ACCRETION DISK TORQUED BY A BLACK HOLE

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ABSTRACT

If a Kerr black hole is connected to a disk rotating around it by a magnetic field, the rotational energy of the Kerr black hole provides an energy source for the radiation of the disk in addition to disk accretion. The black hole exerts a torque on the disk, which transfers energy and angular momentum between the black hole and the disk. If the black hole rotates faster than the disk, energy and angular momentum are extracted from the black hole and transferred to the disk. If the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. With suitable boundary conditions, quasi–steady state solutions are obtained for a thin Keplerian disk magnetically coupled to a Kerr black hole. By “quasi–steady state” we mean that any macroscopic quantity at a given radius in the disk slowly changes with time: the integrated change within one rotation period of the disk is much smaller than the quantity itself. We find that the torque produced by magnetic coupling propagates outward only in the disk, and the total radiation flux of the disk is a superposition of the radiation flux produced by magnetic coupling and that produced by accretion. Interestingly, a disk magnetically coupled to a rapidly rotating black hole can radiate without accretion; the total power of the disk comes from the rotational energy of the black hole. With a simple example that the magnetic field touches the disk at a single radius, we show that the radial radiation profile produced by magnetic coupling can be very different from that of a standard accretion disk: the emissivity index is significantly larger, and most radiation can come from a region that is closer to the center of the disk. While the shape of the radiation flux curve sensitively depends on the extension of the magnetic field in the disk, the spectral signature of magnetic coupling can be robust. The limitations of our model are briefly discussed, which include the assumption of a weak magnetic field, the ignorance of the instabilities of the disk and the magnetic field, and the ignorance of the radiation captured by the black hole and the radiation returning to the disk.

Subject headings: accretion, accretion disks — black hole physics — magnetic fields

1. INTRODUCTION

In the standard theory of an accretion disk around a black hole, it is assumed that there is no coupling between the disk and the central black hole (Pringle & Rees 1972; Shakura & Sunyaev 1973; Novikov & Thorne 1973; Lynden-Bell & Pringle 1974). However, in the presence of a magnetic field, magnetic coupling between the disk and the black hole could exist and play an important role in the balance and transportation of energy and angular momentum (Zeldovich & Schwartzman, quoted in Thorne 1974; Thorne, Price, & Macdonald 1986; Blandford 1998, 1999, 2000; van Putten 1999; Gruzinov 1999; Li 2000c; Li & Paczyński 2000; Brown et al. 2000). Though this issue was commented on by Zeldovich & Schwartzman more than 20 years ago and by many other people afterward, a detailed and quantitative calculation did not appear until Li (2000c). In the absence of magnetic coupling, the energy source for the radiation of the disk is the gravitational energy of the disk (i.e., the gravitational binding energy between the disk and the black hole). But, when magnetic coupling exists and the black hole is rotating, the rotational energy of the black hole provides an additional energy source for the radiation of the disk. With magnetic coupling, the black hole exerts a torque on the disk, which transfers energy and angular momentum between the black hole and the disk. If the black hole rotates faster than the disk, energy and angular momentum are extracted from the black hole and transferred to the disk. If the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates faster than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates faster than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates faster than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk. If the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole, which will lower the efficiency of the disk.

To be specific, we consider a model that a thin Keplerian disk rotates around a Kerr black hole in the equatorial plane and a magnetic field connects the black hole to the disk (Li 2000c). This model is a variant of the standard Blandford-Znajek mechanism (Blandford & Znajek 1977; Macdonald & Thorne 1982; Phinney 1983a, 1983b; Thorne et al. 1986). In the standard Blandford-Znajek mechanism, the magnetic field threading the black hole is assumed to close on a load that could be very far from the black hole. Because of the large scale involved and the fact that the physics in the load region is ill understood, in some sense the Blandford-Znajek model is not well defined and thus the involved physics is extremely complicated. Since the load is far from the black hole, the magnetic field suffers from screw instability (Li 2000b). Because of the lack of knowledge of
the physics in the load region, we do not have a good model for the load. The way people usually adopt is to assume the load is a resistor whose resistance is roughly equal to the resistance of the black hole (Macdonald & Thorne 1982; Phinney 1983a, 1983b; Thorne et al. 1986). Since the load is so far from the black hole that the load cannot be casually connected to the black hole, it is hard to understand how the load can conspire with the black hole to have equal resistances and satisfy the impedance matching condition (Punsly & Coroniti 1990).

Compared with the standard Blandford-Znajek mechanism, the model that we consider here is simpler and relatively well defined. In our model, the magnetic field lines threading the black hole close on the disk rather than a remote load. The disk is much better understood than the remote load, although the magnetohydrodynamics (MHD) of the disk is still very complicated (Balbus & Hawley 1998; Miller & Stone 2000; Hawley & Krolik 2001; Stone & Pringle 2001). In most cases the disk can be treated as fully ionized, and thus its resistance is negligible compared with the resistance of the black hole, which is several hundred ohms. In other words, it is a good approximation to assume that the disk is perfectly conducting and the magnetic field lines are frozen in the disk. Certainly, MHD instabilities have important effects on the dynamics of the disk and the magnetic field—in fact, people believe that the MHD instabilities play an important role in transporting angular momentum within the disk (Balbus & Hawley 1998), but here we choose to ignore this topic since the problem of MHD instabilities is so complicated that a detailed discussion is beyond the scope of this paper.

We assume that the disk is thin and Keplerian, and lies in the equatorial plane of the black hole with the inner boundary being at the marginally stable orbit (Lynden-Bell 1969; Bardeen 1970; Novikov & Thorne 1973). This requires that the magnetic field is so weak that its influence on the dynamics of the particles in the disk is negligible. But, the role played by a weak magnetic field in the balance and transportation of energy and angular momentum may be important. (In § 5 we justify the approximation of a weak magnetic field in detail.) With this “weak magnetic field approximation,” the particles in the disk move around the central black hole in circular orbits with a superposition of a small radial inflow motion. If the inflow timescale is much longer than the dynamical (rotational) timescale of the disk, the disk and the magnetic field can be in a quasi-steady state even though the magnetic field slowly moves toward the central black hole along with the accretion. By “quasi-steady state” we mean that any macroscopic quantity at a given radius in the disk slowly changes with time: the integrated change within one rotational period of the disk is much smaller than the quantity itself. With the assumptions that the disk is thin and Keplerian and the magnetic field and the disk are in a quasi-steady state, we solve the conservation equations of energy and angular momentum, calculate the radiation flux of the disk and the internal viscous torque of the disk, and compare the results with those predicted by the standard theory of an accretion disk.

Recently, several people have considered the magnetic coupling between the disk and the material in the transition region between the disk and the black hole horizon and have argued that such a coupling produces a torque at the inner boundary of the disk (Krolik 1999; Gammie 1999; Agol & Krolik 2000; Hawley & Krolik 2001; Krolik 2001). We emphasize that our model is distinctly different from theirs because of the different topology of the magnetic field lines involved in the two models: (1) In our model, the magnetic field connects the disk to the black hole, and thus the torque on the disk is produced by the black hole; in their model, the magnetic field connects the disk to the material in the transition region, and thus the torque on the disk is produced by the material in the transition region. (2) In our model, the magnetic field lines connect the disk surfaces to the black hole horizon at intermediate latitudes, while in their model, the magnetic field is in the plane of the disk within the inner edge of the disk, and thus magnetic reconnection can easily take place (Blandford 2000). (3) In our model, the magnetic field can touch the disk over a range of radii, while in their model, the magnetic field is attached to the disk only at the inner boundary of the disk (Agol & Krolik 2000). (4) In our model, we show that the torque produced by the black hole propagates outward only, and thus the torque at the inner boundary of the disk is always zero; in their model, the torque is assumed to be nonzero at the inner boundary of the disk and extends into the transition region (Hawley & Krolik 2001; Krolik 2001). (5) In our model, energy and angular momentum can be extracted from the black hole even in the case with no accretion; in their model, in order to extract energy and angular momentum from the black hole, accretion must exist so that negative energy and angular momentum are carried into the black hole horizon by the accretion material (Gammie 1999).

The paper is organized as follows: In § 2 we discuss the transfer of energy and angular momentum between the black hole and the disk arising from the magnetic coupling between them. In § 3 we solve the conservation equations of energy and angular momentum for a quasi-steady and thin Keplerian accretion disk torqued by a black hole and calculate the radiation flux, the internal viscous torque, and the total power of the disk. We will see that, with magnetic coupling to a rapidly rotating black hole, a disk can radiate without accretion. In § 4 we study a simple example, the magnetic field lines touch the disk at a circle of a constant radius, and look for the consequences of magnetic coupling. We show that the radial profile of the radiation flux produced by magnetic coupling is very different from that produced by accretion. In § 5, we justify the assumptions made in the paper and discuss the limitations of our model, which include the ignorance of the instabilities of the disk and the magnetic field and the ignorance of the radiation captured by the black hole and the radiation returning to the disk. In § 6 we draw our conclusions.

Throughout the paper we use the geometrized units \( G = c = 1 \) and the Boyer-Lindquist coordinates \((t, r, \theta, \phi)\) (Misner, Thorne, & Wheeler 1973; Wald 1984).

2. TRANSFER OF ENERGY AND ANGULAR MOMENTUM BY THE MAGNETIC COUPLING

In the presence of a magnetic field, a black hole behaves like a conductor with a surface resistivity \( R_H = 4\pi \approx 377 \) ohms (Znajek 1978; Damour 1978; Carter 1979). So when a black hole rotates in an external magnetic field, an electromotive force (EMF) is induced on its horizon (Macdonald & Thorne 1982; Thorne et al. 1986). As Blandford & Znajek (1977) were first to note, the voltage drop along the magnetic field lines induced by the black hole rotation is huge enough to give rise to a cascade production of electron-
The black hole and the disk form a closed electric circuit: an electric current flows along the magnetic field lines in the magnetosphere and closes itself in the disk and the black hole. Suppose the disk and the black hole rotate in the same direction; then the black hole’s EMF, \( \varepsilon_B \), and the disk’s EMF, \( \varepsilon_D \), have opposite signs. The direction of the electric current, and in turn the direction of the transfer of energy and angular momentum, is determined by the sign of \( \varepsilon_B + \varepsilon_D \). If \( \varepsilon_B + \varepsilon_D > 0 \), the black hole’s EMF dominates the disk’s EMF, so the black hole “charges” the disk: energy and angular momentum are transferred from the black hole to the disk. If \( \varepsilon_B + \varepsilon_D < 0 \), the disk’s EMF dominates the black hole’s EMF, so the disk “charges” the black hole, and energy and angular momentum are transferred from the disk to the black hole. If \( \varepsilon_B + \varepsilon_D = 0 \), the black hole’s EMF balances the disk’s EMF, then no energy and angular momentum are transferred between the black hole and the disk (Li 2000c).

It is straightforward to calculate the electromagnetic power and torque on the disk, using standard electromagnetism. For the specific case that the magnetic field lines touch the disk at a single radius, the calculations are carried out by Li (2000c). In this case, the sign of \( \varepsilon_B + \varepsilon_D \) (and thus the direction of the transfer of energy and angular momentum) is determined by the sign of \( \Omega_B - \Omega_D \), where \( \Omega_B \) is the angular velocity of the black hole, which is constant over the black hole horizon, and \( \Omega_D \) is the angular velocity of the disk at the radius where the magnetic field touches the disk. If \( \Omega_B > \Omega_D \), i.e., if the black hole rotates faster than the disk, energy and angular momentum are transferred from the black hole to the disk. If \( \Omega_B < \Omega_D \), i.e., if the black hole rotates slower than the disk, energy and angular momentum are transferred from the disk to the black hole. If \( \Omega_B = \Omega_D \), there is no transfer of energy and angular momentum between the black hole and the disk. For fixed values of the magnetic flux, the mass and the angular momentum of the black hole, and the resistance of the black hole, the power peaks at \( \Omega_B = \Omega_H/2 \) (Li 2000c).

If the magnetic field is distributed over a differentially rotating disk, the formulae given in Li (2000c) for the EMF of the disk, the power, and the torque on the disk should be replaced with integrations over the radius of the disk. Assume that the magnetic field is stationary and axisymmetric, and touches the disk at radii ranging from \( r_1 \) to \( r_2 \); then, the total EMF induced on the disk is

\[
\varepsilon_D = \frac{1}{2\pi} \int_{r_1}^{r_2} \Omega_D \frac{d\Psi_{HD}}{dr} \, dr,
\]

where \( \Psi_{HD} = \Psi_{HD}(r) \) is the magnetic flux through a surface whose boundary is a circle with a constant \( r \) in the disk. In such a case, an infinite number of adjacent infinitesimal

positional electric current loops flow between the black hole and the disk along the magnetic field lines connecting them. Each infinitesimal current loop produces an infinitesimal power and an infinitesimal torque on the disk, whose sum gives the total power and the total torque on the disk. Thus, assuming that the disk is perfectly conducting, the total power produced by the black hole on the disk is

\[
P_{BD} = \frac{1}{4\pi^2} \int_{r_1}^{r_2} \left( \frac{d\Psi_{HD}}{dr} \right)^2 \Omega_B - \Omega_D \, -d\Psi_{HD}/dr \, dr,
\]

where we have treated the black hole’s resistance \( Z_H \) as a function of the disk’s radius, which is defined by a map from the black hole horizon to the disk surface given by the magnetic field lines. Similarly, the total torque produced by the black hole on the disk is

\[
T_{BD} = \frac{1}{4\pi^2} \int_{r_1}^{r_2} \left( \frac{d\Psi_{HD}}{dr} \right)^2 \Omega_B - \Omega_D \, -d\Psi_{HD}/dr \, dr.
\]

From equations (2) and (3) we have \( dP_{BD} = \Omega_B \, dT_{BD} \), or

\[
P_{BD} = \frac{1}{4\pi^2} \int_{r_1}^{r_2} \Omega_B \, \frac{dT_{BD}}{dr} \, dr,
\]

which follows from the assumption that the disk is perfectly conducting so the magnetic field is frozen in the disk.

For a Kerr black hole of mass \( M_B \) and angular momentum \( J_B = M_B a \), we have (Thorne et al. 1986)

\[
\frac{dZ_H}{d\theta} = \frac{R_H n^2 + a^2 \cos^2 \theta}{2\pi (r_H^2 + a^2) \sin \theta},
\]

where \( r_H = M_B + (M_B^2 - a^2)^{1/2} \) is the radius of the outer horizon of the black hole and \( \theta \) is the polar angle coordinate on the horizon. Then, \( dZ_H/dr \) can be calculated through

\[
\frac{dZ_H}{dr} = \frac{dZ_H}{d\theta} \frac{d\theta}{dr},
\]

where \( \theta = \theta(r) \) is a map between the \( \theta \)-coordinate on the horizon and the \( r \)-coordinate on the disk, which is induced by the magnetic field lines connecting the disk to the horizon. Since \( d\theta/dr < 0 \) and \( dZ_H/dr > 0 \), we have \( dZ_H/dr < 0 \).

3. QUASI–STEADY STATE FOR AN ACCRETION DISK TORQUED BY A BLACK HOLE

Of particular interest is the case of an accretion disk in a steady state, which is simple enough for solving the equations of energy conservation and angular momentum conservation and has been well studied when magnetic coupling between the black hole and disk does not exist (Pringle & Rees 1972; Shakura & Sunyaev 1973; Novikov & Thorne 1973; Lynden-Bell & Pringle 1974; Page & Thorne 1974; Thorne 1974). In a steady state, the material of the disk advected with accretion toward smaller radii is balanced by the material advected from larger radii. Any macroscopic (statistically averaged) quantity of the disk remains (approximately) unchanged for a long period of time during accretion (say, over a time interval > 100 rotation periods of the disk). When there is a magnetic field connecting the black hole to the disk, the disk and the magnetic field cannot be in an exactly steady state unless there is no accretion at all. When there is accretion, the magnetic field frozen in the disk slowly moves toward the central black hole as accretion goes on, which raises the
question of how long a configuration of magnetic connection can be maintained. Because of the complex topologies of the magnetic field, the magnetic field advected with accretion toward smaller radii, which connects the black hole to the disk, does not have to be balanced by the magnetic field advected from larger radii, which also connects the black hole to the disk. However, if the radial velocity of particles in the disk is much smaller than their rotational velocity, the inflow timescale is much larger than the dynamical timescale. Then, it is reasonable to assume that within one rotation period, the global configuration of the magnetic field is almost unchanged, and the overall change in a macroscopic quantity at a given radius in the disk is much smaller than the quantity itself. When this condition is satisfied, we say that the disk and the magnetic field are in a quasi–steady state. In a quasi–steady state, a macroscopic quantity at a given radius may change significantly over a long period of time and the magnetic connection between the black hole and the disk may eventually disappear, but they are approximately unchanged within one rotation period of the disk. In this section we solve the equations for energy conservation and angular momentum conservation for such a quasi–steady state disk magnetically coupled to a black hole.

For a steady, axisymmetric, and thin Keplerian disk around a Kerr black hole, the general relativistic equations of energy conservation and angular momentum conservation have been investigated in detail by Novikov & Thorne (1973), Page & Thorne (1974), and Thorne (1974). Assume that the magnetic field is weak so that its influence on the dynamics of disk particles is negligible; then a thin Keplerian disk is a good approximation. With magnetic coupling between the black hole and the disk being taken into account, for a quasi–steady state the conservation of angular momentum is described by

$$\frac{d}{dr} (M_p L^+ - g) = 4\pi r (F L^+ - H), \quad (7)$$

where $M_p \equiv dM_p/dt$ is the accretion rate of mass (measured by an observer at infinity; we use the convention $M_p > 0$ for accretion), $L^+$ is the specific angular momentum of a particle in the disk, $g$ is the internal viscous torque of the disk, $F$ is the energy flux radiated away from the surface of the disk (measured by an observer corotating with the disk), and $H$ is the flux of angular momentum transferred from the black hole to the disk by the magnetic field. The conservation of energy is described by

$$\frac{d}{dr} (M_p E^+ - g \Omega_p) = 4\pi r (FE^+ - H \Omega_p), \quad (8)$$

where $E^+$ is the specific energy of a particle in the disk. When $H = 0$, i.e., there is no magnetic coupling, equations (7) and (8) return to the equations derived by Novikov & Thorne (1973) and Page & Thorne (1974). For particles moving in circular orbits around the black hole, $E^+$ and $L^+$ are related by (Page & Thorne 1974)

$$\frac{dE^+}{dr} = \Omega_p \frac{dL^+}{dr}. \quad (9)$$

The flux of angular momentum transferred from the black hole to the disk, $H$, is defined by the torque $T_{HD}$ produced by the black hole on the disk:

$$T_{HD} = 2\pi r^2 H r dr. \quad (10)$$

Comparing equation (10) with equation (3), we have

$$H = \frac{1}{8\pi^2} \left( \frac{d\psi_{HD}}{dr} \right)^2 \frac{\Omega_m - \Omega_p}{-rdZ_m/dr}, \quad (11)$$

if the magnetic field is smooth distributed on the disk.

In a quasi–steady state, $M_p$ is constant throughout the disk (the conservation of mass). Then, equations (7) and (8) can be solved for $F$ and $g$ by using equation (9). The solutions are

$$F = \frac{M_p}{4\pi r} f + \frac{1}{4\pi r} \left( \frac{d\Omega_p}{dr} (E^+ - \Omega_p L^+) \right)^{-2} \times \left[ 4\pi \int_{r_m}^r (E^+ - \Omega_p L^+) H r dr + g_{ms} (E_{ms} - \Omega_{ms} L_{ms}) \right], \quad (12)$$

$$g = \frac{E^+ - \Omega_p L^+}{-d\Omega_p/dr} 4\pi r F, \quad (13)$$

where the subscript “ms” denotes the marginally stable orbit that is assumed to be the inner boundary of a thin Keplerian disk, $g_{ms}$ is an integration constant, and $f$ is defined by

$$f \equiv -\frac{d\Omega_p}{dr} (E^+ - \Omega_p L^+)^{-2} \int_{r_m}^r (E^+ - \Omega_p L^+) \frac{dL^+}{dr} dr. \quad (14)$$

The integration of $f$ has been worked out analytically by Page & Thorne (1974) and is given by their equation (15n); we do not display it here since it is lengthy. It is easy to check that $f(r = r_{ms}) = 0$ and $f(r \gg r_{ms}) \approx 3M_b/2r^4$.

Suppose the magnetic field is distributed on the disk from $r = r_1$ to $r = r_2$, where $r_2 > r_1 \geq r_{ms}$; then

$$\int_{r_m}^r (E^+ - \Omega_p L^+ ) H r dr = \begin{cases} 0 & r \leq r_1, \\ \int_{r_1}^{r_2} (E^+ - \Omega_p L^+ ) H r dr & r_1 < r < r_2, \\ \int_{r_1}^{r_2} (E^+ - \Omega_p L^+ ) H r dr & r \geq r_2. \end{cases} \quad (15)$$

At the inner boundary of the disk, where $r = r_{ms} \leq r_1$, we have $g = g_{ms}$. For a thin Keplerian disk, the “no-torque inner boundary condition” is an excellent approximation (Novikov & Thorne 1973; Muchotrzeb & Paczyński 1982; Abramowicz & Kato 1989; Paczyński 2000; Armitage,
The appearance of magnetic coupling between the black hole and the disk does not introduce a torque at the inner boundary. This is because the integration in equation (15) always vanishes at $r = r_{\text{ms}}$ for any $H$. In other words, the torque produced by magnetic coupling propagates outward only in the disk. Therefore, in the following discussion we take $\gamma_{\text{ms}} = 0$. Then we have

$$F = \frac{1}{4\pi} \left[ M_D f + 4\pi \left( -\frac{d\Omega_D}{dr} \right) (E^{+} - \Omega_D L^{+})^{-2} \right] \times \int_{r_{\text{ms}}}^{r} (E^{+} - \Omega_D L^{+}) H r dr. \tag{16}$$

$$g = \frac{E^{+} - \Omega_D L^{+}}{-d\Omega_D/dr} M_D f + 4\pi (E^{+} - \Omega_D L^{+})^{-1} \times \int_{r_{\text{ms}}}^{r} (E^{+} - \Omega_D L^{+}) H r dr. \tag{17}$$

Equation (16) gives the radiation flux of the disk; equation (17) gives the internal viscous torque of the disk. The first terms on the right-hand sides of equations (16) and (17) are the familiar results for a standard accretion disk (Page & Thorne 1974); the second terms represent the contribution of the magnetic coupling.

The integration of equation (8) gives the power (i.e., the luminosity) of the disk, which is the energy radiated by the disk per unit time as measured by an observer at infinity (Thorne 1974):

$$L \equiv 2 \int_{r_{\text{ms}}}^{\infty} E^{+} F 2\pi r dr = \int_{r_{\text{ms}}}^{\infty} \left[ \frac{d}{dr} (M_D E^{+} - g\Omega_D) + 4\pi r H \Omega_D \right] dr = M_D (1 - E^{+}_{\text{ms}}) + 4\pi \int_{r_{\text{ms}}}^{\infty} H \Omega_D r dr, \tag{18}$$

where in the first line on the right-hand side the factor 2 accounts for the fact that a disk has two surfaces, and in the third line we have used the boundary conditions $E^{+}(r \to \infty) = 1, g\Omega_D(r \to \infty) = 0$, and $g\Omega_D(r = r_{\text{ms}}) = 0$. The power of the black hole, which is the energy transferred from the black hole to the disk per unit time as measured by an observer at infinity, is

$$L_{\text{HD}} \equiv 2P_{\text{HD}} = 4\pi \int_{r_{\text{ms}}}^{\infty} H \Omega_D r dr = 4\pi \int_{r_{1}}^{r_{2}} H \Omega_D r dr, \tag{19}$$

so equation (18) can be written as

$$L = M_D \epsilon_0 + L_{\text{HD}}, \tag{20}$$

where

$$\epsilon_0 = 1 - E^{+}_{\text{ms}} \tag{21}$$

is the efficiency of accretion, i.e., the efficiency of the disk when magnetic coupling between the black hole and disk does not exist. For a Schwarzschild black hole, $\epsilon_0 \approx 0.06$; for a canonical Kerr black hole of $s = a/M_D = 0.998$, $\epsilon_0 \approx 0.32$; for an extreme Kerr black hole of $s = 1$, $\epsilon_0 \approx 0.42$ (Thorne 1974). For any thin Keplerian disk around a black hole, $\epsilon_0 \leq 0.42$ always.

Equation (20) describes the global balance of energy for a quasi-steady accretion disk magnetically coupled to a Kerr black hole. The total power of the disk is represented by $L$; $M_D \epsilon_0$ represents the rate of change in the gravitational energy of the disk, as in the standard theory of an accretion disk. The term $L_{\text{HD}}$ represents the rate of energy transferred from the black hole to the disk, which comes from the rotational energy of the black hole and is absent in the standard theory of an accretion disk. Interestingly, with a magnetic connection to a rapidly rotating black hole a disk can radiate without accretion: $M_D = 0$ but $L = L_{\text{HD}} > 0$. For such a nonaccretion disk, the power of the disk comes purely from the rotational energy of the black hole. The radiation flux and the internal viscous torque of a nonaccretion disk are given by equations (16) and (17), with $M_D = 0$.

From equation (20), we can define the total efficiency of the disk:

$$\epsilon = \frac{L}{M_D} = \epsilon_0 + \frac{L_{\text{HD}}}{M_D}. \tag{22}$$

If $L_{\text{HD}} = 0$, i.e., there is no energy transfer between the black hole and the disk by magnetic coupling, as in the case in the standard theory of an accretion disk, $\epsilon = \epsilon_0$, and the power of the disk purely comes from the gravitational energy of the disk. If $L_{\text{HD}} > 0$, which is the case when the black hole rotates faster than the disk, energy is transferred from the black hole to the disk. Then there are two sources for the power of the disk: one is the gravitational energy of the disk (represented by $M_D \epsilon_0$) and the other is the rotational energy of the black hole (represented by $L_{\text{HD}}$). The efficiency of the disk is increased by magnetic coupling: $\epsilon > \epsilon_0$. For very small $M_D$, an efficiency $\epsilon > 1$ can be achieved. If there is no accretion at all, the so-defined total efficiency of the disk is infinite. If $L_{\text{HD}} < 0$, which is the case when the black hole rotates slower than the disk, energy is transferred from the disk to the black hole. Then, the efficiency of the disk is decreased by magnetic coupling: $\epsilon < \epsilon_0$.

Since $L$ cannot be negative, we must have

$$M_D \epsilon_0 \geq -L_{\text{HD}} \tag{23}$$

when $L_{\text{HD}} < 0$.

4. AN EXAMPLE: THE MAGNETIC FIELD TOUCHES THE DISK AT A CIRCLE

Suppose the magnetic field touches the disk at a circle with a radius $r = r_0 > r_{\text{ms}}$; i.e.,

$$H = A_0 \delta(r - r_0), \tag{24}$$

where $A_0$ is a constant and $\delta(x)$ is a Dirac-function that satisfies

$$\delta(x) = 0 \tag{25}$$

for any $x \neq 0$, and

$$\int_{-\infty}^{\infty} y(x)\delta(x)dx = y(0) \tag{26}$$
for any smooth function $\gamma(x)$. Though this is a simple and highly ideal case, it is fundamental in understanding the effect of the magnetic coupling on the energetic process of the disk. For any given distribution of a magnetic field on the disk, the corresponding $H(r)$ can always be written as

$$H(r) = \int_{r_{\text{ms}}}^{r_{\infty}} H(r') \delta(r' - r) dr'. \quad (27)$$

Since the energy flux and the internal torque given by equations (16) and (17) are linear functionals of $H(r)$, the results for any given distribution of a magnetic field can be obtained from the results for the simple case in equation (24) by linear superpositions. Furthermore, for a realistic case of a smooth distribution of a magnetic field with $r_h - r_o \ll r_{\text{ms}}$, where $r_h$ and $r_o$ are the radii of the boundaries of an annular region in the disk within which most of the magnetic field lines are concentrated, equation (24) is a good approximation for $H$ if we take $r_{\text{cor}} \approx (r_h + r_o) / 2$ and

$$A_0 \approx \frac{2}{r_h + r_o} \int_{r_{\text{cor}}}^{r_{\infty}} H r dr. \quad (28)$$

Thus, the simple case given by equation (24) is meaningful not only in mathematics but also in practice.

Inserting equation (24) into equation (10) and comparing the result with equation (3) of Li (2000c), we obtain

$$A_0 = \frac{1}{2\pi r_0} \left[ \frac{\Delta \Psi_{HD}}{2\pi} \right]^2 \frac{\Omega_H - \Omega_0}{Z_H}, \quad (29)$$

where $\Omega_0 \equiv \Omega_0(r_0)$ and $\Delta \Psi_{HD}$ is the magnetic flux connecting the disk to the black hole. The sign of $A_0$ is determined by the sign of $\Omega_H - \Omega_0$: $A_0 > 0$ if $\Omega_H > \Omega_0$; $A_0 < 0$ if $\Omega_H < \Omega_0$. For a thin Keplerian disk around a Kerr black hole, the ratio of $\Omega_0 / \Omega_H$ is

$$\frac{\Omega_0}{\Omega_H} = \frac{2\omega}{s} \left[ 1 + (1 - s^2)^{1/2} \right], \quad (30)$$

where $s \equiv a/M_H$ and $\omega \equiv M_H \Omega_0$ is a function of $s$ and $r_0 / M_H$,

$$\omega = \frac{1}{s + (r_0 / M_H)^{3/2}}. \quad (31)$$

The critical case of $\Omega_H = \Omega_0$ (i.e., $A_0 = 0$) is determined by

$$\omega \left( \frac{s}{M_H} \right) = \frac{s}{2[1 + (1 - s^2)^{1/2}^2]}. \quad (32)$$

For any given value of $r_0 / M_H$, or equivalently, for any given value of $r_0 / r_{\text{ms}}$, equation (32) can be solved for $s$ to obtain the critical spin $s_c$. We have $\Omega_H > \Omega_0$ (thus $A_0 > 0$) for $s > s_c$; $\Omega_H < \Omega_0$ (thus $A_0 < 0$) for $s < s_c$. It is easy to show that $s_c$ is a monotonically decreasing function of $r_0 / r_{\text{ms}}$. When $r_0 / r_{\text{ms}} = 1$, i.e., when the magnetic field touches the disk at the inner boundary, we have $s_c \approx 0.3594$; when $r_0 / r_{\text{ms}}$ increases, $s_c$ decreases quickly. As $r_0 / r_{\text{ms}} \rightarrow \infty$, $s_c \rightarrow 0$.

When $0.3594 < s < 1$, the black hole rotates faster than a particle in the disk at any radius ($r_{\text{ms}}$). When $0 < s < 0.3594$, there exists a corotation radius in the disk defined by equation (32), beyond which the black hole rotates faster than the disk and within which the black hole rotates slower than the disk. When the magnetic field lines touch the disk beyond the corotation radius, energy and angular momentum are transferred from the black hole to the disk. When the magnetic field lines touch the disk inside the corotation radius, energy and angular momentum are transferred from the disk to the black hole. The corotation radius, which we denote by $r_c$, can be solved from equation (32):

$$r_c = M_H \left\{ 2 \left[ 1 + (1 - s^2)^{1/2} \right] - s \right\}^{2/3}, \quad 0 < s < 0.3594. \quad (33)$$

The ratio $r_c / r_{\text{ms}} = (r_c / M_H)(r_{\text{ms}} / M_H)^{-1}$ as a function of $s$ is plotted in Figure 1, from which we see that $r_c / r_{\text{ms}}$ is a monotonically decreasing function of $s$: $r_c \rightarrow r_{\text{ms}}$ as $s \rightarrow 0.3594$, and $r_c \rightarrow \infty$ as $s \rightarrow 0$. The corotation radius does not exist when $s > 0.3594$, for which the black hole always rotates faster than the disk. [Fig. 1 also shows $s_c = s_c(r_0)$ if we replace the label of the horizontal axis with $\lg r_0 / r_{\text{ms}}$.]

Inserting equation (24) into equations (16) and (17), we obtain

$$F = \begin{cases} \frac{1}{4\pi} M_H f & r_{\text{ms}} \leq r < r_0, \\ \frac{1}{4\pi} \left[ M_D f + 4\pi r_0 A_0 (E_0^+ - \Omega_0 L_0^+) \right] & \times \left( -\frac{d\Omega_H}{d\phi} \right) (E^+ - \Omega_D L^+)^{-2} & r > r_0, \end{cases} \quad (34)$$

![Fig. 1.—Ratio of the corotation radius to the marginally stable radius of a thin Keplerian disk, $r_c / r_{\text{ms}}$, as a function of the black hole spin, $s = a / M_H$. When $0 < s < 0.3594$, the corotation radius is given by eq. (33), which is shown with the solid curve. Beyond the corotation radius the black hole rotates faster than the disk; within the corotation radius the black hole rotates slower than the disk. When $s > 0.3594$, the corotation radius does not exist: the black hole always rotates faster than the disk. [If the label of the horizontal axis is replaced with $\lg s$, where $s$ is the critical spin of the black hole defined by eq. (32), the label of the vertical axis is replaced with $\lg (r_c / r_{\text{ms}})$, where $r_c$ is the radius where the magnetic field touches the disk, then the same solid curve gives the relation $s_c = s_c(r_0)$.]](image-url)
Since decreases with radius as $F$, the torque and the radiation flux are always zero at the inner boundary of the disk. This fact is manifested by the step functions in eqs. (36) and (37), which are absent in Agol & Krolik's formula. Thus, as we have pointed out in the introduction, the two models are distinctly different from each other because of the different topology of the magnetic field lines; one cannot treat one as the "infinite efficiency limit" of the other.

4.1. $\dot{M}_D = 0$

Substituting $\dot{M}_D = 0$ into equations (34) and (35) and using the expressions for $E^+ - \Omega_D L^+$ and $d\Omega_D/dr$ given by Page & Thorne (1974), we obtain

$$F = \frac{370}{8\pi} \left( \frac{M_H}{r_d^3} \right)^{1/2} \frac{1}{1 - (3M/r) + 2a(M/r)^{1/2}} S(r - r_0),$$

$$g = \frac{T_0[1 + a(M/r_d^3)^{1/2}]}{[1 - 3M/r + 2a(M/r_d^3)^{1/2}]^{1/2}} S(r - r_0),$$

where $S$ is the step function

$$S(r - r_0) = \begin{cases} 1 & r > r_0, \\ 0 & r < r_0, \end{cases}$$

$$T_0 \equiv 4\pi r_0 A_0(E_0^+ - \Omega_0 L_0^+)$$

$$= 4\pi r_0 A_0 \left[ 1 + a \left( \frac{M}{r_0} \right)^{1/2} \right]^{-1} \times \left[ 1 - \frac{3M}{r_0} + 2a \left( \frac{M}{r_0} \right)^{1/2} \right]^{1/2}.$$  \hspace{1cm} (39)

Since $F$ and $g$ cannot be negative, such nonaccretion solutions can exist only if $A_0 > 0$, which requires that $\Omega_H > \Omega_0$; i.e., the black hole rotates faster than the disk at $r = r_0$. Examples of $F$ and $g$ are plotted in Figures 2 and 3. These figures clearly show that magnetic coupling takes effect only beyond the radius $r = r_0$; i.e., the torque produced by magnetic coupling propagates only outward from $r = r_0$. Both $F$ and $g$ have sharp peaks at $r = r_0$ and are zero for $r < r_0$. Since $\gamma_{in} < r_0$, the torque and the radiation flux are always zero at the inner boundary of the disk. At $r = r_0$, the radiation flux rises suddenly from zero to a sharp peak, then decreases quickly for $r > r_0$. At $r > r_0$, the radiation flux decreases with radius as $F \propto r^{-3.5}$. At $r = r_0$, the internal viscous torque of a thin Keplerian nonaccretion disk coupled to a Kerr black hole with a magnetic field. The model is the same as that in Fig. 2: there is no accretion, the magnetic field touches the disk at a circle of radius $r = r_0 > r_{in}$, and the black hole rotates faster than the disk. The radius is in units of $r_{in}$, and the disk is marginally stable, and the radiation flux is in units of $F_0 \equiv A_0 r_{in}^{-3}$. The inner boundary of the disk is at $r = r_{in}$, as indicated by the vertical dashed line. The radiation flux is shown with the solid curve. For $r < r_0$, the radiation flux is zero. The radiation flux has a sharp peak at $r = r_0$, and decreases quickly for $r > r_0$. The radiation flux approaches $F \propto r^{-3.5}$ at $r > r_0$.
viscous torque rises suddenly from zero to a finite value, then decreases slowly for \( r > r_0 \), and approaches a constant at \( r \rightarrow r_0 \).

The radial profile of the radiation flux produced by magnetic coupling is very different from that produced by accretion. For a standard accretion disk, the radiation flux is zero at \( r = r_{ms} \) (the inner boundary of the disk), gradually rises to a maximum at a radius beyond \( r_{ms} \), then decreases slowly, and approaches \( F \propto r^{-3} \) at large radii\(^8\) (Novikov & Thorne 1973; Page & Thorne 1974; Thorne 1974). While for a nonaccretion disk magnetically coupled to a Kerr black hole, if we assume that the magnetic field touches the disk at the inner boundary, then at \( r = r_{ms} \) the radiation flux suddenly rises from zero to a sharp peak, then decreases quickly and approaches \( F \propto r^{-3.5} \) at large radii. To compare the radiation profile of the magnetic coupling with the radiation profile of accretion, in Figure 4 we plot both the radiation flux of a nonaccretion disk magnetically coupled to a rapidly rotating black hole and the radiation flux of a standard accretion disk rotating around the same black hole. For the nonaccretion disk, the magnetic field is assumed to touch the disk at the inner boundary (the marginally stable orbit). Obviously, the radiation flux of the nonaccretion disk has a much steeper radial profile and a sharp peak closer to the center of the disk, compared to the radiation flux of the standard accretion disk. For the same models, in Figure 5 we show the emissivity index defined by

\[
\alpha \equiv -d \ln F/d \ln r,
\]

which measures the slope of the radial emissivity profile in the disk. We see that throughout the disk the emissivity index for the nonaccretion disk with magnetic coupling is significantly larger than the emissivity index for the standard accretion disk. At large radii, \( \alpha \) approaches 3.5 for the nonaccretion disk and 3 for the standard accretion disk.

Inserting equation (24) into equations (10) and (19), we get

\[
T_{HD} = 2\pi A_0 r_0, \quad P_{HD} = 2\pi A_0 \Omega_0 r_0,
\]

where \( A_0 \) is given by equation (29). As expected, \( P_{HD} = T_{HD} \Omega_0 \). Since a disk has two surfaces, the total power of the disk is

\[
\mathcal{L}_{HD} = 2P_{HD} = 4\pi A_0 \Omega_0 r_0.
\]

The energy radiated per unit time from the region inside a circle of radius \( r > r_0 \) in the disk is

\[
\mathcal{L}_{HD}(r) = 2 \int_{r_{ms}}^{r} E^+ F 2\pi r \, dr - g \Omega D = 4\pi \int_{r_{ms}}^{r} \Omega D r \, dr - g \Omega D = 4\pi A_0 \Omega_0 r_0 \left( 1 - \frac{\Omega D E^+ - \Omega_0 L^+}{\Omega_0 E^+ - \Omega D L^+} \right),
\]

where on the right-hand side in the second line we have used equation (8) (taking \( M_D = 0 \)) and the boundary condition \( g \Omega D (r = r_{ms}) = 0 \) and in the third line we have used equations (24) and (35) (taking \( M_D = 0 \)). We can define a half-light radius \( r_{1/2} \), within which the energy radiated per unit time by the disk is one-half of the total power of the

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\(^8\) At large radii the internal torque approaches \( g \propto r^{1/2} \) for a standard thin accretion disk.
The total power of an accretion disk is which is defined by can be solved
\[ \frac{E^+ - \Omega_D L^+}{\Omega_D} = \frac{2(E_0^+ - \Omega_0 L_0^+)}{\Omega_0}. \]
Similarly, for a standard accretion disk, the energy radiated per unit time from the region inside a circle of radius \( r > r_{ms} \) in the disk is
\[ \mathcal{P}_{acc}(< r) = \dot{M}_D \left( E^+ - E_{ms}^+ - \frac{E^+ - \Omega_D L^+}{-d\Omega_D/dr} \frac{\Omega_D f}{dr} \right). \]
The total power of an accretion disk is \( \mathcal{P}_{acc} = \dot{M}_D(1 - E_{ms}^+) \).
Thus, the half-light radius of a standard accretion disk, which is defined by \( \mathcal{P}_{acc}(< r_{1/2}) = \frac{1}{2} \mathcal{P}_{acc} \), can be solved from
\[ \left[ \frac{E^+ - \frac{E^+ - \Omega_D L^+}{-d\Omega_D/dr} \Omega_D f}{dr} \right]_{r = r_{1/2}} = \frac{1}{2} \left( 1 + E_{ms}^+ \right), \]
where \( f \) is given by equation (15n) of Page & Thorne (1974).

We have calculated the half-light radius of a disk magnetically coupled to a Kerr black hole, assuming that the disk has no accretion and the magnetic field touches the disk at the inner boundary. The results are shown in Figure 6. For comparison, we have also calculated the half-light radius of a standard accretion disk around the same Kerr black hole; the results are also shown in Figure 6, with the dashed curve. From these results we see that for a non-accretion disk magnetically coupled to a Kerr black hole with the magnetic field touching the disk at the inner boundary, most energy radiated by the disk comes from a region closer to the center of the disk, compared to the case of a standard accretion disk. A similar figure is shown by Agol & Krolik (2000, Fig. 1), and very similar results are obtained for them by a disk magnetically coupled to the material in the transition region. But we emphasize that in their model a state with a zero accretion rate and a finite power can never exist, since in their model in order to extract energy from a black hole, material with negative energy must fall into the black hole.

Suppose a Kerr black hole loses its energy and angular momentum through magnetic coupling to a thin Keplerian disk with no accretion, with the magnetic field lines touching the disk at a circle of radius \( r = r_0 \). Then, the evolution of the black hole spin \( s = a/M_H = J_H/M_H^2 \) is given by
\[ \frac{ds}{d\ln M_H} = \frac{1}{\omega} - 2s, \]
where \( \omega \) is defined by equation (31), which is a function of \( s \) and \( r_0/r_{ms} \) (see the discussions below eq. [32]). If we know how \( r_0/r_{ms} \) evolves with \( s \), equation (45) can be integrated to obtain \( M_H = M_H(s) \). As an example, let us consider a Kerr black hole of initial mass \( M_{H,0} \) and initial spin \( s = 0.998 \), the spin of a canonical black hole (Thorne 1974). As the black hole spins down to \( s = s_0 \), the value when \( \Omega_{bh} = 0 \) and thus the transfer of energy and angular momentum between the black hole and the disk stops, which is defined by equation (32), the total amount of energy extracted from the black hole by the disk, \( \Delta M_H = M_H(s = 0.998) - M_H(s = s_0) \), can be calculated by integrating equation (45). Then we can calculate the fraction of energy that can be extracted from the black hole
\[ \eta \equiv \frac{\Delta M_H}{M_{H,0}} = 1 - \exp \left( \int_{s_0}^{s_{max}} \frac{ds}{\omega^{1/2} - 2s} \right), \]
where \( s_0 = 0.998 \). For simplicity, we assume \( r_0/r_{ms} \) keeps constant as \( s \) decreases. The corresponding \( \eta \) is plotted in Figure 7 as a function of \( r_0/r_{ms} \). The figure shows that \( \eta \) decreases quickly with increasing \( r_0/r_{ms} \). The maximum fraction is reached when \( r_0 = r_{ms} \); \( \eta_{max} = 0.152 \). Thus, the magnetic field lines that touch the inner edge of the disk are most efficient in extracting energy from the black hole—in the sense that the largest amount of energy can be extracted from the black hole.\(^9\)

4.2. \( \dot{M}_D \neq 0 \)

In our model there are two torques acting on the disk: one is the external torque produced by the magnetic coupling to the black hole; the other is the internal torque which conveys angular momentum within the disk and dissipates energy. We do not discuss the origin of the internal torque; we assume only that the internal torque exists and the disk automatically adjusts its internal torque so that quasi-steady state solutions exist. If the internal torque balances the external torque exactly, as assumed in § 4.1, which can be true only if the black hole rotates faster than the disk,\(^9\)

\(^9\) We should note that there are two different quantities describing the energetic process for a black hole: the fraction of energy extraction, which describes how much energy can be extracted from a black hole in \textit{total}, and the power, which describes how much energy can be extracted from a black hole per \textit{unit time}. A higher fraction does not imply a higher power, and vice versa. In fact, since \( P_{BH} \propto \Omega_B \Omega_{BH} - \Omega_{BH} P_{BH} \), \( P_{BH} \) peaks when the magnetic field lines touch the disk at the place where \( \Omega_{BH} = \Omega_{BH}/2 \) (Li 2000c).
then a steady state with no accretion is built. In such a nonaccretion case, all the power of the disk comes from the rotational energy of the black hole. If the disk has so much internal torque that cannot be balanced by the external torque, the excess internal torque will produce accretion. In such a case, accretion and magnetic coupling coexist, and the power of the disk comes from both the rotational energy of the black hole and the gravitational energy of the disk.

In the case that there is accretion and the magnetic field touches the disk at a circle of radius $r_0$, the quasi-steady solutions are given by equations (34) and (35). The radiation flux is plotted in Figures 8 and 9, respectively, for the case that the black hole rotates faster than the disk ($\Omega_H > \Omega_0$) and for the case that the black hole rotates slower than the disk ($\Omega_H < \Omega_0$). In both cases, for $r < r_0$, the solutions are the same as those predicted by the standard theory of a thin Keplerian disk (Novikov & Thorne 1973; Page & Thorne 1974; Thorne 1974); in particular, $F$ and $g$ are zero at $r = r_{ms}$. The extensions of the standard solutions to $r > r_0$ are shown with dashed curves, and the positions of $r_0$ are shown with vertical dotted lines. Because of magnetic coupling to the black hole, $F$ and $g$ are modified for $r > r_0$ by superposing the contribution of the magnetic coupling to the standard solutions. For $r >> r_0$, the radiation flux given by the standard theory decreases as $r^{-3}$, while the radiation flux contributed by the magnetic coupling decreases as $r^{-3.5}$. Thus, at large radii the radiation flux $F$ is dominated by the contribution of accretion. This is also true for the viscous torque in the disk; i.e., at large radii the torque in the disk is dominated by the contribution of accretion. When the black hole rotates faster than the disk—as in the case shown in Figure 8—the black hole pumps energy and angular momentum into the disk, the energy is dissipated in the disk and radiated away, and a bright annular “bump” is produced at $r = r_0$. When the black hole rotates slower than the disk—as in the case shown in Figure 9—the black hole extracts energy and angular momentum from the disk, and a dark annular “valley” is produced at $r = r_0$.

When the black hole rotates slower than the disk (i.e., $\Omega_H < \Omega_0$), $A_0$ is negative; thus, energy and angular momentum are transferred from the disk to the black hole. From equation (34), the minimum radiation flux at $r = r_0$ is

$$F_{\text{min}} = \frac{1}{4\pi r_0^2} \times \left[ \dot{M}_D f(r_0) + 4\pi r_0 A_0 \left( -\frac{d\Omega_0}{dr_0} \right)(E_0^+ - \Omega_0 L_0^+)^{-1} \right],$$

(47)

which (and the minimum $g$ at $r = r_0$) is nonnegative if and only if

$$4\pi r_0 |A_0| \leq \left| -\frac{d\Omega_0}{dr_0} \right|^{-1} (E_0^+ - \Omega_0 L_0^+) \dot{M}_D f(r_0),$$

(48)
where we have used the fact that $d\Omega_p/dr < 0$ in the disk. Therefore, when the black hole rotates slower than the disk, quasi-steady solutions exist only if the condition in equation (48) is satisfied. In particular, since $f(r_m) = 0$, no quasi-steady solutions exist if the magnetic field touches the disk at the inner boundary and the black hole rotates slower than the disk (i.e., $a/M_H < 0.3594$).

In a real situation the magnetic field cannot just touch the disk at a single radius. Instead, the magnetic field must be distributed over a range of radii in the disk. To test the sensitivity of our results to the extension of the magnetic field in the disk, we have investigated the following model: the magnetic field connecting the black hole to the disk touches the disk in an annular region bounded by $r = r_0 \pm \Delta r$, where $\Delta r < r_0 - r_{ms}$. We assume that $H \propto r$ for $r_0 - \Delta r < r < r_0 + \Delta r$ and $H = 0$ for $r < r_0 - \Delta r$ and $r > r_0 + \Delta r$, where $H$ is the flux of angular momentum transferred from the black hole to the disk (see eq. [11]). Such a distribution roughly corresponds to a uniform magnetic field in the annular region. We have calculated the radiation flux of the disk and the resulting spectrum seen by a distant observer (using the procedure provided by Cunningham 1975), assuming that the disk radiates like a black-body. The radiation fluxes are shown in Figure 10 for the cases of $r_0 = 2r_{ms}$ and $\Delta r = 0.05r_{ms}, 0.5r_{ms}$, and $r_{ms}$, alternatively. For comparing the results for different values of $\Delta r$, we have normalized the magnetic field so that the total efficiency of the disk, which is defined by equation (22), is $\epsilon = 1$. From Figure 10 we see that the shape of the radiation flux curve sensitively depends on the extension of the magnetic field—i.e., the value of $\Delta r$. As $\Delta r$ increases, the peak of the flux produced by the magnetic coupling spreads in width and eventually merges with the peak of the flux produced by accretion. For the same cases the radiation spectra seen by a distant observer at polar angle $\theta = \pi/3$ from the axis of the black hole are shown in Figure 11. For comparison, the spectrum for a standard accretion disk is also shown in Figure 11, where the standard accretion disk and the accretion disk with magnetic coupling are assumed to have the same luminosity. We see that the magnetic connection tends to harden the radiation of the disk. Though the shape of the radiation flux (Figure 10) is very sensitive to the extension of the magnetic field (i.e., the value of $\Delta r$), the radiation spectrum is not so. Especially, at the low-frequency end the spectrum almost does not depend on the value of $\Delta r$. This demonstrates that the spectral signature of the magnetic coupling can be robust.

5. WEAK MAGNETIC FIELD ASSUMPTION, INSTABILITIES, AND PHOTON CAPTURE

For a nonaccretion disk magnetically coupled to a black hole, a steady state can be established if the magnetic field is frozen in the disk and the angular momentum transferred from the black hole to the disk is steadily conveyed outward by an internal viscous torque. When there is accretion, an exactly steady state cannot exist since the magnetic field frozen in the disk slowly moves toward the central black hole with the accretion flow. However, if the inflow velocity of the disk is much smaller than the rotational velocity, we
can expect the disk and the magnetic field to be in a quasi-steady state, which means that any macroscopic quantity at a given radius in the disk slowly changes with time: the overall change within one rotation period is negligible compared with the quantity itself. The assumption of a quasi-steady state is more or less similar to the assumption of “adiabatic invariance” usually discussed in galactic dynamics and quantum mechanics, where it is assumed that a potential changes very slowly with time so that within one rotation period the potential can be treated as unchanged. For a quasi-steady state, the magnetic connection between the black hole and the disk may evolve with time from the view of a long time interval, but within one rotation period the magnetic connection is approximately unchanged.

In searching for quasi-steady state solutions, we have assumed that the magnetic field is weak so that its influence on the dynamics of particles in the disk is negligible and thus a thin Keplerian disk is a good approximation. This “weak magnetic field assumption” requires that

$$|\nabla (B^2/8\pi)| \ll \rho |g|$$  \hspace{1cm} (49)$$

in the disk, where $\rho$ is the mass density of the disk and $g$ is the gravitational acceleration produced by the black hole. Since $|\nabla (B^2/8\pi)| \sim B^2/r$, $|g| \sim GM_{\text{BH}}/r^2 \sim r_{\text{H}}/r^2$, equation (49) is equivalent to

$$B^2 \ll \rho c^2 \frac{r_{\text{H}}}{r},$$  \hspace{1cm} (50)$$

where we have restored the speed of light in the equation. Equation (50) is the condition for the weak magnetic field assumption.

A weak magnetic field may play an important role in the balance and transportation of energy and angular momentum. From equation (2) of Li (2000c) and $\Psi_{\text{H}} \sim B r^2$, we have

$$L_{\text{H}} = 2P_{\text{H}} \sim B^2 r^2 \left(\frac{r}{r_{\text{H}}}\right)^{1/2},$$  \hspace{1cm} (51)$$

for the case of $a/M_{\text{BH}} \approx 1$, where we have used $\Delta Z_{\text{H}} \sim 1$. The accretion power of the disk is

$$L_{\text{acc}} = M_D \varepsilon_0 \sim \rho r h v_\phi,$$  \hspace{1cm} (52)$$

where $\varepsilon_0$ is the efficiency of accretion ($\approx 0.32$ for $a/M_{\text{BH}} = 0.998$; see eq. [21]) and $h$ is the thickness of the disk. From equations (51) and (52), $L_{\text{H}} \gg L_{\text{acc}}$ if and only if

$$B^2 \gtrsim \rho c^2 \frac{h r_{\text{H}}}{r} v_\phi,$$  \hspace{1cm} (53)$$

where we have used $v_\phi \sim r^{-1/2}$. Equation (53) is the condition for the magnetic field to be important in the balance and transportation of energy and angular momentum.

For a thin and quasi-steady accretion disk, we have $h/r \ll 1$ and $v_\phi/v_\phi \ll 1$. In the disk we always have $r_{\text{H}}/r \lesssim 1$. Therefore, equation (53) is not a too stringent restriction on the values of $B$. To see this, we can compare equation (53) with equation (50). There is much room for $B^2$ to satisfy both equations (50) and (53) if

$$\frac{v_\phi}{v_\phi} \lesssim \frac{r}{r_{\text{H}}},$$  \hspace{1cm} (54)$$

Equation (54) is always satisfied by a thin and quasi-steady disk.

In all our analyses the instabilities of the magnetic field and the disk are ignored, but in practice they may be important. Because of the outwardly decreasing differential rotation, a plasma disk threaded by a weak magnetic field is subject to the Balbus-Hawley instability (Balbus & Hawley 1991, 1998). Within a few rotational periods the magnetic field lines in the disk become chaotic and tangled, and the disk becomes turbulent. This magnetorotational instability is assumed to play an important role in transporting angular momentum within the disk, which is important for producing accretion (Balbus & Hawley 1998; Hawley 2000; Hawley & Krolik 2001; Menou 2000; Stone & Pringle 2001). In our model the Balbus-Hawley instability must also operate in the disk to convey the angular momentum outward. For a thin disk, the effect of magnetic reconnection becomes important if the poloidal magnetic field lines threading the disk are twisted too much by the rotation of the disk and the central object. This may put a restriction on the ampliﬁcation of the toroidal magnetic ﬁeld near the plane of the disk, which in turn restricts the power of the energy and the angular momentum transferred between the central object and the disk (Ghosh & Lamb 1978, 1979a, 1979b; Livio & Pringle 1992; Wang 1995). But because of the fact that a black hole has a large resistance, the situation here may not be as serious as in the case when the central object is a star. The screw instability of the magnetic field plays a similar role in limiting the ampliﬁcation of the toroidal magnetic ﬁeld and in turn the power and torque of the central black hole (Gruzinov 1999; Li 2000b). Thus, con-
considering the effect of magnetic reconnection or the screw instability, the actual power and the torque of the black hole may be somewhat smaller than those given by equations (2) and (3). If the instabilities are strong enough, the situation may change so dramatically that quasi–steady state solutions do not exist. All these effects of instabilities will be addressed in detail in the future.

Not all photons emitted by the disk escape to infinity. Some return to the disk, and some are captured by the black hole, because of the gravity of the black hole and the shape of the disk (Thorne 1974; Cunningham 1976; Agol & Krolik 2000). These effects are especially important in the inner disk region when $a/M$ is close to 1. The radiation returning to the disk is dissipated and reradiated, and eventually reaches infinity or falls into the black hole. The radiation flux and the overall power of the disk are affected by the photons captured by the black hole and the photons returning to the disk. The black hole’s capture cross section is greater for photons of negative angular momentum (angular momentum opposite to the spin of the black hole) than for photons of positive angular momentum (Godfrey 1970; Bardeen 1973). Thus, photon capture tends to spin down the black hole. Considering the effect of photon capture, Thorne (1974) has shown that a black hole will be prevented from spinning up beyond a limiting state with $a/M \approx 0.998$ (the canonical state) through accretion from a thin Keplerian disk.\footnote{However, the limit $s = 0.998$ may be exceeded if a black hole is spun up by an external torque, such as in black hole mergers (Agol & Krolik 2000) or through accretion from a thick disk (Abramowicz & Lasota 1980).}

For a standard accretion disk without magnetic coupling, depending on the spin of the black hole, of the energy radiated by the disk up to 6% is captured by the black hole, and up to 28% returns to the disk. For a disk with magnetic coupling, more energy is captured by the black hole or returns to the disk. In an extreme case when the magnetic field lines touch the disk at the inner boundary and there is no accretion, depending on the spin of the black hole, of the energy radiated by the disk up to 15% is captured by the black hole, up to 58% returns to the disk, as in the “infinite efficiency limit” discussed by Agol & Krolik (2000). Considering the effect of radiation capture, the radiation flux at the inner part of the disk is moderately modified. In the case of a nonaccretion disk magnetically coupled to a Kerr black hole, the returning radiation dominates at large radii since the radiation flux due to the magnetic coupling scales as $r^{-3.5}$ while the returning radiation flux scales as $r^{-3}$ (Cunningham 1976; Agol & Krolik 2000). In our calculations the effects of photon capture (either captured by the black hole or returning to the disk) are ignored; we hope to consider them in the future.

6. CONCLUSIONS

For an accretion disk magnetically coupled to a Kerr black hole, quasi–steady state solutions are obtained by assuming that the inflow timescale of particles in the disk is much longer than the rotational timescale—as adopted in the standard theory of an accretion disk. For a general distribution of the magnetic field connecting the disk to the black hole, the solutions for the radiation flux and the internal viscous torque are given by equations (16) and (17), which are superpositions of the contribution from accretion and the contribution from magnetic coupling. From the view of a long period of time (e.g., with many rotation periods), the radiation flux and the internal torque of the disk may vary with time (the magnetic connection may eventually disappear), but equations (16) and (17) give the instant values of the radiation flux and the internal torque of the disk. These general solutions clearly show that for any distribution of a magnetic field on the disk, the torque produced by the magnetic coupling propagates outward only; thus, the internal torque and the radiation flux are always zero at the inner boundary of the disk. Even for the extreme case that the magnetic field touches the disk at the inner boundary, the internal torque $g$ and the radiation flux $F$ of the disk are also zero at the inner boundary: as $r$ decreases from the outside of the marginally stable orbit, $g$ and $F$ increase but suddenly drop to zero at $r = r_{\text{max}}$. We emphasize that this feature differs from the case of a disk magnetically connected to the material in the transition region, where the magnetic stress is demonstrated to be nonzero at the inner boundary and extends into the transition region (Krolik 1999; Agol & Krolik 2000; Hawley & Krolik 2001; Krolik 2001).

We have discussed in detail a simple but important case: the magnetic field touches the disk at a single radius $r_0$—or equivalently, the magnetic field is distributed on the disk within a narrow region around $r_0$. For this simple case, the radiation flux produced by the magnetic coupling has a sharp peak at $r_0$; it is zero for $r < r_0$ and decreases quickly for $r > r_0$. At large radii, the radiation flux approaches $F \propto r^{-3.5}$. This behavior of the radiation flux is very different from that of a standard accretion disk, where the radiation flux spreads widely over radii and approaches $F \propto r^{-3}$ at large radii. We have compared in detail the radiation flux of a nonaccretion disk with the magnetic field touching the disk at the inner boundary (i.e., the marginally stable orbit) with the radiation flux of a standard accretion disk. The results are summarized in Figures 4–6. Clearly, the radiation profile of the nonaccretion disk is very different from that of the standard accretion disk: the nonaccretion disk magnetically coupled to a rapidly rotating black hole has a bigger emissivity index throughout the disk and a radiation region closer to the center of the disk. We have also shown that the magnetic field lines touching the disk at the inner boundary are most efficient in extracting energy from the black hole, although the power of the black hole peaks at $\Omega_\text{bh} = \Omega_\text{H}/2$.

A nonaccretion state can exist only if the internal torque of the disk exactly balances the external torque on the disk produced by the magnetic coupling. If the disk has so much internal torque that cannot be balanced by the external torque, the excess internal torque will produce accretion. For an accretion disk with the magnetic field touching the disk at a single radius, a specific feature is that the radiation flux of the disk may have two maxima (Figs. 8 and 9). When the black hole rotates faster than the disk, the black hole pumps energy to the disk and the second maximum in the radiation flux is produced by adding a bright “bump” to the standard radiation flux (Fig. 8). When the black hole rotates slower than the disk, the black hole extracts energy from the disk—in such a case the efficiency of the disk is decreased by the magnetic coupling—and the second maximum in the radiation flux is produced by digging a dark “valley” in the standard radiation flux (Fig. 9). The
position of the second maximum is determined by the position where the magnetic field touches the disk. We have tested the sensitivity of the results to the extension of the magnetic field in the disk. We have shown that although the shape of the radiation flux sensitively depends on the extension of the magnetic field, the spectral signature of the magnetic coupling can be robust (Figs. 10 and 11).

For a black hole with $a/M_H \approx 1$, the power of the black hole is approximately

$$P_{HD} \approx B^2 r_H^2 c,$$

(55)

if the magnetic field lines touch the disk close to the inner boundary ($r_{in} = r_{ms} \approx r_H$). If the energy deposited into the disk by the black hole is radiated away locally, either thermally or nonthermally, we can define an effective radiation temperature through

$$T_{eff} = \left( \frac{P_{HD}}{\sigma T_K^4} \right)^{1/4} \approx \left( \frac{B^2 c^4}{\sigma} \right)^{1/4} \approx 5 \times 10^5 K \left( \frac{B}{10^4 G} \right)^{1/2},$$

(56)

where $\sigma$ is the Stephan-Boltzmann constant. Interestingly, this temperature depends only on the strength of the magnetic field on the disk. If the energy is radiated thermally, the radiation is in the UV to soft X-ray domain for $B \approx 10^4 G$.

The recent $XMM$-$Newton$ observation of soft X-ray emission lines from two narrow-line Seyfert 1 galaxies MCG $-6-30-15$ and Mrk 766 shows an extremely large emissivity index ($\sim 4$), which has been suggested to indicate that most of the line emission originates from the inner part of a relativistic accretion disk (Branduardi-Raymont et al. 2001). From the calculations in § 4.1, we have seen that for a nonaccretion disk magnetically coupled to a rapidly rotating black hole with the magnetic field touching the disk at the inner boundary, most radiation comes from a region more concentrated toward the center of the disk and a large emissivity index can be easily realized at any radius, compared to the case for a standard accretion disk. Thus, probably the observational results of $XMM$-$Newton$ can be more easily explained with the model presented in this paper. Another observation that may be relevant to our model is the kilohertz quasi-periodic oscillations (kHz QPOs) in X-ray binaries, which have been suggested to originate from the inner edge of a relativistic accretion disk (van der Klis 2000 and references therein). The magnetic field connecting the black hole to the disk does not have to be axisymmetric. If a bunch of magnetic field lines connect the black hole to a small region in the disk and the black hole rotates faster than the disk, a hot spot will be produced on the disk. The hot spot will corotate with the disk since the magnetic field is frozen in the disk. This could be a natural model for the kHz QPOs.

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