AN ELEMENTARY NOTE ABOUT THE $G$-CONVERGENCE FOR A TYPE OF NONLOCAL $p$-LAPLACIAN OPERATOR

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Abstract. Simple applications of a principle of minimum energy and the property of monotonicity for the corresponding non-local operator, have allowed a direct proof of the $G$-compactness in a weak sense, as well as in the strong sense. The definitions of $H$-convergence and $G$-convergence, weak or strong, have been proved to be equivalent.

1. Introduction

Let $\Omega$ be a bounded open and smooth domain in $\mathbb{R}^N$. We consider $k$, a radial and positive function in $L^1(\mathbb{R}^N)$ such that

$$(1.1) \quad \int_{\mathbb{R}^N \setminus B(0,\varepsilon)} \frac{k(|z|)}{|z|^p} \, dz < \infty,$$

where $1 < p < +\infty$, $B(x,r)$ denoted an open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$, and $\varepsilon$ is any positive number. Additionally, $k$ is supposed to fulfill the estimation

$$(1.2) \quad k(|z|) \geq \frac{c_0}{|z|^{N+(s-1)p}} \text{ for any } z \in \mathbb{R}^N \setminus \{0\},$$

where $c_0 > 0$ and $s \in (0,1)$ are given constants.

The natural frame in which we shall work is the nonlocal energy space

$$(1.3) \quad X = \{ u \in L^p(\mathbb{R}^N) : B(u,u) < \infty \}$$

where $B(\cdot,\cdot)$ is defined by means of the formula

$$(1.4) \quad B(u,v) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k(|x'-x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \, dx' \, dx.$$

We define also the constrained energy space as

$$X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.$$ 

The space $X$ is a reflexive Banach space equipped with the norm

$$\|u\|_X = \|u\|_{L^p(\mathbb{R}^N)} + (B(u,u))^{1/p}.$$ 

The dual of $X$ will be denoted by $X'$ and can be endowed with the norm

$$\|g\|_{X'} = \sup \{ \langle g, w \rangle_{X' \times X} : w \in X, \|w\|_X = 1 \}.$$ 

Analogous definitions applies for the space $X_0$.

There is another functional space that we will use in the formulation of our problem, the one that is formed by the diffusion coefficients of our nonlocal equations. That is

$$\mathcal{H} \doteq \{ h \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N) \mid h(x',x) = h(x',x) \in [h_{\min},h_{\max}] \text{ a.e. } (x',x) \in \mathbb{R}^N \times \mathbb{R}^N \},$$

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where $h_{\text{min}}$ and $h_{\text{max}}$ are positive given constants such that $0 < h_{\text{min}} < h_{\text{max}}$.

Now we are ready to set the nonlocal elliptic problem $(P_h)$: given $f \in X_0'$ and $h \in \mathcal{H}$ we look for a function $u \in X_0$ that solves the nonlocal boundary equation

\[(1.5) \quad -2 \left[ p.v. \int_{\mathbb{R}^N} h(x',x) \frac{k(|x' - x|)|u(x') - u(x)|^{p-2}(u(x') - u(x))}{|x' - x|^p} dx' \right] = f, \; x \in \Omega.\]

That means that, for any $v \in X_0$, $u \in X_0$ satisfies the variational equality

\[(1.6) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(x',x) \frac{k(|x' - x|)|u(x') - u(x)|^{p-2}(u(x') - u(x))(v(x') - v(x))}{|x' - x|^p} dx' dx = \langle f, v \rangle_{X_0' \times X_0} \]

The problem $(1.5)$ shall be written as

\[
(P_h) \begin{cases} 
\mathcal{L}_h u = f \text{ in } \Omega \\
u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Throughout the manuscript, the action of $f$ upon a function $v \in X_0$ is denoted by $\langle f, v \rangle$, and if $u$ is a solution of $(P_h)$, then it can be expressed via the formula

\[
\langle \mathcal{L}_h u, v \rangle = B_h(u,v),
\]

where $B_h(u,v)$ is the left part of $(1.6)$.

**Definition 1** (G-convergence). Let $(h_j)_j$ be a sequence of coefficients from $\mathcal{H}$ and let $(u_j)_j$ be the sequence of the corresponding solution of the problems

\[
(P_j) \begin{cases} 
\mathcal{L}_{h_j} u_j = f \text{ in } \Omega \\
u_j = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

It is said that the sequence of operators $(\mathcal{L}_{h_j})_j$ G-converges to the operator $\mathcal{L}_h$ when $j \to \infty$, if for any $f \in X_0'$, $u_j \rightharpoonup u$ weakly in $X_0$ if $j \to \infty$, where $u$ is the solution to the problem $(P_h)$.

As we shall see, the problems $(P_h)$ or $(P_j)$ are well-posed (see Theorem 1 below). This fact allows us to rewrite the above definition as follows: for any $f \in X_0'$, $\mathcal{L}_h^{-1} f \rightharpoonup \mathcal{L}_h^{-1} f$ weakly in $X_0$ if $j \to \infty$.

We need to define the nonlocal flux associated to the operator $\mathcal{L}_h$: it is

\[
\Psi_h(x',x) = h(x',x) k^{1/p'} (x',x) \frac{|u(x') - u(x)|^{p-2}(u(x') - u(x))}{|x' - x|^{p-1}} \text{ where } u = \mathcal{L}_h^{-1} f
\]

**Definition 2** (H-convergence). Let $(h_j)_j$ be a sequence of coefficients from $\mathcal{H}$. It is said that the sequence of operators $(\mathcal{L}_{h_j})_j$ H-converges to the operator $\mathcal{L}_h$ when $j \to \infty$, if for any $f \in X_0'$ the following convergences hold:

1. $u_j = \mathcal{L}_{h_j}^{-1} f \rightharpoonup u = \mathcal{L}_h^{-1} f$ weakly in $X_0$ if $j \to \infty$.
2. $\Psi_{h_j} \rightharpoonup \Psi_h$ weakly in $L^p' (\mathbb{R}^N \times \mathbb{R}^N)$ if $j \to \infty$.

The type of convergences we have just set up has been the aim of some recent papers. In [7], an H-convergence compactness result, via the oscillating test function method of Tartar, is proved and, a characterization result of $\Gamma$-convergence for the energy functional associated to the problems $(P_j)$, is shown as well. In [3], a characterization of the $H$-limit obtained in [7], is given and, the definitions of G-convergence and H-convergence have been proved to be
equivariant. [13] is a reference where these type convergences are analyzed under a general abstract setting. In [5, 10], interesting advances concerning with Γ-convergence are obtained, and in [2], a strong G-convergence result for the linear case $p = 2$ is obtained.

In the present manuscript, the set up of the problem is slightly different, compared to some of the above references. It includes a kernel so that the nonlocal problem is formulated in a space that may be smaller than $W^{s,p}_0(\Omega)$. As opposed to the detailed and nuanced development elaborated in [7], this work presents a procedure based on much more elementary aspects. Here, the proof of the compactness in $L^p$, and even in stronger norms, have been performed in an simple way. The departure point to develop the study of these convergences is an abstract nonlocal energy criterion. This is a principle of minimum energy which, combined with the monotony of the nonlocal operator, show in a transparent way the steps we have to follow in order to give the proofs on compactness. The exhibited procedure has allowed understanding the reason why, in this case, G-convergence, in the weak sense, is equivalent to G-convergence in the strong sense or $H$-convergence (see [9, 1] and [11] for a general local setting).

We organize the paper as follows: Section 2 contains some essential notes related to the continuity and some fundamental compactness embeddings. It also includes an essential result ensuring the well-posedness of the nonlocal variational problems $(P_j)$ (Theorem 1). In Section 3 Theorem 2 establishes the relative convergence of $\left( \mathcal{L}_{h_j}^{-1}f \right)$ towards $\mathcal{L}^{-1}f$ strongly in the $L^p$-norm. This latter result is improved in Section 4 by an strong $X_0$-compactness result (Theorem 3). After, the equivalences among the definitions of convergence in $X_0$, G-convergence and $H$-convergence, of $\left( \mathcal{L}_{h_j} \right)_j$ towards $\mathcal{L}_h$, have been proved (Theorems 4 and 5). Section 5 is devoted to a brief summary of the procedures and results we have obtained throughout the manuscript.

2. SOME PRELIMINARIES

Some previous remarks concerning the ambient space on which we shall work, have to be commented:

1. We firstly notice $X_0 \subset W^{s,p}_0(\Omega)$, where $W^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}$ and the closure is considered with respect to the classical norm of $W^{s,p}(\mathbb{R}^N)$. The above embedding is true because in the case of $\Omega$ has a Lipschitz boundary, $W^{s,p}_0(\Omega)$ can be identified as the set functions form $W^{s,p}(\mathbb{R}^N)$ vanishing outside of $\Omega$ (see [6]), that is

$$W^{s,p}_0(\Omega) = \{ g \in W^{s,p}(\mathbb{R}^N) : g = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}.$$  

An important fact is the continuous embedding of $X_0$ into $L^p(\Omega)$. Indeed, it is known that there exists a constant $c = c(N, s) > 0$ such that for any $w \in W^{s,p}_0(\Omega)$

$$c \|w\|_{L^p(\Omega)}^p \leq \int_\Omega \int_\Omega \frac{|w(x') - w(x)|^p}{|x' - x|^{N+sp}} \, dx' \, dx$$  

(see [6, Th. 6.7]). By paying attention to the hypotheses on the kernel (1.2), and using (2.1), we ensure there is a positive constant $C$ such that the nonlocal Poincaré inequality

$$C \|w\|_{L^p(\Omega)}^p \leq B \langle w, w \rangle$$

(2.2)

holds for any $w \in X_0$. In particular, the above inequality implies that $X_0$ is a Banach space when it is equipped with the (equivalent) norm defined as $\|w\|_{X_0} = (B \langle w, w \rangle)^{\frac{1}{p}}$. 


The embedding $X_0 \subset L^p(\Omega)$ is compact. To make sure of that we simply take into account (2.2), the compact embedding $W_0^{s,p}(\Omega) \subset L^p(\Omega)$ (see [8, Th. 7.1]) and the fact that $X_0$ is closed with respect to strong convergence in $L^p$. In particular, if $(w_j)_j$ is a sequence uniformly bounded in $X_0$, that is, if there is a constant $C$ such that $B_h(w_j, w_j) \leq C$ for every $j$, then the positiveness of the coefficients and (2.2) guarantee $(w_j)_j$ is uniformly bounded $L^p(\Omega)$. Moreover, the compact embedding $X_0 \subset L^p(\Omega)$ ensures the existence of a subsequence of $(w_j)_j$, still denoted by $(w_j)_j$, such that $w_j \to w \in X_0$ strongly in $L^p(\Omega)$.

(3) Another essential issue of our research is the identification of the above nonlocal boundary problem with a Dirichlet Principle.

Theorem 1. Let $h$ be any given function form $\mathcal{H}$. There exists a unique function $u \in X_0$ solution to the problem (1.6). This function $u$ is the only solution to the minimization problem

\begin{equation}
(2.3) \quad \min_{u \in X_0} J_h(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h(x'x)k(|x' - x|) \frac{|w(x') - w(x)|^p}{|x' - x|^p} dx'dx - \langle f(x), w(x) \rangle.
\end{equation}

See [7] for the proof.

3. Nonlocal energy criterion

Let us examine how is the convergence of the sequence $(u_j)_j$ of solutions of the problem $(P_j)$. We shall pay attention to the fact that $u_j$ is the minimizer of $J_{h_j}(w) = \frac{1}{p}B_{h_j}(w, w) - \langle f, w \rangle$, $w \in X_0$. Now, going into the details, we firstly note that we can extract a subsequence form $(h_j)_j$, still denoted by $(h_j)_j$, such that $h_j \to h$ weakly-* in $L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. In addition, it is a well-known result that $h \in \mathcal{H}$. Let $u$ be solution of $(P_h)$, the one that corresponds to the coefficient $h$. Necessarily $u$ must be the minimizer of $J_h(u) = \frac{1}{p}B_h(w, w) - \langle f, w \rangle$, $w \in X_0$.

Theorem 2. Under the above circumstances, there exists a subsequence of $(u_j)_j$, still denoted by $(u_j)_j$, for which

\[ u_j \to u \text{ strongly in } L^p(\mathbb{R}^N), \]

and

\begin{equation}
(3.1) \quad \lim \min_{u \in X_0} \left\{ \frac{1}{p}B_{h_j}(w, w) - \langle f, w \rangle \right\} = \min_{u \in X_0} \left\{ \frac{1}{p}B_h(w, w) - \langle f, w \rangle \right\}
\end{equation}

In particular

\begin{equation}
(3.2) \quad \lim_{j} B_{h_j}(u_j, u_j) = B_h(u, u).
\end{equation}

Proof. The first consequence derived from the fact $u_j$ is a solution to $(P_j)$ is

\[ C \|u_j\|_{L^p(\Omega)}^p \leq h_{\max} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k(|x' - x|) |u_j(x') - u_j(x)|^p \frac{dx'dx}{|x' - x|^p} \leq \|f\|_{X_0} \|u_j\|_{X_0} \]

for any $j$. This implies the sequences of norms $\|u_j\|_{X_0}$ and $\|u_j\|_{L^p(\mathbb{R}^N)}$ are uniformly bounded. This preliminary serves to ensure that $(u_j)_j$, or at least for a subsequence of it, is weakly
convergent to a function $u^* \in L^p(\mathbb{R}^N)$. But, by exploiting the fact that $u_j$ is also a sequence uniformly bounded in $X_0$, we improve this convergence (see comment 2 in Section 2): $u_j \to u^* \in X_0$ strongly $L^p(\Omega)$. Let $m_j$ be the minimum value of functional $J_{h_j}$, which is attained by $u_j$, and let $m$ be the minimum value of $J_h$, which is attained by $u$. Since $h_j \rightharpoonup h$ weakly-* in $L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and $\frac{k(|x' - x|)u(x') - u(x)}{|x' - x|^p} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ then

$$\lim_j m_j \leq \lim_j \frac{1}{p}B_{h_j}(u, u) - \langle f, u \rangle = \frac{1}{p}B_h(u, u) - \langle f, u \rangle = m.$$ 

And the reverse is also fulfilled:

$$m \leq \frac{1}{p}B_h(u^*, u^*) - \langle f, u^* \rangle \leq \lim_j \frac{1}{p}B_{h_j}(u_j, u_j) - \langle f, u_j \rangle = \lim_j m_j$$ 

Note that to justify the second inequality we have used the following argument: since the function $\frac{k(|x' - x|)u_j(x') - u_j(x)}{|x' - x|^p}$ converges to $\frac{k(|x' - x|)u(x') - u(x)}{|x' - x|^p}$ a.e. in $\mathbb{R}^N \times \mathbb{R}^N$, and $h_j \rightharpoonup h$ weakly-* in $L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, then thanks to a generalized version of Fatou’s Lemma (see [12]) we derive

$$\lim_j B_{h_j}(u_j, u_j) \geq B_h(u, u)$$ 

By linking the above inequalities, we prove $\lim_j m_j = m$, which is factually (3.1). This convergence implies that both $u$ and $u^*$ are solutions to $(P_h)$, so that, thanks to the uniqueness of solution for $(P_h)$ (Theorem 1), $u = u^*$. In addition the limit (3.2) holds.

4. Strong convergence in $X_0$

There are two ingredients for improving the strong convergence in $L^p$. One of them is the monotonicity of the operator $B(\cdot, \cdot)$, which is a consequence of the elementary inequality (4.1) (see below). More specifically, the fact that (4.1) makes the sequence of operators $\left(\mathcal{L}^{-1}_{h_j}\right)_j$ to be uniformly strictly monotone. The second aspect is about the hypothesis concerning the definition of the operator $B$: any of the singular integrals involved in the above definitions, have been understood as the principal value. Compare to the situation of pure convergent integral-Lebesgue, this assumption entails a different treatment of the (a priori nonpositive) singular integrals involved in our analysis. This formulation of the problem allows us to prove a certain formula of integration by parts that eventually, together the strong convergence obtained in Theorem 2 show us, with clarity, the way to achieve strong convergence in $X_0$. However, this assumption concerning the way we have to interpret the singular integrals is not necessary when we deal with the case $p = 2$. In such a case the bilinearity of $B$ and the convergence of energies obtained in (3.2) are enough to prove the aforementioned strong convergence in $X_0$ (see [2]).

4.1. Strong convergence in $X_0$. We are going to prove a $X_0$ strong compactness result. That simply means that from the sequence $\left(\mathcal{L}_{h_j}\right)_j$ we can extract a subsequence, $\left(\mathcal{L}_{h_{jk}}\right)_k$, such that $\mathcal{L}^{-1}_{h_{jk}}f \to \mathcal{L}^{-1}_hf$ strongly in the norm of $X_0$ if $k \to \infty$.

For convenience, in the remainder of the manuscript, we will adopt the notation $v' = v(x')$ for any function $v = v(x)$.

**Theorem 3.** From the given sequence of coefficients $(h_j)_j \subset \mathcal{H}$, we can extract a subsequence, which will not be relabelled, such that $h_j \rightharpoonup h$ weak-* in $L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ and $(u_j)_j$ strongly converges to $u$ strongly in $X_0$ if $j \to \infty$. 

Proof. In a first stage we assume $p \geq 2$. We define the sequences $a_j = u'_j - u_j$ and $b = u' - u$, and we take into account the next elementary inequality: if $1 < p < \infty$, then there exist two positive constants $C = C(p)$ and $c = c(p)$, such that for every $a, b \in \mathbb{R}^N$

\begin{equation}
(4.1) \quad c \{ |a| + |b| \}^{p-2} |a - b|^2 \leq (|a|^{p-2} a - |b|^{p-2} b) \cdot (a - b) \leq C \{ |a| + |b| \}^{p-2} |a - b|^2 .
\end{equation}

(cf. [4]). In particular, there is a constant $C > 0$ such that

\begin{equation}
(4.2) \quad |a - b|^p \leq C \{ |a|^{p-2} a - |b|^{p-2} b \} \cdot (a - b) .
\end{equation}

By applying (4.2) with $a = a_j$ and $b$, we easily get

\begin{equation}
(4.3) \quad \|u_j - u\|_{X_0}^p = B_{h_j} (u_j - u, u_j - u) \leq C \{ B_{h_j} (u_j, u_j - u) - B_{h_j} (u, u_j - u) \} .
\end{equation}

The usage of Theorem 2 ensures $\|u_j - u\|_{L^p} \to 0$ if $j \to \infty$, modulo a subsequence, and therefore

\[ B_{h_j} (u_j, u_j - u) = \langle f, u_j - u \rangle \leq \|f\| \|u_j - u\|_{L^p} \to 0 \text{ if } j \to \infty. \]

Thus, the first term of the right part of (4.3) goes to zero. The second term of (4.3) can be rewritten as

\[ B_{h_j} (u_j, u_j - u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_j(x', x) D(u_j - u)(x', x) \, dx' \, dx, \]

where

\[ \psi_j(x', x) = k^{1/p'} h_j \frac{|u' - u|^{p-2} (u' - u)}{|x' - x|^{p-1}}, \]

and $D(u_j - u)$ is the nonlocal gradient of $u_j - u$,

\[ D(u_j - u)(x', x) = k^{1/p} \frac{(u'_j - u_j - (u' - u))}{|x' - x|}. \]

We also define the nonlocal divergence of $\psi_j$ as

\[ d\psi_j(x) = \int_{\mathbb{R}^N} k^{1/p} \frac{\psi_j(x, x') - \psi_j(x', x)}{|x' - x|} \, dx'. \]

We have the following properties for these functions (see [7] for a detailed account): by the anti-symmetry of $\psi_j$, it is clear that

\[ d\psi_j(x) = \int_{\mathbb{R}^N} k^{1/p} \frac{\psi_j(x, x') - \psi_j(x', x)}{|x' - x|} \, dx' = -2 \int_{\mathbb{R}^N} k^{1/p} \frac{\psi_j(x', x)}{|x' - x|} \, dx'. \]

Besides, the integrability property of the kernel (1.1) and Hölder inequality make automatic the proof of that the sequence $(d\psi_j)$ is uniformly bounded in $L^{p'}$. At this point, one has solely to perform a change of variables to deduce how $d\psi_j(x) \in X_0'$ acts on $u_j - u \in X_0$; it can be expressed through the next formula of integration by parts:

\begin{equation}
(4.4) \quad \int_{\mathbb{R}^N} d\psi_j(x) (u_j - u)(x) \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi_j(x', x) D(u_j - u)(x', x) \, dx' \, dx.
\end{equation}

By gathering all the above comments we derive

\[ |B_{h_j} (u, u_j - u)| \leq \int_{\mathbb{R}^N} d\psi_j(x) (u_j - u)(x) \, dx \leq \left( \int_{\mathbb{R}^N} |d\psi_j(x)|^{p'} \, dx \right)^{1/p'} \left( \int_{\mathbb{R}^N} |u_j - u|^p \, dx \right)^{1/p} \]

\[ \leq C \left( \int_{\mathbb{R}^N} |u_j - u|^p \, dx \right)^{1/p} \to 0 \text{ if } j \to \infty. \]
Thus, we have proved that the right part of (4.3) goes to zero and the desired strong convergence for a subsequence of \( u_j \) toward \( u \) in \( X_0 \) has been checked:

\[
\lim_j \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k h_j \frac{|u_j' - u_j - (u' - u)|^p}{|x' - x|^p} \, dx' \, dx = 0.
\]

If \( 1 < p < 2 \) the procedure is similar. We pay attention to this elementary manipulation: fixed \( m = \frac{(2-p)p}{2} \), we apply Hölder’s inequality. Then, by taking into account that \( k \in L^1 (\mathbb{R}^N) \) and using the left part of the inequality (4.1), we obtain that

\[
\|u_j - u\|^p_{X_0} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |a_j - b|^p \frac{|x' - x|^m}{|x' - x|^p} \, k h_j \, dx' \, dx \\
\leq C \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|a_j - b|^2}{|x' - x|^{2-p}} \, k h_j \, dx' \, dx \right)^{p/2} \\
\leq C \left( B_{h_j} (u_j, u_j - u) - B_{h_j} (u, u_j - u) \right)^{p/2}.
\]

Thus, to finish the proof, it is enough to apply the analysis already employed for the case \( p \geq 2 \). □

4.2. \( G \)-convergence.

**Theorem 4.** From the given sequence of coefficients \( (h_j)_j \subset \mathcal{H} \), we can extract a subsequence, which will not be relabelled, such that \( h_j \rightharpoonup h \) weak-* in \( L^\infty (\mathbb{R}^N \times \mathbb{R}^N) \) and \((\mathcal{L}_{h_j})_j \) \( G \)-converges to \( \mathcal{L}_h \) if \( j \to \infty \).

The proof of \( G \)-convergence is unnecessary because the strong convergence in \( X_0 \) implies weak convergence in that space. Despite this consideration and for the sake of clarity, we will proceed with the proof. So, if we consider any \( f \in X'_0 \) and we fix \( h \in \mathcal{H} \), then we can ensure the existence of a unique solution \( u_f \in X_0 \) to the problem \( \mathcal{L}_h u = f \). Then, it is also clear that

\[
\langle f, u_j \rangle_{H_0' \times H_0} = \langle \mathcal{L}_h u_f, u_j \rangle_{H_0' \times H_0} = B_h (u_f, u_j - u) + B_h (u_f, u).
\]

By taking into account that

\[
B_h (u_f, u_j - u) \leq (B_h (u_f, u_f))^{1/p'} (B_h (u_j - u, u_j - u))^{1/p}
\]

and using Theorem 3, we get the desired \( G \)-convergence, namely

\[
\lim_j \langle f, u_j \rangle_{H_0' \times H_0} = B_h (u_f, u) = \langle f, u \rangle_{H_0' \times H_0}.
\]

Even though the notion of \( G \)-convergence is, by many reasons, not satisfactory for the study of differential equations, it satisfies the basic axiomatic rules (11 [1]). For instance, the \( G \)-limit of a sequence of operator is unique. To prove it, we proceed as follows: assume that for any \( f \in X_0' \), \( \mathcal{L}_h \) and \( \mathcal{L}_g \) are two different \( G \)-limits of \((\mathcal{L}_{h_j})_j \). If the underlying solutions of these operators are \( u_f = \mathcal{L}_h^{-1} f \) and \( u_g = \mathcal{L}_g^{-1} f \), then \( u_j = \mathcal{L}_h^{-1} f \rightharpoonup u_f = \mathcal{L}_h^{-1} f \) and \( u_j \rightharpoonup u_g = \mathcal{L}_g^{-1} f \) weakly in \( X_0 \). Then \( u_f = u_g \), whereby we deduce \( B_h (u_f, w) = \langle f, w \rangle \) and \( B_g (u_f, w) = \langle f, w \rangle \), for any \( w \in X_0 \). Thus, since this equality holds for any arbitrary functional \( f \in X_0' \), then the same equality holds for
arbitrary \( u_f \in X_0 \). If we perform variations with respect to \( u \) in the expression \( B_{h-g} (u,u) = 0 \) we get

\[
\int_{\Omega} \int_{\mathbb{R}^N} (h - g) \left| \frac{( u' - u)^{p-2} ( u' - u)}{| x'|^p} \right| v (x) \, dx' \, dx = 0
\]

(4.6)

for any \( u \in X_0 \) and any \( v \in L^\infty (\Omega) \) with \( \text{supp} \, v \subset \Omega \). Now, if we follow the lines marked in the proof of Proposition 17 from [8], we conclude that the function

\[
V (x',x,w,z) = ( h (x',x) - g (x',x)) \, k \, (|x'|,|x|) \, \left| \frac{z-w}{|x'-x|^p} \right|
\]

is constant in \( z \), for any \((x,x') \in \Omega \times \mathbb{R}^N\) and any \( w \in \mathbb{R}^N \). This clearly implies \( h-g = 0 \) in \((x,x') \in \Omega \times \mathbb{R}^N\). If we use the symmetry property of the coefficients \( h \) and \( g \), then the same is true \( \mathbb{R}^N \times \Omega \). Then \( h = g \) in \((\Omega \times \mathbb{R}^N) \cup (\mathbb{R}^N \times \Omega)\), which suffices to ensure \( L_h \) and \( L_g \) coincide.

### 4.3. \( H \)-convergence.

**Theorem 5.** From a given sequence of coefficients \( (h_j)_j \subset \mathcal{H} \), we can extract a subsequence, which will be not relabelled, such that \( h_j \rightharpoonup h \) weak-* in \( L^\infty (\mathbb{R}^N \times \mathbb{R}^N) \) and \( (L_{h_j})_j \) \( H \)-converges to \( L_h \) if \( j \to \infty \).

**Proof.** We know \( u_j \to u \in X_0 \) strongly in \( X_0 \). It is immediate to prove that the sequence of nonlocal fluxes \( (\Psi_{h_j})_j \) is bounded in \( L^{p'} \). The question we raise here is the following: since there is a subsequence of indexes \( j \) and a function \( \Psi \), such that \( \Psi_j \rightharpoonup \Psi \in L^{p'} (\mathbb{R}^N \times \mathbb{R}^N) \) weakly in \( L^{p'} (\mathbb{R}^N \times \mathbb{R}^N) \), it is then \( \Psi = \Psi_h \)? To answer to this question we examine the limit

\[
I = \lim_j \int \int_j \Psi_j (x',x) G (x',x) \, dx' \, dx,
\]

(4.7)

where \( G \) is any function from \( L^p (\mathbb{R}^N \times \mathbb{R}^N) \). If \( G_k \to G \) strongly in \( L^p (\mathbb{R}^N \times \mathbb{R}^N) \) if \( k \to \infty \), then

\[
\lim_j \int \int_{\mathbb{R}^N} \Psi_{h_j} (x',x) G (x',x) \, dx' \, dx = \lim_j \int \int_{\mathbb{R}^N} \Psi_{h_j} (x',x) G_k (x',x) \, dx' \, dx
\]

\[
+ \lim_j \int \int_{\mathbb{R}^N} \Psi_{h_j} (x',x) (G (x',x) - G_k (x',x)) \, dx' \, dx
\]

and thanks to Hölder’s inequality and the uniform estimation of \( \Psi_{h_j} \) in \( L^{p'} \), we get

\[
\int \int_{\mathbb{R}^N} \Psi_{h_j} (x',x) (G (x',x) - G_k (x',x)) \, dx' \, dx \leq C \left( \int \int_{\mathbb{R}^N} |G (x',x) - G_k (x',x)|^p \, dx' \, dx \right)^{1/p}
\]

if \( k \to \infty \) uniformly in \( j \). So, in the analysis of (4.7), we only shall refer to the case of function \( G \in C (\mathbb{R}^N \times \mathbb{R}^N) \). We firstly consider the case \( p \geq 2 \). We perform the decomposition

\[
I = I_1 + I_2
\]
where
\[ I_1 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} k^{1/p'} h_j \frac{|u'_j - u_j|^p (u'_j - u_j) - |u' - u|^{p-2} (u' - u)}{|x' - x|^{p-1}} G(x', x) \, dx' \, dx, \]
\[ I_2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j k^{1/p'} \frac{|u' - u|^{p-2} (u' - u)}{|x' - x|^{p-1}} G(x', x) \, dx' \, dx. \]

The inequality (4.11) yields \( \left( \frac{1}{p} - \frac{1}{p'} = \frac{p-2}{p} \right) \)
\begin{equation}
I_1 \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j k^{\frac{1}{p'}} \frac{|u'_j - u_j - (u' - u)|}{|x' - x|^{p'}} \left( k^{\frac{1}{p'}} |u'_j - u_j| + k^{\frac{1}{p'}} |u' - u| \right)^{p-2} |G| \, dx' \, dx
\end{equation}
and by invoking Hölder we get
\begin{equation}
I_1 \leq C \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} k h_j \frac{|u'_j - u_j|}{|x' - x|^p} \right)^{1/p'} \times \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} h_j \frac{|u'_j - u_j| k^{\frac{1}{p'}} + k^{\frac{1}{p'}} |u' - u| |(p-2)p'| |G|}{|x' - x|^{(p-2)p'}} \right)^{1/p'}
\end{equation}

Since \( 0 < \frac{(p-2)p}{p-1} \leq p \) and \( \text{supp} G = K \) is a bounded domain in \( \mathbb{R}^N \times \mathbb{R}^N \), we can apply Jensen’s inequality to have the following uniform estimation:
\begin{align*}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j \frac{|u'_j - u_j| k^{\frac{1}{p'}} + k^{\frac{1}{p'}} |u' - u|}{|x' - x|^p} |G| \, dx' \, dx \\
&\leq C \int_{\text{supp} G} h_j \frac{|u'_j - u_j| + |u' - u|}{|x' - x|^p} \, dx' \, dx \\
&\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j k \left( |u'_j - u_j| + |u' - u| \right)^{p-2} |G| \, dx' \, dx \leq C
\end{align*}
After (4.8) and (4.9), is suffices to use (4.1) to deduce \( \lim I_1 = 0. \)

When \( 1 < p < 2 \) the situation does not essentially change because after using (4.1), the estimation
\[ \frac{|u'_j - u_j - (u' - u)|}{|x' - x|} \left( \frac{|u'_j - u_j| + |u' - u|}{|x' - x|^{p-2}} \right)^{p-2} \leq \frac{|u'_j - u_j - (u' - u)|}{|x' - x|^{p-1}} \]
and Hölder’s inequality guarantee
\begin{align*}
I_1 &\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j k^{\frac{1}{p'}} \frac{|u'_j - u_j - (u' - u)|^{p-1}}{|x' - x|^{p-1}} \, dx' \, dx \\
&\leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h_j k^{\frac{1}{p'}} \frac{|u'_j - u_j - (u' - u)|}{|x' - x|^p} \, dx' \, dx,
\end{align*}
and again, thanks to (4.5), we deduce \( \lim I_1 = 0. \)

Now we look at the limit of \( I_2. \) We know \( h_j \rightharpoonup h \) weak-* in \( L^\infty \left( \mathbb{R}^N \times \mathbb{R}^N \right), \) \( k^{1/p'} \frac{|u' - u|^{p-2} (u' - u)}{|x' - x|^{p-1}} \in \)
\( L' \) and \( G \in L^\infty \), then it is evident that 
\[
\lim I_2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} h^{1/p'} \frac{|u' - u|^{p-2}(u' - u)}{|x' - x|^{p-1}} G(x', x) \, dx' \, dx
\]
We have proved \( \Psi_{h_j} \rightharpoonup \Psi_h \) weakly in \( L'(\mathbb{R}^N \times \mathbb{R}^N) \) if \( j \to \infty \).}

We have proved strong convergence in \( X_0 \) implies \( H \) convergence. The reverse implication is also true. Indeed, if we assume \( u_j = L_{h_j} f \to u = L_h f \) weakly in \( X_0 \) and \( \Psi_{h_j} \to \Psi_h \) weakly in \( L' \), then Theorem 2 actually establishes that \( u_j \to u \) strongly in \( L^p \), at least for a subsequence. In addition, the weak convergence of \( (\Psi_{h_j})_j \) in \( L' \) ensures this sequence is uniformly bounded \( L^p \). Then, by using (4.4) we realize that the norm of \( \|u_j - u\|_{X_0} \) tends to zero:
\[
\lim_j B_{h_j} (u_j - u, u_j - u) = \lim_j \int d\Psi_j (x) (u_j - u) (x) \, dx
\leq \lim_j ||d\Psi_j (x)||_{L'} \|u_j - u\|_{L^p} = 0
\]
This amounts to state the strong convergence for the whole sequence \( (u_j)_j \), towards \( u \) strongly in \( X_0 \).

5. Conclusions

Our starting point is the nonlocal energy criterion, Theorem 2. This result establishes \( u_j = L_{h_j}^{-1} f \to u = L_h^{-1} f \) strongly in \( L^p \) (at least for a subsequence). After that, we have improved this convergence by means of Theorem 3. We have proved \( u_j = L_{h_j}^{-1} f \to u = L_h^{-1} f \) strongly in \( X_0 \), at least for a subsequence. From this strong convergence in \( X_0 \) we have inferred, as a particular case, the \( G \)-convergence of \( (L_{h_j})_j \) towards \( L_h \) (Theorem 4). And finally, we have proved that the \( H \)-convergence is equivalent to the strong convergence in \( X_0 \) (see Theorem 5 and the final comments).

We conclude that, as it occurs in the local setting, the appropriate conditions of monotonicity for the nonlocal operator and the use of the corresponding nonlocal energy criterion seem to be the key points to establish the type of convergences we have studied here.

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