A NOTE ON BOURGAIN-MILMAN’S UNIVERSAL CONSTANT

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Abstract. The present note is a result of an on-going investigation into the logarithmic Brunn-Minkowski inequality. We obtain lower estimates on the volume product for convex bodies in \( \mathbb{R}^n \) not necessarily symmetric with respect to the origin from a modified logarithmic Brunn-Minkowski inequality.

1. Introduction

With extraordinary implications which continue to be seen, the classical Brunn-Minkowski theory of convex bodies was placed in a larger theory by Lutwak’s \( L_p \)-Minkowski problem \([15, 16]\). Consequently, many classical results for convex bodies became part of the extended \( L_p \) Brunn-Minkowski-Firey theory, while many other results of the extended theory bring new and original insight in convex geometric analysis. One such strikingly new behavior is due to the \( L_0 \)-Minkowski problem \([3, 20, 22]\), and its version of the Brunn-Minkowski inequality known for technical reasons as the logarithmic Brunn-Minkowski inequality \([4]\). The present note is a result of an on-going investigation into the logarithmic Brunn-Minkowski inequality.

Proved for \( n = 2 \) by Böröczky, Lutwak, Yang and Zhang \([4]\), and conjectured by them for all \( n \)'s where certain cases are known to hold \([18, 23]\), the logarithmic Brunn-Minkowski inequality states the following.

The Logarithmic Brunn-Minkowski Inequality. Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n \), centrally symmetric with respect to the origin. Then the following inequality holds

\[
\int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v} \geq \frac{1}{n} \ln \left( \frac{\text{Vol}(K)}{\text{Vol}(L)} \right),
\]

where \( d\bar{v}_L \) is the cone-volume measure of \( L \) and \( d\bar{v}_L = \frac{1}{\text{Vol}(L)} dv_L \) is its normalization.

Our first result represents a modified logarithmic Brunn-Minkowski:

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Proposition 1.1. Let $K$ and $L$ be convex bodies in $\mathbb{R}^n$ containing the origin in their interior. Then
\[
\int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_1 \geq \ln \left( \frac{V_1(L, K)}{Vol(L)} \right) \geq \frac{1}{n} \ln \left( \frac{Vol(K)}{Vol(L)} \right),
\]
where $d\bar{v}_1$ is a mixed volume measure, namely $\int_{S^{n-1}} d\bar{v}_1 = V([n-1], 1, L, K) =: V_1(L, K)$, and $d\bar{v}_1 = \frac{1}{V_1(L, K)} d\bar{v}_1$ is its normalization. Equality holds if and only if $K$ is homothetic to $L$.

The second inequality of (2) follows immediately from a well-known inequality for mixed volumes due to Minkowski and we included it due to the similarity with the conjectured one. The main focus of this paper is on the implications of the first inequality of (2) for the lower bound of the volume product functional and for the Bourgain-Milman constant well-known from the following theorem.

Theorem (Bourgain-Milman) \[5\] There exists an absolute constant $c > 0$ (thus independent of the dimension $n$) such that, for any centrally symmetric convex body $K$,
\[
Vol(K) \cdot Vol(K^\circ) > c^n \omega_n^2,
\]
where $K^\circ$ is the polar of $K$ with respect to the origin and $\omega_n$ is the volume of the unit ball $B_n^2$ in $\mathbb{R}^n$.

In this paper, we show:

Theorem 1.1. For any $K$ convex body in $\mathbb{R}^n$ containing the origin in its interior, we have
\[
Vol(K) \cdot Vol(K^\circ) > \max \{ evr^n(K), evr^n(K^\circ) \} \omega_n^2,
\]
where $evr(K)$ (and $evr(K^\circ)$) is the exterior volume ratio of $K$ (respectively $K^\circ$). In particular, if $K$ is symmetric about the origin,
\[
Vol(K) \cdot Vol(K^\circ) > \left[ \frac{2^n \Gamma \left( \frac{n}{2} + 1 \right)}{n! \pi^{n/2}} \right] \omega_n^2,
\]
while, for arbitrary convex bodies $K$, we have that
\[
Vol(K) \cdot Vol(K^\circ) > \left[ \frac{(n+1) \frac{n+1}{2} \Gamma \left( \frac{n}{2} + 1 \right)}{n!(n\pi)^{n/2}} \right] \omega_n^2.
\]

The actual lower bound of the volume product is the subject of Mahler conjecture which, although supported by an impressive body of work, remains, except for some cases, still open.
2. Results and Proofs

A convex body in $\mathbb{R}^n$ is a compact convex set in $\mathbb{R}^n$. Let $\mathcal{K}_0^n$ be the set of convex bodies in $\mathbb{R}^n$ containing the origin in their interior.

For $K \in \mathcal{K}_0^n$, we denote by $K^o = \{ x \in \mathbb{R}^n : x \cdot y \leq 1, \forall y \in K \}$ the polar body of $K$, where $x \cdot y$ is the standard inner product of $x$ and $y$ in $\mathbb{R}^n$. Moreover, for $K \in \mathcal{K}_0^n$, consider $h_K : \mathbb{S}^{n-1} \to \mathbb{R}_+$ the support function of $K$ defined by

$$h_K(u) = \max\{x \cdot u, \forall x \in K\}, \quad u \in \mathbb{R}^n, \quad ||u|| = 1.$$  

The volume of a convex body $K$ is the $n$-dimensional Lebesgue measure of $K$ as a compact set of $\mathbb{R}^n$. We call $dS_L$ the surface area measure of $L$, $dv_L = \frac{1}{n} h_L dS_L$ the cone-volume measure of $L$, and $dv_1 = \frac{1}{n} h_K dS_L$ the mixed volume measure of $K$ and $L$, as measures on $\mathbb{S}^{n-1}$, see [19] for a detailed discussion.

Let $\mathcal{F}_n$ denote the convex bodies in $\mathbb{R}^n$ with positive continuous curvature function, and let $\mathcal{V}_n$ denote the convex bodies in $\mathbb{R}^n$ of elliptic type, [14]. If $L$ is convex body in $\mathcal{F}_n$ with curvature function $f_L$, then $\Omega(L) = \int_{\mathbb{S}^{n-1}} f_L^{n+1} d\mu_{\mathbb{S}^{n-1}}$ is the affine surface area of $L$, [14].

Finally, if $K \subset \mathbb{R}^n$ is a convex body, the exterior volume ratio of $K$ is, by definition,

$$(7) \quad evr(K) = \left( \frac{Vol(K)}{Vol(\mathcal{E}_L)} \right)^{1/n} = \max_{K \subseteq \mathcal{E}} \left( \frac{Vol(K)}{Vol(\mathcal{E})} \right)^{1/n},$$

where the maximum is taken after all ellipsoids containing $K$, see for example Ball’s [11] or Barthe’s [2]. The ellipsoid $\mathcal{E}_L$ is called Löwner’s ellipsoid.

John’s theorem [11] states that, for any convex body $K$, its Löwner’s ellipsoid satisfies the inclusions $\frac{1}{n} \mathcal{E}_L \subseteq K \subseteq \mathcal{E}_L$, where $\frac{1}{n} \mathcal{E}_L$, generally denoted by $\mathcal{E}_J$, is also called John’s ellipsoid, the ellipsoid of maximal volume contained in $K$.

**Proof of Proposition 1.1.** Note that

$$\int_{\mathbb{S}^{n-1}} \frac{h_K}{h_L} \ln \left( \frac{h_K}{h_L} \right) dv_L = \int_{\mathbb{S}^{n-1}} \ln \left( \frac{h_K}{h_L} \right) dv_1.$$  

Then the first claim follows from

$$\exp \left[ -\frac{n}{V_1(L,K)} \int_{\mathbb{S}^{n-1}} h_K \ln \left( \frac{h_K}{h_L} \right) dv_L \right] = \lim_{p \to -\infty} \left[ \frac{1}{V_1(L,K)} \int_{\mathbb{S}^{n-1}} \left( \frac{h_K}{h_L} \right)^{\frac{n}{p+n}} dv_L \right]^{p+n}$$

and Hölder’s inequality

$$\left( \int_{\mathbb{S}^{n-1}} \left( \frac{h_K}{h_L} \right)^{\frac{n}{p+n}} dv_L \right)^{\frac{p+n}{p}} \cdot \left( \int_{\mathbb{S}^{n-1}} dv_L \right)^{-\frac{n}{p}} \leq \int_{\mathbb{S}^{n-1}} \frac{h_k}{h_L} dv_L = V_1(L,K),$$

where $\int_{\mathbb{S}^{n-1}} dv_L = Vol(L)$. 

By applying Minkowski's inequality to the first inequality of Proposition 1.1, we obtain the second inequality and that
\[ \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_1 \geq \frac{1}{n} \ln \left( \frac{Vol(K)}{Vol(L)} \right). \]

The first inequality in (2) is a strengthened version of an inequality obtained by Gardner-Hug-Weil [6]. They showed in the larger set-up of the Orlicz-Brunn-Minkowski theory that, if \( L, K \in K^n_0 \) and \( L \subseteq K \), the following inequality holds
\[ \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_L \geq \frac{1}{n} \ln \left( \frac{Vol(K)}{Vol(L)} \right). \]

More precisely, the last inequality follows from Lemma 9.1 of [6] by considering the convex function \( x \mapsto x \ln x \).

Our first inequality in (2) can be written as
\[ \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_L \geq \frac{1}{n} \ln \left( \frac{Vol(K)}{Vol(L)} \right) \]
from which \( L \subseteq K \) and Minkowski's inequality implies (9).

In connection with the conjectured inequality, remark the following fact from information theory: If \( p, q \) are probability density functions on a measure space \( (X, \nu) \), then
\[ \int p \ln p d\nu \geq \int p \ln q d\nu. \]

By taking \( p d\nu = \frac{h_L}{h_K} \cdot \frac{1}{Vol(L)} d\nu_1 \) and \( q d\nu = \frac{1}{V_1(L, K)} d\nu_1 \) (also switching the places of the two measures), we obtain the double inequality
\[ \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_L \leq \ln \left( \frac{V_1(L, K)}{Vol(L)} \right) \leq \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_1. \]

The next proposition is fundamental for our main theorems.

**Proposition 2.1.** For any \( K \in K^n_0 \) and any \( L \in K^n_0 \cap F_n \), we have
\[ Vol(K) \cdot Vol(K^o) \geq \frac{\Omega_n^{n+1}}{\Omega_{n-1}^{n-1}(L)} \cdot \frac{Vol(K)}{Vol(L)} \cdot \exp \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_1^n, \]
with equality if and only if \( K \) and \( L \) are homothetic ellipsoids centered at the origin.

**Proof.** Suppose first that \( K \) and \( L \) are distinct. Rewrite the inequality of Proposition 1.3 as
\[ \exp \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) d\bar{v}_1 \geq \frac{V_1(L, K)}{Vol(L)} = \frac{1}{Vol(L)} \int_{S^{n-1}} h_K f_L d\mu_{S^{n-1}}, \]
where $f_L$ is the curvature function of $\partial L$ as a function on the unit sphere. Since

\[
\int_{S^{n-1}} h_K f_L \, d\mu_{S^n-1} \geq \left( \int_{S^{n-1}} h_K \, d\mu_{S^n-1} \right)^{\frac{1}{n}} \left( \int_{S^{n-1}} f_L^{\frac{n}{n+1}} \, d\mu_{S^n-1} \right)^{\frac{n+1}{n}}
\]

we obtain

\[
\exp \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) \, d\tilde{v}_1 \geq \left( \frac{\Omega(L)}{n^{1+\frac{1}{n}} V ol(L)} \right)^{\frac{n+1}{n}} \left[ \frac{1}{n} \int_{S^{n-1}} \rho_{K^o}^n \, d\mu_{S^n-1} \right]^{\frac{1}{n}},
\]

where $\Omega(L)$ is the affine surface area of $L$ mentioned earlier. Thus, by raising both sides of the previous inequality to the power $n$, we have

\[
\left[ \exp \int_{S^{n-1}} \ln \left( \frac{h_K}{h_L} \right) \, d\tilde{v}_1 \right]^n \geq \left( \frac{\Omega^{n+1}(L)}{n^{n+1} V ol^n(L) V ol(K^o)} \right) \cdot evr_n(K) \cdot \omega_n^2,
\]

from which a rearrangement of terms yields the claim.

If $K = L \in K_0^\circ \cap F_n$, the claim follows directly from the reverse Hölder inequality (14).

Finally, note that the equality holds in (14) if and only if $L$ is of elliptic type and $K$ is its curvature image body (up to a constant), $h_K = \lambda f_L^{-\frac{1}{n+1}}$, for some positive constant $\lambda$. As, in addition, equality holds in Proposition 1.1 if and only if $K$ is homothetic to $L$, we conclude, due to Petty’s lemma, that equality holds in (13) if and only if $K$ and $L$ are homothetic ellipsoids centered at the origin.

\[
\square
\]

**Proof of Theorem 1.1.** Since the left-hand side is invariant under linear transformations, apply to $K$ the linear transformation which transforms its Löwner’s ellipsoid into the unit ball $B_n^2$. For simplicity, keep the same notation for $K$ and apply Proposition 2.1 to $K$ and $E_L$, obtaining, since $h_K \leq h_{E_L}$ in all directions,

\[
V ol(K) \cdot V ol(K^o) \geq evr_n(K) \cdot \omega_n^2.
\]

Of course, the role of $K$ and $K^o$ can be interchanged and we also have $V ol(K) \cdot V ol(K^o) \geq evr_n(K^o) \cdot \omega_n^2$. Therefore, as remarked to us by V. Milman, it is desirable to write explicitly the maximum of the two exterior volume ratios as, for example, for the cross-polytope the exterior volume ratio is extremely small, while for the cube is uniformly bounded from below.

The second part of the claim follows now from the following results due to Barthe [2], see also Ball [1]: For any convex body $K \subset \mathbb{R}^n$, we have $evr(K) \geq evr(\Delta_n)$, where $\Delta_n$ is the regular simplex of $\mathbb{R}^n$ inscribed in the Euclidean ball $B_n^2$. On the other hand, for any convex body $K \subset \mathbb{R}^n$ symmetric with respect to the origin, we have $evr(K) \geq evr(B_n^1)$, where $B_n^1$ is here the $L^1$ ball inscribed in the Euclidean ball, for which the calculations follow immediately.
In the case of the simplex, the calculations for the exterior volume ratio are in Satz 4 of [12] as pointed out to the author by S. Taschuk.

The next corollary is a special case of Proposition 2.1 which we feel justified to state separately.

**Corollary 2.1.** For any \( K \) convex body in \( \mathcal{K}_0^n \cap \mathcal{F}_n \), we have

\[
\text{Vol}(K) \cdot \text{Vol}(K^o) \geq \frac{1}{n^{n+1}} \frac{\Omega^{n+1}(K)}{\text{Vol}^{n-1}(K)},
\]

with equality if and only if \( K \) is a centered ellipsoid.

Note that Corollary 2.1, together with Santaló inequality, implies the famous affine isoperimetric inequality for bodies in \( \mathcal{K}_0^n \cap \mathcal{F}_n \). The affine isoperimetric inequality states that for any convex body \( K \)

\[
\frac{\Omega^{n+1}(K)}{\text{Vol}^{n-1}(K)} \leq n^{n+1} \omega_n^2,
\]

where \( \omega_n = \text{Vol}(\mathbb{B}_n^2) \), see for example [14].

We will use now Proposition 2.1 for the special case of \( L \) the unit ball in \( \mathbb{R}^n \). As before, \( \omega_n \) stands for the volume of the unit ball in \( \mathbb{R}^n \), and \( w(K) = \int_{S^{n-1}} h_K d\mu_{S^{n-1}} \) is, up to the factor of 1/2, the mean width of \( K \). By using the concavity of \( x \mapsto \ln x, x > 0 \), and the fact that the volume product is invariant under (special) linear transformations, we obtain:

**Corollary 2.2.** For any \( K \) convex body in \( \mathcal{K}_0^n \), we have

\[
\text{Vol}(K) \cdot \text{Vol}(K^o) \geq \omega_n \max_{T \in \text{SL}(n)} \frac{\text{Vol}(K)w^n(TK)}{\left(\int_{S^{n-1}} h^2_{TK} d\mu_{S^{n-1}}\right)^n},
\]

where \( K \) and \( K^o \) can be interchanged. Equality occurs above if and only if \( K \) is an ellipsoid centered at the origin.

Thus, for any \( K \in \mathcal{K}_0^n \),

\[
\text{Vol}(K) \cdot \text{Vol}(K^o) \geq \omega_n \min_{L \in \mathcal{K}_0^n} \max_{T \in \text{SL}(n)} \frac{\text{Vol}(L)w^n(TL)}{\left(\int_{S^{n-1}} h^2_{TL} d\mu_{S^{n-1}}\right)^n}.
\]

It is easy to check that if \( \{K_j\}_j \) is a sequence of centered ellipsoids with equal volume whose major axis goes to infinity as \( j \to +\infty \), then

\[
\frac{\text{Vol}(K_j)w^n(K_j)}{\left(\int_{S^{n-1}} h^2_{K_j} d\mu_{S^{n-1}}\right)^n} \to 0.
\]

Thus, taking the maximum after all \( \text{SL}(n) \) transformations in (20) is needed. In fact, we believe that, for a given convex body \( K \) in \( \mathcal{K}_0^n \), the optimal value \( M(K) := \max_{T \in \text{SL}(n)} \frac{\text{Vol}(K)w^n(TK)}{\left(\int_{S^{n-1}} h^2_{TK} d\mu_{S^{n-1}}\right)^n} \).
is reached for some isotropic position of $K$ in the sense of Giannopoulos-Milman. 

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