FROBENIUS-SCHUR THEOREM FOR $C^*$-CATEGORIES

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Abstract. We generalize the Frobenius-Schur theorem to $C^*$-categories. From this category-theoretical point of view, we introduce the notions of real, complex and quaternionic representations of Hopf $C^*$-algebras. Based on these definitions, we give another type of the Hopf-algebraic analogue of the Frobenius-Schur theorem, originally due to Linchenko and Montgomery. Also given are similar results for weak Hopf $C^*$-algebras, table algebras and compact quantum groups.

1. Introduction

1.1. Frobenius-Schur theorem for compact groups. We work over the field $\mathbb{C}$ of complex numbers. Let $G$ be a compact group, and let $V$ be a finite-dimensional irreducible continuous representation of $G$ with character $\chi_V$. We say that $V$ is real if it admits a basis $\{v_i\}_{i=1}^n$ such that the matrix representation $\rho : G \to GL_n(\mathbb{C})$ with respect to $\{v_i\}$ has the following property:

$\rho(G) \subset GL_n(\mathbb{R})$.

Note that if $V$ is real, then $\chi_V(G) \subset \mathbb{R}$. Following, $V$ is said to be pseudo-real, or quaternionic, if it does not admit such a basis but $\chi_V(G) \subset \mathbb{R}$. Finally, $V$ is said to be complex if it is neither real nor quaternionic. The Frobenius-Schur theorem gives a way to determine whether $V$ is real, complex or quaternionic: Define the Frobenius-Schur indicator of $V$ by

$$\nu(V) = \int_G \chi_V(g^2) d\mu(g),$$

where $\mu$ is the normalized Haar measure on $G$. Then the theorem states:

$$\nu(V) = \begin{cases} +1 & \text{if } V \text{ is real,} \\ 0 & \text{if } V \text{ is complex,} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

1.2. Generalizations of the Frobenius-Schur indicator. In this paper, we give a generalization of the Frobenius-Schur theorem to $C^*$-categories. We should note that the Frobenius-Schur indicator has been generalized in various contexts \cite{10, 11, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24}. Before we describe a summary of our results, we briefly review some of other existing generalizations and raise some questions.

We first mention the result of Linchenko and Montgomery \cite{16}. For an irreducible representation $V$ of a finite-dimensional semisimple Hopf algebra $H$ (over $\mathbb{C}$), they defined the Frobenius-Schur indicator of $V$ by

$$\nu(V) = \chi_V(\Lambda_{(1)}\Lambda_{(2)}),$$
where \( \chi_V \) is the character of \( V \) and \( \Lambda_{(1)} \otimes \Lambda_{(2)} \in H \otimes H \) is the comultiplication of the Haar integral \( \Lambda \in H \) (in the Sweedler notation). The main result of [16] is then described as follows:

1. \( \nu(V) \) is one of +1, 0 or \(-1\). Moreover, \( \nu(V) \neq 0 \) if and only if there exists a non-degenerate \( H \)-invariant bilinear form \( \beta : V \times V \to \mathbb{C} \).

2. Suppose \( \nu(V) \neq 0 \). If \( \beta \) is a non-degenerate \( H \)-invariant bilinear form on \( V \), then \( \beta(w, v) = \nu(V)\beta(v, w) \) for all \( v, w \in V \). In other words, \( \beta \) is symmetric if \( \nu(V) = +1 \) and is skew-symmetric if \( \nu(V) = -1 \).

One may wonder if this result is a generalization of (1.2). In the case where \( H \) is the group algebra, it is not so difficult to derive (1.2) from the above result. In general, the notion of a real, a complex and a quaternionic representation of \( H \) is not clear; thus we should rather mention the following problem:

**Problem 1.1.** When can we define a real, a complex and a quaternionic representation of a Hopf algebra? If appropriate definitions are given, how do they relate to the Frobenius-Schur indicator?

Putting this problem aside for a moment, we mention some category-theoretical generalizations of the Frobenius-Schur indicator [10, 11, 18, 19, 20, 21, 24]. In [24], the author introduced the Frobenius-Schur indicator for categories with duality to unify various generalizations of the Frobenius-Schur theorem. However, the results of [24] describe how the Frobenius-Schur indicator relates to the existence of a kind of invariant bilinear forms and, in particular, are not formulated like (1.2) (see [24, Remark 1.4]). The next problem is:

**Problem 1.2.** What is a category-theoretical counterpart of (1.2)?

Namely, we would like to introduce appropriate definitions of ‘real’, ‘complex’ and ‘quaternionic’ objects in some category-theoretical way and then show that the Frobenius-Schur indicator in the sense of [24] detects whether a simple object is real, complex or quaternionic.

### 1.3. Summary of results

Keeping these problems in mind, we now give a summary of our results. Let \( \mathcal{A} \) be a \( \mathbb{C} \)-linear abelian category. Moreover, we assume that \( \mathcal{A} \) is *locally finite-dimensional* in the following sense:

\[
\text{dim}_{\mathbb{C}} \text{Hom}_{\mathcal{A}}(X, Y) < \infty \quad (X, Y \in \mathcal{A}).
\]

In [2] we introduce the notion of a *dual structure* for \( \mathcal{A} \) and an equivalence relation between them. For each dual structure \( \mathcal{D} \) for \( \mathcal{A} \), we define a function

\[
\nu_{\mathcal{D}} : \text{Obj}(\mathcal{A}) \to \mathbb{Z},
\]

where \( \text{Obj}(\mathcal{A}) \) is the isomorphism classes of objects of \( \mathcal{A} \). Following [24], \( \nu_{\mathcal{D}} \) is called the *Frobenius-Schur indicator* with respect to \( \mathcal{D} \). We show that the function \( \nu_{\mathcal{D}} \) depends only on the equivalence class of \( \mathcal{D} \).

Now let \( \mathcal{A} \) be an algebra over \( \mathbb{C} \). In our applications, important are dual structures for the category \( _{\mathcal{A}}\mathcal{M}_{fd} \) of finite-dimensional left \( \mathcal{A} \)-modules. By a *dual structure* for \( \mathcal{A} \), we mean a pair \((\mathcal{S}, g)\) consisting of an anti-algebra map \( \mathcal{S} : \mathcal{A} \to \mathcal{A} \) and an element \( g \in \mathcal{A} \) satisfying certain conditions. Such a pair \((\mathcal{S}, g)\) yields a dual structure for \( _{\mathcal{A}}\mathcal{M}_{fd} \). In [23] we recall from [24] a formula of the Frobenius-Schur indicator with respect to this type of dual structures.
In \cite{3} we introduce the notion of a real structure for \( \mathcal{A} \) and an equivalence relation between them. Given a real structure \( \mathcal{J} \) for \( \mathcal{A} \), the \( \mathcal{J} \)-signature \( \sigma_{\mathcal{J}}(X) \in \{0, \pm 1\} \) is defined for each simple object \( X \in \mathcal{A} \) according to whether there exists a certain type of isomorphisms. Taking the \( \mathcal{J} \)-signature is a function

\[
\sigma_{\mathcal{J}} : \text{Irr}(\mathcal{A}) \to \{0, \pm 1\},
\]

where \( \text{Irr}(\mathcal{A}) \) is the class of isomorphism classes of simple objects of \( \mathcal{A} \). We show that the function \( \sigma_{\mathcal{J}} \) depends only on the equivalence class of \( \mathcal{J} \).

A real form of an algebra \( A \) is an \( \mathbb{R} \)-subalgebra \( A_0 \subset A \) such that \( A = A_0 \oplus 1A_0 \), where \( i = \sqrt{-1} \). In \cite{3}, we first observe that a real form of \( A \) yields a real structure for \( _A\mathcal{M}_{fd} \). We say that a simple module \( V \in _A\mathcal{M}_{fd} \) is said to be real, complex and quaternionic with respect to \( A_0 \) if \( \sigma_{\mathcal{J}}(V) \) is equal to \(+1\), \(0\) and \(-1\), respectively, where \( \mathcal{J} \) is the real structure for \( _A\mathcal{M}_{fd} \) obtained from \( A_0 \). These notions can be characterized in more familiar ways. For example, \( V \) is real if and only if it admits a basis such that the matrix representation \( \rho : A \to M_n(\mathbb{C}) \) with respect to that basis satisfies \( \rho(A_0) \subset M_n(\mathbb{R}) \); see Theorems \( 3.9 \) and \( 3.10 \) for details.

In \cite{3}, we assume moreover that \( \mathcal{A} \) is a \( C^* \)-category. Let \( \text{DS}_*(\mathcal{A}) \) (resp. \( \text{RS}_*(\mathcal{A}) \)) be the equivalence classes of dual (resp. real) structures for \( \mathcal{A} \) which are compatible with the \( * \)-structure (\( * \)-compatibility, see Definition \( 4.3 \)). There exists a bijection

\[
\mathcal{D} : \text{RS}_*(\mathcal{A}) \to \text{DS}_*(\mathcal{A}),
\]

see Theorem \( 4.4 \). Let \( \mathcal{J} \) be a \( * \)-compatible real structure for \( \mathcal{A} \). The existence of the bijection suggests that the \( \mathcal{J} \)-signature can be computed by using the corresponding dual structure for \( \mathcal{A} \) in some way. We show that the way is the Frobenius-Schur indicator: If the equivalence class of a \( * \)-compatible dual structure \( \mathcal{D} \) for \( \mathcal{A} \) corresponds to the equivalence class of \( \mathcal{J} \) via \( (4.4) \), then

\[
\nu_{\mathcal{D}}(X) = \sigma_{\mathcal{J}}(X)
\]

for all simple object \( X \in \mathcal{A} \) (Theorem \( 4.7 \)). This answers to Problem \( 1.2 \) in \cite{5}–\cite{6} we explain why \( (1.5) \) can be considered as a generalization of the Frobenius-Schur theorem.

In \cite{5} we apply \( (1.5) \) to finite-dimensional \( C^* \)-algebras. Let \( A \) be such an algebra, and let \( S : A \to A \) be an anti-algebra map such that

\[
S(S(a)^*)^* = a \quad (a \in A).
\]

Then \( A_0 = \{a \in A \mid S(a)^* = a\} \) is a real form of \( A \) (and, moreover, any real form of \( A \) is obtained in this way). Now we consider the category \( \mathcal{A} := \text{Rep}_{fd}(A) \) of finite-dimensional \( * \)-representations of \( A \). The map \( S \) and the real form \( A_0 \) define a dual structure \( \mathcal{D} \) and a real structure \( \mathcal{J} \) for \( \mathcal{A} \), respectively. We show that \( \mathcal{D} \) and \( \mathcal{J} \) are \( * \)-compatible and their equivalence classes correspond via \( (1.4) \). Thus \( (1.5) \) implies, for all simple \( V \in \text{Rep}(A) \),

\[
(1.7) \quad \nu_{\mathcal{D}}(V) = \begin{cases} +1 & \text{if } V \text{ is real}, \\ 0 & \text{if } V \text{ is complex}, \\ -1 & \text{if } V \text{ is quaternionic (with respect to } A_0). \end{cases}
\]

Now let \( A \) be a finite-dimensional Hopf \( C^* \)-algebra, and let \( S : A \to A \) be the antipode of \( A \) (which is known to satisfy \( (1.6) \)). The right-hand side of \( (1.7) \) is then equal to \( (1.3) \). Hence an answer to Problem \( 1.1 \) is obtained: The Frobenius-Schur indicator for the Hopf algebra \( A \), defined by Linchenko and Montgomery, detects
whether a given simple \( A \)-module \( V \) is real, complex or quaternionic with respect to the real form

\[
A_0 = \{ a \in A \mid S(a)^* = a, \text{ where } S : A \to A \text{ is the antipode} \}.
\]

If \( A = \mathbb{C}G \) is the group algebra of a finite group \( G \), then \( A_0 \) is precisely \( \mathbb{R}G \). A similar result for finite-dimensional weak Hopf \( C^* \)-algebras \([4, 5]\), that for table algebras \([3]\), and their ‘twisted’ versions are also deduced from \((1.7)\).

In \([6]\) we apply \((1.6)\) to compact quantum groups \([23, 8, 15]\). Since, by definition, a representation of a compact quantum group is a comodule over its so-called quantum coordinate algebra, we first prove the results of \([23, 8, 15]\) in coalgebraic settings. Once such results are developed, we easily derive an exact quantum analogue of the Frobenius-Schur theorem for compact groups \((6.10)\).

In Appendix A, we mention a proposition due to Böhm, Nill and Szlachányi \([4]\) (see Proposition \((5.8)\) and related problems. Since their lemma plays a quite important role in \((5.7)\), it is worth to give a new proof from the viewpoint of our theory. We prove Proposition \((5.8)\) by emphasizing the ‘lifting problem’ \((Remark (5.7))\) of a certain functor. A coalgebraic version of Proposition \((5.8)\) is also proved.

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### 2. Dual structures

#### 2.1. Dual structures.

Following Balmer \([1]\), a *category with duality* is a triple \((A, D, \eta)\) consisting of a category \( A \), a contravariant endofunctor \( D \) on \( A \), and a natural isomorphism \( \eta : \text{id}_A \to DD \) satisfying

\[
D(\eta_X) \circ \eta_{D(X)} = \text{id}_{D(X)} \quad (X \in A).
\]

Since we will deal at the same time with many pairs \((D, \eta)\) such that \((A, D, \eta)\) is a category with duality, it is convenient to introduce the following terminology:

**Definition 2.1.** Let \( A \) be a \( \mathbb{C} \)-linear category. A *dual structure* for \( A \) is a pair \((D, \eta)\) of a contravariant \( \mathbb{C} \)-linear endofunctor \( D \) on \( A \) and a natural isomorphism \( \eta : \text{id}_A \to DD \) satisfying \((2.1)\). If \( D = (D, \eta) \) and \( D' = (D', \eta') \) are dual structures for \( A \), then a *morphism* from \( D \) to \( D' \) is a natural transformation \( \xi : D \to D' \) such that the following diagram commutes for all \( X \in A \):

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & DD(X) \\
\downarrow{\eta_X} & & \downarrow{\xi_{D(X)}} \\
D'D'(X) & \xrightarrow{D'\xi_X} & D'D(X)
\end{array}
\]

It is trivial from the definition that \((D, \eta)\) is a dual structure for \( A \) if and only if the triple \((A, D, \eta)\) is a category with duality such that \( D \) is \( \mathbb{C} \)-linear.

Dual structures for \( A \) form a category. If dual structures \( D = (D, \eta) \) and \( D' = (D', \eta') \) are isomorphic in this category, then we say that they are *equivalent* and write \( D \sim D' \). Note that a natural isomorphism \( \xi : D \to D' \) is an isomorphism from \( D \) to \( D' \) if and only if

\[
(\text{id}_A, \xi) : (A, D, \eta) \to (A, D', \eta')
\]

is a strong duality preserving functor in the sense of Calmès and Hornbostel \([6]\).
2.2. Frobenius-Schur indicator. Let \( A \) be a locally finite-dimensional \( \mathbb{C} \)-linear category, and let \( X \in A \) be an object. Given a dual structure \( D = (D, \eta) \) for \( A \), we define a linear map

\[
T_{D|X}: \text{Hom}_A(X, D(X)) \to \text{Hom}_A(X, D(X)), \quad f \mapsto \eta_Y \circ D(f).
\]

**Definition 2.2.** The Frobenius-Schur indicator of \( X \in A \) with respect to the dual structure \( D \) is defined and denoted by \( \nu_D(X) = \text{Tr}(T_{D|X}) \), where \( \text{Tr} \) is the trace of a linear operator.

Let \( \text{Obj}(A) \) be the isomorphism classes of objects of \( A \). The value of \( \nu_D(X) \) depends on the isomorphism class of \( X \). Since the square of \( T_{D|X} \) is the identity, its trace \( \nu_D(X) \) is an integer. Hence, taking the Frobenius-Schur indicator with respect \( D \) defines a map

\[
\nu_D: \text{Obj}(A) \to \mathbb{Z}, \quad X \mapsto \nu_D(X).
\]

Now let \( D' = (D', \eta') \) be another dual structure for \( A \). If \( D \sim D' \), then, since \( (A, D, \eta) \) and \( (A, D', \eta') \) are equivalent as categories with duality via \((2.2)\), we have \( \nu_D(X) = \nu_{D'}(X) \) for all \( X \in A \) [24 Proposition 2.10]. In other words:

**Lemma 2.3.** The function \( \nu_D \) depends only on the equivalence class of \( D \).

We suppose moreover that \( A \) is an abelian category. Let \( X \) be a simple object of \( A \). By Schur’s lemma, \( \text{Hom}_A(X, D(X)) \) is one-dimensional if \( X \cong D(X) \) and zero otherwise. Since \( (T_{D|X})^2 \) is the identity, we have \( \nu_D(X) \in \{0, \pm 1\} \) and

\[
\nu_D(X) \neq 0 \iff X \cong D(X).
\]

If \( X \cong D(X) \), then \( \nu_D(X) \in \{\pm 1\} \) is characterized by the following formula:

\[
D(f) \circ \eta_X = \nu_D(X) \cdot f \quad (f \in \text{Hom}_A(X, D(X)),
\]

see [24 Proposition 2.12] for details.

2.3. Dual structures for an algebra. Given an algebra \( A \) over \( \mathbb{C} \), we denote by \( A\mathcal{M} \) and \( A\mathcal{M}_{fd} \) the \( \mathbb{C} \)-linear abelian category of left \( A \)-modules and its full subcategory consisting of finite-dimensional objects, respectively.

**Definition 2.4.** By a dual structure for \( A \), we mean a pair \((S, g)\) consisting of an anti-algebra map \( S: A \to A \) and an invertible element \( g \in A \) satisfying

\[
S(g) = g^{-1}, \quad S^2(a) = gag^{-1} \quad (a \in A).
\]

A dual structure \((S, g)\) for \( A \) gives rise to a dual structure for \( A\mathcal{M}_{fd} \). To explain, we introduce some notations: We denote by \( X^* \) the set of all linear maps from a vector space \( X \) to \( \mathbb{C} \). Given a linear map \( f: X \to Y \), define

\[
f^*: Y^* \to X^*, \quad f^*(\psi) = \psi \circ f \quad (\psi \in Y^*).
\]

Now, for \( X \in A\mathcal{M}_{fd} \), we define \( D(X) \in A\mathcal{M}_{fd} \) to be the vector space \( X^* \) endowed with the left \( A \)-module structure given by

\[
\langle a \cdot \psi, x \rangle = \langle \psi, S(a)x \rangle \quad (a \in A, \psi \in X^*, x \in X).
\]

For a morphism \( f: X \to Y \) in \( A\mathcal{M}_{fd} \), set \( D(f) = f^* \). \( X \mapsto D(X) \) defines a \( \mathbb{C} \)-linear contravariant endofunctor \( D \) on \( A\mathcal{M}_{fd} \). Moreover, there is a natural isomorphism \( \eta: \text{id} \to DD \) given by

\[
\langle \eta_X(x), \psi \rangle = \langle \psi, gx \rangle \quad (x \in X, \psi \in X^*).
\]
The pair $(D, \eta)$ is a dual structure for $A\mathcal{M}_{fd}$, which will be referred to as the dual structure for $A\mathcal{M}_{fd}$ associated with $(S, \gamma)$.

Recall that a separability idempotent of $A$ is an element $E = \sum_i E_i \otimes E_i'' \in A \otimes A$ such that $\sum_i E_i' E_i'' = 1$ and $\sum_i a E_i' \otimes E_i'' = \sum_i E_i' \otimes E_i'' a$ for all $a \in A$. If such an element exists, then:

**Theorem 2.5** ([24, Theorem 3.8]). For all $V \in A\mathcal{M}_{fd}$, we have

$$\nu_D(V) = \sum_i \chi_V(S(E_i') g E_i''),$$

where $D$ is the dual structure associated with $(S, \gamma)$.

**Remark 2.6.** To express the right-hand side of (2.6) neatly, the following formulas will be used: If $V \in A\mathcal{M}_{fd}$ is simple and $z \in A$ is central, then

$$\chi_V(z a) = \frac{\chi_V(z) : \chi_V(a)}{\chi_V(1)}, \quad \chi_V(z^{-1} a) = \frac{\chi_V(1) : \chi_V(a)}{\chi_V(z)} \quad (a \in A).$$

Here, in the latter formula, $z$ is assumed to be invertible. These formulas follow from that the action of $z$ on $V$ is a scalar multiple by Schur’s lemma.

### 3. Real structures

#### 3.1. Real structures

We say that a functor $F : A \rightarrow B$ between $\mathbb{C}$-linear categories is *anti-linear* if the map $F : \text{Hom}_A(X, Y) \rightarrow \text{Hom}_B(F(X), F(Y))$ induced by $F$ is an anti-linear map for all $X, Y \in A$. Now let $A$ be a $\mathbb{C}$-linear category.

**Definition 3.1.** A *real structure* for $A$ is a pair $\mathcal{J} = (J, i)$ consisting of an anti-linear functor $J : A \rightarrow A$ and a natural isomorphism $i : \text{id}_A \rightarrow JJ$ satisfying

$$i_{J(X)} = J(i_X)$$

for all $X \in A$. If $\mathcal{J} = (J, i)$ and $\mathcal{J}' = (J', i')$ are real structures for $A$, then a *morphism* from $\mathcal{J}$ to $\mathcal{J}'$ is a natural isomorphism $\beta : J \rightarrow J'$ such that the following diagram commutes for all $X \in A$:

$$
\begin{array}{ccc}
X & \xrightarrow{i_X} & JJ(X) \\
\downarrow{i_X} & & \downarrow{\beta_{J(X)}} \\
J'J'(X) & \xleftarrow{\beta_{J(X)}} & J'J(X)
\end{array}
$$

(3.1)

Real structures for $A$ form a category. If $\mathcal{J}$ and $\mathcal{J}'$ are isomorphic in this category, then we say that they are *equivalent* and write $\mathcal{J} \sim \mathcal{J}'$.

Beggs and Majid [2] introduced the notion of *bar categories*. In most situations, a bar category can be thought as a $\mathbb{C}$-linear monoidal category endowed with a real structure compatible with the monoidal structure. Thus many examples of real structures are found in [2].

#### 3.2. Signature

Let $A$ be a locally finite-dimensional $\mathbb{C}$-linear abelian category, and let $\mathcal{J} = (J, i)$ be a real structure for $A$. Given a symbol $\varepsilon \in \{+, -\}$, a $\mathcal{J}_\varepsilon$-structure for $X \in A$ is an isomorphism $j : X \rightarrow J(X)$ satisfying

$$J(j) \circ j = \varepsilon i_X : X \rightarrow JJ(X).$$

**Lemma 3.2.** Let $\mathcal{J} = (J, i)$ be a real structure for $A$. If $X \in A$ is a simple object, then one and only one of the following statements holds:
(1) $X$ is not isomorphic to $J(X)$.
(2) $X$ has a $\mathcal{F}_+$-structure.
(3) $X$ has a $\mathcal{F}_-$-structure.

**Proof.** It is trivial that if (1) holds, then both (2) and (3) do not. Now suppose that (1) does not hold. Let $f : X \to J(X)$ be an isomorphism. By Schur’s lemma, $J(f) \circ f = \alpha i_X$ for some $\alpha \in \mathbb{C}^\times$. By the definition of a real structure,

$$JJ(f) \circ f = JJ(f) \circ \alpha i_X = \alpha i_{J(X)} \circ f = \alpha (i_X) \circ f.$$ 

On the other hand, since the functor $J$ is anti-linear, we compute

$$JJ(f) \circ f = J(J(f) \circ f) = J(\alpha i_X) \circ f = \pi J(i_X) \circ f.$$ 

Hence $\alpha = \pi$ follows. Namely, $\alpha \in \mathbb{R}$. Now we set $j = |\alpha|^{-1/2} f$. Then $j$ is a $\mathcal{F}_+$-structure or a $\mathcal{F}_-$-structure for $X$ according to whether $\alpha > 0$ or $\alpha < 0$. Hence, at least either one of (2) or (3) holds.

To complete the proof, we show that (2) and (3) cannot occur at the same time. Suppose that (2) holds and fix a $\mathcal{F}_+$-structure $j$ for $X$. If $f : X \to J(X)$ is an isomorphism, then, by Schur’s lemma, $f = \lambda j$ for some $\lambda \in \mathbb{C}^\times$. Since

$$J(f) \circ f = J(\lambda j) \circ \lambda j = |\lambda|^2 J(j) \circ j = |\lambda|^2 i_X$$

and $|\lambda^2| > 0$, $f$ cannot be a $\mathcal{F}_-$-structure. Hence, (3) does not hold. In a similar way, we see that (3) implies the negation of (2). \qed

**Definition 3.3.** Let $X \in \mathcal{A}$ be a simple object. The $\mathcal{J}$-signature $\sigma_{\mathcal{J}}(X)$ is defined to be 0, +1 or −1 according to whether (1), (2) or (3) of Lemma 3.2 holds.

Let $\text{Irr}(\mathcal{A})$ denote the isomorphism classes of simple objects of $\mathcal{A}$. It is easy to see that $\sigma_{\mathcal{J}}(X) = \sigma_{\mathcal{J}}(Y)$ whenever $X \cong Y$. Hence, taking the $\mathcal{J}$-signature can be considered as a map

$$\sigma_{\mathcal{J}} : \text{Irr}(\mathcal{A}) \to \mathbb{Z}, \quad X \mapsto \sigma_{\mathcal{J}}(X).$$

**Lemma 3.4.** The function $\sigma_{\mathcal{J}}$ depends on the equivalence class of $\mathcal{J}$.

**Proof.** Let $\mathcal{J} = (J, i)$ and $\mathcal{J}' = (J', i')$ be real structures for $\mathcal{A}$ and suppose that there exists an isomorphism $\beta : \mathcal{J} \to \mathcal{J}'$ of real structures. Then

$$\sigma_{\mathcal{J}}(X) \neq 0 \iff X \cong J(X) \iff X \cong J'(X) \iff \sigma_{\mathcal{J}'}(X) \neq 0$$

for all $X \in \text{Irr}(\mathcal{A})$ by the definition of the signature. In particular,

$$(3.2) \qquad \sigma_{\mathcal{J}}(X) = \sigma_{\mathcal{J}'}(X)$$

holds if $\sigma_{\mathcal{J}}(X) = 0$. Thus we consider the case where $\epsilon := \sigma_{\mathcal{J}}(X) \neq 0$. Then there exists a $\mathcal{F}_\epsilon$-structure $j : X \to J(X)$ for $X$. By (3.1), the morphism

$$j' : X \longrightarrow J(X) \longrightarrow J'(X)$$

is a $\mathcal{F}_\epsilon'$-structure for $X$. Hence (3.2) holds also in this case. \qed
3.3. Real forms of an algebra. Let $A$ be an algebra over $\mathbb{C}$. By a real form of $A$, we mean an $\mathbb{R}$-subalgebra $A_0 \subset A$ such that $A = A_0 \oplus iA_0$. Suppose that a real form $A_0$ of $A$ is given. For $a \in A$, we define

$$\overline{a} = x - iy \quad (x, y \in A_0, a = x + iy)$$

and call $\overline{a}$ the conjugate of $a$ with respect to the real form $A_0$. It is easy to see that the map $\overline{\cdot} : a \mapsto \overline{a}$ is an anti-linear operator on $A$ such that

$$(a, b) \mapsto \overline{ab} = \overline{a} \cdot \overline{b}, \quad \overline{a} = a \quad (a, b \in A).$$

 Conversely, if an anti-linear operator $\overline{\cdot} : A \to A$ satisfies (3.3), then

$$A_0 = \{a \in A \mid \overline{a} = a\}$$

is a real form of $A_0$. Thus giving a real form of $A$ is equivalent to giving an anti-linear operator on $A$ satisfying (3.3).

Given a vector space $X$ over $\mathbb{C}$, we denote by $\overline{X}$ its complex conjugate; namely, $\overline{X} = X$ as an abelian group and the action of $\mathbb{C}$ on $\overline{X}$ is determined by

$$c \cdot \overline{a} = \overline{c} \cdot \overline{a} \quad (c \in \mathbb{C}, x \in X)$$

if we denote by $\overline{a}$ the element $x \in X$ regarded as an element of $\overline{X}$. Now let $A$ be an algebra over $\mathbb{C}$, and let $A_0 \subset A$ be a real form. If $X$ is a left $A$-module, then $\overline{X}$ is also a left $A$-module by the action determined by

$$\overline{a} \cdot \overline{x} = \overline{\overline{a} \cdot x} \quad (a \in A, x \in X).$$

Given a morphism $f : X \to Y$ in $\mathcal{AM}$, we define $\overline{f} : \overline{X} \to \overline{Y}$ in $\mathcal{AM}$ by

$$\overline{f(x)} = \overline{f(x)} \quad (x \in X).$$

We call $\overline{X}$ the conjugate of $X$ with respect to the real form $A_0$. Taking the conjugate of an $A$-module defines an anti-linear functor

$$(3.4) \quad \overline{\cdot} : \mathcal{AM} \to \mathcal{AM}, \quad X \mapsto \overline{X}.$$  

Moreover, there is a natural isomorphism $i : \text{id} \to \overline{\cdot} \circ \overline{\cdot}$ defined by

$$(3.5) \quad i_X : X \to \overline{X}, \quad i_X(x) = \overline{x} \quad (x \in X \in \mathcal{AM}).$$

The pair $\mathcal{J} = (\overline{\cdot}, i)$ is a real structure for $\mathcal{AM}$, which will be referred to as the real structure associated with the real form $A_0$.

We give representation-theoretic interpretations of the $\mathcal{J}$-signature and related notions. Let $V \in \mathcal{AM}$ and $\varepsilon \in \{+, -\}$. By the definition of $\overline{\cdot}$, a $\mathcal{J}_\varepsilon$-structure for $V$ is nothing but an anti-linear map $j : V \to V$ satisfying

$$(3.6) \quad j^2 = \varepsilon \text{id}_V, \quad j(a + b) = \overline{a}j(v) \quad (a, b \in A, v \in V).$$

By a real form of $V$ (with respect to the real form $A_0$), we mean an $A_0$-submodule $V_0 \subset V$ such that $V = V_0 \oplus iV_0$. The proof of the following lemma is omitted.

**Lemma 3.5.** Given a $\mathcal{J}_+\text{-structure} j$ for $V$,

$$(3.7) \quad V_0 = \{v \in V \mid j(v) = v\}$$

is a real form of $V$. Conversely, given a real form $V_0$ of $V$,

$$(3.8) \quad j : V \to V, \quad j(x + iy) = x - iy \quad (x, y \in V_0)$$

is a $\mathcal{J}_+$-structure for $V$. (3.7) and (3.8) establish a bijection between $\mathcal{J}_+$-structures for $V$ and real forms of $V$. 
The following characterization should be noted:

**Lemma 3.6.** Let \( V \in A \mathcal{M}_{fd} \) has a real form if and only if it admits a \( \mathbb{C} \)-basis \( B \) such that the matrix representation \( \rho : A \to M_n(\mathbb{C}) \) with respect to \( B \) has the following property: \( \rho(A_0) \subset M_n(\mathbb{R}) \).

**Proof.** If \( V \) has a real form \( V_0 \), then any \( \mathbb{R} \)-basis \( B \) of \( V_0 \) is a \( \mathbb{C} \)-basis of \( V \) having such a property. Conversely, if \( V \) admits such a basis \( B \), then the \( \mathbb{R} \)-subspace of \( V \) spanned by \( B \) is a real form of \( V \). \( \square \)

Let \( H = \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k \) be the quaternions, i.e., the algebra over \( \mathbb{R} \) generated by \( i, j \) and \( k \) with relations \( i^2 = j^2 = k^2 = ijk = -1 \). Set \( A_H := H \otimes \mathbb{R} A_0 \) and regard \( A (\cong \mathbb{C} \otimes \mathbb{R} A_0) \) as an \( \mathbb{R} \)-subalgebra of \( A_H \). Since \( ji = -ij \), we have \( j \cdot a = \overline{a} \cdot j \) in \( A_H \) for all \( a \in A \). This observation yields:

**Lemma 3.7.** Let \( V \in A \mathcal{M} \). An \( \mathbb{R} \)-linear map \( j : V \to V \) is a \( J \)-structure for \( V \) if and only if the following formula defines an action of \( A_H \):

\[
(a + jb) \cdot v = a + j(bv) \quad (a, b \in A, v \in V)
\]

Thus \( J \)-structures for \( V \) and \( A_H \)-module structures on \( V \) extending the \( A \)-module structure on \( V \) are in bijection.

Given \( V \in A \mathcal{M}_{fd} \), we denote by \( \chi_V : A \to \mathbb{C} \) the character of \( V \). Note that

\[
N \cong M \iff \chi_N = \chi_M
\]

whenever \( N, M \in A \mathcal{M}_{fd} \) are simple \( A \)-modules.

**Lemma 3.8.** Let \( V \in A \mathcal{M}_{fd} \). Then:

1. The character of \( V \) is given by \( \chi_{\overline{V}}(a) = \overline{\chi_V(a)} \) for \( a \in A \).
2. If \( V \) is simple, then \( V \cong \overline{V} \) is equivalent to \( \chi_V(A_0) \subset \mathbb{R} \).

**Proof.** (1) Fix a basis \( \{v_i\}_{i=1}^n \) of \( V \). Note that \( \{\overline{v_i}\}_{i=1}^n \) is a basis of \( \overline{V} \). Now let \( a \in A \) and suppose that the action of \( \overline{a} \) on \( V \) is expressed as

\[
\overline{a} \cdot v_j = c_{1j} v_1 + \cdots + c_{nj} v_n \quad (c_{ij} \in \mathbb{C}, i, j = 1, \ldots, n).
\]

Then \( a \cdot \overline{v_j} = \sum c_{ij} \overline{v_i} \) in \( \overline{V} \). Hence,

\[
\chi_{\overline{V}}(a) = \sum c_{ij} \overline{\chi_V(v_i)} = \overline{\chi_V(a)}.
\]

(2) Let \( V \in A \mathcal{M}_{fd} \) be a simple module. If \( a = x + iy \) \( (x, y \in A_0) \), then, by (1),

\[
\chi_V(a) = \chi_V(x) + i\chi_V(y), \quad \chi_{\overline{V}}(a) = \overline{\chi_V(x) - i\chi_V(y)}.
\]

From this, we see that \( \chi_V(A_0) \subset \mathbb{R} \) if and only if \( \chi_V = \chi_{\overline{V}} \). The latter statement is equivalent to \( V \cong \overline{V} \) since both \( V, \overline{V} \in A \mathcal{M}_{fd} \) are simple. \( \square \)

Now we give characterizations of the \( J \)-signature:

**Theorem 3.9.** Let \( V \in A \mathcal{M}_{fd} \) be a simple module. Then:

1. \( \sigma_J(V) = +1 \) if and only if \( V \) admits a basis such that the corresponding matrix representation \( \rho : A \to M_n(\mathbb{C}) \) has the following property:

\[
\rho(A_0) \subset M_n(\mathbb{R})
\]

2. \( \sigma_J(V) = -1 \) if and only if \( V \) does not admit such a basis but \( \chi_V(A_0) \subset \mathbb{R} \).

3. \( \sigma_J(V) = 0 \) if and only if \( \chi_V(A_0) \not\subset \mathbb{R} \).
Proof. (1) and (3) follow immediately from Lemmas 3.5, 3.6, and 3.8. Once (1) and (3) are proved, we find that the assertion (2) is equivalent to
\[ \sigma_f(V) = -1 \iff \sigma_f(V) \neq +1 \text{ and } \sigma_f(V) \neq 0. \]
This is obvious since \( \sigma_f(V) \in \{0, \pm 1\} \). Thus (2) is proved.

\[ \square \]

**Theorem 3.10.** For a simple module \( V \in \mathcal{AM}_d \),
\begin{equation}
\text{End}_{A_d}(V) \cong \begin{cases} 
M_2(\mathbb{R}) & \text{if } \sigma_f(V) = +1, \\
\mathbb{C} & \text{if } \sigma_f(V) = 0, \\
\mathbb{H} & \text{if } \sigma_f(V) = -1.
\end{cases}
\end{equation}

**Proof.** We first show that one and only one of the following holds:

(1) If \( V \) has a real form and \( E := \text{End}_{A_0}(V) \cong M_2(\mathbb{R}) \).

(2) \( V \) is simple as an \( A_0 \)-module and \( E \cong \mathbb{C} \).

(3) \( V \) is simple as an \( A_0 \)-module and \( E \cong \mathbb{H} \).

To see this, let \( V_0 \) be a simple \( A_0 \)-submodule of \( V \). Then, since \( V_0 + iV_0 \) is closed under the action of \( A_0 \), it is a \( A_0 \)-module. Hence, \( V_0 \) is simple as an \( A_0 \)-module, and since \( iV_0 \subset V \) is also an \( A_0 \)-submodule, \( V_0 \cap iV_0 \) is either one of \( V_0 \) or \( \{0\} \).

If the case is the former, then \( V = V_0 \) is simple as an \( A_0 \)-module. Hence \( E \) is isomorphic to either one of \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \) by Schur’s lemma and the classification of finite-dimensional division algebras over \( \mathbb{R} \). However, since the map
\[ i : V \to V, \quad v \mapsto iv \quad (v \in V) \]
is an element of \( E \) such that \( i^2 = -1 \), \( E \) cannot be isomorphic to \( \mathbb{R} \). Therefore either one of (1) or (3) holds.

If the case is the latter, then \( V = V_0 + iV_0 \) is a direct sum of \( V_0 \) and \( iV_0 \). Hence, in particular, \( V_0 \) is a real form of \( V \). Since \( iV_0 \cong V_0 \) as \( A_0 \)-modules,
\[ E = \text{End}_{A_0}(V_0 \oplus iV_0) \cong \text{End}_{A_0}(V_0 \oplus V_0) \cong M_2(D), \]
where \( D = \text{End}_{A_0}(V_0) \). Since \( V_0 \) is simple as an \( A_0 \)-module, \( D \) is isomorphic to either one of \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \). On the other hand,
\[ \text{End}_{A}(V) = \{ f \in \text{End}_{A_0}(V) \mid i \circ f = f \circ i \} \cong \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in D \right\} \]
via the above isomorphism. Recalling \( \text{End}_{A}(V) \cong \mathbb{C} \), we conclude that \( D \) must be isomorphic to \( \mathbb{R} \). Hence (1) holds.

Now we prove (3.9). If \( \sigma_f(V) = +1 \), then, by Lemma 3.5, \( V \) is not simple as an \( A_0 \)-module. Hence (1) is the only possibility. Conversely, if (1) holds, then \( \sigma_f(V) = +1 \) again by Lemma 3.8. Summarizing:
\begin{equation}
\sigma_f(V) = +1 \iff (1) \text{ holds.}
\end{equation}

If \( \sigma_f(V) = -1 \), then, by Lemma 3.7, \( E \) has \( \mathbb{H} \) as a subalgebra. Hence (3) is the only possibility. Suppose, conversely, that (3) holds. Fix an isomorphism \( \phi : E \to \mathbb{H} \) of \( \mathbb{R} \)-algebras. We put \( u = \phi(i) \) and define \( v \in \mathbb{H} \) to be a pure imaginary quaternion such that \( \langle u, v \rangle = \langle v, v \rangle = 1 \), where \( \langle , \rangle \) is given by
\[ \langle a_1 + a_2d + a_3k + b_1 + b_2i + b_3d + b_4k \rangle = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \]
for \( a_i, b_i \in \mathbb{R} \). Then one can verify that \( u^2 = v^2 = -1 \) and \( vu = -uv \). This implies that \( j = \phi^{-1}(v) \) is a \( \mathcal{J} \)-structure for \( V \). Hence \( \sigma_f(V) = -1 \). Summarizing:
\begin{equation}
\sigma_f(V) = -1 \iff (3) \text{ holds.}
\end{equation}
Finally, we consider the case where $\sigma_J(V) = 0$. Then, by (3.10) and (3.11), (E2) is the only possibility. Hence, $E \cong \mathbb{C}$. The proof is completed. \hfill \Box

We end this section by introducing the following terminologies:

**Definition 3.11.** Let $A$ be a $\mathbb{C}$-algebra with real form $A_0$, and let $J$ be the real structure for $A\text{M}_fd$ associated with $A_0$. Given a simple module $V \in A\text{M}_fd$, we say that $V$ is real, complex, and quaternionic with respect to $A_0$ if $\sigma_J(V)$ is equal to $+1$, $0$ and $-1$, respectively.

**4. Frobenius-Schur theorem for $C^*$-categories**

**4.1. $C^*$-categories.** A $*$-category is a $\mathbb{C}$-linear category $\mathcal{A}$ equipped with an anti-linear contravariant functor $*: \mathcal{A} \to \mathcal{A}$ such that $X^* = X$ for all object $X \in \mathcal{A}$ and $f^{**} = f$ for all morphism $f$ in $\mathcal{A}$. A $C^*$-category [12] is a $*$-category $\mathcal{A}$ satisfying the following three conditions:

1. $\text{Hom}_\mathcal{A}(X, Y)$ is a Banach space for all $X, Y \in \mathcal{A}$ and $\|fg\| \leq \|f\|\|g\|$ holds for all composable morphisms $f$ and $g$ in $\mathcal{A}$.
2. The $C^*$-identity $\|f^*f\| = \|f\|^2$ holds for all morphisms $f$ in $\mathcal{A}$.
3. For any morphism $f$ in $\mathcal{A}$, the morphism $f^*f$ is positive.

Note that, by (1) and (2), $\text{End}_\mathcal{A}(X)$ is a $C^*$-algebra for each $X \in \mathcal{A}$. In (3), that $f^*f$ is positive means that it is a positive element of the $C^*$-algebra $\text{End}_\mathcal{A}(X)$, where $X$ is the source of the morphism $f$.

We prepare some notations for $C^*$-algebras and $C^*$-categories. Given a Hausdorff space $X$, we denote by $C(X)$ the set of ($\mathbb{C}$-valued) continuous functions on $X$. Note that if $X$ is compact, then $C(X)$ is a $C^*$-algebra with the supremum norm. Now let, in general, $\mathcal{A}$ be a unital $C^*$-algebra. If $a \in A$ is a normal element, then there exists a unique unit-preserving $*$-homomorphism $\phi_a : C(\text{sp}(a)) \to A$, where $\text{sp}(a)$ is the spectrum of $a$, mapping the inclusion map $\text{sp}(a) \hookrightarrow \mathbb{C}$ to $a \in A$. For a subset $K \subset \mathbb{C}$ such that $\text{sp}(a) \subset K$, we consider the map

$$\hat{\phi}_a : C(K) \xrightarrow{\text{restriction}} C(\text{sp}(a)) \xrightarrow{\phi_a} A.$$

Given $f \in C(K)$, we write $f(a)$ for the element $\hat{\phi}_a(f) \in A$ (the continuous functional calculus). In particular, the following notation will be used: For a positive element $a \in A$ and $\lambda > 0$, we write $a^\lambda$ for $f(a)$ with $f_\lambda(t) = t^\lambda$ ($t \geq 0)$. If, moreover, $a$ is invertible, then $a^\lambda$ is defined for all $\lambda \in \mathbb{R}$ in a similar way.

Now let $\mathcal{A}$ be a $C^*$-category. A morphism $u$ in $\mathcal{A}$ is said to be unitary if it is invertible and $u^* = u^{-1}$. Given a morphism $f : X \to Y$ in $\mathcal{A}$, we define its absolute value by $|f| = (f^*f)^{1/2}$, $|f|$ is a positive element of the $C^*$-algebra $\text{End}_\mathcal{A}(X)$. The following lemma will be used extensively:

**Lemma 4.1.** $f[f]^{-1}$ is unitary and $f[f]^{-1} = |f^*|^{-1}f$.

Let $\mathcal{C}$ be an arbitrary category, let $F : \mathcal{C} \to \mathcal{A}$ be an arbitrary functor, and let $\xi : F \to F$ be a natural transformation such that $\xi_X : F(X) \to F(X)$ is positive for all $X \in \mathcal{C}$. Given a continuous function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$, we can define $f(\xi_X)$ for each $X \in \mathcal{C}$. If $f$ is a polynomial function, then the family

$$f(\xi) = \{f(\xi_X) : F(X) \to F(X)\}_{X \in \mathcal{C}}$$

is obviously a natural transformation $f(\xi) : F \to F$. By the Weierstrass approximation theorem, one can prove:
Lemma 4.2. \( f(\xi) \) is a natural transformation for all \( f \in C(\mathbb{R}_{\geq 0}) \).

In particular, \( \xi^{1/2} := \{ \xi_X^{1/2} \}_{X \in C} \) is.

4.2. Dual structures versus real structures. We have introduced the notions of dual structures and real structures for a \( \mathbb{C} \)-linear category. For a \( C^* \)-category, it would be reasonable to require them to be compatible with the \( * \)-structure:

Definition 4.3. Let \( \mathcal{F} = (F,u) \) be a dual structure or a real structure for a \( C^* \)-category \( A \). We say that \( \mathcal{F} \) is \( * \)-compatible if \( F \) is a \( * \)-functor (i.e., \( F(f^*) = F(f)^* \) for all morphism \( f \) in \( A \)) and \( u \) is unitary (i.e., \( u_X^* = u_X^{-1} \) for all \( X \in A \)).

Given a \( * \)-compatible real structure \( \mathcal{J} = (J,i) \) for \( A \), we put \( D = J^* \) and \( \eta = i \).

Since \( DD = J * J^* = JJ^* = JJ \), \( \eta \) is a natural isomorphism from \( \text{id}_A \) to \( DD \).

Moreover, \( D* = D\eta \), \( \eta^* = \eta^{-1} \), and

\[
D(\eta_X) \circ \eta_{D(X)} = J(i_X^*) \circ i_{J(X)} = J(i_X^*) = \text{id}_{J(X)} = \text{id}_{D(X)}
\]

for all \( X \in A \). In other words, the pair \( \mathbb{D}(\mathcal{J}) := (D,\eta) \) is a \( * \)-compatible dual structure for \( A \). Conversely, if \( D = (D,\eta) \) is a \( * \)-compatible dual structure for \( A \), then one can verify that the pair \( \mathcal{J}(D) = (D*,\eta^*) \) is a \( * \)-compatible real structure for \( A \). Since \( ** = \text{id}_A \), \( \mathbb{D} \) and \( \mathbb{J} \) are mutually inverse.

The main claim of this subsection is that the bijections \( \mathbb{D} \) and \( \mathbb{J} \) preserve the equivalence relations. Namely, if we denote by \( \mathcal{DS}^*(A) \) (resp. \( \mathcal{RS}^*(A) \)) the class of equivalence classes of \( * \)-compatible dual (resp. real) structures for \( A \), then:

Theorem 4.4. \( \mathbb{J} \to \mathbb{D}(\mathcal{J}) \) induces a well-defined bijection

\[
\mathbb{D} : \mathcal{RS}^*(A) \to \mathcal{DS}^*(A), \quad \mathcal{J} \mapsto [\mathbb{D}(\mathcal{J})]
\]

with the well-defined inverse

\[
\mathbb{J} : \mathcal{DS}^*(A) \to \mathcal{RS}^*(A), \quad [D] \mapsto [\mathbb{J}(D)].
\]

We provide two lemmas to prove this theorem. We say that real structures \( \mathcal{J} \) and \( \mathcal{J}' \) for \( A \) are unitary equivalent if there exists an isomorphism \( \beta : \mathcal{J} \to \mathcal{J}' \) of real structures such that \( \beta_X \) is unitary for all \( X \in A \).

Lemma 4.5. For two \( * \)-compatible real structures \( \mathcal{J} = (J,i) \) and \( \mathcal{J}' = (J',i') \) for \( A \), the following assertions are equivalent:

(1) \( \mathcal{J} \) and \( \mathcal{J}' \) are equivalent.
(2) \( \mathcal{J} \) and \( \mathcal{J}' \) are unitary equivalent.

Proof. The implication (2) \( \Rightarrow \) (1) is trivial. To prove (1) \( \Rightarrow \) (2), let \( \beta : \mathcal{J} \to \mathcal{J}' \) be an isomorphism of real structures. By Lemmas 4.1 and 4.2,

\[
u := \beta \circ |\beta|^{-1} = |\beta^*|^{-1} \circ \beta
\]

is a unitary natural isomorphism from \( \mathcal{J} \) to \( \mathcal{J}' \). We show that \( u \) is a morphism of real structures from \( \mathcal{J} \) to \( \mathcal{J}' \). Let \( X \in A \). By \( \mathbb{J} \) and the \( * \)-compatibility,

\[
\beta_{J(X)} \beta_{J'(X)} = J'(\beta_X)^{-1} (J'(\beta_X)^{-1})^* = J'(\beta_X^* \beta_X)^{-1}.
\]

Let, in general, \( a : V \to V \) be a morphism in \( A \). If \( a \geq 0 \), then \( a = a^* b \) for some \( b \in \text{End}_A(V) \) and thus \( J'(a) = J'(b) J'(b)^* \geq 0 \). Hence, by the uniqueness of the positive square root, \( J'(a^{1/2}) = J'(a)^{1/2} \). Applying this formula to \( J \), we
obtain $|\beta_{i(D)}^*| = J'(\beta_{X})^{-1}$. Now we show that $u : J \to J'$ is a morphism of real structures as follows:

$$J'(u_X)u_{J(X)}i_X = J'(\beta_{X})J'(|\beta_{X}|^{-1})|\beta_{J(X)}^*|^{-1}J'\beta_{J(X)}i_X = i_{\beta_{X}}'. \qedhere$$

Unitary equivalence of dual structures is defined in the same way as unitary equivalence of real structures. The proof of the following lemma is parallel to that of Lemma 4.5.

**Lemma 4.6.** For two $*$-compatible dual structures $D = (D, \eta)$ and $D' = (D', \eta')$ for $A$, the following assertions are equivalent:

1. $D$ and $D'$ are equivalent.
2. $D$ and $D'$ are unitary equivalent.

**Proof.** The implication $(2) \Rightarrow (1)$ is trivial. To prove $(1) \Rightarrow (2)$, let $\xi : D \to D'$ be an isomorphism. We show that the unitary natural isomorphism $$u := \xi \circ |\xi|^{-1} = |\xi^*|^{-1} \circ \xi : D \to D'$$

is a morphism $D \to D'$. Let $X \in A$. By (2.1) and the $*$-compatibility,

$$\xi_{D(X)} \circ \xi_{D'(X)}^* = D'\xi_{D(X)} \circ D'\xi_{D'(X)}^* = D'\xi_{D(X)} \circ \xi_{D(X)}.$$ 

Taking the positive square root of both sides, we obtain $|\xi_{D(X)}| = D'(|\xi_{X}|)$ (cf. the proof of Lemma 4.5). Hence,

$$u_{D(X)}\eta_X = |\xi_{D(X)}|^{-1}\xi_{D'(X)}\eta_X = D'(|\xi_{X}|)^{-1}D'(\xi_{X})\eta_X = D'(u_X)\eta_X. \quad \square$$

**Proof of Theorem 4.4.** Let $J = (J, i)$ and $J' = (J', i')$ be $*$-compatible real structures for $A$, and suppose that there exists an isomorphism $\beta : J \to J'$ of real structures. By Lemma 4.5, we may assume that $\beta$ is unitary. For simplicity, we put $(D, \eta) = D(J)$ and $(D', \eta') = D(J')$. Then $\beta$ is a natural isomorphism from $D$ to $D'$. Moreover, by (3.1),

$$D'(\beta_{X})\eta_X' = J'(\beta_{X}^*)i_X = J'(\beta_{X})^{-1}i_{\beta_{X}}' = \beta_{J(X)}i_X = \beta_{D(X)}\eta_X$$

for all $X \in A$. This means that $\beta : D(J) \to D(J')$ is an isomorphism of dual structures. Hence (4.1) is well-defined. In a similar way, Lemma 4.6 shows that (4.2) is well-defined. It is obvious that (4.1) and (4.2) are mutually inverse. \qed

### 4.3. Frobenius-Schur Theorem

Let $A$ be a locally finite-dimensional abelian $C^*$-category. In (3) we have introduced the Frobenius-Schur indicator $\nu_D : \text{Ob}(A) \to \mathbb{Z}$ with respect to a dual structure $D$ for $A$. By Lemma 2.3 the assignment $D \mapsto \nu_D$ induces a map

$$\nu : DS^*(A) \to \mathbb{Z}^{\text{Ob}(A)}, \quad [D] \mapsto \nu_D.$$

In (3) we have defined the $J$-signature $\sigma_J : \text{Irr}(A) \to \mathbb{Z}$ for each real structure $J$ for $A$. By Lemma 3.4, $J \mapsto \sigma_J$ induces a map

$$\sigma : RS^*(A) \to \mathbb{Z}^{\text{Irr}(A)}, \quad [J] \mapsto \sigma_J.$$

Now we consider the diagram

$$\begin{array}{ccc}
RS^*(A) & \xrightarrow{\sigma} & \mathbb{Z}^{\text{Irr}(A)} \\
\downarrow & & \uparrow \text{restriction} \\
DS^*(A) & \xrightarrow{\nu} & \mathbb{Z}^{\text{Ob}(A)},
\end{array}$$

where $\sigma : RS^*(A) \to \mathbb{Z}^{\text{Irr}(A)}$ is the Frobenius-Schur indicator and $\nu : DS^*(A) \to \mathbb{Z}^{\text{Ob}(A)}$ is the assignment $D \mapsto \nu_D$. The diagram commutes, and hence $\nu$ is a natural transformation. The Frobenius-Schur Theorem then follows from the commutativity of the diagram.
where \( \mathbb{D} : \mathcal{RS}(A) \to \mathcal{DS}(A) \) is the bijection of Theorem 4.3. We formulate the Frobenius-Schur theorem for \( C^* \)-categories as the commutativity of the above diagram. Namely, there holds:

**Theorem 4.7.** Let \( J = (J, i) \) be a \( * \)-compatible real structure for \( A \). If \( D = (D, \eta) \) is a \( * \)-compatible dual structure for \( A \) such that \( D \sim \mathbb{D}(J) \), then

\[
\sigma_J(x) = \nu_D(x)
\]

holds for all \( x \in \text{Irr}(A) \).

**Proof.** Let \( x \in \text{Irr}(A) \). By Lemma 2.3 we may assume \( D = \mathbb{D}(J) \). Then

\[
(4.5) \quad \nu_D(x) \neq 0 \iff x \cong D(x) \iff x \cong J(x) \iff \sigma_J(x) \neq 0.
\]

Here, the first equivalence follows from (2.3), the second from \( D(x) = J(x) \) (the equality of objects of \( A \)), and the last from the definition of \( \sigma_J \). In particular, (4.4) holds in the case where \( \sigma_J(x) = 0 \).

Now we consider the case where \( \varepsilon := \sigma_J(x) \neq 0 \). Put \( \varepsilon' := \nu_D(x) \). By (4.5), \( \varepsilon, \varepsilon' \in \{ \pm 1 \} \). Let \( j : X \to J(X) \) be a \( J \)-structure for \( X \). Since \( J(X) = D(x) \), \( j \) is a morphism from \( X \) to \( D(X) \). By (2.4) with \( f = j \),

\[
(4.6) \quad \varepsilon' \cdot j^* = (D(j)\eta_x)^* = \eta_x^* D(j)^* = \eta_x^* J(j)^* = \eta_x^* \varepsilon^* X(j).
\]

Hence we have

\[
(4.7) \quad \varepsilon \varepsilon' \cdot j^* = \varepsilon \cdot \varepsilon^{-1} X(j) = \varepsilon^2 \cdot \varepsilon^{-1} X = \text{id}_X.
\]

Since \( j^* \) and \( \text{id}_X \) are positive elements of \( \text{End}_A(X) \), \( \varepsilon \varepsilon' \geq 0 \). On the other hand, we have seen that \( \varepsilon, \varepsilon' \in \{ \pm 1 \} \). Therefore \( \varepsilon = \varepsilon' \), i.e., \( (4.4) \) holds.

\[\square\]

5. **Finite-dimensional \( C^* \)-algebras**

### 5.1. Conventions.

Let \( X \) and \( Y \) be Hilbert spaces. We denote by \( \mathcal{B}(X, Y) \) the set of all bounded linear operators from \( X \) to \( Y \). The following notations will be used:

\[
\mathcal{B}(X) := \mathcal{B}(X, X), \quad X^\vee = \mathcal{B}(X, \mathbb{C}).
\]

For \( x \in X \), we put \( \phi_x = \langle x|\cdot \rangle \), where \( \langle \cdot | \cdot \rangle \) is the inner product on \( X \). Our convention is that the inner product on a Hilbert space is anti-linear in the first variable and linear in the second. Thus \( \phi_x \in X^\vee \) for all \( x \in X \) and the map

\[
(5.1) \quad \phi_X : X \to X^\vee, \quad x \mapsto \phi_x \quad (x \in X)
\]

is anti-linear. The Riesz representation theorem states that \( \phi_X \) is bijective. Moreover, \( \phi_X \) is unitary if we define an inner product on \( X^\vee \) by

\[
(5.2) \quad \langle \phi_x | \phi_y \rangle_X = \langle y|x \rangle \quad (x, y \in X).
\]

The complex conjugate \( \overline{X} \) is also a Hilbert space with the inner product given by \( \langle x|y \rangle_{\overline{X}} = \overline{\langle y|x \rangle} \) for \( x, y \in X \). Let \( \mathcal{H} \) be the \( C^* \)-category of Hilbert spaces. The following category-theoretical interpretation of Riesz’s theorem will be important:

**Lemma 5.1.** Given \( X \in \mathcal{H} \), we denote by \( \varphi_X \) the map \( \phi_X : X \to X^\vee \) regarded as a linear map \( \overline{X} \to X^\vee \). Then \( \varphi = \{ \varphi_X \}_{X \in \mathcal{H}} \) is a natural isomorphism

\[
\varphi : - \circ * \to (-)^\vee,
\]

where \( * : \mathcal{H} \to \mathcal{H} \) is the functor taking the adjoint operator, \( - : \mathcal{H} \to \mathcal{H} \) is taking the complex conjugate and \( (-)^\vee : \mathcal{H} \to \mathcal{H} \) is taking the continuous dual.
Proof: We interpret our claim in terms of (5.1) and then find that the claim is equivalent to that the equation \( \phi_X \circ f^* = f^\vee \circ \phi_Y \) holds for all morphism \( f : X \to Y \) in \( \mathcal{H} \). This can be verified straightforward.

5.2. Real forms of a \(*\)-algebra. Let \( A \) be a \(*\)-algebra. By a \(*\)-representation of \( A \), we mean a Hilbert space \( X \) endowed with a \(*\)-homomorphism \( A \to B(X) \). This is the same thing as a Hilbert space \( X \) endowed with a left \( A \)-module structure such that \( x \mapsto ax \ (x \in X) \) is bounded for all \( a \in A \) and

\[
\langle x|ay \rangle = \langle a^*x|y \rangle \ (x, y \in X, a \in A).
\]

We denote by \( \text{Rep}(A) \) the category whose objects are \(*\)-representations of \( A \) and whose morphisms are bounded \( A \)-linear maps between them. \( \text{Rep}_{fd}(A) \) denotes the full subcategory of \( \text{Rep}(A) \) consisting of finite-dimensional objects.

**Proposition 5.2.** \( \text{Rep}(A) \) and \( \text{Rep}_{fd}(A) \) are \( C^\ast \)-categories.

Recall from [3.3] that a real form of \( A \) is an \( \mathbb{R} \)-subalgebra \( A_0 \subset A \) such that \( A = A_0 \oplus iA_0 \). Given a real form \( A_0 \) of \( A \), we define \( S : A \to A \) by

\[
S(a) = (\overline{a})^* \quad (a \in A),
\]

where \( \overline{a} \) is the conjugate of \( a \in A \) with respect to \( A_0 \). Since \( \overline{a} \) and \( * \) are anti-linear maps, \( S \) is linear. Moreover, by (5.3), we obtain:

\[
S(ab) = S(b)S(a), \quad S((S(a))^*) = a \quad (a, b \in A).
\]

Following, we refer the map \( S \) as the anti-algebra map associated with the real form \( A_0 \). Conversely, if such a linear map \( S \) is given, then

\[
A_0 = \{ a \in A \mid \overline{a} = a \}, \quad \text{where} \overline{a} = S(a)^*,
\]

is a real form of \( A \), which will be referred to as the real form associated with \( S \). It is easy to see that (5.3) and (5.5) establish a bijection between real forms of \( A \) and linear maps \( S \) satisfying (5.4).

Note that we do not require a real form \( A_0 \) to be closed under the \(*\)-operation of \( A \); if \( S \) is the anti-algebra map associated with a real form \( A_0 \), then:

\[
a^* \in A_0 \text{ for all } a \in A_0 \iff S \circ * = * \circ S \iff S^2 = \text{id}_A.
\]

We say that \( x \in A \) is positive\(^1\) if \( x = a^*a \) for some \( a \in A \). Now we suppose that there exists a positive element \( g \in A \) such that the pair \( (S, g) \) is a dual structure for \( A \) in the sense of (2.3). For example, we can choose \( g \) to be 1 if one of the equivalent conditions of (5.6) is satisfied.

Let \( X \in \text{Rep}(A) \). Then its continuous dual \( X^\vee \) has a left \( A \)-module structure given by the same formula as (2.3). With respect to this action,

\[
\phi_{\pi x} = a \cdot \phi_x \quad (a \in A, x \in X).
\]

In general, \( X^\vee \) is not a \(*\)-representation of \( A \) with respect to the standard inner product given by (5.2). Following, we define \( D(X) \) to be the left \( A \)-module \( X^\vee \) with the inner product given by

\[
\langle \phi_x | \phi_y \rangle_{D(X)} = \langle y | gx \rangle \quad (x, y \in X).
\]

\(^1\)Since we do not assume \( A \) to be a \( C^\ast \)-algebra, we should clarify the meaning of the positivity of an element of \( A \).
The map \( \phi_X : X \to D(X) \) is not unitary in general but is still bounded. By (5.7), one can check that \( D(X) \) is a \( * \)-representation of \( A \). The assignment \( X \mapsto D(X) \) defines a contravariant endofunctor on \( \text{Rep}(A) \). Now we define

\[
\eta_X : X \to DD(X), \quad \langle \eta_X(x), \lambda \rangle = \langle \lambda, gx \rangle \quad (x \in X, \lambda \in X^\vee)
\]

**Lemma 5.3.** \( D = (D, \eta) \) is a \( * \)-compatible dual structure for \( \text{Rep}(A) \).

**Proof.** It is obvious that \( D \) is a dual structure for \( \text{Rep}(A) \) (cf. (2.3)). We show that it is \( * \)-compatible. If \( f : X \to Y \) is a morphism in \( \text{Rep}(A) \), then

\[
\langle f^\vee(x), \phi_y \rangle_{D(Y)} = \langle \phi_x | f^\vee(\phi_y) \rangle_{D(Y)} = \langle \phi_x | \phi_{f^*(y)} \rangle = \langle f^*(y) | g x \rangle
\]

\[
= \langle y | f(gx) \rangle = \langle y | \phi_{f^**}(x) \rangle = \langle \phi_y | f^**(\phi_x) \rangle
\]

for all \( x \in X \) and \( y \in Y \). This means that \( f^\vee = f^* \). To show that \( \eta \) is unitary, we note that \( \eta \) is expressed as

\[
\eta_X = \phi_{D(X)} \circ \phi_X.
\]

By (5.7) and (5.8), we can show that the bijection \( \eta_X : X \to DD(X) \) preserves the inner product as follows: For \( x, x' \in X \),

\[
\langle \eta_X(x)|\eta_X(x') \rangle_{DD(X)} = \langle \phi_x | \phi_{x'} \rangle_{D(X)} = \langle \phi_{x'}, g x \rangle = \langle x' | g x \rangle = \langle x' | x \rangle. \quad \square
\]

In §3.3 we have introduced the notion of the conjugate \( A \)-module \( \overline{X} \) with respect to the real form \( A_0 \). Given \( X \in \text{Rep}(A) \), we define \( J(X) \) to be the left \( A \)-module \( \overline{X} \) with the inner product given by \( \langle x | y \rangle_{J(X)} = \langle y | gx \rangle \) (\( x, y \in X \)).

Equation (3.3) means that the map \( \varphi_X \) of Lemma 5.1 induces a unitary equivalence of dual structures, i.e.,

\[
\varphi_{J(X)} \circ i_X = D(\varphi_X) \circ \eta_X : X \to DJ(X)
\]

holds for all \( X \in \text{Rep}(A) \). For \( x, y \in X \),

\[
\langle \varphi_{J(X)} i_X(x), y \rangle = \langle \varphi_{J(X)}(x), y \rangle = \langle x | y \rangle_{J(X)} = \langle y | gx \rangle.
\]

\[
\langle D(\varphi_X) \eta_X(x), y \rangle = \langle \eta_X(x), \varphi_X(y) \rangle = \langle \varphi_X(y), gx \rangle = \langle y | gx \rangle.
\]

\[\text{(5.9)}\]

Hence (5.9) is verified. \( \square \)

By Theorem (1.7) and Lemma 5.5 we obtain the following result: If \( V \) is a finite-dimensional irreducible \( * \)-representation of \( A \), then

\[
\nu_D(V) = \sigma_J(V).
\]

\[\text{(5.10)}\]
Theorem 5.7 also implies that the $\mathcal{J}$-signature is equal to the Frobenius-Schur indicator with respect to the dual structure $\mathbb{D}(\mathcal{J})$. Unlike $\mathbb{D}(\mathcal{J})$, the dual structure $\mathcal{D}$ does not have an anti-linear part and is easy to deal with.

Remark 5.6. The following direct proof of (5.10) can be obtained by interpreting the proof of Theorem 4.7 in our context: First observe that

$$\sigma_{\mathcal{J}}(V) \neq 0 \iff V \cong J(V) \iff V \cong D(V) \iff \nu_{\mathcal{D}}(V) \neq 0$$

(cf. (4.3) in the proof of Theorem 4.7). Hence (5.10) is proved in the case where $\varepsilon := \sigma_{\mathcal{J}}(V) = 0$. Now consider the case where $\varepsilon \neq 0$. Then there exists an anti-linear map $j : V \to V$ satisfying (5.6). By using $j$, we define

$$\beta(v,w) = \langle j(v)|w \rangle \quad (v,w \in V).$$

$\beta$ is a non-degenerate bilinear form on $V$ such that $\beta(av, w) = \beta(v,S(a)w)$ for all $v,w \in V$. Hence, by [24, Theorem 3.4], $\beta(w,gv) = \varepsilon' \cdot \beta(v,w)$ for all $v,w \in V$, where $\varepsilon' := \nu_{\mathcal{D}}(V)$. Now we compute:

$$\langle w|gv \rangle = \varepsilon \cdot \langle j^2(w)|gv \rangle = \varepsilon \cdot \beta(j(w),gv) = \varepsilon \cdot \beta(v,j(w)) = \varepsilon \varepsilon' \cdot \langle j(v)|j(w) \rangle$$

for $v,w \in V$ (cf. (4.4) and (4.7)). Recall that $g = a^*a$ for some $a \in A$. For any non-zero element $v \in V$, we have:

$$\varepsilon \varepsilon' = \varepsilon \varepsilon' \cdot \|j(v)\| \cdot \|j(v)\|^{-1} = \langle v|gv \rangle \cdot \|j(v)\|^{-1} \geq 0.$$

Since $\varepsilon, \varepsilon' \in \{\pm 1\}$, $\varepsilon = \varepsilon'$ follows. Thus (5.10) is proved.

Remark 5.7 (Lifting problem). The positive element $g \in A$ plays a key role to define $\mathcal{J}$ and $\mathcal{D}$. Such an element does not always seem to exists. If existence of $g$ is not guaranteed, then we consider $X \mapsto \overline{X}$ as a functor

$$J_0 : \text{Rep}(A) \to \mathcal{AM}, \quad X \mapsto \overline{X}$$

since we do not know how to make $\overline{X}$ into a $*$-representation of $A$. By the same reason, we consider $X \mapsto X^\vee$ as a contravariant functor

$$D_0 : \text{Rep}(A) \to \mathcal{AM}, \quad X \mapsto X^\vee.$$

Let, in general, $F : \text{Rep}(A) \to \mathcal{AM}$ be a (contravariant) functor. By a lift of $F$ on a full subcategory $\mathcal{C} \subset \text{Rep}(A)$, we mean a (contravariant) endo-$*$-functor $\tilde{F}$ on $\mathcal{C}$ such that $U \circ \tilde{F} = F|_C$, where $U : \mathcal{C} \to \mathcal{AM}$ is the functor forgetting the inner product. Under some technical assumptions, we can extend a lift of $D_0$ on $\mathcal{C}$ to a $*$-compatible dual structure for $\mathcal{C}$ (Proposition A.1). In view of this fact, it would be important to study when a lift of $D_0$ exists. We discuss this problem in Appendix A.

5.3. Frobenius-Schur theorem for finite-dimensional $C^*$-algebras. We apply our results to finite-dimensional $C^*$-algebras. The following proposition is due to Böhm, Nill and Szlachányi [3].

Proposition 5.8. Let $A$ be a finite-dimensional $C^*$-algebra. For each linear map $S : A \to A$ satisfying (5.4), there uniquely exists an invertible positive element $g \in A$ satisfying the following two conditions:

1. $S^2(a) = gag^{-1}$ for all $a \in A$.
2. $\chi_V(g) = \chi_V(g^{-1}) > 0$ for all simple left $A$-module $V$.

Moreover, the element $g$ fulfills:

3. $S(g) = g^{-1}$. 

Namely, an element \( g \) as in §5.2 always exists and, moreover, can be chosen in a canonical way. In Appendix A, we will give another proof of Proposition 5.8 from the viewpoint of the lifting problem mentioned in Remark 5.7.

Now we give the Frobenius-Schur theorem for finite-dimensional \( C^* \)-algebras in a form as general as possible:

**Theorem 5.9.** Let \( A \) be a finite-dimensional \( C^* \)-algebra, let \( A_0 \) be a real form of \( A \), and let \( E = \sum_i E_i' \otimes E_i'' \) be a separability idempotent of \( A \). Given a simple left \( A \)-module \( V \), we set

\[
\nu(V) = \sum_i \chi_V(S(E_i')gE_i''),
\]

where \( S : A \to A \) is the anti-algebra map associated with \( A_0 \) and \( g \in A \) is the element given by Proposition 5.8. Then

\[
\nu(V) = \begin{cases} 
+1 & \text{if } V \text{ is real,} \\
0 & \text{if } V \text{ is complex,} \\
-1 & \text{if } V \text{ is quaternionic with respect to the real form } A_0.
\end{cases}
\]

**Proof.** We may assume that \( V \) is a \( * \)-representation since, by the classification theorem of finite-dimensional \( C^* \)-algebras, there exists an isomorphism

\[
(5.12) \quad A \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})
\]

of \( C^* \)-algebras for some \( n_1, \ldots, n_r \).

Define \( D \) and \( J \) as in §5.2 by using the element \( g \). By Theorem 2.5, the left-hand side of (5.11) is equal to the Frobenius-Schur indicator \( \nu_D(V) \). On the other hand, by definition, the right-hand side of (5.11) is equal to the \( J \)-signature \( \sigma_J(V) \). Thus (5.11) follows from (5.10). \( \square \)

**Remark 5.10.** Since we will deal with examples which are not \emph{a priori} a \( C^* \)-algebra, we mention the following characterization: For a finite-dimensional \( * \)-algebra \( A \), the following assertions are equivalent:

1. \( A \) admits a norm making it into a \( C^* \)-algebra.
2. \( A \) has a separability idempotent of the form \( E = \sum_i a_i^* \otimes a_i \), \( a_i \in A \).
3. Any finite-dimensional left \( A \)-module admits an inner product making it into a \( * \)-representation.
4. \( A \) has a faithful \( * \)-representation.

To show (1) \( \Rightarrow \) (2), let \( e^{(k)}_{ij} \in A \) be the element corresponding to the \((i,j)\)-th matrix unit of the \( k \)-th component \( M_{n_k}(\mathbb{C}) \) via the isomorphism (5.12). Then it is straightforward to show that the element

\[
E = \sum_{k=1}^r \sum_{i,j=1}^{n_k} \frac{1}{\sqrt{n_k}} e^{(k)}_{ij} \otimes \frac{1}{\sqrt{n_k}} e^{(k)}_{ji} \in A \otimes_{\mathbb{C}} A
\]

is a separability idempotent of the desired form.

The implication (2) \( \Rightarrow \) (3) is shown as follows: Let \( E = \sum a_i^* \otimes a_i \) be a separability idempotent of such form. Given \( V \in AM_{Jd} \), we fix an inner product \( \langle \cdot | \cdot \rangle_0 \) on \( V \) and then define

\[
\langle v | w \rangle = \sum_{i,j} \langle a_i v | a_j w \rangle_0 \quad (v, w \in V).
\]

This makes \( V \) into a \( * \)-representation. Thus (3) follows. To show (3) \( \Rightarrow \) (4), make the left regular representation of \( A \) into a \( * \)-representation. To show (4) \( \Rightarrow \) (1),
fix a faithful $*$-representation $X$ of $A$ and then realize $A$ as a $*$-subalgebra of the $C^*$-algebra $B(X)$. The operator norm makes $A$ into a $C^*$-algebra.

**Remark 5.11** (cf. Doi [9, §1]). In applications, it is often a problem how to find a separability idempotent. Let $A$ be a finite-dimensional $C^*$-algebra. If we are given an inner product $(\langle \rangle)$ on $A$ making it into a $*$-representation of $A$, then we can express a separability idempotent of $A$ as follows: First, fix an orthonormal basis $\{e_i\}_{i=1}^m$ of $A$ with respect to $(\langle \rangle)$. Consider the linear map

$$\Theta : A \otimes A \to \text{End}_C(A), \quad \Theta(a \otimes b)(x) = \langle b^* \rangle a \cdot x \quad (a, b, x \in A).$$

Since $(\langle \rangle)$ is non-degenerate, $\Theta$ is bijection. Thus we have

$$\sum_{i=1}^m a e_i \otimes e_i^* = \sum_{i=1}^m e_i \otimes e_i^* a \quad (a \in A),$$

since $\Theta(\sum_{i=1}^m a e_i \otimes e_i^*)(x) = ax = \Theta(\sum_{i=1}^m e_i \otimes e_i^* a)(x)$ for all $x \in A$. Now we set $v = \sum_{i=1}^m e_i e_i^*$. It is obvious that $v$ is positive and central. Moreover, $v$ is invertible by [9, Theorem 1.5]. By (5.13), we see that

$$E = \sum_{i=1}^m e_i \otimes e_i^* v^{-1} = \sum_{i=1}^m a_i^* \otimes a_i,$$

where $a_i = e_i^* v^{-1/2}$,

is a separability idempotent.

**5.4. Example I. Weak Hopf $C^*$-algebras.** A weak Hopf algebra [4, 5] is an algebra $A$ over $\mathbb{C}$, which is a coalgebra $(A, \Delta, \varepsilon)$ over $\mathbb{C}$ at the same time, satisfying numerous axioms. To denote the comultiplication, we use Sweedler’s notation:

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \quad (a \in A).$$

We omit the detailed definition but note that a weak Hopf algebra $A$ has a special linear map $S : A \to A$ called the antipode. It is known that the antipode is an anti-algebra map and an anti-coalgebra map:

$$S(ab) = S(b)S(a), \quad S(a_{(1)}) \otimes S(a_{(2)}) = S(a_{(2)}) \otimes S(a_{(1)}) \quad (a, b \in A).$$

A weak Hopf $*$-algebra [4, §4.1] is a weak Hopf algebra $A$ such that whose underlying algebra is a $*$-algebra and there holds

$$(a^*)_1 \otimes (a^*)_2 = (a_{(1)})^* \otimes (a_{(2)})^* \quad (a \in A).$$

If $A$ is a weak Hopf $*$-algebra, then $\varepsilon(a^*) = \overline{\varepsilon(a)}$ and $S(S(a)^*)^* = a$ for all $a \in A$.

In particular, $S$ satisfies [5,4]. The corresponding real form

$$A_0 := \{a \in A \mid S(a)^* = a\}$$

will be referred to as the canonical real form of $A$.

Now let $A$ be a finite-dimensional weak Hopf $C^*$-algebra [4, §4.1], i.e., a weak Hopf algebra whose underlying $*$-algebra is finite-dimensional and satisfies the conditions of Remark 5.11. Since the antipode $S$ of $A$ satisfies (5.3), $A$ has a unique element $g$ satisfying the conditions of Proposition 5.3. The element $g$ is called the canonical grouplike element [4, §4.3] since it satisfies

$$1_{(1)} g \otimes 1_{(2)} g = \Delta(g) = g 1_{(1)} \otimes g 1_{(2)},$$

where $1_{(1)} \otimes 1_{(2)} = \Delta(1_A)$. Define $\varepsilon_L, \varepsilon_R : A \to A$ by

$$\varepsilon_L(a) = \varepsilon(1_{(1)}a)1_{(2)}, \quad \varepsilon_R(a) = \varepsilon(1_{(2)}a)1_{(1)} \quad (a \in A).$$
A Haar integral \([4, §3.6]\) of \(A\) is an element \(\Lambda \in A\) such that
\[
\varepsilon_L(\Lambda) = 1 = \varepsilon_R(\Lambda), \quad a\Lambda = \varepsilon_L(a)\Lambda, \quad \Lambda a = \Lambda \varepsilon_R(a) \quad (a \in A).
\]
Existence and uniqueness of a Haar integral is proved in \([4, §4.2]\). Now let \(\Lambda\) be the Haar integral of \(A\). In the above notations, we propose the following Frobenius-Schur theorem for weak Hopf \(C^*\)-algebras:

**Theorem 5.12.** For a simple left \(A\)-module \(V\),
\[
\chi_V(\Lambda(1)\Lambda(2)) = \frac{\chi_V(1)}{\chi_V(g)} \begin{cases} 
+1 & \text{if } V \text{ is real}, \\
0 & \text{if } V \text{ is complex}, \\
-1 & \text{if } V \text{ is quaternionic}
\end{cases}
\]
with respect to the canonical real form \(A_0 = \{ a \in A \mid S(a)^* = a \}\).

Note that \(\chi_V(1) = \dim_{\mathbb{C}}(V) > 0\). By Remark \(5.10\) we may assume that \(V\) is a \(*\)-representation. Hence, since \(g\) is positive and invertible, \(\chi_V(g) > 0\). From this theorem, we obtain the following result not involving \(g\): The left-hand side of (5.14) is positive, zero or negative according to whether \(V\) is real, complex or quaternionic with respect to the canonical real form.

One can prove Theorem 5.12 by applying Theorem 5.9 to the canonical real form of \(A\). We omit the detail since Theorem 5.12 is a special case of the following twisted version of Theorem 5.12. By an automorphism of \(A\), we mean an automorphism \(\tau: A \to A\) of the underlying algebra such that
\[
\tau(a^*) = \tau(a)^*, \quad \tau(a(1)) \otimes \tau(a(2)) = \tau(a(1)) \otimes \tau(a(2)) \quad (a \in A).
\]
If \(\tau\) is an automorphism of \(A\), then
\[
\varepsilon \circ \tau = \varepsilon, \quad S \circ \tau = \tau \circ S, \quad \tau(g) = g, \quad \tau(\Lambda) = \Lambda
\]
by their uniqueness. Now let \(\tau\) be an involution of \(A\), i.e., an automorphism of \(A\) such that \(\tau^2 = \text{id}_A\). Then:

**Theorem 5.13.** For a simple left \(A\)-module \(V\), we set
\[
\sigma_\tau(V) = \frac{\chi_V(g)}{\chi_V(1)} \chi_V(\Lambda(1)\Lambda(2)).
\]
Then \(\sigma_\tau(V) \in \{0, \pm 1\}\). Moreover, \(\sigma_\tau(V)\) has the following meaning:

1. \(\sigma_\tau(V) = +1\) if and only if \(V\) admits a basis such that the matrix representation \(\rho: A \to M_n(\mathbb{C})\) with respect to that basis fulfills:
\[
\rho(\tau(a)) = \overline{\rho(a)} \quad \text{for all } a \in A_0.
\]

2. \(\sigma_\tau(V) = -1\) if and only if \(V\) does not admit such a basis but satisfies
\[
\chi_V(\tau(a)) = \overline{\chi_V(a)} \quad \text{for all } h \in A_0.
\]

3. \(\sigma_\tau(V) = 0\) if and only if \(\chi_V(\tau(a)) \neq \chi_V(a)\) for some \(a \in A_0\).

Applying this theorem to the group algebra of a finite group, we obtain the result of Kawanaka and Matsuyama \([14]\). This theorem is a generalization of their result to weak Hopf \(C^*\)-algebras (cf. Sage and Vega \([22]\)).
Proof. Since \( S^* := \tau \circ S \) is a linear map satisfying (5.14),
\[
A_0^\tau := \{a \in A \mid S^*(a)^\ast = a\}
\]
is a real form. We define \( \tilde{\sigma}_r(V) \) to be \(+1\), \(0\) or \(−1\) according to whether \( V \) is real, complex or quaternionic with respect to the real form \( A_0^\tau \). By the definition of the Haar integral, we have
\[
(5.15) \quad \Lambda(1) \otimes a \Lambda(2) = S(a) \Lambda(1) \otimes \Lambda(2), \quad \Lambda(1) \otimes \Lambda(2) = \Lambda(1) \otimes \Lambda(2) S(a)
\]
for all \( a \in A \) [4 Lemma 3.2]. This implies that \( E = S(\Lambda(1)) \otimes \Lambda(2) \) is a separability idempotent of \( A \). Applying Theorem 5.9 to \( A_0^\tau \), we obtain
\[
(5.16) \quad \tilde{\sigma}_r(V) = \frac{\chi_V(S^* S(\Lambda(1)) g \Lambda(2))}{\chi_V(\Lambda(1))} = \sigma_r(V).
\]

Theorem 5.12 follows immediately from (5.16) with \( \tau = \text{id}_A \). For general cases, we need to interpret (5.16) in terms of the canonical real form \( A_0 \). For this purpose, let, in general, \( F : A \to M_n(\mathbb{C}) \) be a \( \mathbb{C} \)-linear map. Then:
\[
(5.17) \quad F(A_0^\tau) \subset M_n(\mathbb{R}) \iff F(\tau(z)) = F(z) \quad \text{for all} \ z \in A_0.
\]

Indeed, suppose that \( F(A_0^\tau) \subset M_n(\mathbb{R}) \). For \( a \in A \), we set \( x = 2^{-1}(a + \tau(\overline{a})) \) and \( y = (2i)^{-1}(a - \tau(\overline{a})) \), where \( \overline{a} = S(a)^\ast \). It is easy to check that \( x, y \in A_0^\tau \), \( a = x + iy \) and \( \tau(\overline{a}) = x - iy \). Hence, if \( a \in A_0 \), then:
\[
F(\tau(a)) = F(\tau(\overline{a})) = F(x) - iF(y) = \overline{F(x) + iF(y)} = \overline{F(a)}.
\]

Conversely, suppose that \( F(\tau(z)) = F(z) \) for all \( z \in A_0 \). Let \( a \in A \) and write it as \( a = x + iy \ (x, y \in A_0) \). If \( a \in A_0 \), then:
\[
F(a) = F(\tau(\overline{a})) = F(x) - iF(y)
\]
\[
= \overline{F(x)} + i \cdot \overline{F(y)} = F(x) + iF(x) = F(a).
\]

Now (1) is proved as follows: By (5.16), \( \sigma_r(V) = +1 \) if and only if \( V \) admits a basis such that the matrix representation \( \rho : A \to M_n(\mathbb{C}) \) with respect to that basis has the following property:
\[
(5.18) \quad \rho(A_0^\tau) \subset M_n(\mathbb{R}).
\]

By (5.17) with \( F = \rho \), we see that (5.18) is equivalent to that \( \rho(\tau(a)) = \overline{\rho(a)} \) holds for all \( a \in A_0 \). Hence (1) follows. To show (2) and (3), use (5.17) with \( F = \chi_V \). \qed

5.5. Example II. Table algebras. Let \( A \) be a finite-dimensional \(*\)-algebra with basis \( B = \{b_i\}_{i \in I}, \) where \( I \) is an index set. We suppose that \( B \) is closed under the \(*\)-operation; thus, for each \( i \in I \), we can define \( i^* \in I \) by \( (b_i)^* = b_{i^*} \). For \( i, j, k \in I \), we define \( p_{ij}^k \in \mathbb{C} \) by
\[
b_i \cdot b_j = \sum_{k \in I} p_{ij}^k b_k \quad (i, j \in I).
\]
The pair \( (A, B) \) is called a table algebra if the following conditions are satisfied:

- \( (T0) \quad 1_A \in B; \) the element \( i \in I \) such that \( b_i = 1_A \) will be denoted by \( 0 \).
- \( (T1) \quad p_{ij}^j \in \mathbb{R} \) for all \( i, j, k \in I \).
- \( (T2) \quad p_{ij}^{i^*} = p_{i^*j}^0 > 0 \) and \( p_{ij}^0 = 0 \) whenever \( i \neq j^* \).
Note that in literature the word ‘table algebra’ has been used with several different meanings, see [3]. Our definition is equivalent to [3, Definition 1.16]. One can find many examples of table algebras in [3] and references therein. The Bose-Mesner algebra of an association scheme is an important example.

Now suppose that \((A,B)\) is a table algebra. By (T1), \(A_0 = \text{span}_R(B)\) is a real form of \(A\), which we call the canonical real form of \((A,B)\). Let \(\phi : A \rightarrow \mathbb{C}\) be the linear map determined by \(\phi(b_i) = \delta_{i0}\). By (T2), the sesquilinear form
\[
\langle a|b \rangle = \phi(a^*b) \quad (a, b \in A)
\]
is an inner product on \(A\) making it into a \(*\)-representation \([3, \S 2]\). Note that \(B\) is orthogonal (but not normal in general) with respect to this inner product. Hence, by the arguments of Remark 5.11.

\[
E = \sum_{i \in I} \frac{1}{p_{ii}^0} b_i^* \otimes b_i v^{-1}, \quad \text{where} \quad v = \sum_{i \in I} \frac{1}{p_{ii}^0} b_i^* b_i,
\]
is a separability idempotent of \(A\). Now we formulate the Frobenius-Schur theorem for table algebras as follows:

**Theorem 5.14.** For a simple left \(A\)-module \(V\),

\[
\sum_{i \in I} \frac{1}{p_{ii}^0} \chi_V(b_i^2) = \frac{\chi_V(v)}{\chi_V(1)} \times \begin{cases} +1 & \text{if } V \text{ is real,} \\ 0 & \text{if } V \text{ is complex,} \\ -1 & \text{if } V \text{ is quaternionic} \end{cases}
\]

with respect to the canonical real form \(A_0 = \text{span}_R(B)\).

As we have seen in Remark 5.11, \(\chi_V(v) > 0\). Hence, in particular, the left-hand side of (5.20) is positive, zero or negative according to whether \(V\) is real, complex or quaternionic with respect to \(A_0\).

Instead of proving Theorem 5.14, we prove the following twisted version: By an involution of the table algebra \((A,B)\), we mean an automorphism \(\tau\) of the \(*\)-algebra \(A\) such that \(\tau(B) \subset B\) and \(\tau^2 = \text{id}_A\). Now let \(\tau\) be an involution of \((A,B)\). For each \(i \in I\), we define \(\tau(i) \in I\) by \(b_{\tau(i)} = \tau(b_i)\).

**Theorem 5.15.** For a simple left \(A\)-module \(V\), we set

\[
\sigma_\tau(V) = \frac{\chi_V(1)}{\chi_V(v)} \sum_{i \in I} \frac{1}{p_{ii}^0} \chi_V(b_{\tau(i)} b_i).
\]

Then \(\sigma_\tau(V) \in \{0, \pm 1\}\). Moreover, \(\sigma_\tau(V)\) has the following meaning:

1. \(\sigma_\tau(V) = +1\) if and only if \(V\) admits a basis such that the matrix representation \(\rho : A \rightarrow M_n(\mathbb{C})\) with respect to that basis fulfills:
   \[
   \rho(b_{\tau(i)}) = \overline{\rho(b_i)} \quad \text{for all } i \in I.
   \]
2. \(\sigma_\tau(V) = -1\) if and only if \(V\) does not admit such a basis but satisfies:
   \[
   \chi_V(b_{\tau(i)}) = \overline{\chi_V(b_i)} \quad \text{for all } i \in I.
   \]
3. \(\sigma_\tau(V) = 0\) if and only if \(\chi_V(b_{\tau(i)}) \neq \overline{\chi_V(b_i)}\) for some \(i \in I\).

If \(G\) is a finite group, then the pair \((CG,G)\) is a table algebra. Applying this theorem, we again obtain the result of Kawanaka and Matsuyama [14]. Thus this theorem is another generalization of their result.
Given a real form $C_t := \{ a \in A \mid S^\tau(a)^* = a \}$ is a real form of $A$. We define $\tilde{\sigma}(V)$ to be $+1$, $0$ or $-1$ according to whether $V$ is real, complex or quaternionic with respect to the real form $A_0$. Note that the element $v$ is central. By Theorem 5.9 with $E$ given by (5.19), we obtain

$$\tilde{\sigma}(V) = \sum_{i \in I} \frac{1}{p_i} \chi_V(b_{\sigma(i)}b_i v^{-1}) = \frac{\chi_V(1)}{\chi_V(v)} \sum_{i \in I} \frac{1}{p_i} \chi_V(b_{\sigma(i)}b_i) = \sigma(V).$$

Now (1)–(3) are proved by interpreting this result in terms of the canonical real form $A_0$ in a similar way as the proof of Theorem 6.13. \hfill \square

### 6. Compact Quantum Groups

#### 6.1. Conventions

Given a coalgebra $C = (C, \Delta, \varepsilon)$ over $\mathbb{C}$, we denote by $\mathcal{M}^C_{fd}$ the category of finite-dimensional right $C$-comodules. The coaction of $C$ on $V \in \mathcal{M}^C_{fd}$ is expressed by the Sweedler notation:

$$V \to V \otimes C, \quad v \mapsto v_{(0)} \otimes v_{(1)} \quad (v \in V).$$

If a basis $\{v_i\}_{i=1}^n$ of $V$ is given, then we can define $(c_{ij}) \in M_n(C)$ by

$$(6.1) \quad v_{j(0)} \otimes v_{j(1)} = \sum_{i=1}^n v_i \otimes c_{ij} \quad (j = 1, \ldots, n).$$

The matrix $(c_{ij})$ is called the associated matrix corepresentation with respect to the basis $\{v_i\}$. The subspace spanned by $c_{ij}$’s will be denoted by $C(V)$:

$$C(V) := \text{span}_C \{ c_{ij} \mid i, j = 1, \ldots, n \}.$$  

Note that $C(V)$ does not depend on the choice of the basis $\{v_i\}$. The trace of $(c_{ij})$, $t_V := c_{11} + \cdots + c_{nn} \in C(V)$, also does not depend on the choice of $\{v_i\}$. We call $t_V$ the character of $V$.

#### 6.2. Real forms of a coalgebra

By a real form of a coalgebra $C$, we mean an $\mathbb{R}$-subspace $C_0 \subset C$ such that

$$(6.2) \quad C = C_0 \oplus iC_0, \quad \Delta(C_0) \subset \text{span}_{\mathbb{R}} \{ c' \otimes c'' \mid c', c'' \in C_0 \}.$$  

Given a real form $C_0 \subset C$, we define $x+iy = x-iy (x, y \in C_0)$. For $c \in C$, we call $\overline{c}$ the conjugate of $c$ with respect to the real form $C_0$. By (6.2), taking the conjugate is an anti-linear operator on $C$ such that

$$\overline{c} = c, \quad \Delta(\overline{c}) = \overline{c_{(1)} \otimes c_{(2)}}.$$  

Conversely, if we are given an anti-linear operator $\overline{\cdot} : C \to C$ with this property, then $\{ c \in C \mid \overline{c} = c \}$ is a real form of $C$. Thus giving a real form of $C$ is equivalent to giving an anti-linear operator on $C$ satisfying (6.3).

Fix a real form $C_0 \subset C$. If $V \in \mathcal{M}^C_{fd}$, then $C$ also coacts on $V$ by

$$(6.4) \quad V \to V \otimes C, \quad v \mapsto v_{(0)} \otimes v_{(1)} \quad (v \in V).$$

We call $\overline{V}$ the conjugate of $V$ with respect to $C_0$. In the same way as in (6.3), taking the conjugate defines a real structure for $\mathcal{M}^C_{fd}$, which will be referred to as the real structure for $\mathcal{M}^C_{fd}$ associated with the real form $C_0$. Following (6.3), we introduce the following definition:
Definition 6.1. Let $C$ be a coalgebra with real form $C_0$, and let $\mathcal{J}$ be the real structure for $\mathcal{M}_{fd}^C$ associated with $C_0$. We say that a simple comodule $V \in \mathcal{M}_{fd}^C$ is real, complex and quaternionic with respect to $C_0$ if $\sigma_\mathcal{J}(V)$ is equal to $1$, $0$ and $-1$, respectively.

To characterize the $\mathcal{J}$-signature of a simple comodule, we recall that the dual space $A := C^*$ is an algebra over $\mathbb{C}$ with respect to the convolution product $\ast$ defined by $\langle a \ast b, c \rangle = \langle a, c_{(1)} \rangle \langle b, c_{(2)} \rangle$ ($a, b \in A, c \in C$). The algebra $C^*$ is called the dual algebra of $C$. There is the following relation between real forms of $C$ and those of $A$.

Lemma 6.2. If $C_0$ is a real form of a coalgebra $C$, then

\begin{align}
A_0 &= \{ a \in A \mid a(c) \in \mathbb{R} \text{ for all } c \in C_0 \} \\
\text{is a real form of } A = C^*. \text{ The real form } C_0 \text{ can be recovered from } A_0 \text{ by}
\end{align}

\begin{equation}
C_0 = \{ c \in C \mid a(c) \in \mathbb{R} \text{ for all } a \in A_0 \}.
\end{equation}

Proof. For $a \in A$, we define $\overline{a} \in C$ by $\overline{a}(c) = a(\overline{c})$ ($c \in C$), where $\overline{c}$ is the conjugate of $c$ with respect to $C_0$. Then it is easy to check that

\begin{align}
A_0 &= \{ a \in A \mid \overline{a} = a \}.
\end{align}

Since $\overline{a} = a$ and $\overline{a \ast b} = \overline{a} \ast \overline{b}$ ($a, b \in A$), $A_0$ is a real form. To show \textit{(6.5)}, we denote by $C_1$ the right-hand side of \textit{(6.5)}. $C_0 \subset C_1$ is trivial. Now let $c \in C_1$ and write it as $c = x + iy$ ($x, y \in C_0$). Then, for all $a \in A_0$,

\begin{align}
a(x) \in \mathbb{R}, \quad a(c) = a(x) + ia(y) \in \mathbb{R}.
\end{align}

This implies that $a(y) = 0$ for all $a \in A_0$. Since $A$ is spanned by $A_0$ over $\mathbb{C}$, we see that $a(y) = 0$ for all $a \in A$. Hence $y = 0$. \hfill \square

Let $C$ be a coalgebra, and let $A = C^*$. Given $V \in \mathcal{M}_{fd}^C$, we define $\Phi(V)$ to be the vector space $V$ endowed with the left $A$-action given by $a \cdot v = \langle a, v_{(1)} \rangle v_{(0)}$ for $a \in A$ and $v \in V$. $V \mapsto \Phi(V)$ defines a $\mathbb{C}$-linear functor

\begin{equation}
\Phi : \mathcal{M}_{fd}^C \rightarrow A \mathcal{M}_{fd}, \quad V \mapsto \Phi(V),
\end{equation}

which is well-known to be fully faithful (see, e.g., [4, Chapter 2]). Now let $C_0$ be a real form of $C$, and let $A_0$ be the real form of $A$ given by \textit{(6.5)}. $C_0$ and $A_0$ define real structures for $\mathcal{M}_{fd}^C$ and $A \mathcal{M}_{fd}$, respectively. Abusing notation, we denote them by the same symbol $\mathcal{J} = (J, i)$. It is easy to check that $\Phi \circ J = J \circ \Phi$ and $\Phi(i) = i$. Hence, in particular,

\begin{equation}
\sigma_\mathcal{J}(V) = \sigma_\mathcal{J}(\Phi(V))
\end{equation}

for all simple comodule $V \in \mathcal{M}_{fd}^C$. Now we give the following characterization of the $\mathcal{J}$-signature:

Theorem 6.3. Let $C$ be a coalgebra with real form $C_0$, and let $V \in \mathcal{M}^C$ be a simple comodule with character $t_V$. Then:

1. $V$ is real with respect to $C_0$ if and only if $V$ admits a basis $\{ v_i \}_{i=1}^n$ such that the matrix corepresentation $(c_{ij})$ with respect to $\{ v_i \}$ fulfills:

\begin{align}
c_{ij} \in C_0 \quad \text{for all } i, j = 1, \ldots, n.
\end{align}

2. $V$ is quaternionic with respect to $C_0$ if and only if $V$ does not admit such a basis but $t_V \in C_0$. 
(3) $V$ is complex with respect to $C_0$ if and only if $t_V \not\in C_0$.

**Proof.** Let $A_0$ be the real form of $A := C^*$ given by (6.4). To prove the claim, let, in general, $X \in M^C_{dj}$. Fix a basis $\{v_i\}_{i=1}^n$ of $X$ and define $c_{ij}$’s by (6.1). Then the action of $a \in A$ on $\Phi(X)$ is given by

$$a \cdot v_j = a(v_{j(1)})v_{j(0)} = \sum_{i=1}^n a(c_{ij})v_i \quad (j = 1, \ldots, n).$$

Let $\rho : A \to M_n(\mathbb{C})$ be the matrix representation with respect to the basis $\{v_i\}$ of the left $A$-module $\Phi(X)$. By Lemma 6.2 we have:

$$\rho(A_0) \subset M_n(\mathbb{R}) \iff c_{ij} \in C_0 \text{ for all } i, j = 1, \ldots, n.$$  

Let $t$ and $\chi$ be the characters of $X$ and $\Phi(X)$, respectively. By (6.8),

$$a(t) = a(c_{11}) + \cdots + a(c_{nn}) = \chi(a) \quad (a \in A).$$

Hence, again by Lemma 6.2 we have:

$$\chi(A_0) \subset \mathbb{R} \iff t \in C_0.$$  

By (6.7), $V$ is real, complex and quaternionic with respect to $C_0$ if and only if $\Phi(V)$ is real, complex and quaternionic with respect to $A_0$, respectively. The proof is done by interpreting Theorem 5.9 in terms of $C$ by (6.9) and (6.11). \hfill \Box

### 6.3. Real forms of a *-coalgebra.

A *-coalgebra is a coalgebra $C$ endowed with an anti-linear operator $*: C \to C$ such that

$$c^* = c, \quad (c^*)_{(1)} \otimes (c^*)_{(2)} = (c_{(2)})^* \otimes (c_{(1)})^* \quad (c \in C).$$

Given a real form $C_0$ of a *-coalgebra $C$, we define

$$\varsigma : C \to C, \quad \varsigma(c) = c^* \quad (c \in C),$$

where $-^*$ is the conjugate with respect to $C_0$. We refer the linear map $\varsigma$ as the anti-coalgebra map associated with $C_0$ since, by (6.3), it satisfies

$$\Delta(\varsigma(c)) = \varsigma(c_{(2)}) \otimes \varsigma(c_{(1)}) \quad \varsigma(\varsigma(c)^*) = c \quad (c \in C).$$

Conversely, if a linear map $\varsigma : C \to C$ satisfies (6.12), then

$$C_0 = \{c \in C \mid \varsigma(c^*) = c\}$$

is a real form of $C$. We call $C_0$ the real form associated with $\varsigma$. It is easy to check that (6.12) and (6.14) establish a bijection between real forms of $C$ and linear maps satisfying (6.13).

Note that the dual algebra $A = C^*$ is a *-algebra by

$$\langle a^*, c \rangle = \overline{\langle a, c^* \rangle} \quad (a \in A, c \in C).$$

If $\varsigma : C \to C$ is a linear map satisfying (6.12), then

$$S : A \to A, \quad a \mapsto a \circ \varsigma \quad (a \in A)$$

is a linear map satisfying (5.4). The above constructions for the *-coalgebra $C$ are summarized in the following commutative diagram:

\[
\begin{array}{ccc}
\{\text{real forms of } C\} & \xrightarrow{(6.12)} & \{\varsigma \in \text{End}_C(C) \text{ satisfying (6.13)}\} \\
\downarrow{(6.3)} & & \downarrow{(6.15)} \\
\{\text{real forms of } A = C^*\} & \xrightarrow{(5.4)} & \{S \in \text{End}_C(A) \text{ satisfying (5.4)}\}.
\end{array}
\]
By a \( \ast \)-corepresentation of \( C \), we mean a finite-dimensional Hilbert space \( V \) endowed with a right \( C \)-comodule structure such that

\[
\langle v(0) | w \rangle (v(1))^\ast = \langle v | w(0) \rangle w(1) \quad (v, w \in V).
\]

We denote by \( \text{Corep}_{fd}(C) \) the category whose objects are \( \ast \)-corepresentations of \( C \) and whose morphisms are maps of right \( C \)-comodules.

**Proposition 6.4.** \( \text{Corep}_{fd}(C) \) is a \( C^\ast \)-category.

Now let \( C_0 \) be a real form of \( C \), and let \( \zeta : C \to C \) be the anti-coalgebra map associated with \( C_0 \). Suppose moreover that there exists an invertible positive element \( \gamma \) of the dual \( \ast \)-algebra \( A = C^\ast \) such that

\[
\gamma \circ \zeta = \gamma^{-1}, \quad \zeta^2(c) = \gamma(c(1))c(2)\gamma^{-1}(c(3)) \quad (c \in C).
\]

Given \( X \in \text{Corep}_{fd}(C) \), we define \( D(X) \) to be the dual space \( X^\ast \) endowed with the right \( C \)-comodule structure determined by

\[
(\lambda(0), x)\lambda(1) = \langle \lambda, x(0) \rangle \gamma(x(1)) \quad (\lambda \in X^\ast, x \in X)
\]

and the inner product given by

\[
\langle \phi, \psi \rangle_{D(X)} = \langle \psi(x(0)) \gamma(x(1)) \rangle (x, y \in X).
\]

We also define \( J(X) \) to be the right \( C \)-comodule \( X^\ast \) (i.e., the conjugate comodule with respect to the real form \( C_0 \)) endowed with the inner product given by

\[
\langle \phi, \psi \rangle_{J(X)} = \langle \psi(x(0)) \gamma(x(1)) \rangle (x, y \in X).
\]

\( D(X) \) and \( J(X) \) are \( \ast \)-corepresentations. Moreover, \( D \) and \( J \) are (contravariant) endo-\( \ast \)-functors on \( \text{Corep}_{fd}(C) \). We now define natural isomorphisms \( i : \text{id} \to JJ \) and \( \eta : \text{id} \to DD \) by (6.3)

\[
\langle \eta_X(x), f \rangle = \langle J, x(0) \rangle \gamma(x(1)) \quad (X \in \text{Corep}_{fd}(C), x \in X, f \in X^\ast),
\]

respectively. Then:

**Lemma 6.5.** \( D = (D, \eta) \) and \( J = (J, i) \) are a \( \ast \)-compatible dual structure and a \( \ast \)-compatible real structure for \( \text{Corep}_{fd}(C) \), respectively. Moreover, the dual structures \( D(J) \) and \( D \) are unitary equivalent. Hence, by Theorem 4.4 we obtain

\[
\nu_D(V) = \sigma_J(V).
\]

for all irreducible \( \ast \)-corepresentation \( V \in \text{Corep}_{fd}(C) \).

To verify this lemma, observe that if \( X \in \text{Corep}_{fd}(C) \), then \( \Phi(X) \) is a \( \ast \)-representation of the dual \( \ast \)-algebra \( A \) with respect to the same inner product. Hence (6.6) induces a fully faithful \( \ast \)-functor

\[
\Phi : \text{Corep}_{fd}(C) \to \text{Rep}(A).
\]

Let \( A_0 \) be the real form of \( A \) given by (6.4), let \( S \) be the anti-algebra map associated with \( A_0 \), and let \( g = \gamma \). By the same way as in 5.2, \( A_0, S \) and \( g \) define a real structure and a dual structure for \( \text{Rep}(A) \). Abusing notation, we denote them by \( J = (J, i) \) and \( D = (D, \eta) \), respectively. Then one easily checks that

\[
\Phi \circ J = J \circ \Phi, \quad \Phi(i) = i, \quad \Phi \circ D = D \circ \Phi, \quad \Phi(\eta) = \eta.
\]

Now the lemma follows from the results of 5.2 by regarding \( \text{Corep}_{fd}(C) \) as a full subcategory of \( \text{Rep}(A) \) via (6.17).
6.4. Frobenius-Schur Theorem for Compact Coalgebras. A coseparability idempotent of a coalgebra $C$ is a bilinear map $E : C \times C \to \mathbb{C}$ such that

$$E(c^{(2)}, c^{(2)}) = \varepsilon(c), \quad c^{(1)} \cdot E(c^{(2)}, d) = E(c, d^{(1)}) \cdot d^{(2)} \quad (c, d \in C).$$

If $C$ is finite-dimensional, then a coseparability idempotent of $C$ is nothing but a separability idempotent of the dual algebra $A = C^*$. Let $M_n^c(\mathbb{C})$ be the matrix $*$-coalgebra of degree $n$, i.e., the $\mathbb{C}$-vector space with basis $\{e_{ij}\}_{i,j=1,\ldots,n}$ endowed with the $*$-coalgebra structure determined by

$$\Delta(e_{ij}) = \sum_{k=1}^n e_{ik} \otimes e_{kj}, \quad \varepsilon(e_{ij}) = \delta_{ij}, \quad e_{ij}^* = e_{ji} \quad (i, j = 1, \ldots, n).$$

Following [13, §2], we say that a $*$-coalgebra $C$ is compact if $C = \bigoplus X C(X)$, where $X$ runs over all $*$-corepresentations of $C$. This class of $*$-coalgebras is characterized as follows (see also [13, Lemma 2.1]):

**Proposition 6.6.** For a $*$-coalgebra $C$, the following are equivalent:

1. $C$ is compact.
2. $C = \bigoplus V C(V)$, where $V$ runs over all irreducible $*$-corepresentations.
3. $C$ is isomorphic to a direct sum of matrix $*$-coalgebras.
4. $C$ has a coseparability idempotent $E : C \times C \to \mathbb{C}$ such that

$$E(c^*, d^*) = \overline{E(d, c)}, \quad E(c^*, c) > 0 \quad (c, d \in C, c \neq 0).$$

5. Any finite-dimensional right $C$-comodule admits an inner product making it into a $*$-corepresentation of $C$.

**Proof.** Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a complete set of representatives of isomorphism classes of irreducible $*$-corepresentations of $C$. Note that, by the fundamental theorem of coalgebras, all $V_\lambda$’s are finite-dimensional. By considering the associated matrix corepresentation with respect to an orthonormal basis of $V_\lambda$, we see that $C(V_\lambda)$ is a $*$-subcoalgebra of $C$ isomorphic such that

$$C(V_\lambda) \cong M_{n_\lambda}(\mathbb{C}), \quad \text{where } n_\lambda := \dim_\mathbb{C}(V_\lambda).$$

(1) $\Rightarrow$ (2). Let, in general, $X \in \text{Corep}_{fd}(C)$. If $V \subset X$ is a subcomodule, then its orthogonal complement $V^\perp$ is also a subcomodule and $X = V \oplus V^\perp$. This shows that $X$ is isomorphic to a direct sum of $V_\lambda$’s. Hence $C(X) \subset \bigoplus_{\lambda \in \Lambda} C(V_\lambda)$. By the definition of a compact $*$-coalgebra,

$$C = \bigoplus_X C(X) \subset \bigoplus_{\lambda \in \Lambda} C(V_\lambda) \subset C,$$

where the first sum is taken over all $X \in \text{Corep}_{fd}(C)$.

(2) $\Rightarrow$ (3). By the assumption, $C = \bigoplus_{\lambda \in \Lambda} C(V_\lambda)$. Since $V_\lambda$’s are simple and mutually non-isomorphic, the sum is in fact a direct sum. Hence we obtain an isomorphism of $*$-coalgebras $C \cong \bigoplus_{\lambda \in \Lambda} C(V_\lambda) \cong \bigoplus_{\lambda \in \Lambda} M_{n_\lambda}^c(\mathbb{C})$.

(3) $\Rightarrow$ (4). Fix an isomorphism $C \cong \bigoplus_{\alpha \in A} M_{n_\alpha}^c(\mathbb{C})$ of $*$-coalgebras. Let $e_{ij}^{(\alpha)} \in C$ be the element corresponding to the element $e_{ij}$ of the $\alpha$-th component $M_{n_\alpha}^c(\mathbb{C})$ via the isomorphism. Then the bilinear map $E : C \times C \to \mathbb{C}$ determined by

$$E(e_{ij}^{(\alpha)}, e_{kl}^{(\beta)}) = \delta_{ab}\delta_{a'b'}\delta_{j,k} \quad (a, b \in A; i, j = 1, \ldots, n_a; k, l = 1, \ldots, n_b)$$

is a coseparability idempotent with the desired properties.
(4) ⇒ (5). Let $E$ be such a coseparability idempotent of $C$. Given $X \in \mathcal{M}^C_d$, we fix an inner product $\langle \cdot \rangle_0$ on $X$ and then define

$$\langle x | y \rangle = \langle x(0) | y(0) \rangle_0 \cdot E(x(1), y(1)) \quad (x, y \in X).$$

It is straightforward to verify (6.16). By $E(c^*, d^*) = E(d, c)$, $\langle \cdot \rangle_0$ is Hermitian. To show that $\langle \cdot \rangle$ is positive definite, fix an orthonormal basis $\{e_i\}_{i=1}^n$ of $X$ with respect to $\langle \cdot \rangle_0$ and let $(c_{ij})$ be the matrix corepresentation with respect to $\{e_i\}$. Now let $j \in \{1, \ldots, n\}$. By the counit, we see that $c_{ij} \neq 0$ for some $i$. Hence:

$$\langle e_j | e_j \rangle = \sum_{i,k=1}^n \langle e_i | e_k \rangle E(c^*_{ij}, c_{kj}) = \sum_{i=1}^n E(c^*_{ij}, c_{ij}) > 0.$$

(5) ⇒ (1). The fundamental theorem of coalgebras asserts that $C = \sum X C(X)$, where the sum is taken over all $X \in \mathcal{M}^C_d$. By the assumption, we may assume that all $X$’s are $*$-corepresentation. Thus $C$ is compact.

By Remark 5.10 and the above proposition, a finite-dimensional $*$-coalgebra is compact if and only if its dual $*$-algebra is a $C^*$-algebra. Note that a compact $*$-coalgebra is not necessarily finite-dimensional while an algebra over $\mathbb{C}$ having a separability idempotent is always finite-dimensional.

The following proposition can be considered as the dual of Proposition 5.8:

**Proposition 6.7.** Suppose that a $*$-coalgebra $C$ is compact. Then, for every linear map $\varsigma : C \to C$ satisfying (6.13), there uniquely exists an invertible positive element $\gamma \in C^*$ satisfying the following conditions:

1. $\varsigma^2(c) = \gamma(c(1))c(2)\gamma^{-1}(c(3))$ for all $c \in C$.
2. $\gamma(t) = \gamma^{-1}(t)$ if $t$ is the character of some simple right $C$-comodule.

Moreover, such an element $\gamma$ satisfies the condition

3. $\gamma \circ \varsigma = \gamma^{-1}$.

We omit the proof here; see Appendix A. Now we prove:

**Theorem 6.8.** Let $C$ be a compact coalgebra with real form $C_0$, and let $E$ be a coseparability idempotent for $C$. Then, for all simple comodule $V \in \mathcal{M}^C_d$,

$$\gamma(t_V(2))E(\varsigma(t_V(1)), t_V(3)) = \begin{cases} +1 & \text{if } V \text{ is real}, \\ 0 & \text{if } V \text{ is complex}, \\ -1 & \text{if } V \text{ is quaternionic} \end{cases}$$

with respect to $C_0$, where $\varsigma : C \to C$ is the anti-coalgebra map associated with $C_0$ and $\gamma \in C^*$ is the element of Proposition 6.7.

**Proof.** Define $\mathcal{J}$ and $\mathcal{D}$ as in the previous subsection. By definition, $\sigma_{\mathcal{J}}(V)$ is equal to $+1$, $0$ and $-1$ according to whether $V$ is real, complex or quaternionic with respect to $C_0$. On the other hand, $\sigma_{\mathcal{D}}(V) = \nu_D(V) = \gamma(t_V(2))E(\varsigma(t_V(1)), t_V(3))$ by (6.16) and the coalgebraic version of Theorem 2.5 (see [24] for details).
6.5. **Compact quantum groups.** We apply our results to compact quantum groups proposed by Woronowicz [25]. Instead of his original $C^*$-algebraic approach, we adopt the approach of Koornwinder and Dijkhuizen [15, 8]. See [8, §5] for the comparison between these approaches.

Following [8, Definition 2.2], a compact quantum group algebra (CQG algebra, for short) is a Hopf $*$-algebra $A$ spanned by the matrix elements of the unitary corepresentations of $A$. Here, we should recall that a unitary corepresentation of $A$ is a finite-dimensional Hilbert space $V$ endowed with a right $A$-comodule structure such that $\langle v(0)|w\rangle (v(1))^* = \langle v|w(0)\rangle S(w(1))$ for all $v, w \in V$, where $S$ is the antipode of $A$.

Now let $A$ be a CQG algebra. Note that a Hopf $*$-algebra is not a $*$-coalgebra in the sense of 6.3. Thus we define an anti-linear operator $\dagger$ by

$$a^\dagger = S(a)^* \quad (a \in A).$$

Since $S$ is an anti-coalgebra map, $A$ is a $\dagger$-coalgebra, i.e., a $*$-coalgebra with respect to the $*$-structure $\dagger : A \rightarrow A$. A “$\dagger$-corepresentation” of $A$ is nothing but a unitary corepresentation of $A$. Following, appropriate definitions of real, complex and quaternionic corepresentations of $A$ are:

**Definition 6.9.** Let $V$ be an irreducible corepresentation of the CQG algebra $A$ with character $t_V$.

1. $V$ is real if it admits a basis $\{v_i\}_{i=1}^n$ such that the matrix corepresentation $(c_{ij})$ with respect to $\{v_i\}$ has the following property:

$$c_{ij}^* = c_{ij}.$$

2. $V$ is quaternionic if it does not admit such a basis but $t_V^* = t_V$.

3. $V$ is complex if $t_V^* \neq t_V$.

By Theorem 6.3, $V$ is real, complex and quaternionic if and only if it is real, complex and quaternionic with respect to the real form

$$A_0 = \{a \in A \mid S(a^\dagger) = a\} \quad (= \{a \in A \mid a^* = a\})$$

of the $\dagger$-coalgebra $A$.

By Proposition 6.7, the $\dagger$-coalgebra $A$ is compact. Since $S \circ \dagger \circ S \circ \dagger = \text{id}_A$, there uniquely exists a $\dagger$-positive‘ element $\gamma \in A^*$ satisfying the conditions (1)–(4) of Proposition 6.7. Here, we are saying that $f \in A^*$ is $\dagger$-positive if $f = g^\dagger \ast g$ for some $g \in A^*$, where $g^\dagger(a) = g(a^\dagger) \quad (a \in A)$.

A Haar functional on $A$ is a linear map $h : A \rightarrow \mathbb{C}$ satisfying

$$h(1_A) = 1, \quad h(a_{(1)}) \cdot a_{(2)} = h(a)1_A = h(a_{(2)}) \cdot a_{(1)} \quad (a \in A).$$

See [8] §3 for the existence and the uniqueness of a Haar functional. Now let $h$ be the Haar functional on $A$. It easily follows that

$$h(a_{(1)}b) \cdot a_{(2)} = h(ab_{(1)}) \cdot S(b_{(2)}), \quad a_{(1)} \cdot h(a_{(2)}b) = S(b_{(1)}) \cdot h(ab_{(2)})$$

for all $a, b \in A$. The Frobenius-Schur theorem for compact quantum groups is now stated as follows:

**Theorem 6.10.** If $V$ is an irreducible corepresentation of $A$, then

$$h(t_{V(1)}t_{V(2)}) = \frac{\varepsilon(t_V)}{\gamma(t_V)} \times \begin{cases} 
+1 & \text{if } V \text{ is real}, \\
0 & \text{if } V \text{ is complex}, \\
-1 & \text{if } V \text{ is quaternionic}.
\end{cases}$$
functions on a compact group $G$

travariant endo-

of algebra maps

Remark

Proposition A.1. The forgetful functor.

see also [15].

Remark on why the right-hand side of (6.19) is equal to that of (1.1).

real, complex and quaternionic representations. See [24, Example 4.9] for details for $C$

The real form $A_0 = \{a \in A \mid S(a^1) = a\}$ is precisely the subset of $O(G)$ consisting of $\mathbb{R}$-valued functions. Thus Definition 6.9 agrees with the ordinary definitions of real, complex and quaternionic representations. See [24] Example 4.9] for details on why the right-hand side of (6.19) is equal to that of (1.1).

Remark 6.12. Woronowicz [25] showed that there exists a unique family $\{f_z\}_{z \in \mathbb{C}}$ of algebra maps $f_z : A \to \mathbb{C}$ characterized by numerous properties, including:

\begin{align*}
  f_0 &= \varepsilon, \quad f_w \ast f_z = f_{z+w}, \quad \overline{f_z(a^*)} = f_{z}(a), \quad f_z(S(a)) = f_{-z}(a), \\
  S^2(a) &= f_1(a_{(1)})a_{(2)}f_{-1}(a_{(3)}) \quad (w, z \in \mathbb{C}, a \in A),
\end{align*}

see also [15]. $f_1$ is $\dagger$-positive, since $f_1 = f_{1/2} \ast f_{1/2}$. Going back to the construction of $\{f_z\}$, we see that $f_1$ satisfies the conditions (1) and (2) of Proposition 6.7. Hence, by the uniqueness, $\gamma = f_1$.

APPENDIX A. Duality lifting problems

A.1. A lift yields a dual structure. Let $A$ be a $*$-algebra with real form $A_0$. By using the anti-algebra map $S$ associated with $A_0$, we define a contravariant $\mathbb{C}$-linear functor $D_0 : \text{Rep}(A) \to \mathcal{AM}$ as in [17.2].

Let $\mathcal{C}$ be a full subcategory of $\text{Rep}(A)$. Recall that a lift of $D_0$ on $\mathcal{C}$ is a contravariant endo-$*$-functor $D : \mathcal{C} \to \mathcal{C}$ such that $U \circ D = D_0$, where $U : \mathcal{C} \to \mathcal{AM}$ is the forgetful functor.

Proposition A.1. If there exists a lift $D$ of $D_0$ on $\mathcal{C}$, then there exists a natural isomorphism $\eta : \text{id}_{\mathcal{C}} \to DD$ such that the pair $(D, \eta)$ is a $*$-compatible dual structure for $\mathcal{C}$. 
Proof. For each $X \in \mathcal{C}$, we define $\tilde{\eta}_X = \phi_{D(X)} \circ \phi_X : X \to DD(X)$ (cf. (5.8)). One can show that $\tilde{\eta} = \{\tilde{\eta}_X\}_{X \in \mathcal{C}}$ defines a natural isomorphism $\tilde{\eta} : \text{id}_{\mathcal{C}} \to DD$. Now, for each $X \in \mathcal{C}$, we define

$$\eta_X = \tilde{\eta}_X \circ |\tilde{\eta}_X|^{-1} : X \to DD(X).$$

By Lemmas [1.1] and [1.2], $\eta = \{\eta_X\}_{X \in \mathcal{C}}$ is a unitary natural isomorphism. We show that the pair $(D, \eta)$ is indeed a $*$-compatible dual structure for $\mathcal{C}$. For this purpose, it is sufficient to verify (2.1).

For this purpose, we set $\alpha_X = D(\tilde{\eta}_X) \circ \tilde{\eta}_{D(X)}$ for $X \in \mathcal{C}$. By the naturality of $\tilde{\eta}$, we have $\tilde{\eta}_{DD(X)} \circ \tilde{\eta}_X = DD(\tilde{\eta}_X) \circ \tilde{\eta}_X$. Since $\tilde{\eta}_X$ is an isomorphism, $\tilde{\eta}_{DD(X)} = DD(\tilde{\eta}_X)$. Hence,

(A.1) \[ D(\alpha_X) = D(\tilde{\eta}_{D(X)})DD(\tilde{\eta}_X) = D(\tilde{\eta}_{D(X)})\tilde{\eta}_{DD(X)} = \alpha_D(X). \]

Let $\phi_X : D(X) \to X$ be the adjoint operator of $\phi_X : X \to D(X)$. For $x, y \in X$,

$$(\phi_X^*)^\vee(\phi_x, \phi_y) = \langle \phi_x, \phi_X^*(\phi_y) \rangle = \langle x|\phi_X^*(\phi_y) \rangle_X = \langle \phi_x|\phi_y \rangle_D(X).$$

This implies $(\phi_X^*)^\vee = \phi_D(X)$. By using this result, we get:

(A.2) \[ D(|\tilde{\eta}_X|^2) = D(\tilde{\eta}_X) \circ (\phi_X^*)^\vee \circ (\phi_X^*)^\vee = D(\tilde{\eta}_X)\tilde{\eta}_X = \alpha_X. \]

Hence, by (A.1) and (A.2),

$$D^2(|\tilde{\eta}_X|^2) = D^2(\tilde{\eta}_X \tilde{\eta}_X) = D(\alpha_X) = \alpha_D(X) = D(\tilde{\eta}_{D(X)} \tilde{\eta}_{D(X)}) = D(|\tilde{\eta}_D(X)|^2).$$

Since $D$ is an anti-equivalence, $D(|\tilde{\eta}_X|^2) = |\tilde{\eta}_D(X)|^2$. Taking the square root, we obtain $D(|\tilde{\eta}_X|) = |\tilde{\eta}_{D(X)}|$. Now (2.1) is proved as follows:

$$D(\eta_X)\eta_{D(X)} = D(|\tilde{\eta}_X|^{-1})D(\tilde{\eta}_X)\tilde{\eta}_{D(X)}|\tilde{\eta}_{D(X)}|^{-1} = |\tilde{\eta}_{D(X)}|^2|\tilde{\eta}_{D(X)}|^{-1} = \text{id}_{D(X)}.$$  

Q.E.D.

A.2. Criterion for existence of a lift. Throughout this subsection, we suppose that $\{X_i\}_{i \in I}$ is a family of finite-dimensional irreducible $*$-representations of $A$ such that for all $i \in I$, the dual $X_i^\vee$ is isomorphic to $X_j$ for some $j \in I$. The aim of this subsection is to show that the functor $D_0$ can be lifted on the full subcategory

(A.3) \[ \mathcal{C} := \{X \in \text{Rep}(A) \mid X \cong X_{i_1} \oplus \cdots \oplus X_{i_m} \text{ for some } i_1, \ldots, i_m \in I\}. \]

We may assume that $X_i \not\cong X_j$ whenever $i \neq j$. Then, for $i \in I$, we can define $i^\vee \in I$ so that $X_i^\vee \cong X_{i^\vee}$. One can define a contravariant endofunctor $D$ on $\mathcal{C}$ by extending $X_i \mapsto X_{i^\vee}$. However, it is difficult to check that the functor $D$ so-defined is a $*$-functor since the inner product on $D(X)$ is not given explicitly. Our approach is first to construct a positive operator $g_X : X \to X$ for each $X \in \mathcal{C}$ and then define $D(X)$ to be the left $A$-module $X^\vee$ with the inner product

(A.4) \[ \langle \phi_x|\phi_y \rangle_{D(X)} := (y|g_X(x))_X \quad (x, y \in X). \]

Now we explain how to construct $g_X$. First, let $i \in I$. By the assumption, $X_i^\vee$ has an inner product making it into a $*$-representation. Fix such an inner product $\langle -| - \rangle_0$ and then define $\tilde{g}_i : X_i \to X_i$ by

$$\langle y|\tilde{g}_i(x) \rangle_0 = \langle \phi_x|\phi_y \rangle_0 \quad (x, y \in X_i).$$

Since $(x, y) \mapsto \langle \phi_x|\phi_y \rangle_0$ is an inner product on $X_i$, $\tilde{g}_i$ is positive and invertible. Moreover, $\tilde{g}_i(ax) = S^2(a)\tilde{g}_i(x)$ holds for all $a \in A$ and $x \in X$. Now set

$$g_i = \text{Tr}(\tilde{g}_i^{-1})^{1/2} \text{Tr}(\tilde{g}_i)^{-1/2} \cdot \tilde{g}_i.$$
Then $g_i$ is an invertible linear map such that
\begin{equation}
\tag{A.5} g_i \geq 0, \quad \text{Tr}(g_i) = \text{Tr}(g_i^{-1}) > 0, \quad g_i(ax) = S^2(a)g_i(x) \quad (a \in A, x \in X_i).
\end{equation}
Now let $X \in \mathcal{C}$. By the definition of $\mathcal{C}$, there is a canonical isomorphism
\begin{equation}
\tag{A.6} \bigoplus_{i \in I} X_i \otimes_{\mathbb{C}} \text{Hom}_A(X_i, X) \to X, \quad (x_i \otimes f_i)_{i \in I} \mapsto \sum_{i \in I} f_i(x_i)
\end{equation}
of left $A$-modules. Recall that $\text{End}_A(X_i) = \mathbb{C} \cdot \text{id}_{X_i}$. One can check that \( A.6 \) is unitary if we define an inner product on $\text{Hom}_A(X, X_i)$ by
$$\langle f | g \rangle \cdot \text{id}_{X_i} = f^* g \quad (f, g \in \text{Hom}_A(X_i, X)).$$
By using \( A.6 \), we define $g_X : X \to X$ so that the diagram
\[
\begin{array}{ccc}
X & \longrightarrow & \bigoplus_{i \in I} X_i \otimes_{\mathbb{C}} \text{Hom}_A(X_i, X) \\
g_X & \downarrow & \downarrow (g_i \otimes \text{id})_{i \in I} \\
X & \longrightarrow & \bigoplus_{i \in I} X_i \otimes_{\mathbb{C}} \text{Hom}_A(X_i, X)
\end{array}
\]
commutes. By \( A.5 \) and the unitarity of \( A.6 \), we obtain:
\begin{align}
\tag{A.7} & g_X \geq 0, \quad \text{Tr}(g_X^{-1}) = \text{Tr}(g_X), \\
\tag{A.8} & g_X(ax) = S^2(a)g_X(x)
\end{align}
for all $a \in A$ and $x \in X \in \mathcal{C}$. If $f : X \to Y$ is a morphism in $\mathcal{C}$, then
\begin{equation}
\tag{A.9} g_Y \circ f = f \circ g_X
\end{equation}
by the definition of $g_X$'s.

To summarize results so far, we introduce the following notation: Given a ring homomorphism $\alpha : R_1 \to R_2$, we denote by $\alpha^2 : R_2 \mathcal{M} \to R_1 \mathcal{M}$ the pull-back functor along $\alpha$. Since $S^2 : A \to A$ is an algebra automorphism, it induces a functor
$$S^{2n} : A\mathcal{M} \to A\mathcal{M}.$$ 
Let $U : \mathcal{C} \to A\mathcal{M}$ be the forgetful functor. Then:

**Lemma A.2.** The family $g = \{ g_X \}_{X \in \mathcal{C}}$ defines a natural isomorphism
$$g : U \to S^{2n}U$$
satisfying \( A.7 \). Such a natural isomorphism is unique.

**Proof.** The first sentence is nothing more than a paraphrase of \( A.7 \)-\( A.9 \). To show the uniqueness, let $g' : U \to S^{2n}U$ be another such natural isomorphism. Set $g_i = g_{X_i}$ and $g'_i = g'_{X_i}$. It is sufficient to show that $g_i = g'_i$ for all $i \in I$. Now let $i \in I$. Since $X_i$ and $S^{2n}(X_i)$ are finite-dimensional simple $A$-modules, $g'_i = cg_i$ for some $c \in \mathbb{C}^\times$ by Schur’s lemma. By \( A.7 \),
$$c \text{Tr}(g_i) = \text{Tr}(g'_i) = \text{Tr}(g_i^{-1}) = c^{-1} \text{Tr}(g_i) = c^{-1} \text{Tr}(g_i).$$
By \( A.5 \), $c = \pm 1$. Since $g_i$ and $g'_i$ are positive, $c$ must be $+1$. \( \square \)

The natural isomorphism $g$ plays the role of the element $g$ in \( A.2 \). For $X \in \mathcal{C}$, we define $D(X) \in \mathcal{C}$ to be the left $A$-module $X^\vee$ endowed with the inner product given by \( A.4 \). The following lemma is proved in a similar way to \( A.2 \).

**Lemma A.3.** $X \mapsto D(X)$ extends to a lift of $D_0$ on $\mathcal{C}$. 

Hence, by Proposition \[ A.1 \] there exists a natural isomorphism \( \eta : \text{id}_C \to DD \) such that the pair \( (D, \eta) \) is a \( * \)-compatible dual structure for \( C \). In our situation, we can construct \( \eta \) by using \( g \) of Lemma \[ A.2 \]. We need to prove:

**Lemma A.4.** Let \( g \) be the natural isomorphism of Lemma \[ A.2 \]. Then

\[
g_{D(X)} \circ (g_X)^\vee = \text{id}_{D(X)} \quad (X \in \mathbb{C}).
\]

**Proof.** Set \( \alpha_X = (g_X)^\vee \) and \( \beta_X = (g_{D(X)})^{-1} \) for \( X \in \mathbb{C} \). Since both \( \alpha = \{ \alpha_X \}_{X \in \mathbb{C}} \) and \( \beta = \{ \beta_X \}_{X \in \mathbb{C}} \) are natural isomorphisms \( S^2UD \to UD \), it is sufficient to show that \( \alpha_X = \beta_X \) holds for all \( i \in I \). Let \( i \in I \). By Schur’s lemma, \( \beta_X = c \cdot \alpha_X \), for some \( c \in \mathbb{C}^\times \). By (A.7), we have

\[
\text{Tr}(\alpha_X) = \text{Tr}(g_{X^i}) = \text{Tr}(g_{X^i}^{-1}) = \text{Tr}(\alpha_X^{-1}).
\]

We also obtain \( \text{Tr}(\beta_X) = \text{Tr}(\beta_X^{-1}) \) in a similar way. Hence,

\[
c \cdot \text{Tr}(\alpha_X) = \text{Tr}(c \cdot \alpha_X) = \text{Tr}(\beta_X) = \text{Tr}(\beta_X^{-1}) = \text{Tr}(c^{-1} \alpha_X^{-1}) = c^{-1} \text{Tr}(\alpha_X).
\]

Now we conclude \( c = 1 \) by the same way as the proof of Lemma \[ A.2 \]. \( \square \)

Define \( \eta : \text{id}_C \to DD \) by the same formula as (2.26) but by using the natural isomorphism \( g \) instead of \( q \). We also define an anti-linear functor \( J : \mathbb{C} \to \mathbb{C} \) and a natural isomorphism \( i : \text{id}_C \to JJ \) in a similar manner. Then, again in the same way as (5.2) we prove:

**Proposition A.5.** The pair \( D = (D, \eta) \) is a \( * \)-compatible dual structure for \( C \) and the pair \( J = (J, i) \) is a \( * \)-compatible real structure for \( C \). The dual structures \( \mathbb{D}(J) \) and \( \mathbb{D} \) are unitary equivalent via the natural isomorphism

\[
\varphi_X : J(X) \to D(X), \quad \mathfrak{x} \mapsto \phi_x \quad (x \in X \in \mathbb{C}).
\]

**A.3. Proofs of some propositions.**

**Proof of Proposition 6.3.** Let \( \{ X_i \}_{i \in I} \) be the complete set of representatives of simple left \( A \)-modules. Since \( A \) is assumed to be a finite-dimensional \( C^* \)-algebra, we may assume that all \( X_i \)'s are \( * \)-representations. Applying the arguments of \( \{ A.2 \} \) to \( \{ X_i \} \), we obtain a natural isomorphism \( g \) as in Lemma \[ A.2 \]. Since \( A \) is finite-dimensional, there uniquely exists an element \( g \in A \) such that

\[
g_X(x) = g \cdot x \quad (x \in X \in \text{Rep}_{fd}(A))
\]

Interpreting (A.7) in terms of the element \( g \), we see that the element \( g \) has the desired properties. Uniqueness of such an element follows from Lemma \[ A.2 \]. \( \square \)

**Proof of Proposition 6.4.** Let \( \{ V_i \}_{i \in I} \) be the complete set of representatives of simple \( C \)-comodules. By Proposition \[ 6.6 \] we may assume that all \( V_i \)'s are \( * \)-corepresentations. Now let \( A = C^* \) be the dual algebra of \( C \) and define \( S : A \to A \) by (6.15). Applying the arguments of \( \{ A.2 \} \) to \( \{ X_i \} \), where \( X_i = \Phi(V_i) \in \text{Rep}_{fd}(A) \), we obtain a natural isomorphism \( g \) as in Lemma \[ A.2 \].

Since \( C = \bigoplus_{i \in I} C(V_i) \) (see the proof of Proposition 6.6), we can define uniquely a linear map \( \gamma : C \to \mathbb{C} \) by the following condition:

\[
\langle \gamma, v_{(1)} \rangle v_{(0)} = g_i(v) \quad (i \in I, v \in V_i).
\]

Now the claim is proved by interpreting the properties of the natural isomorphism \( g \) in a similar way as Proposition 5.8. \( \square \)
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