On the Finite Temperature Phase Transition of Scalar QED a

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We analyse the finite temperature phase transition of scalar QED in euclidean spacetime. Analytical solutions of approximations to a Wilsonian renormalisation group equation are discussed. Special emphasis is put on the discussion of the 3d running of the Abelian charge and its effects on thermodynamical quantities. An upper bound for the range of first-order phase transitions is given. Our results are compared to those of resummed perturbation theory.

1 Introduction

The calculation of thermodynamical quantities related to a first-order phase transition necessitates the knowledge of the effective potential at the critical temperature. The latter can be obtained in two steps. Step I integrates-out the so-called heavy and super-heavy modes, i.e. the non-zero Matsubara frequencies modes of all fields, as well as the Debye mode. This reduces the 4d problem to a purely 3d one. Integrating-out the remaining fluctuations is performed in Step II. In the parameter range of interest, Step I can be performed perturbatively. Step II is more involved. Monte-Carlo techniques have been used for large Higgs boson mass \( m_H \) as perturbation theory appears to be applicable only for small \( m_H \). However, the 3d gauge coupling \( \tilde{e}_3 \) is a dimensionful quantity and displays a non-trivial scale dependence which is not accounted for within the standard perturbative approach. The aim of the present note is to give an alternative treatment of Step II clarifying the impact of a running gauge coupling on thermodynamical quantities at the example of scalar QED.

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2 Flow equations

In order to quantify the effect of the running gauge coupling, we will use a Wilsonian renormalisation group method based upon the effective average action. It gives a prescription about how an effective action changes with scale \( k \) when fluctuations with momenta \( p \) have been integrated out down to \( p \approx k \). As a result, one obtains a coupled set of flow equations w.r.t. \( k \) for both the 3d Abelian charge \( \vec{e}_3 \) and the effective average potential \( U_k \). The conventional effective action is obtained in the limit \( k \to 0 \). For simplicity, we will neglect the contributions from the scalar fluctuations to the scale-dependence of the potential. They are known to be subleading compared to those of the gauge field, as long as \( m_H \) is smaller than the photon mass \( M \).

Furthermore, we will employ a sharp cut-off regularisation throughout. The (in-)dependence of physical quantities on the regularisation scheme employed is discussed elsewhere.

Under the above assumptions, the flow equations for \( e_3^2 = \vec{e}_3^2/k \) and \( U_k \) read

\[
\frac{d e_3^2}{dk} = -e_3^2 (1 - \eta_F), \quad \frac{d U_k}{dk} = \frac{k^3}{2\pi} \ln \left( 1 + \frac{2 e_3^2(k) \bar{\rho}}{k} \right). \tag{1}
\]

The anomalous dimension \( \eta_F \) can be approximated by \( \eta_F = e_3^2/e_*^2 \), with \( e_*^2 \) denoting an effective fixed point for the dimensionless gauge coupling. For vanishing scalar and gauge field mass we find \( e_*^2 \approx 6\pi^2 \) in agreement with the \( \epsilon \)-expansion. However, the effective fixed point value \( e_*^2 \) can be different for non-vanishing masses, which is why we will keep \( e_*^2 \) as a free parameter. The running gauge coupling follows from the above as

\[
e_3^2(k) = \frac{e_*^2}{1 + k/k_{tr}}, \quad k_{tr} = \frac{\Lambda e_3^2(\Lambda)}{e_*^2 - e_3^2(\Lambda)}. \tag{2}
\]

The initial scale \( \Lambda \) is proportional to temperature and will be determined below. Two comments are in order. Firstly, note the appearance of a characteristic transition scale \( k_{tr} \), describing the cross-over between the Gaussian and the Abelian fixed point. For \( k > k_{tr} \) the running is very slow and dominated by the Gaussian fixed point, \( \vec{e}_3^2 \sim \text{const.} \), whereas for \( k < k_{tr} \) the running becomes strongly linear and the Abelian fixed point governs the scale dependence, \( \vec{e}_3^2(k) \sim k \). Secondly, the limit \( e_*^2 \to \infty \) corresponds formally to neglecting the running of \( \vec{e}_3^2 \) throughout, because \( \vec{e}_3^2(k) = \Lambda e_3^2(\Lambda) = \text{const.} \) in this case.

The initial conditions to eq. (1), obtained through Step I, read at the scale of dimensional reduction \( \Lambda = \xi T \)

\[
U_\Lambda(\bar{\rho}) = (A T^2 - \frac{1}{2} m_H^2) \bar{\rho} + \frac{1}{2} B T \bar{\rho}^2, \quad e_3^2(\Lambda) = e^2 T, \\
A = e^2/4 + \lambda/6 - e^3/4\sqrt{3}\pi, \quad B = e^4/4\pi^2 - 2e^3/\sqrt{3}\pi + \lambda. \tag{3}
\]
Here, $\lambda$ and $\epsilon$ denote resp. the zero temperature quartic scalar/gauge field coupling. Finally, the effective average potential for scales $k$ smaller than $\Lambda$ follows (with $\bar{\rho} = \phi^* \phi$) as
\begin{equation}
U_k(\bar{\rho}) = U_\Lambda(\bar{\rho}) + \Delta_k(\bar{\rho}) , \quad \Delta_k(\bar{\rho}) = \int_\Lambda^k dk \frac{k^2}{2\pi} \ln \left( 1 + \frac{2\epsilon^2_3(k)\bar{\rho}}{k} \right) .
\end{equation}

The integral in eq. (4) can be solved analytically. Its solution is discussed in the following section.

### 3 Criticality

We are now interested in the characteristics of the first order phase transition, using $\epsilon = 0.3$ and $M = 80.6$ GeV as initial parameters. The latter is related to the coupling $\lambda$ through $\lambda/\epsilon^2 = m_H^2/M^2$. The physical quantities that characterise a first-order phase transition are defined at the critical temperature $T_c$, when the potential has two degenerate minima at $\bar{\rho} = 0$ and a non-trivial one at $\bar{\rho} = \bar{\rho}_0 \neq 0$. Using $x = \bar{\rho}_0/T$ and $\Delta = \Delta/T^3$, the conditions for a degenerate potential read
\begin{equation}
\frac{m_H^2}{T^2_c} - 2A = \frac{2}{x^2} \left[ 2\Delta(x) - x\Delta'(x) \right] , \quad B = \frac{2}{x^2} \left[ \Delta(x) - x\Delta'(x) \right] .
\end{equation}

Note that, for $x > 0$, the r.h.sides are positive, finite, monotonically decreasing and vanishing for $x \to \infty$. They reach their respective maximum at $x = 0$. $B$, as given by eq. (4) and fixed through the 4d parameters of the theory, is positive in the domain under consideration. It follows that a solution to eq. (5) is unique (if it exists). There exists no solution for too large values of $B$. Thus,
\begin{equation}
\frac{m_H^2}{M^2} \approx \frac{8\epsilon^2}{3\pi^2}
\end{equation}
is an upper bound for the scalar mass. For larger $m_H$ the phase transition ceases to be first order. This follows from the very existence of an effective fixed point for the Abelian charge and is in contrast to the perturbative analysis, since a non-running of the gauge coupling ($\epsilon^2 \to \infty$) would predict a first order phase transition for all $m_H$. We interpret this upper limit as a sign for the existence of a tricritical fixed point that marks the endpoint of a line of first order phase transitions. For too large values of $\epsilon^2$, however, this argument is no longer valid as eq. (4) lies outside the domain of validity.

Our results are displayed in the figures 1 – 4. Fig. 1 shows the optimised matching parameter $\xi$ as a function of the Higgs boson mass. It obtains as
the unique solution to the minimum sensitivity condition $d\bar{\rho}_0(\xi)/d\xi = 0$, and ensures that physical quantities do not depend (to leading order) on the details of the matching itself. The magnitude of $\xi$ for small $m_H$ indicates that the two-step matching is not reliable in this region. For larger Higgs boson mass, $\xi$ becomes nearly independent of $m_H$.

Figs. 2 and 3 compare our results with those from resummed perturbation theory. The critical potential at $m_H = 38$ GeV for $\epsilon_* = \sqrt{6}\pi$ (Fig. 2) comes quite close to the two-loop perturbative result. This indicates that the gauge field fluctuations do indeed give the largest contribution to the running potential. We checked also that the leading contributions of the scalar fluctuations give only a small effect. The same holds true for the v.e.v. as a function of $m_H$ (Fig. 3). We wish to emphasize that the latent heat $L = T \partial_T U(\bar{\rho}_0, T_c)$ can be obtained directly from fig. 3, because any solution to eq. (5) automatically fulfills the Clausius-Clapeyron equation $L = \bar{\rho}_0 m^2$. This feature was also observed within a gauge-invariant perturbative calculation, but not within the standard perturbative approach.

The dependence of the critical temperature on the fixed point value can be read off from fig. 4. Note that $T_c$ remains independent of $e^2_\ast$ for $e^2_\ast > 6\pi^2$. Even for small values of $e^2_\ast$, $T_c$ increases only slowly. This is the region where the running of the gauge coupling becomes important due to a sufficiently small value of the fixed point. For large $e^2_\ast$ and small $m_H$ the scale $k_{tr}$ is much smaller.
then the characteristic scales relevant for the first-order phase transition. The scale dependence of the gauge coupling is therefore of no importance in this domain and its neglect seems to be well justified.

To conclude, we have seen that the results of two-loop perturbation theory can be obtained within a simple and systematic approximation to a Wilsonian RG. The very existence of a (partial) fixed point for the gauge coupling seems to imply the existence of a tricritical fixed point. Quantitatively, the effects from the running of the gauge coupling can be neglected for small $m_H$. However, they are getting stronger for larger $m_H$ and smaller $e^2$. With $e^2$ near or above its perturbative value $\approx 6\pi^2$ the effects are of the order of percents or smaller.

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