NONEXISTENCE AND MULTIPLICITY OF SOLUTIONS TO ELLIPTIC PROBLEMS WITH SUPERCRITICAL EXponents

MÓNICA CLAPP, JORGE FAYA, AND ANGELA PISTOIA

ABSTRACT. We consider the supercritical problem

\[-\Delta u = |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,\]

where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\), \(N \geq 3\), and \(p \geq 2^* := \frac{2N}{N-2}\).

Bahri and Coron showed that if \(\Omega\) has nontrivial homology this problem has a positive solution for \(p = 2^*\). However, this is not enough to guarantee existence in the supercritical case. For \(p \geq \frac{2(N-1)}{N-3}\) Passaseo exhibited domains carrying one nontrivial homology class in which no nontrivial solution exists. Here we give examples of domains whose homology becomes richer as \(p\) increases. More precisely, we show that for \(p > \frac{2(N-k)}{N-k-2}\) with \(1 \leq k \leq N-3\) there are bounded smooth domains in \(\mathbb{R}^N\) whose cup-length is \(k + 1\) in which this problem does not have a nontrivial solution.

For \(N = 4, 8, 16\) we show that there are many domains, arising from the Hopf fibrations, in which the problem has a prescribed number of solutions for some particular supercritical exponents.

Key words: Nonlinear elliptic problem; supercritical exponents; existence and nonexistence.

MSC2010: 35J60, 35J20.

1. INTRODUCTION

We consider the problem

\[
\left\{
\begin{array}{ll}
-\Delta u = |u|^{p-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{array}
\right.
\]

where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\), \(N \geq 3\), and \(p \geq 2^*\), where \(2^* := \frac{2N}{N-2}\) is the critical Sobolev exponent.

It is well known that the existence of a solution depends on the domain. Pohozaev’s identity [23] implies that \((\varphi_{2^*})\) does not have a nontrivial solution if \(\Omega\) is strictly starshaped. On the other hand, Kazdan and Warner [14] showed that infinitely many radial solutions exist if \(\Omega\) is an annulus.

For \(p = 2^*\) a remarkable result obtained by Bahri and Coron [3] establishes the existence of at least one positive solution to problem \((\varphi_{2^*})\) in every domain \(\Omega\) having nontrivial reduced homology with \(\mathbb{Z}/2\)-coefficients. Multiplicity results are also available, either for domains which are small perturbations of a given one, as in [13], or for domains which have enough, but possibly finite, symmetries, as in [7]. A more detailed discussion may be found in these papers.

Date: September 2012.

M. Clapp and J. Faya are supported by CONACYT grant 129847 and PAPHT grant IN106612 (Mexico). A. Pistoia is supported by Università degli Studi di Roma "La Sapienza" Accordi Bilaterali "Esistenza e proprietà geometriche di soluzioni di equazioni ellittiche non lineari" (Italy).
Unlike the critical case, in the supercritical case the existence of a nontrivial cohomology class in $\Omega$ does not guarantee the existence of a nontrivial solution to problem (1.1). In fact, for each $1 < k < N - 3$, Passaseo [19, 20] exhibited domains having the homotopy type of a $k$-dimensional sphere $S^k$ in which problem (1.1) does not have a nontrivial solution for any $p \geq 2^*_k := \frac{2(N-k)}{N-k-2}$. We call $2^*_k$ the $(k+1)$-st critical exponent. It is the critical exponent for the Sobolev embedding $H^1(\mathbb{R}^{N-k}) \hookrightarrow L^p(\mathbb{R}^{N-k})$. Nonexistence of bounded positive solutions for $p > 2^*_k$ in a thin enough tubular neighborhood of a $k$-dimensional submanifold of $\mathbb{R}^N$ was recently shown in [17].

The first nontrivial existence result for $p > 2^*$ was obtained by del Pino, Felmer and Musso [9] in the slightly supercritical case, i.e. for $p > 2^*$ but close enough to $2^*$. This case was also considered in [16, 21] where multiplicity was established. In [11] existence was established in a domain with a small enough hole for a.e. $p > 2^*$, whereas in [5, 17] solutions of a particular type were constructed in a tubular neighborhood of fixed radius of an expanding manifold for every $p$. The problem for $p$ slightly below the second critical exponent was considered in [10] where solutions for $p = 2^*_k - \varepsilon$ concentrating at a boundary geodesic as $\varepsilon \to 0$ have been constructed in certain domains. Quite recently, positive and sign changing solutions for $p = 2^*_k - \varepsilon$ which concentrate at $k$-dimensional submanifolds of the boundary as $\varepsilon \to 0$ were exhibited in [1], while in [13] positive and sign changing solutions for $p$ large which concentrate at $(N - 2)$-dimensional submanifolds of the boundary as $p \to +\infty$ have been constructed.

In a recent work Wei and Yan [26] exhibited domains $\Omega$ in which problem (1.1) has infinitely many positive solutions for $p = 2^*_k$. They considered domains $\Omega$ of the form

$$\Omega := \{(y,z) \in \mathbb{R}^{k+1} \times \mathbb{R}^{N-k-1} : (|y|, z) \in \Theta\},$$

where $\Theta$ is a bounded smooth domain in $\mathbb{R}^{N-k}$ with $\overline{\Theta} \subset (0, \infty) \times \mathbb{R}^{N-k-1}$ which satisfies certain geometric assumptions.

For domains of this type we give a geometric condition which guarantees nonexistence.

**Definition 1.1.** We shall say that $\Theta$ is doubly starshaped with respect to $\mathbb{R} \times \{0\}$ if there exist two numbers $0 < t_0 < t_1$ such that $t \in (t_0, t_1)$ for every $(t, z) \in \Theta$ and $\Theta$ is strictly starshaped with respect to $\xi_0 := (t_0, 0)$ and to $\xi_1 := (t_1, 0)$, i.e.

$$\langle x - \xi_i, \nu_{\Theta}(x) \rangle > 0 \quad \forall x \in \partial \Theta \setminus \{\xi_i\},$$

for each $i = 0, 1$, where $\nu_{\Theta}(x)$ is the outward pointing unit normal to $\partial \Theta$ at $x$.

For $\Omega$ as in (1.1) and $K \in C^1(\overline{\Omega})$ we consider the problem

$$\begin{cases}
-\Delta u = K(y, z) |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

We assume $K$ to be strictly positive on $\overline{\Omega}$ and radially symmetric in $y$, i.e. $K(y, z) = K(|y|, z)$. We prove the following result.

**Theorem 1.2.** If $\Theta$ is doubly starshaped with respect to $\mathbb{R} \times \{0\}$ and if $\langle y, \partial_u K(y, z) \rangle \leq 0$ and $\langle z, \partial_z K(y, z) \rangle \leq 0$ for all $(y, z) \in \Omega$, then problem (1.2) does not have a nontrivial solution for $p \geq 2^*_k$ and has infinitely many solutions for $p \in (2, 2^*_k)$, where $0 \leq k \leq N - 3$. 

The domains in Passaseo’s examples \([19, 20]\) are defined as in \([11]\) with \(\Theta\) being a ball centered at some point \((\tau, 0)\), which is obviously doubly starshaped with respect to \(\mathbb{R} \times \{0\}\). We stress that it is not enough for \(\Theta\) to be strictly starshaped to guarantee nonexistence: the domains considered by Wei and Yan \([26]\) are obtained from a domain \(\Theta\) which is not doubly starshaped with respect to \(\mathbb{R} \times \{0\}\), but which may be chosen to be strictly starshaped.

The domains in Passaseo’s examples \([19, 20]\), as well as those in Theorem \([12]\) have the homotopy type of \(S^k\). One may ask whether there are examples of domains having a richer topology for which a similar nonexistence result holds true. We prove the following result.

**Theorem 1.3.** Given \(k = k_1 + \cdots + k_m\) with \(k_1 \in \mathbb{N}\) and \(k \leq N - 3\), and \(\varepsilon > 0\) there exists a bounded smooth domain \(\Omega\) in \(\mathbb{R}^N\), which has the homotopy type of \(S^{k_1} \times \cdots \times S^{k_m}\), in which problem \((\mathcal{P}'')\) does not have a nontrivial solution for \(p \geq 2^*_N + \varepsilon\) and has infinitely many solutions for \(p \in (2, 2^*_N, k)\).

In particular, if we take all \(k_i = 1\), the domain \(\Omega\) is homotopy equivalent to the product of \(k\) circles. So not only the homology of \(\Omega\) is nontrivial but there are \(k\) different cohomology classes in \(H^1(\Omega; \mathbb{Z})\) whose cup-product is the generator of \(H^k(\Omega; \mathbb{Z})\). Hence, the cup-length of \(\Omega\) equals \(k + 1\).

We also obtain an existence result for a different type of domains, arising from the Hopf fibrations. We are specifically interested in the cases where \(N = 4, 8, 16\). In these cases \(\mathbb{R}^N = K \times K\), where \(K\) is either the complex numbers \(\mathbb{C}\), the quaternions \(\mathbb{H}\) or the Cayley numbers \(\mathbb{O}\). The set of units \(S_K := \{z \in K : |z| = 1\}\), which is a group if \(K = \mathbb{C}\) or \(\mathbb{H}\) and a quasigroup with unit if \(K = \mathbb{O}\), acts on \(\mathbb{R}^N\) by multiplication on each coordinate, i.e. \(\zeta(z_1, z_2) := (\zeta z_1, z_2)\). The orbit space of \(\mathbb{R}^N\) with respect to this action turns out to be \(\mathbb{R}^{\dim K + 1}\) and the projection onto the orbit space is the Hopf map \(\pi : \mathbb{R}^N = K \times K \to K \times \mathbb{R} = \mathbb{R}^{\dim K + 1}\) given by

\[
\pi(z_1, z_2) := \left(2z_1z_2, |z_1|^2 - |z_2|^2\right).
\]

We consider domains of the form \(\Omega = \pi^{-1}(U)\) where \(U\) is a bounded smooth domain in \(\mathbb{R}^{\dim K + 1}\). We assume that \(U\) is invariant under the action of some closed subgroup \(G\) of the group \(O(\dim K + 1)\) of linear isometries of \(\mathbb{R}^{\dim K + 1}\). We denote by \(G_x := \{gx : g \in G\}\) the \(G\)-orbit of a point \(x \in \mathbb{R}^{\dim K + 1}\) and by \(#Gx\) its cardinality. Recall that \(U\) is called \(G\)-invariant if \(Gx \subset U\) for all \(x \in U\), and a function \(u : U \to \mathbb{R}\) is called \(G\)-invariant if \(u\) is constant on every \(Gx\).

Fix a closed subgroup \(\Gamma\) of \(O(\dim K + 1)\) and a nonempty \(\Gamma\)-invariant bounded smooth domain \(D\) in \(\mathbb{R}^{\dim K + 1}\) such that \(#\Gamma x = \infty\) for all \(x \in D\). We prove the following result.

**Theorem 1.4.** There exists an increasing sequence \((\ell_m)\) of positive real numbers, depending only on \(\Gamma\) and \(D\), with the following property: If \(U\) contains \(D\) and if it is invariant under the action of a closed subgroup \(G\) of \(\Gamma\) for which

\[
\min_{x \in U} (\#Gx) \frac{|x|^{\dim K + 1}}{2} > \ell_m
\]

holds, then, for \(p = 2^*_N, k\), problem \((\mathcal{P}''')\) has at least \(m\) pairs of solutions \(\pm u_1, \ldots, \pm u_m\) in \(\Omega := \pi^{-1}(U)\), which are constant on \(\pi^{-1}(Gx)\) for each \(x \in U\). In particular, they are \(S_K\)-invariant. Moreover, \(u_1\) is positive and \(u_2, \ldots, u_m\) change sign.
For example, we may fix a bounded smooth domain \( D_0 \) in \( \mathbb{R}^2 \) with \( \overline{D_0} \subset (0, \infty) \times \mathbb{R} \) and set
\[
D := \{ (z, t) \in \mathbb{K} \times \mathbb{R} : (|z|, t) \in D_0 \}.
\]
Then \( D \) is invariant under the action of the group \( \Gamma := \mathbb{S}_\mathbb{C} \) of unit complex numbers on \( \mathbb{K} \times \mathbb{R} \) given by \( e^{i\theta}(z, t) := (e^{i\theta}z, t) \). If \( G_n := \{ e^{2\pi ik/n} : k = 0, \ldots, n - 1 \} \) is the cyclic subgroup of \( \Gamma \) of order \( n \), then \( \#G_n x = n \) for every \( x \in (\mathbb{K} \setminus \{0\}) \times \mathbb{R} \). Therefore, for every \( G_n \)-invariant bounded smooth domain \( U \) in \( \mathbb{K} \times \mathbb{R} \) with
\[
D \subset U \subset (\mathbb{K} \setminus \{0\}) \times \mathbb{R} \quad \text{and} \quad n \frac{\dim K - 1}{\dim K} > \ell_m,
\]
Theorem 1.4 yields at least \( m \) pairs of solutions to problem (1.3) in \( \Omega := \pi^{-1}(U) \) for \( p = 2^*_{N, \dim K - 3} \).

In contrast to [26], where multiplicity is established using Lyapunov-Schmidt reduction, the proof of the Theorem 1.4 uses variational methods. It is based on the following result.

**Proposition 1.5.** Let \( N = 2, 4, 8, 16 \), \( U \) be a bounded smooth domain in \( \mathbb{R}^{\dim K + 1} \) which does not contain the origin, \( a \in \mathbb{R} \), and \( f : \mathbb{R} \to \mathbb{R} \). If \( v \) solves
\[
(1.3) \quad \begin{cases} -\Delta v + \frac{\alpha}{2(|v|)} v = \frac{1}{2(|v|)} f(v) & \text{in } U, \\ v = 0 & \text{on } U, \end{cases}
\]
then \( u := v \circ \pi \) is a solution of
\[
(1.4) \quad \begin{cases} -\Delta u + au = f(u) & \text{in } \Omega := \pi^{-1}(U), \\ u = 0 & \text{on } \partial \Omega, \end{cases}
\]
where \( \pi : \mathbb{R}^N \to \mathbb{R}^{\dim K + 1} \) is the Hopf map. Conversely, if \( u \) is an \( \mathbb{S}_K \)-invariant solution of (1.4) and \( u = v \circ \pi \), then \( v \) solves (1.3).

For \( N = 4 \) this result was proved by Ruf and Srikanth in [24] by direct computation. Here we derive it from the theory of harmonic morphisms (see section 2).

Theorem 1.4 does not apply to the case \( p \in [2^*_{N,k}, 2^*_{N,k} + \epsilon) \). So the question remains open whether there are examples of domains having the homotopy type of a product of spheres for which nonexistence holds true for all \( p \geq 2^*_{N,k} \). We give a partial answer as follows.

**Theorem 1.6.** Let \( N = 4, 8, 16 \). Then there exist bounded smooth domains \( \Omega_n \) in \( \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \), which have the homotopy type of \( S^{\frac{N+2}{2}} \times S^n \) if \( 1 \leq n \leq \frac{N-1}{2} \) and of \( S^{\frac{N+2}{2}} \) if \( n = 0 \), such that problem (1.3) does not have a nontrivial \( \mathbb{S}_K \)-invariant solution for \( p \geq 2^*_{N,k} \) and has infinitely many \( \mathbb{S}_K \)-invariant solutions for \( p < 2^*_{N,k} \) where \( k := \frac{N-2}{2} + n \).

The question remains open as to whether for such domains other solutions exist, which are not \( \mathbb{S}_K \)-invariant, particularly for \( p \geq 2^*_{N,k} \).

This paper is organized as follows: in Section 2 we present the basic notions and results of the theory of harmonic morphisms and prove Proposition 1.3. Section 3 is devoted to proving Theorem 1.4. Theorems 1.2, 1.3 and 1.6 are proved in Section 4.
2. Harmonic morphisms

We recall some basic notions and give examples of harmonic morphisms. A detailed discussion is given e.g. in [4, 12, 27].

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds of dimensions $m$ and $n$ respectively. A smooth map $\pi : M \to N$ is called horizontally weakly conformal if for each $x \in M$ at which $d\pi_x \neq 0$ the differential $d\pi_x : T_x M \to T_{\pi(x)} N$ is surjective and horizontally conformal, i.e. there exists a number $\lambda(x) \neq 0$ such that

$$h(d\pi_x X, d\pi_x Y) = \lambda^2(x) g(X, Y)$$

for all $X, Y \in T^H_x M$, where $T^H_x M$ denotes the orthogonal complement of $\ker (d\pi_x)$. Defining $\lambda(x) = 0$ if $d\pi_x = 0$ we obtain a function $\lambda : M \to [0, \infty)$ called the dilation of $\pi$. It is given by $\lambda^2(x) = \frac{|d\pi_x|^2}{n}$, where $|d\pi_x|$ is the Hilbert-Schmidt norm of $d\pi_x$. Hence, it is a smooth function.

If $\pi$ has no critical points (i.e. $d\pi_x \neq 0$ for all $x \in M$) then it is called a conformal submersion. If $\lambda \equiv 1$ then $\pi : M \to N$ is a Riemannian submersion. Note that, if the dilation is constant and non-zero, then $\pi$ is a Riemannian submersion up to scale, i.e. it is a Riemannian submersion after a suitable homothetic change of metric on the domain or codomain.

The tension field $\tau(\pi)$ of a smooth map $\pi : M \to N$ is defined as

$$\tau(\pi) := \text{Trace}_g \nabla d\pi.$$

Thus, $\tau(\pi)$ is a vector field along $\pi$, i.e. a section of the pullback bundle $\pi^{-1}TN$. In charts,

$$\tau(\pi) = g^{ij} (\nabla_{\partial_i} d\pi)(\partial_j),$$

that is,

$$\tau^\gamma(\pi) = g^{ij} (\nabla d\pi^\gamma)_{ij} + g^{ij} \Gamma^N_{\alpha\beta} \pi^\alpha_i \pi^\beta_j$$

$$= g^{ij} \left( \frac{\partial^2 \pi^\gamma}{\partial x^i \partial x^j} - \Gamma^M_{ij} \frac{\partial \pi^\gamma}{\partial x^k} + \Gamma^N_{\alpha\beta} \pi^\alpha_i \pi^\beta_j \right)$$

$$= -\Delta_M \pi^\gamma + g^{ij} \Gamma^N_{\alpha\beta} \pi^\alpha_i \pi^\beta_j, \quad 1 \leq \gamma \leq n,$$

where $\Delta_M$ is the Laplace-Bertrami operator on $M$ (with the customary sign convention of Riemannian geometry) and $\Gamma^M_{ij} \kappa$ and $\Gamma^N_{\alpha\beta}$ are the Christoffel symbols of $M$ and $N$ respectively. The map $\pi : M \to N$ is called harmonic if $\tau(\pi) \equiv 0$. If, in addition, $\pi$ is horizontally weakly conformal, then $\pi$ is called a harmonic morphism.

The main property of harmonic morphisms is the following one.

**Proposition 2.1.** A smooth map $\pi : M \to N$ is a harmonic morphism with dilation $\lambda$ iff

$$\Delta_M (v \circ \pi) = \lambda^2 (\Delta_N v) \circ \pi$$

for each smooth function $v : V \to \mathbb{R}$ defined on an open subset $V$ of $N$ with $\pi^{-1}(V) \neq \emptyset$.

**Proof.** See [4] Proposition 4.2.3. \qed

**Corollary 2.2.** Let $\pi : M \to N$ be a harmonic morphism with dilation $\lambda$, a $: V \to \mathbb{R}$ be a function defined on an open subset $V$ of $N$ with $\pi^{-1}(V) \neq \emptyset$, and $f : \mathbb{R} \to \mathbb{R}$. 
Assume there exists \( \mu : V \to (0, \infty) \) such that \( \mu \circ \pi = \lambda^2 \) on \( \pi^{-1}(V) \). If \( v : V \to \mathbb{R} \) solves

\[
\Delta_N v + \frac{a(y)}{\mu(y)} v = \frac{1}{\mu(y)} f(v),
\]

then \( u := v \circ \pi : \pi^{-1}(V) \to \mathbb{R} \) solves

\[
\Delta_M u + (a \circ \pi) u = f(u).
\]

Conversely, if \( \pi : \pi^{-1}(V) \to V \) is surjective and \( v : V \to \mathbb{R} \) is such that \( u := v \circ \pi : \pi^{-1}(V) \to \mathbb{R} \) solves \((2.1)\), then \( v \) solves \((2.2)\).

**Proof.** This follows easily from Proposition 2.1. \( \square \)

Next we give some examples of harmonic morphisms.

**Proposition 2.3.** Let \( \pi : M \to N \) be a Riemannian submersion. Then \( \pi \) is a harmonic map iff each fiber \( \pi^{-1}(y) \) is a minimal submanifold of \( M \) (i.e. the mean curvature of \( \pi^{-1}(y) \) in \( M \) is zero).

**Proof.** See [12, (1.12)]. \( \square \)

Consequently, harmonic morphisms with constant non-zero dilation are simply Riemannian submersions with minimal fibres, up to scale. Some interesting examples are the Hopf fibrations.

**Example 2.4.** The Hopf fibrations \( \mathbb{S}^n \to \mathbb{R}P^n, \mathbb{S}^{2n+1} \to \mathbb{C}P^n, \mathbb{S}^{4n+3} \to \mathbb{H}P^n \) and \( \mathbb{S}^{15} \to \mathbb{S}^8 \) are Riemannian submersions (up to scale) with totally geodesic, and so minimal, fibres, see [4, Examples 2.4.14-17].

**Example 2.5.** The Hopf fibration \( \mathbb{S}^{2n+1} \to \mathbb{C}P^n \) factors through the double covering \( \mathbb{S}^{2n+1} \to \mathbb{R}P^{2n+1} \) to give a Riemannian submersion \( \mathbb{R}P^{2n+1} \to \mathbb{C}P^n \) with totally geodesic fibres. Similarly, one obtains a Riemannian submersion \( \mathbb{C}P^{2n+1} \to \mathbb{H}P^n \) with totally geodesic fibres.

The main example for our purposes is the following one.

**Example 2.6.** The Hopf maps \( \pi : \mathbb{R}^N = \mathbb{K} \times \mathbb{K} \to \mathbb{K} \times \mathbb{R} = \mathbb{R}^{\dim \mathbb{K} + 1} \) given by

\[
\pi(z_1, z_2) := (\overline{z_1}z_2, |z_1|^2 - |z_2|^2),
\]

with \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \) respectively, are harmonic morphisms [4, Corollary 5.3.3] with dilation \( \lambda(x, y) = \sqrt{2(|x|^2 + |y|^2)} \). Their restrictions to the unit sphere are the Hopf fibrations of Example 2.4 with \( n = 1 \). A simple computation shows that \( |\pi(x, y)| = |x|^2 + |y|^2 \). Hence, \( \lambda^2(x, y) = 2 |\pi(x, y)| \).

**Proof of Proposition 1.5.** Apply Corollary 2.2 to Example 2.6. \( \square \)

3. **Existence**

Proposition 1.5 suggests considering the problem

\[
(\nu_U^*) \quad \begin{cases} -\Delta v = K(x)|v|^{2^*-2} v & \text{in } U, \\ v = 0 & \text{on } \partial U, \end{cases}
\]

where \( U \) is a bounded smooth domain in \( \mathbb{R}^M \), \( K \in C^0(\mathbb{R}^M) \) is strictly positive on \( U \) and \( 2^* := \frac{2M}{M-2} \) is Sobolev’s critical exponent. We assume that \( U \) and \( K \) are...
G-invariant for some closed subgroup \( G \) of \( O(M) \). Then, the principle of symmetric criticality [15] asserts that the G-invariant solutions of problem \((P)\) are the critical points of the restriction of the functional

\[
J(v) := \frac{1}{2} \int_U |\nabla v|^2 - \frac{1}{2^*} \int_U K(x) |v|^{2^*}
\]

to the space of G-invariant functions

\[
H^1_G(U) := \{ v \in H^1(U) : v(gx) = v(x) \text{ for all } g \in G, x \in U \}.
\]

We shall say that \( J \) satisfies the Palais-Smale condition \((PS)^G_c\) in \( H^1_G(U) \) if every sequence \( (v_n) \) such that

\[
v_n \in H^1_G(U), \quad J(v_n) \to c, \quad \nabla J(v_n) \to 0,
\]
contains a convergent subsequence. Let \( S \) be the best Sobolev constant for the embedding \( D^{1,2}(\mathbb{R}^M) \hookrightarrow L^{2^*}(\mathbb{R}^M) \). The following result was proved in [6, Corollary 2].

**Proposition 3.1.** \( J \) satisfies condition \((PS)^G_c\) in \( H^1_G(U) \) for every

\[
c < \left( \min_{x \in U} \frac{\#Gx}{K(x)^{\frac{M}{M-2}}} \right) \frac{1}{M^{S^{M/2}}}.
\]

In particular, if \( \#Gx = \infty \) for all \( x \in U \), then \( J \) satisfies condition \((PS)^G_c\) in \( H^1_G(U) \) for every \( c \in \mathbb{R} \).

Fix a closed subgroup \( \Gamma \) of \( O(M) \) and a nonempty \( \Gamma \)-invariant bounded smooth domain \( D \) in \( \mathbb{R}^M \) such that \( \#\Gamma x = \infty \) for all \( x \in D \). Then, the following holds.

**Theorem 3.2.** Assume that \( K \) is \( \Gamma \)-invariant. Then, there exists an increasing sequence \( (\ell_m) \) of positive real numbers, depending only on \( \Gamma, D, K \), with the following property: if \( U \) contains \( D \) and if it is invariant under the action of a closed subgroup \( G \) of \( \Gamma \) for which

\[
\min_{x \in U} \frac{\#Gx}{K(x)^{\frac{M}{M-2}}} > \ell_m
\]
holds, then problem \((P)\) has at least \( m \) pairs of \( G \)-invariant solutions \( \pm v_1, \ldots, \pm v_m \) such that \( v_1 \) is positive, \( v_2, \ldots, v_m \) change sign, and

\[
\int_U |\nabla v_k|^2 \leq \ell_k S^{M/2} \text{ for every } k = 1, \ldots, m.
\]

**Proof.** For \( K = 1 \) this was proved in [7, Theorem 1]. The proof for general \( K \) goes through with minor modifications. We sketch it here for the reader’s convenience. Let \( \mathcal{P}_1(D) \) be the set of all nonempty \( \Gamma \)-invariant bounded smooth domains contained in \( D \), and define

\[
\mathcal{P}_k(D) := \{(D_1, \ldots, D_k) : D_i \in \mathcal{P}_1(D), \ D_i \cap D_j = \emptyset \text{ if } i \neq j \},
\]

Note that \( \mathcal{P}_k(D) \neq \emptyset \) for every \( k \in \mathbb{N} \). Since \( \#\Gamma x = \infty \) for all \( x \in D_i \), Proposition 3.1 allows to apply the mountain pass theorem [2] to obtain a nontrivial least energy \( \Gamma \)-invariant solution \( \omega_{D_i} \) to problem \((P_{D_i})\). Extending \( \omega_{D_i} \) by zero outside \( D_i \) we have that \( \omega_{D_i} \in H^1_G(U) \) and

\[
J(\omega_{D_i}) = \max_{t \geq 0} J(t\omega_{D_i}).
\]
We define 
\[ c_k := \inf \left\{ \sum_{i=1}^{k} J(\omega_{D_i}) : (D_1, \ldots, D_k) \in \mathcal{P}(D) \right\} \quad \text{and} \quad \ell_k := \left( \frac{1}{M} S^{M/2} \right)^{-1} c_k. \]

Note that \( c_1 = J(\omega_D) > 0 \) and that \( J(\omega_{D_i}) \geq c_1 \). Therefore, 
\[ c_{k-1} + c_1 \leq \sum_{i=1}^{k} J(\omega_{D_i}) \]
for every \((D_1, \ldots, D_k) \in \mathcal{P}(D), k \geq 2\). It follows that 
\[ c_{k-1} + c_1 \leq c_k \quad \text{and} \quad \ell_{k-1} + \ell_1 \leq \ell_k. \]

Let \( m \in \mathbb{N} \) and let \( \Omega \) be a bounded smooth domain containing \( D \), which is invariant under the action of a closed subgroup \( G \) of \( \Gamma \) for which
\[ \min_{x \in \overline{U}} \frac{\# Gx}{K(x)^{\frac{N}{2}}} > c_m \]
holds. Given \( \varepsilon \in (0, c_1) \) with \( c_m + \varepsilon < \left( \min_{x \in \overline{U}} \frac{\# Gx}{K(x)^{\frac{N}{2}}} \right) \frac{1}{M} S^{M/2} \), we choose \((D_1, \ldots, D_m) \in \mathcal{P}_m(D)\) such that 
\[ c_m \leq \sum_{i=1}^{m} J(\omega_{D_i}) < c_m + \varepsilon. \]

For each \( k = 1, \ldots, m \), let \( W_k \) be the subspace of \( H_0^1(\Omega)^G \) generated by \( \{\omega_{D_1}, \ldots, \omega_{D_k}\} \) and \( d_k := \sup_{W_k} J \). Then, \( \dim W_k = k \) and identity (3.1) implies that
\[ d_k = \sup_{W_k} J \leq \sum_{i=1}^{k} J(\omega_{D_i}) < \left( \min_{x \in \overline{U}} \frac{\# Gx}{K(x)^{\frac{N}{2}}} \right) \frac{1}{M} S^{M/2}. \]

Then, by Proposition 3.1, \( J \) satisfies \((PS)^G_c\) in \( H_0^1(\Omega)^G \) for all \( c \leq d_k \), so the mountain pass theorem [2] yields a positive critical point \( v_1 \in H_0^1(\Omega)^G \) of \( J \) such that \( J(v_1) \leq d_1 \), and Theorem 3.7 in [8], conveniently adapted to the functional we are considering here, yields \( m - 1 \) pairs of sign changing critical points \( \pm v_2, \ldots, \pm v_m \in H_0^1(\Omega)^G \) such that 
\[ J(v_k) \leq d_k \quad \text{for every} \quad k = 2, \ldots, m. \]

The proof that \( v_k \) may be chosen so that \( J(u_k) \leq c_k \) for every \( k = 1, \ldots, m \), follows just as in [7].

**Proof of Theorem 4.3**

This follows from Theorem 3.2 and Proposition 1.5.

\[ \square \]

4. Nonexistence

Fix \( k_1, \ldots, k_m \in \mathbb{N} \cup \{0\} \) with \( k := k_1 + \cdots + k_m \leq N - 3 \) and a bounded smooth domain \( \Theta \) in \( \mathbb{R}^{N-k} \) with \( \overline{\Theta} \subset (0, \infty)^m \times \mathbb{R}^{N-k-m} \). Set
\[ \Omega := \{ (y^1, \ldots, y^m, z) \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{N-k-m} : (|y^1|, \ldots, |y^m|, z) \in \Theta \}. \]

Let \( G := O(k_1+1) \times \cdots \times O(k_m+1) \). We think of \( G \) as a subgroup of \( O(N) \) acting on \( \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{N-k-m} \) in the obvious way, i.e.
\[ (g_1, \ldots, g_m)(y^1, \ldots, y^m, z) := (g_1 y^1, \ldots, g_m y^m, z) \]
for $g_i \in O(k_i + 1)$, $y^i \in \mathbb{R}^{k_i+1}$, $z \in \mathbb{R}^{N-k-m}$. Then $\Omega$ is $G$-invariant. For $K \in C^0(\Omega)$ we consider the problem

\begin{equation}
-\Delta u = K(x)|u|^{p-2}u \quad \text{in } \Omega,
\end{equation}

\begin{equation}
u = 0 \quad \text{on } \partial \Omega.
\end{equation}

**Proposition 4.1.** If $K$ is positive and $G$-invariant in $\Omega$ and $0 \leq k \leq N - 3$, then problem (4.3) has infinitely many $G$-invariant solutions for $p \in (2, 2N, k)$.

**Proof.** A $G$-invariant function $u(y^1, \ldots, y^m, z) = v(|y^1|, \ldots, |y^m|, z)$ solves problem (4.3) if and only if $v$ solves

\begin{equation}
-\Delta v - \sum_{i=1}^{m} \frac{k_i}{x_i} \frac{\partial v}{\partial x_i} = K(x)|v|^{p-2}v \quad \text{in } \Theta, \quad v = 0 \quad \text{on } \partial \Theta.
\end{equation}

This problem can be rewritten as

\begin{equation}
-\text{div}(a(x) \nabla v) = Q(x)|v|^{p-2}v \quad \text{in } \Theta, \quad v = 0 \quad \text{on } \partial \Theta,
\end{equation}

where $a(x_1, \ldots, x_{N-k}) := x_1^{k_1} \cdots x_m^{k_m}$ and $Q(x) := a(x)K(x)$. Note that both $a$ and $Q$ are continuous and strictly positive in $\Theta$. Hence, the norms

\begin{equation}
||v||_a := \left( \int_{\Theta} a(x)|\nabla v|^2 \right)^{1/2} \quad \text{and} \quad ||v||_{Q,p} := \left( \int_{\Theta} Q(x)|v|^p \right)^{1/p}
\end{equation}

are equivalent to those of $H^1_0(\Theta)$ and $L^p(\Theta)$ respectively. Since $H^1_0(\Theta)$ is compactly embedded in $L^p(\Theta)$ for $p < 2N, k$, the functional

\begin{equation}
J(v) := \frac{1}{2} ||v||_a^2 - \frac{1}{p} ||v||_{Q,p}^p, \quad v \in H^1_0(\Theta),
\end{equation}

satisfies the Palais-Smale condition. It clearly satisfies all other hypotheses of the symmetric mountain pass theorem [2]. Hence, it has an unbounded sequence of critical values. The critical values of $J$ are the solutions of (4.4). □

Next, fix $\tau_1, \ldots, \tau_m \in (0, \infty)$, and let $\varphi_i$ be the solution to the problem

\begin{equation}
\left\{ \begin{array}{l}
\varphi_i'(t)t + (k_i + 1)\varphi_i(t) = 1, \quad t \in (0, \infty), \\
\varphi_i(\tau_i) = 0.
\end{array} \right.
\end{equation}

Explicitly, $\varphi_i(t) = \frac{1}{k_i+1} \left[ 1 - (\frac{t}{\tau})^{k_i+1} \right]$. Note that $\varphi_i$ is strictly increasing in $(0, \infty)$. For $y^i \neq 0$ we define

\begin{equation}
\chi(y^1, \ldots, y^m, z) := (\varphi_1(|y^1|)y^1, \ldots, \varphi_m(|y^m|)y^m, z).
\end{equation}

**Lemma 4.2.** $\chi$ has the following properties:

(a) $\text{div} \chi = N - k$,

(b) $\langle \text{div}(y^1, \ldots, y^m, z) \xi, \xi \rangle \leq \max \left\{ 1 - k_1\varphi_1(|y^1|), \ldots, 1 - k_m\varphi_m(|y^m|), 1 \right\} ||\xi||^2$

for every $y^i \in \mathbb{R}^{k_i+1} \setminus \{0\}$, $z \in \mathbb{R}^{N-k-m}$, $\xi \in \mathbb{R}^N$.

**Proof.** (a) Write $y^i = (y^i_1, \ldots, y^i_{k_i+1})$. Then,

\begin{equation}
\text{div} \chi(y^1, \ldots, y^m, z) = m + \sum_{i=1}^{m} \frac{k_i + 1}{\sum_{j=1}^{k_i} \varphi_i'(|y^j|)\left(\frac{y^i_j}{|y^i|}\right)^2 + (k_i + 1)\varphi_i(|y^i|)} + N - k - m = N - k.
\end{equation}

(b) $\chi$ is $G$-equivariant for the $G$-action defined in (1.2), that is,

\begin{equation}
\chi(gy, z) = g\chi(y, z)
\end{equation}
for every $g \in G$, $y = (y_1, \ldots, y_m)$, $y^i \in \mathbb{R}^{k_i+1} \setminus \{0\}$, $z \in \mathbb{R}^{N-k-m}$. Therefore, $g \circ d\chi(y, z) = d\chi(gy, z) \circ g$ and, hence,

$$
\langle d\chi(y, z) [\xi], \xi \rangle = \langle g(d\chi(gy, z) [g\xi]), g\xi \rangle = \langle d\chi(gy, z) [g\xi], g\xi \rangle
$$

for all $\xi \in \mathbb{R}^N$. Thus, it suffices to show that the inequality (b) holds for $y^i = (y_1^i, 0, \ldots, 0)$ with $y_1^i > 0$. Set $\chi_i(y^i) := \varphi_i(\langle y^i \rangle)y^i$. A straightforward computation shows that, for such $y^i$, $d\chi_i(y^i)$ is a diagonal matrix whose diagonal entries are $a_{11} = 1 - k_i \varphi_i(y_1^i)$ and $a_{jj} = \varphi_i(y_1^i)$ for $j = 2, \ldots, k_i + 1$. Since $\varphi_i(t) < \frac{1}{k_i+1}$ for all $t \in (0, \infty)$, (b) follows.

**Proof of Theorem 1.2.** The variational identity (4) in Pucci and Serrin’s paper [22] implies that, if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (1.2) and $\chi \in C^1(\overline{\Omega}, \mathbb{R}^N)$, then

$$
\frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \langle \chi, \nu_\Omega \rangle \, d\sigma = \int_\Omega (\text{div} \chi) \left( - \frac{1}{p} K |u|^p - \frac{1}{2} |\nabla u|^2 \right) \, dx + \frac{1}{p} \int_\Omega |u|^p \langle \chi, \nabla K \rangle \, dx + \int_\Omega \langle d\chi \nabla u, \nabla u \rangle \, dx
$$

(4.6)

where $\nu_\Omega$ is the outward pointing unit normal to $\partial\Omega$. Take $\chi$ to be the vector field defined in (4.3) for $m = 1$, $0 \leq k \leq N - 3$ and $\tau_1 = t_0$ as in Definition 1.1, that is,

$$
\chi(y, z) := \langle \varphi(|y|)y, z \rangle, \quad (y, z) \in (\mathbb{R}^{k+1} \setminus \{0\}) \times \mathbb{R}^{N-k-1}
$$

with $\varphi(t) = \frac{1}{m-1} \left[ 1 - (\frac{t}{m})^k \right]$. Then, by Lemma 4.2

$$
\text{div} \chi = N - k.
$$

(4.7)

Note that, since $\varphi(t) \geq 0$ for $t \in (0, \infty)$ and $|y| > t_0$ if $(y, z) \in \Omega$, we have that

$$
\langle \chi(y, z), \nabla K(y, z) \rangle = \varphi(|y|) \langle y, \partial_y K(y, z) \rangle + \langle z, \partial_z K(y, z) \rangle \leq 0 \quad \forall (y, z) \in \Omega.
$$

Moreover, since $1 - k_\varphi(t) < 1$ for $t \in (0, \infty)$, Lemma 4.2 yields

$$
\langle d\chi(x) [\xi], \xi \rangle \leq |\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^N.
$$

We claim that

$$
\langle \chi(x), \nu_\Omega(x) \rangle > 0 \quad \forall x \in \partial\Omega \setminus \{g\xi_0, g\xi_1 : g \in O(k + 1)\}.
$$

Since $\Omega$ is $O(k + 1)$-invariant, $\nu_\Omega$ is $O(k + 1)$-equivariant. Thus, it suffices to show that

$$
\langle \langle \varphi(t) v(t, z), \nu_\Theta(t, z) \rangle > 0 \quad \text{for all } (t, z) \in \partial\Theta \setminus \{\xi_0, \xi_1\},
$$

where $\nu_\Theta(t, z)$ is the outward pointing unit normal to $\partial\Theta$ at $(t, z)$ which we write as $\nu_\Theta(t, z) = (\nu_1(t, z), \nu_2(t, z)) \in \mathbb{R} \times \mathbb{R}^{N-k-1}$. Let $(t, z) \in \partial\Theta$. Since $\Theta$ is doubly starshaped we have that

$$(t - t_i) \nu_1(t, z) + \langle z, \nu_2(t, z) \rangle > 0 \quad \text{if } (t, z) \neq (t_i, 0), \text{ for } i = 0, 1,$$

with $t_0, t_1$ as in Definition 1.3. Therefore,

$$
\langle \langle \varphi(t) v(t, z), \nu_\Theta(t, z) \rangle = \varphi(t) \nu_1(t, z) + \langle z, \nu_2(t, z) \rangle \rangle > \langle \varphi(t) t - t_i \rangle \nu_1(t, z).
$$

Set $\psi(t) := \varphi(t) t - t$. Note that $\psi'(t) = -k \varphi(t) < 0$ if $t > t_0$. So, since $t \in (t_0, t_1)$ for every $(t, z) \in \Theta$, we have that

$$
\varphi(t_1) t_1 - t_1 = \psi(t_1) \leq \psi(t) \leq \psi(t_0) = -t_0 \quad \forall (t, z) \in \partial\Theta.
$$
If \( \nu_1(t, z) \leq 0 \), then
\[
\langle (\varphi(t)t, z), \nu_\Theta(t, z) \rangle > (\psi(t) + t_0)\nu_1(t, z) \geq 0
\]
and if \( \nu_1(t, z) \geq 0 \), then
\[
\langle (\varphi(t)t, z), \nu_\Theta(t, z) \rangle > (\psi(t) + t_1)\nu_1(t, z) \geq \varphi(t_1)t_1\nu_1(t, z) \geq 0.
\]
This proves (4.11).

Combining properties (4.7), (4.8), (4.9) and (4.10) with identity (4.6) gives
\[
0 < \int_{\Omega} (\text{div} \chi) \left[ \frac{1}{p} |K|^p - \frac{1}{2} |\nabla u|^2 \right] dx + \int_{\Omega} |\nabla u|^2 dx
\]
\[
= (N - k) \left( \frac{1}{p} - \frac{1}{2} + \frac{1}{N - k} \right) \int_{\Omega} |\nabla u|^2 dx
\]
which implies that \( p < 2^*_{N,k} \) if \( u \neq 0 \).

Proposition 4.11 yields infinitely many solutions for \( p < 2^*_{N,k} \).

**Proof of Theorem 1.3** Choose \( \alpha \in \left( 1, \frac{N-k}{2k} \right) \) with \( 2^*_{N,k} + \varepsilon > \frac{2(N-k)}{N-2k} \). Fix \( \tau_1, \ldots, \tau_m \in (0, \infty) \) and, for the given \( k_1, \ldots, k_m \), define \( \chi \) as in (4.5). Let \( 0 < \varphi < \tau_i \) be defined by
\[
\max \{ 1 - k_1\varphi_1(\tau_1 - \varphi), \ldots, 1 - k_m\varphi_m(\tau_m - \varphi) \} = \alpha,
\]
The ball \( B^N_\varrho(\tau) \) be the ball of radius \( \varrho \) centered at \( \tau = (\tau_1, \ldots, \tau_m, 0) \) in \( R^m \times R^{N-k-m} \) and \( \Omega \) be defined as in (4.11). Then \( \Omega \) has the homotopy type of \( S^{k_1} \times \cdots \times S^{k_m} \). Moreover, Lemma 4.12 asserts that
\[
div \chi = N - k \quad \text{and} \quad \langle d\chi(x) [\xi], \xi \rangle \leq \alpha |\xi|^2 \quad \forall x \in \Omega, \ \xi \in R^N.
\]
Since \( \varphi_i(t) < 0 \) if \( t < \tau_i \) and \( \varphi_i(t) > 0 \) if \( t > \tau_i \), we have that, for all but a finite number of points \( (x, z) \in \partial \Omega \),
\[
\langle (\varphi_1(x_1) \ldots, \varphi_m(x_m), m, x, z), \nu_\Theta(t, z) \rangle = \sum_{i=1}^m \varphi_i(x_i) x_i (x_i - \tau_i) + |z|^2 > 0.
\]
Hence,
\[
\langle \chi, \nu_\Omega \rangle > 0 \quad \text{a.e. on } \partial \Omega.
\]
Combining properties (4.12) and (4.13) with identity (4.6) for \( K = 1 \) we obtain
\[
0 < \int_{\Omega} (\text{div} \chi) \left[ \frac{1}{p} |u|^p - \frac{1}{2} |\nabla u|^2 \right] dx + \alpha \int_{\Omega} |\nabla u|^2 dx
\]
\[
= (N - k) \left( \frac{1}{p} - \frac{1}{2} + \frac{\alpha}{N - k} \right) \int_{\Omega} |\nabla u|^2 dx
\]
which implies that \( p < \frac{2(N-k)}{N-2k} \leq 2^*_{N,k} + \varepsilon \) if \( u \neq 0 \). Consequently, problem \( \{\Psi_\rho\} \) does not have a nontrivial solution in \( \Omega \) for \( p \geq 2^*_{N,k} + \varepsilon \), whereas Proposition 4.11 yields infinitely many solutions for \( p < 2^*_{N,k} \).

**Proof of Theorem 1.6** For \( 0 \leq n \leq \dim K - 2 \), let \( \Theta_n \) be a bounded smooth domain in \( R^{\dim K-n+1} \) with \( \Theta_n \subset (0, \infty) \times R^{\dim K-n} \), which is doubly starshaped with respect to \( R \times \{0\} \). Define \( U_0 := \Theta_0 \) and
\[
U_n := \{ (y, z) \in R^{n+1} \times R^{\dim K-n} : (|y|, z) \in \Theta_n \} \subset R^{\dim K+1}
\]
if \( n \geq 1 \). Theorem 1.2 asserts that problem
\[
-\Delta v = \frac{1}{2} \frac{|v|^{p-2} v}{|x|} \quad \text{in } U_n, \quad v = 0 \quad \text{on } \partial U_n,
\]
has infinitely many solutions for \( p < 2_{\dim K+1,n}^* \) and no nontrivial solutions for \( p \geq 2_{\dim K+1,n}^* \). Hence, Proposition 1.5 implies that problem
\[
-\Delta u = |u|^{p-2} u \quad \text{in } \Omega := \pi^{-1}(U_n), \quad u = 0 \quad \text{on } \partial \Omega,
\]
has infinitely many \( S_k \)-invariant solutions if \( p < 2_{N,k}^* \) and does not have a nontrivial \( S_k \)-invariant solution if \( p \geq 2_{N,k}^* \), where \( k := \dim K - 1 + n \).

Finally, since the restriction of the Hopf map \( \pi : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^{\dim K+1} \setminus \{0\} \) is a fibration and \( U_n \) is contractible in \( \mathbb{R}^{\dim K+1} \setminus \{0\} \), the domain \( \Omega_n \) is fiber homotopy equivalent to \( S_k \times U_n \). Hence, it has the homotopy type of \( S_k \times S^n \) if \( n \geq 1 \) and of \( S_k \) if \( n = 0 \). \( \square \)

REFERENCES

[1] N. Ackermann, M. Clapp, A. Pistoia, Boundary clustered layers for some supercritical problems, preprint 2012.
[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973), 349-381.
[3] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988), 253-294.
[4] P. Baird, J.C. Wood, Harmonic morphisms between Riemannian manifolds. London Mathematical Society Monographs. New Series, 29. The Clarendon Press, Oxford University Press, Oxford, 2003.
[5] T. Bartels, M. Clapp, M. Grossi, F. Pacella, Asymptotically radial solutions in expanding annular domains, Math. Ann. 352 (2012), 485-515.
[6] M. Clapp, A global compactness result for elliptic problems with critical nonlinearity on symmetric domains. Nonlinear equations: methods, models and applications (Bergamo, 2001), 117–126, Progr. Nonlinear Differential Equations Appl. 54, Birkhäuser, Basel, 2003.
[7] M. Clapp, J. Faya, Multiple solutions to the Bahri-Coron problem in some domains with nontrivial topology, Proc. Amer. Math. Soc., to appear.
[8] M. Clapp, F. Pacella, Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size, Math. Z. 259 (2008), 575-589.
[9] M. del Pino, P. Felmer, M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries, Bull. London Math. Soc. 35 (2003), 513–521.
[10] M. del Pino, M. Musso, F. Pacard, Bubbling along boundary geodesics near the second critical exponent, J. Eur. Math. Soc. (JEMS) 12 (2010), 1553–1605.
[11] M. del Pino, J. Wei, Supercritical elliptic problems in domains with small holes, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 507–520.
[12] J. Eells, A. Ratto, Harmonic maps and minimal immersions with symmetries. Methods of ordinary differential equations applied to elliptic variational problems. Annals of Mathematics Studies, 130. Princeton University Press, Princeton, NJ, 1993.
[13] Y. Ge, M. Musso, A. Pistoia, Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains, Comm. Partial Differential Equations 35 (2010), 1419-1457.
[14] J. Kazdan, F. Warner, Remarks on some quasilinear elliptic equations, Comm. Pure Appl. Math. 38 (1975), 557-569.
[15] S. Kim, A. Pistoia, Clustered boundary layer sign changing solutions for a supercritical problem, preprint 2012.
[16] R. Molle, D. Passaseo, Positive solutions of slightly supercritical elliptic equations in symmetric domains, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 639–656.
[17] F. Pacard, F. Pacella, B. Sciunzi, Solutions of semilinear elliptic equations in tubes, Journal of Geometric Analysis, at press.
ELLIPITC PROBLEMS WITH SUPERCRITICAL EXPONENTS

[18] R. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979), 19-30.
[19] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains, J. Funct. Anal. 114 (1993), 97–105.
[20] D. Passaseo, New nonexistence results for elliptic equations with supercritical nonlinearity, Differential Integral Equations 8 (1995), 577–586.
[21] A. Pistoia, O. Rey, Multiplicity of solutions to the supercritical Bahri-Coron’s problem in pierced domains, Adv. Differential Equations 11 (2006), 647–666.
[22] P. Pucci, J. Serrin, A general variational identity, Indiana Univ. Math. J. 35 (1986), 681–703.
[23] S.I. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965), 1408-1411.
[24] B. Ruf, P.N. Srikanth, Singularly perturbed elliptic equations with solutions concentrating on a 1-dimensional orbit, J. Eur. Math. Soc. (JEMS) 12 (2010), 413–427.
[25] E.H. Spanier, Algebraic topology. McGraw-Hill Book Co., New York-Toronto, Ont.-London 1966
[26] J. Wei, S. Yan, Infinitely many positive solutions for an elliptic problem with critical or supercritical growth, J. Math. Pures Appl. 96 (2011), 307–333.
[27] J.C. Wood, Harmonic morphisms between Riemannian manifolds, Modern trends in geometry and topology, 397–414, Cluj Univ. Press, Cluj-Napoca, 2006.

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR, C.U., 04510 MÉXICO D.F., MÉXICO
E-mail address: mclapp@matem.unam.mx

INSTITUTO DE MATEMÁTICAS, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR, C.U., 04510 MÉXICO D.F., MÉXICO
E-mail address: jorgefaya@gmail.com

DIPARTIMENTO DI METODI E MODELLI MATEMATICI, UNIVERSITÀ DI ROMA “LA SAPIENZA”, VIA ANTONIO SCARPA 16, 00161 ROMA, ITALY
E-mail address: pistoia@dmmm.uniroma1.it