Lower Bounds for the Exponential Domination Number of $C_m \times C_n$

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Abstract
A vertex $v$ in a porous exponential dominating set assigns weight $(\frac{1}{2})^{\text{dist}(v,u)}$ to vertex $u$. A porous exponential dominating set of a graph $G$ is a subset of $V(G)$ such that every vertex in $V(G)$ has been assigned a sum weight of at least 1. In this paper the porous exponential dominating number, denoted by $\gamma^*_e(G)$, for the graph $G = C_m \times C_n$ is discussed. Anderson et. al. \cite{1} proved that $\frac{mn}{15.875} \leq \gamma^*_e(C_m \times C_n) \leq \frac{mn}{13}$ and conjectured that $\frac{mn}{13}$ is also the asymptotic lower bound. We use a linear programming approach to sharpen the lower bound to $\frac{mn}{13.7619 + \epsilon(m,n)}$.

Keywords. porous exponential domination, domination, grid, linear programming, mixed integer programming

1 Introduction

Given a graph $G$, a weight function of $G$ is a function $w : V(G) \times V(G) \rightarrow \mathbb{R}$. For $u, v \in V(G)$, we say that $u$ assigns weight $w(u,v)$ to $v$. For a set $S \subseteq V(G)$ we denote $w(S,v) := \sum_{s \in S} w(s,v)$.

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Similarly, 

\[ w(v, S) := \sum_{s \in S} w(v, s). \]

For two weight functions of \( G \), \( w \) and \( w' \), we say \( w' \leq w \), if \( w'(u, v) \leq w(u, v) \) for all \( u, v \in V(G) \).

Let \( D \subseteq V(G) \) and \( w \) be a weight function of \( G \). The pair \((D, w)\) dominates the graph \( G \), if for all \( v \in V(G) \), \( w(D, v) \geq 1 \).

The standard definition of domination is where \( w(u, v) \) is 1 if \( v \) is in the closed neighborhood of \( u \), and 0 if it is not. This type of domination has been widely studied (see [4], [7]). Another well-studied type of domination is total domination, in which \( w(u, v) \) is 1 if \( v \) is in the neighborhood of \( u \) and 0 if it is not (see [5], [8]). There is also \( k \)-domination and \( k \)-distance domination (see [2], [6]). In \( k \)-domination \( w(u, v) \) is \( \frac{1}{k} \) if \( v \) is in the neighborhood of \( u \), 1 if \( u = v \), and 0 otherwise. In \( k \)-distance domination \( w(u, v) \) is 1 if the distance from \( u \) to \( v \) is at most \( k \), and 0 otherwise. For exponential domination, \( w(u, v) = \left( \frac{1}{2} \right)^{\text{dist}(u, v)} - 1 \), where \( \text{dist}(u, v) \) represents the length of the shortest path from \( u \) to \( v \).

A porous exponential dominating set of a graph \( G \) is a set \( D \subseteq V(G) \) such that \((D, w)\) dominates \( G \) when \( w(u, v) = \left( \frac{1}{2} \right)^{\text{dist}(u, v)} - 1 \). The exponential domination number of \( G \), denoted by \( \gamma^*_e(G) \), is the cardinality of a minimum exponential dominating set. This type of exponential domination has also been referred to as porous exponential domination. Some work has been done in non-porous exponential domination, where \( \text{dist}(u, v) \) represents the length of the shortest path from \( u \) to \( v \) that does not have any internal vertices that are in the dominating set (see [3]). For the sake of simplicity, we refer to porous exponential domination as exponential domination.

It is described in [3] that applications of exponential domination relate to the passing of information, and specifically models how information can be spread from a speaker through a crowd. Thus, the exponential domination number represents the minimum number of speakers needed to successfully convey a message to every individual within a crowd.

Within exponential domination, there has been research on the exponential domination number of \( C_m \times C_n \), where \( \times \) is the Cartesian product. Anderson et. al. [1], found lower and upper bounds for \( \gamma^*_e(C_m \times C_n) \). The following theorem shows a sharp upper bound for the exponential domination number of the graph \( C_{13m} \times C_{13n} \).
Theorem 1. \[[1]\] For all \( m \) and \( n \),
\[
\frac{\gamma_e^*(C_{13m} \times C_{13n})}{(13m)(13n)} \leq \frac{1}{13}.
\]

The proof of Theorem 1 is constructive. An exponential dominating set is created by choosing a vertex from each row and column of a \( 13 \times 13 \) grid and then periodically tiling \( C_{13m} \times C_{13n} \) with the grid and selecting all the corresponding vertices. This argument was extended to \( C_m \times C_n \), for \( m, n \) arbitrarily large.

Theorem 2. \[[1]\]
\[
\lim_{m,n \to \infty} \frac{\gamma_e^*(C_m \times C_n)}{mn} \leq \frac{1}{13}.
\]

A lower bound can be attained in the following way: Observe that when \( m \) and \( n \) are large enough, for each \( v \in V(C_m \times C_n) \) there exists \( 4i \) vertices \( u \in V(C_m \times C_n) \) such that \( \text{dist}(v,u) = i \), when \( i \) is a positive integer. So
\[
w(v,V(C_m \times C_n)) \leq 2 + \sum_{i=1}^{\infty} 4i2^{1-i} = 18.
\]
This implies
\[
\frac{1}{18} \leq \frac{\gamma_e^*(G)}{mn}.
\]

However this bound can be improved to \( \frac{1}{17} \) by adjusting the weight function so that \( v \) assigns weight 1 to itself, resulting in \( w(v,V(C_m \times C_n)) \leq 17 \). A better lower bound was attained in \[[1]\] by showing that the weight function could be adjusted so that each vertex assigns 2.125 less than in the original weight function.

Theorem 3. \[[1]\] For all \( m, n \geq 3 \),
\[
\frac{1}{15.875} < \frac{\gamma_e^*(G)}{mn}.
\]

The above theorems led to the following conjecture.

Conjecture 1. \[[1]\] For all \( m \) and \( n \),
\[
\frac{1}{13} \leq \frac{\gamma_e^*(C_m \times C_n)}{mn}.
\]
In this paper, a better lower bound for $\gamma^e(C_m \times C_n)$ is produced. In section [2], we use linear programming to minimize how much weight is necessary for each vertex in an exponential dominating set to assign. This leads to improved lower bounds in section [3]. For the remainder of the paper, we refer to $w$ as the weight function

$$w(u, v) := \left(\frac{1}{2}\right)^{\text{dist}(u, v) - 1}.$$

### 2 Linear Program

For the rest of the paper, let $G = C_m \times C_n$ and let $D = \{d_1, d_2, \ldots, d_{|D|}\}$ be an exponential dominating set of $G$. Given an odd positive integer $r$, let $G_v$ be the subgraph of $G$ that is an $r \times r$ grid centered vertex $v \in V(G)$, with $V(G_v) = \{v_1, v_2, \ldots, v_{r^2}\}$. Let $I_v$ be the set of interior vertices of $G_v$.

In this section, a linear program is created where the sum of the weights assigned to the vertices in $I_v$ is minimized. The minimum value attained is of the form $|I_v| + k$, where $0 < k$. A new weight function is then created, which still dominates with $D$ and has $v$ assigning $k$ less weight than before. A sequence of weight functions will be constructed recursively.

#### 2.1 The Grid

First we strategically partition $V(G)$. For each $v_i \in V(G_v)$, define $S_i$ to be the set of vertices $w \in V(G)$ such that the distance between $v_i$ and $w$ is less than the distance between $w$ and any other vertex in $G_v$. Notice that $S_i = \{v_i\}$, if $v_i \in I_v$. For $1 \leq i \leq r^2$, let $x_i = w(S_i, v_i)$. Therefore, if $1 \leq i, j \leq r^2$, then $w(S_i, v_j) = x_i \left(\frac{1}{2}\right)^{\text{dist}(v_i, v_j)}$. We define $\Gamma = V(G) \setminus \bigcup_{i=1}^{r^2} S_i$ and for $1 \leq j \leq r^2$, let $\epsilon_j = w(\Gamma, v_j)$. Thus,

$$w(D, v_j) \leq \sum_{i=1}^{r^2} w(S_i, v_j) + \epsilon_j = \sum_{i=1}^{r^2} x_i \left(\frac{1}{2}\right)^{\text{dist}(v_i, v_j)} + \epsilon_j.$$

Observe that $|V(\Gamma)| \leq m + n - 1$ and $\text{dist}(\Gamma, V(G_v)) \to \infty$ as $m, n \to \infty$. Thus $0 \leq \epsilon_j \leq (m + n - 1) \left(\frac{1}{2}\right)^{\text{dist}(\Gamma, V(G_v)) - 1}$ for each $1 \leq j \leq r^2$. Therefore
assuming that $\epsilon = \sum_{j=1}^{r^2} \epsilon_j$,

$$\epsilon \leq \sum_{j=1}^{r^2} (m + n - 1) \left( \frac{1}{2} \right)^{\text{dist}(\Gamma,V(G_v)) - 1} \leq r^2(m + n - 1) \left( \frac{1}{2} \right)^{\text{dist}(\Gamma,V(G_v)) - 1},$$

which means $\epsilon \to 0$ as $m, n \to \infty$.

**Example 1.** Consider $G = C_6 \times C_8$ as shown in Figure 1. For the simplicity of the figure, we remove the edges of $G$. Choose $r = 3$ and construct $G_{v_5}$ with $V(G_{v_5}) = \{v_1, v_2, \ldots, v_9\}$. We then label the corresponding sets $S_1, S_2, \ldots, S_9, \Gamma$. For instance, observe that $S_3$ consists of all vertices in $G$ whose distance to $v_3$ is smaller than their distance to any other vertex of $G_{v_5}$.

![Figure 1: $C_6 \times C_8$ with edges removed](image)

**2.2 The Program**

Lemma 1 below proves how to get a lower bound for the exponential domination number of a graph $G$, given that each vertex in the dominating set assigns more weight to $V(G)$ than needed.
Lemma 1. Let $D = \{d_1, d_2, \ldots, d_{|D|}\}$ be an exponential dominating set of $G$ and $\rho \in \mathbb{R}$ such that $w(d_j, V(G)) \leq \rho$ for all $j$. If there exists a sequence of weight functions $\{w_j\}_{j=0}^{|D|}$, where $w = w_0$, and the following conditions are satisfied for $1 \leq j \leq |D|$,

1) $w_j < w_{j-1}$,

2) $f(D, w_j)$ dominates $G$, and

3) there exist $k \in \mathbb{R}$ such that $0 < k \leq w_{j-1}(d_j, V(G)) - w_j(d_j, V(G))$,

then

$$\frac{1}{\rho - k} < \frac{|D|}{|V(G)|}.$$ 

Proof. Let $\{w_j\}_{j=0}^{|D|}$ be such a sequence of weight functions for the exponential dominating set $D$. Conditions 1) and 3) imply that $k < w_0(d_j, V(G)) - w_{|D|}(d_j, V(G))$ for all $d_j \in D$. Therefore,

$$k|D| < \sum_{j=1}^{|D|}[w_0(d_j, V(G)) - w_{|D|}(d_j, V(G))].$$

Since condition 2) gives $1 \leq w_{|D|}(D, v)$ for all $v \in V(G)$, then

$$|V(G)| \leq \sum_{j=1}^{|D|} w_{|D|}(d_j, V(G)).$$

Combining these inequalities gives,

$$k|D| + |V(G)| < \sum_{j=1}^{|D|} w_0(d_j, V(G))$$

$$\leq \sum_{j=1}^{|D|} 18 = 18|D|.$$

This implies that

$$\frac{1}{18 - k} < \frac{|D|}{|V(G)|}.$$

$\blacksquare$
We now construct a recursive set of weight functions that satisfy the conditions of Lemma \[\text{[1]}\] for some \(k\). Let \(d_j \in D\) and \(w_{j-1}\) be a weight function such that \((D, w_{j-1})\) dominates \(G\). Let \(G_d\) be the \(r \times r\) grid \(G_{d_j}\) and \(I = I_{d_j}\). Recall that \(x_i = w(S_i, v_i)\). Let \(A\) be the \(r \times r\) matrix such that \([A]_{ij} = \left(\frac{1}{2}\right)^{\text{dist}(v_i, v_j)}\). Let \(\bar{x} = [x_1, x_2, \ldots, x_r]^T\) and \(\bar{w} = [w(D, v_1), w(D, v_2), \ldots, w(D, v_r)]^T\). Thus, \(\bar{w} \leq A\bar{x}\). In fact, if \(w_0 < w\), then \(\bar{w}_0 < A\bar{x}\).

Let \(c\) be the real-valued vector such that \(c^T\bar{x} = \sum_{v_i \in I} w_{j-1}(D, v_i)\).

The objective function in the linear program will be \(c^T\mathbf{x}\), where \(\mathbf{x}\) is a vector of \(r^2\) variables. Since \((D, w_{j-1})\) dominates \(G\), \(1 \leq \bar{w}_{j-1}\), where \(1\) is the all-1s vector. Therefore, \(1 \leq A\bar{x}\); hence, \(1 \leq A\mathbf{x}\) is a constraint. Let \(b\) be the real-valued vector whose \(i\)th entry is \(1 + \left(\frac{1}{2}\right)^{\text{dist}(v_i, d_j)} + \epsilon\) if \(v_i \in I\) and 18 otherwise. The constraint \(A\mathbf{x} \leq b\) will be added to ensure that for each vertex in \(I\) the weight assigned from \(d_j\) can be decreased by the appropriate amount. Consider the following linear program:

\[
\begin{align*}
\min & \quad c^T\mathbf{x} \\
\text{s.t.} & \quad A\mathbf{x} \geq 1 \\
& \quad A\mathbf{x} \leq b \\
& \quad \mathbf{x} \geq 0.
\end{align*}
\]

Define \(\mathbf{x}^*\) to be an optimal solution to the linear program and \(\mathbf{x}_{\text{min}}\) to be the value attained. Obviously, \(|I| + \epsilon < \mathbf{x}_{\text{min}}\), so \(0 < k = \mathbf{x}_{\text{min}} - \epsilon - |I|\). For each \(i\) with \(v_i \in I\), let

\[y_i = \sum_{s=1}^{r^2} x_s \left(\frac{1}{2}\right)^{\text{dist}(v_i, v_s)} - \epsilon_i - 1.\]

Thus, \(0 \leq y_i \leq \left(\frac{1}{2}\right)^{\text{dist}(v_i, d_j)}\) and \(\sum_{v_i \in I} y_i = k\).

**Remark 1.** Note that the weights function \(\{w_j\}_{j=0}^{D}\) satisfy conditions 1), 2), and 3) of Lemma \[\text{[1]}\]. Clearly \(w_j < w_{j-1}\), so 1) is satisfied. For each \(v \in V(G) \setminus I\), \(1 \leq w_{j-1}(D, v) = w_j(D, v)\). For each \(v_i \in I\), \(w_j(D, v) = w_{j-1}(D, v) \cdot k = 1 + \epsilon_i\). This implies \((D, w_j)\) dominates \(G\) so 2) is satisfied. Lastly, \(w_j(d_j, V(G)) = w_{j-1}(d_j, V(G)) - k\), so 3) is satisfied.
3 Main Results

In this section, we use Lemma 1 and the weight functions \( \{w_j\}_{j=0}^{D} \) constructed in Section 2 to attain a lower bound for the exponential domination number of \( C_m \times C_n \).

**Theorem 4.** For all \( m, n \geq 13 \),

\[
\frac{1}{13.7619 + \epsilon} \leq \frac{\gamma_e^*(C_m \times C_n)}{mn},
\]

where \( \epsilon \to 0 \) as \( m, n \to \infty \). Moreover, \( \epsilon = 0 \) when \( m \) and \( n \) are both odd.

**Proof.** Let \( D \) be a minimum exponential dominating set. For each \( v \in D \), let \( G_v \) be the \( 13 \times 13 \) grid centered at \( v \). Recall that \( w(v, V(G)) \leq 18 \) for all \( v \in D \). The solution to the corresponding linear program is \( x_{\min} = 125.2381080608 \). Therefore, it follows that \( k = 125.2381080608 - \epsilon - 121 = 4.2381080608 \), so \( \frac{mn}{13.7618919392} \leq \gamma_e^*(C_m \times C_n) \) by Lemma 1.

The linear program created in Section 2 can be constructed in the form of a mixed integer linear program by adding the constraints \( x_i = 0 \) or 2, when \( v_i \in I_v \). Then the attained \( k \) is 10.94 + \( \epsilon \), by choosing a \( 9 \times 9 \) grid as \( G_v \). However, the weight function can only be adjusted at a vertex \( v \in D \), such that no vertices in \( D \cap I_v \) have been adjusted. Rather than using the linear program for all the vertices in \( D \), we will use it for the vertices in \( D \) that are relatively close together and use the mixed integer linear program for those vertices in \( D \) that are not close to the other vertices of \( D \).

**Theorem 5.** Let \( D \) be an exponential dominating set of \( C_m \times C_n \) and \( \alpha |D| \) be the number of vertices in \( D \) that are not within a \( 7 \times 7 \) grid of any other vertex in \( D \). Then

\[
\frac{1}{13.7619 - 2.8218\alpha - \epsilon} \leq \frac{\gamma(C_m \times C_n)}{mn},
\]

where \( \epsilon \to 0 \) as \( m, n \to \infty \).

**Proof.** Let \( D' \) be the set of vertices that are not within a \( 7 \times 7 \) grid of any other vertex in \( D \); so \( |D'| = \alpha |D| \). Choose \( r = 9 \) in \( G_v \) and let \( b' \) be the real-valued vector whose \( i \)th entry is 0 if \( v_i \in I_v \) and 4 otherwise. By taking geometric sums, it is easy to see that \( x_i \leq 4 \), for all \( i \).
The linear program
\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad A x \geq 1 \\
& \quad A x \leq b \\
& \quad x \leq b' \\
& \quad x \geq 0.
\end{align*}
\]
will attain a minimum of 56.06. So each vertex in \( D' \) can be adjusted by \( 56.06 - \epsilon_2 - 49 = 7.06 - \epsilon_2 \) to 10.94 + \( \epsilon_2 \), for some \( \epsilon_2 \geq 0 \). As before, the vertices of \( D \setminus D' \) can be adjusted to 13.761891392 + \( \epsilon_1 \), for some \( \epsilon_1 \geq 0 \). So \( mn \leq (1 - \alpha)|D|(13.7619 + \epsilon_1) + \alpha|D|(10.94 + \epsilon_2) \), which implies \( mn \leq |D|(13.7619 - 2.8218 \alpha + \epsilon) \).

Corollary 1 is a direct result of combining Theorems 1 and 3.

**Corollary 1.** Let \( D \) be an exponential dominating set of \( C_m \times C_n \). For \( m \) and \( n \) large enough, the number of vertices in \( D \) that are not within a \( 7 \times 7 \) grid of any other vertex in \( D \) is at most \( .27|D| \).

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