Refinements of the Morse stratification of the normsquare of the moment map

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Let $X$ be any nonsingular complex projective variety with a linear action of a complex reductive group $G$, and let $X^{ss}$ and $X^s$ be the sets of semistable and stable points of $X$ in the sense of Mumford’s geometric invariant theory [17]. We can choose a maximal compact subgroup $K$ of $G$ and an inner product on the Lie algebra $\mathfrak{t}$ of $K$ which is invariant under the adjoint action. Then $X$ has a $G$-equivariantly perfect stratification $\{S_\beta : \beta \in \mathcal{B}\}$ by locally closed nonsingular $G$-invariant subvarieties with $X^{ss}$ as an open stratum, which can be obtained as the Morse stratification for the normsquare of a moment map $\mu : X \to \mathfrak{t}^*$ for the action of $K$ on $X$ [9]. In this note the Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is refined to obtain stratifications of $X$ by locally closed nonsingular $G$-invariant subvarieties with $X^s$ as an open stratum. The strata can be defined inductively in terms of the sets of stable points of closed nonsingular subvarieties of $X$ acted on by reductive subgroups of $G$, and their projectivised normal bundles.

These refinements of the Morse stratification are not in general equivariantly perfect; that is, the associated equivariant Morse inequalities are not necessarily equalities. However when $G$ is abelian we can modify the moment map (or equivalently modify the linearisation of the action) by the addition of any constant, since the adjoint action is trivial. Perturbation of the moment map by adding a small constant then provides an equivariantly perfect refinement of the stratification $\{S_\beta : \beta \in \mathcal{B}\}$, and a generic perturbation gives us a refined stratification whose strata can be described inductively in terms of the sets of stable points of linear actions of reductive subgroups of $G$ for which stability and semistability coincide. This is useful even when $G$ is not abelian, since important questions about the cohomology of the Marsden-Weinstein reduction $\mu^{-1}(0)/K$ (or equivalently the geometric invariant theoretic quotient $X//G$) can be reduced to questions about the quotient of $X$ by a maximal torus of $G$.

The same constructions work when $X$ is a compact Kähler manifold with a Hamiltonian action of $K$. Even when $X$ is symplectic but not Kähler, refinements of the Morse stratification for $\|\mu\|^2$ can be obtained by choosing a suitable almost complex structure and Riemannian metric.
The moduli spaces $\mathcal{M}(n, d)$ of holomorphic bundles of rank $n$ and degree $d$ over a Riemann surface $\Sigma$ of genus $g \geq 2$ can be constructed as quotients of infinite dimensional spaces of connections, in a way which is analogous to the construction of quotients in geometric invariant theory; the rôles of the moment map is played by curvature and the rôles of the normsquare of the moment map is played by the Yang-Mills functional. In [13] refinements of the Morse stratification of the Yang-Mills functional are studied using the ideas of this paper. The motivation for this study was the search for a complete set of relations among the standard generators for the cohomology of these moduli spaces $\mathcal{M}(n, d)$ when $n$ and $d$ are coprime and $n > 2$ [4].

The layout of this paper is as follows. §1 reviews some background material and §2 uses the partial desingularisation construction of [10] to define a stratification $\{\Sigma_\gamma : \gamma \in \Gamma\}$ of $X^{ss}$ with $X^s$ as an open stratum. §3 gives an inductive description of the strata $\Sigma_\gamma$ in $X^{ss}$ in terms of the stable and semistable points of linear actions of reductive subgroups and subquotients of $G$ on nonsingular subvarieties of $X$ and their projectivised normal bundles. §4 refines the stratification $\{\Sigma_\gamma : \gamma \in \Gamma\}$ to obtain strata which are described inductively purely in terms of the stable points (not the semistable points) of the linear actions appearing in §3, and in §5 this stratification is used to refine the Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$. In §6 an alternative refinement of this stratification is obtained when $G$ is abelian by perturbing the moment map; in this case the inner product on the Lie algebra of $K$ can also be perturbed to give a refined stratification. Finally §7 discusses applications to the study of the cohomology ring of the Marsden-Weinstein reduction $\mu^{-1}(0)/K$ (or equivalently the geometric invariant theoretic quotient $X//G$), and in particular the relationship between this cohomology ring and the corresponding cohomology ring when $G$ is replaced by a maximal torus.

1 Background

In this section we shall review briefly the material we need from [9] (see also [6, 17, 18]. Let $X$ be a connected nonsingular projective variety embedded in projective space $\mathbb{P}_n$ and let $G$ be a complex reductive group acting linearly on $X$ via a homomorphism $\rho : G \to GL(n + 1; \mathbb{C})$. Then $G$ is the complexification of a maximal compact subgroup $K$, and by rechoosing the coordinates on $\mathbb{P}_n$ if necessary, we can assume that $K$ acts unitarily on $\mathbb{P}_n$ via $\rho : K \to U(n + 1)$.

The geometric invariant theoretic quotient $X//G$ is the projective variety whose homogeneous coordinate ring is the $G$-invariant part of the homogeneous coordinate ring of $X$. A point $x$ of $X$ is called semistable if there exists an invariant homogeneous polynomial $f$ which does not vanish on $x$, and $x$ is called stable if in addition the orbit $Gx$ is a closed subset of the set of stable points $X^{ss}$ and has dimension $\dim G$. There is a rational map from $X$ to $X//G$ which restricts to a $G$-invariant surjective morphism

$$\phi : X^{ss} \to X//G$$

from the open subset $X^{ss}$ of $X$, and every fibre of $\phi$ which meets the set $X^s$ consisting of the points of $X$ which are stable for the action of $G$ is a single $G$-orbit in $X^{ss}$. The image of $X^s$
is an open subset of $X//G$ which can thus be identified via $\phi$ with the quotient $X^s/G$. We shall assume that $X^s \neq \emptyset$ (but see Remark 2.1).

$X$ has a Kähler structure given by the restriction of the Fubini-Study metric on $\mathbb{P}_n$, and the Kähler form $\omega$ is a $K$-invariant symplectic form in $X$. There is a moment map $\mu : X \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of $K$, defined by

$$
\mu(x) = \rho^*((2\pi i \|x^*\|^2)^{-1}x^*\bar{x}^t)
$$

for $x \in X \subseteq \mathbb{P}_n$ represented by $x^* \in \mathbb{C}^{n+1}\setminus\{0\}$, when the Lie algebra of $U(n+1)$ and its dual are both identified with the space of skew-Hermitian $(n+1) \times (n+1)$ matrices in the standard way. Then $\mu^{-1}(0)$ is a subset of $X^{ss}$, and the inclusion induces a homeomorphism from the Marsden-Weinstein reduction (or symplectic quotient) $\mu^{-1}(0)/K$ to the geometric invariant theoretic quotient $X//G$. In fact $X^{ss}$ is the set of points $x$ in $X$ such that $\mu^{-1}(0)$ meets the $G$-orbit of $x$, and $X^s$ is the set of points $x$ in $X$ such that $\mu^{-1}(0)$ meets the $G$-orbit of $x$ in a point which is regular for $\mu$. In the good case when $X^{ss} = X^s$ then $X//G = X^{ss}/G$ and its rational cohomology is isomorphic to the equivariant cohomology of $X^{ss}$.

If we fix an invariant inner product on the Lie algebra $\mathfrak{k}$ then we can consider the function $\|\mu\|^2$ as a Morse function on $X$. It is not in general a Morse function in the classical sense, nor even a Morse-Bott function, since the connected components of its set of critical points may not be submanifolds of $X$, but nonetheless it induces a Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $X$ such that the stratum to which $x \in X$ belongs is determined by the limit set of its path of steepest descent for $\|\mu\|^2$ with respect to the Fubini-Study metric. This stratification can also be defined purely algebraically and has the following properties [9].

**Proposition 1.1** i) Each stratum $S_\beta$ is a $G$-invariant locally closed nonsingular subvariety of $X$.

ii) The unique open stratum $S_0$ is the set $X^{ss}$ of semistable points of $X$.

iii) The stratification is equivariantly perfect over the rationals, so that

$$
\dim H^i_G(X) = \dim H^i_G(X^{ss}) + \sum_{\beta \neq 0} \dim H^i_G(S_\beta)^{\lambda(\beta)}(S_\beta)
$$

where $\lambda(\beta)$ is the real codimension of $S_\beta$ in $X$.

iv) If $\beta \neq 0$ then there is a proper nonsingular subvariety $Z_\beta$ of $X$ acted on by a reductive subgroup $\text{Stab}_\beta$ of $G$ such that

$$
H^*_G(S_\beta) \cong H^*_{\text{Stab}(\beta)}(Z^{ss}_\beta)
$$

where $Z^{ss}_\beta$ is the set of semistable points of $Z_\beta$ with respect to an appropriate linearisation of the action of $\text{Stab}(\beta)$.

**Remark 1.2** i) All cohomology in this paper has rational coefficients.

ii) We shall assume that the invariant inner product chosen on $\mathfrak{k}$ is rational, and we shall use it to identify $\mathfrak{k}^*$ with $\mathfrak{t}$ throughout.

iii) Note that $G$-equivariant cohomology is the same as $K$-equivariant cohomology, since $G$ retracts onto its maximal compact subgroup $K$. 


If we choose a positive Weyl chamber $t_+$ in the Lie algebra $\mathfrak{t}$ of a maximal torus $T$ of $K$, then we can identify the indexing set $B$ with a finite subset of $t_+$ (or equivalently with the set of adjoint orbits of the points in this finite subset of $t_+$, since each adjoint orbit meets $t_+$ in exactly one point). More precisely, let $\alpha_0, ..., \alpha_n$ be the weights of the representation of $T$ on $\mathbb{C}^{n+1}$ and use the restriction to $t$ of the fixed invariant inner product on $\mathfrak{t}$ to identify $t^*$ with $\mathfrak{t}$. Then $\beta \in t_+$ belongs to $B$ if and only if $\beta$ is the closest point to 0 of the convex hull in $t$ of some nonempty subset of $\{\alpha_0, ..., \alpha_n\}$. Moreover $\text{Stab}(\beta)$ is the stabiliser of $\beta$ under the adjoint action of $G$, and $Z_\beta$ is the intersection of $X$ with the linear subspace

$$\{[x_0 : ... : x_n] \in \mathbb{P}_n : x_i = 0 \text{ if } \alpha_i, \beta \neq \|\beta\|^2\}$$

of $\mathbb{P}_n$. Equivalently $Z_\beta$ is the union of those connected components of the fixed point set of the subtorus $T_\beta$ of $T$ generated by $\beta$ on which the constant value taken by the real-valued function $x \mapsto \mu(x).\beta$ is $\|\beta\|^2$. If we use the fixed inner product to identify $t$ with its dual, then the image $\mu(Z_\beta)$ of $Z_\beta$ is contained in the Lie algebra of the maximal compact subgroup $\text{Stab}_K(\beta) = K \cap \text{Stab}(\beta)$ of $\text{Stab}(\beta)$. Since moment maps are only unique up to the addition of a central constant, we can take $\mu - \beta$ as our moment map for the action of $\text{Stab}_K(\beta)$ on $Z_\beta$. This corresponds to a modification of the linearisation of the action of $\text{Stab}(\beta)$ on $Z_\beta$ whose restriction to the complexification $T_\beta^c$ of $T_\beta$ is trivial, and we define $Z_\beta^{ss}$ to be the set of semistable points of $Z_\beta$ with respect to this modified linear action. Equivalently, $Z_\beta^{ss}$ is the stratum labelled by 0 (the minimum stratum) for the Morse stratification of the function $\|\mu - \beta\|^2$ on $Z_\beta$. Then

$$S_\beta = G\bar{Y}_\beta \setminus \bigcup_{|\gamma| > \|\beta\|} G\bar{Y}_\gamma = G\bar{Y}_{\beta^{ss}}$$ \hspace{1cm} (1.1)

where

$$\bar{Y}_\beta = \{x \in X : x_i = 0 \text{ if } \alpha_i, \beta < \|\beta\|^2\}$$

and

$$Y_\beta = \{x \in \bar{Y}_\beta : x_i \neq 0 \text{ for some } i \text{ such that } \alpha_i, \beta = \|\beta\|^2\},$$

while

$$Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$$ \hspace{1cm} (1.2)

where $p_\beta : Y_\beta \to Z_\beta$ is the obvious projection given by

$$p_\beta(x) = \lim_{t \to \infty} \exp(-it\beta)x.$$ \hspace{1cm} (1.3)

**Remark 1.3** In [9] $Z_\beta^{ss}$ is defined as above to be the set of semistable points of $Z_\beta$ with respect to the action of $\text{Stab}(\beta)$ with the linearisation modified in a way which corresponds to replacing the moment map $\mu$ by (a positive integer multiple of) $\mu - \beta$. However we can, if we wish, regard $\mu - \beta$ as a moment map for the action of $\text{Stab}_K(\beta)/T_\beta$ on $Z_\beta$, and then $Z_\beta^{ss}$ is the set of semistable points for the induced linear action of its complexification $\text{Stab}(\beta)/T_\beta^c$. The advantage of this description is that the analogous definition of $Z_\beta^s$ is a useful one, whereas $Z_\beta$ has no points which are stable with respect to the action of $\text{Stab}(\beta)$ when $\beta \neq 0$. Therefore we shall define $Z_\beta^s$ to be the set of stable points for the action of $\text{Stab}(\beta)/T_\beta^c$ on $Z_\beta$, linearised
so that the corresponding moment map is a positive integer multiple of $\mu - \beta$. Equivalently $Z_{\beta}^{ss}$ and $\tilde{Z}_\beta$ are the sets of semistable points of $Z_\beta$ under the action of the subgroup of $\text{Stab}(\beta)$ whose Lie algebra is the complexification of the orthogonal complement to $\beta$ in the Lie algebra of $\text{Stab}_K(\beta)$.

In fact if $B$ is the Borel subgroup of $G$ associated to the choice of positive Weyl chamber $t_+$ and if $P_\beta$ is the parabolic subgroup $B\text{Stab}(\beta)$, then $Y_\beta$ and $Y_{\beta}^{ss}$ are $P_\beta$-invariant and we have

$$S_\beta \cong G \times_{P_\beta} Y_{\beta}^{ss}. \quad (1.4)$$

Moreover $Y_\beta$ is a nonsingular subvariety of $X$ and $p_\beta : Y_\beta \to Z_\beta$ is a locally trivial fibration whose fibre is isomorphic to $\mathbb{C}^{m_\beta}$ for some $m_\beta \geq 0$.

Now an element $g$ of $G$ lies in the parabolic subgroup $P_\beta$ if and only if $\exp(-it\beta)g \exp(it\beta)$ tends to a limit in $G$ as $t \to \infty$, and this limit defines a surjection $q_\beta : P_\beta \to \text{Stab}(\beta)$. Since the surjections $p_\beta : Y_{\beta}^{ss} \to Z_{\beta}^{ss}$ and $q_\beta : P_\beta \to \text{Stab}(\beta)$ are retractions satisfying

$$p_\beta(gx) = q_\beta(g)p_\beta(x) \quad (1.5)$$

for all $g \in P_\beta$ and $x \in Y_{\beta}^{ss}$, this gives us the isomorphism

$$H^*_G(S_\beta) \cong H^*_\text{Stab}(\beta)(Z_{\beta}^{ss})$$

of Proposition 1.1(iv). Moreover since $G = KB$ and $B \subseteq P_\beta$ we have $G\tilde{Y}_\beta = K\tilde{Y}_\beta$, which is compact, and hence

$$\bar{S}_\beta \subseteq G\tilde{Y}_\beta \subseteq S_\beta \cup \bigcup_{\|\gamma\| > \|\beta\|} S_\gamma. \quad (1.6)$$

Note that $x = [x_0, ..., x_n] \in X$ is semistable (respectively stable) for the action of the complex torus $T^c$ if and only if 0 belongs to the convex hull in $t$ of the set of weights $\{\alpha_j : x_j \neq 0\}$ (respectively to the interior of this convex hull), and that $x$ is semistable (respectively stable) for the action of $G$ if and only if every element $gx$ of its $G$-orbit is semistable (respectively stable) for the action of $T^c$. In particular this tells us that

$$S_0 = X^{ss}.$$ 

It also tells us that if $\beta \in \mathcal{B}$ and if $y \in \tilde{Y}_\beta \setminus Y_{\beta}^{ss}$, then there is a subset $S$ of $\{\alpha_j : \alpha_j, \beta \geq \|\beta\|^2\}$ such that

$$y \in \text{Stab}(\beta)Y_{\beta'} \quad (1.7)$$

where $\beta'$ is the closest point to 0 of the convex hull of $S$ and $\beta' \neq \beta$.

**Remark 1.4** Of course the definitions of the subvarieties $Z_\beta$, $Z_{\beta}^{ss}$ and so on in this section depend upon the action of $G$ on $X \subseteq \mathbb{P}_n$ and its linearisation via the representation $\rho : G \to GL(n+1; \mathbb{C})$. When it is necessary to make these explicit in the notation we shall write $Z_\beta(X, \rho)$, $Z_{\beta}^{ss}(X, \rho)$ and so on, but we shall omit $X$ if $X = \mathbb{P}_n$, so that for example

$$Z_\beta(X, \rho) = X \cap Z_\beta(\rho).$$
Remark 1.5 In the special case when every semistable point is stable, the quotient variety $X//G$ is topologically the ordinary quotient $X^{ss}/G$ where $G$ acts with only finite stabilisers on $X^{ss}$, which means that its Betti numbers $\dim H^i(X//G)$ are the same as the equivariant Betti numbers $\dim H^i_G(X^{ss})$ of $X^{ss}$. These can be calculated inductively using Proposition 1.1 together with the fact that the equivariant cohomology of a nonsingular complex projective variety is isomorphic as a vector space to the tensor product of its ordinary cohomology and the equivariant cohomology of a point.

Remark 1.6 If $X$ is any compact symplectic manifold with a Hamiltonian action of a compact Lie group $K$ and a moment map $\mu : X \to \mathfrak{k}^*$, then the Morse stratification of $|\mu|_2$ has essentially the properties described here, with $G$ replaced by $K$ and $P_\beta$ replaced by $K \cap \text{Stab}(\beta)$.

2 Stratifying the set of semistable points

Suppose now that $X$ has some stable points but also has semistable points which are not stable. In [10, 11] it is described how one can blow $X$ up along a sequence of nonsingular $G$-invariant subvarieties to obtain a $G$-invariant morphism $\tilde{X} \to X$ where $\tilde{X}$ is a complex projective variety acted on linearly by $G$ such that $\tilde{X}^{ss} = \tilde{X}^s$. The induced birational morphism $\tilde{X}//G \to X//G$ of the geometric invariant theoretic quotients is then a partial desingularisation of $X//G$ in the sense that $\tilde{X}//G$ has only orbifold singularities (it is locally isomorphic to the quotient of a nonsingular variety by a finite group action) whereas the singularities of $X//G$ are in general much worse. In this section we shall review the construction of the partial desingularisation $\tilde{X}//G$ and use it firstly to stratify $X^{ss}$ and subsequently (in §5) to refine the stratification $\{S_\beta : \beta \in B\}$ of $X$ described in §1.

The set $\tilde{X}^{ss}$ can be obtained from $X^{ss}$ as follows. There exist semistable points of $X$ which are not stable if and only if there exists a non-trivial connected reductive subgroup of $G$ fixing a semistable point. Let $r > 0$ be the maximal dimension of a reductive subgroup of $G$ fixing a point of $X^{ss}$ and $\mathcal{R}(r)$ be a set of representatives of conjugacy classes of all connected reductive subgroups $R$ of dimension $r$ in $G$ such that

$$Z^{ss}_R = \{x \in X^{ss} : R \text{ fixes } x\}$$

is non-empty. Then

$$\bigcup_{R \in \mathcal{R}(r)} GZ^{ss}_R$$

is a disjoint union of nonsingular closed subvarieties of $X^{ss}$. The action of $G$ on $X^{ss}$ lifts to an action on the blow-up $X_{(1)}$ of $X^{ss}$ along $\bigcup_{R \in \mathcal{R}(r)} GZ^{ss}_R$ which can be linearised so that the complement of $X^{ss}_{(1)}$ in $X_{(1)}$ is the proper transform of the subset $\phi^{-1}(\phi(GZ^{ss}_R))$ of $X^{ss}$ where $\phi : X^{ss} \to X//G$ is the quotient map (see [10] 7.17). Moreover no point of $X^{ss}_{(1)}$ is fixed by a reductive subgroup of $G$ of dimension at least $r$, and a point in $X^{ss}_{(1)}$ is fixed by a reductive subgroup $R$ of dimension less than $r$ in $G$ if and only if it belongs to the proper transform of the subvariety $Z^{ss}_R$ of $X^{ss}$.
We can now apply the same procedure to $X^{ss}_{(1)}$ to obtain $X^{ss}_{(2)}$ such that no reductive subgroup of $G$ of dimension at least $r - 1$ fixes a point of $X^{ss}_{(2)}$. If we repeat this process enough times, we obtain $X^{ss}_{(1)} = X^{ss}, X^{ss}_{(1)}, X^{ss}_{(2)}, \ldots, X^{ss}_{(r)}$ such that no reductive subgroup of $G$ of positive dimension fixes a point of $X^{ss}_{(r)}$, and we set $\tilde{X}^{ss} = X^{ss}_{(r)}$. Equivalently we can construct a sequence

$$X^{ss}_{(R_0)} = X^{ss}, X^{ss}_{(R_1)}, \ldots, X^{ss}_{(R_r)} = \tilde{X}^{ss}$$

where $R_1, \ldots, R_r$ are connected reductive subgroups of $G$ with

$$r = \dim R_1 \geq \dim R_2 \geq \cdots \dim R_r \geq 1,$$

and if $1 \leq l \leq r$ then $X^{ss}_{(R_l)}$ is the blow up of $X^{ss}_{(R_{l-1})}$ along its closed nonsingular subvariety $GZ^{ss}_{R_l} \cong G \times_{N_l} Z^{ss}_{R_l}$, where $N_l$ is the normaliser of $R_l$ in $G$. Similarly $\tilde{X} = \tilde{X}^{ss}/G$ can be obtained from $X^{ss}/G$ by blowing up along the proper transforms of the images $Z_{R_l}/N$ in $X^{ss}/G$ of the subvarieties $GZ^{ss}_{R_l}$ of $X^{ss}$ in decreasing order of dim $R$.

If $1 \leq l \leq r$ then there is a $G$-equivariant stratification

$$\{ S_{\beta, l} : \beta \in \mathcal{B}_l \}$$

of $X_{(R_l)}$ by nonsingular $G$-invariant locally closed subvarieties such that one of the strata, indexed by $0 \in \mathcal{B}_l$, coincides with the open subset $X^{ss}_{R_l}$ of $X_{R_l}$. This stratification is constructed exactly as the stratification $\{ S_{\beta} : \beta \in \mathcal{B} \}$ of $X$ was constructed in the last section; note that $X_{(R_l)}$ is in general only quasi-projective rather than projective, but it is shown in [10] that the construction of the stratification still works for $X_{(R_l)}$ and the properties given in Proposition 11 still hold.

There is a partial ordering on $\mathcal{B}_l$ with 0 as its minimal element such that if $\beta \in \mathcal{B}_l$ then the closure in $X_{(R_l)}$ of the stratum $S_{\beta, l}$ satisfies

$$\overline{S}_{\beta, l} \subseteq \bigcup_{\gamma \in \mathcal{B}_l, \gamma \geq \beta} S_{\gamma, l}.$$ 

If $\beta \in \mathcal{B}_l$ and $\beta \neq 0$ then the stratum $S_{\beta, l}$ retracts $G$-equivariantly onto its (tranverse) intersection with the exceptional divisor $E_l$ for the blow-up $X_{(R_l)} \to X^{ss}_{(R_{l-1})}$. This exceptional divisor is isomorphic to the projective bundle $\mathcal{P}(N_l)$ over $GZ^{ss}_{R_l}$, where $Z^{ss}_{R_l}$ is the proper transform of $Z^{ss}_{R_l}$ in $X^{ss}_{(R_{l-1})}$ and $N_l$ is the normal bundle to $GZ^{ss}_{R_l}$ in $X^{ss}_{R_{l-1}}$. The stratification $\{ S_{\beta, l} : \beta \in \mathcal{B}_l \}$ is determined by the action of $R_l$ on the fibres of $N_l$ over $Z^{ss}_{R_l}$ (see [10] §7).

The composition

$$\tilde{X}^{ss} = X_{(R_r)} \to X^{ss}_{(R_{r-1})} \to \cdots X^{ss}_{(R_1)} \to X^{ss}$$

is an isomorphism over the set $X^*$ of stable points of $X$, and the complement of $X^*$ in $\tilde{X}^{ss}$ is just the union of the proper transforms in $X^{ss}$ of the exceptional divisors $E_1, \ldots, E_k$ for the blow-ups $X_{R_l} \to X^{ss}_{(R_{l-1})}$ for $l = 1, \ldots, r$.

We can now stratify $X^{ss}$ as follows. We take as the highest stratum the nonsingular closed subvariety $GZ^{ss}_{R_1}$ whose complement in $X^{ss}$ can be naturally identified with the complement $X_{(R_1)} \setminus E_1$ of the exceptional divisor $E_1$ in $X_{(R_1)}$. Recall that $GZ^{ss}_{R_1} \cong G \times N_1 Z^{ss}_{R_1}$ where $N_1$ is
the normaliser of $R_1$ in $G$, and $Z_{R_1}^{ss}$ is equal to the set of semistable points for the action of $N_1$, or equivalently for the induced action of $N_1/R_1$, on $Z_{R_1}$, which is a union of connected components of the fixed point set of $R_1$ in $X$ (see [10] §5). Since $R_1$ has maximal dimension among those reductive subgroups of $G$ with fixed points in $X^{ss}$, we have

$$Z_{R_1}^{ss} = Z_{R_1}^s$$

where $Z_{R_1}^s$ is the set of stable points for the action of $N_1/R_1$ on $Z_{R_1}$ for $1 \leq l \leq \tau$.

Next we take as strata the nonsingular locally closed subvarieties

$$\{S_{\beta,1}\backslash E_1 : \beta \in B_1, \beta \neq 0\}$$

of $X_{(R_1)}\backslash E_1 = X^{ss}\backslash GZ_{R_1}^{ss}$, whose complement in $X_{(R_1)}\backslash E_1$ is just $X_{(R_1)}^{ss}\backslash E_1 = X_{(R_1)}^{ss}\backslash E_1^s$ where $E_1^s = X_{(R_1)}^{ss}\cap E_1$, and then we take the intersection of $X_{(R_1)}^{ss}\backslash E_1$ with $GZ_{R_2}^{ss}$. This intersection is $GZ_{R_2}^{ss}$ where $Z_{R_2}^s$ is the set of stable points for the action of $N_2/R_2$ on $Z_{R_2}$, and its complement in $X_{(R_1)}^{ss}\backslash E_1$ can be naturally identified with the complement in $X_{(R_2)}$ of the union of $E_2$ and the proper transform $\hat{E}_1$ of $E_1$.

Our next strata are the nonsingular locally closed subvarieties

$$\{S_{\beta,2}\backslash (E_2 \cup \hat{E}_1) : \beta \in B_2, \beta \neq 0\}$$

of $X_{(R_2)}\backslash (E_2 \cup \hat{E}_1)$, whose complement in $X_{(R_2)}\backslash (E_2 \cup \hat{E}_1)$ is $X_{(R_2)}^{ss}\backslash (E_2 \cup \hat{E}_1)$, and the stratum after these is $GZ_{R_2}^{ss}$. Repeating this process gives us strata which are all nonsingular locally closed $G$-invariant subvarieties of $X^{ss}$ indexed by the disjoint union

$$\{R_1\} \cup \{R_1\} \times (B_1\backslash\{0\}) \cup \cdots \cup \{R_\tau\} \cup \{R_\tau\} \times (B_\tau\backslash\{0\}),$$

and the complement in $X^{ss}$ of the union of these strata is just the open subset $X^s$. We take $X^s$ as our final stratum indexed by 0, so that the indexing set for our stratification of $X^{ss}$ is the disjoint union

$$\Gamma = \{R_1\} \cup \{R_1\} \times (B_1\backslash\{0\}) \cup \cdots \cup \{R_\tau\} \cup \{R_\tau\} \times (B_\tau\backslash\{0\}) \cup \{0\}. \quad (2.1)$$

Moreover the given partial orderings on $B_1, \ldots, B_\tau$ together with the ordering in the expression (2.1) above for $\Gamma$ induce a partial ordering on $\Gamma$, with $R_1$ as the maximal element and 0 as the minimal element, such that the closure in $X^{ss}$ of the stratum $\Sigma_\gamma$ indexed by $\gamma \in \Gamma$ satisfies

$$\bigcup_{\delta \in \Gamma, \delta \geq \gamma} \Sigma_\delta \subseteq \Sigma_\gamma. \quad (2.2)$$

Thus this process gives us a stratification

$$\{\Sigma_\gamma : \gamma \in \Gamma\} \quad (2.3)$$

of $X^{ss}$ such that the stratum indexed by the minimal element 0 of $\Gamma$ coincides with the open subset $X^s$ of $X^{ss}$.
Remark 2.1 We have been assuming that \( X^s \neq \emptyset \), but this procedure gives us a stratification of \( X^{ss} \) even when \( X^s \) is empty. The only difference when \( X^s \) is empty is that the procedure terminates at some stage \( l \) when

\[
X^{ss}_{(R_{l-1})} = GZ^{ss}_{R_l} \cong G \times_{N_l} \hat{Z}^{ss}_{R_l}
\]

and gives us a stratification indexed by

\[
\Gamma = \{ R_1 \} \cup \{ R_1 \} \times (B_1 \setminus \{ 0 \}) \cup \cdots \cup \{ R_{l-1} \} \cup \{ R_{l-1} \} \times (B_{l-1} \setminus \{ 0 \}) \cup \{ R_l \}
\]

such that the stratum indexed by the minimal element \( R_l \) of \( \Gamma \) is the open subset \( GZ^{ss}_{R_l} \) of \( X^{ss} \). Note also that \( Z^{ss}_{R_l} \) is nonempty, since otherwise \( Z^{ss}_{R_l} = N_lZ^{ss}_R \) for some \( R \) containing \( R_l \) with \( \dim R > \dim R_l \), and then

\[
GZ^{ss}_R = GZ^{ss}_{R_l} = X^{ss},
\]

so the procedure would have terminated at an earlier stage.

3 Inductive description of the strata \( \Sigma \gamma \) in \( X^{ss} \)

The last section described a stratification \( \{ \Sigma \gamma : \gamma \in \Gamma \} \) of \( X^{ss} \) such that the stratum indexed by the minimal element \( 0 \) of \( \Gamma \) coincides with the open subset \( X^s \) of \( X^{ss} \). In this section we shall study the strata \( \Sigma \gamma \) in more detail.

Note that the strata \( \Sigma \gamma \) with \( \gamma \neq 0 \) fall into two classes. Either \( \gamma = R_l \) for some \( l \in \{ 1, \ldots, \tau \} \), in which case the stratum \( \Sigma \gamma \) is

\[
GZ^{ss}_{R_l},
\]

or else \( \gamma = (R_l, \beta) \) where \( \beta \in B_l \setminus \{ 0 \} \) for some \( l \in \{ 1, \ldots, \tau \} \) and the stratum \( \Sigma \gamma \) is

\[
S_{\beta, l} \setminus (E_l \cup \hat{E}_{l-1} \cup \cdots \cup \hat{E}_1).
\]

In the latter case we know from (1.3) that

\[
S_{\beta, l} = GY^{ss}_{\beta, l} \cong G \times_{P_{\beta, l}} Y^{ss}_{\beta, l}, \tag{3.1}
\]

where \( Y^{ss}_{\beta, l} \) fibres over \( Z^{ss}_{\beta, l} \) via \( p_{\beta} : Y^{ss}_{\beta, l} \to Z^{ss}_{\beta, l} \) with fibre \( \mathcal{C}^{m_{\beta, l}} \) for some \( m_{\beta, l} > 0 \), and

\[
S_{\beta, l} \cap E_l = G(Y^{ss}_{\beta, l} \cap E_l) \cong G \times_{P_{\beta, l}} (Y^{ss}_{\beta, l} \cap E_l) \tag{3.2}
\]

where \( Y^{ss}_{\beta, l} \cap E_l \) fibres over \( Z^{ss}_{\beta, l} \) with fibre \( \mathcal{C}^{m_{\beta, l}-1} \) (see [10] Lemmas 7.6 and 7.11). Thus

\[
S_{\beta, l} \setminus E_l \cong G \times_{P_{\beta, l}} (Y^{ss}_{\beta, l} \setminus E_l) \tag{3.3}
\]
where $Y_{\beta, l}^{ss} \setminus E_{l}$ fibres over $Z_{\beta, l}^{ss}$ with fibre $C^{m_{\beta, l} - 1} \times (C \setminus \{0\})$. Let

$$\pi_l: E_l \cong \mathbb{P}(N_l) \to GZ_{R_l}^{ss}$$

denote the projection. Lemma 7.9 of \cite{10} tells us that if $x \in Z_{R_l}^{ss}$ then the intersection of $S_{\beta, l}$ with the fibre $\pi_l^{-1}(x) = \mathbb{P}(N_{l,x})$ of $\pi_l$ at $x$ is the union of those strata indexed by points in the adjoint orbit $Ad(G)\beta$ in the stratification of $\mathbb{P}(N_{l,x})$ induced by the representation $p_l$ of $R_l$ on the normal $N_{l,x}$ to $GZ_{R_l}^{ss}$ at $x$. Note that we can assume that $R_l \cap K$ is a maximal compact subgroup of $R_l$ and that $R_l \cap T$ is a maximal torus for $R_l \cap K$, and then $Ad(G)\beta$ meets a positive Weyl chamber for $R_l$ in $\text{Lie}(R_l \cap T)$ in a finite number of points

$$\beta = \beta_1 = Ad(w_1)\beta, \beta_2 = Ad(w_2)\beta, \ldots, \beta_{l, r} = Ad(w_{r, l})\beta$$

where $w_1, w_2, \ldots, w_{r, l} \in G$ represent elements of the Weyl group of $G$.

Now if $y \in Z_{\beta, l}^{ss}$ and $\pi_l(y) = gx$ where $g \in G$ and $x \in Z_{R_l}^{ss}$, then $x$ is fixed by $Ad(g^{-1})\beta$ and so $Ad(g^{-1})\beta$ lies in the Lie algebra of $R_l$. Since $R_l \cap T$ is a maximal compact torus of $R_l$, there exists $r \in R_l$ such that $Ad(rg^{-1})\beta \in \text{Lie}(R_l \cap T)$. Then $Ad(rg^{-1})\beta = Ad(w_j)\beta$ for some $j \in \{1, \ldots, r_{\beta, l}\}$, and hence $w_j^{-1}rg^{-1} \in \text{Stab}(\beta)$, so $g \in \text{Stab}(\beta)w_j^{-1}R_l$. Conversely if $g = hw_j^{-1}$ where $h \in \text{Stab}(\beta)$ and $r \in R_l$, then $y \in Z_{\beta, l}^{ss}$ if and only if $h^{-1}y$ lies in $Z_{\beta, l}^{ss}$ where $\pi_l(h^{-1}y) = w_j^{-1}x \in w_j^{-1}Z_{R_l}^{ss}$. Thus

$$Z_{\beta, l}^{ss} = \bigcup_{1 \leq j \leq r_{\beta, l}} \text{Stab}(\beta) \left( Z_{\beta, l}^{ss} \cap w_j^{-1} \pi_l^{-1}(Z_{R_l}^{ss}) \right)$$

$$= \bigcup_{1 \leq j \leq r_{\beta, l}} \text{Stab}(\beta) w_j^{-1} \left( Z_{Ad(w_j)\beta, l}^{ss} \cap \pi_l^{-1}(Z_{R_l}^{ss}) \right).$$

Also if $\text{Stab}(\beta)(Z_{\beta, l}^{ss} \cap w_j^{-1} \pi_l^{-1}(Z_{R_l}^{ss}))$ meets $\text{Stab}(\beta)(Z_{\beta, l}^{ss} \cap w_i^{-1} \pi_l^{-1}(Z_{R_l}^{ss}))$ then $\text{Stab}(\beta)w_j^{-1}Z_{R_l}^{ss}$ meets $w_i^{-1}Z_{R_l}^{ss}$, and since $GZ_{R_l}^{ss} \cong G \times N_l Z_{R_l}^{ss}$ this means that there is some $h \in \text{Stab}(\beta)$ and $n \in N_l$ such that $w_i h = nw_j$, so that $\beta_i = Ad(w_i)\beta \in Ad(N_l)\beta_j$. Conversely if $\beta_i \in Ad(N_l)\beta_j$ then $w_i h = nw_j$ for some $h \in \text{Stab}(\beta)$ and $n \in N_l$, and so

$$\text{Stab}(\beta)(Z_{\beta, l}^{ss} \cap w_j^{-1} \pi_l^{-1}(Z_{R_l}^{ss})) = \text{Stab}(\beta)(Z_{\beta, l}^{ss} \cap h^{-1} w_i^{-1} n \pi_l^{-1}(Z_{R_l}^{ss}))$$

$$= \text{Stab}(\beta)(Z_{\beta, l}^{ss} \cap w_i^{-1} \pi_l^{-1}(Z_{R_l}^{ss})).$$

Thus $Z_{\beta, l}^{ss}$ is a disjoint union

$$Z_{\beta, l}^{ss} = \bigcup_{1 \leq j \leq r_{\beta, l}} \text{Stab}(\beta) \left( Z_{\beta, l}^{ss} \cap w_j^{-1} \pi_l^{-1}(Z_{R_l}^{ss}) \right)$$

where $Ad(w_1)\beta = \beta_1, \ldots, Ad(w_{r_{\beta, l}})\beta$ form a set of representatives for the $Ad(N_l)$ orbits in $Ad(G)\beta$, and $Y_{\beta, l}^{ss}$ and $S_{\beta, l}$ can be expressed similarly as disjoint unions. In fact, since by \cite{13} the fibration

$$p_\beta: Y_{\beta, l}^{ss} \to Z_{\beta, l}^{ss}$$


satisfies $p_\beta(gy) = q_\beta(g)p_\beta(y)$ for all $g \in P_\beta$ and $y \in Y_{\beta,l}^{ss}$ where $q_\beta : P_\beta \rightarrow \text{Stab}(\beta)$ is the projection, we have

$$Y_{\beta,l}^{ss} = \bigsqcup_{1 \leq j \leq s_{\beta,l}} P_\beta p_\beta^{-1} \left( Z_{\beta,l}^{ss} \cap w_j^{-1}\pi_l^{-1}(\hat{Z}_{R_l}^{ss}) \right)$$

and

$$S_{\beta,l} = \bigsqcup_{1 \leq j \leq s_{\beta,l}} G p_\beta^{-1} \left( Z_{\beta,l}^{ss} \cap w_j^{-1}\pi_l^{-1}(\hat{Z}_{R_l}^{ss}) \right).$$

This means that we could, if we wished, replace the indexing set $B \setminus \{0\}$, whose elements correspond to the $G$-adjoint orbits $\text{Ad}(G)\beta$ of elements of the indexing set for the stratification of $\mathbb{P}(N_{l,x})$ induced by the representation $\rho_t$, by the set of their $N_l$-adjoint orbits $\text{Ad}(N_l)\beta$. Then we would still have (3.1) – (3.3), but now

$$Z_{\beta,l}^{ss} = \text{Stab}(\beta)(Z_{\beta,l}^{sl} \cap \pi_l^{-1}(\hat{Z}_{R_l}^{ss})) \cong \text{Stab}(\beta) \times_{N_l \cap \text{Stab}(\beta)} (Z_{\beta,l}^{ss} \cap \pi_l^{-1}(\hat{Z}_{R_l}^{ss}))$$

and

$$Y_{\beta,l}^{ss} \cong P_\beta \times q_\beta p_\beta^{-1} \left( Z_{\beta,l}^{ss} \cap w_j^{-1}\pi_l^{-1}(\hat{Z}_{R_l}^{ss}) \right)$$

where

$$Q_\beta = q_\beta^{-1}(N_l \cap \text{Stab}(\beta))$$

is a subgroup of $P_\beta$, and hence

$$S_{\beta,l} \cong G \times_{P_\beta} Y_{\beta,l}^{ss} \cong G \times q_\beta p_\beta^{-1} \left( Z_{\beta,l}^{ss} \cap w_j^{-1}\pi_l^{-1}(\hat{Z}_{R_l}^{ss}) \right).$$

Furthermore $\pi_l$ now restricts to a fibration

$$\pi_l : Z_{\beta,l}^{ss} \cap \pi_l^{-1}(\hat{Z}_{R_l}^{ss}) \rightarrow \hat{Z}_{R_l}^{ss}$$

whose fibre at $x \in \hat{Z}_{R_l}^{ss}$ is $Z_{\beta}^{ss}(\rho_l)$ defined as at Remark 1.4, where $\rho_l$ is the representation of $R_l$ on the normal $N_{l,x}$ to $G\hat{Z}_{R_l}^{ss}$ at $x$.

If $1 \leq j \leq t - 1$ then the proper transform $\hat{E}_j$ in $X_{(R_l)}^{ss}$ of the exceptional divisor $E_j$ in $X_{(R_j)}^{ss}$ meets the exceptional divisor $E_t \cong \mathbb{P}(N_l)$ transversely, and their intersection is the restriction

$$\mathbb{P}(N_l|_{E_j \cap G\hat{Z}_{R_l}^{ss}})$$

of the projective bundle $\mathbb{P}(N_l)$ over $G\hat{Z}_{R_l}^{ss}$ to the intersection in $X_{(R_{l-1})}^{ss}$ of $G\hat{Z}_{R_l}^{ss}$ with the proper transform of $E_j$ in $X_{(R_{l-1})}^{ss}$ (which by abuse of notation we shall also denote by $\hat{E}_j$). Moreover the complement in $G\hat{Z}_{R_l}^{ss}$ of its intersection with the exceptional divisors $\hat{E}_1, ..., \hat{E}_{l-1}$ is just $G\hat{Z}_{R_l}^{ss}$. Thus

$$\Sigma_l = G Y_{\beta,l}^{\setminus E} \cong G \times P_\beta Y_{\beta,l}^{\setminus E}$$

where

$$Y_{\beta,l}^{\setminus E} = Y_{\beta,l}^{ss} \setminus (E_t \cup \hat{E}_{l-1} \cup ... \cup \hat{E}_1)$$

11
fibres over $\text{Stab}(\beta) \times \mathcal{N}_l \cap \text{Stab}(\beta) \left( Z_{\beta, l}^{ss} \cap \pi_l^{-1}(Z_{R_l}^{ss}) \right)$ with fibre $C_{m_{\beta, l}}$ and $Z_{ss}^{\beta, l} \cap \pi_l^{-1}(Z_{R_l}^{ss})$ fibres over $Z_{(R_l)}$ with fibre $Z_{ss}^{\beta}$. In addition, if we set
\[ Y_{\beta} = Y_{\beta, l} \cap p_{\beta}^{-1} \left( Z_{\beta, l}^{ss} \cap \pi_l^{-1}(Z_{R_l}^{ss}) \right) \]
we have from (3.5) and (3.6) that
\[ Y_{\beta} \cong P_{\beta} \times q_{\beta} Y_{\beta} \]
and hence
\[ \Sigma_{\gamma} \cong G \times q_{\beta} Y_{\beta} \]
where $q_{\beta} = q_{\beta}^{-1}(\mathcal{N}_l \cap \text{Stab}(\beta))$ and $p_{\beta} : Y_{\beta} \rightarrow Z_{ss}^{\beta} \cap \pi_l^{-1}(Z_{R_l}^{ss})$ is a fibration with fibre $C_{m_{\beta, l}} \times (\mathbb{C} \setminus \{0\})$.

Remark 3.1 The moduli space $\mathcal{M}(n, d)$ of semistable holomorphic bundles of rank $n$ and degree $d$ over a fixed Riemann surface of genus $g \geq 2$ can be constructed as a quotient of an infinite dimensional affine space of connections $\mathcal{C}$ by a complexified gauge group $G_c$, in an infinite-dimensional version of the construction of quotients in geometric invariant theory, or equivalently as an infinite dimensional symplectic reduction with curvature as a moment map. When $n$ and $d$ are coprime, semistability coincides with stability and $\mathcal{M}(n, d)$ is the topological quotient of the semistable subset $\mathcal{C}^{ss}$ of $\mathcal{C}$ by the action of $G_c$. The role of the normsquare of the moment map is played by the Yang-Mills functional, which was studied by Atiyah and Bott in their fundamental paper [1]. Atiyah and Bott studied the stratification of $\mathcal{C}$ defined using the Harder-Narasimhan type of a holomorphic bundle, which they expected to be the Morse stratification of the Yang-Mills functional (this was later shown to be the case [2]). The methods of this paper can be used to provide a stratification $\{ \Sigma_{\gamma} : \gamma \in \Gamma \}$ of $\mathcal{C}^{ss}$ with $\mathcal{C}^{ss}$ as the unique open stratum. This stratification of $\mathcal{C}^{ss}$ and induced refinements of the Yang-Mills stratification of $\mathcal{C}$ are studied in detail in [3], where they are related to natural refinements of the notion of the Harder-Narasimhan type of a holomorphic bundle.

4 A refined stratification of $X^{ss}$

We can now iterate the construction of the stratification (2.3) described in §2 and use induction on the dimension of $G$ to define a stratification
\[ \{ \Sigma_{\tilde{\gamma}} : \tilde{\gamma} \in \tilde{\Gamma} \} \]
of $X^{ss}$ by $G$-equivariant nonsingular subvarieties which refines the stratification (2.3). When the dimension of $G$ is zero, so that $X^{ss} = X$, then $\tilde{\Gamma} = \Gamma$ and the stratification has one stratum which is $X$ itself. When $\dim G > 0$ then we shall refine the stratification $\{ \Sigma_{\gamma} : \gamma \in \Gamma \}$ defined at (2.3) as follows. If $\gamma \in \Gamma \setminus \{0, R_1, ..., R_{\tau} \}$ then $\gamma = (R_l, \beta)$ where $\beta \in B_1 \setminus \{0\}$ for some $l \in \{1, ..., \tau\}$, and by (3.8) we have
\[ \Sigma_{\gamma} = G Y_{\beta, l}^{\gamma} \cong G \times p_{\beta} Y_{\beta, l}^{\gamma} \]
finally an induced stratification of \( Z_N \) is 
\[
\{ Stab(x) \}
\]
where \( Y^{\gamma E} \) fibres over \( \text{Stab}(\beta) \times_{N_i \cap \text{Stab}(\beta)} (Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s)) \) with fibre \( \mathbb{C}^{m_{\beta,1} - 1} \times (\mathbb{C} \setminus \{0\}) \), and \( Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s) \) fibres over \( Z_{(R_i)}^s \) with fibre \( Z_{\beta}^{ss}(\rho_l) \). We have a linear action of \( R_i \cap \text{Stab}(\beta)/T_\beta^c \) on \( Z_{\beta}^{ss}(\rho_l) \) which corresponds (up to multiplication by a positive integer) to the moment map \( \mu - \beta \). Therefore by induction on \( \dim G \) we can assume that we have defined a stratification \( \{ \Sigma_{\gamma}^{\beta} : \gamma \in \tilde{\Gamma}_{\gamma} \} \) of \( Z_{\beta}^{ss}(\rho_l) \) by nonsingular \( R_i \cap \text{Stab}(\beta) \)-invariant subvarieties. In fact, since the stabiliser in \( G \) of any \( x \in Z_{R_i}^s \) has connected component \( R_i \), we can assume that we have a stratification of \( Z_{\beta}^{ss}(\rho_l) \) by nonsingular \( \text{Stab}(x) \cap \text{Stab}(\beta) \)-invariant subvarieties. Since \( \text{Stab}(x) \subseteq N_i \) and since the fibration 
\[
\pi_{l} : Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s) \rightarrow Z_{R_i}^s
\]
is \( N_i \cap \text{Stab}(\beta) \)-equivariant with fibre \( Z_{\beta}^{ss}(\rho_l) \), this gives us a stratification of \( Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s) \) by nonsingular \( N_i \cap \text{Stab}(\beta) \)-invariant subvarieties, and hence a stratification of 
\[
\text{Stab}(\beta) \times_{N_i \cap \text{Stab}(\beta)} (Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s))
\]
by nonsingular \( \text{Stab}(\beta) \)-invariant subvarieties. We also have a fibration 
\[
p_{\beta} : Y_{\beta,l}^{\gamma E} = Y_{\beta,1}^{\gamma E}(E_l \cup \hat{E}_{l-1} \cup ... \cup \hat{E}_1) \rightarrow \text{Stab}(\beta) \times_{N_i \cap \text{Stab}(\beta)} (Z_{\beta,1}^{ss} \cap \pi_{-1}^s(Z_{R_i}^s))
\]
with fibre \( \mathbb{C}^{m_{\beta,1} - 1} \times (\mathbb{C} \setminus \{0\}) \), which satisfies \( p_{\beta}(gx) = \hat{q}_{\beta}(g)p_{\beta}(x) \) for all \( g \in P_{\beta} \) and \( x \in Y_{\beta,l}^{\gamma E} \) (see (4.3)). Thus we get an induced stratification of \( Y_{\beta,l}^{\gamma E} \) by \( P_{\beta} \)-invariant subvarieties, and finally an induced stratification of 
\[
\Sigma_{\gamma} \cong G \times P_{\beta} Y_{\beta,l}^{\gamma E}
\]
by nonsingular \( G \)-invariant subvarieties \( \tilde{\Sigma}_{\gamma}^{\beta} \) for \( \gamma \in \tilde{\Gamma}_{\gamma} \). In particular \( \Sigma_{\gamma} \) has an open stratum 
\[
\Sigma_{\gamma}^{s} = \tilde{\Sigma}_{\gamma}^{s}_{0}
\]
(4.2)
corresponding to the open stratum \( Z_{\beta}^{s}(\rho_l) \) of \( Z_{\beta}^{ss}(\rho_l) \) consisting of stable points for the action of \( R_i \cap \text{Stab}(\beta)/T_{\beta}^c \). In this way we obtain a stratification \( \{ \tilde{\Sigma}_{\gamma} : \tilde{\gamma} \in \tilde{\Gamma} \} \) of \( X^{ss} \) indexed by 
\[
\tilde{\Gamma} = \{ \tilde{\gamma} = (\gamma) : \gamma \in \{0, R_1, ..., R_{\tau}\} \} \cup \{ \tilde{\gamma} = (R_1, \beta) : 1 \leq l \leq \tau \text{ and } \beta \in B_i \setminus \{0\} \}
\]
\[
\cup \{ \tilde{\gamma} = (R_1, \beta, \gamma_1, ..., \gamma_l) : t \geq 1 \text{ and } 1 \leq l \leq \tau \text{ and } \beta \in B_i \setminus \{0\} \text{ and } (\gamma_1, ..., \gamma_l) \in \tilde{\Gamma}_{(R_1, \beta)} \setminus \{0\} \},
\]
(4.3)
where \( \tilde{\Gamma}_{(R_1, \beta)} \) is defined inductively as above, and the strata \( \tilde{\Sigma}_{\tilde{\gamma}} \) are given by 
\[
\tilde{\Sigma}_{(0)} = X^s
\]
and if \( 1 \leq l \leq \tau \) and \( \beta \in B_i \setminus \{0\} \) then 
\[
\tilde{\Sigma}_{(R_i)} = GZ_{R_i}^s \text{ and } \tilde{\Sigma}_{(R_i, \beta)} = \Sigma_{(R_i, \beta)},
\]
while if \( \tilde{\gamma} = (R_1, \beta, \gamma_1, ..., \gamma_l) \) then 
\[
\tilde{\Sigma}_{\tilde{\gamma}} = \tilde{\Sigma}_{(R_1, \beta, \gamma_1, ..., \gamma_l)}.
\]
5 The refined Morse stratification

In §2 a stratification \( \{ \Sigma_\gamma : \gamma \in \Gamma \} \) of the set \( X^{ss} \) of semistable points of \( X \) was defined, and in §4 this stratification was refined to give a stratification \( \{ \tilde{\Sigma}_\tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma} \} \) of \( X^{ss} \). Via the inductive description of the strata \( S_\beta \) of the Morse stratification of \( \| \mu \|^2 \) in terms of the semistable points of nonsingular subvarieties of \( X \) given in Proposition 1.1, we can use these stratifications of \( X^{ss} \) to refine the Morse stratification.

For simplicity we shall just discuss the stratification \( \{ \tilde{\Sigma}_\tilde{\gamma} : \tilde{\gamma} \in \tilde{\Gamma} \} \) of \( X^{ss} \) constructed in §4. The construction of this stratification can be applied for each \( \beta \in B \) to the action of \( \text{Stab}(\beta) \) on the nonsingular projective subvariety \( Z_\beta \) of \( X \) which appeared in Proposition 1.1(iv) (or more precisely to the action of the quotient \( \text{Stab}(\beta)/T_\beta^c \) of \( \text{Stab}(\beta) \) by its complex subtorus \( T_\beta^c \) which acts trivially on \( Z_\beta \)) to give a stratification

\[
\{ \tilde{\Sigma}_\tilde{\gamma}^{[\beta]} : \tilde{\gamma} \in \tilde{\Gamma}_{[\beta]} \}
\]

of \( Z_\beta^{ss} \) by nonsingular \( \text{Stab}(\beta) \)-invariant subvarieties, with \( Z_0^{ss} \) as the stratum indexed by (0).

Since \( S_\beta = GY_\beta^{ss} \) satisfies (1.4) and we have a retraction \( p_\beta : Y_\beta^{ss} \to Z_\beta^{ss} \) satisfying (1.5), we can stratify \( S_\beta \) as the disjoint union of strata

\[
Gp_\beta^{-1}(\tilde{\Sigma}_\tilde{\gamma}^{[\beta]}) \cong G \times_{p_\beta} p_\beta^{-1}(\tilde{\Sigma}_\tilde{\gamma}^{[\beta]})
\]

for \( \gamma \in \tilde{\Gamma}_{[\beta]} \). This gives us a stratification

\[
\{ \tilde{S}_\tilde{\beta} : \tilde{\beta} \in \tilde{B} \}
\]

of \( X \) indexed by

\[
\tilde{B} = \tilde{\Gamma} \cup \bigcup_{\beta \in B \setminus \{0\}} \{ \beta \} \times \tilde{\Gamma}_{[\beta]}
\]

where \( \tilde{S}_\tilde{\beta} = \tilde{\Sigma}_\tilde{\beta} \) defined as in §4 if \( \tilde{\beta} \in \tilde{\Gamma} \), and if \( \tilde{\beta} = (\beta_1, ..., \beta_t) \) where \( \beta_1 \in B \setminus \{0\} \) and \( (\beta_2, ..., \beta_t) \in \tilde{\Gamma}_{[\beta]} \) then

\[
\tilde{S}_\tilde{\beta} = Gp_{\beta_1}^{-1}(\tilde{\Sigma}_{(\beta_2, ..., \beta_t)}^{[\beta_1]}).
\]

This stratification \( \{ \tilde{S}_\tilde{\beta} : \tilde{\beta} \in \tilde{B} \} \) refines the original stratification \( \{ S_\beta : \beta \in B \} \) and has the following properties.

**Proposition 5.1**  
\begin{itemize}
  \item[i)] Each stratum \( \tilde{S}_\tilde{\beta} \) is a \( G \)-invariant locally closed nonsingular subvariety of \( X \).
  \item[ii)] The unique open stratum \( \tilde{S}_{(0)} \) is the set \( X^s \) of stable points of \( X \).
  \item[iii)] There is a partial ordering \( > \) on \( \tilde{B} \) such that if \( \tilde{\beta} \in \tilde{B} \) then the closure \( \overline{S_{\tilde{\beta}}} \) in \( X \) of the stratum \( \tilde{S}_{\tilde{\beta}} \) satisfies
    \[
    \overline{S_{\tilde{\beta}}} \subseteq \bigcup_{\tilde{\gamma} \geq \tilde{\beta}} \tilde{S}_{\tilde{\gamma}}.
    \]
\end{itemize}
iv) $\tilde{\mathcal{B}}$ has a subset $\tilde{\Gamma}$ such that $\tilde{\gamma} < \tilde{\beta}$ for all $\tilde{\gamma} \in \tilde{\Gamma}$ and $\tilde{\beta} \in \tilde{\mathcal{B}} \setminus \tilde{\Gamma}$ and

$$\bigcup_{\tilde{\gamma} \in \tilde{\Gamma}} \tilde{S}_{\tilde{\gamma}} = X^{ss}.$$ 

v) If $\tilde{\beta} \in \tilde{\mathcal{B}}$ and $\tilde{\beta} \neq (0)$ then the stratum $\tilde{S}_{\tilde{\beta}}$ can be described inductively in terms of the sets of stable points of certain nonsingular linear sections $Z$ of $X$ and projectivised normal bundles of nonsingular subvarieties of $X$, acted on by reducible subgroups of $G$ and their quotients.

**Remark 5.2** If $X$ is a compact Kähler manifold which has a Hamiltonian action of a compact group $K$ with moment map $\mu : X \to \mathfrak{t}^*$, then the Morse stratification for $|\mu|^2$ can be refined just as in Proposition 5.1. Even when $X$ is symplectic but not Kähler we can construct a similar refinement by choosing a suitable almost complex structure and Riemannian metric on $X$ (cf. [10, 16]).

**Example 5.3** Consider the action of $G = SL(2, \mathbb{C})$ and its maximal compact subgroup $K = SU(2)$ on $X = (\mathbb{P}_1)^n$, with the moment map given by the centre of gravity in $\mathbb{R}^3$ when $\mathbb{P}_1$ is identified suitably with the unit sphere in $\mathbb{R}^3$ and $\mathbb{P}_1$ is identified with the Lie algebra of $SU(2)$. An element $(x_1, \ldots, x_n)$ of $(\mathbb{P}_1)^n$ is semistable (respectively stable) for the action of $G$ if and only if at most $n/2$ (respectively strictly fewer than $n/2$) of the points $x_j$ coincide anywhere on $\mathbb{P}_1$. The Morse stratification for the normsquare of the moment map on $X$ has strata $S_0 = X^{ss}$ and $S_{2j-n}$ for $n/2 < j \leq n$. If $n/2 < j \leq n$ then the elements of $S_{2j-n}$ correspond to sequences of $n$ points on $\mathbb{P}_1$ such that exactly $j$ of these points coincide somewhere on $\mathbb{P}_1$, and $S_{2j-n}$ retracts equivariantly onto the subset of $X$ where $j$ points coincide somewhere on $\mathbb{P}_1$ and the remaining $n - j$ points coincide somewhere else on $\mathbb{P}_1$, which is a single $G$-orbit with stabilizer $\mathbb{C}^*$, for $j < n$, and with stabiliser a Borel subgroup of $G$ when $j = n$ (see [2] §16.1 for more details).

If $n$ is odd then semistability coincides with stability and the refined stratification coincides with the Morse stratification of $X$. Now suppose that $n$ is even, so that semistability and stability do not coincide. The semistable elements of $X$ which are fixed by nontrivial connected reductive subgroups of $G$ are those represented by sequences $(x_1, \ldots, x_n)$ of points of $\mathbb{P}_1$ such that there exist distinct $p$ and $q$ in $\mathbb{P}_1$ with exactly half of the points $x_1, \ldots, x_n$ equal to $p$ and the rest equal to $q$. They form $n!/2((n/2)!)^2$ $G$-orbits, and their stabilisers are all conjugate to the maximal torus $T_c = \mathbb{C}^*$ of $G$. These stabilisers act with weights 2 and $-2$, each with multiplicity $(n/2) - 1$, on the normals to the orbits. We obtain the partial desingularization $\tilde{X} \setminus G$ by blowing up $X \setminus G$ at the points corresponding to these orbits, or equivalently by blowing up $X^{ss}$ along these orbits, removing from the blowup the unstable points (which form the proper transform of the set of $(x_1, \ldots, x_n) \in X^{ss}$ such that exactly half of the points $x_1, \ldots, x_n$ coincide somewhere on $\mathbb{P}_1$) and finally quotienting by $G$. The refined stratification $\{\tilde{S}_{\tilde{\beta}} : \tilde{\beta} \in \tilde{\mathcal{B}}\}$ of $X$ thus has as its strata the set $\tilde{S}_{(0)} = X^s$ of stable points, the set $\tilde{S}_{(T)}$ consisting of points represented by sequences $(x_1, \ldots, x_n)$ in $\mathbb{P}_1$ such that there exist distinct $p$ and $q$ in $\mathbb{P}_1$ with exactly half of $x_1, \ldots, x_n$ equal to $p$ and the rest equal to $q$, the set $\tilde{S}_{(T,2)}$ consisting of points represented by sequences $(x_1, \ldots, x_n)$ in $\mathbb{P}_1$ such that there exists $p$ in $\mathbb{P}_1$ with exactly half of the points $x_1, \ldots, x_n$ equal to $p$ and the rest different from $p$ and not all equal to each other, and finally the strata $S_{2j-n}$ (for $n/2 < j \leq n$) of the Morse stratification.
Example 5.4  A very similar example is given by the action of $G = SL(2; \mathbb{C})$ on $X = \mathbb{P}_n$ identified with the space of unordered sequences of $n$ points in $\mathbb{P}_1$; that is, with the projectivized symmetric product $\mathbb{P}(S^n(\mathbb{C}^2))$ (see [9] §16.2 for more details). The diagonal subgroup $T \cong S^1$ of $K = SU(2)$ acts with weights $n, n-2, n-4, \ldots, 2-n, -n$ on $S^n(\mathbb{C}^2) = \mathbb{C}^{n+1}$. An element $[a_0, \ldots, a_n]$ of $\mathbb{P}_n$ corresponds to the $n$ roots in $\mathbb{P}_1$ of the polynomial in one variable $t$, say, with coefficients $a_0, \ldots, a_n$: it is semistable (respectively stable) for the action of $G$ if and only if at most $n/2$ (respectively strictly fewer than $n/2$) of these roots coincide anywhere on $\mathbb{P}_1$, and the Morse stratification for the normsquare of the moment map on $X$ is essentially the same as in Example 5.3 above. Again when $n = 2m$ is even the semistable elements of $X$ which are fixed by nontrivial connected reductive subgroups of $G$ are those represented by polynomials such that there exist distinct $p$ and $q$ in $\mathbb{P}_1$ with exactly half the roots equal to $p$ and the rest equal to $q$. They form a single $G$-orbit, and the stabiliser $\mathbb{C}^*$ acts with weights $\pm 4, \pm 6, \ldots, \pm n = \pm 2m$ on the normal to the orbit. The refined stratification $\{\tilde{S}_\beta : \beta \in \tilde{B}\}$ of $X$ this time has as its strata the set $\tilde{S}_{(0)} = X^s$ of stable points, the set $\tilde{S}_{(T)}$ represented by polynomials with exactly two distinct roots each with multiplicity $m = n/2$, for $2 \leq k \leq m$ the set $\tilde{S}_{(T;2k)}$ represented by polynomials in the orbit of one of the form

$$a_m t^m + a_{m+k} t^{m+k} + a_{m+k+1} t^{m+k+1} + \cdots + a_{2m} t^{2m}$$

where $a_m$ and $a_{m+k}$ are nonzero, and finally the strata $S_{2j-n}$ (for $n/2 < j \leq n$) of the Morse stratification.

Example 5.5  For a more complicated example consider the action of $G = SL(3; \mathbb{C})$ and its maximal compact subgroup $K = SU(3)$ on $X = (\mathbb{P}_2)^n$. A sequence $(x_1, \ldots, x_n)$ of points in $\mathbb{P}_2$ is semistable if and only if there is no projective line $L$ in $\mathbb{P}_2$ such that

$$\left| \frac{\{j : x_j \in L\}}{2} \right| > \frac{n}{3}$$

(5.6)

and no point $p \in \mathbb{P}_2$ such that

$$|\{j : x_j = p\}| > \frac{n}{3},$$

(5.7)

and is stable if we can replace $>$ with $\geq$ in (5.6) and (5.7) (see for example [9] (16.5)). The stratification $\{S_\beta : \beta \in \mathcal{B}\}$ can be described as follows (see [9] Proposition 16.9). Any $(x_1, \ldots, x_n) \in (\mathbb{P}_2)^n$ which is not semistable determines a unique flag

$$0 = M_0 \subset \cdots \subset M_s = \mathbb{C}^3$$

in $\mathbb{C}^3$ with $s = 2$ or $3$, such that if $1 \leq i \leq s$ then

$$\frac{k_1}{m_1} > \cdots > \frac{k_s}{m_s}$$

where

$$k_i = |\{j : x_j \in M_i \setminus M_{i-1}\}|$$

and $m_i = \dim(M_i/M_{i-1})$. 

16
and moreover if $m_i = 2$ then those $x_j$ lying in $M_i \setminus M_{i-1}$ determine a semistable element of $(\mathbb{P}_1)^{k_i}$ after projection into $\mathbb{P}(M_i/M_{i-1}) \cong \mathbb{P}_1$. Then $(x_1, \ldots, x_n)$ lies in the stratum labelled by the projection into $\text{Lie}(SU(3))$ of the vector

\[ \beta = \left( \frac{k_1}{m_1}, \ldots, \frac{k_s}{m_s} \right) \in \text{Lie}(U(3)) \]

in which each $k_i/m_i$ appears $m_i$ consecutive times. Thus $B$ is the projection into the Lie algebra $\text{Lie}(SU(3))$ of

\[ \left\{ \left( \frac{n}{3}, \frac{n}{3}, \frac{n}{3} \right) \right\} \cup \left\{ \left( \frac{k}{2}, \frac{k}{2}, n-k \right) : \frac{2n}{3} < k \leq n \right\} \cup \left\{ \left( \frac{k}{2}, \frac{n-k}{2}, \frac{n-k}{2} \right) : \frac{n}{3} < k \leq n \right\} \]

\[ \cup \left\{ (k_1, k_2, n-k_1-k_2) : n-k_1-k_2 < k_2 < k_1 \leq n \right\}. \]

With a slight abuse of notation (identifying $\beta \in \text{Lie}(U(3))$ with its projection into $\text{Lie}(SU(3))$) we have

\[ S_{\left( \frac{n}{3}, \frac{n}{3}, \frac{n}{3} \right)} = X^{ss}, \]

while if $2n/3 < k \leq n$ then $(x_1, \ldots, x_n)$ belongs to $S_{(k/2,k/2,n-k)}$ if and only if there is a line $L$ in $\mathbb{P}_2$ containing exactly $k$ of the points $x_j$ and at most $k/2$ of these points coincide anywhere on $L$. If $n/3 < k \leq n$ then $(x_1, \ldots, x_n)$ belongs to $S_{(k,(n-k)/2,(n-k)/2)}$ if and only if exactly $k$ of the points $x_j$ coincide at some $p \in \mathbb{P}_2$, and any line through $L$ contains at most $(n-k)/2$ of the remaining points. Finally if $n-k_1-k_2 < k_2 < k_1 \leq n$ then $(x_1, \ldots, x_n)$ belongs to $S_{(k_1,k_2,n-k_1-k_2)}$ and if and only if there is a line $L$ in $\mathbb{P}_2$ and a point $p \in L$ such that exactly $k_1$ of the points $x_j$ coincide at $p$ and exactly $k_2$ additional points lie on $L$.

The strata $S_\beta$ which need refining to obtain the stratification $\{ S_\beta : \beta \in B \}$ are $S_{(k/2,k/2,n-k)}$ when $k$ is even and $S_{(k,(n-k)/2,(n-k)/2)}$ when $n-k$ is even, together with $X^{ss}$ if $n$ is a multiple of 3.

If $k$ is even we have

\[ S_{\left( \frac{k}{2}, \frac{k}{2}, n-k \right)} = \tilde{S}_{\left( \frac{k}{2}, \frac{k}{2}, n-k \right)} \sqcup \tilde{S}_{\left( \frac{k}{2}, \frac{k}{2}, n-k, T_1 \right)} \sqcup \tilde{S}_{\left( \frac{k}{2}, \frac{k}{2}, n-k, T_1, 3 \right)} \]

where $T_1 = \{ (t, t, t^2) : t \in \mathbb{C}^* \}$. Here $(x_1, \ldots, x_n)$ belongs to the stratum $\tilde{S}_{(k/2,k/2,n-k)}$ if and only if there is a line $L$ in $\mathbb{P}_2$ containing exactly $k$ of the points $x_j$ and at most $k/2 - 1$ of these points coincide anywhere on $L$, while $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(k/2,k/2,n-k,T_1)}$ if exactly $k/2$ of these points coincide somewhere on $L$ and the remaining $k/2$ coincide somewhere else, and $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(k/2,k/2,n-k,T_1,3)}$ if exactly $k/2$ of these points coincide somewhere on $L$ and the remaining $k/2$ on $L$ do not all coincide.

If $n-k$ is even we have

\[ S_{\left( \frac{n-k}{2}, \frac{n-k}{2}, \frac{n-k}{2} \right)} = \tilde{S}_{\left( \frac{n-k}{2}, \frac{n-k}{2}, \frac{n-k}{2} \right)} \sqcup \tilde{S}_{\left( \frac{n-k}{2}, \frac{n-k}{2}, \frac{n-k}{2}, T_2 \right)} \sqcup \tilde{S}_{\left( \frac{n-k}{2}, \frac{n-k}{2}, \frac{n-k}{2}, T_2, 3 \right)} \]

where $T_2 = \{ (t^2, t, t) : t \in \mathbb{C}^* \}$. Here $(x_1, \ldots, x_n)$ belongs to the stratum $\tilde{S}_{(k,(n-k)/2,(n-k)/2)}$ if and only if there is some point $p$ in $\mathbb{P}_2$ where exactly $k$ of the points $x_j$ coincide, and no line through $p$ contains at least $(n-k)/2$ of the remaining points $x_j$, while $(x_1, \ldots, x_n)$ belongs
to $\tilde{S}_{(k,(n-k)/2,(n-k)/2,T_2)}$ if there are lines $L_1$ and $L_2$ meeting at $p$ and each containing exactly $(n-k)/2$ of the remaining points $x_j$, and $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(k,(n-k)/2,(n-k)/2,T_2,3)}$ if there is a line $L$ through $p$ containing exactly $(n-k)/2$ of the remaining points $x_j$ but no other line through $p$ contains at least $(n-k)/2$ of these remaining points.

Finally if $n$ is divisible by 3 then we have

$$X^{ss} = X^* \sqcup \tilde{S}_{(T)} \sqcup \{ \tilde{S}_{(T,\beta)} : \beta \in B_T \setminus \{0\} \} \sqcup \tilde{S}_{(T_1)} \sqcup \tilde{S}_{(T_1,3)} \sqcup \tilde{S}_{(T_1,-3)}$$

where $T$ is the standard maximal torus of $SU(3)$ and $B_T \setminus \{0\}$ can be identified with the set of four vectors

$$\{(1,0,-1), \left(\frac{1}{2},0,-\frac{1}{2}\right), \left(1,-\frac{1}{2},-\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2},-\frac{3}{2}\right)\}.$$ 

Here $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(T)}$ if there are three points $p_1, p_2, p_3 \in \mathbb{P}_2$ such that exactly $n/3$ of the points $x_j$ coincide at $p_i$ for $i = 1, 2, 3$, while $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(T_1)}$ if there is a point $p \in \mathbb{P}_2$ and a line $L$ not containing $p$ such that exactly $n/3$ of the points $x_j$ coincide at $p$ and the remaining $2n/3$ lie on $L$ with strictly fewer than $n/3$ coinciding anywhere on $L$. Also $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(T_1,3)}$ if there is a line $L$ in $\mathbb{P}_2$ containing exactly $2n/3$ of the points $x_j$ and strictly fewer than $n/3$ points coincide anywhere on $\mathbb{P}_2$, while $(x_1, \ldots, x_n)$ belongs to $\tilde{S}_{(T_1,-3)}$ if there is a point $p \in \mathbb{P}_2$ where exactly $n/3$ of the points $x_j$ coincide, and strictly fewer than $2n/3$ of the points $x_j$ lie on any line in $\mathbb{P}_2$.

Finally, if $\beta \in B_T$ and $(x_1, \ldots, x_n) \in \tilde{S}_{(T,\beta)}$ then there is some $p \in \mathbb{P}_2$ and a line $L$ through $p$ such that exactly $n/3$ of the points $x_j$ coincide at $p$ and exactly $2n/3$ lie on $L$, and

(a) when $\beta = (1/2,0,-1/2)$ then $p$ and $L$ are unique;

(b) when $\beta = (1/2,1/2,-1)$ then $L$ is unique but there is another point $p'$ on $L$ where $n/3$ of the points $x_j$ coincide;

(c) when $\beta = (1,-1/2,-1/2)$ then $p$ is unique but there is another line $L'$ through $p$ containing exactly $2n/3$ of the points $x_j$;

(d) when $\beta = (1,0,-1)$ then there is another point $p' \in \mathbb{P}_2 \setminus L$ where exactly $n/3$ of the points $x_j$ coincide, but the $n/3$ points $x_j$ lying on $L \setminus \{p\}$ do not all coincide.

**Example 5.6** We can generalise Examples 5.3 and 5.5 by considering the action of $SL(m; \mathbb{C})$ on a product of Grassmannians

$$X = \prod_{j=1}^r \text{Grass}(l_j, \mathbb{C}^m)$$

where Grass$(l, \mathbb{C}^m)$ denotes the Grassmannian of $l$-dimensional linear subspaces of $\mathbb{C}^m$ and we linearise the action by using the Plücker embedding. The Morse stratification $\{ S_{\beta} : \beta \in B \}$ is described in [9] §16.3, and the refinement $\tilde{S}_{\beta} : \tilde{\beta} \in \tilde{B}$ can be calculated by adapting the methods used in [13] (especially [13] §5) to refine the Yang–Mills stratification.
6 Refinements when G is abelian

The refinements we have considered so far of the Morse stratification \( \{ S_\beta : \beta \in \mathcal{B} \} \) for \( \| \mu \|^2 \) are unfortunately unlikely to be equivariantly perfect. When \( G = T^c \) is abelian there is another way to refine this stratification which does lead to equivariantly perfect stratifications.

If \( G \) is abelian then there is a partial desingularisation \( X//G \) of \( X/G \) obtained by perturbing the linearisation of the action of \( G \) on \( X \), or equivalently by replacing the moment map \( \mu : X \to \mathfrak{t}^* \) by \( \mu - \epsilon \) where \( \epsilon \in \mathfrak{t}^* \) is a generic constant close to 0. Since \( G \) is abelian the coadjoint action of \( K = T \) on \( \mathfrak{t}^* = \mathfrak{t}^c \) is trivial and \( \mu - \epsilon \) is an equivariant moment map for the action of \( K \) on \( X \).

Recall from §1 that the Morse stratification \( \{ S_\beta : \beta \in \mathcal{B} \} \) for \( \| \mu \|^2 \) is indexed by the closest points to 0 in the convex hulls of nonempty subsets of the set of weights \( \\{ \alpha_0, \ldots, \alpha_n \} \), and \( x = [x_0 : \ldots : x_n] \in X \subseteq \mathbb{P}_n \) lies in the stratum \( S_\beta \) indexed by the closest point \( \beta \) to 0 in the convex hull of
\[
\{ \alpha_j : x_j \neq 0 \}.
\]
Similarly the Morse strata \( \{ S^\epsilon_\beta : \beta \in \mathcal{B}^c \} \) for \( \| \mu - \epsilon \|^2 \) correspond to the closest points to \( \epsilon \) in such convex hulls. More precisely \( x = [x_0 : \ldots : x_n] \in X \) lies in the stratum \( S^\epsilon_\beta \) indexed by the closest point \( \beta \) to 0 in the convex hull of \( \{ \alpha_j - \epsilon : x_j \neq 0 \} \); then \( \beta + \epsilon \) is the closest point to \( \epsilon \) in the convex hull of
\[
\{ \alpha_j : x_j \neq 0 \}.
\]
If \( \beta \in \mathcal{B} \) does not lie in the convex hull of a proper subset of \( \{ \alpha_j : \alpha_j, \beta = \| \beta \|^2 \} \), then for sufficiently small \( \epsilon \) we have
\[
S_\beta = S^\epsilon_{\beta(\epsilon)}
\]
for some small perturbation \( \beta(\epsilon) \) of \( \beta \), where \( \beta(\epsilon) + \epsilon \) is the closest point to \( \epsilon \) of the convex hull of \( \{ \alpha_j : \alpha_j, \beta = \| \beta \|^2 \} \). Otherwise for sufficiently small \( \epsilon \) the stratum \( S_\beta \) in the Morse stratification for \( \| \mu \|^2 \) is the union of \( S^\epsilon_{\beta(\epsilon)} \) and other strata in the Morse stratification for \( \| \mu - \epsilon \|^2 \), which correspond to the closest points to \( \epsilon \) of those subsets of \( \{ \alpha_j : \alpha_j, \beta = \| \beta \|^2 \} \) whose closest point to 0 is \( \beta \). Moreover if \( \beta \in \mathcal{B}^c \) and \( \epsilon \) is chosen generically we can assume that \( \beta \) does not lie in the convex hull of any proper subset of \( \{ \alpha_j : (\alpha_j - \epsilon), \beta = \| \beta \|^2 \} \). This means that every point of
\[
Z^\epsilon_\beta = \{ x \in X : x_j = 0 \text{ unless } (\alpha_j - \epsilon), \beta = \| \beta \|^2 \}
\]
which is semistable for the action of the subgroup of \( G \) whose Lie algebra is spanned by
\[
\{ \alpha_j - \epsilon - \beta : (\alpha_j - \epsilon), \beta = \| \beta \|^2 \}
\]
is also stable for the action of this subgroup. Moreover the set of semistable points for the action of this subgroup is the same as the set \( Z^{\epsilon,ss}_\beta \) of semistable points of \( Z^\epsilon_\beta \) under the action of the subgroup of \( G \) whose Lie algebra is the complexification of the orthogonal complement to \( \beta \) in \( \mathfrak{t} \) (cf. Remark 1.3).

Thus we have a refinement of the Morse stratification for \( \| \mu \|^2 \) which is both equivariantly perfect and has the property that all strata can be described in terms of the sets of semistable points of nonsingular closed subvarieties \( Z^\epsilon_\beta \) of \( X \) under suitable linear actions of reductive subgroups of \( G \) for which semistability coincides with stability.
Proposition 6.1 If $G = T^c$ is abelian and $\epsilon \in t^* = \mathfrak{t}^*$ is chosen generically and sufficiently close to 0 then the Morse stratification $\{S^c_\beta : \beta \in \mathcal{B}^s\}$ for $\|\mu - \epsilon\|^2$ is an equivariantly perfect refinement of the Morse stratification $\{S^s_\beta : \beta \in \mathcal{B}\}$ for $\|\mu\|^2$. In addition if $\beta \in \mathcal{B}^s$ then we have

$$S^c_\beta = Y^c,ss_\beta = p^{-1}_\beta(Z^c,ss_\beta)$$

and

$$H^*_G(S^c_\beta) = H^*_G(Z^c,ss_\beta)$$

where

$$Z^c_\beta = \{x \in X : x_j = 0 \text{ unless } (\alpha_j - \epsilon), \beta = \|\beta\|^2\},$$

$$Y^c_\beta = \{x \in X : x_j = 0 \text{ if } (\alpha_j - \epsilon), \beta < \|\beta\|^2 \text{ and } x_j \neq 0 \text{ for some } j$$

such that $(\alpha_j - \epsilon), \beta = \|\beta\|^2$}

and

$$p_\beta(x) = \lim_{t \to \infty} \exp(-it\beta)x.$$ 

Furthermore $Z^c,ss_\beta$ is the set of semistable points for the action on $Z^c_\beta$ of the subgroup of $G$ whose Lie algebra is spanned by

$$\{\alpha_j - \epsilon - \beta : (\alpha_j - \epsilon), \beta = \|\beta\|^2\},$$

and every semistable point for this action is stable.

Remark 6.2 When $G$ is abelian and $X^{ss} = X^s$ we can obtain the same result by perturbing the inner product on $t = \mathfrak{t}$ instead of perturbing the moment map from $\mu$ to $\mu - \epsilon$.

Remark 6.3 Yet another possible procedure when $G = (\mathbb{C}^*)^r$ is abelian is to use induction on $r$ to reduce to the simple case when $G = \mathbb{C}^*$.

Example 6.4 Consider the action of the maximal torus

$$T^c = \{(t, t^{-1}) : t \in \mathbb{C}^*\} \cong \mathbb{C}^*$$

of $G = SL(2; \mathbb{C})$ acting on $X = \mathbb{P}_n$ or $X = (\mathbb{P}_1)^n$ as in Examples 5.3 and 5.5. An element of $X$ represented by a sequence $(x_1, \ldots, x_n)$ of points of $\mathbb{P}_1$ is semistable (respectively stable) for $T^c$ if and only if at most $n/2$ (respectively strictly fewer than $n/2$) of the points $x_j$ coincide at 0 or $\infty$, and the Morse stratification $\{S^T_\beta : \beta \in \mathcal{B}^T\}$ for the norm square of the moment map $\mu_T : X \to t$ is indexed by

$$\{0\} \cup \{2j - n : 0 \leq j \leq n\}$$

with $(x_1, \ldots, x_n)$ representing a point of $S^T_{2j-n}$ when exactly $j$ of the points $x_i$ coincide at 0 if $j > n/2$, and exactly $n - j$ of the points $x_i$ coincide at $\infty$ if $j < n/2$. When $n$ is odd this stratification is unchanged if we perturb the linearisation slightly. When $n$ is even the only change is that $S^T_0$ becomes two strata: the subset represented by $(x_1, \ldots, x_n)$ with exactly $n/2$ of the points coinciding at 0 (or at $\infty$, depending on the sign of the perturbation), and its complement in $S^T_0$. This contrasts with the refinement $\{\tilde{S}^T_\beta : \tilde{\beta} \in \tilde{\mathcal{B}}^T\}$ in which $S^T_0$ is subdivided into three strata when $X = (\mathbb{P}_1)^n$ and more when $X = \mathbb{P}_n$ (cf. Examples 5.3 and 5.0).
Example 6.5 When the maximal torus $T^c \cong (\mathbb{C}^*)^2$ of $SL(3; \mathbb{C})$ acts on $X = (\mathbb{P}_2)^n$ then the Morse stratification $\{S_T^\beta : \beta \in \mathcal{B}^T\}$ for $\|\mu_T\|^2$ and its refinement $\{\tilde{S}_T^\beta : \tilde{\beta} \in \tilde{\mathcal{B}}^T\}$ defined as in §5 can be described in a way closely analogous to the stratifications $\{S_\beta : \beta \in \mathcal{B}\}$ and $\{\tilde{S}_\beta : \tilde{\beta} \in \tilde{\mathcal{B}}\}$ associated to the action of $G = SL(3; \mathbb{C})$ which were described in Example 5.6. The difference is that the points $p$ and lines $L$ which appear in the description are $[1 : 0 : 0]$, $[0 : 1 : 0]$ or $[0 : 0 : 1]$ or one of the lines joining these points. A generic perturbation of the linearisation provides a coarser refinement of $\{S_T^\beta : \beta \in \mathcal{B}^T\}$ than the stratification $\{\tilde{S}_T^\beta : \tilde{\beta} \in \tilde{\mathcal{B}}^T\}$. If $k$ or $n - k$ is even then the strata $S_T^\beta$ with indices $\beta$ in the Weyl group orbit of $(k/2, k/2, n - k)$ or $(k, (n - k)/2, (n - k)/2)$ decompose as the disjoint union of two refined strata instead of the three in the refinement $\{\tilde{S}_T^\beta : \tilde{\beta} \in \tilde{\mathcal{B}}^T\}$, in a way which is directly analogous to the decomposition of $S_0^0$ in Example 6.4. Similarly if $n$ is divisible by 3 then $S_0^0$ decomposes into a smaller number of strata than in the stratification $\{\tilde{S}_T^\beta : \tilde{\beta} \in \tilde{\mathcal{B}}^T\}$; in particular the analogues of the strata $\tilde{S}_{(T)}$ and $\tilde{S}_{(T_1)}$ described in Example 5.9 are amalgamated with higher dimensional strata.

7 Reduction to a maximal torus

The Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of the norm square of the moment map is useful for studying the cohomology of the GIT quotient $X//G$ because it is equivariantly perfect over the rationals; that is, if $\lambda(\beta)$ is the real codimension of $S_\beta$ in $X$ and if $U_\beta$ is the open subset

$$U_\beta = S_\beta \cup \bigcup_{1^\beta < 1^\beta} S_{\beta'}$$

which contains $S_\beta$ as a closed subset, then the Thom–Gysin long exact sequences

$$\ldots \rightarrow H^*_G(U_\beta) \rightarrow H^*_G(U_\beta \setminus S_\beta) \rightarrow \ldots$$

break up into short exact sequences

$$0 \rightarrow H^*_{-\lambda(\beta)}(S_\beta) \rightarrow H^*_G(U_\beta) \rightarrow H^*_G(U_\beta \setminus S_\beta) \rightarrow 0$$

of equivariant cohomology with rational coefficients.

Remark 7.1 This happens because the composition of the Thom–Gysin map $TG_\beta$ with the restriction map $H^*_G(U_\beta) \rightarrow H^*_G(S_\beta)$ is given by multiplication by the equivariant Euler class $e_\beta$ of the normal bundle to $S_\beta$ in $U_\beta$, and it follows from a criterion of Atiyah and Bott §13 that $e_\beta$ is not a zero-divisor in $H^*_G(S_\beta)$. Note that the restriction maps from $H^*_G(X)$ to $H^*_G(U_\beta)$ for $\beta \in \mathcal{B} \setminus \{0\}$ are compositions of restriction maps from $H^*_G(U_{\beta'})$ to $H^*_G(U_{\beta'} \setminus S_{\beta'})$ which are all surjective. In particular the cohomology ring $H^*_G(X^{ss})$ (which in the good case when $X^{ss} = X^*$ is isomorphic to the rational cohomology ring of the geometric invariant theoretic quotient $X//G$ or equivalently the Marsden–Weinstein reduction $\mu^{-1}(0)/K$) is isomorphic to the quotient of $H^*_G(X)$ by the kernel of the restriction map $\rho : H^*_G(X) \rightarrow H^*_G(X^{ss})$. 

21
Unfortunately the refinement \( \tilde{S}_\beta : \beta \in \tilde{B} \) described in §5 of \( \{ S_\beta : \beta \in B \} \) is not in general equivariantly perfect. It is still the case that we have Thom–Gysin long exact sequences for equivariant cohomology, and the kernels of the restriction maps from \( H^*_G(X) \) to \( H^*_G(X^{ss}) \) and to \( H^*_G(X^s) \) can be described in terms of the images of the associated Thom–Gysin maps, but the description is not as clean as in the equivariantly perfect case, and the restriction map to \( H^*_G(X^s) \) is not necessarily surjective (see Example 7.16 below).

However when \( G \) is abelian the alternative refinement \( \{ S'_\beta : \beta \in B' \} \) described in §6 is equivariantly perfect. We can exploit this fact even when \( G \) is not abelian by making use of the close relationship between the restriction maps 
\[
\rho : H^*_G(X) \to H^*_G(X^{ss})
\]
and 
\[
\rho_T : H^*_T(X) \to H^*_T(X^{ss,T})
\]
where \( T^c \) is a maximal torus of \( G \) (we can assume that it is the complexification of \( T = T^c \cap K \) which is a maximal torus of \( K \)) and \( X^{ss,T} \) is the set of \( T^c \)-semistable points of \( X \). Therefore in this final section we shall describe this relationship.

The kernel of \( \rho \) can be described in several different ways. The first follows immediately from the fact that the stratification \( \{ S_\beta : \beta \in B \} \) is equivariantly perfect.

**Lemma 7.2** For \( \beta \in B \setminus \{ 0 \} \) let \( \overline{T_G}_\beta : H^{-\lambda(\beta)}_G(S_\beta) \to H^*_G(X) \) be any lift to \( X \) of the Thom–Gysin map \( T_G_\beta : H^{-\lambda(\beta)}_G(S_\beta) \to H^*_G(U_\beta) \). Then
\[
\ker \rho = \bigoplus_{\beta \in B \setminus \{ 0 \}} \text{im} \overline{T_G}_\beta.
\]

**Remark 7.3** Note that such lifts always exist by Remark 7.1.

Another closely related description is given in [3, 4, 12].

**Lemma 7.4** Let \( \mathcal{R} \) be a subset of \( \ker \rho \) such that for every \( \beta \in B \setminus \{ 0 \} \) and every \( \eta \in H^*_G(S_\beta) \) there exists \( \zeta \in \mathcal{R} \) such that
\[
\zeta|_{S_\gamma} = 0 \text{ unless } \| \gamma \| \geq \| \beta \|
\]
and
\[
\zeta|_{S_\beta} = \eta e_\beta
\]
where \( e_\beta \) is the equivariant Euler class of the normal bundle to \( S_\beta \). Then \( \mathcal{R} \) spans \( \ker \rho \).

In the good case when \( X^{ss} = X^s \) a rather different description follows from the formulas for the intersection pairings in \( H^*(X/G) \) given in [8] (see also [7, 14, 15, 20]).

**Lemma 7.5** If \( X^{ss} = X^s \) and \( \eta \in H^*_G(X) \) then \( \eta \in \ker \rho \) if and only if
\[
\text{res} \left( \mathcal{P}^2 \sum_{F \in \mathcal{F}} e^{hF} \int_F (\eta \zeta)|_F e_F \right) = 0
\]
for all \( \zeta \in H^k_G(X) \) with \( j + k = \dim_R(X//G) \). Here \( \mathcal{F} \) is the set of connected components \( F \) of the fixed point set of \( T \) in \( X \) and \( e_F \in H^*_T(X) \) is the \( T \)-equivariant Euler class of the normal bundle to \( F \) in \( X \). Also \( \mu_F : t \to \mathbb{R} \) denotes the linear function on \( t \) given by the constant value in \( t^* \) taken by the moment map \( \mu \) on \( F \in \mathcal{F} \), and the polynomial \( D : t \to \mathbb{R} \) is defined by

\[
D = \prod_{\gamma > 0} \gamma,
\]

where \( \gamma \) runs over the positive roots of \( K \).

**Proof**: This follows immediately from Poincaré duality for the orbifold \( X//G \) and the surjectivity of

\[
\rho : H^*_G(X) \to H^*_G(X^{ss}) \cong H^*(X//G)
\]

together with [8] Theorem 8.1, which tells us that, up to multiplication by a nonzero constant, the formula in Lemma 7.5 is the intersection pairing

\[
\int_{X//G} \rho(\eta)\rho(\zeta)
\]

of \( \rho(\eta) \) and \( \rho(\zeta) \) in \( X//G \).

**Remark 7.6** The multivariable residue \( \text{res} \) which appears in Lemma 7.5 above is a linear map defined on a class of meromorphic differential forms on \( t \otimes \mathbb{C} \). In order to apply it to the individual terms in the residue formula it is necessary to make some choices which do not affect the residue of the whole sum. Once the choices have been made, many of the terms in the sum contribute zero and the formula can be rewritten as a sum over a subset \( \mathcal{F}_+ \) of the set \( \mathcal{F} \) of components of the fixed point set \( X^T \), consisting of those \( F \in \mathcal{F} \) on which the constant value taken by \( \mu_T \) lies in a certain cone with its vertex at 0. When \( \dim T = 1 \) and \( t \) is identified with \( \mathbb{R} \), we can take

\[
\mathcal{F}_+ = \{ F \in \mathcal{F} : \mu_T(F) > 0 \}
\]

and if we replace \( \mathcal{F} \) with \( \mathcal{F}_+ \) in the formula in Lemma 7.5 then we can interpret \( \text{res} \) as the usual residue at 0 of a rational function in one variable.

**Remark 7.7** When \( G \) is abelian we have analogues of Lemmas 7.2 and 7.4 for the refinement \( \{ S_\beta : \beta \in B' \} \) of the stratification \( \{ S_\beta : \beta \in B \} \) obtained by perturbing the moment map \( \mu \) to \( \mu - \epsilon \) (see §6). We also have the following description of \( \ker \rho \) due to Tolman and Weitsman [19]. We can use these results even when \( G \) is not abelian via Lemma 7.10 below.

**Lemma 7.8** When \( G = T^c \) is abelian and \( X^{ss} = X^s \) then

\[
\ker \rho = \sum_{\xi \in t} \{ \eta \in H^*_T(X) : \eta|_{F \cap X^\xi} = 0 \text{ for all } F \in \mathcal{F} \}
\]

where

\[
X^\xi = \{ x \in X : \mu(x)(\xi) \leq 0 \}
\]

if \( \xi \in t \).
Recall that the Weyl group $W$ of $K$ acts faithfully on the Lie algebra $t$ of $T$ and is generated by reflections. For any $w \in W$ we denote by $(-1)^w$ the determinant of $w$ regarded as an automorphism of $t$. If $W$ acts on a module $M$ then

$$M^W = \{ m \in M : wm = m \text{ for all } w \in W \}$$

is the set of $W$-invariants in $M$ and

$$M^{\text{anti}W} = \{ m \in M : wm = (-1)^w m \text{ for all } w \in W \}$$

is the set of anti-invariants for the action of $W$ on $M$. There is a natural identification of $H^*_G(X)$ with $[H^*_T(X)]^W$, which in the case when $X$ is a point becomes a natural identification of $H^*(BG)$ with $[H^*(BT)]^W$ where $H^*(BT)$ is the $\mathbb{Q}$-algebra of polynomial functions on $t$. There is (up to sign) one fundamental anti-invariant in $H^*(BT)$, which is the product $D$ of the positive roots of $K$, and we have the following wellknown facts (see for example [5] Lemma 1.2).

**Lemma 7.9**

(i) $[H^*(BT)]^{\text{anti}W}$ is a free $[H^*(BT)]^W$-module of rank one generated by $D$.

(ii) $[H^*(BT)]^{\text{anti}W}$ is a direct summand of $H^*(BT)$ as an $[H^*(BT)]^W$-module, with splitting given by

$$\eta \mapsto \frac{1}{|W|} \sum_{w \in W} (-1)^w w\eta.$$

Note that $H^*_T(X)$ has analogous properties, since it is isomorphic to $H^*(BT) \otimes H^*(X)$ as an $H^*(BT)$-module (see [9] Proposition 5.8), though not as a ring.

**Lemma 7.10** Suppose that $X^s = X^s$. If $\eta \in H^*_G(X) \cong [H^*_T(X)]^W$ then the following are equivalent:

(i) $\eta \in \ker \rho$;

(ii) $\eta D \in \ker \rho_T$;

(iii) $\eta D^2 \in \ker \rho_T$.

Moreover multiplication by $D$ induces a bijection

$$\ker \rho \to \ker \rho_T \cap [H^*_T(X)]^{\text{anti}W}$$

with inverse

$$\eta \mapsto \frac{1}{D|W|} \sum_{w \in W} (-1)^w w\eta.$$

**Remark 7.11** The corresponding results are true when $X$ is a compact symplectic manifold with a Hamiltonian action of a compact group $K$, provided that 0 is a regular value of the moment map.
Remark 7.12 Martin [14, 15] gave a direct proof of the equivalence $(i) \iff (iii)$ when $X^{ss} = X^s$, while Guillemin and Kalkman [1] observed that it follows immediately from Lemma 7.9. The equivalence $(i) \iff (ii)$ and the bijection (7.11) goes back in essence at least to Ellingsrud and Stromme [3] §4. Direct proofs of the equivalences $(i) \iff (ii)$ and $(ii) \iff (iii)$ and the bijection (7.11) are given below; the assumption that $X^{ss} = X^s$ is not needed for the equivalence $(i) \iff (ii)$ or for the existence of the bijection (7.11), and in fact the most convenient assumption for the proof below of the equivalence $(ii) \iff (iii)$ is not that $X^{ss} = X^s$ but rather the corresponding assumption $X^{ss,T} = X^{s,T}$ for the action of the maximal torus. Thus Lemma 7.10 is true when either $X^{ss} = X^s$ or $X^{ss,T} = X^{s,T}$, and if (iii) is omitted then it is true without either of these hypotheses.

Note also that the direct proofs of the equivalences $(i) \iff (iii)$ and $(ii) \iff (iii)$ do not require $X$ to be compact; it suffices that $\mu^{-1}(0)$ or equivalently $X//G$ should be compact, and for example they can be applied to the study of moduli spaces of bundles over a compact Riemann surface (cf. [8]).

Before giving a proof of Lemma 7.10 we shall relate the stratifications $\{S_β : β ∈ B\}$ and $\{S_β^T : β ∈ B_T\}$ and the associated Thom–Gysin maps $TG_β$ and $TG_β^T$ for the actions of $G$ and its maximal torus $T^c$ on $X$. Recall from §1 that the indexing set $B_T$ consists of the closest points to 0 of the convex hulls in $t^c \cong t$ of the weights $α_0, \ldots, α_n$ for the linear action of $T^c$ on $X ⊆ P_n$, and that

$$B = B_T \cap t_+ \text{ and } B_T = \{wβ : β ∈ B, w ∈ W\}$$

where $t_+$ is a positive Weyl chamber in $t$. If $β ∈ B$ then in the notation of §1 we have

$$S_β = GY_β^{ss} \text{ and } S_β^T = Y_β^{ss,T}$$

where $Y_β^{ss,T} = p^{-1}_β(Z_β^{ss,T})$ and $Z_β^{ss,T}$ is the set of semistable points for an appropriate linearisation of the action of $T^c$ on $Z_β$. There are induced isomorphisms

$$H^*_G(S_β) \cong H^*_G(β) = H^*_\text{stab}(β)(Z_β^{ss}) \text{ and } H^*_T(S_β^T) \cong H^*_T(Z_β^{ss,T})$$

with surjective restriction maps

$$H^*_{\text{stab}(β)}(Z_β) \cong [H^*_T(Z_β)]^W_β → H^*_{\text{stab}(β)}(Z_β^{ss})$$

and

$$H^*_T(Z_β) → H^*_T(Z_β^{ss,T})$$

where $W_β$ is the Weyl group of $\text{stab}(β)$. Notice that the boundary $S_β^T \setminus S_β^T$ of $S_β^T$ is contained in

$$\bigcup_{β' ∈ B_T: |β'| > |β|} S_{β'} \subseteq \bigcup_{β' ∈ B: |β'| > |β|} S_{β'}.$$

Let $D_β ∈ H^*(BT)$ be the product of the positive roots of $\text{stab}(β)$, and if $β ∈ B$ let $\overline{TC_β}$ be any lift to $H^*_T(X)$ of the Thom–Gysin map $TG_β^T$ associated to the inclusion of $S_β^T$ in

$$U_β^T = S_β^T \cup \bigcup_{β' ∈ B_T: |β'| < |β|} S_{β'}.$$
If $\eta \in H^*_T(Z_\beta)$ then $\eta$ restricts to an element of $H^*_T(Z^s_{ss,T} \sim H^*_T(Y_{ss,T}) = H^*_T(Y_{ss,T})$ and

$$
\sum_{w \in W} (-1)^w w \left( \mathcal{D}_\beta T G_\beta^T(\eta) \right)
$$

is a $W$-anti-invariant element of $H^*_T(X)$. Thus we have a well defined element

$$
\frac{1}{|D|} \sum_{w \in W} (-1)^w (\mathcal{D}_\beta T G_\beta^T(\eta))
$$

of $[H^*_T(X)]^W \cong H^*_G(X)$ (cf. Lemma 7.9), whose restriction to

$$
U_\beta = S_\beta \cup \bigcup_{\beta' \in B, \beta' \neq \beta, |\beta'| \leq |\beta|} S_{\beta'}
$$

is independent of the choice of lift $T G_\beta^T$ of the Thom–Gysin map $T G_\beta^T$ and thus can be expressed as

$$
\frac{1}{|D|} \sum_{w \in W} (-1)^w (\mathcal{D}_\beta T G_\beta^T(\eta)) = \frac{1}{|D|} \sum_{w \in W} (-1)^w \mathcal{D}_w \mathcal{G}(w \eta).
$$

**Lemma 7.13** If $\beta \in B$ and $\eta \in H^*_{\text{Stab}(\beta)}(Z_\beta) \cong [H^*_T(Z_\beta)]^W_{\beta}$ then

$$
T G_\beta(\eta) = \frac{1}{|D|} \sum_{w \in W} (-1)^w (\mathcal{D}_\beta T G_\beta^T(\eta)) = \frac{1}{|D|} \sum_{w \in W} (-1)^w \mathcal{D}_w \mathcal{G}(w \eta).
$$

**Proof:** Note that $\eta \in H^*_{\text{Stab}(\beta)}(Z_\beta)$ represents an element of $H^*_{\text{Stab}(\beta)}(Z^s_{ss}) \cong H^*_G(S_{\beta})$ by restriction from $Z_\beta$ to $Z^s_{ss}$. Since the equivariant Euler class $e_\beta$ of the normal bundle $\mathcal{N}_{\beta}$ to $S_\beta$ is not a zero-divisor in $H^*_G(S_{\beta})$, it suffices to show that

$$
\frac{1}{|D|} \sum_{w \in W} (-1)^w (\mathcal{D}_\beta T G_\beta^T(\eta)) = \frac{1}{|D|} \sum_{w \in W} (-1)^w \mathcal{D}_w \mathcal{G}(w \eta)
$$

restricts to 0 on

$$
\bigcup_{\beta' \neq \beta, |\beta'| \leq |\beta|} S_{\beta'}
$$

and restricts to $\eta e_\beta$ on $S_\beta$. Since $H^*_{\text{Stab}(\beta)}(Z^s_{ss}) \cong H^*_G(S_{\beta})$ it suffices to check that the restriction to $Z^s_{ss}$ is $\eta e_\beta$. But

$$
\overline{S_{w,\beta}} = w(S^T_{\beta}) \subseteq S_\beta \cup \bigcup_{|\beta'| > |\beta|} S_{\beta'}
$$

so $T G^T_{w,\beta}(w \eta)$ restricts to 0 on

$$
\bigcup_{\beta' \neq \beta, |\beta'| \leq |\beta|} S_{\beta'}
$$
as required. Also the composition of \( TG_\beta^T \) with restriction to \( S_\beta^T \) is multiplication by the equivariant Euler class \( e_\beta^T \) of the normal bundle \( N_\beta^T \) to \( S_\beta^T \), and

\[
Z_\beta^{ss} \subseteq Z_\beta^{ss,T} \subseteq S_\beta^T.
\]

Hence the restriction of \( D_\beta TG_\beta^T(\eta) \) to \( Z_\beta^{ss} \) is \( D_\beta \eta e_\beta^T \). Now

\[
S_\beta \cong G \times_{P_\beta} Y^{ss} \text{ and } S_\beta^T = Y^{ss,T}_\beta
\]

where \( Y^{ss} \) is an open subset of \( Y^{ss,T}_\beta \), so their normal bundles \( N_\beta \) and \( N_\beta^T \) are related by

\[
N_\beta^T|_{Y^{ss}_\beta} \cong g/p_\beta \oplus N_\beta|_{Y^{ss}_\beta}
\]

where \( g \) and \( p_\beta \) are the Lie algebras of \( G \) and \( P_\beta \). Therefore on restriction to \( Y^{ss}_\beta \) (or to \( Z_\beta^{ss} \))

\[
e_\beta^T = \frac{D e_\beta}{D_\beta}
\]

so the restriction of \( D_\beta TG_\beta^T(\eta) \) to \( H^*_\text{Stab}(\beta)(Z_\beta^{ss}) \) is

\[
D_\beta \eta e_\beta^T = D \eta e_\beta.
\]

Hence the restriction to \( H^*_G(S_\beta) \) of

\[
\frac{1}{|W|D} \sum_{w \in W} (-1)^w w(D_\beta TG_\beta^T(\eta)) = \frac{1}{|W|D} \sum_{w \in W} (-1)^w D_{w\beta} TG_{w\beta}^T(w\eta)
\]

is

\[
\frac{1}{|W|D} \sum_{w \in W} (-1)^w w(D \eta e_\beta) = \eta e_\beta
\]

as required, since \( e_\beta \) and \( \eta \in H^*_G(S_\beta) \) are \( W \)-invariant.

**Proof of Lemma 7.10** First recall from Lemma 7.9 that the map

\[
p : [H^*_T(X)]^{\text{anti}W} \to [H^*_T(X)]^W \cong H^*_G(X)
\]

defined by

\[
p(\zeta) = \frac{1}{|W|D} \sum_{w \in W} (-1)^w w \zeta
\]

is a bijection whose inverse is given by multiplication by \( D \), since if \( \zeta \) is \( W \)-invariant then \( p(D\zeta) = \zeta \) and if \( \zeta \) is anti-invariant then \( Dp(\zeta) = \zeta \) (cf. [5] (4.3)). Suppose that \( \eta \in H^*_G(X) \).

If \( \eta \in \ker \rho \), then by Lemma 7.2 we can write

\[
\eta = \sum_{\beta \in B \setminus \{0\}} T\tilde{G}_\beta(\eta_\beta)
\]
for some $\eta_{\beta} \in H_{\text{Stab}(\beta)}^*(Z_{\beta})$ representing an element of $H_{\text{Stab}(\beta)}^*(S_{\beta}) \cong H_{\text{Stab}(\beta)}^*(Z_{\beta}^*)$, where $\widetilde{T G}_{\beta} : H_{G}^{*-\lambda(\beta)}(S_{\beta}) \rightarrow H_{G}^{*}(X)$ is any lift to $X$ of the Thom–Gysin map $T G_{\beta} : H_{G}^{*-\lambda(\beta)}(S_{\beta}) \rightarrow H_{G}^{*}(U_{\beta})$. By Lemma 7.13 we can choose $\widetilde{T G}_{\beta}$ so that

$$\widetilde{T G}_{\beta}(\eta_{\beta}) = \frac{1}{\left|W\right|D} \sum_{w \in W} (-1)^w D_{w_{\beta}} T G_{w_{\beta}}^T (w \eta_{\beta})$$

where $\widetilde{T G}_{\beta}^T$ is a lift of the Thom–Gysin map $T G_{\beta}^T$ and the choice of these lifts $\widetilde{T G}_{\beta}$ respects the action of $W$. Then

$$D_{\eta} = \sum_{\beta \in B \setminus \{0\}} \frac{1}{|W|} \sum_{w \in W} (-1)^w D_{w_{\beta}} T G_{w_{\beta}}^T (w \eta_{\beta}) \in \bigoplus_{\beta \in B \setminus \{0\}} \text{im}\widetilde{T G}_{\beta}^T = \ker \rho_T.$$  

Conversely, suppose that $D_{\eta} \in \ker \rho_T$. Then by Lemma 7.2

$$D_{\eta} = \sum_{\beta \in B \setminus \{0\}} \widetilde{T G}_{\beta}^T (\eta_{\beta})$$  

(7.2)

for some $\eta_{\beta} \in H_{T}^*(Z_{\beta})$ representing an element of $H_{T}^*(S_{\beta}^*) \cong H_{T}^*(Z_{\beta}^*)$, where $\widetilde{T G}_{\beta}$ is any lift to $H_{G}^*(X)$ of the Thom–Gysin map $T G_{ \beta}^T$. But $\eta$ is $W$-invariant, so $D_{\eta}$ is anti-invariant, and so

$$D_{\eta} = \frac{1}{|W|} \sum_{w_{1} \in W} (-1)^{w_{1}} w_{1} (D_{\eta}) = \frac{1}{|W|} \sum_{\beta \in B \setminus \{0\}} \sum_{w_{1} \in W} (-1)^{w_{1}} \widetilde{T G}_{w_{1}\beta}^T (w_{1} \eta_{\beta})$$

if the lifts $\widetilde{T G}_{\beta}^T$ are chosen to respect the action of $W$. We have

$$B_{T} = \{w_{\beta} : w \in W, \beta \in B\}$$

and if $w \in W$ and $\beta \in B$ then $w_{\beta} = \beta$ if and only if $w \in W_{\beta}$. Thus

$$D_{\eta} = \frac{1}{|W|} \sum_{\beta \in B \setminus \{0\}} \sum_{w_{2} \in W} \sum_{w_{1} \in W} (-1)^{w_{1}} \frac{1}{|W_{\beta}|} \widetilde{T G}_{w_{1}w_{2}\beta}^T (w_{1} \eta_{w_{2}\beta})$$

$$= \frac{1}{|W|} \sum_{\beta \in B \setminus \{0\}} \sum_{w_{2} \in W} \sum_{w_{1} \in W} (-1)^{w_{1}} (-1)^{w_{2}} \frac{1}{|W_{\beta}|} \widetilde{T G}_{w_{2}^{-1}\beta}^T (w_{2}^{-1} \eta_{w_{2}\beta})$$

$$= \frac{1}{|W|} \sum_{\beta \in B \setminus \{0\}} \sum_{w_{2} \in W} (-1)^{w} \left( \widetilde{T G}_{\beta}^T \left( \sum_{w_{2} \in W} (-1)^{w_{2}} \frac{1}{|W_{\beta}|} w_{2}^{-1} \eta_{w_{2}\beta} \right) \right).$$

If $\tilde{w} \in W_{\beta}$ then

$$\tilde{w} \left( \sum_{w_{2} \in W} (-1)^{w_{2}} \frac{1}{|W_{\beta}|} w_{2}^{-1} \eta_{w_{2}\beta} \right) = \sum_{w_{2} \in W} (-1)^{w_{2}} \frac{1}{|W_{\beta}|} \tilde{w} w_{2}^{-1} \eta_{w_{2}\beta}$$

28
\[ \sum_{w_3 \in W} \frac{(-1)^{w_3}(1)^{\tilde{w}}}{|W_\beta|} w_3^{-1} \eta_{w_3 \beta} = (-1)^{\tilde{w}} \sum_{w_3 \in W} \frac{(-1)^{w_3}}{|W_\beta|} w_3^{-1} \eta_{w_3 \beta}. \]

Since elements of \( H^*_T(Z_\beta) \) which are anti-invariant under the action of \( W_\beta \) are multiples of \( D_\beta \), it follows that

\[ \sum_{w_2 \in W} \frac{(-1)^{w_2}}{|W_\beta|} w_2^{-1} \eta_{w_2 \beta} = D_\beta \zeta_\beta \]

for some \( \zeta_\beta \in [H^*_T(Z_\beta)]^W_\beta \cong H^*_{\text{Stab}(\beta)}(Z_\beta) \) and hence

\[ D\eta = \frac{1}{|W|} \sum_{\beta \in B_T \setminus \{0\}} \sum_{w \in W} (-1)^w w \left( D_\beta \widetilde{G}_\beta^T(\zeta_\beta) \right) \]

so that

\[ \eta \in \bigoplus_{\beta \in B_\{0\}} \text{im} \widetilde{G}_\beta = \ker \rho \]

by Lemmas 7.2 and 7.14. This proves the equivalence \((i) \iff (ii)\), and the same argument shows that the bijection

\[ H^*_G(X) \to [H^*_T(X)]^{\text{anti} W} \]

given by multiplication by \( D \) restricts to a bijection from \( \ker \rho \) to \( \ker \rho_T \cap [H^*_T(X)]^{\text{anti} W} \).

The observation that the equivalence \((i) \iff (iii)\) follows directly from Lemma 7.5 when \( X^{ss} = X^s \) now completes the proof of Lemma 7.10. However it is also easy to show directly that if \( X^{ss,T} = X^{s,T} \) then \((ii) \iff (iii)\) for any \( \eta \in H^*_G(X) \cong [H^*_T(X)]^W \); that is, \( D\eta \in \ker \rho_T \) if and only if \( D^2 \eta \in \ker \rho_T \). This follows from Lemma 7.3 applied to the action of \( T^c \), together with Poincaré duality on \( X/T^c \) and the surjectivity of \( \rho_T \), as

\[ D\eta \in \ker \rho_T \iff \int_{X/T^c} \rho_T(D\eta) \rho_T(\zeta) = 0 \text{ for all } \zeta \in H^*_T(X). \]

Since \( D\eta \) is anti-invariant, this holds for all \( \zeta \in H^*_T(X) \) if and only if it holds for all anti-invariant \( \zeta \); that is, for all \( \zeta \) of the form \( D\xi \) where \( \xi \in [H^*_T(X)]^W \). Thus

\[ D\eta \in \ker \rho_T \iff 0 = \int_{X/T^c} \rho_T(D\eta) \rho_T(D\xi) = \int_{X/T^c} \rho_T(D^2 \eta \xi) \]

for all invariant \( \xi \in H^*_T(X) \), or equivalently (since \( D^2 \eta \) is invariant) for all \( \xi \in H^*_T(X) \). But by Poincaré duality again we have

\[ \int_{X/T^c} \rho_T(D^2 \eta \xi) = 0 \]

for all \( \xi \in H^*_T(X) \) if and only if \( \rho_T(D^2 \eta) = 0 \), as required.
Remark 7.14 These ideas are used in [4] to obtain a complete set of relations between the standard generators of the moduli space $\mathcal{M}(n, d)$ of stable holomorphic vector bundles of rank $n$ and degree $d$ over a fixed compact Riemann surface of genus $g \geq 2$ when $n$ and $d$ are coprime. There the rôle of $X//G$ is played by the moduli space $\mathcal{M}(n, d)$, and the rôle of $X//T^c$ is played by the corresponding moduli space of parabolic bundles where the parabolic structure is associated to a full flag. The generic perturbation of the linearisation used in \S 6 to obtain a refined stratification is played by a generic perturbation of the parabolic weights.

Example 7.15 Suppose that $X = \mathbb{P}_n$ and that as usual the maximal torus $T^c$ of $G$ acts with weights $\alpha_0, \ldots, \alpha_n$. In this case the closure $\bar{S}_T = \bigcup_{\beta} S_T^T$ of any $T^c$-stratum $S_T^T$ is a linear subspace of $\mathbb{P}_n$ and hence is nonsingular. Thus there is an obvious choice of lift $\tilde{T}G_T^T$ to $X$ of the Thom–Gysin map $TG_T^T$ which is given by the Thom–Gysin map associated to the inclusion of $S_T^T$ in $X$. We have

$$H^*_T(X) \cong H^*(BT)[\zeta]/I$$

where $I$ is the ideal in the polynomial ring $H^*(BT)[\zeta]$ generated by the polynomial $(\zeta + \alpha_0) \cdots (\zeta + \alpha_n)$, while

$$H^*_T(\bar{S}_T^T) \cong H^*(BT)[\zeta]/I_{\beta}$$

where $I_{\beta}$ is the ideal generated by

$$\prod_{\alpha_j, \beta = |\beta|^2} (\zeta + \alpha_j),$$

and

$$\tilde{T}G_T^T(I_{\beta} + \eta) = I + \eta \prod_{\alpha_j, \beta < |\beta|^2} (\zeta + \alpha_j)$$

where

$$\prod_{\alpha_j, \beta < |\beta|^2} (\zeta + \alpha_j)$$

represents the equivariant Euler class $c_T^T$ of the normal bundle to $\bar{S}_T^T$ in $X$. Also

$$H^*_T(Z_{\beta}) \cong H^*(BT)[\zeta]/J_{\beta}$$

where $J_{\beta}$ is the ideal generated by

$$\prod_{\alpha_j, \beta = |\beta|^2} (\zeta + \alpha_j).$$

Hence by Lemma 7.13 there are lifts $\tilde{T}G_{\beta}$ to $X$ of the Thom–Gysin maps

$$TG_{\beta} : H^*_{Stab(\beta)}(Z_{\beta}^{ss}) \cong H^*_{G}(S_{\beta}) \to H^*_G(U_{\beta})$$

represented by

$$\tilde{T}G_{\beta}(J_{\beta} + \eta) = I + \frac{1}{|W|D} \sum_{w \in W} (-1)^w w \left( D_{\beta} \eta \prod_{\alpha_j, \beta < |\beta|^2} (\zeta + \alpha_j) \right),$$

where $D_{\beta}$ is the discriminant associated to $\beta$. \hfill $\blacksquare$
and by Lemma 7.2 the equivariant cohomology ring $H^*_G(X^{ss})$ is isomorphic to the quotient of the polynomial ring $H^*(BT)[\zeta]$ by the ideal generated by $(\zeta + \alpha_0) \cdots (\zeta + \alpha_n)$ and all polynomials in $\zeta$ with coefficients in $H^*(BT)$ of the form

$$\frac{1}{|W|D} \sum_{w \in W} (-1)^w w \left( D_\beta \eta \prod_{\alpha_j \in \beta} (\zeta + \alpha_j) \right)$$

for some $\beta \in B \setminus \{0\}$ and $\eta \in H^*(BT)[\zeta]$.

When $G = SL(2; \mathbb{C})$ acts on $X = \mathbb{P}_n$ as in Example 5.3 then the weights are

$$n\alpha, (n-2)\alpha, \ldots, -n\alpha$$

where $\alpha$ is a basis vector for $\mathfrak{t}$ and

$$B = \{(2j-n)\alpha : j > \frac{n}{2}\} \cup \{0\}.$$ 

Moreover $|W| = 2$ and $D = 2\alpha$, and if $\beta = (2j-n)\alpha \in B \setminus \{0\}$ then $D_\beta = 1$ and $\mathcal{T}G_\beta$ sends $\mathcal{J}_\beta + \eta(\zeta, \alpha)$ to

$$\mathcal{I} + \frac{1}{4\alpha} \left( \eta(\zeta, \alpha) \prod_{k>j} (\zeta + (n-2k)\alpha) - \eta(\zeta, -\alpha) \prod_{k>j} (\zeta - (n-2k)\alpha) \right).$$

Thus when $n$ is odd $H^*(X//G)$ is generated as a $\mathbb{Q}$-algebra by $\zeta$ and $\alpha$ with relations given by

$$\frac{1}{\alpha} \left( \eta(\zeta, \alpha) \prod_{k>j} (\zeta + (n-2k)\alpha) - \eta(\zeta, -\alpha) \prod_{k>j} (\zeta - (n-2k)\alpha) \right)$$

for all polynomials $\eta$ in $\zeta$ and $\alpha$.

**Example 7.16** When $G = SL(2; \mathbb{C})$ acts on $X = (\mathbb{P}_1)^n$ the stratification $\{\tilde{S}_\beta : \beta \in \bar{B}\}$ is described in Example 5.3 it differs from the Morse stratification $\{S_\beta : \beta \in B\}$ for $|\mu|^2$ only in that when $n$ is even the open stratum $S_0 = X^{ss}$ is decomposed into the union of three strata, which are $\tilde{S}_{(0)} = X^{ss}$ together with $\tilde{S}_{(T)}$ and $\tilde{S}_{(T,2)}$. Here $H^*(BT) \cong \mathbb{Q}[\alpha]$ where $\alpha$ has degree two and the nontrivial element of the Weyl group sends $\alpha$ to $-\alpha$, while $H^*_T(X)$ is generated by $n+1$ elements $\zeta_1, \ldots, \zeta_n, \alpha$ of degree two subject to the relations

$$(\zeta_1)^2 = \cdots = (\zeta_n)^2 = \alpha^2$$  \hspace{1cm} (7.3)$$

and $H^*_G(X)$ is generated by $\zeta_1, \ldots, \zeta_n$ and $\alpha^2$ subject to the same relations. The connected components $S_J$ of the strata $S_\beta$ for $\beta \in B \setminus \{0\}$ are indexed by subsets $J$ of $\{1, \ldots, n\}$ of size $|J| > n/2$, and their elements are sequences $(x_1, \ldots, x_n) \in (\mathbb{P}_1)^n$ for which there is some $p \in \mathbb{P}_1$ satisfying $x_j = p$ if and only if $j \in J$. The connected components $S_J^T$ of the $T$-strata $S_\beta^T$ are defined in the same way with $p = 0$ or $p = \infty$. We have

$$\overline{S_J^T} \cong (\mathbb{P}_1)^{n-|J|}.$$
and if \( \{1, \ldots, n\} \setminus J = \{i_1, \ldots, i_{n-|J|}\} \) then the associated Thom–Gysin map
\[
\overline{T G}_J : H^{s^2-2|J|}_T(S^J_T) \rightarrow H^*_T(X)
\]
sends a polynomial \( p(\zeta_{i_1}, \ldots, \zeta_{i_{n-|J|}}, \alpha) \in H^*_T((\mathbb{P}_1)^{n-|J|}) \) to
\[
p(\zeta_{i_1}, \ldots, \zeta_{i_{n-|J|}}, \alpha) \prod_{j \in J} (\zeta_j + \alpha) \in H^*_T((\mathbb{P}_1)^n).
\]

Thus by Lemma 7.13 a lift
\[
\overline{T G}_J : H^{s^2-2|J|}_G(S_J) \cong H^{s^2-2|J|+2}(BT) \rightarrow H^*_G(X)
\]
of the Thom–Gysin map \( TG_J \) is given by
\[
\overline{T G}_J(p(\alpha)) = \frac{1}{4\alpha} \left( p(\alpha) \prod_{j \in J} (\zeta_j + \alpha) - p(-\alpha) \prod_{j \in J} (\zeta_j - \alpha) \right).
\]

It follows that \( H^*_G(X^{ss}) \) is generated by \( \zeta_1, \ldots, \zeta_n \) and \( \alpha^2 \) subject to the relations (7.3) and
\[
\frac{1}{\alpha} \left( \prod_{j \in J} (\zeta_j + \alpha) - \prod_{j \in J} (\zeta_j - \alpha) \right) = 0 = \prod_{j \in J} (\zeta_j + \alpha) + \prod_{j \in J} (\zeta_j - \alpha) \tag{7.4}
\]
for all subsets \( J \) of \( \{1, \ldots, n\} \) with \( |J| > n/2 \).

When \( n \) is even then the components of \( \tilde{S}_{(T)} \) are indexed by partitions \( \{1, \ldots, n\} \) into \( J_1 \sqcup J_2 \) where \( |J_1| = |J_2| = n/2 \), and their elements are sequences \( (x_1, \ldots, x_n) \in (\mathbb{P}_1)^n \) for which there are \( p_1 \neq p_2 \) in \( \mathbb{P}_1 \) satisfying \( x_j = p_1 \) if \( j \in J_1 \) and \( x_j = p_2 \) if \( j \in J_2 \). The components of \( \tilde{S}_{(T,2)} \) are indexed by subsets \( J_1 \) of \( \{1, \ldots, n\} \) with \( |J_1| = n/2 \), and their elements are sequences \( (x_1, \ldots, x_n) \in (\mathbb{P}_1)^n \) for which there is some \( p_1 \in \mathbb{P}_1 \) satisfying \( x_j = p_1 \) if and only if \( j \in J_1 \), but no \( p_2 \in \mathbb{P}_1 \) satisfying \( x_j = p_2 \) if \( j \in \{1, \ldots, n\} \setminus J \). We have
\[
H^*_G(S_{\{J_1, J_2\}}) \cong H^*(BT) \cong H^*_G(S_{J_1} \cup S_{\{J_1, J_2\}})
\]
and the restriction map from \( H^*_G(S_{J_1} \cup S_{\{J_1, J_2\}}) \) to \( H^*_G(S_{J_1}) \) is surjective. Lifts to \( H^*_G(X) \) of the Thom–Gysin maps
\[
TG_{\{J_1, J_2\}} : H^{s^2-2n+4}(S_{\{J_1, J_2\}}) \rightarrow H^*_G(X^{ss}) \tag{7.5}
\]
and
\[
TG_{J_1} : H^{s^2-n+2}_G(S_{J_1}) \rightarrow H^*_G(X^{ss} \setminus S_{\{J_1, J_2\}}) \tag{7.6}
\]
are given by
\[
TG_{\{J_1, J_2\}}(p(\alpha)) = \frac{1}{4\alpha} \left( p(\alpha) \alpha^{n/2-1} \prod_{j \in J_1} (\zeta_j + \alpha) - p(-\alpha)(-\alpha)^{n/2-1} \prod_{j \in J_1} (\zeta_j - \alpha) \right)
\]
and
\[
TG_{J_1}(p(\alpha)) = \frac{1}{4\alpha} \left( p(\alpha) \alpha^{n/2-1} \prod_{j \in J_1} (\zeta_j + \alpha) - p(-\alpha)(-\alpha)^{n/2-1} \prod_{j \in J_1} (\zeta_j - \alpha) \right).
\]
and
\[ TG_{J_1}(p(\alpha)) = \frac{1}{4\alpha} \left( p(\alpha) \prod_{j \in J_1} (\zeta_j + \alpha) - p(-\alpha) \prod_{j \in J} (\zeta_j - \alpha) \right). \]

Thus the kernel of the restriction map
\[ H^*_G(X) \to H^*_G(X^s) \]
is generated by the relations (7.4) for all subsets \( J \) of \( \{1, \ldots, n\} \) with \( |J| \geq n/2 \). Note however that although the Thom–Gysin maps (7.5) and (7.6) are injective, the Thom–Gysin map associated to the inclusion of \( S_{J_2} \) in \( X^{ss} \setminus (S_{\{J_1,J_2\}} \cup S_{J_1}) \) (which is just the composition of \( TG_{J_2} \) as at (7.4) with restriction from \( X^{ss} \setminus S_{\{J_1,J_2\}} \) to \( X^{ss} \setminus (S_{\{J_1,J_2\}} \cup S_{J_1}) \) ) is not injective, and the restriction map from \( H^*_G(X) \) to \( H^*_G(X^s) \cong H^*(X^s/G) \) is not surjective when \( n \geq 4 \) is even. For example, when \( n = 4 \) then \( X^s/G \cong \mathbb{P}_1 \setminus \{0, 1, \infty\} \) and so
\[ \dim H^1(X^s/G) = 2 \]
whereas the equivariant cohomology \( H^*_G(X) \) of \( X \) is all in even degrees.

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