TRUNCATED OVERLAP FERMIONS: THE LINK BETWEEN OVERLAP AND DOMAIN WALL FERMIONS

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Abstract. In this talk I will emphasize the role of the Truncated Overlap Fermions in showing the equivalence between the Domain Wall and Overlap Fermions up to an irrelevant factor in the fermionic integration measure. I will also show how Domain Wall type fermions with a finite number of flavors can be used to accelerate propagator calculations of their light partner in the infinite flavor limit.

1. Introduction

It required some time until Domain Wall [1, 2] and Overlap [3] formulations of chiral lattice fermions gained the adequate momentum [6]. A remnant chiral symmetry on the lattice, called the Ginsparg-Wilson relation [9], that was more recently noticed [4, 5], was shown to be the building block of a chiral gauge theory which exists on the lattice [7].

The basic idea of Domain Wall Fermions is an expanded flavor space which may be seen as an extra dimension with left and right handed fermions defined in the two opposite boundaries or walls, as it is sketched schematically below.

Let $N$ be the size of the extra dimension, $D_W$ the Wilson-Dirac operator, and $m$ the bare fermion mass. Then, the theory with Domain Wall Fermions is defined by the action [1, 2]:

$$S_{DW} := \bar{\psi}_5 \mathcal{M}_{DW} \psi = \sum_{i=1}^{N} \bar{\psi}_i \left[ (a_5 D_{i+1} - 1) \psi_i + P_+ \psi_{i+1} + P_- \psi_{i-1} \right]$$

(1)
with boundary conditions given by
\begin{align*}
P_+ (\psi_{N+1} + m \psi_1) &= 0 \\
P_- (\psi_0 + m \psi_N) &= 0
\end{align*}
(2)
where \( \mathcal{M} \) is the five-dimensional fermion matrix of the regularized theory and \( D^\| = M - D_W \) with \( M \in (0, 2) \) being a mass parameter and \( a_5 \) the lattice spacing in the 5\textsuperscript{th} direction.

The theory with \textit{Truncated Overlap Fermions} is defined by \([8, 15]\):
\[
S_{DW} := \bar{\Psi} a_5 \mathcal{M}_{TOV} \Psi = \sum_{i=1}^{N} \bar{\psi}_i \left[ \left( a_5 D^\| - 1 \right) \psi_i + \left( a_5 D^\| + 1 \right) P_+ \psi_{i+1} + \left( a_5 D^\| + 1 \right) P_- \psi_{i-1} \right]
\]  
(3)
In both cases the lattice spacing \( a \) of the four dimensional theory is set to one.

Truncated Overlap Fermions can be formally constructed from Domain Wall Fermions by substituting
\begin{align*}
P_+ \psi_{i+1} &\to \left( a_5 D^\| + 1 \right) P_+ \psi_{i+1} \\
P_- \psi_{i-1} &\to \left( a_5 D^\| + 1 \right) P_- \psi_{i-1}
\end{align*}
(4)
while the boundary conditions remain the same as before.

2. \textbf{Continuum limit in the 5\textsuperscript{th} dimension}

Let me first write down the operator kernels of both theories:
\[
\mathcal{M}_{DW} = D^\| + \frac{1}{a_5} \left( e^{a_5 \gamma_5 \partial_5} - 1 \right) \\
\mathcal{M}_{TOV} = D^\| \left( e^{a_5 \gamma_5 \partial_5} + 1 \right) + \frac{1}{a_5} \left( e^{a_5 \gamma_5 \partial_5} - 1 \right)
\]  
(5)
This form can be easily checked by using the identity:
\begin{align*}
\psi(t_5 + a_5) &= e^{a_5 \partial_5} \psi(t_5) \\
\text{with} \quad \psi_i &\equiv \psi(t_5 = ia_5)
\end{align*}
(6)
Taking the limit $a_5 \to 0$, I get:

\[
\mathcal{M}_{DW} = D\| + \gamma_5 \partial_5 + \frac{a_5}{2} \partial^2_5 \\
\mathcal{M}_{TOV} = D\|(2 + a_5 \gamma_5 \partial_5) + \gamma_5 \partial_5 + \frac{a_5}{2} \partial^2_5
\]  

(7)

Hence, in a continuous flavor space, both theories are unique up to an asymmetric factor remaining from the Truncated Overlap. Therefore, one may conclude that Domain Wall and Truncated Overlap Fermions are discretizations of the same Domain Wall fermion theory in the continuous flavor space $[0, T_5]$:

\[
\text{defined by the following action:}

S = \Psi(D + \gamma_5 \partial_5 - M) \Psi
\]

(8)

with the following boundary conditions:

\[
P_+ [\Psi(\cdot, T_5) + m \Psi(\cdot, 0)] = 0
\\
P_- [\Psi(\cdot, 0) + m \Psi(\cdot, T_5)] = 0
\]

(9)

3. Ginsparg-Wilson relation

A remnant chiral symmetry on the lattice may be possible if one allows a local symmetry breaking for propagating states. This statement is encoded in the Ginsparg-Wilson relation [9]:

\[
\gamma_5 D^{-1} + D^{-1} \gamma_5 = 2a \gamma_5 R,
\]

(10)

where $D$ is a local Dirac operator and $R$ is also a local operator trivial in Dirac space.

An explicit solution of this relation is given by the Overlap Dirac operator [10]. In fact one can show that a Dirac operator obeying the Ginsparg-Wilson symmetry can be derived from the Domain Wall [11, 8], and Truncated Overlap Fermions [8] in the infinite flavor limit.

The situation is unclear when the number of flavors is finite [8]. I present here some preliminary tests of the Ginsparg-Wilson relation on a small number of configurations on a $4^4$ lattice at $\beta = 6$. 

In Figs. 1-2 the locality of the Dirac operator is observed for \( N = 4 \) and \( N = 32 \) number of flavors. The behavior of \( R \) is tested in Figs. 3-4. These suggest that \( R \) tends towards a Kronecker-Delta function as the number of flavors grows and the convergence is faster for Domain Wall Fermions. More data are needed to verify this evidence.

4. Infinitely separated walls

The results of the previous section, although preliminary, are enough to conclude that the infinite limit in the fifth dimension is needed. This may be unrealistic for practical computations, if one would keep working with the whole 5-dimensional theory.

A simple solution is to work in the four dimensional framework of the Overlap Dirac operator [10]:

\[
D = \frac{1 + m}{2} - \frac{1 - m}{2} \gamma_5 \text{sgn}(H)
\]  

(11)

where the Hamiltonian \( H = \gamma_5 D^\parallel \) corresponds to the “evolution” in the fifth dimension of Truncated Overlap Fermions with a transfer matrix given by [8]:

\[
T_{TOV} = \frac{1 + H}{1 - H}
\]  

(12)

For Domain Wall Fermions it is not straightforward to construct “easy to use” Hamiltonians, since the transfer matrix is given by [8]:

\[
T_{DW} = \frac{1}{1 + HP_-}(1 - HP_+)
\]  

(13)

where numerator and denominator do not commute.

In analogy to Truncated Overlap Fermions, I define a Hamiltonian \( \mathcal{H} \) for Domain Wall Fermions, such that the transfer matrices of both theories coincide:

\[
\frac{1 + \mathcal{H}}{1 - \mathcal{H}} = \frac{1}{1 + HP_-}(1 - HP_+)
\]  

(14)

from which I can write down the solution:

\[
\mathcal{H} = \gamma_5 \frac{D^\parallel}{2 - D^\parallel} = H \frac{1}{2 - D^\parallel}
\]  

(15)

where \( a_5 = 1 \) is assumed.

This looks merely a trick, but in fact it is obvious by the definition that \( \mathcal{H} \) derives from the transfer matrix of the Domain Wall Fermions. Therefore, I arrive to the conclusion that
The light fermion operator in the infinite flavor limit of Domain Wall Fermions is given by the Overlap Dirac operator with Hamiltonian $\mathcal{H}$.

Some remarks are in order here:

a) The form of $\mathcal{H}$ suggests that both theories are identical in the limit $a \to 0$. In this case $\mathcal{H} \approx \mathcal{H}$.

b) For finite $a$ any theory with Wilson fermions can be equivalently defined to a theory with a Dirac operator:

$$\frac{D^\parallel}{2-D^\parallel} = \frac{D_W}{1+D_W}$$

for $M = 1$ and $m = 0$ (16)

up to the determinant factor $\det(1 + D_W)$. This is easily seen by the identity:

$$\int \psi \bar{\psi} e^{-\bar{\chi}(\bar{\chi} - \bar{\psi}) - \bar{\psi}D_W\psi} = \det(1 + D_W) e^{-\bar{\chi}D_W \chi}$$

(17)

i.e. the new operator is the Schur complement of the new “effective” theory with free fermions $\chi, \bar{\chi}$. Therefore up to an irrelevant determinant factor, both theories are equivalent for finite $a$.

Computational remarks on $\mathcal{H}$.

It is important to know the computation overhead of $\mathcal{H}$ if one would like to work with Domain Wall Fermions in the infinite flavor limit, i.e. in the Overlap framework.

Practical methods to compute the Overlap operator use the application of $\mathcal{H}$ or $\mathcal{H}^2$ to a vector [12, 13, 14].

It is obvious that the computation of $\mathcal{H}$ is more complex than that of $H$, although the inversion of $2 - D^\parallel$ is well conditioned and can be done fast.

On the other hand, $\mathcal{H}$ is conditioned better than $H$. To illustrate this, I have computed the spectrum of $D^\parallel/(2 - D^\parallel)$ for free fermions on a 16$^4$ lattice and also for a fixed background at $\beta = 6$ on a 4$^4$ lattice. The spectra are shown in Figs. 5-7.

5. Inversion of the Overlap Dirac operator

It has been pointed out that Truncated Overlap Fermions can be used to compute efficiently the inverse of the Overlap Dirac operator [8, 15]. From the discussion above, it can be concluded that Domain Wall Fermions can also be used effectively to compute the propagation of the light fermion in the infinite flavor limit.

The basic idea is a multigrid algorithm, which is illustrated below.
I would like to solve the linear system:

\[ Dz = b \]  

(18)

where \( D \) is the chiral Dirac operator and \( z_0 \) is a first guess. The algorithm may be described as a three step iteration scheme [8, 15]:

I. Compute the quark propagator in the Domain Wall framework, i.e. finite \( N \), which can be interpreted as a coarse lattice propagator:

\[
\mathcal{M} P x_5 = M_1 P b_5 \\
b_5 = (b, 0, \ldots, 0)^T \\
x_5 = (x, y, \ldots, z)^T
\]

(19)

where \( x_5 \) and \( b_5 \) are block-vectors with \( N \) blocks, \( \mathcal{M}_1 \) is the same matrix \( \mathcal{M} \) but with bare quark mass \( m = 1 \), and \( P \) is the following permutation operator:

\[
(P x_5)_i = P_+ (x_5)_i + P_- (x_5)_{i+1}, \quad i = 1, \ldots, N - 1 \]
\[
(P x_5)_N = P_+ (x_5)_N + P_- (x_5)_1
\]

(20)

II. Compute the residual error in the Overlap framework:

\[
z = z_0 + x \\
r = b - Dz
\]

(21)

III. Construct the new residual error of the five dimensional theory and define the new approximate solution:

\[
b_5 \leftarrow (r, 0, \ldots, 0)^T \\
z_0 \leftarrow z
\]

(22)

and go to step I. or otherwise stop.

The scheme is tested on 30 small \( 4^4 \) lattices at \( \beta = 6 \) for the Overlap Fermions. The results are shown in Fig. 8, where the multigrid pattern of the residual norm is clear. For comparison, in Fig. 8 are shown the results of directly applying the Conjugate Residuals (CR) algorithm. The gain is about a factor 10 in this case. More results are needed on larger lattices.

Note that CR is the best, i.e. the optimal algorithm for the Overlap operator, which is a normal operator [16]. To invert the “big” matrix in
I have used the BiCGstab2 algorithm [17] which is almost optimal in most of the cases for the non-normal matrices as its is the matrix \( M \) [16].

6. Conclusions

I have shown the equivalence between Domain Wall and Overlap Fermions up to an irrelevant factor in the fermionic integration measure.

Domain Wall and Truncated Overlap Fermions can be used to accelerate the computation of wall propagators in the infinite flavor limit.

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Figure 1. Norm of the $D$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles) and $N = 32$ (stars).

Figure 2. Norm of the $D$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles) and $N = 32$ (stars).
Figure 3. Norm of the $R$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles), $N = 32$ (stars) and $N = 64$ (crosses).

Figure 4. Norm of the $R$ kernel in spin and color space with the distance $r$ from the origin for $N = 4$ (circles), $N = 32$ (stars) and $N = 64$ (crosses).
\textbf{Figure 5.} The spectrum of the $D^\parallel/(2 - D^\parallel)$ matrix in the complex plane.
Figure 6. The spectrum of the $D || / (2 - D ||)$ matrix in the complex plane.

Figure 7. The spectrum of the $D || / (2 - D ||)$ matrix in the complex plane.
Figure 8. Norm of the residual error vs. the number of $D_W$ multiplications on 30 configurations. Circles stand for the straightforward inversion with CR and stars for the multigrid algorithm.