Derived category of coherent sheaves and
counting invariants

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Abstract. We survey recent developments on Donaldson-Thomas theory, Bridgeland sta-
bility conditions and wall-crossing formula. We emphasize the importance of the counting
theory of Bridgeland semistable objects in the derived category of coherent sheaves to
find a hidden property of the generating series of Donaldson-Thomas invariants.

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1. Introduction

1.1. Moduli spaces and invariants. The study of the moduli spaces is
a traditional research subject in algebraic geometry. They are schemes or stacks
whose points bijectively correspond to fixed kinds of algebro-geometric objects,
say curves, sheaves on a fixed variety, etc. These moduli spaces are interesting
not only in algebraic geometry, but also in connection with other research fields
such as number theory, differential geometry and string theory. In general it is not
easy to study the geometric properties of the moduli spaces. Instead one tries to
construct and study the invariants of the moduli spaces, e.g. their (weighted) Euler
characteristics, virtual Poincaré or Hodge polynomials, integration of the virtual
cycles via deformation-obstruction theory. It has been observed that the best way
to study such invariants is taking the generating series. Sometimes the generating
series defined from the moduli spaces have beautiful forms and properties. Let
us observe this phenomenon for some rather amenable examples. For a quasi-
projective variety $X$ (in this article, we always assume that the varieties are defined
over $\mathbb{C}$), the Hilbert scheme of $n$-points denoted by Hilb$_n(X)$ is the moduli space
of zero dimensional subschemes $Z \subset X$ such that $\chi(O_Z) = n$. It contains an
open subset corresponding to $n$-distinct points in $X$, and the geometric structures
of its complement is in general complicated. Nevertheless if $X$ is non-singular,
the generating series of the Euler characteristics of Hilb$_n(X)$ have the following

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beautiful forms \cite{G00}, \cite{Che96}
\[
\sum_{n \geq 0} \chi(\text{Hilb}_n(X))q^n = \begin{cases} 
(1 - q)^{-\chi(X)}, & \text{dim } X = 1 \\
\prod_{m \geq 1} (1 - q^m)^{-\chi(X)}, & \text{dim } X = 2 \\
\prod_{m \geq 1} (1 - q^m)^{-m\chi(X)}, & \text{dim } X = 3.
\end{cases}
\]

In the case \(X = \mathbb{C}^d\), the torus localization shows that \(\chi(\text{Hilb}_n(\mathbb{C}^d))\) coincides with the number of \(d\)-dimensional partitions of \(n\). The resulting product formulas are consequences of enumerative combinatorics, as in \cite{Sta99}. A general case is reduced to the case \(X = \mathbb{C}^d\).

\subsection*{1.2. Curve counting invariants}

The study of the invariants of the moduli spaces of curves inside a variety is more important and interesting, because of its connection with world sheet counting in string theory. A particularly important case is when \(X\) is a \textit{Calabi-Yau 3-fold}, i.e. \(X\) has a trivial canonical line bundle with \(H^1(X, \mathcal{O}_X) = 0\), as the string theory predicts our universe to be the product of the four dimensional space time with a Calabi-Yau 3-fold. Similarly to \(\text{Hilb}_n(X)\), we denote by \(\text{Hilb}_n(X, \beta)\) the \textit{Hilbert scheme of curves} inside \(X\), that is the moduli space of projective subschemes \(C \subset X\) with \(\text{dim } C \leq 1\), \([C] = \beta\) and \(\chi(\mathcal{O}_C) = n\). The following result was obtained by the author in 2008, and plays a key role in this article:

\textbf{Theorem 1.1} (\cite{Tod10b}, \cite{Tod10a}). Let \(X\) be a smooth projective Calabi-Yau 3-fold. Then for fixed \(\beta \in H_2(X, \mathbb{Z})\), the quotient series
\[
\frac{\sum_{n \in \mathbb{Z}} \chi(\text{Hilb}_n(X, \beta))q^n}{\sum_{n \geq 0} \chi(\text{Hilb}_n(X))q^n}
\]
\textit{is the Laurent expansion of a rational function of }\(q\), \textit{invariant under }\(q \leftrightarrow 1/q\).

Note that the denominator of (2) is given by the formula (1). A typical example of a rational function of \(q\) invariant under \(q \leftrightarrow 1/q\) is \(q/(1+q)^2 = q - 2q^2 + 3q^3 - \cdots\). We remark that the invariance of \(q \leftrightarrow 1/q\) does not say the invariance of the generating series after the formal substitution \(q \mapsto 1/q\), but so after taking the analytic continuation of the function (2) from \(|q| \ll 1\) to \(|q| \gg 1\). The above result was conjectured in \cite{LQ06} as the \textit{unweighted version} of the rationality conjecture of rank one Donaldson-Thomas (DT) invariants by Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) \cite{MNOP06}. The rationality conjecture was proposed in order to formulate the \textit{Gromov-Witten/Donaldson-Thomas correspondence conjecture} comparing two kinds of curve counting invariants on Calabi-Yau 3-folds.

The DT invariant was introduced by Thomas \cite{Tho00}, as a holomorphic analogue of Casson invariants of real 3-manifolds. It counts stable coherent sheaves on a Calabi-Yau 3-fold, and is a higher dimensional generalization of Donaldson invariants on algebraic surfaces. For a Calabi-Yau 3-fold \(X\), an ample divisor \(H\) on \(X\) and a cohomology class \(v \in H^*(X, \mathbb{Q})\), the DT invariant \(\text{DT}_H(v) \in \mathbb{Z}\) is defined to be the degree of the zero dimensional virtual fundamental cycle on the moduli space of \(H\)-stable coherent sheaves \(E\) on \(X\) with \(\text{ch}(E) = v\). It also coincides with
the weighted Euler characteristic with respect to the Behrend’s constructible function on that moduli space \[ \text{Beh09}\]. The DT invariants were later generalized by Joyce-Song \[ JS12\] and Kontsevich-Soibelman \[ KS\] so that they also count strictly \( H\)-semistable sheaves. The generalized DT invariants involve the Behrend functions and the motivic Hall algebras in the definition, and they are \( \mathbb{Q}\)-valued.

The Hilbert scheme of points or curves on a Calabi-Yau 3-fold is also interpreted as a moduli space of stable sheaves, by assigning a subscheme \( C \subset X \) with its ideal sheaf \( I_C \subset \mathcal{O}_X \). The resulting DT invariant is the weighted Euler characteristic of the Hilbert scheme of points or curves, and in particular it is independent of \( H \). In this sense, the invariant \( \chi(\text{Hilb}_n(X, \beta)) \) is the unweighted version of the DT invariant, which coincides with the honest DT invariant up to sign if \( \text{Hilb}_n(X, \beta) \) is non-singular. The result of Theorem 1.1 for the weighted version was later proved by Bridgeland \[ Bri11\]. The rationality property and the invariance of \( q \leftrightarrow 1/q \) of the series (2) are not visible if we just look at the moduli spaces of curves or points. Such hidden properties of the series (2) are visible after we develop new moduli theory and invariants of objects in the derived category of coherent sheaves.

1.3. Derived category of coherent sheaves. Recall that for a variety \( X \), the bounded derived category of coherent sheaves \( D^b\text{Coh}(X) \) is defined to be the localization by quasi-isomorphisms of the homotopy category of the bounded complexes of coherent sheaves on \( X \). The derived category is no longer an abelian category, but has a structure of a triangulated category. It was originally introduced by Grothendieck in 1960’s in order to formulate the relative version of Serre duality theorem, known as Grothendieck duality theorem. Later it was observed by Mukai \[ Muk81\] that an abelian variety and its dual abelian variety, which are not necessary isomorphic in general, have the equivalent derived categories of coherent sheaves. This phenomena suggests that the category \( D^b\text{Coh}(X) \) has more symmetries than the category of coherent sheaves, as the latter category is known to reconstruct the original variety. Such a phenomena has drawn much attention since Kontsevich proposed the Homological mirror symmetry conjecture in \[ Kon95\]. It predicts an equivalence between the derived category of coherent sheaves on a Calabi-Yau manifold and the derived Fukaya category of its mirror manifold, based on an insight that the derived category \( D^b\text{Coh}(X) \) is a mathematical framework of D-branes of type B in string theory. There have been several developments in constructing Mukai type derived equivalences between non-isomorphic varieties \[ Bri02\], \[ Orl97\], \[ BC09\], \[ Kaw02\], and non-trivial autoequivalences \[ ST01\], \[ HT06\], based on the ideas from mirror symmetry. Furthermore such Mukai type equivalences have been discovered beyond algebraic geometry. For instance, derived McKay correspondence \[ BKR01\] gives an equivalence between the derived category of finite group representations and the derived category of coherent sheaves on the crepant resolution of the quotient singularity. This is now interpreted as a special case of equivalences between usual commutative varieties and non-commutative varieties in the context of Van den Bergh’s non-commutative crepant resolutions \[ KB04\]. There also exists an Orlov’s equivalence \[ Orl09\] between the derived category of coherent sheaves on a Calabi-Yau hy-
persurface in the projective space and the category of graded matrix factorizations
of the defining equation of it. This result, called Landau-Ginzbrug/Calabi-Yau
correspondence, was also motivated by mirror symmetry. Now it is understood
that the derived categories have more symmetries than the categories of coherent
sheaves. Our point of view is to make the hidden properties of the generating series
of DT type invariants visible via symmetries in the derived categories.

1.4. Bridgeland stability conditions. The idea of applying derived cat-
egories to the study of generating series of DT type invariants suggests an im-
portance of constructing moduli spaces and invariants of objects in the derived
categories. Note that in constructing the original DT invariants, we need to fix
an ample divisor on a Calabi-Yau 3-fold \(X\), and the associated stability condition
on \(\text{Coh}(X)\) in order to construct a good moduli space of stable sheaves. The
notion of stability conditions on triangulated categories, in particular on derived
categories of coherent sheaves, was introduced by Bridgeland [Bri07] as a mathe-
matical framework of Douglas’s II-stability [Dou02] in string theory. For a trian-
gulated category \(\mathcal{D}\), a Bridgeland stability condition on it roughly consists of data
\(\sigma = (Z, \{P(\phi)\}_{\phi \in \mathbb{R}})\) for a group homomorphism \(Z : K(\mathcal{D}) \to \mathbb{C}\) called the central
charge, and the collection of subcategories \(P(\phi) \subset \mathcal{D}\) for \(\phi \in \mathbb{R}\) whose objects are
called \(\sigma\)-semistable objects with phase \(\phi\). The main result by Bridgeland [Bri07]
is to show that the set of ‘good’ stability conditions on \(\mathcal{D}\) forms a complex mani-
fold. This complex manifold is in particular important when \(\mathcal{D} = D^b\text{Coh}(X)\) for
a Calabi-Yau manifold \(X\). In this case, the space of stability conditions \(\text{Stab}(X)\)
is expected to contain the universal covering space of the moduli space of complex
structures of a mirror manifold of \(X\). So far the space \(\text{Stab}(X)\) has been studied
in several situations, e.g. \(X\) is a curve [Bri07], [Mac07], \(X\) is a K3 surface [Bri08],
\(X\) is a some non-compact Calabi-Yau 3-fold [Bri06], [BM11], [Tod08c], [Tod09b].
On the other hand, there has been a serious issue in studying Bridgeland stability
conditions on projective Calabi-Yau 3-folds which are likely to be the most im-
portant case: we are not able to prove the existence of Bridgeland stability conditions
on smooth projective Calabi-Yau 3-folds. In [BMT14], the existence problem is re-
duced to showing a conjectural Bogomolov-Gieseker type inequality evaluating the
third Chern character of certain two term complexes of coherent sheaves. However
proving that inequality conjecture seems to require a new idea.

1.5. New invariants via derived categories. Let \(X\) be a smooth
projective Calabi-Yau 3-fold. We expect that, for a given \(\sigma \in \text{Stab}(X)\) and
\(v \in H^*(X,\mathbb{Q})\), there exists the DT type invariant \(\text{DT}_\sigma(v) \in \mathbb{Q}\) which counts
\(\sigma\)-semistable objects \(E \in D^b\text{Coh}(X)\) with \(\text{ch}(E) = v\). As we mentioned, there is a
serious issue in constructing a Bridgeland stability condition on projective Calabi-
Yau 3-folds, but let us ignore this for a while. For an ample divisor \(H\) on \(X\),
we expect that the classical \(H\)-stability appears as a certain special limiting point
in \(\text{Stab}(X)\) called the large volume limit. If we take \(\sigma \in \text{Stab}(X)\) near the large
volume limit point, then we expect the equality \(\text{DT}_\sigma(v) = \text{DT}_H(v)\). On the other
hand, suppose that there is an autoequivalence \(\Phi\) of \(D^b\text{Coh}(X)\) and \(\tau \in \text{Stab}(X)\)
so that the equality $\text{DT}_\tau(v) = \text{DT}_\tau(\Phi_* v)$ holds for any $v$. Then the generating series of the invariants $\text{DT}_\tau(v)$ is preserved by the variable change induced by $v \mapsto \Phi_* v$. If we are able to relate $\text{DT}_\tau(v)$ and $\text{DT}_\tau(v)$, then it would imply the hidden symmetry of the generating series of classical DT invariants $\text{DT}_H(v)$ with respect $v \mapsto \Phi_* v$. The relationship between $\text{DT}_\tau(v)$ and $\text{DT}_\tau(v)$ is studied by the wall-crossing phenomena: there should be a wall and chamber structure on the space $\text{Stab}(X)$ so that the invariants $\text{DT}_\tau(v)$ are constant on a chamber but jumps if $*$ crosses a wall. The wall-crossing formula of the invariants $\text{DT}_\tau(v)$ should be described by a general framework established by Joyce-Song [JS12], Kontsevich-Soibelman [KS], using stack theoretic Hall algebras.

However as we mentioned, we are not able to prove $\text{Stab}(X) \neq \emptyset$, so the above story is the next stage after proving the non-emptiness. The idea of proving Theorem [13] was to introduce ‘weak’ Bridgeland stability conditions on triangulated categories, and apply the above story for the space of weak stability conditions on the subcategory of $D^b\text{Coh}(X)$ generated by $\mathcal{O}_X$ and one or zero dimensional sheaves. The latter subcategory is called the category of $D^b\text{-D2-D6 bound states}$. The notion of weak stability conditions is a kind of limiting degenerations of Bridgeland stability conditions, and it is a coarse version of Bayer’s polynomial stability conditions [Bay09], the author’s limit stability conditions [Tod09a]. It is easier to construct weak stability conditions and enough to prove Theorem [13] applying the above story. The derived dual $E \mapsto \mathbf{R}\text{Hom}(E, \mathcal{O}_X)$, an autequivalence of $D^b\text{Coh}(X)$, turned out to be responsible for the hidden symmetric property of $q \leftrightarrow 1/q$ of the series (2) in the above story.

The idea of proving Theorem [13] has turned out to be useful in proving several other interesting properties of DT type invariants, say DT/PT correspondence [Tod10a], [Bri11] conjectured by Pandharipande-Thomas [PT09]. We refer to [Tod13a], [Cal], [Tod08a], [Tod11], [Tod10c], [Sto12], [Nag], [Tod08b], [Tod12b], [Tod13a] for other works relating the above story.

### 1.6. Plan of this article.

In Section 2 we review and survey recent developments of Donaldson-Thomas theory. In Section 3 we survey the developments on Bridgeland stability conditions. In Section 4 we discuss open problems on DT theory and Bridgeland stability conditions.

### 2. Donaldson-Thomas theory

#### 2.1. Moduli spaces of semistable sheaves.

Let $X$ be a smooth projective variety and $H$ an ample divisor on $X$. For an object $E \in \text{Coh}(X)$, its Hilbert polynomial is given by

$$\chi(E \otimes \mathcal{O}_X(mH)) = a_d m^d + a_{d-1} m^{d-1} + \cdots$$

for $a_i \in \mathbb{Q}$ by the Riemann-Roch theorem. Here $a_d \neq 0$ and $d$ is the dimension of the support of $E$. The reduced Hilbert polynomial $\bar{\chi}_H(E, m)$ is defined to be $\chi(E \otimes \mathcal{O}_X(mH))/a_d$. 


Definition 2.1. An object $E \in \text{Coh}(X)$ is $H$-(semi)stable if for any subsheaf $0 \neq F \subseteq E$, we have $\dim \text{Supp}(F) = \dim \text{Supp}(E)$ and the inequality $\chi_H(F, m) < (\leq) \chi_H(E, m)$ holds for $m \gg 0$.

Remark 2.2. Note that if $E$ is torsion free, then $\chi_H(F, m) = m^d + c \cdot \mu_H(E) m^{d-1} + O(m^{d-2})$ where $d = \dim X$, $\mu_H(E) = c_1(E)H^{d-1}/\text{rank}(E)$, and $c$ is some constant. Hence the $H$-(semi)stability is the refinement of $H$-slope (semi)stability defined by the slope function $\mu_H(\ast)$.

Let $\text{Coh}(X)$ be the 2-functor from the category of complex schemes to the groupoid, whose $S$-valued points form the groupoid of flat families of coherent sheaves on $X$ over $S$. The 2-functor $\text{Coh}(X)$ forms a stack, which is known to be an Artin stack locally of finite type, but neither finite type nor separated. The situation becomes better if we consider the substacks for $v \in H^*(X, \mathbb{Q})$

$$\mathcal{M}_H^s(v) \subset \mathcal{M}_H^\text{ss}(v) \subset \text{Coh}(X).$$

Here $\mathcal{M}_H^{s(\text{ss})}(v)$ is the substack of $H$-(semi)stable $E \in \text{Coh}(X)$ with $\text{ch}(E) = v$, which is an open substack of $\text{Coh}(X)$. The stack $\mathcal{M}_H^\text{ss}(v)$ is of finite type but not separated in general. The stack $\mathcal{M}_H^s(v)$ is of finite type, separated, and a $\mathbb{C}^*$-gerb over a quasi-projective scheme $\mathcal{M}_H(v)$. The scheme $\mathcal{M}_H^s(v)$ is projective if $\mathcal{M}_H(v) = \mathcal{M}_H^\text{ss}(v)$.

2.2. Donaldson-Thomas invariants. Let $X$ be a smooth projective 3-fold. We say it is a Calabi-Yau 3-fold if $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$. A typical example is a quintic hypersurface in $\mathbb{P}^4$. Let $H$ be an ample divisor on $X$, $v$ an element in $H^*(X, \mathbb{Q})$, and consider the moduli scheme $\mathcal{M}_H(v)$. A standard deformation theory of sheaves (cf. [HL97]) shows that the tangent space at $[E] \in \mathcal{M}_H^s(v)$ is given by $\text{Ext}^1(E, E)$, and the obstruction space is given by $\text{Ext}^2(E, E)$. The Calabi-Yau condition and the Serre duality implies that the latter space is dual to $\text{Ext}^1(E, E)$. Hence the virtual dimension at $[E]$, defined to be the dimension of the tangent space minus the dimension of the obstruction space, is zero which is independent of $E$. Based on this observation, Thomas [Tho00] constructed two term complex of vector bundles $\mathcal{E}^\bullet$ on $\mathcal{M}_H^s(v)$ and a morphism $\mathcal{E}^\bullet \rightarrow L_{\mathcal{M}_H^s(v)}$ in $D(\mathcal{M}_H^s(v))$, giving a symmetric perfect obstruction theory in the sense of Behrend-Fantechi [BF97]. [BF08]. By the construction in [BF97], there is the associated zero dimensional virtual cycle $[\mathcal{M}_H^s(v)]^\text{vir}$ on $\mathcal{M}_H^s(v)$, and we are able to take its degree if $\mathcal{M}_H^s(v)$ is projective. The DT invariant is defined as follows:

Definition 2.3. If $\mathcal{M}_H^s(v) = \mathcal{M}_H^\text{ss}(v)$ holds, we define $\text{DT}_H(v) \in \mathbb{Z}$ to be the degree of $[\mathcal{M}_H^s(v)]^\text{vir}$.

The above construction via the virtual cycle easily shows that the DT invariant is invariant under deformations of complex structures of $X$. However in practice, it is more convenient to describe the DT invariant in terms of Behrend’s constructible function [Beh09]. The Behrend function is easily described if we use the following result by Joyce-Song [JS12]:
Theorem 2.4 ([JS12]). For any \( p \in M^\nu_H(v) \), there is an analytic open subset \( p \in U \subset M^\nu_H(v) \), a complex manifold \( V \) and a holomorphic function \( f: V \to \mathbb{C} \) such that \( U \) is isomorphic to \( \{ df = 0 \} \).

Using the above result, the Behrend function \( \nu_B \) on \( M^\nu_H(v) \) is described as
\[
\nu_B(p) = (-1)^{\dim V} (1 - \chi(M_p(f)))
\]
where \( M_p(f) \) is the Milnor fiber of \( f \) at \( p \). The function \( \nu_B \) is shown to be a well-defined constructible function on \( M^\nu_H(v) \).

Theorem 2.5 ([Beh09]). If \( M^\nu_H(v) = M^{ss}_H(v) \) holds, we have the equality
\[
\text{DT}_H(v) = \sum_{k \in \mathbb{Z}} k \cdot \chi(\nu_B^{-1}(k)).
\]
(3)

In particular if \( M^\nu_H(v) \) is non-singular and connected, the invariant \( \text{DT}_H(v) \) coincides with \( \chi(M^\nu_H(v)) \) up to sign.

Based on the above description of the DT invariant, Joyce-Song [JS12] and Kontsevich-Soibelman [KS] constructed the generalized DT invariant \( \text{DT}_H(v) \in \mathbb{Q} \) without the condition \( M^\nu_H(v) = M^{ss}_H(v) \). The construction uses the stack theoretic Hall algebra \( H(\text{Coh}(X)) \) of \( \text{Coh}(X) \), and its well-definedness is highly non-trivial.

A very rough description of it may be
\[
\text{DT}_H(v) = \int_{\log M^\nu_H(v)} \nu_B \cdot d\chi.
\]
The ‘log’ is taken in the algebra \( H(\text{Coh}(X)) \). Some more explanation of a specific case is available in [Tod12a].

Remark 2.6. We can define another invariant \( \text{DT}_H^\nu(v) \in \mathbb{Q} \) by formally putting \( \nu_B \equiv 1 \) in the definition of \( \text{DT}_H(v) \). If \( M^\nu_H(v) = M^{ss}_H(v) \), it coincides with the usual Euler characteristic \( \chi(M^\nu_H(v)) \). When we say a result as a weighted (resp. an unweighted) version, it means the result for the invariants \( \text{DT}_H(v) \) (resp. \( \text{DT}_H^\nu(v) \)).

2.3. Rank one DT invariants. In what follows, we identify \( H^4(X, \mathbb{Q}) \), \( H^6(X, \mathbb{Q}) \) with \( H_2(X, \mathbb{Q}) \), \( \mathbb{Q} \) respectively by the Poincaré duality. Given \( \beta \in H_2(X, \mathbb{Z}) \) and \( n \in \mathbb{Z} \), it is easy to show that \( \text{Hilb}_n(X, \beta) \) is isomorphic to \( M^\nu_H(v) \) for \( v = (1, 0, -\beta, -n) \) by the assignment \( C \mapsto I_C \). The resulting invariant
\[
I_{n, \beta} = \text{DT}_H(1, 0, -\beta, -n) \in \mathbb{Z}
\]
is independent of \( H \), and it counts one or zero dimensional subschemes \( C \subset X \) with \( [C] = \beta, \chi(\mathcal{O}_C) = n \). For \( \beta \in H_2(X, \mathbb{Z}) \), the series \( I_\beta(X) \) is defined to be
\[
I_\beta(X) = \sum_{n \in \mathbb{Z}} I_{n, \beta} q^n.
\]
Example 2.7. (i) If $\beta = 0$, we have \cite{Li06, BF08, LP09}

$$I_0(X) = \prod_{k \geq 1} (1 - (-q)^k)^{-k\chi(X)}.$$ 

(ii) If $f : X \to Y$ is a birational contraction whose exceptional locus is $C \cong \mathbb{P}^1$ with normal bundle $\mathcal{O}_C(-1)^{\oplus 2}$, we have \cite{BB07}

$$\sum_{m \geq 0} I_{m[C]}(X)t^m = \prod_{k \geq 1} (1 - (-q)^k)^{-k\chi(X)} \prod_{k \geq 1} (1 - (-q)^k t)^k.$$ 

The above example indicates that the quotient series $I_\beta(X)/I_0(X)$ is the honest curve counting series with homology class $\beta$. The following conjecture was proposed by MNOP \cite{MNOP06}:

Conjecture 2.8 \cite{MNOP06}. (i) The quotient series $I_\beta(X)/I_0(X)$ is the Laurent expansion of a rational function of $q$, invariant under $q \leftrightarrow 1/q$.

(ii) After the variable change $q = -e^{i\lambda}$, we have the equality

$$\sum_{\beta \geq 0} I_\beta(X)t^\beta = \exp \left( \sum_{g \geq 0, \beta > 0} \GW_{g, \beta}(X)\lambda^{2g-2} t^\beta \right).$$

Here $\GW_{g, \beta}(X) \in \mathbb{Q}$ is the Gromov-Witten invariant counting stable maps $f : C \to X$ from projective curves $C$ with at worst nodal singularities with $g(C) = g$, $f_*[C] = \beta$. The variable change $q = -e^{i\lambda}$ makes sense by the rationality conjecture (i). The above conjecture was first proved for toric Calabi-Yau 3-folds in \cite{MNOP06}.

2.4. Developments on MNOP conjecture. As we mentioned in the introduction, the result of Theorem 1.1 is the unweighted version of Conjecture 2.8 (i). The weighted version was proved in \cite{Bri11}. We have the following result \cite{Tod10b, Tod10a} (unweighted version), \cite{Bri11} (weighted version):

Theorem 2.9. There exist invariants $N_{n, \beta} \in \mathbb{Q}$, $L_{n, \beta} \in \mathbb{Q}$ satisfying

- $N_{n, \beta} = N_{-n, \beta} = N_{n + H \beta, \beta}$ for any ample divisor $H$ on $X$,
- $L_{n, \beta} = L_{-n, \beta}$, and it is zero for $|n| \gg 0$,

such that we have the following formula:

$$\sum_{\beta \geq 0} I_\beta(X)t^\beta = \prod_{n \geq 0, \beta \geq 0} \exp((-1)^{n-1}nN_{n, \beta}q^n t^\beta) \left( \sum_{n, \beta} L_{n, \beta}q^n t^\beta \right).$$

Remark 2.10. The proofs for the unweighted version in the author’s papers \cite{Tod10a, Tod10b} can be modified to show the weighted version, if once a similar result of Theorem 2.4 for the moduli spaces of complexes in \cite{Ina02, Lie06} is shown to be true (cf. \cite{Tod12a}). This is also applied for the results below.
The rationality conjecture is an easy consequence of Theorem 2.9.

**Corollary 2.11.** Conjecture 2.8 (i) is true.

There exist geometric meanings of $N_{n,\beta}$ and $L_{n,\beta}$. The former invariant is nothing but the generalized DT invariant $DT_H(0,0,\beta,n)$, which counts one or zero dimensional $H$-semistable sheaves $F$ on $X$ with $[F] = \beta$, $\chi(F) = n$. A priori, $N_{n,\beta}$ is defined using the ample divisor $H$, but the resulting invariant is shown to be independent of $H$. The latter invariant $L_{n,\beta}$ is more interesting. It counts certain two term complexes $E \in D^bCoh(X)$ (indeed they are perverse coherent sheaves in the sense of [Kas04], [Bez]) satisfying $\text{ch}(E) = (1,0,-\beta,-n)$, which are semistable with respect to a derived self dual weak stability condition on it. The result of Theorem 2.9 is proved along with the idea stated in Subsection 1.5.

A similar idea also proves Pandharipande-Thomas conjecture [PT09] relating the quotient series of rank one DT invariants with the invariants counting stable pairs. The definition of stable pairs is given as follows:

**Definition 2.12 ([PT09]).** A stable pair is data $(F,s)$ where $F$ is a pure one dimensional sheaf on $X$, $s: O_X \to F$ is a morphism which is surjective in dimension one.

A typical example of a stable pair is $(O_C(D),s)$, where $C \subset X$ is a smooth curve, $D \subset C$ is an effective divisor and $s$ is a natural composition $O_X \to O_C \subset O_C(D)$. For given $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space $P_n(X,\beta)$ of stable pairs $(F,s)$ with $[F] = \beta$, $\chi(F) = n$ is a projective scheme with a symmetric perfect obstruction theory [PT09]. The PT invariant $P_{n,\beta} \in \mathbb{Z}$ is defined to be the degree of the zero dimensional virtual fundamental cycle $[P_n(X,\beta)]^{vir}$ on $P_n(X,\beta)$. The invariant $P_{n,\beta}$ is deformation invariant, and coincides with the weighted Euler characteristic with respect to the Behrend function on $P_n(X,\beta)$. The following conjecture was proposed by [PT09], its unweighted version was proved in [Tod10a], [ST11], and the weighted version was proved in [Bri11]:

**Theorem 2.13.** For fixed $\beta \in H_2(X,\mathbb{Z})$, we have the equality of the generating series

$$\frac{I_{\beta}(X)}{I_0(X)} = \sum_{n \in \mathbb{Z}} P_{n,\beta} q^n.$$

Finally in [PP], Pandharipande-Pixton proved Conjecture 2.8 (ii) for large class of Calabi-Yau 3-folds including quintic hypersurfaces in $\mathbb{P}^4$.

**Theorem 2.14 ([PP]).** Conjecture 2.8 (ii) is true if $X$ is a complete intersection Calabi-Yau 3-fold in the product of projective spaces.

Indeed what they proved is the correspondence between Gromov-Witten invariants and stable pair invariants. Combined with Theorem 2.13 the result of Theorem 2.14 was proved. Their proof relies on the degeneration formula of GW and PT invariants, and the torus localization formula.
2.5. Non-commutative DT theory and flops. The DT theory can be also constructed for non-commutative varieties or algebras. Let $Y$ be a quasi-projective 3-fold which admits two crepant small resolutions giving a flop:

$$\phi: X \overset{f}{\leftrightarrow} Y \overset{f^*}{\leftrightarrow} X^\dagger. \quad (4)$$

In this situation, Van den Bergh [dB04] constructed sheaf of non-commutative algebras $A_Y$ on $Y$ and derived equivalences

$$D^b\text{Coh}(X^\dagger) \xrightarrow{\Phi} D^b\text{Coh}(A_Y) \xrightarrow{\Psi} D^b\text{Coh}(X) \quad (5)$$

so that their composition gives Bridgeland’s flop equivalence [Bri07]. For $n \in \mathbb{Z}$ and $\beta \in H_2(X, \mathbb{Z})$, let $\text{Hilb}_n(A_Y, \beta)$ be the moduli space of surjections $A_Y \to F$ in $\text{Coh}(A_Y)$ such that $F$ has at most one dimensional support and $[\Phi(F)] = \beta$, $\chi(\Phi(F)) = n$. If $X$ is a smooth projective Calabi-Yau 3-fold, there is a symmetric perfect obstruction theory on $\text{Hilb}_n(A_Y, \beta)$, and the degree of its zero dimensional virtual fundamental cycle defines the non-commutative DT (ncDT) invariant $A_{n,\beta} \in \mathbb{Z}$. Alternatively, $A_{n,\beta}$ is defined to be the weighted Euler characteristic of the Behrend function on $\text{Hilb}_n(A_Y, \beta)$. We set $I_{\beta}(A_Y)$ to be

$$I_{\beta}(A_Y) = \sum_{n \in \mathbb{Z}} A_{n,\beta} q^n.$$

The following result was proved in [Tod13b] for the unweighted version, and [Cal] for the weighted version, basically along with the argument in Subsection 1.5:

**Theorem 2.15.** We have the following identities:

$$\sum_{f, \beta=0} I_{\beta}(A_Y) t^\beta = \prod_{k \geq 1} (1 - (-q)^k) \chi(X) \left( \sum_{f, \beta=0} I_{\beta}(X) t^\beta \right) \left( \sum_{f, \beta=0} I_{-\beta}(X) t^\beta \right).$$

$$\frac{\sum_{f, \beta} I_{\beta}(X) t^\beta}{\sum_{f, \beta=0} I_{\beta}(X) t^\beta} = \frac{\sum_{f, \beta} I_{\beta}(A_Y) t^\beta}{\sum_{f, \beta=0} I_{\beta}(A_Y) t^\beta} = \frac{\sum_{f, \beta} I_{\phi, \beta}(X^\dagger) t^\beta}{\sum_{f, \beta=0} I_{\phi, \beta}(X^\dagger) t^\beta}.$$

**Example 2.16.** Let $Y = (xy + zw = 0) \subset \mathbb{C}^4$ be the conifold singularity, and take two crepant small resolutions [T] by blowing up at the ideals $(x, z)$ and $(x, w)$. In this case, the algebra $A_Y$ is the path algebra of the following quiver:

![Quiver Diagram]

with relation given by the derivations of the super potential $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$. Although $X$ is not projective in this case, the ncDT invariant $A_{n,m}[C] \in \mathbb{Z}$ makes sense, and coincides with the weighted Euler characteristic of the moduli space of framed $A_Y$-representations with dimension vector $(n, m + n)$. The proof...
of Theorem 2.15 also works in this situation. Using Example 2.7 (ii), the first identity of Theorem 2.15 becomes

\[
\sum_{n,m} A_{n,m}[c]q^n t^m = \prod_{k \geq 1} (1 - (-q)^k)^{-2k} \prod_{k \geq 1} (1 - (-q)^{kt})^k \prod_{k \geq 1} (1 - (-q)^{kt-1})^k.
\]

The above formula was first conjectured by Szendrői [Sze08], and later proved by Young [You09], Nagao-Nakajima [NN11].

Remark 2.17. In general for a quiver \( Q \) with a super potential \( W \), we are able to define the ncDT theory for \((Q,W)\). A mutation of the pair \((Q,W)\) defines another quiver with a super potential \((Q^\dagger, W^\dagger)\). The relationship between ncDT invariants on \((Q,W)\) and \((Q^\dagger, W^\dagger)\) is described in terms of cluster transformations. We refer to [KS], [Nag13] for the detail.

3. Bridgeland stability conditions

3.1. Definitions. We recall the definition of Bridgeland stability conditions on a triangulated category \( D \). We fix a finitely generated free abelian group \( \Gamma \) with a norm \( \| \cdot \| \) on \( \Gamma \) together with a group homomorphism \( \text{cl}: K(D) \to \Gamma \). A typical example is that \( D = D^b\text{Coh}(X) \) for a smooth projective variety \( X \), \( \Gamma \) is the image of the Chern character map \( \text{ch}: K(X) \to H^\ast(X, \mathbb{Q}) \), and \( \text{cl} = \text{ch} \). By taking the dual of \( \text{cl} \), we always regard a group homomorphism \( \Gamma \to \mathbb{C} \) as a group homomorphism \( K(D) \to \mathbb{C} \).

Definition 3.1 ([Bri07]). A stability condition on \( D \) is data \( \sigma = (Z, \{ P(\phi) \}_{\phi \in \mathbb{R}} \), where \( Z: \Gamma \to \mathbb{C} \) is a group homomorphism, \( P(\phi) \subset D \) is a full subcategory (called \( \sigma \)-semistable objects with phase \( \phi \)) satisfying the following conditions:

- For \( 0 \neq E \in P(\phi) \), we have \( Z(E) \in \mathbb{R}_{>0} \exp(\sqrt{-1}\pi \phi) \).
- For all \( \phi \in \mathbb{R} \), we have \( P(\phi + 1) = P(\phi)[1] \).
- For \( \phi_1 > \phi_2 \) and \( E_i \in P(\phi_i) \), we have \( \text{Hom}(E_1, E_2) = 0 \).
- (Harder-Narasimhan property): For each \( 0 \neq E \in D \), there is a collection of distinguished triangles \( E_{i-1} \to E_i \to F_i \to E_{i-1}[1], E_N = E, E_0 = 0 \) with \( F_i \in P(\phi_i) \) and \( \phi_1 > \phi_2 > \cdots > \phi_N \).

Another way defining a stability condition is to use a t-structure as follows:

Lemma 3.2 ([Bri07]). Giving a stability condition on \( D \) is equivalent to giving data \( (Z, A) \), where \( Z: \Gamma \to \mathbb{C} \) is a group homomorphism, \( A \subset D \) is the heart of a bounded t-structure, satisfying

\[
Z(A \setminus \{0\}) \in \{ r \exp(i\pi \phi) : r > 0, 0 < \phi \leq 1 \}
\]

(6)
together with the Harder-Narasimhan property: for any $E \in \mathcal{A}$, there exists a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_N = E$ such that $F_i = E_i/E_{i-1}$ is $Z$-semistable with $\arg Z(F_i) > \arg Z(F_{i+1})$ for all $i$. Here $E \in \mathcal{A}$ is $Z$-semistable if for any subobject $0 \neq F \subsetneq E$, we have $\arg Z(F) < (\leq) \arg Z(E)$.

**Proof.** The correspondence is as follows: given $(Z, \{P(\phi)\}_{\phi \in \mathbb{R}})$, the corresponding heart $\mathcal{A}$ is the extension closure of $P(\phi)$ for $0 < \phi \leq 1$. Conversely given $(Z, \mathcal{A})$, the category $\mathcal{P}(\phi)$ is defined to be the category of $Z$-semistable objects $E \in \mathcal{A}$ with $Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$. Other $\mathcal{P}(\phi)$ are defined by the rule $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$. \hfill \Box

**Example 3.3.** Let $C$ be a smooth projective curve, $\mathcal{D} = D^b\text{Coh}(C)$, $\Gamma = \mathbb{Z}^{\oplus 2}$ and $cl = (\text{rank}, \text{deg})$. We set $Z : \Gamma \rightarrow \mathbb{C}$ to be $(r, d) \mapsto -d + ir$. Then $(Z, \text{Coh}(C))$ is a stability condition, whose $Z$-semistable objects coincide with classical semistable sheaves on $C$.

### 3.2. The space of stability conditions

The space of stability conditions is defined as follows:

**Definition 3.4.** We define $\text{Stab}_\Gamma(\mathcal{D})$ to be the set of stability conditions on $\mathcal{D}$ satisfying the support property, i.e. there is a constant $C > 0$ such that $||cl(E)|| / |Z(E)| < C$ holds for any $0 \neq E \in \cup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$.

The main result of Bridgeland [Bri07] shows that the set $\text{Stab}_\Gamma(\mathcal{D})$ has a structure of a complex manifold. If $\mathcal{D} = D^b\text{Coh}(X)$ for a smooth projective variety $X$, $\Gamma = \text{Im}(ch)$ and $cl = ch$, we set $\text{Stab}(X) = \text{Stab}_\Gamma(\mathcal{D})$. Let $\text{Auteq}(X)$ be the group of exact autoequivalences of $D^b\text{Coh}(X)$. The space $\text{Stab}(X)$ admits a left action of $\text{Auteq}(X)$ and a right action of $\mathbb{C}$. The latter action is given by $(Z, \{P(\phi)\}_{\phi \in \mathbb{R}}) \cdot \lambda = (e^{-i\pi\lambda}Z, \{P(\phi + \text{Re}\lambda)\}_{\phi \in \mathbb{R}})$ for $\lambda \in \mathbb{C}$. We are interested in the double quotient stack

$$[\text{Auteq}(X) \backslash \text{Stab}(X) / \mathbb{C}].$$

The conjecture by Bridgeland [Bri09] is that if $X$ is a Calabi-Yau manifold, the above double quotient stack contains the stringy Kähler moduli space of $X$, that is the moduli space of complex structures of a mirror manifold of $X$.

**Example 3.5.** (i) If $C$ is an elliptic curve, then (7) is shown in [Bri07] to be isomorphic to the modular curve $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. This is compatible with the fact that $C$ is self mirror.

(ii) Let $\pi : X \rightarrow \mathbb{P}^2$ be the total space of $\omega_{\mathbb{P}^2}$, which is a non-compact Calabi-Yau 3-fold. In this case, $\text{Stab}(X)$ is defined to be $\text{Stab}_\Gamma(\mathcal{D})$ where $\mathcal{D}$ is the bounded derived category of compact supported coherent sheaves on $X$, $\Gamma$ is the image of $ch \circ \pi_*$ in $H^*(\mathbb{P}^2, \mathbb{Q})$, and $cl = ch \circ \pi_*$. Then the quotient stack (7) contains $[(\mathbb{C} \setminus \mu_3) / \mu_3]$ by [BM11]. The latter stack is the parameter space $\psi^3$ of the mirror family of $X$ given by

$$\{y_0^3 + y_1^3 + y_2^3 - 3\psi y_1 y_2 y_3 = 0\} \subset \mathbb{P}^2.$$
The space (7) is most interesting for projective Calabi-Yau 3-folds, e.g. quintic hypersurfaces in $\mathbb{P}^4$. Even in the quintic 3-fold case, the space (7) is very difficult to study. In this case, Bridgeland’s conjecture [Bri09] is stated in the following way:

**Conjecture 3.6.** Let $X \subset \mathbb{P}^4$ be a smooth quintic 3-fold, and set $\mathcal{M}_K = [(\mathbb{C} \setminus \mu_5)/\mu_5]$. Then there is an embedding

$$\mathcal{M}_K \hookrightarrow [\text{Auteq}(X)\setminus \text{Stab}(X)/\mathbb{C}].$$

The above embedding should be given by the solutions of the Picard-Fuchs equation which the period integrals of the mirror family of $X$ satisfy. Its explicit description is available in [Todk]. However in the projective Calabi-Yau 3-fold case, it is even not known that whether $\text{Stab}(X)$ is non-empty or not. The first issue in solving Conjecture 3.6 is to construct stability conditions, which we discuss in the next subsection.

### 3.3. Existence problem.

It has been a serious issue to construct Bridgeland stability conditions on projective Calabi-Yau 3-folds. Contrary to the one dimensional case, it turns out that there is no stability condition $\sigma \in \text{Stab}(X)$ of the form $\sigma = (Z, \text{Coh}(X))$ if $\dim X \geq 2$. From the arguments in string theory, we expect the following conjecture:

**Conjecture 3.7.** Let $X$ be a smooth projective variety and take $B+i\omega \in H^2(X, \mathbb{C})$ with $\omega$ ample class. Then there exists the heart of a bounded t-structure $\mathcal{A}_{B,\omega}$ on $D^b\text{Coh}(X)$ such that the pair $\sigma_{B,\omega} = (Z_{B,\omega}, \mathcal{A}_{B,\omega})$ determines a point in $\text{Stab}(X)$, where $Z_{B,\omega}$ is given by

$$Z_{B,\omega}(E) = -\int_X e^{-i\omega} \cdot \text{ch}^B(E).$$

Here $\text{ch}^B(E)$ is defined to be $e^{-B}\text{ch}(E)$.

The resulting stability conditions are expected to form a neighborhood at the large volume limit in terms of string theory. The above conjecture is known to hold if $\dim X \leq 2$. In the $\dim X = 2$ case, the heart $\mathcal{A}_{B,\omega}$ is constructed to be a certain tilting of $\text{Coh}(X)$, which we are going to review.

Let $X$ be a $d$-dimensional smooth projective variety and take $B+i\omega \in H^2(X, \mathbb{C})$ with $\omega$ ample. The $\omega$-slope function on $\text{Coh}(X)$ is defined to be

$$\mu_\omega(E) = \frac{\text{ch}_1(E) \cdot \omega^{d-1}}{\text{rank}(E)} \in \mathbb{R} \cup \{\infty\}.$$ 

Here $\mu_\omega(E) = \infty$ if $\text{rank}(E) = 0$.

**Definition 3.8.** An object $E \in \text{Coh}(X)$ is $\mu_\omega$-(semi)stable if for any non-zero subobject $0 \neq F \subsetneq E$, we have $\mu_\omega(F) < (\leq)\mu_\omega(E/F)$.
We define the pair of subcategories \((\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})\) of \(\text{Coh}(X)\) to be
\[
\mathcal{T}_{B,\omega} = \langle E \in \text{Coh}(X) : E \text{ is } \mu_{\omega}\text{-semistable with } \mu_{\omega}(E) > B_{\omega}d^{-1} \rangle
\]
\[
\mathcal{F}_{B,\omega} = \langle E \in \text{Coh}(X) : E \text{ is } \mu_{\omega}\text{-semistable with } \mu_{\omega}(E) \leq B_{\omega}d^{-1} \rangle.
\]
Here \((\ast)\) means the extension closure. The existence of Harder-Narasimhan filtrations with respect to the \(\mu_{\omega}\)-stability implies that the pair \((\mathcal{T}_{B,\omega}, \mathcal{F}_{B,\omega})\) is a torsion pair (cf. [HRS96]) of \(\text{Coh}(X)\). Its tilting defines another heart
\[
\mathcal{B}_{B,\omega} = \langle \mathcal{F}_{B,\omega}[1], \mathcal{T}_{B,\omega} \rangle \subset D^b\text{Coh}(X).
\]
The following result is due to [Bri08], [AB13], [Tod13c].

**Proposition 3.9.** If \(\dim X = 2\), then \((Z_{B,\omega}, \mathcal{B}_{B,\omega}) \in \text{Stab}(X)\).

**Proof.** Here is a rough sketch of the proof: if \(\dim X = 2\), then \(Z_{B,\omega}(E)\) is written as
\[
Z_{B,\omega}(E) = -\text{ch}^B_3(E) + \text{ch}^B_1(E)\omega^2/2 + i(\text{ch}^B_2(E)\omega - \text{ch}^B_0(E)\omega^3/6).
\]
The construction of \(\mathcal{B}_{B,\omega}\) immediately implies \(\text{Im}Z_{B,\omega}(E) \geq 0\) for any \(0 \neq E \in \mathcal{B}_{B,\omega}\). We need to check that \(\text{Im}Z_{B,\omega}(E) = 0\) implies \(\text{Re}Z_{B,\omega}(E) < 0\). This property can be easily deduced from the classical Bogomolov-Gieseker (BG) inequality in Theorem 3.10 below.

The following BG inequality played an important role:

**Theorem 3.10** ([Bog78], [Gie79]). Let \(X\) be a \(d\)-dimensional smooth projective variety, and \(E\) a torsion free \(\mu_{\omega}\)-semistable sheaf on \(X\). Then we have the following inequality:
\[
\left( \text{ch}^B_1(E)^2 - 2\text{ch}^B_0(E)\text{ch}^B_3(E) \right) \cdot \omega^{d-2} \geq 0.
\]

3.4. Double tilting construction for 3-folds. Suppose that \(X\) is a smooth projective 3-fold, and \(B, \omega\) are defined over \(\mathbb{Q}\). In this case, the central charge \(Z_{B,\omega}\) is written as
\[
Z_{B,\omega}(E) = -\text{ch}^B_3(E) + \text{ch}^B_1(E)\omega^2/2 + i\left(\text{ch}^B_2(E)\omega - \text{ch}^B_0(E)\omega^3/6\right).
\]
Contrary to the surface case, the heart \(\mathcal{B}_{B,\omega}\) does not fit into a stability condition with central charge \(Z_{B,\omega}\). In [BMT14], Bayer, Macri and the author constructed a further tilting of \(\mathcal{B}_{B,\omega}\) in order to give a candidate of \(\mathcal{A}_{B,\omega}\) in Conjecture 3.7. The key observation is the following lemma, which also relies on Theorem 3.10.

**Lemma 3.11** ([BMT14]). For any \(0 \neq E \in \mathcal{B}_{B,\omega}\), one of the following conditions hold:

(i) \(\text{ch}^B_1(E)\omega^2 > 0\).

(ii) \(\text{ch}^B_1(E)\omega^2 = 0\) and \(\text{Im}Z_{B,\omega}(E) > 0\).

(iii) \(\text{ch}^B_1(E)\omega^2 = \text{Im}Z_{B,\omega}(E) = 0\) and \(\text{Re}Z_{B,\omega}(E) < 0\).
The above lemma indicates that the vector \((\text{ch}_1^B(E)\omega^2, \text{Im} Z_{B,\omega}(E), -\text{Re} Z_{B,\omega}(E))\) behaves as if it were \((\text{rank}, c_1, \text{ch}_2)\) on coherent sheaves on surfaces. In [BMT14], this observation led to the following slope function on \(B_{B,\omega}\):

\[
\nu_{B,\omega}(E) = \frac{\text{Im} Z_{B,\omega}(E)}{\text{ch}_1^B(E)\omega^2} \in \mathbb{Q} \cup \{\infty\}.
\]

Here \(\nu_{B,\omega}(E) = \infty\) if \(\text{ch}_1^B(E)\omega^2 = 0\). The above lemma shows that \(\nu_{B,\omega}\) satisfies the weak see-saw property, and it defines a slope stability condition on \(B_{B,\omega}\).

In [BMT14], it was called \(\text{tilt-stability}\):

**Definition 3.12.** An object \(E \in B_{B,\omega}\) is tilt (semi)stable if for any subobject \(0 \neq F \subset E\), we have \(\nu_{B,\omega}(F) < (\leq) \nu_{B,\omega}(E/F)\).

We can show the existence of Harder-Narasimhan filtrations with respect to the tilt stability. Similarly to the surface case, the pair of subcategories \((T'_{B,\omega}, F'_{B,\omega})\) of \(B_{B,\omega}\) defined to be

\[
T'_{B,\omega} = \{E \in B_{B,\omega} : E \text{ is tilt semistable with } \nu_{B,\omega}(E) > 0\}
\]

\[
F'_{B,\omega} = \{E \in B_{B,\omega} : E \text{ is tilt semistable with } \nu_{B,\omega}(E) \leq 0\}
\]

is a torsion pair. By tilting, we have another heart

\[
A_{B,\omega} = \langle F'_{B,\omega}[1], T'_{B,\omega} \rangle \subset D^b \text{Coh}(X).
\]

By the construction, we have \(\text{Im} Z_{B,\omega}(E) \geq 0\) for any \(E \in A_{B,\omega}\). In [BMT14], we proposed the following conjecture:

**Conjecture 3.13 ([BMT14]).** If \(\dim X = 3\), we have \((Z_{B,\omega}, A_{B,\omega}) \in \text{Stab}(X)\).

### 3.5. Conjectural BG inequality for 3-folds.

Our double tilting construction led to a BG type inequality conjecture evaluating the third Chern character of tilt semistable objects.

**Conjecture 3.14 ([BMT14]).** Let \(X\) be a smooth projective 3-fold. Then for any tilt semistable object \(E \in B_{B,\omega}\) with \(\nu_{B,\omega}(E) = 0\), i.e. \(\text{ch}_2^B(E)\omega = \frac{\text{ch}_0^B(E)\omega^3}{6}\), we have the inequality

\[
\text{ch}_3^B(E) \leq \frac{1}{18} \text{ch}_1^B(E)\omega^2.
\]

**Remark 3.15.** In order to show \((Z_{B,\omega}, A_{B,\omega})\) satisfies the property \(\mathcal{E}\), it is enough to show the weaker inequality \(\text{ch}_3^B(E) < \text{ch}_1^B(E)\omega^2/2\). If this is true, the existence of HN filtrations is proved in [BMT14], while the support property remains open. The stronger bound in Conjecture 3.14 was obtained by the requirement that the equality is achieved for tilt semistable objects with zero discriminant.

It seems to be a hard problem to show Conjecture 3.14 even in concrete examples. So far, it is proved when \(X = \mathbb{P}^3\) by Macri [Mac]. X is a quadric 3-fold by...
Schmidt \([Sch]\), and \(X\) is a principally polarized abelian 3-fold with Picard rank one by Maciocia-Piyaratne \([MPa]\), \([MPb]\). Another kind of evidence is that assuming Conjecture 3.14 implies some open problems in other research fields. In \([BBMT]\), it was proved that Conjecture 3.14 implies (almost) Fujita conjecture for 3-folds: for any polarized 3-fold \((X, L)\), \(K_X + 4L\) is free and \(K_X + 6L\) is very ample. In \([Tod13a]\), it was also proved that Conjecture 3.14 implies a conjectural relationship between two kinds of DT type invariants inspired by string theory. This result will be reviewed in Theorem 4.5. It may be worth pointing out that, in both of the above applications, assuming a weaker inequality, say \(\text{ch}^B_3(E) < \text{ch}^B_1(E)\omega^2/2\), does not imply anything. The stronger evaluation in Conjecture 3.14 is crucial for the proofs of the applications.

3.6. The space of weak stability conditions. Although the existence of Bridgeland stability conditions on projective Calabi-Yau 3-folds remains open, we are able to modify the definition of Bridgeland stability conditions so that the story in Subsection 1.5 works. The notion of weak stability conditions in \([Tod10a]\) is one of them. In the situation of Subsection 3.1, we further fix a filtration
\[
0 \subsetneq \Gamma_0 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma
\]
such that each subquotient \(\Gamma_j/\Gamma_{j-1}\) is a free abelian group. Instead of considering a group homomorphism \(Z : \Gamma \to \mathbb{C}\), we consider an element
\[
Z = \{Z_i\}_{i=0}^N \in \prod_{j=0}^N \text{Hom}(\Gamma_j/\Gamma_{j-1}, \mathbb{C}). \tag{8}
\]
Given an element \(Z\), we set \(Z(v) \in \mathbb{C}\) for \(v \in \Gamma\) as follows: there is a unique \(0 \leq m \leq N\) such that \(v \in \Gamma_m \setminus \Gamma_{m-1}\), where \(\Gamma_{-1} = \emptyset\). Then \(Z(v)\) is defined to be \(Z_m([v]) \in \mathbb{C}\) where \([v]\) is the class of \(v\) in \(\Gamma_m/\Gamma_{m-1}\).

Definition 3.16. A weak stability condition on \(D\) with respect to the filtration \(\Gamma_\bullet\) is data \((Z, \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}})\), where \(Z\) is as in \(8\), \(\mathcal{P}(\phi) \subset D\) is a full subcategory, satisfying the same axiom in Definition 3.1.

Similarly to Lemma 3.2, giving a weak stability condition is equivalent to giving \((Z, A)\), where \(Z\) is as in \(8\), \(A \subset D\) is the heart of a bounded \(t\)-structure, satisfying the same conditions in Lemma 3.2. We denote by \(\text{Stab}_\bullet(D)\) the set of weak stability conditions on \(D\) with respect to \(\Gamma_\bullet\) with a support property. This set also has a structure of a complex manifold, and coincides with \(\text{Stab}_\bullet(D)\) if \(N = 0\), i.e. the filtration \(\Gamma_\bullet\) is trivial. In general, it is easier to show the non-emptiness for the space \(\text{Stab}_\bullet(D)\) with a non-trivial filtration \(\Gamma_\bullet\). The result of Theorem 2.9 was obtained by the wall-crossing formula in the space of weak stability conditions on the following triangulated category
\[
D_X = \langle \mathcal{O}_X, \text{Coh}_{\leq 1}(X) \rangle_{tr} \subset D^b\text{Coh}(X).
\]
Here \(\text{Coh}_{\leq 1}(X)\) is the category of one or zero dimensional sheaves on \(X\), and \(\langle \ast \rangle_{tr}\) is the triangulated closure. The relevant data is
\[
\Gamma_0 = \mathbb{Z} \oplus H_2(X, \mathbb{Z}) \oplus \{0\} \subset \Gamma_1 = \Gamma = \mathbb{Z} \oplus H_2(X, \mathbb{Z}) \oplus \mathbb{Z}.
\]
with the map $cl$ given by $cl(E) = (ch_3(E), ch_2(E), ch_0(E))$. Here $H_2(X, \mathbb{Q})$ is identified with $H^4(X, \mathbb{Q})$ via Poincaré duality. The result of Theorem 2.9 is proved along with the wall-crossing argument of Subsection 1.5 with respect to the one parameter family of weak stability conditions on $\mathcal{D}_X$

$$\sigma_\theta = (Z_{\omega, \theta}, A_X) \in \text{Stab}_{r_\bullet}(\mathcal{D}_X), \quad 1/2 \leq \theta < 1$$

from $\theta = 1/2$ to $\theta \to 0$. Here $\omega$ is an ample divisor on $X$, $Z_{\omega, \theta, j}$ are given by

$$Z_{\omega, \theta, 0}: \Gamma_0 \ni (n, \beta) \mapsto n - (\omega \cdot \beta), \quad Z_{\omega, \theta, 1}: \mathbb{Z} \ni r \mapsto r \exp(i\pi\theta).$$

The heart $A_X \subset \mathcal{D}_X$ is obtained as the extension closure of objects $\mathcal{O}_X$ and $\text{Coh}_{\leq 1}(X)[-1]$. We are able to construct DT type invariant $DT_{\sigma_\theta}(1, 0, -\beta, -n) \in \mathbb{Q}$ which counts $\sigma_\theta$-semistable objects $E \in A_X$ with $ch(E) = (1, 0, -\beta, -n)$. It is shown that

$$DT_{\sigma_\theta \to 1}(1, 0, -\beta, -n) = P_{n, \beta}, \quad DT_{\sigma_\theta = 1/2}(1, 0, -\beta, -n) = L_{n, \beta}$$

where $L_{n, \beta}$ is the invariant in Theorem 2.9. The wall-crossing formula describes the difference between $P_{n, \beta}$ and $L_{n, \beta}$. A similar wall-crossing phenomena also implies the relationship between $I_{n, \beta}$ and $P_{n, \beta}$ in Theorem 2.13. Combined them, we obtain the result of Theorem 2.9. Some more detail is also available in [Tod12a].

4. Further results and conjectures

4.1. Multiple cover formula conjecture. Although Conjecture 2.8 (i) is proved, a stronger version of the rationality conjecture remains open. It was proposed by Pandharipande-Thomas [PT09], and predicts the product expansion (called Gopakumar-Vafa form) of the generating series of PT invariants:

$$1 + \sum_{n \in \mathbb{Z}, \beta > 0} P_{n, \beta} q^n t^\beta = \prod_{\beta > 0} \left( \prod_{j=1}^{\infty} \left( 1 - (-q^j t^\beta) \right)^n \right) \prod_{g=1}^{2g-2} \prod_{k=0}^{n_g \beta} \left( 1 - (-q^g)^{g-1} t^\beta \right)^{(-1)^{k+g} n_g \beta (2g-2)}$$

for some $n_g \beta \in \mathbb{Z}$. Using Theorem 2.9 and Theorem 2.13 the above strong rationality conjecture is proved in [Tod12a] to be equivalent to the following conjecture:

**Conjecture 4.1** (JS12, Tod12a). We have the following identity:

$$N_{n, \beta} = \sum_{k \in \mathbb{Z}, \beta \in \Gamma(n, \beta)} \frac{1}{k^2} N_{1, \beta / k}.$$ 

The invariant $N_{1, \beta}$ is always integer, and the above conjecture is stronger then the integrality conjecture by Kontsevich-Soibelman [KS].
4.2. Gepner type stability conditions. Let \( W \in A = \mathbb{C}[x_1, \cdots, x_n] \) be a homogeneous polynomial of degree \( d \). By definition, a graded matrix factorization consists of data

\[
P^0 \xrightarrow{p^0} P^1 \xrightarrow{p^1} P^0(d)
\]

where \( P^i \) are graded free \( A \)-modules of finite rank, \( p^i \) are homomorphisms of graded \( A \)-modules, \( P^i \mapsto P^i(1) \) is the shift of the grading, satisfying \( p^1 \circ p^0 = p^0 \circ p^1 = -W \).

The triangulated category \( \text{HMF}(W) \) is defined to be the homotopy category of graded matrix factorizations of \( W \). It has a structure of a triangulated category, and related to \( D^b \text{Coh}(X) \) for the hypersurface \( X = (W = 0) \subset \mathbb{P}^{n-1} \). For instance if \( d = n \), there is an equivalence \( \text{HMF}(W) \sim D^b \text{Coh}(X) \).

As an analogy of Gieseker stability on \( \text{Coh}(X) \), we expect the existence of a natural stability condition on \( \text{HMF}(W) \).

**Conjecture 4.2.** There is a Bridgeland stability condition \( \sigma_G = (Z_G, \{P_G(\phi)\}_{\phi \in \mathbb{R}}) \) on \( \text{HMF}(W) \) whose central charge \( Z_G \) is given by

\[
Z_G \left( \bigoplus_{i=1}^{N} A(m_i) \right) = \sum_{i=1}^{N} \left( e^{2\pi m_i \sqrt{-1}} - e^{-2\pi m_i \sqrt{-1}} \right)
\]

and the set of semistable objects satisfy \( \tau P_G(\phi) = P_G(\phi + 2/d) \), where \( \tau \) is the graded shift functor \( P^\bullet \mapsto P^\bullet(1) \).

If \( n = d = 5 \), i.e. \( X \) is a quintic 3-fold, a stability condition above is expected to correspond to the orbifold point \( 0 \in M_K \) in Conjecture 3.6 called Gepner point. By this reason, a stability condition in Conjecture 4.2 is called Gepner type. Some evidence of Conjecture 4.2 is available in [KST07, Todd]. Suppose that Conjecture 4.2 is true for \( n = d = 5 \). Then as an analogy of Fan-Jarvis-Ruan-Witten theory \( \text{FJR} \) in GW theory, we may define the DT type invariant

\[
\text{DT}_G(\gamma) \in \mathbb{Q}, \quad \gamma \in \text{HH}_0(W)
\]

which counts \( \sigma_G \)-semistable graded matrix factorizations \( P^\bullet \) with \( \text{ch}(P^\bullet) = \gamma \).

Here \( \text{HH}_0(W) \) is the zero-th Hochschild homology of \( \text{HMF}(W) \), and \( \text{ch} \) is the Chern character map on graded matrix factorizations (cf. [PV12]). Because of the property of \( \sigma_G \), the invariant (10) should satisfy \( \text{DT}_G(\gamma) = \text{DT}_G(\tau_* \gamma) \). Under the Orlov equivalence (9), the equivalence \( \tau \) on the LHS corresponds to the equivalence \( \text{ST}_{\mathcal{O}_X} \circ \mathcal{O}_X(1) \) on the RHS, where \( \text{ST}_{\mathcal{O}_X} \) is the Seidel-Thomas twist \( \text{ST01} \) associated to \( \mathcal{O}_X \). Along with the argument in Subsection 1.3, the existence of the invariant (10) should imply a hidden symmetry of the generating series of the original DT invariants on the quintic hypersurface \( X = (W = 0) \) with respect to the equivalence \( \text{ST}_{\mathcal{O}_X} \circ \mathcal{O}_X(1) \).
4.3. S-duality conjecture for DT invariants. Let us recall the original S-duality conjecture by Vafa-Witten [VW94]. It predicts the (at least almost) modularity of the generating series of Euler characteristics of moduli spaces of stable torsion free sheaves on algebraic surfaces with a fixed rank and a first Chern class. We refer to [G09] for the developments on the S-duality conjecture so far.

Instead of stable torsion free sheaves on algebraic surfaces, we consider semistable pure two dimensional torsion sheaves on Calabi-Yau 3-folds, and DT invariants counting them. Let $X$ be a smooth projective Calabi-Yau 3-fold, $H$ an ample divisor on $X$ and fix a divisor class $P \in H^2(X, \mathbb{Z})$. We consider the following generating series

$$DT_H(P) = \sum_{\beta \in H_2(X), n \in \mathbb{Q}} DT_H(0, P, -\beta, -n - P \cdot c_2(X)/24) q^n t^\beta.$$  \hspace{1cm} (11)

Here each coefficient counts $H$-semistable $E \in \text{Coh}(X)$ whose Mukai vector (not Chern character) satisfies $\text{ch}(E) \cdot \sqrt{\text{td}(X)} = (0, P, -\beta, -n - P \cdot c_2(X)/24)$. As a 3-fold version of the S-duality conjecture, we expect that the series (11) satisfies a modular transformation property of (almost) Jacobi forms. (We refer to [EZ85] for a basic of Jacobi forms.) Some computations of the invariants $DT_H(0, P, -\beta, -n)$ are available in [GSa], [GSb]. Also the transformation formula of the series (11) under a flop is obtained in [Toda]. Let us consider a flop diagram (4) with $Y$ projective, and $\omega$ an ample divisor on $Y$. We assume that the exceptional locus $C, C^\dagger$ of $f, f^\dagger$ are isomorphic to $\mathbb{P}^1$ with $p = f(C) = f^\dagger(C^\dagger)$. Let $l$ be the scheme theoretic length of $f^{-1}(p)$ at the generic point of $C$.

**Theorem 4.3** ([Toda]). There exist $n_j \in \mathbb{Z}_{\geq 1}$ for $1 \leq j \leq l$ such that we have the following formula:

$$DT_{f^\dagger*\omega}(\phi_* P) = \phi_* DT_f^* \omega(P) \prod_{j=1}^l \left\{ t^{jP \cdot C} \eta(q)^{-1} \vartheta_{1,1}(q, ((-1)^3 P \cdot t)^j C^\dagger) \right\}^{jn_j P \cdot C}.$$  \hspace{1cm} (12)

Here $\phi_*$ is the variable change $(n, \beta) \mapsto (n, \phi_* \beta)$, $\eta(q)$ is the Dedekind eta function and $\vartheta_{1,1}(q,t)$ is the Jacobi theta function, given as follows:

$$\eta(q) = q^{1/24} \prod_{k \geq 1} (1 - q^k), \quad \vartheta_{1,1}(q, t) = \sum_{k \in \mathbb{Z}} q^{k(k+1)/2} (-t)^{k+1/2}.$$  \hspace{1cm} (13)

Although $f^* \omega$ is not ample, it is shown that the invariants $DT_{f^* \omega}(v)$ are well-defined. Recall that $\eta(q)$ is a modular form of weight $1/2$, $\vartheta_{1,1}(q,t)$ is a Jacobi form of weight $1/2$ and index $1/2$. The result of Theorem 4.3 shows that the series (11) transforms under a flop by a multiplication of a meromorphic Jacobi form, which gives evidence of the S-duality conjecture for DT invariants.

4.4. Mathematical approach toward OSV conjecture. In string theory, the OSV conjecture [OSV04] predicts a certain approximation

$$Z_{\text{BH}} \sim |Z_{\text{top}}|^2.$$  \hspace{1cm} (13)
where the LHS is the partition function of black hole entropy, and the RHS is the partition function of topological string. A version of the above conjecture is mathematically stated as an approximation between the generating series of DT invariants counting torsion sheaves on Calabi-Yau 3-folds, and the generating series of GW invariants. In [DM], Denef-Moore proposed a relationship among the series (11) and the generating series of $I_{n,\beta}$, $P_{n,\beta}$ in order to give a derivation of (13). A mathematical refinement of Denef-Moore conjecture is stated in [Tod13a]. For simplicity, suppose that Pic($X$) is generated by an ample divisor $H$. For $m \in \mathbb{Z}_{>0}$, we define the following cut off series

$$I^m(q,t) = \sum_{(\beta,n) \in C(m)} I_{n,\beta} q^{nt^\beta}, \quad P^m(q,t) = \sum_{(\beta,n) \in C(m)} P_{n,\beta} q^{nt^\beta}.$$ 

Here $C(m) = \{ (\beta,n) : \beta H < mH^3/2, |n| < m^2H^3/2 \}$. Moreover, we define the cut off generating series of D6-anti-D6 brane counting

$$Z_{\text{D6-\overline{D6}}}(q,t,w) = \sum_{m_2 - m_1 = m} q^{H^3(m_1^3 - m_2^3)/6} t^{H^2(m_1^2 - m_2^2)/2} w^{-mH^3/6 + Hc_2(X)m/12} I^m(qw^{-1}, q^{-m_1H}t^{1}w^{-mH}).$$

Conjecture 4.4 ([DM], [Tod13a]). For $m \gg 0$, we have the equality

$$\text{DT}_H(mH) = \frac{\partial}{\partial w} Z_{\text{D6-\overline{D6}}}(q,t,w)|_{w = -1}$$

modulo terms of $q^{nt^\beta}$ with

$$-\frac{H^3}{24} m^3 \left( 1 - \frac{1}{m} \right) \leq n + \frac{(\beta \cdot H)^2}{2mH^3}.$$

In [Tod13a], we proved the following:

Theorem 4.5 ([Tod13a]). The unweighted version of Conjecture 4.4 is true if we assume Conjecture 3.14.

Even if Conjecture 4.4 is proved, still the relationship (13) is not obvious. If we follow the arguments in [DM], at least we need to prove S-duality conjecture for DT invariants in the previous subsection and MNOP conjecture. Moreover we need to make a mathematical understanding of the approximation $\sim$ in (13). Although the relationship (13) is motivated by string theory, it seems to involve deep and interesting mathematics.

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