Duality. Humans are not very good at understanding interacting many body systems. We understand the behavior of free particles, and can sometimes use perturbation theory to incorporate weak interactions between them. But what happens when we couple the particles strongly? We have had to rely largely on experiments to suggest the many interesting collective phenomena that can result\(^1\).

Duality is the circumstance in which cranking up the interactions results in another system we understand, usually described in terms of another set of weakly-coupled particles. You might think this is a failure of imagination (we thought we had two models but really they are the same), or you might think this is evidence that we are doing a good job thinking of all the possibilities (this is probably wrong given our track record). Either way, when this happens, it is a solution to the strong coupling problem. A further reason that theoretical physicists get excited about duality is that it undermines the very notion of an ‘elementary particle’. In terms of the original variables, the dual weakly coupled particles are generally solitons – large, heavy classical objects made up of many quanta. The amazing thing is that solitons can become ‘elementary,’ in the sense that the model can be reformulated with them as the basic constituents. An example of a duality, which we will use below, arises in Maxwell’s theory of electromagnetism, where the duality exchanges electric and magnetic point

\(^{1}\)Examples include: superconductivity, superfluidity, the quantum Hall effects, fractionalization of charge and spin, the wetness of water...
charges, and relates a theory with interaction strength $e$ to one with interaction strength $1/e$. This idea has been generalized to many other field theories, particularly with tools of supersymmetry\footnote{For more examples and references, I recommend \cite{4}. Evidence can be found for the duality symmetry of (3+1)-dimensional abelian gauge theory in \cite{5, 6} (on the lattice) and in \cite{7, 8} (in the continuum, at energies below the electron mass).}

**Classic duality for vortices.** A vortex is a localized object in the plane around which a phase field (a field that takes values on the circle) winds. Dualities relating particles and vortices have a rich history in particle physics and condensed matter physics. The simplest version relates the particles in the normal state of a superfluid to the vortices in a dual superconductor \cite{9, 10, 11, 12, 13}. Specifically, it maps the 2+1-dimensional XY model – a model with a $U(1)$ symmetry, such as particle number – (near its Wilson-Fisher critical point), to the abelian Higgs model – the low-energy description of a superconductor – (near its critical point). The density of particles maps to the density of magnetic flux\footnote{The gauge field $\vec{a}$ can be regarded as a solution of the continuity equation $\nabla \cdot \vec{J} = 0$ for the particle 3-current: $\vec{J} \propto \nabla \times \vec{a}$.}. In the symmetry-broken phase, the Goldstone boson is the photon (which has only one polarization state in 2+1d). The symmetric (Mott insulator) phase is achieved when the proliferation of vortices disorders the condensate of bosons; the resulting gapped state is the Higgs phase of the gauge theory, where the photon eats the phase field and becomes massive. This charge-vortex duality has been a workhorse in the study of two dimensional condensed matter systems, such as fractional quantum Hall states \cite{14}, superconducting thin films \cite{13, 15}, and beyond-Landau critical phenomena \cite{16}.

**Fermionic vortices.** Vortices can be fermions\footnote{It is not so easy to give a simple example where this happens. In type II superconductors, a vortex supports bound fermion modes \cite{17}. The resulting boundstate of a quasiparticle and a vortex is, however, still a boson; this is because the charge and magnetic flux have a long-ranged statistical interaction symptomatic of topological order. So these two exotic phenomena (fermionic modes bound to the vortices and topological order) cancel each other out, and the ordinary bosonic particle-vortex duality applies even in this case.}. A dual description of such a system requires a new ingredient, since it must include a fermionic particle. An appealing proposal for the required duality has been made by Metlitski and Vishwanath \cite{1} and independently by Wang and Senthil \cite{2} (building on earlier work of Son \cite{18}), and further illuminated by Mross, Alicea and Motrunich \cite{3}: the fermionic vortex of QED in 2+1 dimensions is a free Dirac fermion. This deceptively simple statement joins together several disparate lines of progress in theoretical physics: interacting topological insulators, anomalies in quantum field theory, spin liquids, compressible quantum Hall states.
**Interacting topological insulators.** Topological insulators (TIs) (and more generally symmetry-protected topological states) are nearly-trivial states of matter, distinguished only by their edge states\(^5\). The most famous example is a (3+1)-dimensional time-reversal invariant TI, which can host an odd number of Dirac cones (such as one) at the boundary of the sample. A time-reversal-symmetric realization of a single two-dimensional Dirac cone is not possible without the bulk TI, so it provides a signature of the bulk phase. But the Dirac cone is not the only possible signature. Deformations of the boundary conditions can dramatically change the spectrum of the edge states, but they preserve some essential “TI-ness”; this essence is called an *anomaly* in the high energy theory literature.

**Making use of the bulk.** The argument of [1, 2] uses this ambiguity in the edge theory, and the electric-magnetic duality of an auxiliary theory of bulk electromagnetism coupled to the TI. When I said above that (3+1)-dimensional QED had a duality that interchanged the electric and magnetic charges, you could have complained that the electron is a fermion, while a magnetic monopole may not be. However, the authors observe that in a TI, a charge-two monopole is also a fermion [1, 2]. By analyzing the spectrum of electric and magnetic charges, they conclude that the duality exchanges the electron and this double monopole, and moreover exchanges a TI with time-reversal symmetry with a ‘chiral TI’\(^6\) which instead has particle-hole symmetry. But this is just what is required of a symmetry which exchanges a fermionic particle with a fermionic vortex: time reversal symmetry maps the electron to itself, but the vortex to a vortex winding the other way. The exchange of particles and holes, in turn, preserves the vortex. The bulk duality\(^7\) implies a duality between edge theories.

![Diagram of discrete symmetries](image)

Figure: The action of discrete symmetries on the fermionic particle (top) and its dual fermionic vortex (bottom).

The freedom to deform the edge of the TI implies a weak notion of equivalence:

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\(^5\)For a review of interacting TIs, see [19].

\(^6\)That is, it exchanges Class AII and AIII of the tenfold-way classification of TIs [19].

\(^7\)Further strong mathematical evidence for the duality relating electromagnetism coupled to class AII and class AIII TIs was provided in a detailed calculation by [20].
although the edge theories are guaranteed to be the same in terms of topology (charge sectors, symmetry assignments, anomalies), they need not have identical low-energy physics. One side of the duality is a free, massless Dirac particle, but the long-distance physics of (2+1)-dimensional QED is a long-standing problem, which is important in the study of spin liquids\(^8\). Abelian gauge theory in two dimensions tends to develop an energy gap because of the proliferation of monopoles [23]. In the realization proposed in [1, 2], this cannot happen because those monopoles carry electric charge, and indeed the stronger notion of duality as an equivalence of low-energy physics may hold.

**Dimensional bootstrap.** At a more microscopic level, a lattice version of the duality was derived in [3]. Two aspects of this construction are notable: First, it relies on an explicit change of variables, which (like the Jordan-Wigner transformation which makes fermions out of spins in one dimension\(^9\)) can be explicitly checked, but remains arcane. Second, the lattice models in the construction are made out of systems that do not themselves exist except at the edge of a TI, namely an array of wires carrying chiral fermions. The derivation on the lattice provides evidence for the strong form of the duality, but is not a proof since a phase transition may intervene between the lattice model and the strongly-coupled continuum limit.

**Impact.** Furious activity surrounds this development. It immediately solves outstanding puzzles about interacting TIs and their symmetric, gapped topologically-ordered surface states [1, 2, 24]. Moreover, the ‘auxiliary’ bulk gauge theory in the above discussion actually describes a novel 3d spin liquid with time-reversal symmetry [25, 24], which was in fact the starting point for the authors of [2]. Pyrochlore magnets made from rare-earth elements\(^10\) provide candidate materials for such phases.

Perhaps most interestingly, ingredients of this work grew out of the attempt to construct a manifestly particle-hole symmetric theory of a half-filled Landau level [18], the best-understood non-Fermi liquid state. Briefly, the successful theory of this state [26] makes a distinction between half-filled and half-empty\(^11\). The associated particle-hole symmetry is emergent, and an exact local realization can only occur at the edge of a TI. The main outcome is a new understanding of the Dirac-like nature of the composite fermion excitation of this system, which provides a new point of view on the phenomenology [18, 24, 28, 29, 30]\(^12\). This circle of ideas has also led to new insights on 2d superfluid-insulator transitions with nearby metallic phases, where the fermions come alive [32, 33]. More recently, consequences of the fermionic vortex duality have

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\(^8\)See e.g. [21] and for recent progress, see [22].

\(^9\)For a pedagogical treatment I will recommend §2.2.5 of these lecture notes.

\(^10\)four of seventeen of which are named after the same village of Ytterby, Sweden where they were discovered.

\(^11\)This brilliant joke provides the title of [27].

\(^12\)See [31] for commentary on the connection with the half-filled Landau level.
been checked mathematically [20] and numerically [28], and the duality has been fit into the wider web of field theory dualities [34, 35, 36].

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