On fibre bundle formulation of classical and statistical mechanics

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Short title: On bundle classical and statistical mechanics

Basic ideas: → October 1997
Began: → November 6, 1997
Ended: → December 20, 1997
Revised: → August 1999
Last update: → September 16, 2001
Produced: → March 31, 2022

LANL xxx archive server E-print No.: physics/0109033

BOZHOR⃝ TM

Subject Classes:
Classical mechanics, Differential geometry

2000 MSC numbers: 70G99, 70H99, 82C99
2001 PACS numbers: 02.90.+p, 05.20.-y, 05.90.+m

Key-Words:
Classical mechanics, Statistical mechanics,
Fibre Bundles, Liouville equation

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Abstract

Some elements of classical mechanics and classical statistical mechanics are formulated in terms of fibre bundles. In the bundle approach the dynamical and distribution functions are replaced by liftings of paths in a suitably chosen bundle. Their time evolution is described by appropriate linear transports along paths in it or, equivalently, by corresponding invariant bundle equations of motion. In particular, the bundle version of the Liouville equation is derived.
1. Introduction

In the series of papers [1–5], we have reformulated nonrelativistic quantum mechanics in terms of fibre bundles. In the present work, we want to try to apply some ideas and methods from these papers to classical mechanics and classical statistical mechanics. However, as a whole this is scarcely possible because these theories are more or less primary related to the theory of space (space-time) which is taken as a base of the corresponding bundle(s) in the bundle approach and, consequently, it has to be determined by other theory. By this reason, the fibre bundle formalism is only partially applicable to some elements of classical mechanics and classical statistical mechanics.

A different geometrical approach to the statistical mechanics, based on the projective geometry, can be found in [6].

The organization of this paper is the following. In Sect. 2 are recalled some facts of classical Hamiltonian mechanics and fix our notation. In Sect. 3, we give a fibre bundle description of (explicitly time-independent) dynamical functions, representing the observables in classical mechanics. In this approach they are represented by liftings of paths in a suitably chosen bundle. We show that their time evolution is governed by a kind of linear (possibly parallel) transport along paths in this bundle or, equivalently, via the corresponding bundle equation of motion derived here. Sect. 4 is devoted to the bundle (analogue of the) Liouville equation, the equation on which classical statistical mechanics rests. In the bundle description, we replace the distribution function by a lifting of paths in the same bundle appearing in Sect. 3. In it we derive the bundle version of the Liouville equation which turns to be the equation for (linear) transportation of this lifting with respect to a suitable linear transport along paths. The paper closes with some remarks in Sect. 5.

2. Hamilton description of classical mechanics (review)

In classical mechanics [7] the state of a dynamical system is accepted to be describe via its generalized coordinates $q = (q_1, \ldots, q_N) \in \mathbb{R}^N$ and momenta $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ with $N \in \mathbb{N}$ being the number of system’s degree of freedom. The quantities characterizing a dynamical system, the so called dynamical functions or variables, are described by $C^1$ functions in $\mathbb{R}^F = \{f : F \to \mathbb{R}\}$ with $F$ being the system’s phase space. The Poisson bracket of $f, g \in \mathbb{R}^F$ is [7, § 8.4]

$$[f, g]_P := \sum_{i=1}^N \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) \quad (2.1)$$
which is an element of $\mathbb{R}^F$. The subset of $\mathbb{R}^F$ consisting of $C^1$ functions and endowed with the operations addition, multiplication (with real numbers) and forming of Poisson brackets is called dynamical algebra and will be denoted by $\mathcal{D}$ [8, Section 1.2]. The set $\mathcal{D}$ is closed with respect to the mentioned operations and is a special kind of Lie algebra, the Poisson bracket playing the role of Lie bracket.

If $h(q,p;t)$ is the system’s Hamiltonian, the system evolves in time $t \in \mathbb{R}$ according to the (canonical) Hamilton equations [7, chapter 7, §8.5]

$$\dot{q}^i = \frac{\partial h(q,p;t)}{\partial p_i} = [q^i, h]^P_P, \quad \dot{p}_i = -\frac{\partial h(q,p;t)}{\partial q^i} = [p_i, h]^P_P,$$

(2.2)

where $i = 1, \ldots, N$ and the dot means full derivative with respect to time, e.g. $\dot{q}^i := dq^i/dt$. The system’s state is completely known for every instant $t$ if for some $t_0 \in \mathbb{R}$ are fixed the initial values $(q,p)|_{t=t_0} = (q_0,p_0) \in \mathcal{F}$ with $\mathcal{F} = \mathbb{R}^{2N}$ being system’s phase space.

If $g$ is depending on time dynamical function, $g \in \mathbb{R}^{F \times \mathbb{R}}$, then its full time derivative is [3, equation (8.58)]

$$\frac{dg}{dt} := \dot{g} = [g, h]^P + \frac{\partial g}{\partial t}.$$

(2.3)

To any dynamical function $f \in \mathcal{D}$ there corresponds operator $[f]_P : g \mapsto [g,f]^P_P, g \in \mathcal{D}$, i.e.

$$[f]_P := [,f]_P : \mathcal{D} \to \mathcal{D}.$$  

(2.4)

Putting $\xi := (q,p) = (q^1,\ldots,q^N,p_1,\ldots,p_N) \in \mathcal{F}$ and defining the map $\overline{h}: \mathcal{F} \to \mathcal{F}$ by $\overline{h}: (q,p) \mapsto ([h]^P q^1,\ldots,[h]^P q^N,[h]^P p_1,\ldots,[h]^P p_N)$, which map can be called Hamiltonian operator; we see that (2.2) is equivalent to

$$\frac{d\xi}{dt} = \overline{h}(\xi).$$

(2.5)

3. Bundle description of dynamical functions in classical mechanics

At first sight, it seems the solution of (2.5) might be written as $\xi(t) = \mathcal{U}(t,t_0)\xi(t_0)$ with $\mathcal{U}(t,t_0)$ being the Green’s function for this equation. However, this is wrong as generally $h$ depends on $\xi$, $h = h(\xi;t)$, so $\mathcal{U}$ itself must depend on $\xi$. Consequently, we cannot apply to the Hamiltonian equation (2.2) the developed in [1] method for fibre bundle interpretation and reformulation of Schrödinger equation. The basic reason for this is that the Hamilton equation is primary related to the (phase) space while the Schrödinger one is closely related to the ‘space of observables’. This suggests the idea of bundle description of dynamical functions which are the
classical analogue of quantum observables. Below we briefly realize it for

Let \( g \in D \) and \( \partial g / \partial t = 0 \). By (2.3) and (2.4), we have

\[
\frac{dg}{dt} = [h]_P g.
\] (3.1)

Writing for brevity \( g(t) \) instead of \( g(\xi(t); t) = g(\xi(t); t_0) \), we can put

\[
g(t) = V(t, t_0)g(t_0),
\] (3.2)

where \( t_0 \) is a fixed instant of time and the dynamical operator \( V \), the Green function of (3.1), is defined via the initial-value problem

\[
\frac{\partial V(t, t_0)}{\partial t} = [h]_P V(t, t_0), \quad V(t_0, t_0) = 1.
\] (3.3)

(Here \( 1 \) is the corresponding unit operator.)

The explicit form of \( V(t, t_0) \) is

\[
V(t, t_0) = \left( \exp \int_{t_0}^{t} [h(\xi; \tau)]_P d\tau \right)_{\xi = \xi(t_0)}
\] (3.4)

where \( \exp \int_{t_0}^{t} \ldots \) denotes the chronological (called also T-ordered, P-ordered, or path-ordered) exponent. One can easily check the linearity of \( V(t, t_0) \) and the equalities

\[
V(t_3, t_1) = V(t_3, t_2)V(t_2, t_1),
\] (3.5)

\[
V(t_1, t_1) = 1,
\] (3.6)

\[
V^{-1}(t_1, t_2) = V(t_2, t_1),
\] (3.7)

the last of which is a consequence of the preceding two. Here \( t_1, t_2 \) and \( t_3 \) are any three moments of time.

Let \( M \) and \( T \) be the classical Newtonian respectively 3-dimensional space and 1-dimensional time of classical mechanics. Let \( \gamma: J \to M, J \subseteq T \), be the trajectory of some (point-like) observer (if the observer exists for all \( t \in T \), then \( J = T \).)

Now define a bundle \((R, \pi_R, M)\) with a total space \( R \), base \( M \), projection \( \pi_R: R \to M \), and isomorphic fibres \( R_x := \pi_R^{-1}(x) = d_x^{-1}(\mathbb{R}) \) where \( \mathbb{R} \) is regarded as a standard fibre of \((R, \pi_R, M)\) and \( d_x: R_x \to \mathbb{R} \) are (arbitrarily) fixed isomorphisms.

\[ M \] and \( T \) are isomorphic to \( \mathbb{R}^3 \) and \( \mathbb{R} \) respectively. This is insignificant for the following.
To every function \( g : \mathcal{F} \times \mathbb{T} \to \mathbb{R} \), we assign a lifting of paths such that

\[
g : \gamma \mapsto g_{\gamma} : t \mapsto g_{\gamma}(\xi; t) := d_{\gamma(t)}^{-1}(g(\xi; t)) \in R_{\gamma(t)}.
\]

(3.8)

In this way the dynamical algebra \( \mathcal{D} \) becomes isomorphic to a subalgebra of the algebra of liftings of paths (or sections along paths) of \((\mathbb{R}, \pi_{\mathbb{R}}, M)\).

For explicitly time-independent dynamical functions, substituting (3.8) into (3.2), we get

\[
g_{\gamma}(t) = V_{\gamma}(t, t_{0})g_{\gamma}(t_{0}),
\]

(3.9)

where, for brevity, we write

\[
g_{\gamma}(t) := g_{\gamma}(\xi(t); t) = g_{\gamma}(\xi(t); t_{0})
\]

and

\[
V_{\gamma}(t, t_{0}) := \lim_{\varepsilon \to 0} \left\{ \frac{1}{\varepsilon} [V_{\gamma}(t, t + \varepsilon)\lambda(\xi(t + \varepsilon)) - \lambda(\xi(t))] \right\}
\]

(3.14)

The equivalence of (3.13) and the conventional equation of motion (3.1) can easily be verified. Therefore (3.13) represents the bundle equation of motion for dynamical functions.

To conclude, we emphasize on the fact that the application of the bundle approach, developed in [1, 2], to classical mechanics results only in bundle description of dynamical functions.
4. Bundle description of the Liouville equation

In classical statistical mechanics [8] the evolution of a system is described via a distribution (function (on the phase space)) \( \mathcal{P}: \mathcal{F} \times T \to \mathbb{R} \) satisfying the conditions \( \int_{\mathcal{F}} \mathcal{P}(\xi; t) \, d\xi = 1 \) and \( \mathcal{P}(\xi; t) \geq 0, \quad \xi \in \mathcal{F}, \quad t \in T, \) and whose time evolution is governed by the Liouville equation

\[
\frac{\partial \mathcal{P}}{\partial t} = \mathcal{L}\mathcal{P} \tag{4.1}
\]

with \( \mathcal{L} = \mathcal{L}(\xi; t) \) being the Liouville operator (the Liouvillian) of the investigated system [8, § 2.2]. If the system is Hamiltonian, i.e. if it can be described via a Hamiltonian \( h \), its Liouvillian is \( \mathcal{L} = -\{h\}_\mathcal{P} \).

Since equations (3.1) and (4.1) are similar, we can apply the already developed ideas and methods to the bundle reformulation of the basic equation of classical statistical mechanics.

We can write the solution of (4.1) as

\[
\mathcal{P}(\xi; t) = W(\xi; t, t_0)\mathcal{P}(\xi; t_0) \tag{4.2}
\]

where the distribution operator \( W \) is defined by the initial-value problem

\[
\frac{\partial W}{\partial t}(\xi; t, t_0) = \mathcal{L}(\xi; t)W(\xi; t, t_0), \quad W(\xi; t_0, t_0) = 1, \tag{4.3}
\]

i.e. \( W(\xi; t, t_0) = \text{Exp} \int_{t_0}^{t} \mathcal{L}(\xi; \tau) \, d\tau. \)

Since \( W \) satisfies (3.5) and (3.6) with \( W \) instead of \( V \), a fact that can easily be checked, the maps

\[
W(\xi; t, t_0) := d^{-1}_{\gamma(t)} \circ W(\xi; t, t_0) \circ d_{\gamma(t_0)}: R_{\gamma(t_0)} \to R_{\gamma(t)} \tag{4.4}
\]

satisfies (3.11) and (3.12). Therefore these maps define a transport \( W \) along paths in \((R, \pi_R, M)\). It can be called the distribution transport.

Now to any distribution \( \mathcal{P}: \mathcal{F} \times T \to \mathbb{R} \), we assign a (distribution) lifting \( P \) of paths in the fibre bundle \((R, \pi_R, M)\), introduced in Sect. 3, such that

\[
P: \gamma \mapsto P_\gamma: t \mapsto P_\gamma(\xi; t) := d^{-1}_{\gamma(t)}(\mathcal{P}(\xi; t)) \in R_{\gamma(t)}. \tag{4.5}
\]

The so-defined lifting \( P: \gamma \mapsto P_\gamma \) of paths in \((R, \pi_R, M)\) is linearly transported along arbitrary observer’s trajectory \( \gamma \) by means of \( W \). In fact, combining (4.2) and (4.3), using (4.3) for \( t = t_0 \) and (4.4), we get

\[
P_{\gamma}(\xi; t) = W_{\gamma}(\xi; t, t_0)P_{\gamma}(\xi; t_0) \tag{4.6}
\]

which proves our assertion. We want to emphasize on the equivalence of (4.6) and the Liouville equation (4.1), a fact following from the derivation of (4.6).
and the definitions of the quantities appearing in it. This result, combined with [10, proposition 5.3] shows the equivalence of (4.1) with the invariant equation

\[ W^D(P) = 0 \] (4.7)

where \( W^D \) is the derivation along \( \gamma \) corresponding to \( W \) (see (3.14)). The last equation can naturally be called the bundle Liouville equation.

5. Conclusion

In this paper we tried to apply the methods developed in [1–5] for quantum mechanics to classical mechanics and classical statistical mechanics. Regardless that these methods are fruitful in quantum mechanics, they do not work with the same effectiveness in classical mechanics and statistics. The main reason for this is that these mechanics are more or less theories of space (and time), i.e. they directly depend on the accepted space (and time) model. So, since the fibre bundle formalism, we are attempting to transfer from quantum mechanics and statistical to classical ones, is suitable for describing quantities directly insensitive to the space(-time) model, we can realize the ideas of [1–5] in the classical region only partially.

In this work we represented dynamical and distribution functions as liftings of paths of a suitably chosen fibre bundle over space. These liftings, as it was demonstrated, appear to be linearly transported along any observer's trajectory with respect to corresponding (possibly parallel) transports along paths in the bundle mentioned. As a consequence of this fact, the equations of motion for distributions and time-independent dynamical functions have one and the same mathematical form: the derivations, generated by the corresponding transports, of these liftings vanish along observer's trajectory.

Thus, we have seen that (some) quantities arising over space admit natural bundle formulation which is equivalent to the conventional one. We demonstrated this for time-independent dynamical functions in classical Hamiltonian mechanics and distribution functions in classical statistical mechanics. Other classical quantities also admit bundle description.

The fibre bundle formalism is extremely suitable for describing all sorts of fields over space(-time). Therefore it seems naturally applicable to quantum physics. In particular, this is true for nonrelativistic and relativistic quantum mechanics (and statistics) whose full self-consistent bundle (re)formulation we have developed in the series of papers [1–5, 12, 13].

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