Non-linear radial oscillations of neutron stars: Mode-coupling results

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ABSTRACT

The non-linear behavior of oscillation modes in compact stars is a topic of considerable current interest. Accurate numerical studies of such phenomena are likely to require powerful new approaches to both fluid and spacetime computations. We propose that a key ingredient of such methods will be the non-linear evolution of deviations from the background stationary equilibrium star. We investigate the feasibility of this approach by applying it to non-linear radial oscillations of a neutron star, and explore numerically various non-linear features of this problem, for a large range of amplitudes. Quadratic and higher order mode coupling and non-linear transfer of energy is demonstrated and analysed in detail.

1 INTRODUCTION

Since the first realisation that the Cepheids are stars that pulsate radially, stellar oscillations have provided a fertile ground for astrophysicists. Observations obtained with much improved sensitivity have provided a wealth of relevant data in recent years. This provides an important theoretical challenge given that the gathered information can be matched against detailed stellar models. With the likely advent of gravitational-wave astronomy in the next few years, relativists are considering whether a similar program may be feasible for compact stars. This is an exciting idea since the detection of gravitational waves from a pulsating star may shed light on the nature of the equation of state at supra-nuclear densities. Although stellar oscillation theory has been an active field of research for many decades (in particular in the context of Newtonian gravity) and there are several monographs covering the main results, many crucial questions remain open. The main uncertainties concern the behavior in the non-linear regime, e.g., the coupling between different modes, the formation of shocks etcetera. The purpose of the present paper is to demonstrate the accuracy of a new approach to the study of non-linear stellar oscillations. We apply the new method, in which the main focus is on non-linear deviations from the background stationary equilibrium star, to radial oscillations of neutron stars. This is a problem which, given its significance for the stability of the star, has received a lot of attention in the past. Although most results have been obtained in the linear regime, and concern the nature of the various eigenmodes (Chanmugan 1977; Glass & Lindblom 1983; Vath & Chanmugam 1992; Kokkotas & Ruoff 2001), there have also been attempts to study non-linear features in a perturbative way (including quadratic and cubic coupling terms) (Dziembowski 1982; Perdang & Blacher 1983; Perdang 1983; Wentzel 1987; Kumar & Goldreich 1989; van Hoolst 1996). [It is also relevant to mention the recent application of this approach to non-linear effects on inertial modes (Shenk et al 2001) as well as the fully nonlinear simulations by Font, Stergioulas and Kokkotas (2000).] Our new approach provides a powerful complement to these studies. We investigate the main features that appear in the weak to mildly non-linear regimes. This leads to some interesting new results regarding non-linear mode-coupling and sheds light on two of the main questions in this area: i) What is the amplitude at which large amplitude modes saturate?, and ii) How reliable are expansion methods beyond quadratic order in the amplitude? We believe our paper provides the first detailed study of these effects in full general relativity.

2 NON-LINEAR PERTURBATIONS

We model the neutron star as a single component perfect fluid at zero temperature which obeys a polytropic
equations of state \( P = K \rho^\gamma \), where \( K \) and \( \gamma \) are constants. For such a fluid the energy momentum tensor is given by \( T^{\mu \nu} = (\rho + P) u^\mu u^\nu + P g^{\mu \nu} \), where \( u^\nu \) is the four velocity, normalised as \( u^\nu u_\nu = -1 \). We restrict our consideration to spherical stars undergoing radial motions, in which case the four velocity is given by \( u^\nu = [v(t, r), w(t, r), 0, 0] \). Furthermore, in radial gauge and polar slicing the spherically symmetric line element is,

\[
ds^2 = -\lambda^2 dt^2 + \mu dr^2 + r^2 (d\theta^2 + \sin \theta) d\phi^2,
\]
i.e. it depends only on two functions \( \lambda(t, r) \) and \( \mu(t, r) \).

With those assumptions the Einstein field equations \( G_{\mu \nu} = 8\pi T_{\mu \nu} \) and the equations of hydrodynamics \( \nabla_\mu T^{\mu \nu} = 0 \) lead to four independent equations

\[
\lambda_\nu = \frac{\mu^2-1}{2\mu} + 4\pi \tau \mu^2 \left[ \rho + (\rho + P) \mu^2 w^2 \right],
\]

\[
\mu_\nu = -\frac{\lambda^2}{2\lambda} + 4\pi \tau \mu^2 \left[ \rho + (\rho + P) \mu^2 w^2 \right],
\]

\[
\rho, t + \alpha_{11} \rho, r + \alpha_{12} w_r = b_1,
\]

\[
w, t + \alpha_{21} \rho, r + \alpha_{11} w_r = b_2.
\]

Here \( \alpha_{11}, \alpha_{12}, \alpha_{21}, b_1 \) and \( b_2 \) (see Sperhake 2001 for their exact form) are functions of the fundamental variables \( \lambda, \mu, \rho, \) and \( w \), but not of their derivatives, so that Eqs. (2)-(5) form a quasi-linear system. The solution of Eqs. (2)-(5) requires suitable boundary conditions. At the centre of the star we require that \( \mu = 1 \) in order to avoid a conical singularity. The velocity \( w \) vanishes at the origin due to spherical symmetry. At the stellar surface we fix the lapse function by matching the line element to an exterior Schwarzschild metric which implies that \( \lambda_\nu = 1 \). We use typical parameters for the neutron star model: \( K = 150 \text{ km}^2 \) and \( \gamma = 2 \). We set the central density of the equilibrium configuration of the neutron star to \( \rho_s = 0.001224 \text{ km}^{-2} \), which leads to a compactness (mass/radius ratio) \( M/R = 0.19 \).

In our analysis of non-linear mode coupling we approach radial oscillations of neutron stars satisfying equations (2)-(5) in the following manner. We expand the time dependent variables \( \lambda, \mu, \rho, \) and \( P \) into static background contributions and time dependent perturbations according to

\[
f(t, r) = f(r) + \delta f(t, r).
\]

The background quantities are determined by the Tolman-Oppenheimer-Volkoff equations, the static analogues of Eqs. (2)-(5). We use the background equations to eliminate all zero order terms from the resulting perturbative dynamic equations. The resultant equations are equivalent to the original system (2)-(5). In particular, the two evolution equations form a quasilinear system for \( \delta \rho \) and \( w \). By eliminating terms of order zero in the perturbations we obtain numerical accuracy that is determined by the amplitude of the perturbation rather than the static background. This is the key advantage of this new approach. All preliminary tests have verified that this non-linear perturbation scheme provides much enhanced accuracy over a large range of amplitudes. Full details of the new method as well as the numerical code and its calibration are provided by Sperhake (2001).

Normally the surface of the star is defined by the vanishing of the pressure, which in the polytropic case is equivalent to \( \rho = 0 \). However, if one is using an Eulerian framework and the surface of the star is allowed to move, the outer grid boundary does not coincide with the surface of the star and this condition cannot be applied easily. As has been discussed in detail by Sperhake (2001) this leads to severe numerical difficulties, and could trigger artificial shock formation in the surface region. In the present study we want to focus on the non-linear coupling between various oscillation modes. In order to isolate this effect (and avoid any artificial effects due to the surface of the star) we use a fixed (rather than free) boundary condition, i.e. we require \( w = 0 \) at the surface. Furthermore we do not evolve the low density layers of the neutron star, in order to avoid negative total energy densities. The resulting neutron star model contains about 90% of the mass of the original model.

The eigenmodes of a dynamic spherically symmetric neutron star are described by the linearised version of the dynamic equations (2)-(5). It is a well known result that the linearised equations lead to a self adjoint eigenvalue problem in terms of the rescaled displacement vector \( \zeta \), which is related to our variables by \( \zeta_\nu = \tau^2 w_\nu \), cf. chapter 26 in Misner, Thorne and Wheeler (1973). The solutions of the eigenvalue problem form a complete orthonormal system and hence we can expand the time dependent \( \zeta(t, r) \) resulting from fully non-linear evolution in a series of the linear eigenmodes

\[
\zeta(t, r) = \sum_i A_i(t) \zeta_i(r),
\]

where the coefficients \( A_i \) are given in terms of the inner product

\[
A_i(t) = \int_0^R (P + \rho) \frac{\mu^2}{\tau^2} \zeta(t, r) \zeta_i(r) dr.
\]
3 RESULTS

In order to study non-linear mode-coupling we evolve the nonlinear perturbation equations from initial data corresponding to the velocity field \( w_j \) of a single linear eigenmode (of order \( j \)) with amplitude \( K_j = \max|\lambda_j(r)/r^2| \), which represents the maximum radial displacement of a fluid element inside the star. The initial density perturbation \( \delta \rho \) is set to zero, while the initial values for the metric variables follow from the constraints (3)-(4).

3.1 Exciting the fundamental mode

We first consider the case when the initial data correspond to the fundamental radial eigenmode. Having evolved this data, we measure the maximal coefficients \( A_i = \max|A_i(t)| \) obtained over an integration time corresponding to many times the dynamical timescale. In Fig. 2 we show the maximal coefficients obtained for the lowest 10 eigenmodes for excitation amplitudes ranging between 1 cm and 50 m. We observe weak mode-coupling throughout most of this domain. The fundamental mode itself (\( A_1 \)), is seen to grow more or less linearly with the initial amplitude \( K_1 \), which indicates the absence of significant self-interaction. Meanwhile, for higher order modes we can clearly identify two different regimes: For amplitudes below 10 m, all coefficients \( A_2, \ldots, A_{10} \) grow quadratically with the excitation amplitude \( K_1 \). At larger amplitudes all eigenmode coefficients except for \( A_2 \) show a transition to power laws with larger index. We have illustrated this behaviour in Fig. 2 by modelling the coefficients \( A_2, A_3 \) and \( A_4 \) as power series expansions in \( K_1 \) according to

\[
A_i = c_i K_1^2 + d_i K_1^4,
\]

(9)

(the Einstein summation convention is not used here or in similar expressions below) where \( c_i = \{3.6 \times 10^{-7}, 3.4 \times 10^{-8}, 1.0 \times 10^{-8}\} \) and \( d_i = \{0, 9.7 \times 10^{-10}, 1.2 \times 10^{-11}\} \) for \( i = 2, 3, 4 \). The higher order power laws have been obtained by least square fits to the coefficients \( A_i \) after subtracting the quadratic contributions \( c_i K_1^2 \). For the modes \( i = 5 - 10 \) the contribution of the higher order power law is rather weak which makes it difficult to obtain accurate measurements of the corresponding exponents. The steepening of the curves is, however, still obvious in the figure. It is also clear, since the curves for the eigenmode coefficients \( A_i \) do not intersect in Fig. 2, that the coupling strength decreases with the order of the eigenmodes over the whole range of amplitudes.

3.2 Exciting higher order modes

Next we set initial data in the form of second and higher eigenmodes. In this case we still observe the two
In this paper we have applied a new non-linear approach to the study of stellar pulsation, and studied mode-coupling due to non-linear effects by evolving initial data corresponding to a single linear eigenmode with varying amplitude. Concerning the transfer of energy to other modes we have found two distinct regimes, a weakly non-linear regime where the excitation of modes grows quadratically with the initial amplitude and a moderately non-linear regime, which can be reasonably well described by power laws of higher order.

The results for the weakly non-linear regime agree qualitatively with Newtonian perturbative studies. In the analytic study of non-linear mode coupling one normally views the eigenmode coefficients as harmonic oscillators and the non-linear interaction between eigenmodes is represented in the form of driving terms which are quadratic or of higher order in the amplitudes (see for example van Hoolst (1996))

\[
\frac{d^2 A_i}{dt^2} + \omega_i^2 A_i = c_i^{jk} A_j A_k + d_i^{ijkl} A_j A_k A_l + \ldots ,
\]

where the \( c_i^{jk} \), \( d_i^{ijkl} \), \ldots are the quadratic, cubic and higher order coupling coefficients and summation over \( j, k, l \) is assumed. In our analysis the initial data consists of one isolated eigenmode \( j \), so that the right hand side can be approximated by \( c_i A_j^2 + d_i A_j^3 + \ldots \). In analytic studies this series expansion is normally truncated at second or third order. The omission of higher order terms is justified in the weakly non-linear regime, where our fully non-linear simulations confirm that quadratic terms in the initial amplitude dominate the coupling between eigenmodes. This is no longer true, however, in the moderately non-linear regime, where higher order terms are more important. Our results allow us to define the transition to this regime, manifested by the breaks in the curves in Figures 2 and 3. As is clear from the figures, the moderately non-linear regime corresponds to initial mode amplitudes above 10 m or so. We note that the corresponding Mach number is of the order of 0.01. This agrees well with investigations of Newtonian perturbative studies (Kumar & Goldreich 1989) which assume a Mach number of 0.1 as the limit of applicability of semi-analytic mode coupling methods.

Furthermore, our results indicate that the non-linear couplings would be poorly captured by polynomial expansions in the mildly non-linear regime. We observe significant excitation of higher order modes (eighth, tenth etc) which can only emerge from very high-order
couplings. To quantify the associated coupling coefficients in a perturbative calculation would be very difficult.

We have also observed that, given an initial mode $j$, the coupling to modes $nj$ is particularly efficient in the moderately non-linear regime. This is naturally interpreted as a resonance effect. In analogy with the simple problem of a single forced oscillator we can assume that resonance occurs for any mode whose frequency is an integer multiple of the driving frequency $\omega_j$ in the general non-linear case, i.e. we can schematically write the eigenmode coefficients in the form

$$A_i(t) \sim \sum_n \frac{F_n}{\omega_i^2 - (n\omega_j)^2}, \quad (12)$$

where the $F_n$ will depend on the frequencies (cf. Eqs. (18), (19) of van Hoolst (1990)). In our case the external force is provided by the non-linear coupling to the initial mode $j$, as indicated in (12). We therefore obtain resonance if $\omega_i = n\omega_j$. For our simple neutron star model, the eigenfrequencies of radial neutron star oscillations are almost equally spaced and we can use $\omega_j \approx (j\omega_i)/i$ for $i, j \geq 2$ as a reasonable approximation. The condition for resonance then becomes $i = nj$ which is exactly what we have observed.

As one of the main results of this paper, we emphasise the new perturbative approach that enabled us to obtain highly accurate, fully non-linear, evolutions over a large range of amplitudes. This approach was discussed in detail by Sperhake (2001), and we plan to present further results regarding mode-coupling and non-linear shock formation in future papers. In principle, this technique can be applied to any physical problem that involves a non-trivial stationary limit and we expect it to prove a valuable tool in many non-linear problems. We note that the accuracy improvements are independent of the numerical discretization used (here, second order, centred finite differences). In combination with methods suitable for smooth oscillatory solutions (Gourgoulhon 1993, Gourgoulhon et al. 1993), we expect a dramatic expansion of the applicability of non-linear simulations to relativistic stellar pulsations.

A problem for which our new approach may prove useful concerns unstable modes of rotating neutron stars (eg. the r-modes, see Andersson & Kokkotas 2001 for a review). One of the most important questions raised in connection with the r-modes concerns the amplitude at which an unstable mode saturates. Recently, direct three dimensional numerical simulations have been brought to bear on this problem (Stergioulas & Font 2001, Lindblom et al. 2001). The picture that emerges from these studies is, however, not conclusive. Both studies suggest that an unstable r-mode saturates at an extremely large amplitude (corresponding to waves of a height of several hundred meters in a star spinning near the breakup limit) due to shocks forming in the surface region. That such “wave breaking” would occur once a mode reaches a large amplitude is likely, but simulations must isolate the true physical behaviour near the surface of a star, from numerical artifacts associated with the rapid decrease in the density (for a detailed discussion see Sperhake 2001). In fact, it is interesting to contrast these results with those of the present work that suggest that non-linear effects are highly relevant already at wave amplitudes of order 10 m. We believe that a suitable generalisation of the method used in this paper, that provides unprecedented accuracy for a large range of wave amplitudes, could prove extremely useful for the study of unstable non-axisymmetric modes and plan to address such problems in the near future.

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