AN IMPROVED KTNS ALGORITHM FOR THE JOB SEQUENCING AND TOOL SWITCHING PROBLEM

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ABSTRACT

We outline a new Max Pipe Construction Algorithm (MPCA) with the purpose to reduce the CPU time for the classic Keep Tool Needed Soonest (KTNS) algorithm. The KTNS algorithm is applied to compute the objective function value for the given sequence of jobs in all exact and approximating algorithms for solving the Job Sequencing and Tool Switching Problem (SSP). Our MPCA outperforms the KTNS algorithm by at least an order of magnitude in terms of CPU times. Since all exact and heuristic algorithms for solving the SSP spend most of their CPU time on applying the KTNS algorithm we show that our MPCA solves the entire SSP on average 59 times faster for benchmark instances of D compared to current state of the art heuristics.

INTRODUCTION

Currently as far as we are aware a globally optimal solution to the Job Sequencing and Tool Switching Problem (SSP) Tang, Denardo [1] is limited to a couple of dozen jobs with no more than 25 tools in each of them da Silva et al. [2]. This class of problems includes many optimization versions of SSP solved for a large number of small-scale orders (jobs) on conveyor lines (flexible manufacturing system - FMS). We do not consider technical and technological changes (improvements) in the equipment of the FMS in order to increase its productivity. Our goal is to increase the productivity of the conveyor line by finding an optimal sequence of loading the necessary tools, materials and other production resources (hereinafter referred to as tools) including the professional skills of employees, e.g. in order to minimize the conveyor’s downtime. In other words, the methods and means of increasing the productivity of the conveyor in this article are related to an optimal sequencing of jobs each of which requires its specific collection of tools. Since a production line has a limited number of slots (magazine with a limited capacity) a tool switch is necessary to store the tools required to complete all jobs. Here by tool switch we understand the removal of a tool from the magazine (collection of slots) and the insertion of another tool in its place. Thus, the SSP consists in finding a sequence of jobs minimizing the total number of tool switches.

In this paper we assume that the following refined assumptions are satisfied Calmels [6], da Silva et al. [2]:
- There is a set of jobs to be processed and each job requires a fixed set of specific tools;
- The set of jobs and the subset of tools required for each job is known in advance;
- No job requires a set of tools that exceeds the capacity of the machine’s magazine;
- All tools are always available at least outside of the magazine;
- A single machine is available to process all jobs;
- It is an offline version of the problem;
- Once the machine has started processing a job it must be completed;
Exactly one job can be processed at any time unit;

- The processing and completion times of a job are not dependent on the set of tools and do not impact the number of tool switching times;
- The tool sockets (slots) are identical;
- Only one tool switch is done at time;
- Each tool fits in any slot of the magazine and occupies one slot;
- The time associated with removal and insertion (switch) of a tool is independent and constant;
- No breaks, no wear and no maintenance of the tools are considered.

Here, a single machine is an abstraction. It reflects many different interpretations, e.g., huge and small companies, production and service lines, and organizations - educational, medical, governmental and private. For each of these we are able to indicate an input of jobs, their operations including available resources (tools) and an expected output of ordered (sequenced) products (jobs) to be processed Goldengorin, Romanuke [3].

In this paper we are not going to overview the state of the art for mathematical models, methods and algorithms for modeling and solving the SSP. We rather refer the interested reader to the recent literature review Calmels [6], as well as the article da Silva et al. [2]. We would like to emphasize that the SSP is one of the NP-hard problems Tang, Denardo [1]. The relevant mathematical models and solution methods for SSP should take into account that the number of tool switches for the next job depends not only on a single or pair of prior scheduled jobs but, in the worst case, on all jobs scheduled before the pending job Crama et al. [10], Ghiani et al. [9], Ahmadi et al. [4]. Most publications report globally optimal solutions to the entire SSP. However, their conclusions are questionable without any proof of how they take into account the real number of switches. For details we refer to Tables 6 and 7 in da Silva et al. [2].

The SSP can be formulated as follows. We are given the sets of jobs \( J = \{1, \ldots, n\} \) and tools \( T = \{1, \ldots, m\} \), a single magazine \( C \) denoting the maximum number of tools (slots) that can be placed (occupied) in the machine’s magazine, \( C < m \), the set of tools \( T_i \subset T, i = 1, \ldots, n \) required by job \( i \) which should be in the magazine in order to complete the job \( i \), \( |T_i| \leq C \). Thus, for the given sequence of \( n \) jobs we are going to associate \( n \) instants each of which will be presented by the set tools \( M_i \subset T \) in the magazine. A feasible SSP solution is the sequence \( M = \{M_i\}_{i=1}^n \), such that sequence of jobs \( (1, 2, \ldots, n) \) can be completed. An optimal solution to the SSP is a sequence \( M \) that minimizes the total number of tool switches required to move from one job to another and complete all jobs.

As shown by Tang, Denardo [1] SSP can be decomposed into 2 following problems.

1. Tool Loading Problem (TLP) - for a given sequence of jobs, find the optimal sequence of magazine states \( M \) that minimizes the total number of tool switches.

2. Job Sequencing Problem (JeSP) – finding a sequence of jobs such that the number of tool switches is minimal after solving the TLP for this sequence.

Note that most publications devoted to solving the SSP consider heuristics based on completely different classes of metaheuristics, e.g., tabu search, iterated local search, and genetic algorithms [7]. Regardless of the designed heuristic’s nature they try to replace all permutations defined on the entire set of jobs and select the best values of the SSP objective function (OF), i.e. the minimum number of tool switches. One of the first and most popular algorithms to compute the SSP OF, well known for over thirty years, is the Keep Tool Needed Soonest (KTNS) algorithm [1]. The main purpose of our paper is to improve the KTNS algorithm since the efficiency of any exact or
heuristic algorithm is based on a partial enumeration of SSP feasible solutions and depends on the CPU time to compute the SSP OF value.

Our paper is organized as follows. Before describing our new Max Pipe Construction Algorithm (MPCA) we provide a small numerical example to illustrate the pitfalls of slowing down the SSP OF value computation. In the next two sections we describe the solution method and formulate statements to justify the correctness of our MPCA. We provide an example demonstrating the inner workings of MPCA and design an efficient $MPCA - bitwise$ implementation and evaluate its time complexity. The Experimental Results section includes our computational study and the final section contains a summary and future research directions.

**A Numerical Example**

Let's consider an example. $T_1 = \{4, 5, 6\}, T_2 = \{1, 3, 4, 5\}, T_3 = \{1, 2, 7\}, T_4 = \{2, 3, 7\}, T_5 = \{4, 5, 7\}, T_6 = \{1, 2, 3, 6\}$ is a given sequence of jobs represented as sets of tools that they require, $C = 5$ is magazine capacity. Fig. 1 shows the optimal loading of empty slots to perform job in the order of $(1, 2, 3, 4, 5, 6)$. Fig. 2 shows the optimal loading of empty slots for performing job in the order $(1, 2, 5, 3, 4, 6)$. The sequence $(1, 2, 5, 3, 4, 6)$ is found by exhaustive brute force search and is a solution to the problem JeSP. In the Fig. 1, 2, red arcs represent tool switches. Thus, the smallest number of switches for the $(1, 2, 3, 4, 5, 6)$ sequence is 5, and for the $(1, 2, 5, 3, 4, 6)$ sequence is 3.

Crama et al. [10] proved that the JeSP is NP-hard. Usually, TLP is solved with purpose to calculate the SSP objective function value for each job sequence in JeSP. To solve TLP Tang, Denardo [1] proposed the Keep Tool Needed Soonest (KTNS) with time complexity $O(mn)$. Ghiani et al. [9] modified KTNS to Tailored KTNS with the purpose to calculate a lower bound to the number of switches between two jobs which was used to accurately solve JeSP.

**PROBLEM FORMULATION**

Let’s refresh the basic notation for the job Sequencing and tool Switching Problem (SSP) and further illustrate them with examples. $T = \{1, 2, \ldots, m\}$ is the set of tools, where $m$ is the number of tools. $J = \{1, 2 \ldots n\}$ is the set of jobs, where $n$ is the number of jobs. $T_i \subset T$ is the set of tools needed to do the job $i$, where $i \in J$. $C < m$ - capacity (capacity) of the magazine. $M_i \subset T$ is the state of the magazine when the job $i$ is performed i.e. the set of tools located in the magazine at the time of the job $i$, where $i \in J, |M_i| = C, T_i \subseteq M_i$. $S = \{S : S = (\sigma(1), \sigma(2), \ldots, \sigma(n))\}$, where $\sigma$ is a permutation $\}$ is the set of all reordering of jobs. $M(S) = \{M = (M_1, \ldots M_n) : T_i \subseteq M_i, |M_i| = C, i \in J\}$ is the set of all sequences of magazine states such that jobs can be
performed in order $S \in S$. $M = \{M \in M(S) : S \in S\}$ is the set of all possible magazine states. 

$$\text{switches}(M) = \sum_{i=1}^{n-1} C - |M_i \cap M_{i+1}|$$

is the number of switches for the sequence of magazine states, where $M \in M$. The TLP problem is formulated as finding $\underset{M \in M(S)}{\text{argmin}} \{\text{switches}(M)\}$. The problem JeSP is formulated as finding $\underset{S \in S}{\text{argmin}} \{\underset{M \in M(S)}{\text{min}} \{\text{switches}(M)\}\}.$

Table 1: Example of KTNS processing.

| Instants | Tools |
|----------|-------|
| 1        | 2 2 3 6 6 6 |
| 2        | 3 3 3 4 6 6 |
| 3        | 2 2 4 4 6 6 |
| 4        | 1 2 5 5 5 7 |
| 5        | 1 2 5 5 5 7 |
| 6        | 1 6 6 6 6 6 |
| 7        | 3 3 3 4 5 7 |

$T_1 = \{4, 5, 6\}, T_2 = \{1, 3, 4, 5\}, T_3 = \{1, 2, 7\}, T_4 = \{2, 3, 7\}, T_5 = \{4, 5, 7\}, T_6 = \{4, 5, 7\}$ is given sequence of jobs represented as sets of tools that they require (marked in orange). At first KTNS calculates an auxiliary matrix, the element $a_{ij}$ of which denotes the first instant starting from $j$ at which the tool $i$ is needed. For example starting from instant 2 the first instant at which the tool 6 is needed is instant 2 itself, the first instant in which the tool 6 is needed is 6. Starting from instant 6 the tool 4 is never needed again than $a_{4,6} = 7$.

Let’s consider the first point in time. Tools required for the job scheduled at this moment $T_1 = \{4, 5, 6\}$ (marked in orange). We fill 3 slots with these tools. The remaining 2 slots are filled with the tools that are needed soonest. We see in the table (column 1) that the 1 tool will be needed at the moment 2, the 2 tool will be needed at the moment 3, the 3 tool will be needed at the moment 2, the 7 tool will be needed at the moment 3. Choose the 1 and 3 tools since they are needed soonest.

Let’s consider the instant 2. The tools necessary for the job $T_2 = \{1, 3, 4, 5\}$ scheduled at this moment (marked in orange), all these tools were already present at the previous step, so no switches are required. Let’s move on to the next step.
Table 2: Example of KTNS execution.

| tools | instants | instant | instant |
|-------|----------|---------|---------|
| 1     | 2 2 3 6 6 6 6 | | |
| 2     | 3 3 3 4 6 6 | | |
| 3     | 2 2 4 4 6 6 | | |
| 4     | 1 2 5 5 5 7 | | |
| 5     | 1 2 5 5 5 7 | | |
| 6     | 1 6 6 6 6 6 | | |
| 7     | 3 3 3 4 5 7 | | |

Let’s consider the instant 3. Tools required for the job scheduled at this moment $T_3 = \{1, 2, 7\}$ (marked in orange). The 1 tool was already in the magazine in the previous step, and the 7 and 2 tools need to be charged, but the magazine is already full, so you need to remove two tools. Let’s remove the tools that are needed later. We see in the table (column 3) that the 3 tool will be needed at the moment 4, the 4 tool will be needed at the moment 5, the 5 tool will be needed at the moment 5, the 6 tool will be needed at the moment 6. Let’s remove the 4 and 6 tools as they will be needed later than others. Thus, there were two switches of the tool 6 for the tool 2 and the tool 4 for the tool 7.

| tools | instants | instant | instant |
|-------|----------|---------|---------|
| 1     | 2 2 3 6 6 6 | | |
| 2     | 3 3 3 4 6 6 | | |
| 3     | 2 2 4 4 6 6 | | |
| 4     | 1 2 5 5 5 7 | | |
| 5     | 1 2 5 5 5 7 | | |
| 6     | 1 6 6 6 6 6 | | |
| 7     | 3 3 3 4 5 7 | | |

Let’s consider the instant 4. The tools required for the job $T_4 = \{2, 3, 7\}$ scheduled at this moment (marked in orange), all these tools were already present at the previous step, so no switches are required. Let’s move on to the next step.

| tools | instants | instant | instant |
|-------|----------|---------|---------|
| 1     | 2 2 3 6 6 6 | | |
| 2     | 3 3 3 4 6 6 | | |
| 3     | 2 2 4 4 6 6 | | |
| 4     | 1 2 5 5 5 7 | | |
| 5     | 1 2 5 5 5 7 | | |
| 6     | 1 6 6 6 6 6 | | |
| 7     | 3 3 3 4 5 7 | | |

Let’s consider the instant 5. The tools required for the job scheduled at this moment $T_5 = \{4, 5, 7\}$ (marked in orange). The 5, 7 tools were already in the magazine in the previous step, and the tool 4 needs to be charged, but the magazine is already full, so we need to delete one tool. Let’s delete the tool that will be needed later than others. We see in the table (column 4) that the 1 tool will be needed at the instant 6, the 2 tool will be needed at the instant 6, the 3 tool will be needed at the moment 6. Let’s remove the tool 1 as it is needed later than others. Instead of 1, it would be possible to delete 2 or 3, which would not affect the total number of switches. Thus, the 1 tool was switched with a 4 tool.
Table 3: Example of KTNS execution.

Let’s consider the instant 6. The tools needed for the job scheduled at this moment \( T_6 = \{1, 2, 3, 6\} \) (marked in orange), The 2, 3 tools were already in the magazine in the previous step, and the 1, 6 tools need to be charged, but the magazine is already full, so two tools must be removed. Let’s remove the tools that are needed later than others. We see in the table (column 6) that the 5 tool will be needed at the time 7, the 4 tool will be needed at the time 7, the 7 tool will be needed at the time 7. Let’s remove the 4, 5 tools as they will be needed later than others. Instead of removing 4, 5, it would be possible to remove any pair from 4, 5, 7 without affecting the total number of switches. Thus, there were two replacements of the tool 4 for the tool 6 and the tool 4 for the tool 1. KTNS has finished job with 5 switches.

### SOLUTION METHOD

Let’s define \( \mathcal{F}(M) = \{\pi_{s,e}^{\text{tool}} : \text{tool} \in (T_s \cap T_e)\backslash(\bigcup_{i=s+1}^{e-1} T_i), \text{tool} \in \bigcap_{i=s+1}^{e-1} M_i\} \) as the set of all such triplets \((s, e, \text{tool})\), that the tool is used for job at moments \(s, e\), not used for job at moments \(s+1, \ldots, e-1\), however, is present in the magazine at moments \(s+1, \ldots, e-1\), despite the fact that at these moments the tool is not used for job. Let’s consider the main objects of this job - the elements of the set \( \mathcal{F}(M) \), we will call them pipes, denote \( \pi_{\text{start}, \text{end}}^{\text{tool}} \) - pipe with start at \(\text{start}\), end at \(\text{end}\) and tool \(\text{tool}\). Fig. 3 shows examples of pipes. Informally, the pipe is the saving of the tool from the moment \(\text{start}\), where it was used for job (marked in orange) until the moment \(\text{end}\), where it will again be used for job (marked in orange), but at intermediate points in time \(\text{start}+1, \ldots, \text{end} - 1\) it must not be used to job (marked in white). So for example, \(\pi_{1,2}^{9} \) saves tool 1 from time 5 to time 7 even though it is not used for job at time 6. Note that \(\pi_{1,2}^{7} \) is also a pipe, although there are no intermediate times between the times 1, 2. The figure marked with a red pipe is not, since at the moment of time 1 the tool 2 is not used for job, i.e. \(2 \notin T_1\). Let’s call

![Figure 3: Examples of pipes.](image-url)
the capacity of the pipe the number of empty slots that is necessary for the existence of the pipe, which is equal to $end - start - 1$.

**Algorithm 1: MPCA**

```plaintext
pipes_count := 0
M_1 := T_1, M_2 := T_2, ..., M_n := T_n
for end = 2, ..., n do
  for start = end - 1, ..., 1 do
    candidates := \{π_{tool, end} : tool ∈ (T_{start} \cap T_{end}) \setminus (∪_{i=s+1}^{e-1} T_i)\}
    empty_slots := min\{C - |M_i| : i ∈ \{start+, ..., end - 1\}\}
    if |candidates| > empty_slots then
      candidates := arbitrary elements from candidates in the amount of empty_slots.
    end
    for π_{start, end} ∈ candidates do
      add tool to M_{start+1}, ..., M_{start+2}, ..., M_{end-1}
    end
  pipes_count := pipes_count + |candidates|
end
return pipes_count
```

**Theorem 1.** Let $C$ be the capacity of the magazine, $T_1, \ldots T_n$ are the required sets of tools for jobs $1, \ldots n$, $S ∈ S$ is the sequence of jobs, then

$$\min_{M \in M(S)} \{\text{switches}(M)\} = -\max_{M \in M(S)} \{|\mathcal{T}(M)|\} - C + \sum_{i=1}^{n} |T_i|.$$  

**Theorem 2.** Let $C$ be the capacity of the magazine, $T_1, \ldots T_n$ are the required sets of tools for jobs $1, \ldots n$, $S ∈ S$ is the sequence of jobs, $S ∈ S$ - sequence of jobs, then

$$MPCA(S) = MPCA(T_1, \ldots, T_n; C; S) = \max_{M \in M(S)} \{|\mathcal{T}(M)|\}.$$  

From Theorem 1, Theorem 2 it follows that the value of the objective function in the problem $JeSP$ equals

$$\min_{M \in M(S)} \{\text{switches}(M)\} = -MPCA(S) - C + \sum_{i=1}^{n} |T_i|.$$
Table 4: Example of MPCA execution.

$T_1 = \{4, 5, 6\}, T_2 = \{1, 3, 4, 5\}, T_3 = \{1, 2, 7\}, T_4 = \{2, 3, 7\}, T_5 = \{4, 5, 7\}, T_6 = \{1, 2, 3, 6\}$ is given sequence of jobs represented as sets of tools that they require. Let’s depict all pipes of capacity 0. What is implemented in the algorithm line 1.

Let’s try to build all possible pipes with end at $\text{end} = 3$ and beginning at $\text{start} = 1$. There are no such pipes, since there are no tools that are needed both for the job scheduled at the time of 1 and for the job planned at the time of 3, while the time 2 is not needed for the interim memorial, i.e. $(T_1 \cap T_3) \setminus T_2 = \emptyset$.

Let’s try to build all possible pipes with end at $\text{end} = 4$ and beginning at $\text{start} = 2$. $(T_4 \cap T_2) \setminus T_3 = \{3\}$. At the instant 3 there are two empty slots, one of which we will occupy with the tool 3, thereby constructing a pipe $\pi_{2,4}$ with the beginning at the instant 2, the end at the moment 4 and tool 3. Note that pipes with an end at $\text{end} = 3$ can no longer be built, since each tool in this instant is already the end of a pipe and any expression of the form $(T_i \cap T_4) \setminus (T_3 \cup T_2 \cup \ldots) = \emptyset$ since $T_4 \subseteq T_3 \cup T_2$. Then the iteration where $\text{start} = 1$ can be skipped, which is reflected in the MPCA-bitwise in line 9, where the variable $|\text{end_tools}|$ will be equal to zero.
Table 5: Example of \(\text{MPCA}\) execution.

Let’s try to build all possible pipes with end at \(\text{end} = 5\) and beginning at \(\text{start} = 3\). There are no such pipes, since there are no tools that are needed both for the job planned at the time of 3 and for the job planned at the time of 5, while the time 4 is not needed for the intervening instant, that is, \((T_5 \cap T_3) \setminus T_4 = \emptyset\).

Let’s try to build all possible pipes with end at \(\text{end} = 5\) and beginning at \(\text{start} = 2\). \((T_2 \cap T_5) \setminus (T_3 \cup T_4) = \{4, 5\}\). Then the pipes \(\pi_{2,5}^4\) and \(\pi_{2,5}^5\) claim to be constructed. At instant 4 there are two empty slots, At instant 3 there is one empty slot, then only one pipe can be built instead of two. Let’s take one empty slot at these instants 3,4, thereby constructing a pipe \(\pi_{2,5}^5\) with the beginning at the instant 2, the end at the instant 5 and the tool 5. Note that at the instant 3 the magazine is completely full, so no pipes passing through the instant 2 will already be built. Then the iteration \(\text{start} = 1\) can be skipped, which is \(\text{MPCA-bitwise}\) in the algorithm in line 9, when \(\text{fullmag} = 2 > 1 = \text{start}\).

Let’s try to build all possible pipes with end at \(\text{end} = 6\) and beginning at \(\text{start} = 4\). \((T_4 \cap T_6) \setminus T_5 = \{2, 3\}\). Then the pipes \(\pi_{4,6}^2\) and \(\pi_{4,6,3}\) claim to be constructed. At the instant 5 there are two empty slots, we will occupy these slots with tools 2,3, thereby constructing two pipes \(\pi_{4,6,2}\) and \(\pi_{4,6,3}\). Note that at the moment of time 5 the magazine is completely full, so no pipes passing through the instant 5 can no longer be built. Then iterations \(\text{start} = 3, 2, 1\) can be skipped, which is \(\text{MPCA-bitwise}\) in the algorithm in line 9. \(\text{MPCA}\) has finished job. There is 10 pipes than number of switches = \(\sum_{i=1}^{n} |T_i| - C - 10 = 20 - 5 - 10 = 5\).

\(\text{MPCA-bitwise}\) is an efficient implementation of \(\text{MCPA}\), where the sets \(T_i\) and \(M_i\) are encoded by 64 -bit vectors, which will allow performing set operations in \(\lceil \frac{m}{64} \rceil\) operations. Let’s analyze the complexity. Let’s create a sparse table called \textit{possib\_tools}. Time to create it \(O(n \log(n))\), time to call it \(O(1)\). Let \(l\) be the maximum pipe length. The variable \textit{end\_tools} symbolizes the set of \textit{tool} tools such that the pipe \(\pi_{\text{start},\text{end}}^\text{tool}\) can be built, where \(\text{start} \geq \text{fullmag}\). From this variable with
each new start all pipes $\pi_{\text{start,end}}$ are removed, than when at the moment when start = end – l variable end_tools will become equal to empty set and loop from string 8 will finish job because of break from line 10. The loop on line 3 does $n – 1$ iterations, the loop on line 8 does no more than $l$ iterations. The 11 line runs in $\lceil \frac{m}{64} \rceil$ operations. Each execution of the line 21 means loading one slot of the magazine, of which there are only $Cn$, thus 21 will be called at most $Cn$. Then the time complexity $O(l\lceil \frac{m}{64} \rceil n + n\log(n)) = O(lmn + n\log(n))$.

Algorithm 2: MPCA – bitwise

```
1 pipes_count := 0
2 pissib_tools[i][j] := $\bigcup_{k=i}^j T_k$ for all $i < j$
3 for end = 2, ..., n do
4     if empty[end - 1] = 0 then
5         fullmag := end - 1
6     end
7     end_tools := $T_{end} \cap$ pissib_tools[fullmag][end - 1]
8     for start = end - 1, end - 3, ..., 1 do
9         if fullmag > start or $|end\text{tools}| = 0$ then
10            break
11        end
12    candidates := $T_{start} \cap$ end_tools
13    if $|candidates| > 0$ then
14        end_tools := end_tools \ candidates
15        new_pipes_count := $|candidates|$
16        if new_pipes_count > 0 then
17            for $j = start + 1, start + 2, ..., end - 1$ do
18                if empty[j] \leq new_pipes_count then
19                    new_pipes_count := empty[j]
20                    fullmag := j
21                end
22                end
23            end
24            empty[j] := empty[j] - new_pipes_count
25            end
26        end
27    end
28 return pipes_count
```

EXPERIMENTAL RESULTS

To compare the speed of computing the SSP objective function we present our computational study of KTNS, MPCA, KTNS – bitwise, MPCA – bitwise algorithms as an intermediate implementations from KTNS to MPCA – bitwise.

All computations were performed on an Intel® Core™ i5 CPU 2.60 GHz computer with 4 GB or RAM. MPCA – bitwise, KTNS – bitwise, MPCA are implemented in $C++$. The implementation of KTNS was taken from the repository published by Mecler et al.[7] Both MPCA and KTNS was compilated with g++ version 10.3.0 using $-O3$ flag.
$10^5$ random job sequences were generated for each Catanzaro et al. [5] dataset, each dataset contains 10 instances, so each row in Table 1 shows the processing time of $10^6$ problems by KTNS and proposed MPCA. Results of computational experiments is given in Table 7, Figure 1,2. You can see, as mentioned in the previous section, KTNS is accelerating with $C$ growth, while MPCA is decelerating, which can be seen in the graph as non-monotonicity of KTNS. On average, MPCA is 6 times faster than KTNS on type A datasets, 11 times on type B datasets, 28 times on type C datasets, 59 times on type D datasets.

![Graphs showing comparison between KTNS, MPCA, KTNS-bitwise, and MPCA-bitwise for Catanzaro et al. datasets](image1)

![Graph showing comparison between KTNS, MPCA, KTNS-bitwise, and MPCA-bitwise for Catanzaro et al. datasets in logarithmic scale](image2)

Figure 4: Comparison of KTNS[7], MPCA, KTNS-bitwise, MPCA-bitwise for Catanzaro et al. datasets.
Figure 5: Comparison of KTNS[7], MPCA, KTNS-bitwise, MPCA-bitwise for Catanzaro et al. datasets in logarithmic scale.

Table 6: Comparison of KTNS[7], MPCA, KTNS-bitwise, MPCA-bitwise, for Catanzaro et al. datasets.

| dataset | n  | m  | C | KTNS[7] | MPCA | KTNS-bitwise | MPCA-bitwise |
|---------|----|----|---|---------|------|--------------|--------------|
| A1      | 10 | 10 | 4 | 1.377   | 0.401| 0.380        | 0.161        |
| A2      | 10 | 10 | 5 | 1.334   | 0.449| 0.383        | 0.217        |
| A3      | 10 | 10 | 6 | 1.215   | 0.481| 0.376        | 0.256        |
| A4      | 10 | 10 | 7 | 1.049   | 0.496| 0.345        | 0.291        |
| B1      | 15 | 20 | 6 | 5.493   | 0.792| 0.628        | 0.302        |
| B2      | 15 | 20 | 8 | 5.187   | 0.958| 0.651        | 0.449        |
| B3      | 15 | 20 | 10| 4.554   | 1.023| 0.633        | 0.551        |
| B4      | 15 | 20 | 12| 3.969   | 1.063| 0.591        | 0.637        |
| C1      | 30 | 40 | 15| 34.291  | 3.335| 1.278        | 0.744        |
| C2      | 30 | 40 | 17| 31.021  | 3.640| 1.313        | 0.910        |
| C3      | 30 | 40 | 20| 26.690  | 3.817| 1.379        | 1.134        |
| C4      | 30 | 40 | 25| 19.972  | 4.071| 1.333        | 1.592        |
| D1      | 40 | 60 | 20| 82.308  | 5.677| 2.024        | 1.010        |
| D2      | 40 | 60 | 22| 76.167  | 6.149| 1.805        | 1.169        |
| D3      | 40 | 60 | 25| 69.728  | 6.596| 1.909        | 1.389        |
| D4      | 40 | 60 | 30| 60.206  | 6.931| 1.972        | 1.859        |
Summary and Future Research Directions

Our MPCA – bitwise algorithm speeds up 59 times on average compared to KTNS for large-scale datasets type D [5]. In further research we aim to obtain a more accurate time complexity of the algorithm and test it on larger SSP benchmark instances. We also intend to measure the effect of incorporating MPCA – bitwise into exact and approximate algorithms for solving SSP. Another goal is to investigate how the pipe characteristics change when increasing the capacity of the magazine C, number of jobs n, number of tools m, and how this correlates with the optimality of the sequence.

PROOF OF THEOREM 1

Let $S = (1, \ldots n)$ be a sequence of jobs, $M \in M(S)$ - sequence of magazine states, such that at instant $i$ job $i$ can be performed i.e. $T_i \subseteq M_i$, where $i = 1, \ldots, n$.
Let $G_M = (V, A)$ denote graph where $V = \{v^t_i : tool \in M_i\}$ and $A = \{(v^t_i, v^t_{i+1}) : v^t_i, v^t_{i+1} \in V\}$ i.e. arc exists iff tool is planned at instant $i$ and at next instant $i + 1$. $V$ shows the content of each slot of the magazine at each moment of time, since $v^t_i \in V$ iff a tool $t$ is contained in magazine at instant $i$. $(v^t_i, v^t_{i+1}) \in A$ iff no switch of a tool $t$ at instant $i + 1$ is needed. Therefore number of tool switches in $M$ is equal to $C(n-1) - |A|$ e.i. the number of all possible places where switch might be needed minus number of places where switch are not needed.

Figure 6: Example of $G_M$. Blue arcs are arcs of $G_M$, red lines symbolize switches.

Let’s call a needed vertex a vertex $v^t_i$ such that $v^t_i \in T_i$. Let’s call a useless vertex a vertex $v^t_i$ such that $v^t_i \notin T_i$.

$\mathcal{S}(M) := \{\pi^t_{s,e} = (v^t_s, \ldots, v^t_e) : t \in T_i \cap T_e, t \notin \bigcup_{i=s+1}^{e-1} T_i, t \in \bigcap_{i=s+1}^{e-1} M_i\}$ - the set of all pipes in $M$.
Let $\mathcal{H}^*(M) := \{P - path in G_M : P contains no common arcs with pipes\}$

$$\mathcal{H}(M) := \{P \in \mathcal{H}^*(M) : P - inclusion-wise maximal path\}$$

$\mathcal{H}(M) = \mathcal{H}_0(M) \sqcup \mathcal{H}_1(M) \sqcup \cdots \sqcup \mathcal{H}_n(M)$, where $\mathcal{H}_r(M) := \{P \in \mathcal{H}(M) : there are exactly$ $r$ needed vertices $\}$

Lemma 1: $\mathcal{H}(M) = \mathcal{H}_0(M) \sqcup \mathcal{H}_1(M)$

Proof:
Let $P \in \mathcal{H}_r(M), r \geq 2$, $P = (v^t_1, \ldots, v^t_n)$ then let’s consider two needed vertices $v^t_i,v^t_j$ such that $v^t_{i+1},\ldots,v^t_{j-1}$ are useless, then $P$ is a pipe $\pi^t_{i,j}$, therefore $P$ contains a pipe as subpath, therefore $P \notin \mathcal{H}^*(M)$, therefore $P \notin \mathcal{H}(M)$ □.
Lemma 1.2. \( \forall P \neq P' \in \mathcal{H}(M) \ A(P) \cap A(P') = \emptyset \)

**Proof:** Suppose \( P \neq P' \) and \( \exists a \in A(P) \cap A(P') \), then \( P, P' \) belong to one connected subgraph of \( G_M \). Since the arcs are only between adjacent instants and at one instant the tool cannot be present twice, then any connected subgraph of \( G_M \) is a path. \( P, P' \in \mathcal{H}(M) \) then they are inclusion-wise maximal paths, then if they have a common arc \( a \), then \( P = P' \), which leads to a contradiction \( \Box \).

Lemma 1.3. \( \forall \pi \neq \pi' \in \mathcal{T}(M) A(\pi) \cap A(\pi') = \emptyset \)

**Proof:** Suppose \( \pi^{t,s}_{s,e} \neq \pi'^{t',s'}_{s',e'} \) and \( \exists a \in (\pi^{t,s}_{s,e} \cap A(\pi'^{t',s'}_{s',e'})) \), then \( \pi^{t,s}_{s,e}, \pi'^{t',s'}_{s',e'} \) belong to one connected subgraph of \( G_M \). Since \( \pi^{t,s}_{s,e} \) and \( \pi'^{t',s'}_{s',e'} \) have common arc \( a \), then \( t = t' \) (Since arcs only connect vertices with the same tool). Since \( \pi^{t,s}_{s,e} \) is a pipe, then \( v^{t,s}_{s+1} \ldots v^{t,s}_{e-1} \) are useless vertices, then since \( v^t_s \) is needed vertex then \( v'^{t,s}_{s'} = v^t_s \) or \( v'^{t,s}_{s'} = v^{t,s}_{e} \). Since \( \pi^{t,s}_{s,e} \) and \( \pi'^{t',s'}_{s',e'} \) have common arc \( a \), then \( \pi'^{t',s'}_{s',e'} \) starts no later than \( e - 1 \), with implies \( s = s' \). Similar reasoning leads to \( e = e' \). \( \pi^{t,s}_{s,e} \) and \( \pi'^{t',s'}_{s',e'} \) have the same start \( s \), the same end \( e \), the same tool \( t \) and they belong to one connected subgraph of \( G_M \) which is path, then \( \pi^{t,s}_{s,e} = \pi'^{t',s'}_{s',e'} \), which leads to a contradiction \( \Box \).

Lemma 1.4. \( \text{switches}(M) = \sum_{j=1}^{n} |J_j| - C - |\mathcal{T}(M)| + |\mathcal{H}_0(M)| \)

**Proof:**
According to the Lemma 1 \( A(G_M) = A(\mathcal{T}(M)) \cup A(\mathcal{H}(M)) = A(\mathcal{T}(M)) \cup A(\mathcal{H}_0(M)) \cup A(\mathcal{H}_1(M)) \)
\( \lambda := C \cdot n - \sum_{i=1}^{n} |T_i| \) i.e. total number of useless vertices \( G_M \).
\( \lambda = \lambda_0 + \lambda_1 + \lambda_2 \), where \( \lambda_0 \) is number of useless vertices contained in paths from \( \mathcal{H}_0(M) \), \( \lambda_1 \) from \( \mathcal{H}_1(M) \), \( \lambda_2 \) from \( \mathcal{T}(M) \).
Let \( P \in \mathcal{H}_0(M) \), then \( |A(P)| = |V(P)| - 1 = \#(\text{useless vertices in } P) + \#(\text{needed vertices in } P) - 1 = \#(\text{useless vertices in } P) + 0 - 1 = \#(\text{useless vertices in } P) - 1 \)
Then according to the Lemma 2 \( |A(\mathcal{H}_0(M))| = \sum_{P \in \mathcal{H}_0(M)} (\#(\text{useless vertices in } P) - 1) = \lambda_0 - |\mathcal{H}_0(M)| \)
Let \( P \in \mathcal{H}_1(M) \), then \( |A(P)| = |V(P)| - 1 = \#(\text{useless vertices in } P) + \#(\text{needed vertices in } P) - 1 = \#(\text{useless vertices in } P) + 1 - 1 = \#(\text{useless vertices in } P) \)
Then according to the Lemma 2 \( |A(\mathcal{H}_1(M))| = \sum_{P \in \mathcal{H}_1(M)} \#(\text{useless vertices in } P) = \lambda_1 \).
Let \( P \in \mathcal{T}(M) \), then \( |A(P)| = |V(P)| - 1 = \#(\text{useless vertices in } P) + \#(\text{needed vertices in } P) - 1 = \#(\text{useless vertices in } P) + 2 - 1 = \#(\text{useless vertices in } P) + 1 \)
Then according to the Lemma 3 \( |A(\mathcal{T}(M))| = \sum_{\pi \in \mathcal{T}(M)} (\#(\text{useless vertices in } P) + 1) = \lambda_2 + |\mathcal{T}(M)| \).
Then \( |A(G_M)| = |A(\mathcal{T}(M)) \cup A(\mathcal{H}_0(M)) \cup A(\mathcal{H}_1(M))| = |A(\mathcal{H}_0(M))| + |A(\mathcal{H}_1(M))| + |A(\mathcal{T}(M))| = \lambda_0 - |\mathcal{H}_0(M)| + \lambda_1 + \lambda_2 + |\mathcal{T}(M)| = \lambda + |\mathcal{T}(M)| - |\mathcal{H}_0(M)| = C \cdot n - \sum_{j=1}^{n} |T_j| + |\mathcal{T}(M)| - |\mathcal{H}_0(M)| \).
Since \( \text{switches}(M) = C \cdot (n-1) - |A(G_M)| \), then \( \text{switches}(M) = C \cdot (n-1) - (C \cdot n - \sum_{j=1}^{n} |T_j| + |\mathcal{T}(M)| - |\mathcal{H}_0(M)|) = \sum_{j=1}^{n} |T_j| - C - |\mathcal{T}(M)| + |\mathcal{H}_0(M)| \) \( \Box \).

Theorem 1. \( \min_{M \in \mathcal{M}(S)} \{ \text{switches}(M) \} = \sum_{i=1}^{n} |T_i| - C - \max_{M \in \mathcal{M}(S)} \{|\mathcal{T}(M)|\} \).

**Proof:**
Since according to Theorem 4. \( \text{switches}(M) = \sum_{j=1}^{n} |T_j| - C - |\mathcal{T}(M)| + |\mathcal{H}_0(M)| \)
Let’s proof that \( \exists M \in \operatorname{argmax}_{M \in \mathcal{M}(S)} \{|\mathcal{T}(M)|\} \) such that \( |\mathcal{H}_0(M)| = 0 \).
Let’s consider an arbitrary \( M \in \operatorname{argmax}_{M \in \mathcal{M}(S)} \{|\mathcal{T}(M)|\} \). Let \( R_i = \{ t : \exists \text{ path } P \in \mathcal{H}_0(M) \text{ such that } v^t_i \).
in P'.
Then let $M'_1 = M_1 \setminus R_1, \ldots, M'_n = M_n \setminus R_n$ i.e. there is $\sum_{i=1}^n |R_i|$ empty slots in sequence of magazine states $M'$.
If it is proved that it is always possible to fill in an empty slot at instant $i$ so that the added tool $t \in M'_{i-1}$ or $t \in M'_{i+1}$, then after $\sum_{i=1}^n |R_i|$ repetitions of such a procedure all slots will be filled and $\mathcal{K}_0(M)$ will be empty.
Suppose that there is no $M_i$ with empty slot that $t \notin M_i$ and $t \in M_{i-1} \cup M_{i+1}$, then $M_{i-1} \cup M_{i+1} \subseteq M_i$, then $M_{i+1}$ has empty slot too, than $M_i \cup M_{i+2} \subseteq M_{i+1}$, then $M_i = M_{i+1}$. Then $M_1 = M_2 = \cdots = M_n$ and $|M_1| < C$, than $|\bigcup_{i=1}^n M_i| = |M_1| < C$, but $T_i \subseteq M_i$, $i = 1, \ldots, n$, then $\bigcup_{i=1}^n T_i \subseteq \bigcup_{i=1}^n M_i$, then $|\bigcup_{i=1}^n T_i| < C$, but $|\bigcup_{i=1}^n T_i| = m > C$, which leads to a contradiction.
Finally $\exists M^* \in \arg\max_{M \in \mathcal{M}(S)} \{|\mathcal{T}(M)|\}$ such that $|\mathcal{K}_0(M^*)| = 0$, then since $\sum_{i=1}^n |T_i|$ and $C$ are constants and $|\mathcal{K}_0(M)|$ can be decreased to zero, then $\min_{M \in \mathcal{M}(S)} \{\text{switches}(M)\} = \sum_{i=1}^n |T_i| - C - \max_{M \in \mathcal{M}(S)} \{|\mathcal{T}(M)|\} \square.$

**PROOF OF THEOREM 2**

Let $L(S) = \{L = (L_{S_1}, \ldots, L_{S_n}) : \forall i \in J \ T_i \subseteq L_i, |L_i| \geq C\}$ is a magazine state sequence in which job can be performed in the order $S \in S$, where empty slots are allowed (which limits the conditions $|L_i| \geq C$). Note that $M(S) \subseteq L(S)$, since empty slots are not allowed in $M(S)$, i.e. the magazine will pay in full at every moment of time, i.e. $\forall i \in J \ |M_i| = C$.

Let $\text{possib\_pipes}(S) = \{\pi_{s,e}' : \exists L \in L(S) : \pi_{s,e}' \in \mathcal{T}(L)\} = \{\pi_{s,e}' : |T_j| < C, j = s + 1, \ldots, e - 1, t \in (T_s \cap T_e)\ \setminus \bigcup_{i=s+1}^e T_i\}$ denote all possible pipes that can be constructed (not at the same time), where $S$ is a sequence of jobs.
Let $\text{can\_construct}(L) = \{\pi_{s,e}' \in \text{possib\_pipes}(S) : t \notin L_j, |L_j| < C, j = s + 1, \ldots, e - 1\}$ denote all pipes that can be constructed in $L$.
We denote by $\hat{L}(S)$ the set of all possible sequences of the state of the magazine with possible empty slots, empty slots were filled only when constructing pipes.

**Lemma 2.1:** Let $L \in \hat{L}(S)$ then $\text{can\_construct}(L) = \{\pi_{s,e}' \in \text{possib\_pipes}(S) \setminus \mathcal{T}(L) : \forall j \in \{s + 1, \ldots, e - 1\} \ |L_j| < C\}$.

That is if only pipes has been constructed in $L$, than pipe $\pi_{s,e}'$ can be constructed iff $\pi_{s,e}'$ has not been constructed in $L$ and there is enough empty slots to construct $\pi_{s,e}'$.

**Proof:**
Based on the definition of $\text{can\_construct}(\cdot)$, it suffices to prove that if $L \in \hat{L}(S)$ and $\pi_{s,e}' \in \text{possib\_pipes}(S) \setminus \mathcal{T}(L)$, then $\forall j \in \{s + 1, \ldots, e - 1\} \ t \notin L_j$. Suppose $\exists j \in \{s + 1, \ldots, n\} : t \in L_j$, since $\pi_{s,e}'$ is a pipe, then $\forall i \in \{s + 1, \ldots, e - 1\} \ t \notin T_i$, , then $t \notin T_j$, which implies that vertex $\pi_{j}'$ is useless, then since only pipes has been constructed $\exists \pi_{s,e}' \in \mathcal{T}(L) : s' < j < e'$. Since $\pi_{s,e}'$ is a pipe, than $\forall i \in \{s' + 1, \ldots, e' - 1\} \ t \notin T_i$, and since $\pi_{s,e}'$ is a pipe, than $t \in T_s$, than $s' \geq s$. Since $\pi_{s,e}'$ is a pipe, than $\forall i \in \{s + 1, \ldots, e - 1\} \ t \notin T_i$, and since $\pi_{s,e}'$ is a pipe, than $t \in T_s$ than $s' \leq s$, then $s = s'$ and similarly $e = e'$, which implies that $\pi_{s,e}'$ and $\pi_{s,e}'$ are the same pipe, then $\pi_{s,e}' \in \mathcal{T}(L)$ and $\pi_{s,e}' \in \text{possib\_pipes}(S) \setminus \mathcal{T}(L)$, which leads to a contradiction $\square$.

Further we always assume that $L \in \hat{L}(S)$, since we will only talk about constructing and removing pipes and otherwise empty slots will not be filled.
Let $L_{\text{opt}}(S) = \{L \in \hat{L}(S) : |\mathcal{T}(L)| = \max\{|\mathcal{T}(M)| : M \in \mathcal{M}(S)\}\}$, i.e. this is the set $L \in \hat{L}(S)$: $L$ contains the smallest possible number of pipes.
Let $L \not\in \mathbf{L}_{\text{opt}}(S)$, then $L$ does not contain the maximum possible number of pipes, i.e. you can remove some (possibly all) constructed pipes, let them be $r_1$ pieces, and then build another set of pipes of capacity $r_2 > r_1$. $L \not\in \mathbf{L}_{\text{opt}}(S)$ then $\exists K \subseteq \text{possib.pipes}(S):$ if we remove $K$ from $L$, then it will be possible to construct $K' \subseteq \text{pipes}(S): |K'| > |K|$ and vice versa, if there is such a pair $(K, K')$, then $L$ contains not the maximum possible number of pipes, i.e. $L \not\in \mathbf{L}_{\text{opt}}(S)$. Let us define $\mathcal{X}(L)$ as the set of all such pairs $(K, K')$. And therefore $\mathcal{X}(L) = \emptyset \iff L \not\in \mathbf{L}_{\text{opt}}(S)$.

Let $\mathcal{X}_{\text{min}}(L) := \{(K, K') \in \mathcal{X}(L): \exists (K, K') \in \mathcal{X}(L) : K \subseteq K$ and $K' \subseteq K' \text{ and } (K \neq K \text{ or } K' \neq K')\}$ i.e. there are many such pairs in which there are no deletions and constructions that could be excluded (for example, it is pointless to delete and add the same pipe).

Since $\mathcal{X}_{\text{min}}(L) = \emptyset \iff \mathcal{X}(L) = \emptyset$, then

**Lemma 2.2:** $L \in \mathbf{L}_{\text{opt}}(S) \iff \mathcal{X}_{\text{min}}(L) = \emptyset$.

Let $\text{needed.instants}(\pi) = \{s, e\} (s_e \in \mathbb{K} \text{ needed.instants}(\pi))$ i.e. the set of all times in which at least one empty slot is needed to build the pipe $\pi$. Let $\text{needed.instants}(\pi) = \bigcup_{s_e \in \mathbb{K}} \text{needed.instants}(\pi)$. I.e. the set of all times in which at least one empty slot is needed to construct the pipe $\pi$.

**Lemma 2.3:** Let $(K, K') \in \mathcal{X}(L), \pi_{s_e} \in K, \pi'_{s_e} \in K'$, then if

$\text{needed.instants}(\pi_{s_e}) \cap \text{needed.instants}(\pi') \subseteq \text{needed.instants}(\pi_{s_e})$, then $(K, K') \not\in \mathcal{X}_{\text{min}}(L)$

**Proof:**

Let’s denote $\pi = \pi_{s_e}, \pi' = \pi'_{s_e}$. We get $L_0^\pi$ by removing $K$ from $L$. We obtain $L_{\pi}^\tau$ by constructing $\tau$ in $L_0^\pi$. We obtain $L_{\pi}^\tau$ by constructing $\pi$ in $L_0^\pi$. $\pi' := K' \setminus \{\pi\}$. Thus, $\pi'$ can be constructed in $L_0^\pi$, let us prove that $\pi'$ can also be constructed in $L_{\pi}^\tau$. Since $\text{needed.instants}(\pi) \cap \text{needed.instants}(\pi') \subseteq \text{needed.instants}(\pi)$, then $\text{needed.instants}(\pi) \cap \text{needed.instants}(\pi') \subseteq \text{needed.instants}(\pi) \cap \text{needed.instants}(\pi')$. Since $\pi' \subseteq K'$, then $\text{needed.instants}(\pi) \cap \text{needed.instants}(\pi') \subseteq \text{needed.instants}(\pi) \cap \text{needed.instants}(\pi')$, which (according to **Lemma 2.1**) means that $\pi'$ can be constructed in $L_0^\pi$, then $\tau'$ can also be constructed in $L_{\pi}^\tau$ that is $\tau'$ can be excluded from the set of $K$ being removed, and $\tau'$ can be included from the set of $K'$ being added. Then $K := K \setminus \{\pi\}, K' := K' \setminus \{\tau\}$, then $\exists (\tilde{K}, \tilde{K}') \in \mathcal{X}(L): K \subseteq K$ and $\tilde{K}' \subseteq K'$, therefore $(K, K') \not\in \mathcal{X}_{\text{min}}(L)$. □

**Example:** $\text{needed.instants}(\pi) := \{2, 3, 4\}$, $\text{needed.instants}(\tau) := \{3, 4, 5, 6, 7\}$, $\text{needed.instants}(\pi) := \{3, 4\}$. According to **Lemma 2.1** since pipes from $\tilde{K}'$ will not use slots whose time points are from $\{5, 6, 7\}$, then for the possibility of constructing $\tilde{K}'$, it does not matter whether they are filled or not, it is important only is there enough empty slots at the time instants $\text{needed.instants}(\tilde{K}')$, then the "threat" for $\tilde{K}'$ from constructing $\pi$ is $\text{needed.instants}(\tilde{K}') \cap \text{needed.instants}(\tau) = \{3, 4\}$, the "threat" to $\tilde{K}'$ from building $\tau$ is $\text{needed.instants}(\tilde{K}') \cap \text{needed.instants}(\tau) = \{3, 4\}$ i.e. with respect to $\tilde{K}'$ the construction of $\pi$ is no worse than the construction of $\tau$, and if so, if $\pi$ is contained in the set of $K$ to be removed, $\tau$ is contained in the set of $K'$ to be added, then $\pi, \tau$ can be simultaneously removed from these sets by obtaining the sets $(\tilde{K}, \tilde{K}') \in \mathcal{X}(L)$, then by definition $\mathcal{X}_{\text{min}}(L)$ it is true that $\mathcal{X}_{\text{min}}(L)$ does not contain $(K, K')$.

**Theorem 2:** $|\mathcal{F}(\text{MPCA}(S))| = \max\{|\mathcal{F}(M)| : M \in M(S)\}$

**Proof:**

According to **Lemma 2.2**, it will suffice to prove that $\mathcal{X}_{\text{min}}(L) = \emptyset$, where $L = \text{MPCA}(S)$
Suppose the opposite, i.e. \( \exists (K, K') \in \mathcal{X}_{\min}(L) \). Since the Algorithm 1 tries to build all pipes from \( \text{possib.pipes}(S) \) and builds if possible. Then it is impossible to complete one more pipe in \( L \) without deleting one before this, therefore the set of pipes to be removed is always not empty. \( K \neq \emptyset \) and since by definition \( |K'| > |K| \), then \( K' \neq \emptyset \).

\[ \min \_\text{end}(K) := \min \{ \text{end} : \pi^{\text{tool}}_{\text{start}, \text{end}} \in K \} \]

**Case 1:** \( e = \min \_\text{end}(K) = \min \_\text{end}(K') = e' \).
Then let \( \pi^t_{s, e} \in K \), \( \pi^{t'}_{s', e'} \in K' \). 2 cases are possible

1. \( s \geq s' \). Then since \( e' = e \), then \( \text{needed.instants}(\pi^t_{s, e}) \subseteq \text{needed.instants}(\pi^{t'}_{s', e'}) \). Therefore, according to the Lemma 2.3 \( (K, K') \notin \mathcal{X}_{\min}(L) \), which leads to a contradiction.

2. \( s < s' \). Since \( \text{needed.instants}(\pi^{t'}_{s', e'}) \subseteq \text{needed.instants}(\pi^t_{s, e}) \), then after deletion \( \pi^t_{s, e} \) it will be possible to build \( \pi^{t'}_{s', e'} \) and therefore it will be possible to move on to \( (K, K') : K = K' \setminus \{\pi^t_{s, e}\}, K' = K' \setminus \{\pi^{t'}_{s', e'}\} \). It is impossible to endlessly get into **Case 1.2** because each time the cardinality of \( K \) and \( K' \) decreases by 1.

**Case 2:** \( e = \min \_\text{end}(K) < \min \_\text{end}(K') = e' \).
Then let \( \pi^t_{s, e} \in K \), \( \pi^{t'}_{s', e'} \in K' \). 2 cases are possible:

1. \( s \geq s' \). Then since \( e < e' \), then \( \text{needed.instants}(\pi^t_{s, e}) \subseteq \text{needed.instants}(\pi^{t'}_{s', e'}) \). Therefore, according to Lemma 2.3 \( (K, K') \notin \mathcal{X}_{\min}(L) \), which leads to a contradiction.

2. \( s < s' \). Let \( s'' = \min \{ i : \pi^k_{i,j} \in K' \} \) and we will consider the pipe \( \pi^{t''}_{s'', e''} \in K' \), then since \( s' \geq s'' \) either we go to **Case 2.1**, or \( s < s'' \) and from the minimality of \( s'' \) and \( e < e'' \) it follows that \( \text{needed.instants}(\pi^t_{s, e}) \cap \text{needed.instants}(K') \subseteq \text{needed.instants}(\pi^{t''}_{s'', e''}) \). Therefore, according to Lemma 4 \( (K, K') \notin \mathcal{X}_{\min}(L) \), which leads to a contradiction.

**Case 3:** \( e = \min \_\text{end}(K) > \min \_\text{end}(K') = e' \).
Then let \( \pi^{t'}_{s', e'} \in K' \), \( s = \min \{ i : \pi^k_{i,j} \in K, j = \text{end} \}, \pi^t_{s, e} \in K \).

Emptying the slots at each of the times from the set \( \text{needed.instants}(\pi^{t'}_{s', e'}) \cap \text{needed.instants}(K) \) will allow you to build a pipe \( \pi^{t'}_{s', e'} \), otherwise even after removing \( K \) it would not be possible to build the pipe \( \pi^{t'}_{s', e'} \). But note that \( s \) is minimally possible and \( e' > e \), which implies that \( \text{needed.instants}(\pi^{t'}_{s', e'}) \cap \text{needed.instances}(K) \subseteq \text{needed.instants}(\pi^t_{s, e}) \), which means that after removing the pipe \( \pi^t_{s, e} \) it will be possible to construct \( \pi^{t'}_{s', e'} \). Consequently, the Algorithm 1 constructed the pipe \( \pi^{t'}_{s', e'} \) at an iteration earlier than \( \pi^t_{s, e} \), since \( e' < e \), and the Algorithm 1 iterates over the variable \( \text{end} \) in ascending order. At the iteration, when the variable \( \text{end} \) was equal to \( e' \), the pipe \( \pi^{t'}_{s', e'} \) should have been built, since at that moment the pipes \( \pi^t_{s, e} \) didn’t exist yet, which is the same as it was removed. From which it follows that \( \pi^{t'}_{s', e'} \) has already been built, which contradicts the fact that it belongs to \( K' \). Let’s move on to \( (\tilde{K}, \tilde{K}') : \tilde{K} = K \setminus \{\pi^{t'}_{s', e'}\}, \tilde{K}' = K' \setminus \{\pi^{t'}_{s', e'}\} \) and note that after removing \( \tilde{K} \) it will be possible to build \( \tilde{K}' \) i.e. \( (\tilde{K}, \tilde{K}') \in \mathcal{X}(L) \), but \( \tilde{K} \subseteq K, \tilde{K}' \subseteq K' \). Then, by the definition of \( \mathcal{X}_{\min}(L) \), \( (K, K') \notin \mathcal{X}_{\min}(L) \), which leads to a contradiction.

**Case 1, Case 2, Case 3** led to a contradiction, hence the assumption about \( \exists (K, K') \in \mathcal{X}_{\min}(\text{MPCA}(S)) \) is not true, then by Lemma 2.2 we have \( \text{MPCA}(S) \in L_{\text{opt}}(S) \), which means that \( |\mathcal{I}(\text{MPCA}(S))| = \max\{|\mathcal{I}(M)| : M \in M(S)\} \). □
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