THE CHERN-SIMONS SOURCE AS A
CONFORMAL FAMILY
AND
ITS VERTEX OPERATORS

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ABSTRACT

In a previous work, a straightforward canonical approach to the source-free quantum Chern-Simons dynamics was developed. It makes use of neither gauge conditions nor functional integrals and needs only ideas known from QCD and quantum gravity. It gives Witten’s conformal edge states in a simple way when the spatial slice is a disc. Here we extend the formalism by including sources as well. The quantum states of a source with a fixed spatial location are shown to be those of a conformal family, a result also discovered first by Witten. The internal states of a source are not thus associated with just a single ray of a Hilbert space. Vertex operators for both abelian and nonabelian sources are constructed. The regularized abelian Wilson line is proved to be a vertex operator. We also argue in favor of a similar nonabelian result. The spin-statistics theorem is established for Chern-Simons dynamics even though the sources are not described by relativistic quantum fields. The proof employs geometrical methods which we find are strikingly transparent and pleasing. It is based on the research of European physicists about “fields localized on cones.”
1. INTRODUCTION

Theoretical investigations of gauge theories during the past few years have persuasively established that the Chern-Simons (CS) dynamics is a source of creative and fertile ideas from a mathematical as well as a physical perspective. Its significance for the theory of knots and links and for an intrinsic three-dimensional approach to conformal field theories (CFT’s) [1 to 7] is now widely appreciated. It is extremely useful for describing fractional statistics [8, 3, 4, 9], while its central role in theory of quantum Hall effect (QHE) has also been clarified in recent literature [10].

In previous communications [11], which are reviewed in Section 2, we investigated the quantization of abelian and nonabelian source-free CS dynamics by elementary canonical methods borrowed from QCD and quantum gravity. When the spatial slice is a disc $D$, it was shown that the quantum states form a conformal family localized on the boundary $\partial D$, a result discovered first by Witten [1]. These are the edge states of QHE [12, 10] when the gauge group is U(1). It was suggested in that paper that our approach has the virtues of great simplicity and transparency as it does not require scholarship in CFT’s or skills in the manipulation of functional integrals or gauge choices.

In this paper, we extend these considerations by allowing for point sources. When a point source is immersed in the CS field, it is well known that its statistics is affected thereby [8, 3, 4, 9]. As interaction renormalizes statistics, it must renormalize spin as well if, as some of us may conservatively desire, the CS dynamics incorporates the canonical spin-statistics connection. The specific mechanism for spin renormalization is a novel one: the configuration space of particle mechanics is enlarged by a circle $S^1$. A point of $S^1$ can be regarded as parametrizing a tangent direction or an orthonormal frame (although not canonically). A spinless source thus ends up acquiring a configuration space which is that of a two-dimensional rotor with translations added on. What occurs in CS theory
is a conformal quantum field on this $S^1$ with ability to change its location in space and with precisely the right spin to maintain the spin-statistics connection. The necessity for framing the particle has been emphasized before [1, 8, 13]. The qualitative reason for the emergence of this frame is regularization, which surrounds the particle with a tiny hole $H$ which is eventually shrunk to a point. The CS action is then no longer for a disc $D$, but for $D \setminus H$, which is a disc with a hole. In contrast to $D$, the latter has an additional boundary $\partial H$, which is the circle $S^1$ mentioned above. Just as $\partial D$, this boundary as well is associated with a conformal family. The internal states of a CS anyon for a fixed location on $D$ thus form an infinite dimensional conformal family of quantum states and are not described by just a single ray. This remark was first stated by Witten and applies with equal force to the quantum Hall quasiparticle [14] if described in the Chern-Simons framework [10]. It is also noteworthy that the CS source is not a first quantized framed particle, but is better regarded as a “particle” with a first quantized position and a second quantized frame. One intention of this paper is to explain these striking, although not quite original results with hopefully transparent arguments. We undertake this task in Section 2.

The Wilson line integral and its variants have physical and mathematical importance in gauge theories, the CS dynamics being no exception. For instance, the Jones polynomial [15] can be understood as the expectation value of the trace of a Wilson loop with its integration running over a knot or a link [1]. There are also remarks in the literature [3, 7, 14] roughly to the effect that the Wilson line emanating on $\partial D$ and terminating at a point $z$ is a vertex operator [17] of CFT creating a source at $z$. We do not however know of a published proof showing the correspondence between a Wilson line and a CFT vertex operator. We establish it in its abelian form in Section 3 by displaying the regularized Wilson line in terms of the Fubini-Veneziano vertex operator [17].

It has been argued elsewhere [18] that the canonical spin-statistics connection does
not require field theory or relativity for its validity and holds true in any dynamical system incorporating creation-annihilation processes subject to certain rules. Now the CS dynamics as well is consistent with this connection. In Section 4, we give a visual and geometric demonstration of this fact using the properties of the Wilson line. We will show elsewhere that the CS dynamics is compatible with the creation-annihilation rules of [18] and also present further developments of these ideas. Incidentally, arguments similar to those in this proof have previously occurred in work on “fields localized on cones” [19].

Section 5 is our last section and it contains the nonabelian extension of the preceding work. In particular, the vertex operators of the nonabelian Kac-Moody (KM) algebras [17] are constructed in the CS framework and the latter are physically interpreted as creation operators of nonabelian anyons (which are now nonabelian conformal families). We also establish their spin-statistics theorem.

There are several publications including our own which quantize CS sources. These sources do not get associated with conformal families in many of these papers [8, 3, 4, 9]. The reason lies in the rules of quantization employed in these papers: they are not the same as those which lead to CFT’s. Thus, if we choose not to identify observables related by nontrivial gauge transformations on boundaries such as \( \partial D \) or \( \partial H \), we end up with conformal families. If in contrast, we do identify these observables, the connection to CFT’s is essentially lost, and the U(1) source for instance ceases to have internal multiplicity of states for a given location. Lacking intrinsic reasons for preferring one of these approaches to quantization and not the other, we suppose that both will be found to have their uses to account for appropriate phenomenology.
2. CHERN-SIMONS ANYONS AS CONFORMAL FAMILIES

We have previously remarked that a source of the CS field (as treated here) is not the same as a framed particle. For this reason, we refer to the former by phrases such as a CS anyon or a CS source.

Let us first briefly recall the contents of ref.11, confining ourselves to the gauge group $U(1)$ and to a spacetime with a disc $D$ as its time slice. With the convention $\epsilon^{012} = +1$ for the Levi-Civita symbol, the phase space of the CS action

$$S = \frac{k}{4\pi} \int_{D \times \mathbb{R}^1} A dA \quad A = A_\mu dx^\mu \quad A dA = A \wedge dA$$

(2.1)
can be described by the equal time Poisson brackets (PB’s)

$$\{A_i(x), A_j(y)\} = \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y) ;$$

$$i, j = 1, 2 ; \quad x^0 = y^0 ; \quad \epsilon_{12} = -\epsilon_{21} = 1 ; \quad \epsilon_{11} = \epsilon_{22} = 0$$

(2.2)
and the Gauss law constraint

$$g(\Lambda^{(0)}) = \frac{k}{2\pi} \int_D \Lambda^{(0)} dA \approx 0$$

(2.3)
where

$$\Lambda^{(0)}|_{\partial D} = 0 .$$

(2.4)
Dirac’s weak equality is denoted by the symbol $\approx$ while $\partial D$ is the boundary of $D$. The condition (2.4) on the “test function” $\Lambda^{(0)}$ is needed if $g(\Lambda^{(0)})$ is to be differentiable in $A_i$ \[\square\]. The space of such test functions will be denoted by $\mathcal{T}^{(0)}$. Note also that $A_0$ and its conjugate momentum do not occur in this phase space. As is permissible, they will hereafter be regarded as nonexistent.

The observables of the theory are

$$q(\xi) = \frac{k}{2\pi} \int_D d\xi A$$

(2.5)
(and functions of \( q(\xi) \)) where \( \xi \) need not vanish on \( \partial D \). Their PB’s at equal times are

\[
\{q(\xi), q(\eta)\} = \frac{k}{2\pi} \int_{\partial D} \xi d\eta.
\]  

(2.6)

(All variables and operators will hereafter be at equal times). Notice that test functions \( \xi, \xi', ... \) with the same boundary values define the same observable since

\[
q(\xi) - q(\xi') = -g(\xi - \xi').
\]

The PB’s (2.6) are easily quantized. Thus let

\[
\xi_N|_{\partial D}(\theta) = e^{iN\theta}, \quad N \in \mathbb{Z},
\]

\( \theta (\mod 2\pi) \) being an angular coordinate on \( \partial D \) increasing in the anticlockwise sense. Also, for \( D \), we can take \( \xi_0 \) to have the constant value 1 throughout \( D \). Hence \( q(\xi_0) \approx 0 \) \[\text{(1)}\]. We will therefore hereafter assume that \( N \neq 0 \) for \( D \). Calling \( Q_N \) the quantum operator for \( q(\xi_N) \), we find the commutation relation

\[
[Q_N, Q_M] = Nk\delta_{N+M,0}
\]  

(2.7)

which defines a U(1) KM algebra on \( \partial D \). The quantum states are now constructed by treating \( Q_N \) as creation-annihilation operators. They are also annihilated by the quantum version \( G(\Lambda^{(0)}) \) of the Gauss law, \( G(\Lambda^{(0)})|\cdot > = 0 \).

Suppose now that a spinless point source with coordinate \( z \) is coupled to \( A_\mu \) with coupling \( eA_\mu(z(x^0))\dot{z}^\mu \), \( z^0 = x^0 \). The field equation \( \partial_1 A_2 - \partial_2 A_1 = 0 \) is thereby changed to

\[
\partial_1 A_2 - \partial_2 A_1 = -\frac{2\pi e}{k} \delta^2(x - z).
\]  

(2.8)

If \( C \) is a contour enclosing \( z \) with positive orientation, then, by (2.8),

\[
\oint_C A = -\frac{2\pi e}{k}.
\]  

(2.9)
On letting $C$ shrink to a point, it now follows that $A(x) = A_j(x)dx^j$ has no definite limit when $x$ approaches $z$. This singularity of $A$ demands regularization. A good way to regularize is to punch a hole $H$ containing $z$, and eventually to shrink the hole to a point.

Once this hole is made, the action is no longer for a disc $D$, but for $D \setminus H$, a disc with a hole. $D \setminus H$ has a new boundary $\partial H$ and it must be treated exactly like $\partial D$. The Gauss law must accordingly be changed to

$$g(\Lambda^{(1)}) \approx 0 \quad (2.10)$$

where the new test function space $T^{(1)}$ for $\Lambda^{(1)}$ is defined by

$$\Lambda^{(1)}|_{\partial D} = \Lambda^{(1)}|_{\partial H} = 0 \quad . \quad (2.11)$$

The quantum operator $G(\Lambda^{(1)})$ for $g(\Lambda^{(1)})$ annihilates all the physical states.

There are now two KM algebras, one each for $\partial D$ and $\partial H$. The former is defined by observables $q(\xi^{(0)})$ with test functions $\xi^{(0)}$ which vanish on $\partial H$, the latter by observables $q(\xi^{(1)})$ with test functions $\xi^{(1)}$ which vanish on $\partial D$. Let us now define the KM generators for the outer and inner boundaries as

$$q^{(0)}_N \equiv q(\xi^{(0)}_N), \quad \xi^{(0)}_N(\theta)|_{\partial D} = e^{iN\theta}, \quad \xi^{(0)}_N|_{\partial H} = 0;$$

$$q^{(1)}_N \equiv q(\xi^{(1)}_N), \quad \xi^{(1)}_N(\theta)|_{\partial H} = e^{-iN\theta}, \quad \xi^{(1)}_N|_{\partial D} = 0, \quad (2.12)$$

$\theta \ (\text{mod } 2\pi)$ being an angular coordinate on $\partial H$. [The coordinates $\theta$ on both $\partial D$ and $\partial H$ increase, say, in the anticlockwise sense.] The corresponding quantum operators will be denoted by $Q^{(0)}_N$ and $Q^{(1)}_N$. Note that the boundary conditions exclude the choice $\xi^{(0)}_{\alpha} = \text{the constant function on } D \setminus H$. Hence we may not exclude $N = 0$ now.

An interpretation of the observables localized on $\partial H$ is as follows. Let $\theta \ (\text{mod } 2\pi)$ be an angular coordinate on $D$ which reduces to the $\theta$ coordinates we have fixed on $\partial D$ and $\partial H$. A typical $A$ compatible with (2.9) has a blip $-\frac{2\pi k}{\hbar}\delta(\theta - \theta_0)d\theta$ localized on $\partial H$ at $\theta_0$. 

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The behavior of a general $A$ on $\partial H$ can be duplicated by an appropriate superposition of these blips. The observable $q(\xi^{(1)})$ has zero PB with the left side of (2.9) and hence preserves the flux enclosed by $C$. In fact, the finite canonical transformation generated by $q(\xi^{(1)})$ changes $A$ to $A + d\xi^{(1)}$ where the fluctuation $d\xi^{(1)}$ creates zero net flux through $C$. All $A$ compatible with (2.9) can be generated from any one $A$, such as an $A$ with a blip, by these transformations. Thus the KM algebra of observables $Q_{N}^{(1)}$ on $\partial H$ generates all connections on $\partial H$ with a fixed flux from any one of these connections.

We have now reproduced Witten’s observation [1] that the CS anyon or the CS version of the quantum Hall quasiparticle is a conformal family.

In classical CS theory, there exist diffeomorphism (diffeo) generators which perform diffeos of boundaries by canonical transformations. For $\partial D$, it was shown before [11] that they generate the Virasoro algebra and are given by the Sugawara construction after quantization. The same is trivially true for $\partial H$. The $2\pi$ rotation diffeo for $\partial H$, of interest in the next Section, is readily displayed using this algebra.

3. THE WILSON LINE IS A VERTEX OPERATOR

We have seen this statement occasionally in the literature [6, 7, 16]. Its correct meaning and proof are likely to be known to some physicists. Here we demonstrate them in our approach.

The vertex operator [17] acts on a state of zero charge (or “momentum”) and creates a state with charge. So let us first establish that the Wilson line too can be interpreted as playing an analogous role.

For this purpose, take a disc $D$ and pierce a hole $H$ at $z$ as before, but without inserting a charge at $z$. There is then a conformal family with zero charge for this hole, with the “highest weight state” or “vacuum” $|0\rangle$. [ The states of this family can be regarded
as describing spin fluctuations in the quantum Hall effect without corresponding charge fluctuations. Cf. Moore and Read, and Balatsky and Stone [10]. It is annihilated by the charge operator \( Q_0^{(1)} \), which is the quantum version of \( q(\xi_0^{(1)}) \) with \( \xi_0^{(1)}|_{\partial H} = 1 \). It is also annihilated by the Gauss law and by \( Q_N^{(1)} \) for \( Nk > 0 \). We shall furthermore assume that it describes the highest weight state of zero charge or vacuum on \( \partial D \).

Next consider the Wilson line from a point \( P \) on \( \partial D \) to \( z \), the integration being along a line \( L \):

\[
w(z) = \exp ie \int_P^z A.
\]

(3.1)

Its response

\[
w(z) \rightarrow [\exp ie\xi_0^{(1)}(z)]w(z) \exp[-ie\xi_0^{(1)}(P)] = [\exp ie\xi_0^{(1)}(z)]w(z)
\]

to the finite transformation \( A \rightarrow A + d\xi_0^{(1)} \) generated by \( q(\xi_0^{(1)}) \) shows that it creates a state of charge \( e \) at \( z \):

\[
Q_0^{(1)}w(z)|0> = ew(z)|0>
\]

(3.2)

(It also creates charge \( -e \) at \( P \).) If the tangent to \( L \) at \( z \) points in the angular direction \( \theta_0 \), then \( A\theta d\theta \) becomes the blip \( -2\pi e k \delta(\theta - \theta_0)d\theta \) at \( \partial H \) on the state \( w(z)|0> \). We can say that \( w(z) \) creates charge \( e \) localized at \( \theta_0 \) on \( \partial H \).

Now \( w(z) \) involves a quantum field on a line and is not a well defined operator. It requires regularization, a task we can approach as follows.

Let \( \Delta \) be a strip of tiny width \( \delta \) bounded by \( L \) on one side and by another line \( L' \) on the other side (see Figure 1). \([L \) does not quite reach \( z \) here unlike the previous \( L \). But this is immaterial as \( H \) will presently be shrunk to \( z \).]\) Let \( \Lambda \) be a multivalued function which is constant on \( D \setminus (H \cup \Delta) \) and which increases by 1 as \( \Delta \) is crossed from \( L' \) to \( L \).

Then \( d\Lambda \) is a globally defined closed one form on \( D \setminus H \) and its integral over \( C \) is 1. Now consider

\[
\int_{D \setminus H} d\Lambda A.
\]

(3.3)
It is the same as
\[ \int_{D \setminus (H \cup L)} d\Lambda A , \]
\( D \setminus (H \cup L) \) being the disc punctured at \( H \) and cut also along \( L \). \( \Lambda \) is single valued on this punctured, cut disc and
\[ \lim_{H \to z} \int_{D \setminus (H \cup L)} d\Lambda A = \int_L A - \lim_{H \to z} \int_{D \setminus (H \cup L)} \Lambda dA . \]  
(3.4)

In view of the Gauss law, we formally identify (3.3) with
\[ \int_L A \]  
(3.5)
when \( H \) shrinks to \( z \) and \( \delta \to 0 \). [The necessity for the last limit is not perhaps apparent here. Without this limit, the exponential of \( ie \) times (3.3) will create \( A \) with support of size \( \delta \) at \( \partial H \) and not a blip.] In this way we are led to consider
\[ W(z) = \exp(ie\tilde{Q}(\Lambda)), \quad \tilde{Q}(\Lambda) = \frac{2\pi}{k} Q(\Lambda) = \lim_{H \to z} \int_{D \setminus H} d\Lambda A \]  
(3.6)
instead of \( w(z) \). [Here \( Q(\Lambda) \) is the quantum operator for \( q(\Lambda) \).]

Actually let us take one further step away from \( w(z) \), and replace \( W(z) \) by another operator \( \mathcal{W}(z) \) which still creates a blip on \( \partial H \), but creates uniformly distributed charge instead of a blip on \( \partial D \). For this purpose, let us introduce a function \( \Theta \) in the following way. The angular coordinate \( \theta \) of a point in \( D \setminus H \) is multivalued, but it can be made single valued by cutting \( D \setminus H \) along \( L \). The function \( \Theta \) is any single valued determination of \( \theta \) in \( D \setminus (H \cup L) \). On \( \partial H \) for example, we can assume that it increases from \( \theta_0 \) to the right of \( L \) to \( \theta_0 + 2\pi \) to the left of \( L \). The operator \( \mathcal{W}(z) \) is obtained from \( W(z) \) by putting a new function \( \chi \) in place of \( \Lambda \) where i) \( \chi \) equals \( \Lambda \) on \( \partial H \), ii) \( \chi \) becomes \( \frac{\Theta}{2\pi} \) on \( \partial D \), iii) \( \chi \) is single valued in \( D \setminus (H \cup L) \), and iv) \( d\chi \) is a closed globally defined form on \( D \setminus H \). [This means that the integral of \( d\chi \) over any positively oriented closed contour encircling \( z \) once is 1. If \( d\chi \) is replaced by a nonclosed form on \( D \setminus H \) (which coincides with \( d\chi \) on
∂H and ∂D), then the flux through C for the state W(z)|> will depend on the location and shape of C and not just its homology class. We can not then regard W(z) as creating charge localized on ∂H and ∂D.] There clearly exists such a χ.

The next step is Fourier analysis. The function χ − Θ 2π is single valued on D \ H and vanishes as well on ∂D. Only its value on ∂H is relevant because of the Gauss law. Its Fourier decomposition on ∂H when δ → 0 is

$$\chi(\theta) - \frac{\Theta(\theta)}{2\pi} = \sum_{N \in \mathbb{Z}} a_N e^{-iN\theta},$$

$$a_N = \frac{1}{2\pi} \int_{\theta_0}^{\theta_0+2\pi} d\theta \left( \chi(\theta) - \frac{\Theta(\theta)}{2\pi} \right) e^{iN\theta} = -\frac{1}{(2\pi)^2} \int_{\theta_0}^{\theta_0+2\pi} d\theta \theta e^{iN\theta}. \quad (3.7)$$

Thus

$$a_0 = -\left( \frac{1}{2} + \frac{\theta_0}{2\pi} \right),$$

$$a_N = -\frac{1}{2\pi i} e^{iN\theta_0}, \quad N \neq 0. \quad (3.8)$$

Hence

$$\hat{Q}(\chi) \equiv \int d\chi A = \left[ \frac{1}{k} Q(\Theta) - \frac{\pi}{k} Q_0^{(1)} \right] - \frac{\theta_0}{k} Q_0^{(1)} + \frac{i}{k} \sum_{N \neq 0} \frac{1}{N} Q_N^{(1)} e^{iN\theta_0} \quad (3.9)$$

where

$$Q(\Theta) = \frac{k}{2\pi} \int d\Theta A, \quad (3.10)$$

and

$$\left[ \frac{1}{k} Q(\Theta) - \frac{\pi}{k} Q_0^{(1)}, Q_N^{(1)} \right] = i\delta_{N,0}. \quad (3.11)$$

Now \(\hat{Q}(\chi)\) can be written as follows:

$$\hat{Q}(\chi) = \frac{1}{\sqrt{|k|}} \left[ q - \epsilon(k)p\theta_0 + i \sum_{N \neq 0} \frac{\alpha_N}{N} e^{i\epsilon(k)N\theta_0} \right], \quad \epsilon(k) = \frac{k}{|k|}. \quad (3.12)$$

Here

$$q = \epsilon(k) \left[ \frac{Q(\Theta)}{\sqrt{|k|}} - \frac{\pi}{\sqrt{|k|}} Q_0^{(1)} \right],$$
\[ p = \frac{Q_0^{(1)}}{\sqrt{|k|}}, \]
\[ \alpha_N = \frac{Q_{\ell(k)N}}{\sqrt{|k|}}. \quad (3.13) \]

They fulfill the commutation relations
\[ [q, p] = i, \quad [\alpha_N, \alpha_M] = N\delta_{N+M,0}, \quad \] (3.14)
the remaining commutators being zero. Thus \( \sqrt{|k|}\tilde{Q}(\chi) \) is the Fubini-Veneziano coordinate field for the construction of the vertex operator \([17]\) with \( q \) and \( p \) playing the roles of position and momentum. Position and momentum, especially the former, occur naturally in CS theory and need not be introduced by hand.

As in the Fubini-Veneziano construction, we finally define the correctly regularized vertex operator by the normal ordered expression
\[ U(\theta_0) =: \exp \left(i\bar{e}\tilde{Q}(\chi)\right) :. \quad (3.15) \]

In this normal ordering, creation and annihilation operators are treated in the usual way and the \( p \) term is moved to the right of the \( q \) term.

The Wilson line creates a localized charge on \( \partial D \) as well at some angle \( \theta'_0 \). It can also be regularized by the Fourier analysis of \( \Lambda - \frac{\theta_0}{2\pi} \) on \( \partial H \) and \( \partial D \) and normal ordering. We find,
\[ \tilde{Q}(\Lambda) = \left[ \frac{1}{k}Q(\Theta) - \frac{\pi}{k}Q_0^{(1)} \right] - \frac{\theta_0}{k}Q_0^{(1)} + i \frac{1}{k} \sum_{N \neq 0} \frac{1}{N} Q_N^{(1)} e^{iN\theta_0} + \]
\[ - \frac{\pi}{k}Q_0^{(0)} - \frac{\theta_0'}{k}Q_0^{(0)} - i \frac{1}{k} \sum_{N \neq 0} \frac{1}{N} Q_N^{(0)} e^{-iN\theta'_0} \quad (3.16) \]

Note that
\[ [Q_0^{(1)}, Q(\Theta)] = -[Q_0^{(0)}, Q(\Theta)] = -ik. \quad (3.17) \]

We can now imagine splitting up \( Q(\Theta) \) as \( X^{(0)} + X^{(1)} \), \( X^{(0)} \) commuting with \( Q_N^{(1)} \) and \( X^{(1)} \) with \( Q_N^{(0)} \). Then (3.16) becomes the sum of two slightly altered Fubini-Veneziano
fields with corresponding vertex operators, the Wilson line becoming the product of these operators. In the CS theory, however, the aforementioned split is unnatural as this theory contains no operator creating a charge without creating its negative “image” charge elsewhere. It is thus prudent not to insist on the split and write

\[
\text{Regularized Wilson line} = : \exp i e \int_P^z A : \\
= : \exp \left( i e \left[ \bar{Q}(\chi) - \frac{\pi}{k} Q_0^{(0)} - \frac{\theta^0}{k} Q_0^{(0)} - \frac{i}{k} \sum_{N \neq 0} \frac{1}{N} Q_N^{(0)} e^{-iN\theta_0} \right] \right) : .
\] (3.18)

4. SPIN AND STATISTICS

As seen in Section 3, the vertex operator (3.18) which describes the Wilson line creates a pair of localized charges, one at the point \(z\) in the interior of the disc and another, the “image” charge, at the point \(P\) on the outer boundary. In this Section, as a matter of convenience, the image \(P\) for a Wilson line is taken to be situated at a point \(z'\) inside the disc and not on its rim. In order to regularize the connection \(A(x)\) as \(x\) approaches \(z'\), we create a hole around \(z'\) as well and treat it according to the previous Sections. The images for any two distinct Wilson lines are taken to be distinct. Note that the image particles are held fixed and consequently their internal states are not really important in the discussion of this Section.

It is useful, for the purpose of analyzing the topic of this Section, to have a method of comparing the internal states of the particle at distinct spatial locations and a convention which fixes their identity. We can achieve these goals by introducing a connection for transporting tangent directions from point to point. Let us choose it to be the Levi-Civita connection for a Euclidean metric on \(D\). Directions related by parallel transport will then be said to be identical.
A two-particle state with identical internal states at locations \( z^{(i)} \) is

\[
( : \exp ie \int_{L^{(i)}} A : ) ( : \exp ie \int_{L^{(2)}} A : ) |0 > , \tag{4.1}
\]

\( L^{(i)} \) being nonintersecting lines from \( z^{(i)'} \) to \( z^{(i)} \) as in Figure 2. The tangents to \( L^{(i)} \) at \( z^{(i)} \) are parallel, that is, according to the chosen convention, they point in the same direction. The vacuum state here is the tensor product of the oscillator vacua for the holes \( H^{(i)} \) and \( H^{(i)'} \) associated to the two particles and their images. As \( A_j \) along \( L^{(1)} \) and \( L^{(2)} \) commute, we can also write this state as

\[
: \exp ie \int_{L^{(1)}+L^{(2)}} A : |0 > . \tag{4.2}
\]

Some qualitative properties of this state will now be pointed out. If \( L^{(i)} \) are deformed (keeping \( z^{(i)} \), \( z^{(i)'} \) and tangents at those points fixed), the exponentials are changed by exponentials of the Gauss law and so the state is not changed. The lines can even touch as in Figure 3 without affecting the state. It must further be observed that the integral over \( A \) in (4.2) can be run along two lines \( L^{(1)'} \) and \( L^{(2)'} \) without affecting the state, the former being from \( z^{(1)'} \) to \( z^{(2)} \) and the latter from \( z^{(2)'} \) to \( z^{(1)} \), as indicated in Figure 3. But as these lines intersect, (4.2) can be written in the form (4.1), with the product of two vertex factors running over \( L^{(i)'} \), only at the cost of an extra phase.

Consider next the adiabatic exchange of the two particles keeping all the internal states (tangents at \( z^{(i)} \) and \( z^{(i)'} \)) fixed. The result for the initial state of Figure 4a is Figure 4b. It is the same as the successive stages 4c and 4d of Figure 4. As \( L^{(1)} \) and \( L^{(2)} \) do not have points in common, we can write the final state as

\[
: \exp ie \int_{L^{(1)}} A : : \exp ie \int_{L^{(2)}} A : |0 > . \tag{4.3}
\]

As the \( L^{(2)} \) integral is \( 2\pi \) rotation around \( z^{(2)} \) of the \( L^{(2)} \) integral, the spin-statistics theorem has been proved for the state (4.1). Its proof for a general state can now be constructed along the lines indicated towards the end of the next Section.
5. NONABELIAN GROUPS

5.1 The Vertex Operator

We limit our discussion to KM algebras associated with compact simple Lie groups $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$. We identify $\mathfrak{g}$ (or rather $i\mathfrak{g}$) with one of its faithful representations $\gamma$ by hermitean matrices. Let $\{T_\alpha\}$ be a basis for $\gamma$ with $Tr T_\alpha T_\beta = \delta_{\alpha\beta}$. The connection $A = iA_\mu^\alpha dx^\mu T_\alpha$ in our notation is antihermitian ($A_\mu^\alpha(x)^* = A_\mu^\alpha(x)$), the curvature $F = F_{\mu\nu} dx^\mu dx^\nu$ reads $dA + A^2$ and the CS action is

$$ S = -\frac{k}{4\pi} \int_{D \times R^3} Tr \left( AdA + \frac{2}{3} A^3 \right) . \tag{5.1} $$

It leads to the PB’s

$$ \{A_i^\alpha(x), A_j^\beta(y)\} = \delta_{\alpha\beta} \epsilon_{ij} \frac{2\pi}{k} \delta^2(x - y) , \tag{5.2} $$

$$ i, j = 1, 2 ; \quad x^0 = y^0 . $$

The classical and quantum interaction of nonabelian sources with the CS field has been treated elsewhere [3, 4]. It was established there that the presence of a source at $z$ changes the classical Gauss law $F_{12} = 0$ to

$$ F_{12} = -i \frac{2\pi}{k} I \delta^2(x - z) \tag{5.3} $$

where the hermitean internal vector $I$ is valued in $\gamma$. Its range is restricted to an orbit of $G$ in $\mathfrak{g}$ under the adjoint action, the specific orbit fixing the unitary irreducible representation (UIR) characterizing the source in quantum theory.

In the presence of a nonabelian source, the limit of $A$ as $z$ is approached is uncertain just as for an abelian source. This was proved in ref. 3. It is not therefore a matter for surprise that this source term turns into a conformal family in quantum theory, this being what happens in the $\tilde{U}(1)$ problem. Briefly, for quantization, we first pierce a hole
\( H \) containing \( z \) and introduce the test function space \( \mathcal{T}^{(1)} \) following Section 2. Elements \( \Lambda^{(1)} \) of \( \mathcal{T}^{(1)} \) vanish on \( \partial D \) and \( \partial H \). They are now also \( \gamma \) valued. The classical Gauss law is then
\[
g(\Lambda^{(1)}) = -\frac{k}{2\pi} \int_D \text{Tr} (\Lambda^{(1)} F) \approx 0, \quad \Lambda^{(1)} \in \mathcal{T}^{(1)}. \tag{5.4}
\]
The quantum analogue \( G(\Lambda^{(1)}) \) of \( g(\Lambda^{(1)}) \) annihilates all the physical states.

Observables are generated by two KM algebras, one localized on \( \partial D \) and one on \( \partial H \). They are constructed following Section 2 using test functions \( \xi^{(0)} \) and \( \xi^{(1)} \). (For further details, see ref.11.) They vanish on \( \partial H \) and \( \partial D \) respectively and are \( \gamma \) valued. The classical KM observables localized on \( \partial D \) and \( \partial H \) are given by
\[
q(\xi^{(i)}) = \frac{k}{2\pi} \int_D \text{Tr} (-d\xi^{(i)} A + \xi^{(i)} A^2), \quad i = 0, 1. \tag{5.5}
\]

We now focus attention on \( \partial H, \partial D \) having been looked at before. The quantum KM generators \( Q_N^{(1)\alpha} \) localized on \( \partial H \) are the quantum operators for
\[
q(\xi_N^{(1)\alpha}), \quad \xi_N^{(1)\alpha}|_{\partial H} = i e^{-iN\theta} T_\alpha \quad \text{(and } \xi_N^{(1)\alpha}|_{\partial D} = 0). \]
They fulfill the algebraic relation
\[
[Q_N^{(1)\alpha}, Q_M^{(1)\beta}] = i f^{\alpha\beta\gamma} Q_N^{(1)\gamma} + Nk\delta_{N+M,0} \delta_{\alpha\beta} \tag{5.6}
\]
and the hermiticity property \( Q_N^{(1)\alpha} = Q_N^{(1)\alpha} \) in a unitary KM group representation.

Given an algebra such as (5.6), there is a standard approach to finding its irreducible representations \([17]\). Let \( H_i (1 \leq i \leq r), E_\alpha \) be the Cartan or rather the Chevalley basis for \( G \) with \( H_i \) spanning the Cartan subalgebra and \( \alpha > 0 \) denoting the positive roots. Let \( H_N^{(1)i}, E_N^{(1)\alpha} \) denote the quantum operators \( Q(1)(ie^{-iN\theta} H_i), Q(1)(ie^{-iN\theta} E_\alpha) \) for \( q(ie^{-iN\theta} H_i), q(ie^{-iN\theta} E_\alpha) \) where in the arguments, only the boundary values of test functions \( \xi^{(1)} \) on \( \partial H \) have been displayed for simplicity. Then \( H_N^{(1)i}, E_N^{(1)\alpha} (N > 0) \) and \( E_0^{(1)\alpha} (\alpha > 0) \) are the generators associated with the positive roots of the \( \text{extended} \) KM algebra whereas
$H_0^{(1)i}$ span the Cartan subalgebra. A representation is obtained by setting up the highest weight state $|h\rangle$, $h = (h^1, h^2, ..., h^r)$, fulfilling $H_0^{(1)i}|h\rangle = h^i|h\rangle$ and annihilated by all the positive root generators: $H_N^{(1)i}|h\rangle = E_N^{(1)\alpha}|h\rangle = 0$ if $N > 0$ and $E_0^{(1)\alpha}|h\rangle = 0$ if $\alpha > 0$. $h$ is also subject to certain constraints \cite{17}. Repeated applications of the negative roots then generate all the states of the representation. There is a scalar product guaranteeing the hermiticity conditions $H_N^{(1)i\dagger} = H_N^{(1)i} - N$, $E_N^{(1)\alpha\dagger} = E_N^{(1)\alpha} - N$.

In our case, there is a conformal family sitting on $\partial D$ even prior to the insertion of a source. We will assume that $|h\rangle$ is also a highest weight state (say) for $\partial D$ and will not bother to display this fact.

We want to describe the insertion of a source using a vertex operator as in the U(1) CS theory. It is known that a vertex operator exists only for level $k = 1$ KM representations \cite{17}. Let us specialize to $G = SU(2)$ for simplicity. Then $r = 1$ and $h^1$ (the conventional “third component of angular momentum”) for a highest weight state is 0 or $\frac{1}{2}$ for a level 1 representation. (Here and in what follows until (5.19), we have taken the longest root of $SU(2)$ to be normalized to 1.) These two representations are inequivalent. The basic vertex operator we construct generates the $h^1 = \frac{1}{2}$ highest weight state from the $h^1 = 0$ highest weight state. It is associated with the creation of a particular sort of source. It performs this creation in a way similar to that of an abelian vertex operator of Section 3. Other related vertex operators we discuss have analogous properties.

It is convenient to identify $SU(2)$ with $2 \times 2$ unitary unimodular matrices. Then we can set

$$H_1 := \frac{1}{2}\tau_3, \quad E_\alpha(\alpha > 0) := \tau_+ = \frac{1}{2}(\tau_1 + i\tau_2), \quad E_\alpha(\alpha < 0) := \tau_- = \frac{1}{2}(\tau_1 - i\tau_2),$$

$$H_N^{(1)i} := Q^{(1)}(ie^{-iN\theta} \tau_3), \quad E_N^{(1)\alpha}(\alpha > 0) := E_N^{(1)\alpha} = Q^{(1)}(ie^{-iN\theta} \tau_+),$$

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\[
E_N^{(1)\alpha}(\alpha < 0) : = E_N^{(1)\alpha} = Q^{(1)}(ie^{-iN\theta} \tau_\alpha).
\]

(5.7)

\(H_N^{(1)}\) has been renamed here as \(H_N^{(1)3}\) and \(\tau_i\) are Pauli matrices.

We can now go about finding a vertex operator method of source insertion as in the abelian CS theory, partially imitating considerations in KM theory. A hole \(H\) is first made at \(z\) and a highest weight state \(|0\rangle\) with eigenvalue 0 for \(H_0^{(1)3}\) is first erected on \(\partial H\). It is annihilated by the generators for the positive roots and by \(H_0^{(1)3}\).

Now from a KM highest weight state, we can uniquely induce a unitary irreducible representation (UIR) of SU(2). For it is annihilated by its raising operator \(E_0^{(1)+}\) and is its highest weight state as well. As \(E_0^{(1)+}\) and \(H_0^{(1)3}\) kill \(|0\rangle\), this UIR is the trivial one for \(|0\rangle\), and \(E_0^{(1)-}\) as well annihilates it.

Let the internal vector \(I\) of the source determine the “total angular momentum" \(j = \frac{1}{2}\) UIR of SU(2). The KM highest weight state \(|\frac{1}{2}\rangle\) is clearly associated with this SU(2) UIR. We shall later discuss the sense in which (5.3) is enforced on the two states \(|\frac{1}{2}\rangle\), \(E_0^{(1)-}|\frac{1}{2}\rangle\) with \(j = \frac{1}{2}\) and \(H_0^{(1)3} = \pm \frac{1}{2}\) and also how to create \(j\) which differs from \(\frac{1}{2}\). We are now ready to reveal the operator \(V\) which creates \(|\frac{1}{2}\rangle\) from \(|0\rangle\).

Let \(\theta = i\theta \frac{2\pi}{\delta}\) where the value of the function \(\theta\) at a point is the polar coordinate at that point. This function is well defined on \(D \setminus (H \cup L_0)\) where the line \(L_0\) from \(\partial D\) to \(\partial H\) has zero polar angle. Consider

\[
Q(\theta) = \frac{1}{2\pi} \int \text{Tr} (-d\theta A + \theta A^2)
\]

where \(k\) now is being set equal to 1. Its commutator with \(Q(\xi^{(1)})\) is

\[
[Q(\theta), Q(\xi^{(1)})] = -iQ \left( [\theta, \xi^{(1)}] \right) - \frac{i}{2\pi} \int_{\partial H} \text{Tr} \xi^{(1)} d\theta.
\]

(5.9)

Special consequences of this equation are

\[
[Q(\theta), H_N^{(1)3}] = \frac{i}{2} \delta_{N,0} ,
\]

(5.10)

\[
[Q(\theta), E_N^{(1)\pm}] = Q \left( \pm \theta ie^{-iN\theta} \tau_\pm \right).
\]

(5.11)
From $Q(\theta)$, we form the operator

$$V = e^{iQ(\theta)}.$$  \hspace{1cm} (5.12)

It creates a state $V|0 >$ of uniform $\frac{\tau}{2}$ charge $\frac{1}{2}$ on $\partial H$ (and $-\frac{1}{2}$ on $\partial D$) since by (5.10),

$$H_0^{(1)3}V|0 > = \frac{1}{2}V|0 > .$$  \hspace{1cm} (5.13)

Furthermore, by (5.10) and (5.11),

$$V^{-1}H_N^{(1)3}V = H_N^{(1)3}, \ N \neq 0 ,$$

$$V^{-1}E_N^{(1)\pm}V = E_N^{(1)\pm}.$$  \hspace{1cm} (5.14)

Hence,

$$H_0^{(1)3}V|0 > = E_N^{(1)\pm}V|0 > = 0 , \ N > 0 ,$$

$$E_0^{(1)+}V|0 > = 0 .$$  \hspace{1cm} (5.15)

$V|0 >$ is thus the highest weight state $|\frac{1}{2} >$ of the entire KM algebra and induces the $j = \frac{1}{2}$ UIR of SU(2).

The state $|\frac{1}{2} >$ is gauge invariant. (Hence so are all the level 1 KM states.) To see this, let us regard $\Lambda^{(1)} \in T^{(1)}$ in the Gauss law $G(\Lambda^{(1)})$ as valued in the SU(2) spin $\frac{1}{2}$ Lie algebra and write $\Lambda^{(1)} = \Lambda^{(1)}_- \tau_+ + \Lambda^{(1)}_+ \tau_- + i\Lambda^{(1)}_3 \tau_3$. We have

$$V^{-1}G(\Lambda^{(1)})V = G(\overline{\Lambda}^{(1)}) , \ \overline{\Lambda}^{(1)} = \Lambda^{(1)}_- e^{-i\theta} \tau_+ + \Lambda^{(1)}_+ e^{i\theta} \tau_- + i\Lambda^{(1)}_3 \frac{\tau_3}{2} .$$  \hspace{1cm} (5.16)

The function $\overline{\Lambda}^{(1)}$ is well defined on $D \setminus H$ (being the same for $\theta = 0$ and $2\pi$) and also vanishes on its boundaries. Hence $\overline{\Lambda}^{(1)} \in T^{(1)}$, $G(\overline{\Lambda}^{(1)})$ annihilates physical states and $|\frac{1}{2} >$ is gauge invariant.

The standard vertex operator $U(\theta_0)$ is not quite $V$, but an operator creating a blip state from $|0 >$ localized at some angle $\theta_0$. We can make up this operator from a function $\overline{\chi}$ which equals $2\pi \Lambda i\frac{\tau_3}{2}$ on $\partial H$ and $\Theta i\frac{\tau_3}{2}$ on $\partial D$, $\Lambda$ being defined before (3.3) and $\Theta$
after (3.6). The U(1) computation of Section 3 gives us its regularized form on suitably identifying the U(1) generated by \( \frac{\tau}{2} \) with the U(1) of that Section:

\[
U(\theta_0) = e^{iQ(\chi)} : , \quad Q(\chi) = \frac{1}{2\pi} \int \text{Tr}(-d\chi A + \chi A^2).
\] (5.17)

The KM representation associated with \( U(\theta_0)|0\rangle \) is known to be the same as the one defined by \( |\frac{1}{2}\rangle \).

As we remarked in Section 3, the Fubini-Veneziano “position” and “momentum” operators occur naturally in CS theory. This is especially so for \( Q(\theta) \) which is a combination of position and momentum.

There is a construction, the “vertex operator construction,” of all KM generators from (5.17) [17]. As their algebraic properties and action on \( |\frac{1}{2}\rangle \) are exactly the same as those of \( H_N^{(1)\alpha} \), \( E_N^{(1)\alpha} \) defined here, it must be so that both are (weakly) identical. A direct demonstration of this identity will not be attempted in this paper.

The CS diffeo generators for the conformal family at \( \partial H \) (constructed along the lines of ref. [11]) become, in a standard notation [17], the Virasoro generators in quantum theory and are given by the Sugawara construction. A proof of this result can be developed using our U(1) treatment [11]. We will hereafter write \( L_N^{(1)} \) for these \( L_N \) to indicate the fact that they correspond to zero vector fields on \( \partial D \).

Let \( \tilde{U} \) be the operator creating blips at both \( z \in \partial H \) and \( P \in \partial D \). The work in Section 3 shows how to construct it by using the function \( 2\pi \Lambda \frac{\tau_{\alpha}}{2} \) instead of \( \chi \). This operator appears to be related to the regularized version of the path ordered integral

\[
W_{\frac{1}{2}\frac{1}{2}} = [P \exp\left(-\int A\right)]_{\frac{1}{2}\frac{1}{2}}
\] (5.18)

where \( A = iA_i^\alpha \frac{1}{2} \tau_\alpha dx^i \), the integration is along a line \( L \) defined following Section 3 and the matrix elements of the Wilson matrix

\[
W = P \exp\left(-\int A\right)
\]
are between $\frac{\tau_1}{2}$ eigenstates for eigenvalue $\frac{1}{2}$. This conjecture is made plausible by the following: i) The response of a state to $W$ is insensitive to deformations of $L$ (keeping $z$, $P$ and tangents there fixed), a property it shares with $\tilde{U}$. ii) The action of a gauge transformation $g$ on $W$ is $W \rightarrow g(z)Wg^{-1}(P)$. $W$ thus commutes with $G(A^{(1)})$, another property it shares with $\tilde{U}$. iii) This gauge response of $W$ also shows that $W_{\downarrow \downarrow} |0 >$ is the highest weight state for $H_0^{(1)} = \frac{1}{2}$ of the global SU(2) localized on $\partial H$. So obviously is $\tilde{U}|0 >$. iv) We have already seen in Section 3 that there is a precise correspondence between the Wilson integral and $\tilde{U}$ for the group U(1).

The vertex operator construction for all KM algebras based on compact simple groups is known from the point of view of the theory of these algebras. With this knowledge in mind, it is easy to extrapolate the preceding CS approach to the SU(2) KM vertex operator theory to these algebras as well.

We conclude this subsection by enquiring about the sense in which the vertex operator construction describes a source with internal vector $I$ for a general $G$. Towards this end, let us first recall how the U(1) CS constraint $F_{12}(x) = -\frac{2\pi e}{k} \delta^2(x-z)$ is interpreted and enforced in quantum theory. We first rewrite it in the form

$$-\frac{k}{2\pi} \oint_{C} A = e$$

where $C$ is any positively oriented contour enclosing $z$. The left hand side here is next identified with $q_0^{(1)} = q(\xi^{(1)}_0)$ which follows classically from Stokes’ theorem and partial integration of (2.5), since $F_{12}$ is numerically zero on $D \setminus H$. Thus, (5.19) is $q_0^{(1)} = e$. As $q_0^{(1)}$ is the classical charge and the quantum charge is $Q_0^{(1)}$, the quantum version of (5.19) is taken to mean that $Q_0^{(1)} = e$ on all states. We finally verify that this equality is fulfilled on the states obtained by the vertex operator approach.

Somewhat similar tactics can be pursued for nonabelian CS theories. For this purpose, we first rewrite (5.3) in an integral form using the nonabelian Stokes theorem [see ref. 3.
which also contains citations to the original work on this theorem):

\[
P \exp \left( - \int_{\mathcal{C}(x_0)} A \right) = U_{\mathcal{L}}^{-1} \left[ \exp \left( \frac{2\pi}{k} I \right) \right] U_{\mathcal{L}} \tag{5.20}
\]

\(\mathcal{C}(x_0)\) here is a positively oriented contour around \(z\) starting and terminating at \(x_0\), \(\mathcal{L}\) is a line from \(x_0\) to \(z\) and

\[
U_{\mathcal{L}} = P \exp \left( - \int_{\mathcal{L}} A \right). \tag{5.21}
\]

Now the left integral in (5.20) does not change if \(\mathcal{L}\) approaches \(z\) in different directions. Hence the change of \(U_{\mathcal{L}}\) as the angle of approach to \(z\) is changed is of the form \(U_{\mathcal{L}} \rightarrow HU_{\mathcal{L}}\) where \(H\) is in the stability group of \(I : H^{-1}IH = I\). Therefore \(A_\theta\) on \(\partial H\) is in the Lie algebra of the stability group of \(I\).

It is known that all UIR’s of \(G\) can be obtained from \(I\) with stability groups generated by Cartan subalgebras. We therefore assume that the stability group \(T\) of \(I\) has a Cartan subalgebra \(\mathfrak{C}\) as its Lie algebra. \(A_\theta\) is then \(\mathfrak{C}\) valued on \(\partial H\).

Now consider the limit where \(\mathcal{C}(x_0)\) shrinks to \(\partial H\). \(U_{\mathcal{L}}\) tends to an element of \(T\) in that limit and drops out of (5.20) while the path ordering there also becomes redundant. Taking logarithms, we can now rewrite it as

\[
\oint_{\partial H} A = -i \frac{2\pi}{k} I. \tag{5.22}
\]

Another way to get (5.22) is to first restrict (5.3) to \(\partial H\). This alternative also justifies our taking logarithms.

Following our abelian approach, we now declare the quantum version of (5.3) to be the equality

\[
Q_0^{(1)\alpha} = \hat{I}_\alpha, \tag{5.23}
\]

\(\hat{I}_\alpha\) being the quantum operators for the classical \(I_\alpha = \text{Tr} T_\alpha I\). They fulfill the commutation relations appropriate to \(\mathfrak{G}\) and generate the UIR associated with \(I\). For SU(2), if the orbit of \(I\) contains \(j\tau_3\), then \(\hat{I}_\alpha\) generate the spin \(j\) representation.
Let us now specialize to SU(2), considerations for general $G$ being similar. Let us also initially limit ourselves to the orbit of $\frac{1}{2} \tau_3$ so that $j = \frac{1}{2}$. We have seen that the SU(2) UIR for a highest weight KM state has $j = \frac{1}{2}$. On this family of states, we have then an enforcement of the relation (5.23). Repeated application of KM generators on states describing the $\frac{1}{2}$ UIR also generates any $j$ of the form $\frac{1}{2} + K$, $K \in \mathbb{Z}^+$, so that the relation (5.23) for any $j$ is enforced on an appropriate subset of states.

The KM states describe all $j = \frac{1}{2}, \frac{3}{2}, \ldots$ so that (5.23) with a fixed $j$ can not be enforced on all states. This large family of states makes its appearance because of our regularization of the Gauss law (5.3). We may of course discard all states except those with a fixed $j$, but at the cost of losing diffeomorphism invariance on $\partial H$. This is because a diffeo generates $Q_N^{(1)\alpha}$ from $Q_0^{(1)\alpha}$, the commutator $[L_N^{(1)}, Q_0^{(1)\alpha}]$ of the Virasoro generator $L_N^{(1)}$ with $Q_0^{(1)\alpha}$ being proportional to $Q_N^{(1)\alpha}$. [The superscript 1 on $L_N^{(1)}$ is to indicate that it corresponds to the zero vector field on $\partial D$.]

The state $V|0>$ is an eigenstate of $L_0^{(1)}$. The corresponding eigenvalue $\frac{1}{2} + \frac{3}{2} = \frac{1}{4}$ follows from the formula

$$L_0^{(1)} = \frac{1}{2k + c_v} \sum_{\alpha} \sum_N Q_{-N}^{(1)\alpha} Q_N^{(1)\alpha},$$

(5.24)

$c_v$ being the quadratic Casimir operator in the adjoint representation. The operator $L_0^{(1)}$ generates rotation diffeos of $\partial H$. $V|0>$ thus has spin $S = \frac{1}{4}$. It is also associated with the $\frac{1}{2}$ representation of $G = SU(2)$. Note that in the KM representation space, there will in general be multiple occurrences of a given UIR of $G$, possibly differing in spin. Note also that there is a spin $S$– internal symmetry $j$ correlation predicted by a KM UIR which may have some phenomenological use.

States compatible with (5.23) for integer $j$ do not require a vertex operator for construction. They can be obtained by applying polynomials in $Q_N^{(1)\alpha}$ to $|0>$. Remarks similar to the preceding ones can be made about these states as well.
5.2. The Spin-Statistics Theorem

We restrict ourselves to level 1 representations and for clarity consider $G = SU(2)$. The proof is accomplished using the earlier abelian ideas, so we can be brief.

We first examine a two particle state consisting of two identical blips at $z^{(i)}$ (i = 1, 2) in $\tau_3$ direction (and correspondingly, two blips at $z^{(i)'}$). It is obtained by applying vertex operators $\tilde{U}$ to the tensor product of highest weight states with zero $Q^{(i)\alpha}_0$ and corresponds to a state in the tensor product of level 1 representations with $j = \frac{1}{2}$ highest weight states. Figure 2 is a visual display of this state. When the particles are exchanged without disturbing the internal states, we get Figure 4b which as before is equal to Figures 4c and 4d. The canonical spin-statistics connection is thus established for these states.

The general two particle state with identical internal states in this tensor product space is obtained from a preceding state by applying two identical polynomials $\mathcal{P}$ of the KM generators $Q^{(i)\alpha}_{N}$ for the holes at $z^{(i)}$ [and perhaps also taking limits of such states]. The operators $\mathcal{P}(\{Q^{(1)\alpha}_{N}\})$ and $\mathcal{P}(\{Q^{(2)\alpha}_{N}\})$ are commuting tensorial local fields. The spin-statistics connection is thus valid for these more general states.

This remark also shows the validity of this theorem for integer $j$ states obtained from the tensor product of the highest weight KM states $|0 \rangle^{(i)}$ localized at $z^{(i)}$ (and with zero $Q^{(i)\alpha}_0$) by applying $\mathcal{P}(\{Q^{(1)\alpha}_{N}\}) \mathcal{P}(\{Q^{(2)\alpha}_{N}\})$. The exchange operator $\sigma$ is clearly trivial here. Further, the 2$\pi$ rotation for hole 1 say is $\exp(i2\pi L^{(1)}_0)$ where the Virasoro generator $L^{(1)}_0$ is the spin operator for hole 1. It commutes with the polynomials and is one on $|0 \rangle_1 \otimes |0 \rangle_2$. The theorem $\sigma = \exp(i2\pi L^{(1)}_0)$ thus follows.

Suppose next that the internal states of the particles differ and that the two-body state is $A\mathcal{P}(Q^{(1)\alpha}_{N})\mathcal{P}(Q^{(2)\alpha}_{N})|0 \rangle_1 \otimes |0 \rangle_2$. $A$ here is an element of the KM group for hole 1, and hence commutes with $Q^{(2)\alpha}_N$ and is 1 on $|0 \rangle_2$. This state is similar to $|\text{Proton} \rangle |\text{Neutron} \rangle$ in a nucleon model with isospin symmetry, the role of isospin $SU(2)$ being assumed here by the KM group. The exchange operator for this state is
\[ A\sigma A^{-1} = A \exp(i2\pi L_0^{(1)})A^{-1}. \] But \[ \exp(i2\pi L_0^{(1)}) \] commutes with \( A \) and hence so does \( \sigma \). Therefore, the \( 2\pi \) rotations and exchanges are equal to their old versions and mutually equal. A similar result is readily proved for \( A \) times the state involving the vertex operators looked at previously.

A completely general state is a linear combination of states of the sort considered in the preceding paragraphs. We thus have the standard spin-statistics connection for level 0 and 1 KM states. But we do not have a proof for a general KM representation.

There seems to be no particular difficulty in extending this discussion to CS theories for all compact simple groups. The details will be omitted here.

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