ON FUNCTIONS OF QUASI TOEPLITZ MATRICES∗

DARIO A. BINI †, STEFANO MASSEI ‡, AND BEATRICE MEINI §

Abstract. Let \( a(z) = \sum_{i \in \mathbb{Z}} a_i z^i \) be a complex valued continuous function, defined for \( |z| = 1 \), such that \( \sum_{i=-\infty}^{\infty} |a_i| < \infty \). Consider the semi-infinite Toeplitz matrix \( T(a) = (t_{i,j})_{i,j \in \mathbb{Z}^+} \) associated with the symbol \( a(z) \) such that \( t_{i,j} = a_{j-i} \). A quasi-Toeplitz matrix associated with the continuous symbol \( a(z) \) is a matrix of the form \( A = T(a) + E \) where \( E = (e_{i,j}), \sum_{i,j \in \mathbb{Z}^+} |e_{i,j}| < \infty, \) and is called a CQT-matrix. Given a function \( f(x) \) and a CQT matrix \( M \), we provide conditions under which \( f(M) \) is well defined and is a CQT matrix. Moreover, we introduce a parametrization of CQT matrices and algorithms for the computation of \( f(M) \). We treat the case where \( f(x) \) is assigned in terms of power series and the case where \( f(x) \) is defined in terms of a Cauchy integral. This analysis is applied also to finite matrices which can be written as the sum of a Toeplitz matrix and of a low rank correction.

Key words. Matrix functions, Toeplitz matrices, Infinite matrices

1. Introduction. Functions of finite matrices have received a lot of interest in the literature both for their theoretical properties and for the many applications they have in real world problems. We refer the interested reader to the book [16] for more details. Among the available different definitions of matrix function, which are equivalent under mild hypotheses, many rely on the Jordan canonical form of the matrix argument or on its Schur form. As a consequence, they are not directly generalizable to the setting of infinite matrices. However, there are two definitions which apparently seem to be more suited for extending the concept of function to infinite matrices. They rely on the Laurent series expansion, and on the integral representation through the Dunford Cauchy formula [13].

In the set of matrices with infinite size there is a class which is a cornerstone for numerical linear algebra. It is the class of Toeplitz matrices \( T(a) = (t_{i,j}) \) associated with a function \( a(z) = \sum_{i \in \mathbb{Z}} a_i z^i \), called symbol, defined by \( t_{i,j} = a_{j-i} \). Toeplitz matrices are widely analyzed in the literature, from the pioneering papers by O. Toeplitz, to the seminal monograph by Grenander and Szegö [14], until to the more recent and wide contributions given by several international research groups, including, but not limited to, the books [9], [10], [11], [12], [17] and the papers [20], [21], [23], [24] with the references cited therein. For a general view on this topic we refer the reader to the monograph [5] which collects a wide set of contributions and gives the taste of the problems in this area and an idea of their richness.

In some problems, typically encountered in the analysis of stochastic processes, like bi-dimensional random walks in the quarter-plane, one has to deal with matrices of the kind \( A = T(a) + E \), where \( T(a) \) is a semi-infinite Toeplitz matrix, while \( E = (e_{i,j}) \) is a non-Toeplitz correction such that \( \sum_{i,j \in \mathbb{Z}^+} |e_{i,j}| < \infty \). We call this class of matrices Quasi-Toeplitz matrices, in short QT matrices, and denote them with QT.

We consider the subset of QT matrices whose symbol \( a(z) \) is differentiable and both \( a(z) \) and \( a(z)' \) belong to the Wiener class. We call the latter as the set of CQT matrices and we denote it with CQT.

From the computational point of view, one has to solve matrix equations where the coefficients are CQT matrices, or to compute the value of \( f(A) \), where \( f(z) \) is an

∗Work supported by GNCS of INdAM.
†Dipartimento di Matematica, Università di Pisa, (dario.bini@unipi.it)
‡Scuola Normale Superiore, Pisa (stefano.massei@sns.it)
§Dipartimento di Matematica, Università di Pisa, (beatrice.meini@unipi.it)
assigned function, typically the exponential function, and \( A \) is a CQT matrix.

Concerning the recent literature in this research area, it is worth citing the paper [6] where the exponential function of a block-triangular block-Toeplitz matrix is analyzed with application to solving certain fluid queues. In the recent paper [18] the problem of computing the exponential function of finite Toeplitz matrices is investigated and several applications are presented. In [8] the case of the exponential of a semi-infinite CQT matrix is analyzed in depth. Related issues concern the decay of the coefficients of a matrix function [2], [3], [19], and the decay of the singular values of matrices having a displacement rank structure [1].

In this paper, our interest is addressed to analyze the definition and the properties of \( f(A) \), for \( A \) being a CQT matrix. This work continues the analysis recently started in [7] and in [8] where the classes of QT and CQT matrices have been introduced and analyzed, and where the exponential function has been extended to the case of CQT matrices.

Here we deepen this analysis by considering matrix functions assigned either in terms of a Laurent power series or in terms of the Dunford-Cauchy integral. We give conditions on the function \( f(z) \) and on the CQT matrix \( A \) in order to ensure that the semi-infinite matrix \( f(A) \) is well defined and is still a CQT matrix, i.e., it can be written in the form \( f(A) = T(f(a)) + E_{f(a)} \). Moreover, we outline some algorithms for its computation. Finally, we show that the same analysis can be applied to the case where the matrix is finitely large and can still be written as the sum of a Toeplitz matrix and a correction.

A case of interest concerns matrices associated with an analytic symbol \( a(z) \) where the coefficients of the Toeplitz part have an exponential decay. This situation is very convenient from the computational point of view. However, the class of matrices that we obtain this way, which we call analytically quasi-Toeplitz (AQT), is still a matrix algebra, but is not a Banach space with the norm \( \| \cdot \|_{\text{CQT}} \). In the analysis that we carry out, we point out the cases where the result of the computation is still in the class of AQT matrices.

The paper is organized as follows. In Section 2 we recall the representation of semi-infinite CQT matrices and the description of the arithmetic in this matrix algebra. Moreover, we outline the way to extend this definition to the case of finitely large matrices where \( A \) can still be written as \( A = T(a) + E \), and the correction \( E \) has nonzero entries in the upper leftmost corner and in the lower rightmost corner, so that the rank of \( E \) is small with respect to its size.

In Section 3 we generalize the concept of matrix functions to semi-infinite CQT matrices. As first step, we consider the case where the function \( f(x) \) is assigned in terms of a power series and give conditions under which \( f(A) \) is still a CQT matrix. Then we analyze the case where \( f(x) \) is given in terms of a Laurent series. Also in this framework, we give conditions under which \( f(A) \) is a CQT matrix. We examine some computational issues and outline an algorithm for computing a general matrix function of a CQT matrix. Some numerical tests which demonstrate the effectiveness of our approach are presented both for semi-infinite and for finite matrices.

Section 4 is devoted to the extension of the concept of matrix function based on the Dunford-Cauchy integral. Once again, we provide conditions under which \( f(A) \) is a CQT matrix. Finally, we discuss some computational issues, outline an algorithm for computing \( f(A) \) based on the trapezoidal rule, and present a validation of our theory by means of some numerical tests. In Section 5 we draw some conclusions and research lines.
2. Definitions and preliminaries. We indicate with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle of the complex plane $\mathbb{C}$, with $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integer values and the set of positive integers, respectively. We denote by $W$ the Wiener class, i.e., the set of complex valued Laurent series $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ such that $a_i \in \mathbb{C}$. We also denote the set $W_1 := \{a(z) \in W : a'(z) \in W\} \subset W$. Recall that $W$ is a Banach algebra with the norm $\|a\|_W$, i.e., a Banach linear space closed under product of functions. Given $a(z) \in W$ define $a^+(z) = \sum_{i \in \mathbb{Z}}^+ a_i z^i$, $a^-(z) = \sum_{i \in \mathbb{Z}}^- a_i z^i$ so that $a(z) = a_0 + a^-(z) + a^+(z)$, and associate with $a(z)$ and $a^\pm(z)$ the following semi-infinite matrices

$$
T(a) = (t_{i,j}), \quad t_{i,j} = a_{j-i},
$$

$$
H(a^+) = (h^+_{i,j}), \quad h^+_{i,j} = a_{i+j-1},
$$

$$
H(a^-) = (h^-_{i,j}), \quad h^-_{i,j} = a_{-i-j+1}.
$$

Observe that $T(a)$ is Toeplitz while $H(a^+)$ and $H(a^-)$ are Hankel matrices. The function $a(z)$ is said symbol associated with $T(a)$. Moreover, denote by $F$ the set of semi-infinite matrices $F = (f_{i,j})_{i,j \in \mathbb{Z}^+}$ such that $\|f\|_F := \sum_{i,j \in \mathbb{Z}^+} |f_{i,j}| < +\infty$.

2.1. Quasi Toeplitz matrices. Dealing with an infinite amount of data can be an insurmountable problem from the computational point of view. However, such difficulties can be overcome whenever the data to be processed are representable—at an arbitrary precision—with a finite number of parameters. With this motivation, in [7] the authors introduce the following classes of infinite matrices.

**Definition 2.1.** A semi-infinite matrix $A$ is said quasi Toeplitz (QT-matrix) if it can be written in the form

$$
A = T(a) + E,
$$

where $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i \in W$ and $E = (e_{i,j}) \in F$. We refer to $T(a)$ as the Toeplitz part of $A$, and to $E$ as the correction. If $a(z)$ belongs to $W_1$ then we say that $A$ is a CQT matrix. Finally, if $a(z)$ is analytic in an open annulus enclosing $\mathbb{T}$, then the matrix $A$ is said analytically quasi Toeplitz (AQT-matrix). The classes of QT-matrices, CQT-matrices and AQT-matrices are denoted by $\mathcal{QT}$, $\mathcal{CQT}$ and $\mathcal{AQT}$, respectively.

We report some properties of these classes. More details are given in [7] and [8]. The first result shows that $\mathcal{QT}$ is a Banach space, while $\mathcal{CQT} \subset \mathcal{QT}$ is a Banach algebra.

**Theorem 2.2.** The class $\mathcal{QT}$ is a Banach space with the norm $\|T(a) + E\|_\mathcal{QT} := \|a\|_W + \|E\|_F$. The class $\mathcal{CQT}$ is a Banach algebra equipped with the norm $\|T(a) + E\|_{\mathcal{CQT}} := \|a\|_W + \|a'\|_W + \|E\|_F$, where $a'(z)$ is the first derivative of $a(z)$. The class $\mathcal{AQT}$ is a normed algebra equipped with the norm $\|\cdot\|_{\mathcal{AQT}}$. Moreover, $\|AB\|_{\mathcal{CQT}} \leq \|A\|_{\mathcal{CQT}} \|B\|_{\mathcal{CQT}}$ for any $A, B \in \mathcal{CQT}$, and $\|A\|_{\mathcal{AQT}} \leq \|A\|_{\mathcal{CQT}}$ for any $A \in \mathcal{CQT}$.

**Remark 1.** It is interesting to notice that $\mathcal{AQT}$ with the norm $\|\cdot\|_{\mathcal{CQT}}$ is not Banach. In fact, consider the sequence of semi-infinite Toeplitz matrices $\{T(a_n)\}$ with $a_n(z) = \sum_{j=1}^n \frac{1}{j} z^j$, and observe that this is a Cauchy sequence in $\mathcal{AQT}$ with the norm $\|\cdot\|_{\mathcal{CQT}}$, but its limit does not belong to $\mathcal{AQT}$ because the corresponding symbol $a(z) = \sum_{j=1}^\infty \frac{1}{j} z^j$ is not analytic. On the other hand, the completeness of $\mathcal{CQT}$ implies that any Cauchy sequence in $\mathcal{AQT}$ admits limit in $\mathcal{CQT}$. Therefore, we can claim that the limit of a Cauchy sequence in $\mathcal{AQT}$ can still be represented—at an arbitrary precision—with a finite number of parameters.
The next result from [10] provides a representation of the product of two semi-infinite Toeplitz matrices.

**Theorem 2.3.** For $a(z), b(z) \in W$ let $c(z) = a(z)b(z)$. Then we have

$$T(a)T(b) = T(c) - H(a^-)H(b^+).$$

The following result from [8] provides a representation of the matrices $T(a)^i$ and $(T(a) + E)^i$.

**Theorem 2.4.** If $a(z) \in W_1$ then $T(a)^i = T(a^i) + E_i$, where $E_1 = 0$ and $E_i = T(a)E_{i-1} - H(a^-)H((a^{-1})^+) + ET(a^{-1})$, $i \geq 2$. Moreover,

$$\|E_i\|_x \leq \frac{i(i-1)}{2}\|a^\prime\|_{\infty}^{-2} \|a\|_{\infty}^{-2}.$$ 

If $A = T(a) + E \in CQT$ then $A^i = T(a^i) + D_i$, where $D_0 = E$ and

$$D_i = AD_{i-1} - H(a^-)H((a^{-1})^+) + ET(a^{-1})$$

Moreover, for $\alpha = \|a^\prime\|_x^2 + \|E\|_x$, $\beta = \|a^\prime\|_{\infty}^2$ we have

$$\|D_i\|_x \leq \frac{1}{\|E\|_x} \left( \alpha \left( \frac{\|a\|_{\infty} + \|E\|_x}{\|E\|_x} \right)^i - \|a\|_{\infty}^i - \beta \|a^\prime\|_{\infty}^{i-1} \right).$$

### 2.2. Quasi-Toeplitz matrix arithmetic

Since $CQT$ is a matrix algebra then the outcome of a rational computation which takes as input a CQT-matrix and that can be carried out with no breakdown, say caused by singularity, still belongs to $CQT$. This observation explains how to compute functions of semi-infinite CQT-matrices by means of convergent sequences of rational approximations.

For this reason, in [7] the design and the analysis of a matrix arithmetic for CQT matrices has been introduced. In particular, a CQT matrix $A = T(a) + E$ is represented in the following form. The Toeplitz part $T(a)$ is represented up to an arbitrarily small error by means of the finite sequence $\hat{a} = (a_{-n}, \ldots, a_0, \ldots, a_{n})$ of the coefficients of the Laurent polynomial $\hat{a}(z) = \sum_{i=-n}^{n} a_i z^i$ which approximates the function $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ for suitable $n$. The correction $E$ is represented by two finite dimensional matrices $F$ and $G$ such that $FGT$ coincides with the non negligible part of $E$. Here $F$ and $G$ have size $n_f \times r$, $n_g \times r$, respectively where $n_f$ and $n_g$ are the sizes of the leading submatrix of $E$ which contains the non-negligible entries, while $r$ is the rank of this submatrix.

This way, a matrix in $CQT$ is represented by means of the triple $(\hat{a}, E, F)$ up to any controllable small error $\epsilon$. Each arithmetic operation, say, $C = A + B$, is defined by means of a set of equations which provide the components of the triple $(\hat{c}, F_c, G_c)$ associated with the matrix $C$ in terms of the components of the triples $(\hat{a}, F_a, G_a)$ and $(\hat{b}, F_b, G_b)$ associated with $A$ and $B$, respectively. Similarly, matrix inversion is implemented by providing the relation between the triple $(\hat{a}, F_a, G_a)$ associated with $A$ and the triple $(\hat{c}, F_c, G_c)$ associated with $C = A^{-1}$. An operation of compression is introduced to reduce the sizes of $F_c$ and $G_c$ obtained after each operation. More details can be found in [7].
2.2.1. Finite quasi Toeplitz matrix arithmetic. Given a symbol $a(z)$ and $m \in \mathbb{Z}^+$ we indicate with $T_m(a)$, $H_m(a^-)$ and $H_m(a^+)$ the $m \times m$ leading principal submatrices of $T(a)$, $H(a^-)$ and $H(a^+)$, respectively.

The approach used in Section 2.2 can be easily adapted to design a matrix arithmetic for quasi Toeplitz matrices of finite size. The crucial tool for doing this extension is a version of Theorem 2.3 which applies to the finite case.

**Theorem 2.5.** Let $a(z) = \sum_{i=-m+1}^{m-1} a_i z^i$, $b(z) = \sum_{i=-m+1}^{m-1} b_i z^i$ and set $c(z) = a(z)b(z) = \sum_{i=-2m+2}^{2m-2} c_i z^i$. Then,

$$T_m(a)T_m(b) = T_m(c) - H_m(a^-)H_m(b^+) - JH_m(a^+)H_m(b^-)J,$$

where $J$ is the $m \times m$ flip matrix having $1$ on the anti-diagonal and zeros elsewhere.

A pictorial description of the above theorem is given in Figure 2.1 where $a(z)$ and $b(z)$ are Laurent polynomials of degree much less than $m$ so that $T_m(a)$ and $T_m(b)$ are banded.

![Fig. 2.1. Multiplication of two finite dimensional banded Toeplitz matrices](image)

If $a(z) = \sum_{i=-k}^{k} a_i z^i$, $b(z) = \sum_{i=-k}^{k} b_i z^i$ with $k$ much smaller than $m - 1$, then the matrices $H_m(a^-)H_m(b^+)$ and $JH_m(a^+)H_m(b^-)J$ have disjoint supports located in the upper leftmost corner and in the lower rightmost corner, respectively. Thus, $T_m(a)T_m(b)$ can be represented as the sum of the Toeplitz matrix associated with the Laurent polynomial $c(z)$ and of two correction matrices $E^+$ and $E^-$ which collect the finite number of nonzero entries located in the upper leftmost and in the lower rightmost corners, respectively.

Algorithms for dealing with finite quasi Toeplitz matrices can be easily obtained from those presented in [2] just by taking into account the additional lower rightmost corner correction. In the case of a sufficiently large gap between the size $m$ and the bandwidth $k$ of the symbols that come into play, the two corner corrections behave independently of each other and the finite CQT matrix arithmetic becomes more effective. The cost of these operations essentially depends on the bandwidth of $T_m(c)$ and on the sizes and ranks of the correction matrices $E^+$ and $E^-$. The cost remains small as long as the bandwidth and the size of the corrections $E^+$, $E^-$ remain small together with their rank. Whether this condition is not satisfied, the two corrections may spread and overlap. This may cause a slowdown due to the additional operations which are needed in the computation.

3. Function of a CQT matrix: power series representation. In this section we give conditions under which a function $f(x)$, expressed in terms of a power series or a Laurent series, can be applied to matrices $A$ in the class $\mathbb{CQT}$, and prove that under these conditions $f(A)$ still belongs to $\mathbb{CQT}$.

Let $a(z) \in \mathbb{W}_1$ and $A = T(a) + E \in \mathbb{CQT}$. Assume we are given a complex valued function $f(x) = \sum_{i=0}^{+\infty} f_i x^i$ which is analytic on the open disk $\mathcal{D}(\rho) = \{ x \in \mathbb{C} : |x| < \rho \}$. Observe that, if $a(\mathbb{T}) \subseteq \mathcal{D}(\rho)$, then the composed function $f(a(x))$ belongs to $\mathbb{W}_1$. Recall also the following decay property of the coefficients $f_i$ of an analytic function
Define $\varphi_k(x) = \sum_{i=0}^{k} f_i x^i$ and observe that for any integers $h, k$ such that $h > k$ one has $\varphi_h(A) - \varphi_k(A) = \sum_{i=k+1}^{h} f_i A^i$. Thus,

$$\|\varphi_h(A) - \varphi_k(A)\|_{CQT} \leq \sum_{i=k+1}^{h} |f_i| \cdot \|A\|_{CQT}^i.$$  \hspace{1cm} (3.2)

This inequality implies the following result.

**Theorem 3.1.** Let $A = T(a) + E_a \in CQT$ and let $f(x) = \sum_{i=0}^{+\infty} f_i x^i$ be analytic in $\mathbb{D}(\rho)$. If $\|A\|_{CQT} < \rho$ then $f(A) = \sum_{i=0}^{+\infty} f_i A^i$ is well defined, belongs to $CQT$, and

$$f(A) = T(f(a)) + E_{f(a)}.$$

Furthermore, if $A \in AQT$ then $f(A) \in AQT$. More precisely, there exists an annulus $\mathbb{A}(r(R))$ containing $\mathbb{T}$, such that $f(a(z))$ is well defined and analytic for $z \in \mathbb{A}(r(R))$.

**Proof.** We prove that the sequence $\varphi_k(A) = \sum_{i=0}^{k} f_i A^i$ is a Cauchy sequence in $(CQT, \|\cdot\|_{CQT})$. In fact, since $\|A\|_{CQT} < \rho$ there exists $0 < \delta < \rho$ such that $\|A\|_{CQT} = \rho - \delta$. Thus, from (3.2), for $h > k$ we have

$$\|\varphi_h(A) - \varphi_k(A)\|_{CQT} \leq \sum_{i=k+1}^{h} |f_i| (\rho - \delta)^i.$$  \hspace{1cm} (3.1)

On the other hand, in view of equation (3.1) with $\epsilon = \delta/2$, there exists $\gamma$ such that $|f_i| \leq \gamma (\rho - \delta (2)^{-i}$. This implies that $\|\varphi_h(A) - \varphi_k(A)\|_{CQT} \leq \gamma \sum_{i=k+1}^{h} \lambda^i$, $\lambda = (\rho - \delta)/(\rho - \delta/2) < 1$. Thus for sufficiently large values of $h$ and $k$, the latter summation is smaller than any given $\epsilon > 0$ so that the sequence $\varphi_k(A)$ is Cauchy. Since the space $CQT$ is Banach, there exists $F \in CQT$ such that $\lim_k \|\varphi_k(A) - F\|_{CQT} = 0$. That is, $F := f(A)$ is well defined and belongs to $CQT$. Thus, $f(A)$ can be written as $f(A) = T(g) + E_g$ for a suitable $g(z) \in W_1$ and $E_g \in F$. Observe that $\varphi_k(A)$ can be written in the form $\varphi_k(A) = T(\varphi_k(a)) + E_k$ for a suitable $E_k \in F$. Thus, the convergence of $\varphi_k(A)$ to $T(g) + E_g$ in the norm $\|\cdot\|_{CQT}$ implies that $\lim_k \|E_k - E_g\|_{x} = 0$ and $\lim_k \|\varphi_k(a) - g\|_{W} = 0$. Thus we deduce that $g(z) = f(a(z))$. In the case $A \in AQT$, in order to show that $F \in AQT$, it is sufficient to prove that $g(z) = f(a(z))$ is analytic over some annulus $\mathbb{A}(r(R))$. From the condition $\|a\|_{W} \leq \|A\|_{CQT} < \rho$ it follows that for $|z| = 1$, we have $|a(z)| \leq \sum_{i \in \mathbb{Z}} |a_i| \cdot |z|^i = \|a\|_{W} < \rho$. By continuity of $a(z)$ there exists an open annulus $\mathbb{A}(r(R))$ which includes the unit circle $\mathbb{T}$, such that $|a(z)| < \rho$ for $z \in \mathbb{A}(r(R))$. This way, the function $f(a(z))$ is well defined and analytic in $\mathbb{A}(r(R))$ since composition of analytic functions. This shows that $f(A) \in AQT$ and the proof is complete. \(\square\)

Now we consider the problem of determining bounds to $\|E_{f(a)}\|_{x}$. These bounds are useful from the computational point of view since they provide an indication of the mass of information which is stored in the correction part of $f(A)$. Equivalently, they tell us how much the matrix $f(A)$ differs from a Toeplitz matrix. For simplicity, we deal with the case where $A = T(a)$ is Toeplitz. Then we treat the general case of a matrix $A = T(a) + E_a$.

Since $\|a\|_{W} < \rho$, for the analyticity of $f(x)$ in the disk $\mathbb{D}(\rho)$, we may write

$$\|f(a)\|_{W} \leq \sum_{i=0}^{+\infty} |f_i| \cdot \|a\|_{W}^i < \sum_{i=0}^{+\infty} |f_i| \rho^i < \infty.$$
Let $A^k = T(a^k) + E_k$ and decompose $\varphi_k(A)$ as $\varphi_k(A) = G_k + F_k$, where $G_k = \sum_{i=0}^{k} f_i T(a^i)$, $F_k = \sum_{i=0}^{k} f_i E_i$. Then we have $G_k = T(\sum_{i=0}^{k} f_i a^i)$ so that, $\lim_k G_k = T(f(a))$ and $\lim_{k} F_k = f(A) - T(f(a)) = E_f(a)$.

Now we can provide upper bounds for $\|E_f(a)\|_\infty$ in the case of an almost general function $f(z)$.

**Theorem 3.2.** Assume that the function $f(x) = \sum_{i \in \mathbb{Z}^+} f_i x^i$ is analytic on $\mathbb{D}(\rho)$, that $a(z) \in W_1$ and is such that $\|a\|_\infty < \rho$. Let $A = T(a)$ and $f(A) = T(f(a)) + E_f(a)$. Then

$$\|E_f(a)\| \leq \frac{1}{2} \|\alpha\|_\infty^2 \|g''(\|a\|_\infty)\|$$

where $g(z) = \sum_{i=0}^{\infty} |f_i| z^i$.

**Proof.** Recall from Theorem 3.1 that $f(a(z)) \in W_1$. From Theorem 2.4 we have the bound

$$\|E_i\|_\infty \leq \frac{i(i-1)}{2} \|\alpha\|_\infty^2 \|a\|_\infty^{-2}$$

so that for the matrix $E_f(a) = \sum_{i=0}^{\infty} f_i E_i$ we have

$$\|E_f(a)\|_\infty \leq \sum_{i=0}^{\infty} |f_i| \cdot \|E_i\|_\infty \leq \frac{i(i-1)}{2} \|\alpha\|_\infty^2 \|a\|_\infty^{-2} = \frac{1}{2} \|\alpha\|_\infty g''(\|a\|_\infty),$$

where $g''(\|a\|_\infty)$ is well defined and finite since $\|a\|_\infty < \rho$ and $f(z)$ is analytic for $|z| \leq \rho$. This completes the proof. $\square$

Observe that in the case of a power series with non-negative coefficients $f_i$, we have $g(x) = f(x)$. In particular, for $f(x) = e^x$ we get $\|E_{\exp(a)}\|_\infty \leq \frac{1}{2} \|\alpha\|_\infty^2 \exp(\|a\|_\infty)$, which coincides with the bound given in [8].

In the case where $A = T(a) + E_a$, we may prove a similar bound relying on Theorem 2.4 as expressed by the following

**Theorem 3.3.** Assume that the function $f(x) = \sum_{i \in \mathbb{Z}^+} f_i x^i$ is analytic on $\mathbb{D}(\rho)$, that $a(z) \in W_1$, and is such that $\|a\|_\infty < \rho$. Let $A = T(a) + E_a$ and $f(A) = T(f(a)) + E_f(a)$. Then

$$\|E_f(a)\|_\infty \leq \frac{1}{\|E_a\|_\infty} \left( \alpha \left( \|a\|_\infty + \|E_a\|_\infty \right) - \|a\|_\infty \right) - \beta g''(\|a\|_\infty)$$

where $g(z) = \sum_{i=0}^{\infty} |f_i| z^i$ and $\alpha = \|\alpha\|_\infty^2 + \|E_a\|_\infty$, $\beta = \|\alpha\|_\infty^2$.

**Proof.** Recall from Theorem 3.3 that $f(a(z)) \in W_1$ and that $E_{f(a)} = \lim_k F_k$, $F_k = \sum_{i=0}^{k} f_i D_i$ for $A^i = T(a^i) + D_i$. From Theorem 2.4 we have the bound

$$\|D_i\|_\infty \leq \frac{1}{\|E_a\|_\infty} \left( \alpha \left( \|a\|_\infty^i + \|E_a\|_\infty i \right) - \|a\|_\infty^{i-1} \right) - \beta i \|a\|_\infty^{-1}$$

with $\alpha = \|\alpha\|_\infty^2 + \|E_a\|_\infty$, $\beta = \|\alpha\|_\infty^2$ so that for the matrix $E_f(a) = \sum_{i=0}^{\infty} f_i D_i$ we get the bound $\|E_f(a)\|_\infty \leq \sum_{i=0}^{\infty} |f_i| \cdot \|D_i\|_\infty$ which leads to

$$\|E_f(a)\|_\infty \leq \frac{1}{\|E_a\|_\infty} \left( \alpha \left( \|a\|_\infty + \|E_a\|_\infty \right) - \|a\|_\infty \right) - \beta g''(\|a\|_\infty) \right).$$

This completes the proof. $\square$
Observe that, taking the limit for \(\|E_\alpha\|_x \to 0\) in the bound given in the above theorem yields the bound of Theorem \(3.2\).

Next, we consider the case where the function \(f(x)\) is assigned as a Laurent series in the form \(f(x) = \sum_{i \in \mathbb{Z}} f_i x^i\) analytic over the open annulus \(\mathbb{A}(r_f, R_f)\) for \(r_f < R_f\). We recall from [15, Theorem 4.4c] the following decay property of the coefficients \(f_i\):

\[
\forall \epsilon > 0, \exists \gamma > 0 : |f_i| \leq \gamma (R_f - \epsilon)^{-i}, \quad |f_{-i}| \leq \gamma (r_f + \epsilon)^{i}, \quad i > 0. \tag{3.3}
\]

Concerning the existence of \(f(A)\) for \(A \in \mathbb{A}^{\mathbb{C}}\) we have the following

**Theorem 3.4.** Let \(f(x) = \sum_{i \in \mathbb{Z}} a_i x^i\) be an analytic function in the open annulus \(\mathbb{A}(r_f, R_f)\). Let \(a(z) \in W_1\) and consider a matrix \(A = T(a) + E_\alpha \in \mathbb{A}^{\mathbb{C}}\). If \(a(T) \subset \mathbb{A}(r_f, R_f), \|A^{-1}\|_{\mathbb{C}} < r_f^{-1}\) and \(\|A\|_{\mathbb{C}} < R_f\) then

\[
f(A) := \sum_{i \in \mathbb{Z}} a_i A^i = T(f(a)) + E_f(a) \in \mathbb{A}^{\mathbb{C}}.
\]

*Moreover if \(A \in \mathbb{A}^{\mathbb{C}}\) then \(f(A) \in \mathbb{A}^{\mathbb{C}}\).*

**Proof.** The proof follows the same line as the one of Theorem \(3.3\). We consider \(\varphi_k(x) = \sum_{i = -k}^k a_i x^i\) and show that \(\varphi_k(A)\) is a Cauchy sequence in \(\mathbb{C}^{\mathbb{C}}\). Since \(\|A^{-1}\|_{\mathbb{C}} < r_f^{-1}\) and \(\|A\|_{\mathbb{C}} < R_f\), there exists \(0 < \delta < R_f\) such that \(\|A^{-1}\|_{\mathbb{C}} \leq (r_f + \delta)^{-1}\) and \(\|A\|_{\mathbb{C}} \leq R_f - \delta\). Thus, applying the inequality (3.3) with \(\epsilon = \delta/2\), for \(h > k > 0\) we get

\[
\|\varphi_k(A) - \varphi_h(A)\|_{\mathbb{C}} \leq \sum_{i = k+1}^h (|f_i| \cdot \|A\|^i_{\mathbb{C}} + |f_{-i}| \cdot \|A^{-1}\|^i_{\mathbb{C}}) \leq \gamma \sum_{i = k+1}^h \left( \left( \frac{R_f - \delta}{r_f + \delta/2} \right)^{i} + \left( \frac{r_f + \delta/2}{r_f + \delta} \right)^i \right).
\]

The latter quantity converges to 0 for \(k \to \infty\) so that the sequence \(\varphi_k(A)\) is Cauchy in \(\mathbb{C}^{\mathbb{C}}\) and thus there exists \(F \in \mathbb{C}^{\mathbb{C}}\) such that \(\lim_{k \to \infty} \|\varphi_k(A) - F\|_{\mathbb{C}} = 0\). Therefore the matrix \(F\) has the form \(F = T(g) + E_\varphi\) for some function \(g\) in the Wiener class and for \(E_\varphi \in F\). By using the same argument as in the proof of Theorem \(3.1\) we obtain that \(g(z) = f(a(z))\).

Now, consider the case \(a(z)\) analytic. Since \(a(T) \subset \mathbb{A}(r_f, R_f)\), then there exists an open annulus \(\mathbb{A}(r, R)\), which includes the unit circle, such that \(a(\mathbb{A}(r, R)) \subset \mathbb{A}(r_f, R_f)\) so that \(f(a(z))\) is well defined and analytic for \(z \in \mathbb{A}(r, R)\). Thus \(g(z) = f(a(z))\) is analytic for \(z \in \mathbb{A}(r, R)\). Therefore we may conclude that \(F \in \mathbb{A}^{\mathbb{C}}\).

Observe that the two technical hypotheses

\[
\|A^{-1}\|_{\mathbb{C}} < r_f^{-1}, \quad \|A\|_{\mathbb{C}} < R_f,
\]

given in Theorem \(3.4\) are not needed if \(f(x)\) is a Laurent polynomial, i.e., a function of the form \(f(x) = \sum_{i = -n_1}^{n_2} f_i x^i\). If the function is entire on \(\mathbb{C}\), we need no additional assumption. For example we can claim that the exponential function of a \(\mathbb{C}^{\mathbb{C}}\)-matrix is again a \(\mathbb{C}^{\mathbb{C}}\)-matrix.

### 3.1. Computational aspects.

Observe that, if \(f(x) = \sum_{i = 0}^{\infty} f_i x^i\) and \(A = T(a)\), the combination of the two expressions \(\varphi_k(A) = \sum_{i = 0}^k f_i A^i\) and \(A^i = T(a^i) + E_i\), enables one to compute the quantity \(\varphi_k(A)\) at a low computational effort. In fact,
decomposing $\varphi_k(A)$ as $\varphi_k(A) = T(\varphi_k(a)) + F_k$, from $\varphi_{k+1}(A) = \varphi_k(A) + f_{k+1}A^{k+1}$ we deduce the equation

$$F_{k+1} = F_k + f_{k+1}E_k$$

for updating the correction part $F_{k+1}$ in $\varphi_{k+1}(A)$. The above equation is easily implementable, moreover, representing $F_k$ in the form $F_k = Y_kW_k^T$, where $Y_k$ and $W_k$ are matrices with infinitely many rows and a finite number of columns, and providing the same representation for $E_k$ as $E_k = U_kV_k^T$, we may use the updating equation

$$Y_{k+1} = [Y_k | f_{k+1}U_k], \quad W_{k+1} = [W_k | V_k]. \quad (3.4)$$

Moreover, in order to keep low the number of columns in the matrices $Y_{k+1}$ and $W_{k+1}$, one can apply a compression procedure based on the rank-revealing QR factorization and on SVD, to the two matrices in the right hand sides of (3.4). This strategy has been successfully used in [5] in the case of the exponential function.

Updating the Toeplitz part in $\varphi_k(A)$, i.e., computing the coefficients of $\varphi_{k+1}(a(z))$ given those of $\varphi_k(a(z))$, can be performed by means of the evaluation/interpolation technique using as knots the roots of the unity of sufficiently large order. In fact, in this case we may rely on FFT to carry out the computation at a low cost.

A similar computational strategy can be used if $f(x)$ is assigned as a Laurent series in the form $\sum_{i \in \mathbb{Z}} f_i x^i$ so that $f(A)$ takes the form $f(A) = f_0I + \sum_{i=1}^{\infty} (f_i A^i + f_{-i} A^{-i})$. Thus, once the matrix $A^{-1}$ has been written in the form $A^{-1} = T(a^{-1}) + E_{a^{-1}}$, one can apply the above technique. Similar equations can be given in the case the Toeplitz matrix is finite and has a sufficiently large size.

As an example to show the effectiveness of our approach, we performed two numerical experiments. In the first one, we applied the above machinery to compute the exponential of the semi-infinite Toeplitz matrix $T(a)$ associated with the symbol $a(z) = \sum_{i=1}^{10} z^i$ for $k = 1, 2, \ldots, 10$ corresponding to a Toeplitz matrix in Hessenberg form. In table [3.1] we report, besides the CPU time in seconds, the values of the numerical bandwidth of the exponential function, the dimension of the non-negligible part of the correction $E_{\text{exp}}$ and its rank.

We point out that the approximation of $\exp(T(a))$ represented in the CQT form is quite good and that the CPU time needed for this computation is particularly low. We observe also that the rank of the correction has a moderate growth with respect to the band of $T(a)$.

In the second experiment, we consider matrices of finite size extending the CQT-arithmetic as pointed out in Section 2.2.1. More precisely, we applied the power series definition for computing $\exp(A)$, where $A = H^10$ and $H$ is the $m \times m$ matrix $\text{tridiag}(1, 2, 1)/(2 + 2 \cos(\pi/m^2))$. In the numerical test we have chosen increasing values of $m$ as integer powers of 10. Observe that, the matrix $A$ is diagonalizable by means of the sine transform. Therefore, for all the matrices in the algebra generated by $A$ and for any function $f$, it is possible to retrieve a particular column of $f(A)$ with linear cost. In order to validate the results, we report —as residual error— the euclidean norm of the difference between the first column of the outcome and the first column of $\exp(A)$ computed by means of the sine transform. Table [3.2] shows the execution time in seconds, the residual errors, the Toeplitz bandwidth and the features of the correction. Note that, the features of only one correction are reported because, due to the symmetry of $A$, the upper left and lower right corner corrections are equal.
| $k$ | time  | band | rows | columns | rank |
|-----|-------|------|------|---------|------|
| 1   | 2.58 \times 10^{-2} | 37    | 18   | 18      | 7    |
| 2   | 2.58 \times 10^{-2} | 57    | 37   | 37      | 8    |
| 3   | 2.5 \times 10^{-2}  | 86    | 32   | 65      | 9    |
| 4   | 2.61 \times 10^{-2} | 113   | 39   | 91      | 9    |
| 5   | 3.66 \times 10^{-2} | 142   | 46   | 121     | 9    |
| 6   | 3.18 \times 10^{-2} | 174   | 31   | 152     | 9    |
| 7   | 3.21 \times 10^{-2} | 236   | 45   | 210     | 12   |
| 8   | 3.47 \times 10^{-2} | 275   | 45   | 252     | 12   |
| 9   | 3.84 \times 10^{-2} | 316   | 45   | 288     | 12   |
| 10  | 4.15 \times 10^{-2} | 359   | 45   | 429     | 11   |

Table 3.1

Computation of $\exp(T(a))$ where $a(z) = \sum_{i=1}^{k} z^i$.

| Size | time   | error  | band | rows | columns | rank |
|------|--------|--------|------|------|---------|------|
| 100  | 5.58 \times 10^{-2} | 8.51 \times 10^{-16} | 87   | 81   | 49      | 15   |
| 1,000| 5.45 \times 10^{-2} | 2.08 \times 10^{-15} | 87   | 65   | 45      | 15   |
| 10,000| 6.54 \times 10^{-2} | 8.04 \times 10^{-16} | 87   | 65   | 45      | 15   |
| 1 \times 10^5 | 8.22 \times 10^{-2} | 1.87 \times 10^{-15} | 87   | 73   | 65      | 15   |
| 1 \times 10^6 | 7.94 \times 10^{-2} | 1.45 \times 10^{-15} | 87   | 46   | 65      | 15   |
| 1 \times 10^7 | 8.18 \times 10^{-2} | 1.04 \times 10^{-15} | 87   | 46   | 67      | 15   |

Table 3.2

Computation of $\exp(A)$, with $A = H^{10}$ where $H = \text{trid}(1, 2, 1)/(2 + 2 \cos(\frac{\pi}{m+1}))$ is an $m \times m$ tridiagonal matrix.

4. Function of a CQT matrix: the Dunford-Cauchy integral. The definition of $f(A)$ based on the contour integral can be easily extended to infinite matrices which represent bounded operators [22][13].

**Definition 4.1.** Let $A$ be a semi-infinite matrix which represents a bounded linear operator on $l^2(\mathbb{Z}^+)$ and let $\Lambda = \{z \in \mathbb{C} : zI - A \text{ is not invertible} \}$ be its spectrum. Given an analytic function $f(x)$ defined on a compact domain $\Omega \supseteq \Lambda$ having boundary $\partial \Omega$, $f(A)$ is defined as

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Omega} f(z) \mathcal{R}(z) dz$$

where $\mathcal{R}(z) := (zI - A)^{-1}$ is the resolvent.

The integral formula (4.1) allows us to approximate $f(A)$ through a numerical integration scheme. That is, given a differentiable arc length parametrization $\gamma : [a, b] \to \mathbb{C}$ of $\partial \Omega$ we can write

$$\frac{1}{2\pi i} \int_{\partial \Omega} f(z) \mathcal{R}(z) dz = \int_{a}^{b} g(x) dx$$

where $g(x) := \frac{1}{2\pi i} \gamma'(x)f(\gamma(x))\mathcal{R}(\gamma(x))$ is a matrix valued function. The above integral can be approximated by means of a quadrature formula with nodes $x_k$ and weights...
Therefore, it is sufficient to show that this sequence is Cauchy with respect to the approximation scheme for (4.1) we consider the sequence of rational functions in CQT-matrices whose limit, if it exists, has a Toeplitz part with symbol $f$ nodes). That is, we consider the double indexed family \( \{ x_k^{(n)}, w_k^{(n)} \} \) such that:

- \( n \in \mathbb{Z}^+ \) and \( k = 1, \ldots, 2^n + 1 \),
- \( a = x_1^{(n)} < x_2^{(n)} < \cdots < x_{2^n+1}^{(n)} = b \) are equally spaced points in \( [a, b] \) \( \forall n \in \mathbb{Z}^+ \),
- \( w_1^{(n)} = w_{2^n+1}^{(n)} = \frac{b-a}{2^n} \) and \( w_k^{(n)} = \frac{b-a}{2^n}, k = 2, \ldots, 2^n \).

In particular, observe that the nodes at a certain step \( n \) correspond to those with odd indices at step \( n + 1 \).

Using a trapezoidal approximation of the integral (4.1) we can prove that the function of a CQT-matrix is again a CQT-matrix.

**Theorem 4.2.** Let \( A = T(a) + E_a \) be a CQT-matrix with spectrum \( \Lambda \) and symbol \( a(z) \in W_1 \). Let \( f(z) \) be an analytic function defined on the domain \( \Omega \subset \mathbb{C} \) which encloses \( \Lambda \) such that \( a(T) \subset \Omega \). Assume that \( \partial \Omega \) admits a differentiable arc length parametrization \( \gamma : [a, b] \to \partial \Omega \). Then \( f(A) \) is a CQT-matrix.

Moreover, if \( A \in AQ\mathcal{T} \) then \( f(A) \in AQ\mathcal{T} \).

**Proof.** Given the family \( \{ x_k, w_k \} \) of nodes and weights of the trapezoidal approximation scheme for (4.1) we consider the sequence of rational functions in \( A \):

\[
\{ r_n(A) \}_{n \in \mathbb{N}^+} = \left\{ \sum_{k=1}^{2^n+1} w_k^{(n)} g(x_k^{(n)}) \right\}_{n \in \mathbb{N}^+} = \left\{ \frac{b-a}{2^n} \sum_{k=1}^{2^n} g(x_k^{(n)}) \right\}_{n \in \mathbb{N}^+}
\]

where \( g \) is defined according to (4.2) and the latter equality follows from the fact that \( \partial \Omega \) is a closed simple curve, thus \( \gamma(x_1^{(n)}) = \gamma(x_{2^n+1}^{(n)}) \). This sequence is formed by CQT-matrices whose limit, if it exists, has a Toeplitz part with symbol \( f(a(z)) \). Therefore, it is sufficient to show that this sequence is Cauchy with respect to the norm \( \| \cdot \|_{CQT} \).

Consider the difference

\[
r_{n+1}(A) - r_n(A) = \frac{b-a}{2^{n+1}} \sum_{k=1}^{2^n} \left( g(x_{2k}^{(n+1)}) - g(x_{2^k-1}^{(n+1)}) \right)
\]

and observe that (for notational simplicity we omit the superscript \( n+1 \) in the nodes)

\[
g(x_{2k}) - g(x_{2k-1}) = l(x_{2k}) \mathcal{R}(\gamma(x_{2k})) - l(x_{2k-1}) \mathcal{R}(\gamma(x_{2k-1}))
\]

where \( l : [a, b] \to \mathbb{C}, l(x) = \frac{1}{2\pi i} \gamma'(x)f(\gamma(x)) \). Assuming that \( \gamma(x) \) has continuous second derivative, then \( l(x) \) is a Lipschitz function. Indicating with \( L \) the Lipschitz...
constant of $l$ and defining $M := \max_{B} ||\mathcal{R}(z)||_{\mathcal{CQT}}$, $G := \max_{[a,b]} |l(x)|$ we get

\[
\|g(x_{2k})-g(x_{2k-1})\|_{\mathcal{CQT}} \leq \|l(x_{2k})-l(x_{2k-1})\| \cdot ||\mathcal{R}(\gamma(x_{2k}))||_{\mathcal{CQT}} \\
+ |l(x_{2k-1})| \cdot ||\mathcal{R}(\gamma(x_{2k})) - \mathcal{R}(\gamma(x_{2k-1}))||_{\mathcal{CQT}} \\
\leq L|x_{2k} - x_{2k-1}| \cdot ||\mathcal{R}(\gamma(x_{2k}))||_{\mathcal{CQT}} \\
+ |l(x_{2k-1})| \cdot |\gamma(x_{2k}) - \gamma(x_{2k-1})| \cdot ||\mathcal{R}(\gamma(x_{2k}))||_{\mathcal{CQT}} ||\mathcal{R}(\gamma(x_{2k-1}))||_{\mathcal{CQT}} \\
\leq \frac{LM(b-a)}{2n+1} + \frac{GM^2(b-a)}{2n+1}
\]

where we used $|\gamma(x_{2k}) - \gamma(x_{2k-1})| \leq |x_{2k} - x_{2k-1}|$ and the identity $\mathcal{R}(z_1) - \mathcal{R}(z_2) = (z_2 - z_1)\mathcal{R}(z_1)\mathcal{R}(z_2)$.

In particular, we can write

\[
\|r_{n+1}(A) - r_n(A)\|_{\mathcal{CQT}} \leq \frac{b-a}{2n+1} \sum_{k=1}^{2^n} \frac{(LM + GM^2)(b-a)}{2n+1} = c \cdot 2^{-(n+2)}
\]

where $c := (b-a)^2(LM + GM^2)$ is independent of $n$. Therefore, given $n_2 > n_1$, we have

\[
\|r_{n_2}(A) - r_{n_1}(A)\|_{\mathcal{CQT}} \leq \sum_{j=n_1}^{n_2-1} \|r_{j+1}(A) - r_j(A)\|_{\mathcal{CQT}} \\
\leq c \sum_{j=n_1}^{n_2-1} 2^{-(j+2)} \leq c \cdot 2^{-(n_1+1)},
\]

which proves that $\{r_n(A)\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in the Banach algebra of $\mathcal{CQT}$-matrices. By relying on the same arguments used in the proof of Theorem 3.1 we deduce that $g(z) = f(a(z))$. So if $a(z)$ is analytic in a certain annulus $\mathcal{A}(r_a, R_a)$ containing $\mathbb{T}$ then there exists $\mathcal{A}(r, R) \subset \mathcal{A}(r_a, R_a)$ such that $a(\mathcal{A}(r, R)) \subset \mathcal{A}(r_f, R_f)$. Thus the composed function $g(z) = f(a(z))$ is analytic in $\mathcal{A}(r, R)$. This completes the proof. \[\square\]

### 4.1. Computational aspects.
Numerical integration based on the trapezoidal rule at the roots of unity can be easily implemented to approximate a matrix function assigned in terms of a Dunford-Cauchy integral. In fact all the operations involved in the computation reduce to performing matrix additions, multiplication of a matrix by a scalar and matrix inversion. The latter operation is the one with the highest computational cost.

We applied the contour integral definition for computing $\sqrt{I + H^{10}}$ and $\log(I + H^{10})$ where $H$ is the $m \times m$ matrix $H = \text{trid}(1, 2, 1)/(2 + 2 \cos(\frac{\pi}{m})))$ considered in Section 3.1. We used the trapezoidal rule with a doubling strategy for the nodes for integrating on a disc which contains the spectrum of $I + H^{10}$. Since $H$ is rescaled to have spectrum in $[0, 1]$, we selected as center of the disc $1.5$ and radius $1$. Table 4.1 and 4.2 report the execution time, the residuals, the Toeplitz bandwidth and the features of the correction as the size of the argument increases exponentially. Once again, we reported only the features of one correction because, due to the symmetry of $A$, the upper left and lower right corner corrections are equal.

### 5. Conclusions.
We have extended the concept of matrix function to CQT matrices, i.e., infinite matrices of the form $A = T(a) + E$, by showing that, under
Computation of $f(A)$ for $A = I + H^{10}$ where $H = \text{trid}(1, 2, 1)/(2 + 2 \cos(\frac{\pi}{m+1}))$ is an $m \times m$ tridiagonal matrix.

Table 4.1

| Size | time | error   | band | rows | columns | rank |
|------|------|---------|------|------|---------|------|
| 100  | 1.9  | 5.57 $\cdot 10^{-14}$ | 87   | 89   | 90      | 15   |
| 1,000| 1.88 | 5.5 $\cdot 10^{-14}$  | 159  | 90   | 90      | 15   |
| 10,000|1.53|5.57 $\cdot 10^{-14}$|159|89|90|15|
| $1 \cdot 10^5$ | 1.62 | 5.56 $\cdot 10^{-14}$ | 159 | 89 | 89 | 15 |
| $1 \cdot 10^6$ | 1.99 | 5.54 $\cdot 10^{-14}$ | 159 | 90 | 89 | 15 |
| $1 \cdot 10^7$ | 1.65 | 5.56 $\cdot 10^{-14}$ | 159 | 89 | 90 | 15 |

Computation of $\log(A)$ for $A = I + H^{10}$ where $H = \text{trid}(1, 2, 1)/(2 + 2 \cos(\frac{\pi}{m+1}))$ is an $m \times m$ tridiagonal matrix.

Table 4.2

| Size | time | error   | band | rows | columns | rank |
|------|------|---------|------|------|---------|------|
| 100  | 1.9  | 5.57 $\cdot 10^{-14}$ | 87   | 89   | 90      | 15   |
| 1,000| 1.88 | 5.5 $\cdot 10^{-14}$  | 159  | 90   | 90      | 15   |
| 10,000|1.53|5.57 $\cdot 10^{-14}$|159|89|90|15|
| $1 \cdot 10^5$ | 1.62 | 5.56 $\cdot 10^{-14}$ | 159 | 89 | 89 | 15 |
| $1 \cdot 10^6$ | 1.99 | 5.54 $\cdot 10^{-14}$ | 159 | 90 | 89 | 15 |
| $1 \cdot 10^7$ | 1.65 | 5.56 $\cdot 10^{-14}$ | 159 | 89 | 90 | 15 |

suitable mild assumptions, for a CQT matrix $A$ and for a function $f(x)$ expressed either in terms of a power (Laurent) series, or in terms of the Dunford-Cauchy integral, the matrix function $f(A)$ is still a CQT matrix. We have outlined algorithms for the computation of $f(A)$. This approach has been adapted to the case of $f(A_m)$ where $A_m$ is the $m \times m$ leading principal submatrix of $A$.

Among the open issues that will be part of our research interests, it would be interesting to analyze the behavior of the singular values $a_i^{(k)}$ of the $m \times m$ truncation of the correction $E_k$ such that $(T(a) + E)^k = T(a^k) + E_k$, and relate the decay of these values for $i = 1, 2, \ldots, m$ and for $k = 1, 2, \ldots$, to the qualitative properties of the function $a(z)$. In fact, from the numerical experiments that we have performed with several functions $a(z)$, it turns out that the numerical rank of $E_k$ remains bounded by a constant independent of $k$.

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