RELATIONS FOR ANNIHILATING FIELDS OF STANDARD MODULES FOR AFFINE LIE ALGEBRAS

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ABSTRACT. J. Lepowsky and R. L. Wilson initiated the approach to combinatorial Rogers-Ramanujan type identities via the vertex operator constructions of representations of affine Lie algebras. In a joint work with Arne Meurman this approach is developed further in the framework of vertex operator algebras. The main ingredients of that construction are defining relations for standard modules and relations among them. The arguments involve both representation theory and combinatorics, the final results hold only for affine Lie algebras $A_{k}^{(1)}$ and $A_{k}^{(2)}$. In the present paper some of those arguments are formulated and extended for general affine Lie algebras. The main result is a kind of rank theorem, guaranteeing the existence of combinatorial relations among relations, provided that certain purely combinatorial quantities are equal to dimensions of certain representation spaces. Although the result holds in quite general setting, applications are expected mainly for standard modules of affine Lie algebras.

1. Introduction

J. Lepowsky and R. L. Wilson gave in [LW] a Lie-theoretic interpretation and proof of the classical Rogers-Ramanujan identities in terms of representations of the affine Lie algebra $\hat{\mathfrak{sl}}(2, \mathbb{C})$. The identities are obtained by expressing in two ways the principal characters of vacuum spaces for the principal Heisenberg subalgebra of $\mathfrak{g}$. The product sides follow from the principally specialized Weyl-Kac character formula for level 3 standard $\mathfrak{g}$-modules; the sum sides follow from the vertex operator construction of bases of level 3 standard $\mathfrak{g}$-modules, parametrized by partitions satisfying difference 2 conditions. In fact, Lepowsky and Wilson gave a construction of combinatorial bases of all standard $\mathfrak{g}$-modules: the spanning follows in general from the vertex operator “generalized anticommutation relations”, but the Lie-theoretic proof of linear independence given for level 3 modules could not be extended for higher levels $k \geq 4$. In [MP1] a Lie-theoretic proof of linear independence of these bases was given. To be precise, the vertex operator construction of combinatorial bases of maximal submodules of the corresponding Verma modules was given, so that the linear independence was easy, and the spanning followed from the relations among the generalized anticommutation relations. Similar ideas have been used in [FNC] and [C].

In [MP2] these ideas were applied for the affine Lie algebra $\hat{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C})$ in the homogeneous picture by using the methods of vertex operator algebras (cf. [FLM], [FHL]). In this setting the “generalized anticommutation relations” $x_{\theta}(z)^{k+1} = 0$, studied before in [FF], are seen as annihilating fields of standard modules. For

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level $k$ standard $\hat{g}$-modules the annihilating fields are associated with vectors in the maximal ideal $N^1(k\mathfrak{L}_0)$ in the universal vertex operator algebra $N(k\mathfrak{L}_0)$. The maximal ideal is generated by a finite dimensional irreducible $\mathfrak{g}$-module $R$, and the coefficients of annihilating fields associated to elements in $R$ form a loop $\hat{g}$-module $\bar{R}$. The main point is that $\bar{R}$ is a set of defining relations for standard $\hat{g}$-modules: for every Verma module $M(\Lambda)$ with integral dominant $\Lambda$ the maximal submodule $M^1(\Lambda)$ can be written as

$$M^1(\Lambda) = \bar{R}M(\Lambda), \quad L(\Lambda) = M(\Lambda)/\bar{R}M(\Lambda).$$

Since the character of Verma module $M(\Lambda)$ is easily described, a combinatorial description of the character of standard module $L(\Lambda)$ may be obtained from a combinatorial description of $M^1(\Lambda)$. So in Lepowsky-Wilson’s approach a construction of combinatorial basis of $L(\Lambda)$ can be obtained by constructing first a combinatorial basis of $M^1(\Lambda)$. If $v_\Lambda \in M(\Lambda)$ is a highest weight vector, then it is natural to seek for a combinatorial basis of $M^1(\Lambda) = \bar{R}M(\Lambda)$ within the spanning set

$$r x_1 x_2 \ldots x_s v_\Lambda, \quad r \in \bar{R}, \ x_i \in \hat{g}.$$  \hspace{1cm} (1.1)

It is clear that it is enough to consider elements $x_i \in \hat{g}$ from some fixed basis $\hat{B}$ in $\hat{g}$. The corresponding monomial basis $\mathcal{P}$ of the symmetric algebra $S(\hat{g})$ can be used for parametrization of a monomial basis of the universal enveloping algebra $U(\hat{g})$: a monomial basis element $u(\pi) = x_1 x_2 \ldots x_s \in U(\hat{g})$ corresponds to $\pi = x_1 x_2 \ldots x_s \in \mathcal{P}$. The elements $\pi \in \mathcal{P}$ are interpreted as colored partitions.

In the same manner, there is a basis of the space of relations $\bar{R}$ parametrized by elements in $\mathcal{P}$. For a fixed order $\preceq$ on $\mathcal{P}$ every nonzero element $r \in \bar{R}$ is expanded in terms of monomial basis elements

$$r = c_\rho u(\rho) + \sum_{\pi \succ \rho} c_\pi u(\pi),$$

where $c_\rho, c_\pi \in \mathbb{C}, c_\rho \neq 0$. The first nonzero monomial $u(\rho)$ which appears in this expansion is called the leading term and denoted as $\rho = \bar{\ell}(r)$. Then there is a basis $\{r(\rho) | \rho \in \bar{\ell}(\bar{R})\}$ of $\bar{R}$ such that $\bar{\ell}(r(\rho)) = \rho$. So the spanning set (1.1) can be reduced to the spanning set

$$r(\rho) u(\pi) v_\Lambda, \quad \rho \in \bar{\ell}(\bar{R}), \ \pi \in \mathcal{P}.\hspace{1cm} (1.2)$$

In order to reduce this spanning set to a basis, one needs relations among vectors of the form $r(\rho) u(\pi) v_\Lambda$, or relations among operators of the form $r(\rho) u(\pi)$, in $\mathbb{M} \mathbb{P}^2, \mathbb{M} \mathbb{P}^3$ they are called relations among relations. Their precise form is given by [2,13], let us write here simply

$$r(\rho) u(\pi) \sim r(\rho') u(\pi') \quad \text{if} \quad \rho \pi = \rho' \pi'. \hspace{1cm} (1.3)$$

In $\mathbb{M} \mathbb{P}^2$ all relations among relations are found for $\hat{g} = \mathfrak{sl}(2, \mathbb{C})^-$-modules. As a result, the spanning set (1.2) is reduced to a spanning subset parameterized by colored partitions in $\mathcal{P}$ of the form $\rho \pi$, where $\rho \in \bar{\ell}(\bar{R}), \pi \in \mathcal{P}$. The linear independence of such set of vectors in $M(\Lambda)$ is easy to prove.

In a sharp contrast to the general results on algebraic properties of relations $\bar{R}$, the combinatorial relations (1.3) among these relations are very difficult to handle even for $\hat{g} = \mathfrak{sl}(2, \mathbb{C})^-$-modules. In the present paper some arguments and results in $\mathbb{M} \mathbb{P}^2$, Chapters 6 and 8] and $\mathbb{M} \mathbb{P}^3$ are formulated and extended for general
affine Lie algebras. The main result is Theorem 2.12, guaranteeing the existence of combinatorial relations among relations (1.3), provided that
\[
\sum_{\pi \in \mathcal{P}(\ell)(n)} N(\pi) = \dim Q(n).
\]
(1.4)
Here the left hand side is certain purely combinatorial quantity defined for colored partitions in terms of leading terms $\theta(\bar{R})$, and $\mathcal{P}(\ell)(n)$ denotes the set of colored partitions of degree $n$ and length $\ell$. On the other side, $Q(n)$ is the space of coefficients of degree $n$ of vertex operators $Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z)$ associated to elements of $Q$. By assumption $Q$ should be in the kernel of the map $\Phi: \bar{R} \otimes N(k\Lambda_0) \to N(k\Lambda_0)$, $\Phi(u \otimes v) = u_{-1}v$, and such that $Q \subset (\bar{R} \otimes N(k\Lambda_0))^{\ell}$ and $\ell(\mathcal{P}(n)) \subset \mathcal{P}(\ell)(n)$. Roughly speaking, the left hand side of (1.4) counts the number of relations (1.3) needed for combinatorial arguments, and the right hand side counts the number of relations obtained by using representation theory of vertex operator algebras.

The proof of Theorem 2.12 follows the proof of Lemma 4 in [MP3], the novelty is a general construction of relations among relations by using vertex operators $Y(q, z)$ associated to elements $q \in \ker \Phi$. Although the result holds under very mild assumptions on $R$, applications are expected mainly for standard modules of affine Lie algebras in a manner described above.

By Theorem 2.12 the construction of combinatorial basis of $M^1(\Lambda)$ within the spanning set (1.2) is reduced to verifying the equality (1.4). It is hoped that the equality holds in some generality, for now there are only few examples: Besides the motivating example for $\mathfrak{sl}(2, \mathbb{C})$, Ivica Siladić has found in [S] a basis $\tilde{B}$ such that for a level 1 twisted $\mathfrak{sl}(3, \mathbb{C})$-module the equality holds. For another motivating example, the basic $\mathfrak{sl}(3, \mathbb{C})$-module in the homogeneous picture, with a basis $\tilde{B}$ involving root vectors, the equality (1.4) does not hold for certain weight subspaces of $Q$, and one is forced to perform a sort of Gröbner base theory procedure to obtain a combinatorial basis of $M^1(\Lambda)$. It might be that here Theorem 2.12 could be applied as well if one takes a somewhat larger generating subspace $R \subset N^1(k\Lambda_0)$.

Throughout the paper $k \in \mathbb{C}$ is fixed, it will be the level of representations. Some ideas worked out in this paper come from collaboration with Arne Meurman for many years, and some even further from collaboration with Jim Lepowsky. I thank both Arne Meurman and Jim Lepowsky for their implicit contribution to this work. I also thank Ivica Siladić for numerous stimulating discussions.

2. Relations among relations in terms of vertex operators and leading terms

Bases of twisted affine Lie algebras and colored partitions. Let $\mathfrak{g}$ be a simple complex Lie algebra and $\sigma$ an automorphism of $\mathfrak{g}$ of finite order $T$, so that $\sigma^T = 1$. Set $\varepsilon = \exp(\frac{2\pi \sqrt{-1}}{T})$ and $\mathfrak{g}_{[j]} = \{x \in \mathfrak{g} | \sigma x = \varepsilon^j x\}$ for $j \in \mathbb{Z}/T\mathbb{Z}$. Let $B_{[j]}$ be a basis of $\mathfrak{g}_{[j]}$ and
\[
B = \bigcup_{j=0}^{T-1} B_{[j]}
\]
a basis of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\langle \ , \ \rangle$ a symmetric invariant bilinear form on $\mathfrak{g}$. Via this form we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ and we assume that $\langle \theta, \theta \rangle = 2$ for the maximal root $\theta$ (with respect to some fixed basis of the root system).
Remark. As a general rule, we shall write \( U \) both inherit the corresponding automorphism \( \sigma \). We write \( \tilde{\sigma} \) for an object whose construction involves \( \sigma \), and we shall write \( X \) instead of \( U^{\text{id}} \) when \( \sigma = \text{id} \). However, even when we consider the case \( \sigma \neq \text{id} \), we shall usually need both \( \tilde{\sigma} \) and \( \tilde{\sigma}[\sigma] \) and the corresponding objects \( X \) and \( X^{\sigma} \). In that case we define the automorphism \( \sigma \) of \( \tilde{\sigma}[\sigma] \) by \( \sigma(x(s/T)) = (\sigma x)(s/T) = \varepsilon^s x(s/T), \sigma c = c, \sigma d = d \).

In the case when the automorphism \( \sigma \) is the identity on \( \mathfrak{g} \), i.e., \( \sigma = \text{id} \) and \( T = 1 \), we write \( \tilde{\mathfrak{g}} \) instead of \( \tilde{\mathfrak{g}}[\text{id}] \). In this case we identify \( \mathfrak{g} \) with \( \mathfrak{g} \).

Let \( U^{\sigma} = U(\tilde{\mathfrak{g}}[\sigma])/(c - k) \), where \( U(\tilde{\mathfrak{g}}[\sigma]) \) is the universal enveloping algebra of \( \tilde{\mathfrak{g}}[\sigma] \), and for fixed \( k \in \mathbb{C} \) we denoted by \( (c - k) \) the ideal generated by the element \( c - k \). Note that \( \tilde{\mathfrak{g}}[\sigma] \)-modules of level \( k \) are \( U^{\sigma} \)-modules. Note that \( U(\tilde{\mathfrak{g}}[\sigma]) \) is graded by the derivation \( d \), and so is the quotient \( U^{\sigma} \). The associative algebra \( U^{\sigma} \) also inherits from \( U(\tilde{\mathfrak{g}}[\sigma]) \) the filtration \( U^{\sigma}_\ell, \ell \in \mathbb{Z}_{\geq 0} \); let us denote by \( S^{\sigma} \cong S(\tilde{\mathfrak{g}}[\sigma]) \) the corresponding commutative graded algebra.

We fix a basis \( \tilde{B} \) of \( \tilde{\mathfrak{g}}[\sigma] \),

\[
\tilde{B} = \tilde{B} \cup \{c, d\}, \quad \tilde{B} = \bigcup_{j \in \mathbb{Z}} B_j \otimes t^{j/T},
\]

so that \( \tilde{B} \) may also be viewed as a basis of \( \tilde{\mathfrak{g}}[\sigma] = \tilde{\mathfrak{g}}[\sigma]/\mathbb{C}c \). Let \( \preceq \) be a linear order on \( \tilde{B} \) such that

\[
i < j \quad \text{implies} \quad x(i) < x(j).
\]

The symmetric algebra \( S^{\sigma} \) has a basis \( \mathcal{P} \) consisting of monomials in basis elements \( \tilde{B} \). Elements \( \pi \in \mathcal{P} \) are finite products of the form

\[
\pi = \prod_{i=1}^{\ell} b_i(j_i), \quad b_i(j_i) \in \tilde{B},
\]

and we shall say that \( \pi \) is a colored partition of degree \( |\pi| = \sum_{i=1}^{\ell} j_i \in \frac{1}{T} \mathbb{Z} \) and length \( \ell(\pi) = \ell \), with parts \( b_i(j_i) \) of degree \( j_i \) and color \( b_i \). The set of all colored partitions of degree \( n \) and length \( \ell \) is denoted as \( \mathcal{P}^\ell(n) \). We shall usually assume that parts of \( \pi \) are indexed so that

\[
b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell).
\]

The basis element \( 1 \in \mathcal{P} \) we call a colored partition of degree 0 and length 0, we may also denote it by \( \emptyset \), suggesting it has no parts. Note that \( \mathcal{P} \subseteq S^{\sigma} \) is a monoid with the unit element 1, the product of monomials \( \pi \) and \( \rho \) is denoted by \( \pi \rho \).

We shall fix the monomial basis

\[
u(\pi) = b_1(j_1)b_2(j_2)\ldots b_n(j_n), \quad \pi \in \mathcal{P},
\]

Set \( \tilde{\mathfrak{g}}[\sigma] = \bigoplus_{j \in \mathbb{Z}} g_j \otimes t^{j/T} + \mathbb{C}c \). Then \( \tilde{\mathfrak{g}}[\sigma] = \tilde{\mathfrak{g}}[\sigma] + \mathbb{C}d \) is the associated twisted affine Lie algebra (cf. [R]) with the commutator

\[
[x(i), y(j)] = [x, y](i + j) + i\delta_{i+j,0}(x, y)c.
\]

Here, as usual, \( x(s/T) = x \otimes t^{s/T} \) for \( x \in g_j \) and \( s \in \mathbb{Z} \), \( c \) is the canonical central element, and \( [d, x(i)] = ix(i) \). Sometimes we shall denote \( g_j \otimes t^{j/T} \) by \( g(j/T) \). We define the automorphism \( \sigma \) of \( \tilde{\mathfrak{g}}[\sigma] \) by \( \sigma(x(s/T)) = (\sigma x)(s/T) = \varepsilon^s x(s/T), \sigma c = c, \sigma d = d \).

We fix a basis \( \tilde{\mathfrak{g}} \) of \( \tilde{\mathfrak{g}}[\sigma] \),

\[
\tilde{B} = \tilde{B} \cup \{c, d\}, \quad \tilde{B} = \bigcup_{j \in \mathbb{Z}} B_j \otimes t^{j/T},
\]

so that \( \tilde{B} \) may also be viewed as a basis of \( \tilde{\mathfrak{g}}[\sigma] = \tilde{\mathfrak{g}}[\sigma]/\mathbb{C}c \). Let \( \preceq \) be a linear order on \( \tilde{B} \) such that

\[
i < j \quad \text{implies} \quad x(i) < x(j).
\]

The symmetric algebra \( S^{\sigma} \) has a basis \( \mathcal{P} \) consisting of monomials in basis elements \( \tilde{B} \). Elements \( \pi \in \mathcal{P} \) are finite products of the form

\[
\pi = \prod_{i=1}^{\ell} b_i(j_i), \quad b_i(j_i) \in \tilde{B},
\]

and we shall say that \( \pi \) is a colored partition of degree \( |\pi| = \sum_{i=1}^{\ell} j_i \in \frac{1}{T} \mathbb{Z} \) and length \( \ell(\pi) = \ell \), with parts \( b_i(j_i) \) of degree \( j_i \) and color \( b_i \). The set of all colored partitions of degree \( n \) and length \( \ell \) is denoted as \( \mathcal{P}^{\ell}(n) \). We shall usually assume that parts of \( \pi \) are indexed so that

\[
b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell).
\]

The basis element \( 1 \in \mathcal{P} \) we call a colored partition of degree 0 and length 0, we may also denote it by \( \emptyset \), suggesting it has no parts. Note that \( \mathcal{P} \subseteq S^{\sigma} \) is a monoid with the unit element 1, the product of monomials \( \pi \) and \( \rho \) is denoted by \( \pi \rho \).

We shall fix the monomial basis

\[
u(\pi) = b_1(j_1)b_2(j_2)\ldots b_n(j_n), \quad \pi \in \mathcal{P},
\]
of the enveloping algebra $\mathcal{U}^\pi$. Then we have a slight modification of Lemma 6.4.1 in \[MP2\]:

**Lemma 2.1.** Let $\pi \in \mathcal{P}$. Then there exist a restricted $\hat{\mathfrak{g}}[\sigma]$-module $M$ of level $k$, a vector $v_\pi \in M$ and a functional $v_\pi^* \in M^*$ such that

$$\langle u_\pi^*, u(\pi)v_\pi \rangle \neq 0,$$

$$\langle v_\pi^*, u(\pi')v_\pi \rangle = 0 \quad \text{if} \quad \pi' \in \mathcal{P}, \pi' \neq \pi, \ell(\pi') \leq \ell(\pi).$$

**Proof.** Denote by $x^*, x \in B$, the elements of the dual basis of $B$ with respect to $\langle \ , \ \rangle$. For an element $b = x(i) \in \hat{B}$ define $b = x^*(-i)$. Note that for $x(i), y(j) \in \hat{B}$ we have

$$(2.2) \quad [x(i), y^*(-j)] = [x, y^*](i - j) + i\delta_{i,j}g_{x,y}c.$$ 

Set

$$\hat{g}[\sigma]_{>0} = \prod_{j>0} g_{[j]} \otimes t_j^\ell + C\mathsf{d}, \quad \hat{g}[\sigma]_{\geq 0} = \prod_{j \geq 0} g_{[j]} \otimes t_j^\ell + C\mathsf{d}.$$ 

For $s \in \mathcal{C}$ denote by $\mathcal{C}_s$ the one-dimensional $(\hat{\mathfrak{g}}[\sigma]_{\geq 0} + C\mathsf{c})$-module on which $\hat{\mathfrak{g}}[\sigma]_{\geq 0}$ acts trivially and $c$ as multiplication by $s$. Define induced modules

$$W(s) = U(\hat{\mathfrak{g}}[\sigma]) \otimes U(\hat{\mathfrak{g}}[\sigma]_{>0} + C\mathsf{c}) \mathcal{C}_s, \quad W = U(\hat{\mathfrak{g}}[\sigma])/U(\hat{\mathfrak{g}}[\sigma]_{>0} + C\mathsf{c}).$$

Let $\pi = \pi - \pi_0\pi_+$, where $\pi_- \in \mathcal{P}$ has parts of negative degrees, $\pi_0 \in \mathcal{P}$ has parts of degree zero, and $\pi_+ \in \mathcal{P}$ has parts of positive degrees. Let $p = \ell(\pi_-)$, $\ell(\pi_0) = q$, $r = \ell(\pi_+)$, $\ell = \ell(\pi) = p + q + r$ and set

$$M = W(k - r) \otimes (\otimes_{i=1}^p W(0)) \otimes W \otimes (\otimes_{i=1}^r W(1)).$$

Then $M$ is a $\hat{\mathfrak{g}}[\sigma]$-module of level $k$. Obviously $M$ is a restricted $\hat{\mathfrak{g}}[\sigma]$-module, i.e., for every $v \in M$ there is $n$ such that $(g_{[j]} \otimes t_j^\ell)v = 0$ for $j \geq n$.

Recall that $\mathcal{U}^\pi$ is a quotient of $U(\hat{\mathfrak{g}}[\sigma])$ and that $u(\rho) \in \mathcal{U}^\sigma$. Here we define elements $u(\rho) \in U(\hat{\mathfrak{g}}[\sigma])$, $\rho \in \mathcal{P}$, by the same formula (2.1). In $W(0)$ we choose the basis $u(\rho)v_0$, where $\rho \in \mathcal{P}$ has parts of negative degrees; for $W(k - r)$ and $W$ we choose bases in a similar way. For $\rho$ with parts $b_1 \leq \cdots \leq b_n$ of positive degrees we set $\tilde{u}(\rho) = b_1 \cdots b_n$, and in $W(1)$ we choose the basis of the form $\tilde{u}(\rho)v_1$. For a basis of $M$ we choose the corresponding tensor products.

Let $\pi_+ = b_1 \cdots b_r$ and let $v_\pi \in M$ be the basis vector

$$v_\pi = v_{k-r} \otimes (\otimes_{i=1}^p v_0) \otimes 1 \otimes (\otimes_{i=1}^r \tilde{b}_i v_1).$$

Let $v_\pi^*$ be the element of dual basis of $M^*$ corresponding to the vector

$$u = v_{k-r} \otimes (\otimes_{i=1}^p d_i v_0) \otimes u(\pi_0)1 \otimes (\otimes_{i=1}^r v_1),$$

where $d_i \in \hat{B}$ are the parts of $\pi_- = d_1 \cdots d_p$. It is clear from our choice and (2.2) that

$$v_\pi^*(u(\pi)v_\pi) > 0.$$
A filtration on a completion of the enveloping algebra and leading terms. The coefficients of vertex operators may be seen as infinite sums of elements in $U^\sigma$ and in certain arguments it is necessary to use some completion $\overline{U^\sigma}$ where these infinite sums converge. In [MP1, MP2] the topology of pointwise convergence on a certain set (category) of $U$-modules was used. However, it seems that another kind of topology, used in [F1, FF, FZ] and [KL], for example, has several advantages. Here we shall adopt the point of view taken by I. B. Frenkel and Y. Zhu: Let us denote the homogeneous components of the graded algebra $U^\sigma$ by $U^\sigma(n)$, $n \in \frac{1}{\sigma}Z$. We take

$$V_p(n) = \sum_{i \geq p} U^\sigma(n - i)U^\sigma(i), \quad p = 1, 2, \ldots,$$

to be a fundamental system of neighborhoods of $0 \in U^\sigma(n)$. It is easy to see that

$$\bigcap_{p=1}^\infty V_p(n) = \{0\},$$

so we have a Hausdorff topological group $(U^\sigma(n), +)$, and we denote by $\overline{U^\sigma(n)}$ the corresponding completion. Then

$$\overline{U^\sigma} = \prod_{n \in \frac{1}{\sigma}Z} \overline{U^\sigma(n)}$$

is a topological ring. It is clear that we have the action of $\sigma$ on both $\overline{U}$ and $\overline{U^\sigma}$.

Clearly $\bar{B} \subset \mathcal{P}$, viewed as colored partitions of length 1. We assume that on $\mathcal{P}$ we have a linear order $\preceq$ which extends the order $\preceq$ on $\bar{B}$. Moreover, we assume that order $\preceq$ on $\mathcal{P}$ has the following properties:

- $\ell(\pi) > \ell(\kappa)$ implies $\pi \prec \kappa$.
- $\ell(\pi) = \ell(\kappa), \|\pi\| < \|\kappa\|$ implies $\pi \prec \kappa$.
- Let $\ell(\pi) = \ell(\kappa), \|\pi\| = \|\kappa\|$. Let $\pi$ be a partition $b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell)$ and $\kappa$ a partition $a_1(i_1) \preceq a_2(i_2) \preceq \cdots \preceq a_\ell(i_\ell)$. Then $\pi \preceq \kappa$ implies $j_\ell \preceq i_\ell$.
- Let $\ell \geq 0, n \in \frac{1}{\sigma}Z$ and let $S \subset \mathcal{P}$ be a nonempty subset such that all $\pi$ in $S$ have length $\ell(\pi) \leq \ell$ and degree $\|\pi\| = n$. Then $S$ has a minimal element.
- $\mu \preceq \nu$ implies $\pi \mu \preceq \pi \nu$.

**Remark.** An order with these properties is used in [MP2, MP3]; the first three properties are a part of definition, the last two properties are guaranteed by Lemmas 6.2.1 and 6.2.2 in [MP2].

For $\pi \in \mathcal{P}, \|\pi\| = n$, set

$$U^\mathcal{P}_{\|\pi\|} = \overline{C\text{-span}\{u(\pi') \mid \|\pi'\| = \|\pi\|, \pi' \preceq \pi\}} \subset \overline{U^\sigma(n)},$$

$$U^\mathcal{P}_{\|\pi\|} = \overline{C\text{-span}\{u(\pi') \mid \|\pi'\| = \|\pi\|, \pi' \succeq \pi\}} \subset \overline{U^\sigma(n)},$$

the closure taken in $\overline{U^\sigma(n)}$. Set

$$U^\mathcal{P}(n) = \bigcup_{\pi \in \mathcal{P}, \|\pi\| = n} U^\mathcal{P}_{\|\pi\|}, \quad U^\mathcal{P} = \prod_{n \in \frac{1}{\sigma}Z} U^\mathcal{P}(n) \subset \overline{U^\sigma}.$$

As suggested by our notation, the construction of $U^\mathcal{P}$ depends on our choice of $(\mathcal{P}, \preceq)$. Since by assumption $\mu \preceq \nu$ implies $\pi \mu \preceq \pi \nu$, we have that $U^\mathcal{P}$ is a subalgebra of $\overline{U^\sigma}$. Obviously $U^\mathcal{P}(n)$ is filtered by $U^\mathcal{P}_{\|\pi\|}$. As in [MP2, Lemma 6.4.2], by using Lemma 2.1, we obtain:
Lemma 2.2. For \( \pi \in \mathcal{P} \) we have \( U^P_{[\pi]} = \mathcal{C} u(\pi) + U^P_{(\pi)} \). Moreover,
\[
\dim U^P_{[\pi]} / U^P_{(\pi)} = 1.
\]
For \( u \in U^P_{[\pi]} \), \( u \notin U^P_{(\pi)} \) we define the leading term
\[
\theta(u) = \pi.
\]

Proposition 2.3. Every element \( u \in U^P(n) \), \( u \neq 0 \), has a unique leading term \( \theta(u) \).

Proof. Since in the construction of \( U^P(n) \) we used a countable fundamental system of neighborhoods of 0, we may work only with Cauchy sequences, and step (3) in the proof of [MP2, Proposition 6.4.3] can be almost literally applied: The main point to prove is that
\[
\bigcap_{\ell(\pi) = \ell, |\pi| = n} U^P_{[\pi]} = \bigcup_{\ell(\pi) < \ell, |\pi| = n} U^P_{[\pi]}.
\]
The inclusion \( \supset \) is obvious from the assumed properties of our order \( \preceq \). So let \( x \in U^P_{[\pi]} \) for all \( \pi \) such that \( \ell(\pi) = \ell, |\pi| = n \). For each \( \pi \) we may choose a sequence \( (x^\pi_i) \) in \( \mathcal{C}\text{-span}\{u(\pi') | |\pi'| = n, \pi' \succeq \pi \} \) such that \( x^\pi_i \to x \). Let
\[
x^\pi_i = \sum_{\ell(\kappa) = \ell, |\kappa| = n} C_{\kappa,i} u(\kappa) + x^\pi_{i,\ell-1} = x^\pi_{i,\ell} + x^\pi_{i,\ell-1},
\]
where \( x^\pi_{i,\ell-1} \in U^P_{\ell-1}(n) \). For a sequence \( \pi^{(1)} \prec \pi^{(2)} \prec \cdots \) such that \( \ell(\pi^{(s)}) = \ell, |\pi^{(s)}| = n \) we can construct a diagonal sequence
\[
x^{\pi^{(s)}}_{i_s} \in x + V_s(n), \ s = 1, 2, \ldots, \quad i_s > i_{s'} \text{ for } s > s',
\]
so that \( x^{\pi^{(s)}}_{i_s} \to x \) when \( s \to \infty \).

Note that for a given \( M \) there is only finitely many partitions \( n = j_1 + \cdots + j_\ell \), \( j_p \in \frac{1}{2}\mathbb{Z}, j_1 \leq \cdots \leq j_\ell \), such that \( j_\ell \leq M \), and these partitions can be colored with elements in \( B \) in only finitely many ways. Hence for
\[
\pi^{(s)} = ((b^{(s)}_{1}(j^{(s)}_{1}), \leq b^{(s)}_{2}(j^{(s)}_{2}), \leq \cdots \leq b^{(s)}_{e}(j^{(s)}_{e}))
\]
we have that \( j^{(s)}_{\ell} \to \infty \) when \( s \to \infty \). Now it follows from our construction and the assumed properties of our order that \( x^{\pi^{(s)}}_{i_s,\ell} \to 0 \) when \( s \to \infty \). Hence we have the sequence
\[
x_s = x^{\pi^{(s)}}_{i_s,\ell-1} \in U^P_{[\pi_0]} \subset \bigcup_{\ell(\pi) < \ell} U^P_{[\pi]}
\]
such that \( x_s \to x \), where we denoted by \( \pi_0 \) the minimal element among colored partitions \( \pi \) of length \( \ell(\pi) \leq \ell - 1 \) and degree \( |\pi| = n \). So the other inclusion \( \subset \) holds as well.

Let \( x \in U^P_{[\pi]} \). Assume that \( x \notin U^P_{(\pi)} \), where \( \pi' \triangleright \pi \), \( \ell(\pi') = \ell(\pi) = \ell, |\pi'| = |\pi| = n \). Arguing as before we see that the interval \( [\pi, \pi'] \) is finite, so there is \( \pi \preceq \tau \preceq \pi' \) such that \( x \in U^P_{[\tau]} \setminus U^P_{(\tau)} \), and \( x \) has a leading term \( \tau \). On the other hand, if \( x \in U^P_{[\pi]} \) for all \( \pi' \triangleright \pi, \ell(\pi') = \ell, |\pi'| = n \), then we apply the first part of the argument, and in a finite number of steps we see that either \( x = 0 \) or \( x \) has a leading term.

Since for \( \pi \prec \kappa \) we have \( U^P_{[\pi]} \supset U^P_{[\kappa]} \), it is impossible to have \( u \in U^P_{[\pi]} \), \( u \notin U^P_{(\pi)} \) and \( u \in U^P_{[\kappa]} \), \( u \notin U^P_{(\kappa)} \). Hence the leading term is unique. \( \square \)
By Proposition 2.3 every nonzero homogeneous \( u \) has the unique leading term. For a nonzero element \( u \in U^P \) we define the leading term \( \check{\theta}(u) \) as the leading term of the nonzero homogeneous component of \( u \) of smallest degree. For a subset \( S \subset U^P \) set

\[
\check{\theta}(S) = \{ \check{\theta}(u) \mid u \in S, u \neq 0 \}.
\]

Clearly Lemma 2.2 and Proposition 2.3 imply the following:

**Proposition 2.4.** For all \( u, v \in U^P \backslash \{0\} \) we have \( \check{\theta}(uv) = \check{\theta}(u)\check{\theta}(v) \).

**Proposition 2.5.** Let \( W \subset U^P \) be a finite dimensional subspace and let \( \check{\theta}(W) \to W \) be a map such that

\[
\rho \mapsto w(\rho), \quad \check{\theta}(w(\rho)) = \rho.
\]

Then \( \{ w(\rho) \mid \rho \in \check{\theta}(W) \} \) is a basis of \( W \).

**Vertex operators with coefficients in a completion of the enveloping algebra.** The untwisted affine Lie algebra \( \hat{\mathfrak{g}} \) gives rise to the universal vertex operator algebra \( N(k\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0} + \mathbb{C}U_k)} \mathbb{C}u_k \) for \( k \neq -g^\vee \), where \( g^\vee \) is the dual Coxeter number of \( \mathfrak{g} \) (see [FZ] and [L1], we use the notation from [MP2]); it is generated by fields

\[
x(z) = \sum_{n \in \mathbb{Z}} x_n z^{-n-1}, \quad x \in \mathfrak{g},
\]

where we set \( x_n = x(n) \) for \( x \in \mathfrak{g} \). As usual, we shall write \( Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \) for the vertex operator (field) associated with a vector \( v \in N(k\Lambda_0) \). By formal definition, the coefficients \( v_n \) are linear operators on \( N(k\Lambda_0) \), but sometimes, as in [FF], [FZ] and [MP2], [MP3], it is convenient to think of \( v_n \) as elements in a completion of the universal enveloping algebra of \( \hat{\mathfrak{g}} \), i.e., \( v_n \in \mathcal{U} \). Then for any restricted \( \hat{\mathfrak{g}} \)-module \( M \) the elements \( v_n \in \mathcal{U} \) act on \( M \). The construction of a map

\[
Y : N(k\Lambda_0) \to \mathcal{U}[[z, z^{-1}]], \quad v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1},
\]

was given by I. B. Frenkel and Y. Zhu in [FZ] Definition 2.2.2. Their result can be interpreted in another way by using the approach developed by Haisheng Li: As in [FZ] Section 2.2, we may consider regular and mutually local fields \( a(z) \) and \( b(z) \) with coefficients \( a_n, b_n \in \mathcal{U} \) and, as in [L1] Lemma 3.1.4, we may define products \( a(z)_m b(z) \) for \( m \in \mathbb{Z} \). These products are well defined because infinite sums, appearing as coefficients of \( a(z)_m b(z) \), are convergent in \( \mathcal{U} \) (cf. [FZ] Definition 2.2.1). Since the proofs of [L1] Propositions 3.2.7 and 3.2.9 apply literally, we may introduce the notion of local system of vertex operators with coefficients in \( \mathcal{U} \) for which Theorem 3.2.10 in [L1] applies. In particular, we have a vertex operator algebra \( V \) generated by fields \( \mathfrak{g} \). Again by literally repeating the arguments in [L1], we get that Frenkel-Zhu’s map \( N(k\Lambda_0) \to V \) is an isomorphism of vertex operator algebras. To be more precise, by [FZ] Section 2.2 \( V \) is a highest weight \( \hat{\mathfrak{g}} \)-module, and we have a surjective homomorphism \( N(k\Lambda_0) \to V \) of vertex operator algebras; it must be an isomorphism because the adjoint representation of vertex operator algebra is faithful.
In the case when $\sigma$ is not the identity on $\mathfrak{g}$, we may use Haisheng Li’s theory \cite{L2} of local systems of twisted vertex operators. The twisted vertex operators
\[
x(z) = \sum_{n \in \frac{1}{T} \mathbb{Z}} x_n z^{-n-1}, \quad x \in \mathfrak{g},
\]
where we set $x_{s/T} = x(s/T)$ for $x \in \mathfrak{g}\{s\}$, generate a vertex operator algebra $V$, the $m$-products of fields are given by \cite[Definition 3.7]{L2}. The results in \cite{L2} also imply that we have a surjective homomorphism $N(k\Lambda_0) \to V$ of vertex operator algebras given by a map
\[
(2.4) \quad Y^\sigma : N(k\Lambda_0) \to \mathbb{T}[|z^{\frac{1}{T}}, z^{-\frac{1}{T}}|], \quad v \mapsto Y^\sigma(v, z) = \sum_{n \in \frac{1}{T} \mathbb{Z}} v_n z^{-n-1}.
\]
I am not aware of the existence of a faithful $\sigma$-twisted representation of $N(k\Lambda_0)$ in general (see \cite{L2}, Section 5 for inner automorphisms) and I don’t know whether \eqref{2.4} is an isomorphism in general. It is clear that for any restricted $\tilde{\sigma}$ in general (see \cite{L2} Definition 3.7). The results in \cite{L2} also imply that we have a surjective homomorphism $N(k\Lambda_0) \to V$ of vertex operator algebras given by a map
\[
Y^\sigma : N(k\Lambda_0) \to \mathbb{T}[|z^{\frac{1}{T}}, z^{-\frac{1}{T}}|], \quad v \mapsto Y^\sigma(v, z) = \sum_{n \in \frac{1}{T} \mathbb{Z}} v_n z^{-n-1}.
\]

When $\sigma \neq \text{id}$, we have the action of automorphism $\sigma$ on the vertex operator algebra $N(k\Lambda_0)$, as well as on both $\mathcal{U}$ and $\mathcal{U}^\sigma$. It is easy to see that for both untwisted and twisted fields we have $(\sigma v)_n = \sigma(v_n)$.

By following the notation in \cite{FF} we set
\[
U^\sigma_{\text{loc}} = \mathbb{C}\text{-span}\{v_n \mid v \in N(k\Lambda_0), n \in \frac{1}{T} \mathbb{Z}\} \subset \mathbb{T},
\]
where $v_n$ denotes a coefficient in $Y^\sigma(v, z)$. From the commutator formula (cf. \cite{L2} (2.31)) we see that $U^\sigma_{\text{loc}}$ is a Lie algebra. Let us denote by $U^\sigma$ the associative subalgebra of $\mathbb{T}$ generated by $U^\sigma_{\text{loc}}$. Clearly
\[
U^\sigma = \bigoplus_{n \in \frac{1}{T} \mathbb{Z}} U^\sigma(n),
\]
where $U^\sigma(n) \subset U^\sigma$ is the homogeneous subspace of degree $n$.

From the normal order product formula (cf. \cite{L2} (3.6)) we see by induction that
\[
U^\sigma_{\text{loc}} \subset U^\sigma = \bigoplus_{n \in \frac{1}{T} \mathbb{Z}} \left( \bigcup_{|\pi|=n} U^\sigma_{[\pi]} \right).
\]
Since $U^\sigma \subset \mathbb{T}$ is a subalgebra, we have $U^\sigma \subset U^\sigma$. Hence Propositions \ref{2.3} and \ref{2.4} imply that every nonzero element in $U^\sigma$ has the leading term and that $\ell(uv) = \ell(u)\ell(v)$. Note that $\ell(uv) > \ell([u,v])$, so in particular $\ell(uv) < \ell([u,v])$.

We may summarize our constructions with the following:

**Proposition 2.6.**
\[
\mathcal{U} \subset U^\sigma \subset U^\sigma \subset \mathbb{T}.
\]

A combinatorial formulation of relations among relations. From now on we fix $k \neq -g^\vee$. Then $N(k\Lambda_0)$ is a vertex operator algebra, and we denote by $\omega$ the conformal vector and by $L_n$, $n \in \mathbb{Z}$, the elements of the Virasoro algebra. We assume that $N(k\Lambda_0)$ is reducible and we denote by $N^1(k\Lambda_0)$ the maximal ideal.

From now on we also assume that $R \subset N(k\Lambda_0)$ is a nontrivial subspace such that:

- $R$ is finite dimensional,
- $R$ is invariant for $L_0$ and $\sigma$, 

\begin{itemize}
  \item $R$ is invariant for $\bigcap_{n \geq 0} g(n)$,
  \item $R \subset N^1(k\Lambda_0)$.
\end{itemize}

Set

$$\bar{R}^\sigma = \mathbb{C}\text{-span}\{r_n \mid r \in R, n \in \frac{1}{2}\mathbb{Z}\} \subset U^\sigma,$$

where $r_n$ denotes a coefficient in $Y^\sigma(r, z)$. Then $\bar{R}^\sigma$ is a $\tilde{g}[\sigma]$-module for the adjoint action given by the commutator formula

$$[x_m, r_n] = \sum_{i \geq 0} \binom{m}{i} (x_i r)_{m+n-i}, \quad x \in g, \ r \in R.$$

We shall say that $\bar{R}^\sigma$ is a loop module; in general it is reducible. When $\sigma$ is the identity, we shall write $\bar{R}$ instead of $\bar{R}^{ad}$.

Let $M(\Lambda)$ be a Verma module for $\tilde{g}[\sigma]$. Set

$$W(\Lambda) = \bar{R}^\sigma M(\Lambda).$$

Then $W(\Lambda)$ is a $\tilde{g}[\sigma]$-submodule of $M(\Lambda)$. If $W(\Lambda) \neq M(\Lambda)$, then we have a highest weight $\tilde{g}[\sigma]$-module

$$M(\Lambda)/W(\Lambda)$$

which is, by construction, annihilated by all $Y^\sigma(v, z)$ for elements $v$ in the ideal $\bar{R}N(k\Lambda_0)$. We shall call such vertex operators $Y^\sigma(v, z)$ the annihilating fields of the $\tilde{g}[\sigma]$-module $M(\Lambda)/W(\Lambda)$. Although obvious, it should be emphasized that it is the coefficients of annihilating fields that annihilate the module, and, since they define the module, we shall call them relations for $M(\Lambda)/W(\Lambda)$.

**Remark.** In the case of standard modules, i.e., when $k$ is a positive integer, we take a $g$-module $R$ generated by the singular vector $x_\theta(-1)^{k+1}1$. Then $\bar{R}N(k\Lambda_0)$ is the maximal ideal by the result of V. G. Kac [K, Corollary 10.4]. Moreover, for $\Lambda$ integral dominant $M(\Lambda)/W(\Lambda)$ is the standard module $L(\Lambda)$, and the structure of relations for $L(\Lambda)$ is well understood (cf. [DL], [F2], [L1] and [MP2, Chapter 5]). This is also true in a twisted case (cf. [B, Theorem 4.11] and [L2, Section 5]).

Our discussion might be also applicable to other admissible representations $\tilde{\Lambda}$ like the series $k \in \frac{1}{2} + \mathbb{N}$ for affine Lie algebras of type $A^{(1)}_1$ studied in [A], where all irreducible $L(k\Lambda_0)$-modules are of the form $L(\Lambda) = M(\Lambda)/W(\Lambda)$ for admissible weights $\Lambda$ of level $k$.

Since by assumption $R$ is finite dimensional, the space $\bar{R}^\sigma \subset U^\sigma$ is a direct sum of finite dimensional homogeneous subspaces. Hence Proposition 2.3 implies that we can parametrize a basis of $\bar{R}^\sigma$ by the set of leading terms $\theta(\bar{R}^\sigma)$: we fix a map

$$\theta(\bar{R}^\sigma) \to \bar{R}^\sigma, \quad \rho \mapsto r(\rho)$$

such that $r(\rho) \in U^\sigma(\rho), \ \theta(r(\rho)) = \rho$,

then $\{r(\rho) \mid \rho \in \theta(\bar{R}^\sigma)\}$ is a basis of $\bar{R}^\sigma$. We will assume that this map is such that the coefficient $C$ of “the leading term” $a(\rho)$ in “the expansion” of $r(\rho) = Cu(\rho) + \ldots$ is chosen to be $C = 1$. Note that our assumption $R \subset N^1(k\Lambda_0)$ implies that $1 \not\in \theta(\bar{R}^\sigma)$ and that $\theta(\bar{R}^\sigma) \cdot \mathcal{P}$ is a proper ideal in the semigroup $\mathcal{P}$.

Since the character of Verma module $M(\Lambda)$ is easily described, a combinatorial description of the character of $M(\Lambda)/W(\Lambda)$ may be obtained from a combinatorial description of $W(\Lambda)$. So in Lepowsky-Wilson’s approach a construction of combinatorial basis of $M(\Lambda)/W(\Lambda)$ can be obtained by constructing first a combinatorial
basis of $W(\Lambda)$. If $v_{\Lambda} \in M(\Lambda)$ is a highest weight vector, then it is natural to seek for a combinatorial basis of $W(\Lambda)$ within the spanning set
\begin{equation}
(2.5) \quad r(\rho)u(\pi)v_{\Lambda}, \quad \rho \in \hat{\vartheta}(\bar{R}^{\sigma}), \pi \in \mathcal{P}.
\end{equation}
In order to reduce this spanning set to a basis, one needs relations among vectors of the form $r(\rho)u(\pi)v_{\Lambda}$, or relations among operators of the form $r(\rho)u(\pi)$.

For colored partitions $\kappa$, $\rho$ and $\pi = \kappa \rho$ we shall write $\kappa = \pi/\rho$ and $\rho \subset \pi$. We shall say that $\rho \subset \pi$ is an embedding (of $\rho$ in $\pi$), notation suggesting that $\pi$ “contains” all the parts of $\rho$. For an embedding $\rho \subset \pi$, where $\rho \in \hat{\vartheta}(\bar{R})$, we define the element $u(\rho \subset \pi)$ in $U^{\sigma}$ by
\[ u(\rho \subset \pi) = \begin{cases} u(\pi/\rho) r(\rho) & \text{if } |\rho| > |\pi/\rho|, \\ r(\rho) u(\pi/\rho) & \text{if } |\rho| \leq |\pi/\rho|. \end{cases} \]

Instead of (2.5), it is convenient to consider a slightly modified spanning set
\begin{equation}
(2.6) \quad u(\rho \subset \pi)v_{\Lambda}, \quad \rho \in \hat{\vartheta}(\bar{R}^{\sigma}), \pi \in \mathcal{P}.
\end{equation}
Note that, by Proposition 2.4, we have $\hat{\vartheta} (u(\rho \subset \pi)) = \pi$. It is clear that in general we shall have several embeddings in given $\pi$, say $\rho_1 \subset \pi$, $\rho_2 \subset \pi$, ..., giving several vectors in the spanning set (2.6) with the same leading term $\hat{\vartheta} (u(\rho_1 \subset \pi)) = \hat{\vartheta} (u(\rho_2 \subset \pi)) = \cdots = \pi$. If we can prove that for any two embeddings $\rho_1 \subset \pi$, $\rho_2 \subset \pi$ there is a relation of the form
\begin{equation}
(2.7) \quad u(\rho_1 \subset \pi) \in u(\rho_2 \subset \pi) + \overline{\text{span}}\{ u(\rho \subset \pi') \mid \rho \in \hat{\vartheta}(\bar{R}^{\sigma}), \rho \subset \pi', \pi \subset \pi' \},
\end{equation}
then we can reduce the spanning set (2.6) to a spanning set of $W(\Lambda)$ indexed by colored partitions:
\begin{equation}
(2.8) \quad u(\rho \subset \pi)v_{\Lambda}, \quad \pi \in \hat{\vartheta}(\bar{R}^{\sigma}) \cdot \mathcal{P},
\end{equation}
where for each $\pi \in \hat{\vartheta}(\bar{R}^{\sigma}) \cdot \mathcal{P}$ we choose only one embedding $\rho \subset \pi$. Roughly speaking (i.e., if we forget the so-called initial conditions, see [MP1, Chapter 6] for the correct formulation), the spanning set (2.8) should be a basis of $W(\Lambda)$; the linear independence should follow easily from the fact that our vectors $u(\rho \subset \pi)v_{\Lambda} = u(\pi \subset \pi) v_{\Lambda} + \ldots$ are elements of the Verma module $M(\Lambda)$ and that the leading term $\pi = \hat{\vartheta} (u(\rho \subset \pi))$ appears only once in the spanning set (2.8).

We may call a relation of the form (2.7) a combinatorial relation among relations.

A vertex operator algebra formulation of relations among relations. Recall that $N(k\Lambda_0) \otimes N(k\Lambda_0)$ is a vertex operator algebra; the fields are defined by $Y(a \otimes b, z) = Y(a, z) \otimes Y(b, z)$, the conformal vector is $\omega \otimes 1 + 1 \otimes \omega$ (cf. [FH]). In particular, the derivation $D = L_{-1}$ is given by $D \otimes 1 + 1 \otimes D$, the degree operator $-d = L_0$ is given by $L_0 \otimes 1 + 1 \otimes L_0$, and we have the action of $g = g(0)$ given by $x \otimes 1 + 1 \otimes x$. We consider the automorphism $\sigma \otimes \sigma$ of vertex operator algebra $N(k\Lambda_0) \otimes N(k\Lambda_0)$, also denoted by $\sigma$.

For a $\sigma$-twisted $N(k\Lambda_0)$-module $M$ given by twisted vertex operators $Y^\sigma_M(a, z)$ we have the $\sigma$-twisted $(N(k\Lambda_0) \otimes N(k\Lambda_0))$-module $M \otimes M$ given by $\sigma$-twisted vertex operators $Y^\sigma_M(a, z) \otimes Y^\sigma_M(b, z)$. The coefficients $(a \otimes b)_n$ of $Y^\sigma_M(a, z) \otimes Y^\sigma_M(b, z)$ are well defined operators on $M \otimes M$ given by
\begin{equation}
(2.9) \quad (a \otimes b)_n = \sum_{i+j+1=n} a_i \otimes b_j.
\end{equation}
Note that we have a linear map between fields

\[(2.10) \quad Y_M^\sigma(a, z) \otimes Y_M^\sigma(b, z) \mapsto (Y_M^\sigma(a, z))_{-1}(Y_M^\sigma(b, z)) = Y_M^\sigma(a_{-1}b, z),\]

where the \((-1)\)-product for mutually local twisted fields is given by \([\text{L2}, \text{Definition } 3.7]\). Set

\[\Phi: N(k\Lambda_0) \otimes N(k\Lambda_0) \to N(k\Lambda_0), \quad \Phi(a \otimes b) = a_{-1}b.\]

Although we don’t have a state-field correspondence for vertex operators on the module \(M\), the map \(\Phi\) between vectors corresponds to the map \((2.10)\) between fields. It is easy to see that \(\Phi\) intertwines the actions of \(L_{-1}, L_0, g(0)\) and \(\sigma\), so that \(\ker \Phi\) is also invariant for the actions of these operators.

The basic idea for constructing relations for annihilating fields \(Y_M^\sigma(a, z), a \in \hat{N}(k\Lambda_0)\) is to use the following observation:

**Proposition 2.7.** \(\sum a \otimes b \in \ker \Phi \implies \sum (Y_M^\sigma(a, z))_{-1}(Y_M^\sigma(b, z)) = 0.\)

Since it is the coefficients of annihilating fields that are “true” relations, we shall need a slight refinement of the map \((2.10)\). As before, we want to consider vertex operators \(Y^\sigma(a, z)\) with coefficients in \(U^\sigma\). But then we have to make sense of \((2.9)\):

For a fixed \(n \in \frac{1}{\Lambda} \mathbb{Z}\) we have a linear map

\[\chi_n^\sigma: N(k\Lambda_0) \otimes N(k\Lambda_0) \to (U^\sigma \otimes U^\sigma) \frac{1}{\Lambda} \mathbb{Z}, \quad a \otimes b \mapsto (a_i \otimes b_{n-i-1} \mid i \in \frac{1}{\Lambda} \mathbb{Z}).\]

So formally \(\chi_n^\sigma(a \otimes b)\) is a sequence in \(U^\sigma \otimes U^\sigma\), but we should think of it as “the \(n\)-th coefficient \((a \otimes b)_n\) of the vertex operator \(Y^\sigma(a \otimes b, z) = Y^\sigma(a, z) \otimes Y^\sigma(b, z)\),” and we shall formally write

\[\chi_n^\sigma(a \otimes b) = \sum_{i+j+1=n} a_i \otimes b_j \in K_n^\sigma,\]

where \(K_n^\sigma = \chi_n^\sigma(N(k\Lambda_0) \otimes N(k\Lambda_0))\) is a linear subspace of \((U^\sigma \otimes U^\sigma) \frac{1}{\Lambda} \mathbb{Z}\). Now we define a linear map \(\Phi_n^\sigma: K_n^\sigma \to U^\sigma\) given by

\[\Phi_n^\sigma: (a_p \otimes b_{n-p-1} \mid p \in \frac{1}{\Lambda} \mathbb{Z}) \mapsto \sum_{j \geq 0 \atop i+j+1=m} a_{j+r} b_{i+n} + \sum_{j \geq 0 \atop i+j+1=m} b_{i+n} a_{j+r} + \sum_{l \geq 0 \atop i+j+1=m} (-1)^l \binom{\frac{r}{\Lambda}}{l} \sum_{j=-1 \atop i+j+1=m}^{l-1} [a_{j+r} b_{i+n}]\]

for \(a \in N(k\Lambda_0)^*\), \(b \in N(k\Lambda_0)^*\), \(n = m + \frac{r}{\Lambda} + \frac{q}{\Lambda}\). We should see that this map is well defined. So first recall that we have the automorphism \(\sigma\) on \(N(k\Lambda_0)\) and \(U^\sigma\) and that \((\sigma a)_p = \sigma(a_p)\). We also have \(\sigma(a \otimes 1) = 1 \otimes \sigma(a)\) and on \((U^\sigma \otimes U^\sigma) \frac{1}{\Lambda} \mathbb{Z}\) and \(\chi_n^\sigma\) intertwines these actions. Hence \(K_n^\sigma\) is invariant for these actions as well, and the assumption \(a \in N(k\Lambda_0)^*\), \(b \in N(k\Lambda_0)^*\) means that we are defining a map for elements in \(K_n^\sigma\) with eigenvalues \(\varepsilon^\tau\) for \(\sigma \otimes 1\) and \(\varepsilon^s\) for \(1 \otimes \sigma\). Next note that each term that appears on the right hand side is a linear function of some coordinate \(a_p \otimes b_q\), \(p + q + 1 = n\). Since infinite sums are convergent, we have that \(\Phi_n^\sigma\) is well defined linear map, provided that we specify what “finite sum” means. So, finally, note that the right hand side is the expression for the coefficient \((a_{-1}b)_n\) in the \((-1)\)-product \([\text{L2}, \text{Definition } 3.7]\) of mutually local
twisted fields $Y^\sigma(a, z)$ and $Y^\sigma(b, z)$. Hence there is certain $N(a, b)$ such that $a_j b = 0$
for $j > N(a, b)$, and we should take the last term to be
\[
\sum_{i=1}^{N} (-1)^i \binom{r}{i} \sum_{j=-i}^{-1} \binom{l-1}{j+l} [a_j + \frac{r}{T}, b_{j+\frac{r}{T}}]
\]
for any $N \geq N(a, b)$. Our discussion shows that we have a linear map
\[
\Phi_n^{q_\sigma}: K_n^q \rightarrow U^\sigma, \quad \Phi_n^{q_\sigma} \left( \sum_{i+j+1 = n} a_i \otimes b_j \right) = (a_{-1} b)_n,
\]
where $(a_{-1} b)_n$ is the coefficient of the twisted vertex operator $Y^\sigma(a_{-1} b, z)$.

Hence, by construction, we have the following consequence of Proposition 2.7:

**Proposition 2.8.** For each $q \in \ker (\Phi|\tilde{R} \otimes N(kA_0))$ and $n \in \frac{1}{T} \mathbb{Z}$ we have

\[
(2.11) \quad \Phi_n^{q_\sigma}(\chi_n^\sigma(q)) = 0.
\]

It is clear that $\Phi_n^{q_\sigma}(\chi_n^\sigma(q)) = 0$ is a relation among coefficients of annihilating fields, but it will be more convenient to say that $q_n = \chi_n^\sigma(q)$ itself is a relation among coefficients of annihilating fields, or a relation among relations. It will be also more convenient to consider “coefficients $q_n$ of vertex operators $Y^\sigma(q, z)$” in a slightly different way.

**A connection between two formulations of relations among relations.** Let $a \in N(kA_0)$ be a homogeneous element of weight $\text{wt} a$, i.e., $L_0 a = (\text{wt} a) a$. Then the coefficient $a_i \in U^\sigma$ of the twisted vertex operator $Y^\sigma(a, z)$ is of degree $(i + 1 - \text{wt} a)$, i.e., $[d, a_i] = -[L_0, a_i] = (i + 1 - \text{wt} a) a$. Set $a_i = a(i + 1 - \text{wt} a)$. Then we have
\[
Y^\sigma(a, z) = \sum_{i \in \frac{1}{T} \mathbb{Z}} a_i z^{-i-1} = \sum_{n \in \frac{1}{T} \mathbb{Z}} a(n) z^{-n-\text{wt} a},
\]
where $a(n) \in U^\sigma(n)$. Likewise, for a homogeneous element $q = a \otimes b$ in $N(kA_0) \otimes N(kA_0)$ of weight wt $q = \text{wt} a + \text{wt} b$ we shall write
\[
Y^\sigma_{M \otimes M}(q, z) = \sum_{n \in \frac{1}{T} \mathbb{Z}} q(n) z^{-n-\text{wt} q}, \quad q(n) = \sum_{i+j = n} a(i) \otimes b(j),
\]
where $q(n)$ are operators on $M \otimes M$. Again we want to make sense of this formula for $a(i), b(j) \in U^\sigma$, but in a slightly different way than before.

Set
\[
(U^\sigma \otimes U^\sigma)(n) = \prod_{i+j = n} (U^\sigma(i) \otimes U^\sigma(j)), \quad U^\sigma \otimes U^\sigma = \prod_{n \in \frac{1}{T} \mathbb{Z}} (U^\sigma \otimes U^\sigma)(n).
\]

The elements of $U^\sigma \otimes U^\sigma$ are linear combinations of homogeneous sequences in $U^\sigma \otimes U^\sigma$, usually we shall denote them as $\sum_{i+j = n} a(i) \otimes b(j)$. For a fixed $n \in \frac{1}{T} \mathbb{Z}$ we have a linear map
\[
\chi^\sigma(n): N(kA_0) \otimes N(kA_0) \rightarrow (U^\sigma \otimes U^\sigma)(n)
\]
defined for homogeneous elements $a$ and $b$ by
\[
\chi^\sigma(n): a \otimes b \mapsto \sum_{p+r = n} a(p) \otimes b(r).
\]
Again we think of $\chi^\sigma(n)(q)$ as “the coefficient of $q(n)$ of the vertex operator $Y^\sigma(q, z)$”, but now in another space. We shall usually write $q(n) = \chi^\sigma(n)(q)$ for an element $q \in N(kA_0) \otimes N(kA_0)$ and $Q(n) = \chi^\sigma(n)(Q)$ for a subspace $Q \subset N(kA_0) \otimes N(kA_0)$.

Since we have the adjoint action of $\hat{g}[\sigma]$ on $U^\sigma$, we define “the adjoint action” of $\hat{g}[\sigma]$ on $U^\sigma \otimes U^\sigma$ by

$$[x(m), \sum_{p+r=n} a(p) \otimes b(r)] = \sum_{p+r=n} [x(m), a(p)] \otimes b(r) + \sum_{p+r=n} a(p) \otimes [x(m), b(r)].$$

Note that we have the action of $\hat{g}$ on $N(kA_0) \otimes N(kA_0)$ given by

$$x_i(a \otimes b) = (x_i a) \otimes b + a \otimes (x_i b), \quad x \in \mathfrak{g}, \; i \in \mathbb{Z}.$$  

As expected, we have the following commutator formula for $q(n) = \chi^\sigma(n)(q)$:

**Proposition 2.9.** For $x(m) \in \hat{g}[\sigma]$ and homogeneous $q$ we have

$$[x(m), q(n)] = \sum_{i \geq 0} \binom{m}{i} (x_i q)(m + n), \quad (Dq)(n) = -(n + \text{wt } q) q(n).$$

So if a subspace $Q \subset N(kA_0) \otimes N(kA_0)$ is invariant for $\prod_{n \in \mathbb{Z}} Q(n)$ is a loop $\hat{g}[\sigma]$-module, in general reducible.

Let us define the map $\Psi^\sigma: U^\sigma \otimes U^\sigma \rightarrow U^\sigma$ by

$$\Psi^\sigma: \sum_{p+r=n} a(p) \otimes b(r) \rightarrow \sum_{p+r=n} a(p)b(r) + \sum_{p+r=n} b(r)a(p)$$

(compare with the map $\Psi$ in [MP3]). Then for homogeneous $a \in N(kA_0)^r$, $b \in N(kA_0)^s$, $n = m + \frac{T}{r} + \frac{T}{s}$, $n' = n + 1 - \text{wt } a - \text{wt } b$, we have

$$\Phi^\sigma_n(\chi^\sigma_n(a \otimes b)) = \Phi^\sigma_n(\chi^\sigma_n(n')(a \otimes b))$$

$$= - \sum_{j=0}^{\text{wt } a-2} \sum_{i+j+1=m} [a_j^\frac{T}{r}, b_{i+1}^\frac{T}{s}] + \sum_{l \in \mathbb{Z} > 0} (-1)^l \binom{\frac{T}{r}}{l} \sum_{j=-l}^{l-1} \sum_{i+j+1=m} \binom{l-1}{j} [a_{j+\frac{T}{r}}, b_{i+\frac{T}{s}}].$$

By using this formula we can rewrite the relations (2.11) among coefficients of annihilating fields in the following way (compare with Proposition 3 in [MP3]):

**Proposition 2.10.** Let $q \in \ker \Phi$ be written as $q = \sum a \otimes b$ in terms of homogeneous elements, and write $r(a) = r \in \{0, 1, \ldots, T-1\}$ for $a \in N(kA_0)^r$. Then for $n \in \frac{1}{r} \mathbb{Z}$ we have

$$\Psi^\sigma(q(n)) = \sum_{a, b} \sum_{i+j=n} \sum_{1-\text{wt } a \leq i \leq -1} [a(i), b(j)]$$

$$- \sum_{a, b} \sum_{l \in \mathbb{Z} > 0} (-1)^l \binom{\frac{r(a)}{l}}{l} \sum_{i+j=n} \sum_{1-\text{wt } a \leq i \leq -1} \binom{l-1}{l} (-1)^{-l-1} [a(i), b(j)].$$
Now assume that \( q = \sum a \otimes b \) is a homogeneous element in \( \hat{R} \otimes N(k\Lambda_0) \). Note that for \( a \in \hat{R} \) the coefficient \( a(\ell) \) of the corresponding field \( Y^a(a, z) \) can be written as a finite linear combination of basis elements \( r(\rho), \rho \in \hat{R}(R^\sigma) \). Hence each element of the sequence \( q(n) = \chi^a(n)(\sum a \otimes b) \in (U^\sigma \otimes U^\sigma)(n) \), say \( c_i \), can be written uniquely as a finite sum of the form

\[
  c_i = \sum_{\rho \in \hat{R}(R^\sigma)} r(\rho) \otimes b_\rho,
\]

where \( b_\rho \in U^\sigma \). If \( b_\rho \neq 0 \), then it is clear that \( |\rho| + |\hat{\ell}(b_\rho)| = n \). Let us assume that \( q(n) \neq 0 \), and for nonzero \( \ell \)-th component \( c_i \) let \( \pi_i \) be the smallest possible \( \rho \ell \) that appears in the expression for \( c_i \). Denote by \( S \) the set of all such \( \pi_i \).

Since \( q \) is a finite sum of elements of the form \( a \otimes b \), it is clear that there is \( \ell \) such that \( \ell(\pi_i) \leq \ell \). Then, by our assumptions on the order \( \preceq \), the set \( S \) has the minimal element, and we call it the leading term \( \hat{\ell}(q(n)) \) of \( q(n) \). For a subspace \( Q \subset \hat{R} \otimes V \) set

\[
  \hat{\ell}(Q(n)) = \{ \hat{\ell}(q(n)) \mid q \in Q, q(n) \neq 0 \}.
\]

Note that our definition of leading terms depends on \( R^\sigma \), and that we have not defined leading terms for general elements in \( (U^\sigma \otimes U^\sigma)(n) \).

For a colored partition \( \pi \) of set

\[
  N(\pi) = \max\{ \#E(\pi) - 1, 0 \}, \quad E(\pi) = \{ \rho \in \hat{\ell}(\hat{R}) \mid \rho \subset \pi \}.
\]

Note that \( N(k\Lambda_0) \otimes N(k\Lambda_0) \) has a natural filtration \( (N(k\Lambda_0) \otimes N(k\Lambda_0))_\ell, \ell \in \mathbb{Z}_{\geq 0} \), inherited from the filtration \( U_\ell, \ell \in \mathbb{Z}_{\geq 0} \).

**Lemma 2.11** (MP3). Let \( Q \subset \ker(\Phi(\hat{R} \otimes N(k\Lambda_0))_\ell) \) be a finite dimensional subspace and \( n \in \mathbb{Z} \). Assume that \( \hat{\ell}(\pi) = \ell \) for all \( \pi \in \hat{\ell}(Q(n)) \). Then

\[
  (2.12) \quad \sum_{\pi \in \hat{\ell}(Q(n))} N(\pi) \geq \dim Q(n).
\]

Moreover, if in \( (2.13) \) the equality holds, then for any two embeddings \( \rho_1 \subset \pi, \rho_2 \subset \pi \), where \( \rho_1, \rho_2 \in \hat{\ell}(\hat{R}^\sigma) \) and \( \pi \in \hat{\ell}(Q(n)) \), we have a relation

\[
  (2.13) \quad u(\rho_1 \subset \pi) \in u(\rho_2 \subset \pi) + \overline{\text{C-span}}\{ u(\rho \subset \pi') \mid \rho \in \hat{\ell}(\hat{R}^\sigma), \rho \subset \pi', \pi' \preceq \pi \}.
\]

**Proof.** Let \( Q_{[\pi]} = \{ q \in Q(n) \mid \hat{\ell}(q) \geq \pi \} \) and \( Q_{(\pi)} = \{ q \in Q(n) \mid \hat{\ell}(q) > \pi \} \). Let \( \dim Q_{[\pi]}/Q_{(\pi)} = m(\pi) = m \leq \dim Q(n), m \geq 1 \) and let \( \rho_1 \subset \pi, \ldots, \rho_s \subset \pi \) (where \( s = s(\pi) \)) be all possible embeddings in \( \pi \). Let \( \pi^* \) be such that \( Q_{(\pi)} = Q_{[\pi^*]} \). Let us follow the proof of Lemma 4 in MP3; we can write a basis of \( Q_{[\pi]}/Q_{(\pi)} \) in the form

\[
  c_{11}r(\rho_1) \otimes u(\pi/\rho_1) + \cdots + c_{1s}r(\rho_s) \otimes u(\pi/\rho_s) + v_1 + Q_{(\pi)},
  
  c_{21}r(\rho_1) \otimes u(\pi/\rho_1) + \cdots + c_{2s}r(\rho_s) \otimes u(\pi/\rho_s) + v_2 + Q_{(\pi)},
  
  \cdots
  
  c_{m1}r(\rho_1) \otimes u(\pi/\rho_1) + \cdots + c_{ms}r(\rho_s) \otimes u(\pi/\rho_s) + v_m + Q_{(\pi)},
\]

where vectors \( v_i \) are of the form

\[
  v_i = \sum_{\pi \prec \pi^*} \sum_{\rho \in \hat{\ell}(\pi')} r(\rho) \otimes d_{i,\rho,\pi},
  + \sum_{\pi \preceq \pi^*} \sum_{\rho \in \hat{\ell}(\pi')} r(\rho) \otimes e_{i,\rho,\pi'} \in (U^\sigma \otimes U^\sigma)(n),
\]

with \( d_{i,\rho,\pi}, e_{i,\rho,\pi'} \in U^\sigma \), \( \hat{\ell}(d_{i,\rho,\pi}) = \pi'/\rho, \hat{\ell}(e_{i,\rho,\pi'}) = \pi'/\rho \).
Assume that rank \((c_{ij}) < m\). Then the rows are linearly dependent. By taking a nontrivial linear combination of the basis elements we get a vector in \(Q(n)\) of the form

\[
v = \sum_{\pi \preceq \pi'} \sum_{\rho \in \mathcal{E}(\pi')} r(\rho) \otimes d_{\rho, \pi'} + \sum_{\pi' \preceq \pi} \sum_{\rho \in \mathcal{E}(\pi')} r(\rho) \otimes c_{\rho, \pi'}.
\]

The elements \(d_{\rho, \pi'} \in U^\sigma, \rho \in \mathcal{E}(\pi'), \pi \preceq \pi' \prec \pi^\ast,\) must be zero since otherwise \(Q(\pi) \neq Q(\pi^\ast)\). But then \(v \in Q(\pi) = Q(\pi^\ast)\) and our nontrivial linear combination of basis elements is zero in \(Q(\pi)/Q(\pi^\ast)\), a contradiction.

Since elements in \((U^n \otimes U^n)(n)\) are really the sequences, we should check that the above argument makes sense; it works because the “summands” \(r(\rho_1) \otimes u(\pi/\rho_1)\), for example, are always at the same place in a sequence, depending on a degree \(|\rho_1|\) of \(r(\rho_1)\). (Note that the above argument would not make sense for sequences in \((U^n \otimes U^n)^{\pi/\rho}\).)

Hence the rank of matrix \((c_{ij})\) is \(m\) and we have \(s \geq m\). Assume that \(s = m\). Then the matrix \((c_{ij})\) is regular and the vectors of the form \(r(\rho_1) \otimes u(\pi/\rho_1) + v_1 + Q(\pi), \ldots, r(\rho_m) \otimes u(\pi/\rho_m) + v_m + Q(\pi)\) (vectors \(v_i\) as above) are a basis of \(Q(\pi)/Q(\pi^\ast)\). In particular, we have a vector \(u \in Q(n)\) of the form \(u = r(\rho_1) \otimes u(\pi/\rho_1) + v_1\). From the definition of \(\Psi^\sigma\) we have

\[
\Psi^\sigma(u) \in u(\pi) + U^P_{(\pi)}.
\]

On the other hand \(u = q(n)\) for some \(q \in Q \subset \ker \Phi\), and by assumption \(\ell(\theta(u)) = \ell(q(n)) = \ell, \) so Proposition 2.10 and \(\ell(\theta(ab)) > \ell(\theta([a, b]))\) imply

\[
\Psi^\sigma(u) \notin U^P_{(\pi)}.
\]

what is in contradiction with Lemma 2.2.

Hence \(s > m\), i.e. \(N(\pi) = s(\pi) - 1 \geq m(\pi)\). Since \(\dim Q(n) = \sum m(\pi)\), the inequality (2.11) follows.

Let us assume that \(N(\pi) = m(\pi)\). Then there are altogether \(s = m + 1\) possible embeddings \(\rho_1 \subset \pi, \ldots, \rho_s \subset \pi\) and the rank of \((c_{ij})\) is \(m = s - 1\). Let us assume that the first \(s - 1\) columns of \((c_{ij})\) are linearly independent. Then for each \(1 \leq i \leq s - 1\) there is a vector in \(Q(n)\) of the form

\[
r(\rho_i) \otimes u(\pi/\rho_i) + d_i r(\rho_s) \otimes u(\pi/\rho_s) + \sum_{\pi \preceq \pi'} \sum_{\rho \in \mathcal{E}(\pi')} r(\rho) \otimes d_{i, \rho, \pi'}
\]

for some \(d_i \in \mathbb{C}\) and \(d_{ij, \rho, \pi'} \in U^\sigma, \rho \in \mathcal{E}(\pi'), \rho \preceq \rho_s\). As it was seen before, the assumption \(Q \subset \ker \Phi\) implies \(d_i \neq 0\). But then for each \(i, j \in \{1, \ldots, s\}\) there are \(d_{ij} \in \mathbb{C}, d_{ij, j, \rho, \pi'} \in U^\sigma\) such that

\[
r(\rho_1) \otimes u(\pi/\rho_1) + d_{ij} r(\rho_j) \otimes u(\pi/\rho_j) + \sum_{\pi \preceq \pi'} \sum_{\rho \in \mathcal{E}(\pi')} r(\rho) \otimes d_{ij, j, \rho, \pi'} \in Q(n).
\]

By applying \(\Psi^\sigma\) to these vectors, and taking into account Proposition 2.10, we obtain relations (2.13).

Even for low dimensional \(Q\) it is not easy to determine the set \(\theta(Q(n))\) of leading terms of \(Q(n)\). For this reason it is preferable to use the following simple consequence of Lemma 2.11.
Theorem 2.12. Let $Q \subset \ker \left( \Phi(\bar{\mathcal{R}}_1 \otimes N(k\Lambda_0)) \right)$ be a finite dimensional subspace and $n \in \frac{1}{\ell} \mathbb{Z}$. Assume that $\ell(\pi) = \ell$ for all $\pi \in \mathcal{A}(Q(n))$. If

\[
\sum_{\pi \in \mathcal{P}(n)} N(\pi) = \dim Q(n),
\]

then for any two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ in $\pi \in \mathcal{P}(n)$, where $\rho_1, \rho_2 \in \mathcal{A}(\bar{\mathcal{R}}^\sigma)$, we have a relation

\[
u(\rho_1 \subset \pi) \in \nu(\rho_2 \subset \pi) + \mathbb{C}\cdot \text{span}\{\nu(\rho \subset \pi') \mid \rho \in \mathcal{A}(\bar{\mathcal{R}}^\sigma), \rho \subset \pi', \pi < \pi'\}.
\]

Proof. Since our assumptions imply $\mathcal{A}(Q(n)) \subset \mathcal{P}(n)$, we have

\[
\sum_{\pi \in \mathcal{P}(n)} N(\pi) \geq \sum_{\pi \in \mathcal{A}(Q(n))} N(\pi).
\]

Hence our assumption (2.14) implies the equality in (2.12), and the theorem follows from Lemma 2.11. \qed

Remark. By using the same arguments we can see that Theorem 2.12 holds as well for $Q \subset \ker \left( \Phi(\bar{\mathcal{R}}_1 \otimes N(k\Lambda_0)) \right)$.

Remarks. (i) Note that the assumption (2.14) relates purely combinatorial quantity on the left hand side, which depends only on $(\mathcal{P}, \preceq)$ and $\mathcal{A}(\bar{\mathcal{R}}^\sigma)$, with purely algebraic quantity on the right hand side, which depends only on the kernel of the map $\Phi|\bar{\mathcal{R}}_1 \otimes N(k\Lambda_0)$.

It seems that “the right hand side” $\ker(\Phi|\bar{\mathcal{R}}_1 \otimes N(k\Lambda_0))$ is easier to understand; it can be described in terms of the kernel of the $\bar{\mathcal{g}}$-module map $U(\bar{\mathcal{g}}) \otimes_{U(\bar{\mathcal{g}}_{\geq 0})} \bar{\mathcal{R}}_1 \to N(k\Lambda_0)$, $u \otimes v \mapsto uv$ between (induced) $\bar{\mathcal{g}}$-modules.

(ii) If $\ell(\rho) \leq K$ for all $\rho \in \mathcal{A}(\bar{\mathcal{R}}^\sigma)$, then, for constructing a combinatorial basis, it is enough to construct combinatorial relations among relations (2.13) for all $\pi$ such that $\ell(\pi) \leq 2K - 1$ (cf. [MP2] MP3). In particular, for standard modules of level $k$, and $\bar{\mathcal{R}}$ generated by the singular vector $z_{\theta}(-1)^{k+1}1$, it is enough to construct combinatorial relations among relations (2.13) for all $\pi$ such that $\ell(\pi) \leq 2k + 1$.

(iii) The condition that $\ell(\pi) = \ell$ for all $\pi \in \mathcal{A}(Q(n))$ does not require a precise knowledge of $\mathcal{A}(Q(n))$. For standard level $k$ modules, and $\bar{\mathcal{R}}$ as above, this condition is obviously fulfilled for $\ell = k + 2$. In particular, for all level $1$ standard modules this condition is satisfied for $\ell = 3$, and as observed above, it is sufficient to consider this case alone. Hopefully, a precise knowledge of highest weight vectors in $Q$, and the use of Proposition 2.9 with related loop module structure, should make possible to check this condition for general $\ell > k + 2$.

(iv) A special case of Theorem 2.12 for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ is implicit in the proof of MP2, Lemma 9.2. In a general level $k \geq 2$ case the relations (2.13) for $\ell \leq 2k + 1$ are obtained by “explicitly solving” certain systems of equations using the relations for $\ell = k + 2$. Having this example in mind, one may think of Theorem 2.12 as a kind of “rank theorem”, the “existence of solutions” (2.13) follows from calculating two sides of (2.14). Of course, it may be that in general finding the left hand side of (2.14) is just as hard as it is to “explicitly solve” the systems of equations in question.

(v) Assume that $\sigma = \text{id}$ and that the chosen basis $B$ consists of weight vectors for the action of Cartan subalgebra $\mathfrak{h}$. Then we can define $\mathfrak{h}$-weights of colored
partitions and Theorem 2.12 can be refined in a sense that for any \( \mathfrak{h} \)-weight \( \mu \) the equality
\[
\sum_{\pi \in P(n)_\mu} N(\pi) = \dim Q(n)_\mu
\]
implies (2.15) for any two embeddings in \( \pi \in P(n)_\mu \).

(vi) Theorem 2.12 might be useful only if (2.14) holds in some generality. For now there are only examples for twisted \( A^{(1)}_1 \) and \( A^{(1)}_2 \)-modules. In the case \( \sigma = \text{id} \) for the basic \( A^{(1)}_2 \)-module the relation (2.14) holds for all but two weights \( \mu \) (in the sense of previous remark, cf. [MP3]). In the case when \( \sigma \) is a Dynkin diagram automorphism, there is a certain basis \( B \) of \( \mathfrak{sl}(3, \mathbb{C}) \) such that for the \( \sigma \)-twisted level 1 module (2.14) holds (see [S]). In both examples the left hand side of (2.14) is obtained by direct counting of embeddings. For the right hand side the appropriate \( Q \) is guessed correctly, \( \dim Q \) is calculated by using the Weyl dimension formula, and then \( \dim Q(n) \) is deduced from the loop module structure related to \( \bigoplus_{n \in \mathbb{Z}} Q(n) \).

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