Topological one-way fiber of second Chern number

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Optical fiber is a ubiquitous and indispensable component in communications, sensing, biomedicine and many other lightweight technologies and applications. Here we propose topological one-way fibers to remove two fundamental mechanisms that limit fiber performance: scattering and reflection. We design three-dimensional (3D) photonic crystal fibers, inside which photons propagate only in one direction, that are completely immune to Rayleigh and Mie scatterings and significantly suppress the nonlinear Brillouin and Raman scatterings. A one-way fiber is also free from Fresnel reflection, naturally eliminating the needs for fiber isolators. Our finding is enabled by the recently discovered Weyl points in a double-gyroid (DG) photonic crystal. By annihilating two Weyl points by supercell modulation in a magnetic DG, we obtain the photonic analogue of the 3D quantum Hall phase with a non-zero first Chern number ($C_1$). When the modulation becomes helixes, one-way fiber modes develop along the winding axis, with the number of modes determined by the spatial frequency of the helix. These single-polarization single-mode and multi-mode one-way fibers, having nearly identical group and phase velocities, are topologically-protected by the second Chern number ($C_2$) in the 4D parameter space of the 3D wavevectors plus the winding angle of the helixes. This work suggests a unique way to utilize higher-dimensional topological physics without resorting to artificial dimensions.

Topological photonics \cite{1,2} started with the realization of one-way edge waveguides \cite{3–14} as the analog of chiral edge states in quantum Hall effect, where the number and direction of 1D edge modes \cite{15} are determined by the 2D bulk topological invariant: the first Chern number ($C_1$). These Berry monopoles has now been realized in 3D as Weyl points \cite{16}, opening doors to 3D topological phases for photons \cite{17}. Here we show that, by annihilating a single pair of Weyl points with helix modulations, light can be guided unidirectionally in the core of 3D photonic crystal fibers where the number and direction of one-way modes equals the magnitude and sign of the second Chern number ($C_2$) — the topological invariant of complex vector bundles on 4D manifolds. In addition, we provide a definitive way to obtain arbitrary mode number (from $-\infty$ to $+\infty$) in the one-way fibers by varying the helix frequencies. Furthermore, all the modal dispersions have almost identical group and phase velocities, superior for multimode operations. The above aspects indicate that topological one-way fibers are fundamentally distinct and more advantageous than the topological one-way edge waveguides in 2D.

Topological one-way fiber represents a conceptual leap in fiber designs \cite{18–21} (Fig. 1), potentially advancing many aspects of fiber technologies. Firstly, Rayleigh scattering sets the ultimate lower limit for fiber loss. In the absence of Rayleigh scattering, topological one-way fiber only suffer from material absorption and could enable ultra-low loss fibers. Secondly, Mie scattering, occurring at irregular interfaces and miro-bendings, is mostly suppressed in topological fibers — another way of lowering fiber loss. Thirdly, stimulated Brillouin scattering (SBS) back-couples the forward signal and usually restricts the fiber transmitting power to a few mWs. This SBS threshold sets the upper bound on the maximum injection power in fiber communication systems, limiting the signal-to-noise ratio and the transmission distance without amplification. SBS threshold can now be lifted in one-way fibers. Furthermore, half of the Raman scat-

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FIG. 1. Ordinary fiber versus topological one-way fiber. a) The optical mode in an ordinary fiber is confined in the core, where both forward and backward modes exist. The imperfection induces scattering losses into back and radiation channels. (b) The optical mode in a topological one-way fiber is confined by the 3D topological bandgap which spatially separates the forward and backward modes. The one-way mode in the core is immune to scattering losses of any kinds.
Supercell of magnetic double gyroids

Modulated gyroids

FIG. 2. (a) Two cubic unit-cells of the DG photonic crystal magnetized along z. (b) The band structure of a cubic cell shows two Weyl points, which fold into one 3D Dirac point in the Brillouin zone of the supercell. (c) The DG photonic crystal whose volume fraction (colored) is periodically modulated along z. (d) The band structure of the photonic analogue of 3D Chern insulator whose topological gap frequencies are highlighted in green.

Because the bulk spectrum is gapped, \( C_1^z \) cannot change as a function of \( k_z \). When there are \( N \) bulk bands below the bandgap, \( F_{xy} \) is an \( N \times N \) matrix, whose elements are \( F_{xy}^{\alpha\beta} = \partial_y A_{y}^{\alpha\beta} - \partial_x A_{x}^{\alpha\beta} + i [ A_{z}^{\alpha}, A_{y}^{\beta} ]^{\alpha\beta} \), in which \( \alpha, \beta = 1, 2, \cdots, N \). The Berry connection \( A_{y}^{\alpha\beta}(k) = -i \langle \psi^{\alpha}(k) | \frac{\partial}{\partial k} | \psi^{\beta}(k) \rangle \), where \( | \psi^{\alpha}(\beta) \rangle \) are the Bloch eigenfunctions. Note that the trace of the commutator \( \text{Tr} [ A_{x}, A_{y} ] \) always vanishes for the first Chern class.

The topological invariants of our modulated DG is \( C_1 = (0,0,1) \). This can be understood from the original Weyl photonic crystal whose first Chern number is one for half of its Brillouin zone, as illustrated in Fig. 2b. By folding the Brillouin zone to half of its original size, the Chern numbers in different regions add up (Fig. 2b).

3D quantum Hall phase is a weak topological phase whose weak topological invariants are defined in a lower dimension as compared to a strong topological phase with
strong topological invariant. It is theoretically known that a lattice dislocation in a weak topological phase creates a 1D topological defect mode [28]. Unfortunately, in our case, a dislocation induces significant lattice distortion that generates many additional non-topological modes in the bandgap.

Fortunately, we propose and demonstrate below that, for a 3D quantum Hall phase constructed from Weyl crystals, a new approach is available: a smooth helical modulation generates a one-way mode at the core of the helix. The advantage here, compared with the dislocation approach, is the intactness of the lattice that prevents the generation of non-topological modes in the bandgap. Before presenting rigorous calculations, we outline a physical interpretation as follows (see Supplemental Material for details). A supercell modulation couples two Weyl points of opposite chiralities, forming a gapped 3D Dirac point with a mass term that is complex-valued. (A 3D Dirac point consists of two Weyl points of opposite chiralities.) Then a helical modulation amounts to a nonzero winding number of the phase of the Dirac mass around the helical axis, and it was indicated in previous theoretical models that such a topological perturbation can generate topological defect modes in both 2D [29, 30] and 3D systems [31-33].

One-way fiber modes.—Now comes the crucial step in our design of topological one-way fibers. Instead of the plain modulation (fig. 2c), we create a helical modulation by filling the volume defined by the equation:

\[ f(x, y, z) > f_0 + \Delta f \cos(\pi z/a + w\theta) \]  

(2)

The modulation now winds as a function of the angle \( \theta \) [\( \arctan(y/x) \)] in the \( x-y \) plane, whose spatial frequency is controlled by the signed integer \( w \). The sign and magnitude of \( w \) determines the direction and number of the one-way modes on the winding axes. This is illustrated in the upper panels of Fig. 3 for \( w = +1, +2, +3 \), corresponding to single, double and triple helix one-way fibers.

The band structures of the one-way fibers are shown in middle panels of Fig. 3. They were calculated using MIT Photonic Bands on an \( 11 \times 11 \times 2 \) cubic supercell. The spectra exhibit one-way modes within the bulk band gap. The profiles of the topological fiber modes are localized around the helix cores (Fig. 3 lower panel).

Shown in the middle panel of Fig. 3 are all one-way-fiber dispersions (green lines) have very similar phase and group velocities. In the multimode cases, their dispersions almost overlap on top of each other. This is due to the fact that these defect modes originated from the same Weyl bulk bands, so they all share the same Brillouin-zone location and group velocities as that of the original Weyl cones. This behavior is different from that of the one-way-edge waveguides in 2D [15], where the edge modal dispersions have different phase or group velocities. This can be attributed to the fact that the edge environment is distinct from the environment of the 2D bulk lattice, while, here, there is no sharp interfaces in the 3D one-way fibers. This unique feature, of multiple fiber modes having almost identical dispersions, ensures multimode signals propagate at the same speed without intermodal distortion.

Hollow-core one-way fibers can be made by removing the materials on the helix axis, in which the mode confinement in the air core is ensured by the 3D photonic bandgap. The extra air volume in the hollow core may generate trivial fiber modes, which can be tuned away. In contrast, the existence of the non-trivial one-way mode is protected by the topological invariant of the system.

Second Chern number.—It is natural to ask for a topological invariant for the one-way fibers. With the simplest helix modulation of the form of Eq. 2 it is intuitive to guess that \( w \) is the topological invariant, since the number and direction of the one-way modes match the magnitude and sign of \( w \). However, this observation does not work if we consider the modulation of the general form as \( f(x, y, z) > 0.4 + \sum_{w} h_w \cos(\pi z/a + w\theta) \), where \( h_w \) are real-valued constants.

For a lattice dislocation in a 3D Chern insulator (the 3D QHE), it is known [31] that the number of chiral modes is given by \( C_1 \cdot b \), where the dimensionless Burgers vector(b) represents the magnitude and direction of the lattice distortion. However, this approach cannot be applied to our system due to the lack of unique “Burgers vector” other than in the simplest case (as of Eq. 2).

We show that the desired topological invariant of our one-way fibers are the second Chern number \( C_2 \), the strong topological invariant in our system. Note that, far away from the axis of the helix, the Bloch Hamiltonian smoothly varies with \( \theta \), thus is a smooth function of the four variables \( (k_x, k_y, k_z, \theta) \). Since \( (k_x, k_y, k_z, \theta) \) span a four-dimensional parameter space with periodic boundary conditions (a 4D torus), the second Chern number [34-35] can be defined:

\[ C_2 = \frac{1}{4\pi^2} \int d^3kd\theta \text{Tr} [\mathcal{F}_{\alpha \beta} \mathcal{F}_{\alpha \beta} + \mathcal{F}_{\beta \alpha} \mathcal{F}_{\beta \alpha} + \mathcal{F}_{\alpha \gamma} \mathcal{F}_{\beta \gamma}]. \]  

(3)

Similar to the definitions in Eq. 4, \( \mathcal{F}_{\alpha \beta} = \partial_i A_{\alpha}^{\beta} - \partial_j A_{\alpha}^{\beta} + i [A_i, A_j]^{\alpha \beta} \), in which \( \alpha, \beta \) are the band indices. The non-Abelian Berry potential \( A_i^{\alpha \beta}(k, \theta) = -i \langle \psi^{\alpha}(k, \theta) | \partial_i | \psi^{\beta}(k, \theta) \rangle \), where \( | \psi^{\alpha}(\beta) \rangle \) are the eigenfunctions and \( k_i \) runs through \( k_x, k_y, k_z, \theta \). It is notable that this definition of \( C_2 \) involves three variables \( (k_{x,y,z}) \) in the reciprocal space and one variable \( \theta \) in the real space, in contrary to the four momentum variables in the 4D quantum Hall effect [35-38]. Consequently, the Berry curvature \( \mathcal{F}_{\alpha \beta} \) is even while \( \mathcal{F}_{\beta \alpha} \) is odd under time-reversal, where \( i \) or \( j \) represents one of \( x, y \) and \( z \). Although in 4D QHE, \( C_2 \) can be non-zero without breaking time-reversal symmetry, non-zero \( C_2 \) requires time-reversal breaking in our system.

In the Supplemental Material, we carried out the explicit calculations of \( C_2 \), which is consistent with our numerical findings in Fig. 3. The topological protection by the second Chern number indicates that the physical origin of the one-way fiber modes is fundamentally different.
from that of the edge modes of the photonic analog of 2D quantum Hall effect \[3,4\], whose topology is captured by the first Chern number. We note that although, in our system, both the weak indices ($C_1$) and the strong index ($C_2$) are non-zero, it is possible to construct a one-way fiber design with zero $C_1$ and non-zero $C_2$. For example, when the separation between the two Weyl points shrinks zero (forming a 3D Dirac point), one can apply only angular ($\theta$) modulations to obtain a one-way fiber of non-zero $C_2$ but zero $C_1$s.

**Experimental feasibility.**— Various existing technologies can be adopted to realize these magnetic fibers in experiments across various frequencies. At microwave frequencies, the sample can be fabricated by the same drilling and stacking approach as demonstrated in Ref. \[10\] using gyromagnetic materials \[4,15\]. Similar methodology can be used at terahertz wavelengths. Towards optical frequencies, gyroelectrical materials are the choices. For example, paramagnetic Terbium-doped magnetic fibers have already been demonstrated with high Verdet constants \[39,40\]. Ferrimagnetic materials, having a stronger magneto-optical effect than paramagnetic ones, usually suffer higher optical losses. Fortunately, they are continually being improved \[41\] and enhanced \[42\]. A DG fiber can either be made by drawing a 3D-printed preform or potentially by self-assembly \[43,44\] during the drawing process. 3D direct writing \[45\] and interference lithography \[46\] can also be adopted. Finally, the chiral modulation can be created by spinning the fiber during drawing, as demonstrated in the chiral fibers \[47,48\].

**Outlook.**—The proposal of one-way fibers enriches the prospects of device applications for the Weyl materials, topological photonics and topological physics in general. It also brings a new playground on the realization of higher dimensional topological phases. The same phenomena can be realized in other Weyl systems \[49–57\] with time-reversal symmetry breaking. More importantly, such topological fibers can inspire new directions, design principles and applications for fiber technology.

**Acknowledgements.**—We thank the discussion with Jian Wang, Wei Ding and Changyuan Yu on fiber technologies and with Hannah M. Price and Chen Fang on 3D QHE. Z. W. was supported by NSFC under Grant
L. L. was supported in part by the National Thousand-Young-Talents Program of China. Part of the numerical calculations of this work was performed using MIT Photonic Bands on En-Real.com.
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Supplemental Material for “Topological one-way fiber of second Chern number”

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In the main text we have presented our design of the one-way fiber and the results of band-structure calculations. To gain a simple analytical understanding of the one-way modes, we outline below an effective Hamiltonian description, using the low-energy Dirac Hamiltonian. The picture can be summarized as follows. In terms of the effective Dirac Hamiltonian, the modulation corresponds to the presence of a Dirac mass. The helix modulation introduces a topologically nontrivial configuration of the Dirac mass (nonzero winding number of the phase of Dirac mass), which creates topologically one-way modes.

Effective 3D Dirac Hamiltonian

In the absence of modulations, the double gyroid crystal host two Weyl points with opposite chirality, either of which is described by a 2 × 2 effective Weyl Hamiltonian. We can combine them as a 4 × 4 block-diagonal Dirac Hamiltonian:

\[ H_D = -iv(\sigma_2 \partial_2 + \sigma_y \partial_y + \sigma_z \partial_z)\tau_z, \]

where \( \sigma_i, \tau_i \) (\( i = x, y, z \)) are Pauli matrices, \( v \) is the group velocity (for simplicity, we take isotropic group velocities with \( v > 0 \)). Here \( \tau_z = 1 \) and \( \tau_z = -1 \) is associated with the two Weyl points, respectively. As we have seen in the main text, a modulation with periodicity \( 2\alpha \) in the \( z \) direction gaps out the Weyl points. In the effective Dirac Hamiltonian description, the frequency gap is due to the Dirac mass terms. There are only two possible Dirac mass terms that can generate a frequency gap: \( m_1\tau_x \) and \( m_2\tau_y \), both of which anti-commute with the \( \sigma, \tau \) terms in \( H_D \). It is thus expected that the modulation amounts to the presence of these Dirac mass terms in the low energy effective Hamiltonian. In general, both \( m_1 \) and \( m_2 \) can be nonzero, and the mass term can be written as \( m_1\tau_x + m_2\tau_y = m\tau_+ + m^*\tau_- \), with \( \tau_\pm \equiv (\tau_x \pm i\tau_y)/2 \) and \( m \equiv m_1 - im_2 \). The full effective Hamiltonian, with the effect of modulation included as the Dirac mass, can be written as

\[ H_{\text{eff}} = -iv(\sigma_2 \partial_2 + \sigma_y \partial_y + \sigma_z \partial_z)\tau_z + m\tau_+ + m^*\tau_- , \]

from which we can readily see that a frequency gap \( 2|m| \) is generated by the Dirac mass terms.

The phase factor of the complex-valued Dirac mass \( m \) is crucial and deserves more discussions. Suppose that the modulation can be modelled in the effective Hamiltonian by a perturbation \( V(r) = V_Q \exp(iQ \cdot r) + V'_Q \exp(-iQ \cdot r) + \cdots \), where \( Q = (0, 0, \pi/a) \) is the wave vector of the modulation that couples the two Weyl points. We can readily see that \( \exp(iQ \cdot r) \to \tau_\pm \) is valid near the Weyl points, thus we have \( m = V_Q \), in other words, the complex-valued Dirac mass is simply the \( Q \)-component of the perturbation.

Now we have the following important observation. If we displace the modulation by a distance \( d \), then the perturbation becomes \( V(r + d) \), which can be expand as \( V(r + d) = V_Q \exp[iQ \cdot (r + d)] + V'_Q \exp[-iQ \cdot (r + d)] + \cdots \), thus we can see that the displacement causes \( V_Q \to V_Q \exp(iQ \cdot d) \), or equivalently, \( m \to m \exp(iQ \cdot d) \). For the helix-shape modulation along the axis \( r = 0 \), as described in the main text, the displacement \( d \) is a function of \( \theta \) such that \( Q \cdot d = w\theta \), therefore, we have a nonzero winding of the phase of Dirac mass around the axis, namely, \( m(\theta) = m_0 \exp(iw\theta) \), in which \( m_0 \equiv m(\theta = 0) \). The overall phase of \( m_0 \) can be changed by rotating the coordinate systems around the \( r = 0 \) axis, thus we are free to take \( m_0 \) to be real-valued and positive.

Analytic solutions of the one-way modes

For the effective Dirac Hamiltonian with a nonzero winding of the phase of Dirac mass (with winding number \( w \)), which is a consequence of the helix perturbation, we shall show that there exist \( |w| \) topological one-way modes. We can rewrite Eq.(2) in the cylindrical coordinates as

\[ H_{\text{eff}} = -iv(\sigma_2 \partial_2 + \sigma_y \partial_y + \sigma_z \partial_z)\tau_z + \frac{m\rho^2}{\rho^2 + \zeta^2}\tau_+ + \frac{m^*\rho^2}{\rho^2 + \zeta^2}\tau_- , \]
\[ H_{\text{eff}} = \begin{bmatrix} v k_z, & -i v e^{i \theta} \frac{\partial}{\partial r} - i \frac{\partial}{\partial \theta}, & m_0 e^{i w \theta}, & 0, & m_0 e^{i w \theta} \\
-i v e^{i \theta} \left( \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta} \right), & -v k_z, & 0, & m_0 e^{i w \theta}, & -v k_z, \\
m_0 e^{-i w \theta}, & 0, & \frac{i v e^{i \theta}}{r} + \frac{\partial}{\partial r}, & 0, & \frac{i v e^{i \theta}}{r} + \frac{\partial}{\partial r} \end{bmatrix}. \] (3)

where we have taken advantage of the translational symmetry in the \( z \) direction by replacing \(-i \partial_z \) by \( k_z \). For notational simplicity, we shall keep implicit the common factor \( \exp(i k_z z) \) in the eigenfunction. For reason that will become clear shortly, we look for eigenfunctions of the form of \( \psi = [\psi_1, 0, 0, \psi_4]^T \). The eigenvalues are \( E = v k_z \), and the eigenfunctions satisfy

\[
\begin{align*}
-i v e^{i \theta} \left( \frac{\partial}{\partial r} + i \frac{\partial}{\partial \theta} \right) \psi_1 + m_0 e^{i w \theta} \psi_4 &= 0, \\
m_0 e^{-i w \theta} \psi_1 + i v e^{-i \theta} \left( \frac{\partial}{\partial r} - i \frac{\partial}{r \partial \theta} \right) \psi_4 &= 0.
\end{align*}
\] (4)

It is not difficult to observe that the second equation is equivalent to the first one if we take \( \psi_4 = \pm \psi_1^* \). Let us focus on the \( \psi_4 = \psi_1^* \) case first. With this condition, the above two equations are reduced to a single equation

\[
-i v e^{i \theta} \left( \frac{\partial}{\partial r} + i \frac{\partial}{r \partial \theta} \right) \psi_1 + m_0 e^{i w \theta} \psi_1^* = 0
\] (5)

For the \( w = 1 \) case, the common \( \exp(i \theta) \) factors can be eliminated, thus the equation becomes especially simple, and the solution can be found analytically as

\[
|\psi_{w=1}\rangle = \begin{pmatrix} e^{i \pi / 4} \\ 0 \\ 0 \\ e^{-i \pi / 4} \end{pmatrix} e^{-m_0 r}.
\] (6)

For an arbitrary integer \( w \geq 1 \), by analysis analogous to Ref.[1], we can show that there exist \( w \) localized modes. In fact, we can take the following ansatz for Eq.(5):

\[
\psi_1^{(l)} = e^{i \pi / 4} [u_l e^{i \theta} + v_l e^{i(w-1-l) \theta}],
\] (7)

with an integer parameter \( l \), whose acceptable values are to be determined. We first notice that when \( l = (w-1)/2 \), \( e^{i \theta} \) and \( e^{i(w-1-l) \theta} \) are actually equal, and the \( v_l \) term is redundant. Let us first focus on the cases \( l \neq (w-1)/2 \). The special case \( l = (w-1)/2 \), with \( e^{i \theta} = e^{i(w-1-l) \theta} \), will be discussed separately later.

According to Eq.(5), the coefficient functions \( u_l, v_l \) have to satisfy

\[
\begin{align*}
u_l \left( \frac{\partial}{\partial r} - \frac{l}{r} \right) u_l + m_0 v_l &= 0, \\
v_l \left( \frac{\partial}{\partial r} - \frac{w-1-l}{r} \right) u_l + m_0 u_l &= 0.
\end{align*}
\] (8)

The asymptotic behaviors of \( u_l, v_l \) in the \( r \to 0 \) limit can be found as (I) \( u_l \sim r^l, v_l \sim r^{l+1} \) or (II) \( u_l \sim r^{w-l}, v_l \sim r^{w-l-1} \). On the other hand, the asymptotic behaviors in the \( r \to \infty \) limit are (a) \( u_l \to \exp(-\frac{m_0}{r} r), v_l \to \exp(-\frac{m_0}{r} r) \) or (b) \( u_l \to \exp(\frac{m_0}{r} r), v_l \to \exp(\frac{m_0}{r} r) \), only the first of which is normalizable in the \( r \to \infty \) regime. A normalizable solution must have behavior (a) in the \( r \to \infty \) limit, which is generally a superposition of (I) and (II) in the \( r \to 0 \) regime. Therefore, the normalizability of the solution in the \( r \to 0 \) limit requires that both (I) and (II) are normalizable, which leads to the constraint

\[
0 \leq l \leq w - 1.
\] (9)

Thus we have proved that, leaving out the special case \( l \neq (w-1)/2 \) undetermined, there exists one normalizable solution for every integer \( l = 0, 1, 2, \cdots, w-1 \). However, the solutions with \( l > (w-1)/2 \) are redundant, because the solutions for \( l \) and \( l' \) with \( l + l' = w - 1 \) are actually the same one, as can be appreciated from Eq.(7). Therefore, the total number of solutions with \( \psi_4 = \psi_1^* \) is the number of nonnegative integer smaller than \((w-1)/2\), which is \([w/2]\)
(Here, “[···]” denotes the floor function, mapping a real number to the largest previous integer), excluding possible solution with \( l = (w - 1)/2 \).

Now we consider the other choice: \( \psi_4 = -\psi_1^* \). By calculations similar to the case \( \psi_4 = \psi_1^* \), we can obtain equations similar to Eq.(5), except that the “+” sign before \( m_0 \) is replaced by “−”. We adopt the same ansatz as given in Eq.(7), and follows the steps below Eq.(7), solving the case \( l \neq (w - 1)/2 \) first. It is found that the number of solutions is \([w/2]\).

Finally, we study the special case \( l = (w - 1)/2 \) (This case needs consideration only when \( w \) is odd; for \( w \) even, this option is automatically absent). Given this value of \( l \), the second term in Eq.(7) becomes redundant, thus we can take

\[
\psi_1^{(l)} = e^{i\pi/4}u_le^{i\theta}.
\] (10)

Under the choice \( \psi_4 = \pm \psi_1^* \), we obtain the single differential equation

\[
u\left(\frac{\partial}{\partial r} - \frac{l}{r}\right)u_l \pm m_0u_l = 0.
\] (11)

For the choice “+” of the “±”, Eq.(11) has a single normalizable solution with asymptotic behaviors \( u_l \to r^l \) in the \( r \to 0 \) limit and \( u_l \to \exp(-\frac{m_0}{w}r) \) in the \( r \to \infty \) limit. For the choice “−” of the “±”, Eq.(11) leads to \( u_l \to \exp(\frac{m_0}{w}r) \) in the \( r \to \infty \) limit, which is apparently not normalizable. Therefore, there exists a single normalizable localized mode, in the \( \psi_4 = \psi_1^* \) sector, for the special case \( l = (w - 1)/2 \). We also note that, if we take \( m_0 < 0 \) instead of \( m_0 > 0 \), the normalizable solution would be present in the \( \psi_4 = -\psi_1^* \) sector but absent in the \( \psi_4 = \psi_1^* \) sector, thus the total number of solution is the same.

Let us summarize the above calculations as follows. When \( w \) is odd, the total number of normalizable solutions is \( 2[w/2] + 1 = w \); when \( w \) is even, the total number of normalizable solutions is \( 2[w/2] = w \). Therefore, the total number of topological modes is always \( w \), irrespective of the parity (odd/even) of \( w \). Furthermore, our calculation shows that the eigenvalues take the simple form

\[
E(k_z) = vk_z,
\] (12)

thus all these \( w \) modes propagate along the \( +z \) direction, with the same velocity \( v \).

Finally, let us discuss the one-way modes for the integer \( w < 0 \). Solution of the form of \( \psi = [\psi_1, 0, 0, \psi_4]^T \) does not exist in this case, because the condition given in Eq.(9) can never be satisfied. On the other hand, solutions of the form of \( \psi = [0, \psi_2, \psi_3, 0]^T \) can be found. In fact, we can follow the steps above and obtain the equations

\[
-ive^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right)\psi_2 + m_0e^{iw|\theta|}\psi_3 = 0,
\]

\[
m_0e^{-i\theta}\psi_2 + iwe^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)\psi_3 = 0,
\] (13)

whose complex conjugations are

\[
-m_0e^{i\theta}\psi_2 + iwe^{-i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right)(-\psi_3) = 0,
\]

\[
m_0e^{-i\theta}\psi_2 + iwe^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right)(-\psi_3) = 0.
\] (14)

We can see that Eqs.(14) are the same as Eqs.(4) except that \( \psi_2 \) and \( -\psi_3 \) take the place of \( \psi_1 \) and \( \psi_4 \), respectively. Now our previous analysis for Eqs.(4) with \( w \geq 1 \) immediately tells us that the number of one-way modes for \( w < 0 \) is \( |w| \). Because the solutions for \( w < 0 \) take the form of \( \psi = [0, \psi_2, \psi_3, 0]^T \), the dispersion is \( E(k_z) = -vk_z \), thus all the one-way modes propagate in the \( -z \) direction.

**Calculation of the second Chern number \( C_2 \)**

The effective Hamiltonian Eq.(2) takes the form of

\[
H_{\text{eff}} = \sum_{a=1}^{5} d_a \Gamma^a,
\] (15)
where the Dirac matrices $\Gamma^a = \sigma_a \tau_z$ ($a = 1, 2, 3$), $\Gamma^4 = \tau_x$, $\Gamma^5 = \tau_y$, the coefficient functions $d_a = v_k a$ ($a = 1, 2, 3$), $d_4 = \text{Re}(m)$, $d_5 = -\text{Im}(m)$. A straightforward calculation\[4\] of $C_2$ leads to

$$C_2 = \frac{3}{8\pi^2} \int d\theta d^3k \epsilon^{abcde} \hat{d}_a \frac{\partial \hat{d}_b}{\partial k_x} \frac{\partial \hat{d}_c}{\partial k_y} \frac{\partial \hat{d}_d}{\partial k_z} \frac{\partial \hat{d}_e}{\partial \theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{d[\text{arg}(m(\theta))]}{d\theta}, \quad (16)$$

where $\hat{d}_a = d_a / \sqrt{\sum_{b=1}^{5} d_b^2}$. With the Dirac mass $m = m_0 \exp(iw(\theta))$, we have

$$C_2 = w. \quad (17)$$

For a general modulation that combines several different spatial frequency, namely, $m(\theta) = \sum_w m_w \exp(iw(\theta))$, Eq.(16) is not amenable to further simplification in the generic cases, however, we have $C = w_0$ in the cases that $|m_{w_0}| > \sum_{w \neq w_0} |m_w|$, in other words, $C_2$ is determined by the dominant modulation.

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[3] We have used the cylindrical coordinates $(x, y, z) \equiv (r \cos \theta, r \sin \theta, z)$.
[4] The calculation of Chern number for Hamiltonians of this form can be found in Ref.[2].