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Modeling the pandemic trend of 2019 Coronavirus with optimal control analysis

BiBi Fatima a, Gul Zaman a, Manar A. Alqudah b, Thabet Abdeljawad c,d,∗

a Department of Mathematics, University of Malakand, Chakdara Dir (Lower), Khyber Pakhtunkhawa, Pakistan
b Department of Mathematical Sciences, Faculty of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
c Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
d Department of Medical Research, China Medical University, Taichung 40402, Taiwan

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A B S T R A C T

In this work, we propose a mathematical model to analyze the outbreak of the Coronavirus disease (COVID-19). The proposed model portrays the multiple transmission pathways in the infection dynamics and stresses the role of the environmental reservoir in the transmission of the disease. The basic reproduction number \( R_0 \) is calculated from the model to assess the transmissibility of the COVID-19. We discuss sensitivity analysis to clarify the importance of epidemic parameters. The stability theory is used to discuss the local as well as the global properties of the proposed model. The problem is formulated as an optimal control one to minimize the number of infected people and keep the intervention cost as low as possible. Medical mask, isolation, treatment, detergent spray will be involved in the model as time dependent control variables. Finally, we present and discuss results by using numerical simulations.

1. Introduction

The 2019 novel Coronavirus (COVID-19) is a new virus that causes respiratory illness in people. This virus probably, originally, emerged from an animal source but now seems to be spreading from person-to-person. On 12th December 2019, a pandemic case of unknown etiology was reported in Wuhan City, Hubei Province, China, and on 31st December 2019, the disease outbreak was reported to World Health Organization (WHO) [1–4].

Cases have been detected in most countries worldwide and community spread is being detected in a growing number of countries. On March 11, the COVID-19 outbreak was characterized as a pandemic by the WHO [5,6].

This virus is viewed as having zoonotic origin, which is going to be the third zoonotic human Coronavirus that is emerging in the 21st century, after the severe acute respiratory syndrome Coronavirus (SARS-CoV) that emerged in 2003 and (MERS-CoV) that emerged in 2012. The symptoms of COVID-19 including cough, fever, fatigue, breathing difficulties similar to those caused by (SARS-CoV) and (MERS-CoV) infection [7,8].

Mathematical models have played an increasingly important role in predicting the behavior of outbreaks, optimizing control strategies, understanding the immune response and so on [9–15]. At the current stage, there are many unclear aspects about this novel Coronavirus, e.g., the contribution of asymptomatic transmission, the case fatality rate, and the initial source of 2019-nCoV. We believe that modelers can certainly make a substantial contribution to understand the virus and the transmission dynamics of the disease it causes. A number of modeling studies have already been performed for the COVID-19 pandemic. Based on reported data from December 31, 2019 to January 28, 2020, Wu et al. [16] introduced a susceptible-exposed-infectious-recovered (SEIR) model to describe the transmission dynamics, and forecasted the national and global spread of the disease. Read et al. [17] reported a value of 3.1 for the basic reproductive number based on data fitting of a SEIR model. A compartmental model which incorporate the clinical progression of the disease was proposed by Tang et al. [18].

In the current study, we investigate an epidemic model of COVID-19. This model will consist of six epidemiological classes i.e susceptible population \( S(\tau) \), exposed population \( E(\tau) \), infected population \( I(\tau) \), asymptomatic population \( A(\tau) \), hospitalized population \( H(\tau) \), recovered population \( R(\tau) \) and reservoir for COVID-19 \( W(\tau) \). In every disease the role of threshold parameter is very important for the transmission potential of a disease. We find the threshold quantity \( R_0 \) by using

* Corresponding author at: Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia.
E-mail addresses: fatima.uom@uom.edu.pk (B. Fatima), gzaman@uom.edu.pk (G. Zaman), maalqudah@psu.edu.sa (M.A. Alqudah), tabdeljawad@psu.edu.sa (T. Abdeljawad).

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Next generation method. We discuss sensitivity analysis in order to analyze the important of every epidemic parameter in the disease transmission. We use Routh Hurwitz criteria for the local stability of the proposed model, for global stability we use Lyapunov function theory and geometrical approach. We use optimal control strategy to minimize infected people and maximize the number of recovered people in the population. Medical mask, isolation, treatment and detergent will be involved in the model as time dependent control variable t. Finally, all the theoretical results will be verified with the help of numerical simulation for easy understanding.

This article is arrange as follow: In Section 2, we present the flowchart for the transmission of COVID-19 between reservoir and people. In Section 3, we develop COVID-19 virus transmission model. The basic reproductive number along with sensitivity analysis are presented in Section 4. In Sections 5 and 6 we investigate the stability of model (1). Numerical simulation of the stability results are presented in Section 7, to verify our analytical results. In Section 8, we discuss optimal control of the proposed model and its numerical simulation.

2. Flow chart

The reservoir for 2019 nCoronavirus (COVID-19) is denoted as W. In Fig. 1 the population is divided into six compartment: $S_r(t)$ is susceptible people; $E_r(t)$ is the exposed people; $I_r(t)$ is infectious people; infectious but asymptotic class $A_r(t)$; hospitalized $H_r(t)$; remover or recover class $R_r(t)$. b is the birth rate and $\mu_0$ is the death rate. The susceptible people will be infected through contact with W and $I_r$ and the transmission rate were defined by $\beta_r, \beta_a$. The transmissibility of $A_r$ was $a$ times $I_r$ and that of hospitalized was q time $I_r$.

3. Model formulation

This section, describe the (COVID-19) virus transmission model between reservoir and people and from people to people. This model contains a composition of ordinary differential equations. The compartmental deterministic mathematical model can be represented by nonlinear system of ordinary differential equations:

$$\begin{align*}
\frac{dS_r(t)}{dt} &= b - \beta_r I_r S_r - \beta_a A_r S_r - \beta_h H_r S_r - \mu_0 S_r, \\
\frac{dE_r(t)}{dt} &= \beta_r I_r S_r + \beta_a A_r S_r + \beta_h H_r S_r - (\epsilon + \mu_0)E_r, \\
\frac{dA_r(t)}{dt} &= \kappa(1 - \rho)E_r - (\gamma_a + \gamma_1)I_r - \mu_0 A_r, \\
\frac{dH_r(t)}{dt} &= \gamma_a I_r - (\gamma_2 + \mu_0)H_r, \\
\frac{dR_r(t)}{dt} &= \gamma_1 I_r + \gamma_2 H_r - \mu_0 R_r, \\
\frac{dW(t)}{dt} &= \phi_1 I_r + \phi_2 A_r - \delta W,
\end{align*}$$

with initial condition:

$$\begin{align*}
S_r(0) &> 0, \quad E_r(0) \geq 0, \quad I_r(0) \geq 0, \quad A_r(0) \geq 0, \quad H_r(0) \geq 0, \quad R_r(0) \geq 0, \quad W(0) \geq 0.
\end{align*}$$

$\beta_r$ shows transmission per unit time, $q$ shows the approximate transmissibility of hospitalized patient, $k$ is the progression at which individuals go to infectious class, $\rho$ is the moving rate from exposed class $E_r$ to infectious class $I_r$, $(1 - \rho)$ is that of transmissibility to asymptotic class $A_r$. The rate at which infected individuals are hospitalize is $\gamma_1$, and $\gamma_2$ is the recovery rate beyond hospitalization. The recovery rate of hospitalized patient is $\gamma_2$, $\delta$ is the life time of virus reservoir.

4. Equilibria and basic reproductive number

4.1. Equilibria

We discuss qualitative study of the proposed model. For this we find equilibria of the model (1). In order to find the disease free equilibrium of the proposed model (1), we set the right hand side of all equations equal to zero and set $E_r = I_r = A_r = H_r = R_r = W = 0$, we get $F_0$ is given by

$$F_0 = (S_r, 0, 0, 0, 0, 0, 0) \quad \text{and the endemic equilibrium point is represented by } E^* = (S^*_r, E^*_r, I^*_r, A^*_r, H^*_r, R^*_r, W^*)_T$$

and it occur when the disease present in the population where: $S^*_r, E^*_r, A^*_r, H^*_r, R^*_r, W^*$ are given in Box I.

4.2. Basic reproductive number

A simple but effective measure of the transmissibility of an infectious disease is given by the basic reproduction number $R_0$, defined as the total number of secondary infections produced by introducing a single infective in a completely susceptible population. In general, for simple epidemic models, if $R_0$ is greater than unity, an epidemic will occur while if $R_0$ is less than unity, an outbreak will most likely not occur.

To find $R_0$ for our proposed model (1) we use the method of Driessche and Watmough [19], we have

$$F = \begin{pmatrix}
0 & \beta_r S_r E_r & \beta_a S_r A_r & \beta_h S_r H_r & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Thus $R_0$ is the effective reproduction number of the model, a scalar and the approximate spectral radius of next generation matrix $H = FV^{-1}$.

$$R_0 = \frac{\beta_r \kappa \rho S_r(1 - \rho)}{\delta (\epsilon + \mu_0)} + \frac{\beta_a \kappa S_r}{\epsilon (\mu_0 + \delta)} + \frac{Q_1}{\epsilon (\mu_0 + \delta)} + \frac{Q_2}{\epsilon (\mu_0 + \delta)}.$$
\[ S_p = \frac{b(\epsilon + \mu_0)\kappa \rho + (\gamma_2 + \mu_0)\delta}{\kappa \rho} \]

\[ E_p = \frac{\gamma_a + \gamma_1 + \mu_0}{\kappa \rho} I_p^*, \]

\[ A_p = \frac{\kappa(1 - \rho)(\gamma_a + \gamma_1 + \mu_0)}{(\epsilon + \mu_0)\kappa \rho} I_p^*, \]

\[ H_p = \frac{\gamma_a(\epsilon + \mu_0)\kappa(1 - \rho)(\gamma_a + \gamma_1 + \mu_0)I_p^*}{(\gamma_2 + \mu_0)(\epsilon + \mu_0)\kappa}, \]

\[ R_p = \frac{\gamma_1(\gamma_2 + \mu_0)(\epsilon + \mu_0)\kappa + \gamma_a(\epsilon + \mu_0) + \kappa(1 - \rho)(\gamma_a + \gamma_1 + \mu_0)I_p^*}{(\gamma_2 + \mu_0)(\epsilon + \mu_0)}, \]

\[ W_p = \frac{\phi_1(\epsilon + \mu_0)\kappa \rho I_p^* + \phi_2(1 - \rho)(\gamma_a + \gamma_1 + \mu_0)I_p^*}{\rho(\epsilon + \mu_0)\kappa \rho}. \]

Box I.

4.3. Sensitivity analysis

In this section, we present sensitivity analysis of a few parameters which are used in the proposed model (1). This will make it easier for us to know that parameters that have highly effect on the reproductive number. For this analysis we apply the technic given in [20]. Sensitivity index of basic reproductive number \( R_0 \) is given by \( \Delta R_0 = \frac{\partial R_0}{\partial h} \) where \( h \) is parameter. For our model (1) sensitivity analysis is given by:

\[ \Delta R_p = \frac{\partial R_p}{\partial \beta_p} = 0.6015621127, \Delta R_0 = \frac{\partial R_0}{\partial \kappa} = -0.2211558165, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \gamma_1} = -0.2311558165, \Delta R_0 = \frac{\partial R_0}{\partial \beta_1} = 0.9398437885, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \mu_0} = 0.6015621127, \Delta R_0 = \frac{\partial R_0}{\partial \rho} = -0.2211558165, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \phi_1} = 0.9398437885, \Delta R_0 = \frac{\partial R_0}{\partial \phi_2} = 0.9999999997, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \gamma_0} = 0.2741112624, \Delta R_0 = \frac{\partial R_0}{\partial \gamma_a} = -0.6310175812, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \mu_0} = -0.0192053, \Delta R_0 = \frac{\partial R_0}{\partial \rho} = -0.00021125, \]

\[ \Delta R_0 = \frac{\partial R_0}{\partial \phi_1} = -0.0006740313. \]

Parameter Sensitivity indices Parameter Sensitivity indices
\[ \beta_p \quad + \quad \epsilon \quad - \]
\[ \beta_a \quad + \quad \gamma_0 \quad - \]
\[ b \quad - \quad \mu_0 \quad - \]
\[ \phi_1 \quad + \quad \phi_2 \quad + \]
\[ h \quad \epsilon \quad - \]
\[ a \quad + \quad \kappa \quad + \]
\[ \gamma_a \quad - \quad \gamma_2 \quad - \]
\[ \rho \quad - \quad \]

and summarized by:

Fig. 2 show sensitivity analysis of the basic reproductive number \( R_0 \). These indices allow us the importance of different factor involved in the disease transmission.

5. Local stability analysis

We show the local asymptotic stability of disease free equilibrium point and endemic equilibria of the system (1) in the following theorems.

**Theorem 1.** The DFE point \((S_0, 0, 0, 0, 0)\), is locally asymptotically stable if \( R_0 < 1 \), otherwise unstable for \( R_0 > 1 \).

**Proof.** The Jacobian matrix of the system at DFE point \((S_0, 0, 0, 0, 0)\), is given by

\[ J_0 = \begin{pmatrix}
-\mu_0 & 0 & -\beta_p S_p & -\beta_a S_p & -\beta_q S_p & -\beta_w S_p \\
0 & -(\kappa + \mu_0) & \beta_p S_p & \beta_a S_p & \beta_q S_p & \beta_w S_p \\
0 & 0 & \kappa \rho & -(\gamma_a + \gamma_1 + \mu_0) & 0 & 0 \\
0 & 0 & 0 & 0 & -\gamma_a & 0 \\
0 & 0 & 0 & 0 & \phi_1 & -\gamma_2 \\
0 & 0 & 0 & 0 & \phi_2 & 0 & -\delta \\
\end{pmatrix}. \]

**Theorem 2.** If \( R_0 > 1 \), then the endemic equilibrium point \( E^* \) is locally asymptotically stable, unstable for \( R_0 < 1 \).

**Proof.** Linearization of the model (1) around endemic equilibrium point \( E^* \) is given by

\[ J_0 = \begin{pmatrix}
A & 0 & -\beta_p S_p & -\beta_a S_p & -\beta_q S_p & -\beta_w S_p \\
A_1 & -(\kappa + \mu_0) & \beta_p S_p & \beta_a S_p & \beta_q S_p & \beta_w S_p \\
0 & \kappa \rho & -A_1 & 0 & 0 & 0 \\
0 & 0 & -\gamma_a & 0 & -A_2 & 0 \\
0 & 0 & 0 & \phi_1 & \phi_2 & 0 \\
0 & 0 & 0 & 0 & \phi_2 & 0 & -\delta \\
\end{pmatrix}. \]
Fig. 2. The graphs show the variation of different parameters and its effect on the basic reproductive number.
Using elementary row transformation we get the following matrix:

\[
J_x = \begin{bmatrix}
-\alpha & 0 & -\beta_s \rho S_p & -\beta_q S_p & -\beta_w S_p \\
0 & -\beta & -\beta_s \rho S_p & -\beta_q S_p & -\beta_w S_p \\
0 & 0 & -B & -\beta_s \rho S_p & -\beta_q S_p & -\beta_w S_p \\
0 & 0 & 0 & -B & -\beta_s \rho S_p & -\beta_q S_p & -\beta_w S_p \\
0 & 0 & 0 & 0 & -B & -\beta_s \rho S_p & -\beta_w S_p \\
0 & 0 & 0 & 0 & 0 & -B \\
\end{bmatrix},
\]

where \( B = (\kappa + \mu_0)\beta_p I_p + \beta_p \alpha A_p + \beta_q H_p + \mu_0 S_p, \) \( B_1 = (\gamma_a + \gamma_t + \mu_0)(\kappa + \mu_0)A, \) \( B_2(\kappa + \mu_0)(\kappa + \mu_0)(R_0 - 1)A, \) \( B_3(\gamma_a + \gamma_t + \mu_0)\gamma_2 + \mu_0). \) It is clear that all of the eigenvalues \( \lambda_i, \) for all \( i = 1, 2, 3, 4, \) of \( J_x \) have negative real parts for \( R_0 > 1 \) which completes the proof. \( \square \)

6. Global stability analysis

**Theorem 3.** For \( R_0 < 1 \) the disease free equilibrium of the system is stable globally asymptotically, unstable for \( R_0 > 1. \)

**Proof.** We define the following Lyapunov function, and show that this function satisfies the condition of Lyapunov function that is function is positive definite and its derivative is negative definite,

\[
U(t) = \frac{1}{2}((S_p - S^0_p) + E_p + I_p + A_p + H_p + (W - W_0)) (b - \mu_0 N(t)) \\
+ \phi_1 I_p + \phi_2 A_p - \delta W \\
+ Q Q_1((b - (1 - R_0)) - \mu_0 E_p(t)).
\]

By choosing the positive parameter \( d_1 = d_2 = d_3 = Q Q_1, \) \( d_4 = \frac{1}{Q^2}, \) \( d_5 = \mu_0 \) and after interpretation we can get

\[
U'(t) = -((S_p - S^0_p) + E_p + I_p + A_p + H_p + (W - W_0)) (b - \mu_0 N(t)) \\
+ \phi_1 I_p + \phi_2 A_p - \delta W \\
- Q Q_1((b - (1 - R_0)) - \mu_0 E_p(t)).
\]

where \( F^0 = \frac{b}{\mu_0}, \)

\[
U'(t) < 0 \text{ if } S_p > S^0_p \text{ and } R_0 < 1 \text{ and } U'(t) = 0, \text{ if } S_p = S^0_p.
\]

By Lasala inverence principle [23,24], \( E_p = I_p = A_p = H_p = W = 0. \)

All the conditions of Lyapunov function are satisfied that function is positive definite and its derivative is negative definite. Thus the disease free equilibrium is globally asymptotically stable in \( F_0. \) \( \square \)

**Theorem 4.** If \( R_0 > 1, \) then the endemic equilibrium point \( E^* \) is globally asymptotically stable and unstable otherwise.

**Proof.** To prove the global asymptotic stability of the proposed model (1) at endemic equilibrium \( E^*, \) we use castillo chevez method [25,26] let us consider the system of (1):

\[
\begin{align*}
\frac{dS_p(t)}{dt} &= b - \beta_p I_p S_p - \beta_p a A_p S_p - \beta_q H_p S_p - \beta_w W S_p - \mu_0 S_p, \\
\frac{dE_p(t)}{dt} &= \beta_p I_p S_p + \beta_p a A_p S_p + \beta_q H_p S_p + \beta_w W S_p - (\kappa + \mu_0) E_p, \\
\frac{dI_p(t)}{dt} &= \kappa \rho E_p - (\gamma_a + \gamma_t) I_p - \mu_0 I_p.
\end{align*}
\]

Taking the Jacobean as well as the additive compound matrix of order 2 for the above system (8), which may take the form is given by:

\[
J = \begin{bmatrix}
-a_{11} & 0 & -a_{13} \\
-a_{21} & a_{22} & a_{23} \\
0 & 0 & -a_{33}
\end{bmatrix},
\]

\[
J^{[2]} = \begin{bmatrix}
-(a_{11} + a_{22}) & a_{23} & -a_{13} \\
-a_{21} & -a_{23} & -a_{13} \\
-a_{21} & a_{23} & -a_{13}
\end{bmatrix}.
\]

Consider the function \( Q(x) = Q(S_p, E_p, I_p) = \text{diag} \{ S_p, E_p, S_p \}, \) the time derivative of the function, \( Q(f(x)) \), implies that

\[
Q(f(x)) = \text{diag} \{ S_p, E_p, S_p \},
\]

\[
\frac{dQ}{dt} = \begin{bmatrix}
S_p & -S_p E_p & S_p \\
E_p & -E_p S_p & E_p \\
S_p & -S_p E_p & S_p
\end{bmatrix},
\]

\[
\text{and } Q J^{[2]} = J^{[2]}.
\]

\[
A = Q J^{[2]} Q^{-1}, \text{ which can be written as}
\]

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

\[
\begin{align*}
A_{11} &= \frac{S_p}{S_p} - \frac{E_p}{E_p} - \beta_p I_p - \beta_p a A_p - \beta_q H_p - (\kappa + \mu_0), \\
A_{12} &= \beta_p S_p - \beta_p S_p, \quad B_{21} = \{ k \rho \} .
\end{align*}
\]

Let \( (c_1, c_2, c_3) \) be a vector in \( \mathbb{R}^3 \) and the \( ||.|| \) of \( (c_1, c_2, c_3) \) given by,

\[
||c_1, c_2, c_3|| = max \{ ||c_1||, ||c_2|| + ||c_3|| \}.
\]

Now we take the Lozinski measure described by [27], \( \epsilon(A) \leq sup \{ h_1, h_2 \} = sup \{ \epsilon(A_{11}) + ||(A_{12})|| + \epsilon(A_{22}) + ||A_{22}|| \}, \) where \( h_1 = \epsilon(A_0) + ||(A_{12})|| \text{ for } i = 1, 2 \) and \( i \neq j, \) which implies that

\[
h_1 = \epsilon(A_{11}) + ||(A_{12})||, \quad h_2 = \epsilon(A_{22}) + ||A_{22}||.
\]

\[
\epsilon(A_{11}) = \frac{S_p}{S_p} - \frac{E_p}{E_p} - \beta_p I_p - \beta_p a A_p - \beta_q H_p - (\kappa + \mu_0), \quad \epsilon(A_{22}) = \max \{|S_p - E_p| - \beta_p I_p - \beta_p a A_p - \beta_q H_p - (\kappa + \mu_0), \}
\]

\[
\beta_p W = |S_p - E_p - \beta_p I_p - \beta_p a A_p - \beta_q H_p - (\kappa + \mu_0)|, \quad ||(A_{12})|| = |S_p - E_p|
\]

\[
\] and \( ||A_{22}|| = \max \{ k \rho, 0 \} = 0. \) Therefore \( h_1 \) and \( h_2 \) becomes, such that,

\[
h_1 \leq \frac{S_p}{S_p} - 2\mu_0 - k \rho \quad \text{and} \quad h_2 \leq \frac{S_p}{S_p} - 2\mu_0 - \gamma - min(\gamma, k \rho) +, \text{ which show that}
\]

\[
\epsilon(A) \leq \frac{S_p}{S_p} - 2\mu_0 - \min(\gamma, \gamma_t) + \gamma_a.
\]

Hence \( \epsilon(B) \leq \frac{S_p}{S_p} - 2\mu_0. \) Taking integral of \( \epsilon(A), \) we get

\[
\lim_{t \to \infty} sup \int_0^t \epsilon(A) dt < \frac{2\mu_0}{k}.
\]

\[
\kappa = \lim_{t \to \infty} sup \int_0^t \epsilon(A) dt < 0.
\]

Hence model (1) is globally asymptotically stable. \( \square \)
7. Numerical simulation of stability results

We solved the proposed deterministic model using Runge-kutta method of order four [28]. This verify our analytical results. The variable and parameter value in Table 1 were used for simulation. For the purpose of illustrations, we assumed some parameters values. The choice of parameters are taken in the way as to be biologically feasible. The time interval is taken 0 – 250 units with initial population for susceptible people S_0(t), exposed people E_0(t), infected with COVID-19 I_0(t), asymptomatic people A_0(t), hospitalize people H_0(t), and recovered people R_0(t), reservoir for COVID-19 W_0(t).

The application of Runge-kutta method of order 4th on the proposed model leads to the following system:

\[ S_{p}^{i+1} - S_{p}^{i} = b - \beta_{p} I_{p} S_{p}^{i+1} + \beta_{p} A_{p} S_{p}^{i+1} - \beta_{p} q H_{p} S_{p}^{i+1} - \mu_{0} S_{p}^{i+1} \]

\[ E_{p}^{i+1} - E_{p}^{i} = \beta_{p} I_{p} S_{p}^{i+1} + \beta_{p} A_{p} S_{p}^{i+1} + \beta_{p} q H_{p} S_{p}^{i+1} + \mu_{0} W_{p} S_{p}^{i+1} \]

\[ I_{p}^{i+1} - I_{p}^{i} = \kappa_{p} E_{p}^{i+1} - \gamma_{p} I_{p}^{i+1} - \mu_{0} I_{p}^{i+1} \]

\[ A_{p}^{i+1} - A_{p}^{i} = \kappa_{p} (1 - \rho) E_{p}^{i+1} - (\gamma_{p} + \mu_{0}) A_{p}^{i+1} \]

\[ H_{p}^{i+1} - H_{p}^{i} = \gamma_{p} I_{p}^{i+1} + \epsilon_{p} A_{p}^{i+1} - (\gamma_{p} + \mu_{0}) H_{p}^{i+1} \]

\[ R_{p}^{i+1} - R_{p}^{i} = \gamma_{p} I_{p}^{i+1} + \gamma_{p} H_{p}^{i+1} - \mu_{0} R_{p}^{i+1} \]

\[ W_{p}^{i+1} - W_{p}^{i} = \phi_{p} I_{p}^{i+1} + \phi_{p} A_{p}^{i+1} - \delta W_{p}^{i+1} \]

7.1. Algorithm

Step 1: \( S_{0}(0) = 0, E_{0}(0) = 0, I_{0}(0) = 0, H_{0}(0) = 0, R_{0}(0) = 0, W_{0}(0) = 0 \)

Step 2: for \( i = 1, 2, \ldots n - 1 \),

\[ S_{p}^{i+1} = \frac{S_{p}^{i+1}}{1 + (\kappa_{p} S_{p}^{i+1} + \beta_{p} A_{p} S_{p}^{i+1} + \beta_{p} q H_{p} S_{p}^{i+1} + \mu_{0} W_{p}^{i+1}} + \mu_{0} \]

\[ E_{p}^{i+1} = \frac{E_{p}^{i}}{1 + (\kappa_{p} E_{p}^{i} + \beta_{p} a A_{p} + \beta_{p} q H_{p} + \mu_{0} W_{p}^{i}} + \mu_{0} \]

\[ I_{p}^{i+1} = \frac{I_{p}^{i}}{1 + (\gamma_{p} + \gamma_{p} + \mu_{p}) + \mu_{0}} \]

\[ A_{p}^{i+1} = \frac{A_{p}^{i}}{1 + (\mu_{p} + \mu_{0}) + \mu_{0}} \]

\[ H_{p}^{i+1} = \frac{H_{p}^{i}}{1 + (\gamma_{p} + \mu_{p} + \mu_{0}) + \mu_{0}} \]

\[ R_{p}^{i+1} = \frac{R_{p}^{i}}{1 + (\mu_{p} + \gamma_{p} + \mu_{p} + \mu_{0}) + \mu_{0}} \]

\[ W_{p}^{i+1} = \frac{W_{p}^{i}}{1 + \delta + \mu_{0}} \]

Step 3: for \( i = 1, 2, \ldots, n - 1 \), write \( S_{p}^{i+1} = S_{p}^{i}, E_{p}^{i+1} = E_{p}^{i}, I_{p}^{i+1} = I_{p}^{i}, A_{p}^{i+1} = A_{p}^{i}, H_{p}^{i+1} = H_{p}^{i}, R_{p}^{i+1} = R_{p}^{i}, W^{i+1} = W^{i} \)

When we run the above algorithm using Matlab software, we get the graphs presented in Figs. 3 and 4, which represent the dynamics of susceptible population (\( S_{p}(t) \)); exposed population (\( E_{p}(t) \)); infected with COVID-19 (\( I_{p}(t) \)); asymptomatic population (\( A_{p}(t) \)); hospitalized population (\( H_{p}(t) \)); recovered population (\( R_{p}(t) \)); and reservoir (\( W(t) \)).

The biological interpretation of these results show that if \( R_{0} < 1 \), then the susceptible population decreases, then become stable and shows that there will be always stable susceptible population. The dynamics of exposed, infected, asymptomatic, hospitalize, recover and reservoir for COVID-19 conclude that the number of these populations will decrease and reach to zero, which show the stability of the proposed model.

8. Optimal control strategy for COVID-19

We formulate control strategies on the basis of sensitivity analysis and dynamic of the proposed model. The maximum sensitivity index parameter is \( (\beta_{p}, \mu_{p}) \) whose value in (0.9398437, 0.601562) increase in this parameter by 10 percent would increase the threshold quantity by (9.939 and 6.01562). Therefore to control the spread of the disease we need to minimize this parameters by taking the control variable \( u_{1}(t) \) and \( u_{2}(t) \) representing (awareness about medical mask, hand washing and isolation of infected and non infected people). Moreover the parameters \( c, \delta, \mu_{0} \) decrease the threshold quantity by 10 percent by increasing this parameter, to increase this we use the control variables \( u_{3}(t), u_{4}(t) \) representing oxygen therapy, mechanical ventilation and detergent spray.

Our goal here are to reduce COVID-19 in the population through increasing the number of recovered person R(t) and decreasing the number of infectious I(t), and hospitalized H(t), environmental reservoir W(t) by applying the time dependent control variables \( u_{1}(t), u_{2}(t), u_{3}(t), u_{4}(t) \).

i. \( u_{1}(t) \) is the time dependent control variable representing the awareness about medical mask hard washing.

ii. \( u_{2}(t) \) is the time dependent control variable representing of infected people.

iii. \( u_{3}(t) \) is the time dependent control variable representing oxygen therapy mechanical ventilation.

iv. \( u_{4}(t) \) represent the time dependent control variable for environmental reservoir i.e detergent spray.

By using this control variables our optimal control problem which is modified version of (1) become

\[ dS_{p}(t) \frac{dt}{dt} = b - \beta_{p} I_{p} S_{p}(1 - u_{1}(t)) - \beta_{p} a A_{p} S_{p}(1 - u_{1}(t)) - \beta_{p} q H_{p} S_{p}(1 - u_{1}(t)) - \mu_{0} S_{p} \]

\[ -\beta_{p} q H_{p} S_{p}(1 - u_{1}(t)) - \mu_{0} S_{p} \]

\[ dE_{p}(t) \frac{dt}{dt} = \beta_{p} I_{p} S_{p}(1 - u_{1}(t)) + \beta_{p} A_{p} S_{p}(1 - u_{1}(t)) + \beta_{p} q H_{p} S_{p}(1 - u_{1}(t)) + \mu_{0} W_{p} S_{p}(1 - u_{1}(t)) - (\kappa_{p} + \mu_{0} + u_{1}(t)) E_{p} \]

\[ dI_{p}(t) \frac{dt}{dt} = \kappa_{p} E_{p} - (\gamma_{p} + \gamma_{p} + \mu_{0}) I_{p} \]

\[ dA_{p}(t) \frac{dt}{dt} = \kappa_{p} (1 - \rho) E_{p} \]

\[ dH_{p}(t) \frac{dt}{dt} = \gamma_{p} I_{p} + \epsilon_{p} A_{p} - (\gamma_{p} + \mu_{0}) H_{p} \]

\[ dR_{p}(t) \frac{dt}{dt} = \gamma_{p} I_{p} + \gamma_{p} H_{p} - \mu_{0} R_{p} \]

\[ dW(t) \frac{dt}{dt} = \phi_{p} I_{p} + \phi_{p} A_{p} - \delta W - u_{4}(t) W(t) \]

With initial condition

\[ S_{p}(0) > 0, \ E_{p}(0) > 0, \ I_{p}(0) > 0, \ A_{p}(0) > 0, \ H_{p}(0) > 0, \ R_{p}(0) > 0, \ W(0) > 0. \]
The goal here is to show that it is possible to implement time dependent control measures while minimizing the cost of implementation of those techniques [29]. We choose the objective (cost) function by

$$J(u_1, u_2, u_3, u_4) = \int_0^T [v_1 I_p + v_2 A_p + v_2 H_p + v_4 W + \frac{1}{2}(v_3 u_1^2(t) + v_4 u_2^2(t))
+ v_5 u_3^2(t) + v_6 u_4^2(t))] dt.$$ 

(16)

In Eq. (16) $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$, represent weight constant. The weight constant $v_1, v_2, v_3, v_4$, represent relative cost of infectious person $I_p$, asymptomatic person $A_p$, hospitalized person $H_p$ and reservoir $W$ while $v_3, v_5, v_7, v_8$, represents the associated cost of control variables. $v_3 u_1^2(t), v_4 u_2^2(t), v_5 u_3^2(t), v_6 u_4^2(t)$ describes self care, isolation, medical treatment, and detergent spray.

Our purpose is to find an optimal control pair $u_1^*, u_2^*, u_3^*, u_4^*$ such that

$$J(u_1^*, u_2^*, u_3^*, u_4^*) = \min \{J(u_1, u_2, u_3, u_4) | u_1, u_2, u_3, u_4 \in U \}$$

(17)
dependent on system (3), we define the control set,

$$U = \{(u_1, u_2, u_3, u_4) | u(t) \text{ is lebesgue measurable on } [0, 1],
0 \leq u_i(t) \leq 1, i = 1, 2, 3, 4 \}.$$ 

(18)

9. Existence of optimal control problem

Let us take the control system (15) along initial condition at time $t = 0$ and reveal the presence of the control problem. Where as bounded Lebesgue measurable controls, positive initial conditions and positive bounded solutions to the state system occur [30]. To assert the optimal solution, we go back to the optimal control problem (15), (16). First we use the Lagrangian and Hamiltonian considering the optimal control solution, we go back to the optimal control problem (15), (16). First we use the Lagrangian and Hamiltonian considering the optimal control solution, we go back to the optimal control problem (15), (16). First we use the Lagrangian and Hamiltonian considering the optimal control solution, we go back to the optimal control problem (15), (16). First we use the Lagrangian and Hamiltonian considering the optimal control solution, we go back to the optimal control problem (15), (16). First we use the Lagrangian and Hamiltonian considering the optimal control solution, we go back to the optimal control problem (15), (16).

**Theorem 5.** There exist an optimal control $u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \in U$, to the control problem as stated in Eqs. (15)–(16).
control problem is presented by the following equation, problem (15) and (16). Indeed the Lagrangian illustrate the optimal control we use the Lagrangian and Hamiltonian considering the optimal control problem as, \[ L = (I, A, H, W) = \int (v_1 I + v_2 A + v_3 H + v_4 W + \frac{1}{2}(v_5 u_1^2(t) + v_6 u_2^2(t) + v_7 u_3^2(t) + v_8 u_4^2(t))) \]

The plots demonstrate the time dynamics of different compartmental population (Hospitalized, Recovered or Removed and Reservoir for COVID-19).

### 10. Optimality condition

In order to characterize an optimal solution to (15) and (16). First we use the Lagrangian and Hamiltonian considering the optimal control problem (15) and (16). Indeed the Lagrangian illustrate the optimal control problem is presented by the following equation,

\[ L(I, A, H, W, u_1, u_2, u_3, u_4) = (v_1 I + v_2 A + v_3 H + v_4 W + \frac{1}{2}(v_5 u_1^2(t) + v_6 u_2^2(t) + v_7 u_3^2(t) + v_8 u_4^2(t))). \]

We define the associated Hamiltonian (H), therefore using the notion \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \) and \( F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7) \) then, For the smallest value of the Lagrangian, we determine Hamiltonian \( H \) for the optimal control problem as,

\[ H(x, u, \lambda) = L(x, u) + \lambda F(x, u), \]

where,

\[ x = (S, E, I, A, H, R, W), \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7). \]

Following the Pontryagin’ Maximum Principle [31,32] for finding the optimal solution to the proposed control problem (15). Using \( (x^*, u^*) \) as a notation for the optimal solution then,

\[ \frac{dx}{dt} = \frac{\partial H}{\partial x}, \quad 0 = \frac{\partial H}{\partial u} \]

The maximality condition

\[ H(t, x^*(t), u^*(t), \lambda(t))dx = \max_{x_t, u_t, x_t, u_t \in [0, t]} H(x^*(t), u_t, x_t, u_t, \lambda(t)); \]
define the transversality condition as
\[
\lambda(T) = 0
\]  
(21)

**Theorem 6.** Let the optimal state variables and control variables are denoted by \( S^*_t, E^*_t, I^*_t, A^*_t, H^*_t, R^*_t, W^* \) be optimal state \((u^*_1, u^*_2, u^*_3, u^*_4)\) for the optimal control problem (15) and (16). Then the set of adjoint variables \( \lambda(t) \) satisfying
\[
\lambda'_1(t) = (\lambda_1 - \lambda_2)\beta_1 I^*_t + \beta_2 A^*_t + \beta_3 H^*_t + \beta_4 W^*(1 - u_1(t)) + \lambda_1 \mu_1, \\
\lambda'_2(t) = (\lambda_2 - \lambda_3)\kappa + (\lambda_4 - \lambda_5)\kappa \rho + \lambda_2 u_1(t), \\
\lambda'_3(t) = \gamma_1 + (\lambda_1 - \lambda_2)\beta_2 S^*_t (1 - u_1(t)) + (\lambda_3 - \lambda_3)\gamma_6 \\
\quad - \lambda_2 \gamma_6 u_2(t) - \lambda_2 \gamma_2, \\
\lambda'_4(t) = -\gamma_2 + (\lambda_1 - \lambda_2)\beta_2 A^*_t (1 - u_1(t)) + (\lambda_4 - \lambda_5)\kappa \\
\quad + \lambda_4 (\mu_1 + \mu_2(t)) - \lambda_4 \theta_2, \\
\lambda'_5(t) = -\gamma_3 + (\lambda_1 - \lambda_2)\beta_3 H^*_t (1 - u_1(t)) + (\gamma_2 + \gamma_6 + u_1(t))\lambda_5 - \lambda_5 \gamma_6 u_2(t), \\
\lambda'_6(t) = -\gamma_6 + (\lambda_1 - \lambda_6)\beta_4 S^*_t (1 - u_1(t)) + \lambda_6 (\gamma_6 + \gamma_2(t)), \\
\lambda'_7(t) = -\gamma_4 + (\lambda_1 - \lambda_2)\beta_2 A^*_t (1 - u_1(t)) + \lambda_7 (\delta + u_2(t)),
\]  
(22)

the transversality conditions (Boundary conditions) is define as,
\[
\lambda_i = 0 \quad \text{for} \quad i = 1, 2, 3, 4, 5, 6.
\]  
(23)

More over, the controls variables \( u^*_1(t), u^*_2(t), u^*_3(t), u^*_4(t) \) are obtained as:
\[
u^*_1(t) = \max \{ \min \left( \frac{(\lambda_1 I^*_t + \lambda_5 A^*_t + \lambda_4 E^*_t + \lambda_1 H^*_t - \lambda_2 E^*_t)}{\gamma_3}, 1, 0 \right) \right\},
\]  
(24)

**Proof.** The adjoint system (22) comes from the direct application of the Pontryagin Maximum Principle (20), while the transversal conditions are the direct consequences of \( \lambda(T) = 0 \). For the set of optimal functions \( u^*_1, u^*_2, u^*_3, u^*_4 \) we used \( \frac{\partial H}{\partial u} \). We solve the optimality system numerically in the subsequent section. Because it would be easy in understanding for the reader rather than analytical results. The optimality system are characterized by the control system (15), the adjoint system (22), boundary (terminal) conditions, together with the optimal control functions. Clearly, the simulation carried out justified our control strategies to minimize the infected population, asymptomatic population, hospitalize and reservoir, and to maximize the susceptible and recovered population as shown in Figs. 5–6. 

\[\square\]

10.1. Numerical simulation of optimal control analysis

Here we solve the optimal control system (15) to see the impact of medical mask, isolation, treatment and detergent spray by using the Runge–Kutta method of order four. We use forward Runge-Kutta procedure to solve the state system (16) with initial condition in
Fig. 6. The graphical results show the dynamics of the compartmental population hospitalized, recovered, reservoir with and without controls.

11. Conclusion

In this paper we have developed a mathematical model to investigate the current outbreak of Coronavirus disease (COVID-19). The proposed model described the multiple transmission pathways in the infection dynamics, and stressed the role of the environmental reservoir in the transmission of the disease. This model consists of six epidemiological classes i.e susceptible population $S_I(t)$, exposed population $E_I(t)$, infected population $I_I(t)$, asymptomatic population $A_I(t)$, hospitalized population $H_I(t)$, recovered population $R_I(t)$ and reservoir for COVID-19 $W(t)$. In every disease the role of threshold parameter is very important for transmission potential of diseases. We have found threshold quantity $R_0$ by using the next generation matrix method. We have used Routh Hurwitz criteria for the local stability of the proposed model, while for the global stability we have used the Lyapunov function theory and geometrical approach. We further, have used optimal control strategy to minimize infected people and maximize the number of recovered people in the population. Medical mask, isolation, treatment and detergent spray have been involved in the model as time dependent control variable $t$. Finally, all the theoretical results have been verified with the help of numerical simulation for easy understanding.

CRediT authorship contribution statement

Bibi Fatima: Conceptualization, Methodology, Investigation, Visualization, Writing - original draft, Review and editing. Gul Zaman: Conceptualization, Methodology, Investigation, Visualization, Supervision, Review and editing. Manar A. Alqudah: Methodology, Investigation, Visualization, Review, Editing. Thabet Abdeljawad: Methodology, Investigation, Visualization, Review, Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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