APPLICATIONS OF ALGEBRAIC COMBINATORICS TO ALGEBRAIC GEOMETRY

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Dedicated to the memory of Tonny Springer

Abstract. We formulate a number of new results in Algebraic Geometry and outline their derivation from Theorem 2.4 which belongs to Algebraic Combinatorics. Theorem 7.3 is new, while complete proofs of other results are in [7] and [8].

1. Introduction

Let $K$ be a field. It will be important to distinguish between an algebraic variety $X$ defined over $K$ and the set $X := X(K)$ of $K$-points of $X$. We denote by $A$ the affine line which is an one-dimensional algebraic variety whose $K$-points are $K$. To a $K$-vector space $V$ one canonically associates an algebraic $K$-variety $V$ such that $V(K) = V$. To a family $\bar{P} = \{P_1, \ldots, P_c\}$ of polynomials on $V$ one associates a morphism $\bar{P} : V \to A^c$. For any $\bar{t} \in K^c$ we associate a subscheme $F_{\bar{t}}(\bar{P}) := \bar{P}^{-1}(\bar{t}) \subset V$. We write $X_{\bar{P}} := \bar{P}^{-1}(0)$.

Definition 1.1. (1) Let $P$ be a polynomial of degree $d$ on a $K$-vector space $V$. We define the rank $r(P)$ as the minimal number $r$ such that $P$ can be written in the form $P = \sum_{i=1}^{r} Q_i R_i$, where $Q_i, R_i$ are polynomials on $V$ of degrees $< d$.

(2) For a family $\bar{P} = \{P_i\}_{1 \leq i \leq c}$ of polynomials on $V$ we define the rank $r(\bar{P})$ as the minimal rank of polynomials $P_{\bar{a}} := \sum_{i=1}^{c} a_i P_i$, $\bar{a} \in k^c \setminus \{0\}$.

Example 1.2. If $P : V \to K$ is a non-degenerate quadratic form then $\dim(V)/2 \leq r(\bar{P}) \leq \dim(V)$.

We show that morphisms $\bar{P} : V \to A^c$ for families of sufficiently high rank possess a number of nice properties for fields of characteristic $> |d| := \max_i \deg(P_i)$. In particular we show that these morphisms are flat and that their fibers are complete intersections with rationally singularities.

In the case of fields of small characteristic we have to replace the rank $r(\bar{P})$ by the non-classical rank $r_{nc}(\bar{P})$.

Definition 1.3. Let $P$ be a polynomial of degree $d$ on a $K$-vector space $V$.

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(1) We denote by $\tilde{P} : V^d \to k$ the multilinear symmetric form given by $\tilde{P}(h_1, \ldots, h_d) := \Delta_{h_1} \ldots \Delta_{h_d}P : V^d \to k$, where $\Delta_hP(x) := P(x + h) - P(x)$.

(2) We define the non-classical rank (nc-rank) $r_{nc}(P)$ to be the rank of $\tilde{P}$.

(3) For a family $\tilde{P} = \{P_i\}_{1 \leq i \leq c}$ of polynomials on $V$ we define the nc-rank $r_{nc}(\tilde{P})$ as the minimal nc-rank of polynomials $P_\tilde{a} := \sum_{i=1}^{c} a_iP_i$, $\tilde{a} \in k^c \setminus \{0\}$.

Remark 1.4. (1) If $\text{char}(k) > d$ then $r(P) \sim r_{nc}(P)$.

(2) In low characteristic it can happen that $P$ is of high rank while $\tilde{P}$ is of low rank.

Example 1.5. Let $K$ be a field of characteristic 2, $V = A^n$ and $P(x_1, \ldots, x_n) = \sum_{1 \leq i < j < k < l} x_i x_j x_k x_l$ is of rank $\sim n$, but of nc-rank 3, (see [15]).

We will denote finite fields by $k$ and general fields by $K$.

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2. The main tool

Definition 2.1. Let $k := F_q$, let $V$ be a $k$-vector space and $\tilde{P} : V \to k^c$ be a map. We define

(1) $F_\tilde{P}(P) := \{v \in V|\tilde{P}(v) = \bar{t}\}, \bar{t} \in k^c$.

(2) $\nu_P : k^c \to \mathbb{C}$ is the function given by $\nu_P(\bar{t}) := |F_\tilde{P}(P)|/q^{\dim(V)c}$.

(3) A map $\tilde{P}$ is $s$-uniform if $|\nu_P(\bar{t}) - 1| \leq q^{-s}$ for all $\bar{t} \in k^c$.

Lemma 2.2. Let $\tilde{P} = \{P_1, \ldots, P_c\}, P_i : V \to k$ be a family of maps such that all the maps $P_\tilde{a} := \sum_{i=1}^{c} a_iP_i$, $\tilde{a} \in k^c \setminus \{0\}$ are $s$-uniform. Then the map $\tilde{P}$ is $(s - c)$-uniform.

Proof. We start the proof with the following general well known result.

Let $A$ be a finite commutative group and $\Xi$ be the group of characters $\chi : A \to \mathbb{C}^\times$. For a function $f : A \to \mathbb{C}$ we define $\hat{f}(\chi) = 1/|A| \sum_{a \in A} \chi(a)f(a)$.

Claim 2.3. $\sum_{\chi \in \Xi} \hat{f}(\chi)(-a_0) = f(a_0)$ for any $a_0 \in A$.

Let $A = k^c$. We fix a non-trivial additive character $\psi : k \to \mathbb{C}^\times$ and associate with any $\bar{a} \in k^c$ a character $\chi_{\bar{a}} : k^c \to \mathbb{C}^\times$ by $\chi_{\bar{a}}(\bar{t}) = \psi(\bar{a}, \bar{t})$ where $<, > : k^c \times k^c \to k$ is the natural pairing. The map $\bar{a} \to \chi_{\bar{a}}$ is a bijection between $A$ and the group $\Xi$ of characters of $A$.

The Fourier transform of the function $\nu$ is given by $\hat{\nu}(\chi_{\bar{a}}) = q^{-c}\sum_{\bar{t} \in k^c} \nu(\bar{t})\chi(\bar{t}) = q^{-\dim(V)c}\sum_{s \in V} \psi(P_{\bar{a}}(v))$. Since, by the assumptions of Lemma, maps $P_{\bar{a}}$ are $s + c$-uniform for all $\bar{a} \neq 0$ we see that $|\hat{\nu}(\chi_{\bar{a}})| \leq q^{s+c}$ for $\bar{a} \neq 0$. 

As follows from Claim 2.3 we have \(|\hat{\nu}(\chi_{\bar{a}})| \leq q^{-s}\) for \(\bar{a} \neq 0\). On the other hand Claim 2.3 shows that

\[
\nu_{\bar{P}}(\bar{b}) = \sum_{\bar{a} \in k^c} \psi_{\bar{a}}(-\bar{b}) \hat{\nu}_{\bar{P}}(\bar{a}) = 1 + \sum_{\bar{a} \in k^c \setminus \{0\}} \psi_{\bar{a}}(-\bar{b}) \hat{\nu}_{\bar{P}}(\bar{a})
\]

for all \(\bar{b} \in k^c\). So \(|\nu_{\bar{P}}(\bar{b}) - 1| \leq q^{-s}\) \(\square\)

The following result from Algebraic Combinatorics, (see [13] and [5]).

**Theorem 2.4.** There exist explicitly definable functions \(e(d), a(d) > 0\) such that or any \(d\) the following holds. Let \(V\) be a \(F_q\)-vector space. Then any polynomial \(P: V \to k\) of degree \(d\) is \(s\)-uniform if \(r_{nc}(P) \geq e(d)s^{a(d)}\).

**Remark 2.5.**

1. This result from Algebraic Combinatorics the corner stone of our work.
2. A weaker form of this result which shows the existence of functions \(\alpha_d(r)\), \(\lim_{r \to \infty} \alpha_d(r) = \infty\) such that polynomials \(P: V \to k\) of degree \(d\) are \(\alpha_d(r(P))\) uniform was proven earlier in [2].
3. The effectiveness of lower bound for \(r_{nc}(P)\) is important for some of our results.
4. We provide a proof of a generalization of the result in [2] in the Appendix.
5. The theorem was proved for partition rank, but as shown in [7] the partition rank is proportional to the \(nc\)-rank.

**Question 2.6.** Could one assume that \(a(d) = 1\)?

**Definition 2.7.** For \(\bar{d} = (d_1, \ldots, d_c)\) we denote by \(r(\bar{d})\) the minimal number \(r\) such that (in notations of Theorem 2.4) we have \(e_{d'}r^{a_{d'}} \geq c + 2\) for all \(d' \leq |\bar{d}|\).

The following result, to which we often refer, follows immediately for Lemma 2.2 and Theorem 2.4.

**Theorem 2.8 (Uniform).** Let \(\bar{P} = \{P_1, \ldots, P_c\}\) be a family of polynomials of degree \(\bar{d}\) and of \(nc\)-rank \(r \geq r(\bar{d})\) we have \(|a_{\bar{P}}(t) - q^{-c}| \leq q^{-(c+2)}\) for all \(t \in k^c\).

### 3. Irreducibility of Fibers

In this section we show a derivation of the following result from Theorem 2.8

**Theorem 3.1.** For any field \(K\) and a family \(\bar{P} = \{P_1, \ldots, P_c\}\), \(P_i \in K[x_1, \ldots, x_n]\) of degrees \(\bar{d}\) and \(nc\)-rank \(\geq r(\bar{d})\) all the fibers \(\mathbb{F}_t(\bar{P}) := \bar{P}^{-1}(t) \subset A^n, t \in k^c\) are irreducible varieties of dimension \(n - c\).

**Proof.** As well known (see Krull’s principal, [11]) any irreducible component \(Y\) of \(\mathbb{F}_t(\bar{P})\) is of dimension \(\geq (n - c)\). So it is sufficient to show that varieties \(\mathbb{F}_t(\bar{P})\) are irreducible and of dimension \(\leq (n - c)\).

We first consider the case when \(K\) is a finite field when we can use the following result (see [10]).
Let $k := \mathbb{F}_q$, $k_l := \mathbb{F}_{q^l}$, $X$ be an $m$-dimensional algebraic variety defined over $k$ and $c(X)$ be the number of irreducible components of $X$ of dimension $m$ (considered as a variety over the algebraic closure $\bar{k}$ of $k$). We define $\tau_l(X) := \frac{|X(k_l)|}{q^m}$, $l \geq 1$.

**Lemma 3.2.** There exists $u \geq 1$ such that $\lim_{l \to \infty} \tau_{lu}(X) = c(X)$.

To prove Theorem 3.1 in the case when $K = \mathbb{F}_q$ we observe that Lemma 3.2 and Theorem 2.8 imply that $\dim(F_{\bar{t}}(\bar{P})) = n - c$ and $c(F_{\bar{t}}(\bar{P})) = 1$.

Now we consider the case when $K$ is an algebraic closure of $\mathbb{F}_q$. Since $K = \bigcup \mathbb{F}_{q^n}$ we may assume that $\bar{t} \in \mathbb{F}_{q^n}^c$. So $\dim(F_{\bar{t}}(\bar{P})) = n - c$ and $c(F_{\bar{t}}(\bar{P})) = 1$. Therefore Theorem 3.1 is proven in the case when $K$ is an algebraic closure of a finite field.

We start the reduction of the general case of Theorem 3.1 to the case when $K$ is an algebraic closure of a finite field with a reformulation.

**Definition 3.3.** Let $\bar{d} = (d_1, \ldots, d_c)$ and $n \geq 1$. A field $K$ has the property $\star(n, \bar{d})$ if for any family $\bar{P} = \{P_i\}_{i=1}^c$ of polynomials $P_i \in K[x_1, \ldots, x_n]$ of degrees $d_i$ and $nc$-rank $\geq r(\bar{d})$ all varieties $F_{\bar{t}}$, $\bar{t} \in K^c$ are irreducible of dimension $\dim(V) - c$.

**Claim 3.4 (n).** All fields have the property $\star(n, \bar{d})$.

**Proof.** It is clear that it is sufficient to prove this claim for algebraic closed fields. Our proof of Claim 3.4(n) uses the following result from Model theory (see [12]).

**Claim 3.5.** Let $T$ be the theory of algebraically closed fields. Then any first order property in $T$ true for algebraic closure of finite fields is true for all algebraic closed fields.

Since $\star(n, \bar{d})$ is a first order property in $T$, Claim 3.4(n), $n \geq 1$ are proven.

Since the constant $r(\bar{d})$ does not depend on $n$, the validity of Claim 3.4(n) for all $n \geq 1$, implies the validity of Theorem 3.1.

**4. Universal**

**Definition 4.1.** A family $\bar{P} = \{P_i\}_{1 \leq i \leq c}$ of degree $\bar{d} = (d_1, \ldots, d_c)$ of polynomials on a $K$-vector space $V$ is $m$-universal if for any family $\bar{Q} = \{Q_i\}_{1 \leq i \leq c}$ of polynomials $Q_i \in K[x_1, \ldots, x_m]$ of degrees $\bar{d}$ there exists an affine map $\phi : K^m \to V$ such that $Q_i = P_i \circ \phi$ for $1 \leq i \leq c$.

**Theorem 4.2 (Universal).** There exists a function $R(\bar{d}, m)$ such that any family $\bar{P}$ of polynomials degree $\bar{d}$ and $nc$-rank $\geq R(\bar{d}, m)$ is $m$-universal where $K$ is a field which is either finite or algebraically closed.
Proof. To simplify notations we only consider the case when $c = 1$ and therefore $\bar{P} = P$. Let $W$ be the vector space of affine maps $\phi : K^m \to V$ and $L$ be the vector space of polynomials $Q \in K[x_1, \ldots, x_m]$ of degree $\leq d$. Choose a basis $\lambda_i, i \in I$ of the dual space to $L$. For any polynomial $P$ of degree $d$ on $V$ we define the map $\bar{R} : W \to K^I$ given by $\phi \to \{R_i(\phi)\}_{i \in I}$. We have to show the surjectivity of the map $\bar{R}(P)$. This surjectivity follows immediately from the following result (see Claim 3.11 in [7]).

Claim 4.3. For any $r$ there exists $h(r, d, m)$ such that the nc-rank of $\bar{R}(P)$ is $\geq r$ for any polynomial $P$ on $V$ of nc-rank $\geq h(r, d, m)$. □

5. Weakly polynomial functions

We start the next topic with a couple of definitions.

Definition 5.1. (1) Let $K$ be a field, $V$ be a $K$-vector space and $X$ a subset of $V$. A function $f : X \to K$ is weakly polynomial of degree $\leq a$, if for any affine subspace $L \subset X$ the restriction of $f$ on $L$ is a polynomial of degree $\leq a$.

(2) An algebraic $K$-subvariety $X \subset V$ satisfies the condition $\star a$ if any weakly polynomial function of degree $\leq a$ on $X$ is a restriction of a polynomial function of degree $\leq a$ on $V$.

The following example demonstrates the existence of cubic surfaces $X \subset K^2$ which do not have the property $\star 1$ for any field $K \neq \mathbb{F}_2$.

Example 5.2. Let $V = K^2$, $Q = xy(x - y)$. Then $X = X_0 \cup X_1 \cup X_2$ where $X_0 = \{v \in V| x = 0\}, X_1 = \{v \in V| y = 0\}, X_2 = \{v \in V| x = y\}$. The function $f : X \to K$ such that $f(x, 0) = f(0, y) = 0, f(x, x) = x$ is weakly linear but one can not extend it to a linear function on $V$.

Definition 5.3. For $e \geq 1$ we say that a field $K$ is $e$-admissible if $K^*$ contains the subgroup $C$ of size $> e$.

Theorem 5.4 (Extension). There exists an $S = S(a, d)$ such that any hypersurface $Y \subset V$ of degree $d$ and nc-rank $\geq S$ satisfies the condition $\star a$ if $K$ is an ad-admissible field which is either finite or algebraically closed.

Remark 5.5. (1) The main difficulty in a proof of Theorem 5.4 is the non-uniqueness of an extension of $f$ to a polynomial on $V$ in the case when $a > d$.

(2) An analogous statement is true for weakly polynomial functions on subsets $X_\bar{P} = \{P_i\}_{i=1}^n$ where $\bar{P}$ is a family of a sufficiently high nc-rank.

Proof. We fix the degree $d$ of $P$. The proof consists of two steps. In the first step we construct a family $X_m \subset V_m$ of hypersurfaces of degree $d$ and nc-rank $\geq m$.
Definition 5.6.  

1. \( W := \mathbb{A}^d \) and \( \mu : W \to \mathbb{A} \) is the product \( \mu(x^1, \ldots, x^d) := \prod_{j=1}^d x^j \).

2. \( V_m := W^m \) and \( Q_m(w_1, \ldots, w_m) := \sum_{i=1}^m \mu(w_i) \).

3. \( X_m = X_m Q_m \).

Proposition 5.7.  

1. \( r_{nc}(Q_m) \geq m \).

2. For any \( ad \)-admissible field \( K \) the subvariety \( X_m(K) \subset V_m(K) \) has the property \( *_a \).

Proof. The inequality \( r_{nc}(Q_m) \geq m \) follows immediately from Lemma 16.1 of [14].

To outline a proof of the second statement we introduce a number of definitions. We fix \( m \) and write \( X \) instead of \( X_m \). Since our field \( K \) is \( ad \)-admissible the group \( K^* \) contains a finite subgroup \( C \) isomorphic to \( \mathbb{Z}_{ad} \).

Definition 5.8.  

1. \( W_1 := \{(x^1, \ldots, x^d) \in W \mid x^i = 1, i \geq 2 \} \).

2. \( L := W_1^m \cap X \).

3. \( H \subset C^d \subset (K^*)^d \) is the kernel of the product map \( \mu_C : C^d \to C \). The group \( H \) acts on \( W \) by \((c^1, \ldots, c^d)(x^1, \ldots, x^d) = (c^1 x^1, \ldots, c^d x^d)\).

4. \( \Theta \) is the group of characters \( \theta : H^m \to K^* \).

5. We write elements of \( V_n \) in the form \( v = (w_1, \ldots, w_n), 1 \leq i \leq n, w_i \in W \). The group \( H^m \) acts on \( X_m \subset V_m \) by \((h_1, \ldots, h_m)(w_1, \ldots, w_m) = (h_1 w_1, \ldots, h_m w_m) \).

6. \( P_a^w(X) \subset k[X] \) is the subspace of weakly polynomial functions of degree \( \leq a \) on \( X \).

7. \( P_a(X) \subset P_a^w(X) \) is the subspace of functions \( f : X \to k \) which are restrictions of polynomial functions on \( V \) of degree \( \leq a \).

8. For denote by \( P_a^w(X)^\theta \subset P_a^w(X), P_a(X)^\theta \subset P_a(X) \), \( \theta \in \Theta \) the subspaces \( \theta \)-eigenfunctions.

Since \( C \subset K^* \) we have direct sum decompositions \( P_a^w(X) = \oplus_{\theta \in \Theta} P_a^w(X)^\theta \) and \( P_a(X) = \oplus_{\theta \in \Theta} P_a(X)^\theta \). Therefore for a proof of Proposition 5.7 it is sufficient to show the equality \( P_a^w(X)^\theta = P_a(X)^\theta \) for \( \theta \in \Theta \).

Fix \( f \in P_a^w(X)^\theta \). Since \( L \subset V \) is a linear subspace the restriction \( f|_L \) extends to a polynomial on \( V \). So (after the subtraction of a polynomial) we may assume that \( f|_L \equiv 0 \). We show that any weakly admissible function \( f \in P_a^w(X)^\theta \) vanishing on \( L \) is identically 0. So \( f \in P_a(X)^\theta \).  \( \square \)
5.2. The second step. The proof of the general case of Theorem 5.4 is based on the following result.

Proposition 5.9. There exists a function $r(d,a)$ such that the following holds. Let $K$ be a field which is either finite or algebraically closed, $V$ a $K$-vector space, $P$ a polynomial of degree $d$ and $W \subset V$ an affine subspace such that $\text{nc-rank}$ of the restriction of $P$ on $W$ is $P \geq r(d,a)$. Then any weakly polynomial function $f$ on $X$ of degree $\leq a$ such that $f|_{X \cap W}$ extends to a a polynomial on $W$ of degree $\leq a$ is a restriction of a polynomial of degree $a$ on $V$.

Proof. After a subtraction of a polynomial from $f$ we may assume that $f|_{X \cap W} \equiv 0$. Using the induction on the codimension of $W$ we reduce the Proposition to the case when $W \subset V$ is a hyperplane. We fix a direct sum decomposition $V = W \oplus K$ and denote by $t: V \to K$ the projection. Our proof is by induction in $a$.

The function $\bar{g} := f/t$ is defined on $X \setminus X \cap W$. We start with a construction of an extension of $\bar{g}$ to a function $g$ on $X$. Given a point $y \in X \cap W$ consider the set $\mathcal{L}$ of lines $L \subset X$ such that $L \cap W = \{y\}$. Since $f$ is weakly polynomial, the restriction $f_L$ is a polynomial $p_L(t)$ vanishing at 0. We define $g_L(y)$ as the value of $p_L(t)/t$ at 0. It is clear that the following two results imply the validity of Proposition 5.9.

Claim 5.10. Under the conditions of Proposition 5.9 there exist $g(y) \in k$ such that $|\mathcal{L}^-|/|\mathcal{L}| < 1/q$ where $\mathcal{L}^- = \{L \in \mathcal{L} \mid g_L(y) \neq g(y)\}$.

We extend $\bar{g}$ to a function on $X$ whose values on $y \in X \cap W$ are equal to $g(y)$.

Claim 5.11. The function $g : X \to k$ is weakly polynomial of degree $a - 1$.

Now we can finish the proof of Theorem 5.4. Fix $m$ such that $r_{nc}(X_m) \geq r(d,a)$. Let $R(d,m)$ be as Theorem 4.2. I claim that for any admissible field $K$ which is either finite or algebraically closed, any hypersurface $Y \subset V$ of degree $d$ and $\text{nc-rank} \geq R(d,m)$ satisfies $\chi_n^Y$. Really, let $f$ be a weakly polynomial function on $X = X_P$ of degree $\leq a$ where $P : V \to K$ is a polynomial of degree $d$ of $\text{nc-rank} \geq R(d,m)$. Since $r_{nc}(P) \geq R(d,m)$ there exists an affine map $\phi : K^m \to V$ such that $P \circ \phi = Q_m$. It is clear that the function $f \circ \phi$ is a weakly polynomial function on $X_m$ of degree $\leq a$. Therefore it follows from Proposition 5.7 that the restriction of $f$ on $\text{Im}(\phi) \cap X$ extends to a polynomial on $\text{Im}(\phi)$. It follows now from Proposition 5.9 that $f$ extends to a polynomial on $V$.

\Box

6. Nullstellensatz

Let $k$ be a field and $V$ be a finite dimensional $k$-vector space. We denote by $\exists V$ the corresponding $k$-scheme, and by $\mathcal{P}(V)$ the algebra of polynomial functions on $V$ defined over $k$. 

For a finite collection $\bar{P} = (P_1, \ldots, P_c)$ of polynomials on $V$ we denote by $J(\bar{P})$ the ideal in $\mathcal{P}(V)$ generated by these polynomials, and by $X_{\bar{P}}$ the subscheme of $V$ defined by this ideal.

Given a polynomial $R \in \mathcal{P}(V)$, we would like to find out whether it belongs to the ideal $J(\bar{P})$. It is clear that the following condition is necessary for the inclusion $R \in J(\bar{P})$.

(N) $R(x) = 0$ for all $k$-points $x \in X_{\bar{P}}(k)$.

**Proposition 6.1** (Nullstellensatz). Suppose that the field $k$ is algebraically closed and the scheme $X_{\bar{P}}$ is reduced. Then any polynomial $R$ satisfying the condition (N) lies in $J(\bar{P})$.

We will show that the analogous result hold for $k = \mathbb{F}_q$ if $X_{\bar{P}}$ is of high $nc$-rank.

From now on we fix a degree vector $\bar{d} = (d_1, \ldots, d_c)$ and write $D := \prod_{i=1}^c d_i$.

We denote by $\mathcal{P}_d(V)$ the space of $\bar{d}$-families of polynomials $\bar{P} = (P_i)_{i=1}^c$ on $V$ such that $\deg(P_i) \leq d_i$.

**Theorem 6.2.** There exists and an effective bound $r(\bar{d}) > 0$ such that for any finite field $k = \mathbb{F}_q$, of characteristic $> d$, any larger than $r_{nc}(\bar{d})$ and any polynomial $R$ of degree $a$ such that $q > aD$, the vanishing condition (N) implies that $R$ lies in the ideal $J(\bar{P})$.

**Proof.** We start with the following rough bound (see [4]).

**Lemma 6.3.** Let $\bar{P} = \{P_i\}_{i=1}^c \subset \mathbb{F}_q[x_1, \ldots, x_n]$ be a family of polynomials of degrees $d_i, 1 \leq i \leq c$ such that the variety $X \subset \mathbb{A}^n$ is of dimension $n - c$. Then $|X(\mathbb{F}_q)| \leq \prod_{i=1}^c d_i q^{n-c}$.

For a convenience a reader we reproduce the proof.

**Proof.** Let $F$ be the algebraic closure of $\mathbb{F}_q$. Then $X(\mathbb{F}_q)$ is the intersection of $X$ with hypersurfaces $Y_j, 1 \leq j \leq n$ defined by the equations $h_j(x_1, \ldots, x_n) = 0$ where $h_j(x_1, \ldots, x_n) = x_j^q - x_j$.

Let $H_1, \ldots, H_{n-c}$ be generic linear combinations of the $h_j$ with algebraically independent coefficients from an transcendental extension $F'$ of $F$ and $Z_1, \ldots, Z_{c+1} \subset \mathbb{A}^n$ be the corresponding hypersurfaces.

Intersect successively $X$ with $Z_1, Z_2, \ldots$. Inductively we see that for each $j \leq n-c$, each component $C$ of the intersection $X \cap Z_1 \cap \cdots \cap Z_j$ has dimension $n-c-j$. Really passing from $j$ to $j+1$ for $j < n-c$ we have $\dim(C) = n-c-i > 0$. So not all the functions $h_j$ vanish on $C$. Hence by the genericity of the choice of linear combinations $\{H_j\}$ we see that $H_{j+1}$ does not vanish on $C$ and therefore $Z_{j+1} \cap C$ is of pure dimension $n-c-j-1$. Thus the intersection $X \cap Z_1 \cap \cdots \cap Z_{n-c}$ has dimension 0. By Bezout’s theorem we see that $|X \cap Y_1 \cap \cdots \cap Y_{n-c}| \leq \prod_{i=1}^c d_i q^{n-c}$.

Since $X(\mathbb{F}_q) = X \cap Z_1 \cap \cdots \cap Z_n \subset X \cap Y_1 \cap \cdots \cap Y_{n-c}$ we see that $|X(\mathbb{F}_q)| \leq \prod_{i=1}^c d_i q^{n-c}$. \qed
Now we can finish the proof of Theorem 7.2. Let \( R \in \mathbb{F}_q[x_1, \ldots, x_n] \) be a polynomial of degree \( a \) vanishing on the set \( X(\mathbb{F}_q) \). Suppose that \( R \) does not lie in the ideal generated by the \( P_i \), \( 1 \leq i \leq c \). Then the variety \( \mathbb{Z} \) cut out by the \( P_i \) and \( R \) has pure codimension \( c+1 \) and the sum of the degrees of its components is at most the product \( aD = d_1 \ldots d_c \). As follows from Theorem 7.2 there exists an effective bound \( r(\bar{d}) > 0 \) such that the condition \( r_{ne}(\bar{P}) \geq r(\bar{d}) \) implies the inequality \( |X_p(\mathbb{F}_q)| \geq q^{\dim(V) - c} / 2 \).

On the other hand we Lemma 6.3 shows that \( |\mathbb{Z}(\mathbb{F}_q)| \leq Dq^{\dim(V) - c - 1} \). But we assumed that \( X(\mathbb{F}_q) = \mathbb{Z}(\mathbb{F}_q) \). This construction shows that \( R \) belongs to the ideal generated by the \( P_i \), \( 1 \leq i \leq c \).

\[ \square \]

7. Rational singularities

**Definition 7.1.** Let \( X \) be a normal irreducible variety over a field of characteristic zero and \( a : \bar{X} \to X \) a resolution of singularities. We say that \( X \) has rational singularities if \( Ra_i(\mathcal{O}_{\bar{X}}) = \{0\} \) for \( i > 0 \).

**Remark 7.2.** This property of \( X \) does not depend on a choice of a resolution \( a : \bar{X} \to X \).

**Theorem 7.3.** There exists a function \( r(\bar{d}), \bar{d} = (d_1, \ldots, d_c) \) such that the following holds.

Let \( \bar{P} = \{P_i\}_{i=1}^c \subset \mathbb{C}[x_1, \ldots, x_n] \) be a family of polynomials of degrees \( d_i, 1 \leq i \leq c \) of rank \( r(\bar{d}) \). Then the variety \( \bar{X}_P \) has rational singularities.

**Proof.** To simplify the exposition we assume that \( \bar{d} = \{d\} \). So \( \bar{P} = \{P\} \) where \( P \in \mathbb{C}[x_1, \ldots, x_n] \) is a homogeneous polynomial of degree \( d \).

We first consider the case when \( P \in K[x_1, \ldots, x_n] \) where \( K/\mathbb{Q} \) is a finite extension. In this case there exists an infinite set \( S \) of prime ideals in \( \mathcal{O}_K \) such that

1. for any \( \pi \in S \) the completion \( \mathcal{O}_{K,\pi} \) of \( \mathcal{O}_K \) at \( \pi \) is isomorphic to \( \mathbb{Z}_p \) where \( p = \text{char}(\mathcal{O}_K/\pi) \),
2. \( P \in \mathcal{O}_{K,\pi}[x_1, \ldots, x_n] \) and
3. the reduction \( \bar{P} \in \mathbb{F}_p[x_1, \ldots, x_n] \) is of rank \( r(P) \).

We fix \( \pi \in S \) such that \( p > \) the degree of \( P \).

As follows from Theorem A of [1], the inequality

\[
(*) |X(\mathbb{Z}/p^m\mathbb{Z})| - 1 | \leq p^{-1/2}, m \geq 1
\]

would imply that that the variety \( \bar{X}_P \) has rational singularities.

**Definition 7.4.**

1. We fix \( l \geq 1 \) and write \( A_l = \mathbb{Z}/p^l\mathbb{Z} \).
2. For a map \( P : A^n \to A \) and \( a \in A \) we write \( \nu(a) := p^{l(1-n)|P^{-1}(a)|} \).
3. A map \( P \) is \( s \)-uniform if \( |\frac{\nu(a)}{p^m} - 1| \leq p^{-s} \).
4. \( \Xi \) is the group of additive characters \( \chi : A \to \mathbb{C}^* \).
For $\chi \in \Xi$ we denote by $d(\chi)$ the smallest number $d$ such that $\chi|_{p^dA} \equiv 1$.

(6) $\Xi_d \subset \Xi$ is the subset of characters $\chi$ such that $d(\chi) = d$.

For $\chi \in \Xi$ we define $b(P, \chi) := p^{-nd} \sum_{v \in A^n} \chi(P(v)) = p^{-nd} \sum_{a \in A} \nu(a)\chi(a)$.

Claim 7.5. If $|b(P; \chi)| < p^{-sd(\chi)}$ for all $\chi \in \Xi$ then $P$ is $(s - 2)$-uniform.

Proof. The proof is completely analogous to the proof of Lemma 2.2.

We see that the validity of Theorem 7.3 in the case when $P \in K[x_1, \ldots, x_n]$ where $K/Q$ is a finite extension is implied by the following result which is proven in the next section.

Proposition 7.6. There exists a function $r(d, s)$ such that for any polynomial $P : A^n \rightarrow A$ of degree $d$ with the reduction $\tilde{P} : \mathbb{F}_p^n \rightarrow \mathbb{F}_p$ for $p > d$ of rank $\geq r(d, s)$ we have $|b(\tilde{P}; \chi)| < p^{-sd(\chi)}$ for all $\chi \in \Xi_d$.

In the rest of this section show how to derive the general case of Theorem 7.3 from the case when $P \in K[x_1, \ldots, x_n]$ where $K/Q$ is a finite extension.

Definition 7.7. Let $a : X \rightarrow Y$ be a morphism between complex algebraic varieties such that all the fibers $X_y$ are normal and irreducible. We write $Y = Y(\mathbb{C})$ and denote by $Y_a \subset Y$ the subset of points $y$ such that the fiber $X_y$ has rational singularities.

Claim 7.8. If $a$ be a projective morphism defined over $\mathbb{Q}$ then the subset $Y_a \subset Y$ is also defined over $\mathbb{Q}$.

Proof. The proof is by induction in $d = \dim(Y)$. Suppose that Claim is known in the case when the dimension of the base is $< d$. Let $t$ be the generic point of $Y$ and $X_t$ the fiber of $X$ over $t$. Fix a resolution $\tilde{b} : \tilde{X}_t \rightarrow X_t$ over the field $k(t)$ of rational functions on $Y$. Then there exists a non-empty open subset $U \subset Y$ such that $b := \tilde{b} \circ a^{-1} : Y_a \rightarrow U$ is a resolution of $U$ such that $a^{-1}(U)$. Since by definition $Y_a \cap U = \{u \in U | R^i\tilde{a}_*(\mathcal{O}_X)_u = \{0\}\}$ for all $i > 0$. Since the sheaves $R^i\tilde{a}_*(\mathcal{O}_X)$ are coherent we see that the subset $Y_a \cap U$ is defined over $\mathbb{Q}$. On the other hand, the inductive assumption implies that $Y_a \cap (Y \setminus U)$ is defined over $\mathbb{Q}$. □

Now we can finish a proof of Theorem 7.3 in the general case. Consider the trivial fibration $\tilde{a} : \mathbb{A}^n \times Y \rightarrow Y$ where $Y \subset \mathbb{P}^n_d$ is the variety of polynomials of degree $d$ on $\mathbb{A}^n$ and of rank $\geq r(d)$. Let $X \subset \mathbb{A}^n \times Y$ be the hypersurface such that $a^{-1}(P) \cap X = X_P$ and $a : X \rightarrow Y$ be the restriction of $\tilde{a}$ onto $X$. As follows from Theorem 3.1 and Proposition III of [14] all fibers of $a$ are irreducible and normal. For a proof of Theorem 7.3 we have to show that $Y_a = Y$. The validity of Theorem 7.3 in the case when $P \in K[x_1, \ldots, x_n]$, where $K/Q$ is any finite extension, shows that any point of $Y$ defined over a finite extension $K$ of $Q$ belongs to $Y_a$. Since the subset $Y_a$ of $Y$ is is defined over $\mathbb{Q}$ we see that $Y_a = Y$. □
8. PAdIC-BiAs-RAnk

Let $V_i = A_i^N$, and $P : V_i \to A_l$ be of degree $d$. We denote by $\hat{P}$ its reduction mod $p$. Let $\chi : A_l \to \mathbb{C}^*$ be the character $\chi(a) = e(a/p^l) = e^{2\pi a/p^l}$. We assume that $p > d$.

**Proposition 8.1.** Let $\chi : A_l \to \mathbb{C}^*$ be a primitive character. For and $s > 0$, there exists $r = r(d, s)$ such that if $\hat{P}$ is of rank $> r$ then $|b(P; \chi)| < p^{-sl}$.

**Proposition 8.2 (B).** For and $s > 0$, there exists $r = r^B(d, s)$ such that for any polynomial $S$ of degree $< d$, any $m$ if $\hat{P}$ is of rank $> r$ then

$$|\mathbb{E}_{x \in V^d} e(P(x)/p^l + S(x)/p^m)| < p^{-sl}.$$  

*Proof.* We will need the following lemma which we prove in the Appendix.

**Lemma 8.3.** Let $s > 0$. There exists $r_1(d, s)$ be such that if $\hat{P}$ is of rank $> r_1(d, s)$ then for any polynomial $S$ of degree $< d$, any $m$ we have

$$|\mathbb{E}_{x \in V^d} e(P(x)/p^l + S(x)/p^m)| < p^{-ls}.$$  

Since $p > d$ we have $P(x) = \hat{P}(x, \ldots, x)/d!$ where $\hat{P}(h_1, \ldots, h_d)$ is the symmetric tensor associated with $P$ define by $\hat{P} = \Delta_{h_d} \ldots \Delta_{h_1} P$, where $\Delta_h f(x) = f(x + h) - f(x)$. It suffices then to prove the theorem for multilinear symmetric function. We will from now on assume that $P : V^d_l \to A_l$ is a multilinear symmetric function.

From Lemma 8.3 we have that for $\hat{P}$ of rank $> r_1(d, s)$ we have any $S$ of degree $< d$, any $m$,

$$|\mathbb{E}_{x \in V^d} e(\hat{P}(x)/p + S(x)/p^m)| < p^{-s}.$$  

For any function $f : V^d_l \to \mathbb{T}$, we define the $U_d$ Gowers norms:

$$\|e(f)\|_{U_d} = (\mathbb{E}_{x, h_1, \ldots, h_d \in V^d} e(\Delta_{h_d} \ldots \Delta_{h_1} f))^{1/2^d}.$$  

It is well known that the $U_d$ norms for a monotone sequence $\|e(f)\|_{U_1} \leq \|e(f)\|_{U_2} \leq \ldots$ Furthermore if $P$ is a symmetric degree $d$ multilinear function, then

$$\|e(\hat{P}/p)\|_{U_d}^{2^d} = (\mathbb{E}_{x \in V^d} e(\hat{P}(x)/p))^{d!}.$$  

We prove Proposition 8.2 by induction on $d, l$. Let $P$ be so that $\hat{P}$ is of rank $> c(d, s)$ where

$$c(d, s) = \max \{ \max_{a \leq d} r_a(a, s), r_1(d, 2^d(r^B(d - 1, 2s + 1) + 2ds + 1)) \}.$$  

For any $d$ and $l \leq d$ this is known (by definition of $r_a(a, s)$). So we assume $l > d$.

Suppose the Proposition 8.2 holds for all degree $a$ polynomials $1 \leq a < d$, and for degree $d$ polynomials it holds for $d < j < l$. Suppose the proposition does not hold for $l$. Then there exits a degree $d$ multilinear polynomial $P : V^d_l \to A_l$, of
rank > c(d, s) and some polynomial $S : V^d_1 \to A_1$ of degree < d and $m \in \mathbb{N}$ such that:

$$|\mathbb{E}(P(x)/p^t + S(x)/p^m)| \geq 1/p^{ls}.$$  

From this it follows that

$$\frac{1}{p^{2ls}} \leq |\mathbb{E}_{x \in V^d_1}e(P(x)/p^t + S(x)/p^m)|^2$$

$$= \mathbb{E}_{x,y \in V^d_1}e((P(x+y) - P(x))/p^t + (S(x+y) - S(x))/p^m))$$

$$= \mathbb{E}_{x \in V^d_1}e((P(x+y) - P(x))/p^t + (S(x+y) - S(x))/p^m))$$

Fix $t$ and consider the inner average: by shift invariance in $x$ we have

$$a_t = \mathbb{E}_{y \in V^d_1 : y=p(x)} |\mathbb{E}_{x \in V^d_1}e((P(x+y) - P(x))/p^t + (S(x+y) - S(x))/p^m))$$

By the induction hypothesis this is $\leq \frac{1}{p^{l-1-a_t}}$: indeed we have

$$|\mathbb{E}_{y=p(x)}e((P(x+y))/p^t + S(x+y))/p^m)|$$

$$= |\mathbb{E}_{y \in V^d_1 : y=p(x)}e((P(x+y)/p^t + S(x+y))/p^m)|$$

$$= |\mathbb{E}_{y \in V^d_1}e((P(x+t+py))/p^t + S(x+t+py))/p^m)|$$

$$= |\mathbb{E}_{y \in V^d_1}e(P(y)/p^{1-d} + S(y)/p^{m-d} + S_{x,t}(py)/p^m + R_{x,t}(py))/p^t)|$$

where $R_{x,t}(py)$ is of degree $\leq (d-1)$ in $y$. By the induction hypothesis the latter is $\leq \frac{1}{p^{l-1-a_t}}$.

**Claim 8.4.** Suppose $\mathbb{E}_{t \in E} |a_t| \geq \frac{1}{p^{ls}}$ and $|a_t| \leq \frac{1}{p^{l-1-a_t}}$ then for $\frac{1}{p^{2ls}}|E|$ many $t$ we have that $|a_t| \geq \frac{1}{p^{ls}}$.

**Proof.** Let $F$ be the set where $|a_t| \geq \frac{1}{p^{ls}}$. We have that

$$\frac{1}{p^{2ls}}|E| \leq \sum_{t \in E} |a_t| \leq \sum_{t \in F} \frac{1}{p^{2l-1-a_t}} + \sum_{t \in E \setminus F} \frac{1}{p^{2l-1}}$$

Rearranging we get

$$|E| \leq |F|(p^{2ls} - 1/2) + |E| \frac{1}{2}$$

so that $|F| \geq \frac{|E|}{2(p^{2ls} - 1/2)}$.  

We obtain that for $\geq \frac{1}{p^{ls}}|V^d_1|$ many $t \in V^d_1$ we have that

$$|\mathbb{E}_{y \in V_1^d : y=p(x)} |\mathbb{E}_{x \in V^d_1}e((P(x+y) - P(x))/p^t + (S(x+y) - S(x))/p^m))| \geq \frac{1}{p^{2ls}} \geq \frac{1}{p^{l(2s+1)}}.$$
Now $\Delta_y P$ is of degree $< d$ so the induction on the degree $\Delta_t \hat{P}$ is of rank $< r^B(d - 1, 2s + 1)$.

This implies that for $t \geq \frac{1}{2p^2} |V_1^d|$ many $t \in V_1^d$, if $y \equiv t(p)$ then
\[
\left\| \mathbb{E}_{x \in V_1^d} e\left((\hat{P}(x) + y) - \hat{P}(x)\right)/p \right\|_{U_1} \geq \frac{1}{p^{r^B(d - 1, 2s + 1)}}
\]
But this now implies that
\[
\mathbb{E}_{t \in V_1^d} \left\| \mathbb{E}_{y \equiv t(p)} \mathbb{E}_{x \in V_1^d} e\left((\hat{P}(x) + y) - \hat{P}(x)\right)/p \right\|_{U_1} \geq \frac{1}{p^{r^B(d - 1, 2s + 1)} p^{2ds + 1}}
\]
But LHS is bounded by
\[
\|e(\hat{P}(x)/p)\|_{U_1}^2 \leq \|e(\hat{P}(x)/p)\|_{U_1} \leq \left| \mathbb{E}_{x \in V_1} e\left(\hat{P}(x)/p\right) \right|^{1/2d}
\]
and we are given that $\hat{P}$ is of rank $> r_1(d, 2d^2 (r^B - 1, 2s + 1) + 2ds + 1)$, contradiction.

\[\square\]

9. APPENDIX

We use the notation of Section 8. We modify the argument in [2] to obtain the following claim:\footnote{It is likely that one can obtain quantitative bounds for the claim modifying the arguments in [13].}

**Lemma 9.1.** Let $s > 0$. There exists $r_1(d, s)$ be such that if $\hat{P}$ is of rank $> r_1(d, s)$ and $p > d$ then for any polynomial $S$ of degree $< d$, any $m$ we have
\[
\left| \mathbb{E}_{x \in V_1^d} e\left(P(x)/p^l + S(x)/p^m\right) \right| < p^{-ls}.
\]

**Proof.** Let $q = p^l$, and $P : V = V_1^d \rightarrow A_1$ a polynomial of degree $d$. Let $e_q(t) = e^{2\pi i t/q}$. Denote $\mu = |\mathbb{E}_{x \in V} e(P(x))|.

We assume that
\[
q^{-s} \leq \left| \mathbb{E}_{x \in V} e(P(x) + qS(x)/p^m) \right| \leq \left| \mathbb{E}_{x \in V} e(P(x)) \right|_{U_1},
\]
so that
\[
\mu = \left| \mathbb{E}_{x \in V} e(P(x)) \right| \geq q^{-2d s}.
\]
Replacing $s$ with $2^d s$ we may assume that $\mu > q^{-s}$.

Observe that $\mu e_q(-P(x)) = \mathbb{E}_{y \in V} e_q(\Delta_y P(x))$. Indeed, we have
\[
\mathbb{E}_{y \in V} e_q(\Delta_y P(x)) = \mathbb{E}_{y \in V} e_q(P(x + y) - P(x)) = e_q(-P(x)) \mu.
\]

Let $E \subset A_1^k$ be the set defined as follows
\[
E = \{ a \in A_1^k : (a, p) = 1 \}
\]
Then
\[
|E| = q^k \left(1 - 1/p\right)^k.
\]
Fix $x \in V$ and pick $z_1, \ldots, z_k \in V$ uniformly at random. For $a \in E$ write $a \cdot z$ for $\sum_{i=1}^k a_i z_i$. For $a \in E$ let $W_a(z)$ be the random variable defined by

$$W_a(z) = e_q(\Delta_{a \cdot z} P(x))$$

**Claim 9.2.** For $a \in E$ we have $\mathbb{E} W_a = \mu e_q(-P(x))$.

*Proof.* Since $a$ has a coordinate that is coprime to $p$:

$$\mathbb{E}_{z \in V^k} W_a(z) = \mathbb{E}_{z \in V^k} e_q(\Delta_{a \cdot z} P(x)) = \mathbb{E}_y e_q(\Delta_y P(x)) = \mu e_q(-P(x)).$$

\[\square\]

**Claim 9.3.** Let $k \geq 2$. For $(a, b) \in E^2$ outside a set $F$ of size $|E|q(1-1/p)q/p)^{k-1}$ we have that $\mathbb{E} W_a W_b = \mathbb{E} W_a \mathbb{E} W_b$.

*Proof.* Fix $a, b \in E$. We calculate

$$\mathbb{E} W_a W_b = \mathbb{E}_{z \in V^k} W_a(z) W_b(z) = \mathbb{E}_{z \in V^k} e_q(\Delta_{a \cdot z} P(x) - \Delta_{b \cdot z} P(x))$$

$$= \mathbb{E}_{z \in V^k} e_q(P(x + a \cdot z) - P(x + b \cdot z)).$$

Choose an indexes $i$. Now make the change of variable $z_i \to z_i - \sum_{m \neq i} (a_m / a_i) z_m$, and then $z_i \to z_i / a_i$. We get

$$\mathbb{E}_{z \in V^k} e_q(P(x + z_i) - P(x + (b_i / a_i) z_i + \sum_{m \neq i} ((b_m - a_m b_i / a_i) z_m)$$

If for some $j \neq i$ we have that $((b_j - a_j b_i / a_i, p) = 1$ then the later is

$$\mathbb{E}_{z_i} e_q(\Delta_{z_i} P(x)) \mathbb{E}_{z_j} e_q(\Delta_{z_j} P(x)) = \mathbb{E} W_a \mathbb{E} W_b.$$ 

Otherwise for any $i, j$ we have that $((b_j - a_j b_i / a_i, p) = 1$, i.e $p|(b_j a_i - a_j b_i)$ for all $i, j$. We count the number of pairs $(a, b)$ : We have $|E|$ choices for $a$. Once $a$ is chosen we have $q(1-1/p)$ choices for $b_1$, and then $(q/p)^{k-1}$ choices for $(b_2, \ldots, b_k)$. All together $|E|q(1-1/p)(q/p)^{k-1}$ pairs. \[\square\]

Let $\Gamma : E \setminus \{0\} \to A_1$ be defined as follows

$$\Gamma((a)_{a \in E}) = \arg \min \left\{ \frac{1}{|E|} \sum_{a \in E} e_q(a) - e_q(-t) \mu \right\},$$

i.e. the value $t$ for which the expression on the right is minimal.

**Claim 9.4.** Suppose from some $z \in V^k$ we have that

$$\left(\ast\right) \quad \left| \frac{1}{|E|} \sum_{a \in E} W_a(z) - \mathbb{E}_y e_q(\Delta_y P(x)) \right| \leq \frac{1}{2q^{s+1}}.$$

Then

$$P(x) = \Gamma((\Delta_{a \cdot z} P(x))_{a \in E}).$$
Proof. We are given that
\[ \left| \frac{1}{|E|} \sum_{a \in E} e_q(\Delta_{a}(z)P(x)) - \mu e_q(-P(x)) \right| \leq \frac{1}{2q^{s+1}} \]
But since \( \Gamma((\Delta_{a}(z)P(x))_{a \in E}) \) gives the best approximation to the left average we have
\[ \left| \frac{1}{|E|} \sum_{a \in E} e_q(\Delta_{a}(z)P(x)) - \mu e_q(-\Gamma((\Delta_{a}(z)P(x))_{a \in E})) \right| \leq 1/2q^{s+1} \]
Thus we get that
\[ |\mu e_q(-P(x)) - \mu e_q(-\Gamma((\Delta_{a}(z)P(x))_{a \in E}))| \leq 1/q^{s+1} \]
and since \( |\mu| \geq 1/q^s \) we get
\[ |e_q(-P(x)) - e_q(-\Gamma((\Delta_{a}(z)P(x))_{a \in E}))| \leq 1/q \]
Since for any \( t \neq t' \in A_l \) we have that
\[ |e_q(t) - e_q(t')| > 1/q \]
We get the desired result. \( \square \)

We show that (\*) holds for almost all \( z \):
\textbf{Claim 9.5.} For \( k = 2lt + 4l(s + 1) + 4 \) we have
\[ \mathbb{P}\left( \left| \frac{1}{|E|} \sum_{a \in E} W_a(z) - E_y e_q(\Delta_y P(x)) \right| > \frac{1}{2q^{s+1}} \right) < \frac{1}{q^l} \]

Proof. Let \( Y = \frac{1}{|E|} \sum_{a \in E} W_a(z) \). Then \( EY = E_y e_q(\Delta_y P(x)) \). We calculate the variance \( \text{Var}(Y) \):
\[ \mathbb{E}_{z \in V^*} \left| \frac{1}{|E|} \sum_{a \in E} W_a(z) - E_y e_q(\Delta_y P(x)) \right|^2 \]
\[ = \frac{1}{|E|^2} \sum_{(a,b) \in F} \left( \mathbb{E}_z W_a(z) \bar{W}_b(z) - \mathbb{E}_z W_a(z) \mathbb{E}_z \bar{W}_b(z) \right) \leq \frac{2|F|}{|E|^2} = \frac{1}{(p-1)^{k-1}} \]
Thus by Chebychev we have
\[ \mathbb{P}(|Y - EY| > \frac{1}{2q^{s+1}}) \leq \frac{4q^{2(s+1)}}{(p-1)^{(k-1)}} \]
For the claim we need:
\[ \frac{1}{(p-1)^{(k-1)}p^{2-2l(s+1)}} \leq \frac{1}{p^l} \]
Since \( (p-1)^2 > p \) for \( p > 2 \), it suffices to have
\[ \frac{1}{p^{(k-1)/2-2-2l(s+1)}} \leq \frac{1}{p^l} \]
So \( k > 2lt + 4l(s + 1) + 4 \) will do. \( \square \)
Thus we have that for all \( x \in V \) for \( q^{-t} \) a.e. \( z \in V^k \) we have that
\[
P(x) = \Gamma(\Delta_{a,z}P(x) : a \in E).
\]
Changing order of summation we obtain that there exists \( z \in V^k \) such that for \( q^{-t} \) a.e. \( x \in V \) we have that
\[
P(x) = \Gamma(\Delta_{a,z}P(x) : a \in E).
\]
Notice that until now we did not use the fact that \( P \) is a polynomial.

Denote by \( \hat{P}, \hat{x} \) etc the projection mod \( p \).

**Corollary 9.6.** For \( q^{-t} \) a.e. \( \hat{x} \in \mathbb{F}_p^n \) we have that
\[
\hat{P}(\hat{x}) = \hat{\Gamma}(\Delta_{\hat{a},\hat{z}}\hat{P}(\hat{x}) : \hat{a} \in \mathbb{F}_p^k - 0).
\]

**Claim 9.7** (2). Let \( B = \{b \in \mathbb{F}_p^k : \sum_{j=1}^k |b_j| \leq d \} \) then for any \( a \in \mathbb{F}_p^k \) we have that
\[
\Delta_{a,\hat{z}}\hat{P}(\hat{x}) = \sum \lambda_{a,b}\Delta_{b,\hat{z}}\hat{P}(\hat{x})
\]
for some \( \lambda_{a,b} \in \mathbb{F}_p^k \).

**Corollary 9.8.** For \( q^{-t} \) a.e. \( \hat{x} \in \mathbb{F}_p^n \) we have that
\[
(\ast\ast) \quad \hat{P}(\hat{x}) = \hat{\Gamma}(\Delta_{\hat{b},\hat{z}}\hat{P}(\hat{x}) : \hat{b} \in B).
\]

**Claim 9.9** (2). There exists \( \sigma = \sigma(d) > 0 \) such that if (\( \ast\ast \)) holds for \( \sigma \)-a.e. \( \hat{x} \in \mathbb{F}_p^n \) then it holds for all \( \hat{x} \).

It follows that as long as \( q^{-t} < \sigma(d) \) we the corollary holds. But it suffices to choose \( s \) so that \( 2^{-t} < \sigma(d) \), since \( q^{-t} \leq p^{-t} \leq 2^{-t} \). So this is independent of \( p \) and \( k \) (so long as \( p > d \)).

Now we continue as in the analysis in \( \mathbb{F}_p^n \) to obtain that \( \hat{P} \) is of rank \( r_i(d, s) \).

**References**

[1] Aizenbud, A., Avni, N. Counting points of schemes over finite rings and counting representations of arithmetic lattices. Duke Math. J. 167 (2018), no. 14, 2721–2743.

[2] Bhowmick A., Lovett S. Bias vs structure of polynomials in large fields, and applications in effective algebraic geometry and coding theory. arXiv:1506.02047

[3] Bik, A., Draisma, J., Eggermont, R. Polynomials and tensors of bounded strength. arXiv 1805.01816

[4] Hrushovski, E The Elementary Theory of the Frobenius Automorphisms arXiv: 0406514

[5] Janzer, O. Polynomial bound for the partition rank vs the analytic rank of tensors arXiv:1902.11207

[6] Kazhdan, D., Ziegler, T. Extending linear and quadratic functions from high rank varieties. arXiv:1712.01335

[7] Kazhdan, D., Ziegler, T. Properties of high rank subvarieties of affine spaces. arXiv:1902.00767

[8] Kazhdan, D., Ziegler, T. Polynomials as splines Sel. Math. New Ser. (2019) 25: 31.
[9] Kazhdan, D., Ziegler, T. Approximate cohomology. Sel. Math. New Ser. (2018) 24: 499.
[10] Lang, S., Weil A. Number of points of varieties in finite fields. Amer. J. Math. 76 (1954)
[11] Matsumura, H Comutative ring theory. Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1989.
[12] Marker, D. Model Theory : An Introduction Graduate Texts in Mathematics, Vol. 217, 2002
[13] Miličević, L. Polynomial bound for partition rank in terms of analytic rank Geom. Funct. Anal. 29 (2019), no. 5, 1503-1530.
[14] Schmidt, W. M. The density of integer points on homogeneous varieties. Acta Math. 154 (1985), no. 3-4, 243-296.
[15] Tao,T.; Ziegler, T. The inverse conjecture for the Gowers norm over finite fields in low characteristic, Ann. Comb. 16 (1) (2012) 121-188.