Graviton Production by a Thermal Bath

Dario Grasso
Scuola Normale Superiore, P.zza Dei Cavalieri 7, I-56126 PISA, Italy

Abstract

Thermal fluctuations in the early universe plasma and in very hot astrophysical objects are an unavoidable source of gravitational waves (GW). Differently from previous studies on the subject, we approach this problem using methods based on field theory at finite temperature. Such an approach allows to probe the infrared region of the spectrum where dissipative effects are dominant. Incidentally, this region is the most interesting from the point of view of the detectability perspectives. We find significant deviations from a Planck spectrum.

1. Introduction

It is well known that thermal collisions in a plasma give rise to production of gravitational radiation [?]. Graviton emission is a consequence of particle acceleration in the scattering process (gravitational Bremsstrahlung). The emitted power of a non-relativistic gas was determined by Weinberg [?] by summing up the radiated energies per collision under the condition that the wave frequency is much larger than the mean collision frequency ($\omega \gg \omega_c$). This is required for the waves do not interfere. As was noted in [?], for $\omega \ll \omega_c$ the gas behaves as a fluid rather than as a collection of independent particles. The low frequency limit is the most interesting from the point of view of GW detection perspective. In this contribution we address the problem of computing the gravitational emission rate by a relativistic plasma in the low frequency limit. We focus on the contribution to the emission process of collective excitations of the plasma. For definiteness we consider here the case of a QED plasma.

2. Collective excitations in a relativistic plasma

Fluctuations of electric and magnetic fields in a heat bath are related to the dispersive properties of the medium by the fluctuation-dissipation theorem [?, ?]. In a field theory language this theorem states that the photon spectral function $\langle A_\mu A_\nu \rangle (q)$ is related to the retarded and advanced Green’s functions through

$$\langle A_\mu A_\nu \rangle (q) \equiv i(1 - e^{iq0})^{-1} \left[ D_{\mu\nu}(q_0 + i\epsilon, q) - D^*_{\mu\nu}(q_0 + i\epsilon, q) \right], \quad (1)$$
where \( q \) is the photon 4-momentum and \( \beta = 1/T \) is the inverse temperature. In the Feynman gauge this quantity can be decomposed into transversal and longitudinal components \( A_{\mu\nu} = -P_{\mu\nu}A_T - Q_{\mu\nu}A_L \) where

\[
A_{T,L}(q) = -\frac{1}{\pi} \frac{\text{Im} \Pi_{T,L}}{|q^2 - \text{Re} \Pi_{T,L}|^2 + |\text{Im} \Pi_{T,L}|^2} .
\]

The expressions of the transversal (longitudinal) components, \( \Pi_{T(L)} \), of the polarization tensor, as well as the definitions of the projectors \( P_{\mu\nu} \) and \( Q_{\mu\nu} \), are given in [?]). The dispersion relation of transverse (longitudinal) modes is given by \( q^2 + q^2 = \text{Re} \Pi_{T(L)}(q) \). At small momenta \( \Pi_{T(L)}(q) \) is of the order of the plasma frequency which for a relativistic plasma is \( \omega_p \simeq eT/3 \). The propagation of photons is strongly altered for \( q_0, q < \omega_p \); in this case they behave as quasiparticles endowed with an effective mass of order \( eT \). What is more relevant in the present context is that electromagnetic plasma fluctuations below the light cone (\( q_0 < |q| \)) are subject to Landau damping. Graviton emission by electromagnetic excitations take place because of the damping. The effect is analogous to axion production due to the Primakoff effect (see e.g. [?]) and the results we present here can be derived following an approach similar to that used to determine the axion production rate in hot stars.

Using Eqs.\( (1,2) \) one finds [?]

\[
\langle B^2 \rangle_q = \frac{2\pi}{1 - e^{-\beta q_0}} 2q^2 A_T(q) ,
\]

\[
\langle E^2 \rangle_q = \frac{2\pi}{1 - e^{-\beta q_0}} \left[ 2q_0^2 A_T(q) + q^2 A_L(q) \right] .
\]

In the following we will focus on magnetic fluctuations since electric fields are screened in the static limit. Interestingly, it was showed by the authors of Refs.[?, ?] that a large peak is present in the magnetic fluctuations spectrum at nearly zero-frequency. This can be interpreted as “squeezing” of energy toward \( q_0 = 0 \) because of the non-linear coupling at small momenta. The effect give rise to a random quasi-static magnetic field with root-mean-square energy at the scale \( l \) given by

\[
B_t \simeq B_0 \left( \frac{l_p}{l} \right)^{3/2} , \quad B_0 = \left( \frac{T \omega_p^3}{32\pi^9/2} \right)^{1/2} ,
\]

where \( l_p \equiv 2\pi/\omega_p \).

3. Gravitational waves production

It is well known that stochastic magnetic fields can power gravitational waves production [?]. Magnetic anisotropic stresses

\[
a_{ij}^{(B)}(k) = \frac{1}{4\pi} \int d^3q \left( B_i(q)B_j^*(k - q) - \frac{1}{2} B_i(q)B^l*(k - q)\delta_{ij} \right)
\]
act as a source term in the time evolution equation for the spatial components of the metric perturbations. In the Fourier space this equation can be written
\[ \ddot{h}_{ij} + k^2 h_{ij} = 8\pi G \Pi_{ij}^{(B)}, \]
where \( \Pi_{ij}^{(B)} = \left( P_i^a P_j^b - \frac{1}{2} P^a P_{ab} \right) \tau_{ab}^{(B)} \) is the tensor component of \( \tau_{ij}^{(B)} \). Due to the stochastic nature of thermal fluctuations, hence also of the GW background they give rise to, the relevant quantity we are concerned here is the GW spectral function \( \langle h^{ij}(k, t), h_{ij}(k', t) \rangle \). Starting from the general solution of [3] \( h_{ij}(k, t) = \frac{8\pi G}{k} \int_t^{\infty} dt' \Theta(t - t') \sin[k(t - t')]\Pi_{ij}(k', t') \) and replacing ensemble average with time average, one finds [4, 5, 6]
\[ \langle h_{ij}(k, t) h_{ij}^*(k', t) \rangle \simeq \frac{9 \sqrt{2} (16\pi G)^2 \Delta t_s}{16 k^3} \xi^3(k - k') \int d^3 q B^2(q) B^2(|k - q|) \]
where \( \Delta t_s \) is the source duration. Assuming a power-law spectrum \( B^2(k) = A k^n \) for the fluctuations, it can be showed [7, 8] that whenever \( n > -3/2 \) the convolution is dominated by the frequency independent term \( \int d^3 q B^2(q) B^2(|k - q|) \simeq 4\pi A^2 \frac{k^{2n+3}}{2n + 3} \). This give rise to a white-noise GW spectrum. In the previous section we showed that the energy density in quasi-static magnetic fluctuations scales with the scale \( l \) as \( B^2 \propto l^{-3} \). Since \( B^2 \sim B^2(k)k^3|_{k=1/l} \propto l^{-(n+3)} \) it follows that \( n = 0 \) in this case, which fulfills the \( n > -3/2 \) condition. It was recently claimed by the authors of [9] that \( n = 0 \) is unphysical and it should be replaced by \( n = 2 \). Such substitution would not affect our results. In our case \( k_{\text{max}} \) has to be identified with the plasma frequency \( \omega_p \) and \( k_{\text{min}} \) (see below) with the Hubble expansion rate \( H(T) \).

The real-space correlator is given by
\[ \langle h_{ij}(x, t) h^{ij}(x, t) \rangle = \frac{V^2}{(2\pi)^6} \int d^3 k d^3 k' e^{i(k' - k) \cdot x} \langle h_{ij}(k, t) h_{ij}^*(k', t) \rangle. \]

The power-spectrum normalization requires \( A = \frac{B_0^2 \pi^2}{8\pi V} (n + 3) \omega_p^{-(n+3)} \). Then, for \( n = 0 \) we find
\[ \langle h_{ij}(x, t) h_{ij}(x, t) \rangle = \frac{81 \sqrt{2}}{32} G^2 \Delta t B_0^4 \omega_p^{-3} \xi^3(k - k') \int_{H}^{\omega_p/2\pi} \frac{dk}{k} \]

The GW power spectrum is usually parametrized in terms of the characteristic amplitude \( h_c(f) \) of GWs at frequency \( f \), defined by [10] \( \langle h_{ij}(x, t) h_{ij}(x, t) \rangle \equiv 2 \int_0^\infty df h_c^2(f, t) \). The characteristic amplitude measured today at a frequency \( f \) is
\[ h_c(k_0, t_0) = \frac{a}{a_0} h_c(k = \frac{a_0}{a} k_0, t) \simeq 10^{-16} \left( \frac{100}{g_*} \right)^{1/3} \left( \frac{T_{\text{eV}}}{T} \right). \]
It is natural to assume the emission time interval to be \( \Delta t \sim H^{-1} = g_s^{-1/2} \frac{M_P}{T^2} \), where \( M_P \) is the Planck mass and \( T \) the emission temperature. Disregarding the weak dependence on the number of relativistic d.o.f. \( g_s \) we finally find

\[
h_c(f, t_0) \simeq 10^{-24} \kappa^3 \left( \frac{T}{M_P} \right)^3,
\]

where we parametrized the plasma frequency dependence on \( T \) as \( \omega_p = \kappa T \). For QED \( \kappa \simeq 0.1 \) but it can be larger for GUT theories. Furthermore the larger number of gauge bosons predicted by these theories may very well give rise to more than a factor 10 enhancement in the predicted value of \( h_c \). The minimal wavenumber today corresponds to the red-shifted value of \( H(T) \). In term of frequency it is \( f_{\text{min}} \simeq 10^{-4} \text{ Hz} \left( \frac{T}{\text{TeV}} \right) \). It is remarkable that the expected GW signal may be in the detectable range of LISA [?] if gravity becomes strong \( (M_P \sim \text{TeV}) \) at the TeV scale (A specific model should be adopted here. If, for example, modification of gravity take place because of the presence of large extra-dimensions, the role of gravitons escaping in the bulk should be taken into account).

Independently on these speculations, our main result is to have shown that collective excitations in a relativistic plasma give rise to a GW background with a frequency independent spectrum. At low frequencies this background overwhelm that produced by the conventional gravitational bremsstrahlung.

Acknowledgments

The author would like to thank D. Comelli, R. Durrer, T. Kahniashvili, A. Nicolis and M. Pietroni for valuable discussions.

References

[1] Altherr T. 1991, Annals of Phys. 207, 374.
[2] Caprini C. and Durrer R. 2002, Phys. Rev. D65, 023517.
[3] Dolgov D.D., Grasso, D. and Nicolis A. 2002, Phys. Rev. D66, 103505.
[4] Durrer R., Ferreira P.G. and Kahniashvili T. 2000, Phys. Rev. D61, 043001.
[5] Kosowsky A.,Mack A. and Kahniashvili T. 2002, Phys. Rev. D66, 024030.
[6] Lemoine D. 1995, Phys. Rev. D51, 2677.
[7] LISA, Laser Interferometer Space Antenna, [http://lisa.jpl.nasa.gov/](http://lisa.jpl.nasa.gov/).
[8] Maggiore M. 2000, Phys. Rept. 331, 283.
[9] Tajima T. et al., 1992, Astrop. J., 390, 309.

[10] Weinberg S. 1965, Phys. Rev. 140, B516.

[11] Weinberg S, *Gravitation and Cosmology*, Wiley, 1972.