A Precise Determination of $\alpha_s$ From Lattice QCD

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Abstract

We present a new determination of the QCD strong coupling constant based on precise lattice calculations of the $\Upsilon$ spectrum. The largest systematic uncertainty in previous such determinations resulted from the absence of vacuum polarization from light quarks. We substantially reduce this error by including two flavors of dynamical light quarks and extrapolating to three. We find $\alpha_s(3 V(8.2 \text{ GeV}) = 0.196(3)$ for three light flavors, corresponding to $\alpha_s^{\text{MS}}(M_Z) = 0.115(2)$. This is significantly more accurate than previous determinations using this or any other technique.

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In recent years it has become apparent that numerical simulations of lattice quantum chromodynamics (QCD) could provide accurate and reliable determinations of the strong interaction coupling constant. The largest systematic error in previous lattice determinations resulted from the neglect of quark vacuum polarization, whose sizable contribution was estimated perturbatively. In this paper we significantly reduce this systematic error by presenting simulation results that include dynamical light quarks. We find $\alpha_s(8.2 \text{ GeV}) = 0.196(3)$ for the strong coupling constant with $n_f = 3$ flavors of light quarks; $V$ denotes the physical scheme based on the static-quark potential, discussed in Refs. [3][4][5]. The corresponding $\text{MS}$ coupling at the $Z$ mass is $\alpha_s^{\text{MS}}(M_Z) = 0.115(2)$, consistent with the world average of 0.117(5)[6], but significantly more accurate.
In this study we used two different sets of simulations, one without quark vacuum polarization ($n_f = 0$), the other with two flavors of light quarks ($n_f = 2$) \cite{9}. The $n_f = 2$ gauge-field configurations were generated using a staggered-quark action for light quarks with the Hybrid Molecular Dynamics algorithm, and both used the standard Wilson action for gluons. For $b$ quarks, we employed the nonrelativistic formulation of quark dynamics (NRQCD); details are in Refs. \cite{10,11}.

There are two steps in a determination of the coupling constant from lattice simulations. The first is to specify or accurately determine the parameters of the lattice lagrangian, and in particular the lattice spacing $a$. The second is to use this lagrangian to compute nonperturbatively an appropriate short-distance quantity. Comparison with the perturbative expansion for the same quantity fixes the coupling.

Just as for continuum QCD, the bare coupling constant and masses must be provided as input. In the lattice action, the bare coupling $g_{\text{lat}}$ appears in the parameter $\beta = 6/g_{\text{lat}}^2$. The lattice spacing is not an input. Rather, it is specified implicitly by $\beta$; for each $\beta$ there is a corresponding $a$. By expressing all dimensionful quantities in units of $a$, it is scaled out of the action, and so serves to set the overall mass scale. As a result, the simulation produces a value for some particular mass $M$ only in the dimensionless combination $aM$. We must therefore know $a$ before we can compare $M$ to its experimental value. In our simulation, we compute the mass difference between the $\Upsilon$ and $\Upsilon'$ mesons, $a\Delta M(\Upsilon' - \Upsilon)$, and between the $\Upsilon$ and the spin average of the $\chi_b$ states, $a\Delta M(\chi_b - \Upsilon)$. We then divide these by the experimentally measured mass differences to obtain two independent estimates for $a$.

Heavy-quark systems possess several properties which permit us to measure $a$ accurately \cite{2}. They are essentially nonrelativistic; the use of a nonrelativistic effective action to exploit this allows a large portion of the spectrum to be computed efficiently and precisely. They are physically small, and do not suffer from finite-volume errors on modestly sized lattices. Their spin-averaged mass splittings are observed experimentally to be nearly independent of the heavy-quark mass, varying by only a few per cent between the $\Upsilon$ and $\psi$, making our results insensitive to tuning errors in the bare $b$-quark mass. Because including vacuum polarization in simulations from nearly massless quarks is difficult, it is common to use unrealistically large $u$- and $d$-quark masses, and then extrapolate to the correct values. However, due to the small size of the $\Upsilon$ and the large momentum transfers between its constituents, the bare light-quark masses of about 25 MeV in our $n_f = 2$ simulations are negligible, making extrapolation unnecessary. Finally, $\Upsilon$ decay rates are negligible as compared to their energy splittings, making the effect of light-quark mass values on nonanalytic threshold behavior unimportant.

Our results for $n_f = 0$ and 2 are summarized in Table \ref{tab:results}. The $b$-quarks in $\Upsilon$, $\Upsilon'$ and $\chi_b$ mesons typically exchange momenta of order 1 GeV, so that the appropriate number of light flavors to include in a study of their dynamics

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textbf{System} & \textbf{Mass Difference ($aM$)} & \textbf{Experiment} & \textbf{Lattice Simulation} \\
\hline
$\Upsilon - \Upsilon'$ & 52 MeV & 55 MeV & 52 MeV \\
$\Upsilon - \chi_b$ & 44 MeV & 47 MeV & 44 MeV \\
\hline
\end{tabular}
\caption{Comparison of measured and simulated mass differences.}
\end{table}
is three. Having results for \( n_f \) of both 0 and 2 will allow us to accurately extrapolate to \( n_f = 3 \). In addition, we varied the bare \( b \)-quark mass around the correct value of \( aM_0^b = 1.7(1) \). As expected, the splittings showed little sensitivity to \( M_0^b \).

We have determined \( a^{-1} \) both by fitting the splittings separately and simultaneously. Fitting separately produced a discrepancy in \( a^{-1} \) of a couple standard deviations for \( n_f = 0 \), and about half this for \( n_f = 2 \). As we will show, this small discrepancy vanishes when we extrapolate \( n_f \) to three. Insofar as the effect is real, it is likely due to the larger intrinsic momentum transfers for \( S \) states as opposed to \( P \) states. Fitting simultaneously, we find that our simulation data are consistent with \( a^{-1} \approx 2.4 \) GeV. Our results for other low-lying excitations and spin splittings of the \( \Upsilon \) system, using this value for \( a^{-1} \), are displayed in Figures 1 and 2, where they are compared with their experimental values. The excellent agreement supports the reliability of our simulations. We emphasize that these are calculations from first principles; our approximations can be systematically improved. The only inputs are the lagrangians describing gluons and quarks, and the only parameters are the bare coupling constant and quark

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( \beta \) & \( n_f \) & \( aM_0^b \) & \( \Upsilon' - \Upsilon \) & \( \chi_b - \Upsilon \) \\
\hline
6.0 & 0 & 1.71 & .241(11) & 2.34(11) & .171(8) & 2.57(12) \\
1.80 & .239(11) & 2.36(11) & .174(12) & 2.53(18) \\
2.00 & .235(11) & 2.40(11) & .173(10) & 2.54(15) \\
5.6 & 2 & 1.80 & .237(10) & 2.38(10) & .178(5) & 2.47(7) \\
\hline
\end{tabular}
\caption{Lattice QCD simulation results for the difference between the \( \Upsilon' \) and \( \Upsilon \) masses, and between the spin-averaged \( \chi_b \) mass and the \( \Upsilon \) mass. Results are given for different bare gluon couplings \( \beta = 6/g^2_{\text{lat}} \) and bare quark masses \( M_0^b \), and for \( n_f = 0 \) and 2 flavors of light quarks. The correct bare mass for the \( b \)-quark is \( aM_0^b = 1.7(1) \). The splittings are corrected for \( O(a^2) \) errors in the gluon action. The errors shown are statistical and result from our use of Monte Carlo methods in the simulations. Values for \( a^{-1} \) are in GeV and are obtained using \( \Delta M(\Upsilon' - \Upsilon) = 0.563 \) GeV and \( \Delta M(\chi_b - \Upsilon) = 0.440 \) GeV.}
\end{table}

Footnote: Perturbation theory, though not justified at these momenta, provides a qualitative explanation of the effect on the determination of \( a^{-1} \). The centrifugal barrier makes the average separation between the quarks in the \( P \)-state \( \chi_b \) larger than for the \( S \)-state \( \Upsilon \) or \( \Upsilon' \), as is familiar from hydrogen or positronium. Consequently, the typical exchanged momentum for \( \chi_b \) quarks, \( q_{\chi_b} \), is smaller than \( q_{\Upsilon'} \). The perturbative binding energy is given by \( \alpha^2_s(q)C_F^2 M_b/16 \), with \( q = q_{\Upsilon'} \) for \( \Upsilon' \) and \( q_{\chi_b} \) for \( \chi_b \). Since \( q_{\chi_b} < q_{\Upsilon'} \), the \( \chi_b \) is more tightly bound. However, for \( n_f = 0 \), this effect is exaggerated, as \( \alpha^{(3)}_s(q) \) increases more quickly than \( \alpha^{(3)}_s(q) \) with decreasing \( q \). Thus, for \( n_f < 3 \), \( \Delta M(\chi_b - \Upsilon) \) should be underestimated relative to \( \Delta M(\Upsilon' - \Upsilon) \), as is observed. Fitting to data would then require a larger \( a^{-1} \) for \( \Delta M(\chi_b - \Upsilon) \) than for \( \Delta M(\Upsilon' - \Upsilon) \).
Figure 1: NRQCD simulation results for the spectrum of the $\Upsilon$ system, including radial excitations. Dashed lines indicate experimental values for the triplet $S$-states, and for the spin-average of the triplet $P$-states. The energy zero from simulation results is adjusted to give the correct mass to the $\Upsilon(1^3S_1)$. Results are from a simulation with $n_f = 0$ (filled circles) and from one with $n_f = 2$ (open circles), using $a^{-1} = 2.4$ GeV for both. The errors shown are statistical; systematic errors are several tens of MeV.

mass. In particular, these simulations are not based on a phenomenological quark potential model.

The $b$-quark action includes relativistic corrections in the quark velocity $v$ to $O(v^2)$, and corrections to $O(a^2)$ in the lattice spacing. The leading systematic error affecting $a^{-1}$ is almost certainly due to finite-lattice-spacing errors in the gluon action. To correct for this requires an additional $O(a^2)$ interaction. As it is sensitive to short distances, the effect of this interaction can be estimated using perturbation theory, which gives a mass shift of

$$a\Delta M_g = \frac{4\pi \alpha_v(q_s)}{15} a^3 |\psi(0)|^2 .$$

(1)

Here $q_s \approx 5$ GeV is the typical momentum transferred through the interaction, and $\psi(0)$ is the meson wavefunction evaluated at the origin. Using $n_f = 0$ simulation results for $\psi(r)$ and $\alpha_v[10]$, we find that $a\Delta M_g$ is 0.0036 for the $\Upsilon$
Figure 2: NRQCD simulation results for the spin structure of the lowest-lying $P$-states. Dashed lines indicate experimental values for the triplet $P$-states; the dotted line shows their spin average. Masses are relative to the spin-averaged state. Results are from a simulation with $n_f = 0$ (filled circles) and from one with $n_f = 2$ (open circles), using $a^{-1} = 2.4$ GeV for both. The errors shown are statistical; systematic errors are within about 5 MeV.

and 0.0023 for the $\Upsilon'$ when $aM_b^0 = 1.71$; there is no shift for the $P$-state $\chi$'s. The correction increases approximately linearly with mass and is about 30% larger for $n_f = 2$. The splittings in Table 1 include this correction (although Figures 1 and 3 do not). It shifts $a^{-1}$ by almost a standard deviation in the case of the $\chi_b - \Upsilon$ splittings, but is negligible for the $\Upsilon' - \Upsilon$ splittings.

To check the validity of Eq. (1), we examined the perturbative prediction for the hyperfine splitting between the $\Upsilon$ and $\eta_b$, given by the similar formula

$$\Delta M_{\text{hfs}} = \frac{32 \pi \alpha_v (q_b)}{9 M_b^2} |\psi(0)|^2 ,$$

(2)

where $M_b$ is the pole mass of the $b$-quark [1]. This formula gives $a \Delta M_{\text{hfs}} = 0.0122(10)$ when $aM_b^0 = 1.71$ and $n_f = 0$, which compares very well with the nonperturbative result 0.0123(2) we obtain from our simulation [10]. This suggests that our perturbative estimates of the $O(a^2)$ shifts (Eq. (1)) are quite reliable. We have also computed these shifts nonperturbatively using a lattice potential model, and found essentially identical corrections. Other systematic errors are higher order in $a$ or in the quark velocity $v$, and so are most likely negligible. The dominant errors in determining $a^{-1}$ are statistical.

Having determined the lattice spacing, the second step is to use simulation results to extract the coupling constant. Our approach closely parallels determinations based on high-energy phenomenology, with the simulation playing
\[ \beta = 6.0 \quad n_f = 0 \]

| loop | \(c_1\) | \(c_2\) | \(c_3\) | \(a q_{n,m}\) | sim'n | \(\alpha_V^{(n_f)}(3.41/a)\) | \(O(\alpha^2)\) | \(O(\alpha^3)\) |
|------|--------|--------|--------|--------------|-------|----------------|-------------|-------------|
| \(-\ln W_{1,1}\) | 4.19   | -4.96  | 0      | 3.41         | 0.5214(0) | .152           | .1517       |             |
| \(-\ln W_{1,2}\) | 7.22   | -7.57  | 2.6    | 3.07         | 0.9582(1) | .154           | .1522       |             |
| \(-\ln W_{1,3}\) | 10.07  | -9.60  | 5.3    | 3.01         | 1.3757(2) | .155           | .1525       |             |
| \(-\ln W_{2,2}\) | 11.47  | -10.58 | 11.1   | 2.65         | 1.6605(3) | .158           | .1532       |             |

| \(\beta = 5.6 \quad n_f = 2\) |

| loop | \(c_1\) | \(c_2\) | \(c_3\) | \(a q_{n,m}\) | sim'n | \(\alpha_V^{(n_f)}(3.41/a)\) | \(O(\alpha^2)\) | \(O(\alpha^3)\) |
|------|--------|--------|--------|--------------|-------|----------------|-------------|-------------|
| \(-\ln W_{1,1}\) | 4.19   | -5.55  | 0      | 3.41         | 0.5708(1) | .179           | .1785       |             |
| \(-\ln W_{1,2}\) | 7.22   | -8.51  | -      | 3.07         | 1.0522(1) | .181           |             |             |
| \(-\ln W_{1,3}\) | 10.07  | -10.89 | -      | 3.01         | 1.5123(2) | .181           |             |             |
| \(-\ln W_{2,2}\) | 11.47  | -11.84 | -      | 2.65         | 1.8337(3) | .185           |             |             |

Table 2: Perturbative and simulation results for several small Wilson loops [15]. The entries in the final two columns list the values for \(\alpha_V\) extracted by comparing the simulation results for each \(-\ln W\) to its perturbative expansion, correct to second and third order, respectively.

the role of an experimental measurement. We identify short-distance quantities whose perturbative expansions are known at least through second order, and then determine these quantities nonperturbatively using our simulation. By equating the perturbative expansion to the nonperturbative value, we can solve for the strong coupling constant.

One of the most ultraviolet quantities in lattice QCD is the expectation value of the 1 \(\times\) 1 Wilson loop operator. Its perturbative expansion to \(O(\alpha_V^2)\) is [12, 13, 14, 7]

\[^{2}\text{The strong coupling constant } \alpha_V \text{ was defined in Refs. [6, 7] in terms of the static-quark potential. To simplify the discussion of larger Wilson loops that follows, it is convenient to regard Eq. (3), with no higher-order terms in } \alpha_V, \text{ as defining } \alpha_V. \text{ The two definitions differ only at } \mathcal{O}(\alpha_V^3). \text{ The scale } 3.41/a \text{ follows from the technique described in Ref. [7], and indicates the important momentum scale in } W_{1,1}. \text{ It corresponds to } 8 - 9 \text{ GeV for our lattices, confirming that } W_{1,1} \text{ is very ultraviolet.} \]

\[^{3}\text{Simulation results for } -\ln W_{1,1} \text{ are listed in Table 2. From Eq. (3) and the } a^{-1} \text{ extracted from the } \chi_b - \Upsilon \text{ splitting we find} \]

\[^{4}\text{We have } \alpha_V^{(n_f)}(8.76(41) \text{ GeV}) = 0.1517, \text{ and use this value in Eq. (4) to solve for } \alpha_V. \]
\[ \alpha_\nu^{(3)}(8.42(24) \text{ GeV}) = 0.1785 \]  

or equivalently,

\[
\alpha^{(n_f)}_\nu(8.2 \text{ GeV}) = \begin{cases} 
0.1548(23) & \text{for } n_f = 0 \\
0.1800(16) & \text{for } n_f = 2 ,
\end{cases}
\]  

where the errors are due to the statistical errors in \( a^{-1} \). As mentioned above, we must extrapolate our results to \( n_f = 3 \). Perturbation theory suggests that \( 1/\alpha^{(n_f)}_\nu \) is more nearly linear for small changes in \( n_f \) than \( \alpha^{(n_f)}_\nu \), and so we extrapolate the inverse couplings to obtain

\[ \alpha^{(3)}_\nu(8.2 \text{ GeV}) = 0.1959(34). \]  

Since we are extrapolating by only 9%, extrapolation errors are probably smaller than the statistical errors quoted here; perturbation theory indicates that they are about 0.2%, which is negligible.

Repeating this analysis using \( a^{-1} \) from \( \Upsilon' - \Upsilon \) gives

\[
\alpha^{(n_f)}_\nu(8.2 \text{ GeV}) = \begin{cases} 
0.1504(22) & \text{for } n_f = 0 \\
0.1779(23) & \text{for } n_f = 2 ,
\end{cases}
\]  

and

\[ \alpha^{(3)}_\nu(8.2 \text{ GeV}) = 0.1958(46), \]  

which agrees with the \( \chi_b - \Upsilon \) value. We have then, from both \( \chi_b - \Upsilon \) and \( \Upsilon' - \Upsilon \), our primary result:

\[ \alpha^{(3)}_\nu(8.2 \text{ GeV}) = 0.1959(34). \]  

Because the internal momenta transferred by the \( \Upsilon \) constituents are small relative to \( c \) - and \( b \)-quark masses, it would be incorrect to extrapolate \( \alpha_\nu \) obtained with \( n_f = 0 \) and 2 light quarks directly to \( n_f = 4 \) or 5. The correct way to incorporate these heavier flavors is to run the coupling to below the \( c \)-quark threshold, then apply matching conditions as \( \alpha \) is run back up through the \( c \) and \( b \) thresholds. We will use this procedure to obtain \( \alpha^{(3)}_{\overline{\text{MS}}} \) at the \( Z \) mass.

We chose the \( 1 \times 1 \) Wilson loop because we expected nonperturbative effects to be very small due to the large momentum scale it probes. To verify this, we have determined \( \alpha^{(n_f)}_\nu \) using larger Wilson loops and loops with different shapes, which should have significantly larger nonperturbative contributions. In Table \ref{table:1} we give the perturbative expansion coefficients for the smallest Wilson loops. These coefficients are defined by

\[ -\ln W^{(n_f)}_{n,m} = \sum_{i=1}^{c^{(n_f)}_{\nu}(n,m)} \left[ \frac{\alpha^{(n_f)}_\nu(q_{n,m})}{\alpha^{(n_f)}_\nu(q_{n,m})} \right]^i , \]  

\footnote{We expect these two determinations to be statistically correlated, and quote as an error the uncertainty in the \( \chi_b - \Upsilon \) determination, rather than combining the errors as if independent.}
\[ \beta - \ln W_{1,1} \quad a^{-1} \quad q_c \quad \alpha_v^{(0)}(q_c) \quad \alpha_v^{(0)}(8.2 \text{ GeV}) \]

| \beta   | \beta - \ln W_{1,1} | a^{-1} | q_c  | \alpha_v^{(0)}(q_c) | \alpha_v^{(0)}(8.2 \text{ GeV}) |
|---------|----------------------|--------|-----|---------------------|--------------------------|
| 6.4     | 0.4610               | 4.12(63) | 14(2) | 0.1302              | 0.151(7)                 |
| 6.2     | 0.4884               | 3.50(33) | 12(1) | 0.1381              | 0.156(5)                 |
| 6.0     | 0.5214               | 2.57(12) | 8.8(4) | 0.1517              | 0.155(2)                 |

Table 3: Results for \( \alpha_v^{(0)}(8.2 \text{ GeV}) \) with \( \beta = 6.4, 6.2 \) and 6.0 using the \( \chi_b - \Upsilon \) splitting. The scale \( q_c = 3.41/a; \) \( q_c \) and \( a^{-1} \) are in GeV.

with \( \alpha_v^{(n_f)} \) defined by Eq. (3). We also quote simulation results for these quantities, and values for \( \alpha_v^{(n_f)}(3.41/a) \) obtained by matching second- and third-order perturbative expansions to the simulation results. The \( n_f = 0 \) results, when third-order perturbation theory is used, show that the small loops all give the same value for \( \alpha_v^{(0)}(3.41/a) \) to within less than one per cent. This confirms that nonperturbative effects are completely negligible. (In fact, the slight variation is likely due to fourth-order perturbative corrections.) The \( n_f = 2 \) results are consistent with this conclusion, although the test is somewhat less stringent since the third-order perturbative coefficients are not known for \( n_f \neq 0 \).

To verify that additional lattice-spacing errors are under control, in Table 3 we compare our result for \( \alpha_v^{(0)}(8.2 \text{ GeV}) \) computed from a simulation at \( \beta = 6.0 \) with those from \( \beta = 6.2 \) and 6.4 [17]. The scales for the corresponding \( 1 \times 1 \) Wilson loops range from 8 to 14 GeV. That these give consistent values for \( \alpha_v^{(0)}(8.2 \text{ GeV}) \) indicates that, within errors, the coupling constant is scaling correctly, and confirms results found in Ref. [7].

Eq. (10) is our final result. However, to facilitate comparison with other determinations we convert our result to the \( \overline{\text{MS}} \) scheme, which is related to the \( V \) scheme by

\[ \alpha_v^{(n_f)}(Q) = \alpha_v^{(n_f)}(e^{5/6} Q) \left\{ 1 + 2 \alpha_v^{(n_f)} / \pi + \mathcal{O}(\alpha_v^{(n_f)}^2) \right\}. \]  

Our result is then equivalent to

\[ \alpha_v^{(3)}(3.56 \text{ GeV}) = 0.2203(84), \]  

with the error now dominated by the unknown third-order contribution to Eq. (12), which we estimate as \( (\alpha_v^{(0)})^3 = 0.0075 \).

We numerically integrated the third-order perturbative beta function for \( \alpha_v^{(n_f)} \) and applied appropriate matching conditions at quark thresholds to evolve it to several other scales:

\[ \alpha_v^{(n_f)}(Q) = \begin{cases} 
0.304(17) & \text{for } Q = 1.7 \text{ GeV } \approx M_c \text{ and } n_f = 3, 4 \\
0.203(7) & \text{for } Q = 5.0 \text{ GeV } \approx M_b \text{ and } n_f = 4, 5 \\
0.115(2) & \text{for } Q = 91.2 \text{ GeV } = M_Z \text{ and } n_f = 5
\end{cases}. \]
Table 4: Sources of error in $\alpha^{(5)}_{\text{MS}}(M_Z)$.

| Source                                      | Uncertainty |
|---------------------------------------------|-------------|
| Converting from $\alpha^{(3)}_V$ to $\alpha^{(3)}_{\text{MS}}$ | 1.7%        |
| Statistical error in determination of $a^{-1}$ | .9%         |
| Extrapolation in $n_f$                      | .2%         |
| Finite $a$ and $\mathcal{O}(v^4)$ errors    | .2%         |
| Fourth-order evolution of $\alpha_{\text{MS}}$ | .01%        |

The last of these results is consistent with, and significantly more accurate than, the current world average $\alpha^{(5)}_{\text{MS}}(M_Z) = 0.117(5)$.

We believe we have accurately estimated sources of error. We have checked our result using two different mass splittings to determine $a^{-1}$, and four independent Wilson loops to extract $\alpha_{\nu}(3.4/a)$. We have checked for consistency against variations in the quark mass and lattice spacing. Our estimate of the third-order perturbative contribution to the relation between $\alpha_{\nu}$ and $\alpha_{\text{MS}}$ is consistent with first- and second-order contributions, and with the third-order terms in Table 2. Our result also agrees well with earlier determinations based on lattice simulations [1, 4]. These were performed without dynamical light quarks. To estimate the effect of light quarks, $\alpha^{(0)}_{\nu}$, computed at $3.41/a$, was run down to a scale typical of momenta exchanged in the $\Upsilon$ or $\psi$, determined as in Ref. [7, 4]. There it was equated to $\alpha^{(3)}_V$, since it is at this scale that the two are required to produce the same values for splittings, and $\alpha^{(3)}_V$ was then run back up to the desired scale. We may repeat this procedure as an alternative to a direct extrapolation in $n_f$, now using results for both zero and two light flavors. This gives $\alpha^{(3)}_{\text{MS}}(M_Z) = .112(4)$ from $n_f = 0$ data, and .115(3) from $n_f = 2$. So this method yields values consistent with, but not as reliable as, extrapolation in $n_f$.

There are prospects for substantially improving the accuracy of our result fairly soon. Sources of error in our value for $\alpha^{(5)}_{\text{MS}}(M_Z)$ are listed in Table 4. The dominant error is unrelated to our primary result of Eq. (10), but rather is due to the conversion to MS. The total error could be cut in half by computing the third-order correction to Eq. (12), a straightforward perturbative calculation. The error in $a^{-1}$ will decrease with improved statistics; we have now completed a new simulation that should soon reduce it to about .6%. Use of an improved gluon action would remove the need for the $a^2$ correction in the $\chi_b - \Upsilon$ analysis, at little additional cost [10]. Finally, a simulation with either $n_f = 3$ or 4 light quarks would eliminate the extrapolation error and would require roughly the same amount of time as for $n_f = 2$.

There are a variety of additional lattice calculations that would provide a
broader check on the consistency of our result. The first would be to repeat our
determination using charmonium, extending previous studies by including
dynamical light quarks. A similar analysis is also possible using light hadrons.
For reasons outlined in the introduction, systematic errors for light hadrons are
not nearly as well understood as for heavy mesons. It is important that control
of these errors be improved to demonstrate that these systems yield consistent
results. Finally, simulations which include light-quark vacuum polarization are
still relatively rare. It would be very useful to repeat this analysis using an \( n_f \)
other than two, algorithms for generating gauge-field configurations other than
HMD, and Wilson fermions rather than staggered.

In this paper we have demonstrated that lattice simulations provide one of
the simplest, most accurate, and most reliable determinations of the strong
coupling constant. The fact that a lattice simulation of nonperturbative hadronic
structure at scales smaller than 1 GeV agrees with perturbative analyses of
high-energy jet formation is striking confirmation that these diverse phenomena
are governed by a single theory — QCD. Furthermore, this result demonstrates
our growing mastery over both the nonperturbative and perturbative aspects of
the theory.

An independent lattice determination of \( \alpha_S(M_Z) \) with dynamical light
quarks has recently appeared in Ref. [20]. They obtain results consistent with
ours, though with larger errors.

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