IRREDUCIBILITY OF GEOMETRIC GALOIS REPRESENTATIONS
AND THE TATE CONJECTURE FOR A FAMILY OF ELLIPTIC
SURFACES

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Abstract. Using Calegari’s result on the Fontaine-Mazur conjecture, we prove
the irreducibility of a 3-dimensional self-dual Galois representation comes from the
étale cohomology of a surface satisfying some conditions. As a consequence, we
prove that the Tate conjecture holds for a family of elliptic surfaces defined over \( \mathbb{Q} \)
with geometric genus bigger than 1.

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1. Introduction

Let \( \ell \) be a prime and \( K \) be a field. We let \( G_K \) denote the absolute Galois group of \( K \). The study of \( \ell \)-adic representations of \( G_K \) is not only interesting in theoretic research in number theory, but also has important application in arithmetic geometry. There are two natural sources of \( \ell \)-adic Galois representations. The first one arises from the Galois representations attached to algebraic automorphic representations. The second comes from algebraic geometry, i.e., those ones which are the subquotient of \( \ell \)-adic étale cohomology of a smooth projective variety. Both sources of Galois representations have several nice properties. Usually, we call such Galois representations geometric. (For a precise definition of geometric and other basic definitions related to Galois representations appeared in this introduction, we refer the reader to the Section 2 of this paper.) To study the geometric \( \ell \)-adic representations, one natural question we can ask is the following.

**Question 1.1.** How to tell whether a geometric Galois representation is (absolutely) irreducible?

It is a conjecture that the geometric Galois representations associated to algebraic cuspidal automorphic representations are irreducible. One can find some results in this direction (see [CG13] and their discussion of some classical results).

If a Galois representation is a subquotient of the étale cohomology \( H^i_{\text{ét}}(X, \mathbb{Q}_\ell) \) of a smooth projective variety \( X \) over \( K \) for some non-negative integer \( i \), this question is closely related to the Grothendieck’s theory on pure motives. In general, it is hard to give a satisfactory answer. In this paper, we focus our research on surfaces, and answer our question for those surfaces satisfying the conditions (*) and (**) in Theorem 1.1. As a corollary, for an elliptic surface satisfying the same conditions, the Tate conjecture holds. We also verify that a family of elliptic surfaces in [vGT95] satisfies the conditions (*) and (**). Hence we prove their Tate conjecture.
1.1. **Main results.** Let $X$ be a smooth projective variety over $K$. Let $\text{NS}(X_K)$ be the first Néron-Severi group of $X_K$. There is a cycle class map

\[ c^1 : \text{NS}(X_K) \to H^2_{\text{ét}}(X_K, Q_\ell(1)). \]

The image of $\text{NS}(X_K) \otimes \mathbb{Z} Q_\ell$ under $c^1 \otimes Q_\ell$ is called the algebraic part of $H^2_{\text{ét}}(X_K, Q_\ell(1))$. It is an $\ell$-adic subrepresentation of $G_K$. We define the transcendental part $\text{Tran}_\ell(X)$ of $H^2_{\text{ét}}(X_K, Q_\ell(1))$ to be the quotient $H^2_{\text{ét}}(X_K, Q_\ell(1))/(\text{NS}(X_K) \otimes \mathbb{Z} Q_\ell)$. In particular, it is known that when $X$ is an elliptic surface, then $H^2_{\text{ét}}(X_K, Q_\ell(1)) \cong \text{Tran}_\ell(X) \oplus (\text{NS}(X_K) \otimes \mathbb{Z} Q_\ell)$ since the algebraic equivalence classes of such surface is equivalent to its numerical equivalence classes. As $\ell$ varies, the representations $\text{Tran}_\ell(X)$ are in fact related to each other in the sense of weakly compatible system of $\ell$-adic Galois representations.

For an $\ell$-adic representation $\rho$, we will use $\rho^{ss}$ to denoted it semi-simplification. Now we can state our first main theorem.

**Theorem 1.1.** Let $X$ be a surface over $\mathbb{Q}$. Assuming the following conditions.

\begin{itemize}
  \item[(∗)] For some positive integer $s$,
  \[ \{\text{Tran}_\ell(X)^{ss}\}_\ell \cong \bigoplus_{i=1}^{s} \{\rho^{ss}_{\ell,i}\}_\ell, \]
  where each $\{\rho^{ss}_{\ell,i}\}_\ell$ is a regular rank 2 or 3 weakly compatible system of $\ell$-adic representations of $G_\mathbb{Q}$ defined over $\mathbb{Q}$.
  \item[(∗∗)] If, for any $\ell$ and $i$, $\rho^{ss}_{\ell,i}$ is decomposes into irreducible $\overline{\mathbb{Q}}_\ell$-subrepresentations as follows
  \[ \rho^{ss}_{\ell,i} \cong \psi_{\ell,i} \oplus r_{\ell,i}, \]
  with $\dim \psi_{\ell,i} = 1$ and $\dim r_{\ell,i} = 2$, then $\rho^{ss}_{\ell,i}$ is self-dual and $\det r_{\ell,i} = 1$.
\end{itemize}

then, for each $i$, $\rho_{\ell,i}$ is absolutely irreducible for a Dirichlet density one subset of primes $\ell$.

**Remark 1.1.** The condition (∗) requires that the transcendental part of the $H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, Q_\ell(1))$ is a direct sum of several 3-dimensional $\ell$-adic representations. A way to construct families of such elliptic surfaces can be found in [vGT91] and [vGT95].
Remark 1.2. Here we explain that $\det r_{\ell,i} = 1$ in condition (**) is reasonable. The self-dual condition of $\rho^{ss}_{\ell,i}$ implies that $r_{\ell,i}$ is also self-dual, thus $\det r_{\ell,i}$ is a quadratic character. If this quadratic character is nontrivial, it is a conjecture that there exist a CM elliptic curve $E$ such that $\rho^{ss}_{\ell,i} \varepsilon_{\ell}^{-1} \cong \Sym^2 T_\ell E$ (up to a quadratic twist). In this case, $\rho^{ss}_{\ell,i}$ has a finite image 1-dimensional subrepresentation. This is a contradiction with the Tate conjecture (see our discussion about the Tate conjecture later). As a hint, we prove a concrete family of elliptic surfaces in [vGT95] satisfies the condition (⋆) and (⋆⋆).

In order to speak about the application of our main theorem in arithmetic geometry, let us recall the Tate conjecture for divisors. For a smooth projective variety $X$ over $K$, $c^1$ is $G_K$-equivariant. Let $\NS(X)$ be the subgroup of $\NS(X_{\overline{K}})$ spanned by the divisors over $K$. We have an induced map

$$C^1 : \NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \longrightarrow H^2_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{G_K}.$$  

Tate makes the following conjecture [Tat65]

**Conjecture 1.1** (The Tate conjecture for divisors). Let $K$ be a finitely generated field over the prime field. Then the map $C^1$ is an isomorphism.

If $K$ is of characteristic 0, the above conjecture is known to hold for abelian varieties [Fal83], $K3$ surfaces [And96], and, more generally, smooth projective varieties with geometric genus 1 [Moo17]. If $K$ is of positive characteristic, it is known to hold for abelian varieties [Tat66] and $K3$ surfaces [ASD73], [NO85] [Mau14], [Cha13] and [MP15].

We call a $G_K$-invariant class in $H^2_{\et}(X_K, \mathbb{Q}_\ell(1))$ Tate class. Then, roughly speaking, the Tate conjecture claims that the Tate classes are in the algebraic part. If a Tate class is in the transcendental part $\Trans(X)$, it generates an 1-dimensional (trivial) $\ell$-adic subrepresentation. So we can prove the Tate conjecture for $X$ as long as we show that its transcendental part does not have any 1-dimensional subrepresentation.

Following the above idea, we have the following corollary of Theorem 1.1.

**Corollary 1.1.1.** Let $X \to \mathbb{P}^1_{\mathbb{Q}}$ be an elliptic surface over $\mathbb{Q}$ and satisfies the conditions (⋆) and (⋆⋆) in Theorem 1.1. Then, for a Dirichlet density one subset of primes, the
corresponding Tate conjecture for \( X \) is true. Precisely, we have
\[
(1.2) \quad \text{NS}(X) \otimes \mathbb{Z} \mathbb{Q}_\ell \sim H^2_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))^{\mathbb{G}_{\mathbb{Q}}}. 
\]

In [vGT95], van Geemen and Top construct a family of non-isotrivial elliptic surfaces \( S_a \) parameterized by \( a \in \mathbb{P}^1 \). Each member in this family has geometric genus 3. We apply our method to this family and show the following result.

**Theorem 1.2.** For each \( a \in \mathbb{Q} \), if \( a \equiv 2, 3 \mod 5 \), and none of \( 2(1 + a) \) or \( 2(1 - a) \) is a square in \( \mathbb{Q} \), the surface \( S_a \) satisfies the conditions (*) and (**) in Theorem 1.1.

In particular, for a Dirichlet density one subset of primes \( \ell \), the corresponding Tate conjecture for \( S_a \) is true. Precisely, we have
\[
(1.3) \quad \text{NS}(S_a) \otimes \mathbb{Z} \mathbb{Q}_\ell \sim H^2_{\text{ét}}((S_a)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))^{\mathbb{G}_{\mathbb{Q}}}. 
\]

1.2. **Our approach to proving Theorem 1.1.** We show Theorem 1.1 by contradiction. Let \( \rho_\ell \) be one of \( \rho_{\ell,i} \). Suppose \( \rho_\ell \) is not absolutely irreducible, then it either decomposes into a direct sum of characters, or a direct sum of a character and a 2-dimensional absolutely irreducible representation \( r_\ell \). The former case is impossible since the regularity of \( \rho_\ell \) forces this representation to have distinct Hodge-Tate weights, which contracts the result of the Weil conjectures.

Our method for the later case is inspired by the Fontaine-Mazur conjecture. Note that by condition (**), \( \det r_\ell = 1 \). In particular, \( r_\ell \) is even. This violates the Fontaine-Mazur conjecture. However, this conjecture is only known by the work of Kisin [Kis09], Emerton [Eme11], and Pan [Pan18] for odd 2-dimensional Galois representations and by Calegari [Cal12] for even Galois representations with several additional conditions. Thus to prove our theorem, we will show that under self-dual condition, all the additional conditions in Calegari’s result will be fulfilled. And this completes our proof.

1.3. **Our approach to proving Theorem 1.2.** The geometry of \( S_a \) implies that \( \{\text{Tran}_\ell(S_a)\}_\ell \) has a decomposition into three compatible systems of 3-dimensional subrepresentations. One of them is automatically absolutely irreducible, and the rest two are isomorphic self-dual when \( a \neq \pm 1 \). Let \( \rho_\ell \) be one of the two subrepresentations, and assume \( r_\ell \) is a 2-dimensional absolutely irreducible subrepresentation of \( \rho_\ell \). We show that \( r_\ell \) is also self-dual and thus by class field theory, there is a integer \( D \).
such that
\[ \det r_\ell(Frob_p) = \left( \frac{D}{p} \right) \]
for the prime \( p \mid D \). To prove that \( \det r_\ell = 1 \), it is enough to show that \( D \) is 1. For the later, we study the relationship \( \text{tr} \rho_\ell(Frob_p) \) and \( \det r_\ell(Frob_p) \) under the self-dual condition. And Theorem 1.2 follows immediately after combining our results with counting trick used to compute the trace of \( \rho_\ell \).

1.4. **Remark on our method.** We want to talk about our method and its potential generalization in motivic aspect. Suppose \( r_\ell \) exists. By the Tate conjecture, \( r_\ell \) is not motivically defined. But the Fontaine-Mazur conjecture predicts that \( r_\ell \) is motivically defined. This is the fundamental contradiction in our proof. To realize the contradiction in our proof, we make use the oddness condition, which reflects the motivic property of geometric Galois representations. Thus in order to generalize our result to non self-dual representations or higher dimensional representations, we expect (1) a proper analog of the oddness condition for higher dimensional representations as well as a geometric method to check this condition; (2) a generalization of known results which predicts the oddness of geometric Galois representations. Those problems are interesting to the authors. We hope to report a further result in this direction in a future paper.

1.5. **Outline of the paper.** In Section 2, we collect necessary definitions and facts on Galois representations. In particular, a theorem of Calegari about the Fontaine-Mazur conjecture is mentioned. In Section 3 we prove Theorem 1.1. Then, as an application of this theorem, we use it to prove the Tate conjecture for the elliptic surfaces satisfying the conditions (\( \ast \)) and (\( \ast \ast \)). In Section 4, we first recall a family of elliptic surfaces constructed in [vGT95]. Then we verify that the conditions (\( \ast \)) and (\( \ast \ast \)) of Theorem 1.1 are satisfied for about 40% members in this family. Then as a corollary we prove the Tate conjecture for those members.

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Joel Specter and Dingxin Zhang for helpful discussions on the Galois representations and pure motives. We would also like to thank Jeff Achter, Yuan Ren, David Savitt, Shaoyun Yi, for helpful conversations on this project.

1.7. Notations and conventions. For a field $K$, we fix the separable closure $\overline{K}$ of $K$. If $K$ is a number field and $p$ is a finite place of $K$, we let $\text{Frob}_p$ denote the geometric Frobenius.

If $X$ is a $K$-scheme, we let $X_{\overline{K}}$ denote the base-change $X \times_{\text{Spec}K} \text{Spec} \overline{K}$. The symbol $\dim$ in this paper means the dimension over $\mathbb{Q}_\ell$.

For a rational number $a = \frac{m}{n}$ with $\gcd(m, n) = 1$, we say $p$ is a divisor of $a$ if either $p|m$ or $p|n$. And we denote by $\left(\frac{a}{p}\right)$ the classical Legendre symbol $\left(\frac{mn}{p}\right)$.

2. Backgrounds of Galois representations

In this section, we recall some definitions and facts on Galois representations. The reader who are familiar with Galois representations can skip this section.

Let $L$ be a topological field and $V$ be a finite dimensional topological vector space over $L$. A Galois representation (or a $L$-representation of $G_K$) is a continuous linear group action of $G_K$ on $V$. Up to a choice of basis of $V$, we can realize this representation as a continuous homomorphism

$$\rho : G_K \longrightarrow \text{GL}_n(L).$$

Such data is denoted by $\{V, \rho\}$ (or one of $V$ or $\rho$ for simple). If $L$ is a finite extension of $\mathbb{Q}_\ell$ or $\mathbb{Q}_p$, we call $\rho$ an $\ell$-adic representation. If $L$ is a finite extension of $\mathbb{F}_\ell$ or $\overline{\mathbb{F}}_\ell$, we call $\rho$ an mod $\ell$ representation.

2.1. $\ell$-adic representations. We begin with recalling some basic definitions in $\ell$-adic Hodge theory. In this paper, we will only need those definitions formally. We refer the reader to [FO] for details. Suppose $K/\mathbb{Q}_\ell$ is a finite extension. Let $B_{dR}$ be Fontaine’s de Rham periods ring. It is a filtered $K$-algebra with a continuous $K$-linear action of $G_K$.

Definition 2.1. Let $V$ be a $\overline{\mathbb{Q}}_\ell$-representation of $G_K$. We say $V$ is de Rham if

$$\dim_K(B_{dR} \otimes \overline{\mathbb{Q}}_\ell V)^{G_K} = \dim_{\overline{\mathbb{Q}}_\ell} V.$$ 

If $V$ is de Rham, for each $\mathbb{Q}_\ell$-embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_\ell$,

$$\dim_{\mathbb{Q}_\ell}(B_{dR} \otimes \mathbb{Q}_\ell V)^{G_K} = \dim_{\mathbb{Q}_\ell} V.$$ 

If $V$ is de Rham, for each $\mathbb{Q}_\ell$-embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_\ell$,

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$$\dim_{\mathbb{Q}_\ell}(B_{dR} \otimes \mathbb{Q}_\ell V)^{G_K} = \dim_{\mathbb{Q}_\ell} V.$$ 

If $V$ is de Rham, for each $\mathbb{Q}_\ell$-embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_\ell$,

$$\dim_{\mathbb{Q}_\ell}(B_{dR} \otimes \mathbb{Q}_\ell V)^{G_K} = \dim_{\mathbb{Q}_\ell} V.$$ 

If $V$ is de Rham, for each $\mathbb{Q}_\ell$-embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_\ell$,
over \( \overline{\mathbb{Q}}_\ell \), we define the \( \dim_{\overline{\mathbb{Q}}_\ell} \) \( V \)-element multi-set of \( \tau \)-Hodge-Tate weights, \( HT_\tau(V) \), to be the multi-set of integers \( h \) such that

\[
gr^h(B_{dR} \otimes_{K, \tau} V)^{G_K} \neq 0
\]

where \( h \) has multiplicity \( \dim_{\overline{\mathbb{Q}}_\ell} gr^h(B_{dR} \otimes_{K, \tau} V)^{G_K} \).

**Example 2.1.** Let \( \varepsilon_\ell : G_{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell^\times \) be the \( \ell \)-adic cyclotomic character. Then \( \varepsilon_\ell|_{G_{\mathbb{Q}_\ell}} \) is de Rham. The Hodge-Tate weight of \( \varepsilon_\ell \) (more precisely, \( \varepsilon_\ell|_{G_{\mathbb{Q}_\ell}} \)) is \(-1\).

Now we can talk about global \( \ell \)-adic representations.

**Definition 2.2.** Let \( K/\mathbb{Q} \) be a finite extension and \( \rho \) be an \( \ell \)-adic representation of \( G_K \). We say \( \rho \) is geometric if \( \rho \) is unramified almost everywhere and \( \rho|_{G_{\mathbb{Q}_v}} \) is de Rham for every place \( v \) of \( K \) above \( \ell \).

**Example 2.2.** Suppose that \( X \) is a smooth projective variety over a number field \( K \). Then the \( \ell \)-adic representation \( H^i_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell) \) of \( G_K \), for \( 0 \leq i \leq 2 \dim X \), is geometric. Furthermore, any subquotient of the \( \ell \)-adic representation \( H^i_{\et}(X_{\overline{K}}, \mathbb{Q}_\ell) \) is geometric.

If \( V \) is an \( \ell \)-adic representation, we use \( V(n) \) to denote \( V \) tensor the \( \ell \)-adic cyclotomic character to the power \( n \).

**Lemma 2.3.** Let \( K/\mathbb{Q} \) be a totally real field. Suppose \( \rho : G_K \to \overline{\mathbb{Q}}_\ell^\times \) is a geometric \( \ell \)-adic representation. Then

\[
\rho = \tau \cdot \varepsilon_\ell^n|_{G_K}
\]

for some non-negative integer \( n \), where \( \varepsilon_\ell \) is the \( \ell \)-adic cyclotomic character of \( G_{\mathbb{Q}} \) and \( \tau \) is of finite order.

**Proof.** By class field theory, it is enough to study the algebraic Hecke characters of \( \mathbb{A}_{K}^\times/K^\times \). Then this Lemma follow from a classification of such characters [Wei56].

Let \( c \in G_{\mathbb{Q}} \) be the complex conjugation. For an \( \ell \)-adic representation of \( G_{\mathbb{Q}} \), \( \det(\rho(c)) \in \{1, -1\} \).

**Definition 2.3.** We say an \( \ell \)-adic representation \( \rho \) of \( G_{\mathbb{Q}} \) is **odd** (resp. **even** if \( \det(\rho(c)) \) is \(-1\) (resp. \(1\)).
**Definition 2.4.** We say a Galois representation $\rho$ is **self-dual** if

$$\rho \cong \rho^*, \quad \text{where } \rho^* \text{ is dual of } \rho.$$  

One of the main problem concerning the geometric $\ell$-adic representations is the Fontaine-Mazur conjecture. In this paper, an important input is Calegari’s result on Fontaine-Mazur conjecture (see [Cal12]). We state it here for the convenience of readers.

**Theorem 2.4.** Let $r : G_{\mathbb{Q}} \to \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ be an $\ell$-adic representation. Suppose that $\ell > 7$, and, furthermore, that

1. $r$ is geometric, i.e., unramified almost everywhere and de Rham at $\ell$.
2. $r|_{G_{\mathbb{Q}_\ell}}$ has distinct Hodge-Tate weights.
3. $\tau|_{G_{\mathbb{Q}_\ell}}$ is not a twist of a representation of the form $\left( \begin{array}{cc} \varepsilon_\ell & * \\ 0 & 1 \end{array} \right)$
   where $\varepsilon_\ell$ is the mod-$\ell$ cyclotomic character.
4. The residue representation $\tau$ is not of dihedral type.
5. The residue representation $\tau$ is absolutely irreducible.

Then $r$ is modular. In particular, $r$ is odd.

2.2. **Mod $\ell$ representations.** Let us recall a classical computations about mod $\ell$ representations by Serre [Ser87]. Considering the mod $\ell$ continuous character $\varphi : I_\ell \to \overline{\mathbb{F}}_\ell^\times$, where $I_\ell$ is the inertia subgroup of $G_{\mathbb{Q}_\ell}$. By continuity $\varphi$ has a finite image. The smallest $n$ such that $\varphi(I_\ell) \subset \overline{\mathbb{F}}_\ell^n$ is called the **level** of $\varphi$.

**Definition 2.5.** We define the level $n$ **fundamental character**

$$\omega_n : I_\ell \to \overline{\mathbb{F}}_\ell^{\times},$$

$$u \mapsto \frac{u(\nu)}{\nu} \mod \ell,$$

where $\nu$ is an $((\ell^n - 1)^{th}$ root of $\ell$. In particular, $\omega_1 = \varepsilon_\ell|_{I_\ell}$. 


From now on and to the end of this section, we only consider the Galois representations of $G_{\mathbb{Q}}$. Let $\rho : G_{\mathbb{Q}} \to \text{GL}_2(\overline{\mathbb{Q}}_\ell)$ be an $\ell$-adic representation. And assume $\overline{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\overline{\mathbb{F}}_\ell)$ to be its residue representation. Since the tame inertia group is abelian and the wild inertia group $I^w_\ell$ acts on $\overline{\rho}^{ss}_{|G_{\mathbb{Q}}}$ trivially, $\overline{\rho}^{ss}_{|I^w_\ell}$ has to be the direct sum of two characters, i.e.

$$\overline{\rho}^{ss}_{|I^w_\ell} = \varphi_1 \oplus \varphi_2.$$

Suppose $s \in G_{\mathbb{Q}_s}$ is a lift of the geometric Frobenius. Then, if $u \in I_\ell$, it is a fact that $s^{-1}us \in u^{s}I^w_\ell$. So $\overline{\rho}^{ss}_{|I^w_\ell}(s^{-1}us) = \overline{\rho}^{ss}_{|I^w_\ell}(u)$. This implies that either

1. $\varphi_1^\ell = \varphi_1$ and $\varphi_2^\ell = \varphi_2$, or
2. $\varphi_1^\ell = \varphi_2$ and $\varphi_2^\ell = \varphi_1$.

In the former case, $\varphi_1$ and $\varphi_2$ are powers of the level one fundamental character. In this case, we say $\rho$ is of level one. In the later case, $\varphi_1$ and $\varphi_2$ are powers of the level two fundamental character. In this case, we say $\rho$ is of level two. The following lemma will be used later.

**Lemma 2.5.** If $\rho$ is of level two, then $\overline{\rho}$ is absolutely irreducible.

**Proof.** It is enough to show that $\overline{\rho}|_{G_{\mathbb{Q}_s}}$ is absolutely irreducible. Suppose not, then $\overline{\rho}_s$ has a subrepresentation $\phi$. Without loss of generality, we assume $\phi|_{I^w_\ell} = \varphi_1$. This implies that $\varphi_1$ can be extended to a $G_{\mathbb{Q}_s}$-character. In particular, for every $u \in I_\ell$ and $s$ be an arbitrary lifting of the geometric Frobenius, we have $\varphi_1(u)^\ell = \varphi_1(s^{-1}us) = \varphi_1(u)$. So $\varphi_1^\ell(u) = 1$. However, this contradicts the fact that $\rho$ is of level two, thus the proof is done. \hfill $\square$

### 2.3. Weakly compatible system of $\ell$-adic representations.

The following definition of compatible system follows from [BLGGR14, Section 5.1]. For the convenience of readers, we also state it here.

**Definition 2.6.** Let $K$ denoted a number field. A rank $n$ weakly compatible system of $\ell$-adic representations $\mathcal{R}$ of $G_K$ defined over $M$ is a 5-tuple

$$(M, S, \{Q_v(T)\}, \{\rho_\lambda\}, \{H_v\}),$$

where

1. $M$ is a number field.
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(2) $S$ is a finite set of primes of $K$.
(3) for each $v \notin S$, $Q_v(T)$ is a monic degree $n$ polynomial in $M[T]$.
(4) for each prime $\lambda$ of $M$ (with residue characteristic $\ell$)
$$\rho_\lambda : G_K \to \operatorname{GL}_n(\overline{M}_\lambda)$$
is a continuous, semi-simple representation such that
- if $v \notin S$ and $v \nmid \ell$, is a prime of $K$, then $\rho_\lambda$ is unramified at $v$ and $\rho_\lambda(\text{Frob}_v)$ has characteristic polynomial $Q_v(T)$.
- if $v | \ell$ then $\rho_\lambda|_{G_{K,v}}$ is de Rham.
(5) for $\tau : K \hookrightarrow \overline{M}$, $H_\tau$ is a multiset of $n$ integers such that for any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over $M$ we have $\text{HT}_\tau(\rho_\lambda) = H_\tau$.

We will call $\mathcal{R}$ regular if for each $\tau : K \hookrightarrow \overline{M}$ every element of $H_\tau$ has multiplicity 1.

We will sometimes simply write $\{\rho_\lambda\}_\lambda$ for a weakly compatible system $\mathcal{R}$.

**Example 2.6.** Let $X$ be a projective smooth variety over a number field $K$, then for $0 \leq i \leq 2 \dim X$, $\{H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\}_\ell$ is a weakly compatible system of $\ell$-adic representations of $G_K$ defined over $\mathbb{Q}$.

Let $\mathcal{R}_1 = (M, S_1, \{Q_{v,1}(T)\}, \{\rho_{v,1}\}, \{H_{\tau,1}\})$ and $\mathcal{R}_2 = (M, S_2, \{Q_{v,2}(T)\}, \{\rho_{v,2}\}, \{H_{\tau,2}\})$ be two weakly compatible system of $\ell$-adic representations of $G_K$ defined over $M$. We can define direct sum $\mathcal{R}_1 \oplus \mathcal{R}_2$ a new weakly compatible system of $\ell$-adic representations of $G_K$ defined over $M$ by
$$(M, S_1 \cup S_2, \{Q_{v,1}(T)Q_{v,2}(T)\}, \{\rho_{v,1} \oplus \rho_{v,2}\}, \{H_{\tau,1} \cup H_{\tau,2}\}).$$

3. Irreducibility of $\ell$-adic representations

In this section, we will prove Theorem 1.1, i.e., taking $\{\rho_\ell\}_\ell$ to be one of the weakly compatible system $\{\rho_{\ell,i}\}_\ell$ in Theorem 1.1, we will show that up to a zero density subset of rational primes, $\rho_\ell$ are absolutely irreducible. Our strategy is proving by contradiction, i.e., we assume that $\rho_\ell$ is not absolutely irreducible. Immediately $\rho_\ell$ has one of the three decompositions
$$\chi_{\ell,1} \oplus \chi_{\ell,2} \oplus \chi_{\ell,3}, \quad \chi_{\ell,4} \oplus \chi_{\ell,5}, \quad \text{or} \quad \psi_{\ell} \oplus r_\ell,$$
where $\chi_{\ell,i}$ ($i = 1, 2, 3, 4, 5$) and $\psi_{\ell}$ are 1-dimensional $\overline{Q}_\ell$-representations and $r_{\ell}$ is an absolutely irreducible 2-dimensional $\overline{Q}_\ell$-representation. In the following, we will call the first and the second one of type $1 + 1 + 1$ and of type $1 + 1$. And we call the last one of type $1 + 2$. Similarly, we can also define the type of $G_K$-representations for a number field $K$. We will first exclude the possibility for $\rho_{\ell}$ is of type $1 + 1 + 1$ and of type $1 + 1$. Then we show the nonexistence of the “2-dimensional subrepresentation $r_{\ell}$”. So Theorem 1.1 follows. In the end, as an application of the theorem, we proved the related Tate conjecture for divisors.

3.1. Proof of Theorem 1.1. We first assume that $\rho_{\ell} : G_Q \to GL_3(\overline{Q}_\ell)$ is of type $1 + 1 + 1$ or of type $1 + 1$ and try to deduce a contradiction. Note that if this was the case, then for every Galois extension $K/Q$, the restriction $\rho_{\ell}|_{G_K}$ is also of type $1 + 1 + 1$ or of type $1 + 1$.

**Proposition 3.1.** Suppose that $K/Q$ is a totally real extension, then $\rho_{\ell}|_{G_K}$ is not of type $1 + 1 + 1$ and of type $1 + 1$.

**Proof of Proposition 3.1.** First we fix $\ell$ and assume that $\rho_{\ell}|_{G_K} \simeq \chi_{\ell,1} \oplus \chi_{\ell,2} \oplus \chi_{\ell,3}$ is a direct sum of three 1-dimensional $\overline{Q}_\ell$ representations. Then by Corollary 2.3 and use the Riemann hypothesis in the Weil conjectures, we know that $\rho_{\ell,i}$, $i \in \{1, 2, 3\}$, are finite characters. Then, for each embedding $\tau : K \hookrightarrow \overline{Q}_\ell$, the $\tau$-Hodge-Tate weights of $\rho_{\ell}|_{G_K}$ are $\{0, 0, 0\}$. This is a contradiction with our assumption that $\{\rho_{\ell}\}_\ell$ is regular. So $\rho_{\ell}|_{G_K}$ is not of type $1 + 1 + 1$. The proof for $\rho_{\ell}|_{G_K}$ is not of type $1 + 1$ is similar. □

Now we want to show that $\rho_{\ell}$ is not of type $1 + 2$.

**Proposition 3.2.** Under the conditions of Theorem 1.1, except for a density zero subset of all the rational prime integers $\ell$, $\rho_{\ell}$ is not of type $1 + 2$.

The strategy of proving Proposition 3.2 is still proof by contradiction. Suppose Proposition 3.2 is false, then we have an infinity set $J$ of primes which has positive density with respect to the set of all primes and is disjoint from $S$ and for every $\ell \in J$, $\rho_{\ell}$ has a 2-dimensional irreducible subrepresentation $r_{\ell} : G_Q \to GL_2(\overline{Q}_\ell)$. By our condition (***) in Theorem 1.1, $\det r_{\ell} = 1$. In particular, $r_{\ell}$ is odd for each $\ell \in J$. 
In the rest part of this section, we will use Calegari’s theorem on Fontaine-Mazur conjecture (Theorem 2.4) to show that some of such $r_\ell$ are also odd, which is impossible, and hence deduce to contradiction.

In order to use Theorem 2.4, without loss of generality, we can assume all primes in $J$ are greater than 7, then, under the conditions of Theorem 1.1, we need to show that there is a subset $J' \subset J$ consisting of infinity many $\ell$ such that $r_\ell$ satisfies all the five conditions of Theorem 2.4 for $\ell \in J'$.

**Lemma 3.3.** For $\ell \in J$, $r_\ell$ is geometric and has distinct Hodge-Tate weights. Equivalently, conditions (a) and (b) of Theorem 2.4 are true for all primes in $J$.

**Proof.** Every $\rho_\ell$ is geometric, and so is $r_\ell$ since the geometric property is closed under subquotient, this proves the condition (a).

Recall that the weakly compatible system $\{\rho_\ell\}_\ell$ is regular. So $\rho_\ell$ has distinct Hodge-Tate weights. The $r_\ell$ is a subquotient of $\rho_\ell$, hence condition (b) is true. Indeed, one easily see that the Hodge-Tate weights of $r_\ell$ are $\{1, -1\}$ by Riemann Hypothesis. 

**Lemma 3.4.** The representation $\overline{\tau}_\ell|_{G_{Q_\ell}}$ is not a twist of an extension of trivial character by mod $\ell$ cyclotomic character $\overline{\epsilon}_\ell$. In particular, condition (c) of Theorem 2.4 is satisfied by all $\ell$ in $J$.

**Proof.** Note that the mod $\ell$ cyclotomic character $\overline{\epsilon}_\ell$ is a surjective map to $F_\ell^\times$. Taking $g \in G_{Q_\ell}$ such that $\overline{\epsilon}_\ell(g) \in F_\ell^\times$ is not a square. If $\overline{\tau}_\ell|_{G_{Q_\ell}}$ is a twist of an extension of trivial character by mod $\ell$ cyclotomic character, then $\det(\overline{\tau}_\ell|_{G_{Q_\ell}}(g))$ is not a square element. This is a contradiction.

**Lemma 3.5.** For a subset $J'$ of rational primes of Dirichlet density one in $J$, $\overline{\tau}_\ell$ is not of dihedral type. Hence condition (d) of Theorem 2.4 is true for $J'$.

**Proof.** This follows exactly Proposition 2.7 in [CG13].

At last, we need to speak about the irreducibility of $r_\ell$.

**Lemma 3.6.** For all but finitely many $\ell \in J$ the associated residue representation $\overline{\tau}_\ell$ is absolutely irreducible. Equivalently, condition (e) of Theorem 2.4 is true for all but finitely many $\ell \in J$. 
Due to the length of our proof, before we go into detailed discussion, we want to express our idea by a baby-example.

**Example 3.7.** We claim that we cannot find infinity many $\ell \in J$ such that

$$\tau_{\ell}^{ss} \varepsilon_{\ell}^{-1} = \varepsilon_{\ell}^{-2} \oplus \text{id}$$

where $\tau_{\ell}^{ss}$ is the semi-simplification of $\tau_{\ell}$.

In fact, otherwise, by the fact that there is a weakly compatible system $\{\varepsilon_{\ell}^2\}_{\ell}$ we know

$$\text{tr} \ r_{\ell}\varepsilon_{\ell}^{-1}(\text{Frob}_p) \equiv \text{tr} \ \varepsilon_{\ell}^{-2}(\text{Frob}_p) + 1 = p^2 + 1 \pmod{\ell}$$

for all the $\ell \in J$, which means as integers

$$\text{tr} \ r_{\ell}\varepsilon_{\ell}^{-1}(\text{Frob}_p) = \text{tr} \ \varepsilon_{\ell}^{-2}(\text{Frob}_p) + 1.$$

Since this is true for all $p \notin S$ (recall that by Definition 2.6, every weakly compatible system has a finite set $S$ of exceptional primes), we conclude that

$$r_{\ell}\varepsilon_{\ell}^{-1} \simeq \varepsilon_{\ell}^{-2} \oplus \text{id}$$

as representations. Then $\rho_{\ell} \cong \psi_{\ell} \oplus r_{\ell} = \psi_{\ell} \oplus \varepsilon_{\ell}^{-1} \oplus \varepsilon_{\ell}^{-1}$ is of type $1 + 1 + 1$, which is impossible.

However, in general as we will see in the proof that $r_{\ell}$ is of level one, which only means that

$$\tau_{\ell}^{ss} \varepsilon_{\ell}^{-1}|_{I_{\ell}} = \omega_{1}^{-2} \oplus \text{id}$$

where $\omega_{1}$ is the level one fundamental character (Definition 2.5). Hence we have to deduce to the similar arguments as in the above example.

**Proof of Lemma 3.6.** Assume not, then this means that we have an infinite set $I \subset J$ consisting of prime numbers such that for each $\ell \in I$ the corresponding $\tau_{\ell}$ is not absolutely irreducible. In particular, each of these $\tau_{\ell}$ is of level one (cf. Lemma 2.5), i.e., for each of those $\tau_{\ell}$, we have

$$\tau_{\ell}^{ss} = \varepsilon_{\ell}^{a_{\ell}} \eta_{\ell} \oplus \varepsilon_{\ell}^{a'_{\ell}} \eta'_{\ell}$$

where $a_{\ell}$ and $a'_{\ell}$ are two integers, and $\eta_{\ell}$ and $\eta'_{\ell}$ are characters which are unramified at $\ell$, i.e. $\eta_{\ell}|_{I_{\ell}}$ are trivial.
First notice that for all \( \ell \in I \), we have that \( \{a_\ell, a'_\ell\} = \{1, -1\} \) due to the fact that \( r_\ell|_{G_{Q_p}} \) has Hodge-Tate weights \( \{1, -1\} \). Recall also that \( \det r_\ell = 1 \). Hence for those \( \ell \), without lose of generality, we can always write
\[
\tau_\ell^{-1} = \varepsilon_\ell^{-2} |_{G_K} \oplus \id.
\]

Recall that there exists an positive integer \( N \), which only dependents on the weakly compatible system \( \{\rho_\ell\}_\ell \), i.e. independent of the choice of \( \ell \) (see the conditions of Theorem 1.1). We know that \( \eta_\ell|_{I_1} \) is trivial, hence \( \cond(\eta_\ell)|N \).

As an immediate consequence of the above arguments, we know that for every \( \ell \in I \), the character \( \eta_\ell \) factors through \( \Gal(K/Q) \), where \( K := Q(\xi_N) \). In particular, we have
\[
\tau_\ell^{-1}|_{G_K} = \varepsilon_\ell^{-2}|_{G_K} \oplus \id.
\]

for all \( \ell \in I \).

Now, let \( K' \) be the maximal totally real subfield of \( K \), i.e. \( K' = Q(\xi_N + \xi_N^{-1}) \).

The extension of \( K \) over \( K' \) is degree 2. Hence \( \tau_\ell|_{G_{K'}} \) is either trivial or a quadratic character. According to the restriction of \( \eta_\ell \), we can split the infinite set \( I \) into two subsets \( I_1 \) and \( I_2 \), where \( I_1 = \{ \ell \in I : \eta_\ell|_{G_{K'}} \text{ is trivial} \} \) and \( I_2 = \{ \ell \in I : \eta_\ell|_{G_{K'}} \text{ is nontrivial} \} \). And in the following part of the proof, we will show that both \( I_1 \) and \( I_2 \) are finite sets, hence contract to the assumption that \( I \) is an infinity set.

Assume \( I_1 \) is an infinity set. Take \( p \nmid N \) be a prime integer and \( p \) a prime in \( K' \) above \( p \) with norm \( p^{J_p} \). Then for all \( \ell \in I_1 \) and \( p \neq \ell \), we have
\[
\tr r_\ell^s \varepsilon_\ell^{-1}|_{G_{K'}}(\Frob_p) \equiv p^{2J_p} + 1 \mod \ell.
\]

Recall that \( \{\rho_\ell\}_\ell \) is a compatible system, and the sign of \( \psi_\ell \) is the same as that of \( \det r_\ell \) as long as \( \ell \in J \). Hence in (3.1) the trace of \( r_\ell(\Frob_p) \) are independent of the choice of \( \ell \). By our assumption, \( I_1 \) is an infinite set, this means
\[
\tr r_\ell^s \varepsilon_\ell^{-1}|_{G_{K'}}(\Frob_p) = p^{2J_p} + 1 = \tr(\varepsilon_\ell^{-2}|_{G_{K'}} \oplus \id)(\Frob_p).
\]

Now, vary the prime \( p \), we will see that (3.2) is true for almost all \( p \). This means that
\[
r_\ell^s \varepsilon_\ell^{-1}|_{G_{K'}} \simeq \varepsilon_\ell^{-2}|_{G_{K'}} \oplus \id
\]
for every \( \ell \in I_1 \), i.e. \( r_\ell|_{G_{K'}} \) is not absolutely irreducible. This contradicts our irreducibility assumption of \( r_\ell \), hence impossible. Thus \( I_1 \) has to be a finite set.
Now assume $I_2$ to be infinity. With the same notations as above, we have
\begin{align}
(3.3) \quad \text{tr} r^s \varepsilon_{\ell}^{-1}|_{G_{K'}}(\text{Frob}_p) & \equiv \ell^{2f_p} \eta_p(\text{Frob}_p) + \eta_p^{-1}(\text{Frob}_p) \mod \ell \\
\end{align}
where $\eta_p : G_{K'} :\rightarrow \overline{\mathbb{Q}}^\times_p$ is a continuous character such that $\eta_p(g) = 1$ if $g \in G_K$ and $-1$ otherwise. Note that $\eta_p$ is in fact independent of the choice of $\ell \in I_2$. Thus similar to the above arguments, after varying $\ell \in I_2$ we have
\begin{align}
(3.4) \quad \text{tr} r^s \varepsilon_{\ell}^{-1}|_{G_{K'}}(\text{Frob}_p) & = \text{tr}(\varepsilon_{\ell}^{-2}|_{G_{K'}} \eta_p \oplus \eta_p^{-1})(\text{Frob}_p).
\end{align}
And then vary $p$, we get
\begin{align}
& r^s \varepsilon_{\ell}^{-1}|_{G_{K'}} \simeq \varepsilon_{\ell}^{-2}|_{G_{K'}} \eta_p \oplus \eta_p^{-1}
\end{align}
which is also impossible. Hence $I_2$ cannot be infinity.

Now combine all of the above arguments, we know that $I = I_1 \cup I_2$ has to be finite. And this contradicts the assumption we made in the beginning of the proof, and hence finishes the proof of this proposition.

Finally, by combing all the above discussions, we finish the proofs to Proposition 3.2 and Theorem 1.1.

Proof of Proposition 3.2. As we assumed in the beginning of this section, there is a subset $J$ consisting of rational primes $\ell$ whose associated $\rho_\ell$ is of type $1 + 2$. By Lemma 3.3, 3.4, 3.5 and 3.6, there is a density one subset $J'$ of $J$ such that $r_\ell$ satisfies all assumptions of Theorem 2.4 for every $\ell \in J'$. Hence, for $\ell \in J'$, $r_\ell$ is modular, hence odd. However, this is impossible since by our setups, $r_2$ should be even for every $\ell \in J$. Thus by all above, we cannot assume $J$ to have nonzero density, and this completes the proof of Proposition 3.2

Proof of Theorem 1.1. This Theorem follows by combining Proposition 3.1 and Proposition 3.2.

3.2. Application to the Tate conjecture. In this section, we prove Corollary 1.1.1. Notice that in the conditions of this theorem, we assume that all sub $G_{\mathbb{Q}}$-representations of the transcendental part of $H^2_{\text{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ satisfies the conditions of Theorem 1.1. Then it follows immediately that they are all absolutely irreducible. This means the transcendental part has no contribution to the Galois invariant part of $H^2_{\text{et}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$. 

Thus to prove Corollary 1.1.1, it is sufficient to show that

\[(3.5) \quad \text{NS}(X) \times \mathbb{Q}_\ell \to (\text{NS}(X_{\overline{\mathbb{Q}}}) \times \mathbb{Q}_\ell)^{\text{Gal}}\]

is an isomorphism. Indeed this is true when \(X\) is an elliptic surface. This result should be known for experts. But for the convenience of readers, we write a proof in this section. Our proof is not original, it follows the idea in [Huy16, Chapter 17, Section 3], especially its Remark 3.2, although the proof there is for \(K3\) surfaces.

**Proof of Corollary 1.1.1.** As the discussion above, we are reduced to show that the morphism in (3.5) is an isomorphism. Note that \(\text{NS}(X_{\overline{\mathbb{Q}}})\) is generated by the geometric divisors of \(X_{\overline{\mathbb{Q}}}\). This means that we can find a finite Galois extension \(K/\mathbb{Q}\) such that all the generators are defined over \(K\), i.e.

\[
\text{NS}(X_{\overline{\mathbb{Q}}}) = \text{NS}(X_K).
\]

Thus to prove the corollary, it is reduced to show

\[(\text{NS}(X_K) \otimes \mathbb{Q}_\ell)^{\text{Gal}} = \text{NS}(X) \otimes \mathbb{Q}_\ell.
\]

To show this, we note that for elliptic surfaces with section over a base curve of genus 0, we have \(\text{NS}(X_{\overline{\mathbb{Q}}}) = \text{Pic}(X_{\overline{\mathbb{Q}}})\), i.e. the linearly equivalent class and the algebraic equivalent class coincide by the fact that the pull back map from \(\text{Pic}^0(C_{\overline{\mathbb{Q}}})\) to \(\text{Pic}^0(X_{\overline{\mathbb{Q}}})\) is an isomorphism (here \(C\) is the genus 0 base curve of \(X\)). Since \(\text{Pic}(X) = H^1(X, \mathbb{G}_m)\), and similarly for \(\text{Pic}(X_K)\). Then consider the Hochschild-Serre spectral sequence

\[
E_2^{p,q} = H^p(\text{Gal}(K/\mathbb{Q}), H^q(X_K, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m)
\]

and apply the facts that \(H^1(\text{Gal}(K/\mathbb{Q}), K^*) = 0\) (i.e. Hilbert 90) we get

\[
0 \to \text{Pic}(X) \to \text{Pic}(X_K)^{\text{Gal}(K/\mathbb{Q})} \to H^2(\text{Gal}(K/\mathbb{Q}), K^*).
\]

According to the fact that \(H^2(\text{Gal}(K/\mathbb{Q}), K^*)\) is torsion, we have

\[(\text{Pic}(X_K) \otimes \mathbb{Q})^{\text{Gal}} = \text{Pic}(X) \otimes \mathbb{Q}.
\]

This finishes the proof. □
4. The Tate conjecture of the surfaces of van Geemen and Top

In this section, we will apply Theorem 1.1 to the construction of van Geemen and Top [vGT95]. As the result, we show that the Tate conjecture is true in general for a sub family of their construction.

More precisely, we recall the construction of a family of elliptic surfaces $S_a$ in [vGT95], and the properties of the corresponding weakly compatible system $\{V_\ell(1)\}_\ell$ which is constructed by van Geemen and Top in Section 4.1. In Section 4.2, we apply some calculation tricks to work out several technical properties of the trace of $V_\ell$ which can be used in proving Theorem 1.2. Finally, in Section 4.3 we prove Theorem 1.2.

4.1. The surfaces of van Geemen and Top. We simply recall the construction of van Geemen and Top here. Readers who are interested in more details are referred to Section 2 and Section 5 of the original paper.

For each $a \in \mathbb{Q} \setminus \{\pm 1\}$, considering the elliptic surface

\begin{equation}
E_a : Y^2 = X \left( X^2 + 2 \left( \frac{a + 1}{t^2} + a \right) X + 1 \right).
\end{equation}

**Remark 4.1.** The original surface in their paper has two parameters $a$ and $s$, while when $s \in \mathbb{Q}^*$, we can parameterize the equation to get the form as above.

Define the elliptic surfaces $X_a$ and $S_a$ as fiber products of $E_a$ which satisfies the following commutative diagram.

\[
\begin{array}{ccc}
S_a & \longrightarrow & X_a \\
\downarrow & & \downarrow \\
\mathbb{P}_z^1 & \overset{j}{\longrightarrow} & \mathbb{P}_u^1 \overset{h}{\longrightarrow} \mathbb{P}_t^1
\end{array}
\]

where $j : z \mapsto u = (z^2 - 1)/z$ and $h : u \mapsto t = (u^2 - 4)/(4u)$. One can see that $S_a$ is not isotrivial by computing its $j$-invariant.

In this and following sections, we will denote by $E_{a,t}$ (resp. $X_{a,u}$, $S_{a,z}$) the fiber above $t \in \mathbb{P}_t^1$ (resp. $u$, $z$) of the surface $E_a$ (resp. $X$, $S$). However, to simplify the notations and to emphasize on one of the indexes, we will usually drop one or both of the sub-indexes as long as there is no danger of confusion by doing this.

Considering the geometric action on $\mathbb{P}_z^1$ defined by

\[
\sigma : z \mapsto \frac{z + 1}{-z + 1},
\]
which has order 4. Then one easily checks that \( j : \mathbb{P}^1_z \to \mathbb{P}^1_u \) defined above identifies \( \mathbb{P}^1_u \) with the quotient space \( \mathbb{P}^1_z / \langle \sigma^2 \rangle \). This means that both of the two étale cohomology \( H^2_{\text{ét}}(S_{\overline{Q}}, \mathbb{Q}/\ell) \) and \( H^2_{\text{ét}}(X_{\overline{Q}}, \mathbb{Q}/\ell) \) are stable under the \( G \mathbb{Q} \times \langle \sigma \rangle \)-action. Moreover, if we denote by \( A_{\ell}(S) \) the \( \mathbb{Q}/\ell \)-subspace of in \( H^2_{\text{ét}}(S_{\overline{Q}}, \mathbb{Q}/\ell) \) spanned by all components of bad fibers of \( S \to \mathbb{P}^1 \), then \( A_{\ell}(S) \) is also \( G \mathbb{Q} \times \langle \sigma \rangle \)-stable.

Define

\[
W_\ell := H^2_{\text{ét}}(S_{\overline{Q}}, \mathbb{Q}/\ell)/ (H^2_{\text{ét}}(X_{\overline{Q}}, \mathbb{Q}/\ell) + A_{\ell}(S)).
\]

Then \( W_\ell \) has dimension 6 and also equips an \( G \mathbb{Q} \times \langle \sigma \rangle \)-action. The \( \sigma \) has two eigenvalues \( \pm i \) on \( W_\ell \). We take \( V_\ell \) (resp. \( \overline{V}_\ell \) ) to be the 3-dimensional eigenspace corresponding to eigenvalue \( i \) (resp. \( -i \)).

**Remark 4.2.** From now on, we work on the representation \( V_\ell \). But all the following results are also true for \( \overline{V}_\ell \) due to the isomorphism between \( \overline{V}_\ell \) and \( V_\ell \).

**Proposition 4.1.** [vGT95, Proposition 5.2] For \( a \in \mathbb{Q} \setminus \{\pm 1\} \), the corresponding \( V_\ell(1) \) is self-dual (up to semi-simplification).

We also collect some facts about \( S_a, X_a, E_a \) for later use.

**Lemma 4.2.** Denote by \( \mathcal{E} = \mathcal{E}_a \), and let \( p \) be a good prime of \( \mathcal{E} \). For each \( t \in \mathbb{P}^1_{\overline{F}_p} \), we have the following results.

1. \( \mathcal{E}_a \) is of type \( I_8 \).
2. \( \mathcal{E}_{\pm t} \) are of type \( I_1 \), and they are the only two ramified fibre along \( h \) and \( j \).
3. When \( t = \pm \sqrt{\frac{1+a}{1-a}} \), the corresponding fibre are of type \( I_1 \).
4. \( \mathcal{E}_\infty \) is smooth.
5. For a general \( t \) (i.e. when \( t \) is not corresponding to the fibre in (1)-(4)), \( \mathcal{E}_t \) is smooth.

**Proof.** This follows from a calculation by Tate algorithm. \( \Box \)

**Lemma 4.3.** In the above construction,

1. \( \mathcal{E}_a \) are rational elliptic surfaces.
2. \( X_a \) are \( K3 \) surfaces with Picard number 19.
3. \( S_a \) has (complex) Hodge numbers \( h^{2,0} = h^{0,2} = 3 \), \( h^{1,1} = 40 \). And the Picard number for it is 37.

**Proof.** See the proof of Proposition 4.2 and Remark 5.3 in [vGT95]. \( \Box \)
4.2. **Trace of \( \mathcal{V}_\ell(1) \).** Now we want to deduce a trace formula of \( \mathcal{V}_\ell(1) \) for future requirement.

Note that the ramified primes of \( \mathcal{V}_\ell \) are the divisors of \( 2(1 + a)(1 - a) \) (for notation, see Section 1.7). For a given prime integer \( p \) which is not dividing \( 2(1 + a)(1 - a) \), and let \( p \) be a prime ideal in a number field \( K \) and \( p \) be lying above \( p \). Let \( q = Nm_{K/Q}(p) = p^r \) and \( \rho_\ell : G_Q \to \text{GL}_3(\mathbb{Q}_\ell) \) be the semi-simplification of \( \mathcal{V}_\ell(1) \). Then the formula to compute the trace of corresponding geometric Frobenius \( \text{Frob}_p \) attached to \( \mathcal{V}_\ell \) is (see [vGT95, Theorem 3.5] for more details)

\[
\text{tr} \rho_\ell \varepsilon^{-1}_\ell(\text{Frob}_p) = \frac{\#S(F_q) - \#X(F_q)}{2}.
\]

With this formula, we can compute the trace of \( \text{Frob}_p \) modulo an integer \( m \). In order to do this, we compute \( \#S(F_q) - \#X(F_q) \) fiberwisely with respect to \( t \in \mathbb{P}^1_{F_q} \). In the following, we discuss general fibre first, and then special ones. To simplify our arguments, we will assume that \( p > 2 \).

For the general fibre, notice that only when \( h^{-1}(t) \in F_q \), \( \mathcal{E}_t \) has two liftings (with multiplicity) in \( X(F_q) \). And only when \( (j \circ h)^{-1}(t) \in F_q \), \( \mathcal{E}_t \) has four liftings in \( S(F_q) \). Hence in the following, we only care about those \( t \in F_q \) such that \( \mathcal{E}_t \) has contribution to the tr \( \rho_\ell \varepsilon^{-1}_\ell(\text{Frob}_p) \), i.e., those \( t \) satisfying \( \sqrt{t^2 + 1} \in F_q \).

Secondly, due to the defining equation (4.1) we have \( 2\#\mathcal{E}_t(F_q) \) for each smooth fiber. Moreover, when \( t \) is general, (cf. Lemma 4.2), we have \( \mathcal{E}_t \simeq \mathcal{E}_{-t} \), hence \( \#\mathcal{E}_t(F_q) = \#\mathcal{E}_{-t}(F_q) \). Hence for general \( t \in F_q \) we have

\[
4\#\mathcal{E}_t(F_q) + \#\mathcal{E}_{-t}(F_q).
\]

Moreover, when \( \sqrt{2(1 + a)} \in F_q \), we have a 4-torsion point \( \left( 1, \frac{\sqrt{2(1 + a)(t^2 + 1)}}{t} \right) \) in each of \( \mathcal{E}_{\pm t}(F_q) \), thus in this case,

\[
8\#\mathcal{E}_t(F_q) + \#\mathcal{E}_{-t}(F_q)
\]

and the contribution of \( \mathcal{E}_{\pm t} \) as \( F_q \)-points is divisible by 16.

Now we discuss the special fibre.

(1) When \( t = 0 \), the corresponding fiber is

\[
\mathcal{E}_0 : Y^2 = X^2(X + 2(a + 1)).
\]
Hence one can see that the singular point is \((0, 0)\) with two tangent lines 
\[ Y = \pm \sqrt{2(a + 1)}X. \] Moreover, the \(u\)-fibre above \(t = 0\) are \(u = \pm 2\), and the \(z\)-fibre are \(z = \pm 1 \pm \sqrt{2}\). Hence we can tabular the contribution of \(E_0\) in Table 1. In each of the first two columns of this table, we write 1 to indicate the element at the top of this column is in \(\mathbb{F}_q\), and \(-1\) otherwise. If we leave the block unfilled, it means that the contribution is independent with the value. At the last column, we list the contribution (as number of \(\mathbb{F}_q\) points) of each situation.

| \(\sqrt{2(1 + a)}\) | \(\sqrt{2}\) | contribution |
|---------------------|-----------------|--------------|
| 1                   | 1               | \(2q\)       |
| 1                   | -1              | \(-2q\)      |
| -1                  | 1               | \(2(q + 2)\) |
| -1                  | -1              | \(-2(q + 2)\) |

Table 1. Contribution of \(t = 0\).

(2) When \(t = \pm i\), since the corresponding \(u\)-fibre are \(u = \pm 2i\) and the \(z\)-fibre are \(z = \pm i\), we know that \(E_{\pm i}\) have no contribution.

(3) When \(t = \pm \sqrt{\frac{1 + a}{1 - a}}\), the corresponding fiber is 
\[ E_t : Y^2 = X(X + 1)^2. \]

The singular point is \((-1, 0)\) with two tangent lines \(Y = \pm i(X + 1)\). Then the \(u\)-fibre above are \(u = \pm 2\sqrt{\frac{1 + a}{1 - a}} \pm 2\sqrt{\frac{2}{1 - a}}\). The the \(z\)-fibre are \(z = \pm \sqrt{\frac{1 + a}{1 - a}} \pm \sqrt{\frac{2}{1 - a}} \pm \sqrt{\frac{4 + 2\sqrt{2(1 + a)}}{1 - a}}\). Hence by the same manner as above, we have Table 2.

(4) When \(t = \infty\), the corresponding fiber is smooth (recall \(a \neq \pm 1\)). In particular, we have 
\[ E_{\infty} : Y^2 = X(X^2 + 2aX + 1). \]

When \(\sqrt{2(1 + a)} \in \mathbb{F}_q\), we have a 4-torsion point \((1, \sqrt{2(1 + a)})\). Moreover, when \(\sqrt{a^2 - 1} \in \mathbb{F}_q\), then we have three distinct 2-torsion points over \(\mathbb{F}_q\). The \(u\)-fibre above are \(u = 0, \infty\), and the \(z\)-fibre are \(z = 0, \infty, \pm 1\). Hence

(a) When \(\sqrt{2(1 + a)} \notin \mathbb{F}_q\), \(2 \mid \#E_{\infty}(\mathbb{F}_q)\) and \(E_{\infty}\) contributes \(2 \#E_{\infty}(\mathbb{F}_q)\).
(b) When \(\sqrt{2(1 + a)} \in \mathbb{F}_q\), \(4 \mid \#E_{\infty}(\mathbb{F}_q)\) and \(E_{\infty}\) contributes \(2 \#E_{\infty}(\mathbb{F}_q)\).
\[
\begin{array}{cccc|c}
\sqrt{\frac{1+a}{1-a}} & \sqrt{\frac{2}{1-a}} & \sqrt{\frac{4+2\sqrt{2(1+a)}}{1-a}} & i & \text{contribution} \\
1 & 1 & 1 & 1 & 4q \\
1 & 1 & 1 & -1 & 4(q+2) \\
1 & 1 & -1 & 1 & -4q \\
1 & 1 & -1 & -1 & -4(q+2) \\
-1 & -1 & -1 & 1 & 0 \\
-1 & -1 & -1 & 0 & 0 \\
\end{array}
\]

Table 2. Contribution of \( t = \pm \sqrt{\frac{1+a}{1-a}} \).

(c) When \( \sqrt{a^2-1} \in \mathbb{F}_q \), \( 4\#\mathcal{E}_\infty(\mathbb{F}_q) \) and \( \mathcal{E}_\infty \) contributes \( 2\#\mathcal{E}_\infty(\mathbb{F}_q) \).

(d) When \( \sqrt{2(1+a)} \in \mathbb{F}_q \) and \( \sqrt{a^2-1} \in \mathbb{F}_q \), \( 8\#\mathcal{E}_\infty(\mathbb{F}_q) \) and \( \mathcal{E}_\infty \) contributes \( 2\#\mathcal{E}_\infty(\mathbb{F}_q) \).

Proposition 4.4. When \( q = p^2 \), and \( \left( \frac{2(1+a)}{p} \right) = \left( \frac{2(1-a)}{p} \right) = -1 \), then

\[
\text{tr } \rho_{\ell}^{-1}(\text{Frob}_p) = -q \pmod{8}.
\]

Proof. According to the above discussion, we can see that when \( q = p^2 \), then

\[
i, \sqrt{2}, \sqrt{1+a}, \sqrt{1-a} \in \mathbb{F}_q.
\]

This means

(1) The contribution (as trace) of general fibre are \( 0 \pmod{8} \).

(2) \( t = 0 \) contributes \( q \) to trace.

(3) If \( \sqrt{\frac{4+2\sqrt{2(1+a)}}{1-a}} \in \mathbb{F}_q \), then \( t = \pm \sqrt{\frac{1+a}{1-a}} \) contribute \( 2q \) for trace, otherwise \(-2q\).

(4) The contribution of \( t = \infty \) (to trace) \( 0 \pmod{8} \).

To determine whether \( \sqrt{\frac{4+2\sqrt{2(1+a)}}{1-a}} \in \mathbb{F}_q \), we need to tell whether \( \mathbb{F}_p(\sqrt{4 \pm 2\sqrt{2(1+a)}}) \subset \mathbb{F}_q \). Notice that \( \alpha := \sqrt{4 + 2\sqrt{2(1+a)}} \) is a root of the polynomial \( T^4 - 8T^2 + 8(1-a) \).

In fact, let \( \beta = \sqrt{4 - 2\sqrt{2(1+a)}} \), then the four roots of this polynomial are \( \pm\alpha \) and \( \pm\beta \). Now let \( \sigma \in \text{Gal}(\mathbb{F}_p(\alpha)/\mathbb{F}_p) \) be a generator, then if \( \sigma \) has order 2, then it either exchanges \( \alpha \) with \(-\alpha\), or exchanges \( \alpha \) with one of \( \pm\beta \). For the former, we know that it means \( \alpha^2 \in \mathbb{F}_p \), i.e. \( \sqrt{2(1+a)} \in \mathbb{F}_p \). For the later, we know it means \( \alpha\beta \in \mathbb{F}_p \), i.e. \( \sqrt{2(1-a)} \in \mathbb{F}_p \). If \( \sigma \) has order 1, i.e. \( \alpha \in \mathbb{F}_p \), then \( \alpha^2 = 4 - 2\sqrt{2(1+a)} \in \mathbb{F}_p \).
and thus $\sqrt{2(1 + a)} \in \mathbb{F}_p$. Hence under the assumption of the proposition we know that $\sqrt{\frac{4 + 2(1 + a)}{1 - a}} \in \mathbb{F}_q$ is not in $\mathbb{F}_q$, and thus

$$\text{tr} \rho_{\ell}^{-1}(\text{Frob}_p) = q - 2q = -q \mod 8.$$ 

□

4.3. **Proof of Theorem 1.2.** Now we want to prove Theorem 1.2, i.e., $S_a$ satisfies the conditions (\*) and (**) of Theorem 1.1. Recall that

$$H^2_{\text{ét}}((S_a)_{\mathbb{Q}}, \mathbb{Q}_\ell(1))^{\text{ss}} \cong (\text{NS}((S_a)_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell)^{\text{ss}} \oplus \text{Tran}_\ell(S_a)^{\text{ss}}.$$ 

By the construction of $V_\ell$ and $\nabla_\ell$ in Section 4.2 and Lemma 4.3 (2), we have

$$\text{Tran}_\ell(S_a)^{\text{ss}} \cong V_\ell(1)^{\text{ss}} \oplus V_\ell(1)^{\text{ss}} \oplus U_\ell(1)^{\text{ss}},$$

as $\ell$-adic representations of $G_Q$, where $U_\ell(1)^{\text{ss}}$ is the transcendental part of the $H^2_{\text{ét}}((\mathcal{X}_a)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(1))$. Then $\{U_\ell(1)^{\text{ss}}\}_\ell$ is a rank 3 weakly compatible system of $\ell$-adic representations of $G_Q$ defined over $\mathbb{Q}$. Note that the intermediate surface $\mathcal{X}_a$ is a $K3$ surface with Picard rank 19. This means $\mathcal{X}_a$ admits a Shioda-Inose structure ([Mor84, § 6]) corresponding to two isogenous non-CM elliptic curves $E$ and $E'$. In particular, this means over a finite Galois extension $L/\mathbb{Q}$, we have an isomorphism of $\ell$-adic representations of $G_L$

$$U_\ell \simeq \text{Sym}^2(T_\ell(E)).$$

where $\text{Sym}^2(T_\ell(E))$ is the symmetry square of the Tate module of $E$. So $U_\ell$ is irreducible as $G_L$-representation (note that the Tate conjecture is known for abelian surfaces), and hence irreducible as $G_Q$-representation.

**Proposition 4.5.** For each $a \in \mathbb{Q} \setminus \{\pm 1\}$, The surface $S_a$ satisfies the condition (\*) of Theorem 1.1.

**Proof.** By the above discussion, $\{U_\ell(1)\}_\ell$ is a rank 3 regular weakly compatible system of $\ell$-adic representations of $G_Q$ defined over $\mathbb{Q}$.
Considering the representations $V_\ell$ and $\overline{V}_\ell$. They are motivically defined, and the complex Hodge number are $h^{2,0} = h^{1,1} = h^{0,2} = 1$ (see [vGT95, proof of Proposition 4.2]). So $\{V_\ell(1)_{ss}\}_\ell$ and $\{\overline{V}_\ell(1)_{ss}\}_\ell$ are also rank 3 regular weakly compatible system of $\ell$-adic representations of $G_{\mathbb{Q}}$ defined over $\mathbb{Q}$. □

Before we think about the condition of $(\ast\ast)$ of Theorem 1.1, we need to state a lemma for latter use.

**Lemma 4.6.** Let $\rho : G_{\mathbb{Q}} \to \text{GL}_3(\overline{\mathbb{Q}}_\ell)$ be an $\ell$-adic representation and $\rho \cong \psi \oplus r$ decomposes into the direct sum of two irreducible $\overline{\mathbb{Q}}_\ell$-subrepresentations with $\dim \psi = 1$ and $\dim r = 2$. Then, for an element $g$, $\det r = 1$ in anyone of the following cases:

1. $\text{tr} \rho(g^2) \neq 3 \pmod{m}$ for some integer $m \geq 5$.
2. $\text{tr} \rho(g) \neq \pm 1$.

**Proof.** Since $\rho$ is self-dual, $\psi \oplus r \cong \rho \cong \rho^* \cong \psi^* \oplus r^*$. Note that $\psi$ and $r$ have different dimension. By Jordan-Holder Theorem, $\psi \cong \psi^*$ and $r \cong r^*$, i.e., $\psi$ and $r$ are self-dual. In particular, $\psi$ and $\det r$ are quadratic characters.

Also notice that, by $\rho$ is self-dual, the image is in the orthogonal group $O_3(\overline{\mathbb{Q}}_\ell)$. Then, for any $g \in G_{\mathbb{Q}}$, $\rho(g)$ is a diagonalizable matrix. Assume that the Jordan canonical form of $\rho(g)$ is $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$. Since $\rho(g)$ is similar to $(\rho(g)^{-1})^t$. Then there are two cases:

1. $\alpha = \pm 1$, $\beta = \pm 1$, and $\gamma = \pm 1$.
2. $\alpha \beta = 1 (\alpha \neq \pm 1)$, and $\gamma = \pm 1$.

Observe that, in case (2), $\det r(g) = \alpha \beta = 1$ since $\psi$ is a quadratic character. And, in case (1), if $\alpha = \beta = \gamma = \pm 1$, $\det r(g) = 1$. Then this lemma follows. □

**Proposition 4.7.** For each $a \in \mathbb{Q}$, if $a \equiv 2, 3 \mod{5}$, and none of $2(1 + a)$ or $2(1 - a)$ is a square in $\mathbb{Q}$, the surface $S_a$ satisfies the condition $(\ast\ast)$ of Theorem 1.1.

We denote $V_\ell(1)$ by $\rho_\ell$ for any prime $\ell$. By Proposition 4.2, $\rho_\ell^{ss}$ is self-dual. If $\rho_\ell^{ss}$ is decomposes into irreducible $\overline{\mathbb{Q}}_\ell$-subrepresentations as following

$$
\rho_\ell^{ss} \cong \psi_\ell \oplus r_\ell
$$

with $\dim \psi_\ell = 1$ and $\dim r_\ell = 2$. We want to prove that $\det r_\ell = 1$. 
Proof. By the proof of Lemma 4.6, we know that \( \det r_\ell \) is a quadratic character. By class field theory, there is an integer \( D \) such that

\[
\det r_\ell(Frob_p) = \left( \frac{D}{p} \right)
\]

for prime \( \ell \nmid D \).

We first prove that, if neither \( 2(1+a) \) or \( 2(1-a) \) is a square in \( \mathbb{Q} \), then \( D \) is 1 or \( 1-a^2 \) (up to a square). Suppose not the case, then by Chinese reminder theorem we can find a prime integer \( p \) such that

\[
\left( \frac{D}{p} \right) = -1, \quad \left( \frac{2(1+a)}{p} \right) = \left( \frac{2(1-a)}{p} \right) = -1.
\]

So \( \det r_\ell(Frob_p) = \left( \frac{D}{p} \right) = -1 \). On the other hand, by Proposition 4.4, \( \left( \frac{2(1+a)}{p} \right) = \left( \frac{2(1-a)}{p} \right) = -1 \) implies that \( \text{tr} \rho_\ell(Frob_p^2) = -1 \pmod{8} \). Then according to Lemma 4.6, we have \( \det r_\ell(Frob_p) = 1 \). This is a contradiction. So \( D \) is 1 or \( 1-a^2 \) (up to a square).

Secondly, we want to show that, if \( a \equiv 2, 3 \pmod{5} \) and neither \( 2(1+a) \) or \( 2(1-a) \) is a square in \( \mathbb{Q} \), \( D \) is 1 up to a square. Suppose not, then \( \det r \) has to be \( \left( \frac{1-a^2}{p} \right) \).

Now let \( p = 5 \), then \( 1-a^2 = 2 \in \mathbb{F}_p \), hence \( \det r(Frob_5) = \left( \frac{1-a^2}{p} \right) = -1 \). Then by part (2) of Lemma 4.6, we know that \( \text{tr} \rho_\ell(Frob_5) \) has to be \( \pm 1 \), or equivalently, \( \text{tr} \rho_\ell \epsilon^{-1}(Frob_5) = \pm 5 \). In the following, we will use the trace formula 4.2 to show that this is impossible by calculation.

Notice that over \( \mathbb{F}_5 \), the general fibre (cf. Lemma 4.2) are \( t = 1, 4 \). None of them contribute to the trace \( \text{tr} \rho_\ell \epsilon^{-1}(Frob_5) \) since \( t^2 + 1 = 2 \in \mathbb{F}_5 \) is not a square. So we only need to consider the special fibre \( t = 0 \) and \( \infty \). Suppose \( a \equiv 2 \pmod{5} \), then by Table 1 we know that the fiber \( t = 0 \) contributes \( -5 \) to \( \text{tr} \rho_\ell \epsilon^{-1}(Frob_5) \). By point counting, we know that the fiber \( t = \infty \) contributes \( 8 \) to \( \text{tr} \rho_\ell \epsilon^{-1}(Frob_5) \). Hence now we have

\[
\text{tr} \rho_\ell \epsilon^{-1}(Frob_5) = 8 - 5 = 3 \neq \pm 5.
\]

Similarly, suppose \( a = 3 \), then the fiber \( t = 0 \) contributes \( -7 \), and \( t = \infty \) contributes \( 8 \) to \( \text{tr} \rho_\ell(Frob_5) \). Hence

\[
\text{tr} \rho_\ell \epsilon^{-1}(Frob_5) = 8 - 7 = 1 \neq \pm 5.
\]
By all above, one sees that for each $a = 2$ or $3 \pmod{5}$, we have $\text{tr} \rho_\ell \varepsilon_\ell^{-1}(\text{Frob}_p) \neq \pm 5$, hence we obtain a contradiction. So we are done. □

Remark 4.3. In fact, $\left(\frac{2}{5}\right) = -1$ in Theorem 1.2 is only a technical condition and seems is easy to generalize. For instance one can also show that if $a = 3$ or $4 \pmod{7}$, then $\det \rho_2$ is trivial.

Now combine all the results above, we are able to give a proof to Theorem 1.2.

Proof of Theorem 1.2. By Propositions 4.5 and 4.7, the surface $S_a$ satisfies all the conditions of Theorem 1.1.

Then by Theorem 1.1 and Corollary 1.1.1, for a Dirichlet density one subset of primes $\ell$, the corresponding Tate conjecture for $S_a$ is true. □

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