TENSOR PRODUCTS OF
HOMOTOPY GERSTENHABER ALGEBRAS

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Abstract. On the tensor product of two homotopy Gerstenhaber algebras we construct a Hirsch algebra structure which extends the canonical dg algebra structure. Our result applies more generally to tensor products of “level 3 Hirsch algebras” and also to the Mayer–Vietoris double complex.

1. Introduction

Let \( R \) be a commutative unital ring and \( A \) an augmented associative differential graded (dg) algebra over \( R \). A Hirsch algebra structure on \( A \) is a (possibly non-associative) multiplication in the normalized bar construction \( \bar{B}A \) of \( A \) which is a morphism of coalgebras and has the counit \( 1 \in \bar{B}A \) as a unit. It is uniquely determined by its associated twisting cochain

\[
E : \bar{B}A \otimes \bar{B}A \to A.
\]

Because the map \( a_1 \otimes b_1 \mapsto E([a_1], [b_1]) \) is essentially a \( \cup_1 \) product for \( A \) (without strict Hirsch formulas), the product of a Hirsch algebra is always commutative up to homotopy in the naive sense.

Let \( a = [a_1] \cdots [a_k] \in \bar{B}kA \) and \( b = [b_1] \cdots [b_l] \in \bar{B}lA \). A Hirsch algebra with \( E(a, b) = 0 \) for all \( k > 1 \) is called a “level 3 Hirsch algebra” in [5]. It is a homotopy Gerstenhaber algebra (or homotopy G-algebra) if in addition the resulting multiplication is associative. Important examples of homotopy Gerstenhaber algebras are the cochain complex of a simplicial set or topological space [1], the Hochschild cochains of an associative algebra [4], [3, Sec. 5.1], [8] and the cobar construction of a dg bialgebra over \( \mathbb{Z}_2 \) [5].

Let \( A' \) and \( A'' \) be two Hirsch algebras. Then \( A' \otimes A'' \) is a dg algebra, again commutative up to homotopy in the naive sense. In this paper we address the question of whether such a homotopy is part of a system of higher homotopies. We obtain the following result:

**Theorem 1.1.** Let \( A' \) and \( A'' \) be two level 3 Hirsch algebras. Then \( A' \otimes A'' \) is a Hirsch algebra in a natural way. Moreover, the shuffle map \( \bar{B}A' \otimes \bar{B}A'' \to \bar{B}(A' \otimes A'') \) is multiplicative.

The paper is organized as follows: In Section 2 we introduce the notation needed for the later parts. The Hirsch algebra structure of \( A = A' \otimes A'' \) is constructed in Section 3. Example 3.1 shows how our twisting cochain \( E : \bar{B}A \otimes \bar{B}A \to A \) looks like in small degrees, and Example 3.2 illustrates a general recipe for computing it explicitly. Section 4 contains the proof that \( E \) is well-defined and that the shuffle
map is multiplicative. We conclude by reformulating our result in an operadic language and applying it to the Mayer–Vietoris double complex in Section \[\text{Acknowledgements.}\] The author would like to thank Tornike Kadeishvili for helpful discussions.

2. Notation

We work in a cohomological setting, so that differentials are of degree +1. We denote the desuspension of a complex $C$ by $s^{-1}C$, and the canonical chain map $s^{-1}C \to C$ of degree 1 by $\sigma$. Anticipating the definition of the bar construction, we also write $\sigma^{-1}(c) = [c]$ for $c \in C$. The differential on $s^{-1}C$ is given by $d[c] = -[dc]$.

Let $A$ be an augmented, unital associative dg algebra over $R$ with multiplication $\mu_A: A \otimes A \to A$ and augmentation $\varepsilon_A: A \to R$. Denote the augmentation ideal of $A$ by $\tilde{A}$, so that $A = R \oplus \tilde{A}$ canonically.

Let that there are canonical isomorphisms of complexes
\begin{align}
(2.1a) & \quad s^{-1}A' \otimes A'' \to s^{-1}(A' \otimes A''), \quad [a'] \otimes a'' \mapsto [a' \otimes a''], \\
(2.1b) & \quad A' \otimes s^{-1}A'' \to s^{-1}(A' \otimes A''), \quad a' \otimes [a''] \mapsto (-1)^{|a'|}|a' \otimes a''|.
\end{align}

Although we are mostly interested in the normalized bar construction $\bar{BA}$ of $A$, it will be convenient to consider the unnormalized bar construction $BA$ as well. This is the tensor coalgebra of the desuspension of $A$ (instead of $\tilde{A}$),
\begin{equation}
(2.2) \quad BA = T(s^{-1}A) = \bigoplus_{k \geq 0} (s^{-1}A)^{\otimes k}.
\end{equation}

We write $B_k A = (s^{-1}A)^{\otimes k}$ and for elements $[a_1] \cdots [a_k] \in B_k A$. The differential on $BA$ is the sum of the tensor product differential $d_\otimes$ and the differential
\begin{equation}
(2.3) \quad \partial = \bigoplus_{i=1}^{k-1} 1^{\otimes i-1} \otimes \tilde{\mu} \otimes 1^{\otimes k-i-1} : B_k A \to B_{k-1} A.
\end{equation}

Here $\tilde{\mu}$ denotes the desuspension of $\mu$,
\begin{equation}
(2.4) \quad \tilde{\mu} = \sigma^{-1} \mu(\sigma \otimes \sigma) : s^{-1}A \otimes s^{-1}A \to s^{-1}A.
\end{equation}

We write $1 \in B_0 A$ for the counit of $BA$ and $\alpha$ for the canonical twisting cochain
\begin{equation}
(2.5) \quad \alpha : BA \to B_1 A = s^{-1}A \xrightarrow{\sigma} A.
\end{equation}

Let $M$ be a right dg-$A$-module and $N$ a left dg-$A$-module with structure maps $\mu_M: M \otimes A \to M$ and $\mu_N: A \otimes N \to N$, respectively. The two-sided bar construction of the triple $(M, A, N)$ is
\begin{equation}
(2.6) \quad B(M, A, N) = M \otimes BA \otimes N
\end{equation}
with differential $d_{B(M,A,N)} = d_{M \otimes BA \otimes N} + \partial'$, where
\begin{equation}
(2.7) \quad \partial' = (\mu_M(1 \otimes \alpha) \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1) - (1 \otimes 1 \otimes \mu_N(\alpha \otimes 1))(1 \otimes \Delta \otimes 1),
\end{equation}
and with augmentation
\begin{align}
(2.8a) & \quad \varepsilon_{B(M,A,N)} : B(M, A, N) \to M \otimes_A N, \\
(2.8b) & \quad m[a_1] \cdots [a_k]n \mapsto \begin{cases} m \otimes n & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{align}
We write repeated (co)associative maps in the form
\[\mu^{(k)}: A^\otimes k \to A,\]
\[\Delta^{(k)}: T(s^{-1}A) \to T(s^{-1}A)^\otimes k,\]
for instance, and we agree that \(\mu^{(0)}\) is the unit map \(\nu: R \to A\).

We will also need the concatenation operator
\[\nabla: BA \otimes BA \to BA,\]
\[\alpha_1 | \cdots | \alpha_k \otimes \beta_1 | \cdots | \beta_l \mapsto [\alpha_1] \cdots [\alpha_k][\beta_1] \cdots [\beta_l],\]
which satisfies
\[d(\nabla) = \nabla(\tilde{\mu} \otimes 1)(1 \otimes \alpha \otimes \alpha \otimes 1)(\Delta \otimes \Delta)\]
and
\[(\alpha \otimes 1)\Delta \nabla = (\alpha \otimes \nabla)(\Delta \otimes 1) + \varepsilon_{BA} \otimes (\alpha \otimes 1)\Delta\]
\[= (1 \otimes \nabla)((\alpha \otimes 1)\Delta \otimes 1) + \varepsilon_{BA} \otimes (\alpha \otimes 1)\Delta,\]
\[(1 \otimes \alpha)\Delta \nabla = (\nabla \otimes \alpha)(1 \otimes \Delta) + (1 \otimes \alpha)\Delta \otimes \varepsilon_{BA}\]
\[= (\nabla \otimes 1)(1 \otimes (1 \otimes \alpha)\Delta) + (1 \otimes \alpha)\Delta \otimes \varepsilon_{BA}.\]

On both the unnormalized and the normalized bar construction, we will only consider multiplications which are coalgebra maps and have the counit 1 as a (two-sided) unit. We do not require the multiplication to be associative.

Any such multiplication \(f: BA \otimes BA \to BA\) is uniquely determined by its twisting cochain \(E = \alpha f\), which satisfies
\[d(E) = E \cup E,\]
\[E(1, -) = E(-, 1) = \alpha.\]

We will only consider twisting cochains \(E\) satisfying both conditions.

Any multiplication on the normalized bar construction \(\bar{B}A \subset BA\) can be extended to \(BA\) in a canonical way: Define \(E([1], 1) = E(1, [1]) = 1\) and, for \(a = [a_1] \cdots [a_k], b = [b_1] \cdots [b_l] \in BA,\) set \(E(a, b) = 0\) if \(k + l > 1\) and some \(a_i = 1\) or some \(b_j = 1\). Then \(E(a, b) \in \bar{A}\) whenever \(k + l > 1\). We call a twisting cochain having these additional properties normalized. Any normalized twisting cochain \(E: BA \otimes BA \to A\) comes from a unique multiplication on \(\bar{B}A\).

For a map \(E: BA \otimes BA \to A\) and \(a \in BA\) we define
\[E_a: BA \to A,\]
\[b \mapsto E(a, b).\]

In this notation, the properties of a multiplication on \(BA\) become
\[d(E_a) = -E_{d a} + \sum_{i=0}^{k} (-1)^{|a_1| \cdots |a_i|} \mu(E_{[a_1] \cdots [a_i]} \otimes E_{[a_{i+1}] \cdots [a_k]}) \Delta\]
\[E_1(b) = \alpha(b),\]
\[E_a(1) = \alpha(a)\]
for \(a = [a_1] \cdots [a_k]\) and \(b = [b_1] \cdots [b_l] \in BA\). If \(E\) is normalized, then one additionally has
\[E_a(b) = 0\]
\[\varepsilon(E_a(b)) = 0\]
for \(k + l > 1\) and some \(a_i = 1\) or some \(b_j = 1\).
If $E$ is of level 3, then condition (2.15a) is equivalent to the two identities

\begin{align}
(2.17a) & \quad d(E_{[a_1]}) = -E_{d[a_1]} + \mu(\alpha \otimes E_{[a_1]} + (-1)^{|a_1|-1}E_{[a_1]} \otimes \alpha)\Delta, \\
(2.17b) & \quad E_{[a_1,a_2]} = (-1)^{|a_1|-1}\mu(E_{[a_1]} \otimes E_{[a_2]})\Delta.
\end{align}

3. Construction of the twisting cochain

Let $A'$ and $A''$ be two level 3 Hirsch algebras with twisting cochains $E'$ and $E''$, respectively. Set $A = A' \otimes A''$. We are going to inductively define maps $G_a : BA \to B(A, A, A)$ of degree $|a| + 1$ for $a \in BA$ and then set $E_a = \varepsilon_{B(\mathbb{A}, \mathbb{A})}G_a$. In Section 4 we will show that this defines a twisting cochain $E : BA \otimes BA \to A$, hence a multiplication in $BA$. Moreover, if both $E$ and $E''$ are normalized, then so is $E$.

For the construction as well as for the proof, it is convenient to identify $B(A, A, A)$ with $A \otimes BA \otimes A$. This is an isomorphism of graded $R$-modules; the difference between the two differentials is given by (2.7). We write $a = [a_1] \cdots [a_k] \in BA$ with $a_i = a'_i \otimes a''_i$.

For $k = 0$ we set $E_1 = \alpha$ as required by (2.15b). We define for $k = 1$

\begin{equation}
(3.1) \quad G_{[a_1]} = (\langle E'_{[a'_1]} \otimes \mu_{A''} \rangle \otimes 1 \otimes (\mu_{A'} \otimes E''_{[a''_1]}))\Delta^{(3)}
\end{equation}

and for $k > 1$

\begin{equation}
(3.2) \quad G_a = M(E'_{[a'_1]}, E''_{[a''_1]}, G_{[a_2] \cdots [a_k]}).
\end{equation}

Here we have used the abbreviation

\begin{equation}
(3.3) \quad M(\tilde{E}', \tilde{E}'', \tilde{G}) = (1 \otimes 1 \otimes \mu_A)((\tilde{E}' \otimes \mu_{A''}) \otimes 1 \otimes (\mu_{A'} \otimes \tilde{E}'') \otimes 1) \\
(1 \otimes \Delta\nabla^{(3)} \otimes 1)(1 \otimes 1 \otimes (\sigma^{-1} \otimes 1 \otimes 1)\tilde{G})\Delta^{(3)}
\end{equation}

for maps $\tilde{E}' : BA' \to A'$, $\tilde{E}'' : BA'' \to A''$ and $\tilde{G} : BA \to A \otimes BA \otimes A$. Moreover, by $\tilde{E}' \otimes \mu_{A''} : BA \to A$ we mean the map

\begin{equation}
(3.4) \quad \left[ b_1 \right] \cdots \left[ b_k \right] \mapsto \left( \prod_{i > j} (-1)^{|b'_i||b'_j|} \right) \tilde{E}' (\left[ b'_1 \right] \cdots \left[ b'_k \right]) \otimes \mu_{A''}(b''_1 \otimes \cdots \otimes b''_k),
\end{equation}

and similarly by $\mu_{A'} \otimes \tilde{E}'' : BA \to A$

\begin{equation}
(3.5) \quad \left[ b_1 \right] \cdots \left[ b_k \right] \mapsto \left( \prod_{i > j} (-1)^{|b''_i||b''_j|} \right) \mu_{A'}(b'_1 \otimes \cdots \otimes b'_k) \otimes \tilde{E}'' (\left[ b''_1 \right] \cdots \left[ b''_k \right]).
\end{equation}

By identities (2.1), the differentials of these maps are

\begin{equation}
(3.6) \quad d(\tilde{E}' \otimes \mu_{A''}) = d(\tilde{E}') \otimes \mu_{A''}, \quad d(\mu_{A'} \otimes \tilde{E}'') = \mu_{A'} \otimes d(\tilde{E}'').
\end{equation}

Figures 1 and 2 visualize the definitions of $G_{[a_1]}$ and of $M(\tilde{E}', \tilde{E}'', \tilde{G})$. 
Example 3.1. The following list shows $E(a, b)$ for $a \in B_k A$ and $b \in B_l A$ with $k \leq 2$ and $l \leq 2$. We are ignoring signs here.

(3.7a) \quad E([a_1], [b_1]) = a'_1 b'_2 \otimes E''([a''_1], [b''_1]) + E'([a'_1], [b'_1]) \otimes b''_2 a''_2,

(3.7b) \quad E([a_1], [b_1|b_2]) = a'_1 b'_2 b'_2 \otimes E''([a''_1], [b''_1|b''_2])

(3.7c) \quad + E'([a'_1], [b'_1|b_2]) \otimes b''_2 a''_1

(3.7d) \quad + E'([a'_1], [b'_1|b_2]) \otimes b''_1 b''_2 a''_2

(3.7e) \quad E([a_1|a_2], [b_1]) = a'_1 E'([a'_2], [b'_1]) b'_2 \otimes E''([a''_1], [b''_1]) E''([a''_2], [b''_2])

(3.7f) \quad E([a_1|a_2], [b_1|b_2]) = a'_1 E'([a'_2], [b'_1]) b'_2 \otimes E''([a''_1], [b''_1]) E''([a''_2], [b''_2])

(3.7g) \quad + a'_1 E'([a'_2], [b'_1]) b'_2 \otimes E''([a''_1], [b''_1|b''_2]) a''_2

(3.7h) \quad + a'_1 E'([a'_2], [b'_1|b_2]) \otimes E''([a''_1], [b''_1|b''_2]) a''_2

(3.7i) \quad + a'_1 b'_2 E'([a'_2], [b'_2]) \otimes E''([a''_1], [b''_1|b''_2]) a''_2

(3.7j) \quad + E'([a'_1], [b'_1|b_2]) E'([a'_2], [b'_2]) \otimes b''_1 b''_2 E''([a''_1], [b''_2]) a''_2.

Example 3.2. We give a general recipe for computing $E(a, b)$ as in Example 3.1. To show all features of the algorithm, we illustrate it with $a = [a'_1 \otimes a''_2 | a''_1 \otimes a'_2]$ and $b = [b'_1 \otimes b''_2 | \cdots | b'_2 \otimes b''_1]$. We are going to explain how to obtain the terms $c' \otimes c'' \in A' \otimes A''$ appearing in $E(a, b)$, again ignoring signs for simplicity.

We start by looking at the component $c' \in A'$. Take $[b'_1|\cdots|b'_2]$ and cut it into $2k$ pieces such that the pieces at positions 3, 5, \ldots, $2k - 1$ have length at least 1. In our example, one such decomposition is

\begin{equation}
[b'_1] \otimes [b'_2|b'_3] \otimes [b'_4|b'_5] \otimes 1.
\end{equation}

(The last piece has length 0.) Now apply $E'_{[a'_1]}$ to the $(2i - 1)$-th group and then multiply everything together:

\begin{equation}
E'_{[a'_1]}([b'_1|b'_2]) \cdot b'_2 b'_3 \cdot E'_{[a'_2]}([b'_4|b'_5]) \cdot 1 = E'([a'_1], [b'_1|b'_2]) b'_2 b'_3 E'([a'_2], [b'_4|b'_5]) = c'.
\end{equation}
These are the possible factors \( c' \in A' \) of the terms \( c' \otimes c'' \) appearing in \( E(a, b) \).

For each such factor, we now describe which factors \( c'' \in A'' \) appear: Switch from primed to doubly primed variables and multiply the components within the odd-numbered groups together to obtain

\[
\begin{bmatrix} b''_1 & b''_2 & b''_3 & b''_4 & b''_5 \end{bmatrix}.
\]

Take the first factor of the tensor product (in the example, \( [b''_1] \)) apart. Cut the rest

\[
\begin{bmatrix} b''_2 & b''_3 & b''_4 & b''_5 \end{bmatrix}
\]

into \( k \) pieces. Only cuts satisfying the following condition are allowed: If some \( b''_j \) appears as argument to \( E'(\alpha) \), then the corresponding element \( b''_j \) can only appear
in the \((i - 1)\)-th piece or earlier. In our example, this forces the second piece to be empty, hence the first piece is everything. Now plug the \(i\)-th piece into \(E''_{[a_j]}\) and multiply everything together, including the first factor we have put apart earlier:

\[
\text{(3.12)} \quad b''_1 \cdot E''_{[a_j]}([b'_2, b'_4 [b''_5]]) \cdot E''_{[a_j]}(1) = b''_1 E''([a'_1], [b'_2, b'_4 [b''_5]]) a''_1 = c''.
\]

Summing up,

\[
\text{(3.13)} \quad E'([a'_1], [b'_1]) b''_2 b''_3 E'([a'_2], [b'_4 [b''_5]]) \otimes b''_4 E''([a''_1], [b''_2 [b''_4 [b''_5]]) a''_1
\]

is one term appearing in \(E(a, b)\). (There are 70 terms altogether.)

The reason for the length condition imposed in the first step is the following: The recursive definition of \(G_a\) together with the assignment \(E_a = \varepsilon G_a\) force everything that “runs through” \(E''_{[a_j]} \otimes \mu, i > 1\), to “go through” some \(\mu \otimes E''_{[a'_j]}\) with \(j < i\) as well. Because \((E''_{[a'_j]} \otimes \mu)(1) = a'_i \otimes 1\) and \(E''_{[a'_j]}(1) = 0\), the length of the argument of \(E''_{[a'_j]}\) must therefore be at least 1 if \(i > 1\).

**Remark 3.3.** The multiplication in \(\overline{B}(A' \otimes A'')\) is not associative in general, not even if it is so in \(\overline{B}A'\) and \(\overline{BA'}\) (which means that \(A'\) and \(A''\) are homotopy Gerstenhaber algebras). In the latter case one has

\[
\text{(3.14)} \quad ([a] \cdot [b]) \cdot [c] + [a] \cdot ([b] \cdot [c]) = d(h)([a], [b], [c])
\]

for \(a = a' \otimes a''\), \(b = b' \otimes b''\), \(c = c' \otimes c'' \in A' \otimes A''\) and

\[
\text{(3.15a)} \quad h([a], [b], [c]) = [a' E([b'], [c']) \otimes E([a''], [c'' | b''])]
\]

\[
\text{(3.15b)} \quad + [E([a'], [b'|c']) \otimes E([b''], [c'']) a''].
\]

(We are again ignoring signs here.)

**Question 3.4.** Is \(\overline{B}(A' \otimes A'')\) an \(A_{\infty}\)-algebra if \(A'\) and \(A''\) are homotopy Gerstenhaber algebras?

## 4. Proof of the Main Result

In Section 3 we constructed a map \(G_a : BA \to B(A, A, A)\) for each \(a \in BA\). They can be assembled into a map \(G : BA \otimes BA \to B(A, A, A)\). We now study its differential.

Denote the left and right action of \(A\) on \(B(A, A, A)\) by \(\mu_L\) and \(\mu_R\), respectively, and let \(\beta\) be the twisting cochain

\[
\text{(4.1)} \quad \beta = \varepsilon_{BA} \otimes \alpha_{BA} : BA \otimes BA \to R \otimes A = A.
\]

**Proposition 4.1.** The differential of \(G\) is

\[
d(G) = \mu_L(\beta \otimes G) \Delta_{BA \otimes BA} + \mu_R(G \otimes (E - \beta)) \Delta_{BA \otimes BA}.
\]
Proof. We again identify \( B(A, A, A) \) with \( A \otimes BA \otimes A \). Taking equation (4.7) into account, we have to show

\[(4.2a) \quad d(G_a) = -G_{d_a} \]
\[(4.2b) \quad + (\mu \otimes 1 \otimes 1)(\alpha \otimes G_a) \Delta_{BA} \]
\[(4.2c) \quad + \sum_{i=1}^{k} (-1)^{|[a_i]|} (1 \otimes 1 \otimes \mu)(G_{[a_i]} \otimes E_{[a_i]} \otimes E_{[a_{i+1}]} \otimes E_{[a_k]}) \Delta_{BA} \]
\[(4.2d) \quad - (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G_a \]
\[(4.2e) \quad + (1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)G_a \]

for all \( a = [a_1] \cdots [a_k] \in BA \). We proceed by induction on \( k \). Write \( \tilde{E}' = E_{[a_1]}' \) and \( \tilde{E}'' = E_{[a_k]''} \). Recall that we have

\[(4.3) \quad |E_{[a_1]}'| = |a_1|, \quad |E_{[a_k]''}| = |a_k|, \quad |G_a| = |a| + 1.\]

For \( k = 1 \), i.e., \( a = [a_1] \in s^{-1}A \), we have

\[(4.4a) \quad d(G_a) = \left( (d(E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''})) \right) \Delta^{(3)} \]
\[(4.4b) \quad + (-1)^{|a_1|} ((E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes d(E_{[a_k]''})) \right) \Delta^{(3)} \]

using formula (4.4a).

\[(4.4c) \quad = \left( (E_{[a_1]}' d_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''}) \right) \Delta^{(3)} \]
\[(4.4d) \quad - (-1)^{|a_1|} ((E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''}')) \Delta^{(3)} \]
\[(4.4e) \quad + (\mu \otimes 1 \otimes 1)(E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''}) \Delta^{(4)} \]
\[(4.4f) \quad + (-1)^{|a_1|-1}(\mu \otimes 1 \otimes 1)((E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''}')) \Delta^{(4)} \]
\[(4.4g) \quad + (-1)^{|a_1|}(1 \otimes 1 \otimes \mu)((E_{[a_1]}' \otimes \mu) \otimes 1 \otimes \alpha \otimes (\mu \otimes E_{[a_k]''}')) \Delta^{(4)} \]
\[(4.4h) \quad + (-1)^{|a_1|+|a_k|-1}(1 \otimes 1 \otimes \mu) \]
\[(4.4i) \quad ((E_{[a_1]}' \otimes \mu) \otimes 1 \otimes (\mu \otimes E_{[a_k]''}')) \Delta^{(4)} \]
\[(4.4j) \quad = -G_{d_a} \]
\[(4.4k) \quad + (\mu \otimes 1 \otimes 1)(\alpha \otimes G_{[a_1]}) \Delta \]
\[(4.4l) \quad + (-1)^{|a_1|}(1 \otimes 1 \otimes \mu)(G_{[a_1]} \otimes E_1) \Delta \]
\[(4.4m) \quad - (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G_a \]
\[(4.4n) \quad + (1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)G_a \].

For \( k > 1 \), we write \( \tilde{a} = [a_2] \cdots [a_k] \) and \( \tilde{G} = G_{\tilde{a}} \). Then, using definition (3.3),

\[(5.1a) \quad d(G_a) = d(M(\tilde{E}', \tilde{E}'', \tilde{G})) \]
\[(5.1b) \quad = M(d(\tilde{E}'), \tilde{E}'', \tilde{G}) + (-1)^{|a_1|} M(\tilde{E}', d(\tilde{E}'', \tilde{G})) \]
\[(5.1c) \quad + (-1)^{|a_1|+|a_k|-1}(\mu \otimes \mu \otimes \mu \otimes \mu \otimes \mu)(\tilde{E}' \otimes \mu \otimes \mu \otimes \mu \otimes \mu) \Delta^{(1)} \]
\[(5.1d) \quad + 1 \otimes \Delta d(\nabla^{(3)} \otimes 1)(1 \otimes 1 \otimes 1 \otimes 1) \Delta^{(3)} \]
\[(5.1e) \quad + (-1)^{|a_1|+|a_k|-1} M(\tilde{E}', \tilde{E}'', d(\tilde{G})) \].
using (2.12a), \( \tilde{\mu}(1 \otimes \sigma^{-1}) = \sigma^{-1}\mu(\sigma \otimes 1) \) and \( \tilde{\mu}(\sigma^{-1} \otimes 1) = -\sigma^{-1}\mu(1 \otimes \sigma) \)

\[
\begin{align*}
(4.5f) \quad & = M(d(\tilde{E}''), \tilde{E}'', \tilde{G}) + (-1)^{|a_1|}M(\tilde{E}', d(\tilde{E}''), \tilde{G}) \\
(4.5g) \quad & + (-1)^{|a_1|}M(\tilde{E}', \tilde{E}'', (\mu \otimes 1 \otimes 1)(\alpha \otimes \tilde{G})\Delta) \\
(4.5h) \quad & + (-1)^{|a_1|-1}M(\tilde{E}', \tilde{E}'', (\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)\tilde{G}) \\
(4.5i) \quad & + (-1)^{|a_1|-1}M(\tilde{E}', \tilde{E}'', d(\tilde{G})) ;
\end{align*}
\]

\[
\begin{align*}
(4.6a) \quad & G_{d\otimes a} = M(E_{d[a_1]}, \tilde{E}'', \tilde{G}) + (-1)^{|a_1|}M(\tilde{E}', E_{d[a_1]}, \tilde{G}) \\
(4.6b) \quad & + (-1)^{|a_1|-1}M(\tilde{E}', \tilde{E}'', G_{d\otimes a}) ;
\end{align*}
\]

\[
\begin{align*}
(4.7a) \quad & \sum_{i=2}^{k} M(\tilde{E}', \tilde{E}'', (1 \otimes 1 \otimes \mu)(G_{[a_2]\cdots[a_i]} \otimes E_{[a_{i+1}][a_k]}))\Delta \\
(4.7b) \quad & = \sum_{i=2}^{k} (1 \otimes 1 \otimes \mu)(G_{[a_1]\cdots[a_i]} \otimes E_{[a_{i+1}][a_k]})\Delta ;
\end{align*}
\]

\[
\begin{align*}
(4.8) \quad & M(\tilde{E}', \tilde{E}'', G_{d\otimes a}) = \sum_{i=2}^{k-1} (-1)^{|a_2|-|a_i|}G_{[a_1]\cdots[a_i-1][a_{i+1}][a_k]} ;
\end{align*}
\]

\[
\begin{align*}
(4.9) \quad & M(\mu(\alpha \otimes \tilde{E}')\Delta, \tilde{E}'', \tilde{G}) = (\mu \otimes 1 \otimes 1)(\alpha \otimes \tilde{G})\Delta ;
\end{align*}
\]

\[
\begin{align*}
(4.10) \quad & M(\tilde{E}', \mu(\alpha \otimes \tilde{E}'')\Delta, \tilde{G}) = (-1)^{|a_1|}(1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes \alpha \otimes 1)(1 \otimes \Delta \otimes 1)\tilde{G} ;
\end{align*}
\]

and

\[
\begin{align*}
(4.11a) \quad & M(\mu(\tilde{E}' \otimes \alpha)\Delta, \tilde{E}'', \tilde{G}) \\
(4.11b) \quad & = (-1)^{|a_1|}(\mu \otimes 1 \otimes \mu)((\tilde{E}' \otimes \mu) \otimes 1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1) \\
(4.11c) \quad & (1 \otimes 1 \otimes \Delta \nabla(3) \otimes 1)(1 \otimes (\alpha \otimes 1)\Delta \otimes (\sigma^{-1} \otimes 1 \otimes 1)\tilde{G})\Delta (3)
\end{align*}
\]

using (2.12b) and the fact that \( \tilde{G} \) maps to \( \mathcal{A} \otimes BA \otimes \mathcal{A} \)

\[
\begin{align*}
(4.11d) \quad & = (-1)^{|a_1|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)M(\tilde{E}', \tilde{E}'', \tilde{G}) \\
(4.11e) \quad & + (-1)^{|a_1|}(1 \otimes 1 \otimes \mu) \\
(4.11f) \quad & (1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1)(\mu \otimes \Delta \otimes 1)((\tilde{E}' \otimes \mu) \otimes \tilde{G})\Delta
\end{align*}
\]

We consider the case \( k = 2 \) first.

\[
\begin{align*}
(4.11g) \quad & = (-1)^{|a_1|}(\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)M(\tilde{E}', \tilde{E}'', \tilde{G}) \\
(4.11h) \quad & + (-1)^{|a_1|}(1 \otimes 1 \otimes \mu) \\
(4.11i) \quad & (1 \otimes 1 \otimes (\mu \otimes \tilde{E}'') \otimes 1)(\mu \otimes \Delta \otimes 1)((\tilde{E}' \otimes \mu) \otimes G_{[a_2]})\Delta
\end{align*}
\]
using (2.17b) in the form \( \mu((E'_{a_1'}) \otimes \mu) (E'_{a_2'}) \otimes \mu) \Delta = (-1)^{|a'_1|-1} E'_{a_1'a_2'} \otimes \mu \)

\[(4.11j) \quad = (-1)^{|a'_1|}((\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \]
\[(4.11k) \quad + (-1)^{|a_1|+|a'_1||a'_1|-1}(1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes (\mu \otimes E''_{a_1'}(a_1')) \otimes 1) \]
\[(4.11l) \quad (1 \otimes \Delta \otimes 1)G_{[a'_1a'_2a'_2]} \Delta \]

using (2.17b) in the form \( \mu((\mu \otimes E''_{a_1'}(a_1'))(\mu \otimes E''_{a_2'}(a_2'))) \Delta = (-1)^{|a'_1|-1} \mu \otimes E''_{a_1'a_2'} \)

\[(4.11m) \quad = (-1)^{|a'_1|}((\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \]
\[(4.11n) \quad + (-1)^{|a_1|+|a'_1||a'_1|}G_{[a'_1a'_2a'_2]} \]

\[(4.11o) \quad \text{Continuing at } (4.11e) \text{ for } k > 2 \text{ and using the same identities as before,} \]
\[(4.11p) \quad \text{So the result is the same for all } k \geq 2. \]

\[(4.11q) \quad = (-1)^{|a'_1|}((\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \]
\[(4.11r) \quad + (-1)^{|a_1|+|a'_1||a'_1|-1}(1 \otimes 1 \otimes \mu)(1 \otimes 1 \otimes (\mu \otimes E''_{a_1'}(a_1')) \otimes 1) \]
\[(4.11s) \quad (1 \otimes \Delta \otimes 1)M(E'_{a_1'a_2'}, E''_{a_2'}(a_2'), G_{[a_3]...[a_k]}) \]
\[(4.11t) \quad = (-1)^{|a'_1|}((\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \]
\[(4.11u) \quad + (-1)^{|a_1|+|a'_1||a'_2|}M(E'_{a_1'a_2'}, E''_{a_2'a_2'}, G_{[a_3]...[a_k]}) \]
\[(4.11v) \quad = (-1)^{|a'_1|}((\mu \otimes 1 \otimes 1)(1 \otimes \alpha \otimes 1 \otimes 1)(1 \otimes \Delta \otimes 1)G \]
\[(4.11w) \quad + (-1)^{|a_1|}G_{[a_1a_2a_3]...[a_k]}. \]

\[(4.11x) \quad \text{Putting all terms together finishes the proof.} \]

**Proposition 4.2.** The map \( E : BA \otimes BA \to A \) is a twisting cochain. Moreover, if \( E' \) and \( E'' \) are normalized, then so is \( E \).
Proof. To verify (2.15a), we compute:

\[
\begin{align*}
(4.13a) \quad d(E_a) &= \varepsilon d(G_a) \\
(4.13b) &= -\varepsilon G_{da} + \mu(\alpha \otimes E_a)\Delta \\
(4.13c) &= -\varepsilon G_{da} + \sum_{i=0}^{k} (-1)^{|a_1| \cdots |a_k|} \mu(E_{[a_1] \cdots |a_k|} \otimes E_{[a_{i+1}] \cdots |a_k|})\Delta \\
(4.13d) &= -E_{da} + \sum_{i=0}^{k} (-1)^{|a_1| \cdots |a_k|} \mu(E_{[a_1] \cdots |a_k|} \otimes E_{[a_{i+1}] \cdots |a_k|})\Delta.
\end{align*}
\]

Condition (2.15b) holds by definition. Condition (2.15c) holds for \( k = 1 \) because \( G_{[a_1]}(1) = (a_1' \otimes 1) \otimes 1 \otimes 1 \). For \( k > 1 \), one similarly has \( G_a(1) \in (A' \otimes 1) \otimes BA \otimes A \), hence \( \varepsilon(G_a(1)) = 0 \) by condition (2.15c) for \( E'' \). (This is related to the length condition in Example 3.2.)

Assume now that \( E' \) and \( E'' \) are normalized. For the proof of (2.16a) one inductively shows \( G_a(b) \in \bigoplus_{m \geq 1} A \otimes B_m A \otimes A \) if some \( a_i = 1 \) or some \( b_j = 1 \).

We now turn to the shuffle maps

\[
\begin{align*}
(4.14a) \quad \nabla : BA' \otimes BA'' &\to B(A' \otimes A''), \\
(4.14b) \quad \nabla : BA' \otimes BA'' &\to \tilde{B}(A' \otimes A''),
\end{align*}
\]

cf. [7 Sec. 7.1].

**Proposition 4.3.** The shuffle maps (4.14) are multiplicative.

**Proof.** It suffices to consider the unnormalized bar construction. We have to show that the diagram

\[
\begin{array}{ccc}
(BA' \otimes BA'') \otimes (BA' \otimes BA'') & \xrightarrow{\nabla \otimes \nabla} & B(A' \otimes A'') \otimes B(A' \otimes A'') \\
\downarrow \quad \mu \otimes \mu & & \downarrow \\
BA' \otimes BA'' & \xrightarrow{\nabla} & B(A' \otimes A'')
\end{array}
\]

commutes. Because all maps are morphisms of coalgebras, it is enough to verify that the two associated twisting cochains coincide.

Take two elements \( a = a' \otimes a'' \in B_{p'} A' \otimes B_{p''} A'' \), \( b = b' \otimes b'' \in B_{q'} A' \otimes B_{q''} A'' \). The twisting cochain of the composition via \( BA' \otimes BA'' \) vanishes unless \( p' = q' = 0 \) or \( p'' = q'' = 0 \). Consider now the twisting cochain of the other composition. It follows from properties (2.15b) and (2.15c) and the inductive definition of \( G_a \) that for \( p' > 0 \) this twisting cochain vanishes if \( p'' > 0 \) or \( q'' > 0 \). The case \( p'' > 0 \) is analogous. It is therefore enough to check the two cases \( a = a' \otimes 1 \), \( b = b' \otimes 1 \) and \( a = 1 \otimes a'' \), \( b = 1 \otimes b'' \). That both twisting cochains agree follows again inductively from the definition of \( G_a \). \( \square \)
5. Operadic reformulation

It is useful to translate Theorem 1.1 into the language of operads. Let $Ass$ be the operad of associative augmented unital $R$-algebras. We write $\mu \in Ass(2)$ for the multiplication, $\varepsilon \in Ass(1)$ for the augmentation and $\iota \in Ass(0)$ for the unit. An operad under $Ass$ is a morphism of operads $Ass \to P$.

We define the Hirsch operad $H$ to be the dg operad under $Ass$ generated by operations $E_{kl} \in H(k+l)_1, k,l \geq 0$ subject to the relations (2.15) and (2.16) (modulo the desuspension) plus the generators and relations for $Ass$. A Hirsch algebra then is the same as an algebra over $H$.

Let $H_3$ be the dg operad under $Ass$ describing level 3 Hirsch algebras. It is the quotient of $H$ by the relations $E_{kl} = 0$ for $k > 1$. Equivalently, it is generated by operations $E_{1k} \in H_3(1+k)$ and $E_{01}$ subject to the relations (2.15) and (2.16) with (2.15a) replaced by (2.17), and of course again plus the generators and relations for $Ass$.

**Theorem 5.1.** The construction in Section 3 defines a morphism $f: H \to H_3 \otimes H_3$ of dg operads under $Ass$.

**Proof.** Let $P$ be the free dg operad under $Ass$ generated by the operations $E_{kl}$. It is clear that our construction defines a morphism of dg operads under $Ass$

\[(5.1)\quad P \to H_3 \otimes H_3.\]

Moreover, we know that the relations for $H$ hold whenever $H_3 \otimes H_3$ acts on a tensor product of two $H_3$-algebras $A'$ and $A''$. More precisely, we have proven that the composed map

\[(5.2)\quad P \to H_3 \otimes H_3 \to End(A') \otimes End(A'')\]

factors through $H$. Because $A'$ and $A''$ can be free $H_3$-algebras (cf. [6, Sec. I.1.4]), this implies that the necessary relations hold already in $H_3 \otimes H_3$.

**Example 5.2.** The homotopy Gerstenhaber algebra structure on the cochain complex $C^*(X)$ of a simplicial set $X$ is constructed by dualizing a “homotopy Gerstenhaber coalgebra” structure on the chain complex $C(X)$. Therefore, for simplicial sets $X$ and $Y$ there is a natural action of $H_3 \otimes H_3$ on the complex dual to $C(X) \otimes C(Y)$, and the canonical map

\[(5.3)\quad C^*(X) \otimes C^*(Y) \to (C(X) \otimes C(Y))^*\]

is a morphism of $H_3 \otimes H_3$-algebras, hence of $H$-algebras. Note however that the dual of the shuffle map

\[(5.4)\quad C^*(X \times Y) \xrightarrow{\nabla^*} (C(X) \otimes C(Y))^*\]

is not a morphism of Hirsch algebras. ($\nabla^*$ already fails to commute with the operation (3.7a).)

An analogous remark applies to Hochschild cochains.

**Example 5.3.** Let $A$ be a cosimplicial $H_3$-algebra. By this we mean a collection $A^q$, $q \geq 0$, of $H_3$-algebras together with morphisms $d_i: A^q \to A^{q+1}$, $0 \leq i \leq q + 1$, satisfying the usual coface relations, cf. [7, Def. 8.40]. Then the associated total complex $Tot A^*$ is an algebra over $H_3 \otimes H_3$ in the following way: Let $E \otimes E' \in H_3(m) \otimes H_3(m')$, and $a_i \in A^{q_i}$ for $1 \leq i \leq m$. Set $q = \sum q_i - n'$. 


Via the coface operators, $E'$ determines morphisms $\phi_i : A^{q_i} \to A^q$ in the same way as it acts on the unnormalized cochains of a simplicial set. We can therefore set
\[(5.5) \quad (E \otimes E')(a_1, \ldots, a_m) = E(\phi_1(a_1), \ldots, \phi_m(a_m)) \in A^q.\]
(If $q < q_i$ for some $i$, we define the result to be 0.)

An important special case of this is the Mayer–Vietoris double complex
\[(5.6) \quad C^{pq}(U) = \prod_{i_0 < \cdots < i_q} C^p(U_{i_0} \cap \cdots \cap U_{i_q}; R)\]
associated to an ordered cover $U = (U_i)_{i \in I}$ of a simplicial set, cf. [2, §§8, 14] for instance. In this case Theorem 5.1 says that $\text{Tot} C^{**}(U)$ has the structure of a Hirsch algebra which extends the familiar dg algebra structure. Note also that the canonical inclusion map
\[(5.7) \quad C^*(X; R) \to \text{Tot} C^{**}(U), \quad \alpha \mapsto (\alpha|_{U_i})_{i \in I} \in C^{**}(U; R)\]
is a morphism of Hirsch algebras because for $n' > 0$ the maps $\phi_1, \ldots, \phi_m$ described above vanish on the image of the inclusion map, and for $n' = 0$ they must all be the identity map.

Remark 5.4. Assume $R = \mathbb{Z}_2$ and let $\tau = (12) \in S_2$. Note that $\mu$ is basis of $H_3(2)_0$ over $R[S_2]$, and $E_{11}$ is one for $H_3(2)_{-1}$. A direct computation shows that up to applying $\tau$ and transposing the factors, $h = \mu \otimes E_{11} + E_{11} \otimes \tau \mu \in (H_3 \otimes H_3)(2)_{-1}$ is the only solution to $d(h) = \mu \otimes \mu + \tau \mu \otimes \tau \mu$. Hence, our definition (5.7) of $f(E_{11})$ is essentially the only possible choice. Together with $d(f(E_{21})) \neq 0$, this also proves that one cannot hope for a morphism $H_3 \to H_3 \otimes H_3$ of dg operads under $\text{Ass}$ because condition (2.17b) never holds.

But of course one may ask:

Question 5.5. Is $\mathcal{H}$ a dg Hopf operad under $\text{Ass}$? In other words, is the tensor product of two Hirsch algebras again a Hirsch algebra?

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