NEW PÓLYA-SZEGÖ-TYPE INEQUALITIES
AND AN ALTERNATIVE APPROACH
TO COMPARISON RESULTS FOR PDE’S

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Abstract. We prove some Pólya-Szegö type inequalities which involve couples of functions and their rearrangements. Our inequalities reduce to the classical Pólya-Szegö principle when the two functions coincide. As an application, we give a different proof of a comparison result for solutions to Dirichlet boundary value problems for Laplacian equations proved in [1].

Key words: Pólya-Szegö principle, Steiner symmetrization, elliptic equations, comparison results

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1. Introduction

The celebrated Pólya-Szegö Principle asserts that Dirichlet type integrals do not increase under Schwarz symmetrization. In its simplest form it states that, if $u$ is a compactly supported function which belongs to $W^{1,2}(\mathbb{R}^n)$ then also its spherically symmetric rearrangement $u^*$ is in $W^{1,2}(\mathbb{R}^n)$ and

\[ \int_{\mathbb{R}^N} |\nabla u(z)|^2 \, dz \geq \int_{\mathbb{R}^N} |\nabla u^*(z)|^2 \, dz. \]

The interest in the Pólya-Szegö principle is due to its multitude of applications in analysis and physics. For instance, it is the main tool in proving isoperimetric inequalities for capacities and of Faber-Krahn type, as well as apriori estimates for solutions to boundary value problems for PDEs (see e.g. [6], [27], [28], [29], [33] and references therein). The topic has attracted the attention of many authors, and it has been developed in various directions since the middle of last century. For instance, more general functionals of the gradient under different types of symmetrizations have been investigated (see, for example, [4], [9], [11], [13], [20], [22], [23], [32] and references therein). More recently, the equality case and the stability in these inequalities have been studied (see, for example, [12], [26]).
In this paper we prove a Pólya-Szegö type inequality which - unlike the classical case (1.1) - involves two functions $u, w$ and their rearrangements. Our inequality reduces to (1.1) when $u = w$. We focus on the Steiner symmetrization, and we will analyze the differences which appear when Steiner symmetrization is replaced by Schwarz symmetrization.

The proofs of Pólya-Szegö type inequalities are typically based on the isoperimetric inequality in Euclidean space, while our approach relies on two further well-known tools from the theory of rearrangements: the Hardy-Littlewood inequality and the Riesz inequality. The Hardy-Littlewood states that

$$
\int_{\mathbb{R}^N} u(z) w(z) dz \leq \int_{\mathbb{R}^N} u^\#(z) w^\#(z) dz ,
$$

for any couple of measurable nonnegative functions. Here $u^\#$ and $w^\#$ are the Steiner rearrangements of $u$ and $w$ respectively defined in Section 2.

The main result of the paper is

**Theorem 1.1.** Assume that $u$ and $w$ are Lipschitz-continuous nonnegative functions with compact support defined in $\mathbb{R}^N$ and

$$
\int_{\mathbb{R}^N} u(z) w(z) dz = \int_{\mathbb{R}^N} u^\#(z) w^\#(z) dz .
$$

Then the following inequalities hold

$$
\int_{\Omega} \nabla_x u(z) \cdot \nabla_x w(z) dz \geq \int_{\Omega^\#} \nabla_x u^\#(z) \cdot \nabla_x w^\#(z) dz ,
$$

and that, for each $i = 1, \ldots, m$

$$
\int_{\Omega} u_{y_i}(z) \cdot w_{y_i}(z) dz \geq \int_{\Omega^\#} u_{y_i}^\#(z) \cdot w_{y_i}^\#(z) dz
$$

and, hence

$$
\int_{\mathbb{R}^N} \nabla u(z) \cdot \nabla w(z) dz \geq \int_{\mathbb{R}^N} \nabla u^\#(z) \cdot \nabla w^\#(z) dz .
$$

Note that, if $u = w$, then equation (1.3) is in force, since symmetrization preserves the $L^2$ norm. As previously mentioned, in such case (1.6) reduces to the standard Pólya-Szegö inequality (1.1).

The proof of Theorem 1.1 is based on a discretization of the gradient and the Riesz Inequality.

Inequality (1.7) in our next Theorem is related to (1.6). The difference is that we allow $w$ to be not weakly differentiable, but instead we require more regularity for $u$. 

Theorem 1.2. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ and let $u \in C^2(\Omega) \cap C(\Omega)$ be a nonnegative function satisfying $u = 0$ on $\partial \Omega$. Further, let $W \in W^{1,\infty}_0(\Omega^\#)$ be a nonnegative function such that $W = W^\#$. Then, if $w$ is any function satisfying (1.3) with $w^\# = W$, we have that

$$-\int_{\Omega} w(x) \Delta u(x)dx \geq \int_{\Omega^\#} \nabla u^\#(x) \cdot \nabla W(x)dx.$$  

As an application of the previous two Theorems, we recover a comparison result proved in [1], (see also [5], [7], [15], [24], [25] and the references therein). More precisely, we consider the following linear homogeneous Dirichlet problem

$$\begin{align*}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$  

where $\Omega$ is a bounded domain of $\mathbb{R}^N$. We decompose every $z \in \mathbb{R}^N$ by $z = (x, y)$ with $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $n + m = N$. Accordingly, let $\Omega_y = \{x \in \mathbb{R}^n : (x, y) \in \Omega\}$ denote the $y$-section of $\Omega$. By $B_r(0)$ we denote the $n$-dimensional ball centered at the origin with radius $r$, and by $\Omega^\#$ the subset of $\mathbb{R}^N$ such that, for any $y \in \mathbb{R}^m$, its $y$-section $(\Omega^\#)_y = \{x \in \mathbb{R}^n : (x, y) \in \Omega^\#\}$ is the $n$-dimensional ball centered at zero which has the same $\mathcal{L}^n$-measure as $\Omega_y$. Then the following result holds.

Theorem 1.3. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ satisfying the exterior sphere condition and $f \in L^q(\Omega), q > \frac{N}{2}$. Further, let $u \in W^{1,2}_0(\Omega)$ be a weak solution to problem (1.8), and let $v \in W^{1,2}_0(\Omega^\#)$ be the weak solution to the symmetrized problem

$$\begin{align*}
-\Delta v &= f^\# \quad \text{in } \Omega^\#, \\
v &= 0 \quad \text{on } \partial \Omega^\#.
\end{align*}$$  

Then we have for all $r \in [0, \infty)$ and for a.e. $y \in \mathbb{R}^m$

$$\int_{B_r(0)} u^\#(x, y)dx \leq \int_{B_r(0)} v^\#(x, y)dx.$$  

Note that all the previous results hold also for Schwarz symmetrization, with appropriate modifications. In such a case the absence of the $y$-variables allows to recover the well-known pointwise comparison result which is due to Talenti (see [31]). The case of Schwarz symmetrization will be treated in Section 4 where also nonlinear problems are considered.
2. Notation and preliminary results

In this section we introduce some notations, and we recall some well-known results which will be used in the sequel.

Let $\mathbb{R}^N$, $N \geq 1$, be the Euclidean space and let $E$ be a measurable subset of $\mathbb{R}^N$. The $N$-dimensional Lebesgue measure of the set $E$ is denoted by $\mathcal{L}^N(E)$, while for any $d \geq 0$, $\mathcal{H}^d(E)$ denotes its $d$-dimensional Hausdorff measure. The notation $| \cdot |$ denotes the standard Euclidean norm, independently from the dimension of the space.

Let $\Omega$ be an open subset of $\mathbb{R}^N$, $N \geq 1$, and let $u$ be a nonnegative measurable function on $\Omega$. Its distribution function is given by

$$
\mu_u(t) = \mathcal{L}^N\left(\{x \in \Omega : u(x) > t\}\right), \quad t \in [0, +\infty),
$$

and its decreasing rearrangement is defined as

$$
u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}, \quad s \in (0, \mathcal{L}^N(\Omega)].
$$

We denote by $\Omega^*$ the ball of $\mathbb{R}^n$ centered at the origin and having the same $\mathcal{L}^n$-measure as $\Omega$. The Schwarz rearrangement of $u$, is given by

$$u^*(x) = u^*(\omega_n|x|^n) \quad x \in \Omega^*,
$$

where $\omega_n$ is the measure of the $n$-dimensional unit ball.

If $N \geq 2$, let $n, m \in \mathbb{N}$ be such that $n + m = N$, and decompose every $z \in \mathbb{R}^N$ by $z = (x, y)$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Accordingly, the gradient $\nabla u$ of a function $u$ is the pair $(\nabla_x u, \nabla_y u)$, where $\nabla_x u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$ and $\nabla_y u = \left(\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_m}\right)$.

For any $y \in \mathbb{R}^m$, let $\Omega_y$ be the $y$-section of $\Omega$ which is defined by

$$\Omega_y := \{x \in \mathbb{R}^n : (x, y) \in \Omega\}, \quad y \in \mathbb{R}^m.
$$

The distribution function (in codimension $n$) of $u$ and its decreasing rearrangement (in codimension $n$) are defined as

$$
\mu_u(t, y) = \mathcal{L}^n\left(\{x \in \Omega_y : u(x, y) > t\}\right), \quad (t, y) \in [0, +\infty) \times \mathbb{R}^m,
$$

and

$$u^*(s, y) = \sup\{t \geq 0 : \mu_u(t, y) > s\}, \quad (s, y) \in (0, \mathcal{L}^n(\Omega_y)] \times \mathbb{R}^m,
$$

respectively. By $\Omega^y$ we denote the open set in $\mathbb{R}^N$ such that, for any $y \in \mathbb{R}^m$, its $y$-section $(\Omega^y)_y$ is the $n$-dimensional ball centered at the origin and having the same $\mathcal{L}^n$-measure as $\Omega_y$. The Steiner symmetrization (in codimension $n$) of $u$, is given by

$$
(2.1) \quad u^\#(x, y) = (u(\cdot, y))^\#(x) = u^*(\omega_n|x|^n, y) \quad (x, y) \in \Omega^y.
$$

It is well known that if $u \in W^{1,p}_0(\Omega)$, for some $1 \leq p \leq \infty$, then also $u^\# \in W^{1,p}_0(\Omega^\#)$, and the $L^p$ norm is preserved while the $W^{1,p}$ norm is reduced (see for example [4, 8, 9, 14] and the references therein).
The Hardy-Littlewood inequality with respect to the Schwarz rearrangement states that if $u$ and $w$ are nonnegative measurable functions on a bounded open set $\Omega$ of $\mathbb{R}^n$, $n \geq 1$, then

\begin{equation}
\int_{\Omega} u(z)w(z)dz \leq \int_{\Omega^*} u^*(z)w^*(z)dz.
\end{equation}

Furthermore, the Riesz inequality states that

\begin{equation}
\int_{\mathbb{R}^2} u(x)w(z)h(x-z)dx dz \leq \int_{\mathbb{R}^2} u^*(x)w^*(z)h^*(x-z)dx dz.
\end{equation}

for any triple $u, w, h$ of nonnegative measurable functions on $\mathbb{R}^N$ for which the right-hand side is finite.

In the following we are interested in the situation where equality in (2.2) is achieved. Let $u$ and $W = W^*$ be two given nonnegative measurable functions, defined in $\Omega$ and $\Omega^*$, respectively. We will say that a function $w$, satisfying $w^* = W$, is an extremal for (2.2), if it produces equality in (2.2), that is

\begin{equation}
\int_{\Omega} u(x)w(x)dx = \int_{\Omega^*} u^*(x)W(x)dx.
\end{equation}

Extremals of (2.2) have been completely characterized (see, for example [2, 14, 17, 18]). In particular, an extremal $w$ always exists. However, it is not unique in general. Furthermore, equality (2.4) holds if and only if the level sets of $u$ and the level sets of $w$ are mutually nested, that is, for any choice of values $t, \tau$ there holds

\begin{center}
\begin{align*}
\text{either} & \quad \{x : u(x) > t\} \subset \{x : w(x) > \tau\}, \\
\text{or} & \quad \{x : w(x) > \tau\} \subset \{x : u(x) > t\}.
\end{align*}
\end{center}

An equivalent condition is

\begin{center}
\begin{align*}
(u(x) - u(x'))(w(x) - w(x')) \geq 0 \quad \text{for a.e. } (x, x') \in \Omega \times \Omega.
\end{align*}
\end{center}

The Hardy-Littlewood inequality (1.2) for Steiner symmetrization can be easily recovered by the one for Schwarz symmetrization. Indeed, if $N \geq 2$, we easily deduce from (2.2),

\begin{equation}
\int_{\Omega_y} u(x,y)w(x,y)dx \leq \int_{\Omega^*_y} (u^*(\cdot, y))(w^*(\cdot, y))^*(x)dx \quad \text{for a.e. } y \in \mathbb{R}^m,
\end{equation}

which immediately implies (1.2), thanks to (2.1).

Finally, we recall the following well-known result of [2].

**Proposition 2.1.** Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, and let $u, v \in L^1(\Omega)$ be two nonnegative functions. Then we have for a.e. $y \in \mathbb{R}^m$,

\begin{equation}
\int_0^s u^*(s, y)ds \leq \int_0^s v^*(s, y)ds \quad \text{for } s \in [0, \mathcal{L}^m(\Omega_y)],
\end{equation}
if and only if
\[ \int_{\Omega^#} u^*(x,y) h(x,y) dxdy \leq \int_{\Omega^#} v^*(x,y) h(x,y) dxdy, \]
for every nonnegative function \( h = h^\# \) belonging to \( L^\infty(\Omega^#) \).

3. New Pólya-Szegő type inequalities for Steiner symmetrization

In this section we prove the Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let us first show inequality (1.4). For convenience, we extend \( u \) and \( w \) by zero outside of \( \Omega \). Let \( h \in \mathbb{R}^n \) be such that \( |h| \leq 1 \). Since \( u \) is a Lipschitz function, it is differentiable a.e. Hence

\[ \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon h, y) - u(x,y)}{\varepsilon} = \nabla_x u(x,y) \cdot h, \quad \text{for a.e. } x \in \mathbb{R}^n, \]

and

\[ \frac{|u(x+\varepsilon h, y) - u(x,y)|}{\varepsilon} \leq \|\nabla_x u\|_{L^\infty(\mathbb{R}^N)} |h|, \quad 0 < \varepsilon < \varepsilon_0, \]

for a suitable \( \varepsilon_0 > 0 \), and analogously for \( w \). Let \( B_1(0) \) denote the unit ball in \( \mathbb{R}^n \), and let \( \phi \in C_0^\infty(B_1(0)) \) be a radial and radially nonincreasing function. By (3.1), (3.2) and the Dominated Convergence Theorem it follows that

\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{B_1(0)} \frac{(u(x+\varepsilon h, y) - u(x,y))(w(x+\varepsilon h, y) - w(x,y))}{\varepsilon^2} \phi(h) dh dxdy \]

\[ = \int_{\mathbb{R}^N} \int_{B_1(0)} (\nabla_x u(z) \cdot h)(\nabla_x w(z) \cdot h) \phi(h) dh dz \]

\[ = \sum_{i,j=1}^n \int_{\mathbb{R}^N} u_{x_i}(z) w_{x_j}(z) \left( \int_{B_1(0)} \phi(h) h_i h_j dh \right) dz. \]

Since \( \phi \) is radial, we deduce that

\[ \int_{B_1(0)} \phi(h) h_i h_j dh = 0 \quad \text{for } i \neq j, \]

and for \( i = 1, \ldots, n \) we have

\[ \int_{B_1(0)} \phi(h) h_i^2 dh = \frac{1}{n} \int_{B_1(0)} \phi(h) |h|^2 dh = \frac{C}{n}, \]

where

\[ C := \int_{B_1(0)} \phi(h) |h|^2 dh. \]
From (4.15)-(3.5) we obtain

\begin{equation}
\int_{\mathbb{R}^N} \nabla_x u(z) \cdot \nabla_x w(z) \, dz
\end{equation}
\begin{align*}
\;&= \frac{C}{n} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N} \int_{B_1(0)} \frac{(u(x + \epsilon h, y) - u(x, y))(w(x + \epsilon h, y) - w(x, y))}{\epsilon^2} \phi(h) \, dh \, dx \, dy \\
\end{align*}

On the other hand, since \( \phi = \phi^* \) we get by Riesz’ inequality for a.e. \( y \in \mathbb{R}^m \)

\begin{align*}
\int_{\mathbb{R}^m} \int_{B_1(0)} u(x + \epsilon h, y) w(x, y) \phi(h) \, dh \, dx \\
&\leq \int_{\mathbb{R}^m} \int_{B_1(0)} (u(\cdot, y))^* (x + \epsilon h)(w(\cdot, y))^* \phi(h) \, dh \, dx.
\end{align*}

By integrating this w.r.t. \( y \) and recalling the definition of (2.1), this leads to

\begin{align*}
\int_{\mathbb{R}^N} \int_{B_1(0)} u(x + \epsilon h, y) w(x, y) \phi(h) \, dh \, dx \\
&\leq \int_{\mathbb{R}^N} \int_{B_1(0)} u^*(x + \epsilon h, y) w^*(x, y) \phi(h) \, dx \, dy \, dh.
\end{align*}

Similarly, we deduce

\begin{align*}
\int_{\mathbb{R}^N} \int_{B_1(0)} u(x, y) w(x + \epsilon h, y) \phi(h) \, dh \, dx \\
&\leq \int_{\mathbb{R}^N} \int_{B_1(0)} u^*(x, y) w^*(x + \epsilon h, y) \phi(h) \, dx \, dy \, dh.
\end{align*}

Furthermore, since \( u \) and \( w \) satisfy (1.3), we find

\begin{align*}
\int_{\mathbb{R}^N} u(x + \epsilon h, y) w(x + \epsilon h, y) \, dx \, dy \\
&= \int_{\mathbb{R}^N} u^*(x + \epsilon h, y) w^*(x + \epsilon h, y) \, dx \, dy,
\end{align*}

and similarly,

\begin{align*}
\int_{\mathbb{R}^N} u(x, y) w(x, y) \, dx \, dy = \int_{\mathbb{R}^N} u^*(x, y) w^*(x, y) \, dx \, dy.
\end{align*}

Collecting (3.8)-(3.10), we get

\begin{align*}
\int_{\mathbb{R}^N} \int_{B_1} \frac{(u(x + \epsilon h, y) - u(x, y))(w(x + \epsilon h, y) - w(x, y))}{\epsilon^2} \phi(h) \, dh \, dx \, dy \\
\geq \int_{\mathbb{R}^N} \int_{B_1} \frac{(u^*(x + \epsilon h, y) - u^*(x, y))(w^*(x + \epsilon h, y) - w^*(x, y))}{\epsilon^2} \phi(h) \, dh \, dx \, dy.
\end{align*}
Finally, passing to the limit $\epsilon \to 0$ and using (3.6) we obtain (1.4).
It remains to prove (1.5). Fix $i \in \{1, \ldots, m\}$, and let $e_i$ denote the unit vector of $\mathbb{R}^m$ in the positive $y_i$-direction. Then

$$
\int_{\Omega} w_{y_i}(x, y) w_{y_i}(x, y) \, dx \, dy
= \lim_{\epsilon \to 0} \int_{\Omega} \frac{(u(x, y + \epsilon e_i) - u(x, y)) (w(x, y + \epsilon e_i) - w(x, y))}{\epsilon^2} \, dx \, dy.
$$

An analogous relation holds for $u^#$ and $w^#$ in place of $u$ and $w$. Now, (1.2) yields

$$
\int_{\Omega} u(x, y + \epsilon e_i) w(x, y) \, dx \, dy \leq \int_{\Omega} u^#(x, y + \epsilon e_i) w^#(x, y) \, dx \, dy
$$

and

$$
\int_{\Omega} u(x, y) w(x, y + \epsilon e_i) \, dx \, dy \leq \int_{\Omega} u^#(x, y) w^#(x, y + \epsilon e_i) \, dx \, dy,
$$

while (1.3) gives

$$
\int_{\Omega} u(x, y + \epsilon e_i) w(x, y + \epsilon e_i) \, dx \, dy
= \int_{\Omega} u^#(x, y + \epsilon e_i) w^#(x, y + \epsilon e_i) \, dx \, dy.
$$

Now inequality (1.5) follows from (3.11)-(3.14).

For the proof of Theorem 1.2 we will need the following result.

**Theorem 3.1.** Under the assumptions of Theorem 1.2, there holds

$$
- \int_{\Omega} w(x, y) \Delta_x u(x, y) \, dx \, dy \geq \int_{\Omega^*} \nabla_x W(x, y) \cdot \nabla_x u^#(x, y) \, dx \, dy,
$$

and for any $i = 1, \ldots, m$,

$$
- \int_{\Omega} w(x, y) u_{y_i} u_{y_i}(x, y) \, dx \, dy \geq \int_{\Omega^*} W_{y_i}(x, y) \cdot u^#_{y_i}(x) \, dx \, dy.
$$

**Proof.** Let us first prove inequality (3.15). Let $\phi \in C^\infty_c(\mathbb{R}^n)$ be a radial function, compactly supported in the unit ball of $\mathbb{R}^n$ and let $C$ be the constant defined in (3.4). Then

$$
\frac{C}{n} \Delta_x u(x, y)
= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \frac{u(x + \epsilon h, y) - 2u(x, y) + u(x - \epsilon h, y)}{\epsilon^2} \phi(h) \, dh,
$$

and

$$
\int_{\Omega} w(x, y) \Delta_x u(x, y) \, dx \, dy
\geq \int_{\Omega^*} \nabla_x W(x, y) \cdot \nabla_x u^#(x, y) \, dx \, dy.
$$

Now inequality (1.5) follows from (3.11)-(3.14).
for any \((x, y) \in \Omega\), with uniform convergence on compact subsets of \(\Omega\). To see that, let \((x, y) \in \Omega\) and choose \(\varepsilon_0 > 0\) small enough such that \(x + \varepsilon h \in \Omega\) and \(x - \varepsilon h \in \Omega\) for every \(\varepsilon \in (0, \varepsilon_0)\) and for every \(h \in B_1(0)\). Then a Taylor expansion gives

\[
D^2_x u(x, y, \varepsilon) = \frac{u(x + \varepsilon h, y) - 2u(x, y) + u(x - \varepsilon h, y)}{\varepsilon^2} = \sum_{i,j=1}^{n} u_{x_i x_j}(x, y) h_i h_j + o(1),
\]

with uniform convergence on compact subsets of \(\Omega\). (Here: \(\lim_{\varepsilon \to 0} o(1) = 0\).) Since (3.4) holds for every nonnegative radial function \(\phi\), we have

\[
\int_{B_1(0)} D^2_x u(x, y, \varepsilon) \phi(h) dh = \sum_{i=1}^{n} \int_{B_1(0)} u_{x_i x_i}(z) h_i^2 \phi(h) dh + o(1)
\]

\[
= \frac{C}{n} \Delta u(x, y) + o(1),
\]

from which (3.17) immediately follows on letting \(\varepsilon\) go to zero. For any \(\delta > 0\), let \(\delta'\) be such that

\[
\{W > \delta\} \subset \subset \{u^\# > \delta'\},
\]

and set \(W_\delta = (W - \delta)_+\) and \(w_\delta = (w - \delta)_+\). It is easy to check that \(w_\delta\) is compactly supported, \((w_\delta)^\# = W_\delta\) and

\[
\int_{\Omega} u(z) w_\delta(z) dz = \int_{\Omega^*} u^\#(z) W_\delta(z) dz.
\]

In view of the uniform convergence on compact sets in (3.17), we deduce

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \int_{B_1(0)} D^2_x u(x, y, \varepsilon) w_\delta(x, y) \phi(h) dxdydh = - \int_{\Omega} w_\delta(z) \Delta x u(z) dz.
\]

On the other hand, arguing as in the proof of Theorem 1.1, we get

\[
\int_{\Omega} \int_{B_1(0)} D^2_x u(x, y, \varepsilon) w_\delta(x, y) \phi(h) dxdydh \leq \int_{\Omega^*} \int_{B_1(0)} D^2_x u^\#(x, y, \varepsilon) W_\delta(x, y) \phi(h) dxdydh.
\]
Furthermore, a standard change of variables gives

\[
\int_{\Omega^\#} \int_{B_1(0)} D^2_\mathbf{x} u^\#(x, y, \epsilon) W_\delta(x, y) \phi(h) \, dxdydh
= \int_{\Omega^\#} \int_{B_1(0)} \frac{(u^\#(x + \epsilon h, y) - u^\#(x, y))(W_\delta(x + \epsilon h, y) - W_\delta(x, y))}{\epsilon^2} \phi(h) \, dxdydh.
\]

Finally, collecting (3.18)-(3.20) we obtain

\[-\int_{\Omega} w_\delta(z) \Delta_x u(z) \, dz \geq \int_{\Omega^\#} \nabla u^\# \cdot \nabla W_\delta \, dz,
\]

which leads to the thesis on letting \( \delta \) go to zero. Inequality (3.16) follows in a similar way. \( \square \)

Finally, it remains to prove the comparison result of Theorem 1.3.

Proof of Theorem 1.3. By a standard approximation argument it is enough to prove our result when the datum \( f \) is analytic (see, for example, [21], [16]). This implies that the solution \( u \) is analytic too. Let \( h \in C^\infty(\Omega^\#) \), be such that \( h = h^\# \) and consider the solution to the problem

\[
\begin{cases}
-\Delta W = h & \text{in } \Omega^\# \\
W = 0 & \text{on } \partial\Omega^\#.
\end{cases}
\]

We have that \( W \in C^\infty(\Omega^\#) \), \( W = W^\# \) and, by (4.4), we get

\[
\int_{\Omega^\#} f^\#(x, y) W(x, y) \, dxdy = \int_{\Omega^\#} \nabla v(x, y) \cdot \nabla W(x, y) \, dxdy.
\]

On the other hand, if \( w \) is a function satisfying (1.3) such that \( w^\# = W \), then by Theorem 1.2 and (1.8) we get

\[
\int_{\Omega} f(x, y) w(x, y) \, dxdy = -\int_{\Omega} \Delta u(x, y) w(x, y) \, dxdy
\geq \int_{\Omega^\#} \nabla u^\#(x, y) \cdot \nabla W(x, y) \, dxdy.
\]

Collecting (3.22) and (3.23), we obtain by the Hardy-Littlewood inequality,

\[
\int_{\Omega^\#} \left[ \nabla u^\#(x, y) - \nabla v(x, y) \right] \cdot \nabla W(x, y) \, dxdy \leq 0,
\]
or, equivalently, by (3.21),
\[
\int_{\Omega^*} \left[ u^\#(x, y) - v(x, y) \right] ( - \Delta W(x, y)) \, dx \, dy \\
= \int_{\Omega^*} \left[ u^\#(x, y) - v(x, y) \right] h(x, y) \, dx \, dy \leq 0.
\]
By the arbitrariness of \( h \) we deduce the thesis, applying Proposition 2.1. □

4. Schwarz symmetrization for nonlinear problems: a new approach

In this Section we will adapt and modify the previous tools for Schwarz symmetrization. In this case, the gradients of the functions \( u^* \) and \( w^* \) are parallel, a fact which simplifies the approach a great deal. An analogue of Theorem 1.1 for Schwarz symmetrization states as follows.

**Theorem 4.1.** Let \( \Omega \) be an open set of \( \mathbb{R}^n, \, n \geq 1 \), and let \( u, w \in W^{1,\infty}_0(\Omega) \) be nonnegative functions such that

\[
(4.1) \quad \int_{\Omega} u(x) w(x) \, dx = \int_{\Omega^*} u^*(x) w^*(x) \, dx,
\]

then

\[
(4.2) \quad \int_{\mathbb{R}^n} \nabla u(x) \cdot \nabla w(x) \, dx \geq \int_{\mathbb{R}^n} |\nabla u^*(x)| \cdot |\nabla w^*(x)| \, dx.
\]

As we already mentioned in Section 2, functions which satisfy (4.1), have been completely characterized. In particular, it has been observed in \( [18] \), Theorem 1.1, that the extremal functions \( w \) in (4.1) are unique if and only if \( u^* \) is strictly monotone. As a consequence, any extremal \( w \) is uniquely determined outside the flat zones of \( u \), and it is given by

\[
(4.3) \quad w(x) = w^* (\mu_u(u(x))),
\]

for a.e. \( x \), such that \( u(x) \) is a point of continuity for \( \mu_u \). In particular, we deduce the uniqueness of such extremal if \( w^* \) is constant where \( u^* \) is constant.

Furthermore, if \( w^* \) is a smooth function, then the classical result of Vallée-Poussin on differentiability of composite functions tells us, that any function satisfying (4.1) is differentiable at any \( x \) such that \( u(x) \) is a point of differentiability for \( \mu_u \) and

\[
(4.4) \quad \nabla w(x) = (w^*)'(\mu_u(u(x))) (\mu_u)'(u(x)) \nabla u(x).
\]
The differentiability properties of $\mu_u$ have been studied in [12, 20]. In particular, if $u \in W^{1,1}_0(\Omega)$ and

$$\left\{ s \in (0, L^n(\Omega)) : (u^*)'(s) = 0 \quad 0 < u^*(s) < \text{ess sup } u \right\} = 0,$$

then $\mu_u \in W^{1,1}((0, +\infty))$ and

$$\mu'_u(t) = \frac{1}{(u^*)'(s)|_{s=\mu_u(t)}} = -\frac{n \omega_n^n \mu_u(t)^{1-\frac{1}{n}}}{|\nabla u^*(x)|_{\{x: u^*(x) = t\}}}.$$

A stronger assumption than (4.5), which ensures the differentiability of $\mu_u$, is

$$\left\{ x \in \Omega : |\nabla u(x)| = 0 \quad 0 < u(x) < \text{ess sup } u \right\} = 0.$$

The following Lemma gives sufficient conditions on $w^*$ which ensure the uniqueness and regularity of the extremal $w$.

**Lemma 4.2.** Let $\Omega$ be an bounded domain of $\mathbb{R}^n$ and let $u \in W^{1,p}_0(\Omega)$, $1 \leq p < \infty$. Let $W : (0, L^n(\Omega)) \to \mathbb{R}$ be a nonincreasing function belonging to $W^{1,p}(a, L^n(\Omega))$ for every $a > 0$, such that $W(L^n(\Omega)) = 0$ and

$$-W'(s) \leq C(-u^*)'(s) \quad \text{for a.e. } s \in (0, L^n(\Omega)),$$

for some positive constant $C$. Then there exists only one function $w \in W^{1,p}_0(\Omega)$ satisfying $w^* = W$ and (2.4). Moreover, $W \circ \mu_u : [0, +\infty) \to \mathbb{R}$ is Lipschitz-continuous and

$$\nabla w(x) = W'(\mu_u(u(x)))(\mu_u)'(u(x))\nabla u(x) \quad \text{for a.e. } x \in \Omega.$$

**Proof.** Hypothesis (4.8) ensures that $W$ is constant where $u^*$ is constant, so that the uniqueness of the extremal satisfying (2.4) easily follows. Moreover such an extremal is given by (4.3), so it remains to prove that $w \in W^{1,p}_0(\Omega)$.

To this aim, let us consider $\phi \in C_c^1(0, +\infty)$. By the absolute continuity of $W$ and since $W(L^n(\Omega)) = 0$, we get

$$\int_0^{+\infty} W(\mu_u(t))\phi'(t)dt = \int_0^{+\infty} \left( \int_{\mu_u(t)}^{L^n(\Omega)} (-W'(s))ds \right) \phi'(t)dt.$$

Further, the distribution function $\mu_u$ is a right-continuous and decreasing function and, moreover, it is continuous if and only if $u^*$ is strictly decreasing and

$$\mu_u(u^*(s)) = s \quad \text{for a.e. } s \in (0, L^n(\Omega)).$$
Since \( u^* \) is the distribution function of \( \mu_u \), \( u^* \) is continuous if and only if \( \mu_u \) is strictly decreasing, and in such case we have

\[
(4.11) \quad u^*(\mu_u(t)) = t \quad \text{for a.e. } t \in (0, +\infty)
\]

and

\[
\{(s, t) \in (0, \mathcal{L}^n(\Omega)) \times (0, +\infty) : s \geq \mu_u(t)\} = \{(s, t) \in (0, \mathcal{L}^n(\Omega)) \times (0, +\infty) : t \geq u^*(s)\}.
\]

By Fubini’s Theorem, it follows that

\[
\int_0^{+\infty} \left( \int_{\mu_u(t)}^{\mathcal{L}^n(\Omega)} (-W'(s)) ds \right) \phi'(t) dt
= \int_0^{\mathcal{L}^n(\Omega)} \left( \int_{u^*(s)}^{+\infty} \phi'(t) dt \right) (-W'(s)) ds
= -\int_0^{+\infty} \phi(u^*(s))(-W'(s)) ds.
\]

Assumption (4.8) ensures that

\[
\int_0^{+\infty} \phi(u^*(s))(-W'(s)) ds
= \int_{[0, \mathcal{L}^n(\Omega)] \cap \{ s : (-u^*)'(s) \neq 0 \}} \phi(u^*(s))(-W'(s)) ds.
\]

Therefore we obtain, by the coarea formula and (4.6),

\[
\int_0^{+\infty} \phi(u^*(s))(-W'(s)) ds
= \int_0^{+\infty} \phi(t) \left( \int_{\{s : t = u^*(s)\}} (-W'(s)) \frac{1}{(-u^*)'(s)} d\mathcal{H}^0 \right) dt
= \int_0^{+\infty} (-W'(\mu_u(t))(-\mu'_u(t))\phi(t) dt.
\]

We deduce that \( W \circ \mu_u \) has a distributional derivative which is given by

\[
(W \circ \mu_u)'(t) = (-W'(\mu_u(t))(-\mu'_u(t)),
\]

and (4.8) ensures that this derivative is bounded. This implies that \( W \circ \mu_u \) is a Lipschitz continuous function. Observe that when \( x \in \partial \Omega \) then \( u(x) = 0 \) and \( W(\mu_u(0)) = W(\mathcal{L}^n(\Omega)) = 0 \). By a classical result on composite functions in Sobolev spaces we conclude that \((W \circ \mu_u) \circ u \in W^{1,p}_0(\Omega)\) and that its gradient can be evaluated through the chain rule. This proves (4.9). \( \square \)
The last regularity result also allows to establish a nonlinear version of Theorem 4.1.

We will make use of the following nonlinear version of the classical Pólya-Szegö inequality (see e.g. \[10\], \[23\], \[19\], \[16\], \[9\]),

$$\int_{\Omega} A(u(x))|\nabla u(x)|^p \, dx \geq \int_{\Omega^*} A(u^*(x))|\nabla u^*(x)|^p \, dx,$$

which holds for every nonnegative function \(u \in W^{1,p}_0(\Omega)\), \(1 \leq p < \infty\), and for every bounded and Borel measurable function \(A : [0, +\infty) \rightarrow [0, +\infty)\).

**Theorem 4.3.** Let \(\Omega\) be a bounded domain of \(\mathbb{R}^n\). If \(u \in W^{1,p}_0(\Omega)\), \(1 \leq p < \infty\), and if \(W\) is a function as in Lemma 4.2, then there exists only one function \(w \in W^{1,p}_0(\Omega)\) satisfying \(w^* = W\) and (2.4). Furthermore, there holds

$$\int_{\Omega} |\nabla u|^p - 2\nabla u \cdot \nabla w \, dx \geq \int_{\Omega^*} |\nabla u^*|^p - 2\nabla u^* \cdot \nabla w^* \, dx.$$

**Proof.** By Lemma 4.2, we deduce that there exists a unique extremal \(w \in W^{1,1}_0(\Omega)\) of (2.2), which can be represented by (4.3). Moreover, the gradient of \(w\) is given by (4.9). Since \(W \circ \mu_u\) is Lipschitz continuous, this implies that \(w \in W^{1,p}_0(\Omega)\). Hence we have that

$$\int_{\Omega} |\nabla u|^p - 2\nabla u \cdot \nabla w \, dx = \int_{\Omega} W'(\mu_u(u(x)))\mu_u'(u(x))|\nabla u|^p \, dx = \int_{\Omega} \mathcal{A}(u(x))|\nabla u|^p \, dx,$$

where \(\mathcal{A}(t) = W'(\mu_u(t))\mu_u'(t)\) is a nonnegative bounded function. Applying (4.12), and since \(\mu_u(u^*(x)) = \omega_n|x|^n\) and \(w^*(x) = W(\omega_n|x|^n)\), the assertion follows from (4.14). \(\square\)

Using our method, we can recover a classical comparison result for Schwarz symmetrization which is due to Talenti (see [30]).

**Theorem 4.4.** Let \(u \in H^1_0(\Omega)\) be a weak solution to problem (1.8), and let \(v \in H^1_0(\Omega^*)\) the weak solution to the symmetrized problem

$$\begin{cases} -\Delta v = f^* & \text{in } \Omega^*, \\ v = 0 & \text{on } \partial \Omega^*, \end{cases}$$

then

$$|\nabla u^*(x)| \leq |\nabla v(x)| \quad \text{for a.e. } x \in \Omega^*.$$
Proof. We adapt the previous proof to the case of Schwarz symmetrization. Since the vectors $\nabla u^*, \nabla v$ and $\nabla W$ are parallel, the inequality (3.24) is equivalent to
\[
\int_{\Omega^*} ||\nabla u^*(x)| - |\nabla v(x)|| |\nabla W(x)|dx \leq 0.
\]
By the arbitrariness of $W$, (4.15) follows.

Finally, a similar result can be also obtained for nonlinear differential operators. More precisely, we consider the following homogeneous Dirichlet problem for the $p$-Laplacian,
\[
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is an bounded domain of $\mathbb{R}^N$, $N \geq 2$, $1 < p < \infty$ and $f$ is a measurable function belonging to $L^{(p^*)'}(\Omega)$, where $(p^*)' := \frac{np}{np-n+p}$.

The Polya-Szégo type inequality proved in the previous section allows us to give a different proof of a comparison result which is due to Talenti (see [31]).

**Theorem 4.5.** Let $u \in W^{1,p}_0(\Omega)$ be the weak solution to problem (4.16) and let $v \in W^{1,p}_0(\Omega^*)$ be the weak solution to the symmetrized problem
\[
\begin{cases}
-\text{div}(|\nabla v|^{p-2}\nabla v) = f^* & \text{in } \Omega^* \\
v = 0 & \text{on } \partial \Omega^*.
\end{cases}
\]
Then
\[u^*(x) \leq v(x) \quad \text{for a.e. } x \in \Omega.\]

**Proof.** Let $h \in L^\infty(0, +\infty)$ be a nonnegative function and consider the function
\[\Phi(t) = \int_0^t h(\tau)d\tau, \quad t \geq 0.\]
Since $\Phi$ is Lipschitz continuous and $\Phi(0) = 0$, we can choose $w = \Phi(u)$ as a test function in (4.16) to get
\[
\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla w dx = \int_{\Omega} f w dx.
\]
Further, since $\Phi$ is non decreasing, we have $[\Phi(u)]^* = \Phi(u^*)$ and, arguing as above, we can choose $w^* = \Phi(u^*)$ as test function in (4.17). By (4.18) and the Hardy-Littlewood inequality we obtain
\[
\int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla w dx \leq \int_{\Omega^*} f^* w^* dx = \int_{\Omega^*} |\nabla v|^{p-2}\nabla v \cdot \nabla w^* dx.
\]
On the other hand, the function $W := w^*$ satisfies the assumptions of Theorem 4.3 and $w$ is the function which realizes equality (2.4). Hence, applying (4.19), we have
\begin{equation}
\int_{\Omega^*} |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla w^* \, dx \leq \int_{\Omega^*} |\nabla v|^{p-2} \nabla v \cdot \nabla w^* \, dx .
\end{equation}
In view of the coarea formula and since
\[ |\nabla w^*(x)| = \Phi'(u^*)|\nabla u^*(x)| = h(u^*)|\nabla u^*(x)| , \]
equation (4.20) can be equivalently written as
\begin{equation}
\int_{\Omega^*} h(u^*(x))|\nabla u^*(x)| \left[ |\nabla u^*(x)|^{p-1} - |\nabla v(x)|^{p-1} \right] \, dx
= \int_0^{+\infty} h(t) \int_{\{x : \omega_n|x|^n = \mu_u(t)\}} \left[ |\nabla u^*(x)|^{p-1} - |\nabla v(x)|^{p-1} \right] \, d\mathcal{H}^{n-1} \, dt \leq 0 .
\end{equation}
Furthermore, since both $u^*$ and $v$ are radial functions, $|\nabla u^*(x)|$ and $|\nabla v(x)|$ are constant on $\{x \in \Omega^* : \omega_n|x|^n = \mu_u(t)\}$ and these constants are given by
\[ |\nabla u^*| \{x : \omega_n|x|^n = \mu_u(t)\} = n \omega_n^{1/n} \mu_u(t)^{1-1/n} \left( -\frac{du^*}{ds} \mu_u(t) \right) , \]
\[ |\nabla v| \{x : \omega_n|x|^n = \mu_u(t)\} = n \omega_n^{1/n} \mu_u(t)^{1-1/n} \left( -\frac{dv^*}{ds} \mu_u(t) \right) . \]
Together with (4.21) this implies
\[ \int_0^{+\infty} h(t) \left[ \left( -\frac{du^*}{ds} \mu_u(t) \right)^{p-1} - \left( -\frac{dv^*}{ds} \mu_u(t) \right)^{p-1} \right] \left( n \omega_n^{1/n} \mu_u(t) \right)^p \, dt \leq 0 , \]
for any nonnegative function $h \in L^\infty(0, +\infty)$. By the arbitrariness of $h$, we deduce
\[ \left( -\frac{du^*}{ds} \mu_u(t) \right)^{p-1} \leq \left( -\frac{dv^*}{ds} \mu_u(t) \right)^{p-1} , \quad \text{for a.e. } t \in (0, \infty) , \]
from which we easily deduce the thesis.

\[ \square \]

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