Frozen shuffle update for an asymmetric exclusion process on a ring

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Abstract. We introduce a new rule of motion for a totally asymmetric exclusion process (TASEP) representing pedestrian traffic on a lattice. Its characteristic feature is that the positions of the pedestrians, modeled as hard-core particles, are updated in a fixed predefined order, determined by a phase attached to each of them. We investigate this model analytically and by Monte Carlo simulation on a one-dimensional lattice with periodic boundary conditions. At a critical value of the particle density a transition occurs from a phase with ‘free flow’ to one with ‘jammed flow’. We are able to analytically predict the current–density diagram for the infinite system and to find the scaling function that describes the finite size rounding at the transition point.

Keywords: phase transitions into absorbing states (theory), stochastic particle dynamics (theory), traffic and crowd dynamics, zero-range processes
1. Introduction

Let a set of hard-core particles, labeled by indices \( i = 1, 2, \ldots, N \), move unidirectionally from site to site on a one-dimensional lattice. We imagine that the particles represent pedestrians all walking at the same pace but not necessarily in phase with each other. This leads us to formulate the following rule of motion, that we state as the update scheme of a Monte Carlo simulation. Each particle \( i \) is assigned a phase \( \tau_i \in [0, 1) \), permanently attached to it, and during each time step (that is, each unit time interval) all \( N \) particles make a forward hopping attempt in the order of increasing phases. For a closed system with a fixed number of particles (but not for an open one) the phase assignment \( \{\tau_i\} \equiv \{\tau_i \mid i = 1, \ldots, N\} \) is just equivalent to a permutation of the particles.

We stipulate that an attempted hop will always succeed if the target site is empty. Hence for given \( \{\tau_i\} \) the dynamics considered in this paper is deterministic.

This model is an instance of what are commonly called totally asymmetric simple exclusion processes (TASEP); its novelty resides in its update rule. Before continuing the discussion, we mention some connections to existing work.

Processes of particles moving stochastically on—often one-dimensional—lattices serve on the one hand as archetypes of out-of-equilibrium systems, and on the other hand as modeling tools to study transport in various systems, ranging from road and pedestrian traffic to intracellular traffic [1]. The particle motion may take place according to a
large diversity of hopping rules. By the ‘exclusion’ principle one imposes the hard-core condition (at most one particle per site); the ‘total asymmetry’ forbids backward hops; and the process is called ‘simple’ when hops are only between nearest-neighbor sites.

Given these three properties that are characteristic of a TASEP, it is still possible to choose from a variety of update schemes. In particular, the following update schemes have been studied: parallel update [2]–[4], random sequential update, sequential update ordered backward or forward in space [5]–[8], [2], sublattice update [9]–[12], and random shuffle update [13]–[15]. The properties of the system depend on the update scheme [16] and the choice of the scheme should be determined by the application.

The most common update schemes are the random sequential and the parallel updates. Random sequential update produces a dynamics very close to that defined by a master equation in continuous time. A time step is defined as a succession of \( N \) elementary updates, each associated with a time interval of length \( 1/N \), and each allowing only a single particle, chosen at random, to make a hopping attempt. With this dynamics considerable fluctuations occur, since the same particle may be updated several times in the same time step whereas another one may be ignored during several time steps.

With parallel update particles make hopping attempts only at integer values of time but then do so simultaneously. Parallel update is used in particular for applications to road traffic [17,18]: all vehicles are moving at the same time and the time step of the scheme is then supposed to represent a reaction time. Fluctuations are reduced, but parallel update can create conflicts—that should be settled by additional rules—when more than one particle tries to hop onto the same target site. This may occur in particular in applications to pedestrian traffic, which usually takes place in two-dimensional space.

In order to overcome the limitations of these two types of updates, the so-called ‘shuffle update’ has been proposed for modeling pedestrian flow. In the random shuffle update [13,15], before each time step the particles are pre-arranged in a randomly chosen order and then each of them is updated once, exactly in that order. This update scheme was used, for example, in [19] for large-scale simulations of pedestrians.

In the present paper we explore a variant of the shuffle update for which the order in which the particles are arranged, that is, the updating order, is fixed once and for all\(^1\). This scheme is therefore appropriately characterized by the name ‘frozen shuffle update’. Frozen and random shuffle update are dynamical analogs of quenched and annealed systems.

For a closed system this is easy to implement; a random phase \( \tau_i \) is drawn for each of the \( N \) particles independently, for example from the uniform distribution on \([0,1)\). In each time step all \( N \) particle positions are updated once, one after the other, according to increasing values of their phases. The phases \( \tau_i \) do not change during the whole simulation and may be considered as frozen variables of the motion. The set \( \{ \tau_i \} \) determines a random permutation of the particles; for uniformly distributed \( \tau_i \), all permutations have the same probability.

A closed system is expected to evolve toward a stationary state. We must be prepared to envisage that the final stationary state may depend (and, as we shall see,\(^1\) This frozen variant was mentioned, but not studied, in the conclusion of [13].

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indeed does depend) on the precise permutation that fixes the updating order of the particles. An average over all permutations is therefore appropriate and is analogous to the averages on quenched disorder variables standardly performed in statistical physics. The term ‘disorder average’ will therefore denote below the average over all random assignments \( \{ \tau_i \} \).

For an open system the frozen shuffle update requires that by a suitable algorithm we fix the phase of each particle the moment it enters. The equivalence of the set \( \{ \tau_i \} \) to a simple permutation may then no longer hold. The case of open boundary conditions will not be considered here but is studied in a forthcoming paper [23].

In section 2 we introduce some terminology that actually already is the expression of several model properties. In section 3 we consider the TASEP with frozen shuffle update on a ring with particle density \( \rho \). We show that a phase transition occurs at a critical density \( \rho_c \) which separates a low density regime with ‘free flow’ from a high density regime with ‘jammed flow’. We determine the current-versus-density curve \( J_L(\rho) \) analytically, first for an infinite system (section 3.1), where \( J(\rho) = \lim_{L \to \infty} J_L(\rho) \) has a cusp, and then for a system of finite size \( L \) (section 3.2), where the finite size rounding of \( J_L(\rho) \) is described by a scaling function that depends only on the product variable \( (\rho - \rho_c)L^{1/2} \). Monte Carlo simulations show very good agreement with theory. In section 4, by way of a supplement, we show that under ‘free flow’ conditions the TASEP with frozen shuffle update is equivalent to a system of noninteracting particles in continuous space and time. In section 5 we conclude.

2. Free flow and jammed flow

We introduce here the concepts that characterize the structures formed by the particles as they result from the frozen shuffle update scheme. The discussion below is independent of any boundary conditions that may be imposed at a later stage. The most important point is the identification of two distinct stable flow states that we call the free flow state and the jammed state.

2.1. Well-ordered and ill-ordered pairs

Let the flow direction be to the right, let the particles be numbered \( \ldots, i-1, i, i+1, \ldots \) from right to left (see figure 1), and let their phases \( \ldots, \tau_{i-1}, \tau_i, \tau_{i+1}, \ldots \) be given. The pair of successive particles \((i, i+1)\), not necessarily on adjacent sites, will be called well-ordered if \( \tau_{i+1} > \tau_i \) and ill-ordered in the opposite case. The time evolution of well- and ill-ordered pairs under the frozen shuffle update scheme has the following properties, illustrated in figure 1.

If a well-ordered pair \((i, i+1)\) occupies two adjacent sites, then at each time step particle \( i \) will move first and particle \( i+1 \) will move next; hence, when the time step is completed, the two particles are still adjacent and have advanced one lattice distance to the right. Their speed is \( v = 1 \) in units of lattice distances per time step.

If an ill-ordered pair occupies two adjacent sites, the two particles cannot move in the same time step; particle \( i+1 \), having \( \tau_{i+1} < \tau_i \), will attempt first to move but finds itself blocked by particle \( i \). Hence the two particles of an ill-ordered pair move at speed \( v = 1 \) only if they are separated by at least one empty site.
Figure 1. Lattice sites are represented by squares that may be either empty or occupied by a single particle. A configuration involving four particles is shown at two successive times $t = s$ and $s + 1$; the flow is in the direction of the arrow. Particles 1 and 2 form a well-ordered pair. They can move at the same time step, as particle 1 is updated before particle 2. Particles 3 and 4 form an ill-ordered pair. During the time step from $t = s$ to $s + 1$ the update attempt of particle 4 is performed at time $s + \tau_4$, but remains unsuccessful, since its target site is still occupied by particle 3. When subsequently particle 3 is updated at time $s + \tau_3$, it moves forward. Thus a hole is inserted between the two particles. In the case $\tau_3 < \tau_2$, particles 2 and 3 also form an ill-ordered pair, but they are not adjacent and thus can hop independently.

2.2. Free flow configuration

A particle configuration will be said to satisfy the free flow (FF) condition when each ill-ordered pair has its two members separated by at least one hole. In view of the above, such a configuration is again identical to itself at the end of each time step except for a translation by one lattice distance to the right. This corresponds to a free flow with speed $v = 1$, and hence for an FF configuration we have

$$J = \rho,$$

where $J$ is the current and $\rho$ the particle density. It is tacitly understood here that these quantities refer to time averages in a stationary state.

2.3. Rising sequences and platoons

Going along the lattice from right to left one may divide the particles encountered into sequences of increasing phases (for short: rising sequences). The set of particles $(i, i+1, \ldots, i')$ will be said to constitute a rising sequence if $\tau_i < \tau_{i+1} < \cdots < \tau_{i'}$ but $\tau_{i-1} > \tau_i$ and $\tau_{i'} > \tau_{i'+1}$. Examples are shown in figure 2.

Let a set of particles occupy consecutive sites and have phases that increase from right to left. If this set corresponds to a full rising sequence, it will be called a platoon. If it corresponds to only part of a rising sequence, it will be called a subplatoon. One may say that a platoon (a subplatoon) is a fully compacted rising sequence (part of a rising sequence). Platoons and subplatoons are limited on both ends either by holes or by ill-ordered pairs. A rising sequence is composed either of a single platoon or of several subplatoons, an isolated particle being considered as a (sub-)platoon of length 1.
Figure 2. A six particle configuration is shown at two successive times $t = s$ and $s+1$. Because of the inequalities between their phases, the set of particles $(1, 2, 3)$ forms a rising sequence, and so does $(4, 5, 6)$. The inequality $\tau_3 > \tau_4$ defines the separation between these two sequences. The fact that particles 3 and 6 are the last ones of their rising sequences is marked by a heavy (red) line segment delimiting their lattice site to the left. At the $(s+1)$th time step all particles will move except number 4. It so happens that at time $t = s+1$ particles 4, 5, and 6 have formed a platoon, i.e. the rising sequence $(4, 5, 6)$ has been compacted.

Under the frozen shuffle update platoons and subplatoons obey the following simple rules.

(i) If in a given time step the first particle of a (sub-)platoon can move, then all its other particles will also move; hence (sub-)platoons move as single entities.

(ii) When two subplatoons merge, they can never separate again; hence the length of a subplatoon can only grow until it includes the whole rising sequence in which it is embedded.

2.4. Jammed configuration

A configuration of particles will be called jammed if all its rising sequences are compacted into platoons and if consecutive platoons are separated by at most one hole. The reason for this definition will become clear in section 3. Figure 3 shows an example of a jammed configuration. At each time step the evolution of a jammed configuration may be simply described in terms of the motion of its platoons. The rules follow directly from those above.

(i) A platoon preceded by a hole advances by one site as a single entity; this amounts to a position exchange of the platoon and the hole.

(ii) A platoon not preceded by a hole is blocked and does not advance.

Let $\nu$ stand for the average platoon length in a jammed configuration that is statistically homogeneous in space. Noting that $1 - \rho$ is the hole density and that only
Figure 3. A jammed configuration involving seven particles is shown at two successive time steps $t = s$ and $s + 1$. All particles are grouped together in platoons; particles belonging to the same platoon are labeled by the same letter. The last particle of each platoon is indicated by a heavy (red) line segment to its left. Successive platoons are separated by either zero holes or a single hole, that is, the particles are in a jammed configuration. During the $(s + 1)$th time step platoons $a$ and $b$ move one lattice distance to the right, but $c$ and $d$ are blocked. Inversely, one may describe this dynamics as a motion of holes that jump at each time step across the platoon to their left.

3. Phase transition on a ring

After the preliminaries of section 2 we are now ready to study a concrete system. We consider a ring, that is, a lattice of $L$ sites with periodic boundary conditions. Let $N$ be the number of particles and hence $\rho = N/L$ the particle density. We set ourselves the task of determining the particle current $J_L(\rho)$ as a function of the particle density $\rho$ in the stationary state that will result from a given initial state. At the initial time $t = 0$ the particles are placed at distinct but otherwise random positions on the lattice. They are numbered $i = 1, 2, \ldots, N$ from right to left (clockwise around the lattice) and their direction of motion is from left to right (anticlockwise). The particles are assigned random phases $\tau_i$ which we take independently and uniformly distributed on $[0, 1)$. This assignment determines the updating order of the particles.

The initial configuration does not necessarily satisfy the FF condition. If it does, then the particle configuration at time $t = s$ is obtained from that at time $t = 0$ by rotating all particle positions by $s$ steps along the ring. If the FF condition is not satisfied, then after a transient period the system will reach a stationary state which may or may not be of the FF type. We will investigate below the conditions for the realization of each of these possibilities, and the ensuing consequences for the particle current.
3.1. Infinite system limit

The infinite system limit is easiest to discuss, since we may apply the law of large numbers and formulate statements that in that limit are true with probability 1. Let us first ask up to what value of the density $\rho$ it is still possible to have free flow.

For a given set $\{\tau_i\}$ the densest possible FF configuration occurs when all rising sequences are compacted into platoons separated by a single hole. This corresponds precisely to the aforementioned special case of a configuration which is both FF and jammed. The maximum density of the FF phase thus is $\rho_c = \nu/(\nu + 1)$, where as before $\nu$ is the average platoon length. It may be shown (see [20] or the appendix) that in the infinite system limit one has $\nu = 2$ when the phases $\tau_i$ are uniformly distributed, and therefore $\rho_c = \frac{2}{3}$.

For $\rho \leq \rho_c$ any arbitrary initial configuration—tacitly understood to be statistically homogeneous in space—will, after a transient, be converted into an FF configuration. Indeed, whenever an ill-ordered pair of particles occupies two successive sites, the second one will not be able to move and a hole will naturally be included between them. When in this way all ill-ordered pairs have come to include a hole, an FF configuration is obtained. The current $J(\rho) = \lim_{L \to \infty} J_L(\rho)$ is then given by its FF value (1),

$$J(\rho) = \rho, \quad \rho \leq \rho_c.$$  

(3)

For $\rho > \rho_c$ the time evolution will produce two effects. It will compact rising sequences into platoons and it will distribute the available holes such that each platoon is separated from its predecessor by at most a single hole. However, the number of holes is less than the number of platoons. The number of platoons that move in a given time step has thus been maximized and is equal to the number of holes, the other platoons being blocked at that time step. This corresponds to the definition of a jammed phase given in section 2.4, whence upon applying (2) with $\nu = 2$ we obtain

$$J(\rho) = 2(1 - \rho), \quad \rho \geq \rho_c.$$  

(4)

Equations (3) and (4) lead to the cusped current–density diagram shown in figure 4.

The work most closely related to ours is due to Wölki et al [13,14], who studied the same model with random shuffle update\(^2\). The theoretical curve for random shuffle update is not exactly known, but we have reproduced in figure 4 the result of an excellent approximate theory [13]; it exhibits a critical point for a density $\rho_c = \frac{1}{2}$. For $\rho \leq \frac{1}{2}$ the $J$ versus $\rho$ relation is the same for random as for frozen shuffle update. At $\rho = \frac{1}{2}$ random shuffle update has a critical point and its $J(\rho)$ begins to decrease with, initially, a horizontal slope. The fact that the particle current for random shuffle update is reduced with respect to the one for frozen update is due to the absence, in the former case, of a mechanism that would allow long-lived compact platoons to form and to advance at unit speed.

Figure 4 also demonstrates the agreement between our exact theoretical result and data from finite size Monte Carlo simulations, averaged over a representative number of randomly drawn assignments $\{\tau_i\}$. Although the agreement is already quite good for system sizes from $L = 12$ up, finite size effects are visible around the maximum, as shown in the inset of figure 4. In subsection 3.2 we shall refine the theory to account for this rounding of the transition.

\(^2\) Called simply ‘shuffle update’ in that work.

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3.2. Finite system

We consider in this subsection a finite ring of size $L$ containing exactly $N$ particles; throughout we set $\rho = N/L$. Our interest is in the density dependent particle current, which in this finite system we shall denote by $J_L(\rho)$ and whose definition we shall render precise.

By the mechanism described above, the system, whatever its initial configuration, will evolve so as to maximize the number of ill-ordered pairs that include a hole. For densities $\rho \leq 1/2$, there is enough space in the system to place a hole between each pair of particles. Then the FF condition can be fulfilled with certainty and the stationary state is an FF state. Denoting the current in the stationary state by $J_{NL}$ we have

$$J_{NL} = \frac{N}{L} = \rho, \quad \rho \leq \frac{1}{2}. \quad (5)$$

For densities $\rho > 1/2$ it may or may not be possible to converge toward an FF configuration, depending on the random assignment $\{\tau_i\}$. The considerations of section 2.2 show that the discriminating quantity is the number of ill-ordered particle pairs in the initial state. We denote by $n^w (n^i)$ the number of well-ordered (ill-ordered) pairs, so that $n^w + n^i = N$. It will be convenient to work with the difference variable

$$n(\{\tau_i\}) = n^w - n^i, \quad (6)$$

Figure 4. Disorder averaged current $J$ as a function of the density $\rho$ for the frozen shuffle update with periodic boundary conditions. Solid line: theoretical prediction for an infinite system. Data points: Monte Carlo simulations for systems of sizes $L = 12, 52, \text{and} 102$. The inset is a zoom around the maximum of the curve. Dashed line: for comparison, the curve of [13] for random shuffle update.
of which we shall henceforth suppress the argument. Because of the periodic boundary conditions, there is always at least one ill-ordered pair and one well-ordered pair in the system, so that \( n \) may take the values \( n = 1, 2, \ldots, N - 1 \). A necessary and sufficient condition to fulfil the FF condition in the stationary state is to have at least one empty site available for each ill-ordered pair, that is, \( n^w + 2n^i \leq L \) or equivalently

\[
n \geq \frac{3}{2}N - L. \tag{7}
\]

The expression for the stationary state current now involves the variable \( n \) and we will denote it by \( J_{NL_n} \). Two cases have to be distinguished. First, if inequality (7) is satisfied, the system evolves toward an FF state and for this subset of realizations the current is

\[
J_{NL_n} = \frac{N}{L} = \rho, \quad \rho > \frac{1}{2}, \quad \frac{n}{N} \geq \frac{3}{2} - \rho^{-1}. \tag{8}
\]

Secondly, we consider realizations \( \{\tau_i\} \) for which inequality (7) is violated. The stationary state then only has isolated holes\(^3\), and all rising sequences are compacted into platoons. At each time step only the platoons headed by one of the \( L - N \) holes move forward, which means that the instantaneous current per time step fluctuates with time. However, averaged over time each platoon will move in a fraction \((L - N)/N\) of all time steps. Using the fact that \( N/n^i = 2/(1 - n/N) \) is the average length of a platoon, we therefore find after time averaging for the current \( J_{NL_n} \) the expression

\[
J_{NL_n} = \frac{L - N}{L} \times \frac{N}{n^i} = \frac{2(1 - N/L)}{1 - n/N}, \quad \rho > \frac{1}{2}, \quad \frac{n}{N} < \frac{3}{2} - \rho^{-1}. \tag{9}
\]

For finite systems and for densities \( \frac{1}{2} < \rho < 1 \), there will always exist realizations of the \( \tau_i \) that converge toward FF stationary states with a current \( \rho \), and others that do not and have a current less than \( \rho \) and given by (9). In this density regime we will denote by \( J_{NL} \) the current \( J_{NL_n} \) of (8) and (9) averaged with respect to \( n \), that is,

\[
J_{NL} = \sum_{n=1}^{N-1} P_N(n)J_{NL_n}, \quad \rho > \frac{1}{2}, \tag{10}
\]

in which \( P_N(n) \) is the probability distribution of \( n(\{\tau_i\}) \) and remains to be determined. Since \( n \) is determined by \( \{\tau_i\} \), the current \( J_{NL} \) in (10) deserves the name ‘disorder averaged current’.

We note that for given \( \{\tau_i\} \) the stationary state currents determined in this section are independent of the initial particle positions.

### 3.3. Finite size effects near the transition point

The probability distribution \( P_N(n) \) was studied by Oshanin and Voituriez [20] for the case—which is also ours—where the \( \tau_i \) are drawn independently from a uniform distribution on \([0, 1)\). These authors showed, among other things, that in the limit of large \( N \) and with \( n \) scaling as \( \sim N^{1/2} \) the variable \( x = n/N^{1/2} \) has the probability distribution

\[
\Pi(x) = (3/2\pi)^{1/2} \exp\left(-\frac{3}{2}x^2\right). \tag{11}
\]

---

\(^3\) The same behavior appears with random shuffle update.
It is symmetric in $n$, as dictated by the left–right symmetry of the phase assignment. In the appendix we derive equation (11) in a more direct way.

From here on we shall consider the equations of the preceding subsection in the limit of large but finite $N$, $n$ and $L$, and fixed ratios $\rho = N/L$ and $x = n/N^{1/2}$. We conform to usage and take the system size $L$, rather than $N$, as the independent large variable. In the limit in question we shall write $J_{N,L,n} = J_L(\rho, x)$ and $J_{NL} = J_L(\rho)$. We may then reexpress the disorder averaged current (10) as

$$J_L(\rho) = \int_{-\infty}^{\infty} dx \Pi(x) J_L(\rho, x).$$  \hfill (12)

The expression for $J_L(\rho, x)$ is derived from (8) or (9), depending on the value of $x$, that is,

$$J_L(\rho, x) = \begin{cases} 
\frac{2(1 - \rho)}{1 - x(\rho L)^{-1/2}}, & x < x_c(\rho), \\
\rho, & x \geq x_c(\rho),
\end{cases}$$  \hfill (13)

in which

$$x_c(\rho) = (\rho L)^{1/2}(\rho_c^{-1} - \rho^{-1}),$$  \hfill (14)

where $\rho_c = \frac{2}{3}$. We observe parenthetically that in the limit $L \to \infty$ the $x$ dependence of (13) disappears and we recover $\lim_{L \to \infty} J_L(x, \rho) = J(\rho)$, where $J(\rho)$ is the infinite system current of equations (3) and (4). Since (11) is valid in the limit in which $x$ remains finite as $L \to \infty$, we conclude that the present approach is valid for densities

$$\rho = \rho_c + \Delta \rho$$  \hfill (15)

such that $\Delta \rho$ is on the scale of $L^{-1/2}$. Remembering this and expanding in powers of $L^{-1/2}$ we find from (14) and (13)

$$x_c(\rho) = \left(\frac{2}{3}\right)^{3/2} L^{1/2} \Delta \rho + \mathcal{O}(L^{-1/2}),$$

$$J_L(\rho, x) = \begin{cases} 
\rho_c - 2 \Delta \rho + (3L/2)^{-1/2}x + \mathcal{O}(L^{-1}), & x < x_c(\rho), \\
\rho_c + \Delta \rho, & x \geq x_c(\rho).
\end{cases}$$  \hfill (16)

We introduce the scaling variable $y = L^{1/2} \Delta \rho$, which in the limit of interest should be of order unity. Substitution of (16) in (12) then yields

$$J_L(\rho) = \rho_c + L^{-1/2} y - L^{-1/2} \int_{-\infty}^{(3/2)^{3/2} y} dx \Pi(x) \left[3y - \left(\frac{2}{3}\right)^{1/2}x\right] + \mathcal{O}(L^{-1}).$$  \hfill (17)

When using in (17) the explicit expression (11) for $\Pi(x)$ we may evaluate the $x$ integral and obtain, up to corrections of higher order in $L^{-1/2}$,

$$J_L(\rho) = \rho_c + L^{-1/2} \Phi(L^{1/2} \Delta \rho),$$  \hfill (18)

valid in the limits $\Delta \rho = \rho - \rho_c \to 0$ and $L \to \infty$ with $L^{1/2} \Delta \rho$ fixed, and in which the scaling function $\Phi(y)$ is given by

$$\Phi(y) = -\frac{y}{2} - \frac{3}{2} \mathrm{erf}\left(\frac{y}{2}\right) - (9\pi)^{-1/2} \exp\left(-\frac{81}{4} y^2\right).$$  \hfill (19)
Figure 5. Solid line: the theoretical scaling function $\Phi((\rho - \rho_c)L^{1/2})$ of equation (19), representing the average current $J_L(\rho) - J_c$ as a function of the particle density $\rho$ in a finite system of size $L$ near criticality. The dashed lines are the asymptotes for $(\rho - \rho_c)L^{1/2} \to \pm \infty$. The simulation data for large system sizes $L$ are seen to collapse very well onto the theoretical curve. Each point corresponds to an average over 10 000 or 100 000 time steps and over 1000 realizations of the disorder.

This function is negative and such that

$$
\Phi(y) \simeq y, \quad y \to -\infty, \\
\Phi(y) \simeq -2y, \quad y \to \infty,
$$

which ensures the correct limit behavior of (18) for $|\rho - \rho_c| \gg L^{-1/2}$. We have plotted $\Phi(y)$ in figure 5 together with simulation data for different system sizes $L$. The data are seen to collapse very well onto the theoretical curve.

4. Mapping to a continuous model and interpretation for pedestrian motion

In this section we point out that under free flow conditions the time evolution defined by the frozen shuffle update for the particle system on a lattice may be seen as a sequence of snapshots taken at integer instants of time $t = \ldots, s - 1, s, s + 1, \ldots$, of a system that itself evolves in continuous time $t$ and space $x$.

To show this we consider a collection of non-overlapping hard rods all moving continuously to the right at speed $v = 1$ along the $x$ axis, as depicted in figure 6. If we associate lattice sites with the integer axis positions $x = \ldots, k - 1, k, k + 1, \ldots$, then at any given instant of continuous time, each rod covers exactly one site. The mapping is performed by placing on that site a particle associated with that rod. Let figure 6

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Figure 6. Hard rods labeled by an index \( i \) move at constant speed \( v = 1 \) along the \( x \) axis. The integer axis positions have been labeled by an index \( k \).

represent the rod positions at time \( t = 0 \) (or for that matter at any other integer instant of time). The particle labeled \( i \) and corresponding to rod \( i \) occupies lattice site \( k \) and therefore gives rise, at \( t = 0 \), to a particle on site \( k \). The distance between site \( k \) and the tail of rod \( i \) has been indicated as a time interval\(^4\) \( \tau_i \), this being the time still needed for the tail of that rod to cross the point \( k \) during its continuous motion along the \( x \) axis. This crossing therefore occurs at time \( t = \tau_i \), and that is the time at which particle \( i \) will hop from site \( k \) to site \( k+1 \). Particle \( i \) will execute its subsequent hops at times \( t = s + \tau_i \), where \( s \) is an integer. This is exactly the frozen shuffle update scheme.

We remark that the mapping defined here yields only the FF configurations of the discrete model. If we try to perform the inverse mapping, i.e. from the discrete to the continuous model, then in the case of a jammed configuration the non-overlapping condition for rods cannot be enforced anymore. This may actually still have some physical relevance, if one adopts the view that a rod represents not only a pedestrian but also some ‘private’ space around him. In free flow pedestrians are not willing to approach each other too closely and they avoid entering each other’s ‘private’ space, whereas at increasing densities they tolerate smaller distances.

5. Conclusion

We have introduced in this paper a new update scheme for the TASEP, namely the frozen shuffle update, which should be appropriate, in particular, for the modeling of pedestrians.

We have characterized the behavior of the TASEP with frozen shuffle update for a closed one-dimensional lattice of \( L \) sites and \( N \) particles. The time evolution under frozen shuffle update is deterministic\(^5\); it is fully determined by the initial particle positions and by the set \( \{\tau_i\} \) of their phases. The latter are quenched random variables that at each time step determine the update order of the particles. We showed that the analysis of the particle motion and their interaction may be fruitfully carried out in terms of the concepts of well/ill-ordered pairs and of platoons. Two principal types of flow may then be distinguished, ‘free flow’ and ‘jammed flow’.

We were able to predict completely the fundamental diagram, that is, the current \( J_L(\rho) \) as a function of density \( \rho = N/L \), for both the infinite \( (N, L \to \infty) \) and the finite systems.

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\(^4\) Because \( v = 1 \), times and distances may be identified.

\(^5\) By this we mean that the hopping probability is always unity when the target site is empty.
We found that for increasing particle density \( \rho \) the passage from a free flow phase to a jammed phase takes place via a phase transition at a critical density \( \rho = \rho_c \). This contrasts with random sequential update, for which with increasing density the system becomes gradually more and more congested. Critical points, however, were observed in the fundamental diagram in other instances of deterministic motion, namely with parallel update \([21,22]\) and with random shuffle update \([13,14]\). In the latter case, although the particle–hole symmetry is broken, the critical point was still found at the symmetric point \( \rho_c = 1/2 \); by contrast, for the present frozen shuffle update we find \( \rho_c = 2/3 \), i.e. the asymmetry between holes and particles is still enhanced. Another difference is that for random shuffle update the critical point is already present in finite systems, whereas for frozen shuffle update the transition is rounded and becomes sharp only in the limit of infinite system size.

A mapping with a continuous model of hard rods is proposed, which is exact for free flow configurations, and may be useful for the interpretation of the results in terms of pedestrian motion.

Two final remarks about open questions are in place here.

First, the deterministic time evolution studied in this paper entails that, if the target site is empty, particles hop with probability \( p = 1 \). While in the case of random sequential update the hopping probability \( p \) can be modified through a simple rescaling of time, here such a rescaling is not possible. We therefore expect a qualitatively different behavior of the current \( J(\rho) \) when the hopping probability \( p \) is strictly less than one. We leave the analysis of this case for future work.

Second, this work has been exclusively concerned with a closed system having periodic boundary conditions. New types of questions arise when one applies frozen shuffle update to open systems. In a companion paper \([23]\) we shall address the case of open boundary conditions and determine in particular the phase diagram.

**Appendix. Random walk generated by a random permutation of \( N \) integers**

We arrange the integers \( 1, 2, 3, \ldots, N \) on the sites of a circular lattice and permute them randomly, all permutations having the same probability. Suppose that when going clockwise along the lattice in \( N \) steps, we encounter \( n^w \) well-ordered and \( n^i \) ill-ordered pairs in the sense of section 2.1. Obviously \( n^w \) and \( n^i \) are random integers that depend on the permutation, and are such that \( n^w + n^i = N \). Let \( n = n^w - n^i \). We ask what the probability distribution \( P_N(n) \) of \( n \) is in the limit of large \( N \).

This question was first asked by Oshanin and Voituriez \([20]\), who obtained the distribution \( \Pi(x) \) given in (11). It is possible to arrive at the same result in a different and, we believe, simpler way that we present here. It is based on establishing a recursion in \( N \). Suppose that the integers \( 1, 2, \ldots, N \) have been permuted and placed on the sites of a circular \( N \)-site lattice. A permutation of \( 1, 2, \ldots, N+1 \) on an \((N+1)\)-site lattice is obtained by inserting between two randomly chosen neighboring sites a new site carrying the integer \( N+1 \). The probability \( p^w_N \) (or \( p^i_N \)) to perform the insertion on a well-ordered (or on an ill-ordered) pair is

\[
p^w_i(n) = \frac{1}{2}[1 \pm n/N]. \tag{A.1}
\]

In either case the original pair disappears and, since the newly inserted integer \( N+1 \) is
necessarily larger than its two neighbors, is replaced with the succession of a well- and an ill-ordered pair. Hence we have the recursion

$$P_{N+1}(n) = p_w P_N(n-1) P_N(n) + p_i P_N(n+1), \quad (A.2)$$

valid for $n = -N + 1, -N + 3, \ldots, N - 1$ (which are the only values of $n$ that can occur) and with the convention that $P_N(-N) = P_N(N) = 0$. We substitute (A.1) in (A.2) and set

$$x = \frac{n}{N^{1/2}}, \quad P_N(n) = \frac{1}{N^{1/2}} \Pi_N \left( \frac{n}{N^{1/2}} \right), \quad (A.3)$$

expecting that in the large-$N$ limit the variables $x$ and $N$ may be treated as continuous.

On the expression thus obtained we perform a standard expansion in negative powers of $N$. The result is the Fokker–Planck equation

$$\frac{\partial \Pi_N(x)}{\partial N} = \frac{3}{2} \frac{\partial x \Pi_N(x)}{\partial x} + \frac{1}{2} \frac{\partial^2 \Pi_N(x)}{\partial x^2} \quad (A.4)$$

of which (11) is the stationary solution, that is, the one solving $\partial \Pi_N(x)/\partial N = 0$.

We also note that the average length of the platoons

$$\nu = \frac{N}{n^i} = \frac{2N}{N - n} \quad (A.5)$$

tends to $\nu = 2$ when $N$ becomes large, as $n$ typically scales as $N^{1/2}$.

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