Abstract: Let $X$ be a smooth variety or orbifold and let $Z \subseteq X$ be a complete intersection defined by a section of a vector bundle $E \to X$. Originally proposed by Givental, quantum Serre duality refers to a precise relationship between the Gromov–Witten invariants of $Z$ and those of the dual vector bundle $E^\vee$. In this paper we prove a quantum Serre duality statement for quasimap invariants. In shifting focus to quasimaps, we obtain a comparison which is simpler and which also holds for non-convex complete intersections. By combining our results with the wall-crossing formula developed by Zhou, we recover a quantum Serre duality statement in Gromov-Witten theory without assuming convexity.

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1. Introduction

Let $X$ be a smooth projective variety or Deligne–Mumford stack and let $Z \subset X$ be a smooth complete intersection, defined by the vanishing of a section of a vector bundle $E \to X$. Quantum Serre duality refers to a relationship between the genus-zero Gromov-Witten invariants of $Z$ and those of the dual vector bundle $E^\vee$. In this paper we investigate this correspondence in the context of quasimap invariants.

1.1. History. Quantum Serre duality was first described in mathematics by Givental in [20], for the case of $X = \mathbb{P}^n$. The correspondence is given as a relation between generating functions of genus-zero Gromov–Witten invariants of $Z$ and $E^\vee$ via a complicated change of variables and a non-equivariant limit. Since then, quantum Serre duality has been generalized and reformulated in a number of ways. In [14], Coates–Givental employ Givental’s symplectic formalism [21] to show that twisted overruled Lagrangian cones for $E$ and $E^\vee$ may be identified by a symplectic isomorphism. When $E$ is convex, this implies that generating functions of Gromov–Witten invariants of $Z$ may be recovered from those of $E^\vee$. In [25], Iritani–Mann–Mignon observe that quantum Serre duality may be cast as an isomorphism of quantum $D$-modules, and is compatible with a Fourier–Mukai transform in $K$-theory. This result was refined and extended to orbifolds by the second author in [35].

The utility of the correspondence is based on the fact that in many cases the geometry of $E^\vee$ is simpler than that of $Z$. For instance, if $X$ is a toric variety then $E^\vee$ is as well. It was noted in [14] that by using quantum Serre duality, the mirror theorem for $Z$ follows from the mirror theorem for $E^\vee$. In recent years, quantum Serre duality has been employed to prove other correspondences in Gromov–Witten theory. Applications have included the crepant transformation conjecture [15, §6], the LG/CY correspondence [29, 34], and the Gromov–Witten theory of extremal transitions [32].

A theme which persists through all of the formulations of quantum Serre duality described above is that in order to observe a correspondence, the Gromov–Witten invariants of $Z$ and $E^\vee$ must be packaged in a clever way (Lagrangian cones, $D$-modules, etc...). There is no simple relation between the individual Gromov–Witten invariants of $Z$ and those of $E^\vee$.

1.2. Quasismaps. Let $W$ be an affine variety, acted on by a reductive algebraic group $G$, and let $\theta$ be a character of $G$ such that $W^{ss}(\theta) = W^s(\theta)$. Denote by $X$ the GIT stack quotient $[W^{ss}(\theta)/G]$. The moduli stacks $Q_{g,k}^{0}(X, \beta)$ of stable quasimaps to $X$ (which depends implicitly on the GIT presentation) provide an alternative to Kontsevich’s space of stable maps. Generalizing the stable quotient spaces of [30], quasimaps were first introduced for toric varieties in [9] and generalized to GIT quotients in [12, 7].

In contrast to stable maps, a quasimap $f : C \to X$ generally defines only a rational map. For $X = [W^{ss}(\theta)/G]$, a quasimap to $X$ is a morphism...
from a (orbi-)curve $C$ to the stack $[W/G]$ such that the preimage of
the unstable locus is a finite set of points, disjoint from the nodes and markings
of $C$. Under certain mild conditions on $W$, $G$, and $\theta$, the moduli space
of $(0^+-\text{stable})$ quasimaps $Q_{g,k}^0(X, \beta)$ is (relatively) proper, and carries
a canonical virtual fundamental class (see §2.3 for details). As in Gromov–
Witten theory, quasimap invariants may be defined by integrating over the
virtual fundamental class.

The relationship between quasimap invariants and Gromov–Witten in-
vvariants is now well understood in many cases; these results are called $\varepsilon$-wall-
crossing [11, 10, 13, 38]. In principal, one can determine Gromov–Witten
invariants from quasimap invariants and vice versa.

1.3. Results. Let $X = [W^{ss}(\theta)/G]$ be as in the previous section. Denote by $X$ the stack quotient $[W/G]$. A choice of representation $\tau$ in $\text{Hom}(G, \text{GL}(r, \mathbb{C}))$
determines vector bundles

$$E := [W^{ss}(\theta) \times \mathbb{C}^r/G] \to X$$
$$\mathcal{E} := [W \times \mathbb{C}^r/G] \to X.$$  

Assume $\mathcal{E}$ is weakly convex (Definition 3.8). Let $s \in \Gamma(X, E)$ be a section of $E$ defined as in §2.2.1, and let $Z = Z(s) \subset X$ be the complete intersection
defined by $s$.

In this paper we compare the two-pointed genus-zero quasimap invariants
of $Z$ with those of $E^\vee$. In contrast to the case of Gromov–Witten theory,
here we obtain a direct relation between invariants, as well as a statement
at the level of virtual classes.

The first step is to identify the relevant state spaces. Consider the diagram

$$\begin{array}{c}
IE^\vee \\
\uparrow_{p} \\
IZ \\
\downarrow_{j} \\
IX
\end{array}$$

of inertia stacks. Define the ambient cohomology of $Z$ by

$$H^*_{\text{CR,amb}}(Z) := \text{im}(j^*)$$

and the cohomology of compact type of $E^\vee$ by

$$H^*_{\text{CR,ct}}(E^\vee) := \text{im}(i_*).$$

Under mild assumptions, there exists (Lemma 6.5) an isomorphism

$$\tilde{\Delta} : H^*_{\text{CR,ct}}(E^\vee) \to H^*_{\text{CR,amb}}(Z),$$

characterized by the fact that $\tilde{\Delta}(i_*(\alpha)) = e^{\pi i \text{age}_j(\mathcal{E})} j^*(\alpha)$ for $\alpha \in H^*(X_g)$.

Our first result is that $\tilde{\Delta}$ identifies two-pointed genus-zero quasimap invariants of $Z$ and $E^\vee$ up to a sign.
Theorem 6.8. Given elements $\gamma_1, \gamma_2 \in H^*_\text{CR} (E^\vee)$, we have the equality

$$\left\langle \tilde{\Delta}(\gamma_1) \psi_1^{a_1}, \tilde{\Delta}(\gamma_2) \psi_2^{a_2} \right\rangle^Z_{0, \beta} = e^{\pi i (\text{det} E + \text{rank}(E))} \langle \gamma_1 \psi_1^{a_1}, \gamma_2 \psi_2^{a_2} \rangle^E_{0, \beta}.$$

Our focus on two-pointed invariants arises from their role in solutions to the quantum differential equation. We use the superscripts $\infty$ and $0+$ to distinguish between Gromov–Witten theory and quasimap theory respectively. Recall the Dubrovin connection in Gromov–Witten theory

$$\nabla^X_{i, \infty} f := \frac{\partial}{\partial t_i} f + \frac{1}{z} T_i \bullet^X_{t, \infty} f,$$

where $\{T_i\}_{i \in I}$ is a basis of $H^*_{\text{CR}}(X)$ and $\bullet^X_{t, \infty}$ denotes the quantum product. The fundamental solution is given by

$$L^X_{i, \infty}(t, z)(\alpha) := \alpha + \sum_{i \in I} \left\langle \left\langle \frac{\alpha}{-z - \psi}, T_i \right\rangle \right\rangle_{0, \beta}^X_{t, \infty}(t) T_i,$$

where the double bracket is defined in Definition 2.18.

In quasimap theory, one can define an analogous product $\bullet^X_{t, 0+}$ and connection $\nabla^X_{t, 0+}$, replacing Gromov–Witten invariants with quasimap invariants in the definition. As in Gromov-Witten theory, $\nabla^X_{t, 0+}$ is flat and its solution is given by $L^X_{t, 0+}(t, z)$. Restricting to $t = 0$, we obtain the generating function

$$(10) \quad L^X_{0+}(z)(\alpha) := L^X_{0+}(0, z)(\alpha) = \alpha + \sum_{i \in I} \sum_{\beta \in \text{Eff}} q^\beta \left\langle \frac{\alpha}{-z - \psi}, T_i \right\rangle_{0, \beta}^X_{0+} T_i,$$

of two-pointed genus-zero quasimap invariants. Our main theorem is an identification of $L^Z_{0+}(z)$ and $L^{E^\vee}_{0+}(z)$.

Theorem 6.9. The transformation $\tilde{\Delta}$ identifies the operators $L^Z_{0+}(z)$ and $L^{E^\vee}_{0+}(z)$ up to a change of variables in the Novikov parameter:

$$L^Z_{0+}(z) \circ \tilde{\Delta} = \tilde{\Delta} \circ L^{E^\vee}_{0+}(z) |_{q^\beta \rightarrow e^{\pi i \beta (\text{det} E)}} q^\beta.$$

This theorem provides a quasimap analogue of [35, Proposition 6.13], proven by the second author in the context of Gromov–Witten theory. Note that here the comparison statement is simpler and holds for more general vector bundles $E^\vee \rightarrow X$.

Our strategy of proof is different than the arguments used to prove similar results in Gromov–Witten theory. By working with quasimaps as opposed to stable maps, we are able to recover a direct relationship between virtual classes. As in Gromov–Witten theory, we first relate the virtual fundamental classes $[Q^0_{0,2}(Z, \beta)]^\text{vir}$ and $[Q^0_{0,2}(E^\vee, \beta)]^\text{vir}$ to certain twisted virtual classes on $Q^0_{0,2}(X, \beta)$. We then show, using (non-quantum) Serre duality, that these twisted virtual classes agree up to a sign (Theorem 6.2). The cycle-valued statement is interesting in its own right, as its failure in Gromov–Witten
theory is exactly what accounts for the change of variables appearing in previous results.

In the last section, we combine our main theorem with the recent $\varepsilon$-wall-crossing results of Zhou [38, Theorem 1.12.2] to recover a quantum Serre–Witten theory when $E^\vee$ is a GIT quotient.

**Corollary 6.12.** The operator $\tilde{\Delta}$ identifies the fundamental solutions $L_{Z,\infty}$ and $L_{E^\vee,\infty}$ up to a change of variables:

$$L_{Z,\infty}(\mu_{Z,\geq 0^+}(q, -\psi), z) \circ \tilde{\Delta} = \tilde{\Delta} \circ L_{E^\vee,\infty}(\mu_{E^\vee,\geq 0^+}(q, -\psi), z)|_{q^\beta \mapsto e^{\pi i \beta (\det E)} q^\beta},$$

where $\mu_{Z,\geq 0^+}(q, -\psi), z)$ and $\mu_{E^\vee,\geq 0^+}(q, -\psi), z)$ are changes of variables defined in terms of the $I$-functions for $Z$ and $E^\vee$ respectively [38, §1.11].

We may interpret Theorem 6.9 and Corollary 6.12 together as evidence that quantum Serre duality arises naturally in the setting of quasimaps, at least when the base $X$ can be expressed as a GIT quotient. The complicated change of variables appearing in previous results in Gromov–Witten theory is not inherent to the correspondence itself, but rather a remnant of the $\varepsilon$-wall-crossing formula arising in the passage from $0^+$-stable quasimaps to $\infty$-stable maps. A further benefit to this approach is that we no longer require $E$ be convex, as we describe below.

### 1.4. Removing the convexity hypothesis.

In previous results on quantum Serre duality in Gromov–Witten theory, the vector bundle $E \to X$ was required to be convex. In the case that $X$ is a variety this holds whenever $E$ is semi-positive (Definition 3.5). However, it was observed in [16] that when $X$ is an orbifold, semi-positivity of $E$ does not imply convexity. This has been a serious obstacle in the computation of Gromov–Witten invariants of orbifold complete intersections. By working with quasimap invariants, we avoid the convexity requirement. Our results require only weak convexity (Definition 3.8), which once again holds whenever $E$ is semi-positive (and in fact holds slightly more generally, see Definition 3.7).

Recent works by Guéré [23] and Wang [37] also address the computation of genus-zero Gromov–Witten theory of complete intersections when $E$ is not convex. Wang’s work also features quasimaps, and is based on a similar philosophy to that used in this paper.

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2. Preliminaries

In this section we define quasimap invariants and introduce the generating functions which we will use in the rest of the paper. This section also serves to set notation.

2.1. Orbifold cohomology. We refer the reader to [2] for an introduction to Chen-Ruan orbifold cohomology. Denote by \( X \) an oriented orbifold which admits a finite good cover.

**Definition 2.1.** Define the inertia stack of \( X \), written \( IX \), by the fiber square

\[
\begin{array}{ccc}
IX & \rightarrow & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Delta} & X \times X.
\end{array}
\]

For a stack quotient \( X = [U/G] \) the inertia stack is composed of a disjoint union of suborbifolds \( X_g = [U_g/C(g)] \) where \( U_g \) are the elements of \( U \) fixed by \( g \in G \). Let \( S \) be a set of representatives of each conjugacy class of \( G \), then we may write the inertia stack as

\[
IX = \bigsqcup_{g \in S} X_g.
\]

We refer to each \( X_g \) as a twisted sector and call the twisted sector corresponding to the identity the untwisted sector.

Let \((x, g)\) be a point in a twisted sector \( X_g \). The tangent space \( T_x X \) splits as the direct sum of eigenspaces

\[
T_x X = \bigoplus_{0 \leq f < 1} (T_x X)_f,
\]

where \( g \) acts on the fiber \((T_x X)_f\) by multiplication by \( e^{2\pi i f} \). The age shift for \( X_g \) is defined to be

\[
\iota_g = \sum_{0 \leq f < 1} f \dim \mathbb{C}(T_x X)_f.
\]

**Definition 2.2.** [6, 2] The \( d \)th Chen-Ruan cohomology group of \( X \) is

\[
H_{CR}^d(X) := \bigoplus_{g \in S} H^{d-2\iota_g}(X_g; \mathbb{C})
\]

and

\[
H_{CR}^*(X) := \bigoplus_{d \in \mathbb{Q}_{\geq 0}} H_{CR}^d(X).
\]

Unless specified otherwise we assume cohomology groups have complex coefficients. Let \( \iota \) be the involution of \( IX \) that maps \( X_g \) to \( X_g^{-1} \). Denote the compactly supported Chen-Ruan cohomology of \( X \) by \( H_{CR,cs}^*(X) \).
Definition 2.3. [2, §4] The Chen-Ruan Poincaré pairing of the classes \( \alpha \in H_{\text{CR}}^*(X) \) and \( \beta \in H_{\text{CR,cs}}^*(X) \) is given by the integral

\[
(\alpha, \beta)^X := \int_{IX} \alpha \cup \iota^* \beta.
\]

2.2. GIT stack quotients. We follow the notations and definitions developed in [7]. Let \( W \) be an irreducible affine variety with a right action by a reductive algebraic group \( G \). Let \( \theta \) be a character of \( G \) and \( \mathbb{C}_\theta \) be the corresponding one-dimensional \( G \)-representation. We also denote the linearization \( W \times \mathbb{C}_\theta \) as \( \mathbb{C}_\theta \). Denote the semistable locus by \( W^{ss}(\theta) \) and the stable locus by \( W^s(\theta) \). Observe the following diagram of quotients:

\[
\begin{array}{ccc}
X := [W^{ss}(\theta)/G] & \longrightarrow & \mathcal{X} := [W/G] \\
\downarrow & & \downarrow \\
\mathcal{X} := W/\mathbb{C}_\theta & \longrightarrow & X_0 := \text{Spec}(\mathbb{C}[W]^G).
\end{array}
\]

We refer to \( X \) as the GIT stack quotient, \( \mathcal{X} \) as the stack quotient, \( \mathcal{X} \) as the underlying coarse space or GIT quotient with respect to \( \theta \) and \( X_0 \) as the affine quotient.

Assumption 2.4. We assume \( W^{ss}(\theta) = W^s(\theta) \) so that \( X \) is a quasi-compact Deligne-Mumford stack.

Definition 2.5. Fix a representation \( \tau : G \to \text{GL}(r, \mathbb{C}) \) for some integer \( r \). Define the vector bundle \( E = E_\tau \to \mathcal{X} \) as

\[
E := [W \times \mathbb{C}^r/G],
\]

where \( g \in G \) acts on the \( \mathbb{C}^r \) factor by multiplication by \( \tau(g) \). For simplicity we will omit \( \tau \) from the notation for the representation \( \mathbb{C}^r \) and the vector bundle \( E \) throughout the paper.

The vector bundle \( E \) restricts to a vector bundle on \( X \), which we denote by \( E \). It may be realized as the quotient \( E := [W^{ss}(\theta) \times \mathbb{C}^r/G] \).

Definition 2.6. Let \( (x, g) \) be a point in a twisted sector \( \mathfrak{X}_g \). The fiber \( E_x \) splits as the direct sum of eigenspaces

\[
E_x = \bigoplus_{0 \leq f < 1} (E_x)_f,
\]

where \( g \) acts on \( (E_x)_f \) by multiplication by \( e^{2\pi i f} \). Define the age of \( E \) at \( g \) as

\[
age_g(E) := \sum_{0 \leq f < 1} f \dim_{\mathbb{C}}(E_x)_f.
\]

2.2.1. A local complete intersection in \( X \). Let \( \mathcal{E} \) and \( E \) be as defined in Definition 2.5. We do not assume \( E \) is pulled back from the coarse space \( \mathcal{X} \), hence the isotropy groups \( G_x \) for \( x \) in \( X \) may act nontrivially on the fibers of \( E \).
Let \( s \) be a \( G \)-invariant global section of \( \Gamma(W, W \times \mathbb{C}^r)^G \) such that the zero locus \( Z(s) \) is an irreducible complete intersection intersecting the semistable locus \( W^{ss}(\theta) \) non-trivially. Assume \( Z(s) \cap W^{ss}(\theta) \) is non-singular.

**Definition 2.7.** Define \( \mathcal{Z} := [Z(s)/G] \) to be the a closed substack of \( X \) given by the section \( s \). Define \( Z \) as the GIT stack quotient 
\[
Z := [(Z(s) \cap W^{ss}(\theta))/G].
\]

The semistable locus of \( Z(s) \) is exactly the intersection of \( Z(s) \) and \( W^{ss}(\theta) \). The section \( s \) restricts to a section of \( W^{ss}(\theta) \times \mathbb{C}^r \) whose zero locus is exactly \( Z(s) \cap W^{ss}(\theta) \). Equivalently, \( \mathcal{Z} \) and \( Z \) are defined as the zero loci of the sections of \( E \) and \( E \) induced by \( s \). We will also sometimes denote these sections by \( s \).

**Remark 2.8.** If we assume the representation \( \mathbb{C}^r \) splits as the direct sum of one-dimensional representations \( \bigoplus_{i=1}^{r} \mathbb{C}_{\tau_i} \) for characters \( \tau_i : G \to \mathbb{C}^* \), then \( E \) and \( E \) split and \( Z \) defines a complete intersection of \( X \).

**Definition 2.9.** For a fixed \( g \) in \( S \), denote
\[
E_g := [(W^{ss}(\theta) \times \mathbb{C}^r)^g/C(g)].
\]

The inertia stack \( IE \) may be written as \( IE = \bigsqcup_{g \in S} E_g \). Note that \( E_g \) is not usually equal to \( E|_{X_g} \). When \( g \) acts nontrivially on the representation \( \mathbb{C}^r \), the twisted sector \( E_g \) equals \( X_g \). If \( g \) acts trivially, then \( E_g \) is a vector bundle over \( X_g \).

The section \( s \) induces a section \( s_g \in \Gamma(X_g, E_g) \) for each \( g \in S \). The twisted sector \( Z_g \subset IZ \) may be realized as the vanishing locus of \( s_g \). Denote by \( j : IZ \to IX \) and \( 0_E : IX \to IE \) the inclusion and zero section respectively. Then we have a fiber square
\[
\begin{array}{ccc}
IZ & \xrightarrow{j} & IX \\
\downarrow{j} & & \downarrow{s} \\
IX & \xrightarrow{0_E} & IE.
\end{array}
\]

By abuse of notation, we will also denote by \( j, s, \) and \( 0_E \) the corresponding maps between the rigidified inertia stacks.

**Definition 2.10.** The image of the cohomology of \( X_g \) under the pullback via the inclusion \( j_g : Z_g \to X_g \) is called the *ambient cohomology* of \( Z_g \) and denoted \( H^*_{\text{amb}}(Z_g) \). The *ambient Chen-Ruan cohomology* of \( Z \) is
\[
H^*_{\text{CR,amb}}(Z) := H^*_{\text{amb}}(IZ) = \bigoplus_{g \in S} H^*_{\text{amb}}(Z_g).
\]

Restricting the computation of quasimap invariants of \( Z \) to the ambient cohomology is common and, as Proposition 4.6 will suggests, natural.
Assumption 2.11. Following [25] and [35], we assume the Poincaré pairing on $H^*_{\text{CR,amb}}(Z)$ is non-degenerate. This is equivalent to assuming the cohomology of $Z$ splits as the direct sum:

$$H^*_{\text{CR}}(Z) = \text{im}(j^*) \oplus \ker(j_*).$$

2.2.2. The total space $E^\vee$. With the setting as in §2.2.1, consider the vector bundles

$$E^\vee := [W \times \mathbb{C}^r/G], \quad E_g^\vee := \left[(W^{ss}(\theta) \times \mathbb{C}^r)/G\right].$$

Note that $IE^\vee = \bigsqcup_{g \in S} E^\vee_g$. In this section we describe the Chen–Ruan cohomology of compact type of $E^\vee$. A more detailed introduction to cohomology of compact type appears in §2 of [35]. Note that in [35], the cohomology of compact type is referred to as the narrow cohomology.

Definition 2.12. [35, Definition 2.1] The cohomology of compact type of $E^\vee_g$ is the image of the natural homomorphism

$$\phi : H^*_{\text{cs}}(E^\vee_g) \to H^*(E^\vee_g)$$

from compactly supported cohomology to cohomology. The Chen-Ruan cohomology of compact type is

$$H^*_{\text{CR,ct}}(E^\vee) := H^*_{\text{ct}}(IE^\vee) = \bigoplus_{g \in S} H^*_{\text{ct}}(E^\vee_g).$$

Given a class $\gamma \in H^*_{\text{CR,ct}}(E^\vee)$, we call $\gamma \in H^*_{\text{CR,cs}}(E^\vee)$ a lift of $\gamma$ if $\phi(\gamma) = \gamma$. It is proven in [35, Lemma 2.6] that for $\gamma \in H^*_{\text{CR,ct}}(E^\vee)$ and $\kappa \in \ker(\phi) \subset H^*_{\text{CR,cs}}(E^\vee)$, $\kappa \cup \gamma$ is zero. From this fact one can check that the following pairing is well-defined and nondegenerate.

Definition 2.13. Define the compact type pairing on $E^\vee$ as follows. For $\alpha, \beta \in H^*_{\text{CR,ct}}(E^\vee)$,

$$\langle \alpha, \beta \rangle_{E^\vee,\text{ct}} := \langle \alpha, \beta \rangle_{E^\vee} = \int_{IE^\vee} \alpha \cup i^*(\beta),$$

where $\beta \in H^*_{\text{CR,cs}}(E^\vee)$ is a lift of $\beta$.

Let $p_g : E^\vee_g \to X_g$ denote the vector bundle projection for each $g \in S$. Define the linear map

$$e(p_g^*E^\vee_g) \cup - : H^*(E^\vee_g) \to H^*(E^\vee_g)$$

by cupping with the Euler class $e(p_g^*E^\vee_g)$. Denote by $i : IX \hookrightarrow IE^\vee$ the inclusion induced by the zero section of $E^\vee$ and $i_g : X_g \hookrightarrow E^\vee_g$ the inclusion induced by the zero section of $E^\vee_g$.

Lemma 2.14. [35, Proposition 2.15] If $X$ is compact, then the following groups are equal:

$$(2)\quad H^*_{\text{CR,ct}}(E^\vee) = \text{im} \left(i_* : H^*_{\text{CR}}(X) \to H^*_{\text{CR}}(E^\vee)\right) = \bigoplus_{g \in S} \text{im} \left(e(p^*E^\vee_g) \cup -\right).$$
Proof. We will prove the result for a given twisted sector $E^\vee_g$. If $g$ fixes only the origin of $W \times C^r$, then the equality holds trivially since $X$ is compact. Let $i_{g*}^\text{cs}$ denote the pushforward on compactly supported cohomology induced by $i_g$. Note that $i_{g*}^\text{cs}$ factors as $i_{g*} = i_{g*}^\phi \circ i_{g*}^\text{cs}$.

Thus, by [5, Equation (6.11)], $i_{g*}^\text{cs}$ is an isomorphism. We obtain the first equality of (2):

$$\text{im} \left( i_{g*} : H^*(X_g) \to H^*(E^\vee_g) \right) = \text{im} \left( \phi \circ i_{g*}^\text{cs} \right) = H^*_{\text{ct}}(E^\vee_g).$$

Now, let $\alpha$ be an element of $H^*_{\text{ct}}(E^\vee_g)$. Then

$$e(p_g^*E^\vee_g) \cup \alpha = i_{g*}(1) \cup \alpha$$

$$= i_{g*}(1 \cup i_g^*(\alpha))$$

$$= i_{g*}(i_g^*(\alpha)).$$

The first equality holds because $X_g$ can be viewed as the zero locus of the tautological section of $p_g^*E^\vee_g$ and the second is the projection formula.

Since the pullback via $i_g$ is an isomorphism, the second equality of (2) holds.

Later we will use Lemma 2.14 to express classes in the Chen-Ruan cohomology of compact type of $E^\vee_g$ in terms of classes in the Chen-Ruan cohomology of $X$. For $\gamma$ an element of $H^*_{\text{ct}}(E^\vee_g)$, there exist a class $\alpha$ in $H^*(X_g)$ such that

$$(3) \quad \gamma = i_{g*}(\alpha) = p_g^*(e(E^\vee_g) \cup \alpha).$$

2.3. Quasimaps to a GIT stack quotient. In this section we recall the definition of $\varepsilon$-stable quasimap invariants and define the generating functions that will appear in our later theorems. Let $X$ be a GIT stack quotient as in §2.2.

Definition 2.15. [1, §4] Over an algebraically closed field, a twisted curve is a connected, one-dimensional Deligne-Mumford stack which is étale locally a nodal curve, and a scheme outside the marked points and the singular locus.

Definition 2.16. [7, Definition 2.1] Let $(C,x_1,\ldots,x_k)$ be a $k$-pointed, genus-$g$ twisted curve and let $\varphi : (C,x_1,\ldots,x_k) \to (\mathcal{C},\mathcal{z}_1,\ldots,\mathcal{z}_k)$ be its rigidification.

- A $k$-pointed, genus-$g$ quasimap to $X$ is a twisted curve $(C,x_1,\ldots,x_k)$ together with a representable morphism $[u] : C \to \mathcal{X}$ such that $|u|^{-1}(\mathcal{X} \setminus X)$, referred to as the base locus, is purely zero-dimensional.

  We denote this quasimap as $(C,x_1,\ldots,x_k,[u])$ and call $(C,x_1,\ldots,x_k)$ or $C$ its source curve.
• The class $\beta$ of the quasimap is an element of $\text{Hom}(\text{Pic} X, \mathbb{Q})$ defined by
  \[ \beta : \text{Pic} X \to \mathbb{Q}, \quad \mathcal{L} \mapsto \deg ([u]^*(\mathcal{L})). \]
  The degree of the quasimap $[u]$ is given by the rational number $\beta(\mathcal{L}_\theta)$.

• If the base locus of a quasimap $(C, x_1, \ldots, x_k, [u])$ does not contain marked or nodal gerbes, then we call the quasimap prestable.

• Let $e$ be the least common multiple of $|\text{Aut}(p)|$ for all geometric points $p \to X$ with isotropy groups $\text{Aut}(p)$. Fix a rational number $\varepsilon$.
  A prestable quasimap is called $\varepsilon$-stable if
  (1) the $\mathbb{Q}$-line bundle
    \[ \omega_C \left( \sum_{i=1}^k x_i \right) \otimes (\varphi_*([u]^*\mathcal{L}_\theta^\otimes e))^\varepsilon \]
    on the coarse curve $C$ is ample and
   (2) for all $x \in C$, $\varepsilon l(x) \leq 1$,
   where $l(x)$ is the length at $x$ defined in [12, §7.1].

We say a prestable quasimap is $0^+$-stable, if (4) is ample for all rational $\varepsilon > 0$.

**Definition 2.17.** The moduli space of $k$-pointed, genus-$g$, $\varepsilon$-stable quasimaps to $X$ of degree $\beta$, denoted by $Q_{g,k}^\varepsilon(X,\beta)$, is the space of isomorphism classes of $k$-pointed, genus-$g$, $\varepsilon$-stable quasimaps to $X$ of degree $\beta$.

By Theorem 2.7 of [7], $Q_{g,k}^\varepsilon(X,\beta)$ is a proper Deligne-Mumford stack over the affine quotient $X_0$. Furthermore, when the singularities of $W$ are at worst local complete intersections and $W^{ss}(\theta)$ is nonsingular, $Q_{g,k}^\varepsilon(X,\beta)$ carries a canonical perfect obstruction theory.

When $\varepsilon$ is sufficiently large, the moduli space of $\varepsilon$-stable quasimaps $Q_{g,k}^\varepsilon(X,\beta)$ coincides with the moduli space of stable maps $\overline{M}_{g,n}(X,\beta)$ from Gromov-Witten theory. We adopt the notation of [12, 7] and define
\[ Q_{g,k}^\infty(X,\beta) := \overline{M}_{g,n}(X,\beta). \]

2.4. **The Dubrovin connection.** Since the base locus of a $\varepsilon$-stable quasimap is disjoint from the marked gerbes for $i = 1, \ldots, k$, there exists evaluation maps to the rigidified inertia stack $\overline{IX}$:
\[ ev_i : Q_{g,k}^\varepsilon(X,\beta) \to \overline{IX}. \]
We refer the reader to [1, §4.4] for an introduction to such maps. There is a canonical map $\tilde{\omega} : IX \to \overline{IX}$ inducing an isomorphism of cohomology
\( \bar{\omega}_* : H^*_CR(X) \to H^*(\bar{I}X) \). By composing \( ev_i^* \) with \( \bar{\omega}_*^{-1} \), we may define quasimap invariants using cohomology class insertions from \( H^*_CR(X) \).

Let \( \psi_l \) represent the first Chern class of the universal cotangent line over \( Q^e_{g,k}(X, \beta) \) whose fiber over a point \( (C, x_1, \ldots, x_k, [u]) \) is given by the cotangent space of the underlying coarse curve \( C \) at the \( i \)th marked point. Denote by \( [Q^e_{g,k}(X, \beta)]^{vir} \) the virtual fundamental class from [12, §4.5] and [7, §2.4.5]. By [7, Theorem 2.7], \( Q^e_{g,k}(X, \beta) \) is proper over \( X_0 \). By [36, Lemma 01W6], this implies each evaluation map \( ev_i \) is proper.

**Definition 2.18.** Assume \( X \) is proper. For non-negative integers \( a_i \) and classes \( \alpha_i \in H^*_CR(X) \), \( \varepsilon \)-quasimap invariants or simply quasimap invariants are given by integrals

\[
\langle \alpha_1 \psi^{a_1}_1, \ldots, \alpha_k \psi^{a_k}_k \rangle^{X,\varepsilon}_{g,\beta} := \int_{[Q^e_{g,k}(X, \beta)]^{vir}} \prod_{i=1}^k ev_i^*(\alpha_i) \psi^{a_i}_i.
\]

Fix a basis \( \{ T^i \}_{i \in I} \) of \( H^*_CR(X) \) and let \( t = \sum_{i \in I} t_i T^i \). Define the double-bracket

\[
\langle \langle \alpha_1 \psi^{a_1}_1, \ldots, \alpha_k \psi^{a_k}_k \rangle \rangle^{X,\varepsilon}_{0,\beta} (t) := \sum_{\beta \in \text{Eff}} \sum_{m \geq 0} \frac{q^\beta}{m!} \langle \alpha_1 \psi^{a_1}_1, \ldots, \alpha_k \psi^{a_k}_k, t, \ldots, t \rangle^{X,\varepsilon}_{0,\beta}.
\]

If \( X \) is not proper (i.e. \( X_0 \) is not a point), then slightly more care must be taken to define quasimap invariants. If at least one class \( \alpha_j \) lies in compactly supported cohomology \( H^*_{CR,cs}(X) \) for \( 1 \leq j \leq k \) then we define

\[
\langle \alpha_1 \psi^{a_1}_1, \ldots, \alpha_k \psi^{a_k}_k \rangle^{X,\varepsilon}_{g,\beta}
\]

exactly as in (5).

**Definition 2.19.** We also define \( \varepsilon \)-quasimap invariants whenever at least two insertion classes are compact type. For simplicity assume \( \alpha_j \) and \( \alpha_k \) are in \( H^*_{CR,ct}(X) \) for \( j \leq k \). Define

\[
\langle \alpha_1 \psi^{a_1}_1, \ldots, \alpha_k \psi^{a_k}_k \rangle^{X,\varepsilon}_{g,\beta} := \langle \bar{ev}_k^* \left( \prod_{i=1}^{k-1} ev_i^*(\alpha_i) \psi^{a_i}_i \cup \psi^{a_k}_k \cap [Q^e_{0,k}(X, \beta)]^{vir} \right), \alpha_k \rangle^{X,ct},
\]

where \( \langle -,- \rangle^{X,ct} \) is the compact type pairing of Definition 2.12 and \( \bar{ev}_k = \iota^* ev_k \). Define the double bracket exactly as in (6), but replacing (5) with (7).

By [35, Proposition 2.5], pullback and pushforward via a proper map each preserve the compact type subspace of \( H^*_CR(X) \), therefore the pushforward in the right hand side of (7) lies in \( H^*_{CR,ct}(X) \) as desired.

Let \( \{ T_i \}_{i \in I} \subseteq H^*_{CR,cs}(X) \) denote the dual basis to \( \{ T^i \}_{i \in I} \) in compactly supported cohomology. Whether or not \( X \) is proper, the expression

\[
\bar{ev}_k^* \left( \prod_{i=1}^{k-1} ev_i^*(\alpha_i) \psi^{a_i}_i \cup \psi^{a_k}_k \cap [Q^e_{0,k}(X, \beta)]^{vir} \right)
\]
is equal to
\[ \sum_{i \in I} \langle \alpha_1 \psi_1^{a_1}, \ldots, T_i \psi_k^{a_k} \rangle_{g, \beta} X, \varepsilon T^i. \]

**Definition 2.20.** [8, §2] The \( \varepsilon \)-quantum product of \( \alpha_1 \) and \( \alpha_2 \), written \( \alpha_1 \bullet_{t} X, \varepsilon \alpha_2 \), is the sum
\[ \alpha_1 \bullet_{t} X, \varepsilon \alpha_2 := \sum_{i \in I} \langle \langle \alpha_1, \alpha_2, T_i \rangle \rangle_{X, \varepsilon 0}(t) T^i. \]

**Definition 2.21.** Let \( z \) be a formal variable. The \( \varepsilon \)-Dubrovin connection on \( H^*_{\text{CR}}(X) \otimes \mathbb{C}[\{t_i, q\}]_{i \in I}[z, z^{-1}] \) is defined by
\[ \nabla X, \varepsilon = \frac{\partial}{\partial t_i} + \frac{1}{z} T_i \bullet_{t} X, \varepsilon. \]

\[ L^X, \varepsilon(t, z)(\alpha) := \alpha + \sum_{i \in I} \langle \langle \frac{\alpha}{-z - \psi}, T_i \rangle \rangle_{0, \beta} X, \varepsilon T^i, \]
for all \( \alpha \in H^*_{\text{CR}}(X) \).

**Proposition 2.22.** The Dubrovin connection \( \nabla X, \varepsilon \) is flat, with fundamental solution given by the operator \( L^X, \varepsilon(t, z) \). For \( i \in I \) and \( \alpha \in H^*_{\text{CR}}(X) \) we have the equality
\[ \nabla X, \varepsilon(L^X, \varepsilon(t, z)(\alpha)) = 0. \]

**Proof.** The proof is identical to the Gromov-Witten theory case in [17, 24]. The key ingredient is the topological recursion relation for quasimaps, which appears in [8, Corollary 2.3.4]. \( \square \)

Denote by \( L^X, \varepsilon(z) \) the restriction of \( L^X, \varepsilon(t, z) \) to \( t = 0 \):
\[ L^X, \varepsilon(z)(\alpha) := L^X, \varepsilon(0, z)(\alpha) = \alpha + \sum_{i \in I} \sum_{\beta \in \text{Eff}} q^\beta \langle \langle \frac{\alpha}{-z - \psi}, T_i \rangle \rangle_{0, \beta} X, \varepsilon T^i \]
\[ = \alpha + \sum_{\beta \in \text{Eff}} q^\beta \tilde{e}v_{2*} \left( \frac{ev^*_1(\alpha)}{-z - \psi_1} \cap [Q^*_{0, 2}(X, \beta)]^\text{vir} \right). \]

The operator \( L^X, \varepsilon(z) \) records all two-pointed genus-zero \( \varepsilon \)-stable quasimap invariants with one primary insertion and one descendants insertion. In this paper we restrict our attention to \( L^X, \varepsilon^+ (z) \).

### 3. Two-pointed genus-zero quasimaps

In this section we define weak forms of convexity and concavity and show that both are equivalent. We also prove that weakly semi-positive vector bundles \( \mathcal{E} \rightarrow X \) (Definition 3.7) are weakly convex.
3.1. **Source curves.** Here we describe the source curves of two-pointed genus-zero $0^+$-stable quasimaps.

**Lemma 3.1.** For a point $(C, x_1, x_2, [u])$ in $Q_{0,2}^0(X, \beta)$, the underlying coarse curve $(C, x_1, x_2)$ is an at worst nodal curve such that each irreducible component has exactly two special points. Furthermore, the degree of $[u]$ is positive on every rational component.

**Proof.** For a point $(C, x_1, x_2, [u])$ in $Q_{0,2}^0(X, \beta)$, the dual graph of its source curve $(C, x_1, x_2)$ is a tree because the curve is genus zero. The stability condition (4) states that the line bundle

$$\omega_C(x_1 + x_2) \otimes (\phi_*([u]^\otimes L \otimes e^\theta)^\varepsilon$$

has positive degree on each rational component of the underlying coarse curve $(C, x_1, x_2)$ for all $\varepsilon > 0$. Hence, every rational component of $(C, x_1, x_2)$ with a single node must contain a marked point. Since there are only two marked points, there are at most two rational components with a single node. That is, $(C, x_1, x_2)$ must be a chain of rational components with marked points on the terminal components.

Therefore each rational component of $C$ has exactly two special points. The stability condition then implies that the degree of $[u]$ must be positive on every component of $(C, x_1, x_2)$. \hfill \Box

---

![Figure 1: The underlying coarse curves $(C, x_1, x_2)$](image)

For integers $c, d$, denote by $\mathbb{P}_{[c, d]}$ a smooth twisted curve whose coarse space is $\mathbb{P}^1$ with two marked points such that the isotropy groups at the marked points are $\mu_c$ and $\mu_d$. By Lemma 3.1, each component of a source curve in $Q_{0,2}^0(X, \beta)$ is given by some $\mathbb{P}_{[c, d]}$.

Denote by $l$, $a$, and $b$ the integers $l = \gcd(c, d)$, $a = c/l$, and $b = d/l$. We can rewrite $l$ as the product of integers $l_1$ and $l_2$ such that $\gcd(l_1, l_2) = 1$, $\gcd(l_1, b) = 1$, and $\gcd(l_2, a) = 1$. In the next lemma we give a uniform way of expressing $\mathbb{P}_{[c, d]}$ as a quotient stack.

**Lemma 3.2.** Let $\chi : \mu_1 \times \mu_2 \times \mathbb{C}^* \to \mathbb{C}^*$ be the character given by

$$\chi(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda) = \lambda.$$  

The twisted curve $\mathbb{P}_{[c, d]}$ may be represented as a GIT stack quotient

$$\mathbb{C}^2 \sslash_\chi (\mu_1 \times \mu_2 \times \mathbb{C}^*)$$

with action

$$(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda) \cdot (x, y) = (\lambda^a_1x e^{2\pi i \frac{m_1}{l_1}}, \lambda^b_2 y e^{2\pi i \frac{m_2}{l_2}}),$$

where $a_1$ and $b_2$ are the integers $a$ and $b$, respectively.
for $0 \leq m_1 < l_1$, $0 \leq m_2 < l_2$, $\lambda \in \mathbb{C}^*$, and $(x, y) \in \mathbb{C}^2$.

**Proof.** It suffices to show that the orbifold curve (11) contains exactly two orbifold points with isotropy groups of orders $c$ and $d$.

We first calculate the generic isotropy. Fix $x, y \neq 0$ and let $\left( e^{2\pi i m_1 \frac{l_1}{l}}, e^{2\pi i m_2 \frac{l_2}{l}}, \lambda \right)$ be an element of $\mu_1 \times \mu_2 \times \mathbb{C}^*$ which fixes $(x, y)$. Then we have

\begin{equation}
\lambda^a e^{2\pi i \frac{m_1}{l_1}} = 1
\end{equation}

and

\begin{equation}
\lambda^b e^{2\pi i \frac{m_2}{l_2}} = 1.
\end{equation}

Equality (12) implies $\lambda = e^{2\pi i \frac{ln-m_1}{al_1}}$ for some integer $n$. So, the product $\lambda^b e^{2\pi i \frac{m_2}{l_2}}$ equals $e^{2\pi i \left( \frac{m_2}{l_2} + \frac{bl_1n bm_1}{al_1} \right)}$ and (13) implies

$$
\frac{m_2}{l_2} + \frac{bl_1n bm_1}{al_1} = \frac{al_1 m_2 + bl_1 l_2 n - bl_2 m_1}{al_1 l_2}
$$

is an integer.

If $al_1 l_2$ divides $al_1 m_2 + bl_1 l_2 n - bl_2 m_1$, then $l_2$ divides $al_1 m_2$. However, the greatest common divisor of $al_1$ and $l_2$ is one, and $m_2$ is a non-negative integer strictly less than $l_2$. Thus, $m_2$ must equal 0. Similar reasoning shows $m_1$ also equals 0.

In this case, (12) and (13) reduce to the equation $\lambda^a = \lambda^b = 1$. By assumption, the greatest common divisor of $a$ and $b$ is one. Therefore (12) holds if and only if $\left( e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda \right)$ is the identity.

An element $\left( e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda \right)$ in $\mu_1 \times \mu_2 \times \mathbb{C}^*$ fixes $(1, 0)$ if and only if $\lambda^a e^{2\pi i \frac{m_1}{l_1}} = 1$. Hence, $\mu_2$ fixes $(1, 0)$ and the subgroup of $\mu_1 \times \mathbb{C}^*$ which fixes $(1, 0)$ is

\begin{equation}
\left\langle \left( e^{-2\pi i \frac{l_2}{l}}, e^{2\pi i \frac{m_1}{l}} \right) \right\rangle \cong \mu_{al_1}.
\end{equation}

This gives us the following isotropy group:

$$
G_{(1, 0)} \cong \mu_{al_1} \times \mu_{l_2} \cong \mu_{al_1 l_2} = \mu_c.
$$

The second isomorphism follows from the fact that $\gcd(al_1, l_2) = 1$. The final equality results from the equalities $l = l_1 l_2$ and $a = c/l$.

An identical argument shows that the order of the isotropy group at $(0, 1)$ is $d$. One easily checks that the $\chi$-stable locus is $\mathbb{C}^2 \setminus \{(0, 0)\}$ as desired. $\square$

The GIT stack quotient $\left[ \mathbb{C}^2 \sslash \chi (\mu_1 \times \mu_2 \times \mathbb{C}^*) \right]$ may be written as a global quotient of a weighted projective space by a finite cyclic group. Recall that $l = l_1 l_2$ and $\gcd(l_1, l_2) = 1$. We will sometimes write $\left[ \mathbb{C}^2 \sslash \chi (\mu_1 \times \mu_2 \times \mathbb{C}^*) \right]$ simply as $\mathbb{P}(a, b)/\mu_l$, where the action of $\mu_l = \mu_{al_1} \times \mu_{l_2}$ is understood to be as described.
Corollary 3.3. Let \((C, x_1, x_2)\) be a source curve of a two-pointed genus-zero 0+-stable quasimap to a GIT stack quotient. Then the rational components of \((C, x_1, x_2)\) are isomorphic to a quotient of a weighted projective \(\mathbb{P}(a, b)/\mu_l\) as above.

Proof. The result follows immediately from Lemmas 3.1 and 3.2. \(\square\)

We conclude our discussion of two-pointed genus-zero (0+-)stable quasimap source curves with a remark about line bundles over such curves.

Remark 3.4. Let \((C, x_1, x_2, [u])\) be a quasimap in the moduli space \(Q_{0+0,2}^0(X, \beta)\). By Lemma 3.1, Lemma 3.2, and [31, Proposition 2.2], line bundles over \(C\) restricted to an irreducible component are isomorphic to a GIT stack quotient

\[
\left[\left(\mathbb{C}^2 \setminus \{0\}\right) \times \mathbb{C} \right]/\chi (\mu_1 \times \mu_2 \times \mathbb{C}^*)
\]

with the action given by

\[
\left(e^{2\pi i \frac{m_1}{n_1}}, e^{2\pi i \frac{m_2}{n_2}}, \lambda\right) \cdot (x, y, z) = \left(\lambda^a x e^{2\pi i \frac{m_1}{n_1}}, \lambda^b y e^{2\pi i \frac{m_2}{n_2}}, \lambda^d z e^{2\pi i \frac{k_1 k_2 m_1 + k_2 m_2}{n_1 n_2}}\right),
\]

for some integers \(k_1, k_2,\) and \(d\). Denote this GIT quotient by \(O_{k_1, k_2}^{b_1, b_2}(a, b)/\mu_l(d)\). We omit the superscripts \(k_1\) and \(k_2\) when \(l\) equals 1.

3.2. Weak convexity. We now define (weak) semi-positivity for a vector bundle over \(X\) and prove that it implies a weak form for convexity.

Definition 3.5. For a character \(\tau : G \to \mathbb{C}^*\) with one-dimensional representation \(\mathbb{C}_\tau\), denote by \(L = [W \times \mathbb{C}_\tau/G]\) the corresponding line bundle over \(X\). We say \(L\) is positive if \(\beta(L) > 0\) for all \(\beta \in \text{Eff}(W, G, \theta)\) and semi-positive if \(\beta(L) \geq 0\) for all \(\beta \in \text{Eff}(W, G, \theta)\).

A vector bundle \(E\) over \(X\) is positive (semi-positive) if it splits as the direct sum of positive (semi-positive) line bundles.

Remark 3.6. Our definition of a positive (semi-positive) line bundle \(L\) agrees with the definition in [12, §6.2] of a positive (semi-positive) character \(\tau\).

The authors of [31] generalize the Birkhoff-Grothendieck theorem to orbifolds whose coarse space is \(\mathbb{P}^1\) with only two points with nontrivial isotropy and chains of projective lines meeting at nodal singularities. These are exactly the source curves of \(Q_{0+2}^0(X, \beta)\) that we are considering.

Fix a quasimap \((C, x_1, x_2, [u]) \in Q_{0+2}^0(X, \beta)\). By Lemma 3.1 and [31], vector bundles over \(C\) split as line bundles. Hence, for any vector bundle \(E \to X\) as in Definition 2.5, the pullback \([u]^*E\) splits as the direct sum of line bundles \(\oplus_i \mathcal{L}_i\), regardless of whether \(E\) splits. This allows us to make the following definition:

Definition 3.7. The vector bundle \(E\) is weakly semi-positive if, for any \(\beta \in \text{Eff}(W, G, \theta)\) and \((C, x_1, x_2, [u]) \in Q_{0+2}^0(X, \beta)\), the pullback \([u]^*E\) splits
as the direct sum of line bundles $\oplus_{i=1}^r L_i$ such that $\deg(L_i) \geq 0$ for all $1 \leq i \leq r$.

If $X$ is a smooth variety, convexity follows from semi-positivity of $E$. This is no longer the case when $X$ is an orbifold (See Remark 4.1 for more details). In this section we consider a weaker notion, which we term weak convexity.

**Definition 3.8.** A vector bundle $E$ over $X$ is **weakly convex** if $H^1(C, [u]^*E(-x_2))$ vanishes for all $(C, x_1, x_2, [u]) \in Q_0^0(X, \beta)$ and $\beta \in \text{Eff}(W, G, \theta)$.

Assume the vector bundle $E$ is weakly semi-positive. For a quasimap $(C, x_1, x_2, [u]) \in Q_0^0(X, \beta)$, the group $H^1(C, [u]^*E)$ splits as

$$H^1(C, [u]^*E) = \bigoplus_{i=1}^r H^1(C, L_i).$$

The line bundles $L_i$ have non-negative degree by assumption. Furthermore, each $L_i$ has non-negative degree on every irreducible component of $C$. Let $l, a,$ and $b$ be as in §3.1. As noted in Remark 3.4, the restriction of $L_i$ to an irreducible component of $C$ is then isomorphic to $O_{P(a,b)/\mu_l}^{k_1,k_2}(d)$ for some integers $k_1, k_2,$ and $d$ with $d \geq 0$.

We will show that $H^1(C, L)$ vanishes whenever $C$ is a two-pointed genus-zero $0+$-stable quasimap source curve and $L \to C$ has non-negative degree on each irreducible component. We begin with the case that $C$ is smooth.

For Lemmas 3.9 and 3.10, let $(x, y)$ be the homogeneous coordinates of $P(a,b)/\mu_l$ and label $x_1 = (1, 0)$ and $x_2 = (0, 1)$. Also fix a non-negative integer $d$.

**Lemma 3.9.** The cohomology group $H^1\left(P(a,b)/\mu_l, O_{P(a,b)/\mu_l}^{k_1,k_2}(d)\right)$ vanishes.

**Proof.** First consider the case $\mu_l$ is trivial. Then $P(a,b)/\mu_l$ is a weighted projective space $P(a,b)$ and $O_{P(a,b)/\mu_l}^{k_1,k_2}(d)$ is simply $O_{P(a,b)}(d)$. Let $\varphi$ denote the rigidification map from $P(a,b)$ to $P^1$.

The points $x_1$ and $x_2$ are the only points of $P(a,b)$ with nontrivial isotropy. The generator of the isotropy group at $L|_{x_j}$ acts by multiplication by $e^{2\pi i \frac{w_j}{ab}}$ for some weight $0 \leq w_j < ab$. Orbifold Reimann-Roch gives the following:

$$\deg(\varphi_*O_{P(a,b)}(d)) = \frac{d}{ab} - \left(\frac{w_1}{ab} + \frac{w_2}{ab}\right)$$

$$> \frac{d}{ab} - 2$$

$$> -2.$$

Since the degree of $\varphi_*O_{P(a,b)}(d)$ is an integer no less than $-1$, we conclude that

$$H^1(P(a,b), O_{P(a,b)}(d)) = H^1(P^1, \varphi_*O_{P(a,b)}(d)) = 0.$$
Now let $\mu_l$ be an arbitrary finite cyclic group. We have the fiber diagram:

$$
\begin{array}{c}
\mathcal{O}_{\mathbb{P}(a,b)}(d) \\ \downarrow \\
\mathbb{P}(a,b)
\end{array} \xrightarrow{\mu_l} 
\begin{array}{c}
\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d) \\ \downarrow \\
\mathbb{P}(a,b)/\mu_l.
\end{array}
$$

So, $H^1\left(\mathbb{P}(a,b)/\mu_l, \mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)\right)$ is the $\mu_l$-invariant part of $H^1\left(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)}(d)\right)$. Equation (15) implies $H^1\left(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)}(d)\right)$ vanishes when $d \geq 0$. Therefore $H^1\left(\mathbb{P}(a,b)/\mu_l, \mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)\right)$ vanishes. \hfill \Box

**Lemma 3.10.** If the isotropy at $x_j$ acts nontrivially on the fibers of $\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)$, then $H^0\left(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)\right)_{x_j}$ vanishes. Otherwise, there exists a section of $\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)$ that is nonzero at $x_j$ and zero at the other point.

**Proof.** The first claim is immediate. For the second, assume the isotropy at $x_1$ acts trivially.

By (14), the isotropy group at $x_1$ is

$$G_{x_1} = \langle \left(e^{2\pi i \frac{1}{a_1}}, e^{2\pi i \frac{1}{a_2}}, e^{-2\pi i \frac{1}{a_1}}\right) \rangle. $$

The action of $\left(e^{2\pi i \frac{1}{a_1}}, e^{2\pi i \frac{1}{a_2}}, e^{-2\pi i \frac{1}{a_1}}\right)$ on $\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)_{x_1}$ is given by

$$\left(e^{2\pi i \frac{1}{a_1}}, e^{2\pi i \frac{1}{a_2}}, e^{-2\pi i \frac{1}{a_1}}\right) \cdot (1,0,z) = \left(1,0,ze^{2\pi i (a_2 k_2 + a_1 k_2 - l_2 d)/a_1 a_2}\right).$$

This action is trivial, therefore $al_1l_2$ divides $al_2k_1 + al_1k_2 - l_2 d$. Thus $a$ divides $l_2 d$. Since $l_2$ and $a$ are coprime by the assumption preceding Lemma 3.2, this implies $a$ divides $d$. Rewrite $d$ as the product $ac$ for some integer $c$. Similar arguments show that $l_2$ divides $k_2$ and $l_1$ divides $(k_1 - c)$. The latter ensures $k_1$ is congruent to $c$ modulo $l_1$.

Consider the map $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(x,y) = x^c$. We claim $f$ descends to a section of $\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)$. Note that

$$f\left(\lambda^a y e^{2\pi i \frac{\mu_1}{a_1}}, \lambda^b ye^{2\pi i \frac{\mu_2}{a_2}}\right) = \lambda^{ac} ye^{2\pi i \frac{\mu_1}{a_1}} = \lambda^d e^{2\pi i \frac{kl_1 m_1 + kl_2 m_2}{a_1 a_2}} f(x,y).$$

The second equality holds because $k_1$ is congruent to $c$ modulo $l_1$, $d$ equals $ac$, and $l_2$ divides $k_2$ so $\frac{k_2 l_2 m_2}{l_1 l_2}$ is an integer for all integers $0 \leq m_2 < l_2$.

This verifies the map $(x,y) \mapsto (x,y,f(x,y))$ is $G$-equivariant, and therefore $f$ descends to a section $s$ of $\mathcal{O}_{\mathbb{P}(a,b)/\mu_l}^{k_1,k_2}(d)$. One sees immediately that $s(x_1) \neq 0$ and $s(x_2) = 0$. 
An identical argument shows that if the action of the isotropy group at \( x_2 \) is trivial, then there exists a section \( \tilde{s} \) of \( \mathcal{O}_{\mathbb{P}^{(a,b)}/\mu_l}(d) \) such that \( \tilde{s}(x_1) = 0 \) and \( \tilde{s}(x_2) \neq 0 \).

**Proposition 3.11.** Let \( (C, x_1, x_2) \) be the source curve of a genus-zero 0+-stable quasimap to a GIT stack quotient and \( \mathcal{L} \to C \) a line bundle with non-negative degree on each irreducible component of \( C \). Then \( H^1(C, \mathcal{L}(-x_2)) \) vanishes.

**Proof.** By Lemma 3.1, the underlying coarse curve \( \overline{C} \) is a chain of rational components such that the marked points \( x_1 \) and \( x_2 \) lie on the end components. Suppose \( C \) has \( k \) irreducible components \( \{C_j\}_{j=1}^k \), labeled so that \( x_1 \) lies on \( C_1 \), \( x_2 \) lies on \( C_k \), and the curves \( C_j \) and \( C_{j+1} \) intersect at the node \( n_j \).

We first show \( H^1(C, \mathcal{L}) \) vanishes. Consider the normalization sequence:

\[
0 \longrightarrow \mathcal{O}_C \xrightarrow{k} \bigoplus_{j=1}^k \mathcal{O}_{C_j} \xrightarrow{k-1} \bigoplus_{j=1}^{k-1} \mathcal{O}_{C|n_j} \longrightarrow 0.
\]

Tensoring by \( \mathcal{L} \) and taking cohomology, we obtain

\[
0 \longrightarrow H^0(C, \mathcal{L}) \xrightarrow{k} \bigoplus_{j=1}^k H^0(C_j, \mathcal{L}|_{C_j}) \xrightarrow{k-1} \bigoplus_{j=1}^{k-1} H^0(n_j, \mathcal{L}|_{n_j}) \longrightarrow
\]

\[
\longrightarrow H^1(C, \mathcal{L}) \longrightarrow 0.
\]

Here \( \bigoplus_{j=1}^k H^1(C_j, \mathcal{L}|_{C_j}) \) is zero by Corollary 3.3 and Lemma 3.9.

For sections \( s_j \) in \( H^0(C_j, \mathcal{L}|_{C_j}) \), the map \( F \) is defined as

\[
F(s_1, \ldots, s_k) = (s_1(n_1) - s_2(n_1), s_2(n_2) - s_3(n_2), \ldots, s_{k-1}(n_{k-1}) - s_k(n_{k-1})).
\]

If the isotropy group \( G_{n_j} \) acts nontrivially on the fiber \( \mathcal{L}|_{n_j} \), then \( H^0(n_j, \mathcal{L}|_{n_j}) \) is zero. Otherwise, Lemma 3.10 implies there exists a section \( \tilde{s}_j \) in \( H^0(C_j, \mathcal{L}|_{C_j}) \) such that \( \tilde{s}_j(n_j) \neq 0 \) and, if \( j > 1 \), \( \tilde{s}_j(n_{j-1}) = 0 \). Thus

\[
F\left(0, \ldots, \frac{1}{\tilde{s}_j(n_j)} \tilde{s}_j, \ldots, 0\right) = (0, \ldots, 1, \ldots, 0),
\]

where the vectors above have a zero in every entry except the \( i \)th one. This shows \( F \) is surjective. As a result, \( H^1(C, \mathcal{L}) \) vanishes.

Tensoring the divisor exact sequence of \( C \) for the marked point \( x_2 \) by \( \mathcal{L} \) and pushing forward gives the long exact sequence:

\[
0 \longrightarrow H^0(C, \mathcal{L}(-x_2)) \longrightarrow H^0(C, \mathcal{L}) \longrightarrow H^0(x_2, \mathcal{L}|_{x_2}) \longrightarrow
\]

\[
\longrightarrow H^1(C, \mathcal{L}(-x_2)) \longrightarrow 0.
\]
If $G_{x_2}$ acts nontrivially on the fiber of $\mathcal{L}$ at $x_2$, then $H^0(x_2, \mathcal{L}|_{x_2})$ vanishes. So, the cohomology group $H^1(C, \mathcal{L}(-x_2))$ also vanishes.

If $G_{x_2}$ acts trivially on the fiber of $\mathcal{L}$ at $x_2$, we need to show that the map $H^0(C, \mathcal{L}) \to H^0(x_2, \mathcal{L}|_{x_2})$ is surjective. By Lemma 3.10, there exists a section $s$ of $\mathcal{L}|_{C_k} \to C_k$ such that $s(n_{k-1})$ is zero and $s(x_2)$ is nonzero. The section $s$ can be extended by the zero section on the other components to get a section in $H^0(C, \mathcal{L})$ that does not vanish at $x_2$. Therefore the map $H^0(C, \mathcal{L}) \to H^0(x_2, \mathcal{L}|_{x_2})$ is surjective, so $H^1(C, \mathcal{L}(-x_2))$ is zero. \hfill $\square$

**Theorem 3.12.** If a vector bundle $\mathcal{E} \to \mathfrak{X}$ is weakly semi-positive, then it is weakly convex.

**Proof.** Consider a 0+-stable quasimap $(C, x_1, x_2, [u])$ in $Q_{0,2}^+(X, \beta)$. Since $[u]^*\mathcal{E}$ is a direct sum of line bundles, we may write

$$H^1(C, [u]^*\mathcal{E}(-x_2)) = \bigoplus_{i=1}^r H^1(C, \mathcal{L}_i(-x_2)).$$

Each $\mathcal{L}_i$ satisfies the hypothesis of Proposition 3.11. Hence, the cohomology group $H^1(C, [u]^*\mathcal{E}(-x_2))$ vanishes. \hfill $\square$

### 3.3. Weak concavity.

In this section we define weak concavity. We then prove the relative log canonical bundle over the universal curve $\pi : C \to Q_{0,2}^+(X, \beta)$ is trivial and use this result to prove a vector bundle $\mathcal{E} \to \mathfrak{X}$ is weakly convex if and only if its dual $\mathcal{E}^\vee \to \mathfrak{X}$ is weakly concave.

**Definition 3.13.** A vector bundle $\mathcal{E}^\vee$ over $\mathfrak{X}$ is *weakly concave* if $H^0(C, [u]^*\mathcal{E}^\vee(-x_1))$ vanishes for all 0+-stable quasimaps $(C, x_1, x_2, [u]) \in Q_{0,2}^{0+}(X, \beta)$ and $\beta \in \text{Eff}(W, G, \theta)$.

**Lemma 3.14.** The log canonical bundle over a source curve $(C, x_1, x_2)$ of a 0+-stable quasimap $(C, x_1, x_2, [u])$ in $Q_{0,2}^{0+}(X, \beta)$ is trivial.

**Proof.** If $C$ is smooth, the canonical bundle of $\mathbb{P}(a, b)/\mu_l$ is isomorphic to $\mathcal{O}(-[0] - [\infty])$. The result is then immediate. When the source curve is nodal, we again use Lemma 3.3. In this case we will show that $\omega_{C}(x_1 + x_2)$ has a nowhere vanishing section, which gives an isomorphism $\omega_{C}(x_1 + x_2) \cong \mathcal{O}_C$.

Let $C$ have $k$ rational components and $k - 1$ nodes, labeled as $\{C_i\}_{i=1}^k$ and $\{n_i\}_{i=1}^{k-1}$ as in the proof of Proposition 3.11. For convenience, relabel $x_1$ by $n_0$ and $x_2$ by $n_k$. Tensoring the normalization sequence (16) by the log canonical bundle gives rise to the following long exact sequence:

$$0 \to H^0(C, \omega_C(x_1 + x_2)) \overset{\partial}{\to} \bigoplus_{i=1}^k H^0(C_i, \omega_{C_i}(n_{i-1} + n_i)) \overset{\partial}{\to} \bigoplus_{i=1}^{k-1} H^0(n_i, \omega_C(x_1 + x_2)|_{n_i}) \to \cdots.$$

Note that the map $\partial$ is a sum of residue maps. There exists nowhere vanishing sections $s_i$ of each $\omega_{C_i}(n_{i-1} + n_i)$ that have residue 1 near $n_{i-1}$ and $-1$ near $n_i$. The tuple

$$(s_1, \ldots, s_k) \in \bigoplus_{i=1}^k H^0(C_i, \omega_{C_i}(n_{i-1} + n_i))$$
lies in kernel of $F$, which is isomorphic to $H^0(C, \omega_C(x_1 + x_2))$. Hence, the sections $s_1, \ldots, s_k$ glue together to give a nowhere vanishing global section of $\omega_C(x_1 + x_2)$. Thus the log canonical bundle is trivial.

**Proposition 3.15.** The relative log canonical bundle over the universal curve $\pi : C \to Q^0_{0,2}(X, \beta)$ is trivial.

**Proof.** Tensoring the divisor exact sequence of $C$ for the divisor corresponding to the first marked point by $\omega_\pi(x_2)$ and pushing forward, we obtain the long exact sequence

$$0 \longrightarrow \mathbb{R}^0\pi_*\omega_\pi(x_2) \longrightarrow \mathbb{R}^0\pi_*\omega_\pi(x_1 + x_2) \longrightarrow \mathbb{R}^0\pi_* (\omega_\pi(x_1 + x_2)|_{x_1}) \longrightarrow$$

$$\longrightarrow \mathbb{R}^1\pi_*\omega_\pi(x_2) \longrightarrow \mathbb{R}^1\pi_*\omega_\pi(x_1 + x_2) \longrightarrow \mathbb{R}^1\pi_* (\omega_\pi(x_1 + x_2)|_{x_1}) \longrightarrow 0.$$

Lemma 3.14 and Serre duality imply that for every $0+$-stable quasimap $(C, x_1, x_2, [u])$ in $Q^0_{0,2}(X, \beta)$, the cohomology groups $H^0(C, \omega_C(x_2))$ and $H^1(C, \omega_C(x_2))$ vanish. Therefore $\mathbb{R}^0\pi_*\omega_\pi(x_2)$ and $\mathbb{R}^1\pi_*\omega_\pi(x_2)$ are both zero. Hence, the map

$$\mathbb{R}^0\pi_*\omega_\pi(x_1 + x_2) \longrightarrow \mathbb{R}^0\pi_* (\omega_\pi(x_1 + x_2)|_{x_1})$$

is an isomorphism.

The sheaf $\mathbb{R}^0\pi_* (\omega_\pi(x_1 + x_2)|_{x_1})$ is canonically trivialized by the residue map. Thus $\mathbb{R}^0\pi_*\omega_\pi(x_1 + x_2)$ is a trivial line bundle.

The constant function $1 \in \Gamma(Q^0_{0,2}(X, \beta), \mathbb{R}^0\pi_* (\omega_\pi(x_1 + x_2)|_{x_1}))$ is the image of a section $s \in \Gamma(Q^0_{0,2}(X, \beta), \mathbb{R}^0\pi_*\omega_\pi(x_1 + x_2))$. The section $s$ corresponds to a nonzero section $\tilde{s}$ of $\omega_\pi(x_1 + x_2)$. The restriction of $\omega_\pi(x_1 + x_2)$ to a fiber $(C, x_1, x_2)$ of the universal curve $C$ is canonically trivial by Lemma 3.14. The section $\tilde{s}|_C$ is nonzero and hence nowhere vanishing on each fiber $(C, x_1, x_2)$ in $C$. Therefore $\tilde{s}$ is nowhere vanishing.

Hence, $\omega_\pi(x_1 + x_2)$ has a nowhere vanishing section and so it is trivial. □

**Theorem 3.16.** The vector bundle $E$ is weakly convex if and only if $E^\vee$ is weakly concave.

**Proof.** For a point $(C, x_1, x_2, [u])$ in the moduli space $Q^0_{0,2}(X, \beta)$, the log canonical bundle $\omega_C(x_1 + x_2)$ is canonically trivial by Lemma 3.14. Hence, the canonical bundle $\omega_C$ is isomorphic to $\mathcal{O}_C(-x_1 - x_2)$. We now have the following:

$$H^0(C, [u]^*E^\vee(-x_1)) = H^1(C, [u]^*E(x_1) \otimes \omega_C)^\vee = H^1(C, [u]^*E(-x_2))^\vee.$$

The first equality is Serre duality and the second is because of the isomorphism $\omega_C \cong \mathcal{O}_C(-x_1 - x_2)$. If the left hand side vanishes for all $0+$-stable two-pointed quasimaps then so does the right, and vice versa. This completes the proof. □
4. QUANTUM LEFSCHETZ

The quantum Lefschetz hyperplane theorem compares the genus-zero Gromov-Witten theory of a space \( X \) with that of a complete intersection \( Z \subset X \) defined by a section of a vector bundle \( E \to X \) \([14, 3, 26, 4, 28, 19]\). The same proof applies to quasimaps under certain conditions on \( E \) \([12]\).

In this section we use a modification of the quantum Lefschetz theorem for two-pointed genus-zero quasimaps to relate the generating function \( L_{Z,0+}(z) \) to a twisted version of \( L_{X,0+}(z) \).

Remark 4.1 (quantum Lefschetz for orbifolds). Let \( Z \) be a closed subset of \( X \) cut out by a section of a vector bundle \( E \to X \). In the case when \( X \) is a smooth variety, the quantum Lefschetz theorem holds as long as \( E \) is the direct sum of semi-positive line bundles. However this can fail when \( X \) is an orbifold. For example, the line bundle \( \mathcal{O}_{\mathbb{P}(1,1,2,2)}(1) \to \mathbb{P}(1,2,2) \) is positive, but quantum Lefschetz fails for the hypersurface \( \mathbb{P}(1,2,2) \) defined by a section of \( \mathcal{O}_{\mathbb{P}(1,1,2,2)}(1) \) \([16]\).

As observed in \([16]\), the general setting in which one should expect a quantum Lefschetz statement is not when \( E \) is semi-positive, but rather when \( E \) is convex, that is, when for every stable map \( f : C \to X \) from a genus-zero curve \( C \), the cohomology group \( H^1(C, f^*(E)) \) is zero. If \( X \) is a variety, then if \( E \to X \) is semi-positive it is also convex. This no longer holds when \( X \) is an orbifold.

For two-pointed genus-zero quasimaps, we will see that in fact weak convexity is sufficient for a quantum Lefschetz statement.

4.1. Weak convexity and twisted invariants. Let \( X, X, \) and \( \mathcal{E} \) be as in §2.2.1. For the remainder of the paper we assume that \( X \) is proper and that \( \mathcal{E} \) is weakly convex.

Lemma 4.2. The cohomology group \( H^1(C, [u]^*\mathcal{E}) \) vanishes for all points \((C, x_1, x_2, [u])\) in \( Q^0_{0,2}(X, \beta) \) and classes \( \beta \in \text{Eff}(W,G,\theta) \).

Proof. Fix a point \((C, x_1, x_2, [u])\) in \( Q^0_{0,2}(X, \beta) \). Consider the divisor exact sequence

\[
0 \longrightarrow \mathcal{O}_C(-x_2) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C|_{x_2} \longrightarrow 0.
\]

Tensoring this by \([u]^*\mathcal{E}\) and pushing forward gives rise to a long exact sequence in cohomology

\[
0 \longrightarrow H^0(C, [u]^*\mathcal{E}(-x_2)) \longrightarrow H^0(C, [u]^*\mathcal{E}) \longrightarrow H^0(x_2, [u]^*\mathcal{E}|_{x_2}) \longrightarrow
\]

\[
\longrightarrow H^1(C, [u]^*\mathcal{E}(-x_2)) \longrightarrow H^1(C, [u]^*\mathcal{E}) \longrightarrow 0.
\]

By Theorem 3.12, \( H^1(C, [u]^*\mathcal{E}(-x_2)) \) vanishes. Therefore \( H^1(C, [u]^*\mathcal{E}) \) also vanishes. \( \square \)
Denote by $\pi$ the projection from the universal curve $C$ to $Q_{0,2}^+(X, \beta)$. Let $[u]$ be the universal map from $C$ to $X$. By Lemma 4.2, $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ is a vector bundle on $Q_{0,2}^+(X, \beta)$. Let $0_X$ denote the zero section of $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ and $\tilde{s} \in \Gamma(Q_{0,2}^+(X, \beta), \mathbb{R}^0\pi_*[u]^*\mathcal{E})$ denote the section induced by $s$.

**Theorem 4.3.** There is a fiber square

$$
\begin{array}{ccc}
Q_{0,2}^+(Z, \beta) & \xrightarrow{j} & Q_{0,2}^+(X, \beta) \\
\downarrow{j} & & \downarrow{\tilde{s}} \\
Q_{0,2}^+(X, \beta) & \xrightarrow{0_X} & \mathbb{R}^0\pi_*[u]^*\mathcal{E}.
\end{array}
$$

The virtual classes are related by

$$0_X^! \left[Q_{0,2}^+(X, \beta)\right]^{vir} = \left[Q_{0,2}^+(Z, \beta)\right]^{vir}.$$

**Proof.** The proof of the theorem in [27] for Gromov-Witten theory extends to quasimaps. \qed

Theorem 4.3 implies the more familiar quantum Lefschetz statement from [12].

**Proposition 4.4.** [12] The following classes are equal in the Chow group of $Q_{0,2}^+(X, \beta)$:

$$\tilde{j}_* \left[Q_{0,2}^+(Z, \beta)\right]^{vir} = e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}) \cap \left[Q_{0,2}^+(X, \beta)\right]^{vir}.$$

The definitions and arguments that follow are similar to those appearing in [33, §2.1] to define a twisted quantum product.

**Definition 4.5.** In the Chow group of $Q_{0,2}^+(X, \beta)$, define the twisted virtual class

$$\left[Q_{0,2}^+(X, \beta)\right]^{vir}_{X/Z,2} := e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q_{0,2}^+(X, \beta)\right]^{vir}.$$

Define the operator $L^{X/Z,0+}(z)$ by

$$L^{X/Z,0+}(z)(\alpha) := \alpha + \sum_{\beta \in \text{Eff}} q^\beta \bar{e}\psi_2 \left( \frac{e\psi_1^*(\alpha)}{-z - \psi_1} \cap \left[Q_{0,2}^+(X, \beta)\right]^{vir}_{X/Z,2} \right)$$

for $\alpha \in H^*_{CR}(X)$.

Now, we compare $L^{X/Z,0+}(z)$ and $L^{Z,0+}(z)$.

**Proposition 4.6.** The operators $L^{X/Z,0+}(z)$ and $L^{Z,0+}(z)$ are related as follows

$$j^* \circ L^{X/Z,0+}(z) = L^{Z,0+}(z) \circ j^*.$$

The analogous statement in Gromov-Witten theory appears in [24, Proposition 2.4]. The proof in this case is very similar.
Proof. To avoid confusion, we use superscripts to distinguish the evaluation maps \( ev^X_1 : Q_{0,2}^{0+}(X, \beta) \to \bar{I}X \) and \( ev^Z_1 : Q_{0,2}^{0+}(Z, \beta) \to \bar{I}Z \). For \( \alpha \) in \( H^*(\bar{I}X) \), we can express \( j^* \circ L^{X,0+}(z)(\alpha) \) as
\[
(18) \quad j^* \alpha + \sum_{\beta \in \text{Eff}} q^\beta j^*_X \tilde{ev}^X_2 \left( \frac{ev^X_1(\alpha)}{-z - \psi_1} \cup e([\mathbb{R}^0\pi_* u] \ast \mathcal{E}(x_2)) \cap \left[ Q_{0,2}^{0+}(X, \beta) \right]_{\text{vir}} \right)
\]
and \( L^{Z,0+}(z) \circ j^*(\alpha) \) as
\[
(19) \quad j^*(\alpha) + \sum_{\beta \in \text{Eff}} q^\beta \tilde{ev}^Z_2 \left( \frac{ev^Z_1 j^*(\alpha)}{-z - \psi_1} \cap \left[ Q_{0,2}^{0+}(Z, \beta) \right]_{\text{vir}} \right).
\]

To prove the claim we compare (18) and (19) term by term.

Define \( Q \) to be the zero locus of the section \( ev^X_2 s \in \Gamma(Q_{0,2}^{0+}(X, \beta), ev^X_2 E) \) and denote the evaluation maps of \( Q \) by \( ev^Q_1 : Q \to \bar{I}X \) and \( ev^Q_2 : Q \to \bar{I}Z \). Label the following projections \( \pi_1^X, \pi_2^X : \bar{I}X \times \bar{I}X \to \bar{I}X \), \( \pi_1^X \times \bar{I}Z \to \bar{I}X \), and \( \pi_2^X : \bar{I}X \times \bar{I}Z \to \bar{I}Z \). Consider the following diagram, where the rectangles are fiber squares
\[
Q_{0,2}^{0+}(Z, \beta) \xrightarrow{j''} Q \xrightarrow{j'} Q_{0,2}^{0+}(X, \beta) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\bar{I}X \times \bar{I}Z \xrightarrow{id \times j} \bar{I}X \times \bar{I}X \\
\downarrow \pi_2 \quad \downarrow \pi_2^X \\
\bar{I}Z \xrightarrow{j} \bar{I}X,
\]

where \( ev^X = (ev^X_1, ev^X_2) \), \( ev^Q = (ev^Q_1, ev^Q_2) \), and \( ev^Z = (j \circ ev^Z_1, ev^Z_2) \).
Using the diagram above, we rewrite each term in the sum of (18):

\[
\begin{aligned}
&j^*ev_{2*}^X\left(\frac{ev_{1*}^X(\alpha)}{-z - \psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q_{0,2}^{0+}(X, \beta)\right]_{\text{vir}}\right) \\
= &ev_{2*}^Q j^!\left(\frac{ev_{1*}^X(\alpha)}{-z - \psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q_{0,2}^{0+}(X, \beta)\right]_{\text{vir}}\right) \\
= &t_*\pi_{2*} ev_{2*}^Q (id \times j)^!\left(\frac{ev_{1*}X(\pi_1)^{X*}(\alpha)}{-z - \psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q_{0,2}^{0+}(X, \beta)\right]_{\text{vir}}\right) \\
= &\left(t_*\pi_{2*}\left(\pi_1^*(\alpha) \cup ev_{2*}^Q j^!\left(\frac{e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q_{0,2}^{0+}(X, \beta)\right]_{\text{vir}}}{-z - \psi_1}\right)\right)\right).
\end{aligned}
\]

Equality two is by [18, Theorem 6.2(c)] and (20). Equality four is the projection formula.

Let 0_X denote the zero section of $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ over $Q_{0,2}^{0+}(X, \beta)$. We rewrite each term in (19):

\[
\begin{aligned}
&ev_{2*}^Z\left(\frac{ev_{1*}^Z j^*(\alpha)}{-z - \psi_1} \cap \left[Q_{0,2}^{0+}(Z, \beta)\right]_{\text{vir}}\right) \\
= &t_*\pi_{2*} ev_{2*}^Z\left(\frac{ev_{1*}Z(\pi_1)^Z(\alpha)}{-z - \psi_1} \cap \left[Q_{0,2}^{0+}(Z, \beta)\right]_{\text{vir}}\right) \\
= &t_*\pi_{2*}\left(\pi_1^*(\alpha) \cup ev_{2*}^Z\left(\frac{Q_{0,2}^{0+}(Z, \beta)}{-z - \psi_1}\right)\right) \\
= &t_*\pi_{2*}\left(\pi_1^*(\alpha) \cup ev_{2*}^Q j^!0_X\left(\frac{Q_{0,2}^{0+}(X, \beta)}{-z - \psi_1}\right)\right).
\end{aligned}
\]

The third equality follows by factoring $ev_{2*}^Z$ as $ev_{2*}^Q \circ j''$, Theorem 4.3, and [18, Proposition 6.3].

Compare (21) and (22). We may factor $\psi$-classes out of Gysin maps by [18, Proposition 6.3]. Hence, to complete the proof it suffices to show the
following equality in the Chow group of $Q$:
\[
(23) \quad j''_* 0_X^! \left( \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right) = j^! \left( e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(x_2)) \cap \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right).
\]

Consider the fiber diagram (also constructed in [24, §2]):
\[
(24) \quad \begin{array}{ccc}
Q_{0,2}^+(Z, \beta) & \xrightarrow{j''} & Q \\
\downarrow^j & & \downarrow^\tilde{s}_Q \\
Q & \xrightarrow{0_Q} & \mathbb{R}^0\pi_*[u]^*\mathcal{E}(x_2)|_Q \\
\downarrow^f & & \downarrow^\tilde{s} \\
Q_{0,2}^+(X, \beta) & \xrightarrow{0^\prime_X} & \mathbb{R}^0\pi_*[u]^*\mathcal{E}(x_2) \\
& \xrightarrow{h} & \mathbb{R}^0\pi_*[u]^*\mathcal{E}.
\end{array}
\]

The morphisms $0_Q$ and $0^\prime_X$ are the zero sections, $f$ and $h$ are the natural inclusions, and $\tilde{s}_Q$ is the section induced by $s$. We then have the following:
\[
\begin{align*}
j''_* 0_X^! \left( \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right) &= j''_* 0_Q^! h^! \left( \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right) \\
&= 0^\prime_* \tilde{s}_Q^* h^! \left( \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right) \\
&= h^! \left( e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(x_2)) \cap \left[ Q_{0,2}^+(X, \beta) \right]^{\vir} \right).
\end{align*}
\]

The first equality is [18, Theorem 6.5]. The second is [18, Theorem 6.2(a)].

To verify (23) it now suffices to show that $h^!$ equals $j^!$. This can be seen from the following commutative diagram, where the front, back, top, and bottom faces are fiber squares:
\[
\begin{array}{ccc}
\mathbb{R}^0\pi_*[u]^*\mathcal{E}(x_2) & \xrightarrow{h} & \mathbb{R}^0\pi_*[u]^*\mathcal{E} \\
\downarrow^j & & \downarrow^\tilde{s} \\
IZ & \xrightarrow{j} & IX \\
\downarrow^0_E & & \downarrow^s \\
IX & \xrightarrow{0^\prime_X} & IE.
\end{array}
\]

The morphism $\tilde{s}|_{x_2}$ is defined by the composition
\[
\mathbb{R}^0\pi_*[u]^*\mathcal{E} \rightarrow ev_v^X \mathcal{E} \rightarrow IE.
\]
Applying [18, Theorem 6.2(c)] twice completes the proof,

\begin{align*}
h^! \left( e(R^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[ Q_{0,2}^+(X, \beta) \right]^{vir} \right) \\
= 0^!_E \left( e(R^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[ Q_{0,2}^+(X, \beta) \right]^{vir} \right) \\
= j^! \left( e(R^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[ Q_{0,2}^+(X, \beta) \right]^{vir} \right).
\end{align*}

□

5. The total space

In this section we consider quasimaps to $E^\vee$. Similar to the previous section, we will compute the quasimap invariants of $E^\vee$ in terms of integrals over the moduli space of $0+$-stable quasimaps to $X$.

**Definition 5.1.** Let $\mathcal{E}^\vee \to \bar{X}$ be a vector bundle. Define the moduli space of $k$-pointed, genus-$g$, $0+$-stable quasimaps to $E^\vee$ of degree $\beta$ as

$$Q_{0+k}^+(E^\vee, \beta) := \text{tot}(\pi_* [u]^* \mathcal{E}^\vee) := \text{Spec Sym} R^1 \pi_* ([u]^* \mathcal{E} \otimes \omega_\pi).$$

There is a natural perfect obstruction theory on $Q_{0+k}^+(E^\vee, \beta)$ obtained by pulling back the obstruction theory on $Q_{0+k}^+(X, \beta)$ and taking the direct sum with $(R^* \pi_* [u]^* \mathcal{E}^\vee)^\vee$. This yields a virtual fundamental class $[Q_{0+k}^+(E^\vee, \beta)]^{vir}$.

When the evaluation maps $ev_i : Q_{0+k}^+(E^\vee, \beta) \to IE^\vee$ are proper, we can define quasimap invariants as in Definition 2.19.

As in the previous section we will assume that $\mathcal{E}$ is weakly convex. Then recall by Theorem 3.16 that $E^\vee \to \bar{X}$ is weakly concave.

On the universal curve over $Q_{0+2}^+(X, \beta)$, consider the divisor given by the first marked point. Tensor the divisor exact sequence by $[u]^* \mathcal{E}^\vee$ and apply $R^0 \pi_*(-)$ to get the long exact sequence

\begin{equation}
0 \longrightarrow R^0 \pi_* [u]^* \mathcal{E}^\vee \longrightarrow R^0 \pi_* [u]^* \mathcal{E}^\vee|_{x_1} \longrightarrow \end{equation}

\begin{equation}
R^1 \pi_* [u]^* \mathcal{E}^\vee(-x_1) \longrightarrow R^1 \pi_* [u]^* \mathcal{E}^\vee \longrightarrow 0.
\end{equation}

Fix an open and closed subspace $ev_1^{-1}(X_g^\vee) \subset Q_{0,2}^+(X, \beta)$. Let $G_{x_1}$ denote the isotropy group at $x_1$ for a point $(C, x_1, x_2, [u])$ in this subspace. Recall that $[u]^* \mathcal{E}$ splits as a direct sum of line bundles $\bigoplus_{i=1}^r L_i$. The line bundle $R^0 \pi_* L_i|_{x_1}$ is nonzero if and only if $G_{x_1}$ acts trivially on the fiber of $L_i^\vee$ at $x_1$. We conclude that on a given open and closed subset $ev_1^{-1}(X_g^\vee) \subset Q_{0,2}^+(X, \beta)$,

\begin{equation}
R^0 \pi_* [u]^* \mathcal{E}^\vee|_{x_1} = ev_* E_g^\vee.
\end{equation}
Combining (25) and (26), we obtain the following equality in $K$-theory:

$$
ev_{g}^{*} E_{g}^{\prime} \oplus \mathbb{R} \pi_{*}[u]^{*} \mathcal{E}^{\prime} = \mathbb{R} \pi_{*}[u]^{*} \mathcal{E}^{\prime}(-x_{1}).$$

**Proposition 5.2.** If $\mathcal{E}^{\prime}$ is weakly concave, then the evaluation maps

$$ev_{i} : Q_{0}^{0+}(E^{\prime}, \beta) \to \bar{IE}^{\prime}$$

are proper.

**Proof.** For notational simplicity we consider $ev_{1}$. By (25) and (26), there is a closed immersion

$$Q_{g,k}^{0+}(E^{\prime}, \beta) = \text{tot}(\pi_{*}[u]^{*} \mathcal{E}^{\prime}) \hookrightarrow \text{tot}(\pi_{*}([u]^{*} \mathcal{E}^{\prime}|_{x_{1}})) = (ev_{1}^{X})^{*} \bar{IE}^{\prime},$$

where $(ev_{1}^{X})^{*} \bar{IE}^{\prime}$ denotes the fiber product of $ev_{1}^{X} : Q_{g,k}^{0+}(X, \beta) \to \bar{I}X$ and $\bar{I}E^{\prime} \to \bar{I}X$. The evaluation map $ev_{1}^{E^{\prime}}$ factors as

$$Q_{g,k}^{0+}(E^{\prime}, \beta) \hookrightarrow (ev_{1}^{X})^{*} \bar{I}E^{\prime} \to \bar{I}E^{\prime}.$$

This composition is proper because $ev_{1}^{X}$ is. \qed

By weak concavity, the sheaf $\mathbb{R} \pi_{*}[u]^{*} \mathcal{E}^{\prime}(-x_{1})$ is a vector bundle on $Q_{0}^{0+}(X, \beta)$. This allows the following definition.

**Definition 5.3.** Define the $\mathcal{E}^{\prime}$-twisted virtual class to be

$$[Q_{0}^{0+}(X, \beta)]_{\mathcal{E}^{\prime},1}^{\text{vir}} := e(\mathbb{R} \pi_{*}[u]^{*} \mathcal{E}^{\prime}(-x_{1})) \cap [Q_{0}^{0+}(X, \beta)]^{\text{vir}}.$$  

Define the operator $L^{X/E^{\prime},0+}(z)$ by

$$L^{X/E^{\prime},0+}(z)(\alpha) := \alpha + \sum_{\beta \in \text{Eff}} q^{\beta} ev_{2*} \left( \frac{e^{X}_{1}(\alpha)}{-z - \psi} \cap \left[ Q_{0}^{0+}(X, \beta) \right]^{\text{vir}}_{X/E^{\prime},1} \right),$$

for $\alpha \in H_{CR}^{*}(X)$.

We will make use of the commutative diagram

$$
\begin{array}{ccc}
Q_{0}^{0+}(X, \beta) & \xrightarrow{ev_{1}^{X}} & Q_{0}^{0+}(E^{\prime}, \beta) \\
\bar{I}X & \xrightarrow{i} & \bar{I}E^{\prime}, \\
\end{array}
$$

where the $i$th evaluation maps on $Q_{0}^{0+}(X, \beta)$ and $Q_{0}^{0+}(E^{\prime}, \beta)$ are denoted by $ev_{1}^{X}$ and $ev_{i}^{E^{\prime}}$ respectively.

**Proposition 5.4.** Given $\gamma_{1}, \gamma_{2} \in H_{CR,ct}^{*}(E^{\prime})$, choose $\alpha_{1}, \alpha_{2} \in H_{CR}^{*}(X)$ such that $\gamma_{i} = i_{*}(\alpha_{i})$. We have the following equality:

$$
\langle \gamma_{1} \psi_{1}^{a_{1}} \gamma_{2} \psi_{2}^{a_{2}} \rangle_{0, \beta}^{E^{\prime},0+} = \int_{[Q_{0}^{0+}(X, \beta)]_{X/E^{\prime},1}^{\text{vir}}} \ev_{1}^{X_{*}}(\alpha_{1}) \psi_{1}^{a_{1}} \cup \ev_{2}^{X_{*}}(\alpha_{2} \cup e(E_{g,k})) \psi_{2}^{a_{2}}.
$$
The claim is most easily seen via virtual localization. Consider the $\mathbb{C}^*$-action on $E^\vee$ given by scaling fibers. This induces actions on $IE^\vee$ and $Q^0_{0,2}(E^\vee, \beta)$, with fixed loci $IX$ and $Q^0_{0,2}(X, \beta)$ respectively. All the maps in Diagram (29) are $\mathbb{C}^*$-equivariant. By (1) of [22], we have the equality

$$\left[Q^0_{0,2}(E^\vee, \beta)\right]_{\mathbb{C}^*, \text{vir}} = \tilde{z}_* \left(\frac{[Q^0_{0,2}(X, \beta)]_{\mathbb{C}^*, \text{vir}}}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*E^\vee)}\right),$$

where $[-]_{\mathbb{C}^*, \text{vir}}$ denotes the $\mathbb{C}^*$-equivariant virtual fundamental class.

Choose equivariant lifts of $\gamma_1$, $\gamma_2$, $\alpha_1$, and $\alpha_2$ (by abuse of notation we will not change their labels). Using the fact that $\gamma_2$ is the pushforward of a class $\alpha_2$ supported on $IX$, a localization argument shows that the left hand side of (30) is equal to the non-equivariant limit of

$$\int_{\left[Q^0_{0,2}(E^\vee, \beta)\right]_{\mathbb{C}^*, \text{vir}}} e_1^{E, \gamma_1}(\gamma_1) \psi^{\alpha_1} \cup e_2^{E, \gamma_2}(\gamma_2) \psi^{\alpha_2}.$$

This can be rewritten as

$$\int_{\left[Q^0_{0,2}(X, \beta)\right]_{\mathbb{C}^*, \text{vir}}} e_1^{X^*, \gamma_1}(\alpha_1 \cup e_{\mathbb{C}^*}(E_g)) \psi^{\alpha_1} \cup e_2^{X^*, \gamma_2}(\alpha_2 \cup e_{\mathbb{C}^*}(E_g)) \psi^{\alpha_2} \frac{e_{\mathbb{C}^*}(ev_1^{X^*E^\vee})}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*E^\vee)}$$

$$= \int_{\left[Q^0_{0,2}(X, \beta)\right]_{\mathbb{C}^*, \text{vir}}} e_1^{X^*, \gamma_1}(\alpha_1) \psi^{\alpha_1} \cup e_2^{X^*, \gamma_2}(\alpha_2) \psi^{\alpha_2} \frac{e_{\mathbb{C}^*}(ev_1^{X^*E^\vee})}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*E^\vee)}$$

$$= \int_{\left[Q^0_{0,2}(X, \beta)\right]_{\mathbb{C}^*, \text{vir}}} e_1^{X^*, \gamma_1}(\alpha_1) \psi^{\alpha_1} \cup e_2^{X^*, \gamma_2}(\alpha_2) \psi^{\alpha_2} \frac{e_{\mathbb{C}^*}(ev_1^{X^*E^\vee})}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*E^\vee)}$$

The first line is a result of (31), the projection formula, and (29). The second is immediate. The third line follows from (27). Taking the non-equivariant limit yields the desired equality. \hfill $\Box$

A similar argument yields the following.

**Proposition 5.5.** The operators $L^{X/E^\vee, 0+}(z)$ and $L^{E^\vee, 0+}(z)$ are related by pushforward along $i$,

$$i_* \circ L^{X/E^\vee, 0+}(z) = L^{E^\vee, 0+}(z) \circ i_*.$$

**Proof.** For $\alpha \in H^*_{\mathbb{C}R}(X)$, we can rewrite the left-hand-side of (32) applied to $\alpha$ as

$$i_* \circ L^{X/E^\vee, 0+}(z)(\alpha) = i_*(\alpha) + \sum_{\beta \in \text{Eff}} q^\beta i_* \psi^{E^\vee}_{2*} \left(\frac{ev_1^{X^*}(\alpha)}{-z - \psi} \cap \left[Q^0_{0,2}(X, \beta)\right]_{\mathbb{C}^*, \text{vir}}\right)_{X/E^\vee, 1}$$

and the right-hand-side as

$$L^{E^\vee, 0+}(z) \circ i_*(\alpha) = i_*(\alpha) + \sum_{\beta \in \text{Eff}} q^\beta \psi^{E^\vee}_{2*} \left(\frac{ev_1^{X^*}(\alpha)}{-z - \psi} \cap \left[Q^0_{0,2}(E^\vee, \beta)\right]_{\mathbb{C}^*, \text{vir}}\right).$$
We equate (33) and (34) term by term. Again we apply via virtual localization. Fix a degree $\beta$ and choose a class $\alpha$ in $H^*_C(X)$. Consider the following sequence of equalities in the localized equivariant cohomology ring of $\bar{I}E^\vee$:

$$
e^\nu_{2*} \left( \frac{ev_1^* \nu_i^* (\alpha)}{z - \psi} \cap \left[ Q_{0,2}^{0+} (E^\vee, \beta) \right]^{\mathbb{C}*} \right)
$$

$$
= e^\nu_{2*} \left( \frac{ev_1^* \nu_i^* (\alpha)}{z - \psi} \cap \left[ Q_{0,2}^{0+} (X, \beta) \right]^{\mathbb{C}*} \right)
$$

$$
= i_* \nu_2^* \left( \frac{ev_1^* (\alpha \cup e_C^* (E^\vee))}{z - \psi} \cap \left[ Q_{0,2}^{0+} (X, \beta) \right]^{\mathbb{C}*} \right)
$$

$$
= i_* \nu_2^* \left( \frac{ev_1^* (\alpha \cup e_C^* (E^\vee))}{z - \psi} \cup \frac{e_C^* (ev_1^* E^\vee)}{e_C^* (R \pi_* [u]^* E^\vee)} \cap \left[ Q_{0,2}^{0+} (X, \beta) \right]^{\mathbb{C}*} \right)
$$

The first equality is (31). The second line is obtained by two applications of the projection formula. The third line is immediate. The forth follows from (27).

Taking the non-equivariant limit of the first and last terms in the chain of equalities and recalling Definition 5.3 completes the proof. □

6. QUASIMAP QUANTUM SERRE DUALITY

In this section we prove a quantum Serre duality statement in three contexts. Specifically, we compare the twisted virtual classes

$$
\left[ Q_{0,2}^{0+} (X, \beta) \right]^{\mathbb{C}*} \quad \text{and} \quad \left[ Q_{0,2}^{0+} (X, \beta) \right]^{\mathbb{C}*}
$$

defined in Sections 4 and 5 to get a cycle-valued statement. From this we compute a direct relationship between the two-pointed genus-zero quasimap invariants of $Z$ and $E^\vee$. We then rephrase this as a comparison between the operators $L^Z,0^+ (z)$ and $L^{E^\vee,0^+} (z)$.

6.1. Cycle-valued statement. Let $E \rightarrow \mathcal{X}$ be a weakly convex vector bundle. For elements $g_1, g_2 \in S$, denote by $Q_{0,2}^{0+} (X, \beta)$ the open and closed subset of $Q_{0,2}^{0+} (X, \beta)$ defined by the conditions $\text{im}(ev_1) \subset X_{g_1}$ and $\text{im}(ev_2) \subset X_{g_2}$.

**Lemma 6.1.** On $Q_{0,2}^{0+} (X, \beta)$ the vector bundle $R^1 \pi_* [u]^* E^\vee (-x_1)$ has rank $\beta(\det E) - \text{age}_{g_1} (E) + \text{age}_{g_2} (E^\vee)$, where $\det E = \bigwedge_{i=1}^r E$ is the determinant line bundle.
Proof. Fix a quasimap \((C, x_1, x_2, [u]) \in Q^{0^+}_{0, g_1, g_2}(X, \beta)\) and let \(r_j\) be the order of the isotropy group at \(x_j\) for \(j = 1, 2\). Recall from §3.2 that there exists line bundles \(\{L_i \to C\}_{i=1}^r\) such that \([u]^*E = \oplus_{i=1}^r L_i\). Write \(R_1^*\pi^*[u]^*\mathcal{E}^\vee(-x_1)\) as the direct sum \(\oplus_{i=1}^r R_1^*\pi^*L_i^\vee(-x_1)\). Using orbifold Riemann-Roch we compute:

\[
\text{rank} (R_1^*\pi^*L_i^\vee(-x_1)) = h^1 (C, L_i^\vee(-x_1)) \\
= h^0 (C, L_i(-x_2)) \\
= \deg(L_i(-x_2)) + 1 - \text{age}_{x_1}(L_i) - \text{age}_{x_2}(L_i(-x_2)) \\
= \deg(L_i) - \frac{1}{r_2} + 1 - \text{age}_{x_1}(L_i) - \text{age}_{x_2}(L_i(-x_2)),
\]

where \(\text{age}_{x_j}(\cdot)\) denotes the age with respect to cyclic generator which acts on a local chart by multiplication by \(e^{2\pi i \frac{1}{r_j}}\). The second equality is Serre duality and Lemma 3.14. The third equality follows from assuming \(E\) is weakly convex.

We claim that

\[
-\frac{1}{r_2} + 1 - \text{age}_{x_2}(L_i(-x_2)) = \text{age}_{x_2}(L_i^\vee).
\]

To see this, consider three cases.

First, assume \(G_{x_2}\) acts nontrivially on the fiber \(L_i(-x_2)|_{x_2}\). Then we observe that

\[
-\frac{1}{r_2} + 1 - \text{age}_{x_2}(L_i(-x_2)) = -\frac{1}{r_2} + \text{age}_{x_2}(L_i^\vee(x_2)) \\
= \text{age}_{x_2}(L_i^\vee),
\]

where the second equality follows because the age of \(\mathcal{O}_C(x_2)\) at \(x_2\) is \(1/r_2\).

For the second case, assume \(G_{x_2}\) is nontrivial but acts trivially on the fiber \(L_i(-x_2)|_{x_2}\). Then the age of \(L_i\) and \(L_i^\vee\) at \(x_2\) is \(1/r_2\) and \((r_2-1)/r_2\) respectively. We have the following:

\[
-\frac{1}{r_2} + 1 - \text{age}_{x_2}(L_i(-x_2)) = \frac{r_2 - 1}{r_2} \\
= \text{age}_{x_2}(L_i^\vee).
\]

Finally, consider the case \(g_2\) equals the identity. Then \(G_{x_2}\) is trivial, hence \(r_2\) equals 1. We obtain

\[
-\frac{1}{r_2} + 1 - \text{age}_{x_2}(L_i(-x_2)) = 0 = \text{age}_{x_2}(L_i^\vee).
\]

This proves the claim.

Note that

\[
\sum_{i=1}^r \deg(L_i) = \deg([u]^*(\det \mathcal{E})) = \beta(\det \mathcal{E})
\]
and
\[ \sum_{i=1}^{r} \text{age}_{x_j}(L_i) = \text{age}_{x_j}([u]^*\mathcal{E}) = \text{age}_{g_j}(\mathcal{E}). \]

Summing (35) and (36) over \( i \) then completes the proof. \( \square \)

We conclude this section with the following equality of twisted virtual classes. This may be interpreted as a cycle-valued formulation of quantum Serre duality.

**Theorem 6.2.** [Cycle-valued quantum Serre duality.] Let \( \mathcal{E} \) be weakly convex. On the connected component \( Q^0_{g_1,g_2}(X,\beta) \), the virtual classes
\[ [Q^0_{g_1,g_2}(X,\beta)]_{X/Z,2}^\text{vir} \]
and
\[ [Q^0_{g_1,g_2}(X,\beta)]_{X/E^\vee,1}^\text{vir} \]
are equal up to sign,
\[ [Q^0_{g_1,g_2}(X,\beta)]_{X/Z,2}^\text{vir} = (1)^{-\beta(\det \mathcal{E}) - \text{age}_{g_1}(\mathcal{E}) + \text{age}_{g_2}(\mathcal{E}^\vee)} [Q^0_{g_1,g_2}(X,\beta)]_{X/E^\vee,1}^\text{vir}. \]

**Proof.**
We defined the twisted virtual classes in 4.5 and 5.3 as
\[ [Q^0_{g_1,g_2}(X,\beta)]_{X/Z,2}^\text{vir} := e(\mathbb{R}^0 \pi_* [u]^*\mathcal{E}(-x_2)) \cap [Q^0_{g_1,g_2}(X,\beta)]_{X/Z,2}^\text{vir}, \]
\[ [Q^0_{g_1,g_2}(X,\beta)]_{X/E^\vee,1}^\text{vir} := e(\mathbb{R}^1 \pi_* [u]^*\mathcal{E}^\vee(-x_1)) \cap [Q^0_{g_1,g_2}(X,\beta)]_{X/E^\vee,1}^\text{vir}. \]

Via Proposition 3.15,
\[ \mathcal{O}_C = \omega_\pi(x_1 + x_2). \]

Tensoring (37) by \([u]^*\mathcal{E}(-x_2)\), pushing forward via \( \pi_* \), and applying Serre duality yields:
\[ \mathbb{R}^0 \pi_* [u]^*\mathcal{E}(-x_2) = (\mathbb{R}^1 \pi_* [u]^*\mathcal{E}^\vee(-x_1))^\vee. \]

This implies the Euler class identity
\[ e(\mathbb{R}^0 \pi_* [u]^*\mathcal{E}(-x_2)) = e((\mathbb{R}^1 \pi_* [u]^*\mathcal{E}^\vee(-x_1))^\vee) = (-1)^\beta(\det \mathcal{E}) - \text{age}_{g_1}(\mathcal{E}) + \text{age}_{g_2}(\mathcal{E}^\vee) e((\mathbb{R}^1 \pi_* [u]^*\mathcal{E}^\vee(-x_1))^\vee), \]
where the second equality follows from Lemma 6.1. \( \square \)

### 6.2. Quantum Serre duality for quasimap invariants.

In this section we use the comparison of twisted virtual cycles of Theorem 6.2 to compare quasimap invariants for \( Z \) and \( E^\vee \) when \( \mathcal{E} \) is weakly convex. We obtain a simple relationship between the generating functions \( L^{Z,0+}(z) \) and \( L^{E^\vee,0+}(z) \).

For the remainder of the paper we assume:
- Assumption 2.11;
- the GIT stack quotient \( X \) is proper;
- the vector bundle \( \mathcal{E} \) is weakly convex.

In particular, the third case holds whenever \( \mathcal{E} \) is weakly semi-positive.
Definition 6.3. [35, Definition 6.9] Given a class $\gamma \in H^*_{\text{CR,ct}}(E^\vee)$, denote by $\overline{\gamma}$ a lift of $\gamma$ to the compactly supported cohomology. Define the linear map $\Delta : H^*_{\text{CR,ct}}(E^\vee) \rightarrow H^*_{\text{CR,amb}}(Z)$ by
\[
\Delta(\gamma) := j^* \circ p^\text{cs}_*(\overline{\gamma}),
\]
where $p^\text{cs}_* : H^*_{\text{CR,cs}}(E^\vee) \rightarrow H^*_{\text{CR}}(X)$ is the pushforward of compactly supported cohomology.

Lemma 6.4. [35, Lemma 6.10] Assuming 2.11, the transformation
\[
\Delta : H^*_{\text{CR,ct}}(E^\vee) \rightarrow H^*_{\text{CR,amb}}(Z)
\]
is well defined.

Proof. We work over a given twisted sector $X_g$ for $g \in S$. Let $\gamma$ be an element of $H^*_{\text{ct}}(E^\vee_g)$. A lift $\overline{\gamma} \subset H^*(E^\vee_g)$ is defined up to an element of the kernel of $\phi_g : H^*_{\text{cs}}(E^\vee_g) \rightarrow H^*(E^\vee_g)$. To show that $\Delta$ is well defined, we must show $p^\text{cs}_g(\ker(\phi_g)) \subseteq \ker(j^*_g)$. Let $\kappa$ be an element of $\ker(\phi_g)$. The pushforward $i^\text{cs}_g : H^*(X_g) \rightarrow H^*_{\text{cs}}(E^\vee_g)$ is an isomorphism. By the compactness of $X$, there exists $\alpha \in H^*(X_g)$ such that $i^\text{cs}_g(\alpha) = \kappa$. Note that $p^\text{cs}_g(\kappa) = p^\text{cs}_g \circ i^\text{cs}_g(\alpha) = \alpha$. We see that
\[
0 = \phi_g(\kappa) = \phi_g \circ i^\text{cs}_g(\alpha) = i^\text{cs}_g(\alpha) = p^*(e(E^\vee_g) \cup \alpha),
\]
where the last equality is (3). Assumption 2.11 implies $j^*_g(\alpha) = 0$ if and only if $j^*_g \circ j^*_g(\alpha) = 0$. The projection formula implies $j^*_g \circ j^*_g = p^*(e(E^\vee_g) \cup \alpha)$. Up to a sign this is equal to
\[
e(\gamma) := i^\text{cs}_g(p^*(e(E^\vee_g) \cup \alpha)) = 0.
\]
Therefore $p^\text{cs}_g(\kappa) = \alpha$ lies in $\ker(j^*_g)$. \hfill $\square$

Recall Lemma 2.14, which states that any element $\gamma$ in $H^*_{\text{ct}}(E^\vee_g)$ may be written as $p^*_g(e(E^\vee_g) \cup \alpha)$ for some $\alpha$ in $H^*(X_g)$.

Lemma 6.5. [35, Lemma 6.11] Given $\gamma \in H^*_{\text{ct}}(E^\vee_g)$, choose $\alpha \in H^*(X_g)$ such that $\gamma = p^*_g(e(E^\vee_g) \cup \alpha)$, then
\[
\Delta(\gamma) = j^*_g(\alpha).
\]
Furthermore, the transformation $\Delta : H^*_{\text{CR,ct}}(E^\vee) \rightarrow H^*_{\text{CR,amb}}(Z)$ is an isomorphism.

Proof. If $g$ fixes only the origin of $W \times \mathbb{C}^r$, then $E^\vee_g$ equals $X_g$ and the result is immediate. When that is not the case, the proof is similar to that in Lemma 6.11 of [35].

By Equation (3), we have
\[
\gamma = p^*_g(e(E^\vee_g) \cup \alpha) = i^\text{cs}_g(\alpha).
\]
Noting that $i_{gs}$ factors as $\phi \circ \bar{i}_{gs}^c$ gives the following:

$$\Delta(\gamma) = \Delta \circ \phi \circ \bar{i}_{gs}^c(\alpha)$$
$$= j_g^c \circ p_{gs}^c \circ \bar{i}_{gs}^c(\alpha)$$
$$= j_g^c(\alpha).$$

To prove the second claim, we observe

$$H^*_{ct}(E^\vee_g) = p_g^* \left( \text{im}(E^\vee_g) \cup - \right)$$
$$= \text{im}(E_g) \cup -$$
$$\cong j_g^c \left( \text{im}(j_g^c) \right)$$
$$\cong \text{im}(j_g^c)$$
$$= H_{amb}^*(Z_g).$$

The first equality follows from Lemma 2.14, the isomorphism in line four is from the projection formula, and the fifth isomorphism is from Assumption 2.11.

It will be useful to consider a modification of $\Delta$.

**Definition 6.6.** Define the linear transformation $\tilde{\Delta}$ by

$$\tilde{\Delta}|_{H^*_{ct}(E^\vee_g)} := e^{\pi i \text{age}_g(E)} \Delta|_{H^*_{ct}(E^\vee_g)}.$$

**Lemma 6.7.** The transformation $\tilde{\Delta}$ identifies the Chen-Ruan Poincaré pairings up to a sign:

$$\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2) \rangle^Z = (-1)^{\text{rank}(E)} \langle \gamma_1, \gamma_2 \rangle^{E^\vee, ct},$$

for any $\gamma_1$ and $\gamma_2$ in $H^*_{ct}(E^\vee_g)$.

**Proof.** Assume $\gamma_1$ is supported on $E^\vee_{g_1^c}$ and $\gamma_2$ is supported on $E^\vee_{g_2^c}$. The pairings $\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2) \rangle^Z$ and $\langle \gamma_1, \gamma_2 \rangle^{E^\vee, ct}$ equal zero unless $\iota(E^\vee_{g_1}) = E^\vee_{g_2^c}$.

It therefore suffices to consider the case $\gamma_1 \in H^*_{ct}(E^\vee_g)$ and $\gamma_2 \in H^*_{ct}(E^\vee_{g_1^c})$. Choose $\alpha_1, \alpha_2 \in H^*_{CR}(X)$ such that $\gamma_1 = i_g^c(\alpha_1)$ and $\gamma_2 = i_{g_1^c}(\alpha_2)$. Then, applying the projection formula and Lemma 6.5, we obtain

$$\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2) \rangle^Z = (-1)^{\text{age}_g(E)+\text{age}_{g_1^c}(E)} \int_{IZ} j_g^c(\alpha_1) \cup \iota^*(j_{g_1^c}^c(\alpha_2))$$
$$= (-1)^{\text{rank}(E)-\text{rank}(E_g)} \int_{IX} \alpha_1 \cup \iota^*(\alpha_2 \cup e(E_g))$$
$$= (-1)^{\text{rank}(E)} \int_{IX} \alpha_1 \cup \iota^*(i_{g_1^c}^c(i_{g_1^c}^c(\alpha_2)))$$
$$= (-1)^{\text{rank}(E)} \int_{IE^\vee} i_{gs}^c(\alpha_1) \cup \iota^*(i_{g_1^c}(\alpha_2))$$
$$= (-1)^{\text{rank}(E)} \langle \gamma_1, \gamma_2 \rangle^{E^\vee, ct}.$$
there is a short exact sequence
\[ \text{age}_g(E) + \text{age}_g(E^\vee) = \text{rank}(E) - \text{rank}(E_g). \]

**Theorem 6.8.** Given elements \( \gamma_1, \gamma_2 \in H^*_{CR,c}(E^\vee) \), we have the equality
\[
\left\langle \hat{\Delta} (\gamma_1) \psi_1^{a_1}, \hat{\Delta} (\gamma_2) \psi_2^{a_2} \right\rangle_{0,\beta}^{Z,0+} = e^{\pi i (\beta(\det E) + \text{rank}(E))} \left\langle \gamma_1 \psi_1^{a_1}, \gamma_2 \psi_2^{a_2} \right\rangle_{0,\beta}^{E^\vee,0+}.
\]

**Proof.** Let \( g_1 \) and \( g_2 \) be elements in \( S \). Without loss of generality, choose \( \gamma_1 \in H^*_c(E^\vee_{g_1}), \gamma_2 \in H^*_c(E^\vee_{g_2}) \). Choose \( \alpha_1 \in H^*(X_{g_1}) \), and \( \alpha_2 \in H^*(X_{g_2}) \) such that
\[
\gamma_1 = p^*_{g_1}(\alpha_1 \cup e(E^\vee_{g_1})) = \iota^*_{g_1}(\alpha_1) \\
\gamma_2 = p^*_{g_2}(\alpha_2 \cup e(E^\vee_{g_2})) = \iota^*_{g_2}(\alpha_2).
\]

By Proposition 3.11, our assumption that \( E \) is weakly convex, and (26) there is a short exact sequence
\[
0 \longrightarrow \mathbb{R}^0 \pi_*[u]^* E(-x_2) \longrightarrow \mathbb{R}^0 \pi_*[u]^* E \longrightarrow \mathbb{R}^0 \pi_*[u]^* E \longrightarrow 0.
\]

Also note that the following diagram commutes:
\[
\begin{array}{ccc}
Q^0_{0,2} (Z, \beta) & \xrightarrow{j} & Q^0_{0,2} (X, \beta) \\
\downarrow e_{v_1}^Z & & \downarrow e_{v_1}^X \\
IZ & \xrightarrow{j} & IX.
\end{array}
\]

Expanding the integral in the left-hand-side of the statement gives us
\[
\left\langle \hat{\Delta} (\gamma_1) \psi_1^{a_1}, \hat{\Delta} (\gamma_2) \psi_2^{a_2} \right\rangle_{0,\beta}^{Z,0+}
= e^{\pi i (\text{age}_{g_1}(E) + \text{age}_{g_2}(E))} \int_{[Q^0_{0,2}(Z, \beta)]_{\text{vir}}} e^{v_1^Z} j^*_g (\alpha_1) \psi_1^{a_1} \cup e^{v_2^Z} j^*_g (\alpha_2) \psi_2^{a_2}
= e^{\pi i (\text{age}_{g_1}(E) + \text{age}_{g_2}(E))} \int_{[Q^0_{0,2}(X, \beta)]_{\text{vir}}} e^{v_1^X} (\alpha_1) \psi_1^{a_1} \cup e^{v_2^X} (\alpha_2) \psi_2^{a_2} \cup e^{(\mathbb{R}^0 \pi_*[u]^* E)}
= e^{\pi i (\text{age}_{g_1}(E) + \text{age}_{g_2}(E))} \int_{[Q^0_{0,2}(X, \beta)]_{\text{vir}}} e^{v_1^X} (\alpha_1) \psi_1^{a_1} \cup e^{v_2^X} (\alpha_2 \cup e(E_{g_2})) \psi_2^{a_2} \cup e^{(\mathbb{R}^0 \pi_*[u]^* E(-x_2))}
= e^{\pi i (\text{age}_{g_1}(E) + \text{age}_{g_2}(E))} \text{rank}(E_{g_2}) \int_{[Q^0_{0,2}(X, \beta)]_{\text{vir}}} e^{v_1^X} (\alpha_1) \psi_1^{a_1} \cup e^{v_2^X} (\alpha_2 \cup e(E_{g_2}^\vee)) \psi_2^{a_2}
= e^{\pi i (\beta(\det E) + \text{rank}(E))} \int_{[Q^0_{0,2}(X, \beta)]_{\text{vir}}} e^{v_1^X} (\alpha_1) \psi_1^{a_1} \cup e^{v_2^X} (\alpha_2 \cup e(E_{g_2}^\vee)) \psi_2^{a_2}
= e^{\pi i (\beta(\det E) + \text{rank}(E))} \left\langle \gamma_1 \psi_1^{a_1}, \gamma_2 \psi_2^{a_2} \right\rangle_{0,\beta}^{E^\vee,0+}.
\]
The second equality follows from (40), the projection formula, and Proposition 4.4. The third equality is by (39). The fifth equality is by (38) and Theorem 6.2. The final equality is by Proposition 5.4. □

**Theorem 6.9.** The transformation \( \tilde{\Delta} \) identifies the operators \( L^{Z,0+}(z) \) and \( L^{E^\vee,0+}(z) \) up to a change of variables in the Novikov parameter:

\[
L^{Z,0+}(z) \circ \tilde{\Delta} = \tilde{\Delta} \circ L^{E^\vee,0+}(z)|_{q^\beta \to \gamma \pi_i(\det E) q^\beta}.
\]

**Proof.** By Assumption 2.11, the Chen-Ruan Poincaré pairing on the ambient cohomology of \( Z \) is non-degenerate and by Lemma 6.5, \( \Delta \) is an isomorphism. It therefore suffices to show that

\[
\langle \tilde{\Delta}(\gamma_2), L^{Z,0+}(z) \circ \tilde{\Delta}(\gamma_1) \rangle_Z = \langle \tilde{\Delta}(\gamma_2), \tilde{\Delta} \circ L^{E^\vee,0+}(z)(\gamma_1)|_{q^\beta \to \gamma \pi_i(\det E) q^\beta} \rangle_Z,
\]

for any \( \gamma_1 \) and \( \gamma_2 \) in \( H^*_{CR,ct}(E^\vee) \).

Assume \( \gamma_1 \) is in \( H^*_c(E^\vee_{g_1}) \) and \( \gamma_2 \) is in \( H^*_c(E^\vee_{g_2}) \). Expand the left-hand-side of (42) to obtain

\[
\langle \tilde{\Delta}(\gamma_2), \tilde{\Delta}(\gamma_1) \rangle^Z + \sum_{\beta \in \text{Eff}} q^\beta \left( \langle \tilde{\Delta}(\gamma_2), \tilde{\Delta} \circ \tilde{v}_{2*} \left( \frac{e\nu^*_{1}(\gamma_1)}{-z - \psi_1} \cap \left[ Q^{0+}_{0,2}(Z, \beta) \right]_{\text{vir}} \right) \rangle^Z \right)
\]

The right-hand-side of (42) expands as

\[
\langle \tilde{\Delta}(\gamma_2), \tilde{\Delta}(\gamma_1) \rangle^Z + \sum_{\beta \in \text{Eff}} e^{\pi i(\det E)} q^\beta \left( \langle \tilde{\Delta}(\gamma_2), \tilde{\Delta} \circ \tilde{v}_{2*} \left( \frac{e\nu^*_{1}(\gamma_1)}{-z - \psi_1} \cap \left[ Q^{0+}_{0,2}(E^\vee, \beta) \right]_{\text{vir}} \right) \rangle^Z \right).
\]

For \( \beta \in \text{Eff}(W, G, \theta) \), we have

\[
\langle \tilde{\Delta}(\gamma_2), \tilde{v}_{2*} \left( \frac{e\nu^*_{1}(\gamma_1)}{-z - \psi_1} \cap \left[ Q^{0+}_{0,2}(Z, \beta) \right]_{\text{vir}} \right) \rangle^Z
\]

\[
= e^{\pi i(\beta(\det E) + \text{rank}(E))} \langle \frac{\gamma_1}{-z - \psi_1}, \gamma_2 \rangle^Z_{0,\beta} \]

\[
= e^{\pi i(\beta(\det E) + \text{rank}(E))} \langle \gamma_2, \tilde{v}_{2*} \left( \frac{e\nu^*_{1}(\gamma_1)}{-z - \psi_1} \cap \left[ Q^{0+}_{0,2}(E^\vee, \beta) \right]_{\text{vir}} \right) \rangle^Z_{E^\vee,ct}
\]

\[
= e^{\pi i(\beta(\det E))} \langle \tilde{\Delta}(\gamma_2), \tilde{v}_{2*} \left( \frac{e\nu^*_{1}(\gamma_1)}{-z - \psi_1} \cap \left[ Q^{0+}_{0,2}(E^\vee, \beta) \right]_{\text{vir}} \right) \rangle^Z.
\]
The first equality is the projection formula. The second is Theorem 6.8. The third is another application of the projection formula. The fourth is Lemma 6.7.

Each coefficient of $q^{\beta}$ in (43) and (44) is equal, completing the proof. □

6.3. Quantum Serre duality for Gromov-Witten invariants. In this section we combine the above results with the wall-crossing formulas proven by Zhou in [38] to prove a quantum Serre duality statement for Gromov-Witten invariants without assuming convexity.

Theorem 1.12.2 of [38] states

\begin{equation}
J_{\alpha}^{X, \infty}(t + \mu^{X, \geq 0+}(q, -z), q, z) = J_{\alpha}^{X, 0+}(t, q, z),
\end{equation}

where $J_{\alpha}^{X, \epsilon}(t, q, z)$ is defined in [38, §1.12] and $\mu^{X, \geq 0+}(q, -z)$ is defined in [38, §1.11]. Similar to the $J_{\alpha}^{X, \epsilon}$-function defined in [38], we may generalize the operator $L_{\alpha}^{X, \epsilon}(t, z)$ of (9) by including $\psi$-classes in the $t$ insertions of (6) and (9). Let $f$ be a cohomology-valued polynomial. We redefine the double-bracket as

\[ \langle \langle \alpha_1 \psi_1^{a_1}, \ldots, \alpha_k \psi_k^{a_k} \rangle \rangle_{X, \epsilon}^{J}(f(\psi)) \]

for classes $\alpha_1, \ldots, \alpha_k \in H^*_{CR}(X)$ and non-negative integers $a_1, \ldots, a_k$. Then define $L_{\alpha_1, \ldots, \alpha_k}^{X, \epsilon}(f(\psi), z)$ as in (9) by replacing $t$ with $f(\psi)$. We now use (45) to relate $L_{\alpha}^{X, 0+}(z)$ and $L_{\alpha}^{X, \infty}(\mu^{X, \geq 0+}(q, -\psi), z)$.

**Lemma 6.10.** The operators $L_{\alpha}^{X, 0+}(z)$ and $L_{\alpha}^{X, \infty}(\mu^{X, \geq 0+}(q, -\psi), z)$ are equal.

**Proof.** Let $\{T_i\}_{i \in I}$ be a basis of $H^*_{CR}(X)$ with dual basis $\{T^i\}_{i \in I}$. For $p \in I$, observe that

\begin{equation}
z \frac{\partial}{\partial t_p} J_{\alpha}^{X, 0+}(t, q, z)|_{t=0} = T_p + \sum_{\beta \in \text{Eff}} \sum_{i \in I} q^\beta \left\langle T_p, \frac{T_i}{z - \psi} \right\rangle_{X, 0+}^{X, 0+} T_i.
\end{equation}

Pairing (46) with $\alpha \in H^*_{CR}(X)$, we obtain

\begin{equation}
\left\langle \left( \frac{\partial}{\partial t_p} J_{\alpha}^{X, 0+}(t, q, z) \right)|_{t=0}, \alpha \right\rangle^X = \left\langle T_p, \alpha \right\rangle + \sum_{\beta \in \text{Eff}} q^\beta \left\langle T_p, \frac{\alpha}{z - \psi} \right\rangle_{0, \beta}^{X, 0+} = \left\langle T_p, L_{\alpha}^{X, 0+}(-z)(\alpha) \right\rangle^X.
\end{equation}

Similarly, note that

\begin{equation}
z \frac{\partial}{\partial t_p} J_{\alpha}^{X, \infty}(t + \mu^{X, \geq 0+}(q, -z), q, z)|_{t=0}
\end{equation}

\[ = T_p + \sum_{i \in I} \sum_{\beta \in \text{Eff}} \sum_{m \geq 0} q^\beta \left\langle T_p, \frac{T_i}{z - \psi}, \mu^{X, \geq 0+}(q, -\psi), \ldots, \mu^{X, \geq 0+}(q, -\psi) \right\rangle_{0, m+2, \beta}^{X, \infty} T_i.\]
and

\[
\left\langle \frac{\partial}{\partial t_p} J_{X,\infty}(t + \mu^{X, \geq 0+}(q, -\psi), q, z)\big|_{t=0, \alpha}\right\rangle^X \\
= \langle T_p, \alpha \rangle + \sum_{\beta \in \text{Eff}} \sum_{m \geq 0} \frac{q^\beta}{m!} \left\langle T_p, \frac{\alpha}{z - \psi}, \mu^{X, \geq 0+}(q, -\psi), \ldots, \mu^{X, \geq 0+}(q, -\psi)\right\rangle_{0, m+2, \beta}^{X, \infty}
= \langle T_p, L^{X, \infty}(\mu^{X, \geq 0+}(q, -\psi), -z)(\alpha)\rangle^X.
\]

By (45), equations (47) and (49) are equal for all \( p \in I \). By the nondegeneracy of the Poincaré pairing on \( X \), it follows that

\[ L^{X, 0+}(z)(\alpha) = L^{X, \infty}(\mu^{X, \geq 0+}(q, -\psi), z)(\alpha). \]

Proposition 4.6 and Lemma 6.10 imply the following relationship between two-pointed quasimap invariants of \( X \) and Gromov-Witten invariants of \( Z \).

**Corollary 6.11.** The operators \( L^{X/Z, 0+}(z) \) and \( L^{Z, \infty}(t, z) \) are related by

\[ j^* \circ L^{X/Z, 0+}(z) = L^{Z, \infty}(\mu^{Z, \geq 0+}(q, -\psi), z) \circ j^*. \]

Assume that the total space of the vector bundle \( E^\vee \to X \) is realized as the GIT stack quotient

\[ [W \times \mathbb{C}^r \sslash \theta G]. \]

In other words, we require that

\[ (W \times \mathbb{C}^r)^{ss} = W^{ss}(\theta) \times \mathbb{C}^r. \]

This guarantees that the moduli space \( \mathcal{Q}_{0,k}^{\delta+}(E^\vee, \beta) \) of Definition 5.1 coincides with the space \( \mathcal{Q}_{0,k}^{\delta+}([W \times \mathbb{C}^r \sslash \theta G], \beta) \) of Definition 2.17 (as do their respective virtual classes). In particular, this allows us to combine the wall-crossing results of [38] with Theorem 6.9 to deduce the following corollary.

**Corollary 6.12.** The operator \( \tilde{\Delta} \) identifies the fundamental solutions \( L^{Z, \infty} \) and \( L^{E^\vee, \infty} \) up to a change of variables:

\[ L^{Z, \infty}(\mu^{Z, \geq 0+}(q, -\psi), z) \circ \tilde{\Delta} = \tilde{\Delta} \circ L^{E^\vee, \infty}(\mu^{E^\vee, \geq 0+}(q, -\psi), z)|_{q^\beta \mapsto e^{\pi i \beta (\det \varepsilon)} q^\beta}, \]

where \( \mu^{Z, \geq 0+}(q, -\psi) \) and \( \mu^{E^\vee, \geq 0+}(q, -\psi) \) are the changes of variables from [38, §1.11] for \( Z \) and \( E^\vee \) respectively.

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