A class of difference schemes uniformly convergent on a modified Bakhvalov mesh

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Abstract

In this paper we consider the numerical solution of a singularly perturbed one-dimensional semilinear reaction-diffusion problem. We construct a class of finite-difference schemes to discretize the problem and we prove that the discrete system has a unique solution. The central result of the paper is second-order convergence uniform in the perturbation parameter, which we obtain for the discrete approximate solution on a modified Bakhvalov mesh. Numerical experiments with two representatives of the class of difference schemes show that our method is robust and confirm the theoretical results.

Keywords: Singular perturbation, nonlinear, boundary layer, Bakhvalov mesh, layer-adapted mesh, uniform convergence.

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1. Introduction

We consider the boundary value problem

\[ \varepsilon^2 y''(x) = f(x, y) \text{ on } [0, 1], \]
\[ y(0) = 0, \quad y(1) = 0, \]

where \( 0 < \varepsilon < 1 \) is a perturbation parameter and \( f \) is a non-linear function. We assume that the nonlinear function \( f \) is continuously differentiable, i.e. for \( k \geq 2, \ f \in C^k([0, 1] \times \mathbb{R}), \) and that it has a strictly positive derivative with respect to \( y \)

\[ \frac{\partial f}{\partial y} = f_y \geq m > 0 \text{ on } [0, 1] \times \mathbb{R} \ (m = \text{const}). \]
The boundary value problem (1.1)–(1.2) under the condition (1.3) has a unique solution, see Lorenz [21]. Differential equations with the small parameter $\varepsilon$ multiplying the highest order derivative terms are said to be singularly perturbed.

Singularly perturbed equations occur frequently in mathematical models of various areas of physics, chemistry, biology, engineering science, economics and even sociology. These equations appear in analysis of practical applications, for example in fluid dynamics (aero and hydrodynamics), semiconductor theory, advection-dominated heat and mass transfer, theory of plates, shells and chemical kinetics, seismology, geophysics, nonlinear mechanics and so on.

A common features of singularly perturbed equations is that their solutions have tiny boundary or interior layers, in which there is a sudden change of the solution’s values of these equations. Such sudden changes occur e.g. in physics when viscous gas flows at high speed and has contact with a solid surface, then in chemical reaction, in which besides the reactants, a catalyst is also involved.

Using classical numerical methods such are finite difference methods and finite element methods, which do not take into account the appearance of the boundary or inner layer, we get results which are unacceptable from the standpoint of stability, the value of the error or the cost of calculation.

Our goal is to construct a numerical method to overcome the previously listed problems, i.e., to construct an $\varepsilon$–uniformly convergent numerical method for problem (1.1)–(1.3).

The numerical method is said to be an $\varepsilon$–uniformly convergent in the maximum discrete norm of the order $r$, if

$$
\| y - \bar{y} \|_\infty \leq CN^{-r},
$$

where $y$ is the exact solution of the original continuous problem, $\bar{y}$ is the numerical solution of a given continuous problem, $N$ is the number of mesh points, and $C$ is a constant which does not depend of $N$ nor $\varepsilon$.

Many authors have analyzed and made a great contribution to the study of the problem (1.1)–(1.3) with different assumptions about the function $f$; and as well as more general nonlinear problems.

There were many constructed $\varepsilon$–uniformly convergent difference schemes of order 2 and higher (Herceg [6], Herceg and Surla [11], Herceg and Miloradović [10], Herceg and Herceg [7], Kopteva and Linß [15], Kopteva and Stynes [17, 18], Kopteva, Pickett and Purtill [16], Linß, Roos and Vulanović [20], Sun and Stynes [30, 31], Stynes and Kopteva [29], Surla and Uzelac [33], Vulanović [34, 35, 36, 37, 38, 39], Kopteva [14] etc.).

The numerical method which we are going to construct and analyze in this paper is a synthesis of the two approaches in numerical solving the singular perturbation boundary value problem and the use of a layer–adaptive mesh. As mentioned above, the exact solutions of the singular perturbation boundary value problems usually exhibits sharp boundary or interior layers.

The first approach in numerical solving the singular perturbation boundary value problem is a method of fitted operators. Construction and analysis of these exponentially fitted differences schemes for solving linear singular–perturbation problems can be seen in Roos [26], O’Riordan and Stynes [23] etc, while the appropriate schemes for nonlinear problems can be seen in Nijima [22], O’Riordan and Stynes [24], Stynes [28] and others. The above mentioned fitted exponential difference schemes are uniformly convergent. In order to obtain an $\varepsilon$–uniformly convergent method, we need to use a appropriate layer-adapted mesh.

Shishkin mesh [27] and their modification [32, 39, 19] and others, Bakhvalov mesh [1] and their modification [6, 12, 9, 10, 34, 37] and others are the most used layer–adapted meshes.

The method, appropriate for our purpose, was first presented by Boglaev [2], where the discretisation of the problem (1.1)–(1.3) on a modified Bakhvalov mesh was analysed and first order uniform convergence with respect to $\varepsilon$ was demonstrated.
Using the method of [2], authors constructed new difference schemes in papers [3] and [4] for the problem (1.1)–(1.3) and carried out numerical experiments. In [5, 13] authors constructed new difference schemes and proved the uniqueness of the numerical solution and an $\varepsilon$–uniform convergence on a modified Shishkin mesh, and at the end presented numerical experiments.

In order to obtain better results, instead of Shishkin mesh, we will use a modification of Bakhvalov mesh. We have decided to use the modification of Bakhvalov mesh constructed by Vulanović [37]. This mesh has the features that we need in our analysis of the numerical method value of the error.

Shishkin mesh is much simpler than Bakhvalov mesh, but difference schemes applied to Bakhvalov mesh show better results. In order to get better results we used a modification of Bakhvalov mesh.

This paper consists of six parts and it has the following structure. The first part is Introduction. Next, in Section 2 a class of difference schemes are constructed, and it is proven the theorem of existence and uniqueness of the numerical solution. Mesh construction is in Section 3. In Section 4, it is showed and proven the theorem of $\varepsilon$–uniform convergence. In Section 5 are numerical experiments which confirm the theoretical results. The last two sections are Conclusion and Acknowledgments.

We use $\mathbb{R}^{N+1}_{\ast}$ to denote the real $(N+1)$–dimensional linear space of all column vectors

$$u = (u_0, u_1, \ldots, u_N)^T.$$

We equip space $\mathbb{R}^{N+1}$ with usual maximum vector norm

$$\|u\|_\infty = \max_{0 \leq i < N} |u_i|.$$ 

The induced norm of a linear mapping $A = (a_{ij}) : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is

$$\|A\|_\infty = \max_{0 \leq i < N} \sum_{j=0}^N |a_{ij}|.$$ 

Remark 1.1. Throughout this paper we let $C$, sometimes subscripted, denote a generic positive constant that may take different values in different formulas, but it is always independent of $N$ and $\varepsilon$.

2. Construction of the scheme

We will use the well–known Green’s function for the operator $L_{\varepsilon}y := \varepsilon^2 y'' - \gamma y$, for the construction of the difference scheme, where $\gamma$ is a constant. The value of $\gamma$ will be determined later in this section.

This method, as we mentioned in Introduction, first was introduced by Boglaev in his paper [2]. Detailed construction of difference schemes done by this method, can be found in [5, 13]. In [13] was obtained the following equality

$$\beta \frac{y_{i-1}}{\sinh(\beta h_{i-1})} - \left( \frac{\beta}{\tanh(\beta h_{i-1})} + \frac{\beta}{\tanh(\beta h_i)} \right) y_i + \frac{\beta}{\sinh(\beta h_i)} y_{i+1} = \frac{1}{\varepsilon^2} \left[ \int_{x_{i-1}}^{x_i} u_{i-1}^H(s) \psi(s, y(s)) ds + \int_{x_i}^{x_{i+1}} u_i^H(s) \psi(s, y(s)) ds \right],$$

where

$$y_0 = 0, \quad y_N = 0, \quad i = 1, 2, \ldots, N - 1,$$
\[ \psi(x, y(x)) = f(x, y(x)) - \gamma y(x), \]
\[ 0 = x_0 < x_1 < x_2 < \cdots < x_N = 1, \] (2.2)
is an arbitrary mesh on \([0, 1]\), \(h_i = x_{i+1} - x_i\)
\[ \beta = \frac{\sqrt{\gamma}}{\varepsilon}, \] (2.3)
functions \(u_i^I\) and \(u_i^{II}\) are the solutions of the next boundary value problem
\[
L_i y = 0 \text{ on } (x_i, x_{i+1}), \quad \text{and} \quad L_i y = 0 \text{ on } (x_i, x_{i+1}),
\]
\[ u_i(x_i) = 1, \quad u_i(x_{i+1}) = 0, \quad i = 0, 1, \ldots, N - 1, \]
\[ u_i(x_i) = 0, \quad u_i(x_{i+1}) = 1, \quad i = 0, 1, \ldots, N - 1. \]
\[
\begin{align*}
\quad i = 0, 1, 2, \ldots, N - 1. \\
\end{align*}
\]
We cannot, in general, explicitly compute the integrals on the right-hand side of (2.1). In order to get a simple enough difference scheme, we approximate the function \(\psi\) on \([x_{i-1}, x_i] \cup [x_i, x_{i+1}]\) using
\[ \frac{\psi(x_{i-1}, \bar{y}_{i-1}) + q\psi(x_i, \bar{y}_i) + \psi(x_{i+1}, \bar{y}_{i+1})}{q + 2}, \] (2.4)
where \(q \in \mathbb{R}^+\), while \(\bar{y}_i\) are approximate values of the solution \(y\) of the problem (1.1)–(1.3) at mesh points \(x_i\).
Finally, from (2.1), using (2.4), we get the following difference scheme
\[
\begin{align*}
[(q + 1)a_i + d_i + \Delta d_{i+1}] (\bar{y}_{i-1} - \bar{y}_i) - [(q + 1)a_{i+1} + d_{i+1} + \Delta d_i] (\bar{y}_i - \bar{y}_{i+1})
- \frac{f(x_{i-1}, \bar{y}_{i-1}) + qf(x_i, \bar{y}_i) + f(x_{i+1}, \bar{y}_{i+1})}{\gamma} (\Delta d_i + \Delta d_{i+1}) &= 0, \\
\gamma \bar{y}_0 &= 0, \quad \bar{y}_N = 0, \quad i = 1, 2, \ldots, N - 1, \\
\end{align*}
\] (2.5)
where \(a_i = \frac{1}{\sinh(\beta h_{i-1})}, \quad d_i = \frac{1}{\tanh(\beta h_{i-1})}, \quad \Delta d_i = d_i - a_i.\)
Let us introduce the discrete problem of the problem (1.1)–(1.3), using (2.5) on the mesh (2.2) we can write
\[
F \bar{y} = (F_0 \bar{y}, F_1 \bar{y}, \ldots, F_N \bar{y})^T = 0, \] (2.6)
where are
\[
\begin{align*}
F_0 \bar{y} := \bar{y}_0 &= 0, \\
F_i \bar{y} &:= \gamma \frac{\bar{y}}{\Delta d_i + \Delta d_{i+1}} \left[ (q + 1)a_i + d_i + \Delta d_{i+1} \right] (\bar{y}_{i-1} - \bar{y}_i) \\
&- [(q + 1)a_{i+1} + d_{i+1} + \Delta d_i] (\bar{y}_i - \bar{y}_{i+1}) \\
&- \frac{f(x_{i-1}, \bar{y}_{i-1}) + qf(x_i, \bar{y}_i) + f(x_{i+1}, \bar{y}_{i+1})}{\gamma} (\Delta d_i + \Delta d_{i+1}) \right], \\
F_N \bar{y} &:= \bar{y}_N = 0. \\
\end{align*}
\] (2.7)
Theorem 2.1. The discrete problem (2.6) for $\gamma \geq f_{\gamma}$, has the unique solution $\overline{y}$, where $\overline{y} = (\overline{y}_0, \overline{y}_1, \overline{y}_2, \ldots, \overline{y}_{N-1}, \overline{y}_N)^T \in \mathbb{R}^{N+1}$. Moreover, for any $v, w \in \mathbb{R}^{N+1}$, the following stability inequality holds
\[
\|w - v\|_{\infty} \leq \frac{1}{m} \|Fw - Fv\|_{\infty} .
\] (2.8)

Proof. We use a technique from [10, 37], the proof of existence of the solution of $F\overline{y} = 0$ is based on the proof of the following relation: $\|(F')\overline{y}\|_{\infty} \leq C$, where $F'\overline{y}$ is a Fréchet derivative of $F$.

The Fréchet derivative $H := F'\overline{y}$ is a tridiagonal matrix. Let $H = [h_{ij}]$. The non-zero elements of this tridiagonal matrix are
\[
h_{0,0} = h_{N,N} = 1, \\
h_{ij} = \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ -q(a_i + a_{i+1}) - 2(d_i + d_{i+1}) - \frac{q}{\gamma} \frac{\partial f(v_i, y_i)}{\partial y_i} (\Delta d_i + \Delta d_{i+1}) \right] < 0, \\
h_{i,i-1} = \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ (\Delta d_i + \Delta d_{i+1}) \left( 1 - \frac{1}{\gamma} \frac{\partial f(v_i, y_i)}{\partial y_i} \right) + (q + 2)a_i \right] > 0, \\
h_{i,i+1} = \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ (\Delta d_i + \Delta d_{i+1}) \left( 1 - \frac{1}{\gamma} \frac{\partial f(v_i, y_i)}{\partial y_i} \right) + (q + 2)a_{i+1} \right] > 0, \\
i = 1, 2, \ldots, N - 1.
\] (2.9)

Hence $H$ is an $L$-matrix. Let us show that $H$ is an $M$-matrix. Now, we have
\[
|h_{ii} - |h_{i,i-1} - |h_{i-1,i}|
\]
\[
= \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ (\Delta d_i + \Delta d_{i+1}) \left( \frac{\partial f(v_i, y_i)}{\partial y_i} + \frac{q}{\gamma} \frac{\partial f(v_i, y_i)}{\partial y_i} + \frac{\partial f(v_i, y_i)}{\partial y_i} \right) \right] \geq (q + 2)m. \] (2.10)

Based on (2.10), we have proved that $H$ is an $M$-matrix. Since $H$ is an $M$-matrix, now we obtain
\[
\|H^{-1}\|_{\infty} \leq \frac{1}{(q + 2)m}. \] (2.11)

Finally, by the Hadamard Theorem (5.3.10 from [25]), the first statement of our theorem follows.

The second part of the proof is based on the part of the proof of [6]. We have that
\[
Fw - Fv = (F' u)(w - v), \text{ for some } u = (u_0, u_1, \ldots, u_N)^T \in \mathbb{R}^{N+1}.
\] (2.12)

and
\[
w - v = (F' u)^{-1} (Fw - Fv) . \] (2.13)

Now, based on (2.11), we have that
\[
\|w - v\|_{\infty} = \|(F' u)^{-1} (Fw - Fv)\|_{\infty}
\]
\[
\leq \frac{1}{(q + 2)m} \|Fw - Fv\|_{\infty} \leq \frac{1}{m} \|Fw - Fv\|_{\infty}. \] (2.14)

□
3. Mesh construction

The exact solution $y$ of the problem (1.1)–(1.2) has boundary layers of exponential type near the points $x = 0$ and $x = 1$. In order to achieve an $\varepsilon$–uniformly convergence of the numerical method, it is necessary to use a layer-adapted mesh. In the construction of this mesh the occurrence of boundary or inner layers needs to be taken into account. We will use the modified Bakhvalov mesh from [37], which has a sufficiently smooth generating function, that is, going to provide the necessary characteristics of the mesh that we need for further analysis.

The mesh $\Delta : x_0 < x_1 < \ldots < x_N$ is generated by $x_i = \varphi(t_i)$, $t_i = i/N = ih$, $h = 1/N$, $i = 0, 1, \ldots, N; N = 2m$, $m \in \mathbb{N}\setminus\{1\}$, with the mesh generating function

$$
\varphi(t) = \begin{cases} \kappa(t) := \frac{\mu(t)}{p-1}, & t \in [0, \alpha], \\
\pi(t) := \omega(t-\alpha)^3 + \frac{\kappa''(\alpha)(t-\alpha)^2}{2} + \kappa'(\alpha)(t-\alpha) + \kappa(\alpha), & t \in [\alpha, 1/2], \\
1 - \varphi(1-t), & t \in [1/2, 1], \end{cases}
$$

(3.1)

here $p$ is an arbitrary parameter from $(\varepsilon^*)^{1/3}, 1/2$, $\varepsilon \in (0, \varepsilon^*)$ and $\alpha = p - \varepsilon^{1/3} > 0$, where we assume that $\varepsilon^* < \frac{1}{3}$. The coefficient $\omega$ is determined from $\pi\left(\frac{1}{2}\right) = \frac{1}{2}$, we get

$$
\omega = \left(\frac{1}{2} - \alpha\right)^{-3} \left\{ \frac{1}{2} - a \left[ p \left(\frac{1}{2} - \alpha\right)^2 + p \left(\frac{1}{2} - \alpha\right) \varepsilon^{1/3} + \alpha \varepsilon^{2/3} \right] \right\},
$$

and $a$ is chosen such that $\omega > 0$, (such $a$, independent of $\varepsilon$, obviously exist).

By this choice of $\alpha$ and $\pi$ we get

$$
\varphi \in C^2\left(\left[0, 1\right] \setminus \left\{ \frac{1}{2} \right\}\right),
$$

(3.2a)

$$
|\varphi'(t)| \leq C, \ t \in [0, 1],
$$

(3.2b)

and

$$
|\varphi''(t)| \leq C, \ t \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}.
$$

(3.2c)

Values of the mesh sizes $h_i$, and values of differences $h_{i+1} - h_i$, will be given in the next lemma.

**Lemma 3.1.** The mesh sizes $h_i = x_{i+1} - x_i$, defined by the generating function (3.1), satisfy

$$
h_i \leq CN^{-1}, \ i = 0, 1, \ldots, N - 1,
$$

(3.3a)

and

$$
|h_i - h_{i-1}| \leq CN^{-2}, \ i = 1, 2, \ldots, N - 1.
$$

(3.3b)

**Proof.** Due to (3.2b), we have

$$
h_i = \int_{i/N}^{(i+1)/N} \varphi'(t) \ dt \leq C \int_{i/N}^{(i+1)/N} \varphi'(t) \ dt \leq CN^{-1}.
$$

(3.4)

Let us divide the proof of (3.3b), based on (3.2c), into three parts.

Firstly, when $i \in \{1, \ldots, N-1\} \setminus \{N/2 - 1, N/2, N/2 + 1\}$, based on (3.2c), we have

$$
|h_i - h_{i-1}| = \int_{i/N}^{(i+1)/N} \int_{i-1/N}^{t} \varphi''(s) \ ds \ dt \leq C \int_{i/N}^{(i+1)/N} \int_{i-1/N}^{t} \ ds \ dt \leq CN^{-2}.
$$

(3.5)

Secondly, for $i = N/2$, we get
\[ |h_{N/2} - h_{N/2-1}| = 1 - \varphi(1 - (N/2 + 1)/N) - \frac{1}{2} - \left(\frac{1}{2} - \varphi((N/2 - 1)/N)\right)\]
\[ = \varphi((N/2 - 1)/N) - \varphi(1 - (N/2 + 1)/N)\]
\[ = 0,\]  
and finally, for \( i = N/2 - 1 \) or \( i = N/2 + 1 \), we get
\[ |h_{N/2-1} - h_{N/2-2}| = |h_{N/2+1} - h_{N/2}| \leq \frac{6\omega}{N^3} + \frac{\mu''(\alpha) + (3 - 6\alpha)\omega}{N^2} \leq \frac{C}{N^2}. \]  

Now, using (3.4), (3.5), (3.6) and (3.7) the inequalities (3.3a) and (3.3b) are proven. \( \square \)

4. Uniform convergence

In this section we prove the theorem on \( \varepsilon \)-uniform convergence of the discrete problem (2.6). The proof of the theorem is based on relation \( \|y - \overline{y}\|_\infty \leq C \|Fy - F\overline{y}\|_\infty \). Stability of the difference scheme is proven in Theorem 2.1, and as \( F\overline{y} \equiv 0 \), it is enough to estimate the value of the expression \( \|Fy\|_\infty \).

The proof uses the decomposition of the solution \( y \) to the problem (1.1)–(1.2) to a layer \( s \) and a regular component \( r \), given in the following assertion.

**Theorem 4.1.** [34] The solution \( y \) to problem (1.1)–(1.2) can be represented in the following way:

\[ y = r + s, \]

where for \( j = 0, 1, \ldots, k + 2 \) and \( x \in [0, 1] \) we have

\[ |x^{(j)}(x)| \leq C, \]  
and

\[ |s^{(j)}(x)| \leq Ce^{-j}(e^{\frac{k}{2} \sqrt{m}} + e^{-\frac{1-k}{2} \sqrt{m}}). \]  

**Proof.** See in Vulanović [34]. \( \square \)

Note that \( e^{-\frac{k}{2} \sqrt{m}} \geq e^{-\frac{1-k}{2} \sqrt{m}}, \forall x \in [0, 1/2] \) and \( e^{-\frac{k}{2} \sqrt{m}} \leq e^{-\frac{1-k}{2} \sqrt{m}}, \forall x \in [1/2, 1] \). These inequalities and the estimate (4.2) imply that the analysis of the error value can be done for \( \|Fy\|_\infty \) on the part of the mesh which corresponds to \([0, 1/2]\) omitting the function \( e^{-\frac{k}{2} \sqrt{m}} \), keeping in mind that on this part of the mesh we have that \( h_{i-1} \leq h_i \). An analogous analysis holds for the part of the mesh which corresponds to \( x \in [1/2, 1] \), but with the omission of the function \( e^{-\frac{k}{2} \sqrt{m}} \) and using the inequality \( h_{i-1} \geq h_i \).

In order to simplify our analysis, let us write \( F_i y, i = 1, 2, \ldots, N - 1 \), in the following form

\[ F_i y = y (y_{i-1} - 2y_i + y_{i+1}) + \frac{a_i(y_{i-1} - y_i) - a_{i+1}(y_i - y_{i+1}) - f(x_i, y_i)}{\Delta d_i + \Delta d_{i+1}} \]
\[ + \frac{\Delta d_i + \Delta d_{i+1}}{\Delta d_i + \Delta d_{i+1}} \]
\[ + 2\gamma \frac{a_i(y_{i-1} - y_i) - a_{i+1}(y_i - y_{i+1})}{\Delta d_i + \Delta d_{i+1}} \]
\[ - \frac{f(x_{i-1}, y_{i-1}) + f(x_{i+1}, y_{i+1})}{\Delta d_i + \Delta d_{i+1}} \]
\[ i = 1, \ldots, N - 1. \]  

\[ \square \]
Using marks $P_i$, $Q_i$, $R_i$, we have

$$F_i y = P_i + Q_i + R_i, \quad i = 1, 2, \ldots, N - 1, \tag{4.4}$$

where

$$P_i = \gamma (y_{i-1} - 2y_i + y_{i+1}), \tag{4.5}$$

and using Taylor expansions for $y_{i-1}$ and $y_{i+1}$, we get

$$Q_i = \gamma q \left( \frac{h_i}{2} \frac{h_i \sinh(\beta h_{i-1}) - h_{i-1} \sinh(\beta h_i)}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} + \frac{y_i'}{2} + \frac{y_i''}{6} \right)$$

$$+ \gamma y_i''' \left[ \frac{1}{2} \frac{h_i^2}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} - \frac{1}{\beta^2} \frac{h_i^2}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right]$$

and

$$R_i = 2\gamma y_i' \frac{h_i \sinh(\beta h_{i-1}) - h_{i-1} \sinh(\beta h_i)}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)}$$

$$+ 2\gamma y_i' \left( \frac{1}{2} \frac{h_i^2}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} + \frac{h_i^2}{\frac{1}{\beta^2} \sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right)$$

$$+ \gamma y_i''' \left[ \frac{3}{2} \frac{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right]$$

$$+ \gamma y_i''' \frac{y_i'\left(\zeta_{i-1}^- \mu_{i-1}^- h_i^2 \sinh(\beta h_i) + y_i'' \left(\zeta_{i-1}^- \mu_{i-1}^- \right) h_i^2 \sinh(\beta h_i) \right)}{24}$$

$$+ 2\gamma \frac{y_i'\left(\zeta_i^+ \mu_i^+ \right) h_i^2 \sinh(\beta h_i)}{24} + \frac{y_i'' \left(\zeta_i^+ \mu_i^+ \right) h_i^2 \sinh(\beta h_i)}{24}$$

$$- \frac{c}{2} \left[ y_i' \left(\mu_{i-1}^- h_i^2 \sinh(\beta h_i) + y_i'' \left(\mu_{i-1}^- \right) h_i^2 \sinh(\beta h_i) \right) \right]$$

$i = 1, 2, \ldots, N - 1$ and $\zeta_{i-1}^-, \mu_{i-1}^- \in (x_{i-1}, x_i), \ z_i^+, \mu_i^+ \in (x_i, x_{i+1})$.

**Lemma 4.2.** Let the mesh size $h_i = x_{i+1} - x_i$ be defined by the generating function (3.1). It holds the estimate

$$\left| y_{i-1} - y_i - (y_i - y_{i+1}) \right| \leq C \left( \left| y_i' \right| \left( h_i - h_{i-1} \right) + \frac{\left| y_i'' \right|}{2} h_{i-1}^2 + \frac{\left| y_i''' \right|}{2} h_i^2 \right), \tag{4.8}$$

where are $\delta_{i-1} \in (x_{i-1}, x_i), \delta_i^+ \in (x_i, x_{i+1})$.

**Proof.** The proof is trivial, using Taylor expansions for $y_{i-1}$ and $y_{i+1}$ we obtain (4.8). □
Lemma 4.3. Let the mesh size $h_i = x_{i+1} - x_i$, be defined by the generating function (3.1) and $\beta$ defined by (2.3). It holds estimate

$$|y_i'| \leq C \frac{|h_i \sinh(\beta h_i) - h_{i-1} \sinh(\beta h_{i-1})|}{\sinh(\beta h_i) \cosh(\beta h_{i-1}) - 1 + \sinh(\beta h_{i-1}) \cosh(\beta h_i) - 1} \leq C |y_i'| (h_i - h_{i-1}), \quad i = 1, \ldots, N/2. \quad (4.9)$$

Proof. We have that

$$|y_i'| = \frac{|h_i \sinh(\beta h_i) - h_{i-1} \sinh(\beta h_{i-1})|}{\sinh(\beta h_i) \cosh(\beta h_{i-1}) - 1 + \sinh(\beta h_{i-1}) \cosh(\beta h_i) - 1}$$

$$= \sum_{n=1}^{+\infty} \frac{\beta_n h_{i-1} h_i}{(2n+1)!} \sinh(\beta h_i) \sinh(\beta h_{i-1}) \sum_{n=0}^{+\infty} \frac{\beta_n h_{i-1-n}}{(2n+2)!} \sinh(\beta h_{i-1-n}) \cosh(\beta h_{i-1-n}) - 1 + \sinh(\beta h_{i-1-n}) \cosh(\beta h_i) - 1$$

$$\leq 2 |y_i'| \frac{\beta h_{i-1} h_i (h_i - h_{i-1})}{4 \sinh(\beta h_i/2) \sinh(\beta h_{i-1}/2) \sinh(\beta h_{i-1})/2}$$

$$= 2 |y_i'| \frac{\beta h_{i-1} h_i (h_i - h_{i-1})}{4 \sinh(\beta h_i/2) \sinh(\beta h_{i-1}/2) \sinh(\beta h_{i-1})/2} \leq C |y_i'| (h_i - h_{i-1}). \quad (4.10)$$

Remark 4.4. It is true that $\sum_{n=0}^{+\infty} \frac{\beta_n h_{i-1-n}}{(2n+2)!} = \frac{\cosh x - 1}{x} \cosh x - 1 = 2 \sinh^2 \frac{x}{2}$

and $\sinh x(\cosh y - 1) + \sinh y(\cosh x - 1) = 4 \sinh \frac{x}{2} \sinh \frac{y}{2} \sinh \frac{x+y}{2}$. 

Lemma 4.5. Let the mesh size $h_i = x_{i+1} - x_i$, be defined by the generating function (3.1) and $\beta$ defined by (2.3). We have the following estimate

$$|y_i''| \leq C \frac{h_i^2 \sinh(\beta h_i) + h_{i-1}^2 \sinh(\beta h_{i-1})}{\sinh(\beta h_i) \cosh(\beta h_{i-1}) - 1 + \sinh(\beta h_{i-1}) \cosh(\beta h_i) - 1} \leq C |y_i''| h_i^2, \quad i = 1, \ldots, N/2. \quad (4.11)$$
Proof. We have that

\[ \left| y_i'' \right| = \left| y_i'' \right| \frac{1}{2} \left[ \frac{h_i^3 \sinh(\beta h_i) + h_i^2 \sinh(\beta h_{i-1})}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)} \right] \]

\[ = \left| y_i'' \right| \left[ \frac{\sinh(\beta h_i) \left( \frac{h_i^3}{2} - \frac{\cosh(\beta h_i) - 1}{\beta^3} \right) + \sinh(\beta h_{i-1}) \left( \frac{h_i^2}{2} - \frac{\cosh(\beta h_i) - 1}{\beta^2} \right)}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right] \]

\[ = \left| y_i'' \right| \left[ \frac{h_i^2 \sinh(\beta h_i) \left( \frac{\beta^2 h_i^2}{4!} + \frac{\beta h_i^4}{6!} + \cdots \right) + \sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1) \right] \]

\[ \leq \left| y_i'' \right| \frac{h_i^2}{2} \left[ \frac{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right] \]

\[ \leq \left| y_i'' \right| \frac{h_i^2}{2} \left[ \frac{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \right] \]

\[ \leq C \left| y_i''' \right| h_i. \]

\[ \square \]

Remark 4.6. It is true that \( \sum_{n=0}^{+\infty} \frac{x^{2n+2}}{(2n+4)!} = \frac{\cosh x - 1 - \frac{x^2}{2}}{x^2} \), \( 0 < \frac{\cosh x - 1 - \frac{x^2}{2}}{x^2} < 1 \), and \( \frac{\cosh x - 1 - \frac{x^2}{2}}{x^2} \leq C(\cosh x - 1) \).

Lemma 4.7. Let the mesh size \( h_i = x_{i+1} - x_i \) be defined by the generating function (3.1) and \( \beta \) defined by (2.3). We have the following estimate

\[ \left| y_i'''' \right| \left[ \frac{h_i^3 \sinh(\beta h_i) - h_i^3 \sinh(\beta h_{i-1})}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)} \right] \]

\[ \leq C \left| y_i''' \right| \left( t^2 + h_i^2 \right) (h_i - h_{i-1}), \quad i = 1, \ldots, N/2. \]
Proof. We have that

\[
\begin{align*}
|y''_i| & \leq \left| y''_i \right| \frac{h_i^3 \sinh(\beta h_i) - h_{i-1}^3 \sinh(\beta h_{i-1})}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)} \\
& \leq \left| y''_i \right| \frac{\beta h_i^3 h_{i-1}(h_i + h_{i-1})h_{i-1}}{\beta h_i^3 h_{i-1}^3 + \beta h_{i-1}^3 \frac{\beta h_{i-1}}{2}} \frac{\beta^3 h_{i-1}^3 h_i^3 (h_i^2 - h_{i-1}^2)}{\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1)} \\
& \leq \left| y''_i \right| \frac{\beta h_i^3 h_{i-1}(h_i - h_{i-1})h_{i-1}}{4 \sinh(\frac{\beta h_i}{2}) \sinh(\frac{\beta h_{i-1}}{2}) \sinh(\frac{\beta(h_i + h_{i-1})}{2})} \\
& \leq \left| y''_i \right| \frac{\beta h_i^3 (h_i - h_{i-1})}{4 \sinh(\frac{\beta h_i}{2}) \sinh(\frac{\beta h_{i-1}}{2}) \sinh(\frac{\beta(h_i + h_{i-1})}{2})} \\
& \leq \left| y''_i \right| \frac{\beta h_i^3 (h_i - h_{i-1})}{4 \sinh(\frac{\beta h_i}{2}) \sinh(\frac{\beta h_{i-1}}{2}) \sinh(\frac{\beta(h_i + h_{i-1})}{2})} \cdot \frac{\cosh(\beta h_i) - 1 - \frac{\beta^2 h_i^2}{2}}{2} \\
& \leq C \left| y''_i \right| \left( \varepsilon^2 + h_{i-1}^2 \right) |h_i - h_{i-1}|.
\end{align*}
\]

(4.14)

Remark 4.8. It is true that \(\sum_{n=0}^{\infty} \frac{\beta^{2n+4} h_i^{2n+4}}{(2n+4)!} = \cosh(\beta h_i) - 1 - \frac{\beta^2 h_i^2}{2}\) and

\[
0 < \frac{\cosh(\beta h_i) - 1 - \frac{\beta^2 h_i^2}{2}}{\sinh(\frac{\beta h_i}{2}) \sinh(\frac{\beta(h_i + h_{i-1})}{2})} < 2.
\]

Lemma 4.9. Let the mesh size \(h_i = x_{i+1} - x_i\) be defined by the generating function (3.1) and \(\beta\) defined by (2.3). We have the following estimate

\[
\begin{align*}
\left| y^{(i)}(\zeta^-_{i-1}) h_{i-1}^4 \sinh(\beta h_i) + y^{(i)}(\zeta^+_{i}) h_i^4 \sinh(\beta h_{i-1}) \right| \\
\sinh(\beta h_i)(\cosh(\beta h_i) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_{i-1}) - 1) \\
& \leq C \varepsilon^2 \left| y^{(i)}(\zeta^-_{i-1}) \right| h_{i-1}^2 + \left| y^{(i)}(\zeta^+_{i}) \right| h_i^2, \ i = 1, \ldots, N/2.
\end{align*}
\]

(4.15)
Proof. We have that

\[
\frac{y^{(i)}(\zeta_{i-1}^-)h_i^- \sinh(\beta h_i) + y^{(i)}(\zeta_i^+)h_i^+ \sinh(\beta h_{i-1})}{\sinh(\beta h_i)(\cosh(\beta h_{i-1}) - 1) + \sinh(\beta h_{i-1})(\cosh(\beta h_i) - 1)} \leq \frac{|y^{(i)}(\zeta_{i-1}^-)h_i^- \sinh(\beta h_i)|}{\sinh(\beta h_i)(\beta^2h_i^2 - 1) + \sinh(\beta h_{i-1})(\beta^2h_i^2 - 1)} + \frac{|y^{(i)}(\zeta_i^+)h_i^+ \sinh(\beta h_{i-1})|}{\sinh(\beta h_i)(\beta^2h_i^2 - 1) + \sinh(\beta h_{i-1})(\beta^2h_i^2 - 1)}
\]

(4.16)

Let us continue with the following lemma that will be further used in the proof of the \(\varepsilon\)-uniform convergence theorem. In the lemma quite rough estimate of \(F_i\) is given but it is fairly enough for our needs.

Lemma 4.10. Let the mesh size \(h_i = x_{i+1} - x_i\) be defined by the generating function (3.1) and \(\beta\) defined by (2.3). We have the following estimate

\[
|F_i y| \leq C \left[ |s_{i-1}| \left( \frac{\varepsilon^2}{h_{i-1}^2} + \frac{\varepsilon^2}{h_i^2} + 4\gamma + q + 2 \right) \frac{1}{\sqrt{N^2}} \right], \quad i = 1, 2, \ldots, N/2.
\]

(4.17)

Proof. For \(F_i y\), \(i = 1, 2, \ldots, N/2\) holds

\[
|F_i y| = \frac{\varepsilon^2}{\triangle d_i + \triangle d_{i+1}} \left[ \left( \frac{(q + 1)a_i + d_i + \Delta d_{i+1}}{\gamma} \right) (y_{i-1} - y_i) - \left( \frac{(q + 1)a_{i+1} + d_{i+1} + \Delta d_i}{\gamma} \right) (y_i - y_{i+1}) - \left( \frac{(x_{i-1}y_{i-1} + y(x_i, y_i) + f(x_i, y_{i+1})}{\gamma} \right) (\triangle d_i + \triangle d_{i+1}) \right]
\]

(4.18)

\[
\leq \frac{\varepsilon^2}{\triangle d_i + \triangle d_{i+1}} \left[ \left( \frac{(q + 1)a_i + d_i + \Delta d_{i+1}}{\gamma} \right) (r_{i-1} - r_i) - \left( \frac{(q + 1)a_{i+1} + d_{i+1} + \Delta d_i}{\gamma} \right) (r_i - r_{i+1}) - \epsilon^2 \left| \frac{2s_{i-1} - 2s_i + s_{i+1}}{\triangle d_i + \triangle d_{i+1}} + \epsilon^2 \right| \left| s_{i-1} - 2s_i + s_{i+1} \right| \right]
\]

For the layer component \(s\), due to (4.2), we have
\[ \gamma (q + 2) \frac{a_i(s_{i-1} - s_i) - a_{i+1}(s_i - s_{i+1})}{\Delta d_i + \Delta d_{i+1}} + \gamma |s_{i-1} - 2s_i + s_{i+1}| + \varepsilon^2 \left| s''_{i-1} + \tilde{q}s''_{i} + s''_{i+1} \right| = \gamma (q + 2) \frac{1}{\sinh(\beta h_i) - 1} \left( \frac{s_{i-1} - s_i}{\sinh(\beta h_i) - 1} \right) \right| \\
\begin{aligned}
&+ \gamma |s_{i-1} - 2s_i + s_{i+1}| + \varepsilon^2 \left| s''_{i-1} + \tilde{q}s''_{i} + s''_{i+1} \right|

\leq & \gamma (q + 2) \left( \frac{s_{i-1} - s_i}{\cosh(\beta h_{i-1}) - 1} \right) + \frac{1}{\cosh(\beta h_{i-1}) - 1} (s_i - s_{i+1}) \right| \\
&+ \gamma |s_{i-1} - 2s_i + s_{i+1}| + \varepsilon^2 \left| s''_{i-1} + \tilde{q}s''_{i} + s''_{i+1} \right|

\leq & Ce^{-\frac{\gamma(q+1)\sqrt{\varepsilon}}{h_i^2}} \left( \frac{\varepsilon^2}{h_i^2} + \frac{\varepsilon^2}{h_{i+1}^2} + 4\gamma + q + 2 \right). \quad (4.19)
\end{aligned}
\]

Now, for the regular component \( r \), due to mesh sizes (3.3a), (3.3b), the estimate (4.1), expansions (4.4), (4.5) (4.6), (4.7), and Lemma 4.2–Lemma 4.9, we have

\[ \left| \frac{\gamma}{\Delta d_i + \Delta d_{i+1}} \left[ ((q + 1)a_i + d_i + \Delta d_{i+1}) (r_{i-1} - r_i) \right. \right. \left. \left. \left. - \left[(q + 1)a_{i+1} + d_{i+1} + \Delta d_i \right) (r_i - r_{i+1}) - \varepsilon^2 \frac{r_{i-1} + q r_{i+1}}{\gamma} (\Delta d_i + \Delta d_{i+1}) \right]\right| \leq \frac{C}{N^2}. \quad (4.20) \]

Using (4.19) and (4.20) completes the proof of the lemma. \( \Box \)

Now we can state and prove the main theorem on \( \varepsilon \)-uniform convergence.

**Theorem 4.11.** The discrete problem (2.7) on the Bakhvalov–type mesh from Section 3 is uniformly convergent with respect to \( \varepsilon \) and

\[ \max_{0 \leq i \leq N} \left| y(x_i) - \bar{y} \right| \leq \frac{C}{N^2}, \]

where \( y \) is a solution of the problem (1.1)–(1.3), \( \bar{y} \) is the corresponding solution of (2.7) and \( C > 0 \) is a constant independent of \( N \) and \( \varepsilon \).

**Proof.** From (4.4), due to Lemma 4.2, Lemma 4.3, Lemma 4.5, Lemma 4.7 and Lemma 4.9 we have

\[ |G_i y| \leq C \left( \left| y'_{i} \right| (h_i - h_{i-1}) + \frac{|y''(\zeta_{i-1})|}{2} h_{i-1}^2 + \frac{|y''(\zeta_{i})|}{2} h_i^2 + \left| y^{(\beta)} \right| h_i^2 \right. \left. + \left| y^{(\beta)} \right| \left( \varepsilon^2 + h_{i-1}^2 \right) \right) (h_i - h_{i-1}) + \varepsilon^2 \left| \left| y^{(i)}(\zeta_{i-1}) \right| + \left| y^{(i)}(\zeta_{i}) \right| \right| h_{i-1}^2 \]

\[ + \varepsilon^2 \left| y^{(i)}(\zeta_{i-1}) \right| + \varepsilon^2 \left| y^{(i)}(\zeta_{i}) \right| h_i^2 \right), \quad i = 1, 2, \ldots N/2. \quad (4.21) \]

The statement of the theorem for the regular component due to (4.20) is proved.

For the layer component \( s \), holds the estimate (4.2). On the observed part of mesh, we have that \( e^{-\frac{\gamma(q+1)\sqrt{\varepsilon}}{h_i^2}} \geq e^{-\frac{\gamma(q+1)\sqrt{\varepsilon}}{h_{i+1}^2}} \), now it is enough to estimate \( e^{-\frac{\gamma(q+1)\sqrt{\varepsilon}}{h_{i+1}^2}} \), on this part of mesh.

We use a technique from [1] and [6, 34, 37].
Case I
Let \( t_{i-1} \geq \tau \), we have that

\[
e^{-\frac{\gamma - 1}{2} \sqrt{m}} \leq e^{-\frac{\gamma(t)}{2} \sqrt{m}} = e^{-\frac{\gamma}{2} \sqrt{m}} \leq e^{-\frac{\gamma}{2} \sqrt{m}}.
\]

Now for the layer component \( s \) due to (4.21), holds

\[
|s_i^t| (h_i - h_{i-1}) + \frac{|\epsilon''(\pi_i^t)| h_i^2}{2} + \frac{|\epsilon'''(\pi_i^t)| h_i^2}{2} + s_i^t h_i^2

+ \frac{|\epsilon''''(\pi_i^t)| (h_i - h_{i-1}) + \epsilon^2 |s_i^t(\pi_i^-)_{i-1} + s_i^t(\mu_{i-1}) h_{i-1}^2

+ \epsilon^2 |s_i^t(\zeta_i) + s_i^t(\mu_i^+) h_i^2

\leq C_1 e^{-\frac{\gamma}{2} \sqrt{m}} (h_i - h_{i-1}) + C_2 e^{-\frac{\gamma}{2} \sqrt{m}} h_i^2

+ C_3 e^{-\frac{\gamma}{2} \sqrt{m}} (h_i^2 + h_{i-1}^2) (h_i - h_{i-1}) \leq \frac{C}{N^2}.
\]

Case II
Let \( t_{i-1} < \tau \) if \( t_{i-1} \leq p - 3h, h = 1/N \). From \( t_{i-1} \leq p - 3h \Leftrightarrow p - t_{i-1} \geq 3h \Leftrightarrow p - t_{i+1} \geq h \) and \( p - t_{i-1} = p - t_{i+1} + 2h \), we have that

\[
p - t_{i+1} \geq \frac{p - t_{i-1}}{3}.
\]

Also, there holds the following estimate

\[
e^{-\frac{\gamma - 1}{2} \sqrt{m}} = e^{-\frac{\gamma}{2} \sqrt{m}} \leq C e^{-\frac{\gamma}{2} \sqrt{m}}.
\]

From the construction of the mesh (3.1), it implies

\[\kappa^{b}(\tau) = \pi^{(k)}(\tau), \ k \in \{0, 1, 2\}\]

and

\[\kappa''''(t) - \pi''''(t) = \frac{6aep}{(p - t)^4} - 6\omega \geq \frac{6aep}{(p - \tau)^4} - 6\omega = \frac{ap}{\sqrt{\epsilon}} - \omega, \ t \in [\tau, p).
\]

Hence, for sufficiently small \( \epsilon \), we have

\[\frac{ap}{\sqrt{\epsilon}} - \omega > 0,
\]

\[\kappa''''(t) - \pi''''(t) > 0, \ t \in [\tau, p),\]

and, due to (4.26) and (4.28b), we get

\[\kappa^{(k)}(t) > \pi^{(k)}(t), \ k \in \{0, 1, 2\}, \ t \in [\tau, p).
\]

Now, because of (3.1), (4.24) and (4.28c) we have

\[h_i \leq \int_{i/N}^{(i+1)/N} \kappa'(t) \, dt \leq \frac{\kappa'(t_{i+1})}{N} = \frac{aep}{(p - t_{i+1})^2} \cdot \frac{1}{N} \leq \frac{9ae}{(p - t_{i+1})^2} \cdot \frac{1}{N}
\]

and

\[h_i - h_{i-1} \leq \int_{i/N}^{(i+1)/N} \int_{i-1/N}^{i} \kappa''(s) \, ds \, dt
\]

\[\leq \frac{\kappa''(t_{i+1})}{N^2} = \frac{2aep}{(p - t_{i+1})^3} \cdot \frac{1}{N^2} \leq \frac{54ae}{(p - t_{i+1})^3} \cdot \frac{1}{N^2}.
\]
Finally, for the layer component $s$ from (4.21), due to (4.25), (4.29a) and (4.29b), we obtain the estimate

$$
|s'_i(h_i - h_{i-1}) + \frac{|\nu^{(\sigma)}|}{2}h_{i-1}^2 + \frac{|s''(\nu)|}{2}h_i^2 + |s''(\nu)|h_i^2 + s''(\nu)(\nu_{i-1}^\mu + s(\nu)\mu_{i-1}^\nu)h_{i-1}^2 + s''(\nu)(\nu_i^\mu + s(\nu)\mu_i^\nu)h_i^2
\leq C_1 e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \sqrt{m} (h_i - h_{i-1}) + C_2 e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \frac{1}{N^2} + 2 \frac{81a^2 x^2 p^2}{(p - t_{i-1})^3} + \frac{1}{N^2}
$$

$$
+ C_3 e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \frac{54a^2 x^2 p^2}{(p - t_{i-1})^3} + \frac{1}{N^2} + C_4 e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \frac{81a^2 x^2 p^2}{(p - t_{i-1})^3} + \frac{1}{N^2}
$$

$$
\leq C_6 \frac{e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}}}{(p - t_{i-1})^3} + \frac{e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}}}{(p - t_{i-1})^4} + \frac{e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}}}{(p - t_{i-1})^4} \frac{1}{N^2}
$$

$$
\leq \frac{C}{N^2}.
$$

(4.30)

**Case III**

At the end, let $p - 3h < t_{i-1} < \tau$, there holds

$$
e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} = e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} = C e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}}.
$$

(4.31)

On the observed part of the mesh is $h_{i-1} \leq h_i$, and for the mesh sizes $h_{i-1}$, we have

$$
h_{i-1} = \kappa(t_i) - \kappa(t_{i-1}) = \kappa'(t_{i-1})h = \frac{aeph}{(p - \theta_{i-1})^2} \geq \frac{aeph}{(p - (p - 3h))^2} = \frac{aep}{9h}, \quad \theta_{i-1} \in [t_{i-1}, t_i].
$$

(4.32)

Now, due to (4.31), (4.32) and (4.17) we get

$$
|G_i| \leq C_1 \left[ e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \left( \frac{e^2}{h_{i-1}^2} + \frac{e^2}{h_i^2} + 4\gamma + \tilde{q} + 2 \right) + \frac{1}{N^2} \right]
\leq C_2 \left[ e^{\frac{\nu_{i-1}^\mu}{\nu_{i-1}^\nu}} \left( 2h^2 + 4\gamma + \tilde{q} + 2 \right) + \frac{1}{N^2} \right] \leq \frac{C}{N^2}.
$$

(4.33)

Case $3h \leq \sqrt{e}$ is proved in Case I and Case II.

According to (4.20), (4.21), (4.23), (4.30) and (4.33), the proof of the theorem is complete. □
5. Numerical results

In this section we present the numerical results to confirm the uniform accuracy of the discrete problem (2.6) using the mesh from Section 3. To demonstrate the efficiency of the method, we present two examples having boundary layers.

**Example 5.1.** Consider the following problem from [37]

\[ \varepsilon^2 y'' = y - 1 \quad \text{for } x \in [0, 1], \]  
\[ y(0) = y(1) = 0. \]  
\[
(5.1a) \\
(5.1b)
\]

The exact solution of this problem (5.1a)–(5.1b) is given by

\[ y(x) = 1 - \frac{e^{-\frac{x}{\varepsilon}} + e^{-\frac{1-x}{\varepsilon}}}{1 + e^{-\frac{1}{\varepsilon}}}. \]  
\[
(5.2)
\]

Because of the fact that the exact solution is known, we define the computed error \( E_N \) and the computed rate of convergence \( \text{Ord} \) in the usual way

\[ E_N = \| y - \bar{y}^N \|_\infty \]  
\[
(5.3)
\]

and

\[ \text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln 2}. \]  
\[
(5.4)
\]

Other values of constants are \( m = 1, q = 4, a = 1 \) and \( p = 0.4 \).

The values of \( E_N \) and \( \text{Ord} \) are given at the of the paper in Appendix (Table 1).

**Example 5.2.** Consider the following problem

\[ \varepsilon^2 y'' = (y - 1)(1 + (y - 1)^2) \quad \text{for } x \in [0, 1], \]  
\[ y(0) = y(1) = 0. \]  
\[
(5.5a) \\
(5.5b)
\]

The exact solution of the problem (5.5a)–(5.5b) is unknown. The system of nonlinear equations is solved by Newton’s method with initial guess \( y0 = (0, y0_1, \ldots, y0_{N-1}, 0)^T, \) \( y_0 = 1, i = 1, \ldots, N - 1, \) by Newton’s method. The value of the constant \( \gamma = 1 \) has been chosen so that the condition \( \gamma \geq f_c(x, y), \forall (x, y) \in [0, 1] \times [y_L, y_U] \subset [0, 1] \times R \) is fulfilled, where \( y_L \) and \( y_U \) are the lower and the upper solutions of the test problem (5.1a)–(5.1b) and their values are \( y_L = 0 \) and \( y_U = 1 \).

Because of the fact that we do not know the exact solution, we replace \( y \) by \( \hat{y} \) in order to calculate \( E_N \) and \( \text{Ord} \), where \( \hat{y} \) is the numerical solution of (5.5a) – (5.5b) obtained by using \( N = 16384 \), (same procedure is applied in [12], [8]).

Now, we calculate the value of error \( E_N \) and the the rate of convergence on the following way

\[ E_N = \| \hat{y} - \bar{y}^N \|_\infty \]  
\[
(5.6)
\]

and

\[ \text{Ord} = \frac{\ln E_N - \ln E_{2N}}{\ln 2}. \]  
\[
(5.7)
\]

Other values of constants are \( m = 1, q = 4, a = 1 \) and \( p = 0.3 \).

The values of \( E_N \) and \( \text{Ord} \) are given at the of the paper in Appendix (Table 2).
6. Conclusion

In this paper we presented a discretization of an one-dimensional semilinear reaction–diffusion problem, with suitable assumptions that have ensured the existence and uniqueness of the continuous problem. We constructed a class of difference schemes, and we proved the existence and uniqueness of the numerical solution, after which we proved the $\varepsilon$–uniform convergence using a suitable layer–adaptive mesh. Finally we performed a numerical experiments to confirm the theoretical results.

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Appendix

| $N$ | $E_N$ | Ord | $E_N$ | Ord | $E_N$ | Ord |
|-----|-------|-----|-------|-----|-------|-----|
| $2^6$ | $2.4133e-04$ | 2.00 | $1.0062e-03$ | 1.98 | $1.3294e-03$ | 1.96 |
| $2^7$ | $6.0436e-05$ | 2.00 | $2.5429e-04$ | 1.99 | $3.4128e-04$ | 1.99 |
| $2^8$ | $1.5095e-05$ | 2.00 | $6.3802e-05$ | 2.00 | $8.5934e-05$ | 2.00 |
| $2^9$ | $3.7691e-06$ | 1.97 | $1.5961e-05$ | 2.00 | $2.1523e-05$ | 2.00 |
| $2^{10}$ | $9.6433e-07$ | 2.00 | $3.9909e-06$ | 2.00 | $5.3833e-06$ | 2.00 |
| $2^{11}$ | $2.4108e-07$ | 2.00 | $9.9777e-07$ | 2.00 | $1.3460e-06$ | 2.00 |
| $2^{12}$ | $6.0271e-08$ | 2.00 | $2.4945e-07$ | 2.00 | $3.3651e-07$ | 2.00 |
| $2^{13}$ | $1.5068e-08$ | – | $6.2363e-08$ | – | $8.4127e-08$ | – |

| $\varepsilon$ | $2^{-3}$ | $2^{-5}$ | $2^{-10}$ |
|----------------|----------|---------|---------|
| $2^6$ | $1.3243e-03$ | 1.96 | $1.3243e-03$ | 1.96 |
| $2^7$ | $3.3945e-04$ | 1.99 | $3.3945e-04$ | 1.99 |
| $2^8$ | $8.5413e-05$ | 2.00 | $8.5413e-05$ | 2.00 |
| $2^9$ | $2.1388e-05$ | 2.00 | $2.1388e-05$ | 2.00 |
| $2^{10}$ | $5.3493e-06$ | 2.00 | $5.3493e-06$ | 2.00 |
| $2^{11}$ | $1.3375e-06$ | 2.00 | $1.3375e-06$ | 2.00 |
| $2^{12}$ | $3.3438e-07$ | 2.00 | $3.3438e-07$ | 2.00 |
| $2^{13}$ | $8.3595e-08$ | – | $8.3595e-08$ | – |

| $\varepsilon$ | $2^{-15}$ | $2^{-25}$ | $2^{-30}$ |
|----------------|----------|---------|---------|
| $2^6$ | $1.3243e-03$ | 1.96 | $1.3243e-03$ | 1.96 |
| $2^7$ | $3.3945e-04$ | 1.99 | $3.3945e-04$ | 1.99 |
| $2^8$ | $8.5413e-05$ | 2.00 | $8.5413e-05$ | 2.00 |
| $2^9$ | $2.1388e-05$ | 2.00 | $2.1388e-05$ | 2.00 |
| $2^{10}$ | $5.3493e-06$ | 2.00 | $5.3493e-06$ | 2.00 |
| $2^{11}$ | $1.3375e-06$ | 2.00 | $1.3375e-06$ | 2.00 |
| $2^{12}$ | $3.3438e-07$ | 2.00 | $3.3438e-07$ | 2.00 |
| $2^{13}$ | $8.3595e-08$ | – | $8.3595e-08$ | – |

Table 1: Error $E_N$ and convergence rates Ord for approximate solution.
| $N$ | $E_1$  | $E_2$  | $E_3$  | $E_4$  |
|-----|--------|--------|--------|--------|
| 1   | 1.2281 | 1.2276 | 1.2276 | 1.2276 |
| 2   | 3.5293 | 3.5247 | 3.5247 | 3.5247 |
| 3   | 9.1947 | 9.1749 | 9.1749 | 9.1749 |
| 4   | 2.3235 | 2.3180 | 2.3180 | 2.3180 |
| 5   | 5.8267 | 5.8140 | 5.8140 | 5.8140 |
| 6   | 1.4544 | 1.4544 | 1.4544 | 1.4544 |
| 7   | 3.6449 | 3.6366 | 3.6366 | 3.6366 |
| 8   | 9.1127 | 9.0921 | 9.0921 | 9.0921 |

Table 2: Error $E_N$ and convergence rates Ord for approximate solution.

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