We consider a class of fourth-order elliptic equations of Kirchhoff type with critical growth in \( \mathbb{R}^N \). By using constrained minimization in the Nehari manifold, we establish sufficient conditions for the existence of nodal (that is, sign-changing) solutions.

1. Introduction

In this article we studies the existence of nodal solutions to the fourth-order elliptic equations of Kirchhoff type with critical growth in \( \mathbb{R}^N \),

\[
\Delta^2 u - \left( 1 + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x) u = \lambda f(u) + |u|^{2^* - 2} u, \quad x \in \mathbb{R}^N, \tag{1.1}
\]

where \( \Delta^2 u \) is the biharmonic operator, \( 2^* = \frac{2N}{N - 4} \) is the critical Sobolev exponent with \( 5 \leq N < 8 \), and \( b \) and \( \lambda \) are positive parameters. The continuous functions \( V(x) \) and \( f(u) \) satisfy the following conditions:

(A1) \( V \in C(\mathbb{R}^N, \mathbb{R}) \) satisfies \( \inf_{x \in \mathbb{R}^N} V(x) \geq V_0 > 0 \), where \( V_0 \) is a positive constant.

For each \( M > 0 \), \( \text{meas}\{x \in \mathbb{R}^N : V(x) \leq M\} < \infty \), where \( \text{meas}(\cdot) \) denotes the Lebesgue measure on \( \mathbb{R}^N \);

(A2) \( f \in C^1(\mathbb{R}, \mathbb{R}) \) and \( f(u) = o(|u|) \), as \( u \to 0 \);

(A3) There exists \( p \in (4, 2^*) \) such that \( \lim_{u \to \infty} f(u)/u^{p-1} = 0 \);

(A4) \( \lim_{u \to \infty} F(u)/u^4 = +\infty \), where \( F(u) = \int_0^u f(t) \, dt \);

(A5) \( f(u)/|u|^3 \) is a strictly increasing function for \( u \in \mathbb{R} \setminus \{0\} \).

Problem (1.1) originates from the Kirchhoff equation

\[
- \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u) \quad \text{in} \ \Omega, \tag{1.2}
\]

where \( \Omega \in \mathbb{R}^N \) is a bounded domain, \( a > 0, \ b \geq 0 \), and \( u \) satisfies certain boundary conditions. The above equation stems from a typical model proposed by Kirchhoff [11],

\[
u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), \tag{1.3}
\]

which serves as a generalization of the classical D’Alembert wave equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, u),
\]

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by taking into account the effects of changes in the length of strings during vibrations. The nonlocal term thus appears. See for example [6, 24] for more background on such problems. Thanks to the pioneering work of Lions [19] on problem (1.3), a lot of attention has been drawn to these nonlocal problems during the last decade. That was followed by some interesting results on the existence of various solutions to (1.2), including positive solutions, multiple solutions, bound state solutions, multibump solutions, and semiclassical state solutions, both on bounded domains and on the entire space. For more results on the Kirchhoff-type equations we refer to [7, 15, 16, 13, 23, 27] and the references therein. Problem (1.2) with critical nonlinearity, however, is seldom covered, mainly because of the challenge - the lack of compactness - presented by the presence of the critical Sobolev exponent. We also refer the interested readers to [9, 18, 25, 34, 33, 35] on the fractional Kirchhoff type problems.

Recently, various approaches have been adopted for considering the fourth-order elliptic equations of the Kirchhoff type,

$$\Delta^2 u - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), \quad x \in \Omega,$$

$$u = \Delta u = 0, \quad x \in \partial \Omega,$$

where $\Delta^2 u$ is the biharmonic operator, with different hypotheses on the nonlinearity. For instance, Ma [21] studied the existence and multiplicity of positive solutions to the fourth-order equation with the fixed point theorems in cones of ordered Banach spaces. Wang et al. [30] applied the mountain pass and the truncation methods to get the existence of nontrivial solutions to the fourth-order elliptic equations of the Kirchhoff type with one parameter $\lambda$. Liang and Zhang [17] used the variational methods to obtain the existence and multiplicity of solutions to the fourth-order elliptic equations of the Kirchhoff type with critical growth in $\mathbb{R}^N$.

The motivation for this paper comes from [26, 28, 29, 36]. In [26], the existence was proved of one least energy nodal solution $u_b$ to problem (1.2), with its energy strictly larger than the ground state energy. Meanwhile, the asymptotic behavior of $u_b$, as the parameter $b \searrow 0$, was investigated as well. Later, under some weak assumptions on $f$ (especially, with the Nehari type monotonicity condition removed), Tang and Cheng [28] improved and generalized some results obtained in [26] with some new analytical skills and the non-Nehari manifold method. In [29], the authors obtained the existence of least energy nodal solutions to the Kirchhoff-type equation with critical growth in bounded domains by using the constraint variational method and the quantitative deformation lemma. In [36], the authors studied the fourth-order elliptic equation of the Kirchhoff-type,

$$\Delta^2 u - \left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N,$$

$$u \in H^2(\mathbb{R}^N),$$

where $a > 0$ and $b \geq 0$ are constants. By the constraint variational method and the quantitative deformation lemma, they proved that the problem possesses one least energy nodal solution. For more results on nodal solutions to the Kirchhoff-type equations, please refer to [8, 12, 20, 37] and the references therein. However, to the best of our knowledge, there are no such results concerning the existence of nodal solutions of the problem (1.1) involving critical nonlinearities in the whole space.
The purpose of this article is to study the existence, energy estimates and convergence properties of the least energy nodal solutions to the fourth-order elliptic equation (1.1). The novelty of this paper is that problem (1.1) concerns the critical case on the entire space. Based on these facts, the problem turns out to be extremely complicated and more difficult than the one without critical nonlinearities in bounded domains. Since problem (1.1) involves critical exponents in the nonlinearity, it is rather difficult to show that the energy functional reaches a lower infimum on the Nehari manifold because of the lack of compactness caused by the critical term. As we will see, this problem prevents us from using the approach in [2, 26, 28, 36]. So we need some new ideas to overcome the above difficulties. Moreover, we use the constraint variational method, the topological degree theory and the quantitative deformation lemma to prove our main results. Thus, our main results generalize papers [2, 26, 28, 36] in several directions.

Before stating our main results, we define

$$H^2(\mathbb{R}^N) := \{ u \in L^2(\mathbb{R}^N) : |\nabla u|, \Delta u \in L^2(\mathbb{R}^N) \},$$

deedowed with the norm

$$\|u\|_{H^2(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} (|\Delta u|^2 + (\nabla u)^2 + u^2) \, dx \right)^{1/2}.$$ 

Now, we introduce the space

$$E := \{ u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < \infty \}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + V(x)uv) \, dx$$

and the norm

$$\|u\| = \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2) \, dx.$$ 

Under condition (A1), it is known that the embedding $E \hookrightarrow H^2(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \in (2, 2^{**})$ is compact, and continuous for $p \in [2, 2^{**}]$ (see [4]), and

$$S_p|u|_p \leq \|u\|, \quad \text{for every } u \in E. \quad (1.4)$$

In particular, the best Sobolev constant for the embedding $E \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$ is

$$S = \inf \{ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx : \int_{\mathbb{R}^N} |u|^{2^{**}} \, dx = 1 \}.$$ 

**Definition 1.1.** We say that $u \in E$ is a weak solution to problem (1.1), if

$$\int_{\mathbb{R}^N} (\Delta u \phi + \nabla u \nabla \phi + V(x)u\phi) \, dx + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx$$

$$= \lambda \int_{\mathbb{R}^N} f(u)\phi \, dx + \int_{\mathbb{R}^N} |u|^{2^{**}-2} u\phi \, dx,$$

for every $\phi \in E$. 


For convenience, we will omit the term weak throughout this paper. The corresponding energy functional $I_b^\lambda : E \to \mathbb{R}$ to problem (1.1) is defined by

$$I_b^\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)|u|^2) \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 - \lambda \int_{\mathbb{R}^N} F(u) \, dx - \frac{1}{2^{**}} \int_{\mathbb{R}^N} |u|^{2^{**}} \, dx. \quad (1.5)$$

It is easy to see that $I_b^\lambda$ belongs to $C^1(E, \mathbb{R})$ and the critical points of $I_b^\lambda$ are the solutions to (1.1). For every $u \in E$ we can write

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}.$$  

Then every solution $u \in E$ to problem (1.1) with the property that $u^\pm \neq 0$ is a nodal solution to problem (1.1).

Our objective is to find the least energy nodal solutions to problem (1.1). There exist several interesting studies on the following typical semilinear equation, which is related to problem (1.1) (see [3, 4]),

$$-\Delta u + V(x)u = f(x, u) \quad \text{in} \ \mathbb{R}^N. \quad (1.6)$$

These methods, however, depend heavily upon the decompositions:

$$J(u) = J(u^+) + J(u^-), \quad (1.7)$$

$$\langle J'(u), u^+ \rangle = \langle J'(u^+), u^+ \rangle \quad \text{and} \quad \langle J'(u), u^- \rangle = \langle J'(u^-), u^- \rangle, \quad (1.8)$$

where $J$ is the energy functional of (1.3), given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx.$$  

However, if $b > 0$, the energy functional $I_b^\lambda$ cannot be decomposed in the same way as it is done in (1.7) and (1.8). In fact, we have

$$I_b^\lambda(u) = I_b^\lambda(u^+) + I_b^\lambda(u^-) + \frac{b}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx;$$

if $u^+ \neq 0$, then

$$\langle (I_b^\lambda)'(u), u^+ \rangle = \langle (I_b^\lambda)'(u^+), u^+ \rangle + b \int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx \int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx > \langle (I_b^\lambda)'(u^+), u^+ \rangle;$$

if $u^- \neq 0$, then

$$\langle (I_b^\lambda)'(u), u^- \rangle = \langle (I_b^\lambda)'(u^-), u^- \rangle + b \int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx \int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx > \langle (I_b^\lambda)'(u^-), u^- \rangle.$$

Therefore, the methods used for obtaining nodal solutions to the local problem (1.1) do not seem applicable to problem (1.1). In this paper, we follow the approach in [2] by defining the constrained set

$$N_b^\lambda = \{ u \in E : u^\pm \neq 0, \langle (I_b^\lambda)'(u), u^\pm \rangle = 0 \} \quad (1.9)$$

and considering a minimization problem of $I_b^\lambda$ on $N_b^\lambda$. Shuai [26] proved that $N_b^\lambda \neq \emptyset$, in the absence of the nonlocal term, by applying the parametric method and the implicit theorem. However, it is the nonlocal terms in problem (1.1), the biharmonic operator and the nonlocal term involved, that add to our difficulties. Roughly speaking, compared to the general Kirchhoff type problem (1.2), decompositions (1.7) and (1.8) corresponding to $I_b^\lambda$, are much more complicated, which accounts for some technical difficulties during the proof of the nonemptiness of $N_b^\lambda$. Moreover,
the parametric method and implicit theorem are not applicable to problem \((1.1)\) because the complexity of the nonlocal problem there. Hence, inspired by \([1]\), we follow a different path, specifically, we resort to a modified Miranda’s theorem (see \([22]\)). It is also feasible to prove that the minimizer of the constrained problem is also a nodal solution via the quantitative deformation lemma and degree theory. We can now present our first main result.

**Theorem 1.2.** Assume that \((A1)\)–\((A5)\) hold. Then there exists \(\lambda^* > 0\) such that for all \(\lambda \geq \lambda^*\), problem \((1.1)\) has a least energy nodal solution \(u_0 \in N_b^\lambda\) such that \(I_b^\lambda(u_0) = \inf_{u \in N_b^\lambda} I_b^\lambda(u).\)

Another goal of this paper is to establish the so-called energy doubling property (cf. \([31]\)), i.e., the energy of any nodal solution to problem \((1.1)\) is strictly larger than twice the ground state energy. The conclusion is trivial for the semilinear equation problem \((1.6)\). When \(b > 0\), a similar result was obtained by Shuai \([26]\) in a bounded domain \(\Omega\). We are also interested in whether energy doubling property still holds for problem \((1.1)\). To answer this question, we prove the following result.

**Theorem 1.3.** Assume that \((A1)\)–\((A5)\) hold. Then there exists \(\lambda^{**} > 0\) such that for all \(\lambda \geq \lambda^{**}\), \(c^* := \inf_{u \in M_b^\lambda} I_b^\lambda(u) > 0\) is achieved, and \(I_b^\lambda(u) > 2c^*\), where \(M_b^\lambda = \{ u \in E \setminus \{ 0 \} : \langle (I_b^\lambda)'(u), u \rangle = 0 \}\) and \(u\) is the least energy nodal solution obtained in Theorem 1.2. In particular, \(c^* > 0\) is achieved either by a positive or a negative function.

It is obvious that the energy of the nodal solution \(u_0\) obtained in Theorem 1.2 depends on \(b\). Next, we establish a convergence property of \(u_0\) as \(b \to 0\), which demonstrates a relationship between \(b > 0\) and \(b = 0\) for problem \((1.1)\).

**Theorem 1.4.** Assume that \((A1)\)–\((A5)\) hold. Then for any sequence \(\{ b_n \}\) with \(b_n \to 0\) as \(n \to \infty\), there exists a subsequence, still denoted by \(\{ b_n \}\), such that \(\{ u_n \}\) strongly converges to \(u_0\) in \(E\) as \(n \to \infty\), where \(u_0\) is a least energy nodal solution to the problem

\[
\Delta^2 u - \Delta u + V(x) = \lambda f(u) + |u|^{2^* - 2} u \quad \text{in } \mathbb{R}^N. \quad (1.10)
\]

The structure of this article is as follows: Section 2 contains the proof of the achieving the least energy for the constraint problem \((1.1)\). While section 3 is devoted to the proofs of our main theorems.

Throughout this paper, we use standard notation. For simplicity, we use “\(\to\)” and “\(\rightharpoonup\)” to denote the strong and weak convergence in the related function space, respectively. By \(C\) and \(C_i\) we denote various positive constants, and by “\(:=\)” definitions. To simplify the notation, we denote a subsequence of a sequence \(\{ u_n \}\) also as \(\{ u_n \}_n\), unless otherwise specified.

2. SOME TECHNICAL LEMMAS

To begin, fix \(u \in E\) with \(u^\pm \neq 0\). Consider the function \(\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}\) and the mapping \(W : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^2\), where

\[
\varphi(\alpha, \beta) = I_b^\lambda(\alpha u^+ + \beta u^-),
\]

\[
W(\alpha, \beta) = (\langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \alpha u^+ \rangle, \langle (I_b^\lambda)'(\alpha u^+ + \beta u^-), \beta u^- \rangle).
\]
For brevity, we define the quantities
\[ A^+(u) = \int_{\mathbb{R}^N} |\nabla u^+|^2 \, dx, \quad A^-(u) = \int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx, \quad B(u) = \int_{\mathbb{R}^N} \Delta u^+ \Delta u^- \, dx. \]

**Lemma 2.1.** Assume that (A1)-(A5) hold. Then for any \( u \in E \) with \( u^\pm \neq 0 \), there is the unique maximum point pair of positive numbers \((\alpha_u, \beta_u)\) such that \( \alpha_u u^+ + \beta_u u^- \in \mathcal{N}_0^\lambda \).

**Proof.** Our proof consists in verifying three claims.

**Claim 1.** There exists a pair of positive numbers \((\alpha_u, \beta_u)\) such that \( \alpha_u u^+ + \beta_u u^- \in \mathcal{N}_0^\lambda \), for any \( u \in E \) with \( u^\pm \neq 0 \). Note that
\[
(I_1^\lambda)'(\alpha u^+ + \beta u^-) = \int_{\mathbb{R}^N} \Delta(\alpha u^+ + \beta u^-) \Delta (\alpha u^+ + \beta u^-) \, dx + \int_{\mathbb{R}^N} |\nabla (\alpha u^+ + \beta u^-)|^2 \, dx + \int_{\mathbb{R}^N} V(x)|\alpha u^+|^2 \, dx
\]
and
\[
(I_2^\lambda)'(\alpha u^+ + \beta u^-) = \int_{\mathbb{R}^N} \Delta(\alpha u^+ + \beta u^-) \Delta (\alpha u^+ + \beta u^-) \, dx + \int_{\mathbb{R}^N} |\nabla (\alpha u^+ + \beta u^-)|^2 \, dx + \int_{\mathbb{R}^N} V(x)|\beta u^-|^2 \, dx
\]
By a direct computation we obtain that
\[
(I_1^\lambda)'(\alpha u^+ + \beta u^-) = \alpha^2 \|u^+\|^2 + \alpha^2 \beta^2 b A^+(u) A^- (u) + \alpha^4 b (A^-(u))^2 + \alpha \beta B(u) - \lambda \int_{\mathbb{R}^N} f(\alpha u^+) \alpha u^+ \, dx - \int_{\mathbb{R}^N} |\alpha u^+|^2 \, dx 
\]
and
\[
(I_2^\lambda)'(\alpha u^+ + \beta u^-) = \beta^2 \|u^-\|^2 + \alpha^2 \beta^2 b A^+(u) A^- (u) + \beta^4 b (A^- (u))^2 + \alpha \beta B(u) - \lambda \int_{\mathbb{R}^N} f(\beta u^-) \beta u^- \, dx - \int_{\mathbb{R}^N} |\beta u^-|^2 \, dx. 
\]
By assumptions (A2) and (A3), we have
\[
\int_{\mathbb{R}^N} f(\alpha u^+) \alpha u^+ \, dx \leq \varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^2 \, dx + C_\varepsilon \varepsilon \int_{\mathbb{R}^N} |\alpha u^+|^p \, dx.
\]
Choose \( \varepsilon > 0 \) small enough such that \( (1 - \lambda \varepsilon C_\varepsilon) > 0 \), which together with (2.5) and (2.3), yields
\[
(I_1^\lambda)'(\alpha u^+ + \beta u^-) \geq (1 - \lambda \varepsilon C_\varepsilon) \alpha^2 \|u^+\|^2 + \alpha^2 \beta^2 b A^+(u) A^- (u)
\]
Since $2^{**} > 4$, we have $(I^\lambda_b)'(\alpha u^+ + \beta u^-), \alpha u^+) > 0$ for a small enough $\alpha$ and for all $\beta \geq 0$.

Similarly, according to (2.3) and (2.4), we get $(I^\lambda_b)'(\alpha u^+ + \beta u^-), \beta u^-) > 0$, for small enough $\beta$ and all $\alpha \geq 0$. Hence, there exists $r > 0$ such that

$$(I^\lambda_b)'(ru^+ + \beta u^-), ru^-) > 0 \quad \text{and} \quad (I^\lambda_b)'(\alpha u^+ + ru^-), ru^-) > 0,$$  

(2.6)

for all $\alpha, \beta \geq 0$.

On the other hand, by (A3) and (A4), we have

$$f(t)t > 0, t \neq 0; \quad F(t) \geq 0, \quad t \in \mathbb{R}.$$  

(2.7)

Now, choose $R > r$. For sufficiently large $R$, and by (2.3), (2.4), (2.7), we have

$$(I^\lambda_b)'(Ru^+ + \beta u^-), Ru^-) < 0 \quad \text{and} \quad (I^\lambda_b)'(\alpha u^+ + Ru^-), Ru^-) < 0,$$  

(2.8)

for all $\alpha, \beta \in [r, R]$. Invoking Miranda’s theorem [22], together with (2.6) and (2.8), we can conclude that there exists $(\alpha_u, \beta_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $W(\alpha_u, \beta_u) = (0, 0)$, i.e., $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_\lambda^0$.

**Claim 2.** The pair $(\alpha_u, \beta_u)$ is unique.

- Case $u \in \mathcal{N}_\lambda^0$. Then we have

$$(I^\lambda_b)'(u), u^+ = 0 \quad \text{and} \quad (I^\lambda_b)'(u), u^- = 0,$$

that is,

$$|u^+|^2 + B(u) + bA^+(u)(A^+(u) + A^-(u))$$

$$= \lambda \int_{\mathbb{R}^N} f(u^+)u^+dx + \int_{\mathbb{R}^N} |u^+|^{2^{**}}dx$$

(2.9)

and

$$|u^-|^2 + B(u) + bA^-(u)(A^+(u) + A^-(u))$$

$$= \lambda \int_{\mathbb{R}^N} f(u^-)u^-dx + \int_{\mathbb{R}^N} |u^-|^{2^{**}}dx.$$  

(2.10)

By Claim 1, we know that there exists at least one positive pair $(\alpha_0, \beta_0)$ satisfying $\alpha_0 u^+ + \beta_0 u^- \in \mathcal{N}_\lambda^0$.

Next we show that $(\alpha_0, \beta_0) = (1, 1)$ is the unique pair of numbers. Without loss of generality, let us assume that $\alpha_0 \leq \beta_0$. It follows from (2.8) that

$$\alpha_0^2 (|u^+|^2 + B(x)) + \alpha_0^2 bA^+(u)(A^+(u) + A^-(u))$$

$$= \lambda \int_{\mathbb{R}^N} f(\alpha_0 u^+)\alpha_0 u^+dx + \int_{\mathbb{R}^N} |\alpha_0 u^+|^{2^{**}}dx.$$  

(2.11)

If $\alpha_0 < 1$, then from (2.10), (2.11) and (A5), we have

$$0 < [(\alpha_0)^{-2} - 1] (|u^+|^2 + B(u))$$

$$\leq \lambda \int_{\mathbb{R}^N} \left( \frac{f(x, \alpha_0 u^+)}{(\alpha_0 u^+)^3} - \frac{f(u^+)}{(u^+)^3} \right)(u^+)^4 dx$$

$$+ [(\alpha_0)^{2^{**}-4} - 1] \int_{\mathbb{R}^N} |u^+|^{2^{**}}dx < 0,$$

(2.12)

which is a contradiction. Hence, $1 \leq \alpha_0 \leq \beta_0$.

Adopting a similar approach, we can see that $\beta_0 \leq 1$, which implies that $\alpha_0 = \beta_0 = 1$. 

$$+$$
• Case \( u \notin N_\lambda^b \). Assume there exist two other pairs of positive numbers \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) such that
\[
\sigma_1 = \alpha_1 u^+ + \beta_1 u^- \in N_\lambda^b \quad \text{and} \quad \sigma_2 = \alpha_2 u^+ + \beta_2 u^- \in N_\lambda^b.
\]
Then
\[
\sigma_2 = \left(\frac{\alpha_2}{\alpha_1}\right) \alpha_1 u^+ + \left(\frac{\beta_2}{\beta_1}\right) \beta_1 u^- = \left(\frac{\alpha_2}{\alpha_1}\right) \sigma_1^+ + \left(\frac{\beta_2}{\beta_1}\right) \sigma_1^- \in N_\lambda^b.
\]
Since \( \sigma_1 \in N_\lambda^b \), it is clear that
\[
\frac{\alpha_2}{\alpha_1} = \frac{\beta_2}{\beta_1} = 1,
\]
which means that \( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \).

**Claim 3.** The pair \((\alpha_u, \beta_u)\) is the unique maximum point of the function \( \varphi \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \). We know from the above that \((\alpha_u, \beta_u)\) is the unique critical point of \( \varphi \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \). By definition and (2.5), we have
\[
\varphi(\alpha, \beta) = I_\lambda^b (\alpha u^+ + \beta u^-)
\]
\[
= \frac{\alpha^2}{2} ||u^+||^2 + \frac{\beta^2}{2} ||u^-||^2 + \alpha \beta B(u) + \frac{\alpha^4 b}{4} (A^+(u))^2 + \frac{\beta^4 b}{4} (A^-(u))^2
\]
\[
+ \frac{\alpha^2 \beta^2 b}{2} A^+(u) A^- (u) - \lambda \int_{\mathbb{R}^N} F(\alpha u^+) \, dx - \lambda \int_{\mathbb{R}^N} F(\beta u^-) \, dx
\]
\[
- \frac{\alpha^{2**}}{2**} \int_{\mathbb{R}^N} |u^+|^{2**} \, dx - \frac{\beta^{2**}}{2**} \int_{\mathbb{R}^N} |u^-|^{2**} \, dx
\]
\[
< \frac{\alpha^2}{2} ||u^+||^2 + \frac{\beta^2}{2} ||u^-||^2 + \alpha \beta B(u) + \frac{\alpha^4 b}{4} (A^+(u))^2 + \frac{\beta^4 b}{4} (A^-(u))^2
\]
\[
+ \frac{\alpha^2 \beta^2 b}{2} A^+(u) A^- (u) - \frac{\alpha^{2**}}{2**} \int_{\mathbb{R}^N} |u^+|^{2**} \, dx - \frac{\beta^{2**}}{2**} \int_{\mathbb{R}^N} |u^-|^{2**} \, dx,
\]
as \(|(\alpha, \beta)| \to \infty\). This implies that \( \lim_{|(\alpha, \beta)| \to \infty} \varphi(\alpha, \beta) = -\infty \), because \( 2^{**} > 4 \).

Hence, it suffices to show that the maximum point cannot be achieved on the boundary of \( \mathbb{R}_+ \times \mathbb{R}_+ \).

We carry out the proof by contradiction. Assuming \((0, \bar{\beta})\) is the global maximum point of \( \varphi \) with \( \bar{\beta} \geq 0 \), we have
\[
\varphi(\alpha, \bar{\beta}) = \frac{\alpha^2}{2} ||u^+||^2 + \frac{\bar{\beta}^2}{2} ||u^-||^2 + \alpha \bar{\beta} B(u) + \frac{\alpha^4 b}{4} (A^+(u))^2 + \frac{\bar{\beta}^4 b}{4} (A^-(u))^2
\]
\[
+ \frac{\alpha^2 \bar{\beta}^2 b}{2} A^+(u) A^- (u) - \lambda \int_{\mathbb{R}^N} F(\alpha u^+) \, dx - \lambda \int_{\mathbb{R}^N} F(\bar{\beta} u^-) \, dx
\]
\[
- \frac{\alpha^{2**}}{2**} \int_{\mathbb{R}^N} |u^+|^{2**} \, dx - \frac{\bar{\beta}^{2**}}{2**} \int_{\mathbb{R}^N} |u^-|^{2**} \, dx.
\]
Hence, it is clear that
\[
\varphi'(\alpha, \bar{\beta}) = \alpha ||u^+||^2 + \bar{\beta} B(u) + \alpha \bar{\beta} b (A^+(u))^2 + \alpha \bar{\beta}^2 b A^+(u) A^- (u)
\]
\[
- \lambda \int_{\mathbb{R}^N} f(\alpha u^+) u^+ \, dx - \alpha^{2**-1} \int_{\mathbb{R}^N} |u^+|^{2**} \, dx > 0,
\]
for small enough \( \alpha \). This means that \( \varphi \) is an increasing function with respect to \( \alpha \) if \( \alpha \) is small enough, which is a contradiction. In a similar way, we can deduce
that $\varphi$ cannot achieve its global maximum at $(\alpha,0)$ with $\alpha \geq 0$. Thus, we have completed the proof. \hfill \square

**Lemma 2.2.** Assume that (A1) — (A5) hold. Then for any $u \in E$ with $u^\pm \neq 0$ such that $\langle (I_b^\lambda)'(u), u^\pm \rangle \leq 0$, the unique maximum point pair of $\varphi$ on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfies $0 < \alpha_u, \beta_u \leq 1$.

**Proof.** Without loss of generality, we may assume that $\alpha_u \geq \beta_u > 0$. Since $\alpha_u u^+ + \beta_u u^- \in \mathcal{N}_b^\lambda$, we have

$$
\begin{align*}
\alpha_u^2 \|u^+\|^2 + \alpha_u \beta_u B(u) + \alpha_u^2 \beta_u^2 b A^+(u) A^-(u) + \alpha_u \beta_u (A^+(u))^2
= \lambda \int_{\mathbb{R}^N} f(\alpha_u u^+) u^+ dx + \int_{\mathbb{R}^N} |\alpha_u u^+|^2 \ast dx.
\end{align*}
$$

(2.13)

Furthermore, since $\langle (I_b^\lambda)'(u), u^+ \rangle \leq 0$, we have

$$
\begin{align*}
\|u^+\|^2 + B(u) + b (A^+(u))^2 + b A^+(u) A^-(u) \leq \lambda \int_{\mathbb{R}^N} f(u^+) u^+ dx + \int_{\mathbb{R}^N} |u^+|^2 \ast dx.
\end{align*}
$$

Then by (2.13), we have

$$
\begin{align*}
[(\alpha_u)^{-1} - 1] \left( \|u^+\|^2 + B(u) \right)
\geq \lambda \int_{\mathbb{R}^N} \left( \frac{f(\alpha_u u^+)}{(\alpha_u u^+)^{\lambda}} - \frac{f(u^+)}{(u^+)^{\lambda}} \right) (u^+)^4 dx + [(\alpha_u)^{2\ast} - 1] \int_{\mathbb{R}^N} |u^+|^2 \ast dx.
\end{align*}
$$

(2.14)

Obviously, the left hand side of (2.14) is negative for $\alpha_u > 1$ whereas the right hand side is positive, which is a contradiction. Therefore $0 < \alpha_u, \beta_u \leq 1$. \hfill \square

**Lemma 2.3.** Suppose that $c^\lambda_b = \inf_{u \in \mathcal{N}_b^\lambda} I_b^\lambda(u)$. Then $\lim_{\lambda \to \infty} c^\lambda_b = 0$.

**Proof.** For every $u \in \mathcal{N}_b^\lambda$, we have $\langle (I_b^\lambda)'(u), u \rangle = 0$, thus

$$
\begin{align*}
\|u^+\|^2 + \|u^-\|^2 + 2B(u) + b (A^+(u) + A^-(u))^2 = \lambda \int_{\mathbb{R}^N} f(u) u dx + \int_{\mathbb{R}^N} |u|^2 \ast dx.
\end{align*}
$$

Then, by (2.5), we have

$$
\begin{align*}
\|u\|^2 \leq \lambda \int_{\mathbb{R}^N} f(u^+) u^+ dx + \int_{\mathbb{R}^N} |u^+|^2 \ast dx
\leq \lambda \varepsilon \int_{\mathbb{R}^N} |u^+|^2 dx + \lambda C\int_{\mathbb{R}^N} |u^+|^p dx + \int_{\mathbb{R}^N} |u^+|^2 \ast dx.
\end{align*}
$$

(2.15)

Choose $\varepsilon$ small so that $\lambda \varepsilon \int_{\mathbb{R}^N} |u^+|^2 dx \leq \frac{1}{2} \|u^+\|^2$. Then we can claim that there exists $\rho > 0$ such that

$$
\|u^\pm\|^2 \geq \rho \quad \text{for all } u \in \mathcal{N}_b^\lambda,
$$

(2.16)

since $4 < 2\ast$. Next, by (A5), we have for $t \neq 0$ that

$$
F(t) := tf(t) - 4F(t) \geq 0,
$$

and $F(t)$ is increasing when $t > 0$, and decreasing when $t < 0$. Therefore,

$$
\begin{align*}
I_b^\lambda(u) = I_b^\lambda(u) - \frac{1}{4} \langle (I_b^\lambda)'(u), u \rangle
= \frac{1}{4} \|u\|^2 + \left( \frac{1}{4} + \frac{1}{2\ast} \right) \int_{\mathbb{R}^N} |u|^2 \ast dx + \frac{\lambda}{4} \int_{\mathbb{R}^N} [f(u) u - 4F(u)] dx
\geq \frac{1}{4} \|u\|^2 \geq \frac{\rho}{4} > 0.
\end{align*}
$$

(2.17)
So we have $I_b^\lambda(u) > 0$ for all $u \in \mathcal{N}_b^\lambda$, which means that $c_b^\lambda = \inf_{u \in \mathcal{N}_b^\lambda} I_b^\lambda(u)$ is well-defined.

Fix $u \in E$ with $u^\pm \neq 0$. According to Lemma 2.1, for each $\lambda > 0$, there exist $\alpha_\lambda, \beta_\lambda > 0$ such that $\alpha_\lambda u^+ + \beta_\lambda u^- \in \mathcal{N}_b^\lambda$. Therefore,

$$0 \leq c_b^\lambda = \inf_{u \in \mathcal{N}_b^\lambda} I_b^\lambda(u) \leq I_b^\lambda(\alpha_\lambda u^+ + \beta_\lambda u^-) \leq \frac{1}{2} \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla (\alpha_\lambda u^+ + \beta_\lambda u^-)|^2 \, dx \right)^2,$$

$$= \frac{\alpha_\lambda^2}{2} \|u^+\|^2 + \frac{\beta_\lambda^2}{2} \|u^-\|^2 + \alpha_\lambda \beta_\lambda B(u) + \frac{\alpha_\lambda^4 b}{4} (A^+(u))^2 + \frac{\beta_\lambda^3 b}{2} (A^-(u))^2 + \frac{\alpha_\lambda^2 \beta_\lambda^2 b}{2} A^+(u) A^-(u).$$

It suffices to prove that $\alpha_\lambda \to 0$ and $\beta_\lambda \to 0$, as $\lambda \to \infty$. Let

$$\mathcal{T} = \{(\alpha_\lambda, \beta_\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ : W(\alpha_\lambda, \beta_\lambda) = (0, 0), \lambda > 0\},$$

where $W$ is defined as in (2.2). Then

$$\alpha_\lambda^2 \int_{\mathbb{R}^N} |u^+|^{2^*} \, dx + \beta_\lambda^2 \int_{\mathbb{R}^N} |u^-|^{2^*} \, dx \leq \alpha_\lambda^{2^*} \int_{\mathbb{R}^N} |u^+|^{2^*} \, dx + \beta_\lambda^{2^*} \int_{\mathbb{R}^N} |u^-|^{2^*} \, dx + \lambda \int_{\mathbb{R}^N} f(\alpha_\lambda u^+) \alpha_\lambda u^+ \, dx + \lambda \int_{\mathbb{R}^N} f(\beta_\lambda u^-) \beta_\lambda u^- \, dx = \|\alpha_\lambda u^+ + \beta_\lambda u^-\|^2 + b \left( \alpha_\lambda^2 A^+(u) + \beta_\lambda^2 A^-(u) \right)^2.$$

Therefore, $\mathcal{T}$ is bounded, since $4 < 2^{*\star}$. Let $\{\lambda_n\} \subset (0, \infty)$ be such that $\lambda_n \to \infty$, as $n \to \infty$. Then there exist $\alpha_0$ and $\beta_0$ such that $(\alpha_{\lambda_n}, \beta_{\lambda_n}) \to (\alpha_0, \beta_0)$, as $n \to \infty$.

Now we claim that $\alpha_0 = \beta_0 = 0$. Assume, to the contrary, that $\alpha_0 > 0$ or $\beta_0 > 0$. Since $\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^- \in \mathcal{N}_{b_{\lambda_n}}$, then for any $n \in \mathbb{N}$, we have

$$\|\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\|^2 + b \left( \alpha_{\lambda_n}^2 A^+(u) + \beta_{\lambda_n}^2 A^-(u) \right)^2 = \lambda_n \int_{\mathbb{R}^N} f(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) \, dx + \int_{\mathbb{R}^N} |\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-|^{2^*} \, dx.$$  (2.18)

Then, invoking $\alpha_{\lambda_n} u^+ \to \alpha_0 u^+, \beta_{\lambda_n} u^- \to \beta_0 u^-$ in $E$ and the Lebesgue dominated convergence theorem, we have

$$\int_{\mathbb{R}^N} f(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-)(\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-) \, dx \to \int_{\mathbb{R}^N} f(\alpha_0 u^+ + \beta_0 u^-)(\alpha_0 u^+ + \beta_0 u^-) \, dx > 0,$$

as $n \to \infty$. This contradicts (2.18), given that $\lambda_n \to \infty$, as $n \to \infty$ and that $\{\alpha_{\lambda_n} u^+ + \beta_{\lambda_n} u^-\}$ is bounded in $E$. Therefore, $\alpha_0 = \beta_0 = 0$, which implies $\lim_{\lambda \to \infty} t_b^\lambda = 0$. \hfill \Box
Lemma 2.4. There exists $\lambda^* > 0$ such that the infimum $c^N_b$ is achieved for all $\lambda \geq \lambda^*$.

Proof. According to the definition of $c^N_b$, there exists a sequence $\{u_n\} \subset N^N_b$ such that $\lim_{n \to \infty} I^N_b(u_n) = c^N_b$. Clearly, $\{u_n\}$ is bounded in $E$. By Lemma 2.1 and the properties of $L^p$ space, up to a subsequence, we have

$$u^\pm_n \to u^\pm \text{ in } E,$$

$$u^\pm_n \to u^\pm \text{ in } L^p(\mathbb{R}^N) \text{ for } p \in [2, 2^{**}),$$

$$u^\pm_n \to u^\pm \text{ a.e. in } \mathbb{R}^N.$$

In view of Lemma 2.1, we also have

$$I^N_b(\alpha u^+_n + \beta u^-_n) \leq I^N_b(u_n),$$

for all $\alpha, \beta \geq 0$. So, by the Brézis-Lieb lemma, Fatou’s lemma and the weak lower semicontinuity of norm, we can conclude that

$$\liminf_{n \to \infty} I^N_b(\alpha u^+_n + \beta u^-_n) \geq \frac{\alpha^2}{2} \lim_{n \to \infty} \|u^+_n - u^+\|^2 + \frac{\beta^2}{2} \lim_{n \to \infty} \|u^-_n - u^-\|^2 + \frac{\alpha^2 b}{4} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla u^+_n - \nabla u^+ \right)^2 \int_{\mathbb{R}^N} \|\nabla u^+\|^2 dx + \frac{\beta^2 b}{4} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} \nabla u^-_n - \nabla u^- \right)^2 \int_{\mathbb{R}^N} \|\nabla u^-\|^2 dx$$

$$- \frac{\alpha^{2^*}}{2} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^+_n - u^+|^{2^*} dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^-|^{2^*} dx \right)$$

$$- \frac{\beta^{2^*}}{2} \left( \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^-_n - u^-|^{2^*} dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^-|^{2^*} dx \right)$$

$$- \lambda \int_{\mathbb{R}^N} F(\alpha u^+) dx - \lambda \int_{\mathbb{R}^N} F(\beta u^-) dx$$

$$+ \frac{\alpha^2 \beta^2 b}{2} \liminf_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u^+_n|^2 dx \int_{\mathbb{R}^N} |\nabla u^-|^2 dx$$

$$\geq I^N_b(\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_2^2 + \frac{\alpha^4 b}{2} A_3 A^+(u) - \frac{\alpha^{2^*}}{2^{**}} B_1$$

$$+ \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^4 b}{2} A_4 A^-(u) - \frac{\beta^{2^*}}{2^{**}} B_2,$$

where

$$A_1 = \lim_{n \to \infty} \|u^+_n - u^+\|^2, \quad A_2 = \lim_{n \to \infty} \|u^-_n - u^-\|^2,$$

$$A_3 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u^+_n - \nabla u^+|^2 dx, \quad A_4 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u^-_n - \nabla u^-|^2 dx,$$

$$B_1 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^+_n - u^+|^{2^*} dx, \quad B_2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u^-_n - u^-|^{2^*} dx.$$
That is,

\[
\begin{align*}
\lambda^\lambda & \geq I_b^\lambda (\alpha u^+ + \beta u^-) + \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 + \frac{\alpha^4 b}{2} A_4 A^+ (u) - \frac{\alpha^{2**}}{2} B_1 \\
& \quad + \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^4 b}{2} A_4 A^- (u) - \frac{\beta^{2**}}{2} B_2,
\end{align*}
\]

for all \( \alpha, \beta \geq 0 \).

**Step 1:** \( u^\pm \neq 0 \). We carry out our proof by contradiction. Assume that \( u^+ = 0 \). Letting \( \beta = 0 \) in (2.19), we have

\[
\lambda^\lambda \geq \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2**}}{2} B_1 := \phi(\alpha),
\]

for all \( \alpha \geq 0 \).

**Case 1:** \( B_1 = 0 \). If \( A_1 = 0 \), then \( u^+ \to 0 \) in \( E \). By (2.15), we obtain \( \|u^+\| > 0 \), which contradicts our assumption. If \( A_1 > 0 \), then by (2.20), we have \( \lambda^\lambda \geq \frac{\alpha^2}{2} A_1 \) for all \( \alpha \geq 0 \), which contradicts Lemma 2.3.

**Case 2:** \( B_1 > 0 \). From the definition of \( S \) and Lemma 2.3, there exists \( \lambda^* > 0 \) such that

\[
\lambda^\lambda < \frac{2}{N} S^{-2/N}
\]

for all \( \lambda \geq \lambda^* \). According to the Sobolev embedding and the fact that \( B_1 > 0 \), we obtain \( A_1 > 0 \). By (2.20), we have

\[
\frac{2}{N} S^{-2/N} \leq \frac{2}{N} \left[ \frac{A_1^2}{B_1} \right]^{\frac{2}{2**}} \leq \max_{\alpha \geq 0} \left\{ \frac{\alpha^2}{2} A_1 - \frac{\alpha^{2**}}{2**} B_1 \right\} \leq \max_{\alpha \geq 0} \left\{ \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2**}}{2**} B_1 \right\} \leq \lambda^\lambda,
\]

which is a contradiction. Hence, we can conclude that \( u^+ \neq 0 \). Similarly, we get that \( u^- \neq 0 \).

**Step 2:** \( B_1 = B_2 = 0 \). Given that the proof of \( B_2 = 0 \) is analogous, we just prove \( B_1 = 0 \). By contradiction, assume \( B_1 > 0 \).

**Case 1:** \( B_2 > 0 \). Since \( B_1, B_2 > 0 \), we get \( A_1, A_2 > 0 \). Clearly, \( \phi(\alpha) > 0 \) for \( \alpha \) small enough, where \( \phi(\alpha) \) is given by (2.20), and \( \phi(\alpha) < 0 \) for \( \alpha \) sufficiently large. Therefore, by continuity of \( \phi(\alpha) \), there exists \( \alpha > 0 \) such that

\[
\alpha^2 A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2**}}{2**} B_1 = \max_{\alpha \geq 0} \left( \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 - \frac{\alpha^{2**}}{2**} B_1 \right).
\]

Similarly, there exists \( \beta > 0 \) such that

\[
\beta^2 A_2 + \frac{\beta^4 b}{4} A_4^2 - \frac{\beta^{2**}}{2**} B_2 = \max_{\beta \geq 0} \left( \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 - \frac{\beta^{2**}}{2**} B_2 \right).
\]

In view of the compactness of \([0, \alpha] \times [0, \beta]\) and the continuity of \( \phi \), there exists \( (\alpha, \beta, u) \in [0, \alpha] \times [0, \beta] \) such that

\[
\varphi(\alpha, \beta) = \max_{(\alpha, \beta) \in [0, \alpha] \times [0, \beta]} \varphi(\alpha, \beta),
\]

where \( \varphi \) is defined as in Lemma 2.1.
Now we prove that \((\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \bar{\beta})\). Note that if \(\beta\) is small enough, then we have
\[
\varphi(\alpha, 0) = I^\lambda_b(\alpha u^+) < I^\lambda_b(\alpha u^-) + I^\lambda_b(\beta u^-) \leq I^\lambda_b(\alpha u^+ + \beta u^-) = \varphi(\alpha, \beta),
\]
for all \(\alpha \in [0, \bar{\alpha}]\). Thus, there exists \(\beta_0 \in [0, \bar{\beta}]\) such that \(\varphi(\alpha, 0) \leq \varphi(\alpha, \beta_0)\), for all \(\alpha \in [0, \bar{\alpha}]\). That is, \((\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{0\}\). With a similar method, we can show that \((\alpha_u, \beta_u) \notin \{0\} \times [0, \bar{\beta}]\).

It is obvious that
\[
\frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3 + \frac{\alpha^4 b}{2} A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}} B_1 > 0, \quad \alpha \in (0, \bar{\alpha})
\]
and
\[
\frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^+ + \frac{\beta^4 b}{2} A_4 A^- (u) - \frac{\beta^{2^{**}}}{2^{**}} B_2 > 0, \quad \beta \in (0, \bar{\beta}).
\]
Thus we obtain
\[
\frac{2}{N} S^{-2/N} \leq \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 + \frac{\alpha^{2^{**}}}{2} B_1 + \frac{\alpha^4 b}{2} A_3 A^+(u) + \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^{2^{**}}}{2} B_2
\]
and
\[
\frac{2}{N} S^{-2/N} \leq \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^{2^{**}}}{2} B_2 + \frac{\beta^4 b}{2} A_4 A^- (u) + \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3^2 + \frac{\alpha^{2^{**}}}{2} B_1,
\]
for all \(\alpha \in [0, \bar{\alpha}]\), \(\beta \in [0, \bar{\beta}]\).

From the these inequalities and (2.19), we obtain \(\varphi(\bar{\alpha}, \bar{\beta}) \leq 0\), \(\varphi(\alpha, \bar{\beta}) \leq 0\) for all \(\alpha \in [0, \bar{\alpha}]\), \(\beta \in [0, \bar{\beta}]\). Therefore, \((\alpha_u, \beta_u) \notin \{\bar{\alpha}\} \times [0, \bar{\beta}]\) and \((\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{\bar{\beta}\}\), which means \((\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \bar{\beta})\). It follows that \((\alpha_u, \beta_u)\) is a critical point of \(\varphi\).

So, \(\alpha_u u^+ + \beta_u u^- \in N^\lambda_b\). By (2.19), we have
\[
\tilde{c}_b^\lambda \geq I^\lambda_b(\alpha u^+ + \beta u^-) \geq \frac{\alpha^2}{2} A_1 + \frac{\alpha^4 b}{4} A_3 + \frac{\alpha^4 b}{2} A_3 A^+(u) - \frac{\alpha^{2^{**}}}{2^{**}} B_1 + \frac{\beta^2}{2} A_2 + \frac{\beta^4 b}{4} A_4^2 + \frac{\beta^{2^{**}}}{2^{**}} B_2 > I^\lambda_b(\alpha u^+ + \beta u^-) \geq \tilde{c}_b^\lambda,
\]
which is a contradiction. Therefore \(B_1 = 0\).

**Case 2:** \(B_2 = 0\). In this case, we can maximize in \([0, \bar{\alpha}] \times [0, \infty)\). It is possible to show that there exists \(\beta_0 \in [0, \infty)\) satisfying
\[
I^\lambda_b(\alpha u^+ + \beta_0 u^-) \leq 0 \quad \text{for all } (\alpha, \beta) \in [0, \bar{\alpha}] \times [\beta_0, \infty).
\]
Then there is \((\alpha_u, \beta_u) \in [0, \bar{\alpha}] \times [0, \infty)\) such that
\[
\varphi(\alpha_u, \beta_u) = \max_{(\alpha, \beta) \in [0, \bar{\alpha}] \times [0, \infty)} \varphi(\alpha, \beta).
\]

We claim that \((\alpha_u, \beta_u) \in (0, \bar{\alpha}) \times (0, \infty)\). Indeed, \(\varphi(\alpha, 0) < \varphi(\alpha, \beta)\) for \(\alpha \in [0, \bar{\alpha}]\)
and \(\beta\) small enough, while \(\varphi(0, \beta) < \varphi(\alpha, \beta)\) for \(\beta \in [0, \infty)\) and \(\alpha\) sufficiently small, which implies \((\alpha_u, \beta_u) \notin [0, \bar{\alpha}] \times \{0\}\) and \((\alpha_u, \beta_u) \notin \{0\} \times [0, \infty)\).
Note that
\[
\frac{2}{N^2} S^{-2/N} \leq \hat{\alpha}^4 b A_2^3 - \hat{\alpha}^{2*} \frac{2^{*}}{A_2} B_1 + \frac{\hat{\alpha}^4 b}{2} A_3 A^+(u) + \frac{\beta^2}{4} A^+ (u) + \frac{3^4 b}{2} A_4 A^-(u),
\]
for every \( \beta \in [0, \infty) \). Therefore, we have \( \varphi(\hat{\alpha}, \beta) \leq 0 \) for all \( \beta \in [0, \infty) \), which means \( (\alpha, \beta) \notin \{ \hat{\alpha} \} \times [0, \infty) \). Based on the above, we get \( (\alpha, \beta) \in (0, \hat{\alpha}) \times (0, \infty) \), that is, \( (\alpha, \beta) \) is an inner maximizer of \( \varphi \) in \([0, \hat{\alpha}) \times [0, \infty) \). Therefore, \( \alpha u^+ + \beta u^- \in N^\lambda_b \). In that case, by (2.22), we have
\[
c^b_\lambda \geq I_b^\lambda (\alpha u^+ + \beta u^-) + \frac{\hat{\alpha}^2}{2} A_1 + \frac{\hat{\alpha}^4 b}{4} A_3 - \frac{\hat{\alpha}^{2*}}{2^{*}} B_1
\]
\[
+ \frac{\hat{\alpha}^4 b}{2} A_3 A^+(u) + \frac{\beta^2}{4} A_2 + \frac{3^4 b}{2} A_4 A^-(u)
\]
\[
> I_b^\lambda (\alpha u^+ + \beta u^-) \geq c^b_\lambda,
\]
which is a contradiction. Hence, we have \( B_1 = B_2 = 0 \).

**Step 3:** \( c^b_\lambda \) is achieved. Given \( u^\pm \neq 0 \), according to Lemma 2.1 there exists \( \alpha, \beta > 0 \) such that \( \hat{u} := \alpha u^+ + \beta u^- \in N^\lambda_b \). Moreover, \( \langle (I_b^\lambda)'(u), u^\pm \rangle \leq 0 \). By Lemma 2.2 we have \( 0 < \alpha, \beta, a \leq 1 \).

Combining \( u_n \in N^\lambda_b \) and Lemma 2.1 we obtain
\[
I_b^\lambda (\alpha u^+_n + \beta u^-_n) \leq I_b^\lambda (u^+_n + u^-_n) = I_b^\lambda (u_n).
\]

Taking into consideration \( B_1 = B_2 = 0 \) and the semicontinuity of the norm, we obtain
\[
c^b_\lambda \leq I_b^\lambda (\hat{u})
\]
\[
= I_b^\lambda (\hat{u}) - \frac{1}{4} \langle (I_b^\lambda)'(\hat{u}), \hat{u} \rangle
\]
\[
= \frac{1}{4} ||\hat{u}||^2 + \left( \frac{1}{4} - \frac{1}{2^{*}} \right) \int_{\mathbb{R}^N} |\hat{u}|^{2^{*}} dx + \frac{\lambda}{4} \int_{\mathbb{R}^N} [f(\hat{u})\hat{u} - 4F(\hat{u})] dx
\]
\[
\leq \frac{1}{4} ||u||^2 + \left( \frac{1}{4} - \frac{1}{2^{*}} \right) \int_{\mathbb{R}^N} |u|^{2^{*}} dx + \frac{\lambda}{4} \int_{\mathbb{R}^N} [f(u)u - 4F(u)] dx
\]
\[
\leq \liminf_{n \to \infty} \left[ I_b^\lambda (u_n) - \frac{1}{4} \langle (I_b^\lambda)'(u_n), u_n \rangle \right] \leq c^b_\lambda.
\]

Hence, we can conclude that \( \alpha = \beta = 1 \), and \( c^b_\lambda \) is achieved by \( u_b := u^+ + u^- \in N^\lambda_b \).

### 3. Proofs of main results

**Proof of Theorem 1.2**. Thanks to Lemma 2.4 we only need to prove that the minimizer \( u_b \) for \( c^b_\lambda \) is indeed a nodal solution to problem (1.1).

Because \( u_b \in N^\lambda_b \), we have \( \langle (I_b^\lambda)'(u_b), u_b^+ \rangle = \langle (I_b^\lambda)'(u_b), u_b^- \rangle = 0 \). In view of Lemma 2.1 for \( (\alpha, \beta) \in (\mathbb{R}_+ \times \mathbb{R}_+) \setminus (1, 1) \), we have
\[
I_b^\lambda (\alpha u_b^+ + \beta u_b^-) < I_b^\lambda (u_b^+ + u_b^-) = c^b_\lambda.
\]

Now we proceed by contradiction. Suppose \( (I_b^\lambda)'(u_b) \neq 0 \), then there exist \( \delta > 0 \) and \( \theta > 0 \) such that
\[
|| (I_b^\lambda)'(v) || \geq \theta \quad \text{for all } ||v - u_b|| \leq 3\delta.
\]
Choose $\tau \in (0, \min\{1/2, \frac{\delta}{\sqrt{2\|u_b\|}}\})$, and define

$$D := (1 - \tau, 1 + \tau) \times (1 - \tau, 1 + \tau),$$

and for all $(\alpha, \beta) \in D.$

By (3.1), we have

$$\tilde{\epsilon}_\lambda := \max_{\partial D} (I_\lambda^b \circ g) < c_\lambda^b. \tag{3.2}$$

Let $\varepsilon := \min\{(c_\lambda^b - \tilde{\epsilon}_\lambda)/2, \theta \delta/8\}$ and $S_\delta := B(u_b, \delta)$. By Lemma 2.3, there exists a deformation $\eta \in C([0, 1] \times D, D)$ such that

(a) $\eta(1, v) = v$ if $v \notin (I_\lambda^b)^{-1}([c_\lambda^b - 2\varepsilon, c_\lambda^b + 2\varepsilon] \cap S_\delta),$

(b) $\eta(1, (I_\lambda^b)^{c_\lambda^b + \varepsilon} \cap S_\delta) \subset (I_\lambda^b)^{c_\lambda^b - \varepsilon},$

(c) $I_\lambda^b(\eta(1, v)) \leq I_\lambda^b(v)$ for all $v \in E.$

Clearly,

$$\max_{(\alpha, \beta) \in D} I_\lambda^b(\eta(1, g(\alpha, \beta))) < c_\lambda^b. \tag{3.3}$$

Therefore we claim that $\eta(1, g(D)) \cap N_\lambda^b \neq \emptyset,$ which contradicts the definition of $c_\lambda^b.$

We define $h(\alpha, \beta) := \eta(1, g(\alpha, \beta)),$

$$\Phi_0(\alpha, \beta) := \langle (I_\lambda^b)'(g(\alpha, \beta)), u_b^+ \rangle, \langle (I_\lambda^b)'(g(\alpha, \beta)), u_b^- \rangle \rangle = \langle \langle (I_\lambda^b)'(\alpha u_b^+ + \beta u_b^-), u_b^+ \rangle, \langle (I_\lambda^b)'(\alpha u_b^+ + \beta u_b^-), u_b^- \rangle \rangle$$

and

$$\Phi_1(\alpha, \beta) := \langle \frac{1}{\alpha} \langle (I_\lambda^b)'(h(\alpha, \beta)), \langle (h(\alpha, \beta))^+, \frac{1}{\beta} \langle (I_\lambda^b)'(h(\alpha, \beta)), h(\alpha, \beta)^- \rangle \rangle \rangle.$$

With an approach similar to [14], we use degree theory to obtain $\deg(\Phi_0, D, 0) = 1.$ Then by (3.2), we obtain

$$g(\alpha, \beta) = h(\alpha, \beta) \text{ on } \partial D,$$

as a result of which, we have $\deg(\Phi_1, D, 0) = \deg(\Phi_0, D, 0) = 1.$ Hence, $\Phi_1(\alpha_0, \beta_0) = 0$ for some $(\alpha_0, \beta_0) \in D$ so that

$$\eta(1, g(\alpha_0, \beta_0)) = h(\alpha_0, \beta_0) \in N_\lambda^b,$$

which contradicts (3.3). Hence, $(I_\lambda^b)'(u_b) = 0$, which implies $u_b$ is a critical point of $I_\lambda^b.$ Thus, we can deduce that $u_b$ is a nodal solution to problem (1.1). \hfill \Box

By Theorem 1.2 we obtain a least energy nodal solution $u_b$ to problem (1.1), contributing to the establishment of Theorem 1.3 where we shall prove that the energy of $u_b$ is strictly larger than twice the ground state energy.

**Proof of Theorem 1.3.** As in the proof of Lemma 2.3, there exists $\lambda_1^* > 0$ such that for all $\lambda \geq \lambda_1^*$, and for each $b > 0$, there exists $v_b \in M_b^\lambda$ such that $I_\lambda^b(v_b) = c^* > 0.$ By standard arguments (see [10], Corollary 2.13), the critical points of the functional $I_\lambda^b$ on $M_b^\lambda$ are critical points of $I_\lambda^b$ in $E$, so we obtain $(I_\lambda^b)'(v_b) = 0.$ That is, $v_b$ is a ground state solution to problem (1.1).

As stated in Theorem 1.2 $u_b$ is known as a least energy nodal solution to problem (1.1), which changes sign only once when $\lambda \geq \lambda^*.$

Let $\lambda^{**} = \max\{\lambda^*, \lambda_1^*\}$ and assume $u_b = u_b^+ + u_b^-.$ Adopting the same approach as in Lemma 2.1 we claim there exist $\alpha_{u_b^+}^b > 0$ and $\beta_{u_b^-}^b > 0$ such that $\alpha_{u_b^+}^b u_b^+ \in M_b^\lambda$ and
and \( \beta_{u^+}u^- \in M^b \). Then, by Lemma 2.2 we obtain \( \alpha_{u^+}, \beta_{u^-} \in (0, 1) \). Hence, thanks to Lemma 2.1, we have

\[
2c^* \leq I^b_0(\alpha_{u^+}u^+ + \beta_{u^-}u^-) \leq I^b_0(\alpha_{u+}u^+ + \beta_{u^-}u^-) \leq I^b_0(u^+ + u^-) = c^*_b.
\]

It follows that \( c^* > 0 \) cannot be achieved by a nodal function.

We complete this section with the proof of Theorem 1.4. In the sequel, we regard \( b > 0 \) as a parameter in problem (1.1).

**Proof of Theorem 1.4** In 3 steps, we analyze the convergence property of \( u_b \) as \( b \to 0 \), where \( u_b \) is the least energy nodal solution obtained in Theorem 1.2

**Step 1.** For any sequence \( \{b_n\} \), we prove that \( \{u_{b_n}\} \) is bounded in \( E \), if \( b_n \searrow 0 \). Let \( \chi \in C_0^\infty(\mathbb{R}^N) \) be a nonzero function with \( \chi^\pm \neq 0 \) fixed. Analogous to the argument in Lemma 2.1 for any \( b \in [0, 1] \), there exists a pair of positive numbers \( (\alpha_1, \lambda_2) \) independent of \( b \), such that

\[
(\langle I^b_0\rangle'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_1\chi^+) < 0 \quad \text{and} \quad (\langle I^b_0\rangle'(\lambda_1\chi^+ + \lambda_2\chi^-), \lambda_2\chi^-) < 0.
\]

Then according to Lemma 2.2 for any \( b \in [0, 1] \), there exists a unique pair \( (\alpha_\chi(b), \lambda_\chi(b)) \in (0, 1) \times (0, 1) \) such that \( \overline{\chi} := \alpha_\chi(b)\lambda_1\chi^+ + \lambda_\chi(b)\lambda_2\chi^- \in N^\lambda_b \). Therefore, by (2.5), it follows that, for any \( b \in [0, 1] \),

\[
I^b_0(u_b) \leq I^b_0(\overline{\chi}) = I^b_0(\overline{\chi}) - \frac{1}{4}\langle I^b_0\rangle'(\overline{\chi}, \overline{\chi})
\]

\[
= \frac{1}{4}\|\chi\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}|\overline{\chi}|^2 \varphi dx + \frac{\lambda}{4}\int_{\mathbb{R}^N}[f(\overline{\chi})\overline{\chi} - 4F(\overline{\chi})] dx
\]

\[
\leq \frac{1}{4}\|\chi\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}|\overline{\chi}|^2 \varphi dx + \frac{\lambda}{4}\int_{\mathbb{R}^N}(C_1|\overline{\chi}|^2 + C_2|\overline{\chi}|^p) dx
\]

\[
\leq \frac{1}{4}\|\lambda_1\chi^+\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}|\lambda_1\chi^+|^2 \varphi dx
\]

\[
+ \frac{\lambda}{4}\int_{\mathbb{R}^N}(C_1|\lambda_1\chi^+|^2 + C_2|\lambda_1\chi^+|^p) dx
\]

\[
+ \frac{\lambda}{4}\int_{\mathbb{R}^N}(C_1|\lambda_2\chi^-|^2 + C_2|\lambda_2\chi^-|^p) dx
\]

\[
:= C^*,
\]

where \( C^* \) is a positive constant independent of \( b \). Thus, as \( n \to \infty \), it follows that

\[
C^* + 1 \geq I^b_n(u_{b_n}) = I^b_n(u_{b_n}) - \frac{1}{4}\langle I^b_n\rangle'(u_{b_n}, u_{b_n}) \geq \frac{1}{4}\|u_{b_n}\|^2,
\]

that is, \( \{u_{b_n}\} \) is bounded in \( E \).

**Step 2.** In this step, we prove that problem (1.10) possesses one nodal solution \( u_0 \). Since \( \{u_{b_n}\} \) is bounded in \( E \), thanks to Step 1, up to a subsequence, there exists \( u_0 \in E \) such that

\[
u_{b_n} \to u_0 \quad \text{in} \quad E,
\]

\[
u_{b_n} \to u_0 \quad \text{in} \quad L^p(\mathbb{R}^N) \quad \text{for} \quad p \in [2, 2^{**}),
\]

\[
u_{b_n} \to u_0 \quad \text{a.e. in} \mathbb{R}^N.
\]
Given that \( \{u_{b_n}\} \) is a weak solution to (1.1) with \( b = b_n \), we have

\[
\int_{\mathbb{R}^N} (\Delta u \Delta \phi + \nabla u \nabla \phi + V(x)u \phi) \, dx + b_n \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} \nabla u \nabla \phi \, dx
= \lambda \int_{\mathbb{R}^N} f(u) \phi \, dx + \int_{\mathbb{R}^N} |u|^{2^*-2}u \phi \, dx
\]

for all \( \phi \in C_0^\infty(\mathbb{R}^N) \).

Combining (3.4), (3.5) and Step 1, we find that

\[
\int_{\mathbb{R}^N} (\Delta u_0 \Delta \phi + \nabla u_0 \nabla \phi + V(x)u_0 \phi) \, dx + b_n \int_{\mathbb{R}^N} |\nabla u_0|^2 \, dx \int_{\mathbb{R}^N} \nabla u_0 \nabla \phi \, dx
= \lambda \int_{\mathbb{R}^N} f(u_0) \phi \, dx + \int_{\mathbb{R}^N} |u_0|^{2^*-2}u_0 \phi \, dx
\]

for all \( \phi \in C_0^\infty(\mathbb{R}^N) \), which in turn implies that \( u_0 \) is a weak solution to (1.10).

Analogous to the process of Lemma 2.3, we obtain that \( u_0^\pm \neq 0 \). Thus, we have completed the proof of this step.

**Step 3.** In this step, we prove that problem (1.10) possesses a least energy nodal solution \( v_0 \), and that there exists a unique pair \((\alpha_{b_n}, \beta_{b_n}) \in \mathbb{R}^+ \times \mathbb{R}^+ \) satisfying \( \alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in N_{b_n}^\lambda \). Also we prove that \((\alpha_{b_n}, \beta_{b_n}) \to (1, 1) \) as \( n \to \infty \).

Similar to the proof of Theorem 1.2, we can reach the conclusion that problem (1.10) possesses a least energy nodal solution \( v_0 \), where \( \lambda_0^{(1)}(v_0) = c_0 \) and \((I_0^{(1)})'(v_0) = 0 \). Then, in view of Lemma 2.1, we can obtain with ease the existence and uniqueness of the pair \((\alpha_{b_n}, \beta_{b_n}) \) such that \( \alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in N_{b_n}^\lambda \). Besides, we know \( \alpha_{b_n} > 0 \) and \( \beta_{b_n} > 0 \). To complete the proof, we just establish that \((\alpha_{b_n}, \beta_{b_n}) \to (1, 1) \) as \( n \to \infty \). Actually, given that \( \alpha_{b_n} v_0^+ + \beta_{b_n} v_0^- \in N_{b_n}^\lambda \), we have

\[
\alpha_{b_n}^2 \|v_0^+\|^2 + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx
+ \beta_{b_n}^2 \|v_0^-\|^2 + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx
\]

and

\[
\beta_{b_n}^2 \|v_0^-\|^2 + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx
+ \beta_{b_n}^2 \|v_0^-\|^2 + \alpha_{b_n} \beta_{b_n} \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx
\]

a Since \( b_n \searrow 0 \), we conclude that the sequences \( \{\alpha_{b_n}\} \) and \( \{\beta_{b_n}\} \) are bounded. Assume, up to a subsequence, \( \alpha_{b_n} \to \alpha_0 \) and \( \beta_{b_n} \to \beta_0 \). Then by (3.7) and (3.8), we have

\[
\alpha_0^2 \|v_0^+\|^2 + \alpha_0 \beta_0 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx = \lambda \int_{\mathbb{R}^N} f(\alpha_0 v_0^+)\alpha_0 v_0^+ \, dx + \int_{\mathbb{R}^N} |\alpha_0 v_0^+|^{2^*} \, dx \quad (3.9)
\]

and

\[
\beta_0^2 \|v_0^-\|^2 + \alpha_0 \beta_0 \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx = \lambda \int_{\mathbb{R}^N} f(\beta_0 v_0^-)\beta_0 v_0^- \, dx + \int_{\mathbb{R}^N} |\beta_0 v_0^-|^{2^*} \, dx \quad (3.10)
\]
Noticing that $v_0$ is a nodal solution to problem (1.10), we obtain
\[
\|v_0^+\|^2 + \int_{\mathbb{R}^N} \Delta v_0^+ \Delta v_0^- \, dx = \lambda \int_{\mathbb{R}^N} f(v_0^+) v_0^+ \, dx + \int_{\mathbb{R}^N} |v_0^+|^{2^*} \, dx, \tag{3.11}
\]
\[
\|v_0^-\|^2 + \int_{\mathbb{R}^N} \Delta v_0^- \Delta v_0^+ \, dx = \lambda \int_{\mathbb{R}^N} f(v_0^-) v_0^- \, dx + \int_{\mathbb{R}^N} |v_0^-|^2^{2^*} \, dx. \tag{3.12}
\]

Therefore, from (3.9)-(3.12), we can easily obtain that $(\alpha_0, \beta_0) = (1, 1)$, and thus Step 3 follows.

We can now complete the proof of Theorem 1.4. We claim that $u_0$ obtained in Step 2 is a least energy solution to problem (1.10). In fact, according to Step 3 and Lemma 2.1, we see that
\[
I_{\lambda_0}^0(v_0) \leq I_{\lambda_0}^0(u_0) = \lim_{n \to \infty} I_{\lambda_0}^0(u_{b_n}) \leq \lim_{n \to \infty} I_{\lambda_0}^0(\alpha_{b_n} v_{0^+} + \beta_{b_n} v_{0^-}) = \lim_{n \to \infty} I_{\lambda_0}^0(v_0^+ + v_0^-) = I_{\lambda_0}^0(v_0),
\]
which yields completes the proof of Theorem 1.4.

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