Recoil and Ring–Down Effects in Gravitation

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Abstract

We construct a model of a relativistic fireball or light–like shell of matter by considering a spherically symmetric moving gravitating mass experiencing an impulsive deceleration to rest. We take this event to be followed by the mass undergoing a deformation leading to the emission of gravitational radiation while it returns to a spherically symmetric state. We find that the fireball is accompanied by an impulsive gravitational wave.

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1 Introduction

In a recent paper [1] we constructed a model in general relativity of a spherically symmetric, moving gravitating mass which experiences an impulsive deceleration to rest, as viewed by a distant observer. To the distant observer the mass, which is moving rectilinearly with uniform 3–velocity $v$, suddenly halts. This event is accompanied by the emergence of a spherical light–like shell whose total energy measured by the observer is found to be the same as the relative kinetic energy of the source before it stops. The shell may be considered a ‘relativistic fireball’ and such objects are believed to constitute the ‘inner engine’ of the fireball model of gamma–ray bursts [2]. The deceleration phenomenon is a recoil effect and may be thought of as a limiting case of a Kinnersley rocket [3], [4]. It has an electromagnetic analogue in which the mass is replaced by a point charge and the light–like shell is replaced by a spherical impulsive electromagnetic wave [1].

It was pointed out in [1] that the concept of an extended spherical body remaining rotationally symmetric after suddenly decelerating to rest is a strong idealisation. It would be more realistic to expect the body to experience some deformation following the deceleration which would lead to the emission of gravitational radiation before eventually settling back to a spherically symmetric state. This is the question we address in the present paper. To do so we make use of the Schwarzschild space–time, with mass parameter $m$ and with parameter $v$ representing the uniform 3–velocity of the spherical source viewed by a distant observer, and the Robinson space–time [5] describing the vacuum gravitational field of an axially symmetric, isolated body which rapidly evolves into a Schwarzschild space–time due to the emission of gravitational radiation. Both of these space–times contain future null–cones (i.e. null hypersurfaces generated by shear–free, expanding null geodesics) and we attach them to each other in an appropriate manner on one such null–cone $\mathcal{N}$, with the two–parameter Schwarzschild space–time to the past of $\mathcal{N}$ and the Robinson space–time to the future of $\mathcal{N}$. We thus create the more realistic model of sudden deceleration to rest which we set out to do and the remaining question is: what is the physical nature of the light–like signal whose history in space–time is the future null–cone $\mathcal{N}$? We show that $\mathcal{N}$ is the history of a light–like shell accompanied by an impulsive gravitational wave. The total energy of the shell measured by the distant observer referred to above is calculated as a perturbation of its value obtained in [1] when the deceleration was instantly followed by a Schwarzschild space–time with $v = 0$. If this energy is less than the original kinetic energy of the source then the sign of a small parameter involved in the approximate calculation is determined and this in turn means that the signal front is a
prolate spheroid which, in the ring–down phase, quickly becomes a sphere and the static Schwarzschild space–time is then established.

The paper is organised as follows: in section 2 the gravitational radiation–free model of sudden deceleration to rest of a Schwarzschild source having uniform 3–velocity \( v \) is described. This is followed in section 3 by the more realistic model in which, after deceleration to rest, the source is an axially symmetric radiating source whose gravitational field is described by the Robinson space–time. The paper ends with a discussion in which the non–spherical shape of the wave front in the ring–down phase is examined.

2 The Gravitational Radiation–Free Model

To set the scene we review the spherically symmetric model of a recoil effect \([1]\). Consider first the Schwarzschild line–element in the form

\[
ds^2 = k^2 r^2 \left\{ \frac{d\xi^2}{1 - \xi^2} + (1 - \xi^2) d\phi^2 \right\} - 2du dr - \left( 1 - \frac{2m}{r} \right) du^2 , \tag{2.1}\]

with \( k^{-1} = \gamma (1 - v \xi) \), \( \gamma = (1 - v^2)^{-1/2} \) and \( 0 \leq v < 1 \). Here the coordinate ranges are \(-1 \leq \xi \leq +1\) , \( 0 \leq \phi < 2 \pi \) , \( -\infty < u < +\infty \) and \( 0 \leq r < +\infty \). Instead of using the polar angle \( \theta \) we have used \( \xi = \cos \theta \) as a coordinate. This proves to be useful for all subsequent calculations. The constant \( m \) is the mass of the source and the constant \( v \) is the 3–velocity of the source which appears to be moving rectilinearly to a distant observer. The line–element \((2.1)\) can be transformed to the usual 1–parameter Schwarzschild form

\[
ds^2 = \bar{r}^2 \left\{ \frac{d\bar{\xi}^2}{1 - \xi^2} + (1 - \xi^2) d\bar{\phi}^2 \right\} - 2d\bar{u} d\bar{r} - \left( 1 - \frac{2}\bar{m}\right) d\bar{u}^2 , \tag{2.2}\]

by the transformation \([1]\)

\[
\bar{\xi} = \frac{\xi - v}{1 - \xi} \bar{\phi} = \phi , \quad \bar{r} = r , \quad \bar{u} = u . \tag{2.3}\]

From the form \((2.2)\) we conclude that \( m \) is the Bondi mass or ‘mass aspect’ in this case of the source at rest \([3]\). By writing \((2.1)\) in Bondi form we will find the mass aspect associated with a mass \( m \) moving along the symmetry axis with 3–velocity \( v \) \([cf. [4], equation (72)]\). This confirms the physical interpretation of the parameter \( v \). To see this we begin with the exact transformation

\[
r' = kr \varphi , \tag{2.4}\]

\[
r = \bar{r} \quad \xi = \frac{\xi - v}{1 - \xi} \bar{r} = r , \quad \bar{u} = u . \tag{2.5}\]
\[ u' = \gamma u + k r (1 - \varphi) , \quad \xi' = \varphi^{-1} (\chi + \xi) , \quad \phi' = \phi , \quad (2.5) \]

\[ \xi' = \phi - 1 (\chi + \xi) , \quad (2.6) \]

\[ \phi' = \varphi , \quad (2.7) \]

with

\[ \varphi = \left(1 + 2 \chi \xi + \chi^2 \right)^{1/2} , \quad \chi = \frac{\gamma v u}{k r} . \quad (2.8) \]

The transformation (2.4)–(2.8) appeared first in [1] and is utilised there for a different purpose. Its effect on (2.1) is to yield

\[ ds^2 = r'^2 \left\{ \frac{d\xi'^2}{1 - \xi'^2} + (1 - \xi'^2) d\phi'^2 \right\} - 2du' dr' - du'^2 + \frac{2 m k \varphi}{r'} dr' du'^2 . \quad (2.9) \]

We will simplify the last term here assuming \( r' \) is large. To this end we first find that for large \( r' \),

\[ \varphi = 1 + \frac{\gamma v u \xi'}{r'} + O \left(\frac{1}{r'^2}\right) , \quad (2.10) \]

and so we obtain

\[ \xi = \xi' - \frac{\gamma v k' u' (1 - \xi'^2)}{r'} + O \left(\frac{1}{r'^2}\right) , \quad (2.11) \]
\[ \phi = \phi' , \quad (2.12) \]
\[ r = \frac{r'}{k'} + \gamma^2 v k' u' (v - \xi') + O \left(\frac{1}{r'}\right) , \quad (2.13) \]
\[ u = k' u' + O \left(\frac{1}{r'}\right) , \quad (2.14) \]

with \( k'^{-1} = \gamma (1 - v \xi') \). Now (2.9) becomes

\[ ds^2 = r'^2 \left\{ \frac{d\xi'^2}{1 - \xi'^2} + (1 - \xi'^2) d\phi'^2 \right\} - 2du' dr' - du'^2 + \frac{2 m k \varphi}{r'} (du' + k' u' \sqrt{1 - \xi'^2})^2 + O \left(\frac{1}{r'^2}\right) . \quad (2.15) \]

A further transformation \( r' \rightarrow r'' \) given by

\[ r'' = r' + \frac{m k'^5 \gamma v^2 u'^2 (1 - \xi'^2)}{2 r'^2} + O \left(\frac{1}{r'^3}\right) , \quad (2.16) \]

puts the line-element (2.15) in Bondi form [3],

\[ ds^2 = r''^2 \left\{ \frac{e^{2\lambda}}{(1 - \xi'^2)} \left( d\xi' + U \sqrt{1 - \xi'^2} du' \right)^2 + e^{-2\lambda} (1 - \xi'^2) d\phi'^2 \right\} - 2 e^{2\lambda} du' dr'' - r''^{-1} V e^{2\beta} dr'^2 , \quad (2.17) \]
with

\[
\lambda = \frac{M \gamma^2 v^2 u' k' v}{2 r'^3} (1 - \xi'^2) + O \left(\frac{1}{r'^4}\right), \quad (2.18)
\]

\[
U = \frac{2 M \gamma v u' k' \sqrt{1 - \xi'^2}}{r'^3} + O \left(\frac{1}{r'^4}\right), \quad (2.19)
\]

\[
\beta = O \left(\frac{1}{r'^2}\right), \quad (2.20)
\]

\[
V = r'' - 2 M + O \left(\frac{1}{r''}\right), \quad (2.21)
\]

and

\[
M = m k'^3. \quad (2.22)
\]

Thus \(M\) is the ‘mass aspect’ of the moving source. This is precisely the mass aspect associated with a mass \(m\) moving with velocity \(v\) along the axis of symmetry \([6]\).

To set up our model describing the instantaneous deceleration to rest of a Schwarzschild source we consider the Schwarzschild space–time \(\mathcal{M}\) subdivided into two halves \(\mathcal{M}^+\) and \(\mathcal{M}^-\) each with boundary the future null–cone \(\mathcal{N}(u = 0)\). For the line–element of the region \(\mathcal{M}^-\), which we take to be the past of \(\mathcal{N}\), we use (2.1). For the line–element of the region \(\mathcal{M}^+\) to the future of \(\mathcal{N}\) we take

\[
ds^2_+ = r_+^2 \left\{ \frac{d\xi^2_+}{1 - \xi^2_+} + (1 - \xi^2_+) d\phi^2 \right\} - 2du dr_+ - \left(1 - \frac{2m}{r_+}\right) du^2. \quad (2.23)
\]

These space–times are attached on \(\mathcal{N}\) with the matching conditions

\[
\xi_+ = \xi, \quad r_+ = k r, \quad (2.24)
\]

with \(k^{-1} = \gamma (1 - v \xi)\), which will ensure that the induced line–element on \(\mathcal{N}\) from its embedding in \(\mathcal{M}^+\) agrees with the induced line–element on \(\mathcal{N}\) from its embedding in \(\mathcal{M}^-\). In this way we have constructed the space–time denoted \(\mathcal{M}^- \cup \mathcal{M}^+\). Other ways of mapping \(\mathcal{N}\) to itself preserving the induced line–element are possible and lead to different models. The particular matching (2.25) is motivated by the electromagnetic analogue of this gravitational deceleration problem in \([4]\).

We now use the theory of light–like signals in general relativity developed by Barrabès and Israel (denoted BI) \([7]\) to discover the physical properties of the signal with history \(\mathcal{N}\). In general the Einstein tensor and the Weyl tensor of the space–time \(\mathcal{M}^- \cup \mathcal{M}^+\) will each have a Dirac \(\delta(u)\)–term and the BI theory enables us to calculate its coefficient using (2.1), (2.24) and the
matching conditions (2.25). If the coefficient of $\delta(u)$ in the Einstein tensor is non–zero then it is simply associated with a surface stress–energy for the light–like shell. If the coefficient of $\delta(u)$ in the Weyl tensor has a part which is type N in the Petrov classification with degenerate principal null direction coinciding with the direction of the null normal to $u = 0$ then the signal with history $N(u = 0)$ contains an impulsive gravitational wave. This is discussed in [7], [8]. For the reader familiar with the BI theory we now provide a guide through the present application.

The local coordinate system in $M^-$ with line–element (2.1) is denoted $\{x^\mu\} = \{\xi, \phi, r, u\}$ while the local coordinate system in $M^+$ with line–element (2.23) is denoted $\{x'^\mu\} = \{\xi_+, \phi, r_+, u\}$. As normal to $N$ we take the null vector field with components $n^\mu$ given via the 1–form $n^\mu dx^\mu = -du$. Since we want the physical properties of $N$ observed by the observer using the plus coordinates we take as intrinsic coordinates on $N$, $\{\xi^a\} = \{\xi_+, \phi, r_+\}$ with $a = 1, 2, 3$. A set of three linearly independent tangent vector fields to $N$ is $\{e^{(1)} = \partial/\partial \xi_+, e^{(2)} = \partial/\partial \phi, e^{(3)} = \partial/\partial r_+\}$. The components of these vectors on the plus side of $N$ are $e^{(a)}_\mu = \delta^{(a)}_\mu$. The components of these vectors on the minus side of $N$ are

$$e^{(a)}_\mu|_- = \frac{\partial x^-_\mu}{\partial \xi^a}, \quad (2.25)$$

with the relation between $\{x^-_\mu\}$ and $\{\xi^a\}$ given by the matching conditions (2.24). Hence we find that

$$e^{(1)}_\mu|_- = (1, 0, -r_+ \gamma v, 0), \quad (2.26)$$
$$e^{(2)}_\mu|_- = (0, 1, 0, 0), \quad (2.27)$$
$$e^{(3)}_\mu|_- = (0, 0, \gamma (1 - v \xi_+), 0). \quad (2.28)$$

We need a transversal on $N$ consisting of a vector field on $N$ which points out of $N$. A convenient such (covariant) vector expressed in the coordinates $\{x'^\mu\}$ is $^+N_\mu = (0, 0, 1, \frac{1}{2} - \frac{m}{r_+})$. Thus since $n^\mu = \delta_3^3$ we have $^+N_\mu n^\mu = 1$. We next construct the transversal on the minus side of $N$ with covariant components $^-N_\mu$. To ensure that this is the same vector on the minus side of $N$ as $^+N_\mu$ when viewed on the plus side we require

$$^+N_\mu e^{(a)}_\mu|_+ = -N_\mu e^{(a)}_\mu|-, \quad ^+N_\mu N_\mu = -N_\mu N_\mu. \quad (2.29)$$

The latter scalar product is zero as we have chosen to use a null transversal. We find that

$$^-N_\mu = \left(\frac{r_+ v}{1 - v \xi_+}, 0, \frac{1}{\gamma (1 - v \xi_+)}, D\right), \quad (2.30)$$
with
\[ D = \frac{v^2(1 - \xi_+^2)}{2(1 - v \xi_+)} + \frac{1}{2\gamma(1 - v \xi_+)} - \frac{m}{\gamma^2(1 - v \xi_+)^2 r_+}. \] (2.31)

Next the transverse extrinsic curvature on the plus and minus sides of \( \mathcal{N} \) is given by
\[ \pm K_{ab} = -\pm N_\mu \left( \frac{\partial e^\mu_{(a)\pm}}{\partial \xi^b} \pm \pm \Gamma_{\alpha\beta}^{\mu} e^\alpha_{(a)\pm} e^\beta_{(b)\pm} \right), \] (2.32)
where \( \pm \Gamma_{\alpha\beta}^{\mu} \) are the components of the Riemannian connection associated with the metric tensor of \( \mathcal{M}^+ \) or \( \mathcal{M}^- \) evaluated on \( \mathcal{N} \). The key quantity we need is the jump in the transverse extrinsic curvature across \( \mathcal{N} \) given by
\[ \sigma_{ab} = 2 \left( \pm K_{ab} - K_{ab} \right). \] (2.33)

This jump is independent of the choice of transversal on \( \mathcal{N} \). We find that in the present application \( \sigma_{ab} = 0 \) except for
\[ \sigma_{11} = \frac{2}{1 - \xi_+^2} \left( mk^3 - m \right), \quad \sigma_{22} = 2 \left( 1 - \xi_+^2 \right) \left( mk^3 - m \right), \] (2.34)
with \( k^{-1} = \gamma(1 - v \xi) \). Now \( \sigma_{ab} \) is extended to a 4–tensor field on \( \mathcal{N} \) with components \( \sigma_{\mu\nu} \) by padding–out with zeros (the only requirement on \( \sigma_{\mu\nu} \) is \( \sigma_{\mu\nu} e^\mu_{(a)\pm} e^\nu_{(b)\pm} = \sigma_{ab} \)). With our choice of future–pointing normal to \( \mathcal{N} \) and past–pointing transversal, the surface stress–energy tensor components are \( -S_{\mu\nu} \) with \( S_{\mu\nu} \) given by
\[ 16\pi S_{\mu\nu} = 2 \sigma_{(\mu n_\nu)} - \sigma_{n_\mu n_\nu} - \gamma^\mu \gamma_{\mu\nu}, \] (2.35)
with
\[ \sigma_{\mu} = \sigma_{\mu\nu} n^\nu, \quad \sigma^\dagger = \sigma_{\mu} n^\mu, \quad \sigma = g^{\mu\nu} \gamma_{\mu\nu}. \] (2.36)

In the present case \( \sigma_{\mu} = 0 \) and thus \( \sigma^\dagger = 0 \) and the surface stress–energy tensor takes the form
\[ -S_{\mu\nu} = \rho n_\mu n_\nu. \] (2.37)

Hence the energy density of the light–like shell measured by the distant observer using the plus coordinates is
\[ \rho = \frac{\sigma}{16\pi} = \frac{1}{4\pi r_+^2} \left( mk^3 - m \right). \] (2.38)

Thus the null–cone \( \mathcal{N} \) is the history of a light–like shell with surface stress–energy given by (2.37). We note that \( mk^3 \) is the “mass aspect” (cf. (2.22))
on the minus side of $\mathcal{N}$. A calculation of the singular $\delta$–part of the Weyl tensor for $\mathcal{M}^− \cup \mathcal{M}^+$ reveals that it vanishes. Hence there is no possibility of the light–like signal with history $\mathcal{N}$ containing an impulsive gravitational wave. We note that $\rho$ is a monotonically increasing function of $\xi^+$. Thus on the interval $-1 \leq \xi^+ \leq +1$, $\rho$ is maximum at $\xi^+ = +1$ (in the direction of the motion) and $\rho$ is minimum at $\xi^+ = -1$. This is as one would expect. A burst of null matter predominantly in the direction of motion is required to halt the mass. In this sense the model we have constructed here could be thought of as a limiting case of a Kinnersley rocket \[3\].

By integrating (2.38) over the shell with area element $dA_+ = r^2_+ d\xi_+ d\phi$ and with $-1 \leq \xi^+ \leq +1, 0 \leq \phi < 2\pi$ we obtain the total energy $E_+$ of the shell measured by the distant observer who sees the mass $m$, moving rectilinearly with 3–velocity $v$ in the direction $\xi^+ = +1$, suddenly halted. Thus

$$E_+ = \frac{1}{4\pi} \int_0^{2\pi} d\phi_+ \int_{-1}^{+1} (mk^3 - m) \, d\xi_+ .$$

(2.39)

This results in

$$E_+ = m(\gamma - 1) .$$

(2.40)

Thus all of the kinetic energy of the mass $m$ before stopping is converted into the relativistic shell. This is a satisfactory result from the point of view of energy conservation. It is interesting that an exact formula from special relativity has emerged from an exact calculation in general relativity.

### 3 Recoil with Ring–Down

In order to achieve a more realistic model we take the space–time $\mathcal{M}^+$ to be a Robinson space–time rather than the Schwarzschild space–time with line–element (2.23). Thus (2.23) is replaced by the line–element \[5\]

$$ds^2_+ = r^2_+ \{f_+^{-1}(d\xi_+ + a_+ f_+ du)^2 + f_+ d\phi^2\} - 2du\, dr_+ - c_+ du^2 ,$$

(3.1)

where $f_+, a_+$ are functions of $\xi_+, u$ and

$$c_+ = K_+ - 2H_+r_+ - \frac{2m}{r_+} ,$$

(3.2)

with $m$ a constant,

$$H_+ = \frac{1}{2} (a_+ f_+)' , \quad K_+ = -\frac{1}{2} f_+'' ,$$

(3.3)

and

$$a_+' = -f_+^{-2} \dot{f}_+ , \quad \left(6m a_+ f_+ + f_+ K_+ '\right)' = 0 ,$$

(3.4)
with the prime denoting differentiation with respect to \( \xi^+ \) and the dot denoting differentiation with respect to \( u \). Here the coordinates \( \{\xi^+, \phi, r^+\} \) have the ranges \(-1 \leq \xi^+ \leq +1\), \(0 \leq \phi < 2\pi\), \(0 < r^+ < +\infty\), \(0 \leq u < +\infty\) in \( \mathcal{M}^+ \). The hypersurfaces \( u = \text{constant} \) are future null–cones (in the sense used above) and \( \mathcal{M}^+ \) is attached to \( \mathcal{M}^- \), with line–element (2.1), on the null–cone \( \mathcal{N}(u = 0) \) with the natural generalisation of the matching (2.24) given by

\[
 r^+ = \frac{1}{g} r , \quad \frac{d\xi^+}{d\xi} = \frac{f^+}{1 - \xi^2} ,
\]

with

\[
 g = k^{-1} \left( \frac{f^+}{(1 - \xi^2)} \right)^{1/2} ,
\]
evaluated on \( u = 0 \) with \( k^{-1} = \gamma (1 - v \xi) \) as in (2.1). The function \( f^+(\xi^+, u) \) satisfies the conditions

\[
f^+(\pm 1, u) = 0 , \quad f'^+(+1, u) = 2 = -f'_+(+1, u) .
\]

The space–time with line–element (3.1) and with (3.2)–(3.4) and (3.7) holding is a vacuum space–time containing gravitational waves from a bounded source having smooth wave fronts which are homeomorphic to a 2–sphere. Robinson [5] has shown very elegantly how such a space–time will evolve with increasing \( u \) into a Schwarzschild space–time with mass \( m \). He has in addition given an exact solution in the form of a power series for the function \( f^+ \) in powers of \( \exp \left( -\frac{2u}{m} \right) \), with coefficients which are polynomials in \( \xi^+ \). The derived functions \( a^+, K^+, H^+ \) are then also given as similar power series in this variable. The function \( f^+ \) has the general form

\[
f^+ = (1 - \xi^2) \left\{ 1 + (1 - \xi^2) \hat{f}(\xi^+, u) \right\} ,
\]

with

\[
 \hat{f}(\xi^+, u) = \sum_{n=0}^{\infty} c_n \hat{f}_n(\xi^+) e^{-2(n+1)u/m} ,
\]

where \( c_n \) are arbitrary constants for \( n = 0, 1, 2, \ldots \) and \( \hat{f}_n(\xi^+) \) are known polynomials. We shall assume that when \( u = 0 \) the constants \( c_n \) are sufficiently small to ensure that the series of polynomials \( \hat{f}^+(\xi^+, 0) \) converges uniformly for \(-1 \leq \xi^+ \leq +1\). The leading terms in the power series for \( f^+, a^+, K^+, H^+ \) for the Robinson solution are

\[
f^+ = 1 - \xi^2 + c_0(1 - \xi^2)^2 e^{-2u/m} + \ldots ,
\]

\[
a^+ = \frac{2c_0\xi^+}{m} e^{-2u/m} + \ldots ,
\]
\[ K_+ = 1 - 2 c_0 (3 \xi_+^2 - 1) e^{-2u/m} + \ldots , \quad (3.12) \]
\[ H_+ = \frac{c_0}{m} (1 - 3 \xi_+^2) e^{-2u/m} + \ldots . \quad (3.13) \]

The calculation using the BI theory to establish the physical nature of the signal having as history the null-cone \( N(u = 0) \) parallels that given in section 2. We will outline the differences between the results given in section 2 and the calculations in the present case. Firstly (2.26) remains unchanged while (2.27)–(2.29) are now replaced by

\[ e^{\mu}_{(1)}|_- = \left( k^{-2} g^2, 0, r_+ g', 0 \right), \quad (3.14) \]
\[ e^{\mu}_{(2)}|_- = (0, 1, 0, 0), \quad (3.15) \]
\[ e^{\mu}_{(3)}|_- = (0, 0, g, 0), \quad (3.16) \]

with the prime as always denoting differentiation with respect to \( \xi_+ \). We note that \( \mathcal{M}^+ \) becomes the Schwarzschild space–time when \( c_n = 0 \) (\( n = 0, 1, 2, \ldots \)). In this case, by (3.7), \( f_+ = 1 - \xi_+^2 \) and so (3.3) becomes \( r_+ = k r_+ \), \( \xi_+ = \xi \) in agreement with (2.24). We note that now \( g = k^{-1} \) in (3.3) and so (3.14)–(3.16) reduce to (2.27)–(2.29) in this case. Next we take

\[ +N_\mu = \left( 0, 0, 1, \frac{c_+}{2} \right), \quad (3.17) \]

with \( c_+ \) given by (3.2) and now (2.30) for \( -N_\mu \) is replaced by

\[ -N_\mu = \left( -r_+ k^2 g g', 0, g^{-1}, \frac{1}{2 g} \left( 1 - \frac{2 m}{g r_+} + (g')^2 f_+ \right) \right). \quad (3.18) \]

We can now calculate the components of the transverse extrinsic curvature on the plus and minus sides of \( N \) using the formula (2.32). The only non–vanishing components \( +\mathcal{K}_{ab} \), in intrinsic coordinates \( \{ \xi^a \} = \{ \xi_+, \phi, r_+ \} \) with \( a = 1, 2, 3 \), on \( N \) are

\[ +\mathcal{K}_{11} = f_+^{-2+} \mathcal{K}_{22} = -\frac{m}{f_+} - \frac{r_+ f''_+}{4 f_+}, \quad (3.19) \]

and the non–vanishing components \( -\mathcal{K}_{ab} \) are

\[ -\mathcal{K}_{11} = -\frac{m}{g^3 f_+} - r_+ \frac{g''}{g} - r_+ \left[ \frac{g'}{g} \left( \frac{k'}{k} + \frac{f'}{2 f} + \frac{g'}{2 g} \right) - \frac{1}{2 g^2 f_+} \right], \quad (3.20) \]
\[ -\mathcal{K}_{22} = \frac{m f_+}{g^3} - r_+ f_+^2 \left[ \frac{g'}{g} \left( \frac{k'}{k} + \frac{f'}{2 f} + \frac{g'}{2 g} \right) - \frac{1}{2 g^2 f_+} \right], \quad (3.21) \]
with \( f = 1 - \xi^2 \), and with the prime denoting differentiation with respect to \( \xi \) and \( d/d\xi = (k g)^{-2}d/d\xi \) by (3.5) and (3.6). Next the jump \( \sigma_{ab} \) in the transverse extrinsic curvature (2.33) is calculated, followed by the surface stress–energy tensor \( S_{\mu\nu} \) given by (2.35). This again has the form given in (2.37) but with the energy density (2.38) replaced by

\[
\rho = \frac{\sigma}{16 \pi} = \frac{1}{4\pi r^2_+} \left( \frac{m}{g^3} - m \right) .
\]

To proceed further we need \( g(\xi) \) given by (3.6) and this requires \( f(\xi_+, 0) \).

Assuming that the sequence of small constants \( \{c_n\} \) appearing as coefficients in the expansion (3.9) is decreasing we shall take as an approximation to \( f(\xi_+, 0) \), using (3.10),

\[
f_+ = 1 - \xi^2 + c_0 (1 - \xi^2)^2 + O_2 , \tag{3.23}
\]

where we are treating the constant \( c_0 \) as small of first order \( (O_1) \) and the constant \( c_1 \) as small of second order \( (O_2) \) etc.. Now (3.7) gives the following approximate relation between \( \xi_+ \) and \( \xi \) on \( u = 0 \) (we assume that \( \xi = \pm 1 \) corresponds to \( \xi_+ = \pm 1 \) respectively)

\[
\xi = \xi_+ - c_0 \xi_+(1 - \xi^2_+) + O_2 . \tag{3.24}
\]

It now follows from (3.6) that \( g(\xi) \) is given approximately by

\[
g(\xi) = \gamma (1 - v \xi) \left[ 1 + \frac{c_0 (1 - 3 \xi^2)}{2 (1 - v \xi)} + \frac{c_0 v \xi + (1 + \xi^2)}{2 (1 - v \xi)} + O_2 \right] . \tag{3.25}
\]

This must now be substituted into (3.22) and the total energy \( E_+ \) of the light–like shell measured by the distant observer who sees the mass \( m \) suddenly stop is then found from

\[
E_+ = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{+1} m (g^{-3} - 1) d\xi_+ . \tag{3.26}
\]

This works out as

\[
E_+ = m (\gamma - 1) + \frac{3 m c_0}{4 \gamma^3 v^3} \left[ \frac{2 v (5v^2 - 3)}{3 (1 - v^2)^2} + \log \left( \frac{1 + v}{1 - v} \right) \right] + O_2 . \tag{3.27}
\]

The coefficient of \( c_0 \) here is always positive for \( 0 < v < 1 \) and vanishes if \( v = 0 \). In the special case of (3.23) when \( f_+ = 1 - \xi^2_+ \) only the first term in (3.27) survives and thus (2.40) is recovered. The first term on the right hand side of (3.27) is the kinetic energy of the mass \( m \) before stopping and it would
be natural in this model to assume that $c_0 < 0$ so that some of the original kinetic energy is available to be converted into gravitational radiation. The coefficient of $\delta(u)$ in the Weyl tensor of $\mathcal{M}^- \cup \mathcal{M}^+$ is non–vanishing and has a radiative part \[\hat{\Psi}_4\], denoted $\hat{\Psi}_4$ in Newman–Penrose notation, indicating the presence of an impulsive gravitational wave in the signal with history $\mathcal{N}$. Here $\hat{\Psi}_4$ is given by

\[
\hat{\Psi}_4 = \frac{3c_0}{r_+} (1 - \xi_+^2) + O_2 .
\] (3.28)

Thus with $c_0 < 0$ one can visualise that some of the kinetic energy in the original moving mass $m$ is converted into an impulsive gravitational wave accompanying the relativistic fireball (light–like shell) and the rest supplies the energy for the gravitational radiation present in the ring–down phase.

4 Discussion

The line–element induced on the 2–surfaces $r_+ = 1$, $u = \text{constant} \geq 0$ in the space–time with line–element $\text{(3.1)}$ is given by

\[
dl^2 = f_+^{-1}d\xi_+^2 + f_+ d\phi^2 ,
\] (4.1)

with $f_+$ given by $\text{(3.10)}$. It is easy to see that this 2–surface can be embedded in three dimensional Euclidean space by rotating a curve in the plane $z = 0$ the $x$–axis given parametrically by

\[
x(\xi_+) = \xi_+ \left\{ 1 - \frac{c_0}{2} (1 + \xi_+^2) e^{-2u/m} + O_2 \right\} ,
\] (4.2)

\[
y(\xi_+) = \sqrt{1 - \xi_+^2} \left\{ 1 + \frac{c_0}{2} (1 - \xi_+^2) e^{-2u/m} + O_2 \right\} ,
\] (4.3)

with $-1 \leq \xi_+ \leq +1$. The distance from the origin to any point on this 2–surface is

\[
R(\xi_+) = \sqrt{x^2 + y^2} = 1 - c_0 P_2(\xi_+) e^{-2u/m} + O_2 ,
\] (4.4)

with $P_2(\xi_+)$ the Legendre polynomial of degree 2 in the variable $\xi_+$. The sign of the small parameter $c_0$ was determined to be negative on physical grounds following $\text{(3.27)}$ and so we see from $\text{(4.4)}$ that this means that in the ring–down phase $u > 0$ the wave fronts of the gravitational waves are prolate spheroids with a common axis of symmetry coinciding with the original direction of motion of the moving mass $m$. 
Acknowledgment

This collaboration has been funded by the Ministère des Affaires Étrangères, D.C.R.I. 220/SUR/R.

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