A COUNTEREXAMPLE RELATED TO THE REGULARITY OF THE $p$-STOKES PROBLEM

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ABSTRACT. In this paper we construct a solenoidal vector field $u$ belonging to $W^{2,q}(\Omega) \cap W^{1,s}_0(\Omega)$, $s \in (1, \infty)$, $q \in (1, n)$, such that $(1 + |Du|)^{p-2}$, $p \in (1, 2) \cup (2, \infty)$, does not belong to the Muckenhoupt class $A_\infty(\Omega)$. Thus, one cannot use the Korn inequality in weighted Lebesgue spaces to prove the natural regularity of the $p$-Stokes problem.

Keywords. Regularity of $p$-Stokes problem, symmetric gradient, boundary regularity, counterexample, Muckenhoupt weights.

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Dedicated to V.V. Zhikov

1. Introduction

The question of full natural regularity for the $p$-Stokes system
\begin{align*}
- \text{div } S(Du) + \nabla \pi &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega,
\end{align*}
(1)
is still not completely solved in the three- and higher-dimensional situation. Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain. The unknowns are the velocity vector field $u = (u_1, \ldots, u_n)^\top$ and the scalar pressure $\pi$, while the external body force $f = (f_1, \ldots, f_n)^\top$ is given. The effective stress tensor $S$ depends only on $Du := \frac{1}{2}(\nabla u + \nabla u^\top)$, the symmetric part of the velocity gradient $\nabla u$. Physical interpretation and discussion of some non-Newtonian fluid models can be found, e.g., in [14], [25], [26]. The relevant example which we have in mind is:
\begin{align*}
S(Du) &= \mu(\delta + |Du|)^{p-2}Du, \quad (2)
\end{align*}
with $p \in (1, \infty)$, $\delta > 0$, and $\mu > 0$. We restrict ourselves to the case $\mu = \delta = 1$. However, the arguments in our paper can be adapted such that the results also hold for $\delta > 0$.

Analogously to the regularity theory of nonlinear elliptic systems (cf. [1], [22], [23]), the natural regularity for solutions of (1) is
\begin{align*}
F(Du) &\in W^{1,2}(\Omega), \quad (3)
\end{align*}
where
\begin{align*}
F(Du) &:= (1 + |Du|)^{\frac{p-2}{2}}Du. \quad (4)
\end{align*}
Note that for $n = 2$ the regularity problem for (1), and even for the $p$-Navier–Stokes problem, is solved (cf. [24]). In the following, we concentrate on the three-dimensional situation, i.e., $n = 3$. Using the standard approach of divided differences, one can show that weak solutions of (1) satisfy $F(Du) \in W^{1,2}_{\text{loc}}(\Omega)$. From this it immediately follows that (3) is fulfilled.
for periodic boundary conditions. Unfortunately, it is the only situation where this optimal result is known. Despite various efforts (cf. [19], [4], [20], [5], [16], [8], [15], [6], [7], [10], [9], [13]), full regularity near the boundary is still an open problem for the $p$-Stokes system (1) completed with zero Dirichlet boundary conditions. Many of the existing papers essentially prove that, under appropriate assumptions on the regularity of the boundary, for some $q < 2$ depending on $p$ there holds

$$\xi \partial_{\nu} F(Du) \in L^2(\Omega), \quad \xi \partial_{\nu} F(Du) \in L^q(\Omega),$$

(5)

where $\partial_{\nu}$ is the local normal derivative, $\partial_\nu$ a local tangential derivative and $\xi \in C_0^\infty(\mathbb{R}^3)$ an appropriate cut-off function with support near the boundary $\partial \Omega$. The precise definition of these quantities using a local description of the boundary as a graph can be found, e.g., in [13]. Note that one can easily derive from (5) that $u \in W^{2,r}(\Omega)$, for some $r$ depending on $p$, i.e., regularity in terms of Sobolev spaces. Some of the above mentioned papers prove the Sobolev space regularity directly. All of these papers also contain certain statements about the regularity of the pressure gradient.

The reason for the non-optimality of the results near the boundary lies in a subtle interplay between the dependence of the elliptic operator on the symmetric part of the velocity gradient and the divergence constraint resulting in the appearance of the pressure gradient. If these difficulties are separated, one can prove better results. If (1) is considered without the divergence constraint and the resulting pressure gradient, it is proved in [29], [12] that (3) holds for $p \in (1, 2)$. If (1) is considered with $S$ depending on the full velocity gradient $\nabla u$, it is proved in [17] that $u \in W^{2,r}(\mathbb{R}^3) \cap W^{1,p}_0(\mathbb{R}^3)$ for some $r > 3$, provided $p < 2$ is very close to 2. On the other hand, it is pointed out in [9], [13] that a Korn-type inequality

$$\int_{\Omega} (1 + |Du|)^{p-2} |\nabla \partial_\nu u|^2 \, dx \leq c \int_{\Omega} (1 + |Du|)^{p-2} |D \partial_\nu u|^2 \, dx$$

(6)

would yield the desired optimal results. However, the validity of (6) is not known. It would be granted if a stronger assertion held true, namely the Korn inequality in the weighted Lebesgue space $L^2_\omega(\Omega)$ with $\omega = (1 + |Du|)^{p-2}$. The validity of the Korn inequality in Lebesgue spaces is proved in [28], and in weighted Lebesgue spaces with Muckenhoupt weights in [18]. In the weighted case, the proof is based on the continuity of the maximal operator in weighted Lebesgue spaces. This, in turn, is equivalent to the condition that the weight is a Muckenhoupt weight (cf. [27]). The purpose of this paper is to construct a solenoidal vector field $u \in W^{2,p}_0(\Omega)$ satisfying (3) but such that $(1 + |Du|)^{p-2}$ does not belong to any Muckenhoupt class for any $p \neq 2$ (cf. Theorem 5).

2. Counterexample

Throughout the paper we use the standard notation for Lebesgue and Sobolev spaces. We write $f \lesssim g$ if there exists a positive constant $c$, depending only on irrelevant parameters, such that $f \leq c g$. Moreover, we write $f \approx g$ if both $f \lesssim g$ and $g \lesssim f$ are satisfied.

Let us start with a technical lemma which loosely follows the ideas of standard covering theorems of Vitali or Besicovitch (cf. [21, Section 1.5]) It will ensure that the constructed vector field will be a counterexample on every open subset of $\Omega$.

**Lemma 1.** Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded or unbounded domain. Then there exists a countable system of nonempty balls $\{E_i\}_{i \in \mathbb{N}}$ contained in $\Omega$ and satisfying the following conditions:
(i) the set of all centers of the balls $E_i$ is dense in $\Omega$;
(ii) $\sup_{t \in \mathbb{N}} \frac{\operatorname{diam} E_{t+1}}{\operatorname{diam} E_t} < \frac{1}{2}$;
(iii) if $k, l \in \mathbb{N}$ satisfy $\lfloor \sqrt{l+1} \rfloor \leq k < l$, then $E_k \cap E_l = \emptyset$.

Proof. Let $\{x^i\}_{i \in \mathbb{N}}$ be a dense sequence of points in $\Omega$. Choose a sufficiently small $\varrho \in (0, \frac{1}{4})$ so that $B(x^1, \varrho) \subset \Omega$ and $\Omega \setminus B(x^1, \varrho) \neq \emptyset$, and define $E_1 := B(x^1, \varrho)$. Next, let $l \in \mathbb{N}$ and suppose that the balls $E_k$ are defined for all $k = 1, \ldots, l$ so that the open set

$$G_l := \Omega \setminus \bigcup_{k=\lfloor \sqrt{l+1} \rfloor}^{l} E_k$$

is not empty. Since $\{x^i\}$ is dense in $\Omega$ and $G_l \subset \Omega$, there exists the minimal index $i \in \mathbb{N}$ such that $x^i \in G_l$ and $x^i$ is not a center of any ball $E_k$, $k = 1, \ldots, l$. The set $G_l$ is open, hence we can choose a sufficiently small number $\delta \in (0, \varrho \operatorname{diam} E_l)$ such that $B(x^i, \delta) \subset G_l$ and $G_l \setminus B(x^i, \delta) \neq \emptyset$. Then we define $E_{l+1} := B(x^i, \delta)$. Notice that

$$G_{l+1} := \Omega \setminus \bigcup_{k=\lfloor \sqrt{l+2} \rfloor}^{l+1} E_k \supset \Omega \setminus \bigcup_{k=\lfloor \sqrt{l+1} \rfloor}^{l+1} E_k = G_l \setminus E_{l+1} \neq \emptyset.$$ 

The process continues by induction, generating the system $\{E_l\}_{l \in \mathbb{N}}$.

From the construction it follows that $\operatorname{diam} E_{l+1} \leq \varrho \operatorname{diam} E_l$ for all $l \in \mathbb{N}$. Since $\varrho < \frac{1}{4}$, this gives the property (ii). As a consequence, we also get $\operatorname{diam} E_l \downarrow 0$ as $l \to \infty$.

In the $(l+1)$-st step of the construction, $E_{l+1}$ is defined so that $E_{l+1} \subset G_l$, hence $\{E_l\}_{l \in \mathbb{N}}$ has the property (iii).

It remains to prove (i). Let $x \in \Omega$ and $\varepsilon > 0$. Thanks to the density of $\{x^i\}_{i \in \mathbb{N}}$ there exists an $i \in \mathbb{N}$ such that $|x - x^i| < \varepsilon$. If $x^i$ is a center of a ball from the system $\{E_l\}$, we are finished.

Suppose that it is not the case. We claim that this means that there exists $l_0 = l_0(i) \in \mathbb{N}$ such that for every $l \geq l_0$ we have $x^i \notin E_l$. Indeed, if $x^i \in E_l$ and $x^i$ is not chosen as the center of $E_{l+1}$, then there is an $x^{m} \in G_l$ with $m < i$ which was not yet chosen as the center of a ball $E_k$ and which becomes the center of $E_{l+1}$. This situation may occur only $(i-1)$ times before $i$ becomes the smallest index such that $x^i$ was not yet used as the center of some $E_k$.

Hence, for all sufficiently large $l \in \mathbb{N}$ we get $x^i \notin G_l$. It yields $x^i \in \overline{E_{k_0}}$ for some $k_0 \geq \lfloor \sqrt{l+1} \rfloor$. This and $\operatorname{diam} E_l \downarrow 0$, imply that for a sufficiently large $l$ we get $|x^i - y| < \varepsilon$, where $y$ is the center of $E_{k_0}$. Therefore, $|x - y| \leq |x - x^i| + |x^i - y| < 2\varepsilon$ and the proof is finished. \hfill \square

The following proposition is a special case of [2, Lemma 2.3].

**Proposition 2.** Assume that $2 \leq n < q < \infty$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain. Let $g \in W^{1-\frac{1}{q}, q} (\partial \Omega)$ be a vector field such that $g \cdot \nu = 0$ on $\partial \Omega$, where $\nu$ denotes the outer unit normal of $\partial \Omega$. Then there exists a vector field $h \in W^{2, q}(\Omega)$ such that $\operatorname{div} h = 0$ in $\Omega$, $\operatorname{Tr}_{\partial \Omega} h \equiv 0$ and $\partial_t h = g$ on $\partial \Omega$.

**Proposition 3.** Assume that $2 \leq n < r < \infty$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ domain. Then there exists a vector field $w : \Omega \to \mathbb{R}^n$ such that $w \in W^{2, r}(\Omega) \cap W^{1, \infty}_0(\Omega)$, both $|\nabla w| \leq 1$ and $\operatorname{div} w = 0$ hold in $\Omega$, and $\operatorname{Tr}_{\partial \Omega} \nabla w \neq 0$.

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\footnote{We use the notation $|z| := \max \{n \in \mathbb{N} \mid n \leq z\}$.}
Proof. At any point $x = (x_1, \ldots, x_n)^T \in \partial \Omega$ define $\varphi(x) := x_1$. Following [3, Examples 4.2.2, 4.2.4], one can verify that $\varphi : \partial \Omega \to \mathbb{R}$ is a $C^{1,1}$ function and as such it satisfies that the gradient vector field $\nabla \varphi \in C^{0,1}(\partial \Omega)$ and $\nabla \varphi \cdot \nu = 0$ on $\partial \Omega$. (Notice that in [3] only the smooth situation for surfaces in $\mathbb{R}^2$ is treated. However, the regularity conditions treated here may be obtained by a careful tracking of the calculations in [3], which can be easily generalized to the general case of a hypersurface in $\mathbb{R}^n$.) Moreover, $\varphi$ is not constant on its domain. (This may be verified using the fact that $\Omega$ is a (nonempty) bounded $C^{1,1}$ domain.) Now we can take $0 \neq g := \nabla \varphi \in C^{0,1}(\partial \Omega)$, and get

$$\|g\|_{W^{1-\frac{1}{r}, r}} = \left(\|g\|_{L^r(\partial \Omega)} + \int_{\partial \Omega} \int_{\partial \Omega} \frac{|g(x) - g(y)|^r}{|x - y|^{r + n - 2}} \, dS(x) \, dS(y)\right)^{\frac{1}{r}} \leq \left(\|g\|_{L^\infty(\partial \Omega)} + C_g \int_{\partial \Omega} \int_{\partial \Omega} \frac{1}{|x - y|^{n - 2}} \, dS(x) \, dS(y)\right)^{\frac{1}{r}}.$$

A standard calculation yields that the expression on the second line is bounded by a constant depending on $\|g\|_{L^\infty(\partial \Omega)}$ and the Lipschitz constant $C_g$ of $g$. Hence, we have shown that $g \in W^{1-\frac{1}{r}, r}(\partial \Omega)$.

Proposition 2 now provides a function $h \in W^{2,r}(\Omega) \cap W^{1,\infty}(\Omega)$ satisfying $\text{div} \, h = 0$ in $\Omega$ and $\partial_{\nu} h = g \neq 0$ on $\partial \Omega$. (We have used the embedding $W^{2,r} \subset W^{1,\infty}$. Then the function $w := \frac{h}{\|\nabla h\|_{\infty}}$ satisfies $|\nabla w| \leq 1$ in $\Omega$. (This indeed can be stated at every point of $\Omega$ since $\nabla w \in W^{1,r}(\Omega)$ admits a continuous representative therein.) Finally, since $\partial_{\nu} w = \frac{g}{\|\nabla w\|_{\infty}} \neq 0$ on $\partial \Omega$, it is verified that $\text{Tr}_{\partial \Omega} \nabla w \neq 0$ on $\partial \Omega$.

Before formulating the main result we recall when a weight $\omega$ belongs to a Muckenhoupt $A^p$ class.

**Definition 4.** Let $\Omega \subset \mathbb{R}^n$, $p \in (1, \infty)$, $p' := \frac{p}{p-1}$ and let $\omega$ be a weight function, i.e., a locally integrable non-negative function on $\mathbb{R}^n$. We say that $\omega$ satisfies the $A_p$ condition on $\Omega$ if

$$\sup_{B \subseteq \Omega} \int_B \omega(x) \, dx \left(\int_B \omega^{-p'}(x) \, dx\right)^{p-1} < \infty,$$

where $B \subseteq \Omega$ are balls. We denote by $A_p(\Omega)$ the class of all weights satisfying this condition. Furthermore, we say that $\omega \in A_\infty(\Omega)$ if $\omega \in \bigcup_{p > 1} A_p(\Omega)$.

**Theorem 5.** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Then there exists a vector field $u : \Omega \to \mathbb{R}^n$ such that:

(i) $u \in W^{2,q}(\Omega) \cap W^{1,\infty}_0(\Omega)$ for any $1 < q < n$ and $1 < s < \infty$;
(ii) $\text{Tr}_{\partial \Omega} \nabla u \neq 0$;
(iii) $\text{div} \, u \equiv 0$ in $\Omega$;
(iv) $(1 + |Du|)^{p-2} \notin A_\infty(\Omega_0)$ for any $p \in (1, \infty)$, $p \neq 2$ and any open set $\Omega_0 \subset \Omega$.

**Remark 6.** In the assertion (iv), the function $(1 + |Du|)^{p-2}$ is understood as extended by zero to the whole $\mathbb{R}^n$ so that it corresponds to the formal definition of a weight. Notice that we intentionally prove the stronger property $(1 + |Du|)^{p-2} \notin A_\infty(\Omega_0)$ for any open

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2 The gradient vector field $\nabla \varphi$ of a function $\varphi : \partial \Omega \to \mathbb{R}$ is, at each point $x \in \partial \Omega$, the Riesz representative of the differential $d_x \varphi$ of $\varphi$ at $x$ with respect to the standard scalar product in $\mathbb{R}^3$ restricted to the tangent plane $T_x \partial \Omega$. 
Ω₀ ⊂ Ω, to exclude the possibility that a localisation of the weight near the boundary would be a Muckenhoupt weight, which in turn would imply the validity of a localized version of (6).

Proof of Theorem 5. By Lemma 1, we find a system of balls {Eₖ}ₖ∈ℕ, Eₖ = B(xᵏ, 2rₖ) such that \( \bigcup_{k \in \mathbb{N}} E_k \subset \Omega \); \( \{x^k\}_{k \in \mathbb{N}} \) is dense in \( \Omega \); there exists an \( \varrho \in (0, 1/2) \) such that \( r_{k+1} \leq \varrho r_k \) for all \( k \in \mathbb{N} \); if \( k, l \in \mathbb{N} \) satisfy \( \lfloor \sqrt{7} \rfloor \leq k < l \), then \( E_k \cap E_l = \emptyset \). Without loss of generality we can suppose that \( r_1 \leq \varrho \). Let us denote \( B_k := B(x^k, r_k) \) for each \( k \in \mathbb{N} \), and consider that each point \( x^k \) is represented by \( x^k = (x^k_1, \ldots, x^k_n)^T \).

For every \( k \in \mathbb{N} \), \( 3 \leq m \leq n \) and \( x \in \Omega \) set

\[
\begin{align*}
    u^k_1(x) &= \frac{k}{r_k} \chi_{B_k}(x)(x_2 - x^k_2) \left( \frac{r_k}{2} + \frac{|x - x^k|^2}{2r_k} - |x - x^k| \right), \\
    u^k_2(x) &= -\frac{k}{r_k} \chi_{B_k}(x)(x_1 - x^k_1) \left( \frac{r_k}{2} + \frac{|x - x^k|^2}{2r_k} - |x - x^k| \right), \\
    u^k_m(x) &= 0,
\end{align*}
\]

and define the vector field \( u^k : \overline{\Omega} \to \mathbb{R}^n \) with compact support in \( \Omega \) by \( u^k := (u^k_1, \ldots, u^k_n)^T \).

Let \( k \in \mathbb{N} \), and \( i, j \in \{1, \ldots, n\} \). The classical partial derivatives of \( u^k \) can be computed as

\[
\begin{align*}
    \partial_i u^k_1(x) &= \frac{k}{r_k} \chi_{B_k}(x) \left[ \delta_{2i} \left( \frac{r_k}{2} + \frac{|x - x^k|^2}{2r_k} - |x - x^k| \right) \\
    &\quad + (x_2 - x^k_2)(x_i - x^k_i) \left( \frac{1}{r_k} - \frac{1}{|x - x^k|} \right) \right], \quad x \in \Omega \setminus \{x^k\}, \\
    \partial_i u^k_2(x) &= -\frac{k}{r_k} \chi_{B_k}(x) \left[ \delta_{1i} \left( \frac{r_k}{2} + \frac{|x - x^k|^2}{2r_k} - |x - x^k| \right) \\
    &\quad + (x_1 - x^k_1)(x_i - x^k_i) \left( \frac{1}{r_k} - \frac{1}{|x - x^k|} \right) \right], \quad x \in \Omega \setminus \{x^k\}, \\
    \partial_j \partial_i u^k_1(x) &= \frac{k}{r_k} \chi_{B_k}(x) \left[ \left( \delta_{2i} x_j - \delta_{2j} x_i \right) \delta_{2i} x_j + \delta_{2j} x_i - \delta_{ji} x_2 \right] \left( \frac{1}{r_k} - \frac{1}{|x - x^k|} \right) \\
    &\quad + \frac{(x_2 - x^k_2)(x_i - x^k_i)(x_j - x^k_j)}{|x - x^k|^3}, \quad x \in \Omega \setminus \{x^k \} \cup \partial B_k, \\
    \partial_j \partial_i u^k_2(x) &= -\frac{k}{r_k} \chi_{B_k}(x) \left[ \left( \delta_{1i} x_j - \delta_{1j} x_i \right) \delta_{1i} x_j + \delta_{1j} x_i - \delta_{ji} x_1 \right] \left( \frac{1}{r_k} - \frac{1}{|x - x^k|} \right) \\
    &\quad + \frac{(x_1 - x^k_1)(x_i - x^k_i)(x_j - x^k_j)}{|x - x^k|^3}, \quad x \in \Omega \setminus \{x^k \} \cup \partial B_k, \\
    \partial_i u^k_m(x) &= \delta_j \partial_i u^k_m(x) \equiv 0, \quad 3 \leq m \leq n, \quad x \in \Omega.
\end{align*}
\]

Let \( i, j, m \in \{1, \ldots, n\} \). The function \( u^k_m \) is continuous in \( \overline{\Omega} \), the function \( \partial_i u^k_m \) is continuous in \( \Omega \setminus \{x^k\} \), and \( \partial_j \partial_i u^k_m \) is continuous in \( \Omega \setminus \{x^k \} \cup \partial B_k \). Recalling that if \( x \in B_k \), then
\[ |x_i - x_i^k| \leq |x - x^k| \leq r_k, \] we obtain the following estimates:

\[ |u_m^k(x)| \leq k r_k \chi_{B_k}(x), \quad x \in \Omega, \tag{7} \]
\[ |\partial_i u_m^k(x)| \leq 2k \chi_{B_k}(x), \quad x \in \Omega \setminus \{x^k\}, \tag{8} \]
\[ |\partial_j \partial_i u_m^k(x)| \leq \frac{4k}{r_k} \chi_{B_k}(x), \quad x \in \Omega \setminus (\{x^k\} \cup \partial B_k). \tag{9} \]

The function \( \partial_i u_m^k \) belongs to \( L^\infty(\Omega) \). Hence, using a classical argument, based on the approximation of \( \Omega \) by

\[ \Omega \setminus B(x^k, \varepsilon), \quad \varepsilon \to 0_+, \]
and the continuity of \( u_m^k \) in \( \Omega \), one can show that \( \partial_i u_m^k \) is the weak derivative of \( u_m^k \) on \( \Omega \). Next, also \( \partial_j \partial_i u_m^k \) belongs to \( L^\infty(\Omega) \). Again, an argument using the approximation of \( \Omega \) by

\[ \Omega \setminus (B(x^k, \varepsilon) \cup \{x \in \Omega \mid r_k - \varepsilon \leq |x - x^k| \leq r_k + \varepsilon\}), \quad \varepsilon \to 0_+, \]
and the continuity of \( \partial_i u_m^k \) in \( \Omega \setminus \{x^k\} \), shows that the function \( \partial_j \partial_i u_m^k \) is the second-order weak derivative of \( u_m^k \) on \( \Omega \).

Let \( l \in \mathbb{N} \). Using (7) and the assumption \( r_1 \leq \varrho \), we get

\[ \sum_{k=1}^{l} \sup_{x \in \Omega} |u_m^k(x)| \leq \sum_{k=1}^{l} k r_k \leq \sum_{k=1}^{l} k g_k < \sum_{k \in \mathbb{N}} k g_k < \infty. \]

Hence, there exists a function \( u^0 \in C(\Omega) \) such that \( \sum_{k=1}^{l} u^k \to u^0 \) absolutely uniformly on \( \Omega \) as \( l \to \infty \).

Let \( s > 1 \). Applying (8), we have

\[ \sum_{k=1}^{l} \left( \int_{\Omega} |\partial_i u_m^k(x)|^s \, dx \right)^\frac{1}{s} \leq 2 \sum_{k=1}^{l} \left( \int_{\Omega} \chi_{B_k}(x) \, dx \right)^\frac{1}{s} \]
\[ = 2 \sum_{k=1}^{l} k |B_k|^{\frac{1}{s}} \leq 2 \sum_{k=1}^{l} k r_k^{\frac{n}{s}} \leq \sum_{k \in \mathbb{N}} k g_k^{\frac{n}{s}} < \infty. \]

For every \( l \in \mathbb{N} \) we have \( \sum_{k=1}^{l} u^k \in W^{1,s}_0(\Omega) \), and

\[ \sum_{k \in \mathbb{N}} \|u^k\|_{W^{1,s}_0(\Omega)} \approx \sum_{k \in \mathbb{N}} \|\nabla u^k\|_{L^s(\Omega)} \approx \sum_{k \in \mathbb{N}} k g_k^{\frac{n}{s}} < \infty. \]

Therefore, \( \{ \sum_{k=1}^{l} u^k \}_{l \in \mathbb{N}} \) is a Cauchy sequence in \( W^{1,s}_0(\Omega) \). Since \( \sum_{k=1}^{l} u^k \overset{l \to \infty}{\to} u^0 \) point-wise in \( \Omega \), it is also true that \( \sum_{k=1}^{l} u^k \overset{l \to \infty}{\to} u^0 \) in \( W^{1,s}_0(\Omega) \). Moreover, \( \text{div} u^0 = 0 \) since \( \text{div} \sum_{k=1}^{l} u^k \equiv 0 \) for all \( l \in \mathbb{N} \).

Next, let \( q \in (1, n) \). Using (9), we get

\[ \sum_{k=1}^{l} \left( \int_{\Omega} |\partial_j \partial_i u_m^k(x)|^q \, dx \right)^\frac{1}{q} \leq 4 \sum_{k=1}^{l} r_k \left( \int_{\Omega} \chi_{B_k}(x) \, dx \right)^\frac{1}{q} = 4 \sum_{k=1}^{l} \frac{k |B_k|^{\frac{1}{q}}}{r_k} \]
\[ \leq \sum_{k=1}^{l} kr_k^{\frac{n}{q}-1} \leq \sum_{k=1}^{l} k g_k \left( \frac{n}{q} - 1 \right) k < \sum_{k \in \mathbb{N}} k g_k \left( \frac{n}{q} - 1 \right) k < \infty. \]
For every \( l \in \mathbb{N} \) one has \( \sum_{k=1}^{l} u^k \in W^{2,q}_0(\Omega) \), and
\[
\sum_{k=\mathbb{N}} \| u^k \|_{W^{2,q}_0(\Omega)} \approx \sum_{k=\mathbb{N}} \| \nabla^2 u^k \|_{L^q(\Omega)} \lesssim \sum_{k=\mathbb{N}} k \rho \left( \frac{\rho}{q} - 1 \right)^k < \infty.
\]

It implies that \( \{ \sum_{k=1}^{l} u^k \}_{l \in \mathbb{N}} \) is a Cauchy sequence in \( W^{2,q}_0(\Omega) \). Taking the pointwise convergence into account, we obtain that \( \sum_{k=1}^{l} u^k \xrightarrow{l \to \infty} u^0 \) in \( W^{2,q}_0(\Omega) \).

We have shown that \( u^0 \in C(\mathbb{R}) \cap W^{1,s}_0(\Omega) \cap W^{2,q}_0(\Omega) \) for any \( s \in (1, \infty) \) and \( q \in (1, n) \). Now consider the field \( w \in W^{2,r}(\Omega) \cap W^{1,\infty}_0(\Omega) \subset W^{2,q}(\Omega) \cap W^{1,\infty}_0(\Omega) \) (where \( r > n \)) granted by Proposition 3, and define \( u := u^0 + w \). Obviously, the field \( u \) satisfies (i), (ii) and (iii).

It remains to verify (iv). There exists a constant \( \varepsilon \in (0, 1) \) dependent only on \( n \) and such that for any \( l \in \mathbb{N} \) one has
\[
|G_l| \geq \frac{2}{3} |B_l|,
\]
where
\[
G_l := \left\{ x \in B_l \mid |x_1 - x_1'| \geq r_l \varepsilon, |x_2 - x_2'| \geq r_l \varepsilon, |x - x'| \leq r_l (1 - \varepsilon) \right\}.
\]

For each \( l \in \mathbb{N} \) define \( M_l := G_l \setminus \bigcup_{k=l+1}^{\infty} B_k \). Recall that \( u^0 = \sum_{k=\mathbb{N}} u^k \) (convergence in \( W^{1,s} \)) and \( u^k = u^k \chi_{B_k} \) for every \( k \in \mathbb{N} \). Hence,
\[
Du^0(x) = \sum_{k=1}^{l} Du^k(x) \chi_{B_k}(x) \quad \text{for a.e. } x \in \Omega \setminus \bigcup_{k=l+1}^{\infty} B_k.
\]

Let \( l \in \mathbb{N}, l \geq 2 \). From the properties of the system \( \{ E_k \} \) it follows that \( B_k \cap M_l \subset E_k \cap E_l = \emptyset \) for all \( k, l \in \mathbb{N} \) such that \( \sqrt[3]{l} < k < l \). Together with (10) it gives
\[
Du^0(x) = Du'(x) + \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} Du^k(x) \quad \text{for a.e. } x \in M_l.
\]

Therefore, we may write
\[
|Du^0(x)| = |Du'(x) + \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} Du^k(x)| \geq |Du'(x)| - \left| \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} Du^k(x) \right|
\geq |Du'(x)| - \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} |\nabla u^k(x)| \geq |\partial_1 u_1'(x)| - n \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} \max_{1 \leq i, j \leq n} |\partial_i u_j^k(x)|.
\]

for a.e. \( x \in M_l \). Since \( x \in M_l \subset G_l \), we may write
\[
|\partial_1 u_1'(x)| = \frac{l}{r_l} |x_2 - x_2'| |x_1 - x_1'| \left( \frac{1}{|x_1 - x_1'|} - \frac{1}{r_l} \right) \geq \varepsilon^2 l r_l \left( \frac{1}{n(1 - \varepsilon)} - \frac{1}{r_l} \right) \geq \frac{\varepsilon^3 l}{1 - \varepsilon}.
\]

Using this estimate and (8), we obtain
\[
|Du_1'(x)| \geq |\partial_1 u_1'(x)| - n \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} \max_{1 \leq i, j \leq n} |\partial_i u_j^k(x)| \geq \frac{\varepsilon^3 l}{1 - \varepsilon} - 2n \sum_{k=1}^{\left\lfloor \sqrt[3]{l} \right\rfloor} k \geq \frac{\varepsilon^3 l}{1 - \varepsilon} - 2nl^{\frac{2}{3}}
\]
(12)
for a.e. \( x \in M_l \). Moreover, since \( q^n < 2^{-n} \leq \frac{1}{3} \), we have

\[
\left| \bigcup_{k=l+1}^{\infty} B_k \right| \leq \sum_{k=l+1}^{\infty} \frac{|B_k|}{|B_l|} = \sum_{k=l+1}^{\infty} q^n(k-l) \leq \frac{1}{3}.
\]

This in turn yields

\[
|M_l| \geq |G_l| - \left| \bigcup_{k=l+1}^{\infty} B_k \right| \geq \frac{1}{3} |B_l| = \frac{1}{3 \cdot 2^n} |E_l|.
\]

Similarly, we can also derive the estimate \( \left| \bigcup_{k=l}^{\infty} B_k \right| \leq \frac{4}{3} |B_l| \), from which it follows that

\[
\left| E_l \setminus \bigcup_{k=l}^{\infty} B_k \right| \geq |E_l| - \frac{4}{3} |B_l| = \left( 1 - \frac{1}{3 \cdot 2^{n-2}} \right) |E_l|. \tag{13}
\]

Since \( u^l = u^l \chi_{B_l} \), by using (11) we obtain the estimate

\[
|Du^0(x)| = \sum_{k=1}^{\hat{M}} |Du^k(x)| \leq n \sum_{k=1}^{\hat{M}} \max_{1 \leq i,j \leq n} |\partial_i u^k_j(x)| \leq 2n \sum_{k=1}^{\hat{M}} k \leq 2n l^{\frac{2}{3}}, \tag{14}
\]

valid for a.e. \( x \in E_l \setminus \bigcup_{k=l}^{\infty} B_k \). Furthermore, considering that \( |\nabla w| \leq 1 \) a.e. in \( \Omega \), we also have the relation

\[
1 + |Du| = 1 + |Du^0 + Dw| \geq 1 + |Du^0| - |Dw| \geq 1 + |Du^0| - |\nabla w| \geq |Du^0|
\]
a.e. in \( \Omega \). Now we have prepared everything to show that \((1 + |Du|)^{p-2}\) does not satisfy the \( A_\alpha \) condition for any \( \alpha > 1 \). First, for \( p > 2 \) we have

\[
\int_{E_l} (1 + |Du(x)|)^{p-2} \, dx \geq \int_{E_l} |Du^0(x)|^{p-2} \, dx
\]

\[
\geq \int_{M_l} |Du^0(x)|^{p-2} \, dx
\]

\[
\geq \left( \frac{\varepsilon^3 l}{1 - \varepsilon} - 2nl^{\frac{2}{3}} \right)^{p-2} |M_l|
\]

\[
\geq \frac{1}{3 \cdot 2^n} \left( \frac{\varepsilon^3 l}{1 - \varepsilon} - 2nl^{\frac{2}{3}} \right)^{p-2} |E_l|.
\]

Hence,

\[
\int_{E_l} (1 + |Du(x)|)^{p-2} \, dx \geq C_1(C_2 l - l^{\frac{2}{3}})^{p-2}, \tag{15}
\]
where $C_1, C_2$ are positive constants depending only on $n$ and $p$. Next, applying (14) and (15) we obtain for $\alpha > 1$

$$\int_{E_l} (1 + |Du(x)|)^{(p-2)(1-\alpha')} \, dx \geq \int_{E_l \setminus \bigcup_{k=l}^{\infty} B_k} (1 + |Du(x)|)^{(p-2)(1-\alpha')} \, dx$$

$$\geq \int_{E_l \setminus \bigcup_{k=l}^{\infty} B_k} (1 + |\nabla w(x)| + |Du^0(x)|)^{(p-2)(1-\alpha')} \, dx$$

$$\geq (2 + 2n l^{\frac{2}{\alpha'}})^{(p-2)(1-\alpha')} |E_l \setminus \bigcup_{k=l}^{\infty} B_k|$$

Thus,

$$\left( \int_{E_l} (1 + |Du(x)|)^{(p-2)(1-\alpha')} \, dx \right)^{\alpha-1} \geq C_3^{\alpha-1} (C_4 + l^{\frac{2}{\alpha'}})^{p-2}$$

where $C_3, C_4$ are positive constants depending only on $n$ and $p$. By combining the obtained estimates we get

$$\int_{E_l} (1 + |Du(x)|)^{p-2} \, dx \left( \int_{E_l} (1 + |Du(x)|)^{(p-2)(1-\alpha')} \, dx \right)^{\alpha-1} \geq \frac{C_1}{C_3^{\alpha-1}} \left( \frac{C_2 l - l^{\frac{2}{\alpha'}}}{C_4 + l^{\frac{2}{\alpha'}}} \right)^{p-2} \xrightarrow{l \to \infty} \infty.$$  \hspace{1cm} (16)

Let $\Omega_0 \subset \Omega$ be an open set. Since the set of centers of the balls $E_l$ is dense in $\Omega$ and $\text{diam } E_l \downarrow 0$, there exists an infinite subsequence $\{E_{l_k}\}_{k \in \mathbb{N}}$ such that $E_{l_k} \subset \Omega_0$ for all $k \in \mathbb{N}$. The property (16) then implies that the function $(1 + |Du|)^{p-2}$ does not satisfy the $A_\alpha$ condition on $\Omega_0$ for any $p > 2$ and $\alpha > 1$.

Now, let $p \in (1, 2)$ and $\alpha > 1$. Define $p_0 := \frac{p-2}{1-\alpha} + 2$. Since $p_0 > 2$ now, from (16) we get

$$\int_{E_l} (1 + |Du(x)|)^{p-2} \, dx \left( \int_{E_l} (1 + |Du(x)|)^{(p-2)(1-\alpha')} \, dx \right)^{\alpha-1} \geq \left( \int_{E_l} (1 + |Du(x)|)^{(p_0-2)(1-\alpha')} \, dx \right)^{\alpha-1} \xrightarrow{l \to \infty} \infty.$$  \hspace{1cm} (17)

By the same argument as in the previous part of the proof, we conclude that the function $(1 + |Du|)^{p-2}$ does not belong to the $A_{\alpha'}(\Omega_0)$ class for any $p \in (1, 2)$, $\alpha > 1$ and any open set $\Omega_0 \subset \Omega$. This completes the proof. \hfill \Box

Remark 7. Estimates (12) and (14) should clarify the role of the bound $|\nabla^3 f|$ from Lemma 1. This bound was chosen with the purpose of making the right-hand side in the final estimate (16) divergent.

Remark 8. Notice that the function $u$ from Theorem 5 satisfies (3). One easily sees that $F(Du) \in L^2(\Omega)$ is equivalent to $u \in W^{1,p}(\Omega)$. This is satisfied since $u \in W^{1,s}(\Omega)$, $1 < s < \infty$. Moreover, one can show (cf. [11]) that

$$|\nabla F(Du)|^2 \approx (1 + |Du|)^{p-2} |\nabla Du|^2 \approx (1 + |Du|)^{p-2} |\nabla^2 u|^2,$$

where we also used the algebraic identity

$$\partial_j \partial_k v^l = \partial_j D_{ik} v + \partial_k D_{ij} v - \partial_i D_{jk} v,$$
valid for sufficiently smooth vector fields $v$. Thus, it is sufficient to verify that
\[ \int_{\Omega} (1 + |Du|)^{p-2} |\nabla^2 u|^2 \, dx < \infty \]
for any $p \in (1, \infty)$. If $1 < p \leq 2$, then
\[ \int_{\Omega} (1 + |Du|)^{p-2} |\nabla^2 u|^2 \, dx \leq \int_{\Omega} |\nabla^2 u|^2 \, dx < \infty, \]
since $\nabla^2 u \in L^q(\Omega)$ for any $1 < q < n$, in particular $q = 2$. If $p > 2$, then
\[ \int_{\Omega} (1 + |Du|)^{p-2} |\nabla^2 u|^2 \, dx \leq 2^p \int_{\Omega} |\nabla^2 u|^2 \, dx + 2^p \left( \int_{\Omega} |\nabla^2 u|^{\frac{q}{2}} \, dx \right) \frac{4}{5} \left( \int_{\Omega} |Du|^{5(p-2)} \, dx \right)^{\frac{1}{5}} < \infty, \]
since $u \in W^{2,q}(\Omega)$ for any $1 < q < n$. More precisely, one can choose the $q \in \left[ \frac{5}{2}, n \right)$ so that
\[ \frac{2q}{n-q} \geq 5(p-2) \] and thus $\nabla^2 u \in L^{\frac{5}{2}}(\Omega)$ and $Du \in W^{1,q}(\Omega) \subset L^{\frac{5q}{n-q}}(\Omega) \subset L^{5(p-2)}(\Omega)$.

Remark 9. Let $u$ be the vector field constructed in Theorem 5 in the case $n = 3$. Setting $f := -\text{div} S(Du)$, where $S$ is given by (2) with $\delta = 1$, we see that $u$ and $\pi \equiv 0$ are a weak solution of the $p$-Stokes system (1) with zero Dirichlet boundary condition for any $p \in (1, \infty)$. From Remark 8 we know that $u$ possesses the natural regularity (3). Computations similar to the ones in Remark 8 show that $f \in L^2(\Omega)$ for any $p \in (1, \infty)$. For $p \geq 2$, this regularity of the right-hand side $f$ is exactly the requirement posed in the papers dealing with the regularity of (1) which were mentioned in the introduction. For $p \in (1, 2]$, the usual requirement on the right-hand side of (1) in the regularity investigations is $f \in L^{\frac{5}{2}}(\Omega)$. This is fulfilled for the above defined $f$ for $p \in \left( \frac{3}{2}, 2 \right]$.

Remark 10. Using the same notation as in the previous remark, one sees that $u$ is for any $p \in (1, \infty)$ also a solution of (1) without the divergence constraint and the resulting pressure gradient, but equipped with zero Dirichlet boundary condition. From Remark 8 we know that $u$ possesses the natural regularity (3). However, for this modified problem this regularity property is proved in [29], [12] for any weak solution and any right-hand side $f \in L^{\frac{5}{2}}(\Omega)$.

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