On Exponential Synchronization Rates for High-Dimensional Kuramoto Models With Identical Oscillators and Digraphs

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Abstract—For a high-dimensional Kuramoto model with identical oscillators under a general digraph that has a directed spanning tree, although the exponential synchronization has been proved under some initial state constraints, exponential synchronization rates have not been described exactly until now. In this article, the supremum of exponential synchronization rates is precisely determined as the smallest real part of the nonzero Laplacian eigenvalues of the digraph. Our obtained result extends the existing results from the special case of strongly connected balanced digraphs to the condition of general digraphs owning directed spanning trees, which is the weakest condition for synchronization from the aspect of network structure. Moreover, our adopted method is completely different from and much more elementary than the previous differential geometry method.

Index Terms—Directed graph, exponential synchronization rate, high-dimensional Kuramoto model.

I. INTRODUCTION

The well-known Kuramoto model proposed by Kuramoto [1] is one of the most successful mathematical models to describe collective behaviors of complex dynamical networks. Kuramoto model and its various generalized forms have many applications in engineering, neuroscience, physics, and so on [2]. For the Kuramoto model, exponential synchronization is a typical collective behavior and an interesting theoretical issue [3]–[6]. The dynamics of the general Kuramoto model is described by

$$\dot{\theta}_i = \omega_i + k \sum_{j=1}^{m} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, m$$

(1)

where the $m \times m$ matrix $(a_{ij})$ is the adjacency matrix of the interconnecting graph, $\theta_i$ is the $i$th oscillator’s phase angle and $\omega_i$ is the natural frequency. The $n$-dimensional vector-valued Kuramoto model is described as follows:

$$\dot{r}_i = \Omega_i r_i + k \sum_{j=1}^{m} a_{ij} \left( r_j - \frac{r_i^T r_j}{r_i^T r_i} r_i \right), \quad i = 1, 2, \ldots, m$$

(2)

where $r_i \in \mathbb{R}^n$ is the $i$th oscillator’s state, and $\Omega_i$ is an $n \times n$ skew-symmetric matrix [7]. When $n = 2$ and $r_i = [\cos \theta_i, \sin \theta_i]^T$, by the $n$-dimensional Kuramoto model (2), it is easy to deduce the original Kuramoto model (1). If (2) has the identical oscillators, i.e., $\Omega_i = \Omega$ for $i = 1, 2, \ldots, m$, and the initial states of all oscillators are limited on the unit sphere $S^{n-1}$, the dynamical network (2) with $k = 1$ is reduced to

$$\dot{r}_i = \Omega r_i + \sum_{j=1}^{m} a_{ij} (r_j - (r_i^T r_j) r_i), \quad i = 1, 2, \ldots, m$$

(3)

When $\Omega = 0$, (3) is just the swarm model on spheres, and can be used to solve the max-cut problem [8]. Due to Lohe’s pioneering literatures, such as [9] and [10], the high-dimensional Kuramoto model is also called Lohe model, which has potential applications to quantum systems [11]. More general Kuramoto models defined on some matrix manifolds can be seen in [12], [13], and [14].

For the case of complete graphs, many theoretical results on the synchronization of the high-dimensional Kuramoto model have been achieved, such as [8], [15], and [16]. It has been proved that a Cartesian product of $m$ open hemispheres is a set of the synchronization region of (3), please refer to [7], [17], and [18]. If the states of particles are not limited on an open half-sphere, some almost global synchronization results can also be obtained for the case of undirected connected graphs [19], [20]. For general directed graphs, the synchronization of the high-dimensional Kuramoto model is proved in [17] by using a differential geometry method. However, in our paper [18], a simpler approach based on LaSalle invariance principle is adopted to demonstrate the synchronization under the general directed graph condition. Just like the original Kuramoto model, the high-dimensional Kuramoto model also has the dynamical property of exponential synchronization (see [15], [17], [21], and [22]). But so far, the exact description of exponential synchronization rates is only obtained for a very special kind of digraphs, i.e., strongly connected balanced digraphs [17]. In our recent paper [22], the exponential synchronization is achieved for a general digraph admitting a spanning tree, but only a rough exponential synchronization rate is obtained.

In this article, for the high-dimensional Kuramoto model under a general digraph containing a spanning tree, it is proved that the supremum of exponential synchronization rates is just the smallest real part of nonzero eigenvalues of Laplacian matrix $L$. The required graph condition is the weakest and the adopted methods are simple.

The rest of this article is organized as follows. Section II gives some preliminaries and the problem statement. Section III includes our main results. Section IV gives an example to illustrate the obtained main result. Finally, Section V concludes this article.
II. PRELIMINARIES AND PROBLEM STATEMENT

For the high-dimensional Kuramoto model (3), the exponential synchronization is defined as follows.

**Definition 1:** It is said that the exponential synchronization for (3) is achieved if there exist a \( \mu > 0 \) and a constant \( c(r_0) > 0 \) such that

\[
||r_i(t) - r_j(t)|| \leq c(r_0)e^{-\mu t} \quad \forall i, j, \quad i = 1, 2, \ldots, m
\]

and \( \mu \) is called an exponential synchronization rate of the system.

The previous definition is similar to the concept of exponential decay rate for ordinary differential equations. We first give two basic results on exponential decay rates of ordinary differential equations.

**Lemma 1 ([23, Th. 7.1]):** Denote the eigenvalues of \( \lambda \in \mathbb{R}^{m \times m} \) by \( \alpha_j, 1 \leq j \leq m \). The system \( \dot{x} = Ax \) is globally asymptotically stable if and only if \( \operatorname{Re}(\alpha_j) < 0 \) for all \( j \). Moreover, in this case, there is a constant \( C \) for every \( \alpha < \min\{\operatorname{Re}(\alpha_j)\}_{j=1}^{m} \) such that \( ||e^{At}|| \leq Ce^{-\alpha t} \).

**Lemma 2 ([24, Corollary 4.3]):** Let \( x(0) = x_0 \) be an equilibrium point of the nonlinear system \( \dot{x} = f(x) \), where \( f(x) \) is continuously differentiable in some neighborhood of \( x = 0 \). If \( A = \partial f / \partial x(0) \). Then, \( x(0) \) is an exponentially stable equilibrium point for the nonlinear system if and only if \( A \) is Hurwitz.

Although Lemma 2 shows the equivalence between the exponential stability of a nonlinear system and that of its linearized system, the relationship between the two exponential decay rates is not given yet. Actually, by the stable manifold theorem (see [23, Th. 7.4]), the nonlinear system can also get the same exponential decay rates as Lemma 1:

**Lemma 3:** Consider the nonlinear system

\[
\dot{x}(t) = Ax(t) + g(x(t)), \quad x(0) = x_0
\]

where \( A \in \mathbb{R}^{m \times m} \) is Hurwitz, \( g(x) \) is a smooth function satisfying \( ||g(x)|| \leq k_1 ||x||^2 \) for a constant \( k_1 > 0 \) and any \( x \in \mathbb{R}^m \). Then, for any \( \alpha < \min\{\operatorname{Re}(\alpha_j)\}_{j=1}^{m} \) and \( x(0) \), there exists a constant \( C(\alpha, x(0)) \) such that \( ||x(t)|| \leq C(\alpha, x(0))e^{-\alpha t} \).

Different from Definition 1, in [17], the exponential stability of the set

\[
\Delta = \{ (r_1, r_2, \ldots, r_m) | r_1 = r_2 = \cdots = r_m \in S^{n-2} \}
\]

is used to describe the exponential synchronization, where \( \Delta \) is called the synchronization manifold.

In order to conveniently introduce the problem of this article from the results of [17], we review some basic differential geometric notions and some contributions of [17]. Let \( \phi_t \) be a flow on a smooth Riemannian manifold \( M \). Let \( \mathcal{N} \) be a compact submanifold of \( M \), which is invariant under \( \phi_t \). Denote by \( T_xM \) the tangent bundle of \( M \) over \( \mathcal{N} \), assume that \( T_xM = T_x\mathcal{N} \oplus N_x \), where \( T_x\mathcal{N} \) is the tangent bundle of \( \mathcal{N} \), and \( N_x \) is a vector bundle over \( \mathcal{N} \) with fibers \( N_x \subset T_x\mathcal{N} \). For the vector bundle \( N \), define two generalized Lyapunov-type numbers as follows:

\[
\nu(x) = \inf \left\{ a : \frac{||u||}{D\phi_{-t}w} \left/ a^t \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad w \in N_x \right. \right\}
\]

\[
\sigma(x) = \inf \left\{ b : \frac{||u||^b}{||v||} \left/ \frac{D\phi_{-t}w}{D\phi_{-t}v} \right. \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad v \in T_x\mathcal{N}, \quad w \in N_x \right. \right\}
\]

where \( D\phi_t(x) \) is the differential of \( \phi_t \), \( \Pi_x \) is the orthogonal projection of \( T_xM \) onto \( N_x \), and \( || \cdot || \) denotes the norm on the tangent spaces of \( M \) induced by the Riemannian metric.

In the following proposition, one considers the special case of \( M = (S^{n-1})^m, \mathcal{N} = \Delta \) and \( N \) being the normal bundle of \( \Delta \) in \( (S^{n-1})^m \).

**Proposition 1 ([17, Th. 1]):** Consider the high-dimensional Kuramoto model (3). If the interconnecting graph is strongly connected and balanced, then for all \( x \in (S^{n-1})^m \), we have

\[
\nu(x) \leq e^{-\operatorname{Re}(\lambda_2)} \quad \text{and} \quad \sigma(x) = 0
\]

where \( \lambda_2 \) is a non-zero eigenvalue with the smallest real part of Laplacian \( L \) of the interconnecting graph.

**Proposition 2 ([17, Corollary 1]):** Consider the high-dimensional Kuramoto model (3). If the interconnecting graph is strongly connected and balanced, then the synchronization manifold \( \Delta \) is locally asymptotically stable and the rate of convergence is exponential.

The results of [17] are significant since the exponential synchronization problem of the high-dimensional Kuramoto model for a class of digraphs is solved for the first time and the exponential convergence rate to the synchronization manifold is determined as the smallest real part of nonzero Laplacian eigenvalues of interconnecting graphs. However, there are still three important issues worthy to be studied further.

1) The interconnecting graphs are assumed to be strongly connected and balanced, which is much stronger than the usually used graph condition (i.e., the existence of a directed spanning tree) in multi-agent systems and complex dynamical networks. How to get exponential synchronization rates under the weakest graph condition is an interesting problem.

2) Comparing (7), (8), and (9) with Definition 1, one can see that the concept of exponential synchronization rate described by Definition 1 is much simpler than that in Proposition 1. So, another question is how to compute exponential synchronization rates directly using the synchronization errors instead of the generalized Lyapunov-type numbers \( \nu(x) \) and \( \sigma(x) \).

3) The last problem is whether one can use some elementary tools, such as linear algebra and ordinary differential equation instead of differential geometry, to obtain the exact description of exponential synchronization rates.

The abovementioned problems are just the research topics of this paper. Actually, the exponential synchronization problem of the high-dimensional Kuramoto model has been solved for digraphs admitting directed spanning trees in our early paper [22]. However, the exact exponential synchronization rate has not been achieved yet. We will investigate the exact exponential synchronization rate in the framework of matrix Riccati differential equation with respect to state error functions, which is proposed in our paper [22].
for all \( i, j = 1, 2, \ldots, m \). Let \( E(t) = (e_{ij}(t)) \in \mathbb{R}^{m \times m} \). Then, the dynamics of \( E(t) \) is described by matrix Riccati differential equation

\[
\dot{E} = -LE - EL^T - \alpha(E)T - \Lambda(E)E + \Lambda(E)E + \Lambda(E)E
\]  

(10)

where \( L \) is the Laplacian matrix of the digraph

\[
\alpha(E) = (\alpha_1(E), \alpha_2(E), \ldots, \alpha_m(E))^T \in \mathbb{R}^m
\]  

(11)

with each \( \alpha_i(E) = \sum_{j=1}^{m} a_{ij}e_{ij} \), and \( \Lambda(E) = \text{diag}(\alpha(E)) \) is the diagonal matrix with the diagonal elements composed of \( \alpha_1(E), \alpha_2(E), \ldots, \alpha_m(E) \).

For the linear space \( \mathbb{R}^{m \times m} \), denote by \( S_0^m \) the subspace composed of all the symmetric real matrices with all the diagonal entries being 0. Then, the synchronization error equation (10) is the dynamics restricted on \( S_0^m \). Let \( S_m \) and \( K_m \) be the \( m \)-order symmetric matrix subspace and the \( m \)-order skew-symmetric matrix subspace, respectively. Then,

\[
\mathbb{R}^{m \times m} = S_m + K_m, \quad S_m = S_0^m + D_m
\]  

(12)

where \( D_m \) is the \( m \)-order diagonal matrix subspace and \( \oplus \) denotes the direct sum.

### III. MAIN RESULTS

In our early paper [22], Lyapunov functions are designed by using the left eigenvector of the Laplacian matrix of the digraph and consequently the exponential synchronization is proved. However, we cannot get the exact exponential synchronization rate by the approach of Lyapunov functions.

In this section, we turn to Lyapunov’s first method, i.e., the approximate linearization method to get the exact description of synchronization rates.

It is straightforward to check that the approximate linearized system of (10) can be rewritten as

\[
\dot{E} = -(LE + EL^T - \dot{\text{vec}}(E)T - \text{vec}(E))^T + \text{vec}(E)^T + \text{vec}(E) + \text{vec}(E)
\]  

(13)

where \( \text{vec}(.): \mathbb{R}^{m \times m} \to \mathbb{R}^{m^2} \) denotes the column-stacking operator

\[
\dot{L} = \begin{bmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{bmatrix}, \quad L_i = \text{Row}_i(L), \quad i = 1, 2, \ldots, m.
\]

(14)

Considering (13), we define a linear transformation on \( \mathbb{R}^{m \times m} \) by

\[
T(X) = LX + XL^T - \dot{\text{vec}}(X)T - \text{vec}(X)^T + \text{vec}(X)
\]

(15)

So the exponential decay rate of \( E(t) \) is just the smallest real part of eigenvalues of \( T(X) \) restricted on \( S_0^m \). It is not easy to directly compute the eigenvalues of \( T(X) \). We first investigate the properties of \( T(X) \) defined on \( \mathbb{R}^{m \times m} \).

**Proposition 3:** Consider the linear transformation \( T(X) \) defined by (14) and the Lyapunov mapping

\[
S(X) = LX + XL^T
\]

(16)

For the subspaces \( S_0^m, D_m, \) and \( K_m \), the following statements hold:

(i) if \( X \in S_m \), then \( T(X) \in S_0^m \);

(ii) if \( X \in K_m \), then the projection of \( T(X) \) onto \( K_m \) is just \( S(X) \).

**Proof:** (i) If \( X = X^T \), it is easy to check that \( (T(X))^T = T(X) \). Moreover, a straightforward computation shows that

\[
\delta_i^T T(X) \delta_i = 2 \left( \delta_i^T LX \delta_i - \delta_i^T \dot{\text{vec}}(X) T_0 \delta_i \right) = 2 \left( L_i X_i - \delta_i^T \dot{\text{vec}}(X) \right) = 0
\]

where \( \delta_i \) is the \( m \)-dimensional vector whose \( i \)-th entry is one and the others are zero. So \( T(X) \in S_0^m \).

(ii) If \( X = -X^T \), then

\[
(S(X))^T = X^T L^T + LX^T = -S(X).
\]

Thus, \( S(X) \in K_m \). From (14), it follows that:

\[
T(X) = S(X) - \dot{\text{vec}}(X)T + \text{vec}(X)^T T^T
\]

(17)

where \( S(X) \in K_m \), \( \dot{\text{vec}}(X)T + \text{vec}(X)^T T^T \in S_0^m \). So the projection of \( T(X) \) onto \( K_m \) is just \( S(X) \).

Denote by \( B_1, B_2, \) and \( B_3 \) the base of \( S_0^m, D_m, \) and \( K_m \), respectively. By conclusion (i) of Proposition 3, we have \( T(S_0^m) \subset S_0^m \) and \( T(D_m) \subset S_0^m \). So

\[
T[B_1, B_2, B_3] = [B_1, B_2, B_3] \begin{bmatrix} T_{1, 1} & T_{1, 2} & T_{1, 3} \\ T_{2, 1} & 0 & T_{2, 3} \\ T_{3, 1} & T_{3, 2} & T_{3, 3} \end{bmatrix}
\]

(18)

By the properties of the Lyapunov mapping \( S(\cdot) \) shown in [25] or [26], we know that both \( S_m \) and \( K_m \) are invariant subspaces of the Lyapunov mapping. Thus,

\[
S[B_1, B_2, B_3] = [B_1, B_2, B_3] \begin{bmatrix} S_{1, 1} & S_{1, 2} & 0 \\ S_{2, 1} & S_{2, 2} & 0 \\ S_{3, 1} & S_{3, 2} & S_{3, 3} \end{bmatrix}
\]

(19)

From conclusion (ii) of Proposition 3, it follows that \( T_{3, 3} = S_{3, 3} \). Fortunately, the eigenvalues of \( S_{3, 3} \), i.e., the spectrum of the Lyapunov mapping \( S(\cdot) \) restricted on \( K_m \), have been revealed as follows.

**Lemma 4 ([26]):** Assume that the eigenvalues of \( L \) are \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m \). Then, the eigenvalues of the Lyapunov mapping \( S(\cdot) \) restricted on \( K_m \) are \( \{\lambda_1 + \lambda_j | 1 \leq i < j \leq m\} \).

Therefore, if all the eigenvalues of \( T(\cdot) \) on \( \mathbb{R}^{m \times m} \) are obtained, then by (15) and Lemma 4 we can get the eigenvalues of \( T_{1, 1} \), i.e., the eigenvalues of \( T(\cdot) \) restricted on \( S_0^m \). So, the rest content of this article is just pursing the eigenvalues of \( T(\cdot) \) on \( \mathbb{R}^{m \times m} \). To this end, we use the isomorphic mapping \( \text{vec}(X): \mathbb{R}^{m \times m} \to \mathbb{R}^{m^2} \) to deal with \( T(\cdot) \).

**Lemma 5 ([27]):** Let \( A, B, \) and \( C \) be \( m \times n, n \times s, \) and \( s \times t \) matrices, respectively. Then,

\[
\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B)
\]

Applying Lemma 5 to \( T(\cdot) \) described by (14) yields

\[
\text{vec}(T(X)) = (L \otimes I_m + I_m \otimes L - I_m \otimes \hat{L} - \hat{L} \otimes I_m)\text{vec}(X)
\]

So the eigenvalues are just those of the matrix

\[
T = L \otimes I_m + I_m \otimes L - I_m \otimes \hat{L} - \hat{L} \otimes I_m
\]

(20)

Before the main results, we first give a lemma as follows.

**Lemma 6:** Let \( A, B, \) and \( C \in \mathbb{R}^{m \times n} \) satisfy \( AB = BC \) and \( B^2 = 0 \). Then, the eigenvalue set of \( A + B \) is the same as that of \( A \).

**Proof:** Let \( \text{rank}(B) = r \). When \( r = 0 \), the assertion of the lemma is obviously right. When \( r > 0 \), there is an \( n \)-by-\( r \) matrix \( M \) and a \( r \)-by-\( n \) matrix \( N \) such that

\[
B = MN, \quad \text{rank}(M) = \text{rank}(N) = r
\]

(21)

Since \( M \) has a full column rank, one can construct a nonsingular matrix \( T = [M, T_1] \). Let \( T^{-1} = [P^T, Q^T]^T \). Then,

\[
\begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix} T^{-1} T = \begin{bmatrix} PM & PT_1 \\ QM & QT_1 \end{bmatrix}
\]

(22)

which implies that

\[
QM = 0, \quad QB = QMN = 0
\]

(23)
Since $N$ has a full row rank and $BMN = B^2 = 0$, we have that $BM = 0$. Thus,

$$T^{-1}BT = \begin{bmatrix} P \\ Q \end{bmatrix} B [M \ T_1] = \begin{bmatrix} 0 & PBT_1 \\ 0 & 0 \end{bmatrix}.$$  

(21)

From $AMN = AB = BC$ and (20), it follows that

$$QAM = QBCNT(NNT)^{-1} = 0.$$  

(22)

Thus,

$$T^{-1}AT = \begin{bmatrix} P \\ Q \end{bmatrix} A [M \ T_1] = \begin{bmatrix} PAM \ PAT_1 \\ 0 \end{bmatrix}.$$  

(23)

By (21) and (23), the proof is complete. $\blacksquare$

**Proposition 4:** Assume that the Laplacian matrix $L$ satisfies rank$(L) = m - 1$ and the eigenvalues of $L$ are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$. Then, the characteristic polynomial of $T(\cdot)$ is

$$s^m \prod_{i=2}^m (s - \lambda_i) \prod_{j=2}^m (s - 2\lambda_j) \prod_{2 \leq q < p \leq m} (s - \lambda_p - \lambda_q)^2.$$  

(24)

**Proof:** Since rank$(L) = m - 1$ and $L1_m = 0$, there exists a nonsingular matrix $P = [1_m, P_1]$, which results in the Jordan canonical form of $L$ as follows:

$$J = P^{-1}LP = \begin{bmatrix} 0 \\ \lambda_2 b_2 \\ \vdots \\ \lambda_{m-1} b_{m-1} \lambda_m \end{bmatrix}.$$  

(25)

where every $\lambda_i$ is nonzero and each $b_i$ is one or zero. It is easy to check that

$$(P^{-1} \otimes P^{-1})(L \otimes I_m + I_m \otimes L - 1_m \otimes \hat{L})(P \otimes P)$$

$$= J \otimes I_m + I_m \otimes J - \delta_1 \otimes P^{-1}\hat{L}(P \otimes P)$$

$$= J \otimes I_m + I_m \otimes J - \delta_1 \otimes P^{-1}\hat{L}(1_m \otimes P, P_1 \otimes P)$$

$$= J \otimes I_m + I_m \otimes J - \delta_1 \otimes [P^{-1}LP, P^{-1}\hat{L}(P_1 \otimes P)]$$

$$= J \otimes I_m + I_m \otimes J - \delta_1 \otimes [J, P^{-1}\hat{L}(P_1 \otimes P)]$$

$$= \begin{bmatrix} 0_{m \times m} \\ J \otimes I_m \end{bmatrix} \begin{bmatrix} J \\ I_{m-1} \otimes J \end{bmatrix} - \begin{bmatrix} J \\ P^{-1}\hat{L}(P_1 \otimes P) \end{bmatrix}$$

$$= \begin{bmatrix} 0_{m \times m} \\ J \otimes I_m \end{bmatrix} - P^{-1}\hat{L}(P_1 \otimes P) \cdot J \otimes I_m + I_m \otimes J.$$  

(26)

From (26), it follows that the characteristic polynomial of $L \otimes I_m + I_m \otimes L - 1_m \otimes \hat{L}$ is just (24). In the following, we use Lemma 6 to complete the proof. Let

$$A = L \otimes I_m + I_m \otimes L - 1_m \otimes \hat{L}$$

$$B = \hat{L} \otimes 1_m = \begin{bmatrix} 1_m L_1 \\ 1_m L_2 \\ \vdots \\ 1_m L_m \end{bmatrix}.$$  

From $L1_m = 0$, it follows that $B^2 = 0$, $(I_m \otimes L)B = 0$ and $(1_m \otimes \hat{L})B = 0$. Now, we construct a matrix $C$ such that $(L \otimes I_m)B = BC,$ which is equivalent to

$$l_{ij}1_m L_j = 1_m L_C ij \forall 1 \leq i, j \leq m.$$  

(27)

When $L_i = 0$, we have $l_{ij} = 0$, which means that $C_{ij}$ can be any $m$-by-$m$ matrix. When $L_i \neq 0$, it is easy to check that $C_{ij} = l_{ij}L_1 m^T (L_1 m^T)^{-1}L_i$ satisfies (27). So, there exists a matrix $C$ such that $AB = BC$. Therefore, by Lemma 6, $A - B$ and $A$ have the same characteristic polynomial. The proof is complete. $\blacksquare$

By Proposition 4, we have obtained all the eigenvalues of the linear transformation $T(\cdot)$. Now we can determine the exponential synchronization rate.

**Theorem 1:** Consider the high-dimensional Kuramoto model (3) limited on the unit sphere $S^{n-1}$. If the interconnecting digraph with the adjacency matrix $(a_{ij})$ has a spanning tree, then the supremum of exponential synchronization rates is the smallest real part of the nonzero eigenvalues of the Laplacian matrix $L$.

**Proof:** Since the interconnecting digraph with the adjacency matrix $(a_{ij})$ has a spanning tree, the Laplacian matrix $L$ satisfies rank$(L) = m - 1$ [28]. So by Proposition 4, we get the characteristic polynomial of $T(\cdot)$ as shown in (24), where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $L$. By Lemma 5 and $\lambda_2 = 0$, we get the characteristic polynomial of $T_{33}$ in (15) as follows:

$$\prod_{i=2}^m (s - \lambda_i) \prod_{2 \leq q < p \leq m} (s - \lambda_p - \lambda_q).$$  

(28)

From (15), (24), and (28), it follows that the characteristic polynomial of $T_{11}$ is

$$\prod_{j=2}^m (s - 2\lambda_j) \prod_{2 \leq q < p \leq m} (s - \lambda_p - \lambda_q).$$  

(29)

Without loss of generality, we assume that $\lambda_2$ has the smallest real part of nonzero eigenvalues of $L$. Therefore, by (29) and Lemma 3, we conclude that the supremum of exponential decay rates of the synchronization error dynamics is $2\Re(\lambda_2)$. Considering $|e_{ij}(t)| = e_{ij}(t) = \frac{1}{2}||r_i(t) - r_j(t)||^2$, we conclude that the supremum of the exponential synchronization rates of the high-dimensional Kuramoto model (3) is $\Re(\lambda_2)$.

**Remark 1:** The graph condition that the digraph has a directed spanning tree is the weakest condition for synchronization. Actually, if the digraph has no a directed spanning tree, there exist at least two independent strongly connected components. Since there is no any information interaction between the two independent strongly connected components, it is impossible to achieve synchronization for all the possible initial states. Compared with the assumption that the graph is strongly connected and balanced [17], our result significantly weakens the graph condition.

**Remark 2:** In [21], the considered high-dimensional Kuramoto model with a complete graph is described as follows:

$$\dot{r}_i = \Omega r_i + \frac{K}{m} \sum_{j=1}^m (r_j - (r_i^T r_j)r_i), \quad i, 1, 2, \ldots, m$$  

(30)

where $K > 0$ is a constant. From (30), it follows that $a_{ij} = K/m$ for all $i \neq j$. So the Laplacian matrix is

$$L = KL_m - K/m1_m 1_m^T$$

whose characteristic polynomial is

$$\det(sI_m - L) = (s - K)^{m-1} s.$$
Thus, by Theorem 1, the supremum of exponential synchronization rates is just $K$. By [21, Th. 3.1], $K$ is an exponential synchronization rate. So $K$ is just the maximum exponential synchronization rate.

**Remark 3:** Compared with linear multi-agent systems defined in liner spaces, the high-dimensional Kuramoto model as a nonlinear multi-agent system limited on the unit sphere is more difficult and challenging. We have proved that the supremum of exponential synchronization rates is just the minimum real part of nonzero eigenvalues of $L$, which is consistent with the case of linear consensus models. But for a linear multi-agent system, exponential synchronization rates can be easily calculated by using the Jordan form of $L$ (see [29, Th. 3]).

**Remark 4:** If the oscillators are nonidentical, it is usually difficult to achieve complete synchronization and exponential synchronization. In this case, practical synchronization can be considered [15], and the exponential rate of convergence to a ball in [30] may be useful for generalizing Theorem 1 to the case of nonidentical oscillators. Moreover, if there are transmission delays in the interconnection [29], [31], [32], [33], it is worth further studying the exponential synchronization.

**IV. SIMULATIONS**

In this section, we give an example to illustrate the obtained main result. Consider the high-dimensional Kuramoto model (3) with $n = 3$ and $m = 5$. The digraph is shown in Fig. 1, which has a directed spanning tree. From Fig. 1, we obtain the Laplacian matrix as follows:

$$
L = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
0 & -1 & -1 & 0 & 2
\end{bmatrix}.
$$

The high-dimensional Kuramoto model is described by

$$
\begin{align*}
\dot{r}_1 &= \Omega r_1 + r_5 - r_1^2 r_5 r_1 \\
\dot{r}_2 &= \Omega r_2 + r_1 - r_2^2 r_1 r_2 \\
\dot{r}_3 &= \Omega r_3 + r_2 - r_3^2 r_2 r_3 \\
\dot{r}_4 &= \Omega r_4 + r_3 - r_4^2 r_3 r_4 \\
\dot{r}_5 &= \Omega r_5 + r_2 - r_5^2 r_2 r_5 + r_3 - r_5^2 r_3 r_5
\end{align*}
$$

(31)
By Theorem 1, $1$ is the supremum of the exponential synchronization rates. It is easy to calculate that the smallest real part of nonzero eigenvalues of $L$ is $1$. By Theorem 1, $1$ is the supremum of the exponential synchronization rates. The time-response curves of the synchronization errors are drawn in Fig. 6, from which the exponential synchronization rate is reflected. In all the simulations, the initial states of the oscillators are set to

$$r_i(0) = [0.1622, 0.1622, 0.9733]^T$$

$$r_2(0) = [0.4816, -0.3655, 0.7965]^T$$

$$r_3(0) = [0.4199, -0.1462, 0.8957]^T$$

$$r_4(0) = [0.2182, 0.4364, 0.8729]^T$$

$$r_5(0) = [0.4407, -0.5806, 0.6845]^T$$

respectively.

V. CONCLUSION

For the high-dimensional Kuramoto model with identical oscillators and a general digraph admitting a spanning tree, the supremum of the exponential synchronization rates has been accurately determined by using the matrix Riccati differential equation of the synchronization error dynamics. In our future work, we will try to generalize the main results and method to nonidentical Kuramoto oscillators and some other generalized high-dimensional Kuramoto models.

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