Homoclinic classes for generic $C^1$ vector fields

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Abstract

We prove that homoclinic classes for a residual set of $C^1$ vector fields $X$ on closed $n$-manifolds are maximal transitive, and depend continuously on periodic orbit data. In addition, $X$ does not exhibit cycles formed by homoclinic classes. We also prove that a homoclinic class of $X$ is isolated if and only if it is $\Omega$-isolated, and it is the intersection of its stable set with its unstable set. All these properties are well known for structural stable Axiom A vector fields.

1 Introduction

We show some properties of homoclinic classes for generic $C^1$ flows on closed $n$-manifolds. By homoclinic class we mean the closure of the transversal homoclinic points associated to a hyperbolic periodic orbit. So, homoclinic classes are transitive and the closure of its periodic orbits [16, Chapter 2, §8].

For structural stable Axiom A vector fields it is known that homoclinic classes are maximal transitive and depend continuously on the periodic orbit data. In addition, if $H$ is a homoclinic class of $X$ then it is saturated, that is, $H = W^s_X(H) \cap W^u_X(H)$, where $W^s_X(H)$ is the stable set of $H$ and $W^u_X(H)$ is the unstable set of $H$ [2, p. 371]. Moreover, such vector fields do not exhibit cycles formed by homoclinic classes. In this paper we shall prove these properties for generic $C^1$ vector fields on closed $n$-manifolds $M$ (neither structural stability nor Axiom A is assumed). Furthermore, we prove that generically a homoclinic class is isolated if and only if it is isolated from the nonwandering set. In particular, all the mentioned properties hold for a dense set of $C^1$ vector fields on $M$. It is interesting to observe that neither structural stability nor Axiom A is dense in the space of $C^1$ vector fields on $M$, $\forall n \geq 3$.

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To state our results in a precise way we use the following notation. \( M \) is a compact boundaryless \( n \)-manifold, \( \mathcal{X}^1(M) \) is the space of \( C^1 \) vector fields endowed with the \( C^1 \) topology. Given \( X \in \mathcal{X}^1(M) \), \( X_t \) denotes the flow induced by \( X \). The \( \omega \)-limit set of \( p \) is the set \( \omega_X(p) \) of accumulation points of the positive orbit of \( p \). The \( \alpha \)-limit set of \( p \) is \( \alpha_X(p) = \omega_{-X}(p) \), where \(-X\) denotes the time-reversed flow of \( X \). The nonwandering set \( \Omega(X) \) of \( X \) is the set of \( p \) such that for every neighborhood \( U \) of \( p \) and \( T > 0 \) there is \( t > T \) such that \( X_t(U) \cap U \neq \emptyset \). Clearly \( \Omega(X) \) is closed, nonempty and contains any \( \omega \)-limit (\( \alpha \)-limit) set. A compact invariant set \( B \) of \( X \) is \( \Omega \)-isolated if \( \Omega(X) \setminus B \) is closed. \( B \) is isolated if \( B = \cap_{t \in \mathbb{R}} X_t(U) \) for some compact neighborhood \( U \) of \( B \) (in this case \( U \) is called isolating block). We denote by \( \text{Per}(X) \) the union of the periodic orbits of \( X \) and \( \text{Crit}(X) \) the set formed by the union of \( \text{Per}(X) \) and the singularities of \( X \).

A set is transitive for \( X \) if it is the \( \omega \)-limit set of one of its orbits. A transitive set \( \Lambda \) of \( X \) is maximal transitive if it contains every transitive set \( T \) of \( X \) satisfying \( \Lambda \cap T \neq \emptyset \). Note that a maximal transitive set is maximal with respect to the inclusion order. In [4, 5] it was asked whether every homoclinic class \( H_f(p) \) of a generic diffeomorphism \( f \) satisfies the property that if \( T \) is a transitive set of \( f \) and \( p \in T \), then \( T \subset H_f(p) \). In [4], M. C. Arnaud also considered homoclinic classes for \( C^1 \) diffeomorphisms on \( M \), and in particular she gives a positive answer to this question [4, Corollary 40]. On the other hand, item (1) of Theorem [A] below states that generically any transitive set of a \( C^1 \) vector field intersecting the homoclinic class is included in it, and thus the diffeomorphism version of it extends this result of M. C. Arnaud.

If \( \Lambda \) is a compact invariant set of \( X \), we denote

\[
W^s_X(\Lambda) = \{ q \in M : \text{dist}(X_t(q), \Lambda) \to 0, t \to \infty \}
\]

and

\[
W^u_X(\Lambda) = \{ q \in M : \text{dist}(X_t(q), \Lambda) \to 0, t \to -\infty \},
\]

where dist is the metric on \( M \). These sets are called respectively the stable and unstable set of \( \Lambda \). We shall denote \( W^s_X(p) = W^s_X(\mathcal{O}_X(p)) \) and \( W^u_X(p) = W^u_X(\mathcal{O}_X(p)) \) where \( \mathcal{O}_X(p) \) is the orbit of \( p \). We say that \( \Lambda \) is saturated if \( W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda \).

A cycle of \( X \) is a finite set of compact invariant sets \( \Lambda_0, \Lambda_1, \ldots, \Lambda_n \) such that \( \Lambda_n = \Lambda_0 \), and \( \Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1} \) are disjoint, and

\[
(W^s_X(\Lambda_i) \setminus \Lambda_i) \cap (W^s_X(\Lambda_{i+1}) \setminus \Lambda_{i+1}) \neq \emptyset
\]

for all \( i = 0, \ldots, n - 1 \).

A compact invariant set \( \Lambda \) of \( X \) is hyperbolic if there is a continuous tangent bundle decomposition \( E^s \oplus E^X \oplus E^u \) over \( \Lambda \) such that \( E^s \) is contracting, \( E^u \)
is expanding and $E^X$ denotes the direction of $X$. We say that $p \in \text{Crit}(X)$ is hyperbolic if $O_X(p)$ is a hyperbolic set of $X$.

The Stable Manifold Theorem \cite{10} asserts that $W^s_X(p)$ is an immersed manifold tangent to $E^s \oplus E^X$ for every $p$ in a hyperbolic set $\Lambda$ of $X$. Similarly for $W^u_X(p)$. This remark applies when $\Lambda = O_X(p)$ for some $p \in \text{Crit}(X)$ hyperbolic. As already defined, the homoclinic class associated to a hyperbolic periodic orbit $p$ of $X$, $H_X(p)$, is the closure of the transversal intersection orbits in $W^s_X(p) \cap W^u_X(p)$.

We say that $X$ is Axiom A if $\Omega(X)$ is both hyperbolic and the closure of Crit($X$). The non wandering set of a nonsingular Axiom A flow splits in a finite disjoint union of homoclinic classes \cite[Chapter 0, p. 3]{10}.

Another interesting property of homoclinic classes for Axiom A vector fields is their continuous dependence on the periodic orbit data, that is, the map $p \in \text{Per}(X) \to H_X(p)$ is upper-semicontinuous.

In general, we say that a compact set sequence $\Lambda_n$ accumulates on a compact set $\Lambda$ if for every neighborhood $U$ of $\Lambda$ there is $n_0 > 0$ such that $\Lambda_n \subset U$ for all $n \geq n_0$. Note that this kind of accumulation is weaker than the usual Hausdorff metric accumulation.

If $\mathcal{Y}$ denotes a metric space, then $R \subset \mathcal{Y}$ is residual in $\mathcal{Y}$ if $R$ contains a countable intersection of open-dense subsets of $\mathcal{Y}$. Clearly a countable intersection of residual subsets of $\mathcal{Y}$ is a residual subset of $\mathcal{Y}$. For example, the set of Kupka–Smale vector fields $\mathcal{KS}^1(M)$ on $M$ is a residual subset of $\mathcal{X}^1(M)$ \cite[Chapter 3, §3]{3}. Recall that a vector field is Kupka–Smale if all its periodic orbits and singularities are hyperbolic and the invariant manifolds of such elements intersect transversally.

**Theorem A.** The following properties hold for a residual subset of vector fields $X$ in $\mathcal{X}^1(M)$:

1. The homoclinic classes of $X$ are maximal transitive sets of $X$. In particular, different homoclinic classes of $X$ are disjoint.

2. The homoclinic classes of $X$ are saturated.

3. The homoclinic classes of $X$ depends continuously on the periodic orbit data, that is, the map $p \in \text{Per}(X) \to H_X(p)$ is upper-semicontinuous.

4. A homoclinic class of $X$ is isolated if and only if it is $\Omega$-isolated.

5. The hyperbolic homoclinic classes of $X$ are isolated.

6. There is no cycle of $X$ formed by homoclinic classes of $X$.

7. $X$ has finitely many homoclinic classes if and only if the union of the homoclinic classes of $X$ is closed and every homoclinic class of $X$ is isolated.
When $M$ has dimension three we obtain the following corollaries using Theorem \[13\] and \[14\]. Recall that an isolated set $\Lambda$ of a $C^r$ vector field $X$ is $C^r$ robust transitive ($r \geq 1$) if it exhibits an isolating block $U$ such that, for every vector field $Y \in C^r$ close to $X$, $\cap_{t \in \mathbb{R}} Y_t(U)$ is both transitive and nontrivial for $Y$.

**Corollary 1.1.** The properties below are equivalent for a residual set of nonsingular 3-dimensional $C^1$ vector fields $X$ and every nontrivial homoclinic class $H_X(p)$ of $X$:

1. $H_X(p)$ is hyperbolic.
2. $H_X(p)$ is isolated.
3. $H_X(p)$ is $C^1$ robust transitive for $X$.

**Corollary 1.2.** The properties below are equivalent for a residual set of nonsingular 3-dimensional $C^1$ vector fields $X$:

1. $X$ is Axiom A.
2. $X$ has finitely many homoclinic classes.
3. The union of the homoclinic classes of $X$ is closed and every homoclinic class of $X$ is isolated.

The equivalence between the Items (1) and (2) of the above corollary follows from \[13, 14\]. It shows how difficult is to prove the genericity of vector fields exhibiting finitely many homoclinic classes. The equivalence between (2) and (3) follows from Theorem \[13\]-(7).

To prove Theorem \[A\] we show in Section 3 that homoclinic classes $H_X(p)$ for a residual set of $C^1$ vector fields $X$ satisfy $H_X(p) = \Lambda^+ \cap \Lambda^-$, where $\Lambda^+$ is Lyapunov stable for $X$ and $\Lambda^-$ is Lyapunov stable for $-X$. The main technical tool to prove such result is Lemma 3.6, a stronger version of Hayashi’s $C^1$ Connecting Lemma \[8\], recently published in [19, Theorem E, p. 5214] (see also \[1, 7, 9\]). In Section 2 we study compact invariant sets $\Lambda$ of $X$ satisfying $\Lambda = \Lambda^+ \cap \Lambda^-$, where $\Lambda^\pm$ is Lyapunov stable for $\pm X$. The proof of Theorem \[A\] and Corollary 1.1 will be given in the final section using the results of Sections 2 and 3.

**Remark 1.3.** We observe that Theorem \[A\] is valid for a residual set of $C^1$ diffeomorphisms on any $n$-manifold $M$ by the usual method of suspension.

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2 Lyapunov stability lemmas

In this section we shall establish useful properties of Lyapunov stable sets. A reference for Lyapunov stability theory is [3].

Recall we have denoted by \( X_t, t \in \mathbb{R} \) the flow generated by \( X \in \mathcal{X}^1(M) \). Given \( A \subseteq M \) and \( R \subseteq \mathbb{R} \) we set \( X_R(A) = \{ X_t(q) : (q, t) \in A \times R \} \). We denote \( \text{Cl}(A) \) the closure of \( A \), and \( \text{int}(A) \) the interior of \( A \). If \( \epsilon > 0 \) and \( q \in M \) we set \( B_\epsilon(q) \) the \( \epsilon \)-ball centered at \( q \).

A compact subset \( A \subseteq M \) is Lyapunov stable for \( X \) if for every open set \( U \) containing \( A \) there exists an open set \( V \) containing \( A \) such that \( X_t(V) \subseteq U \) for every \( t \geq 0 \). Clearly a Lyapunov stable set is forward invariant.

The following lemma summarizes some classical properties of Lyapunov stable sets (see [3, Chapter V]).

**Lemma 2.1.** Let \( \Lambda^+ \) be a Lyapunov stable set of \( X \). Then,

1. If \( x_n \in M \) and \( t_n \geq 0 \) satisfy \( x_n \to x \in \Lambda^+ \) and \( X_{t_n}(x_n) \to y \), then \( y \in \Lambda^+ \);
2. \( W^u_X(\Lambda^+) \subseteq \Lambda^+ \);
3. if \( \Gamma \) is a transitive set of \( X \) and \( \Gamma \cap \Lambda^+ \neq \emptyset \), then \( \Gamma \subseteq \Lambda^+ \).

We are interested in invariant compact sets \( \Lambda = \Lambda^+ \cap \Lambda^- \) of \( X \), where \( \Lambda^+ \) is Lyapunov stable set for \( X \) and \( \Lambda^- \) is Lyapunov stable set for the reversed flow \( -X \). We shall call such sets neutral for the sake of simplicity. As we shall see in the next section, homoclinic classes are neutral sets for generic \( C^1 \) vector fields on closed \( n \)-manifolds.

Elementary properties of neutral sets are given in the lemma below.

**Lemma 2.2.** Let \( \Lambda \) be a neutral set of \( X \). Then,

1. \( \Lambda \) is saturated;
2. \( \Lambda \) is transitive for \( X \) if and only if \( \Lambda \) is maximal transitive for \( X \). In particular, different transitive neutral sets of \( X \) are disjoint.

**Proof:** Let \( \Lambda = \Lambda^+ \cap \Lambda^- \) with \( \Lambda^\pm \) being Lyapunov stable for \( \pm X \). Clearly \( W^u_X(\Lambda) \subseteq \Lambda^+ \) by Lemma 2.1(2). Similarly, \( W^s_X(\Lambda) \subseteq \Lambda^- \). Hence

\[
W^u_X(\Lambda) \cap W^s_X(\Lambda) \subseteq \Lambda^+ \cap \Lambda^- = \Lambda.
\]

Conversely, \( \Lambda \subseteq W^u_X(\Lambda) \cap W^s_X(\Lambda) \) since \( \Lambda \) is invariant. This proves (1).

Now, by Lemma 2.1(3), if \( \Gamma \) is a transitive set intersecting \( \Lambda \), then \( \Gamma \subseteq \Lambda^+ \) and \( \Gamma \subseteq \Lambda^- \). Thus, \( \Gamma \subseteq \Lambda^+ \cap \Lambda^- = \Lambda \), and so, \( \Lambda \) is maximal transitive. The
converse is obvious. Different transitive neutral sets of $X$ are maximal transitive, and so, they are necessarily disjoint. This finishes the proof. □

Note that a Smale horseshoe with a first tangency is an example of a maximal transitive set which is not neutral, see Proposition 2.6. This example also provides a hyperbolic homoclinic class which is not neutral (compare with Theorem 3.1).

**Proposition 2.3.** There is no cycle of $X$ formed by transitive neutral sets.

**Proof:** By contradiction suppose that there exists a cycle $\Lambda_0, \ldots, \Lambda_n$ of $X$ such that every $\Lambda_i$ is a transitive neutral set of $X$. Recall $\Lambda_n = \Lambda_0$.

Set $\Lambda_i = \Lambda_i^+ \cap \Lambda_i^-$ where each $\Lambda_i^\pm$ is Lyapunov stable for $\pm X$. Choose

$$x_i \in (W^u_X(\Lambda_i) \setminus \Lambda_i) \cap (W^s_X(\Lambda_{i+1}) \setminus \Lambda_{i+1})$$

according to the definition.

We claim that $x_i \in \Lambda_0^-$ for every $i$. Indeed, as $W^s_X(\Lambda_0) \subset \Lambda_0^-$ one has $x_{n-1} \in \Lambda_0^-$. Assume by induction that $x_i \in \Lambda_0^-$ for some $i$. As $x_i \in W^u_X(\Lambda_i)$, the backward invariance of $\Lambda_0^-$ implies

$$\Lambda_0^- \cap \Lambda_i \supset \alpha_X(x_i) \neq \emptyset.$$

By Lemma 2.1-(3) one has $\Lambda_i \subset \Lambda_0^-$ since $\Lambda_i$ is transitive. In particular, $W^u_X(\Lambda_i) \subset \Lambda_0^-$ by Lemma 2.1-(2) applied to $-X$. As $x_{i-1} \in W^u_X(\Lambda_i)$, one has $x_{i-1} \in \Lambda_0^-$. The claim follows by induction.

By the claim $x_0 \in \Lambda_0^-$. As $W^u_X(\Lambda_0) \subset \Lambda_0^+$ and $x_0 \in W^u_X(\Lambda_0)$ (by definition) one has $x_0 \in \Lambda_0^+ \cap \Lambda_0^-=\emptyset$. This contradicts $x_0 \in W^u_X(\Lambda_0) \setminus \Lambda_0$ and the proposition is proved. □

**Lemma 2.4.** If $\Lambda$ is neutral for $X$, then for every neighborhood $U$ of $\Lambda$ there exists a neighborhood $V \subset U$ of $\Lambda$ such that

$$\Omega(X) \cap V \subset \cap_{t \in \mathbb{R}} X_t(U).$$

**Proof:** Let $U$ be a neighborhood of a neutral set $\Lambda$ of $X$. Choose $U' \subset \text{Cl}(U') \subset U$ with $U'$ being another neighborhood of $\Lambda$. We claim that there is a neighborhood $V \subset U'$ of $\Lambda$ so that:

1. $t \geq 0$ and $p \in V \cap X_{-t}(V) \Rightarrow X_{[0,t]}(p) \subset U'$.
2. $t \leq 0$ and $p \in V \cap X_t(V) \Rightarrow X_{[-t,0]}(p) \subset U'$.

Indeed, it were not true then there would exist a neighborhood $U$ of $\Lambda$ and sequences $p_n \to \Lambda$, $t_n > 0$ such that $X_{[0,t_n]}(p_n) \to \Lambda$ but $X_{[0,t_n]}(p_n) \not\subset U'$. Choose $q_n \in X_{[0,t_n]}(p_n) \setminus U'$. Write $q_n = X_{t'_n}(p_n)$ for some $t'_n \in [0,t_n]$ and assume that $q_n \to q$ for some $q \notin U'$. Let $\Lambda = \Lambda^+ \cap \Lambda^-$ with $\Lambda^\pm$ Lyapunov stable for $\pm X$. 

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Since $\Lambda^+$ is Lyapunov stable for $X$ and $t'_n > 0$, Lemma 2.4(1) implies $q \in \Lambda^+$. On the other hand, we can write $q_n = X_{t'_n-t_n}(X_{t_n}(p_n))$ where $t'_n - t_n > 0$ and $X_{t_n}(p_n) \to \Lambda$ and using again Lemma 2.4(1) we have that $q \in \Lambda^-$. This proves that $q \in \Lambda$, a contradiction since $q \notin U'$. This proves the claim.

Next we prove that $\Omega(X) \cap V \subseteq \cap_{t \in \mathbb{R}} X_t(U)$. Indeed, choose $q \in \Omega(X) \cap V$. By contradiction, we assume that there is $t > t_0$ such that $B_t(q) \cap X_{t}(B_t(q)) = \emptyset$. Pick $p \in B_t(q) \cap X_{t}(B_t(q))$. By (1) above one has $X_{t_0}(p) \subseteq U' \subset B_t(q) \subset V$. This contradicts $X_{t_0}(p) \in X_{t_0}(B_t(q)) \subseteq \cap_{t \in \mathbb{R}} X_t(U') = \emptyset$. The proof is completed.

A first consequence of the above lemma is the following corollary. Given compact subsets $A, B \subset M$ we denote $\text{dist}(A, B) = \inf\{\text{dist}(a, b); a \in A, b \in B\}$.

**Corollary 2.5.** If $\Lambda$ is a neutral set of $X$ and $\Lambda_n$ is a sequence of transitive sets of $X$ such that $\text{dist}(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$, then $\Lambda_n$ accumulates on $\Lambda$.

**Proof:** Let $\Lambda_n$ and $\Lambda$ as in the statement. Fix a neighborhood $U$ of $\Lambda$ and let $V \subset U$ be the neighborhood of $\Lambda$ obtained by the previous lemma. As $\text{dist}(\Lambda_n, \Lambda) \to 0$ as $n \to \infty$ we have that $\Lambda_n \cap V \neq \emptyset$ for every $n$ large. Let $q_n$ the dense orbit of $\Lambda_n$. Clearly $q_n \in \Omega(X)$. We can assume that $q_n \in V$ for $n$ large, and so, $q_n \in \Omega(X) \cap V$. Then, $X_t(q_n) \in U$ for every $t$. In particular, $\Lambda_n = \omega_X(q_n) \subset \text{Cl}(U)$. This proves the corollary since $U$ is arbitrary.

**Proposition 2.6.** A neutral set is isolated if and only if it is $\Omega$-isolated.

**Proof:** We first claim that any saturated $\Omega$-isolated set $\Lambda$ of $X$ is isolated. Indeed, since $\Lambda$ is $\Omega$-isolated, there is $U \supset \Lambda$ open such that $\text{Cl}(U) \cap \Omega(X) = \Lambda$. This $U$ is an isolating block for $\Lambda$. For if $x \in \cap_{t \in \mathbb{R}} X_t(U)$, then $\omega_X(x) \cup \alpha_X(x) \subset \text{Cl}(U) \cap \Omega(X) \subset \Lambda$. So, $x \in W^X_\Lambda(\Lambda) \cap W^X_\Lambda(\Lambda) = \Lambda$. This proves that $\cap_{t \in \mathbb{R}} X_t(U) \subset \Lambda$. The opposite inclusion follows since $\Lambda$ is invariant. The claim follows.

To prove that invariant $\Omega$-isolated neutral set are isolated we use the above claim and Lemma 2.2(1). To prove that isolated neutral sets are $\Omega$-isolated we use Lemma 2.4. 

**Proposition 2.7.** Transitive hyperbolic neutral sets are isolated.

**Proof:** By Proposition 2.6 it is suffices to show that transitive neutral hyperbolic sets $\Lambda$ are $\Omega$-isolated.

Suppose by contradiction that $\Lambda$ is not $\Omega$-isolated. Then, there is a sequence $p_n \in \Omega(X) \setminus \Lambda$ converging to $p \in \Lambda$. Fix $U$ a neighborhood of $\Lambda$ and let $V$ be given in Lemma 2.4 for $U$. We can assume that $p_n \in V$ for every $n$. As $p_n$ is non wandering for $X$, for every $n$ there are sequences $q_i \in V \to p_n$ and $t_i > 0$ such that $X_{t_i}(q_i) \to p_n$ as $i \to \infty$. By (1) in the proof of Lemma 2.4 we have $X_{[0, t_i]}(q_i) \subset U$.
for every $i$. So, we can construct a periodic pseudo orbit of $X$ arbitrarily close to $p_n$. By the Shadowing Lemma for Flows ([11, Theorem 18.1.6, p. 569]) applied to the hyperbolic set $\Lambda$, such a periodic pseudo orbit can be shadowed by a periodic orbit. This proves that $p_n \in \text{Cl(Per}(X))$. As the neighborhood $U$ is arbitrary, we can assume that $p_n \in \text{Per}(X)$ for every $n$. Note that $O_X(p_n)$ converges to $\Lambda$ by Corollary 2.5.

As $\Lambda$ is transitive we have that if $E^s \oplus E^X \oplus E^u$ denotes the corresponding hyperbolic splitting, then $\dim(E^s) = s$ and $\dim(E^u) = u$ are constant in $\Lambda$. Clearly neither $s = 0$ nor $u = 0$ since $\Lambda$ is not $\Omega$-isolated. As $O_X(p_n)$ converges to $\Lambda$ both the local stable and unstable manifolds of $p_n$ have dimension $s$, $u$ respectively. Moreover, both invariant manifolds have uniform size as well. This implies that $W^s_X(p_n) \cap W^u_X(p) \neq \emptyset$ and $W^s_X(p_n) \cap W^u_X(p) \neq \emptyset$ for $n$ large. As $p_n \in \text{Per}(X)$ and $p \in \Lambda$, we conclude by the Inclination Lemma [6] that $p_n \in \text{Cl}(W^s_X(p) \cap W^u_X(p))$. As $p \in \Lambda$, $W^s_X(p) \subseteq W^s_X(\Lambda)$. So, $p_n \in \text{Cl}(W^s_X(\Lambda) \cap W^u_X(\Lambda))$. As $\Lambda$ is saturated, $W^s_X(\Lambda) \cap W^u_X(\Lambda) = \Lambda$ and hence $p_n \in \text{Cl}(\Lambda) = \Lambda$. But this is impossible since $p_n \in \Omega(X) \setminus \Lambda$ by assumption. This concludes the proof.

Denote by $\mathcal{F}$ the collection of all isolated transitive neutral sets of $X$.

**Proposition 2.8.** A sub collection $\mathcal{F}'$ of $\mathcal{F}$ is finite if and only if $\bigcup_{\Lambda \in \mathcal{F}'} \Lambda$ is closed.

**Proof:** Obviously $\bigcup_{\Lambda \in \mathcal{F}'} \Lambda$ is closed if $\mathcal{F}'$ is finite. Conversely, suppose that $\bigcup_{\Lambda \in \mathcal{F}'} \Lambda$ is closed. If $\mathcal{F}'$ were infinite then, it would exist sequence $\Lambda_n \in \mathcal{F}'$ of (different) sets accumulating some $\Lambda \in \mathcal{F}'$. By Corollary 2.3 we have $\Lambda_n \subseteq U$ for some isolating block $U$ of $\Lambda$ and $n$ large. And then, we would have that $\Lambda_n = \Lambda$ for $n$ large, a contradiction.

### 3 Homoclinic classes

The main result of this section is

**Theorem 3.1.** There is a residual subset $\mathcal{R}$ of $\mathcal{X}^1(M)$ such that every homoclinic class of every vector field in $\mathcal{R}$ is neutral.

**Corollary 3.2.** The following properties are equivalent for $X \in \mathcal{R}$ and every compact invariant set $\Lambda$ of $X$:

1. $\Lambda$ is a transitive neutral set with periodic orbits of $X$.
2. $\Lambda$ is a homoclinic class of $X$.
3. $\Lambda$ is a maximal transitive set with periodic orbits of $X$. 

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Proof: That (2) implies (1) follows from Theorem 3.1. That (1) implies (3) follows from Lemma 2.4. Let us prove that (3) implies (2). If Λ is as in (3) and \( p \in \text{Per}(X) \cap \Lambda \), then \( \Lambda \cap H_X(p) \neq \emptyset \). By Theorem 3.1 we can assume \( H_X(p) \) is neutral, and so it is maximal transitive (using (1) \( \Rightarrow \) (3)). As both \( \Lambda \) and \( H_X(p) \) are maximal transitive we conclude \( \Lambda = H_X(p) \) and the proof follows.

Corollary 3.3. For \( X \in \mathcal{R} \), a non singular compact isolated set of \( X \) is neutral and transitive if and only if it is a homoclinic class.

Proof: The converse follows from Theorem 3.1. To prove the direct, denote \( \Lambda \) a transitive isolated neutral set of a generic \( C^1 \) vector field \( X \). By Proposition 2.6 it follows that \( \Lambda \) is also \( \Omega \)-isolated. Since \( \Lambda \) is transitive we have \( \Lambda \subseteq \Omega(X) \). Thus, by [17] it follows that \( \Lambda = \text{Cl}(\Lambda \cap \text{Per}(X)) \), and so, \( \Lambda \cap \text{Per}(X) \neq \emptyset \). Then the conclusion follows from the previous corollary.

The proof of Theorem 3.1 follows immediately from the two lemmas below.

Lemma 3.4. There exists a residual set \( \mathcal{R} \) of \( X^1(M) \) such that, for every \( Y \in \mathcal{R} \) and \( \sigma \in \text{Crit}(Y) \), \( \text{Cl}(W^u_X(\sigma)) \) is Lyapunov stable for \( X \) and \( \text{Cl}(W^s_X(\sigma)) \) is Lyapunov stable for \( -X \).

Lemma 3.5. There exists a residual set \( \mathcal{R} \) in \( X^1(M) \) such that every \( X \in \mathcal{R} \) satisfies

\[
H_X(p) = \text{Cl}(W^u_X(p)) \cap \text{Cl}(W^s_X(p))
\]

for all \( p \in \text{Per}(X) \).

Lemma 3.4 was proved in [13, Theorem 6.1, p. 372] when \( \sigma \) is a singularity and the same proof works when \( \sigma \) is a periodic orbit. We shall give another proof of this lemma in the Appendix for completeness.

Before the proof of Lemma 3.4, let us introduce some notation. Recall that \( M \) is a closed \( n \)-manifold, \( n \geq 3 \). We denote \( 2^M \) the space of all compact subsets of \( M \) endowed with the Hausdorff topology. Recall that \( K\mathcal{S}^1(M) \subset X^1(M) \) denotes the set of Kupka–Smale \( C^1 \) vector fields on \( M \).

Given \( X \in X^1(M) \) and \( p \in \text{Per}(X) \) we denote \( \Pi_X(p) \) the period of \( p \). We set \( \Pi_X(p) = 0 \) if \( p \) is a singularity of \( X \).

If \( T > 0 \) we denote

\[
\text{Crit}_T(X) = \{ p \in \text{Crit}(X) : \Pi_X(p) < T \}.
\]

If \( p \in \text{Crit}(X) \) is hyperbolic, then there is a continuation \( p(Y) \) of \( p \) for \( Y \) close enough to \( X \) so that \( p(X) = p \).
Note that if $X \in KS^1(M)$ and $T > 0$, then
\[ \text{Crit}_T(X) = \{p_1(X), \cdots, p_k(X)\} \]
is a finite set. Moreover,
\[ \text{Crit}_T(Y) = \{p_1(Y), \cdots, p_k(Y)\} \]
for every $Y$ close enough to $X$.

Let $\mathcal{Y}$ be a metric space. A set-valued map $\Phi : \mathcal{Y} \to 2^M$ is lower semi-continuous at $Y_0 \in \mathcal{Y}$ if for every open set $U \subset M$ one has $\Phi(Y_0) \cap U \neq \emptyset$ implies $\Phi(Y) \cap U \neq \emptyset$ for every $Y$ close to $Y_0$. Similarly, we say that $\Phi$ is upper semi-continuous at $Y_1 \in \mathcal{Y}$ if for every compact set $K \subset M$ one has $\Phi(Y_1) \cap K = \emptyset$ implies $\Phi(Y) \cap K = \emptyset$ for every $Y$ close to $Y_1$. We say that $\Phi$ is lower semi-continuous if it is lower semi-continuous at every $Y_0 \in \mathcal{Y}$. A well known result [12, Corollary 1, p. 71] asserts that if $\Phi : \mathcal{X}^1(M) \to 2^M$ is a lower semi-continuous map, then it is upper semi-continuous at every $Y$ in a residual subset of $\mathcal{X}^1(M)$.

The lemma below is the flow version of [19, Theorem E, p. 5214] (see also [1, 7, 8, 9]).

**Lemma 3.6.** Let $Y \in \mathcal{X}^1(M)$ and $x \notin \text{Crit}(Y)$. For any $C^1$ neighborhood $\mathcal{U}$ of $Y$ there are $\rho > 1$, $L > 0$ and $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$ and any two points $p, q \in M$ satisfying
\begin{enumerate}[(a)]  
  \item $p, q \notin B_\epsilon(Y_{[-L,0]}(x))$,  
  \item $\mathcal{O}_Y^+(p) \cap B_\epsilon(x) \neq \emptyset$, and  
  \item $\mathcal{O}_Y^-(q) \cap B_\epsilon(x) \neq \emptyset$,  
\end{enumerate}
there is $Z \in \mathcal{U}$ such that $Z = Y$ off $B_\epsilon(Y_{[-L,0]}(x))$ and that $q \in \mathcal{O}_Z^+(p)$.

**Proof of Lemma 3.6:** Given $X \in \mathcal{X}^1(M)$ we denote by $\text{Per}_T(X)$ the set of periodic orbits of $X$ with period $< T$.

We first prove a local version of Lemma 3.6.

**Lemma 3.7.** If $X \in KS^1(M)$ and $T > 0$ then there are a neighborhood $\mathcal{V}_{X,T} \ni X$ and a residual subset $\mathcal{P}_{X,T}$ of $\mathcal{V}_{X,T}$ such that if $Y \in \mathcal{P}_{X,T}$ and $p \in \text{Per}_T(Y)$ then $H_Y(p) = \text{Cl}(W^u_Y(p)) \cap \text{Cl}(W^s_Y(p))$. 

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Proof: There is a neighborhood $\mathcal{V}_{X,T} \ni X$ such that

$$\text{Per}_T(Y) = \{\sigma_1(Y), \ldots, \sigma_m(Y)\} \quad \forall \quad Y \in \mathcal{V}_{X,T}.$$ 

For each $1 \leq i \leq m$, let $\Psi_i : \mathcal{V}_{X,T} \ni Y \mapsto H_Y(\sigma_i(Y)) \in 2^M_c$. Note that $\Psi_i, \forall i$, is lower semi-continuous by the persistence of transverse homoclinic orbits. So, there is a residual subset $\mathcal{P}_{X,T}^i$ of $\mathcal{V}_{X,T}$ such that $\Psi_i$ is upper semi-continuous in $\mathcal{P}_{X,T}^i$. Set $\mathcal{P}_{X,T} = KS^1(M) \cap (\cap \mathcal{P}_{X,T}^i) \cap \mathcal{R}$, where $\mathcal{R}$ is the residual set given in Lemma 3.4. Then $\mathcal{P}_{X,T}$ is residual in $\mathcal{V}_{X,T}$.

Let us prove that $\mathcal{P}_{X,T}$ satisfies the conclusion of the lemma. For this, let $\sigma \in \text{Per}_T(Y)$ for some $Y \in \mathcal{P}_{X,T}$. Then $\sigma = \sigma_i(Y)$ for some $i$, and so $\Psi_i(Y) = H_Y(\sigma)$. Suppose, by contradiction, that $H_Y(\sigma) \neq \text{Cl}(W^u_r(\sigma)) \cap \text{Cl}(W^s_l(\sigma))$. Then there is $x \in \text{Cl}(W^u_r(\sigma)) \cap \text{Cl}(W^s_l(\sigma)) \setminus H_Y(\sigma)$.

We have either

(a) $x \notin \text{Crit}(Y)$ or

(b) $x \in \text{Crit}(Y)$.

It is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As $Y$ is Kupka-Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink or a source and so $W^u_r(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$ and $W^s_l(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$. Note that $\text{Cl}(W^u_r(\sigma))$ is Lyapunov stable since $Y \in \mathcal{R}$. As $x \in \text{Cl}(W^u_r(\sigma))$ we conclude that $W^u_r(x) \subseteq \text{Cl}(W^u_r(\sigma))$. As $x \in \text{Cl}(W^u_r(\sigma))$, there is $x' \in \text{Cl}(W^u_r(\sigma)) \cap (W^u_r(x) \setminus \mathcal{O}_Y(x))$ arbitrarily close to $x$ (for this use the Grobman–Hartman Theorem as in the proof of Lemma 3.4 in the Appendix). Obviously $x' \notin \text{Crit}(Y)$. If $x' \in H_Y(\sigma)$ we would have that $x \in H_Y(\sigma)$ since $\alpha_Y(x') = \mathcal{O}_Y(x)$ contradicting $x \notin H_Y(\sigma)$. Henceforth $x' \in \text{Cl}(W^u_r(\sigma)) \cap \text{Cl}(W^s_l(\sigma)) \setminus H_Y(\sigma)$ and $x' \notin \text{Crit}(Y)$. Then we conclude as in case (a) replacing $x$ by $x'$.

Now we prove the lemma in case (a).

As $x \notin H_Y(\sigma)$, there is a compact neighborhood $K$ of $x$ such that $K \cap H_Y(\sigma) = \emptyset$. As $\Psi_i$ is upper semi-continuous at $Y$, there is a neighborhood $U$ of $Y$ such that

$$K \cap H_Z(\sigma(Z)) = \emptyset, \quad (1)$$

for all $Z \in U$.

Let $\rho, L, \epsilon_0$ be the constants in Lemma 3.4 for $Y \in \mathcal{X}^1(M)$, $x$, and $U$ as above. As $x \notin \text{Crit}(Y)$, $Y_{[−L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset$. Then, there is $0 < \epsilon < \epsilon_0$ such that $\mathcal{O}_Y(\sigma) \cap B_\epsilon(Y_{[−L,0]}(x)) = \emptyset$ and $B_\epsilon(x) \subseteq K$.

Choose an open set $V$ containing $\mathcal{O}_Y(\sigma)$ such that $V \cap B_\epsilon(Y_{[−L,0]}(x)) = \emptyset$. As $x \in \text{Cl}(W^u_r(\sigma))$, one can choose $p \in W^u_r(\sigma) \setminus \{\sigma\} \cap V$ such that

$$\mathcal{O}_Y^+(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset.$$
Similarly, as \( x \in \text{Cl}(W^s_Y(\sigma)) \), one can choose \( q \in W^s_Y(\sigma) \setminus \{ \sigma \} \cap V \) such that
\[
\mathcal{O}^-_Y(q) \cap B_{\epsilon/p}(x) \neq \emptyset.
\]

We can assume that \( \mathcal{O}^-_Y(p) \subset V \) and \( \mathcal{O}^+_Y(q) \subset V \). Henceforth
\[
(\mathcal{O}^-_Y(p) \cup \mathcal{O}^+_Y(q)) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \tag{2}
\]

Observe that \( q \notin \mathcal{O}^+_Y(p) \) for, otherwise, \( p \) would be a homoclinic orbit of \( Y \) passing through \( K \) contradicting (1).

By construction \( \epsilon, p, q \) satisfy (b) and (c) of Lemma 3.6.

As \( p, q \in V \) and \( V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset \) we have that that \( \epsilon, p, q \) also satisfy (a) of Lemma 3.6.

Then, by Lemma 3.6, there is \( Z \in U \) such that \( Z = Y \) off \( B_\epsilon(Y_{[-L,0]}(x)) \) and \( q \in \mathcal{O}^+_Y(p) \).

Clearly \( \sigma(Z) = \sigma \) and by (2) we have \( p \in W^s_Z(\sigma) \) and \( q \in W^s_Z(\sigma) \) since \( Z = Y \) off \( B_\epsilon(Y_{[-L,0]}(x)) \).

Hence \( \mathcal{O} = \mathcal{O}^-_Z(p) = \mathcal{O}^-_Z(q) \) is a homoclinic orbit of \( \sigma \).

As \( q \notin \mathcal{O}^+_Y(p) \), we have that \( \mathcal{O} \cap B_\epsilon(x) \neq \emptyset \).

Perturbing \( Z \) we can assume that \( \mathcal{O} \) is transverse, i.e. \( \mathcal{O} \subseteq H_Z(\sigma) \).

As \( \mathcal{O} \cap B_\epsilon(x) \neq \emptyset \) and \( B_\epsilon(x) \subset K \) we would obtain \( K \cap H_Z(\sigma(Z)) \neq \emptyset \) contradicting (1).

This finishes the proof. \( \square \)

**Proof of Lemma 3.5.** Fix \( T > 0 \). For any \( X \in KS^1(M) \) consider \( \mathcal{V}_{X,T} \) and \( \mathcal{P}_{X,T} \) as in Lemma 3.7.

Choose a sequence \( X^n \in KS^1(M) \) such that \( \{ X^n : n \in \mathbb{N} \} \) is dense in \( X^1(M) \) (recall that \( X^1(M) \) is a separable metric space). Denote \( \mathcal{V}_{n,T} = \mathcal{V}_{X^n,T} \) and \( \mathcal{P}_{n,T} = \mathcal{P}_{X^n,T} \).

Define
\[
\mathcal{O}^T = \bigcup_n \mathcal{V}_{n,T} \quad \text{and} \quad \mathcal{P}^T = \bigcup_n \mathcal{P}_{n,T}.
\]

Clearly \( \mathcal{O}^T \) is open and dense in \( X^1(M) \).

We claim that \( \mathcal{P}^T \) is residual in \( \mathcal{O}^T \). Indeed, for any \( n \) there is a sequence \( D_{k,n,T}, k \in \mathbb{N} \), such that
\[
\mathcal{P}_{n,T} = \bigcap_k D_{k,n,T},
\]
and \( D_{k,n,T} \) is open and dense in \( \mathcal{V}_{n,T} \) for any \( k \). As
\[
\mathcal{P}^T = \bigcup_n \mathcal{P}_{n,T} = \bigcup_n (\bigcap_k D_{k,n,T}) = \bigcap_k (\bigcup_n D_{k,n,T})
\]
and \( \bigcup_n D_{k,n,T} \) is open and dense in \( \bigcup_n \mathcal{V}_{n,T} = \mathcal{O}^T \) we conclude that \( \mathcal{P}^T \) is residual in \( \mathcal{O}^T \). This proves the claim.
In particular, \( P^T \) is residual in \( \mathcal{X}^1(M) \) for every \( T \). Set \( P = \cap_{N \in \mathbb{N}} P^N \). It follows that \( P \) is residual in \( \mathcal{X}^1(M) \). Choose \( X \in P \), \( p \in \text{Per}(X) \) and \( N_0 \in \mathbb{N} \) bigger than \( \Pi_X(p) + 1 \). By definition \( X \in P^{N_0} \), and so, \( X \in P_{X, N_0} \) for some \( n \). As \( N_0 > \Pi_X(p) \) we have \( p \in \text{Per}_{N_0}(X) \). Then \( H_X(p) = \text{Cl}(W^u_X(p)) \cap \text{Cl}(W^s_X(p)) \) by Lemma 3.7 applied to \( X^n \) and \( T = N_0 \). This completes the proof of the lemma.

4 Proof of Theorem A and Corollary 1.1

Proof of Theorem A: By Theorem 3.1 we have that homoclinic classes for a residual subset of \( C^1 \) vector fields on closed manifolds are neutral sets. This leads us to apply the results in Section 2. Thus, Theorem A-(1) follows from Lemma 2.2-(2). Theorem A-(2) follows from Lemma 2.2-(1). Theorem A-(3) follows from Corollary 2.3. Theorem A-(4) follows from Proposition 2.6. Similarly, Theorem A-(5) follows from Proposition 2.7. Theorem A-(6) follows from Proposition 2.3

To prove Theorem A-(7) we proceed as follows. If \( X \) has finitely many homoclinic classes, then the union of the homoclinic classes of \( X \) is obviously closed. By [17] it follows that \( \Omega(X) \) is the union of the homoclinic classes of \( X \) (recall that \( X \) is \( C^1 \) generic). This implies that every homoclinic class of \( X \) is \( \Omega \)-isolated, and so, they are isolated by Theorem A-(4). Conversely, suppose that the union of the homoclinic classes of \( X \) is closed and that every homoclinic class of \( X \) is isolated. Let \( F' \) be the collection of all homoclinic classes of \( X \). As every homoclinic class of \( X \) is isolated by assumption one has \( F' \subset F \) (recall the notation in Proposition 2.8). We have that \( \bigcup_{\Lambda \in F'} \Lambda \) is closed by hypothesis. Then, Proposition 2.8 implies that \( F' \) is finite and the proof follows.

Proof of Corollary 1.1: That (1) implies (2) follows from Theorem A-(5). To prove that (2) implies (1) we proceed as follows. If \( H_X(p) \) is isolated, \( H_X(p) \) is \( \Omega \)-isolated by Theorem A-(4). In particular, \( H_X(p) \) is not in the closure of the sinks and sources of \( X \) unless it is either a sink or a source of \( X \) and we would be done. By [13], [4], as \( M \) is 3-dimensional and \( X \) is nonsingular and generic, one has that \( H_X(p) \) is hyperbolic. That (1) implies (3) follows from the hyperbolic theory. Indeed, if \( H_X(p) \) is hyperbolic, then \( H_X(p) \) is isolated, transitive and hyperbolic. In other words, \( H_X(p) \) is a basic set of \( X \). Then, the conclusion follows from the structural stability of basic sets [16]. That (3) implies (1) follows from [13], [18] since \( H_X(p) \) has no singularities (recall \( X \) has no singularities by hypothesis).

5 Appendix

Here we give a proof of Lemma 3.4 using Lemma 3.6. The proof we gave in [13, Theorem 6.1, p. 372] uses a different version of the \( C^1 \) Closing Lemma rather
Lemma 5.1. If $X \in K\mathcal{S}^1(M)$ and $T > 0$, then there is a neighborhood $U_{X,T}$ of $X$ and a residual subset $\mathcal{R}_{X,T}$ of $U_{X,T}$ such that if $Y \in \mathcal{R}_{X,T}$ and $p \in \text{Crit}_T(Y)$, then $\text{Cl}(W^u_Y(p))$ is Lyapunov stable for $Y$ and $\text{Cl}(W^s_Y(p))$ is Lyapunov stable for $-Y$.

**Proof:** Recall that $\text{Crit}(Y) = \{p_1(Y), \ldots, p_k(Y)\}$ for every $Y$ in some neighborhood $U_{X,T}$ of $X$, where $p_i(Y)$, $1 \leq i \leq k$, is either a periodic orbit or a singularity of $Y$.

For any $i \in \{1, \ldots, k\}$ we define $\Phi_i : U_{X,T} \to 2^M$ by

$$\Phi_i(Y) = \text{Cl}(W^u_Y(p_i(Y))).$$

By the continuous dependence of unstable manifolds we have that $\Phi_i$ is a lower semi-continuous map, and so, $\Phi_i$ is also upper semi-continuous for every vector field in some residual subset $\mathcal{R}_i$ of $U_{X,T}$. Set $\mathcal{R}_{X,T} = K\mathcal{S}^1(M) \cap (\cap_i \mathcal{R}_i)$. Then $\mathcal{R}_{X,T}$ is residual in $U_{X,T}$. Let us prove that $\mathcal{R}_{X,T}$ satisfies the conclusion of the lemma.

Let $\sigma \in \text{Crit}_T(Y)$ for some $Y \in \mathcal{R}_{X,T}$. Then, $\sigma = p_i(Y)$ for some $i$, and so, $\Phi_i(Y) = \text{Cl}(W^u_Y(\sigma))$.

Suppose by contradiction that $\text{Cl}(W^u_Y(\sigma))$ is not Lyapunov stable for $Y$.

Then, there are an open set $U$ containing $\text{Cl}(W^u_Y(\sigma))$ and two sequences $x_n \to x \in \text{Cl}(W^u_Y(\sigma))$, $t_n \geq 0$ such that

$$Y_{t_n}(x_n) \notin U.$$

As in the proof of Lemma 3.3 we have either

(a) $x \notin \text{Crit}(Y)$ or

(b) $x \in \text{Crit}(Y)$.

Again it is enough to prove the lemma in case (a). Indeed, suppose that case (b) holds. As $Y$ is Kupka–Smale we have that $\mathcal{O}_Y(x)$ is hyperbolic. Clearly $\mathcal{O}_Y(x)$ is neither a sink or a source and so $W^s_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$ and $W^u_Y(x) \setminus \mathcal{O}_Y(x) \neq \emptyset$. Let $V \subset U$ be a small neighborhood of $x$ given by the Grobman–Hartman Theorem 3 such that $\partial(W^u_Y(x,V)) = D^u_Y(x)$ is a fundamental domain for $W^u_Y(x)$ (here $W^s_Y(x,V)$ denotes the connected component of $V \cap W^s_Y(x)$ containing $\mathcal{O}_X(x)$).

Note that $D^u_Y(x) \subset W^u_Y(x) \setminus \mathcal{O}_Y(x)$. As $x_n \to x$, we can assume $x_n \in \text{int}(V)$ for all $n$. As $Y_{t_n}(x_n) \notin U$, we have that $x_n \notin W^s_Y(x)$. So, there is $s_n > 0$ such that $x'_n = Y_{s_n}(x_n) \in \partial V$ and $Y_s(x_n) \in \text{int}(V)$ for $0 \leq s < s_n$. Since $Y_{t_n}(x_n) \notin U$ we
have that \( Y_{t_n}(x_n) \notin \text{Cl}(V) \) for all \( n \). From this we conclude that \( s_n < t_n \) for all \( n \). On the other hand, as \( x_n \to x \), passing to a subsequence if necessary, we can assume that \( x'_n \to x' \) for some \( x' \in D^u_Y(x) \subseteq W^u_Y(x) \setminus \mathcal{O}_Y(x) \). Now we have the following claim.

**Claim 5.2.** \( x' \in \text{Cl}(W^u_Y(\sigma)) \).

**Proof:** As \( x \in \text{Cl}(W^u_Y(\sigma)) \), using the Connecting Lemma \( \ref{connecting} \), there is \( Z \) \( C^1 \) near \( Y \) such that \( W^s_Z(\sigma(Z)) \cap W^u_Z(x(Z)) \neq \emptyset \). In other words there is a saddle-connection between \( \sigma(Z) \) and \( x(Z) \). Breaking this saddle-connection as in the proof of Lemma 2.4 in \([3, \text{p. 101}]\), using the Inclination Lemma \([3]\), we can find \( Z' \) \( C^1 \) close to \( Z \) so that \( W^u_{Z'}(\sigma(Z')) \) passes close to \( x' \). This contradicts the upper-semicontinuity of \( \Phi \) at \( Y \). Thus, \( x' \in \text{Cl}(W^u_Y(\sigma)) \) and the Claim is proved. \( \square \)

As \( x' \in D^u_Y(x) \) we have that \( x' \notin \text{Crit}(Y) \). As \( x'_n \to x' \) and \( Y_{t_n-s_n}(x'_n) \notin U \) with \( t_n - s_n > 0 \), we conclude as in case (a) replacing \( x \) by \( x', x_n \) by \( x'_n \) and \( t_n \) by \( t_n - s_n \).

Now we prove the lemma in case (a).

As \( \text{Cl}(W^u_Y(\sigma)) \subseteq U \) and \( \Phi \) is upper semi-continuous, there is a \( C^1 \) neighborhood \( U \subseteq U_{X,T} \) of \( Y \) such that

\[
\text{Cl}(W^u_Z(\sigma(Z))) \subseteq U, \quad (3)
\]

for all \( Z \in U \).

Let \( \rho, L, \epsilon_0 \) as in Lemma \( \ref{lemma3.6} \) for \( X = Y \), \( x \), and \( U \) as above.

As \( x \notin \text{Crit}(Y) \), \( Y_{[-L,0]}(x) \cap \mathcal{O}_Y(\sigma) = \emptyset \).

As \( x \in \text{Cl}(W^u_Y(\sigma)) \), \( Y_{[-L,0]}(x) \subseteq \text{Cl}(W^u_Y(\sigma)) \) and so \( Y_{[-L,0]}(x) \subseteq U \). Then, there is \( 0 < \epsilon \leq \epsilon_0 \) such that \( B_\epsilon(Y_{[-L,0]}(x)) \cap \mathcal{O}_Y(\sigma) = \emptyset \) and \( B_\epsilon(Y_{[-L,0]}(x)) \subseteq U \).

Choose an open set \( V \) containing \( \mathcal{O}_Y(\sigma) \), \( V \subset \text{Cl}(V) \subset U \), such that \( V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset \).

For \( n \) large, we have \( x_n \in B_{\epsilon/\rho}(x) \) and we set \( q = Y_{t_n}(x_n) \notin U \).

As \( x_n \to x \) and \( t_n > 0 \) we have \( \mathcal{O}_Y^+(q) \cap B_{\epsilon/\rho}(x) \neq \emptyset \).

As \( x \in \text{Cl}(W^u_Y(\sigma)) \), there is \( p \in (W^u_Y(\sigma) \setminus \{\sigma\}) \cap V \) such that \( \mathcal{O}_Y^+(p) \cap B_{\epsilon/\rho}(x) \neq \emptyset \) and \( \mathcal{O}_Y^+(p) \subseteq V \).

By construction, \( \epsilon, p, q \) satisfy (b) and (c) of Lemma \( \ref{lemma3.6} \).

As \( V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset \) and \( q \notin U \), we have that \( \epsilon, p, q \) also satisfy (a) of Lemma \( \ref{lemma3.6} \).

Then, by Lemma \( \ref{lemma3.6} \), there is \( Z \in U \) such that \( Z = Y \) off \( B_\epsilon(Y_{[-L,0]}(x)) \) and \( q \in \mathcal{O}_Z^+(p) \).

As \( V \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset \) and \( \mathcal{O}_Y(p) \subset V \) we have

\[
\mathcal{O}_Y(p) \cap B_\epsilon(Y_{[-L,0]}(x)) = \emptyset. \quad (4)
\]
Now, (3), together with $V \cap B_{e}(Y_{[-L,0]}(x)) = \emptyset$ and $Z = Y$ off $B_{e}(Y_{[-L,0]}(x))$ imply that $\sigma(Z) = \sigma$ and $p \in W_{Z}^{s}(\sigma)$. As $q \notin U$ and $q \in W_{Z}^{s}(\sigma)$ (recall $p \in W_{Z}^{s}(\sigma)$ and $q \in O_{Z}^{+}(p)$), we have a contradiction by (3). This finishes the proof.

\[\square\]

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