DUAL \( \phi \)-DIVERGENCES ESTIMATION IN NORMAL MODELS

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Abstract. A class of robust estimators which are obtained from dual representation of \( \phi \)-divergences, are studied empirically for the normal location model. Members of this class of estimators are compared, and it is found that they are efficient at the true model and offer an attractive alternative to the maximum likelihood, in term of robustness.

Key words and phrases: Minimum divergence estimators; Efficiency; Robustness; \( M \)-estimators; Influence function.

1. Introduction

Divergences of probability distributions are widely used in a number of theoretical and applied statistical inference, they are also of key importance in data processing problems, see Basseville (2010). The \( \phi \)-divergence modeling has proved to be a flexible and provided a powerful statistical modeling framework in a variety of applied and theoretical contexts, see Liese and Vajda (2006, 1987), Pardo (2006), Broniatowski and Keziou (2009) and the recent monograph by Basu et al. (2011).

Recently introduced, the minimum divergences estimation method based on a dual representation of the divergence between probability measures, is an appealing estimation method, it avoids the use of non-parametric density estimation and the complications related to the bandwidth selection. The estimators are defined in an unified way for both continuous and discrete models. They do not require any prior smoothing and include the classical maximum likelihood estimators as a benchmark. These estimators called “dual \( \phi \)-divergence estimators” (Do\( \phi \)DE’s), was shown by Keziou (2003) and Broniatowski and Keziou (2009), under suitable conditions, to be consistent, asymptotically normal and asymptotically full efficient at the true model.
Application of dual representation of $\phi$-divergences have been considered by many authors, we cite among others, Keziou and Leoni-Aubin (2008) for semi-parametric two-sample density ratio models, robust tests based on saddlepoint approximations in Toma and Leoni-Aubin (2010), bootstrapped $\phi$-divergences estimates are considered in Bouzebda and Cherfi (2011), extension of dual $\phi$-divergences estimators to right censored data are introduced in Cherfi (2011), for estimation and tests in copula models we refer to Bouzebda and Keziou (2010) and the references therein.

The $\phi$-divergences estimators are motivated by the fact that a suitable choice of the divergence may lead to an estimate more robust than the MLE one, see Jiménez and Shao (2001). Toma and Broniatowski (2010) studied the robustness of the D$\phi$DE’s through the influence function approach. However the fact that the D$\phi$DE’s have unbounded influence functions in the normal location model, causes some concern that these estimators are non-robust. In this article we show that the $\epsilon$-influence functions are more suitable to capture the robustness properties of the D$\phi$DE’s. We also give some insights about the choice of the escort parameter. Simulation results indicate that the D$\phi$DE’s attain the dual goal of robustness and efficiency, they are competitive with Huber $M$-estimator without the loss of efficiency at the true model.

The rest of the paper is organized as follows. In Section 2 we provide background material concerning the dual $\phi$-divergences estimators. Section 3 is devoted to the choice of the escort parameter. Section 4 deals with robustness. In section 5, we illustrate the performance of the method in real data example. A simulation study described in Section 6 investigates the asymptotic properties of the estimators. Section 7 contains some concluding remarks.

2. Background and estimators definition

The class of dual divergences estimators has been recently introduced by Keziou (2003), Broniatowski and Keziou (2009). In the following, we shortly recall their context and definition.

Recall that the $\phi$-divergence between a bounded signed measure $Q$ and a probability $P$ on $\mathcal{D}$, when $Q$ is absolutely continuous with respect to $P$, is defined by

$$D_{\phi}(Q, P) := \int_{\mathcal{D}} \phi \left( \frac{dQ}{dP}(x) \right) dP(x),$$

where $\phi$ is a convex function from $]-\infty, \infty[ \to [0, \infty]$ with $\phi(1) = 0$.

Well-known examples of divergences are the Kullback-Leibler associated to the function $\phi(x) = x \log x - x + 1$, modified Kullback-Leibler for
\( \phi(x) = -\log x + x - 1 \), the \( \chi^2 \)-divergence with \( \phi(x) = \frac{1}{2}(x-1)^2 \) and the Hellinger distance given by \( \phi(x) = 2(\sqrt{x} - 1)^2 \). All these divergences belong to the class of the so called “power divergences” introduced in Cressie and Read (1984) (see also Liese and Vajda (1987) chapter 2). They are defined through the class of convex functions

\[
 x \in [0, +\infty[ \mapsto \phi_\gamma(x) := \frac{x^\gamma - \gamma x + \gamma - 1}{\gamma(\gamma - 1)} \quad (1)
\]

if \( \gamma \in \mathbb{R} \setminus \{0, 1\} \), \( \phi_0(x) := -\log x + x - 1 \) and \( \phi_1(x) := x \log x - x + 1 \). (For all \( \gamma \in \mathbb{R} \), we define \( \phi_\gamma(0) := \lim_{x \downarrow 0} \phi_\gamma(x) \)). So, the KL-divergence is associated to \( \phi_1 \), the \( KL_m \) to \( \phi_0 \), the \( \chi^2 \) to \( \phi_2 \) and the Hellinger distance to \( \phi_1/2 \). We refer to Liese and Vajda (1987) for an overview on the origin of the concept of divergences in statistics. The reader interested in other class of \( \phi \)-divergences can refer to the recent paper of Kus et al. (2008), which propose a simple method of construction of new families of \( \phi \)-divergences.

Let \( X_1, \ldots, X_n \) be an i.i.d. sample with p.m. \( P_{\theta_0} \). Consider the problem of estimating the population parameters of interest \( \theta_0 \) when the underlying identifiable model is given by \( \mathcal{P} = \{ P_\theta : \theta \in \Theta \} \) with \( \Theta \) a subset of \( \mathbb{R}^d \). We denote \( \lambda \) a dominating measure for \( \mathcal{P} \) and the resulting density of any \( P_\theta \) is denoted \( p_\theta \).

Let \( \phi \) be a function of class \( C^2 \), strictly convex and satisfies

\[
 \int \left| \phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) \right| dP_\theta(x) < \infty. \quad (2)
\]

By Lemma 3.2 in Broniatowski and Keziou (2006), if the function \( \phi \) satisfies: There exists \( 0 < \eta < 1 \) such that for all c in \( [1 - \eta, 1 + \eta] \), we can find numbers \( c_1, c_2, c_3 \) such that

\[
 \phi(cx) \leq c_1 \phi(x) + c_2 |x| + c_3, \quad \text{for all real } x, \quad (3)
\]

then the assumption (2) is satisfied whenever \( D_\phi(P_\theta, P_\alpha) \) is finite. From now on, we suppose that there exists a neighborhood \( U \) of \( \theta_0 \) for which \( D_\phi(P_\theta, P_\alpha) < \infty \) whatever \( \theta \) and \( \alpha \) in \( U \). Note that all the real convex functions \( \phi_\gamma \) pertaining to the class of power divergences defined in (1) satisfy the condition (3).

Under the above conditions, the \( \phi \)-divergence:

\[
 D_\phi(P_\theta, P_{\theta_0}) = \int \phi \left( \frac{p_\theta(x)}{p_{\theta_0}(x)} \right) dP_{\theta_0},
\]

can be represented as the following form:

\[
 D_\phi(P_\theta, P_{\theta_0}) = \sup_{\alpha \in U} \int h(\theta, \alpha) dP_{\theta_0}, \quad (4)
\]
where \( h(\theta, \alpha) : x \mapsto h(\theta, \alpha, x) \) and
\[
h(\theta, \alpha, x) := \int \phi' \left( \frac{p_{\theta}}{p_{\alpha}} \right) dP_{\theta} - \left[ \frac{p_{\theta}(x)}{p_{\alpha}(x)} \phi' \left( \frac{p_{\theta}}{p_{\alpha}} \right) - \phi \left( \frac{p_{\theta}}{p_{\alpha}} \right) \right]. \tag{5}
\]
Since the supremum in (4) is unique and is attained in \( \alpha = \theta_0 \), independently upon the value of \( \theta \), define the class of estimators of \( \theta_0 \) by
\[
\hat{\alpha}_{\phi}(\theta) := \arg \sup_{\alpha \in U} \int h(\theta, \alpha) dP_n, \quad \theta \in \Theta, \tag{6}
\]
where \( h(\theta, \alpha) \) is the function defined in (5). This class is called “dual \( \phi \)-divergences estimators” (D\( \phi \)DE’s).
The corresponding estimating equation for the unknown parameter is then given by
\[
\int \frac{\partial}{\partial \alpha} h(\theta, \alpha) \ dP_n = 0. \tag{7}
\]
Remark that the maximum likelihood estimate belongs to the class of estimates (6). Indeed, it is obtained when \( \phi(x) = -\log x + x - 1 \), that is as the dual modified KL-divergence estimate. Observe that
\[
\int h(\theta, \alpha) dP_n = - \int \log \left( \frac{p_{\theta}}{p_{\alpha}} \right) dP_n.
\]
Hence keeping in mind definitions (6), we get
\[
\hat{\alpha}_{KL_m}(\theta) = \arg \sup_{\alpha \in \Theta} - \int \log \left( \frac{p_{\theta}}{p_{\alpha}} \right) dP_n = \arg \sup_{\alpha \in \Theta} \int \log(p_{\alpha}) dP_n = MLE,
\]
independently upon \( \theta \).
Formula (4) defines a family of M-estimators indexed by some instrumental value of the parameter \( \theta \) and by the function \( \phi \) defining the divergence. In the sequel we call \( \theta \) the “escort parameter”, the choice of \( \theta \) appears as a major feature in the estimation procedure, see Section 3 below.
Recall that an M-estimator of \( \psi \)-type is the solution of the vector equation:
\[
\int \psi(x; \alpha) \ dP_n = 0, \tag{8}
\]
where the elements of \( \psi(x; \alpha) \) represent the partial derivatives of \( h(\theta, \alpha, x) \) with respect to the components of \( \alpha \). For more details about M-estimators we may refer to Huber (1981), Hampel et al. (1986), Maronna et al. (2006) and the references therein.
We apply the dual representation of \( \phi \)-divergences (4), which we specialize to the present setting. Consider now, the normal model with density:

\[
p_{\theta, \sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( \frac{x - \theta}{\sigma} \right)^2 \right\},
\]

(9)

and the power divergences family (1). Observe that, for \( \gamma \in \mathbb{R} \setminus \{0, 1\} \),

\[
\frac{1}{\gamma - 1} \int \left( \frac{p_{\theta, \sigma}(x)}{p_{\alpha, \tilde{\sigma}}(x)} \right)^{\gamma - 1} p_{\theta, \sigma}(x) \, dx = \frac{1}{\gamma - 1} \frac{\tilde{\sigma}^\gamma \sigma^{-(\gamma - 1)}}{\sqrt{\tilde{\sigma}^2 - (\gamma - 1)\sigma^2}} \exp \left\{ \frac{\gamma(\gamma - 1)(\theta - \alpha)^2}{2(\gamma \tilde{\sigma}^2 - (\gamma - 1)\sigma^2)} \right\}.
\]

Hence,

\[
D_{\gamma}(P_{\theta, \sigma}, P_n) = \sup_{\alpha, \tilde{\sigma}} \left\{ \frac{1}{\gamma - 1} \frac{\tilde{\sigma}^\gamma \sigma^{-(\gamma - 1)}}{\sqrt{\tilde{\sigma}^2 - (\gamma - 1)\sigma^2}} \exp \left\{ \frac{\gamma(\gamma - 1)(\theta - \alpha)^2}{2(\gamma \tilde{\sigma}^2 - (\gamma - 1)\sigma^2)} \right\} - \frac{1}{\gamma \sigma} \sum_{i=1}^n \left( \frac{\tilde{\sigma}}{\sigma} \right)^\gamma \exp \left\{ -\frac{1}{2} \left( \frac{X_i - \theta}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{X_i - \alpha}{\sigma} \right)^2 \right\} \right\}.
\]

For \( \gamma = 0 \),

\[
D_{KL_m}(P_{\theta, \sigma}, P_n) = \sup_{\alpha, \tilde{\sigma}} \left\{ \frac{1}{2n} \sum_{i=1}^n \left( \frac{X_i - \theta}{\sigma} \right)^2 - \left( \frac{X_i - \alpha}{\sigma} \right)^2 \right\} - \log \left( \frac{\tilde{\sigma}}{\sigma} \right).
\]

For \( \gamma = 1 \),

\[
D_{KL}(P_{\theta, \sigma}, P_n) = \sup_{\alpha, \tilde{\sigma}} \left\{ \frac{1}{2} \left( 1 - \left( \frac{\tilde{\sigma}}{\sigma} \right)^2 - \left( \frac{\theta - \alpha}{\tilde{\sigma}} \right)^2 \right) - \log \left( \frac{\tilde{\sigma}}{\sigma} \right) - \frac{1}{n} \sum_{i=1}^n \exp \left\{ -\frac{1}{2} \left( \frac{X_i - \theta}{\sigma} \right)^2 - \frac{1}{2} \left( \frac{X_i - \alpha}{\sigma} \right)^2 \right\} + 1 \right\}.
\]

For the normal family \( P_{\theta} \equiv \mathcal{N}(\theta, 1) \), with the location parameter \( \theta \) and scale \( \sigma = 1 \). It follows that, for \( \gamma \in \mathbb{R} \setminus \{0, 1\} \),

\[
D_{\gamma}(P_{\theta}, P_n) := \sup_{\alpha} \int h(\theta, \alpha) \, dP_n
\]

\[
= \sup_{\alpha} \left\{ \frac{1}{\gamma - 1} \exp \left\{ \frac{\gamma(\gamma - 1)(\theta - \alpha)^2}{2} \right\} - \frac{1}{\gamma \sigma} \sum_{i=1}^n \exp \left\{ -\frac{\gamma}{2}(\theta - \alpha)(\theta + \alpha - 2X_i) \right\} - \frac{1}{\gamma(\gamma - 1)} \right\}.
\]
For $\gamma = 0$,
\[
D_{KL_{m}}(P_\theta, P_n) := \sup_\alpha \int h(\theta, \alpha) \, dP_n = \sup_\alpha \left\{ \frac{1}{2n} \sum_{i=1}^{n} (\theta - \alpha)(\theta - \alpha - 2X_i) \right\}.
\]

For $\gamma = 1$,
\[
D_{KL}(P_\theta, P_n) := \sup_\alpha \int h(\theta, \alpha) \, dP_n = \sup_\alpha \left\{ -\frac{1}{2} (\theta - \alpha)^2 - \frac{1}{n} \sum_{i=1}^{n} \exp \left\{ -\frac{1}{2} (\theta - \alpha)(\theta + \alpha - 2X_i) \right\} + 1 \right\}.
\]

We remark that the above optimizations are a feasible computationally closed-form expressions and can be performed by any standard non linear optimization code.

3. On the choice of the escort parameter

The very peculiar choice of the escort parameter defined through $\theta = \theta_0$ has same limit properties as the MLE. The DΦDE $\hat{\alpha}_\phi(\theta_0)$, in this case, has variance which indeed coincides with the MLE, see for instance (Keziou, 2003, Theorem 2.2, (1) (b)). This result is of some relevance, since it leaves open the choice of the divergence, while keeping good asymptotic properties. For data generated from the distribution $N(0, 1)$, Figure 1 shows that the global maximum of the empirical criterion $P_n h(\hat{\theta}_n, \alpha)$ is zero, independently of the value of the escort parameter $\hat{\theta}_n$ (the sample mean $\bar{X}$ in Figure 1(a) and the median in Figure 1(b)) for all the considered divergences.

Unlike the case of data without contamination, the choice of the escort parameter is crucial in the estimation method in the presence of outliers. We plot in Figure 2 the empirical criterion $P_n h(\hat{\theta}_n, \alpha)$, where the data are generated from the distribution
\[
(1 - \epsilon)N(\theta_0, 1) + \epsilon \delta_{10},
\]
where $\epsilon = 0.1$ and $\theta_0 = 0$. Figure 2(a) illustrates the empirical criterion under contamination, when we take the empirical mean $\bar{X}$ as the value of the escort parameter $\theta$, it shows how the global maximum of the empirical criterion $P_n h(\hat{\theta}_n, \alpha)$ shifts from zero to the contamination point. In Figure 2(b), the choice of the median as escort parameter value leads to the position of the global maximum remains close to
Figure 1. Criterion for the normal location model.

Figure 2. Criterion for the normal location model under contamination.
\(\alpha = 0\) for Hellinger (\(\gamma = 0.5\)), \(\chi^2\) (\(\gamma = 2\)) and \(KL\)-divergence (\(\gamma = 1\)), while the criterion associated to the \(KL_m\)-divergence (\(\gamma = 0\), the maximum is the MLE) stills affected by the presence of outliers.

In practice, the consequence is that the escort parameter should be chosen as a robust estimator of \(\theta_0\), say \(\hat{\theta}_n\).

Observe that for the power divergences family (1), the estimating equation (7) reduces to

\[-\int \left( \frac{p_{\theta}(x)}{p_{\alpha}(x)} \right)^{\gamma-1} \frac{\hat{p}_\alpha(x)}{p_{\alpha}(x)} p_{\theta}(x) \, dx + \frac{1}{n} \sum_{i=1}^{n} \left( \frac{p_{\theta}(X_i)}{p_{\alpha}(X_i)} \right)^{\gamma} \hat{p}_\alpha(X_i) \frac{p_{\theta}(X_i)}{p_{\alpha}(X_i)} = 0, \quad (10)\]

and the estimate \(\hat{\alpha}_{\phi}(\theta)\) is the solution in \(\alpha\) of (10). An improvement of the present estimate results in the plugging of a preliminary robust estimate of \(\theta_0\), say \(\hat{\theta}_n\), as an adaptive escort parameter \(\theta\) choice.

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**Figure 3.** Behaviour of the ratio \(\frac{p_{\hat{\theta}_n}(x)}{p_{\theta}(x)}\) under contamination, for a randomly generated Normal sample \(\mathcal{N}(\theta_0, 1)\) of size 100 with 10\% of contamination by 10.

Let \(x\) be some outlier, the role of the outlier \(x\) in (10) appears in the term

\[\left( \frac{p_{\hat{\theta}_n}(x)}{p_{\theta}(x)} \right)^{\gamma} \frac{\hat{p}_\alpha(x)}{p_{\alpha}(x)}. \quad (11)\]

The estimate \(\hat{\alpha}_{\phi}(\theta)\) is robust if this term is stable. That is, if it is small when \(\alpha\) is near \(\theta_0\). If the escort parameter \(\hat{\theta}_n\) is not a robust estimator,
the ratio \( \frac{p_{\theta_n}(x)}{p_{\theta_0}(x)} \) can be very large, see Figure 3. This is due to the fact that the outlier \( x \) will be more likely under \( P_{\theta_n} \), that is \( \hat{\theta}_n \) will lead to an over evaluation of \( p_{\theta_n}(x) \) with respect to the expected value under \( \theta_0 \), say \( p_{\theta_0}(x) \). To guard against such situations, compensate through the choice of \( \gamma \), this requires further investigations.

4. Robustness

One method of assessing the robustness of an estimator is by considering its influence function (IF), with the usual interpretation being that a robust estimator will have a bounded influence function. However this requirement makes the corresponding estimator deficient at the model in relation to the maximum likelihood estimator which have unbounded influence function for most common models. As we will see in the following, our class of dual \( \phi \)-divergences estimators have strong robustness features in spite of having the same influence function as the maximum likelihood estimator. On the other hand, being equal to the influence function of the MLE, the influence function of the D\( \phi \)DE's is potentially unbounded. Thus the robustness of the D\( \phi \)DE's cannot be described through the traditional bounded influence approach. Hampel (1974) claims that the use of the \( \epsilon \)-IF (before the limit) is preferable to the use of the influence function to assess estimator robustness. The limiting form is often used because it is usually easier to evaluate, and it does not depend on \( \epsilon \). Beran (1977) claims that for evaluating the robustness of a functional with respect to a gross-error model, one should consider the \( \epsilon \)-influence function instead of the influence function, unless the former converges to the latter uniformly. For more details, see also Section 4.7 of Basu et al. (2011).

In this section we present the \( \epsilon \)-influence function technique and show that there is no intrinsic conflict between the robustness of our estimators and optimal model efficiency.

The statistical functional associated with the estimate \( \hat{\alpha}_\phi(\theta) \) is given by

\[
T_{\theta}(P_{\theta_0}) = \arg \sup_{\alpha \in \Theta} \int h(\theta, \alpha) \ dP_{\theta_0},
\]

which is Fisher consistent, namely \( T_{\theta}(P_{\alpha}) = \alpha \) for all \( \alpha \in \Theta \), keeping in mind (6).

The influence function of the functional \( T_{\theta} \) is defined as

\[
\text{IF}(x; T, P_{\theta_0}) = \lim_{\epsilon \to 0} \frac{T_{\theta}((1 - \epsilon)P_{\theta_0} + \epsilon \delta_x) - T_{\theta}(P_{\theta_0})}{\epsilon},
\]
in which \( \delta_x \) is the Dirac measure at point \( x \). The \( \epsilon \)-influence function is the quotient

\[
\epsilon - \text{IF}(x) = \frac{T_\theta ((1 - \epsilon)P_{\theta_0} + \epsilon \delta_x) - \theta_0}{\epsilon}.
\]

Using existing theory on \( M \)-estimators, see Hampel et al. (1986), p. 230, see also Proposition 1 in Toma and Broniatowski (2010), the influence function of the functional \( T_\theta \) corresponding to an estimator \( \hat{\alpha}_\phi(\theta) \) is given by

\[
\text{IF}(x; T_\theta, P_{\theta_0}) := -S^{-1}\left\{ \int \phi'' \left( \frac{p_\theta}{p_{\theta_0}} \right) \frac{p_\theta}{p_{\theta_0}} \dot{p}_\theta_0 dP_\theta \\
- \phi'' \left( \frac{p_\theta(x)}{p_{\theta_0}(x)} \right) \frac{p_\theta^2(x)}{p_{\theta_0}(x)} \dot{p}_\theta_0(x) \right\}.
\] (13)

When \( \theta = \theta_0 \), it reduces to the influence function of the MLE given by

\[
\text{IF}(x; T_{\theta_0}, P_{\theta_0}) = I_{\theta_0}^{-1} \dot{p}_{\theta_0}(x) / p_{\theta_0}(x),
\]

where \( I_{\theta_0} \) is the information matrix defined by

\[
I_{\theta_0} = \int \frac{\dot{p}_{\theta_0} \dot{p}_{\theta_0}^T}{p_{\theta_0}} d\lambda.
\]

Using (13), the influence function for the normal location model is, for \( \gamma \in \mathbb{R} \setminus \{0, 1\} \),

\[
\text{IF}(x; T_\theta, P_{\theta_0}) := \frac{(x - \theta_0) \exp \left\{ -\frac{\gamma}{2} (\theta - \theta_0) (\theta + \theta_0 - 2x) \right\} + \gamma (\theta_0 - \theta) \exp \left\{ \frac{\gamma(\gamma-1)(\theta - \theta_0)^2}{2} \right\}}{(1 + \gamma^2 (\theta - \theta_0)^2) \exp \left\{ \frac{\gamma(\gamma-1)(\theta - \theta_0)^2}{2} \right\}}.
\]

For \( \gamma = 0 \) (\( KL_m \)),

\[
\text{IF}(x; T_\theta, P_{\theta_0}) := x - \theta_0.
\]

For \( \gamma = 1 \) (\( KL \)),

\[
\text{IF}(x; T_\theta, P_{\theta_0}) := \frac{(x - \theta_0) \exp \left\{ -\frac{1}{2} (\theta - \theta_0) (\theta + \theta_0 - 2x) \right\} + (\theta_0 - \theta)}{1 + (\theta - \theta_0)^2}.
\]

Remark that when \( \theta = \theta_0 \), the influence functions of \( T_\theta \) coincide for all \( \gamma \) and are equal to the influence function of the MLE,

\[
\text{IF}(x; \text{MLE}, P_{\theta_0}) := x - \theta_0.
\]

Figure 4 presents the influence functions of the dual \( \phi \)-divergences estimators in the normal location model, with \( \theta = 0.1 \) and \( \theta = 0.5 \). We can see that the influence functions are unbounded. Thus, the \( D\phi \text{DES} \)'s are examples of estimators for which the limiting form of their influence

functions does not reliably provide information about the form of the \( \epsilon \)-IF’s.

Empirical \( \epsilon \)-IF’s were calculated for \( \epsilon \)'s= 0.1, 0.2. This was done by generating a sample size of 100 for Normal distribution \( \mathcal{N}(\theta_0, 1) \) with \( \theta_0 = 1 \). After the data set was sorted, the largest \( \epsilon n \) values were then iteratively reassigned along a grid of values ranging from 0.01 to 25. At each grid value the D\( \phi \)DE’s were computed. This was repeated 1000 times. The mean of the D\( \phi \)DE’s at each grid were plotted against the grid. Below are the plots of the empirical average value of the D\( \phi \)DE’s, see Figures 5 and 6.

Figure 5(a) illustrates the empirical \( \epsilon \)-IF, when we take the mean, \( \hat{\theta}_n = \bar{X} \), as the value of the escort parameter \( \theta \), it shows that the D\( \phi \)DE’s \( \hat{\alpha}_\phi(\theta) \) have the same empirical \( \epsilon \)-IF as the MLE. In Figure 5(b), the choice of the median as escort parameter value improve considerably the robustness properties of our estimators. The D\( \phi \)DE’s, with robust escort parameter perform very well, outliers tend have less influence on the D\( \phi \)DE’s. Clearly, we can see from Figure 6 that the MLE is greatly affected by the value of the outlier.

**Figure 4.** Influence functions of the D\( \phi \)DE’s for the normal location model.
Figure 5. Empirical $\epsilon$-influence functions, $\epsilon = 0.1$.

5. A Real data example

This example involves Newcombs light speed data (Stigler (1977), Table 5). The data were also analysed by Brown and Hwang (1993), who were trying to fit the best approximating normal distribution to the corresponding histogram. The data set shows a nice unimodal structure, and the normal model would have provided an excellent fit to the data except for the two large outliers.

For the dataset, Table 1 gives the values of the $D\phi$DE’s. These estimators exhibit strong outlier resistance properties. The histogram, and the normal fits using the maximum likelihood estimate and the $D\phi$DE’s are presented in Figure 7, all the normal densities fit the main body of the histogram quite well, except the MLE. Note that the values of the escort parameters considered in this example are the median for $\theta$ and the mad for $\sigma$.

6. Simulation

In this section, we present results of a simulation study which was conducted to explore the properties of the $D\phi$DE’s.
Figure 6. Empirical $\epsilon$-influence functions, $\epsilon = 0.2$.

| $\gamma$ | 0  | 0.5 | 1   | 2   |
|----------|----|-----|-----|-----|
| $\hat{\alpha}$ | 26.21 | 27.67 | 27.00 | 27.64 |
| $\hat{\sigma}$ | 10.66 | 5.16 | 4.47 | 4.84 |

Table 1. Estimated parameters for the Newcomb data under the normal model.

A sample is generated from $\mathcal{N}(\theta_0, 1)$ with $\theta_0 = 0$. The $\phi$DE’s are calculated for samples of sizes 25, 50, 75, 100, 200 and the hole procedure is repeated 1000 times. In the examination of the robustness of the $\phi$DE’s, the data are generated from the distribution

$$(1 - \epsilon)\mathcal{N}(\theta_0, 1) + \epsilon \delta_{10},$$

where $\epsilon = 0.05, 0.1, 0.2$. The value of escort parameter $\theta$ is taken to be the median.
Figure 7. A histogram of the Newcomb data with normal fits.

The values of $\gamma$ are chosen to be 0, 0.5, 1, 2 which correspond to the well known standard divergences: $KL_m$—divergence, the Hellinger distance, $KL$ and the $\chi^2$—divergence respectively. These estimators are also compared with some other methods, including maximum likelihood estimator (MLE) and Huber estimator (see Huber (1964)) using the $\psi$-function

$$
\psi(t) = \begin{cases} 
t & \text{if } |t| < \tau \\
\tau \text{sgn}(t) & \text{if } |t| \geq \tau
\end{cases}
$$

for some constant $\tau > 0$. A value of 1.4 was used for $\tau$, for location estimation which is in the range of values shown to perform well in the Princeton Robustness Study (see Andrews et al. (1972)). We carried out this analysis within the R Language R Development Core Team (2009).

Under the true model, from Table 2, as expected, the MLE produces most efficient estimators in this case. The D$\phi$DE’s seem to be a good competitors to the MLE in terms of MSE. Recall that, theoretically, the
Table 2. MSE of the estimates under the true model.

| $\gamma$ | 25     | 50     | 75     | 100    | 150    | 200    |
|----------|--------|--------|--------|--------|--------|--------|
| 0        | 0.0386 | 0.0187 | 0.0129 | 0.0096 | 0.0064 | 0.0055 |
| 0.5      | 0.0386 | 0.0187 | 0.0129 | 0.0096 | 0.0064 | 0.0055 |
| 1        | 0.0386 | 0.0187 | 0.0129 | 0.0096 | 0.0064 | 0.0055 |
| 2        | 0.0440 | 0.0190 | 0.0129 | 0.0096 | 0.0064 | 0.0055 |
| $M$-Estimator | 0.0400 | 0.0199 | 0.0136 | 0.0100 | 0.0067 | 0.0057 |

Table 3. MSE of the estimates under 10% of contamination

| $\gamma$ | 25     | 50     | 75     | 100    | 150    | 200    |
|----------|--------|--------|--------|--------|--------|--------|
| 0        | 1.3999 | 1.1696 | 1.1041 | 1.0941 | 1.0505 | 1.0360 |
| 0.5      | 0.2004 | 0.1610 | 0.1509 | 0.1475 | 0.1447 | 0.1402 |
| 1        | 0.1280 | 0.0874 | 0.0780 | 0.0726 | 0.0689 | 0.0649 |
| 2        | 0.1393 | 0.0843 | 0.0731 | 0.0680 | 0.0638 | 0.0598 |
| $M$-Estimator | 0.2460 | 0.0902 | 0.0724 | 0.0646 | 0.0593 | 0.0540 |

Table 4. MSE of the estimates under 20% of contamination

| $\gamma$ | 25     | 50     | 75     | 100    | 150    | 200    |
|----------|--------|--------|--------|--------|--------|--------|
| 0        | 4.6936 | 4.3133 | 4.1415 | 4.1344 | 4.1282 | 4.0724 |
| 0.5      | 0.4935 | 0.4351 | 0.3947 | 0.3802 | 0.3812 | 0.3701 |
| 1        | 0.3253 | 0.2673 | 0.2325 | 0.2178 | 0.2163 | 0.2067 |
| 2        | 0.3092 | 0.2433 | 0.2072 | 0.1929 | 0.1906 | 0.1812 |
| $M$-Estimator | 2.6171 | 1.5906 | 1.1710 | 0.9353 | 0.7837 | 0.6220 |

D\(\phi\)DE’s are asymptotically efficient. It is interesting to note that the asymptotics appear to take effect for a sample sizes of 75 or more but the results for sample sizes below 75 are also good, thus the D\(\phi\)DE’s perform as well as the MLE at the true model.

We now turn to the comparison of these various estimators under contamination. We can see from Table 3 that for small sample sizes the D\(\phi\)DE’s yield clearly the most robust estimates. As \(n\) increases, the $M$-Estimator obtain slightly higher performance. For high amount of contamination, we can see from Tables 4 and 5 that the D\(\phi\)DE with \(\gamma = 2\) has the smallest MSE over all other D\(\phi\)DE’s and outperform the
Table 5. MSE of the estimates under 25% of contamination

\[
\begin{array}{ccccccc}
\gamma & n & 25 & 50 & 75 & 100 & 150 & 200 \\
0 & 7.1318 & 6.6138 & 6.5256 & 6.3594 & 6.3376 & 6.3436 \\
0.5 & 1.1670 & 0.6124 & 0.5920 & 0.5563 & 0.5596 & 0.5533 \\
1 & 0.9946 & 0.3963 & 0.3754 & 0.3442 & 0.3454 & 0.3383 \\
2 & 0.9921 & 0.3561 & 0.3347 & 0.3047 & 0.3051 & 0.2980 \\
M-Estimator & 5.7497 & 4.3835 & 3.9891 & 3.7579 & 3.5211 & 3.3328 \\
\end{array}
\]

MLE substantially and the performance of the M-Estimator decreases dramatically.

7. Conclusions

The aim of this paper was to investigate the new estimation procedure based on the dual representation of \( \phi \)-divergences for the univariate normal location model. The estimators are easily computed and exhibit appropriate asymptotic behavior. First, we evaluated the impact of the choice of the escort parameter on the estimates and provided a practical choice of the escort parameter. Second, we have shown that there is no intrinsic conflict between the robustness of our estimators and optimal model efficiency, by considering empirical \( \epsilon \)-influence functions. The simulation results presented here provide solid evidence that the \( D^\phi \)DE’s in the normal location setting, obtain full efficiency at the true model while keeping good performances as robust estimators, they demonstrated unexpected empirical robustness for high amounts of contamination. Thus they provide good alternative to maximum likelihood.

In this paper we have limited ourself to the \( D^\phi \)DE’s associated to the standard divergences, which are widely used in statistical inference. It will be interesting to investigate theoretically the problem of the choice of the divergence which leads to an “optimal” estimate in terms of efficiency and robustness, we leave this study open for future research.

References

Andrews, D. F., Bickel, P. J., Hampel, F. R., Huber, P. J., Rogers, W. H., and Tukey, J. W. (1972). Robust estimates of location: Survey and advances. Princeton University Press, Princeton, N.J.

Basseville, M. (2010). Divergence measures for statistical data processing.
Basu, A., Shioya, H., and Park, C. (2011). *Statistical Inference: The Minimum Distance Approach*, volume 120 of *Monographs on Statistics & Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL.

Beran, R. (1977). Minimum Hellinger distance estimates for parametric models. *Ann. Statist.*, **5**(3), 445–463.

Bouzebda, S. and Cherfi, M. (2011). General bootstrap for dual $\phi$-divergences estimates. *Arxiv preprint arXiv:1106.2246*.

Bouzebda, S. and Keziou, A. (2010). Estimation and tests of independence in copula models via divergences. *Kybernetika*, **46**(1), 178–201.

Broniatowski, M. and Keziou, A. (2006). Minimization of $\phi$-divergences on sets of signed measures. *Studia Sci. Math. Hungar.*, **43**(4), 403–442.

Broniatowski, M. and Keziou, A. (2009). Parametric estimation and tests through divergences and the duality technique. *J. Multivariate Anal.*, **100**(1), 16–36.

Brown, L. D. and Hwang, J. T. G. (1993). How to approximate a histogram by a normal density. *Amer. Statist.*, **47**(4), 251–255.

Cherfi, M. (2011). Dual divergences estimation for censored survival data. *Arxiv preprint arXiv:1106.2627*.

Cressie, N. and Read, T. R. C. (1984). Multinomial goodness-of-fit tests. *J. Roy. Statist. Soc. Ser. B*, **46**(3), 440–464.

Hampel, F. R. (1974). The influence curve and its role in robust estimation. *J. Amer. Statist. Assoc.*, **69**, 383–393.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J., and Stahel, W. A. (1986). *Robust statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. The approach based on influence functions.

Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.*, **35**, 73–101.

Huber, P. J. (1981). *Robust statistics*. John Wiley & Sons Inc., New York. Wiley Series in Probability and Mathematical Statistics.

Jiménez, R. and Shao, Y. (2001). On robustness and efficiency of minimum divergence estimators. *Test*, **10**(2), 241–248.

Keziou, A. (2003). Dual representation of $\phi$-divergences and applications. *C. R. Math. Acad. Sci. Paris*, **336**(10), 857–862.

Keziou, A. and Leoni-Aubin, S. (2008). On empirical likelihood for semiparametric two-sample density ratio models. *J. Statist. Plann. Inference*, **138**(4), 915–928.

Küs, V., Morales, D., and Vajda, I. (2008). Extensions of the parametric families of divergences used in statistical inference. *Kybernetika (Prague)*, **44**(1), 95–112.
Liese, F. and Vajda, I. (1987). Convex statistical distances, volume 95 of Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]. BSB B. G. Teubner Verlagsgesellschaft, Leipzig. With German, French and Russian summaries.

Liese, F. and Vajda, I. (2006). On divergences and informations in statistics and information theory. IEEE Trans. Inform. Theory, 52(10), 4394–4412.

Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). Robust statistics. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester. Theory and methods.

Pardo, L. (2006). Statistical inference based on divergence measures, volume 185 of Statistics: Textbooks and Monographs. Chapman & Hall/CRC, Boca Raton, FL.

R Development Core Team (2009). R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0.

Stigler, S. M. (1977). Do robust estimators work with real data? Ann. Statist., 5(6), 1055–1098. With discussion and a reply by the author.

Toma, A. and Broniatowski, M. (2010). Dual divergence estimators and tests: robustness results. Journal of Multivariate Analysis, 102(1), 20–36.

Toma, A. and Leoni-Aubin, S. (2010). Robust tests based on dual divergence estimators and saddlepoint approximations. Journal of Multivariate Analysis, 101(5), 1143–1155.

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