Confinement for all values of the coupling in four-dimensional SU(2) gauge theory

E. T. Tomboulis

Department of Physics, UCLA, Los Angeles, CA 90095-1547

Abstract

A derivation is given from first principles of the fact that the SU(2) gauge theory is in a confining phase for all values of the coupling $0 < g < \infty$ defined at lattice spacing (UV regulator) $a$, and space-time dimension $d \leq 4$. The strategy is to employ approximate RG decimation transformations of the potential moving type which give both upper and lower bounds on the partition function at each successive decimation step. By interpolation between these bounds an exact representation of the partition function is obtained on progressively coarser lattices. In the same manner, one obtains a representation of the partition function in the presence of external center flux. Under successive decimations the flow of the effective action in these representations is constrained by that in the upper and lower bounds which are easily explicitly computable. Confining behavior for the vortex free energy order parameter (ratio of partition functions with and without external flux), hence ‘area law’ for the Wilson loop, is the result for any initial coupling. Keeping the string tension fixed determines the dependence $g(a)$, which is such that $g(a) \to 0$ for $a \to 0$. 

\footnote{e-mail: tombouli@physics.ucla.edu}
1 Introduction

Four-dimensional $SU(N)$ gauge theory at zero temperature is known to be in a confining phase for all values of the bare coupling. A very large amount of work has been performed over the last decade in an effort to isolate the types of configurations in the functional measure responsible for maintaining one confining phase for arbitrarily weak coupling [1], [2]. Nevertheless, a direct derivation of this unique feature of $SU(N)$ theories (shared only by non-abelian ferromagnetic spin systems in 2 dimensions) has remained elusive.

The origin of the difficulty is clear. It is the multi-scale nature of the problem: passage from a short distance ordered regime, where weak coupling perturbation theory is applicable, to a long distance strongly coupled disordered regime, where confinement and other collective phenomena emerge. Systems involving such dramatic change in physical behavior over different scales are hard to treat. Hydrodynamic turbulence, involving passage from laminar to turbulent flow, is another well-known example, which, in fact, shares some striking qualitative features with the confining QCD vacuum.

The natural framework for addressing the problem from first principles is a Wilsonian renormalization group (RG) block-spinning procedure bridging short to long scales. The use of lattice regularization, i.e. the framework of lattice gauge theory (LGT) [3], is virtually mandatory in this context. There is no other known usable non-perturbative formulation of gauge theory that gives the path integral in closed form preserving non-perturbative gauge invariance and positivity of the transfer matrix (reflection positivity). Attempts at exact blocking constructions towards the ‘perfect action’ along the Wilsonian renormalized trajectory [4], however, turn out, not surprisingly, to be exceedingly complicated.

There are, nonetheless, approximate RG decimation procedures that can provide bounds on judicially chosen quantities. The basic idea in this paper is to obtain both upper and lower bounds for the partition function and the partition function in the presence of external center flux. The bounds are obtained by employing approximate decimations of the ‘potential moving’ type [5], [6], which can be explicitly computed to any accuracy by simple algebraic operations. This leads to a rather simple construction constraining the behavior of the exact partition functions in the presence and in the absence of center flux; and, through them, the exact vortex free energy order parameter. The latter is the ratio of these two partition functions. It is thus shown to exhibit confining behavior for all values $0 < \beta < \infty$, of the inverse coupling $\beta = 4/g^2$ defined at lattice spacing $a$ (UV cutoff). An earlier outline of the argument was given in [7].

As it will become clear in the following, there are two main ingredients here that allow this type of result to be obtained. The first is the use of approximate decimations that are easily explicitly computable at every step, while correctly reflecting the nature of RG flow in the exact theory. The second is to consider only partition functions, or (differences of) free energies, rather than the RG evolution of a full effective action that would allow computation of any observable at different scales. This more narrowly focused approach results into tremendous simplification compared to a general RG blocking construction.

The presentation is for the most part quite explicit. Some simple propositions, mostly containing basic bounds, serve as building blocks of the argument. They are enumerated by roman numerals in the text below. Most proofs have been relegated to a series of appendices.
so as not to clutter what is essentially a simple construction. Only the case of gauge group 
\( SU(2) \) is considered explicitly here. The same development, however, can be applied to 
other groups, and, most particularly, to \( SU(3) \) which exhibits identical behavior under the 
approximate decimations.

It will be helpful at this point to provide an outline of the steps in the argument develop-
ed in the rest of the paper. In section 2 starting with the pure \( SU(2) \) LGT with partition 
function defined on a lattice of spacing \( a \), we define a class of approximate decimation trans-
formations to a coarser lattice of spacing \( ba \). In section 3 the resulting partition function on 
this decimated lattice is shown to be an upper bound on the partition function on the original 
lattice. A similar rule can be devised for obtaining a partition function on the decimated 
lattice which gives a lower bound on the original partition function. One then interpolates 
between these bounds. For some appropriate value of the interpolating parameter, one thus 
obtains an exact integral representation of the original partition function. This representation 
is in terms of an effective action defined on the decimated lattice of spacing \( ab \) plus a bulk 
free energy contribution resulting from the blocking \( a \rightarrow ab \). Now, any such interpolation is 
not unique, and it is indeed expedient to consider different interpolation parametrizations. 
The resulting partition function representation is then invariant under such parametrization 
variations in its effective action. The other important ingredient is that the effective action 
in this representation is constrained between the effective actions corresponding to the upper 
and lower bound partition functions. Iterating this procedure in successive decimations, a 
representation of the partition function is obtained on progressively coarser lattices of spacing 
\( a \rightarrow ab \rightarrow ab^2 \rightarrow \cdots \rightarrow ab^n \).

In section 4 we consider the partition function in the presence of external center flux. This 
is the flux of a center vortex, introduced by a \( Z(2) \) ‘twist’ in the action, and rendered topo-
logically stable by winding around the lattice torus. The decimation-interpolation procedure 
just outlined for the partition function can be applied also in the presence of the external 
flux. A representation of the twisted partition function on progressively coarser lattices can 
then be obtained in a completely analogous manner.

The ratio of the twisted to the untwisted partition function is the vortex free energy 
order parameter. Its behavior as a function of the size of the system characterizes the sys-
tem’s possible phases. By known correlation inequalities it can, furthermore, be related to 
the Wilson and t’Hooft order parameters. Our representations of the twisted and untwisted 
partition functions may now be used to represent the ratio (section 5). One may exploit 
the parametrization invariance of these representations to ensure that the bulk free energy 
contributions resulting in each decimation step \( ab^{m-1} \rightarrow ab^m \) explicitly cancel between nu-
merator and denominator in the ratio. One is then left with a representation of the vortex 
free energy solely in terms of an effective action defined on a lattice of spacing \( ab^n \).

Now this effective action is constrained by the effective actions corresponding to the upper 
and lower bounds. The latter are easily explicitly computable by straightforward iteration 
of the potential-moving decimation rules. Under successive transformations they flow, for 
space-time dimension \( d \leq 4 \) and any original coupling \( g \) defined at spacing \( a \), to the strong 
coupling regime. This is the regime where the coefficients in the character expansion of the 
exponential of the action become sufficiently small for the strong coupling cluster expansion 
to converge. Confining behavior is the immediate result for the vortex free energy, and, hence, 
‘area law’ behavior for the Wilson loop (section 6).
As it is well-known the theory contains only one free parameter, a physical scale which is conventionally taken to be (some multiple of) the string tension. This fact comes out in a natural way in the context of RG decimations, as we will see in the following. Fixing this scale then determines the dependence $g(a)$. The fact that $g(a) \to 0$ as $a \to 0$ is an essentially qualitative consequence of the flow exhibited by the decimations.

Some concluding remarks are given in section 7.

2 Decimations

We work on a hypercubic lattice $\Lambda \subset \mathbb{Z}^d$ of length $L_\mu$ in the $x^\mu$-direction, $\mu = 1, \ldots, d$, in units of the lattice spacing $a$. Individual bonds, plaquettes, 3-cubes, etc are generically denoted by $b$, $p$, $c$, etc. More specific notations such as $b_\mu$ or $p_{\mu\nu}$ are used to indicate elementary $m$-cells of particular orientation. We use the standard framework and common notations of LGT with gauge group $G$. Group elements are generically denoted by $U$, and the bond variables by $U_b \in G$. In this paper we take $G = SU(2)$.

We start with some appropriate plaquette action $A_p$ defined on $\Lambda$, which, for definiteness, is taken to be the Wilson action

$$A_p(U_p, \beta) = \frac{\beta}{2} \text{Re} \chi_{1/2}(U_p) , \quad U_p = \prod_{b \in \partial p} U_b , \quad (2.1)$$

with $\beta = 4/g^2$ defining the lattice coupling $g$. The character expansion of the exponential of the plaquette action function is given by

$$\exp \left( A_p(U, \beta) \right) = \sum_j d_j F_j(\beta) \chi_j(U) \quad (2.2)$$

with Fourier coefficients:

$$F_j(\beta) = \int dU \exp \left( A_p(U, \beta) \right) \frac{1}{d_j} \chi_j(U) . \quad (2.3)$$

Here $dU$ denotes Haar measure on $G$, and $\chi_j$ the character of the $j$-th representation of dimension $d_j$. So, for SU(2), the only case considered explicitly here, all characters are real, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, and $d_j = (2j + 1)$. (2.3) implies that $F_0 \geq F_j$, all $j \neq 0$. Explicitly, one finds

$$F_j(\beta) = \frac{2}{\beta} I_{d_j}(\beta) \quad (2.4)$$

in terms of the modified Bessel function $I_\nu$.

It will be convenient to work in terms of normalized coefficients:

$$c_j(\beta) = \frac{F_j(\beta)}{F_0(\beta)} , \quad (2.5)$$

so that

$$\exp \left( A_p(U, \beta) \right) = F_0 \left[ 1 + \sum_{j \neq 0} d_j c_j(\beta) \chi_j(U) \right]$$

$$\equiv F_0 f_p(U, \beta) . \quad (2.6)$$
The (normalized) partition function on lattice $\Lambda$ is then

$$Z_\Lambda(\beta) = \int dU_\Lambda \prod_{p \in \Lambda} f_p(U_p, \beta) \equiv \int d\mu_\Lambda^0,$$

(2.7)

where $dU_\Lambda \equiv \prod_{b \in \Lambda} dU_b$, and expectations are computed with the measure $d\mu_\Lambda = d\mu_\Lambda^0/Z_\Lambda(\beta)$. The action (2.1) is such that

$$F_j(\beta) \geq 0,$$

hence $1 \geq c_j(\beta) \geq 0$ all $j$, (2.8)

which implies that the measure defined by (2.7) satisfies reflection positivity (RP) both in planes without sites and in planes with sites. Note that $\lim_{\beta \to \infty} c_j(\beta) = 1$.

Let $\Lambda^{(n)}$ be the hypercubic lattice of spacing $b^n a$, with integer $b \geq 2$, and $Z_{\Lambda^{(n)}}\{c_j(n)\}$ denote a partition function of the form (2.7) defined on $\Lambda^{(n)}$ in terms of some given set of coefficients $\{c_j(n)\}$:

$$Z_{\Lambda^{(n)}}\{c_j(n)\} = \int dU_{\Lambda^{(n)}} \prod_{p \in \Lambda^{(n)}} \left[ 1 + \sum_{j \neq 0} d_j c_j(n) \chi_j(U_p) \right]$$

$$\equiv \int dU_{\Lambda^{(n)}} \prod_{p \in \Lambda^{(n)}} f_p(U_p, n) \equiv \int d\mu_{\Lambda^{(n)}}^0,$$

(2.9)

where $dU_{\Lambda^{(n)}} \equiv \prod_{b \in \Lambda^{(n)}} dU_b$. We also employ the notations

$$g_p(U, n) \equiv f_p(U, n) - 1 = \sum_{j \neq 0} d_j c_j(n) \chi_j(U),$$

(2.10)

and $\| \cdot \|$ for the $\| \cdot \|_\infty$-norm:

$$\|g(n)\| = \sum_{j \neq 0} d_j^2 c_j(n).$$

(2.11)

One has the simple but basic result:

**II.1** For $Z_{\Lambda^{(n)}}\{c_j(n)\}$ given by (2.9) with $c_j(n) \geq 0$ for all $j$, and periodic boundary conditions,

(i) $Z_{\Lambda^{(n)}}\{c_j(n)\}$ is an increasing function of each $c_j(n)$:

$$\partial Z_{\Lambda^{(n)}}\{c_i(n)\}/\partial c_j(n) \geq 0;$$

(2.12)

(ii)

$$Z_{\Lambda^{(n)}}\{c_j(n)\} \geq \left[ 1 + \sum_{j \neq 0} d_j^2 c_j(n)^6 \right]^{\left| \Lambda^{(n)} \right|},$$

(2.13)

(2.12) is an immediate consequence of RP in planes without sites. The proof of (2.13), also based on RP, is given in Appendix A. Strict inequality in fact holds in (2.12) and (2.13), with equality only in the trivial case where all $c_j(n)$’s vanish. In particular, one has

$$Z_{\Lambda^{(n)}}\{c_j(n)\} > 1.$$

(2.14)

Simple as (2.14) is, it is not trivial, as it requires non-negativity of $c_j(n)$’s, and will be useful in the following.
2.1 Construction of decimation transformations

To perform an RG transformation $a \to ba$, the lattice is partitioned into $d$-dimensional decimation cells of side length $ba$. Various approximate decimation transformations may be devised involving the ‘weakening’, i.e. decreasing the $c_j$’s of interior plaquettes, while compensating by ‘strengthening’, i.e. increasing $c_j$’s of boundary plaquettes of each cell. The simplest such scheme [5], which is adopted in the following, implements complete removal of interior plaquettes. This may be pictured [6] as moving the potentials due to interior plaquette interactions to the boundary.

This ‘potential moving’ may be performed as the composition of elementary steps. The elementary potential moving step is defined in terms of a 3-dimensional cell of side length $ba$ in a given decimation direction, say the $x^\kappa$-direction, and length $a$ in the other two directions $\mu, \nu$. Two such 3-cells adjacent along the $\kappa$-direction are shown in Figure 1. The $(b - 1)$ interior plaquettes in each cell perpendicular to $x^\kappa$ (shaded) are removed, i.e.

$$A_p(U_p) \to 0 \quad (2.15)$$

for the action at their original location, and displaced (arrows) in the positive $x^\kappa$ direction to the position of the corresponding plaquette (bold) on the cell boundary. There the displaced interior plaquettes are combined with the boundary plaquette into one plaquette $p$ with action ‘renormalized’ by some appropriate amount $\zeta_0$:

$$A_p(U) \to \zeta_0 A_p(U) \quad (2.16)$$

A complete transformation consists of performing this elementary operation successively in every lattice direction $\kappa = 1, \ldots, d$ in such a way that eventually one is left only with plaquette interactions on a lattice of spacing $ba$. In practice, there is no reason for a choice other than $b = 2$, but, for clarity, we keep general (integer) $b$. The result of a complete transformation is given by equations (2.18)-(2.22) below, to which a reader may turn directly.

To describe this process in more detail, let the lattice be partitioned into $d$-dimensional hypercubic decimation cells $\sigma^d$ of side length $ba$ in each lattice direction. Plaquettes interior to a $\sigma^d$ are defined as those not wholly contained in its $(d - 1)$-dimensional boundary $\partial \sigma^d$. Consider the effect of successive application of the elementary moving operation to plaquettes

\footnote{One may take this renormalization factor to depend on the move direction, but we need not consider these more general transformations here.}

![Figure 1: Basic plaquette moving operation, $b = 2$](image)
of fixed orientation, say $[\mu \nu]$. There are $(d - 2)$ normal directions $\kappa_i \neq \mu, \nu, \ i = 1, \ldots, d - 2$, in which a plaquette $p_{\mu \nu}$ can be moved. Interior $p_{\mu \nu}$'s in each $\sigma^d$ are first moved to the cell boundary $\partial \sigma$ in groups of $(b - 1)$ parallel plaquettes, along, say, the positive $\kappa_1$-direction (as in Figure 1). They end up in the face $\sigma^{(d-1)}_\kappa \subset \partial \sigma^d$ perpendicular to the $\kappa_1$-axis. There each group is identified with the plaquette present at that location and merged in one plaquette $p_{\mu \nu} \in \sigma^{(d-1)}_\kappa$ with a ‘renormalized’ action (2.16)). Similarly, $p_{\mu \nu}$ plaquettes in each face $\sigma^{(d-1)}_{\kappa_2 \kappa_i} \subset \partial \sigma^d_\kappa$, with $i \neq 1, \mu, \nu$ are moved along the $\kappa_1$-axis in groups of $(b - 1)$ to the face $\sigma^{(d-2)}_{\kappa_1 \kappa_i} \subset \partial \sigma^{(d-1)}_\kappa_i$ normal to the $\kappa_1$ and $\kappa_i$ directions, where they are merged and renormalized.

There are now $(d - 3)$ directions inside the $(d - 1)$-dimensional face $\sigma^{(d-1)}_{\kappa_1}$ in which a $[\mu \nu]$-plaquette can move. Thus in proceeding to apply the elementary moving operation successively in all directions, the once-moved-renormalized $p_{\mu \nu}$'s in $\sigma^{(d-1)}_{\kappa_1}$ are next moved, in groups of $(b - 1)$ plaquettes in the positive $\kappa_2$-direction, to the face $\sigma^{(d-2)}_{\kappa_1 \kappa_2} \subset \partial \sigma^{(d-1)}_{\kappa_1}$. Similarly, the once-moved-renormalized $p_{\mu \nu}$'s inside a face $\sigma^{(d-2)}_{\kappa_1 \kappa_i}$ are moved provided $\kappa_2$ is among the $(d - 4)$ available directions normal to a $[\mu \nu]$-plaquette inside $\sigma^{(d-2)}_{\kappa_1 \kappa_i}$.

Continuing this process in the remaining directions $\kappa_i, \ i = 3, \ldots, (d - 2)$, the set of $[\mu \nu]$-plaquettes on the initial lattice ends up in the 2-dimensional faces $\sigma^{(d-2)}_{\kappa_1 \kappa_2 \ldots \kappa_{(d-2)}} \subset \partial \sigma^{(d-1)}_{\kappa_1 \kappa_2 \ldots \kappa_{(d-3)}} \subset \ldots \subset \partial \sigma^{(d-1)}_{\kappa_1}$.

The above process, described for plaquettes of one fixed orientation $[\mu \nu]$, is carried out for each of the $d(d - 1)/2$ possible choices of plaquette orientation $[\mu \nu]$.

The end result of the process is then a lattice having elementary 2-faces of side length $ba$, each tiled by $b^2$ plaquettes of side length $a$. The action of each of these $b^2$ plaquettes has been renormalized according to (2.16) by a total factor of

$$\zeta^{(d-2)}_0 \equiv \zeta.$$  (2.17)

This is expressed by (2.22) below.

The integrations over the bonds interior to each 2-face of side length $ba$ are now carried out. This merges the $a^2$ tiling plaquettes into a single plaquette of side length $ba$. These integrations are exact and do not change the value of the partition function that resulted after the completion of the plaquette moving operations. We, however, allow further renormalizing the result of these integrations by introducing, in addition to $\zeta_0$, another parameter, $r$ (cf. (2.19) below). This completes the decimation transformation to a hypercubic lattice of spacing spacing $ba$.

The important feature of this decimation transformation is that it preserves the original one-plaquette form of the action, so the result can again be represented in the form (2.6). The transformation rule for successive decimations

$$a \rightarrow ba \rightarrow b^2a \rightarrow \ldots \rightarrow b^{n-1}a \rightarrow b^n a \rightarrow \ldots$$

$$\Lambda \rightarrow \Lambda^{(1)} \rightarrow \Lambda^{(2)} \rightarrow \ldots \rightarrow \Lambda^{(n-1)} \rightarrow \Lambda^{(n)} \rightarrow \ldots,$$

is then:

$$f_p(U, n - 1) \rightarrow F_0(n) f_p(U, n) = F_0(n) \left[1 + \sum_{j \neq 0} d_j c_j(n) \chi_j(U)\right].$$  (2.18)
The \( n \)-th step coefficients \( F_0(n), c_j(n) \) are obtained from the coefficients \( c_j(n-1) \) of the previous step by

\[
    c_j(n) = \hat{c}_j(n) b^2 r, \quad (2.19)
\]

\[
    F_0(n) = \hat{F}_0(n) b^2, \quad (2.20)
\]

where

\[
    \hat{c}_j(n) \equiv \hat{F}_j(n) / \hat{F}_0(n) \leq 1, \quad j \neq 0, \quad (2.21)
\]

and

\[
    \hat{F}_j(n) = \int dU \left[ f(U, n-1) \right]^{\frac{\zeta}{d_j}} \chi_j(U). \quad (2.22)
\]

The \( n = 0 \) coefficients are the coefficients \( c_j(\beta) \) on the original lattice \( \Lambda \). (2.22) encodes the end result of the plaquette moving-renormalization operations described above, with \( \zeta \) of the form (2.17); and (2.19), (2.20) that of the subsequent 2-dimensional integrations, and further renormalization by the parameter \( r \).

It is easily seen that \( f_p(U, n) > 0 \) given that this holds for \( n = 0 \) (cf. (2.2), (2.6)). The effective plaquette action on lattice \( \Lambda(n) \) of spacing \( b^n a \) is then

\[
    f_p(U, n) = \left[ 1 + \sum_{j \neq 0} d_j c_j(n) \chi_j(U) \right] \quad (2.23)
\]

\[
    \equiv \exp \left( A_p(U, n) \right), \quad (2.24)
\]

with effective couplings defined by the character expansion

\[
    A_p(U, n) = \beta_0(n) + \sum_{i \neq 0} \beta_i(n) d_i \chi_i(U). \quad (2.25)
\]

A point on notation. In the above we used the notations \( F_0(n), c_j(n), \beta_j(n) \), etc, which do not display the full set of explicit or implicit dependences of these quantities. Thus, a more complete notation is:

\[
    c_j(n) = c_j(n, b, \zeta, r, \{c_j(n-1)\}),
\]

\[
    F_0(n) = F_0(n, b, \zeta, \{c_j(n-1)\}). \quad (2.26)
\]

Dependence on the original coupling \( \beta \) comes, of course, iteratively through the coefficients \( \{c_j(n-1)\} \) of the preceding step. Because of the iterative nature of many of the arguments in this paper several explicit and implicit dependences propagate to most of the quantities used in the following. To prevent notation from getting out of hand we generally employ short-hand notations such as those on the l.h.s. of (2.26), unless specific reference to particular dependences is required.

The resulting partition function after \( n \) such decimation steps is:

\[
    Z_\Lambda(\beta, n) = \prod_{m=1}^{n} F_0(m)^{\lambda(m)} b^{m d} Z_{\Lambda(m)}(\{c_j(n)\}), \quad (2.27)
\]

with \( Z_{\Lambda(m)}(\{c_j(n)\}) \) of the form (2.10) and coefficients (2.26) resulting after \( n \) steps according to (2.19) - (2.22). The bulk free energy density resulting from decimating from scale \( a \) to \( b^m a \) is then \( \sum_{m=1}^{n} \ln F_0(m) / b^{m d} \), each term in this sum representing the contribution from \( b^{(m-1)} a \to b^m a \) as specified by (2.18). The partition function (2.27) is, of course, not equal to the original partition function \( Z_\Lambda(\beta) \) of (2.7) since the decimation transformation is not exact. How they are related will be addressed below.

8
2.2 Some properties of the decimation transformations

The transformation rule specified by (2.18)-(2.22) is meaningful for real positive \( \zeta \). Here, however, a basic distinction can be made. As it is clear from (2.22), for integer \( \zeta \) the important property of positivity of the Fourier coefficients in (2.18) is maintained at each decimation step:

\[
F_0(n) \geq 1, \quad 1 \geq c_j(n) \geq 0 \quad \text{(integer } \zeta) .
\]

(2.28)

This means that reflection positivity is maintained at each decimation step. This clearly is not guaranteed to be the case for non-integer \( \zeta \). Thus non-integer \( \zeta \) results in transformations that, in general, violate the reflection positivity of the theory (assuming a reflection positive original action).

It is important in this connection that, after each decimation step, the resulting action retains the original one-plaquette form, but will generally contain all representations in (2.25). Furthermore, among the effective couplings \( \beta_j(m) \) negative ones will occur. These features are present in general, even after a single decimation step \( a \rightarrow ba \) starting, as we did, with the single (fundamental) representation Wilson action (2.1). For integer \( \zeta \), however, the resulting effective action (2.25), even in the presence of some negative couplings, still defines a reflection positive measure, since, as just noted, the expansion of its exponential (2.23) gives positive coefficients (2.28).

It is also worth noting that, given a set of initial coefficients (2.5), the transformation rule (2.19) - (2.22) with integer \( \zeta \) ≥ 1 and \( r = 1 \) can be explicitly evaluated, to any desired accuracy, by purely algebraic operations, namely repeated application of the KG reduction rule

\[
\chi_i \chi_j = \sum_{k=|i-j|}^{i+j} \chi_k
\]

in (2.22) and character orthogonality – no actual integrations need be carried out.

The choice (cf. (2.17))

\[
\zeta_0 = 1 + (b - 1) \quad \Rightarrow \quad \zeta = b^d - 2
\]

(2.30)
is special. It increases the couplings of receiving plaquettes, at each basic moving step, by an amount exactly equal to that of the corresponding displaced plaquettes. This, together with \( r = 1 \), is essentially the original choice in [5] as reformulated in [6], and will be referred to as MK decimation. It will be important in the following.

There are various other interesting properties of the decimations that can be derived from (2.19) - (2.22). The following one is particularly important. The norm (2.11) of the coefficients obtained by application of (2.19) - (2.22) with integer \( \zeta \geq 1 \) and \( r = 1 \) satisfies (Appendix D):

\[
\|g(n + 1)\| \leq \left[ \zeta \|g(n)\| \right] b^d \left[ 1 + \|g(n)\| \right]^{(\zeta - 1)b^2}.
\]

(2.31)

Assume now that

\[
\|g(n)\| \leq \exp(-C_n) , \quad C_n > 0
\]

(2.32)

\footnote{It is worth noting in this context that in numerical investigations of the standard MK recursions in gauge theories [5] fractional \( b \), \( 1 < b < 2 \), which by (2.30) corresponds to non-integer \( \zeta \), has often been used.}
for some \( n \). Then
\[
\|g(n + 1)\| \leq \left[\zeta \|g(n)\|\right] b^2 \exp\left((\zeta - 1) b^2\right) \leq \exp \left[ - (C_n - k) b^2 \right],
\]
where \( k = \ln \zeta + (\zeta - 1) \). The recursion
\[
C_{n+1} = C_n b^2 - k b^2
\]
gives
\[
C_{n+m} = \left[C_n - \frac{b^2 k}{b^2 - 1}\right] b^{2m} + \frac{b^2 k}{b^2 - 1}.
\]

II.2 If for some \( n \) the norm of coefficients (2.17) obeys (2.32) with
\[
C_n > \frac{b^2 k}{b^2 - 1},
\]
then, under iteration of the decimation transformation (2.19) - (2.22), \( ||g(n + m)|| \rightarrow 0 \) as \( m \rightarrow \infty \) according to (2.35).

This fall-off behavior is immediately recognizable as “area-law”.

If one assumes that \( c_j(n) \) are small enough so that the theory is within the strong coupling regime, this behavior can be immediately deduced for the leading coefficient \( c_{1/2}(n) \) directly from (2.19) - (2.22):
\[
c_{1/2}(n + 1) = c_{1/2}(n) b^2 \exp \left( [\ln \zeta + O(c_{1/2}(n))] b^2 \right).
\]
The result (2.33) gives then an estimate of the corrections due to all higher representations. What is noteworthy here, however, is that the condition (2.36) is rather weaker than the commonly stated conditions for being inside the convergence radius of the strong coupling cluster expansion (cf. section IV).

We note two further properties of the decimation transformations (2.18) - (2.22). The first is that with \( r = 1 \) they become exact in space-time dimension \( d = 2 \) since then, from (2.17), \( \zeta = 1 \). The second is that, with \( \zeta = b^{(d-2)} \), vanishing coupling \( g = 0 \) is a fixed point in any \( d \), i.e. MK decimation is exact at zero coupling. This follows simply from the fact that
\[
\lim_{\beta \rightarrow \infty} \left[ \int d\nu(x) \ e^{\beta f(x)} \right]^{1/\beta} = \text{ess. sup} \ e^{f(x)} \equiv \|e^f\|
\]
for any normalized measure \( d\nu(x) \). Applying this to the result of performing the plaquette moving operation starting from (2.6), and with \( p' \in \Lambda \) labeling the plaquettes tiling the plaquettes \( p \in \Lambda^{(1)} \), one has
\[
\lim_{\beta \rightarrow \infty} \left[ \int dU \prod_{p \in \Lambda^{(1)}} \prod_{p' \in p} \exp \left( \beta b^{(d-2)} \frac{1}{2} \chi_{1/2}(U_{p'}) \right) \right]^{1/\beta} = \prod_{p \in \Lambda^{(1)}} \|\exp \left( b^{(d-2)} \frac{1}{2} \chi_{1/2} \right) ||b^{d^2} = e^{|\Lambda|} = \lim_{\beta \rightarrow \infty} \left[ \int dU \prod_{p \in \Lambda} \exp \left( \beta \frac{1}{2} \chi_{1/2}(U_p) \right) \right]^{1/\beta} .(2.38)
\]
This clearly holds also for \( r \neq 1 \), as is evident from the fact that \( \lim_{\beta \rightarrow \infty} c_j(\beta) = 1 \). This fixed point is easily seen to be unstable.
3 Partition function

Since our decimations are not exact RG transformations, the partition function does not in general remain invariant under them. The subsequent development hinges on the following two basic propositions that relate partition functions under such a decimation.

3.1 Upper and lower bounds

Consider a partition \( Z_{\Lambda}^{(n-1)} \) on lattice \( \Lambda^{(n-1)} \) of the form (2.9) given in terms of some set of coefficients \( \{c_j(n-1)\} \). Apply a decimation transformation (2.18) - (2.22) performed with \( \zeta = b^{(d-2)} \). Denote the resulting coefficients by \( c_j^U(n,r) \), \( F_0^U(n) \), i.e.

\[
\begin{align*}
c_j^U(n,r) &\equiv c_j(n,b, \zeta = b^{(d-2)}, r, \{c_j(n-1)\} ) \\
F_0^U(n) &\equiv F_0(n,b, \zeta = b^{(d-2)}, \{c_j(n-1)\} ) .
\end{align*}
\]

Note that

\[
c_j^U(n,r) = c_j^U(n,1)^r .
\]

III.1 For \( Z_{\Lambda}^{(n-1)} \) of the form (2.9), a decimation transformation (2.18) - (2.22) with \( \zeta = b^{(d-2)} \) and \( 0 < r \leq 1 \) results in an upper bound on \( Z_{\Lambda}^{(n-1)} \):

\[
Z_{\Lambda}^{(n-1)}(\{c_j(n-1)\}) \leq F_0^U(n)^{|\Lambda^{(n)}|} Z_{\Lambda^{(n)}}(\{c_j^U(n,r)\}) .
\]

The r.h.s. in (3.4) is a monotonically decreasing function of \( r \) on \( 0 < r \leq 1 \).

Given partition function \( Z_{\Lambda}^{(n-1)} \) on lattice \( \Lambda^{(n-1)} \) of the form (2.9) in terms of some set of coefficients \( \{c_j(n-1)\} \), let

\[
\begin{align*}
c_j^L(n) &\equiv c_j(n-1)^6 \\
F_0^L(n) &\equiv 1 .
\end{align*}
\]

III.2 For \( Z_{\Lambda^{(n-1)}} \), \( Z_{\Lambda^{(n)}} \) of the form (2.9):

\[
Z_{\Lambda^{(n)}}(\{c_j^L(n)\}) \leq Z_{\Lambda^{(n-1)}}(\{c_j(n-1)\}) .
\]

The proof of III.1 is given in Appendix A, where somewhat stronger results than (3.3) are actually obtained. III.2 is a corollary of (2.13) (Appendix A). For the argument in the rest of this paper, the precise form of the lower bound is in fact not important. By II.1(i) a further lower bound is obtained by replacing \( c_j^L(n) \) in III.2 by, for example,

\[
c_j^L(n) \equiv c_j(n-1)^6 c_j^U(n,r)
\]

since \( 0 \leq c_j^U(n,r) \leq 1 \). Another choice is to simply set

\[
c_j^L(n) = 0 ,
\]
which is a restatement of (2.14).

A related lower bound, which, in analogy to the upper bound in III.1, can be formulated directly in terms of the transformations (2.18) - (2.22), is obtained by taking \( c_{Lj}^{j}(n) \) in III.2 to be given by:

\[
\begin{align*}
    c_{Lj}^{j}(n) & \equiv c_j(n, b, \zeta = 1, r = 1, \{c_j(n-1)\}) \\
    &= c_j(n-1)^{b^2}, \\
    F_0^{L}(n) & \equiv F_0(n, b, \zeta = 1, \{c_j(n-1)\}) \\
    &= 1. 
\end{align*}
\]

(3.10)

(3.11)

With this choice of \( c_{Lj}^{j}(n) \), note that III.1 - III.2 imply the fact that the decimations (2.18) - (2.21) become exact for \( d = 2 \) and \( r = 1 \). III.1 says that, after removal of interior plaquettes, modifying the couplings of the remaining plaquettes by taking \( \zeta = b^{d-2} \) (and \( r \leq 1 \)) results into overcompensation. III.2 says that decimating plaquettes while leaving the couplings of the remaining plaquettes unaffected (\( \zeta = 1, r = 1 \)) results in undercompensation. The proof of III.2 for \( c_{Lj}^{j}(n) \) given by (3.10) is similar to that of II.1, but need not be given here, since the weaker bounds above will suffice.

In the following it will in fact be more convenient to take (3.8) or (3.9) for the definition of the lower bound coefficients \( c_{Lj}^{j}(m) \). Use of the stronger lower bounds above may be preferable for numerical investigations, but does not contribute anything further to the argument in this paper.

III.1 and III.2 give upper and lower bounds on the partition function after a decimation step. It is then natural to interpolate between these bounds.

### 3.2 Interpolation between upper and lower bounds

Introducing a parameter \( \alpha \in [0, 1] \), we define coefficients \( \tilde{c}_j(m, \alpha, r) \) interpolating between \( c_{Lj}^{j} \) at \( \alpha = 0 \) and \( c_{Uj}^{j} \) at \( \alpha = 1 \):

\[
\tilde{c}_j(m, \alpha, r) = (1 - w(\alpha)) c_{Lj}^{j}(m) + w(\alpha) c_{Uj}^{j}(m, r), \quad 0 < r \leq 1. 
\]

(3.12)

with

\[
\begin{align*}
    w(0) &= 0, \\
    w(1) &= 1, \\
    w'(\alpha) &> 0. 
\end{align*}
\]

(3.13)

For example,

\[
w(\alpha) = \frac{e^\alpha - 1}{e - 1}. 
\]

(3.14)

There is clearly a variety of other choices than (3.12) for these interpolating coefficients. We always require that

\[
\partial \tilde{c}_j(m, \alpha, r)/\partial \alpha > 0, 
\]

(3.15)

which is satisfied by (3.12) - (3.13).

Similarly, we define coefficients interpolating between (3.6) and (3.2). For our purposes it will be convenient to take

\[
\tilde{F}_0(m, h, \alpha, t) = F_0^{U}(m)^{h_t(\alpha)}, 
\]

(3.16)
where \( h_t \) denote a family of monotonically increasing smooth functions of \( \alpha \), labeled by a parameter \( t \in [t_a, t_b] \), and such that

\[
\begin{align*}
    h_t(0) &= 0, & h_t(1) &= 1. \tag{3.17}
\end{align*}
\]

We write \( h_t(\alpha) \equiv h(\alpha, t) \). Examples are\(^4\)

\[
    h(\alpha, t) = \exp\left(-\sigma(t) \frac{1-\alpha}{\alpha}\right), \quad h(\alpha, t) = \alpha^{\sigma(t)}, \quad h(\alpha, t) = \tanh\left(\frac{\alpha}{\sigma(t)(1-\alpha)}\right),
\]

where \( 0 < \alpha \leq 1 \), \( 0 < t_a \leq t \leq t_b < \infty \), \( \sigma(t) \) is a smooth monotonically increasing positive function on \([t_a, t_b]\), e.g. \( \sigma(t) = t \).

The interpolating partition function on \( \Lambda^{(m)} \) constructed from \( \tilde{c}_j \) and \( \tilde{F}_0 \) is now defined by

\[
    \tilde{Z}_{\Lambda^{(m)}}(\beta, h, \alpha, t, r) = \tilde{F}_0(m, h, \alpha, t)^{\Lambda^{(m)}} Z_{\Lambda^{(m)}}(\{\tilde{c}_j(m, \alpha, r)\}) \tag{3.19}
\]

where

\[
    Z_{\Lambda^{(m)}}(\{\tilde{c}_j(m, \alpha, r)\}) = \int dU_{\Lambda^{(m)}} \prod_{p \in \Lambda^{(m)}} \left[ 1 + \sum_{j \neq 0} d_j \tilde{c}_j(m, \alpha, r) \chi_j(U_p) \right]
    = \int dU_{\Lambda^{(m)}} \prod_{p \in \Lambda^{(m)}} f_p(U_p, m, \alpha, r). \tag{3.20}
\]

Combining II.1, (3.15) and the fact that \( \tilde{F}_0 \) is, by definition, also an increasing function of \( \alpha \) one has

**III.3** The interpolating free energies \( \ln Z_{\Lambda^{(m)}}(\{\tilde{c}_j(m, \alpha, r)\}) \) and \( \ln \tilde{Z}_{\Lambda^{(m)}}(\beta, h, \alpha, t, r) \) are increasing functions of \( \alpha \):

\[
    \partial \ln Z_{\Lambda^{(m)}}(\{\tilde{c}_j(m, \alpha, r)\}) / \partial \alpha > 0. \tag{3.21}
\]

Equality in (3.21) applies only in the trivial case were all the coefficients \( \tilde{c}_j \)'s vanish.

In terms of (3.14), III.1 and III.2 give

\[
    \tilde{Z}_{\Lambda^{(m)}}(\beta, h, 0, t, r) \leq Z_{\Lambda^{(m-1)}}(\beta, h, 1, t, r). \tag{3.22}
\]

Now \( \tilde{Z}_{\Lambda^{(m)}}(\beta, h, \alpha, t, r) \) is continuous in \( \alpha \). It follows from (3.22) that there exist a value of \( \alpha \) in \((0, 1)\):

\[
    \alpha(m, h, t, r, \{c_j(m-1)\}, b, \Lambda) \equiv \alpha_{\Lambda, h}^{(m)}(t, r) \tag{3.23}
\]

such that

\[
    \tilde{Z}_{\Lambda^{(m)}}(\beta, h, \alpha_{\Lambda, h}^{(m)}(t, r), t, r) = Z_{\Lambda^{(m-1)}}. \tag{3.24}
\]

In other words, at each given value of \( t, r \), there exist a value of \( \alpha \) at which the partition function on \( \Lambda^{(m)} \), resulting from a decimation transformation \( \Lambda^{(m-1)} \rightarrow \Lambda^{(m)} \), equals the

---

\(^4\)Supplementing these definitions at \( \alpha = 0 \) as needed is understood. Thus, \( h(\alpha, t) = 0 \) on \( \alpha \leq 0 \) in the first example in (3.13), and standard smoothing in the second example: replace \( \alpha \) in \( \gamma \) by \( g_\alpha(\alpha) = \int \rho_\alpha(\alpha - x)g(x)dx \), where \( g(x) = x \) for \( x > 0 \), \( g(x) = 0 \) for \( x \leq 0 \), and \( \rho_\alpha(x) \) is \( C^\infty \), has support inside \(|x|^2 \leq \epsilon^2\) and satisfies \( \rho_\epsilon \geq 0 \) and \( \int \rho_\epsilon = 1 \).
partition function on $\Lambda^{(m-1)}$. This value is unique by III.3. By construction, $\alpha_{(m)}^{(m)}(t, r)$ is such that (3.24) remains invariant under variation of $t, r$ in their domain of definition, i.e. $\alpha_{(m)}^{(m)}(t, r)$ represents the level surface of the function $Z_{\Lambda^{(m)}}(\beta, h, \alpha, t, r)$ fixed by the value $Z_{\Lambda^{(m-1)}}$. The parametrization invariance under varying $t$ will be important later.

We now examine the dependence on $t, r$ in (3.23) more closely. Given $Z_{\Lambda^{(m-1)}}$ and some interpolation $h$, assume that (3.24) is satisfied at the point $(t_0, r_0, \alpha = \alpha_{(m)}^{(m)})$. Then, by the implicit function theorem, applicable by III.3, there is a function $\alpha_{(m)}^{(m)}(t, r)$ with continuous derivatives such that $\alpha_{(m)}^{(m)}(t_0, r_0) = \alpha_{(m)}^{(m)}$, and uniquely satisfies (3.24) in a sufficiently small neighborhood of $(t_0, r_0, \alpha_{(m)}^{(m)})$. But since a solution to (3.24) exists for each choice of $t, r$ in their domain of definition, this neighborhood can be extended by a standard continuity argument to all points of this domain. $\alpha_{(m)}^{(m)}(t, r)$ then represents the regular level surface of the function (3.24) fixed by (3.24). Furthermore,

$$
\frac{\partial \alpha_{(m)}^{(m)}(t, r)}{\partial t} = v(\alpha_{(m)}^{(m)}(t, r), t, r) ,
$$

(3.25)

where

$$
v(\alpha, t, r) \equiv -\frac{\partial h(\alpha, t)/\partial t}{\partial h(\alpha, t)} + A_{(m)}(\alpha, r)
$$

(3.26)

with

$$
A_{(m)}(\alpha, r) \equiv \frac{1}{\ln F_U(m)} \frac{1}{|A^{(m)}|} \frac{\partial}{\partial \alpha} \ln Z_{\Lambda^{(m)}} \left( \{ c_j(m, \alpha, r) \} \right) > 0 .
$$

(3.27)

We will always assume that $h$ is chosen such that $\partial h/\partial t$ is negative. This is the case with the examples (3.18). Then, from (3.26), $v > 0$ on $0 < \alpha < 1$, with $v = 0$ at $\alpha = 0$ and $\alpha = 1$.

It is also useful to equivalently view $\alpha_{(m)}^{(m)}(t, r)$ as the solution to the ODE

$$
d\alpha/dt = v(\alpha, t, r) , \quad \alpha \in (0, 1) ,
$$

(3.28)

$$
\alpha(t_0) = \alpha_{(m)}^{(m)} > 0 , \quad t_0 \in [t_a, t_b] .
$$

(3.28)

Then standard results of ODE theory imply the existence of a unique solution in a neighborhood of $\alpha_{(m)}^{(m)} > 0$, which can in fact be extended indefinitely forward for all $t \geq t_0$.

A short computation using (3.25) gives

$$
\frac{dh(\alpha_{(m)}^{(m)}(t, r), t)}{dt} = -\frac{\partial \alpha_{(m)}^{(m)}(t, r)}{\partial t} A_{(m)}(\alpha_{(m)}^{(m)}(t, t), r) ,
$$

(3.29)

as it should for consistency with (3.24), (3.26) and (3.25) make apparent what the effect of a parametrization change due to a shift in $t$ is. Increasing (decreasing) $t$ increases (decreases) the contribution of $\ln Z_{\Lambda^{(m)}}(\{ c_j(m, \alpha_{(m)}^{(m)}(t, r), r) \})$ while decreasing (increasing) by an equal amount the contribution from $\ln F_U(m) h(\alpha_{(m)}^{(m)}(t, r), t) |A^{(m)}|$, so that the sum stays constant and equal to $\ln Z_{\Lambda^{(m-1)}}$ in accordance with (3.24).

5Indeed, $v$ is differentiable on $\alpha_{(m)}^{(m)} \leq \alpha \leq 1$ and vanishes at $\alpha = 1$. 

14
The derivative w.r.t. \( r \) is similarly given by (B.1) - (B.2) in Appendix B. Now, by (III.1), the upper bound in (3.22) is optimized for \( r = 1 \), which would appear to make consideration of other \( r \) values unnecessary. The reason one may want, however, to vary \( r \) away from unity is the following.

The values \( \alpha_{\Lambda,h}^{(m)}(t, r) \) lie in the interval \((0,1)\). Consider the possibility that one finds that \( \alpha_{\Lambda,h}^{(m)}(t_m, 1) \) differs from 1 only by terms that vanish as the lattice size grows. This means that, since \( v \geq 0 \) in (3.25), \( \alpha_{\Lambda,h}^{(m)}(t, 1) \) is, to within such terms, a constant function of \( t \) for all \( t \geq t_m \). For the purposes of the argument in the following sections we want to exclude this possibility, and ensure that, at least in some neighborhood of a chosen \( t \) value, the derivative (3.25) is non-vanishing by an amount independent of lattice size.

We require that
\[
\delta' < \alpha_{\Lambda,h}^{(m)}(t, r) < 1 - \delta , \tag{3.30}
\]
with \( \delta > 0, \delta' > 0 \) independent of the lattice size \( |\Lambda^{(m)}| \). The lower bound requirement is easily shown (Appendix B) to be automatically satisfied by combining II.1 and (3.24). As it is also shown in Appendix B, one may always ensure that the upper bound requirement in (3.30) holds by choosing the decimation parameter \( r \) to vary, if necessary, away from unity in the domain
\[
1 \geq r \geq 1 - \epsilon , \tag{3.31}
\]
where \( 0 < \epsilon \ll 1 \) with \( \epsilon \) independent of \( |\Lambda^{(m)}| \).

With (3.30) in place, (3.25) and (3.29) imply (Appendix B) that
\[
\frac{\partial \alpha_{\Lambda,h}^{(m)}(t, r)}{\partial t} \geq \eta_1(\delta) > 0 , \quad -\frac{dh}{dt}(\alpha_{\Lambda,h}^{(m)}(t, r), t) \geq \eta_2(\delta) > 0 , \tag{3.32}
\]
where \( \eta_1, \eta_2 \) are lattice-size independent. Furthermore, if (3.30) already holds for \( r = 1 \), it also holds for any \( r \) in (3.31). We may as well then simplify matters in the following by setting the parameter \( r \) to the value \( r = 1 - \epsilon \) with some fixed small \( \epsilon \). This \( \epsilon \) may eventually be taken as small as one pleases after a sufficiently large number of decimations have been performed. This has an obvious meaning in the context of iterating the decimation transformation as pointed out in subsection 3.4 below. We accordingly simplify notation by dropping explicit reference to \( r \), except on occasions when a statement is made for general \( r \) values. Thus we write \( \alpha_{\Lambda,h}^{(m)}(t) \equiv \alpha_{\Lambda,h}^{(m)}(t, 1 - \epsilon), \ c_j^{(m)}(m) \equiv c_j^{(m)}(m, 1 - \epsilon) \), etc.

### 3.3 Representation of the partition function on decimated lattices

So, starting on the original lattice spacing \( a \), with partition function given in terms of coefficients \( \{c_j(\beta)\} \), one may iterate the procedure represented by (3.22) - (3.24). Taking the same interpolation family \( h \) in every cycle, an iteration cycle consists of the following steps.

(i) A decimation transformation \( \Lambda^{(m-1)} \rightarrow \Lambda^{(m)} \) given by the rules (2.18) - (2.22) applied to the coefficients in \( \tilde{Z}_{\Lambda^{(m-1)}} \), and resulting into the upper bound coefficients on \( \Lambda^{(m)} \) according to (3.1) - (3.2) and (3.4). Similarly, a lower bound on \( \Lambda^{(m)} \) is obtained according to (3.7) with lower bound coefficients given by (3.6) and (3.8) or (3.9).
(ii) Interpolation between the resulting upper and lower bound partition functions on $\Lambda^{(m)}$ according to (3.12), (3.16), and (3.19), (3.20).

(iii) Fixing the value $0 < \alpha^{(m)}_{\Lambda,h}(t) < 1$, eq. (3.23), so that the $(m-1)$-th step partition function $Z^{(m-1)}_{\Lambda}$ is preserved, eq. (3.24).

(iv) Picking a value of the parameter $t = t_m$, to fix the coefficients $\{\tilde{c}_j(m, \alpha^{(m)}_{\Lambda}(t_m))\}$ of the resulting partition function $Z^{(m)}_{\Lambda}$, and return to step (i).

This scheme for the coefficients in $Z^{(m)}_{\Lambda}$ may be depicted as follows:

\[ c_j(\beta) \]
\[ \{c_L^j(1)\} \leq \{\tilde{c}_j(1, \alpha^{(1)}_{\Lambda,h}(t))\} \leq \{c_U^j(1)\} \]
\[ \{c_L^j(2)\} \leq \{\tilde{c}_j(2, \alpha^{(2)}_{\Lambda,h}(t))\} \leq \{c_U^j(2)\} \]
\[ \vdots \]

The result after $n$ iterations is then:

\[ Z^{(m)}_{\Lambda}(\beta) = \int dU^{\Lambda} \prod_{p \in \Lambda} f_p(U, \beta) \]
\[ = \left[ \prod_{m=1}^{n} \hat{F}_0(m, h, \alpha^{(m)}_{\Lambda,h}(t_m), t_m)^{|\Lambda|/\beta^{mod}} \right] Z^{(n)}_{\Lambda} \left( \{\tilde{c}_j(n, \alpha^{(n)}_{\Lambda,h}(t_n))\} \right). \] (3.35)

(3.35) is an exact integral representation on the decimated lattice $\Lambda^{(n)}$ of the partition function $Z^{(n)}_{\Lambda}$ originally defined on the undecimated lattice $\Lambda$ by the integral representation (2.7) or (3.34).

III.3 allows the iterative procedure leading to (3.35) to be implemented in a slightly different manner, one that turns out later to be more convenient for our purposes. Since by III.3

\[ Z^{(m)}_{\Lambda} \left( \{\tilde{c}_j(m, \alpha^{(m)}_{\Lambda,h}(t_m))\} \right) \leq Z^{(m)}_{\Lambda} \left( \{\tilde{c}_j(m, 1)\} \right) = Z^{(m)}_{\Lambda} \left( \{c_U^j(m)\} \right), \] (3.36)

an upper bound for each successive iteration step is also obtained by applying III.1 to the r.h.s. rather than the l.h.s. of the inequality sign in (3.36). The only resulting modification in the above procedure is in step (i): the upper bound coefficients $c_U^j(m)$ and $F_U^0(m)$ on $\Lambda^{(m)}$ are computed according to (3.1) and (3.2), but now using the set $\{\tilde{c}_j(m-1, \alpha^{(m-1)}_{\Lambda,h}(t_{m-1}))\}$ rather than the set $\{\tilde{c}_j(m-1, \alpha^{(m-1)}_{\Lambda,h}(t_{m-1}))\}$ as the coefficient set of the previous step.

The same alternative can be applied to the lower bounds in (3.33). Since, again by III.3, one has

\[ Z^{(m)}_{\Lambda} \left( \{c_L^j(m)\} \right) = Z^{(m)}_{\Lambda} \left( \{\tilde{c}_j(m, 0)\} \right) \leq Z^{(m)}_{\Lambda} \left( \{\tilde{c}_j(m, \alpha^{(m)}_{\Lambda,h}(t_m))\} \right), \] (3.37)
a lower bound for each successive iteration step is also obtained by applying III.2 to the l.h.s. rather than the r.h.s. of the inequality sign in (3.37). If one adopts (3.39), this makes no difference since the lower bound coefficients equal zero at every step. If one uses (3.8), the resulting modification to (3.33) is that in step (i) the lower bound coefficients $c^L_j(m)$ on $\Lambda^m(t_{m-1})$ are now computed using the set $\{c^L_j(m-1)\}$ rather than $\{\tilde{c}_j(m-1, \alpha^{(m-1)}_{A,h}(t_{m-1}))\}$ as the coefficient set of the previous step.

One may adopt either or both modifications following from (3.36) or (3.37). Adopting both, the iterative scheme for the coefficients in $Z_{\Lambda^m}$ replacing (3.33) is:

\[
\begin{align*}
\{c^L_j(1)\} & \leq \{\tilde{c}_j(1, \alpha^{(1)}_A(t_1))\} \leq \{c^U_j(1)\} \\
\{c^L_j(2)\} & \leq \{\tilde{c}_j(2, \alpha^{(2)}_A(t_2))\} \leq \{c^U_j(2)\} \\
\vdots & \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]  

(3.38)

This again leads, after $n$ iterations, to the representation (3.35). Note, however, that the actual numerical value of $\alpha^{(m)}_{A,h}(t_{m})$ in (3.35), fixed at each step by requiring (3.24), will, in general, be different depending on whether scheme (3.33) or (3.38) is used for the iteration. Also note that the upper bounds $c^U_j(m)$ in (3.38) are not optimal compared to those in (3.33). The scheme (3.38), however, turns out to be more convenient for our purposes in the following.

3.4 Discussion of the representation (3.35)

As indicated by the notation, on any finite lattice, the $\alpha^{(m)}_{A,h}$ values possess a lattice size dependence. This weak dependence enters as a correction that vanishes inversely with lattice size. Indeed, by the standard results on the existence of the thermodynamic limit of lattice systems, for a partition function $Z_{\Lambda^m}(\{c_j\})$ of the form (2.9) on lattice $\Lambda^m$ with torus topology (periodic boundary conditions):

\[
\ln Z_{\Lambda^m}(\{c_j\}) = |\Lambda^m| \varphi(\{c_j\}) + \delta \varphi_{\Lambda^m}(\{c_j\}),
\]  

(3.39)

$\varphi(\{c_j\})$ being the free energy per unit volume in the infinite volume limit, and $\delta \varphi_{\Lambda^m}(\{c_j\}) \leq O(\text{constant})$. From this and (3.24) it is straightforward to show that

\[
\alpha^{(m)}_{A,h}(t, r) = \alpha^{(m)}_h(t, r) + \delta \alpha^{(m)}_{A,h}(t, r) \quad (3.40)
\]

That is, there are no ‘surface terms’ for torus topology. In fact surface terms arising with other, e.g. free, boundary conditions can be precisely defined as the difference in the free energies computed with periodic versus such other boundary conditions [11].
with $\delta \alpha_{\Lambda, h}^{(m)}(t, r) \to 0$ as some inverse power of lattice size in the large volume limit. In fact, we have already established the presence of a lattice-size independent contribution in $\alpha_{\Lambda, h}^{(m)}(t, r)$ in an alternative manner through (3.30), i.e. the fact that in (3.40) one must have

$$\alpha_{\Lambda, h}^{(m)}(t, r) > \delta' .$$  

(3.41)

An explicit expression for $\delta'$ is given by (3.17), (3.19).

At weak and strong coupling the $\alpha_{\Lambda, h}^{(m)}(t, r)$ values may be estimated analytically by comparison with the weak and strong coupling expansions, respectively. In general, starting from (3.22), the location of $\alpha_{\Lambda, h}^{(m)}$ satisfying (3.24) may be formulated as the fixed point of a contraction mapping. This allows in principle its numerical determination, for given values of all other parameters, to any desired accuracy.

For our purposes here, however, the actual numerical values of the $\alpha_{\Lambda, h}^{(m)}$'s, beyond the fact that they are fixed between 0 and 1, will not be directly relevant. The main application of the representation (3.35) in this paper will be to relate the behavior of the exact theory to that of the easily computable approximate decimations bounding it without explicit knowledge of the actual $\alpha_{\Lambda, h}^{(m)}$ values.

It is important to be clear about the meaning of (3.35). The partition function $Z_{\Lambda}(\beta)$ is originally given by its integral representation (3.34) on lattice $\Lambda$ of spacing $a$. (3.35) then gives another integral representation of $Z_{\Lambda}(\beta)$ in terms of an integrand defined on the coarser lattice $\Lambda^{(n)}$ of spacing $b^n a$ plus a total bulk free energy contribution resulting from decimating between scales $a$ and $b^n a$. The action $A_p(U, n, \alpha_{\Lambda, h}^{(n)})$ in $\{c_j(n, \alpha_{\Lambda, h}^{(n)})\}$ is constructed to reproduce this one physical quantity, i.e. the free energy $\ln Z_{\Lambda}(\beta)$, nothing more and nothing less. In particular, it is not implied that this action on $\Lambda^{(n)}$ can also be used to exactly compute any other observable. For that one would need to attempt the previous development from scratch with the corresponding operator inserted in the integrand.

Recall that, by (3.19), the coefficients $c_j(n, \alpha, r)$’s are increasing in $\alpha$, and $c_j(n, 1, r) = c_j^U(n, r)$, $c_j(n, 0, r) = c_j^L(n)$:

$$c_j^L(n) < c_j(n, \alpha_{\Lambda, h}^{(n)}(t)) < c_j^U(n) , \quad 0 < \alpha_{\Lambda, h}^{(n)}(t) < 1 .$$  

(3.42)

Thus, the coefficients $c_j(n, \alpha_{\Lambda, h}^{(n)}(t))$ in the representation (3.35) are bounded from above by $c_j^U(n)$ no matter what the actual values of $\alpha_{\Lambda, h}^{(n)}(t)$ are.

When considering the implications of this bound under successive decimations the advantage of employing scheme (3.35), rather than (3.33), becomes clear. The coefficients $c_j^U(n)$ on the r.h.s. column in (3.38) are obtained by straightforward iteration of the decimation rules (2.19)-(2.22) with $\zeta = b^{-d/2}$; i.e. only knowledge of the $c_j^U(n-1)$, not of the $c_j(n-1, \alpha_{\Lambda, h}^{(n-1)}(t_{n-1}))$, is required to obtain the $c_j^U(n)$ at the $n$-th step. The flow of these $c_j^U(n)$ coefficients then constrains the flow of the exact representation coefficients $c_j(n, \alpha_{\Lambda, h}^{(n)}(t_n))$ according to (3.42) from above. In particular, if the $c_j^U(n)$’s on the r.h.s. in column in (3.38) approach the strong coupling fixed point, i.e.

$$F_0^U(n) \to 1, \quad c_j^U(n) \to 0, \quad as \ n \to \infty ,$$  

(3.43)
so must the $\tilde{c}_j(n, \alpha^{(n)}_{A,h})$’s in the representation (3.35).

Now the coefficients $c^U_j(n, r)$ at $r = 1$ are the MK decimation coefficients (cf. section 2.2). As it is well-known, the MK decimations for $SU(2)$ (and also $SU(3)$) are found by explicit evaluation to indeed flow to the strong coupling fixed point (3.43) for all starting $\beta < \infty$ and $d \leq 4$. Above the critical dimension $d = 4$, the decimations result in free spin wave behavior ($c^U_j(n, 1) \to 1$ as $n \to \infty$) starting from any $\beta > \beta_0$, where $\beta_0 = O(1)$.

Here, for reasons discussed at the end of section 3.2, we take $r$ in the range (3.31). This may be viewed as fixing the direction from which the point $\zeta = b^{(d-2)}$, $r = 1$ in the parameter space of the iteration (2.19) - (2.22) is approached. This is actually irrelevant for the flow behavior of the $c^U_j(n, 1 - \epsilon) \equiv c^U_j(n)$ since, in the case of $SU(2)$ considered here, this point is a structurally stable point of the iteration.

Note that zero lattice coupling, $g = 0$, is a fixed point as it is for the MK decimations. This is also evident from $\lim_{\beta \to \infty} c_j(\beta) = 1$ and III.2.

What does (3.43) combined with (3.42) imply about the question of confinement in the exact theory? The fact that the long distance part, $Z_{\Lambda}^{(n)}(\{\tilde{c}_j(n, \alpha^{(n)}_{A,h})\})$, in (3.35) flows in the strong coupling regime does not suffice to answer the question. It is the combined contributions from all scales between $a$ and $b^\beta a$ in (3.35) that add up to give the exact free energy $\ln Z_{\Lambda}(\beta)$. Indeed, recall that, by a parametrization change by shifts in $t$ at each decimation step, one can shift the relative amounts assigned to these various contributions keeping the total sum fixed (cf. remarks immediately following (3.29). This parametrization freedom will in fact be important in the following. On the other hand, the fact that by (3.42) the flow of $\tilde{c}_j(n, \alpha^{(n)}_{A,h}(t_n))$ to the strong coupling regime is independent of such parametrization changes is strongly suggestive. At any rate, to unambiguously determine the long distance behavior of the theory one needs to consider appropriate long distance order parameters.

4 ‘Twisted’ partition function

The above derivation leading to the representation (3.35) for the partition function cannot be applied in the presence of observables without modification. Thus, in the presence of operators involving external sources, such as the Wilson or ‘t Hooft loop, translation invariance is lost. Reflection positivity is also reduced to hold only in the plane bisecting a rectangular loop. Fortunately, there are other order parameters that can characterize the possible phases of the theory while avoiding most of these complications. They are the well-known vortex free energy, and its transform with respect to the center of the gauge group (electric flux free energy). They are in fact the natural order parameters in the present context since they are constructed out of partition functions, i.e. partition functions in the presence of external fluxes.

Let $Z_{\Lambda}(\tau_{\mu\nu}, \beta)$ denote the partition function with action modified by the ‘twist’ $\tau_{\mu\nu}$, i.e. an element of the group center, for every plaquette on a coclosed set of plaquettes $V_{\mu\nu}$ winding

---

7 To strictly draw the same conclusion from the alternative scheme (3.33) requires an additional step, such as showing that the $c^U_j(n)$’s computed according to the scheme (3.33) flow to the strong coupling regime if those computed according to (3.35) do.

8 It is, however, very much relevant in cases where this point is not structurally stable, e.g. in $U(1)$. 

---
through the periodic lattice in the \((d-2)\) directions perpendicular to the \(\mu\), and \(\nu\)-directions, i.e. winding through every \([\mu\nu]\)-plane for fixed \(\mu, \nu\):

\[
A_p(U_p) \rightarrow A_p(\tau_{\mu\nu} U_p) , \quad \text{if} \quad p \in \mathcal{V}_{\mu\nu} .
\]  

(4.1)

A nontrivial twist \((\tau_{\mu\nu} \neq 1)\) represents a discontinuous gauge transformation on the set \(\mathcal{V}_{\mu\nu}\) with multivaluedness in the group center. Thus, for group \(SU(N)\), it introduces vortex flux characterized by elements of \(\pi_1(SU(N)/Z(N)) = Z(N)\). The vortex is rendered topologically stable by being wrapped around the lattice torus.

In the case of \(SU(2)\) explicitly considered here, there is only one nontrivial element, \(\tau_{\mu\nu} = -1\). As indicated by the notation \(Z_{\Lambda}(\tau_{\mu\nu}, \beta)\), the twisted partition function depends only on the directions in which \(\mathcal{V}_{\mu\nu}\) winds through the lattice, not the exact shape or location of \(\mathcal{V}_{\mu\nu}\). This expresses the mod 2 conservation of flux. Indeed, a twist \(\tau_{\mu\nu} = -1\) on the plaquettes forming a coclosed set \(\mathcal{V}_{\mu\nu}\) can be moved to the plaquettes forming any other homologous coclosed set \(\mathcal{V}'\) by the change of variables \(U_b \rightarrow -U_b\) for each bond \(b\) in a set of bonds cobounded by \(\mathcal{V} \cup \mathcal{V}'\), leaving \(Z_{\Lambda}(\tau_{\mu\nu}, \beta)\) invariant. By the same token, \(Z_{\Lambda}(\tau_{\mu\nu}, \beta)\) is invariant under changes mod 2 in the number of homologous coclosed sets in \(\Lambda\) carrying a twist. In the following, for definiteness, we fix, say, \(\mu = 1, \nu = 2\), and drop further explicit reference to the \(\mu, \nu\) indices. Also, we write \(Z_{\Lambda}(-1, \beta) \equiv Z^{(-)}_{\Lambda}(\beta)\).

Equation (4.1) implies that \(Z^{(-)}_{\Lambda}\) is obtained from \(Z_{\Lambda}\) by the replacement

\[
f_p(U_p, a) \rightarrow f_p(-U_p, a) = \left[ 1 + \sum_{j \neq 0} (-1)^{2j} d_j c_j(\beta) \chi_j(U_p) \right], \quad \text{for each} \quad p \in \mathcal{V},
\]  

(4.2)

in (2.7), (2.6), i.e. only half-integer representations on plaquettes in \(\mathcal{V}\) are affected. In general, then, the twisted version of the partition function (2.9) on \(\Lambda^{(n)}\) is

\[
Z^{(-)}_{\Lambda^{(n)}}(\{c_j(n)\}) = \int dU_{\Lambda^{(n)}} \prod_{p \in \Lambda^{(n)}} f_p^{(-)}(U_p, n),
\]  

(4.3)

with

\[
f_p^{(-)}(U_p, n) = \left[ 1 + \sum_{j \neq 0} (-1)^{2j} S_p[\mathcal{V}] d_j c_j(n) \chi_j(U_p) \right].
\]  

(4.4)

\(S_p[\mathcal{V}]\) denotes the characteristic function of the plaquette set \(\mathcal{V}\), i.e. \(S_p[\mathcal{V}] = 1\) if \(p \in \mathcal{V}\), and \(S_p[\mathcal{V}] = 0\) otherwise. A simple result (Appendix A) of obvious physical significance is:

**IV.1** With \(c_j(n) \geq 0\), all \(j\),

\[
Z^{(-)}_{\Lambda^{(n)}}(\{c_j(n)\}) \leq Z_{\Lambda^{(n)}}(\{c_j(n)\}).
\]  

(4.5)

Strict inequality holds in fact in (4.5) for any nonvanishing \(\beta\) on any finite lattice.

Application of the decimation operation defined in section 2 on some given \(Z^{(-)}_{\Lambda^{(m-1)}}\) of the form (4.3) results in the rule

\[
f_p^{(-)}(U, m - 1) \rightarrow F_0(m) f_p^{(-)}(U, m) = F_0(m) \left[ 1 + \sum_{j \neq 0} (-1)^{2j} S_p[\mathcal{V}] d_j c_j(m) \chi_j(U) \right],
\]  

(4.6)
with coefficients $F_0(m)$, $c_j(m)$ computed according to the rules (2.19) - (2.22). Starting on
lattice $\Lambda$, the twisted partition function resulting after $n$ such steps is

$$Z^{(-)}_{\Lambda}(\beta, n) = \prod_{m=1}^{n} F_0(m)^{|\Lambda|/b^m} Z^{(-)}_{\Lambda(n)}(\{c_j(n)\}).$$

(4.7)

Note that the flux is carried entirely in $Z^{(-)}_{\Lambda(n)}$. Indeed, bulk free energy contributions from
each $\Lambda^{(m-1)} \rightarrow \Lambda^{(m)}$ decimation step arise from local moving-integration operations within
cells of side length $b$ on $\Lambda^{(m-1)}$, i.e. topologically trivial subsets, and are thus insensitive to
the flux presence. The evolution with $n$ of the effective action in $Z^{(-)}_{\Lambda(n)}$ then determines the
manner in which flux spreads, which is characteristic of the phase the system is in.

4.1 Upper and lower bounds

In the presence of the flux, the measure in (4.3) possesses the property of reflection positivity
only in $(d-1)$-dimensional planes perpendicular to any one of the directions $\rho \neq 1, 2$ in which
$V$ winds around the lattice. One way of dealing with this is to simply consider the quantity

$$Z^+_{\Lambda(n)}(\{c_j(n)\}) \equiv \frac{1}{2} \left( Z_{\Lambda(n)}(\{c_j(n)\}) + Z^{(-)}_{\Lambda(n)}(\{c_j(n)\}) \right)$$

(4.8)

instead of $Z^{(-)}_{\Lambda(n)}$. It is indeed easily checked that reflection positivity holds for the measure
in $Z^+_{\Lambda(n)}$ in all planes. A direct consequence of this (Appendix A) is then the analog of II.1:

IV.2 For $Z^+_{\Lambda(n)}(\{c_j(n)\})$ given by (4.8) with $c_j(n) \geq 0$ for all $j$, and periodic boundary
conditions,

(i) $Z^+_{\Lambda(n)}(\{c_j(n)\})$ is an increasing function of each $c_j(m)$:

$$\partial Z^+_{\Lambda(n)}(\{c_i(n)\})/\partial c_j(n) \geq 0;$$

(4.9)

(ii) $Z^+_{\Lambda(n)}(\{c_j(n)\}) \geq \left[ 1 + \sum_{j \neq 0} d_j^2 c_j(n)^6 \right]|\Lambda(n)|$.

(4.10)

Again, in these bounds equality holds only in the trivial case where all $c_j(n)$’s vanish. In
particular, one has

$$Z^+_{\Lambda(n)}(\{c_j(n)\}) > 1.$$  

(4.11)

Note that these bounds are identical to those in II.1. This signifies the obvious fact that
they bound from below by underestimating the bulk free energies proportional to the lattice volume, whereas the lattice size dependence of the free energy discrepancy between
$Z_{\Lambda(n)}(\{c_j(n)\})$ and $Z^{(-)}_{\Lambda(n)}(\{c_j(n)\})$ is much weaker.

Upper and lower bound statements analogous to III.1 and III.2 can be obtained for $Z^+_{\Lambda(n)}$.
One has:

IV.3 For $Z^+_{\Lambda(n-1)}$ of the form (4.8), a decimation transformation (4.6), (2.19) - (2.22) with
$\zeta = b^{d-2}$ and $0 < r \leq 1$ results in an upper bound on $Z^+_{\Lambda(n-1)}$:

$$Z^+_{\Lambda(n-1)}(\{c_j(n-1)\}) \leq F_0^U(n)|\Lambda(n)| Z^+_{\Lambda(n)}(\{c_j^U(n, r)\}).$$

(4.12)
The r.h.s. in (4.12) is a monotonically decreasing function of $r$ on $0 < r \leq 1$.

**IV.4** For $Z_{\Lambda_{(n-1)}}^+ \leq Z_{\Lambda_{(n-1)}}^+$ of the form (4.8):

\[
Z_{\Lambda_{(n)}}^+\{\{c_j^L(n)\}\} \leq Z_{\Lambda_{(n-1)}}^+\{\{c_j(n-1)\}\},
\]  

(4.13)

where the coefficients $c_j^L(n)$ are given by (3.3).

The proof of IV.3, as well as that of IV.4, an easy corollary of IV.2, are given in Appendix A. It then follows from (4.9) that (4.13) holds also with coefficients $\tilde{c}_j(n)$ given by (3.8) or (3.9). Again, in analogy to III.2, IV.4 also holds with $c_j^L$ given by (3.10), but this form will not be used here.

**4.2 Representation of $Z_{\Lambda} + Z_{\Lambda}^(-)$ on decimated lattices**

The procedure of section 3 leading to the representation (3.35) for $Z_{\Lambda}$ can now be applied to $Z_{\Lambda}^+ = (Z_{\Lambda} + Z_{\Lambda}^-)/2$. One introduces the interpolating coefficients $\tilde{c}_j(m, \alpha, r)$ given by eq. (3.12), and $F_0(m, h, \alpha, t)$ given by eq. (3.16) for some choice of interpolation function $h$ such as given by the examples (3.18). The quantity corresponding to (3.19) is then given by

\[
\tilde{Z}_{\Lambda(m)}^+(\beta, h, \alpha, t, r) = F_0(m, h, \alpha, t)|_{\Lambda(m)} Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\}
\]

(4.14)

where

\[
Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\} = \frac{1}{2}\left( Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\} + Z_{\Lambda(m)}^-\{\{\tilde{c}_j(m, \alpha, r)\}\}\right)
\]

(4.15)

with $Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\}$ given by (3.20) and $Z_{\Lambda(m)}^-\{\{\tilde{c}_j(m, \alpha, r)\}\}$ given by (3.3) - (4.4) with coefficients $\tilde{c}_j(m, \alpha, r)$. We then have the analog of III.3:

**IV.5** The interpolating free energies $\ln Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\}$ and $\ln \tilde{Z}_{\Lambda(m)}^+(\beta, h, \alpha, t, r)$ are increasing functions of $\alpha$:

\[
\partial \ln Z_{\Lambda(m)}^+\{\{\tilde{c}_j(m, \alpha, r)\}\}/\partial \alpha > 0.
\]

(4.16)

In terms of (4.12), IV.3 and IV.4 give

\[
\tilde{Z}_{\Lambda(m)}^+(\beta, h, 0, t, r) \leq Z_{\Lambda(m-1)}^+(\beta, h, 1, t, r) \leq \tilde{Z}_{\Lambda(m)}^+(\beta, h, 1, t, r).
\]

(4.17)

which implies that there exist a value of $\alpha$ in $(0, 1)$:

\[
\alpha^+(m, h, t, r; \{c_j(m-1)\}, b, \Lambda) \equiv \alpha_{\Lambda, h}^+(m)(t, r)
\]

(4.18)

such that

\[
\tilde{Z}_{\Lambda(m)}^+(\beta, h, \alpha_{\Lambda, h}^+(m)(t, r), t, r) = Z_{\Lambda(m-1)}^+(\beta, h, t, r).
\]

(4.19)

This value is unique, for given values of $t, r$, by IV.5. $\alpha_{\Lambda, h}^+(m)(t, r)$ gives the regular level surface of the function $\tilde{Z}_{\Lambda(m)}^+(\beta, h, \alpha, t, r)$ fixed by the value $Z_{\Lambda(m-1)}^+$. 

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All the considerations concerning the dependence on the parameters \( t, r \) in the previous section carry over directly to \( \alpha_{\Lambda,h}^+(m)(t,r) \). In particular, one has

\[
\frac{\partial \alpha_{\Lambda,h}^+(m)(t,r)}{\partial t} = v^+(\alpha_{\Lambda,h}^+(m)(t,r), t, r),
\]

where

\[
v^+(\alpha, t, r) \equiv -\frac{\partial h(\alpha, t)/\partial t}{\partial \alpha + A_{\Lambda(m)}^+(\alpha, r)},
\]

with

\[
A_{\Lambda(m)}^+(\alpha, r) \equiv \frac{1}{\ln F_0^U(\alpha)} \left( \frac{1}{\partial \alpha} \right) \left( \ln Z_{\Lambda(m)}^+(\{\tilde{c}_j(n, \alpha, r)\}) > 0 \right).
\]

Again, we always assume that \( h \) is chosen such that \( \partial h/\partial t \) is negative. Then, from (4.24), \( v^+ > 0 \) on \( 0 < \alpha < 1 \), with \( v^+ = 0 \) at \( \alpha = 0 \) and \( \alpha = 1 \). Also

\[
\frac{dh(\alpha_{\Lambda,h}^+(m)(t,r), t)}{dt} = \frac{\partial \alpha_{\Lambda,h}^+(m)(t,r)}{\partial t} A_{\Lambda(m)}^+(\alpha_{\Lambda,h}^+(m)(t,r), r).
\]

The derivative w.r.t. \( r \) is similarly given by (3.3).

The values (4.15) obey

\[
\delta^+ < \alpha_{\Lambda,h}^+(m)(t,r) < 1 - \delta^+
\]

with lattice-size independent, positive \( \delta^+ \) and \( \delta^+ \). Again, the lower bound is automatically satisfied, whereas the upper bound is ensured by letting the parameter \( r \) vary, if necessary, in (3.3) (cf. Appendix B). From this it follows that the analog of (3.32):

\[
\frac{\partial \alpha_{\Lambda,h}^+(m)(t,r)}{\partial t}(t, r) \geq \eta^+_1(\delta^+) > 0, \quad \frac{dh(\alpha_{\Lambda,h}^+(m)(t,r), t)}{dt}(t, r) \geq \eta^+_2(\delta^+) > 0,
\]

holds for some lattice-size independent \( \eta^+_1, \eta^+_2 \). Since, furthermore, (4.24) holds for any \( r \) if it already holds for \( r = 1 \), we may again set \( r = 1 - \epsilon \), and, according to the convention introduced in the previous section, write \( \alpha_{\Lambda,h}^+(m)(t) \equiv \alpha_{\Lambda,h}^+(m)(t, 1 - \epsilon) \), etc.

As in the last section, one may iterate this procedure of performing a decimation transformation to produce upper and lower bounds according to (4.17), and then fixing the value (4.18) of the interpolating parameter \( \alpha \) according to (4.19). Assume that we choose the same interpolation family \( h \) at every step. Then starting from the original lattice, after \( n \) iterations one obtains

\[
Z_\Lambda^+(\beta) = \frac{1}{2} \left( Z_\Lambda(\beta) + Z_\Lambda(-)(\beta) \right)
\]

\[
= \left[ \prod_{m=1}^n F_0(m, h, \alpha_{\Lambda,h}^+(m)(t_m), t_m)^{|\Lambda|/h^{md}} \right] Z_\Lambda^{+}(\{\tilde{c}_j(n, \alpha_{\Lambda,h}^+(m)(t_n))\}).
\]

The discussion in subsection 3.4 concerning the representation (3.35) of \( Z_\Lambda \) applies equally well to (4.26). In particular, note that again the existence of the large volume limit implies that

\[
\alpha_{\Lambda,h}^+(m)(t,r) = \alpha_{h}^+(m)(t,r) + \delta \alpha_{\Lambda,h}^+(m)(t,r)
\]
with $\delta \alpha_{A,h}^{(+m)}(t,r) \to 0$ as some inverse power of lattice size in the $|\Lambda^{(m)}| \to \infty$ limit. Alternatively, (4.24) already implies that one must have a lattice-size independent contribution $\alpha_{h}^{(+m)}(t,r) > \delta^{+\prime} > 0$ in (4.27) (cf. Appendix B).

Again, either scheme (3.33) or (3.38) may be used to obtain (4.26). For the reasons already noted, however, the latter scheme is more convenient for our considerations. Note, furthermore, that the bounding coefficients $c_{j}^{U}(m)$ and $c_{j}^{L}(m)$ in this scheme are the same for $Z_{\Lambda}$ and $Z_{\Lambda}^{+}$ since they do not depend on $\alpha_{A,h}^{(m)}(t,r)$ or $\alpha_{A,h}^{(+m)}(t,r)$. We, therefore, adopt it in what follows as the common iteration scheme for $Z_{\Lambda}$ and $Z_{\Lambda}^{+}$:

\[
\begin{align*}
\{c_{j}^{L}(1)\} & \leq \{\tilde{c}_{j}(1, \alpha_{A,h}^{(1)}(t_{1}))\}, \{\tilde{c}_{j}(1, \alpha_{A,h}^{(+1)}(t_{1}))\} \leq \{c_{j}^{U}(1)\} \\
\{c_{j}^{L}(2)\} & \leq \{\tilde{c}_{j}(2, \alpha_{A,h}^{(2)}(t_{2}))\}, \{\tilde{c}_{j}(2, \alpha_{A,h}^{(+2)}(t_{2}))\} \leq \{c_{j}^{U}(2)\} \\
& \vdots \ 
\end{align*}
\]

In (4.28) and in the following, the more detailed notation $h^{+}$ and $t^{+}$ is used for the choice of interpolation and $t$-parameter values occurring in (4.26) whenever they need be distinguished from those used in the representation (3.35) for $Z_{\Lambda}$, which can, of course, be chosen independently.

As indicated by the notation, even for common choice of interpolation $h = h^{+}$ and of all other parameters, the values of $\alpha_{A,h}^{(+m)}(t,r)$ fixed by the requirement (4.19) are a priori distinct from those of $\alpha_{A,h}^{(m)}(t,r)$ fixed by (3.24). It is easily seen, however, that for sufficiently large lattice volume they must nearly coincide. We examine this difference more precisely below.

## 5 The ratio $Z_{\Lambda}^{(-)} / Z_{\Lambda}$

We may now compare $Z_{\Lambda}$ and $Z_{\Lambda} + Z_{\Lambda}^{(-)}$ by means of their representations (3.35) and (4.26) on successively decimated lattices. Consider then the ratio of $Z_{\Lambda} + Z_{\Lambda}^{(-)}$ and $Z_{\Lambda}$ as given by (4.26) and (3.35) with common choice of interpolation $h = h^{+}$ after one decimation:

\[
\left(1 + \frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}}\right) = \frac{2 \tilde{Z}_{\Lambda}^{(+)}(\beta, h, \alpha_{A,h}^{(+1)}(t^{+}), t^{+})}{\tilde{Z}_{\Lambda}^{(1)}(\beta, h, \alpha_{A,h}^{(1)}(t), t)}
\]

\[
= \left(\frac{\tilde{Z}_{\Lambda}^{(1)}(\beta, h, \alpha_{A,h}^{(+1)}(t^{+}), t^{+})}{\tilde{Z}_{\Lambda}^{(1)}(\beta, h, \alpha_{A,h}^{(1)}(t), t)}\right) \left(1 + \frac{Z_{\Lambda}^{(-)}(\{\tilde{c}_{j}(1, \alpha_{A,h}^{(+1)}(t^{+}))\})}{Z_{\Lambda}^{(1)}(\{\tilde{c}_{j}(1, \alpha_{A,h}^{(1)}(t))\})}\right)
\]

\[
(5.1)
\]

\[
(5.2)
\]

\[
24
\]

By construction, the r.h.s. is invariant under independent variations of $t$ and $t^+$. Now since by IV.1

$$1 < \left(1 + \frac{Z_{-}^{(-)}}{Z_{\Lambda}}\right) < 2 \quad \text{and} \quad 1 < \left(1 + \frac{Z_{-}^{(-)}}{Z_{\Lambda}} \left(\{\tilde{c}_j(1, \alpha_{\Lambda, h}^{(1)}(t^+))\}\right)\right) < 2 ,$$

it follows that

$$\frac{1}{2} < \frac{\tilde{Z}_{\Lambda}(\beta, h, \alpha_{\Lambda, h}(t^+), t^+)}{\tilde{Z}_{\Lambda}(\beta, h, \alpha_{\Lambda, h}(t), t)} < 2 .$$

(5.4)

Though the bounds (5.3) are rather crude, the resulting constraint (5.4) is quite informative. First, it says that if in the equality (3.24), i.e. Though the bounds (5.3) are rather crude, the resulting constraint (5.4) is quite informative. First, it says that if in the equality (3.24), i.e.

$$\tilde{Z}_{\Lambda}(\beta, h, \alpha_{\Lambda, h}(t), t) = Z_{\Lambda},$$

one substitutes for $\alpha_{\Lambda, h}(t)$ the wrong level surface $\alpha_{\Lambda, h}^{(1)}(t)$, the resulting discrepancy in the free energy per unit volume is at most $O(1/|\Lambda(1)|)$. Furthermore, (5.4) constrains by how much $\alpha_{\Lambda, h}^{(1)}(t)$ can differ from $\alpha_{\Lambda, h}(t^+)$ at $t = t^+$. From the definition (3.19) and III.3, the change in $\tilde{Z}_{\Lambda(1)}(\beta, h, \alpha, t, r)$ under a shift $\delta \alpha$ in $\alpha$ satisfies

$$| \delta \ln \tilde{Z}_{\Lambda(1)}(\beta, h, \alpha, t, r) | > | \delta \alpha | | \Lambda(1) | \ln F_0^U(1) \frac{\partial h(\alpha, t)}{\partial \alpha}. $$

(5.5)

When combined with (5.3), the constraint (5.4), taken at general $r$, implies that one must have

$$| \alpha_{\Lambda, h}^{(1)}(t, r) - \alpha_{\Lambda, h}(t, r) | \leq O\left(\frac{1}{|\Lambda(1)|}\right) .$$

(5.6)

This implies that in (4.40), (4.27) one has $\alpha_{h}^{(1)}(t) = \alpha_{h}^{(1)}(t)$, i.e. any difference occurs only in the parts $\delta \alpha_{\Lambda, h}^{(1)}$, $\delta \alpha_{\Lambda, h}^{+1}$ that vary inversely with lattice size. Thus, in the large volume limit, this difference becomes unimportant if one is interested only in the computation of partition functions, or bulk free energies. This, however, is not the case for free energy differences such as the ratio (5.1). Indeed, any discrepancy of the size (5.6) means that the first factor in (5.5) is never as large as the second factor in round brackets. Thus the expression for the ratio of the twisted to the untwisted partition function given by (5.1) is not immediately useful for extracting this ratio on the coarser lattice.

To address this issue one may make use of the $t$-parametrization invariance of (5.2). First the cancellation of the bulk energies generated in the integration from scale $a$ to $ba$ is made explicit as follows. For any given $t^+_1$, choose $t_1$ in $\tilde{Z}_{\Lambda(1)}(\beta, \alpha_{\Lambda, h}(t_1), t_1)$ so that

$$h(\alpha_{\Lambda, h}(t_1), t_1) = h(\alpha_{\Lambda, h}^{(1)}(t^+_1), t^+_1).$$

(5.7)

This is clearly always possible by (3.32) and (4.25), and by (5.6); in fact, $t_1 - t^+_1 = O(1/|\Lambda(1)|)$. Then (5.1) assumes the form

$$\left(1 + \frac{Z_{-}^{(-)}}{Z_{\Lambda}}\right) = \frac{2Z_{\Lambda}^{(+)} (\{\tilde{c}_j(1, \alpha_{\Lambda, h}^{(1)}(t_1))\})}{Z_{\Lambda^{(1)}} (\{\tilde{c}_j(1, \alpha_{\Lambda, h}(t_1))\})}.$$ 

(5.8)

(5.4) clearly holds for general values of the parameter $r$, not just for the values (3.31) used in (5.2).
We may now iterate this procedure performing \((n - 1)\) decimation steps according to the scheme \((4.28)\), at each step choosing \(t_m, t^+_m\) such that

\[
h(\alpha_{\Lambda,h}^m(t_m), t_m) = h(\alpha_{\Lambda,h}^{+(m)}(t^+_m), t^+_m), \quad m = 1, \ldots (n - 1). \tag{5.9}
\]

Carrying out a final \(n\)-th decimation step one obtains

\[
\left(1 + \frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}}\right) = \frac{2 \tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t^+, t^+))}{\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t), t)} = \frac{\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t^+, t^+))}{\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t), t)} \left(1 + \frac{Z_{\Lambda}^{(-)}(\{ \tilde{c}_j(n, \alpha_{\Lambda,h}^{+(n)}(t^+)) \})}{Z_{\Lambda}(\{ \tilde{c}_j(n, \alpha_{\Lambda,h}^{+(n)}(t^+)) \})}\right) \tag{5.10}
\]

The argument for \(n = 1\) (eq. \((5.2)\)) above may now be applied to \((5.11)\) to conclude

\[
|\alpha_{\Lambda,h}^{+(n)}(t, r) - \alpha_{\Lambda,h}^{+(n)}(t, r)| \leq O\left(\frac{1}{|\Lambda|}\right). \tag{5.12}
\]

Any such discrepancy between \(\alpha_{\Lambda,h}^{+(n)}(t)\) and \(\alpha_{\Lambda,h}^{+(n)}(t)\) in \((5.11)\) presents the same problem for extracting the ratio at scale \(b^n a\) as at scale \(b a\). In this sense \((5.11)\) is not qualitatively different from the \(n = 1\) case \((5.2)\). Transferring the discrepancy to large \(n\), however, allows a technical simplification as we see below.

Next, consider \((5.10)\) rewritten as

\[
\left(1 + \frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}}\right) = \left(1 + \frac{Z_{\Lambda}^{+(n-1)}}{Z_{\Lambda}^{+(n)}}\right) \left(1 + \frac{Z_{\Lambda}^{(-)}(\{ \tilde{c}_j(n, \alpha_{\Lambda,h}^{+(n)}(t)) \})}{Z_{\Lambda}(\{ \tilde{c}_j(n, \alpha_{\Lambda,h}^{+(n)}(t)) \})}\right) \tag{5.13}
\]

by use of \((4.19)\). By construction (cf. \((3.24)\)), \(\alpha_{\Lambda,h}^{+(n)}(t)\) is such that the r.h.s. in \((5.10)\), hence in \((5.13)\), is invariant under changes in the parameter \(t\); but note that the two \(\alpha_{\Lambda,h}^{+(n)}\)-dependent factors in round brackets on the r.h.s. in \((5.13)\) are not separately invariant. If, for some given \(t\), \(\alpha_{\Lambda,h}^{+(n)}(t)\) is larger (smaller) than \(\alpha_{\Lambda,h}^{+(n)}(t)\), then, by IV.5, \(\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t), t)\) is larger (smaller) than \(\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{+(n)}(t), t)\), and the second factor in round brackets on the r.h.s. of \((5.13)\) overestimates (underestimates) the ratio \(Z_{\Lambda}^{(-)}/Z_{\Lambda}\). It is then natural to ask whether there exist a value \(t = t_{\Lambda,h}^{((n)}\) such that

\[
\tilde{Z}_{\Lambda}^{+(n)}(\beta, h, \alpha_{\Lambda,h}^{(n)}(t_{\Lambda,h}^{(n)}), t_{\Lambda,h}^{(n)})) = Z_{\Lambda}^{+(n-1)}. \tag{5.14}
\]

Note that the graphs of \(\alpha_{\Lambda,h}^{(n)}(t)\) and \(\alpha_{\Lambda,h}^{+(n)}(t)\) must intersect at \(t_{\Lambda,h}^{(n)}\).

A unique solution to \((5.14)\) indeed exists as shown in Appendix C provided

\[
A_{\Lambda^{(n)}}(\alpha, r) \geq A_{\Lambda^{(n)}}^{+(\alpha)}(\alpha, r) \tag{5.15}
\]

with \(r\) in \((3.31)\). An equivalent statement to \((5.15)\) is

\[
A_{\Lambda^{(n)}}(\alpha, r) \geq A_{\Lambda^{(n)}}^{(-\alpha)}(\alpha, r), \tag{5.16}
\]

26
where \( A_{\Lambda(n)}^{(-)}(\alpha, r) \) is defined by (4.22) but with \( Z_{\Lambda(n)}^{+} \) replaced by \( Z_{\Lambda(n)}^{(-)} \). Assume now that under successive decimations the coefficients \( c_{j}^{(m)} \) in (4.28) evolve within the convergence radius of the strong coupling cluster expansion. Taking then \( n \) in (5.10) sufficiently large, we need establish inequality (5.15) only at strong coupling. Within this expansion it is a straightforward exercise to establish the validity of (5.15), with strict inequality on any finite lattice.

We summarize the above development in the following:

**V.1** Consider \( n \) successive decimation steps performed according to the scheme (4.28). Assume that there is an \( n_{0} \) such that the upper bound coefficients \( c_{j}^{+(n)} \) become sufficiently small for \( n \geq n_{0} \).

Then the ratio of the twisted to the untwisted partition function on lattice \( \Lambda^{(n)} \), of spacing \( b^{n}a \) and \( n \geq n_{0} \), given by:

\[
\frac{Z_{\Lambda}^{(-)}(\beta)}{Z_{\Lambda}(\beta)} = \frac{Z_{\Lambda(n)}^{(-)}\left(\{c_{j}(n, \alpha)\}\right)}{Z_{\Lambda(n)}\left(\{c_{j}(n, \alpha)\}\right)}, \tag{5.17}
\]

where

\[
\alpha_{\Lambda}^{*}(n) \equiv \alpha_{\Lambda,h}(t_{\Lambda,h}^{(n)}). \tag{5.18}
\]

Here, the function \( \alpha_{\Lambda,h}(t) \) is defined by (3.24), i.e. is the solution for \( \alpha \) to

\[
\tilde{Z}_{\Lambda(n)}(\beta, h, \alpha, t) = Z_{\Lambda(n-1)}, \tag{5.19}
\]

and \( t_{\Lambda,h}^{(n)} \) is defined by (5.14), i.e. is the solution for \( t \) to the equation

\[
\tilde{Z}_{\Lambda(n)}^{+}(\beta, h, \alpha_{\Lambda,h}(t), t) = Z_{\Lambda(n-1)}^{+}. \tag{5.20}
\]

As indicated by the notation in (5.18), any dependence on \( h \) must cancel in \( \alpha_{\Lambda}^{*}(n) \). Indeed, Cauchy’s form of the intermediate value theorem gives

\[
\ln \frac{Z_{\Lambda(n)}^{(-)}\left(\{c_{j}(n, \alpha)\}\right)}{Z_{\Lambda(n)}\left(\{c_{j}(n, \alpha)\}\right)} - \ln \frac{Z_{\Lambda(n)}^{(-)}\left(\{c_{j}(n, \alpha^{*}(n))\}\right)}{Z_{\Lambda(n)}\left(\{c_{j}(n, \alpha^{*}(n))\}\right)} = \frac{A_{\Lambda(n)}^{(-)}(\xi)}{A_{\Lambda(n)}^{+}(\xi)} \leq 1, \tag{5.21}
\]

for some \( \xi \) between \( \alpha^{*}(n) \) and \( \alpha \), and use of (5.16) was made to obtain the last inequality. Setting \( \alpha \) equal to 1 in (5.21), combining with (5.17), and using III.3, IV.5, gives

\[
\frac{Z_{\Lambda}^{(-)}}{Z_{\Lambda}} \geq \frac{Z_{\Lambda(n)}^{(-)}\left(\{c_{j}(n)\}\right)}{Z_{\Lambda(n)}\left(\{c_{j}(n)\}\right)}. \tag{5.22}
\]

The upper bound coefficients in (4.28) then, which correspond to upper bounds for the partition functions \( Z_{\Lambda} \) and \( Z_{\Lambda}^{(-)} \), give a lower bound for the ratio \( Z_{\Lambda}^{(-)}/Z_{\Lambda} \).

\[\text{[10] For Abelian systems, comparison inequalities of the type (5.15) either follow from Griffith’s inequalities, or can be approached by the same methods. All such known methods fail in the non-Abelian case.}\]

\[\text{[11] This result was first stated a long time ago in [12].}\]
in (5.21), similarly yields an upper bound. Thus:

**V.2** With the same conditions as in V.1 the ratio of the twisted to the untwisted partition function on lattice $\Lambda$ of spacing $\alpha$ is bounded on lattice $\Lambda^{(n)}$ of spacing $b^n\alpha$ by:

$$\frac{Z_{\Lambda^{(n)}}^{-}(\{ c^{-}_{j}(n) \})}{Z_{\Lambda^{(n)}}^{-}(\{ c^{U}_{j}(n) \})} \geq \frac{Z_{\Lambda^{(n)}}^{-}(\{ c^{U}_{j}(n) \})}{Z_{\Lambda^{(n)}}^{-}(\{ c^{U}_{j}(n) \})} \geq \frac{Z_{\Lambda^{(n)}}^{-}(\{ c^{U}_{j}(n) \})}{Z_{\Lambda^{(n)}}^{-}(\{ c^{U}_{j}(n) \})}. \quad (5.23)$$

Now, the ratio of the interpolating partition functions (4.15) and (3.20) interpolates monotonically between the upper and lower bounds in (5.23) since

$$\frac{d}{d\alpha} Z_{\Lambda^{(n)}}^{-}(\{ c^{-}_{j}(n, \alpha) \}) < 0 \quad (5.24)$$

by (5.16). It follows that there exist a unique value $\alpha^{*}_{\Lambda^{(n)}}$ of $\alpha$ at which this ratio of the interpolating partition functions equals $Z_{\Lambda}^{-}/Z_{\Lambda}$. This is a restatement of (5.17), but makes explicit the fact that this value is independent of $h$. In fact, it shows that all dependence on parametrization choices, i.e. the choice of parameters $t_{m}$ made in successive decimations, eventually cancels in $\alpha^{*}_{\Lambda^{(n)}}$. Indeed, the latter can depend only on the number of decimations $n$ and the initial coupling $\beta$, since this is all the upper and lower bounds in (5.23) depend on. This, in retrospect, is as expected, since all bulk free-energy contributions depending on such choices were canceled in finally arriving at (5.17), but V.2 makes it manifest.

(5.23) was obtained as a corollary of (5.17). An alternative approach would be to proceed in the reverse direction, i.e. establish (5.23) directly, from which (5.17) would follow by interpolation between the upper and lower bounds as in the previous paragraph. In other words, follow also in the case of the ratio of the partition functions the approach followed separately for the untwisted and twisted partition functions in the previous sections. This is further discussed in Appendix C.

### 6 Confinement

#### 6.1 Order parameters

The vortex free energy $F_{\Lambda}^{-}(\beta)$ is defined by the ratio of partition functions considered in the previous section:

$$\exp(-F_{\Lambda}^{-}(\beta)) = \frac{Z_{\Lambda}^{-}(\beta)}{Z_{\Lambda}(\beta)}. \quad (6.1)$$

It represents the free energy cost for adding a vortex to the vacuum, the $Z(2)$ flux of the inserted vortex being rendered stable by wrapping around the toroidal lattice. As has been discussed in the literature, all possible phases of gauge theory (Higgs, Coulomb, or confinement) can be characterized by the behavior of (6.1) as one lets the lattice become large. In particular, having taken the vortex to wind through the lattice in the directions $\kappa = 3, \ldots, d$, a confining phase is signaled by the asymptotic behavior

$$F_{\Lambda}^{-}(\beta) \sim L \exp(-\tilde{\sigma}(\beta)|A|), \quad (6.2)$$
where \( L \equiv \prod_{\kappa \neq 1,2} L_\kappa \), and \( A \equiv L_1 L_2 \). (6.2) represents exponential spreading of the flux introduced by the twist on the set \( \mathcal{V} \) in the transverse directions (creation of mass gap), with \( \hat{\sigma}(\beta) \) giving the exact string tension. Note that, according to (6.2), \( F_A^{(-)}(\beta) \to 0 \) as \( |\Lambda| \to \infty \) in any power-law fashion, i.e. one has ‘condensation’ of the vortex flux. The behavior (6.2) is dictated by physical reasoning [9], [10], and explicitly realized within the strong coupling expansion. As such free energies differences are generally notoriously difficult to measure accurately, demonstration of the behavior (6.2) by numerical simulations at large \( \beta \)'s has been achieved only relatively recently [13], [14].

The \( Z(2) \) Fourier transform of (6.1)

\[
\exp(-F_A^q(\beta)) = \frac{1}{2} \left( 1 - \frac{Z_A^{(-)}(\beta)}{Z_A(\beta)} \right) \quad (6.3)
\]

gives the corresponding dual (w.r.t. the gauge group center) order parameter, the color electric free energy. (6.1) and (6.3) are ideal pure long-range order parameters. They do not suffer from the physically irrelevant but technically quite bothersome complications, such as loss of translational invariance, or mass renormalization and other short range contributions, that arise from the explicit introduction of external sources. Such external current sources are introduced in the definition of the Wilson and t’Hooft loops. Furthermore, the behavior of the latter can be bounded by that of (6.1) and (6.3) [15]. In particular, the following relation holds. Let \( C \) be a rectangular loop of minimal area \( S \) lying in a 2-dimensional [12]-plane. Then [15]:

\[
\langle W[C] \rangle_\Lambda \leq \left[ \exp(-F_A^q) \right]^{S/A}, \quad (6.4)
\]

where \( W[C] = \chi_{1/2} \left( \prod_{b \in C} U_b \right) \) is the usual Wilson loop observable. It follows from (6.4) that confining behavior (6.2) of the vortex free energy implies confining behavior (‘area-law’) for the Wilson loop.

### 6.2 Strong coupling cluster expansion and confinement

We now return to our considerations at the end of section 3 regarding the flow of the coefficients \( \tilde{c}_j(n, \alpha^{(n)}_{\Lambda,A,h}(t)) \) in our partition function representations (3.35) and (4.26). This flow is bounded from above by that of the MK coefficients \( c^U_j(m) \) regardless of the specific value assumed by the \( \alpha^{(m)}_{A,h}(t_m) \)'s at each decimation step (cf (3.42)). Furthermore, by explicit evaluation under the iteration rules (2.19) - (2.20), one finds that \( c^U_j(n) \to 0 \) as \( n \to \infty \) for any initial \( \beta \), provided \( d \leq 4 \). Thus, given any initial \( \beta \), one may always take the number of iterations \( n \) large enough so that the coefficients \( c^U_j(n) \) become small enough to be within the region of convergence of the strong coupling expansion. Then by V.1:

\[
\exp(-F_A^{(-)}(\beta)) = \frac{Z_A^{(-)}(\{ \tilde{c}_j(n, \alpha^{*}_A(n)) \})}{Z_A(\{ \tilde{c}_j(n, \alpha^{*}_A(n)) \})}. \quad (6.5)
\]

The vortex free energy may then be evaluated in terms of the coefficients \( \tilde{c}_j(n, \alpha^{*}_A(n)) \) directly on lattice \( \Lambda^{(n)} \) of spacing \( b^n a \) within a convergent strong coupling polymer expansion.
Recall that, in the pure lattice gauge theory context, a polymer is a set $Y$ of connected plaquettes containing no ‘free’ bond, i.e. no bond belonging to only one plaquette in $Y$ (see e.g. [10]). The activity of a polymer $Y$ is defined by
\[ z(Y) = \int \prod_{b \in Y} dU_b \prod_{p \in Y} g_p(U, n), \]
where
\[ g_p(U, n) = \sum_{j \neq 0} d_j \tilde{c}_j(n, \alpha^*(n)) \chi_j(U_p). \]

The polymer expansion is then
\[ \ln Z_{\Lambda} = \sum_{X \subset \Lambda(n)} a(X) \prod_{Y_i \in X} z(Y_i)^{n_i}, \]
where the sum is over all linked clusters of polymers in $\Lambda(n)$, each cluster $X$ consisting of a connected set of polymers $Y_i$, $i = 1, \ldots, k_X$ with multiplicities $n_i$. The combinatorial factor $a(X)$ is given by
\[ a(X) = \sum_{G(X)} (-1)^{l(G)}, \]
where the sum is over all connected graphs on $X$ (full set (including multiplicities) $\{ Y_i \}$ as vertices with a line connecting overlapping polymers) and $l(G)$ is the number of lines in the graph.

In the case of $Z^{(-)}_{\Lambda(n)}$, the presence of the flux enters the activities through the replacement $\tilde{c}_j(n, \alpha^*(n)) - \tilde{c}_j(n, \alpha^*(n))$. We denote the resulting activities by $\tilde{c}^{(-)}(Y)$. This replacement does not affect polymers that are wholly contained in a simply connected part of $\Lambda(n)$, since, in this case, the flux can be removed by a change of variables in the integrals in (6.6). Only clusters that contain at least one non-simply connected polymer forming a topologically non-trivial closed surface can be affected. Thus, one has
\[ \ln Z^{(-)}_{\Lambda(n)} - \ln Z_{\Lambda(n)} = \sum_{X \subset \Lambda(n)} a(X) \left( \prod_{Y_i \in X} z^{(-)}(Y_i)^{n_i} - \prod_{Y_i \in X} z(Y_i)^{n_i} \right), \]
where the sum is only over all such topologically nontrivial linked clusters, the contribution of all other clusters canceling in the difference. The minimal cluster of this type consists of a single polymer which is a 2-dimensional plane $\Pi : x_\mu = \text{const.}, \mu = 3, \ldots, d$ on $\Lambda(n)$, thus of size $A(n) = L_1^{(n)} L_2^{(n)}$, and activity
\[ z(\Pi) = \sum_{\text{half-int. } j \geq 1/2} \tilde{c}_j(n, \alpha^*(n)) A^{(n)} = \tilde{c}_{1/2}(n, \alpha^*(n)) A^{(n)} \left[ 1 + \sum_{\text{half-int. } j \geq 3/2} \left( \frac{\tilde{c}_j(n, \alpha^*(n))}{\tilde{c}_{1/2}(n, \alpha^*(n))} \right)^{A^{(n)}} \right]. \]
(Note that the terms from the higher representations in $\tilde{c}_{1/2}(n, \alpha^*(n))$ become utterly negligible in the large volume limit.) There are $L_1^{(n)} = \prod_{k \neq 1, 2} L_k^{(n)}$ such minimal clusters giving the leading contribution in (6.10). This leading contribution is thus seen to give the confining behavior $\tilde{c}_{1/2}(n, \alpha^*(n))$. Nonleading contributions come from nonminimal clusters consisting of $\Pi$ with
or without ‘decorations’, and additional polymers touching Π. Such corrections have been evaluated in terms of the character expansion coefficients (the \(\tilde{c}_j(n, \alpha_\Lambda^{(n)})\)’s in our case) to quite high order \[16\]. They can be shown to exponentiate, so that

\[
\frac{1}{L} F^{(-)}_\Lambda(\beta) = \exp(-\sigma \Lambda A)
\]  

(6.12)

with

\[
\hat{\sigma}_\Lambda = \frac{1}{b^{2n}} \kappa_\Lambda(n, \alpha_\Lambda^{(n)})
\]

\[
= \frac{1}{b^{2n}} \left[ \kappa(n, \alpha_\Lambda^{(n)}) + O \left( (\tilde{c}_{j+1/2}/\tilde{c}_{1/2})^{L(\mu)} \right) + O(n/A^{(n)}) \right],
\]

(6.13)

where \[16\]

\[
\kappa(n, \alpha_\Lambda^{(n)}) = \left[ -\ln \tilde{c}_{1/2}(n, \alpha_\Lambda^{(n)}) - 4 \tilde{c}_{1/2}(n, \alpha_\Lambda^{(n)})^4 + 8 \tilde{c}_{1/2}(n, \alpha_\Lambda^{(n)})^6 + \ldots \right].
\]

(6.14)

By the convergence of the expansion \[17\], the large volume limit exists and is given by \(\hat{\sigma} = \kappa(n, \alpha_\Lambda^{(n)})/b^{2n}\), where \(\alpha_\Lambda^{(n)}\) is the lattice independent part of \(\alpha_\Lambda^{(n)}\) (cf. \(3.40\)).

The number of iterations \(n\) in the above expressions is taken large enough so that, given some initial \(\beta\) on \(\Lambda\), the resulting \(\alpha_U^{(n)}(n)\) are within the expansion convergence regime, and one can write the representation \[6.5\] by V.1. This implies the existence of a scale, a point to which we return below. Otherwise, \(n\) is arbitrary. By construction, our procedure is such that the ratio \(6.1\) is reproduced under successive decimations. Thus, given \(6.5\) at some \(n\), suppose one performs one more decimation to lattice \(\Lambda^{(n+1)}\). The condition that determines \(\alpha_\Lambda^{(n+1)}\) such that \(6.5\) is preserved is

\[
\kappa_\Lambda(n, \alpha_\Lambda^{(n)}) = \frac{1}{b^{2n}} \kappa_\Lambda(n + 1, \alpha_\Lambda^{(n+1)}),
\]

(6.15)

which then results in constant string tension \(\hat{\sigma}_\Lambda\) under successive decimations. Using \[6.14\] with \[3.12\] and \[2.37\], it is an easy exercise to solve \[6.15\], at least to leading approximation, for \(\alpha_\Lambda^{(n+1)}\). The \(t\)-parameter value \(t_\hbar^{(n+1)}\) this \(\alpha_\Lambda^{(n+1)}\) corresponds to can then also be easily obtained, if desired, from

\[
h(\alpha^{(n)}, t) \ln F_0^U(n + 1)|\Lambda^{(n+1)}| + \ln Z_{\Lambda^{(n+1)}} \left\{ \tilde{c}_j(n + 1, \alpha_\Lambda^{(n+1)}) \right\} = \ln Z_{\Lambda^{(n)}} \left\{ \tilde{c}_j(n, \alpha^{(n)}) \right\},
\]

with \(\ln Z_{\Lambda^{(n)}}\), \(\ln Z_{\Lambda^{(n+1)}}\) given by \[5.8\] - in fact, to leading approximation, \(\ln Z_{\Lambda^{(n+1)}}\) can be ignored. Note that this amounts to replacing the set of the two equations \[5.19\] - \[5.20\] in V.1 by their ratio and one of them. This is indeed the most convenient procedure once \[6.5\] has been achieved.

### 6.3 String tension and asymptotic freedom

\(\kappa(n, \alpha^{(n)})\) is the string tension in lattice units of lattice \(\Lambda^{(n)}\). It is a complicated, but well-defined function of the original coupling \(\beta = 4/g^2\) defined on lattice \(\Lambda\), eq. \[2.1\] (cf. remarks following \[5.21\]). We write

\[
\kappa(n, \alpha^{(n)}) \equiv \hat{\sigma}(n, g).
\]

(6.16)
In dimensional units the asymptotic string tension in \( d = 4 \) \( \text{(6.13) - (6.14)} \) is then

\[
\sigma = \frac{1}{a^2 b^{2n}} \sigma(n, g) \quad (6.17)
\]

\[
= \frac{1}{a^2} \hat{\sigma}(g). \quad (6.18)
\]

Here, as remarked above, \( n \) is assumed greater than some required smallest \( n(g) \). This (dynamically generated) physical scale, or some chosen multiple of it, is the only parameter in the theory. Fixing it specifies how the coupling \( g \) must vary with changes of the (unphysical) lattice spacing \( a \).

It is convenient, and customary, to introduce a fixed scale \( \Lambda_0 \) serving as an arbitrary unit of physical scales. Setting

\[
\Lambda_0^{-1} = ab^n, \quad (6.19)
\]

determines the lattice spacing \( a \) such that it takes \( n \) steps to reach length scale \( 1/\Lambda_0 \):

\[
n = \frac{1}{\ln b} \ln \frac{1}{a \Lambda_0}. \quad (6.20)
\]

Fixing the string tension, given in units of \( \Lambda_0 \):

\[
\sigma = k\Lambda_0^2, \quad (6.21)
\]

implies

\[
\hat{\sigma}(n, g) = k \quad (6.22)
\]

for some constant \( k \). \( \text{(6.22)} \) specifies the dependence of the bare coupling \( g \) on \( n \), hence, through \( \text{(6.20)} \), the dependence on the lattice spacing \( a \). It gives then the value \( g(a) \) specified by the value of the string tension. (This is, of course, equivalent to fixing \( \text{(6.18)} \) directly.)

Since

\[
\hat{\sigma}(n + 1, g + \Delta g)^{1/2} - \hat{\sigma}(n, g)^{1/2} = b [\hat{\sigma}(n, g + \Delta g)^{1/2} - \hat{\sigma}(n, g)^{1/2}] + (b - 1)\hat{\sigma}(n, g)^{1/2}
\]

and \( \Delta a = -(b - 1)a/b \) for \( \Delta n = 1 \), one has from \( \text{(6.22)} \):

\[
\frac{\Delta \sqrt{\hat{\sigma}}}{\Delta g} (a \Delta g / \Delta a) = \sqrt{\hat{\sigma}} \quad (6.23)
\]

If \( (a \Delta g / \Delta a) \equiv \beta(g) \), the ‘beta-function’, is known, \( \text{(6.23)} \) can be integrated directly for \( \hat{\sigma}(g) \). (This introduces a dimensional integration constant which can serve as the scale \( \Lambda_0 \)). This is in fact the familiar textbook argument were one assumes the existence of a string tension so as to get \( \text{(6.23)} \), in which the standard weak coupling perturbative expression for the beta function is then used.

For us, however, the existence of a non-zero string tension is the outcome of the process of successive decimations to coarser scales as developed above. This process embodies all relevant information in the theory. In particular, it also supplies the specification of the function \( g(a) \).

One can indeed construct the function \( g(a) \) directly as follows:

(i) Starting with some initial value of \( \beta = 4/g^2 \) perform successive decimations following the
flow into the strong coupling regime with resulting string tension \( \kappa(n, \alpha^{(n)}_\Lambda) \), eq. (6.14), at some \( n = n_0 \). Let \( k \) denote the value of this string tension. The corresponding value of the lattice spacing \( a_0 \) is given by (5.20), and \( g = g(a_0) \).

(ii) Fix the string tension as in (6.22). This is then satisfied at \( n_0, g(a_0) \).

(iii) Vary \( g \) away from \( g(a_0) \) to determine \( g \) such that, under successive decimations following the flow into the strong coupling regime, the resulting string tension satisfies (6.22) for \( n = n_0 + 1 \).

(iv) Repeat (iii) for \( n = n_0 + 2, n_0 + 3, \ldots, n_0 - 1, \ldots \).

This provides the functional relation \( g(a) \). In particular, for \( b = 2 \), it gives the sequence of values \( g(a_0/2^l), l = 1, 2, \ldots \), starting from some value \( g(a_0) \).\(^{12}\)

Note that, according to (i) above, the number of decimations \( n_0 \) at which one chooses to apply V.1 to obtain (6.5), (6.14) amounts to fixing the string tension. This is the only physical parameter in the theory. A specification of \( \Lambda_0 \) is a specification of the value \( g(a_0) \) at spacing \( a_0 \), which is a convention of no physical import.

One then has in principle a constructive method for obtaining \( g(a) \) by a sequence of simple algebraic operations. This is the coupling \( g(a) \) as defined in the physical non-perturbative renormalization scheme specified by keeping the string tension fixed.

A straightforward illustration of the method is provided by setting all \( \alpha^{(n)}_{\Lambda} = 1 \), i.e. apply it to the flow according to the upper bound coefficients \( c_{1/2}^U \) in (3.33). This yields \( g(a) \) as given by MK decimations. We cannot apply it explicitly to the case of interest, i.e. the flow following the middle column coefficients in (3.38), since we do not determine them explicitly in this paper. The qualitative features at strong and weak coupling, however, are readily discernible.

At strong coupling, i.e. small initial \( \beta \), the number of decimations needed to reach a given string tension is of order unity, i.e. the lattice spacing \( a \) is large: \( a = O(\Lambda_0^{-1}) \), and one is very far from any continuum limit. Successive decimations, by construction, reproduce the behavior seen within the strong coupling expansion, and the familiar strong coupling variation given by \( \beta(g) \sim g \ln g \) is the result, as can be checked by a short computation.

The opposite limit of large initial \( \beta \) corresponds to large number of decimations, hence \( a \ll \Lambda_0^{-1} \). Indeed, recall that \( g = 0 \) is a fixed point of the decimations. Hence, for \( \beta \to \infty \), one necessarily has \( n \to \infty \) in order for, say, the leading upper bound coefficient \( c_{1/2}^U(n) \) to reach any prescribed value \( < 1 \). Thus, \( a\Lambda_0 \to 0 \). Note that this limit is well-defined by construction since everything is bounded and continuous under successive decimations. Asymptotic freedom, i.e. the statement that \( g(a) \to 0 \) as \( a \to 0 \), is then a direct qualitative consequence of the flow produced by the decimations.

It is instructive to examine the actual manner in which \( g(a) \to 0 \) under the upper bound decimations, i.e. the \( c_{1/2}^U(m) \)'s in (3.38). Comparing two \( g \) values that differ by one decimation step \( (b = 2) \), one finds

\[
\frac{1}{g^2(a)} = \frac{1}{g^2(2a)} + 2b_0 \ln 2 + O(g^2)
\]

(6.24)

for sufficiently small \( g(a) \). The constant \( b_0 = (1 - 1/b^2)/(24 \ln b) \) underestimates the value \( \frac{11}{24} \pi^2 \) obtained in a continuum perturbative calculation by only about 3%.

\(^{12}\)This is the analog in the present context of the ‘staircase’ procedure in [18].
The actual flow (middle column in (3.38)) is faster, corresponding to somewhat larger $b_0$. According to RG lore, a beta-function defined by other means, such as fixing some renormalized coupling within weak coupling perturbation theory, should coincide, in its universal first two terms, with that defined by the above physical non-perturbative scheme. This, however, is outside the scope of, and not of direct relevance for the main argument in this paper.

To reiterate, the above procedure completely specifies the dependence $g(a)$ in the physical renormalization scheme defined by keeping the string tension fixed, and this dependence is necessarily such that $g(a) \to 0$ as $a \to 0$.

7 Concluding remarks

In summary, we obtained a representation of the vortex free energy, originally defined on a lattice of spacing $a$, in terms of partition functions on a lattice of spacing $ab^n$. The effective action in this representation is bounded by the corresponding effective action resulting from potential moving decimations (MK decimations) from spacing $a$ to spacing $ab^n$. The latter are explicitly computable. Confining behavior is the result, starting from any initial coupling $g$ on spacing $a$, by taking the number of decimation $n$ large enough.

It is worth remarking again that in an approach based on RG decimations the fact that the only parameter in the theory is a physical scale emerges in a natural way. Picking a number of decimations can be related to fixing the string tension. That this can be done only after flowing into the strong coupling regime reflects the fact that this dynamically generated scale is an ‘IR effect’. The coupling $g(a)$ is completely determined in its dependence on $a$ once the string tension is fixed. In particular, $g(a) \to 0$ as $a \to 0$. Note that this implies that there is no physically meaningful or unambiguous way of non-perturbatively viewing the short distance regime independently of the long distance regime. Computation of all physical observable quantities in the theory must then give a multiple of the string tension or a pure number. In the absence of other interactions, this scale provides the unit of length; there are in fact no free parameters.

There is a variety of other results related to the approach in this paper that could not be included here. We note, in particular, that the same procedure can be immediately transcribed to the Heisenberg $SU(2)$ spin model. Also, apart from analytical results, the considerations in this paper may be combined with Monte Carlo RG techniques to constrain the numerical construction of improved actions at different scales, a subject of perennial interest to the practicing lattice gauge theorist. We hope to report on these matters elsewhere.

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A Appendix

In this appendix we obtain the lower and upper bounds II.1, IV.2, and III.1, III.2, IV.3, IV.4, and also IV.1.

\[13\] This is part of the meaning of the common saying “QCD is the perfect theory”.
§1. To prove II.1 take the lattice $\Lambda^{(n)}$ to have length $L^{(n)}_\mu = 2^{m_\mu}$, with integer $m_\mu$ in each direction $\mu = 1, \ldots, d$, and torus topology (periodic boundary conditions) in all directions. Choose a hyperplane $\pi$ without sites perpendicular to, say, the $x^1$-direction and bisecting $\Lambda^{(n)}$, so that $\Lambda^{(n)} = \Lambda^{(n)}_L \cup \Lambda^{(n)}_R$ with $\Lambda^{(n)}_R = R[\Lambda^{(n)}_L]$, where $R$ denotes reflection in $\pi_1$. Dropping now the terms in (2.9) coming from all the plaquettes bisected by $\pi_1$, all non-negative by reflection positivity, gives

$$Z_{\Lambda^{(n)}}(\{c_j(n)\}) = \int d\mu^0_{\Lambda^{(n)}} \geq \left( \int d\mu^0_{\Lambda^{(n)}} \right)^2.$$  \hspace{1cm} (A.1)

Note that the ‘half-lattice’ $\Lambda^{(n)}_L$ has a boundary with resulting free boundary conditions for $d\mu^0_{\Lambda^{(n)}}$ in the $x^1$-direction. ($d\mu^0_{\Lambda^{(n)}}$ still has periodic boundary conditions in all the other directions.) Next take a plane $\pi_1'$ bisecting $\Lambda^{(n)}_L$ so that $\Lambda^{(n)}_L = \Lambda^{(n)}_L' \cup \Lambda^{(n)}_R'$ with $\Lambda^{(n)}_R' = R'[\Lambda^{(n)}_L']$, where $R'$ denotes reflection in $\pi_1'$. Removing now all plaquettes bisected by $\pi'$ on the r.h.s. of (A.1) gives

$$\int d\mu^0_{\Lambda^{(n)}} \geq \left( \int d\mu^0_{\Lambda^{(n)}_L} \right)^4.$$  \hspace{1cm} (A.2)

Proceeding in this manner one arrives at

$$\int d\mu^0_{\Lambda^{(n)}} \geq \left( \int d\mu^0_{\Lambda^{(n)}_L(1)} \right)^{|L^{(n)}_1|}.$$  \hspace{1cm} (A.3)

In (A.3) $\Lambda^{(n)}_L(1)$ denotes the lattice resulting from $\Lambda^{(n)}$ by reducing its extent to one lattice spacing in the $x^1$-direction, and $d\mu^0_{\Lambda^{(n)}_L(1)}$ is computed with free boundary conditions in the $x^1$-direction. One next chooses a hyperplane normal to one of the remaining directions bisecting $\Lambda^{(n)}_L(1)$. Iterating this procedure in each successive direction, one eventually arrives at

$$\int d\mu^0_{\Lambda^{(n)}} \geq \left( \int d\mu^0_h \right)^{|\Lambda^{(n)}|},$$  \hspace{1cm} (A.4)

where $h$ denotes a hypercube, i.e. a subset of $L^{(n)}$ with $2^d$ sites. But, as it is easily seen,

$$\int d\mu^0_h \geq \left[ 1 + \sum_{j \neq 0} d^2 c^6 \right].$$  \hspace{1cm} (A.5)

Inserting (A.5) in (A.4) completes the proof of II.1. Note that, since the number of bonds is larger than the number of plaquettes, $|\Lambda^{(n)}|$ in (A.4) may be replaced by the latter.

§2. To obtain IV.2 let

$$P_\Psi^+ = \frac{1}{2} \left[ 1 + \prod_{p \in \Psi} \frac{f(-)(U_p, n)}{f(U_p, n)} \right] = \frac{1}{2} \left[ 1 + \prod_{p \in \Psi} \exp \left[ A^-(U_p, n) - A(U_p, n) \right] \right].$$  \hspace{1cm} (A.6)

\footnote{Actually a pair of hyperplanes because of the toroidal lattice topology. Following common practice, when a pair is actually meant, it will not be explicitly pointed out for brevity.}
Then
\[ Z_{\Lambda(n)}^+(\{c_j(n)\}) = \int d\mu_{\Lambda(n)}^0 P_\mathcal{V}_+^+ \]  \hspace{1cm} (A.7)
\[ = \int d\mu_{\Lambda(n)}^0 P_\mathcal{V}_+^+ P_{\mathcal{V}}^+, \]  \hspace{1cm} (A.8)
where in the second equality \( \mathcal{V}' \) is any other coclosed plaquette set homologous to \( \mathcal{V} \) (cf. remarks preceding eq. (4.2)). As in the text, we take \( \mathcal{V} \) to wind around the lattice in the directions perpendicular to \( x^1 \)- and \( x^2 \)-directions. The measures defined by \( Z_{\Lambda(n)} \) and \( Z_{\Lambda(n)}^{(-)} \), and hence \( Z_{\Lambda(n)}^+ \), are clearly reflection positive in hyperplanes perpendicular to any one of the direction \( \mu \neq 1,2 \). (A.8) makes it clear that the measure in \( Z_{\Lambda(n)}^+ \) is also reflection positive in planes normal to \( \mu = 1 \) or \( \mu = 2 \): simply take \( \mathcal{V}' = \mathcal{R}[\mathcal{V}] \) where \( \mathcal{R} \) denotes reflection in such a plane. Thus, the measure in \( Z_{\Lambda(n)}^+ \) possesses RP in all planes. IV.2 (i) is a trivial consequence of this fact.

To establish IV.2 (ii) let a hyperplane \( \pi \) normal to the \( x^1 \)-axis bisect the lattice \( \Lambda(n) \) so that \( \Lambda(n) = \Lambda_L(n) \cup \Lambda_R(n) \). Take \( \mathcal{V} \subset \Lambda_L(n) \), and \( \mathcal{V}' = \mathcal{R}[\mathcal{V}] \subset \Lambda_R(n) \). Dropping the terms in (A.8) coming from all plaquettes bisected by \( \pi_1 \), all of which are non-negative by RP, gives then
\[ Z_{\Lambda(n)}^+(\{c_j(n)\}) \geq \left( \int d\mu_{\Lambda(n)}^0 P_\mathcal{V}_+^+ \right)^2. \]  \hspace{1cm} (A.9)

But \( \Lambda_L(n) \) has a boundary with resulting free boundary conditions for \( d\mu_{\Lambda_L(n)}^0 \) in the \( x^1 \)-direction. This implies that
\[ \int d\mu_{\Lambda_L(n)}^0 P_\mathcal{V}_+^+ = \int d\mu_{\Lambda_L(n)}^0, \]  \hspace{1cm} (A.10)
since one may, by a shift of integration variables, move the location of the twist-carrying set \( \mathcal{V} \) to this boundary and, by virtue of the free boundary conditions there, remove it from \( \Lambda_L(n) \). The rest of the argument then proceeds exactly as in §1 above. One thus arrives at (A.10).

§3. The lower bounds III.2 and IV.4 are simple corollaries of II.1 and IV.2. Let \( |\Lambda(n)| \) denote the number of plaquettes in lattice \( \Lambda(n) \). One has
\[ \left[ 1 + \sum_{j \neq 0} d^2_j c_j(n-1)^6 \right] |\Lambda^{(n-1)}| \geq \left[ 1 + \sum_{j \neq 0} d^2_j c_j(n-1)^6 \right] |\Lambda^{(n)}| \]  \hspace{1cm} (A.11)
since \( |\Lambda^{(n-1)}| > |\Lambda^{(n)}| \). So, by II.1(ii)
\[ Z_{\Lambda(n-1)}(\{c_j(n-1)\}) \geq \left[ 1 + \sum_{j \neq 0} d^2_j c_j(n-1)^6 \right] |\Lambda^{(n)}| \]
\[ \geq \int dU_{\Lambda^{(n)}} \prod_{p \in \Lambda^{(n)}} \left[ 1 + \sum_{j \neq 0} d_j c^L_j(n) \chi_j(U_p) \right] \]
\[ = Z_{\Lambda(n)}(\{c^L_j(n)\}), \]  \hspace{1cm} (A.12)
which gives III.2.
Similarly, from (A.11), IV.2(ii) and using (4.3)

\[
Z_{\Lambda(n-1)}^{+}(\{c_j(n-1)\}) \geq \frac{1}{2} \left[ \int dU_{\Lambda(n)} \prod_{p \in \Lambda(n)} \left[ 1 + \sum_{j \neq 0} d_j c_j^L(n) \chi_j(U_p) \right] \right. \\
+ \left. \int dU_{\Lambda(n)} \prod_{p \in \Lambda(n)} \left[ 1 + \sum_{j \neq 0} (-1)^{2j} d_j c_j^L(n) \chi_j(U_p) \right] \right] \\
= Z_{\Lambda(n)}^{+}(\{c_j^L(n)\}),
\]

which gives IV.4. Note that, as it clear from (A.11), both (A.12) and (A.13) are strict inequalities except in the trivial case where all \(c_j(n-1)\) vanish.

\section{IV. To obtain III.1 consider the decimation operation on the partition function \(Z_{\Lambda(m)}(\{c_j(m)\})\) of the form (2.9). Let \(Q_{\mu\nu}\) denote the set of all \([\mu\nu]-\)plaquettes in \(\Lambda(m)\). Consider the set of all 3-cells in \(\Lambda(m)\) of side length \(b\) (in units of lattice spacing) in the \(\kappa\)-direction and unit side length in the \(\mu\)- and \(\nu\)-directions. The basic moving operation consists of moving the \((b-1)\) interior \([\mu\nu]-\)plaquettes of each cell along the positive \(\kappa\)-direction to the location of the \([\mu\nu]-\)plaquette in the boundary of the cell (Figure 1). Then \(Q_{\mu\nu} = Q_{\mu\nu}^- \cup Q_{\mu\nu}^+\), where \(Q_{\mu\nu}^-\) is the set of all moved \([\mu\nu]-\)plaquettes and \(Q_{\mu\nu}^+\) the set of all ‘receiving’ \([\mu\nu]-\)plaquettes on the 3-cell boundaries. The action of the receiving boundary plaquettes is renormalized as in (2.16). Given \(Z_{\Lambda(m)}(\{c_j(m)\})\) with corresponding action \(A_p(U(m),\xi)\), eq. (2.24), define the action

\[
A_{\Lambda(m)}(U, m, \xi) = \sum_{p \in \Lambda(m)} A_p(U_p, m) + \sum_{p \in Q_{\mu\nu}^+} \xi (\zeta_0 - 1) A_p(U_p, m) - \sum_{p \in Q_{\mu\nu}^-} \xi A_p(U_p, m),
\]

interpolating between the action before (\(\xi = 0\)) and after (\(\xi = 1\)) the move. Then

\[
\frac{d}{d\xi} Z_{\Lambda(m)}(m, \xi)|_{\xi=0} = \frac{d}{d\xi} \left[ \int dU_{\Lambda(m)} \exp A_{\Lambda(m)}(U, m, \xi) \right] |_{\xi=0} = (\zeta_0 - 1) \sum_{p \in Q_{\mu\nu}^+} \left< A_p(U_p, m) \right>_{0, \Lambda(m), \xi=0} - \sum_{p \in Q_{\mu\nu}^-} \left< A_p(U_p, m) \right>_{0, \Lambda(m), \xi=0},
\]

where \(<-\) denotes the unweighted expectation with measure defined by (A.14).

Now, given a \(p \in Q_{\mu\nu}^\pm\), one notes that \(\left< A_p(U_p, m) \right>_{0, \Lambda(m), \xi=0}\) in (A.15) does not vary with \(p\) along the \(\kappa\)-direction i.e. along the direction of the move. Hence, if

\[
\zeta_0 = b
\]

(A.16) gives

\[
\frac{d}{d\xi} Z_{\Lambda}(m, \xi)|_{\xi=0} = 0,
\]

(A.17)

since there are \((b-1)\) plaquettes in \(Q_{\mu\nu}^-\) for each plaquette in \(Q_{\mu\nu}^+\). But

\[
\frac{d^2}{d\xi^2} Z_{\Lambda(m)}(m, \xi) = \left< \left( \sum_{p \in Q_{\mu\nu}^+} (\zeta_0 - 1) A_p(U_p, m) - \sum_{p \in Q_{\mu\nu}^-} A_p(U_p, m) \right)^2 \right>_{0, \Lambda(m), \xi} \geq 0.
\]

(A.18)
(A.17)-(A.18) then imply that, with condition (A.16), $Z_{A(m)}(m, \xi)$, and $\ln Z_{A(m)}(m, \xi)$, are increasing convex functions of $\xi$ on $0 \leq \xi \leq 1$. Thus, $Z_{A(m)}(m, 0) \leq Z_{A(m)}(m, 1)$.

A complete decimation $\Lambda^{(m)} \to \Lambda^{(m+1)}$ is performed by repeating the above basic moving-renormalization step in each of the available normal directions for each possible plaquette orientation as described in section 2.1. One need only observe that, in carrying out each such successive step, translation invariance at $\xi = 0$ for a given plaquette along the direction of the move holds regardless of any previous moves performed along other directions as schematically depicted in Figure 2. Each step then with $\zeta_0 = b$ results in a further upper bound on $Z_{A(m)}$.

Figure 2: Translational invariance along undecimated direction $x^\kappa$ at $\xi = 0$ (see text) after decimation along $x^\mu$.

The completion of all the moving-renormalization steps results into (3.4) at $r = 1$, i.e.

$$Z_{A(m)}(\{c_j(m)\}) \leq F^U_0(m + 1)^{|A^{(m+1)}|} Z_{A(m+1)}(\{c^U_j(m + 1, 1)\}) \, .$$

(A.19)

with $F^U_0(m + 1), c^U_j(m + 1, 1)$ given by (2.19) - (2.22) at $\zeta = c_5^{(d-2)} = b^{(d-2)}$. Furthermore, it follows from II.1(i), (3.3) and the fact that $0 \leq c^U_j(m + 1, 1) \leq 1$ that replacing $c^U_j(m + 1, 1)$ by $c^U_j(m + 1, r)$ in the r.h.s. in (A.19) gives a decreasing function in $r$ on $0 < r \leq 1$. This completes the proof of III.1.

The proof of III.1 can be used essentially unaltered to obtain IV.3. Introducing the analog of (A.14) for the action (4.4) in the presence of flux:

$$A_{A(m)}^{-}(U, m, \xi) = \sum_{p \in A^{(m)}} A_p^{-}(U_p, m) + \sum_{p \in Q^{+}_{[\mu \nu]}} \xi (\zeta_0 - 1) A_p^{-}(U_p, m) - \sum_{p \in Q^{-}_{[\mu \nu]}} \xi A_p^{-}(U_p, m) \, .$$

(A.20)

Indeed, the second derivative of $\ln Z_{A(m)}(m, \xi)$ w.r.t. $\xi$ is also positive by the usual convexity of the free energy, a statement that follows generally by an application of Hölder’s inequality.
one has

\[
\frac{d}{d\xi} Z_{\Lambda(m)}^{(-)}(m, \xi)|_{\xi=0} = (\xi_0 - 1) \sum_{p \in Q_{[\mu\nu]}^+} \left\langle A_p^{(-)}(U_p, m) \right\rangle_{0, \Lambda(m), \xi=0}^{(-)} - \sum_{p \in Q_{[\mu\nu]}^-} \left\langle A_p^{(-)}(U_p, m) \right\rangle_{0, \Lambda(m), \xi=0}^{(-)}, \tag{A.21}
\]

where \( < - >_{0, \Lambda(m), \xi} \) denotes the unweighted expectation with action \( \Lambda \). The observation that, having moved plaquettes in certain directions, there is still translational invariance at \( \xi = 0 \) for plaquettes to be moved in the remaining undecimated directions (Figure 2) holds also in the presence of the flux. This is a consequence of the basic property (cf. section 4) that \( Z_{\Lambda(m)}^{(-)} \) does not depend on the location but only the homology class of \( \mathcal{V} \). One may, for example, always bring the set \( \mathcal{V} \) by a change of variables to occupy the exact same location with respect to the plaquette \( p \) in each expectation \( \left\langle A_p^{(-)}(U_p, m) \right\rangle_{0, \Lambda(m), \xi=0}^{(-)} \) in \( \Lambda \). Thus it is again the case that each such expectation in \( \Lambda \) does not vary with \( p \in Q_{[\mu\nu]}^+ \) along the undecimated directions normal to \( p \). Hence, when \( \Lambda \) holds,

\[
\frac{d}{d\xi} Z_{\Lambda(m)}^{(-)}(m, \xi)|_{\xi=0} = 0 , \quad \text{whereas} \quad \frac{d^2}{d\xi^2} Z_{\Lambda(m)}^{(-)}(m, \xi) \geq 0 ,
\]

which gives the analog of \( \Lambda \) in the presence of flux:

\[
Z_{\Lambda(m)}^{(-)}\left(\{c_j(m)\}\right) \leq F_0^U (m + 1)^{\Lambda(m+1)} | Z_{\Lambda(m+1)}^{(-)}\left(\{c_j^U(m + 1, 1)\}\right) . \tag{A.22}
\]

Combining \( \Lambda \) with \( \Lambda \) and IV.2(i), (3.3) then gives IV.3.

§5. To establish IV.1, take \( \mathcal{V} \) such that all \( p \in \mathcal{V} \) are bisected by a hyperplane \( \pi \) perpendicular to, say, \( x^1 \). Let

\[
S_p^{1/2} = \sum_{j=\text{half-int.}} d_j c_j(n) \chi_j(U_p) , \quad S_p^1 = 1 + \sum_{j=\text{int.} \atop j \neq 0} d_j c_j(n) \chi_j(U_p) \tag{A.23}
\]
denote the sums over half-integer and integer representations, respectively. Now

\[
\prod_{p \in \mathcal{V}} \left[ S_p^{1/2} + S_p^1 \right] - \prod_{p \in \mathcal{V}} \left[ -S_p^{1/2} + S_p^1 \right] = 2 \sum_{Q \subset \mathcal{V}} \prod_{p \in Q \atop |Q| = \text{odd}} S_p^{1/2} \prod_{p \in \mathcal{V} \setminus Q} S_p^1 , \tag{A.24}
\]

where the sum is over all subsets of plaquettes \( Q \) in \( \mathcal{V} \) with odd number of plaquettes \( |Q| \geq 1 \). Inserting \( \Lambda \) in \( \Lambda - Z_{\Lambda(m)}^{(-)} \) one has

\[
Z_{\Lambda(n)} - Z_{\Lambda(n)}^{(-)} = 2 \sum_{Q \subset \mathcal{V} \atop |Q| = \text{odd}} \int dU_{\Lambda(n)} \prod_{p \in \Lambda(n) \setminus \mathcal{V}} \left[ 1 + \sum_{j \neq 0} d_j c_j^U(n) \chi_j(U_p) \right] \prod_{p \in Q} S_p^{1/2} \prod_{p \in \mathcal{V} \setminus Q} S_p^1 . \tag{A.25}
\]

Since \( c_j(n) \geq 0 \), all \( j \), every term in the sum in \( \Lambda \) is manifestly non-negative by RP in \( \pi \), which proves IV.1.
B Appendix

In this Appendix we give some simple estimates concerning the variation of the level surfaces \( \alpha_{\Lambda,h}(t,r) \) and \( \alpha_{\Lambda,h}^+(t,r) \) w.r.t. the parameters \( t,r \).

§1. The derivative of \( \alpha_{\Lambda,h}(t,r) \) w.r.t. \( r \) is given by

\[
\frac{\partial \alpha_{\Lambda,h}(t,r)}{\partial r} = -\left[ \frac{B_{\Lambda}(m)}{\partial \alpha} + A_{\Lambda}(m) \right] \alpha_{\Lambda,h}(t,r),
\]

where

\[
B_{\Lambda}(m) = \frac{1}{\ln F_0^U(m)} \frac{\partial}{\partial r} \ln Z_{\Lambda}(m) \left( \{ \tilde{c}_j(m,\alpha,r) \} \right) < 0 .
\]

The derivative of \( \alpha_{\Lambda,h}^+(t,r) \) w.r.t. \( r \) is similarly given by

\[
\frac{\partial \alpha_{\Lambda,h}^+(t,r)}{\partial r} = -\left[ \frac{B_{\Lambda}^+(m)}{\partial \alpha} + A_{\Lambda}^+(m) \right] \alpha_{\Lambda,h}^+(t,r),
\]

with

\[
B_{\Lambda}^+(m) = \frac{1}{\ln F_0^U(m)} \frac{\partial}{\partial r} \ln Z_{\Lambda}^+(m) \left( \{ \tilde{c}_j(m,\alpha,r) \} \right) < 0 .
\]

§2. In this Appendix we use the short-hand notation \( Z_{\Lambda}(m) = Z_{\Lambda}(m) \left( \{ \tilde{c}_j(m,\alpha,r) \} \right) \) and \( Z_{\Lambda}^+(m) = Z_{\Lambda}^+(m) \left( \{ \tilde{c}_j(m,\alpha,r) \} \right) \); and also, given any set of plaquettes \( P \subset \Lambda(m) \), we define:

\[
d\tilde{\mu}_P = \prod_{b \in \Lambda(m)} dU_b \prod_{p \in P} f_p(U_p,m,\alpha,r).
\]

By translational invariance

\[
A_{\Lambda}(m) = \frac{1}{\ln F_0^U(m)} \sum_{j \neq 0} d_j \frac{\partial \tilde{c}_j(m,\alpha,r)}{\partial \alpha} \frac{1}{Z_{\Lambda}(m)} \int d\tilde{\mu}_{\Lambda(m)\setminus p} \chi_j(U_p).
\]

Since

\[
Z_{\Lambda}(m) \geq \int d\tilde{\mu}_{\Lambda(m)\setminus p} \chi_j(U_p) \geq 0
\]

by RP in a plane bisecting \( p \), one has

\[
A_{\Lambda}(m) \leq \frac{1}{\ln F_0^U(m)} \sum_{j \neq 0} d_j^2 \frac{\partial \tilde{c}_j(m,\alpha,r,r')}{\partial \alpha} = \frac{1}{\ln F_0^U(m)} \| \frac{\partial g_p(m,\alpha,r)}{\partial \alpha} \|.\]
In the same manner one obtains

\[ |B_{\Lambda^{(m)}}(\alpha, r)| \leq \frac{1}{\ln F_0^U(m)} \sum_{j \neq 0} d_j^2 \left| \frac{\partial \tilde{c}_j(m, \alpha, r)}{\partial r} \right| \]

\[ = \frac{1}{\ln F_0^U(m)} \left| \frac{\partial g_p(m, \alpha, r)}{\partial r} \right|. \]  \hspace{1cm} (B.8)

By translational invariance for plaquettes of the same orientation in the presence of flux, the same bounds can similarly be shown to hold for the quantities \(A_{\Lambda^{(m)}}^+(\alpha, r)\) and \(B_{\Lambda^{(m)}}^+(\alpha, r)\):

\[ A_{\Lambda^{(m)}}^+(\alpha, r) \leq \frac{1}{\ln F_0^U(m)} \left| \frac{\partial g_p(m, \alpha, r)}{\partial \alpha} \right| \]  \hspace{1cm} (B.9)

and

\[ |B_{\Lambda^{(m)}}^+(\alpha, r)| \leq \frac{1}{\ln F_0^U(m)} \left| \frac{\partial g_p(m, \alpha, r)}{\partial r} \right|. \]  \hspace{1cm} (B.10)

Note that all these upper bounds are independent of the lattice size \(|\Lambda^{(m)}|\).

Let \(p\) be a fixed plaquette, and \(c\) a 3-cube having \(p\) in its boundary \(\partial c\) and protruding from \(p\) in one of the \(d - 2\) directions normal to \(p\). Let \(Q\) denote the set of plaquettes sharing a bond with \(\partial c\) but not belonging to \(\partial c\). Then

\[ \frac{1}{Z_{\Lambda^{(m)}}} \int d\tilde{\mu}_0^{\Lambda^{(m)}(\alpha)} \chi_j(U_p) = \frac{1}{Z_{\Lambda^{(m)}}} \int d\tilde{\mu}_0^{\Lambda^{(m)}(\alpha)} \chi_j(U_p) \prod_{p' \in Q \cup \partial c \setminus p} f_p(U_{p'}, m, \alpha, r) \]

\[ \geq \frac{1}{Z_{\Lambda^{(m)}}} \int d\tilde{\mu}_0^{\Lambda^{(m)}(\alpha)} \chi_j(U_p) \prod_{p' \in \partial c \setminus p} f_p(U_{p'}, m, \alpha, r) \]

\[ = \frac{1}{Z_{\Lambda^{(m)}}} \int d\tilde{\mu}_0^{\Lambda^{(m)}(\alpha)} \chi_j(U_p) \prod_{p' \in \partial c \setminus p} f_p(U_{p'}, m, \alpha, r) \]

\[ \geq \frac{1}{|f_p(m, \alpha, r)||Q|} \left| 1 + \sum_{j \neq 0} d_j^2 \tilde{c}_j(m, \alpha, r)^5 \right|. \]  \hspace{1cm} (B.11)

RP in each of the two planes bisecting the plaquette \(p\) shows that the terms involving plaquettes in the set \(Q\) in the first line are all positive which results in the inequality in the second line. Using (B.11) in (B.6) gives the lower bound

\[ A_{\Lambda^{(m)}}(\alpha, r) \geq \frac{1}{|f_p||Q| \ln F_0^U(m)} \sum_{j \neq 0} \frac{\partial \tilde{c}_j(m, \alpha, r)}{\partial \alpha} \left| 1 + \sum_{i \neq 0} d_i^2 \tilde{c}_i(m, \alpha, r)^6 \right| \]

\[ = \frac{1}{|f_p||Q| \ln F_0^U(m)} \frac{\partial}{\partial \alpha} \left| 1 + \sum_{j \neq 0} d_j^2 \tilde{c}_j(m, \alpha, r)^6 \right|. \]  \hspace{1cm} (B.12)

Similarly, one obtains a lower bound on \(|B_{\Lambda^{(m)}}(\alpha, r)|\)

\[ |B_{\Lambda^{(m)}}(\alpha, r)| \geq \frac{1}{|f_p||Q| \ln F_0^U(m)} \sum_{j \neq 0} \left| \frac{\partial \tilde{c}_j(m, \alpha, r)}{\partial r} \right| \left| 1 + \sum_{i \neq 0} d_i^2 \tilde{c}_i(m, \alpha, r)^6 \right| \]

\[ = \frac{1}{|f_p||Q| \ln F_0^U(m)} \frac{\partial}{\partial r} \left| 1 + \sum_{j \neq 0} d_j^2 \tilde{c}_j(m, \alpha, r)^6 \right|. \]  \hspace{1cm} (B.13)
Again, these lower bounds are manifestly lattice-size independent.

Since, as it is easily seen (apply (A.8)), (B.11) holds also when \(d\mu_\Lambda^{0+}(m)\) is replaced by \(d\mu_\Lambda(0)\), the r.h.s. of (B.12) and (B.13) also give lower bounds for \(A^+\) and \(|B^+|\), respectively.

§3. The lower bound in (3.30), i.e.

\[
\delta' < \alpha^{(m)}(t, r)
\]  

is a consequence of II.1 and (3.24). From the first inequality in (A.12) (excluding the trivial case of all \(c_j\) vanishing) one has

\[
Z_\Lambda^{(m-1)} > \left[ 1 + \sum_{j \neq 0} d_j^2 c_j(m-1)^6 \right] |\Lambda^{(m)}| ;
\]  

(B.15)

whereas from (3.24), with the short-hand notation \(\alpha^{(m)}(t, r) = \alpha^{(m)}(t, r)\),

\[
Z_\Lambda^{(m-1)} \leq F_0^U(m)^{h_\Lambda(t, r)} \left[ 1 + \sum_{j \neq 0} d_j^2 \tilde{c}_j(m, \alpha^{(m)}(t, r), r) \right] |\Lambda^{(m)}| .
\]  

(B.16)

Combining (B.15) and (B.16) gives

\[
\left[ 1 + \sum_{j \neq 0} d_j^2 c_j(m-1)^6 \right] < F_0^U(m)^{h_\Lambda(t, r)} \left[ 1 + \sum_{j \neq 0} d_j^2 \tilde{c}_j(m, \alpha^{(m)}(t, r), r) \right] ,
\]  

(B.17)

which, together with (3.17) and (3.8) or (3.9), shows that it cannot be that \(\alpha^{(m)}(t, r) \rightarrow 0\) in any fashion with increasing lattice size \(|\Lambda^{(m)}|\).

An explicit lower bound on \(\alpha^{(m)}(t, r)\) is easily obtained from (B.17) by taking (3.12) with, for example, (3.9) and \(h(t) = \alpha^{(s)}(t), s(t) \geq 1\). Using the elementary inequality

\[
x^q - 1 \leq q(x - 1) , \quad x \geq 0, \quad 0 \leq q \leq 1 ,
\]  

(B.18)

(B.17) gives

\[
1 \geq \alpha^{(m)}(t, r) > \frac{\sum_{j \neq 0} d_j^2 c_j(m-1)^6}{\left( F_0^U(m) - 1 \right) + F_0^U(m) \sum_{j \neq 0} d_j^2 c_j(m, r)} \equiv \delta' > 0 .
\]  

(B.19)

Note that, from (D.5) below, \(\left( F_0^U(m) - 1 \right) > \sum_{j \neq 0} d_j^2 c_j(m-1)^2 > \sum_{j \neq 0} d_j^2 c_j(m-1)^2\).

Similar expressions can be obtained for other choices of \(h(t)\).

The lower bound in (4.24) is similarly seen to hold by combining IV.2 and (4.19).

To satisfy the upper bound requirement in (3.30), i.e.

\[
\alpha^{(m)}(t, r) < 1 - \delta
\]  

(B.20)

it suffices to let the decimation parameter \(r\) vary, if necessary, away from unity in the domain (3.31). To see this, suppose that, performing the \(m\)-th decimation step, one finds that

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\( \alpha^{(m)}_{\Lambda, h}(t_m, 1) = 1 - \delta^{(m)}_{\Lambda} \), where, say, \( \delta^{(m)}_{\Lambda} \leq O(1/|\Lambda^{(m)}|) \). Using the bounds (B.7), (B.13) on \( A^{(m)}_\Lambda, B^{(m)}_\Lambda \), and the boundedness of the derivatives of \( h(\alpha, t) \), in eq. (B.1) then gives

\[
\partial \alpha^{(m)}_{\Lambda, h}(t, r)/\partial r \geq \theta > 0 \tag{B.21}
\]

for some constant \( \theta \) independent of \( |\Lambda^{(m)}| \) for all \( 0 < r \leq 1 \). Hence, for \( t \geq t_m \), and some \( \xi \) between 0 and 1,

\[
\alpha^{(m)}_{\Lambda, h}(t_m, 1 - \epsilon) \leq \alpha^{(m)}_{\Lambda, h}(t, 1 - \epsilon) \leq \alpha^{(m)}_{\Lambda, h}(t, 1) - \epsilon \left( \frac{\partial \alpha^{(m)}_{\Lambda, h}(t, r)}{\partial r} \right)_{r=1-\xi \epsilon} \leq \alpha^{(m)}_{\Lambda, h}(t, 1) - \epsilon \theta < 1 - \delta(\epsilon) \tag{B.22}
\]

with \( \delta(\epsilon) = \epsilon \theta/2 \), and (B.20) is satisfied.

Given (3.30), the bounds (B.7), (B.12) and the properties of the interpolation \( h \) (cf. (3.17), (3.18)), it follows from (3.25) and (3.29) that

\[
\frac{\partial \alpha^{(m)}_{\Lambda, h}(t, r)}{\partial t} \geq \eta_1(\delta) > 0, \quad -\frac{\partial h_1(\alpha^{(m)}_{\Lambda, h}(t, r), t)}{\partial t} \geq \eta_2(\delta) > 0, \tag{B.23}
\]

where \( \eta_1, \eta_2 \) are lattice-size independent, and \( r \) in the domain (3.31).

Since the same upper and lower bounds apply to \( A^{+}_{\Lambda^{(m)}}, |B^{+}_{\Lambda^{(m)}}| \), the same considerations again show that eqs. (4.24) and (4.25) are always ensured to hold by letting \( r \) vary, if necessary, in (3.31).

C Appendix

§1. Under the conditions in V.1, given \( \alpha^{(n)}_{\Lambda, h}(t) \) satisfying (3.24), one seeks a solution \( t = t^{(n)}_{\Lambda, h} \) to (5.20), i.e.

\[
\tilde{Z}^{+}_{\Lambda^{(n)}}(\beta, h, \alpha^{(n)}_{\Lambda, h}(t), t) = Z^{+}_{\Lambda^{(n-1)}}. \tag{C.1}
\]

First note that, if a solution exists, it is unique since, as it is easily checked, \( \tilde{Z}^{+}_{\Lambda^{(n)}}(\beta, h, \alpha^{(n)}_{\Lambda, h}(t), t) \) is monotonic in \( t \).

To show that a solution exists, we proceed as follows. By (1.19), given any suitable interpolation \( h \), there is a function \( \alpha^{+^{(m)}}_{\Lambda, h}(t) \) such that

\[
\tilde{Z}^{+}_{\Lambda^{(n)}}(\beta, h, \alpha^{+^{(n)}}_{\Lambda, h}(t), t) = Z^{+}_{\Lambda^{(n-1)}} \tag{C.2}
\]

for all allowed values of the parameter \( t \). We simplify notation in the following by omitting the fixed labels \( n, \Lambda \), and write

\[
\alpha_h(t) \equiv \alpha^{(n)}_{\Lambda, h}(t), \quad \alpha^+_h(t) \equiv \alpha^{+^{(n)}}_{\Lambda, h}(t). \]

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Assume that there is a $t = t_I$ at which $\alpha_h(t_I) > \alpha_h^+(t_I)$. Then let $t_0$ be such that
$$h(\alpha_h^+(t_I), t_I) = h(\alpha_h(t_0), t_0), \quad t_0 \in [t_a, t_b]. \quad (C.3)$$
It is always possible to have such a $t_0$ by virtue of (3.32) and (4.25), which imply (Figure 3(a)) that $t_0 > t_I$ and
$$\alpha_h(t) > \alpha_h^+(t_0) > \alpha_h^+(t_I), \quad t \geq t_0. \quad (C.4)$$

![Diagram](image)

Figure 3: Fixing $t_0$ in the neighborhood of some initial $t_I$. Distance between curves in the figure is greatly exaggerated.

With the notation
$$\Phi^{+}_{\Lambda^{(n)}}(\alpha) \equiv \frac{1}{\ln F_0^{+}(n)} \frac{1}{\ln Z_{\Lambda^{(n)}}^{+}(\{\tilde{e}_j(n, \alpha)\})} \quad (C.5)$$
we now define
$$\Psi(\lambda, t) \equiv h(\alpha_h(t), t) + (1 - \lambda) \Phi^{+}_{\Lambda^{(n)}}(\alpha_h^+(t_I)) + \lambda \Phi^{+}_{\Lambda^{(n)}}(\alpha_h(t)) - \Phi^{+}_{\Lambda^{(n-1)}} \quad (C.6)$$
and consider the equation
$$\Psi(\lambda, t) = 0. \quad (C.7)$$
At $\lambda = 0$ eq. (C.7) is solved by setting $t = t_0$ since there, by (C.3), it reduces to (3.2) evaluated at $t_I$. At $\lambda = 1$, (C.7) becomes the equation to be solved (C.1). A solution to (C.7) then determines implicitly a function $t(\lambda)$ with the property $t(0) = t_0$. If this function can be extended on $0 \leq \lambda \leq 1$, it gives the branch of solutions of (C.1) through $(0, t_0)$, and $t(1)$ will be the solution to the original problem (C.1) (method of imbedding or continuity [19]).
By the implicit function theorem, if grad $\Psi$ is continuous and $(\partial \Psi / \partial t)(0, t_0) \neq 0$, there exists a branch $t(\lambda)$ through $(0, t_0)$ on a sufficiently small interval around $\lambda = 0$. One then extends $t(\lambda)$ by a standard argument. Denoting partial derivatives by subscripts following a comma, one has

$$t_{\lambda} = -\frac{\Psi_{,\lambda}(\lambda, t)}{\Psi_{,t}(\lambda, t)}.$$ 

(C.8)

For sufficiently small $\Delta \lambda$ then, and by the mean value theorem, one can write

$$t(0 + \Delta \lambda) = t(0) + t_{\lambda}(\xi \Delta \lambda) \Delta \lambda = t(0) + t_{\lambda}(0) \Delta \lambda + O(\Delta \lambda^2)$$

for some $0 < \xi < 1$. One can then use (C.8) to find $t_{\lambda}(0 + \Delta \lambda)$, and repeat the procedure to obtain $t(0 + 2\Delta \lambda)$, and so on. If grad $\Psi$ is well-behaved throughout the relevant $\lambda - t$ domain, this procedure constructs the desired branch away from the initial point as long as $\Psi_{,t} \neq 0$ along the branch. The existence of a solution $t(1)$ is therefore guaranteed by basic existence theorems (see e.g. [19]) if this condition is satisfied throughout the interval $0 \leq \lambda \leq 1$.

Now

$$\Psi_{,t}(\lambda, t) = \left[ -\frac{h(a_h, t) + \lambda A^+(a_h)}{h(a_h, t) + A(a_h)} + 1 \right] h(a_h, t) > 0,$$

(C.9)

by (5.15) and since $h(a_h, t) < 0$. On the other hand,

$$\Psi_{,\lambda}(\lambda, t) = \left[ \Phi_{\lambda}(a_h(t)) - \Phi_{\lambda}(\alpha_h(t)) \right] > 0,$$

(C.10)

by (C.4) and IV.5. Thus, from (C.8), $t(\lambda)$ is an increasing function of $\lambda$, and extends to the solution $t(1) > t_0$.

Conversely, if there is a $t_I$ such that $a_h(t_I) < \alpha_h^+(t_I)$, one can find $t_0$ such that (C.3) is satisfied (Figure 3(b)), where now $t_0 < t_I$ and

$$\alpha_h^+(t_I) > \alpha_h^+(t_0) > \alpha_h(t), \quad t \leq t_0.$$ 

(C.11)

Now (C.9) remains unchanged, but

$$\Psi_{,\lambda}(\lambda, t) = \left[ \Phi_{\lambda}(a_h(t)) - \Phi_{\lambda}(\alpha_h(t)) \right] < 0,$$

(C.12)

by (C.11) and IV.5. It follows that $t(\lambda)$ is now a decreasing function of $\lambda$, i.e. if $a_h(t_I) < \alpha_h^+(t_I)$ at the starting point one moves backwards in $t$ to hit the point $t(1)$ where $a_h(t(1)) = \alpha_h^+(t(1))$ and (C.11) is satisfied. This concludes the demonstration of the existence of a solution to (5.14).

§2. Going back to (5.10), assume there is a $t_I$ such that $a_h^{(n)}(t_I) < \alpha_h^{(n)}(t_I)$. Then, setting $t = t^+ = t_I$, one has

$$1 + \frac{Z^{(-)}_{\lambda}}{Z_{\lambda}} = \frac{2\tilde{Z}_{\lambda}^{(n)}(\beta, h, \alpha_h^{(n)}(t_I), t_I)}{\tilde{Z}_{\lambda}^{(n)}(\beta, h, \alpha_h^{(n)}(t_I), t_I)}.$$
Hence, with \( b = \alpha \) upper bound at \( j, k, l \) with \( \Delta \), \( \alpha \) paragraph of section 5. One way to complete it would be to show that, for some interpolation \( F \) from (2.22) with integer \( i \), there is at least one value \( \sum_{i \neq 0} \sum_{j_k} \sum_{l_{i_k}} \Delta(j_1, j_2, l_2) \Delta(l_2, j_3, l_3) \cdots \Delta(l_{k-1}, j_k, i) \) holds. This value is unique by monotonicity from (5.24).

This is (5.23) again (for the case \( c_T(n) = 0 \)). It follows from (C.15) that there exist a value \( \alpha^*_{\Lambda} \) such that (5.17) holds. This value is unique by monotonicity from (5.24).

This is an alternative way of treating the \( \alpha_h(t_I) < \alpha^+_{\Lambda}(t_I) \) case in §1 above.

(C.14) - (C.15) partially implement the alternative approach to (5.17) outlined in the last paragraph of section 5. One way to complete it would be to show that, for some interpolation \( h \), there is at least one value \( t_I \) such that \( \alpha^*_{\Lambda, h}(t_I) < \alpha^+_{\Lambda, h}(t_I) \).

### D Appendix

From (2.22) with integer \( \zeta > 1 \) one has

\[
\hat{F}_i(n + 1) = \delta_{0,i} + \left(1 - \delta_{0,i}\right) \zeta c_i(n) + \sum_{k=2}^{\zeta} \left(\frac{\zeta}{k}\right) I_i(k),
\]

where

\[
I_i(k) = \frac{1}{d_i} \sum_{\{j_s| 0 < s \leq k\}} d_{j_1} c_{j_1}(n) \cdots d_{j_k} c_{j_k}(n) \sum_{l_1, \ldots, l_{k-1}} \Delta(j_1, j_2, l_2) \Delta(l_2, j_3, l_3) \cdots \Delta(l_{k-1}, j_k, i)
\]

with \( \Delta(j, k, l) = 1 \) if \( j, k, l \) form the ‘angular momentum addition triangle’ relation, i.e. \( l = |j - k|, \ldots, j + k \), and 0 otherwise. Then

\[
\sum_{i \neq 0} d_i I_i(k) = \sum_{\{j_s| 0 < s \leq k\}} d_{j_1} c_{j_1}(n) \cdots d_{j_k} c_{j_k}(n) \sum_{i \neq 0, l_1, \ldots, l_{k-1}} \Delta(j_1, j_2, l_2) \Delta(l_2, j_3, l_3) \cdots \Delta(l_{k-1}, j_k, i)
\]

\[
\leq \sum_{\{j_s| 0 < s \leq k\}} d_{j_1} c_{j_1}(n) \cdots d_{j_k} c_{j_k}(n) d_{j_2} d_{j_3} \cdots d_{j_k}
\]

\[
\leq ||g(n)||^k.
\]

Hence, with \( b \geq 2 \),

\[
\sum_{i \neq 0} d_i^2 \hat{F}_i(n + 1)^{b^2} = \sum_{i \neq 0} d_i^2 \left[ \zeta c_i(n) + \sum_{k=2}^{\zeta} \left(\frac{\zeta}{k}\right) I_i(k) \right]^{b^2}
\]
\[ \begin{align*}
\leq & \left[ \zeta \sum_{i \neq 0} d^2 c_i(n) + \sum_{k=2}^{\zeta} \left( \frac{\zeta}{k} \sum_{i \neq 0} d_i I_i(k) \right) \right] b^2 \\
\leq & \left[ \zeta \|g(n)\| + \sum_{k=2}^{\zeta} \left( \frac{\zeta}{k} \|g(n)\| \right)^k \right] b^2 \\
= & \left( 1 + \|g(n)\| \right)^\zeta - 1 \right] b^2 \\
\leq & \left[ \zeta \|g(n)\| \right]^{\zeta} \left[ 1 + \|g(n)\| \right]^{(\zeta-1)b^2}.
\end{align*} \] (D.4)

Also, from (D.1), (D.2)

\[ \hat{F}_0(n+1) \geq 1 + \frac{\zeta(\zeta-1)}{2} I_0(2) = 1 + \frac{\zeta(\zeta-1)}{2} \sum_{j \neq 0} d^2 c_j(n)^2 \] (D.5)

\[ > 1, \] (D.6)

whereas also from (2.22)

\[ \hat{F}_0(n+1) \leq \left( 1 + \|g(n)\| \right)^\zeta. \] (D.7)

Combining (D.4) and (D.6) and taking \( r = 1 \) gives (2.31).

For \( r \neq 1 \), (D.4) and, hence, (2.31) hold with the replacement \( b^2 \to b^2 r \), provided \( b^2 r > 2 \).

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