A BEAM MODEL FOR HYDRAULIC FRACTURING

F. Tzschichholz$^{1,2,3}$, H.J. Herrmann$^2$, H. E. Roman$^{3,4}$ and M. Pfuff$^1$

$^1$ Institut für Werkstofforschung
GKSS-Forschungszentrum, Postfach 1160, D-21502 Geesthacht, Germany

$^2$ HLRZ, KFA Jülich, Postfach 1913, D-52428 Jülich, Germany

$^3$ I. Institut für Theoretische Physik
Universität Hamburg, D-20355 Hamburg, Germany

$^4$ Dipartimento di Fisica, Università di Milano
Via Celoria 16, I-20133 Milano, Italy

Abstract

We investigate numerically the shape of cracks obtained in hydraulic fracturing at constant pressure using a square lattice beam model with disorder. We consider the case in which only beams under tension can break, and discuss the conditions under which the resulting cracks may develop fractal patterns. We also determine the opening volume of the crack and the elastic stress field in the bulk, quantities which are accessible experimentally.

PACS numbers:

46.30, 91.60.-x
Hydraulic fracturing is a technique of great importance in soil mechanics and is used systematically to improve oil recovery. In particular, it has a considerable importance for the functioning of geothermal wells\[^1\]. An incompressible fluid, in general water, is pushed with pressure deep inside a solid, in the case of soil by injecting it into a deep perforation. The fluid penetrates into the solid by opening long cracks that radially go from the injection hole into the material.

In two dimensions controlled experiments have been performed recently\[^2\]. Water or air is pushed from above into the center of a circular Hele Shaw cell filled with clay. While on long time scales clay behaves like a fluid for high injection pressures it fractures like a solid (viscoelastic fluid). In this fracturing regime the resulting cracks display a disordered ramified structure which appears to obey self-similarity with a fractal dimension of 1.4 - 1.5. Compared to the fractal structures observed in Laplacian systems (dielectric breakdown, viscous fingering or diffusion-limited aggregation (DLA) ) not only the fractal dimension is lower but also the angles between branches are about three times larger (i.e. about 90°) and no tip splitting is observed.

These interesting observations remain largely unexplained. While for the Laplacian case the underlying instabilities (Saffman-Taylor, tip splitting, side branching) have been heavily investigated\[^3\] and numerical clusters of tens of millions of particles have been generated\[^4\], in the case of fracture much work still needs to be done. A recent stability analysis of the shape of a circular hole with either internal pressure or in a stretched membrane has shown that contrary to the common belief a large difference can be expected in the shape of a crack between the two cases\[^5\]. The origin of this is the non-linear dependence of the growth velocity of the crack surface due to the threshold in cohesion force that must be overcome to break the material. In the Laplacian case these differences do not exist in the limit of large clusters\[^6\].
Numerically the breaking of a material from a central hole was first investigated by Louis and Guinea\textsuperscript{[7]} using a triangular network of springs and stretching the network radially on the outer boundary into the six directions of a hexagon. They observed fractal cracks having a fractal dimension close to that of DLA. These findings were studied subsequently by various groups\textsuperscript{[8]}. It became clear that the patterns depend very much on the type of applied displacements (shear, uni-axial, radial). Since networks of springs need more extended structures to transfer momenta it turned out to be more efficient to use a beam model\textsuperscript{[9]} which is no longer a central force model. For more details on the work that has been done we refer to the book of Ref. 10.

While in previous works imposed displacements were applied only at the external surface of the solid (Dirichlets boundary value problem), in this paper we introduce and study a model in which the imposed load represents a pressure that acts along the whole (inner) surface of the crack in a direction perpendicular to the surface (von Neumann boundary value problem). In this way, the point of application of the imposed load varies during the growth of the crack, a situation that more realistically describes the case of hydraulic fracturing.

We consider the beam model (as defined in p. 232 of Ref. 10) on a square lattice of linear size $L$. Each of the lattice sites $i$ carries three real variables: the two translational displacements $x_i$ and $y_i$ and a rotational displacement $\varphi_i$. Next neighbouring sites are rigidly connected by elastic beams of length $l$\textsuperscript{[11]}. When a site is rotated $\varphi_i \neq 0$ then the beams must bend accordingly (Fig. 1a) and continue to tangentially form $90^\circ$ angles with each other. In this way local momenta are taken into account.

Since most materials typically crack under tension and in much less degree under compression we will assume that only beams that are under tension are allowed to break, i.e. to be irreversibly removed from the forces and momenta
balance equations\cite{12}. This means that the cohesion force against compression is actually infinite, i.e. the beams can be compressed but not broken by compression.

At the place into which the incompressible fluid is supposed to be injected (center of the lattice) one vertical beam connecting sites $i$ and $j$ is removed. Since we want to simulate the loading of a crack by the injected fluid an invariant double force $\vec{F}_i$ and $\vec{F}_j$ conserving momentum is applied at the sites $i$ and $j$ pointing from the hole (removed beam) into the elastic bulk (Fig.1b). Thus, force densities are replaced by discrete force vectors $\vec{F} = (F_x, F_y)$,

$$
\vec{F}_i = F_0 (0, 1), \quad \vec{F}_j = -F_0 (0, 1). \quad (1)
$$

The forces for a broken horizontal bond are defined correspondingly. The acting pressure $P$ is just the force $F_0$ per beam thickness $d$ and beam length $l$ and its value is kept fixed during the fracture process.

Each time a beam is removed a new force dipole is applied at its corresponding nearest-neighbor sites $i$ and $j$ which destroys the balance of forces existing previously. Consequently the lattice must be relaxed and the new static equilibrium must be obtained again. This is done in our case using conjugate gradient\cite{13}. After the unique displacement field $\vec{u}_i$ corresponding to the new boundary conditions is determined one can decide which is the next beam to be broken. Due to this procedure we only consider the case of an instantaneous relaxation, i.e. we assume that the physical process of stress redistribution is much faster than the actual fracture process.

For simplicity, only beams along the surface of the inner hole are considered in the breaking process\cite{14}. For each of these beams the force $f_{ij}$ acting along its axis is determined and only if this $f_{ij}$ is positive, i.e. a tension, the beam can be broken (active beams). In addition, since we are interested in random structures we need to incorporate some stochastic mechanism or disorder in the model. The disorder may be either quenched, i.e. the beams may have different thresholds for breaking,
or it may be annealed, i.e. one selects the next broken beam randomly according to some probability. In the following we will employ the second procedure.

Among the active beams one is chosen with a probability $p_{ij}$, which is determined in analogy to dielectric breakdown\cite{15} to be

$$p_{ij} \propto f_{ij}^\eta$$

if $f_{ij}$ is larger than a certain cohesion force threshold $f_{coh} \geq 0$ and zero otherwise\cite{16}. Here, $0 \leq \eta < \infty$ is a parameter which determines the local breaking properties of the material. If $\eta \to \infty$ only the most stretched beam breaks, i.e. the one with the largest $f_{ij}$. The opposite case corresponds to $\eta = 0$ where all active beams can break with the same probability, i.e. to a situation in which heterogeneities are more important than tension forces.

Let us start discussing our results for the case $\eta = 1$ (similar results are obtained for $\eta > 1$) and $f_{coh} = 0$. For the Laplacian case, the value $\eta = 1$ corresponds to DLA with a fractal dimension $d_f \approx 1.7$ in two dimensions\cite{15}. For the present beam model, however, topologically linear cracks develop ($d_f = 1$) as shown in Fig. 2a. This indicates that the distribution of $p_{ij}$ is much narrower than for DLA and essentially only the most stretched beams (located at the tips of the crack) break. Thus, already for $\eta = 1$, only a single branch persist on large length scales. The colors in Fig.2 represent the stress field. They show that the strongest gradients are indeed around the tips and that regions other than the tips are under constant compression of magnitude $P$.

Also surprising is the result obtained when $\eta = 0$, i.e. the case in which all beams under tension can be selected for breaking with the same probability. For the Laplacian case, one obtains compact clusters of spherical shape and fluctuations occur only at their surface\cite{15}. In the present case, the resulting cracks (shown in Fig. 2b) display ramifications which persist up to scales comparable to the system size $L$, and it seems plausible that they might be fractal. The col-
ors show that the strongest gradients occur again around the crack tips and that the regions between crack branches are virtually under constant compression of magnitude $P$. One clearly sees the opening of the crack being larger on certain main cracks than on the side branches. Although only weakly visible the system of Fig. 2b is no perfect square anymore but has small bulges\cite{17}.

To characterize quantitatively the shape of the cracks obtained when $\eta = 0$, we have plotted the number $N$ of broken beams against the radius of gyration $R_G$ of the crack\cite{18}. The results are shown in the log-log plot of Fig. 3 for the case of free external boundaries. We see a powerlaw behaviour $N \sim R_G^{d_f}$ over nearly two decades and the slope gives a fractal dimension of $d_f = 1.56 \pm 0.05$. Similar results were obtained for the case of periodic boundary conditions.

Our model is different with respect to the Laplacian one in that all the beams under compression cannot break. The fact that in our case the cracks become fractal when $\eta = 0$ shows that only at very few points, namely at the crack tips, the crack surface is under tension and everywhere else under compression.

It is also interesting to consider a fractal structure through its intrinsic metric by using as geodesics only the shortest paths that are entirely on the cluster. In this topological or “chemical” space \cite{19}, one can calculate the number of broken bonds $N_\ell$ that are found within a chemical distance $\ell$ \cite{20}. We have obtained a powerlaw behavior $N_\ell \sim \ell^{d_\ell}$ with $d_\ell = 1.4 \pm 0.05$, for the same cracks considered in Fig. 3, somewhat smaller than $d_f$. The interest in the “chemical dimension” $d_\ell$ is because for loopless structures, such as the cracks described here, both $d_f$ and $d_\ell$ completely determine the scaling behavior of diffusion controlled transport phenomena \cite{21} such as e.g. the diffusion of chemical substances which can be present in the fluid contained within the cracks.

For a finite cohesion threshold $f_{coh} > 0$ and $\eta = 0$ the cracks tend to grow more anisotropically since, initially, only the most stressed beams can break\cite{16}.
However, much larger system sizes than the ones considered here are required in order to reach the asymptotic shape of the cracks when $f_{\text{coh}} > 0$. This is because a finite ratio $f_{\text{coh}}/P$ introduces an additional length scale in the system\cite{22}, and the scaling behavior obtained when $f_{\text{coh}} = 0$ is recovered only asymptotically.

Our simulations are performed at constant pressure and the volume $V$ of the crack opening which corresponds to the amount of fluid that has penetrated into the soil is a measurable variable. We measure this volume by taking into account the actual displacements of the lattice sites as the crack grows. These displacements are given from the elastic solution $\vec{u}_i = (x_i, y_i, \varphi_i)$, and the volume elements $\Delta V_{ij}$ connecting sites $i$ and $j$ are obtained simply as $\Delta V_{ij} = (x_i - x_j)ld$ for horizontal broken beams, with $l$ and $d$ being the beam length and thickness, respectively. Similar expressions hold for vertical broken beams. The total crack opening volume $V$ is just the sum of all the volume elements $\Delta V_{ij}$.

In Fig. 4 we show the crack volume $V$ for free and periodic boundary conditions as a function of the radius of gyration $R_G$ in a log-log plot. The slope gives an exponent consistent with two, i.e. the spatial dimension. This agrees with the observation that a finite amount of fluid actually enters into the hole. When the crack approaches the boundary too much the curve has an artificial increase in slope due to boundary effects. For comparison, we also measured the volume of a deterministic straight crack as shown in Fig. 4. For such a crack in an infinite plate it is well known from elasticity theory that the crack opening volume is proportional to the squared crack length (in two dimensions) \cite{23}. This shows that (1) although the crack surface (number of broken beams) is fractal the volume of the crack is not, (2) the relationship $V \sim R_G^2$ is independent of the crack structure itself and (3) particular external boundary conditions play a minor role as long as the “typical length” $R_G$ describing the crack structure is much smaller than the system size.
We have presented a discrete fracture model in two dimensions adapted to the conditions of hydraulic cracking of soils as typically used for instance to create additional heat exchange surface for geothermal energy recovery. Only tensile forces break the solid and the heterogeneities are considered to be dominant [24]. Under these circumstances the crack patterns are fractal and we determined the fractal dimension to be less than that of DLA. Our results for \( \eta = 0 \) reproduce some of the conspicuous features observed in the 2d experiments of Orléans for high injection rates and rigid pastes [2]: The crack patterns are more kinky than DLA and the fractal dimension is lower, their (large) error actually overlap. Thus, the basic assumptions of our model, namely that the material does not open under compression and that for \( \eta = 0 \) the heterogeneities are more relevant than the actual value of the tension force, may describe the case of pastes with large clay/water ratios used in Orléans.

We also have clear numerical evidence that at constant pressure the opening volume of the crack grows proportionally to the area spanned by the crack, i.e. the square of its radius of gyration, independent of the crack structure itself.

Our model should be considered as yet quite preliminary concerning the understanding of real industrial hydrostatic fracturing. On one hand real soils are three dimensional and in fact we do not know whether in three dimensions the patterns are finger-like as in two dimensions or rather consisting of randomly attached fracture planes. The later case seems more likely has yet never been seen numerically up to now. Also the role of the heterogeneities must be investigated closer. In soils the disorder is quenched, i.e. the randomness (in breaking threshold, modulus, etc) are assigned before cracking starts. We have used “annealed” randomness: random numbers are used during the breaking process in order to select beams with a certain probability. Also realistic, e.g. Weibull distributions, should be considered. Other important ingredients that should be taken
into account when trying to make the model more realistic are plasticity, pressure gradients and hydrodynamic effects in the fluid, stress corrosion and short time effects, like shock waves.

Acknowledgements

H.E.R. gratefully acknowledges financial support from the Alexander von Humboldt Stiftung (Feodor Lynen program).

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11. The beams are assumed to have the same shape and the same elastic behavior, governed by three material dependent constants \( a = l/(EA) \), \( b = l/(GA) \) and \( c = l^3/(EI) \) as described in Ref. 10. Here \( l \) is the beam length (assumed to be unity), \( E \) and \( G \) are the Young and shear moduli, \( A \) is the area of the beam section and \( I \) is the moment of inertia for flexion. We used for all simulations the values \( a = 1.0 \), \( b = 0.017 \) and \( c = 12.0 \). The constant pressure had a value of 0.01.

12. For example for a horizontal beam between sites \( i \) and \( j \) we have for the longitudinal force acting at site \( i \), \( -F_i = \alpha(x_i - x_j) \), for the shear force \( -S_i = \beta(y_i - y_j) + \frac{\beta}{2}l(\varphi_i + \varphi_j) \), and for the flexural torque at site \( i \), \( -M_i = \frac{\beta}{2}l(y_i - y_j + l\varphi_j) + \delta l^2(\varphi_i - \varphi_j) \). We have used the abbreviations \( \alpha = 1/a \), \( \beta = 1/(b + c/12) \) and \( \delta = \beta(b/c + 1/3) \) with respect to the definitions of Ref. 10.

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14. In this case only a single crack is generated. More generally, bonds not belonging to the crack surface can be allowed to break. For these bonds the external pressure does not act directly since the fluid is supposed to invade only the connected hole. This leads to further complication of the model which will not be considered here.

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16. In the case of vanishing cohesion force for tension, \( f_{coh} = 0 \), there exists always active bonds (i.e. bonds which can break) with \( p_{ij} > 0 \) since an infinitesimal external load is sufficient to break a bond. When \( f_{coh} > 0 \),
however, a minimum initial crack length is required to initiate the fracture process. Since the external load increases with the number of broken bonds, at which the force dipoles are applied, the process continues indefinitely.

17. The calculations were performed on IBM RS/6000-320H workstations using for each system of $L = 150$ roughly 100 hours to break $N = 650$ beams.

18. For a single sample the radius of gyration is usually defined as

\[ R_G^2(N) = \frac{1}{N} \sum_{i=1}^{N} (r_i - r_0)^2, \quad r_0 = \frac{1}{N} \sum_{i=1}^{N} r_i, \]

with $r_i$ being the position vector of the $i$th broken beam with respect to the undistorted lattice.

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Figure Captions

Fig. 1 The beam model on the square lattice. (a) A beam connecting sites $i$ and $j$ is shown to display the rotational displacements $\varphi$ at both sites. (b) Diagram of the pair of external forces $\vec{F}$ (arrows) applied at sites $i$ and $j$ corresponding to the initially removed beam.

Fig. 2a A typical crack obtained with a beam model for hydraulic fracturing at constant pressure when $\eta = 1$, on a square lattice of $200 \times 200$ sites and free external boundary conditions. The crack consists of 310 broken beams, and only beams under tension are allowed to break. The different colors represent the intensity of the hydrostatic stress field $|f_{ij}|$. The full range for the stress field is linearly mapped onto twenty color-circle cycles starting at Magenta going through Blue, Cyan, Green, Yellow and ending at Red.

Fig. 2b Same as in Fig. 2a for $\eta = 0$, on a square lattice of $150 \times 150$ sites. This crack consists of 680 broken beams.

Fig. 3 Plot of the number of broken beams $N$ as a function of the radius of gyration $R_G$ of the cracks obtained when $\eta = 0$ and free boundary conditions. Averages over 27 samples were performed. The inset shows the successive slopes $d_f$ of the data, and indicate an average value $d_f = 1.56 \pm 0.05$.

Fig. 4 Plot of the crack opening volume $V$ as a function of the radius of gyration $R_G$, obtained by averaging over 12 samples using free boundary conditions (triangles) and averaging 4 samples using periodic boundary conditions (squares). For comparison, we also show the results for a straight crack (circles) obtained in the case $\eta \to \infty$ under free boundary conditions. In all three cases we find the behavior $V \sim R_G^2$. 

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