Detailed balance and ultraviolet stability of scalar field in Horava-Lifshitz gravity

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Detailed balance and projectability conditions are two main assumptions when Horava recently formulated his theory of quantum gravity - the Horava-Lifshitz (HL) theory. While the latter represents an important ingredient, the former often believed needs to be abandoned, in order to obtain an ultraviolet stable scalar field, among other things. In this paper, because of several attractive features of this condition, we revisit it, and show that the scalar field can be stabilized, if the detailed balance condition is allowed to be softly broken. Although this is done explicitly in the non-relativistic general covariant setup of Horava-Melby-Thompson with an arbitrary coupling constant \( \lambda \), generalized lately by da Silva, it is also true in other versions of the HL theory. With the detailed balance condition softly breaking, the number of independent coupling constants can be still significantly reduced. It is remarkable to note that, unlike other setups, in this da Silva generalization, there exists a master equation for the linear perturbations of the scalar field in the flat Friedmann-Robertson-Walker background.

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I. INTRODUCTION

With the perspective that Lorentz symmetry may appear as an emergent symmetry at low energies, but can be fundamentally absent at high energies, Horava considered a gravitational system whose scaling at short distances exhibits a strong anisotropy between space and time [1],

\[
\mathbf{x} \to \ell \mathbf{x}, \quad t \to \ell^2 t.
\]  

(1.1)

In \((d+1)\)-dimensions, in order for the theory to be power-counting renormalizable, the critical exponent \( z \) needs to be \( z \geq d + 2 \) [2]. At long distances, all the high-order curvature terms are negligible, and the linear terms become dominant. Then, the theory is expected to flow to the relativistic fixed point \( z = 1 \), whereby the Lorentz invariance is "accidentally restored."

The special role of time can be realized with the Arnowitt-Deser-Misner decomposition [3],

\[
ds^2 = -N^2c^2 dt^2 + g_{ij} \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right),
\]

\((i, j = 1, 2, 3)\).  

(1.2)

Under the rescaling \((\Box)\) with \( z = d = 3 \), a condition we shall assume in the rest of this paper, \( N, N^i \) and \( g_{ij} \) scale, respectively, as,

\[
N \to N, \quad N^i \to \ell^{-2} N^i, \quad g_{ij} \to g_{ij}.
\]

(1.3)

The gauge symmetry of the system now is reduced to the foliation-preserving diffeomorphisms \( \text{Diff}(M, \mathcal{F}) \),

\[
\tilde{t} = t - f(t), \quad \tilde{x}^i = x^i - \zeta^i(t, x),
\]

(1.4)

under which, \( N, N^i \) and \( g_{ij} \) change as,

\[
\delta g_{ij} = \nabla_i \zeta_j + \nabla_j \zeta_i + f \delta g_{ij},
\]

\[
\delta N_i = N_k \nabla_i \dot{k} + \zeta^k \nabla_k N_i + g_{ik} \zeta^j + \dot{N}_i f + N_i \dot{f},
\]

\[
\delta N = \zeta^k \nabla_k N + \dot{N} f + N \dot{f},
\]

(1.5)

where \( \dot{f} \equiv df/dt \), \( \nabla_i \) denotes the covariant derivative with respect to the 3-metric \( g_{ij} \), \( N_i = g_{ik} N^k \), and \( \delta g_{ij} = \delta g_{ij}(t, x^k) - g_{ij}(t, x^k) \), etc. Eq. (1.5) shows clearly that the lapse function \( N \) and the shift vector \( N^i \) play the role of gauge fields of the \( \text{Diff}(M, \mathcal{F}) \) symmetry. Therefore, it is natural to assume that \( N \) and \( N^i \) inherit the same dependence on spacetime as the corresponding generators, while the dynamical variables \( g_{ij} \) in general depend on both time and spatial coordinates, i.e.,

\[
N = N(t), \quad N_i = N_i(t, x), \quad g_{ij} = g_{ij}(t, x).
\]

(1.6)

This is often referred to as the projectability condition, and clearly preserved by the \( \text{Diff}(M, \mathcal{F}) \). It should be noted that, although this condition is not necessary, breaking it often leads to inconsistence theories [4].

Abandoning the Lorentz symmetry, on the other hand, gives rise to a proliferation of independently coupling constants, which could potentially limit the prediction powers of the theory. Inspired by condensed matter systems [5], Horava assumed that the gravitational potential \( \mathcal{L}_V \) can be obtained from a superpotential \( W_g \) via the relations,

\[
\mathcal{L}_{V,detailed} = w^2 E_{ij} \mathcal{G}^{ijkl} E_{kl},
\]

(1.7)

where \( w \) is a coupling constant, and \( \mathcal{G}^{ijkl} \) denotes the generalized DeWitt metric, defined as \( \mathcal{G}^{ijkl} = \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) / 2 - \lambda g^{ij} g^{kl} \), with \( \lambda \) being another coupling constant. The 3-tensor \( E_{ij} \) is obtained from \( W_g \) by

\[
E_{ij} = \frac{1}{\sqrt{g}} \frac{\delta W_g}{\delta g_{ij}}.
\]

(1.8)
In (3+1)-dimensional spacetimes, $W_g$ is given by \(^1\),
\[
W_g = \int \omega_3(\Gamma),
\]
where $\omega_3(\Gamma)$ denotes the gravitational Chern-Simons term,
\[
\omega_3(\Gamma) = \text{Tr}\left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right).
\]

Despite of many remarkable features of the theory \([6–10]\), it is plagued with three major problems: ghost, strong coupling and instability. Although they are different one from another, their origins are all the same: because of the breaking of the Lorentz symmetry,
\[
\tilde{x}^\mu = x^\mu - \zeta^\mu(t, x), \quad (\mu = 0, 1, 2, 3).
\]
In particular, due to such a breaking, the kinetic part of the gravitational field in general takes the form,
\[
\mathcal{L}_K = K_{ij} K^{ij} - \lambda K^2,
\]
where the extrinsic curvature $K_{ij}$ is defined as,
\[
K_{ij} = \frac{1}{2N} (-\ddot{g}_{ij} + \nabla_i N_j + \nabla_j N_i).
\]

The coupling constant $\lambda$ is subjected to radiative corrections, and its values are expected to be different at different energy scales. This is different from Lorentz-invariant theories, where $\lambda = 1$ is protected by the Lorentz symmetry \((1.11)\) even in the quantum level. Then, considering linear perturbations, one can show that the kinetic part of the gravitational sector is proportional to $(\lambda - 1)/(3\lambda - 1)$ \([1, 11–14]\). Thus, to avoid the ghost problem, $\lambda$ has to be either $\lambda \geq 1$ or $\lambda < 1/3$. It is still an open question how $\lambda$ runs from its ultraviolet (UV) fixed point to its relativistic one, $\lambda_{IR} = 1$. To answer this question, one way is to study the corresponding renormalization group (RG) flows. However, since the problem is so much mathematically involved, the RG flows have not been explicitly worked out, yet, although some preliminary work has been already initiated \([12, 13]\).

Strong coupling problem is also closely related to the fact that $\lambda$ is generically different from one, though manifests itself in a different manner \([16]\). It appears in the self-interaction of the gravitational sector (as well as in the interactions of gravity with matter fields). In the framework of linear perturbations, it can be shown that some coupling coefficients of third-order actions are inversely proportional to the powers of $(\lambda - 1)$ \([17–21]\).

As the order increases, the powers in terms of $1/(\lambda - 1)$ also increase \([1]\). Then, when $\lambda$ runs from its UV fixed point to its relativistic one, the coupling coefficients become larger and larger. When energy is greater than the strong coupling energy scale, $\Lambda_{SC}(\lambda)$, the coefficients become much bigger than unit, and the theory enters the strong coupling regime. Typically, one can show that $\Lambda_{SC} \approx M_{pl}|\lambda - 1|^{3/4}$, where $M_{pl}$ is the Planck mass. Together with other observational constraints, it implies $|\lambda - 1| \approx 10^{-24}$ \([21]\). Clearly, this gives rise to the issue of fine-tuning. To solve this problem, two different approaches have been proposed. One is the Blas-Pujolas-Sibiryakov (BPS) mechanism \([17]\), in which an effective energy scale $M_s$ is introduced. By properly choosing the coupling constants involved in the theory, BPS showed that $M_s$ can be lower than $\Lambda_{SC}$. As a result, the perturbative theory becomes invalid before $\Lambda_{SC}$ is reached, whereby the problem is circumvented. While this seems a very attractive mechanism, it turns out \([21]\) that it may apply only to the version \([22]\) of the HL theory without projectability condition, a setup that also faces other challenges, including the one with a large number $(> 60)$ of coupling constants \([20]\). The other approach is to provoke the Vainshtein mechanism \([23]\), as showed recently in the spherical static \([7]\) and cosmological \([21]\) spacetimes.

Instability, on the other hand, can appear in both of the gravitational and matter sectors. In the gravitational sector, due to the restricted diffeomorphisms \([14]\), a spin-0 graviton appears, which is unstable in the Minkowski background \([11, 12, 24, 25]\) (but, stable in the de Sitter one, as shown explicitly in \([21, 25]\)), as far as the Sotiriou, Visser and Weinfurtner (SVW) generalization with projectability condition \([11]\) is concerned \([2]\). This is potentially dangerous, and needs to decouple in the IR, in order to be consistent with observations. It is still an open question whether this is possible or not \([7]\).

The instability of matter sector has been mainly found for a scalar field in the UV, due to the detailed balance condition \([20]\). Because of this and the non-existence of relativistic limit \([27]\), it is generally believed that this condition should be abandoned \([6, 9]\). However, the detailed balance condition has several remarkable features \([10]\), which provoke a great desire to revisit it. In particular, it is in the same spirit of the AdS/CFT correspondence \([28]\), where a string theory and gravity defined on one space is equivalent to a quantum field theory without gravity defined on the conformal boundary of this space, which has one or more lower dimension(s). Yet, in the non-equilibrium thermodynamics, the counter-part of the super potential $W_g$ plays the role of entropy, while the term $E^{3/3}$ the entropic forces \([29]\). This might shed light on the nature of the gravitational forces, as proposed re-

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\(^1\) One can include all the possible relevant operators by adding $\int d^3 x \sqrt{\gamma} (R - 2\Lambda_W)$ to $W_g$. For detail, see \([3]\).
cently by Verlinde [30]. To overcome the above problems, recently Horava and Melby-Thompson (HMT) [31] extended the foliation-preserving-diffeomorphisms $\text{Diff}(M, \mathcal{F})$ to include a local $U(1)$ symmetry,
\begin{equation}
U(1) \ltimes \text{Diff}(M, \mathcal{F}), \tag{1.14}
\end{equation}
and then showed that, similar to GR, the spin-0 graviton is eliminated [31, 32]. Thus, the instability of the spin-0 gravity is automatically fixed. A remarkable by-production of this “non-relativistic general covariant” setup is that it forces the coupling constant $\lambda$ to take exactly its relativistic value $\lambda_{IR} = 1$. Since both of the ghost and strong coupling problems are precisely due to the deviation of $\lambda$ from 1, as shown above, this in turn implies that these two problems are also resolved.

However, it was soon challenged by da Silva [33], who argued that the introduction of the Newtonian pre-potential is so strong that actions with $\lambda \neq 1$ also has the $U(1)$ symmetry. Although the spin-0 graviton is eliminated even in the da Silva generalization, as shown explicitly in [32] for de Sitter and anti-de Sitter backgrounds, and in [14] for the Minkowski, the ghost and strong coupling problems rise again, since now $\lambda$ can be different from one. Indeed, it was shown [14] that to avoid the ghost problem, $\lambda$ must satisfy the same constraints, $\lambda \geq 1$ or $\lambda < 1/3$, as found previously. In addition, the coupling becomes strong for a process with energy higher than $M_{pl}[\lambda - 1]^{5/4}$ in the flat Friedmann-Robertson-Walker (FRW) background, and $M_{pl}[\lambda - 1]^{3/2}$ in a static weak gravitational field. It must be noted that this does not contradict to the fact that the spin-0 graviton does not exist even for $\lambda \neq 1$. In fact, when one counts the degrees of freedom of the gravitational excitations, one needs to consider free gravitational fields. Another way to count the degrees of freedom is to study the structure of the Hamiltonian constraints [31, 34]. On the other hand, to study the ghost and strong coupling problems, one needs to consider the cases in which the gravitational perturbations are different from zero. This can be realized by the presence of matter fields $^3$. This is exactly what was done in [14].

It should also be noted that the HMT setup (with $\lambda = 1$) and its da Silva generalization (with any $\lambda$) are applicable to the cases with or without detailed balance condition. As a matter of fact, in [31, 33] the authors were mainly concerned with the theory with detailed balance condition, while in [14, 32] the cases without detailed balance condition were studied.

Assuming that the strong coupling problem can be solved in certain ways, in this paper we re-consider the stability of a scalar field with detailed balance condition, and show explicitly that it can be stabilized in all energy scales, including the UV and IR, by softly breaking the detailed balance condition. We show this explicitly in the da Silva generalization, although it can be easily generalized to other versions of the HL theory. It should be noted that softly breaking detailed balance condition was already considered by Horava in his seminal work [1], and later was studied by many others, mainly in the versions without projectability condition in order to find static solutions that has relativistic limit [35, 36]. We also note that static spacetimes were studied recently in the HMT setup (with $\lambda = 1$) [37, 38].

The rest of the paper is organized as follows: In Sec. II we briefly review the da Silva generalization, and present the general action of the gravitational sector by softly breaking the detailed balance condition. With such a breaking, the number of independent coupling constants can be still significantly reduced. In fact, only in the gravitational sector, the number is already reduced from eleven to seven [cf. Eq. (2.9)]. In Sec. III we study the coupling of gravity with a scalar field, and construct its most general action with the same requirement: softly breaking detailed balance condition. In Sec. IV, we investigate the stability of the scalar field with our new action of the scalar field constructed in the last section, and show explicitly that it is indeed stabilized in all the energy scales. It is remarkable to note, unlike other versions of the HL theory [33], now there exists a master equation for the linear perturbations of a scalar field in the flat FRW background for any given $\lambda$. Our main conclusions are presented in Sec. V. There are also three appendices, A, B and C. In Appendix A, the field equations are given, and in Appendix B, the linear scalar perturbations of the flat FRW universe are presented for any matter fields, while in Appendix C, the scalar perturbations for $\lambda = 1$ and $c_1 \neq 1$ are studied.

## II. NON-RELATIVISTIC GENERAL COVARIANT THEORY WITH ANY $\lambda$

In order to limit the spin-0 graviton, HMT introduced two new fields, the $U(1)$ gauge field $A$ and the Newtonian pre-potential $\varphi$. Under $\text{Diff}(M, \mathcal{F})$, these fields transfer as,
\begin{align}
\delta A &= \zeta^i \partial_i A + \dot{f} A + f \dot{A}, \\
\delta \varphi &= f \dot{\varphi} + \zeta^i \partial_i \varphi, \tag{2.1}
\end{align}

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$^3$ A similar situation also happens in GR, in which gravitational scalar perturbations of the FRW universe in general do not vanish, although the only degrees of freedom of the gravitational sector are the spin-2 massless gravitons.

$^4$ Note that the notations used in this paper are slightly different to those adopted in [33]. In particular, we have $\varphi = -\varphi^{\text{HMT}}, K_{ij} = -K_{ij}^{\text{HMT}}, A = -\tilde{A}^{\text{HMT}}, \lambda_0 = \Omega^{\text{HMT}}, \tilde{g}_{ij} = \tilde{g}_{ij}^{\text{HMT}}$, where quantities with super indice “HMT” are the ones used in [31].
while under $U(1)$, characterized by the generator $\alpha$, they, together with $N, N^i$ and $g_{ij}$, transfer as
\[
\delta_\alpha A = \alpha - N^i \nabla_i \alpha, \quad \delta_\alpha \varphi = - \alpha, \\
\delta_\alpha N_i = N \nabla_i \alpha, \quad \delta_\alpha g_{ij} = 0 = \delta_\alpha N.
\]
For the detail, we refer readers to \[14, 31, 32\].

As mentioned above, HMT considered only the case $\lambda = 1$. Later, da Silva generalized it to the cases with any $\lambda$ \[33\], in which the total action can be written in the form \(1.3 \) \[40\], where \[2.5\], for which the detailed balance condition, but such a breaking is soft, time that in the UV these lower dimension operators become negligible, and only the marginal term of Eq. (2.9) remains, whereby the detailed balance condition is restored. Then, including all the relevant terms, the most general potential with the detailed balance condition softly breaking takes the form \[10\],
\[
\mathcal{L}_V = \zeta^2 \gamma_0 + \gamma_1 R + \frac{1}{\zeta^2} \left( \gamma_2 R^2 + \gamma_3 R^{ij} \right) + \frac{2}{\zeta^4} \gamma_4 R^{ij} R_k^l + \frac{\gamma_5}{\zeta^4} \mathcal{C}_{ij} C^{ij},
\]
where the coupling constants $\gamma_s (s = 0, 1, 2, \ldots 5)$ are all dimensionless, and $\zeta_5 \equiv \nu^2 \zeta^4$. The relativistic limit in the IR requires
\[
\lambda = 1, \quad \gamma_1 = - 1, \quad \zeta^2 = \frac{1}{16 \pi G}.
\]
The existence of the $\gamma_4$ term explicitly breaks the parity, which could have important observational consequences on primordial gravitational waves \[41\]. The corresponding field equations are given in Appendix A.

It should be noted that, even softly breaking the detailed balance condition, the number of the independent coupling constants is still considerably reduced. In fact, without detail balance condition, the number is eleven \[11, 32\] (when the parity is allowed to be broken), while Eq. (2.9) shows that now only seven of them remain.

### III. COUPLING OF SCALAR FIELD

To construct the action of a scalar field $\chi$, following what was done for the gravitational sector in the last section, we assume that: (i) the scalar field respects the detailed balance condition in the UV; and (ii) it breaks the condition only softly. With these in mind, let us first consider the action,
\[
\hat{S}_\chi = \frac{1}{2} \int d^3 x \sqrt{g} \left[ \frac{f(\lambda)}{2 N^2} \left( \chi - N^i \nabla_i \chi \right)^2 + \left( \frac{\delta W_\chi}{\delta \chi} \right)^2 \right],
\]
where
\[
W_\chi = - \frac{1}{2} \int d^3 x \sqrt{g} \left[ \sigma_3 \chi (\Delta)^{3/2} \chi + \sigma_2 \chi \Delta \chi - m \chi^2 \right],
\]
adopted in [26] is replaced by a function, $f$, and a fixed point. The field should be reduced to the relativistic one at the IR limit [42]. In fact, inserting Eq.(3.2) into Eq.(3.3) we obtain,

$$
\hat{S}_X = \frac{1}{2} \int dt d^3x \sqrt{g} \left[ \frac{f}{2N^2} \left( \chi - N' \nabla \chi \right)^2 + \sum_{A=2}^{6} \beta_A P_A + m^2 \chi^2 \right],
$$

(3.3)

where

$$
P_A = \chi A^2, \quad \beta_6 = \sigma^2_2, \quad \beta_5 = 2 \sigma_2 \sigma_3, \
\beta_4 = \sigma^2_2, \quad \beta_3 = -2 \sigma_3, \quad \beta_2 = -2 \sigma_2.
$$

(4.4)

Clearly, the mass term now has a wrong sign. Thanks to the softly breaking condition, this can be fixed by adding all the relevant terms into Eq.(3.3). After doing so, we find that the scalar field can be written in the form,

$$
S_X^{(0)}(N, N^i, g_{ij}, \chi) = \int dt d^3x \sqrt{g} L_X^{(0)},
$$

(3.5)

where

$$
L_X^{(0)} = \frac{f}{2N^2} \left( \chi - N' \nabla \chi \right)^2 - V, \\
V = V(\chi) + \left( \frac{1}{2} + V_1(\chi) \right)(\nabla \chi)^2 + V_2(\chi)P_i^2 + V_3(\chi)P_iP_j + V_4(\chi)(\nabla \chi)^2P_i
$$

(4.6)

with $V(\chi)$ and $V_n(\chi)$ being arbitrary functions of $\chi$, and

$$
P_n = \Delta n, \quad V_6 = -\sigma_3^2.
$$

(3.7)

It should be noted that the $\lambda$-dependent factor $(3\lambda - 1)$ adopted in [26] is replaced by a function, $f(\lambda)$, which, subjected to some physical restrictions, is otherwise arbitrary. Those constraints include that the scalar field must be ghost-free in all the energy scales. By properly choosing it, we also assume that the speed of the scalar field should be reduced to the relativistic one at the IR fixed point.

From Eqs. (3.5) and (3.6), one can see that the scalar field couples directly only to the metric components, $N$, $N^i$, and $g_{ij}$. To have it also coupled with the gauge fields $A$ and $\varphi$, we borrow the recipe of [33] by the replacement,

$$
S_X^{(0)}(N, N^i, g_{ij}, \chi) \rightarrow S_X(N, N^i, g_{ij}, A, \varphi; \chi),
$$

(3.8)

where

$$
S_X = S_X, A(\chi, A) + S_X^{(0)}(N, N^i + N \nabla \varphi, g_{ij}, \chi),
$$

(3.9)

and

$$
S_{X, A} = \int dt d^3x \sqrt{g} \left[ c_1(\chi)\Delta_X + c_2(\chi)(\nabla \chi)^2 \nabla \nabla \chi \right] \\
\times (A - A),
$$

(3.10)

$$
A \equiv - \dot{\varphi} + N' \nabla \varphi + \frac{1}{2} N (\nabla \varphi)^2.
$$

(3.11)

Therefore, the total action of the scalar field in the HMT setup can be finally written in the form,

$$
S_X = \int dt d^3x \sqrt{g} L_X,
$$

(3.12)

with $L_X^{(0)}$ being given by Eq.(3.6). Variation of $S_X$ with respect to $\chi$ yields the generalized Klein-Gordon equation,

$$
\frac{f}{N \sqrt{g}} \left\{ \sqrt{g} \left[ \chi - (N^i + N \nabla \varphi) \nabla \chi \right] \right\} =
$$

(3.13)

$$
= \frac{f}{N^2} \nabla \chi \left\{ \left[ \chi - (N^i + N \nabla \varphi) \nabla \chi \right] (N^i + N \nabla \varphi) \right\}
$$

$$
+ \frac{g_{ij}}{N^2} \nabla \chi \left[ \left[ (A - A) c_1 - 2(A - A) c_2 \nabla \chi \right] \right]
$$

$$
+ \frac{A - A}{N} \left[ C_1(\Delta X + C_2(\nabla \chi)^2 \right]
$$

$$
+ \nabla \chi \left[ (1 + 2V_1 + 2V_2P_2) \nabla \chi \right]
$$

$$
- V_1 \chi - \Delta (V_1) - \Delta^2 (V_2),
$$

(3.14)

IV. STABILITY OF SCALAR FIELD

In this section, we consider the problem of stability of the scalar field in a flat FRW background,

$$
\dot{N} = a(\eta), \quad \dot{N}_i = 0, \quad \dot{g}_{ij} = a^2(\eta) \delta_{ij}.
$$

(4.1)
Since in this paper we are working with the conformal time \( \eta \), we use symbols with hats to denote the quantities of background, in order to distinguish from the ones used in the coordinates \((t, x^i)\) \([12, 14, 32]\), where \( t \) denotes the cosmic time. The relations between the two different coordinate systems are given explicitly in \([14, 32]\).

The flat FRW universe and its linear perturbations with any matter fields are presented in Appendix B. In this section, as well as in the next, we shall apply those formulas to the case where the only source is a scalar field, constructed in the last section.

In particular, without loss of generality, we assume that \( A = \hat{A}(\eta) \), \( \dot{\varphi} = \tilde{\varphi}(\eta) \). However, using the \( U(1) \) gauge freedom of Eq. \((4.2)\), we can always set one of them zero. In this paper, we choose the gauge

\[
\dot{\varphi} = 0. \tag{4.2}
\]

Thus, to zero-order, from Eq. \((A.17) - (A.22)\) we find that

\[
\begin{align*}
\dot{J}^t & = -2 \left( \frac{f}{2a^2} \dot{\chi}^2 + V(\chi) \right), \\
\dot{J}^i & = 0, \quad \dot{A} = 0, \\
\dot{\tau}_{ij} & = \left( \frac{f}{2a^2} \dot{\chi}^2 - V(\chi) \right) a^2 \delta_{ij},
\end{align*}
\tag{4.3}
\]

where a prime denotes the ordinary derivative with respect to its indicated argument, for example, \( \dot{\chi}' = d\chi/d\eta \), \( V' = dV/d\chi \), etc. Hence, the generalized Friedmann equation \((B.3)\) and the conservation law of energy \((B.7)\) become, respectively,

\[
\begin{align*}
\frac{\dot{H}^2}{a^2} & = \frac{8\pi G}{3} \left( \frac{1}{2a^2} \dot{\chi}^2 + V(\chi) \right), \\
\ddot{\chi}'' + 2\dot{H} \dot{\chi}' + a^2 \dot{V}'(\chi) & = 0,
\end{align*}
\tag{4.4}
\]

where

\[
\dot{G} \equiv \frac{2fG}{3\lambda - 1}, \quad \dot{V} \equiv \frac{V}{f}. \tag{4.6}
\]

It should be noted that in writing Eqs. \((4.4)\) and \((4.5)\), we had set \( \Lambda = 0 \), where \( \Lambda \equiv \gamma_0 c^2/2 \). From Eqs. \((B.3)\) and \((B.5)\), on the other hand, we also obtain \( \Lambda = 0 \). Note that Eq. \((4.5)\) can be obtained from the generalized Klein-Gordon equation \((B.13)\). In addition, the gauge field \( A \) is not determined. One may set \( A = 0 \). However, in this paper we shall leave this possibility open.

It is remarkable that Eqs. \((4.4)\) and \((4.5)\) are the same as those given in GR, after replacing \((G, \dot{V})\) by \((G, V)\). Thus, all the results obtained there are equally applicable to the present case, as far as only the background is concerned. These include the slow-roll conditions for inflation \([43]\),

\[
\epsilon_V, \ |\eta_V| \ll 1, \tag{4.7}
\]

where

\[
\epsilon_V \equiv \frac{1}{16\pi G} \frac{\dot{V}^2}{V^2}, \quad \eta_V \equiv \frac{1}{8\pi G} \frac{\dot{V}}{V}. \tag{4.8}
\]

However, because of the presence of high-order spatial derivatives here, the perturbations are expected to be dramatically different, as shown below.

The linear scalar perturbations are given by

\[
\begin{align*}
\delta N & = a\phi, \quad \delta N_I = a^2 B, \\
\delta g_{ij} & = -2a^2 (\psi\delta_{ij} - E_{ij}), \\
A & = \hat{A} + \delta A, \quad \varphi = \varphi + \delta \varphi. \tag{4.9}
\end{align*}
\]

Using the \( U(1) \) gauge freedom \((B.9)\), we can further set,

\[
\delta \varphi = 0. \tag{4.10}
\]

In addition, we also adopt the quasi-longitudinal gauge \([12]\).

\[
\phi = 0 = E. \tag{4.11}
\]

With the above gauge choice, it can be shown that the gauge freedom is completely fixed. Then, the linear perturbations for any matter fields are given in Appendix B. When the scalar field is the only source of the spacetime, to first-order, Eqs. \((A.17) - (A.22)\) yield

\[
\begin{align*}
\delta \mu & = \frac{f}{a^2} \dot{\chi}' \delta \chi' + \left( V' + \frac{V}{a^4} \dot{\chi}' \right) \delta \chi, \\
\delta J^i & = \frac{f}{a^3} \dot{\chi}' \delta \chi, \\
\delta J_A & = \frac{2}{a} c_1 \delta \chi, \\
\delta \tau_{ij} & = \left[ f \dot{\chi}' \left( \delta \chi' - \tilde{\varphi}' \right) - a^2 \left( V' \delta \chi - 2V \tilde{\varphi} \right) \right] \delta_{ij}, \\
\delta J_\varphi & = \frac{1}{a} \left( ac_1 \delta \chi \right)' - \frac{f}{a} \dot{\chi}' \delta \chi.
\end{align*}
\tag{4.12}
\]

Then, Eqs. \((B.14)\) and \((B.16) - (B.19)\) reduce, respectively, to

\[
\begin{align*}
(3\lambda - 1)\psi' + (\lambda - 1)\ddot{B} & = 8\pi f G \dot{\chi}' \delta \chi, \tag{4.13}
\end{align*}
\]

\[
2\dot{H} \ddot{\psi} - (\lambda - 1) \delta^2 (3\dot{\varphi}' + \ddot{\varphi})
= 8\pi G \left[ \left( c_1 \dot{\varphi}' + \mathcal{H} c_1 \right) \delta^2 \chi + c_1 \delta^2 \dot{\chi} \right], \tag{4.14}
\]

\[
\psi = 4\pi G c_1 \delta \chi, \tag{4.15}
\]

\[
\psi' + 2\dot{H} \psi' + \frac{\lambda - 1}{3\lambda - 1} \delta^2 (B' + 2\dot{H} B)
= \frac{8\pi G}{3\lambda - 1} \left( f \dot{\chi}' \delta \chi' - a^2 V' \delta \chi \right), \tag{4.16}
\]

\[
B' + 2\dot{H} B = \left( \frac{a - \hat{A}}{a} + \frac{8\gamma_2 + 3\gamma_3}{\zeta^2 a^2} a^2 \right) \psi \delta \chi \tag{4.17}
\]

It can be shown that Eq. \((4.11)\) is not independent, and can be derived from the rest.
On the other hand, the Hamiltonian constraint Eqs. (B.13) and the conservation law of energy (B.21) become,
\[
\int dx^3 \left[ \partial^2 \psi - \frac{1}{2} (3\lambda - 1) \mathcal{H} (3\psi' + \partial^2 B) \right] = 4\pi G \int dx^3 \left[ f \capl{\chi} + a^2 V' \capl{\chi} + \frac{V}{a^2} \partial^4 \capl{\chi} \right], \tag{4.18}
\]
\[
\int dx^3 \left\{ f a^2 \left( \capl{\chi} - \capl{\chi} \capl{\psi} - 2WH \capl{\psi} - 3\capl{\psi}^2 \right) + (V' + 3\mathcal{H} V) \partial^2 \capl{\chi} - (ac_1 \dot{A} - V_4 \partial^2 \capl{\chi} \right) + \left[ a^4 \capl{\chi} V' - a (\capl{\chi} c_1 + 3 \mathcal{H} c_1) \dot{A} \partial^2 \capl{\chi} \right] \right\} = 0. \tag{4.19}
\]
The conservation law of momentum (B.22) is satisfied identically, and the Klein-Gordon equation (3.13) takes the form,
\[
f \left[ \capl{\chi} + 2\mathcal{H} \capl{\chi} - \capl{\chi} (3\psi' + \partial^2 B) \right] + a^2 V'' \capl{\chi} = \frac{c_1}{a} \partial^2 \partial A + \frac{1}{a} \left[ a (1 + 2V_1) + 2 \dot{A} (c_1 - c_2) \right] \partial^2 \capl{\chi}
\]
\[
- \frac{2}{a^2} (V_2 + V_4) \partial^4 \capl{\chi} + \frac{2 \sigma_2 a^4}{a^2} \partial^6 \capl{\chi}. \tag{4.20}
\]
To further study it, it is found convenient to consider the cases \( \lambda = 1 \) and \( \lambda \neq 1 \) separately.

### A. \( \lambda \neq 1 \)

When \( \lambda \neq 1 \), substituting Eqs. (1.13), (1.15) and (1.17) into Eq. (4.20), we find that in the moment space it can be cast in the form,
\[
u_k'' + \left( \omega_k^2 - \frac{a''}{a} \right) \nu_k = 0, \tag{4.21}
\]
where \( \delta \chi = \nu/a \), and
\[
\omega_k^2 = \frac{a''}{a} + \frac{4a^2 a_3 + 2a_3 a'_3 - a^2 - 2a_3 a'_1}{4a^3}, \quad a_1 \equiv 2 \mathcal{H} + 8\pi G \frac{3\lambda - 1}{f (\lambda - 1)} c_1 \left( \chi c_1 + c_1 \mathcal{H} \right),
\]
\[
a_2 \equiv \frac{2 \sigma_2^2 k^6}{fa^4} + \frac{2 k^4}{fa^2} \left[ V_2 + V_4 + 2 \pi Gc_1^2 \frac{8\gamma_2 + 3\gamma_3}{\xi^2} \right] + \frac{k^2}{fa} \left[ a (1 + 2V_1) - \dot{A} (c'_1 - c_2) - 4 \pi Gc_1^2 (a - \dot{A}) \right] - \frac{8 \pi G}{\lambda - 1} \left[ c_1 \chi'' + \left( f - c_1' \right) \chi' \right] - \frac{3 \lambda - 1}{2f} c_1 c_1' \chi'
\]
\[
+ 2c_1 \mathcal{H} \chi' + e^{\frac{a^2 V''}{f}}, \quad a_3 \equiv 1 + 4 \pi Gc_1^2 \frac{3\lambda - 1}{f (\lambda - 1)} \tag{4.22}
\]

It is remarkable that, in contrast to other version of the HL theory [29], now a master equation exists. In the UV \( (k \gg \mathcal{H}) \), we have \( \omega_k^2 \sim 2 \sigma_2^2 k^6 / (fa^4) > 0 \), and the scalar field is indeed stabilized. In the IR \( (k \ll \mathcal{H}) \), the last term in the expression of \( a_2 \) dominates. Since \( V'' = 2m^2_\chi > 0 \), where \( m_\chi \) denotes the mass of the scalar field, one can see that it is stable also in the IR.

In particular, in the extreme slow-roll (de Sitter) limit, we take \( \chi' \approx V'(\chi) \approx 0 \) and \( \alpha \approx -1 / (\mathcal{H} \eta) \), where \( \mathcal{H} = [8\pi G \mathcal{V}(\chi_0) / 3]^{1/2} \). Then, Eq. (4.22) reduces to,
\[
\omega_k^2 = \frac{1}{a_3} \left\{ \frac{2\sigma_2^2 k^6}{fa^4} + \frac{2 k^4}{fa^2} \left[ V_2 + V_4 + 2 \pi Gc_1^2 \frac{8\gamma_2 + 3\gamma_3}{\xi^2} \right] + \frac{k^2}{fa} \left[ a (1 + 2V_1) - \dot{A} (c'_1 - c_2) - 4 \pi Gc_1^2 (a - \dot{A}) \right] + \frac{a^2 V''}{f} \right\}, \quad (\chi' = 0), \tag{4.23}
\]

which clearly shows that the scalar field is stable in all the energy scalar, by properly choosing the potential terms \( V_i \).

### B. \( \lambda = 1 \)

In this case, if \( c_1 = 0 \), Eqs. (1.13) and (1.15) imply \( \chi' = 0 = \psi \), that is, to zero-order the scalar field must take its vacuum expectation value, \( \chi = \chi_0 \), where \( V'(\chi_0) = 0 \), and the corresponding background is necessarily de Sitter. Then, Eqs. (4.17) reads,
\[
\delta A = \frac{1}{a} (a^2 B)' \tag{4.24}
\]

The Klein-Gordon equation (4.20) can be also written in the form of Eq. (4.21), but now with
\[
\omega_k^2 = \frac{V''}{f H^2 \eta^2} + \frac{1}{f} \left[ (1 + 2V_1) + 2c_2 H \dot{A} \eta \right] k^2
\]
\[
+ \frac{2 H^2 \eta^2}{f} \left[ (V_2 + V_4) + \sigma_2^2 H^2 \eta^2 k^2 \right] k^4. \tag{4.25}
\]

Again, since \( V'' = 2m^2_\chi > 0 \), the above show that the scalar field is stable in the IR, by properly choosing the potential term \( V_1 \). In the UV, the \( \sigma_2^2 \) dominates, and is strictly positive, so it is also stable in this regime. In fact, by properly choosing the potentials \( V_2 \) and \( V_4 \), it can be made stable in all the energy scales.

It is interesting to note that in this case, similar to the background, the gauge field \( A \) is not determined by the field equations.

When \( c_1 \neq 0 \), one can find explicit solutions for \( \delta \chi \), \( \psi \), \( B \) and \( \delta A \), and are given in Appendix C, which seem not physically much interesting. So, we shall not consider them further.
V. CONCLUSIONS

In this paper, we have studied the stability of a scalar field in the case where the detailed balance condition is softly broken. This is done in the da Silva generalization \[7\] of the HMT setup \[33\].

In Sec. III, we have first constructed the most general action of a scalar field with softly breaking the detailed balance condition, while in Sec. IV, we have studied its stability and shown explicitly that the scalar field becomes stable in all the energy scales, including the UV and IR. It is remarkable to note that, unlike other versions of the HL theory \[39\], there exists a master equation for the linear perturbations of a scalar field in the flat FRW universe in this setup.

It should be also noted that our conclusions do not contradict to the ones obtained earlier by Calcagni \[26, 42\], who showed that a scalar field is not stable in the UV when the detailed balance condition is imposed. This is mainly due to two facts: First, we have chosen a different sign in the front of the super-potential of the scalar field [cf. \eqref{eq:3.1}]. This helps to improve the UV stability of the scalar field, but usually leads to undesired behavior in the IR \[12\]. To fix the latter, we allow the detailed balance condition to be softly broken, so that the behavior of the scalar field in the IR is healthy. Such a breaking is also desired by other considerations \[24, 35, 36\], including the solar system tests. In addition, it still provides a very effective mechanism to reduce significantly the number of independent coupling constants of the marginal terms.

For example, in the gravitational sector, this condition reduces the number from five \((g_1, g_5, ..., g_8)\) to one, denoted by \(\gamma_5\) in Eq.\,(\ref{eq:2.9}). Moreover, it may also shed light on the nature of the gravitational forces \[10, 28, 50\].

In addition, our conclusions regarding to the stability of a scalar field can be easily generalized to other versions of the HL theory \[3, 8, 10\], including the SVW setup \[11\].

Finally, we note that there does not exist the ghost problem for the scalar field \(\chi\), provided that \(f(\lambda) > 0\). This can be seen from Eqs.\,(\ref{eq:3.6}), \,(\ref{eq:3.11}) and \,(\ref{eq:3.12}), where the kinetic part (in the gauge \(\varphi = 0\)) reads

\[
L^{(K)} = \frac{f}{2N^2} \left( \dot{\chi} - N^i \nabla_i \chi \right)^2 \approx \frac{f}{2a^2} \left( \dot{\chi}^2 + \delta \chi^2 \right)^2,
\]

to the first-order approximations. Clearly, it is always positive for \(f > 0\).

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Appendix A: The field Equations

Variation of the total action \(\ref{eq:2.3}\) with respect to \(N(t)\) yields the Hamiltonian constraint,

\[
\int d^4x \sqrt{g} \left[ L_K + L_V - \varphi G^{ij} \nabla_i \nabla_j \varphi - (1 - \lambda) (\Delta \varphi)^2 \right] = 8\pi G \int d^4x \sqrt{g} J^i,
\]

\(\text{(A.1)}\)

where

\[
J^i \equiv 2 \frac{\delta (NL_M)}{\delta N}.
\]

\(\text{(A.2)}\)

Variation with respect to \(N^i\) yields the super-momentum constraint,

\[
\nabla^j \left[ \pi_{ij} - \varphi G_{ij} - (1 - \lambda) g_{ij} \Delta \varphi \right] = 8\pi G J_i,
\]

\(\text{(A.3)}\)

where the super-momentum \(\pi_{ij}\) and matter current \(J_i\) are defined as

\[
\pi_{ij} \equiv - K_{ij} + \lambda K g_{ij},
\]

\(\text{(A.4)}\)

\[
J_i \equiv - N \frac{\delta L_M}{\delta N^i}.
\]

\(\text{(A.4)}\)

Similarly, variations of the action with respect to \(\varphi\) and \(A\) yield,

\[
\mathcal{G}^{ij} \left( K_{ij} + \nabla_i \nabla_j \varphi \right) + (1 - \lambda) \Delta \left( K + \Delta \varphi \right) = 8\pi G J_\varphi,
\]

\(\text{(A.5)}\)

\[
R - 2\Lambda_g = 8\pi G J_A,
\]

\(\text{(A.6)}\)

where

\[
J_\varphi \equiv - \frac{\delta L_M}{\delta \varphi}, \quad J_A \equiv 2 \frac{\delta (NL_M)}{\delta A}.
\]

\(\text{(A.7)}\)

On the other hand, the dynamical equations now read,

\[
\frac{1}{N\sqrt{g}} \left\{ \sqrt{g} \left[ \pi_{ij} - \varphi G^{ij} - (1 - \lambda) g_{ij} \Delta \varphi \right] \right\}_{,t} = - \frac{1}{4} \left( K^2 \right)^{ij} + 2\lambda K K^{ij} + \frac{1}{N} \nabla_k \left[ N^k \pi_{ij} - 2\pi^{k(i} N^{j)} \right] - 2(1 - \lambda) \left[ (K + \Delta \varphi) \nabla^i \nabla^j \varphi + K^{ij} \Delta \varphi \right] + (1 - \lambda) \left[ 2 \nabla^i (F^j_\varphi) - g_{ij} \nabla_k F^k_\varphi \right] + \frac{1}{2} \left( L_K + L_\varphi + L_A + L_\lambda \right) g^{ij} + F^{ij} + F^2_\varphi + F^{ij}_A + 8\pi G \tau^{ij},
\]

\(\text{(A.8)}\)

where \((K^2)^{ij} \equiv K^{i} K^{j}_{i}, f_{(ij)} \equiv (f_{ij} + f_{ji})/2, \) and

\[
\tau^{ij} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ij}} \left( \sqrt{g} L_M \right).
\]

\(\text{(A.9)}\)
The quantities $F^i_j$, $F^i_{ij}$, $F^i_{ij}$ and $F^i_{A}$ are defined as,

$$F^i_j = \frac{1}{\sqrt{g}} \frac{\delta (-\sqrt{g}\mathcal{L}_\psi)}{\delta g_{ij}} = \sum_{\nu=0}^{5} \gamma_{\nu} \zeta^{\nu} (F_\nu)^{ij},$$

$$F^i_i = (K + \Delta \varphi) \nabla^i \varphi + \frac{N^i}{N} \Delta \varphi,$$

$$F^i_{ij} = \sum_{n=1}^{3} F_{ij}^{\varphi, n},$$

$$F^A_{ij} = \frac{1}{N} \left[ A R^{ij} - \left( \nabla^i \nabla^j - g^{ij} \Delta \right) A \right], \quad (A.10)$$

with $n = (2, 0, -2, -2, -2, -4)$ and

$$(F_0)_{ij} = (F_0)_{ij}, \quad (F_1)_{ij} = (F_1)_{ij},$$

$$(F_2)_{ij} = (F_2)_{ij}, \quad (F_3)_{ij} = (F_3)_{ij},$$

$$(F_4)_{ij} = -\kappa^{jmn} \left\{ \nabla_m \nabla_n \nabla R_{jk} + g_{mn} \nabla R_{jk} - R_{m} \nabla_n R_{jk} \right\}, \quad (A.11)$$

$$(F_5)_{ij} = \frac{1}{2} (F_4)_{ij} - \frac{5}{2} (F_5)_{ij} + 3 (F_6)_{ij} - \frac{3}{8} (F_7)_{ij} + (F_8)_{ij}, \quad (A.12)$$

where $(F_\nu)_{ij}$ are given in (12).

The $F^i_{ij}$'s are given by

$$F^i_{ij} = \frac{1}{2} \varphi \left\{ (2K + \Delta \varphi) R^{ij} - 2 \left( 2K_k + \nabla^j \nabla_k \varphi \right) R^{ik} - 2 \left( 2K_k + \nabla^i \nabla_k \varphi \right) R^{jk} - 2 \left( 2\Delta \varphi - R \right) \left( 2K^i + \nabla^i \varphi \right) \right\},$$

$$F^i_{ij} = \frac{1}{2} \varphi \left\{ \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^j \varphi \right) + \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi G^{jk} \left( \frac{2N^k}{N} + \nabla^k \varphi \right) \right\},$$

$$F^i_{ij} = \frac{1}{2} \left\{ 2 \nabla_k \nabla_i f^{kl}_{\varphi} k - \Delta f^{ij}_{\varphi} - (\nabla_k \nabla_i f^{kl}_{\varphi}) g^{ij} \right\},$$

$$F^i_{(\varphi, 1)} = \frac{1}{2} \varphi \left\{ \left( 2K + \Delta \varphi \right) R^{ij} - 2 \left( 2K_k + \nabla^j \nabla_k \varphi \right) R^{ik} - 2 \left( 2K_k + \nabla^i \nabla_k \varphi \right) R^{jk} - \left( 2\Delta \varphi - R \right) \left( 2K^i + \nabla^i \varphi \right) \right\},$$

$$F^i_{(\varphi, 2)} = \frac{1}{2} \varphi \left\{ \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^j \varphi \right) + \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi G^{jk} \left( \frac{2N^k}{N} + \nabla^k \varphi \right) \right\},$$

$$F^i_{(\varphi, 3)} = \frac{1}{2} \left\{ 2 \nabla_k \nabla_i f^{kl}_{\varphi} k - \Delta f^{ij}_{\varphi} - (\nabla_k \nabla_i f^{kl}_{\varphi}) g^{ij} \right\},$$

$$F^i_{(\varphi, 2)} = \frac{1}{2} \varphi \left\{ \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^j \varphi \right) + \varphi G^{jk} \left( \frac{2N^i}{N} + \nabla^i \varphi \right) - \varphi G^{jk} \left( \frac{2N^k}{N} + \nabla^k \varphi \right) \right\},$$

The matter, on the other hand, satisfies the conservation laws,

$$\int d^3 \mathcal{Q} \left[ \frac{\delta}{\delta J_t} - \frac{\varphi G^{jk}}{\sqrt{g}} \left( \frac{2N^i}{N} + \nabla^j \varphi \right) \right]_{,t} = 0, \quad (A.15)$$

$$\nabla^k \tau_{jk} - \frac{1}{N} \frac{\varphi G^{jk}}{\sqrt{g}} \nabla_j \varphi \nabla_i \varphi - \frac{J_{A}}{2N} \nabla_i \varphi = 0. \quad (A.16)$$

When the scalar field defined by Eqs. (3.11) and (3.12) is the only source, we find that

$$J^t = -2 \left( \frac{f}{2N^2} \left( \chi - N^k \nabla_k \chi \right)^2 + \mathcal{V} \right) - \left[ c_1 \Delta \chi + c_2 (\nabla \chi)^2 \right] (\nabla \varphi)^2 + \left[ (\nabla^k \varphi) \left( \nabla_k \chi \right) \right]^2, \quad (A.17)$$

$$J^i = \frac{f}{N} \left[ \chi - \left( N^k + N \nabla^k \varphi \right) \left( \nabla_k \chi \right) \right] \nabla_i \chi + \left[ c_1 \Delta \chi + c_2 (\nabla \chi)^2 \right] \nabla^i \varphi, \quad (A.18)$$

$$J_{\varphi} = \frac{1}{N} \varphi \left\{ \sqrt{g} \left[ c_1 \Delta \chi + c_2 (\nabla \chi)^2 \right] \right\}_{,t}$$
where \( \eta \) perturbations. To have our results as much applicable as possible, we find that \( \hat{\rho} = \hat{\rho}_i = 0 \). Then, we find

\[
\hat{K}_{ij} = -a \hat{H} \delta_{ij}, \quad \hat{R}_{ij} = 0, \quad \hat{F}^i_j = 0, \quad \hat{F}^i_A = \hat{F}^j_A = 0, \quad \hat{F}^i_\nu = 0,
\]

where \( \mathcal{H} = a'/a \), \( \Lambda = \zeta^2 \gamma_0/2 \), and \( a' \equiv da/d\eta \). Hence,

\[
\hat{\mathcal{L}}_K = 3(1 - 3\lambda) \frac{\hat{H}^2}{a^2}, \quad \hat{\mathcal{L}}_\phi = 0 = \hat{\mathcal{L}}_\lambda,
\]

\[
\hat{\mathcal{L}}_A = 2 \Lambda a \hat{A}, \quad \hat{\mathcal{L}}_V = 2 \Lambda. \quad (B.2)
\]

It can be shown that the super-momentum constraint (A.3) is satisfied identically for \( J^i = 0 \), while the Hamiltonian constraint (A.1) yields,

\[
\frac{1}{2} (3\lambda - 1) \frac{\hat{H}^2}{a^2} = 8\pi G \frac{\hat{\rho}}{3} + \Lambda \quad \text{(B.3)}
\]

where \( \hat{J}^i \equiv -\hat{\rho} \). On the other hand, Eqs. (A.5) and (A.6) give, respectively,

\[
\Lambda_5 \mathcal{H} = -\frac{8\pi Ga}{3} \hat{J}_\phi, \quad (B.4)
\]

\[
\Lambda_5 = -4\pi G J_A, \quad (B.5)
\]

while the dynamical equation (A.8) reduces to

\[
\frac{1}{2} (3\lambda - 1) \left( \frac{a''}{a} - \frac{\hat{H}^2}{a^2} \right) = -\frac{4\pi G}{3} (\hat{\rho} + 3\hat{\dot{\rho}}) + \frac{1}{3} \Lambda - \frac{1}{2a} \Lambda_5 \hat{A}, \quad (B.6)
\]

where \( \hat{\rho}_{ij} = \hat{\rho} \hat{g}_{ij} \). The conservation law of momentum (A.10) is satisfied identically, while the one of energy (A.15) reads,

\[
\hat{\rho}' + 3 \mathcal{H} (\hat{\rho} + \hat{\dot{\rho}}) = \hat{A} \hat{J}_\phi, \quad (B.7)
\]

which can be also obtained from Eqs. (B.3) and (B.6).

**B. Linear perturbations**

The linear scalar perturbations are given by Eq. (4.11). Under the gauge transformations (1.4), the perturbations transform as

\[
\delta \phi = \phi - \mathcal{H} \xi^0 - \xi^0, \quad \delta \psi = \psi + \mathcal{H} \xi^0, \quad \delta \bar{B} = B + \xi^0 - \xi', \quad \delta \bar{E} = E - \xi, \quad \delta \bar{\varphi} = \delta \varphi - \xi^0 \varphi', \quad \delta \bar{A} = \delta A - \xi^0 \bar{A}' - \xi^0 \bar{A}, \quad (B.8)
\]

where \( f = -\xi^0, \xi^i = -\xi^i, \mathcal{H} \equiv a'/a \), and \( \bar{A} \) denotes the ordinary derivative with respect to \( \eta \). Under the \( U(1) \) gauge transformations, on the other hand, we find that

\[
\tilde{\delta} \phi = \tilde{\delta} \phi, \quad \tilde{\hat{\psi}} = \tilde{\hat{E}} = \tilde{E}, \quad \tilde{\hat{\psi}} = \tilde{\hat{\psi}} = \tilde{\xi}, \quad \tilde{\hat{B}} = \tilde{\hat{B}} = B - \xi, \quad \tilde{\delta} \varphi = \delta \varphi + \epsilon, \quad \tilde{\delta} A = \delta A - \epsilon, \quad (B.9)
\]

where \( \epsilon = -\alpha \). Then, the gauge transformations of the whole group \( U(1) \ltimes \text{Diff}(M, \mathcal{F}) \) will be the linear combination of the above two. Out of the six unknown, one
can construct three gauge-invariant quantities,
\[ \Phi = \phi - \frac{1}{a} \frac{1}{a} \frac{1}{\varphi'} (a \sigma - \delta \varphi)' \]
\[ - \frac{1}{(a - \varphi')^2} (\varphi' - H \varphi') (a \sigma - \delta \varphi), \]
\[ \Psi = \psi + \frac{\delta A}{a - \varphi'} (a \sigma - \delta \varphi), \]
\[ \Gamma = \delta A + \left[ \frac{a \delta \varphi - \varphi' \sigma}{a - \varphi'} \right]' \equiv E, \] \( (B.10) \)
where \( \sigma \equiv E' - B. \) For the background, we have chosen the gauge (4.2), for which Eq. (B.10) reduces to
\[ \Phi = \phi - \frac{1}{a} \frac{1}{a} (a \sigma - \delta \varphi)', \]
\[ \Psi = \psi - \frac{\delta A}{a} (a \sigma - \delta \varphi), \]
\[ \Gamma = \delta A + \left[ \frac{\delta \varphi - \frac{\delta}{a} (a \sigma - \delta \varphi)}{a - \varphi'} \right]' \equiv E, \] \( (B.11) \)
Then, with the gauge choice,
\[ \delta \varphi = \phi = E = 0, \] \( (B.12) \)
to first-order it can be shown that the Hamiltonian and momentum constraints become, respectively,
\[ \int d^3 x \left\{ \delta^2 \psi - \frac{1}{2} (3 \lambda - 1) H (3 \psi' + \delta^2 B) \right. \]
\[ - 4 \pi G a^2 \delta \mu \left. \right\} = 0, \] \( (B.13) \)
\[ (3 \lambda - 1) \psi' + (\lambda - 1) \delta^2 B = 8 \pi G a q, \] \( (B.14) \)
where
\[ \delta \mu \equiv - \frac{1}{2} \delta J^i, \quad \delta J^i \equiv \frac{1}{a} q^i. \] \( (B.15) \)
On the other hand, the linearized equations (A.5) and (A.6) reduce, respectively, to
\[ 2 H \delta^2 \psi + \left[ \Lambda \rho a^2 + (1 - \lambda) \delta^2 \right] (3 \psi' + \delta^2 B) \]
\[ = 8 \pi G a^3 \delta J_x, \] \( (B.16) \)
\[ \delta^2 \psi = 2 \pi G a^2 \delta J_A. \] \( (B.17) \)
The linearized dynamical equations can be divided into the trace and traceless parts. The trace part reads,
\[ \psi'' + 2 H \psi' + \frac{1}{3} \delta^2 \left( B' + 2 H B \right) = \frac{1}{3 \lambda - 1} \left\{ 8 \pi G a^2 \delta P \right. \]
\[ + \frac{2}{3} \left[ \frac{a - \dot{A}}{a} + \frac{8 \gamma_2 + 3 \gamma_3}{\zeta^2 a^2} \right] \delta^2 \psi \]
\[ + \frac{1}{a} \left( \Lambda \rho a^2 + \frac{2}{3} \delta^2 \right) \delta A \right\}, \] \( (B.18) \)
while the traceless part is given by
\[ B' + 2 H B = \frac{a - \dot{A}}{a} + \frac{8 \gamma_2 + 3 \gamma_3}{\zeta^2 a^2} \delta^2 \psi \]
\[ - \frac{1}{a} \delta A = - 8 \pi G a^2 \Pi, \] \( (B.19) \)
where
\[ \delta \tau^{ij} = \frac{1}{a^2} \left( \delta P + 2 \dot{\rho} \psi \right) \delta^{ij} + \Pi_{<ij>}, \]
\[ \Pi_{<ij>} = \Pi^{ij} - \frac{1}{3} \delta^{ij} \delta^2 \Pi. \] \( (B.20) \)
The conservation laws (A.15) and (A.16) to first order are given by,
\[ \int d^3 x \left\{ 3 H \delta \mu + 3 \dot{\rho} \psi + 3 H \delta P \right. \]
\[ + \frac{1}{2 a} \left\{ 3 \dot{A} \dot{J}_A \psi' - \dot{A} (\delta J_A + 3 H \delta J_A) \right. \]
\[ - 2 a \dot{J}_A \delta A \right\} = 0, \] \( (B.21) \)
\[ q' + 3 H q + \frac{1}{2 a} \dot{J}_A \delta A = a \left( \delta P + \frac{2}{3} \delta^2 \Pi \right), \] \( (B.22) \)
where \( \dot{J}_A \) is given by Eq. (B.22).

This completes the general description of linear scalar perturbations in the flat FRW background in the framework of the HMT setup with detailed balance condition softly breaking and any given \( \lambda \), generalized recently by da Silva [33].

**Appendix C: Linear Perturbations for \( \lambda = 1 \) and \( c_1 \neq 0 \)**

When \( \lambda = 1 \) and \( c_1 \neq 0 \), Eqs. (4.13) and (4.15) yields,
\[ c_1 \delta \chi' + (c'_1 - f \chi') \delta \chi = 0, \] \( (C.1) \)
which has the solution,
\[ \delta \chi = \frac{1}{c_1} \exp \left[ \delta \chi_0 (\eta) + \delta \chi_1 (x) \right], \] \( (C.2) \)
where \( \delta \chi_0 (\eta) = \int f (\chi' / c_1) d \eta \), and \( \delta \chi_1 (x) \) is an arbitrary function of \( x^i \) only. Inserting the above solution into Eqs. (4.14), (4.16) and (4.17), one finds,
\[ \psi = 4 \pi G \exp \left[ \delta \chi_0 (\eta) + \delta \chi_1 (x) \right], \]
\[ B_k = \frac{e^{-\delta \chi_0}}{a^2} \left\{ \int Q e^{\delta \chi_0} d \eta + R (x^i) \right\}, \]
\[ \delta A_k = - \frac{a}{c_1} (f \chi' B_k + P \delta \chi), \] \( (C.3) \)
where $R(x)$ is another arbitrary function of $x^i$ only, and

$$P \equiv \frac{f}{k^2} \left[ \frac{\dot{\chi}^2}{c_1^2} + (2c_1^2 + f^2 - 3f c_1' - c_1') \right]$$

$$+ \frac{\dot{\chi}'' + 2\dot{\chi}' (f - c_1') - 12 \pi G f \dot{\chi}^2}{c_1}$$

$$+ \frac{1}{a} \left[ \alpha (1 + 2V_1) + 2\dot{A} (c_1' - c_2) \right]$$

$$+ \frac{a^2}{k^2} V'' + \frac{2k^2}{a^2} \left( V_2 + V_2' \right) + \frac{2\sigma^2 k^4}{a^4}.$$