A Reconstructing Minimization Approach for Total Variation Based Transient Electromagnetic Inversion

Liting Rao*
School of Electronic Engineering, Xi’an Shiyou University, Xi’an, Shaanxi, 710065, China
*Corresponding author’s e-mail: ltrao@xsyu.edu.cn

Abstract. Transient electromagnetic (TEM) data are conveniently inverted to visualized underground structure with the widely used Occam’s inversion. However, in sedimentary environments, Occam’s inversion performs poorly in reproducing the sharp boundaries. This issue can be overcome by adopting total variation (TV) regularization. The existing computational methods for TV-based regularization are more complex than Occam’s inversion, which limits the application of TV-based regularization in TEM inversion. Here, we develop a reconstructing minimization (RM) approach for TV-based transient electromagnetic inversion. The TV stabilizer is reconstructed as the weighted squared L2-norm of a kernel function. The obtained iterative model update expression of TV-based inversion is extremely close to that of Occam’s inversion, which is rather simple to implement. Synthetic examples validate the efficiency and accuracy of the proposed RM approach.

1. Introduction

Transient electromagnetic (TEM) method is a well-established geophysical tool with applications in mineral exploration, geological mapping, unexploded ordnance detection, etc. [1]. In the TEM survey, a primary electromagnetic (EM) impulse is transmitted to the underground and then a secondary EM field induced by the eddy current can be measured. These secondary field data contain abundant geoelectrical information and we can transform these data into visualized underground structure by means of inversion method [2-3].

As with most geophysical inverse problems, the TEM inverse problem is always ill-posed. In order to reduce the non-uniqueness and stabilize the solution, a priori geologic information can be included in the inversion process and formalized by means of the stabilizer, which is classified as regularization method [4]. Currently, one of the most widely used stabilizers in TEM inversion is the maximum smoothness stabilizer and the corresponding inversion method is what we refer to as Occam’s inversion [5], which is rather simple to implement. However, Occam’s inversion produces smoothed results of subsurface geophysical properties [6], and therefore sometimes it looks geologically unrealistic, especially for sedimentary areas. This issue can be overcome by adopting another stabilizer, called total variation (TV) stabilizer. The TV-based regularization was originally developed to reconstruct noisy images [7], which can produce better quality images for blocky structures than the maximum smoothness stabilizer. The success of TV-based regularization relies on its ability of favouring the bounded variation, without penalizing discontinuities. With the advantage over Occam’s inversion, TV-based regularization should also be widely used in TEM inversion, while the fact is not quite so. Due to the highly nonlinearity of TV stabilizer, the optimization problem of TV-based regularization is theoretically more difficult to solve. So far, compared with Occam’s inversion,
widely recognized computation methods, the fixed-point iteration scheme [7], and the Split-Bregman iterative method [8–9], are still quite complex so that the application of TV-based regularization is limited in TEM data inversion, and, more broadly, in geophysical data inversion.

Here, we proposed a reconstructing minimization (RM) approach for TV-based TEM inversion, which has a computational complexity comparable to Occam’s inversion. We reconstruct the TV stabilizer as the weighted squared L2-norm of a kernel function and linearize the TV stabilizer by fixing its weight value at each iteration. The obtained iterative model update expression of TV-based inversion is extremely close to that of Occam’s inversion. In this sense, the computational complexity of TV-based inversion is comparable to that of Occam’s inversion. Moreover, the reconstruction idea of TV stabilizer also provides a new perspective of better understanding how the stabilizers differ from each other.

In the following text, we first explain the forward modeling and inversion procedure in details. Then we validate the proposed RM approach and compare the capabilities of TV-based inversion against Occam’s inversion by means of synthetic TEM data sets.

2. Formulation

2.1. TEM Forward Modeling
In TEM method, the transmitter first creates a primary time varying EM field, which causes eddy currents in the subsurface conductors, then a secondary EM field is generated by these eddy currents. One can infer the underground electrical distribution by inverting the measured secondary field data. Up to date, basing forward modeling and inversion on the stratified model is still a routine practice. In this letter, we implement our modeling and inversion on layered earth model. The circular loop source is adopted for the transmitter. The system parameters, including the radius of transmitting loop \( a \), the transmitting current \( I \), the height of transmitter \( h_0 \), the height of the receiver \( z \), and the center offset between the transmitter loop and receiver coil \( \rho \) are known in general. The received vertical magnetic field can be calculated by [10]

\[
H_z = \frac{Ia}{2} \int_{0}^{\infty} \frac{\lambda^2}{\mu_0} \left[ e^{-u_i(z+h)} + r_{TE} e^{-u_i(z-h)} \right] J_0(\lambda \rho) J_1(\lambda a) d\lambda
\]  

where \( J_0 \) and \( J_1 \) are Bessel functions of the first kind, \( \lambda \) is the beam of electromagnetic wave, and \( r_{TE} \) is the reflection coefficient which can be iteratively calculated as

\[
r_{TE} = \frac{Y_0 - Y_i}{Y_0 + Y_i}
\]  

where

\[
\hat{Y}_i = Y_i \frac{Y_{i+1} + \tanh(u_i h)}{Y_i + Y_{i+1} \tanh(u_i h)}, \quad i = N - 1, N - 2, \ldots 1
\]  

starts from \( \hat{Y}_N = Y_N \), \( Y_i = u_i / (i\mu_0) \), \( u_i = (\lambda^2 + i\omega \mu_0 \sigma)^{1/2} \).

Fast Hankel and Fourier transformation methods are widely used to calculate the received magnetic field in the frequency domain and transform it to time domain. These methods are not described in detail here and can be referred to [2] and [11].

2.2. Inversion
Introducing specially selected stabilizing functional to narrow the class of models within the correctness set is a simple and efficient way to overcome the ill-posedness of inverse problem. For TEM inversion, the objective functional can be constructed as

\[
P^\alpha(m) = \phi(m) + \alpha s(m)
\]  

(4)
where $m$ represents the model parameters, $\phi(m)$ is the misfit functional that determines the difference between observed data and theoretical calculation, $s(m)$ is a stabilizing functional (or a stabilizer) and $\alpha$ is a regularization parameter.

In this paper the misfit functional is calculated based on least-squares norm, which can be expressed as

$$\phi(m) = \| W_d (F(m) - d) \|_{l_2}^2$$

(5)

where $F(\cdot)$ represents the forward modeling operator, $W_d$ is the data weight matrix, $d$ is the observed TEM data set. Here $F(\cdot)$ represents the forward modeling procedure described in last section and the model parameters consist of each layer’s resistivity and thickness. Note that we usually apply logarithmic data and logarithmic model parameters in TEM inversion, to minimize nonlinearity and to impose positivity [12].

For TV-based inversion, we select the TV stabilizer to constrain the model parameters, which is essentially the $L_1$-norm of the gradient of the model parameters:

$$s_{TV}(m) = \| \nabla m \|_{l_1}$$

(6)

The minimization of TV stabilizer is the key part in solving the optimization problem of objective functional. To deal with the highly nonlinearity of TV term, a lot of research on efficient algorithms for computing optimal of nearly optimal solutions have been conducted. However, compared with widely used Occam’s inversion for TEM data, these recognized methods for TV stabilizer are still quite complex. In order to reduce the computational complexity and make the inversion easy to implement, we propose a reconstructing minimization (RM) approach for TV-based inversion to solve the optimization problem.

First, reconstruct TV stabilizer as the weighted squared $L_2$-norm of a kernel function:

$$s_{TV}(m) = \| W_{TV}(m) k(m) \|_{l_2}^2$$

(7)

where $w_{TV}(m)$ is weight function, $k(m)$ is kernel function, and their specific expressions are as follows:

$$w_{TV}(m) = \frac{\sqrt{\nabla m}}{\sqrt{| \nabla m |^2 + \epsilon^2}}$$

(8)

$$k(m) = \nabla m$$

(9)

where $\epsilon$ is a very small number for reconstruction.

Now the objective functional with reconstructed TV stabilizer can be expressed as:

$$P^o(m) = \| W_d (F(m) - d) \|_{l_2}^2 + \alpha \| W_{TV} (m) \nabla m \|_{l_2}^2$$

(10)

Suppose that $m$ is a vector with length $N$ and $d$ is a vector with length $M$. Rewrite the objective functional using matrix notations, we have

$$P^o(m) = [W_d (F(m) - d)]^T [W_d (F(m) - d)] + \alpha [W_{TV} G m]^T W_{TV} G m$$

(11)

where $W_{TV}$ is the diagonal matrix representation of the weight function, the symbol $T$ denotes a transposed complex conjugate matrix, and $G$ is the matrix representation of gradient operator $\nabla$ which is written as
\[
G = \begin{bmatrix}
0 & 0 \\
-1 & 1 \\
-1 & 1 \\
\vdots \\
0 & -1 & 1
\end{bmatrix}_{N \times N}
\] (12)

Suppose an initial model \( m_0 \) and assume the forward modeling operator is differentiable at \( m_1 = m_0 + \delta m \) for sufficiently small vector \( \delta m \), we have
\[
F(m_1) = F(m_0) + J_0(m_1 - m_0) + o\left(\delta m\right)^2
\] (13)

Where \( J_0 \) is the Jacobian matrix evaluated at the vector \( m_0 \), \( o\left(\delta m\right)^2 \) is the higher order remainder of Taylor expansion.

Then approximate the \( F(m_1) \) by dropping the remainder term and substitute its approximate expression into equation (12). Meanwhile, one can linearize the stabilizer by substituting \( m_0 \) into the weight function to obtain the weight matrix \( W_{TV0} \). Under this approximation and linearization, we have returned to a linear problem of \( m_1 \). Assume \( m_1 \) as the model that minimizes the objective functional, we have
\[
\nabla_{m_0} P^m(m_1) = 0
\] (14)

Then the model vector \( m_1 \) satisfies the equation:
\[
(W_{d}J_0)^T W_d \left[ F(m_0) + J_0(m_1 - m_0) - d \right] + \alpha(W_{TV0}G)^T (W_{TV0}G)m_1 = 0
\] (15)

After some algebraic operations, we get
\[
m_1 = \left( (W_{d}J_0)^T W_d J_0 + \alpha(W_{TV0}G)^T W_{TV0}G \right)^{-1} (W_dJ_0)^T W_d \hat{d}_0
\] (16)
\[
\hat{d}_0 = d + J_0 m_0 - F(m_0)
\] (17)

Through \( n \) steps of iteration,
\[
m_{n+1} = \left( (W_{d}J_n)^T W_d J_n + \alpha(W_{TV0}G)^T W_{TV0}G \right)^{-1} (W_dJ_n)^T W_d \hat{d}_n
\] (18)
\[
\hat{d}_n = d + J_n m_n - F(m_n)
\] (19)

Although the reconstruction seems to make the expression for TV stabilizer more complicated in the first place, it turns out that one can linearize the TV stabilizer by fixing its weight value at each iteration and the iterative model update expression is extremely close to that of Occam’s inversion. The difference between their expressions can be considered as replacing the identity matrix in Occam’s inversion with the diagonal matrix \( W_{TV} \) determined by weight function. In this sense, the computational complexity of TV-based inversion is comparable to that of Occam’s inversion. Moreover, it should be noted that the weight function depends on the gradient of model parameters rather than the model parameters themselves since the core of the minimization of TV stabilizer is the gradient of model parameters, and a false reconstruction of weight function directly using the model parameters leads to wrong results.

To examine whether the updated \( m_n \) meets the requirement, the rms misfit is defined as follow:
\[
RMS = \sqrt{\frac{1}{M} \sum_{n=1}^{M} \left( \frac{F(m_n) - d}{d} \right)^2}
\] (20)

The iteration procedure continues until the rms misfit reaches an acceptable level.
3. Numerical examples
We demonstrate the application of TV-based inversion using the RM approach on the synthetic TEM data sets. In particular, we compare the capabilities of TV-based inversion against Occam’s inversion which is based on maximum smoothness model. The goals are to validate the RM approach and understand the differences between TV-based inversion and traditional Occam’s inversion.

In the following cases, the synthetic data are simulated by means of (1) for a ground-based TEM instrument. We set the transmitting loop with radius 100 m and current strength 20 A. The receiver coil is located at the center of transmitting loop. Noise with Gaussian distribution \((0, 0.01)\text{nV/m}^2\) is added into the synthetic data. For all the inversions, a 100 Ohm\(\text{m}\) uniform half-space is used as a starting model, and the ground in vertical direction is discretized with 50 layers, with the first layer thickness 5m and increasing ratio 1.05. The choice of regularization parameter is significant for results of the inversion procedure. Our selection strategy of regularization parameter is similar with that of Occam’s inversion, which emphasizes more on misfit functional. At each iteration, the ratio of the misfit functional to the stabilizing functional is calculated as the initial value of regularization parameter. The optimal regularization parameter is chosen based on the minimization of rms misfit with a 1-D linear search and the zoom range for regularization parameter is set linearly partitioned in the log10 domain from \(10^{-3}\) to \(10^0\). The target rms misfit is set to 1%.

3.1. 1D inversion
Four synthetic models describing different underground structures are considered. Table 1 shows the model parameters of these models. Their forward responses with noise perturbation are presented in figure 1, which will then be used for inversions. Comparing their responses, we find that the response of model A coincides with that of model C in the early time, while their response curves differ from each other in the late time. This phenomenon also occurs on the response curves of model B and model D. The reason for this phenomenon is that the resistivity values of their shallow layers are set the same and the values of deep layers become different. According to the physical principle of TEM method, the early stage response mainly reflects the property of shallow layer while the late time response is determined by the deep structure.

| Models | Resistivity (\(\Omega\text{m}\)) | Thickness (m) |
|--------|-------------------------------|--------------|
| Model A | [100, 20, 300] | [50, 50, \(\infty\)] |
| Model B | [100, 300, 20, 100] | [50, 100, 100, \(\infty\)] |
| Model C | [100, 20, 300, 10, 100] | [50, 50, 100, 100, \(\infty\)] |
| Model D | [100, 300, 20, 300, 10, 100] | [50, 100, 100, 200, 200, \(\infty\)] |

Figure 1. Responses of four synthetic models, with noises added.
The results of TV-based inversion and Occam’s inversion are shown in figure 2. Iteration times of Occam’s inversion for these four models are 5, 4, 8, and 8, respectively. Iteration times of TV-based inversion are 5, 4, 8, and 9, respectively. Average running time of each iteration for both methods is basically the same. Comparing the inversion results, we find that both inversions retrieve the main features of the true models reasonably well. However, from the specific details of the recovered structures, TV-based inversion, represented by black solid lines, has better quality than Occam’s inversion. The differences between the two inversions are discussed in terms of the boundary location and resistivity value. In the case of Occam’s inversion, the recovered boundaries appear less steep than they actually are due to the slow transition between the high resistivity interface and the low resistivity interface, and the retrieved resistivity presents spurious oscillation around the true resistivity value especially in homogenous top layer and bottom layer. As for TV-based inversion, the boundaries in TV-based inversion are more precisely retrieved which presents the capability to reproduce sharp boundary, and the recovered resistivity values of each layer are better determined and also more homogenously reproduced. As can be seen in all the examples in figure. 2, the spurious oscillation in Occam’s inversion has been almost completely removed by the TV-based inversion. Clearly, the TV-based inversion enhances the sharp features and also eliminates spurious oscillations. Combining the above statements, in terms of efficiency and quality, we have validated the proposed RM approach in solving the optimization problem of TV-based inversion.

Figure 2. Results of TV-based inversion and Occam inversion for different structures.
Besides, in figure 2c and 2d, we notice that the retrieved resistivity level of the middle high resistivity layer is underestimated in both inversion methods. This is because the TEM method is more sensitive to conductive abnormal formation than high resistance formation, meanwhile the middle high resistivity layer locates at relatively deeper area which further aggravates the shielding effect of the conductive layer. In other words, the local underestimation of high resistivity layer results from the insufficient data sensitivity.

3.2. Quasi-2D inversion

Here the considered synthetic model (see figure 3a) is generated by using 35 five-layered models to form a 2D section. As shown in figure 3a, the true model represents a five-layered sedimentary environment. The first layer is characterized by the resistivity of 90 Ωm to 95 Ωm and the thickness about 40 m to 50m. The second layer represents the conductive structure with the resistivity of 18 Ωm to 22 Ωm and the thickness ranging from 30 m to 60 m. The third layer has the resistivity of 80 Ωm to 85 Ωm and the thickness about 100 m. The fourth layer is another conductive structure with the resistivity of 8 Ωm to 9 Ωm and the thickness ranging from 90 m to 140 m. The bottom layer has a relatively high resistivity of 100 Ωm to 105 Ωm.

![Figure 3. True model and results of TV-based inversion and Occam inversion. (a) The true model, (b) Occam’s inversion, (c) TV-based inversion.](image-url)
The inversion results for both methods are shown in figure 3b and 3c. The average iteration times for Occam’s inversion and TV-based inversion are 6.94 and 6.87, respectively, which are basically the same on efficiency. From figure 3b, we find that Occam’s inversion is fairly successful in recovering the overall section: the outline of the anomalous area can be seen, the mean resistivity value of the conductive structure is well reconstructed, the overall resistivity level of the background is slightly overshooting and not homogeneous enough, and the lower boundary is quite blurred especially in the deeper area. Similar to that in figure 2, the Occam’s inversion is prone to have spurious oscillation around the true resistivity value, which results in inhomogeneous reproduced background. In the case of TV-based inversion (see figure 3c), we again find that the recovered section is significantly improved: the location of the boundary is more accurate, and the resistivity values of the conductive structure and the background are better recovered and more homogeneously reproduced. Along the entire section, we can notice the bottom boundary is less clear than the upper boundary and this fuzziness is aggravating in deeper area for both inversion methods. This is normal because the resolution of TEM method will decline along with the increasing of depth essentially. Even if you increase the number of layers for inversion, this phenomenon still exists.

4. Conclusion
In this paper, we have developed a reconstructing minimization (RM) approach for TV-based transient electromagnetic inversion. The computational cost of the RM approach is comparable to that of classical Occam’s inversion, which is simpler to be implemented than previous computational methods for TV-based inversion. Numerical examples have verified the effectiveness of the RM approach and demonstrated the superior capability of TV-based inversion in resolving sharp resistivity changes. Concerning future applications, the proposed RM approach will broaden the application of TV-based regularization in the processing of TEM data or other geophysical data.

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