Novel Methods to acceleration Simpson’s 3/8 Rule

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Abstract. The main goal of this report is to find new accelerations of Newton-Coates methods. Especially the Simpson 3/8, Boole's and Weddle's rules and use them to find numerical results for continuous integrations. The results are faster and more accurately. The error rate is less than previous methods. The derivation of new accelerated laws is trigonometric, hyperbolic and exponential for each of the Simpson 3/8, Boole's and Weddle's rules.

Keywords: Numerical integration, Simpson’s 3/8 Rule.

1. Introduction

Numerical integration is one of the topics of numerical analysis that involves finding approximate values of the definite integral. Numerical integration is also used to solve many physical problems such as finding the velocity, acceleration, volumes and areas. Numerical integration is needed in spite of the appearances of the error rate. There are many studies and developed issues of numerical integration. In 1967, Fox L. studied the way of Romberg which is the class of individual integrals which takes into account the conditions of the correction obtained and uses them together with the extrapolation of three quadratic formulas [2]. In 1970 the same scientist clarified some concepts with a study on the definite integral of a singular integral. He used the power of h by using the newton-cot’s rules [3]. In 1972 Shanks JA, using Romberg's tables of single integration, demonstrated accelerated methods and he proved Fox's work with quadratic correction equations based on the foundations of the laws of error and produced the same result in shorter and easier ways [7]. In 2017, AL-Karamy N. Scholar N. studied a new acceleration equation to improve the results of numerical integration based on Romberg's integrated formula and obtained excellent results and better than the previous methods [1]. In 2018, a scholar AL-Sharify studied the new rule of induction to increase the accuracy of the results of numerical integrals, and in the same year, Ali Hassan and Asmahan Abed presented a new acceleration method for methods of accelerating the base of the trapezoid and the midpoint to
improve the values of integrals numerically, reduce the error rate, and reach better results and they called it Al-Tememe acceleration methods of the first type [4].

In 2019, Ali Hassan and Shatha H. Theyab introduced a new accelerated method based on Simpson 1/3 rule, which was helpful in reducing error rate, and they called it Al-Tememe Acceleration Methods of the second type [5].

In our research paper, we will use the series of thoughts of Ali Hassan and Shatha H. Diab, which was used in creating Al-Tememe Acceleration, and we seek to apply it to the Simpson 3/8 rule to improve the results of this method and reach the lowest error rate in shorter methods of continuous integration.

2. Basic concepts and notations

The basic concepts of Simpson’s 3/8 rule as follows

Let's assume that integration $I$ is:

$$ I = \int_{x_0}^{x_3} f(x)dx $$

(2.1)

Such that $f(x)$ is a continuous integral located above the x-axis in the interval $[x_0, x_3]$.

In general terms, the Newton-Cots of integration (2.1) can be written as follows

$$ I = \int_{x_0}^{x_3} f(x)dx = S^*(h) + E_s^*(h) + R_s^* $$

(2.2)

Where $S^*(h)$ is lagranjian approximation of the value of integral approximately. $E_s^*(h)$ is a sequence of correction term that can be added to $S^*(h)$, and $R_s^*$ is the remainder. the Simpson’s 3/8 rule value $S^*(h)$ that is referred to by $S^*(h)$ is:

$$ S^*(h) = \frac{3h}{8} [f(a) + 3f(a + h) + 3f(a + 2h) + 2f(a + 3h) + 3f(a + 4h) + \cdots + 2f(n - 3h) + 3f(n - 2h) + 3f(n - h) + f(n)] $$

Where $h = \frac{x_3-x_0}{3n}$. The general form for $E_s^*(h)$ is as follows

$$ E_s^*(h) = \frac{3(x_3-x_0)}{80} h^4 f^iv(\bar{x}) $$

(2.3)

Notice that the error form in Simpson's 3/8 rule is almost equal to the error form in Simpson's 1/3 rule, both of them are using of the power of four.

Depending on Fox [1] when the integral is continuous and its derivatives continuous at each point of integration over the interval $[x_0, x_3]$ the error form of Simpson 3/8 can be written as follows:

$$ E = I - S^*(h) \equiv A_1 h^4 + A_2 h^6 + A_3 h^8 + \cdots $$

(2.3)

Where as $A_1, A_2, A_3, \ldots$ are constants whose value does not depend on $h$, but it depends on the values of their derivatives at the end of the integration period.
3. Accelerating the Trigonometric Functions of the Simpson’s 3/8 rule.

In this section we will derive a new acceleration to the Simpson’s 3/8 rule. From eq. (2.3) we can write

\[ E \equiv h^3(A_1h + A_2h^3 + A_3h^5 + \cdots) \]  \hspace{1cm} (3.1)

Since

\[ \sin(h) = h - \frac{h^3}{6} + \frac{h^5}{120} - \cdots \] \hspace{1cm} (3.2)

[8]

Let \( S'(h) \) is the approximate value of Simpson’s 3/8 rule.

From eq. (2.3), (3.1) and (3.2) can be written

\[ I - S'(h) \equiv h^3 \sin(h) \] \hspace{1cm} (3.3)

If we assume that we calculate two values for I numerically based on Simpson’s 3/8 rule as:

\[ S'(h_1) \] when \( h = h_1 \), \( S'(h_2) \) when \( h = h_2 \) so,

\[ I - S'(h_1) \equiv h_1^3 \sin(h_1) \] \hspace{1cm} (3.4)

\[ I - S'(h_2) \equiv h_2^3 \sin(h_2) \] \hspace{1cm} (3.5)

From the equations (3.4) and (3.5), we get:

\[ A^*_{\sin} \equiv \frac{(h_1^3 \sin h_1)S'(h_2) - (h_2^3 \sin h_2)S'(h_1)}{h_1^3 \sin h_1 - h_2^3 \sin h_2} \quad ; \sin h_1 \neq \sin h_2 \] \hspace{1cm} (3.6)

The formula (3.6) is called trigonometric acceleration sine for Simpson’s 3/8.

Similarly, we can write the equation (3.1) as follows:

\[ E \equiv h^4(A_1 + A_2h^2 + A_3h^4 + \cdots) \equiv h^4 \cos(h) \]

since

\[ \cos(h) = 1 - \frac{h^2}{2} + \frac{h^4}{24} - \frac{h^6}{72} + \cdots \] \hspace{1cm} [8]

In the same way as the previous acceleration we get;

\[ I - S'(h_1) \equiv h_1^4 \cos(h_1) \] \hspace{1cm} (3.7)

\[ I - S'(h_2) \equiv h_2^4 \cos(h_2) \] \hspace{1cm} (3.8)

From the equations (3.7) and (3.8) we get:

\[ A^*_{\cos} \equiv \frac{(h_1^4 \cosh h_1)S'(h_2) - (h_2^4 \cosh h_2)S'(h_1)}{h_1^4 \cosh h_1 - h_2^4 \cosh h_2} \quad ; \cosh h_1 \neq \cosh h_2 \] \hspace{1cm} (3.9)

The formula (3.9) is called trigonometric acceleration cosine for Simpson’s 3/8 rule.

In the same way, the third acceleration law, which is tan, can be found as follows.

Equation (3.3) can be written in the following formulas:
\[ I - S^*(h) \equiv h^3 \tan(h), \quad \text{since} \tan(h) = h + \frac{h^3}{3} + \frac{2h^5}{15} + \cdots \quad [8] \]

With the same steps;

\[ I - S^*(h_1) \equiv h_1^3 \tan(h_1) \quad (3.10) \]

\[ I - S^*(h_2) \equiv h_2^3 \tan(h_2) \quad (3.11) \]

From the equations (3.10) and (3.11) we get:

\[
A^*_{\tan} = \frac{(h_1^3 \tanh_1) S^*(h_2) - (h_2^3 \tanh_2) S^*(h_1)}{h_1^3 \tanh_1 - h_2^3 \tanh_2} \quad ; \tanh_1 \neq \tanh_2 \quad (3.12)
\]

4. Examples

Below, there are some continuous integrations in the integration period using these accelerated methods and the results will be improved for the Simpson’s 3/8 rule.

1. \( \int_{0.4}^{1} \frac{1}{x} \, dx \) Its analytical value 0.28768207245178
2. \( \int_{0.2}^{1} e^{(1)} \, dx \) its analytical value 1.50170674769234
3. \( \int_{0}^{0.5} e^{x^2} \, dx \) its analytical value 0.54498710418362

These results (1,2,3) were found by Matlab which were approximating 14 decimal places, also, these results will compare with the new accelerations and Simpson’s 3/8 rule in the next tables.
| N  | Simpson3/8  | true error | A⁺sin | true error | A⁺cos | true error | A⁺tan | true error |
|----|-------------|------------|-------|------------|-------|------------|-------|------------|
| 3  | 0.28768939  | 393939 E-07 | | | | | | |
| 6  | 0.28768255  | 4.800 E-07  | 0.28768209 | 468606 E-08 | 0.28768209 | 125291 E-08 | 0.28768209 | 977323 E-08 |
| 9  | 0.28768216  | 9.573 E-08  | 0.28768207 | 350433 E-09 | 0.28768207 | 335209 E-09 | 0.28768207 | 373200 E-09 |
| 12 | 0.28768210  | 3.039 E-08  | 0.28768207 | 259193 E-10 | 0.28768207 | 257197 E-10 | 0.28768207 | 262181 E-10 |
| 12 | 0.28768208  | 1.247 E-08  | 0.28768207 | 248283 E-11 | 0.28768207 | 247844 E-11 | 0.28768207 | 248942 E-11 |
| 18 | 0.28768207  | 6.018 E-09  | 0.28768207 | 246110 E-12 | 0.28768207 | 245979 E-12 | 0.28768207 | 246307 E-12 |
| 21 | 0.28768207  | 3.250 E-09  | 0.28768207 | 245520 E-12 | 0.28768207 | 245472 E-12 | 0.28768207 | 245592 E-12 |
| 24 | 0.28768207  | 1.906 E-09  | 0.28768207 | 245323 E-12 | 0.28768207 | 245303 E-12 | 0.28768207 | 245354 E-12 |
| 27 | 0.28768207  | 1.190 E-09  | 0.28768207 | 245247 E-13 | 0.28768207 | 245237 E-13 | 0.28768207 | 245261 E-13 |
| 30 | 0.28768207  | 7.809 E-10  | 0.28768207 | 245213 E-13 | 0.28768207 | 245208 E-13 | 0.28768207 | 245221 E-13 |

Table (1) Integration Calculation \( \int_{3}^{1} \frac{1}{x} \, dx = 0.28768207245178 \), Numerically using The Simpson 3/8 rule with the trigonometric acceleration.
Table (2) Integration Calculation $\int_{\frac{1}{3}}^{1} e^{\left(\frac{1}{x}\right)} \, dx = 1.50170674769234$, Numerically using The Simpson 3/8 rule with the Accelerating Triangular.

| N  | Simpson3/8  | true error | $A_{\text{cos}}^*$ | true error | $A_{\text{sin}}^*$ | true error | $A_{\text{tan}}^*$ | true error |
|----|-------------|------------|---------------------|------------|---------------------|------------|---------------------|------------|
| 3  | 1.5017142  | 7.4596e-06 | 1.5017073753663     | 6.2767e-06 | 1.50170758179587    | 8.3410e-06 | 1.5017078764019    | 1.13e-06  |
| 6  | 1.5017082  | 1.5142e-06 | 1.50170678415464    | 3.6464e-06 | 1.50170679369806    | 4.6008e-06 | 1.5017068078374    | 6.01e-08  |
| 9  | 1.5017072  | 4.8390e-07 | 1.50170675282404    | 5.1340e-09 | 1.50170675409009    | 6.4000e-09 | 1.50170675597848    | 8.29e-09  |
| 1  | 1.5017069  | 1.9914e-07 | 1.50170674885547    | 1.1654e-09 | 1.50170674913562    | 1.4456e-09 | 1.50170674955448    | 1.86e-09  |
| 2  | 1.5017068  | 2e-08      | 1.5017067480454     | 3.554e-10  | 1.50170674812944    | 4.3944e-10 | 1.5017067482552     | 5.65e-10  |
| 1  | 1.5017067  | 5.2055e-08 | 1.50170674782281    | 1.3281e-10 | 1.50170674785365    | 1.6365e-10 | 1.50170674789983    | 2.1e-10   |
| 2  | 1.5017067  | 3.054e-08  | 1.50170674477476    | 5.7860e-11 | 1.50170674477609    | 7.0920e-11 | 1.50170674477805    | 9.05e-11  |
| 7  | 1.5017067  | 1.9084e-08 | 1.5017067477186     | 2.86e-11   | 1.50170674772477    | 3.4770e-11 | 1.50170674773401    | 4.4e-11   |
| 3  | 1.5017067  | 1.2528e-08 | 1.50170674477058    | 1.584e-11  | 1.50170674477090    | 1.9010e-11 | 1.50170674477135    | 2.38e-11  |
Table (3) Integration Calculation \( \int_{0}^{0.5} e^{x^2} \, dx = 0.54498710418362 \), Numerically using The Simpson 3/8 rule with the Accelerating Triangular.

| N  | Simpson/3/8 | true error | \( A^x_{\sin} \) | true error | \( A^x_{\cos} \) | true error | \( A^x_{\tan} \) | true error |
|----|-------------|------------|------------------|------------|------------------|------------|------------------|------------|
| 3  | 0.54506775  | 970166     | 0.54498738       | 2.759      | 0.54498734       | 2.381      | 0.54498743       | 3.319      |
| 6  | 0.54499242  | 129159     | 0.54498711       | 1.283      | 0.54498711       | 1.115      | 0.54498711       | 1.536      |
| 9  | 0.54498816  | 544987     | 0.54498710       | 1.700      | 0.54498710       | 1.479      | 0.54498710       | 2.031      |
| 12 | 0.5449744   | 121286     | 0.54498710       | 3.757      | 0.54498710       | 3.270      | 0.54498710       | 4.487      |
| 15 | 0.5449724   | 246718     | 0.54498710       | 1.124      | 0.54498710       | 9.779      | 0.54498710       | 1.342      |
| 18 | 0.5449717   | 093343     | 0.54498710       | 1.249      | 0.54498710       | 4.28179    | 0.54498710       | 1.342      |
| 21 | 0.5449714   | 023377     | 0.54498710       | 4.098      | 0.54498710       | 3.564      | 0.54498710       | 4.900      |
| 24 | 0.5449712   | 532328     | 0.54498710       | 1.715      | 0.54498710       | 1.488      | 0.54498710       | 2.054      |
| 27 | 0.5449711   | 738431     | 0.54498710       | 7.891      | 0.54498710       | 6.824      | 0.54498710       | 9.490      |
| 30 | 0.544971    | 284615     | 0.54498710       | 3.865      | 0.54498710       | 3.318      | 0.54498710       | 4.685      |

5. The Results

1. From Table (1) we get nine correct decimal places in accelerating (\( A^x_{\sin}, A^x_{\cos}, A^x_{\tan} \)) when \( n=12 \) while we got eight correct decimal places only in Simpson's 3/8 rule at \( n=30 \). This means that we got a better result than Simpson 3/8 with only half the number of divisions. It is observed that the new acceleration received an additional three integer decimal places at \( n = 30 \) compared to Simpson 3/8 rule.

2. From Table (2) we get seven correct decimal places in accelerating (\( A^x_{\sin}, A^x_{\cos}, A^x_{\tan} \)) when \( n=15 \) while we got six correct decimal places only in Simpson's 3/8 rule at \( n=30 \). This means that we got a better result than Simpson 3/8 with only half the number of divisions. It is observed that the new acceleration received an additional two integer decimal places at \( n = 30 \) compared to Simpson 3/8 rule.

3. From Table (3) we get eight correct decimal places in accelerating (\( A^x_{\sin}, A^x_{\cos}, A^x_{\tan} \)) when \( n=12 \) while we got seven correct decimal places only in Simpson's 3/8 rule at \( n=30 \). If we
compare the accelerated results at n = 30, we will find that it outperforms the Simpson’s 3/8 rule by four correct ranks.

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