Research Article

General Decay of a Nonlinear Viscoelastic Wave Equation with Balakrishnân-Taylor Damping and a Delay Involving Variable Exponents

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This paper was aimed at investigating the stability of the following viscoelastic problem with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term

\[ u_{tt} - \mathcal{M}(\|u\|_p^2)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)\,ds + \mu_1|u|^p(t-x)^2u_t + \mu_2|u_t(t-x)|^p(t-x)^2u_t = 0 \]

in \( \Omega \times (0, \infty) \),

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \]

\[ u_t(x, t) = j_0(x, t-x), \]

\[ u(x, t) = 0, \]

in \( \Omega \times (0, \infty) \),

on \( \partial \Omega \times (0, \infty) \),

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \), \( p(\cdot) : \Omega \rightarrow \mathbb{R} \) is a measurable function, \( g > 0 \) is a memory kernel that decays exponentially, \( \alpha \geq 0 \) is the potential, and \( \mathcal{M}(\|u\|_p^2) = a + b\|u(t-x)|^2 + \sigma \int_\Omega \nabla u \nabla u_t \,dx \) for some constants \( a > 0, b \geq 0, \) and \( \sigma > 0 \). Under some assumptions on the relaxation function, we use some suitable Lyapunov functionals to derive the general decay estimate for the energy. The problem considered is novel and meaningful because of the presence of the flutter panel equation and the spillover problem including membrane and variable-exponent time delay control. Our result generalizes and improves previous conclusion in the literature.

1. Introduction

In recent years, much attention has been paid to the study systems with variable exponents of nonlinearities which are models of hyperbolic, parabolic, and elliptic equations. These models may be nonlinear over the gradient of unknown solutions and have nonlinear variable exponents. Researches of these systems usually use the imbedding of Lebesgue and Sobolev spaces with variable exponents (see, e.g., [1, 2]). Or see [3–14] and the references therein for more details of relevant problems.

In this paper, we concentrate on the asymptotic behavior of weak solutions for the following weakly damped viscoelastic wave equation with Balakrishnân-Taylor damping and variable-exponent nonlinear time delay term
where \( u : \bar{\Omega} \times [0,\infty) \rightarrow \mathbb{R} \) is an unknown function, \( \mu_1 \geq 0, \mu_2 \) is a real number, \( \tau > 0 \) is the time delay, \( g > 0 \) is a memory kernel, and \( \alpha > 0 \) is the potential.

Much attention has been paid to the simulation of phenomena such as the vibration of elastic strings and elastic plates, when \( g = 0 \), and \( \mu_1 = \mu_2 = 0 \); equation (1) degrades into the Kirchhoff’s original equation

\[
\rho \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{E h}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + \frac{f}{L}, \quad 0 \leq x \leq L, \quad t \geq 0,
\]

which was first introduced to study the oscillations of stretched strings and plates in [15]. In addition, equation (2) is also said to be the wave equation of Kirchhoff type, where the unknown function \( u = u(x, t) \) represents lateral deflection and \( E, \rho, h, L, p_0 \), and \( f \) respectively, denote Young’s modulus, mass density, cross-section area, length, initial axial tension, and external force. The Kirchhoff equation has been investigated in a lot of articles due to its abundant physical background. At the present paper, we try to mention some considerable efforts on this topic.

There are many important results, such as the local solutions in time, well-posedness, and solvability; for the Kirchhoff type, equation (2) in general dimensions and domains has been obtained in lots of articles (see, e.g., [16–24] and the references therein).

When \( p > 1 \) identically equals to a constant, problem (1) with the Balakrishnan-Taylor damping term \( (\sigma > 0) \) is related to the flutter panel equation and the spillover problem involving time delay term. Balakrishnan and Taylor in [25] and Bass and Zes in [26] introduced Balakrishnan-Taylor damping, which arises from a wind tunnel experiment at supersonic speeds (see, e.g., [27–32]).

On damping terms, we point out several excellent works: Lian and Xu in [33] studied a class of nonlinear wave equations with weak and strong damping terms, and they established the existence of weak solutions and related blow-up results under three different initial energy levels and different conditions. Yang et al. [34] investigated the exponential stability of a system with locally distributed damping. Lian et al. [35] were interested in a fourth-order wave equation with strong and weak damping terms; they obtained the local solution, the global existence, asymptotic behavior, and blow-up of solutions under some condition.

Time delays are common phenomena in many physical, chemical, biological, thermal, and so on (see [36–38] for more details). Several authors have investigated existence and stability of the solutions to the viscoelastic wave equation involving delay term under some appropriate conditions on \( \mu_1, \mu_2 \), and \( g \) (see, e.g., [39]). For other related problems, one can also refer to [40–44]. The terminology variable exponents mean that \( p(\cdot) \) is a measurable function and not a constant. This term \( \mu_1 |u_t|^{p(\cdot)-2} u_t + \mu_2 |u_x(t-\tau)|^{p(\cdot)-2} u_x(t-\tau) \) is a generalization of \( \mu_1 u_t + \mu_2 u_x(t-\tau) \), which corresponds to \( p(\cdot) > 1 \). In fact, (1) is also an extension of the second-order viscoelastic wave equation under variable growth conditions

\[
u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \alpha(t) \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_x(t-\tau) \tag{3}
\]

which is obtained when considering \( \mu_1 |u_t|^{p(\cdot)-2} u_t + \mu_2 |u_x(t-\tau)|^{p(\cdot)-2} u_x(t-\tau) \). Equation (3) is a well-known electro-rheological fluid model that appears in fluid dynamic treatment (see in [45]). However, the researches related to the viscoelastic wave equation possessing delay terms, Balakrishan-Taylor damping, and variable growth conditions are not sufficient, and the results about these equations are relatively rare (see [46]). In particular, in [40], the authors considered this class of equations under some suitable assumptions; they used suitable Lyapunov functionals to derive general energy decay results, and one see similar work in [44]. Mingione and Rădulescu [47] were concerned with the regularity theory of elliptic variational problems under nonstandard growth conditions.

This paper devotes to generalize some previous results. In particular, in this case, we will use the relaxation function, the specified initial data, and a special Lyapunov functional, which depends on the behavior of the relation function and is not necessary to decay in some polynomial or exponential form, to get a general decay estimate of the energy.

In addition to the introduction, this paper is divided into two parts. In Section 2, we review some basic definitions about Lebesgue and Sobolev spaces with variable exponents and give some related properties. At the end of this section, we present our main results. In Section 3, we prove our results, showing that a solution of (1) possesses a general decay with small initial values \((u_0, u_1)\).

2. Functional Setting and Main Results

In this section, we will give some preliminaries and our main results.

Without loss of generality, hereinafter, we suppose \( \Omega \subseteq \mathbb{R}^n \) \((n \geq 1)\) is a bounded domain with smooth boundary \( \Gamma \). Moreover, let \( p : \bar{\Omega} \rightarrow (1, +\infty) \) be a measurable function and denote

\[
\begin{align*}
p^- := \inf_{x \in \Omega} [p(x)], \\
p^+ := \sup_{x \in \Omega} [p(x)].
\end{align*}
\]

As in [1, 48, 49], we define the following variable-exponent Lebesgue spaces and Sobolev spaces. The first one is the variable-exponent space \( L^{p(\cdot)}(\Omega) \):

\[
L^{p(\cdot)}(\Omega) = \left\{ \psi : \Omega \rightarrow \mathbb{R} \text{ measurable}, |\psi|_{L^{p(\cdot)}(\Omega)} = \int_{\Omega} |\psi(x)|^{p(\cdot)}\,dx < +\infty \right\},
\]

and it is obvious a Banach space with the following
Luxemburg norm

\[ \| \psi \|_{p(\cdot),\Omega} = \inf \left\{ v > 0 \mid \int_{\Omega} \frac{|u(x)|^{p(x)}}{v} \, dx \leq 1 \right\}. \quad (6) \]

Actually, in many respects, variable-exponent Lebesgue spaces are very similar to classical Lebesgue spaces (see [49]). In particular, from the above definition of the norm, we can directly get the following results:

\[ \min \left\{ \| u \|_{p'_{(\cdot)},\Omega}^{p'_{(\cdot)}}, \| u \|_{p_{(\cdot)},\Omega}^{p_{(\cdot)}} \right\} \leq q_{p(\cdot),\Omega}(u) \leq \max \left\{ \| u \|_{p'_{(\cdot)},\Omega}^{p'_{(\cdot)}}, \| u \|_{p_{(\cdot)},\Omega}^{p_{(\cdot)}} \right\}. \quad (7) \]

For any measurable function \( p : \Omega \rightarrow [p^-, p^+] \subset (2,\infty) \), where \( p^\pm \) are constants, we define the second space and the variable-exponent Lebesgue space

\[ L^{p(\cdot)}(\Omega) = \left\{ \phi : \Omega \rightarrow \mathbb{R} : \phi \text{ is measurable on } \Omega, \int_{\Omega} |\phi(x)|^{p(x)} \, dx < \infty \right\}, \]

which is a Banach space with the following Luxemburg norm:

\[ \| u \|_{p_{(\cdot)}} = \inf \left\{ v > 0 \mid \int_{\Omega} \frac{|u(x)|^{p(x)}}{v} \, dx \leq 1 \right\}. \quad (9) \]

We also assume that \( p \) satisfies the following Zhikov-Fan condition for the local uniform continuity: there exist a constant \( M > 0 \) such that for all points \( x, y \in \Omega \) with \( |x - y| < 1/2 \), we have the inequality

\[ |p(x) - p(y)| \leq \frac{M}{\log |x - y|}. \quad (10) \]

In addition, \( \| \cdot \|_p \) and \( \| \cdot \|_{L^p(\Omega)} \) denote the usual \( L^p(\Omega) \) norm and \( H^1(\Omega) \) norm.

In order to obtain the main results, we give the following lemma firstly.

**Lemma 1** (see [1]).

(1) If

\[ 2 \leq p^ - = \underset{x \in \Omega}{\operatorname{ess \ inf}} p(x) \leq p(x) \leq p^ + = \underset{x \in \Omega}{\operatorname{ess \ sup}} p(x) < \infty, \]

then

\[ \min \left\{ \| u \|_{p'_{(\cdot)},\Omega}^{p'_{(\cdot)}}, \| u \|_{p_{(\cdot)},\Omega}^{p_{(\cdot)}} \right\} \leq \int_{\Omega} |u(x)|^{p(x)} \, dx \leq \max \left\{ \| u \|_{p'_{(\cdot)},\Omega}^{p'_{(\cdot)}}, \| u \|_{p_{(\cdot)},\Omega}^{p_{(\cdot)}} \right\} \quad (12) \]

for any \( u \in L^{p(\cdot)}(\Omega) \).

(2) Assume that \( m, n, p : \Omega \rightarrow (1,\infty) \) are measurable functions satisfying

\[ \frac{1}{m(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{n(\cdot)}. \quad (13) \]

Then, for all functions \( u \in L^{p(\cdot)}(\Omega) \) and \( v \in L^{n(\cdot)}(\Omega) \), we have \( uv \in L^{m(\cdot)}(\Omega) \) with

\[ \| uv \|_{m(\cdot)} \leq \mathcal{C} \| u \|_{p(\cdot)} \| v \|_{n(\cdot)}. \quad (14) \]

**Lemma 2.** Suppose that \( p : \Omega \rightarrow [p^-, p^+] \subset [1,\infty) \) is a measurable function satisfying

\[ \operatorname{ess \ sup} p(x) < p_* \leq \frac{2n}{n - 2} \text{ with } p_* = \frac{np(x)}{\operatorname{ess \ sup} (n - p(x))}. \quad (15) \]

Then, the embedding \( H^1_{0}(\Omega) = W^{1,2}_{0}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega) \) is continuous and compact, and there is a constant \( c_* = c_*(\Omega, p^-, p^+) \) such that

\[ \| \phi \|_{p_{(\cdot)}} \leq c_* \| \nabla \phi \|_2 \text{ for } \phi \in H^1_0(\Omega). \quad (16) \]

We assume that the relaxation function \( g \) and the potential \( \alpha \) satisfy the following assumptions:

Hypothesis \( g, \alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are nonincreasing differentiable functions such that

\[ g(s) \geq 0, \quad l_0 = \int_0^{\infty} g(s) \, ds < \infty, \quad a_0(\Omega) > 0, \quad a_0(\Omega) \right\} \quad (17) \]

Hypothesis \( \xi_0, \xi_1, \xi_2, \xi_3 \): there exist a positive differentiable function \( \xi \) satisfying

\[ g'(t) \leq -\xi(t) g(t), \text{ for } t \geq 0, \lim_{t \to \infty} \frac{\alpha(t)}{\xi(t)} = 0 \quad (18) \]

Hypothesis \( \rho(\cdot) \): the function \( \rho(\cdot) \) satisfies

\[ p^- \geq 2, \text{ if } n = 1, 2, 2 < p^- \leq p(x) \leq p^+ < \frac{n + 2}{n - 2} \text{ if } n \geq 3 \quad (19) \]

Hypothesis \( \mu_1 \) and \( \mu_2 \): the constants \( \mu_1 \) and \( \mu_2 \) satisfy

\[ |\mu_2| < \rho(\cdot) \mu_1 \quad (20) \]

Calculating \( (d/dt)\alpha(t)(g \ast u)(t) \) with respect to \( t \), it
shows that
\[
\begin{align*}
\alpha(t)\int_0^t g(t-s)\int_\Omega u(s)du_x(t)dx &= \\
&\quad -\frac{\alpha(t)}{2}g(t)\|u(t)\|^2 - \frac{d}{dt}\left[\frac{\alpha(t)}{2}g(u(t)) - \frac{\alpha(t)}{2}\|u(t)\|^2\right]_0^t + \frac{\alpha(t)}{2}(g \ast u')(t) + \frac{\alpha'(t)}{2}(g \ast u)(t) - \frac{\alpha'(t)}{2}\|u(t)\|^2\int_0^t g(s)ds,
\end{align*}
\]
where
\[
(g \ast u)(t) = \int_0^t g(t-s)\|u(t) - u(s)\|^2ds.
\]

As in [38, 43], we present a new time-dependent variable to deal with the time delay term:
\[
z(x, \rho, t) = u_0(x, t - \rho), x \in \Omega, \rho \in (0, 1), t > 0.
\]

Consequently, we have
\[
\tau z_\rho(x, \rho, t) + z_\rho(x, \rho, t) = 0, \text{ in } \Omega \times (0, 1) \times (0, \infty).
\]

Therefore, problem (1) can be transformed into
\[
u_{tt} - \mathcal{M}(|\nabla u|^2\mu)\Delta u + \alpha(t)\int_0^t g(t-s)\Delta u(s)ds + \mu_1|u_t|^{p(x)} - \mu_2|z(1, t)|^{p(x)}z(1, t) = 0, \text{ in } \Omega \times \mathbb{R}^+,
\]
\[
\tau z_\rho(\rho, t) + z_\rho(\rho, t) = 0, \text{ in } (0, 1) \times (0, \infty),
\]
\[
z(0, t) = u_t, \text{ in } (0, +\infty),
\]
\[
z(\rho, 0) = j_0(-\rho(t + 1)), \text{ in } (0, 1),
\]
\[
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), x \in \Omega.
\]

By the standard methods as in Section 3 of [50], we can easily prove the well-posedness of problem (1) presented as follows.

**Theorem 3.** Let (17)–(20) be in force and \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega), j_0 \in L^2(\Omega) \times (0, 1))\). Then, problem (1) possesses a unique local solution \(u\) such that
\[
u \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)), u_t \in C([0, T]; H^1_0(\Omega)) \cap L^2([0, T] \times (\Omega)).
\]

(26)

### 3. Main Asymptotic Theorem

Next, we will give the proof of Theorem 4.

The functional \(E\) of problem (25) is as follows:
\[
E(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left[a - \alpha(t)\int_0^t g(s)ds\right]\|\nabla u\|^2
\]
\[
\quad + \frac{b}{4}\|\nabla u\|^4 + \xi\left[\frac{1}{2} \int_\Omega \rho(x) \int_{t-\tau}^t e^{\lambda(x-s)}|u_t(x, s)|^{p(x)}dsdx
\]
\[
+ \frac{1}{2}\alpha(t)(g \ast V(u))(t),
\]
(27)

To prove this theorem, the following technical lemmas are necessary.
Lemma 5. If $u$ is a solution of problem (25). Then,

\[
E'(t) \leq -\sigma \left( \frac{1}{2} \left\| \nabla u \right\|^2 \right) + \frac{1}{2} a(t) \left( g' \nabla u \right)(t)
- \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \int_0^t g(s) ds - \frac{1}{2} a(t) g(t) \left\| \nabla u \right\|^2 + \frac{1}{2} \alpha'(t) \left( g \nabla u \right)(t)
- \left( \frac{\mu_1}{p} - \frac{|\mu_2|}{p^2} \right) \int_{\Omega} |u_i|^{(p+1)} dx
- \left( \frac{\xi}{p^3} e^{-\lambda t} - |\mu_2| \right) \int_0^t |z(1, t)|^{(p+1)} dx
- \lambda \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, s)|^{(p+1)} ds dx \right).
\]
\hspace{1cm} (30)

Proof. Using the same idea as in [50], multiply the first equation in (25) by $u_i$ and then integrate in $\Omega$. Similarly, multiply the second equation in (25) by $\xi z e^{-\lambda t} u_i$ and integrate in $(0, 1) \times \Omega$. Summarizing the above, we can obtain

\[
E'(t) = -\sigma \left( \frac{1}{2} \left\| \nabla u \right\|^2 \right) + \frac{1}{2} a(t) \left( g' \nabla u \right)(t)
- \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \int_0^t g(s) ds - \frac{1}{2} a(t) g(t) \left\| \nabla u \right\|^2 + \frac{1}{2} \alpha'(t) \left( g \nabla u \right)(t)
+ \frac{\alpha'(t)}{2} \left( g \nabla u \right)(t) - \mu_1 \int_{\Omega} |u_i|^{(p+1)} dx
- \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, t-s)|^{(p+1)} ds dx \right)
- \mu_2 \int_{\Omega} |z(1, t)|^{(p+1)} dx
+ \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, t-s)|^{(p+1)} ds dx \right)
- \lambda \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, s)|^{(p+1)} ds dx \right).
\]
\hspace{1cm} (31)

By $z(1, t) = u_i(t - t)$ and the Young inequality, we get

\[
- \mu_2 \int_{\Omega} |z(1, t)|^{(p+1)} z(1, t) u_i dx
\leq |\mu_2| \frac{p^2 - 1}{p} \int_{\Omega} |z(1, t)|^{(p+1)} dx + \frac{|\mu_2|}{p} \int_{\Omega} |u_i|^{(p+1)} dx.
\]
\hspace{1cm} (32)

From (23), we have

\[
- \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, t-s)|^{(p+1)} ds dx \right)
\leq - \frac{\xi}{p} e^{-\lambda t} \int_0^t |z(1, t)|^{(p+1)} dx.
\]
\hspace{1cm} (33)

Comparing (31) and (32), we obtain

\[
E'(t) \leq -\sigma \left( \frac{1}{2} \left\| \nabla u \right\|^2 \right) + \frac{1}{2} a(t) \left( g' \nabla u \right)(t)
- \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \int_0^t g(s) ds - \frac{1}{2} a(t) g(t) \left\| \nabla u \right\|^2 + \frac{1}{2} \alpha'(t) \left( g \nabla u \right)(t)
+ \frac{\alpha'(t)}{2} \left( g \nabla u \right)(t) - \left( \frac{\mu_1}{p} - \frac{|\mu_2|}{p^2} \right) \int_{\Omega} |u_i|^{(p+1)} dx
- \left( \frac{\xi}{p^3} e^{-\lambda t} - |\mu_2| \right) \int_0^t |z(1, t)|^{(p+1)} dx
- \lambda \xi \int_{\Omega} \frac{1}{\Omega} \left( \frac{1}{\Omega} \int_{1-t}^t e^{\lambda(t-s)} |u_i(x, s)|^{(p+1)} ds dx \right).
\]
\hspace{1cm} (34)

Setting

\[
c_0 = \mu_1 - \frac{\xi}{p^3} e^{-\lambda t},
\]
\hspace{1cm} (35)

\[
c_1 = \frac{\xi}{p^3} e^{-\lambda t} - |\mu_2| \frac{p^2 - 1}{p},
\]

by condition (28), we derived the desired inequality (30). □

Remark 6. If

\[
- \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \int_0^t g(s) ds \geq 0
\]
\hspace{1cm} (36)

holds, $E(t)$ may not be nonincreasing.

Lemma 7. Assume that $u$ be a solution of problem (25). Then,

\[
\frac{1}{2} \frac{l_0}{l_1} e^{l_0(l_0/l_1)\alpha(t)}, t \geq 0,
\]
\hspace{1cm} (37)

where $l_0$ and $l_1$ as in (17).

Proof. From (27) and (30), we have

\[
E'(t) \leq - \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \int_0^t g(s) ds \leq \frac{1}{2} a'(t) \left\| \nabla u \right\|^2 \leq \frac{1}{2} \alpha'(t) E(t).
\]
\hspace{1cm} (38)

Integrating the above inequality in $(0, t)$, we get

\[
E(t) \leq E(0) e^{-\frac{l_0(l_0/l_1)\alpha(t)}{l_1}} \leq E(0) e^{\frac{l_0(l_0/l_1)\alpha(t)}{l_1}}.
\]
\hspace{1cm} (39)

From (27), we see that

\[
\frac{1}{2} \frac{l_0}{l_1} \leq \frac{2}{l_1} E(t).
\]
\hspace{1cm} (40)

Combining it with (39), it gives (37).
Now, we give a modified functional:

\[ L(t) = NE(t) + \varepsilon_1 \alpha(t) \varphi(t) + \varepsilon_2 \alpha(t) \psi(t), \]  

(41)

\[ \varphi(t) = \frac{\sigma}{4} \| \nabla u \|_{L_2}^4, \]  

(42)

\[ \psi(t) = -\int t_0 u(t-s) u(s) ds + \| u(t) \|_{L_2}^4, \]  

(43)

where \( \varepsilon_1, \varepsilon_2, \) and \( N \) are positive constants. In fact, \( L \) is equivalent to \( E \) by the following lemma.

**Lemma 8.** There exists \( C_1, C_2 > 0 \) such that

\[ C_1 E(t) \leq L(t) \leq C_2 E(t), t \geq 0. \]  

(44)

**Proof.** By the Poincaré theorem and Young inequality, we have the following results through integrating by parts:

\[ |L(t) - NE(t)| = \left| \varepsilon_1 \alpha(t) \int \Omega u(t) u(t) dx + \varepsilon_2 \alpha(t) \frac{\sigma}{4} \| \nabla u \|_{L_2}^4 + \varepsilon_2 \alpha(t) \psi(t) \right| \]

\[ \leq \varepsilon_1 |\alpha(t)| \int \Omega |u(t)| |u(t)| dx + \varepsilon_2 \frac{\sigma}{4} |\alpha(t)| \| u \|_{L_2}^4 + \varepsilon_2 |\alpha(t)| \| u \|_{L_2}^4 \]

\[ + \varepsilon_2 \frac{\sigma}{4} |\alpha(t)| \| u \|_{L_2}^4 + \varepsilon_2 \| u \|_{L_2}^4 \]

\[ \leq C(\varepsilon_1 + \varepsilon_2) E(t), \]

(45)

where \( C_4 \) as in Lemma 1, taking \( C_1 = N - C(\varepsilon_1 + \varepsilon_2) \) and \( C_2 = N + C(\varepsilon_1 + \varepsilon_2) \), provided \( \varepsilon_1 \) and \( \varepsilon_2 \) are sufficiently small, and the proof is completed.

**Lemma 9.** There exists \( c_1, C_1 > 0 \) fulfilling

\[ \varphi'(t) \leq \| u \|_{L_2}^2 \frac{1}{2} \| \nabla u \|_{L_2}^2 - b \| \nabla u \|_{L_2}^2 + \alpha(t) \frac{a}{2} (g \nabla u)(s) \]

\[ + c_1 \left( \int \Omega |u|^p(s) ds + \int \Omega |z(1, t)|^p(s) ds \right) \]

\[ + C_1 \| u \|_{L_2}^4. \]  

(46)

**Proof.** By the first equation of (25), we differentiate (42), and then we have

\[ \varphi'(t) = \| u \|_{L_2}^2 + \int \Omega \nu + \sigma \| \nabla u \|_{L_2}^2 \int \Omega \nabla u \cdot dx \]

\[ = \| u \|_{L_2}^2 - a \| \nabla u \|_{L_2}^2 + \alpha(t) \int \Omega g(t-s) \nabla u(s) \cdot dx \]

\[ - \mu_1 \int \Omega |u|^p(s) ds + \mu_2 \int \Omega |z(1, t)|^p(s) ds = \| u \|_{L_2}^2 - \| \nabla u \|_{L_2}^2 - b \| \nabla u \|_{L_2}^2 + \alpha(t) \frac{a}{2} (g \nabla u)(t) \]

\[ - \mu_1 \| u \|_{L_2}^2 + \mu_2 |z(1, t)|^2 (1, t) \| u \|_{L_2}^2 = \| u \|_{L_2}^2 - \mu_1 \| u \|_{L_2}^2 + \mu_2 |z(1, t)|^2 (1, t) \| u \|_{L_2}^2 =\]

(47)

By the Hölder inequality, Sobolev-Poincaré inequalities, and (17), we estimate the second part of the right-hand side in (47).

\[ l_1 = a(t) \int \Omega g(t-s) \nabla u(s) \cdot dx \]

\[ \leq a(t) \frac{\sigma}{4} \| \nabla u \|_{L_2}^4 \]

\[ \leq a(t) \frac{\sigma}{4} \| \nabla u \|_{L_2}^4 \]

\[ \leq a(t) \frac{\sigma}{4} \| \nabla u \|_{L_2}^4 \]

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\[ \leq a(t) \frac{\sigma}{4} \| \nabla u \|_{L_2}^4 \]

(48)

For every \( \eta > 0 \), using the Young inequality and (17), we deduce

\[ \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 + \frac{\alpha(t)}{2} (\nabla u(s) - \nabla u(t))^2 ds \]

\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

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\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

\[ \leq \frac{\alpha(t)}{2} \int_\Omega g(t-s) (\nabla u(s) - \nabla u(t))^2 ds \]

(49)

Summarizing the above estimates, (48) and (49), we obtain

\[ \alpha(t) \int_\Omega (g(t-s) \nabla u(s))^2 dx \]

\[ = (2 + \eta) \frac{\alpha(t)}{2} \| \nabla u \|_{L_2}^2 \]

\[ + \frac{\alpha(t)}{2} \left( 1 + \frac{1}{\eta} \right) (g \nabla u)(t). \]

(50)
Setting \(\eta = \frac{l}{l(a - l)}\), it is easy to obtain

\[
|I_1| \leq \alpha(t) \int_0^t g(t - s)\nabla u(s)ds \nabla u dx \\
\leq \left( a - \frac{1}{2} \right) \|\nabla u\|_2^2 + \frac{a}{2\eta} \alpha(t)(g\nabla u)(t),
\]

and by means of the Young inequality, we have

\[
|I_2| \leq c_1 \int_\Omega |u_i|^{p(s)} dx + \varepsilon \max \left( \mu_\ell^p, \mu_\ell^p \right) \int_\Omega |u_i|^{p(s)} dx = c_1 \int_\Omega |u_i|^{p(s)} dx \\
+ \varepsilon c_2 \int_\Omega |u_i|^{p(s)} dx,
\]

\[
|I_3| \leq c_3 \int_\Omega |z(1, t)|^{p(s)} dx \\
+ \varepsilon \max \left( \mu_\ell^p, \mu_\ell^p \right) \int_\Omega |u_i|^{p(s)} dx = c_3 \int_\Omega |z(1, t)|^{p(s)} dx \\
+ \varepsilon c_3 \int_\Omega |u_i|^{p(s)} dx.
\]

Substituting (51)–(53) into (47), we deduce

\[
\varphi'(t) \leq \|u_i\|_2^2 - \frac{l}{2} \|\nabla u\|_2^2 + C_\varepsilon \int_\Omega |u_i|^{p(s)} dx - b\|\nabla u\|_2^2 \\
+ \frac{a}{2\eta} \alpha(t)(g\nabla u)(t) + c_\varepsilon \left( \int_\Omega |u_i|^{p(s)} dx + \int_\Omega |z(1, t)|^{p(s)} dx \right). \tag{54}
\]

Set \(C_\varepsilon = \varepsilon (c_2 + c_3) > 0\), for \(\varepsilon\) sufficiently small.

**Lemma 10.** There exists positive constants \(\delta\) and \(c_\delta\) satisfying

\[
\varphi'(t) \leq -\left( \int_0^t g(s)ds \right) \|u_i\|_2^2 + \delta \left( a + 2(a - l)^2 \alpha(t) \right) \|\nabla u\|_2^2 \\
+ \delta b\|\nabla u\|_2^2 + \delta \frac{2aeE(0)}{l} e^{(l_j)/u(t)} \left( \frac{1}{2\delta} \right) \|\nabla u\|_2^2 \\
+ \left( \delta \right) \left( \frac{2\delta + \frac{1}{4\delta}}{2} \right) (a - l) \alpha(t) (g\nabla u)(t) \\
+ \varepsilon \delta \left( \int_\Omega |u_i|^{p(s)} dx + \int_\Omega |z(1, t)|^{p(s)} dx \right) \\
- \frac{g(0)\varepsilon^2}{4\delta} \left( g' \nabla u \right)(t). \tag{55}
\]

**Proof.** Similar to Lemma 9 by the first equation (25), we dif-
\[ |I_4| \leq \varepsilon \int_{\Omega} |u(t)|^{p(x)} \, dx + \delta \max \left( \mu \right) \left( \int_{\Omega} g(t-s)(u(t) - u(s)) \, ds \right) \]
\[ \leq \varepsilon \int_{\Omega} |u(t)|^{p(x)} \, dx + \delta \max \left( \mu \right) \left( \int_{\Omega} g(t-s) \, ds \right) \]
\[ \int_{\Omega} \left| \nabla (u(t-s))(u_s) \right|^{p(x)} \, dx \]
\[ \leq \varepsilon \int_{\Omega} |u(t)|^{p(x)} \, dx + \delta \max \left( \mu \right) \left( \int_{\Omega} g(t-s) \, ds \right) \]
\[ \int_{\Omega} \left| \nabla (u(t-s))(u_s) \right|^{p(x)} \, dx \]
\[ (g \circ \nabla u)(t) = \varepsilon \int_{\Omega} |u(t)|^{p(x)} \, dx + \delta \varepsilon (g \circ \nabla u)(t). \]
\[ (60) \]

Similarly,
\[ |I_5| \leq \varepsilon \int_{\Omega} |z(1,t)|^{p(x)} \, dx + \delta \varepsilon (g \circ \nabla u)(t), \]
\[ |I_6| \leq \delta |u|^{2} + \frac{g(0)}{4\delta} \left( g' \circ \nabla u \right)(t). \]
\[ (61) \]

Comparing these above estimates (57)–(61), we have
\[ \psi'(t) \leq -\left( \int_{0}^{t} g(s) \, ds - \delta \right) |u|^{2} + \delta \left( a + 2\alpha \right) |u| |\nabla u|^{2} \]
\[ + \delta b |\nabla u|^{2} + \frac{2a}{4\delta} e^{(l/\alpha)(\alpha)(0)} \left( \frac{1}{2} \frac{d}{dt} |\nabla u|^{2} \right)^{2} \]
\[ + \left\{ C_{\delta} + \left( 2\delta + \frac{1}{4\delta} \right) \left( g \circ \nabla u \right)(t) \right\} \]
\[ + \left( \int_{\Omega} |u|^{p(x)} \, dx \right) + \delta \left( \int_{\Omega} |z(1,t)|^{p(x)} \, dx \right) \]
\[ - \frac{g(0)}{4\delta} \left( g' \circ \nabla u \right)(t), \]
\[ (62) \]

where \( C_{\delta} = \left\{ \alpha l/4\delta + (bl_{0}E(0)/2\delta l) e^{(l/\alpha)(0)} + \sigma l_{0}/4\delta + \delta \left( \varepsilon \right) + c_{j} \right\}. \]

Lemma 11. There exists positive constants \( C_{3}, C_{4}, \text{ and } t_{0} \) satisfying
\[ L'(t) \leq -C_{2} \alpha(t) \left( g \circ \nabla u \right)(t), \]
\[ t > t_{0}. \]
\[ (63) \]

Proof. Since \( g > 0 \) and is continuous, then for any \( t \geq t_{0} > 0 \), we get
\[ \int_{0}^{t} g(s) \, ds \geq \int_{0}^{t_{0}} g(s) \, ds = g_{0} > 0. \]
\[ (64) \]

Differentiate (41), and using Lemmas 9 and 10, we get
\[ L'(t) = NE'(t) + \varepsilon_{1} a'(t) \psi(t) + \varepsilon_{1} a(t) \psi'(t) + \varepsilon_{2} a'(t) \psi(t) + \varepsilon_{2} a(t) \psi'(t) \]
\[ \geq -a(t) \left( \varepsilon_{1} C_{3} - \varepsilon_{1} \delta (a + 2\alpha \right) |u| |\nabla u|^{2} \]
\[ - a(t) \left( b(\varepsilon_{1} - \varepsilon_{2} \delta) |u|^{2} \right. \]
\[ - a(t) \left( c_{0} \right. \]
\[ - a(t) \left( \frac{e_{0}(t)}{2} - e_{1} C_{4} - \varepsilon_{2} C_{5} \right) \left( g \circ \nabla u \right)(t) \]
\[ - a(t) \left( \frac{N}{2} - \frac{g(0)}{4\delta} \right) \left( g' \circ \nabla u \right)(t) - a(t) \]
\[ \left( \int_{\Omega} |z(1,t)|^{p(x)} \, dx \right) \]
\[ (65) \]

Indeed,
\[ \alpha'(t) \int_{\Omega} u_{x} \, dx + \alpha'(t) \int_{\Omega} \left( g(t-s)(u(t) - u(s)) \right) \, dx \]
\[ \leq -\alpha'(t) \frac{g(0)}{2} |\nabla u|^{2} - \alpha'(t) |\nabla u|^{2} - \alpha'(t) \frac{g(0)}{2} \left( \int_{0}^{t} g(s) \, ds \right) \]
\[ (66) \]

Thus,
\[ L'(t) \leq -a(t) \left( \varepsilon_{1} C_{3} - \varepsilon_{1} \delta (a + 2\alpha \right) |u|^{2} \]
\[ \geq \left( \frac{C_{2} a(t)}{2a(t)} \right) \left( \int_{\Omega} g(s) \, ds \right) + \frac{c_{2} a'(t)}{2a(t)} \left( g \circ \nabla u \right)(t) \]
\[ \frac{d}{dt} |\nabla u|^{2} \]
\[ + a(t) \left( \frac{e_{0}(t)}{2} - e_{1} C_{4} - \varepsilon_{2} C_{5} \right) \left( g \circ \nabla u \right)(t) - a(t) \]
\[ \left( \int_{\Omega} |z(1,t)|^{p(x)} \, dx \right) \]
\[ (67) \]

Fix \( \delta > 0 \) such that
\[ g_{0} - \delta > \frac{1}{2} g_{0} \left( a + 2\alpha \right) |\nabla u|^{2} < \frac{1}{4} g_{0}. \]
\[ (68) \]
and take \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough to satisfy
\[
\frac{\varepsilon_0}{4} \varepsilon_1 < \varepsilon_1 < \varepsilon_2 \frac{\varepsilon_0}{2},
\]
\[
\varepsilon_5 = \varepsilon_2 (\varepsilon_0 - \delta) - \varepsilon_1 > 0,
\]
\[
\varepsilon_6 = \varepsilon_1 \varepsilon_4 - \varepsilon_2 \delta (a + 2b_0) a(0) > 0.
\]

Select \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough to make (44) and (67) hold, and moreover
\[
\frac{\varepsilon_0}{a(0)} - \varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_6 > 0, \quad \frac{\varepsilon_1}{a(0)} - \varepsilon_1 \varepsilon_4 - \varepsilon_2 \varepsilon_6 > 0.
\]

Hence, for a generic positive constant \( c_1 \), (67) is equal to the following results:
\[
L'(t) \leq -\alpha(t) \left( c + \frac{\alpha'(t)}{\alpha(t)} \right) \| u \|_2^2 - \alpha(t) \left( c + \frac{\alpha'(t)}{2 \alpha(t)} \right) \left( \int_0^t g(s) \, ds \right) \| \nabla u \|_2^2 + \alpha(t) \left( c - \frac{\gamma^2 g \alpha'(t)}{2 \alpha(t)} \right) (g \nabla u)(t), \forall t \geq t_0.
\]

Noticing that \( \lim_{t \to +\infty} \alpha'(t)/\alpha(t) = 0 \), so choose \( t_1 > t_0 \), we see
\[
L'(t) \leq -\alpha(t) \left( c \| u \|_2^2 + C \| \nabla u \|_2^2 + c (g \nabla u)(t) \right) \leq -C_3 \alpha(t) E(t) + C_4 \alpha(t) (g \nabla u)(t), \forall t \geq t_1,
\]
where \( C_3 \) and \( C_4 \) are positive constants.

Now, we are in the position to prove Theorem 4.

**Proof of Theorem 4.** According to Lemma 5, Lemma 11, and (17), we have
\[
\zeta(t)L'(t) \leq -C_3 \alpha(t) \zeta(t) E(t) + C_4 \alpha(t) \zeta(t) (g \nabla u)(t)
\leq -C_3 \alpha(t) \zeta(t) E(t) - C_4 \alpha(t) \left( g \nabla u \right)(t)
\leq -C_3 \alpha(t) \zeta(t) E(t)
\leq C_4 \left( 2E'(t) + \alpha'(t) \left( \int_0^t g(s) \, ds \right) \| \nabla u \|_2^2 \right).
\]

Since \( \zeta(t) \) is nonincreasing, by assumption (17) and the definition of \( E(t) \), we get
\[
\frac{1}{2} \| \nabla u \|_2^2 \leq E(t), \quad \frac{d}{dt} \zeta(t) L(t) + 2C_4 E(t) \leq -C_3 \alpha(t) \zeta(t) E(t) - C_4 \alpha'(t) \left( \int_0^t g(s) \, ds \right) \| \nabla u \|_2^2,
\]
which leads to
\[
\frac{d}{dt} \zeta(t) E(t) + 2C_4 E(t) \leq -C_3 \alpha(t) \zeta(t) E(t)
\leq -C_3 \alpha(t) \zeta(t) E(t)
\leq \frac{2C_4 E(t)}{t} \alpha'(t) \left( \int_0^t g(s) \, ds \right) \| \nabla u \|_2^2.
\]

Since \( \lim_{t \to +\infty} \alpha'(t)/\alpha(t) = 0 \), we can choose \( t_1 \geq t_0 \) such that \( C_3 + 2C_4 \alpha'(t)/\alpha(t) \zeta(t) > 0 \) for \( t \geq t_1 \). Hence, if we let
\[
L(t) = \zeta(t) L(t) + 2C_4 E(t),
\]
then it is obvious that \( L(t) \) is equivalent to \( E(t) \) and satisfies
\[
L'(t) \leq -k \zeta(t) \alpha(t) L(t) \text{ for } t \geq t_1.
\]

Consequently, to integrate (77) over \( (t_1, t) \), it yields
\[
L(t) \leq L(t_1) e^{-\int_{t_1}^t k \zeta(t) \alpha(t) \, dt} \geq t_0.
\]

Thus, the desired result yields from the equivalence relations of \( L(t), L(t), \text{ and } E(t) \).

**Data Availability**

No data is used in the manuscript.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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**References**

[1] L. Diening, P. Hasto, and P. Harjulehto, *Lebesgue and Sobolev Spaces With Variable Exponents*, Springer Lecture Notes; 2017, Springer-Verlag, Heidelberg, 2011.

[2] L. Diening and M. Růžička, “Calderon Zygmund operators on generalized Lebesgue spaces \( L^{p(x)}(\Omega) \) and problems related to
fluid dynamics,” *Journal für die reine und angewandte Mathematik* (Crelles Journal), no. 563, 2003 pages, 2003.

[3] R. Aboulach, D. Meskine, and A. Souissi, “New diffusion models in image processing,” *Computers & Mathematics with Applications*, vol. 56, no. 4, pp. 874–882, 2008.

[4] S. Antontsev and S. Shmarev, “Blow-up of solutions to parabolic equations with nonstandard growth conditions,” *Journal of Computational and Applied Mathematics*, vol. 234, no. 9, pp. 2633–2645, 2010.

[5] S. Antontsev and S. Sergey, *Handbook of Differential Equations: Stationary Partial Differential Equations*, 2006.

[6] Y. Chen, S. Levine, and M. Rao, “Variable exponent, linear growth functionals in image restoration,” *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[7] S. Lian, W. Gao, C. Cao, and H. Yuan, “Study of the solutions to a model porous medium equation with variable exponent of nonlinearity,” *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 27–38, 2008.

[8] G. Yunzhu, G. Bin, and G. Wenjie, “Weak solutions for a high-order pseudo-parabolic equation with variable exponents,” *Applicable Analysis*, vol. 93, no. 2, pp. 322–338, 2014.

[9] S. Antontsev, “Wave equation with \( p(x,t) \)-laplacian and damping term: existence and blow-up,” *Differential Equations & Applications*, vol. 3, pp. 503–525, 2011.

[10] G. Yunzhu and G. Wenjie, “Existence of weak solutions for viscoelastic hyperbolic equations with variable exponents,” *Boundary Value Problems*, vol. 2013, no. 1, 2013.

[11] M. Liao, B. Guo, and X. Zhu, “Bounds for blow-up time to a viscoelastic hyperbolic equation of Kirchhoff type with variable sources,” *Acta Applicandae Mathematicae*, vol. 170, no. 1, pp. 755–772, 2020.

[12] A. Rahmouni, “Existence and asymptotic stability for the semilinear wave equation with variable exponent nonlinearities,” *Journal of Mathematical Physics*, vol. 60, no. 12, p. 122701, 2019.

[13] A. Rahmouni, “On the existence, uniqueness and stability of solutions for semi-linear generalized elasticity equation with general damping term,” *Acta Mathematica Sinica, English Series*, vol. 33, no. 11, pp. 1549–1564, 2017.

[14] A. Rahmouni, “Semilinear hyperbolic boundary value problem associated to the nonlinear generalized viscoelastic equations,” *Acta Mathematica Vietnamica*, vol. 43, no. 2, pp. 219–238, 2017.

[15] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.

[16] P. D’Ancona, “Global solvability for the degenerate Kirchhoff equation with real analytic data,” *Inventiones Mathematicae*, vol. 108, no. 1, pp. 247–262, 1992.

[17] P. D’Ancona and Y. Shibata, “On global solvability of nonlinear viscoelastic equations in the analytic category,” *Mathematical Methods in the Applied Sciences*, vol. 17, no. 6, pp. 477–486, 1994.

[18] X. He and W. Zou, “Existence and concentration behavior of positive solutions for a Kirchhoff equation in \( \mathbb{R}^2 \),” *Journal of Differential Equations*, vol. 252, no. 2, pp. 1813–1834, 2012.

[19] J. Jin and X. Wu, “Infinitely many radial solutions for Kirchhoff-type problems in \( \mathbb{R}^3 \),” *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 564–574, 2010.

[20] Y. Li, F. Li, and J. Shi, “Existence of a positive solution to Kirchhoff type problems without compactness conditions,” *Journal of Differential Equations*, vol. 253, no. 7, pp. 2285–2294, 2012.

[21] X. Wu, “Existence of nontrivial solutions and high energy solutions for Schrodinger Kirchhoff-type equations in \( \mathbb{R}^3 \),” *Nonlinear Analysis: Real World Applications*, vol. 12, no. 2, pp. 1278–1287, 2011.

[22] T. Takeshi, “Existence and asymptotic behaviour of solutions to weakly damped wave equations of Kirchhoff type with nonlinear damping and source terms,” *Journal of Mathematical Analysis and Applications*, vol. 361, no. 2, pp. 566–578, 2010.

[23] G. Li, L. Hong, and W. Liu, “Global nonexistence of solutions for viscoelastic wave equations of Kirchhoff type with high energy,” *Journal of Function Spaces and Applications*, vol. 2012, article 530861, pp. 1–15, 2012.

[24] K. Ono, “Global existence, decay, and blow-up of solutions for some mildly degenerate nonlinear Kirchhoff strings,” *Journal of Differential Equations*, vol. 137, no. 2, pp. 273–301, 1997.

[25] A. V. Balakrishnān and L. W. Taylor, “Distributed parameter nonlinear damping models for flight structures,” in *Proceedings “Damping 89”, Flight Dynamics Lab and Air Force Wright Aeronautical Labs, WPAPF*, 1989.

[26] R. W. Bass and D. Zes, “Spillover and nonlinearity, and structures,” in *The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems*, L. W. Taylor, Ed., vol. 10065, pp. 1–14, NASA Conference Publication, 1991.

[27] C. Mu and J. Ma, “On a system of nonlinear wave equations with Balakrishnān-Taylor damping,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 65, no. 1, pp. 91–113, 2014.

[28] S. Park, “Arbitrary decay of energy for a viscoelastic problem with Balakrishnān-Taylor damping,” *Taiwanese Journal of Mathematics*, vol. 20, no. 1, pp. 129–141, 2016.

[29] G. Liu, C. Hou, and X. Guo, “Global nonexistence for nonlinear Kirchhoff systems with memory term,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 65, pp. 1–16, 2013.

[30] Z. Yang and Z. Gong, “Blow-up of solutions for viscoelastic equations of Kirchhoff type with arbitrary positive initial energy,” *Electronic Journal of Differential Equations*, vol. 332, pp. 1–8, 2016.

[31] B. Gheraibia and N. Boumaza, “General decay result of solutions for viscoelastic wave equation with Balakrishnan–Taylor damping and a delay term,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 71, no. 6, 2020.

[32] J. Hao and F. Wang, “General decay rate for weak viscoelastic wave equation with Balakrishnan–Taylor damping and time-varying delay,” *Computers & Mathematics with Applications*, vol. 78, no. 8, pp. 2632–2640, 2019.

[33] W. Lian and R. Xu, “Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term,” *Advances in Nonlinear Analysis*, vol. 9, pp. 613–632, 2020.

[34] F. Yang, Z. H. Ning, and L. Chen, “Exponential stability of the nonlinear Schrödinger equation with locally distributed damping on compact Riemannian manifold,” *Advances in Nonlinear Analysis*, vol. 10, pp. 569–583, 2021.

[35] W. Lian, V. D. Rădulescu, R. Xu, Y. Yang, and N. Zhao, “Global well-posedness for a class of fourth-order nonlinear strongly damped wave equations,” *Advances in Calculus of Variations*, vol. 14, no. 4, pp. 589–611, 2021.

[36] E. Fridman, S. Nicaise, and J. Valein, “Stabilization of second order evolution equations with unbounded feedback with time-dependent delay,” *SIAM Journal on Control and Optimization*, vol. 48, no. 8, pp. 5028–5052, 2010.
[37] N. Serge, P. Cristina, and V. Julie, "Exponential stability of the wave equation with boundary time-varying delay," *Discrete & Continuous Dynamical Systems - S*, vol. 4, no. 3, pp. 693–722, 2011.

[38] S. Nicaise and C. Pignotti, "Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1561–1585, 2006.

[39] Q. Dai and Z. Yang, "Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 65, no. 5, pp. 885–903, 2014.

[40] M. J. Lee, J. Y. Park, and Y. H. Kang, "Asymptotic stability of a problem with Balakrishnan-Taylor damping and a time delay," *Computers & Mathematics with Applications*, vol. 70, no. 4, pp. 478–487, 2015.

[41] S. T. Wu, "Asymptotic behavior for a viscoelastic wave equation with a delay term," *Taiwanese Journal of Mathematics*, vol. 17, no. 3, pp. 765–784, 2013.

[42] Z. Yang, "Existence and energy decay of solutions for the Euler–Bernoulli viscoelastic equation with a delay," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 66, no. 3, pp. 727–745, 2015.

[43] S. Park, "Decay rate estimates for a weak viscoelastic beam equation with time-varying delay," *Applied Mathematics Letters*, vol. 31, pp. 46–51, 2014.

[44] J. H. Hao and F. Wang, "General decay rate for weak viscoelastic wave equation with Balakrishnan–Taylor damping and time-varying delay," *Computers & Mathematics with Applications*, vol. 78, pp. 2632–2640, 2018.

[45] M. Ruzicka, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2002.

[46] G. Autuori, P. Pucci, and M. C. Salvatori, "Global nonexistence for nonlinear Kirchhoff systems," *Archive for Rational Mechanics and Analysis*, vol. 196, no. 2, pp. 489–516, 2010.

[47] G. Mingione and V. D. Rădulescu, "Recent developments in problems with nonstandard growth and nonuniform ellipticity," *Journal of Mathematical Analysis and Applications*, vol. 501, no. 1, article 125197, 2021.

[48] Y. Fu, "The existence of solutions for elliptic systems with nonuniform growth," *Studia Mathematica*, vol. 151, no. 3, pp. 227–246, 2002.

[49] O. Kovácik and J. Rákosník, "On spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$," *Czechoslovak Mathematical Journal*, vol. 41, 1991.

[50] A. Rahmoune, "General decay for a viscoelastic equation with time-varying delay in the boundary feedback condition," *Mathematics and Mechanics of Complex Systems*, vol. 9, no. 2, pp. 127–142, 2021.