CHARACTER VARIETIES OF HIGHER DIMENSIONAL REPRESENTATIONS AND SPLITTINGS OF 3-MANIFOLDS

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Abstract. In 1983 Culler and Shalen established a way to construct essential surfaces in a 3-manifold from ideal points of the $SL_2$-character variety associated to the 3-manifold group. We present in this article an analogous construction of certain kinds of branched surfaces (which we call essential tribranched surfaces) from ideal points of the $SL_n$-character variety for a natural number $n$ greater than or equal to 3. Further we verify that such a branched surface induces a nontrivial presentation of the 3-manifold group in terms of the fundamental group of a certain 2-dimensional complex of groups.

0. Introduction

In their notable work [CS83] Culler and Shalen established a method to construct essential surfaces in a 3-manifold from information of the $SL_2(\mathbb{C})$-character variety of its fundamental group. The method is based upon the interplay among hyperbolic geometry, the theory of incompressible surfaces and the theory on the structure of subgroups of the special linear group $SL(2)$ of degree 2. Culler-Shalen theory provides a basic and powerful tool in low-dimensional topology, and it has given fundamentals for many significant breakthroughs as follows. For example, Culler and Shalen themselves proved the generalised Smith conjecture as a special case of their main results in [CS83]. Morgan and Shalen [MS84, MS88a, MS88b] proposed new understandings of Thurston’s results: the characterisation of 3-manifolds with the compact space of hyperbolic structures [T86] and a compactification of the Teichmüller space of a surface [T88]. Further Culler, Gordon, Luecke and Shalen [CGLS87] proved the cyclic surgery theorem on Dehn fillings of knots. We refer the reader to the exposition [Sh02] for more literature and related topics on Culler-Shalen theory.

The aim of this article is to present a theory analogous to Culler and Shalen’s for higher dimensional representations of the 3-manifold group. We first introduce a special kind of branched surfaces embedded in a 3-manifold, which we call an essential tribranched surface (see Definition 2.2 for details), and observe that it induces a nontrivial presentation of the 3-manifold group in terms of the fundamental group of a certain 2-dimensional complex of groups (see Section 1 for the definition of complexes of groups). Then we show that an essential tribranched surface is constructed from an ideal point of an affine algebraic curve in the $SL_n(\mathbb{C})$-character variety of the 3-manifold group. Note that in our terminology an essential surface (in the usual sense) can be regarded as an essential tribranched surface without any branched points.

We here explain our strategy to construct an essential tribranched surface in more detail. Let $M$ be a compact, connected, irreducible and orientable 3-manifold. We suppose that the $SL_n(\mathbb{C})$-character variety $X_n(M)$ of $\pi_1(M)$ is of positive dimension, and let $\bar{x}$ be an ideal point of an affine algebraic curve $C$ in $X_n(M)$. By construction $X_n(M)$ is obtained as the (geometric

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invariant theoretical) quotient of the affine algebraic set $\text{Hom}(\pi_1(M), \text{SL}_n(\mathbb{C}))$, and we may take a lift $D$ of $C$ in $\text{Hom}(\pi_1(M), \text{SL}_n(\mathbb{C}))$. Let $\bar{y}$ be a lift of $\bar{x}$ in $D$, which is also an ideal point of $D$. We denote by $\mathbb{C}(D)$ the field of rational functions on $D$. The construction of an essential tribranched surface from $\bar{x}$ is divided into the following three steps. Firstly, on the basis of the theory of Bruhat-Tits buildings elaborated by Iwahori and Matsumoto [IM65], and Bruhat and Tits [BT72, BT84], we may associate to the discrete valuation of $\mathbb{C}(D)$ at $\bar{y}$ a canonical action of $\text{SL}_n(\mathbb{C}(D))$ on an $(n-1)$-dimensional Euclidean building $B_{n,\bar{y}}$ (see Section 4.2 for details). Pulling back this canonical action by the tautological representation $\pi_1(M) \to \text{SL}_n(\mathbb{C}(D))$, we obtain an action of $\pi_1(M)$ on $B_{n,\bar{y}}$. Secondly, we prove that this action is nontrivial, that is, the isotropic subgroup at each vertex of $B_{n,\bar{y}}$ with respect to this action is a proper subgroup of $\pi_1(M)$. The important point to note here is that in the case of $n = 2$ this step is an algebraic heart of Culler and Shalen’s original work [CS83, Theorem 2.2.1]. Thirdly, we show that one can construct an essential tribranched surface in general from a nontrivial action of $\pi_1(M)$ on an Euclidean building. In this step we consider certain modifications of classical techniques due to Stallings and Waldhausen for constructing an essential surface as a Euclidean building. In this step we consider certain modifications of classical techniques due to Stallings and Waldhausen for constructing an essential surface as a dual of a nontrivial action of $\pi_1(M)$ on a tree.

Now let $B_{n,\bar{y}}(2)$ denote the 2-skeleton of the Bruhat-Tits building $B_{n,\bar{y}}$ and let $Y(B_{n,\bar{y}}(2)/\pi_1(M))$ denote the 1-dimensional subcomplex of the first barycentric subdivision of the quotient complex $B_{n,\bar{y}}(2)/\pi_1(M)$ consisting of all the barycentres of 1- and 2-simplices and all the edges connecting them. We say that an ideal point $\bar{x}$ of an affine curve in $X_n(M)$ gives a tribranched surface $\Sigma$ if there exists a map $f : M \to B_{n,\bar{y}}/\pi_1(M)$ such that the tribranched surface $\Sigma$ coincides with the inverse image of $Y(B_{n,\bar{y}}(2)/\pi_1(M))$ under $f$. The main theorem of this article is as follows:

**Main Theorem** (Theorem 4.9). Let $n$ be a natural number greater than or equal to 3, and assume that the boundary $\partial M$ of $M$ is non-empty when $n$ is strictly greater than 3. Then an ideal point of an affine algebraic curve in $X_n(M)$ gives an essential tribranched surface in $M$.

The assumption on the boundary of $M$ comes from a certain technical reason required in the proof of the main result. See the proof of Theorem 4.7 for details.

This article is organised as follows. In Section 1 we give a brief exposition on complexes of groups. Section 2 is devoted to introduce the notion of essential tribranched surfaces and to describe splittings of the 3-manifold groups induced by an essential tribranched surface. In Section 3 we review fundamentals on Bruhat-Tits buildings, in particular, for the special linear groups. In Section 4 the main theorem stated above is proved. We first review several standard facts on $\text{SL}_n(\mathbb{C})$-character varieties in Section 4.1. We then show in Section 4.2 that the action of the 3-manifold group on the Bruhat-Tits building associated to an ideal point is nontrivial, and construct an essential tribranched surface from such a nontrivial action in Section 4.3. Section 5 provides an application of the theory to small Seifert manifolds. In Section 6 we raise several questions to be further studied.

The contents of Sections 1, 2.2 and 2.3 (concerning complexes of groups associated to essential tribranched surfaces) are rather independent of other parts of this article, and hence readers who are only interested in the construction of nontrivial essential tribranched surfaces may skip these sections and proceed to Section 4.
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1. Preliminaries on complexes of groups

The theory of graphs of groups due to Bass and Serre [Se77] has been naturally generalised to the theory of complexes of groups introduced independently by Jon Michael Corson [Co92] (mainly for 2-complexes of groups) and André Haefliger [Hae91] (in general). We shall briefly recall the definitions of complexes of groups and their fundamental groups. Here we adopt a combinatorial approach proposed in [BH99 Chapter III.C] rather than a topological approach based upon the concept of complexes of spaces especially when we define the fundamental groups of complexes of groups (see [Co92, Hae91] for details of the latter approach). One of the great virtues of the combinatorial approach is that one may explicitly describe generators and relations of the fundamental group of a complex of groups, as we shall see later in Section 1.2. Furthermore we shall consider complexes of groups over scwols rather than complexes of groups over combinatorial CW-complexes (the latter notion is introduced in [Co92, Section 2, Definition]). One may readily observe that these two concepts of complexes of groups essentially coincide by considering scwols associated to combinatorial CW-complexes (essentially it is equivalent to consider the “first barycentric subdivision” of a combinatorial CW-complex).
We shall briefly explain how to associate a scwol to a combinatorial CW-complex (of dimension 2) later in Section 2.2.

1.1. Scwols and their fundamental groups. Recall that a scwol \( \mathcal{Y} \) (an abbreviation of a small category without loops) consists of two sets \( V(\mathcal{Y}) \) and \( E(\mathcal{Y}) \) equipped with set-theoretical maps \( i: E(\mathcal{Y}) \to V(\mathcal{Y}) \) and \( t: E(\mathcal{Y}) \to V(\mathcal{Y}) \) satisfying the following properties:

(Scw1) an element of \( E(\mathcal{Y}) \) denoted by \( ab \) is associated to each pair \((a, b)\) of elements of \( E(\mathcal{Y}) \) satisfying \( i(a) = t(b) \) (The element \( ab \) is called the composition of the composable pair \((a, b)\) in \( E^2(\mathcal{Y}) \));

(Scw2) we have \( i(ab) = i(b) \) and \( t(ab) = t(a) \) for a composable pair \((a, b)\) in \( E^2(\mathcal{Y}) \);

(Scw3) the composition law is associative, that is, the composition \( (ab)c \) coincides with \( (a(bc)) \) for composable pairs \((a, b)\) and \((b, c)\) in \( E^2(\mathcal{Y}) \);

(Scw4) the elements \( i(a) \) and \( t(a) \) are distinct for each \( a \) in \( E(\mathcal{Y}) \).

Here \( E^k(\mathcal{Y}) \) denotes the set of \( k \)-sequences \((a_1, \ldots, a_k)\) of elements of \( E(\mathcal{Y}) \) satisfying the equality \( i(a_j) = t(a_{j+1}) \) for each \( 1 \leq j \leq k - 1 \). Elements of \( V(\mathcal{Y}) \) are called vertices of \( \mathcal{Y} \), and those of \( E(\mathcal{Y}) \) are called (oriented) edges of \( \mathcal{Y} \). For an edge \( a \) of \( \mathcal{Y} \), the vertices \( i(a) \) and \( t(a) \) are respectively called the initial and terminal vertices of \( a \). A scwol \( \mathcal{Y} \) has its geometric realisation \( |\mathcal{Y}| \) defined in an appropriate way, which is a (polyhedral) complex all of whose cells are simplices; see \[BH99\] Chapter III.C Section 1.3 for details. A morphism \( f: \mathcal{Y} \to \mathcal{Y}' \) of scwols consists of (set-theoretical) maps \( V(\mathcal{Y}) \to V(\mathcal{Y}') \) and \( E(\mathcal{Y}) \to E(\mathcal{Y}') \) which are compatible with the scwol structures of \( \mathcal{Y} \) and \( \mathcal{Y}' \) (refer to \[BH99\] Chapter III.C Section 1.5 for the precise definition).

In order to introduce the fundamental group of a scwol, we here summarise basic notion on edge paths. Let \( E^* (\mathcal{Y}) \) (resp. \( E^-(\mathcal{Y}) \)) denote the set consisting of an element denoted by \( a^+ \) (resp. \( a^- \)) for each edge \( a \) of \( \mathcal{Y} \), whose initial and terminal vertices are determined by \( i(a^+) = t(a) \) and \( t(a^-) = i(a) \) (resp. \( i(a^-) = t(a) \) and \( t(a^+) = i(a) \)). We denote by \( E^*(\mathcal{Y}) \) the disjoint union of \( E^*(\mathcal{Y}) \) and \( E^-(\mathcal{Y}) \). We also set \( (a^\pm)^{-1} = a^- \) (double sign in the same order). An edge path in \( \mathcal{Y} \) is a finite sequence \( l = (e_1, \ldots, e_n) \) of elements of \( E^*(\mathcal{Y}) \) satisfying \( t(e_j) = i(e_{j+1}) \) for each \( 1 \leq j \leq n - 1 \). The vertices \( i(e_1) \) and \( t(e_n) \) are called the initial and terminal vertices of the edge path \( l \), and denoted by \( i(l) \) and \( t(l) \) respectively. We define the concatenation \( l * l' \) of edge paths \( l = (e_1, \ldots, e_m) \) and \( l' = (e'_1, \ldots, e'_n) \) by \( l * l' = (e_1, \ldots, e_m, e'_1, \ldots, e'_n) \) when the pair \((l, l')\) satisfies \( t(l) = i(l') \), and we also define the inverse edge path \( l^{-1} \) of an edge path \( l = (e_1, \ldots, e_n) \) by \( l^{-1} = (e_n^{-1}, e_{n-1}^{-1}, \ldots, e_1^{-1}) \). An edge path is called an edge loop when its initial vertex coincides with its terminal vertex; in the case its initial (and hence also terminal) vertex is called its base vertex.

Now let \( \mathcal{Y} \) be a connected scwol (in the sense that arbitrary two vertices of \( \mathcal{Y} \) are connected by an edge path in \( \mathcal{Y} \)) and let us consider the set of all edge loops with base vertex \( \sigma_0 \). We endow this set with an equivalence relation \( \sim \) (called homotopy equivalence) generated by the following two elementary relations:

i) \( (e_1, \ldots, e_{j-1}, e_j, e_{j+1}, e_{j+2}, \ldots, e_n) \sim (e_1, \ldots, e_{j-1}, e_{j+2}, \ldots, e_n) \) if \( e_{j+1} \) coincides with \( e_j^{-1} \);

ii) if \( (a, b) \) is a composable pair in \( E^2(\mathcal{Y}) \), we impose

\[
(e_1, \ldots, e_{i-1}, e_i = a^+, e_{i+1} = b^+, e_{i+2}, \ldots, e_m) \sim (e_1, \ldots, e_{i-1}, (ab)^+, e_{i+2}, \ldots, e_m)
\]

and

\[
(e_1, \ldots, e_{j-1}, e_j = b^-, e_{j+1} = a^-, e_{j+2}, \ldots, e_n) \sim (e_1, \ldots, e_{j-1}, (ab)^-, e_{j+2}, \ldots, e_n).
\]
The set \( \pi_1(\mathcal{Y}, \sigma_0) \) of all (homotopy) equivalence classes of edge loops with base vertex \( \sigma_0 \) is indeed equipped with a group structure whose group law is induced by the concatenation of edge loops: \( [c] \ast [c'] = [c \ast c'] \). For an edge loop \( c \), the inverse of \( [c] \) is given by the homotopy class \( [c^{-1}] \) of the inverse loop \( c^{-1} \) of \( c \), and the unit element is given by the homotopy class of the constant loop \( [c_{\sigma_0}] \) at \( \sigma_0 \) (by definition the constant loop \( c_{\sigma_0} \) corresponds to the “empty word” \( c_{\sigma_0} = () \) both of whose initial and terminal vertices are defined as \( \sigma_0 \)). We call \( \pi_1(\mathcal{Y}, \sigma_0) \) the fundamental group of the scwol \( \mathcal{Y} \) at \( \sigma_0 \). We can construct an isomorphism \( \pi_1(\mathcal{Y}, \sigma_0) \to \pi_1(\mathcal{Y}, \sigma_0') \); \([c] \mapsto [l_{\sigma_0, \sigma_0'}^{-1} \ast c \ast l_{\sigma_0, \sigma_0'}] \) as in the case of usual fundamental groups, where \( l_{\sigma_0, \sigma_0'} \) is an edge path with initial vertex \( \sigma_0 \) and terminal vertex \( \sigma_0' \) (obviously this isomorphism is not a canonical one).

We end this subsection by quoting the following classical fact.

**Proposition 1.1.** Let \( \mathcal{Y} \) be a connected scwol and \( \sigma_0 \) a vertex of \( \mathcal{Y} \). Then the fundamental group \( \pi_1(\mathcal{Y}, \sigma_0) \) of the scwol \( \mathcal{Y} \) is canonically isomorphic to the fundamental group (in the usual sense) \( \pi_1(|\mathcal{Y}|, \sigma_0) \) of its geometric realisation. In particular, a connected scwol \( \mathcal{Y} \) is simply connected (in the sense that its fundamental group is trivial) if and only if its geometric realisation \( |\mathcal{Y}| \) is simply connected in the usual sense.

For details, see [BH99, Chapter III.C Section 1.8] and [Ma91].

1.2. **Complexes of groups and their fundamental groups.** A complex of groups \( G(\mathcal{Y}) \) over a scwol \( \mathcal{Y} \) consists of three types of data: a group \( G_\sigma \) for each vertex \( \sigma \) of \( \mathcal{Y} \) called the local group at \( \sigma \), an injective group homomorphism \( \psi_a : G_{\rho(a)} \to G_{\rho(a)} \) for each edge \( a \) of \( \mathcal{Y} \), and a specific element \( g_{a,b} \) of \( G_{\rho(a)} \), called a twisting element, for each composable pair \( (a,b) \) in \( E^2(\mathcal{Y}) \). We impose the following two constraints on these data:

- (twisted commutativity) the equality \( g_{a,b}\psi_{ab}(x)g_{a,b}^{-1} = \psi_a \circ \psi_b(x) \) holds for each composable pair \( (a,b) \) in \( E^2(\mathcal{Y}) \) and every element \( x \) of \( G_{\rho(b)} \);
- (cocycle condition) the equality \( \psi_{a}(g_{b,c})g_{a,b,c} = g_{a,b}g_{ab,c} \) holds for each pairwisely composable triple \( (a,b,c) \) in \( E^3(\mathcal{Y}) \).

A complex of groups \( G(\mathcal{Y}) \) over \( \mathcal{Y} \) is called simple if all the twisting elements are trivial, that is, the element \( g_{a,b} \) equals the unit of \( G_{\rho(a)} \) for each composable pair \( (a,b) \) in \( E^2(\mathcal{Y}) \).

**Remark 1.2.** We here remark that for a complex of groups of dimension at most 2, that is, for a complex of groups whose geometric realisation is of dimension at most 2, the cocycle condition among twisting elements introduced above is just the empty condition. Later we shall mainly study complexes of groups associated to essential tribranched surfaces in a 3-manifold, which we shall define in Section 2.2. Obviously by construction they are of dimension at most 2, and hence we do not have to consider the cocycle conditions whenever we are concerned with complexes of groups associated to essential tribranched surfaces.

Let \( f : \mathcal{Y} \to \mathcal{Y}' \) be a morphism of scwols, and let \( G(\mathcal{Y}) \) and \( G(\mathcal{Y}') \) be complexes of groups over \( \mathcal{Y} \) and \( \mathcal{Y}' \) respectively. A morphism of complexes of groups (over \( f \)) \( \phi : G(\mathcal{Y}) \to G(\mathcal{Y}') \) consists of two types of data: a group homomorphism between local groups \( \phi_\sigma : G_\sigma \to G_{\rho(\sigma')} \) for each vertex \( \sigma \) of \( \mathcal{Y} \) and a specific element \( \phi(a) \) of \( G_{\rho(f(a))} \), called a twisting element, for each edge \( a \) of \( \mathcal{Y} \). We impose the following two constraints on these data:

- (twisted commutativity) the equality \( \phi(f(a))\psi_{f(a)} \circ \phi_{\rho(a)}(x)\phi(a)^{-1} \) of \( G_{\rho(f(a))} \) coincides with \( \phi_{\rho(a)} \circ \psi_a(x) \) for each edge \( a \) of \( \mathcal{Y} \) and each element \( x \) of \( G_{\rho(a)} \).
- (compatibility among twisting elements) the element $\phi_{\sigma}(g_{ab})\phi(ab)$ of $FG_{n(f(\sigma))}$ coincides with $\phi(a)\psi_f(\sigma)(\phi(b))g_{f(\sigma),f(b)}$ for each composable pair $(a, b)$ in $E^2(\Upsilon)$.

A morphism $\phi$ of complexes of groups is said to be an isomorphism if its local group homomorphism $\phi_\sigma$ is an isomorphism for each vertex $\sigma$ of $\Upsilon$. By regarding an abstract group $G$ as a complex of groups over the trivial scwol (that is, the scwol consisting of a single vertex) whose local group at its unique vertex is $G$, we may also consider a morphism $\phi: G(\Upsilon) \to G$ from a complex of groups $G(\Upsilon)$ to an abstract group $G$.

Next we introduce the notion of the fundamental group of a complex of groups. Let $G(\Upsilon)$ be a complex of groups over a scwol $\Upsilon$. A $G(\Upsilon)$-path in $\Upsilon$ is a finite sequence $l = (g_0,e_1,g_1,\ldots,e_n,g_n)$ where $(e_1, \ldots, e_n)$ is an edge path in $\Upsilon$, $g_0$ is an element of the local group $G_{i(e_1)}$ at $i(e_1)$ and $g_j$ is an element of the local group $G_{t(e_j)}$ at $t(e_j)$ for each $1 \leq j \leq n$.

For $G(\Upsilon)$-paths we define their initial and terminal vertices, concatenations and inverse paths similarly to those of edge paths as follows.

**Initial and terminal vertices:** for a $G(\Upsilon)$-path $l = (g_0,e_1,g_1,\ldots,e_n,g_n)$, set $i(l) = i(e_1)$ and $t(l) = t(e_n)$.

**Concatenation:** for $G(\Upsilon)$-paths $l = (g_0,e_1,g_1,\ldots,e_m,g_m)$ and $l' = (g_0',e_1',g_1',\ldots,e_n',g_n')$ satisfying $t(l) = i(l')$, set $l \ast l' = (g_0,e_1,g_1,\ldots,e_m,g_m,e_1',g_1',\ldots,e_n',g_n')$.

**Inverse path:** for a $G(\Upsilon)$-path $l = (g_0,e_1,g_1,\ldots,e_n,g_n)$, define its inverse $G(\Upsilon)$-path $l^{-1}$ as $l^{-1} = (g_n^{-1}, e_{n-1}^{-1}, \ldots, e_1^{-1}, g_0^{-1})$.

Now let $FG(\Upsilon)$ be the universal group associated to $G(\Upsilon)$ which is defined by the following generators and relations.

**Generators:** elements of all local groups $G_\sigma$ and elements of $E^\pm(\Upsilon)$.

**Relations:** we impose on the generators the following four types of relations:
- the group relations for each $G_\sigma$;
- $(a^\pm)^{-1} = a^\mp$ for each edge $a$ in $\Upsilon$ (double sign in the same order);
- $a^x b^y = g_{ab}(ab)^x$ for each composable pair $(a, b)$ in $E^2(\Upsilon)$;
- $\psi_a(x) = a^*x a^{-*}$ for each edge $a$ of $\Upsilon$ and each element $x$ of $G_{\sigma(a)}$.

Then it is easy to check that the morphism $\iota: G(\Upsilon) \to FG(\Upsilon)$, which consists of a group homomorphism $\iota_{\sigma}: G_\sigma \to FG(\Upsilon); g \mapsto g$ for each vertex $\sigma$ of $\Upsilon$ and a twisting element $\iota(a) = a^*$ for each edge $a$ of $\Upsilon$, has a universal property among morphisms from $G(\Upsilon)$ to abstract groups. More specifically, for every morphism $\phi: G(\Upsilon) \to G$ from $G(\Upsilon)$ to an abstract group $G$ we obtain a unique group homomorphism $F\phi: FG(\Upsilon) \to G$ which satisfies $\phi = F\phi \circ \iota$ (see [BH99] Chapter III.C Section 3.2 for details).

We associate to each $G(\Upsilon)$-loop $c = (g_0,e_1,g_1,\ldots,e_n,g_n)$ with base vertex $\sigma_0$ an element $[c]$ of $FG(\Upsilon)$ which is by definition the image of the word $g_0e_1g_1\ldots e_ng_n$ in $FG(\Upsilon)$. The image of $[\cdot]$ (as a map from the set of $G(\Upsilon)$-loops with base vertex $\sigma_0$) is equipped with a group structure induced by concatenations, which we denote by $\pi_1(G(\Upsilon),\sigma_0)$ and call the fundamental group of $G(\Upsilon)$. We remark that the definition of the fundamental group $\pi_1(G(\Upsilon),\sigma_0)$ of a complex of groups $G(\Upsilon)$ (of higher dimension) introduced here is a direct generalisation of the definition of the fundamental group of a graph of groups due to Bass and Serre [Se77] Section 5.1].

**1.3. Group actions on scwols and developability.** Let $X$ be a scwol and $G$ an abstract group. An action of $G$ on $X$ is a group homomorphism $G \to \text{Aut}(X)$ satisfying the following two conditions:

i) the vertex $g.\iota(a)$ does not equal $t(a)$ for each edge $a$ of $X$ and each element $g$ of $G$;
ii) if \( g.i(a) = i(a) \) holds for an edge \( a \) of \( X \) and an element \( g \) of \( G \), we have \( g.a = a \).

Here \( g.\sigma \) (resp. \( g.a \)) denotes the image of a vertex \( \sigma \) of \( X \) (resp. an edge \( a \) of \( X \)) under the automorphism of \( X \) induced by \( g \). For such an action of \( G \) on \( X \), we may construct the quotient scwol \( Y = G \backslash X \) by setting \( V(Y) = G \backslash V(X) \) and \( E(Y) = G \backslash E(X) \) (initial and terminal vertices and compositions in \( Y \) are determined in obvious manners; that is, \( i(G.a) \) and \( t(G.a) \) are defined as \( G.i(a) \) and \( G.t(a) \) respectively for an edge \( a \) of \( X \), and the composition \((G.a)(G.b)\) is defined as \( G.ab \) for each composable pair \((a, b)\) in \( E^2(X) \)).

We may endow the quotient scwol \( Y = G \backslash X \) with the structure of a complex of groups in the following way. For each vertex \( \sigma \) of \( Y \), we choose a lift \( \tilde{\sigma} \) of \( \sigma \) to \( X \) (that is, \( \tilde{\sigma} \) is a vertex of \( X \) whose \( G \)-orbit coincides with \( \sigma \)). The condition (ii) of the group action on a scwol implies that for each edge \( a \) of \( Y \) with initial vertex \( \sigma \), there exists a unique lift \( \tilde{a} \) of \( a \) to \( X \) with initial vertex \( \tilde{\sigma} \). Let us choose an element \( h_a \) of \( G \) satisfying \( h_a.t(\tilde{a}) = t(a) \). We define the local group \( G_a \) at a vertex \( \sigma \) of \( Y \) as the isotropy subgroup \( G_{\tilde{\sigma}} \) of \( G \) at \( \tilde{\sigma} \) with respect to the group action of \( G \) on \( X \). For each edge \( a \) of \( Y \), we define a group homomorphism \( \psi_a : G_{i(a)} \to G_{i(a)} \) by \( \psi_a(g) = h_agh_{a}^{-1} \).

Finally for each composable pair \((a, b)\) in \( E^2(Y) \), we define a twisting element \( g_{ab} \) as \( h_a h_b h_{ab}^{-1} \). It is easy to verify that these data determine the structure of a complex of groups \( G(Y) \) over the quotient scwol \( Y \), which we call the complex of groups associated to the group action of \( G \) on \( X \). Note that if we choose a different lift \( \tilde{\sigma}' \) for each vertex \( \sigma \) of \( Y \) and a different element \( h'_a \) for each edge \( a \) of \( Y \), the resultant complex of groups \( G'(Y) \) is still isomorphic to \( G(Y) \); see [BH99, Chapter III.C Section 2.9 (2)] for details. When a complex of groups \( G(Y) \) is obtained as an action of \( G \) on a scwol \( X \) by the construction explained above, we may associate to it a morphism \( \phi : G(Y) \to G \) by setting \( \phi_\sigma(g) = g \) for each vertex \( \sigma \) and \( \phi(a) = h_a \) for each edge \( a \).

A complex of groups \( G(Y) \) over a scwol \( Y \) is called developable if there exists a scwol \( X \) equipped with an action of a group \( G \) such that \( G(Y) \) is isomorphic to the complex of groups associated to the group action of \( G \) on \( X \). Unlike graphs of groups, complexes of groups of higher dimension are not always developable. The following proposition proposes a necessary and sufficient condition for a complex of groups to be developable.

**Proposition 1.3** ([BH99, Chapter III.C Corollary 2.15]). A complex of groups \( G(Y) \) is developable if and only if there exists a morphism from \( G(Y) \) to a certain (abstract) group \( G \) which is injective on each local group \( G_\sigma \) of \( G(Y) \).

In fact if \( G(Y) \) admits a morphism \( \phi : G(Y) \to G \) which is injective on all the local groups of \( G(Y) \), we may construct in a canonical manner a scwol \( D(Y, \phi) \) equipped with a group action of \( G \) (which is called the development of \( Y \) with respect to \( \phi \)) by setting

\[
V(D(Y, \phi)) = \{ (g\phi_\sigma(G_\sigma), \sigma) \mid \sigma \in V(Y), g\phi_\sigma(G_\sigma) \in G/\phi_\sigma(G_\sigma) \}
\]

\[
E(D(Y, \phi)) = \{ (g\phi_{i(a)}(G_{i(a)}), a) \mid a \in E(Y), g\phi_{i(a)}(G_{i(a)}) \in G/\phi_{i(a)}(G_{i(a)}) \}
\]

and

\[
i((g\phi_{i(a)}(G_{i(a)}), a)) = (g\phi_{i(a)}(G_{i(a)}), i(a))
\]

\[
t((g\phi_{i(a)}(G_{i(a)}), a)) = (g\phi(a)^{-1}\phi_{i(a)}(G_{i(a)}), t(a))
\]

for each \((g\phi_{i(a)}(G_{i(a)}), a)\) in \( E(D(Y, \phi)) \). The group \( G \) acts on \( D(Y, \phi) \) in a natural way; namely, an element \( x \) of \( G \) acts as

\[
x.(g\phi_\sigma(G_\sigma), \sigma) = (xg\phi_\sigma(G_\sigma), \sigma), \quad x.(g\phi_{i(a)}(G_{i(a)}), a) = (xg\phi_{i(a)}(G_{i(a)}), a).
\]

For details, see [BH99, Chapter III.C Theorem 2.13].
2. TRIBRANCHED SURFACES AND COMPLEXES OF GROUPS

We introduce in this section the notion of tribranched surfaces and essential tribranched surfaces which shall play key roles throughout this article. It is a certain generalisation of the concepts of surfaces (contained in a 3-manifold) and essential surfaces (see, for example, [Sh02, Definition 1.5.1] for the definition of essential surfaces). After proposing the definitions of tribranched surfaces and essential tribranched surfaces in Section 2.1, we observe that essential tribranched surfaces behave compatibly with the theory of complexes of groups in Sections 2.2 and 2.3 (as the notion of essential surfaces is well adapted to Bass and Serre’s theory on graphs of groups [Se77] in the original work of Culler and Shalen [CS83]).

2.1. Tribranched surfaces and essential tribranched surfaces. Let \( M \) be a 3-manifold with possibly nonempty boundary. Let \( \Sigma \) be a compact subset of \( M \) such that the pair \( (M, \Sigma) \) is locally homeomorphic to \( (\overline{\mathbb{H}}, Y \times [0, \infty)) \), where \( \overline{\mathbb{H}} \) and \( Y \) are defined by

\[
\overline{\mathbb{H}} = \{ (z, s) \in \mathbb{C} \times \mathbb{R} \mid s \geq 0 \}, \quad Y = \{ r e^{\sqrt{-1} \theta} \in \mathbb{C} \mid r \in \mathbb{R}_{\geq 0} \text{ and } \theta = 0, \pm 2\pi/3 \}.
\]

We denote by \( C(\Sigma) \) the set of branched points of \( \Sigma \) corresponding to \( \{0\} \times [0, \infty) \subset Y \times [0, \infty) \), by \( S(\Sigma) \) the complement of a sufficiently small tubular neighbourhood of \( C(\Sigma) \) in \( \Sigma \), and by \( M(\Sigma) \) the complement of a regular neighbourhood of \( \Sigma \) in \( M \). The subsets \( C(\Sigma) \) and \( S(\Sigma) \) are a properly embedded 1-submanifold and a subsurface of \( M \) respectively. See Figure 1 for a local picture of \( \Sigma \).

**Figure 1. Tribranched surface \( \Sigma \)**

**Definition 2.1** (Tribranched surfaces). Let \( (M, \Sigma) \) be as above. We call \( \Sigma \) a tribranched surface in \( M \) if the following conditions are satisfied:

- (TBS1) the intersection of \( \Sigma \) and a sufficiently small tubular neighbourhood of \( C(\Sigma) \) in \( M \) is homeomorphic to \( Y \times C(\Sigma) \);
- (TBS2) the subsurface \( S(\Sigma) \) is orientable.

In the following, we will suppress the base point in the notation of fundamental groups unless specifically noted.

**Definition 2.2** (Essential tribranched surfaces). A tribranched surface \( \Sigma \) in \( M \) is said to be essential if the following conditions are satisfied:
(ETBS1) for any component \( N \) of \( M(\Sigma) \), the homomorphism \( \pi_1(N) \to \pi_1(M) \) induced by the natural inclusion map \( N \hookrightarrow M \) is not surjective;

(ETBS2) for any components \( C, S, N \) of \( C(\Sigma), S(\Sigma), M(\Sigma) \) respectively, if the homomorphisms \( \pi_1(C) \to \pi_1(S) \) and \( \pi_1(S) \to \pi_1(N) \) are induced by the natural inclusion maps, they are injective;

(ETBS3) no component of \( \Sigma \) is contained in a 3-ball in \( M \) or a collar of \( \partial M \).

**Remark 2.3.** An essential surface (in the usual sense) in \( M \) is regarded as an essential tribranched surface without any branched points.

2.2. **Complexes of groups associated to essential tribranched surfaces.** It is well known that one may associate a graph of groups to an essential surface embedded in a 3-manifold \( M \), which gives a splitting of the 3-manifold group \( \pi_1(M) \) (we refer the readers to [Sh02 Sections 1.4 and 1.5]). Then the concept of essential tribranched surfaces, which is a more general notion including essential surfaces, should be closely related to the theory of complexes of groups of higher dimension. Here we shall discuss the relation between them.

Now let \( M \) be a 3-manifold which is compact, connected, irreducible and orientable. Suppose that \( M \) contains a tribranched surface \( \Sigma \).

The dual 2-complex associated to \( \Sigma \). A cellular map between CW-complexes \( f : X \to Y \) is said to be combinatorial if it maps each open cell of \( X \) homeomorphically to an open cell of \( Y \), and a CW-complex \( X \) is said to be combinatorial if, for each cell \( e_i \) of \( X \) of dimension \( n_i \), the characteristic map \( \varphi_i : D^{n_i} \to X^{(n_i-1)} \) of \( e_i \) is a combinatorial cellular map with respect to a certain cellular complex structure on the \( n_i \)-dimensional closed unit ball \( D^{n_i} \). In this paragraph we associate to the pair \((M, \Sigma)\) a combinatorial CW-complex \( Y_{\Sigma} = Y_{(M, \Sigma)} \) of dimension 2. The construction of \( Y_{\Sigma} \) which we shall explain below is a natural generalisation of a well-known construction of the dual graph of a bicollared surface contained in a 3-manifold. The readers are referred to the exposition [Sh02 Section 1.4], for example, for details on the classical construction of dual graphs.

Recall that \( C(\Sigma) \) denotes the set of branched points of \( \Sigma \). Let \( C \) be a connected component of \( C(\Sigma) \), and let \( D^2 \) (resp. \( \tilde{D}^2 \)) denote the closed unit disk \( \{ z \in \mathbb{C} \mid |z| \leq 1 \} \) (resp. the open unit disk \( \{ z \in \mathbb{C} \mid |z| < 1 \} \)). For each \( C \), there exists a tubular neighbourhood \( h_C : C \times D^2 \to M \) by virtue of the condition (TBS1) of tribranched surfaces; more specifically, \( h_C \) induces a homeomorphism of \( C \times D^2 \) onto a neighbourhood of \( C \) in \( M \) and satisfies \( h_C(x, 0) = x \) for each point \( x \) of \( C \). Furthermore \( h_{|C \times (\tilde{D}^2 \cap Y)} \) induces a homeomorphism of \( C \times (\tilde{D}^2 \cap Y) \) onto a regular neighbourhood of \( C \) in \( \Sigma \). We choose and fix such a tubular neighbourhood \( h_C \) for each connected component \( C \) of \( C(\Sigma) \). We denote by \( U_C \) (resp. \( \tilde{U}_C \)) the open tubular neighbourhood \( h_C(C \times \tilde{D}^2) \) (resp. the closed tubular neighbourhood \( h_C(C \times D^2) \)) of \( C \) in \( M \).

Next let \( S \) be an arbitrary connected component of \( S(\Sigma) = \Sigma \setminus \bigcup_{C \in C(\Sigma)} U_C \). The condition (TBS2) combined with the theory of regular neighbourhoods provides us with a homeomorphism \( h_S : S \times [-1, 1] \to M \) onto a regular neighbourhood of \( S \) in \( M \setminus \bigcup_{C \in C(\Sigma)} U_C \); namely \( h_S \) satisfies \( h_S(x, 0) = x \) for each point \( x \) of \( S \) and \( h_S(\partial S \times [-1, 1]) \) coincides with the intersection of \( h_S(S \times [-1, 1]) \) and \( \partial M \setminus \bigcup_{C \in C(\Sigma)} \partial U_C \). We also choose and fix such a bicollared neighbourhood \( h_S \) for each connected component \( S \) of \( S(\Sigma) \). We further assume that the closed sets \( h_S(S \times [-1, 1]) \) are pairwise disjoint after replacing them by thinner ones if necessary. We denote by \( U_S \) (resp. \( \tilde{U}_S \)) the subset \( h_S(S \times (-1, 1)) \) (resp. \( h_S(S \times [-1, 1]) \)) of \( M \) which is an open (resp. a closed) bicollared neighbourhood of \( S \) in \( M \setminus \bigcup_{C \in C(\Sigma)} U_C \).
We denote by $M(\Sigma)$ the complement of $\bigcup_{C \in \pi_0(M)} U_C \cup \bigcup_{S \in \pi_0(S(\Sigma))} U_S$ in $M$. Note that all of $\pi_0(C(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(M(\Sigma))$ are finite sets due to the compactness of $M$. We thus obtain a partition of $M$ into disjoint subsets:

\begin{equation}
M = \bigcup_{N \in \pi_0(M(\Sigma))} N \sqcup \bigcup_{S \in \pi_0(S(\Sigma))} h_S(S \times \{t\}) \sqcup \bigcup_{C \in \pi_0(C(\Sigma))} h_C(C \times \{s\}).
\end{equation}

We use the notation $x \sim_\Sigma y$ to indicate that both of two points $x$ and $y$ of $M$ are contained in one of the disjoint subsets occurring in the right hand side of (2.1). Obviously $\sim_\Sigma$ defines an equivalence relation on $M$. Set $Y_\Sigma = Y(M,\Sigma) = M/\sim_\Sigma$ and endow $Y_\Sigma$ with the quotient topology. One then easily observes that $Y_\Sigma$ is a combinatorial CW-complex of dimension 2 whose 0-cells, 1-cells and 2-cells are labeled by elements of $\pi_0(M(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(C(\Sigma))$ respectively. Moreover, for each 2-cell $e_C$ of $Y_\Sigma$, the characteristic map $\varphi_C : D^2 \to Y_\Sigma^{(1)}$ is a combinatorial cellular map with respect to the following cellular complex structure on $D^2$:

$$D^2 = \emptyset \sqcup \bigcup_{a=1}^3 \left\{ e^{\sqrt{1} \theta} \left| \frac{2}{3} (a - 1) \pi < \theta < \frac{2}{3} a \pi \right. \right\} \sqcup \bigcup_{a=1}^3 \left\{ e^{\frac{2}{3} a \pi} \right\}.$$ 

Roughly speaking, this implies that each 2-cell of $Y_\Sigma$ may be identified with a 2-simplex whose boundaries are appropriately glued to the 1-skeleton $Y_\Sigma^{(1)}$ of $Y_\Sigma$. Figure 2 illustrates a local picture of the dual 2-complex $Y_\Sigma$ associated to a tribranched surface $\Sigma$.

![Figure 2](image)

**Figure 2.** The dual 2-complex $Y_\Sigma$ associated to a tribranched surface $\Sigma$

**Remark 2.4.** The combinatorial CW-complex $Y_\Sigma$ is nothing but a $(M_0)$-polyhedral complex of dimension 2 in the sense of [BH99, Chapter I.7, Definition 7.37] all of whose cells are (Euclidean) simplices. One often requires in many other references, however, that the intersection of two polytopes of a polyhedral complex should consist of a single common face of them unless it is empty. Therefore we here adopt the term a “combinatorial CW-complex of dimension 2” rather than a “polyhedral complex of dimension 2” in order to avoid terminological confusion.

It is straightforward to check that a 1-cell $e_S$ (labeled by an element $S$ of $\pi_0(S(\Sigma))$) occurs in the boundary of a 2-cell $e_C$ (labeled by an element $C$ of $\pi_0(C(\Sigma))$) if and only if the intersection of $\bar{U}_S$ and $\bar{U}_C$ is nonempty. Similarly a 0-cell $e_N$ (labeled by an element $N$ of $\pi_0(M(\Sigma))$) occurs in the boundary of a 1-cell $e_S$ (resp. a 2-cell $e_C$) if and only if the intersection of $N$ and $\bar{U}_S$ (resp. $\bar{U}_C$) is nonempty. We call $Y_\Sigma$ the dual (2-)complex associated to the tribranched surface $\Sigma$. 


The scwol associated to $\Sigma$. We next associate a scwol $\mathcal{Y}_\Sigma = \mathcal{Y}_{(M, \Sigma)}$ to the dual complex $Y_\Sigma$ in a canonical way (refer also to [BH99], Chapter III.C, Section 1, Example 1.4 (2)). By identifying each 2-cell of $Y_\Sigma$ with a 2-simplex, we may consider the first barycentric subdivision of the dual complex $Y_\Sigma$ associated to $\Sigma$. We define the vertex set $V(\mathcal{Y}_\Sigma)$ of $\mathcal{Y}_\Sigma$ as the set of all cells of $Y_\Sigma$ (or equivalently, the set of the barycentres of all cells in $Y_\Sigma$). Therefore every element of $V(\mathcal{Y}_\Sigma)$ is labeled by an element of the disjoint union of $\pi_0(C(\Sigma))$, $\pi_0(S(\Sigma))$ and $\pi_0(M(\Sigma))$, which we denote by $\Lambda$ and regard as the index set. We also define the edge set $E(\mathcal{Y}_\Sigma)$ of $\mathcal{Y}_\Sigma$ as the set of all 1-cells of the first barycentric subdivision of $Y_\Sigma$. One then readily observes that there exists an edge $a$ of $\mathcal{Y}_\Sigma$ connecting two vertices $\sigma_\lambda$ and $\sigma_\mu$ (labeled by elements $\lambda$ and $\mu$ of $\Lambda$ respectively) if and only if the cell $e_\mu$ of $Y_\Sigma$ labeled by $\mu$ occurs in the boundary of the cell $e_\lambda$ labeled by $\lambda$ or vice versa (in particular $\sigma_\lambda$ and $\sigma_\mu$ are distinct). For an edge $a$ of $\mathcal{Y}_\Sigma$ connecting vertices $\sigma_\lambda$ and $\sigma_\mu$, we set $i(a) = \sigma_\lambda$ and $t(a) = \sigma_\mu$ if the cell $e_\mu$ of $Y_\Sigma$ corresponding to $\sigma_\mu$ occurs in the boundary of the cell $e_\lambda$ corresponding to $\sigma_\lambda$. There exists a natural composition law among edges of $\mathcal{Y}_\Sigma$: namely $ab = c_{a,b}$ for each composable pair $(a, b)$ in $E(2)(\mathcal{Y}_\Sigma)$. Here $c_{a,b}$ denotes a unique edge with $i(c_{a,b}) = i(b)$ and $t(c_{a,b}) = t(a)$ such that all of $a$, $b$ and $c_{a,b}$ occur in the boundary of a single 2-cell in the first barycentric subdivision of $Y_\Sigma$ (see Figure 3 for details). Note that if a pair $(a, b)$ of edges of $\mathcal{Y}_\Sigma$ is composable, $i(b)$ is labeled by an element of $\pi_0(C(\Sigma))$, $t(b) = i(a)$ is labeled by an element of $\pi_0(S(\Sigma))$, and $t(a)$ is labeled by an element of $\pi_0(M(\Sigma))$ respectively. It is obvious that $\mathcal{Y}_\Sigma = (V(\mathcal{Y}_\Sigma), E(\mathcal{Y}_\Sigma))$ equipped with the structures explained above satisfies all the conditions (Scw1)–(Scw4) of scwols (note that (Scw3) is now the empty condition).

![Figure 3. The scwol structure of $\mathcal{Y}_\Sigma$ on a 2-simplex of $Y_\Sigma$](image)

The complex of groups associated to $\Sigma$. We now endow $\mathcal{Y}_\Sigma$ with the natural structure of a complex of groups. Let us choose and fix a point $x_\lambda$ in $\lambda$ and define the local group $G^\Sigma_\lambda = G^\Sigma_{\sigma_\lambda}$ at $\sigma_\lambda$ as the fundamental group $\pi_1(\lambda, x_\lambda)$ (in the usual sense) for each element $\lambda$ of $\Lambda$ (recall that the label set $\Lambda$ consists of connected subspaces of $M$). We next associate a group homomorphism $\psi_a^\Sigma : G^\Sigma_{i(a)} \to G^\Sigma_{t(a)}$ to each edge $a$. Let $\lambda$ and $\mu$ be elements of $\Lambda$ satisfying $i(a) = \sigma_\lambda$ and $t(a) = \sigma_\mu$. The existence of the edge $a$ implies that the cell $e_\mu$ of $Y_\Sigma$ corresponding to $\sigma_\mu$ occurs in the boundary of the cell $e_\lambda$ corresponding to $\sigma_\lambda$, and in particular the intersection of $\bar{U}_\lambda$ and $\bar{U}_\mu$ is nonempty as we have already remarked (with the convention $U_N = \bar{U}_N = N$ for each element $N$ of $\pi_0(M(\Sigma))$). We may thus take a path $l_{\lambda,\mu} : [0, 1] \to \bar{U}_\lambda \cup \bar{U}_\mu$ satisfying $l_{\lambda,\mu}(0) = x_\lambda$ and $l_{\lambda,\mu}(1) = x_\mu$. We choose and fix such a path $l_{\lambda,\mu}$ for each edge $a$ with $i(a) = \sigma_\lambda$ and $t(a) = \sigma_\mu$. We may readily verify that $\mu$ is a deformation retract of $\bar{U}_\lambda \cup \bar{U}_\mu$ by the definition of $\bar{U}_\lambda$ as a tubular or bicollar neighbourhood, and therefore we may define a group homomorphism
\[ \psi_a^\Sigma : G^\Sigma_1 \rightarrow G^\Sigma_\mu \] as the composition
\[ G^\Sigma_A = \pi_1(\Lambda, x_\Lambda) \rightarrow \pi_1(\bar{u}_\Lambda \cup \bar{u}_\mu, x_\Lambda) \xrightarrow{(\bar{b})} \pi_1(\bar{u}_\Lambda \cup \bar{u}_\mu, x_\mu) \sim \pi_1(\mu, x_\mu) = G^\Sigma_\mu \]
where the first map is induced from the natural inclusion \( \lambda \hookrightarrow \bar{u}_\Lambda \cup \bar{u}_\mu \) and the last isomorphism is induced from a deformation retraction from \( \bar{u}_\Lambda \cup \bar{u}_\mu \) to \( \mu \). The middle map \((\bar{b})\) is the change of base points with respect to the path \( l_{1,\mu} \), or in other words, the map defined by \([c] \mapsto [l_{1,\mu}^{-1}cl_{1,\mu}]\).

Here we define the concatenation \( l_1l_2 \) of two paths \( l_1, l_2 : [0, 1] \rightarrow X \) in a topological space \( X \) with \( l_1(1) = l_2(0) \) as follows:
\[ l_1l_2(t) = \begin{cases} l_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ l_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \]

We finally define a twisting element \( g_{a,b}^\Sigma \) for each composable pair \((a, b)\) in \( E^{(2)}(Y_\Sigma) \). Suppose that the vertices \( i(b), t(b) = i(a) \) and \( t(a) \) are labeled by elements \( C \) of \( \pi_0(C(\Sigma)), S \) of \( \pi_0(S(\Sigma)) \) and \( N \) of \( \pi_0(M(\Sigma)) \) respectively. Then we define \( g_{a,b}^\Sigma \) as the image of \([l_{1,\mu}^{-1}l_{1,\mu}^{-1}]\) under the map \( \pi_1(N \cup \bar{u}_S \cup \bar{u}_C, x_N) \rightarrow \pi_1(N, x_N) = G_N^\Sigma \) induced by a deformation retraction from \( N \cup \bar{u}_S \cup \bar{u}_C \) to \( N \). The twisted commutativity
\[ g_{a,b}^\Sigma \psi_a^\Sigma([c])g_{a,b}^{-1} = \psi_a^\Sigma \circ \psi_b^\Sigma([c]) \]
straightforwardly holds for each element \([c]\) of \( G_N^\Sigma = \pi_1(C, x_C) \). We have now verified, combining Remark [1.2] with the calculations above, that \( G(Y_\Sigma) = (Y_\Sigma, \{\psi^\Sigma_{a,b} \}_{a,b} \in E^{(2)}(Y_\Sigma)) \) satisfies all the conditions of complexes of groups over \( Y_\Sigma \) except for injectivity of each \( \psi^\Sigma_a \). If we further assume that the tribranched surface \( \Sigma \) under consideration is essential, we readily observe that every \( \psi^\Sigma_a \) is injective due to the condition (ETBS2) and the twisted commutativity (2.2). As a consequence, the triple \( G(Y_\Sigma) \) is indeed a 2-complex of groups over the scwol \( Y_\Sigma \) when \( \Sigma \) is essential, which we call the complex of groups associated to the essential tribranched surface \( \Sigma \).

Let us choose and fix a point \( x_0 \) in \( M(\Sigma) \) and a path \( l_\Lambda : [0, 1] \rightarrow M \) for each element \( \lambda \) of \( \Lambda \) such that \( l_\Lambda(0) = x_0 \) and \( l_\Lambda(1) = x_\lambda \). We define a morphism \( \phi_\Sigma : G(Y_\Sigma) \rightarrow \pi_1(M, x_0) \) as follows. For each label \( \lambda \), we define a group homomorphism \( \phi_{\Sigma,\lambda} : G^\Sigma_A = \pi_1(\Lambda, x_\Lambda) \rightarrow \pi_1(M, x_0) \),
\[ G^\Sigma_A = \pi_1(\Lambda, x_\Lambda) \rightarrow \pi_1(\bar{u}_\Lambda \cup \bar{u}_\mu, x_\Lambda) \xrightarrow{(\bar{b})} \pi_1(\bar{u}_\Lambda \cup \bar{u}_\mu, x_\mu) \sim \pi_1(\mu, x_\mu) = G^\Sigma_\mu \]
where the first map is induced by the natural inclusion \( \lambda \hookrightarrow M \) and the second map \((\bar{b})\) is the change of the base point with respect to the path \( l_\lambda \), that is, the map defined as \([c] \mapsto [l_{1,\mu}^{-1}l_{1,\mu}]\).

We also associate an element \( \phi_\Sigma(a) \) of \( \pi_1(M, x_0) \) defined as \([l_{1,\mu}^{-1}l_{1,\mu}^{-1}]\) to each edge \( a \) of \( Y_\Sigma \) when \( i(a) \) and \( t(a) \) are denoted by \( \sigma_a \) and \( \sigma_\mu \) respectively. Then the twisted commutativity
\[ \phi_\Sigma(a)\phi_{\Sigma,\lambda}([c])\phi_\Sigma(a)^{-1} = \phi_{\Sigma,\mu} \circ \psi_a^\Sigma([c]) \]
straightforwardly holds for each element \([c]\) of \( G_N^\Sigma = \pi_1(\Lambda, x_\Lambda) \) by the construction of \( \phi_\Sigma(a) \). Furthermore, for each composable pair \((a, b)\) of \( E^{(2)}(Y_\Sigma) \) satisfying \( i(b) = \sigma_C, t(b) = i(a) = \sigma_5 \) and \( t(a) = \sigma_N \), one may readily verify the equation 
\[ \phi_{\Sigma,N}(g_{a,b}^\Sigma)\phi_\Sigma(ab) = \phi_\Sigma(a)\phi_\Sigma(b) \]
by direct calculation. Therefore \( \phi_\Sigma = \{\phi_{\Sigma,\lambda} \}_{\lambda \in \Lambda}, \{\phi_\Sigma(a) \}_{a \in E(Y_\Sigma)} \) defines a morphism from \( G(Y_\Sigma) \) to \( \pi_1(M, x_0) \).

Let \( \sigma_0 \) denote the unique vertex of \( Y_\Sigma \) whose corresponding connected component \( N_0 \) of \( M(\Sigma) \) contains \( x_0 \). Then the morphism \( \phi_\Sigma : G(Y_\Sigma) \rightarrow \pi_1(M, x_0) \) induces a homomorphism
Kampen theorem then implies that the canonical homomorphisms $\phi_{\Sigma,*}: \pi_1(G(Y_\Sigma),\sigma_0) \to \pi_1(M,x_0)$ on the fundamental groups (refer to [BH99, Chapter III. Proposition 3.6]).

**Proposition 2.5.** Let $\Sigma$ be an essential tribranched surface in $M$. Then the homomorphism $\phi_{\Sigma,*}: \pi_1(G(Y_\Sigma),\sigma_0) \to \pi_1(M,x_0)$ induced form the morphism $\phi_{\Sigma}: G(Y_\Sigma) \to \pi_1(M,x_0)$ is surjective.

**Proof.** Let $\ast_{i\in A} \pi_1(\lambda, x_i)$ denote the free product of all local groups $\pi_1(\lambda, x_i)$. The Seifert-van Kampen theorem then implies that the canonical homomorphisms $\pi_1(\lambda, x_i) \to \pi_1(M, x_0)$ induce an surjection $\ast_{i\in A} \pi_1(\lambda, x_i) \to \pi_1(M, x_0)$. On the other hand, there exists a natural quotient map $\ast_{i\in A} \pi_1(\lambda, x_i) \to \pi_1(G(Y_\Sigma),\sigma_0)$ by the construction of $\pi_1(G(Y_\Sigma),\sigma_0)$. The homomorphism $\phi_{\Sigma,*}$ is compatible with these homomorphisms, and thus $\phi_{\Sigma,*}$ is also surjective.

**Remark 2.6.** The surjectivity of $\phi_{\Sigma,*}$ implies that the development $D(Y_\Sigma,\phi_{\Sigma})$ is connected (see [BH99 Chapter III.C 3.14]).

Due to the surjectivity of $\phi_{\Sigma,*}$, we may say that the 3-manifold group $\pi_1(M, x_0)$ has a nontrivial presentation in terms of $\pi_1(G(Y_\Sigma),\sigma_0)$; in other words, $\pi_1(M, x_0)$ admits a splitting with respect to the fundamental group $\pi_1(G(Y_\Sigma),\sigma_0)$ of the 2-complex of groups $G(Y_\Sigma)$. In the next subsection we study when the induced homomorphism $\phi_{\Sigma,*}$ is injective (and is thus an isomorphism).

### 2.3. Strongly essential tribranched surfaces.

In order to describe a condition for the induced homomorphism $\phi_{\Sigma,*}$ to be injective, we here introduce the notion of strongly essential tribranched surfaces.

**Definition 2.7** (Strongly essential tribranched surfaces). Let $\Sigma$ be an essential tribranched surface contained in $M$. We say that $\Sigma$ is **strongly essential** if it satisfies the following additional condition besides the conditions (ETBS1), (ETBS2) and (ETBS3) in Definition2.2

(ETBS4) for each connected component $N$ of $M(\Sigma)$, the natural functorial homomorphism $\pi_1(N) \to \pi_1(M)$ is injective.

In the rest of this section we consider a strongly essential tribranched surface $\Sigma$ contained in a 3-manifold $M$. Due to the condition (ETBS4) and the twisted commutativity $[2.4]$ one readily verifies that the morphism $\phi_{\Sigma}: G(Y_\Sigma) \to \pi_1(M, x_0)$ defined in the previous subsection is injective on each local group $G(\Sigma)^\Sigma$, and thus the 2-complex of groups $G(Y_\Sigma)$ is developable due to Proposition1.3. In the following we shall verify that the development $D(Y_\Sigma,\phi_{\Sigma})$ of $G(Y_\Sigma)$ with respect to the morphism $\phi_{\Sigma}$ is not only connected but also simply connected by reconstructing it in another geometric manner (compare to the construction of trees associated to hypersurfaces in [Sh02 Section 1.4]).

**Geometric construction of a development.** Consider the universal cover $\tilde{M}$ of $M$ and let $\tilde{\Sigma}$ denote the preimage of $\Sigma$ under the universal covering map $p_{\tilde{M}}: \tilde{M} \to M$. Then one readily shows by using covering space theory that $\tilde{\Sigma}$ is also a tribranched surface, and the preimage $C(\tilde{\Sigma})$ of $C(\Sigma)$ under $p_{\tilde{M}}$ coincides with the set of branched points of $\tilde{\Sigma}$. Furthermore, for each connected component $\tilde{C}$ of $C(\tilde{\Sigma})$ in the preimage of a connected component $C$ of $C(\Sigma)$ under $p_{\tilde{M}}$, there exists a unique tubular neighbourhood $h_{\tilde{C}}: \tilde{C} \times D^2 \to \tilde{M}$ of $\tilde{C}$ in $\tilde{M}$ satisfying $p_{\tilde{M}}(h_{\tilde{C}}(x,t)) = h_\Sigma(p_{\tilde{M}}(x),t)$. We define $U_{\tilde{C}}$ as an open subspace $h_{\tilde{C}}(\tilde{C} \times \bar{D}^2)$ and set $S(\tilde{\Sigma}) = \tilde{\Sigma} \setminus \bigcup_{\tilde{C} \in \pi_0(C(\tilde{\Sigma}))} U_{\tilde{C}}$. Then, for each connected component $\tilde{S}$ of $S(\tilde{\Sigma})$ in the
preimage of a connected component $S$ of $S(\Sigma)$ under $p_\tilde{M}$, there exists a unique bicollar neighbourhood $h_\Sigma : \tilde{S} \times [-1, 1] \to \tilde{M} \setminus \bigcup \tilde{\mathcal{C}}_{\text{end}(\Sigma)} U_\Sigma$ of $S$ in $\tilde{M} \setminus \bigcup \tilde{\mathcal{C}}_{\text{end}(\Sigma)} U_\Sigma$ satisfying $p_\tilde{M}(h_\Sigma(x, t)) = h_\Sigma(p_\tilde{M}(x), t)$. We define $U_\Sigma$ as an open subspace $h_\Sigma(S \times (-1, 1))$, and define $M(\Sigma)$ as the complement of $\bigcup \tilde{\mathcal{C}}_{\text{end}(\Sigma)} U_\Sigma \cup \bigcup \tilde{\mathcal{C}}_{\text{end}(\Sigma)} U_\Sigma$ in $\tilde{M}$. We remark that $S(\Sigma)$ and $M(\Sigma)$ coincide with the preimages of $S(\Sigma)$ and $M(\Sigma)$ under $p_\tilde{M}$ respectively. We now endow $\tilde{M}$ with an equivalence relation $\sim_\Sigma$ and construct a combinatorial CW-complex $Y_\Sigma$ of dimension 2 as the quotient space $Y_\Sigma = \tilde{M} / \sim_\Sigma$, in the completely same manner as the construction of $Y_\Sigma$. By definition there exists a quotient map $r_\Sigma : \tilde{M} \to Y_\Sigma$, and it is easy to construct a continuous map $i_\Sigma : Y_\Sigma \to \tilde{M}$ such that $r_\Sigma \circ i_\Sigma$ is homotopic to the identity map on $Y_\Sigma$. The composition of the induced maps

$$\pi_1(Y_\Sigma) \xrightarrow{i_\Sigma} \pi_1(\tilde{M}) \xrightarrow{r_\Sigma} \pi_1(Y_\Sigma)$$

is thus the identity map. On the other hand the fundamental group $\pi_1(\tilde{M})$ of the universal cover $\tilde{M}$ is trivial. Consequently $\pi_1(Y_\Sigma)$ is also trivial, or in other words, $Y_\Sigma$ is simply connected. Note that the simply connected combinatorial CW-complex $Y_\Sigma$ admits an action of $\pi_1(M, x_0)$ induced from its natural action on $\tilde{M}$. Moreover one readily checks by construction that the induced action of $\pi_1(M, x_0)$ on $Y_\Sigma$ satisfies the following property;

$$(\star) \quad \text{an element } \gamma \text{ of } \pi_1(M, x_0) \text{ pointwise fixes a cell } e_\lambda \text{ of dimension 1 or 2 if it stabilises } e_\lambda.$$  

Now let $Y_\Sigma$ denote the scwol associated to $Y_\Sigma$, which is constructed in the same manner as $Y_\Sigma$. Due to $(\star)$, the action of $\pi_1(M, x_0)$ on $Y_\Sigma$ induces its action on the scwol $Y_\Sigma$.

**Proposition 2.8.** The 2-complex of groups $G(Y_\Sigma)$ is isomorphic to the complex of groups associated to the action of $\pi_1(M, x_0)$ on the scwol $Y_\Sigma$ constructed as above, and the morphism $\phi_\Sigma : G(Y_\Sigma) \to \pi_1(M, x_0)$ coincides with the morphism associated to this action (up to homotopy). In particular, the scwol $Y_\Sigma$ is $\pi_1(M, x_0)$-equivariantly isomorphic to the development $D(Y_\Sigma, \phi_\Sigma)$ of $G(Y_\Sigma)$ with respect to $\phi_\Sigma$.

**Proof.** Recall that $p_\tilde{M} : \tilde{M} \to M$ denotes the universal cover of $M$. Take an arbitrary point $\tilde{x}_0$ from $p_\tilde{M}^{-1}(x_0)$. For each $\lambda$ in $\Lambda$, let $\tilde{l}_\lambda$ denote a unique lift of $l_\lambda$ to $\tilde{M}$ satisfying $\tilde{l}_\lambda(0) = \tilde{x}_0$. We set $\tilde{x}_\lambda = \tilde{l}_\lambda(1)$ and denote by $\tilde{l}_\lambda$ a unique connected component of $p_\tilde{M}^{-1}(\lambda)$ containing $\tilde{x}_\lambda$. Note that $\tilde{x}_\lambda$ is a lift of $x_\lambda$ to $\tilde{M}$. We shall verify that all the data of which the complex of groups $G(Y_\Sigma)$ consists (specifically the local groups $G^x_\lambda$, the local homomorphisms $\psi_\lambda^x$ and the twisting elements $s_\alpha^\Sigma$) are obtained from the action of $\pi_1(M, x_0)$ on the scwol $Y_\Sigma$.

Via the monodromy homomorphism and the parallel translation, we may identify $\pi_1(M, x_0)$ with the automorphism group of $p_\tilde{M}^{-1}(x_\lambda)$. On the other hand, since $\tilde{l}_\lambda \to \lambda$ is also a covering space, one readily observes that the isotropy subgroup $\pi_1(M, x_0)_\lambda \cong \text{Aut}(p_\tilde{M}^{-1}(x_\lambda))_\lambda$ at $\tilde{l}_\lambda$ coincides with the image of $\pi_1(\lambda, x_\lambda)$ in $\pi_1(M, x_0)$ under the map (2.3). The morphism (2.3) is injective due to the condition (ETBS2), and we may thus conclude that $G^x_\lambda = \pi_1(\lambda, x_\lambda)$ is the isotropy subgroup of $\pi_1(M, x_0)$ at $\tilde{l}_\lambda$ (or $\sigma_\lambda$) with respect to the natural action of $\pi_1(M, x_0)$ on $\tilde{M}$ (or on $Y_\Sigma$).

Next let $a$ be an edge of $Y_\Sigma$ and denote its initial and terminal vertices by $\sigma_\lambda$ and $\sigma_\mu$ respectively. Let $\tilde{a}$ be a unique edge of $Y_\Sigma$ which is a lift of $a$ and satisfies $i(\tilde{a}) = \sigma_\lambda$. We may identify
with a unique lift \( \tilde{I}_{a,\mu} \) of \( I_{a,\mu} \) to \( \tilde{M} \) satisfying \( \tilde{I}_{a,\mu}(0) = \tilde{a} \) up to homotopy. Then by construction, an element \( h_a \) of \( \text{Aut}(\pi_1(M, x_0)) \) satisfying \( h_a \cdot t(\tilde{a}) = \sigma_\mu \) is none other than the parallel translation along the path \( I_{a,\mu}^{-1} \tilde{I}_{a,\mu} \). Via the change of base point \( \pi_1(M, x_0) \to \pi_1(M, x_0) \) appearing in (2.3), we may regard \( h_a \) as an element of \( \pi_1(M, x_0) \) defined as \( [I_{a,\mu}^{-1} I_{a,\mu}] \). Furthermore the image of an element \( \xi \) of \( \tilde{G}^\Sigma = \pi_1(\lambda, x_0) \) in \( \pi_1(M, x_0) \) under the map (2.3) is \( [\lambda] \xi [\lambda]^{-1} \), and we may thus calculate

\[
 h_a \xi h_a^{-1} = [\lambda_I^{-1} \lambda_{a,\mu}^{-1}] ([I_{\lambda_I} \xi [\lambda_I^{-1}]) [I_{a,\mu} I_{\lambda_I}^{-1}] = [\lambda_I^{-1} I_{a,\mu} I_{\lambda_I}^{-1}] \xi [I_{a,\mu} I_{\lambda_I}^{-1}],
\]

which is regarded as an element of \( \pi_1(\mu, x_\mu) \) defined by \( [\lambda_I^{-1} I_{a,\mu} I_{\lambda_I}^{-1}] \xi [I_{a,\mu} I_{\lambda_I}^{-1}] \) via the injection (2.3). Similarly we may calculate as

\[
 h_a h_b h_a^{-1} = [\lambda_I^{-1} I_{\lambda_I}^{-1}] ([I_{\lambda_I} \xi [\lambda_I^{-1}]) [I_{a,\mu} I_{\lambda_I}^{-1}] = [\lambda_I^{-1} I_{\lambda_I}^{-1}] \xi [I_{a,\mu} I_{\lambda_I}^{-1}],
\]

for composable edges \( a \) and \( b \). Here \( C, S \) and \( N \) denote elements of \( \pi_0(C(\Sigma)), \pi_0(S(\Sigma)) \) and \( \pi_0(M(\Sigma)) \) respectively such that \( i(b) = \sigma_C, t(b) = i(a) = \sigma_S \) and \( t(a) = \sigma_N \) hold. Moreover the equality

\[
 \phi_\Sigma(a) = [\lambda_I^{-1} I_{a,\mu} I_{\lambda_I}^{-1}] = h_a
\]

obviously holds for an edge \( a \) with \( i(a) = \sigma_\lambda \) and \( t(a) = \sigma_\mu \). Therefore, under the specific choices of a lift \( \tilde{\sigma} \) of each vertex \( \sigma \) of \( \Sigma \) and an element \( h_a \) of \( \pi_1(M, x_0) \) for each edge \( a \) of \( \Sigma \) as

\[
 \tilde{\sigma}_\lambda = \sigma_\lambda \quad (\lambda \in \Lambda), \quad h_a = [\lambda_I^{-1} I_{a,\mu} I_{\lambda_I}^{-1}],
\]

for an edge \( a \) with \( i(a) = \sigma_\lambda, t(a) = \sigma_\mu \), the complex of groups \( G(\Sigma) \) is indeed the one associated to the action of \( \pi_1(M, x_0) \) on the scwol \( \Sigma \), and \( \phi_\Sigma \) is the associated morphism.

The rest of the statement is then a direct consequence of [BH99] Chapter III.C Theorem 2.13 (2). □

Note that the combinatorial CW-complex \( Y_\Sigma \) of dimension 2 is regarded as the geometric realisation of the scwol \( \Sigma \). Since the geometric realisation \( |\Sigma| = Y_\Sigma \) of \( \Sigma \) is simply-connected as we have observed, so is \( Y_\Sigma \) itself due to Proposition 1.1. Consequently the scwol \( Y_\Sigma \) is connected and simply connected, and Proposition 2.8 implies that the development \( D(\Sigma, \phi_\Sigma) \) of \( G(\Sigma) \) with respect to \( \phi_\Sigma \) is also connected and simply connected. Then by basic facts of covering space theory on complexes of groups (see [BH99] Chapter III.C 3.14 (2) for details), we obtain the following result.

**Theorem 2.9.** Let \( \Sigma \) be a strongly essential tribranched surface contained in a compact, connected, irreducible and orientable 3-manifold \( M \). Then the morphism \( \phi_\Sigma : G(\Sigma) \to \pi_1(M, x_0) \) constructed in Section 2.2 induces a group isomorphism \( \phi_{\Sigma, *} : \pi_1(G(\Sigma), \sigma_0) \to \pi_1(M, x_0) \).

3. **Bruhat-Tits buildings**

**Bruhat-Tits buildings** are combinatorial and topological objects associated to reductive algebraic groups defined over non-archimedean valued fields, which behave as Riemannian symmetric spaces in differential geometry; in particular they admit natural “transitive” actions of the algebraic groups (to be precise, the natural group actions on the Bruhat-Tits buildings
are strictly transitive; see the end of Section [3.1] for the definition of strict transitivity). The theory of Bruhat-Tits buildings has its origin in the study of Nagayoshi Iwahori and Hideya Matsumoto on the generalised Bruhat decomposition of $p$-adic Chevalley groups [IM65], and then it has been elaborated by François Bruhat and Jacques Tits in a systematic and axiomatic way [BT72, BT84]. The Bruhat-Tits tree, which appears in the work of Culler and Shalen [CS83], is none other than the Bruhat-Tits building associated to the special linear group $SL(2)$ of degree 2, and the Bruhat-Tits buildings associated to the special linear groups of higher degree play crucial roles in our extension of Culler and Shalen’s results. In this section we shall summarise basic notion on Bruhat-Tits buildings and their fundamental properties especially for the special linear groups.

3.1. Euclidean buildings and their contractibility. We first review the axiomatic definition of (Euclidean) buildings after Tits and basic properties of Euclidean buildings. Refer, for instance, to [AB08, Ga97] for details of the contents of this subsection.

Definition 3.1 (Chamber complexes). Let $\Sigma$ be an abstract simplicial complex of finite dimension (that is, every simplex of $\Sigma$ is of finite dimension). We call $\Sigma$ a chamber complex if the following two conditions are fulfilled:

(CC1) every maximal simplex of $\Sigma$ has the same dimension $n$;
(CC2) every two maximal simplices $C$ and $C'$ are connected by a gallery; that is, there exists a sequence of maximal simplices $C_0 = C$, $C_1, \ldots, C_r = C'$ of $\Sigma$ such that $C_{i-1}$ and $C_i$ are adjacent for each $1 \leq i \leq r$.

Here we say that maximal simplices $C$ and $C'$ of $\Sigma$ are adjacent if $C$ and $C'$ are distinct and contain a common $(n-1)$-dimensional face. A maximal simplex of $\Sigma$ is called a chamber of $\Sigma$. The dimension of $\Sigma$ is defined as the (same) dimension $n$ of a chamber of $\Sigma$. A chamber complex $\Sigma$ of dimension $n$ is said to be thin if every $(n-1)$-dimensional simplex of $\Sigma$ is a face of exactly two chambers.

Definition 3.2 (Buildings). Let $\Delta$ be an abstract simplicial complex. We call $\Delta$ a (simplicial, thick) building of dimension $n$ if there exists a family $\mathcal{A}$ of $n$-dimensional thin chamber subcomplexes of $\Delta$ and the pair $(\Delta, \mathcal{A})$ satisfies the following axioms:

(B0) the complex $\Delta$ is (set-theoretically) expressed as the union of all elements of $\mathcal{A}$, and each $(n-1)$-dimensional simplex of $\Delta$ is a face of at least three maximal simplices of $\Delta$ (which are of dimension $n$);
(B1) every two simplices of $\Delta$ are contained in a single chamber subcomplex of $\Delta$ belonging to $\mathcal{A}$;
(B2) if $\Sigma$ and $\Sigma'$ are elements of $\mathcal{A}$ both of which contain two simplices $\sigma$ and $\tau$, there exists an isomorphism $\Sigma \cong \Sigma'$ of chamber complexes which fixes all the vertices of $\sigma$ and $\tau$.

A thin chamber subcomplex $\Sigma$ of $\Delta$ belonging to $\mathcal{A}$ is called an apartment of $\Delta$, and a maximal simplex of $\Delta$ is called a chamber of $\Delta$. Among families of thin chamber subcomplexes of $\Delta$ satisfying all the axioms (B0), (B1) and (B2), there exists a unique maximal one $\mathcal{A}^{cpl}$ which is called the complete system of apartments of $\Delta$.

It is well known that a building $\Delta$ of dimension $n$ is a colorable chamber complex; namely there exists a (set-theoretical) $I_{n+1}$-valued function $\tau$ on the vertices of $\Delta$ such that the vertices
of each chamber of $\Delta$ are mapped bijectively onto $I_{n+1}$, where $I_{n+1}$ denotes a finite set of cardinality $n + 1$. Such a function $\tau$ is called a type function on $\Delta$ (with values in $I_{n+1}$). We refer the reader to [AB08, Proposition 4.6] for details.

**Definition 3.3** (Euclidean buildings). A building $\Delta$ of dimension $n$ is said to be a Euclidean building (or a building of affine type) if the geometric realisation of each apartment of $\Delta$ is isomorphic to the standard tessellation of the $n$-dimensional (real) Euclidean space by equilateral $n$-dimensional simplices (more precisely, we require that each apartment should be isomorphic to a Euclidean Coxeter complex).

Now let $\Delta$ be a Euclidean building. For arbitrary two points $x$ and $y$ of the geometric realisation $|\Delta|$ of $\Delta$, there exists an apartment $\Sigma_{(x,y)}$ of $\Delta$ whose geometric realisation $|\Sigma_{(x,y)}|$ contains both of $x$ and $y$ due to the axiom (B1) of buildings. We equip $|\Sigma_{(x,y)}|$ with the standard Euclidean metric $d_{\Sigma_{(x,y)}}$, and define a real-valued function $d_{|\Delta|}$ on $|\Delta| \times |\Delta|$ by

$$d_{|\Delta|}: |\Delta| \times |\Delta| \to \mathbb{R}_{\geq 0}; \ (x, y) \mapsto d_{\Sigma_{(x,y)}}(x, y).$$

Then $d_{|\Delta|}$ is a metric on the geometric realisation $|\Delta|$ of $\Delta$ which is well defined independently of the choice of an apartment $\Sigma_{(x,y)}$ due to the axiom (B2) of buildings. One readily checks that the topology of $|\Delta|$ determined by the metric $d_{|\Delta|}$ coincides with the weak topology endowed on $|\Delta|$. Bruhat and Tits have verified that the metric space $(|\Delta|, d_{|\Delta|})$ is a CAT(0) space; in particular $|\Delta|$ is contractible (refer to [BT72, Propositions 2.5.3. et 2.5.16] for details; see also [AB08, the proof of Theorem 11.16]). The contractibility of Euclidean buildings shall play a crucial role in the construction of tribranched surfaces in Section 4.2.

We shall end this subsection by presenting several notion concerning group actions on buildings. Let $G$ be an abstract group and $\Delta$ a building on which $G$ acts. One easily verifies that the action of $G$ on $\Delta$ induces actions of $G$ both on the complete system of apartments $\mathcal{A}^{pl}$ of $\Delta$ and on the set of all the chambers of $\Delta$. An action of a group $G$ on a building $\Delta$ is said to be strictly transitive if $G$ acts transitively on the set of all pairs $(\Sigma, C)$ consisting of an apartment $\Sigma$ (belonging to $\mathcal{A}^{pl}$) and a chamber $C$ contained in $\Sigma$, and said to be type-preserving if an arbitrary element $g$ of $G$ maps a vertex of $\Delta$ to one of the same type (with respect to a certain type function on $\Delta$).

**3.2. Bruhat-Tits buildings associated to the special linear groups.** One of the most significant aspects in the theory of Euclidean buildings is the fact that one may associate in a canonical manner a Euclidean building $\mathcal{B}(G_F)$ to a reductive algebraic group $G$ defined over a non-archimedean valued field $F$. Furthermore $\mathcal{B}(G_F)$ admits a natural, strictly transitive action of $G(F)$. The existence of such Euclidean buildings was first observed in the pioneering work of Iwahori and Matsumoto [IM65] for Chevalley groups (which are in particular split, semisimple and simply connected algebraic groups) defined over $p$-adic fields. Then Bruhat and Tits established construction of such Euclidean buildings in [BT72, BT84] for general reductive algebraic groups. The Euclidean building $\mathcal{B}(G_F)$ attached to $G_F$ is therefore called the Bruhat-Tits building associated to $G_F$.

Bruhat and Tits’s construction of $\mathcal{B}(G_F)$ utilising “valuated root data” is rather abstract and complicated, but limiting ourselves to the Bruhat-Tits building $\mathcal{B}(G_F)$ associated to the special

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1More precisely, Iwahori and Matsumoto have constructed a (generalised) BN pair with respect to the Iwahori subgroup $B$ of a $p$-adic Chevalley group in [IM65, Proposition 2.2, Theorem 2.22]. Although they have never mentioned buildings in [IM65], it is well known that one may associate buildings to such BN-pairs in a canonical way; see [AB08, Theorem 6.56] for example.
linear group $G = SL(n)$ defined over a discrete valuation field (which is a $p$-adic Chevalley group and thus has been already dealt with by Iwahori and Matsumoto in [IM65]), we may explicitly describe the combinatorial structure of $\mathcal{B}(G/F)$ and the effect of the action of $G(F)$ on $\mathcal{B}(G/F)$ without introducing any root datum. We propose in this subsection a combinatorial description of the Bruhat-Tits building $\mathcal{B}(SL(n)/F)$ associated to the special linear group $SL(n)/F$, mainly following [Ga97, Chapter 19]. We shall only utilise the Bruhat-Tits buildings $\mathcal{B}(SL(n)/F)$ associated to the special linear groups in our later applications.

Let $F$ be a field equipped with a (normalised) discrete valuation $w: F^× \rightarrow \mathbb{Z}$. We do not require that the base field $F$ is complete with respect to the multiplicative valuation $|·|_w$ associated to $w$ (indeed we shall later apply results of this subsection to a case where the base field is not complete). We denote the valuation ring of $F$ with respect to $w$ by $O_w$. We fix a uniformiser $\sigma_w$ of the discrete valuation field $(F,w)$; in other words, we choose and fix a generator $\sigma_w$ of the maximal ideal of $O_w$ (which is known to be a principal ideal due to basic facts of valuation theory).

Let $V_n$ denote an $n$-dimensional vector space over $F$ equipped with a basis $\{e_1, \ldots, e_n\}$. We identify $V_n$ with $F^n$ (the $F$-vector space of $n$-dimensional column vectors) with respect to the specified basis $\{e_i\}_{i=1}^n$, and regard the special linear group $SL_n(F)$ as a subgroup of $\text{Aut}_F(V_n)$. An $O_w$-submodule $L$ of $V_n$ is called a lattice of $V_n$ if $L$ spans $V_n$ over $F$: $\langle L \rangle_F = V_n$. Every lattice of $V_n$ is then a free $O_w$-module of rank $n$ by elementary divisor theory. Two lattices $L$ and $L'$ of $V_n$ are said to be homothetic if there exists a nonzero element $a$ of $F$ such that $L$ coincides with $aL'$ (as an $O_w$-submodule of $V_n$). The homothety relation is an equivalence relation on the set of all lattices of $V_n$, and we define the vertex set $V(\mathcal{B}(SL(n)/(F,w)))$ of the Bruhat-Tits building $\mathcal{B}(SL(n)/(F,w))$ as the set of homothety classes of lattices of $V_n$. We say that two distinct elements $v$ and $v'$ of $V(\mathcal{B}(SL(n)/(F,w)))$ are adjacent if there exist lattices $L$ and $L'$ representing the homothety classes $v$ and $v'$ respectively such that

$$\sigma_w L' \subseteq L \subseteq L'$$

holds (as $O_w$-submodules of $V_n$). We then define $\mathcal{B}(SL(n)/(F,w))$ as an abstract simplicial complex each of whose simplices is a finite subset $\{v_1, \ldots, v_r\}$ of $V(\mathcal{B}(SL(n)/(F,w)))$ consisting of vertices adjacent to each other; in other words, a set $\{v_1, \ldots, v_r\}$ of $r$ vertices of $\mathcal{B}(SL(n)/(F,w))$ forms an $r$-simplex if and only if there exists a lattice $L_i$ representing $v_i$ for each $1 \leq i \leq r$ such that

$$\sigma_w L_r \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_r$$

holds (after appropriate relabeling of the subindices). For an arbitrary $F$-basis $f = \{f_1, \ldots, f_n\}$ of $V_n$, consider a subcomplex $\Sigma_f$ of $\mathcal{B}(SL(n)/(F,w))$ generated by the homothety classes of lattices of the form $\sum_{j=1}^n O_w \sigma_w^m f_j$ (each $m_j$ takes an arbitrary integer). The subcomplex $\Sigma_f$ is indeed a thin chamber complex of dimension $n - 1$. Denote by $\mathcal{A}$ the family of the subcomplexes $\Sigma_f$ of $\mathcal{B}(SL(n)/(F,w))$ indexed by an $F$-basis $f$ of $V_n$. Then we may readily verify that the pair $(\mathcal{B}(SL(n)/(F,w)), \mathcal{A})$ satisfies all the axioms (B0), (B1) and (B2) of buildings; see [Ga97, Chapter 19.2] for details. The special linear group $SL_n(F)$ acts on the set of lattices of $V_n$ in an obvious manner; namely, for a lattice $L = \sum_{j=1}^n O_w f_j$ with an $O_w$-basis $\{f_1, \ldots, f_n\}$, we define $gL$ as an $O_w$-submodule of $V_n$ spanned by $\{g(f_1), \ldots, g(f_n)\}$ (here we regard $g$ as an element of $\text{Aut}_F(V_n)$). This defines an action of $SL_n(F)$ on $V(\mathcal{B}(SL(n)/(F,w)))$, which is naturally extended to an action of $SL_n(F)$ on $\mathcal{B}(SL(n)/(F,w))$. One of the significant features of the action of $SL_n(F)$ on $\mathcal{B}(SL(n)/(F,w))$ is that it is a strictly transitive and type-preserving action. In particular, an element $\gamma$ of $SL_n(F)$ fixes all the vertices of a chamber $C$ whenever $\gamma$ stabilises $C$. 


In order to see that it is type-preserving, one has only to check that an association of a value \( \tau(v) = (w(\det g_v) \mod n) \) to each vertex \( v \) of \( \mathcal{B}(SL(n)/(F_w)) \) defines a type function \( \tau \) on \( \mathcal{B}(SL(n)/(F_w)) \) with values in \( \mathbb{Z}/n\mathbb{Z} \). Here \( g_v \) is an element of \( \text{Aut}_F(V_n) \) satisfying \( L = g_v(L_0) \) for a certain lattice \( L \) representing \( v \), and \( L_0 \) denotes the standard lattice of \( V_n \) defined as \( L_0 = \sum_{j=1}^n \mathbb{Q} e_j \). Then the type of a vertex of \( \mathcal{B}(SL(n)/(F_w)) \) does not change under the action of an element \( g \) of \( SL_n(F) \) since one has
\[
\tau(gv) = (w(\det(g_v)) \mod n) = (w(\det g) \mod n) + \tau(v) = \tau(v)
\]
by using \( \det(g) = 1 \).

**Remark 3.4.** The Bruhat-Tits building \( \mathcal{B}(GL(n)/(F_w)) \) associated to the general linear group \( GL(n)/(F_w) \) is completely the same one as \( \mathcal{B}(SL(n)/(F_w)) \). However, the natural action of \( GL_n(F) \) on \( \mathcal{B}(GL(n)/(F_w)) \) does not preserve the type function \( \tau(v) = (w(\det g_v) \mod n) \) introduced above since the \( \mathbb{Z} \)-valued function \( w \circ \det \) on \( GL_n(F) \) takes arbitrary value (indeed \( GL_n(F) \) acts transitively on the vertex set \( V(\mathcal{B}(GL(n)/(F_w))) \)). In order to guarantee that the natural action on the Bruhat-Tits building is type-preserving, we deal with the Bruhat-Tits building associated to the special linear group \( SL(n) \) rather than the Bruhat-Tits building associated to the general linear group \( GL(n) \). We shall effectively utilise the type-preserving property of the action when we consider the quotient complex \( \mathcal{B}_{n,\mathbb{D}}/\pi_1(M, x_0) \) in Section 4.3.

**Example 3.5 (Bruhat-Tits trees).** In the case where \( n \) equals 2, the construction of \( \mathcal{B}(SL(2)/(F_w)) \) explained above is none other than the classical construction of the Bruhat-Tits tree associated to \( SL(2, F) \), which is, for example, presented in [Se77, Chapitre II, Section 1]. Note that the Bruhat-Tits trees play crucial roles in the original work of Culler and Shalen [CS83].

4. **Construction of essential tribranched surfaces**

We shall establish our construction of essential tribranched surfaces in this section. There are two technical hearts in the construction. One is to obtain a nontrivial type-preserving action of the 3-manifold group on the Bruhat-Tits building associated to the special linear group \( SL(n) \) by utilising geometry of character varieties of higher degree. After a brief review on character varieties of higher degree in Section 4.1, we explain how to obtain such a nontrivial action in Section 4.2. The other is to construct a non-empty tribranched surfaces from such a nontrivial action. In Section 4.3, we put this procedure in practice, and then modify them to be essential by certain local surgery.

4.1. **\( SL_n(\mathbb{C}) \)-character variety.** We begin with briefly reviewing the \( SL_n(\mathbb{C}) \)-character variety of a finitely generated group. See Lubotzky and Magid [LM85] for more details.

Let \( \pi \) be a finitely generated group. We denote by \( R_n(\pi) \) the set \( \text{Hom}(\pi, SL_n(\mathbb{C})) \) of all the \( SL_n(\mathbb{C}) \)-representations of \( \pi \), which is an affine algebraic set. The algebraic group \( SL_n(\mathbb{C}) \) acts on \( R_n(\pi) \) by conjugation. We denote by \( X_n(\pi) \) the geometric invariant theoretical quotient of \( R_n(\pi) \) with respect to this action, which is called the \( SL_n(\mathbb{C}) \)-character variety of \( \pi \). We define the character \( \chi_\rho : \pi \to \mathbb{C} \) of a \( SL_n(\mathbb{C}) \)-representation \( \rho : \pi \to SL_n(\mathbb{C}) \) as \( \chi_\rho(\gamma) = \text{tr} \rho(\gamma) \) for each element \( \gamma \) of \( \pi \). The quotient variety \( X_n(\pi) \) is known to be realised as the set of characters \( \chi_\rho \) (in the set-theoretical sense), and under this identification the quotient map \( R_n(\pi) \to X_n(\pi) \) is regarded as the map which sends \( \rho \) to \( \chi_\rho \). For an element \( \gamma \) of \( \pi \), we define the trace function \( I_\gamma : X_n(\pi) \to \mathbb{C} \) associated to \( \gamma \) as \( I_\gamma(\chi_\rho) = \text{tr} \rho(\gamma) \), which is a regular function on \( X_n(\pi) \).

The following theorem is a direct consequence of the result of Procesi [P76].
Theorem 4.1 ([P76]). Let $\gamma_1, \ldots, \gamma_n$ be a generator system of $\pi$. Then the trace functions $\{I_{\gamma_1, \ldots, \gamma_k} : 1 \leq k \leq 2^n - 1\}$ give affine coordinates of $X_n(\pi)$.

For a compact 3-manifold $M$ we abbreviate $X_n(\pi_1(M))$ as $X_n(M)$ to simplify notation.

Remark 4.2. Let $M$ be a hyperbolic 3-manifold with $l$ torus cusps. Then we may consider a lift $\rho_0 : \pi_1(M) \to SL_2(\mathbb{C})$ of the holonomy representation with respect to the hyperbolic structure of $M$ [CS83, Proposition 3.1.1]. Menal-Ferrer and Porti [MFP12a, MFP12b] showed for general $n$ the following facts:

i) the character variety $X_n(M)$ is smooth at $X_{\omega_0, \rho_0}$;

ii) the irreducible component of $X_n(M)$ containing $X_{\omega_0, \rho_0}$ is of dimension $l(n - 1)$.

Here $\iota_n : SL_2(\mathbb{C}) \to SL_n(\mathbb{C})$ denotes an (arbitrary) irreducible representation. They also gave explicit local coordinates around $X_{\omega_0, \rho_0}$ [MFP12b]. When $n$ equals 2, these results had been already proved by Kapovich [K01] (see also Bromberg [Br04]).

Definition 4.3 (Ideal points). Suppose that $X_n(\pi)$ is of positive dimension and let us take an affine curve $C$ contained in $X_n(\pi)$. Let $\bar{C} \to C$ denote a desingularisation of a projective completion of $C$ so that $\bar{C}$ is the smooth projective model of $C$. A closed point $\bar{x}$ of $\bar{C}$ is called an ideal point of $C$ if the birational map $\bar{C} \to C$ above is undefined at $\bar{x}$.

Note that the notion of ideal points does not depend on the choices of projective completions and desingularisations in the definition (see [CS83 Section 1.3] for details). We also remark that there are only finitely many ideal points of $C$ on $\bar{C}$.

4.2. Nontrivial actions on Bruhat-Tits buildings. We discuss in this subsection how to obtain a nontrivial, type-preserving action of a finitely generated group $\pi$ on a Euclidean building. Such a nontrivial action gives rise to a nontrivial splitting of $\pi$, which shall play a central role in the construction of tribranched surfaces when $\pi$ is a 3-manifold group. Similarly to the arguments in [CS83 Section 2.2], we utilise geometry of the character variety associated to $\pi$ in order to obtain such an action.

Assume that the character variety $X_n(\pi)$ is of positive dimension and consider an affine curve $C$ in $X_n(\pi)$. Then we may take a lift $D$ of $C$ in $R_n(\pi)$. Namely $D$ is an affine curve contained in the inverse image of $C$ under the natural projection $pr_n : R_n(\pi) \to X_n(\pi)$ such that the restriction $pr_n|_D$ is not a constant morphism. The projection $pr_n|_D : D \to C$ induces a (surjective) regular morphism $pr_n|_D : D \to \bar{C}$ on the smooth projective models of $C$ and $D$, which sends the ideal points of $D$ to those of $\bar{C}$.

Recall that, by the definition of $R_n(\pi)$, each closed point $y$ of the affine variety $R_n(\pi)$ corresponds to a $SL_n(\mathbb{C})$-representation $\rho_n : \pi \to SL_n(\mathbb{C})$. We denote by $\mathbb{C}[R_n(\pi)]$ the affine coordinate ring of $R_n(\pi)$. Let $\rho_{\text{taut}} : \pi \to SL_n(\mathbb{C}[R_n(\pi)])$ denote the tautological representation of $\pi$; namely $\rho_{\text{taut}}(\gamma)$ is a regular $SL_n(\mathbb{C})$-valued function on $R_n(\pi)$ for each element $\gamma$ of $\pi$ whose value at a closed point $y$ of $R_n(\pi)$ is $\rho_n(y)$. Let $\rho_{\bar{D}} : \pi \to SL_n(\mathbb{C}(D))$ denote the composition of the tautological representation $\rho_{\text{taut}} : \pi \to SL_n(\mathbb{C}[R_n(\pi)])$ with

$$SL_n(\mathbb{C}(D)) \to SL_n(\mathbb{C}[D]) \hookrightarrow SL_n(\mathbb{C}(D)),$$

where the first map is induced by the natural embedding $D \hookrightarrow R_n(\pi)$. In the construction of $\rho_{\bar{D}}$, we identify $\mathbb{C}(D)$ with the field of rational functions of $\bar{D}$ due to the fact that $\bar{D}$ is birational to $D$ (this gives justification to the notation $\rho_{\bar{D}}$). We call $\rho_{\bar{D}}$ the tautological representation associated to the affine curve $D$. Now recall that a closed point $y$ of the smooth projective curve
\(\tilde{D}\) (possibly an ideal point of \(D\)) determines a discrete valuation \(w_x : \mathbb{C}(D)^* \to \mathbb{Z}; f \mapsto \text{ord}_x(f)\) on the field of rational functions \(\mathbb{C}(D)\) of \(\tilde{D}\) (that is, the order function at \(y\)). The Bruhat-Tits building associated to \((\tilde{D}, y)\) is then defined as \(\mathcal{B}_{n,\tilde{D},y} = \mathcal{B}(SL(n)/\mathbb{C}(D), w_y)\), which admits a canonical action of \(SL_n(\mathbb{C}(D))\). We thus obtain an action of \(\pi\) on the Bruhat-Tits building \(\mathcal{B}_{n,\tilde{D},y}\)

\[
\pi \overset{\rho_{\tilde{D}}}{\longrightarrow} SL_n(\mathbb{C}(D)) \overset{\text{canonical}}{\longrightarrow} \text{Aut}(\mathcal{B}_{n,\tilde{D},y})
\]

which is automatically type-preserving as we have already remarked in Section [3.2].

The following theorem is an analogue of Culler and Shalen’s “Fundamental Theorem” [CS83, Theorem 2.2.1] for representations of \(\pi\) of higher dimension.

**Theorem 4.4.** Let \(pr_n|_{\tilde{D}} : \tilde{D} \to \tilde{C}\) be as above and let \(y\) be a closed point of \(\tilde{D}\). Set \(x = pr_n|_{\tilde{D}}(y)\). Then the trace function \(I_y\) associated to an element \(\gamma\) of \(\pi\) is holomorphic at \(x\) if \(\gamma\) fixes a certain vertex of the Bruhat-Tits building \(\mathcal{B}_{n,\tilde{D},y}\) associated to \((\tilde{D}, y)\).

**Proof.** We first claim that \(I_y\) is holomorphic at \(x\) if and only if \(\text{tr} \rho_{\tilde{D}}(\gamma)\) is contained in the valuation ring \(O_x\) of \(\mathbb{C}(D)\) with respect to the valuation \(w_x = \text{ord}_x\). Indeed we may easily check that \(I_y\) coincides with \(\text{tr} \rho_{\tilde{D}}(\gamma)\) as an element of \(\mathbb{C}(C)(\mathbb{C}(D))\), and the holomorphy of \(I_y\) at \(x\) is equivalent to the non-negativity of the order of \(I_y\) at \(x\). The claim easily follows from these observations combined with the elementary fact that, for each element \(f\) of \(\mathbb{C}(C)\), the order \(\text{ord}_x(f)\) of \(f\) at \(x\) is non-negative if and only if \(w_x(f) = \text{ord}_x(f)\) is non-negative.

Let \(v_0\) denote the vertex of \(\mathcal{B}_{n,\tilde{D},y}\) represented by the standard lattice \(\sum_{j=1}^n O_y e_j\). The isotropy subgroup of \(SL_n(\mathbb{C}(D))\) at \(v_0\) is then calculated as \(Z(SL_n(\mathbb{C}(D)))SL_n(O_y)\). Here \(Z(SL_n(\mathbb{C}(D)))\) denotes the centre of \(SL_n(\mathbb{C}(D))\) and consists of scalar matrices \(aI_n\) where \(a\) is an \(n\)-th root of unity contained in \(\mathbb{C}(D)\). But the group of \(n\)-th roots of unity \(\mu_n(\mathbb{C}(D))\) contained in \(\mathbb{C}(D)\) is indeed contained in \(O_y\) because \(O_y\) is integrally closed in \(\mathbb{C}(D)\). Hence \(Z(SL_n(\mathbb{C}(D)))\) is a subgroup of \(SL_n(O_y)\) and the isotropic subgroup at \(v_0\) exactly coincides with \(SL_n(O_y)\).

Now assume that \(\gamma\) fixes a vertex \(v\) of \(\mathcal{B}_{n,\tilde{D},y}\). Then there exists an element \(g\) of \(\text{Aut}_G(D(V_n))\) satisfying \(gv_0 = v\) (recall that \(V_n\) denotes the \(n\)-dimensional \(\mathbb{C}(D)\)-vector space \(\sum_{j=1}^n \mathbb{C}(D)e_j\)). The isotropic subgroup of \(SL_n(\mathbb{C}(D))\) at \(v\) then coincides with \(gSL_n(O_y)g^{-1}\), and hence \(\rho_{\tilde{D}}(\gamma)\) is contained in the conjugate \(gSL_n(O_y)g^{-1}\) of \(SL_n(O_y)\). The trace function is invariant under conjugation, and we may thus conclude that \(\text{tr} \rho_{\tilde{D}}(\gamma)\) is contained in \(O_y\) as desired. \(\Box\)

As a direct consequence of Theorem 4.4, we may verify that the action of \(\pi\) associated to an ideal point of \(X_n(\pi)\) is nontrivial. Recall that an action of a group \(G\) on a simplicial complex \(\Delta\) is said to be nontrivial if, for every vertex \(v\) of \(\Delta\), the isotropic subgroup \(G_v\) of \(G\) at \(v\) is a proper subgroup of \(G\).

**Corollary 4.5.** Let \(\tilde{x}\) be an ideal point of an affine curve \(C\) contained in \(X_n(\pi)\) and \(\tilde{y}\) a lift of \(\tilde{x}\) (namely, an ideal point of a lift \(D\) of \(C\) satisfying \(pr_n|_{\tilde{D}}(\tilde{y}) = \tilde{x}\)). Then the associated action of \(\pi\) on \(\mathcal{B}_{n,\tilde{D},y}\) is nontrivial.

**Proof.** Let \(D\) be a lift of \(C\) in \(R_n(\pi)\). Striving for a contradiction, suppose that the action of \(\pi\) induced on \(\mathcal{B}_{n,\tilde{D},y}\) is trivial, or in other words, suppose that there exists a vertex \(v\) of \(\mathcal{B}_{n,\tilde{D},y}\) at which the isotropic subgroup of \(\pi\) coincides with the whole group \(\pi\). Theorem 4.4 then implies that the trace function \(I_y\) does not have a pole at \(\tilde{x}\) for every element \(\gamma\) of \(\pi\). In particular every affine coordinate function of \(C\) is holomorphic at \(\tilde{x}\) due to Theorem 4.1. The last assertion contradicts the fact that at least one coordinate function must have a pole at \(\tilde{x}\) (recall that we have chosen \(\tilde{x}\) from ideal points of \(C\)). \(\Box\)
we can extend a simplicial map to be a simplicial map by subdividing boundary. Take an arbitrary identified with a closed simplex as a CW-complex. We now define each vertex 1-dimensional subcomplex of the first barycentric subdivision of of exist a map of construct a for the purpose of this article.

Let us take a triangulation of a compact, connected, irreducible and orientable 3-manifold. We consider a type-ideal point gives an essential tribranched surface under certain conditions. Let be a triangulation of of M is non-empty when is strictly greater than 1, so that a tribranched surface in M.

Proof. The proof is divided into two parts. In the first part we show that the action of the 1-skeleton of a simplicial complex naturally induces the combinatorial CW-complex structureof M-equivariant simplicial map of as follows. First consider the case of dimension 2 is identified with a closed simplex as a CW-complex. We now define gives a tribranched surface Σ if there exists a map such that the tribranched surface Σ coincides with the inverse image of under f.

Theorem 4.7. Let n be a natural number greater than or equal to 3, and assume that the boundary of M is non-empty when n is strictly greater than 3. Then a nontrivial type-preserving action of on a Euclidean building B gives a tribranched surface Σ if there exists a map such that the tribranched surface Σ coincides with the inverse image of under f.

Proof. The proof is divided into two parts. In the first part we show that the action of on B gives a non-empty tribranched surface which is not necessarily essential, and in the second part we modify such a tribranched surface given by the action to be essential by local surgery.

Let us take a triangulation of M and consider the triangulation on M induced from it. We construct a -equivariant simplicial map as follows. First consider the case of n = 3 (in the case the 2-skeleton of B coincides with B itself since it is of dimension 2). For each vertex v of M, we choose a lift v of v in M and a vertex w of B. Then we define as for so that is -equivariant. Now assume that we have already constructed a -equivariant simplicial map on the (i - 1)-skeleton of M, and let us take an arbitrary i-simplex o of M. We may extend the restriction of onto oσ to a map on oσ due to the contractibility of the Euclidean building B. Moreover we can take to be a simplicial map by subdividing M (and M) if necessary. By continuing this procedure, we can extend to simplicial maps on , , and inductively, and obtain a desired simplicial map Next consider the case of n ≥ 4. Since M is non-empty by the

Remark 4.6. In the case where n equals 2, Culler and Shalen have also verified the converse of Theorem [4.4] in [CS83, Theorem 2.2.1]; namely, they have proved that if is holomorphic at x (or equivalently, if tr(ρ_D)(γ) is contained in O), there exists a vertex of B_n,D_3 which is fixed by the action of γ. When n is greater than or equal to 3, the converse of Theorem 4.4 does not hold at all in general (indeed one readily observes that the proof given in [CS83, Theorem 2.2.1] clearly collapses for matrices of higher rank). Theorem 4.4 is, however, sufficient to construct tribranched surfaces, and the failure of the converse of Theorem 4.4 does not cause any harm for the purpose of this article.

4.3. Ideal points of character varieties and tribranched surfaces. Now we show that an essential tribranched surface in a 3-manifold is constructed from a nontrivial type-preserving action of its fundamental group on a Euclidean building. Such an action is obtained from an ideal point of an affine curve in the character variety as in Section 4.2 and, consequently, an ideal point gives an essential tribranched surface under certain conditions.

Let B be a compact, connected, irreducible and orientable 3-manifold. We consider a type-preserving action of on a Euclidean building B. Then the simplicial complex structure of B naturally induces the combinatorial CW-complex structure of B/ on M, where, for each non-negative integer i, we denote by the i-skeleton of a simplicial complex K. In particular, each closed cell of the combinatorial CW-complex B/ of dimension 2 is identified with a closed simplex as a CW-complex. We now define Y(B/ to be the 1-dimensional subcomplex of the first barycentric subdivision of B consisting of all the barycentres of 1- and 2-simplices and all the edges connecting them. We say that a type-preserving action of on a Euclidean building B gives a tribranched surface Σ if there exists a map f: M → B/ such that the tribranched surface Σ coincides with the inverse image of Y(B/) under f.

Theorem 4.7. Let n be a natural number greater than or equal to 3, and assume that the boundary of M is non-empty when n is strictly greater than 3. Then a nontrivial type-preserving action of on a Euclidean building B of dimension n − 1 gives an essential tribranched surface in M.
assumption, we can take a 2-dimensional subcomplex $V$ which is a deformation retract of $M$. Denote by $\tilde{V}$ the preimage of $V$ under the universal covering map $\tilde{M} \to M$. We define $f|_{\tilde{V}(0)}$ and, by subdividing $M$ if necessary, we extend it to a $\pi_1(M)$-equivariant simplicial map $\tilde{f}|_{\tilde{V}}: \tilde{V} \to \mathcal{B}^{(2)}$ similarly to the case of $n = 3$. Note that the image of the extended map $\tilde{f}|_{\tilde{V}}$ is contained in the 2-skeleton $\mathcal{B}^{(2)}$ of $\mathcal{B}$ since $\tilde{V}$ is of dimension 2. By composing $\tilde{f}|_{\tilde{V}}$ with a deformation retraction $\tilde{M} \to \tilde{V}$, we obtain a desired map $\tilde{f}: \tilde{M} \to \mathcal{B}^{(2)}$.

We can slightly modify the above construction so that the restriction of $\tilde{f}: \tilde{M} \to \mathcal{B}^{(2)}$ to each simplex is a linear map. Denote by $f: M \to \mathcal{B}^{(2)}/\pi_1(M)$ the quotient of the simplicial map $\tilde{f}: \tilde{M} \to \mathcal{B}^{(2)}$ by $\pi_1(M)$, and set $\Sigma = f^{-1}(\mathcal{B}^{(2)}/\pi_1(M))$. We show that $\Sigma$ is a tribranched surface. It follows from the linearity that $(M, \Sigma)$ is locally homeomorphic to $(\mathbb{R}^3, Y \times [0, \infty))$ and that $\Sigma$ satisfies (TBS2). Let $C$ be an arbitrary component of $\tilde{C}(\Sigma)$, and consider a sufficiently small tubular neighbourhood $\nu(C)$ of $C$ in $M$ (note that $\nu(C)$ is denoted as $U_C$ in Section 2.2). The intersection $\nu(C) \cap \Sigma$ of $\nu(C)$ and $\Sigma$ naturally admits the structure of a fibre bundle over $C$ whose fibre is $Y$ (recall that we define the topological space $Y$ as

$$Y = \{ re^{\sqrt{-1} \theta} \in \mathbb{C} \mid r \in \mathbb{R}_{\geq 0} \text{ and } \theta = 0, \pm 2\pi/3 \}$$

in Section 2.1). We may identify $f(\nu(C) \cap \Sigma)$ with $Y$ so that $f(C)$ corresponds to $\{0\}$. Then since the inverse image of $\{0\}$ under $f$ is $C$, the topological space $f((\nu(C) \cap \Sigma) \setminus C)$ has 3 components, and so does $(\nu(C) \cap \Sigma) \setminus C$ by continuity of $f$. Therefore the fibre bundle $\nu(C) \cap \Sigma \to C$ above must be trivial, which implies that $\Sigma$ satisfies (TBS1). Note that the above construction of $\Sigma$ is far from being canonical since it depends on choices, for instance, of a triangulation of $M$ and a $\pi_1(M)$-equivariant simplicial map $\tilde{f}$.

Next we show that $\Sigma$ satisfies (ETBS1), which, in particular, implies that $\Sigma$ is non-empty. Striving for a contradiction, suppose that there exists a component $N$ of $M(\Sigma)$ such that the homomorphism $\pi_1(N) \to \pi_1(M)$ induced by the natural inclusion $N \hookrightarrow M$ is surjective. Let $N_0$ be a component of the preimage of $N$ under the universal covering map $\tilde{M} \to M$. Since $\tilde{f}(N_0)$ does not intersect $\mathcal{B}^{(2)}$ by construction, it is contained in the open star of a vertex $v$ of $\mathcal{B}^{(2)}$ in its barycentric subdivision. Obviously $N_0$ is a covering space over $N$, and thus the fundamental group $\pi_1(N)$ stabilises $N_0$. The image of the homomorphism $\pi_1(N) \to \pi_1(M)$ then also stabilises the open star of $v$ containing $\tilde{f}(N_0)$ due to the $\pi_1(M)$-equivariance of $\tilde{f}$, and it is, in particular, contained in the isotropic subgroup $\pi_1(M)_v$ of $\pi_1(M)$ at $v$. Hence we conclude that $\pi_1(M)_v$ coincides with the whole group $\pi_1(M)$, combining the arguments above with the assumption on the surjectivity of the homomorphism $\pi_1(N) \to \pi_1(M)$, which contradicts nontriviality of the action of $\pi_1(M)$ on $\mathcal{B}$.

As we have already mentioned at the beginning of the proof, the tribranched surface $\Sigma$ itself might not be essential. From now on we modify $\Sigma$ to be essential as the second part of the proof. For a tribranched surface $\Sigma$ given by the action of $\pi_1(M)$ on $\mathcal{B}$, we set

$$l(\Sigma) = \text{the number of components of } C(\Sigma),$$

$$m(\Sigma) = \sum_S (2 - \chi(S))^2,$$

$$n(\Sigma) = \text{the number of components of } \Sigma,$$

where the sum in the second equation runs over all components $S$ of $S(\Sigma)$. We see at once that these integers are all non-negative. We consider the triple $(l(\Sigma), m(\Sigma), n(\Sigma)) \in \mathbb{Z}^3$ with respect to the lexicographical order of $\mathbb{Z}^3$ as a complexity of a non-empty tribranched surface.
\[\Sigma.\] In the following we show that if \(\Sigma\) is not essential, there are operations of replacing \(\Sigma\) by another tribranched surface with lower complexity. Therefore a tribranched surface of minimal complexity given by the action of \(\pi_1(M)\) on \(\mathcal{B}\) must be essential.

Let us consider the case where \(\Sigma\) does not satisfy (ETBS2). First assume that there exists a pair of components \(C\) and \(S\) of \(C(\Sigma)\) and \(S(\Sigma)\) respectively such that the natural inclusion map between them induces a homomorphism \(\pi_1(C) \rightarrow \pi_1(S)\) which is not injective. This implies that \(S\) is a disk. Let \(S_1\) and \(S_2\) be the other components of \(S(\Sigma)\) whose boundary contain parallel copies of \(C\) as components (the surfaces \(S_1\) and \(S_2\) might coincide). Take a small neighbourhood \(B\) of \(S\) which is homeomorphic to a ball and intersects \(S_1\) and \(S_2\) in the collars of \(C\). Figure 4 illustrates a local picture of the neighbourhood \(B\). Choose properly embedded disks \(D_1\) and \(D_2\) in \(B\) bounding \(S_1 \cap \partial B\) and \(S_2 \cap \partial B\) respectively and not intersecting \(S\). We construct a map \(g : B \rightarrow \mathcal{B}(2)/\pi_1(M)\) such that \(g_{|\partial B} = f_{|\partial B}\) and that \(g^{-1}(Y(\mathcal{B}(2)/\pi_1(M))) = D_1 \cup D_2\) as follows.

Since \(g(\partial D_1)\) and \(g(\partial D_2)\) are contained in open edges of \(Y(\mathcal{B}(2)/\pi_1(M))\) near the vertex \(f(C)\), the maps \(g_{|\partial D_1}\) and \(g_{|\partial D_2}\) extend to \(D_1\) and \(D_2\) respectively so that \(g(D_1)\) and \(g(D_2)\) are also contained in the same open edges. The ball \(B\) is divided into 3 balls \(B_1, B_2\) and \(B_3\) by \(D_1\) and \(D_2\), where \(\partial B_1\) contains not \(D_2\) but \(D_1\), \(\partial B_2\) contains not \(D_1\) but \(D_2\), and \(\partial B_3\) contains both disks. There exists a unique 2-simplex of \(\mathcal{B}(2)\) which contains \(f(C)\) as its barycentre, and the open star of each of its 3 vertices contains one of \(g(\partial B_1 \setminus D_1)\), \(g(\partial B_2 \setminus D_2)\) and \(g(\partial B_3 \setminus (D_1 \cup D_2))\). We can thus extend \(g_{|\partial B_1}, g_{|\partial B_2}\) and \(g_{|\partial B_3}\) to \(B_1, B_2\) and \(B_3\) respectively so that all of \(g(B_1 \setminus D_1), g(B_2 \setminus D_2)\) and \(g(B_3 \setminus (D_1 \cup D_2))\) do not intersect \(Y(\mathcal{B}(2)/\pi_1(M))\). Then we see at once that the inverse images of \(Y(\mathcal{B}(2)/\pi_1(M))\) under the maps \(g_{|B_1}, g_{|B_2}\) and \(g_{|B_3}\) are \(D_1, D_2\) and \(D_1 \cup D_2\) respectively. Figure 5 illustrates the image \(g(B)\) of the neighbourhood \(B\) of \(S\). We now define \(f' : M \rightarrow \mathcal{B}(2)/\pi_1(M)\) so that \(f'_{|M \setminus B} = f{|M \setminus B}\) and \(f'|_B = g\). Then \(f'^{-1}(Y(\mathcal{B}(2)/\pi_1(M)))\) is another tribranched surface and has a lower complexity since \(l(\Sigma)\) decreases.

Next assume that there exists a pair of components \(S\) and \(N\) of \(S(\Sigma)\) and \(M(\Sigma)\) respectively such that the natural inclusion map between them induces a homomorphism \(\pi_1(S) \rightarrow \pi_1(N)\) which is not injective. By Dehn’s lemma, there exists a compressing disk \(D\) of \(S\) in \(N\). Take a small neighbourhood \(B\) of \(D\) which is homeomorphic to a ball and intersects an annulus in \(S\). Figure 6 illustrates a local picture of the neighbourhood \(B\). Choose properly embedded disks
Since $g(\partial D_1)$ and $g(\partial D_2)$ are contained in the open star of a vertex of $\mathcal{B}^{(2)}$ in its barycentric subdivision, we can extend $g|_{\partial D_1}$ and $g|_{\partial D_2}$ to $D_1$ and $D_2$ respectively so that $g(D_1)$ and $g(D_2)$ are contained in the same star. The ball $B$ is divided into 3 balls $B_1$, $B_2$, and $B_3$ by $D_1$ and $D_2$, where $\partial B_1$ contains not $D_2$ but $D_1$, $\partial B_2$ contains not $D_1$ but $D_2$, and $\partial B_3$ contains both disks. Since $g(\partial B_1 \setminus D_1)$ and $g(\partial B_2 \setminus D_2)$ are contained in the open star of a vertex of $\mathcal{B}^{(2)}$ in its barycentric subdivision, and since $g(\partial B_3 \setminus (D_1 \cup D_2))$ is contained in that of another vertex, we can extend $g|_{\partial B_1}$, $g|_{\partial B_2}$ and $g|_{\partial B_3}$ to $B_1$, $B_2$, and $B_3$ respectively so that $g(B_1 \setminus D_1)$, $g(B_2 \setminus D_2)$ and $g(B_3 \setminus (D_1 \cup D_2))$ do not intersect $Y(\mathcal{B}^{(2)}/\pi_1(M))$. Then we see at once that the inverse images of $Y(\mathcal{B}^{(2)}/\pi_1(M))$ under the maps $g|_{B_1}$, $g|_{B_2}$, and $g|_{B_3}$ are $D_1$, $D_2$, and $D_1 \cup D_2$ respectively. Figure 7 illustrates the image $g(B)$ of the neighbourhood $B$ of the compression disk $D$. Now we define $f': M \to \mathcal{B}^{(2)}/\pi_1(M)$ so that $f'|_{M\setminus B} = f|_{M\setminus B}$ and $f'|_B = g$. Set $\Sigma' = f'^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M)))$, which is another tribranched surface with the same $l(\Sigma')$ as $l(\Sigma)$. We show in the followings
that \( m(\Sigma') \) is strictly less than \( m(\Sigma) \), which implies that \( \Sigma' \) has a lower complexity than \( \Sigma \). Set \( S' = (S \setminus B) \cup D_1 \cup D_2 \). First suppose that \( S' \) is connected. Then we can calculate as

\[
m(\Sigma) - m(\Sigma') = (2 - \chi(S))^2 - (2 - \chi(S'))^2 = 4 + 4(2 - \chi(S')) > 0
\]

by using \( \chi(S') = \chi(S) + 2 \leq 2 \). Next suppose that \( S' \) has two components \( S'_1 \) and \( S'_2 \). Note that neither \( S'_1 \) nor \( S'_2 \) is a sphere. Then we can calculate as

\[
m(\Sigma) - m(\Sigma') = (2 - \chi(S))^2 - (2 - \chi(S')_1)^2 - (2 - \chi(S')_2)^2 = 2(2 - \chi(S'_1))(2 - \chi(S'_2)) > 0
\]

by using \( \chi(S'_1) + \chi(S'_2) = \chi(S) + 2, \chi(S'_1) < 2 \) and \( \chi(S'_2) < 2 \). In both the cases \( m(\Sigma') \) decreases from \( m(\Sigma) \), as desired.

Finally, we consider the case where \( \Sigma \) does not satisfy (ETBS3). Then we see as follows that, after eliminating a component of \( \Sigma \) contained in a ball in \( M \) or a collar of \( \partial M \), the resultant tribranched surface is also given by the action of \( \pi_1(M) \) on \( \mathcal{B} \). If there is a component of \( \Sigma \) contained in a ball \( B \), we can construct a map \( f' : M \to \mathcal{B}^{(2)}/\pi_1(M) \) such that \( f'|_{M \setminus B} = f|_{M \setminus B} \) and that \( f'(B) \) does not intersect \( Y(\mathcal{B}^{(2)}/\pi_1(M)) \) since \( f(\partial B) \) is contained in a contractible component of the complement of \( Y(\mathcal{B}^{(2)}/\pi_1(M)) \) in \( \mathcal{B}^{(2)}/\pi_1(M) \). If there is one contained in a collar of \( \partial M \), we set \( f' : M \to \mathcal{B}^{(2)}/\pi_1(M) \) to be the composition of a deformation retraction from \( M \) to the complement of the collar with the restriction of \( f \) to it. In both the cases the complexity of a new tribranched surface defined as \( f'^{-1}(Y(\mathcal{B}^{(2)}/\pi_1(M))) \) is lower than the original one’s, since \( l(\Sigma) \) and \( m(\Sigma) \) do not increase and \( n(\Sigma) \) decreases. The proof is now completed. \( \square \)

Remark 4.8. Since a tribranched surface of minimal complexity given by the action of \( \pi_1(M) \) on a Euclidean building is not necessarily unique, the construction of an essential tribranched surface in the proof is far from being canonical.

Now let us return to the settings in Section 4.2. Let \( \tilde{x} \) be an ideal point of a curve \( C \) in \( X_n(M) \) and let \( \tilde{y} \) be a lift of \( \tilde{x} \), which is an ideal point of a lift \( D \) of \( C \). We say that \( \tilde{x} \) gives an tribranched surface \( \Sigma \) if the associated action of \( \pi_1(M) \) on \( \mathcal{B}_{n,\tilde{D},\tilde{y}} \) gives \( \Sigma \). The following is the main theorem of this article, which is now a direct consequence of Corollary 4.5 and Theorem 4.7.
Theorem 4.9. Let \( n \) be a natural number greater than or equal to 3, and assume that the boundary \( \partial M \) of \( M \) is non-empty when \( n \) is strictly greater than 3. Then an ideal point of an affine algebraic curve in \( X_n(M) \) gives an essential tribranched surface in \( M \).

5. An application to small Seifert manifolds

An advantage of extending Culler-Shalen theory to higher dimensional representations is that we may apply the extended theory also to a non-Haken 3-manifold, that is, a 3-manifold which does not contain any essential surfaces. Here we describe an application of Theorem 4.9 to a class of 3-manifolds called small Seifert manifolds, which contains non-Haken 3-manifolds. We remark that all the homology groups appearing in this section are singular homology groups.

A Seifert manifold is a compact, orientable 3-manifold admitting the structure of a Seifert fibred space whose base orbifold is a compact surface with cone points. A small Seifert manifold is a Seifert manifold with at most 3 singular fibres. We refer the reader to [J80, Chapter IV] for details on Seifert manifolds.

Let \( p, q \) and \( r \) be natural numbers greater than or equal to 3. We denote by \( S^2(p, q, r) \) the 2-sphere with three cone points whose cone angles are \( 2\pi/p, 2\pi/q \) and \( 2\pi/r \), and consider a small Seifert manifold \( M \) with the base orbifold \( S^2(p, q, r) \). Such a 3-manifold is known to be irreducible, and it is Haken if and only if its first homology group \( H_1(M, \mathbb{Z}) \) is infinite. The fundamental group \( \pi_1(M) \) has a presentation of the form

\[
\langle x, y, h \mid h: \text{central}, x^a = h^c, y^b = h^c, (xy)^r = h^c \rangle
\]

for certain integers \( a, b, c \) satisfying \( (a, p) = (b, q) = (c, r) = 1 \). The orbifold fundamental group \( \pi_1^{\text{orb}}(S^2(p, q, r)) \) of \( S^2(p, q, r) \) is isomorphic to the group \( \Delta(p, q, r) \) of the form

\[
\langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle,
\]

and by identifying \( \pi_1^{\text{orb}}(S^2(p, q, r)) \) with \( \Delta(p, q, r) \), we may regard the natural homomorphism \( \pi_1(M) \rightarrow \pi_1^{\text{orb}}(S^2(p, q, r)) \) induced by the projection \( M \rightarrow S^2(p, q, r) \) as the homomorphism which maps \( x \) and \( y \) identically and sends \( h \) to the unit (in particular it is a surjection). It is easily seen that the first homology group \( H_1(M, \mathbb{Z}) \) is infinite if and only if the equality

\[
\frac{a}{p} + \frac{b}{q} = \frac{c}{r}
\]

holds. From this observation we thus find that \( M \) tends to be non-Haken in most cases.

In the case where \( M \) is Haken, we may readily construct an affine curve in \( X_3(M) \) consisting of abelian characters because the first homology group \( H_1(M, \mathbb{Z}) \) is infinite. In the following we verify that \( X_3(M) \) contains an affine curve also in the case where \( M \) is non-Haken. It thus follows from Theorem 4.9 that an ideal point of the curve gives an essential tribranched surface \( \Sigma \) contained in \( M \), which one can never obtain by utilising classical Culler-Shalen theory (since the \( SL_2(\mathbb{C}) \)-character variety \( X_3(M) \) is of dimension 0 in the case).

The group \( \Delta(p, q, r) \) is regarded as a subgroup of index 2 of the Schwartzian triangle group \( \Gamma(p, q, r) \) of the form

\[
\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \rangle
\]

under the identification that \( x \) and \( y \) correspond with \( ab \) and \( bc \) respectively. It follows from the argument in [Go88, Section 6] that there exists a family of \( SL_2(\mathbb{C}) \)-representations
$\rho_3: \Gamma(p, q, r) \to SL_3(\mathbb{C})$ with parameter $s$ defined by

$$\rho_3(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2s \cos \frac{\pi}{p} & -1 & 0 \\ -2 \cos \frac{\pi}{r} & 0 & -1 \end{pmatrix},$$

$$\rho_3(b) = \begin{pmatrix} -1 & -2s^{-1} \cos \frac{\pi}{p} & 0 \\ 0 & 1 & 0 \\ 0 & -2 \cos \frac{\pi}{q} & -1 \end{pmatrix},$$

$$\rho_3(c) = \begin{pmatrix} -1 & 0 & -2 \cos \frac{\pi}{r} \\ 0 & -1 & -2 \cos \frac{\pi}{q} \\ 0 & 0 & 1 \end{pmatrix}.$$

(the representations above are minor modifications of the ones introduced in [Go88], where $\cos \frac{\pi}{p}$, $\cos \frac{\pi}{q}$ and $\cos \frac{\pi}{r}$ are replaced by $\cos \frac{2\pi}{p}$, $\cos \frac{2\pi}{q}$ and $\cos \frac{2\pi}{r}$ respectively in the matrices). A simple computation enables us to obtain the equation

$$\text{tr} \, \rho_3(abac) = 8(s + s^{-1}) \cos \frac{\pi}{p} \cos \frac{\pi}{q} \cos \frac{\pi}{r} + 16 \cos^2 \frac{\pi}{p} \cos^2 \frac{\pi}{q} + 4 \cos^2 \frac{\pi}{r} - 1,$$

which shows that the restrictions of $\rho_3$ to $\Delta(p, q, r)$ define a nontrivial curve contained in $X_3(\Delta(p, q, r))$. Since the natural homomorphism $\pi_1(M) \to \pi_1^{\text{orb}}(S^2(p, q, r))$ is surjective, the morphism $X_3(\pi_1^{\text{orb}}(S^2(p, q, r))) \to X_3(M)$ induced on the character varieties is an embedding. Therefore one readily sees that, by identifying $X_3(\Delta(p, q, r))$ with $X_3(\pi_1^{\text{orb}}(S^2(p, q, r)))$, the character variety $X_3(M)$ also contains a nontrivial curve.

6. Questions

We conclude with a list of questions. Let $M$ be a compact, connected, irreducible and orientable 3-manifold. It is known by Boyer and Zhang [BZ98], Motegi [Mo88], and Schanuel and Zhang [SZ01] that there exists an essential surface not given by any ideal points of any affine curves in $X_3(M)$ for a certain 3-manifold $M$. We may now propose the following important question:

**Question 6.1.** Does there exist an essential surface (without branched points) not given by any ideal points of any affine curves in $X_2(M)$ but given by an ideal point of an affine curve in $X_n(M)$ for $n \geq 3$?

Recall that an essential surface (without branched points) is also an essential tribranched surface in our terminology.

In Subsection 2.3 we have defined and discussed strongly essential tribranched surfaces.

**Question 6.2.** Under the same assumption as Theorem 4.9 is a strongly essential tribranched surface in $M$ also given by an ideal point of an affine curve in $X_n(M)$?

**Question 6.3.** Does the same conclusion as Theorem 4.9 hold without the assumption that the boundary $\partial M$ is non-empty when $n$ is strictly greater than 3?

Let $M$ be a small Seifert manifold whose base orbifold is $S^2(p, q, r)$ with $p, q, r \geq 3$ (recall the definitions from Section 5). Let us consider a $\theta$-graph $\Theta$ in $S^2(p, q, r)$, which has 2 vertices and 3 edges connecting them, so that all the cone points are separated by $\Theta$. Then it is straightforward
to see that the preimage $\Sigma_0$ of $\Theta$ in $M$ under the projection $M \to S^2(p, q, r)$ is an essential tribranched surface.

**Question 6.4.** Is an essential tribranched surface, which is given by an ideal point of the non-trivial curve considered in Section 5, isotopic to $\Sigma_0$?

The following question is concerning the characterisation of the class of 3-manifolds containing essential tribranched surfaces.

**Question 6.5.** Does every aspherical 3-manifold contain any essential surfaces or any strongly essential surfaces?

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