Combinatorial $n$-fold Integer Programming and Applications

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Abstract

Integer Linear Programming is a famous NP-complete problem. Lenstra showed that in the case of small dimension, it can be solved in polynomial time. This algorithm became a ubiquitous tool, especially in the design of parameterized algorithms for NP-complete problems, where we wish to isolate the hardness of an instance to some parameter. However, it was discovered that in many cases using Lenstra’s algorithm has two drawbacks. First, the dependence of the resulting algorithms is often doubly-exponential in the parameter, and second, an ILP formulation in small dimension can not easily express problems which involve many different costs.

Inspired by the work of Hemmecke, Onn and Romanchuk [Math. Prog. 2013], we develop a single-exponential algorithm for so called combinatorial $n$-fold integer programs, which are remarkably similar to prior ILP formulations for various problems, but unlike them, also allow variable dimension. We then apply our algorithm to a few representative problems like CLOSEST STRING, SWAP BRIBERY, WEIGHTED SET MULTICOVER etc., and obtain exponential speedups in the dependence on the respective parameters, the input size, or both.

Unlike Lenstra’s algorithm, which is essentially a bounded search tree algorithm, our result uses the technique of augmenting steps. At its heart is a deep result stating that in combinatorial $n$-fold IPs an existence of an augmenting step implies an existence of a “local” augmenting step, which can be found using dynamic programming. Our results provide an important insight into many problems by showing that they exhibit this phenomenon, and highlights the importance of augmentation techniques.

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Introduction

The INTEGER LINEAR PROGRAMMING (ILP) problem is fundamental because it models many problems. Since it is NP-complete, we naturally ask about the complexity of special cases. Famously, Lenstra proved in 1983 that ILP is polynomial when the number of variables (the dimension) $d$ is fixed [35], which makes it a natural tool to prove that the complexity of some special cases of other NP-hard problems is also polynomial.

A systematic way to study the complexity of “special cases” of NP-hard problems has been developed in the past 25 years in the field of parameterized complexity. There, along
with the size of an instance $n$, we also distinguish some parameter $k$, and then we study the complexity of our problem in terms of both $n$ and $k$. Of central importance are algorithms with runtime of the form $f(k)n^O(1)$, called FPT algorithms (for “fixed parameter tractable”). For more background on parameterized complexity we refer the reader to monograph [11].

Kannan’s improvement [29] of Lenstra’s algorithm has complexity $d^{O(k \log k)} n$, making it an FPT algorithm with parameter $d$. An algorithm pioneering the application of Lenstra’s algorithm solves the CLOSEST STRING problem [23], and has become widely known after being pointed out by Niedermeier [39]:

\[ \ldots \] it remains to investigate further examples besides CLOSEST STRING where the described ILP approach turns out to be applicable. More generally, it would be interesting to discover more connections between fixed-parameter algorithms and (integer) linear programming.

Many applications of Lenstra’s algorithm to various problems have been proposed, proving its ubiquitousness. However, many of them [7, 10, 15, 27, 34] share a common trait with the algorithm for CLOSEST STRING: they have a doubly-exponential dependence on the parameter. Moreover, it is difficult to apply Lenstra’s algorithm where the input contains many objects with varying cost functions, such as in SWAP BRIEBRY [6, Challenge #2].

1.1 Our contributions

We show that a certain form of ILP, which is closely related to the previously used formulations for CLOSEST STRING and other problems, can be solved in single-exponential time and in variable dimension. For example, Gramm et al.’s [23] algorithm for CLOSEST STRING runs in time $2^{O(k \log k)} \log L$ and has not been improved since 2003, while our algorithm runs in time $O(k^2 \log L)$. Moreover, our algorithm has a strong combinatorial flavor and is based on different notions than are typically encountered in parameterized complexity, most importantly augmenting steps.

As an example of our form of ILP, consider the following ILP formulation of the CLOSEST STRING problem. We are given $k$ strings $s_1, \ldots, s_k$ of length $L$ that come (after some preprocessing) from alphabet $[k]$, and an integer $d$. The goal is to find a string $y \in [k]^L$ such that, for each $s_i$, the Hamming distance $d_H(y, s_i)$ is at most $d$, if such $y$ exists. For $i \in [L]$, $(s_1[i], \ldots, s_k[i])$ is the $i$-th column of the input. Clearly there are at most $k^k$ different column types in the input, and we can represent the input succinctly with multiplicities $b_f$ of each column type $f \in [k]^k$. Moreover, there are $k$ choices for the output string $y$ in each column. Thus, we can encode the solution by, for each column type $f \in [k]^k$ and each output character $e \in [k]$, describing how many solution columns are of type $(f, e)$. This is the basic idea behind the formulation of Gramm et al. [23], as depicted on the left.

\[
\begin{align*}
\sum_{e \in [k]} \sum_{f \in [k]} d_H(e, f_j) x_{f, e} &\leq d \quad \forall j \in [k] \\
\sum_{e \in [k]} x_{f, e} &= b_f \\
x_{f, e} &\geq 0 \\
\sum_{(f', e) \in [k]^{k+1}} d_H(e, f_j) x_{f', e} &\leq d \\
x_{f', e} &= b_{f'} \\
x_{f, e} &\geq 0 \\
x_{f', e} &\geq 0 \\
0 \leq x_{f, e} &\leq b_f \\
x_{f', e} &\geq 0 \\
\forall f' \neq f, \forall e \in [k] \\
\forall f \in [k]^k
\end{align*}
\]
Let $\mathbf{1} = (1 \cdots 1) = \mathbf{1}^\top$ be a row vector of all ones. Then we can view the above as

$$
\begin{array}{cccccccc}
D_1 & D_2 & \cdots & D_{k^4} & \leq d & D & D & \cdots & D & \leq d \\
\mathbf{1}^\top & 0 & \cdots & 0 & = b^1 & \mathbf{1}^\top & 0 & \cdots & 0 & = b^1 \\
0 & \mathbf{1}^\top & \cdots & 0 & = b^2 & 0 & \mathbf{1}^\top & \cdots & 0 & = b^2 \\
\vdots & \vdots & \ddots & \vdots & = \vdots & \vdots & \vdots & \ddots & \vdots & = \vdots \\
0 & 0 & \cdots & \mathbf{1}^\top & = b^{k^4} & 0 & 0 & \cdots & \mathbf{1}^\top & = b^{k^4},
\end{array}
$$

where $D = (D_1 \ D_2 \ \cdots \ D_{k^4})$. The formulation on the right is clearly related to the one on the left, but contains “dummy” variables which are always zero. This makes it seem unnatural at first, but notice that it has the nice form

$$
\min \left\{ f(\mathbf{x}) \mid E^{(n)} \mathbf{x} = \mathbf{b}, \ 1 \leq \mathbf{x} \leq \mathbf{u}, \ \mathbf{x} \in \mathbb{Z}^{nt} \right\}, \text{ where } E^{(n)} := \begin{pmatrix} D & D & \cdots & D \\ A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}. \tag{1}
$$

Here, $r, s, t, n \in \mathbb{N}$, $\mathbf{u}, \mathbf{l} \in \mathbb{Z}^{nt}$, $\mathbf{b} \in \mathbb{Z}^{r+ns}$ and $f : \mathbb{Z}^{nt} \rightarrow \mathbb{Z}$ is a separable convex function, $E^{(n)}$ is an $(r + ns) \times nt$-matrix, $D \in \mathbb{Z}^{r \times t}$ is an $r \times t$-matrix and $A \in \mathbb{Z}^{s \times t}$ is an $s \times t$-matrix. We call $E^{(n)}$ the $n$-fold product of $E = (\mathcal{D})$. This problem is known as $n$-fold integer programming $(IP)_{E^{(n)}, b, l, u, f}$. Building on a dynamic program of Hemmecke, Onn and Romanchuk [25] and a so-called proximity technique of Hemmecke, Köppe and Weismantel [21], Knop and Koutecký [31] prove that:

**Proposition 1.** ([31 Theorem 7]). **There is an algorithm that, given $(IP)_{E^{(n)}, b, l, u, f}$ encoded with $L$ bits, solves it in time $a^{O(tr+ts)} \cdot n^3L$, where $a = \max\{\|D\|_\infty, \|A\|_\infty\}$.**

However, since the IP on the right has $t = k^4$, applying Proposition 1 gives no advantage over applying Lenstra. We overcome this by focusing on a special case with $A = (1 \cdots 1) = \mathbf{1}^\top \in \mathbb{Z}^{1 \times t}$, $(b^1, \ldots, b^n) \geq \mathbf{0}$, $\mathbf{u} = \mathbf{0}$, such that for all $i \in [n]$ and $j \in [t]$, $u^i_j \in \{0, \|\mathbf{b}\|_\infty\}^2$ and, $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$, i.e., the objective is linear. We denote $f^i(\mathbf{x}^i) = \mathbf{w}^i \mathbf{x}^i$. We call this form combinatorial $n$-fold IP and achieve an exponential speed-up in $t$:

**Theorem 2.** Let $(IP)_{E^{(n)}, b, 0, u, w}$ be a combinatorial $n$-fold IP instance with $L = \langle \mathbf{b}, \mathbf{0}, \mathbf{u}, \mathbf{w} \rangle$ and $a = \|D\|_\infty$. Then it can be solved in time $t^{O(r)(ar)^{O(r^2)}n^3L}$.

Observe that, when applicable, our algorithm is not only faster than Lenstra’s, but works even if the number $n$ is variable (not parameter).

By applying this result to a few selected problems we obtain exponential improvements in the dependence on the parameter, the length of the input, or both, as presented in Table 1.

**Stringology.** A typical problem from stringology is to find a string $y$ satisfying certain distance properties with respect to $k$ strings $s_1, \ldots, s_k$. All previous results we are aware of for parameter $k$ rely on Lenstra’s algorithm, or were not known at all (e.g. **OPTIMAL**).
Consensus [2] was open for $k \geq 4$). Interestingly, Boucher and Wilkie [5] show the counterintuitive fact that Closest String is easier to solve when $k$ is large, which makes the parameterization by $k$ even more significant. Finding an algorithm with runtime single-exponential in $k$ was a repeatedly posed open problem, e.g. [8, Challenge #1] and [3, Problem 7.1]. By applying our result, we close this gap for a wide range of problems.

▶ Theorem 3. The problems
- Closest String, Farthest String, Distinguishing String Selection, Neighbor String, Closest String with Wildcards, Closest to Most Strings, $c$-HRC and Optimal Consensus are solvable in time $k^{O(k^2 \log L)}$, and,
- $d$-Mismatch is solvable in time $k^{O(k^2) L^2 \log L}$, where $k$ is the number of input strings, $L$ is their length, and we are assuming that the input is presented succinctly by multiplicities of identical columns.

Computational Social Choice. A typical problem in computational social choice involves an election with voters ($V$) and candidates ($C$). A natural and much studied parameter is the number of candidates $|C|$. For a long time, only algorithms double-exponential in $|C|$ were known, and this was posed as an open problem [6, Challenge #1]. Very recently, this gap was closed [32] using Proposition 1; however, while the old results depended logarithmically on the number of voters, the new result depends on it cubically. Moreover, its dependence on the number of candidates is still quite large, namely $|C|^{O(|C|^6)}$. We improve as follows:

▶ Theorem 4. $R$-Swap Bribery can be solved in time
- $|C|^{O(|C|^2)} T^3 (\log |V| + \log \sigma_{\text{max}})$ for $R$ any natural scoring protocol, and,
- $|C|^{O(|C|^1)} T^3 (\log |V| + \log \sigma_{\text{max}})$ for $R$ any $C1$ rule,
where $T \leq |V|$ is the number of voter types and $\sigma_{\text{max}}$ is the maximum cost of a swap.

Weighted Set Multicover. Bredereck et al. [7] points out the Weighted Set Multicover problem. It is a significant generalization of the Set Cover problem, and for example models problems from computational social choice and optimization problems on graphs [16, 19, 34, implicit in]. Bredereck et al. [7] design a double-exponential algorithm for WSM using Lenstra’s algorithm.

Again, applying our result yields an exponential improvement over Bredereck et al. [6] both in the dependence on the parameter and the size of the instance:

▶ Theorem 5. Weighted Set Multicover can be solved in time $k^{O(k^2)} W^3 (\log n + \log w_{\text{max}})$, where $W$ is the number of different weights and $w_{\text{max}}$ is the maximum weight.

Huge $n$-fold IP. Onn [42] introduces a high-multiplicity version of the standard $n$-fold IP problem [1]. It is significant for example because of its connection to the Bin Packing problem in the case of few item sizes, as studied by Goemans and Rothvoss [22]. Previously, Huge $n$-fold IP was shown to be FPT when $D = I$ and $A$ is totally unimodular; using our result, we show that it is also FPT when $D$ and $A$ are arbitrary, but the size of variable domains is bounded by a parameter.

This list is not meant to be exhaustive. In fact, we believe that for any Lenstra-based result in the literature which only achieves double-exponential runtimes, there is a good chance that it can be sped up using our algorithm. The only significant obstacle seem to be large coefficients in the constraint matrix. We provide further insights and discussion in Appendix ??.
1.2 Related work

Our main inspiration are augmentation methods based on Graver bases, especially an FPT algorithm for $n$-fold IP of Hemmecke, Onn and Romanchuk [25]. Our result improves the runtime of their algorithm for a special case. All the following related work is orthogonal to ours in either the achieved result, or the parameters used for it.

In fixed dimension, Lenstra’s algorithm [35] was generalized for arbitrary convex sets and quasiconvex objectives by Khachiyan and Porkolab [30]. The currently fastest algorithm of this kind is due to Dadush et al. [9]. The first notable FPT algorithm for a non-convex objective is due to Lokshtanov [38], who shows that optimizing a quadratic function over the integers of a polytope is in $\tilde{O}(k)$. The first notable FPT algorithm for a non-convex objective is due to Dadush et al. [9]. The first notable FPT algorithm for a non-convex objective is due to Dadush et al. [9]. The first notable FPT algorithm for a non-convex objective is due to Dadush et al. [9].

Besides fixed-parameter tractability, there is interest in the (non)existence of kernels of ILPs, which formalize the (im)possibility of various preprocessing procedures. Jansen and Kratsch [28] show that ILPs containing parts with simultaneously bounded treewidth and bounded domains are amenable to kernelization, unlike ILPs containing totally unimodular parts. Kratsch [33] studies the kernelizability of sparse ILPs with small coefficients.

1.3 Remarks

Comparison with Lenstra’s algorithm. The basic idea behind Lenstra’s algorithm is the following. Given a system $Ax \leq b$ it is possible to compute its volume and determine that it is either too large not to contain an integer point, or too small not to be flat in some direction. In the first case we are done; in the second case we can take $d$ slices of dimension $d-1$ and recurse into them, achieving a $d^{O(d)}n^{O(1)}$ runtime. Note that we only decide feasibility; optimization can be then done by binary search. On the other hand, the basic idea behind our algorithm is the following. We only focus on optimizing and later show that testing feasibility reduces to it. Starting from some feasible solution, the crucial observation is that if there is a step improving the objective, there is one which does not modify many variables, and can be found quickly by dynamic programming. Moreover, if the current solution is far from the optimum, then it is possible to make a long step, and polynomially many long steps will reach the optimum.

More concretely, consider the run of these two algorithms on an instance of Closest String. Lenstra’s algorithm essentially either determines that the bounds are loose enough that there must exist a solution, or (oversimplifying) determines that there is a column type $f \in [k]^k$ and a character $e \in [k]$ such that there are at most $k^k$ competitive choices for
how many times the solution contains character $e$ at a column of type $f$. Then, we recurse, obtaining a $2^{O(k \log k)}$ algorithm. On the other hand, our algorithm views the problem as an optimization problem, so we think of starting with a string of all blanks which is trivially at distance 0 from any string, and the goal is to fill in all blanks such that the result is still in distance at most $d$ from the input strings. An augmenting step is a set of character swaps that decreases the number of blanks. The crucial observation is that if an augmenting step exists, then there is also one only changing few characters, and it can be found in time $k^{O(k^2)}$. Thus (omitting details), we can iteratively find augmenting steps until we reach the optimum.

**Succinctness.** A common aspect shared by all of our applications is that bounding some parameter of the instance makes it preferable to view the instance in a succinct way (following the terminology of Faliszewski et al. [14]; Oum [42, 43] calls these problems huge and Goemans and Rothvoss [22] call them high multiplicity). The standard way of viewing an instance is that the input is a collection of individual objects (bricks, matrix columns, voters, covering sets etc.). The succinct way of viewing an instance is by saying that identical objects are of the same type, giving a bound $T$ on the number of distinct types, and then presenting the input as numbers $n_1, \ldots, n_T$ such that $n_i$ is the number of objects of type $i$. Clearly, any standard instance can be converted to a succinct instance of roughly the same size (the number of objects is an upper bound on $T$), but the converse is not true as the numbers $n_i$ might be large. Also, it is sometimes not trivial (see Section 4.3) that the output can be represented succinctly; still, in all cases which we study it can.

In our applications we always state what are the types and what is the upper bound $T$ on the number of types; we assume some arbitrary enumeration of the types. We also assume that the input is presented succinctly and thus we do not include the time needed to read and convert a standard instance into a succinct instance in the runtime of our algorithms.

**Connections between Stringology and Computational Social Choice.** Challenge #3 of Bulteau [8] asks for connections between problems in Stringology and Computational Social Choice. We have clearly demonstrated that in both fields combinatorial $n$-fold IP is an important tool. An important feature of both Bribery-like problems and Closest String-like problems is that permuting voters or characters does not have any effect. This fits well the $n$-fold IP format, which does not allow any interaction between the bricks. It seems that this feature is important, because when it is taken away, such as in the Closest Substring problem, the problem becomes W[1]-hard [38], even when parameterized by both $d$ and $k$.

Another common feature is that both types of problems naturally admit ILP formulations for succinct variants of the problems, as mentioned above. Moreover, it was precisely this fact that made all previous algorithms doubly-exponential — the natural succinct formulation has exponentially many (in the parameter) variables and thus applying Lenstra’s algorithm leads to a doubly-exponential runtime.

## 2 Preliminaries

Let $m$ and $n$ be integers. We define $[m : n]$ to be the set of all integers between $m$ and $n$, that is, $[m : n] = \{m, \ldots, n\}$. For a positive integer $n$ we define $[n] = [1 : n]$. For a graph $G$ we denote by $V(G)$ the set of its vertices. We write vectors in boldface (e.g., $\mathbf{x}, \mathbf{y}$ etc.) and their entries in normal font (e.g., the $i$-th entry of $\mathbf{x}$ is $x_i$). Given an integer matrix $A \in \mathbb{Z}^{m \times n}$, vectors $\mathbf{b} \in \mathbb{Z}^m$, $\mathbf{l}, \mathbf{u} \in \mathbb{Z}^n$ and a function $f : \mathbb{Z}^n \to \mathbb{Z}$, we denote by $(IP)_{A, \mathbf{b}, \mathbf{l}, \mathbf{u}, f}$
the problem

\[ \min \{ f(x) \mid Ax = b, \quad 1 \leq x \leq u, \quad x \in \mathbb{Z}^n \}. \]

We say that \( \mathbf{x} \) is feasible for \((IP)\) if \( \mathbf{Ax} = \mathbf{b} \) and \( 1 \leq \mathbf{x} \leq \mathbf{u} \). If we want to talk about any such IP, we simply denote it as \((IP)\).

**Graver Bases and Augmentation.** Let us now introduce Graver bases, how they can be used for optimization, and also the special case of \( n \)-fold IPs. For background, we refer to the books of Onn [41] and De Loera et al. [36].

Given two \( n \)-dimensional integer vectors \( \mathbf{x} \) and \( \mathbf{y} \), we say they are sign-compatible if they lie in the same orthant, or equivalently, if for each \( i \in [n] \), the sign of \( x_i \) and \( y_i \) is the same. We say \( \sum_i g_i^t \) is a sign-compatible sum if all \( g_i^t \) are pair-wise sign-compatible. Moreover, we write \( \mathbf{y} \subseteq \mathbf{x} \) if \( \mathbf{x} \) and \( \mathbf{y} \) are sign-compatible and \( |y_i| \leq |x_i| \) for each \( i \in [n] \), and write \( \mathbf{y} \sqcap \mathbf{x} \) if at least one of the inequalities is strict. Clearly, \( \sqsubseteq \) imposes a partial order called “conformal order” on \( n \)-dimensional vectors. For an integer matrix \( A \in \mathbb{Z}^{m \times n} \), its Graver basis \( \mathcal{G}(A) \) is the set of \( \sqsubseteq \)-minimal non-zero elements of the lattice of \( A \), \( \ker_2(A) = \{ z \in \mathbb{Z}^n \mid Az = 0 \} \). An important property of \( \mathcal{G}(A) \) is the following.

**Proposition 6 ([41] Lemma 3.2).** Every integer vector \( \mathbf{x} \neq 0 \) with \( \mathbf{Ax} = 0 \) is a sign-compatible sum \( \mathbf{x} = \sum_i g_i^t \) of Graver basis elements \( g_i^t \in \mathcal{G}(A) \), with some elements possibly appearing with repetitions.

Given a feasible solution \( \mathbf{x} \) to an \((IP)\), we call \( \mathbf{g} \) a feasible step if \( \mathbf{x} + \mathbf{g} \) is feasible in \((IP)\). Moreover, we call \( \mathbf{g} \) an augmenting step if it is a feasible step and \( f(\mathbf{x} + \mathbf{g}) < f(\mathbf{x}) \). Given a feasible solution \( \mathbf{x} \) to \((IP)\), we call a tuple \((\mathbf{g}, \alpha)\) with \( \alpha \in \mathbb{Z} \) a Graver-best step if \( \mathbf{g} \) is an augmenting step and \( \forall \tilde{g} \in \mathcal{G}(A) \) and \( \forall \alpha' \in \mathbb{Z}, f(\mathbf{x} + \alpha \mathbf{g}) \leq f(\mathbf{x} + \alpha' \tilde{g}) \). We call \( \alpha \) the step length. The Graver-best augmentation procedure for an \((IP)\) and a given feasible solution \( \mathbf{x}_0 \) works as follows:

1. If there is no Graver-best step for \( \mathbf{x}_0 \), return it as optimal.
2. If a Graver-best step \((\alpha, \mathbf{g})\) for \( \mathbf{x}_0 \) exists, set \( \mathbf{x}_0 := \mathbf{x}_0 + \alpha \mathbf{g} \) and go to 1.

**Proposition 7 ([36] implicit in Theorem 3.4.1).** Given a feasible solution \( \mathbf{x}_0 \) and a separable convex function \( f \), the Graver-best augmentation procedure finds an optimum in at most \( 2n - 2 \log M \) steps, where \( M = f(\mathbf{x}_0) - f(\mathbf{x}^*) \) and \( \mathbf{x}^* \) is any minimizer.

**n-fold IP.** The structure of \( E^{(n)} \) (in problem \((1)\)) allows us to divide the \( nt \) variables of \( \mathbf{x} \) into \( n \) bricks of size \( t \). We use subscripts to index within a brick and superscripts to denote the index of the brick, i.e., \( x^t_j \) is the \( j \)-th variable of the \( i \)-th brick with \( j \in [t] \) and \( i \in [n] \).

## 3 Combinatorial \( n \)-fold IPs

This section is dedicated to proving Theorem 3. We fix an instance of combinatorial \( n \)-fold IP, that is, \( n, D, \mathbf{b}, \mathbf{u}, \) and \( \mathbf{w} \).

### 3.1 Graver complexity of combinatorial \( n \)-fold IP

The key property of the \( n \)-fold product \( E^{(n)} \) is that, for any \( n \in \mathbb{N} \), the number of nonzero bricks of any \( \mathbf{g} \in \mathcal{G}(E^{(n)}) \) is bounded by some constant \( g(E) \) called the Graver complexity of \( E \). The proof is given for example in [11] Lemma 4.3 and goes roughly as follows. Consider any \( \mathbf{g} \in \mathcal{G}(E^{(n)}) \) and take its restriction to its nonzero bricks \( \tilde{\mathbf{g}} \). By Proposition 6, each
brick $\mathbf{g}^j$ can be decomposed into elements from $\mathcal{G}(A)$, giving a vector $\mathbf{h}$ whose bricks are elements of $\mathcal{G}(A)$. Then, consider a compact representation $\mathbf{v}$ of $\mathbf{h}$ by counting how many times each element from $\mathcal{G}(A)$ appears. Since $\mathbf{g} \in \mathcal{G}(E(n))$ and $\mathbf{h}$ is a decomposition of its nonzero bricks, we have that $\sum_j \mathbf{D} \mathbf{h}^j = 0$. Let $\mathbf{G}$ be a matrix with the elements of $\mathcal{G}(A)$ as columns. It is not difficult to show that $\mathbf{v} \in \mathcal{G}(\mathbf{G})$. Since $\|\mathbf{v}\|_1$ is an upper bound on the number of bricks of $\mathbf{g}$ and thus of nonzero bricks of $\mathbf{g}$ and clearly does not depend on $n$, $g(E) = \max_{\mathbf{v} \in \mathcal{G}(\mathbf{G})} \|\mathbf{v}\|_1$ is finite. Let us make precise two observations from this proof.

Lemma 8 ([26 Lemma 3.1], [41 implicit in proof of Lemma 4.3]). Let $(\mathbf{g}^1, \ldots, \mathbf{g}^n) \in \mathcal{G}(E(n))$. Then, for all $i \in [n]$ there exist vectors $\mathbf{h}^{i,1}, \ldots, \mathbf{h}^{i,n_i} \in \mathcal{G}(A)$ such that $\mathbf{g}^i = \sum_{k=1}^{n_i} \mathbf{h}^{i,k}$, and $\sum_{i=1}^n n_i \leq g(E)$.

Lemma 9 ([26 Lemma 6.1], [41 implicit in proof of Lemma 4.3]). Let $\mathbf{D} \in \mathbb{Z}^{r \times t}$, $\mathbf{A} \in \mathbb{Z}^{s \times t}$, $\mathbf{G} \in \mathbb{Z}^{r \times p}$ be the matrix whose columns are elements of $\mathcal{G}(A)$ and $p = |\mathcal{G}(A)| \leq \|\mathbf{A}\|_\infty^t$, and let $\mathbf{E} = (\frac{\mathbf{A}}{1})$. Then $g(E) \leq \max_{\mathbf{v} \in \mathcal{G}(\mathbf{G})} \|\mathbf{v}\|_1 \leq \|\mathbf{A}\|_\infty^t : (r \|\mathbf{D}\|_\infty)\sp{t}$.

Notice that this bound on $g(E)$ is exponential in $t$. Our goal now is to exploit the fact that the matrix $\mathbf{A}$ in a combinatorial $n$-fold IP is very simple and thus get a better bound.

Lemma 10. Let $\mathbf{D} \in \mathbb{Z}^{r \times t}$, $\mathbf{E} = (\frac{\mathbf{D}}{1})$, and $a = \|\mathbf{D}\|_\infty$. Then, $g(E) \leq t^2 (2ra)^t$.

To see this, we will need to understand the structure of $\mathcal{G}(\mathbf{1}^T)$:

Lemma 11. It holds that $\mathcal{G}(\mathbf{1}^T) = \{ \mathbf{g} \mid \mathbf{g} \text{ has one } 1 \text{ and one } -1 \text{ and } 0 \text{ otherwise} \} \subseteq \mathbb{Z}^t$, $|\mathcal{G}(\mathbf{1}^T)| = t(t-1)$, and for all $\mathbf{g} \in \mathcal{G}(\mathbf{1}^T)$, $\|\mathbf{g}\|_1 = 2$.

Proof. Observe that the claimed set of vectors is clearly $\geq$-minimal in $\ker_{\mathbb{Z}}(\mathbf{1}^T)$. We are left with proving there is no other non-zero $\geq$-minimal vector in $\ker_{\mathbb{Z}}(\mathbf{1}^T)$. For contradiction assume there is such a vector $\mathbf{h}$. Since it is non-zero, it must have a positive entry $h_i$. On the other hand, since $\mathbf{1}^T \mathbf{h} = 0$, it must also have a negative entry $h_j$. But then $\mathbf{g}$ with $g_i = 1$, $g_j = -1$ and $g_k = 0$ for all $k \notin \{i, j\}$ is $\mathbf{g} \supseteq \mathbf{h}$, a contradiction. The rest follows.

Proof of Lemma 10. We simply plug into the bound of Lemma 9. By Lemma 11, $p = t(t-1) \leq t^2$. Also, $\|\mathbf{D}\|_\infty \leq \max_{\mathbf{g} \in \mathcal{G}(\mathbf{1}^T)} \{ \|\mathbf{D}\|_\infty, \|\mathbf{g}\|_1 \} \leq 2a$ where the last inequality follows from $\|\mathbf{g}\|_1 = 2$ for all $\mathbf{g} \in \mathcal{G}(\mathbf{1}^T)$, again by Lemma 11.

3.2 Dynamic programming

Hemmecke, Onn and Romanchuk [25] devise a clever dynamic programming algorithm to find augmenting steps for a feasible solution of an $n$-fold IP. Lemma [8] is key in their approach, as they combine by building a set $Z(E)$ of all sums of at most $g(E)$ elements of $\mathcal{G}(A)$ and then use it to construct the dynamic program. However, such a set $Z(E)$ would clearly be of size exponential in $t$, which we cannot afford. Our insight here is to build a different dynamic program. In [25], the layers of the dynamic program correspond to partial sums of elements of $\mathcal{G}(A)$; in our dynamic program, the layers will correspond directly to elements of $\mathcal{G}(A)$. This makes it impossible to enforce feasibility with respect to lower and upper bounds in the same way as done in [25]; however, we work around this by exploiting the special structure of $\mathcal{G}(A) = \mathcal{G}(\mathbf{1}^T)$ and simpler lower and upper bounds and enforce them by varying the number of layers of given types. Additionally, we also differ in how we enforce feasibility with respect to the upper rows ($D \mathbf{D} \cdots \mathbf{D}$).

Given a brick $i \in [n]$ and $j \in [t]$, let $\mathcal{H}_j^i = \{ \mathbf{h} \in \mathcal{G}(\mathbf{1}^T) \mid h_j = -1, h \leq \mathbf{u}^i \} \cup \{0 \in \mathbb{Z}^t\}$; $\mathcal{H}_j^i$ represents the steps which can decrease coordinate $x_j^i$. Observe that $|\mathcal{H}_j^i| \leq t$. Let
\[ \Sigma(E) = \prod_{i=1}^{t} [-2g(E)a : 2g(E)a] \] be the signature set of \( E \) whose elements are signatures. Essentially, we will use the signature set to keep track of partial sums of selected elements from \( h \in \mathcal{G}(1^T) \) to ensure that a resulting vector \( g \) satisfies \( Dg = 0 \). However, we notice that to ensure \( Dg = 0 \), it is sufficient to remember the partial sum of elements \( Dh \) for \( h \in \mathcal{G}(1^T) \), thus shrinking them to dimension \( r \). This is another insight which allows us to avoid the exponential dependence on \( t \). Note that \( |\Sigma(E)| \leq (1 + 4g(E)a)^T \). Given \( x \) with \( 0 \leq x \leq u \), we define an index function \( \mu \): for \( i \in [n], j \in [t] \) and \( \ell \in [x_j^i] \) let \( \mu(i,j,\ell) := \left( \sum_{k=1}^{i-1} \|x^k\|_1 \right) \left( \sum_{j' = 1}^{j-1} x_{j'}^i \right) + \ell \). In the following text, we consider any vector \( x \) satisfying \( 0 \leq x \leq u \) even though it would be natural to consider a feasible solution. This is deliberate, as we will later show that we need these claims to hold also for vectors \( x \) derived from feasible solutions which, however, need not be feasible solutions themselves.

**Definition 12** (Augmentation Graph). Given a vector \( x \) with \( 0 \leq x \leq u \), we define the augmentation graph \( DP(x) \) to be the following vertex weighted directed layered graph.

There are two distinguished vertices \( S \) and \( T \) in \( DP(x) \), called the source and the sink. We split the remaining vertices of \( DP(x) \) into \( M = \|x\|_1 \) layers, denoted \( \mathcal{L}(1), \ldots, \mathcal{L}(M) \). With \( i \in [n], j \in [t] \) and \( \ell \in [x_j^i] \) we associate the layer \( \mathcal{L}(1) = \{(1,h,Dh) \mid h \in \mathcal{H}_1 \} \) if \( \mu(i,j,\ell) = 1 \) and \( \mathcal{L}(\mu(i,j,\ell)) = \{\mu(i,j,\ell)\} \times \mathcal{H}_1 \times \Sigma(E) \) otherwise. Let \( L = \max_{\ell=1,\ldots,M} |\mathcal{L}(\ell)| \). A vertex \((\mu(i,j,\ell),h,\sigma)\) has weight \( f^\ell(h + x^i) - f^\ell(x^i) \).

There are the following edges in \( DP(x) \). From \( S \) to every vertex in the first layer \( \mathcal{L}(1) \). Let \( u \in \mathcal{L}(\ell) \) and \( v \in \mathcal{L}(\ell + 1) \) be vertices in consecutive layers with \( u = (\ell,h^\ell,\sigma^\ell) \) and \( v = (\ell + 1,h^{\ell+1},\sigma^{\ell+1}) \). If \( \sigma^{\ell+1} = \sigma^\ell + Dh^{\ell+1} \), then there is an edge oriented from \( u \) to \( v \). Finally, there is an edge from every vertex \( u \in \mathcal{L}(M) \) to \( T \) if \( u = (M,h,0) \).

Note that by the bounds on \(|\mathcal{G}(1^T)|\) (Lemma \[11\]) and \( g(E) \) (Lemma \[10\]), there are at most \( L \leq t (t^2(2ra)^r)^T \) vertices in each layer of \( DP(x) \). For an overview of the augmentation graph refer to Fig. \[1\].

Let \( P \) be an \( S-T \) path in \( DP(x) \) and let \( h^\ell \in \mathcal{G}(1^T) \) be such that \((\ell,h^\ell,\sigma)\) is its \((\ell + 1)\)-st vertex. For each \( i \in [n] \), let \( g^i = \sum_{j=1}^{t} \sum_{\ell=1}^{i} h^{a(i,j,\ell)} \). We say that \( h = (h^1,\ldots,h^M) \) is the \( P \)-augmentation vector and that \( g = (g^1,\ldots,g^n) \) is the compression of \( h \) (denoted by \( g = h^1 \)). Conversely let \( g \in \mathcal{G}(E^{(n)}) \) and recall that \( M \) is the number of layers of \( DP(x) \). By

\[ \text{Figure 1} \] A schema of the augmentation graph \( DP(x) \).
Lemma 8] for all $i \in [n]$ there exist vectors $h^{i,1}, \ldots, h^{i,n_i} \in G(1^T)$ such that $g^i = \sum_{k=1}^{n_i} h^{i,k}$, and $\sum_{i=1}^{n_i} n_i \leq g(E)$. For each $i \in [n]$ and $j \in [t]$, let $m^i_j$ be the number of $h^{i,k}$ with $h^i_j = -1$. The expansion of $g$ is $h = (h^1, \ldots, h^M)$ defined as follows (we denote this as $h = g^1$). Fix $i \in [n]$ and $j \in [t]$. Assign the distinct $m^i_j$ vectors $h^{i,k}$ with $h^i_j = -1$ to $h^{i(j,k,\ell)}$ for $\ell \in [m^i_j]$, and let $h^{i(j,k,\ell)} = 0$ for $\ell \in [m^i_j + 1 : x^i_j]$. Essentially, we pad the vector $h$ obtained by Lemma 8 with $0$ bricks to construct an $h = g^1$. Also notice that an $S$–$T$ path $P$ such that $h$ is a $P$-augmentation vector can be constructed by choosing appropriate $\sigma \in \Sigma(E)$ for each brick of $h$.

Let $0 \leq x \leq u$. We say that $g$ is a solution of $DP(x)$ if $0 \leq x + g \leq u$ and there exists an $S$–$T$ path $P$ with $P$-augmentation vector $h$ and $g = h^1$; the weight $w(g)$ is then defined as the weight of the path $P$; note that $w(g) = f(x + g) - f(x)$. A solution $g$ is called a minimal solution of $DP(x)$ if it is a solution of minimal weight. The following lemma relates solutions of $DP(x)$ to potential feasible steps in $G(E^{(n)})$.

Lemma 13. Let $x \in \mathbb{Z}^{nt}$ satisfy $0 \leq x \leq u$ and let $g$ be a solution of $DP(x)$. It holds that $0 \leq x + g \leq u$ and $E^{(n)} g = 0$.

Proof. It follows from the definition that there are exactly $x^i_j$ layers in which it is possible to select a vector $h$ such that $h^i_j = -1$. Observe further that all other layers that are derived from the $i$-th brick can only increase the value of $g^i_j$. It follows that $x + g \geq 0$.

Recall that $u^i_j \in \{0, \|b\|_\infty\}$. If $u^i_j = 0$, then we have excluded all vectors $h$ with $h^i_j = 1$ from $H^i_k$ for all $k \in [t]$. Thus $x^i_j + g^i_j = x^i_j \leq 0$ as claimed. On the other hand, if $u^i_j = \|b\|_\infty$, then observe that $\sum_{k=1}^{t} g^i_k = 0$ and because $x + g \geq 0$, we conclude that 

$$(x^i_j + g^i_j) \leq b^i_j \leq \|b\|_\infty = u^i_j.$$ 

Let $(h_k, \sigma_k)$, for each $k \in [M]$, be the vertex from the $k$-th layer of path $P$ corresponding to $g^1$. Note that $\sigma_M = 0$. If follows that $\sigma_{\ell+1} = \left(\sum_{k=1}^{\ell} D h_k\right) + D h_{\ell+1}$ for all $1 \leq \ell \leq M - 1$. Thus $\sum_{k=1}^{M} D h_k = 0$ and because we have $A h_k = 0$ from the definition of $H^i_j$ we conclude that $E^{(n)} g = 0$. □

Lemma 14. Let $x \in \mathbb{Z}^{nt}$ satisfy $0 \leq x \leq u$. Every $\tilde{g} \in G(E^{(n)})$ with $0 \leq x + \tilde{g} \leq u$ is a solution of $DP(x)$.

Proof of Lemma 14 Let $\tilde{g} \in G(E^{(n)})$ satisfy $0 \leq x + \tilde{g} \leq u$. From Lemma 8 it follows that the expansion $h = g^\top$ contains at most $g(E)$ nonzero elements from $G(1^T)$. Note that this implies that $h$ corresponds to an $S$–$T$ path $P$ in $DP(x)$ as the signature of all vertices along the path exists, i.e., $\sum_{k=1}^{\ell} D h^i_j \in \Sigma(E)$ for all $1 \leq k \leq n$ and $1 \leq \ell \leq n_i$. □

We define the $g(E)$-truncation of $x$ as the vector $\mathcal{X}$ given by $\mathcal{X}^i_j = \min\{x^i_j, g(E)\}$.

Lemma 15. Let $x \in \mathbb{Z}^{nt}$ satisfy $0 \leq x \leq u$. Every $\tilde{g} \in G(E^{(n)})$ with $0 \leq x + \tilde{g} \leq u$ is a solution of $DP(x)$.

Proof of Lemma 15 Let $P$ be an $S$–$T$ path in $DP(x)$ for which $\tilde{g}$ is the compression of the $P$-augmentation vector. Lemma 11 together with Lemma 8 imply that, for all $i \in [n]$, $\|g^i\|_\infty \leq g(E)$. This means that, for all $i \in [n]$ and $j \in [t]$, $P$ contains $\min\{x^i_j, g(E)\}$ vertices $(\mu(i,j,\ell), h^{\mu(i,j,\ell),\sigma})$ with $h^{\mu(i,j,\ell),\sigma} \neq 0$ for some $\ell \in N$ and $\sigma \in \Sigma(E))$. Thus, there must exist a path $\tilde{P}$ in $DP(\mathcal{X})$ with $w(\tilde{P}) = w(P)$. The rest of the statement follows from Lemma 12. □

Clearly our goal is then to find the lightest $S$–$T$ path in the graph $DP(\mathcal{X})$. However, there will be edges with negative weights. Still, finding the lightest path can be done in a
Thus, we have to prove that if \( f(x) < f(x + g) \), or decide there is none such \( g \), in time \( \|x\|_1 \cdot L^2 \leq t^{O(r)}(ar)^{O(r^2)}n \).

Proof. It follows from Lemma 13 that all solutions of \( DP(x) \) fulfill the first two conditions. Observe that if we take \( \tilde{g} \) to be a minimal solution of \( DP(x) \), then either \( f(x) = f(x + g) \) or \( f(x) < f(x + g) \). Due to Lemma 15 the set of solutions of \( DP(x) \) contains all \( \tilde{g} \in G(E^{(\alpha)}) \) with \( 0 \leq \tilde{g} \leq u \). Thus, by Proposition 7 if \( f(x) = f(x + g) \), no \( g \) satisfying all three conditions exist.

Now simply plug in our bounds on \( \|x\|_1 \) and \( L \) and compute a minimal \( S-T \) path:

\[
L \leq |G(1^T)| \cdot |\Sigma(E)| \leq t^2 \cdot (1 + 4t^2a(2ra)^r)^r \leq t^{O(r)}(ar)^{O(r^2)}
\]

is the maximum size of a layer and \( \|x\|_1 \leq nt \cdot g(E) \leq nt \cdot t^2(2ra)^r \leq O(t^2)(ar)^{O(r)}n \) is the number of layers.

### 3.3 Long steps

So far, we are able to find an augmenting step in time independent of \( M \); however, each step might only bring an improvement of \( O(1) \) and thus possibly many improving steps would be needed. Now, given a step length \( \alpha \in \mathbb{N} \), we will show how to find a feasible step \( g \) such that \( f(x + \alpha g) \leq f(x + \alpha \bar{g}) \) for any \( \bar{g} \in G(E^{(\alpha)}) \). Moreover, we will show that there are not too many step lengths that need to be considered in order to find a Graver-best step which, by Proposition 7, leads to a good bound on the required number of steps.

Let \( \alpha \in \mathbb{N} \) and let \( x \) with \( 0 \leq x \leq u \). We define \( x_\alpha \) to be the \( \alpha \)-reduction of \( x \), \( x_\alpha = \lceil \frac{x}{\alpha} \rceil \).

This operation takes priority over the truncation operation, that is, by \( \overline{x_\alpha} \) we mean the \( g(E) \)-truncation of vector \( x_\alpha \) (i.e., \( x_\alpha = (x_\alpha) \)). Note that for large enough \( \alpha \), \( DP(x_\alpha) \) contains only two vertices \( S \) and \( T \) and no arcs and thus there is no \( S-T \) path and no solutions.

**Lemma 17.** Let \( \alpha \in \mathbb{N} \) and let \( x \) with \( 0 \leq x \leq u \). Every \( \tilde{g} \in G(E^{(\alpha)}) \) with \( 0 \leq x + \alpha \tilde{g} \leq u \) is a solution of \( DP(\overline{x_\alpha}) \).

**Proof of Lemma 17.** By Lemma 15 it suffices to focus on \( DP(x_\alpha) \) instead of \( DP(\overline{x_\alpha}) \). Thus, we have to prove that if \( \tilde{g} \in G(E^{(\alpha)}) \) fulfills \( 0 \leq x + \alpha \tilde{g} \leq u \), then \( \tilde{g} \) is a solution of \( DP(x_\alpha) \).

In order to see this, observe that \( 0 \leq x + \alpha \tilde{g} \leq u \) implies \( 0 \leq \lceil \frac{\tilde{g}}{\alpha} \rceil \leq \lceil \frac{x}{\alpha} \rceil + \tilde{g} = x_\alpha + \tilde{g} \leq \lceil \frac{x + \alpha \tilde{g}}{\alpha} \rceil \leq \lceil \frac{u}{\alpha} \rceil \leq u \) and thus, by Lemma 14 a path corresponding to \( \tilde{g} \) exists in \( DP(x_\alpha) \).

However, a Graver-best step might still be such that its step length \( \alpha \) is large and thus we cannot afford to find a minimal solution of \( DP(x_\alpha) \) for all possible step lengths. We need another observation to see that many step lengths need not be considered. Let the state of \( x_\alpha \), \( \psi(x_\alpha) \in \{0, 1, 2\}^{\lceil \alpha \rceil \times i} \), be defined by:

- \( \psi(x_\alpha)_{ij} = 0 \) if \( \langle x_\alpha \rangle_j = 0 \)
- \( \psi(x_\alpha)_{ij} = 1 \) if \( \langle x_\alpha \rangle_j < g(E) \), and,
- \( \psi(x_\alpha)_{ij} = 2 \) if \( \langle x_\alpha \rangle_j \geq g(E) \).

Given a feasible solution \( x \), we call a step length \( \alpha \) interesting if \( \overline{x_\alpha} \neq \overline{x_{\alpha+1}} \) and boring otherwise. Moreover, \( \alpha \) is irrelevant if there is no Graver-best step with step length \( \alpha \).
Lemma 18. If \( \alpha \) is boring, then it is irrelevant.

Proof of Lemma 18 Let \( g \) be a minimal solution of \( DP(x^-) \). Because \( x^- = x^{a+1} \), we have that \( DP(x^-) = DP(x^{a+1}) \). Since the objective function \( f(x) \) is linear, we have that \( f(x + (\alpha + 1)g) \leq f(x + \alpha g) \). Moreover, if \( g \) is an augmenting step, the inequality is strict, making \( \alpha \) irrelevant.

Definition 19 (Candidate step lengths \( \Gamma \)). Let \( \Gamma \) be a set of candidate step lengths constructed iteratively as follows:

Input: vector \( x \) with \( 0 \leq x \leq u \) and \( g(E) \)

Computes: set of candidate steps \( \Gamma \)

\( \Gamma \leftarrow \{1\} \) and \( \gamma \leftarrow 2 \)

while \( x_\gamma > 0 \) do

foreach \( i, j \) with \( \psi(x_\gamma \mid x_{ij}) = 1 \) do

\( \Gamma_j \leftarrow \{(k, \lfloor x_j^\prime / k \rfloor) \mid k \in \mathbb{N}, 0 < \lfloor x_j^\prime / k \rfloor < (x_{\gamma-1})^j \} \)

\( \gamma_j \leftarrow k \) such that \( (k, q) \in \Gamma_j \), \( q \) is maximal, and secondary to this \( k \) also maximal

\( \gamma_1 \leftarrow \min \{ \gamma_j \mid \psi(x_\gamma \mid x_{ij}) = 1 \} \)

\( \gamma_2 \leftarrow \min \{ \lfloor x_j^\prime / k \rfloor \mid \psi(x_\gamma \mid x_{ij}) = 2 \} \)

add \( \min\{\gamma_1, \gamma_2\} \) to \( \Gamma \)

\( \gamma \leftarrow \max \Gamma + 1 \)

return \( \Gamma \)

Example 20. For example take \( x = (0, 7, 1000) \) and \( g(E) = 7 \). Now

\( \psi(x_1) = (0, 2, 2) \),
\( \psi(x_2) = \cdots = \psi(x_7) = (0, 1, 2) \),
for \( \gamma = 2 \) yield \( \gamma_1 = 2 \) \((x_2)_2 = 3 \) and \( (x)_2 = 7 \) and \( \gamma_2 = 142 \) and thus \( \Gamma = \{1, 2\} \)
for \( \gamma = 3 \) yield \( \gamma_1 = 3 \) and \( \gamma_2 = 142 \) and this results into \( \Gamma = \{1, 2, 3\} \)
for \( \gamma = 4 \) yield \( \gamma_1 = 7 \) and \( \gamma_2 = 142 \) and this results into \( \Gamma = \{1, 2, 3, 7\} \)
\( \psi(x_8) = \cdots = \psi(x_{142}) = (0, 0, 2) \), this results in \( \Gamma = \{1, 2, 3, 7, 142\} \),
\( \psi(x_{143}) = \cdots = \psi(x_{1000}) = (0, 0, 1) \), and
as \( (x_{166})_3 = 6 \) and \( (x_{167})_3 = 5 \), we add 166 to \( \Gamma \),
as \( (x_{200})_3 = 5 \) and \( (x_{201})_3 = 4 \), we add 200 to \( \Gamma \),
and so on, finally,
\( 0 = \psi(x_{1001}) = \cdots \).

This implies that \( \Gamma = \{1, 2, 3, 7, 142, 166, 200, 250, 333, 500, 1000\} \).

Lemma 21. If \( \alpha \) is the step length of a Graver-best step, then \( \alpha \in \Gamma \).

Proof. We will prove that \( \Gamma \) contains all interesting step lengths. Consider an \( \alpha \not\in \Gamma \). Either \( x_\alpha = 0 \) and clearly in that case \( DP(x_\alpha) \) does not yield an augmenting step since it has no layers and thus no weighted vertices, and thus \( \alpha \) is irrelevant.

Otherwise, take \( \gamma := \min\{\gamma' \mid \gamma' \in \Gamma, \gamma' \geq \alpha\} \). Because of the minimality of \( \gamma \) with respect to all of the \( \min\{\cdot\} \) clauses of the algorithm of Definition 19 we have that \( x^- = x^- \gamma \) and thus \( \alpha \) is boring and by Lemma 18 irrelevant.

Since \( \Gamma \) contains all remaining step lengths, it also contains all interesting steps and must contain the step length for any Graver-best step.
Lemma 22. $|\Gamma| \leq O(nt \cdot g(E))$ and $\Gamma$ can be constructed in time $O(|\Gamma| \cdot \log \|x\|_{\infty})$.

Proof. Fix a coordinate $x'_j$ of $x$ and consider a run of the algorithm of Definition 19. If $x'_j > g(E)$, $\hat{\gamma}_1 := \left\lfloor \frac{x'_j}{g(E)} \right\rfloor$ is added to $\Gamma$ at some point. For every $\gamma > \hat{\gamma}_2$ we have that $(x,\gamma)_j < g(E)$ and thus we need not consider the min\{\} clause for $\psi(x,\gamma)_j = 2$.

Consider a step of the algorithm which adds $\hat{\gamma}_1$, and observe that $\hat{\gamma}_1$ is chosen such that $(x,\hat{\gamma}_1)_j > (x,\hat{\gamma}_1+1)_j$. But since $(x,\hat{\gamma}_1+1)_j < g(E)$, such situation can occur at most $g(E)$ times.

Thus we have added at most $O(g(E))$ different step lengths to $\Gamma$ per coordinate, $O(nt \cdot g(E))$ step lengths in total.

Regarding the time it takes to construct $\Gamma$, we perform $O(|\Gamma|)$ arithmetic operations, and since we are dealing with numbers of size at most $\|x\|_{\infty}$, each operation takes time $O(\log \|x\|_{\infty})$, concluding the proof.

Lemma 23 (Graver-best computation). Given a feasible solution $x$ of combinatorial $n$-fold IP, it is possible to find a Graver-best step $(\alpha, g)$ in time $t^{O(r)}(ar)^{O(r^2)n^2}$, or decide that none exists.

Proof. For $\gamma \in \Gamma$ let $g^\gamma$ be a minimal solution of $DP(x^\gamma)$ and let $\alpha := \arg \min_{\gamma \in \Gamma} \{ f(x + \gamma g^\gamma) \}$. Finally, let $g := g^\gamma$. Then we claim that $(\alpha, g)$ is a Graver-best step.

By Lemma 17 for all $\hat{g} \in G(E^{(n)})$ it holds that $f(x + \alpha \hat{g}) \leq f(x + \alpha g)$. Moreover, by Lemma 21 if there exists a Graver-best step with step length $\gamma$, then $\gamma \in \Gamma$, and thus by the construction of $\alpha$, $(\alpha, g)$ is Graver-best step.

Regarding the time complexity, to obtain $g$ we need to solve $DP(x^\gamma)$ for each $\gamma \in \Gamma$ by Lemma 16 requiring time $|\Gamma| \cdot t^{O(r)}(ar)^{O(r^2)n^2}$ to find a Graver-best step.

3.4 Finishing the proof

Proof of Theorem 2 In order to prove Theorem 2 we need to put the pieces together. First, let us assume that we have an initial feasible solution $x_0$. In order to reach the optimum, by Proposition 7 we need to make at most $(2nt - 2) \cdot O(L)$ Graver-best steps, where $L = \langle b, 0, u, w \rangle$; this is because $O(L)$ is an upper bound on $f(x^0) - f(x^*)$ for some minimum $x^*$. By Lemma 23 it takes time $t^{O(r)}(ar)^{O(r^2)n^2}$ to find a Graver-best step.

Now we are left with the task of finding a feasible solution. We follow along the lines of Lemma 3.8 and solve an auxiliary combinatorial $n$-fold IP given by the bimatrix $\tilde{E} = \begin{pmatrix} \tilde{D} \\ \tilde{A} \end{pmatrix}$ with $\tilde{D} := (D I_r - I_r 0)$ and $\tilde{A} := (A 1_{2r+1}) = 1^T \in \mathbb{Z}^{t+2r+1}$, where $I_r$ is the identity matrix of dimension $r$, $0$ is a column vector of length $r$ and $1_{2r+1}$ is the vector of all $1$s of length $2r + 1$.

The variables $\tilde{x}$ of this problem have a natural partition into $nt$ variables $x$ corresponding to the original problem and $n(2r + 1)$ auxiliary variables $\tilde{x}$. Keep the original lower and upper bounds on $x$ and introduce a lower bound $0$ and upper bound $\|b\|_{\infty}$ on each auxiliary variable. Finally, let the new linear objective $\tilde{w}^T \tilde{x}$ be the sum of the auxiliary variables. Observe that it is easy to construct an initial feasible solution by setting $x = 0$ and computing $\tilde{x}$ accordingly, as $\tilde{x}$ serve the role of slack variables.

Then, applying the algorithm described previously either finds a solution with objective value $0$, implying $\tilde{x} = 0$, and thus $x$ is feasible for the original problem, or no such solution exists, meaning that the original problem is infeasible.
4 Application

In applications, it is practical to use combinatorial \( n \)-fold IP formulations which contain inequalities. Given an \( n \)-fold IP (in particular a combinatorial \( n \)-fold IP), we call the upper rows \((D D \cdots D)x = b^0\) \textit{globally uniform constraints}, and the lower rows \(Ax^t = b^t\), for all \( i \in [n] \), \textit{locally uniform constraints}. In Appendix ??, we show that introducing inequalities into a combinatorial \( n \)-fold IP is possible, however we need a slightly different approach than in a standard \( n \)-fold IP to keep the rigid format of a combinatorial \( n \)-fold IP.

\textbf{Inequalities in locally uniform constraints.} We add \( n \) variables \( x_{i+1}^t \) for all \( i \in [n] \) and we replace \( D \) with \((D 0)\). For each row \( i \) where we wish to enforce \( 1^T x^i \leq b^i \), we set the upper bound on \( u_{i+1}^t = \|b\|_\infty \). For the remaining rows we set \( u_{i+1}^t = 0 \), enforcing \( 1^T x^i = b^i \).

\textbf{Inequalities in globally uniform constraints.} We replace \( D \) with \((D I_r - I_r)\), where \( I_r \) is the \( r \times r \) identity matrix. Thus, we have introduced \( 2r \) new variables \( x_{i+j}^t \) with \( i \in [n] \) and \( j \in [2r] \); however, we enforce them all to be 0 by setting \( u_{i+j}^t = 0 \) for all \( i \in [n] \) and \( j \in [2r] \). Next, we introduce an \((n + 1)-st\) brick, set \( u_{n+1}^t = 0 \) for all \( j \in [r] \) and set \( b_{n+1}^t = \|D\|_\infty \cdot \|(b^1, \ldots, b^n)\|_1 \). Then, for each row \( i \in [r] \) where we wish to enforce a “\( \leq \)" inequality, we set \( u_{i+1}^t = \|b\|_\infty \) and \( u_{i+1}^t = 0 \), vice versa for a “\( \geq \)" inequality, and \( u_{i+1}^t = u_{i+r+1}^t = 0 \) for equality. To enforce a strict inequality “\( < \)" we proceed as for “\( \leq \)" and increase the right hand side by one; similarly for “\( > \)".

\textbf{Weighted Set Multicover.} We demonstrate Theorem 2 on the following problem:

\textbf{Weighted Set Multicover}

\textbf{Input:} A universe of size \( k \), \( U = [k] \), a set system represented by a multiset \( \mathcal{F} = \{F_1, \ldots, F_n\} \subseteq 2^U \), weights \( w_1, \ldots, w_n \in \mathbb{N} \), demands \( d_1, \ldots, d_k \in \mathbb{N} \).

\textbf{Find:} A multisubset \( \mathcal{F}' \subseteq \mathcal{F} \) minimizing \( \sum_{F_i \in \mathcal{F}'} w_i \) and satisfying \( |\{i \mid F_i \in \mathcal{F}', j \in F_i\}| \geq d_j \) for all \( j \in [k] \).

\textbf{Proof of Theorem 5.} Observe that since \( W \) is the number of different weights, and there can be at most \( 2^k \) different sets \( F \in 2^U \), each pair \((F, w)\) on the input is of one of \( T \leq W 2^k \) different types; let \( n_1, \ldots, n_T \in \mathbb{N} \) be a succinct representation of the instance.

We shall construct a combinatorial \( n \)-fold IP to solve the problem. Let \( x_\tau^t \) for each \( f \in 2^U \) and each \( \tau \in [T] \) be a variable. Let \( u_\tau^t = 0 \) for each \( f \in 2^U \) such that \( f \neq F^\tau \), and let \( u_\tau^t = \max n_\tau \) for \( f = F^\tau \). The variable \( x_\tau^t \) with \( f = F^\tau \) represents the number of sets of type \( \tau \) in the solution. The formulation is straightforward and reads

\[
\min \sum_{\tau=1}^T \sum_{f \in 2^U} w_\tau' x_\tau^f
\]

\[
\sum_{\tau=1}^T \sum_{f \in 2^U} f_i x_\tau^f \geq d_i \quad \text{for all } i \in [k]
\]

\[
\sum_{f \in 2^U} x_\tau^f \leq n_\tau \quad \text{for all } \tau \in [T];
\]

note that \( f_i \) is 1 if \( i \in f \) and 0 otherwise. Let us determine the parameters \( \hat{a}, \hat{r}, \hat{t}, \hat{n} \) and \( \hat{L} \) of this combinatorial \( n \)-fold IP instance. Clearly, the largest coefficient \( \hat{a} \) is 1, the number of globally uniform constraints \( \hat{r} \) is \( k \), the number of variables per brick \( \hat{t} \) is \( 2^k \), the number of bricks \( \hat{n} \) is \( T \), and the length of the input \( \hat{L} \) is at most \( \log n + \log w_{\max} \). \( \blacksquare \)
4.1 Stringology

To show that Theorem 2 can be used to speed up many previous results, we show a single-exponential algorithm for an artificial “meta-problem” called $\delta$-MULTI STRINGS which generalizes many previously studied problems:

**$\delta$-MULTI STRINGS**

**Input:** A set of strings $S = \{s_1, \ldots, s_k\}$, each of length $L$ over alphabet $\Sigma \cup \{\star\}$, distance lower and upper bounds $d_1, \ldots, d_k \in \mathbb{N}$ and $D_1, \ldots, D_k \in \mathbb{N}$, distance function $\delta : \Sigma^* \times \Sigma^* \to \mathbb{N}$ and a binary parameter $b \in \{0, 1\}$.

**Find:** An output string $y \in \Sigma^L$ satisfying
- $d_i \leq \delta(y, s_i) \leq D_i$ for each $s_i \in S$, and,
- $b \cdot (\sum_{i=1}^k \delta(y, s_i))$ is minimized.

We say a distance function $\delta : \Sigma^* \times \Sigma^* \to \mathbb{N}$ is a *wildcard-compatible character-wise function* if $\delta(x, y) = \sum_{i=1}^L \delta(x[i], y[i])$ for any two strings $x, y \in \Sigma^L$, and $\delta(c, \star) = 0$ for all $c \in \Sigma$.

**Theorem 24.** The $\delta$-MULTI STRINGS problem can be solved in time $K^{O(k^2)} \log L$, where $K = \max\{|\Sigma|, k, \max_{e, f \in \Sigma} \delta(e, f)\}$ and $\delta$ is any wildcard-compatible character-wise function.

When $\delta$ is the Hamming distance $d_H$, it is standard to first normalize the input and obtain an equivalent instance over the alphabet $[k]$ [27] Lemma 1. Thus, for $\delta = d_H$ we get rid of the dependence on $|\Sigma|$.

**Theorem 25.** The $d_H$-MULTI STRINGS problem can be solved in time $k^{O(k^2)} \log L$.

Then, Theorem 3 is a simple corollary of Theorem 25 and the fact that $\delta$-MULTI STRINGS generalizes all the listed problems; see problem definitions and Table 2 below.

**Proof of Theorem 24** Let us fix an instance of $\delta$-MULTI STRINGS. We shall create an instance of combinatorial $n$-fold IP and show how solving it corresponds to solving the original $\delta$-MULTI STRINGS problem.

As is standard, we represent the input as an $L \times k$ matrix $C$ with entries from $\Sigma \cup \{\star\}$ whose rows are the input strings $s_1, \ldots, s_k$. There are at most $T = (|\Sigma| + 1)^k$ different *input column types*; let $n_e$ be the number of columns of type $e \in (\Sigma \cup \{\star\})^k$ and denote $T_e = (\Sigma \cup \{\star\})^k$ the set of input column types. A solution can be represented as an $L \times (k + 1)$ matrix with entries from $\Sigma \cup \{\star\}$ whose last row does not contain any $\star$ symbol. Thus, there are at most $(|\Sigma| + 1)^k \cdot |\Sigma|$ *solution column types* $\alpha = (e, f) \in (\Sigma \cup \{\star\})^k \times \Sigma$ and we denote $T_\alpha = ((\Sigma \cup \{\star\})^k \times \Sigma)$ the set of all solution column types. We say that an input column type $e \in T_e$ is compatible with a solution column type $\alpha \in T_\alpha$ if $\alpha = (e, f)$ for some $f \in \Sigma$.

Let us describe the combinatorial $n$-fold IP formulation. We have a variable $x^e_\alpha$ for each $\alpha \in T_\alpha$ and each $e \in T_e$. Intuitively, the variable $x^e_\alpha$ encodes the number of columns $\alpha$ in the solution; however, to obey the format of combinatorial $n$-fold IP, we need a copy of this variable for each brick, hence the upper index $e$. We set an upper bound $n^e_\alpha = \|b\|_\infty$ for each two compatible $e$ and $\alpha$, and we set $n^e_\alpha = 0$ for each pair which is not compatible. The locally uniform constraints are simply $\sum_{\alpha \in T_\alpha} x^e_\alpha = n_e$ for all $e \in T_e$. The globally uniform constraints are

\[
\sum_{e \in T_e} \sum_{\alpha \in (e', f) \cap T_e} \delta(f, e') x^e_\alpha \geq d_i \quad \text{for all } s_i \in S
\]

\[
\sum_{e \in T_e} \sum_{\alpha \in (e', f) \cap T_e} \delta(f, e') x^e_\alpha \leq D_i \quad \text{for all } s_i \in S
\]
Huge \( n \)-fold integer programming and applications.

and the objective is

\[
\min b \cdot \left( \sum_{i=1}^{k} \sum_{e \in T_e} \sum_{\alpha=(e',f) \in T_e} \delta(f,e') x_{e\alpha}^i \right).
\]

In order to apply Theorem 2, we need to determine its parameters \( \hat{a}, \hat{r}, \hat{t}, \hat{n} \) and \( \hat{L} = \langle b, 0, u, w \rangle \). Here,
- \( \hat{a} \) is the largest coefficient in \( D \), which is \( \max_{e, f \in \Sigma} \delta(e, f) \leq K \),
- \( \hat{r} \) is the number of globally uniform constraints, which is \( 2k \),
- \( \hat{t} \) is the number of variables in each brick, which is \( |T_e| \leq (|\Sigma| + 1)^k \cdot |\Sigma| \leq K^{2k} \),
- \( \hat{n} \) is the number of bricks, which is \( |T_e| \leq (|\Sigma| + 1)^k \leq K^k \), and,
- \( \hat{L} \) is the size of the input \( \langle b, 0, u, w \rangle \) \leq \log L.

Plugging in finishes the proof.

The problem definitions follow. Some problems reduce to solving polynomially (in \( L \)) or \( k^k \) many instances of CLOSEST STRING. In such a case, we mention the fact after introducing the problem, and say that the problem poly-reduces or FPT-reduces to CLOSEST STRING.

**CLOSEST STRING**

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, d \in \mathbb{N} \).

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \leq d \) for all \( s_i \in S \).

**FARDEST STRING**

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, d \in \mathbb{N} \).

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \geq d \) for all \( s_i \in S \).

**d-MISMATCH**

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, d \in \mathbb{N} \).

**Find:** A string \( y \in \Sigma^{L'} \) with \( L' \leq L \) and a position \( p \in [L-L'] \) such that \( d_H(y, s_i, p, L') \geq d \) for all \( s_i \in S \), where \( s_i, p, L' \) is the substring of \( s_i \) of length \( L' \) starting at position \( p \).

**Note.** Gramm et al. [23] observe that d-MISMATCH poly-time reduces to CLOSEST STRING.

**DISTINGUISHING STRING SELECTION (DSS)**

**Input:** Bad strings \( S = \{ s_1, \ldots, s_k \} \), good strings \( S' = \{ s'_1, \ldots, s'_{k_2} \}, k = k_1 + k_2, d_1, d_2 \in \mathbb{N} \), all strings of length \( L \) over alphabet \( \Sigma \).

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \leq d_1 \) for each bad string \( s_i \) and \( d_H(y, s'_i) \geq L - d_2 \) for each good string \( s'_i \).

**NEIGHBOR STRING**

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, d_1, \ldots, d_k \in \mathbb{N} \).

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \leq d_i \) for all \( i \in [k] \).

**Note.** NEIGHBOR STRING is studied by Nishimura and Simjou [40]. It generalizes DSS: given an instance of DSS, create an instance of NEIGHBOR STRING with \( d_i = 0 \) and \( D_i = d_1 \) for all bad strings, and \( d_i = d^2 \) and \( D_i = L \) for all good strings.

**CLOSEST STRING WITH WILDCARDS**

**Input:** Strings \( s_1, \ldots, s_k \in (\Sigma \cup \{\ast\})^L, d \in \mathbb{N} \).

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \leq d \) for all \( i \in [k] \), where \( d_H(e, \ast) = 0 \) for any \( e \in \Sigma \).
CLOSEST TO MOST STRINGS (also known as CLOSEST STRING WITH OUTLIERS)

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, \) \( o \in [k], \) \( d \in \mathbb{N}. \)

**Find:** A string \( y \in \Sigma^L \) and a set of outliers \( O \subseteq \{s_1, \ldots, s_k\} \) such that \( d_H(y, s_i) \leq d \) for all \( s_i \in O \) and \( |O| \leq o. \)

**Note.** CLOSEST TO MOST STRINGS is FPT-reducible to CLOSEST STRING with parameter \( k \) [23].

c-HAMMING RADIUS CLUSTERING (c-HRC)

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, \) \( d \in \mathbb{N}. \)

**Find:** A partition of \( \{s_1, \ldots, s_k\} \) into \( S_1, \ldots, S_c \) and output strings \( y_1, \ldots, y_c \in \Sigma^L \) such that \( d(y_i, s_j) \leq d \) for all \( i \in [c] \) and \( s_j \in S_i. \)

OPTIMAL CONSENSUS

**Input:** Strings \( s_1, \ldots, s_k \in \Sigma^L, \) \( d \in \mathbb{N}. \)

**Find:** A string \( y \in \Sigma^L \) such that \( d_H(y, s_i) \leq d \) for all \( i \in [k] \) and \( \sum_{i \in [k]} d_H(y, s_i) \) is minimal.

See Table 2 for a summary of our improvements for the problems introduced above.

| Problem                        | Specialization of \( \delta \)-MULTI STRINGS | Previous best run time / hardness |
|--------------------------------|---------------------------------------------|----------------------------------|
|-closest string                | \( D_i = d \) for all \( i \in [k] \)         | \( 2^{O(\log^2 k)} \log^\Omega(1) n \) [23] |
| Farthest string               | \( d_i = d \) for all \( i \in [k] \)         | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] “implicit” |
| d-Mismatch                    | poly-reduces to CLOSEST STRING [23]           | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] |
| Distinguishing String Selection | special case of Neighbor String             | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] “implicit” |
| Neighbor String               | \( D_i = d \) for all \( i \in [k] \)         | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] “implicit” |
| Closest String with Wildcards | \( D_i = d \) for all \( i \in [k] \)         | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [24] |
| Closest to Most Strings       | FPT-reduces to CLOSEST STRING [1]             | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] implicit |
| c-HRC                         | FPT-reduces to CLOSEST STRING [1]             | \( 2^{O((\log k) \log n)} \log^\Omega(1) n \) [23] implicit |
| Optimal Consensus             | \( D_i = d \) for all \( i \in [k], b = 1 \) | FPT for \( k = 3 \), open for \( k > 3 \) [23] |

Table 2: If the “specialization” row does not contain a value, it means its “default” value is assumed. The default values are \( \delta = d_H, b = 0, \) and for all \( i \in [k], d_i = 0, D_i = L. \) In each row corresponding to a problem, the last column gives the run time of the algorithm with the slowest-growing dependency on \( k. \) Most problems either reduce to CLOSEST STRING and thus derive their time complexity from the result of Gramm et al. even though the original paper does not mention these problems; in that case, we write [23] implicit. For some problems, no FPT algorithm was known, but it is not difficult to see that the ILP formulation of Gramm et al. could be modified to model these problems as well; in that case, we write [23] “implicit”.

4.2 Computational Social Choice

For simplicity, we only show how Theorem 2 can be applied to speed up the \( \mathcal{R} \)-SWAP BRIBERY for two representative voting rules \( \mathcal{R}. \) Let us first introduce the necessary definitions and terminology.

**Elections.** An election \( (C, V) \) consists of a set \( C \) of candidates and a set \( V \) of voters, who indicate their preferences over the candidates in \( C, \) represented via a preference order \( \succ_v \) which is a total order over \( C. \) For ranked candidates \( c \) we denote by \( \text{rank}(c, v) \) their rank in \( \succ_v; \) then \( v \)’s most preferred candidate has rank 1 and their least preferred candidate has
rank $|C|$. For distinct candidates $c, c' \in C$, we write $c \succ_v c'$ if voter $v$ prefers $c$ over $c'$. To simplify notation, we sometimes identify the candidate set $C$ with $\{1, \ldots, |C|\}$, in particular when expressing permutations over $C$. We sometimes identify a voter $v$ with their preference order $\succ_v$, as long as no confusion arises.

**Swaps.** Let $(C,V)$ be an election and let $\succ_v \in V$ be a voter. For candidates $c, c' \in C$, a swap $s = (c,c')_v$ means to exchange the positions of $c$ and $c'$ in $\succ_v$; denote the perturbed order by $\succ_v$. A swap $(c,c')_v$ is admissible in $\succ_v$ if rank$(c,v) = $ rank$(c',v) - 1$. A set $S$ of swaps is admissible in $\succ_v$ if they can be applied sequentially in $\succ_v$, one after the other, in some order, such that each one of them is admissible. Note that the perturbed vote, denoted by $\succ_v^S$, is independent from the order in which the swaps of $S$ are applied. We also extend this notation for applying swaps in several votes and denote it $V^S$. We specify $v$’s cost of swaps by a function $\sigma^v : C \times C \to \mathbb{Z}$.

**Voting rules.** A voting rule $R$ is a function that maps an election $(C,V)$ to a subset $W \subseteq C$, called the winners. Let us define two significant classes of voting rules:

Scoring protocols. A scoring protocol is defined through a vector $s = (s_1, \ldots, s_{|C|})$ of integers with $s_1 \geq \ldots \geq s_{|C|} \geq 0$. For each position $p \in \{1, \ldots, |C|\}$, value $s_p$ specifies the number of points that each candidate $c$ receives from each voter that ranks $c$ as $p$th best. Any candidate with the maximum number of points is a winner. Examples of scoring protocols include the Plurality rule with $s = (1, 0, \ldots, 0)$, the $d$-Approval rule with $s = (1, 1, \ldots, 1, 0, \ldots, 0)$ with $d$ ones, and the Borda rule with $s = (|C| - 1, |C| - 2, \ldots, 1, 0)$. Throughout, we consider only natural scoring protocols for which $s_1 \leq |C|$; this is the case for the aforementioned popular rules.

$C1$ rules A candidate $c \in C$ is a Condorcet winner if any other $c' \in C \setminus \{c\}$ satisfies $|\{v \in V \mid c \succ_v c'\}| > |\{v \in V \mid c' \succ_v c\}|$. A voting rule is Condorcet-consistent if it selects the Condorcet winner in case there is one. Fishburn [17] classified all Condorcet-consistent rules as $C1$, $C2$ or $C3$, depending on the kind of information needed to determine the winner. For candidates $c, c' \in C$, let $v(c,c')$ be the number of voters who prefer $c$ over $c'$, that is, $v(c,c') = |\{v \in V \mid c \succ_v c'\}|$; we write $c <_M c'$ if $c$ beats $c'$ in a head-to-head contest, that is, if $v(c,c') > v(c',c)$.

A rule $R$ is $C1$ if knowing $<_M$ suffices to determine the winner, that is, for each pair of candidates $c, c'$ we know whether $v(c,c') > v(c',c)$, $v(c,c') < v(c',c)$ or $v(c,c') = v(c',c)$. An example is the Copeland$^\alpha$ rule for $\alpha \in [0,1]$, which specifies that for each head-to-head contest between two distinct candidates, if some candidate is preferred by a majority of voters then they obtain one point and the other candidate obtains zero points, and if a tie occurs then both candidates obtain $\alpha$ points; the candidate with largest sum of points wins.

**R-Swap Bribery**

**Input:** An election $(C,V)$, a designated candidate $c^* \in C$ and swap costs $\sigma^v$ for $v \in V$.

**Find:** A set $S$ of admissible swaps of minimum cost so that $c^*$ wins the election $(C,V^S)$ under the rule $R$.

We say that two voters $v$ and $v'$ are of the same type if $\succ_v = \succ_{v'}$, and $\sigma^v = \sigma^{v'}$.

**Proof of Theorem**[14] Let $n_1, \ldots, n_T$ be the numbers of voters of given types. Let $x^i_j$ for $j \in [|C|!]$ and $i \in [T]$ be a variable encoding the number of voters of type $i$ that are bribed to be of order $j$ in the solution. With slight abuse of notation, we denote $\sigma^i(j, i)$ the cost of bribery for a voter of type $i$ to change order to $j$ (as by [13] Proposition 3.2) this cost is fixed). Regardless of the voting rule $R$, the objective and the locally uniform constraints are
identical:

$$\min \sum_{i=1}^{T} \sum_{j=1}^{[C]!} \sigma^j(i, j)x^j_i$$

$$\sum_{j=1}^{[C]!} x^j_i = n_i \quad \text{for all } i \in [T]$$

The number of variables per brick $\hat{t}$ is $[C]!$, the number of bricks $\hat{n}$ is $T$, and the size of the instance $\hat{L}$ is $\log n + \log([C]! \sigma_{max})$, because at most $[C]!^2$ swaps suffice to permute any order $i \in [[C]!]$ to any other order $j \in [[C]!]$ [13, Proposition 3.2]. Let us now describe the globally uniform constraints separately for the two classes of voting rules we study here.

**Natural scoring protocol.** Let $s = (s_1, \ldots, s_{[C]!})$ be a natural scoring protocol, i.e., $s_1 \geq \cdots \geq s_{[C]!}$ and $\|s\|_{\infty} \leq |C|$. With slight abuse of notation, we denote $s_j(c)$, for $j \in [[C]!]$ and $c \in C$, the number of points obtained by candidate $c$ from a voter of order $j$. The globally uniform constraints then enforce that $c^*$ gets at least as many points as any other candidate $c$:

$$\sum_{i=1}^{T} \sum_{j=1}^{[C]!} s_j(c)x^j_i \leq \sum_{i=1}^{T} \sum_{j=1}^{[C]!} s_j(c^*)x^j_i \quad \text{for all } c \in C, c \neq c^* .$$

The number $\hat{r}$ of these constraints is $|C| - 1$, and the largest coefficient in them is $\hat{a} = \|s\|_{\infty} \leq |C|$; the last inequality follows from $s$ being a natural scoring protocol.

**Any C1 rule.** Let $\alpha_j(c, c')$ be 1 if a voter with order $j \in [[C]!]$ prefers $c$ to $c'$ and 0 otherwise. Recall that a voting rule is C1 if, to determine the winner, it is sufficient to know, for each pair of candidates $c, c'$, whether $v(c, c') > v(c', e), v(c, c') < v(c', c)$ or $v(c, c') = v(c', c)$; here $v(c, c') = |\{v \mid c >_v c'\}|$. We call a vector $<_{M} \in \{<,=,>\}^{[C]!}$ a scenario. Thus, a C1 rule can be viewed as partitioning the set of all scenarios into those that select $c^*$ as a winner and those that do not. Then, it suffices to enumerate all the at most $3^{[C]!^2}$ scenarios $<_{M}$ where $c^*$ wins, and for each of them to solve a combinatorial $n$-fold IP with globally uniform constraints enforcing the scenario $<_{M}$:

$$\sum_{i=1}^{T} \sum_{j=1}^{[C]!} \alpha_j(c, c')x^j_i > \sum_{i=1}^{T} \sum_{j=1}^{[C]!} \alpha_j(c', c)x^j_i \quad \text{for all } c, c' \in C \text{ s.t. } c <_{M} c'$$

$$\sum_{i=1}^{T} \sum_{j=1}^{[C]!} \alpha_j(c, c')x^j_i = \sum_{i=1}^{T} \sum_{j=1}^{[C]!} \alpha_j(c', c)x^j_i \quad \text{for all } c, c' \in C \text{ which are incomparable} .$$

The number $\hat{r}$ of these constraints is $\binom{|C|}{2} \leq |C|^2$, and the largest coefficient in them is $\hat{a} = 1$. The proof is finished by plugging in the values $\hat{a}, \hat{r}, \hat{t}, \hat{n}$ and $\hat{L}$ into Theorem 2.

### 4.3 Huge $n$-fold integer programming with small domains

Onn introduces the HUGe n-{fold integer programming} problem [12, 43], which concerns problems that can be formulated as an $n$-fold IP with the number of bricks $n$ given in binary. Bricks are thus represented not explicitly, but succinctly by their multiplicity. It is at first unclear whether there even exists an optimal solution of such problem which can be encoded in polynomial space, but this is possible by a theorem of Eisenbrand and Shmonin [12, Theorem 2], as pointed out by Onn [42, Theorem 1.3 (1)]. Thus, the problem is as follows:
Let $E = (\hat{\mathbb{Z}})$ be a bimatrix. Let $T$ be a positive integer representing the number of types of bricks. We are given $T$ positive integers $n_1, \ldots, n_T$ with $n = \sum_{i=1}^{T} n_i$ and vectors $b^0 \in \mathbb{Z}^r$ and $l', u' \in \mathbb{Z}^r$ and $b' \in \mathbb{Z}^s$, and a separable convex function $f_i$, for every $i \in [T]$. For $i \in [T]$ and $\ell \in [n_i]$, we define the index function $\nu_i(i, \ell) := (\sum_{j=1}^{\ell} n_j) + \ell$.

We call an $n$-fold IP instance given by the constraint matrix $E(n)$ with the right hand side $\hat{b}$ defined by $\hat{b}^{i, \ell}(i, \ell) := b^i$ and $\hat{b}^0 := b^0$, lower and upper bounds defined by $\hat{l}^{i, \ell} := l'$ and $\hat{u}^{i, \ell} := u'$ with the objective function $f(x) := \sum_{i=1}^{T} \sum_{\ell=1}^{n_i} f_i(x^{i, \ell})$ the huge instance.

For $i \in [T]$ and $\ell \in [n_i]$, and a vector $x$ which is a feasible solution of the huge instance, we say that the brick $x^{i, \ell}$ is of type $i$, and we say that $x^{i, \ell}$ has configuration $c \in \mathbb{Z}^r$ with $\hat{l}^i \leq c \leq \hat{u}^i$ if $x^{i, \ell} = c$. The succinct representation of $x$ is the set of tuples

$$\{(x^{i, \ell}, m^{i, \ell}) \mid x \text{ has } m^{i, \ell} \text{ bricks of type } i \text{ with configuration } x^{i, \ell}\}.$$ 

HUGE $n$-FOLD INTEGER PROGRAMMING

**Input:** Bimatrix $E = (\hat{\mathbb{Z}})$, positive integers $n_1, \ldots, n_T$, vectors $l', u' \in \mathbb{Z}^r$ and $b' \in \mathbb{Z}^s$ and a separable convex function $f_i$.

**Find:** Solve the huge instance and if an optimal solution exists, return its succinct representation.

In the special case of small domains we obtain the following:

**Theorem 26.** Let $d_1, \ldots, d_t \in \mathbb{N}$ be such that $d_j = \max_{x \in [n]} w_j - l_j$, $d_{\max} = \max_{j \in [d]} d_j$ and let $\delta = \prod_{j=1}^{d} 1$. Then the huge $n$-fold IP problem can be solved in time $\delta^{O(r)}(\delta_{\max}\|D\|\hat{r})^{O(r^2)}T^{3log n}$.}

This result is useful for the following. Using the framework of extended $n$-fold IPs introduced by Knop et al., $n$-fold IP formulations with small domains are obtained for the very general $\mathcal{R}$-MULTI BRIBERY problem. Together with Theorem 26 this immediately implies an exponential speedup in the number of bricks, without having to reformulate these problems as combinatorial $n$-fold IPs. However, there are still benefits in using Theorem 2 directly as it leads to better dependence on the respective parameters.

**Proof of Theorem 26** Let $E, n_1, \ldots, n_T, b, l, u, f$ be an instance $I$ of huge $n$-fold IP. First, we shall prove that we can restrict our attention to the case where $l = 0$ and $u^i \leq (d_1, \ldots, d_t) = d$ for all $i \in [T]$. Consider a variable $x^i_j$ with $l^i_j \neq 0$ and any row $e \in b$ of the system $E(x) = b$. Because the contribution of $x^i_j$ to the right hand side is $e_j x_j$, we have that

$$e \in b \Leftrightarrow e - e l^i_j = b - e l^i_j.$$ 

Let $I'$ be an instance of huge $n$-fold IP obtained from $I$ by, for every row $e \in b$ and every variable $x^i_j$, changing the right hand side from $b$ to $b - e l^i_j$, and setting $u^i_j := w^i_j - l^i_j$ and $l^i_j := 0$. Clearly there is a bijection between the feasible solutions of $I$ and $I'$ such that if $x - l$ is a feasible solution of $I$, $x$ is a feasible solution of $I'$, and thus minimizing $f(x - l)$ over $I'$ is equivalent to minimizing $f(x)$ over $I$. Thus, we shall assume from now on that $I$ satisfies $l = 0$ and $u^i_j \leq d$ for all $i \in [T]$. Let $C^i$ for $i \in [T]$ be the set of all possible configurations of a brick of type $i$, defined as

$$C^i = \{c \in \mathbb{Z}^r \mid Ac = b^i, 0 \leq c \leq u^i\}$$ 

and let $C = \prod_{i=1}^{d} [0 : d]_{ij}$ be the set of all configurations.
Clearly $C^i \subseteq C$ for all $i \in [T]$, and $|C| = \delta$. Let $C \in \mathbb{Z}^{t \times \delta}$ be a matrix whose columns are all configurations from $C$.

We shall give a combinatorial $n$-fold IP formulation solving the huge $n$-fold IP instance $I$. We have a variable $y^i_c$ for each $c \in C$ and each $i \in [T]$ encoding how many bricks of type $i$ have configuration $c$ in the solution of $I$. The formulation then is

\[
\begin{aligned}
\min & \quad \hat{f}(y) = \sum_{i=1}^{T} \sum_{c \in C} f^i(c) y^i_c \\
\text{s.t.} & \quad D \mathbf{y} = \mathbf{b}^0 \\
& \quad 1^i \mathbf{y}^i = n_i \quad \text{for all } i \in [T] \\
& \quad y^i_c = 0 \quad \text{for all } c \not\in C^i, i \in [T] \\
& \quad y^i_c \leq \|b\|_{\infty} \quad \text{for all } c \in C^i, i \in [T].
\end{aligned}
\]

It remains to verify that the formulation above corresponds to the huge $n$-fold IP instance $I$. The objective (2) clearly has the same value. Consider the globally uniform constraints (3). In the huge $n$-fold IP instance, a configuration $c \in C^i$ of a brick of type $i \in [T]$ contributes $Dc$ to the right hand side in the first $r$ rows. This corresponds in our program to the column $Dc$ of the matrix $DC$. The locally uniform constraints (4) simply state that the solution needs to contain exactly $n_i$ bricks of type $i$. Finally, since a brick of type $i$ can never have a configuration $c \not\in C^i$ we set all variables $y^i_c$ with $c \not\in C^i$ to zero with the upper bound (5), and place no restrictions on $y^i_c$ with $c \in C^i$ (6).

Let us determine the parameters $\hat{a}, \hat{r}, \hat{t}, \hat{n}$ and $\hat{L}$ of the resulting combinatorial $n$-fold IP:
- $\hat{a}$ is the largest coefficient in the upper matrix $\hat{D} = DC$, which is $\|DC\|_{\infty} \leq td_{\max} \|D\|_{\infty}$,
- the number of globally uniform constraints $\hat{r} = r$,
- the number of variables in a brick $\hat{t} = \delta$,
- the number of bricks $\hat{n} = T$, and,
- the input length $\hat{L} = \langle \mathbf{b}, 0, \hat{u}, \hat{f} \rangle \leq \log n \cdot (\max_{i \in [T]} \max_{c \in C^i} f^i(c))$.

## 5 Open problems

Can our result be extended to minimizing a separable convex function $f'$? Is HUGE $n$-FOLD IP (see Appendix 4.3) in FPT (with respect to $r, s, t$ and $a$) in the general case? It is not difficult to see that optimality certification is in FPT using ideas similar to Onn [42]; however, one possibly needs exponentially many augmenting steps.

For most of our applications, lower bounds have not been studied, because getting a good upper bound was the important question. The situation is different now: we have an upper bound of $k^{O(k^2)}$ on the dependence on parameter $k$ for various problems, such as CLOSEST STRING, WEIGHTED SET MULTICOVER, SCORE-SWAP BRIBERY or even MAKESPAN MINIMIZATION [31]. Is this just a common feature of our algorithm, or are there hidden connections between some of these problems? And what are their actual complexities? All we know so far is a trivial ETH-based $2^{\omega(k)}$ lower bound for CLOSEST STRING based on its reduction from SATISFIABILITY [18].

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