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RIGIDITY OF AMALGAMATED PRODUCT IN NEGATIVE CURVATURE

GÉRARD BESSON, GILLES COURTOIS, SYLVAIN GALLOT

Abstract. Let $\Gamma$ be the fundamental group of a compact riemannian manifold $X$ of sectional curvature $K \leq -1$ and dimension $n \geq 3$. We suppose that $\Gamma = A \ast_C B$ is the free product of its subgroups $A$ and $B$ over the amalgamated subgroup $C$. We prove that the critical exponent $\delta(C)$ of $C$ satisfies $\delta(C) \geq n - 2$. The equality happens if and only if there exist an embedded compact hypersurface $Y \subset X$, totally geodesic, of constant sectional curvature $-1$, whose fundamental group is $C$ and which separates $X$ in two connected components whose fundamental groups are $A$ and $B$. Similar results hold if $\Gamma$ is an HNN extension, or more generally if $\Gamma$ acts on a simplicial tree without fixed point.

1. Introduction

In [15], Y. Shalom proved the following theorem which says that for every lattice $\Gamma$ in the hyperbolic space and for any decomposition of $\Gamma$ as an amalgamated product $\Gamma = A \ast_C B$, the group $C$ has to be “big”. In order to measure how “big” $C$ is, let us define the critical exponent of a discrete group $C$ acting on a Cartan Hadamard manifold by

$$\delta(C) = \inf\{s > 0 \mid \Sigma_{\gamma \in \Gamma} e^{-sd(\gamma x, x)} < +\infty\}.$$ 

Theorem 1.1. Let $\Gamma$ be a lattice in $PO(n, 1)$. Assume that $\Gamma$ is an amalgamated product of its subgroups $A$ and $B$ over $C$. Then, the critical exponent $\delta(C)$ of $C$ satisfy $\delta(C) \geq n - 2$.

An example is given by any $n$-dimensional hyperbolic manifold $X$ which contains a compact separating connected totally geodesic hypersurface $Y$. The Van Kampen theorem then says that the fundamental group $\Gamma$ of $X$ is isomorphic to the free product of the fundamental groups of the two halves of $X - Y$ amalgamated over the fundamental group $C$ of the incompressible hypersurface $Y$. Such examples do exist in dimension 3 thanks to the W. Thurston’s hyperbolization theorem. In any dimension, A. Lubotzky showed that any standard arithmetic lattice of $PO(n, 1)$ has a finite cover whose fundamental group is an amalgamated product, cf. [11]. In fact, A. Lubotzky proved that any standard arithmetic lattice $\Gamma$ has a finite index subgroup $\Gamma_0$ which is mapped onto a nonabelian free group. A nonabelian free group can be written in infinitely many ways as an amalgamated product, so one get infinitely many decomposition of $\Gamma_0$ as an amalgamated product.
product by pulling back the amalgamated decomposition of the nonabelian free group.

In these cases there is equality in theorem 1.1, i.e. \( \delta(C) = n - 2 \) where \( C \) is the fundamental group of \( Y \), and Y. Shalom suggested in [15] that the equality case in the theorem 1.1 happens only in that case.

The aim of this paper is to show that the theorem 1.1 still holds when \( \Gamma \) is the fundamental group of a compact riemannian manifold of variable sectional curvature less than or equal to \(-1\), and characterize the equality case.

**Theorem 1.2.** Let \( X \) be an \( n \)-dimensional compact riemannian manifold of sectional curvature \( K \leq -1 \). We assume that the fundamental group \( \Gamma \) of \( X \) is an amalgamated product of its subgroups \( A \) and \( B \) over \( C \) and that neither \( A \) nor \( B \) equals \( \Gamma \). Then, the critical exponent \( \delta(C) \) of \( C \) satisfy \( \delta(C) \geq n - 2 \). Equality \( \delta(C) = n - 2 \) happens if and only if \( C \) cocompactly preserves a totally geodesic isometrically embedded copy \( \mathbb{H}^{n-1} \) of the hyperbolic space of dimension \( n - 1 \). Moreover, in the equality case, the hypersurface \( Y^{n-1} := \mathbb{H}^{n-1}/C \) is embedded in \( X \) and separates \( X \) in two connected components whose fundamental groups are respectively \( A \) and \( B \).

**Remark 1.3.** (i) By the assumption on \( A \) or \( B \) not being equal to \( \Gamma \) we exclude the trivial decomposition \( \Gamma = \Gamma \ast_C C \) where \( A = \Gamma \) and \( C = B \) can be an arbitrary subgroup of \( \Gamma \), for example any cyclic subgroup, in which case the conclusion of theorem 1.2 fails. Also note that because of this assumption on \( A \) and \( B \), we have \( A \neq C \) and \( B \neq C \).

(ii) Let us recall that standard arithmetic lattices in \( PO(n, 1) \) have finite index subgroup with infinitely many non-equivalent decompositions as amalgamated products, cf. [11]. In fact, among these decompositions, all but finitely many of them are such that \( \delta(C) > n - 2 \). Indeed, by theorem 1.2, if \( \delta(C) = n - 2 \) then \( C \) is the fundamental group of an embedded totally geodesic hypersurface in \( X \), but there are only finitely many totally geodesic hypersurfaces by [21].

When a group is an amalgamated product, it acts on a simplicial tree without fixed point and theorem 1.1 is a particular case of the

**Theorem 1.4.** ([13], theorem 1.6). Let \( \Gamma \subset SO(n, 1) \), \( n \geq 3 \), be a lattice. Suppose \( \Gamma \) acts on a simplicial tree \( T \) without fixed vertex. Then there is an edge of \( T \) whose stabilizer \( C \) satisfies \( \delta(C) \geq n - 2 \).

In the case \( \Gamma \) is cocompact, the conclusion of theorem 1.4 holds for the stabilizer of any edge which separates the tree \( T \) in two unbounded components, and the proof of this is exactly the same as the proof of theorem 1.2. In particular, when the action of \( \Gamma \) on \( T \) is minimal, (i.e. there is no proper subtree of \( T \) invariant by \( \Gamma \)), the conclusion of theorem 1.4 holds for every edge of \( T \), in the variable curvature setting, and we are able to handle the equality case.
Theorem 1.5. Let $\Gamma$ be the fundamental group of an $n$-dimensional compact riemannian manifold $X$ of sectional curvature less than or equal to $-1$. Suppose $\Gamma$ acts minimally on a simplicial tree $T$ without fixed point. Then, the stabilizer $C$ of every edge of $T$ satisfies $\delta(C) \geq n-2$. The equality $\delta(C) = n-2$ happens if and only if there exist a compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, $Y$ with its induced metric has constant sectional curvature $-1$.

Another interesting case contained in theorem 1.5 is the case of HNN extension. Let us recall the definition of an HNN extension. Let $A$ and $C$ be groups and $f_1 : C \to A$, $f_2 : C \to A$ two injective morphisms of $C$ into $A$. The HNN extension $A \ast_C$ is the group generated by $A$ and an element $t$ with the relations $tf_1(\gamma)t^{-1} = f_2(\gamma)$. For example, let $X$ be a compact manifold containing a non separating compact incompressible hypersurface $Y \subset X$. Let $A$ be the fundamental group of the manifold with boundary $X-Y$ obtained by cutting $X$ along $Y$ and let $C$ be the fundamental group of $Y$. The boundary of $X-Y$ consists in two connected components $Y_1 \subset X-Y$ and $Y_2 \subset X-Y$ homeomorphic to $Y$. By the incompressibility assumption, these inclusions give rise to two embeddings of $C$ into $A$, and the fundamental group of $X$ is the associated HNN extension $A \ast_C$.

Theorem 1.6. Let $\Gamma$ be the fundamental group of a compact riemannian manifold $X$ of dimension $n$ and sectional curvature less than or equal to $-1$. Suppose that $\Gamma = A \ast_C$ where $A$ is a proper subgroup of $\Gamma$. Then, we have $\delta(C) \geq n-2$ and equality $\delta(C) = n-2$ if and only if there exist a non separating compact totally geodesic hypersurface $Y \subset X$ with fundamental group $\pi_1(Y) = C$. Moreover, in that case, $Y$ with its induced metric is of constant sectional curvature $-1$, and the HNN decomposition arising from $Y$ is the one we started with.

Let us summarize the ideas of the proof of theorem 1.2. We work on $\tilde{X}/C$. The amalgamation assumption provides an essential hypersurface $Z$ in $\tilde{X}/C$, namely $Z$ is homologically non trivial in $\tilde{X}/C$. The volume of all hypersurfaces homologous to $Z$ is bounded below by a positive constant because their systole are bounded away from zero. We then construct a smooth map $F : \tilde{X}/C \to \tilde{X}/C$, homotopic to the identity which contracts the volume of all compact hypersurfaces $Y$ by the factor $\left(\frac{\delta(C)}{n-2}\right)^{n-1}$, namely $\text{vol}_{n-1}F(Y) \leq \left(\frac{\delta(C)}{n-2}\right)^{n-1}\text{vol}_{n-1}Y$. This contracting property together with the lower bound of the volume of hypersurfaces in the homology class of $Z$ gives the inequality $\delta(C) \geq n-2$. This map is different from the map constructed in [3], in particular it can be defined under the single condition that the limit set of $C$ is not reduced to one point. Moreover, its derivative has an upper bound depending only on the critical exponent of $C$.

The equality case goes as follows. When $\delta(C) = n-2$, the map $F : \tilde{X}/C \to \tilde{X}/C$ contracts the $(n-1)$-dimensional volumes, ie. $|\text{Jac}_{n-1}F| \leq 1$. 
This contracting property is infinitesimally rigid in the following sense. Let us consider a lift \( \tilde{F} \) of \( F \). If \( |\text{Jac}_{n-1} \tilde{F}(x)| = 1 \) at some point \( x \in \tilde{X} \), then \( \tilde{F}(x) = x \), there exists a tangent hyperplane \( E \subset T_x \tilde{X} \) such that \( D\tilde{F}(x) \) is the orthogonal projector of \( T_x \tilde{X} \) onto \( E \) and the limit set \( \Lambda_C \) is contained in the topological equator \( E(\infty) \subset \partial \tilde{X} \) associated to \( E \). By topological equator \( E(\infty) \subset \partial \tilde{X} \), we mean the set of end points of those geodesic rays starting at \( x \) tangently to \( E \).

We then prove the existence of a point \( x \in \tilde{X} \) such that

\[
(1.1) \quad |\text{Jac}_{n-1} \tilde{F}(x)| = 1.
\]

If there would exist a minimizing cycle in the homology class of \( Z \) in \( \tilde{X}/C \), any point of such a cycle would satisfy (1.1). As no such minimizing cycle a priori exists because of non compactness of \( \tilde{X}/C \), we prove instead the existence of a \( L^2 \) harmonic \((n-1)\)-form dual to \( Z \), which is enough to prove existence of a point \( x \) such that (1.1) holds.

At this stage of the proof, there is a big difference between the constant curvature case and the variable curvature case.

In the constant curvature case, any topological equator bounds a totally geodesic hyperbolic hypersurface \( \mathbb{H}^{n-1} \), and therefore, as the group \( C \) preserves \( \Lambda_C \subset E(\infty) = \partial \mathbb{H}^{n-1} \), it is not hard to see that \( C \) also preserves \( \mathbb{H}^{n-1} \) and acts cocompactly on it, and the hypersurface of the equality case in theorem 1.2 is \( \mathbb{H}^{n-1}/C \).

In the variable curvature case, we first show the existence of a \( C \)-invariant totally geodesic hypersurface \( \tilde{Z}_\infty \subset \tilde{X} \) whose boundary at infinity coincides with \( \Lambda(C) \), and then we show that \( \tilde{Z}_\infty \) is isometric to the real hyperbolic space. We then show that \( Y =: \mathbb{H}^{n-1}_R/C \), which is compact, injects in \( X = \tilde{X}/\Gamma \) and separates \( X \) in two connected components whose fundamental groups are \( A \) and \( B \) respectively.

In order to show the existence of such a totally geodesic hypersurface \( \tilde{Z}_\infty \), we first prove that \( C \) is a convex cocompact group, i.e., the convex hull of the limit set of \( C \) in \( \tilde{X} \) has a compact quotient under the action of \( C \), and that the limit set of \( C \) is homeomorphic to an \((n-2)\)-dimensional sphere.

The convex cocompactness property of \( C \) and the fact that the limit set \( \Lambda(C) \) of \( C \) is homeomorphic to an \((n-2)\)-dimensional topological sphere are the two key points in the equality case.

This compactness property then allows us to prove the existence of a minimizing current in the homology class of the essential hypersurface \( Z \subset \tilde{X}/C \).

By regularity theorem this minimizing current \( \tilde{Z}_\infty \) is a smooth manifold except at a singular set of codimension at least 8. By the contracting properties of our map \( F \), \( \tilde{Z}_\infty \) is fixed by \( F \) and the geometric properties of \( F \) at fixed points where the \((n-1)\)-jacobian of \( F \) equals 1 allows us to prove that \( \tilde{Z}_\infty \) is totally geodesic and isometric to the hyperbolic space.

Let us now briefly describe the proof of the convex cocompactness property of \( C \) in the equality case.
The group $C$ (or a finite index subgroup of it) actually globally preserves a smooth cocompact hypersurface $\tilde{Z} \subset \tilde{X}$ which separates $\tilde{X}$ into two connected components and whose boundary $\partial \tilde{Z} \subset \partial X$ coincides with $\Lambda_C \subset E(\infty)$. In the case where $C$ wouldn’t be convex cocompact, we are able to find an horoball $HB(\theta_0)$ centered at some point $\theta_0 \in \Lambda_C$ in the complementary of which lies the hypersurface $\tilde{Z}$.

The contradiction then comes from the following.

Consider a sequence of points $\theta_i \in \partial \tilde{X}$ converging to $\theta_0$ and geodesic rays $\alpha_i$ starting from the point $x \in \tilde{X}$ at which $|Jac_{n-1}(x)| = 1$ and ending up at $\theta_i$. These geodesic rays have to cross $\tilde{Z}$ at points $z_i$ which are at bounded distance from the orbit $Cx$ of $x$, therefore the shadows $O_i$ of balls centered at these $z_i$ enlighted from $x$ have to contain points of $\Lambda_C$ by the shadow lemma of D. Sullivan. On the other hand, we show that it is possible to choose the sequence $\theta_i$ in such a way that these shadows $O_i$ don’t meet $\Lambda_C$. This property $O_i \cap \Lambda_C = \emptyset$ comes from a choice of $\theta_i$ such that the distance between $z_i$ and the set $H$ of all geodesics rays at $x$ tangent to $E \subset T_x\tilde{X}$ tends to $\infty$. Intuitively, in order to chose $z_i$ as far as possible from $H$, the points $\theta_i$ have to be chosen tranversally to $\Lambda_C$. This transversality condition is not well defined because the limit set $\Lambda_C$ might be highly non regular. Thus, in order to prove that such a choice is possible, we argue again by contradiction. If for any choice of a sequence $\theta_i$ converging to $\theta_0$, the distance between $z_i$ and $H$ stays bounded, then the Gromov distances $d(\theta_i, \theta_0)$ between $\theta_i$ and $\theta_0$ satisfy $d(\theta_i, \Lambda_C) = o(d(\theta_i, \theta_0))$, and therefore any tangent cone of $\Lambda_C$ at $\theta_0$ would coincide with a tangent cone of $\partial \tilde{X}$ at $\theta_0$, which is known to be topologically $\mathbb{R}^{n-1}$. But on the other hand, the existence of a point $x$ such that $|Jac_{n-1}(x)| = 1$ and the fact that $C$ acts uniformly quasiconformally with respect to the Gromov distance on $\partial \tilde{X}$ imply that the Alexandroff compactification of the above tangent cone of $\Lambda_C$ at $\theta_0$ is homeomorphic to $\Lambda_C$ which is contained in a topological sphere $S^{n-2}$, leading to a contradiction.

From convex cocompactness of $C$ and the fact that the limit set of $C$ is a topological $(n-2)$-dimensional sphere, there is an alternative proof of the existence of a totally geodesic $C$-invariant copy of the hyperbolic space $\mathbb{H}^{n-1}_\mathbb{C} \subset \tilde{X}$ which consists in observing that the topological dimension and the Hausdorff dimension of the limit set $\Lambda(C)$ are equal to $n-2$ and then use the following result of M. Bonk and B. Kleiner (which we quote in the riemannian manifold setting although it remains true for $CAT(-1)$ spaces) instead of the (simpler) minimal current argument.

**Theorem 1.7.** Let $X$ be a Cartan Hadamard $n$-dimensional manifold whose sectional curvature satisfy $K \leq -1$, and $C$ a convex cocompact discrete subgroup of isometries of $X$ with limit set $\Lambda_C$. Let us assume that the topological dimension and the Hausdorff dimension (with respect to the Gromov distance on $\partial \tilde{X}$) of $\Lambda_C$ coincide and are equal to an integer $p$. Then, $C$
preseves a totally geodesic embedded copy of the real hyperbolic space $\mathbb{H}^{p+1}$, with $\partial \mathbb{H}^{p+1} = \Lambda_C$.

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2. Essential hypersurfaces

Let $\Gamma$ be a discrete cocompact group of isometries of a $n$-dimensional Cartan-Hadamard manifold $(\tilde{X}, \tilde{g})$ whose sectional curvature satisfies $K_{\tilde{g}} \leq -1$. Let us assume that the compact manifold $X = \tilde{X}/\Gamma$ is orientable. Let us also assume that $\Gamma = A \ast_C B$ is an amalgamated product of its subgroups $A$ and $B$ over $C$.

We first reduce to the case where $[\Gamma : C]$ is infinite. Namely, if $[\Gamma : C] < \infty$, then the critical exponent $\delta(C) = \delta(\Gamma) \geq n - 1$, and the equality in theorem 1.2 holds.

We then can assume that $[\Gamma : C] = \infty$.

Lemma 2.1. Let $\Gamma = A \ast_C B$ be as above, with $[\Gamma : C] = \infty$. If neither $A$ nor $B$ equals $\Gamma$, then $H_{n-1}(\tilde{X}/C, \mathbb{Z}) \neq 0$.

Proof : The Mayer-Vietoris sequence coming from the decomposition $\Gamma = A \ast_C B$ writes cf. [6], Corollary 7.7,

$$H_n(\tilde{X}/C, \mathbb{Z}) \to H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z}) \to H_n(\tilde{X}/\Gamma, \mathbb{Z}) \to \ldots$$

$$\ldots \to H_{n-1}(\tilde{X}/C, \mathbb{Z}) \to \ldots$$

As $[\Gamma : C] = \infty$, $H_n(\tilde{X}/C, \mathbb{Z}) = 0$ thus, if $H_{n-1}(\tilde{X}/C, \mathbb{Z}) = 0$, we deduce from the Mayer-Vietoris sequence that $H_n(\tilde{X}/A, \mathbb{Z}) \oplus H_n(\tilde{X}/B, \mathbb{Z})$ is isomorphic to $H_n(\tilde{X}/\Gamma, \mathbb{Z})$. As $H_{n-1}(\tilde{X}/\Gamma, \mathbb{Z}) = \mathbb{Z}$, we then deduce that either $[\Gamma : A] = \infty$ and $B = \Gamma$, or $[\Gamma : B] = \infty$ and $A = \Gamma$. □

In fact in the sequel of the paper we will make use of a smooth essential hypersurface $Z$ in $\tilde{X}/C$.

Definition 2.2. A compact smooth orientable hypersurface $Z$ of an $n$-dimensional manifold $Y$ is essential in $Y$ if $i_*([Z]) \neq 0$ where $[Z] \in H_{n-1}(Z, \mathbb{R})$ denotes the fundamental class of $Z$ and $i_* : H_{n-1}(Z, \mathbb{R}) \to H_{n-1}(Y, \mathbb{R})$ the morphism induced by the inclusion $i : Z \to Y$.

The end of this section is devoted to finding such an hypersurface $Z$ in $\tilde{X}/C$.

Let us recall a few facts about amalgamated products and their actions on trees, following [14]. Let $\Gamma = A \ast_C B$ be an amalgamated products of
its subgroups $A$ and $B$ over $C$. Then, $\Gamma$ acts on a simplicial tree $\tilde{T}$ with a fundamental domain $T \subset \tilde{T}$ being a segment, i.e. an edge joining two vertices. Let us describe this tree $\tilde{T}$. There are two orbits of vertices $\Gamma v_A$ and $\Gamma v_B$, the stabilizer of the edge $v_A$ (resp. $v_B$) being $A_i$ (resp. $B$). There is one orbit of edges $\Gamma e_C$, the stabilizer of the edge $e_C$ being $C$. The fundamental domain $T$ can be chosen as the edge $e_C$ joining the two vertices $v_A$ and $v_B$. The set of vertices adjacent to $v_A$, (resp. $v_B$), is in one to one correspondence with $A/C$, (resp. $B/C$). Note that as neither $A$ nor $B$ are equal to $\Gamma$, then $[A : C] \neq 1$ and $[B : C] \neq 1$, therefore for an arbitrary point $t_0$ on the edge $e_C$ we see that $\tilde{T} - t_0$ is a disjoint union of two unbounded connected components. This fact will be used later on.

Let us consider a continuous $\Gamma$-equivariant map $\tilde{f} : \tilde{X} \to T$ where $T$ is the Bass-Serre tree associated to the amalgamation $\Gamma = A *_C B$. One regularizes $\tilde{f}$ such that it is smooth in restriction to the complementary of the inverse image of the set of vertices of $T$. Let $t_0$ a regular value of $\tilde{f}$ contained in that edge of $T$ which is fixed by the subgroup $C$ and define $\tilde{Z} = \tilde{f}^{-1}(t_0)$. $\tilde{Z}$ is a smooth orientable possibly not connected hypersurface in $\tilde{X}$, globally $C$-invariant. Let us write $Z = \tilde{Z}/C$. We will show $Z \subset \tilde{X}/C$ is compact and that one of the connected components of $Z$ is essential.

Lemma 2.3. $Z \subset \tilde{X}/C$ is compact.

Proof: Let us show that for any sequence $z_n \in \tilde{Z}$, there exists a subsequence $z_{n_k}$ and $\gamma_k \in C$ such that $\gamma_k z_{n_k}$ converges. As $\Gamma$ is cocompact, there exists $g_n \in \Gamma$ such that the set $(g_n z_n)$ is relatively compact. Let $g_n z_{n_k}$ a subsequence which converges to a point $z \in \tilde{X}$. By continuity, the sequence $f(g_n z_{n_k})$ converges to $f(z)$, and by equivariance we get

$$g_n \tilde{f}(z_{n_k}) = g_n t_0 \to \tilde{f}(z)$$

when $k$ tends to $\infty$. As $\Gamma$ acts in a simplicial way on the tree $T$ and transitively on the set of edges, the sequence $g_n t_0$ is stationary, i.e. $g_n t_0 = t_0' = g t_0$ for $k$ large enough. Thus $g^{-1} g_n = \gamma_k \in C$ for $k$ large enough since it fixes $t_0$ and $\gamma_k z_{n_k} = g^{-1} g_n z_{n_k}$ converges to $g^{-1}(z)$ $\square$

The smooth compact hypersurface $Z$ we constructed might be not connected. Let us write $Z = Z_1 \cup Z_2 \cup \ldots \cup Z_k$ where the $Z_j$’s are the connected components of $Z$. Each $Z_j$ is a compact smooth oriented hypersurface of $\tilde{X}/C$.

The aim of what follows is to prove that at least one component $Z_i$ of $Z$ is essential.

Lemma 2.4. There exists $i \in [1, k]$ such that $Z_i$ is essential in $\tilde{X}/C$.

Proof: If there exists a $Z_i$ which doesn’t separate $\tilde{X}/C$ in two connected components, then $Z_i$ is essential in $\tilde{X}/C$. So we can assume that every $Z_j$, $j = 1, \ldots k$, does separate $\tilde{X}/C$ in two connected components. In that case
we will show that there exists a $Z_i$ which separates $\tilde{X}/C$ in two unbounded connected components which easily implies that $Z_i$ is essential.

Let us denote $U_l$, $l = 1, 2, \ldots, p$, the connected components of $\tilde{X}/C - \bigcup_{j=1}^k \tilde{Z}_j$.

Claim : at least two components $U_m, U_{m'}$ are unbounded.

Assuming the claim let us finish the proof of the lemma. For each $Z_j$ we denote $V_j$, $V_j'$ the two connected components of $\tilde{X}/C - Z_j$. Then $U_m = W_1 \cap W_2 \cap \ldots \cap W_k$ where for each $j$, $W_j = V_j$ or $W_j = V_j'$. In the same way, $U_{m'} = W_1' \cap W_2' \cap \ldots \cap W_k'$. As $U_m \cap U_{m'} = \emptyset$, there exists $i \in [1, k]$ such that $W_i \cap W_i' = \emptyset$, thus $U_m \subset V_i$ and $U_{m'} \subset V_i'$ or $U_m \subset V_i'$ and $U_{m'} \subset V_i$ so $Z_i$ separates $\tilde{X}/C$ into two unbounded components. This proves the lemma.

Let us prove the claim.

We have already noticed that $T - \{t_0\}$ is the disjoint union of two unbounded connected components $T_1$ and $T_2$. As $C$ acts on $T$ isometrically and simplicially then $T/C - \{t_0\} = T_1/C \cup T_2/C$ is the disjoint union of two unbounded connected components. Let $\bar{f} : \tilde{X}/C \to T/C$ the quotient map of $\bar{f}$. For each component $U_i$, we have $\bar{f}(U_i) \subset T_1/C$ or $\bar{f}(U_i) \subset T_2/C$, thus we can conclude the claim because $\bar{f}$ is onto $\square$

Let $\pi : \tilde{X} \to \tilde{X}/C$ be the natural projection. For any $i = 1, 2, \ldots, k$, let us denote $\{\tilde{Z}_i\}_{j \in J}$ the set of connected components of $\tilde{Z}_i := \pi^{-1}(Z_i)$.

For each $i \in [1, k]$, we claim that $C$ acts transitively on the set $\{\tilde{Z}_i\}_{j \in J}$. Namely, let us consider $\tilde{Z}_i$, $\tilde{Z}_i'$, $\tilde{z} \in \tilde{Z}_i$, $\tilde{z}' \in \tilde{Z}_i'$, and write $z = \pi \tilde{z} \in Z_i$ and $z' = \pi \tilde{z}' \in Z_i$. Let $\alpha$ be a continuous path on $Z_i$ such that $\alpha(0) = z$ and $\alpha(1) = z'$, and $\hat{\alpha}$ the lift of $\alpha$ such that $\hat{\alpha}(0) = \tilde{z}$. We have $\pi \hat{\alpha}(1) = z'$ and therefore $\tilde{c} \tilde{Z}_i = \tilde{Z}_i'$. $\square$

Let us denote $C_i^j$ the stabilizer of $\tilde{Z}_i$, and $Z_i^j = \tilde{Z}_i^j/C_i^j \subset \tilde{X}/C_i^j$. Let us write $p : \tilde{X}/C_i^j \to \tilde{X}/C$ the natural projection.

**Lemma 2.5.** The restriction of $p$ to $Z_i^j$ is a diffeomorphism onto $Z_i$. In particular, $Z_i^j$ is compact.

**proof :** Let $z$ and $z'$ be two points in $Z_i^j$ such that $p(z) = p(z')$. Let $\tilde{z}$ and $\tilde{z}'$ be lifts of $z$ and $z'$ in $\tilde{X}$. These two points $\tilde{z}$ and $\tilde{z}'$ which are in $\tilde{Z}_i$ actually belong to the same connected component $\tilde{Z}_i$ because for $j \neq j'$, $\tilde{Z}_i/C_i^j \cap \tilde{Z}_i/C_i^{j'} = \emptyset$. As $p(z) = p(z')$, there exits $c \in C$ such that $z' = c \tilde{z}$, and thus $c \in C_i^j$, and $z = z'$, therefore the restriction of $p$ to $Z_i^j$ is injective. The surjectivity comes from the fact that $\pi^{-1}Z_i = \bigcup_{j \in J} \tilde{Z}_i^j$ and $C$ acts transitively on the set $\{\tilde{Z}_i^j\}_{j \in J}$. $\square$

Let us consider the integer $i \in [1, k]$ as in lemma 2.4, i.e. such that $Z_i \hookrightarrow \tilde{X}/C$ is essential, and choose $\tilde{Z}_i^j$ one component of $\pi^{-1}(Z_i)$. 

After possibly replacing $C_l$ by an index two subgroup, we may assume that $C_l$ globally preserves each of the two connected components $U_l^i$ and $V_l^i$ of $\tilde{X} - \tilde{Z}_l^i$.

**Lemma 2.6.** Let $i$, $l$ and $C_l^i$ be chosen as above. The compact hypersurface $Z_l^i = \tilde{Z}_l^i/C_l^i$ is essential in $\tilde{X}/C_l^i$. Moreover the two connected components $U_l^i/C_l^i$ and $V_l^i/C_l^i$ of $\tilde{X}/C_l^i - Z_l^i$ are unbounded.

**Proof:** Let us consider $p : \tilde{X}/C_l^i \to \tilde{X}/C_i$. By lemma 2.5, the restriction of $p$ to $Z_l^i$ is a diffeomorphism onto $Z_l^i$, therefore $Z_l^i$ is essential in $\tilde{X}/C_l^i$ because $Z_l^i$ is essential in $\tilde{X}/C_i$. As $C_l^i$ preserves $U_l^i$ and $V_l^i$, $Z_l^i$ separates $\tilde{X}/C_l^i$ into two connected components $U_l^i/C_l^i$ and $V_l^i/C_l^i$, and as $Z_l^i$ is essential in $\tilde{X}/C_l^i$, $U_l^i/C_l^i$ and $V_l^i/C_l^i$ are unbounded. □

In the sequel of the paper we will denote $\tilde{Z}' = \tilde{Z}_l^i$, $C' = C_l^i$ and $Z' = Z_l^i = \tilde{Z}_l^i/C_l^i$.

### 3. Isosystolic inequality

In this section we summarize facts and results due to M. Gromov. Let $Z$ be a $p$-dimensional compact orientable manifold and $i : Z \hookrightarrow Y$ an embedding of $Z$ into $Y$ where $Y$ is an aspherical space. We suppose that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the morphism induced by the embedding $Z \hookrightarrow Y$. Let us fix a riemannian metric $g$ on $Z$. For each $z \in Z$ we consider the set $C_z$ of those loops $\alpha$ at $z$ such that $i \circ \alpha$ is homotopically non trivial in $Y$.

Let us define the systole of $(Z, g, i)$ at the point $z$ by

**Definition 3.1.** $\text{sys}_i(Z, g, z) = \inf \{\text{length}(\alpha), \alpha \in C_z\}$

and the systole of $(Z, g, i)$ by

**Definition 3.2.** $\text{sys}(Z, g) = \inf \{\text{sys}_i(Z, g, z), z \in Z\}$.

The following isosystolic inequality, due to M. Gromov says that the volume of any essential submanifold $Z$ of an aspherical space $Y$ relatively to any riemannian metric on $Z$ is universally bounded below by it’s systole.

**Theorem 3.3.** There exists a constant $C_p$ such that for each $p$-dimensional riemannian manifold $(Z, g)$ and any embedding $Z \hookrightarrow Y$ into an aspherical space $Y$ such that $i_*([Z]) \neq 0$ where $i_* : H_p(Z, \mathbb{R}) \to H_p(Y, \mathbb{R})$ is the induced morphism in homology, then $\text{vol}_p(Z, g) \geq C_p(\text{sys}(Z, g))^p$.

We will apply this volume estimates to the essential hypersurface $i : Z \hookrightarrow \tilde{X}/C$ that we constructed in lemma 2.6.

The following lemma is immediate.
Lemma 3.4. Let \( C \) be a discrete group acting on a simply connected manifold \( \tilde{X} \), \( \tilde{Z} \) a \( C \)-invariant hypersurface of \( \tilde{X} \) and \( i : Z = \tilde{Z}/C \hookrightarrow \tilde{X}/C \) the natural inclusion. Let \( g \) any riemannian metric on \( Z \) and \( \tilde{g} \) the lift of \( g \) to \( \tilde{Z} \). Then, for any \( z \in Z \) we have,

\[
sys_i(Z, g, z) = \inf \{ d_{\tilde{g}}(\tilde{z}, \gamma \tilde{z}), \gamma \in C \}
\]

where \( \tilde{z} \in \tilde{Z} \) is a lift of \( z \in Z \) and \( d_{\tilde{g}} \) is the distance induced by \( \tilde{g} \) on \( \tilde{Z} \).

Proof: Let \( \alpha \in C \) a loop based at \( z \in Z \). As \( i \circ \alpha \) is an homotopically non trivial loop at \( i(z) = z \) in \( \tilde{X}/C \), its lift \( \tilde{i} \circ \alpha \) at some \( \tilde{z} \in \tilde{Z} \) ends up at \( \gamma \tilde{z} \) for some \( \gamma \in C \).

4. Volume of hypersurfaces in \( \tilde{X}/C \)

Let \((\tilde{X}, \tilde{g})\) be a \( n \)-dimensional Cartan-Hadamard manifold whose sectional curvature satisfies \( K_{\tilde{g}} \leq -1 \) and \( C \) a discrete group of isometries of \((\tilde{X}, \tilde{g})\). We assume that the group \( C \) is non elementary, namely \( C \) fixes neither one nor two points in the geometric boundary \( \partial \tilde{X} \) of \((\tilde{X}, \tilde{g})\).

The aim of this section is to construct a map \( F : \tilde{X}/C \to \tilde{X}/C \) such that for any compact hypersurface \( Z \) of \( \tilde{X}/C \), we have

\[
vol_{n-1}(F(Z)) \leq \left( \frac{\delta + 1}{n-1} \right)^{n-1} vol_{n-1}(Z)
\]

where \( \delta \) is the critical exponent of \( C \) and \( vol_{n-1}(Z) \) stands for the \((n-1)\)-dimensional volume of the metric on \( Z \) induced from \( g \). For every subgroup \( C' \subset C \) and any hypersurface \( Z' \) of \( \tilde{X}/C' \) the lift \( F' : \tilde{X}/C' \to \tilde{X}/C' \) of \( F \) will also verify

\[
vol_{n-1}(F'(Z')) \leq \left( \frac{\delta + 1}{n-1} \right)^{n-1} vol_{n-1}(Z').
\]

In order to construct the map \( F \) we need a few preliminaries. We consider a finite positive Borel measure \( \mu \) on the boundary \( \partial \tilde{X} \) whose support contains at least two points. Let us fix an origin \( o \in \tilde{X} \) and denote \( B(x, \theta) \) the Busemann function defined for each \( x \in \tilde{X} \) and \( \theta \in \partial \tilde{X} \) by

\[
B(x, \theta) = \lim_{t \to \infty} \text{dist}(x, c(t)) - t
\]

where \( c(t) \) is the geodesic ray such that \( c(0) = o \) and \( c(+\infty) = \theta \).

Let \( D_\mu : \tilde{X} \to \mathbb{R} \) the function defined by

\[
D_\mu(y) = \int_{\partial \tilde{X}} e^{B(y, \theta)} d\mu(\theta)
\]

A computation shows that

\[
DdD_\mu(y) = \int_{\partial \tilde{X}} (DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta)) e^{B(y, \theta)} d\mu(\theta).
\]
When $K_g \leq -1$ the Rauch comparison theorem says that for every $y \in \tilde{X}$, and $\theta \in \partial \tilde{X}$,

\begin{equation}
DdB(y, \theta) + DB(y, \theta) \otimes DB(y, \theta) \geq \tilde{g}.
\end{equation}

We then get

\begin{equation}
DdD\mu(y) \geq \mu(y)\tilde{g},
\end{equation}

thus $DdD\mu(y)$ is positive definite and $D\mu$ is strictly convex.

**Lemma 4.1.** We have $\lim_{y_k \to \partial \tilde{X}} D\mu(y) = +\infty$.

**Proof:** Let $y_k \in \tilde{X}$ a sequence such that

\begin{equation}
\lim_{k \to \infty} y_k = \theta_0 \in \partial \tilde{X}.
\end{equation}

As $\text{supp}(\mu) \cap (\partial \tilde{X} - \{\theta_0\}) \neq \emptyset$, there exists a compact subset $K \subset \partial \tilde{X} - \{\theta_0\}$ such that $\mu(K) > 0$ thus,

\begin{equation}
\int_{\partial \tilde{X}} e^{B(y_k, \theta)} d\mu \geq \int_{K} e^{B(y_k, \theta)} d\mu \to +\infty.
\end{equation}

\hfill \Box

**Corollary 4.2.** Let $\mu$ a finite borel measure on $\partial \tilde{X}$ whose support contains at least two points. The function $D\mu$ has a unique minimum. This minimum will be denoted by $C(\mu)$.

Let us now consider some discrete subgroup $C \subset Isom(\tilde{X}, \tilde{g})$. Recall that a family of Patterson measures $(\mu_x)_{x \in \tilde{X}}$ associated to $C$ is a set of positive finite measures $\mu_x$ on $\partial \tilde{X}$, $x \in \tilde{X}$, such that the following holds for all $x \in \tilde{X}$, $\gamma \in C$,

\begin{equation}
\mu_{\gamma x} = \gamma_* \mu_x
\end{equation}

\begin{equation}
\mu_x = e^{-\delta B(x, \theta)} \mu_o,
\end{equation}

where $o \in \tilde{X}$ is a fixed origin, $B$ the Busemann function associated to $o$ and $\delta$ the critical exponent of $C$.

We assume now that $\text{supp}(\mu_o)$ contains at least two points and define the map $\tilde{F} : \tilde{X} \to \tilde{X}$ for $x \in \tilde{X}$ by

\begin{equation}
\tilde{F}(x) = C(e^{-B(x, \theta)} \mu_x).
\end{equation}

Here are a few notations. For a subspace $E$ of $T_x \tilde{X}$, we will write $\text{Jac}_E \tilde{F}(x)$ the determinant of the matrix of the restriction of $D\tilde{F}(x)$ to $E$ with respect to orthonormal bases of $E$ and $D\tilde{F}(x)E$. For an integer $p$,
we denote by $Jac_p\tilde{F}(x)$ the supremum of $|Jac_E\tilde{F}(x)|$ as $E$ runs through the
set of $p$-dimensional subspaces of $T_x\tilde{X}$.

**Lemma 4.3.** The map $\tilde{F}$ is smooth, homotopic to the Identity and verifies
for all $x \in \tilde{X}$, $\gamma \in C$ and $p \in [2, n = \dim(X)]$,

$(i)$ $\tilde{F}(\gamma x) = \gamma \tilde{F}(x)$

$(ii)$ $|Jac_p\tilde{F}(x)| \leq \left(\frac{(\delta + 1)^p}{p}\right)^p$.

**Proof :**

The map

$$(x, y) \rightarrow \int_{\partial X} e^{B(y, \theta) - B(x, \theta)} d\mu_x(\theta) = \int_{\partial X} e^{B(y, \theta) - (\delta + 1)B(x, \theta)} d\mu_\gamma(\theta)$$

is smooth because $y \rightarrow B(y, \theta)$ is smooth.

For all $x$ the map $y \rightarrow \int_{\partial X} e^{B(y, \theta) - (\delta + 1)B(x, \theta)} d\mu_x(\theta)$ is strictly convex by
(4.4) and tends to infinity when $y$ tend to $\partial X$ (cf. lemma 4.1), thus the unique minimum $\tilde{F}(x)$ is a smooth function. The equivariance of $\tilde{F}$ comes from the cocycle relation $B(\gamma y, \gamma \theta) - B(\gamma x, \gamma \theta) = B(y, \theta) - B(x, \theta)$.

For each $x \in \tilde{X}$ let $c_x$ be the geodesic in $\tilde{X}$ such that $c_x(0) = x$, $c_x(1) = \tilde{F}(x)$ and which is parametrized with constant speed. The map $\tilde{F}_t : X \rightarrow \tilde{X}$ defined by $\tilde{F}_t(x) = c_x(t)$ is a $C$-equivariant homotopy between $Id_\tilde{X}$ and $\tilde{F}$.

It remains to prove (ii).

The point $\tilde{F}(x)$ is characterized by

$(4.10)$

$$\int_{\partial X} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0.$$  

In order to simplify the notations we will write $B(x, \theta)$ instead of $B(x, \theta)$
and we will denote $\nu_x$ the measure $e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x$. We will also write
$D\tilde{F}(u)$ instead of $D\tilde{F}(x)(u)$.

The differential of $\tilde{F}$ is characterized by the following: for $u \in T_{\tilde{F}(x)}\tilde{X}$ and $v \in T_{\tilde{F}(x)}\tilde{X}$, one has

$$(4.11) \int_{\partial X} DB(\tilde{F}(x), \theta)(v) DB(\tilde{F}(x), \theta)(D\tilde{F}(u)) d\nu_x(\theta) = (\delta + 1) \int_{\partial X} DB(\tilde{F}(x), \theta)(v) DB(\tilde{F}(x), \theta)(u) d\nu_x(\theta).$$

We define the quadratic forms $k$ and $h$ for $v \in T_{\tilde{F}(x)}\tilde{X}$ by

$(4.12) k(v, v) = \int_{\partial X} DB(\tilde{F}(x), \theta)(v) + (DB(\tilde{F}(x), \theta)(v))^2 d\nu_x(\theta).$
and

\begin{equation}
(4.13) \quad h(v, v) = \int_{\partial \hat{X}} DB(\tilde{f}(x, \theta))(v)^2 d\nu_x(\theta).
\end{equation}

The relation (4.11) writes, for \( u \in T_x\hat{X} \) and \( v \in T_{\tilde{f}(x)}\hat{X} \):

\begin{equation}
(4.14) \quad k(D\tilde{f}(u), v) = (\delta + 1) \int_{\partial \hat{X}} DB(\tilde{f}(x, \theta))(v)DB(x, \theta)(u) d\nu_x(\theta).
\end{equation}

We defines the quadratic form \( h' \) on \( T_x\hat{X} \) for \( u \in T_x\hat{X} \) by

\begin{equation}
(4.15) \quad h'(u, u) = \int_{\partial \hat{X}} DB(x, \theta)(u)^2 d\nu_x(\theta),
\end{equation}

and one derives from (4.14)

\begin{equation}
(4.16) \quad |k(D\tilde{f}(x)(u), v)| \leq (\delta + 1)h(v, v)^{1/2}h'(u, u)^{1/2}.
\end{equation}

One now can estimate \( \text{Jac}_p\tilde{F}(x) \). Let \( P \subset T_x\hat{X}, \dim P = p \). If \( D\tilde{F}(P) \) has dimension lower than \( p \), then there is nothing to be proven. Let us assume that \( \dim D\tilde{F}(P) = p \). Denote by the same letters \( H' \) [resp. \( H \) and \( K \)] the selfadjoint operators (with respect to \( \tilde{g} \)) associated to the quadratic forms \( h' \) [resp. \( h \), \( k \)] restricted to \( P \) [resp. \( D\tilde{F}(P) \)].

Let \( (v_i)_{i=1}^p \) an orthonormal basis of \( D\tilde{F}(P) \) which diagonalizes \( H \) and \( (u_i)_{i=1}^p \) an orthonormal basis of \( P \) such that the matrix of \( K \circ D\tilde{F}(x): P \rightarrow D\tilde{F}(P) \) is triangular. Then,

\begin{equation}
(4.17) \quad \det K \cdot |\text{Jac}_p\tilde{F}(x)| \leq (\delta + 1)^p(\Pi_{i=1}^p h(v_i, v_i)^{1/2})(\Pi_{i=1}^p h'(u_i, u_i)^{1/2})
\end{equation}

thus,

\begin{equation}
(4.18) \quad \det K \cdot |\text{Jac}_p\tilde{F}(x)| \leq (\delta + 1)^p \left( \frac{\text{Trace}H}{p} \right)^{p/2} \left( \frac{\text{Trace}H'}{p} \right)^{p/2}.
\end{equation}

In these inequalities one can normalize the measures

\[ \nu_x = e^{B(\tilde{f}(x), \theta) - B(x, \theta)} \mu_x \]

such that their total mass equals one, which gives

\begin{equation}
(4.19) \quad \text{trace}H = \Sigma_{i=1}^p h(v_i, v_i) \leq 1,
\end{equation}

the last inequality coming from the fact that for all \( \theta \in \partial \hat{X} \),

\begin{equation}
(4.20) \quad \Sigma_{i=1}^p DB(\tilde{f}(x), \theta)(v_i)^2 \leq ||\nabla B(\tilde{f}(x), \theta)||^2 = 1
\end{equation}

and from the previous normalization.
Similarly,

\[(4.21) \quad \text{trace} H' = \sum_{i=1}^p h'(u_i, u_i) \leq 1.\]

We then obtain with (4.18)

\[(4.22) \quad \det K. |\text{Jac}_{P} \tilde{F}(x)| \leq \left( \frac{\delta + 1}{p} \right)^p.\]

Thanks to (4.3), we have \(\det K \geq 1\), so

\[(4.23) \quad |\text{Jac}_{P} \tilde{F}(x)| \leq \left( \frac{\delta + 1}{p} \right)^p.\]

We get (ii) by taking the supremum in \(P \). □

As the map \(\tilde{F} : \tilde{X} \to \tilde{X}\) is \(C\)-equivariant, then for every subgroup \(C' \subset C\), \(\tilde{F}\) gives rise to a map \(F' : \tilde{X}/C' \to \tilde{X}/C'\) and so does the homotopy \(\tilde{F}_t\) between \(\tilde{F}\) and \(\text{Id}_{\tilde{X}}\).

**Corollary 4.4.** The map \(F' : \tilde{X}/C' \to \tilde{X}/C'\) is homotopic to the Identity map and verifies for all \(x \in \tilde{X}/C'\) and \(p \in [2, n = \text{dim} X]\)

\[|\text{Jac}_{P} F_p(x)| \leq \left( \frac{\delta + 1}{p} \right)^p.\]

Let \(C \subset \text{Isom} (\tilde{X}, \tilde{g})\) as above, ie such that the support of the Patterson-Sullivan measures contains at least two points and with critical exponent \(\delta\).

Let \(C' \subset C\) be a subgroup.

Let us consider an compact hypersurface \(Z' \subset \tilde{X}/C'\).

Denote \(F^{\otimes k} = F^i \circ F^i \circ \ldots \circ F^i\) the composition of \(F^i\) \(k\)-times.

Let us write \(g_k = (F^{\otimes k})^* g\), where \(g\) is the metric on \(\tilde{X}/C'\) induced by \(\tilde{g}\).

The symmetric 2-tensor \(g_k\) may not be a riemannian metric on \(\tilde{X}/C'\) nor its restriction to \(Z'\), so we have to modify it. For \(\epsilon > 0\), the following symmetric 2-tensor \(g_{\epsilon,k}\) is a riemannian metric on \(\tilde{X}/C'\) and so is its restriction \(h_{\epsilon,k}\) to \(Z'\).

\[(4.24) \quad g_{\epsilon,k} = g_k + \epsilon^2 g.\]

**Lemma 4.5.** Let \(h_{\epsilon,k}\) be the restriction of \(g_{\epsilon,k}\) to the hypersurface \(Z'\) and \(g_{Z'}\) the restriction of \(g\) to \(Z'\). Let \(\Phi_{\epsilon,k} : Z' \to \mathbb{R}\) the density defined for all \(x \in Z'\) by \(d\nu_{h_{\epsilon,k}}(x) = \Phi_{\epsilon,k}(x) d\nu_{g_{Z'}}(x)\). For any sequence \(\epsilon_k\) such that \(\lim_{k \to \infty} \epsilon_k = 0\), there exists a sequence \(\epsilon'_k\), \(\lim_{k \to \infty} \epsilon'_k = 0\), such that for all \(x \in Z'\),

\[0 < \Phi_{\epsilon'_k, k}(x) \leq |\text{Jac}_{n-1} F^{\otimes k}(x)| + \epsilon_k.\]

In particular,

\[\Phi_{\epsilon'_k, k}(x) \leq \left( \frac{\delta + 1}{n-1} \right)^{k(n-1)} + \epsilon_k.\]
and
\[ \text{vol}(Z', h'_{\epsilon, k}) \leq \left[ \left( \frac{\delta + 1}{n - 1} \right)^{k(n-1)} + \epsilon_k \right] \text{vol}(Z', g'_{Z'}). \]

**Corollary 4.6.** Under the above assumptions, if \( \delta < n - 2 \) there exists a sequence \( \epsilon_k' \) such that \( \lim_{k \to \infty} \epsilon_k' = 0 \), and \( \lim_{k \to \infty} \text{vol}(Z', h'_{\epsilon, k}) = 0 \).

**Proof of lemma 4.5:**
Let us fix \( k \) an integer. Let \( x \in Z' \) and \( u \in T_x Z' \). We have \( g_{e,k}(u, u) = h_{e,k}(u, u) = g(DF^k(x)(u), DF^k(x)(u)) + \epsilon^2 g(u, u) \) thus \( h_{e,k}(u, u) = g(A_{x, e} u, u) \) where \( A_{x, e} \in \text{End}(T_x Z') \) is the self-adjoint operator \( A_x = DF^k(x)^* \circ DF^k(x) + \epsilon^2 I_d \), with \( DF^k(x)^* \) the adjoint of \( DF^k(x) : (T_x Z', g(x)) \to (DF^k(x)(T_x Z'), g(F^k(x))) \).

By compactness of \( Z' \) and continuity of \( A_{x, e} \), there exist \( \epsilon'_k \) such that
\[ \Phi_{\epsilon'_k, k}(x) = \text{det} A^{1/2}_{x, \epsilon'_k} \leq \text{det} A_{x, 0} + \epsilon_k, \]
thus
\[ \Phi_{\epsilon'_k, k}(x) \leq |\text{Jac}_{n-1} F^k(x)| + \epsilon_k. \]

The lemma then follows from Corollary 4.4. \( \square \)

**5. Proof of Theorem 1.2**

This section is devoted to the proof of the Theorem 1.2. Let \( \Gamma \) be a discrete cocompact group of isometries of a \( n \)-dimensional Cartan-Hadamard manifold \((\bar{X}, \bar{g})\) whose sectional curvature satisfies \( K_{\bar{g}} \leq -1 \). We assume that \( \Gamma = \Gamma \ast C. B \). At the end of section 2 we constructed a subgroup \( C' \subset C \) and an orientable hypersurface \( \tilde{Z}' \subset \bar{X} \) such that \( C', \tilde{Z}' = \tilde{Z}' \) and \( Z' = \tilde{Z}' / C' \) is compact in \( \bar{X} / C' \). Moreover \( Z' \) is essential in \( \bar{X} / C' \) i.e \( i_* ([\tilde{Z}']) \neq 0 \) where \( i_* : H_{n-1}(\tilde{Z}', \mathbb{R}) \to H_{n-1}(\bar{X} / C', \mathbb{R}) \) is the morphism induced on homology groups by the inclusion \( i : Z' \to \bar{X} / C' \) and \([\tilde{Z}']\) the fundamental class of \( Z' \).

**5.1 Proof of the inequality**

We now prove the inequality in the theorem 1.2. Let us assume that \( \delta < n - 2 \) and derive a contradiction. Let \( h'_{\epsilon, k} \) the sequence of metric defined on \( Z' \) in lemma 4.5, then by corollary 4.6 we have
\[ \lim_{k \to \infty} \text{vol}(Z', h'_{\epsilon, k}) = 0. \]

We now show that the systole of the metric \( h'_{\epsilon, k} \) on \( Z' \) is bounded below independently of \( k \). Recall that the systole of \( i : Z' \to \bar{X} / C' \) at a point \( z \in Z' \) with respect to a metric \( h_{e,k} \) can be defined by
\[ \text{syst}_{i}(Z', h_{e,k}, z) = \inf_{\gamma \in C \ast \tilde{C}} \text{dist}_{(\tilde{Z}', h_{\tilde{e},k})}(\tilde{z}, \gamma \tilde{z}) \]
where \( \tilde{z} \) is any lift of \( z \) and \( \tilde{h}_{e,k} \) the lift on \( \tilde{Z}' \) of \( h_{e,k} \) (cf. lemma 3.4).
Let \( \alpha(t) \) be a minimizing geodesic between \( \tilde{z} \) and \( \gamma \tilde{z} \) on \((Z', \tilde{h}_{\epsilon,k})\). By definition of \( \tilde{h}_{\epsilon,k} \) we have

\[
(5.3) \quad \text{dist}(\tilde{Z}', \tilde{h}_{\epsilon,k})(\tilde{z}, \gamma \tilde{z}) \geq l_{\tilde{g}}(\tilde{F}_k \circ \alpha)
\]

where \( l_{\tilde{g}} \) stands for the length with respect to \( \tilde{g} \) on \( \tilde{X} \).

We get

\[
(5.4) \quad \text{dist}(\tilde{Z}', \tilde{h}_{\epsilon,k})(\tilde{z}, \gamma \tilde{z}) \geq \text{dist}(\tilde{X}, \tilde{g})(\tilde{F}_k(\tilde{z}), \gamma \tilde{F}_k(\tilde{z})) \geq \rho
\]

where \( \rho \) is the injectivity radius of \( \tilde{X}/C' \).

We then have

\[
(5.5) \quad \text{sys}(Z', h_{\epsilon,k}) \geq \rho,
\]

and thanks to the Theorem 3.3 we obtain

\[
(5.6) \quad \text{vol}(Z', h_{\epsilon',k}) \geq C_n \rho^{n-1}
\]

which contradicts (5.1).

\( \square \)

5.2 Proof of the equality case

There will be several steps.

**Step 1:** The limit set of \( C \) is contained in a topological equator.

**Step 2:** The weak tangent to \( \partial \tilde{X} \) and \( \Lambda_{C'} \).

**Step 3:** The limit set \( \Lambda_{C'} \) of \( C' \) and the limit set \( \Lambda_C \) of \( C \) are equal to a topological equator.

**Step 4:** \( C' \) and \( C \) are convex cocompact.

**Step 5:** \( C \) preserves a copy of the real hyperbolic space \( \mathbb{H}^{n-1}_\mathbb{R} \) totally geodesically embedded in \( \tilde{X} \).

**Step 6:** Conclusion

**Step 1:** The limit set \( \Lambda_C \) of \( C \) is contained in a topological equator.

Let \( x \in \tilde{X} \) and \( E \subset T_x \tilde{X} \) a codimension one subspace. For each \( u \in T_x \tilde{X} \), \( \tilde{g}(u, u) = 1 \), one considers the geodesic \( c_u \) defined by \( c_u(0) = x \) and \( \dot{c}_u(0) = u \). We define the equator \( E(\infty) \) associated to \( E \) as the subset of \( \partial \tilde{X} \)

\[
(5.7) \quad E(\infty) = \{ c_u(+\infty) \mid u \in E \}
\]
Our goal is to prove the existence of a point $x \in \tilde{X}$ and an hyperplane $E \subset T_x \tilde{X}$ such that the limit set $\Lambda_C$ satisfies $\Lambda_C \subset E(\infty)$.

Recall that $C' \subset C$ globally preserves an hypersurface $\tilde{Z}'$ such that $\tilde{Z}'/C' \subset \tilde{X}/C'$ is compact and essential.

Let us also recall that we have constructed a $C$-equivariant map $\tilde{F} : \tilde{X} \to \tilde{X}$ such that, for all $x \in \tilde{X}$,

$$|\text{Jac}_{n-1}\tilde{F}(x)| \leq \left(\frac{\delta + 1}{n - 1}\right)^{n-1} \tag{5.8}$$

where the critical exponent $\delta$ of $C$ satisfies $\delta = n - 2$, thus

$$|\text{Jac}_{n-1}\tilde{F}(x)| \leq 1. \tag{5.9}$$

The step 1 follows from the two following Propositions.

**Proposition 5.1.** Let $x \in \tilde{X}$ such that $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$. Then there exists $E \subset T_x \tilde{X}$ such that the limit set $\Lambda_C$ satisfies $\Lambda_C \subset E(\infty)$. Moreover, $\tilde{F}(x) = x$ and $D\tilde{F}(x)$ is the orthogonal projector onto $E$.

**Proposition 5.2.** There exists $x \in \tilde{X}$ such that $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$.

**Proof of Proposition 5.1**

Let $x \in \tilde{X}$ such that $|\text{Jac}_{n-1}\tilde{F}(x)| = 1$. By definition we have a subspace $E \subset T_x \tilde{X}$ such that $|\text{Jac}_E\tilde{F}(x)| = 1$. By (4.18) and $\det K \geq 1$ we have,

$$|\text{Jac}_E\tilde{F}(x)| \leq (n - 1)^{n-1} \left(\frac{\text{trace}H}{n - 1}\right)^{\frac{n-1}{2}} \left(\frac{\text{trace}H'}{n - 1}\right)^{\frac{n-1}{2}}$$

$$\leq (n - 1)^{n-1} \left(\frac{1}{n - 1}\right)^{n-1}. \tag{5.10}$$

In particular as $|\text{Jac}_E\tilde{F}(x)| = 1$, we have equality in the inequalities (5.10), thus, $\text{trace}H = \text{trace}(h) = 1$, and

$$H = \frac{1}{n - 1} Id_{D\tilde{F}(x)(E)}. \tag{5.11}$$

Let us recall that the quadratic form $h$ is defined by

$$h(v, v) = \int_{\partial \tilde{X}} DB(\tilde{F}(x), \theta)(v)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)$$

where $\mu_x$ is the Patterson-Sullivan measure of $C$ normalized by

$$\int_{\partial \tilde{X}} e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 1. \tag{5.12}$$
We then have

\[ 1 = \text{trace}(h) = \text{trace}H = \Sigma_{i=1}^{n-1} h(v_i, v_i) = \]

\[ = \int_{\partial X} \Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \]

\[ \leq \int_{\partial X} ||DB(\tilde{F}(x), \theta)||^2 e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \leq 1, \]

because \( \Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 \leq ||DB(\tilde{F}(x), \theta)||^2 = 1 \)

for all \( \theta \in \partial X \).

Therefore for \( \mu_x \)-almost all \( \theta \in \text{supp}(e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta)) = \text{supp}(\mu_x) \), we have

\[ (5.13) \quad \Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 = ||DB(\tilde{F}(x), \theta)||^2 = 1. \]

In (5.12), \( \Sigma_{i=1}^{n-1} DB(\tilde{F}(x), \theta)(v_i)^2 \) represents the square of the norm of the projection of \( \nabla B(\tilde{F}(x), \theta) \) on \( E \).

By continuity of \( B(x, \theta) \) in \( \theta \) one then gets \( \Lambda_C = \text{supp}(\mu_x) \subset E(\infty) \).

Let us now prove that \( \tilde{F}(x) = x \). When \( \text{Jac}\tilde{F}_E(x) = 1 \), we have equality in the Cauchy-Schwarz inequality (4.16), therefore for each \( i = 1, \ldots, n-1 \) and \( \theta \in \Lambda_C \) we get \( DB(\tilde{F}(x), \theta)(v_i) = DB(x, \theta)(u_i) \). Therefore we deduces from (5.13) that \( \nabla B(x, \theta) = \Sigma_{i=1}^{n-1} DB(x, \theta)(u_i) u_i \), which imply with (4.10) that

\[ (5.14) \quad \int_{\partial X} DB(x, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0. \]

On the other hand, as \( \int_{\partial X} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0 \) and \( H = \frac{1}{n-1} \int_{\partial X} DB(\tilde{F}(x), \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) \), the support of \( \mu_x \) cannot be just a pair of points, therefore the barycenter of the measure \( e^{B(\tilde{F}(x), \theta) - B(x, \theta)} \mu_x \) defined in (5.14) is well defined and characterized as the point \( z \in \tilde{X} \) such that

\[ \int_{\partial X} DB(z, \theta) e^{B(\tilde{F}(x), \theta) - B(x, \theta)} d\mu_x(\theta) = 0, \]

thus (5.14) and (4.10) imply \( x = \tilde{F}(x) \). \( \Box \)

**Proof of Proposition 5.2 :**

If we knew that there exists a minimizing hypersurface \( Z_0 \) in the homology class of \( Z \), then every points \( x \in Z_0 \) would verify \( |\text{Jac}_{n-1}\tilde{F}(x)| = 1 \). We unfortunately don’t know if there exists such a minimizing hypersurface nor a minimizing current in the homology class of \( Z \). Instead we will consider an \( L^2(\tilde{X}/C') \) harmonic \((n-1)\)-form dual to the homology class of \( Z \).

We need the following lemmas in order to prove the existence of such a dual form.
Let $\lambda_1(\tilde{X}/C')$ be the bottom of the spectrum of the Laplacian on $(\tilde{X}, \tilde{g})$, ie.

\begin{equation}
\lambda_1(\tilde{X}/C') = \inf_{u \in C_0^\infty(\tilde{X}/C')}\left\{ \int_{\tilde{X}/C'} |du|^2 \right\}.
\end{equation}

**Lemma 5.3.** Let $C \subset Isom(\tilde{X}, \tilde{g})$ a discrete group of isometries with critical exponent $\delta = n - 2$ where $(\tilde{X}, \tilde{g})$ is an $n$-dimensional Cartan-Hadamard manifold of sectional curvature $K_{\tilde{g}} \leq -1$. Then for any subgroup $C' \subset C$ we have $\lambda_1(\tilde{X}/C') \geq n - 2$.

**Proof:**

Thanks to a theorem of Barta, cf.\[17\] Theorem 2.1, the lemma boils down to finding a positive function $c : \tilde{X}/C' \to \mathbb{R}^+$ such that $\Delta c(x) \geq (n-2)c(x)$. Here, the laplacian $\Delta$ is the positive operator ie. $\Delta c = -\text{trace}Dc$. We consider the smooth function $\tilde{c} : \tilde{X} \to \mathbb{R}^+$ defined by $\tilde{c}(x) = \mu_x(\partial\tilde{X})$ where $\{\mu_x\}_{x \in \tilde{X}}$ is a family of Patterson-Sullivan measure of $C$. The function $\tilde{c}$ is $C$-equivariant therefore it defines a map $c : \tilde{X}/C' \to \mathbb{R}^+$ for any subgroup $C' \subset C$. Let us show

\begin{equation}
\Delta \tilde{c}(x) \geq \delta(n-1-\delta)c(x) = n-2.
\end{equation}

We have

\[\tilde{c}(x) = \int_{\partial\tilde{X}} e^{-\delta B(x, \theta)} d\mu_\alpha(\theta)\]

therefore

\[\Delta \tilde{c}(x) = \int_{\partial\tilde{X}} [-\delta \Delta B(x, \theta) - \delta^2] d\mu_\alpha(\theta).\]

The sectional curvature $K_{\tilde{g}}$ of $(\tilde{X}, \tilde{g})$ satisfies $K_{\tilde{g}} \leq -1$ we thus have $-\Delta B(x, \theta) \geq n - 1$ and as $\delta = n - 2$ we get

\[\Delta \tilde{c}(x) \geq [\delta(n-1) - \delta^2] \tilde{c}(x) = (n-2)\tilde{c}(x).\]

\[\square\]

The following lemma is due to G.Carron and E.Pedon, \[8\]. For a complete riemannian manifold $Y$, we denote $H^1_c(Y, \mathbb{R})$ the first cohomology group generated by differential forms with compact support.

**Lemma 5.4 (\[8\], Lemme 5.1).** Let $Y$ be a complete riemannian manifold all ends of whose having infinite volume and such that $\lambda_1(Y) > 0$, then the natural morphism

\[H^1_c(Y, \mathbb{R}) \to H^1_{L^2}(Y, \mathbb{R})\]

is injective. In particular any $\alpha \in H^1_c(Y, \mathbb{R})$ admits a representative $\bar{\alpha}$ which is in $L^2(Y, \mathbb{R})$. 
Corollary 5.5. Let $C'$ be as above and assume that there exists a compact essential hypersurface $Z' \subset \tilde{X}/C'$. Then there exists an harmonic $n-1$-form $\omega$ in $L^2(\tilde{X}/C')$ such that $\int_{Z'} \omega \neq 0$.

Proof:

Let $\alpha \in H^1_c(\tilde{X}/C', \mathbb{R})$ a Poincaré dual of $[Z'] \in H_{n-1}(\tilde{X}/C', \mathbb{R})$. By definition of $\alpha$, for any $\beta \in H^{n-1}(\tilde{X}/C', \mathbb{R})$, one has

\begin{equation}
\int_{Z'} \beta = \int_{\tilde{X}/C'} \beta \wedge \alpha,
\end{equation}

(5.17) p.51, note that $\tilde{X}/C'$ has a ” finite good cover”).

After Lemma 5.4, $\alpha$ admits a non trivial harmonic representative $\bar{\alpha}$ in $L^2(\tilde{X}/C')$. (In order to apply the Lemma 5.4, one has to check that all ends of $\tilde{X}/C'$ have infinite volume, ie for a compact $K \subset \tilde{X}/C'$ each unbounded connected component of $\tilde{X}/C' - K$ has infinite volume: this comes from the fact that the injectivity radius of $\tilde{X}/C'$ is bounded below by the injectivity radius of $X = \tilde{X}/T$ and the sectional curvature bounded above by $-1$.)

The $(n-1)$-harmonic form $\omega = *\bar{\alpha}$, where $*$ is the Hodge operator, is in $L^2(\tilde{X}/C')$ and verifies after (5.17)

\begin{equation}
\int_{Z'} \omega = \int_{\tilde{X}/C'} \omega \wedge \bar{\alpha} = \int_{\tilde{X}/C'} \omega \wedge *\omega = ||\omega||^2_{L^2(\tilde{X}/C')} \neq 0.
\end{equation}

(5.18)

We can now prove the proposition 5.2. Let us briefly describe the idea.

We consider the iterates $F^{\cdot k}$ of $F' : \tilde{X}/C' \to \tilde{X}/C'$. As $F'$ is homotopic to the identity map, $F^{\cdot k}(Z')$ is homologous to $Z'$ and if $\omega$ is the harmonic form of the corollary 5.5 we have

\begin{equation}
\int_{Z'} (F^{\cdot k})^* \omega = \int_{Z'} \omega = a \neq 0.
\end{equation}

(5.19)

We don’t know if $F^{\cdot k}(Z')$ converges or stays in a compact subset of $\tilde{X}/C'$ but we will show that $F^{\cdot k}(Z')$ cannot entirely diverge in $\tilde{X}/C'$ and that there exists a $z' \in Z'$ such that $F^{\cdot k}(z')$ subconverges to a point $x \in \tilde{X}/C'$ with $|J\text{act}_{n-1} F' (x)| = 1$.

From (5.19) one gets

\begin{equation}
0 < |a| = |\int_{Z'} (F^{\cdot k})^* (\omega)| \leq \int_{Z'} |J\text{act}_{Z'} F^{\cdot k}(z)| |\omega(F^{\cdot k}(z))| dz
\end{equation}

(5.20)

where

$|J\text{act}_{Z'} F^{\cdot k}(z)| = ||DF^{\cdot k}(z)(u_1) \wedge DF^{\cdot k}(z)(u_2) \wedge ... \wedge DF^{\cdot k}(z)(u_{n-1})||$

and $(u_1, ..., u_{n-1})$ is an orthonormal basis of $T_z(Z')$. 

Let us define $\mathcal{B} = \{ z \in Z', |\text{Jac}_Z(z)| \text{ does not converge to } 0 \}$. For $z \in Z'$ we define the sequence $z_k$ by $z_0 = z$ and $z_k = F'(z_{k-1}) = F^k(z) \in \tilde{X}/C'$.

**Lemma 5.6.** There exists $z \in \mathcal{B}$ and a subsequence $z_{k_j}$ such that $z_{k_j}$ converges to a point $x \in \tilde{X}/C'$ with $|\text{Jac}_{n-1}(x)| = 1$.

**Proof:**

We first remark that $\lim_{x \to \infty} ||\omega(x)|| = 0$. This follows the following facts: $\omega$ is harmonic, $\omega \in L^2(\tilde{X}/C')$ and the injectivity radius of $\tilde{X}/C'$ is bounded below by a positive constant.

Let us assume that for all $z \in \mathcal{B}$ the sequence $z_k$ diverges in $\tilde{X}/C'$. Then we have, for all $z \in \mathcal{B}$,

\begin{equation}
(5.21) \quad ||\omega(z_k)|| = ||\omega(F^{nk}(z))|| \to 0
\end{equation}

whenever $k$ tends to $\infty$ because of the previous remark.

On the other hand, as $||\omega(F^{nk}(z))|| \leq C$ and $|\text{Jac}_{n-1}F^k| \leq 1$, it follows from (5.20)

\begin{equation}
\lim_{k \to \infty} \int_{Z'} (F^{nk})^*(\omega) \leq 0
\end{equation}

which contradicts our assumption.

Thus there exists a point $z \in Z'$ such that

\begin{equation}
(5.22) \quad |\text{Jac}_{Z'}F^{nk}(z)| \to \alpha \neq 0
\end{equation}

and such that there exists a subsequence $z_{k_j} = F^{nk_j}(z)$ with

\begin{equation}
(5.23) \quad \lim_{j \to \infty} z_{k_j} = x \in \tilde{X}/C'.
\end{equation}

The property (5.22) comes from the fact that the sequence $|\text{Jac}_{Z'}F^{nk}(z)|$ doesn’t tend to zero and is decreasing (because $|\text{Jac}_{n-1}F^l| \leq 1$).

Let us define

\[ E_0 = T_zZ', E_1 = DF'(z)(E_0) \]

and

\[ E_k = DF'(z_{k-1})(E_{k-1}) \subset T_{z_k}(\tilde{X}/C'). \]

As $z_{k_j} \to x$ we can assume, after extracting again a subsequence, that $E_{k_{j}} \to E \subset T_x(\tilde{X}/C')$. On the other hand we also have

\begin{equation}
(5.24) \quad |\text{Jac}_{Z'}F^{nk}(z)| = |\text{Jac}_{E_{k-1}}F'(z_{k-1})||\text{Jac}_{E_{k-2}}F'(z_{k-2})|...|\text{Jac}_{E_0}F'(z)|
\end{equation}
We know that $|\text{Jac}_{E_k} F'(z_k)| = 1 - \epsilon_k$, where $0 \leq \epsilon_k < 1$. As $z \in B$, we have

$$\lim_{k \to \infty} \pi_{j=1}^k (1 - \epsilon_j) = \alpha > 0$$

therefore $\lim_{k \to \infty} \epsilon_k = 0$ and by continuity we have $|\text{Jac}_{E} F'(x)| = 1$. □

Now we can finish the proof of the step 1. Let consider a lift of $x$ in $\tilde{X}$ and $E$ in $T\tilde{X}$ that we again call $x$ and $E$. Then we have $|\text{Jac}_{E} \tilde{F}'(x)| = 1$. □

Let us remark that corollary 4.4 and (5.19) give another proof of the inequality $\delta \geq n - 2$, which does not use the isosystolic inequality, i.e. Theorem 3.3.

**Step 2 : The weak tangent of $\partial \tilde{X}$ and $\Lambda_C$**

We first recall the definition of the Gromov-distance on $\partial \tilde{X}$. For two arbitrary points $\theta$ and $\theta'$ in $\partial \tilde{X}$ let us define

$$(5.25) \quad l(\theta, \theta') = \inf\{t > 0/\text{dist}(\alpha_{\theta}(t), \alpha_{\theta'}(t)) = 1\}$$

and

$$(5.26) \quad d(\theta, \theta') = e^{-l(\theta, \theta')}$$

then $d$ is a distance on $\partial \tilde{X}$.

We now recall a few definitions following [4]. A complete metric space $(S, \bar{d})$ is a weak tangent of a metric space $(Z, d)$ if there exist a point $0 \in S$, a sequence of points $z_k \in Z$ and a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(Z, \lambda_k d, z_k)$ converges in the pointed Gromov-Hausdorff topology to $(S, \bar{d}, 0)$ where $(Z, \lambda_k d)$ stands for the set $Z$ endowed with the rescaled metric $\lambda_k d$.

Let us recall that the sequence of metric spaces $(Z_k, d_k, z_k)$ converges to $(S, \bar{d}, 0)$ in the pointed Gromov-Hausdorff topology if the following conditions hold, (cf. (B-B-I), definition 8.1.1).

**Definition 5.7.** We say that the sequence of metric spaces $(Z_k, d_k, z_k)$ converges to $(S, \bar{d}, 0)$ if for any $R > 0$, $\epsilon > 0$ there exists $k_0$ such that for any $k \geq k_0$ there exists a (non necessary continuous) map $f : B(z_k, R) \to S$ such that

(i) $f(z_k) = 0$,

for any two points $x$ and $y$ in $B(z_k, R)$,

(ii) $|\bar{d}(f(x), f(y)) - d_k(x, y)| \leq \epsilon$,

and

(iii) the $\epsilon$-neighborhood of the set $f(B(z_k, R))$ contains $B(0, R - \epsilon)$.

In the previous definition, $B(z_k, R)$ stands for the ball of radius $R$ centered at the point $z_k$ in $(Z_k, d_k)$. 
For a metric space \((Z, d)\) we will denote \(WT(Z, d)\) the set of weak tangents of \((Z, d)\).

Let \(\Gamma\) a cocompact group of isometries of \((\tilde{X}, \tilde{g})\) a \(n\)-dimensional Cartan Hadamard manifold of sectional curvature \(K_{\tilde{g}} \leq -1\) and \(C'\) a subgroup of \(\Gamma\). The limit set \(\Lambda_{\Gamma}\) of \(\Gamma\) is the full boundary \(\partial \tilde{X}\) namely a topological \((n-1)\)-dimensional sphere \(S^{n-1}\). We endow \(\partial \tilde{X}\) with the Gromov distance \(d\) defined in (5.26). In [4] Lemma 5.2, M.Bonk and B.Kleiner show among other properties the following

**Lemma 5.8.** For any weak tangent space \((S, \tilde{d})\) in \(WT((\partial \tilde{X}, d))\), \(S\) is homeomorphic to \(\partial \tilde{X}\) less a point, thus to \(\mathbb{R}^{n-1}\).

In fact the crucial assumption in the above lemma, coming from the co-compacness of \(\Gamma\), is the property that any triple of points in \(\partial \tilde{X}\) can be uniformly separated by an element of \(\Gamma\), i.e. there is \(\delta > 0\) such that for any three points \(\theta_1, \theta_2, \theta_3 \in \partial \tilde{X}\) there exists a \(\gamma \in \Gamma\) such that \(d(\gamma \theta_i, \gamma \theta_j) \geq \delta\) for all \(1 \leq i \neq j \leq 3\). Following the argument of M.Bonk and B.Kleiner one can show that if \(C'\) is a subgroup of \(\Gamma\) such that one weak tangent \((S, \tilde{d})\) of \((\Lambda_{C'}, d)\) is in \(WT(\partial \tilde{X}, d)\) and enough triples of points of \(\Lambda_{C'}\) can be uniformly separated by elements of \(C'\), then \(S\) is homeomorphic to \(\Lambda_{C'}\) less a point. In particular, \(\Lambda_{C'}\) is homeomorphic to \(\partial \tilde{X}\).

**Lemma 5.9.** Let \(\mathcal{L} \subset \partial \tilde{X}\) be a closed \(C'\)-invariant set and \(\theta_0 \in \mathcal{L}\). We assume that there exist a sequence of positive real numbers \(\lambda_k \to \infty\) such that the sequence of pointed metric spaces \((\mathcal{L}, \lambda_k d, \theta_0)\) converges in the pointed Gromov-Hausdorff topology to \((S, \tilde{d}, 0)\) where \((S, \tilde{d}, 0)\) is a weak tangent of \((\partial \tilde{X}, d)\). We also assume that there exist positive constants \(C\) and \(\delta\), a sequence of points \(\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{L}\) and a sequence of elements \(\gamma_k \in C'\) such that \(C^{-1} \lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C \lambda_k^{-1}\) and \(d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta\) for all \(0 \leq i \neq j \leq 2\). Then, \(S\) is homeomorphic to \(\mathcal{L}\) less a point. In particular \(\mathcal{L}\) is homeomorphic to \(\partial \tilde{X}\).

The proof of this lemma is postponed in the Appendix.

**Step 3 : The limit set \(\Lambda_{C'}\) of \(C'\) and the limit set \(\Lambda_{C}\) of \(C\) are equal to a topological equator.**

We have shown in step 1 that \(\Lambda_{C}\) is a subset of some topological equator \(E(\infty)\).

Let \(o \in \tilde{X}\) and \(E \subset T_o \tilde{X}\) be such that \(|Jac_{\tilde{E}} \tilde{F}(o)| = 1\) and \(E(\infty)\) is the equator associated to \(E\).

Recall that there exists a subgroup \(C'\) of \(C\) which globally preserves an hypersurface \(\tilde{Z}' \subset \tilde{X}\) and that \(\tilde{Z}'/C' \subset \tilde{X}/C'\) is compact. Furthermore \(\tilde{Z}'\) separates \(\tilde{X}\) into two connected components \(\tilde{U}\) et \(\tilde{V}\). We can assume that \(\tilde{U}\) and \(\tilde{V}\) are globally invariant by \(C'\) after having replaced \(C'\) by an index 2 subgroup.

The limit set \(\Lambda_{C'}\) of \(C'\) is contained in \(\Lambda_{C}\), therefore \(\Lambda_{C'} \subset E(\infty)\).
We will show that $\Lambda_{C'} = E(\infty)$.
For any subset $W \in \tilde{X}$ we define the boundary at infinity $\partial W$ of $W$ by

$$\partial W = \text{Cl}(W) \cap \partial \tilde{X}$$

(5.27) where $\text{Cl}(W)$ stands for the closure of $W$ in $\tilde{X} \cup \partial \tilde{X}$.

As $Z'/C'$ is compact, $\tilde{Z}'$ is at bounded distance of the orbit $C'z$ of some point $z$ in $\tilde{Z}'$, thus

$$\text{Cl}({\tilde{Z}'}) \cap \partial \tilde{X} = \Lambda_{C'}$$

(5.28) where $\text{Cl}$ is a sphere, any two points of $\partial \tilde{X} - \Lambda_{C'}$ can be joined by a continuous path contained in $\partial \tilde{X} - \Lambda_{C'}$ and so does $\zeta$ and $\theta$, joined by such a path $\alpha$.

The set $Z' \cup \Lambda_{C'}$ is a closed subset of $\tilde{X} \cup \partial \tilde{X}$ thus there is an open connected neighborhood $W$ of $\alpha$ in $\tilde{X} \cup \partial \tilde{X}$ contained in the complementary of $Z' \cup \Lambda_{C'}$.

As $\zeta$ and $\theta$ can be approximated by points in $\tilde{U}$ and $\tilde{V}$ respectively there exist points $x \in \tilde{U} \cap W$ and $y \in \tilde{V} \cap W$ that can be joined by a continuous path by connectedness of $W$, which leads to a contradiction. □

Remark 5.11. In fact, we are going to show that under the assumption $\delta(C') = n - 2$, it is impossible to have $\partial \tilde{U} = \Lambda_{C'}$ or $\partial \tilde{V} = \Lambda_{C'}$.

For any $x \in \tilde{X}$ and $\theta \in \partial \tilde{X}$ let us denote $HB(x, \theta)$ the open horoball centered at $\theta$ and passing through $x$.

Lemma 5.12. Let us assume that $\partial \tilde{U} = \Lambda_{C'}$. Then there exist $\theta_0 \in \Lambda_{C'}$ and $\zeta' \in \tilde{Z}'$ such that $HB(\zeta', \theta_0) \subset \tilde{U}$.

Proof : Let us recall that $\tilde{X}/C' - \tilde{Z}' = U \cup V$ where $U = \pi(\tilde{U})$ $V = \pi(\tilde{V})$ and $\pi : \tilde{X} \rightarrow \tilde{X}/C'$ is the projection.

We know that $U$ and $V$ are unbounded. Let $x_n$ a sequence of points in $U$ such that $\text{dist}(x_n, \tilde{Z}') \rightarrow \infty$. Let $z_n \in \tilde{Z}'$ such that $\text{dist}(x_n, \tilde{Z}') = \text{dist}(x_n, z_n)$. We consider a fundamental domain $D \subset \tilde{Z}'$ of $C'$. There exist lifts $\tilde{z}_n \in D$ and $\tilde{x}_n \in \tilde{U}$ such that $\text{dist}(\tilde{x}_n, \tilde{Z}') = \text{dist}(\tilde{x}_n, \tilde{z}_n)$ tends to infinity.

By compactness we can assume that a subsequence $\tilde{x}_{n_j}$ converges to a point $\tilde{0} \in \partial \tilde{X}$ and $\tilde{z}_{n_j}$ also converges to a point $\tilde{z} \in D$. Furthermore
the sequence of open balls \( B(\tilde{x}_{n_j}, \text{dist}(\tilde{x}_{n_j}, \tilde{z}_{n_j})) \subset \tilde{U} \) converges to the open horoball \( H\tilde{B}(\theta_0, \tilde{z}) \in \tilde{U}. \)

**Proposition 5.13.** \( \Lambda_{C'} = E(\infty) \) and \( \Lambda_C = E(\infty). \)

We first describe the idea of the proof and next state some facts we will need in order to do it.

As \( \Lambda_{C'} \subset \Lambda_C \subset E(\infty) \) the proposition boils down to proving that \( \Lambda_{C'} = E(\infty). \) Let us assume \( \Lambda_{C'} \neq E(\infty) \) and find a contradiction.

We will show that for any sequence \( \theta_i \) converging to \( \theta_0 \) the geodesic starting at a point \( o \) such that \( |Jac_{\gamma_{\infty}}\tilde{F}(o)| = 1 \) and ending at \( \theta_i \) crosses the hypersurface \( \tilde{Z}' \) in a point \( z_i. \) For an appropriate choice of such a sequence \( \theta_i \) (roughly speaking, the sequence \( \theta_i \) is chosen to be converging to \( \theta_0 \) "transversally to \( \Lambda_{C'}\)"), the shadow (defined below) projected from \( o \) through some geodesic ball \( B(z_i, r) \) will not intersect \( \Lambda_{C'}. \) On the other hand this shadow has to meet the limit set \( \Lambda_{C'} \) because of the shadow lemma of D. Sullivan, which leads to a contradiction.

Precisely, by Lemma (5.10) and Lemma (5.12) we know that \( \partial \tilde{U} = \Lambda_{C'} \) and that there exists an open horoball \( H\tilde{B}(\theta_0, \tilde{z}) \subset \tilde{U} \) centered at a point \( \tilde{z} \in \Lambda_{C'}, \) and whose closure contains a point \( \tilde{Z}' \).

Let \( o \in \tilde{X} \) and \( E \in T_o\tilde{X} \) an hyperplane such that \( \tilde{F}(o) = o, \) \( |Jac_E\tilde{F}(o)| = 1, \) and \( \Lambda_{C'} \subset E(\infty) \) where \( E(\infty) \) is the topological equator associated to \( E. \)

For each \( \theta \in \partial \tilde{X} \) we denote by \( \alpha_\theta \) the geodesic starting from \( o \) and such that \( \alpha_\theta(+\infty) = \theta. \)

Let \( \theta_i \in \partial \tilde{X} - E(\infty) = \partial \tilde{V} - \partial \tilde{U} \) be a sequence converging to \( \theta_0. \) By continuity, for each \( i \) large enough, the geodesic \( \alpha_{\theta_i} \) spends some time inside the horoball \( H\tilde{B}(\theta_0, \tilde{z}) \subset \tilde{U} \) and ends up inside \( \tilde{V} \) because \( \theta_i \) converges to \( \theta_0 \) and \( \theta_i \) belongs to \( \partial \tilde{V} - \partial \tilde{U}. \)

Thus \( \alpha_{\theta_i} \) eventually crosses \( \tilde{Z}'. \) Let \( z_i \in \alpha_{\theta_i} \cap \tilde{Z}'. \) As \( \tilde{Z}'/C' \) is compact, there is an element \( \gamma_i \in C' \) such that \( z_i = \gamma_i(x_i) \) where \( x_i \) is a point in the closure \( \tilde{D} \) of a fundamental domain \( D \) for the action of \( C' \) on \( \tilde{Z}'. \) The points \( \gamma_i(x_i) \) and \( \gamma_i(o) \) stay at bounded distance because \( dist(\gamma_i(x_i), \gamma_i(o)) = dist(x_i, o) \leq dist(o, D) + diamD. \) In particular, \( \lim_{i \to \infty} \gamma_i(o) = \theta_0. \)

We have proved the

**Lemma 5.14.** Let \( \theta_i \in \partial \tilde{X} - E(\infty) \) be a sequence which converges to \( \theta_0. \)
There exists a constant \( A \) such that for \( i \) large enough there exists \( z_i \in \tilde{Z}' \cap \alpha_{\theta_i} \) and \( \gamma_i \in C' \) such that \( dist(z_i, \gamma_i(o)) \leq A \) and both \( z_i \) and \( \gamma_i(o) \) converge to \( \theta_0. \)

Let \( x \) and \( y \) two points in \( \tilde{X}. \)

We define the shadow \( \mathcal{O}(x, y, R) \subset \partial \tilde{X} \) of the ball \( B(y, R) \) enlightened from the point \( x \) by

\[ (5.29) \quad \mathcal{O}(x, y, R) = \{ \alpha(\infty) \} \]
where $\alpha$ runs through the set of geodesic rays starting from $x$ and meeting $B(y, R)$.

Let $\{\mu_x\}_x$ be a family of Patterson measures associated to the discrete group $C'$ with critical exponent $\delta' = \delta(C')$.

The following shadow lemma is due to D. Sullivan.

**Lemma 5.15.** [18], [13], [20]. There exist positive constants $C$ and $R$ such that for any $y$ in $\tilde{X}$, $\nu_y(\mathcal{O}(y, \gamma(y), R)) \geq C e^{\delta' d(y, \gamma(y))}$

**Corollary 5.16.** Let $z_i$ be defined in lemma (5.14), then we have $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} \neq \emptyset$ for $i$ large enough.

We now prove that for a good choice of $\theta_i$, the shadow $\mathcal{O}(o, z_i, R + A)$ (with $z_i$ associated to $\theta_i$ as in lemma 5.14) never meet $\Lambda_{C'}$ for all large $i$’s, i.e. for any $\theta \in \Lambda_{C'}$ the geodesic $\alpha_\theta$ does not cross $B(z_i, R + A)$. We have no control on the radius $R$ coming from the shadow lemma nor on the constant $A$ but we will show

**Proposition 5.17.** There exists a sequence $\theta_i \in \partial \tilde{X} - \Lambda_{C'}$ such that $\theta_i$ converges to $\theta_0$ and

$$\lim_{i \to \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = +\infty$$

where $z_i = \tilde{Z}' \cap \alpha_{\theta_i}$ has been constructed in lemma (5.14).

**Corollary 5.18.** For $i$ large enough, $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} = \emptyset$.

The corollary (5.16) and the corollary (5.18) lead to a contradiction, which ends the proof of the proposition (5.13).

The end of the paragraph is devoted to proving the proposition (5.17).

**Lemma 5.19.** Let $\theta_i$ be a sequence of points in $\partial \tilde{X}$ converging to $\theta_0$ and $z_i$ constructed in lemma (5.14). Assume that

$$\liminf_{i \to \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) = C < +\infty$$

then $\lim_{i \to \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$.

**Proof :** We first show that

\begin{equation}
(5.30) \lim_{i \to \infty} \text{dist}(z_i, \alpha_{\theta_0}) = \infty
\end{equation}

Recall that, for any $z \in \tilde{X}$ and $\theta \in \partial \tilde{X}$, $B(z, \theta)$ equals the decreasing limit as $t$ tends to infinity of $\text{dist}(z, \alpha_\theta(t)) - \text{dist}(o, \alpha_\theta(t))$ where $\alpha_\theta(t)$ is the geodesic ray joining $o$ to $\theta$. Therefore, as the points $z_i \in \tilde{Z}$ belongs to the complementary of the fixed horoball $HB(\tilde{z}, \theta_0)$, we have,

\begin{equation}
(5.31) \text{dist}(z_i, \alpha_{\theta_0}(T_i)) \geq T_i + B(\tilde{z}, \theta_0)
\end{equation}

where $\text{dist}(z_i, \alpha_{\theta_0}(T_i)) = \text{dist}(z_i, \alpha_{\theta_0})$.

On the other hand, as $z_i$ tends to $\theta_0$, $T_i$ tends to infinity so (5.30) is proven.
Let $t_i$ be such that $z_i = \alpha_{\vartheta_i}(t_i)$. By (5.30) we have

(5.32) \[ \lim_{i \to \infty} \text{dist}(\alpha_{\vartheta_i}(t_i), \alpha_{\vartheta_0}(t_i)) = \infty. \]

Let $u_i$ be such that

(5.33) \[ \text{dist}(\alpha_{\vartheta_i}(u_i), \alpha_{\vartheta_0}(u_i)) = 1, \]

then in particular $u_i \leq t_i$ for $i$ large enough and by the triangle inequality we have

(5.34) \[ \text{dist}(\alpha_{\vartheta_i}(t_i), \alpha_{\vartheta_0}(t_i)) \leq 2(t_i - u_i) + 1. \]

By (5.32) we get

(5.35) \[ \lim_{i \to \infty} (t_i - u_i) = +\infty. \]

Let us assume there exists a sequence $\theta'_i \in \Lambda_{C'}$ and a constant $C$ such that

(5.36) \[ \text{dist}(z_i, \alpha_{\theta'_i}) \leq C < +\infty. \]

We can assume that $C \geq 1$.

Let $v_i$ be such that

(5.37) \[ \text{dist}(z_i, \alpha_{\theta'_i}) = \text{dist}(z_i, \alpha_{\theta'_i}(v_i)). \]

By triangle inequality,

(5.38) \[ |t_i - v_i| \leq C \]

and

(5.39) \[ \text{dist}(\alpha_{\theta'_i}(t_i), \alpha_{\theta_i}(t_i)) \leq 2C. \]

On the other hand, as the curvature of $\tilde{X}$ is bounded above by $-1$, a classical comparison theorem gives for any $t \in [0, t_i]$,

(5.40) \[ \sinh\left(\frac{\text{dist}(\alpha_{\theta'_i}(t), \alpha_{\theta_i}(t))}{2}\right) \leq \sinh C \frac{\sinh t}{\sinh t_i}. \]

Let $s_i$ be such that

(5.41) \[ \text{dist}(\alpha_{\theta'_i}(s_i), \alpha_{\theta_i}(s_i)) = 1. \]
There are two cases. Either \( s_i \geq t_i \) or \( s_i < t_i \). If \( s_i < t_i \), we get from (5.39) and (5.40) the existence of a constant \( A \) such that for any \( i \),

\[
(5.42) \quad s_i \geq t_i - A,
\]

and this inequality also holds when \( s_i \geq t_i \).

\( \text{From (5.42) we get} \)

\[
(5.43) \quad \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = e^{-s_i + u_i} \leq e^A e^{-t_i + u_i},
\]

therefore, thanks to (5.35) we obtain

\[
(5.44) \quad \lim_{i \to \infty} \frac{d(\theta_i, \theta'_i)}{d(\theta_i, \theta_0)} = 0.
\]

which ends the proof of lemma (5.19). \( \square \)

**Lemma 5.20.** Let us assume that for every sequence \( \theta_i \) of points in \( \partial \tilde{X} \) converging to \( \theta_0 \), \( \lim_{i \to \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0 \). Let \( \lambda_k \to \infty \) be such that the sequence of spaces \( (\partial \tilde{X}, \lambda_k d, \theta_0) \) converges to a space \( (S, \bar{d}, 0) \) in the pointed Gromov-Hausdorff topology, then the sequence of spaces \( (\Lambda_{C'}, \lambda_k d, \theta_0) \) also converges to \( (S, \bar{d}, 0) \).

**Proof:** Let us define

\[
(5.45) \quad \epsilon_k =: \frac{R r(R/\lambda_k)}{\lambda k}.
\]

The assumption says that

\[
(5.46) \quad \lim_{k \to \infty} \epsilon_k = 0.
\]

For an arbitrary metric space \( (Y, d) \) and \( Y' \) a subset of \( Y \), let us denote \( B_{\lambda_k}(Y, d)(y, R) \) the closed ball of \( (Y, d) \) of radius \( R \) centered at \( y \in Y \), and \( U_{\epsilon_k}^{(Y, d)}(Y') \) the \( \epsilon \)-neighborhood of \( Y' \) in \( (Y, d) \). For a metric space \( (Y, d) \) and a positive number \( \lambda \), let us denote \( \lambda Y \) the rescaled space \( (Y, \lambda d) \).

By definition of the function \( r \), we have for any \( R \),

\[
B_{\lambda_k\partial \tilde{X}}(\theta_0, R) \subset U_{\epsilon_k}^{\lambda_k\partial \tilde{X}} B_{\lambda_k\Lambda_{C'}}(\theta_0, R + \epsilon_k)
\]

\[
\subset B_{\lambda_k\partial \tilde{X}}(\theta_0, R + 2\epsilon_k).
\]

where \( \epsilon_k =: R r(R/\lambda_k) \).

Let us fix \( \alpha > 0 \). By definition 5.7, for any \( R > 0, \epsilon > 0 \), there exist a map \( f : B_{\lambda_k\partial \tilde{X}}(\theta_0, R + \alpha) \to S \) such that for \( k \geq k_0 \),

(i) \( f(\theta_0) = 0 \),

for any two points \( x \) and \( y \) in \( B_{\lambda_k\partial \tilde{X}}(\theta_0, R + \alpha) \).
(ii) \(|\tilde{d}(f(x), f(y)) - \lambda_k d(x, y)| \leq \epsilon\), and 
(iii) \(B_{(S,\delta)}(0, R + \alpha - \epsilon) \subset U_k^{(S,\tilde{\delta})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha))\).

Moreover let us prove:
(iv) \(B_{(S,\delta)}(0, R - 2\epsilon) \subset U_k^{(S,\tilde{\delta})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R))\).

Indeed, let \(z \in B_{(S,\delta)}(0, R - 2\epsilon)\). By (iii), there exists \(\theta \in B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha)\) such that \(\tilde{d}(z, f(\theta)) \leq \epsilon\). Since \(\tilde{d}(z, 0) \leq R - 2\epsilon\), we thus deduce from triangle inequality \(\tilde{d}(f(\theta), 0) \leq R - \epsilon\), and therefore we get, thanks to (i) and (ii), \(\lambda_k d(\theta, \theta_0) \leq R\).

By (5.45), for \(\epsilon\) small enough, there exists \(k_1 \geq k_0\) such that for any \(k \geq k_1\), then \(2\epsilon_k \leq \epsilon\) and

\[B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha)\]

Therefore, by (5.46) and the above properties (i),(ii),(iii) and (iv) of the map \(f\), and the triangle inequality we get,

\[B_{(S,\delta)}(0, R - 2\epsilon) \subset U_k^{(S,\tilde{\delta})} f(B_{\lambda_k \partial \tilde{X}}(\theta_0, R))\]

\[\subset U_k^{(S,\tilde{\delta})} f(U_k^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k))\]

\[\subset U_k^{(S,\tilde{\delta})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)).\]

About the second inclusion above let us remark that the set \(U_k^{\lambda_k \partial \tilde{X}} B_{\lambda_k \Lambda_{C'}}(\theta_0, R + \epsilon_k)\) is contained in \(B_{\lambda_k \partial \tilde{X}}(\theta_0, R + 2\epsilon_k) \subset B_{\lambda_k \partial \tilde{X}}(\theta_0, R + \alpha)\), so that we can apply \(f\) to this set.

From the above inclusions we obtain

\[B_{(S,\delta)}(0, R - 3\epsilon) \subset U_k^{(S,\tilde{\delta})} f(B_{\lambda_k \Lambda_{C'}}(\theta_0, R))\]

which implies the convergence of \((\Lambda_{C'}, \lambda_k d, \theta_0)\) to \((S, \tilde{d}, 0)\).

\(\square\)

**Corollary 5.21.** Let us assume that for every sequence \(\theta_i\) of points in \(\partial \tilde{X}\) converging to \(\theta_0\) and \(z_i\) the sequence of points constructed in lemma (5.14), 
\(\liminf_{\theta_i \to \infty} \inf f_{\theta_i \Lambda C'} \text{dist}(z_i, \alpha_\theta) < +\infty\). Let \(\lambda_k \to \infty\) be such that the sequence of spaces \((\partial \tilde{X}, \lambda_k d, \theta_0)\) converges to the space \((S, \delta, 0)\) in the pointed Gromov-Hausdorff topology, then the sequence of spaces \((\Lambda_{C'}, \lambda_k d, \theta_0)\) also converges to \((S, \delta, 0)\).

We will show now that there exist a sequence of points \(\theta_k^1, \theta_k^2 \in \Lambda_{C'}\) converging to \(\theta_0\), such that the mutual distances \(d(\theta_k^1, \theta_k^2), d(\theta_k^1, \theta_0), d(\theta_k^2, \theta_0)\) is tending to zero at the same rate, and the triple \(\theta_k^1, \theta_k^2, \theta_0\) can be uniformly separated by elements \(\gamma_k \in C'\).
Lemma 5.22. Assume that every weak tangent of \((\partial X, d)\) at \(\theta_0\) belongs to \(WT(\Lambda_{C'}\), \(d)\), then there exist positive constants \(c, \delta\), a sequence \(\epsilon_k\) tending to 0 when \(k\) tends to \(\infty\), a sequence \(\gamma_k \in C'\), a sequence of points \(\theta_k, \theta_0\in \Lambda_{C'}\) such that for \(i = 1, 2\),
\[ c^{-1}\epsilon_k \leq d(\theta_k, \theta_0) \leq c\epsilon_k, \]
\[ c^{-1}\epsilon_k \leq d(\theta_k, \theta_0) \leq c\epsilon_k \quad \text{and} \quad \]
\[ d(\gamma_k, \gamma_0) \geq \delta, \quad d(\gamma_k, \gamma_0) \geq \delta. \]

**Proof:** For any \(x \in \tilde{X} \cup \partial \tilde{X}\) and \(y \in \tilde{X} \cup \partial \tilde{X}\) let us define \(\alpha_{x,y}\) the geodesic ray joining \(x\) and \(y\). Let \(o \in \tilde{X}\) and \(E \in T_o\tilde{X}\) be such that \(|Jac_E\tilde{F}(o)| = 1\) and \(E(\infty)\) the equator associated to \(E\). Let \(\gamma_k \in C'\) be a sequence such that \(\gamma_k(o)\) converges to the point \(\theta_0\) where \(\theta_0 \in \Lambda_{C'}\) is the point coming from lemma 5.12. In particular, according to that lemma, there exist a point \(\tilde{z} \in \tilde{Z}'\) such that the hypersurface \(\tilde{Z}'\) is contained in the complementarity of the open horoball \(HB(\tilde{z}, \theta_0)\). We define \(D := dist(\tilde{z}, o)\).

As \(\tilde{Z}'\) lies outside the open horoball \(HB(\tilde{z}, \theta_0)\), the points \(\gamma_k(o)\) belong to the complementarity of the open horoball \(HB(\alpha_k, \theta_0(D), \theta_0)\). By standard triangle comparison argument (comparison with the hyperbolic case) the angle \(\text{Angle}(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o})\) between the two geodesic rays \(\alpha_{\gamma_k(o), \theta_0}\) and \(\alpha_{\gamma_k(o), o}\) satisfies:

\[
\lim_{k \to \infty} \text{Angle}(\alpha_{\gamma_k(o), \theta_0}, \alpha_{\gamma_k(o), o}) = 0.
\]

By equivariance we have \(\Lambda_{C'} \subset (\gamma_k E)(\infty)\) where \(\gamma_k E \subset T_{\gamma_k(o)}\tilde{X}\). For any \(v \in T\tilde{X}\) let \(\alpha_v\) be the geodesic ray such that \(\alpha_v(0) = v\). Let us denote by \(u_k\) the unit vector in \(\gamma_k E\) such that \(\alpha_{u_k}(\infty) = \theta_0\) and let us choose some \(w_k \in \gamma_k E\) such that \(< u_k, w_k > = 0\) (this is possible because \(n - 1 \geq 2\)).

We claim now that there exist \(v_k \in \gamma_k E\) such that the angle between \(v_k\) and \(w_k\) is not too far from 0 or \(\pi\), namely

\[
< v_k, w_k > | \geq \frac{1}{(n - 1)^{1/2}},
\]

and \(\alpha_{v_k}(\infty) \in \Lambda_{C'}\) or \(\alpha_{v_k}(-\infty) \in \Lambda_{C'}\).

Let us prove this claim.

According to Proposition 5.1 and to (5.11), the restriction to \(\gamma_k E\) of the quadratic form \(h(u) = \int DB(\gamma_k(o), \theta)(u)^2 d\mu_{\gamma_k(o)}(\theta)\) verifies

\[
h_{\gamma_k E}(u) = \frac{||u||^2}{n - 1}.
\]

Therefore, if for all \(u \in \gamma_k E\) such that \(\alpha_u(\infty) = \theta \in \Lambda_{C'}\) we had \(< u, w_k > | < \frac{1}{(n - 1)^{1/2}},\) then one would get \(h(w_k) < \frac{1}{n - 1}\), which contradicts (5.49) and proves the claim.
In particular the angle between $u_k$ and $v_k$ is not too far from $\pi/2$ for $k$ large enough, i.e.

\[(5.50) \quad | \langle u_k, v_k \rangle | \leq \left( \frac{n-2}{n-1} \right)^{1/2},\]

and thanks to (5.47), we have for $k$ large enough

\[(5.51) \quad | \langle \dot{\alpha}_{\gamma_k(0),o}(0), v_k \rangle | \leq \left( \frac{n-3}{2n-1} \right)^{1/2}.\]

Let us now assume for example that $\theta_k = \alpha_{e_k}(+\infty) \in \Lambda_{C'}$. Let us show that

\[(5.52) \quad \lim_{k \to \infty} d(\theta_0, \theta_k) = 0.\]

Assume that (5.52) is not true. Then, one can assume after extracting a subsequence that $\theta_k$ converges to $\theta \neq \theta_0$. Therefore the geodesics $\alpha_{\gamma_k(0),o}$ and $\alpha_{e_k}$ would converge to the geodesics $\alpha_{\theta_0,o}$ and $\alpha_{\theta_0,\theta}$ and thus the angle $\text{Angle}(\alpha_{\gamma_k(0),o}, \alpha_{e_k})$ would converge to 0. But this would contradict (5.51).

Let us now denote $\epsilon_k := d(\theta_0, \theta_k)$. According to (5.52), $\lim_{k \to \infty} \epsilon_k = 0$. We now consider the following sequence of pointed metric space $(\tilde{\partial}X, \epsilon_k^{-1}d, \theta_0)$, a subsequence of which being converging to some metric space $(S, \delta)$, cf.

For convenience we still denote by the same index $k$ the subsequence. By the corollary 5.21, the sequence $(\Lambda_{C'}, \epsilon_k^{-1}d, \theta_0)$ also converges to $(S, \delta)$. According to lemma 5.8, the space $S$ is homeomorphic to $\mathbb{R}^{n-1}$. In particular there exist a sequence of points $\theta^1_k \in \Lambda_{C'}$ and a constant $c$ such that

\[(5.53) \quad c^{-1} \epsilon_k \leq d(\theta_k, \theta^1_k) \leq c \epsilon_k,\]

\[(5.54) \quad c^{-1} \epsilon_k \leq d(\theta^1_k, \theta_0) \leq c \epsilon_k.\]

Thus, the points $\theta^1_k = \theta_k$ and $\theta^2_k = \theta'_k$ satisfy the two first properties of lemma 5.22.

In order to complete the proof of lemma 5.22, we will show that the elements $\eta_k := \gamma_k^{-1}$ uniformly separate $\theta_0, \theta^1_k$ and $\theta^2_k$.

Thanks to (5.50) the angle at $\gamma_k(o)$ between $\theta^1_k$ and $\theta_0$ is uniformly bounded away from 0 and $\pi$ and so does the angle at $o$ between $\gamma_k^{-1}(\theta^1_k)$ and $\gamma_k^{-1}(\theta_0)$. Therefore, as the angle is Hölder-equivalent to the distance $d$, cf. [10], there is a constant $c$ such that

\[(5.55) \quad d(\gamma_k^{-1}(\theta^1_k), \gamma_k^{-1}(\theta_0)) \geq c.\]
Now the cocompact group $\Gamma$ acts uniformly quasi-conformally on $(\partial \tilde{X}, d)$, (and [19] Theorem 5.2), and so does $C' \subset \Gamma$, therefore

\begin{align}
(5.56) \quad & d(\gamma_k^{-1}(\theta_k^1), \gamma_k^{-1}(\theta_k^2)) \geq c, \\
\text{and} \\
(5.57) \quad & d(\gamma_k^{-1}(\theta_k^2), \gamma_k^{-1}(\theta_0)) \geq c.
\end{align}

which ends the proof of lemma 5.22. □

Proof of Proposition 5.17 :

Let us assume that for every sequence $\theta_i$ of points in $\partial \tilde{X}$ converging to $\theta_0$, \liminf_{i \to \infty} \inf_{\theta \in \Lambda_C'} \text{dist}(z_i, \alpha_\theta) < +\infty$, then by corollary 5.21 and lemma 5.22 there exist a positive constant $c$, a sequence $\epsilon_k$ tending to 0 when $k$ tends to $\infty$, a sequence $\epsilon_k \in C'$, a sequence of points $\theta_k^1, \theta_k^2 \in \Lambda_C'$ such that

\begin{align}
& \epsilon_k \leq d(\theta_k^1, \theta_k^2) \leq c \epsilon_k, \\
& \epsilon_k \leq d(\theta_k^1, \theta_0) \leq c \epsilon_k \quad \text{and} \\
& d(\gamma_k \theta_k^1, \gamma_k \theta_k^2) \geq \delta, \quad d(\gamma_k \theta_k^1, \gamma_k \theta_0) \geq \delta.
\end{align}

Applying lemma 5.9 for $L = \Lambda_C'$ and $\lambda_k = \epsilon_k^{-1}$ we conclude that $\Lambda_C'$ is homeomorphic to $\partial \tilde{X}$, which is impossible because $\Lambda_C'$ is contained in a topological equator $E(\infty)$. □

Step 4 : $C'$ and $C$ are convex cocompact.

We first define convex cocompactness. For a discrete group $C$ of isometries acting on a Cartan Hadamard manifold of negative sectional curvature with limit set $\Lambda_C$, one defines the geodesic hull $G(\Lambda_C)$ of $\Lambda_C$ as the set of all geodesics both ends of whose belong to $\Lambda_C$.

The geodesic hull of $\Lambda_C$ is a $C$ invariant set. One says that $C$ is convex cocompact if $G(\Lambda_C)/C$ is compact.

Lemma 5.23. $C'$ is convex cocompact.

Proof : Let us denote $\pi : \tilde{X} \to \tilde{X}/C'$ the projection. Assume that $C'$ is not convex cocompact. Then, there exist a sequence $x_n \in G(C')$ such that $x_n$ tends to infinity. In particular $\text{dist}(x_n, Z') \to +\infty$, where $Z' = \tilde{Z}'/C'$ is the compact hypersurface which separates $\tilde{X}/C'$ in two unbounded connected components. There exist lifts $\tilde{x}_n$ of $x_n$ such that

\begin{align}
(5.58) \quad & \tilde{x}_n \to \theta_0 \in \Lambda_{C'} \\
(5.59) \quad & \text{dist}(\tilde{x}_n, \tilde{Z}') = \text{dist}(\tilde{x}_n, \tilde{z}_n)
\end{align}

where $\tilde{z}_n \in \tilde{Z}'$ is bounded. Therefore there exist $\tilde{z} \in \tilde{Z}'$ such that $HB(\tilde{z}, \theta_0) \subset \tilde{U}$, where $\tilde{U}$ is one of the two connected components of $\tilde{X} - \tilde{Z}'$, the other being $\tilde{V}$.
We recall that $\mathcal{M}, \mathcal{N}$ are the two connected components of $\partial \tilde{X} - \Lambda_{C'}.$

We also have $\partial \tilde{Z}' = \Lambda_{C'} = E(\infty)$, and after possibly replacing $C'$ by an index two subgroup, we can assume that $C'$ preserves $\tilde{U}$ and $\tilde{V}$.

**Claim:** There are the two following cases.

Either one of the two boundaries $\partial \tilde{U}$ or $\partial \tilde{V}$ is equal to $\Lambda_{C'}$ (in this case the other boundary is equal to $\partial \tilde{X}$), or $\partial \tilde{U} = \tilde{M}$ and $\partial \tilde{V} = \tilde{N}$, where $\tilde{M}$ and $\tilde{N}$ are the closure of $\mathcal{M}$ and $\mathcal{N}$.

Let us prove the claim. We first remark that if there exist $\theta \in \partial \tilde{U} \cap \mathcal{M}$, then $\tilde{M} \subset \partial \tilde{U}$. Namely, let $\xi$ be any other point in $\mathcal{M}$ and $\alpha$ a continuous path in $\mathcal{M}$ joining $\theta$ and $\xi$. Since the set $\tilde{Z}' \cup \Lambda_{C'}$ is a closed subset in $X \cup \partial X$, there exist an open connected neighborhood $W$ of $\alpha$ in $X \cup \partial X$ contained in the complementary of $\tilde{Z}' \cup \Lambda_{C'}$. Therefore, as $W \cap \tilde{U} \neq \emptyset$, we have $W \cap \tilde{X} \subset \tilde{U}$ and $\xi \in \partial \tilde{U}$. Let us assume that neither $\partial \tilde{U} \cap \mathcal{M}$ nor $\partial \tilde{V} \cap \mathcal{M}$ is equal to $\Lambda_{C'}$. Then, each boundary $\partial \tilde{U}$ and $\partial \tilde{V}$ contains $\mathcal{M}$ or $\mathcal{N}$. But on the other hand, since the set $\tilde{Z}' \cup \Lambda_{C'}$ is closed, $(\partial \tilde{U} - \Lambda_{C'}) \cap (\partial \tilde{V} - \Lambda_{C'}) = \emptyset$ thus we have $\partial \tilde{U} = \mathcal{M}$ and $\partial \tilde{V} = \mathcal{N}$ or the other way around and the claim is proved.

**Case 1:** $\partial \tilde{U} = \Lambda_{C'}$ and $\partial \tilde{V} = \partial \tilde{X}$ or the other way around.

In this case, we are in the situation of the step 3, which leads to a contradiction, cf. remark 5.11.

**Case 2:** $\partial \tilde{U} = \tilde{M}$ and $\partial \tilde{V} = \tilde{N}$.

In that case, assuming $C'$ is not convex-cocompact, there exist an open horoball $\text{HB}(\theta_0, \tilde{z}) \subset \tilde{U}$ where $\theta_0 \in \Lambda_{C'}$, $\tilde{z} \in \tilde{Z}'$, $\partial \tilde{U} = \tilde{M}$ and $\partial \tilde{V} = \tilde{N}$. We will find a contradiction in a similar way as in case 1, i.e. step 3. We consider a point $o \in X$ and an hyperplane $E \subset T_o X$ such that $|Jac_E F(o)| = 1$ and $\Lambda_{C'} = E(\infty)$.

Let $\theta_i \in \mathcal{N}$ be a sequence which converge to $\theta_0$. By continuity, for $i$ large enough, the geodesic ray $\alpha_{o, \theta_i}$ spends some time in $\text{HB}(\theta_0, \tilde{z}) \subset \tilde{U}$ and ends up in $\tilde{V}$ because $\theta_i$ converges to $\theta_0$ and $\theta_i$ belongs to $\mathcal{N} = \partial \tilde{V} - \Lambda_{C'}$. Therefore, $\alpha_{o, \theta_i}$ eventually crosses $\tilde{Z}'$. Let $z_i$ be some point in $\tilde{Z}' \cap \alpha_{o, \theta_i}$.

We will prove the following Proposition, similar to the Proposition 5.18,

**Proposition 5.24.** There exist a sequence $\theta_i \in \mathcal{N}$ such that $\theta_i$ converges to $\theta_0$ and

$$\lim_{i \to \infty} \inf_{\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_{\theta}) = +\infty$$

where $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$.

**Remark 5.25.** The difference between the propositions 5.24 and 5.18 is that we are looking for a sequence $\theta_i \in \mathcal{N}$ instead of $\theta_i \in \partial \tilde{X} - \Lambda_{C'}$.

Assuming the Proposition 5.24 we find a contradiction in the same way as in step 3. Namely, as $\tilde{Z}' / C'$ is compact, the points $z_i \in \tilde{Z}' \cap \alpha_{\theta_i}$ stay at
bounded distance from the $C'$-orbit of a fixed point, say, $o$, thus there exist a constant $A > 0$ and elements $\gamma_i \in C'$ such that for any $i$,

\[(5.60)\]

\[\text{dist}(z_i, \gamma_i o) \leq A.\]

From (5.60) and the shadow lemma 5.15, we obtain $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} \neq \emptyset$, and on the other hand, from the proposition 5.24, we have $\mathcal{O}(o, z_i, R + A) \cap \Lambda_{C'} = \emptyset$, which gives the contradiction. It remains to prove the Proposition 5.24.

**Proof of the proposition 5.24**: We argue by contradiction, like in the proof of the proposition 5.17. Let us assume that there exist a constant $C > 0$ such that for any sequence $\theta_i \in \mathcal{N}$ converging to $\theta_0$, $\liminf_{i \to \infty} \inf_{f_\theta \in \Lambda_{C'}} \text{dist}(z_i, \alpha_\theta) \leq C$, then by lemma 5.19, we have for any such sequence $\theta_i \in \mathcal{N}$

\[(5.61)\]

\[\lim_{i \to \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0.\]

The proof of the following lemma is the same as the proof of lemma 5.20.

**Lemma 5.26.** Let us assume that for any sequence $\theta_i \in \mathcal{N}$ converging to $\theta_0$, $\lim_{i \to \infty} \frac{d(\theta_i, \Lambda_{C'})}{d(\theta_i, \theta_0)} = 0$. Let $\{\lambda_k\}$ be a sequence of positive numbers tending to $+\infty$ such that the sequence of spaces $(\partial X, \lambda_k d, \theta_0)$ converges to a space $(S, \delta, 0)$ in the pointed Gromov-Hausdorff topology, then, $(\mathcal{M}, \lambda_k d, \theta_0)$ also converges to $(S, \delta, 0)$.

**Proof**: Since $\Lambda_{C'} \subset \bar{\mathcal{M}}$, the assumption implies that $\lim_{\epsilon \to 0} r(\epsilon) = 0$ where

\[r(\epsilon) = \sup\left\{\frac{d(\theta, \mathcal{M})}{d(\theta, \theta_0)}, \theta \neq \theta_0, \theta \in \mathcal{N}, d(\theta, \theta_0) \leq \epsilon\right\}\]

and the proof goes the same way as in lemma 5.20 replacing $\Lambda_{C'}$ by $\mathcal{M}$.

Similarly to the lemma 5.22, we have the

**Lemma 5.27.** Let us assume that every weak tangent of $(\partial X, d)$ at $\theta_0$ belongs to $\text{WT}(\mathcal{M}, d)$. There exist positive constant $c, \delta$, a sequence $\epsilon_k$ tending to 0 when $k$ tends to $+\infty$, a sequence of $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\}$,

\[c^{-1} \epsilon_k \leq d(\theta_0^k, \theta_1^k) \leq c \epsilon_k\]

and

\[d(\gamma_k \theta_0^k, \gamma_k \theta_1^k) \geq \delta.\]

We can now end the proof of the proposition 5.24. Let us assume that there exist a constant $C > 0$ such that for every sequence $\theta_i$ of points in $\mathcal{N}$ converging to $\theta_0$, $\liminf_{i \to \infty} \inf_{f_\theta \in \Lambda_{C'}} \text{dist}(\theta_i, \alpha_\theta) \leq C$, then by (5.61), lemma 5.26 and lemma 5.27, there exist a sequence $\epsilon_k$ tending to 0 when $k$ tends to $\infty$, a sequence $\gamma_k \in C'$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in \mathcal{M}$ such that for $i \neq j \in \{0, 1, 2\},$
Lemma 5.29. Let \( \lambda_k \) be such that \( Z_{\mathcal{C}} \) is convex cocompact, and that their limit set \( \Lambda \) is a totally geodesic hypersurface embedded in \( \tilde{X} \). This ends the proof of the proposition 5.24. □

Corollary 5.28. \( C \) is convex cocompact.

Proof: The subgroup \( C' \) of \( C \) is convex cocompact and the limit sets of \( C' \) and \( C \) coincide by step 3, therefore \( C \) is convex cocompact. □.

Step 5: \( C \) preserves a copy of the \((n-1)\)-dimensional hyperbolic space \( \mathbb{H}^{n-1} \) totally geodesically embedded in \( \tilde{X} \).

From the steps 1-4, we know that the groups \( C \) and \( C' \) are convex cocompact, and that their limit set \( \Lambda_C \) and \( \Lambda_{C'} \) are equal to a topological equator \( E(\infty) \).

Let us consider the essential hypersurface \( Z' \subset \tilde{X}/C' \). We will show that there exist a minimizing current representing the class of \( Z' \) in \( H_{n-1}(\tilde{X}/C', \mathbb{R}) \) and that this minimizing current lifts to a totally geodesic hypersurface embedded in \( X \). We will then show that this totally geodesic hypersurface is eventually hyperbolic.

We work in \( \tilde{X}/C' \) and consider the essential hypersurface \( Z' \subset \tilde{X}/C' \). We will now prove that there exist a minimal current representing the class of \( Z' \) in \( H_{n-1}(\tilde{X}/C', \mathbb{R}) \). Let \( \{Z_k\} \) be a minimizing sequence of currents homologous to \( Z' \). The orthogonal projection onto the convex core of \( \tilde{X}/C' \) is distance nonincreasing and thus volume nonincreasing. Therefore we can assume that the \( Z_k \)'s are in the the convex core of \( \tilde{X}/C' \), which is compact. By [1] (5.5), the sequence \( \{Z_k\} \) subconverges to a minimal current \( Z_\infty \) in \( \tilde{X}/C' \). By [1] (8.2), \( Z_\infty \) is a manifold with possible singularities of codimension greater than or equal to 8. By corollary (4.4) and minimality we get that \( |\text{Jac}_{n-1}F(x)| = 1 \) at every regular points \( x \in Z_\infty \). We will use the fact that \( |\text{Jac}_{n-1}F(x)| = 1 \) at every regular points \( x \in Z_\infty \) in order to prove that \( Z_\infty \) is a totally geodesic hypersurface.

Lemma 5.29. Let \( x \) and \( y \) two distinct points in \( \tilde{X} \) and \( E_x \subset (T_x \tilde{X}, E_y \subset (T_y \tilde{X}) \) such that \( \text{Jac}_{n-1}F'(x) = \text{Jac}_{E_x}F'(x) = 1 \) and \( \text{Jac}_{n-1}F'(y) = \text{Jac}_{E_y}F'(y) = 1 \). Then, the geodesic \( \alpha_{x,y} \) (resp. \( \alpha_{y,x} \)) joining \( x \) and \( y \) (resp. \( y \) and \( x \)) satisfies \( \tilde{\alpha}_{x,y}(0) \in E_x \), (resp. \( \tilde{\alpha}_{y,x}(0) \in E_y \)). In particular, \( \alpha_{x,y}(+\infty) \) and \( \alpha_{y,x}(+\infty) \) belong to \( \Lambda_{C'} \).

Proof: Let \( S_x \) and \( S_y \) be the unit spheres of \( E_x \) and \( E_y \). For any unit tangent vector \( u \in T_z \tilde{X} \) at some point \( z \), we define \( \theta_u \in \partial \tilde{X} \) by \( \tilde{\alpha}_{x,y}(0) = u \). By step 3, \( \Lambda_{C'} = E_x(\infty) = E_y(\infty) \), therefore for every \( u \in S_x \), \( \theta_u \in \Lambda_{C'} \) and there exist \( v \in E_y \) such that \( \theta_u = \theta_v \). As \( E_y \) is a vector space, \( \theta_{-v} \) belongs to
\begin{align*}
\Lambda_{C'} \text{ therefore there exist } w \in E_x \text{ such that } \theta_w = \theta_{-w}. \text{ The map } f : S_x \to S_x \text{ defined by } f(u) = w \text{ is a continuous map. The lemma then boils down to proving that there exist } u \in S_x \text{ such that } f(u) = -u \text{ because in that case, } x, y \text{ and } \theta_u \text{ are on the same geodesic } \alpha_{x, \theta_u}. \\
\text{The following properties of } f \text{ are obvious.} \\
\text{(i) For every } u \in S_x, f(u) \neq u. \\
\text{(ii) } f \circ f = \text{Id.} \\
\text{So } f \text{ is an involution of the sphere without fixed point and for any such map, we claim that there exist } u \text{ in the sphere such that } f(u) = -u. \text{ In order to prove the claim, we follow a very similar argument in } [R], \text{ theorem 1. We argue by contradiction. Let us assume that for every } u \in S_x, f(u) \neq -u. \\
\text{The map } g : S_x \to S_x \text{ defined by } g(u) = \frac{f(u)+u}{\|f(u)+u\|} \text{ is then well defined and continuous. Let us remark that as for every } u \in S_x, f(u) \neq -u, \text{ then } f \text{ is homotopic to the Identity, and so is } g. \text{ Moreover by (ii) we clearly have } g \circ f = g, \text{ thus the map } g \text{ factorizes through } S_x/G_f \text{ where } G_f \text{ is the group generated by the involution } f. \text{ By (i) } f \text{ has no fixed point thus } S_x/G_f \text{ is a manifold and the projection } p : S_x \to S_x/G_f \text{ is a degre } 2 \text{ map. Therefore, the induced endomorphism } g_* \text{ on } H_{n-1}(S_x, \mathbb{Z}_2) \text{ is trivial, which contradicts the fact that } g \text{ is homotopic to the Identity.} \quad \square
\end{align*}

Corollary 5.30. Let \( \mathcal{H}^{n-1} \subset \tilde{X} \) be an hypersurface with possibly non empty boundary \( \partial \mathcal{H}^{n-1} \), such that for any \( x \in \mathcal{H}^{n-1} \), \( \text{Jac}_{x} F(x) = \frac{1}{2^n} \) where \( E_x \) is the tangent space of \( \mathcal{H}^{n-1} \) at \( x \). Let us consider \( x \in \mathcal{H}^{n-1} \) such that \( \text{dist}_{\tilde{X}}(x, \partial \mathcal{H}^{n-1}) = r > 0 \). Then, for any \( x' \in \mathcal{H}^{n-1} \) with \( \text{dist}_{\tilde{X}}(x, x') < r \), the geodesic \( \alpha_{x, x'} \) joining \( x \) and \( x' \) is contained in \( \mathcal{H}^{n-1} \). In particular, \( \mathcal{H}^{n-1} \) is locally convex.

Proof of the corollary: Let us fix \( \theta \in \Lambda_{C'} \) and consider the vector field \( \nabla B(y, \theta) \) in \( \tilde{X} \). Let \( x \in \mathcal{H}^{n-1} \). As \( \text{Jac}_{E_x} F'(x) = 1 \), we have by step 3 \( \Lambda_{C'} = E_x(\infty) \). Then, for any \( x \in \mathcal{H}^{n-1} \), \( \nabla B(x, \theta) \) is tangent to \( \mathcal{H}^{n-1} \), therefore the geodesic \( \alpha_{x, \theta} \) satisfies \( \alpha_{x, \theta}(t) \in \mathcal{H}^{n-1} \) for all \( t \in [0, r] \). Let \( x' \in \mathcal{H}^{n-1} \). By lemma 5.29, \( \alpha_{x, \theta}(0) \in E_x \), therefore \( \alpha_{x, x'} = \alpha_{x, \theta} \) and \( \alpha_{x, x'}(t) \in \mathcal{H}^{n-1} \) for all \( t \in [0, r] \). \quad \square

We now prove that \( Z_{\infty} \) is a totally geodesic hypersurface in \( \tilde{X}/C' \). Let us recall that \( Z_{\infty} \) is a manifold which is smooth except at a singular subset of codimension at least 7. Let us consider a lift \( \tilde{Z}_{\infty} \subset \tilde{X} \) of \( Z_{\infty} \) and denote \( \tilde{Z}_{\infty}^{\text{reg}} \) (resp. \( \tilde{Z}_{\infty}^{\text{sing}} \)) the set of regular (resp.) singular points of \( \tilde{Z}_{\infty} \).

Lemma 5.31. \( Z_{\infty} \) is a totally geodesic hypersurface in \( \tilde{X} \).

Proof: Let us consider a regular point \( x \in \tilde{Z}_{\infty}^{\text{reg}} \). We shall show that for every point \( x' \in \tilde{Z}_{\infty}^{\text{reg}} \) the geodesic segment joining \( x \) and \( x' \) is contained in \( \tilde{Z}_{\infty} \), and as the set of regular points is dense in \( \tilde{Z}_{\infty} \) (as the complementary of a subset of codimension at least 8), this will show that \( Z_{\infty} \) is totally geodesic. We claim that there exist a sequence \( y_k \in \tilde{Z}_{\infty}^{\text{reg}} \) such that \( \lim_{k \to \infty} y_k = x' \) and the geodesic segment joining \( x \) and \( y_k \) is contained in \( \tilde{Z}_{\infty} \).
The claim immediately implies that the geodesic segment joining \( x \) and \( x' \) is contained in \( \tilde{Z}_\infty \).

Let us prove the claim.

For \( y \in \tilde{Z}_\infty \) we consider \( \alpha_{x,y} \) the geodesic joining \( x \) and \( y \) and define

\[
(5.62) \quad t_y = \inf \{ t > 0, \alpha_{x,y}(t) \notin \tilde{Z}_\infty \}
\]

As \( x \) is a regular point, by corollary 5.30, there exist \( \epsilon > 0 \) such that \( t_y > \epsilon \).

In order to prove the claim, we argue by contradiction. Let us assume that there exist \( r > 0 \) such that for any \( y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty \), \( t_y < \text{dist}(x, y) \).

By corollary 5.30 applied to \( \tilde{Z}_\infty \), we have \( \alpha_{x,y}(t_y) \in \tilde{Z}_\infty \). As the set of regular points is an open subset of \( \tilde{Z}_\infty \), if \( r \) is small enough we have \( B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty = B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty \). We choose such an \( r \) and we consider the set \( S \) of all singular points contained in the union of all geodesic segments joining \( x \) to a point \( y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty \). Let us consider the map defined on \( S \) by

\[
p(y) = \alpha_{x,y}(\epsilon).
\]

As we already saw, for any \( y \in B_{\tilde{X}}(x', r) \cap \tilde{Z}_\infty \), we have \( t_y > \epsilon \), therefore the map \( p \) is distance decreasing and by assumption \( p \) is surjective onto an open subset of the sphere \( p(S) \) is homeomorphic to an open subset of \( \mathbb{R}^{n-1} \), therefore the Hausdorff dimension of \( S \) is greater than or equal to \( n-1 \), which contradicts the fact that the singular set has codimension at least 8 in \( \tilde{Z}_\infty \). \( \Box \)

The totally geodesic hypersurface \( \tilde{Z}_\infty \subset \tilde{X} \) is preserved by \( C \), and \( \tilde{Z}_\infty / C \) is of minimal volume in its homology class.

Let us prove that \( \tilde{Z}_\infty \) is isometric to the hyperbolic space \( \mathbb{H}^{n-1}_R \).

**Lemma 5.32.** \( \tilde{Z}_\infty \) is isometric to the hyperbolic space \( \mathbb{H}^{n-1}_R \).

**Proof:**

As \( \tilde{Z}_\infty / C \) is of minimal volume in its homology class, we have by Proposition 5.1, for all \( x \in \tilde{Z}_\infty \), \( \text{Jac}_{\tilde{E}_x} \tilde{F}(x) = 1 \) and \( \tilde{F}(x) = x \), where \( \tilde{E}_x \) is the tangent space of \( \tilde{Z}_\infty \) at \( x \). Moreover, we saw in the proof of proposition 5.1 that

\[
H = \frac{1}{n-1} Id_{D\tilde{F}(x)(E_x)} = \frac{1}{n-1} Id_{E_x},
\]

therefore we get from (4.11) and \( \tilde{F}(x) = x \), that for all \( u, v \in T_x \tilde{Z}_\infty \),

\[
\int_{\partial \tilde{X}} [DdB_{(x, \theta)}(u, v) + DB_{(x, \theta)}(u)DB_{(x, \theta)}(v)]d\nu_x(\theta)
\]

\[
= \tilde{g}(u, v)
\]

\[(5.63)\]
where \( \tilde{g} \) is the metric on \( \tilde{X} \). As \( \tilde{Z}_\infty \) is totally geodesic, the relation (5.62) remains true with the Busemann function \( B_{\tilde{Z}_\infty} \) of \( \tilde{Z}_\infty \) instead of the Busemann function \( B \) of \( X \):

\[
\int_{\partial X} [\text{II}(\tilde{Z}_\infty)](u,v) + \text{II}(\tilde{Z}_\infty)(u)\text{II}(\tilde{Z}_\infty)(v) = \tilde{g}(u,v).
\]

(5.64)

On the other hand, as \( \tilde{Z}_\infty \) is totally geodesic, its sectional curvature is less than or equal to \(-1\), thus by Rauch comparison theorem, we have

\[
\text{II}(\tilde{Z}_\infty) + \text{II}(\tilde{Z}_\infty) \otimes \text{II}(\tilde{Z}_\infty) \geq \tilde{g}_{\tilde{Z}_\infty}(x,\theta) \geq \tilde{g}(x,\theta)
\]

for all \( \theta \in \partial \tilde{Z}_\infty = \Lambda_C \), where \( \tilde{g}_{\tilde{Z}_\infty} \) is the restriction of \( \tilde{g} \) to \( \tilde{Z}_\infty \).

As the support of the measure \( \nu_x \) is \( \partial \tilde{Z}_\infty = \Lambda_C \) (by convex cocompactness of \( C \)) and the Busemann function is continuous, we get from (5.64) and (5.65) that for all \( x \in \tilde{Z}_\infty \) and all \( \theta \in \tilde{Z}_\infty \)

\[
\int_{\partial \tilde{X}} [\text{II}(\tilde{Z}_\infty)](u,v) + \text{II}(\tilde{Z}_\infty)(u)\text{II}(\tilde{Z}_\infty)(v) = \tilde{g}_{\tilde{Z}_\infty}(x).
\]

(5.66)

and this last relation is characteristic of the hyperbolic space. \( \square \)

**Step 6: Conclusion**

So far we have shown that \( C \) preserves a totally geodesic copy of the hyperbolic space \( \mathbb{H}^{n-1}_R \subset \tilde{X} \) such that \( \mathbb{H}^{n-1}_R/C \) is compact.

Our goal now is to show that \( Y := \mathbb{H}^{n-1}_R/C \) injects diffeomorphically in \( X = \tilde{X}/\Gamma \) and separates \( X \) in two connected components \( R \) and \( S \) such that \( \pi_1(R) = \bar{A} \) and \( \pi_1(S) = \bar{B} \).

In order to do this, we will consider the \( \Gamma \) orbit of \( \mathbb{H}^{n-1}_R \) in \( \tilde{X} \) and the two connected components \( U \) and \( V \) of \( \tilde{X} = \tilde{X} - \Gamma \mathbb{H}^{n-1}_R \) which are adjacent to \( \mathbb{H}^{n-1}_R \).

The stabilizers \( \bar{A} \), \( \bar{B} \) and \( \bar{C} \) of \( U \), \( V \) and \( \mathbb{H}^{n-1}_R \) contain respectively \( A \), \( B \) and \( C \) and the hypersurface \( \mathbb{H}^{n-1}_R/C \) injects in \( X = \tilde{X}/\Gamma \) and separates \( X \) in two connected components \( R \) and \( S \) such that \( \pi_1(R) = \bar{A} \) and \( \pi_1(S) = \bar{B} \). We then show that \( \bar{C} = C \), \( \bar{A} = A \) and \( \bar{B} = B \).

Let \( \bar{C} \) be the stabilizer of \( \mathbb{H}^{n-1} \), namely \( \bar{C} = \{ \gamma \in \Gamma : \gamma \mathbb{H}^{n-1} = \mathbb{H}^{n-1} \} \).

We have \( C \subset \bar{C} \) and as \( \mathbb{H}^{n-1}/C \) is compact, so is \( \mathbb{H}^{n-1}/\bar{C} \) and thus \( [C : \bar{C}] \) is compact.

Let \( p : \tilde{X}/C \rightarrow X = \tilde{X}/\Gamma \) and \( \bar{p} : \tilde{X}/\bar{C} \rightarrow X = \tilde{X}/\Gamma \) the natural projections. We now show that the restriction of \( p \) to \( \mathbb{H}^{n-1}/C \) is an embedding, thus \( Y := p(\mathbb{H}^{n-1}/C) \) is a compact totally geodesic hypersurface of \( X \).
In the section 2, we constructed a $C$-invariant hypersurface $\tilde{Z} \subset \tilde{X}$ such that $Z = \tilde{Z}/C \subset \tilde{X}/C$ is compact. The hypersurface is defined as $\tilde{Z} = \tilde{f}^{-1}(t_0)$ where $\tilde{f} : \tilde{X} \to T$ is an equivariant map onto the Bass-Serre tree associated to the amalgamation $A \ast_C B$, and $t_0$ belongs to that edge of $T$ which is fixed by $C$.

Let us first show two lemmas.

**Lemma 5.33.** The restriction of $p$ to $\tilde{Z}/C$ is an embedding into $X = \tilde{X}/\Gamma$.

**Proof:** Let $\gamma \in \Gamma$, $z, z' \in \tilde{Z}$ such that $z' = \gamma z$. By equivariance,

$$\tilde{f}(\gamma z) = \gamma \tilde{f}(z) = \gamma t_0 = \tilde{f}(z') = t_0,$$

thus $\gamma \in C$. $\square$

**Lemma 5.34.** The restriction of $p$ to $\mathbb{H}^{n-1}/C$ is an embedding into $X = \tilde{X}/\Gamma$.

**Proof:** Let us assume that there is a $\gamma \in \Gamma - C$ such that $\gamma \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1} \neq \emptyset$ and choose an $x \in \mathbb{H}^{n-1} \cap \mathbb{H}^{n-1}$. As $\gamma \notin C$, there exist $u \in T_x \gamma \mathbb{H}^{n-1} - T_x \mathbb{H}^{n-1}$. We consider $c_u$, the geodesic ray such that $c_u(0) = u$.

We know that $\tilde{Z}$ is contained in an $\epsilon$-neighbourhood $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$. The $\epsilon$-neighbourhood $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ of $\mathbb{H}^{n-1}$ separates $X$ in two connected components $U$ and $V$ and for $t > 0$ large enough, we have, say, $c_u(t) \in U$ and $c_u(-t) \in V$.

Let $\tilde{Z}'$ be the connected component of $\tilde{Z}$ that we constructed at the end of section 2, whose stabilizer (or an index two subgroup of it) $C'$ is such that $\tilde{Z}'/C'$ separates $\tilde{X}/C'$ in two unbounded connected components $U'/C'$ and $V'/C'$ where $U'$ and $V'$ are the two connected components of $\tilde{X} - \tilde{Z}'$.

We claim that $U \subset U'$ and $V \subset V'$ or the other way around. Indeed if not, $U$ and $V$ would be both contained in $U'$. But in that case, $V'$ would be contained in $\mathcal{U}_\epsilon \mathbb{H}^{n-1}$ and therefore $V'/C'$ would be bounded, which is a contradiction.

As $\gamma \tilde{Z}$ lies in the $\epsilon$ neighborhood of $\gamma \mathbb{H}^{n-1}$, there exist sequences $z_k, z'_k$ in $\gamma \tilde{Z}$ such that $dist(z_k, c_u(k)) \leq \epsilon$ and $dist(z'_k, c_u(-k)) \leq \epsilon$. By proposition 5.13 and lemma 5.23, $C'$ also acts cocompactly on $\mathbb{H}^{n-1}$, thus $C'$ is of finite index in $C$, and therefore there are finitely many connected components of $\tilde{Z}$ and the same holds for $\gamma \tilde{Z}$. We thus can assume that the $z_k$’s and $z'_k$’s belong to a single connected component of $\gamma \tilde{Z}$. Let us consider a continuous path $\alpha \subset \gamma \tilde{Z}$ joining $z_k$ and $z'_k$.

By construction the distance between $c_u(k)$ [resp. $c_u(-k)$] and $\mathbb{H}^{n-1}$ tends to infinity and thus, for $k$ large enough, $z_k \in U$ and $z'_k \in V$ or the other way around. By the claim, we then have $z_k \in U'$ and $z'_k \in V'$, therefore the path $\alpha$ has to cross $\tilde{Z}'$ which contradicts the lemma (5.29) and ends the proof of the lemma 5.30. $\square$

As we already saw, $\tilde{Z}$ has finitely many connected components, and so does $\tilde{X} - \tilde{Z}$. Let us write $\{W_j\}_{j=1,\ldots,m}$ the connected components of $\tilde{X} - \tilde{Z}$.

As $C$ acts cocompactly on $\tilde{Z}$ and $\mathbb{H}^{n-1}$ there exist $\epsilon > 0$ such that $\mathbb{H}^{n-1} \subset \tilde{X} - \tilde{Z}$.
\[ \mathcal{U}_t \tilde{Z} \] and \[ \tilde{Z} \subset \mathcal{U}_t \mathbb{H}^{n-1} \]. Moreover \[ \mathcal{U}_t \mathbb{H}^{n-1} \] separates \( \tilde{X} \) in two connected components \( U \) and \( V \).

**Lemma 5.35.** Let us consider \( \epsilon \) such that \( \tilde{Z} \subset \mathcal{U}_t \mathbb{H}^{n-1} \) and \( U \) and \( V \) the two connected components of \( \tilde{X} - \mathcal{U}_t \mathbb{H}^{n-1} \). There are two distinct connected components \( W_1 \) and \( W_2 \) of \( \tilde{X} - \tilde{Z} \) such that \( U \subset W_1 \) and \( V \subset W_2 \). Moreover, \( f(W_1) \subset \tilde{T}_1 \) and \( f(W_2) \subset \tilde{T}_2 \), where \( \tilde{T}_1 \) and \( \tilde{T}_2 \) are the two connected components of \( \tilde{T} - \{ t_0 \} \).

**Proof:** We argue by contradiction. Let us assume that \( U \) and \( V \) are contained in the same connected component \( W \). Then, all other components \( W_j, j \neq 1 \), satisfy \( W_j \subset \mathcal{U}_t \mathbb{H}^{n-1} \subset \mathcal{U}_t \tilde{Z} \). Therefore, as \( C \) acts cocompactly on \( \mathcal{U}_t \tilde{Z} \), there exist a constant \( D \) such that for any \( j \neq 1 \), \( \max_{w \in W_j} \text{dist}_T(f(w), t_0) \leq D \). Thus, \( f(W_1) \) is contained in one connected component of \( \tilde{T} - \{ t_0 \} \) and \( f(U_{j \neq 1} W_j) \), contained in the ball \( B_{\tilde{T}}(t_0, D) \) of \( \tilde{T} \) of radius \( D \) centered at \( t_0 \), is bounded. This is clearly impossible because \( \tilde{T} - \{ t_0 \} \) has two unbounded connected components and \( f \) is onto. \( \square \)

Let us denote \( \mathcal{A} = A \mathbb{H}^{n-1} \) the \( A \)-orbit of the \( C \)-invariant totally geodesic copy of the real hyperbolic space \( \mathbb{H}^{n-1} \), and \( \mathcal{A} \) the stabilizer of \( \mathcal{A} \), ie. \( \mathcal{A} = \{ \gamma \in \Gamma, \gamma \mathcal{A} = \mathcal{A} \} \). We define in a similar way \( \mathcal{B} = B \mathbb{H}^{n-1} \) and \( \mathcal{B} = \{ \gamma \in \Gamma, \gamma \mathcal{B} = \mathcal{B} \} \).

Let us recall that \( \mathcal{C} \) is the stabilizer of \( \mathbb{H}^{n-1} \) in \( \Gamma \). We now prove the following

**Lemma 5.36.** We have \( \tilde{A} = A \tilde{C} \) and \( \tilde{B} = B \tilde{C} \). Moreover, \( \tilde{A} \) and \( \tilde{B} \) are charactized by \( \tilde{A} = \{ \gamma \in \Gamma, \gamma \mathbb{H}^{n-1} \in \mathcal{A} \} \) and \( \tilde{B} = \{ \gamma \in \Gamma, \gamma \mathbb{H}^{n-1} \in \mathcal{B} \} \).

**Proof:** Let \( \gamma' \in \tilde{A} \), then \( \gamma' \mathbb{H}^{n-1} \in \mathcal{A} \) and thus there exist \( \gamma \in \mathcal{A} \) such that \( \gamma' \mathbb{H}^{n-1} = \gamma \mathbb{H}^{n-1} \), therefore \( \gamma^{-1} \gamma' \in \mathcal{C} \), which proves the first part of the lemma.

Let us prove the second part of the lemma.

Let \( \gamma' \in \Gamma \) be such that \( \gamma' \mathbb{H}^{n-1} \in \mathcal{A} \). Then there exist \( \gamma \in \mathcal{A} \) such that \( \gamma' \mathbb{H}^{n-1} = \gamma \mathbb{H}^{n-1} \), thus \( \gamma^{-1} \gamma' \in \mathcal{C} \) and therefore \( \gamma' \in \mathcal{C} \subset \mathcal{A} \tilde{C} = \tilde{A} \). This proves one inclusion, the other inclusion being obvious. \( \square \)

For each \( \gamma \in \Gamma \), \( \gamma \mathbb{H}^{n-1} \) separates \( \tilde{X} \) in two connected components \( U_\gamma \) and \( V_\gamma \).

Let us now prove the following lemma.

**Lemma 5.37.** (i) Let \( \gamma \in \mathcal{A} \), [resp. \( \gamma \in \mathcal{B} \)]. Then, we have \( \mathcal{A} - \{ \gamma \mathbb{H}^{n-1} \} \subset U_\gamma \) or \( \mathcal{A} - \{ \gamma \mathbb{H}^{n-1} \} \subset V_\gamma \), [resp. \( \mathcal{B} - \{ \gamma \mathbb{H}^{n-1} \} \subset U_\gamma \) or \( \mathcal{B} - \{ \gamma \mathbb{H}^{n-1} \} \subset V_\gamma \)]

(ii) Let \( \gamma \) be an element of \( \Gamma - \mathcal{A} \), [resp. \( \Gamma - \mathcal{B} \)]. Then \( \mathcal{A} \subset U_\gamma \) or \( \mathcal{A} \subset V_\gamma \), [resp. \( \mathcal{B} \subset U_\gamma \) or \( \mathcal{B} \subset V_\gamma \)].

**Proof:** (i) We argue by contradiction. Let us consider \( \gamma' \mathbb{H}^{n-1} \), \( \gamma'' \mathbb{H}^{n-1} \) and \( \gamma''' \mathbb{H}^{n-1} \) three distinct elements in \( \mathcal{A} \) such that \( \gamma' \mathbb{H}^{n-1} \subset U_\gamma \) and \( \gamma'' \mathbb{H}^{n-1} \subset V_\gamma \). By equivariance we can assume \( \gamma \) is the identity. Let us recall that \( U \) and \( V \) are the two connected components of \( \tilde{X} - \mathcal{U}_t \mathbb{H}^{n-1} \).
We then have $\gamma \mathbb{H}^{n-1} \cap U \neq \emptyset$ and $\gamma'' \mathbb{H}^{n-1} \cap V \neq \emptyset$, which implies $\gamma' \mathcal{Z} \cap U \neq \emptyset$ and $\gamma'' \mathcal{Z} \cap V \neq \emptyset$.

By lemma (5.31), $U \subset W_1$ and $V \subset W_2$ where $W_1$ and $W_2$ are two connected components of $\mathcal{X} - \mathcal{Z}$ and $\tilde{f}(U) \subset \tilde{T}_1$ and $\tilde{f}(V) \subset \tilde{T}_2$, therefore, $\tilde{f}(U)$ contains $\gamma' t_0 \in \tilde{T}_1$ and $\tilde{f}(V)$ contains $\gamma'' t_0 \in \tilde{T}_2$. This is impossible because for all elements $\gamma'$ and $\gamma''$ in $A$, $\gamma' t_0$ and $\gamma'' t_0$ belong to the same connected component of $\tilde{T} - \{t_0\}$.

(ii) Let us consider $\gamma \in \Gamma - A$. We argue by contradiction. Let us assume there exist $\gamma$, $\gamma'$ in $A$ such that

$$\gamma' \mathbb{H}^{n-1} \subset U_{\gamma}$$  

$$\gamma'' \mathbb{H}^{n-1} \subset V_{\gamma}$$  

Let $\epsilon > 0$ such that $\mathcal{Z} \subset U_{\mathbb{H}^{n-1}}$ and $U$ and $V$ the connected component of $\mathcal{X} - \mathbb{H}^{n-1}$. By lemma (5.31) we have $\tilde{f}(\gamma U) \subset \gamma \tilde{T}_1$ and $\tilde{f}(\gamma V) \subset \gamma \tilde{T}_2$, where $\gamma \tilde{T}_1$ and $\gamma \tilde{T}_2$ are the two connected components of $\tilde{T} - \{\gamma t_0\}$. By assumption (5.62), we have

$$\gamma' \mathbb{H}^{n-1} \cap \gamma U \neq \emptyset$$ and $$\gamma'' \mathbb{H}^{n-1} \cap \gamma V \neq \emptyset,$$

which implies $\gamma' \mathcal{Z} \cap \gamma U \neq \emptyset$ and $\gamma'' \mathcal{Z} \cap \gamma V \neq \emptyset$, therefore $\gamma' t_0 \in \gamma \tilde{T}_1$ and $\gamma'' t_0 \in \gamma \tilde{T}_2$, which is impossible because in the tree $\tilde{T}$, the points $\gamma' t_0$ and $\gamma'' t_0$ belong to two adjacent edges.

By lemma (5.33) (i), for every $\gamma$ in $A$, [resp. $B$], we can define $U_{\gamma}$ as the connected component of $\mathcal{X} - \mathbb{H}^{n-1}$ which contains all $\gamma' \mathbb{H}^{n-1}$ for all $\gamma'$ in $A$, [resp. $B$], and $\gamma' \mathbb{H}^{n-1} \neq \gamma'' \mathbb{H}^{n-1}$.

Let us define

$$U_A := \cap_{\gamma \in A} U_{\gamma}.$$  

By definition, $U_A$ [resp. $U_B$] is a convex set in $\mathcal{X}$ whose boundary is the collection $A$, [resp. $B$], of $\gamma \mathbb{H}^{n-1}$, $\gamma$ in $A$ [resp. $B$] and by lemma (5.33) (ii), $U_A$ and $U_B$ are two disjoint connected components of $\mathcal{X} - \Gamma \mathbb{H}^{n-1}$.

In fact, $U_A$ [resp. $U_B$], is the convex hull of $A$, [resp. $B$], and $A$, [resp. $B$], is the stabilizer of $U_A$, [resp. $U_B$].

Lemma 5.38. The closures of $U_A$ and $U_B$ intersect along $\mathbb{H}^{n-1}$ and $\bar{A} \cap \bar{B} = \bar{C}$. Moreover, no element of $\Gamma$ sends $U_A$ on $U_B$ nor the other way around.

Proof : The convex set $U_A$ is the intersection of open half spaces $U_{\gamma}$, $\gamma \in A$, and is delimited by the disjoint union of hyperplanes $\gamma \mathbb{H}^{n-1}$, for some $\gamma \in A$. The same is true for $U_B$ and as $U_A \cap U_B = \emptyset$, the closures of $U_A$ and $U_B$ can intersect only along one of the connected components of their boundaries, thus along $\mathbb{H}^{n-1}$ which is obviously in both closures. This proves the first part of the lemma, let us prove the second part. By lemma 5.32, $\bar{C} \subset \bar{A} \cap \bar{B}$. Conversely, let us take $\gamma \in \bar{A} \cap \bar{B}$, then $\gamma$ preserves
the closures of $U_A$ and $U_B$, thus it preserves their intersection $\mathbb{H}^{n-1}$, and therefore $\gamma \in \bar{C}$.

Let us prove the last part of the lemma. Let $\gamma$ be an element such that $\gamma U_A = U_B$. As $\mathbb{H}^{n-1}$ is one component of the boundary $B$ of $U_B$, there exist one component $\gamma' \mathbb{H}^{n-1} \in A$, $\gamma'$ being in $\bar{A}$, such that $\gamma(\gamma' \mathbb{H}^{n-1}) = \mathbb{H}^{n-1}$. Therefore, $\gamma \gamma' \in \bar{C}$, thus $\gamma \in \bar{A}$. The same argument yields $\gamma^{-1} \in B$, so $\gamma \in \bar{A} \cap \bar{B} = \bar{C}$ and $\gamma$ preserves $U_A$ and $U_B$, which contradicts our choice of $\gamma$.

The $\Gamma$-orbit of the closure of $U_A \cup U_B$ covers $\bar{X}$. Let us construct a tree $\bar{T}$ embedded in $\bar{X}$ in the following way: the set of vertices is the set of the connected components of $\bar{X} - \Gamma \mathbb{H}^{n-1}$ and two vertices are joined by an edge if the boundaries of their corresponding connected components intersect non trivially in $\bar{X}$. By construction $\Gamma$ acts on $\bar{T}$, the stabilizers of the vertices $a$ and $b$ corresponding to $U_A$ and $U_B$ are $\bar{A}$ and $\bar{B}$, the stabilizer of the edge between $a$ and $b$ is $\bar{C}$ and a fundamental domain for this action is the segment joining $a$ and $b$. By [4], 1, 4, Theorem 6, the group $\Gamma$ is the amalgamated product of $\bar{A}$ and $\bar{B}$ over $\bar{C}$.

We now claim that $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$.

As $A$, $B$ and $C$ are subgroups of $\bar{A}$, $\bar{B}$, and $\bar{C}$, the corresponding Mayer-Vietoris sequences of $A \ast_C B$ and $\bar{A} \ast_{\bar{C}} \bar{B}$ are related by the following commutative diagram

$$
\begin{array}{ccc}
H_n(A, \mathbb{R}) \oplus H_n(B, \mathbb{R}) & \rightarrow & H_n(\Gamma, \mathbb{R}) \\
\downarrow & & \downarrow \\
H_n(\bar{A}, \mathbb{R}) \oplus H_n(\bar{B}, \mathbb{R}) & \rightarrow & H_n(\bar{\Gamma}, \mathbb{R}) \\
\end{array}
$$

We know that the index $[\bar{C} : C]$ is finite, and by the lemma 5.32, the indices of $A$, $B$, and $C$ in $\bar{A}$, $\bar{B}$ and $\bar{C}$ are finite and equal. On the other hand, the indices $[\Gamma : A]$ and $[\Gamma : B]$ are infinite by assumption, thus the previous diagram becomes

$$
\begin{array}{ccc}
0 & \rightarrow & H_n(\Gamma, \mathbb{R}) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H_n(\bar{\Gamma}, \mathbb{R})
\end{array}
$$

Moreover, the map $H_n(\Gamma, \mathbb{R}) \rightarrow H_n(\bar{\Gamma}, \mathbb{R})$ is bijective. Namely, the injectivity comes from the above diagram and the surjectivity from the fact that that the hypersurface $\mathbb{H}^{n-1}/\bar{C}$ bounds in $\mathbb{H}^n/A$ and $\mathbb{H}^n/B$ so that the map $H_{n-1}(\bar{C}, \mathbb{R}) \rightarrow H_{n-1}(A, \mathbb{R}) \oplus H_{n-1}(B, \mathbb{R})$ is trivial. Therefore the index $[\bar{C} : C] = 1$ and we get $\bar{A} = A$, $\bar{B} = B$ and $\bar{C} = C$. □

6. Proof of the Theorems 1.5 and 1.6.

The proof of theorem 1.5 is exactly the same as the proof of theorem 1.2. The actions of $\Gamma$ on $(X)$ and $T$ give rise to a continuous $\Gamma$-equivariant map $\bar{f} : \bar{X} \rightarrow T$. Like in section 2, we build an hypersurface $\bar{f}^{-1}(t_0)$ where $t_0$ is
a regular value of $\tilde{f}$ belonging the interior of an edge. As the edge separates the tree in two unbounded components, the section 2 applies and we get a subgroup $C'$ of $C$, and an hypersurface $\tilde{Z}' \subset X/C'$ which is essential. Now, if the action of $\Gamma$ is minimal, every edge separates $T$ in two unbounded components. □

7. Appendix

The goal of this section is to give a proof of lemma 5.9. This lemma is contained in lemma 2.1, 5.1 and 5.2 of [4], but our situation being not exactly the same, we reproduce it down here for sake of completeness.

Let us restate the lemma 5.9.

**Lemma 7.1.** Let $L \subset \partial \tilde{X}$ be a closed $C'$-invariant subset and $\theta_0 \in L$. We assume that there exist a sequence of positive real numbers $\lambda_k \to \infty$ such that the sequence of pointed metric spaces $(L, \lambda_k d, \theta_0)$ converges in the pointed Gromov-Hausdorff topology to $(S, \delta, 0)$ where $(S, \delta, 0)$ is a weak tangent of $(\partial \tilde{X}, d)$. We also assume that there exist positive constants $C$ and $\delta$, a sequence of points $\theta_0^k = \theta_0, \theta_1^k, \theta_2^k \in L$ and a sequence of elements $\gamma_k \in C'$ such that $C^{-1}\lambda_k^{-1} \leq d(\theta_1^k, \theta_2^k) \leq C\lambda_k^{-1}$ and $d(\gamma_k, \gamma_k^0) \geq \delta$ for all $0 \leq i \neq j \leq 2$. Then, $L$ is homeomorphic to the one point compactification $\hat{S}$ of $S$. In particular $L$ is homeomorphic to $\partial \tilde{X}$.

We first give a definition of pointed Hausdorff-Gromov convergence which is equivalent to the definition 5.7. We follow [4], paragraph 4.

A sequence of metric spaces $(Z_k, d_k, z_k)$ converges to the metric space $(S, \delta, 0)$ if for every $R > 0$, and every $\epsilon > 0$, there exist an integer $N$, a subset $D \subset B_S(0, R)$, subsets $D_k \subset B_{Z_k}(z_k, R)$ and bijections $f_k : D_k \to D$ such that for $k \geq N$,

(i) $f_k(z_k) = 0$,
(ii) the set $D$ is $\epsilon$-dense in $B_S(0, R)$, and the sets $D_k$ are $\epsilon$-dense in $B_{Z_k}(z_k, R)$,
(iii) $|d_{Z_k}(x, y) - d_Z(f_k(x), f_k(y))| < \epsilon$,
where $x, y$ belong to $D_k$.

Let us describe now the lemmas 2.1 and 5.1 following [4].

For a metric space $(Z, d)$ the cross ratio of four points $\{z_i\}, i = 1, \ldots, 4$, is the quantity

\[
[z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}
\]

Given two metric spaces $X$ and $Y$, an homeomorphism $\eta : [0, \infty) \to [0, \infty)$, and an injective map $f : X \to Y$, we say that $f$ is an $\eta$-quasi-Möbius map if for any four points $\{x_i\}, i = 1, \ldots, 4$, in $X$, we have

\[
[f(x_1), f(x_2), f(x_3), f(x_4)] \leq \eta([x_1, x_2, x_3, x_4]).
\]
For example, any discrete cocompact group of isometries of $\hat{X}$, where $\hat{X}$ is a Cartan-Hadamard manifold with sectional curvature $K \leq -1$, is acting on the ideal boundary $(\partial \hat{X}, d)$ endowed with the Gromov distance by $\eta$-quasi-Möbius transformations for some $\eta$.

**Lemma 7.2** ([4], Lemma 2.1 ). Let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces, and for any integer $k$, $g_k : \tilde{D}_k \to Y$ an $\eta$-quasi-Möbius map defined on a subset $\tilde{D}_k$ of $X$. We assume that the Hausdorff distance between $\tilde{D}_k$ and $X$ satisfies

$$\lim_{k \to \infty} \text{dist}_H(\tilde{D}_k, X) = 0$$

and that for any integer $k$, there exist points $(x^k_1, x^k_2, x^k_3)$ in $D_k$ and $(y^k_1, y^k_2, y^k_3)$ in $Y$, such that $g_k(x^k_i) = y^k_i$ for $i \in \{1, 2, 3\}$, $d_X(x^k_i, x^k_j) \geq \delta$ and $d_Y(y^k_i, y^k_j) \geq \delta$ for $i, j \in \{1, 2, 3\}, i \neq j$, where $\delta$ is independant of $k$. Then a subsequence of $g_k$ converges uniformly to a quasi-Möbius map $f : X \to Y$, i.e. $\lim_{k_j \to \infty} \text{dist}_H(g_{k_j}, f|_{\tilde{D}_{k_j}}) = 0$. If in addition, we suppose that

$$\lim_{k \to \infty} \text{dist}_H(g_k(\tilde{D}_k), Y) = 0,$$

then the sequence $\{g_{k_j}\}$ converges uniformly to a quasi-Möbius homeomorphism $f : X \to Y$.

Before stating the second lemma, let us define a metric space $Z$ to be uniformly perfect if there exist a constant $\lambda > 1$ such that for every $z \in Z$ and $0 < R < \text{diam}Z$, we have $B(z, R) - B(z, R/\lambda) \neq \emptyset$.

**Lemma 7.3** ([4], lemma 5.1 ). Let $Z$ be a compact uniformly perfect metric space and $G$ an $\eta$-quasi-Möbius action on $Z$. Suppose that for each integer $k$ we are given a set $D_k$ in a ball $B_k = B(z, R_k) \subset Z$ that is $(\epsilon_k R_k)$-dense in $B_k$, where $\epsilon_k > 0$, distinct points $x^k_1, x^k_2, x^k_3 \in B(z, \lambda_k R_k)$, where $\lambda_k > 0$, with

$$d_Z(x^k_i, x^k_j) \geq \delta_k R_k$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where $\delta_k > 0$, and groups elements $\gamma_k \in G$ such that for $y^k_i := \gamma_k(x^k_i)$ we have,

$$d_Z(y^k_i, y^k_j) \geq \delta'$$

for $i, j \in \{1, 2, 3\}, i \neq j$, where $\delta'$ is independant of $k$. Let $D'_k = \gamma_k(D_k)$, and suppose that $\lambda_k \to 0$ when $k \to \infty$, and the sequence $\frac{\delta}{\delta_k}$ is bounded. Then $\lim_{k \to \infty} \text{dist}_H(D'_k, Z) = 0$.

Let us go back to the proof of lemma 6.1. By definition of convergence, there exist a subsequence of $\{\lambda_k\}$, which we still denote by $\{\lambda_k\}$, subsets $\tilde{D}_k \subset B_S(0, k)$, $D_k \subset B_{\lambda_k L}(0, k)$, where $\tilde{D}_k$ and $D_k$ are minimal $1/k$-dense
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subsets of $B_S(0, k)$ and $B_\lambda(\mathcal{L}, \lambda d)(\theta_0, k)$, and bijections $f_k : \tilde{D}_k \to D_k$ such that for all $x, y \in \tilde{D}_k$,

\begin{equation}
\frac{1}{2} \delta(x, y) \leq \lambda_k d(f_k(x), f_k(y)) \leq 2\delta(x, y),
\end{equation}

cf. [4], (5.4).

We can suppose that the points $\theta_0^k := \theta_0$, $\theta_1^k$, and $\theta_2^k$ in lemma 6.1 belong to the set $D_k$. By assumption there exist elements $\gamma_k \in C'$ and a constant $\delta$ such that

\begin{equation}
d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta
\end{equation}

for all $i, j \in \{0, 1, 2\}$.

The lemma 6.1 is a direct consequence of the lemma 6.2 applied to $(\mathcal{X}, d\mathcal{X}) = (\tilde{S}, \delta)$ and $(\mathcal{Y}, d\mathcal{Y}) = (\mathcal{L}, d)$ and to the sequence of maps $g_k := \gamma_k \circ f_k$, where $\tilde{S}$ is the one point compactification of $S$ and $\delta$ the distance on $\tilde{S}$ associated to $\delta$, cf. [4] Lemma 2.2.

Let us denote $x_0^k, x_1^k, x_2^k$ be the points in $S$ such that $f_k(x_0^k) = \theta_0^k$, for $i \in \{0, 1, 2\}$.

Let us check that the assumptions of lemma 6.2 are verified.

The fact that $\lim_{k \to \infty} \text{dist}_H(\tilde{D}_k, S) = 0$ comes the same way as in [4], (5.5).

By (6.3), we have, $\delta(x_i^k, x_j^k) \geq \frac{1}{2} d(\theta_i^k, \theta_j^k)$ and by assumption we then get

\begin{equation}
\delta(x_i^k, x_j^k) \geq \frac{1}{2C}.
\end{equation}

We then get the separation assumption on triples of points by choosing $\delta := \inf \{D, \frac{1}{2C} \}$.

It remains to check the assumption on $g_k(\tilde{D}_k) = \gamma_k \circ f_k(\tilde{D}_k) = \gamma_k(D_k)$, namely,

\begin{equation}
\lim_{k \to \infty} \text{dist}_H(\gamma_k(D_k), \Lambda C') = 0.
\end{equation}

In order to prove the property (6.6), we want to apply the lemma 6.3, but as the set $(\mathcal{L}, d)$ is a priori not uniformly perfect, we shall replace the uniform perfectness by the fact that $(\mathcal{L}, \lambda_k d, \theta_0)$ converges to a space $(S, d_0, 0)$, which is uniformly perfect, cf. ()

We will show the

**Lemma 7.4.** We consider the subsets $\tilde{D}_k \subset B_S(0, k)$ and $D_k \subset B_\lambda(\mathcal{L}, \lambda d)(\theta_0, k)$, where $\tilde{D}_k$ and $D_k$ are $1/k$-dense subsets of $B_S(0, k)$ and $B_\lambda(\mathcal{L}, \lambda d)(\theta_0, k)$, and the bijections $f_k : \tilde{D}_k \to D_k$ coming from the convergence of the sequence of pointed metric spaces $(\mathcal{L}, \lambda d, \theta_0)$ to $(S, d, 0)$ where $(S, d, 0)$ is a weak tangent of $(\partial X, d)$. We also assume that there exist positive constants $C$ and $\delta$, a
sequence of points \( \theta_i^k, \theta_j^k \in \Lambda_C \) and a sequence of elements \( \gamma_k \in C' \) such that
\[
C^{-1} \lambda_k^{-1} \leq d(\theta_i^k, \theta_j^k) \leq C \lambda_k^{-1} \quad \text{and} \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta \quad \text{for all } 0 \leq i \neq j \leq 2.
\]
Then, the Hausdorff distance \( \text{dist}_H(\gamma_k D_k, \mathcal{L}) \) tends to 0 as \( k \) tends to infinity.

**Proof:** The proof is word by word the same as the proof of lemma 6.3, i.e. lemma 5.1 (i) of [4] with a difference in case 2).

We have \( B_{\lambda_k} \varepsilon(\theta_0, k) = B_{\lambda_k}(\theta_0, k) \) and \( D_k \subset B_{\lambda_k} \varepsilon(\theta_0, k) \) an \( \frac{1}{k} \)-dense subset, for the metric \( \lambda_k d \). In term of the distance \( d \), the set \( D_k \) is \( (\varepsilon_k R_k) \)-dense in \( B_{\lambda_k}(\theta_0, R_k) \), where \( R_k := \frac{1}{\lambda_k} \) and \( \varepsilon_k := \frac{1}{k} \). By assumption, the points \( \theta_0^k = \theta_0, \theta_1^k, \theta_2^k \) belong to \( B_{\lambda_k}(\theta_0, \varepsilon_k R_k) \), and satisfy
\[
(7.7) \quad d(\theta_i^k, \theta_j^k) \geq \delta_k R_k
\]
where \( \delta_k := \frac{1}{\varepsilon_k} \), and \( \varepsilon_k := \frac{C}{k} \).

The points \( \gamma_k \theta_i^k \) satisfy
\[
(7.8) \quad d(\gamma_k \theta_i^k, \gamma_k \theta_j^k) \geq \delta,
\]
and \( \frac{\delta_k}{k} = C^2 \) is bounded.

Let us consider a point \( \theta \in \mathcal{L} \). We want to approximate it by a point of \( \gamma_k D_k \).

We can write \( \theta = \gamma_k \theta_k \), for some \( \theta_k \in \Lambda_{C'} \). There are two cases.

Case 1). For infinitely many indices \( k \), \( \theta_k \in B_{\lambda_k}(\theta_0, R_k) \). We work in that case for these indices \( k \), thus there are points \( \theta'_k \in D_k \cap B_{\lambda_k}(\theta_0, R_k) \), with \( d(\theta, \theta'_k) \leq \varepsilon_k R_k \).

Since the distance between the \( \theta'_k \)'s is bounded below by \( \delta_k R_k \), we can find at least two of them which we call \( a_k \) and \( b_k \), such that
\[
d(\theta_k, b_k) \geq \frac{\delta_k R_k}{2}
\]
and,
\[
d(\theta'_k, a_k) \geq \frac{\delta_k R_k}{2}.
\]

As \( C' \) is contained in the cocompact group \( \Gamma \), it acts in a quasi-Möbius way on \( (\partial X, d) \) thus,
\[
d(\gamma_k \theta'_k, \gamma_k \theta_k) \leq \eta(\gamma_k \theta'_k, \gamma_k \theta_k, b_k) \leq \eta(\gamma_k \theta'_k, \gamma_k \theta_k, a_k, b_k)
\]
for some homeomorphism \( \eta : [0, \infty) \to [0, \infty) \). This implies
\[
(7.11) \quad d(\gamma_k \theta'_k, \gamma_k \theta_k) \leq \frac{(\text{diam} \mathcal{L})^2 \eta(\theta_k, \theta'_k, \delta_k)}{\delta},
\]
therefore \( d(\gamma_k \theta'_k, \gamma_k \theta_k) \) tends to zero as \( k \) tends to infinity.

Case 2). For all but finitely many indices \( k \), \( \theta_k \notin B_{\lambda_k}(\theta_0, R_k) \).
We work with these indices \( k \) such that \( \theta_k \notin B_L(\theta_0, R_k) \).

We know that \( \epsilon_k/\delta_k^2 \) is bounded above independently of \( k \), and by assumption, \( \delta_k \leq 2\mu_k \).

We claim that there exist \( \xi_k \in D_k \) and a positive constant \( c_0 \) such that for all \( k \),

\[
\frac{d(\xi_k, \theta_0)}{R_k} \geq c_0
\]

let us prove the claim.

On one hand, as \((\partial \tilde{X}, d)\) is uniformly perfect, and so is its weak tangent \((S, \delta)\) because the one point compactification \((\hat{S}, \hat{\delta})\) of \((S, \delta)\) is quasi-Möbius homeomorphic to \((\partial \tilde{X}, d)\), therefore there exist a constant \( C_0 \in [0, 1) \) such that for every \( x \in S \) and \( 0 < R < \text{diam}S \), we have

\[
\bar{B}(S, \delta)(0, R - \epsilon) \subset B(S, \delta)(0, C_0 R).
\]

On the other hand, \((\mathcal{L}, \lambda_k, \theta_0)\) converges to \((S, \delta, 0)\). After reindexing the sequence \( \{\lambda_k\} \), we have for each \( \epsilon > 0 \) a map \( g_k : B_{\lambda_k, \mathcal{L}}(\theta_0, k) \to S \) such that

(i) \( g_k(\theta_0) = 0 \),

for any two points \( \theta \) and \( \theta' \) in \( B_{\lambda_k, \mathcal{L}}(\theta_0, k) \),

(ii) \( |\delta(g_k(\theta), g_k(\theta')) - \lambda_k d(\theta, \theta')| \leq \epsilon \),

(iii) the \( \epsilon \)-neighborhood of \( g_k(B_{\lambda_k, \mathcal{L}}(\theta_0, k)) \) contains \( B_{(S, \delta)}(0, k - \epsilon) \).

By (iii), we have

\[
\bar{B}_{(S, \delta)}(0, k - \epsilon) \subset U^{(S, \delta)} g_k(B_{\lambda_k, \mathcal{L}}(\theta_0, k)).
\]

By (6.13) there exist \( y_k \in \bar{B}_{(S, \delta)}(0, k - \epsilon) - B_{(S, \delta)}(0, C_0(k - \epsilon)) \), and by (6.14) there exist \( \xi'_k \in B_{\lambda_k, \mathcal{L}}(\theta_0, k) \) such that

\[
\delta(y_k, g_k(\xi'_k)) \leq \epsilon.
\]

We now evaluate \( d(y_k, g_k(\xi'_k)) \). By the above properties (i), (ii), (6.15) and the triangle inequality we have

\[
\lambda_k d(\xi'_k, \theta_0) \geq \delta(g_k(\xi'_k), 0) - \epsilon \geq \delta(y_k, 0) - \delta(y_k, g_k(\xi'_k)) - \epsilon
\]

\[
\geq C_0(k - \epsilon) - 2\epsilon.
\]

As \( D_k \) is \( \epsilon_k R_k \)-dense in \( B_{(\mathcal{L}, d)}(\theta_0, k/\lambda_k) \), there exist \( \xi_k \in D_k \) such that

\[
d(\xi_k, \xi'_k) \leq \epsilon_k R_k = \frac{k \epsilon_k}{\lambda_k}.
\]
Let us denote $c_0 = C_0/2$. For $k$ large enough we have
\[
\frac{C_0(k-\varepsilon) - 2\varepsilon - k\lambda}{\lambda_k} \geq \frac{c_0}{\lambda_k},
\]
therefore by (6.16) we get
\[
(7.17) \quad d(\xi_k, \theta_0) \geq d(\xi_k', \theta_0) - d(\xi_k', \xi_k) \geq c_0 R_k,
\]
which proves the claim.

We can assume that for $k$ large enough, $\mu_k < c_0/2 < 1/2$.

We choose $a_k = \theta_k^1$ and $b_k = \theta_k^2$, and we get
\[
\frac{d(\gamma_k \xi_k, \gamma_k \theta_k) d(\gamma_k a_k, \gamma_k b_k)}{d(\gamma_k \xi_k, \gamma_k b_k) d(\gamma_k a_k, \gamma_k \theta_k)} \leq \eta\left( \frac{d(\xi_k, \theta_k) d(a_k, b_k)}{d(\xi_k, b_k) d(\theta_k, a_k)} \right)
\leq \eta\left( \frac{4 \mu_k d(\theta_k, \theta_k^0)}{(d(\theta_k, \theta_k^0) - \mu_k R_k)} (c_0 - \mu_k) \right)
\]
\[
(7.18) \quad \leq \eta(16 \mu_k/c_0).
\]

We get
\[
d(\gamma_k \theta_k, \gamma_k \xi_k) \leq (\text{diam} \Lambda C) \eta(16 \mu_k/c_0)/\delta.
\]

\[\square\]

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