Equivariant Khovanov homology associated with symmetric links

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Abstract

Let $\Delta$ be a trivial knot in the three-sphere. For every finite cyclic group $G$ of odd order, we construct a $G$-equivariant Khovanov homology with coefficients in the field $\mathbb{F}_2$. This homology is an invariant of links up to isotopy in $(S^3, \Delta)$. Another interpretation is given using the categorification of the Kauffman bracket skein module of the solid torus. Our techniques apply in the case of graphs as well to define an equivariant version of the graph homology which categorifies the chromatic polynomial.

Key words. Khovanov homology, group action, equivariant Jones polynomial, skein modules.

MSC. 57M25.

1- Introduction

In the late nineties, M. Khovanov introduced an invariant of isotopy classes of oriented links in the three-sphere, now widely known as the Khovanov homology. This invariant takes the form of bigraded homology groups whose polynomial Euler characteristic is the Jones polynomial.

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Namely, if \( L \) is an oriented link and \( H^{\ast \ast}(L) \) is its Khovanov homology with integral coefficients, then the Jones polynomial of \( L \) is given by the following formula

\[
V(L)(q) = \sum_{i,j} (-1)^i q^j \text{rank} H^{i,j}(L),
\]

where \( V(L)(q) \) is the augmented version of the Jones polynomial, equal to \( (q + q^{-1}) \) times the original Jones invariant \([6]\). The original definition of Khovanov homology is complicated and overloaded with algebraic details. Viro \([13]\) suggests an elementary combinatorial approach to define the Khovanov homology. This approach has proved to be useful in several works. For instance, it was used in \([1]\) to construct an homology theory for framed links in \( I\)-bundles over surfaces. This theory categorifies the Kauffman bracket skein module \([10]\).

Quantum invariants of links have proved to be a powerful tool in the study of the symmetry of links. For instance, the Jones and the HOMFLY polynomials satisfy certain necessary conditions which helped determine the symmetries of some links \([9, 12, 11]\). Our main goal in this paper is to investigate the behavior of the Khovanov homology of links with \( \mathbb{Z}/p\mathbb{Z} \)-symmetry. Let \( \Delta \) be a trivial knot in \( S^3 \) and let \( L \) be a link in \( S^3 \) such that \( L \) does not intersect \( \Delta \). Let \( \tilde{L} \) be the covering link of \( L \) in the \( p \)-fold cyclic cover branched over \( \Delta \). Obviously, the group \( G = \mathbb{Z}/p\mathbb{Z} \) acts on \( (S^3, \tilde{L}) \). Let \( D \) be a diagram of the link \( L \) and let \( \tilde{D} \) be a symmetric diagram of \( \tilde{L} \). We prove that the action of \( G \) extends naturally to the Khovanov chain complex of \( \tilde{D} \) with coefficients in \( \mathbb{F}_2 \). The homology of the quotient chain complex is called here the \( G \)-equivariant Khovanov homology of \( D \), we shall denote it by \( H^\ast_{\ast} G(D) \). Throughout this paper, two links in \((S^3, \Delta)\) are isotopic if they are related by an isotopy of \( S^3 \) keeping the knot \( \Delta \) fixed. If \( E \) is a vector space on which the group \( G \) acts, then we set \( E^G \) to be the subset of fixed points under this action.

**Theorem 1.** If the order of \( G \) is odd, then the \( G \)-equivariant Khovanov homology \( H^\ast_{\ast} G \) is an invariant of ambient isotopy of oriented links in \((S^3, \Delta)\). In addition, \( H^\ast_{\ast} G(L) \) is isomorphic to the subspace of fixed points \( H^\ast_{\ast} G(\tilde{L}) \).

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The polynomial Euler characteristic of $H_G^{*,*}$ is an invariant of ambient isotopy of links in $S^3$ which we call here the $G-$equivariant Jones polynomial:

$$V_G(L)(q) = \sum_{i,j} (-1)^i q^j \dim H_G^{i,j}(L).$$

**Corollary 1.** If $V_G(L) \neq V(\tilde{L})$, then the action of $G$ in homology is not trivial.

In [1], Asaeda, Przytycki and Sikora constructed an homology theory which categorifies the Kauffman bracket skein modules of $I-$bundles over surfaces. In the case of the solid torus, this homology is an invariant of framed links which associates to each framed link $L$ homology groups $H^{*,*,*}(L)$, where scripts are integers. Let $L$ be a framed link in the solid torus $S^1 \times I \times I$ and let $D$ be a diagram of $L$ in the annulus. Let $\tilde{L}$ be the pre-image of $L$ in the $p-$fold cyclic cover of the solid torus. Let $\tilde{D}$ be a symmetric diagram of $\tilde{L}$ in the annulus. We prove that the finite cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ acts on the chain complex $(C^{*,*,*}(\tilde{D}), d)$, where coefficients are taken in $\mathbb{F}_2$. Thus we construct a $G-$equivariant Khovanov homology $H_G^{*,*,*}(D)$ and we prove that this homology defines an invariant of framed links.

**Theorem 2.** If the order of $G$ is odd, then the $G-$equivariant Khovanov homology $H_G^{*,*,*}$ is an invariant of framed links in the solid torus.

**Examples.** Computing the Khovanov homology of a link is not an easy task in general. The computation of the equivariant Khovanov homology is even more difficult. We give here some easy examples with $G = \mathbb{Z}/3\mathbb{Z}$.

If $L$ is a trivial knot such $\Delta \cup L$ is a trivial link, then the only non trivial homology spaces are $H_G^{0,3}(L) = H_G^{0,-3}(L) = \mathbb{F}_2$ and $H_G^{0,1}(L) = H_G^{0,-1}(L) = \mathbb{F}_2$. Since $H_G^{0,3}(\tilde{L}) = H_G^{0,-3}(\tilde{L}) = \mathbb{F}_2$, $H_G^{0,1}(\tilde{L}) = H_G^{0,-1}(\tilde{L}) = (\mathbb{F}_2)^3$, then the equivariant homology of $L$ is different from the Khovanov homology of $\tilde{L}$. The equivariant Jones polynomial of $L$ is different from the Jones polynomial of $\tilde{L}$, as we have $V_G(L) = q^3 + q + q^{-1} + q^{-3} \neq V(\tilde{L}) = q^3 + 3q + 3q^{-1} + q^{-3}$. In conclusion, the Khovanov homology of $L$, the Khovanov homology of $\tilde{L}$ and the $G-$equivariant Khovanov homology of $L$ are different.

Now, we consider the knot $L$ depicted by the picture below, where the linking number of $\Delta$ and $L$ is equal to 2. The covering link $\tilde{L}$ is the trefoil knot. Computations show that the $G-$equivariant
Khovanov homology of \( L \) is equal to the Khovanov homology of \( \tilde{L} \), the non-trivial homology spaces are listed below: \( H_G^{0,1}(L) = H_G^{0,3}(L) = H_G^{2,5}(L) = H_G^{2,7}(L) = H_G^{3,7}(L) = H_G^{3,9}(L) = \mathbb{F}_2 \). The equivariant Jones polynomial of \( L \) is \( V_G(L) = -q^9 + q^5 + q^3 + q \) which is equal to the Jones polynomial of \( \tilde{L} \).

Here is an outline of our paper. In Section 2, we review the construction of Khovanov homology following [13]. Section 3 discusses the Khovanov homology of symmetric links. In Section 4, we review some basic properties of the transfer map in homology needed in the sequel. The proof of Theorem 1 is given in Section 5. Section 6 and 7 discusses extension of our construction to framed links in the solid torus and to graph homology.

2- Khovanov homology

This section is to review the definition of the Khovanov homology of links following the elementary combinatorial construction introduced by Viro [13]. Note that coefficients will be always in \( \mathbb{F}_2 \) and will usually be dropped from the notation except when desired for stress.

Let \( D \) be a link diagram with \( n \) crossings. A Kauffman state of \( D \) is an assignment of \(+1\) marker or \(-1\) marker to each crossing of \( D \). In a Kauffman state the crossings of \( D \) are smoothed according to the following convention

\[
\begin{array}{c}
\begin{array}{c}
\times \quad \text{+1 marker} \\
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\quad \text{+1 marker} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\times \quad \text{-1 marker}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Figure 1

to obtain a collection of circles \( D_s \). Let \( |s| \) be the number of circles in \( D_s \) and let

\[
\sigma(s) = \#\{+1 \text{ markers}\} - \#\{-1 \text{ markers}\}.
\]
The augmented version of Kauffman bracket of $D$ is the Laurent polynomial in the indeterminate $A$ given by the following formula:

$$\langle D \rangle (A) = \sum_{\text{states } s \text{ of } D} (-A)^{\sigma(s)}(-A^2 - A^{-2})^{\mid s \mid}.$$ 

An *enhanced Kauffman state* $S$ of $D$ is a Kauffman state $s$ with an assignment of a $+$ or $-$ sign to each circle in $D_s$. We set $\tau(S)$ to be the algebraic sum of signs associated to the circles of $D_s$.

If $D$ is given an orientation, then let $w(D)$ stands for the writhe of $D$. Now, we define:

$$i(S) = \frac{w(D) - \sigma(s)}{2} \quad \text{and} \quad j(S) = \frac{3w(D) - \sigma(s) + 2\tau(S)}{2}.$$ 

One may check easily that both $i(S)$ and $j(S)$ are integers. Let $i$ and $j$ be two integers, we define $S_{i,j}^D$ to be the set of states of $D$ with $i(S) = i$ and $j(S) = j$. The Khovanov chain space $C_{i,j}^D$ is defined to be the vector space over $\mathbb{F}_2$ having $S_{i,j}^D$ as a basis.

It remains to define the differential. Assume that $v$ is a crossing of $D$, we define the partial differential $d_v$ as follows

$$d_v^{i,j} : C_{i,j}^D \longrightarrow C_{i+1,j}^D$$

\[ S \mapsto \sum_{\text{All states } S'} (S : S')_v S' \]

where $(S : S')_v$ is

- $1$ if $S$ and $S'$ differ only at the crossing $v$, where $S$ has a $+1$ marker, $S'$ has a $-1$ marker, all the common circles in $D_S$ and $D_{S'}$ have the same signs and in a neighborhood of $v$, $S$ and $S'$ are as in figure 2,

- $(S : S')$ is zero otherwise.
The differential $d$ is defined by:

$$d^{i,j} : C^{i,j}(D) \rightarrow C^{i+1,j}(D)$$

$$S \mapsto \sum_v d^{i,j}_v(S)$$

The homology $H^{*,*}(D)$ of the chain complex $(C^{*,*}(D), d^{*,*})$ is called the Khovanov homology of $D$. This homology is conserved under Reidemeister moves. Hence, it is an invariant of ambient isotopy of links. If $L$ is an oriented link in $S^3$, then we denote its Khovanov homology by $H^{*,*}(L)$. As we have mentioned in the introduction, the Jones polynomial of $L$ is obtained as the polynomial Euler characteristic of $H^{*,*}(L)$.

**Framed Khovanov homology.** As it is the case for the Jones polynomial. It is sometimes more convenient to work with framed links when studying Khovanov homology. Viro [13] showed that one may define a Khovanov homology which categorifies the Kauffman bracket polynomial. Let $D$ be a nonoriented link diagram. With respect to the notations of section 2, we set $p(S) = \tau(S)$ and $q(S) = \sigma(S) - 2\tau(S)$. Let $C_{p,q}(D)$ be the vector space generated by all enhanced states with $p(S) = p$ and $q(S) = q$. We have a chain complex $(C_{*,*}(D), d)$, where $d : C_{p,q}(D) \rightarrow C_{p-1,q}(D)$ is defined as in the previous paragraph. If $D$ is oriented, then we get the Khovanov homology of $C^{*,*}(D)$ by shifting the degrees in the homology of $C_{*,*}(D)$. The advantage of this framed version of Khovanov homology is that there is a short exact sequence which categorifies the Kauffman bracket skein relation [13]. Let $D$, $D_0$ and $D_\infty$ be three link diagrams.
diagrams which are identical except in a small disk where they are like in the following picture

![Figure 3.](image)

The following short sequence is exact:

$$0 \longrightarrow C_{p,q}(\big) \stackrel{\alpha}{\longrightarrow} C_{p,q-1}(\big) \stackrel{\beta}{\longrightarrow} C_{p,q-2}(\big) \longrightarrow 0$$

where $\alpha$ is the chain map defined by: $\big) \mapsto \big$ and $\beta$ is defined by the following correspondence

$$\big \mapsto 0$$

$$\big \mapsto \big.$$

### 3- Symmetric links and equivariant Khovanov homology

This section is concerned with the natural question of whether the Khovanov homology reflects the symmetry of links. In other words, does the invariance of a link by some finite group action on the three-sphere induce some group action on the Khovanov homology of the link?

A link $L$ in $S^3$ is said to be $p$–periodic if and only if there exists an orientation preserving diffeomorphism $\varphi$ of $S^3$ such that $\varphi$ is of order $p$, the set of fixed points of $\varphi$ is a knot disjoint from $L$ and $\varphi(L) = L$. By the positive solution of the smit conjecture, we may assume without loss of generality that $\varphi$ is a rotation of $2\pi/p$–angle around a trivial knot. Consequently, a $p$–periodic knot admits a planar diagram which is invariant by a planar rotation of the same angle.

Let $\Delta$ be a trivial knot and let $L$ be a link in the three-sphere such that $L \cap \Delta = \emptyset$. Let $\tilde{L}$ be the covering link of $L$ in the $p$–fold cyclic cover branched over $\Delta$. Let $\tilde{D}$ be a diagram of $\tilde{L}$.
which is invariant by a planar rotation. Such a diagram exists since the link $\tilde{L}$ is $p$–periodic. Let $D$ be the quotient diagram of $\tilde{D}$ under the action of the group $G = \mathbb{Z}/p\mathbb{Z} = \langle \varphi \rangle$. The rest of this section is devoted to study the Khovanov chain complex $(C^{*,*}(\tilde{D}), d^{*,*})$.

One can see easily that the action of the rotation on the diagram $\tilde{D}$ extends naturally to an action of the cyclic group $G$ on the set of enhanced Kauffman states of $\tilde{D}$. In addition, for all enhanced state $S$ we have:

$$i(\varphi^k(S)) = i(S) \text{ and } j(\varphi^k(S)) = j(S) \text{ for all } 1 \leq k \leq p.$$ 

Consequently, the group $G$ acts on the set $S_{\tilde{D}}^{i,j}$. Since $S_{\tilde{D}}^{i,j}$ is a basis for $C^{*,*}(\tilde{D})$, then this action extends naturally to an action of $G$ on $C^{*,*}(\tilde{D})$. It remains now to check if the action of $G$ commutes with the differential.

**Lemma 3.1.** We have: $\varphi \circ d = d \circ \varphi$.

**Proof:** Let $S$ be an enhanced state, one can easily see that for every crossing in $\tilde{D}$ we have $(S : S')_v = (\varphi(S), \varphi(S'))_{\varphi(v)}$. Thus:

$$\varphi(d_v(S)) = \varphi(\sum_{\text{All states } S'} (S : S')_vS')$$

$$= \sum_{\text{All states } S'} (S : S')_v\varphi(S')$$

$$= \sum_{\text{All states } S'} (\varphi(S) : \varphi(S'))_{\varphi(v)}\varphi(S')$$

$$= \sum_{\text{All states } T} (\varphi(S) : T)_{\varphi(v)}T$$

$$= d_{\varphi(v)} \circ \varphi(S).$$

Finally, we get $\sum_{\text{crossings } v} \varphi(d_v^{i,j}(S)) = \sum_{\text{crossings } v} d_v^{i,j}(\varphi(S))$ which means that $\varphi \circ d = d \circ \varphi$. This ends the proof of the lemma. 

Let $(\overline{C^{*,*}(D)}, \overline{d})$ be the quotient chain complex of $(C^{*,*}(\tilde{D}), d)$ by the action of $G$. The homology of the quotient chain complex $(\overline{C^{*,*}(D)}, \overline{d})$ is called the $G$–equivariant homology of $D$. We denote this homology by $H^{G,*}_{*,*}(D)$.

**Remark 3.1.** Similarly, a framed version of equivariant Khovanov homology can be defined for non oriented diagrams. We shall denote it by here $H^{G,*}_{*,*}$. 

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Remark 3.2. If we consider Khovanov homology with coefficients in $\mathbb{Z}$ as in the original definition [8]. We still have an action of the group $G$ on the Khovanov chain groups but that action does not commute with the differential. This is due to the signs that appear in the definition of the differential. Actually, this is the raison for which we choose to work with coefficients in $\mathbb{F}_2$.

4- The transfer in homology

In this section, we review some properties of the transfer map. Let $G = < \varphi >$ be the finite cyclic group of order $p$ and let $(C^*, d)$ be a chain complex with coefficients in some field $F$. Assume that $G$ acts on the chain complex $(C^*, d)$ and set $(\overline{C^*}, \overline{d})$ to be the quotient chain complex. We denote by $\pi$ the canonical surjection with respect to the action of $G$. Let $t$ be the map from $(\overline{C^*}, \overline{d})$ to $(C^*, d)$ defined by $t(S) = S + \varphi(S) + ... + \varphi^{p-1}(S)$. The map $t$ induces a map $t_*$ from the homology of $(\overline{C^*}, \overline{d})$ to the homology of $(C^*, d)$. This map called the transfer has been useful in the study of homological properties of topological transformation groups. The following properties are extracted from [3].

**Theorem 4.1.** The composition $\pi_* t_*$ is the multiplication by $p$. It is an isomorphism if the field $F$ is of characteristic zero or prime to $p$.

Obviously, the action of $G$ on the chain complex $(C^*, d)$ induces an action of $G$ on the homology. We have:

**Theorem 4.2.** If the field $F$ is of characteristic zero or prime to $p$, then:

$$
\pi_* : H((C^*, d))^G \longrightarrow H((\overline{C^*}, \overline{d})) \quad \text{is an isomorphism, as is}
$$

$$
t_* : H((\overline{C^*}, \overline{d})) \longrightarrow H((C^*, d))^G.
$$

5- Proof of Theorem 1

In this section, we shall prove that if the order of $G$ is odd, then the $G$–equivariant Khovanov homology does not change under Reidemeister moves $R1$, $R2$ and $R3$. Note that as we consider isotopy in $(S^3, \Delta)$, then we should consider only Reidemeister moves which are performed in a
a three ball which does not intersect $\Delta$.

5.1- Invariance under first Reidemeister move

Let $D$ and $D'$ be two link diagrams which are related by a Reidemeister move $R1$. Assume that $D$ is the diagram in the middle of figure 4 and that $D'$ is the right twisted diagram.

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{figure4.png}
\end{array}
\]

Figure 4.

Let $\tilde{D}$ and $\tilde{D}'$ be the two covering diagrams. Obviously these two diagrams differ by $p$ Reidemeister moves of type 1 performed along an orbit of the action of $G$.

Following Viro [13], if two diagrams differ by a Reidemeister move $R1$, there is a chain map $h_v$ ($v$ is the crossing which appears in $D'$ but not in $D$) between the two complexes which induces an isomorphism in homology. This map is defined by

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{equation1.png}
\end{array}
\]

Lemma 5.1. The linear map $h : C^{*,*}(\tilde{D}) \rightarrow C^{*,*}(\tilde{D}')$ induces an isomorphism in homology. In addition $h$ is $G$–equivariant.

Proof. The induced map $h_\ast$ is an isomorphism in homology because it is the composition of isomorphisms. It is $G$–equivariant due to the two elementary facts: $h_w \circ h_w' = h_w' \circ h_w$ and $\varphi \circ h_w = h_{\varphi(w)} \circ \varphi$. □

According to Lemma 5.1, the map $h$ induces a map $\overline{h}$ from $(\overline{C^{*,*}(\tilde{D})}, \overline{d})$ to $(\overline{C^{*,*}(\tilde{D}')}, \overline{d})$. We are going to prove that this map induces an isomorphism in homology. Note that we have a commutative diagram
which induces a commutative diagram in homology

\[
\begin{array}{ccc}
C^{*,*}(\tilde{D}) & \xrightarrow{h_*} & C^{*,*}(\tilde{D}') \\
\downarrow \pi & & \downarrow \pi' \\
C^{*,*}(\tilde{D}) & \xrightarrow{\pi_*} & C^{*,*}(\tilde{D}')
\end{array}
\]

where \( t_* \) (respectively \( t'_* \)) stands for the transfer map corresponding to the action of \( G \) on \( \tilde{D} \) (respectively \( \tilde{D}' \)). Since we are working with coefficients in \( \mathbb{F}_2 \) and the order of \( G \) is odd, then both \( \pi_* t_* \) and \( \pi'_* t'_* \) are isomorphisms. In addition, the commutative diagram implies that \( h_* t_* = t'_* \pi_* \). Using the fact that \( h_* \) is an isomorphism we should be able to conclude that \( \pi_* \) is injective. A similar argument using the fact that \( \pi_* \) and \( \pi'_* \) are onto implies that \( \pi_* \) is surjective. Finally, the equivariant homologies of \( D \) and \( D' \) are isomorphic.

The invariance under the left twisted first Reidemeister move is proved in a similar way.

### 5.2- Invariance under the second Reidemeister move

We will switch to framed links for a while. Let \( D \) and \( D' \) be two link diagrams related by a single second Reidemeister move and assume that \( D' \) is the one that has more crossings, see figure 5. Let \( \tilde{D} \) and \( \tilde{D}' \) be the two covering diagrams. We shall prove that the equivariant homologies are isomorphic. Here we consider the framed version \( H^G_{*,*}(D) \) and \( H^G_{*,*}(D') \).

\[
\begin{array}{ccc}
\begin{array}{c}
\text{\text{\text{\text{\text{}}}}}
\end{array}
& \xrightarrow{\text{\text{\text{\text{\text{}}}}}} & \begin{array}{c}
\text{\text{\text{\text{\text{}}}}}
\end{array}
\end{array}
\]

Figure 5.

Following [1], we define two maps \( \overline{\pi} \) and \( \overline{\beta} \)

\[
\overline{\beta} : C_{p,q-2}(\text{\text{\text{\text{\text{}}}}}) \rightarrow C_{p,q-1}(\text{\text{\text{\text{\text{}}}}}) \\
\text{\text{\text{\text{\text{}}}}} \quad \rightarrow \quad \text{\text{\text{\text{\text{}}}}}
\]

Following [1], we define two maps \( \overline{\pi} \) and \( \overline{\beta} \)
and
\[ \tau : C_{p,q-1}(\bigotimes) \to C_{p,q}(\bigotimes) \]
\[ \bigotimes \mapsto 0 \]
\[ \bigotimes \mapsto \bigotimes \]

Now, we set: \( \gamma = \alpha d_v \beta \) which is chain map from \( C_{p,q}(\bigotimes) \to C_{p-1,q+2}(\bigotimes) \). We define two maps \( f \) and \( g \) as follows:
\[ f : C_{p,q}(\bigotimes) \to C_{p,q}(D') \]
\[ \bigotimes \mapsto \bigotimes \]

and
\[ g : C_{p,q}(\bigotimes) \to C_{p+1,q-2}(D') \]
\[ \bigotimes \mapsto \bigotimes \bigotimes \]

Let \( \rho \) be the chain map \( \rho = f + g \circ \gamma : C_{p,q}(D) \to C_{p,q}(D') \).

**Theorem 5.2.** \([1]\) The map \( \rho \) induces an isomorphism in homology.

The diagrams \( \tilde{D} \) and \( \tilde{D}' \) differ by \( p \) Reidemeister moves of type 2. To each move we associate a map \( \rho \) as explained earlier. Let us label these maps by \( \rho_v, \rho_{\varphi(v)}, \ldots, \rho_{\varphi^{p-1}(v)} \). By composing these maps we define a map \( \Phi = \rho_v \circ \rho_{\varphi(v)} \circ \ldots \circ \rho_{\varphi^{p-1}(v)} : C_{p,q}(\tilde{D}) \to C_{p,q}(\tilde{D}') \). It is easy to see that we have \( \rho_w \circ \rho_{w'} = \rho_{w'} \circ \rho_w \) and \( \varphi \circ \rho_w = \rho_{\varphi(w)} \circ \varphi \). Consequently, \( \Phi \) is \( G \)-equivariant. Thus, it induces a map \( \overline{\Phi} \) between the quotient chain complexes. Arguments similar to those used in the case of the invariance under first Reidemeister move should enable us to conclude that \( \overline{\Phi} \) induces an isomorphism between the equivariant Khovanov homologies of \( D \) and \( D' \).

**5.3- Invariance under the third Reidemeister move**

In this paragraph, we shall prove the invariance of our equivariant homology under the third Reidemeister move. Once again, we are going to work with the framed version. Let \( D \) and \( D' \) be two diagrams which differ by a single third Reidemeister move as in the following picture:
\[ \bigotimes \mapsto \bigotimes \]
Our proof is based both on the construction in [1] and the techniques we have developed in the previous paragraph. It could be helpful if the reader has a copy of [1] with him. Let us consider the following diagrams $D_+ = \bigcirc \bigcirc$, $D_- = \bigcirc \bigcirc$ and $D_{++} = \bigcirc \bigcirc$. We define diagrams $D'_+$, $D'_-$ and $D'_{++}$ in the same way. Note that the signs in the subscripts refer to the marker associated to the considered crossing.

The diagrams $D_{++}$ and $D_+$ differ by a single Reidemeister move of type 2. As we have explained in the previous paragraph there exists a map $\Phi : C_*(\tilde{D}_{++}) \to C_*(\tilde{D}_+)$ which is $G$–equivariant and induces an isomorphism in homology. Now, let $C'_*(\tilde{D}_+) = \Phi(\tilde{C}_*(\tilde{D}_{++}))$ and consider the map $\tilde{i} : C'_*(\tilde{D}_+^2) \to C_*(\tilde{D}_+)$. We set $\tilde{\beta} : C_*(\tilde{D}) \to C_*(\tilde{D}_+^2)$ to be the map define by:

\[
\begin{array}{c}
\bigcirc \ldots \bigcirc \\
\end{array}
\to
\begin{array}{c}
\bigcirc \ldots \bigcirc \\
\end{array}
\]

and zero otherwise. Let $C'_*(\tilde{D}) = \tilde{\beta}^{-1}(C'_*(\tilde{D}_+))$. We have the following [1] Lemma 5.3.

**Lemma 5.3.** The maps $\tilde{i} : C'_*(\tilde{D}_+^2) \to C_*(\tilde{D}_+)$ (resp. $\tilde{i}' : C'_*(\tilde{D}_+) \to C_*(\tilde{D}_+^2)$ ) and $\tilde{j} : C'_*(\tilde{D}) \to C_*(\tilde{D}_+^2)$ (resp. $\tilde{j}' : C'_*(\tilde{D}_+) \to C_*(\tilde{D}_+^2)$) induce isomorphisms in homology.

**Proof.** The induced map $\tilde{i}_*$ is an isomorphism in homology because it is a composition of isomorphisms, see [1] Proposition 11.10]. Same argument applies for $\tilde{j}_*$. \hfill \square

Similarly to the case of the second Reidemeister move discussed earlier, by composition of the maps of type $\rho_{III}$ defined in [1] we should be able to construct a map $\tilde{\Psi} : C'_*(\tilde{D}) \to C'_*(\tilde{D}_+^2)$ which is $G$–equivariant and induces an isomorphism in homology. This map induces an isomorphism in homology $\overline{\Psi}$ between the homology of $C'_*(\tilde{D})$ and the homology of $C'_*(\tilde{D}_+^2)$. Consequently, $\overline{j}' \circ \overline{\Psi} \circ \overline{j}^{-1} : H^*_G(D) \to H^*_G(D')$ is an isomorphism. This completes the proof of the invariance under the third Reidemeister move.

Finally we use Theorem 4.2 to prove that $H^*_G(L)$ is isomorphic to $H^*_G(\tilde{L})$. This completes the proof of Theorem 1.
6- Equivariant Khovanov homology for framed links in the solid torus

In this section, we show how our equivariant construction can be described in the context of the categorification of the Kauffman bracket skein module of the solid torus [1]. Everything here is done similarly to what we have discussed in the previous sections. Consequently, we are going to omit the details and describe things briefly. We first review the notion of skein modules.

Let $M$ be an oriented compact three-manifold. A framed link in $M$ is an embedding of a finite family of annuli into the interior of $M$. Let $L$ be the set of all isotopy classes of framed links in $M$ including the empty link. Let $\mathbb{Z}[A^\pm]L$ be the free module generated by $L$. The Kauffman bracket skein module of $M$, denoted here by $\mathcal{K}(M)$, is defined as the quotient of $\mathbb{Z}[A^\pm]L$ by the smallest submodule generated by all elements of the following form

1) $L \cup \varnothing + (A^2 + A^{-2})L$, where $L$ is any framed link in $M$, and $L \cup \varnothing$ is the disjoint union of $L$ with a trivial component,

2) $L_+ - AL_0 - A^{-1}L_\infty$, where $L_+$, $L_0$ and $L_\infty$ are three links which are identical except in a small ball where they look like in figure 3.

The existence and the uniqueness of the Jones polynomial is equivalent to the fact that $\mathcal{K}(S^3)$ is isomorphic to $\mathbb{Z}[A^\pm]$ with the empty link as a basis. If $F$ is an oriented surface, then the skein module of $F \times I$ admits an algebra structure [4]. In particular, the skein algebra of the solid torus $S^1 \times I \times I$ is isomorphic to the polynomial algebra $\mathbb{Z}[A^\pm][z]$, where $z$ is represented by a nontrivial curve in the annulus as in the following picture

Let $L$ be a link in the solid torus. Let $D$ be diagram of $L$ in the annulus and let $(C^\ast,\ast,\ast(D), d)$ be the chain complex of $D$ with coefficients in $\mathbb{F}_2$. The skein module of the solid torus has a basis made up of links of the form $z^n$ ($n$ parallel copies of $z$). Thus, we shall use $n$ as the third script instead of $z^n$ as in original definition [1]. The homology of $(C^\ast,\ast,\ast(D), \mathbb{F}_2)$ defines an invariant of framed links in the solid torus.

Let $\tilde{L}$ be the covering link of $L$ in the $p$-fold cyclic cover of the solid torus. Let $\tilde{D}$ be a
symmetric diagram of $\tilde{L}$. Arguments similar to those used in section 3 show that the action of the rotation on the diagram $\tilde{D}$ extends to an action of the finite cyclic group $G$ on the chain complex $(C^{*,*,*}(\tilde{D}),d)$. The homology $H^{*,*,*}_G$ of the quotient complex $(C^{*,*,*}(\tilde{D}),\overline{d})$ is called the $G$–equivariant Khovanov homology of $D$. The proofs in the previous section extend straightforward to conclude that $H^{*,*,*}_G$ is an invariant of framed links in the solid torus.

### 7- Equivariant graph homology

In this section, we explain how one may extend our link equivariant homology to graphs. Let us first fix notations and review some definitions. Throughout the rest of this paper, a graph is a 1-dimensional finite CW-complex. Let $\mathcal{G}$ be a graph with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$. The chromatic polynomial of $\mathcal{G}$ is a one variable polynomial $P(\mathcal{G}) \in \mathbb{Z}[\lambda]$ which when evaluated at an integer $m$ gives the number of colorings of the vertices of $\mathcal{G}$ by a palette of $m$–colors satisfying the property that vertices which are connected by a edge have different colors. Now, we shall briefly review the definition of graph homology following [5]. We consider homology with coefficients in $\mathbb{F}_2$. Take a set of colors $\{1, x\}$ and define a product $\star$ as in $\mathbb{Z}[x]/x^2$. For each $s \subseteq E(\mathcal{G})$, we set $[G : s]$ to be the graph whose vertex set is $V(\mathcal{G})$ and whose edge set is $s$. An enhanced state of $\mathcal{G}$ is $S = (s, c)$ where $s \subseteq E(\mathcal{G})$ and $c$ is an assignment of 1 or $x$ to each connected component of the spanning subgraph $[G : s]$. If $S$ is an enhanced state then we set $i(S)$ to be the number of edges in $S$ and we set $j(S)$ to be the number of $x$'s in $c$. Now, we define $C^{i,j}(\mathcal{G})$ to be the vector space generated by all enhanced states of $\mathcal{G}$ with $i(S) = i$ and $j(S) = j$. The differential is defined by

$$d : C^{i,j}(\mathcal{G}) \rightarrow C^{i+1,j}(\mathcal{G})$$

$$S \mapsto \sum_{e \in E(\mathcal{G} - s)} S_e,$$

where $S_e$ is any enhanced state obtained from $S$ by adding an edge not in $s$ and adjusting the sign according to the product $\star$, see [5 Page 1375] for more details. The homology of $C^{*,*}(\mathcal{G},d)$ is an invariant of $\mathcal{G}$. The chromatic polynomial is the Euler characteristic of $H^{*,*}(\mathcal{G})$ evaluated at $q = \lambda - 1$. 
Let \( \tilde{G} \) be a graph on which the finite cyclic group \( G \) acts. The action of \( G \) on \( \tilde{G} \) extends to an action on the set of enhanced states. Thus, the group \( G \) acts on \( C^{i,j}(\tilde{G}) \). The action of \( G \) commutes with the differential for the same reason as in the proof of Lemma 3.1. We obtain a quotient complex, its homology is called the \( G \)-equivariant homology of \( \tilde{G} \).

References

[1] M. M. Asaeda, J. H. Przytycki, A. S. Sikora. *Categorification of the Kauffman bracket skein module of I–bundles over surfaces*. Algebraic and Geometric Topology 4, (2004), 1177-1210.

[2] H. Bass and J. W. Morgan. *The Smith conjecture*. Pure and App. Math. 112, New York Academic Press (1994).

[3] G. E Bredon. *Introduction to compact transformation groups*. Academic Press (1972)

[4] D. Bullock. *A finite set of generators for the Kauffman bracket skein algebra*. Math. Z. 231, (1999), pp. 91-101.

[5] L. Hemle-Guizon and Y. Rong. *A categorification for the chromatic polynomial*. Algebraic and Geometric Topology 5, (2005), 1365-1388.

[6] V. F. R. Jones. *A polynomial invariant for knots via von Neumann algebras*. Bull. Amer. Math. Soc. (N.S.) 12 (1985), no. 1, 103–111.

[7] L. H. Kauffman. *An invariant of regular isotopy*. Trans. Amer. Math. Soc. 318 (1990), no. 2, 417–471.

[8] M. Khovanov. *Categorification of the Jones polynomial*. Duke Math. J. 87 (1997), 409-480.

[9] K. Murasugi. *The Jones polynomials of periodic links*. Pacific J. Math. 131 (1988) pp. 319-329.
[10] J. H. Przytycki. *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci.: Math., 39, 1-2 (1991), 91-100.

[11] J. H. Przytycki. *On Murasugi’s and Traczyk’s criteria for periodic links*. Math. Ann., 283, (1989), pp. 465-478.

[12] P. Traczyk. *10_{101} has no period 7: A criterion for periodicity of links*. Proc. Amer. Math. Soc. 108, (1990), pp. 845-846.

[13] O. Viro. *Khovanov homology, its definitions and ramifications*. Fund. Math. 184 (2004), 317-342
