ON COMPLEXES OF FINITE COMPLETE INTERSECTION DIMENSION

PETTER ANDREAS BERGH

Abstract. We study complexes of finite complete intersection dimension in
the derived category of a local ring. Given such a complex, we prove that the
thick subcategory it generates contains complexes of all possible complexities.
In particular, we show that such a complex is virtually small, answering a
question raised by Dwyer, Greenlees and Iyengar.

1. Introduction

In [DGI], the authors raised the question whether every nonzero homologically
finite complex of finite complete intersection dimension over a local ring is virtually
small. In other words, given such a ring \(A\) and such a complex \(M \in D(A)\), is the
intersection
\[
\text{thick}_{D(A)}(M) \cap \text{thick}_{D(A)}(A)
\]
nonzero? We give an affirmative answer to this question, by showing that the
thick subcategory generated by \(M\) contains a nonzero complex of complexity zero.
In fact, we show that if the complexity of \(M\) is \(c\), then \(\text{thick}_{D(A)}(M)\) contains a
nonzero complex of complexity \(t\) for every \(0 \leq t \leq c\). The homologically finite com-
plexes of complexity zero are precisely the complexes of finite projective dimension.
Moreover, a homologically finite complex belongs to \(\text{thick}_{D(A)}(A)\) if and only if its
projective dimension is finite. Thus, the results mentioned indeed settle the above
question.

2. Notation and terminology

Let \((A, \mathfrak{m}, k)\) be a local (meaning commutative Noetherian local) ring, and denote
by \(D(A)\) the derived category of (not necessarily finitely generated) \(A\)-modules. A
complex
\[
M : \cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots
\]
in \(D(A)\) is bounded below if \(M_n = 0\) for \(n \ll 0\), and bounded above if \(M_n = 0\) for
\(n \gg 0\). The complex is bounded if it is both bounded below and bounded above,
and finite if it is bounded and degreewise finitely generated. The homology of \(M\),
denoted \(H(M)\), is the complex with \(H(M)_n = H_n(M)\), and with trivial differentials.
When \(H(M)\) is finite, then \(M\) is said to be homologically finite.

As shown for example in [Rob], when the complex \(M\) is homologically finite, then
it has a minimal free resolution. Thus, there exists a quasi-isomorphism \(F \simeq M\),
where \(F\) is a bounded below complex
\[
\cdots \to F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \to \cdots
\]

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of finitely generated free modules, and where \( \text{Im } d_n \subseteq m F_{n-1} \). This resolution is unique up to isomorphism, and so for each integer \( n \) the rank of the free module \( F_n \) is a well defined invariant of \( M \). This is the \( n \)-th Betti number \( \beta_n(M) \) of \( M \), and the corresponding generating function \( \sum_{n=0}^{\infty} \beta_n(M)t^n \) is the Poincaré series \( P(M, t) \) of \( M \). The complexity of \( M \), denoted \( \text{cx } M \), is defined as

\[
\text{cx } M \overset{\text{def}}{=} \inf \{ t \in \mathbb{N} \cup \{ 0 \} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq at^{n-1} \text{ for } n \gg 0 \}.
\]

The complexity of a homologically finite complex is not necessarily finite. In fact, by a theorem of Gulliksen (cf. [Gul]), finiteness of the complexities of all homologically finite complexes in \( D(A) \) is equivalent to \( A \) being a complete intersection ring.

Suppose that our complex \( M \) is homologically finite, with a minimal free resolution \( F \). Given a complex \( N \in D(A) \), the complex \( \text{Hom}_A(F, N) \) is denoted by \( R \text{Hom}_A(M, N) \). Up to quasi-isomorphism, this complex is well defined, hence so is the cohomology group

\[
\text{Ext}^n_A(M, N) \overset{\text{def}}{=} H_{-n}(R \text{Hom}_A(M, N))
\]

for every integer \( n \). The projective dimension of \( M \), denoted \( \text{pd}_A(M) \), is defined as

\[
\sup \{ n \mid \text{Ext}^n_A(M, k) \neq 0 \},
\]

which is the same as the supremum of all integers \( n \) such that \( F_n \) is nonzero. Namely, since the complex \( F \) is minimal, the differentials in the complex \( \text{Hom}_A(F, k) \) are trivial, and \( \beta_n(M) = \dim_k \text{Ext}^n_A(M, k) \) for all \( n \). In particular, the projective dimension of \( M \) is finite if and only if its complexity is zero.

The derived category \( D(A) \) is triangulated, the suspension functor \( \Sigma \) being the left shift of a complex. Given complexes \( M \) and \( N \) as above, for each \( n \) we may identify the cohomology group \( \text{Ext}^n_A(M, N) \) with \( \text{Hom}_{D(A)}(M, \Sigma^n N) \). For complexes \( X \) and \( Y \) in \( D(A) \), denote the graded \( A \)-module \( \bigoplus_{n=0}^{\infty} \text{Ext}^n_A(X, Y) \) by \( \text{Ext}^*_A(X, Y) \). Using composition of maps in \( D(A) \), the graded \( A \)-module \( \text{Ext}^*_A(M, M) \) becomes a ring, and \( \text{Ext}^*_A(M, N) \) becomes a graded right \( \text{Ext}^*_A(M, M) \)-module.

We end this section by recalling the notion of nearness in an arbitrary triangulated category \( T \) with a suspension functor \( \Sigma \). A subcategory of \( T \) is thick if it is a full triangulated subcategory closed under direct summands. Now let \( \mathcal{C} \) and \( \mathcal{D} \) be subcategories of \( T \). We denote by \( \text{thick}^1_T(\mathcal{C}) \) the full subcategory of \( T \) consisting of all the direct summands of finite direct sums of shifts of objects in \( \mathcal{C} \). Furthermore, we denote by \( \mathcal{C} * \mathcal{D} \) the full subcategory of \( T \) consisting of objects \( M \) such that there exists a distinguished triangle

\[
C \to M \to D \to \Sigma C
\]

in \( T \), with \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \). Now for each \( n \geq 2 \), define inductively \( \text{thick}^n_T(\mathcal{C}) \) to be \( \text{thick}^1_T( \text{thick}^{n-1}_T(\mathcal{C}) * \text{thick}^1_T(\mathcal{C}) ) \), and denote \( \bigcup_{n=1}^{\infty} \text{thick}^n_T(\mathcal{C}) \) by \( \text{thick}_T(\mathcal{C}) \). This is the smallest thick subcategory of \( T \) containing \( \mathcal{C} \).

3. Complexes of finite CI-dimension

Let \( A \) be a local ring. Recall that a quasi deformation of \( A \) is a diagram \( A \to R \leftarrow Q \) of local homomorphisms, in which \( A \to R \) is faithfully flat, and \( R \to Q \) is surjective with kernel generated by a regular sequence. A homologically finite complex \( M \in D(A) \) has finite complete intersection dimension if there exists such a quasi deformation for which \( \text{pd}_Q(R \otimes_A M) \) is finite. From now on, we write “CI-dimension” instead of “complete intersection dimension”.

The notion of CI-dimension was first introduced for modules in [ACP]. The terminology reflects the fact that a local ring is a complete intersection precisely when all its finitely generated modules have finite CI-dimension. The same holds if we replace “modules” with “homologically finite complexes”.

In order to prove the main result, we need the following lemma. It shows that finite CI-dimension is preserved in thick subcategories.

**Lemma 3.1.** Let $A$ be a local ring, and let $M \in D(A)$ be a homologically finite complex of finite CI-dimension. Then every complex in $\text{thick}_{D(A)}(M)$ has finite CI-dimension.

**Proof.** Let $A \to R \to Q$ be a quasi-deformation of $A$ such that $\text{pd}_Q(R \otimes_A M)$ is finite. We show by induction that for all $n \geq 1$, every complex $X \in \text{thick}^n_{D(A)}(M)$ satisfies $\text{pd}_Q(R \otimes_A X) < \infty$. If $X$ is a direct summand of finite direct sums of shifts of $M$, then this clearly holds. Therefore $\text{pd}_Q(R \otimes_A X)$ is finite for all complexes $X \in \text{thick}_{D(A)}(M)$. Next, suppose the claim holds for all $1, \ldots, n$, and let $X$ be a complex in $\text{thick}^n_{D(A)}(M) \ast \text{thick}_1^{D(A)}(M)$. Then there exists a triangle

$$Y \to X \to Z \to \Sigma Y$$

in $D(A)$, where $Y$ and $Z$ are complexes in $\text{thick}^n_{D(A)}(M)$ and $\text{thick}_1^{D(A)}(M)$, respectively. By induction, $\text{pd}_Q(R \otimes_A Y)$ and $\text{pd}_Q(R \otimes_A Z)$ are both finite, hence so is $\text{pd}_Q(R \otimes_A X)$. Since

$$\text{thick}^{n+1}_{D(A)}(M) = \text{thick}^1_{D(A)}\left(\text{thick}^n_{D(A)}(M) \ast \text{thick}_1^{D(A)}(M)\right),$$

the proof is complete. □

Next, we prove the main result; the thick subcategory generated by a complex of finite CI-dimension contains complexes of all possible complexities.

**Theorem 3.2.** Let $A$ be a local ring, and let $M \in D(A)$ be a nonzero homologically finite complex of finite CI-dimension. Then for every $0 \leq t \leq \text{cx} M$, there exists a nonzero complex in $\text{thick}_{D(A)}(M)$ of complexity $t$.

**Proof.** The proof is by induction on the complexity $c$ of $M$. If $c = 0$, then there is nothing to prove, so suppose that $c$ is nonzero. We shall construct a nonzero complex in $\text{thick}_{D(A)}(M)$ of complexity $c - 1$.

Let $A \to R \hookrightarrow Q$ be a quasi-deformation of $A$ such that $\text{pd}_Q(R \otimes_A M)$ is finite, and let $Z_R(R \otimes_A M)$ be the graded $R$-subalgebra of $\text{Ext}^*_R(R \otimes_A M, R \otimes_A M)$ generated by the central elements in degree two. By [AvS Corollary 5.1], the $Z_R(R \otimes_A M)$-module $\text{Ext}^*_R(R \otimes_A M, R \otimes_A k)$ is Noetherian, where $k$ is the residue field of $A$. Now let $Z_A(M)$ be the graded $A$-subalgebra of $\text{Ext}^*_A(M, M)$ generated by the central elements in degree two. Then by [AGP Theorem 4.9], we may identify $Z_R(R \otimes_A M)$ with $R \otimes_A Z_A(M)$, and so by faithfully flat descent the $Z_A(M)$-module $\text{Ext}^*_A(M, k)$ is Noetherian (see also [AV] Section 7).

By [BKO Lemma 2.5], there exists a positive degree element $\eta \in Z_A(M)$ with the property that scalar multiplication

$$\text{Ext}^*_A(M, k) \xrightarrow{\cdot \eta} \text{Ext}^*_A(M, k)$$

is injective for $n \gg 0$. This element corresponds to a map $M \xrightarrow{f_n} \Sigma^{|\eta|} M$ in $D(A)$, and completing this map we obtain a triangle

$$M \xrightarrow{f_n} \Sigma^{|\eta|} M \to K \to \Sigma M$$

in $\text{thick}_{D(A)}(M)$. Note that the object $K$ is nonzero; if not, then $M$ would be isomorphic to $\Sigma^{|\eta|} M$, and this is impossible. The triangle induces a long exact sequence

$$\cdots \to \text{Ext}^n_A(K, k) \to \text{Ext}^{n-|\eta|}_A(M, k) \xrightarrow{(-1)^n \cdot \eta} \text{Ext}^n_A(M, k) \to \text{Ext}^{n+1}_A(K, k) \to \cdots$$
in cohomology, hence for some integer $n_0$ the equality $\beta_{n+1}(K) = \beta_{n}(M) - \beta_{n-|\eta|}(M)$ holds for $n \geq n_0$. The Poincaré series of $K$ is then given by

$$P(K, t) = \sum_{n=0}^{\infty} \beta_n(K) t^n$$

$$= \sum_{n=0}^{n_0} \beta_n(K) t^n + \sum_{n=n_0+1}^{\infty} \beta_n(K) t^n$$

$$= \sum_{n=0}^{n_0} \beta_n(K) t^n + \sum_{n=n_0+1}^{\infty} (\beta_{n-1}(M) - \beta_{n-|\eta|-1}(M)) t^n$$

$$= g(t) + (t - t^{n_0+1})P(M, t)$$

$$= g(t) + t(1 + t + \cdots + t^{n_0-1})(1 - t)P(M, t),$$

where $g(t)$ is a polynomial in $\mathbb{Z}[t]$.

By [S-W] Corollary 3.10(iv), the Poincaré series $P(M, t)$ of $M$ is rational, hence so is $P(K, t)$. Therefore, the complexities of $M$ and $K$ equal the orders of the poles at $t = 1$ of $P(M, t)$ and $P(K, t)$, respectively. Consequently, the complexity of $K$ is $\text{cx} M - 1$. By Lemma 3.1, the complex $K$, being an object in $\text{thick}_{D(A)}(M)$, has finite CI-dimension. By induction, for every $0 \leq t \leq \text{cx} K$, there exists a nonzero complex in $\text{thick}_{D(A)}(K)$ of complexity $t$. This completes the proof.

Using [DGI] 3.8, the following corollary follows immediately from Theorem 3.2.

It settles the question, raised in [DGI], whether every nonzero homologically finite complex of finite CI-dimension over a local ring is virtually small.

**Corollary 3.3.** Let $A$ be a local ring, and let $M \in D(A)$ be a nonzero homologically finite complex of finite CI-dimension. Then $\text{thick}_{D(A)}(M) \cap \text{thick}_{D(A)}(A)$ is nonzero.

We also obtain the following criterion for a local ring to be Gorenstein.

**Corollary 3.4.** Let $A$ be a local ring, and suppose that for every homologically finite complex $M \in D(A)$ the thick subcategory $\text{thick}_{D(A)}(M)$ contains a nonzero complex of finite CI-dimension. Then $A$ is Gorenstein.

**Proof.** Follows immediately from Corollary 3.3 and [DGI] Theorem 9.11. □

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Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

E-mail address: bergh@math.ntnu.no