CURVATURE PROPERTIES OF VAIHYA METRIC

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Dedicated to Professor Prahalad Chunnilal Vaidya on his 100th birthday

Abstract. As a generalization of the Schwarzschild solution \cite{22}, in \cite{72} Vaidya presented a radiating metric to develop a model of the exterior of a star including its radiation field, called Vaidya metric. The present paper deals with the investigation on the curvature properties of Vaidya metric. It is shown that Vaidya metric can be considered as a model of different pseudosymmetric type curvature conditions, namely, \( C \cdot C = \frac{1}{r^3}Q(g,C) \), \( R \cdot R - Q(S,R) = \frac{1}{r^3}Q(g,C) \) etc. It is also shown that Vaidya metric is Ricci simple, vanishing scalar curvature and its Ricci tensor is Riemann-compatible. As a special case of the main result, we obtain the curvature properties of Schwarzschild metric. Finally, we compare the curvature properties of Vaidya metric with another radiating metric, namely, Ludwig-Edgar pure radiation metric (\cite{27}, \cite{57}).

1. Introduction

Let \( M \) be a semi-Riemannian manifold of dimension \( n \) endowed with a semi-Riemannian metric \( g \) and let \( \nabla, R, S \) and \( \kappa \) be respectively the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of \( M \). The additional restriction(s) on the curvature tensor(s) of a specific manifold help us to realize the geometry of that manifold. For example, if on a manifold \( R = 0 \) (resp., \( \nabla R = 0 \)) then it is flat (resp., locally symmetric). Hence it is very important to investigate the curvature restricted geometric structures on a manifold.

In the literature of differential geometry we find various generalizations of locally symmetric manifolds (\cite{5}, \cite{6}) in several directions, such as recurrent manifold by Ruse (\cite{40}, \cite{41}, \cite{42}, \cite{74}), some generalized recurrent manifolds by Shaikh and his coauthors (\cite{19}, \cite{59}, \cite{58}, \cite{60}, \cite{44}, \cite{62}), semisymmetric manifolds by Cartan (\cite{7}) and Sinyukov (\cite{67}, \cite{68}) (which were latter classified by Szabó (\cite{63}, \cite{64}, \cite{65})), generalized semisymmetric manifold by Mikeš (\cite{34}, \cite{35}, \cite{36}, \cite{37}, \cite{38}, \cite{39}), pseudosymmetric manifolds by Chaki (\cite{8}), pseudosymmetric manifolds by Deszcz (\cite{1}, \cite{11}, \cite{16}), weakly symmetric manifolds by Tamássy and Binh (\cite{70}), manifolds of recurrent curvature 2-forms (\cite{3}, \cite{26}) etc.

In his theory of general relativity, Einstein made a bridge between the geometrical and physical quantities of a spacetime (a connected 4-dimensional Lorentzian manifold) by presenting the famous field equation

\[
S - \frac{\kappa}{2}g + \Lambda g = \frac{8\pi G}{c^4} T,
\]

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where $\Lambda$ is the cosmological constant, $G$ is Newton’s gravitational constant, $c$ is the speed of light in vacuum and $T$ is the energy-momentum tensor. For instance, a spacetime represent perfect fluid if and only if it is quasi-Einstein, i.e., its Ricci tensor satisfies the condition $S = \alpha g + \beta \Pi \otimes \Pi$, where $\alpha, \beta$ are some scalars and $\Pi$ is an 1-form \cite{18}. Therefore the study of curvature restricted geometric structures become essential both physically and geometrically in the investigation of a spacetime.

According to Birkhoff’s theorem \cite{1}, the vacuum spherically symmetric spacetime with zero cosmological constant is the asymptotically flat solution given by Schwarzschild \cite{22}

$$ds^2 = -(1 - \frac{2m}{r}) du^2 - 2drdu + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (1.1)$$

where $m$ is an arbitrary parameter. To develop a model of the exterior of a star, which includes its radiation field, there arose an important generalization of the Schwarzschild solution by Vaidya \cite{72}. In terms of $(r, t, \theta, \phi)$-coordinates Vaidya metric \cite{72} can be written as

$$ds^2 = \left(1 - \frac{2m}{r}\right)^{-1} \left[dr^2 - \frac{(m_r)^2}{(m_r)^2}dt^2\right] + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $m = m(r, t)$. In fact, the above metric can be expressed in a much more useful form (for outgoing radiation) by introducing a null coordinate $u$, as follows:

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 - 2drdu + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (1.2)$$

If $m(u) = constant$, then the Vaidya metric \cite{1.2} reduces to the Schwarzschild metric \cite{1.1}. Hence \cite{1.2} is a generalization of the Schwarzschild metric \cite{1.1}. In this case, $m(u)$ is an arbitrary non-increasing function of the retarded null coordinate $u$.

The main object of the present paper is to investigate the geometric structures admitted by the Vaidya metric \cite{1.2}. It is interesting to note that such a metric admits several important geometric structures, such as, it is Ricci simple, manifold of vanishing scalar curvature, its Ricci tensor is Riemann-compatible and its conformal curvature 2-forms are recurrent. Moreover Vaidya metric satisfies various pseudosymmetric type curvature conditions, such as pseudosymmetric Weyl conformal curvature tensor, the difference tensor $R \cdot R - Q(S, R)$ is linearly dependent with $Q(g, C)$. As a special case of the main result, we obtain the curvature properties of Schwarzschild metric. It is shown that Schwarzschild metric is of harmonic curvature and pseudosymmetric in the sense of Deszcz.

The paper is organized as follows. Section 2 deals with the different geometric structures due to various curvature tensors. Section 3 is concerned with the calculations of components of various tensors of Vaidya spacetime. Section 4 is devoted to the conclusion on the geometric structures admitted by Vaidya metric and Schwarzschild metric. Finally, in section 5, we compare the curvature properties of Vaidya metric with Ludwig-Edgar pure radiation metric.
2. Curvature Restricted Geometric Structures

A curvature restricted geometric structure on a semi-Riemannian manifold $M$ is a geometric structure obtained by imposing some restriction(s) on some curvature tensor(s) of $M$ by means of covariant derivatives of first order or higher orders. In this section we will explain some useful notations and definitions of various curvature restricted geometric structures.

Let $A$ and $E$ be two symmetric $(0,2)$-tensors. The Kulkarni-Nomizu product $A \wedge E$ of $A$ and $E$ is defined by (see e.g. [15], [20])

$$
(A \wedge E)(X_1, X_2, X_3, X_4) = A(X_1, X_4)E(X_2, X_3) + A(X_2, X_3)E(X_1, X_4)
- A(X_1, X_3)E(X_2, X_4) - A(X_2, X_4)E(X_1, X_3),
$$

where $X_1, X_2, X_3, X_4 \in \chi(M)$, the Lie algebra of all smooth vector fields on $M$. Throughout the paper we will consider $X, Y, X_1, X_2, \ldots \in \chi(M)$. Again for a symmetric $(0,2)$-tensor $E$, we can define two endomorphisms $E$ and $(X \wedge E)Y$ as ([7], [15])

$$
g(E X_1, X_2) = E(X_1, X_2) \quad \text{and} \quad (X \wedge E)Y X_1 = E(Y, X_1)X - E(X, Y).$$

Now in terms of Kulkarni-Nomizu product $\wedge$ and the endomorphism $(X \wedge E)Y$, the Gaussian curvature tensor $\mathfrak{S}$, the projective curvature tensor $P$, conharmonic curvature tensor $K$, concircular curvature tensor $W$ and the conformal curvature tensor $C$ are given by ([23], [73])

$$
\mathfrak{S} = \wedge g, \quad P = R - \frac{1}{n - 1}(\wedge S), \quad K = R - \frac{1}{n - 2}(g \wedge S),
W = R - \frac{r}{n(n - 1)}\mathfrak{S}, \quad C = K + \frac{r}{(n - 2)(n - 1)}\mathfrak{S}.
$$

For a $(0,4)$-tensor $D$ we can define the corresponding $(1,3)$-tensor $\mathcal{D}$ and the endomorphism $\mathcal{D}(X, Y)$ as follows:

$$
\mathcal{D}(X_1, X_2, X_3, X_4) = g(\mathcal{D}(X_1, X_2)X_3, X_4), \quad \mathcal{D}(X, Y)X_1 = \mathcal{D}(X, Y)X_1.
$$

Now operating $\mathcal{D}(X, Y)$ and $(X \wedge A)Y$ on a $(0,k)$-tensor $H$, $k \geq 1$, we get two $(0,k + 2)$-tensors $D \cdot H$ and $Q(A, H)$ respectively as follows (see [12], [13], [17], [40], [50], [69] and also references therein)

$$
D \cdot H(X_1, X_2, \cdots, X_k, X, Y) = -H(\mathcal{D}(X_1, X_2)X_3, X_4, \cdots, X_k) - \cdots - H(X_1, X_2, \cdots, \mathcal{D}(X, Y)X_k).
$$

and \]

$$
Q(A, H)(X_1, X_2, \cdots, X_k, X, Y) = ((X \wedge A)Y)H(X_1, X_2, \cdots, X_k)
= A(X, X_1)H(Y, X_2, \cdots, X_k) + \cdots + A(X, X_k)H(X_1, X_2, \cdots, Y)
- A(Y, X_1)H(X_2, \cdots, X_k) - \cdots - A(Y, X_k)H(X_1, X_2, \cdots, X).
$$

**Definition 2.1.** ([1], [7], [11], [50], [54], [55], [61], [63]) A semi-Riemannian manifold $M$ is said to be $H$-semisymmetric type if $D \cdot H = 0$ and it is said to be $H$-pseudosymmetric type if $\left(\sum_{i=1}^{k} c_i D_i\right) \cdot H = 0$ for some scalars $c_i$’s, where $D$ and each $D_i$, $i = 1, \ldots, k$, ($k \geq 2$), are $(0,4)$ curvature tensors.
In particular, if $D = R$ and $H = R$ (resp., $S, C, W$ and $K$), then $M$ is called semisymmetric (resp., Ricci, conformally, concircularly and conharmonically semisymmetric). Again, if $i = 2$, $D_1 = R$, $D_2 = \mathcal{G}$ and $H = R$ (resp., $S, C, W$ and $K$), then $M$ is called Deszcz pseudosymmetric (resp., Ricci, conformally, concircularly and conharmonically pseudosymmetric). Especially, if $i = 2$, $D_1 = C$, $D_2 = \mathcal{G}$ and $H = C$, then $M$ is called a manifold of pseudosymmetric Weyl conformal curvature tensor.

We note that the pp-wave metric [45] is semisymmetric and the Robinson-Trautman metric [43] is pseudosymmetric.

**Definition 2.2.** A semi-Riemannian manifold $M$ is said to be quasi-Einstein if $S = \alpha g + \beta \Pi \otimes \Pi$ for some scalars $\alpha$ and $\beta$, and an 1-form $\Pi$. In particular, if $\beta \equiv 0$ (resp., $\alpha \equiv 0$), then a quasi-Einstein manifold is called Einstein [3] (resp., Ricci simple).

We mention that the Robertson-Walker spacetimes are quasi Einstein [2], Kaigorodov metric [24] is Einstein, and Gödel spacetime [18] is Ricci simple.

In 1983, Derdzinski and Shen [10] showed that if $E$ is a Codazzi tensor such that $V_\lambda$ and $V_\mu$ are two eigenspaces corresponding to the eigenvalues $\lambda$ and $\mu$ of the operator $\mathcal{E}$, then the subspace $V_\lambda \wedge V_\mu$ is invariant under the action of the curvature operator $\mathcal{R}(X, Y)$. Recently, Mantica and Molinari [28] extended their theorem by replacing the condition ‘Codazzi type’ by ‘Riemann-compatibility’.

**Definition 2.3.** Let $D$ be a $(0, 4)$-tensor and $E$ be a symmetric $(0, 2)$-tensor on $M$. Then $E$ is said to be $D$-compatible ([14], [29], [30]) if

$$D(\mathcal{E}X_1, X, X_2, X_3) + D(\mathcal{E}X_2, X, X_3, X_1) + D(\mathcal{E}X_3, X, X_1, X_2) = 0$$

holds, where $\mathcal{E}$ is the endomorphism corresponding to $E$. Again an 1-form $\Pi$ is said to be $D$-compatible if $\Pi \otimes \Pi$ is $D$-compatible.

We note that the Ricci tensor of Gödel metric [18] and Som-Raychaudhuri metric [53] are Riemann-compatible.

The curvature 2-forms $\Omega^m_{(D)\parallel}$ ([3], [26]) associated to a $(0, 4)$-curvature tensor $D$ and the 1-forms $\Lambda(Z)_{\parallel}$ [66] associated to a symmetric $(0, 2)$-curvature tensor $Z$ are defined as follows:

$$\Omega^m_{(D)\parallel} = D^m_{jk\ell}dx^j \wedge dx^k \text{ and } \Lambda(Z)_{\parallel} = Z_{\ell m}dx^m,$$

where $\wedge$ indicates the exterior product. Now $\Omega^m_{(D)\parallel}$ (resp., $\Lambda(Z)_{\parallel}$) is recurrent if

$$\mathcal{D}\Omega^m_{(D)\parallel} = \Pi \wedge \Omega^m_{(D)\parallel} \text{ (resp., } \mathcal{D}\Lambda(Z)_{\parallel} = \Pi \wedge \Lambda(Z)_{\parallel}),$$

where $\mathcal{D}$ is the exterior derivative and $\Pi$ is the associated 1-form. Recently Mantica and Suh ([31], [32], [33]) showed that $\Omega^m_{(D)\parallel}$ are recurrent if and only if

$$(\nabla_X D)(X_2, X_3, X, Y) + (\nabla_X D)(X_3, X_1, X, Y) + (\nabla_X D)(X_1, X_2, X, Y) =$$

$$\Pi(X_1)D(X_2, X_3, X, Y) + \Pi(X_2)D(X_3, X_1, X, Y) + \Pi(X_3)D(X_1, X_2, X, Y)$$
and $\Lambda_{(Z)t}$ are recurrent if and only if
\[
(\nabla_X Z)(X_2, X) - (\nabla_{X_2} Z)(X_1, X) = \Pi(X_1)Z(X_2, X) - \Pi(X_2)Z(X_1, X)
\]
for an 1-form $\Pi$.

### 3. Various tensors of Vaidya metric

In terms of $(u, r, \theta, \phi)$-coordinates, the Vaidya metric (1.2) is given by
\[
g = \begin{pmatrix}
-1 + \frac{2m}{r} & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & r^2 & 0 \\
0 & 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}.
\]

Then the non-zero components of its Riemann-Christoffel curvature tensor $R$, Ricci tensor $S$ and scalar curvature $\kappa$ of (1.2) are given by
\[
R_{1212} = -\frac{2m}{r^3}, \quad R_{1313} = -\frac{2m^2 + r^2m' - mr}{r^2}, \quad R_{1424} = \frac{m \sin^2 \theta}{r},
\]
\[
R_{1323} = \frac{m}{r}, \quad R_{1414} = -\frac{(2m^2 + r^2m' - mr) \sin^2 \theta}{r^2}, \quad R_{3434} = 2mr \sin^2 \theta.
\]

The non-zero components of its conformal curvature tensor $C$ are given by
\[
C_{1212} = -\frac{2m}{r^3}, \quad C_{1313} = -\frac{m(2m - r)}{r^2}, \quad C_{1323} = \frac{m}{r},
\]
\[
C_{1414} = -\frac{m(2m - r) \sin^2 \theta}{r^2}, \quad C_{1424} = \frac{m \sin^2 \theta}{r}, \quad C_{3434} = 2mr \sin^2 \theta.
\]

Now from above we can easily evaluate the non-zero components of $R \cdot R$, $C \cdot R$, $R \cdot C$, $C \cdot C$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(S, C)$ as follows:
\[
-2R \cdot R_{121313} = R \cdot R_{131312} = \frac{4mm'}{r^3}, \quad R \cdot R_{121323} = -R \cdot R_{122313} = \frac{3m^2}{r^4},
\]
\[
-2R \cdot R_{121412} = R \cdot R_{141412} = \frac{4mm' \sin^2 \theta}{r^3}, \quad R \cdot R_{121424} = -R \cdot R_{122414} = \frac{3m^2 \sin^2 \theta}{r^4},
\]
\[-R \cdot R_{133414} = R \cdot R_{143413} = \frac{m(6m^2 + 4r^2m' - 3mr) \sin^2 \theta}{r^3},
\]
\[R \cdot R_{133424} = -R \cdot R_{143423} = R \cdot R_{233414} = -R \cdot R_{243413} = \frac{3m^2 \sin^2 \theta}{r^4},
\]
\[R \cdot C_{121313} = -\frac{3mm'}{r^3}, \quad R \cdot C_{121323} = -R \cdot C_{122313} = \frac{3m^2}{r^4},
\]
\[R \cdot C_{121414} = -\frac{3mm' \sin^2 \theta}{r^3}, \quad R \cdot C_{121424} = -R \cdot C_{122414} = \frac{3m^2 \sin^2 \theta}{r^4},
\]
\[-R \cdot C_{133414} = R \cdot C_{143413} = \frac{3m(2m^2 + r^2m' - mr) \sin^2 \theta}{r^3}.
\]
\[ R \cdot C_{13424} = -R \cdot C_{14324} = R \cdot C_{23414} = -R \cdot C_{24314} = \frac{3m^2 \sin^2 \theta}{r^2}, \]
\[ 4C \cdot R_{121313} = C \cdot R_{131312} = \frac{4mm'}{r^3}, \quad C \cdot R_{121323} = -C \cdot R_{122313} = \frac{3m^2}{r^4}, \]
\[ 4C \cdot R_{121414} = C \cdot R_{141412} = \frac{4mm' \sin^2 \theta}{r^3}, \quad C \cdot R_{121424} = -C \cdot R_{122414} = \frac{3m^2 \sin^2 \theta}{r^4}, \]
\[-C \cdot R_{133414} = C \cdot R_{143413} = m(6m^2 + r^2m' - 3mr) \sin^2 \theta, \]
\[ C \cdot R_{13424} = -C \cdot R_{14324} = C \cdot R_{23414} = -C \cdot R_{24314} = \frac{3m^2 \sin^2 \theta}{r^2}, \]
\[ C \cdot C_{121323} = -C \cdot C_{122313} = \frac{3m^2}{r^4}, \quad C \cdot C_{121424} = -C \cdot C_{122414} = \frac{3m^2 \sin^2 \theta}{r^4}, \]
\[-C \cdot C_{133414} = C \cdot C_{143413} = \frac{3m^2(2m - r) \sin^2 \theta}{r^3}, \]
\[ C \cdot C_{13424} = -C \cdot C_{14324} = C \cdot C_{23414} = -C \cdot C_{24314} = \frac{3m^2 \sin^2 \theta}{r^2}, \]
\[ 2Q(g, R)_{121313} = -Q(g, R)_{131312} = 2m', \quad Q(g, R)_{121323} = -Q(g, R)_{122313} = \frac{3m}{r}, \]
\[ 2Q(g, R)_{12414} = -Q(g, R)_{141412} = 2m' \sin^2 \theta, \quad Q(g, R)_{124124} = -Q(g, R)_{122414} = \frac{3m \sin^2 \theta}{r}, \]
\[-Q(g, R)_{133414} = Q(g, R)_{143413} = (6m^2 + r^2m' - 3mr) \sin^2 \theta, \]
\[ Q(g, R)_{13424} = -Q(g, R)_{14324} = Q(g, R)_{23414} = -Q(g, R)_{24314} = 3mr \sin^2 \theta, \]
\[-2Q(S, R)_{121313} = Q(S, R)_{131312} = \frac{4mm'}{r^3}, \quad -2Q(S, R)_{121414} = Q(S, R)_{141412} = \frac{4mm' \sin^2 \theta}{r^3}, \]
\[-Q(S, R)_{133414} = Q(S, R)_{143413} = \frac{4mm' \sin^2 \theta}{r}, \]
\[ Q(g, C)_{121323} = -Q(g, C)_{122313} = \frac{3m}{r}, \quad Q(g, C)_{121424} = -Q(g, C)_{122414} = \frac{3m \sin^2 \theta}{r}, \]
\[-Q(g, C)_{133414} = Q(g, C)_{143413} = 3m(2m - r) \sin^2 \theta, \]
\[ Q(g, C)_{13424} = -Q(g, C)_{14324} = Q(g, C)_{23414} = -Q(g, C)_{24314} = 3mr \sin^2 \theta, \]
\[-2Q(S, C)_{121313} = Q(S, C)_{131312} = \frac{4mm'}{r^3}, \quad -2Q(S, C)_{121414} = Q(S, C)_{141412} = \frac{4mm' \sin^2 \theta}{r^3}, \]
\[-Q(S, C)_{133414} = Q(S, C)_{143413} = \frac{4mm' \sin^2 \theta}{r}. \]

Now from Einstein’s field equations with zero cosmological constant, the energy momentum tensor is given by
\[ T = \frac{c^4}{8\pi G} \left[ S - \frac{\kappa}{2} g \right], \]
where \( c = \) speed of light in vacuum, \( G = \) gravitational constant. Then the only non-zero component (upto symmetry) of \( T \) is given by
\[
T_{11} = \frac{2c^4m'}{8\pi Gr^2}.
\] (3.1)

Then the non-zero components of \( \nabla T \) are given by
\[
T_{11,1} = \frac{c^4 (r^2 m'' + 2mm')}{4\pi Gr^4}, \quad T_{13,3} = -\frac{c^4 m'}{4\pi Gr},
\]
\[
T_{11,2} = -\frac{c^4 m'}{2\pi Gr^3}, \quad T_{14,4} = -\frac{c^4 \sin^2(\theta)m'}{4\pi Gr}.
\]

4. Curvature properties of Vaidya metric

From above we can conclude that the metric (1.2) fulfills the following curvature restricted geometric structures.

**Theorem 4.1.** The Vaidya metric given in (1.2) possesses the following curvature properties:

(i) neither Ricci flat nor Ricci symmetric but its scalar curvature is zero and hence \( R = W \) and \( C = K \).

(ii) neither conformally semisymmetric nor Deszcz pseudosymmetric but \( C \cdot C = \frac{m'}{r} Q(g, C) \).

(iii) neither Ricci generalized pseudosymmetric nor Weyl pseudosymmetric but satisfies the pseudosymmetric type conditions \( R \cdot R - Q(S, R) = \frac{m'}{r^2} Q(g, C) \) and \( R \cdot C + C \cdot R = \frac{2m'}{r^3} Q(g, C) + Q(S, C) \).

(iv) neither conformally recurrent nor the curvature 2-forms \( \Omega_{(R)}^m(n) \) nor the Ricci 1-forms \( \Lambda_{(S)}^m(n) \) are recurrent but the conformal 2-forms \( \Omega_{(C)}^m(n) \) are recurrent for the 1-form \( \Pi = \{ \frac{m'}{r}, 0, 0, 0 \} \).

(v) not Einstein but it is Ricci simple, such that \( S = \beta (\eta \otimes \eta) \), where \( \beta = 2m' \) and \( \eta = \{ \frac{1}{r}, 0, 0, 0 \} \) with \( ||\eta|| = 0 \). Hence \( S \wedge S = 0 \) and \( S^2 = 0 \).

(vi) Ricci tensor is neither cyclic parallel nor Codazzi type but Riemann compatible as well as Weyl compatible.

(vii) the general form of the compatible tensors for \( R \) and \( P \) are given by
\[
\begin{pmatrix}
  a_{11} & a_{12} & 0 & 0 \\
  a_{12} + \frac{rm'}{m} a_{22} & a_{22} & 0 & 0 \\
  0 & 0 & a_{33} & a_{43} \\
  0 & 0 & a_{43} & a_{44}
\end{pmatrix},
\]
where \( a_{ij} \) being arbitrary scalars.

(viii) the general form of the compatible tensors for \( C \) and \( K \) are given by
\[
\begin{pmatrix}
  a_{11} & a_{12} & 0 & 0 \\
  a_{12} & a_{22} & 0 & 0 \\
  0 & 0 & a_{33} & a_{34} \\
  0 & 0 & a_{34} & a_{44}
\end{pmatrix}.
\]

**Remark 4.1.** Some curvature properties of Vaidya metric have already shown in [25].
Remark 4.2. From (3.1), we see that the energy momentum tensor of the Vaidya metric (1.2) can be expressed as
\[ T = \frac{2c^4 m'}{8\pi Gr^2} (\eta \otimes \eta), \quad \text{where} \quad \eta = \{1, 0, 0, 0\}. \]
Now \( ||\eta|| = 0 \) and hence the Vaidya metric is a pure radiation metric [72].

Remark 4.3. From the value of the local components (presented in Section 3) of various tensors of the Vaidya metric (1.2), we can easily conclude that the metric does not fulfill the following geometric structures:
(i) super generalized recurrent ([56], [62])
(ii) weakly symmetric ([49], [70]) for \( R, C \) and \( P \)
(iii) weakly cyclic Ricci symmetric [47],
(iv) generalized Roter type ([51], [52]),
(v) \( R \)-space, \( C \)-space or \( P \)-space by Venzi [73],
(vi) harmonic or conformal harmonic.

From (1.1) and (1.2), it is clear that the Vaidya metric reduces to Schwarzschild metric (1.1) if \( m(u) \) is a non-zero constant. Hence from Theorem 4.1, we can conclude the following about the curvature properties of the Schwarzschild metric (1.1).

Corollary 4.1. The Schwarzschild metric (1.1) possesses the following curvature properties:
(i) The metric is Ricci flat and hence \( R = P = W = C = K \).
(ii) The metric is harmonic, i.e., \( \text{div} R = 0 \).
(iii) It is Deszcz pseudosymmetric manifold satisfying \( R \cdot R = \frac{m}{p^2} Q(g, R) \).
(iv) The general form of the compatible tensors for \( R \) is given by
\[
\begin{pmatrix}
a_{11} & a_{12} & 0 & 0 \\
a_{12} & a_{22} & 0 & 0 \\
0 & 0 & a_{33} & a_{34} \\
0 & 0 & a_{34} & a_{44}
\end{pmatrix},
\]
where \( a_{ij} \) being arbitrary scalars.

Remark 4.4. From (3.1), we see that for Schwarzschild metric (1.1), \( T = 0 \) and hence the Schwarzschild metric is a vacuum solution of Einstein’s field equations.

5. Comparisons between Vaidya metric and Ludwig-Edgar pure radiation metric

In 1997 Ludwig and Edgar [27] presented a pure radiation metric which is conformally related to Vacuum space or Ricci flat space. The pure radiation metric of Ludwig and Edgar ([27], [57]) in \((u; r; x; y)\)-coordinates is given by
\[
ds^2 = \left( xw - p^2 \frac{r^2}{x^2} \right) du^2 + 2 du dr - \frac{4r}{p^2} dx dy - \frac{1}{p^2} (dx^2 + dy^2), \tag{5.1}
\]
where \( w \) is a smooth function of \( u, x, y, \) and \( p \) is a constant.

Recently, Shaikh et al. [57] have studied the curvature properties of the Ludwig-Edgar pure radiation metric (5.1). Since both the Vaidya metric (1.2) and Ludwig-Edgar metric (5.1) represent pure radiation fields, in this section we are mainly interested to make a comparisons between the
geometric properties of Vaidya metric (1.2) and Ludwig-Edgar pure radiation metric (5.1).

A. Similarities:

(i) Ricci tensors are neither Codazzi type nor cyclic parallel but the scalar curvatures are zero
(ii) Ricci simple
(iii) Ricci tensors are Riemann compatible as well as Weyl compatible
(iv) conformal curvature 2-forms are recurrent

B. Dissimilarities:

| Ludwig-Edgar pure radiation metric (5.1) | Vaidya metric (1.2) |
|----------------------------------------|--------------------|
| (i) semisymmetric                       | (i) pseudosymmetric Weyl conformal curvature tensor |
| (ii) weakly Ricci symmetric             | (ii) not weakly Ricci symmetric |
| (iii) R-space by Venzi                  | (iii) not R-space by Venzi |
| (iv) energy momentum tensor is parallel | (iv) energy momentum tensor is not parallel |

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