Recovering Velocity Distributions via Penalized Likelihood

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ABSTRACT

Line-of-sight velocity distributions are crucial for unravelling the dynamics of hot stellar systems. We present a new formalism based on penalized likelihood for deriving such distributions from kinematical data, and evaluate the performance of two algorithms that extract $N(V)$ from absorption-line spectra and from sets of individual velocities. Both algorithms are superior to existing ones in that the solutions are nearly unbiased even when the data are so poor that a great deal of smoothing is required. In addition, the discrete-velocity algorithm is able to remove a known distribution of measurement errors from the estimate of $N(V)$. The formalism is used to recover the velocity distribution of stars in five fields near the center of the globular cluster ω Centauri.

1. Introduction

The most complete kinematical information obtainable for a distant stellar system is the distribution of line-of-sight velocities at every point in the image. Velocity distributions are crucial for understanding the dynamical states of slowly-rotating stellar systems like elliptical galaxies and globular clusters, since velocity dispersions alone place almost no constraints on the form of the potential unless one is willing to make ad hoc assumptions about the shape of the velocity ellipsoid (Dejonghe & Merritt 1992). Velocity distributions are also useful when searching for kinematically distinct subcomponents (e.g. Franx & Illingworth 1988; Rix et al. 1992).

The velocity distribution at point $\mathbf{R}$ in the image of a stellar system, $N(\mathbf{R}, V)$, can be related to the data in different ways depending on the nature of the observations. In a system like a globular cluster, for which the data usually consist of individual stellar
velocities, the velocity distribution is just the frequency function of stellar velocities defined by those stars with apparent positions near to $R$. Since measured velocities are always in error, the observed and true $N(V)$’s are related via a convolution integral. In a distant, unresolved galaxy, one typically measures the integrated spectrum of many stars along a line of sight. The observed spectrum is then a convolution of the velocity distribution of these stars with the broadening function of the spectrograph, and the spectrum of a typical star.

With both sorts of data, the goal is to find a function $N(V)$, at some set of points $R$, such that $\sum_i L\{Y_i; N(V)\}$ – the log likelihood of observing the data $Y_i$ given $N$ – is large. Maximizing this quantity over the space of all possible functions $N(V)$ is unlikely to yield useful results, however, since any $N(V)$ that maximizes the likelihood (assuming it exists, which it often will not) is almost certain to be extremely noisy. This is obviously true if the data are related to the model via a convolution, since the process of deconvolution will amplify the errors in the data. But it is equally true if $N(V)$ is simply the frequency function of observed velocities, since the most likely distribution corresponding to an observed set of $V$’s is just a sum of delta functions at each of the measured velocities. One is therefore forced to place smoothness constraints on the solution.

But smoothing always introduces a bias, i.e. a systematic deviation of the solution from the true $N(V)$. The nature of the bias is obvious when the smoothing is carried out by imposing a rigid functional form on $N(V)$, since the true function will almost certainly be different from this assumed form. But even nonparametric smoothing generates a bias since it effectively averages the data over some region. Furthermore, because the required degree of smoothing increases with the amplitude of the noise in the data, the error from the bias goes up as the quality of the data falls. An ideal algorithm for estimating $N(V)$ would therefore be one in which the bias introduced by the smoothing was effectively minimized, so that the derived $N(V)$ was close to the true function even when the data were so poor that a great deal of smoothing was required.

One way to accomplish this is to make use of prior knowledge about the likely form of $N(V)$. Many studies of stellar and galactic systems have shown that $N(V)$ is often close to a Gaussian. This fact suggests that we infer $N(V)$ by maximizing a quantity like

$$\log L_p = \sum_i L\{Y_i; N(V)\} - \alpha P(N),$$

(1)

the “penalized log likelihood,” where the penalty functional $P(N)$ is large for any $N(V)$ that is noisy and zero for any $N(V)$ that is Gaussian. A natural choice for such a penalty
functional has been suggested by Silverman (1982):

\[
P_G(N) = \int_{-\infty}^{+\infty} [(d/dV)^3 \log N(V)]^2 dV. \tag{2}
\]

This functional assigns zero penalty to any \( N(V) = N_0 \exp[-(V - V_0)^2/2\sigma^2] \), i.e. any Gaussian velocity distribution, and a large penalty to any \( N(V) \) that is rapidly varying. The limiting estimate as the smoothing parameter \( \alpha \) tends to infinity is the normal distribution that best corresponds, in a maximum-likelihood sense, to the data. Thus varying \( \alpha \) takes one from an estimate of \( N(V) \) that is very noisy but that reproduces the data well, to the “infinitely smooth” maximum likelihood Gaussian fit to \( N(V) \). When the data are copious and accurate, the degree of smoothing required, i.e. the value of \( \alpha \), will be small and the inferred \( N(V) \) will be close to the true function. When the data are poor, \( \alpha \) must be increased to deal with the noise; however even a very large value of \( \alpha \) will yield an \( N(V) \) that is nearly Gaussian and that is therefore likely to be close to the true velocity distribution. Regardless of the quality of the data, there will always be a formally optimal choice of \( \alpha \) that yields a solution that is closest in some sense to the true \( N(V) \) – neither too noisy nor too biased.

Here we apply penalized likelihood methods to the recovery of velocity distributions from kinematical data of two sorts: Doppler-broadened spectra, and discrete velocities. There are of course a large number of excellent algorithms already in use for extracting velocity distributions from absorption-line spectra (Franx & Illingworth 1988; Bender 1990; Rix & White 1992; Kuijken & Merrifield 1993; van der Marel & Franx 1993; Saha & Williams 1994). Is another approach really needed? Many of the existing algorithms are essentially nonparametric and hence yield unbiased estimates of \( N(V) \) when the signal-to-noise ratio of the data is large. Their behavior given data with small S/N is more variable, however, due to the different ways in which they carry out the smoothing. For instance, in algorithms that represent the unknown \( N(V) \) via a basis-set expansion,

\[
N(V) = \sum_{k=1}^{K} C_k N_k(V), \tag{3}
\]

the smoothing is accomplished by truncating the expansion after a finite number of terms. Adding more terms decreases the bias by giving the algorithm more freedom to match the true \( N(V) \); however the level of fluctuations in the solution increases as the bias falls since the high-order terms will always try to reproduce the noise in the data. The optimal estimate for data of a given quality will therefore contain only a limited number of terms, and if the data are poor, this number will be small; hence the solution will be biased toward the functional form selected for \( N_1(V) \). Of course one can choose \( N_1 \) to be the normal distribution (e.g. van der Marel & Franx 1993) but its mean and variance must be
specified, and the optimal choices for both parameters will depend in some complex way on
the number of terms retained in the expansion and on the level of noise in the data. Worse,
one does not have complete freedom in this approach to adjust the smoothing to its optimal
level since the basis set is fixed and its terms are discrete.

Most of the spectral deconvolution schemes currently in use have continuously-
adjustable smoothing and so do not suffer from this latter defect. However few of these
schemes incorporate any prior knowledge about the likely form of $N(V)$ and so the
solutions they return given noisy data tend to be unphysical, with functional forms that
are determined primarily by the mechanics of the smoothing. (In fact it is a common
practice to fall back on parametric methods when the data are poor.) For instance, both
the Wiener-filtered algorithm of Rix & White (1992) and the kernel-based algorithm of
Kuijken & Merrifield (1993) yield $N(V) \approx \text{constant}$ in the limit of infinite smoothing. Since
real velocity distributions are unlikely to be well approximated by constant functions, the
estimates produced by these schemes can be substantially biased.

The approach advocated here is neither better nor worse than the existing ones when
the data are of high quality. However it has an advantage when the data are poor, since
even a large degree of smoothing is likely to bias the solution only slightly. Furthermore
the amount of smoothing can be adjusted with infinite precision by varying $\alpha$. In practice
it may be difficult to find the optimal choice of $\alpha$ given noisy data, but there are bootstrap
techniques for choosing $\alpha$ that work well in most cases; and one can always elect to be
guided by physical intuition when deciding how smooth $N(V)$ should be. Most important,
even a gross overestimate of $\alpha$ will yield no worse than a Gaussian fit to the velocity
distribution, which is perhaps the best that can be hoped for when the signal-to-noise ratio
or the number of observed velocities is small.

(Much of the uncertainty in $N(V)$’s derived from absorption line spectra is due to
systematic errors such as incorrect continuum subtraction, template mismatch, limited
resolution, etc. These problems afflict every spectral deconvolution scheme and have been
treated at length by other authors. We have nothing new to say about these systematic
effects and will simply ignore them – our focus is on the uncertainty in $N(V)$ that is
generated by noise in the data rather than by systematic errors.)

Below we evaluate the performance of penalized-likelihood algorithms for estimating
$N(V)$ from spectral data ($\S$2) and from discrete velocities ($\S$3). The former sort of data
are routinely obtained for distant galaxies and the latter for globular clusters, clusters of
galaxies and systems of emission-line objects around galaxies. The same formalism works
equally well in both cases; essentially all that needs to be changed is the form of the matrix
that relates the observable quantities to $N(V)$. We also show how the cross-validation
score can be used to estimate the optimal degree of smoothing from the data. We then (§4) apply the formalism to the recovery of the velocity distribution near the center of the globular cluster ω Centauri using a new sample of stellar radial velocities from an imaging spectrophotometer.

2. Spectra

The optical spectrum at any point in the image of a galaxy is made up of the sum of the spectra of its component stars along the line of sight, Doppler shifted according to their line-of-sight velocities \( V \). If the galaxy spectrum is \( I(\lambda) \) and the spectrum of a single star – after convolution with the instrumental response – is \( T(\lambda) \), then

\[
I(\ln \lambda) = \int_{-\infty}^{+\infty} N(V)T(\ln \lambda - V/c) \, dV = N \circ T,
\]

the convolution of \( N(V) \) with the template. We accordingly seek a function \( N = \hat{N}(V) \) such that

\[
-\log \mathcal{L}_p = \sum_i [I(\lambda_i) - (N \circ T)_i]^2 + \alpha P_G(N)
\]

is minimized, where \( I(\lambda_i) \) is the observed spectral intensity at wavelength \( \lambda_i \).

We first define a discrete grid in velocity space, \( V_1, ..., V_m \), where \( V_j = V_1 + (j - 1)\delta \) and \( \delta = (V_m - V_1)/(m - 1) \). Typically \( V_1 \approx \bar{V} - 4\sigma \) and \( V_1 \approx \bar{V} + 4\sigma \), where \( \bar{V} \) is the mean velocity of the system and \( \sigma \) is the expected velocity dispersion. The number of grid points \( m \) should be large enough that the numerical solution to the minimization problem is essentially independent of \( m \), yet small enough that the minimization — which requires a computational time that scales as \( \sim m^3 \) — does not take inordinately long. We mostly used \( m = 50 \) velocity grid points in what follows. A discrete representation of the right hand side of Eq. (4) on this grid is

\[
\sum_i [I(\lambda_i) - (N \circ T)_i]^2 + \alpha \delta^{-5} \sum_{j=2}^{m-2} [-\log N_{j-1} + 3\log N_j - 3\log N_{j+1} + \log N_{j+2}]^2.
\]

In the first term, \( N \circ T \) is computed by fitting an interpolating spline to the template spectrum and assuming a linear dependence of the (unknown) function \( N \) on \( V \) between the grid points. The convolution can then be represented as a matrix multiplication, i.e.

\[
(N \circ T)_i \approx \sum_j A_{ij} N_j,
\]

\[
A_{ij} = \frac{V_{j+1}I_{ij} - J_{ij}}{V_{j+1} - V_j} - \frac{V_{j-1}I_{i,j-1} - J_{i,j-1}}{V_j - V_{j-1}},
\]
\[ I_{ij} = \int_{V_j}^{V_{j+1}} T \left( \ln \lambda_i - V/c \right) dV, \]
\[ J_{ij} = \int_{V_j}^{V_{j+1}} V T \left( \ln \lambda_i - V/c \right) dV. \]

One then varies the \( m \) parameters \( N_j \) until a minimum is obtained; the use of \( \log N \) in the penalty functional guarantees that the solution will remain everywhere positive without the need to impose additional positivity constraints. Minimization was carried out using the NAG routine E04JAF and required about one minute on a DEC Alpha 300/700 machine.

If the true \( N(V) \) is a Gaussian, the optimal estimates will be obtained from this algorithm by choosing \( \alpha \) to be very large – the algorithm becomes essentially parametric in this limit, returning the normal distribution that best approximates the data. But we would like to show that the algorithm works well even in cases where \( N(V) \) deviates strongly from a Gaussian. The crucial problem, of course – both for this algorithm and any other – lies in choosing the correct degree of smoothing.

Figure 1 shows three estimates of \( N(V) \) based on pseudo-data generated from the very non-Gaussian velocity distribution

\[ N(V) = \frac{1}{\sqrt{2\pi}\sigma} \left( 0.7 \exp\left[-(V - V_1)^2/2\sigma^2\right] + 0.3 \exp\left[-(V - V_2)^2/2\sigma^2\right] \right), \]

with \( V_1 = 140 \text{ km s}^{-1}, V_2 = -140 \text{ km s}^{-1} \) and \( \sigma = 80 \text{ km s}^{-1} \). This \( N(V) \) was found by Merrifield & Kuijken (1994) to provide a good description of the stars in the disk of the Sab galaxy NGC 7217. Merrifield & Kuijken’s broadening function was convolved with a template spectrum from a K0III star, kindly provided by T. Williams; to this broadened spectrum was added Gaussian random noise with amplitude \( \sigma_N \) such that \( I/\sigma_N = S/N = 20 \). The resultant spectrum was then sampled at 800 points in the wavelength range \( 4800 \, \text{Å} < \lambda < 5500 \, \text{Å} \) and these “data” were given to the minimization routine. Figure 1 illustrates the famous “bias-variance” tradeoff of solutions to ill-conditioned problems. When \( \alpha \) is small, the estimated \( N(V) \) matches the true function in an average sense, but fluctuates strongly from grid point to grid point. For larger \( \alpha \), the estimate is nearly correct, with fluctuations of only a few percent around the true \( N(V) \). When \( \alpha \) is increased still more, the solution is driven toward a single Gaussian with roughly the same mean and dispersion as that of the true \( N(V) \). All of these estimates for \( N(V) \) reproduce the input spectrum with acceptable accuracy, but \( N(V) \) itself is correctly recovered only when \( \alpha \) is chosen appropriately – a characteristic feature of ill-conditioned problems. Nevertheless an appropriate choice of \( \alpha \) allows one to recover even this very non-Gaussian \( N(V) \) with good accuracy.
Figure 1 also shows three attempts to recover the broadening function

\[ N(V) = \frac{1}{\sqrt{2\pi}} \left[ \frac{0.4}{\sigma_1} \exp \left( -V^2/2\sigma_1^2 \right) + \frac{0.6}{\sigma_2} \exp \left( -V^2/2\sigma_2^2 \right) \right] \]  

(12)

with \( \sigma_1 = 200 \text{ km s}^{-1} \), \( \sigma_2 = 50 \text{ km s}^{-1} \). This \( N(V) \) is more peaked than a Gaussian, and is designed to represent the velocity distribution in the halo of a spherical galaxy dominated by radial orbits (Dejonghe 1987). Once again the nature of the solution depends on \( \alpha \), with large values of \( \alpha \) generating a Gaussian approximation to the true broadening function. But the optimal \( \alpha \) produces an estimate of \( N(V) \) that is again quite close to the true function, and the velocity dispersion derived from the estimated \( N(V) \) remains close to the true dispersion even when \( \alpha \) is far from its optimal value.

The examples just discussed were based on data with a high signal-to-noise ratio. A more stringent test is the recovery of a non-Gaussian \( N(V) \) from noisy data. Since a larger degree of smoothing will be required, the penalty functional will tend to drive the solution away from the true \( N(V) \) as the level of the noise increases. Figure 2 shows how the average error in the estimated \( N(V) \) varies with the signal-to-noise ratio of the data. For each value of \( S/N \), 300 random realizations of the same spectrum were generated from the Merrifield-Kuijken broadening function and the value of \( \alpha \) that minimized the average, integrated square deviation between the estimates \( \hat{N} \) and the true function \( N \) was found. The integrated square error was defined as

\[ \text{ISE} = \frac{\int \left[ N(V) - \hat{N}(V) \right]^2 dV}{\int N^2(V) dV} \]  

(13)

the normalizing function in the denominator was added so that a value \( \text{ISE} \approx 1 \) corresponds to an rms deviation of roughly 100% between \( N \) and \( \hat{N} \). The MISE was then defined as the mean value of the ISE over the 300 realizations using this “optimal” value of \( \alpha \). Figure 2 shows that the MISE falls roughly as a power law with \( S/N \), with a logarithmic slope of approximately \( -1.3 \). The same information is presented in more detail in Figure 3, which shows the average estimate and its 95% variability bands for each value of \( S/N \). This figure suggests that the existence of two peaks in \( N(V) \) is recoverable for \( S/N > 5 \).

In these examples, the optimal choice of \( \alpha \) and its associated MISE could be computed since the correct form of \( N(V) \) was known. In general one does not know \( N(V) \), of course, and the choice of \( \alpha \) must be based somehow on information contained within the data itself. One way of doing this is via the “cross-validation score” (Green & Silverman 1994, p. 30; Wahba 1990, p. 47). One seeks the value of \( \alpha \) that minimizes

\[ \text{CV}(\alpha) = \frac{1}{n} \sum_i \left[ I(\lambda_i) - (N[i] \circ T)_i \right]^2, \]  

(14)
where $N^{[i]}$ is the minimizer of Eq. (6) with the $i$th data point left out. In effect, the CV measures the degree to which the spectral intensities predicted by $N(V)$ are consistent with the observed intensities. This is not quite the same as asking how close the estimated $N(V)$ is to the true $N(V)$, but it is probably the best that one can do in practice (Wahba 1980).

Figure 4 shows the result of two attempts to recover the optimal $\alpha$ from fake spectra generated using the Merrifield-Kuijken broadening function with S/N = 5 and 10. The values of $\alpha$ that minimize the CV are tolerably close to the values that actually minimize the ISE. More to the point, the ISE of the $N(V)$’s generated using the CV estimates of $\alpha$ differ only negligibly from those obtained using the optimal $\alpha$’s. These examples suggest that one can indeed hope to recover a useful estimate of the optimal smoothing parameter from the data alone. At the very least, such an estimate would provide a starting point when searching for the $\alpha$ that produces the physically most appealing estimate of $N(V)$.

### 3. Discrete Velocities

In many stellar systems, information about $N(V)$ is most naturally obtained in the form of discrete velocities. If the velocities are measured with negligible error, $N(V)$ is simply their frequency function, which can be defined as the function that maximizes the penalized log-likelihood

$$\log \mathcal{L}_p = \sum_{i=1}^{n} \log N(V_i) - \alpha P_G(N)$$  \hspace{1cm} (15)

subject to the constraints

$$\int N(V)dV = 1, \quad N(V) \geq 0$$ \hspace{1cm} (16)

(Thompson & Tapia 1990, p. 102). The penalty functional is needed since, in its absence, the optimal estimate would be a set of delta-functions at the measured velocities.

But the uncertainty in the measured velocities is sometimes comparable to the width of $N(V)$. For instance, radial velocities of faint stars in globular clusters may have measurement errors of a few km s$^{-1}$ compared to intrinsic velocity dispersions of $\sim 10$ km s$^{-1}$. The observable function is then not $N(V)$ but rather its convolution with the error distribution, $N \circ P$. Assuming that the errors have a normal distribution, with dispersion $\sigma_N$, we have

$$N \circ P = \frac{1}{\sqrt{2\pi}\sigma_N} \int_{-\infty}^{+\infty} N(V') \exp \left[ -\frac{(V - V')^2}{2\sigma^2} \right] dV'$$ \hspace{1cm} (17)

and one accordingly seeks the $N(V)$ that maximizes

$$\log \mathcal{L}_p = \sum_{i=1}^{n} \log(N \circ P)_i - \alpha P_G(N)$$ \hspace{1cm} (18)
subject to the same constraints on \( N \). Following Silverman (1982), we find the solution to this constrained optimization problem as the unconstrained maximizer of the functional

\[
\sum_{i=1}^{n} \log(N \circ P)_i - \alpha P_G(N) - n \int N(V) dV.
\]  

(19)

This problem is formally very similar to the deconvolution problem solved above, and a discrete representation of (19) on a grid in velocity space is

\[
\sum_{i=1}^{n} \log(N \circ P)_i + \alpha \delta^{-5} \sum_{j=2}^{m-2} [- \log N_{j-1} + 3 \log N_j - 3 \log N_{j+1} + \log N_{j+2}]^2 - n \sum_{j=1}^{m} \epsilon_j N_j,
\]  

(20)

with \( \epsilon_1 = \epsilon_m = \delta/2 \), \( \epsilon_j = \delta, j = 2, \ldots, m - 1 \). The convolution of \( N \) with \( P \) is again represented as a matrix, Eq. (7), with

\[
I_{ij} = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{V_j}^{V_{j+1}} \exp \left[ \frac{-(V - V_j)^2}{2\sigma^2} \right] dV,
\]  

(21)

\[
J_{ij} = \frac{1}{\sqrt{2\pi \sigma^2}} \int_{V_j}^{V_{j+1}} V \exp \left[ \frac{-(V - V_j)^2}{2\sigma^2} \right] dV,
\]  

(22)

which can be expressed in terms of the error function.

Noise in the data now comes from two sources: measurement errors, as described by \( \sigma_N \); and finite-sample fluctuations due to the limited number \( n \) of measured velocities. Figure 5 shows how the MISE depends on \( n \) for data sets generated from the velocity distribution

\[
N(V) = \frac{0.5}{\sqrt{2\pi \sigma}} \left( \exp[-(V - V_1)^2/2\sigma^2] + \exp[-(V - V_2)^2/2\sigma^2] \right),
\]  

(23)

with \( V_1 = 9 \text{ km s}^{-1}, V_2 = -9 \text{ km s}^{-1} \) and \( \sigma = 8 \text{ km s}^{-1} \). This flat-topped velocity distribution was designed to mimic \( N(V) \) near the projected center of a globular cluster containing an abundance of nearly-radial orbits. (Examples of such models and their velocity distributions may be found in Merritt 1989).

The two sets of points in Figure 5 correspond to pseudo-data generated with zero velocity errors (circles) and with \( \sigma_N = 4 \text{ km s}^{-1} \) (squares); the latter value is roughly one-third of the intrinsic velocity dispersion. The MISE falls off roughly as \( n^{-0.9} \) for both types of data, almost as steep as the \( n^{-1} \) dependence of parametric estimators. Of course the mean error is larger, at a given \( \alpha \), for the sample with nonzero \( \sigma_N \); however an increase in sample size from \( n = 200 \) to \( n = 300 \) produces the same decrease in the MISE as a reduction in the measurement uncertainties from 4 to 0 km s\(^{-1}\). Thus even relatively large
measurement errors can be overcome by a modest increase in sample size (assuming, of course, that the distribution of errors is well understood). Figure 6 shows the average estimates obtained with the optimal smoothing parameters and their 95% variability bands. The non-Gaussian nature of $N(V)$ is surprisingly well reproduced even for $n = 200$, but the two peaks only begin to be clearly resolved for $n = 1000$.

Various techniques, including a version of the cross-validation score described above, can be used to estimate optimal smoothing parameters for data like these. The “unbiased” or “least-squares” cross-validation score (Scott 1992, p. 166; Silverman 1986, p. 48) is defined as

$$UCV(\alpha) = \int (\hat{N} \circ P)^2 dV - \frac{2}{n} \sum_{i=1}^{n} (\hat{N}^{[i]} \circ P)_i$$

where $\hat{N}^{[i]}$ is an estimate of $N$ obtained by omitting the $i$th velocity. The value of $\alpha$ that minimizes the UCV is an estimate of the value that minimizes the ISE of $\hat{N} \circ P$. Figure 7 shows the dependence of the UCV on $\alpha$ for two data sets, with $n = 200$ and $n = 500$, generated from the velocity distribution of Eq. (23) with $\sigma_N = 0$. The minimum in both curves occurs at a value of $\alpha$ close to the value that actually minimizes the ISE.

The results just presented suggest that increasing the number of measured velocities in globular clusters may be a greater priority than reducing measurement errors if the goal is to determine $N(V)$, since existing techniques can already extract stellar radial velocities with greater precision than the uncertainty $\sigma_N = 4 \text{ km s}^{-1}$ adopted here. Because one would like to estimate $N(V)$ at several different points in an image, the total number of velocities required for a single globular cluster would be in the thousands at least.

4. Application to $\omega$ Centauri

Fortunately, data sets of this size are now becoming available for a number of stellar systems. Here we analyze radial velocities of a new sample of 4200 stars near the center of the globular cluster $\omega$ Centauri. The velocities were measured using the Rutgers Fabry-Perot interferometer on the CTIO 1.5m telescope and were kindly made available by C. Pryor, who also carried out an analysis of the measurement errors. Figure 8 shows the spatial distribution of the observed stars. The effective field of view of the Fabry-Perot is about 2.75 arc minutes in radius and is offset by about 1.1 arc minutes E/SE from the cluster center as determined by Meylan et al. (1995). The core radius of $\omega$ Centauri is around 2.5 arc minutes (Peterson & King 1975) so the observed velocities lie mostly within the projected core.
The core of a globular cluster is perhaps not a very auspicious place to look for non-Gaussian velocity distributions. However \(\omega\) Centauri is fairly young in a collisional-relaxation sense, with an estimated central relaxation time of a few billion years (Meylan et al. 1995). Thus its velocity distribution might still be non-Maxwellian. Furthermore there is evidence for two, chemically distinct populations in \(\omega\) Centauri (Norris et al. 1996) which may have formed at different epochs and hence may have different kinematics. Finally, we note that the theoretical expectation of Maxwellian velocity distributions in globular clusters has rarely been tested by direct determination of line-of-sight velocity distributions. For this reason alone, it seems worthwhile to estimate \(N(V)\) in \(\omega\) Centauri.

About 20 stars were removed from the original sample because their Fabry-Perot line profiles showed contamination by H\(\alpha\) emission. Velocity uncertainties of the remaining stars were estimated using standard procedures (e.g. Gebhardt et al. 1994); a typical estimated error was 3-5 km s\(^{-1}\). Although the estimated errors were found to correlate with stellar magnitude, this fact was ignored in the analysis and \(\sigma_N\) was simply set equal to the rms estimated uncertainty of all the stars in each subsample, about 4 km s\(^{-1}\) in each case. For comparison, the central velocity dispersion of \(\omega\) Centauri is 15-20 km s\(^{-1}\) (Meylan et al. 1995).

One would like to estimate \(N(V)\) independently at many different points in this observed field of view. However Figure 6 suggests that sample sizes less than a few hundred are not very useful for detecting departures from normality. The compromise, as illustrated in Figure 8, was to identify five, partially overlapping fields of one arc minute radius containing about one-half of the observed stars. Two fields lie along the estimated rotation axis of the cluster, at a position angle of 30° east from north with their centers displaced one arc minute from the cluster center. The three additional fields were situated along a perpendicular line. The orientation of the rotation axis was estimated from this sample, but is consistent with a determination based on \(\sim\)500 stars with velocities measured by CORAVEL (Meylan 1996). The offset between the Fabry-Perot field of view and the cluster center was used to advantage by centering the fifth field at a distance of 2.5 arc minutes from the cluster center along the direction of maximal rotation.

The number of stars in each of the subsamples, along with their mean velocity and velocity dispersion (with estimated errors removed) in km s\(^{-1}\), are given in Table 1. The average number of stars per field was 653 for the four inner fields, with 456 stars in Field 5.

The velocity distributions in Fields 1 and 5 exhibit the greatest apparent departures from normality. Figure 9 shows the dependence of the inferred \(N(V)\) in these two fields on the value of the smoothing parameter \(\alpha\). Figure 10 displays estimates of \(N(V)\) in all five fields, for one choice of \(\alpha\), and their estimated 95% confidence bands. The confidence bands
were computed in the usual way via the bootstrap (Scott 1992, p. 259); the choice of $\alpha$ is justified below. Also shown are the normal distributions with the same mean and variance as the inferred $N(V)$’s.

None of the recovered velocity distributions is strikingly non-Gaussian, although the confidence bands in Fields 1 and 5 do barely exclude the normal distribution at one or more points. The inferred $N(V)$ for Field 5 is more centrally peaked than a Gaussian and has what might be described as a tail or bump at large positive velocities. It is tempting to interpret this curve as resulting from the superposition of two normal distributions with different means and variances, though such an interpretation seems extravagant given the relatively small amplitude of the deviations.

The choice of the optimal $\alpha$ for these estimates presented certain difficulties. When dealing with random samples drawn from the normal distribution, the value of $\alpha$ that minimizes the error in an estimate of $N(V)$ would clearly be very large, since an infinite $\alpha$ always returns a normal distribution. Because the velocity distributions in $\omega$ Centauri do appear to be quite Gaussian, we might expect the optimal $\alpha$’s to be “nearly infinite” and hence difficult to estimate from the data alone. In fact the UCV score (Eq. 24) was found not to have a minimum at any value of $\alpha$ in any of these five sub-samples; instead the function $UCV(\alpha)$ was always found to asymptote to a constant value as $\alpha$ was increased. A similar result was obtained using the “likelihood cross-validation” score (Silverman 1986, p. 53). These results certainly do not imply that the distribution of velocities in $\omega$ Centauri must be exactly Gaussian, since cross-validation is not a precise prescription for determining $\alpha$ and often fails to give an extremum. But it appears that these velocity distributions are close enough to Gaussian that the cross-validation technique can not find a significant difference between the estimates made with finite and infinite $\alpha$.

The following alternative scheme was adopted for selecting the optimal $\alpha$. A crude estimate of $N \circ P$ can be obtained by replacing each of the measured velocities by a kernel function of fixed width. If $N \circ P$ is exactly Gaussian, and if the kernel is also Gaussian, the optimal window width (i.e. dispersion) for the kernel may be shown to be

$$h_{opt} = 1.06\sigma n^{-1/5}$$  \hspace{1cm} (25)  

(Silverman 1986, p. 45). Fixed-kernel estimates of $N \circ P$ in each of the five fields were generated from the data using this optimal $h$, and compared to estimates of $N \circ P$ using the penalized-likelihood algorithm with various values of $\alpha$. These comparisons could not be made precise since the fixed-kernel estimates have a larger bias and do not compensate for the velocity measurement errors. Nevertheless it is reasonable to assume that the degree of “roughness” in the optimal kernel-based estimates ought to be similar to that in the
penalized-likelihood estimates made with the optimal value of $\alpha$. The plots of $N(V)$ in Fig. 10 were made using these estimates of the optimal $\alpha$’s.

We conclude from this analysis that the evidence for non-Gaussian velocity distributions in our five fields near the center of $\omega$ Centauri is marginal at best. We note that the strongest deviations appear in the field that is farthest from the center. Perhaps a sample of stars even farther from the core – where the relaxation time exceeds the age of the universe – would show even larger departures from normality.

The $\omega$ Centauri radial velocities analyzed here were obtained by K. Gebhardt, J. Hesser, C. Pryor and T. Williams using the Rutgers Fabry-Perot interferometer. C. Pryor devoted considerable time to reducing the observations and estimating the measurement uncertainties in this sample. Conversations with C. Joseph, M. Merrifield, H.-W. Rix, P. Saha, R. van der Marel and T. Williams were very helpful for understanding the ins and outs of spectral deconvolution. C. Pryor read parts of the manuscript and made useful suggestions for improvements. This work was supported by NSF grant AST 90-16515 and NASA grant NAG 5-2803.
Table 1: Fields in ω Centauri

| Field | N  | ⟨V⟩ | σ   |
|-------|----|-----|-----|
| 1     | 655| 0.  | 16.0|
| 2     | 670| -2.7| 16.7|
| 3     | 624| 0.1 | 16.8|
| 4     | 663| 2.2 | 16.6|
| 5     | 456| 6.4 | 14.6|
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Fig. 1.—

Bias-variance tradeoff in estimates of non-Gaussian $N(V)$’s from absorption line spectra. Input spectra were generated using two different broadening functions (thin curves) with added noise of amplitude $S/N=20$. (a) - (c): Merrifield-Kuijken broadening function, Eq. (11); (d) - (f): broadening function of Eq. (12). $N(V)$ was then estimated (thick curves) using three different values of $\alpha$. (a), (d): undersmoothed; (b), (e): optimally smoothed; (c), (f): oversmoothed. In the limit of large $\alpha$, i.e. infinite smoothing, the estimates tend toward a Gaussian with approximately the same mean and dispersion as the true $N(V)$.

Fig. 2.—

Dependence of the mean integrated square error of the recovered broadening function on the signal-to-noise ratio of the spectrum, for spectra generated from the Merrifield-Kuijken broadening function (Eq. (11). For each value of $S/N$, 300 noise realizations of the same spectrum were generated and the value of $\alpha$ that minimized the average square deviation between the true and estimated $N(V)$’s was found. The ordinate is the MISE of the estimates using this optimal $\alpha$.

Fig. 3.—

Average estimates of $N(V)$, and their 95% variance bands, based on 300 noise realizations of spectra generated from the Merrifield-Kuijken broadening function (11). As the signal-to-noise ratio increases, the average estimate tends toward the true broadening function and the variance of the estimates decreases.

Fig. 4.—

Dependence of the cross-validation score (CV) on the smoothing parameter $\alpha$ for two spectra generated using the Merrifield-Kuijken broadening function (11), with $S/N = 10$ and $S/N = 5$. The value of $\alpha$ at the minimum of the CV curve is an estimate of the value of the smoothing parameter that minimizes the integrated square error of the estimated $N(V)$. Arrows indicate the values of $\alpha$ that actually minimize the ISE of the broadening functions inferred from these two spectra.

Fig. 5.—

Dependence of the mean integrated square error of the recovered $N(V)$ on the number of velocities $n$, for pseudo-data generated from the flat-topped velocity distribution of Eq. (23). Squares: $\sigma_N = 4$ km s$^{-1}$; circles: $\sigma_N = 0$. 
Fig. 6.—

Average estimates of $N(V)$, and their 95% variance bands, based on 300 samples of discrete velocities generated from the flat-topped velocity distribution of Eq. (23).

Fig. 7.—

Dependence of the unbiased cross validation score (UCV) on the smoothing parameter $\alpha$ for data generated from the velocity distribution of Eq. (23), with $\sigma_N = 0$. The value of $\alpha$ at the minimum of the UCV curve is an estimate of the value that minimizes the integrated square error of $\hat{N}(V)$. Arrows indicate the values of $\alpha$ that actually minimize the ISE of the velocity distributions inferred from these two data sets.

Fig. 8.—

Map of stars with observed radial velocities in the globular cluster $\omega$ Centauri. The origin of the coordinate system is the center of the cluster as defined by Meylan & Mayor (1986); major tick marks are separated by one arc minute. The Fabry-Perot field of view is offset by approximately 1.1 arc minute from the origin. The dashed line is an estimate of the projected rotation axis of the cluster. Velocity distributions were computed for stars in the five circular fields shown.

Fig. 9.—

Penalized-likelihood estimates of $N(V)$ in two fields in $\omega$ Centauri. The value of the smoothing parameter $\alpha$ increases from (a) to (c). Thin lines are normal $N(V)$’s with the same mean velocity and velocity dispersion as the estimated $N(V)$’s.

Fig. 10.—

Penalized-likelihood estimates of $N(V)$ in five fields in $\omega$ Centauri. Heavy solid lines are the estimates; dashed lines are 95% bootstrap confidence bands; thin solid lines are the normal distributions with the same mean velocity and velocity dispersion as the estimated $N(V)$’s. Arrows indicate mean velocities.
