Ward Identities and Anomalies in Pure $W_4$ Gravity

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Abstract

$W_4$ gravity is treated algebraically, represented by a set of transformations on classical fields. The Ward identities of the theory are determined by requiring the algebra to close. The general forms for the anomalies are found by looking for solutions to the Wess-Zumino consistency conditions, and some specific cases are considered.

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1 Introduction

Since conformal symmetries play such a large role in modern physics, it may be of some interest to see to what degree such symmetries may be extended. One such extension to consider is the role of $\mathcal{W}$-algebras in 2-dimensional physics \[1\]. A $\mathcal{W}_N$ algebra is, simply put, any consistent algebra which contains the Virasoro algebra, as well as a tower of Virasoro-primary currents such that $N$ is the highest conformal weight. In general, for $N \in \mathbb{Z}$, this requires that there are also primary fields with weights $3, 4, \ldots, N$. Thus, we have not only the usual spin 2 field which appears in 2-dimensional conformal field theory, but also fields with spins up to $N$. What makes $\mathcal{W}$-algebras particularly notable is that the commutation relations are generally not linear; the commutator between two generators can be any polynomial function of the basis elements, rather than just a linear combination of generators. The resulting algebra is thus no longer Lie but rather Poisson. (For a comprehensive overview of the general subject, see \[2\]).

The minimal extension of the Virasoro algebra in this manner is the $\mathcal{W}_3$ algebra, which has been treated extensively in far too many works to fully catalogue here (see \[3\] and references therein, although \[4\] deserves special mention as a major influence on the present work, both in notation and approach). In this case, the nonlinearity of the theory shows up in the fact that the generators of the Virasoro algebra appear not only linearly, but quadratically. $\mathcal{W}_4$ is the first instance in which not only do the Virasoro generators appear cubically, but the additional $\mathcal{W}_3$ currents themselves show up quadratically. The structure of this particular algebra (and $\mathcal{W}$ algebras in general) has been examined both from BRST \[5, 6\] and OPE \[7, 8\] points of view.

$\mathcal{W}_N$ gravity \[10, 9\] is the gauged version of a $\mathcal{W}_N$ theory: Additional fields, one for each current, are introduced such that the $\mathcal{W}_N$ algebra is preserved even when the gauge symmetry (i.e. coordinate transformations) is included. However, as in regular conformal theories, this results in the appearance of gravitational Ward identities and anomalies. Most of the work treating these in a $\mathcal{W}$-algebra context have been for $\mathcal{W}_3$ gravity, and fall into three basic categories: Actually finding the effective action, Ward identities and anomalies arising from a $\mathcal{W}$-symmetric quantum field theory \[11, 4, 12\]; explicitly constructing a free-field realisation of the algebra \[13\]; or examining the BRST algebra \[14, 15\]. When the anomalies arising from these approaches are compared, it is not always obvious they are the same. However, each result turns out to be a specific case of a more general expression for the anomalies of the theory, one obtained purely on algebraic grounds, as shown (for the first and last approaches) in \[16\].

In this work we attempt to apply the algebraic approach just cited to $\mathcal{W}_4$: First, we introduce the set of transformations on classical fields which gives a representation of the algebra, and then extend the theory to $\mathcal{W}_4$ gravity by gauging the original $\mathcal{W}_4$ theory. In doing so, we find that conditions have to be
placed on the fields to close the algebra, which we call ‘Ward identities’, despite
the fact that our theory is a purely algebraic one, with no a priori connection to
an effective field theory or the like. In fact, it is for this very reason that we call
the theory ‘pure $W_4$ gravity’. The form of these Ward identities suggests a basis
for the anomalies, and after checking which particular combinations of these
basis elements fulfill the Wess-Zumino consistency conditions, the general forms
are found. Finally, we find the expressions for the anomalies for several specific
effects, and comment on their possible connections with known theories.

2 $W_4$ Algebra

We start with a 2-dimensional space $\Sigma$ with coordinates $z$ and $\bar{z}$ (and cor-
responding derivatives $\partial$ and $\bar{\partial}$); the classical fields which appear in $W_4$ are
the familiar energy-momentum tensor $T(z, \bar{z})$ and the two $W$-currents, $W_3(z, \bar{z})$
and $W_4(z, \bar{z})$. By definition, the $W_4$ transformations must include the conformal
transformation $\delta_2$, which on $T$ has the form

$$\delta_2 (\epsilon) T = \left( \frac{c}{12} \right) \partial^3 \epsilon + \epsilon \partial T + 2 \partial \epsilon T,$$

(2.1)

where $c$ is the central charge and $\epsilon(z, \bar{z})$ the conformal variation parameter. This
gives a representation of the Virasoro algebra (which, in an abuse of terminology,
will occasionally be referred to as the ‘$W_2$ algebra’):

$$[\delta_2 (\epsilon_1), \delta_2 (\epsilon_2)] = \delta_2 (\{\epsilon_1, \epsilon_2\})_2,$$

(2.2)

with the Poisson bracket between the $W_2$ variation parameters defined as

$$\{\epsilon_1, \epsilon_2\}_2 := \partial \epsilon_1 \epsilon_2 - \epsilon_1 \partial \epsilon_2.$$

(2.3)

A ‘primary field’ in this language is a field $\phi_h(z, \bar{z})$ which transforms as

$$\delta_2 (\epsilon) \phi_h = \epsilon \partial \phi_h + h \partial \epsilon \phi_h,$$

(2.4)

and what we’d like to do is to find a way to introduce $W_3$ and $W_4$ and their
$W_3$ and $W_4$ transformation laws in a way such that they are Virasoro primary,
$W_2$ is a subalgebra, and the entire algebra closes. The first and second of these
criteria are easily fulfilled by saying

$$\delta_2 (\epsilon) W_3 = \epsilon \partial W_3 + 3 \partial \epsilon W_3,$$
$$\delta_2 (\epsilon) W_4 = \epsilon \partial W_4 + 4 \partial \epsilon W_4,$$

(2.5)

The last criterion can be satisfied uniquely as well (otherwise this would be
a very short paper): First, the $W_3$ transformations are given by

$$\delta_3 (\lambda) T = 2 \lambda \partial W_3 + 3 \partial \lambda W_3,$$
\[ \delta_3 (\lambda) W_3 = \left( \frac{c}{12} \right) \partial^3 \lambda + 2 \lambda \partial^3 T + 9 \partial \lambda \partial^2 T + 15 \partial^2 \lambda \partial T + 10 \partial^3 \lambda T \]
\[ + \lambda \partial W_4 + 2 \partial \lambda W_4 + \left( \frac{12}{c} \right) (16 \lambda \partial T + 16 \partial \lambda T^2), \]
\[ \delta_3 (\lambda) W_4 = \lambda \partial^3 W_3 + 6 \partial \lambda \partial^2 W_3 + 14 \partial^2 \lambda \partial W_3 + 14 \partial^3 \lambda W_3 \]
\[ + \left( \frac{12}{c} \right) (18 \lambda T \partial W_3 + 25 \lambda \partial T W_3 + 52 \partial \lambda T W_3). \] (2.6)

The algebra of these transformations does not close even if we include the conformal transformations, because unlike \( W_2 \), \( W_3 \) is not a subalgebra of \( W_4 \), so \( \mathcal{W}_3 \) transformations' is something of a misnomer. To complete the algebra, we have to include the \( W_4 \) transformations

\[ \delta_4 (\xi) T := 3 \xi \partial W_4 + 4 \partial \xi W_4, \]
\[ \delta_4 (\xi) W_3 := 5 \xi \partial^3 W_3 + 20 \partial \xi \partial^2 W_3 + 28 \partial^2 \xi \partial W_3 + 14 \partial^3 \xi W_3 \]
\[ + \left( \frac{12}{c} \right) (34 \xi T \partial W_3 + 27 \xi W_3 \partial T + 52 \partial \xi T W_3), \]
\[ \delta_4 (\xi) W_4 := \left( \frac{c}{12} \right) \partial^3 \xi + 3 \xi \partial^3 T + 20 \partial \xi \partial^2 T + 56 \partial^2 \xi \partial^2 T + 84 \partial^3 \xi \partial T \]
\[ + 70 \partial^4 \xi \partial T + 28 \partial^5 \xi T - \xi \partial^3 W_4 - 5 \partial \xi \partial^2 W_4 - 9 \partial^2 \xi \partial W_4 - 6 \partial^3 \xi W_4 \]
\[ + \left( \frac{12}{c} \right) \left[ \xi (177 \partial T \partial^2 T + 78 T \partial^3 T) + \partial \xi (352 T \partial^2 T + 295 (\partial T)^2) \right] \]
\[ + 588 \partial^2 \xi T \partial T + 196 \partial^3 \xi T^2 - 14 \xi \partial T W_4 - 14 \xi \partial T W_4 - 28 \partial \xi T W_4 + 75 \xi W_3 \partial W_3 + 75 \partial \xi W_3^2 \]
\[ + \left( \frac{12}{c} \right)^2 (432 \xi T^2 \partial T + 288 \partial \xi T^3). \] (2.7)

Not surprisingly, the conformal transformations take the \( \mathcal{W} \) transformations into themselves:

\[ [\delta_2 (\epsilon), \delta_3 (\lambda)] = \delta_3 ([\epsilon, \lambda])_3, \]
\[ [\delta_2 (\epsilon), \delta_4 (\xi)] = \delta_4 ([\epsilon, \xi])_4, \] (2.8)
where

\[ [\epsilon, \lambda]_3 := 2 \partial \epsilon \lambda - \epsilon \partial \lambda, \]
\[ [\epsilon, \xi]_4 := 3 \partial \epsilon \xi - \epsilon \partial \xi. \] (2.9)

At this point, it may be useful to make a comment on the notation: Throughout this work, \( \mathcal{W}_2 \), \( \mathcal{W}_3 \) and \( \mathcal{W}_4 \) transformation parameters will always be referred to by \( \epsilon \), \( \lambda \) and \( \xi \) respectively. Furthermore, the subscripts on the Poisson brackets will indicate what type of variation parameter the resulting bracket is,
e.g. \( \{ \epsilon, \xi \} \) takes a \( W_2 \) parameter and a \( W_4 \) parameter and spits back a \( W_4 \) parameter.

That being said, the rest of the algebra and the other Poisson brackets can now be written: The commutators between the transformations are

\[
\begin{align*}
[\delta_3 (\lambda_1), \delta_3 (\lambda_2)] &= \delta_2 (\{\lambda_1, \lambda_2\}_2) + \delta_4 (\{\lambda_1, \lambda_2\}_4), \\
[\delta_3 (\lambda), \delta_4 (\xi)] &= \delta_2 (\{\lambda, \xi\}_2) + \delta_3 (\{\lambda, \xi\}_3), \\
[\delta_4 (\xi_1), \delta_4 (\xi_2)] &= \delta_2 (\{\xi_1, \xi_2\}_2) + \delta_3 (\{\xi_1, \xi_2\}_3) + \delta_4 (\{\xi_1, \xi_2\}_4), (2.10)
\end{align*}
\]

with the Poisson brackets

\[
\begin{align*}
\{\lambda_1, \lambda_2\}_2 &= 2\lambda_2 \partial^3 \lambda_1 - 3\partial\lambda_2 \partial^2 \lambda_1 + 3\partial^2 \lambda_2 \partial \lambda_1 - 2\partial^3 \lambda_2 \lambda_1 \\
&\quad + \left( \frac{12}{c} \right) 16 (\lambda_2 \partial \lambda_1 - \partial \lambda_2 \lambda_1) T, \\
\{\lambda, \xi\}_2 &= \left( \frac{12}{c} \right) (27 \partial \lambda \xi W_3 - 25 \lambda \partial \xi W_3 - 7 \lambda \xi \partial W_3), \\
\{\xi_1, \xi_2\}_2 &= 3\xi_2 \partial^3 \xi_1 - 5\partial \xi_2 \partial^2 \xi_1 + 6\partial^2 \xi_2 \partial \xi_1 - 6\partial^3 \xi_2 \xi_1 + 5\partial^4 \xi_2 \partial \xi_1 - 3\partial^5 \xi_2 \xi_1 \\
&\quad + \left( \frac{12}{c} \right) [\delta_3 (\xi_2 \partial \xi_1 - \partial \xi_2 \xi_1) \partial T + 57 (\xi_2 \partial^2 \xi_1 - \partial^2 \xi_2 \xi_1) \partial T \\
&\quad + (78 \xi_2 \partial^3 \xi_1 - 118 \partial \xi_2 \partial^2 \xi_1 + 118 \partial^2 \xi_2 \partial \xi_1 - 78 \partial^3 \xi_2 \xi_1) T \\
&\quad - 14(\xi_2 \partial \xi_1 - \partial \xi_2 \xi_1) W_4] + \left( \frac{12}{c} \right)^2 432(\xi_2 \partial \xi_1 - \partial \xi_2 \xi_1) T^2, \\
\{\lambda, \xi\}_3 &= 5\xi \partial^3 \lambda - 5\partial \xi \partial^2 \lambda + 3\partial^2 \xi \partial \lambda - \partial^3 \xi \lambda \\
&\quad + \left( \frac{12}{c} \right) (34 \xi \partial \lambda T - 18 \xi \lambda T + 7 \xi \lambda \partial T), \\
\{\xi_1, \xi_2\}_3 &= \left( \frac{12}{c} \right) 75(\xi_2 \partial \xi_1 - \partial \xi_2 \xi_1) W_3, \\
\{\lambda_1, \lambda_2\}_4 &= \lambda_2 \partial \lambda_1 - \partial \lambda_2 \lambda_1, \\
\{\xi_1, \xi_2\}_4 &= -\xi_2 \partial^3 \xi_1 + 2\partial \xi_2 \partial^2 \xi_1 - 2\partial^2 \xi_2 \partial \xi_1 + \partial^3 \xi_2 \xi_1 \\
&\quad - \left( \frac{12}{c} \right) 14(\xi_2 \partial \xi_1 - \partial \xi_2 \xi_1) T. (2.11)
\end{align*}
\]

Therefore, we get a three-dimensional representation of the full \( W_4 \) algebra on \( T, W_3 \) and \( W_4 \). (In fact, since \( W_4 \) itself has three generators, this is just the adjoint representation.)

## 3 \( W_4 \) Gravity and Ward Identities

We now want to introduce gravity by gauging the \( W_4 \) theory just discussed. We call it ‘gravity’ because we interpret our three \( W \)-transformations as arising
from coordinate transformations, so we need to include three ‘metrics’ \( h \), \( g_3 \) and \( g_4 \), each changing inhomogeneously under the \( \mathcal{W}_{2,3,4} \) variations respectively, but still giving a representation of \( \mathcal{W}_4 \). However, up until now, we have said very little about the geometrical nature of our transformations, merely defining them on a set of classical fields and then blindly manipulating them algebraically. We can no longer use this approach if we want to talk about gravity, and must now be more specific.

As we did above, we start with the Virasoro algebra: Recall that a conformal transformation is a coordinate transformation on \( \Sigma \) under which the invariant length \( ds^2 = h_{ab}(x)dx^a dx^b \) rescales by an overall factor, where \( x^{1,2} \) are the coordinates and \( h_{ab} \) the metric on \( \Sigma \). The definition of an object \( \phi_h(x) \) of conformal weight \( h \) (not to be confused with the metric) in this picture is one which under a conformal transformation \( x \mapsto x' \) satisfies

\[
\phi'_h(x') = \det h \left( \frac{\partial x'}{\partial x} \right) \phi_h(x),
\]

which, for an infinitesimal transformation \( x' := x - \epsilon(x) \), becomes

\[
\delta_2(\epsilon) \phi_h = \epsilon^a \partial_a \phi_h + h \partial_a \epsilon^a \phi_h.
\]

Looking at our original definition \( g_{3,4} \), it is straightforward to see that in the chosen coordinate system \( (z, \bar{z}) \), the transformations are simply \( z \mapsto z - \epsilon(z, \bar{z}) \) and \( \bar{z} \mapsto \bar{z} \). In order for this transformation to truly be conformal, we need to choose the light-cone gauge, in which case the metric takes the form \( ds^2 := dz \bar{d}z + h(z, \bar{z}) d\bar{z}^2 \) for some quantity \( h(z, \bar{z}) \) \[17\]. In this gauge, \( ds^2 \mapsto (1 - \partial \epsilon) ds^2 \) provided that \( h \) transforms according to

\[
\delta_2(\epsilon) h = \bar{\partial} \epsilon + \epsilon \partial h - h \partial \epsilon.
\]

This fits right in with the criteria we’d like for the gauge field: It is primary (of weight -1) except for the inhomogeneous first term, so we take it as the field to include in the formulation of \( W_4 \) gravity.

Now, on to the other two ‘metrics’, \( g_3 \) and \( g_4 \): Note that \( g_{3,4} \) can also be written as \( \bar{\partial} \epsilon + \{ h, \epsilon \}_2 \). The appearance of the Poisson bracket in a linear and local way, plus the fact that we would like \( g_{3,4} \) (like \( W_{3,4} \)) to be primary, motivates the choices

\[
\delta_2(\epsilon) g_3 := \{ g_3, \epsilon \}_3 = c \partial g_3 - 2 \partial \epsilon g_3,
\]

\[
\delta_2(\epsilon) g_4 := \{ g_4, \epsilon \}_4 = c \partial g_4 - 3 \partial \epsilon g_4,
\]

so \( g_{3,4} \) are primary of weight \(-2\) and \(-3\), and therefore we will still have a representation of the Virasoro algebra.
Continuing along this train of thought, we construct the $\mathcal{W}_3$ and $\mathcal{W}_4$ transformations of $h$, $g_3$ and $g_4$ with the following requirements: First, in analogy to $\delta_2 (c) h$, $\delta_3 (\lambda) g_3$ and $\delta_4 (\xi) g_4$ must include a $\partial \lambda$ and $\partial \xi$ respectively; and secondly, all other pieces of the variations must be expressible purely in terms of the Poisson brackets, but only linearly and locally. These criteria, together with the ever-present demand that $\mathcal{W}_2$ remains a subalgebra, lead to

$$\begin{align*}
\delta_3 (\lambda) h &= \{ g_3, \lambda \}_2 + \{ g_4, \lambda \}_2, \\
\delta_3 (\lambda) g_3 &= \partial \lambda + \{ h, \lambda \}_3 + \{ g_4, \lambda \}_3, \\
\delta_3 (\lambda) g_4 &= \{ g_3, \lambda \}_4, \\
\delta_4 (\xi) h &= \{ g_3, \xi \}_2 + \{ g_4, \xi \}_2, \\
\delta_4 (\xi) g_3 &= \{ g_3, \xi \}_3 + \{ g_4, \xi \}_3, \\
\delta_4 (\xi) g_4 &= \partial \xi + \{ h, \xi \}_4 + \{ g_4, \xi \}_4,
\end{align*}$$

or, more explicitly,

$$\begin{align*}
\delta_3 (\lambda) h &= 2\lambda \partial^3 g_3 - 3\partial \lambda \partial^2 g_3 + 3\partial^2 \lambda \partial g_3 - 2\partial^3 g_3 \\
&\quad+ \left( \frac{12}{c} \right) \left( 16\lambda T \partial g_3 - 16\partial \lambda T g_3 + 25\lambda W_3 \partial g_4 + 7\lambda \partial W_3 g_4 - 27\partial \lambda W_3 g_4 \right), \\
\delta_3 (\lambda) g_3 &= \partial \lambda + 2\lambda \partial h - \partial \lambda h + \lambda \partial^3 g_4 - 3\partial \lambda \partial^2 g_4 + 5\partial^2 \lambda \partial g_4 - 5\partial^3 g_4 \\
&\quad+ \left( \frac{12}{c} \right) \left( 18\lambda T \partial g_4 - 7\lambda \partial T g_4 - 34\partial \lambda T g_4 \right), \\
\delta_3 (\lambda) g_4 &= \lambda \partial g_3 - \partial \lambda g_3,
\end{align*}$$

and

$$\begin{align*}
\delta_4 (\xi) h &= 3\xi \partial^5 g_4 - 5\partial \xi \partial^4 g_4 + 6\partial^2 \xi ^3 \partial^2 g_4 - 6\partial^3 \xi \partial^2 g_4 + 5\partial^4 \xi \partial g_4 - 3\partial^5 \xi g_4 \\
&\quad+ \left( \frac{12}{c} \right) \left[ 57 (\xi \partial g_4 - \partial \xi g_4) \partial^2 T + 57 (\xi \partial^2 g_4 - \partial^2 \xi g_4) \partial T \\
&\quad+ (78\xi \partial^3 g_4 - 118\partial \xi \partial^2 g_4 + 118\partial^2 \xi \partial g_4 - 78\partial^2 \xi g_4) T \\
&\quad+ 27(5 \partial^2 g_3 - 25\partial \xi W_3 g_3 - 7\xi \partial W_3 g_3 - 14 (\xi \partial g_4 - \partial \xi g_4) W_4) \\
&\quad+ \left( \frac{12}{c} \right)^2 432 (\xi \partial g_4 - \partial \xi g_4) T^2, \\
\delta_4 (\xi) g_3 &= 5\xi \partial^3 g_3 - 5\partial \xi \partial^2 g_3 + 3\partial^2 \xi \partial g_3 - \partial^3 g_3 \\
&\quad+ \left( \frac{12}{c} \right) \left[ 34\xi T \partial g_3 - 18\partial \xi T g_3 + 7\xi \partial T g_3 + 75 (\xi \partial g_4 - \partial \xi g_4) W_3 \right], \\
\delta_4 (\xi) g_4 &= \partial \xi + 2\xi \partial h - \partial \xi h - \xi \partial^3 g_4 + 2\partial \xi \partial^2 g_4 - 2\partial^2 \xi \partial g_4 + \partial^3 \xi g_4 \\
&\quad- \left( \frac{12}{c} \right) 14 (\xi \partial g_4 - \partial \xi g_4) T.
\end{align*}$$

Unfortunately, now the algebra no longer closes; this is a result of the fact that not all the Poisson brackets respect $\partial$, i.e. $\partial \{ \alpha, \beta \}_i - \{ \partial \alpha, \beta \}_i - \{ \alpha, \partial \beta \}_i$. 

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does not always vanish ($\alpha$ and $\beta$ are arbitrary transformation parameters). For instance,

$$[\delta_3 (\lambda_1), \delta_3 (\lambda_2)] h = (\delta_2 (\{\lambda_1, \lambda_2\},_2) + \delta_4 (\{\lambda_1, \lambda_2\},_4)) h - \left(\frac{12}{c}\right) 16 (\lambda_2 \partial \lambda_1 - \partial \lambda_2 \lambda_1) \omega_2,$$

(3.8)

where $\omega_2$ is the quantity

$$\omega_2 := \tilde{\partial} T - \left(\frac{c}{12}\right) \partial^3 h - h \partial T - 2 \partial h T - 2 g_3 \partial W_3 - 3 \partial g_3 W_3 - 3 g_4 \partial W_4 - 4 \partial g_4 W_4.$$  

(3.9)

Given our assumptions as to the forms of the variations, the above transformations seem to be the only choices, and there would appear to be a problem. Luckily, there is a resolution: Notice that if we define the operator *nabla* as

$$\nabla := \tilde{\partial} - \delta_2 (h) - \delta_3 (g_3) - \delta_4 (g_4),$$

(3.10)

then $\omega_2 \equiv \nabla W_{3,4}$. We can therefore define two new quantities $\omega_{3,4}$ as $\nabla W_{3,4}:

$$\omega_3 := \tilde{\partial} W_3 - \left(\frac{c}{12}\right) \partial^3 g_3 - h \partial W_3 - 3 \partial h W_3 - 2 g_3 \partial^3 T - 9 \partial g_3 \partial^2 T - 15 \partial^2 g_3 \partial T$$

$$- 10 \partial^3 g_3 - 10 \partial h W_4 - 2 g_3 W_4 - 5 g_4 g_3 \partial^3 T - 20 g_4 g_3 \partial^2 T - 28 g_4 g_3 \partial T$$

$$- 14 \partial^3 g_4 W_3 - \left(\frac{12}{c}\right) (16 g_3 T \partial T + 16 \partial g_3 T^2 + 34 g_4 T \partial W_3 + 27 g_4 \partial T W_3$$

$$+ 52 g_4 T W_3),$$

$$\omega_4 := \tilde{\partial} W_4 - \left(\frac{c}{12}\right) \partial^3 g_4 - h \partial W_4 - 4 \partial h W_4 - g_3 \partial^3 W_3 - 6 \partial g_3 \partial^2 W_3 - 14 \partial^2 g_3 \partial W_3$$

$$- 14 \partial^3 g_3 W_3 - 3 g_4 \partial^2 T - 20 g_4 g_3 \partial^2 T - 56 \partial^2 g_4 \partial^3 T - 84 \partial^3 g_4 \partial^2 T - 70 \partial^4 g_4 \partial T$$

$$- 28 \partial^5 g_4 T + g_4 \partial^3 W_4 + 5 g_4 \partial^2 W_4 + 9 \partial^2 g_4 \partial W_4 + 6 \partial^3 g_4 W_4$$

$$- \left(\frac{12}{c}\right) [18 g_3 T \partial W_3 + 25 g_3 \partial T W_3 + 52 g_3 T W_3 + g_4 (177 T \partial^2 T$$

$$+ 78 T \partial^3 T) + \partial g_4 (352 T \partial^2 T + 295 (\partial T)^2) + 588 \partial^2 g_4 T \partial T + 196 \partial^3 g_4 T^2$$

$$- 14 g_4 \partial T W_4 - 14 g_4 \partial T W_4 - 28 \partial g_4 T W_4 + 75 g_4 W_3 \partial W_3 + 75 g_4 W_3]$$

$$- \left(\frac{12}{c}\right)^2 (432 g_4 T^2 \partial T + 288 g_4 T^3).$$

(3.11)

The usefulness of introducing these quantities lies in the fact that they change into one another under the $W_4$ transformations, e.g.

$$\delta_3 (\lambda) \omega_4 = \lambda \partial^3 \omega_3 + 6 \partial \lambda \partial^2 \omega_3 + 14 \partial^2 \lambda \partial \omega_3 + 14 \partial^3 \lambda \omega_3$$

$$+ \left(\frac{12}{c}\right) (18 \lambda W_3 + 25 \partial \omega W_3 + 52 \partial \lambda W_3$$

$$+ 18 \lambda T \partial \omega_3 + 25 \lambda \partial T \omega_3 + 52 \partial \lambda T \omega_3).$$

(3.12)
Hence, since $\omega_2, \omega_3, \omega_4$ span an invariant subspace of the space of fields, we can mod them out, i.e. take them to vanish identically. This means that all the commutation relations of the $\mathcal{W}_4$ transformations now are satisfied on all of the fields, and the algebra closes.

Algebraically, we should have no qualms in setting the $\omega$s to zero, but what’s the interpretation of this requirement from a physics point of view? Simply that the vanishing of these quantities give the Ward identities (WIs) of the physical theory: We can think of the classical fields we deal with here as resulting from the quantisation of a $\mathcal{W}_4$ invariant quantum field theory, and that the WIs are necessary to ensure that the $\mathcal{W}_4$ algebra is preserved at the level of the effective theory, precisely as we have just seen. So in reality, the vanishing of (3.9) and (3.11) implies that we have only three physical degrees of freedom rather than six. Since the $\omega$s span an invariant subspace, setting them equal to zero amounts to imposing a set of first-class constraints on our theory.

As a slight digression, we notice that a geometrical interpretation can also be provided: If $\mathcal{M}$ is the space of functions on $\Sigma$, then $\bar{\nabla}$ is just $\bar{\partial}$ on elements of $\mathcal{M}$. Furthermore, the definitions of the transformation laws of the metrics automatically imply that they are all annihilated by $\bar{\nabla}$, and therefore $\bar{\nabla}\omega_2, \omega_3, \omega_4$ are each linear in the $\omega$s; thus, we can consistently impose $\omega_2, \omega_3, \omega_4 \equiv 0$, i.e. that $\bar{\nabla}$ vanishes on $T$ and $W_3, W_4$. This means that we can define the nilpotent exterior derivative $\bar{D} := d\bar{\partial}$ and immediately see that $H^0(\mathcal{M}, \mathbb{C}; \bar{D})$ is the space of holomorphic functions on $\Sigma$ and that all physical fields (multiplied by $d\bar{\partial}$) must belong to $H^1(\mathcal{M}, \mathbb{C}; \bar{D})$ (with, in the latter case, the equivalence relation $D\alpha \sim 0$, $\alpha \in \mathcal{M}$ ensuring that our currents are not simply functions on $\Sigma$).

4 Anomalies

4.1 General $\mathcal{W}_4$ Anomalies

The purpose of this subsection is to try to find the most general possible forms of the anomalies in $\mathcal{W}_4$ gravity. As with the rest of this work, this will be done in an algebraic manner: Suppose we have an algebra represented on classical fields $\{\phi_i\}$ by means of a transformation rule $\phi_i \mapsto \phi_i + \delta(\alpha)\phi_i$, such that $[\delta(\alpha_1), \delta(\alpha_2)] = \delta(\{\alpha_1, \alpha_2\})$ for some Poisson bracket $\{\cdot, \cdot\}$. In this context, an
anomaly $\Delta[\alpha]$ is a functional of the variation parameter of the form

$$\Delta[\alpha] \equiv \int_\Sigma d^2 z \alpha A (\{ \phi \}) \quad (4.1)$$

which satisfies the Wess-Zumino consistency condition (WZCC)

$$\delta (\alpha_1) \Delta [\alpha_2] - \delta (\alpha_2) \Delta [\alpha_1] = \Delta [\{ \alpha_1, \alpha_2 \}]. \quad (4.2)$$

The WZCC is satisfied iff the algebra closes, of course. However, if there is an invariant subspace with basis $\{ \omega_m \}$ such that the commutation relations have the form

$$([\delta (\alpha_1), \delta (\alpha_2)] - \delta (\{ \alpha_1, \alpha_2 \})) \phi_i = \sum_m F_{im} (\alpha_1, \alpha_2; \{ \phi \}) \omega_m, \quad (4.3)$$

then the right-hand side of (4.2) will pick up a term whose integrand is a linear combination of the $\omega$s. If we then close the algebra by modding out by this subspace, this extraneous term will vanish, and the WZCC will work after all. Thus, in looking for the most general possible form of the anomaly, we must allow for such terms to pop up when checking the WZCC.

To try to find generic expressions for the $\mathcal{W}_4$ anomalies, we propose to start with a basis of functionals and see which particular combinations satisfy the nine WZCCs. However, as just argued, we should only wait until all the computations are done before imposing the WIs, so we allow the $\omega$s to show up in the integrands of our proposed anomalies. Now, notice that $(\frac{12}{c})$ serves throughout all our discussions as a sort of expansion parameter, and therefore it follows that the anomalies themselves are expressible as a sum of terms of various powers of $(\frac{12}{c})$. Furthermore, unlike $\partial$, $\bar{\partial}$ never appears with more than unit degree in any of the transformation laws. With all this in mind, and looking at the expressions (3.9) and (3.11) for the $\omega$s, we propose the following basis for the $\mathcal{W}_2$ anomalies:

$$\Delta_2^{(1)} [\epsilon] := \frac{c}{12} \int_\Sigma d^2 z \epsilon \partial^3 h,$$

$$\Delta_2^{(2)} [\epsilon] := \int_\Sigma d^2 z \epsilon \bar{\partial} T,$$

$$\Delta_2^{(3)} [\epsilon] := \int_\Sigma d^2 z \epsilon (h \partial T + 2 \partial h T + 2 g_3 \partial W_3 + 3 \partial g_3 W_3$$

$$+ 3 g_4 \partial W_4 + 4 \partial g_4 W_4); \quad (4.4)$$

for the $\mathcal{W}_3$ anomalies:

$$\Delta_3^{(1)} [\lambda] := \frac{c}{12} \int_\Sigma d^2 z \lambda \bar{\partial} g_3,$$
\[ \Delta_3^{(3)} [\lambda] := \int_\Sigma d^2 z \lambda \partial W_3, \]
\[ \Delta_3^{(4)} [\lambda] := \int_\Sigma d^2 z \lambda \left( h \partial W_3 + 3 \partial h W_3 + 2 g_3 \partial^3 T + 9 \partial g_3 \partial^2 T + 15 \partial^2 g_3 \partial T + 10 \partial^3 g_3 T + g_3 \partial W_4 + 2 \partial g_3 W_4 + 5 \partial g_4 \partial^3 W_3 + 20 \partial g_4 \partial^2 W_3 + 28 \partial^2 g_4 \partial W_3 + 14 \partial^3 g_4 W_3 \right), \]
\[ \Delta_3^{(4)} [\lambda] := \left( \frac{12}{c} \right) \int_\Sigma d^2 z \lambda \left( 16 g_4 T \partial T + 16 \partial g_4 T^2 + 34 g_4 T \partial W_3 + 27 \partial g_4 \partial T W_3 + 52 \partial g_4 T W_3 \right); \quad (4.5) \]

and finally for the \( \mathcal{W}_4 \) anomalies:
\[ \Delta_4^{(1)} [\xi] := \left( \frac{c}{12} \right) \int_\Sigma d^2 z \xi \partial^4 g_4, \]
\[ \Delta_4^{(2)} [\xi] := \int_\Sigma d^2 z \xi \partial W_4, \]
\[ \Delta_4^{(3)} [\xi] := \int_\Sigma d^2 z \xi \left( h \partial W_4 + 4 \partial h W_4 + g_3 \partial^3 W_3 + 6 \partial g_3 \partial^2 W_3 + 14 \partial^2 g_3 \partial W_3 + 14 \partial^3 g_3 W_3 + 3 g_4 \partial^3 T + 20 \partial g_4 \partial^2 T + 56 \partial^2 g_4 \partial T + 84 g_4 \partial T + 28 \partial g_4 T - g_4 \partial^3 W_4 - 5 \partial g_4 \partial^2 W_4 - 9 \partial^2 g_4 \partial W_4 - 4 \partial^3 g_4 W_4 \right), \]
\[ \Delta_4^{(4)} [\xi] := \left( \frac{12}{c} \right) \int_\Sigma d^2 z \xi \left[ 18 g_4 T \partial W_3 + 25 g_4 \partial T W_3 + 52 \partial g_4 T W_3 + 9 \left( 177 \partial T \partial^2 T + 78 T \partial^3 T \right) + g_4 \left( 52 T \partial^2 T + 295 (\partial T)^2 \right) + 588 \partial^2 g_4 T \partial T + 316 \partial g_4 T^2 - 14 g_4 T \partial W_4 - 14 \partial g_4 \partial T W_4 - 28 \partial g_4 T W_4 + 75 g_4 W_3 \partial W_3 + 75 \partial g_4 W_3^2 \right], \]
\[ \Delta_4^{(5)} [\xi] := \left( \frac{12}{c} \right)^2 \int_\Sigma d^2 z \xi \left( 432 g_4 T^2 \partial T + 288 \partial g_4 T^3 \right). \quad (4.6) \]

This is obviously not the only basis possible, and for the purposes of our discussion it’s more convenient to introduce the six functionals \( \delta_{2,3,4} L \) and \( \Omega_{2,3,4} \) defined by
\[ \delta_2 L [\epsilon] := -\Delta_2^{(1)} [\epsilon] - \Delta_2^{(2)} [\epsilon], \]
\[ \delta_3 L [\lambda] := -\Delta_3^{(1)} [\lambda] - \Delta_3^{(2)} [\lambda] + \Delta_3^{(4)} [\lambda], \]
\[ \delta_4 L [\xi] := -\Delta_4^{(1)} [\xi] - \Delta_4^{(2)} [\xi] + \Delta_4^{(4)} [\xi] + 2 \Delta_4^{(5)} [\xi], \]
\[ \Omega_2 [\epsilon] := \Delta_2^{(2)} [\epsilon] - \Delta_2^{(1)} [\epsilon] - \Delta_2^{(3)} [\epsilon], \]
\[ \Omega_3 [\lambda] := \Delta_3^{(2)} [\lambda] - \Delta_3^{(1)} [\lambda] - \Delta_3^{(3)} [\lambda] - \Delta_3^{(4)} [\lambda], \]
\[ \Omega_4 [\xi] := \Delta_4^{(2)} [\xi] - \Delta_4^{(1)} [\xi] - \Delta_4^{(3)} [\xi] - \Delta_4^{(4)} [\xi] - \Delta_4^{(5)} [\xi]. \quad (4.7) \]

These choices are not arbitrary; the first three are, respectively, the \( \mathcal{W}_2, \mathcal{W}_3 \) and \( \mathcal{W}_4 \) variations of the quantity
\[ L := \int_\Sigma d^2 z \left( Th + W_3 g_3 + W_4 g_4 \right). \quad (4.8) \]
The other three are simply the integrals of the products between the appropriate 
\( \omega \)s and parameters, e.g. \( \Omega_2 \equiv \int_{\Sigma} d^2 z \sqrt{\omega_2} \).

To show why these are useful, we stick these into the WZCCs. First of all, 
a straightforward computation shows that
\[
\begin{align*}
\delta_2 (\epsilon_1) \Omega_2 [\epsilon_2] &= \Omega_2 [\{\epsilon_1, \epsilon_2\}_2], \\
\delta_2 (\epsilon) \Omega_3 [\lambda] &= \Omega_3 [\{\epsilon, \lambda\}_3], \\
\delta_2 (\epsilon) \Omega_4 [\xi] &= \Omega_4 [\{\epsilon, \xi\}_4].
\end{align*}
\]  
(4.9)

Similar computations show that the variations of the \( \Omega \)s parallel the algebra
itself; in other words, if we write the algebra symbolically as
\[
\begin{align*}
\delta_1 (\epsilon) \delta_j (\xi) &= f_{ij}^k \delta_k,
\end{align*}
\]  
where \( f_{ij}^k \) gives the structure 'constants' (actually Poisson brackets) of \( \mathcal{W}_4 \),
then we find \( \delta_i \Omega_j = f_{ij}^k \Omega_k \).

The same holds true for the \( \delta L \)s, namely, they respect the commutation 
relations for the algebra, except for the fact that when a Poisson bracket has 
explicit field dependence, an additional \( \Omega \) term is picked up. For example,
\[
\delta_3 (\lambda_1) \delta_3 L [\lambda_2] - \delta_3 (\lambda_2) \delta_3 L [\lambda_1] - \delta_2 L [\{\lambda_1, \lambda_2\}_2] - \delta_4 L [\{\lambda_1, \lambda_2\}_4] = \\
\Omega_2 \left[ \frac{\partial}{\partial T} \{\lambda_1, \lambda_2\}_2 \right],
\]  
(4.10)

where the \( T \) derivative will just pick out the \( O \left( \frac{1}{\tau} \right) \) piece of \( \{\lambda_1, \lambda_2\}_2 \). Of course,
when the \( \omega \)s are modded out, these \( \Omega \) terms vanish.

So these six functionals automatically satisfy the WZCCs when the WIs are 
imposed, and therefore we can eliminate two each of (4.4), (4.5) and (4.6) in 
favour of them. We choose \( \Delta_1^{(2)} \) and \( \Delta_4^{(3)} \), because the former are the only ones 
with \( \delta \tilde{s} \), and the latter the most complicated.

Now, to finish finding the most general anomalies, we have to check the 
WZCCs for each of the twelve proposed basis anomalies. This involves some 
lengthy computations which we will not include here; suffice it to say that we 
find that apart from the \( \Omega \)s and \( \delta L \)s, only three other specific combinations of the \( \Delta \)s work:
\[
\begin{align*}
\Delta_2^{(0)} [\epsilon] &:= \Delta_2^{(1)} [\epsilon], \\
\Delta_3^{(0)} [\lambda] &:= \Delta_3^{(1)} [\lambda] - \Delta_3^{(4)} [\lambda], \\
\Delta_4^{(0)} [\xi] &:= \Delta_4^{(1)} [\xi] - \Delta_4^{(4)} [\xi] - 2\Delta_4^{(5)} [\xi].
\end{align*}
\]  
(4.11)

These form a set satisfying the WZCCs, in exactly the same manner as the \( \delta L \)s 
do, except with the opposite sign in front of the \( \Omega \), i.e. (4.10) with \( \delta_i L \rightarrow \Delta_i^{(0)} \) and \( \Omega_i \rightarrow -\Omega_i \).

So, finally, we conclude that the most general possible forms for the \( \mathcal{W}_2, \mathcal{W}_3 \) and \( \mathcal{W}_4 \) anomalies satisfying the WZCCs, modulo the Ward identities, are
\[
\begin{align*}
\Delta_2 [\epsilon] &= a\Delta_2^{(0)} [\epsilon] + b\delta_2 L [\epsilon] + r_2 \Omega_2 [\epsilon], \\
\Delta_3 [\lambda] &= a\Delta_3^{(0)} [\lambda] + b\delta_3 L [\lambda] + r_3 \Omega_3 [\lambda], \\
\Delta_4 [\xi] &= a\Delta_4^{(0)} [\xi] + b\delta_4 L [\xi] + r_4 \Omega_4 [\xi].
\end{align*}
\]  
(4.12)
where $a$, $b$ and $r_{2,3,4}$ are constants.

### 4.2 ‘Ward-Free’ Anomalies

At this point, we’ve done all we can algebraically. To say anything more about the anomalies requires that we put further conditions on what constitutes an ‘acceptable anomaly’ in addition to the WZCCs. For instance, if we demand that we never have to invoke the WIs at all, then that leads to the unique choice of $a = b$ and $r_{2,3,4} = 0$. This follows from the facts that the $\Delta_i^{(0)}$s and the $\delta_i L$s violate the WZCCs by the ‘opposite amounts’ of the $\Omega$s, and because the $\Omega$s themselves satisfy the WZCCs iff they are put to zero afterward. Therefore, by taking the sum of the former ($a = b$) and eliminating the latter ($r_{2,3,4} = 0$), the anomalies obtained,

$$\Delta_2[\epsilon] = \int d^2z \epsilon \bar{\partial} T, \quad \Delta_3[\lambda] = \int d^2z \lambda \bar{\partial} W_3, \quad \Delta_4[\xi] = \int d^2z \xi \bar{\partial} W_4,$$

satisfy the WZCCs. So even though we must demand that the $\omega$s vanish to close the algebra, there do in fact exist anomalies which respect the WZCCs without invoking this condition.

### 4.3 BRST Anomalies

The fact that we had to impose a set of first-class constraints to close the $\mathcal{W}_4$ gravity algebra ($\omega_{2,3,4} \equiv 0$) suggests that an interesting case to look at might be where we actually have a BRST algebra: This amounts to introducing three fermionic ghost fields $b_{2,3,4}$ and a new operator $Q$, the BRST charge, defined by

$$Q := \delta_2 (b_2) + \delta_3 (b_3) + \delta_4 (b_4).$$

$Q^2$ vanishes identically on $T$, $W_3$ and $W_4$ provided $Q$ acts on the ghosts as

$$Q b_2 := - \frac{1}{2} \{b_2, b_2\}_2 + \frac{1}{2} \{b_3, b_3\}_2 + \frac{1}{2} \{b_4, b_4\}_2 - \{b_3, b_4\}_2,$$

$$Q b_3 := - \{b_2, b_3\}_3 + \{b_3, b_4\}_3 - \{b_4, b_4\}_3,$$

$$Q b_4 := - \{b_2, b_4\}_4 + \frac{1}{2} \{b_3, b_3\}_4 + \frac{1}{2} \{b_4, b_4\}_4,$$

(4.15)

where the Poisson brackets are those obtained when $b_{2,3,4}$ are treated as $W_{2,3,4}$ variation parameters respectively. These definitions also lead to the vanishing of $Q^2$ on the ghosts themselves as well. (Note that since the ghosts are fermionic, the ordering in the above Poisson brackets is important, so the reader is advised to use (2.3), (2.9) and (2.11) exactly as written, e.g. $\{b_2, b_2\}_2 = \partial b_2 b_2 - b_2 \partial b_2 = -2 b_2 \partial b_2$.)
\( Q^2 \) vanishes on the ghosts as well, but not on the metrics; for example,

\[
Q^2 h = \left( \frac{12}{c} \right) \left[ 16b_3 \partial b_3 \omega_2 + 57b_4 \partial b_4 \partial^2 \omega_2 + 57b_4 \partial^2 b_4 \partial \omega_2 + (78b_4 \partial^3 b_4 \\
-118 \partial b_4 \partial^2 b_4) \omega_2 - 27b_3 b_4 \omega_3 + 25b_3 \partial b_4 \omega_3 + 7b_3 b_4 \partial \omega_3 - 14b_3 \partial b_4 \omega_4 \right] \\
+ \left( \frac{12}{c} \right)^2 864b_4 \partial b_4 T \omega_2.
\]  

(4.16)

This should come as no surprise at all, because the nilpotency of the BRST charge is dependent upon the fact that the structure constants of the symmetry algebra satisfy the Jacobi identity, which is not true for a \( \mathcal{W} \)-algebra, due to the field dependence of the Poisson brackets. The condition for the satisfaction of the Jacobi identities, i.e. the closure of the algebra, is merely the imposition of the WIs, as we have seen before, so \( Q \) is nilpotent iff the \( \omega \)s vanish. Or, if we reverse the argument, we could have introduced the BRST transformations and imposed nilpotency of \( Q \) to find the WIs.

To find the BRST anomalies, we look for functionals of our fields with unit ghost number which are \( Q \)-closed. Such functionals must be integrals where the integrand is linear in one of the \( b \)s, so it makes sense to look at our basis of anomalies where the arguments are replaced by the ghost fields. It is a straightforward exercise to show

\[
Q (\delta_2 L [b_2] + \delta_3 L [b_3] + \delta_4 L [b_4]) = 0,
\]

\[
Q (\Omega_2 [b_2] + \Omega_3 [b_3] + \Omega_4 [b_4]) = -\Omega_2 [Qb_2] - \Omega_3 [Qb_3] - \Omega_4 [Qb_4],
\]

\[
Q \left( \Delta^{(0)}_2 [b_2] + \Delta^{(0)}_3 [b_3] + \Delta^{(0)}_4 [b_4] \right) = 0,
\]

(4.17)

so upon imposing the WIs, all of these fit the bill, so the most general BRST anomaly has the form

\[
\Delta = \Delta_1 [b_1] + \Delta_2 [b_2] + \Delta_3 [b_3],
\]

(4.18)

for arbitrary \( a, b \) and \( r_{2,3,4} \).

### 4.4 Anomalies from Effective Action

Another case we might want to consider is where we think of our anomalies as arising from an effective \( \mathcal{W}_4 \) symmetric field theory, in which case the anomalies will be the variations of the effective action. To be precise, if we start with a theory of \( \mathcal{W}_4 \) gravity with metrics \( h', g'_3 \) and \( g'_4 \) and invariant action \( S' \), then by introducing currents \( T, W_3 \) and \( W_4 \) we find the generating partition function

\[
Z := \int [dh'] [dg'_3] [dg'_4] e^{iS' + i \int_{\Sigma} d^2 z (Th' + W_3 g'_3 + W_4 g'_4)}.
\]

(4.19)
The effective metrics $h$, $g_3$ and $g_4$ are, respectively, the functional derivatives of $-i \ln Z$ with respect to $T$, $W_3$ and $W_4$, and so the effective action is obtained via the usual Legendre transformation $S := -i \ln Z - L$, where $L$ is the same functional from $\mathcal{L}$. Now, if we compute the variation of $S$ under a $W_4$ transformation, the definitions of the effective metrics immediately lead to

$$\delta_i S = - \int_{\Sigma} d^2 z \left( T \delta_i h + W_3 \delta_i g_3 + W_4 \delta_i g_4 \right).$$  \hfill (4.20)

If the variations are put in explicitly, we get the result that $\delta_i S = \Delta^{(0)}_i + \Omega_i$. However, recall that this is all in the context of the currents being functions of the effective metrics, which is simply the statement that the WIs are satisfied. Thus, a $\mathcal{W}_4$ symmetric field theory has $a = 1$, $b = r_{2.3.4} = 0$:

$$\delta_2 (\epsilon) S = \int_{\Sigma} d^2 z \left( \frac{c}{12} \epsilon \partial^3 h, \right.$$  

$$\delta_4 (\lambda) S = \int_{\Sigma} d^2 z \lambda \left[ \left( \frac{c}{12} \right) \partial^5 g_3 \right.$$  

$$- \left( \frac{12}{c} \right) \left( 16g_3 T \partial T + 16g_3 T^2 + 34g_4 T W_3 + 27g_4 T W_3 + 52g_4 T W_3 \right), \right.$$  

$$\delta_4 (\xi) S := \int_{\Sigma} d^2 z \xi \left\{ \left( \frac{c}{12} \right) \xi \partial^7 g_4 \right.$$  

$$- \left( \frac{12}{c} \right) \left[ 18g_3 T \partial W_3 + 25g_3 \partial TW_3 + 52g_3 TW_3 + g_4 (177\partial T \partial^2 T + 78T \partial^3 T) \right.$$  

$$+ \partial g_4 (352T \partial^2 T + 295(\partial T)^2) + 588g_2 T \partial T + 196\partial^3 g_4 T^2 \right.$$  

$$- 14g_4 T \partial W_4 - 14g_4 T W_4 - 28g_4 TW_4 + 75g_4 W_3 \partial W_3 + 75g_4 W_3^2 \right]. \left( 864g_4 T^2 \partial T + 576g_4 T^3 \right). \right\}.$$  \hfill (4.21)

5 Conclusions

In this paper we have shown that $\mathcal{W}_4$, represented as a set of transformations on classical fields, can be gauged into ‘pure $\mathcal{W}_4$ gravity’ provided that the three currents and three metrics satisfy relations which may be identified with the Ward identities of the theory. (Alternatively, this may be thought of as saying that we have a theory on a fibre bundle with base space $\Sigma$, structure group $\mathcal{W}_4$ and fibre $H^1(\mathcal{M}, C; \overline{D})$.) The form of these identities suggests a basis for the anomalies of the theory, from which we can find the particular linear combinations which satisfy the Wess-Zumino consistency conditions. Once found, we can then restrict our theory in various ways (e.g. as arising from a BRST algebra) to determine the anomalies specific to those cases.
5.1 Connection to Toda Theory

One possible avenue of further exploration might be a deeper examination of one of the cases for which the anomalies were found, namely, the one in which our theory arises from an underlying $\mathcal{W}_4$-symmetric quantum field theory: Recall the usual conformal case, where the WI is just

$$\partial T = \left(\frac{c}{12}\right) \partial^3 h + h \partial T + 2 \partial h T. \quad (5.1)$$

This gives a relation between $T$ and $h$, so in reality there is only one physical degree of freedom, call it $f_2$. This appears explicitly when we solve the above WI, via

$$h = \bar{\partial} f_2, \quad T = \left(\frac{c}{12}\right) \frac{\partial f_2 \partial^3 f_2 - \frac{4}{3} (\partial^2 f_2)^2}{(\partial f_2)^2}. \quad (5.2)$$

The conformal variations on $T$ and $h$ become

$$\delta \Delta = \left(\frac{c}{12}\right) \int_{\Sigma} d^2 z \epsilon \partial^3 h$$

$$= \left(\frac{c}{12}\right) \int_{\Sigma} d^2 z \frac{\delta_2 (\epsilon) f_2}{\partial f_2} \partial^3 \left(\frac{\bar{\partial} f_2}{\partial f_2}\right). \quad (5.3)$$

This must be a conformal variation of an effective action, and when we integrate it, we find

$$S[f_2] = - \left(\frac{c}{24}\right) \int_{\Sigma} d^2 z \frac{\bar{\partial} f_2}{\partial f_2} \partial^2 \ln \partial f_2. \quad (5.4)$$

Actually, we can add any conformally invariant term to this, since it will not effect the anomaly. The obvious one is simply some multiple of the volume of $\Sigma$, which, since the determinant of the metric is unity, is simply $\int_{\Sigma} d^2 z$.

Suppose we now change coordinates to $\sigma^+ := f_2(z, \bar{z})$ and $\sigma^- := \bar{z}$, and define a new field $\phi_2(\sigma) := - \ln \partial f_2(z(\sigma))$ (so $\delta_2 (\epsilon) \phi_2 = - \partial \epsilon + \epsilon \partial \phi_2$); the metric is now the conformal one, $d s^2 = e^{\phi_2} d\sigma^+ d\sigma^-$, and the action takes the form

$$S[\phi_2] = \left(\frac{c}{24}\right) \int_{\Sigma} d^2 \sigma \left(\partial^+ \phi_2 \partial^- \phi_2 - \Lambda e^{\phi_2}\right) \quad (5.5)$$

(where we have chosen the constant multiplying the volume of $\Sigma$ to be $-\frac{c}{24}$).

This is instantly recognisable as the action for Liouville gravity, which is no great revelation, since both (5.4) and (5.5) are just specific examples of the general gravitational action

$$S[h] = - \left(\frac{c}{24}\right) \int_{\Sigma} d^2 x \sqrt{h} \left(\frac{1}{\Lambda} R + \Lambda\right) \quad (5.6)$$
evaluated in the appropriate coordinate systems.

So how does this connect to $\mathcal{W}_4$ gravity as examined here? In principle, we can follow the same technique as just presented, by finding three fields $f_2$, $f_3$ and $f_4$ such that the currents and metrics may be given as functions of these when the WIs are solved. Then the three anomalies could be written as functionals of the same fields, and hopefully an effective action leading to them could be found. Since this involves solving three coupled nonlinear partial differential equations, it is obviously easier said than done, but if possible, then an appropriate change of coordinates $z \rightarrow \sigma$ and field redefinitions $f_i \rightarrow \phi_i$ may bring the action into a Liouville-like form.

But in principal, this may have already been done: Recall that there already exists a generalisation of the Liouville theory which exhibits $\mathcal{W}$-symmetry, namely, Toda theory \cite{5,12}, so it seems very likely that the action obtained would be the one for a particular Toda theory. Furthermore, since the number of fields in a Toda theory with associated simple Lie algebra $g$ is equal to the rank of $g$, it follows that since our theory has three degrees of freedom after solving the WIs, we would be expecting a $A_3$, $B_3$, $C_3$, $D_3$ or $F_3$ Toda theory. However, the only one of these containing fields of weights 2, 3 and 4 is the $A_3 = SU(4)$ case, so presumably this would be the result of the computation just outlined. As of this writing, this equivalence has not yet been shown, but may serve as a basis for future work.

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