Smooth Quantum Dynamics of Mixmaster Universe

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Introduction. The Friedmann-Robertson-Walker model is successfully used to describe the data of observational cosmology (see e.g. [1, 2]). Nevertheless, the isotropy of space is dynamically unstable towards the big-bang singularity [3]. On the other hand, if the present Universe originated from an inflationary phase, then the pre-inflationary universe is supposed to have been both inhomogeneous and anisotropic. As evidence suggests (see [4, 5]), the dynamics of such universe backwards in time becomes ultralocal and effectively identical with the homogeneous but anisotropic one at each spatial point. In both cases quantization of the isotropic models alone appears to be insufficient. Hence the quantum version of an anisotropic model, comprising the Friedmann model as a particular case, is expected to be better suited for describing the earliest Universe.

In the Letter we advocate a new quantization method of the dynamics of a vacuum Bianchi type IX geometry, the Mixmaster universe. We identify a soluble sector of this model, which lies deeply in the quantum domain and, as we show, contains relevant physics.

The Mixmaster universe exhibits a complex behavior [6]. As it collapses, the universe enters chaotic oscillations producing an infinite sequence of distortions from its spherical shape [7]. Those distortions essentially correspond to the level of anisotropy and may be viewed as an effect of a gravitational wave evolving in an isotropic background [8]. Dynamics of this wave is nonlinear, and its interaction with the isotropic background fuels the gravitational contraction. Not surprisingly, the quantization of Bianchi IX model is a difficult task. Some formulations can be found in literature, including the Wheeler-DeWitt equation [6] or, more recently, a formulation based on loop quantum cosmology [9, 10]. However, the search for solutions within these formulations is quite challenging [11, 12] leaving the near big-bang dynamics largely unexplored. The most recent developments, e.g. [13], do not address the singularity resolution.

To fill this gap we propose a quantum Mixmaster dynamics, which originates from affine coherent state (ACS) quantization that was recently used to obtain the quantum Friedmann model [14]. It was shown that ACS quantization causes some new terms to appear in the quantum Hamiltonian, producing a strong repulsive force counteracting the contraction of universe. The capacity to resolve the singularity constitutes the basic advantage of our quantization method. In order to solve the dynamics in the present, more complex setting, we employ the adiabatic approximation widely utilized in quantum molecular physics [15, 16]. This approach is reasonable when the vibrations of the shape of the universe are significantly faster than the contraction of its volume.

The main result is a semiclassical Friedmann-like equation obtained from expectation values in ACS, a description peculiar to our approach. In that equation, the expansion of the universe is governed by two terms of quantum origin. The first one is proper to the quantum Mixmaster model and corresponds to the energy of the wave in an eigenstate. It is proportional to the energy level number. The other one, which is more universal, corresponds to the repulsive potential preventing the singularity. The lowest energy eigenstates of this system are interpreted as the quantum Friedmann universe supplemented with vacuum fluctuations of the anisotropy.

Beyond issues of singularity resolution, the Friedmann-like equation describes two novel and rather surprising properties of the quantum dynamics. Firstly, the anisotropic degrees of freedom remain in their lowest energy states during the quantum phase consistent with our approximation. This implies that the quantum Friedmann model, unlike its classical counterpart, is in fact stable with respect to the anisotropy. Therefore, the classical chaos is suppressed within the considered domain. Secondly, during the contraction the quantum energy of anisotropy grows much slower than it does on the classical level. Namely, it effectively gravitates as radiation leading to a significant reduction in the overall gravitational pull from anisotropy due to quantum effects.
tended to anisotropic universes, with \( c = 1 = 8\pi G \), reads:

\[
H^2 + \frac{1}{6} 3R - \frac{1}{6} \Sigma^2 = \frac{1}{3} \rho, \tag{1}
\]

where \( H = \dot{a}/a \) (dot denotes the derivative with respect to the cosmological time) is the expansion parameter, \( \rho \) is the energy density of matter, and \( 3R \) is the spatial curvature. The additional term \( \Sigma^2 \) is the total shear of the spatial section and is non-vanishing for anisotropic models. Due to its negative sign, the shear drives the gravitational collapse and it eventually dominates the dynamics. For this reason we neglect the attraction of matter by putting \( \rho = 0 \), that is, we restrict our considerations to the vacuum Bianchi IX, whose dynamics near the singularity is effectively the same as in the presence of perfect fluid [4].

The Mixmaster describes the space-time metric \( ds^2 = -dt^2 + a^2 (e^{2\beta} \gamma_{ij} \sigma^i \sigma^j) \), where \( a \) is an averaged scale factor and \( \sigma^i \) are differential forms on a three-sphere (covering group of the rotation group) satisfying \( d\sigma^i = \frac{1}{2} \epsilon_{ijk} \sigma^j \wedge \sigma^k \). The diagonal form of the metric is assumed \( (e^{2\beta})_{ij} := \text{diag} (e^{2(\beta_+ + \sqrt{3}\beta_-)}, e^{2(\beta_+ - \sqrt{3}\beta_-)}, e^{4\beta_+}) \), where \( \beta_\pm \) are distortions parameters [4].

In terms of these variables, the shear is the kinetic energy of anisotropic distortion: \( \Sigma^2 = (p_+^2 + p_-^2)/24a^6 \), where the momenta \( p_\pm \) are canonical conjugates to \( \beta_\pm \). The spatial curvature \( 3R \) grows due to the overall contraction of space, but decreases due to the growth of anisotropy. This last circumstance leads to the back-reaction from spatial curvature on the shear and, as a result, oscillations in \( \beta_\pm \) occur. As there is no matter content in our model, \( \beta_\pm \) describe a sort of gravitational wave. The curvature can be split into isotropic and anisotropic parts: \( 3R = 3(1 - V(\beta))/2a^2 \), where \( V(\beta) \) is the anisotropy curvature potential [3]:

\[
V(\beta) = \frac{e^{4\beta_+}}{3} \left( \left( e^{-6\beta_+} - 2 \cosh(2\sqrt{3}\beta_-) \right)^2 - 4 \right) + 1.
\]

As shown in Fig. 1, this potential has three “open” \( C_3 \) symmetry directions. One can view them as three deep “canyons”, increasingly narrow until their respective wall edges close up at the infinity whereas their respective bottoms tend to zero. Due to its (almost) confining shape, \( V \) is expected to produce a discrete energy spectrum on the quantum level.

The generalized Friedmann equation [1] may be rewritten as

\[
H^2 + \frac{1}{3a^2} = \frac{1}{6} \Sigma^2 - V(\beta) - \frac{3}{4} \rho \tag{2}
\]

where the isotropic background geometry on the l.h.s. is pulled by the energy of anisotropic oscillations. The energy of oscillations scales roughly as \( a^{-6} \).

It follows that the Hamiltonian constraint to be quantized reads in canonical variables:

\[
C = \frac{3}{16} p^2 + \frac{3}{4} q^{2/3} - H_q, \tag{3}
\]

where \( q = a^{3/2} \) and \( p^2 = 16a^2 \) are more suitable to ACS quantization, and where

\[
H_q = \frac{1}{12q^2} (p_+^2 + p_-^2) + \frac{3}{4} q^{2/3} V(\beta) \tag{4}
\]

is the anisotropy energy. The closed Friedmann-Robertson-Walker (FRW) geometry is obtained by putting \( p_\pm = 0 \) and \( \beta_\pm = 0 \), or simply \( H_q = 0 \).

The Hamiltonian constraint [3] resembles a diatomic molecular Hamiltonian with the pairs \( (q, p) \) and \( (\beta_\pm, p_\pm) \) playing the rôle of the reduced nuclear and electronic variables, respectively. In molecules, the motion of nuclei is slow enough in comparison with electrons so the motion of electrons may be approximated as becoming immediately adjusted to varying positions of nuclei. However, the coupling between the nucleus-like and electron-like degrees of freedom in the present model differs from the usual molecular case for which the validity of the approximation rests upon the smallness of the ratio between the nuclear and electron masses. In the present case, described by Eqs. [3] and [4], the “mass” of the degrees of freedom \( \beta_\pm \) evolves as \( q^2 \). Thus, it goes to zero near the singularity, \( q = 0 \). On the other hand, the “mass” of the degree of freedom \( q \) in Eq. [4] is constant. Thus, close to singularity the latter may be regarded as “heavy” in comparison with the anisotropic variables that can be treated as “light”.

**Quantum Hamiltonian.** The six-dimensional phase space of the Mixmaster universe is quantized as follows: (A) The isotropic variables form the canonical pair \( (q, p) \) living in a half-plane. That half-plane can be viewed as the affine group. We resort to one of its two unitary irreducible representations, denoted by \( U \), to build from a suitable fiducial vector \( |\nu\rangle \) (where \( \nu > 0 \) is a free parameter) a family of affine coherent states (i.e., wavelets) \( |q, p\rangle := U(q, p)|\nu\rangle \). These ACS’s are then used to consistently quantize the isotropic variables. While the method
provides the usual $\dot{p} = -i\hbar \partial_q$, and $\dot{q}$ defined as the multiplication by $q$, its interest lies in the regularization of the Hamiltonian [17]. This approach together with $|\nu\rangle$ was introduced for cosmological models in [13]. Next, we use the ACS’s to obtain a semiclassical description, which enables us to analyze the effective dynamics of isotropic variables. (B) For the anisotropic variables, each canonical pair $(\beta_+, p_+)$ lives in the plane. Thus, it is natural to proceed with the usual canonical quantization which yields $\hat{p}_\pm = -i\hbar \partial_{\beta_\pm}$, and $\beta_\pm$ being the multiplication by $\beta_\pm$.

The quantized Hamiltonian corresponding to $\hat{C}$ and issued from quantizations (A) and (B) above reads

$$\hat{C} = \frac{3}{16} \left( \dot{p}_0^2 + \frac{\hbar^2 \hat{R}_1}{q^2} \right) + \frac{3}{4} \hat{R}_3 q^{2/3} - \hat{H}_q ,$$  

where

$$\hat{H}_q = \frac{1}{12} \hat{R}_1 \dot{q}_0^2 + \frac{\dot{p}_0^2}{q^2} + \frac{3}{4} \hat{R}_3 q^{2/3} V(\dot{\beta}).$$  

The $\hat{R}_i$’s are purely positive numerical constants dependent on the choice of the ACS. With the choice made in our previous paper [13] all these constants are simple rational functions of modified Bessel functions $K_i(\nu)$. We note in [17] the appearance of the repulsive centrifugal potential term $\hbar^2 \hat{R}_1 q^{-2}$. It is the signature of the ACS quantization, which is consistent with the half-plane geometry, and it regularizes the quantum Hamiltonian for small $q$. As the universe approaches the singularity, $q \to 0$, this centrifugal term sharply grows in dynamical significance.

We consider the oscillations of $\beta_\pm$ fast in comparison with the contraction of the universe. It legitimates the adiabatic approximation, in a way analogous to the Born-Oppenheimer approximation (BOA) [15, 16] widely used in molecular physics. Due to the confining property of $V$, the operator $\hat{H}_q$ at fixed $q$ has a discrete spectrum. In accordance with BOA we assume that the anisotropy degrees of freedom $\beta_\pm$ are frozen in some eigenstate of $\hat{H}_q$ with eigenenergy $e_q^{(N)} (N = 0, 1, \ldots)$ evolving adiabatically. Thus, the light degrees of freedom $\beta_\pm$ can be averaged leading to the Hamiltonian:

$$\hat{C}_A = \frac{3}{16} \left( \dot{p}_0^2 + \frac{\hbar^2 \hat{R}_1}{q^2} \right) + \frac{3}{4} \hat{R}_3 q^{2/3} - e_q^{(N)} .$$  

Focusing on the deep quantum domain, we look at the first energy levels near the ground state of $\hat{H}_q$. Therefore we essentially investigate the domain near the minimum of $V(\dot{\beta})$. Within the harmonic approximation $V(\dot{\beta}) \approx \delta(\dot{\beta}_+^2 + \dot{\beta}_-^2)$ near its minimum $\dot{\beta}_\pm = 0$, the eigenenergies are found to be $e_q^{(N)} \approx h q^{-2/3} \sqrt{2 \hat{R}_q \hat{R}_1} (N + 1)$ with $N = n_+ + n_-$. The quantum numbers $n_\pm$ correspond to the independent harmonic oscillations in $\beta_+$ and $\beta_-$. Consequently, the approximation of the eigenvalues of $\hat{H}_q$ may be written as [18]:

$$e_q^{(N)} \approx \frac{\hbar}{q^{2/3}} \sqrt{2 \hat{R}_3 \hat{R}_1} (N + 1) .$$  

The expression for $e_q^{(N)}$ is rather a rough approximation for large values of $N$, since $V(\dot{\beta})$ is highly nonharmonic far away from its minimum. But for small values of $N$, this expression is valid at any value of $q$.

**Validity of approximation.** We notice in Eq. [8] that the discrete spectrum part in Eq. [7] becomes a small perturbation at large $q$, a range for which BOA possibly loses its validity, whereas it gains all its value at small $q$. From the mass criterion, our approach based on BOA is legitimate as $q$ assumes its values near the singularity $q = 0$. Furthermore, our procedure of quantization generates a supplementary repulsive potential that prohibits the system to access the singularity neighborhood $q \in (0, q_m)$ with some very small bound $q_m > 0$, which depends on the initial state.

Furthermore, calculations made in molecular physics beyond BOA (the so-called vibronic approximation) show that the mass criterion is in fact too strong: a significant breakdown of BOA only occurs when different eigenenergy curves $q \mapsto e_q^{(N)}$ of $\hat{H}_q$ are crossing. In our approach these crossings do not occur, at least for the lowest levels of $\hat{H}_q$ in Eq. [8].

This reasoning based on molecular physics is robust, but qualitative in our case, due to the coupling between the $q$ and $\beta_\pm$ degrees of freedom which is not of molecular type. In [18] we weaken the adiabatic condition by allowing the quantized oscillations to be excited by the semiclassical dynamics of the isotropic background described below, for a fixed $N$. We find that the excitation is indeed very limited, which justifies our approach.

**Semiclassical Hamiltonian.** We introduce a semiclassical observable associated with the quantum Hamiltonian [7] as its expectation value $\hat{C}_A(q, p) := \langle q, p | \hat{C}_A | q, p \rangle$ in the ACS state $| q, p \rangle$ peaked on the classical phase space point $(q, p)$ in the plane,}

$$\hat{C}_A(q, p) = \frac{3}{16} \left( p_0^2 + \frac{\hbar^2 \hat{R}_4}{q^2} \right) + \frac{3}{4} \hat{R}_5 q^{2/3} - \frac{\hbar}{q^{2/3}} \hat{R}_0 (N + 1) ,$$  

where the $\hat{R}_i$’s are positive numerical constants [18] which are also simple rational functions of modified Bessel functions $K_i(\nu)$. With our choice of $|\nu\rangle$, at large $\nu$, $\hat{R}_i(\nu) \sim 1$, $i \neq 4$ and $\hat{R}_4(\nu) \sim \nu/2$. For the consistency of our semiclassical description we have rescaled the fiducial vector so that $\langle q, p | \hat{q} | q, p \rangle = q$ and $\langle q, p | \hat{p} | q, p \rangle = p$.

The Hamiltonian constraint imposed at the semiclassical level, $\hat{C}_A(q, p) = 0$, leads to the semiclassical Friedmann-like equation:

$$H^2 - \frac{4\pi^2 G^2 \hat{R}_4}{c^4} - \frac{\hat{R}_5}{c^4} \left( \frac{c}{a} \right)^2 = \frac{8\pi G \hat{R}_0}{3c} (N + 1) \frac{\hat{R}_0}{a^2} .$$  


where we have restored physical constants and the standard cosmological variables.

The above semi-classical constraint admits smooth trajectories for all values of $N$ only if $\nu \in (0, 7.19)$. For $\nu > 7.19$, Eq. (9) has no solution for the smallest values of $N$. The solution of (9) for $a$ is a periodic function, $a \in [a_-, a_+]$ with $a_+ > 0$ and $a_- < \infty$, and resolves the cosmological singularity of the Mixmaster universe. In Fig. 2 we plot a few trajectories in the half-plane $(a, H)$.

FIG. 2. Three periodic semiclassical trajectories in the half-plane $(a, H)$ from Eq. (9). They are smooth plane curves. We use standard units $8\pi G = c = \hbar = 1$ and choose $\nu = 1$. Blue dotted curve for $N = 0$, green dotdashed for $N = 1$ and red dashed for $N = 2$.

The classical closed FRW model is recovered at $\nu = 0$. In our harmonic approximation to check if there is a significant excitation of the wave’s energy level during the semiclassical evolution of the background geometry. The method is essentially the same as the one used to discuss the generation of primordial power spectra in inflationary cosmology. We show that the wave in fact remains in its lowest energy states during the quantum phase. It confirms that the quantum FRW universe, unlike its classical version, is dynamically stable with respect to the small isotropy perturbation. Therefore, it seems that the quantum closed Friedmann model may be successfully used to describe also the earliest Universe, provided that the corresponding Hamiltonian is supplemented with the effect of the zero-point energy generated by the quantized anisotropy degrees of freedom of the Mixmaster universe.

Discussion. Our semi-classical analysis of the Mixmaster universe leads to the modified Friedmann equation [9]. The left-hand side describes the isotropic part of geometry. The Hubble parameter squared is accompanied by the repulsive potential of purely quantum origin, which grows as $a^{-6}$ during the contraction. At small volumes, it efficiently counteracts the attraction of common forms of matter, forcing the collapsing universe to rebound. The third term is the usual isotropic spatial curvature.

The right-hand side of Eq. (9) describes the quantized energy of the anisotropy oscillations. The energy is discrete and increases linearly with integer $N$, as expected in our harmonic approximation. Within the adiabatic approximation, that quantum number is conserved during the evolution. The energy of the quantum oscillator evolves due to the $a$-dependent coefficients in front of its kinetic and potential terms given in Eq. (6). The ratio between the coefficients determines the oscillator’s frequency, which is proportional to $a^2$. The energy of the oscillations at the quantum level is multiplied by the frequency and consequently it scales as $a^{-4}$. This becomes a poor approximation for high values of $N$, due to the breakdown of the harmonic approximation. It is quite a contrast to the classical wave, whose total energy is approximately unaffected by its time-dependent frequency, and therefore scales as $a^{-6}$. Thus, the growth of the attractive force induced by the anisotropy is significantly reduced in the semiclassical dynamics. This essential dissimilarity between the classical and semiclassical dynamics is due to the fact that on the quantum level the contraction of space is driven by a quantum average.

Let us note that the energy of the wave does not vanish even in the ground state $N = 0$ due to the zero-point quantum fluctuations corresponding to the classical state $\beta_{\pm} = p_{\pm} = 0$. In [18] we go beyond the adiabatic approximation to check if there is a significant excitation of the wave’s energy level during the semiclassical evolution of the background geometry. The method is essentially the same as the one used to discuss the generation of primordial power spectra in inflationary cosmology. We show that the wave in fact remains in its lowest energy states during the quantum phase. It confirms that the quantum FRW universe, unlike its classical version, is dynamically stable with respect to the small isotropy perturbation. Therefore, it seems that the quantum closed Friedmann model may be successfully used to describe also the earliest Universe, provided that the corresponding Hamiltonian is supplemented with the effect of the zero-point energy generated by the quantized anisotropy degrees of freedom of the Mixmaster universe.

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