Application of the analytical discrete velocities method for calculating mass and heat fluxes of rarefied gas in the problem of the Poiseuille flow

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Abstract. Analytical version of the discrete ordinates method is used to for calculating the gas macroparameters in a plane channel in the problem of Poiseuille flow. The system of finite-difference equations that determine the general solution of a homogeneous integro-differential equation after separation of variables is written in a matrix form. Householder algorithm is used for reducing the coefficient matrix to a tridiagonal one. QL algorithm with implicit shifts is applied for searching eigenvalues. The Gauss method with a permutation of the rows is used to determine the coefficients in the expansion of the solution of the required boundary value problem with allowance for the boundary conditions on the channel walls. The software implementation of the algorithm is developed using the C ++ programming language. The mass velocity profile and the heat flow profile of the gas, the flow rate and heat flow through the cross section of the channel are calculated for a wide range of the Knudsen number. The comparison with similar results presented in the open press is performed.

1. Introduction
Actively developing micro- and nanotechnologies require conducting systematic fundamental research of the processes taking place in them. Given the size of existing microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS), conducting experimental studies is often difficult or even impossible [1]. Therefore, the main direction of research is the use of mathematical modeling methods, which complexity is related to the fact that in practice most of them operate in a sufficiently wide range of Knudsen numbers. The latter circumstance leads to either using several models of gas flow for different flow regimes and their subsequent matching, or applying universal approaches, which include gas flow simulation based on the Boltzmann kinetic equation. The difficulty of its numerical solution lies in the presence of a six-dimensional collision integral. One of the possible ways to construct solutions is to simplify the right integral part of this equation [2].

The BGK (Bhatnagar, Gross and Krook) model of Boltzmann kinetic equation and the Shakhov model [2] are the most famous models for rarefied monatomic gas. Despite the fact that the right integral part of equation is reduced to a threefold integral in velocity space, the construction of its solution is a rather complicated computational problem even in the case of problems with the simplest cross section geometry [3].
There is a huge number of studies, which are devoted to the numerical solution of the Boltzmann equation, references can be found in [2]. There are three main directions:

- direct statistical modeling methods built on the basis of some random Markovian process capable of approximating the Boltzmann dynamics;
- finite-element methods using basis decomposition in a certain functional space;
- discrete velocity methods implying a fixed set of available molecular velocities.

Direct statistical modeling methods are widely used in many applied fields because of flexibility and simplicity, but sometimes inherent fluctuations severely limit the accuracy of the results obtained. On the contrary, finite-element methods have the best ratio of errors to the dimension of the approximation space, but in a fairly narrow class of solutions. Discrete ordinate method (first used in [4]) combines the merits of both groups of methods. The essence of this method lies in the discrete approximation of the kinetic equation and the desired solution on a fixed grid in the phase space and on the calculation of the collision integrals at the nodes of this grid with the replacement of the collision integral by a suitable quadrature formula [5]. A modification of the discrete velocity method, the so-called analytical discrete ordinate method (ADO), a discrete analogue of the Case method [6], was presented in [7–12].

In the present work we use ADO method to solve the problem of a Poiseuille flow in a plane channel with infinite parallel walls. We calculated the specific mass flow of gas and heat (per channel width unit) for arbitrary values of the distance between the channel walls and the nature of the interaction of gas molecules with the channel walls. Also we compared our findings with similar results from open sources.

2. Formulation of the problem
Consider the flow of gas in a channel with walls located on the planes $x' = \pm a'$ in a rectangular Cartesian coordinate system. Assume that the temperature of the channel walls is constant everywhere and equals $T$, and the motion of the gas is due to a constant pressure gradient $G'_n = (1/p)(dp/dz')$. We direct the axis $Oz'$ of the Cartesian coordinate system along the pressure gradient. Then the BGK model of the Boltzmann kinetic equation in the chosen coordinate system is written in the form [1]

$$v_x \frac{\partial f}{\partial x'} + v_z \frac{\partial f}{\partial z'} = \frac{p}{\eta} (f_{eq} - f).$$

(1)

Here $f(x', y', v)$ is the gas molecules distribution function in coordinates and velocities, $v_x$ and $v_y$ are the projections of the gas molecules velocities, $p$ and $\eta$ are the pressure and the coefficient of dynamic viscosity of the gas, $f_{eq}(x', y', v)$ is the local equilibrium Maxwellian with parameters given on the channel walls. The relative pressure drop over the mean free path of the gas molecules $l_g$ is assumed to be small. This allows us to solve the problem in a linearized form. The distribution function may be represented in the form

$$f(x', y', v) = n(z) \left( \frac{\beta}{\pi} \right)^{3/2} [1 + C_z G_n Z(x, C_z)].$$

(2)

Here $n(z)$ is molecular concentration of gas, $x = x'/l_g$ and $z = z'/l_g$ are dimensionless coordinates, $\beta = m/2k_b T$, $C_z = \beta^{1/2} v_z$, $G_n$ is dimensionless velocity component and pressure gradient. As a boundary condition on the walls of the channel we adopt diffusive reflection Maxwell’s boundary conditions. In this case

$$f^+(r_s, v) = (1 - \alpha) f^-(r_s, v - 2n(vv)) + \alpha f(r_s, v).$$

(3)
Here \( f^+(r_s,v) \) and \( f^-(r_s,v-2n(nv)) \) are the distribution functions of the molecules reflected from the channel walls and falling on it, \( r_s \) is the radius vector of the points of the channel walls, \( \alpha \) is the accommodation coefficient of the tangential momentum of the gas molecules by the channel walls, \( n \) is the normal vector to the channel walls directed toward the gas [3]. Substituting (2) in (1) and (3) for finding \( Z(x,C_x) \), we arrive at the boundary-value problem \((\mu = C_x)\)

\[
\mu \frac{\partial Z}{\partial x} + Z(x,\mu) + 1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\tau^2)Z(x,\tau) d\tau, \tag{4}
\]

\[
Z(\pm a, \mp \mu) = (1 - \alpha) Z(\pm a, \pm \mu), \quad \mu > 0. \tag{5}
\]

The general solution of (4) may be written in the form

\[
Z(x,\mu) = x^2 - 2x\mu + 2\mu^2 - a^2 - \frac{1}{\mu} \int_{-\infty}^{+\infty} \exp(-\tau^2) \left[ Y(x,\tau) + Y(x,-\tau) \right] d\tau. \tag{6}
\]

Substituting (6) into (4) to find \( Y(x,\mu) \) we get the homogeneous equation

\[
\mu \frac{\partial Y}{\partial x} + Y(x,\mu) = \int_{-\infty}^{+\infty} \exp(-\tau^2) \left[ Y(x,\tau) + Y(x,-\tau) \right] d\tau. \tag{7}
\]

Here \( x \in [-a; a], \mu \in (-\infty; +\infty), \Psi(\tau) = \exp(-\tau^2)/\sqrt{\pi} \). Notice that

\[
\int_{-\infty}^{+\infty} \Psi(\tau) Y(x,\tau) d\tau = \int_{-\infty}^{+\infty} \Psi(\tau) Y(x,\tau) d\tau + \int_{0}^{+\infty} \Psi(\tau) Y(x,\tau) d\tau = \int_{0}^{+\infty} \Psi(\tau) [Y(x,\tau) + Y(x,-\tau)] d\tau.
\]

Taking this into account, equation (7) can be presented in the form

\[
\mu \frac{\partial Y}{\partial x} + Y(x,\mu) = \int_{0}^{+\infty} \Psi(\tau) [Y(x,\tau) + Y(x,-\tau)] d\tau. \tag{8}
\]

If we change \( \mu \) on \(-\mu\) in (8), we arrive at equation

\[
-\mu \frac{\partial Y}{\partial x} + Y(x,-\mu) = \int_{0}^{+\infty} \Psi(\tau) [Y(x,\tau) + Y(x,-\tau)] d\tau. \tag{9}
\]

3. Applying analytical discrete ordinate method

Following [2], we pass from (8) and (9) to the finite-difference equations in the velocity space

\[
\mu_i \frac{d}{dx} Y(x,\mu_i) + Y(x,\mu_i) = \sum_{k=1}^{N} \omega_k \Psi(\mu_k) [Y(x,\mu_k) + Y(x,-\mu_k)], \tag{10}
\]
\[-\mu_i \frac{d}{dx} Y(x, -\mu_i) + Y(x, -\mu_i) = \sum_{k=1}^{N} \omega_k \Psi(\mu_k) [Y(x, \mu_k) + Y(x, -\mu_k)]. \tag{11}\]

Here \(\omega_k\) is the weight coefficients of the quadrature formula used to find the value of the integrals in the right-hand sides of (10) and (11). In the present paper 6-point Newton-Cotes quadrature formulas were used to calculate the integrals. Therefore, \(\omega_{m+1} = \omega_{m+6} = 19/288, \omega_{m+2} = \omega_{m+4} = 75/288, \omega_{m+3} = 50/288, m = 0, 1, \ldots, n, N = 6n + 1, \mu_i > 0, i = 1, 2, \ldots, N\). Similarly to the Case method [3], we seek the solution (10) and (11) in the form

\[Y(x, \pm \mu_i) = \phi(\nu, \pm \mu_i)e^{-x/\nu}. \tag{12}\]

Here \(\nu\) is spectral parameter. The substitution of (12) into (10) and (11) leads to two matrix equations

\[\frac{1}{\nu} M \Phi_+ = (I - W) \Phi_+ - W \Phi_-, \tag{13}\]

\[-\frac{1}{\nu} M \Phi_- = (I - W) \Phi_- - W \Phi_. \tag{14}\]

Here \(M = \text{diag} \{\mu_1, \mu_2, \ldots, \mu_N\}, \Phi_\pm = [\phi(\nu, \pm \mu_1), \phi(\nu, \pm \mu_2), \ldots, \phi(\nu, \pm \mu_N)]^T, I\) is identity matrix, \(W_{ij} = \omega_j \Psi(\mu_j)\). If we subtract and summarize the relations (13) and (14), we obtain

\[\frac{1}{\nu} M (\Phi_+ + \Phi_-) = \Phi_+ - \Phi_-, \tag{15}\]

\[\frac{1}{\nu} M (\Phi_+ - \Phi_-) = (I - 2W) (\Phi_+ + \Phi_-). \tag{16}\]

If we let

\[U = \Phi_+ + \Phi_-\]

then the system of matrix equations (15) and (16) can be written in the form of a single matrix equation

\[\frac{1}{\nu^2} MMU = (I - 2W) U. \tag{17}\]

Transforming equation (17), we successively find

\[\frac{1}{\nu^2} MU = M^{-1} (I - 2W) M^{-1} MU, \]

\[\frac{1}{\nu^2} MU = (D - 2W M^{-1}) MU, \]

where \(D = \text{diag} \{\mu_1^{-2}, \mu_2^{-2}, \ldots, \mu_N^{-2}\}\).

Multiplying the last equation from the left by some diagonal matrix we rewrite the resulting equality in the form

\[T (D - 2W M^{-1}) T^{-1} TMU = \frac{1}{\nu^2} TMU\]

or

\[(D - 2V) X = \lambda X. \tag{18}\]

Here \(V = T M^{-1} W T^{-1} M^{-1}, X = TMU, \lambda = 1/\nu^2\) and took into account that the product of diagonal matrix is commutative. We choose the matrix \(T\) so that the matrix \(V\) is symmetric.
In this way we can find

\[
M^{-1} T = \begin{pmatrix}
\frac{1}{\mu_1} & 0 & \ldots & 0 \\
0 & \frac{1}{\mu_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{\mu_N}
\end{pmatrix}
\begin{pmatrix}
T_1 & 0 & \ldots & 0 \\
0 & T_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & T_N
\end{pmatrix}
= \begin{pmatrix}
T_1 & 0 & \ldots & 0 \\
0 & T_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & T_N
\end{pmatrix}
\mu_N.
\]

\[
M^{-1} TW = \begin{pmatrix}
\frac{T_1}{\mu_1} & 0 & \ldots & 0 \\
0 & \frac{T_2}{\mu_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{T_N}{\mu_N}
\end{pmatrix}
\begin{pmatrix}
\omega_1 \Psi(\mu_1) & \omega_2 \Psi(\mu_2) & \ldots & \omega_N \Psi(\mu_N) \\
\omega_1 \Psi(\mu_1) & \omega_2 \Psi(\mu_2) & \ldots & \omega_N \Psi(\mu_N) \\
\omega_1 \Psi(\mu_1) & \omega_2 \Psi(\mu_2) & \ldots & \omega_N \Psi(\mu_N) \\
\omega_1 \Psi(\mu_1) & \omega_2 \Psi(\mu_2) & \ldots & \omega_N \Psi(\mu_N)
\end{pmatrix}
= \begin{pmatrix}
\frac{T_1}{\mu_1} \omega_1 \Psi(\mu_1) & \frac{T_1}{\mu_1} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_1}{\mu_1} \omega_N \Psi(\mu_N) \\
\frac{T_2}{\mu_2} \omega_1 \Psi(\mu_1) & \frac{T_2}{\mu_2} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_2}{\mu_2} \omega_N \Psi(\mu_N) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{T_N}{\mu_N} \omega_1 \Psi(\mu_1) & \frac{T_N}{\mu_N} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_N}{\mu_N} \omega_N \Psi(\mu_N)
\end{pmatrix}
\mu_N.
\]

\[
V = M^{-1} TW T^{-1} M = \begin{pmatrix}
\frac{T_1}{\mu_1} \omega_1 \Psi(\mu_1) & \frac{T_1}{\mu_1} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_1}{\mu_1} \omega_N \Psi(\mu_N) \\
\frac{T_2}{\mu_2} \omega_1 \Psi(\mu_1) & \frac{T_2}{\mu_2} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_2}{\mu_2} \omega_N \Psi(\mu_N) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{T_N}{\mu_N} \omega_1 \Psi(\mu_1) & \frac{T_N}{\mu_N} \omega_2 \Psi(\mu_2) & \ldots & \frac{T_N}{\mu_N} \omega_N \Psi(\mu_N)
\end{pmatrix}
\mu_N.
\]

Thus, the matrix \( V \) will be symmetric under the condition

\[
\frac{T_i \omega_i \Psi(\mu_i)}{\mu_i \mu_j T_j} = \frac{T_j \omega_j \Psi(\mu_j)}{\mu_i \mu_j T_i}.
\]

Hence

\[
\frac{T_i}{T_j} = \sqrt{\frac{\omega_i \Psi(\mu_i)}{\omega_j \Psi(\mu_j)}}.
\]

Thus, the matrix \( V \) can be represented as

\[
V = ZZ^T.
\]
Here
\[ z^T = \left( \frac{\sqrt{\omega_1 \Psi(\mu_1)}}{\mu_1}, \frac{\sqrt{\omega_2 \Psi(\mu_2)}}{\mu_2}, \ldots, \frac{\sqrt{\omega_N \Psi(\mu_N)}}{\mu_N} \right)^T \]
and the symbol \( T \) means transposition.

Thus, finding the values of the spectral parameter \( \nu \) from expression (12) reduces to finding the eigenvalues of the system of equations (18). Since the matrix of the system of equations (18) is symmetric, its eigenvalues are real and different. In order to find them we have used the Householder algorithm, which made it possible to move from a symmetric matrix to a tridiagonal one. The search for the eigenvalues of the tridiagonal matrix was carried out using a QL algorithm with implicit shifts. Hence, the problem of finding the spectral parameter values is solved. Now we turn to finding the eigenfunctions corresponding to the found spectral parameters. We use the normalization condition
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\left(-\tau^2\right) Z(x, \tau) d\tau = 1,
\]
or which is the same for the finite-difference equations (10) and (11)
\[
\sum_{k=1}^{N} \omega_k \Psi(\mu_k) [Y(x, \mu_k) + Y(x, -\mu_k)] = 1. \tag{19}
\]

Substituting (12) into (10) and (11) with mentioned normalization condition, we find
\[
\phi(\nu, \mu_i) = \frac{\nu}{\nu - \mu_i},
\]
\[
\phi(\nu, -\mu_i) = \frac{\nu}{\nu + \mu_i}.
\]

Then the solutions of equations (8) and (9) can be written as a linear combination of the constructed solutions
\[
Y(x, \pm \mu_i) = \sum_{j=1}^{N} \left[ A_j \nu_j \frac{e^{-a-x}/\nu_j}{\nu_j \mp \mu_i} + B \frac{\nu_j}{\nu_j \pm \mu_i} e^{-a-x}/\nu_j \right]. \tag{20}
\]
A direct substitution shows that the constant \( A \) and the function \( B(x, \mp \mu_i) \) are also solutions of equations (8) and (9). Thus, the general solutions (8) and (9) are written in the form
\[
Y(x, \pm \mu_i) = A + B(x, \mp \mu_i) + \sum_{j=1}^{N-1} \left[ A_j \nu_j \frac{e^{-a-x}/\nu_j}{\nu_j \mp \mu_i} + B \frac{\nu_j}{\nu_j \pm \mu_i} e^{-a-x}/\nu_j \right]. \tag{21}
\]

By virtue of condition (19), the values of the spectral parameter tend to zero for an infinite increase \( N \) and, as a consequence, the particular solutions corresponding to them in (20) and do not contribute to \( Y(x, \pm \mu_i) \). With this in mind when writing (21), we excluded from (20) particular solutions that correspond to the minimum of the found values of the spectral parameter. Substituting (6) and (21) into the boundary conditions (5), we arrive at a system of linear algebraic equations for determining the unknown constants \( A, B, A_j \) and \( B_j \)
\[
\sum_{j=1}^{N-1} \left\{ M_{ij} A_j + N_{ij} B_j e^{-2a/\nu_j} \right\} + \alpha A - B [\alpha a + \mu_i (2 - \alpha)] = \alpha \mu_i^2 + \alpha \mu_i (2 - a), \tag{22}
\]
and
\[ \sum_{j=1}^{N-1} \left( M_{ij} B_j + N_{ij} A_j e^{-2a/\nu_j} \right) + \alpha A + B \left[ \alpha a + \mu_i (2 - a) \right] = \alpha \mu_i^2 + \alpha \mu_i (2 - a). \] (23)

Here \( i = 1, 2, \ldots, N, \)
\[ M_{ij} = \nu_j \left[ \frac{\alpha \nu_j + \mu_i (2 - \alpha)}{\nu_j^2 - \mu_i^2} \right], \]
\[ N_{ij} = \nu_j \left[ \frac{\alpha \nu_j - \mu_i (2 - \alpha)}{\nu_j^2 - \mu_i^2} \right]. \]

We have used the Gauss method with a permutation of the rows to solve the system of equations (22) and (23). A solution of the system of equations (22) and (23) completes the solution of the boundary value problem (4) and (5). Proceeding from the statistical meaning of the distribution function [3], we finally find the mass velocity profile of the gas in the channel \( U(x) \)
\[ U(x) = \int_{-\infty}^{+\infty} \Psi(\mu) Z(x, \mu) d\mu, \] (24)
and the flow rate through the cross section of the channel \( J_M \) [1]
\[ J_M = -\frac{1}{2a^2} \int_{-a}^{a} U(x) dx. \] (25)

Substituting the obtained results in (24) and (25), we finally find the mass velocity profile of the gas in the channel
\[ U(x) = \frac{1}{2} \left( 1 - a^2 + x^2 \right) - \left[ A + Bx + \sum_{j=1}^{N-1} \left( A_j e^{-(a-x)/\nu_j} + B_j e^{-(a-x)/\nu_j} \right) \right] \] (26)
and the flow rate
\[ J_M = \frac{1}{2a^2} \left[ 2Aa + \sum_{j=1}^{N-1} \nu_j (A_j + B_j) \left( 1 - e^{-2a/\nu_j} \right) \right] - \frac{1}{2a} \left( 1 - \frac{2}{3} a^2 \right). \] (27)

Similarly, we find the heat flow profile of the gas
\[ q(x) = \int_{-\infty}^{+\infty} \Psi(\mu) \left( \mu^2 - \frac{1}{2} \right) Z(x, \mu) d\mu, \] (28)
and the heat flow through the cross section of the channel \( J_Q \) [1]
\[ J_Q = -\frac{1}{2a^2} \int_{-a}^{a} q(x) dx. \] (29)
Table 1. The mass velocity profile of the gas in the channel $U(x)$ for $2a = 2$.

| $x$  | $\alpha = 0.8 \ [9]$ | $\alpha = 0.8 \ (26)$ | $\alpha = 1 \ [9]$ | $\alpha = 1 \ (26)$ |
|------|---------------------|----------------------|-------------------|-------------------|
| 0.0  | -2.31962            | -2.32430             | -1.87458          | -1.87855          |
| 0.1  | -2.31215            | -2.31682             | -1.86706          | -1.87103          |
| 0.2  | -2.28964            | -2.29430             | -1.84440          | -1.84836          |
| 0.3  | -2.25176            | -2.25640             | -1.80627          | -1.81020          |
| 0.4  | -2.19790            | -2.20250             | -1.75206          | -1.75595          |
| 0.5  | -2.12707            | -2.13163             | -1.68078          | -1.68462          |
| 0.6  | -2.03767            | -2.04216             | -1.59082          | -1.59459          |
| 0.7  | -1.92699            | -1.93140             | -1.47952          | -1.48321          |
| 0.8  | -1.79004            | -1.79435             | -1.34193          | -1.34551          |
| 0.9  | -1.61528            | -1.61946             | -1.16676          | -1.17020          |
| 1.0  | -1.34037            | -1.34551             | -0.89392          | -0.89824          |

Table 2. The flow rate through the cross section of the channel $J_M$.

| $2a$ | $\alpha = 0.8 \ [9]$ | $\alpha = 0.8 \ (27)$ | $\alpha = 1 \ [9]$ | $\alpha = 1 \ (27)$ |
|------|---------------------|----------------------|-------------------|-------------------|
| 0.05 | 3.08971             | 3.15899              | 2.30226           | 2.37010           |
| 0.10 | 2.70744             | 2.74444              | 2.03271           | 2.06814           |
| 0.30 | 2.24477             | 2.25934              | 1.70247           | 1.71605           |
| 0.50 | 2.10227             | 2.11224              | 1.60187           | 1.61095           |
| 0.70 | 2.03877             | 2.04671              | 1.55919           | 1.56629           |
| 0.90 | 2.00924             | 2.01603              | 1.54180           | 1.54779           |
| 1.00 | 2.00187             | 2.00825              | 1.53868           | 1.54427           |
| 2.00 | 2.04139             | 2.04587              | 1.59486           | 1.59863           |
| 5.00 | 2.43823             | 2.44150              | 1.99077           | 1.99335           |
| 7.00 | 2.74611             | 2.74914              | 2.29493           | 2.29727           |
| 9.00 | 3.06346             | 3.06635              | 2.60925           | 2.61145           |

Substituting the obtained results in (28) and (29), we finally come to

$$q(x) = \frac{1}{2} \left( 1 + \sum_{j=1}^{N-1} A_j e^{-\alpha x/\nu_j} + B_j e^{-(\alpha - x)/\nu_j} \right),$$

(30)

$$J_Q = -\frac{1}{4a^2} \left[ 2a + \sum_{j=1}^{N-1} \nu_j (A_j + B_j) \left( 1 - e^{-2a/\nu_j} \right) \right].$$

(31)

4. Results and discussion

The software implementation for calculating the values of $U(x)$, $q(x)$, $J_M$ and $J_Q$ have been performed using the C ++ programming language. The results of the calculations for various values of the accommodation coefficient of the tangential momentum $\alpha$ are given in tables 1–4.
Table 3. The heat flow profile of the gas in the channel $q(x)$ for $2a = 2$.

| $x$ | $\alpha = 0.8$ [12] | $\alpha = 0.8$ (30) | $\alpha = 1$ [12] | $\alpha = 1$ (30) |
|-----|---------------------|---------------------|-------------------|-------------------|
| 0.0 | 0.24205             | 0.26786             | 0.24126           | 0.26190           |
| 0.1 | 0.24138             | 0.26663             | 0.24046           | 0.26064           |
| 0.2 | 0.23935             | 0.26286             | 0.23802           | 0.25680           |
| 0.3 | 0.23587             | 0.25641             | 0.23833           | 0.25022           |
| 0.4 | 0.23080             | 0.24696             | 0.22775           | 0.24000           |
| 0.5 | 0.22391             | 0.23403             | 0.21946           | 0.22743           |
| 0.6 | 0.21482             | 0.21679             | 0.20853           | 0.20992           |
| 0.7 | 0.20291             | 0.19392             | 0.19420           | 0.18673           |
| 0.8 | 0.18704             | 0.16289             | 0.17511           | 0.15538           |
| 0.9 | 0.16469             | 0.11794             | 0.14824           | 0.11022           |
| 1.0 | 0.12276             | 0.02847             | 0.09806           | 0.02174           |

Table 4. The heat flow through the cross section of the channel $J_Q$.

| $2a$ | $\alpha = 0.8$ [12] | $\alpha = 0.8$ (31) | $\alpha = 1$ [12] | $\alpha = 1$ (31) |
|------|---------------------|---------------------|-------------------|-------------------|
| 0.05 | -1.08087            | -1.02414            | -0.84529          | -0.78808          |
| 0.10 | -0.86598            | -0.83829            | -0.69449          | -0.66684          |
| 0.30 | -0.57121            | -0.56263            | -0.48420          | -0.47538          |
| 0.50 | -0.45517            | -0.45030            | -0.39850          | -0.39346          |
| 0.70 | -0.38646            | -0.38314            | -0.34638          | -0.34293          |
| 0.90 | -0.33924            | -0.33677            | -0.30970          | -0.30712          |
| 2.00 | -0.32050            | -0.31832            | -0.29490          | -0.29262          |
| 5.00 | -0.11166            | -0.11137            | -0.11426          | -0.11397          |
| 7.00 | -0.08533            | -0.08515            | -0.08850          | -0.08832          |
| 9.00 | -0.06908            | -0.06895            | -0.07219          | -0.07206          |

Since the integrand in (24) and (28) contains a factor proportional to $\exp(-\mu^2)$ the integral in $U(x)$ and $q(x)$ converges sufficiently fast. Taking this into account, the interval $[0; 3.5]$ was considered in computing the values of $U(x)$ and $q(x)$ instead of infinite integration interval. The time and accuracy of the calculations depended on the number of points of the partition. The calculations showed that the accuracy computing $10^{-4}$ can be achieved at $N = 1600$. The computation time on an Intel Core i5 processor was 51 seconds when solving the system of equations (22) and (23) by the Gauss method and 14 seconds when calculating the eigenvalues of the system of equations (18). As it can be seen from the tables, the obtained values $U(x)$, $q(x)$, $J_M$ and $J_Q$ are in good agreement with the analogous results from [9].

References

[1] Rudyak V Y, Belkin A A, Ivanov D A and Andrushenko V A 2011 Self-Diffusion and viscosity coefficients of fluids in nanochannels Proc. of 3rd Micro and Nano Flows Conference (Thessaloniki) 174

[2] Sharipov F and Seleznev V 1998 Data on Internal Rarefied Gas Journal of Physical and Chemical Reference Data 3 27657–706

[3] Kloss Y Y, Shuvalov P V and Tcheremissine F G 2012 Solving Boltzmann equation on GPU Procedia Computer Science 1 108391
[4] Nordsieck A and Hicks B L 1966 *Monte Carlo evaluation of the Boltzmann collision integral: tech. rep.* (Urbana, Illinois: University of Illinois)

[5] Dodulad O I, Kloss Y Y, Potapov A P et al 2016 Simulation of rarefied gas flows on the basis of the Boltzmann kinetic equation solved by applying a conservative projection method *Computational Mathematics and Mathematical Physics* 6 56 996–1011

[6] Latyshev A V and Yushkanov A A 2004 *Analytical solutions of boundary-value problems for kinetic equations* (Moscow: Moscow State University) p 286

[7] Siewert C E, Garcia R D M and Granjean P 1980 *Journal of Mathematical Physics* 21 2760–63

[8] Barichello L B and Siewert C E 1999 *Z. Angew. Math. Phys.* 50 972–81

[9] Barichello L B, Camargo M, Rodrigues P and Siewert C E 2001 *Z. Angew. Math. Phys.* 52 517–34

[10] Siewert C E 2002 *European Journal of Mechanics B/Fluids* 21 579–97

[11] Siewert C E and Sharipov F 2002 *Phys. Fluids* 14 4123–29

[12] Siewert C E 2003 *Z. Angew. Math. Phys.* 54 273–303