0. Introduction.

This short Note falls within the context of the references [1], [2], [3] and [4] cited below and establishes that the counter-example provided in [4] to the second inequality of Corollary (19.10) in the Clay Institute Monograph by J. Morgan and G. Tian [1] entitled "Ricci Flow and the Poincare Conjecture" still stands after the correction published in [2]. The constant $C_1$ in Lemma 0.4 of [2] can be taken to be zero.

The problem lies deeper.

1. Preliminaries.

We assume in the sequel that the curve-shortening flow, starting from a given curve, defines a piece of (immersed) surface $\Sigma$. This happens for example when $k(c(x_0, 0))$, the norm of the curve-shortening flow deformation vector $H(c(x, 0))$ as in eq [1], is non-zero at a given point $x_0$ of a smooth immersed curve $c(x, 0)$. Extending in section to the curve-shortening flow, we find that an open set $U$ in $M$ is parameterized as $\{c_\mu(x, t)\}$, $\mu$ an extra-parameter, with $\frac{\partial c_\mu(x,t)}{\partial t} = H((c_\mu(x, t)) = \nabla^{g(t)}_S S(c_\mu(x, t)), S$ is the unit vector of $x \rightarrow c_\mu(x, t)$, $(t, \mu)$ frozen) for the metric $g(t)$ evolving as in [1] through the Ricci flow.

$U$ is now mapped into $M \times [0, \epsilon)$ through the map $c_\mu(x, t) \rightarrow (c_\mu(x, t), t), t \in [0, \epsilon)$.

This is the framework of [2], with the metric $\hat{g}$ on $M \times [0, \epsilon)$. The image of $M$ through this map will be denoted $M_1$ in the sequel.

2. $C_1$ of [2] is zero.

a. The estimate $(\nabla^{\hat{g}}_H H, H)_{(c(x,t),t)} = -Ric^{g(t)}(H, H) = O(k^2)$.

Consider the identity:

$$(\nabla^{\hat{g}}_H H, H) = (\nabla^{g(t)}_H H, H)_{g(t)} - Ric^{g(t)}(H, H)$$

This identity is derived from the two ways to compute $\frac{\partial^2}{\partial t^2}$, the first one using the metric and connection over $M \times [0, \epsilon)$, see [2], the second one by differentiating directly as in (19.1) of [1] and using the connection of the metric $g(t)$. The proof of (19.1) is repeated in section 3, below, for the sake of completeness.

$\nabla^{\hat{g}}_H H$ above is viewed as covariant differentiation along the curve $(c(x, t), t)$ of $M_1$ (see section 1). Along this curve, $H = \nabla^{g(t)}_S S$, with $S = \frac{\partial c(t)}{\partial x} |_{g(t)}$. Since this quantity depends only on the value of $H$ along the curve $(c(x, t), t)$, $H$ can be extended in this subsection to the $(c(x, t), s)$s as $H(c(x, t), s) = \nabla^{g(t)}_S S, S = \frac{\partial c(t,s)}{\partial x} |_{g(t)}$. This is understood as covariant differentiation of $S$ along the curve parametrized in the variable $x$ as $(c(x, t), s)$, with $t$ and $s$ frozen, hence as covariant differentiation along $S$.

With $H = \nabla^{g(t)}_S S, S = \frac{\partial c(t)}{\partial x} |_{g(t)}$, the expression above $(\nabla^{g(t)}_H H, H)_{g(t)},$ at $t = t_0$, is in fact $[\nabla^{g(t_0)}_H H, H]_{g(t_0)}|_{t=t_0}$, see the proof below in section 3.

Observe now that, since $H$ is horizontal:

$$\{[\nabla^{g(t_0)}_H H, H]_{g(t_0)}|_{(c(x,t))}\}_{t=t_0} = (\nabla^{\hat{g}}_H H, H)_{(c(x,t_0),t_0)}$$

$C_1$ IN [2] IS ZERO

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We use now the fact that \( \dot{H} = \frac{\partial}{\partial s} + H \). We can then split \( \nabla_{\dot{H}} \) into \( \nabla_H + \nabla_{\frac{\partial}{\partial s}} \) in the derivation formula above. Using this splitting, we derive that
\[
(\nabla_{\frac{\partial}{\partial s}} H, H)_{(c(x,t),t)} = -Ricg(t)(H, H) = O(k^2)
\]

This estimate can also be derived directly, without comparing the two formulae, but with working on \( M \times [0, \epsilon) \), with the connection \( \nabla \) and a suitable coordinate frame, using \( \frac{\partial}{\partial s}, \frac{\partial x(t)}{\partial x} \), \( H(c(x,t), s) = \nabla_S^{g(t)} S \), with \( S(c(x,t), s) = \frac{\partial c(x,t)}{\partial x} |_{g(t)} \) and an additional suitable vector-field which adds the parameter \( \mu \) of section 1. We may assume that, with this additional vector-field, we find a frame that is orthogonal (not orthonormal) at \( (c(x,t_0), t_0) \). The computation at \( (c(x,t_0), t_0) \) becomes straightforward: \( H \) has a component on itself equal to 1; any derivative of this component is zero. Then, \( (\nabla_{\frac{\partial}{\partial s}} H, H)_{(c(x,t_0), t_0)} = \Gamma^H_{\frac{\partial}{\partial s}H} (c(x,t_0), t_0)(H, H)_{g(t_0,c(x,t_0))} \). The Lie bracket \( [\frac{\partial}{\partial s}, H] \) is zero\(^1\); furthermore, the base point \( c(x,t) \) for the computation of the dot-products in \( g(t) \) does not move under the action of the one-parameter group of \( \frac{\partial}{\partial s} \). The only parameter that changes under the action of the one parameter group of \( \frac{\partial}{\partial s} \) is the second parameter \( s \) in \( (c(x,t), s) \); over a time equal to \( \tau \), this involves a change of the metric from \( g(t, c(x,t)) \) into \( g(t+\tau, c(x,t)) \). \( H \) is unchanged. Thus, we find that as above:
\[
(\nabla_{\frac{\partial}{\partial s}} H, H)_{(c(x,t_0), t_0)} = -Ricg(t)(H, H) = O(k^2)
\]

Since \( t_0 \) is arbitrary, this new direct computation confirms the former one. We have skipped some details in this construction and in this computation; they do not present any real difficulty.

Observe that the estimate above \( O(k^2) \) does not depend on \( \Sigma \). It depends only on the ambient metric and on the value of \( k \) of course.

b. \( C_1 \) in Lemma 0.4 of [2] can be taken to be zero.

Coming back to the formula of the correction [2], we focus on the addition of the two terms (observe the change of order in \( S \) and \( H \) for the two last arguments in \( Rm \) with respect to [2] in our notation. This is only a matter of notation: the content, which is derived through the formula using the curvature operator \( (R(A,B,C,D)) = \tilde{Rm}(A,B,C,D) \) in our notations and \( (R(A,B,C,D)) = \tilde{Rm}(A,B,D,C) \) in the notations of [1] and [2] is unchanged) \( \tilde{Rm}(H, S, S, H) + (\nabla_S \nabla_S \frac{\partial}{\partial s}, H) \). These are the two terms (multiplied by \( 2 \)) contributing to the constant \( C_1 \) of Lemma 0.4 of [2]\(^2\).

We need to be careful with the definition of \( S \): \( S \), within the framework of the metric \( \tilde{g} \) of [2], is equal to \( S_2(c(x,t), s) = \frac{\partial c(x,t)}{\partial x} |_{g(t)} \). We will modify, without loss of generality for the computation, its definition in the expression for \( \tilde{Rm} \) below.

Observe that the one parameter group of \( \frac{\partial}{\partial s} \), used over the time \( \tau \), maps \( (c(x,t), s) \) into \( (c(x,t), s + \tau) \). It follows that commutation of \( \frac{\partial}{\partial s} \) with \( X = \frac{\partial c(x,t)}{\partial x} \) does occur, so that \( [\frac{\partial}{\partial s}, X] = 0 \).

We now replace \( S_2 \), in \( \tilde{Rm} \) only, by \( S(c(x,t), s) = \frac{\partial c(x,t)}{\partial x} |_{g(t)} \), over \( M \times [0, \epsilon) \); since we are completing this computation at points of \( M_1 \), where the value of \( S_2 \) is indeed \( \frac{\partial c(x,t)}{\partial x} |_{g(t)} \), this modification is legitimate in \( \tilde{Rm} \):

\(^1\)This follows from the action of the one parameter group of \( \frac{\partial}{\partial s} \); this action is only on the second factor of the couple \( (c(x,t), s) \), observe that \( H(c(x,t), s) = \nabla_S^{g(t)} S \), the value of \( S \) being \( S(c(x,t), s) = \frac{\partial c(x,t)}{\partial x} |_{g(t)} \), does not depend on \( s \). This is the framework, which we defined above, for our computation.

\(^2\)The other terms displayed in [2] which could contribute to \( C_1 \) in Lemma 0.4, such as \( 2(\nabla_S [Ricg(S,S)\frac{\partial}{\partial s}, H])_{g} \), are in fact \( O(k^2) \). For example, for this latter term, we know that \( H \) and \( \frac{\partial}{\partial s} \) are orthogonal and we know that \( \nabla_S \frac{\partial}{\partial s} \) is zero. \( \bar{H} \) can be broken into \( \frac{\partial}{\partial s} + H \). The contribution of \( \frac{\partial}{\partial s} \) in the expression above is zero using our observations and the contribution of \( H \) is then \( O(k^2) \) since there is an additional dot product with \( H \) in this expression.
value of \( \hat{R}m(\hat{H}, S_2, S_2, H) \) at a point of \( M_1 \) is the same than the value of \( \hat{R}m(\hat{H}, S, S, H) \), \( S_2 \) and \( S \) as above, at the same point. This follows from the tensor properties of \( \hat{R}m \).

As in [2], with new notations \( S_2 \) is a notation which is not used in [2]- \( S_2 \) in the other term \( (\nabla_S \nabla_S \frac{\partial}{\partial t}) \) remains \( S_2(\gamma, t, s) = \frac{\partial c(\gamma, t)}{\partial t} \). Observe that \( S_2 = \gamma S, \gamma = 1, S_2, \gamma = 0 \) on \( M_1 \), so that \( \hat{\nabla}_S \hat{\nabla}_S = \nabla_S \nabla_S \) on \( M_1 \), see section 1, above for the definition of \( M_1 \).

We need only to consider the terms which are not \( O(k^2) \) and, therefore, the above expression can be changed into \( \hat{R}m(\frac{\partial}{\partial t}, S, S, H) + (\nabla_{S_2} \nabla_{S_2} \frac{\partial}{\partial t}, H) \).

\( H \) has been extended, without loss of generality, in both terms, over \( M \times [0, \epsilon] \) as \( H(\gamma(x, t), s) = \nabla^{\gamma(t)}_S S \), the covariant derivative along the unit vector \( S \) for \( g(t) \) to the curve \( x \rightarrow (\gamma(x, t), s, (t, s) \) frozen.

Writing the first term with the use of the riemannian tensor on \( \frac{\partial}{\partial t} \) and \( S \), this is \( (\nabla_{\frac{\partial}{\partial t}} \nabla_S S - \nabla_S \nabla_{\frac{\partial}{\partial t}} S - \nabla_{\frac{\partial}{\partial t}} \nabla_S S, H) \). This first term is thereby divided itself into three further terms, which we now discuss one by one: We use the fact that \( [\frac{\partial}{\partial t}, S] = \theta S \), with \( \theta \) bounded. This allows to see that the term \( (\nabla_{\frac{\partial}{\partial t}} S, H) \) is \( O(k^2) \). The second term, after the use of the commutation relation \( [\frac{\partial}{\partial t}, S] = \theta S \) and the fact that \( S \) and \( H \) are orthogonal, cancels with \( (\nabla_{S_2} \nabla_{S_2} \frac{\partial}{\partial t}, H) \) (use our observation above about \( \gamma, S_2, \gamma \) on \( M_1 \)) leaving \( O(k^2) \).

Using the fact that \( \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0 \) and the fact that \( \frac{\partial}{\partial t} \) and \( H \) are orthogonal, we find that the first term is \( (\nabla_{\frac{\partial}{\partial t}} H, H) \), which is, from our reasoning above, \( O(k^2) \).

The conclusion follows. \( C_1 \) in [2] can be taken to be zero (we assume that we have a piece of surface \( \Sigma \); the estimate, as pointed out above does not depend on \( \Sigma \). When \( k = 0 \) and there is no immersed \( \Sigma \); locally, the estimate of [2] contains \( C_1 k \) and our argument does not work as is. It is however then clear that \( (\nabla_{\frac{\partial}{\partial t}} H, H)_{g(t)} = 0 = O(k^2) \) at such points. It is also clear that \( C_1 k \) is zero at such points and can be forgotten. Using then density and continuity, the assumption that \( \Sigma \) immersed exists can be removed). The additional terms in \( k \) in the correction hide a cancellation.

3. Proof of (19.1) of [1].

For completion, we add here the computation that shows that (19.1) of [1] holds: Let \( \gamma(t) = g(t) - g(t_0) \) be the bilinear symmetric 2-tensor form defined by difference. We write on the image (piece of) surface in \( M \), which we assume to be immersed:

\[
(H, H)_{g(t)} = (H, H)_{g(t_0)} + (H, H)_{\gamma(t)}
\]

\( H \) is here equal to \( H = \nabla^{\gamma(t)}_S S \) the unit vector tangent to the curve \( x \rightarrow c(x, t) \), with respect to the metric \( g(t) \).

Differentiating with the use of the connection along the surface induced by \( g(t_0) \), we find, since \( \frac{\partial}{\partial t} = H \) on the surface:

\[
\left[ \frac{\partial k^2}{\partial t} \right]_{t=t_0} = 2(\nabla_H^\Sigma H, H)_{g(t_0)} + \left( \frac{\partial((H, H)_{\gamma(t)})}{\partial t} \right)_{t=t_0}
\]

\( \Sigma \) is our piece of surface.

Since \( H \) is tangent to \( \Sigma \), we may replace \( \nabla^\Sigma \) with \( \nabla^{g(t_0)} \) the connection for \( g(t_0) \) on \( M \). For the derivative of the second term, we use local coordinates on \( \Sigma \) and we find that this is \( (H, H)_{g(t_0)} \), which yields the term \( -2 \text{Ric}(H, H) \).

(19.1) follows. Observe that, with \( H_1 = \nabla^{H_{g_1}} S_1, S_1 = \frac{\partial c(x, s)}{\partial t} \):

\[
[\left( \nabla_H^{g(t_0)} \nabla_S^{g(t_0)} S, H \right)_{g(t_0)}]_{t=t_0} = \left[ \left( \nabla_H^{g(t_0)} H_1 \right)_{s=t, H} \right]_{t=t_0}
\]
4. The Clay Institute computation [1], p442 for $\frac{\partial k^2}{\partial t}$, slightly modified in order to make it more transparent.

The Clay Institute monograph [1], besides the division by $k$ pointed out in [3] and corrected in [2] (counter-example still standing, see [4]), can be modified so that the computation of $\nabla_H H$, p442, is more elementary and transparent. The final result is unchanged: When computing $\nabla_H H$ as in p442 of [1], it is preferable to take the connection $\nabla^{g(t)}$ at a fixed value of $t$, $t = t_0$. This can be done, taking $H_2$ to be $\nabla^{g(t_0)} S$, where $S$ is $\frac{\partial x(t)}{\partial x(t_0)}$. The computation is carried as in [1], except that the Lie bracket $[H_2, S]$ is not $\theta S$ anymore, since $H$ has been changed into $H_2$. However, with $H_2$, $[H_2, S] = \theta S + O(t - t_0)$, where $O(t - t_0)$ is small, as well as all its spatial derivatives as $t$ tends to $t_0$. There is no $t$-derivative in the computation of [1]. The result for $\nabla^{g(t_0)} H_2$, p 442, reads $(H_2, S$ as described above):

$$\nabla^{g(t_0)} H_2 = \nabla_S^{g(t_0)} \nabla_S^{g(t_0)} H_2 + 2(k^2 + \text{Ric}(S, S))H + S.(k^2 + \text{Ric}(S, S))S + \text{Riem}^{g(t_0)}(H, S)S + O(t - t_0)$$

In this computation now, $H_2$ can be replaced by $H = \nabla_S^{g(t)} S$ in the left hand side. The result is unchanged since there is no $t$-derivative in this formula, the corrective terms are dropped into $O(t - t_0)$. $g(t_0)$ can be replaced by $g(t)$ in the right hand side, where the covariant derivatives are along $S$. Taking now the dot product with $H$, for the metric $g(t_0)$ for the left hand side and for the metric $g(t)$ for the right hand side, with $H = H_2$ there, we find ($(S, H_2)_{g(t)} = O(t - t_0))$:

$$(\ast) \quad (\nabla^{g(t)} H, H)_{g(t_0)} = (\nabla_S^{g(t)} \nabla_S^{g(t)} H_2 + 2(k^2 + \text{Ric}(S, S))H + \text{Riem}^{g(t)}(H, S)S + O(t - t_0), H_2)_{g(t)}$$

Understanding $\nabla_S^{g(t)} \nabla_S^{g(t)} H_2$ as covariant differentiation along the curve $c(x, t)$, ie along $S$, we find that

$$(\nabla_S^{g(t)} \nabla_S^{g(t)} H_2, H_2)_{g(t_0)} = (\nabla_S^{g(t)} \nabla_S^{g(t)} H, H)_{g(t)} + O(t - t_0)$$

Then, after entering this and further replacing $H_2$ with $H$ in $(\ast)$, taking then this modified identity at $t = t_0$, we find that the computation of [1] for $\frac{\partial k^2}{\partial t}$ holds at $t = t_0$. But $t_0$ is arbitrary.

5. Conclusion.

In view of the present note and in view of the counter-example in [4], the problem lies deeper.
References

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