COUPLED VORTEX EQUATIONS AND MODULI: DEFORMATION THEORETIC Approach AND KÄHLER GEOMETRY

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Abstract. We investigate differential geometric aspects of moduli spaces parametrizing solutions of coupled vortex equations over a compact Kähler manifold $X$. These solutions are known to be related to polystable triples via a Kobayashi–Hitchin type correspondence. Using a characterization of infinitesimal deformations in terms of the cohomology of a certain elliptic double complex, we construct a Hermitian structure on these moduli spaces. This Hermitian structure is proved to be Kähler. The proof involves establishing a fiber integral formula for the Hermitian form. We compute the curvature tensor of this Kähler form. When $X$ is a Riemann surface, the holomorphic bisectional curvature turns out to be semi–positive. It is shown that in the case where $X$ is a smooth complex projective variety, the Kähler form is the Chern form of a Quillen metric on a certain determinant line bundle.

1. Introduction

A holomorphic vector bundle together with a holomorphic section of it will be called a pair. More generally, a pair of holomorphic vector bundles on a compact Kähler manifold together with a holomorphic homomorphism between them will be called a triple. Pairs and triples have been investigated extensively by Bradlow and García-Prada (they introduced these objects) and others [A-G, Br1, Br2, Ga1, Ga2, B-D-G-W, B-G, B-G-G].

The notion of stability of a usual vector bundle generalizes to the context of triples; the definition of a stable triple is recalled in Section 4.1. The stable pairs are related, by a Kobayashi–Hitchin type correspondence, to the solutions of vortex equation, and the stable triples are known to be related to the solutions of the coupled vortex equations. The coupled vortex equations are recalled in Section 4.1 and the Kobayashi–Hitchin correspondence between the stable triples and the solutions of the coupled vortex equations is recalled in Section 4.2.

Fix a compact Kähler manifold $X$ equipped with a Kähler form $\omega$. Take a triple $(E_1, E_2, \phi)$ over $X$, where $E_1$ and $E_2$ are holomorphic vector bundles over $X$ and $\phi : E_2 \longrightarrow E_1$ is a holomorphic homomorphism of vector bundles. Associated to this triple we have a complex of $\mathcal{O}_X$–modules

$$C^\bullet : 0 \longrightarrow C^0 := \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2) \xrightarrow{\Delta} C^1 := \text{Hom}_{\mathcal{O}_X}(E_2, E_1) \longrightarrow 0,$$

with

$$\Delta(\psi_1, \psi_2) = \psi_1 \circ \phi - \phi \circ \psi_2 =: [\psi_1, \psi_2].$$

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where $\text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)$ is at the 0–th position. It is easy to see that the space of all endomorphisms of the triple $(E_1, E_2, \phi)$ coincides with the 0–th hypercohomology $H^0(C^*)$. The first hypercohomology $H^1(C^*)$ parametrizes all infinitesimal deformations of the triple, while the second hypercohomology contains the obstructions to deformations.

Since the homomorphism $\phi$ is holomorphic, it intertwines the Dolbeault resolutions of $E_1$ and $E_2$. Therefore, we obtain a double complex of $C^\infty$ flabby sheaves which gives a resolution of $C^\bullet$. Consequently, the cohomologies associated to this double complex coincide with the hypercohomologies of $C^\bullet$.

Now, if the vector bundles $E_1$ and $E_2$ admit Hermitian structures that give solutions of the coupled vortex equations for $(E_1, E_2, \phi)$, then the Hermitian metrics on $E_1$ and $E_2$ and the Kähler form $\omega$ together define Hermitian structures on all the terms of the above mentioned double complex associated to $C^\bullet$. Consequently, we obtain harmonic representatives of the hypercohomologies of $C^\bullet$.

This immediately induces an $L^2$ metric on the hypercohomologies of $C^\bullet$, in particular, on the first hypercohomology. Thus the infinitesimal deformations of the triple $(E_1, E_2, \phi)$ are equipped with a Hermitian structure. This gives us a Hermitian structure on the moduli space of solutions of the coupled vortex equations. We call this Hermitian structure on the moduli space the \textit{vortex-moduli} metric.

We prove the following theorem.

\textbf{Theorem 1.1.} The vortex–moduli Hermitian metric is actually a Kähler metric in the orbifold sense.

The following theorem is used in our investigation of the vortex–moduli metric.

\textbf{Theorem 1.2.} Let $T = (E_1, E_2, \phi)$ be a stable triple over a compact Kähler manifold $(X, \omega_X)$. Let $S$ be a reduced complex space with a base point $s_0 \in S$, and let $T = (E_1, E_2, \Phi)$ be a deformation of $T$ over $(S, s_0)$. Let $h_1$ and $h_2$ be Hermitian metrics on $E_1$ and $E_2$ respectively that solve the coupled vortex equations for $T$. Then there exists a neighborhood $U$ of $s_0$ such that the solutions can be extended to the fibers $T_s$, $s \in U$, in a $C^\infty$ way.

The above theorem says that, in a holomorphic family of stable triples, a solution of the coupled vortex equations can be extended in a unique way to the neighboring fibers. As an application of Theorem 1.2, we get the existence of a moduli space of solutions of the coupled vortex equations on stable triples in the category of (not necessarily Hausdorff) complex spaces; see Section 4.4.

In Theorem 8.1 we compute the curvature of the vortex–moduli metric.

Theorem 8.1 has the following corollary (see Corollary 8.2):

\textbf{Corollary 1.3.} \textit{If $X$ is a Riemann surface, then the holomorphic bisectional curvature of the vortex–moduli metric is semi–positive.}

Under the extra assumption that $X$ is complex projective, we construct a certain holomorphic Hermitian line bundle over the moduli space of stable triples whose curvature coincides with the vortex–moduli form. The line bundle in question is a determinant bundle associated to direct images, and the Hermitian structure is given by a construction due to Quillen and Bismut–Gillet–Soulé.
We now give a very brief description of the contents of the individual sections.

In Section 2 we collect basic definitions and notations. In Section 3 deformations of triples are studied. In Section 4 the coupled vortex equations are investigated, and the deformation theoretic approach introduced in Section 3 is pursued further. In Section 5 the associated elliptic complex is investigated. In Section 6 the vortex–moduli metric is constructed. In Section 7 a fiber integral formula for this metric is established, and it is shown that the vortex–moduli metric is Kähler. In Section 8 we compute the curvature of the vortex–moduli metric.

2. Basic definitions

Let $X$ be a compact, connected Kähler manifold, of complex dimension $n$, equipped with a Kähler form $\omega_X$. We will write

$$\omega_X = \sqrt{-1} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta = \sqrt{-1} \sum_{\alpha,\beta=1}^{n} g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta$$

with respect to local holomorphic coordinates $(z^1, \ldots, z^n)$, and we will always use the summation convention. In the sequel, we identify locally free coherent analytic sheaves on Kähler manifolds with holomorphic vector bundles on them.

We will use the following conventions. The Kähler form $\omega_X$ gives rise to a connection on $X$, which we will, given any complex space $S$, extend in a flat way (in the direction of $S$) to $X \times S$. As above, we will denote by $z^\alpha, z^\gamma, \ldots$ holomorphic local coordinates on $X$ together with the conjugates $\overline{z}^\alpha, \overline{z}^\gamma, \ldots$. We will denote by $s^i, s^k, \ldots$ and $\overline{s}^\overline{i}, \overline{s}^\overline{k}, \ldots$ respectively similar coordinates on $S$ if $S$ is smooth. If $S$ is not smooth, then $s^i, s^k, \ldots$ and $\overline{s}^\overline{i}, \overline{s}^\overline{k}, \ldots$ will denote coordinates on an ambient smooth space into which a neighborhood, of $S$, of a given base point in $S$ is minimally embedded. We use the semi–colon notation $()$ for covariant derivatives of sections, and also of differential forms or tensors, with values in the holomorphic Hermitian vector bundles with respect to connections induced by the Kähler metric on $X$ and the Hermitian connection on the holomorphic Hermitian vector bundles. Let the Hermitian connection $\theta_E$ on any holomorphic vector bundle $E$ over $X$ be given locally by matrix–valued $(1,0)$–forms $\{\theta_\alpha\}_{\alpha=1}^{n}$ with respect to some local trivialization of $E$. Let $\sigma$ be a locally defined section of $E$, which is a vector–valued function with respect to the trivialization of $E$. We use

$$\frac{\partial \sigma}{\partial z^\alpha} = \partial_\alpha \sigma = \sigma |_\alpha,$$

and set

$$\sigma |_\alpha = \nabla_\alpha \sigma = \sigma |_\alpha + \theta_\alpha \circ \sigma$$

and

$$\sigma |_{\overline{\beta}} = \sigma |_{\overline{\beta}}.$$

Hence

$$\sigma |_{\alpha\overline{\beta}} = \sigma |_{\overline{\beta}\alpha} - R_{\alpha\overline{\beta}} \circ \sigma,$$

where $R_{\alpha\overline{\beta}}$ denote the components of the curvature form $\Omega_{\alpha\overline{\beta}} = \theta_{\alpha\overline{\beta}}$ of the connection $\theta_E$. For tensors with values in the endomorphism bundle $\text{End}_{\mathcal{O}_X} E$, we also have the
contributions that arise from the Kähler connection on the base. For any differentiable homomorphism of vector bundles
\[ \psi : E_2 \longrightarrow E_1, \]
where \((E_i, h_i)\) are holomorphic Hermitian vector bundles, with curvature tensors \(R^i_{\alpha \beta}\), we have
\[ \psi_{;\alpha \beta} = \psi_{;\beta \alpha} - R^1_{\alpha \beta} \circ \psi + \psi \circ R^2_{\alpha \beta}, \]
and we will write
\[ (2.1) \quad \psi_{;\alpha \beta} = \psi_{;\beta \alpha} - [R_{\alpha \beta}, \psi] \]
for short.

3. DEFORMATIONS

Let \(E_1\) and \(E_2\) be holomorphic vector bundles, and let
\[ \phi : E_2 \longrightarrow E_1 \]
be an \(\mathcal{O}_X\)-linear homomorphism. By definition, an automorphism of the triple
\[ T = (E_1, E_2, \phi) \]
consists of a pair of automorphisms \(\psi_1\) and \(\psi_2\), of \(E_1\) and \(E_2\) respectively, such that
\[ \phi \circ \psi_2 = \psi_1 \circ \phi. \]

A holomorphic family of such triples over a complex parameter space \(S\) consists of a triple
\[ T = (\mathcal{E}_1, \mathcal{E}_2, \Phi) \]
on \(X \times S\). For any point \(s \in S\), the fiber \(T_s\) is just the restriction of \((\mathcal{E}_1, \mathcal{E}_2, \Phi)\) to \(X \times \{s\} \simeq X\). Using the notion of a holomorphic family, we can derive the notion of a deformation of an object \(T\) over a space \((S, s_0)\) with a distinguished base point \(s_0 \in S\) in the usual way, which is done by fixing an isomorphism \(T \sim T_{s_0}\). Isomorphism classes of deformations of such triples \((E_1, E_2, \phi)\) satisfy the Schlessinger condition \([Sc]\), and semi-universal deformations exist by the general theory.

Isomorphism classes of infinitesimal deformations of \((E_1, E_2, \phi)\) over the double point
\[ D = \mathbb{C}[\varepsilon]/\varepsilon^2 = (\mathbb{C} \oplus \varepsilon \mathbb{C}, 0) \]
with \(\varepsilon^2 = 0\) can be identified with the equivalence classes of extensions of the homomorphism
\[ \phi : E_2 \longrightarrow E_1 \]
by itself, i.e., the equivalence classes of diagrams of the following type:
\[ (3.1) \quad \begin{array}{cccccc}
0 & \longrightarrow & \varepsilon E_2 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & E_2 & \longrightarrow & 0 \\
& \phi & \downarrow & \Phi & \downarrow & \phi & & & \\
0 & \longrightarrow & \varepsilon E_1 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & E_1 & \longrightarrow & 0
\end{array} \]
Here \(\varepsilon E_j \hookrightarrow \mathcal{E}_j, j = 1, 2\), refers to the \(\mathcal{O}_D\)-module structure of the \(\mathcal{E}_j\) (cf. [B-R], [B-G-G]).
In order to describe infinitesimal deformations and infinitesimal automorphisms respectively of such triples \((E_1, E_2, \phi)\), we use the following complex of \(\mathcal{O}_X\)-modules:

\[
(3.2) \quad C^\bullet : 0 \longrightarrow C^0 := \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2) \xrightarrow{\Delta} C^1 := \text{Hom}_{\mathcal{O}_X}(E_2, E_1) \longrightarrow 0,
\]

where

\[
\Delta(\psi_1, \psi_2) = \psi_1 \circ \phi - \phi \circ \psi_2 =: [\psi, \phi],
\]

and \(\text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)\) is in the 0-th position. The hypercohomology of the complex \(C^\bullet\) can be computed from the short exact sequence

\[
0 \longrightarrow A^\bullet \longrightarrow C^\bullet \longrightarrow B^\bullet \longrightarrow 0,
\]

where

\[
A^\bullet : 0 \longrightarrow 0 \longrightarrow \text{Hom}_{\mathcal{O}_X}(E_2, E_1) \longrightarrow 0
\]

and

\[
B^\bullet : 0 \longrightarrow \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2) \longrightarrow 0 \longrightarrow 0.
\]

So \(A^1 = C^1\) and \(B^0 = C^0\). We have a long exact sequence of hypercohomologies

\[
0 \longrightarrow \mathbb{H}^0(C^\bullet) \longrightarrow H^0(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \xrightarrow{\Delta} H^0(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1))
\]

\[
\longrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow H^1(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \xrightarrow{\Delta} H^1(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1))
\]

\[
\longrightarrow \mathbb{H}^2(C^\bullet) \longrightarrow \ldots
\]

The hypercohomology groups \(\mathbb{H}^0(C^\bullet)\) and \(\mathbb{H}^1(C^\bullet)\) parametrize respectively the endomorphisms and the infinitesimal deformations of \((E_1, E_2, \phi)\), whereas the obstructions of infinitesimal deformations live in \(\mathbb{H}^2(C^\bullet)\).

Let

\[
T \longrightarrow S
\]

with \(T = (\mathcal{E}_1, \mathcal{E}_2, \Phi)\) be a holomorphic family of triples \(T_s, s \in S\), parametrized by \(S\). Suppose that \(h_i, i = 1, 2\), is a Hermitian metric on \(\mathcal{E}_i\), such that the restrictions of \(h_1\) and \(h_2\) to \(X \times \{s\}\) are solutions of the coupled vortex equations; the coupled vortex equations are recalled in (4.3) and (4.4).

Let \(\Omega^i\) be the curvature form of the Hermitian connection for \(h_i\) on \(\mathcal{E}_i\), \(i = 1, 2\), over \(X \times S\) with curvature tensor \(R^i\). So the contractions

\[
\Omega^i \downarrow \frac{\partial}{\partial s_k} \quad \text{coincides with} \quad R^i_{k\bar{\jmath}}dz^k.
\]

4. Stability, coupled vortex equation and Kobayashi–Hitchin correspondence

4.1. Notions. Let

\[
T = (E_1, E_2, \phi)
\]

be a triple. A sub–triple \(T'\) of \(T\) consists of coherent torsionfree subsheaves \(E'_i \subset E_i,\)

\(i = 1, 2\), such that \(\phi' := \phi|E'_2\) maps \(E'_2\) to \(E'_1\). A sub–triple is called proper if it is not equal to \(T\).
For a real number $\alpha$, the $\alpha$–degree and $\alpha$–slope of $T$ are defined as follows:

$$\deg_\alpha(T) := \deg(E_1) + \deg(E_2) + \alpha \cdot \text{rk}(E_2)$$

$$\mu_\alpha(T) := \frac{\deg_\alpha(T)}{\text{rk}E_1 + \text{rk}E_2}.$$  

The degree of a coherent analytic sheaf on $X$ is defined in terms of the Kähler metric $\omega_X$.

The $\alpha$–degree and $\alpha$–slope of a subtriple of $T$ is defined exactly as done in (4.1) and (4.2) respectively.

**Definition 4.1.** A triple $T = (E_1, E_2, \phi)$ is called $\alpha$–stable if for any nonzero proper subtriple $T' = (E'_1, E'_2, \phi')$, with $\text{rk}E'_1 + \text{rk}E'_2 < \text{rk}E_1 + \text{rk}E_2$, the inequality

$$\mu_\alpha(T') < \mu_\alpha(T)$$

holds.

If the weaker inequality $\mu_\alpha(T') \leq \mu_\alpha(T)$ holds for any non–zero proper subtriple, then $T$ is called $\alpha$–semistable.

An $\alpha$–semistable triple is called $\alpha$–polystable if it is a direct sum of $\alpha$–stable triples.

Let $h^i$ be Hermitian metrics on $E_i$ with curvature forms $\Omega^i$, where $i = 1, 2$. Assume that the $\omega_X$–volume of $X$ is normalized to $2\pi$. Denote by $\Lambda_X$ the operator dual to the exterior multiplication by $\omega_X$ of differential forms with values in vector bundles.

**Definition 4.2.** The coupled vortex equations for $((E_1, h_1), (E_2, h_2), \phi)$ read as

$$\sqrt{-1} \Lambda_X \Omega^1 + \phi \phi^* = \tau_1 \cdot \text{Id}_{E_1}$$

$$\sqrt{-1} \Lambda_X \Omega^2 - \phi \phi^* = \tau_2 \cdot \text{Id}_{E_2}$$

or equivalently

$$g^{\alpha \overline{\beta}} R^1_{\alpha \beta} + \phi \phi^* = \tau_1 \cdot \text{Id}_{E_1}$$

$$g^{\alpha \overline{\beta}} R^2_{\alpha \beta} - \phi \phi^* = \tau_2 \cdot \text{Id}_{E_2}$$

where $\tau_1$ and $\tau_2$ are some real numbers.

### 4.2. Kobayashi–Hitchin correspondence.

For any Hermitian metrics $h_1$ and $h_2$ on $E_1$ and $E_2$ respectively satisfying (4.3) and (4.4), integrating traces of the equations over $X$ yields

$$\deg E_1 + \deg E_2 = \tau_1 \text{rk}E_1 + \tau_2 \text{rk}E_2.$$  

We recall the *Kobayashi–Hitchin correspondence* for triples. It states that a triple $T = (E_1, E_2, \phi)$ is $\alpha$–polystable (see Definition 4.1) if and only if the following holds:

- $E_1$ and $E_2$ admit Hermitian metrics satisfying (4.3) and (4.4) with $\alpha = \tau_1 - \tau_2$, and
- (4.7) holds.

The Kobayashi–Hitchin correspondence is proved in [A-G, p. 182, Theorem 3.1]. We also note that in [Ga2], the Kobayashi–Hitchin correspondence was proved under the assumption that $\text{rk}E_2 = 1$ with $X$ being an arbitrary compact Kähler manifold, while in [B-G] it was proved under the assumption that $\dim X = 1$.  

4.3. **Deformation theoretic approach.** All automorphisms of a stable triple

\[ T = (E_1, E_2, \phi) \]

are of the form \( \lambda(\text{Id}_{E_1} \oplus \text{Id}_{E_2}) \), where \( \lambda \in \mathbb{C}^* \). Therefore, any automorphism of a stable triple can be extended to neighboring fibers in a holomorphic family of triples. So semi-universal deformations are universal. In this section, we prove the unique extendibility of solutions of the coupled vortex equations in a holomorphic family.

**Theorem 4.3.** Let \( T = (E_1, E_2, \phi) \) be a stable triple over a compact Kähler manifold \((X, \omega_X)\). Let \( S \) be a reduced complex space with a base point \( s_0 \in S \), and let

\[ T = (\mathcal{E}_1, \mathcal{E}_2, \Phi) \]

be a deformation of \( T \) over \((S, s_0)\). Let \( h_1 \) and \( h_2 \) be Hermitian metrics on \( E_1 \) and \( E_2 \) respectively that solve the coupled vortex equations for \( T \). Then there exists a neighborhood \( U \) of \( s_0 \) such that the solutions can be extended to the fibers \( T_s, s \in U \), in a \( C^\infty \) way.

**Proof.** We will use an approach, which is slightly different from the usual one involving the action of the complexified gauge group.

Let \((E, h)\) stand for any Hermitian holomorphic vector bundle. If \( \sigma \) and \( \tau \) are sections, then we write any other Hermitian metric \( \tilde{h} \) on \( E \) in the following form:

\[ \tilde{h}(\sigma, \tau) = h(\psi \sigma, \tau), \]

where \( \psi \in \text{End}(E) \) is a differentiable section which is self-adjoint with respect to \( h \), that is,

\[ \psi^* = \psi. \]

We rephrase \((4.8)\) in local coordinates. Let \( \{e_i\} \) be a set of local frames of \( E \) and denote by

\[ \sigma = \sum_i \sigma^i e_i, \quad \tau = \sum_i \tau^i e_i \]

sections of \( E \). Then

\[ h(\sigma, \tau) = \sum_{i,j} \sigma^i h_{ij} \tau^j, \]
\[ \psi(\sigma) = \sum_{i,j} \sigma^i \psi^k_{ij} e_k. \]

(Observe that for compositions of morphisms, the order of the corresponding matrix multiplications is reversed.) For the induced connection \( \theta = \theta_\alpha dz^\alpha \) we have the notation

\[ \theta \sigma = \sigma^i \theta_{\alpha}^k e_k dz^\alpha, \]

where

\[ \theta = \partial h \cdot h^{-1}, \quad \text{i.e.,} \quad \theta_{\alpha}^k = h_{ij\alpha} \cdot h^{jk}. \]

Now \((4.8)\) reads as

\[ \tilde{h}_{ij} = \psi^k_{ij} h_{kj}. \]

The induced connections are

\[ \tilde{\theta} = \theta + (\partial \psi - [\theta, \psi]) \cdot \psi^{-1} = \theta + (\partial \psi) \cdot \psi^{-1}, \]
where $\partial_\theta$ is the covariant exterior derivative. (Here, in this section, we need to use the matrix notation instead of the notations of endomorphisms, which accounts for the above sign.)

If $\chi$ is any differentiable endomorphism of $E$, define $\chi^*$ by

$$\chi^* = \tilde{h}^{ij} \tilde{\chi}^*_j h^{jk}.$$

The following identity for the adjoint $\tilde{\chi}^*$ of $\chi$ with respect to the Hermitian structure $\tilde{h}$ holds

$$\tilde{\chi}^* = \psi^{-1} \chi^* \psi,$$

and $\chi = \tilde{\chi}^*$, if and only if $\psi \chi$ is self–adjoint with respect to $h$.

Now the curvatures are (again in terms of the holomorphic structure on $E$)

$$\tilde{\Omega} = \partial \tilde{\theta},$$

and $\Omega = \bar{\partial} \theta$. We note that

$$(\psi \tilde{\Omega})^* = \tilde{\psi} \tilde{\Omega}.$$

In order to prove Theorem 4.3 we extend a pair of Hermitian metrics $(h_1, h_2)$, which solve the coupled vortex equations for the triple $T$, as differentiable families of Hermitian metrics $(h_{1,s}, h_{2,s})_{s \in S}$ for $\{T_s\}_{s \in S}$. Applying self–adjoint differentiable automorphisms $\psi_{1,s}$ and $\psi_{2,s}$ respectively of $E_{1,s}$ and $E_{2,s}$, we get Hermitian metrics $\tilde{h}_{1,s}$ and $\tilde{h}_{2,s}$ depending differentiably upon the parameter $s$.

For the induced curvature forms

$$\tilde{\Omega}^j = \Omega^j + \bar{\partial} (\partial_{\theta_j} (\psi_j) \cdot \psi_j^{-1}); \ j = 1, 2$$

hold for any fixed $s \in S$.

We consider the assignments

$$F_s : (\psi_{1,s}, \psi_{2,s}) \mapsto \left( \psi_{1,s} (\sqrt{-1} \Lambda X \tilde{\Omega}_{1,s}^1 + \phi_s \tilde{\phi}_{s}^* - \tau_1 \text{Id}_{E_{1,s}}), \right.$$  

$$\left. \psi_{2,s} (\sqrt{-1} \Lambda X \tilde{\Omega}_{2,s}^2 - \phi_s \tilde{\phi}_{s}^* - \tau_2 \text{Id}_{E_{2,s}}), \int_X (\text{tr} \psi_{1,s} + \text{tr} \psi_{2,s}) g \, dV \right);$$

(Here, we denote by $\tilde{\phi}$ the adjoint with respect to $(\tilde{h}_{1,s}, \tilde{h}_{2,s})$.)

The first two components of $F_s(\psi_{1,s}, \psi_{2,s})$ are self–adjoint with respect to $(h_{1,s}, h_{2,s})$. We specify the domain of the $F_s$.

For sufficiently large $k$, and some $0 < \alpha' < 1$, we have Banach manifolds

$$W_s = \{(\psi_{1,s}, \psi_{2,s}) \in W^{k+2, \alpha'}(\text{Aut}(E_{1,s})) \oplus W^{k+2, \alpha'}(\text{Aut}(E_{2,s})) \mid \psi_{1,s}^* = \psi_{1,s}, \psi_{2,s}^* = \psi_{1,s}\}$$

which give rise to a Banach manifold $W$ together with a smooth map

$$\pi : W \longrightarrow S,$$

and we have

$$V_s = \{ (\eta_{1,s}, \eta_{2,s}) \in W^{k, \alpha'}(\text{End}(E_{1,s})) \oplus W^{k, \alpha'}(\text{End}(E_{1,s})) \mid \eta_{1,s}^* = \eta_{1,s}, \eta_{1,s}^* = \eta_{2,s} \}$$
inducing a morphism $\nu : V \to S$ of Banach manifolds. (Here, we assume for simplicity that $S$ is smooth. However, in case of a reduced base space $S$ we get local submersions $\pi$ and $\nu$, and all arguments can be carried over.) The above maps $F_s$ give rise to a diagram (4.9)

\[
\begin{array}{ccc}
W & \xrightarrow{F} & V \times \mathbb{R} \\
\downarrow{\pi} & & \downarrow{\nu} \\
S. & &
\end{array}
\]

Lemma 4.4. After restricting $F$ to suitable neighborhoods of 
\[(\text{Id}_{E_1}, \text{Id}_{E_2}) \in W\]
and of $(0, 0, 0) \in V \times \mathbb{R}$ respectively, the map $F$ in (4.9) is an isomorphism of $W$ onto a Banach submanifold $F(W) \subset V \times \mathbb{R}$ of codimension one.

Proof. We set $E_j := E_{j,s}$ for $j = 1, 2$, and $W_0 := W_{s_0}$ as well as $V_0 := V_{s_0}$ and otherwise drop the index $s_0$ from now on. Let $t$ be a complex parameter and 
\[\psi(t) = (\psi_1(t), \psi_2(t)) \in W_0\]
a differentiable curve with 
\[\psi(0) = (\psi_1(0), \psi_2(0)) = (\text{Id}_{E_1}, \text{Id}_{E_2}).\]
We compute the derivative of $F_0(\psi(t))$.

We have in matrix notation 
\[\phi^* = h_1 \widetilde{\phi} h_2^{-1}, \quad \tilde{\phi}^* = \tilde{\phi} h_2^{-1} = \psi_1 \phi^* \psi_2^{-1}.\]
We return to the endomorphism notation, and get 
\[\frac{d}{dt} \phi^* \bigg|_{t=0} = \phi^* \dot{\psi}_1 - \dot{\psi}_2 \phi^*,\]
where the dot stands for the $t$–derivative at $t = 0$. Furthermore 
\[\frac{d}{dt} \sqrt{-1} \Lambda \tilde{\Omega} \bigg|_{t=0} = \frac{d}{dt} \sqrt{-1} \Lambda (\Omega^j + \overline{\partial} (\partial_j (\psi_j) \cdot \psi_j^{-1})) \bigg|_{t=0} = \partial^*_j \partial_j \psi_j \]
since $\partial_j (\text{Id}_{E_j}) = 0$. Altogether the derivative 
\[DF_0 : T_{\text{Id}} W_0 \rightarrow T_0 V_0 \oplus \mathbb{R}\]
is given by 
\[
(4.10) \quad DF_0(\chi_1, \chi_2) = \left( \partial^*_1 \partial_1 \chi_1 + \phi (\phi^* \chi_1 - \chi_2 \phi^*) , \partial^*_2 \partial_2 \chi_2 - (\phi^* \chi_1 - \chi_2 \phi^*) \phi \cdot \int_X (\text{tr} \chi_1 + \text{tr} \chi_2) g dV \right).
\]
We will see that this map is an injection onto $V_0^0 + \mathbb{R}$, where $V_0^0 \subset T_0 V_0$ denotes the sum of spaces of trace free endomorphisms.

Suppose that $(\chi_1, \chi_2)$ is in the kernel of $DF_0$. Then 
\[\|\partial_1 \chi_1\|^2 + \langle \phi^* \chi_1 - \chi_2 \phi^* , \phi^* \chi_1 \rangle = 0 \]
\[\|\partial_2 \chi_2\|^2 - \langle \phi^* \chi_1 - \chi_2 \phi^* , \chi_2 \phi^* \rangle = 0 \]
which implies that
\[ \| \partial_\theta_1 \chi_1 \|^2 + \| \partial_\theta_2 \chi_2 \|^2 + \| \phi^* \chi_1 - \chi_2 \phi^* \|^2 = 0. \]
In particular the endomorphisms $\chi_j$ are parallel, and self-adjoint, hence holomorphic. Furthermore $\chi_1 \phi = \phi \chi_2$. So the pair
\[ \chi = (\chi_1, \chi_2) \]
defines a holomorphic endomorphism of the given stable triple. Hence $\chi$ is a real multiple of the identity. Since the third component in (4.10) equals zero, we conclude that $\chi$ must be zero. By Hodge theory the rest follows. \qed

We return to the proof of Theorem 4.3. Denote by $\{0\} \times S$ the zero section of $\pi: V \rightarrow S$. We know that $F(W_0)$ intersects $\mathbb{R} \times \{s_0\} \times \{0\}$ transversally at the origin; here $\mathbb{R}$ is considered as a subset of $V_0$. By the above Lemma 4.4, the image $F(W)$ intersects $\mathbb{R} \times \{s_0\} \times \{0\}$ transversally in a differentiable section of $\nu: V \rightarrow S$, whose pull-back under $F$ is the desired solution. \qed

The deformation theoretic approach for the usual moduli space of irreducible Hermite–Einstein connections can be found in [F-S].

4.4. Moduli spaces. As a consequence of Theorem 4.3 the moduli space of $\alpha$–stable triples exists in the category of reduced, complex spaces, which are not necessary Hausdorff. It should be emphasized that the approach of [A-G] actually gives a construction of the moduli space of $\alpha$–stable triple as well as that of irreducible solutions of the coupled vortex equations; we recall that in [A-G] the dimension reduction techniques are employed. More precisely, from [A-G] it follows that a moduli space of $\alpha$–stable triples on a compact Kähler manifold $X$ is realized as an analytic subspace of a moduli space of stable vector bundles on $X \times \mathbb{P}_1$. Similarly, a moduli space of solutions of the coupled vortex equations on $X$ is realized as an analytic subspace of a moduli space irreducible solutions of the Hermite–Einstein equation on $X \times \mathbb{P}_1$.

Theorem 4.5. Given a compact Kähler manifold $X$, the moduli space of objects of the form $((E_1, h_1), (E_2, h_2), \phi)$, where $(E_1, E_2, \phi)$ is a stable triple over $X$, and $h_i, i = 1, 2$, are Hermitian structures on $E_i$ satisfying the coupled vortex equations, exists.

5. Elliptic complex

Take any triple $(E_1, E_2, \phi)$. As before, let $C^\bullet$ be the complex associated to it (see (3.2)). In order to use the theory of elliptic operators, we observe that the Dolbeaulsbol complexes provide a resolution $C^{**}$ of $C^\bullet$:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2) & \longrightarrow & \mathcal{A}^{0,0}(\text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) & \longrightarrow & \cdots \\
& & \downarrow \Delta & & \downarrow \Delta & & \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(E_2, E_1) & \longrightarrow & \mathcal{A}^{0,0}(\text{Hom}_{\mathcal{O}_X}(E_2, E_1)) & \longrightarrow & \cdots
\end{array}
\]
\[ \mathcal{A}^{0,i}(End_{\mathcal{O}_X}(E_1) \oplus End_{\mathcal{O}_X}(E_2)) \xrightarrow{\Delta} \mathcal{A}^{0,i+1}(End_{\mathcal{O}_X}(E_1) \oplus End_{\mathcal{O}_X}(E_2)) \xrightarrow{\Delta} \cdots \]

where \( C^{0,i} = \mathcal{A}^{0,i}(End_{\mathcal{O}_X}(E_1) \oplus End_{\mathcal{O}_X}(E_2)) \) and \( C^{1,i} = \mathcal{A}^{0,i}(Hom_{\mathcal{O}_X}(E_2, E_1)) \).

It follows immediately that \( \Delta \circ \partial = \partial \circ \Delta \).

Let \( \tilde{C}^\bullet \) be the complex which is constructed in the following way. Define

\[ \tilde{C}^i := C^{0,i} \oplus C^{1,i-1} \]

with the convention that \( C^{1,-1} = 0 \). The homomorphisms

\[ (\partial, (-1)^i \Delta) : C^{0,i} \rightarrow C^{0,i+1} \oplus C^{1,i} \]

and

\[ (0, \partial) : C^{1,i-1} \rightarrow C^{0,i+1} \oplus C^{1,i} \]

together give the homomorphisms \( \tilde{C}^i \rightarrow \tilde{C}^{i+1} \) that define the complex \( \tilde{C}^\bullet \).

We note that the cohomology of \( C^\bullet \) can be identified with the cohomology of the single complex \( \tilde{C}^\bullet \) associated to \( C^{\bullet\bullet} \).

Following the construction in [S-T], one can see that the global tensors \( \Omega^i \) and \( \Phi \) over \( X \times S \) already describe the infinitesimal deformations. (This is in fact true for any pair of Hermitian metrics.) In other words, we have the following lemma:

**Lemma 5.1.** Let

\[ \rho_{s_0} : T_{s_0}S \rightarrow H^1(\tilde{C}^{\bullet\bullet}) \]

be the Kodaira–Spencer mapping. Then

\[ \mu_i = \left( -\Phi_i, (R_{ij}dz^j, R_{ij}^2dz^j) \right) \]

\[ \in \mathcal{A}^{0,0}(X, Hom_{\mathcal{O}_X}(E_2, E_1)) \oplus \mathcal{A}^{0,1}(X, End_{\mathcal{O}_X}(E_1) \oplus End_{\mathcal{O}_X}(E_2)) \]

represents the class

\[ \rho_{s_0} \left( \left. \frac{\partial}{\partial s_i} \right|_{s=s_0} \right) \in H^1(\tilde{C}^{\bullet\bullet}) . \]

Given Hermitian metrics \( h^i \) on \( E_i, i = 1, 2 \), together with \( \omega_X \), the single complex associated with (5.1) is elliptic.

We consider the complex

\[ \Gamma(\tilde{C}^\bullet) : 0 \rightarrow C^{0,0}(X) \xrightarrow{d^0} C^{1,0}(X) \oplus C^{0,1}(X) \xrightarrow{d^1} C^{1,1}(X) \oplus C^{0,2}(X) \rightarrow \cdots \]
More precisely,
\[ \Gamma(\tilde{C}^\cdot) : 0 \to \mathcal{A}^{0,0}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \mathcal{A}^{0,1}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \mathcal{A}^{0,2}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \ldots \]

\[ \mathcal{A}^{0,i-1}(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1)) \oplus \mathcal{A}^{0,i}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \mathcal{A}^{0,i}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \ldots \]

\[ \mathcal{A}^{0,n-1}(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1)) \oplus \mathcal{A}^{0,n}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)) \to \mathcal{A}^{0,n}(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1)) \to 0 \]

with

\[ d^0(f) = (\Delta f, \overline{\partial}^0 f); \quad f = (f_1, f_2) \]

\[ d^1(a, b) = (\overline{\partial}^1 a - \Delta b, \overline{\partial}^1 b); \quad b = (b_1, b_2). \]

The following lemma is evident.

**Lemma 5.2.** The adjoint operators

\[ d^1 : C^1(X) \to C^0(X), \quad d^2 : C^2(X) \to C^1(X) \]

are given by

\[ d^1(a, b) = (a\phi^* + \overline{\partial}^* b_1, -\phi^* a + \overline{\partial}^* b_2) \]

\[ d^2(u, v) = (\overline{\partial}^* u, (-u\phi^* + \overline{\partial}^* v_1, \phi^* u + \overline{\partial}^* v_2)) \]

We return to the situation of Lemma 5.1.

**Proposition 5.3.** The forms \( \mu_i \) are the harmonic representatives of the Kodaira–Spencer classes \( \rho(\partial/\partial s|_{s_0}) \):

\[ d\mu_i = 0 \]

\[ d^* \mu_i = 0 \]

**Proof.** We know that \( d\mu_i = 0 \). We refer to Lemma 5.2 for \( d^* \mu_i = 0 \) and use the coupled vortex equations (4.5) and (4.6): The first component equals

\[ (d^* \mu_i)_1 = -\Phi_{i,j} \cdot \Phi^* + \overline{\partial}^* (R_{i\overline{3}}^1 d^3 z^3) = -\Phi_{i,j} \cdot \Phi^* - g_{\overline{3} \overline{a}} R_{i\overline{3} \overline{a}}^1 = -\Phi_{i,j} \Phi^* + (\Phi \Phi^*)_i = 0, \]

and the second follows in the same way. \( \square \)

### 6. Hermitian structure on the moduli space

In this section we will construct a vortex–moduli Hermitian metric on the moduli space of \( \alpha \)-stable triples. The corresponding inner product is given by a natural Hermitian metric on the base of a universal deformation. Then it will be shown that the vortex–moduli Hermitian metric is actually a Kähler metric. The elliptic complex from Section 5 plays a key role in the construction.

Take an effective holomorphic family of holomorphic triples such that each holomorphic triple in the family is equipped with Hermitian structures satisfying the coupled vortex
Let \((s_i)\) be holomorphic coordinates on \(S\) around a point \(s_0\) in the sense of Section 2.

Now, we are in a position to introduce a vortex–moduli metric on the parameter space \(S\) for a family of stable triples. The vortex–moduli metric is an inner product \(G^{VM}\) on the tangent spaces \(T_sS\) of the bases of holomorphic families, which is positive definite for effective families, and it is defined in terms of the tensors \(\mu_i\) representing the Kodaira–Spencer classes. (The superscript “VM” stands for “vortex–moduli metric”.) This is possible, also in the case where \(S\) is singular because the family of holomorphic homomorphisms and the curvature forms for the connection on vector bundles still exist on the first order infinitesimal neighborhood. The latter fact follows from the approach described above.

**Definition 6.1.** A Hermitian structure on the tangent space \(T_{s_0}S\) is given by

\[
G^{VM} \left( \frac{\partial}{\partial s^i}_{s_0}, \frac{\partial}{\partial s^j}_{s_0} \right) := G^{VM}_{ij} := \langle \mu_i, \mu_j \rangle
\]

\[
= \int_X \text{tr}(\Phi^*_i \Phi^*_j) g dV + \int_X \text{tr}(g^{R^1_{\mu \nu} R^1_{\mu \nu}}) g dV + \int_X \text{tr}(g^{R^2_{\mu \nu} R^2_{\mu \nu}}) g dV.
\]

We set

\[
\omega^{VM} = \sqrt{-1} G^{VM}_{ij}(s) ds^i \wedge ds^j
\]

and call this Hermitian structure the vortex–moduli metric.

**7. Fiber integral and Quillen metric**

We will use the following notion of a Kähler space. Let \(W\) be a polydisk together with a Kähler form \(\omega_W\), and \(S \subset W\) a closed reduced analytic subspace. Then all tangent spaces \(T_sS, s \in S\), carry an induced Hermitian metric. If \(S\) is any reduced complex space together with a family of Hermitian metrics on all tangent spaces \(T_sS, s \in S\), which is locally of the above kind, then the induced real \((1,1)\)-form \(\omega_S\) on \(S\) is called a Kähler form. Observe that Kähler forms on reduced complex spaces possess local \(\partial\bar{\partial}\)-potentials of class \(C^\infty\). Clearly the restriction of a Kähler form on a complex space to a complex analytic subspace is again a Kähler form.

We begin with a general remark about fiber integrals. For any projection of differentiable manifolds

\[
Z \times R \rightarrow R,
\]

where \(Z\) is compact of dimension \(m\), the push forward of an \((m+k)\)-form \(\chi\) of class \(C^\infty\) is a \(C^\infty\)-form of degree \(k\) given as a fiber integral

\[
\int_Z \chi := \int_{(Z \times R)/R} \chi.
\]

This process of fiber integration applied to proper smooth, holomorphic maps is type preserving in the sense that for fibers of complex dimension \(n\) the fiber integral of any \((n+k, n+\ell)\)-form of class \(C^\infty\) is a differentiable \((k, \ell)\)-form. In our case the base space \(S\) is not necessarily smooth, but the given \((n+1, n+1)\)-forms will be the local restrictions,
from a smooth ambient space, of $\sqrt{-1}\partial\bar{\partial}$-exact $C^\infty$ forms. This will yield $(1,1)$-forms with local $\sqrt{-1}\partial\bar{\partial}$-potential of class $C^\infty$ in the sense described above. (For details and generalizations see [Va].)

In place of earlier sub–indices (as in $E_i$), in the following proposition we use the super–indices in order not to confuse with the sub–indices for coordinates.

**Proposition 7.1.** Denote by $\Omega^\nu$ the curvature form of $(E^\nu, h^\nu)$, $\nu = 1, 2$. Then the following fiber integral formula for the vortex–moduli form holds

\[
    \omega^{VM} = \frac{1}{2} \sum_{\nu=1,2} \left( \int_X \text{tr}(\Omega^\nu \wedge \Omega^\nu) \wedge \frac{\omega^{n-1}_X}{(n-1)!} + \tau_\nu \int_X \text{tr}(\Omega^\nu \omega^n_X \nu^{-1}) \right) + \sqrt{-1}\partial\bar{\partial} \int_X \text{tr}(\Phi \Phi^*) \frac{\omega^n_X}{n!}
\]

**Proof.** We compute

\[
    \zeta = \frac{1}{2} \sum_{\nu=1,2} \int_X \text{tr}(\Omega^\nu \wedge \Omega^\nu) \wedge \frac{\omega^{n-1}_X}{(n-1)!} = -\frac{1}{2} \sum_{\nu=1,2} \int_X \text{tr}(\sqrt{-1}\Omega^\nu \wedge \sqrt{-1}\Omega^\nu) \wedge \frac{\omega^{n-1}_X}{(n-1)!}
\]

\[
    = \sqrt{-1} \sum_{\nu=1,2} \int_X \text{tr}(R^\nu_{\alpha \beta} \cdot R^\nu_{\alpha \beta} - R^\nu_{\alpha \beta} \cdot R^\nu_{\alpha \beta} g_{\alpha \beta} g dV d\bar{s}^i \wedge d\bar{s}^\tau)
\]

\[
    = \sqrt{-1} \left( \sum_{\nu=1,2} \int_X \text{tr}(R^\nu_{\alpha \beta} \cdot R^\nu_{\alpha \beta}) + \int_X \text{tr}(\Phi \Phi^* R_{\alpha \beta} - \Phi^* \Phi R_{\alpha \beta}) g dV \right) d\bar{s}^i \wedge d\bar{s}^\tau
\]

by (4.5) and (4.6).

Now, on $X \times \{s\}$ we have

\[
    \text{tr}(\Phi^*(-R^1_{\alpha \beta} \Phi + \Phi R^2_{\alpha \beta})) = \text{tr}(\Phi^*(\Phi + \Phi)) = \text{tr}(\Phi^* \Phi)
\]

as $\Phi$ is a holomorphic section on the total space. So

\[
    \int_X \text{tr}(\Phi \Phi^* R_{\alpha \beta} - \Phi^* \Phi R_{\alpha \beta}) g dV = -\int_X \text{tr}(\Phi \Phi^* g dV
\]

\[
    = \int_X \text{tr}(\Phi \Phi^* g dV - \frac{\partial^2}{\partial s_i \partial s_\tau} \int_X \text{tr}(\Phi \Phi^*) g dV.
\]

Furthermore

\[
    \int_X \text{tr}(\Omega^\nu \frac{\omega^n_X}{n!} = \sqrt{-1} \int_X \text{tr}(R^\nu) g dV d\bar{s}^i \wedge d\bar{s}^\tau.
\]

Combining these the proof of the proposition is complete. \qed

Next, we will express the above in terms of Chern character forms described as

\[
    \text{ch}(E, h) = \sum_{k=0}^n \frac{1}{k!} \text{tr} \left( \frac{\sqrt{-1} \Omega \wedge \cdots \wedge \sqrt{-1} \Omega}{2\pi} \right)
\]

with

\[
    \text{ch}_2(E, h) = \frac{1}{2} \left( c_1^2(E, h) - 2c_2(E, h) \right).
\]

In terms of Chern character forms and Chern forms Formula (7.1) reads
\[ (7.2) \quad \frac{1}{4\pi^2} \omega_{VM} = - \int_X \text{ch}_2(\mathcal{E}_1 \oplus \mathcal{E}_2, h^1 \oplus h^2) \wedge \frac{\omega_{X}^{n-1}}{(n-1)!} + \]
\[ + \frac{\tau_1}{2\pi} \int_X c_1(\mathcal{E}, h^1) \wedge \frac{\omega_{X}^n}{n!} + \frac{\tau_2}{2\pi} \int_X c_1(\mathcal{E}_2, h^2) \wedge \frac{\omega_{X}^n}{n!} + \]
\[ + \frac{\sqrt{-1}}{4\pi^2} \partial \bar{\partial} \int_X \text{tr}(\Phi \Phi^*) \wedge \frac{\omega_{X}^n}{n!}. \]

From now on, till the end of this section, we assume that \( X \) is a Kähler manifold whose Kähler form is the Chern form
\[ \omega_X = c_1(\mathcal{L}, h_{\mathcal{L}}) \]
of a positive Hermitian line bundle \((\mathcal{L}, h_{\mathcal{L}})\), in particular, \( X \) is a complex projective manifold.

Given a proper, smooth holomorphic map
\[ f : \mathcal{X} \rightarrow S \]
and a locally free sheaf \( \mathcal{F} \) on \( \mathcal{X} \), the determinant line bundle of \( \mathcal{F} \) on \( S \) is by definition \( \det Rf_*\mathcal{F} \) [K-M, B-G-S].

The generalized Riemann–Roch theorem by Bismut, Gillet and Soulé [B-G-S] applies to Hermitian vector bundles \((\mathcal{F}, h)\) on \( \mathcal{X} \). It states that the determinant line bundle of \( \mathcal{F} \) on \( S \) carries a Quillen metric, whose Chern form equals the fiber integral
\[ \int_{\mathcal{X}/S} \text{ch}(\mathcal{F}, h) \text{td}(\mathcal{X}/S, \omega_X), \]
where \( \text{ch}(\mathcal{F}, h) \) and \( \text{td}(\mathcal{X}/S, \omega_X) \) denote respectively the Chern character form for \((\mathcal{F}, h)\) and the Todd character form for the relative tangent bundle; see [B-G-S, Theorem 0.1], and also [Z-T] for \( \dim \mathcal{X} = 1 \).

Let \( \mathcal{E} \) stand for one of the Hermitian vector bundles \( \mathcal{E}_1 \) or \( \mathcal{E}_2 \). Let \( h \) denote the Hermitian metric on \( \mathcal{E} \). We first mention
\[ (7.3) \quad \text{ch}(\text{End}(\mathcal{E})) = r^2 + 2r \text{ch}_2(\mathcal{E}) - c_1^2(\mathcal{E}) + \ldots \]
where \( r \) is the rank of \( \mathcal{E} \), so that for the virtual bundle \( \text{End}(\mathcal{E}) - \mathcal{O}^{r^2} \) the identity
\[ \text{ch}(\text{End}(\mathcal{E}) - \mathcal{O}^{r^2}) = 2r \text{ch}_2(\mathcal{E})^2 - c_1^2(\mathcal{E}) + \ldots \]
holds.

Now
\[ \text{ch} \left( (\text{End}(\mathcal{E}), h) \otimes ((\mathcal{L}, h_{\mathcal{L}}) - (\mathcal{L}^{-1}, h_{\mathcal{L}}^{-1}))^{\otimes(n-1)} \right) \]
\[ = \text{ch}_2(\text{End}(\mathcal{E}), h) \cdot 2^{n-1} \omega_X^{n-1} + \ldots \]
\[ = \left( 2r \left( \frac{1}{2} \text{tr} \left( \frac{\sqrt{-1}}{2\pi} \Omega \wedge \frac{\sqrt{-1}}{2\pi} \Omega \right) \right) - \left( \text{tr} \frac{\sqrt{-1}}{2\pi} \Omega \right) \right)^2 2^{n-1} \omega_X^{n-1} + \ldots \]
\[ = 2^{n-1} \left( r \cdot \text{tr} \left( \frac{\sqrt{-1}}{2\pi} \Omega \wedge \frac{\sqrt{-1}}{2\pi} \Omega \right) - \left( \text{tr} \frac{\sqrt{-1}}{2\pi} \Omega \right) \right)^2 \omega_X^{n-1} + \ldots \]
The highest exterior power $\Lambda^r \mathcal{E}$ carries the induced Hermitian metric $\hat{h}$, for which the following identity holds:

$$
\text{ch} \left( \left( \Lambda^r \mathcal{E}, \hat{h} \right) - \left( \Lambda^r \mathcal{E}, h_{\mathcal{E}} \right) \right) = 2^{n+1} c_1^2(\mathcal{E}, h) \cdot (L - L^{-1})^{n-1} + \ldots
$$

Hence we have the following theorem:

**Theorem 7.2.** The vortex–moduli metric defined in (6.2) has the following expression:

$$
\frac{1}{4\pi^2} \omega_{VM} = \sum_{\nu=1,2} \left( \frac{1}{2^{n+r_{\nu}}(n-1)!} \int \text{ch} \left( \text{End}(\mathcal{E}_\nu) \otimes (L - L^{-1})^{\otimes (n-1)} \right) 
- \frac{1}{2^{n+2+r_{\nu}}(n-1)!} \int \text{ch} \left( (\Lambda^r \mathcal{E}_\nu - (\Lambda^r \mathcal{E}_\nu)^{-1})^{\otimes 2} \otimes (L - L^{-1})^{\otimes (n-1)} \right) 
+ \frac{\lambda}{2\pi n!} \int \text{ch} \left( (\Lambda^r \mathcal{E}_\nu - (\Lambda^r \mathcal{E}_\nu)^{-1}) \otimes (L - L^{-1})^{\otimes n} \right) \right) 
+ \frac{1}{4\pi^2} \sqrt{-1} \partial \bar{\partial} \int \text{tr}(\Phi \wedge \Phi^*) \wedge \frac{\omega_X^n}{n!}.
$$

Using Theorem 7.2, we will express the vortex–moduli Kähler form as the curvature form of a holomorphic Hermitian line bundle.

Let

$$q : X \times S \to S$$

be the canonical projection, where $S$ stands for the base space of a universal deformation of a stable triple with solution of the coupled vortex equations. Since our construction is functorial, the construction descends to the moduli space $\mathcal{M}$ (after taking suitable powers of the line bundles on the base).

We introduce the following determinant line bundles $\delta_{j\nu}$ where $j = 1, 2, 3$ and $\nu = 1, 2$, equipped with Quillen metrics $h^Q_{j\nu}$:

$$
\delta_{1\nu} = \det Rq_\ast \left( \text{End}(\mathcal{E}_\nu) \otimes (L - L^{-1})^{\otimes (n-1)} \right),
\delta_{2\nu} = \det Rq_\ast \left( (\Lambda^r \mathcal{E}_\nu - (\Lambda^r \mathcal{E}_\nu)^{-1})^{\otimes 2} \otimes (L - L^{-1})^{\otimes (n-1)} \right),
\delta_{3\nu} = \det Rq_\ast \left( (\Lambda^r \mathcal{E}_\nu - (\Lambda^r \mathcal{E}_\nu)^{-1}) \otimes (L - L^{-1})^{\otimes n} \right).
$$

Setting

$$
\chi = \int \text{tr}(\Phi \wedge \Phi^*) \wedge \frac{\omega_X^n}{n!}
$$

we equip the trivial bundle $\mathcal{O}_{\mathcal{M}H}$ with the Hermitian metric $e^\chi$.

Combining Theorem 7.2 and [B-G-S, Theorem 0.1] we have the following theorem:
Theorem 7.3. The vortex–moduli Kähler form is a linear combination of the \((1,1)\)-forms \(c_1(\delta_{j\nu}, h_{j\nu}^Q), \ j = 1, 2, 3, \ \nu = 1, 2, \) and \(c_1(\mathcal{O}_{\mathcal{M}_H}e^x)\). For rational \(\tau^\nu\), a multiple of the vortex–moduli form is equal to the Chern form of an Hermitian line bundle.

We note that this vortex–moduli Kähler metric coincides with the one constructed in [A-G] and [Ga2], where the moduli spaces of triples and pairs respectively have been constructed as Kähler quotients.

8. Curvature of the vortex–moduli metric

In this section \(X\) will be a compact Kähler manifold.

We begin by establishing a collection of identities for the harmonic Kodaira–Spencer tensors
\[
\mu_i \in \mathcal{A}^{0,0}(X, \text{Hom}_{\mathcal{O}_X}(E_2, E_1)) \oplus \mathcal{A}^{0,1}(X, \text{End}_{\mathcal{O}_X}(E_1) \oplus \text{End}_{\mathcal{O}_X}(E_2)).
\]
In particular, we need to understand covariant derivatives with respect to the base directions.

The symmetry
\[
(\mathcal{F}_i)_{\cdot \cdot ; k} = \mu_{i;k}
\]
follows immediately from the definition.

We set \(R_{i\overline{j}} = (R^1_{i\overline{j}}, R^2_{i\overline{j}})\) et cetera. We compute the components of \(d\mu_{i;k}\): Because of
\[
\Phi_{i;k\overline{j}} - R^1_{i\overline{j}}\Phi_i + \Phi R^2_{i\overline{j}} = (\Phi_{i\overline{j}} - R^1_{i\overline{j}} + R^2_{i\overline{j}})k - R^1_{k\overline{j}}\Phi_i + \Phi R^2_{k\overline{j}},
\]
we have
\[
(\mu_{i;k})_1 = (\Phi_{i;k\overline{j}} - R^1_{i\overline{j}}\Phi_i + \Phi R^2_{i\overline{j}})d\overline{z} = (R^1_{i\overline{j}}\Phi_i - \Phi_i R^2_{i\overline{j}} + R^1_{k\overline{j}}\Phi_i - \Phi_i R^2_{k\overline{j}})d\overline{z}.
\]
Furthermore, from the identity
\[
R^1_{i\overline{j};\cdot \cdot ; k} = R^1_{i\overline{j},k\overline{\ell}} - [R^1_{k\overline{\ell}}, R^1_{i\overline{j}]},
\]
we have
\[
(\mu_{i;k})_2 = (\Phi_{i;k\overline{j}} - R^1_{i\overline{j}}\Phi_i + \Phi R^2_{i\overline{j}})d\overline{z} = (R^1_{i\overline{j}}\Phi_i - \Phi_i R^2_{i\overline{j}} + R^1_{k\overline{j}}\Phi_i - \Phi_i R^2_{k\overline{j}})d\overline{z}.
\]
The last term equals
\[
(\overline{R}^1_{\overline{j}\overline{k}}d\overline{z}, \overline{R}^1_{\overline{k}\overline{j}}d\overline{z}),
\]
(involving a symmetric product of one–forms with values in an endomorphism bundle).

Now we can introduce an exterior product on \(\overline{C}^*\), in particular, we define a symmetric exterior product \(\overline{C}^1 \times \overline{C}^1 \rightarrow \overline{C}^2:\)
\[
[\mu_i \cdot \cdot \mu_k] := \left( (-\Phi_{i;k} R^1_{k\overline{j}} + R^1_{k\overline{j}}\Phi_{i;k} - \Phi_{i;k} R^2_{k\overline{j}}, [R^1_{k\overline{j}}dz^\beta, R^1_{k\overline{j}}d\overline{z}]), [R^2_{k\overline{j}}dz^\beta, R^2_{k\overline{j}}d\overline{z}] \right),
\]
where \(\mu_i = (-\Phi_{i;i}, (R^1_{i\overline{j}}d\overline{z})), \mu_k = (-\Phi_{k;k}, (R^1_{k\overline{j}}dz^\beta, R^2_{k\overline{j}}d\overline{z})).\)
Next we compute
\[ d^* (\mu_{i;k}) = (-\Phi_{;ik}\Phi^* - R_{\gamma\eta}^1 \Phi^* g^{\gamma\eta}, \Phi^* \Phi_{;ik} - R_{\gamma\eta}^2 \Phi g^{\gamma\eta}). \]

Because of the coupled vortex equations (4.5) and (4.6) we get
\[ d^* (\mu_{i;k}) = 0. \]

We note that
\[ \mu_{i;\gamma} = (-\Phi_{;i\gamma}, (R_{i\gamma}^1 dz^\gamma, R_{i\gamma}^2 dz^\gamma)). \]

Because of
\[ \Phi_{;i\gamma} = \Phi_{;i\gamma} - R_{i\gamma}^1 \Phi - \Phi R_{i\gamma}^2 = -\Delta R_{i\gamma}, \]

we have
\[ \mu_{i;\gamma} = (\Delta R_{i\gamma}, \partial R_{i\gamma}), \]

which means that
\[ \mu_{i;\gamma} = dR_{i\gamma}, \]

where the tensor \( R_{i\gamma} \) is considered as a section of \( \widetilde{C}^0 \).

We can use Hodge theory on the complex \( \widetilde{C}^\bullet \). So
\[ d^* (\mu_{i;\gamma}) = d^* dR_{i\gamma} = \square R_{i\gamma}. \]

For any section \( f = (f_1, f_2) \) of \( \widetilde{C}^0 \), we compute
\[ d^* df = d^* (\Delta f, \partial f) = (((\partial^2 f)\Phi^* + \Phi \partial^2 f_1, -\Phi^* \Delta f - \Phi \partial^2 f_2) = \]
\[ = (f_1 \Phi \Phi^* - \Phi f_2 \Phi^* - g^{\gamma\alpha} f_1 \Phi^* \Phi f_2 - g^{\gamma\alpha} f_2 \Phi^* \Phi f_1 - \Phi^* \Phi f_2 - g^{\gamma\alpha} f_2 \Phi^* \Phi f_1). \]

We apply the formula for \( f = R_{i\gamma} \). It involves (again by (4.5) and (4.6))
\[ g^{\gamma\alpha} R_{i\gamma}^1 = g^{\gamma\alpha} (R_{i\gamma}^1 + [R_{i\gamma}^1, R_{i\gamma}^1]) = -(\Phi^*)_{;i} + g^{\gamma\alpha} [R_{i\gamma}^1, R_{i\gamma}^1], \]
\[ = -\Phi_{;i} \Phi^* - (\Phi_{;\gamma} - R_{i\gamma}^1 \Phi - \Phi R_{i\gamma}^2) \Phi^* + g^{\gamma\alpha} [R_{i\gamma}^1, R_{i\gamma}^1] \]

so that
\[ (d^* dR_{i\gamma})_1 = \Phi_{;i} \Phi^*_{;\gamma} - g^{\gamma\alpha} [R_{i\gamma}^1, R_{i\gamma}^1]. \]

In a similar way
\[ (d^* dR_{i\gamma})_2 = -\Phi^*_{;i} \Phi_{;\gamma} - g^{\gamma\alpha} [R_{i\gamma}^2, R_{i\gamma}^2]. \]

There is a natural (pointwise) inner product
\[ \widetilde{C}^1 \times \widetilde{C}^1 \rightarrow \widetilde{C}^0. \]

defined for sections \( (a, b) = (a_1 dz^\gamma, b_1 dz^\gamma) \) and \( (a', b') = (a'_1, b'_1) \) of
\[ \mathcal{A}^{0,0}(\text{Hom}(E_2, E_1)) \oplus \mathcal{A}^{0,1}(\text{End}(E_1) \oplus \text{End}(E_2)) \]

by
\[ (a, b) \cdot (a', b') := \left( a a'^* + g^{\gamma\alpha} [b_{1\gamma}, b'_1], -a'^* a + g^{\gamma\alpha} [b_{2\gamma}, b_{2\alpha}]. \right. \]
Then the equality
\[(8.7)\]
\[d^* dR = \mu_i \cdot \mu_j^*\]
holds.

Now, we are in a position to compute the curvature tensor of the vortex–moduli metric. We refer to Definition 6.1 for the metric tensor \(G_{VM}^{ij}(s)\).

First, we compute first partial derivatives of the metric tensor. We claim
\[(8.8)\]
\[G_{VM}^{ij}(s) = \int_{\mathcal{X} \times \{s\}} \text{tr} (\mu_i \cdot \mu_j^*) g dV\]

Proof. We have
\[\int \text{tr} (\mu_i \cdot \mu_j^*) g dV = \langle \mu_i, dR_{ij} \rangle = \langle d^* \mu_i, R_{ij} \rangle = 0\]
because of (5.7) and (8.5).

At a given point \(s_0 \in S\) we introduce holomorphic normal coordinates of the second kind, which means that the Kähler form coincides with the constant one in terms of the coordinate chart, up to order two at \(s_0\), or equivalently, the partial derivatives of the local expression of the Hermitian metric vanish at \(s_0\). From (8.8) it follows that this condition is equivalent to the condition that the harmonic projections of all \(\mu_i; k\) vanish (at \(s_0\)):
\[(8.9)\]
\[H(\mu_i; k(s_0)) = 0.\]

We denote by \(G\) the (abstract) Green’s operator. Then (8.9) and (8.4) imply that, for \(s = s_0\),
\[\mu_i; k = G d^* d\mu_i; k = d^* G d\mu_i; k.\]
Together with (8.3) we get
\[(8.10)\]
\[\mu_i; k(s_0) = -d^* G[\mu_i \wedge \mu_k].\]

Now (in terms of normal coordinates)
\[R_{ij}^{VM} = G_{ij}^{VM} = \int \text{tr} (\mu_i \cdot \mu_j^*) g dV + \int \text{tr} (\mu_i \cdot \mu_j^*) g dV.\]
According to (8.10) the second integral equals
\[\int \text{tr} (G([\mu_i \wedge \mu_k]) \cdot [\mu_j^* \wedge \mu_j^*]) g dV.\]
We compute the first integral: It equals
\[\int \text{tr} (\Phi_{i j k} \cdot \Phi_j^*) g dV + \sum_{\nu=1,2} \int \text{tr}(R_{\nu i j k}^\nu R_{\nu j i}^\nu) g dV.\]
Therefore, we have
\[I = I_0 + I_1 + I_2,\]
where
\[ I := \int \text{tr}(\mu_{i,j,k}\cdot \mu_{j}^*) \, g \, dV, \quad I_0 := \int \text{tr}(\Phi_{i,j,k}\cdot \Phi_{j}^*) \, g \, dV \]
and
\[ I_i := \sum_{\nu=1,2} \int \text{tr}(R_{i,j,k}^{\nu} R_{\nu,j}^{\nu}) \, g \, dV, \]
where \( i = 1, 2 \).

Now with convention (2.1)
\[ \Phi_{i,j,k} = (\Phi_{i} - [R_{i}, \Phi_i])_{i,j} - [R_{j}, \Phi_i] \]
so that
\[ I_0 = -\int \text{tr} \left( (R_{i} \cdot [\Phi_i] + [R_{j}, \Phi_i] + [R_{k}, \Phi_i]) \Phi_i^* \right) \, g \, dV. \]
Next,
\[ R_{i,j,k}^\nu = R_{i,j,k}^{\nu} + [R_{i}^{\nu}, R_{j}^{\nu}] + [R_{j}^{\nu}, R_{k}^{\nu}] \quad \nu = 1, 2 \]
We compute the contributions of \( R_{i,j,k}^{\nu} \) to the integrals \( I_\nu \); \( \nu = 1, 2 \) and get
\[ -\sum_{\nu=1,2} \int \text{tr} \left( g^\alpha R_{i,j,k}^{\nu} g^* \right) \, g \, dV = \int \text{tr} (\Phi_i^* R_{i,j,k}^{1} - \Phi_i^* R_{i,j,k}^{2}) \, g \, dV. \]
These cancel out together with corresponding terms of \( I_0 \). Hence
\[ I = -\int \text{tr} \left( R_{i}^{1} (-\Phi_{i} \Phi_{j} + g^* [R_{i}^{1}, R_{i}^{1}]) \right) + R_{i}^{2} (\Phi_{j}^* \Phi_{i}) + g^* [R_{i}^{2}, R_{i}^{1}] \, g \, dV \]
\[ -\int \text{tr} \left( R_{i}^{1} (-\Phi_{i} \Phi_{j} + g^* [R_{i}^{1}, R_{i}^{1}]) \right) + R_{i}^{2} (\Phi_{j}^* \Phi_{i}) + g^* [R_{i}^{2}, R_{i}^{1}] \, g \, dV. \]
By (8.7) these terms read
\[ I = \int \text{tr} (\Box R_{i} \cdot R_{i}) \, g \, dV + \int \text{tr} (\Box R_{i} \cdot R_{i}) \, g \, dV \]
where again \( R_{i} = (R_{i}^{1}, R_{i}^{2}) \). Since our aim is an expression in terms of the harmonic Kodaira–Spencer tensors, again we use the Green’s operator (which is here the inverse \( \Box^{-1} \) of the Laplacian, restricted to the space of differentiable endomorphisms such that the mean trace vanishes). So
\[ I = -\int \text{tr} \left( (\mu_{i} \cdot \mu_{j}) G(\mu_{k} \cdot \mu_{j}^*) \right) \, g \, dV - \int \text{tr} \left( (\mu_{k} \cdot \mu_{j}^*) G(\mu_{i} \cdot \mu_{j}^*) \right) \, g \, dV. \]
We observe that the result of our curvature computation is independent of the choice of normal coordinates.

**Theorem 8.1.** The curvature tensor of the vortex–moduli metric on the moduli space of solutions of the coupled vortex equations equals

\[ R^{VM}_{i,j,k}(s) = -\int_{X \times \{ s \}} \text{tr} \left( [\mu_{i} \cup \mu_{k}] \cdot G[\mu_{j}^* \cup \mu_{j}^*] \right) \, g \, dV \]

\[ +\int_{X \times \{ s \}} \text{tr} \left( (\mu_{i} \cdot \mu_{j}^*) G(\mu_{k} \cdot \mu_{j}^*) \right) \, g \, dV + \int_{X \times \{ s \}} \text{tr} \left( (\mu_{k} \cdot \mu_{j}^*) G(\mu_{i} \cdot \mu_{j}^*) \right) \, g \, dV \]

(8.11)
If \( \dim_{\mathbb{C}} X = 1 \), then only the second term in the expression of \( R_{VM}^{\mathcal{M}}(s) \) given in Theorem 8.1 is present.

Therefore, Theorem 8.1 has the following corollary:

**Corollary 8.2.** The vortex–moduli metric on any moduli space of stable triples over a compact Riemann surface has semi–positive holomorphic bisectional curvature.

Finally, we note that, if the results \([S-T]\) can be generalized to stable pairs, then the methods of \([A-G]\) and \([Ga2]\) would give an alternative approach to the computation of the curvature of the vortex–moduli metric.

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