Splitting gradient algorithms for solving monotone equilibrium problems

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1 Introduction

Let \( H \) be a real Hilbert space endowed with weak topology defined by the inner product \( \langle \cdot, \cdot \rangle \) and its induced norm \( \| \cdot \| \). Let \( C \subseteq H \) be a nonempty closed convex subset and \( f : H \times H \rightarrow \mathbb{R} \cup \{+\infty\} \) a bifunction such that \( f(x, y) < +\infty \) for every \( x, y \in C \). The equilibrium problem defined by the Nikaido-Isoda-Fan inequality that we are going to consider in this paper is given as

\[
 Find \ x \in C : f(x, y) \geq 0 \ \forall \ y \in C. \tag{EP}
\]

This inequality first was used in 1955 by Nikaido-Isoda [20] in convex game models. Then in 1972 Ky Fan [7] called this inequality a minimax one and established existence theorems for Problem (EP). After the appearance of the paper by Blum and Oettli [4] Problem (EP) has been contracted much attention of researchers. It has been shown in [3, 4, 17] that some important problems such as optimization, variational inequality, Kakutani fixed point and Nash equilibrium can be formulated...
in the form of \((EP)\). Many papers concerning the solution existence, stabilities as well as algorithms for Problem \((EP)\) have been published (see e.g. [8, 10, 15, 18, 22, 23, 24] and the survey paper [3]). A basic method for Problem \((EP)\) is the gradient (or projection) one, where the sequence of iterates is defined by taking

\[
x^{k+1} = \min \left\{ \lambda_k f(x^k, y) + \frac{1}{2} \| y - x^k \|^2 : y \in C \right\},
\]

with \(\lambda_k\) is some appropriately chosen real number. Note that in the variational inequality case, where \(f(x, y) := \langle F(x), y - x \rangle\), the iterate \(x^{k+1}\) defined by \((1)\) becomes

\[
x^{k+1} = P_C \left( x^k - \lambda_k F(x^k) \right),
\]

where \(P_C\) stands for the metric projection onto \(C\).

It is well known that under certain conditions on the parameter \(\lambda_k\), the projection method is convergent if \(f\) is strongly pseudomonotone or paramonotone [10, 23]. However when \(f\) is monotone, it may fail to converge. In order to obtain convergent algorithms for monotone, even pseudomonotone, equilibrium problems, the extragradient method first proposed by Korpelevich [14] for the saddle point and related problems has been extended to equilibrium problems [23]. In this extragradient algorithm, at each iteration, it requires solving the two strongly convex programs

\[
y^k = \min \left\{ \lambda_k f(x^k, y) + \frac{1}{2} \| y - x^k \|^2 : y \in C \right\},
\]

\[
x^{k+1} = \min \left\{ \lambda_k f(x^k, y) + \frac{1}{2} \| y - y^k \|^2 : y \in C \right\},
\]

which may cause computational cost. In order to reduce the computational cost, several convergent algorithms that require solving only one strongly convex program or computing only one projection at each iteration have been proposed. These algorithms were applied to some classes of bifunctions such as strongly pseudomonotone and paramonotone ones, with or without using an ergodic sequence (see e.g. [2, 5, 24]). In another direction, also for the sake of reducing computational cost, some splitting algorithms have been developed (see e.g. [1, 4, 16]) for monotone equilibrium problems where the bifunctions can be decomposed into the sum of two bifunctions. In these algorithms the convex subprograms (resp. regularized subproblems) involving the bifunction \(f\) can be replaced by the two convex subprograms (resp. regularized subproblems), one for each \(f_i\) \((i = 1, 2)\) independently.

In this paper we modify the projection algorithm in [24] to obtain a splitting convergent algorithm for paramonotone equilibrium problems. The main future of this algorithm is that at each iteration, it requires solving only one strongly convex program. Furthermore, in the case when \(f = f_1 + f_2\), this strongly convex subprogram can be replaced by the two strongly convex subprograms, one for each \(f_1\) and \(f_2\) as the algorithm in [1, 4], but for the convergence we do not require any additional conditions such as Hölder continuity and Lipschitz type condition as in [1, 4]. We also show that the ergodic sequence defined by the algorithm’s iterates converges to a solution without paramonotonicity property. We apply the ergodic algorithm for solving a Cournot-Nash model with joint constraints. The computational results and experiences show that the ergodic algorithm is efficient for this model with a restart strategy.

The remaining part of the paper is organized as follows. The next section lists preliminaries containing some lemmas that will be used in proving the convergence of the proposed algorithm. Section 3 is devoted to the description of the algorithm and its convergence analysis. Section 4 shows an application of the algorithm in solving a Cournot-Nash model with joint constraints. Section 5 closed the paper with some conclusions.
2 Preliminaries

We recall from [3] the following well-known definition on monotonicity of bifunctions.

**Definition 1** A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is said to be

(i) strongly monotone on $C$ with modulus $\beta > 0$ (shortly $\beta$-strongly monotone) if

$$f(x, y) + f(y, x) \leq -\beta\|y - x\|^2 \quad \forall x, y \in C;$$

(ii) monotone on $C$ if

$$f(x, y) + f(y, x) \leq 0 \quad \forall x, y \in C;$$

(iii) strongly pseudo-monotone on $C$ with modulus $\beta > 0$ (shortly $\beta$-strongly pseudo-monotone) if

$$f(x, y) \geq 0 \implies f(y, x) \leq -\beta\|y - x\|^2 \quad \forall x, y \in C;$$

(iv) pseudo-monotone on $C$ if

$$f(x, y) \geq 0 \implies f(y, x) \leq 0 \quad \forall x, y \in C.$$

(v) paramonotone on $C$ with respect to a set $S$ if

$$x^* \in S, x \in C \text{ and } f(x^*, x) = f(x, x^*) = 0 \text{ implies } x \in S.$$

Obviously, (i) $\implies$ (ii) $\implies$ (iv) and (i) $\implies$ (iii) $\implies$ (iv). Note that a strongly pseudo-monotone bifunction may not be monotone. Paramonotone bifunctions have been used in e.g. [24, 23]. Some properties of paramonotone operators can be found in [11]. Clearly in the case of optimization problem when $f(x, y) = \varphi(y) - \varphi(x)$, the bifunction $f$ is paramonotone on $C$ with respect to the solution set of the problem $\min_{x \in C} \varphi(x)$.

The following well known lemmas will be used for proving the convergence of the algorithm to be described in the next section.

**Lemma 1** (see [26] Lemma 1) Let $\{\alpha_k\}$ and $\{\sigma_k\}$ be two sequences of nonnegative numbers such that $\alpha_{k+1} \leq \alpha_k + \sigma_k$ for all $k \in \mathbb{N}$, where $\sum_{k=1}^{\infty} \sigma_k < \infty$. Then the sequence $\{\alpha_k\}$ is convergent.

**Lemma 2** (see [21]) Let $\mathcal{H}$ be a Hilbert space, $\{x^k\}$ a sequence in $\mathcal{H}$. Let $\{r_k\}$ be a sequence of nonnegative number such that $\sum_{k=1}^{\infty} r_k = +\infty$ and set $z^k := \sum_{i=1}^{k} r_i x^i / \sum_{i=1}^{k} r_i$. Assume that there exists a nonempty, closed convex set $S \subset \mathcal{H}$ satisfying:

(i) For every $z \in S$, $\lim_{n \to \infty} \|z^k - z\|$ exists;

(ii) Any weakly cluster point of the sequence $\{z^k\}$ belongs to $S$.

Then the sequence $\{z^k\}$ weakly converges.

**Lemma 3** (see [23]) Let $\{\lambda_k\}, \{\delta_k\}, \{\sigma_k\}$ be sequences of real numbers such that

(i) $\lambda_k \in (0, 1)$ for all $k \in \mathbb{N}$;

(ii) $\sum_{k=1}^{\infty} \lambda_k = +\infty$;

(iii) $\limsup_{k \to +\infty} \delta_k \leq 0$;

(iv) $\sum_{k=1}^{\infty} |\sigma_k| < +\infty$.

Suppose that $\{\alpha_k\}$ is a sequence of nonnegative real numbers satisfying

$$\alpha_{k+1} \leq (1 - \lambda_k)\alpha_k + \lambda_k \delta_k + \sigma_k \quad \forall k \in \mathbb{N}.$$

Then we have $\lim_{k \to +\infty} \alpha_k = 0$. 
3 The problem, algorithm and its convergence

3.1 The problem

In what follows, for the following equilibrium problem

\[ \text{Find } x \in C : f(x, y) \geq 0 \quad \forall y \in C \]  

(EP)

we suppose that \( f(x, y) = f_1(x, y) + f_2(x, y) \) and that \( f_i(x, x) = 0 \) (\( i = 1, 2 \)) for every \( x, y \in C \). The following assumptions for the bifunctions \( f, f_1, f_2 \) will be used in the sequel.

(A1) For each \( i = 1, 2 \) and each \( x \in C \), the function \( f_i(x, \cdot) \) is convex and sub-differentiable, while \( f(\cdot, y) \) is weakly upper semicontinuous on \( C \);

(A2) If \( \{x^k\} \subset C \) is bounded, then for each \( i = 1, 2 \), the sequence \( \{g^k_i\} \) with \( g^k_i \in \partial f_i(x^k, x^k) \) is bounded;

(A3) The bifunction \( f \) is monotone on \( C \).

Assumption (A2) has been used in e.g. [25]. Note that Assumption (A2) is satisfied if the functions \( f_1 \) and \( f_2 \) are jointly weakly continuous on an open convex set containing \( C \) (see [27] Proposition 4.1).

The dual problem of (EP) is

\[ \text{find } x \in C : f(y, x) \leq 0 \quad \forall y \in C. \]

(DEP)

We denote the solution sets of (EP) and (DEP) by \( S(C, f) \) and \( S^d(C, f) \), respectively. A relationship between \( S(C, f) \) and \( S^d(C, f) \) is given in the following lemma.

**Lemma 4** (see [13] Proposition 2.1) (i) If \( f(\cdot, y) \) is weakly upper semicontinuous and \( f(x, \cdot) \) is convex for all \( x, y \in C \), then \( S^d(C, f) \subset S(C, f) \).

(ii) If \( f \) is pseudomonotone, then \( S(C, f) \subset S^d(C, f) \).

Therefore, under the assumptions (A1)-(A3) one has \( S(C, f) = S^d(C, f) \). In this paper we suppose that \( S(C, f) \) is nonempty.

3.2 The algorithm and its convergence analysis

The algorithm below is a gradient one for paramonotone equilibrium problem (EP). The stepsize is computed as in the algorithm for equilibrium problem in [24].

**Algorithm 1** A splitting algorithm for solving paramonotone or strongly pseudo-monotone equilibrium problems.

**Initialization:** Seek \( x^0 \in C \). Choose a sequence \( \{\beta_k\}_{k \geq 0} \subset \mathbb{R} \) satisfying the following conditions

\[ \sum_{k=0}^{\infty} \beta_k = +\infty, \quad \sum_{k=0}^{\infty} \beta_k^2 < +\infty. \]

**Iteration** \( k = 0, 1, \ldots \):

Take \( g^k_1 \in \partial_2 f_1(x^k, x^k), g^k_2 \in \partial_2 f_2(x^k, x^k) \).

Compute

\[ \eta_k := \max\{\beta_k, \|g^k_1\|, \|g^k_2\|\}, \quad \lambda_k := \frac{\beta_k}{\eta_k}, \]

\[ y^k := \arg \min_{y \in C} \{\lambda_k f_1(x^k, y) + \frac{1}{2} \|y - x^k\|^2 \}, \]

\[ x^{k+1} := \arg \min_{y \in C} \{\lambda_k f_2(x^k, y) + \frac{1}{2} \|y - y^k\|^2 \}. \]
Theorem 1 In addition to the assumptions (A1), (A2), (A3) we suppose that \( f \) is paramonotone on \( C \), and that either \( \text{int } C \neq \emptyset \) or for each \( x \in C \) both bifunctions \( f_1(x, \cdot), f_2(x, \cdot) \) are continuous at a point in \( C \). Then the sequence \( \{x^k\} \) generated by the algorithm converges weakly to a solution of (EP). Moreover, if \( f \) is strongly pseudomonotone, then \( \{x^k\} \) strongly converges to the unique solution of (EP).

Proof First, we show that, for each \( x^* \in S(f, C) \), the sequence \( \{\|x^k - x^*\|\} \) is convergent.

Indeed, for each \( k \geq 0 \), for simplicity of notation, let

\[
h_1^k(x) := \lambda_k f_1(x^k, x) + \frac{1}{2} \|x - x^k\|^2,
\]

\[
h_2^k(x) := \lambda_k f_2(x^k, x) + \frac{1}{2} \|x - y^k\|^2.
\]

By Assumption (A1), the functions \( h_1^k \) is strongly convex with modulus 1 and subdifferentiable, which implies

\[
h_1^k(y^k) + \langle u^k_1, x - y^k \rangle + \frac{1}{2} \|x - y^k\|^2 \leq h_1^k(x) \quad \forall x \in C
\]

(4)

for any \( u^k_1 \in \partial h_1^k(y^k) \). On the other hand, from the definition of \( y^k \), using the regularity condition, by the optimality condition for convex programming, we have

\[0 \in \partial h_1^k(y^k) + NC(y^k)\]

In turn, this implies that there exists \(-u^k_1 \in \partial h_1^k(y^k)\) such that \( \langle u^k_1, x - y^k \rangle \geq 0 \) for all \( x \in C \).

Hence, from (4), for each \( x \in C \), it follows that

\[
h_1^k(y^k) + \frac{1}{2} \|x - y^k\|^2 \leq h_1^k(x),
\]

i.e.,

\[
\lambda_k f_1(x^k, y^k) + \frac{1}{2} \|y^k - x^k\|^2 + \frac{1}{2} \|x - y^k\|^2 \leq \lambda_k f_1(x^k, x) + \frac{1}{2} \|x - x^k\|^2,
\]

or equivalently,

\[
\|y^k - x\|^2 \leq \|x^k - x\|^2 + 2\lambda_k (f_1(x^k, x) - f_1(x^k, y^k)) - \|y^k - x^k\|^2.
\]

(5)

Using the same argument for \( x^{k+1} \), we obtain

\[
\|x^{k+1} - x\|^2 \leq \|y^k - x\|^2 + 2\lambda_k (f_2(x^k, x) - f_2(x^k, x^{k+1})) - \|x^{k+1} - y^k\|^2.
\]

(6)

Combining (5) and (6) yields

\[
\|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 - \|y^k - x^k\|^2 - \|x^{k+1} - y^k\|^2
\]

\[
+ 2\lambda_k \left(f_1(x^k, x) + f_2(x^k, x) - 2\lambda_k (f_1(x^k, y^k) + f_2(x^k, x^{k+1}))\right)
\]

\[
= \|x^k - x\|^2 - \|y^k - x^k\|^2 - \|x^{k+1} - y^k\|^2
\]

\[
+ 2\lambda_k f(x^k, x) - 2\lambda_k (f_1(x^k, y^k) + f_2(x^k, x^{k+1}))
\]

(7)

From \( g^k_1 \in \partial f_1(x^k, x^k) \) and \( f_1(x^k, x^k) = 0 \), it follows that

\[
f_1(x^k, y^k) - f_1(x^k, x^k) \geq (g^k_1, y^k - x^k),
\]

which implies

\[-2\lambda_k f_1(x^k, y^k) \leq -2\lambda_k (g^k_1, y^k - x^k).
\]

(8)

By using the Cauchy-Schwarz inequality and the fact that \( \|g^k_1\| \leq \eta_k \), from (8) one can write

\[-2\lambda_k f_1(x^k, y^k) \leq 2\beta_k \frac{\eta_k}{\eta_k} \|y^k - x^k\| = 2\beta_k \|y^k - x^k\|.
\]

(9)
By the same argument, we obtain
\[ -2\lambda_k f_2(x^k, x^{k+1}) \leq 2\beta_k \|x^{k+1} - x^k\|. \quad (10) \]
Replacing (9) and (10) to (7) we get
\[ \|x^{k+1} - x\|^2 \leq \|x^k - x\|^2 + 2\lambda_k f(x^k, x) + 2\beta_k \|y^k - x^k\| \]
\[ + 2\beta_k \|x^{k+1} - x^k\| - \|y^{k+1} - x^k\|^2 - \|x^{k+1} - y^k\|^2 \]
\[ = \|x^k - x\|^2 + 2\lambda_k f(x^k, x) + 2\beta_k \|x^{k+1} - x^k\| \]
\[ + 2\beta_k \|y^k - x^k\| - (\|y^k - x^k\| - \beta_k)^2 - (\|x^{k+1} - x^k\| - \beta_k)^2 \]
\[ \leq \|x^k - x\|^2 + 2\lambda_k f(x^k, x) + 2\beta_k^2. \quad (11) \]
Note that by definition of \( x^* \in S(f, C) = S^2(f, C) \) we have \( f(x^*, x^*) \leq 0 \). Therefore, by taking \( x = x^* \) in (10) we obtain
\[ \|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + 2\lambda_k f(x^k, x^*) + 2\beta_k^2 \]
\[ \leq \|x^k - x^*\|^2 + 2\beta_k^2. \quad (12) \]
Since \( \sum_{k=0}^{\infty} \beta_k^2 < +\infty \) by assumption, in virtue of Lemma 11 it follows from (12) that the sequence \( \{\|x^k - x^*\|^2\} \) is convergent.

Next, we prove that any cluster point of the sequence \( \{x^k\} \) is a solution of (EP).
Indeed, from (12) we have
\[ -2\lambda_k f(x^k, x^*) \leq \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2 + 2\beta_k^2 \quad \forall k \in \mathbb{N}. \quad (13) \]
By summing up we obtain
\[ 2 \sum_{i=0}^{\infty} \lambda_i (-f(x^i, x^*)) \leq \|x^0 - x^*\|^2 + 2 \sum_{i=0}^{\infty} \beta_i^2 < \infty. \]
On the other hand, by Assumption (A2) the sequences \( \{g_1^k\}, \{g_2^k\} \) are bounded. This fact, together with the construction of \( \{\beta_k\} \), implies that there exists \( M > 0 \) such that \( \|g_1^k\| \leq M, \|g_2^k\| \leq M, \beta_k \leq M \) for all \( k \in \mathbb{N} \). Hence for each \( k \in \mathbb{N} \) we have
\[ \eta_k = \max\{\beta_k, \|g_1^k\|, \|g_2^k\|\} \leq M, \]
which implies \( \sum_{i=0}^{\infty} \lambda_i = \infty \). Thus, from \( f(x^i, x^*) \leq 0 \), it holds that
\[ \limsup f(x^k, x^*) = 0 \quad \forall x^* \in S(C, f). \]
Fixed \( x^* \in S(C, f) \) and let \( \{x^{k_j}\} \) be a subsequence of \( \{x^k\} \) such that
\[ \limsup f(x^k, x^*) = \lim f(x^{k_j}, x^*) = 0. \]
Since \( \{x^{k_j}\} \) is bounded, we may assume that \( \{x^{k_j}\} \) weakly converges to some \( \bar{x} \). Since \( f(\cdot, x^*) \) is weakly upper semicontinuous by assumption (A1), we have
\[ f(\bar{x}, x^*) \geq \lim f(x^{k_j}, x^*) = 0. \quad (14) \]
Then it follows from the monotonicity of \( f \) that \( f(x^*, \bar{x}) \leq 0 \). On the other hand, since \( x^* \in S(C, f) \), by definition we have \( f(x^*, \bar{x}) \geq 0 \). Therefore we obtain \( f(x^*, \bar{x}) = 0 \). Again, the monotonicity of \( f \) implies \( f(\bar{x}, x^*) \leq 0 \), and therefore, by (14) one has \( f(\bar{x}, x^*) = 0 \). Since \( f(x^*, \bar{x}) = 0 \) and \( f(\bar{x}, x^*) = 0 \), it follows from paramonotonicity of \( f \) that \( \bar{x} \) is a solution to (EP). Since \( \|x^{k_j} - \bar{x}\| \) converges, from the fact that \( x^{k_j} \) weakly converges to \( \bar{x} \), we can conclude that the whole sequence \( \{x^k\} \) weakly converges to \( \bar{x} \).
Note that if \( f \) is strongly pseudomonotone, then Problem (EP) has a unique solution (see Proposition 1). Let \( x^\ast \) be the unique solution of (EP). By definition of \( x^\ast \) we have
\[
f(x^\ast, x) \geq 0 \quad \forall x \in C,
\]
which, by strong pseudomonotonicity of \( f \), implies
\[
f(x, x^\ast) \leq -\beta \|x - x^\ast\|^2 \quad \forall x \in C.
\]
(15)

By choosing \( x = x^k \) in (15) and then applying to (11) we obtain
\[
\|x^{k+1} - x^\ast\|^2 \leq (1 - 2\beta \lambda_k)\|x^k - x^\ast\|^2 + 2\beta_k^2 \quad \forall k \in \mathbb{N},
\]
which together with the construction of \( \beta_k \) and \( \lambda_k \), by virtue of Lemma 3 with \( \delta_k \equiv 0 \), implies that
\[
\lim_{k \to +\infty} \|x^k - x^\ast\|^2 = 0,
\]
i.e., \( x^k \) strongly converges to the unique solution \( x^\ast \) of (EP). \( \square \)

The following simple example shows that without paramonotonicity, the algorithm may not be convergent. Let us consider the following example, taken from [6], where \( f(x, y) := \langle Ax, y - x \rangle \) and \( C := \mathbb{R}^2 \) and
\[
A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
Clearly, \( x^\ast = (0, 0)^T \) is the unique solution of this problem. It is easy to check that this bifunction is monotone, but not paramonotone. An elementary computation shows that
\[
x^{k+1} = x^k - \lambda_k Ax^k = (x_1^k - \lambda_k x_2^k, x_2^k + \lambda_k x_1^k)^T.
\]
Thus, \( \|x^{k+1}\|^2 = (1 + \lambda_k^2)\|x^k\|^2 > \|x^k\|^2 \) if \( x^k \neq 0 \), which implies that the sequence \( \{x^k\} \) does not converge to the solution \( x^\ast = 0 \) for any starting point \( x^0 \neq 0 \).

To illustrate our motivation let us consider the following optimization problem
\[
(\text{OP}) \quad \min \varphi(x) := \frac{1}{2} x^T Q x - \sum_{i=1}^n \ln(1 + \max\{0, x_i\})
\]
subject to \( x_i \in [a_i, b_i] \subseteq \mathbb{R} \quad (i = 1, \ldots, n), \)
where \( Q \in \mathbb{R}^{n \times n} \) is a positive semidefinite matrix. This problem is equivalent to the following equilibrium problem
\[
\text{Find } x^\ast \in C \text{ such that } f(x^\ast, y) \geq 0 \quad \forall y \in C,
\]
where \( C := [a_1, b_1] \times \cdots \times [a_n, b_n] \), and \( f(x, y) := \varphi(y) - \varphi(x) \). We can split the function \( f(x, y) = f_1(x, y) + f_2(x, y) \) by taking
\[
f_1(x, y) = \frac{1}{2} y^T Q y - \frac{1}{2} x^T Q x,
\]
and
\[
f_2(x, y) = \sum_{i=1}^n (\ln(1 + \max\{0, x_i\}) - \ln(1 + \max\{0, y_i\})).
\]
Since \( Q \) is a positive semidefinite matrix and \( \ln(\cdot) \) is concave on \((0, +\infty)\), the functions \( f_1, f_2 \) are equilibrium functions satisfying conditions (A1)-(A3). Clearly, \( f_1(x, \cdot) \) is convex quadratic, not necessarily separable, while \( f_2(x, \cdot) \) is separable, not necessarily differentiable, but their sum does not inherit these properties.

In order to obtain the convergence without paramonotonicity we use the iterate \( x^k \) to define an ergodic sequence by taking
\[
x^k := \frac{\sum_{i=0}^{k} \lambda_i x^i}{\sum_{i=0}^{k} \lambda_i}.
\]
Then we have the following convergence result.
Theorem 2 Under the assumption in Theorem 1, the ergodic sequence \( \{z^k\} \) converges weakly to a solution of \((EP)\).

Proof In the proof of Theorem 1, we have shown that the sequence \( \{\|x^k - x^*\|\} \) is convergent. By the definition of \( z^k \), the sequence \( \{\|z^k - x^*\|\} \) is convergent too. In order to apply Lemma 2, now we show that all weakly cluster points of \( \{z^k\} \) belong to \( S(f, C) \). In fact, using the inequality (12), by taking the sum of its two sides over all indices we have

\[
2 \sum_{i=0}^{k} \lambda_i f(x, x^i) \leq \sum_{i=0}^{k} \left( \|x^i - x\|^2 - \|x^{i+1} - x\|^2 + 2\beta_i^2 \right)
\]

\[
= \|x^0 - x\|^2 - \|x^{k+1} - x\|^2 + 2 \sum_{i=0}^{k} \beta_i^2
\]

\[
\leq \|x^0 - x\|^2 + 2 \sum_{i=0}^{k} \beta_i^2.
\]

By using this inequality, from definition of \( z^k \) and convexity of \( f(x, \cdot) \), we can write

\[
f(x, z^k) = f \left( x, \sum_{i=0}^{k} \lambda_i x^i \right)
\]

\[
\leq \frac{\sum_{i=0}^{k} \lambda_i f(x, x^i)}{\sum_{i=0}^{k} \lambda_i}
\]

\[
\leq \frac{\|x^0 - x\|^2 + 2 \sum_{i=0}^{k} \beta_i^2}{2 \sum_{i=0}^{k} \lambda_i}.
\]

As we have shown in the proof of Theorem 1 that

\[
\lambda_k = \frac{\beta_k}{\eta_k} \geq \frac{\beta_k}{M}.
\]

Since \( \sum_{k=0}^{\infty} \beta_k = +\infty \), we have \( \sum_{k=0}^{\infty} \lambda_k = +\infty \). Then, it follows from (10) that

\[
\lim_{k \to \infty} \inf f(x, z^k) \leq 0.
\]

(17)

Let \( \bar{z} \) be any weakly cluster of \( \{z^k\} \). Then there exists a subsequence \( \{z^{k_j}\} \) of \( \{z^k\} \) such that \( z^{k_j} \to \bar{z} \). Since \( f(x, \cdot) \) is lower semicontinuous, it follows from (17) that

\[
f(x, \bar{z}) \leq 0.
\]

Since this inequality hold for arbitrary \( x \in C \), it means that \( \bar{z} \in S^d(f, C) = S(f, C) \). Thus it follows from Lemma 2 that the sequence \( \{z^k\} \) converges weakly to a point \( z^* \in S(f, C) \), which is a solution to \((EP)\). □

Remark 1 In case that \( \mathcal{H} \) is of finite dimension, we have \( \|z^{k+1} - z^k\| \to 0 \) as \( k \to \infty \). Since \( \sum_{k=0}^{\infty} \lambda_k^2 < +\infty \), at large enough iteration \( k \), the value of \( \lambda_k \) closes to 0, which makes the intermediate iteration points \( y^k, z^{k+1} \) close to \( x^k \). In turn, the new generated ergodic point \( z^{k+1} \) does not change much from the previous one. This slows down the convergence of the sequence \( \{z^k\} \). In order to enhance the convergence of the algorithm, it suggests a restart strategy by replacing the starting point \( x^0 \) with \( x^k \) whenever \( \|z^{k+1} - z^k\| \leq \tau \) with an appropriate \( \tau > 0 \).
4 Numerical experiments

We used MATLAB R2016a for implementing the proposed algorithms. All experiments were conducted on a computer with a Core i5 processor, 16 GB of RAM, and Windows 10.

As we have noted in Remark 1 to improve the performance of our proposed algorithm, we reset $x^0$ to $x^k$ whenever $\|z^{k+1} - z^k\| \leq \tau$ with an appropriate $\tau > 0$ and then restart the algorithm from beginning with the new starting point $x^0$ if the stopping criterion $\|z^{k+1} - z^k\| \leq \epsilon$ is still not satisfied. In all experiments, we set $\tau := 10^{-3}$, and terminated the algorithm when either the number of iterations exceeds $10^4$, or the distance between the two consecutive generated ergodic points is less than $\epsilon := 10^{-4}$ (i.e., $\|z^{k+1} - z^k\| < 10^{-4}$). All the tests reported below were solved within 60 seconds.

We applied Algorithm 1 to compute a Nash equilibrium of a linear Cournot oligopolistic model with some additional joint constraints on the model’s variables. The precise description of this model is as follows.

There are $n$ firms producing a common homogeneous commodity. Let $x_i$ be the production level of firm $i$, and $x = (x_1, \ldots, x_n)$ the vector of production levels of all these firms. Assume that the production price $p_i$ given by firm $i$ depends on the total quantity $\sigma = \sum_{i=1}^n x_i$ of the commodity as follows

$$p_i(\sigma) = \alpha_i - \delta_i \sigma \quad (\alpha_i > 0, \delta_i > 0, i = 1, \ldots, n).$$

Let $h_i(x_i)$ denote the production cost of firm $i$ when its production level is $x_i$ and assume that the cost functions are affine of the forms

$$h_i(x_i) = \mu_i x_i + \xi_i \quad (\mu_i > 0, \xi_i \geq 0, i = 1, \ldots, n).$$

The profit of firm $i$ is then given by

$$q_i(x_1, \ldots, x_n) = x_i p_i(x_1 + \ldots + x_n) - h_i(x_i) \quad (i = 1, \ldots, n).$$

Each firm $i$ has a strategy set $C_i \subset \mathbb{R}_+$ consisting of its possible production levels, i.e., $x_i \in C_i$. Assume that there are lower and upper bounds on quota of the commodity (i.e., there exist $\sigma, \bar{\sigma} \in \mathbb{R}_+$ such that $\underline{\sigma} \leq x_i \leq \bar{\sigma}$). So the set of feasible production levels can be described by

$$\Omega := \{x \in \mathbb{R}_n^+ \mid x_i \in C_i(i = 1, \ldots, n), \sum_{i=1}^n x_i \in [\underline{\sigma}, \bar{\sigma}]\}.$$  

Each firm $i$ seeks to maximize its profit by choosing the corresponding production level $x_i$ under the presumption that the production of the other firms are parametric input. In this context, a Nash equilibrium point for the model is a point $x^* \in \Omega$ satisfying

$$q_i(x^*[x_i]) \leq q_i(x^*) \quad \forall x \in \Omega, i = 1, \ldots, n,$$

where $x^*[x_i]$ stands for the vector obtained from $x^*$ by replacing the component $x^*_i$ by $x_i$. It means that, if some firm $i$ leaves its equilibrium strategy while the others keep their equilibrium positions, then the profit of firm $i$ does not increase. It has been shown that the unique Nash equilibrium point $x^*$ is also the unique solution to the following equilibrium problem

$$\text{Find } x \in \Omega \text{ such that } f(x, y) := (\bar{B} x + \mu - \alpha)^T (y - x) + \frac{1}{2} y^T B y - \frac{1}{2} x^T B x \geq 0 \quad \forall y \in \Omega, \quad (EP1)$$

where $\mu = (\mu_1, \ldots, \mu_n)^T, \alpha = (\alpha_1, \ldots, \alpha_n)^T$, and

$$\bar{B} = \begin{bmatrix} 0 & \delta_1 & \delta_1 & \cdots & \delta_1 \\ \delta_2 & 0 & \delta_2 & \cdots & \delta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta_n & \delta_n & \delta_n & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2\delta_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\delta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 2\delta_n \end{bmatrix}.$$  

Note that $f(x, y) = f_1(x, y) + f_2(x, y)$ in which

$$f_1(x, y) = (\bar{B} x + \mu - \alpha)^T (y - x),$$

and

$$f_2(x, y) = \frac{1}{2} y^T B y - \frac{1}{2} x^T B x.$$
\[
f_2(x, y) = \frac{1}{2} y^T B y - \frac{1}{2} x^T B x.
\]

It is obvious that \( f, f_1, f_2 \) are equilibrium functions satisfying conditions (A1)-(A3).

For numerical experiments, we set \( C_i = [10, 50] \) for \( i = 1, \ldots, n \), \( \sigma = 10n + 10 \), and \( \tau = 50n - 10 \). The initial guess was set to \( x_i^0 = 30 (i = 1, \ldots, n) \). We tested the algorithm on problem instances with different numbers \( n \) of companies but having the following fixed values of parameters \( \alpha_i = 120, \delta_i = 1, \mu_i = 30 \) for \( i = 1, \ldots, n \). Table 1 reports the outcomes of Algorithm 1 with restart strategy applied to these instances for different values of dimension \( n \) and appropriate values of parameters \( \beta_k \).

| \( n \) | \( \beta_k \) | Total number of iterations | Number of restarts | Number of iterations from the last restart |
|---|---|---|---|---|
| 2 | \( 10/(k + 1) \) | 2 | 0 | 2 |
| 3 | \( 10/(k + 1) \) | 639 | 2 | 9 |
| 4 | \( 10/(k + 1) \) | 911 | 2 | 4 |
| 5 | \( 10/(k + 1) \) | 1027 | 2 | 2 |
| 10 | \( 10/(k + 1) \) | 1201 | 1 | 2 |
| 10 | \( 100/(k + 1) \) | 266 | 1 | 2 |
| 15 | \( 10/(k + 1) \) | 2967 | 2 | 2 |
| 15 | \( 100/(k + 1) \) | 408 | 1 | 2 |
| 20 | \( 10/(k + 1) \) | 5007 | 2 | 2 |
| 20 | \( 100/(k + 1) \) | 539 | 1 | 2 |

Table 1: Performance of Algorithm 1 in solving linear Cournot oligopolistic model with additional joint constraints.

On one hand, the results reported in Table 1 show the applicability of Algorithm 1 for solving linear Cournot-Nash oligopolistic model with joint constraints. On the other hand, it follows from this table that the choice of parameter \( \beta_k \) is crucial for the convergence of the algorithm, since changing the value of this parameter may significantly reduce the number of iterations. Furthermore, the last two columns of Table 1 show that, by applying our suggested restart strategy, we can find ‘good’ starting points from that the algorithm terminated after few iterations.

5 Conclusion

We have proposed splitting algorithms for monotone equilibrium problems where the bifunction is the sum of the two ones. The first algorithm uses an ergodic sequence ensuring convergence without extragradient (double projection). The second one is for paramonotone equilibrium problems ensuring convergence without using the ergodic sequence. A restart strategy has been used to enhance the convergence of the proposed algorithms.

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