General solution of equations of motion for a classical particle in 9-dimensional Finslerian space

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Abstract

A Lagrangian description of a classical particle in a 9-dimensional flat Finslerian space with a cubic metric function is constructed. The general solution of equations of motion for such a particle is obtained. The Galilean law of inertia for the Finslerian space is confirmed.

1 Introduction

In the works [1, 2], the theory of Finslerian N-spinors (hyperspinors) was developed. In particular, it was shown that the simplest nontrivial case of the theory for \( N = 2 \) reproduced a formalism of standard Weyl 2-spinors in the 4-dimensional Minkowski space. However, in case of \( N = 3 \), the theory deals with geometrical objects in the 9-dimensional flat Finslerian space with the cubic metric function. Therefore, it is interesting to study the behaviour of a classical point particle in such a space.

In this paper, we construct the Lagrangian description of a free particle moving in the above-mentioned Finslerian space. It should be noted that the corresponding equations of motion are strongly nonlinear ordinary differential equations. Nevertheless, those are integrable in elementary functions. The main purpose of the paper is to obtain the general solution of the equations of motion for a classical particle in the 9-dimensional Finslerian space.

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2 Equations of motion and their general solution

We start with a description of the background Finslerian geometry. Let $\mathcal{A}^9$ be a 9-dimensional affine space over $\mathbb{R}$ and $\{O, E_0, E_1, \ldots, E_8\}$ be a coordinate system in it. Then any point $M \in \mathcal{A}^9$ is uniquely characterized by its radius vector $\mathbf{X} \equiv \vec{OM}$ so that $\mathbf{X} = X^A E_A$, where $X^A \in \mathbb{R}$ ($A = 0, 1, \ldots, 8$) are components of $\mathbf{X}$ with respect to the basis $\{E_0, E_1, \ldots, E_8\}$ of the associated vector space. The Finslerian length $|\mathbf{X}|$ of the radius vector is defined by the homogeneous cubic form

$$|\mathbf{X}|^3 = G_{ABC} X^A X^B X^C = [(X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2] X^8$$

$$- X^0 [(X^4)^2 + (X^5)^2 + (X^6)^2 + (X^7)^2]$$

$$+ 2 X^1 [X^4 X^6 + X^5 X^7] + 2 X^2 [X^4 X^5 - X^4 X^7]$$

$$+ X^3 [(X^4)^2 + (X^5)^2 - (X^6)^2 - (X^7)^2],$$

(1)

where $G_{ABC}$ are components of a symmetric “metric tensor” with respect to the basis $\{E_0, E_1, \ldots, E_8\}$ and $A, B, C = 0, 1, \ldots, 8$ [2].

Let us consider the linear transformations

$$X'^A = L(D_3)^A_B X^B, \quad L(D_3)^A_B = \frac{1}{2} \text{Tr}(\lambda^A D_3 \lambda_B D_3^+),$$

(2)

where $D_3 \in \text{SL}(3, \mathbb{C})$ is an arbitrary unimodular $3 \times 3$ matrix over $\mathbb{C}$, $\lambda^A = \lambda_A$ ($A = 0, 1, \ldots, 7$), $\lambda^8 = 2 \lambda_8$,

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and “$^+$” denotes Hermitian conjugating. Note that $\lambda_1, \lambda_2, \ldots, \lambda_7$ coincide with the well-known Gell-Mann matrices. According to [2], the form (1) is invariant under the coordinate transformations (2), i.e., $G_{ABC} X'^A X'^B X'^C = G_{ABC} X^A X^B X^C$. 
It is important that the transformations (2) have a consistent 4-dimensional limit. Indeed, the $3 \times 3$ matrices
\[
\hat{D}_3 = \begin{pmatrix} D_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_2 \in \text{SL}(2, \mathbb{C})
\]
form a subgroup of $\text{SL}(3, \mathbb{C})$ and split (2) into the independent subtransformations:
\[
\begin{align*}
X'^\alpha &= L(\hat{D}_3)^\alpha_\beta X^\beta & (\alpha, \beta = 0, 1, 2, 3), \\
X'^i &= L(\hat{D}_3)^i_j X^j & (i, j = 4, 5, 6, 7), \\
X'^8 &= X^8.
\end{align*}
\]
It is shown in [2] that (3) is a transformation law of a Lorentz 4-vector, while (4) is that of a Majorana 4-spinor. Because of (5), $X^8$ is a Lorentz scalar. These facts will be necessary below.

Let a point particle $M$ move along a “world line” $X^A(\tau), \tau \in \mathbb{R}$ in the Finslerian space $\mathcal{A}^9$. In general, the evolution parameter $\tau$ may be arbitrary (in particular, $\tau \sim X^0$). However, it is convenient to have a description which is explicitly invariant under the coordinate transformations (2). Therefore, $\tau = \text{inv}$ in all our considerations. By analogy with the standard relativistic theory [3], we assume that the action for a free particle is proportional to the Finslerian length of its “world line”
\[
S = \int_{\tau_i}^{\tau_f} L \, d\tau = \kappa \int_{\tau_i}^{\tau_f} (G_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C)^{1/3} \, d\tau,
\]
where $\tau_i$ and $\tau_f$ correspond to the initial and final “world points” in $\mathcal{A}^9$, $L$ is a Lagrangian, $\kappa$ is a constant characterizing physical properties of the particle, the dots denote differentiating with respect to $\tau$, and the formula (1) is applied to $|\dot{X}|$.

It is evident that the action (6) is invariant under both the coordinate transformations (2) and the continuously differentiable reparametrizations:
\[
\tau' = \tau'(\tau), \quad \frac{d\tau'}{d\tau} \neq 0 \quad (\tau_i \leq \tau \leq \tau_f).
\]

The action (6) implies the following equations of motion
\[
\frac{\delta S}{\delta X^A} = -\frac{d}{d\tau} P_A = 0 \quad (A = 0, 1, \ldots, 8),
\]
where the canonical momenta have the form

\[
\begin{align*}
    P_0 &= \frac{\partial L}{\partial \dot{X}^0} = \frac{2}{3} \left( \dot{X}^0 \dot{X}^8 - (\dot{X}^4)^2 - (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2 \right), \\
    P_1 &= \frac{\partial L}{\partial \dot{X}^1} = \frac{2}{3} \left( -\dot{X}^1 \dot{X}^8 + \dot{X}^4 \dot{X}^6 + \dot{X}^5 \dot{X}^7 \right), \\
    P_2 &= \frac{\partial L}{\partial \dot{X}^2} = \frac{2}{3} \left( -\dot{X}^2 \dot{X}^8 + \dot{X}^5 \dot{X}^6 - \dot{X}^4 \dot{X}^7 \right), \\
    P_3 &= \frac{\partial L}{\partial \dot{X}^3} = \frac{2}{3} \left( -2 \dot{X}^3 \dot{X}^8 + (\dot{X}^4)^2 + (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2 \right), \\
    P_4 &= \frac{\partial L}{\partial \dot{X}^4} = \frac{2}{3} \left( -\dot{X}^0 \dot{X}^4 + \dot{X}^1 \dot{X}^6 - \dot{X}^2 \dot{X}^7 + \dot{X}^3 \dot{X}^4 \right), \\
    P_5 &= \frac{\partial L}{\partial \dot{X}^5} = \frac{2}{3} \left( -\dot{X}^0 \dot{X}^5 + \dot{X}^1 \dot{X}^7 + \dot{X}^2 \dot{X}^6 + \dot{X}^3 \dot{X}^5 \right), \\
    P_6 &= \frac{\partial L}{\partial \dot{X}^6} = \frac{2}{3} \left( -\dot{X}^0 \dot{X}^6 + \dot{X}^1 \dot{X}^4 + \dot{X}^2 \dot{X}^5 - \dot{X}^3 \dot{X}^6 \right), \\
    P_7 &= \frac{\partial L}{\partial \dot{X}^7} = \frac{2}{3} \left( -\dot{X}^0 \dot{X}^7 + \dot{X}^1 \dot{X}^5 - \dot{X}^2 \dot{X}^4 - \dot{X}^3 \dot{X}^7 \right), \\
    P_8 &= \frac{\partial L}{\partial \dot{X}^8} = \frac{2}{3} \left( (\dot{X}^0)^2 - (\dot{X}^1)^2 - (\dot{X}^2)^2 - (\dot{X}^3)^2 \right).
\end{align*}
\]

Thus, (9) are the first integrals of the equations (8). Since the Lagrangian is a homogeneous function of \(\dot{X}^A\), another first integral (the canonical energy)

\[ E = (\partial L/\partial \dot{X}^A) \dot{X}^A - L = L = L \equiv 0. \]

As a consequence of (9), we obtain the very important condition:

\[ \dot{G}_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C \neq 0 \quad (\tau_i \leq \tau \leq \tau_f). \quad (10) \]

Because of (10), the “world line” of the particle is “nonisotropic” in \(A \).

A method of solving the equations (8)–(9) is based on the remarkable matrix identity

\[
\begin{pmatrix}
    \dot{X}^0 + \dot{X}^3 & \dot{X}^1 - i\dot{X}^2 & \dot{X}^4 - i\dot{X}^5 \\
    \dot{X}^1 + i\dot{X}^2 & \dot{X}^0 - \dot{X}^3 & \dot{X}^6 - i\dot{X}^7 \\
    \dot{X}^4 + i\dot{X}^5 & \dot{X}^6 + i\dot{X}^7 & \dot{X}^8 \\
\end{pmatrix}
\begin{pmatrix}
    P_0 + P_3 & P_1 - iP_2 & P_4 - iP_5 \\
    P_1 + iP_2 & P_0 - P_3 & P_6 - iP_7 \\
    P_4 + iP_5 & P_6 + iP_7 & 2P_8 \\
\end{pmatrix}
= \frac{2}{3} \left( G_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C \right)^{1/3}
\begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
\end{pmatrix}
\quad \forall \dot{X}^A \in \mathbb{R}.
\]

\[ (11) \]
which can be proved by direct computations with the help of (9). It should be noted that

\[
\begin{bmatrix}
\dot{X}^0 + \dot{X}^3 & \dot{X}^1 - i\dot{X}^2 & \dot{X}^4 - i\dot{X}^5 \\
\dot{X}^1 + i\dot{X}^2 & \dot{X}^0 - \dot{X}^3 & \dot{X}^6 - i\dot{X}^7 \\
\dot{X}^4 + i\dot{X}^5 & \dot{X}^6 + i\dot{X}^7 & \dot{X}^8
\end{bmatrix} = G_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C. \quad (12)
\]

Due to invariance of the equations (8)–(9) with respect to the reparametrizations (7), it is possible to use any convenient evolution parameter \( \tau \) in the expressions (9) for the canonical momenta \( P_A \). If we choose the arc length \( s(\tau) = \int_{\tau_0}^{\tau} (G_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C)^{1/2} d\tau \) of the “world line” \( X^A(\tau) \) as the evolution parameter, then

\[
G_{ABC} \dot{X}^A \dot{X}^B \dot{X}^C = 1 \quad (\tau = s, s_i = 0, |s| \leq |s_f|). \quad (13)
\]

This choice considerably simplifies the equations of motion, so that, instead of (8)–(9), we obtain the following system of ordinary differential equations

\[
\frac{\kappa}{3} [2\dot{X}^0 \dot{X}^8 - (\dot{X}^4)^2 - (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2] = P_0(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^1 \dot{X}^8 + \dot{X}^4 \dot{X}^6 + \dot{X}^5 \dot{X}^7] = P_1(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^2 \dot{X}^8 + \dot{X}^5 \dot{X}^6 - \dot{X}^4 \dot{X}^7] = P_2(0),
\]

\[
\frac{\kappa}{3} [-2\dot{X}^3 \dot{X}^8 + (\dot{X}^4)^2 + (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2] = P_3(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^0 \dot{X}^4 + \dot{X}^1 \dot{X}^6 - \dot{X}^2 \dot{X}^7 + \dot{X}^3 \dot{X}^4] = P_4(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^0 \dot{X}^5 + \dot{X}^1 \dot{X}^7 + \dot{X}^2 \dot{X}^6 + \dot{X}^3 \dot{X}^5] = P_5(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^0 \dot{X}^6 + \dot{X}^1 \dot{X}^4 + \dot{X}^2 \dot{X}^5 - \dot{X}^3 \dot{X}^6] = P_6(0),
\]

\[
\frac{\kappa}{3} [2 - \dot{X}^0 \dot{X}^7 + \dot{X}^1 \dot{X}^5 - \dot{X}^2 \dot{X}^4 - \dot{X}^3 \dot{X}^7] = P_7(0),
\]

\[
\frac{\kappa}{3} [(\dot{X}^0)^2 - (\dot{X}^1)^2 - (\dot{X}^2)^2 - (\dot{X}^3)^2] = P_8(0), \quad (14)
\]

where \( P_A(0) \) are arbitrary real constants (initial momenta).

In order to apply the existence and uniqueness theorem from the theory of ordinary differential equations [4] to the system (14), it is necessary to solve (14) with respect to the derivatives \( X^A \), i.e., to represent it in the normal form. From the geometric point of view, we should find the intersection of the nine hyperquadrics defined by the equations (14) in \( \mathbb{R}^9 \).
Substituting (14) into the identity (11) and using (13), we obtain the matrix equation

\[
\begin{pmatrix}
\dot{X}^0 + \dot{X}^3 & \dot{X}^1 - i\dot{X}^2 & \dot{X}^4 - i\dot{X}^5 \\
\dot{X}^1 + i\dot{X}^2 & \dot{X}^0 - \dot{X}^3 & \dot{X}^6 - i\dot{X}^7 \\
\dot{X}^4 + i\dot{X}^5 & \dot{X}^6 + i\dot{X}^7 & \dot{X}^8
\end{pmatrix}
\begin{pmatrix}
P_0(0) + P_3(0) \\
P_1(0) + iP_2(0) \\
P_4(0) + iP_5(0)
\end{pmatrix}
= \frac{2\kappa}{3} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(15)

for the explicit determination of \( \dot{X}^A \). Since any solution \( X^A(s) \) of the system (14) must also satisfy the condition (13), the constants \( P_A(0) \) are dependent. Indeed, it follows from (12), (13), and (15) that

\[
\begin{vmatrix}
P_0(0) + P_3(0) & P_1(0) - iP_2(0) & P_4(0) - iP_5(0) \\
P_1(0) + iP_2(0) & P_0(0) - P_3(0) & P_6(0) - iP_7(0) \\
P_4(0) + iP_5(0) & P_6(0) + iP_7(0) & 2P_8(0)
\end{vmatrix} = \left( \frac{2\kappa}{3} \right)^3.
\]

(16)

Because of (16), the corresponding \( 3 \times 3 \) matrix is nonsingular (\( \kappa \neq 0 \)), so that (15) implies

\[
\begin{pmatrix}
\dot{X}^0 + \dot{X}^3 & \dot{X}^1 - i\dot{X}^2 & \dot{X}^4 - i\dot{X}^5 \\
\dot{X}^1 + i\dot{X}^2 & \dot{X}^0 - \dot{X}^3 & \dot{X}^6 - i\dot{X}^7 \\
\dot{X}^4 + i\dot{X}^5 & \dot{X}^6 + i\dot{X}^7 & \dot{X}^8
\end{pmatrix}
\begin{pmatrix}
P_0(0) + P_3(0) \\
P_1(0) + iP_2(0) \\
P_4(0) + iP_5(0)
\end{pmatrix}
= \frac{2\kappa}{3} \begin{pmatrix}
P_0(0) + P_3(0) & P_1(0) - iP_2(0) & P_4(0) - iP_5(0) \\
P_1(0) + iP_2(0) & P_0(0) - P_3(0) & P_6(0) - iP_7(0) \\
P_4(0) + iP_5(0) & P_6(0) + iP_7(0) & 2P_8(0)
\end{pmatrix}^{-1}.
\]

(17)

With the help of (16) and (17), it is not difficult to express \( \dot{X}^A \) in terms of \( P_A(0) \). As a result, we have the system of ordinary differential equations

\[
\begin{align*}
\dot{X}^0 &= \frac{\kappa}{3} [P_{11}^{-1}(0) + P_{22}^{-1}(0)], \\
\dot{X}^1 &= \frac{\kappa}{3} [P_{12}^{-1}(0) + P_{21}^{-1}(0)], \\
\dot{X}^2 &= \frac{i\kappa}{3} [P_{12}^{-1}(0) - P_{21}^{-1}(0)], \\
\dot{X}^3 &= \frac{\kappa}{3} [P_{11}^{-1}(0) - P_{22}^{-1}(0)], \\
\dot{X}^4 &= \frac{\kappa}{3} [P_{13}^{-1}(0) + P_{31}^{-1}(0)], \\
\dot{X}^5 &= \frac{i\kappa}{3} [P_{13}^{-1}(0) - P_{31}^{-1}(0)].
\end{align*}
\]
\[ X^6 = \frac{\kappa}{3} [ P_{23}^{-1}(0) + P_{32}^{-1}(0)], \]
\[ \dot{X}^7 = \frac{i\kappa}{3} [ P_{23}^{-1}(0) - P_{32}^{-1}(0)], \]
\[ \dot{X}^8 = \frac{2\kappa}{3} P_{33}^{-1}(0), \]

where the complex constants

\[ P_{11}^{-1}(0) = \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) - P_3(0) & P_6(0) - iP_7(0) \\ P_6(0) + iP_7(0) & 2P_5(0) \end{vmatrix}, \]
\[ P_{12}^{-1}(0) = - \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_1(0) - iP_2(0) & P_4(0) - iP_5(0) \\ P_6(0) + iP_7(0) & 2P_5(0) \end{vmatrix}, \]
\[ P_{13}^{-1}(0) = \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_1(0) - iP_2(0) & P_4(0) - iP_5(0) \\ P_0(0) - P_3(0) & P_6(0) - iP_7(0) \end{vmatrix}, \]
\[ P_{21}^{-1}(0) = - \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_6(0) - iP_7(0) \\ P_4(0) + iP_5(0) & 2P_5(0) \end{vmatrix}, \]
\[ P_{22}^{-1}(0) = \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_4(0) - iP_5(0) \\ P_4(0) + iP_5(0) & 2P_5(0) \end{vmatrix}, \]
\[ P_{23}^{-1}(0) = - \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_4(0) - iP_5(0) \\ P_1(0) + iP_2(0) & P_6(0) - iP_7(0) \end{vmatrix}, \]
\[ P_{31}^{-1}(0) = \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_6(0) - P_3(0) \\ P_4(0) + iP_5(0) & P_6(0) + iP_7(0) \end{vmatrix}, \]
\[ P_{32}^{-1}(0) = - \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_1(0) - iP_2(0) \\ P_4(0) + iP_5(0) & P_6(0) + iP_7(0) \end{vmatrix}, \]
\[ P_{33}^{-1}(0) = \left( \frac{2\kappa}{3} \right)^{-3} \begin{vmatrix} P_0(0) + P_3(0) & P_1(0) - iP_2(0) \\ P_1(0) + iP_2(0) & P_6(0) - P_3(0) \end{vmatrix}. \]

form the Hermitian \(3 \times 3\) matrix \(\|P_{ab}^{-1}(0)\|\).

It is evident that the systems (14) and (18)–(19) are equivalent (the implications (14) \(\Rightarrow\) (15) \(\Rightarrow\) (17) \(\Rightarrow\) (18) are reversible). However, (18)–(19) is a trivial linear system. Therefore, the general solution of the system (14) can be immediately written as

\[ X^0(s) = \dot{X}^0(0)s + X^0(0), \quad X^1(s) = \dot{X}^1(0)s + X^1(0), \]
\[ X^2(s) = \dot{X}^2(0)s + X^2(0), \quad X^3(s) = \dot{X}^3(0)s + X^3(0), \]
\[ X^4(s) = \dot{X}^4(0)s + X^4(0), \quad X^5(s) = \dot{X}^5(0)s + X^5(0), \]
\[ X^6(s) = \dot{X}^6(0)s + X^6(0), \quad X^7(s) = \dot{X}^7(0)s + X^7(0), \]
\[ X^8(s) = \dot{X}^8(0)s + X^8(0), \]

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\[ X^8(s) = \dot{X}^8(0)s + X^8(0) \quad (0 \leq |s| \leq |s_f|), \]  

(20)

where \( X^A(0) \) are arbitrary real constants (initial coordinates) and

\[ \dot{X}^0(0) = \frac{\kappa}{3} [P^{-1}_{11}(0) + P^{-1}_{22}(0)], \quad \dot{X}^1(0) = \frac{\kappa}{3} [P^{-1}_{12}(0) + P^{-1}_{21}(0)], \]

\[ \dot{X}^2(0) = \frac{i\kappa}{3} [P^{-1}_{12}(0) - P^{-1}_{21}(0)], \quad \dot{X}^3(0) = \frac{\kappa}{3} [P^{-1}_{11}(0) - P^{-1}_{22}(0)], \]

\[ \dot{X}^4(0) = \frac{\kappa}{3} [P^{-1}_{13}(0) + P^{-1}_{31}(0)], \quad \dot{X}^5(0) = \frac{i\kappa}{3} [P^{-1}_{13}(0) - P^{-1}_{31}(0)], \]

\[ \dot{X}^6(0) = \frac{\kappa}{3} [P^{-1}_{23}(0) + P^{-1}_{32}(0)], \quad \dot{X}^7(0) = \frac{i\kappa}{3} [P^{-1}_{23}(0) - P^{-1}_{32}(0)], \]

\[ \dot{X}^8(0) = \frac{2\kappa}{3} P^{-1}_{33}(0) \]  

(21)

are initial velocities. Due to (16), (17), and (19), the velocities (21) satisfy the condition

\[ \left| \begin{array}{cccccc}
\dot{X}^0(0) + \dot{X}^3(0) & \dot{X}^1(0) - i\dot{X}^2(0) & \dot{X}^4(0) - i\dot{X}^5(0) \\
\dot{X}^1(0) + i\dot{X}^2(0) & \dot{X}^0(0) - \dot{X}^3(0) & \dot{X}^6(0) - i\dot{X}^7(0) \\
\dot{X}^4(0) + i\dot{X}^5(0) & \dot{X}^6(0) + i\dot{X}^7(0) & \dot{X}^8(0)
\end{array} \right| = 1. \]  

(22)

Thus, the Galilean law of inertia for the Finslerian space \( \mathcal{A}^9 \) with the cubic metric (1) is confirmed: a free particle moves with a constant velocity or, in other words, its "world line" (20) is straight.

The solution (20), (22) was found as a geodesic of \( \mathcal{A}^9 \) in [5]. Such a geodesic for the \( N^2 \)-dimensional (\( N \geq 3 \)) flat Finslerian space with the \( N \)-ic metric was constructed in [6].

Returning to an arbitrary evolution parameter \( \tau \) and taking into account invariance of (8)–(9) with respect to the reparametrizations (7), we obtain the general solution of the equations of motion for a free classical particle in the form

\[ X^A = X^A(s(\tau)) \quad (\tau_i \leq \tau \leq \tau_f), \]  

(23)

where \( X^A(s) \) are the functions (20), while \( s = s(\tau) \) is a continuously differentiable function such that \( ds/d\tau \neq 0, s(\tau_i) = 0, \) and \( s(\tau_f) = s_f \). The general solution (23) contains 17 arbitrary real constants. Those are the 9 initial coordinates \( X^0(0), X^1(0), \ldots, X^8(0) \) and any 8 of the 9 initial velocities (21) satisfying the condition (22).

Up to this moment, we have not been interested in the physical meaning of the dimensional constant \( \kappa \) from (6). The time to connect \( \kappa \) with the characteristics of a real particle. According to the principle of correspondence, the action (6) must have the consistent 4-dimensional limit. This requirement will allow us to determine the constant \( \kappa \).
Let the evolution parameter $\tau$ be dimensionless. It is necessary to impose a constraint on the velocities $\dot{X}^A(\tau)$ in such a way that the action (6) coincides with the standard action

$$S = -mc \int_{\tau_i}^{\tau_f} (g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta)^{1/2} d\tau \quad (\alpha, \beta = 0, 1, 2, 3)$$

(24)

for a relativistic point particle in the Minkowski space. Here, $m$ is the mass of the particle, $c$ is the speed of light, and $\|g_{\alpha\beta}\| = \text{diag}(1, -1, -1, -1)$. Rewriting the action (6) in the explicit form

$$S = \kappa \int_{\tau_i}^{\tau_f} (g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta \dot{X}^8 - \dot{X}^0[(\dot{X}^4)^2 + (\dot{X}^5)^2 + (\dot{X}^6)^2 + (\dot{X}^7)^2]$$

$$+ 2\dot{X}^1[\dot{X}^4\dot{X}^6 + \dot{X}^5\dot{X}^7] + 2\dot{X}^2[\dot{X}^5\dot{X}^6 - \dot{X}^4\dot{X}^7]$$

$$+ \dot{X}^3[(\dot{X}^4)^2 + (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2])^{1/3} d\tau$$

(25)

and comparing (25) with (24), we obtain $\kappa = -mc$ under the condition that the following nonholonomic constraint

$$g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta \dot{X}^8 - \dot{X}^0[(\dot{X}^4)^2 + (\dot{X}^5)^2 + (\dot{X}^6)^2 + (\dot{X}^7)^2]$$

$$+ 2\dot{X}^1[\dot{X}^4\dot{X}^6 + \dot{X}^5\dot{X}^7] + 2\dot{X}^2[\dot{X}^5\dot{X}^6 - \dot{X}^4\dot{X}^7]$$

$$+ \dot{X}^3[(\dot{X}^4)^2 + (\dot{X}^5)^2 - (\dot{X}^6)^2 - (\dot{X}^7)^2] = (g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta)^{3/2}$$

(26)

is fulfilled for any $\tau \in [\tau_i, \tau_f]$.

Because of (3), (4), and (5), the constraint (26) is Lorentz invariant. Moreover, solving (26) with respect to $\dot{X}^8$ and substituting the result into (25), we get the action (24) if and only if $\kappa = -mc$. Thus, our theory has the consistent 4-dimensional limit when $\kappa = -mc$.

### 3 Conclusion

In this paper, we have considered a free point particle moving in the 9-dimensional flat Finslerian space $\mathcal{A}^9$ with the cubic metric (1). The corresponding action (6) is invariant under the reparametrizations (7). Using this invariance, we simplify the equations of motion (8)–(9) and solve them. We prove that (20)–(21) are the general solution of the system (14). This solution shows the validity of the Galilean law of inertia for the free motion in $\mathcal{A}^9$. 
Finally, we verify that the developed theory has the consistent 4-dimensional limit.

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