SLIT-STRIP ISING BOUNDARY CONFORMAL FIELD THEORY 2: 
SCALING LIMITS OF FUSION COEFFICIENTS

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ABSTRACT. This is the second in a series of three articles about recovering the full algebraic structure of a boundary conformal field theory (CFT) from the scaling limit of the critical Ising model in slit-strip geometry. Here we study the fusion coefficients of the Ising model in the lattice slit-strip, with locally monochromatic boundary conditions. The fusion coefficients are certain renormalized limits of boundary correlation functions at the three extremities of the truncated lattice slit-strips, in a basis of random variables whose correlation functions have an essentially exponential dependence on the truncation heights. The key technique is to associate operator valued discrete 1-forms to certain discrete holomorphic functions. This provides a direct analogy with currents in boundary conformal field theory. For two specific applications of this technique, we use distinguished discrete holomorphic functions from the first article of the series. First, we rederive the known diagonalization of the Ising transfer matrix in a form that parallels boundary conformal field theory. Second, we characterize the Ising model fusion coefficients by a recursion written purely in terms of inner products of the distinguished discrete holomorphic functions. The convergence result for the discrete holomorphic functions proven in the first part can then be used to derive the convergence of the fusion coefficients in the scaling limit. In the third article of the series, it will be shown that up to a transformation that accounts for our chosen slit-strip geometry, the scaling limits of the fusion coefficients become the structure constants of the vertex operator algebra of a fermionic conformal field theory.

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1. Introduction

1.1. Critical Ising model on the slit-strip. This article is the second in a three part series about the boundary conformal field theory describing the critical Ising model with locally monochromatic boundary conditions in its scaling limit. It is this second part which concerns the Ising model itself — the first part was primarily about necessary discrete holomorphic functions and their limits, and the last part will be primarily about the algebraic formulation of the conformal field theory.

The Ising model will be defined precisely in Sections 5.1 – 5.3, but we describe it informally right away to be able to emphasize aspects that are consequential for the overall picture. For more comprehensive treatments of the Ising model, see [McWu73, Pal07], and for some of the notable recent progress in rigorous conformal invariance results for it, see [Sm10a, Hon10, Izy11, Dub11, ChSm12, HoSm13, HoKy13, HKV13, CDHKS14, CH15, CGN15, CGN16, BDH16, KeSm16, KeSm17b, Izy17, PeWu18, BeHo19, GHP19, KeSm19, CH121] or the review [Che18].

Figure 1.1. Ising model samples in the two geometries.
The Ising model on a graph $G = (V, E)$ with vertex set $V$ and edge set $E$ is a random assignment of $\pm 1$ spins to the vertices of the graph: the sample space consists of possible spin configurations $\sigma = (\sigma_z)_{z \in V}$, where $\sigma_z \in \{-1, +1\}$ denotes the spin at vertex $z \in V$. Given an inverse temperature parameter $\beta \geq 0$ (the temperature is $\propto 1/\beta$), the Ising model probability measure $P_{\beta, G}$ is informally such that

$$P_{\beta, G}[\{\sigma\}] \propto \exp \left( \beta \sum_{\{z, w\} \in E} \sigma_z \sigma_w \right),$$

so that configurations with more alignment among neighboring spins have higher probabilities, and the strength of this local alignment tendency increases with $\beta$ (which is to say that it decreases with the temperature). This preference for local alignment is also why the model was originally introduced as a model of ferromagnetism [Len20, Isi25]. With physics developments in renormalization group and universality [Wil71], this model was understood to correctly describe the critical behavior and phase transition in uniaxial ferromagnets (not all ferromagnets), as well as a variety of a priori different phenomena such as the liquid-vapor transition. For finite graphs the formula above directly serves as the definition of the Ising model probability measure, but for infinite graphs the proper definition requires a limit from finite graphs instead.

Specifically, we consider the Ising model on graphs $G$, which are square grid approximations of the infinite strip and the infinite slit-strip domains illustrated in Figures 2.1. On the infinite square grid, a phase transition occurs at the critical point $\beta_{Z_{2\infty}} = \frac{1}{2} \log(\sqrt{2}+1)$ [Pei36, KrWa41]: at high temperatures $\beta < \beta_{Z_{2\infty}}$ the spins decorrelate exponentially with distance (paramagnetic phase), whereas in low temperatures $\beta > \beta_{Z_{2\infty}}$ a uniformly positive correlation among spins remains at arbitrarily large distances (ferromagnetic phase). Conformal field theory should describe the behavior at the critical point $\beta = \beta_{Z_{2\infty}}$, so we focus on this case throughout.

Boundary conditions are imposed by declaring a subset $\partial V \subset V$ of vertices as the boundary vertices, and by conditioning on the values of the spins of these. We mainly impose boundary conditions on the following separate segments of the boundaries of the strip and slit-strip. The left and the right boundaries of the strip are considered the separate boundary segments of the strip geometry. In the slit-strip geometry, the left, the right, and the slit are considered the separate boundary segments. A plus boundary condition on a segment of the boundary amounts to conditioning the spins on that segment to $+1$, minus boundary condition similarly to $-1$, and by locally monochromatic boundary conditions we mean merely conditioning on the spins being constant on each boundary component, so that any combinations of plus and minus boundary values on the separate segments of the boundary can occur. Figures 1.1 illustrate typical samples of the Ising model on strip and slit-strip graphs, at the critical inverse temperature $\beta = \beta_{Z_{2\infty}}$, and with plus boundary conditions on all of the above boundary segments.

Since the Ising model consists of a random spin configuration $\sigma = (\sigma_z)_{z \in V}$, random variables in the model are functions

$$X = f(\sigma)$$
of the configuration. Given real-valued random variables \(X_1 = f_1(\sigma), \ldots, X_n = f_n(\sigma)\), the expected value of their product

\[ E[X_1 \cdots X_n] \]

with respect to the probability measure \(P = P_{\beta, G}\) (or appropriately conditioned version of it if boundary conditions are imposed) is called a correlation function. We will consider in particular correlation functions of the following type, which capture the idea of three point boundary correlation functions associated to the three infinite extremities of the slit-strip geometry. The Ising model is considered on square grid approximations of the slit-strip of fixed width \(\ell\), truncated from both above and below at finite heights \(\pm h\). We take random variables

\[ T = f_T(\sigma), \quad R = f_R(\sigma), \quad L = f_L(\sigma), \]

where the function \(f_T\) depends only on the spins on the top row of the truncated slit-strip graph, \(f_R\) depends only on the spins on the right half of the bottom row of the truncated slit-strip graph, and \(f_L\) depends only on the spins on the left half of the bottom row of the truncated slit-strip graph. The correlation functions

\[ E^{(\ell h)}[T R L] \]

of such triples of random variables at the three boundary segments are the physical quantities of interest to us. Among all possible random variables of the above type, we choose a particular basis for which these correlation functions (or more precisely the numerators, when the correlation functions are expressed so that the partition function appears as their denominator) have a purely exponential dependence on the truncation height \(h\). For such basis random variables \(T_\alpha = f_T^{(\alpha)}(\sigma), R_{\alpha R} = f_R^{(\alpha R)}(\sigma), L_{\alpha L} = f_L^{(\alpha L)}(\sigma)\), the fusion coefficients \(\Phi_{\alpha; \alpha R, \alpha L}^{(\ell)}\) of the Ising model are defined so that they capture the renormalized limits of the boundary correlation functions in the sense that

\[ \lim_{h \to \infty} e^{ah} E^{(\ell h)}[T_\alpha L_{\alpha L} R_{\alpha R}] = \frac{\Phi_{\alpha; \alpha R, \alpha L}^{(\ell)}}{Z}, \]

where \(a = a(\alpha, \alpha R, \alpha L)\) is the appropriate rate to renormalize the exponential dependence, and the factor \(Z\) itself includes a certain simple fusion coefficient (see Section 5.9 for details). If one instead considers for example plus boundary conditions or any mixed monochromatic boundary conditions (i.e., prescribed spin values on each of the boundary components), some details need to be adjusted: the rate \(a = a(\alpha, \alpha R, \alpha L)\) of exponential dependence and the overall normalizing constant \(Z\) depend on the boundary conditions, and naturally only the subspace of functions that are supported on configurations allowed by the boundary conditions remains relevant. However the limits obtained for such boundary correlation functions are still proportional to the same fusion coefficients. For this reason, we view the fusion coefficients as giving the proper description of boundary correlation functions at the three infinite extremities of the slit-strip.

Our general goal in this series of articles is to show that the full algebraic structure of a certain conformal field theory can be recovered from these fusion coefficients \(\Phi_{\alpha; \alpha R, \alpha L}^{(\ell)}\) in the scaling limit \(\ell \to \infty\). In the remaining part of this introduction we try to provide context for why such a result should be expected in the first place, what have been some
of the mathematical difficulties in formulating and proving such a result, which details of the probabilistic questions are believed to be consequential for the correct statement, and to what extent similar results should hold more generally for other models of statistical physics.

1.2. Statistical physics and quantum field theory. In the paradigm of constructive quantum field theory, one seeks to obtain rigorous constructions of specific quantum field theories via probabilistic and analytical techniques [ItDr91, FFS92]. If a probability measure on fields in a Euclidean space can be constructed subject to certain axioms, an analytic continuation of one spatial dimension from real to imaginary provides the physical time dimension, and yields an actual quantum field theory in Minkowski space-time [OsSc73, OsSc75]. The construction of suitable probability measures on fields is itself a major undertaking in mathematical physics. A typical approach is to start from lattice discretizations which serve to regularize the field theories and make them a priori well-defined, and to then try to show the existence and desired properties of their scaling limits. This makes the constructive field theory approach at its core essentially equivalent to scaling limit questions in mathematical statistical mechanics.

The underlying idea that quantum field theory and statistical physics can be done in the same formalism still justifiably retains an element of surprise, although it has long had the status of quite uncontroversial folklore [McC95]. The idea was certainly present in the work of Wilson on the renormalization group [Wil71]. The archetype of lattice model of statistical physics, the square-lattice Ising model, was in fact formulated in an essentially quantum field theoretic formalism as early as in 1949 by Kaufman [Kau49].

In two dimensions, massless quantum field theories can be argued by general grounds to enjoy conformal invariance properties, and consequently be very stringently algebraically constrained [BPZ84a]. In view of the general connection between statistical physics and quantum field theories, conformal field theory (CFT) is thus argued to apply to a wide variety of planar statistical physics models at their critical points of continuous phase transitions [BPZ84b]. The critical planar Ising model is the prime example of this picture. Conformal field theory predicts values of critical exponents, functional forms of scaling limits of multi-point correlation functions, as well as a vast number of other intricate features of the Ising model at and near criticality [DMS97, Mus09]. Such predictions are also not just excellent approximations: due to their algebraic underpinnings, they are supposed to be exactly correct. The entirety of CFT predictions forms a sound and appealing overall picture, whose specifics agree exquisitely with numerics, simulations, and alternative methods of theoretical physics. Moreover, some of these predictions for the critical Ising model have even been verified rigorously with the sophisticated mathematical methods that have been developed in the century of research — including with the transfer matrix formalism, dimer representations, Kac-Ward matrices, and recently by discrete complex analysis methods. Indeed, such is the success of the CFT picture for the Ising model — and perhaps the familiarity of this most prominent example case of the general picture — that it has become necessary to carefully specify whether by the Ising model one means the probabilistic lattice model or a certain conformal field theory!
Yet, the vast majority of the conformal field theory ideas have remained very elusive to a rigorous mathematical approach, even in the case of the Ising model — despite the spectacular progress in rigorous conformal invariance results in the past decade. It is still difficult to find even a precisely phrased conjecture in the literature of the totally commonplace assertion that the scaling limit of the critical Ising model is a conformal field theory — let alone a proof. Our goal in this series is to give a precise formulation and proof of such a statement, which at least recovers the full algebraic structure of a boundary conformal field theory in the scaling limit of the Ising model. The conclusion of our main result will be stated in the last part of the series, and it is only there that we really need the detailed definitions about conformal field theory. We nevertheless should address at least what type of a mathematical object we mean when talking about conformal field theory.

1.3. On definitions of conformal field theory. Axiomatic approaches to conformal field theory incorporate the strong constraints that arise from conformal invariance into the general properties of quantum field theories. A number of different mathematical ways of doing so have been put forward — we focus on two, which have arguably been the most successful and the most influential.

The notion of the chiral symmetry algebra of a CFT was in essence formulated already in the pioneering physics literature, and the mathematical definition of a vertex operator algebra (VOA) \cite{FLM88,Kac97,LeLi04} building from the work of Borcherds and Frenkel–Lepowsky–Meurman fully captures its precise meaning. An algebraic approach to the definition of conformal field theory can then be formulated in terms of representation theory of VOAs \cite{Hua12}. Despite being somewhat involved, VOAs are nevertheless sufficiently concrete that many relevant examples of them can be constructed, and while one can not yet conclude all the desired properties of interest to physics, there is steady progress towards using VOAs as a starting point for the more quantum field theoretically formulated CFTs as well.

By contrast, the functorial axiomatization of CFTs by G. Segal \cite{Seg88,Seg04} places focus on the geometry of the space-time (or the space in Euclidean formulation), and on the time evolution semigroup of operators (or the semigroup generated by the transfer matrix in Euclidean lattice formulation) as well as operators that generalize these. This axiomatization is thus closer to the language of quantum field theory generally, and far reaching conclusions can be derived starting from it. Verifying these functorial axioms in specific cases is unfortunately difficult, and few examples of CFTs in this sense are known to exist.

The very definition of conformal field theory therefore still poses mathematical challenges. The choice of the appropriate definition involves matters of mathematical taste, as well as trade-offs between where the difficulties should lie: in constructing examples of CFTs, or in deriving desired conclusions about them. Our choice in this series is to draw geometric inspiration from Segal’s functorial approach, but to accept VOAs as the definition of (the algebraic structure of) conformal field theories.

Whatever is taken as the mathematical axiomatization of CFTs, a fundamental question is to connect the constructive quantum field theory approach to the definition of CFTs, which

\footnote{The very recent work \cite{HGS21} starts to address the question of the proper mathematical formulation of this general statement seriously.}
incorporates all the structure that makes CFTs so remarkably powerful. In other words, one should show that starting from a given probabilistic lattice model, by passing to the scaling limit in which the lattice spacing is let tend to zero, one recovers objects satisfying the axioms of a (specific) conformal field theory.

1.4. The correct CFT for the scaling limit of the Ising model. In the above we have still disregarded all subtle issues stemming from the fact that conformal field theory is supposed to apply in a few different general situations, which require adapting the definitions appropriately. The algebraic approach of VOAs and the functorial approach by Segal thus represent just the two main frameworks of definitions.

A specific issue relevant to the present work, and generally to any application of CFT to statistical physics, is the difference between a boundary CFT and a bulk CFT. From the statistical physics point of view the difference is whether we consider the models in domains which have physical boundaries or not. Algebraically the difference is the same as that of a chiral CFT and of a full CFT: the latter has separate holomorphic and antiholomorphic chiral algebras while the former only has one. Both the holomorphic and antiholomorphic fields of a CFT should be equally meaningful in domains with and without physical boundaries, but in the presence of boundaries the associated holomorphic and antiholomorphic chiral algebras become coupled ultimately due to the boundary conditions in the statistical physics model. Our results will feature a boundary CFT.

A general definition of CFTs should of course admit many specific instances; in particular to any critical statistical physics model there should correspond a CFT specifically describing its scaling limit. The folklore about the Ising model turns out to be somewhat curious regarding this point. In any reasonable sense (for example as VOAs) there are in fact two different CFTs routinely claimed to do the job. These two CFTs of the Ising model are the unitary $(4,3)$-minimal model (with the rational Virasoro VOA as its chiral algebra) and the massless free fermion (with a certain simple super-VOA as its chiral algebra). Both are well-known to be “the” scaling limit of the same model, while unmistakably not the same CFTs! Serious consideration of whether the scaling limit of the Ising model is in fact an interacting bosonic quantum field theory or a free fermionic one could get philosophical. Ultimately it necessarily comes down to what questions do we seek to answer, i.e., what specific quantities are considered in the scaling limit.

In this series of articles, we will recover specifically the super-VOA of the free fermionic boundary CFT in the scaling limit from the Ising model fusion coefficients. So in our setup, what is it that dictates that the correct CFT is the free fermion rather than the unitary $(4,3)$-minimal model? In a similar vein, what dictates that we obtain the VOA only, and not also some (twisted) modules for it, as generally expected to be the case?

In fusion coefficients we allow for the most general observables depending on spins on the parts of the boundary representing the three infinite extremities of the slit-strip. The

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Some readers will undoubtedly view this as too generous, because actual physical fields are typically of neither chirality strictly. Only in a formal sense does the physical field behave as if it was a product of a holomorphic and an antiholomorphic part. An acceptable less forgiving reading of this sentence is therefore that the holomorphic and antiholomorphic fields are equally meaningless both with and without boundaries... The rest of the conclusion remains unaffected.
rest of the domain boundary is taken to have locally monochromatic boundary conditions. With other choices of boundary conditions or other quantities considered, the details of the statement and conclusion should be modified. We want to highlight exactly that: in any precise formulation of the scaling limit of any statistical physics model as a CFT, such choices (boundaries or not, allowed boundary conditions, allowed observables) will inevitably be involved, and the correct formulation of the statement itself has to depend on these.

1.5. The role of the slit-strip geometry. As stated in the very beginning, we specifically consider the Ising model on graphs \( G \), which are square lattice approximations of the infinite slit and the infinite slit-strip domains illustrated in Figure 2.1. Two reasons in particular stand out for why we use such a slit-strip geometry. First, in a Segal-type approach to boundary CFT, the slit-strip plays the role analogous to the pair-of-pants surface in bulk CFT: it is the fundamental building block of more general geometries. Second, we expect that for critical models of statistical physics in general, the vertex operator algebra (as well as Segal type vertex operators) is the scaling limit of the slit-strip transfer matrix operators in a manner entirely parallel to our main result. Indeed, for certain loop models, the transfer matrix formalism in lattice discretizations of the slit-strip has already been successfully used to probe fusion products of modules of the conjecturally associated very intricate CFTs \([\text{GJRSV13]}\). With the idea expected to be so generally valid, working out the case of the Ising model in full detail should provide a valuable prototype.

1.6. Related work and original contribution. We will analyze the fusion coefficients using the very classical method of transfer matrices, the idea of which in fact goes back to Ising’s original work on the one-dimensional Ising model \([\text{Isi25]}\), and which for the two-dimensional Ising model was used by Onsager in his foundational work \([\text{Ons44]}\). With the locally monochromatic boundary conditions that we specifically use, the transfer matrix has been analyzed in \([\text{AbMa73]}\). We rely particularly on the fermionic nature of the transfer matrix formalism for the Ising model in that our main calculations are done with certain Clifford algebra valued discrete one-forms. The fermionic nature of the transfer matrix was first observed by Kaufmann \([\text{Kau49]}\), and was also the subject of the influential work of Schultz & Mattis & Lieb \([\text{SML64]}\). The transfer matrix formalism in its most common form is used for calculations in rectangles, strips, cylinders, and tori, but it quite readily adapts to our calculations in the slit-strip as well. Underlying our calculations is the recently observed close connection \([\text{HKZ14]}\) between the transfer matrix and the analytic continuation of certain discrete holomorphic functions solving a Riemann boundary value problem, i.e., the

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3To put it bluntly, the often encountered statement that the scaling limit of the Ising model is a particular CFT (pick your favorite) is meaningless without further elaboration!

4As emphasized by the renormalization group picture, the core features of a scaling limit should be universal, and independent of details such as the choice of lattice or even the the microscopic interactions in the model itself. A CFT scaling limit describes such a universality class. Our insistence on the importance of certain other details (for example the allowed observables) is merely the statement that those details are not among the irrelevant ones for the determination of the universality class.

5The exact VOA should be changed to the VOA appropriate for the CFT in question, and (depending on boundary conditions) other modules for the VOA and intertwining operators between them should in general appear also, besides the VOA itself.
key techniques pioneered by Smirnov \cite{Smi06} which have enabled a breakthrough in rigorous progress and spectacular mathematical results on conformal invariance of the scaling limit of the Ising model during the past decade. Earlier uses of discrete complex analysis in the Ising model include \cite{KaCe71, Per80, Mer01}, and \cite{Smi10b, CCK17, Che18} can serve as reviews of various aspects of the more recent uses of discrete holomorphicity method.

Apart from a few auxiliary calculations contained in the first article \cite{AKPR20} of this series and in \cite{HKZ14}, we provide a self-contained diagonalization of the critical Ising model transfer matrix in the strip with locally monochromatic boundary conditions. This result is certainly not new, it has been obtained in \cite{AbMa73} and is reviewed in, e.g., \cite{Pal07} — and it is in essence merely a variation on the theme of \cite{Kau49, SML64}. We write the calculation in slightly different terms, and the power of this approach becomes evident in domains of more general geometries than just the strip — in particular in the slit-strip. Our derivation is based on Clifford algebra valued discrete one-forms, which have two crucial properties: closedness and slidability along vertical boundaries. We show that such forms can be constructed using s-holomorphic functions with Riemann boundary values as coefficients of two natural fermion field components. With such forms, contour deformation arguments entirely parallel to boundary conformal field theory can be performed. Thus our method is first of all directly reminiscent of conformal field theory. Moreover, with it the diagonalization of the transfer matrix immediately reduces to a simple question of discrete complex analysis: namely of finding the s-holomorphic solutions to Riemann boundary value problem in the strip which are eigenfunctions of vertical translations.

The real advantage of our method of closed and vertically slidable Clifford algebra valued discrete one-forms, however, only becomes evident in the slit-strip. At first sight, the fusion coefficients appear to involve such one-forms, which are only locally defined. But with the discrete complex analysis results in the first part of the series \cite{AKPR20}, we can trade unwanted singular parts of such one-forms to globally defined one-forms, which can then be contour deformed to those extremities of the slit-strip in which they have no singularities. This is at the core of our characterization of the fusion coefficients in terms of a recursion. The coefficients in the recursion involve inner products among distinguished discrete holomorphic functions, and from the results \cite{AKPR20} about the convergence of these functions in the scaling limit, we deduce the convergence of the fusion coefficients themselves in the scaling limit. The scaling limit statement for the fusion coefficients may have been foreseen by experts for many decades, but its derivation seems to genuinely require the novel techniques of discrete complex analysis \cite{Smi06, ChSm11}.

1.7. Organization of this work. Part one \cite{AKPR20} of this series was concerned with spaces of holomorphic functions with Riemann boundary values in the strip and the slit-strip, as well as lattice analogues of these. In this article, the necessary results from the first part are recalled as they are needed: Sections 2, 4, and 6 recall the definitions of discrete complex analysis, specific discrete holomorphic functions, and scaling limits of such functions, respectively.

The original results of this article are contained in Sections 3, 5, and 7.

In Section 3 we first review the Clifford algebra acting on the transfer matrix state space, and select a basis of Clifford generators consisting of discrete holomorphic and discrete
anti-holomorphic fermions. Quoting the result of the conjugation of these fermions by the transfer matrix from [HKZ14], we present a construction of closed and vertically slidable Clifford algebra valued 1-forms from s-holomorphic solutions to the Riemann boundary value problem. The integration and contour deformation of such operator valued 1-forms is a direct discrete counterpart of the way that current modes of the chiral symmetry algebra in boundary conformal field theory are treated. This section can therefore be regarded as an exact lattice realization of a key algebraic technique of boundary conformal field theory. It is crucial that the technique works not only in the lattice strip, but also in other domains, including the lattice slit-strip, so it can ultimately be applied to the fusion coefficients.

In Section 5 we define in detail our setups for the Ising model in the strip and the slit-strip. In this section we include a review how the transfer matrix allows for the calculation of partition functions and correlation functions in the strip and the slit-strip. This part is straightforward and mostly well-known, so we do not provide proofs here. We instead make sure that the definitions and statements are self-contained and sufficient for our applications, and we arrange the sequence of statements so that even a reader without prior familiarity with the transfer matrix formalism should be able to fill in the missing proofs with relative ease. In the remaining part of the section, the transfer matrix method is used in conjunction with the method of Clifford-algebra valued discrete 1-forms of Section 3. As a first illustration of the method, we use discrete holomorphic vertical translation eigenfunctions as coefficient functions in the forms, and obtain a self-contained diagonalization of the Ising transfer matrix in the strip. As the main result of the section, we use discrete holomorphic functions adapted to the slit-strip geometry to obtain a recursion that determines the Ising model fusion coefficients.

In Section 7 we introduce continuum fusion coefficients as certain explicit integrals with Pfaffian-form kernels and coefficient functions which are quarter-integer Fourier modes, i.e., vertical translation eigenfunctions in the continuum. We derive a recursion for these continuum fusion coefficients, entirely parallel to the recursion for the Ising model fusion coefficients. Numerical constants in the two recursions only differ because the continuum version involves inner products among distinguished continuum holomorphic functions, while the original Ising model version involves inner products among distinguished discrete holomorphic functions. The convergence in the scaling limit of the inner products among the functions is then sufficient to conclude that the Ising model fusion coefficients converge in the scaling limit to these continuum fusion coefficients.

In the final part [KPR21] of our series it will be shown that the continuum fusion coefficients are essentially (i.e., up to a transformation of necessitated by the slit-strip geometry) equal to the structure constants of the vertex operator algebra of a fermionic conformal field theory. Combined with the result of this second part, we can conclude that the VOA structure constants — and thus indeed the full algebraic structure of the associated boundary conformal field theory — can be recovered from the Ising model fusion coefficients in the scaling limit, and vice versa.

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2. LATTICE DOMAINS AND NOTIONS OF DISCRETE COMPLEX ANALYSIS

In this section we define the infinite strip and slit-strip lattice domains, and the two notions of discrete complex analysis that we rely on: s-holomorphicity and Riemann boundary values \cite{Smi06,ChSm11}. Our specific conventions are as in \cite{AKPR20}.

2.1. The infinite lattice strip and slit-strip graphs. All the graphs we consider are essentially subgraphs of the square grid \( \mathbb{Z}^2 \) (with its graph structure of nearest neighbor edges). We consider the square grid as embedded in the complex plane, \( \mathbb{Z}^2 \subset \mathbb{C} \). Without comment, we identify vertices of any of our graphs with the corresponding complex numbers, edges with the complex numbers that are the midpoints of their two endpoint vertices\(^{6}\) and plaquettes with the complex numbers at the midpoint of the corresponding faces of the square grid.

Fix \( a, b \in \mathbb{Z} \) such that \( a < 0 < b \), and denote \( \ell = b - a \in \mathbb{N} \). The parameters \( a \) and \( b \) will serve as the horizontal positions of the left and the right boundaries of the vertical lattice strip and slit-strip, respectively; \( 0 \) will be the horizontal position of the slit when appropriate; \( \ell \) will be the total width of the strip or slit-strip. Although our lattice definitions depend on \( a, b \), we usually only indicate the discretization by a superscript \( \ell \), for simplicity.

The vertically infinite lattice strip is a graph with vertex set

\[
S^{(\ell)} := [a, b] \times \mathbb{Z},
\]

where \( [a, b] = \{a, a+1, \ldots, b-1, b\} \) is the integer interval from \( a \) to \( b \). This is considered as an induced subgraph of the square grid \( \mathbb{Z}^2 \), i.e., with nearest neighbor edges

\[
E(S^{(\ell)}) = \left\{ \{u,v\} \mid u, v \in S^{(\ell)}, \|v - u\| = 1 \right\}.
\]

Figure 2.1(a) illustrates the graph \( S^{(\ell)} \).

The vertically infinite lattice slit-strip is a multi-graph with the same vertex set

\[
S_{\text{slit}}^{(\ell)} := [a, b] \times \mathbb{Z}.
\]

It is also taken to have only nearest neighbor edges, but the edge between \(-i\ y\) and \(-i\ (y+1)\) for \( y \in \mathbb{Z}_{\geq 0} \) is doubled, with one of the edges interpreted to be on the left and the other on the right of the negative imaginary axis. The set of edges of this multi-graph is denoted by \( E(S_{\text{slit}}^{(\ell)}) \). Figure 2.1(b) illustrates the graph \( S_{\text{slit}}^{(\ell)} \), with the “slit” along the negative imaginary axis.

2.2. Functions on the lattice domains. The discrete functions of interest to us will be defined in the lattice strip \( S^{(\ell)} \) and the lattice slit-strip \( S_{\text{slit}}^{(\ell)} \). More precisely, we consider complex-valued functions

\[
F : E \to \mathbb{C}
\]

\(^{6}\)The slit-strip is a multi-graph with doubled edges between certain vertices, so it is a slight abuse of notation to sometimes identify two edges with the same complex number, but we trust the correct choice is always clear from the context.
Figure 2.1. The vertically infinite lattice strip and slit-strip.

defined on the set $E$ of edges of the corresponding graph, $E = E(S^{(\ell)})$ or $E = E(S_{\text{slit}}^{(\ell)})$. Occasionally we also consider functions only locally defined on one of the following three pieces of the lattice slit-strip,

$$S_{\text{slit}}^{T; (\ell)} = [a, b] \times \mathbb{Z}_{\geq 0}, \quad S_{\text{slit}}^{L; (\ell)} = [a, 0] \times \mathbb{Z}_{\leq 0}, \quad S_{\text{slit}}^{R; (\ell)} = [0, b] \times \mathbb{Z}_{\leq 0},$$

interpreted as subgraphs of the lattice slit-strip so that their nearest neighbor edge sets are

$$E(S_{\text{slit}}^{T; (\ell)}), \ E(S_{\text{slit}}^{R; (\ell)}), \ E(S_{\text{slit}}^{L; (\ell)}) \subset E(S_{\text{slit}}),$$

(the only overlap is on horizontal edges at height 0).

Key notions to us are suitable holomorphicity properties and boundary conditions of such functions. Specifically, we will use discrete holomorphicity in a specific $\mathbb{R}$-linear sense of $s$-holomorphicity (and in Section 3 we will encounter two complexifications of this real-linear notion). We will also use Riemann boundary conditions which specify the arguments of the functions (up to multiples of $\pi$) on the boundaries.

**S-holomorphicity.** Let $E$ be the set of edges of $S^{(\ell)}$ or $S_{\text{slit}}^{(\ell)}$, or of any of the subgraphs (2.1). A function

$$F : E \to \mathbb{C}$$
is said to be \textbf{s-holomorphic}, if for any two edges \( z_1, z_2 \in E \) adjacent to the same vertex \( v \) and face \( p \), we have
\begin{equation}
F(z_1) + \frac{i |v - p|}{v - p} F(z_1) = F(z_2) + \frac{i |v - p|}{v - p} F(z_2).
\end{equation}

\textit{Riemann boundary values.} The Riemann boundary values we impose require the values of the functions on boundary points \( z \) of the domain to be real multiples of \( i \tau(z)^{-1/2} \), where \( \tau(z) \) is the tangent to the boundary of the domain, with positive (i.e., counterclockwise) orientation. Since all of the boundaries of interest to us are vertical (with tangents \( \tau(z) = \pm i \)), we can phrase this boundary conditions explicitly as follows.

A discrete function in the strip
\[ F : E(S^{(\ell)}) \to \mathbb{C} \]
is said to have \textbf{Riemann boundary values} if on the left and the right boundaries its values satisfy
\begin{equation}
F(a + iy') \in e^{-i\pi/4} \mathbb{R} \quad \text{and} \quad F(b + iy') \in e^{+i\pi/4} \mathbb{R}.
\end{equation}
In the slit-strip, boundary conditions are also applied along the slit: if \( E \) is the set of edges of the slit-strip \( S_{\text{slit}}^{(\ell)} \) or one of the subdomains \eqref{2.1}, then a function
\[ F : E \to \mathbb{C} \]
is said to have \textbf{Riemann boundary values} if it satisfies \eqref{2.3} on the edges of the left and the right boundaries, and in addition
\begin{equation}
F(0^- + iy') \in e^{+i\pi/4} \mathbb{R} \quad \text{and} \quad F(0^+ + iy') \in e^{-i\pi/4} \mathbb{R}
\end{equation}
where \( 0^\pm + iy' \) stand for the edges along the slit on the two sides.

\textit{Restrictions to a cross-section.} We focus particularly on the restriction of the functions to the horizontal cross-section at height 0, and we form real vector spaces of such functions, as in [AKPR20].

In the discrete setting, we therefore study functions defined on the set
\begin{equation}
\mathcal{I}^* = \left[ a, b \right]^* = \left\{ a + \frac{1}{2}, a + \frac{3}{2}, \ldots, b - \frac{3}{2}, b - \frac{1}{2} \right\}
\end{equation}
of horizontal edges on the cross-section. We use the real vector space
\begin{equation}
\mathcal{F}^{(\ell)} := \mathbb{C}\mathcal{I}^*
\end{equation}
of complex-valued functions on \( \mathcal{I}^* \), which is of dimension \( \dim_{\mathbb{R}}(\mathcal{F}^{(\ell)}) = 2\ell \). We equip it with the real Hilbert space structure such that inner product and norm are
\begin{equation}
\langle f, g \rangle := \Re \left( \sum_{x' \in \mathcal{I}^*} f(x') \overline{g(x')} \right), \quad \|f\| := \left( \sum_{x' \in \mathcal{I}^*} |f(x')|^2 \right)^{1/2}.
\end{equation}
We furthermore define a unitary transformation \( R : \mathcal{F}^{(\ell)} \to \mathcal{F}^{(\ell)} \) by
\begin{equation}
(Rf)(x') = -i \overline{f(x')} \quad \text{for} \ x' \in \mathcal{I}^*.
\end{equation}
This transformation $R$ has the interpretation of a reflection across the horizontal cross-section; see [AKPR20] Remark 3.2.

3. Clifford algebra valued discrete 1-forms

In this section we develop discretizations of boundary conformal field theory contour integration and contour deformation manipulations [Car84, Car86, Car89] (see also [DMS97, Car06]) relevant for the Ising model on the lattice strip and slit-strip.

We start this section by briefly reviewing the Clifford algebra action on the transfer matrix state space, and recall the notion of discrete holomorphic and antiholomorphic fermions $\psi, \psi^*$ from [HKZ14]. These form a pair of Clifford generator valued functions on the edges of the lattice strip or lattice slit-strip, which satisfy complexifications of the s-holomorphicity and Riemann boundary values.

The main result in this section is that if one introduces discrete 1-forms with values in the space of Clifford generators, using coefficient functions for $\psi$ and $\psi^*$, which form a pair that is imaginary complexified s-holomorphic (ICSH) and has imaginary complexified Riemann boundary values (ICRBV), then the 1-forms are closed and have vanishing integrals along vertical boundaries. This enables contour deformation manipulations exactly analogous to boundary conformal field theory.

The simplest application of this observation is a reformulation of the calculation to diagonalize the Ising transfer matrix in a strip, which we will present in Section 5.6. This diagonalization becomes straightforward by choosing coefficient functions in the 1-forms that are eigenfunctions of vertical translations, as given in Section 4.

A more interesting and novel prospect, however, is choosing coefficient functions in the 1-forms to be globally defined functions in the slit-strip, which have prescribed singularities, see Section 4. Employing such coefficient functions, the results of this section will be used to derive a recursive characterization of the fusion coefficients of the Ising model in Section 5.9.

3.1. Clifford algebra action on a state space. In this section we consider operators and operator valued forms on the state space of the Ising model transfer matrix formalism. We use only rudimentary Clifford algebra theory; e.g., [Pal07] is an appropriate reference. In this first subsection, we define the state space and introduce a Clifford algebra action on it.

The state space. The state space for the transfer matrix is associated to the horizontal cross-section $I = [a, b]$ of the lattice strip (or slit-strip). It has basis vectors $u_\rho$ indexed by configurations $\rho \in \{\pm 1\}^I$ of $\pm 1$-spins in a cross-section row $I$. The (full) state space $\tilde{V}$ is defined as the complex vector space with basis $\{u_\rho\}_{\rho \in \{\pm 1\}^I}$, i.e.,

$$\tilde{V} = \mathbb{C}^{(\{\pm 1\}^I)} = \text{span}_\mathbb{C} \left\{ u_\rho \mid \rho \in \{\pm 1\}^I \right\}. \quad (3.1)$$

More general lattice domains could be considered with only additional complications to the notation. We nevertheless focus only on the strip and the slit-strip, where convergence results of distinguished functions can be used to reconstruct the vertex operator algebra structure in the scaling limit.
At various stages, we employ different subspaces of the state space $\widetilde{V}$. For the purposes of the present section, the most important is the subspace $V \subset \widetilde{V}$ spanned by the basis vectors, whose rightmost spin is $+1$. We call $V$ the irreducible state-space, because it will be an irreducible representation of the Clifford algebra action below.

We make the state space (and its subspaces) a complex inner product space in such a way that the basis vectors $u_\rho, \rho \in \{\pm 1\}$ are orthonormal. The conjugate transpose with respect to this basis is denoted by superscript $\dagger$. In order to clearly distinguish the complex inner product on the state space from the real inner products used in the function spaces in Section 4 and [AKPR20], we will write the inner product of vectors $v_1, v_2 \in \widetilde{V}$ in matrix notation as $v_1^\dagger v_2$.

In this notation, orthonormality of the basis in particular amounts to $u_\rho^\dagger u_\rho = \delta_{\tau,\rho}$ (Kronecker delta).

**Clifford generators.** The state space carries a representation of a Clifford algebra. We directly choose a convenient basis of generators of the Clifford algebra for the present purposes by introducing discrete holomorphic and discrete antiholomorphic fermion operators on the state space.

For any $x' \in I^* = [a, b]^*$, we define the involution
\[
\text{fold}_{x'}: \{\pm 1\}^I \to \{\pm 1\}^I
\]
which flips the values of the spins to the left of $x'$:
\[
\text{fold}_{x'}(\rho) = \rho', \quad \text{where} \quad \rho' = \begin{cases} -\rho_x & \text{if } x < x' \\ \rho_x & \text{if } x > x' \end{cases}.
\]

With the help of this, we define linear maps on the state space
\[
\psi_{x'}, \psi^*_{x'}: \widetilde{V} \to \widetilde{V}
\]
by setting their values on the basis vectors $u_\rho, \rho \in \{\pm 1\}^I$, to be
\[
\psi_{x'} u_\rho = \frac{-\rho_x' - \frac{1}{2} + i \rho_x' + \frac{1}{2}}{\sqrt{2}} u_{\text{fold}_{x'}(\rho)} \quad \text{and} \quad \psi^*_{x'} u_\rho = \frac{-i \rho_x' - \frac{1}{2} + \rho_x' + \frac{1}{2}}{\sqrt{2}} u_{\text{fold}_{x'}(\rho)}.
\]

When $\text{fold}_{x'}(\rho) = \rho'$, we have $\rho_b = \rho'_b$, so these are also well-defined linear maps of the irreducible state space $V$,
\[
\psi_{x'}, \psi^*_{x'}: V \to V.
\]

The linear span of these operators,
\[
\text{CliffGen} = \text{span}_C \left( \{ \psi_{x'} \mid x' \in I^* \} \cup \{ \psi^*_{x'} \mid x' \in I^* \} \right)
\]
is called the space of Clifford generators. We can interpret either $\text{CliffGen} \subset \text{End}(\widetilde{V})$ or $\text{CliffGen} \subset \text{End}(V)$ — all relevant properties remain identical in both cases.
Properties of the Clifford generators. On the state space we use the inner product with respect to which the basis \((u_\rho)_{\rho \in \{\pm 1\}^x}\) is orthonormal. The Hilbert space adjoints of the above operators are easily calculated: \(\psi^*_x\) is self-adjoint and \(\psi^*_{x'}\) anti-self-adjoint.

**Lemma 3.1.** For any \(x' \in \mathcal{I}^*\), we have
\[
(\psi_{x'})^\dagger = -\psi_{x'} \quad \text{and} \quad (\psi^*_{x'})^\dagger = \psi^*_{x'}.
\]

The anticommutator of two linear operators \(A, B : V \to V\) on a vector space \(V\) is
\[
[A, B]_+ = AB + BA.
\]

The anticommutators among the operators above are also straightforwardly calculated.

**Lemma 3.2.** For any \(x'_1, x'_2 \in \mathcal{I}^*\), we have
\[
[\psi_{x'_1}, \psi_{x'_2}]_+ = -2\delta_{x'_1, x'_2} \text{id}, \quad [\psi^*_{x'_1}, \psi^*_{x'_2}]_+ = +2\delta_{x'_1, x'_2} \text{id}, \quad [\psi_{x'_1}, \psi^*_{x'_2}]_+ = 0.
\]

This explicitly shows that the anticommutator defines a nondegenerate bilinear form \((\cdot, \cdot)\) on \(\text{CliffGen}\) via
\[
[\phi_1, \phi_2]_+ = (\phi_1, \phi_2) \text{id} \quad \text{for } \phi_1, \phi_2 \in \text{CliffGen}.
\]

It follows that the subalgebra \(\text{Cliff} \subset \text{End}(V)\) generated by the elements of \(\text{CliffGen}\) is a Clifford algebra of dimension \(\dim(\text{Cliff}) = 2^\ell\).

A pair of subspaces \(\text{Cre}, \text{Ann} \subset \text{CliffGen}\) is said to be a polarization if
\[
[\text{Cre}, \text{Cre}]_+ = 0, \quad [\text{Ann}, \text{Ann}]_+ = 0.
\]

Given any polarization, the unique irreducible representation of the Clifford algebra \(\text{Cliff}\) can be identified with the exterior algebra of the subspace \(\text{Cre} \subset \text{CliffGen}\),
\[
\bigwedge \text{Cre} = \bigoplus_{d=0}^\ell \bigwedge^d \text{Cre},
\]
and the one-dimensional subspace \(\bigwedge^0 \text{Cre} \subset \bigwedge \text{Cre}\) within this exterior algebra consists of vectors annihilated simultaneously by all of \(\text{Ann}\). A non-zero vector with this property is called a vacuum with respect to the polarization. The unique irreducible representation in particular has dimension \(2^\ell\), and consequently the state space \(V\) must be isomorphic to this irreducible representation.

**Remark 3.3.** Note that although we have the direct sum decomposition
\[
\text{CliffGen} = \text{span}_C \{ \psi_x \mid x' \in \mathcal{I}^* \} \oplus \text{span}_C \{ \psi^*_x \mid x' \in \mathcal{I}^* \},
\]
this pair of subspaces is not a polarization — the reason for our choice of basis was altogether different; see Section 3.2, Proposition 3.5 in particular.
3.2. The strip transfer matrix and fermions in the lattice strip. For the purposes of defining a physically relevant Clifford algebra generator valued functions on the lattice strip, we need the Ising transfer matrix. We fix a parameter
\[ \beta = \beta_c^{\mathbb{Z}^2} = \frac{1}{2} \log(\sqrt{2} + 1), \]
the critical inverse temperature of the Ising model on the square grid.

The transfer matrix is a $\mathbb{C}$-linear map
\[ T: \tilde{V} \to \tilde{V} \]
constructed out of two constituent matrices, a diagonal matrix $T_{\text{hor}}^{1/2}: \tilde{V} \to \tilde{V}$ (accounting for the Ising model interactions on horizontal edges, with weight half), and a symmetric matrix $T_{\text{ver}}: \tilde{V} \to \tilde{V}$ (accounting for the Ising model interactions on vertical edges as well as locally monochromatic boundary conditions on the left and right boundaries). The matrix $T_{\text{hor}}^{1/2}$ is defined by the matrix elements
\[ u_x^\dagger T_{\text{hor}}^{1/2} u_\rho = \delta_{\tau,\rho} \exp \left( \frac{\beta}{2} \sum_{x=a}^{b-1} \rho_x \rho_{x+1} \right), \]
and the matrix $T_{\text{ver}}$ is defined by the matrix elements
\[ u_x^\dagger T_{\text{ver}} u_\rho = \exp \left( \frac{\beta}{2} \sum_{x=a}^{b} \rho_x \tau_x \right) \delta_{\tau_a,\rho_a} \delta_{\tau_b,\rho_b}. \]

The transfer matrix $T$ is then defined as
\[ T = T_{\text{hor}}^{1/2} T_{\text{ver}} T_{\text{hor}}^{1/2}. \]

Conjugation by the transfer matrix preserves the space of Clifford generators. This is made explicit in the following.

**Proposition 3.4.** Denote $\lambda = \frac{1+i}{\sqrt{2}}$. For $x' \in \mathcal{I}^*$, we have
\[
\begin{align*}
T^{-1} \psi_{x'} T &= \begin{cases} 
(1 + \frac{1}{\sqrt{2}}) \psi_{a'} + (\lambda^3 + \frac{\lambda^3}{\sqrt{2}}) \psi_{a' + 1} + \frac{\lambda^3}{\sqrt{2}} \psi_{a' + 2} + \frac{1}{\sqrt{2}} \psi_{a' + 3} & \text{if } x' = a' \\
2 \psi_{x'} - \sqrt{2} \psi_{x' + 1} + \frac{\lambda}{\sqrt{2}} \psi_{x' + 2} + \frac{\lambda}{\sqrt{2}} \psi_{x' + 3} + \frac{\lambda^3}{\sqrt{2}} \psi_{x' - 1} + \frac{1}{\sqrt{2}} \psi_{x' - 2} & \text{if } x' \neq a', b' \\
(1 + \frac{1}{\sqrt{2}}) \psi_{b'} + (\lambda^3 + \frac{\lambda^3}{\sqrt{2}}) \psi_{b' + 1} + \frac{\lambda^3}{\sqrt{2}} \psi_{b' + 2} + \frac{1}{\sqrt{2}} \psi_{b' + 3} & \text{if } x' = b' \\
2 \psi_{x'} - \sqrt{2} \psi_{x' - 1} + \frac{\lambda}{\sqrt{2}} \psi_{x' - 2} + \frac{\lambda}{\sqrt{2}} \psi_{x' - 3} + \frac{\lambda^3}{\sqrt{2}} \psi_{x' + 1} + \frac{1}{\sqrt{2}} \psi_{x' + 2} & \text{if } x' \neq a', b' \\
(1 + \frac{1}{\sqrt{2}}) \psi_{b'} + (\lambda^3 + \frac{\lambda^3}{\sqrt{2}}) \psi_{b' - 1} + \frac{\lambda^3}{\sqrt{2}} \psi_{b' - 2} + \frac{1}{\sqrt{2}} \psi_{b' - 3} & \text{if } x' = b'. 
\end{cases}
\end{align*}
\]

**Proof.** The proof is an explicit calculation. The details of the case of general inverse temperature $\beta$ can be found in [HKZ14], and upon specializing to the critical case $\beta = \beta_c^{\mathbb{Z}^2}$, they yield the above formulas. \( \square \)

---

8The transfer matrices make sense and can be used at any inverse temperature $\beta \geq 0$, but the discrete complex analysis properties are specific to the critical point.

9For conjugation, we already use the simple fact that the transfer matrix $T$ is invertible. We explicitly state this later along with other well-known properties, in Theorem 5.1(i).
Holomorphic and antiholomorphic fermions in the strip. We now define two Clifford algebra generator valued functions on the midpoints of the edges of the discrete strip $\mathcal{S}(\ell)$ of Figure 1(a). The horizontal edges at height zero are naturally identified with points $x' \in \mathcal{I}^*$ of the dual cross-section, and horizontal edges at height $y \in \mathbb{Z}$ are identified with points $x' + iy$ with $x' \in \mathcal{I}^*$. On such horizontal edges we define

$$
\psi(x' + iy) = T^{-y} \psi_{x'} T^y \quad \text{and} \quad \psi^*(x' + iy) = T^{-y} \psi_{x'}^* T^y.
$$

At zero height, we therefore simply have $\psi(x') = \psi_{x'}$ and $\psi^*(x') = \psi_{x'}^*$, and extending to other heights $y$ is done as usual in the transfer matrix formalism; compare, e.g., with (5.5).

It follows from Proposition 3.4 that we have $\psi(x' + iy), \psi^*(x' + iy) \in \text{CliffGen}$ in general.

Vertical edges of the strip $\mathcal{S}(\ell)$, likewise, can be identified with their midpoints, which are of the form $x + iy'$, with $x \in \mathcal{I}$ and $y' \in \mathbb{Z} + \frac{1}{2}$. The vertical edges with $x = a$ constitute the left boundary of the strip $\mathcal{S}(\ell)$, and those with $x = b$ constitute the right boundary. The following proposition summarizes the key principle behind our extension of the definitions of $\psi$ and $\psi^*$ from horizontal edges to vertical edges.

**Proposition 3.5.** There exists unique extensions $\psi, \psi^*: E(\mathcal{S}(\ell)) \to \text{CliffGen}$ of (3.6) to vertical edges with the following properties:

The pair $(\psi, \psi^*)$ is complexified s-holomorphic (CSH) in the sense that for any edges $z_1, z_2 \in E(\mathcal{S}(\ell))$ adjacent to a vertex $v$ and a face $p$ we have

$$
\psi(z_1) + \frac{i}{v - p} |v - p| \psi^*(z_1) = \psi(z_2) + \frac{i}{v - p} |v - p| \psi^*(z_2),
$$

and it has complexified Riemann boundary values (CRBV) in the sense that

$$
\psi(L) + i \psi^*(L) = 0 \quad \text{and} \quad \psi(R) - i \psi^*(R) = 0.
$$

We will prove this statement by giving a number of formulas for the fermions on vertical edges in terms of those on horizontal edges, and showing that the various formulas agree. There will be differences between the treatment of the boundary edges and the rest, stemming ultimately from the cases in Proposition 3.4.

Before delving into the details, let us emphasize the following. Having these discrete complex analysis properties, (3.7) and (3.8), valid simultaneously with the propagation (3.6), is a nontrivial property of the transfer matrix $T$ of the critical Ising model and of the chosen basis $\{\psi_{x'} | x \in [a, b]^*\} \cup \{\psi_{x'}^* | x \in [a, b]^*\}$ of the Clifford generators. The relationship of the transfer matrix formalism to discrete complex analysis was observed in [HKZ14].

Let us then look at the calculations needed for the extension of the fermions to vertical edges.
Let $x \in \mathcal{I}$ and $y' \in \mathbb{Z} + \frac{1}{2}$. Consider the vertical edge $z = x + iy'$, and denote

\begin{align*}
\text{NW} &= z - \frac{1}{2} + \frac{i}{2} \\
\text{NE} &= z + \frac{1}{2} + \frac{i}{2} \\
\text{SW} &= z - \frac{1}{2} - \frac{i}{2} \\
\text{SE} &= z + \frac{1}{2} - \frac{i}{2}
\end{align*}

as in Figure 3.1(a). Note that NW and SW are horizontal edges of $S^{(\ell)}$ whenever $x \neq a$, and NE and SE are horizontal edges of $S^{(\ell)}$ whenever $x \neq b$.

Denote $\lambda = \frac{1+i}{\sqrt{2}}$. If $x \neq b$, let

\begin{align*}
\psi_{(E)}(z) &= \frac{\lambda}{\sqrt{2}} \psi_{(NE)} + \frac{1}{\sqrt{2}} \psi^*_{(NE)} - \frac{\lambda^3}{\sqrt{2}} \psi_{(SE)} - \frac{1}{\sqrt{2}} \psi^*_{(SE)}, \\
\psi^*_{(E)}(z) &= \frac{\lambda^{-1}}{\sqrt{2}} \psi^*_{(NE)} + \frac{1}{\sqrt{2}} \psi_{(NE)} - \frac{\lambda^{-3}}{\sqrt{2}} \psi^*_{(SE)} - \frac{1}{\sqrt{2}} \psi_{(SE)},
\end{align*}

and if $x \neq a$, let

\begin{align*}
\psi_{(W)}(z) &= \frac{\lambda^{-1}}{\sqrt{2}} \psi_{(NW)} + \frac{1}{\sqrt{2}} \psi^*_{(NW)} - \frac{\lambda^{-3}}{\sqrt{2}} \psi_{(SW)} - \frac{1}{\sqrt{2}} \psi^*_{(SW)}, \\
\psi^*_{(W)}(z) &= \frac{\lambda}{\sqrt{2}} \psi^*_{(NW)} + \frac{1}{\sqrt{2}} \psi_{(NW)} - \frac{\lambda^3}{\sqrt{2}} \psi^*_{(SW)} - \frac{1}{\sqrt{2}} \psi_{(SW)}. \tag{3.11, 3.12}
\end{align*}

If $x = a$, let also

\begin{align*}
\psi_{(L)}(z) &= \frac{1-\lambda}{2\sqrt{2}} \left( \lambda \psi_{(NE)} + \psi^*_{(NE)} \right) + \frac{1-\lambda^{-1}}{2\sqrt{2}} \left( \lambda^{-1} \psi_{(SE)} - \psi^*_{(SE)} \right), \\
\psi^*_{(L)}(z) &= \frac{1-\lambda^{-1}}{2\sqrt{2}} \left( \lambda^{-1} \psi^*_{(NE)} + \psi_{(NE)} \right) + \frac{1-\lambda}{2\sqrt{2}} \left( \lambda \psi_{(SE)} - \psi_{(SE)} \right), \tag{3.13, 3.14}
\end{align*}

and if $x = b$, let also

\begin{align*}
\psi_{(R)}(z) &= \frac{1-\lambda^{-1}}{2\sqrt{2}} \left( \lambda^{-1} \psi_{(NW)} + \psi^*_{(NW)} \right) + \frac{1-\lambda}{2\sqrt{2}} \left( \lambda \psi_{(SW)} - \psi^*_{(SW)} \right), \\
\psi^*_{(R)}(z) &= \frac{1-\lambda}{2\sqrt{2}} \left( \lambda \psi^*_{(NW)} + \psi_{(NW)} \right) + \frac{1-\lambda^{-1}}{2\sqrt{2}} \left( \lambda^{-1} \psi^*_{(SW)} - \psi_{(SW)} \right). \tag{3.15, 3.16}
\end{align*}
Lemma 3.6. If $x \neq a, b$, then at the vertical edge $z = x + iy'$ the following equalities hold:

$$ψ_{(E)}(z) = ψ_{(W)}(z), \quad ψ^*_{(E)}(z) = ψ^*_{(W)}(z).$$

If $x = a$, then at the edge $z = x + iy'$ of the left boundary the following equalities hold:

$$ψ_{(E)}(z) = ψ_{(L)}(z), \quad ψ^*_{(E)}(z) = ψ^*_{(L)}(z).$$

If $x = b$, then at the edge $z = x + iy'$ of the right boundary the following equalities hold:

$$ψ_{(W)}(z) = ψ_{(R)}(z), \quad ψ^*_{(W)}(z) = ψ^*_{(R)}(z).$$

Proof. The last two terms in each of the expressions (3.9) – (3.16) involve horizontal edge fermions in the row at height $y' - \frac{1}{2}$, while the first two terms correspondingly involve horizontal edge fermions in the row above, at height $y' + \frac{1}{2}$. Using the definition (3.6) and the explicit expression of conjugation of Clifford generators by $T$ in Proposition 3.4, we may write the fermions at height $y' + \frac{1}{2}$ in terms of those at height $y' - \frac{1}{2}$.

For $x \neq a, b$, we explicitly find that (3.9) and (3.11) are both equal to

$$\frac{λ^{-1}}{\sqrt{2}}ψ(SW) + \frac{λ}{\sqrt{2}}ψ(SE) + \frac{i}{\sqrt{2}}ψ^*(SW) - \frac{i}{\sqrt{2}}ψ^*(SE),$$

and that (3.9) and (3.11) are both equal to

$$\frac{λ}{\sqrt{2}}ψ^*(SW) + \frac{λ^{-1}}{\sqrt{2}}ψ^*(SE) - \frac{i}{\sqrt{2}}ψ(SW) + \frac{i}{\sqrt{2}}ψ(SE),$$

proving the first two asserted equalities.

Similarly, for $x = a$, we find that (3.9) and (3.13) are both equal to

$$(i + λ^{-1})ψ(SE) + (-1 + λ^{-1})ψ^*(SE),$$

while (3.10) and (3.14) are both equal to

$$(-i + λ)ψ^*(SE) + (-1 + λ)ψ(SE).$$

Finally, for $x = b$, we find that (3.11) and (3.15) are both equal to

$$(-i + λ)ψ(SW) + (-1 + λ)ψ^*(SW),$$

while (3.12) and (3.16) are both equal to

$$(i + λ^{-1})ψ^*(SW) + (-1 + λ^{-1})ψ(SW).$$

These direct calculations prove the assertion. □

Proof of Proposition 3.5. The uniqueness of the extension of $ψ, ψ^*$ to vertical edges is clear: Equations (3.7) and (3.8) can be used to solve for the values $ψ(z)$ and $ψ^*(z)$ on a vertical edge $z$ in terms of the fermions on any two adjacent horizontal edges. Equations (3.9) – (3.16) are (some of) the expressions thus obtained.
Conversely, since the various expressions agree by Lemma 3.6, we can use (3.9) – (3.16) to extend the definition of the holomorphic and antiholomorphic fermions $\psi, \psi^*$ from horizontal edges (3.6) to the vertical edges at heights $y', y' \in \mathbb{Z} + \frac{1}{2}$, by setting

$$\psi(x + iy') = (3.9) = (3.13), \quad \psi^*(x + iy') = (3.10) = (3.14), \quad \text{if } x = a$$

$$\psi(x + iy') = (3.11) = (3.15), \quad \psi^*(x + iy') = (3.12) = (3.16), \quad \text{if } x = b.$$

It is straightforward to verify that the equalities of the various expressions above are equivalent to the discrete complex analysis properties (3.7) and (3.8). □

3.3. Clifford algebra valued 1-forms in the strip. We next define Clifford generator valued discrete 1-forms and their discrete line integrals. The crucial observation is that such forms are closed when their coefficient functions satisfy a property closely related to $s$-holomorphicity, and they have vanishing integrals along vertical boundaries when the coefficient functions satisfy a property closely related to Riemann boundary values.

**Clifford algebra generator valued 1-forms.** By a Clifford generator valued discrete 1-form on the lattice strip $\mathbb{S}^{(\ell)}$ we mean a formal expression

$$\mathbf{m}(z) \psi(z) d^2z + \mathbf{m}^*(z) \psi^*(z) d^2\bar{z},$$

where $\mathbf{m}, \mathbf{m}^*: E(\mathbb{S}^{(\ell)}) \to \mathbb{C}$ are two complex-valued functions on the edges of the strip. A discrete contour on $\mathbb{S}^{(\ell)}$ is an ordered finite sequence $\gamma = (w_0, w_1, \ldots, w_m) \in V(\mathbb{S}^{(\ell)})$ such that $z_j = \{w_{j-1}, w_j\} \in E(\mathbb{S}^{(\ell)})$ for all $j = 1, \ldots, m$. We define the discrete contour integral along $\gamma$ of a Clifford generator valued 1-form as

$$\int_{\gamma} \left( \mathbf{m}(z) \psi(z) d^2z + \mathbf{m}^*(z) \psi^*(z) d^2\bar{z} \right)$$

$$:= \sum_{j=1}^{m} \left( \mathbf{m}(z_j) \psi(z_j) (w_j - w_{j-1}) + \mathbf{m}^*(z_j) \psi^*(z_j) (\bar{w}_j - \bar{w}_{j-1}) \right) \in \text{CliffGen.}$$

The integral and differential notations $\int^2$ and $d^2$ are meant to emphasize that integration is done on the square grid. Cases where integrals are taken along discrete contours that start and end at the same vertex are furthermore highlighted using the notation $\hat{\int}^2$. 
Closed 1-forms. If \( w_1, w_2, w_3, w_4 \in V(\mathcal{S}^\ell) \) are the vertices of a square face of \( \mathcal{S}^\ell \) in counterclockwise order, then the discrete contour \( \gamma_\square = (w_0, w_1, w_2, w_3, w_4) \) with \( w_0 = w_4 \) is said to be a counterclockwise oriented plaquette of \( \mathcal{S}^\ell \); see Figure 3.2. We say that a Clifford generator valued discrete 1-form \( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \) is closed if for all counterclockwise oriented plaquettes \( \gamma_\square \) of \( \mathcal{S}^\ell \) we have

\[
\oint_{\gamma_\square} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right) = 0.
\]

It turns out that a simple discrete complex analysis property of the coefficient functions ensures the closedness of the one-form. We say that a pair \( (m, m^*) \) of functions \( E(\mathcal{S}^\ell) \to \mathbb{C} \) is **imaginary complexified s-holomorphic (ICSH)**, if whenever \( z_1, z_2 \in E \) are edges adjacent to a vertex \( v \) and a face \( p \), we have

\[
m(z_1) - \frac{i |v - p|}{v - p} m^*(z_1) = m(z_2) - \frac{i |v - p|}{v - p} m^*(z_2).
\]

The terminology is explained by the following.

**Lemma 3.7.** If \( F : E(\mathcal{S}^\ell) \to \mathbb{C} \) is s-holomorphic and we set

\[
m = iF \quad \text{and} \quad m^* = -iF,
\]

then \( (m, m^*) \) is ICSH.

**Proof.** This follows directly from the definition (2.2) of s-holomorphicity. \(\square\)

The relevance of ICSH pairs stems from the following result, whose proof essentially exploits the same algebraic relations as the quintessential trick with s-holomorphic functions: the “(well-definedness of the) imaginary part of the integral of the square” [ChSm12].

**Proposition 3.8.** If \( (m, m^*) \) is ICSH, then the Clifford generator valued discrete 1-form \( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \) is closed.

**Proof.** Consider a plaquette centered at face \( p \), with vertices \( v_1, v_2, v_3, v_4 = v_0 \) in counterclockwise orientation. For \( j = 1, \ldots, 4 \), let \( z_j = \frac{v_{j-1} + v_j}{2} \) denote the edge from \( v_{j-1} \) to \( v_j \), and use again the cyclic interpretation \( z_0 = z_4 \). The integral around the plaquette is

\[
\oint_{\gamma_\square} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right)
\]

\[
= \sum_{j=1}^{4} \left( m(z_j) \psi(z_j) (v_j - v_{j-1}) + m^*(z_j) \psi^*(z_j) (v_j - v_{j-1}) \right)
\]

\[
= \sum_{j=0}^{3} \left( (v_j - p) \left( m(z_j) \psi(z_j) - m(z_{j+1}) \psi(z_{j+1}) \right) + (v_j - p) \left( m^*(z_j) \psi^*(z_j) - m^*(z_{j+1}) \psi^*(z_{j+1}) \right) \right).
\]
Regarding the first of the two terms in the sum, note that we can write
\[
\mathbf{m}(z_j)\psi(z_j) - \mathbf{m}(z_{j+1})\psi(z_{j+1})
\]
\[
= \frac{1}{2} \left( \left( \mathbf{m}(z_j) - \mathbf{m}(z_{j+1}) \right) (\psi(z_j) + \psi(z_{j+1})) + \left( \mathbf{m}(z_j) + \mathbf{m}(z_{j+1}) \right) (\psi(z_j) - \psi(z_{j+1})) \right),
\]
where the last expression indicates how the assumed ICSH equation (3.18) for the coefficient functions and the CSH equation (3.7) for the fermions are used next. Namely, with this, we simplify one contribution to the integral around the plaquette,
\[
\sum_{j=0}^{3} (v_j - p) \left( \mathbf{m}(z_j)\psi(z_j) - \mathbf{m}(z_{j+1})\psi(z_{j+1}) \right)
\]
\[
= \frac{\hat{\mathbf{i}}}{2\sqrt{2}} \sum_{j=0}^{3} \left( \mathbf{m}^*(z_j)\psi(z_j) + \mathbf{m}^*(z_{j+1})\psi(z_{j+1}) - \mathbf{m}^*(z_j)\psi(z_j) - \mathbf{m}^*(z_{j+1})\psi(z_{j+1}) \right)
\]
\[
= \frac{\hat{\mathbf{i}}}{2\sqrt{2}} \sum_{j=0}^{3} \left( \mathbf{m}^*(z_j)\psi(z_j) + \mathbf{m}^*(z_{j+1})\psi(z_{j+1}) - \mathbf{m}^*(z_j)\psi(z_j) - \mathbf{m}^*(z_{j+1})\psi(z_{j+1}) \right),
\]
where in the last step we observed telescopic cancellations.

By similar calculations the other contribution to the integral can be written as
\[
\sum_{j=0}^{3} (v_j - p) \left( \mathbf{m}^*(z_j)\psi^*(z_j) - \mathbf{m}^*(z_{j+1})\psi^*(z_{j+1}) \right)
\]
\[
= \frac{\hat{\mathbf{i}}}{2\sqrt{2}} \sum_{j=0}^{3} \left( -\mathbf{m}^*(z_j)\psi(z_{j+1}) + \mathbf{m}^*(z_{j+1})\psi(z_j) - \mathbf{m}(z_j)\psi^*(z_{j+1}) + \mathbf{m}(z_{j+1})\psi^*(z_j) \right).
\]
We then see that the two contributions cancel, yielding the desired conclusion
\[
\oint_{\gamma_0} \left( \mathbf{m}(z)\psi(z)\,d^2z + \mathbf{m}^*(z)\psi^*(z)\,d^2\bar{z} \right) = 0.
\]

**Vertically slidable 1-forms.** A one-step discrete contour \( \gamma_{\parallel} = (w_0, w_1) \) is said to be an oriented vertical boundary edge of \( S^{(\ell)} \), if \( w_0, w_1 \in V(S^{(\ell)}) \) are such that \( \Re(w_0) = \Re(w_1) \in \{a, b\} \) and \( |\Im(m(w_0) - \Im(m(w_1))| = 1 \); see Figure 3.3. We say that a Clifford generator valued
Figure 3.3. A vertical boundary edge.

discrete 1-form \( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \) is **vertically slidable** if for all oriented vertical boundary edges \( \gamma \) of \( S^{(t)} \) we have

\[
(3.19) \quad \int_{\gamma} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right) = 0.
\]

If \( (m, m^*) \) is a pair of functions \( E(S^{(t)}) \to \mathbb{C} \), such that for all vertical edges \( L = a + iy' \) of the left boundary and all vertical edges \( R = b + iy' \) of the right boundary we have

\[
(3.20) \quad m(L) - i m^*(L) = 0 \quad \quad m(R) + i m^*(R) = 0,
\]

then we say that the pair \( (m, m^*) \) has **imaginary complexified Riemann boundary values (ICRBV)**.

The terminology is explained by the following.

**Lemma 3.9.** If \( F : E(S^{(t)}) \to \mathbb{C} \) has Riemann boundary values and we set \( m = i F \) and \( m^* = -i \overline{F} \), then \( (m, m^*) \) has ICRBV.

**Proof.** This follows directly from the definition (2.3) of Riemann boundary values. \( \square \)

The relevance of pairs having ICRBV stems from the following.

**Proposition 3.10.** If \( (m, m^*) \) has ICRBV, then the Clifford generator valued discrete 1-form \( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \) is vertically slidable.

**Proof.** Consider, e.g., an edge \( L = a + iy' \) on the left boundary. For the upwards oriented vertical edge \( \gamma = (L - \frac{i}{2}, L + \frac{i}{2}) \), we get directly from the ICRBV equation \( m(L) - i m^*(L) = 0 \) and the property \( \psi(L) + i \psi^*(L) = 0 \) of the fermions that

\[
\int_{\gamma} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right) = i m(L) \psi(L) - i m^*(L) \psi^*(L)
\]

\[
= i m(L) \psi(L) - i m(L) \psi(L) = 0.
\]

The case of edges on the right boundary is similar. \( \square \)
Integrals of one-forms across the strip. Denote by $\gamma_y$ the discrete contour crossing the strip from left to right at height $y \in \mathbb{Z}$,

$$\gamma_y := (a + iy, (a + 1) + iy, \ldots, (b - 1) + iy, b + iy)$$

as in Figure 3.4. When $(m, m^*)$ is a pair of functions $E(S^{(\ell)}) \to \mathbb{C}$, we consider in particular the integral across the strip at zero height

$$\int_{\gamma_0}^z \left( m(z) \psi(z) d^2 z + m^*(z) \psi^*(z) d^2 \bar{z} \right) \in \text{CliffGen}. \quad (3.21)$$

The Hilbert space adjoint of such an operator is an operator of the same form, given explicitly below.

**Proposition 3.11.** The Hilbert space adjoint of

$$\phi = \int_{\gamma_0}^z \left( m(z) \psi(z) d^2 z + m^*(z) \psi^*(z) d^2 \bar{z} \right)$$

is

$$\phi^\dagger = \int_{\gamma_0}^z \left( -\overline{m(z)} \psi(z) d^2 z + \overline{m^*(z)} \psi^*(z) d^2 \bar{z} \right)$$

**Proof.** On the contour $\gamma_0$ at height zero, i.e., for $z \in \mathcal{I}^*$, we have the anti-self-adjointness $(\psi(z))^\dagger = -\psi(z)$ of the holomorphic fermions and the self-adjointness $(\psi^*(z))^\dagger = \psi^*(z)$ of the antiholomorphic fermions according to Lemma 3.1. The assertion follows directly from these and conjugate linearity of the Hilbert space adjoint and the fact that the steps of the contour $\gamma_0$ are horizontal (so $d^2 z, d^2 \bar{z}$ are real). \qed

The anticommutators of such operators are scalar multiples of the identity, where the scalar is obtained as a discrete integral of products of the coefficient functions.
Proposition 3.12. Let
\[ \phi_1 = \int_{\gamma_0}^{\gamma_1} \left( m_1(z) \psi(z) \, dz + m_1^*(z) \psi^*(z) \, d\bar{z} \right) \]
\[ \phi_2 = \int_{\gamma_0}^{\gamma_1} \left( m_2(z) \psi(z) \, dz + m_2^*(z) \psi^*(z) \, d\bar{z} \right). \]

The anticommutator of these operators is
\[ [\phi_1, \phi_2]^+ = \left( \int_{\gamma_0}^{\gamma_1} \left( -2 m_1(z) m_2(z) \, dz + 2 m_1^*(z) m_2^*(z) \, d\bar{z} \right) \right) \text{id}. \]

Proof. Lemma 3.2 gives the anticommutators among the holomorphic fermions \( \psi(z) \) and the antiholomorphic fermions \( \psi^*(z) \),
\[ [\psi(z_1), \psi(z_2)]^+ = -2 \delta_{z_1, z_2} \text{id}, \quad [\psi^*(z_1), \psi^*(z_2)]^+ = +2 \delta_{z_1, z_2} \text{id}. \]
The assertion follows directly from these and the bilinearity of the anticommutators. \( \square \)

The above integrals were taken at zero height, \( y = 0 \). However, under the assumptions on the coefficient functions \( m, m^* \) discussed in Section 3.3 the integral across the strip at any other height \( y \in \mathbb{Z} \) also yields the same result.

Proposition 3.13. Suppose that the pair \((m, m^*)\) of coefficient functions in the strip \( S^{(\ell)} \) is ICSH and has ICRBV. Then for any \( y \in \mathbb{Z} \) we have
\[ \int_{\gamma_y}^{\gamma_y} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right) = \int_{\gamma_0}^{\gamma_1} \left( m(z) \psi(z) \, dz + m^*(z) \psi^*(z) \, d\bar{z} \right). \]

Proof. Using discrete contour deformation, the difference of the two integrals in the asserted equality can be written as a sum of integrals around plaquettes between heights 0 and \( y \) and of integrals along vertical boundary edges between these heights. According to Proposition 3.8 the former vanish under the assumption of ICSH, and according to Proposition 3.10 the latter vanish under the assumption of ICRBV. \( \square \)

Remark 3.14. A convenient way to use the integrals across the strip, and phrase the properties anticommutator and Hilbert space adjoint properties is the following. Given an s-holomorphic function \( F : \mathbb{E}(S^{(\ell)}) \to \mathbb{C} \) with Riemann boundary values, note that the pair \( m = \frac{1}{2} e^{i\pi} F, \ m^* = \frac{1}{2} e^{-i\pi} F \) is ICSH and has ICRBV — by virtue of Lemmas 3.7 and 3.9 and the complex linearity of these conditions. The integral (3.21) across the strip with these coefficient functions only depends on the restriction \( f = F|_{T^*} \in \mathcal{F}^{(\ell)} \) of \( F \) to the cross-section \( T^* \), so let us denote it by
\[ \phi(f) := \frac{e^{i\pi}}{2} \int_{\gamma_0}^{\gamma_1} \left( \hat{z} f(z) \psi(z) \, dz - \hat{\bar{z}} f(z) \psi^*(z) \, d\bar{z} \right). \]

The Hilbert space adjoint calculated in Proposition 3.11 is then easily seen to yield the function similarly associated with the function \( Rf = -\hat{\bar{z}} f \) reflected using (2.8), i.e.,
\[ \phi(f)^\dagger = \phi(Rf). \]
Given two such functions with respective restrictions \( f, g \in \mathcal{F}(\ell) \), the anticommutator calculated in Proposition 3.12 is correspondingly simplified to the form

\[
[\phi(f), \phi(g)]_+ = \langle Rf, g \rangle \text{id}
\]

which involves the inner product (2.7) and the reflection (2.8).

3.4. The formalism in the slit-strip. We next turn to the slit-strip \( S^{\ell}_{\text{slit}} \) of Figure 1(b) and find analogous results. The statements and their proofs are mostly almost identical to the preceding sections, so we mainly content ourselves with indicating what needs to change.

First of all, in the slit-strip the Ising model interactions are different above and below the cross-section at height zero, and we need separate transfer matrices for the strip-like top half, and the bottom half with the slit.

Minor differences arise since the slit-strip \( S^{\ell}_{\text{slit}} \) is now considered as a multi-graph: it has double edges between adjacent vertices on the slit — one edge interpreted as the left side of the slit and the other as the right side, as in Figure 3.5(a). Plaquettes adjacent to the slit must be defined so that they are using the correct one of these edges, and contours must not traverse the slit. To keep the presentation palatable, we trust such conventions to be evident and we furthermore abuse the notation by “identifying” also the edges on the slit with their embedded positions in the complex plane \( \mathbb{C} \) (although two different edges are thus “identified” with the same point).

In the context of the slit-strip, we moreover consider one-forms which are only required to be locally defined. This could of course have been done for the strip already, but it really starts to play an essential role when we introduce the creation and annihilation operators associated to the three extremities of the slit-strip. Specifically, we will typically define the coefficient functions of the one-forms on one of the three subgraphs in (2.1): the top half \( S^{\ell}_{\text{slit}} \), the left leg \( S^{L;\ell}_{\text{slit}} \), or the right leg \( S^{R;\ell}_{\text{slit}} \). If we insisted on coefficient functions defined globally on the whole slit-strip \( S^{\ell}_{\text{slit}} \), then functions relevant to the diagonalization of the transfer matrix would fail to possess scaling limits. What we do instead can be interpreted as using naturally chosen local coordinates around each of the extremities, while certain key calculations then require changing bases to globally defined functions. This is ultimately how geometric notions enter our algebraic calculations.

The slit-strip transfer matrix. In the lattice slit-strip \( S^{\ell}_{\text{slit}} \), the local interactions of the Ising model above the vertical position \( y = 0 \) are just like in the strip. We correspondingly use the same transfer matrix (3.5) for this part. To clearly distinguish it from the one used in the slit part, we now denote it by \( T^{[\ell]} := T \).

Below the vertical position \( y = 0 \), we have to take into account locally monochromatic boundary conditions on the slit, and we therefore use a different transfer matrix \( T^{[L;\ell R]} \). It is again a \( \mathbb{C} \)-linear map

\[
T^{[L;\ell R]} : \widehat{V} \to \widehat{V},
\]

constructed out of two constituent matrices. The diagonal matrix \( T^{1/2}_{\text{hot}} : \widehat{V} \to \widehat{V} \) accounting for interactions on horizontal edges (with weight half) remains exactly the same as before.
The symmetric matrix $T_{\text{ver}}^{\text{slit}} : \tilde{V} \to \tilde{V}$ still accounts for interactions on vertical edges and locally monochromatic boundary conditions on the left and right boundaries, and now additionally for locally monochromatic boundary conditions on the slit. It is defined by the matrix elements

$$u^\dagger_{\tau} T_{\text{ver}}^{\text{slit}} u_{\rho} = \exp\left(\beta \sum_{x=a}^{b} \rho_x \tau_x\right) \delta_{\tau_x \rho_x} \delta_{\tau_0 \rho_0} \delta_{\tau_b \rho_b}.$$  \hfill (3.25)

The transfer matrix $T^{[\ell_L,\ell_R]}$ for the slit part is then defined as

$$T^{[\ell_L,\ell_R]} = T_{\text{hor}}^{1/2} T_{\text{ver}}^{\text{slit}} T_{\text{hor}}^{1/2}. \hfill (3.26)$$

**Holomorphic and antiholomorphic fermions in the slit-strip.** Analogously to Section \ref{sect:spin}, we define two Clifford algebra generator valued functions on the edges of the discrete slit-strip $S_{\text{slit}}^{(t)}$. We use a hat in the notation to distinguish them from the earlier introduced operators in the discrete strip.

Again the horizontal edges at height zero are naturally identified with points $x' \in I^*$ of the dual cross-section, and the horizontal edges at height $y \in \mathbb{Z}$ are identified with points $x' + i\, y$ with $x' \in I^*$. As with the spin operators in the slit-strip \ref{sect:spin}, there is now a difference between the top half of the strip, $y \geq 0$, and the bottom half, $y \leq 0$ (zero height could be thought of as belonging to either the top or bottom). Let therefore $x' \in I^*$, and $y \in \mathbb{Z}_{\geq 0}$.

For horizontal edges in the top half we set

$$\hat{\psi}(x' + i\, y) = (T^{[\ell]}_{\ell_L,\ell_R})^{-y} \psi_{x'} (T^{[\ell]}_{\ell_L,\ell_R})^{y}, \quad \hat{\psi}^\ast(x' + i\, y) = (T^{[\ell]}_{\ell_L,\ell_R})^{-y} \psi_{x'}^\ast (T^{[\ell]}_{\ell_L,\ell_R})^{y}, \hfill (3.27)$$

and for horizontal edges in the bottom half we set

$$\hat{\psi}(x' - i\, y) = (T^{[\ell_L,\ell_R]}_{\ell_L,\ell_R})^{y} \psi_{x'} (T^{[\ell_L,\ell_R]}_{\ell_L,\ell_R})^{-y}, \quad \hat{\psi}^\ast(x' - i\, y) = (T^{[\ell_L,\ell_R]}_{\ell_L,\ell_R})^{y} \psi_{x'}^\ast (T^{[\ell_L,\ell_R]}_{\ell_L,\ell_R})^{-y}. \hfill (3.28)$$

From these values of the fermions on horizontal edges, the principle of extending to vertical edges is to require the following discrete complex analysis properties.
Proposition 3.15. There exists unique extensions
\[ \hat{\psi}, \hat{\psi}^*: \text{E}(\mathcal{S}_{\text{slit}}^{(l)}) \to \text{CliffGen} \]
of (3.27) – (3.28) to vertical edges with the following properties:
The pair \( (\hat{\psi}, \hat{\psi}^*) \) is complexified s-holomorphic (CSH) in the sense that for any edges \( z_1, z_2 \in \text{E}(\mathcal{S}_{\text{slit}}^{(l)}) \) adjacent to a vertex \( v \) and a face \( p \) we have
\[
\hat{\psi}(z_1) + \frac{i |v - p|}{v - p} \hat{\psi}^*(z_1) = \hat{\psi}(z_2) + \frac{i |v - p|}{v - p} \hat{\psi}^*(z_2),
\]
and it has complexified Riemann boundary values (CRBV) in the sense that
\[
\hat{\psi}(L) + i \hat{\psi}^*(L) = 0
\]
and
\[
\hat{\psi}(R) - i \hat{\psi}^*(R) = 0.
\]
for any left boundary edge \( L \) (including edges on the slit which are the left boundary of the right substrip) and any right boundary edge \( R \) (including edges on the slit which are the right boundary of the left substrip).

The proof is similar to that of Proposition 3.5: formulas analogous to (3.9) – (3.16) (with just \( \hat{\psi}, \hat{\psi}^* \) in place of \( \psi, \psi^* \)) are used to extend the definition of the holomorphic and antiholomorphic fermions \( \hat{\psi}, \hat{\psi}^* \) from horizontal edges to vertical edges.

Clifford algebra valued 1-forms in the slit-strip. As in Section 3.3 we define Clifford generator valued discrete 1-forms and their discrete line integrals.

By a Clifford generator valued discrete 1-form in the slit-strip \( \mathcal{S}_{\text{slit}}^{(l)} \), we again mean a formal expression
\[
\mathfrak{m}(z) \hat{\psi}(z) d\overline{z} z + \mathfrak{m}^*(z) \hat{\psi}^*(z) d\overline{z},
\]
where \( \mathfrak{m}, \mathfrak{m}^*: \text{E} \to \mathbb{C} \) are two complex valued functions on a set of edges \( \text{E} \). For globally defined 1-forms we take \( \text{E} = \text{E}(\mathcal{S}_{\text{slit}}^{(l)}) \), and for locally defined 1-forms we take \( \text{E} = \text{E}(\mathcal{S}_{\text{slit}}^{T(l)}), \) \( \text{E} = \text{E}(\mathcal{S}_{\text{slit}}^{R(l)}), \) or \( \text{E} = \text{E}(\mathcal{S}_{\text{slit}}^{L(l)}) \).

For a discrete contour \( \gamma \) in the slit-strip \( \mathcal{S}_{\text{slit}}^{(l)} \) (or in one of the subgraphs \( \mathcal{S}_{\text{slit}}^{T(l)}, \mathcal{S}_{\text{slit}}^{L(l)}, \mathcal{S}_{\text{slit}}^{R(l)} \)), the integral
\[
\int_{\gamma} \left( \mathfrak{m}(z) \hat{\psi}(z) d\overline{z} z + \mathfrak{m}^*(z) \hat{\psi}^*(z) d\overline{z} \right) \in \text{CliffGen}
\]
is defined by exactly the same formula as in Section 3.3 but now instead using the slit-strip fermions \( \hat{\psi}(z) \) and \( \hat{\psi}^*(z) \) defined above.

The two crucial notions of such Clifford generator valued 1-forms are defined basically as before. The 1-form is said to be **closed** if for all counterclockwise oriented plaquettes \( \gamma_\Box \) of \( \mathcal{S}_{\text{slit}}^{(l)} \) (or of \( \mathcal{S}_{\text{slit}}^{T(l)}, \mathcal{S}_{\text{slit}}^{L(l)}, \mathcal{S}_{\text{slit}}^{R(l)} \)), its integral vanishes as in (3.17). The 1-form is said to be **vertically slidable** if for all oriented vertical boundary edges \( \gamma_\mid \) of \( \mathcal{S}_{\text{slit}}^{(l)} \) (or of \( \mathcal{S}_{\text{slit}}^{T(l)}, \mathcal{S}_{\text{slit}}^{L(l)}, \mathcal{S}_{\text{slit}}^{R(l)} \)) its integral vanishes as in (3.19).

Also the two notions of coefficient functions for the 1-forms are essentially as before. A pair \( (\mathfrak{m}, \mathfrak{m}^*) \) of complex-valued functions on the edges of \( \mathcal{S}_{\text{slit}}^{(l)}, \mathcal{S}_{\text{slit}}^{T(l)}, \mathcal{S}_{\text{slit}}^{L(l)}, \mathcal{S}_{\text{slit}}^{R(l)} \), is said to be **ICSH** (in the slit-strip or a subgraph) if Equations (3.18) hold for each plaquette...
the slit-strip, from left to middle and from middle to right,

Integrals of one-forms across the slit-strip.

in the subgraph

the same contours

form

m

Proposition 3.19.

If

m

Proposition 3.18.

If

Also the implications of ICSH and ICRBV are as in Propositions 3.8 and 3.10.

Lemma 3.17.

If

Lemma 3.16.

The reasons for the terminology are just as in Lemmas 3.7 and 3.9.

Similarly

or a subgraph) if Equations (3.20) hold for all left boundary vertical edges L and all right boundary vertical edges R (of the corresponding graph).

The Hilbert space adjoints work out exactly as in Proposition 3.11 and the anticommutators as in Proposition 3.12; the proofs go through verbatim.

Under the assumptions of ICSH and ICRBV for coefficient functions, the independence of the integrals on the chosen height \( y \) can again be derived by the closedness and vertical slidability. The precise statements are as follows.

Proposition 3.20. Let \( \ast \) stand for either T, L or R. Suppose that the pair \( (m, m^\ast) \) of complex-valued coefficient functions on the edges of the subgraph \( S^\ast_{\triangleleft}(\ell) \) is ICSH and has ICRBV. Then we have

\[
\int_{\gamma_0^T} (m(z) \hat{\psi}(z) d^2 z + m^\ast(z) \hat{\psi}^\ast(z) d\bar{z}) = \int_{\gamma_0^L} (m(z) \hat{\psi}(z) d^2 z + m^\ast(z) \hat{\psi}^\ast(z) d\bar{z}) = \int_{\gamma_0^R} (m(z) \hat{\psi}(z) d^2 z + m^\ast(z) \hat{\psi}^\ast(z) d\bar{z})
\]
for any \( y \in \mathbb{Z}_{\geq 0} \) if \( \star = T \) and for any \( y \in \mathbb{Z}_{\leq 0} \) if \( \star = L, R \).

The proof is as in Proposition 3.13

4. Distinguished functions on the lattice

In this section we discuss certain distinguished s-holomorphic functions in the lattice strip and slit-strip, constructed in [AKPR20]. Clifford algebra valued discrete 1-forms of Section 3 with these distinguished functions as the coefficient functions will be our main tools for the analysis of the Ising model in Section 5.

Indexing sets. Certain indexing sets of half-integers will be used throughout the rest of this article, as well as the subsequent one [KPR21].

Let

\[
\mathcal{K} := [0, \infty) \cap \left( \mathbb{Z} + \frac{1}{2} \right) = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\}
\]

be the set of all positive half-integers, and

\[
\pm \mathcal{K} := \mathcal{K} \cup (-\mathcal{K}) = \mathbb{Z} + \frac{1}{2}
\]

the set of all half-integers.

In the discrete setting, the width parameter \( \ell \) serves as a truncation, so we denote

\[
\mathcal{K}^{(\ell)} := [0, \ell) \cap \left( \mathbb{Z} + \frac{1}{2} \right) = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \ell - \frac{1}{2} \right\}
\]

and

\[
\pm \mathcal{K}^{(\ell)} := \mathcal{K}^{(\ell)} \cup (-\mathcal{K}^{(\ell)}).
\]

We also sometimes use the substrip widths \( \ell_R = b \) and \( \ell_L = -a \) in place of the width \( \ell \). Moreover, the set \( \{ \alpha \subset \mathcal{K}^{(\ell)} \} \) of all subsets of \( \mathcal{K}^{(\ell)} \) will be our indexing set for the transfer matrix eigenvectors (see Section 5.6), and the sets \( \{ \alpha_L \subset \mathcal{K}^{(\ell_R)} \} \) and \( \{ \alpha_R \subset \mathcal{K}^{(\ell_L)} \} \) will play an analogous role for the transfer matrix of the slit part (see Section 5.8).

4.1. Distinguished functions in the strip. The following distinguished functions in the lattice strip were introduced in [AKPR20].

For any \( k \in \mathcal{K}^{(\ell)} \), according to [AKPR20] Lemma 3.6, there exists a unique

\[
\omega_k^{(\ell)} \in \left( \left( k - \frac{1}{2} \right) \pi / \ell, k \pi / \ell \right)
\]

satisfying

\[
\frac{\cos \left( (\ell + \frac{1}{2}) \omega_k^{(\ell)} \right)}{\cos \left( (\ell - \frac{1}{2}) \omega_k^{(\ell)} \right)} = 3 - 2 \sqrt{2}.
\]

In terms of it, we define

\[
\Lambda_{\pm_k}^{(\ell)} := \left( 2 - \cos(\omega_k^{(\ell)}) + \sqrt{\left( 3 - \cos(\omega_k^{(\ell)}) \right) \left( 1 - \cos(\omega_k^{(\ell)}) \right)} \right) ^{\pm 1}.
\]

Note that \( \Lambda_k^{(\ell)} > 1 \) and \( \Lambda_{-k}^{(\ell)} = 1 / \Lambda_k^{(\ell)} < 1 \) for \( k \in \mathcal{K}^{(\ell)} \).
Proposition 4.1 ([AKPR20, Proposition 3.7]). For \( k \in \mathcal{K}(\ell) \), there exists unique functions \( \mathfrak{F}_k, \mathfrak{F}_{-k} : E(S^{(\ell)}) \to \mathbb{C} \) which are \( s \)-holomorphic in the lattice strip \( S^{(\ell)} \), have Riemann boundary values on the left and right boundaries, have the vertical translation eigenfunction property

\[
\mathfrak{F}_{\pm k}(z + i h) = (\Lambda_{\pm k}^{(\ell)})^h \mathfrak{F}_{\pm k}(z)
\]

for all \( z \in E(S^{(\ell)}) \) and \( h \in \mathbb{Z} \), and satisfy the following normalization conditions: the arguments on the left boundary are \( \mathfrak{F}_{\pm k}(a + i y') \in e^{-i\pi/4} \mathbb{R}^+ \), for \( y' \in \mathbb{Z} + \frac{1}{2} \), and the restrictions

\[
\mathfrak{f}_{\pm k} := \mathfrak{F}_{\pm k} |_{I^*} \in \mathcal{F}^{(\ell)}
\]

to the cross-section \( I^* \) have unit norms, \( \|\mathfrak{f}_{\pm k}\| = 1 \).

The following reflection relations hold for these functions:

\[
(4.6) \quad \mathfrak{f}_{-k}(x') = -i \mathfrak{F}_k(x'), \quad \mathfrak{F}_{-k}(x + i y) = -i \mathfrak{F}_k(x - i y).
\]

Moreover, the collection \( (\mathfrak{f}_k)_{k \in \pm \mathcal{K}(\ell)} \) of functions forms an orthonormal basis of the real Hilbert space \( \mathcal{F}^{(\ell)} = \mathbb{C}^{|I^*|} \) of (2.6).

Decomposition to poles and zeroes at the top. The functions \( \mathfrak{f}_{+k} \), for \( k \in \mathcal{K}(\ell) \), are exponentially growing in the upwards direction of the lattice strip \( S^{(\ell)} \), since \( \Lambda_{+k}^{(\ell)} > 1 \), while the functions \( \mathfrak{f}_{-k} \) are exponentially decaying. We interpret the growing ones as having a pole at the top extremity, and the decaying ones as having a zero.

We decompose the discrete function space \( \mathcal{F}^{(\ell)} \) into the corresponding subspaces. More precisely, we define

\[
(4.7) \quad \mathcal{F}_{T;\text{pole}}^{(\ell)} = \text{span}\left\{ \mathfrak{f}_{+k} \mid k \in \mathcal{K}(\ell) \right\}, \quad \mathcal{F}_{T;\text{zero}}^{(\ell)} = \text{span}\left\{ \mathfrak{f}_{-k} \mid k \in \mathcal{K}(\ell) \right\},
\]

and we denote by

\[
\Pi_{T;\text{pole}}^{(\ell)} : \mathcal{F}^{(\ell)} \to \mathcal{F}_{T;\text{pole}}^{(\ell)}, \quad \Pi_{T;\text{zero}}^{(\ell)} : \mathcal{F}^{(\ell)} \to \mathcal{F}_{T;\text{zero}}^{(\ell)},
\]

the orthogonal projections onto these subspaces.

4.2. Distinguished functions in the slit-strip. In the slit-strip there are three infinite extremities: the top, the right leg, and the left leg. In the left and right substrips there are natural functions defined analogously to the whole strip, which allow us to define poles and zeroes, and obtain corresponding decompositions. The distinguished functions in the slit-strip will then be globally defined functions which have zeroes (i.e., regular behavior) in two of the three infinite extremities, and have a pole of a given order in the third (i.e., a prescribed singular part).
Decomposition to poles and zeroes in the left and right legs. We can apply Proposition 4.1 in the left and right substrips (replacing one of \(a, b\) by 0) to obtain s-holomorphic functions

\[
\mathcal{F}_L;\pm k : E(S_{\text{slit}}^L) \to \mathbb{C} \quad \text{for} \quad k \in \mathcal{K}^{(\ell)}_L, \quad \mathcal{F}_R;\pm k : E(S_{\text{slit}}^R) \to \mathbb{C} \quad \text{for} \quad k \in \mathcal{K}^{(\ell)}_R,
\]

with Riemann boundary values in the corresponding substrips, and with vertical translation eigenvalues \(\Lambda^{(\ell)}_{\pm k}\) and \(\Lambda^{(\ell)}_{\pm k}\), respectively. Splitting the cross-section \(\mathcal{I}^* = [a, b]^*\) to two halves \(\mathcal{I}_L := [a, 0]^*\) and \(\mathcal{I}_R := [0, b]^*\), the discrete function space splits naturally to orthogonally complementary subspaces

\[
\mathcal{F}_L^{(\ell)} := \mathbb{C}\mathcal{I}_L \subset \mathcal{F}^{(\ell)}, \quad \mathcal{F}_R^{(\ell)} := \mathbb{C}\mathcal{I}_R \subset \mathcal{F}^{(\ell)},
\]

consisting of functions supported on one of the two halves of the cross-section. Note that for functions in \(\mathcal{F}_L^{(\ell)}, \mathcal{F}_R^{(\ell)}\), we use the norm and inner product defined exactly analogously to \((2.7)\), and these coincide with the norm and inner product they inherit as subspaces of \(\mathcal{F}^{(\ell)}\) when the functions on one half of the cross-section are extended as zero to the other half.

Applying Proposition 4.1 to the left and right substrips, one obtains functions

\[
\mathcal{f}_L;\pm k := \mathcal{F}_L;\pm k \big|_{\mathcal{I}_L^*} \in \mathcal{F}_L^{(\ell)}, \quad \mathcal{f}_R;\pm k := \mathcal{F}_R;\pm k \big|_{\mathcal{I}_R^*} \in \mathcal{F}_R^{(\ell)},
\]

which form orthonormal bases \((\mathcal{f}_L;k)_{k \in \pm \mathcal{K}^{(\ell)}_L}\) and \((\mathcal{f}_R;k)_{k \in \pm \mathcal{K}^{(\ell)}_R}\) of the subspaces \(\mathcal{F}_L^{(\ell)}\) and \(\mathcal{F}_R^{(\ell)}\), respectively. In terms of these, we define the subspaces

\[
(4.8) \quad \mathcal{F}_L^{(\ell)};\text{pole} = \text{span} \left\{ \mathcal{f}_L;\pm k \big| k \in \mathcal{K}^{(\ell)}_L \right\}, \quad \mathcal{F}_R^{(\ell)};\text{pole} = \text{span} \left\{ \mathcal{f}_R;\pm k \big| k \in \mathcal{K}^{(\ell)}_R \right\},
\]

\[
\mathcal{F}_L^{(\ell)};\text{zero} = \text{span} \left\{ \mathcal{f}_L;\pm k \big| k \in \mathcal{K}^{(\ell)}_L \right\}, \quad \mathcal{F}_R^{(\ell)};\text{zero} = \text{span} \left\{ \mathcal{f}_R;\pm k \big| k \in \mathcal{K}^{(\ell)}_R \right\},
\]

and we denote by

\[
\Pi^{(\ell)}_{L;\text{pole}} : \mathcal{F}^{(\ell)} \to \mathcal{F}_L^{(\ell)};\text{pole}, \quad \Pi^{(\ell)}_{R;\text{pole}} : \mathcal{F}^{(\ell)} \to \mathcal{F}_R^{(\ell)};\text{pole},
\]

\[
\Pi^{(\ell)}_{L;\text{zero}} : \mathcal{F}^{(\ell)} \to \mathcal{F}_L^{(\ell)};\text{zero}, \quad \Pi^{(\ell)}_{R;\text{zero}} : \mathcal{F}^{(\ell)} \to \mathcal{F}_R^{(\ell)};\text{zero},
\]

the orthogonal projections onto these subspaces.

Distinguished discrete functions in the slit-strip. For a function \(f \in \mathcal{F}^{(\ell)}\), we call the projections \(\Pi^{(\ell)}_{T;\text{pole}}(f), \Pi^{(\ell)}_{T;\text{pole}}(f), \Pi^{(\ell)}_{T;\text{pole}}(f)\) its singular parts in the top, right, and left, respectively. When a singular part vanishes, we say that the function admits a regular extension in the corresponding extremity. The distinguished functions are characterized by having only one non-vanishing singular part as follows.

\[\text{Note that poles and zeroes in the left and right legs are determined by the exponential growth or decay in the } \downarrow \text{ direction, and thus the signs of the indices are the opposite compared to } (4.7).\]
Proposition 4.2 ([AKPR20, Section 3.4]). For \(k_T \in \mathcal{K}^{(\ell)}, k_R \in \mathcal{K}^{(\ell_R)}, k_L \in \mathcal{K}^{(\ell_L)}\), there exists unique functions \(p_{T;k_T}, p_{R;k_R}, p_{L;k_L} \in \mathcal{F}(\ell)\) such that

\[
\begin{align*}
\Pi^{(\ell)}_{T;pole}(p_{T;k_T}) &= f_{+k_T}, \\
\Pi^{(\ell)}_{L;pole}(p_{T;k_T}) &= 0, \\
\Pi^{(\ell)}_{R;pole}(p_{T;k_T}) &= 0, \\
\Pi^{(\ell)}_{T;pole}(p_{L;k_L}) &= 0, \\
\Pi^{(\ell)}_{L;pole}(p_{L;k_L}) &= f_{L_i - k_L}, \\
\Pi^{(\ell)}_{R;pole}(p_{L;k_L}) &= 0, \\
\Pi^{(\ell)}_{T;pole}(p_{R;k_R}) &= 0, \\
\Pi^{(\ell)}_{L;pole}(p_{R;k_R}) &= 0, \\
\Pi^{(\ell)}_{R;pole}(p_{R;k_R}) &= f_{R_i - k_R}.
\end{align*}
\]

These functions are the restrictions to the cross-section \(\mathcal{I}^*\) of unique s-holomorphic functions

\[\mathcal{P}_{T,k_T}, \mathcal{P}_{L,k_L}, \mathcal{P}_{R,k_R} : E(\mathbb{S}^{(\ell)}_{\text{slit}}) \to \mathbb{C}\]

with Riemann boundary values.

Note that these discrete pole functions \(\mathcal{P}_{T,k_T}, \mathcal{P}_{L,k_L}, \mathcal{P}_{R,k_R}\) are defined globally on the whole lattice slit-strip \(\mathbb{S}_{\text{slit}}^{(\ell)}\).

5. ISING MODEL AND THE TRANSFER MATRICES

In this section we define the Ising model in the lattice strip and lattice slit-strip, and review the transfer matrix formalism to the calculation of correlation functions. We then show how the method of Clifford algebra valued 1-forms can be used first of all to diagonalize the transfer matrix in the strip \(\mathbb{S}^{(\ell)}\) and, more interestingly, to calculate certain renormalized boundary correlation functions in the slit strip \(\mathbb{S}_{\text{slit}}^{(\ell)}\).

The Ising model on a graph is a random assignment of \(\pm 1\) spins to the vertices of the graph. Configurations of spins with more alignment among the spins of neighboring vertices are given relatively higher probabilities, and the strength of this tendency of local alignment is controlled by a parameter \(\beta > 0\) interpreted as the inverse temperature. We always consider the Ising model with unit coupling constants and no external magnetic field. On a given graph, therefore, \(\beta\) is the only parameter in the model, and we moreover take it to be the critical value for the square lattice.

The definition of the Ising model probability measure is straightforward when the graph is finite. Almost the only subtlety to pay attention to is the choice of boundary conditions. Our choice will be locally monochromatic boundary conditions on vertical boundaries of the strip and slit-strip. The commonly used plus and minus boundary conditions can also be obtained straightforwardly from our results.

On infinite graphs such as the lattice strip \(\mathbb{S}^{(\ell)}\) and the lattice slit-strip \(\mathbb{S}_{\text{slit}}^{(\ell)}\), the construction of the Ising model probability measure requires approximating the infinite graph with a sequence of increasingly large finite subgraphs, and to consider the weak limit of the associated probability measures. The existence of such infinite volume limits is usually established using correlation inequalities (FKG-inequality in the case of plus boundary conditions, or Griffiths’ inequality in the case of free boundary conditions), but in our setup also follows easily from the transfer matrix formalism (Section 5.5).

Figure 1.1 illustrates samples of Ising spin configurations in a lattice strip and a lattice slit-strip, with locally monochromatic boundary conditions and at the critical inverse temperature. The two colors represent the two possible values \(\pm 1\) of the spins.
We take the point of view that the fundamental quantities about the Ising model are its correlation functions, i.e., expected values of suitable random variables of the spin configuration. For instance, the weak limit defining infinite volume Ising probability measure is itself formulated by means of such correlation functions. The key quantities featuring in our main result (and ultimately leading to vertex operator algebra structure constants) will be certain renormalized limits of boundary correlation functions, which we will call fusion coefficients.

5.1. *Ising model on finite graphs.* We begin by defining the Ising model on a general finite graph, and discussing boundary conditions. After the general definitions, we specialize to the cases where the finite graph is taken to be a truncated lattice strip or a truncated lattice slit-strip, and locally monochromatic boundary conditions on vertical boundary components are used.

*Ising model without boundary conditions.* Let \( G = (V(G), E(G)) \) be a graph with a finite set \( V(G) \) of vertices, and a (finite) set \( E(G) \) of edges. Let also the inverse temperature parameter \( \beta > 0 \) be fixed.

The sample space for the Ising model on \( G \) is the set of \( \pm 1 \)-valued configurations on the vertices, 
\[
\Omega = \{ \pm 1 \}^{V(G)}.
\]

For a spin configuration \( \sigma \in \Omega \), 
\[
\sigma = (\sigma_z)_{z \in V(G)}, \quad \text{with} \quad \sigma_z \in \{ \pm 1 \} \quad \text{for each} \quad z \in V(G),
\]
the energy (Hamiltonian) is defined as 
\[
\mathcal{H}_G(\sigma) = - \sum_{\{z,w\} \in E(G)} \sigma_z \sigma_w.
\]

The Ising model probability measure \( P_{\beta;G} \) on \( \Omega \) is then defined by setting the probabilities of spin configurations \( \sigma \in \{ \pm 1 \}^{V(G)} \) proportional to their Boltzmann weights, i.e.,
\[
(5.1) \quad P_{\beta;G}[\{\sigma\}] = \frac{1}{\mathcal{Z}_G(\beta)} e^{-\beta \mathcal{H}_G(\sigma)},
\]
where the partition function
\[
(5.2) \quad \mathcal{Z}_G(\beta) = \sum_{\sigma \in \{ \pm 1 \}^{V(G)}} e^{-\beta \mathcal{H}_G(\sigma)}
\]
normalizes the total mass of \( P_{\beta;G} \) to one.

*Imposing boundary conditions.* A priori, the Ising model probability measure \( P_{\beta;G} \) is defined according to (5.1). Imposing boundary conditions amounts to modifying the measure by appropriate conditioning.
Let a subset of vertices \( \partial G \subset V(G) \) of a finite graph be declared as boundary. The Ising model on \( G \) with plus boundary conditions is then the conditional probability measure, conditioned on the event

\[
\Omega^+_{\partial G} := \left\{ \sigma \in \Omega \mid \sigma|_{\partial G} \equiv +1 \right\}
\]

that all the spins on the boundary are +1. This conditional probability measure is denoted by \( P^+_{\beta;G}[\cdot] = P_{\beta;G}[\cdot \mid \Omega^+_{\partial G}] \), and it is explicitly characterized by the probabilities

\[
P^+_{\beta;G}[\{\sigma\}] = \frac{1}{Z^+_G(\beta)} e^{-\beta H_G(\sigma)} \quad \text{for} \ \sigma \in \Omega^+_{\partial G},
\]

where the partition function for plus boundary conditions is the sum

\[
Z^+_G(\beta) = \sum_{\sigma \in \Omega^+_{\partial G}} e^{-\beta H_G(\sigma)}
\]

which is obtained from (5.2) by keeping only the terms that correspond to the required boundary conditions.

Minus boundary conditions can be imposed similarly by conditioning on the event \( \Omega^-_{\partial G} \subset \Omega \) that the values of spins in the subset \( \partial G \subset V(G) \) (declared as boundary) are all \(-1\).

It is also possible to impose mixed boundary conditions, plus boundary conditions on one part of the boundary and minus boundary conditions on another part simultaneously. More precisely, this amounts to conditioning on the event \( \Omega^{+/-}_{\partial G,\partial -G} \subset \Omega \) that the values of spins in the subset \( \partial^+ G \subset V(G) \) are all +1 and those in a subset \( \partial^- G \subset V(G) \) are all \(-1\).

We mostly use the following locally monochromatic boundary conditions. Given finitely many disjoint subsets \( \partial^1 G, \ldots, \partial^n G \subset V(G) \), we condition on the event

\[
\Omega_{\partial^1 G, \ldots, \partial^n G}^{\text{mono}} = \left\{ \sigma \in \Omega \mid \sigma|_{\partial j G} \equiv \text{const. for each } j = 1, \ldots, n \right\}
\]

that the spins are constant on each of these. The conditioned probability measure is

\[
P^{\text{mono}}_{\beta;G}[\cdot] = P_{\beta;G}[\cdot \mid \Omega_{\partial^1 G, \ldots, \partial^n G}^{\text{mono}}],
\]

and it is explicitly characterized by the probabilities

\[
P^{\text{mono}}_{\beta;G}[\{\sigma\}] = \frac{1}{Z^{\text{mono}}_G(\beta)} e^{-\beta H_G(\sigma)} \quad \text{for} \ \sigma \in \Omega_{\partial^1 G, \ldots, \partial^n G}^{\text{mono}},
\]

where the partition function for plus boundary conditions is the sum

\[
Z^{\text{mono}}_G(\beta) = \sum_{\sigma \in \Omega_{\partial^1 G, \ldots, \partial^n G}^{\text{mono}}} e^{-\beta H_G(\sigma)}.
\]

Note that, by virtue of the tower property of conditioning, it is possible to recover any mixture of plus and minus boundary conditions on the segments \( \partial^1 G, \ldots, \partial^n G \) by further conditioning the locally monochromatic probability measure \( P^{\text{mono}}_{\beta;G} \) on the event of having the desired specific constant values on the segments. This at least partly justifies focusing on the case of locally monochromatic boundary conditions, as we will do.
5.2. **Ising model in the truncated strip and slit-strip.** The finite graphs of interest to us are truncations of the lattice strip $S^{(\ell)}$ and lattice slit-strip $S^{(\ell)}_{\text{slit}}$. We now consider the Ising model with locally monochromatic boundary conditions on these, and introduce in particular notation for the main results of the article.

All of the graphs we consider in what follows are subgraphs of the square lattice $\mathbb{Z}^2$. We will therefore from here on without explicit mention fix the inverse temperature to the critical value for the square lattice,

$$\beta = \beta_{\mathbb{Z}^2} = \frac{1}{2} \log(\sqrt{2} + 1),$$

and for simplicity we omit explicit references to $\beta$ from the notation.

The truncated strip. We consider the positions $a, b \in \mathbb{Z}$ of the left and right boundaries fixed throughout so that $a < 0 < b$, and we denote by $\ell = b - a \in \mathbb{N}$ the width of the strip. Let $h_T, h_B \in \mathbb{Z}_{>0}$ be given truncation heights for the top and bottom parts. The truncated
strip, illustrated in Figure 5.1(a) is defined as
\[
S^{(\ell,h_T,h_B)} := \lbrack a, b \rbrack \times \lbrack -h_B, h_T \rbrack,
\]
where \(\lbrack a, b \rbrack = \{a, a + 1, \ldots, b - 1, b \}\) and \(\lbrack -h_B, h_T \rbrack = \{-h_B, -h_B + 1, \ldots, h_T - 1, h_T \}\) are integer intervals. We view the truncated strip \(S^{(\ell,h_T,h_B)}\) as a subgraph \(S^{(\ell,h_T,h_B)} \subset S^{(\ell)}\) of the lattice strip (induced subgraph, also with nearest neighbor edges), and note that as the truncation heights increase, \(h_T, h_B \to \infty\), the subgraphs exhaust the whole lattice strip \(S^{(\ell)}\).

Boundary conditions will be imposed on the (disjoint) subsets
\[
(5.4) \quad \partial_L S^{(\ell,h_T,h_B)} = \{a\} \times \lbrack -h_B, h_T \rbrack \quad \quad \partial_R S^{(\ell,h_T,h_B)} = \{b\} \times \lbrack -h_B, h_T \rbrack
\]
(constituting the vertical boundaries of the truncated strip. In this setup, we denote the probability measure with locally monochromatic boundary conditions on the left and right sides simply by
\[
P^{(\ell,h_T,h_B)},
\]
and expected values with respect to it by \(E^{(\ell,h_T,h_B)}\).

The truncated slit-strip. Similarly, the truncated slit-strip \(S^{(\ell,h_T,h_B)}_{\text{slit}}\) of Figure 5.1(b) is taken to consist of the same vertices as \(S^{(\ell,h_T,h_B)}\), but in this case we furthermore interpret the subset
\[
\partial_{\text{slit}} S^{(\ell,h_T,h_B)} = \{0\} \times \lbrack -h_B, 0 \rbrack
\]
as a part of the boundary where we will impose boundary conditions. Again the truncated slit-strip \(S^{(\ell,h_T,h_B)}_{\text{slit}}\) is viewed as an induced subgraph \(S^{(\ell,h_T,h_B)}_{\text{slit}} \subset S^{(\ell)}_{\text{slit}}\) of the lattice slit-strip, and as the truncation heights increase, \(h_T, h_B \to \infty\), these subgraphs exhaust the whole lattice slit-strip \(S^{(\ell)}_{\text{slit}}\).

In this setup, we denote by
\[
P^{(\ell,h_T,h_B)}_{\text{slit}}
\]
the probability measure with locally monochromatic boundary conditions on the left and right sides as well as on the slit, and by \(E^{(\ell,h_T,h_B)}_{\text{slit}}\) the expected values with respect to it. Arbitrary mixtures of plus and minus boundary conditions on the left, right, and slit can again be straightforwardly recovered by further conditioning.

We remark that for the Ising model with locally monochromatic boundary conditions on the (truncated) slit-strip, it makes no difference whether or not doubled edges along the slit part are used. In a slight departure from the convention of the previous sections, we therefore occasionally draw pictures without the doubled edges.
5.3. **Ising model in the infinite strip and infinite slit-strip.** In the infinite lattice strip $S_\ell$, the Ising model is defined by an appropriate limit of the above models on truncated strips $S_{\ell,h_T,h_B}$ as $h_T, h_B \to \infty$. The inclusion $V(S_{\ell,h_T,h_B}) \subset V(S_\ell)$, allows us to interpret the sample space $\{\pm 1\}^{V(S_{\ell,h_T,h_B})}$ for the truncated strip as a subset of the sample space $\Omega = \{\pm 1\}^{V(S_\ell)}$ for the infinite strip, by arbitrarily extending the spin configurations (say as constant $+1$ outside the truncated strip). The space $\Omega = \{\pm 1\}^{V(S_\ell)}$ is a countable product of finite sets, naturally equipped with product topology. We thus consider Borel probability measures on this space, and their weak convergence.

The basic infinite strip case is the Ising model probability measure on $S_\ell$ with locally monochromatic boundary conditions, defined as the following weak limit

$$P_\ell = \lim_{h_T, h_B \to \infty} P_{\ell,h_T,h_B}.$$

Expected values with respect to this measure are denoted by $E_\ell$. The existence of the weak limit above could be proven using the Griffiths’ correlation inequality, whereas with plus boundary conditions the existence of the weak limit could be proven using the FKG correlation inequality. The existence of weak limits with all boundary conditions we consider is also straightforwardly obtained with the transfer matrix formalism, as in Corollary 5.3 and Remark 5.4.11

Similarly in the infinite lattice slit-strip $S_{\ell,\text{slit}}$, the Ising model with locally monochromatic boundary conditions is defined as

$$P_{\ell,\text{slit}} = \lim_{h_T, h_B \to \infty} P_{\ell,h_T,h_B,\text{slit}}$$

and the expected value with respect to it by $E_{\ell,\text{slit}}$. Similar comments apply to the existence of the weak limit, see Corollary 5.12 below, in particular.

Note that literal partition functions are not meaningful for the Ising model on infinite domains such as $S_\ell$ and $S_{\ell,\text{slit}}$ — their defining sum (5.2) would diverge and could not be used to normalize probabilities as in (5.1).

Note also that while in finite volume, monochromatic and mixed boundary conditions can be obtained by further conditioning from the locally monochromatic boundary conditions, the same may fail in the infinite volume limit if the corresponding events can have zero probability — and indeed for the genuinely mixed boundary conditions they do. Therefore, while focusing on the case of locally monochromatic boundary conditions, we occasionally make remarks about what to modify for plus, minus, and mixed boundary conditions. We moreover emphasize that our main objects of interest, the fusion coefficients, actually do contain information about all of these boundary conditions; this is ensured by the suitably chosen renormalization in their very definition.

5.4. **Correlation functions.** We next briefly introduce the correlation functions of the Ising model that will be of primary interest to us. The discussion is separated to bulk and boundary correlation functions, and we moreover separately address the slit and the
slit-strip cases. Our main objects of interest, the fusion coefficients, have an interpretation as boundary correlation functions at the infinite extremities of the slit-strip.

**Bulk correlation functions in the strip.** Consider first the Ising model with locally monochromatic boundary conditions in the truncated strip $S^{(ℓ,h_T,h_B)}$. Given (finitely many) vertices
\[ z_1, \ldots, z_n \in V(S^{(ℓ,h_T,h_B)}), \]
the associated spin correlation function is the quantity
\[ E^{(ℓ,h_T,h_B)} \left[ \prod_{j=1}^n \sigma_{z_j} \right]. \]
Apparently more generally, we could take an arbitrary (complex-valued) random variable on the probability space $Ω$, i.e., an arbitrary function $g: \{\pm 1\}^{V(S^{(ℓ,h_T,h_B)})} \to \mathbb{C}$, and consider the expected value $E^{(ℓ,h_T,h_B)}[g(\sigma)]$. However, any such function $g$ is a linear combination of the functions of the form $\sigma \mapsto \prod_{j=1}^n \sigma_{z_j}$, so focusing on the spin correlation functions entails no essential loss of generality in the truncated strip.

In the infinite strip $S^{(ℓ)}$, we still focus on similarly defined spin correlation functions, i.e.,
\[ E^{(ℓ)} \left[ \prod_{j=1}^n \sigma_{z_j} \right] = \lim_{h_T,h_B \to \infty} E^{(ℓ,h_T,h_B)} \left[ \prod_{j=1}^n \sigma_{z_j} \right], \]
and moreover the weak limit $P^{(ℓ)} = \lim_{h_T,h_B \to \infty} P^{(ℓ,h_T,h_B)}$ is characterized by such limits. Exactly similar comments apply to other boundary conditions.

In the transfer matrix formalism, bulk correlation functions in the strip involving a spin at $z = x + iy$ are expressed in terms of the following operators on the state space (3.1). For $x \in I = [a, b]$, let
\[ S_x: \tilde{V} \to \tilde{V} \]
be the diagonal matrix with matrix elements
\[ u_{\tau}^\dagger S_x u_{\rho} = \delta_{\tau,\rho} \rho_x u_{\rho}. \]
For $x \in I = [a, b]$ and $y \in \mathbb{Z}$, define the spin operator as
\[ S(x + iy) = T^{-y} S_x T^y. \]

The precise statement of how to express bulk correlation functions in the truncated strip $S^{(ℓ,h_T,h_B)}$ in terms of the transfer matrix and these operators will be given in Theorem 5.1(6), and a similar formula for bulk correlations in the infinite strip $S^{(ℓ)}$ will be given in Corollary 5.2(2).
Bulk correlation functions in the slit-strip. The definition of bulk correlation functions in the slit-strip is exactly parallel. In the transfer matrix formalism, bulk correlation functions involving the spin at \( z = x + iy \) in the slit-strip are expressed in terms of the following operators. For \( x \in \mathcal{I} \) and \( y \in \mathbb{Z}_{\geq 0} \), we now define

\[
\hat{S}(x + iy) = (T^{[\ell]}_y)^{-y} S_x (T^{[\ell]}_y)^y \quad \text{and} \quad \hat{S}(x - iy) = (T^{[\ell]}_y)^{+y} S_x (T^{[\ell]}_y)^{-y}.
\]

An expression for bulk correlation functions in the truncated slit-strip \( S_{\text{slit}}^{(\ell; h_T, h_B)} \) in terms of the transfer matrices and these operators will be given in Theorem 5.10(6), and a similar formula for bulk correlations in the infinite slit-strip \( S_{\text{slit}}^{(\ell)} \) will be given in Corollary 5.11(2).

Boundary correlation functions in the strip. Proper boundary correlation functions are defined again in the finite, truncated strip and slit-strip — in the infinite strip and slit-strip one needs to form suitably renormalized limits. For definiteness, consider first the Ising model in the truncated strip, with locally monochromatic boundary conditions on the left and right vertical boundary parts. In addition to the left and right vertical boundaries, the truncated strip has two horizontal boundary components

\[
\partial_T S^{(\ell; h_T, h_B)} = \{a, b\} \times \{h_T\} \quad \text{(top boundary)} \quad \text{and} \quad \partial_B S^{(\ell; h_T, h_B)} = \{a, b\} \times \{-h_B\} \quad \text{(bottom boundary)},
\]

on which we have not imposed boundary conditions. The configurations of all spins on the top and bottom boundaries \( \partial_T S^{(\ell; h_T, h_B)} \) and \( \partial_B S^{(\ell; h_T, h_B)} \) are the random elements \( (\sigma_{x+h_T})_{x \in \mathcal{I}} \) and \( (\sigma_{x-h_B})_{x \in \mathcal{I}} \) of the row spin configuration space \( \{\pm 1\}^{\mathcal{I}} \), where \( \mathcal{I} = \{a, b\} \). Resorting to only a slight abuse of notation, we write these as

\[
\sigma \big|_{\partial_T S^{(\ell; h_T, h_B)}} \in \{\pm 1\}^{\mathcal{I}} \quad \text{and} \quad \sigma \big|_{\partial_B S^{(\ell; h_T, h_B)}} \in \{\pm 1\}^{\mathcal{I}}.
\]

Complex-valued random variables depending only on the boundary spins on the top , for example, are therefore of the form

\[
f(\sigma \big|_{\partial_T S^{(\ell; h_T, h_B)}}), \quad \text{where} \quad f : \{\pm 1\}^{\mathcal{I}} \to \mathbb{C} \text{ is a function}.
\]

By a boundary correlation function in the Ising model on the truncated strip \( S^{(\ell; h_T, h_B)} \) with locally monochromatic boundary conditions, we mean a quantity

\[
E^{(\ell; h_T, h_B)} \left[ f_T \left( \sigma \big|_{\partial_T S^{(\ell; h_T, h_B)}} \right) f_B \left( \sigma \big|_{\partial_B S^{(\ell; h_T, h_B)}} \right) \right],
\]

where

\[
f_T, f_B : \{\pm 1\}^{\mathcal{I}} \to \mathbb{C}
\]

are two given functions.

In the transfer matrix formalism, boundary correlation functions in the strip are expressed in terms of the following vectors in the state space (3.1). For any \( \rho \in \{\pm 1\}^{\mathcal{I}} \), let us denote the corresponding diagonal entry of the matrix \( T_{\text{hor}}^{1/2} \) by \( c_\rho = \exp \left( \frac{\beta}{2} \sum_{x=a}^{b-1} \rho_x \rho_{x+1} \right) > 0 \), so that we have \( T_{\text{hor}}^{1/2} u_\rho = c_\rho u_\rho \). For any function \( f : \{\pm 1\}^{\mathcal{I}} \to \mathbb{C} \), introduce the vector

\[
v^{(f)} := \sum_{\rho \in \{\pm 1\}^{\mathcal{I}}} c_\rho f(\rho) u_\rho \in \mathbb{V}.\]
Denote by $1$ the constant function $1$ on $\{\pm 1\}^T$. The corresponding vector $v^{(1)} \in \tilde{V}$ plays a special role, in that it essentially encodes the free boundary conditions (i.e., the absence of conditioning) on the top and bottom horizontal boundaries. Precisely how to express the boundary correlation functions in the truncated strip $S^{(\ell,h_T,h_B)}$ in terms of the transfer matrix and such vectors will be stated in Theorem 5.1(7). Renormalized limits boundary correlation functions are addressed in Corollary 5.2(3).

**Boundary correlation functions in the slit-strip.** For the case of the truncated slit-strip, the top horizontal boundary is as above, but the bottom horizontal boundary is naturally split to two halves (which overlap at one vertex on the slit)

$$\partial_{B;L} S^{(\ell,h_T,h_B)} = [a,0] \times \{-h_B\} \quad \partial_{B;R} S^{(\ell,h_T,h_B)} = [0,b] \times \{-h_B\}.$$

(b) Now boundary correlation functions of the Ising model with locally monochromatic boundary conditions in the slit-strip

$$E^{(\ell,h_T,h_B)}\left[f_T(\sigma|_{\partial_T S^{(\ell,h_T,h_B)}}) f_{B;L}(\sigma|_{\partial_{B;L} S^{(\ell,h_T,h_B)}}) f_{B;R}(\sigma|_{\partial_{B;R} S^{(\ell,h_T,h_B)}})\right]$$

are defined (with similar abuse of notation as above) given three functions

$$f_T: \{\pm 1\}^T \to \mathbb{C}, \quad f_{B;L}: \{\pm 1\}^I_L \to \mathbb{C}, \quad f_{B;R}: \{\pm 1\}^I_R \to \mathbb{C},$$

where $I = [a,b]$, $I_L = [a,0]$, and $I_R = [0,b]$. For expressing boundary correlation functions in the slit-strip in the transfer matrix formalism, we need in addition vectors of the following form. Given two functions $f_L: \{\pm 1\}^I_L \to \mathbb{C}$ and $f_R: \{\pm 1\}^I_R \to \mathbb{C}$, we introduce the vector

$$u^{(f_L,f_R)} := \sum_{\rho \in \{\pm 1\}^I} c_\rho f_L(\rho|_{I_L}) f_R(\rho|_{I_R}) u_\rho \in \tilde{V}.$$ 

Precisely how to express the boundary correlation functions in the truncated slit-strip $S^{(\ell,h_T,h_B)}$ in terms of the transfer matrix and such vectors will be stated in Theorem 5.10(7). Forming renormalized limits boundary correlation functions in the slit-strip is addressed in Corollary 5.11(3).

5.5. **Transfer matrix formalism in the strip.** The classical method of transfer matrices is very well suited for the study of the Ising model in the lattice strip $S^{(\ell)}$ and slit-strip $S^{(\ell)}_{\text{slit}}$, and it features crucially in our main result. Transfer matrix methods for the two-dimensional Ising model have a rich history, the pioneering early contributions to which include [KrWa41, Ons44, Kau49, Yan52]. Formulation of the transfer matrix formalism in terms of Clifford algebra and fermions is due to Kaufman [Kau49]; see also Schultz & Mattis & Lieb [SML64]. Transfer matrices for the Ising model with plus boundary conditions and the natural generalization of locally monochromatic boundary conditions were studied in [AbMa73]. The textbook [Pal07] can serve as a good reference about the transfer matrix.

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12 Other boundary conditions (plus, minus, or mixed) on the top and bottom can be achieved by selecting a different vector as stated in the remark after Theorem 5.1.
method for the two-dimensional Ising model specifically, while [Bax82] gives a broader view of transfer matrix methods for many different statistical mechanics models.

Mainly, our goal is to apply the Clifford algebra valued 1-forms, with coefficient functions related to those in Section 4, to obtain formulas for the renormalized boundary correlation functions in the slit-strip that we call the fusion coefficients. As an illustration of our approach, which in our view has the advantage of being conceptually close to boundary conformal field theory, we also include a self-contained proof of the diagonalization of the transfer matrix, although the result itself is well-known [AbMa73, Pal07].

Subspaces of the state space for fixed boundary conditions in the strip. The state space \( \tilde{V} = \text{span}_\mathbb{C}\{u_\rho \mid \rho \in \{\pm 1\}^T \} \) defined in (3.1) decomposes into four subspaces \( \tilde{V} = \tilde{V}^{++} \oplus \tilde{V}^{+-} \oplus \tilde{V}^{-+} \oplus \tilde{V}^{--} \) defined by

\[
\tilde{V}_{\epsilon_L \epsilon_R} = \text{span}_\mathbb{C}\{u_\rho \mid \rho \in \{\pm 1\}^T, \rho_a = \epsilon_L, \rho_b = \epsilon_R \},
\]

which are relevant for different monochromatic boundary conditions on the left and right vertical boundaries (e.g., \( \tilde{V}^{+-} \) corresponds to minus boundary conditions on the left and plus boundary conditions on the right). Note that the irreducible state space (3.2) considered in Section 3 can be written as \( V = \tilde{V}^{++} \oplus \tilde{V}^{--} \).

Main properties of the strip transfer matrix. The following well-known theorem summarizes some key properties of the transfer matrix, and how it is used in calculations for the Ising model in the truncated strip.

**Theorem 5.1.** The transfer matrix \( T \) defined by (3.5) has the following properties.

1. \( T \) is an invertible symmetric matrix and its entries are non-negative.
2. Each of the subspaces \( \tilde{V}_{\epsilon_L \epsilon_R} \subset \tilde{V} \) is invariant for \( T \), and consequently also the irreducible state space \( V \subset \tilde{V} \) is invariant for \( T \). We may thus consider \( T \) as an operator on any of these subspaces, by restriction.
3. The restriction of \( T \) to any of the subspaces \( \tilde{V}_{\epsilon_L \epsilon_R} \) satisfies the conditions of the Perron-Frobenius theorem. In particular, it has a unique normalized eigenvector \( v_{\epsilon_L \epsilon_R}^{PF} \in \tilde{V}_{\epsilon_L \epsilon_R} \) with non-negative entries. The corresponding eigenvalue \( \mu_{\epsilon_L \epsilon_R}^{PF} > 0 \) is the maximal eigenvalue of \( T \) on \( \tilde{V}_{\epsilon_L \epsilon_R} \), and it has multiplicity one.
4. The Perron-Frobenius eigenvalues satisfy \( \mu_{++}^{PF} \geq \mu_{+-}^{PF} \) and \( \mu_{-+}^{PF} = \mu_{+-}^{PF} \), and \( \mu_{+-}^{PF} = \mu_{--}^{PF} \). In particular, \( \mu_{++}^{PF} \) is the maximal eigenvalue of \( T \) on \( \tilde{V} \) and \( V \).
5. The Ising model partition functions in the truncated strip \( S^{(\ell,h_T,h_B)} \) with locally monochromatic boundary conditions equals

\[
\mathcal{Z}_{\ell,h_T,h_B} = (v^{(1)})^\dagger T^{h_T+h_B} v^{(1)}.
\]

6. Let \( z_1, \ldots, z_n \in V(S^{(\ell,h_T,h_B)}) \) be such that \( \Im(z_1) \leq \cdots \leq \Im(z_n) \). Then we have the following expressions for spin correlation functions of the Ising model in the
truncated strip with locally monochromatic boundary conditions

\[
E^{(\ell,h_T,h_B)} \left[ \prod_{j=1}^{n} \sigma_{z_j} \right] = \frac{(v^{(1)})^\dagger T^{h_T} S(z_n) \cdots S(z_1) T^{h_B} v^{(1)}}{(v^{(1)})^\dagger T^{h_T} T^{h_B} v^{(1)}}.
\]

(7) Let \( f_T, f_B : \{\pm 1\}^T \to \mathbb{C} \) be two functions. Then we have the following expressions for boundary correlation functions of the Ising model in the truncated strip with locally monochromatic boundary conditions:

\[
E^{(\ell,h_T,h_B)} \left[ f_T(\sigma)_{\partial_1 S(\ell,h_T,h_B)} f_B(\sigma)_{\partial_B S(\ell,h_T,h_B)} \right] = \frac{(v^{(1)})^\dagger T^{h_T+h_B} v(f_B)}{(v^{(1)})^\dagger T^{h_T+h_B} v^{(1)}}.
\]

The expressions in the last three items above make it clear that the transfer matrix and its spectrum give detailed information about the limit \( h_B, h_T \to \infty \) of infinite strip.

For the most succinct statements about the infinite height limit, it is useful to note that in (4) we in fact have the strict inequality \( \mu^{PF}_{++} > \mu^{PF}_{--} \) for any fixed width \( \ell \in \mathbb{N} \). This strict inequality form of the statement will be obtained in Proposition \ref{prop:strict} below. With it, the following corollary about the infinite height limits is easily obtained.

**Corollary 5.2.** There exists an \( \varepsilon = \varepsilon(\ell) > 0 \) such that the infinite height asymptotics of the partition function and correlation functions of the Ising model in the strip with locally monochromatic boundary conditions are given by the following:

(1) As \( h_B, h_T \to \infty \) we have

\[
Z^{(\ell)}_{h_B, h_T} = (\mu^{PF})^{h_B+h_T} \left( A + O(e^{-\varepsilon \min(h_B,h_T)}) \right),
\]

where \( A = 2 \left( (v^{PF}_+)^\dagger v^{(1)} \right)^2 > 0 \).

(2) Let \( z_1, \ldots, z_n \in V(S(\ell)) \) be such that \( \Im(z_1) \leq \cdots \leq \Im(z_n) \). If \( n \) is even, then as \( h_B, h_T \to \infty \) we have

\[
E^{(\ell,h_T,h_B)} \left[ \prod_{j=1}^{n} \sigma_{z_j} \right] = \frac{(v^{PF}_+)^\dagger S(z_n) \cdots S(z_1) v^{PF}_+}{(v^{PF}_+)^\dagger v^{PF}_+} + O(e^{-\varepsilon \min(h_B,h_T)})
\]

If \( n \) is odd, the expression on the left vanishes by parity considerations.

(3) If \( f_T, f_B : \{\pm 1\}^T \to \mathbb{C} \) are such that \( v(f_T), v(f_B) \) are eigenvectors of \( T \) with eigenvalue \( \mu \) then as \( h_B, h_T \to \infty \) we have

\[
E^{(\ell,h_T,h_B)} \left[ f_T(\sigma)_{\partial_1 S(\ell,h_T,h_B)} f_B(\sigma)_{\partial_B S(\ell,h_T,h_B)} \right] = \left( \frac{\mu}{\mu^{PF}} \right)^{h_T+h_B} \left( \frac{(v^{(1)})^\dagger v(f_B)}{2 \left( (v^{PF}_+)^\dagger v^{(1)} \right)^2} + O(e^{-\varepsilon \min(h_B,h_T)}) \right).
\]

\[\text{In fact, an efficient proof of part (4) of Theorem 5.1 already uses the property}
\[
\mathcal{P}^{(\ell,h_T,h_B)} \left[ \sigma_{\partial_1 S(\ell,h_T,h_B)} \right] = \mathcal{P}^{(\ell,h_T,h_B)} \left[ \sigma_{\partial_B S(\ell,h_T,h_B)} \right] = \mathcal{P}^{(\ell,h_T,h_B)} \left[ \sigma_{\partial_B S(\ell,h_T,h_B)} \right],
\]

which is a consequence of the FKG inequality, together with infinite height limits in the latter expressions.
If the eigenvectors have different eigenvalues, then the expression on the left vanishes by orthogonality considerations.

Since the spin correlation functions cover a convergence determining collection for Borel probability measures on \( \{ \pm 1 \}^V(S^{(\ell)}) \), property (2) above in particular ensures the existence of the infinite volume limit probability measure.

**Corollary 5.3.** As \( h_T, h_B \to \infty \), the weak limit \( P^{(\ell)} = \lim P^{(\ell, h_T, h_B)} \) exists.

**Remark 5.4.** Statements entirely parallel to Theorem 5.1(5-7), Corollary 5.2, and Corollary 5.3 hold with plus boundary conditions, minus boundary conditions, and mixed boundary conditions (e.g., minus on the left and plus on the right boundary). They merely require replacing the vector \( v^{(1)} \) by its projection to the appropriate subspace \( \tilde{V}_{L,R} \), and using the corresponding Perron-Frobenius eigenvalue \( \mu_{PF}^{L,R} \). For these “pure” boundary conditions, we do not even need the inequalities among the Perron-Frobenius eigenvalues in the different subspaces, and the factors 2 and parity requirements in Corollary 5.2 are absent. Also imposing boundary conditions on the top and bottom boundaries is straightforward to handle by using different vectors in place of \( v^{(1)} \).

5.6. **Diagonalization of the strip transfer matrix.** Having stated above the relevance of the transfer matrix and its leading eigenvectors to the correlation functions of the Ising model in the infinite strip, we now show how to diagonalize the transfer matrix with the help of the Clifford algebra valued 1-forms of Section 3 and the vertical translation eigenfunctions of Section 4. The diagonalization of this transfer matrix is a well-known result [AbMa73, Pal07], which we include mainly as a simple illustration of the use of the boundary conformal field theory inspired method of Section 3.

**Discrete fermion modes.** We now define specific operators of the general form (3.21), i.e., line integrals of the Clifford algebra valued 1-forms across the strip. For \( k \in \pm K^{(\ell)} \), let

\[
\mathfrak{F}_k : E(S^{(\ell)}) \to \mathbb{C}
\]

denote the function on the edges of the discrete strip \( S^{(\ell)} \) as in Proposition 4.1. Recall that \( \mathfrak{F}_k \) is \( s \)-holomorphic, has Riemann boundary values, and is an eigenfunction of vertical translations with eigenvalue \( \Lambda_k^{(\ell)} \). Using this function, we define the corresponding pair \( (m_k, m_k^*) \) of coefficient functions by

\[
m_k(z) = \hat{i} \mathfrak{F}_k(z) \quad \text{and} \quad m_k^*(z) = -\hat{i} \mathfrak{F}_k(z).
\]

The pair \( (m_k, m_k^*) \) is ICSH and has ICRBV by virtue of Lemmas 3.7 and 3.9 and the \( s \)-holomorphicity and Riemann boundary values of \( \mathfrak{F}_k \). We then define the \( k \)th discrete fermion mode as anticipated in Remark 3.14,

\[
f_k^{(\ell)} = \frac{\hat{i}}{2} \int_{\gamma_0}^z \left( m_k(z) \psi(z) \, dz + m_k^*(z) \psi^*(z) \, d\bar{z} \right) \in \text{CliffGen}.
\]

Let us first record the anticommutators and Hilbert space adjoints of the fermion modes.
**Proposition 5.5.** For \( k \in \pm K^{(\ell)} \), we have
\[
(f_k^{(\ell)})^\dagger = f_{-k}^{(\ell)},
\]
and for \( k_1, k_2 \in \pm K^{(\ell)} \), we have
\[
[f_k^{(\ell)}, f_{k_2}^{(\ell)}]_+ = \delta_{k_1, -k_2} \text{id}.
\]

**Proof.** By \((4.6)\) we get the formula \( R_{f_k} = f_{-k} \) for the restrictions to the cross-section of the functions \( \mathcal{F}_k \) and \( \mathcal{F}_{-k} \) under the reflection \([2.8]\). The first asserted formula then follows from Equation \((3.23)\) in Remark \(3.14\). The second asserted formula follows from Equation \((3.24)\) in the same remark, using furthermore that \( \langle R_{f_{k_1}}, f_{k_2} \rangle = \langle f_{-k_1}, f_{k_2} \rangle = \delta_{k_1, -k_2} \) by the orthonormality in Proposition \(4.1\). \(\square\)

The vertical translation eigenfunction property gives the following.

**Lemma 5.6.** For \( k \in \pm K^{(\ell)} \), we have
\[
T f_k^{(\ell)} T^{-1} = \Lambda_k^{(\ell)} f_k^{(\ell)},
\]
where \( \Lambda_k^{(\ell)} \) is as in \((4.5)\).

**Proof.** Directly from the definition \((3.6)\) of fermions on horizontal edges, we get that
\[
T \psi(z) T^{-1} = \psi(z - \hat{i}) \quad \text{and} \quad T \psi^*(z) T^{-1} = \psi^*(z - \hat{i}).
\]

With this observation and a change of variables by one lattice step vertically, the left hand side of the assertion becomes an integral across the cross-section \( \gamma_{-1} \) at height \(-1\)
\[
T f_k^{(\ell)} T^{-1} = \frac{e^{i\pi}}{2} \int_{\gamma_{-1}} \left( m_k(z) \psi(z - \hat{i}) \, dz + m_k^*(z) \psi^*(z - \hat{i}) \, dz \right) = \frac{e^{i\pi}}{2} \int_{\gamma_{-1}} \left( m_k(z + \hat{i}) \psi(z) \, dz + m_k^*(z + \hat{i}) \psi^*(z) \, dz \right).
\]
Now the vertical translation eigenfunction property of \( \mathcal{F}_k \) directly implies
\[
m_k(z + \hat{i}) = \Lambda_k^{(\ell)} m_k(z) \quad \text{and} \quad m_k^*(z + \hat{i}) = \Lambda_k^{(\ell)} m_k^*(z),
\]
which then allows us to write the left hand side as
\[
T f_k^{(\ell)} T^{-1} = \Lambda_k^{(\ell)} \frac{e^{i\pi}}{2} \int_{\gamma_{-1}} \left( m_k(z) \psi(z) \, dz + m_k^*(z) \psi^*(z) \, dz \right).
\]
The assertion \( T f_k^{(\ell)} T^{-1} = \Lambda_k^{(\ell)} f_k^{(\ell)} \) then follows by recalling from Proposition \(3.13\) that the chosen integration height is irrelevant, since the coefficient functions \( (m_k, m_k^*) \) are ICSH and have ICRBV. \(\square\)

Due to the lemma above, the fermion modes shift eigenvalues of the Ising transfer matrix: if \( v \) is an eigenvector of \( T \) with eigenvalue \( \mu \), then \( f_k^{(\ell)} v \) either vanishes or is an eigenvector of \( T \) with eigenvalue \( \Lambda_k^{(\ell)} \mu \).

**Lemma 5.7.** If \( v \in V \) satisfies \( T v = \mu v \), then \( f_k^{(\ell)} v \in V \) satisfies \( T (f_k^{(\ell)} v) = (\Lambda_k^{(\ell)} \mu) (f_k^{(\ell)} v) \).
Proof. This follows from the calculation
\[ T(f_k^{(ℓ)} v) = (T f_k^{(ℓ)} T^{-1})(T v) = (\Lambda_k^{(ℓ)} f_k^{(ℓ)}) (\mu v) = (\Lambda_k^{(ℓ)} \mu) (f_k^{(ℓ)} v). \]
\[ \square \]

Since \( \Lambda_k^{(ℓ)} > 1 \) for \( k > 0 \) and \( \Lambda_k^{(ℓ)} < 1 \) for \( k < 0 \), the positive modes increase the magnitude of the transfer matrix eigenvalue and the negative modes decrease it. The positive modes are called annihilation operators (the rationale for the terminology is Proposition 5.8 below) and the negative modes are called creation operators. Observe that indeed,
\[ \text{CliffGen} = \text{span}_C \left\{ f_k^{(ℓ)} \left| k \in \mathcal{K}^{(ℓ)} \right. \right\} \oplus \text{span}_C \left\{ f_{-k}^{(ℓ)} \left| k \in \mathcal{K}^{(ℓ)} \right. \right\} \]
(5.8)

is a polarization of the Clifford generators: for \( k_1, k_2 \in \mathcal{K}^{(ℓ)} \) we have \([f_{k_1}^{(ℓ)}, f_{k_2}^{(ℓ)}]_+ = 0 \) and \([f_{-k_1}^{(ℓ)}, f_{-k_2}^{(ℓ)}]_+ = 0 \) by Proposition 5.5.14

Diagonalization of the transfer matrix. In this section we combine the observations so far to provide a self-contained diagonalization of the transfer matrix \( T \) of the Ising model with locally monochromatic boundary conditions.

Denote by
\[ v_\emptyset := v_{PF}^{++} \in \widetilde{V}_{++} \subset V \subset \widetilde{V} \]
the Perron-Frobenius eigenvector of the Ising transfer matrix \( T \) in the plus-plus monochromatic sector \( \widetilde{V}_{++} \), as in Theorem 5.1(3), and by \( \mu^{(ℓ)}_\emptyset := \mu_{PF}^{++} \) the corresponding eigenvalue.

The discrete fermion modes \( f_k^{(ℓ)} \) with positive indices \( k > 0 \) are called annihilation operators for the following reason.

**Proposition 5.8.** For any \( k \in \mathcal{K}^{(ℓ)} \), we have
\[ f_k^{(ℓ)} v_\emptyset = 0. \]

Proof. According to Theorem 5.1(5), the Perron-Frobenius eigenvector \( v_\emptyset = v_{PF}^{++} \) in \( \widetilde{V}_{++} \) has the largest eigenvalue \( \mu^{(ℓ)}_\emptyset \) for \( T \), in all of \( V \). Now if \( f_k^{(ℓ)} v_\emptyset \) were non-zero, then by Lemma 5.7 it would be an eigenvector with a larger eigenvalue \( \Lambda_k^{(ℓ)} \mu^{(ℓ)}_\emptyset > \mu^{(ℓ)}_\emptyset \), which is a contradiction. \[ \Box \]

Given a subset \( \alpha \subset \mathcal{K}^{(ℓ)} \), write it in the form
\[ \alpha = \{k_1, \ldots, k_m\} \quad \text{with} \quad 0 < k_1 < \cdots < k_m. \]
(5.9)

We then set
\[ v_\alpha := f_{-k_m}^{(ℓ)} \cdots f_{-k_1}^{(ℓ)} v_\emptyset. \]
(5.10)

14 This polarization property could alternatively be concluded directly from Lemma 5.7 since multiples of the identity operator can neither increase nor decrease the magnitude of eigenvalues.
Proposition 5.9. The collection \((v_\alpha)_{\alpha \subset K^{(\ell)}}\) is an orthonormal basis of \(V\) consisting of eigenvectors of \(T\). Moreover, each \(v_\alpha \in V\) is an eigenvector of \(T\) with eigenvalue

\[
\mu_\alpha^{(\ell)} = \frac{\mu_\emptyset^{(\ell)}}{\prod_{k \in \alpha} \Lambda_k^{(\ell)}},
\]

where \(\Lambda_k^{(\ell)}\) are given by (4.5).

Proof. Since (5.8) is a polarization and \(f^{(\ell)}_k v_\emptyset = 0\) for all \(k \in K^{(\ell)}\), the collection \((v_\alpha)_{\alpha \subset K^{(\ell)}}\) is a basis of a Fock representation of the Clifford algebra, with vacuum vector \(v_\emptyset\). Since \(v_\alpha \in V\) for each \(\alpha\) and the cardinality \(#\{\alpha \subset K^{(\ell)}\} = 2^{\ell}\) of the indexing set coincides with the dimension \(\dim_C(V) = 2^{\ell}\), it follows that the collection is also a basis of \(V\). Orthonormality follows by a routine calculation using the two properties in Proposition 5.5 repeatedly.

Recalling that \(\Lambda_k^{(\ell)} = 1/\Lambda_k^{(\ell)}\), a repeated application of Lemma 5.7 gives

\[
T v_\alpha = \mu_\alpha^{(\ell)} v_\alpha,
\]

so the basis vectors are indeed eigenvectors with the stated eigenvalues. \(\square\)

A global spin flip (the linear map defined by \(u_\rho \mapsto u_{-\rho}\)) relates the eigenvectors of \(T\) in the irreducible state space \(V = \tilde{V}_{++} \oplus \tilde{V}_{-+} \oplus \tilde{V}_{+-} \oplus \tilde{V}_{--}\) to the ones in \(\tilde{V}_{-+} \oplus \tilde{V}_{++} \oplus \tilde{V}_{--} \oplus \tilde{V}_{+-}\), and one thus easily obtains an orthonormal basis of eigenvectors of \(T\) in \(\tilde{V}\) as well. By having found the eigenvectors (5.10) and their eigenvalues (5.11), we have diagonalized the transfer matrix \(T\).

5.7. Transfer matrix formalism in the slit-strip. Recall from Section 3.4 that in the slit-strip, we use two separate transfer matrices: \(T^{[L|R]}: \tilde{V} \to \tilde{V}\) given by (3.26) for the bottom half with the slit, and \(T^{[\ell]}: \tilde{V} \to \tilde{V}\) still given by (3.5) for the top half.

Subspaces of the state space for fixed boundary conditions in the strip. When considering the bottom part of the slit-strip, it is natural to decompose the state space \(\tilde{V}\) to even further subspaces

\[
\tilde{V} = \tilde{V}_{++} \oplus \tilde{V}_{-+} \oplus \tilde{V}_{+-} \oplus \tilde{V}_{--} \oplus \tilde{V}_{++} \oplus \tilde{V}_{-+} \oplus \tilde{V}_{+-} \oplus \tilde{V}_{--}
\]

defined by

\[
\tilde{V}_{\epsilon_L\epsilon_S\epsilon_R} = \text{span}_C \left\{ u_\rho \mid \rho \in \{\pm 1\}^2, \, \rho_0 = \epsilon_L, \, \rho_0 = \epsilon_S, \, \rho_0 = \epsilon_R \right\},
\]

which account for different monochromatic boundary conditions on the left and right vertical boundaries as well as on the slit.

\[\text{Only the somewhat arbitrary overall multiplicative constant } \mu_\emptyset^{(\ell)} \text{ has not been explicitly determined here, but a short calculation would also yield its value } \mu_\emptyset^{(\ell)} = \sqrt{2/3} \left( \frac{2 + \sqrt{2}}{2} \right)^{\ell} \prod_{k \in K^{(\ell)}} (1 + 1/\Lambda_k^{(\ell)})^{-1}.\]
Main properties of the slit-strip transfer matrices. The following theorem, closely parallel to Theorem \[5.1\] summarizes some key properties of the two transfer matrices above, and how they are used in calculations for the Ising model in the truncated slit-strip.

**Theorem 5.10.** The transfer matrices \( T^{[\ell]} \) and \( T^{[\ell')} = T \) have the following properties.

1. \( T^{[\ell]} \) is an invertible symmetric matrix and its entries are non-negative.
2. Each of the subspaces \( \tilde{V}_{L,S,R} \subset \tilde{V} \) is invariant for \( T^{[\ell]} \), and consequently also the irreducible state space \( V \subset \tilde{V} \) is invariant for \( T^{[\ell]} \). We may thus consider \( T^{[\ell]} \) as an operator on any of these subspaces, by restriction.
3. The restriction of \( T^{[\ell]} \) to any of the subspaces \( \tilde{V}_{L,S,R} \) satisfies the conditions of the Perron-Frobenius theorem. In particular, it has a unique normalized eigenvector \( \nu_{PF}^{L,S,R} \in \tilde{V}_{L,S,R} \) with non-negative entries. The corresponding eigenvalue \( \mu_{PF}^{L,S,R} > 0 \) is the maximal eigenvalue of \( T^{[\ell]} \) on \( \tilde{V}_{L,S,R} \), and it has multiplicity one.
4. The maximal eigenvalue of \( T^{[\ell]} \) on \( \tilde{V} \) and \( V \) is \( \mu_{PF}^{++} \).
5. The Ising model partition function in the truncated slit-strip \( S_{slit}^{(\ell,h_B)} \) with locally monochromatic boundary conditions can be expressed as follows:

\[
Z_{slit}^{(\ell,h_B)} = (\nu^{(1)})^{T} T^{[\ell]} T^{[\ell]} (S_{slit}^{(\ell,h_B)}) \nu^{(1)}.
\]

6. Let \( z_1, \ldots, z_n \in V(S^{(\ell,h_B)}) \) be such that \( \Im(z_1) \leq \cdots \leq \Im(z_n) \). Then we have the following expressions for spin correlation functions of the Ising model in the truncated slit-strip with locally monochromatic boundary conditions:

\[
E_{slit}^{(\ell,h_B)} \left[ \prod_{j=1}^{n} \sigma_{s_j} \right] = \frac{(\nu^{(1)})^{T} T^{[\ell]} S(z_n) \cdots S(z_1) (T^{[\ell]} T^{[\ell]} B) \nu^{(1)}}{(\nu^{(1)})^{T} T^{[\ell]} T^{[\ell]} B \nu^{(1)}}
\]

7. Let \( f_T : \{\pm 1\}^{T} \rightarrow \mathbb{C} \), \( f_{B,L} : \{\pm 1\}^{T_L} \rightarrow \mathbb{C} \), \( f_{B,R} : \{\pm 1\}^{T_R} \rightarrow \mathbb{C} \) be functions. Then we have the following expressions for boundary correlation functions of the Ising model in the truncated slit-strip, with locally monochromatic boundary conditions:

\[
E_{slit}^{(\ell,h_B)} \left[ f_T(\sigma_{\mu \in [\ell,h_B]}^{(\ell,h_B)}) f_{B,L}(\sigma_{\partial_{B,L,S}(\ell) B}^{(\ell,h_B)}) f_{B,R}(\sigma_{\partial_{B,R,S}(\ell) B}^{(\ell,h_B)}) \right]
\]

\[
\frac{(\nu^{(1)})^{T} T^{[\ell]} B \nu^{(1)}}{(\nu^{(1)})^{T} T^{[\ell]} B \nu^{(1)}}
\]

The expressions in the last three items above again allow for considering the limit \( h_B, h_T \rightarrow \infty \) of infinite slit-strip. In order to simplify the statement, we make use of the fact that \( \mu_{PF}^{++} = \mu_{+++} \) are strictly larger than the Perron-Frobenius eigenvalues in the other sectors, which is seen as in Proposition \[5.9\].

**Corollary 5.11.** There exists an \( \varepsilon = \varepsilon(\ell) > 0 \) such that the infinite height asymptotics of the partition function and correlation functions of the Ising model in the slit-strip with locally monochromatic boundary conditions are given by the following:
(1) As $h_B, h_T \rightarrow \infty$ we have

$$Z_{\text{slit}}^{(\ell,h_T,h_B)} = (\mu_{++}^{PF})^{h_T} (\mu_{++}^{PF,\text{slit}})^{h_B} \left( A + \mathcal{O}(e^{-\varepsilon \min(h_B,h_T)}) \right),$$

where $A = 2 \left( (v^{(1)} \dagger)^{PF} \right) \left( (v^{(1)} \dagger)^{PF,\text{slit}} \right) > 0$.

(2) Let $z_1, \ldots, z_n \in V(S^{(\ell)})$ be such that $\Re m(z_1) \leq \cdots \leq \Re m(z_n)$. If $n$ is even, then as $h_B, h_T \rightarrow \infty$, we have

$$E_{\text{slit}}^{(\ell,h_T,h_B)} \left[ \prod_{j=1}^n \sigma_{z_j} \right] = \left( (v_{++}^{PF})^{\dagger} \tilde{S}(z_1) \cdots \tilde{S}(z_n) (v_{++}^{PF})^{\dagger} v_{++}^{PF,\text{slit}} \right) + \mathcal{O}(e^{-\varepsilon \min(h_B,h_T)}).$$

If $n$ is odd, then the expression on the left vanishes by parity considerations.

(3) If $f_T: \{\pm 1\}^{Z} \rightarrow \mathbb{C}$ is such that $v^{(f_T)}$ is an eigenvector of $T$ with eigenvalue $\mu(f_T)$, and if $f_{B,L}: \{\pm 1\}^{Z} \rightarrow \mathbb{C}$, $f_{B,R}: \{\pm 1\}^{Z} \rightarrow \mathbb{C}$ are such that $v^{(f_{B,L};f_{B,R})}$ is an eigenvector of $T^{[\ell]}$ with eigenvalue $\mu(f_{B,L};f_{B,R})$, then as $h_B, h_T \rightarrow \infty$ we have

$$E_{\text{slit}}^{(\ell,h_T,h_B)} \left[ f_T(\sigma_{\partial T S^{(\ell,h_T,h_B)}}) f_{B,L}(\sigma_{\partial B L S^{(\ell,h_T,h_B)}}) f_{B,R}(\sigma_{\partial B R S^{(\ell,h_T,h_B)}}) \right] = \left( \frac{\mu(f_T)}{\mu_{++}^{PF}} \right)^{h_T} \left( \frac{\mu(f_{B,L};f_{B,R})}{\mu_{++}^{PF,\text{slit}}} \right)^{h_B} \left( (v^{(f_T)} \dagger)^{PF} (v^{(f_{B,L};f_{B,R})} \dagger (v^{(1)} \dagger)^{PF} (v_{++}^{PF,\text{slit}}) + \mathcal{O}(e^{-\varepsilon \min(h_B,h_T)}) \right).$$

Again since the spin correlation functions cover a convergence determining collection for Borel probability measures on $\{\pm 1\}^{V(S^{(\ell)})}$, property (2) above in particular ensures the existence of the infinite volume Ising probability measure and characterizes it.

**Corollary 5.12.** As $h_T, h_B \rightarrow \infty$, the weak limit $P_{\text{slit}}^{(\ell)} = \lim P_{\text{slit}}^{(\ell,h_T,h_B)}$ exists.

Plus, minus, and mixed boundary conditions on the vertical boundary components, and boundary conditions on the top and bottom horizontal boundaries can again be handled as indicated in Remark 5.4.

### 5.8. Fermionic operators for the slit-strip transfer matrix formalism

We now introduce various specific choices of integrals (3.29) of Clifford algebra valued 1-forms across the slit-strip. First of all, there will be three sets of discrete fermion modes, one for each extremity of the slit-strip, that are exactly analogous to the creation and annihilation operators in the strip, and they similarly yield a diagonalization of the transfer matrix $T^{[\ell]}$ of the slit-part. The more novel ones are operators better adapted to the slit-strip geometry, defined using coefficient functions related to the globally defined distinguished functions on the slit-strip given in Proposition 4.2. They will be crucial in our characterization of the fusion coefficients in the next subsection.

**Discrete fermion modes in the three extremities.** We first introduce fermion modes associated to each of the three extremities of the slit-strip $S^{(\ell)}_{\text{slit}}$.

For the top extremity $S^{T,(\ell)}_{\text{slit}}$, we will in fact use just the fermion modes $f^{(\ell)}_k$, $k \in \pm K^{(\ell)}$, introduced already in Section 5.6. The functions $\mathcal{F}_k$ and the associated coefficient functions $(m_k, m^*_k)$ are defined on the edges of $S^{T,(\ell)}_{\text{slit}}$, and for $y \in \mathbb{Z} \geq 0$ the integration along the
contour $\gamma_y = \gamma^T_y$ from left to right across the slit-strip defines the appropriate fermion mode as in (5.7).

It therefore remains to introduce fermion modes associated with the left leg $S_{\text{slit}}^L(\ell)$ and the right leg $S_{\text{slit}}^R(\ell)$ of the slit-strip. Let

$$\mathcal{F}_{L,k} : E(S_{\text{slit}}^L(\ell)) \to \mathbb{C} \quad \text{for } k \in \pm \mathcal{K}^{(\ell_L)}$$

$$\mathcal{F}_{R,k} : E(S_{\text{slit}}^R(\ell)) \to \mathbb{C} \quad \text{for } k \in \pm \mathcal{K}^{(\ell_R)}$$
denote the functions as in Section 6.2. Recall that each of these is s-holomorphic, and has Riemann boundary values. These are also effectively eigenfunctions of vertical translations, in that the equations

$$\mathcal{F}_{L,k}(z + \hat{1}) = \Lambda^{(\ell_L)}_k \mathcal{F}_{L,k}(z) \quad \mathcal{F}_{R,k}(z + \hat{1}) = \Lambda^{(\ell_R)}_k \mathcal{F}_{R,k}(z)$$

hold whenever both $z$ and $z + \hat{1}$ are in the domain of definition of the function. In terms of these, we define coefficient functions by

$$m_{L,k}(z) = \hat{1} \mathcal{F}_{L,k}(z) \quad m_{R,k}(z) = \hat{1} \mathcal{F}_{R,k}(z)$$

$$m^*_{L,k}(z) = -\hat{1} \mathcal{F}_{L,k}(z) \quad m^*_{R,k}(z) = -\hat{1} \mathcal{F}_{R,k}(z).$$

The pairs $(m_{L,k}, m^*_{L,k})$ and $(m_{R,k}, m^*_{R,k})$ are ICSH and have ICRBV by virtue of Lemmas 3.16 and 3.17. We then define the discrete fermion modes in the left and right extremities as

$$f^{(\ell)}_{L,k} = \frac{e^{\frac{i}{2} \frac{\pi}{\gamma_y}}}{2} \int_{\gamma_y}^z \left( m_{L,k}(z) \hat{\psi}(z) \, dz + m^*_{L,k}(z) \hat{\psi}^*(z) \, dz \right) \quad \text{for } k \in \pm \mathcal{K}^{(\ell_L)}$$

$$f^{(\ell)}_{R,k} = \frac{e^{\frac{i}{2} \frac{\pi}{\gamma_y}}}{2} \int_{\gamma_y}^z \left( m_{R,k}(z) \hat{\psi}(z) \, dz + m^*_{R,k}(z) \hat{\psi}^*(z) \, dz \right) \quad \text{for } k \in \pm \mathcal{K}^{(\ell_R)}.$$

The anticommutators and Hilbert space adjoints of the fermion modes are as before.

**Proposition 5.13.** We have

$$(f^{(\ell)}_{L,k})^\dagger = f^{(\ell)}_{L,-k} \quad (f^{(\ell)}_{R,k})^\dagger = f^{(\ell)}_{R,-k}$$

for $k \in \pm \mathcal{K}^{(\ell_L)}$ and $k \in \pm \mathcal{K}^{(\ell_R)}$, respectively. Also, we have

$$[f_{L,k_1}^{(\ell)} f_{L,k_2}^{(\ell)}]_+ = \delta_{k_1,-k_2} \text{id} \quad [f_{R,k_1}^{(\ell)} f_{R,k_2}^{(\ell)}]_+ = \delta_{k_1,-k_2} \text{id},$$

for $k_1, k_2 \in \pm \mathcal{K}^{(\ell_L)}$ and $k_1, k_2 \in \pm \mathcal{K}^{(\ell_R)}$, respectively. Finally, we have

$$[f_{L,k}^{(\ell)} f_{R,k'}^{(\ell)}]_+ = 0$$

for $k \in \pm \mathcal{K}^{(\ell_L)}$ and $k' \in \pm \mathcal{K}^{(\ell_R)}$.

**Proof.** The proofs of the first two assertions are similar to those of Proposition 5.3. For the last assertion, note that $f_{L,k}^{(\ell)}$ is by construction a linear combination of $\psi_{x'}$ and $\psi_{x'}^*$ for $x' \in [a, 0]^*$, whereas $f_{R,k'}^{(\ell)}$ uses $x' \in [0, b]^*$ instead. These anticommute by Lemma 3.2. □

The vertical translation eigenfunction property gives the following.
Lemma 5.14. We have
\[ T^{[\ell_L,|\ell_R]} f_{L,k}^{(\ell)} T^{[\ell_L,|\ell_R]} = \Lambda_k^{(\ell_L)} f_{L,k}^{(\ell)} \]
\[ T^{[\ell_L,|\ell_R]} f_{R,k}^{(\ell)} T^{[\ell_L,|\ell_R]} = \Lambda_k^{(\ell_R)} f_{R,k}^{(\ell)} \]
for \( k \in \pm K^{(\ell_L)} \) and \( k \in \pm K^{(\ell_R)} \), respectively.

Proof. The proof is similar to Lemma 5.6. \( \square \)

Due to this lemma, the fermion modes of the left and right leg shift eigenvalues of the transfer matrix \( T^{[\ell_L,|\ell_R]} \) for the slit part.

Lemma 5.15. If \( v \in V \) satisfies \( T^{[\ell_L,|\ell_R]} v = \mu v \), then \( f_{L,k}^{(\ell)} v \in V \) and \( f_{R,k}^{(\ell)} v \in V \) satisfy
\[ T^{[\ell_L,|\ell_R]} (f_{L,k}^{(\ell)} v) = (\Lambda_k^{(\ell_L)} \mu) (f_{L,k}^{(\ell)} v) \]
\[ T^{[\ell_L,|\ell_R]} (f_{R,k}^{(\ell)} v) = (\Lambda_k^{(\ell_R)} \mu) (f_{R,k}^{(\ell)} v) \]
for positive \( k > 0 \) and negative \( k < 0 \) respectively.

Proof. The proof is similar to that of Lemma 5.7. \( \square \)

Observe that
\[ \text{CliffGen} = \text{span}_C \left( \left\{ f_{L,k}^{(\ell)} \mid k \in K^{(\ell_L)} \right\} \cup \left\{ f_{R,k}^{(\ell)} \mid k \in K^{(\ell_R)} \right\} \right) \]
\[ \oplus \text{span}_C \left( \left\{ f_{L,-k}^{(\ell)} \mid k \in K^{(\ell_L)} \right\} \cup \left\{ f_{R,-k}^{(\ell)} \mid k \in K^{(\ell_R)} \right\} \right) \]
is a polarization of the Clifford generators, by Proposition 5.13. The modes \( f_{L,k}^{(\ell)} \) and \( f_{R,k}^{(\ell)} \) for positive \( k > 0 \) are again interpreted as annihilation operators and the modes for negative \( k < 0 \) as creation operators: they now respectively raise and lower the magnitude of the eigenvalue of the transfer matrix \( T^{[\ell_L,|\ell_R]} \) of the slit part, according to Lemma 5.15.

**Diagonalization of the transfer matrix for the slit part.** With the fermion modes of the left and the right legs, we obtain the following diagonalization of the transfer matrix \( T^{[\ell_L,|\ell_R]} \) for the slit part.

Let
\[ v_{\ell_L}^{\text{slit}} = v_{\ell_L}^{\text{PF;slit}} \in \tilde{V}_{++} \subset V \subset \tilde{V} \]
denote the Perron-Frobenius eigenvector of the transfer matrix \( T^{[\ell_L,|\ell_R]} \) in the plus-plus-plus monochromatic sector \( \tilde{V}_{++} \), as in Section 5.

The modes with positive indices \( k > 0 \) serve as annihilation operators.

**Proposition 5.16.** We have
\[ f_{L,k}^{(\ell)} v_{\ell_L}^{\text{slit}} = 0 \]
\[ f_{R,k}^{(\ell)} v_{\ell_R}^{\text{slit}} = 0 \]
for \( k \in K^{(\ell_L)} \) and \( k \in K^{(\ell_R)} \), respectively.

Proof. This follows directly from the maximality of the eigenvalue of the \( v_{\ell_L}^{\text{slit}} = v_{\ell_L}^{\text{PF;slit}} \), Theorem 5.10(5). \( \square \)
Given two subsets $\alpha_L \subset K^{(\ell_L)}$, $\alpha_R \subset K^{(\ell_R)}$, write them in the form

$$\alpha_L = \{k_1, \ldots, k_m\} \quad 0 < k_1 < \cdots < k_m$$

$$\alpha_R = \{k'_1, \ldots, k'_{m'}\} \quad 0 < k'_1 < \cdots < k'_{m'}.$$

We then set

$$v_{\alpha_R; \alpha_L}^{\text{slit}} := f_{R; -k'_m}^{(\ell)} \cdots f_{R; -k'_1}^{(\ell)} f_{L; -k_m}^{(\ell)} \cdots f_{L; -k_1}^{(\ell)} v_{\emptyset; \emptyset}^{\text{slit}}$$

(note that we fix a specific ordering here, the choice of which affects some signs later).

**Proposition 5.17.** The collection

$$\left(v_{\alpha_R; \alpha_L}^{\text{slit}}\right)_{\alpha_L \subset K^{(\ell_L)}, \alpha_R \subset K^{(\ell_R)}}$$

is an orthonormal basis of $V$ consisting of eigenvectors of $T^{[\ell_L; \ell_R]}$. Moreover, each $v_{\alpha_R; \alpha_L}^{\text{slit}} \in V$ is an eigenvector of $T^{[\ell_L; \ell_R]}$ with eigenvalue

$$\mu_{\alpha_R; \alpha_L}^{(\ell_L; \ell_R)} = \frac{\mu_{\emptyset; \emptyset}^{(\ell_L; \ell_R)}}{\prod_{k \in \alpha_L} \Lambda_k^{(\ell_L)} \prod_{k' \in \alpha_R} \Lambda_{k'}^{(\ell_R)}}.$$

**Proof.** In view of the polarization (5.12), the proof is similar to that of Proposition 5.9 \qed

**Slit-strip adapted decompositions in the Clifford algebra.** For performing calculations with the creation and annihilation operators associated with each of the three extremities, we introduce yet another set of operators for each extremity. Informally speaking, these are constructed so that they have a specified creation operator part, while the annihilation operator part is chosen so as to make the underlying Clifford generator valued one-form globally defined even in the scaling limit (for the top extremity we in fact care about creation part for the contragredient action, so we actually specify the annihilation part instead). The primary advantage for calculations stems from the fact that global contour deformation is then possible, unlike with the creation and annihilation operators themselves, which used locally defined coefficient functions.

Let

$$\mathfrak{F}_{T; k}: E(S^{(\ell)}_{\text{slit}}) \to \mathbb{C} \quad \mathfrak{F}_{L; k}: E(S^{(\ell)}_{\text{slit}}) \to \mathbb{C} \quad \mathfrak{F}_{R; k}: E(S^{(\ell)}_{\text{slit}}) \to \mathbb{C}$$

for $k \in K^{(\ell)}$ for $k \in K^{(\ell_L)}$ for $k \in K^{(\ell_R)}$

denote the functions as in Proposition 4.2. Recall that each of these is s-holomorphic, has Riemann boundary values in the whole slit-strip $S^{(\ell)}_{\text{slit}}$.

From each of these s-holomorphic functions, we construct globally defined coefficient function pairs as before, as in, e.g., Lemmas 3.16 and 3.17. For consistency with creation and annihilation operators, we moreover multiply by $e^{i\pi/2}$ before integration. The associated operators are defined as integrals across the full cross section contour $\gamma_0$ at zero height: we
consider
\[ p_{T,k}^{(\ell)} := \frac{e^{i\pi}}{2} \int_{\gamma_0}^{\gamma_0} \left( i \hat{\mathcal{P}}_{T,k}(z) \hat{\psi}(z) \, dz - i \hat{\mathcal{P}}_{T,k}(z) \hat{\psi}^*(z) \, dz \right) \quad \text{for } k \in \mathcal{K}^{(\ell)}, \]
\[ p_{L,k}^{(\ell)} := \frac{e^{i\pi}}{2} \int_{\gamma_0}^{\gamma_0} \left( i \hat{\mathcal{P}}_{L,k}(z) \hat{\psi}(z) \, dz - i \hat{\mathcal{P}}_{L,k}(z) \hat{\psi}^*(z) \, dz \right) \quad \text{for } k \in \mathcal{K}^{(\ell_L)}, \]
\[ p_{R,k}^{(\ell)} := \frac{e^{i\pi}}{2} \int_{\gamma_0}^{\gamma_0} \left( i \hat{\mathcal{P}}_{R,k}(z) \hat{\psi}(z) \, dz - i \hat{\mathcal{P}}_{R,k}(z) \hat{\psi}^*(z) \, dz \right) \quad \text{for } k \in \mathcal{K}^{(\ell_R)}. \]

Since the integrations are a priori done on the cross-section at height zero, these in fact only involve the restrictions
\[ \mathcal{P}_{T,k} \big|_{\mathcal{I}^*} = p_{T,k}, \quad \mathcal{P}_{L,k} \big|_{\mathcal{I}^*} = p_{L,k}, \quad \mathcal{P}_{R,k} \big|_{\mathcal{I}^*} = p_{R,k}, \]
which are elements of the real Hilbert space \( \mathcal{F}^{(\ell)} = \mathbb{C}^{\mathcal{I}^*} \) of complex-valued functions on the cross-section, \( \mathcal{I}^* \).

Similarly, e.g., \( f_{k}^{(\ell)} \) for a given \( k > 0 \) is defined by an integral \( \langle 5.7 \rangle \) across the cross-section at height zero, and only the restriction
\[ \mathfrak{F}_k \big|_{\mathcal{I}^*} = f_k \]
appears in the coefficient functions on this cross section. The modes \( f_{L;-k}^{(\ell)} \) and \( f_{R;-k}^{(\ell)} \) are defined by integrals across the left and right halves of the cross-section at zero height, and only the restrictions
\[ \mathfrak{F}_{L;-k} \big|_{\mathcal{I}_L^*} = f_{L;-k}, \quad \mathfrak{F}_{R;-k} \big|_{\mathcal{I}_R^*} = f_{R;-k}, \]
to the two halves, \( \mathcal{I}_L^* = [a, 0]^* \) and \( \mathcal{I}_R^* = [0, b]^* \), appear in the coefficient functions. Furthermore, we can extend the integration to range across the full cross-section provided that the coefficient functions are extended as zero on the complementary half. This allows us to still view the restrictions as elements of the function space \( \mathcal{F}^{(\ell)} \).

Decompositions of the coefficient functions in the function space \( \mathcal{F}^{(\ell)} \) therefore obviously yield corresponding decompositions of the operators. The following lemma phrases the decompositions in such a way as to allow replacing any creation operators with linear combinations of annihilation operators. This, in turn, will yield recursions for the fusion coefficients that we will study soon.

**Lemma 5.18.** For any \( k \in \mathcal{K}^{(\ell)} \), the operator \( f_{k}^{(\ell)} \) can be decomposed as
\[ f_{k}^{(\ell)} = - \sum_{k' \in \mathcal{K}^{(\ell)}} \langle f_{-k'}, p_{T,k} \rangle f_{-k'}^{(\ell)} + \sum_{k' \in \mathcal{K}^{(\ell_L)}} \langle f_{L;k'}, p_{T,k} \rangle f_{L;k'}^{(\ell)} + \sum_{k' \in \mathcal{K}^{(\ell_R)}} \langle f_{R;k'}, p_{T,k} \rangle f_{R;k'}^{(\ell)}. \]
For any \( k \in \mathcal{K}^{(\ell_L)} \), the operator \( f_{L;-k}^{(\ell)} \) can be decomposed as
\[ f_{L;-k}^{(\ell)} = \sum_{k' \in \mathcal{K}^{(\ell)}} \langle f_{-k'}, p_{L,k} \rangle f_{-k'}^{(\ell)} - \sum_{k' \in \mathcal{K}^{(\ell_L)}} \langle f_{L;k'}, p_{L;k} \rangle f_{L;k'}^{(\ell)} - \sum_{k' \in \mathcal{K}^{(\ell_R)}} \langle f_{R;k'}, p_{L;k} \rangle f_{R;k'}^{(\ell)}. \]
For any \( k \in \mathcal{K}^{(\ell n)} \), the operator \( f_{R;\neg k}^{(\ell)} \) can be decomposed as

\[
(5.17) \quad f_{R;\neg k}^{(\ell)} = \sum_{k' \in \mathcal{K}^{(\ell)}} \langle f_{-k'}, p_{R;\ell} \rangle f_{-k'} - \sum_{k' \in \mathcal{K}^{(\ell)}(L)} \langle f_{L;\ell}, p_{R;\ell} \rangle f_{L;\ell} - \sum_{k' \in \mathcal{K}^{(\ell)}(R)} \langle f_{R;\ell}, p_{R;\ell} \rangle f_{R;\ell}.
\]

**Proof.** By Proposition 4.2, the functions \( f_k \) and \( p_{T;\ell} \) have the same singular part in the top extremity. We can therefore expand their difference in terms of functions \( f_{-k'}, k' > 0 \), which are regular in the top extremity,

\[
f_k - p_{T;\ell} = \sum_{k' \in \mathcal{K}^{(\ell)}} c_{k,k'} f_{-k'},
\]

with certain real coefficients \( c_{k,k'} \in \mathbb{R} \). Taking inner products of both sides with \( f_{-k'}, k' \in \mathcal{K}^{(\ell)} \), we see that the coefficients are given by \( c_{k,k'} = -\langle f_{-k'}, p_{T;\ell} \rangle \).

In view of the defining formula (5.7) and the above expansion, we find

\[
f_k^{(\ell)} = p_{T;\ell}^{(\ell)} - \sum_{k' \in \mathcal{K}^{(\ell)}} \langle f_{-k'}, p_{T;\ell} \rangle f_{-k'}^{(\ell)}.
\]

On the other hand, the function \( p_{T;\ell} \) (or its extension \( \mathfrak{p}_{T;\ell} \)) has no singularities in the left and right bottom extremities of the slit-strip. We can therefore expand the restrictions of \( p_{T;\ell} \) to the left and right halves of the cross section in terms of the functions \( f_{L;\ell} \) and \( f_{R;\ell} \), \( k' > 0 \), which are regular in the respective extremities,

\[
p_{T;\ell}|_{\Gamma_L} = \sum_{k' \in \mathcal{K}^{(\ell)}(L)} \langle f_{L;\ell}, p_{T;\ell} \rangle f_{L;\ell}, \quad p_{T;\ell}|_{\Gamma_R} = \sum_{k' \in \mathcal{K}^{(\ell)}(R)} \langle f_{R;\ell}, p_{T;\ell} \rangle f_{R;\ell}.
\]

If we now split the integration in \( p_{T;\ell}^{(\ell)} \) across the cross section \( \gamma_0 \) in two parts, \( \gamma_0^L \) across the left substrip and \( \gamma_0^R \) across the right substrip, and use the above decompositions, we get

\[
p_{T;\ell}^{(\ell)} = \sum_{k' \in \mathcal{K}^{(\ell)}(L)} \langle f_{L;\ell}, p_{T;\ell} \rangle f_{L;\ell}^{(\ell)} + \sum_{k' \in \mathcal{K}^{(\ell)}(R)} \langle f_{R;\ell}, p_{T;\ell} \rangle f_{R;\ell}^{(\ell)}.
\]

Combining the above, we obtain the first asserted formula. The other two are similar. \( \square \)

**5.9. Fusion coefficients.** In Theorem 5.10(7), boundary correlation functions in the truncated slit-strip were written in terms of quantities

\[
(5.18) \quad v_{\text{out}}^{\dagger}(T^{[\ell]} h_T(T^{[\ell]} h_T v_{\text{in}})^h_B v_{\text{in}},
\]

where \( v_{\text{in}}, v_{\text{out}} \in \mathcal{V} \) were suitably chosen vectors encoding either the boundary conditions or the correlation functions in question.

In the infinite volume limit \( h_T, h_B \to \infty \), the quantities (5.18) are dominated by the eigenvectors of \( T^{[\ell]} \) and \( T^{[\ell]} h_T \) of largest eigenvalues onto which \( v_{\text{out}} \) and \( v_{\text{in}} \) have non-vanishing projections. Specifically, if we take \( v_{\text{out}} \) and \( v_{\text{in}} \) to be such eigenvectors, explicitly given in Propositions 5.9 and 5.17,

\[
v_{\text{out}} = v_{\alpha} \quad \text{and} \quad v_{\text{in}} = v_{\alpha_{\ell}}^{\text{slit}},
\]

then the matrix element (5.18) is simply

\[
v_{\alpha}^{\dagger}(T^{[\ell]} h_T(T^{[\ell]} h_T)^h_B v_{\alpha_{\ell}}^{\text{slit}} = (\mu_{\alpha}^{(\ell)} h_T(\mu_{\alpha_{\ell}^{(\ell)} h_T} h_B v_{\alpha}^{\dagger} v_{\alpha_{\ell}}^{\text{slit}}.
\]

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The eigenvalues give the rate of exponential growth of the quantity, while the height-independent constant factor

\begin{equation}
\Phi_{\alpha;\alpha_R,\alpha_L}^{(\ell)} := v_{\alpha}^\dagger v_{\alpha_R} v_{\alpha_L}^{\text{slit}} = v_{\alpha}^\dagger \left( \prod_{k \in \alpha} f_k^{(\ell)} \right) \left( \prod_{k \in \alpha_R} f_{\ell_1-k}^{(\ell)} \right) v_{\alpha_R}^\dagger v_{\alpha_L}^{\text{slit}} \right)
\end{equation}

quantifies how the incoming states at the bottom of the left and right leg of the slit-strip (indexed by \(\alpha_L, \alpha_R\), respectively) combine to produce an outgoing state (indexed by \(\alpha\)) in the slit-strip geometry. We call these factors \((5.19)\) the **fusion coefficients** of the Ising model in the slit-strip.

Let us concretely exemplify the interpretations of the fusion coefficients as renormalized boundary correlation functions at the three extremities of the slit-strip. Fix \(\alpha \in \mathcal{K}^{(\ell)}, \alpha_R \subset \mathcal{K}^{(\ell)}, \) and \(\alpha_L \subset \mathcal{K}^{(\ell)}\). If \(f_T: \{\pm 1\}^Z \to \mathbb{C}\) is the (unique) function such that \(v^{(f_T)} = v_\alpha\), and if \(f_{B;R}: \{\pm 1\}^{2\ell} \to \mathbb{C}\) and \(f_{B;L}: \{\pm 1\}^{2\ell} \to \mathbb{C}\) are the functions (unique up to cancelling multiplicative constants in both) such that \(v^{(f_{B;R};f_{B;L})} = v^{\text{slit}}_{\alpha_R;\alpha_L}\), then by Theorem 5.10(7), the associated boundary correlation function with locally monochromatic boundary conditions in the slit-strip has the following infinite slit-strip renormalized limit

\[
\lim_{\ell_T,\ell_B \to \infty} \mathbb{E}_{\text{slit}}^{(\ell_T;\ell_B)} \left[ f_T(\sigma|_{\partial_T \mathcal{S}(\ell_T;\ell_B)}) f_{B;R}(\sigma|_{\partial_{B;R} \mathcal{S}(\ell_T;\ell_B)}) f_{B;L}(\sigma|_{\partial_{B;L} \mathcal{S}(\ell_T;\ell_B)}) \right] \left( \mu_\alpha(\ell_T,\ell_B) / \mu_\emptyset(\ell_T,\ell_B) \right)^{\ell_T} \left( \mu_{\alpha_R}(\ell_T,\ell_B) / \mu_\emptyset(\ell_T,\ell_B) \right)^{\ell_B} = \frac{1}{\chi_{\text{mono}}^{(\ell)}} \Phi_{\alpha;\alpha_R,\alpha_L}^{(\ell)} \Phi_{\emptyset;\emptyset,\emptyset}^{(\ell)},
\]

where \(\chi_{\text{mono}}^{(\ell)} = 2 v_\emptyset^{(\ell)} v_\emptyset^{(\ell)} \). Note that the constant prefactors are positive numbers: \(\chi_{\text{mono}}^{(\ell)} > 0 \) and \(\Phi_{\emptyset;\emptyset,\emptyset}^{(\ell)} > 0\), since they involve inner products of Perron-Frobenius eigenvectors with other vectors with nonnegative components, and there are overlapping non-zero components in each case. For other boundary conditions we get slightly different formulas, but with the same structure. For example, consider the minus-plus-plus boundary conditions (minus on the left boundary \(\partial_L \mathcal{S}(\ell_T;\ell_B)\), plus on the slit \(\partial_{\text{slit}} \mathcal{S}(\ell_T;\ell_B)\), plus on the right boundary \(\partial_R \mathcal{S}(\ell_T;\ell_B)\)). Assume furthermore that \(\alpha, \alpha_R, \alpha_L\) are chosen so that the functions \(f_T, f_{B;R}, f_{B;L}\) are supported on the spin configurations allowed by these boundary conditions (this assumption obviously causes no loss of generality for nontrivial boundary correlation functions, and it lets us avoid certain projections). Then, modifying Theorem 5.10(7) as indicated in Remark 5.4, one finds

\[
\lim_{\ell_T,\ell_B \to \infty} \mathbb{E}_{-++ \text{-slit}}^{(\ell_T;\ell_B)} \left[ f_T(\sigma|_{\partial_T \mathcal{S}(\ell_T;\ell_B)}) f_{B;R}(\sigma|_{\partial_{B;R} \mathcal{S}(\ell_T;\ell_B)}) f_{B;L}(\sigma|_{\partial_{B;L} \mathcal{S}(\ell_T;\ell_B)}) \right] \left( \mu_\alpha(\ell_T,\ell_B) / \mu_\emptyset(\ell_T,\ell_B) \right)^{\ell_T} \left( \mu_{\alpha_R}(\ell_T,\ell_B) / \mu_\emptyset(\ell_T,\ell_B) \right)^{\ell_B} = \frac{1}{\chi_{-++}^{(\ell)}} \Phi_{\alpha;\alpha_R,\alpha_L}^{(\ell)} \Phi_{\emptyset;\emptyset,\emptyset}^{(\ell)},
\]
where $z^{(\ell)}_{++} = v_1^\dagger \cdot v^{(1)} \cdot (v^{(1)})^\dagger \cdot v_{\{\frac{1}{2}\}}^{\text{slit}}$. Note again that the constant prefactors are positive numbers: $z^{(\ell)}_{++} > 0$ and $\Phi^{(\ell)}_{\{\frac{1}{2}\};\emptyset,\emptyset} > 0$.

As we vary $\alpha, \alpha_R, \alpha_L$, the associated functions $f_T, f_{B;R}, f_{B;L}$ form a basis of functions on the spin configurations on the three boundary components, and the above formulas illustrate the general property that (ratios of) the fusion coefficients capture renormalized limits of boundary correlation functions with both the locally monochromatic as well as any of the fixed monochromatic boundary conditions. A minor difference due to the boundary conditions remains in the prefactors $z_{\text{mono}}$ and $\Phi^{(\ell)}_{\{\frac{1}{2}\};\emptyset,\emptyset}$ versus $z^{(\ell)}_{++}$ and $\Phi^{(\ell)}_{\{\frac{1}{2}\};\emptyset,\emptyset}$ in the two examples above. More drastically, even the appropriate exponential renormalizations (the denominators on the left hand sides) that are needed to form nontrivial limits of the boundary correlation functions are different. Despite that, the (ratios of) fusion coefficients capture the suitably renormalized limits of boundary correlation functions in all cases.

Our main result of this second part of the series will be that the fusion coefficients (5.19) converge in the limit $\ell \to \infty$ of infinite strip width; this will be proven in Section 7. In the final part of the series we will show that this limit of the fusion coefficients fully recovers the algebraic structure of a (chiral) conformal field theory. We now proceed to give a characterization of the fusion coefficients, which lends itself to those purposes.

**Recursion for the fusion coefficients.** Recall from Propositions 5.9 and 5.17 that the eigenvectors $v_\alpha$ (5.10) of the Ising transfer matrix $T^{[\ell]}$ are indexed by subsets

$$\alpha \subset \mathcal{K}^{(\ell)} = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{\ell - 1}{2} \right\},$$

and the eigenvectors $v^{\text{slit}}_{\alpha_L;\alpha_R}$ (5.13) of the Ising transfer matrix $T^{[\ell_L,\ell_R]}$ for the slit part are indexed by pairs of subsets

$$\alpha_L \subset \mathcal{K}^{(\ell_L)} = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{\ell_L - 1}{2} \right\}, \quad \alpha_R \subset \mathcal{K}^{(\ell_R)} = \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, \frac{\ell_R - 1}{2} \right\}.$$

We write the elements of the subsets in increasing order as in (5.9),

$$\alpha = \{ k_1, \ldots, k_m \} \quad \text{with} \quad 0 < k_1 < \cdots < k_m,$$

and similarly for $\alpha_L = \{ k^L_1, \ldots, k^L_{m_L} \}$ and $\alpha_R = \{ k^R_1, \ldots, k^R_{m_R} \}$. The number of elements in such a subset $\alpha = \{ k_1, \ldots, k_m \}$ of positive half-integers will be denoted by

$$\# \alpha = m.$$
We also use the signed indicator notation

\[ \epsilon_\alpha(k) = \begin{cases} (-1)^{m-j} & \text{if } k = k_j \\ 0 & \text{if } k \notin \alpha, \end{cases} \]

whenever \( \alpha = \{k_1, \ldots, k_m\} \) is a subset of positive half-integers as above and \( k \) is a given positive half-integer, and we write \( \alpha \setminus \{k\} \) for the subset where \( k \) has been removed from \( \alpha \). Such notations simplify, for example, the following calculations with eigenvectors of the transfer matrices \( T^{[\ell,L]} \) and \( T^{[\ell]} \).

**Lemma 5.19.** Let \( \alpha_L \subset \mathcal{K}^{(\ell_L)} \) and \( \alpha_R \subset \mathcal{K}^{(\ell_R)} \).

For \( k \in \mathcal{K}^{(\ell_L)} \) we have

\[ f_{L,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = (-1)^{\# \alpha_R} \epsilon_\alpha(k) v_{\alpha_L \setminus \{k\}; \alpha_R}^{\text{slit}}, \]

and for \( k \in \mathcal{K}^{(\ell_R)} \) we have

\[ f_{R,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = \epsilon_\alpha(k) v_{\alpha_L \setminus \{k\}; \alpha_R}^{\text{slit}}. \]

**Lemma 5.20.** For any \( \alpha \subset \mathcal{K}^{(\ell)} \) and \( k \in \mathcal{K}^{(\ell)} \) we have

\[ v_\alpha f_{L,k} = \epsilon_\alpha(k) v_\alpha \]

*Proofs of Lemmas 5.19 and 5.20.* The proofs of all three formulas (5.21), (5.22), and (5.23) are similar and completely standard. We do the first of these below.

From the defining formula (5.13) of \( v_{\alpha_R;\alpha_L}^{\text{slit}} \), we get

\[ f_{L,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = f_{L,k} f_{R,-k_R} f_{R,-k_{m_R}} \cdots f_{R,-k_{1_R}} f_{L;-k_{m_L}} f_{L;-k_{1_L}} \]

Now by Proposition 5.13 \( f_{L,k} \) anticommutes with each \( f_{R,-k}\), so we can first of all rewrite

\[ f_{L,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = (-1)^{m_R} f_{R,-k_R} \cdots f_{R,-k_{1_R}} f_{L;-k_{m_L}} f_{L;-k_{1_L}} v_{\alpha_R;\alpha_L}^{\text{slit}}. \]

Moreover, \( f_{L,k} \) anticommutes with \( f_{L;-k_{1_L}} \) except if \( k_{1_L} = k \), so further anticommuting it to the right, we get

\[ f_{L,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = \sum_{j=1}^{m_L} (-1)^{m_R+m_L-j} \delta_{k,k_{1_L}} f_{R,-k_{m_R}} \cdots f_{R,-k_{1_R}} f_{L;-k_{m_L}} \cdots f_{L;-k_{1_L}} v_{\alpha_R;\alpha_L}^{\text{slit}}. \]

The last term vanishes in view of the annihilation property \( f_{L,k} v_{\alpha_R;\alpha_L}^{\text{slit}} = 0 \) of Proposition 5.16. Since \( k_{1_L}, \ldots, k_{m_L} \) are distinct, at most one term in the sum over \( j \) can be non-vanishing, and this happens if \( k \in \alpha_L \). In that case the sign of the term is \((-1)^{m_R} (-1)^{m_L-j} = (-1)^{\# \alpha_R} \epsilon_\alpha(k)\), and the remaining creation operators applied to the vacuum yield the vector \( v_{\alpha_L \setminus \{k\}; \alpha_R}^{\text{slit}} \). Formula (5.21) is thus established. \( \square \)

We now show how the decompositions of Lemma 5.18 yield recursions for the fusion coefficients, which in fact uniquely determine them apart from an overall multiplicative constant that cancels in the ratios that we are interested in.
Theorem 5.21. The collection \((\Phi^{(i)}_{\alpha;\alpha_L,\alpha_R})_{\alpha \in \mathcal{K}^{(i)},\alpha_L \in \mathcal{K}^{(\ell_L)},\alpha_R \in \mathcal{K}^{(\ell_R)}}\) of all fusion coefficients satisfies the following properties, which furthermore characterize the collection up to an overall positive multiplicative constant:

(REC0\(^{(i)}\)) We have \(\Phi^{(i)}_{\emptyset;\emptyset,\emptyset} > 0\).

(REC\(^{(i)}\)\(_T\)) If \(\alpha \subset \mathcal{K}^{(i)},\alpha_L \subset \mathcal{K}^{(\ell_L)},\alpha_R \subset \mathcal{K}^{(\ell_R)}\) and \(\alpha' = \alpha \cup \{k\}\) with \(\max(\alpha) < k < \ell\), then we have

\[
\Phi^{(i)}_{\alpha';\alpha_R,\alpha_L} = \sum_{k' \in \alpha_L} \langle f_{L,k'}, p_T; k \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L}(k') \Phi^{(i)}_{\alpha;\alpha_L \setminus \{k\},\alpha_R} + \sum_{k' \in \alpha_R} \langle f_{R,k'}, p_T; k \rangle \epsilon_{\alpha_R}(k') \Phi^{(i)}_{\alpha;\alpha_L,\alpha_R \setminus \{k\}} - \sum_{k' \in \alpha} \langle f_{-k'}, p_T; k \rangle \epsilon_{\alpha}(k') \Phi^{(i)}_{\alpha \setminus \{k'\};\alpha_R,\alpha_L}.
\]

(REC\(^{(i)}\)\(_L\)) If \(\alpha \subset \mathcal{K}^{(i)},\alpha_L \subset \mathcal{K}^{(\ell_L)},\alpha_R \subset \mathcal{K}^{(\ell_R)}\) and \(\alpha'_L = \alpha_L \cup \{k\}\) with \(\max(\alpha_L) < k < \ell_L\), then we have

\[
\Phi^{(i)}_{\alpha;\alpha'_L,\alpha_R} = -\sum_{k' \in \alpha_L} \langle f_{L,k'}, p_L; k \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L}(k') \Phi^{(i)}_{\alpha;\alpha_L \setminus \{k\},\alpha_R} - \sum_{k' \in \alpha_R} \langle f_{R,k'}, p_L; k \rangle \epsilon_{\alpha_R}(k') \Phi^{(i)}_{\alpha;\alpha_L,\alpha_R \setminus \{k\}} + \sum_{k' \in \alpha} \langle f_{-k'}, p_L; k \rangle \epsilon_{\alpha}(k') \Phi^{(i)}_{\alpha \setminus \{k'\};\alpha_R,\alpha_L}.
\]

(REC\(^{(i)}\)\(_R\)) If \(\alpha \subset \mathcal{K}^{(i)},\alpha_L \subset \mathcal{K}^{(\ell_L)},\alpha_R \subset \mathcal{K}^{(\ell_R)}\) and \(\alpha'_R = \alpha_R \cup \{k\}\) with \(\max(\alpha_R) < k < \ell_R\), then we have

\[
\Phi^{(i)}_{\alpha;\alpha_L,\alpha'_R} = -\sum_{k' \in \alpha_L} \langle f_{L,k'}, p_R; k \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L}(k') \Phi^{(i)}_{\alpha;\alpha_L \setminus \{k\},\alpha_R} - \sum_{k' \in \alpha_R} \langle f_{R,k'}, p_R; k \rangle \epsilon_{\alpha_R}(k') \Phi^{(i)}_{\alpha;\alpha_L,\alpha_R \setminus \{k\}} + \sum_{k' \in \alpha} \langle f_{-k'}, p_R; k \rangle \epsilon_{\alpha}(k') \Phi^{(i)}_{\alpha \setminus \{k'\};\alpha_R,\alpha_L}.
\]

Proof. Let us first address the uniqueness statement. The three recursions, (REC\(^{(i)}\)\(_T\)), (REC\(^{(i)}\)\(_L\)), and (REC\(^{(i)}\)\(_R\)), allow one to express any coefficient \(\Phi^{(i)}_{\alpha;\alpha_R,\alpha_L}\) as a linear combination of the coefficients with total size strictly less than \(#\alpha + \#\alpha_L + \#\alpha_R\). The total size is a non-negative integer, so inductively these properties allow to write any \(\Phi^{(i)}_{\alpha;\alpha_R,\alpha_L}\) as a multiple of the only coefficient \(\Phi^{(i)}_{\emptyset;\emptyset,\emptyset}\) with total size zero. The initial condition (RECO\(^{(i)}\)) specifies the sign of this coefficient, and therefore the whole collection \((\Phi^{(i)}_{\alpha;\alpha_R,\alpha_L})_{\alpha,L,R}\) gets determined up to a positive overall multiplicative factor \(\Phi^{(i)}_{\emptyset;\emptyset,\emptyset}\).

The positivity property (RECO\(^{(i)}\)) of the initial coefficient \(\Phi^{(i)}_{\emptyset;\emptyset,\emptyset}\) follows directly from the defining formula (5.19), once one notices that the vectors involved in the inner products are
Perron-Frobenius eigenvectors with non-negative entries, and there is non-empty overlap of components where the entries are non-vanishing.

Consider then \((\text{REC}_{T}^\ell)\). We must calculate

\[
\Phi_{\alpha';\alpha_R;\alpha_L}^{(\ell)} = v^\dagger_{\alpha'} v_{\alpha_R;\alpha_L}^{\text{slit}} = (f_{-k}^{(\ell)} v_{\alpha}^{\dagger}) v_{\alpha_R;\alpha_L}^{\text{slit}} = v^\dagger_k f_{\alpha}^{(\ell)} v_{\alpha_R;\alpha_L}^{\text{slit}}.
\]

To this end, we use formula (5.15) of Lemma 5.18 for \(f_k^{(\ell)}\),

\[
f_k^{(\ell)} = - \sum_{k' \in K^{(i)}} \langle f_{-k'}, p_{T;k} \rangle f_{-k'}^{(\ell)} + \sum_{k' \in K^{(iL)}} \langle f_{L;k'}, p_{T;k} \rangle f_{L;k'}^{(\ell)} + \sum_{k' \in K^{(iR)}} \langle f_{R;k'}, p_{T;k} \rangle f_{R;k'}^{(\ell)}.
\]

To find the contribution from the second sum above to the fusion coefficient \(\Phi_{\alpha';\alpha_R;\alpha_L}^{(\ell)}\), we use (5.21) from Lemma 5.19 in

\[
\sum_{k' \in K^{(iL)}} \langle f_{L;k'}, p_{T;k} \rangle v^\dagger_{\alpha} v_{\alpha_R;\alpha_L}^{\text{slit}} = \sum_{k' \in \alpha_L} \langle f_{L;k'}, p_{T;k} \rangle (-1)^{\# \alpha_R} \epsilon_{\alpha_L}(k') v^\dagger_{\alpha} v_{\alpha_L \setminus \{k'\}; \alpha_R}^{\text{slit}}
\]

\[
= \sum_{k' \in \alpha_L} \langle f_{L;k'}, p_{T;k} \rangle (-1)^{\# \alpha_R} \epsilon_{\alpha_L}(k') \Phi_{\alpha;\alpha_L \setminus \{k'\}; \alpha_R}^{(\ell)}.
\]

Similarly using (5.21), the contribution of the third becomes

\[
\sum_{k' \in K^{(iR)}} \langle f_{R;k'}, p_{T;k} \rangle v^\dagger_{\alpha} v_{\alpha_R;\alpha_L}^{\text{slit}} = \sum_{k' \in \alpha_R} \langle f_{R;k'}, p_{T;k} \rangle \epsilon_{\alpha_R}(k') \Phi_{\alpha;\alpha_R \setminus \{k'\}; \alpha_L}^{(\ell)}
\]

For the contribution of the first sum we use (5.23) from Lemma 5.20 and get

\[
- \sum_{k' \in K^{(i)}} \langle f_{-k'}, p_{T;k} \rangle v^\dagger_{\alpha} f_{-k'}^{(\ell)} v_{\alpha_R;\alpha_L}^{\text{slit}} = - \sum_{k' \in \alpha} \langle f_{-k'}, p_{T;k} \rangle \epsilon_{\alpha}(k') \Phi_{\alpha;\alpha_L \setminus \{k'\}; \alpha_R}^{(\ell)}
\]

Combining the terms, we obtain the formula asserted in \((\text{REC}_{T}^\ell)\).

The proofs of \((\text{REC}_{T}^\ell)\) and \((\text{REC}_{R}^\ell)\) are similar. \(\square\)

6. Scaling limits of distinguished functions

In this section we discuss the continuum analogues of the distinguished lattice functions of Section 4 and recall the relevant scaling limit results from [AKPR20].

The appropriate continuum domains are the following two simply connected open sets of the complex plane: the vertical strip

\[
S = \left\{ z \in \mathbb{C} \mid -\frac{1}{2} < \Re(z) < \frac{1}{2} \right\}
\]

and the vertical slit-strip

\[
S_{\text{slit}} = S \setminus \{ iy \mid y \leq 0 \}.
\]
These are illustrated in Figures 6.1(a) and 6.1(b) respectively. We consider holomorphic functions $F$ on these domains, occasionally also just locally defined functions on one of the following three subsets

\[ S_{\text{slit}}^T = \{ z \in S_{\text{slit}} \mid \Im(z) \geq 0 \}, \]

\[ S_{\text{slit}}^L = \{ z \in S_{\text{slit}} \mid \Im(z) \leq 0, \Re(z) < 0 \}, \]

\[ S_{\text{slit}}^R = \{ z \in S_{\text{slit}} \mid \Im(z) \leq 0, \Re(z) > 0 \}. \]

We again focus on the restrictions of such functions $F$ to the horizontal cross-section

\[ I = \left[ -\frac{1}{2}, \frac{1}{2} \right]. \]

The continuum analogue of the discrete function space (2.6) is the real Hilbert space

\[ \mathcal{L}^2 = L^2_{\mathbb{R}}(I, \mathbb{C}) \]

of complex-valued square-integrable functions on $I$, with the inner product and norm given by

\[ \langle f, g \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Re \left( f(x) \overline{g(x)} \right) \, dx, \quad \| f \|^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^2 \, dx. \]

The boundary of the strip $S$ has two components: the left boundary $\{ -\frac{1}{2} + iy \mid y \in \mathbb{R} \}$ and the right boundary $\{ \frac{1}{2} + iy \mid y \in \mathbb{R} \}$. The boundary of the slit-strip $S_{\text{slit}}$ additionally has the slit component $\{ iy \mid y \leq 0 \}$, and in fact each of the points $iy$ with $y < 0$ corresponds to two prime ends, which we denote $0^- + iy$ and $0^+ + iy$, and interpret as the boundaries as seen from the left and right legs of the slit-strip. By the right boundary of the slit strip, we then mean $\{ \frac{1}{2} + iy \mid y \in \mathbb{R} \} \cup \{ 0^- + iy \mid y < 0 \}$ and by the left boundary correspondingly $\{ -\frac{1}{2} + iy \mid y \in \mathbb{R} \} \cup \{ 0^+ + iy \mid y < 0 \}$. A holomorphic function $F$ in any of these domains or their subdomains is said to have Riemann boundary values if $F$ has a continuous extension to the left and right boundaries, and the values on the left boundary are in $e^{-\frac{i\pi}{2}} \mathbb{R}$ and the values on the right boundary are in $e^{\frac{i\pi}{2}} \mathbb{R}$, and additionally the (possible) singularity of $F$ at the “tip” $0 \in \partial S_{\text{slit}}$ of the slit is such that the restriction to the cross-section remains square-integrable, $f = F|_I \in \mathcal{L}^2$.

Indexing sets. The appropriate indexing sets for the continuum functions are the set $\mathcal{K}$ of positive half-integers (4.1) and the set $\pm \mathcal{K} = \mathbb{Z} + \frac{1}{2}$ of all half-integers. In the next section, the continuum fusion coefficients will be indexed by the set $\{ \alpha \in \mathcal{K} \}$ of all finite subsets of positive half-integers.

6.1. Distinguished functions in the continuum strip. The following quarter-integer Fourier modes and their analytic continuations with Riemann boundary values in the continuum strip $S$ are the appropriate continuum counterparts of the discrete functions $\mathfrak{f}_k, \mathfrak{g}_k$ from Section 4 Proposition 4.1.

For $k \in \pm \mathcal{K}$ a half-integer, let $C_k = e^{i\pi(-k/2-1/4)}$ and define

\[ E_k : S \to \mathbb{C} \quad E_k(x + iy) := C_k \exp \left( -i\pi k x + \pi ky \right), \]

\[ e_k : I \to \mathbb{C} \quad e_k(x) := C_k e^{-i\pi k x}, \]
so that \( e_k = E_k|_I \). Note that we have the vertical translation eigenfunction property, boundary conditions, and normalization

\[
E_k(z + i h) = e^{\pi kh} E_k(z), \quad E_k \left( \frac{-1}{2} + i y \right) \in e^{-i\pi/4} \mathbb{R}^+, \quad \| e_k \| = 1.
\]  

This collection \((e_k)_{k \in \pm \mathcal{K}}\) of functions forms an orthonormal basis of \( L^2 \) [AKPR20, Proposition 2.1]. In analogy with the reflection formulas (4.6), we also have

\[
e^{-k}(x) = -i e_k(x), \quad E^{-k}(x + i y) = -i E_k(x - iy).
\]  

It was shown in [AKPR20, Theorem 4.3] that the functions \( f_k, \tilde{f}_k \) of Proposition 4.1 converge in the scaling limit to \( e_k, E_k \), respectively (uniformly on compact subsets, for example).

**Decomposition to poles and zeroes at the top.** The functions \( e_{+k} \), for \( k \in \mathcal{K} \), are exponentially growing in the upwards direction of the continuum strip \( \mathbb{S} \), while the functions \( e_{-k} \) are exponentially decaying. We interpret the growing ones and having a pole at the top extremity, and the decaying ones as having a zero.

We decompose the function space \( L^2 \) into the corresponding subspaces. In analogy with (4.7), this decomposition amounts to defining the closed subspaces

\[
L^2_{T;\text{pole}} = \text{span} \left\{ e_{+k} \mid k \in \mathcal{K} \right\}, \quad L^2_{T;\text{zero}} = \text{span} \left\{ e_{-k} \mid k \in \mathcal{K} \right\}.
\]
and the corresponding orthogonal projections
\[ \Pi_{T;\text{pole}} : \mathcal{L}^2 \to \mathcal{L}_{T;\text{pole}}^2, \quad \Pi_{T;\text{zero}} : \mathcal{L}^2 \to \mathcal{L}_{T;\text{zero}}^2. \]

6.2. Distinguished functions in the slit-strip. In the slit-strip there are three infinite extremities: the top, the right leg, and the left leg. In the left and right substrips there are natural functions defined analogously to the whole strip, which allow us to define poles and zeroes, and obtain corresponding decompositions. The most important distinguished functions in the slit-strip will then be globally defined functions which have zeroes (i.e., regular behavior) in two of the three infinite extremities, and have a pole of a given order in the third (i.e., a prescribed singular part).

Poles and zeroes in the left and right legs. In the continuum setting we define the two halves \( I_L := [-\frac{1}{2}, 0] \) and \( I_R := [0, \frac{1}{2}] \) of the cross-section, and the closed subspaces
\[ \mathcal{L}_L^2 := L^2_{\mathbb{R}}(I_L, \mathbb{C}) \subset \mathcal{L}^2, \quad \mathcal{L}_R^2 := L^2_{\mathbb{R}}(I_R, \mathbb{C}) \subset \mathcal{L}^2, \]
of functions with support on one of the two halves. For \( k \in \pm \mathbb{K} \), let \( C_k^L = \sqrt{2} e^{i\pi(-k-1/4)} \) and \( C_k^R = \sqrt{2} e^{-i\pi/4} \), and define the substrip functions by
\begin{align*}
(6.10) \quad E_k^L(x + iy) &= C_k^L \exp \left( -i 2\pi k x + 2\pi ky \right), \quad c_k^L(x) = C_k^L e^{-i2\pi k x}, \\
(6.11) \quad E_k^R(x + iy) &= C_k^R \exp \left( -i 2\pi k x + 2\pi ky \right), \quad c_k^R(x) = C_k^R e^{-i2\pi k x},
\end{align*}
where \( x \in I_L \) and \( x \in I_R \) in (6.10) and (6.11), respectively, and \( y \leq 0 \) in both. The normalization constants \( C_k^L, C_k^R \) are chosen so as to ensure unit norm and Riemann boundary values (in the respective substrips). The collections \((c_k^L)_{k \in \pm \mathbb{K}}\) and \((c_k^R)_{k \in \pm \mathbb{K}}\) form orthonormal bases of the subspaces \( \mathcal{L}_L^2 \) and \( \mathcal{L}_R^2 \), respectively. In terms of these, we define the closed subspaces\(^{18}\)
\begin{align*}
(6.12) \quad \mathcal{L}_{L;\text{pole}}^2 &= \text{span} \left\{ e_{-k}^L \mid k \in \mathbb{K}^{(\ell)} \right\}, \quad \mathcal{L}_{R;\text{pole}}^2 = \text{span} \left\{ e_{-k}^R \mid k \in \mathbb{K}^{(\ell)} \right\}, \\
\mathcal{L}_{L;\text{zero}}^2 &= \text{span} \left\{ e_{+k}^L \mid k \in \mathbb{K}^{(\ell)} \right\}, \quad \mathcal{L}_{R;\text{zero}}^2 = \text{span} \left\{ e_{+k}^R \mid k \in \mathbb{K}^{(\ell)} \right\},
\end{align*}
and we denote by
\begin{align*}
\Pi_{L;\text{pole}}^{(\ell)} : \mathcal{F}^{(\ell)} &\to \mathcal{F}_{L;\text{pole}}^{(\ell)}, \\
\Pi_{L;\text{zero}}^{(\ell)} : \mathcal{F}^{(\ell)} &\to \mathcal{F}_{L;\text{zero}}^{(\ell)}, \\
\Pi_{R;\text{pole}}^{(\ell)} : \mathcal{F}^{(\ell)} &\to \mathcal{F}_{R;\text{pole}}^{(\ell)}, \\
\Pi_{R;\text{zero}}^{(\ell)} : \mathcal{F}^{(\ell)} &\to \mathcal{F}_{R;\text{zero}}^{(\ell)},
\end{align*}
the orthogonal projections onto these subspaces.

Distinguished functions in the continuum slit-strip. The continuum case is exactly parallel to the above discrete case. For a function \( f \in \mathcal{L}^2 \), we call the projections \( \Pi_{T;\text{pole}}(f) \), \( \Pi_{T;\text{pole}}(f) \), \( \Pi_{T;\text{pole}}(f) \) its singular parts in the top, right, and left, respectively. When a singular part vanishes, we say that the function admits a regular extension in the corresponding extremity. The continuum distinguished functions are characterized as follows.

\(^{18}\)Just like in (4.7), the signs of the indices in (6.12) are again the opposite compared to (6.9).
Proposition 6.1 ([AKPR20, Proposition 2.6]). For all positive half-integers $k \in \mathcal{K}$, there exist unique functions $p_k^T, p_k^L, p_k^R \in \mathcal{L}^2$ such that

$$
\begin{align*}
\Pi_{T;\text{pole}}(p_k^T) &= e_k, & \Pi_{L;\text{pole}}(p_k^T) &= 0, & \Pi_{R;\text{pole}}(p_k^T) &= 0, \\
\Pi_{T;\text{pole}}(p_k^L) &= 0, & \Pi_{L;\text{pole}}(p_k^L) &= e_{-k}^L, & \Pi_{R;\text{pole}}(p_k^L) &= 0, \\
\Pi_{T;\text{pole}}(p_k^R) &= 0, & \Pi_{L;\text{pole}}(p_k^R) &= 0, & \Pi_{R;\text{pole}}(p_k^R) &= e_{-k}^R.
\end{align*}
$$

(6.13)

These functions are the restrictions to the cross-section $I$ of unique holomorphic functions $P_k^T, P_k^L, P_k^R : \mathcal{S}_{\text{slit}} \to \mathbb{C}$ with Riemann boundary values.

It is important that these continuum pole functions $p_k^T, p_k^L, p_k^R$ are defined globally on the whole slit-strip domain $\mathcal{S}_{\text{slit}}$.

6.3. Convergence of the inner products of distinguished functions. In [AKPR20] it was shown that the discrete distinguished functions converge in the scaling limit to their continuum counterparts uniformly on compact subsets. We do not need this form of convergence, but we crucially use its corollary that the inner products among the discrete distinguished functions converge as $\ell \to \infty$ to the inner products of the continuum counterparts.

Corollary 6.2 ([AKPR20, Corollary 4.8]). Choose sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ of integers $a_n, b_n \in \mathbb{Z}$ such that

- $a_n < 0 < b_n$ for all $n$;
- $\ell_n := b_n - a_n \to +\infty$ as $n \to \infty$;
- $a_n/\ell_n \to -\frac{1}{2}$ and $b_n/\ell_n \to +\frac{1}{2}$ as $n \to \infty$.

For $k \in \mathcal{K}$, let

$$
\begin{align*}
p^{(\ell_n)}_{T;k}, p^{(\ell_n)}_{L;k}, p^{(\ell_n)}_{R;k}, f^{(\ell_n)}_{T;\pm k}, f^{(\ell_n)}_{L;\pm k}, f^{(\ell_n)}_{R;\pm k} \in \mathcal{F}^{(\ell_n)}
\end{align*}
$$

denote the functions defined in Section 3 in the lattice strips with $a = a_n$ and $b = b_n$. Correspondingly, let

$$
\begin{align*}
P_k^T, P_k^L, P_k^R, e_{\pm k}, e_{\pm k}^L, e_{\pm k}^R \in \mathcal{L}^2
\end{align*}
$$

be the continuum functions defined above.

Then as $n \to \infty$, we have the convergence of all inner products in $\mathcal{F}^{(\ell_n)}$ to the corresponding ones in $\mathcal{L}^2$:

$$
\begin{align*}
\langle f^{(\ell_n)}_{\star;k}, f^{(\ell_n)}_{\star';k} \rangle \to \langle e_{\star;k}, e_{\star';k} \rangle & \quad \text{for } \star, \star' \in \{T, L, R\} \text{ and } k, k' \in \pm \mathcal{K}, \\
\langle p^{(\ell_n)}_{\star;k}, p^{(\ell_n)}_{\star';k} \rangle \to \langle p_{\star;k}, p_{\star';k} \rangle & \quad \text{for } \star, \star' \in \{T, L, R\} \text{ and } k \in \mathcal{K}, k' \in \pm \mathcal{K}, \\
\langle p^{(\ell_n)}_{\star;k}, p^{(\ell_n)}_{\star';k} \rangle \to \langle p^{T}_{\star;k}, p^{T}_{\star';k} \rangle & \quad \text{for } \star, \star' \in \{T, L, R\} \text{ and } k, k' \in \mathcal{K}
\end{align*}
$$

(where the notation is to be interpreted so that $f^{(\ell_n)}_{T;k} = f^{(\ell_n)}_k$ and $e^T_k = e_k$).
7. Continuum fusion coefficients

We next introduce and study a continuum analogue of the fusion coefficients. The fusion coefficients of the Ising model considered in Section 5 will be shown to converge to these in the scaling limit $\ell \to \infty$. In the final part [KPR21] of this series, these continuum analogues of the fusion coefficients will moreover be related to the structure constants of the vertex operator algebra which underlies the conformal field theory conjectured to describe the Ising model.

The convergence of the fusion coefficients of the Ising model to their continuum analogues (introduced below) could in principle be derived starting from the defining Equations (5.7), (5.10), and (5.19), using [AKPR20, Theorem 4.3] about the convergence of the $s$-holomorphic vertical translation eigenfunctions to the quarter-integer Fourier modes together with the known convergence results for the discrete fermion multipoint correlation functions [Hon10, HoSm13] (although handling the contributions from near the boundaries would require slightly strengthened formulations). We choose a slightly different route, however: we prove that the continuum fusion coefficients satisfy a recursion with similar structure as the fusion coefficients, and we prove convergence of the coefficients in these recursions. Indeed, the coefficients in these two recursions are inner products of distinguished functions in the function spaces of Sections 6 and 2, respectively. Therefore the scaling limit result for the fusion coefficients will be a consequence of the convergence results for the (inner products of the) distinguished functions. The reasons for our choice of strategy are twofold. First of all, while the convergence of the $s$-holomorphic functions uses largely the same discrete complex analysis technology as [Hon10, HoSm13], our analysis for instance avoids the notion of $s$-holomorphic singularities in the bulk (singularities only appear in asymptotics) and is in this sense simpler. Secondly, we view the emphasis on the recursion itself natural for our main goal of this series, since a recursion with a basically similar structure arises from the Jacobi identity in the vertex operator algebra that describes the conformal field theory.

Remark 7.1. In our normalizations, a few constants in the convergence results end up being not particularly elegant, so let us pause to mention the conventions eventually dictating them. The definitions of the discrete holomorphic and antiholomorphic fermions in the discrete setting of Section 3 yielded first of all the straightforward complexified $s$-holomorphicity (3.7) and Riemann boundary values (3.8) of the fermions, and moreover relatively simple algebraic and functional analytic properties (in particular in view of the adjoints and anticommutators in Lemmas 3.1 and 3.2). In the conformal field theory, on the other hand, the fields of interest have a conventional normalization which fixes the residue of the holomorphic fermion two-point function to unit value. In addition, we have chosen to use the unit width continuum strip and corresponding lattice spacing $\ell^{-1}$. With these conventional choices, the unpleasant constants are inevitable: after a renormalization by the square root of the lattice spacing, the discrete holomorphic fermion field basically converges

\footnote{Perhaps the most straightforward simplification of the unpleasant constants could be achieved by using the strips of width $2\pi$ and lattice spacing $\frac{2\pi}{\ell}$ instead. However, by our judgement the simplicity of the lattice spacing is more important.}
(a) The slit-strip $S_{\text{slit}}$.

(b) Conformal mapping of $S_{\text{slit}}$.

**Figure 7.1.** The continuum slit-strip.

(whatever the precise notion of convergence) to $e^{-i\pi/4}/\sqrt{\pi}$ times the holomorphic fermion field of the conformal field theory.

Before starting, let us fix two conventions for the whole section.

Throughout this section, we denote by

$$\varphi: S_{\text{slit}} \to \mathbb{H}$$

the conformal map from the slit-strip $S_{\text{slit}}$ to the upper half-plane $\mathbb{H}$ under which the images of the top extremity and the two bottom extremities are $\infty$, $-\frac{1}{2}$, and $+\frac{1}{2}$, respectively. This mapping is illustrated in Figure 7.1(b) and an explicit formula for it is $\varphi(z) = \frac{1}{2} \sqrt{1 - e^{-2\pi z}}$, where the choice of the branch of square roots is as detailed in [AKPR20, Sec. 2].

Formulas in this section involve many complex conjugations of sometimes unwieldy expressions. To reduce the resulting notational mess, we use $(\cdots)^*$ to denote complex conjugation. In particular, $\mathbb{H}^*$ denotes the lower half-plane. As the main exception, in integrations we retain the notation $d\overline{z}$ for the antiholomorphic one-form corresponding to the coordinate $z$. 
7.1. **Two-point kernels in slit-strip.** Now for any given functions $\mathcal{R}_1, \mathcal{R}_2$, each defined on the slit-strip or a subset thereof, introduce the following four two-point kernels:

\begin{align}
(7.1) \quad k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\circ}(z_1, z_2) &= \frac{\mathcal{R}_1(z_1) \sqrt{\varphi'(z_1)}}{\varphi(z_1) - \varphi(z_2)} \mathcal{R}_2(z_2) \sqrt{\varphi'(z_2)} \\
(7.2) \quad k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\ast}(z_1, z_2) &= \frac{\mathcal{R}_1(z_1) \sqrt{\varphi'(z_1)}}{\varphi(z_1) - \varphi(z_2)} \left( \mathcal{R}_2(z_2) \sqrt{\varphi'(z_2)} \right)^* \\
(7.3) \quad k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\circ}(z_1, z_2) &= \frac{\left( \mathcal{R}_1(z_1) \sqrt{\varphi'(z_1)} \right)^* \mathcal{R}_2(z_2) \sqrt{\varphi'(z_2)}}{\varphi(z_1)^* - \varphi(z_2)^*} \\
(7.4) \quad k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\ast}(z_1, z_2) &= \frac{\left( \mathcal{R}_1(z_1) \sqrt{\varphi'(z_1)} \right)^* \left( \mathcal{R}_2(z_2) \sqrt{\varphi'(z_2)} \right)^*}{\varphi(z_1)^* - \varphi(z_2)^*}
\end{align}

One can immediately note some antisymmetry properties of the kernels above: for example $k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\ast}(z_1, z_2) = -k_{\mathcal{R}_2, \mathcal{R}_1}^{\circ\ast}(z_2, z_1)$, etc.

The following lemma is a vertical slidability property for the two-point kernels. Despite a few substantial differences, an analogue with Proposition 3.10 should be apparent.

**Lemma 7.2.** Suitable Riemann boundary values on the coefficient function $\mathcal{R}_1$ imply the vanishing of integrals of certain combinations of the above kernels along left and right vertical boundaries as follows.

(a) Let $B^L \subset \partial S_{\text{slit}}$ be a boundary segment of the left boundary of the slit-strip. Suppose that the function $\mathcal{R}_1$ extends continuously to $B^L$ and satisfies

$$\mathcal{R}_1(z_1) \in e^{-i\frac{\pi}{2}} \mathbb{R} \quad \text{for all } z_1 \in B^L.$$  

Then for any $z_2 \in S_{\text{slit}}$ we have

\begin{align*}
&\int_{B^L} \left( k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\circ}(z_1, z_2) \, dz_1 + k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\ast}(z_1, z_2) \, d\bar{z}_1 \right) = 0 \\
&\int_{B^L} \left( k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\circ}(z_1, z_2) \, dz_1 + k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\ast}(z_1, z_2) \, d\bar{z}_1 \right) = 0.
\end{align*}

(b) Let $B^R \subset \partial S_{\text{slit}}$ be a boundary segment of the right boundary of the slit-strip. Suppose that the function $\mathcal{R}_1$ extends continuously to $B^R$ and satisfies

$$\mathcal{R}_1(z_1) \in e^{i\frac{\pi}{2}} \mathbb{R} \quad \text{for all } z_1 \in B^R.$$  

Then for any $z_2 \in S_{\text{slit}}$ we have

\begin{align*}
&\int_{B^R} \left( k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\circ}(z_1, z_2) \, dz_1 + k_{\mathcal{R}_1, \mathcal{R}_2}^{\circ\ast}(z_1, z_2) \, d\bar{z}_1 \right) = 0 \\
&\int_{B^R} \left( k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\circ}(z_1, z_2) \, dz_1 + k_{\mathcal{R}_1, \mathcal{R}_2}^{\ast\ast}(z_1, z_2) \, d\bar{z}_1 \right) = 0.
\end{align*}

20 The expressions here involve an even number of square roots of the derivative, so the branch choice of $\sqrt{\varphi'}$ is inconsequential. For definiteness, we could follow the convention of [AKPR20 Sec. 2].
Proof. The proofs of the two cases are virtually identical, so let us only prove part (a).

Observe that since \( \varphi \) maps the slit-strip \( \mathcal{S}_{\text{slit}} \) to the upper half plane \( \mathbb{H} \), then a segment of the boundary \( \partial \mathcal{S}_{\text{slit}} \) gets mapped to an interval on the real axis \( \mathbb{R} \). Considering orientations, we see that for \( z_1 \in B^L \) on a segment of the left boundary, the direction of the derivative is \( \varphi'(z_1) \in i \mathbb{R}_+ \). Therefore the square root of the derivative satisfies \( \sqrt{\varphi'(z_1)} \in e^{i\pi} \mathbb{R} \). By assumption on the coefficient function \( \mathcal{R}_1(z_1) \in e^{-i\pi} \mathbb{R} \), so we get that

\[
\sqrt{\varphi'(z_1)} \mathcal{R}_1(z_1) \in \mathbb{R}.
\]

In view of the defining formulas (7.1) – (7.4), this implies that

\[
k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) = k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \quad k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) = k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2)
\]

for \( z_1 \in B^L \) and any \( z_2 \in \mathcal{S}_{\text{slit}} \). Since \( d\tilde{z}_1 = -dz_1 \) along the vertical boundary segment \( B^L \), we get cancellations which ensure the vanishing of the integrals as asserted. \( \square \)

The following lemma summarizes the holomorphicity and residues at poles for the two-point kernels, as well as antiholomorphic variants. An analogue to Proposition 3.8 can be noted.

**Lemma 7.3.** Suppose that the coefficient function \( \mathcal{R}_1 \) is holomorphic. Then the two-point kernels have the following holomorphicity/antiholomorphicity properties and poles.

(a) The function \( z_1 \mapsto k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \) is holomorphic except at \( z_1 = z_2 \), where it has a simple pole with residue

\[
\frac{1}{2\pi i} \oint_{\partial B^L(z_2)} k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \, dz_1 = \mathcal{R}_1(z_2) \mathcal{R}_2(z_2).
\]

(b) The function \( z_1 \mapsto k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \) is holomorphic.

(c) The function \( z_1 \mapsto k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \) is antiholomorphic.

(d) The function \( z_1 \mapsto k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \) is antiholomorphic except at \( z_1 = z_2 \), where it has an antiholomorphic simple pole with residue

\[
\frac{-1}{2\pi i} \oint_{\partial B^L(z_2)} k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \, dz_1 = \overline{\mathcal{R}_1(z_2)} \overline{\mathcal{R}_2(z_2)}.
\]

Proof. Properties (c) and (d) are obtained from (b) and (a) by complex conjugation, so it suffices to prove the first two statements.

For (b), observe that the numerator of (7.2) is holomorphic as a function of \( z_1 \), since \( z_1 \mapsto \sqrt{\varphi'(z_1)} \) and \( z_1 \mapsto \mathcal{R}_1(z_1) \) are. Also the denominator of (7.2) is holomorphic as a function of \( z_1 \), since \( z_1 \mapsto \varphi(z_1) \) is. Moreover, the denominator is non-vanishing, since \( \varphi(z_1) \in \mathbb{H} \) but \( \varphi(z_2)^* \in \mathbb{H}^* \). Therefore the function \( z_1 \mapsto k_{\mathcal{R}_1, \mathcal{R}_2}^\circ(z_1, z_2) \) defined by (7.2) is indeed holomorphic.

For (a), observe again that the numerator and denominator of (7.1) are both holomorphic, as functions of \( z_1 \). Since we have \( \varphi(z_2) \in \mathbb{H} \), and since \( \varphi : \mathcal{S}_{\text{slit}} \to \mathbb{H} \) is conformal, the denominator only vanishes at \( z_1 = z_2 \), and it has a first order zero at that point. The

\[\text{The boundary of the slit-strip consists of analytic arcs, so continuous and analytic extension of the conformal map to the boundary is possible.}\]
function $z_1 \mapsto k^{\infty}_{q_1,q_2}(z_1, z_2)$ correspondingly is holomorphic except for a simple pole at $z_1 = z_2$. At this point the denominator has a Taylor expansion

$$\varphi(z_1) - \varphi(z_2) = 0 + (z_1 - z_2) \varphi'(z_2) + O((z_1 - z_2)^2),$$

while the numerator takes the value

$$\sqrt{\varphi'(z_2)} R_1(z_2) \sqrt{\varphi'(z_2)} R_2(z_2) = \varphi'(z_2) R_1(z_2) R_2(z_2).$$

After cancelling the common factor $\varphi'(z_2)$, the assertion about the residue follows. \hfill \square

### 7.2. Multi-point kernels in slit-strip

In free fermionic theories, multi-point correlations of fermions are obtained from two-point correlation functions by Pfaffians. We start by recalling the needed properties of Pfaffians, then construct the multi-point kernels from the two-point kernels of the previous section, and finally define and study integrated multi-point kernels, which (in special cases) give the continuum analogues of the fusion coefficients.

**Pfaffians.** Suppose that $A = (A_{ij})_{i,j=1}^{m} \in \mathbb{C}^{m \times m}$ is a square matrix, which is skew-symmetric, $A_{ij} = -A_{ji}$ for all $i, j = 1, \ldots, m$. If the dimension is even, $m = 2n$ for $n \in \mathbb{N}$, we define the **Pfaffian** of $A$ as

$$\text{Pf}(A) = \frac{1}{2^n n!} \sum_{\varsigma \in S_m} \text{sgn}(\varsigma) \prod_{p=1}^{n} A_{(2p-1)\varsigma(2p)}. \tag{7.5}$$

If $m$ is odd, we set $\text{Pf}(A) = 0$.

There is an alternative (less self-explanatory but often more practical) expression\footnote{This is obtained by combining repeated terms in the sum (7.5). The permutations $\varsigma \in S_m = S_{2n}$ can be partitioned into equivalence classes of size $2^n n!$ each, with equivalent permutation contributing equal terms. Representatives of the equivalence classes become naturally labeled by pair partitions $P$ of the set $\{1, \ldots, m\} = \{1, \ldots, 2n\}$.} for the Pfaffian,

$$\text{Pf}(A) = \sum_{P} \text{sgn}(P) \prod_{\{i,j\} \in P, i < j} A_{ij},$$

where the sum is over pair partitions $P$ of $\{1, \ldots, m\}$, and $\text{sgn}(P)$ denotes the signature of any permutation $\varsigma \in S_n$ such that $\{\varsigma(2p-1), \varsigma(2p)\} \in P$ and $\varsigma(2p-1) < \varsigma(2p)$ for all $p = 1, \ldots, n$.

From the latter expression, for instance the following recursion of Pfaffians becomes evident.

**Lemma 7.4.** Let $m = 2n$, and let $A \in \mathbb{C}^{m \times m}$ be a skew-symmetric matrix. For any $1 \leq i < j \leq m$, let $\hat{A}^{(ij)} \in \mathbb{C}^{(m-2) \times (m-2)}$ denote the matrix obtained by erasing rows and columns with indices $i$ and $j$ from $A$. Then for any fixed $1 \leq i \leq m$, we have

$$\text{Pf}(A) = \sum_{j \neq i} \text{sgn}(i-j) (-1)^{i-j} A_{ij} \text{Pf}(\hat{A}^{(ij)}). \tag{7.6}$$

With the convention\footnote{There exists a unique permutation of zero indices and a unique pair partition of the empty set. The empty product is one.} that the Pfaffian of a $0 \times 0$-matrix is 1, this recursion is sufficient (and efficient) for computing Pfaffians. This recursion also features crucially below.
Multi-point kernels. Let $\mathcal{R}_1, \ldots, \mathcal{R}_m$ be functions, each defined on the slit-strip or a subset thereof. In terms of the two-point kernels of Section 4, we then define the $m \times m$ matrix $K_{\mathcal{R}_1, \ldots, \mathcal{R}_m}(z_1, \ldots, z_m)$ with entries

$$K_{\mathcal{R}_1, \ldots, \mathcal{R}_m}(z_1, \ldots, z_m)_{ij} = \begin{cases} k_{\mathcal{R}_i, \mathcal{R}_j}^0(z_i, z_j) + k_{\mathcal{R}_i, \mathcal{R}_j}^0(z_i, z_j) + k_{\mathcal{R}_i, \mathcal{R}_j}^0(z_i, z_j) + k_{\mathcal{R}_i, \mathcal{R}_j}^0(z_i, z_j) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

(7.7)

This matrix is mix skew-symmetric by virtue of the antisymmetry properties of the two-point kernels (7.1) – (7.4). We then define the $m$-point kernel as its Pfaffian

$$k_{\mathcal{R}_1, \ldots, \mathcal{R}_m}(z_1, \ldots, z_m) = \text{Pf}(K_{\mathcal{R}_1, \ldots, \mathcal{R}_m}(z_1, \ldots, z_m)).$$

(7.8)

This multipoint kernel is totally antisymmetric in the sense that for any permutation $\varsigma \in \mathcal{S}_m$ we have $k_{\mathcal{R}_{\varsigma(1)}, \ldots, \mathcal{R}_{\varsigma(m)}}(z_{\varsigma(1)}, \ldots, z_{\varsigma(m)}) = \text{sgn}(\varsigma) k_{\mathcal{R}_1, \ldots, \mathcal{R}_m}(z_1, \ldots, z_m)$.

As the functions $\mathcal{R}_i$, we will use the various distinguished continuum functions from Section 4. Primarily, we use the functions $E_k : \mathcal{S} \to \mathbb{C}$ in the strip (in fact restricted to the top half of it), and the functions $E_k^L : \mathcal{S} \to \mathbb{C}$ and $E_k^R : \mathcal{S} \to \mathbb{C}$ in the left and the right half-strips. However, also the continuous pole functions $P_k^T, P_k^R, P_k^L : \mathcal{S}_{\text{slit}} \to \mathbb{C}$ will be used in the derivation of the main recursion.

Now let $k, k', k'' \in \bigcup_{m=0}^{\infty} (-\mathcal{K})^m$ be three tuples of half-integers,

$$k = (k_1, \ldots, k_m), \quad k' = (k'_1, \ldots, k'_{m'}) , \quad k'' = (k''_1, \ldots, k''_{m''}).$$

Let

$$0 < y_m < \cdots < y_2 < y_1, \quad y'_1 < y'_2 < \cdots < y'_{m'} < 0, \quad y''_1 < y''_2 < \cdots < y''_{m''} < 0.$$  

(7.9)

For any $x_1, \ldots, x_m \in (-\frac{1}{2}, \frac{1}{2}), x'_1, \ldots, x'_{m'} \in (-\frac{1}{2}, 0), \text{ and } x''_1, \ldots, x''_{m''} \in (0, \frac{1}{2})$, use the abbreviated notation

$$(x + \hat{i} y, x' + \hat{i} y', x'' + \hat{i} y'') := (x + \hat{i} y_1, \ldots, x_m + \hat{i} y_m, x'_1 + \hat{i} y'_1, \ldots, x'_{m'} + \hat{i} y'_{m'}, x''_1 + \hat{i} y''_1, \ldots, x''_{m''} + \hat{i} y''_{m''})$$

for the $(m + m' + m'')$-tuple of points in $\mathcal{S}_{\text{slit}}$. Also use the abbreviated notation

$$K_{E_k, E_{k'}^L, E_{k''}^R} := K_{E_k, E_{k'}^L, E_{k''}^R}$$

for the function of type (7.8), with the specific choice of coefficient functions $\mathcal{R}_1 = E_{k_1}, \ldots, \mathcal{R}_{m+m'+m''} = E_{k''_{m''}}$.

Integrated multipoint kernels. We now consider integrated versions of the above multipoint kernels, with integration contours illustrated in Figure 7.2. Define the following integrated
Figure 7.2. Integration contours for the multipoint kernels.

\[ (m + m' + m'') \text{-point kernel} \]

\[
(7.10) \quad \Psi_{k,k',k''} := \left( \frac{-i}{2\sqrt{\pi}} \right)^{m+m'+m''} \int_{[\frac{-1}{2},\frac{1}{2})^m} \int_{[\frac{-1}{2},0)^{m'}} \int_{[0,\frac{1}{2})^{m''}} \hat{d}^{m}x \hat{d}^{m'}x' \hat{d}^{m''}x'' \frac{1}{k_{E_{k},E_{k}'},E_{k}''}(x + \hat{i}y;x', + \hat{i}y',x'' + \hat{i}y''),
\]

where we also abbreviated \( \hat{d}^{m}x = \prod_{i=1}^{m} dx_i \), etc. A priori, the levels \( y_1, \ldots, y_{m''} \) are free variables on the right hand side, but the following lemma shows that the precise choice of them plays no role.

**Lemma 7.5.** The quantity \( \Psi_{k,k',k''} \) does not depend on \( (y_i)_{i=1}^{m}, (y'_i)_{i=1}^{m'}, (y''_i)_{i=1}^{m''}, \) as long as these satisfy the ordering constraints \((7.9)\).

**Proof.** Consider, e.g., changing one \( y_i \) to another value \( \tilde{y}_i \), but still so that \( \tilde{y}_i < y_{i-1} \) (if \( i > 1 \)) and \( \tilde{y}_i > y_{i+1} \) (if \( i < m \)). The new integral over the \( x_i \) variable which we must consider in \((7.10)\) differs from the original integral by

\[
(7.11) \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_i k_{\cdots,E_{k_i},\cdots}(\cdots,x_i + \hat{i}\tilde{y}_i,\cdots) - \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_i k_{\cdots,E_{k_i},\cdots}(\cdots,x_i + \hat{i}y_i,\cdots).
\]

Applying the recursion for Pfaffians, Lemma \(7.4\), we can isolate factors in the integrand \( k_{\cdots,E_{k_i},\cdots}(\cdots,z_i,\cdots) \) which depend on the variable \( z_i \), and write this integrand as a sum of
terms of the form

\[
(7.12) \quad \left( k_{E_{k_i,R}}^\circ(z_i, z) + k_{E_{k_i,R}}^{\bullet\circ}(z_i, z) + k_{E_{k_i,R}}^{\bullet\bullet}(z_i, z) + k_{E_{k_i,R}}^{\circ\bullet}(z_i, z) \right) \times K
\]

where \( K \) is a Pfaffian of a smaller matrix with no dependence on \( z_i \), and \( z \) is one of the other variables. In particular by the assumed orderings of the imaginary parts, \( z \) is always at a positive distance from the rectangle \((-\frac{1}{2}, \frac{1}{2}) \times (y_i, \tilde{y}_i) \subset S_{\text{slit}}\). If we denote by \( \eta(y) \) the horizontal path from \(-\frac{1}{2} + iy \) to \( \frac{1}{2} + iy \), then the contribution to \( (7.11) \) from the first two terms of \( (7.12) \) is

\[
(7.13) \quad K \times \left( \int_{\eta(y_i)} (k_{E_{k_i,R}}^\circ(z_i, z) + k_{E_{k_i,R}}^{\bullet\circ}(z_i, z)) \, dz_i - \int_{\eta(\tilde{y}_i)} (k_{E_{k_i,R}}^\circ(z_i, z) + k_{E_{k_i,R}}^{\bullet\circ}(z_i, z)) \, dz_i \right),
\]

since for the integrals along a horizontal segments we can write \( dz_i = dz_i \). The integrand here is holomorphic in the rectangle \((-\frac{1}{2}, \frac{1}{2}) \times (y_i, \tilde{y}_i) \subset S_{\text{slit}}\), and the paths \( \eta(y_i) \) and \( \eta(\tilde{y}_i) \) are the horizontal segments of the boundary of this rectangle. Due to the vanishing by Cauchy’s theorem of the integral along the whole boundary of the rectangle, the contribution \( (7.13) \) can be written alternatively in terms of the vertical boundary parts \( B^L \) and \( B^R \) of the rectangle (from \( \pm \frac{1}{2} + iy_i \) to \( \pm \frac{1}{2} + iy_i \)), as

\[
(7.14) \quad K \times \left( + \int_{B^L} (k_{E_{k_i,R}}^\circ(z_i, z) + k_{E_{k_i,R}}^{\bullet\circ}(z_i, z)) \, dz_i - \int_{B^R} (k_{E_{k_i,R}}^\circ(z_i, z) + k_{E_{k_i,R}}^{\bullet\circ}(z_i, z)) \, dz_i \right).
\]

Similarly the contribution to \( (7.11) \) from the last two terms of \( (7.12) \) is

\[
(7.15) \quad K \times \left( \int_{\eta(y_i)} (k_{E_{k_i,R}}^{\bullet\bullet}(z_i, z)) \, dz_i - \int_{\eta(\tilde{y}_i)} (k_{E_{k_i,R}}^{\bullet\bullet}(z_i, z)) \, dz_i \right),
\]

since for the integrals along horizontal segments we can write \( dz_i = dz_i \). By antiholomorphicity of the integrand here, we can rewrite this contribution in terms of the vertical parts of the boundary of the rectangle as

\[
(7.16) \quad K \times \left( + \int_{B^L} (k_{E_{k_i,R}}^{\bullet\circ}(z_i, z) + k_{E_{k_i,R}}^{\bullet\bullet}(z_i, z)) \, dz_i - \int_{B^R} (k_{E_{k_i,R}}^{\bullet\circ}(z_i, z) + k_{E_{k_i,R}}^{\bullet\bullet}(z_i, z)) \, dz_i \right).
\]

By virtue of Lemma \( 7.2 \) and the Riemann boundary values for \( E_{k_i} \), we get that the vertical integrals in \( (7.14) \) and \( (7.16) \) cancel each other. This proves that \( (7.11) \) is zero, and consequently that the integrated multi-point kernel \( \Psi_{E_{k_i},E_{k_i},E_{k_i}} \) does not depend on \( y_i \). The proof that \( \Psi_{E_{k_i},E_{k_i},E_{k_i}} \) does not depend on \( y_i \) and \( y_i' \) is similar. \( \Box \)
The following anticommutation and annihilation properties of the integrated multi-point kernels will be a key tool in deriving the main recursion.

**Lemma 7.6.** The integrated multi-point kernels $\Psi_{k,k',k''}$ satisfy the following.

(a) If $k_1 < 0$ or $k_1' > 0$ or $k_1'' > 0$, then we have $\Psi_{k,k',k''} = 0$.

(b) Suppose that $\tilde{k}$ is obtained from $k$ by interchanging the indices at positions $i$ and $i+1$, and $\tilde{k}$ is obtained from $\tilde{k}$ by removing the indices at positions $i$ and $i+1$, i.e.,

$$\tilde{k} = (k_1, \ldots, k_i-1, k_i, k_{i+1}, k_{i+2}, \ldots, k_m)$$

$$\tilde{\tilde{k}} = (k_1, \ldots, k_{i-1}, k_i, k_i, k_{i+1}, k_{i+2}, \ldots, k_m)$$

$$\tilde{k} = (k_1, \ldots, k_i-1, k_i+1, \ldots, k_m).$$

Then we have

$$\Psi_{\tilde{k},\tilde{k}'',\tilde{k}'''} + \Psi_{\tilde{k}',\tilde{k}'',\tilde{k}'''} = \begin{cases} 
\Psi_{\tilde{k},\tilde{k}'',\tilde{k}'''} & \text{if } k_i + k_{i+1} = 0 \\
0 & \text{if } k_i + k_{i+1} \neq 0.
\end{cases}$$

(c) Similarly, if $\tilde{k}'$ is obtained from $k'$ by interchanging the indices at positions $i$ and $i+1$, and $\tilde{\tilde{k}'}$ by removing the indices at positions $i$ and $i+1$, then we have

$$\Psi_{\tilde{k},\tilde{k}',\tilde{k}'''} + \Psi_{\tilde{k}',\tilde{k}',\tilde{k}'''} = \begin{cases} 
\Psi_{\tilde{k},\tilde{k}',\tilde{k}'''} & \text{if } k_i' + k_{i+1}' = 0 \\
0 & \text{if } k_i' + k_{i+1}' \neq 0.
\end{cases}$$

(d) Similarly, if $\tilde{k}'''$ is obtained from $k'''$ by interchanging the indices at positions $i$ and $i+1$, and $\tilde{\tilde{k}'''}$ by removing the indices at positions $i$ and $i+1$, then we have

$$\Psi_{\tilde{k},\tilde{k}',\tilde{k}'''} + \Psi_{\tilde{k},\tilde{k}',\tilde{\tilde{k}''}} = \begin{cases} 
\Psi_{\tilde{k},\tilde{k}',\tilde{k}'''} & \text{if } k_i'' + k_{i+1}''' = 0 \\
0 & \text{if } k_i'' + k_{i+1}''' \neq 0.
\end{cases}$$

**Proof.** Consider, e.g., the case $k_1 < 0$ in the “annihilation property” (a). By the independence on the level choices, we can take $y_1 \to +\infty$ — ordering (7.9) is preserved, as $y_1$ was anyway the highest level. Observe that because of the decaying vertical translation eigenfunction $E_{k_1} (x_1 + iy_1) = e^{x_1 y_1} e_{k_1} (x_1)$, the two point kernels $k_{E_{k_1},y_1}^\infty(x_1 + iy_1, z)$ etc. tend to zero exponentially as $y_1 \to +\infty$ (from the asymptotics given in [AKPR20, Sec. 2]), one sees that the factor $\varphi(x_1 + iy_1) - \varphi(z)$ in the denominator grows faster than the factor $\sqrt{\varphi'(x_1 + iy_1)}$ in the numerator). Separate the dependence on $x_1 + iy_1$ of the integrand in $\Psi_{\tilde{k},\tilde{k}',\tilde{k}''}$ by writing it as a sum of terms of the form (7.12) (with $i = 1$). This expression shows that the integrand is tending to zero exponentially. Thus also the whole integral $\Psi_{\tilde{k},\tilde{k}',\tilde{k}''}$ tends to zero as $y_1 \to +\infty$, and by its independence on $y_1$ (Lemma 7.5), it in fact must be zero for any $y_1$. This concludes one of the three cases in (a). The cases $k_1' > 0$ and $k_1'' > 0$ are similarly handled by taking $y_1' \to -\infty$ and $y_1'' \to -\infty$, respectively. The proofs of the “anticommutation properties” (b), (c), and (d) are similar, so we only prove (b). Let us start by considering the integral of the kernel

$$\kappa_{E_{k_1},E_{k_i},E_{k_{i+1}}} (x_1 + iy_1, x_1 + iy_1', x_1 + iy_1'', \ldots) = \kappa_{E_{k_i},E_{k_{i+1}},\ldots}(x_1 + iy_1, x_{i+1} + iy_1, \ldots)$$
over the $x_i$ variable, and how it changes when the level $y_i$ is changed to another level $\tilde{y}_i$ which satisfies $\tilde{y}_i < y_{i+1}$ and $\tilde{y}_i > y_{i+2}$ (if $i + 2 \leq m$). Such a change does not preserve the ordering of the levels, so the argument of Lemma 7.5 has to be modified. Consider the difference

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} dx_i k_{..., E_{k_i}, E_{k_{i+1}}, ...}(x_i + i y_i, x_{i+1} + i y_{i+1}, ...) - \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_i k_{..., E_{k_i}, E_{k_{i+1}}, ...}(x_i + i \tilde{y}_i, x_{i+1} + i \tilde{y}_{i+1}, ...).
$$

Just as in Lemma 7.5, a recursion of the type (7.6) can be used to isolate the dependence on $z_i$ by expressing the integrand as a sum of terms of the form (7.12). Except only for the term which has $z = z_{i+1} := x_{i+1} + i y_{i+1}$, all these terms can be treated exactly as in Lemma 7.5, and their total contribution to the difference (7.17) is zero. Thus we are left to consider the term

$$
\left( k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) + k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) + k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) + k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) \right) \times K
$$

in the integrand, where $K$ is a Pfaffian of type (7.8) with the indices $i$ and $i + 1$ omitted. The only difference to the argument of Lemma 7.5 here is that the first term has a pole at the point $z_{i+1}$ inside the rectangle, and the last term has a similar antiholomorphic pole. Compared to (7.14), the residue of the first term at the pole $z_i = z_{i+1}$ introduces an extra term

$$
-K \times \oint_{z_{i+1}} k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) \, dz_i = -2\pi i K E_{k_i}(z_{i+1}) E_{k_{i+1}}(z_{i+1}),
$$

where we used Lemma 7.3 to explicitly calculate the residue. Similarly compared to (7.16), the antiholomorphic residue of the last term at the pole $z_i = z_{i+1}$ introduces an extra term

$$
-K \times \oint_{z_{i+1}} k_{E_{k_i}, E_{k_{i+1}}}(z_i, z_{i+1}) \, d\bar{z}_i = +2\pi i K \bar{E}_{k_i}(z_{i+1}) \bar{E}_{k_{i+1}}(z_{i+1}).
$$

In the end, the difference (7.17) therefore simplifies to

$$
(7.17) = -2\pi i K E_{k_i}(z_{i+1}) E_{k_{i+1}}(z_{i+1}) - E_{k_i}(z_{i+1}) E_{k_{i+1}}(z_{i+1})
$$

Denoting $z_{i+1} = x_{i+1} + i y_{i+1}$, and using the vertical translation eigenfunction properties of both $E_{k_i}$ and $E_{k_{i+1}}$ as well as the property $E_{k_{i+1}}(z_{i+1}) = -i e^{2\pi k_{i+1} \Im (z_{i+1})} E_{-k_{i+1}}(z_{i+1})$ from (6.8), we rewrite this as

$$
(7.17) = -2\pi i K e^{2\pi k_{i+1} \Im (z_{i+1})} \left( -i E_{k_i}(z_{i+1}) E_{-k_{i+1}}(z_{i+1}) - i E_{k_i}(z_{i+1}) E_{-k_{i+1}}(z_{i+1}) \right)
$$

$$
= -4\pi K e^{\pi (k_i + k_{i+1}) y_{i+1}} \Re \left( \frac{e_{k_i}(x_{i+1}) e_{-k_{i+1}}(x_{i+1})}{e_{k_i}(x_{i+1}) e_{-k_{i+1}}(x_{i+1})} \right).
$$
The difference (7.17) is still to be integrated over the other variables, including in particular \(x_{i+1}\). In view of the above, the integral over \(x_{i+1}\) becomes simply
\[
\int_{-\frac{1}{2}}^{+\frac{1}{2}} dx_{i+1} = -4\pi K e^{\pi (k_i + k_{i+1}) y_{i+1}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \mathcal{R}e \left( e^{k_i} (x_{i+1}) e^{-k_{i+1}} (x_{i+1}) \right) \, dx_{i+1}
\]
\[
= -4\pi K \delta_{k_i, -k_{i+1}},
\]
using the orthonormality of the functions \(e^{k}k\). The relabeling of the integration variables needed to recover the desired ordering of remaining variables \(x_j, j \neq i, i + 1\) yields
\[
-4\pi \delta_{k_i, k_{i+1}, 0} \left( 2\sqrt{\pi} \right)^{m + m' + m'' - 2} \Phi_{k_i, k', k''}.
\]
On the other hand, we can instead integrate the difference (7.17) directly over all the remaining variables \(x_j, j \neq i\). Calculating this way, we obtain the sum
\[
(2\sqrt{\pi} \right)^{m + m' + m''} \left( \Phi_{k_i, k', k''} + \Phi_{-k_i, -k', -k''} \right),
\]
since the relabeling of the integration variables needed to recover the desired ordering of levels in the second term is an odd permutation (transposition of \(i\) and \(i + 1\)), which introduces a sign change of the completely antisymmetric integral kernel. By comparing these two equal expressions we conclude the asserted anticommutation property. \(\square\)

7.3. Scaling limit of the fusion coefficients. Using the properties of Lemma 7.6 it is possible to rewrite any integrated multi-point kernel \(\Phi_{k_i, k', k''}\) in terms of only those, where the indices in the tuple \(k\) are positive and in increasing order, and the indices in the tuples \(k'\) and \(k''\) are negative and in decreasing order. In this form they are more directly analogous to the fusion coefficients of the Ising model, and we therefore give the following definition.

Let \(\alpha, \alpha_R, \alpha_L \in \mathcal{K}\). Write these as
\[
\alpha = \{k_1, \ldots, k_m\}, \quad \alpha_R = \{k'_1, \ldots, k'_m\}, \quad \alpha_L = \{k''_1, \ldots, k''_m\}
\]
with
\[
0 < k_1 < \cdots < k_m, \quad 0 < k'_1 < \cdots < k'_m, \quad 0 < k''_1 < \cdots < k''_m.
\]
Introduce the corresponding tuples \(\underline{k} = (k_1, \ldots, k_m), \quad -\underline{k}' = (-k'_1, \ldots, -k'_m), \) and \(-\underline{k}'' = (-k''_1, \ldots, -k''_m). The corresponding continuum fusion coefficient is defined as
\[
\Phi_{\alpha; \alpha_R, \alpha_L} := \Phi_{\underline{k}; -\underline{k}', -\underline{k}''}.
\]
These continuum fusion coefficients satisfy a recursion analogous to Theorem 5.21. Here \(\epsilon_\alpha(k)\) again denotes the signed indicator given by (5.20).

**Theorem 7.7.** The collection \((\Phi_{\alpha; \alpha_R, \alpha_L})_{\alpha, \alpha_R, \alpha_L \in \mathcal{K}}\) of all continuum fusion coefficients satisfies the following properties, which furthermore uniquely characterize the collection:

**(REC0)** We have \(\Phi_{\emptyset, \emptyset, \emptyset} = 1.\)
(REC\textsubscript{T}) If $\alpha, \alpha_R, \alpha_L \in \mathcal{K}$ and $\alpha' = \alpha \cup \{k\}$ with $\max(\alpha) < k$, then we have
\[
\Phi_{\alpha';\alpha_L,\alpha_R} = \sum_{k' \in \alpha_L} \langle e_{k'}, p_k^* \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L} (k') \Phi_{\alpha;\alpha_L\setminus\{k\},\alpha_R} \\
+ \sum_{k' \in \alpha_R} \langle e_{k'}, p_k^* \rangle \epsilon_{\alpha_R} (k') \Phi_{\alpha;\alpha_L,\alpha_R\setminus\{k\}} \\
- \sum_{k' \in \alpha} \langle e_{-k'}, p_k^* \rangle \epsilon_{\alpha} (k') \Phi_{\alpha\setminus\{k\};\alpha_L,\alpha_R}.
\]

(REC\textsubscript{L}) If $\alpha, \alpha_R, \alpha_L \in \mathcal{K}$ and $\alpha'_L = \alpha_L \cup \{k\}$ with $\max(\alpha_L) < k$, then we have
\[
\Phi_{\alpha;\alpha'_L,\alpha_R} = \sum_{k' \in \alpha_L} \langle e_{k'}, p_k^* \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L} (k') \Phi_{\alpha;\alpha_L\setminus\{k\},\alpha_R} \\
- \sum_{k' \in \alpha_R} \langle e_{k'}, p_k^* \rangle \epsilon_{\alpha_R} (k') \Phi_{\alpha;\alpha_L,\alpha_R\setminus\{k\}} \\
+ \sum_{k' \in \alpha} \langle e_{-k'}, p_k^* \rangle \epsilon_{\alpha} (k') \Phi_{\alpha\setminus\{k\};\alpha_L,\alpha_R}.
\]

(REC\textsubscript{R}) If $\alpha, \alpha_R, \alpha_L \in \mathcal{K}$ and $\alpha'_R = \alpha_R \cup \{k\}$ with $\max(\alpha_R) < k$, then we have
\[
\Phi_{\alpha;\alpha_L,\alpha'_R} = \sum_{k' \in \alpha_L} \langle e_{k'}, p_k^* \rangle (-1)^{\#\alpha_R} \epsilon_{\alpha_L} (k') \Phi_{\alpha;\alpha_L\setminus\{k\},\alpha_R} \\
- \sum_{k' \in \alpha_R} \langle e_{k'}, p_k^* \rangle \epsilon_{\alpha_R} (k') \Phi_{\alpha;\alpha_L,\alpha_R\setminus\{k\}} \\
+ \sum_{k' \in \alpha} \langle e_{-k'}, p_k^* \rangle \epsilon_{\alpha} (k') \Phi_{\alpha\setminus\{k\};\alpha_L,\alpha_R}.
\]

The idea of the proof is exactly parallel to that of Theorem 5.21, so we content ourselves to sketching the strategy. The uniqueness of the solution to the recursion goes through verbatim, and the initial condition (REC0) is direct by definition/convention, so the main task is to prove the three recursive properties (REC\textsubscript{T}), (REC\textsubscript{R}), and (REC\textsubscript{L}). Each of these is proved by replacing a singular Fourier mode of index $\pm k$ in the appropriate extremity by the corresponding continuous pole function, up to regular Fourier modes. The integrations of the regular Fourier modes can be anticommutated and eventually annihilated with the help of Lemma 7.6. These anticommutations result in one of the sums on the right hand side. Then the integration of the pole function can first of all be considered at height zero by virtue of Lemma 7.5. It can be split to two contributions, over the crosscuts of the two other extremities. The pole function can moreover be expanded in regular Fourier modes in each of these other extremities. The integrations of these regular Fourier modes can then again be anticommutated and eventually annihilated, and the anticommutations from each extremity produces one of the two remaining terms on the right hand side. To finish the calculation one needs to notice that the expansion coefficients of the pole functions are suitable inner products with regular Fourier modes, and that there are signs resulting from the anticommutations, the integration directions, and sign changes needed in the kernel when permuting the integration variables to the desired order.
We can now derive the following scaling limit result for the Ising model fusion coefficients.

**Theorem 7.8.** Choose sequences \((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}\) of integers \(a_n, b_n \in \mathbb{Z}\) such that

- \(a_n < 0 < b_n\) for all \(n\);
- \(\ell_n := b_n - a_n \to +\infty\) as \(n \to \infty\);
- \(a_n/\ell_n \to -\frac{1}{2}\) and \(b_n/\ell_n \to +\frac{1}{2}\) as \(n \to \infty\).

Denote by \((\Phi^{(\ell_n)}_{\alpha; a_R, a_L})\) the Ising model fusion coefficients \((5.19)\) on the discrete slit-strip with left and right boundaries at horizontal positions \(a_n\) and \(b_n\), respectively, and by \((\Phi_{\alpha; a_R, a_L})\) the continuum fusion coefficients \((7.19)\). Then for any given \(\alpha, \alpha_R, \alpha_L \in \mathcal{K}\), the fusion coefficient \(\Phi^{(\ell_n)}_{\alpha; a_R, a_L}\) is defined for all sufficiently large \(n\), and we have

\[
\lim_{n \to \infty} \frac{\Phi^{(\ell_n)}_{\alpha; a_R, a_L}}{\Phi_{\emptyset; \emptyset, \emptyset}} = \Phi_{\alpha; a_R, a_L}.
\]

**Proof.** Fix \(\alpha, \alpha_R, \alpha_L \in \mathcal{K}\). The assumptions imply \(-a_n, b_n, \ell_n \to +\infty\) as \(n \to \infty\). The fusion coefficient \(\Phi^{(\ell_n)}_{\alpha; a_R, a_L}\) is defined as soon as \(-a_n > \max \alpha_L, b_n > \max \alpha_R\), and \(\ell_n > \max \alpha\); in particular for all sufficiently large \(n\).

By Theorem 5.21, a finite recursion allows to express \(\Phi^{(\ell_n)}_{\alpha; a_R, a_L}\) in terms of the overall multiplicative factor \(\Phi^{(\ell_n)}_{\emptyset; \emptyset, \emptyset}\) (coming from the initial condition) times a polynomial in the inner products \(\langle \mathfrak{f}_{-k'}, \mathfrak{p}_{T,k} \rangle, \langle \mathfrak{f}_{R,-k'}, \mathfrak{p}_{T,k} \rangle, \langle \mathfrak{f}_{L,-k'}, \mathfrak{p}_{T,k} \rangle, \langle \mathfrak{f}_{R,-k'}, \mathfrak{p}_{R,k} \rangle, \langle \mathfrak{f}_{L,-k'}, \mathfrak{p}_{R,k} \rangle, \langle \mathfrak{f}_{-k'}, \mathfrak{p}_{L,k} \rangle, \langle \mathfrak{f}_{R,-k'}, \mathfrak{p}_{L,k} \rangle, \langle \mathfrak{f}_{L,-k'}, \mathfrak{p}_{L,k} \rangle\) in the discrete function space \(\mathbb{C}^{[a_n, b_n]}\) of type \((2.6)\).

Similarly, by Theorem 7.7, a finite recursion allows to express \(\Phi_{\alpha; a_R, a_L}\) as a polynomial in the inner products of \(\langle e_{k'}^R, p_k^T \rangle, \langle e_{k'}^L, p_k^T \rangle, \langle e_{k'}^R, p_k^R \rangle, \langle e_{k'}^L, p_k^R \rangle, \langle e_{k'}^R, p_k^L \rangle, \langle e_{k'}^L, p_k^L \rangle\) in the function space \(L^2\) of \((6.4)\) (the multiplicative factor from the initial condition is just \(\Phi_{\emptyset; \emptyset, \emptyset} = 1\) here).

The two recursions have exactly the same structure, so the polynomials in both cases are the same. The asserted convergence therefore follows from Corollary 6.2, which gives the convergence of the former inner products to the latter ones.

8. Conclusions and outlook

In this article we developed tools based on Clifford algebra valued discrete 1-forms for the analysis of the critical Ising model with locally monochromatic boundary conditions. These tools are transparent counterparts of objects in boundary conformal field theory. We showed how to use them for calculations in the Ising model; in particular in order to obtain a characterization of the fusion coefficients that arise as renormalized boundary correlation functions for the Ising model in the lattice slit strip. Discrete complex analysis techniques and judiciously chosen s-holomorphic solutions to the Riemann boundary value problem in the lattice strip and slit strip \([AKPR20]\) played a key role, and enabled in particular the proof of our main result, Theorem 7.8, stating the convergence of the fusion coefficients in the scaling limit.
In the last part of this series, we will relate the continuum fusion coefficients \( \Phi_{\alpha; \alpha R, \alpha L} \) to the structure constants

\[
\langle v'_\alpha, Y(v_{\alpha R}, \xi) v_{\alpha L} \rangle \bigg|_{\xi=1}
\]

of the appropriate fermionic vertex operator algebra \((V, Y, 1, \omega)\) (this still involves a transformation from the slit strip geometry to the half-plane geometry, since the VOA axioms in their standard formulation are only directly appropriate for the latter). With that connection to the correct VOA, Theorem 7.8 rigorously relates the scaling limit of boundary correlation functions of the Ising model with locally monochromatic boundary conditions to the algebraic axiomatization of the boundary conformal field theory.

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