ON THE CONE OF EFFECTIVE 2-CYCLES ON $\overline{M}_{0,7}$

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ABSTRACT. Fulton’s question about effective $k$-cycles on $\overline{M}_{0,n}$ for $1 < k < n - 4$ can be answered negatively by appropriately lifting to $\overline{M}_{0,n}$ the Keel-Vermeire divisors on $\overline{M}_{0,k+1}$. In this paper we focus on the case of 2-cycles on $\overline{M}_{0,7}$, and we prove that the 2-dimensional boundary strata together with the lifts of the Keel-Vermeire divisors are not enough to generate the cone of effective 2-cycles. We do this by providing examples of effective 2-cycles on $\overline{M}_{0,7}$ that cannot be written as an effective combination of the aforementioned 2-cycles. These examples are inspired by a blow up construction of Castravet and Tevelev.

INTRODUCTION

An open problem in the birational geometry of $\overline{M}_{0,n}$, the moduli space of stable $n$-pointed rational curves, is the F-conjecture. This conjecture claims that the cone $\text{Eff}_1(\overline{M}_{0,n})$ of effective curves, is generated by the numerical equivalence classes of 1-dimensional boundary strata, which are obtained by intersecting boundary divisors. This is known to be true if $n \leq 7$ (see [KM]).

A similar question (which is known as Fulton’s question) was stated in [KM] also for the cone $\text{Eff}_k(\overline{M}_{0,n})$ of effective $k$-cycles with $1 < k < n - 3$:

Is the cone $\text{Eff}_k(\overline{M}_{0,n})$ generated by the $k$-dimensional boundary strata?

Denote by $V_k(\overline{M}_{0,n})$ the cone generated by the numerical equivalence classes of the $k$-dimensional boundary strata. Then the question is whether or not $\text{Eff}_k(\overline{M}_{0,n})$ is equal to $V_k(\overline{M}_{0,n})$. As Keel and Vermeire pointed out in the case of divisors (see [GKM], [V]), the cone $V_{n-4}(\overline{M}_{0,n})$ is strictly contained in $\text{Eff}_{n-4}(\overline{M}_{0,n})$, and one can see that $V_k(\overline{M}_{0,n}) \subseteq \text{Eff}_k(\overline{M}_{0,n})$ for all $1 < k < n - 4$ by appropriately lifting to $\overline{M}_{0,n}$ the Keel-Vermeire divisors on $\overline{M}_{0,k+1}$ (see Section 3, in particular Corollary 3.3). So the problem is to understand what lies in $\text{Eff}_k(\overline{M}_{0,n}) \setminus V_k(\overline{M}_{0,n})$ (see [HT], [C], [CT13], [DGJ], [O] for the codimension one case). Recently, a lot of work has been done in order to understand the cones of effective and pseudoeffective cycles of higher codimension on projective varieties (see [DELV], [F], [L], [FL] and [CC]).

We work over an algebraically closed field $\mathbb{K}$ of any characteristic. The main result of this paper (Theorem 5.8) can be synthesized in the following statement.

Theorem. The 2-dimensional boundary strata on $\overline{M}_{0,7}$ together with the lifts of the Keel-Vermeire divisors on $\overline{M}_{0,6}$ are not enough to generate the cone $\text{Eff}_2(\overline{M}_{0,7})$.

The lifts of the Keel-Vermeire divisors are defined as the pushforwards with respect to the natural inclusion $D_{ab} \hookrightarrow \overline{M}_{0,7}$ of the Keel-Vermeire divisors on the boundary divisor $D_{ab}$ (which is isomorphic to $\overline{M}_{0,6}$) for any $\{a, b\} \subset \{1, \ldots, 7\}$. In this way we produce 315 extremal rays of $\text{Eff}_2(\overline{M}_{0,7})$ which lie outside of $V_2(\overline{M}_{0,7})$ (see Proposition 4.4). Denote with $V_{2\text{KV}}(\overline{M}_{0,7})$ the cone generated by $V_2(\overline{M}_{0,7})$ and by these lifts.
Examples of effective 2-cycles on $\overline{M}_{0,7}$ whose numerical equivalence classes do not lie in $V_2^{KV}(\overline{M}_{0,7})$ are produced using the following blow up construction of Castravet and Tevelev (see [CT12, Theorem 3.1]): take seven labeled points in $\mathbb{P}^2$ which do not lie on a line or a (possibly reducible) conic. Then the blow up of $\mathbb{P}^2$ at these points can be embedded in $\overline{M}_{0,7}$ as an effective 2-cycle. Using this construction and considering particular arrangements of seven labeled points in $\mathbb{P}^2$, we define what we call special hypertree surfaces on $\overline{M}_{0,7}$ (see Definition 5.6), which are related to Castravet and Tevelev hypertrees (see [CT13]). In Theorem 5.8 we prove that the numerical equivalence class of a special hypertree surface does not lie in the cone $V_2^{KV}(\overline{M}_{0,7})$. This implies that $V_2^{KV}(\overline{M}_{0,7}) \not\subseteq \text{Eff}_2(\overline{M}_{0,7})$, which is our main result. An example of 7-points arrangement in $\mathbb{P}^2$ which gives rise to a special hypertree surface on $\overline{M}_{0,7}$ is the following:

![Diagram of 7-points arrangement in $\mathbb{P}^2$ which gives a special hypertree surface on $\overline{M}_{0,7}$](image)

Figure 1. 7-points arrangement in $\mathbb{P}^2$ which gives a special hypertree surface on $\overline{M}_{0,7}$.

All the other special hypertree surfaces are obtained by permuting the labels of the points arrangement in Figure 1. In Section 5.3 we show that there are 210 (resp. 30) distinct numerical equivalence classes of special hypertree surfaces on $\overline{M}_{0,7}$ if the characteristic of the base field is different from 2 (resp. equal to 2).

Summing up, if we denote with $V_2^{KV+CT}(\overline{M}_{0,7})$ the cone generated by $V_2^{KV}(\overline{M}_{0,7})$ and by the numerical equivalence classes of the embedded blow ups of $\mathbb{P}^2$ at seven points, we have the following chain of containments

$$V_2(\overline{M}_{0,7}) \subseteq V_2^{KV}(\overline{M}_{0,7}) \subseteq V_2^{KV+CT}(\overline{M}_{0,7}) \subseteq \text{Eff}_2(\overline{M}_{0,7}).$$

The second main result of this paper is an explicit description of the intersection theory of the 2-dimensional boundary strata on $\overline{M}_{0,7}$. In Proposition 2.4 and Proposition 2.5 we give formulas that compute the intersection number of two 2-dimensional boundary strata on $\overline{M}_{0,7}$. Then we study the numerical equivalence classes of these 2-cycles (see Proposition 2.6), and this, together with some recent results of Chen and Coskun in [CC], allows us to give a complete description of the cone $V_2(\overline{M}_{0,7})$ (see Corollary 2.8). We also fully describe the bilinear form $N_2(\overline{M}_{0,7}) \times N_2(\overline{M}_{0,7}) \rightarrow \mathbb{R}$ given by the intersection product (see Propositions 2.9 and 2.10).
In Section 1 we recall some basic facts and notations about $\overline{M}_{0,n}$ that are used in the paper. Section 2 contains the formulas for the intersection of two 2-dimensional boundary strata on $\overline{M}_{0,7}$, and the complete study of the cone $V_2(\overline{M}_{0,7})$. In Section 3 there is a detailed description of the lifting technique, which is immediately applied in Section 4 to describe the lifts to $\overline{M}_{0,7}$ of the Keel-Vermeire divisors on $\overline{M}_{0,6}$. Section 5 is where we discuss the embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,7}$ and where we prove our main theorem. In Section 6 we generalize the construction of the two cones $V_{2}^{KV}(\overline{M}_{0,7})$ and $V_{2}^{KV+CT}(\overline{M}_{0,7})$ to any $\overline{M}_{0,n}$ for $n > 7$. We also state some questions that will be the object of further investigation.

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1. Preliminaries: boundary strata on $\overline{M}_{0,n}$

In this section we review some of the main definitions and facts about the boundary strata on $\overline{M}_{0,n}$. For a more detailed discussion, see for example [KM]. Equivalence between $k$-cycles on $\overline{M}_{0,n}$ refers to numerical equivalence, which is the same as rational equivalence and algebraic equivalence by [Ke].

**Definition 1.1.** The irreducible components of the locus of points on $\overline{M}_{0,n}$ parametrizing stable $n$-pointed rational curves with at least $n-3-k$ nodes, have dimension $k$ and are called $k$-boundary strata. Codimension one (resp. 1-dimensional) boundary strata are also called boundary divisors (resp. $F$-curves).

**Definition 1.2.** Given $n \geq 3$ and $0 \leq k \leq n-3$, define $V_{k}(\overline{M}_{0,n})$ to be the cone generated by the equivalence classes of the $k$-boundary strata on $\overline{M}_{0,n}$ ($V$ stands for “vital cycles”, as they were called in [KM]).

**Notation.** If $n$ is a positive integer, then $[n]$ denotes the set $\{1, \ldots, n\}$. 
Combinatorial description of boundary divisors. There is a bijection between boundary divisors and partitions \( I \sqcup I^c = [n] \), with \( 2 \leq |I| \leq n-2 \). \( D_I = D_{I^c} \) denotes the boundary divisor corresponding to the partition \( I \sqcup I^c = [n] \). \( \delta_I = \delta_{I^c} \) denotes the equivalence class of \( D_I \). For simplicity, the equivalence class of a boundary divisor will be called just boundary divisor.

Combinatorial description of equivalence classes of F-curves. There is a bijection between equivalence classes of F-curves and partitions of \([n] := \{1, \ldots, n\} \) into four nonempty subsets. Given a partition \( I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4 = [n] \), we denote by \( F_{I_1,I_2,I_3,I_4} \) the equivalence class of the F-curves corresponding to that partition.

Every boundary stratum on \( \overline{M}_{0,n} \) can be realized as the complete intersection of all the boundary divisors containing it as follows. Let \( B \) be a boundary stratum and let \( C(B) \) be the stable \( n \)-pointed rational curve corresponding to the generic point of \( B \) (\( C(B) \) has as many nodes as the codimension of \( B \)). Given \( p \in \text{Sing}(C(B)) \), let \( T_p \) be the set of markings that are over one of the two connected components of the normalization of \( C(B) \) at \( p \). Then we have that \( B = \bigcap_{p \in \text{Sing}(C(B))} D_{T_p} \).

Moreover, since the boundary of \( \overline{M}_{0,n} \) has normal crossings, we have that the equivalence class of \( B \) is the product of all the \( \delta_{T_p} \) as \( p \) varies among the nodes of \( C(B) \).

The last thing we want to recall is [Ke, Fact 4]: given two boundary divisors \( D_I, D_J \) on \( \overline{M}_{0,n} \), then \( D_I \cap D_J \neq \emptyset \iff I \ast J \ast \), which by definition means \( I \subseteq J \) or \( I \subseteq J^c \) or \( I \supset J \) or \( I \supset J^c \).

2. The cone of 2-boundary strata on \( \overline{M}_{0,7} \)

The main object of our study is \( \text{Eff}_2(\overline{M}_{0,7}) \), which is a subcone of the real vector space \( N_2(\overline{M}_{0,7}) \) (in Section 2.5 we show that \( \dim_{\mathbb{R}} N_2(\overline{M}_{0,7}) = 127 \)). We start by analyzing the subcone \( V_2(\overline{M}_{0,7}) \subseteq \text{Eff}_2(\overline{M}_{0,7}) \). The first thing we want to do is to give a combinatorial description of the 2-boundary strata on \( \overline{M}_{0,7} \). After this, we study their intersections and their equivalence classes.

2.1. Combinatorial description of the 2-boundary strata on \( \overline{M}_{0,7} \). According to Definition 1.1 a 2-boundary stratum on \( \overline{M}_{0,7} \) is the closure of the locus of points parametrizing stable 7-pointed rational curves of the following shape:

![Figure 2. Stable 7-pointed rational curve parametrized by the generic point of a 2-boundary stratum.](image-url)
where $I \sqcup J \sqcup K$ is a given partition of $[7]$. Stability imposes that $2 \leq |I| \leq 4$, $1 \leq |J| \leq 3$ and $2 \leq |K| \leq 4$. Therefore:

There is a bijection between set-theoretically distinct 2-boundary strata, and partitions $I \sqcup J \sqcup K$ of $[7]$, with $2 \leq |I| \leq 4$, $1 \leq |J| \leq 3$ and $2 \leq |K| \leq 4$, modulo the equivalence relation $I \sqcup J \sqcup K \sim K \sqcup J \sqcup I$.

With $s_{I,J,K} \subset \overline{M}_{0,7}$ we denote the 2-boundary stratum corresponding to the partition $I \sqcup J \sqcup K$ of $[7]$. The equivalence class of $s_{I,J,K}$ is denoted by $\sigma_{I,J,K}$. Obviously, we have that $\sigma_{I,J,K} = \delta_I \cdot \delta_K$. An easy combinatorial count tells us that there are 490 set-theoretically distinct 2-boundary strata $s_{I,J,K}$. A similar description applies for codimension 2 boundary strata on $\overline{M}_{0,n}$ for $n \geq 8$. For general results about boundary strata of codimension 2 on $\overline{M}_{0,n}$, see [CC, Section 6].

2.2. Intersection of two distinct 2-boundary strata. Given $\sigma_{I,J,K}$ and $\sigma_{L,M,N}$, our goal is to compute the intersection $\sigma_{I,J,K} \cdot \sigma_{L,M,N} = \delta_I \cdot \delta_K \cdot \delta_L \cdot \delta_N$. This intersection is clearly zero, unless we require that the condition defined here below is satisfied.

**Definition 2.1.** Consider two 2-boundary strata $s_{I,J,K}$ and $s_{L,M,N}$. Assume that $I \ast \ast L$ and $I \ast \ast N$ and $K \ast \ast L$ and $K \ast \ast N$.

If this condition is satisfied, we write $s_{I,J,K} \ast \ast s_{L,M,N}$.

**Lemma 2.2.** Let $D_{I_1}, D_{I_2}$ and $D_{I_3}$ be three distinct boundary divisors on $\overline{M}_{0,7}$ such that $I_a \ast \ast I_b$ for all $\{a, b\} \subset \{1, 2, 3\}$. Then $D_{I_1} \cap D_{I_2} \cap D_{I_3}$ is an F-curve.

**Proof.** Assume without loss of generality that $I_1 \cap I_2 = \emptyset$. We know that $I_3 \ast \ast I_1$ and $I_3 \ast \ast I_2$, therefore

$$(I_3 \subset I_1 \text{ or } I_3 \subset I_1^c \text{ or } I_3 \supset I_1 \text{ or } I_3 \supseteq I_1^c) \text{ and } (I_3 \subset I_2 \text{ or } I_3 \subset I_2^c \text{ or } I_3 \supset I_2 \text{ or } I_3 \supseteq I_2^c).$$

Among these 16 cases, the only possible are

$(I_3 \subset I_1 \text{ and } I_3 \subset I_2^c) \text{ or } (I_3 \subset I_1^c \text{ and } I_3 \subset I_2)$ or $(I_3 \subset I_1^c \text{ and } I_3 \subset I_2^c)$ or $(I_3 \subset I_1 \text{ and } I_3 \subset I_2)$ or $(I_3 \supset I_1 \text{ and } I_3 \supset I_2)$ or $(I_3 \supset I_1 \text{ and } I_3 \supset I_2^c)$ or $(I_3 \supset I_1^c \text{ and } I_3 \subset I_2)$.

Up to changing $I_3$ with $I_3^c$, we just need to consider

$(I_3 \subset I_1 \text{ and } I_3 \subset I_2^c) \text{ or } (I_3 \subset I_1^c \text{ and } I_3 \subset I_2)$ or $(I_3 \subset I_1 \text{ and } I_3 \supset I_2^c)$ or $(I_3 \subset I_1^c \text{ and } I_3 \supset I_2)$.

Now, inspecting each one of these four cases, it is easy to see that $D_{I_1} \cap D_{I_2} \cap D_{I_3}$ is an F-curve. \qed

**Lemma 2.3.** Let $s_{I,J,K}$ and $s_{L,M,N}$ be two distinct 2-boundary strata on $\overline{M}_{0,7}$ satisfying the condition $s_{I,J,K} \ast \ast s_{L,M,N}$. Then we can write $\sigma_{I,J,K} \cdot \sigma_{L,M,N} = \delta_{I_1} \cdot \delta_{I_2} \cdot \delta_{I_3} \cdot \delta_{I_4}$ where, either the four boundary divisors $\delta_{I_1}, \delta_{I_2}, \delta_{I_3}$ and $\delta_{I_4}$ are pairwise distinct, or exactly two of them are equal. In the latter case, we assume that $I_3 = I_4$. In any case, we assume that $I_1 \cap I_2 = \emptyset$ and $|I_1| \leq |I_2|$. \vspace{2cm}

**Proof.** Write $\sigma_{I,J,K} \cdot \sigma_{L,M,N} = \delta_I \cdot \delta_K \cdot \delta_L \cdot \delta_N$. Obviously $\delta_I \neq \delta_K$ and $\delta_L \neq \delta_N$. If two boundary divisors among $\delta_I, \delta_K, \delta_L$ and $\delta_N$ are equal, assume without loss of generality that $\delta_K = \delta_L$. Then we must have that $\delta_N \neq \delta_I$, or we would have $s_{I,J,K} = s_{L,M,N}$. Also, $\delta_N \neq \delta_K = \delta_L$. This proves that there can be at most two boundary divisors among $\delta_I, \delta_K, \delta_L$ and $\delta_N$ that are equal. So, let us write $\delta_I \cdot \delta_K \cdot \delta_L \cdot \delta_N = \delta_A \cdot \delta_B \cdot \delta_{I_3} \cdot \delta_{I_4}$, where $\{I, K, L, N\} = \{A, B, I_3, I_4\}$ and $I_3 = I_4$ in...
Let us make some preliminary observations. We have that

Proof. Let us make some preliminary observations. We have that $s_{I,J,K} \cdot s_{I,L,M,N}$ be two distinct 2-boundary strata on $\overline{M}_{0,7}$ such that $s_{I,J,K} \ast s_{I,L,M,N}$ (otherwise, the intersection number $\sigma_{I,J,K} \cdot \sigma_{I,L,M,N}$ is trivially zero). Write $\sigma_{I,J,K} \cdot \sigma_{I,L,M,N} = \delta_{I_1} \cdot \delta_{I_2} \cdot \delta_{I_3} \cdot \delta_{I_4}$ as prescribed by Lemma 2.3 (recall that in this lemma we assumed, among other things, that $|I_1| \leq |I_2|$). Then

$$\sigma_{I,J,K} \cdot \sigma_{I,L,M,N} = \begin{cases} 
-1 & \text{if } \delta_{I_3} = \delta_{I_4}, \ |I_1| = 2 \text{ and } |I_2| \in \{2,4\} \\
1 & \text{if } \delta_{I_1}, \delta_{I_2}, \delta_{I_3} \text{ and } \delta_{I_4} \text{ are pairwise distinct} \\
0 & \text{otherwise.}
\end{cases}$$

Now, for $j = 3, 4$, $i^*\delta_{I_j} = [D_{I_1} \cap D_{I_2} \cap D_{I_j}]$, where $D_{I_1} \cap D_{I_2} \cap D_{I_j}$ is an F-curve by Lemma 2.2. So $i^*\delta_{I_3}$ and $i^*\delta_{I_4}$ are two equivalence classes of F-curves on the 2-boundary stratum $S$. There are two possibilities for $S$ up to isomorphism.

(i) If $|I_1| = 2$ and $|I_2| \in \{2,4\}$, then $S \cong \overline{M}_{0,5}$. By Kapranov’s blow up construction of $\overline{M}_{0,n}$ (see [Ka]), we know that $\overline{M}_{0,5}$ is isomorphic to the blow up of $\mathbb{P}^2$ at four points in general linear position. Moreover, the F-curves of $\overline{M}_{0,5}$ correspond to the exceptional divisors of the blow up, and the strict transforms of the lines spanned by the blown up points.

(ii) If $|I_2| = 3$ and $|I_1| \in \{2,3\}$, then $S \cong \overline{M}_{0,4} \times \overline{M}_{0,4}$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. An F-curve on $S$ corresponds to a line on $\mathbb{P}^1 \times \mathbb{P}^1$ in the form $\{p\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{p\}$ for some point $p \in \mathbb{P}^1$.

Observe that in case (i) (resp. case (ii)) the self-intersection of an F-curve is $-1$ (resp. 0), and in both cases two F-curves intersect at exactly one point if and only if their intersection number is 1.

Now, let us prove our intersection formula for $\sigma_{I,J,K} \cdot \sigma_{I,L,M,N}$.

- $|I_1| = 2$ and $|I_2| = 2$. Up to permuting the labels, we have that

$$S \cong \overline{M}_{0,\{1,2,x\}} \times \overline{M}_{0,\{x,3,4,5,y\}} \times \overline{M}_{0,\{y,6,7\}} \cong \overline{M}_{0,\{x,3,4,5,y\}},$$

where $x$ and $y$ are the nodes of the stable 7-pointed rational curve corresponding to the generic point of $S$. If $\delta_{I_3} = \delta_{I_4}$, then $(i^*\delta_{I_3}) \cdot (i^*\delta_{I_4})$ is equal to the self-intersection of an F-curve on $S \cong \overline{M}_{0,\{x,3,4,5,y\}}$, which gives $\sigma_{I,J,K} \cdot \sigma_{I,L,M,N} = -1$. So let us assume that $\delta_{I_1}, \delta_{I_2}, \delta_{I_3}$ and $\delta_{I_4}$ are pairwise distinct. Given $j = 3, 4$, since $I_1 \ast I_j$ and $I_2 \ast I_j$, then $i^*\delta_{I_j}$ is equal to one of the following boundary divisors on $\overline{M}_{0,\{x,3,4,5,y\}}$

$$\delta_{34}, \delta_{35}, \delta_{45} \text{ or } \delta_{345}.$$
Proposition 2.5. Let $\sigma_{I,J,K}$ be the equivalence class of a 2-boundary stratum with $|I| \leq |K|$. Then

$$\sigma_{I,J,K}^2 = \begin{cases} 
0 & \text{if } |I| = 2 \text{ and } |J| = 1 \\
2 & \text{if } |J| = 3 \\
1 & \text{otherwise.}
\end{cases}$$
2.4. Equivalence classes of 2-boundary strata. So far, we considered set theoretically distinct 2-boundary strata. However, we are interested in studying distinct equivalence classes of 2-boundary strata.

Proposition 2.6. Consider $\sigma_{I,J,K}$ and $\sigma_{L,M,N}$ with $|I| \leq |K|$, $|L| \leq |N|$ and $s_{I,J,K} \neq s_{L,M,N}$. Then $\sigma_{I,J,K} = \sigma_{L,M,N} \iff I \cup J = L \cup M$ and $|I \cup J| = 3$. 

Proof. Up to relabeling the markings, it is enough to prove that $\sigma^2_{12,3,4,5,6,7} = 0$, $\sigma^2_{123,4,5,6,7} = \sigma^2_{12,34,5,6,7} = 1$ and $\sigma^2_{12,34,5,6,7} = 2$.

- $\sigma^2_{12,3,4,5,6,7} = \delta_{12} \cdot \delta_{4567} \cdot \delta_{12} \cdot \delta_{4567}$. Let us use the boundary relation

$$
\sum_{\delta_1 \in S} \delta_1 = \sum_{\delta_2 \in S} \delta_2 \Rightarrow \delta_1 = \delta_1 + \delta_2 = \delta_1 + \delta_2$$

$$+ \delta_1 + \delta_2 - \delta_1 - \delta_2 = \delta_1 + \delta_2$$

But now, if $\delta_T$ is one of the boundary divisors that appear in the expression we just found for $\delta_1$, then $\{1,2\} \ast T$ is false or $\{4,5,6,7\} \ast *T$ is false. Hence, $\sigma^2_{12,3,4,5,6,7} = 0$.

- $\sigma^2_{123,4,5,6,7} = \delta_{123} \cdot \delta_{567} \cdot \delta_{123} \cdot \delta_{567}$. Consider

$$
\sum_{\delta_1 \in S} \delta_1 = \sum_{\delta_2 \in S} \delta_2 \Rightarrow \delta_1 = \delta_1 + \delta_2 = \delta_1 + \delta_2$$

$$+ \delta_1 + \delta_2 - \delta_1 - \delta_2 = \delta_1 + \delta_2$$

After replacing $\delta_{123}$ with the new expression and distributing, we get $\sigma^2_{123,4,5,6,7} = -\delta_{123} \cdot \delta_{567} \cdot \delta_{12} \cdot \delta_{567} = -\delta_{12} \cdot \delta_{4567} \cdot \delta_{567} \cdot \delta_{567} = -(-1) = 1$.

- $\sigma^2_{12,34,5,6,7} = \delta_{12} \cdot \delta_{567} \cdot \delta_{12} \cdot \delta_{567}$. We use the following boundary relation

$$
\sum_{\delta_1 \in S} \delta_1 = \sum_{\delta_2 \in S} \delta_2 \Rightarrow \delta_1 = \delta_1 + \delta_2 = \delta_1 + \delta_2$$

$$+ \delta_1 + \delta_2 - \delta_1 - \delta_2 = \delta_1 + \delta_2$$

$$\sigma^2_{12,34,5,6,7} = -\delta_{12} \cdot \delta_{567} \cdot \delta_{124} \cdot \delta_{567} = -\delta_{12} \cdot \delta_{3567} \cdot \delta_{567} \cdot \delta_{567} = 1.$$
Proof.

(⇐) Assume \( \{a, b, c, d, e, f, g\} = \{7\} \) and let \( I \cup J = \{a, b, c\} \). Consider \( D_{abc,defg} \cong \mathbb{P}^1 \times \mathbf{M}_{0,5} \cong \mathbb{P}^1 \times \mathbf{M}_{0,5} \), and let \( \pi : \mathbb{P}^1 \times \mathbf{M}_{0,5} \to \mathbb{P}^1 \) be the usual projection morphism. If \( C \) is the stable 7-pointed rational curve corresponding to the generic point of \( D_{abc,defg} \), assume the node of \( C \) and the labels \( b, c \) fixed on the twig which contains \( a, b \) and \( c \). So we can think of \( a \) as parametrizing \( \mathbb{P}^1 \), and therefore \( \pi^{-1}(b) = s_{ab,c,defg} \), \( \pi^{-1}(c) = s_{ac,b,defg} \). In conclusion, \( s_{ab,c,defg} \) and \( s_{ac,b,defg} \) are rationally equivalent.

(⇒) Let us prove the contrapositive. We proceed by enumerating all the possible cases.

(i) \(|I| = 3\). Then \( 2 = \sigma_{I,J,K} \cdot \sigma_{I,J,K} = \sigma_{L,M,N} \cdot \sigma_{I,J,K} \in \{-1, 0, 1\} \Rightarrow \sigma_{I,J,K} \neq \sigma_{L,M,N} \).

(ii) \(|I| = |J| = 2\). Up to relabeling, we can assume \( \sigma_{I,J,K} = \sigma_{12,34,567} \). \( \sigma_{L,M,N} \) can be in one of the following forms

\[
\sigma_{ab,cd,efg}, \sigma_{abc,d,efg} \text{ or } \sigma_{ab,c,defg}
\]

\( (\sigma_{ab,cde,fg} \text{ is excluded because of what we just discussed in (i))} \). In any case, we can write \( \sigma_{L,M,N} = \delta_S \cdot \delta_T \) with \(|S| = 4 \). Therefore, \( \sigma_{12,34,567} \cdot \sigma_{L,M,N} = \delta_{12} \cdot \delta_{1234} \cdot \delta_S \cdot \delta_T \) can be equal just to 0 or \(-1\) (more in detail, if \( S = \{123, \{12\}, 4\} \) and the intersection can be either 0 or \(-1\)). However \( \sigma_{12,34,567} \cdot \sigma_{12,34,567} = 1 \), so \( \sigma_{12,34,567} \neq \sigma_{L,M,N} \).

(iii) \(|I| = |J| = 3\). This case uses the same strategy we adopted in (ii).

(iv) \(|I| = |J| = 2\). We can assume \( \sigma_{I,J,K} = \sigma_{12,3,4567} \). Because of what we proved so far, we can assume that \( s_{L,M,N} = s_{abc,defg} \). By our hypotheses, we also have that \( s_{ab,c,defg} \) has to be different from \( s_{1,3,2,4,5,6,7} \) and \( s_{23,1,4,5,6,7} \). But now, up to permuting \( \{4, 5, 6, 7\} \) and \( \{1, 2\} \) (which leave \( \sigma_{12,3,4567} \) unchanged), there are few possibilities for \( \sigma_{abc,defg} \), which are

\[
\sigma_{12,4,3567}, \sigma_{14,3,2567}, \sigma_{14,5,2367}, \sigma_{45,1,2367}, \sigma_{45,3,1267} \text{ or } \sigma_{45,6,1237}.
\]

In each case, it is easy to see that \( \sigma_{12,3,4567} \neq \sigma_{abc,defg} \) by intersecting with an appropriate 2-cycle class:

\[
\begin{align*}
1 & = \sigma_{12,3,4567} \cdot \sigma_{45,123,67} \neq \sigma_{12,4,3567} \cdot \sigma_{45,123,67} = 0, \\
1 & = \sigma_{12,3,4567} \cdot \sigma_{45,123,67} \neq \sigma_{14,3,2567} \cdot \sigma_{45,123,67} = 0, \\
1 & = \sigma_{12,3,4567} \cdot \sigma_{45,123,67} \neq \sigma_{14,5,2367} \cdot \sigma_{45,123,67} = 0, \\
1 & = \sigma_{12,3,4567} \cdot \sigma_{46,123,57} \neq \sigma_{45,1,2367} \cdot \sigma_{46,123,57} = 0, \\
1 & = \sigma_{12,3,4567} \cdot \sigma_{46,123,57} \neq \sigma_{45,3,1267} \cdot \sigma_{46,123,57} = 0, \\
1 & = \sigma_{12,3,4567} \cdot \sigma_{46,123,57} \neq \sigma_{45,6,1237} \cdot \sigma_{46,123,57} = 0.
\end{align*}
\]

□

Now, an easy count tells us that there are 420 distinct equivalence classes of 2-boundary strata on \( \mathbf{M}_{0,7} \). In addition, recent work of Chen and Coskun shows that these 420 equivalence classes generate extremal rays of \( \text{Eff}_2(\mathbf{M}_{0,7}) \).

**Theorem 2.7 (CC Theorem 6.1).** Equivalence classes of boundary strata of codimension 2 on \( \mathbf{M}_{0,n} \) are extremal in \( \text{Eff}_2(\mathbf{M}_{0,n}) \).

The next corollary completely describes the cone \( V_2(\mathbf{M}_{0,7}) \) and sums up what we know about \( \text{Eff}_2(\mathbf{M}_{0,7}) \) so far.
Corollary 2.8. The cone \( \text{Eff}_2(\overline{M}_{0,7}) \) has at least 420 extremal rays, which are spanned by the distinct equivalence classes of the 2-boundary strata on \( \overline{M}_{0,7} \). In particular, the closed cone \( V_2(\overline{M}_{0,7}) \) has exactly 420 extremal rays.

2.5. The intersection form \( N_2(\overline{M}_{0,7}) \times N_2(\overline{M}_{0,7}) \to \mathbb{R} \). The real vector space \( N_2(\overline{M}_{0,7}) \) is equipped with a symmetric bilinear form \( Q : N_2(\overline{M}_{0,7}) \times N_2(\overline{M}_{0,7}) \to \mathbb{R} \) given by the intersection between equivalence classes of 2-cycles. Since \( Q \) is nondegenerate, then \( Q \) has rank equal to \( \dim_{\mathbb{R}} N_2(\overline{M}_{0,7}) \).

Proposition 2.9. \( \dim_{\mathbb{R}} N_2(\overline{M}_{0,7}) = 127 \).

Proof. Let \( K \) be our base field. We know that the equivalence classes of 2-boundary strata span \( N_2(\overline{M}_{0,7}) \) in any characteristic. Moreover, the linear dependence relations between the equivalence classes of 2-boundary strata on \( \overline{M}_{0,7} \), only depend on the combinatorics of the intersection between the 2-boundary strata (that we just studied in Proposition 2.4 and Proposition 2.5), and all this does not depend on \( \text{char}(K) \). Hence, \( \dim_{\mathbb{R}} N_2(\overline{M}_{0,7}) \) does not depend on the characteristic, and we can assume that \( K = \mathbb{C} \).

As a complex variety, \( \overline{M}_{0,7} \) is an HI scheme (see [Ke, Appendix]). It follows that the Chow group \( \text{CH}_2(\overline{M}_{0,7}) \) is isomorphic to the homology group \( H_4(\overline{M}_{0,7}; \mathbb{Z}) \). Therefore, the dimension of \( N_2(\overline{M}_{0,7}) \cong \text{CH}_2(\overline{M}_{0,7}) \otimes_{\mathbb{Z}} \mathbb{R} \) as a real vector space is equal to \( b_4 \), the 4-th Betti number of \( \overline{M}_{0,7} \).

We can find \( b_4 \) by computing \( P_{\overline{M}_{0,7}}(q) = \sum_{j \geq 0} b_j q^j \), the Poincaré polynomial of \( \overline{M}_{0,7} \). We compute this polynomial by using a recursive formula in [CGK, Section 5], which gives the Poincaré polynomial of the space \( T_{d,n} \), the compact moduli space of stable \( n \)-pointed rooted trees of \( d \)-dimensional projective spaces. In our case, \( \overline{M}_{0,7} = T_{1,6} \) (see [CGK, Proposition 3.4.3]), and one can compute that \( P_{\overline{M}_{0,7}}(q) = 1 + 42q^2 + 127q^4 + 42q^6 + q^8 \). \( \square \)

Proposition 2.10. The bilinear form \( Q \) has signature \((86, 41)\).

Proof. The 420 equivalence classes of 2-boundary strata span \( N_2(\overline{M}_{0,7}) \). Therefore, we can choose 127 of these 2-cycle to form a basis for \( N_2(\overline{M}_{0,7}) \), and a matrix representation for \( Q \) is given by the intersection matrix of these 127 equivalence classes of 2-boundary strata. Since this matrix just depends on the combinatorics of the intersection between the 2-boundary strata, we have that the signature of \( Q \) is independent of the characteristic of the base field. So let \( \mathbb{C} \) be our base field.

With this assumption, we have that \( \overline{M}_{0,7} \) is an HI scheme and a smooth manifold, implying that \( N_2(\overline{M}_{0,7}) \cong H^4(\overline{M}_{0,7}; \mathbb{R}) \). Using the Hodge-Riemann bilinear relations (see [GH, Chapter 0]), one has that

\[ I(\overline{M}_{0,7}) = \sum_{p+q \text{ is even}} (-1)^p h^{p,q}, \]

where \( I(\overline{M}_{0,7}) \) is the index of \( \overline{M}_{0,7} \) (i.e. the number of positive eigenvalues minus the number of negative eigenvalues in a matrix representation of \( Q \)), and \( h^{p,q} \) the Hodge numbers of \( \overline{M}_{0,7} \).

Now, knowing that the Poincaré polynomial of \( \overline{M}_{0,7} \) is \( P_{\overline{M}_{0,7}}(q) = 1 + 42q^2 + 127q^4 + 42q^6 + q^8 \) (see the proof of Proposition 2.9), and using the Hodge decomposition, we can compute that

\[ I(\overline{M}_{0,7}) = 2h^{0,0} + 4h^{2,0} - 2h^{1,1} + 2h^{4,0} - 2h^{3,1} + h^{2,2} = 2 + 0 - 84 + 0 - 0 + 127 = 45, \]

implying that the signature of \( Q \) is \((86, 41)\). \( \square \)
Under a more arithmetic perspective, we can view $Q$ as a bilinear form on $H_1(\overline{M}_{0,7}; \mathbb{Z})$ (which is torsion-free). In this case, $Q$ is unimodular by Poincaré duality and odd by Proposition 2.5.

### 3. Lift of effective cycles

The technique we are about to describe allows to construct an effective $k$-cycle on $\overline{M}_{0,n+1}$ given an effective $k$-cycle on $\overline{M}_{0,n}$.

#### 3.1. Let $\pi: \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ be the map forgetting the $(n+1)$-th label. Consider the boundary divisor $D_{n,n+1}$ and let $i: D_{n,n+1} \to \overline{M}_{0,n+1}$ be the inclusion morphism. The following varieties can be naturally identified

$$\overline{M}_{0,n} \equiv \overline{M}_0[n-1]\cup\{x\} \times \overline{M}_0\{n,n+1,x\} \equiv D_{n,n+1},$$

and therefore we have a commutative diagram

$$\begin{array}{ccc}
\overline{M}_{0,n} & \xrightarrow{i} & \overline{M}_{0,n+1} \\
\text{id} & \downarrow & \pi \\
\overline{M}_{0,n} & \xrightarrow{} &
\end{array}$$

**Definition 3.1.** If $\alpha \in \text{Eff}_k(\overline{M}_{0,n})$, then $i_*\alpha \in \text{Eff}_k(\overline{M}_{0,n+1})$ will be called the lift of $\alpha$ to $\overline{M}_{0,n+1}$.

Observe that, instead of just considering $D_{n,n+1}$, one can do a similar construction with any $D_{ab}$, $\{a,b\} \subset [n+1]$. As the following lemma explains, some of the properties of $\alpha$ are preserved after we lift it.

**Lifting Lemma.** Let $k$ and $n$ be positive integers such that $0 < k < n - 3$. Let $\alpha$ be the equivalence class of an effective $k$-cycle on $\overline{M}_{0,n}$. Consider the maps $i: \overline{M}_{0,n} \to \overline{M}_{0,n+1}$ and $\pi: \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ as above. Then

(i) if $\alpha \in \text{Eff}_k(\overline{M}_{0,n}) \setminus V_k(\overline{M}_{0,n})$, then $i_*\alpha \in \text{Eff}_k(\overline{M}_{0,n+1}) \setminus V_k(\overline{M}_{0,n+1})$;

(ii) if $\alpha$ is extremal in $\text{Eff}_k(\overline{M}_{0,n})$, then $i_*\alpha$ is extremal in $\text{Eff}_k(\overline{M}_{0,n+1})$.

**Proof.**

(i) Assume by contradiction that $i_*\alpha \in V_k(\overline{M}_{0,n+1})$. Therefore we can write $i_*\alpha$ as an effective sum of $k$-boundary strata on $\overline{M}_{0,n+1}$. But then $\alpha = \text{id}_*\alpha = \pi_*i_*\alpha \in V_k(\overline{M}_{0,n})$, which cannot be.

(ii) Assume that $i_*\alpha = \sum_{j=1}^m r_j[Z_j]$, where $r_j \in \mathbb{R}_{>0}$ and $Z_j \subset \overline{M}_{0,n+1}$ are irreducible and effective $k$-cycles. We prove that $[Z_j]$ is proportional to $i_*\alpha$ for all $j = 1, \ldots, m$.

Consider the reduction morphism $f_A: \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ where $A = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 1, 1)$ (see [11]). The exceptional locus of $f_A$ is exactly $D_{n,n+1} = i(\overline{M}_{0,n})$ and $f_A(D_{n,n+1})$ is a point. In particular $f_Ai_*\alpha = 0$, implying that

$$\sum_{j=1}^m r_j f_A[Z_j] = 0.$$

Since $\overline{M}_{0,n}$ is projective, we have that $f_A[Z_j] = 0$ for all $j$, which is equivalent to $\dim(f_A(Z_j)) < k$ for all $j$. This implies that, given any $j$, $Z_j \subset D_{n,n+1}$. Define $Z_j' = \overline{M}_{0,n} - i_*\alpha \equiv V_k(\overline{M}_{0,n})$.


Given this result, one can prove that efficiently in \( \text{Eff}(\mathbb{M}_{0,n}) \).

\[
i_\ast \sum_{j=1}^m r_j[Z_j] = \sum_{j=1}^m r_j[Z_j] = i_\ast \alpha.
\]

The pushforward morphism \( i_\ast \) is injective on \( k \)-cycles, because \( \pi_\ast \circ i_\ast \) is the identity. It follows that \( \alpha = \sum_{j=1}^m r_j[Z_j] \), and hence each \( [Z_j] \) is proportional to \( \alpha \) by the extremality of \( \alpha \) in \( \text{Eff}(\mathbb{M}_{0,n}) \). In particular, each \( [Z_j] \) has to be proportional to \( i_\ast \alpha \) for all \( j \).

\[
\square
\]

Alternatively, the following proposition can be used to prove the second part of the lifting lemma.

**Proposition 3.2** ([CC Proposition 2.5]). Let \( \gamma : Y \to X \) be a morphism between two projective varieties. Assume that \( A_k(Y) \to N_k(Y) \) is an isomorphism and that the composite \( \gamma_\ast : A_k(Y) \to A_k(X) \to N_k(X) \) is injective. Moreover, assume that \( f : X \to W \) is a morphism to a projective variety \( W \) whose exceptional locus is contained in \( \gamma(Y) \). If \( f \) is extremal in \( \text{Eff}(Y) \) and if \( \dim(\gamma(Z)) - \dim(f(\gamma(Z))) > 0 \), then \( \gamma(Z) \) is also extremal in \( \text{Eff}(X) \).

Given this result, one can prove that \( i_\ast \alpha \) is extremal by taking \( Y = \mathbb{M}_{0,n} \), \( X = \mathbb{M}_{0,n+1} \), \( \gamma = i \), \( W = \mathbb{M}_{0,4} \), with \( A = (\frac{1}{n-1}, \ldots, \frac{1}{n-1}, 1, 1) \), \( f = f_A \) and \( [Z] = \alpha \).

### 3.2. Fulton’s question.

The following question is attributed to Fulton:

**Question** ([KM Question 1.1]). Let \( 0 < k < n - 3 \). Is it true that \( V_k(\mathbb{M}_{0,n}) = \text{Eff}(\mathbb{M}_{0,n}) \)?

Following [GKM] and [V] notation, denote the previous question with \( F_k(0, n) \) (observe that the analogue question for \( k = 0 \) or \( k = n - 3 \) is trivial). \( F_1(0, 5) \) is answered positively because \( \mathbb{M}_{0,5} \) is a del Pezzo of degree 5. The answer to \( F_1(0, 6) \) and \( F_1(0, 7) \) is also yes, but this is a deep result of Keel and McKernan (see [KM]). \( F_1(0, n) \) for \( n > 7 \) is an open question, and the conjecture that says \( F_1(0, n) \) has a positive answer for \( n > 7 \) is called the F-conjecture.

Keel and Vermeire showed that \( F_{n-4}(0, n) \) has a negative answer for all \( n \geq 6 \) (see [GKM], [V]). This result, combined with the lifting lemma, clearly shows what is the answer to \( F_k(0, n) \) for \( 1 < k < n - 4 \).

**Corollary 3.3.** If \( 1 < k < n - 4 \), then \( F_k(0, n) \) has a negative answer, or in other words \( V_k(\mathbb{M}_{0,n}) \not\subset \text{Eff}(\mathbb{M}_{0,n}) \).

**Proof.** We know that \( V_k(\mathbb{M}_{0,k+1}) \not\subset \text{Eff}(\mathbb{M}_{0,k+1}) \) from [GKM], [V]. Therefore, using the lifting lemma, we see that \( V_k(\mathbb{M}_{0,k+2}) \not\subset \text{Eff}(\mathbb{M}_{0,k+2}) \). Now, by iterating this argument, we obtain that \( V_k(\mathbb{M}_{0,n}) \not\subset \text{Eff}(\mathbb{M}_{0,n}) \). \( \square \)

### 4. Lifts to \( \mathbb{M}_{0,7} \) of the Keel-Vermeire divisors on \( \mathbb{M}_{0,6} \)

The following description of the Keel-Vermeire divisors on \( \mathbb{M}_{0,6} \) is convenient for us.

**Definition 4.1.** Assume \([6] = \{i, j, k, \ell, m, q\}\). A divisor on \( \mathbb{M}_{0,6} \) in the from

\[
\delta_{m_0,i,j}^{\text{KV}} := \delta_{m_0,i} + \delta_{j,m_0} + \delta_{k,q} + \delta_{q,\ell} + 2\delta_{ijm} - \delta_{mq},
\]

is called a Keel-Vermeire divisor on \( \mathbb{M}_{0,6} \).
Properties of the Keel-Vermeire divisors on $\overline{M}_{0,6}$. A Keel-Vermeire divisor $\delta_{mq,ij}^{KV}$ on $\overline{M}_{0,6}$ is effective and cannot be written as an effective sum of boundary divisors. This is proved in [V] in characteristic zero, and one can see that it actually holds in any characteristic. It is also important for us to know that the Keel-Vermeire divisors on $\overline{M}_{0,6}$ are extremal in the cone $\text{Eff}_2(\overline{M}_{0,6})$ in any characteristic. A proof of this can be found in [CT13]. Observe that $\delta_{mq,ij}^{KV} = \delta_{im,jm}^{KV} = \delta_{qm,ij}^{KV} = \delta_{mq,jm}^{KV} = \delta_{mq,kl}^{KV}$, therefore there are 15 Keel-Vermeire divisors on $\overline{M}_{0,6}$.

One more property (but we do not use it in this paper) is that the Keel-Vermeire divisors together with the boundary divisors generate the cone $\text{Eff}_2(\overline{M}_{0,6})$. This was first proved by Hassett and Tschinkel in [HT]. An alternative proof of this fact can be found in [C] (actually, in [C] it is proved that the Cox ring of $\overline{M}_{0,6}$ is generated by the sections of these divisors, which is a stronger condition).

Now we want to lift the Keel-Vermeire divisors to $\overline{M}_{0,7}$ and give a combinatorial description of these lifts.

**Proposition 4.2.** Let $[7] = \{a, b, i, j, k, \ell, m\}$. Then any lift to $\overline{M}_{0,7}$ of a Keel-Vermeire divisor on $\overline{M}_{0,6}$ can be written as the following linear combination of 2-boundary strata

\[ \sigma_{im,jk\ell,ab} + \sigma_{jm,ik\ell,ab} + \sigma_{ijkm,\ell,ab} + 2\sigma_{ijm,kl,ab} - \sigma_{ijk\ell,m,ab}. \]

**Proof.** Let us choose a boundary divisor $D_{ab}$ on $\overline{M}_{0,7}$. This can be identified with $\overline{M}_{0,([7] \cup \{x\}) \setminus \{a, b\}}$, where $x$ is an extra label. So, if we write $([7] \cup \{x\}) \setminus \{a, b\} = \{i, j, k, l, m, x\}$, a Keel-Vermeire divisor on $\overline{M}_{0,([7] \cup \{x\}) \setminus \{a, b\}}$ is in the form

\[ \delta_{mx,ij}^{KV} = \delta_{im} + \delta_{jm} + \delta_{kx} + \delta_{\ell x} + 2\delta_{ijm} - \delta_{mx}. \]

If $\iota: D_{ab} \hookrightarrow \overline{M}_{0,7}$ is the natural inclusion, then the lift to $\overline{M}_{0,7}$ of $\delta_{mx,ij}^{KV}$ is by definition

\[ \iota_* \delta_{mx,ij}^{KV} = \iota_* \delta_{im} + \iota_* \delta_{jm} + \iota_* \delta_{kx} + \iota_* \delta_{\ell x} + 2\iota_* \delta_{ijm} - \iota_* \delta_{mx}. \]

Now, each one of the pushforwards appearing in the right hand side of the previous identity, can be computed by attaching along $x$ a rational tail with the labels $\{x, a, b\}$. By doing so, we obtain the claimed 2-cycle on $\overline{M}_{0,7}$. \qed

**Notation.** We use $\sigma_{ab,m,ij}^{KV}$ to denote the following lift to $\overline{M}_{0,7}$ of a Keel-Vermeire divisor on $\overline{M}_{0,6}$

\[ \sigma_{ab,m,ij}^{KV} := \sigma_{im,jk\ell,ab} + \sigma_{jm,ik\ell,ab} + \sigma_{ijkm,\ell,ab} + 2\sigma_{ijm,kl,ab} - \sigma_{ijk\ell,m,ab}. \]

The next lemma will be used several times.

**Lemma 4.3.** Let $\pi_y: \overline{M}_{0,7} \to \overline{M}_{0,([7] \setminus \{y\})}$ be the map forgetting the label $y \in [7]$, and let $\sigma_{ab,m,ij}^{KV}$ be a lift to $\overline{M}_{0,7}$ of a Keel-Vermeire divisor on $\overline{M}_{0,6}$. Then

\[ \pi_y \sigma_{ab,m,ij}^{KV} = \begin{cases} \delta_{im}^{KV} & \text{if } y = a \\ \delta_{ma,ij}^{KV} & \text{if } y = b \\ \delta_{ab} & \text{otherwise}. \end{cases} \]

**Proof.** First observe that

\[ \pi_* \sigma_{ab,m,ij}^{KV} = \pi_* (\sigma_{im,jk\ell,ab} + \sigma_{jm,ik\ell,ab} + \sigma_{ijkm,\ell,ab} + 2\sigma_{ijm,kl,ab} - \sigma_{ijk\ell,m,ab}) = \delta_{im} + \delta_{jm} + \delta_{kx} + \delta_{\ell x} + 2\delta_{ijm} - \delta_{mb} = \delta_{mb,ij}^{KV}. \]

In the same way, one can prove that $\pi_y \sigma_{ab,m,ij}^{KV} = \delta_{ma,ij}^{KV}$. \qed
Let $y \in [7] \setminus \{a,b\}$. Up to relabeling, we can assume that $\sigma_{ab,m,ij}^{KV} = \sigma_{ab,5,12}^{KV}$. Moreover, by the symmetries of the Keel-Vermeire divisors, we just have to prove our claim when $y = 5$ or $y = 1$. In the former case,
\[
\pi_5^*\sigma_{67,5,12}^{KV} = \pi_5^*(\sigma_{15,234,67} + \sigma_{25,134,67} + \sigma_{1245,3,67} + \sigma_{1235,4,67} + 2\sigma_{125,34,67} - \sigma_{1234,5,67}) = \\
\delta_{67} + \delta_{67} + 0 + 0 + 0 - \delta_{67} = \delta_{67}.
\]
Finally, if $y = 1$, we have that
\[
\pi_5^*\sigma_{67,5,12}^{KV} = \delta_{67} + 0 + 0 + 0 + 0 - 0 = \delta_{67}.
\]
\[\square\]

Since we have $\binom{7}{2}$ choices for $D_{ab}$ and 15 choices for a Keel-Vermeire divisor inside $D_{ab}$, in total we have 315 lifts of Keel-Vermeire divisors to $\overline{M}_{0,7}$. The question now is whether these 315 equivalence classes generate different extremal rays of $\text{Eff}_2(\overline{M}_{0,7})$. This is what we are about to prove.

**Proposition 4.4.** The 315 lifts to $\overline{M}_{0,7}$ of the Keel-Vermeire divisors on $\overline{M}_{0,6}$ generate distinct extremal rays of $\text{Eff}_2(\overline{M}_{0,7})$ that lie outside of $V_2(\overline{M}_{0,7})$.

**Proof.** The extremality of these rays and the fact that they lie outside of the cone $V_2(\overline{M}_{0,7})$ follow from our lifting lemma in Section 3. Let us prove that these rays are all distinct.

Consider two lifts of Keel-Vermeire divisors in the form $\sigma_{ab,m,ij}^{KV}$, $\sigma_{ab,m',i'j'}^{KV}$. Assume that $\sigma_{ab,m,ij} = r\sigma_{ab,m',i'j'}^{KV}$ for some $r \in \mathbb{R}_{>0}$. Then $\delta_{mb,ij}^{KV} = \pi_a^*\sigma_{ab,m,ij}^{KV} = \pi_a^* r\sigma_{ab,m',i'j'}^{KV}$, which implies that $\delta_{mb,ij}^{KV} = \delta_{m'b'i'j'}^{KV}$ because different Keel-Vermeire divisors generate different rays. In particular, $\sigma_{ab,m,ij} = \delta_{ab,m',i'j'}^{KV}$. From this we conclude that lifts of different Keel-Vermeire divisors which are contained in the same boundary divisor give rise to distinct rays of $\text{Eff}_2(\overline{M}_{0,7})$.

Let us consider two distinct boundary divisors $D_{ab}$ and $D_{cd}$. Consider two lifts $\sigma_{ab,m,ij}^{KV}$ and $\sigma_{cd,m',i'j'}^{KV}$. Assume by contradiction that $\sigma_{ab,m,ij}^{KV} = r\sigma_{cd,m',i'j'}^{KV}$ for some $r \in \mathbb{R}_{>0}$. Since $D_{ab}$ and $D_{cd}$ are distinct, we can assume without loss of generality that $a \notin \{c,d\}$. It follows that
\[
\delta_{mb,ij}^{KV} = \pi_a^*\sigma_{ab,m,ij} = \pi_a^* r\sigma_{cd,m',i'j'} = r\delta_{cd},
\]
which is a contradiction because a Keel-Vermeire divisor cannot be proportional to a boundary divisor. \[\square\]

**Definition 4.5.** Define $V_2^{KV}(\overline{M}_{0,7}) \subseteq \text{Eff}_2(\overline{M}_{0,7})$ to be the cone spanned by the 2-boundary strata on $\overline{M}_{0,7}$ and the lifts to $\overline{M}_{0,7}$ of the Keel-Vermeire divisors on $\overline{M}_{0,6}$.

**Corollary 4.6.** The cone $\text{Eff}_2(\overline{M}_{0,7})$ has at least 735 extremal rays: 420 are spanned by 2-boundary strata and 315 are spanned by lifts of Keel-Vermeire divisors. In particular, the closed cone $V_2^{KV}(\overline{M}_{0,7})$ has exactly 735 extremal rays.

Now our goal is to describe $\text{Eff}_2(\overline{M}_{0,7})$ outside of the cone $V_2^{KV}(\overline{M}_{0,7})$. The first question that one may ask is whether or not $V_2^{KV}(\overline{M}_{0,7})$ is equal to $\text{Eff}_2(\overline{M}_{0,7})$. In what follows, we establish that these two cones are not equal.
5. Embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,n}$

5.1. The blow up construction. In [CT12], Castravet and Tevelev give a way to embed $\text{Bl}(\mathbb{P}^2)$ in $\overline{M}_{0,n}$, where the embedding and the blow-up depend on the choice of $n$ points in $\mathbb{P}^2$. Moreover, they tell us how boundary divisors pullback under this embedding. Here is their construction.

**Theorem 5.1 ([CT12 Theorem 3.1])**. Suppose $p_1, \ldots, p_n \in \mathbb{P}^2$ are distinct points, and let $U \subset \mathbb{P}^2$ be the complement of the union of the lines spanned by these points. Consider the morphism

$$F: U \to M_{0,n}$$

defined as follows: given $p \in U$, let $F(p) = [(\mathbb{P}^1; \varphi_p(p_1), \ldots, \varphi_p(p_n))]$, where $\varphi_p: \mathbb{P}^2 \to \mathbb{P}^1$ is the projection from $p$. Then $F$ extends to a morphism

$$F: \text{Bl}_{p_1,\ldots,p_n} \mathbb{P}^2 \to \overline{M}_{0,n}.$$ 

If the points $p_1, \ldots, p_n$ do not lie on a line or a (possibly reducible) conic, then $F$ is a closed embedding. In this case the boundary divisors $\delta_I$ of $\overline{M}_{0,n}$ pullback as follows: for each line $L$ in our line arrangement, if $I \subseteq [n]$ is such that $p_i \in L \Rightarrow i \in I$, then $F^*\delta_I = \widehat{L}_I$ (the strict transform of $L_I$) and (assuming $|I| \geq 3$) $F^*\delta_{I \setminus \{k\}} = E_k$, where $k \in I$ and $E_k$ is the exceptional divisor over $p_k$. Other boundary divisors pullback trivially.

In [CT12], this theorem is used to embed curves in $\overline{M}_{0,n}$ that are possible candidate to be counterexamples to the F-conjecture (later on in the paper, they show that these curves actually are not counterexamples by means of the “arithmetic break” technique).

From now on, our attention is focused on this kind of embedded surfaces. Let us give a name to them.

**Definition 5.2.** Consider $n$ points $p_1, \ldots, p_n \in \mathbb{P}^2$ that do not lie on a line or a (possibly reducible) conic. Then, using Theorem 5.1 the embedded surface $F: \text{Bl}_{p_1,\ldots,p_n} \to \overline{M}_{0,n}$ will be called an embedded blow up of $\mathbb{P}^2$ in $\overline{M}_{0,n}$. The points $p_1, \ldots, p_n$ will be called the points associated to the embedded blow up.

**Observation 5.3.** If $\sigma$ is the equivalence class of an embedded blow up of $\mathbb{P}^2$ in $\overline{M}_{0,n}$, observe that the intersection properties of $\sigma$ can be studied using the projection formula. Let $\sigma_{I,J,K}$ be a 2-boundary stratum on $\overline{M}_{0,n}$. Then

$$\sigma \cdot \sigma_{I,J,K} = F_*(\text{Bl}_{p_1,\ldots,p_n}(\mathbb{P}^2)) \cdot (\delta_I \cdot \delta_K) = [\text{Bl}_{p_1,\ldots,p_n}(\mathbb{P}^2)] \cdot F^*(\delta_I \cdot \delta_K) = (F^*\delta_I) \cdot (F^*\delta_K).$$

Now, the intersection $(F^*\delta_I) \cdot (F^*\delta_K)$ is easy to compute because the two divisors $F^*\delta_I$ and $F^*\delta_K$ can be either zero, an exceptional divisor, or the strict transform of a line that is spanned by the $n$ points in $\mathbb{P}^2$.

It will be crucial to know that the Keel-Vermeire divisors on $\overline{M}_{0,6}$ can be realized as particular embedded blow ups of $\mathbb{P}^2$. This is proved by Castravet and Tevelev in [CT13 Section 9] using irreducible hypertrees.

**Definition 5.4.** Let $n \geq 3$ and $d \geq 1$. A hypertree $\Gamma = \{\Gamma_1, \ldots, \Gamma_d\}$ on the set $[n]$ is a collection of subsets of $[n]$ such that the following conditions are satisfied:

- Any subset $\Gamma_j$ has at least three elements;
- Any $i \in [n]$ is contained in at least two subsets $\Gamma_j$;
Remark 5.5. It turns out that, up to a change of labels, there is a unique irreducible hypertree on the set [6], and a planar realization for this is given by the intersection points of four lines in \( \mathbb{P}^2 \) in general linear position. In [CT13 Section 9] is proved that the embedding in \( \overline{M}_{0,6} \) of the blow up of \( \mathbb{P}^2 \) at the six points of this planar realization gives a Keel-Vermeire divisor. Moreover, we can actually obtain all the 15 Keel-Vermeire divisors by labeling the six points appropriately. The reason why we are interested in these embedded blow up of \( \mathbb{P}^2 \) in \( \overline{M}_{0,7} \) is because they allow us to provide examples of effective 2-cycles whose equivalence classes do not lie in the cone \( V_2^{KV}(\overline{M}_{0,7}) \). The examples we discuss are given by what we call special hypertree surfaces, which are related to Castravet and Tevelev irreducible hypertrees.

5.2. Special hypertree surfaces on \( \overline{M}_{0,7} \).

Definition 5.6. An embedded blow up of \( \mathbb{P}^2 \) in \( \overline{M}_{0,7} \) with associated points \( p_1, \ldots, p_7 \) will be called a hypertree surface on \( \overline{M}_{0,7} \) if there exists \( i \in [7] \) such that \( p_1, \ldots, \hat{p}_i, \ldots, p_7 \) is a planar realization for an irreducible hypertree on the set \( [7] \setminus \{i\} \). A hypertree surface will be called special if we can find three distinct such \( i \in [7] \).

Lemma 5.7. Let \( h \in \text{Eff}_2(\overline{M}_{0,7}) \) be the equivalence class of a hypertree surface on \( \overline{M}_{0,7} \). Then \( h \notin V_2(\overline{M}_{0,7}) \).

Proof. Let \( p_1, \ldots, p_7 \in \mathbb{P}^2 \) be the points associated to \( h \) and assume without loss of generality that the points \( p_1, \ldots, p_6 \) form a planar realization for an irreducible hypertree on the set \([6]\). Arguing by contradiction, let \( h = \sum \alpha_{I,J,K} \sigma_{I,J,K} \) for some coefficients \( \alpha_{I,J,K} \in \mathbb{R}_{\geq 0} \). If \( \pi_7: \overline{M}_{0,7} \to \overline{M}_{0,6} \) is the morphism forgetting the 7-th label, we have that \( \pi_7 \ast h = \sum \alpha_{I,J,K} \pi_7 \ast \sigma_{I,J,K} \). Now, \( \pi_7 \ast \sigma_{I,J,K} \) can be either zero (for example \( \pi_7 \ast \sigma_{12,34,567} \)), or a boundary divisor (for example \( \pi_7 \ast \sigma_{12,345,67} \)). Therefore \( \pi_7 \ast h \) is an effective sum of boundary divisors on \( \overline{M}_{0,6} \). However, \( \pi_7 \ast h \) can be thought of as the equivalence class of the surface in \( \overline{M}_{0,6} \) obtained by embedding the blow up of \( \mathbb{P}^2 \) at \( p_1, \ldots, p_6 \). But then \( \pi_7 \ast h \) has to be a Keel-Vermeire divisor (see Remark 5.5), implying that \( \pi_7 \ast h \) cannot be written as an effective sum of 2-boundary strata. This gives a contradiction.

Theorem 5.8. Let \( h \in \text{Eff}_2(\overline{M}_{0,7}) \) be the equivalence class of a special hypertree surface on \( \overline{M}_{0,7} \). Then \( h \notin V_2^{KV}(\overline{M}_{0,7}) \).

Proof. Let \( p_1, \ldots, p_7 \in \mathbb{P}^2 \) be the points associated to \( h \). Up to relabeling, we can assume that \( i = 5, 6, 7 \) are such that \( p_1, \ldots, \hat{p}_i, \ldots, p_7 \) is a planar realization for an irreducible hypertree on
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the set $[7] \setminus \{i\}$. Assume by contradiction that we can find nonnegative coefficients $\alpha_{I,J,K}$, $\beta_{ab,m,ij}$ such that

$$h = \sum \alpha_{I,J,K} \sigma_{I,J,K} + \sum_{\{a,b\} \subset [7]} \sum \beta_{ab,m,ij} \sigma_{ab,m,ij}.$$  

Fix any $\beta_{a',b',m',i',j'}$. At least one between 5, 6 and 7 is not contained in $\{a', b'\}$. Assume without loss of generality that $7 \notin \{a', b'\}$. If we consider the morphism $\pi_7: \overline{M}_{0,7} \to \overline{M}_{0,6}$ forgetting the 7-th label, using Lemma 4.3 we obtain that

$$\pi_7^* h = \sum \alpha_{I,J,K} \pi_7^* \sigma_{I,J,K} + \sum_{\{a,b\} \subset [7]} \sum \beta_{ab,m,ij} \pi_7^* \sigma_{ab,m,ij} =$$

$$\sum \alpha_{I,J,K} \pi_7^* \sigma_{I,J,K} + \sum_{b \in [6]} \sum \beta_{b,m,ij} \pi_7^* \sigma_{b,m,ij} + \sum_{\{a,b\} \subset [6]} \sum \beta_{ab,m,ij} \pi_7^* \sigma_{ab,m,ij} =$$

(1)$$\sum \alpha_{I,J,K} \pi_7^* \sigma_{I,J,K} + \sum_{b \in [6]} \sum \beta_{b,m,ij} \delta_{b,m,ij} + \sum_{\{a,b\} \subset [6]} \sum \beta_{ab,m,ij} \delta_{ab},$$

where $\pi_7^* h$ is a Keel-Vermie divisor. Therefore, the total coefficient of the boundary divisor $\delta_{a',b'}$ in (1) must be zero, because Keel-Vermie divisors are extremal in $\text{Eff}_2(\overline{M}_{0,6})$. Since the sum is effective, it follows that $\beta_{a',b',m',i',j'} = 0$, and hence every $\beta_{ab,m,ij}$ must be zero because $\beta_{a',b',m',i',j'}$ was arbitrary. But then we must have that $h \in V_2(\overline{M}_{0,7})$, which contradicts Lemma 5.7. □

5.3. Classification of the special hypertree surfaces on $\overline{M}_{0,7}$. Let us find all the possible special hypertree surfaces on $\overline{M}_{0,7}$. We start by fixing a planar realization $p_1, \ldots, p_6 \in \mathbb{P}^2$ for the irreducible hypertree given by $\Gamma = \{\{1, 4, 5\}, \{1, 3, 6\}, \{2, 3, 5\}, \{2, 4, 6\}\}$. We consider permutations of these labels later on. Observe that the points $p_1, \ldots, p_6$ span seven lines: three of them contain exactly two labeled points, and the remaining four contain exactly three labeled points. Let $X$ be the union of these seven lines. If $\text{char}(\mathbb{K}) \neq 2$, this points and lines arrangement is shown below ($\mathbb{K}$ is our base field).

![Figure 3. Line arrangement spanned by a planar realization for $\Gamma$ if $\text{char}(\mathbb{K}) \neq 2$.](image-url)
The characteristic 2 case is discussed separately at the end of this section. Therefore, for now assume that \( \text{char}(\mathbb{K}) \neq 2 \).

Let us add a seventh point \( p_7 \) to this configuration. Take \( p_7 \in \mathbb{P}^2 \setminus X \). Then we cannot have a special hypertree surface, because if we drop a label \( i \in [6] \), the points \( p_1, \ldots, \hat{p_i}, \ldots, p_7 \) span at least five lines containing exactly two labeled points. Therefore we must have \( p_7 \in X \).

Doing similar considerations, one can easily prove that \( p_7 \) must lie in the intersection of at least two lines in \( X \). Since \( p_7 \) is distinct from \( p_1, \ldots, p_6 \), we have three possibilities for \( p_7 \) (the lines in \( X \) intersect in 9 points). All three of these cases give a special hypertree surface, as shown in the pictures below. The arrows show the three points \( p_i \) that can be dropped in order to get an irreducible hypertree on the set \( [7] \setminus \{i\} \).

\[
\begin{align*}
\text{Figure 4. Points arrangements in } \mathbb{P}^2 \text{ which give special hypertree surfaces on } \overline{M}_{0,7}.
\end{align*}
\]

Consider the action \( S_7 \rtimes \text{Eff}_2(\overline{M}_{0,7}) \) induced by the natural action \( S_7 \rtimes \overline{M}_{0,7} \). When a permutation \( \tau \in S_7 \) acts on \( \sigma \in \text{Eff}_2(\overline{M}_{0,7}) \), we write \( \tau \star \sigma \). Let \( h_1 \) be the equivalence class of the special hypertree surface obtained by using the top left points configuration in Figure 4. Similarly, define
\( h_2 \) to be the equivalence class of the special hypertree surface obtained by using the top right configuration, and \( h_3 \) the one obtained by using the bottom configuration in the same figure.

First, observe that \( h_2 \) belongs to the orbit of \( h_1 \) under the \( S_7 \)-action because \( h_2 = ((36)(45)) \ast h_1 \).

Also \( h_3 \) belongs to the orbit of \( h_1 \), because \( h_3 = ((35)(56)(26)(24)(15)) \ast h_2 \). Therefore, it is enough to consider the \( S_7 \)-action on \( h_1 \).

Let us find the stabilizer of \( h_1 \) under the \( S_7 \)-action. It is easy to find the following subgroup of \( \text{Stab}_{S_7}(h_1) \)

\[
G_1 := \{ \text{id}, (34)(65), (47)(16), (37)(15), (156)(473), (165)(374) \},
\]

which is isomorphic to the dihedral group \( D_3 \). It is less obvious to notice this other subgroup of the stabilizer

\[
G_2 := \{ \text{id}, (12)(56), (25)(16), (26)(15) \},
\]

which is isomorphic to the Klein group \( (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \). To see why \( G_1, G_2 \subseteq \text{Stab}_{S_7}(h_1) \), just take any \( \tau \in G_1 \cup G_2 \) and observe that \( \tau \ast h_1 \) and \( h_1 \) have the same intersection properties with respect to the boundary divisors of \( \overline{M_{0,7}} \).

To show that \( \text{Stab}_{S_7}(h_1) \) is actually generated by \( G_1 \) and \( G_2 \), take any \( \tau \in \text{Stab}_{S_7}(h_1) \). Thinking of \( \tau \) as a bijection \( \tau : [7] \rightarrow [7] \), then \( \tau \{\{3,4,7\}\} = \{3,4,7\} \). This is true because, in order to preserve the intersections with the boundary divisors, we need to send a labeled point that lies on a line containing exactly two labeled points to a labeled point having the same property. In particular, we must have that \( \tau \{\{1,2,5,6\}\} = \{1,2,5,6\} \). Therefore \( \tau \) acts by permuting the two sets \( \{3,4,7\} \) and \( \{1,2,5,6\} \) separately. Now there are two cases: \( \tau \) fixes 2 or not. In the first case, the only possibility for \( \tau \) is to be an element of \( G_1 \). If \( \tau \) does not fix 2, then assume that \( \tau \) is the identity on \( \{3,4,7\} \) (we can assume this up to composing with an element of \( G_1 \)). In this case, one can check that \( \tau \) must be an element of \( G_2 \) in order to preserve the intersections with the boundary divisors of \( \overline{M_{0,7}} \). Therefore, we just deduced that \( \text{Stab}_{S_7}(h_1) = \langle G_1, G_2 \rangle \).

An easy count tells us that \( \langle G_1, G_2 \rangle = 24 \), and therefore the orbit of \( h_1 \) has \( 7! \cdot 24 = 210 \) distinct equivalence classes. As one can easily check, these classes generate distinct rays in \( \text{Eff}_2(\overline{M_{0,7}}) \). The next proposition summarizes what we proved so far.

**Proposition 5.9.** In characteristic different from 2, there are 210 distinct equivalence classes of special hypertree surfaces on \( \overline{M_{0,7}} \). These classes generate 210 distinct rays of \( \text{Eff}_2(\overline{M_{0,7}}) \) which lie outside of the cone \( V_{2}^{KV}(\overline{M_{0,7}}) \).

**Classification in characteristic 2.** The discussion in characteristic 2 is essentially the same, but with the following exceptions. First of all, in Figure 3 the seven lines intersect in seven points (one of which is unlabeled), giving the well known Fano configuration. Also in this case, \( p_7 \) has to be the unlabeled point at the intersection of three lines, and therefore we produced only one special hypertree surface. Now, if we consider the \( S_7 \)-action, it is straightforward to see that the stabilizer of the special hypertree surface we found is \( \text{PGL}(3,F_2) \), which has 168 elements. So, the analogue of Proposition 5.9 in characteristic 2 is the following.

**Proposition 5.10.** In characteristic 2, there are 30 distinct equivalence classes of special hypertree surfaces on \( \overline{M_{0,7}} \). These classes generate 30 distinct rays of \( \text{Eff}_2(\overline{M_{0,7}}) \) which lie outside of the cone \( V_{2}^{KV}(\overline{M_{0,7}}) \).

**Remark 5.11.** We do not know yet if the rays generated by the equivalence classes of the special hypertree surfaces are extremal in \( \text{Eff}_2(\overline{M_{0,7}}) \), and certainly a proof or a disproof of the extremality of these rays would be a further step toward the understanding of the cone \( \text{Eff}_2(\overline{M_{0,7}}) \).
Remark 5.12. We observed that the equivalence classes of the special hypertree surfaces are invariant with respect to a certain subgroup of $S_7$. Given a subgroup $G$ of $S_n$, the idea of considering $G$-invariant sub-loci of $\overline{M}_{0,n}$ intersecting the interior $M_{0,n}$ recently appeared in [MS]. Also, the same idea was previously used to describe the Keel-Vermeire divisors (see [V, Section 3]).

Remark 5.13. Consider the moduli space $\overline{M}_{0,n}^{S_n}$, which is the quotient of $\overline{M}_{0,n}$ by the natural action $S_n \acts \overline{M}_{0,n}$. As we studied $\text{Eff}_2(\overline{M}_{0,7})$, one can also consider $\text{Eff}_2(\overline{M}_{0,7}^{S_n})$. For a study of the pseudoeffective cone $\text{Eff}_2(\overline{M}_{0,7}^{S_n})$ see [FL, Section 7.3].

Let us define the following subcone of $\text{Eff}_2(\overline{M}_{0,7})$.

Definition 5.14. Define $V_2^{KV+CT}(\overline{M}_{0,7}) \subseteq \text{Eff}_2(\overline{M}_{0,7})$ to be the cone generated by the equivalence classes of the 2-boundary strata, the lifts of the Keel-Vermeire divisors on $\overline{M}_{0,6}$ and the embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,7}$.

It follows from what we proved that we have strict inclusions:

$$V_2(\overline{M}_{0,7}) \subset V_2^{KV}(\overline{M}_{0,7}) \subset V_2^{KV+CT}(\overline{M}_{0,7}).$$

6. Generalization to $\overline{M}_{0,n}$ for any $n > 7$ and further questions

We can generalize our constructions for 2-cycles on $\overline{M}_{0,7}$ to any $\overline{M}_{0,n}$ with $n > 7$. First, define $V_2^{KV}(\overline{M}_{0,n})$ inductively to be the subcone of $\text{Eff}_2(\overline{M}_{0,n})$ generated by $V_2(\overline{M}_{0,n})$ and by the lifts of the effective 2-cycles in $V_2^{KV}(\overline{M}_{0,n-1})$. Similarly, we can define $V_2^{KV+CT}(\overline{M}_{0,n})$ inductively to be the subcone of $\text{Eff}_2(\overline{M}_{0,n})$ generated by $V_2(\overline{M}_{0,n})$, by the lifts of the effective 2-cycles in $V_2^{KV+CT}(\overline{M}_{0,n-1})$ and by the embedded blow up of $\mathbb{P}^2$ in $\overline{M}_{0,n}$. Since we already know that $V_2(\overline{M}_{0,7}) \subset V_2^{KV}(\overline{M}_{0,7}) \subset V_2^{KV+CT}(\overline{M}_{0,7})$, it is not hard to see that we have the following strict inclusions

$$V_2(\overline{M}_{0,n}) \subset V_2^{KV}(\overline{M}_{0,n}) \subset V_2^{KV+CT}(\overline{M}_{0,n}).$$

At this point, one can ask the following questions.

Question 1. Is $V_2^{KV+CT}(\overline{M}_{0,7})$ equal to $\text{Eff}_2(\overline{M}_{0,7})$?

Question 2. Is it possible to give examples of embedded blow ups of $\mathbb{P}^2$ in $\overline{M}_{0,7}$ that generate extremal rays of $\text{Eff}_2(\overline{M}_{0,7})$?

Question 3. Is $\text{Eff}_2(\overline{M}_{0,7})$ equal to $\overline{\text{Eff}_2(\overline{M}_{0,7})}$?

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