A Reasonable Solution for Abstract Decision Problems Based on Hodge Decomposition

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Abstract

Decision science is an important interdisciplinary field requiring more advanced mathematical tools to enhance the reasonability. In this paper, a new solution for abstract decision problems, called the Hodge potential choice, is designed by involving Hodge decomposition on graph to analyze the dominance relation. The new solution is proved to degenerate into the classical Copeland winner set for tournaments. For general cases, Hodge potential choice satisfying three prevalent axiomatic properties, including neutrality, strong monotonicity, cycle independence. Meanwhile, the new solution keeps discrimination to deuces and empty rounds, which avoids the main weakness of conventional solutions. Besides a feasible algorithm and theoretical analyses, this paper provides various examples and results from digital experiments to show the originality and advantages of the new solution.

1 Introduction

Decision science is one of the most essential fields for people to understand human society and analyze various social behaviors by using technics from natural science and engineering. Decision science is a highly interdisciplinary fields which involves social science, economics, game theory, psychology, etc. Along the data science blooming, increasingly advanced and deep mathematical tools should be introduced into decision science to solve problems with higher sizes and complexities. The main object studied in this field is called the abstract decision problem, which provides canonical models for many social problems. An abstract decision problem is presented by an ordered pair \( (X, R) \) where \( X \) is a set of conceivable alternatives from which a selection has to be made by an individual or a collectivity, and \( R \) is an irreflexive dominance relation over \( X \) which reflects
assessments or preferences concerning alternatives. We read $xRy$ as $x$ dominates $y$ for any $x, y$ in $X$. In general, a dominance relation may be interpreted in several ways which take into account of all the characteristics of the two alternatives. For instances, it may be seen as a social preference of some collectivity. It also could be generated by some coalition consisting of members of the organization who are together capable of enforcing their preference of an alternative over another one, if only those two alternatives were considered. We do not impose asymmetry or completeness on $R$. We will leave aside the question how a dominance relation is formed, and simply assume that it is an exogenous variable. Many decision problems can be modelled as such a pair. Examples are tournaments and majority voting problems in [2] and [3], coalition formation problems in [4] and [5], page rank problems in [6], multi-criteria decision problems in [7] and [8] and exchange market problems with indivisible goods in [9] and finite coalitional games with non-transferable utility in [10].

A solution for an abstract decision problem is a mapping which assigns a single alternative or a subset of alternatives to a given decision problem. Generally, it is the outcome of a choice behaviour which may be based on some reasonable criteria, a chance mechanism, a religious code, or a mysterious prophecy of an oracle, etc. As the dominance relation is associated with the evaluation of alternatives, it is logical to postulate a link between the information contained in this relation and the solution of abstract decision problems. If there exists an alternative that dominates any other alternative and if it is not dominated by any other alternative, referred to as the best alternative, then such an alternative should be chosen certainly. The key point is that a dominance relation might be intransitive or cyclic, or even incomplete. In these cases, a best alternative need not exist. The central question therefore is which alternatives should be selected on the basis of a dominance relation when a best alternative does not exist?

A series of set solutions were proposed in the literature to answer this central question based on the additional information contained in the dominance relation. These set solutions can be roughly divided into three categories: set solutions based on the core ideas, set solutions based on stability conditions ideas and set solutions based on some auxiliary model construction. Gillies [11] proposed the core solution which consists of alternatives that are not dominated by any other alternatives. Since the core might be empty when the dominance relation is a cyclic, the variations of the cores were proposed in the literature. Noted examples are the Copeland winner set in [12], the Slater set in [13], the uncovered set in [14] and [15] and the Banks set in [16]. See other noted examples in [3, 17, 18]. The notion of stable sets proposed by [19] which specifies a subset of alternatives $V$ that satisfies the properties: no alternative in $V$ dominates another alternative in $V$; any alternative $y$ outside $V$ can be dominated by one alternative in $V$. It can be verified that stable set may not exist at all. As for the existence conditions for the core and for stable sets, see [20]

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Footnotes:

1. It has been shown in [1] that any irreflexive and asymmetric relation can be obtained as a relation induced by pairwise majority rule.

2. [10] has shown that any irreflexive relation can be obtained as a relation induced by a finite coalitional games with non-transferable utility.
for an excellent review of the state of knowledge at that time. For this reason, a series of more general solutions considered as variations of stable set were formulated to deal with every possible abstract decision problem. See [21], [22] and [23] for reformulations of the stable set solution. Besides, using retention ideas or the construction of auxiliary models are also common methods for solving the abstract decision problems. Noted examples are the tournament equilibrium set in [24], the Bipartisan set in [25], the minimal stable sets in [26] and the minimal extending sets in [27]. See [3] and [18] for other examples. So far, existing research after these variations restrictively assume that the dominance relation is either complete or asymmetric. Whenever the dominance relation is relaxed to incomplete or not necessarily asymmetric situations, the existing variations usually cannot maintain its original nature and characteristics. Moreover, they are either poorly discriminative or even not discriminative at all over the alternatives under consideration when the size of alternative set is large.

On the other hand, from the viewpoint of methodology, the above mentioned solutions might fail to establish a mathematical model precise enough to describe an abstract decision problem. Taking Copeland winner set as an example, it describes an abstract decision problem by a digraph without weights, and it determines the solution as the set of alternatives with the highest net degree on the digraph. However, although this model present its natural motivations and good properties, it fails to distinguish deuces against empty rounds in an incomplete decision problem (see Fig.1-1). In this paper, we will involve cochain complexes on undirected graphs to give a more precise description for abstract decisions and determine solutions by studying cochain complexes topologically and geometrically.
From views of above discussions, we present a new set solution called the Hodge potential choice set by using the Hodge decomposition[28]. Hodge potential choice is designed by following such a natural idea that we decompose an abstract decision problem into two subproblems and ignore the part representing the equalization among alternatives, and if the remained part depends definitely on a score sequence for alternatives, the maximal set of the scores might determine the solution. To realize this goal, de Rham cohomology theory [29] provides a perfect mathematical tool: Hodge decomposition, which can be discretized for cochain complexes on graphs. The classical Hodge decomposition claims that every differential k-form f on a manifold M has a unique decomposition as
\[ f = dP + \delta Q + H, \]
where \( P \) is a \((k-1)\)-form while \( Q \) is a \((k+1)\)-form and \( H \) is a harmonic \( k \)-form satisfying Laplace-Beltrami equation \( \Delta H = 0 \) [30]. When the Hodge decomposition is used to solve an abstract decision problem, we focus on the decomposition of the 1-cochain \( R \) on the undirected graph \( G \), and (1.1) can be simplified to
\[ R = dP + H, \]
where the \( dP \) represents a subproblem depended on the difference of a score sequence \( P \) while \( H \) satisfying Graph-Laplacian equation[31] \( L(G) \cdot H = 0 \) represents an equalized subproblem. The second term in (1.1) \( \delta Q \) vanishes in (1.2) due to the nonexistence of cells with dimension higher than 1 on graph \( G \). This decomposition can be constructed directly by solving a linear equation,
which ensures the feasibility of the Hodge potential choice.

Compared with conventional solutions, the Hodge potential choice will be showed that it maintains several axiomatic properties in general cases, including neutrality, the strong monotonicity and the cycle independence [32]. In addition, the Hodge potential choice set will be proved to degenerated as the Copeland winner set when the abstract decision is complete, and it can be extended onto decision problems with marginal utilities smoothly. Furthermore, in order to present its discrimination, especially for complicated decision problems with huge amounts of alternatives, we design digital experiments and show the advantages of Hodge potential choice from the perspectives of computations and statistics.

This passage goes as follows. In section 2, we introduce basic concepts of abstract decision problems and knowledge of Hodge decomposition on graph. Section 3 contains main results of the new solution by providing algorithms and theoretical analysis for the Hodge potential choice. Section 4 presents the results from digital experiments, which shows the computational and statistical characteristics of the Hodge potential choice. In section 5, the Hodge potential choice will be extended to solve decision problems with marginal utilities, and conclusion and some open ideas about future work will be provided.

2 Preliminary

This section is dedicated to necessary notations, concepts, techniques and data structures concerning the Hodge potential choice of abstract decision problems.

2.1 The foundation of graphs

The notations of abstract decision problems in this paper mainly come from [33] and relevant topological concepts follow [34].

Graph is the general terminology of digraph and undirected graph. A digraph \( G_D \) is a set pair \((V(G_D), E(G_D))\), where \( V(G_D) \) called the vertexes and \( E(G_D) \subseteq V(G_D) \times V(G_D) \) called edges. An undirected graph \( G_U \) is a pair \((V(G_U), E(G_U))\), \( E(G_U) \subseteq V(G_U) \times V(G_U)/\sim \), where the relation \( \sim \) means the identity \( (x, y) \sim (y, x) \) for any \( x, y \in V(G_U) \). Any digraph \( G_D \) can be regarded as an undirected graph by ignoring all directions, which means identifying \( (x, y) \sim (y, x) \in E(G) \).

The cardinalities of vertexes for a graph \( G \) is noted as \( n(G) := |V(G)| \). For any vertex \( x \in V(G) \), \( D(x) := |\{(x_1, x_2) \in V(G) | x_1 = x \text{ or } x_2 = x \}| \) is called as the degree of \( x \) while if \( G \) is a digraph \( D^+(x) := |\{x_1 \in V(G) | (x, x_1) \in E(G) \}| \) and \( D^-(x) := |\{x_1 \in V(G) | (x_1, x) \in E(G) \}| \) are respectively called outdegree and indegree of \( x \), and \( D^0(x) := D^+(x) - D^-(x) \) is called the net degree of \( x \).

A graph is said to be

- irreflexive: \( \forall x \in V(G) \), if \( (x, x) \notin E(G) \);
- complete: \( \forall x, y \in V(G) \), if either \( (x, y) \in E(G) \) or \( (y, x) \in E(G) \);

- connected: \( \forall x, y \in X \), if \( \exists x_1, \ldots, x_m \in X \) s.t. \( \{(x, x_1), \ldots, (x_m-1, x_m), (x_m, y)\} \subseteq E(G) \);

- unconnected: if it is not connected;

- regular: if it is a digraph and \( \forall x, y \in X, D^0(x) = D^0(y) \).

Next, we introduce chain complexes and cochain complexes with the real coefficient \( \mathbb{R} \) for graphs. They are two essential algebraic topological structures that pave the way to involve Hodge decompositions on graphs.

The chain complex \( C_\ast(G, \mathbb{R}) \) of a graph \( G \) is constructed by two linear spaces \( C_0(G, \mathbb{R}) \), \( C_1(G, \mathbb{R}) \) and a homomorphism \( \partial \) from \( C_1(G, \mathbb{R}) \) to \( C_0(G, \mathbb{R}) \), called the boundary operator, where

- \( C_0(G, \mathbb{R}) := \bigoplus_{v \in V(G)} \mathbb{R} \);

- \( C_1(G, \mathbb{R}) := \bigoplus_{e \in E(G)} \mathbb{R} \) with identity \( k(x, y) \sim -k(y, x) \), if \( k \in \mathbb{R} \) and \( (x, y) \) or \( (y, x) \) \( \in E(G) \);

- the boundary operator \( \partial : C_1(G, \mathbb{R}) \to C_0(G, \mathbb{R}) \) by \( \partial(k(x, y)) := ky - kx, \forall k \in \mathbb{R} \).

We default the coefficient \( \mathbb{R} \), and the whole complex \( C_\ast(G, \mathbb{R}) \) can be written as

\[
0 \longrightarrow C_1(G) \xrightarrow{\partial} C_0(G) \longrightarrow 0,
\]

where the first zeros in the chain means nonexistence of cells with dimension higher than 1 in \( G \).

The cochain complex \( C^\ast(G, \mathbb{R}) \) of \( G \) is the dual of chain complex \( C_\ast(G, \mathbb{R}) \), and it is still constructed by two linear spaces \( C^0(G, \mathbb{R}) \), \( C^1(G, \mathbb{R}) \) and a homomorphism \( d \) from \( C^0(G, \mathbb{R}) \) to \( C^1(G, \mathbb{R}) \), called differential operator or coboundary, where

- \( C^0(G, \mathbb{R}) := \text{Hom}(C_0(G, \mathbb{R}), \mathbb{R}) \);

- \( C^1(G, \mathbb{R}) := \text{Hom}(C_1(G, \mathbb{R})) \)

- the differential operator \( d : C^1(G, \mathbb{R}) \to C^0(G, \mathbb{R}) \) by \( d(\phi) := \phi \circ \partial, \forall \phi \in C^0(G, \mathbb{R}) \).

By above definitions, it is easy to check that, for any linear function \( \phi \in C^0(G, \mathbb{R}) \),

\[
d\phi(k(x, y)) = k\phi(y) - k\phi(x), \forall (x, y) \text{ or } (y, x) \in E(G), \tag{2.3}
\]

The whole cochain complex \( C^\ast(G, \mathbb{R}) \) can be written as

\[
0 \longrightarrow C^0(G) \xrightarrow{d} C^1(G) \longrightarrow 0.
\]

From considerations of computation, we use adjacent matrix \( W(G) \) with size of \( n(G) \times n(G) \) to express a graph \( G \). The data structure of a 1-cochain \( \psi \in C^1(G) \) is an antisymmetric
matrix with size of \( n(G) \times n(G) \), i.e., \( \psi_{ij} := \psi(v_i, v_j) = -\psi(v_j, v_i) = -\psi_{ji} \) for any \((v_i, v_j)\) or \((v_j, v_i)\) \( \in \mathcal{E}(G) \), while other positions equaling to 0. A 0-cochain \( \phi \) is expressed by a vector with size of \( n(G) \times 1 \), where \( \phi_i := \phi(v_i) \) for any \( v_i \in \mathcal{V}(G) \).

### 2.2 The foundation of and abstract decision problems

The notations of abstract decision problems in this paper mainly come from [4, 23, 36]. An abstract decision problem is denoted by \((X, R)\) where \( X \) is a nonempty finite set of alternatives and \( R \) is an irreflexive binary relation on \( X \) (i.e, \( xRx \) for no \( x \in X \)). Let \( \Omega(X) \) be the collection of these abstract decision problems with same set of alternatives \( X \) and \( \Omega(\cdot) \) be the collection of all abstract decision problems.

For any \( x \in X \), let \( B(x) \) be the set of alternatives that dominate \( x \) and \( D(x) \) be the set of alternatives that are dominated by \( x \), i.e.,

\[
B(x) = \{ y \in X \mid yRx \} \quad \text{and} \quad D(x) = \{ y \in X \mid xRy \}.
\]

The Cardinalities of \( B(x) \) and of \( D(x) \) are denoted by \( |B(x)| \) and \( |D(x)| \) respectively. Let \( cs(x) = |D(x)| - |B(x)| \) which is the **Copeland score** of \( x \in X \) in \((X, R)\).

A dominance relation \( R \) is said to be

- **asymmetric**: \( \forall x, y \in X \), if \( xRy \), then not \( yRx \);

- **connected**: \( \forall x, y \in X \), if \( \exists x_1, \ldots, x_m \in X \) such that \( xTx_1, \ldots, x_{m-1}Tx_m, x_mTy \), where \( xTy \) means \( xRy \) or \( yRx \) \( \in \mathcal{R} \);

- **complete**: \( \forall x, y \in X \), if \( x \neq y \), then either \( xRy \) or \( yRx \);

- **a tournament**: if it is complete;

- **transitive**: \( \forall x, y, z \in X \), if \( xRy \) and \( yRz \), then \( xRz \);

- **regular**: \( \forall x, y \in X \), \( cs(x) = cs(y) \).

- **irregular**: it is not regular.

There exists some concepts about abstract decision problems, including

- **deuce**: a pair \((x, y) \in X \times X \), if \( xRy \) and \( yRx \);

- **empty round**: a pair \((x, y) \in X \times X \), \( x \neq y \), if neither \( xRy \) nor \( yRx \);

- **cycle**: a sequence \([x_1, x_2, \ldots, x_m] \subseteq X \), such that \( x_1Rx_2, \ldots, x_{m-1}Rx_m, x_mRx_1 \).

1. Abstract decision problems are also known as abstract games or decision systems in [37] and [38].
In this paper, we focus exclusively on abstract decision problem with nonempty finite \( X \) and irreflexive \( R \). Neither asymmetric nor completeness of \( R \) are required. Conformed to conventions, \((X, R) \in \Omega(X)\) should have been represented by a digraph where \( X \) is a vertex set and \( R \) is the set of directed edges. However, we will adopt a new data structure \((G, R)\) for an abstract decision problem \((X, R)\) from topological viewpoints, where \( G \) is an undirected graph with \( X \) as the vertex set and \( R \) is a 1-cochain on \( G \). More specifically, \( V(G) := X, E(G) := \{(x, y) \sim (y, x) | xRy \text{ or } yRx\} \) and the 1-cochain \( R \) satisfies that \( R(x, y) = -1 \) and \( R(y, x) = 1 \) if \( xRy \) but not \( yRx \), while \( R(x, y) = 0 \) if \( xRy \) and \( yRx \). It is illegal to ask \( R(x, y) \) if \( (x, y) \notin E(G) \).

We attempt to use the new data structure to avoid the confusion of deuces and empty rounds and remain the symmetry of data structures meanwhile.

![Figure 2-2 AD_1 can be presented by a digraph (X, R) or by a 1-cochain R on an undirected graph G. The numbers aside edges record the values of the 1-cochain R.](image)

**Example 2.1.** A abstract decision problem \( AD = (X, R) \) with four alternatives \( X = \{a, b, c, d\} \) and \( R = \{aRb, aRd, bRc, cRb, cRd, dRb\} \) (see Fig. 2-2). In this decision problem, we say \( (b, c) \) is a deuce and \( (a, c) \) is an empty round. We can use two data structures to present \( AD \)

1) \((X, R)\), where \( R \) is a digraph;

2) \((G, R)\), where \( G \) is an undirected graph and \( R \) is a 1-cochain on \( G \)

The adjacent matrix of the digraph \( R \), adjacent matrix of undirected graph \( G \) and the 1-cochain \( R \) are written as follows,

\[
1) \quad R = \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}, \quad 2) \quad W(G) = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & -1 & 0 & -1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
1 & -1 & 1 & 0
\end{bmatrix}.
\]
A solution is a mapping \( F : \Omega(X) \to 2^X \), for any \( X \), i.e., \( F(X, R) \subseteq X \) for any \((X, R)\). Take any two solutions \( F \) and \( F' \) for \( \Omega(X) \), where \( F' \) is called a refinement of \( F \) if \( F'(X, R) \subseteq F(X, R) \) for any \((X, R) \in \Omega() \).

**Example 2.2.** Prestigious solution called the Copeland winner set was proposed in [12] that was frequently used to solve the abstract decision problems. The Copeland winner set of \((X, R) \in \Omega(X)\) is defined as the maximal set of Copeland scores \( CW(X, R) = \{ x \in X \mid cs(y) \leq cs(x) \ \forall y \in X \} \). The Copeland winner set \( CW(AD) \) for abstract decision problem \( AD \) in Example 2.1 is \( \{a\} \).

To avoid ambiguities, we shall give a corresponding table of terminologies from fields of decision science and mathematics (see Table 1).

| Decision Science                              | Graph Theory                      | Topology                   |
|-----------------------------------------------|-----------------------------------|----------------------------|
| Abstract decision problem \( AD = (X, R) \)  | \( AD = (X, \text{Digraph} R) \) | \( AD = (\text{Undirected Graph } G, 1\text{-cochain} R) \) |
| Alternatives \( X \)                         | Vertices \( V(R) = X \)           | 0-cells \( V(G) = X \)       |
| Dominance relation \( R \)                   | Edges \( E(R) \)                  | 1-cells \( E(G) \) and 1-cochain \( R \) |
| Dominating: \( xRy \) in \( R \)             | Directed edge \( (x, y) \in E(R) \) | Edge \( (x, y) \in E(G) \) and \( R(x, y) = -1, R(y, x) = 1 \) |
| Deuce: \( xRy, yRx \)                        | \( (x, y), (y, x) \in E(R) \)     | \( (x, y) \in E(G) \) and \( R(x, y) = R(y, x) = 0 \) |
| Empty round                                   | \( (x, y) \notin E(R) \) and \( (y, x) \notin E(R) \) | \( (x, y) \notin E(G) \) |
| Tournament                                    | \( \forall x, y \in X, (x, y) \notin E(R) \) | \( G \) is a complete graph |
| Connected                                     | Weakly connected                  | \( G \) is a connected graph |
| Transitive                                    | Strongly connected                |                            |
| Score \( \phi \)                             | Vertex function \( \phi \)        | \( 0\)-cochain \( \phi \) |
| Copeland score \( cs \)                      | Net degree \( D^0 \)              | \( \psi \cdot [1, \ldots, 1]^T \) |
| Regular                                       | \( D^0 \equiv 0 \)                | \( \psi \cdot [1, \ldots, 1]^T = 0 \) |

### 2.3 Hodge decomposition on graphs

In this part, we will introduce the essential mathematical tool, the Hodge decomposition, for our new solution. The concepts and knowledge about differential geometry appearing in this paper mainly come from [30, 39, 40, 41].

In subsection 2.1, we have introduced the differential operator \( d \) from \( C^0(G) \) to \( C^1(G) \) for any graph \( G \). Besides, there exists a naturally dual operator \( \delta \) of (2.3) from \( C^1(G) \) to \( C^0(G) \), defined as

\[
\delta \psi(v_i) = \sum_{k=1}^{n(G)} \psi(v_i, v_k) \cdot \varepsilon_{ik}, \ \forall v_i \in V(G),
\]

(2.4)

where \( \varepsilon_{ik} = 1 \) if \( (v_i, v_k) \) or \( (v_k, v_i) \in E(G) \) and , \( \varepsilon_{ik} = 0 \) for \( (v_i, v_j) \) and \( (v_j, v_i) \notin E(G) \).

**Remark 2.1.** In Riemannian geometry, \( d \) and \( \delta \) are defined for differential forms \( \{\mathcal{F}^k\} \) on manifold \( M \) [30]. Our definitions can be regarded as some discretization. The original definition of \( \delta \) operator
on a manifold depends on metric while \( d \) is more natural and independent on metric choices. A metric \( g \) determines a series of isomorphisms \(*\), called Hodge star operator, between \( F^k \) and \( F^{n-k} \) (\( n \) is the dimension of \( M \)). Then \( \delta := (-1)^{kn+n} d^* \). Our definition (2.4) is exactly derived from a special defaulted metric. For simplicity, we ignore those complicated mathematical discussions and give new definitions directly.

It is easy to check that \( \delta \) operator is a linear map (or it is a homomorphism). We say \( \delta \) is the dual operator of \( d \) because of the following lemma.

**Lemma 2.2. (Integral equality)** For any \( \phi, \phi' \in C^0(G) \) and \( \psi, \psi' \in C^1(G) \), if

\[
\int_{V(G)} \phi \phi' := \sum_{i=1}^{n(G)} \phi(v_i) \cdot \phi'(v_i), \quad \int_{E(G)} \psi \psi' := \sum_{e_j \in E(G)} \psi(e_j) \cdot \psi'(e_j),
\]

we have

\[
\int_{V(G)} \delta \psi \cdot \phi = -\int_{E(G)} \psi \cdot d\phi.
\]

**Proof.** By definitions, we check that

\[
\sum_{e \in E(G)} \psi(e) \cdot d\phi(e) := \sum_{i=1}^{n(G)} \sum_{j=i+1}^{n(G)} \psi(v_i, v_j) \cdot d\phi(v_i, v_j) \cdot \varepsilon_{ij}
\]

\[
= \frac{1}{2} \sum_{i=1}^{n(G)} \sum_{j=1}^{n(G)} \psi(v_i, v_j) \cdot d\phi(v_i, v_j) \cdot \varepsilon_{ij} = \frac{1}{2} \sum_{i=1}^{n(G)} \sum_{j=1}^{n(G)} \psi(v_i, v_j) \cdot (\phi(v_j) - \phi(v_i)) \cdot \varepsilon_{ij},
\]

\[
= -\frac{1}{2} \sum_{i=1}^{n(G)} \phi(v_i) \sum_{j=1}^{n(G)} \psi(v_i, v_j) \cdot \varepsilon_{ij} + \sum_{j=1}^{n(G)} \phi(v_j) \sum_{i=1}^{n(G)} \psi(v_j, v_i) \cdot \varepsilon_{ij}
\]

\[
= -\sum_{i=1}^{n(G)} \delta \psi \cdot \phi(v_i).
\]

\( \square \)

With differential operator \( d \) and operator \( \delta \), we consider the following two sequences, and introduce Laplacian operators on graph.

\[
C^0(G, \mathbb{R}) \xrightarrow{d} C^1(G, \mathbb{R}) \xrightarrow{\delta} C^0(G, \mathbb{R}),
\]

(2.6)

\[
C^1(G, \mathbb{R}) \xrightarrow{\delta} C^0(G, \mathbb{R}) \xrightarrow{d} C^1(G, \mathbb{R}),
\]

(2.7)

where \( d \circ \delta \) and \( \delta \circ d \) are endomorphisms on \( C^0(G) \) and \( C^1(G) \) respectively. According to the definitions (2.3) and (2.4), we can regard \( \delta \) as divergence \( \text{div} \) on graphs and \( d \) as the counterpart of gradient \( \nabla \) on graphs. Following this idea, the classical Laplacian operator defined as \( \Delta := \text{div} \circ \nabla \)
can be involved onto graphs by the two endomorphisms. We define the Laplacian operators $\Delta$ on graph $G$ by $\Delta^0 := \delta \circ d$, $\Delta^1 := d \circ \delta$, and a cochain $\phi \in C^*(G, \mathbb{R})$ is harmonic iff $\phi \in \ker(\Delta^*)$.

In computing science, there exists another Laplacian operator on graphs called Graph-Laplacian matrix [31] defined as

$$L(G) := D(G) - W(G), \quad (2.8)$$

which plays a essential role in machine learning and other branches of computer science. We will show that the two definitions are compatible to each other.

**Lemma 2.3.** For any graph $G$, $\phi \in C^0(G)$, $\Delta^0 \phi = -L(G) \cdot \phi$.

*Proof.* From definitions, for any $v_i \in V(G)$,

$$\Delta^0 \phi(v_i) := \delta d \phi(v_i) = \sum_{k=1}^{n(G)} d \phi(v_i, v_k) \cdot \varepsilon_{ik}$$

$$= \sum_{k=1}^{n(G)} (\phi(v_k) - \phi(v_i)) \cdot \varepsilon_{ik} = -D(G)_{ii} \cdot \phi(v_i) + \sum_{k=1}^{n(G)} (W(G)_{ki} \phi(v_k))$$

$$= - \sum_{k=1}^{n(G)} L(G)_{ik} \cdot \phi_k = (L(G) \cdot \phi)_i.$$

From the arbitrary of $v_i$, we complete the proof of lemma 2.3. \hfill \Box

Combining the definition of Laplacian operators with the integral lemma 2.2, we can derive the following lemma directly.

**Lemma 2.4.** For any graph $G$, $\phi \in C^0(G)$, $\Delta^0 \phi = 0$, and $\psi \in C^1(G)$, $\phi \in \ker \Delta^0$ iff $d \phi = 0$, and $\psi \in \ker \Delta^1$ iff $\delta \psi = 0$.

*Proof.* By definition, the two backward directions are obvious. To prove the forward directions, we consider the equation from $\Delta^0 \phi = 0$ and $\Delta^1 \psi = 0$,

$$\int_{V(G)} \phi \cdot \Delta^0 \phi = 0 = \int_{E(G)} \psi \cdot \Delta^1 \psi,$$

and according the integral lemma 2.2, we have

$$\int_{G(G)} d \phi \cdot d \phi = 0 = \int_{V(G)} \delta \psi \cdot \delta \psi,$$

and by definition (2.5), it is easy to check that $d \phi = 0$ and $\delta \psi = 0$. \hfill \Box

It is obvious that $\phi \in \ker(d)$ meaning that $\phi$ is **locally constant** (constant on every connected
component) on $G$. Therefore, lemma 2.3 and lemma 2.4 show us

$$\text{rank}(L(G)) = n(G) - \text{dim}(\ker(d)) = n(G) - k(G), \quad (2.9)$$

where $k(G)$ is the number of connected components of $G$.

**Remark 2.5.** This fact corresponds to a result in algebraic topology that the cohomology group $H^0(G, \mathbb{R})$ of graph $G$ is isomorphic to $\bigoplus_{k(G)} \mathbb{R}$.

Along above necessary preparations for presenting what should be concerned about cochains on graphs, we introduce the Hodge decomposition on graphs by the following theorem.

**Theorem 2.6.** (Hodge decomposition on graph): For any $\psi \in C^1(G)$ on graph $G$, there exist $P \in C^0(G)$ and a unique harmonic $H \in \ker \Delta^1$ such that

$$\psi = dP + H, \quad (2.10)$$

where we call $P$ as the **Hodge potential** 0-chain while $H$ as the harmonic part of $\psi$.

**Proof.** The proof focuses on the connected graph while the proof can be extended for unconnected cases by considering connected components respectively.

First, we admit the exitance and then prove the uniqueness. We suppose that there are two decompositions

$$\psi = dP + H = dP' + H'. \quad (2.10)$$

According to lemma 2.3 and lemma 2.4, if we use $\delta$ to operate both sides of the equation $d(P - P') = H' - H$, we get that

$$L(G) \cdot (P - P') = 0.$$

By lemma 2.4, $(P - P') \in \ker(d)$, we get $dP = dP'$ and $H = H'$. It proves the uniqueness.

Now we prove the existence constructively by giving the algorithm for potential function $P$. If $\psi \in \ker \Delta^1$, it is a decomposition of itself. For $\psi \not\in \ker \Delta^1$, by definition (2.4), $\delta \psi = \psi \cdot [1, \cdots, 1]^T$ and by the antisymmetry of $\psi$, we have

$$\langle [1, \cdots, 1], \delta \psi \rangle := [1, \cdots, 1] \cdot \psi \cdot [1, \cdots, 1]^T = 0. \quad (2.11)$$

On the other hand, by (2.9) and the connectedness of $G$, we have known that rank($L(G)$) = $n(G) - 1$, and the null space of $L(G)$ is exactly formed by harmonic 0-cochain which is locally constant. Therefore $\ker L(G) = \{k \cdot [1, \cdots, 1]^T \mid \forall k \in \mathbb{R}\}$. Combining with (2.11), the linear equation

$$-L(G) \cdot X = \delta \psi \quad (2.12)$$

can be solved due to rank($L(G)$) = rank($[L(G), \delta \psi]$) [42]. The general solutions have the form as
$X = P_0 + k \cdot \phi_0$, where $P_0$ is a particular solution. Then $H := \psi - dP_0 = \psi - dX$ is harmonic naturally.

3 The Solution of Hodge Potential Choice

With the strong mathematical tools such as cochains and their Hodge decompositions on graphs, we develop the new solution, Hodge Potential Choice, for abstract decision problems with natural motivations, feasible algorithm, and its advantages and axiomatic prosperities.

3.1 Hodge potential choice set

To solve a connected abstract decision problem $AD = (X, R)$, we regard $R$ as a cochain complex on the connected undirected graph $G$ and present $AD$ by $(G, R)$, and we can decompose $AD$ into two subproblems, $AD_1 = (G, dP)$ and $AD_2 = (G, H)$ by using Hodge decomposition of the cochain complex $R = dP + H$. $AD_2 = (G, H)$ can be regarded a regular decision problem, which represents the equalization among $X$, and $AD_1 = (G, dP)$ is totally induced by a score sequence $P$. It is natural enough to believe that the score $P$ might describe the absolute advantage for each alternative in $X$. Along with this idea, we design the new solution: the Hodge potential choice set. Now, we directly give out the constructive definition of the Hodge potential choice set and post its algorithm.

For any connected abstract decision problem $AD = (X, R) = (G, R)$ on $X$ is called as **Hodge potential score** if the reminder $H$ of $R - dP_R$ satisfies $(G, R)$ is regular, i.e., $H = R - dP_R \in C^1(G)$ is harmonic. There might be different Hodge potential scores for a fixed irregular abstract decision problem, but theorem 2.6 ensures the existence of Hodge potential scores and the unique existence of the harmonic remainder $H$.

**Hodge potential choice set** $\mathcal{HP}(AD)$ for a connected abstract decision problem $AD = (X, R)$ is defined by the maximal set of score sequence $P_R$, if $P_R$ is one of the Hodge potential scores of $AD$, i.e.,

$$\mathcal{HP}(AD) = \{x \in X | P_R(y) \leq P_R(x) \, \forall y \in X \, \exists P_R \in X\}.$$

It is notable that the existence of more than one Hodge potential scores would not make confusions to define the Hodge potential choice set, because the proof of theorem 2.6 claims that the difference between two Hodge potential scores $P_R$ and $P'_R$ is locally constant, which means $P_R$ and $P'_R$ have the same maximal set on a connected component of $G$. Therefore, the connected condition ensures that (3.13) is well defined. In practice, we shall require the irregular condition of a connected abstract decision problem $AD$ which derives the existence of a nonzero $P_R$ promising the meaningfulness of $\mathcal{HP}(AD)$. On the contrary, the Hodge potential choice set $\mathcal{HP}(AD) = X$, if $AD = (X, R)$ is connected and regular.

The algorithm to search the Hodge potential choice set for an abstract decision connected and irregular is attributed to solve the linear Laplacain equation as algorithm 1. The algorithm comes
Algorithm 1 Hodge potential choice (HPC)

**Input:** an abstract decision problem presented by adjacency matrix $W(G)$ and the antisymmetric matrix $R$

**Output:** Hodge potential choice set $\mathcal{HP}$

1: get degree matrix $D(G)$ and Graph-Laplacian $L = D(G) - W(G)$ by (2.8)
2: compute the Copeland score $cs = -\delta R = -R \cdot [1, \cdots, 1]^T$
3: solve linear equation $L \cdot P_R = cs$ to get a particular solution $P_R$
4: return the maximal index set $\mathcal{HP} = \text{index}(\max(P_R))$ of $P_R$.

from the constructive proof of existence of Hodge decomposition in the proof of theorem 2.6.

**Example 3.1.** The abstract decision problem $AD = (G, R)$ presented in example 2.1 can be solved by Hodge potential choice by

![Step 1: L(G)](2x2)

![Step 2: cs = 2](2x2)

![Step 3: PR = \begin{bmatrix} 7 \\ 8 \\ -1 \\ 4 \\ 3 \\ 8 \\ 0 \end{bmatrix}](2x2)

return: $\mathcal{HP} = \{a\}$

If someone cares about how the decision problem has been decomposed (see fig. 3-3) rather than merely get the solution set, $R = dP + H$ where $H$ is regular should be check. Actually, we have

$$R = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 0 & -9 & 0 & -7 \\ 9 & 0 & 5 & 2 \\ 0 & -5 & 0 & -3 \\ 7 & -2 & 3 & 0 \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -5 & 6 \\ 0 & 5 & 0 & -5 \\ 1 & -6 & 5 & 0 \end{bmatrix} = dP_R + H,$$

where $H$ is obviously harmonic due to its all row summations equaling to 0.
Figure 3-3 $AD_1$ can be decomposed into two subproblems $dP_R + H$. $dP_R$ induced by the Hodge potential score $P_R$ as the red number aside alternatives. $P_R(a) = \max(P_R) > P_R(k \neq a)$ determines the Hodge potential choice set $\mathcal{H}(AD_1) = \{a\}$. The values of the 1-cochain $dP_R$ are classical differences of $P_R$ noted as green numbers is in first part. $H$ is a harmonic 1-cochain showed by the net scores for all alternatives in second part equalling to 0 constantly. We can recall original problem $AD_1$ (see Fig. 2-2) by superposing these two subproblems.

3.2 Properties of Hodge potential choice

In this subsection, we show the main properties of Hodge potential choice including its degeneration as Copeland winner set, three prevalent axiomatic properties which the Hodge potential choice satisfies, and its discrimination to deuces and empty rounds.

The first essential characteristic of Hodge potential choice set is that, in completed cases, it will degenerate into Copeland winner set for tournaments.

**Theorem 3.1.** For a tournament $T = (G_c, R)$ where $G_c$ is a graph complete, any Hodge potential score $P_R$ is multiple of the Copeland score $cs$, i.e., $P_R = k \cdot cs$. Thus, $\mathcal{H}(T) = \mathcal{C}(T)$, for any tournament.

**Proof.** Because of the completeness of $G_c$, the Graph-Laplacain $L(G_c) = nI - \text{Ones}(n)$, $n := n(G_c)$ where $I$ is identity matrix and $\text{Ones}(n)$ is composed by 1. Therefore, $L(G_c)$ has one eigenvalue equal 0 and $n - 1$ eigenvalues equal $n$. It is easy to check that $v_0 = [1, \cdots, 1]^T$ is the eigenvector of 0 and any vector $v$ satisfying $\langle v, v_0 \rangle = 0$ is the eigenvector of $n$. By (2.11), we know $\langle \delta R, v_0 \rangle = 0$. Thus, the Copeland score $cs := -R \cdot v_0$ satisfies $-L(G_c) \cdot cs = n\delta R$. The vector $\frac{1}{n}cs$ can be
regarded as a particular solution of equation (2.12) which becomes a Hodge potential score of \((G_c, R)\). Consequently, \(cs\) and \(\frac{1}{n}cs\) share same maximal set, which derives \(\mathcal{HP}(T) = CW(T)\). □

However, the equivalence \(\mathcal{HP}(T) = CW(T)\) may not be holden for general (uncompleted) cases. In the following examples, we may find that Hodge potential choice is more precise than Copeland winners sometimes.

**Example 3.2.** For the abstract decision problem \(AD_1 = (G_1, R_1)\) with eight alternatives \(X_1 = \{A, B, B, D, E, F, G, H\}\) and dominance depicted in Fig. 3-4, we have its Copeland winner set as \(CW(AD_1) = \{D, E, F\}\) by its Copeland score as \([-4, 0, 1, 2, 2, -1, 2]\), but the Hodge potential choice set \(\mathcal{HP}(AD_1) = \{E\}\) due to a Hodge potential score \(P_{R_1} \approx [0, 0.4, 0.8, 1.1, 1.5, 0.7, 0.2, 0.1]\). In this example \(\mathcal{HP}(AD_1) \subset CW(AD_1)\).

Besides it degenerates into a prestigious solution Copeland winner set for tournaments, Hodge potential choice satisfies three widespread axiomatic properties for solutions of decision problem, including neutrality, strong monotonicity and cycle independence. These facts indicate that the Hodge potential choice would be a convincible solution.

**Theorem 3.2.** (Neutrality) For any connected abstract decision problem \(AD = (X, R)\), and \(p\) is a permutation of \(X\), a new decision problem \(AD' = (X, R')\) induced by \(x_{R'y}^{p^{-1}(x)}R_{p^{-1}(y)}\), then we have the Hodge potential choice set \(\mathcal{HP}(AD') = p(\mathcal{HP}(AD))\).
Proof. The proof of the neutrality attributes to the permutation \( p \) regarded as the elementary transformation. Without loss of generality, we suppose that \( p([..., x_i, ..., x_j, ...]) = [..., x_j, ..., x_i, ...] \) so that \( p = p^{-1} = p^T \) and \( R' = p^T R p, \ L(G') = p^T L(G)p, \ \delta R' = p^T \delta R. \) Therefore, \( -L(G') p^T P_R = -p^T L(G)p p^T P_R = p^T \delta R = \delta R', \) where \( p^T P_R \) can be regarded as a Hodge potential score for \( AD'. \) Consequently, \( \mathcal{HP}(AD') = \) is maximal set of \( p^T P_R. \)

Neutrality means the Hodge potential choice is an even solution without bias.

**Theorem 3.3. (Strong monotonicity)** For any connected abstract decision problem \( AD = (X, R) \) that \( \exists x, y \in X \) such that \( x \in \mathcal{HP}(AD) \) and \( yRx \in R \) but \( xRy \notin R, \) and if \( AD' = (X, R') \) satisfying \( R' = R \setminus \{ yRx \} \cup \{ xRy \}, \) then we have \( \mathcal{HP}(AD') = \{ x \}. \)

**Proof.** For simplicity, we present two notations that \( T_{xy} \) is an antisymmetric matrix with 1 on position \((y, x)\) and \(-1\) on \((x, y)\) while else equaling to 0, and \( t_{xy} \) is a vector with 1 on position \( x \) and \(-1\) on \( y \) while else equaling to 0.

By assumptions, we have \( R' - R = 2T_{xy} \) and \( \delta R' - \delta R = 2t_{xy}. \) According to the linearity of equation (2.12), we have

\[-L(G)(P_{R'} - P_R) = 2t_{xy},\]

where we denote \( P_{R'} - P_R \) as \( \tilde{P} \) in following. We present above equation by its components and translate them by \( L = D - W, \) then

\[
\sum_{k}^{n(G)} L(G)_{xk} \tilde{P}_k = 2, \quad \tilde{P}_x - \text{mean}\{\tilde{P}_k | kRx or xRk\} = 2,
\]

\[
\sum_{k}^{n(G)} L(G)_{yk} \tilde{P}_k = -2, \quad \implies \tilde{P}_y - \text{mean}\{\tilde{P}_k | kRy or yRk\} = -2,
\]

\[
\sum_{k}^{n(G)} L(G)_{ik} \tilde{P}_k = 0, \forall i : x \neq i \neq y, \quad \tilde{P}_i - \text{mean}\{\tilde{P}_k | kRi or iRk\} = 0.
\]

The second equations from right side directly shows us that \( \tilde{P}_y \) cannot be \( \max(\tilde{P}). \) By contradiction, if we suppose \( \max(\tilde{P}) = \tilde{P}_j > \tilde{P}_x, \) \( j : x \neq j \neq y, \) the third equation tells us that \( \tilde{P}_k = \tilde{P}_j = \max(\tilde{P}) \) iff \( kRi \) or \( iRk, \) and the connectedness of \( G \) points inductively that \( \tilde{P} \) is constant, which derives the contradictory. Therefore, we have that \( \tilde{P}_x = \max(\tilde{P}) \) and \( x \) is the unique element in maximal set of \( \tilde{P}. \) Thus, \( \tilde{P}_x > \tilde{P}_{i \neq x} \) and \( P_R(x) + \tilde{P}_x > \max(P_R) + \tilde{P}_{i \neq x} \geq P_{R' \setminus i \neq x}, \) which means \( \mathcal{HP}(AD') = \{ x \}. \)

This axiomatic property, the strong monotonicity, describes a phenomena that if we decide the championship for a game by adopting Hodge potential choice as the decision rule, one of the shared champions may enjoy success exclusively if he could turn defeat into victory in any round.

Subsequently, we consider how cycles and deuces influence the Hodge potential choice.
Theorem 3.4. (Cycle independence) For a cycle $C = [x_1, x_2, ..., x_m] \subseteq X$ in a connected abstract decision problem $AD = (X, R)$, if $AD' = (X, R')$ is constructed by reversing $C$ in $(X, R)$ to be $C^{-1} = [x_m, x_{m-1}, ..., x_1]$ in $(X, R')$, we still have $\mathcal{HP}(AD') = \mathcal{HP}(AD)$.

Proof. By definitions, it is obvious that $AD = (G, R)$, $AD' = (G', R')$ where $G = G'$, and for any $[x_k, x_{k+1}] \subseteq C$, we have $(R' - R)(x_k, x_{k+1}) = 2$ and $(R' - R)(x_{k+1}, x_k) = -2$ while values of $R' - R$ on other $e \in E(G)$ equal to 0. Then $\delta(R' - R) = 0$. According to lemma 2.4, we see that $R' - R \in \ker \Delta^1$ is harmonic. Therefore, according to the uniqueness of Hodge decomposition in theorem 2.6, we have $R' = dP_R + (H_R + R' - R)$, where the last three terms are all harmonic. Thus, $\exists P_R = P'_R$ and then $\mathcal{HP}(AD') = \mathcal{HP}(AD)$. \qed

Remark 3.5. If deuces are admitted, any cycle $C$ concerted to a deuces sequence, i.e., $\forall [x_k, x_{k+1}]$ inducing $x_kRx_{k+1}$ and $x_{k+1}Rx_k$, will not change the Hodge potential choice set. However, it will change the underlying graph $G$, if someone tends to vanish a cycle. To maintain Hodge potential choice, the cycle vanishing is not promised.

Cycle independence in theorem 3.4 points that the direction of a cyclic dominance is not important for the Hodge potential choice. In some sense, a cycle describes the local counterbalance among some alternatives, which will not influence the choice of the global winners. Therefore, a cycle reversed might still deliver the same information of the local counterbalance but cycle vanishing erases it.

Example 3.3. For abstract decision problem $AD_1 = (X, R_1)$ with $X = \{A, B, C, D, E, F, G, H\}$ and $R_1$ showed in Fig. 3-5, there exists a cycle $C = [A, F, H]$. We reverse $C$ to obtain $AD_2 = (X, R_2)$ and vanish $C$ to obtain $AD_3 = (X, R_3)$. The three problems have the same Copeland winner set $CW = \{A, H\}$. However, the Hodge potential choice sets $\mathcal{HP}(AD_1) = \mathcal{HP}(AD_2) = \{H\}$, but $\mathcal{HP}(AD_3) = \{A\}$. 

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Similarly, the Hodge potential choice can distinguish whether deuces which the information about the counterbalance of pairs in a connected decision problem disappear or not. Therefore, our new solution avoids the confusion of deuces and empty rounds, one of the main problems of conventional solutions such as Copeland winner set. The key point keeping the Hodge potential choice discriminative is that the counterbalance information from deuces is saved in the undirected $G$ of $AD = (G, R)$ and presented by Graph-Laplacian matrix $L(G)$ during Hodge potential choice. Deuces added or vanished will change the graph $G$ which is the underlying space for all following steps. Let us recall the example provides us in Fig. 1-1.

**Example 3.4.** For two decision problems $AD = (X, R)$ and $AD' = (X, R')$ presented in Fig.1-1, theorem 3.1 derives $\mathcal{HP}(AD') = CW(AD') = \{A, B\}$ from the completeness, and we can use HPC algorithm 1 to obtain a $P_R \sim [1.5, 0.6, -0.8, -0.1]$, then $\mathcal{HP}(AD) = \{A\}$.

With above arguments, we claim Hodge potential choice is a convincible solution for abstract decision problems. Despite of no mathematical evidence pointing that Hodge potential choice prevails existing solutions or it will be a refinement for some conventional solutions, we have shown the strong advantages of the new solution and we will provide positive results to support the new solution from digital experiments in next section.

4 Digital Experiments

In this section, we focus on the effect of the algorithm 1. For convenience, we denote the two Hodge potential choices as HPC and Copeland winner as CW for briefness.
First, we analyse the complexity of algorithm HPC. It is easy to find that the first two steps of HPC, containing only additions without multiply and matrix operating, are actually the processing of CW. Hence the main complexity comes from the step 3: solving the Laplacian equation (2.12). Hence HPC has its complexity approximating to the counterpart of computational solutions for linear equations, such as Jacobi method and Gauss-Seidel method [43], which means HPC has complexity about $O(n^2)$ while it can be reduced when the dominance relation $R$ is spare. The complexity is not the key point we focus on in this paper.

The main purpose of this section is to provide some statistical and computational conversations pointing that HPC has its advantages against traditional methods. We will take CW as a representation of conventional solution for abstract decision problems to compare with HPC, because CW is one of the most refined solution among existing solutions in some aspect [44].

In last section, we present some example to show that HPC is more refined than CW in some cases. Although there is no strict mathematical theorem to illustrate that inclusion relationships exist between CW and HPC, we find that HPC has considerable possibilities to be a refinement of CW by digital experiments with large sample, and this possibilities will change dependently on the sizes $n(AD) = |X|$ of abstract decision problems. Another main statistical observation is that even though the equivalence of CW and HPC will be holden for tournaments, our experiments show that this possibilities have no relevant relation with completeness $u(AD)$ defined as follows.

**Definition 4.1.** For a connected abstract decision problem $AD = (X, R) = (G, R)$, $n(G) = |X|$, we define
\[ u(AD) := \frac{2|E(G)|}{n(G)^2 - n(G)} \]
as the completeness of $G$ and $AD$ while $1 - u(G)$ as the incompleteness.

In our experiments, we generate randomly 100 antisymmetric matrices $G(n, u)$ of distinct parameter pairs $(n, u)$, where $n$ is the size of the matrix while $u$ is the incompleteness, as samples of abstract decision problems. By computing and comparing the Copeland winner set $CW$ and Hodge potential choice set $HP$, we compile the following frequencies of three cases for every fixed pair $(n, u)$, including

1. the frequency $R_1(n, u)$ of cases that $CW \subset HP$
2. the frequency $R_2(n, u)$ of cases that $CW \supset HP$
3. the frequency $R_3(n, u)$ of cases that $CW = HP$

We have that $\sum_{i=1,2,3} R_i(n, u) \leq 1$. If the experiment shows $R_1(n, u) \gg R_2(n, u) \approx 0$ always holden, we may claim that HPC is more precise than CW from statistical and computational views [45].

Fig. 4-6 shows the conditional averages of $R_1, R_2, R_3$ among fixed $u$ and changing $n$ or in reverse.
The red parts in Fig.4-6, the values of average $R_2$, are sensibly larger than the blue parts which almost vanish. From this view, HPC is more precise than CW statistically.

If we drop out the data for $R_2$, we can observe the tendency of $R_2$ changing along completeness and size. It can be found in Fig.4-7 that $R_2$ changing depends on size rather than completeness.

By the way, we record the average cardinalities of solution sets $|\mathcal{CW}|(n, u)$ and $|\mathcal{HP}|(n, u)$ and present their additional averages along $n$ or $u$ changing in Fig.4-8, which will indicate that the Hodge potential choice set $\mathcal{HP}$ rarely produces shared winners but Copeland winner set $\mathcal{CW}$ can not achieve this. This phenomenon shows that HPC is more precise than CW as well.

Figure 4-6 Proportions changing of $R_1$, $R_2$ and $R_3$:

(a) averages for various completenesses  
(b) averages for various sizes

Figure 4-7 Tendency of $R_2$ changing: (a) shows that there is no clear linear relation between completeness and $R_2$, but $R_2$ decreases along problem sizes increasing, which means HPC is more likely to be a refinement of CW when the number of alternatives is relatively smaller.

(a) average $R_2$ for various completenesses  
(b) average $R_2$ for various sizes
Figure 4-8 Changing of ℒ(W) and ℒ(H) along completeness and size varying: The blue bars show that the cardinalities of Copeland winner sets will decrease along the completeness or the sizes of decision problems increasing, while the counterparts of Hodge potential choice sets approximately stay around 1, which seldom do shared winners exist under HPC.

Although this paper has not concentrate on quantitative analyse for the result of the digital experiment, it is evidently to see HPC has relative advantages against CW.

5 Discussion

5.1 Extension onto decisions with marginal utilities

Hodge potential choice can be naturally extended to solve decision problems with marginal utilities. A decision problem with marginal utilities \( MG = (G, R_m) \) is still a pair of an undirected graph \( G \) combined with a cochain \( R_m \) on it, where \( R_m(x, y) \) equals to the marginal utility or relative net score of alternative \( y \) against alternative \( x \). A decision with marginal utilities \( MD = (G, R_m) \) can degenerate into an abstract decision problem \( AD = (G, R) \) with all utilities forgotten by \( R(x, y) := \frac{R_m(x, y)}{|R_m(x, y)|} \). In an intuitive word, if we concern the net score between two players rather than merely consider the winner relationship, the decision problem will be regarded as a decision problem with marginal utilities \( MD \).

By taking \( R_m \) as the input in algorithm 1, we can obtain the Hodge potential choice set \( \mathcal{HP}(MD) \) for \( MD \). The original version of Hodge potential choice can be regard as to solve decision problems with special marginal utilities. Therefore, there is no essential difference from decision with marginal utilities to abstract decisions in the perspective of the Hodge potential choice. For general cases, the Hodge potential choice maintains the neurility and cycle independence, and strong monotonicity that any extra dominance or less inferiority in any round may led one of shared winners to be a exclusive winner.
Example 5.1. The decision problem with marginal utilities $MD_1$ presented in (Fig. 5-9) has a Hodge potential score $P_{R_m} \approx [0.8, 0.1, 0.6, 0, 0.7, 1.3, 0.4, 0.8]$ and the Hodge potential choice set $MD_{\infty p}(MG) = \{F\}$.

![Figure 5-9 The decision problem with marginal utilities $MD_1$ noted as green numbers aside edges can be solved by the Hodge potential choice, where the Hodge potential score is presented by red numbers aside alternatives, and alternative $F$ in yellow occupies the Hodge potential set exclusively.](image)

More details about the general Hodge potential choice are not contained in this paper.

5.2 Conclusion & Future work

We develop a new solution, the Hodge potential choice, for abstract decision problems by involving the Hodge decomposition on graphs to analyse the dominance relation. We provide a feasible algorithm to realize the solution set searching. Meanwhile, we prove some important properties theoretically as well. In addition, this paper includes several examples and digital experiments with large sample which shows the advantage of our new solution from statistical viewpoints.

However, there still exists a large space for developing along this idea. Geometrically, we are curious about the performance of the Hodge potential choice as nontrivial metrics are introduced on graphs. From viewpoints of game theory, it is an interesting and open question that how the Hodge potential choice behaves along a dynamic processing.
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