A wall-distance-free version of the SST turbulence model

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The present work aims to retain the predictive quality of the 2003 SST model while replacing its dependence on explicit wall distance with a local representation thereof, thereby avoiding the need to repeatedly compute wall distance arrays in flow cases involving nonstationary grids or in those involving multimillion-cell meshes. The approach proposed here is a local, algebraic-based representation of wall distance, using $k$ and $\omega$. Several examples are given, showing that the proposed alternative to explicit wall distance usually enables as good predictions as those of the 2003 SST model and in some cases even better. The paper also provides a proposal to treat free shear flows and includes a jet flow example.

Keywords: turbulence; turbulence model; SST; wall-distance-free; local; pointwise

Introduction

The SST turbulence closure of Menter (1994), Menter, Kuntz, and Langtry (2003) contains two blending functions, $F_1$ and $F_2$, which bridge the near-wall $k-\omega$ formulation with the away-from-wall $k-\varepsilon$ formulation (expressed in terms of $k$ and $\omega$). These blending functions involve explicit measure of wall distance. Over years of SST model application, an increasing number of CFD users have asked for a wall-distance-free (wdf) version of this closure, mainly to avoid calculation of the wall distance array which is very time-consuming in the case of large mesh, complex 3D topologies. Furthermore, increasing usage of nonstationary grids in engineering flow problems requires re-computing the varying wall distance array at every time-step, a daunting demand on time- and budget-restricted engineers.

The present work aims to retain the predictive quality of the 2003 SST model while replacing its dependence on actual wall distance with a local algebraic representation thereof, based on $k$ and $\omega$, thereby avoiding the need to repeatedly compute wall distance arrays.

A differential approach, based on $\omega$ and its normal-to-surface derivative, has been proposed by Masson and Gleize (2004). While also avoiding wall distance computation, the necessity to determine local normal-to-surface direction should be avoided in unstructured, massively parallel modern CFD solvers. In this respect the proposed local algebraic approach has a significant advantage. As an example, if the topology includes a finite step, the algebraic approach is the only nonambiguous method, being dependent strictly on local values of $k$ and $\omega$; the differential proposal (Masson & Gleize, 2004) will suffer from ambiguity at the step corners.

Following a description of the local-based wall distance representation several examples are provided, showing that the wdf approach enables as good (and sometimes better) predictions as those by the original closure. The discussion extends to treatment of free shear flows, including an example thereof.

Mathematical model

The 2003 SST model’s transport equations are given by (Menter et al., 2003)

$$
\begin{align*}
\frac{D (\rho k)}{Dt} &= \bar{P}_k - \beta^* \rho k \omega + \nabla \cdot \left[ (\mu + \sigma_k \mu_t) \nabla k \right] \quad (1) \\
\frac{D (\rho \omega)}{Dt} &= \gamma \frac{P_k}{V_i} - \beta \rho \omega^2 + \nabla \cdot \left[ (\mu + \sigma_\omega \mu_t) \nabla \omega \right] \\
&\quad + 2 \left(1 - F_1\right) \rho \sigma_\omega \frac{1}{\omega} \nabla k \cdot \nabla \omega \quad (2)
\end{align*}
$$

Where the eddy viscosity is

$$
\nu_t = \frac{\mu_t}{\rho} = \frac{a_1 k}{\max\{a_1 k, SF_2\}} \quad (3)
$$

$$
\tilde{\nu}_t = \max\{\nu_t, 10^{-8}\} \quad (4)
$$

And the production terms are

$$
P_k = \left[ \mu_t \left( U_{i,j} + U_{j,i} - \frac{2}{3} U_{k,k} \delta_{ij} \right) - \frac{2}{3} \rho k \delta_{ij} \right] U_{i,j} \quad (5)
$$

$$
\bar{P}_k = \min\{P_k, 10 \beta^* \rho k \omega\} \quad (6)
$$

$F_1$ and $F_2$ are the blending functions which depend on wall distance (to be modified in this paper) and all other symbols are given in the list of symbols below.
Local approach to wall distance representation

The functions $F_1$ and $F_2$ (Equations 2 and 3) use wall distance to blend the near-wall $k$-$\omega$ model with the away-from-wall $k$-$\varepsilon$ closure (recast into $k$-$\omega$ variables), this being a fundamental attribute of the SST model. In the present work these two functions are rederived in a form which replaces wall distance with a local (pointwise) representation thereof. The following steps are taken for this purpose:

(a) The turbulence velocity-scale is chosen as:

$$V_t = \sqrt{k}$$  \hspace{1cm} (7)

(b) The realizable time-scale is based on the maximum of the large eddy and Kolmogorov scales (see Goldberg & Apsley, 1996):

$$\tau_t = \frac{C_3^4}{\kappa} \cdot \max \left\{ \frac{k}{\varepsilon}, \sqrt{\frac{2\nu}{\varepsilon}} \right\}.$$  \hspace{1cm} (8)

(c) Finally, turbulence length-scale (used to represent wall distance) is given by

$$\ell_t = V_t \tau_t.$$  \hspace{1cm} (9)

In $k$-$\omega$ terms $[\omega = \varepsilon/(C_\mu k)]$:

$$\ell_t = \frac{C_\mu^2}{\kappa} \cdot \max \left\{ \sqrt{k}, \sqrt{\frac{2\nu}{\varepsilon}} \right\}.$$  \hspace{1cm} (10)

This formulation is now evaluated in the limits of near-wall and defect regions of a turbulent boundary layer.

(a) Viscous sublayer and buffer layer:

In this dissipative eddy region (Wilcox, 1998)

$$k = A_k (u_t y^+)^2,$$  \hspace{1cm} (11)

$$\varepsilon = 2A_k u_t^4 / \nu,$$  \hspace{1cm} (12)

so that the time-scale, from Equation 8, is

$$\tau_t = \frac{C_\mu^2}{\kappa} \sqrt{\frac{2\nu}{\varepsilon}}.$$  \hspace{1cm} (13)

and, from Equation 10, the following linear behavior is obtained:

$$\ell_t = \frac{C_\mu^3}{\kappa} y \cong 0.4y.$$  \hspace{1cm} (14)

(b) Logarithmic overlap:

Here (Wilcox, 1998)

$$k = u_t^2 / \sqrt{C_\mu},$$  \hspace{1cm} (15)

and Equation 8 now yields the time scale

$$\tau_t = \frac{C_\mu^3}{\kappa} \frac{k}{\varepsilon}$$  \hspace{1cm} (17)

whence Equation 10 gives

$$\ell_t = y.$$  \hspace{1cm} (18)

(c) Upper portion of boundary layer (defect layer):

Based on Wilcox’s (1998) asymptotic analysis,

$$\ell_t = \max \left\{ \frac{1 + k_1 \eta \ln \eta}{1 + \omega_1 \eta \ln \eta}, \sqrt{\frac{2C_\mu}{\nu \omega_1 \ln \eta} \frac{\kappa}{\nu} y^+} \right\}.$$  \hspace{1cm} (19A)

where

$$k_1 = \frac{\beta_T / \kappa}{\sigma^* \alpha^* - 1}, \quad \beta_T = \frac{\delta^*}{\tau_w} \frac{d\rho}{dx},$$  \hspace{1cm} (19B)

$$\sigma^* = 1/\sigma_k, \quad \alpha^* = \frac{\sqrt{C_\mu}}{\kappa},$$  \hspace{1cm} (19C)

$$\omega_1 = \frac{\sigma^* k_1^2 (2\alpha^*)}{1 - \beta / (\alpha C_\mu)} k_1,$$  \hspace{1cm} (19D)

and $d\rho/dx$ is streamwise pressure gradient. Since $y^+ \gg 1$, the second term in Equation 19A is negligible compared to the first, leading to

$$\ell_t = \sqrt{\frac{1 + k_1 \eta \ln \eta}{1 + \omega_1 \eta \ln \eta} y}.$$  \hspace{1cm} (20A)

which, under zero pressure gradient (ZPG) conditions, becomes (see Equations 19B and C)

$$\ell_t = \frac{y}{\Delta}, \quad \Delta = U_e \delta^* / u_t,$$  \hspace{1cm} (20B)

retaining the direct connection to wall distance as in the logarithmic overlap (Equation 18).

The above approach requires the following change of constant in the new version of the $F_2$ blending function of Menter’s (1994) SST model:

$$F_2 = \tanh \left[ \frac{\max \left( \frac{C_\mu^2}{\varepsilon^2} \sqrt{k}, \frac{500 \nu}{\varepsilon^2} \right)}{C_\mu^3 \omega_\ell^t} \right]^2.$$  \hspace{1cm} (21A)

where

$$C = \begin{cases} 2.0 & \text{original SST} \\ 0.54 & \text{algebraic wdf} \end{cases}$$  \hspace{1cm} (21B)

This value is based on matching the original SST (Menter, 1994) results for the axisymmetric bump flow case (2nd example below.)
The blending function, $F_1$, of the SST model (Menter et al., 2003) retains its formulation with the wall distance, $d$, replaced by $\ell_t$:

$$F_1 = \tanh(\psi^4)$$  \hspace{1cm} (22A)

$$\psi = \min \left\{ \max \left\{ \frac{\sqrt{k}}{C_\mu \omega \ell_t}, \frac{500v}{\omega \ell_t^2}, \frac{4\sigma_{\omega} \varphi k}{\varphi \ell_t^2} \right\}, 500 \nu \omega \ell_t^2 \right\}$$  \hspace{1cm} (22B)

$$\varphi = \max \left\{ 2\sigma_{\omega} \frac{\partial k}{\partial x}, \frac{1}{10} \right\}$$  \hspace{1cm} (22C)

Selecting the logarithmic overlap as a representation of a turbulent boundary layer and using Equations 15, 16, 18 and 21B [with $\omega = \varepsilon/(C_\mu k)$], the new functions, $F_1$ and $F_2$, can be expressed as follows:

$$F_1 = \tanh \left( \left[ \max \left\{ 2.5, \frac{61.5}{y^+} \right\} \right]^4 \right)$$  \hspace{1cm} (23)

$$F_2 = \tanh \left( \left[ \max \left\{ 1.35, \frac{61.5}{y^+} \right\} \right]^2 \right)$$  \hspace{1cm} (24)

Note that this is strictly a limited representation of the blending functions in the logarithmic region and not a general indication that they can be expressed in terms of $y^+$.

Equation 23 indicates that $F_1 = 1$, hence the k-ω constants will be used in the logarithmic overlap region. From Equation 24, $1 \geq F_2 \geq 0.95$, which retains the basic SST-like eddy viscosity formula. In the original SST closure $F_1 = F_2 = 1$ in the viscous sublayer and the logarithmic overlap, diminishing toward 0 in the defect layer. A comparison between the wdf and original model’s $F_1$ and $F_2$ behaviors is shown in the backward-facing step example below.

**Free shear flow treatment**

In the case of free shear flows, the blending function $F_2$ is set to zero in order to enforce the k-ω model’s eddy viscosity, $\nu_1 = k/\omega$.

**Flow examples**

Several flow cases are presented below. Wall-bounded test cases begin with a basic flat plate flow followed by an axisymmetric transonic case, a backward-facing low-speed flow and a 3D transonic wing/fuselage test case. These are followed by a supersonic axisymmetric jet problem. All 2D cases were validated for grid independence and the 3D wing/fuselage test example already has an optimized mesh which was imported from the NASA data base (cf3d.larc.nasa.gov) (Rumsey, http://turbmodels.larc.nasa.gov). Residual histories are provided in three cases.

Verification and validation of CFD++, the employed commercial CFD solver (Chakravarthy, 1999) were published by NASA (Rumsey, http://turbmodels.larc.nasa.gov/).

**Flat plate**

A Mach 0.2 flow over a flat plate was computed using the 2003 SST turbulence model and its local (wdf) variant. The plate is 0.61 m long and the Reynolds number is $Re_x = 5 \times 10^6$ at $x = 0.305$ m. A $273 \times 193$ grid, provided by NASA (Rumsey), was used with an off-wall $\Delta y_{min} = 3.0 \times 10^{-7}$ m ($y^+ = 0.1$). Figure 1 shows the level of agreement between the computed $C_f$ and the White-Christoph correlation (White, 1974). Whereas SST tends to underpredict flat plate skin friction (Rumsey, http://turbmodels.larc.nasa.gov/flatplate_val_sst.html), the wdf method is observed to moderately improve on the 2003 SST model’s prediction due to elevated eddy viscosity (see backstep case, Figure 3c).

**Transonic flow over an axisymmetric bump (Bachalo & Johnson, 1979)**

A normal shock, interacting with the boundary layer, causes flow detachment over the bump, at $x/c \approx 0.7$, with subsequent reattachment on the downstream cylindrical portion. Figure 2a shows geometry and main flow features. Inflow conditions are: $M_\infty = 0.875$, $Re_\infty = 1.36 \times 10^7$ at $x = 1$ m, $p_\infty = 57,935$ Pa, $T_\infty = 255.6$ K, $Tu = 1\%$ and $\mu / \mu_0 = 20$. The turbulence models were used on a 12,000 cell grid with $y^+ \leq 1$ to permit direct solution to the wall. Figure 2b compares wall pressure prediction with data, showing accurate capture of shock location by both versions of the turbulence closure. Since the original SST model (Menter, 1994) was particularly calibrated based on this flow case, the constant in Equation (21B) was deemed...
to be best calibrated to this case, since the two models share the same formulation except for the two blending functions. Once calibrated, C is frozen and independent of the particular flow under investigation. Figure 2c shows residual history with the wdf model; 18 orders of magnitude reduction in residuals are observed.

**Backward-facing step**

Experiments with various configurations of a backward-facing step were reported in Driver and Seegmiller (1985). The present results are shown for the case of parallel upper wall ($\alpha = 0$ in Figure 3a). Flow details: $M_\infty = 0.128$, $T_\infty = 297$ K, $p_\infty = 1$ atm., $Re_h = 36,000$, $Tu = 1\%$ and $\ell_T = 2.5$ cm. A turbulent boundary layer inflow, with $\delta = 1.9$ cm on both top and bottom walls, was imposed. A 92,350 cell grid with $y^+ \leq 0.3$ permitted direct solution to the walls. Figure 3b compares skin friction prediction along the step-side wall with data (Driver & Seegmiller, 1985). Both the extent of the separation bubble and the post-reattachment behavior are reasonably well captured by the two model versions. Reattachment point is best predicted by the 2003 SST closure, closely followed by that
of the wdf version. However, $C_f$ in the post-reattachment region is better predicted by the latter due to elevated eddy viscosity as seen in Figure 3c, taken at the step-side portion of the domain exit. This example is important since it confirms the performance of the wdf approach under large flow separation conditions and abrupt topological changes. Comparison of the $F_1$ and $F_2$ blending functions profiles, within the separation bubble region, is seen in Figure 3d. While the behavior is not expected to be the same, trend similarities between the corresponding blending functions are observed. The different behavior is expected in view of the fact that the new blending functions do not involve physical wall distance.

**ARA M100 wing/fuselage test case**

This 3D wing/body test case (Peigin & Epstein, 2004) is at $\alpha = 2.87^\circ$, $M_\infty = 0.803$ and chord Reynolds number $Re_c = 13.1 \times 10^6$. The mesh (cfl3d.larc.nasa.gov) (Rumsey, http://turbmodels.larc.nasa.gov/) has 860,000 cells with an off-wall $y^+$ distribution as follows: $y^+_\text{wing} = 0.8$, $0.1 \leq y^+_\text{fuselage} \leq 30$. Consequently a wall function was used over the fuselage and direct solve-to-wall was employed on the wing. Freestream turbulence levels are $Tu = 0.5\%$ and $\mu_t/\mu = 25$. Figure 4a shows pressure contours on the fuselage and upper wing surface. The normal shock footprint on the wing’s suction side is observed. Figure 4b is a wing slice showing the normal shock and its interaction with the turbulent boundary layer. Both contour plots are from the wdf model calculation. $C_p$ profiles at two wing sections are seen in Figures 4c and 4d. Comparisons with experimental data show that the shock location, predicted by the wdf model, is modestly closer to the data than that by the 2003 SST closure. The observed tendency of SST to underpredict skin friction (see the flat plate example) indicates a corresponding underprediction of eddy viscosity which leads to an upstream travel of both shock-boundary layer interaction location and the start of flow separation, leading to the observed shock prediction upstream of the data. As in previous cases, due to elevated eddy viscosity, the wdf model predicts the shock further downstream.

Figure 4e shows residual convergence history using the wdf variant. The residuals show a drop of 3-5 orders of magnitude which is acceptable in 3D flows. The levels are high due to the fact that this transonic case was solved dimensionally.

**Supersonic axisymmetric jet flow**

A NASA-supplied mesh (cfl3d.larc.nasa.gov/Cfl3dv6/cfl3dv6_testcases.html), consisting of 31,000 quadrilateral cells, was used to compute an axisymmetric supersonic jet
Figure 4a. Wing/fuselage pressure contours.

Figure 4b. Wing section showing normal shock/boundary layer interaction.

Figure 4c. $C_p$ profile comparisons at the $\eta = 0.455$ wing section.

Figure 4d. $C_p$ profile comparisons at the $\eta = 0.935$ wing section.

Figure 4e. Residual history.

flow case (Eggers, 1966). The following boundary conditions were applied: Ambient inflow and far-field: $p = 1$ atm, $T = 292$ K, axis of symmetry: x-axis symmetry, outflow: $p = 1$ atm, nozzle inflow: $p_{tot} = 11.0$ atm, $T_{tot} = 292$ K, nozzle walls: adiabatic wall function. Turbulence inflow and freestream conditions were set as follows: $\text{Tu} = 1\%$, $\mu_t/\mu = 50$.

The computation used the compressible RANS equation set with the 2003 SST model and its algebraic wdf variant. A compressibility correction was also invoked for this predominantly supersonic jet. Figure 5a shows Mach contours obtained with the wdf closure. Figure 5b compares axial centerline velocity profiles, predicted by the two closures, with experimental data. The 2003 SST
model follows the near-jet development better than the wdf closure does since the latter behaves like the $k-\omega$ model in free shear flows, which tends to underpredict eddy viscosity levels at the start of a jet, hence delaying proper mixing with the surrounding fluid, as well as spreading, till further downstream (Rumsey, http://turbmodels.larc.nasa.gov/delvilleshear_val_w06.html). In addition, the $k-\omega$ model’s level of free shear mixing is very sensitive to $\omega_\infty$ whereas the SST model behaves like the $k-\epsilon$ closure, entirely insensitive to $\epsilon_\infty$.

The two predictions overlap starting at $x/r \approx 65$. Finally, Figure 5c shows convergence history of mass balance resulting from the wdf calculation.

**Summary**

The present paper proposes a variant of the 2003 SST turbulence model, in which physical wall distance, used in the blending functions, is replaced by a local (pointwise) representation thereof. By recalibrating the constant in the $F_2$ blending function, the new variant is able to predict a variety of flow cases as well (and sometimes even better) as predictions by the 2003 SST closure. The current proposal extends also to free shear flows where it behaves like the $k-\omega$ model. It is noted that, due to the lack of similarity between the original and newly proposed blending functions ($F_1$ and $F_2$), the current model may not be considered an SST-type closure.

The principal aim in the present work is to avoid calculation of wall distance arrays which are very time-consuming in case of large, complex 3D topologies, particularly for nonstationary grids which require re-computing wall distance arrays at every time-step, a daunting demand on time- and budget-restricted engineers.

The proposed turbulence model is promising, however, additional validations are necessary in order to better establish its merits and limitations. Specifically, complex flows, involving reactive and multiphase phenomena, must be computed to gain confidence in the proposed approach.
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List of symbols

| Symbol | Definition |
|--------|------------|
| a₁ = 0.31 | Bradshaw constant |
| A₀ = 0.0266 | viscous sublayer parameter |
| c | chord length [m] |
| C₁ | skin friction coefficient |
| C_p | pressure coefficient |
| C_μ = 0.09 | eddy viscosity coefficient |
| D | diameter [m] |
| evr | eddy viscosity ratio, μ/μ |
| F₁, F₂ | blending functions |
| h | step height [m] |
| k | turbulence kinetic energy [(m/s)²] |
| ℓ | length-scale [m] |
| M | Mach number |
| p | pressure [Pa] |
| P_k | production of turbulent kinetic energy [m²/s³] |
| r | nozzle exit radius |
| Re | Reynolds number |
| RHS | Right-Hand-Side |
| S | mean strain [1/s] |
| t | time [s] |
| T | temperature [K] |
| Tu = (√2k/3)/U_∞ | turbulence intensity |
| U | mean velocity [m/s] |
| u | friction velocity [m/s] |
| V | velocity magnitude |
| wdf | wall-distance-free |
| x, y, z (or x_j) | Cartesian streamwise, normal and transverse coordinates [m] |
| α = 5/9 | turbulent inner layer nondimensional coordinate |
| β, γ | coefficients in the k-ω model |
| δ⁺ | boundary layer thickness [m] |
| δ | boundary layer displacement thickness [m] |
| δ_ij | Kronecker delta (1 if i = j, 0 otherwise) |
| ε | turbulence kinetic energy dissipation rate [m²/s³] |
| κ ≈ 0.41 | von Karman Constant |
| μ | dynamic molecular viscosity [kg/(m s)] |
| μ₀ | dynamic eddy viscosity [kg/(m-s)] |
| ν = μ/ρ | kinematic molecular viscosity [m²/s] |
| ρ | density [kg/m³] |
| σ_k | Prandtl number in k-equation diffusion term |
| σ_ω | Prandtl number in ω-equation diffusion term |
| σ_w² = 0.856 | time-scale [s] or shear stress [Pa] |
| τ | turbulent inverse time-scale [s⁻¹] |

Subscripts

∞ evaluated at freestream
ε evaluated at outer edge of shear layer
i,j j-derivative of i-component
t or T turbulent
tot total, stagnation
w evaluated at the wall
x evaluated at streamwise location x