). Since the iterates of the random transfer operator can only be approximate\(^3\) by one dimensional operators depending on \(\omega\) (see \([11, 39, 43, 28]\)), the iterates of the global transfer operator cannot converge in general towards a one dimensional operator, and instead they can only be approximated by operators depending on \(\omega\). We note that in \([3]\) the authors presented an appropriate type of martingale approximation using the averaged transfer operator,

\(^3\)Namely, a random Ruelle-Perron-Frobenius theorem holds true.
which, as discussed before, does not seem to have ramifications beyond the case of iid maps since it takes advantage of the quasi compactness of the averaged transfer operator, which, as discussed in the previous sections, only describes the asymptotic behavior in the iid case (moreover, we consider functions \( \varphi \) which depend on \( \omega \)).

1.3.3. Martingale-coboundary decomposition for skew products with mixing base maps and deterministic centering. To address the above issues, and to obtain the appropriate martingale-coboundary decomposition in the context of annealed dynamics with mixing base maps, we consider the case when the coordinates \( \omega_j \) are either \( \phi \) or \( \psi \) mixing (see (2.10) and (2.11) for the relevant definitions).

Consider the sub-\( \sigma \)-algebra \( F_0 \) of \( \Omega \times X \) generated by the projection \( \pi_0(\omega, x) = ((\omega_j)_{j \geq 0}, x) \), where \( \omega = (\omega_j)_{j \in \mathbb{Z}} \). Then \( \tau \) preserves \( F_0 \) since \( T_{\omega} = T_{\omega_0} \) depends only on \( \omega_0 \), and \( F_0 \) can be viewed as a sub-system (or a factor) given by \( (\Omega \times X, F_0, \mu, \tau) \). Our main argument is that, under quite mild \( \phi \) or \( \psi \) mixing rates of the coordinates \( \omega_j \), the iterates \( K^n \varphi \) of the transfer operator \( K \) corresponding to this system converge fast enough in \( L^\infty(\mu) \) towards \( \mu(\varphi) \mathbf{1} \), where \( \mathbf{1} \) is the function taking the constant value 1, and \( \varphi \) is our given observable. This convergence can be established for every “scaled” observable, or for any observable with \( \operatorname{esssup}_{\omega \in \Omega} \| \varphi(\omega, \cdot) \|_{BV} < \infty \) when the maps \( T_\omega \) are uniformly expanding and not only expanding on the average. We stress that in any case this is not a spectral result (even under exponential mixing), since the regularity required from the observable \( \varphi \) (i.e. that it has a bounded variation) is much stronger than just \( L^\infty \).

Once an \( L^\infty \) martingale-coboundary decomposition is achieved, as usual, we can apply the Azuma-Hoeffding inequality together with Chernoff’s bounding method and obtain exponential concentration inequalities of the form

\[
P(|S_n| \geq tn + c_1) \leq c_2 e^{-c_3 nt^2}, \quad t > 0
\]

where \( c_1, c_2, c_3 \) are positive constants. These concentration inequities are better than the ones we obtain using the method of cumulants, although they involve the stronger notions of \( \phi \) or \( \psi \) mixing instead of \( \alpha \)-mixing\(^4\). Another immediate consequence is moment estimates of the form \( \| S_n \varphi \|_{L^p(\mu)} = O(n^{1/2}) \) which hold for every \( p \geq 1 \).

Using the above annealed martingale-coboundary decomposition we also obtain an almost sure invariance principle with the same \( O(n^{1/4}(\log n)^{1/2}(\log \log n)^{1/4}) \) rates. In addition to the estimates on \( \| K^n \varphi - \mu(\varphi) \|_{L^\infty} \), this requires estimates on expressions of the form \( \| K^i(\varphi K^j \varphi) - \mu(K^i(\varphi K^j \varphi)) \|_{L^\infty} \).

These estimates, as well as the estimates on \( \| K^n(\varphi) - \mu(\varphi) \|_{L^\infty(\mu)} \), rely on the identification of \( L^\infty \) as the dual of \( L^1 \), and their proof is sketched in Section 2.2.3.

Finally, we also prove an annealed vector-valued almost sure invariance principle for uniformly expanding random maps via the method of Gouëzel [23]. We note that this method was applied for non iid maps in [4], however, as the author of [4] mentions his conditions are quite strong and should be verified only in special cases, such as Gordin-Denker systems considered in the last section of [4]. In a final section we also discuss a few extensions such as more general mixing base maps such as Young towers or Gibbs-Markov maps, application of the method of cumulants for nonconventional sums of the form \( S_n = \sum_{m=1}^n \prod_{j=1}^n \varphi_j \circ \tau^{q_j(m)} \), for polynomial \( q_j(m) \), as well as extension of the results for Gibbs states. We also believe that our martingale constructions can lead to other results such as smooth approximation of stochastic differential equations in the sense of [35], but in order not to overload the paper such results will be considered in a different manuscript.

2. Preliminaries main results

2.1. Skew products with expanding on the average base maps: basic assumptions. Let \( (\xi_n) \) be a two sided stationary sequence taking values on some measurable space \( Y \). Let \( (\Omega, F, \mathbb{P}, \sigma) \)

\(^4\)However, they only require summable \( \phi \) or \( \psi \) mixing coefficients and not stretched exponential ones.
be the corresponding shift system, where $\Omega = \mathcal{Y}^\mathbb{Z}$. Namely, if $\pi_0 : \Omega \to \mathcal{Y}$ denotes the 0-th coordinate projection, then $(\xi_n)$ has the same distribution as $(\pi_0 \circ \sigma^n)$. Next, let $(X, \mathcal{B}, m)$ be a probability space. Let $v(\cdot)$ be a notion of variation of functions on $X$ so that $v(cf) = |c|v(f)$ for all $c \in \mathbb{R}$ and $f : X \to \mathbb{C}$ and $v(fg) \leq \sup |f|v(g) + \sup |g|v(f)$ for every $g, f : X \to \mathbb{C}$. We also assume here that $sup \{f \leq C_0(v(f) + \|f\|_{L^1(m)})\}$ for some constant $C_0$ and that constant functions have zero variation.

In applications there will be additional requirements, but these will be absorbed in our next set of assumptions, described below. Let us define a norm on functions $f : X \to \mathbb{C}$ by setting

$$\|f\|_{BV} = \|f\|_{L^1(m)} + v(f).$$

Let $T_g : X \to X, g \in \mathcal{Y}$ be a family of measurable maps. Let us abuse the notation and write $T_\omega = T_{\omega_0}$ for $\omega = (\omega_k) \in \Omega$. Consider the random iterates given by

$$T^n_\omega = T_{\sigma^n-1\omega} \circ \cdots \circ T_{\sigma\omega} \circ T_\omega = T_{\omega_n-1} \circ \cdots \circ T_1 \circ T_0.$$

Let us define the skew product $\tau : \Omega \times X \to \Omega \times X$ by

$$\tau(\omega, x) = (\sigma\omega, T_\omega x).$$

We will always assume that $\tau$ is measurable. The system $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is often referred to as the driving system, and for future reference we will call $(\xi_n)$ the driving process (since it generates the driving system in our case).

Let $\mathcal{L}_\omega$ be the transfer operator which maps a function on $X$ to another function on $X$ and is defined by the following duality relation

$$\int g \cdot (f \circ T_\omega)dm = \int (\mathcal{L}_\omega g) \cdot f dm, \quad g \in L^1(m), \quad f \in L^\infty(m).$$

Namely, $\mathcal{L}_\omega$ is the dual of the Koopman operator $g \to g \circ T_\omega$ with respect to the measure $m$. Then $\mathcal{L}_\omega$ depends only on the 0-th coordinate $\omega_0$, which for future reference will be written as $\mathcal{L}_\omega = \mathcal{L}_{\omega_0}$ (again, abusing the notations). Let use also define

$$\mathcal{L}^n_\omega = \mathcal{L}_{\sigma^n-1\omega} \circ \cdots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega = \mathcal{L}_{\omega_n-1} \circ \cdots \circ \mathcal{L}_1 \circ \mathcal{L}_0.$$

We recall next that a random variable $K(\omega)$ is called tempered if

$$\lim_{n \to \infty} \frac{1}{n} \ln K(\sigma^n\omega) = 0.$$

Recall next the following result.

2.1. **Proposition.** [3 Proposition 4.3.3.] Let $K : \Omega \to (0, +\infty)$ be a tempered random variable. For each $\varepsilon > 0$, there exists a tempered random variable $K_\varepsilon : \Omega \to (1, +\infty)$ such that

$$\frac{1}{K_\varepsilon(\omega)} \leq K(\omega) \leq K_\varepsilon(\omega) \quad \text{and} \quad K_\varepsilon(\omega)e^{-\varepsilon |n|} \leq K_\varepsilon(\sigma^n\omega) \leq K_\varepsilon(\omega)e^{\varepsilon |n|},$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$.

Our abstract requirements from the random operators are summarized in the following.

2.2. **Assumption.** (1) There is a nonnegative measurable function $h(\omega, x)$ on $\Omega \times X$ so that the measure $\mu$ on $(\Omega \times X, \mathcal{F} \times \mathcal{B})$ given by

$$\mu(\Gamma) = \int_{\Gamma} h(\omega, x)dm(x)d\mathbb{P}(\omega)$$

is a $\tau$ invariant probability measure. Moreover, if we set $h_\omega = h(\omega, \cdot)$ then $\|h_\omega\|_{BV} < \infty$, $\int h_\omega dm = 1$ and $h_\omega > 0$ for $\mathbb{P}$-a.e. $\omega$. Moreover, $\mathcal{L}_\omega h_\omega = h_{\sigma_\omega}$.

(2) There is a tempered random variable $K(\omega) \geq 1$ and a constant $\lambda > 0$ so that

$$\|\mathcal{L}^n_\omega 1 - h_{\sigma^n\omega}\|_{BV} \leq K(\omega)e^{-\lambda n}.$$

\[\text{If follows that the measure } \mu_\omega = h_\omega dm \text{ is equivariant, namely } \mu_\omega \circ T_\omega = \mu_{\sigma_\omega} \text{ for } \mathbb{P}$-a.e. $\omega.\]
where \( 1 \) is the function taking the constant value 1. Moreover, we have
\[
(2.2) \quad v(g \circ T_\omega) \leq v(g) K(\omega)
\]
for every function \( g : X \to \mathbb{C} \). Furthermore, if we set \( L_\omega(g) = L_\omega(gh_\omega)/h_\sigma_\omega \) then, with
\[
L_\omega^n = L_{\sigma^{-n}} \circ \cdots \circ L_{\sigma} \circ L_\omega,
\]
we have
\[
(2.3) \quad \|L_\omega^n g - \mu_\omega(g)1\|_{BV} \leq K(\omega)\|g\|_{BV} e^{-\lambda n}.
\]

For our martingale-coboundary decomposition we also need to assume that
\[
(2.4) \quad \|1/h_\omega\|_{BV} \leq K(\omega).
\]
Note that by possibly increasing \( K \) we can always assume that it satisfies the second property described in Proposition 2.1, namely that
\[
K(\omega) e^{-|n|} \leq K(\sigma^n, \omega) \leq K(\omega) e^{|n|}.
\]
In this paper it will be enough to take \( \varepsilon < \frac{1}{3} \lambda \). All the above conditions hold true in the setup described\(^6\) in [19], and we would like to refer the readers to the specific examples [19] Examples 5 and 6], but in order not to overload the presentation by repeating the corresponding section in [19] we will obtain our results under the above set of abstract assumptions. For future reference, we will call the case “uniformly random” or “uniformly expanding” if the above conditions hold true with a constant random variable \( K \).

2.2. Limit theorems for mixing base maps. Let us take a measurable \( \varphi : \Omega \times X \to \mathbb{R} \) so that \( \int \varphi d\mu = 0 \). The main goal in this section is to obtain limit theorems for the sequence of functions
\[
S_n = S_n \varphi = \sum_{j=0}^{n-1} \varphi \circ \tau^j
\]
under certain mixing assumptions on the driving sequence \( (\xi_n) \) and the observable \( \varphi \).

2.2.1. Limit theorems for stretched exponentially fast \( \alpha \)-mixing driving processes. Let \((\Omega_0, \mathcal{F}, \mathbf{P})\) be the probability space on which \((\xi_n)\) is defined. We recall that the \( \alpha \)-mixing (dependence) coefficient between two sub-\(\sigma\)-algebras \( \mathcal{G}, \mathcal{H} \) of \( \mathcal{F} \) is given by
\[
\alpha(\mathcal{G}, \mathcal{H}) = \sup\{ |\mathbf{P}(A \cap B) - \mathbf{P}(A)\mathbf{P}(B)| : A \in \mathcal{G}, B \in \mathcal{H} \}.
\]
The \( \alpha \)-dependence coefficients of \((\xi_n)\) are defined by
\[
(2.5) \quad \alpha_n = \sup_k \alpha(\mathcal{F}_{-\infty, k}, \mathcal{F}_{k+n, \infty}) = \alpha(\mathcal{F}_{-\infty, 0}, \mathcal{F}_{n, \infty})
\]
where \( \mathcal{F}_{-\infty, k} \) is the \( \sigma \)-algebra generated by \( \xi_j, j \leq k \) and \( \mathcal{F}_{k+n, \infty} \) is generated by \( \xi_j, j \geq k + n \). The last equality holds true due to stationarity. In what follows we will consider an observable \( \varphi \) satisfying the scaling condition
\[
\text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi_\omega\|_{BV}) < \infty
\]
which was first introduced in [18]. In the uniformly random case \( K(\omega) \) is a constant, and so the scaling condition reads
\[
\text{esssup}_{\omega \in \Omega}\|\varphi_\omega\|_{BV} < \infty.
\]
However, for expanding on the average maps the scaling condition is necessary for limit theorems, see [19] Appendix\(^\ast\). In any case, our results are also new in the uniformly random case, and the readers who would prefer can just consider this case together with the assumption \( \text{esssup}_{\omega \in \Omega}\|\varphi_\omega\|_{BV} < \infty \).

Next, let us consider the following class of mixing assumptions on the base map:
\footnote{\textsuperscript{6}The setup in [19] essentially coincides with the setup of [17], apart from the additional log-integrability assumption \( \log(\text{essinf}_{x \in X} L_\omega 1(x)) \in L^1(\Omega, \mathcal{F}) \).}
2.3. **Assumption** (Stretched exponential $\alpha$ mixing rates). There exist positive constants $c_1, c_2$ and $\eta$ so that $\alpha_n \leq c_1 e^{-c_2 n^\alpha}$ for every $n$.

Our first result concerns the variance of $S_n$ and the central limit theorem (with rates).

2.4. **Theorem.** Under Assumption 2.2 we have the following. Let $\varphi$ be an observable so that $\|\varphi\|_K := \esssup_{\omega \in \Omega} (K(\omega)\|\varphi_\omega\|_{BV}) < \infty$, where $\varphi_\omega = \varphi(\omega, \cdot)$. Suppose that $\sum_n n \alpha_n < \infty$. Then the limit

$$s = \lim_{n \to \infty} n^{-1/2} \|S_n\|_{L^2(\mu)}$$

exists and it vanishes if and only if $\phi = r \circ \tau - r$ for some $r \in L^2(\mu)$. If in addition Assumption 2.3 is satisfied then $n^{-1/2} S_n$ converges in distribution to $sZ$, where $Z$ is a standard normal random variable. Moreover, there is a constant $C > 0$ so that for all $n \in \mathbb{N},$

$$\sup_{t \in \mathbb{R}} \left| \mu(S_n \leq ts \sqrt{n}) - \Phi(t) \right| \leq C n^{-\frac{1}{2+\eta}}$$

where $\gamma = 1/\eta$ and $\Phi$ is the standard normal distribution function. The constant $C$ depends only on $c_1, c_2, \eta$, $\|\varphi\|_K$ and the constant $C_0$ specified earlier, and an explicit formula for $C$ can be recovered from the proof.

Next, let us discuss our results concerning moderate deviations and exponential concentration inequalities.

2.5. **Theorem.** Let Assumption 2.2 hold true, and let $\varphi$ be an observable so that $\|\varphi\|_K = \esssup_{\omega \in \Omega} (K(\omega)\|\varphi_\omega\|_{BV}) < \infty$. Let Assumption 2.3 hold and set $\gamma = \frac{1}{\eta}$. Then there exist constants $a_1, a_2 > 0$ so that for every $x > 0$ and $n \in \mathbb{N},$

$$P(S_n \geq x) \leq \exp \left( - \frac{x^2}{2(a_1 + a_2 xn^{-\frac{1}{2+\eta}})^{1+\eta}} \right).$$

All the constants depend only on $c_1, c_2, \eta$, $\|\varphi\|_K$ and $C_0$ specified earlier, and an explicit formula for them can be recovered from the proof.

We will also prove the following

2.6. **Theorem.** Suppose that Assumption 2.2 holds, and let $\varphi$ be an observable so that $\|\varphi\|_K = \esssup_{\omega \in \Omega} (K(\omega)\|\varphi_\omega\|_{BV}) < \infty$. Let Assumption 2.3 hold and set $\gamma = \frac{1}{\eta}$. Set $v_n = \sqrt{\text{Var}(S_n)}$ and when $v_n > 0$ also set $Z_n = \frac{S_n}{v_n}$. Let $\Phi$ be the standard normal distribution function. Let us also assume that the asymptotic variance is positive. Then there exist constants $s_3, s_4, s_5 > 0$ so that for every $n \geq a_3$ we have $v_n > 0$ and for every $0 \leq x < a_4 n^{-\frac{1}{2+\eta}},$

$$\left| \ln \frac{P(Z_n \geq x)}{1 - \Phi(x)} \right| \leq s_5 (1 + x^3) n^{-\frac{1}{2+\eta}} \text{ and }$$

$$\left| \ln \frac{P(Z_n \leq -x)}{\Phi(-x)} \right| \leq s_5 (1 + x^3) n^{-\frac{1}{2+\eta}}.$$

The constants $a_4, a_5$ depend only on $c_1, c_2, \eta$, $\|\varphi\|_K$ and $C_0$, and an explicit formula for them can be recovered from the proof.

Moreover, let $a_n, n \geq 1$ be a sequence of real numbers so that

$$\lim_{n \to \infty} a_n = \infty \text{ and } \lim_{n \to \infty} a_n n^{-\frac{1}{2+\eta}} = 0.$$\n
Then the sequence $W_n = (sn^2 a_n)^{-1} S_n$, $n \geq 1$ satisfies the moderate deviations principle with speed $s_n = a_n^2$ and the rate function $I(x) = \frac{x^2}{2}.$ Namely, for every Borel measurable set $\Gamma \subset \mathbb{R},$

$$- \inf_{x \in \Gamma^0} I(x) \leq \liminf_{n \to \infty} \frac{1}{a_n^2} \ln \mu(W_n \in \Gamma) \leq \limsup_{n \to \infty} \frac{1}{a_n^2} \ln \mu(W_n \in \Gamma) \leq - \inf_{x \in \overline{\Gamma}} I(x)$$

where $\Gamma^0$ is the interior of $\Gamma$ and $\overline{\Gamma}$ is its closure.
We also obtain the following Rosenthal type moments estimates.

2.7. **Theorem.** Let Assumption [2.2] hold true and suppose again that if \( \|\varphi\|_K = \text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi_\omega\|_{BV}) < \infty \). Then under Assumptions [2.3] there exist a constant \( c_0 \) so that with \( \gamma = 1/\eta \) for every \( p \geq 1 \) we have

\[
E[(S_n)^p] - (\text{Var}(S_n))^{\frac{p}{2}} \leq (c_0)^p(\gamma^{1+p}) \sum_{1 \leq u \leq \frac{E(Z^2)}{10^{1/2}}} n^u \frac{p^u}{(u!)^2} = O(n^{(p-1)/2})
\]

where \( Z \) be a standard normal random variable. In particular, \( \|S_n\|_{L^p} = O(\sqrt{n}) \) for every \( p \). As in the previous theorems, the constant \( c_0 \) depends (explicitly) only on \( c_1, c_2, \eta_0, \|\varphi\|_K \) and \( C_0 \).

Finally, let us consider the random function \( S_n(t) = n^{-1/2}S_{[nt]} \) on \([0,1]\). We also obtain a functional CLT for these random functions.

2.8. **Theorem.** Let Assumption [2.2] hold. Suppose that \( \text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi_\omega\|_{BV}) < \infty \) and that Assumption [2.3] holds true. Then the random function \( S_n \) converges in distribution towards the distribution of \( \{sW_t\} \) where \( W \) is a standard Brownian motion (restricted to \([0,1]\)) and \( s^2 \) is the asymptotic variance.

2.2.2. **An almost sure invariance principle and exponential concentration inequalities for \( \phi \) and \( \psi \)-mixing driving processes (via martingale methods).** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the probability space on which \((\xi_n)\) is defined. We recall that the \( \phi \)-mixing and \( \psi \) (dependence) coefficient between two sub-\( \sigma \)-algebras \( \mathcal{G}, \mathcal{H} \) of \( \mathcal{F} \) is given by

\[
\phi(\mathcal{G}, \mathcal{H}) = \sup \left\{ |\mathbb{P}(B|A) - \mathbb{P}(B)| : A \in \mathcal{G}, B \in \mathcal{H}, \mathbb{P}(A) > 0 \right\}
\]

and

\[
\psi(\mathcal{G}, \mathcal{H}) = \sup \left\{ \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)\mathbb{P}(B)} - 1 : A \in \mathcal{G}, B \in \mathcal{H}, \mathbb{P}(A)\mathbb{P}(B) > 0 \right\}.
\]

The reverse \( \phi \)-dependence coefficients of \((\xi_n)\) are defined by

\[
\phi_n(R) = \sup_k \phi(\mathcal{F}_{n+k}, \mathcal{F}_n, k) = \phi(\mathcal{F}_{n+k}, \mathcal{F}_n, 0)
\]

while the \( \psi \)-dependence coefficients of \((\xi_n)\) are defined by

\[
\psi_n = \sup_k \psi(\mathcal{F}_{n+k}, \mathcal{F}_n) = \psi(\mathcal{F}_n, \mathcal{F}_n)
\]

where \( \mathcal{F}_n \) is the \( \sigma \)-algebra generated by \( \xi_j, j \leq k \) and \( \mathcal{F}_{n+k} \) is generated by \( \xi_j, j \geq k + n \). Then it is clear from the definitions that

\[
\alpha_n \leq \phi_n(R) \leq \psi_n.
\]

2.9. **Theorem** (Exponential concentration and maximal inequalities). **Under Assumption [2.2] the following holds.** Suppose the observable satisfies that \( \text{esssup}_{\omega \in \Omega}(K(\omega)^2\|\varphi_\omega\|_{BV}) < \infty \). Moreover, assume that the condition [2.3] holds true.

Let \( \mathcal{F}_0 \) be the \( \sigma \) algebra generated by the the map \( \pi(\omega, x) = ((\omega_j)_{j \geq 0}, x) \), namely the one generated by \( \mathcal{B} \) and the coordinates with non-negative indexes in the \( \omega \) direction. If either \( \text{ess-inf}_{x} h_\omega(x) > 0 \) and \( \sum_n \phi_n < \infty \) or \( \sum_n \psi_n < \infty \) then there is an \( \mathcal{F}_0 \)-measurable function \( \chi \in L^\infty(\mu) \) so that if we set \( u = \varphi + \chi \circ \tau - \chi \) then \( u \circ \tau^n \) is a reverse martingale difference with respect to the reverse filtration \( \{\tau^{-n}\mathcal{F}_0\} \). As a consequence:

(i) there are constants \( a_1, a_2, a_3 > 0 \) (which can be recovered from the proof) so that the following exponential concentration inequality holds true for every \( t > 0 \)

\[
\mathbb{P}(|S_n| \geq tn + a_1) \leq a_2e^{-a_3nt^2}.
\]

The constants \( a_1, a_2, a_3 \) depend only on \( \Phi = \sum_n \phi_n < \infty \) and \( c \) (or \( \Psi = \sum_n \psi_n < \infty \)), the constant \( C_0 \) and \( \|\varphi\|_{K,2} = \text{esssup}_{\omega \in \Omega}(K(\omega)^2\|\varphi_\omega\|_{BV}) \), and an explicit formula for them can be recovered from the proof.
For every $p \geq 2$ we have

$$\left\Vert \max_{1 \leq k \leq n} |S_k| \right\Vert_{L_p} \leq C_p n^{1/2}$$

where $C_p = C_p$ is a constant (which can be recovered from the proof and depends only on $p$ and above constants).

Our next result is an almost sure invariance principle.

**Theorem 2.10. (ASIP)** Let Assumption 2.2 hold and suppose that the observable satisfies that \(e^{\text{esssup}_{\omega \in \Omega} (K(\omega))^2 \| \varphi_\omega \|_{BV}} \leq \infty\). Moreover, assume that the condition 2.3 holds true. When \(e^{\text{essinf}_{x} h_\omega(x)} > 0\) we set \(\gamma_n = \phi_{R,n}\), while otherwise we set \(\gamma_n = \psi_n\). In both cases, assume that

$$\sum_{n \geq 2} n^{5/2} (\log n)^{3/4} < \infty \quad \text{and} \quad \sum_{n \geq 2} n (\log n)^{3/4} \gamma_n^2 < \infty$$

and

$$\sum_{n \geq 2} \frac{(\log n)^3}{n^2} \left( \sum_{k=0}^{n} (k+1) \gamma_k \right)^2 < \infty.$$

Then the limit

$$s^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[S_n^2]$$

exists and the following version of the almost sure invariance principle holds true: there is a coupling of \((\varphi \circ \tau^n)\) with a sequence of iid Gaussian random variables \(Z_j\) with zero mean and variance \(s^2\) so that

$$\sup_{1 \leq k \leq n} |S_k - \sum_{j=1}^{k} Z_j| = O(n^{1/4} (\log n)^{1/2} (\log \log n)^{1/4}) \text{ almost surely.}$$

**Remark 2.11.** The ASIP implies the functional CLT, see [48]. Thus, Theorem 2.10 yields better results than Theorem 2.8 for \(\phi\) or \(\psi\) mixing driving sequences (which are not necessarily stretched-exponentially mixing).

**2.2.3. A sketch of the proof of Theorems 2.2 and 2.10** Notice that \(\tau\) preserves \(F_0\). We consider now the sub-system \((\Omega \times X, F_\mu, \tau)\), which can be viewed as a factor. Let \(K\) be the “transfer operator” corresponding to \(\tau\), namely the one satisfying the duality relation

$$\int (Kg) \cdot f d\mu = \int g \cdot (f \circ \tau), \quad g \in \mathcal{L}^1(\Omega \times X, F_\mu, f \in \mathcal{L}^\infty(\Omega \times X, F_\mu).$$

The first step in the proof is to show that

$$\|K^n \varphi\|_{L^\infty} = \|K^n \varphi - \mu(\varphi)\|_{L^\infty} = O(n^{\delta + \gamma_n/2})$$

where \(\gamma_n\) is either \(\psi_n\) or \(\phi_{n,R}\) depending on the case, and \(\delta \in (0, 1)\). Once this is established we can take

$$\chi = \sum_{n \geq 1} K^n \varphi,$$

and Theorem 2.9(i) follows from the Azuma–Hoeffding inequality together with Chernoff’s bounding method. Theorem 2.9(ii) follows from the so called, Rio’s inequality [50] (see [44] Proposition 7).

The estimate 2.14 relies on the identification \(L^\infty\) as the dual of \(L^1\), which reduces the problem to estimating the integrals \(\int g \cdot (K^n \varphi) d\mu\) for \(g \in \mathcal{L}^1(\Omega \times X, F_\mu)\) such that \(\|g\|_{L^1} = 1\). In (1.4) we show that

$$\int g \cdot (K^n \varphi) d\mu = \int \varphi \cdot (g \circ \tau^n) d\mu = \int \left( \int (L_{d\mu}^{n} f_{\mu} \cdot \mu_{d\mu} \cdot \omega) \right) d\mu(\omega).$$
Thus, in order to estimate the left hand side of (2.15) we use again the duality between (2.2.4). A vector valued almost sure invariance principle in the uniformly random case for exponentially fast (2.15). We recall next that the $k$-th cumulant of a random variable $\varphi = (\varphi_1, ..., \varphi_d) : \Omega \times X \rightarrow \mathbb{R}^d$ so that $\varphi_\omega = \varphi(\omega, \cdot)$ depend on $\omega$ only through $\omega_0$ and $\text{esssup}_{\omega \in \Omega}(K(\omega)||\varphi_\omega||_{BV}) < \infty$ for all $1 \leq i \leq d$. Let us also assume that $\mu(\varphi_i) = 0$ for every $i$.

Set $S_n = \sum_{j=1}^{n-1} \varphi \circ \tau^j$.

2.12. **Theorem.** Suppose that $\alpha_n = O(n^\alpha)$ for some $\alpha \in (0, 1)$. Then there is a positive semi definite matrix $\Sigma^2$ so that

$$\Sigma^2 = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Cov}(S_n).$$

Moreover, $\Sigma^2$ is positive definite if and only if there exists no unit vector $v$ so that $\varphi \circ v = r - r \circ \tau$ for some $r \in L^2(\mu)$.

Assume now that there are constants $C > 0$ and $\delta \in (0, 1)$ so that

$$\|L_n^\alpha 1 - h_n\sigma_\omega\|_{BV} \leq C5^n$$

namely, that $K(\omega)$ is a bounded random variable. Then there is a coupling of $(\varphi \circ \tau^n)$ with a sequence of independent Gaussian centered random vectors $(Z_n)$ so that $\text{Cov}(Z_n) = \Sigma^2$ and for every $\varepsilon > 0$,

$$\left| S_n - \sum_{j=1}^{n} Z_j \right| = o(n^{1/4+\varepsilon}), \text{ almost surely.}$$

3. **Limit Theorems via the Method of Cumulants for $\alpha$-Mixing Driving Processes**

We recall next that the $k$-th cumulant of a random variable $W$ with finite moments of all orders is given by

$$\Gamma_k(W) = \frac{1}{i^k} \frac{d^k}{dt^k} \left( \ln \mathbb{E}[e^{itW}] \right)|_{t=0}.$$ 

Note that $\Gamma_1(W) = \mathbb{E}[W]$, $\Gamma_2(W) = \text{Var}(W)$ and that $\Gamma_k(aW) = a^k \Gamma_k(W)$ for any $a \in \mathbb{R}$ and $k \geq 1$.

The main result in this section is the following.

3.1. **Theorem.** Suppose that Assumptions 2.2 and 2.3 hold true and that $\|\varphi\|_K = \text{esssup}_{\omega \in \Omega}(K(\omega)||\varphi_\omega||_{BV}) < \infty$. Then, with $\gamma = 1/\eta$, there exists a constant $c_0$ which depends only on $\|\varphi\|_K$ and the constants from Assumption 2.3 so that for any $k \geq 3$,

$$\left| \Gamma_k(S_n) \right| \leq n(k!)^{1+\gamma}(c_0)^{k-2}.$$
We will prove Theorem 3.1 by applying the following Proposition 3.2 which appears in [25] as Corollary 3.2.

Let $V$ be a finite set and $\rho : V \times V \to [0, \infty)$ be so that $\rho(v, v) = 0$ and $\rho(u, v) = \rho(v, u)$ for all $u, v \in V$. For every $A, B \subset V$ set

$$\rho(A, B) = \min\{\rho(a, b) : a \in A, b \in B\}.$$  

Let $X_v, v \in V$ be a collection of centered random variables with finite moments of all orders, and for each $v \in V$ and $t \in (0, \infty]$ let $g_{v,t} \in (0, \infty]$ be so that $\|X_v\|_t \leq g_{v,t}$. Set $W = \sum_{v \in V} X_v$.

3.2. Proposition (Corollary 3.2, [25]). Let $0 < \delta \leq \infty$. Suppose that for any $k \geq 1$, $b > 0$ and a finite collection $A_j, j \in J$ of (nonempty) subsets of $V$ so that $\min_{i \neq j} \rho(A_i, A_j) \geq b$ and $r := \sum_{j \in J} |A_j| \leq k$ we have

\[
(3.1) \quad \left| \sum_{j \in J} \sum_{i \in A_j} X_i - \sum_{j \in J} \sum_{i \in A_j} X_i \right| \leq (r - 1) \left( \prod_{j \in J} \prod_{i \in A_j} g_{i,(1+\delta)k} \right) \gamma_\delta(b, k)
\]

where $\gamma_\delta(b, r)$ is some nonnegative number which depends only on $\delta, b$ and $r$, and $|\Delta|$ stands for the cardinality of a finite set $\Delta$.

In addition, suppose that there exist $c_0 \geq 1$ and $u_0 \geq 0$ so that

\[
(3.2) \quad |\{u \in V : \rho(u, v) \leq s\}| \leq c_0 s^{u_0}
\]

for all $v \in V$ and $s \geq 1$. Assume also that

$$\tilde{\gamma}_\delta(m, k) := \max\{\gamma_\delta(m, r) : 1 \leq r \leq k\} \leq de^{-am^a}$$

for some $a, \eta > 0, d \geq 1$ and all $k, m \geq 1$. Then there exists a constant $c$ which depends only on $c_0, a, u_0$ and $\eta$ so that for every $k \geq 2$,

\[
(3.3) \quad |\Gamma_k(W)| \leq d^k |V|^k c^k(k!)^{1+\delta} (M^k_k + M^k_{(1+\delta)k})
\]

where for all $q > 0$, $M_q = \max\{g_{v,q} : v \in V\}$ and $M^k_q = (M^k_q)^k$.

When the $X_v$’s are bounded and (3.1) holds true with $\delta = \infty$ we can always take $g_{v,t} = g_{v,\infty}$, $t > 0$ and then for any $k \geq 2$,  

\[
(3.4) \quad |\Gamma_k(W)| \leq 2d^k |V|^k M^k_{\infty} c^k(k!)^{1+\delta}.
\]

When $\delta < \infty$ and there exist $\theta \geq 0$ and $M > 0$ so that

\[
(3.5) \quad (g_{v,k})^k \leq M^k(k!)^\theta
\]

for any $v \in V$ and $k \geq 1$, then for any $k \geq 2$,

\[
(3.6) \quad |\Gamma_k(W)| \leq 3C^{\frac{\theta}{1+\delta}} d^k |V|^k c^k(1+\delta) M^k_k (k!)^{1+\delta} + \theta
\]

where $C$ is some absolute constant.

Theorem 3.1 will follow from the following result, which is proved in the next section.

3.3. Proposition. Under Assumption 2.2, we have the following. Fix some $n$ and set $V = \{0, 1, \ldots, n-1\}$ and $X_v = \varphi \circ \tau^v$. Set also $\rho(x, y) = |x - y|$, and let $t = \delta = \infty$, $\gamma_\infty(b, k) = \gamma_b = e^{-(\lambda - 3\varepsilon)/2 + \alpha|b|/3}$. Then condition (3.1) holds true with the above choices and with

$$\theta_{v,\infty} = A_0 \max(\text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi\|_{BV}), \|\varphi\|_{L^\infty})$$

where $A_0$ is a constant which depends only on $\lambda - 3\varepsilon$ and on the constant $C$ so that $\sup |g| \leq C\|g\|_{BV}$ for every function $g : X \to \mathbb{C}$ (and the dependence can be easily recovered from the proof).

If in addition Assumption 2.3 holds then the conditions of Proposition 3.2 hold true with $u_0 = 1$, $c_0 = 2$ and $\gamma = 1/\eta$. 

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3.1. Multiple correlation estimates: proof of Proposition [3.3] Our goal is to show that (3.1) holds true with the desired upper bounds. We first need the following result.

3.4. Lemma. For every two measurable functions \( g, h \) on \( \mathcal{Y}^n \) with \( g, h \in L^\infty \) (w.r.t to the law of \( (\xi_n) \)) and all \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \) we have

\[
(3.7) \quad |\mathbb{E}[g(\ldots, \xi_{k-1}, \xi_k)h(\xi_{k+n}, \xi_{k+n+1}, \ldots)] - \mathbb{E}[g(\ldots, \xi_{k-1}, \xi_k)] \cdot \mathbb{E}[h(\xi_{k+n}, \xi_{k+n+1}, \ldots)]| \\
\leq \frac{1}{4} \|g(\ldots, \xi_{k-1}, \xi_k)\|_{L^\infty} \|h(\xi_{k+n}, \xi_{k+n+1}, \ldots)\|_{L^\infty} \alpha_n.
\]

Proof. By [11] Ch. 4], we have

\[
\alpha(G, \mathcal{H}) = \frac{1}{4} \sup\{\|\mathbb{E}[h|G] - \mathbb{E}[h]\|_{L^1} : h \in L^\infty(\Omega, \mathcal{G}, \mathbb{P}), \|h\|_{L^\infty} \leq 1\}.
\]

Taking \( g = g(\ldots, \xi_{k-1}, \xi_k) \) and \( h = h(\xi_{k+n}, \xi_{k+n+1}, \ldots) \), \( G = \mathcal{F}_{-\infty, k} \) and \( \mathcal{H} = \mathcal{F}_{k+n, \infty} \) we get

\[
|\mathbb{E}[h|G] - \mathbb{E}[h|\mathcal{H}]| = |\mathbb{E}[(h|G) - h]| \leq \frac{1}{4} \alpha(G, \mathcal{H}) \|g\|_{L^\infty} \|h\|_{L^\infty}.
\]

\[ \square \]

Next, it is clearly enough to prove Proposition [3.3] when \( \|\varphi\|_{L^\infty} \) and \( \text{esssup}_{\omega \in \Omega}(K(\omega)\|\varphi_\omega\|_{BV}) \) do not exceed 1, for otherwise we can just divide \( \varphi \) by the maximum between the two. Recall also our assumption that that \( K(\omega)e^{-\varepsilon|\omega|} \leq K(\sigma^m, \omega) \leq K(\omega)e^{\varepsilon|m|} \) for some \( \varepsilon < \lambda/3 \).

The first step in the proof of Proposition [3.3] is the following result.

3.5. Lemma. [Fiberwise multiple correlation estimates] For every two measurable functions \( g, h \) on \( \mathcal{Y}^n \) with \( g, h \in L^\infty \) (w.r.t to the law of \( (\xi_n) \)) and all \( k \in \mathbb{Z} \) and \( n \in \mathbb{N} \) we have

\[
|\mathbb{E}[g(\ldots, \xi_{k-1}, \xi_k)h(\xi_{k+n}, \xi_{k+n+1}, \ldots)] - \mathbb{E}[g(\ldots, \xi_{k-1}, \xi_k)] \cdot \mathbb{E}[h(\xi_{k+n}, \xi_{k+n+1}, \ldots)]| \\
\leq \frac{1}{4} \|g(\ldots, \xi_{k-1}, \xi_k)\|_{L^\infty} \|h(\xi_{k+n}, \xi_{k+n+1}, \ldots)\|_{L^\infty} \alpha_n.
\]

Proof. The proof will be carried out by induction on \( m \). Let us first prove the lemma in the case \( m = 2 \). We first note that for all functions \( \varphi_0, \varphi_1, \ldots, \varphi_q \) we have

\[
\var{\prod_{k=0}^q \varphi_k} \leq \prod_{k=0}^q \var{\varphi_k} \cdot \prod_{k<s} \var{\varphi_s} \|g_s\|_{\infty}
\]

where \( \|g_s\|_{\infty} = \sup |g_s| \), and hence

\[
(3.8) \quad \|\prod_{k=0}^q \varphi_k \cdot T^k_{\omega}\|_{BV} \leq \|g_k\|_{\infty} + \sum_{k=0}^q \prod_{k<s} \|g_s\|_{\infty} \cdot (\prod_{s=0}^{k-1} N_{\sigma^k, \omega} \var{\varphi_s}) \prod_{k<s} \|g_s\|_{\infty}
\]

where we have used (2.2). Let us write \( B_0 = \{0, 1, \ldots, d\} \). Taking \( g_k = f_k \) for \( 0 \leq k \leq d = q \) and noting that \( K(\sigma^m) \|g_s\|_{\infty} \leq C \) for some constant \( C \) which depends only on the space \( X \) we conclude that

\[
\|F_1\|_{BV} \leq C(d + 1) \leq 2Ck.
\]

Now, if we write \( B_2 = \{d + n, d + n + 1, \ldots, d + n + L\} \) then

\[
\mu_\omega(F_1F_2) = \mu_\omega(F_1 \cdot G_2 \circ T^2_{\omega}) = \mu_\omega(F_1 \cdot G_2 \circ T^2_{\omega} \cdot G_2L^{d+n}_{\omega}F_1)
\]

\[ \text{where } C \text{ is a constant which satisfies that } \|g\|_{\infty} \leq C \|g\|_{BV} \text{ for every complex function on } X. \]
where

\[ G_2 = \prod_{u \in B_2} f_u \circ T_{n-u}^{n-d}. \]

By (2.3) we have,

\[ \left\| L_\omega^{d+n} F_1 - \mu_\omega(F_1) \right\| \leq K(\omega) \left\| F_1 \right\| e^{-\lambda(d+n)} \leq 2dCK(\omega)e^{-\lambda(d+n)}. \]

Therefore, using also that \( \mu_\omega \) is an equivariant family and that (since \( n + d \in B_2 \))

\[ \sup |G_2| \leq \sup |f_{n+d}| \leq CK(\sigma^{n+d} \omega)^{-1} \]

we get that

\[ |\mu_\omega(F_1F_2) - \mu_\omega(F_1)\mu_\omega(F_2)| = |\mu_{\sigma^n + \omega}(G_2 L_\omega^{n+d} F_1) - \mu_\omega(F_1)\mu_{\sigma^n + \omega}(G_2)| \]

\[ = \left| \int (L_\omega^{d+n} F_1 - \mu_\omega(F_1))G_2 d\mu_{\sigma^n + \omega} \right| \leq 2dCK(\omega)e^{-\lambda(d+n)} \sup |G_2| \]

\[ \leq 2dCK(\omega)e^{-\lambda(d+n)} K(\sigma^{n+d} \omega)^{-1} \leq 2dC^2 e^{-\lambda(d-n)} = (2C^2 d) e^{-\lambda d}. \]

This proves the lemma for \( m = 2 \).

Next, let us complete the induction step. Let \( d \) be the right end point of \( B_{m-1} \). Then \( d + d_m \) is the left end point of \( B_m \) and we can write

\[ \mu_\omega(\prod_{k} F_k) = \mu_\omega(\prod_{k < m} F_k \cdot (G_m \circ T_{d+d_m}^{d+n})) = \mu_{\sigma^n + m \omega}(L_{\omega}^{d+n} \prod_{k < m} F_k \cdot G_m) \]

where \( G_m \) is some function. Now we observe that

\[ \left\| \prod_{k < m} F_k \right\|_{BV} \leq C(d+1) \leq 2Cd \]

which is proved exactly as in the previous case (even though there are gaps between the blocks \( B_j \), we can set \( g_i = 1 \) when \( i \) does not belong to one of the \( B_j \)’s, and then \( \text{var}(g_i) = 0 \)). Thus, as in the case \( m = 2 \), we have

\[ |\mu_\omega(\prod_{k} F_k) - \mu_\omega(G_m)\mu_\omega(\prod_{k < m} F_k)| \leq (2C^2 d) e^{-\lambda d_m}. \]

The induction is completed by the above inequality, taking into account that \( |\mu_\omega(G_m)| \leq 1 \).

Integrating over \( \omega \) yields the following corollary of Lemma 3.5.

3.6. Corollary. Let \( \tau \) be the skew product. Let \( B_j, 1 \leq j \leq m \) be blocks as in Lemma 3.5. Set \( G_j = \prod_{i \in B_j} \varphi \circ \tau^i \). Let us denote by \( b_j \) the left end point of \( B_j \). Then

\[ (3.9) \quad \left| \int_{j=1}^{m} \int_{j=1}^{m} G_j d\mu - \int_{i \in B_j} \varphi_{\sigma^i \omega} \circ T_{\sigma^i \omega}^{d-b_j} \right| = A \sum_{j=1}^{d} e^{-\lambda d_j}. \]

The next step of the proof is to estimate the second term inside the absolute value on the left hand side of (3.9). To obtain appropriate estimates, we first need the following lemma:

3.7. Lemma. Let us fix some \( k \in \mathbb{N} \) and set

\[ F_\omega = \prod_{j=0}^{k} \varphi_{\sigma^j \omega} \circ T_\omega^k \]

Then for every \( n \in \mathbb{N} \) and for \( \mathbb{P} \) a.e. \( \omega \) we have

\[ |\mu_\omega(F_\omega) - m(F_\omega L_{n-\omega}^{n} 1)| \leq Ce^{-n(\lambda-d)}. \]

where \( C \) is such that \( \sup |g| \leq C \|g\|_{BV} \) for every function \( g \) on \( X \).
Proof. Using (2.4), that $K(\sigma^{-n}\omega) \leq e^{cn}K(\omega)$ and that $|F_\omega| \leq |\phi_\omega| \leq C\|\phi_\omega\|_{BV} \leq CK(\omega)^{-1}$ we obtain that

\[
|\mu_\omega(F_\omega) - m(F_\omegaL_{\sigma^{-n}\omega}^{\alpha})| = \int (h_\omega - L_{\sigma^{-n}\omega}^{\alpha})F_\omega dm \leq CK(\omega)^{-1} \int |h_\omega - L_{\sigma^{-n}\omega}^{\alpha}| dm \\
\leq K(\omega)^{-1}e^{-\lambda n}K(\sigma^{-n}\omega) \leq Ce^{-n(\lambda - \varepsilon)}.
\]

□

Taking into account that $|\mu_\omega(F_\omega)| \leq 1$, that $|m(F_\omegaL_{\sigma^{-n}\omega}^{\alpha})| = |m(F_\omega \circ T_{\sigma^{-n}\omega})| \leq 1$ and that $|\prod \alpha_j - \prod \beta_j| \leq \sum |\alpha_j - \beta_j|$ for all numbers $\alpha_j, \beta_j$ so that $|\alpha_j|, |\beta_j| \leq 1$ we get the following result directly from Corollary 3.6 and Lemma 3.7.

3.8. Corollary. Let $b_j$ be the left end point of the block $B_j$. Let us also set $r_j = d_j/3$ and $r_0 = r_1$.

Then there exists a constant $A_1 > 0$ which does not depend on $\omega$ or on the blocks so that in the notations of the previous corollary and the previous lemma we have

\[
\left| \int \prod_{j=1}^{m} G_j dm - \int \left( \prod_{j=0}^{d} m(\phi_{\omega,j}L_{\sigma^{\alpha_j} - \delta_j \omega}^{\beta_j} | 1) \right) d\mu(\omega) \right| \leq A_1 \sum_{j=1}^{m-1} e^{-(\lambda - \varepsilon)r_j}
\]

where

\[
\phi_{\omega,j} = \prod_{j \in B_j} \phi_{\sigma^{\alpha_j} \circ T_{\sigma^{\alpha_j} \omega}}
\]

Now, we observe that $m(\phi_{\omega,j}L_{\sigma^{\alpha_j} - \delta_j \omega}^{\beta_j} | 1)$ is a function of $\xi_{b_j - r_j}, ..., \xi_{b_{j+1} - r_j}$ (i.e. of the coordinates $\omega_{b_j - r_j}, ..., \omega_{b_{j+1} - r_j})$. Namely, in distribution it can be written as

\[
m(\phi_{\omega,j}L_{\sigma^{\alpha_j} - \delta_j \omega}^{\beta_j} | 1) = f_j(\xi_{b_j - r_j}, ..., \xi_{b_{j+1} - r_j})
\]

for some measurable function $f_j$. Since $m(\phi_{\omega,j}L_{\sigma^{\alpha_j} - \delta_j \omega}^{\beta_j} | 1) = m(\phi_{\omega,j} \circ T_{\sigma^{\alpha_j} - \delta_j \omega})$ and $|\phi_{\omega,j}| \leq 1$, we can insure that $|f_j| \leq 1$. Using [25] (2.20) and Corollary 3.8 we conclude that:

3.9. Corollary. Let $G_j, 1 \leq j \leq m$ be as in Corollary 3.6 (defined by some blocks $B_j$ with gaps $d_j$).

There are constants $A > 1$ and $\delta_0 \in (0, 1)$ which do not depend on the blocks so that

\[
\left| \int \prod_{j=1}^{m} G_j dm - \int \prod_{j=1}^{m} G_j dm \right| \leq A \sum_{j=1}^{m} (e^{\delta_j} + \alpha(|r_j|)).
\]

All the is left is to notice that Corollary 3.9 is a reformulation of Proposition 3.3 using the notations of this section.

3.2. Limit theorems via the method of cumulants.

3.2.1. The CLT: proof of Theorem [2.4]

First, a particular consequence of Proposition 3.3 is that $|E_\mu[\phi \cdot \phi^n]| = O(\delta^n + \alpha_n)$ for some $\delta \in (0, 1)$. Hence, if $\sum n\alpha_n < \infty$ then $\sum n|E_\mu[\phi \cdot \phi^n]| < \infty$ and the results concerning the asymptotic variance $s^2$ follow from the general theory of (weakly) stationary processes (see [33] and Lemma 3.10 below).

Now, suppose that $s^2 = \lim_{\sigma \to \infty} \frac{1}{\text{Var}_\sigma(S_n)} > 0$, where $S_n = S_n \phi$. To prove the CLT and the convergence rate (2.6), applying [53] Corollary 2.1, taking into account Theorem 3.1 we get the CLT and the rate (2.6) for $S_n/\sqrt{\text{Var}(S_n)}$. To get the same rate for $S_n/\sqrt{n}$ we need the following general fact from the theory of stationary real-valued sequences, which for the sake of convenience is stated as a lemma.
3.10. **Lemma.** Let $Y_n$ be a centered weakly stationary sequence of square integrable random variables. Set $b_n = E[Y_0 Y_n]$ and $S_n = \sum_{j=1}^n Y_j$. Suppose that $\sum_k k |b_k| < \infty$. Then

$$\lim_{n \to \infty} \frac{1}{n} E[S_n^2] = b_0 + 2 \sum_{n \geq 1} b_n := s^2$$

and

$$\left| \frac{1}{n} E[S_n^2] - s^2 \right| \leq 2n^{-1} \sum_{k=1}^\infty k |b_k|.$$ 

Let us give a remainder of the short proof. We have $\frac{1}{n} E[S_n^2] = \sum_{k=1}^{n-1} (1 - k/n) b_k + b_0$ and so

$$\left| \frac{1}{n} E[S_n^2] - s^2 \right| = \left| \sum_{k=1}^{n-1} b_k + 2n^{-1} \sum_{k=1}^{n-1} k b_k \right| \leq 2n^{-1} \left( \sum_{k=1}^{n-1} k |b_k| + \sum_{k=1}^{n-1} k |b_k| \right) = 2n^{-1} \sum_{k=1}^{n-1} k |b_k|.$$ 

Using this lemma together with [26] Lemma 3.3 with $a = 2$ and that

$$\left\| \frac{S_n}{\sqrt{\text{Var}(S_n)}} - \frac{S_n}{s} \right\|_{L^2} = \|S_n\|_{L^2} \left| \frac{1}{\sqrt{\text{Var}(S_n)}} - \frac{1}{s} \right| = O(n^{1/2}) \cdot O(n^{-3/2}) = O(n^{-1})$$

we obtain (2.6).

3.2.2. **A moderate deviations principle, stretched exponential concentration inequalities and Rosenthal type estimates: proof of Theorems 2.5, 2.6 and 2.7.** First, Theorem 2.5 follows from Theorem 3.1 and [14, Lemma 2.3]. The first part of Theorem 2.6 follows from Theorem 3.1 and [14, Lemma 2.3]. The moderate deviations principle stated in Theorem 2.6 follows from Theorem 3.1 and [14, Theorem 1.1].

3.3. **The functional CLT via the method of cumulants: proof of Theorem 2.8.** Let us first show that the sequence $S_n$ is tight. By Theorem 2.7 we have that

$$\|S_n\|_4 = O(\sqrt{n})$$

where $\| \cdot \|_4 = \| \cdot \|_{L^4}$, and therefore, using also stationarity and the Hölder inequality we get that for all $t_1 < t_2 < r_1 < r_2$,

$$\mathbb{E} \left[ (S_n(t_2) - S_n(r_1))^2 (S_n(t_2) - S_n(t_1))^2 \right] \leq \|S_n(t_2) - S_n(r_1)\|_4^2 \|S_n(t_2) - S_n(t_1)\|_4^2 \leq C \left( \frac{|r_2n - t_1n|}{n} \right)^{2/n}.$$ 

Thus, by [29, Ch.15], $S_n(\cdot)$ is a tight sequence in the Skorokhod space $D[0,1]$.

Now let show that the finite-dimensional distributions converge. Let us fix some $t_1 < t_2 < ... < t_d$. Set $X_k = \phi \circ \tau^k$. By considering characteristic function, it is enough to show that any linear combination of $S_n(t_j)$ converges towards a centered normal random variable with an appropriate variance. More precisely, let $a_1, ..., a_d \in \mathbb{R}$. Then we need to show that $\sum_{j=1}^d a_j S_n(t_j)$ converges in distribution towards a centered normal random variable with variance

$$s^2 \sum_{j=1}^d (a_j + ... + a_d)^2 (t_j - t_{j-1})$$

where $t_0 = 0$ and $s^2 = \lim_{n \to \infty} \frac{1}{n} E[S_n^2]$. We first notice that

$$\sum_{j=1}^d S_n(t_j) = n^{-1/2} \sum_{j=1}^d (a_j + ... + a_d) S_n(t_{nj}).$$
Thus, using stationarity, we have
\[
\mathbb{E} \left[ \left( \sum_{j=1}^{d} r_j S_n(t_j) \right)^2 \right] = n^{-1} \sum_{j=1}^{d} (a_j + \ldots + a_d)^2 \mathbb{E}[S_{[nt_j]-[nt_j-1]}^2]
+ 2n^{-1} \sum_{1 \leq j_1 < j_2 \leq d} (a_{j_1} + \ldots + a_{j_2})(a_{j_2} + \ldots + a_{j_d}) \mathbb{E} \left[ (S_{[nt_{j_2}]} - S_{[nt_{j_2-1}]}) (S_{[nt_{j_1}]} - S_{[nt_{j_1-1}]}) \right].
\]

Now, the first summand on the above right hand side converges to
\[
s^2 \sum_{j=1}^{d} (a_j + \ldots + a_d)^2 (t_j - t_{j-1}),
\]
while the second summand (the double sum) converges to 0 because \(|\mathbb{E}[\phi \cdot \varphi \circ \tau^n]|\) converges to 0 stretched exponentially fast. Therefore, the asymptotic variance of \(\sum_{j=1}^{d} a_j S_n(t_j)\) has the desired form. Now, let us consider the following array of random variables. Set
\[
Y_k = Y_k^{(n,a_1,\ldots,a_d)} = (a_1 + \ldots + a_j) \varphi \circ \tau^k \text{ if } [nt_{j-1}] \leq k < [nt_j].
\]

Then the same estimates on the multiple correlations between the \(Y_k\)'s as in Proposition 3.9 hold true, namely Proposition 3.1 also holds for \(\sum_{k=0}^{[nt_d]-1} Y_k\). Applying again [53 Corollary 2.1] we get that
\[
\sum_{k=0}^{[nt_d]-1} Y_k^{(n,a_1,\ldots,a_d)} / w_n
\]
converges towards the standard normal distribution, where \(w_n\) is the standard deviation of the numerator. Note that, as we have shown, \(w_n / n \to s^2 \sum_{j=1}^{d} (a_1 + \ldots + a_d)^2 (t_j - t_{j-1})\), which is positive unless either \(s = 0\) or \(a_1 = \ldots = a_d = 0\), which are both trivial cases. Thus, in any case we obtain the convergence of the linear combination
\[
\sum_{j=1}^{d} S_n(t_j)
\]
and the proof of Theorem 2.8 is complete.

4. LIMIT THEOREMS VIA MARTINGALE APPROXIMATION FOR \(\phi\) AND \(\psi\) MIXING DRIVING PROCESSES

4.1. Some expectation estimates using mixing coefficients. In the course of the proof of Theorem 2.10 we will need the following two relatively simple lemmas.

4.1. Lemma. Let \(\mathcal{G}, \mathcal{H}\) be two sub-\(\sigma\)-algebras of a given \(\sigma\)-algebra on some space measure space. Let \(g\) a real-valued bounded \(\mathcal{G}\) measurable function and \(h\) be a \(\mathcal{H}\) real-valued integrable function. Then
\[
|\mathbb{E}[hg] - \mathbb{E}[h]\mathbb{E}[g]| \leq \frac{1}{2} \|h\|_{L^\infty} \|g\|_{L^1} \phi(\mathcal{G}, \mathcal{H})
\]

Proof. By [10] Ch. 4 we have
\[
\|\mathbb{E}[hg] - \mathbb{E}[h]\|_{L^\infty} \leq \frac{1}{2} \|h\|_{L^\infty} \phi(\mathcal{G}, \mathcal{H})
\]
which clearly implies the lemma. \(\square\)

The next result is:
4.2. **Lemma.** Let \( G, \mathcal{H} \) be two sub-\( \sigma \)-algebras of a given \( \sigma \)-algebra on some space measure space. Let \( g \) a real-valued bounded \( G \)-measurable function and \( h \) be a \( \mathcal{H} \)-valued integrable function. Suppose also that \( \psi = \psi(G, \mathcal{H}) < 1 \). Then
\[
|E[hg] - E[h]E[g]| \leq \|hg\|_{L^1} C_\psi \psi
\]
where \( C_\psi = (1 - \psi)^{-1} \).

**Proof.** By [10, Ch.4] we have,
\[
\|E[hG] - E[h]\|_{L^\infty} \leq \|h\|_{L^1} \psi(G, \mathcal{H}).
\]
Hence
\[
|E[hg] - E[h]E[g]| \leq \|h\|_{L^1} \|g\|_{L^1} \psi.
\]
Taking \( h, g \geq 0 \) we get that
\[
|E[hg] - E[h]E[g]| \leq E[h]E[g] \psi.
\]
Thus,
\[
E[h]E[g] \leq (1 - \psi)^{-1} E[hg] = C_\psi E[hg]
\]
Therefore, for nonnegative functions we have
\[
|E[hg] - E[h]E[g]| \leq C_\psi \psi E[hg].
\]
Now the general result follows by writing \( h = h^+ - h^- \) and \( g = g^+ - g^- \) where \( h^\pm \) and \( g^\pm \) are nonnegative functions so that \( h^+ + h^- = |f| \) and \( g^+ + g^- = |g| \), and using that both \((g, h) \rightarrow E|g|E|h|\) and \((g, h) \rightarrow E|hg|\) are bilinear in \((g, h)\).

4.2. **Convergence of the iterates of the transfer operator with respect to a sub-\( \sigma \)-algebra.**

Let \( F_0 \) be the \( \sigma \) algebra generated by the the map \( \pi(\omega, x) = ((\omega_j)_{j \geq 0}, x) \), namely the one generated by \( B \) and the coordinates with non-negative indexes in the \( \omega \) direction. Then \((\tau^{-k} F_0)_{k \geq 0}\) is a decreasing sequence of \( \sigma \)-algebras and \( \tau^{-k} F_0 \) is generated by \( \tau^k \) and the coordinates \( \omega_j \) for \( j \geq k \). In particular \( \tau \) preserves \( F_0 \).

Next, let us define a transfer operator with respect to \( F_0 \). For each function \( g \in L^1(\mu) \) there is a unique \( F_0 \) measurable function \( G \) so that
\[
E[g|\tau^{-1} F_0] = G \circ \tau.
\]
Let us define \( K g = G \), where we formally set \( G \) to be 0 outside the image of \( \tau \) (if \( \tau \) is not onto).

Then
\[
E[g|\tau^{-1} F_0] = K g \circ \tau.
\]
Notice that for \( g \in L^1(\Omega \times X, F_0, \mu) \), \( f \in L^\infty(\Omega \times X, F_0, \mu) \) we have
\[
\int (K g) f d\mu = \int (K g \circ \tau) f \circ \tau d\mu = \int [g|\tau^{-1} F_0] \cdot f \circ \tau d\mu = \int g \cdot f \circ \tau d\mu
\]
and therefore \( K \) can also be defined using the usual duality relation. That is, it is the transfer operator of \( \tau \) with respect to \((\Omega \times X, F_0, \mu)\).

The proof of Theorems [2.9 and 2.10] is based on the following result.

4.3. **Lemma.** Under the assumptions of Theorems [2.9 and 2.10] we have the following:

(i) If \( h_\omega \geq c^{-1} > 0 \) for some constant \( c > 1 \) we have
\[
\|K^n \varphi\|_{L^\infty} \leq C (e^{-(\lambda - 2c)n/2} + \phi^{(n/2)_R}) =: C \gamma_{1,n}
\]
while in this lower bound we have
\[
\|K^n \varphi\|_{L^\infty} \leq C (e^{-(\lambda - 2c)n/2} + \psi^{(n/2)}):= C \gamma_{2,n}.
\]
Here \( C = C_\varphi \) is a constant having the form \( C_\varphi = A C_0 \text{esssup}_{\omega \in \Omega} (K(\omega)^2 \varphi_{\omega} \|\varphi\|_{BV}) \) where \( A \) is an absolute constant and \( C_0 \) is a constant satisfying \( \sup |g| \leq C_0 g \|g\|_{BV} \) and \( \|f g\|_{BV} \leq C_0 \|g\|_{BV} \|f\|_{BV} \) for all functions \( g, f : X \rightarrow \mathbb{C} \).
(ii) If \( h_\omega \geq c^{-1} > 0 \) for some constant \( c > 1 \) we have

\[
\| \mathcal{K}^i(\varphi \mathcal{K}^j \varphi) - \mu(\mathcal{K}^i(\varphi \mathcal{K}^j \varphi)) \|_{L^\infty} \leq C \gamma_{1,\max(i,j)}
\]

while without this lower bound we have

\[
\| \mathcal{K}^i(\varphi \mathcal{K}^j \varphi) - \mu(\mathcal{K}^i(\varphi \mathcal{K}^j \varphi)) \|_{L^\infty} \leq C \gamma_{2,\max(i,j)}.
\]

Proof of Theorem 2.10 based on Lemma 4.3. The proof of the ASIP (Theorem 2.10) follows now from [12, Theorem 3.2], while the proof of Theorem 2.9 (i) follows since if we set \( \chi = \sum_{n=1}^{\infty} K^n \varphi \) and \( u = \varphi + \chi \circ \tau - \chi \), then \( u \circ \tau^n \) is a reverse martingale difference with respect to the reverse filtration \( \tau^{-n} \mathcal{F}_0 \). Moreover, the differences \( u \circ \tau^n \) are uniformly bounded. Thus by the Azuma-Hoeffding inequality for every \( \lambda > 0 \) we have,

\[
\mathbb{E}_\mu[e^{\lambda \sum_{j=1}^{n-1} u \circ \tau^j}] \leq e^{\lambda^2 n \|u\|_{L^\infty}^2}.
\]

Now the proof proceeds by using the Chernoff bounding method: by the Markov inequality for all \( t > 0 \) we have

\[
\mu \left\{ n-1 \sum_{j=0}^{n-1} u \circ \tau^j \geq tn \right\} \leq e^{-\lambda tn} e^{\lambda^2 n \|u\|_{L^\infty}^2}.
\]

Taking \( \lambda_t = \frac{t}{2 \|n\|_{L^\infty}} \) and replacing \( u \) with \(-u\) we get that

\[
\mu \left\{ \pm n-1 \sum_{j=0}^{n-1} u \circ \tau^j \geq tn \right\} \leq e^{-\frac{tn^2}{4 \|n\|_{L^\infty}^2}}.
\]

The proof of Theorem 2.9 (i) is completed now by noticing that

\[
(4.3) \quad \left\| S_n \varphi - \sum_{j=0}^{n-1} u \circ \tau^j \right\|_{L^\infty} \leq 2 \|u\|_{L^\infty}.
\]

Finally, the proof of Theorem 2.9 (ii) is completed by applying [14, Proposition 7] with the reverse martingale \( u \circ \tau^n \) and using (4.3). \( \square \)

Proof of Lemma 4.3. (i) Since \( L^\infty(\mu) \) is the dual of \( L^1(\mu) \) and \( \varphi \) and \( K^n \varphi \) are \( \mathcal{F}_0 \) measurable, it is enough to show that for every \( g \in L^1(\Omega \times X, \mathcal{F}_0, \mu) \) so that \( \|g\|_{L^1} \leq 1 \) we have

\[
\left| \int g \cdot (K^n \varphi) d\mu \right| \leq \gamma_n \|g\|_{L^1(\mu)}
\]

where \( \gamma_n \) is one of the desired upper bounds. To achieve that let us first note that \( K^n \) is the dual of the restriction of the Koopman operator \( f \rightarrow \tau^n \) acting on \( \mathcal{F}_0 \)-measurable functions. Thus,

\[
(4.4) \quad \int g \cdot (K^n \varphi) d\mu = \int \varphi \cdot (g \circ \tau^n) d\mu = \int \left( \int \varphi_{\omega} \cdot (g_{\sigma^n \omega} \circ T^n_{\omega}) d\mu_{\omega} \right) d\mathbb{P}(\omega)
\]

\[
= \int \left( \int (L^n_{\omega} \varphi_{\omega}) \cdot g_{\sigma^n \omega} d\mu_{\sigma^n \omega} \right) d\mathbb{P}(\omega).
\]

Now, using (2.3) and that \( \|\varphi\|_K = \text{esssup}_{\omega \in \Omega} \|K(\omega)\|_{BV} < \infty \) we get that

\[
\sup |L^n_{\omega} \varphi_{\omega} - \mu_{\omega}(\varphi_{\omega})| \leq C_0 \|\varphi\|_K e^{-\lambda n}.
\]

Hence, using also the \( \sigma \)-invariance of \( \mathbb{P} \),

\[
\int g \cdot (K^n \varphi) d\mu = \int \mu_{\omega}(\varphi_{\omega}) \mu_{\sigma^n \omega} (g_{\sigma^n \omega}) d\mathbb{P}(\omega) + I
\]

where \( |I| \leq C e^{-\lambda n} \|g\|_{L^1(\mu)} \). Next, let us write

\[
\mu_{\sigma^n \omega} (g_{\sigma^n \omega}) = m(g_{\sigma^n \omega} h_{\sigma^n \omega}).
\]
By (2.1) we have
\[
\sup |\mathcal{L}^{n-[n/2]}_\sigma 1| \leq C_0 K(\sigma^{[n/2]} \omega) e^{(-\lambda n/2)} \leq C_0 K(\omega) e^{-(\lambda-\varepsilon)n/2}.
\]
Observe next that since \(1/h_\omega \|_{BV} \leq K(\omega)\) we have
\[
m(|g|) = \mu_{\sigma \omega}(|g|/h_\sigma \omega) \leq C_0 K(\sigma \omega) \mu_\omega(|g|)
\]
for every function \(g\) and recall that \(K(\sigma \omega) \leq K(\omega) e^{\varepsilon n}\). Combining this with the previous estimates we get that
\[
(4.5) \quad |m(g_{\sigma \omega} h_{\sigma \omega}) - m(g_{\sigma \omega} \mathcal{L}^{n-[n/2]}_\sigma 1)| \leq C_0 K(\omega) e^{-(\lambda-\varepsilon)n/2} m(|g_{\sigma \omega}|)
\]
\[
\leq C K(\omega)^2 \mu_{\sigma \omega}(|g_{\sigma \omega}|) e^{-(\lambda-\varepsilon)n/2}.
\]
Therefore,
\[
(4.6) \quad \int g \cdot (K^n \varphi) d\mu = \int \mu_\omega(\varphi) m(g_{\sigma \omega} \mathcal{L}^{n-[n/2]}_\sigma 1) d\bar{P}(\omega) + I + J
\]
where \(|I| \leq C e^{-\lambda n} \|g\|_{L^1(\mu)}\) and \(|J| \leq C' e^{-(\lambda-\varepsilon)n/2} \|g\|_{L^1(\mu)}\) and we have used that \(K(\omega)^2 \|\varphi\|_{BV}\) is bounded.

Next, using (2.1) and that \(K(\omega)\) is tempered we have \(h_\omega = \lim_{n \to \infty} \mathcal{L}^n_{\sigma \omega} 1\), and therefore \(h_\omega\) depends only on the coordinates \(\omega_j\) for \(j \leq 0\). Thus
\[
\mu_\omega(\varphi) = F(\omega_j; j \leq 0)
\]
for some measurable function \(F\) so that \(|F| \leq \|\varphi\|_{L^1(\mu)}\). Observe also that the random variable
\[
G_n(\omega) = m(g_{\sigma \omega} \mathcal{L}^{n-[n/2]}_\sigma 1)
\]
depends only on \(\omega_j, j \geq [n/2]\) since \(g_\omega(x)\) is a function of \(x\) and \(\omega_j, j \geq 0\) (i.e. factors through \(\pi_0\)). In the case when \(h_\omega \geq c^{-1} > 0\) for some constant \(c > 0\) we have
\[
|G_n(\omega)| = \left|\mu_{\sigma \omega}(g_{\sigma \omega} \mathcal{L}^{n-[n/2]}_\sigma 1)ight| \leq c \mu_{\sigma \omega}(|g_{\sigma \omega}|).
\]
Thus, using also lemma 4.1 we see that there is a constant \(C > 0\) so that
\[
\left|\int \mu_\omega(\varphi) m(g_{\sigma \omega} \mathcal{L}^{n-[n/2]}_\sigma 1) d\bar{P}(\omega)\right| \leq C \phi_{[n/2], R} \int |G_n(\omega)| d\bar{P}(\omega) \leq c C \phi_{[n/2], R} \|g\|_{L^1(\mu)}
\]
where we have taken into account that \(\int \mu_\omega(\varphi) d\bar{P}(\omega) = \mu(\varphi) = 0\). This, together with (4.6) and the previous estimates on \(I, J\) proves (4.1).

To prove (4.2), we first use (4.3) and obtain that
\[
(4.7) \quad |G_n(\omega)| \leq C \mu_{\sigma \omega}(|g_{\sigma \omega}|) (1 + C K^2(\omega) e^{-(\lambda-\varepsilon)n/2}) \leq C \mu_{\sigma \omega}(|g_{\sigma \omega}|) K(\omega)^2.
\]
Taking into account that \(\text{esssup}_{\omega \in \Omega} |\varphi(\omega) K(\omega)^2| < \infty\) we conclude that \(G_n(\omega) \mu_\omega(\varphi)\) is integrable. Now, we would like to apply Lemma 4.2 but the problem is that \(G_n\) is not bounded. To overcome that, for each \(M > 0\) set \(G_n^{(M)}(\omega) = G_n(\omega) \psi(\Omega)\|G_n(\omega)\|_{L^1(\mu)} \leq M\). Then, since \(G_n(\omega) \mu_\omega(\varphi)\) is integrable, by the dominated convergence theorem we have
\[
\int \mu_\omega(\varphi) G_n(\omega) d\bar{P}(\omega) = \lim_{M \to \infty} \int \mu_\omega(\varphi) G_n^{(M)}(\omega) d\bar{P}(\omega).
\]
Now, taking \(n\) so that \(\psi_{[n/2]} \leq 1/2\) and using that \(\mu(\varphi) = 0\) we get from Lemma 4.2 that
\[
\left|\int \mu_\omega(\varphi) G_n^{(M)}(\omega) d\bar{P}(\omega)\right| \leq 2 \left(\int |G_n^{(M)}(\omega)\mu_\omega(\varphi)| d\bar{P}(\omega)\right) \psi_{[n/2]} \leq 2 \left(\int |G_n(\omega)\mu_\omega(\varphi)| d\bar{P}(\omega)\right) \psi_{[n/2]}.
\]
Using also (4.7) and that \( \text{esssup}_{\omega \in \Omega} (\sup |\varphi_\omega|K(\omega)^2) < \infty \), we conclude that
\[
\left| \int \mu_\omega(\varphi_\omega)G_n(\omega) d\mathbb{P}(\omega) \right| \leq 2 \left( \text{esssup}_{\omega \in \Omega} \left( K(\omega)^2 \| \varphi_\omega \|_{BV} \right) \right) C^\prime \|g\|_{L_1\psi(n/2)}
\]
and the (4.2) follows (using also (4.6)).

(ii) First, since \( \mathcal{K} \) weakly contracts the \( L^\infty \) norm (being defined through conditional expectations) and \( \varphi \) is bounded we have
\[
\|\mathcal{K}^j(\varphi \mathcal{K}^j \varphi) - \mu(\mathcal{K}^j(\varphi \mathcal{K}^j \varphi))\|_{L^\infty} \leq 2 \|\varphi\|_{L^\infty} \|\mathcal{K}^j \varphi\|_{L^\infty}.
\]
This provides the desired estimate when \( j \geq i \). The estimate in the case \( i > j \) is carried out similarly to the proof of (i). Let \( g \in L^1(\Omega \times X, \mu, \mathcal{F}_0) \). Let us first show that
\[
(4.8) \quad \int \mathcal{K}^i(\varphi \mathcal{K}^j \varphi) g d\mu = \int \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma_i \omega} \circ T^j_\omega)) \mu_{\sigma^{i+j}_\omega}(g_{\sigma^{i+j}_\omega}) d\mathbb{P}(\omega) + I
\]
where \( |I| \leq C_2 e^{-\lambda_1} \), and \( C_2 \) is some constant.

In order to prove (4.8), using that \( \mathcal{K} \) satisfies the duality relation and the disintegration \( \mu = \int \mu_\omega d\mathbb{P}(\omega) \) we first have
\[
\int \mathcal{K}^i(\varphi \mathcal{K}^j \varphi) g d\mu = \int (\varphi \mathcal{K}^j \varphi) \cdot g \circ \tau^i d\mu = \int \mathcal{K}^j \varphi \cdot (g \circ (g \circ \tau^i)) d\mu = \int (g \circ (\varphi \circ \tau^i)) \cdot g \circ \tau^{i+j} d\mu
\]
\[
(4.9) = \int \left( \int \varphi_\omega \cdot (\varphi_{\sigma_i \omega} \circ T^j_\omega) \cdot (g_{\sigma^{i+j}_\omega} \circ T^{i+j}_\omega) d\mu_\omega \right) d\mathbb{P}(\omega)
\]
Next, since \( L^\infty_{\omega}(f \circ T^j_\omega) = f \) for every function \( f \) and \( n \), we have
\[
L^{i+j}_{\sigma^i_\omega}(\varphi \circ (\varphi_{\sigma^i_\omega} \circ T^j_\omega)) = L^{i+j}_{\sigma^i_\omega}(\varphi_{\sigma^i_\omega} L^j_\omega \varphi).
\]
By (2.3) we have
\[
\|L^{i+j}_\omega \varphi - \mu_\omega(\varphi_\omega)\|_{BV} \leq K(\omega)\|\varphi_\omega\|_{BV} e^{-\lambda_j}
\]
In particular,
\[
\|L^{i+j}_\omega \varphi - \mu_\omega(\varphi_\omega)\|_{BV} \leq C K(\omega)\|\varphi_\omega\|_{BV}
\]
for some constant \( C \). Since \( \|u v\|_{BV} \leq C_0\|u\|_{BV}\|v\|_{BV} \) for every two functions \( u, v \) we have
\[
\|\varphi_{\sigma^i_\omega} L^{i+j}_\omega \varphi\|_{BV} \leq C_0 K(\omega)\|\varphi_{\sigma^i_\omega}\|_{BV}\|\varphi_{\sigma^i_\omega}\|_{BV}.
\]
Thus by (2.3),
\[
\|L^{i+j}_{\sigma^i_\omega}(\varphi_{\sigma^i_\omega} L^j_\omega \varphi) - \mu_{\sigma^i_\omega}(\varphi_{\sigma^i_\omega} L^j_\omega \varphi)\|_{BV} \leq C_0 C K(\omega)\|\varphi_{\sigma^i_\omega}\|_{BV}\|\varphi_{\sigma^i_\omega}\|_{BV} e^{-\lambda_j} \leq C_0 C \|\varphi\|_K^2 e^{-\lambda_i},
\]
where \( \|\varphi\|_K = \text{esssup}_{\omega \in \Omega} (K(\omega)^2 \| \varphi_\omega \|_{BV}) \). Observe next that
\[
\mu_{\sigma^i_\omega}(\varphi_{\sigma^i_\omega} L^j_\omega \varphi) = \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma^i_\omega} \circ T^j_\omega)).
\]
The desired inequality (4.8) follows from the above estimates.

Observe that the function \( \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma^i_\omega} \circ T^j_\omega)) \) depends only on \( \omega_k \) for \( k \leq j \) and that it is bounded by \( C K^{-2}(\omega) \) for some constant \( C > 0 \) (since \( \text{esssup}_{\omega \in \Omega} (K(\omega)^2 \| \varphi_\omega \|_{BV}) < \infty \)). Therefore, the same arguments in the proof of (i) yield that
\[
\int \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma^i_\omega} \circ T^j_\omega)) \mu_{\sigma^{i+j}_\omega}(g_{\sigma^{i+j}_\omega}) d\mathbb{P}(\omega) = \int \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma^i_\omega} \circ T^j_\omega)) d\mathbb{P}(\omega) \cdot \int \mu_{\sigma^{i+j}_\omega}(g_{\sigma^{i+j}_\omega}) d\mathbb{P}(\omega) + J
\]
where \( |J| \leq \gamma_i\|g\|_{L^1} \) and \( \gamma_i \) is one of the right hand sides on the upper bounds in (i) (depending on the case) with \( n \) replaced by \( i \). Notice next that
\[
\int \mu_\omega(\varphi_\omega \cdot (\varphi_{\sigma^{i+j}_\omega} \circ T^j_\omega)) d\mathbb{P}(\omega) = \int \mathcal{K}^i(\varphi \mathcal{K}^j \varphi) d\mu
\]
5.1. Lemma. \(\|L^T\| \) where 
\[
\left| \int (\mathcal{K}^i(\varphi \mathcal{K}^j \varphi) - \mu(\mathcal{K}^i(\varphi \mathcal{K}^j \varphi)) \, gd\mu \right| \leq C(e^{-\lambda t} + \gamma_i)\|g\|_{L^2}
\]
and the desired estimates follow again since \(L^\infty\) is the dual of \(L^1\).

\[\square\]

5. A VECTOR VALUED ANNEALED ASIP UNDER UNIFORM RANDOMNESS

Let us first explain why the matrix \(\Sigma^2\) exists. For a fixed vector \(v\) the limit \(\mathcal{S}_v^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[(S_n \cdot v)]^2\) exists, by considering the real-valued observable \(\varphi \cdot v\). Then the matrix \(\Sigma^2\) from Theorem \[2.12\] is given by \((\Sigma^2)_{i,j} = \frac{1}{2}(s_{e_i + e_j}^2 - s_{e_i}^2 - s_{e_j}^2)\). This matrix satisfies \(\Sigma^2 v \cdot v = s_v^2\) and so it is not positive definite if and only if \(\varphi \cdot v\) is a coboundary for some unire vector \(v\). Note that this part does not require \(T_\omega\) to be uniformly expanding.

We assume next that there exist constants \(C > 0\) and \(\delta \in (0, 1)\) so that for \(\mathbb{P}\) a.e. \(\omega\) we have
\[
(5.1) \quad \|L_n^\alpha - h_{\sigma_n\omega}\|_{BV} \leq C\delta^n.
\]

Theorem \[2.12\] follows from \[23\] Theorem 1.2] applied an arbitrary large \(p\).

5.1. Lemma. There exists \(\varepsilon_0 > 0, c, C > 0\) such that for any \(n, m > 0, b_1 < b_2 < ... < b_{n+m+1}, k > 0\) and \(t_1, ..., t_{n+m} \in \mathbb{R}^d\) with \(|t_j| \leq \varepsilon_0\) we have
\[
(5.2) \quad \mathbb{E}_\mu(e^{i \sum_{j=1}^{n} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i) + i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i)}) - \mathbb{E}_\mu(e^{i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i)}) \leq C^{n+m} e^{-ck},
\]
where \(B_\ell = \varphi \circ \tau^\ell\).

Proof. First, denoting by \(\mathbb{E}_\omega\) the expectation with respect to \(\mu_{\omega}\), by \[19\] Lemma 24] there are \(\varepsilon_0 > 0, c, C > 0\) with the property that for every \(n, m > 0, b_1 < b_2 < ... < b_{n+m+1}, k > 0\) and \(t_1, ..., t_{n+m} \in \mathbb{R}^d\) such that \(|t_j| \leq \varepsilon_0\),
\[
(5.3) \quad \mathbb{E}_\omega(e^{i \sum_{j=1}^{n} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i) + i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i) + i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i)}) - \mathbb{E}_\omega(e^{i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i)}) \leq C^{n+m} e^{-ck},
\]
where \(\mathbb{E}(g) = \int gh_{\omega} dm\) and \(A_\ell := \varphi_{\sigma^\ell\omega} \circ T_{\omega}^\ell, \quad \ell \in \mathbb{N}\).

Let
\[
G(\omega) = \mathbb{E}_\omega(e^{i \sum_{j=1}^{n} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i)})
\]
and
\[
F(\omega) = \mathbb{E}_\omega(e^{i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} A_i)}).
\]

Then with \(B_\ell = \varphi \circ \tau^\ell\) we have
\[
(5.4) \quad \mathbb{E}_\mu(e^{i \sum_{j=1}^{n} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i) + i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i)}) - \mathbb{E}_\mu(e^{i \sum_{j=n+1}^{m} t_j (\sum_{i=b_j}^{b_{j+1}-1} B_i)}) \leq C^{n+m} e^{-ck} + |\text{Cov}_\mathbb{P}(G, F)|.
\]
Using (5.1) and that \((T)_{\omega} \mu_\omega = \mu_{\sigma \omega}\) we get that there are \(k_0 \in \mathbb{Z}\) and functions \(G_1\) and \(F_1\) so that
\[
\|G(\omega) - G_1(\ldots, \omega_{k_0-1}, \omega_{k_0+[k/4]}])\|_\infty \leq C' \delta^{k/4}
\]
and
\[
\|G(\omega) - G_1(\omega_{k_0+k-[k/4]}, \omega_{k_0+k-[k/4]+1}, \ldots)\|_\infty \leq C' \delta^{k/4}.
\]
Thus,
\[
|\text{Cov}_\nu(G, F)| \leq |\text{Cov}_\nu(G_1, F_1)| + C'' \delta^{k/4}
\]
where we have used that \(G_1, G_2, G\) and \(F\) are uniformly bounded (so the above constants \(C', C''\) do not depend on the choice of \(b_j, t_j\) etc.). On the other hand, by (3.7),
\[
|\text{Cov}_\nu(G_1, F_1)| \leq C''\alpha k/2.
\]
Thus,
\[
\left| \mathbb{E}_\mu \left( e^{i \sum_{j=1}^{n} t_j \left( \sum_{k=j}^{j+1} B_k \right) + i \sum_{j=n+1}^{n+m} t_j \left( \sum_{k=j}^{j+k} B_k \right) } \right) \right| \leq C^{n+m} e^{-ck} + C'' \delta^{k/4} + C''\alpha k/2.
\]

6. Several extensions and generalizations and additional results, a short discussion

In this section we will describe a few additional results which we believe can also be obtained using the methods of the current paper. In order not to overload the paper the section is presented in a form of a discussion rather than explicit formulations of theorems.

6.1. More general mixing base maps for continuous in \(\omega\) transfer operators, a short discussion. Let \((\xi_n)_{n \in \mathbb{Z}}\) be a stationary process taking values on a metric space \((\mathcal{Y}, d)\) satisfying the following approximation and mixing conditions:

There are subsigma algebras \(\mathcal{G}_{n,m}\) on the underlying probability space so that \(\mathcal{G}_{n,m} \subset \mathcal{G}_{n_1,m_1}\) if \([n, m] \subset [n_1, m_1]\) and for each \(r\) and \(n\) there is an \(\mathcal{G}_{n-r,n+r}\) measurable random variable \(\xi_{n,r}\) so that:

1. approximation: \(\|d(\xi_n, \xi_{n,r})\| \leq A_1 \beta^r, \beta \in (0, 1)\)
2. mixing: the sequences \((\xi_{2nr,r})_{n \in \mathbb{Z}}\) are \(\alpha\) (or \(\phi_R\) or \(\varphi\)) mixing uniformly in \(r\).

We note that the above uniform approximation by \(\alpha\)-mixing sequences applies to Young towers, when \(\alpha_n = O(n^{-(p-2)})\) is the tails of the tower are \(O(n^{-p})\) for some \(p \geq 3\). We can also take several classes of smooth maps on the interval or Gibbs-Markov maps \([1]\) for which such an approximation holds with \(\psi_n = O(\delta^\alpha)\) for some \(\delta \in (0, 1)\).

Let \((\Omega, \mathcal{F}, \mathbb{P}, \sigma)\) be the shift systems constructed as before. Then all the results stated in the paper hold true when \(\omega \to \mathcal{L}_\omega\) and \(\omega \to \varphi_\omega\) are Hölder continuous in \(\omega\) (on a set with probability 1). The main point is that Lemma 3.7 and the similar approximations used in the construction of the martingale (i.e. in the proof of Lemma 5.3. The main reason we did not include such results in the body of the paper is that it would make the notations more complicated, and that the additional global regularity assumptions on the transfer operators are relatively restrictive.

6.2. Extension to Gibbs states. Let us consider now the random expanding maps as in \([43]\), under the additional assumption that the random subspaces \(\mathcal{E}_\omega \subset X\) so that \(T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\sigma \omega}\) satisfy \(\mathcal{E}_\omega = X\). Let \(\nu_\omega = h_\omega \nu_\omega\) be a random Gibbs measure corresponding to a given random logarithmically \(\alpha\)-Hölder continuous potential, and let \(\lambda_\omega\) be the logarithm of the random pressure. Namely, if \(\mathcal{L}_\omega\) is the transfer operator corresponding to the random potential, then
\[
\mathcal{L}_\omega h_\omega = \lambda_\omega h_{\sigma \omega}, (\mathcal{L}_\omega)^* \nu_\omega = \lambda_\omega \nu_\omega.
\]
We will explain below how to obtain the results discussed in the paper for these maps in the uniformly expanding case. In fact, we believe that an extension to the expanding on the average case is also possible using Oseledets theorem, but it seems that there is a small hole in the argument, and so we prefer not to fully claim that the non-uniformly random expanding case is also covered.

First, as discussed in [19] Remark 11 under the conditions of an appropriate version of Oseledets theorem, there is a tempered random variable $K(\omega)$ so with $\hat{L}_\omega = \mathcal{L}_\omega / \lambda_\omega$ we have
\[
\| \hat{L}_\omega^n - \nu_\omega \otimes h_{\nu_\omega} \|_{\text{Holder}} \leq K(\omega) e^{-\lambda n}
\]
where $\| \cdot \|_{\text{Holder}}$ is the usual Hölder norm corresponding to the exponent $\alpha$ and $\nu \otimes h(g) = \nu(g) h$. Plugging in $g = 1$ we get the same estimate we had in (2.4)
\[
\| \hat{L}_\omega^n - h_{\nu_\omega} \|_{\text{Holder}} \leq K(\omega) e^{-\lambda n}
\]
Now, arguing as in [19] we get the second estimate (2.3). We remark that for uniformly expanding maps there is no need to use Oseledets theorem, and instead we can use the explicit limiting expressions of $h_\omega$ as in [25] Ch. 4-5), and in the non-uniform case we believe that this might be obtained by an appropriate application of the Oseledets theorem.

6.3. Extension to nonconventional sums (multiple recurrences). Let us consider partial “nonconventional” sums of the form
\[
S_n \varphi = \sum_{m=1}^{n} \prod_{j=1}^{\ell} \varphi \circ \tau_{q_j(m)}
\]
where $\ell$ is an integer and $q_j(n)$ are positive integer valued sequences. The statistical properties of such sums were extensively studied for several classes of expanding or hyperbolic maps (in particular), see [40] [41] [25] and references therein. When all $q_j$’s are polynomials, we believe that all the results obtained using the method of cumulants (i.e. Theorems 2.4, 2.5, 2.6 and an appropriate version of Theorem 2.8) can be obtained for such sums exactly as in [25], relying on Proposition 3.2 applied with $\rho(n,m) = \max_{1 \leq i,j \leq \ell} |q_i(m) - q_j(n)|$. The main idea is that by induction on the number of blocks we can show that the conditions of Proposition 3.2 hold true for
\[
X_m = \prod_{j=1}^{\ell} \varphi \circ \tau_{q_j(m)}.
\]
That is, by an inductive argument similar to the one in [25] Corollary 1.3.11], we can prove the following result.

6.1. Lemma. Let $r \in \mathbb{N}$ and let let $B_1, B_2, ..., B_k$ be finite subsets of $\mathbb{N}$ so that the distance between $B_j$ and $B_{j+1}$ is $d_j$. Set $r_j = [d_j / 3]$. Let $C = \{ C_j : 1 \leq j \leq s \}$ be a partition of $\{ 1, 2, ..., k \}$ are set $Y_j = \prod_{k \in C_j} \prod_{\nu \in B_k} \varphi \circ \tau^\nu$. Then, assuming that $\| \varphi \|_{L^\infty} \leq 1$ and that $\text{esssup}_{\omega \in \Omega} (K(\omega) \| \varphi_\omega \|_{BV}) \leq 1$, there is an absolute constant $A > 1$ so that
\[
\left| \mathbb{E}_\mu \left[ \prod_{j=1}^{s} Y_j \right] - \prod_{j=1}^{s} \mathbb{E}_\mu [ Y_j ] \right| \leq A^m \sum_{j=1}^{m} (\delta^{r_j} + \alpha([r_j]))
\]
where $\delta = e^{-(\lambda - 3\epsilon) / 2} \in (0, 1)$. 
We note that in order to prove a version of the functional CLT for the sums above we first need to use the arguments in [11, 24] to compute the variance of the limiting Gaussian, which for general polynomials might differ from a Brownian motion, and this can also be done by using the above lemma.

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