ON REGULARITY OF THE BEREZIN TRANSFORM ON
SMOOTH PSEUDOCONVEX DOMAINS

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ABSTRACT. In this short note we improve some of recent results of Čučković
and Şahutoğlu [1] concerning regularity of the Berezin transform for a class of
smooth pseudoconvex domains.

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. As usual $A^2(\Omega)$ denotes
the Bergman space of square integrable holomorphic functions on $\Omega$. Let $k_w, w \in \Omega$ denote
the normalized Bergman reproducing kernel. Then given a bounded
operator $S : A^2(\Omega) \to A^2(\Omega)$ its Berezin transform is defined as $B(S)(z) = \langle S(k_z), k_z \rangle$. The Berezin transform has been an important tool in the study of
Toeplitz operators. Recall that given $f \in L^\infty(\Omega)$, its Toeplitz operator $T_f : A^2(\Omega) \to A^2(\Omega)$ is defined as the composition of the multiplication by $f$ followed
by the orthogonal projection $L^2(\Omega) \to A^2(\Omega)$. We need also Hankel operators $H_f = mf - T_f : A^2(\Omega) \to A^2(\Omega)_\perp$, where $mf$ is the multiplication by $f$. One
defines the Berezin transform of a function $f$ as $B(T_f)$.

Hereafter, $T(\Omega)$ denotes the algebra generated by all Toeplitz operators with
symbols continuous on $\overline{\Omega}$, and $K(\Omega) \subset T(\Omega)$ denotes the ideal of compact oper-
ators.

In a recent paper [1], Čučković and Şahutoğlu introduces and studied the notion
of a BC-regular domain: A domain $\Omega \subset \mathbb{C}^n$ is called BC-regular if for any
$S \in T(\Omega)$, its Berezin transform $B(S)$ can be continuously extended on $\overline{\Omega}$. The
authors went to prove that (among other results) a bounded smooth convex
domain with no analytic discs in the boundary is a BC-domain.

To state our results we recall a well-known fact that if $\partial \Omega$ is smooth, then for
any $w \in \partial \Omega, k_z \to 0$ weakly as $z \to w$. Therefore, if $S$ is a compact operator then
$B(S)$ vanishes on the boundary of $\Omega$.

It will be convenient to use the following definition.

Condition. Let $\Omega$ be a smooth bounded pseudoconvex domain. Then $B_\Omega \subset \partial \Omega$
is defined as the set of all $w \in \partial \Omega$ such that for any $f \in C(\overline{\Omega})$ we have

$$\lim_{z \to w} B(H_f^*H_f)(z) = 0.$$ 

It is well-known that all strongly pseudoconvex points belong to $B_\Omega$. It is immediate that if $H_f$ is compact for all $f \in C(\overline{\Omega})$, then $B_\Omega = \partial \Omega$. Recall that if
the $\bar{\partial}$-Neumann operator is compact, then all $H_f$ are compact operators for any $f \in C(\overline{\Omega})$ and hence $B_\Omega = \partial \Omega$.

Next we need to recall the following result of Salinas, Sheu and Upmeier about the maximal commutative quotient of $T(\Omega)$.

**Theorem 0.1** ([2], Theorem 1.4). Let $I$ denote the commutator ideal in $T(\Omega)$. Assume that $\partial(\overline{\Omega}) = \partial \Omega$. Then there is an isometry of $C^*$ algebras $\eta : T(\Omega)/I \cong C(X)$, where $X \subset \partial \Omega$ is a closed subset. If in addition $H_f$ is compact for all $f \in C(\overline{\Omega})$, then $K = I$ and $X = \partial(\Omega)$.

It follows from the proof that the surjective homomorphism $\eta : T(\Omega) \rightarrow C(X)$ is uniquely determined by the restriction property $\eta(T_f) = f|_X$.

We show the following result.

**Theorem 0.2.** Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain. Then for any $S \in T(\Omega)$, its Berezin transform $B(S)$ can be continuously extended to $\Omega \cup B_\Omega$ and the Berezin transform induces a surjective homomorphism on Banach algebras $B : T(\Omega)/K \rightarrow C(B_\Omega)$ so that $B(T_f) = f|_{B_\Omega}$. If moreover, $H_f$ is compact for all $f \in C(\overline{\Omega})$, then the above homomorphism is an isometry of $C^*$-algebras.

**Proof.** At first, recall the following relation between semi-commutators of Toeplitz operators and Hankel operators

$$T_{fg} - T_f T_g = H_f^* H_g, \quad f, g \in C(\overline{\Omega}).$$

Hence, it follows from our assumptions that

$$B(T_{fg})|_{B_\Omega} = B(T_f^* H_g)|_{B_\Omega}.$$ 

Let $f, g \in \mathbb{C}[z]$, then $B(T_{fg}) = f \bar{g}$. Therefore, for any $\phi \in \mathbb{C}[z, \bar{z}]$ we have $B(T_\phi)|_{B_\Omega} = \phi|_{B_\omega}$. Hence, using the Stone-Weierstass theorem we get

$$B(T_\psi)|_{B_\Omega} = \psi|_{B_\Omega}, \quad \psi \in C(\overline{\Omega}).$$

Combining this with the above formulas, we conclude that for any $\phi_1, \ldots, \phi_m \in C(\overline{\Omega})$, we have

$$B(T_{\phi_1} \cdots T_{\phi_m})|_{B_\omega} = (\phi_1 \cdots \phi_m)|_{B_\Omega}.$$ 

Thus, we have a continuous algebra homomorphism

$$B : T(\Omega)/K \rightarrow C(B_\Omega)$$

such that $B(T_f) = f|_{B_\Omega}$.

If $H_f$ is compact for all $f \in C(\overline{\Omega})$, then $B_\Omega = \partial \Omega$ and our homomorphisms coincides with the one from Theorem [1,1]. In particular, it is an isometry of $C^*$-algebras.

**Corollary 0.1.** Let $\Omega \subset \mathbb{C}^n$ a smooth bounded pseudoconvex domain such that $H_f$ is compact for all $f \in C(\overline{\Omega})$. Then $\Omega$ is BC-regular and the essential norm of any $S \in T(\Omega)$ equals to $L^\infty(\partial \Omega)$ norm of $B(S)|_{\partial \Omega}$. 

The above corollary generalizes theorems 1, 4, and part of theorem 5 from [1].

As shown in [1, Theorem 3] if \( \Omega \) is a convex domain with a disc in the boundary and dense strongly pseudoconcex points, then \( \Omega \) is not BC-regular. The following simple result shows a dichotomy for domains with dense strongly pseudoconvex points.

**Proposition 0.1.** Let \( \Omega \) be a smooth pseudoconvex domain. Then \( B_\Omega = \partial \Omega \) if and only if \( \Omega \) is BC-regular and the map \( S \to B(S)|_{\partial \Omega}, S \in T \) is multiplicative.

Suppose that \( B_\Omega \) is dense in \( \partial \Omega \) (in particular this is the case if strongly pseudoconvex points are dense in \( \partial \Omega \).) Then \( \Omega \) is BC-regular if and only if \( B_\Omega = \partial \Omega \).

**Proof.** Suppose that \( B : T(\Omega)/K \to C(\partial \Omega) \) is multiplicative. Then \( B([T_f, T_f]) = 0 \) for a holomorphic \( f \). Which implies that \( B(H_f^* H_f) = 0 \). So, \( B_\Omega = \partial \Omega \) as desired.

If \( B_\Omega = \partial \Omega \) then Theorem 0.2 implies that \( \Omega \) is BC-regular.

Now, suppose that \( \Omega \) is BC-regular and \( B_\Omega \) is dense in \( \partial \Omega \). Then for any \( f \in C(\Omega) \) we have that \( B(f)|_{B_\Omega} = f|_{B_\Omega} \). Since by the assumption \( B(f) \) is continuous up to the boundary, we get that \( B(f) = f \). Now it follows that

\[
B(H_f^* H_f) = B(|f|^2) - |f|^2
\]

vanishes on the boundary for all anti-holomorphic \( f \). Hence \( \partial \Omega = B_\Omega \).

In view of the above result, it would be interesting to know an example of a smooth BC-regular domain \( \Omega \) for which the \( \partial \)-Neumann operator is not compact. If the smoothness assumption is dropped, then a polydisc is an example of such a domain [1, Corollary 1].

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**References**

[1] Z. Ćučković, S. Şahutoğlu, Berezin regularity of domains in \( \mathbb{C}^n \) and the essential norms of Toeplitz operators, to appear in Trans. of AMS (2020) arXiv:1909.09221.

[2] N. Salinas, A. Sheu, H. Upmeier, Toeplitz operators on pseudoconvex domains and foliations of \( C^* \)-algebras, Annals of Math. (1989) 531–565.

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