2-elementary subgroups of the space Cremona group

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Abstract We give a sharp bound for orders of elementary abelian 2-groups of birational automorphisms of rationally connected threefolds.

1 Introduction

Throughout this paper we work over $k$, an algebraically closed field of characteristic 0. Recall that the Cremona group $\text{Cr}_n(k)$ is the group of birational transformations of the projective space $\mathbb{P}_k^n$. We are interested in finite subgroups of $\text{Cr}_n(k)$. For $n = 2$ these subgroups are classified basically (see [DH09] and references therein) but for $n \geq 3$ the situation becomes much more complicated. There are only a few, very specific classification results (see e.g. [Pr12], [Pr11], [Pr13c]).

Let $p$ be a prime number. A group $G$ is said to be $p$-elementary abelian of rank $r$ if $G \cong \left(\mathbb{Z}/p\mathbb{Z}\right)^r$. In this case we denote $r(G) := r$. A. Beauville [Be07] obtained a sharp bound for the rank of $p$-elementary abelian subgroups of $\text{Cr}_2(k)$.

Theorem 1.1 ([Be07]). Let $G \subset \text{Cr}_2(k)$ be a 2-elementary abelian subgroup. Then $r(G) \leq 4$. Moreover, this bound is sharp and such groups $G$ with $r(G) = 4$ are classified up to conjugacy in $\text{Cr}_2(k)$.

The author [Pr11] was able to get a similar bound for $p$-elementary abelian subgroups of $\text{Cr}_3(k)$ which is sharp for $p \geq 17$.

In this paper we improve this result in the case $p = 2$. We study 2-elementary abelian subgroups acting on rationally connected threefolds. In particular, we obtain
a sharp bound for the rank of such subgroups in $\text{Cr}_3(k)$. Our main result is the following.

**Theorem 1.2.** Let $Y$ be a rationally connected variety over $k$ and let $G \subset \text{Bir}_k(Y)$ be a 2-elementary abelian group. Then $r(G) \leq 6$.

**Corollary 1.3** Let $G \subset \text{Cr}_3(k)$ be a 2-elementary abelian group. Then $r(G) \leq 6$ and the bound is sharp (see Example 3.4).

Unfortunately we are not able to classify all the birational actions $G \hookrightarrow \text{Bir}_k(Y)$ as above attaining the bound $r(G) \leq 6$ (cf. [Be07]). However, in some cases we get a description of these “extremal” actions.

The structure of the paper is as follows. In Sect. 3 we reduce the problem to the study of biregular actions of 2-elementary abelian groups on Fano-Mori fiber spaces and investigate the case of non-trivial base. A few facts about actions of 2-elementary abelian groups on Fano threefolds are discussed in Sect. 4. In Sect. 5 (resp. 6) we study actions on non-Gorenstein (resp. Gorenstein) Fano threefolds. Our main theorem is a direct consequence of Propositions 3.2, 5.1, and 6.1.

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## 2 Preliminaries

**Notation.**

- For a group $G$, $r(G)$ denotes the minimal number of generators. In particular, if $G$ is an elementary abelian $p$-group, then $G \simeq (\mathbb{Z}/p\mathbb{Z})^{r(G)}$.
- $\text{Fix}(G,X)$ (or simply $\text{Fix}(G)$ if no confusion is likely) denotes the fixed point locus of an action of $G$ on $X$.

**Terminal singularities.** Recall that the index of a terminal singularity $(X \ni P)$ is a minimal positive integer $r$ such that $K_X$ is a Cartier divisor at $P$.

**Lemma 2.1** Let $(X \ni P)$ be a germ of a threefold terminal singularity and let $G \subset \text{Aut}(X \ni P)$ be a 2-elementary abelian subgroup. Then $r(G) \leq 4$. Moreover, if $r(G) = 4$, then $(X \ni P)$ is not a cyclic quotient singularity.

**Proof.** Let $m$ be the index of $(X \ni P)$. Consider the index-one cover $\pi: (X^\sharp \ni P^\sharp) \to (X \ni P)$ (see [Re87]). Here $(X^\sharp \ni P^\sharp)$ is a terminal point of index 1 (or smooth) and $\pi$ is a cyclic cover of degree $m$ which is étale outside of $P$. Thus $X \ni P$ is the quotient of $X^\sharp \ni P$ by a cyclic group of order $m$. If $m = 1$, we take $\pi$ to be the identity map. We may assume that $k = \mathbb{C}$ and then the map $X^\sharp \setminus \{P^\sharp\} \to X \setminus \{P\}$ can be regarded as the topological universal cover. Hence there exists a natural lifting $G^\sharp \subset \text{Aut}(X^\sharp \ni P^\sharp)$ fitting to the following exact sequence
where \( C_m \simeq \mathbb{Z}/m\mathbb{Z} \). We claim that \( G^2 \) is abelian. Assume the converse. Then \( m \geq 2 \).

The group \( G^1 \) permutes eigenspaces of \( C_m \). Let \( T_{\mathfrak{p}}^{\mathfrak{e}, X^2} \) be the tangent space and let \( n := \dim T_{\mathfrak{p}, X^2} \) be the embedded dimension. By the classification of three-dimensional terminal singularities \([\text{Mo85}], \text{Re87}\) we have one of the following:

1. \( \frac{1}{m}(a, -a, b), \quad n = 3, \quad \gcd(a, m) = \gcd(b, m) = 1; \)
2. \( \frac{1}{m}(a, -a, b, 0), \quad n = 4, \quad \gcd(a, m) = \gcd(b, m) = 1; \) \quad (**)  
3. \( \frac{1}{4}(a, -a, b, 2), \quad n = 4, \quad \gcd(a, 2) = \gcd(b, 2) = 1, \quad m = 4, \)

where \( \frac{1}{n}(a_1, \ldots, a_n) \) denotes the diagonal action

\[
x_k \mapsto \exp(2\pi i a_k/m) \cdot x_k, \quad k = 1, \ldots, n.
\]

Put \( T = T_{\mathfrak{p}, X^2} \) in the first case and denote by \( T \subset T_{\mathfrak{p}, X^2} \) the three-dimensional subspace \( x_4 = 0 \) in the second and the third cases. Then \( C_m \) acts on \( T \) freely outside of the origin and \( T \) is \( G^2 \)-invariant. By (*) we see that the derived subgroup \([G^2, G^2]\) is contained in \( C_m \). In particular, \([G^2, G^2]\) is abelian and also acts on \( T \) freely outside of the origin. Assume that \([G^2, G^2]\) \( \neq \{1\} \). Since \( \dim T = 3 \), this implies that the representation of \( G^2 \) on \( T \) is irreducible (otherwise \( T \) has a one-dimensional invariant subspace, say \( T_1 \), and the kernel of the map \( G^2 \to GL(T_1) \simeq \mathbb{R}^* \) must contain \([G^2, G^2]\)). In particular, the eigenspaces of \( C_m \) on \( T \) have the same dimension. Since \( T \) is irreducible, the order of \( G^2 \) is divisible by \( 3 = \dim T \) and so \( m > 2 \). In this case, by the above description of the action of \( C_m \) on \( T_{\mathfrak{p}, X^2} \) we get that there are exactly three distinct eigenspaces \( T_i \subset T \). The action of \( G^2 \) on the set \( \{T_i\} \) induces a transitive homomorphism \( G^2 \to \mathfrak{S}_3 \) whose kernel contains \( C_m \). Hence we have a transitive homomorphism \( G \to \mathfrak{S}_3 \). Since \( G \) is a 2-group, this is impossible.

Thus \( G^2 \) is abelian. Then

\[
r(G) \leq r(G^2) \leq \dim T_{\mathfrak{p}, X^2}.
\]

This proves our statement. \( \square \)

Remark 2.2. If in the above notation the action of \( G \) on \( X \) is free in codimension one, then \( r(G) \leq \dim T_{\mathfrak{p}, X^2} - 1 \).

For convenience of references, we formulate the following easy result.

Lemma 2.3 Let \( G \) be a 2-elementary abelian group and let \( X \) be a \( G \)-threefold with isolated singularities.

(i) If \( \dim \operatorname{Fix}(G) > 0 \), then \( \dim \operatorname{Fix}(G) + r(G) \leq 3 \).

(ii) Let \( \delta \in G \setminus \{1\} \) and let \( S \subset \operatorname{Fix}(\delta) \) be the union of two-dimensional components. Then \( S \) is \( G \)-invariant and smooth in codimension 1.

Sketch of the proof. Consider the action of \( G \) on the tangent space to \( X \) at a general point of a component of \( \operatorname{Fix}(G) \) (resp. at a general point of \( \operatorname{Sing}(S) \)). \( \square \)
3 **$G$-equivariant minimal model program.**

**Definition 3.1.** Let $G$ be a finite group. A $G$-variety is a variety $X$ provided with a biregular faithful action of $G$. We say that a normal $G$-variety $X$ is $G\mathbb{Q}$-factorial if any $G$-invariant Weil divisor on $X$ is $\mathbb{Q}$-Cartier.

The following construction is standard (see e.g. [Pr12]). Let $Y$ be a rationally connected three-dimensional algebraic variety and let $G \subset \text{Bir}(Y)$ be a finite subgroup. Taking an equivariant compactification and running an equivariant minimal model program we get a $G$-variety $X$ and a $G$-equivariant birational map $\theta : Y / \text{axis} \rightarrow X$, where $X$ has a structure a $G$-Fano-Mori fibration $f : X \rightarrow B$. This means that $X$ has at worst terminal $G\mathbb{Q}$-factorial singularities, $f$ is a $G$-equivariant morphism with connected fibers, $B$ is normal, $\dim B < \dim X$, the anticanonical Weil divisor $-K_X$ is ample over $B$, and the relative $G$-invariant Picard number $\rho(X)^G$ equals to one. Obviously, in the case $\dim X = 3$ we have the following possibilities:

- **(C)** $B$ is a rational surface and a general fiber $f^{-1}(b)$ is a conic;
- **(D)** $B \simeq \mathbb{P}^1$ and a general fiber $f^{-1}(b)$ is a smooth del Pezzo surface;
- **(F)** $B$ is a point and $X$ is a $G\mathbb{Q}$-Fano threefold, that is, $X$ is a Fano threefold with at worst terminal $G\mathbb{Q}$-factorial singularities and such that $\text{Pic}(X)^G \simeq \mathbb{Z}$. In this situation we say that $X$ is $G$-Fano threefold if $X$ is Gorenstein, that is, $K_X$ is a Cartier divisor.

**Proposition 3.2.** Let $G$ be a 2-elementary abelian group and let $f : X \rightarrow B$ be a $G$-Fano-Mori fibration with $\dim X = 3$ and $\dim B > 0$. Then $r(G) \leq 6$. Moreover, if $r(G) = 6$ and $Z \simeq \mathbb{P}^1$, then a general fiber $f^{-1}(b)$ is a del Pezzo surface of degree 4 or 8.

**Proof.** Let $G_f \subset G$ (resp. $G_B \subset \text{Aut}(B)$) be the kernel (resp. the image) of the homomorphism $G \rightarrow \text{Aut}(B)$. Thus $G_B$ acts faithfully on $B$ and $G_f$ acts faithfully on the generic fiber $F \subset X$ of $f$. Clearly, $G_f$ and $G_B$ are 2-elementary groups with $r(G_f) + r(G_B) = r(G)$. Assume that $B \cong \mathbb{P}^1$. Then $r(G_B) \leq 2$ by the classification of finite subgroups of $\text{PGL}_2(k)$. By Theorem 1.1 we have $r(G_f) \leq 4$. If furthermore $r(G) = 6$, then $r(G_f) = 4$ and the assertion about $F$ follows by Lemma 3.3 below. This proves our assertions in the case $B \cong \mathbb{P}^1$. The case $\dim B = 2$ is treated similarly. \[\Box\]

**Lemma 3.3 (cf. [Be07])** Let $F$ be a del Pezzo surface and let $G \subset \text{Aut}(F)$ be a 2-elementary abelian group with $r(F) \geq 4$. Then $r(F) = 4$ and one of the following holds:

- (i) $K^2_F = 4$, $\rho(F)^G = 1$;
- (ii) $K^2_F = 8$, $\rho(F)^G = 2$.

**Proof.** Similar to [Be07] §3. \[\Box\]
Example 3.4. Let $F \subset \mathbb{P}^4$ be the quartic del Pezzo surface given by $\sum x_i^2 = \sum \lambda_i x_i^2 = 0$ with $\lambda_i \neq \lambda_j$ for $i \neq j$ and let $G_f \subset \text{Aut}(F)$ be the 2-elementary abelian subgroup generated by involutions $\sigma \mapsto -\sigma$. Consider also a 2-elementary abelian subgroup $G_B \subset \text{Aut}(\mathbb{P}^1)$ induced by a faithful representation $Q_8 \to GL_2(\mathbb{Z})$ of the quaternion group $Q_8$. Then $r(G_f) = 4$, $r(G_B) = 2$, and $G := G_f \times G_B$ naturally acts on $X := F \times \mathbb{P}^1$. Two projections give us two structures of $G$-Fano-Mori fibrations of types (D) and (C). This shows that the bound $r(G) \leq 6$ in Proposition 3.2 is sharp. Moreover, $X$ is rational and so we have an embedding $G \subset \text{Cr}_3(\mathbb{C})$.

4 Actions on Fano threefolds

Main assumption. From now on we assume that we are in the case (F), that is, $X$ is a $G\mathbb{Q}$-Fano threefold.

Remark 4.1. The group $G$ acts naturally on $H^0(X, -K_X)$. If $H^0(X, -K_X) \neq 0$, then there exists a $G$-semi-invariant section $s \neq 0 \in H^0(X, -K_X)$ (because $G$ is an abelian group). This section defines an invariant member $S \in |-K_X|$.

Lemma 4.2 Let $X$ be a $G\mathbb{Q}$-Fano threefold, where $G$ is a 2-elementary abelian group with $r(G) \geq 5$. Let $S$ be an invariant effective Weil divisor such that $-(K_X + S)$ is nef. Then the pair $(X, S)$ is log canonical (lc). In particular, $S$ is reduced. If $-(K_X + S)$ is ample, then the pair $(X, S)$ is purely log terminal (plt).

Proof. Assume that the pair $(X, S)$ is not lc. Since $S$ is $G$-invariant and $\rho(X)^G = 1$, we see that $S$ is numerically proportional to $K_X$. Hence $S$ is ample. We apply quite standard connectedness arguments of Shokurov [Sho93] (see, e.g., [MP09 Prop. 2.6]): for a suitable $G$-invariant boundary $D$, the pair $(X, D)$ is lc, the divisor $-(K_X + D)$ is ample, and the minimal locus $V$ of log canonical singularities is also $G$-invariant. Moreover, $V$ is either a point or a smooth rational curve. By Lemma 2.1 we may assume that $G$ has no fixed points. Hence, $V \cong \mathbb{P}^1$ and we have a map $\zeta : G \to \text{Aut}(\mathbb{P}^1)$. By Lemma 2.3 $r(\ker \zeta) \leq 2$. Therefore, $r(\zeta(G)) \geq 3$. This contradicts the classification of finite subgroups of $PGL_2(\mathbb{Z})$.

If $-(K_X + S)$ is ample and $(X, S)$ has a log canonical center of dimension $\leq 1$, then by considering $(X, S' = S + \varepsilon B)$, where $B$ is a suitable invariant divisor and $0 < \varepsilon \ll 1$, we get a non-lc pair $(X, S')$. This contradicts the above considered case.

Corollary 4.3 Let $X$ be a $G\mathbb{Q}$-Fano threefold, where $G$ is a 2-elementary abelian group with $r(G) \geq 6$ and let $S$ be an invariant Weil divisor. Then $-(K_X + S)$ is not ample.

Proof. If $-(K_X + S)$ is ample, then by Lemma 4.2 the pair $(X, S)$ is plt. By the adjunction principle [Sho93] the surface $S$ is irreducible, normal and has only quotient singularities. Moreover, $-K_S$ is ample. Hence $S$ is rational. We get a contradiction by Theorem 1.1 and Lemma 2.3(i).
Lemma 4.4 Let $S$ be a K3 surface with at worst Du Val singularities and let $\Gamma \subset \text{Aut}(S)$ be a 2-elementary abelian group. Then $r(\Gamma) \leq 5$.

Proof. Let $\tilde{S} \to S$ be the minimal resolution. Here $\tilde{S}$ is a smooth K3 surface and the action of $\Gamma$ lists to $\tilde{S}$. Let $\Gamma_s \subset \Gamma$ be the largest subgroup that acts trivially on $H^{2,0}(\tilde{S}) \simeq \mathbb{C}$. The group $\Gamma/\Gamma_s$ is cyclic. Hence, $r(\Gamma/\Gamma_s) \leq 1$. According to [Ni80, Th. 4.5] we have $r(\Gamma_s) \leq 4$. Thus $r(\Gamma) \leq 5$. □

Corollary 4.5 Let $X$ be a $G\mathbb{Q}$-Fano threefold, where $G$ is a 2-elementary abelian group. Let $S \in |-K_X|$ be a $G$-invariant member. If $r(\Gamma) \geq 7$, then the singularities of $S$ are worse than Du Val.

Proposition 4.6 Let $X$ be a $G\mathbb{Q}$-Fano threefold, where $G$ is a 2-elementary abelian group with $r(\Gamma) \geq 6$. Let $S \in |-K_X|$ be a $G$-invariant member and let $G_\bullet \subset G$ be the largest subgroup that acts trivially on the set of components of $S$. One of the following holds:

(i) $S$ is a K3 surface with at worst Du Val singularities, then $S \subset \text{Fix}(\delta)$ for some $\delta \in G \setminus \{1\}$ and $G/\langle \delta \rangle$ faithfully acts on $S$. In this case $r(G) = 6$.

(ii) The surface $S$ is reducible (and reduced). The group $G$ acts transitively on the components of $S$ and $G_\bullet$ acts faithfully on each component $S_i \subset S$. There are two subcases:

(a) any component $S_i \subset S$ is rational and $r(G_\bullet) \leq 4$.

(b) any component $S_i \subset S$ is birationally ruled over an elliptic curve and $r(G_\bullet) \leq 5$.

Proof. By Lemma 4.2 the pair $(X, S)$ is lc. Assume that $S$ is normal (and irreducible). By the adjunction formula $K_S \sim 0$. We claim that $S$ has at worst Du Val singularities. Indeed, otherwise by the Connectedness Principle [Sho93, Th. 6.9] $S$ has at most two non-Du Val points. These points are fixed by an index two subgroup $G' \subset G$. This contradicts Lemma 2.1. Taking Lemma 4.4 into account we get the case (i).

Now assume that $S$ is not normal. Let $S_i \subset S$ be an irreducible component (the case $S_i = S$ is not excluded). If the action on components $S_i \subset S$ is not transitive, there is an invariant divisor $S' \subset S$. Since $X$ is $G\mathbb{Q}$-factorial and $\rho(X)^G = 1$, the divisor $-\langle K_X + S' \rangle$ is ample. This contradicts Corollary 4.3 By Lemma 2.3(ii) the action of $G_\bullet$ on each component $S_i$ is faithful.

If $S_i$ is a rational surface, then $r(G_\bullet) \leq 4$ by Theorem 1.1. Assume that $S_i$ is not rational. Let $\nu: S' \to S_i$ be the normalization. Write $0 \sim \nu^*(K_X + S) = K_{S'} + D'$, where $D'$ is the different, see [Sho93] [3]. Here $D'$ is an effective reduced divisor and the pair is lc [Sho93, 3.2]. Since $S$ is not normal, $D' \neq 0$. Consider the minimal resolution $\mu: \tilde{S} \to S'$ and let $\tilde{D}$ be the crepant pull-back of $D'$, that is, $\mu_*\tilde{D} = D'$ and

$$K_S + \tilde{D} = \mu^*(K_{S'} + D') \sim 0.$$ 

Here $\tilde{D}$ is again an effective reduced divisor. Hence $\tilde{S}$ is a ruled surface. Consider the Albanese map $\alpha: \tilde{S} \to C$. Let $\tilde{D}_1 \subset \tilde{D}$ be an $\alpha$-horizontal component. By the
adjunction formula \( D_1 \) is an elliptic curve and so \( C \) is. Let \( \Gamma \) be the image of \( G_* \) in \( \text{Aut}(C) \). Then \( r(\Gamma) \leq 3 \) and so \( r(G_*) \leq 5 \). So, the last assertion is proved. \( \square \)

5 Non-Gorenstein Fano threefolds

Let \( G \) be a 2-elementary abelian group and let \( X \) be \( G\mathbb{Q} \)-Fano threefold. In this section we consider the case where \( X \) is non-Gorenstein, i.e., it has at least one terminal point of index \( \geq 1 \). We denote by \( \text{Sing}'(X) = \{P_1, \ldots, P_M\} \) the set of non-Gorenstein points and by \( B = B(X) \) the basket of singularities \([\text{Re87}]\). By \( B(X, P_i) \) we denote the basket of singularities at a point \( P_i \in X \).

**Proposition 5.1** Let \( X \) be a non-Gorenstein Fano threefold with terminal singularities. Assume that \( X \) admits a faithful action of a 2-elementary abelian group \( G \) with \( r(G) \geq 6 \). Then \( r(G) = 6 \), \( G \) transitively acts on \( \text{Sing}'(X) \), \( \{-K_X\} \neq \emptyset \), and the configuration of non-Gorenstein singularities is described below:

1. \( M = 8 \), \( B(X, P_i) = \{ \frac{1}{3}(1,1,1) \} \);
2. \( M = 8 \), \( B(X, P_i) = \{ \frac{1}{4}(1,1,2) \} \);
3. \( M = 4 \), \( B(X, P_i) = \{ 2 \times \frac{1}{4}(1,1,1) \} \);
4. \( M = 4 \), \( B(X, P_i) = \{ 2 \times \frac{1}{4}(1,1,2) \} \);
5. \( M = 4 \), \( B(X, P_i) = \{ 3 \times \frac{1}{6}(1,1,1) \} \);
6. \( M = 4 \), \( B(X, P_i) = \{ \frac{1}{3}(-1,1), \frac{1}{2}(1,1,1) \} \).

**Proof.** Let \( P^{(1)}, \ldots, P^{(n)} \in \text{Sing}'(X) \) be representatives of distinct \( G \)-orbits and let \( G_i \) be the stabilizer of \( P^{(i)} \). Let \( r := r(G), r_i := r(G_i) \), and let \( m_{i,1}, \ldots, m_{i,v_i} \) be the indices of points in the basket of \( P^{(i)} \). We may assume that \( m_{i,1} \geq \cdots \geq m_{i,v_i} \). By the orbifold Riemann-Roch formula \([\text{Re87}]\) and a form of Bogomolov-Miyaoka inequality \([\text{Ka92}],[\text{KM2T}]\) we have

\[
\sum_{i=1}^{n} 2^{r-r_i} \sum_{j=1}^{v_i} \left( m_{i,j} - \frac{1}{m_{i,j}} \right) < 24. \tag{***}
\]

If \( P \) is a cyclic quotient singularity, then \( v_i = 1 \) and by Lemma 2.1, \( r_i \leq 3 \). If \( P \) is not a cyclic quotient singularity, then \( v_i \geq 2 \) and again by Lemma 2.1, \( r_i \leq 4 \). Since \( m_{i,j} - 1/m_{i,j} \geq 3/2 \), in both cases we have

\[
2^{r-r_i} \sum_{j=1}^{v_i} \left( m_{i,j} - \frac{1}{m_{i,j}} \right) \geq 3 \cdot 2^{r-4} \geq 12.
\]

Therefore, \( n = 1 \), i.e. \( G \) transitively acts on \( \text{Sing}'(X) \), and \( r = 6 \).

If \( P \) is not a point of type \( eAx/4 \) (i.e. it is not as in (3) of (**)), then by the classification of terminal singularities \([\text{Re87}]\) \( m_{1,1} = \cdots = m_{1,v_i} \) and (***) has the form
\[ 24 > 2^{6-r_1} v_1 \left( m_{1,1} - \frac{1}{m_{1,1}} \right) \geq 8 \left( m_{1,1} - \frac{1}{m_{1,1}} \right). \]

Hence \( r_1 \geq 3, v_1 \leq 3, m_{1,1} \leq 3, \) and \( 3 \cdot 2^{r_1-3} \geq v_1 m_{1,1}. \) If \( r_1 = 3, \) then \( v_1 = 1. \) If \( r_1 = 4, \) then \( v_1 \geq 2 \) and \( v_1 m_{1,1} \leq 6. \) This gives us the possibilities (1)-(5).

Assume that \( P \) is a point of type \( cAx/4. \) Then \( m_{1,1} = 4, v_1 > 1, \) and \( m_{1,j} = 2 \) for \( 1 < j \leq v_1. \) Thus (***) has the form

\[ 24 > 2^{6-r_1} \left( \frac{15}{4} + \frac{3}{2} (v_1 - 1) \right) = 2^{4-r_1} (9 + 6v_1). \]

We get \( v_1 = 2, r_1 = 4, \) i.e. the case (6).

Finally, the computation of \( \dim |-K_X| \) follows by the orbifold Riemann-Roch formula [Re87]

\[ \dim |-K_X| = -\frac{1}{2} K_X^3 + 2 - \sum_{p \in B(X)} \frac{b_p(m_p - b_p)}{2m_p}. \]

6 Gorenstein Fano threefolds

The main result of this section is the following:

**Proposition 6.1** Let \( G \) be a 2-elementary abelian group and let \( X \) be a (Gorenstein) \( G \)-Fano threefold. Then \( r(G) \leq 6. \) Moreover, if \( r(G) = 6, \) then \( \text{Pic}(X) = \mathbb{Z} \cdot K_X \) and \( -K_X^3 \geq 8. \)

Let \( X \) be a Fano threefold with at worst Gorenstein terminal singularities. Recall that the number

\[ t(X) := \max \{ i \in \mathbb{Z} \mid -K_X \sim iA, A \in \text{Pic}(X) \} \]

is called the *Fano index* of \( X. \) The integer \( g = g(X) \) such that \( -K_X^3 = 2g - 2 \) is called the *genus* of \( X. \) It is easy to see that \( \dim |-K_X| = g + 1 \) [IP99, Corollary 2.1.14]. In particular, \( |-K_X| \neq 0. \)

**Notation.** Throughout this section \( G \) denotes a 2-elementary abelian group and \( X \) denotes a Gorenstein \( G \)-Fano threefold. There exists an invariant member \( S \in |-K_X| \) (see 4.1). We write \( S = \sum_{i=1}^{N} S_i, \) where the \( S_i \) are irreducible components. Let \( G_\bullet \subset G \) be the kernel of the homomorphism \( G \to \mathfrak{S}_N \) induced by the action of \( G \) on \( \{ S_1, \ldots, S_N \}. \) Since \( G \) is abelian and the action of \( G \) on \( \{ S_1, \ldots, S_N \} \) is transitive, the group \( G_\bullet \) coincides with the stabilizer of any \( S_i. \) Clearly, \( N = 2^{r(G)} - r(G_\bullet). \) If \( r(G) \geq 6, \) then by Proposition 4.6 we have \( r(G_\bullet) \leq 5 \) and so \( N \geq 2^{r(G)} - 5. \)

**Lemma 6.2** Let \( G \subset \text{Aut}(\mathbb{P}^n) \) be a 2-elementary subgroup and \( n \) is even. Then \( G \) is conjugate to a diagonal subgroup. In particular, \( r(G) \leq n. \)
Lemma 6.4
Let $G \subset SL_{n+1}(k)$ be the lifting of $G$ and let $G' \subset G^2$ be a Sylow 2-subgroup. Then $G' \simeq G$. Since $G'$ is abelian, the representation $G' \to SL_{n+1}(k)$ is diagonalizable.

Corollary 6.3
Let $Q \subset \mathbb{P}^4$ be a quadric and let $G \subset \text{Aut}(Q)$ be a 2-elementary subgroup. Then $r(G) \leq 4$.

Lemma 6.4
Let $G \subset \text{Aut}(\mathbb{P}^3)$ be a 2-elementary subgroup. Then $r(G) \leq 4$.

Proof. Assume that $r(G) \geq 5$. Take any element $\delta \in G \setminus \{1\}$. By Lemma 2.1 the group $G$ has no fixed points. Since the set $\text{Fix}(\delta)$ is $G$-invariant, $\text{Fix}(\delta) = L_1 \cup L_2$, where $L_1, L_2 \subset \mathbb{P}^3$ are skew lines.

Let $G_1 \subset G$ be the stabilizer of $L_1$. There is a subgroup $G_2 \subset G_1$ of index 2 having a fixed point $P \in L_1$. Thus $r(G_2) \geq 3$ and the “orthogonal” plane $\Pi$ is $G_2$-invariant. By Lemma 6.2 there exists an element $\delta' \in G_2$ that acts trivially on $\Pi$, i.e. $\Pi \subset \text{Fix}(\delta')$. But then $\delta'$ has a fixed point, a contradiction. □

Lemma 6.5
If $Bs|−K_X| \neq \emptyset$, then $r(G) \leq 4$.

Proof. By [Shi89] the base locus $Bs|−K_X|$ is either a single point or a rational curve. In both cases $r(G) \leq 4$ by Lemmas 2.1 and 2.3. □

Lemma 6.6
If $−K_X$ is not very ample, then $r(G) \leq 5$.

Proof. Assume that $r(G) \geq 6$. By Lemma 6.3 the linear system $|−K_X|$ is base point free. It is easy to show that $|−K_X|$ defines a double cover $\phi : X \to Y \subset \mathbb{P}^{g+1}$ (cf. [Is80] Ch. 1, Prop. 4.9)). Here $Y$ is a variety of degree $g−1$ in $\mathbb{P}^{g+1}$, a variety of minimal degree. Let $G$ be the image of $G$ in $\text{Aut}(Y)$. Then $r(G) \geq r(G) − 1$. If $g = 2$ (resp. $g = 3$), then $Y = \mathbb{P}^3$ (resp. $Y \subset \mathbb{P}^4$ is a quadric) and $r(G) \leq 5$ by Lemma 6.4 (resp. by Corollary 6.3). Thus we may assume that $g \geq 4$. If $Y$ is smooth, then according to the Enriques theorem (see, e.g., [Is80] Th. 3.11) $Y$ is a rational scroll $\mathbb{P}_{1}([\delta])$, where $[\delta]$ is a rank 3 vector bundle on $\mathbb{P}^1$. Then $X$ has a $G$-equivariant projection to a curve. This contradicts $\rho(X)^G = 1$. Hence $Y$ is singular. In this case, $Y$ is a projective cone (again by the Enriques theorem). If its vertex $O \in Y$ is zero-dimensional, then $\dim T_O Y \geq 5$. On the other hand, $X$ has only hypersurface singularities. Therefore the double cover $X \to Y$ is not étale over $O$ and so $G$ has a fixed point on $X$. This contradicts Lemma 2.1. Thus $Y$ is a cone over a curve with vertex along a line $L$. As above, $L$ must be contained in the branch divisor and so $L' := \phi^{-1}(L)$ is a $G$-invariant rational curve. Since the image of $G$ in $\text{Aut}(L')$ is a 2-elementary abelian group of rank $\leq 2$, by Lemma 2.3 we have $r(G) \leq 4$. □

Remark 6.7.
Recall that for a Fano threefold $X$ with at worst Gorenstein terminal singularities one has $r(X) \leq 4$. Moreover, $r(X) = 4$ if and only if $X \simeq \mathbb{P}^3$ and $r(X) = 3$ if and only if $X$ is a quadric in $\mathbb{P}^4$ [IP92]. In these cases we have $r(G) \leq 4$ by Lemma 6.4 and Corollary 6.3 respectively. Assume that $r(X) = 2$. Then $X$ is so-called del Pezzo threefold. Let $A := −\frac{1}{2}K_X$. The number $d := A^3$ is called the degree of $X$. 
Lemma 6.8 Assume that the divisor $-K_X$ is very ample, $r(G) \geq 6$, and the action of $G$ on $X$ is not free in codimension 1. Let $\delta \in G$ be an element such that $\dim \text{Fix}(\delta) = 2$ and let $D \subset \text{Fix}(\delta)$ be the union of all two-dimensional components. Then $r(G) = 6$ and $D$ is a Du Val member of $|-K_X|$. Moreover, $r(X) = 1$ except, possibly, for the case where $r(X) = 2$ and $-\frac{1}{2}K_X$ is not very ample.

Proof. Since $G$ is abelian, $\text{Fix}(\delta)$ and $D$ are $G$-invariant and so $-K_X \sim \lambda D$ for some $\lambda \in \mathbb{Q}$. In particular, $D$ is a $\mathbb{Q}$-Cartier divisor. Since $X$ has only terminal Gorenstein singularities, $D$ must be Cartier. Clearly, $D$ is smooth outside of $\text{Sing}(X)$. Further, $D$ is ample and so it must be connected. Since $D$ is a reduced Cohen-Macaulay scheme with $\dim \text{Sing}(D) \leq 0$, it is irreducible and normal.

Let $X \hookrightarrow \mathbb{P}^{g+1}$ the anticanonical embedding. The action of $\delta$ on $X$ is induced by an action of a linear involution of $\mathbb{P}^{g+1}$. There are two disjointed linear subspaces $V_+, V_- \subset \mathbb{P}^{g+1}$ of $\delta$-fixed points and the divisor $D$ is contained in one of them. This means that $D$ is a component of a hyperplane section $S \subset -K_X$ and so $\lambda \geq 1$. Since $r(G) \geq 6$, by Corollary 4.3 we have $\lambda = 1$ and $-K_X \sim D$ (because $\text{Pic}(X)$ is a torsion free group). Since $D$ is irreducible, the case (i) of Proposition 4.6 holds.

Finally, if $r(X) > 1$, then by Remark 6.7 we have $r(X) = 2$. If furthermore the divisor $A$ is very ample, then it defines an embedding $X \hookrightarrow \mathbb{P}^N$ so that $D$ spans $\mathbb{P}^N$. In this case the action of $\delta$ must be trivial, a contradiction. □

Lemma 6.9 If $\rho(X) > 1$, then $r(G) \leq 5$.

Proof. We use the classification of $G$-Fano threefolds with $\rho(X) > 1$ [Pr13b]. By this classification $\rho(X) \leq 4$. Let $G_0$ be the kernel of the action of $G$ on $\text{Pic}(X)$.

Consider the case $\rho(X) = 2$. Then $|G : G_0| = 2$. In the cases (1.2.1) and (1.2.4) of [Pr13b] the variety $X$ has a structure of $G_0$-equivariant conic bundle over $\mathbb{P}^2$. As in Proposition 3.2 we have $r(G_0) \leq 4$ and $r(G) \leq 5$ in these cases. In the cases (1.2.2) and (1.2.3) of [Pr13b] the variety $X$ has two birational contractions to $\mathbb{P}^3$ and a quadric $Q \subset \mathbb{P}^4$, respectively. As above we get $r(G) \leq 5$ by Lemma 6.4 and Corollary 6.3.

Consider the case $\rho(X) = 3$. We show that in this case $\text{Pic}(X)^G \not\cong \mathbb{Z}$ (and so this case does not occur). Since $G$ is a 2-elementary abelian group, its action on $\text{Pic}(X) \otimes \mathbb{Q}$ is diagonalizable. Since $\text{Pic}(X)^G = \mathbb{Z} \cdot K_X$, the group $G$ contains an element $\tau$ that acts on $\text{Pic}(X) \simeq \mathbb{Z}^3$ as the reflection with respect to the orthogonal complement to $K_X$. Since the group $G$ preserves the natural bilinear form $\langle x_1, x_2 \rangle := x_1 \cdot x_2 \cdot K_X$, the action must be as follows

$$\tau : x \mapsto x - \lambda K_X, \quad \lambda = \frac{2x \cdot K_X^2}{K_X^3}.$$

Hence $\lambda K_X$ is an integral element for any $x \in \text{Pic}(X)$. This gives a contradiction in all cases (1.2.5)-(1.2.7) of [Pr13b] Th. 1.2. For example, in the case (1.2.5) of [Pr13b] Th. 1.2 our variety $X$ has a structure (non-minimal) del Pezzo fibration of degree 4 and $-K_X^2 = 12$. For the fiber $F$ we have $F \cdot K_X^2 = K_F^2 = 4$ and $\lambda K_X$ is not integral, a contradiction.
Finally, consider the case \( \rho(X) = 4 \). Then according to \( \text{[Pr13a]} \) \( X \) is a divisor of multidegree \((1, 1, 1, 1)\) in \( \mathbb{P}^1_X. \) All the projections \( \varphi_i : X \to \mathbb{P}^1, i = 1, \ldots, 4 \) are \( G_0 \)-equivariant. We claim that natural maps \( \varphi_i : G_0 \to \text{Aut}(X) \) are injective. Indeed, assume that \( \varphi_{i}(\vartheta) \) is the identity map in \( \text{Aut}(X) \) for some \( \vartheta \in G. \) This means that \( \vartheta \circ \varphi_i = \varphi_i. \) Since \( \text{Pic}(X)^G = \mathbb{Z} \), the group \( G \) permutes the classes \( \varphi_i^* \mathcal{O}_{\mathbb{P}^1}(1) \in \text{Pic}(X) \). Hence, for any \( i = 1, \ldots, 4 \), there exists \( \sigma_i \in G \) such that \( \varphi_i = \varphi_i \circ \sigma_i. \) Then
\[
\vartheta \circ \varphi_i = \vartheta \circ \varphi_i \circ \sigma_i = \varphi_i \circ \sigma_i = \varphi_i.
\]
Hence, \( \varphi_i(\vartheta) \) is the identity for any \( i. \) Since \( \varphi_1 \times \cdots \times \varphi_4 \) is an embedding, \( \vartheta \) must be the identity as well. This proves our claim. Therefore, \( r(G_0) \leq 2 \). The group \( G/G_0 \) acts on \( \text{Pic}(X) \) faithfully. By the same reason as above, an element of \( G/G_0 \) cannot act as the reflection with respect to \( K_X. \) Therefore, \( r(G/G_0) \leq 2 \) and \( r(G) \leq 4. \)

**Lemma 6.10** If \( t(X) = 2 \), then \( r(G) \leq 5. \)

**Proof.** By Lemma 6.9 we may assume that \( \rho(X) = 1. \) Let \( d \) be the degree of \( X. \) Since \( \rho(X) = 1 \), we have \( d \leq 5 \) (see e.g. \( \text{[Pr13a]} \)). Consider the possibilities for \( d \) case by case. We use the classification (see \( \text{[Shi89]} \) and \( \text{[Pr13a]} \)).

If \( d = 1 \), then the linear system \( |A| \) has a unique base point. This point is smooth and must be \( G \)-invariant. By Lemma 2.1 \( r(G) \leq 3. \) If \( d = 2 \), then the linear system \( |A| \) defines a double cover \( \varphi : X \to \mathbb{P}^3. \) Then the image of \( G \) in \( \text{Aut}(\mathbb{P}^3) \) is a 2-elementary group \( \hat{G} \) with \( r(\hat{G}) \geq r(G) - 1, \) where \( r(\hat{G}) \leq 4 \) by Lemma 6.4. If \( d = 3 \), then \( X = X_3 \subset \mathbb{P}^4 \) is a cubic hypersurface. By Lemma 6.2 \( r(G) \leq 4. \) If \( d = 5 \), then \( X \) is smooth, unique up to isomorphism, and \( \text{Aut}(X) \simeq \text{PGL}_2(k) \) (see \( \text{[IP99]} \)).

Finally, consider the case \( d = 4. \) Then \( X = Q_1 \cap Q_2 \subset \mathbb{P}^3 \) is an intersection of two quadrics (see e.g. \( \text{[Shi89]} \)). Let \( \mathcal{Q} \) be the pencil generated by \( Q_1 \) and \( Q_2. \) Since \( X \) has a isolated singularities and it is not a cone, a general member of \( \mathcal{Q} \) is smooth by Bertini’s theorem and for any member \( Q \in \mathcal{Q} \) we have \( \dim \text{Sing}(Q) \leq 1. \) Let \( D \) be the divisor of degree 6 on \( \mathcal{Q} \simeq \mathbb{P}^1 \) given by the vanishing of the determinant. The elements of \( \text{Supp}(D) \) are exactly degenerate quadrics. Clearly, for any point \( P \in \text{Sing}(X) \) there exists a unique quadric \( Q \in \mathcal{Q} \) which is singular at \( P. \) This defines a map \( \pi : \text{Sing}(X) \to \text{Supp}(D). \) Let \( Q \in \text{Supp}(D). \) Then \( \pi^{-1}(Q) = \text{Sing}(Q) \cap X = \text{Sing}(Q) \cap Q', \) where \( Q' \in \mathcal{Q}, Q' \neq Q. \) In particular, \( \pi^{-1}(Q) \) consists of at most two points. Hence the cardinality of \( \text{Sing}(X) \) is at most 12.

Assume that \( r(G) \geq 6. \) Let \( S \in | - K_X | \) be an invariant member. We claim that \( S \supset \text{Sing}(X) \) and \( \text{Sing}(X) \neq \emptyset. \) Indeed, otherwise \( S \cap \text{Sing}(X) = \emptyset. \) By Proposition 4.7 \( S \) is reducible: \( S = S_1 + \cdots + S_N, N \geq 2. \) Since \( t(X) = 2, \) we get \( N = 2 \) and \( S_1 \sim S_2, \) i.e. \( S_i \) is a hyperplane section of \( X \subset \mathbb{P}^5. \) As in the proof of Corollary 4.3 we see that \( S_i \) is rational. This contradicts Proposition 4.6(ii). Thus \( \emptyset \neq \text{Sing}(X) \subset S. \)

By Lemma 6.8 the action of \( G \) on \( X \) is free in codimension 1. By Remark 2.2 for the stabilizer \( G_P \) of a point \( P \in \text{Sing}(X) \) we have \( r(G_P) \leq 3. \) Then by the above estimate the variety \( X \) has exactly 8 singular points and \( G \) acts on \( \text{Sing}(X) \) transitively.

Note that our choice of \( S \) is not unique: there is a basis \( S^{(1)}, \ldots, S^{(x+2)} \in H^0(X, - K_X) \) consisting of eigensections. This basis gives us \( G \)-invariant divisors \( S^{(1)}, \ldots, S^{(x+2)} \) generating \( | - K_X |. \) By the above \( \text{Sing}(X) \subset S^{(i)} \) for all \( i. \) Thus \( \text{Sing}(X) \subset \cap S^{(i)} = B S | - K_X |. \) This contradicts the fact that \( - K_X \) is very ample. \( \square \)
Example 6.11. The bound \( r(G) \leq 5 \) in the above lemma is sharp. Indeed, let \( X \subset \mathbb{P}^3 \) be the variety given by \( \sum x_i^2 = \sum \lambda_i x_i^2 = 0 \) with \( \lambda_i \neq \lambda_j \) for \( i \neq j \) and let \( G \subset \text{Aut}(X) \) be the 2-elementary abelian subgroup generated by involutions \( x_i \mapsto -x_i \). Then \( r(G) = 5 \).

From now on we assume that \( \text{Pic}(X) = \mathbb{Z} \cdot K_X \). Let \( g := g(X) \).

**Lemma 6.12** If \( g \leq 4 \), then \( r(G) \leq 5 \). If \( g = 5 \), then \( r(G) \leq 6 \).

**Proof.** We may assume that \( -K_X \) is very ample. Automorphisms of \( X \) are induced by projective transformations of \( \mathbb{P}^{g+1} \) that preserve \( X \subset \mathbb{P}^{g+1} \). On the other hand, there is a natural representation of \( G \) on \( H^0(X, -K_X) \) which is faithful. Thus the composition

\[
\text{Aut}(X) \hookrightarrow GL(H^0(X, -K_X)) = GL_{g+2}(k) \to PGL_{g+2}(k)
\]

is injective. Since \( G \) is abelian, its image \( \tilde{G} \subset GL_{g+2}(k) \) is contained in a maximal torus and by the above \( \tilde{G} \) contains no scalar matrices. Hence, \( r(G) \leq g + 1 \). \( \square \)

Example 6.13. Let \( G \) be the 2-torsion subgroup of the diagonal torus of \( PGL_2(k) \). Then \( X \) faithfully acts on the Fano threefold in \( \mathbb{P}^6 \) given by the equations \( \sum x_i^2 = \sum \lambda_i x_i^2 = \sum \mu_i x_i^2 = 0 \). This shows that the bound \( r(G) \leq 6 \) in the above lemma is sharp. Note however that \( X \) is not rational if it is smooth \([Be77]\). Hence in this case our construction does not give any embedding of \( G \) to \( \text{Cr}_3(k) \).

**Lemma 6.14** If in the above assumptions \( g(X) \geq 6 \), then \( X \) has at most 29 singular points.

**Proof.** According to \([Na97]\) the variety \( X \) has a smoothing. This means that there exists a flat family \( X \to \mathcal{X} \) over a smooth one-dimensional base \( \mathcal{X} \) with special fiber \( X = \mathcal{X}_0 \) and smooth general fiber \( X_1 = \mathcal{X}_1 \). Using the classification of Fano threefolds \([Is80]\) (see also \([IP99]\)) we obtain \( h^{1,2}(X_i) \leq 10 \). Then by \([Na97]\) we have

\[
\#\text{Sing}(X) \leq 21 - \frac{1}{2} \text{Eu}(X_i) = 20 - \rho(X_i) + h^{1,2}(X_i) \leq 29.
\]

**Proof of Proposition 6.1** Assume that \( r(G) \geq 7 \). Let \( S \in |-K_X| \) be an invariant member. By Corollary 4.5 the singularities of \( S \) are worse than Du Val. So \( S \) satisfies the conditions (ii) of Proposition 4.6. Write \( S = \sum_{i=1}^N S_i \). By Proposition 4.6 the group \( G_\bullet \) acts on \( S_i \) faithfully and

\[
N = 2^{r(G) - r(G_\bullet)} \geq 4.
\]

First we consider the case where \( X \) is smooth near \( S \). Since \( \rho(X) = 1 \), the divisors \( S_i \)’s are linear equivalent to each other and so \( t(X) \geq 4 \). This contradicts Lemma 6.10.

Therefore, \( S \cap \text{Sing}(X) \neq \emptyset \). By Lemma 6.8 the action of \( G \) on \( X \) is free in codimension 1 and by Remark 2.2 we see that \( r(G_F) \leq 3 \), where \( G_F \) is the stabilizer
of a point $P \in \text{Sing}(X)$. Then by Lemma 6.14 the variety $X$ has exactly 16 singular points and $G$ acts on $\text{Sing}(X)$ transitively. Since $S \cap \text{Sing}(X) \neq \emptyset$, we have $\text{Sing}(X) \subset S$. On the other hand, our choice of $S$ is not unique: there is a basis $s_i^{(1)}, \ldots, s_i^{(g+2)} \in H^0(X, -K_X)$ consisting of eigensections. This basis gives us $G$-invariant divisors $S_i^{(1)}, \ldots, S_i^{(g+2)}$ generating $|-K_X|$. By the above $\text{Sing}(X) \subset S_i^{(i)}$ for all $i$. Thus $\text{Sing}(X) \subset \cap S_i^{(i)} = Bs|^{-K_X}$. This contradicts Lemma 6.6. □

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