CORRECTNESS OF THE OPTIMAL CONTROL PROBLEMS FOR DISTRIBUTED PARAMETER SYSTEMS
(SURVEY)

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The problem of existence of optimal control for nonlinear processes, in contradistinction to linear, is investigated a little. This problem, for the processes, described by the ordinary differential equations, is studied, as a rule under Philipov condition ([1]) - at condition of convexity of admissible speeds set. However, it is known that, this condition covers very narrow class of nonlinear problems, even in some cases of ”almost linear” problems. For multivariate variational problems and the optimal control problems in the processes described by the equation with the partial derivatives, in this direction we note works [2-7].

The basic idea of these works consists of providing weak semi-continuity of the functional on weekly compact set.

The second direction in the existence of the solution is the proof of ‘individual theorems” of the existence which take into account specificity of particularly considered problem. In this direction are received the most important results in the works [8-10].

The essence of this direction is connected to necessary conditions of an optimality.

The problem of existence of optimal control is investigated also in [11-13].

In the classes of problems considered in this works, the functional depends on parameter and existence of optimal control is proved for values of parameter from dense set. Though such type results have been obtained at enough general conditions, the fulfillment of these conditions for particularly taken parameter, generally speaking, is difficult.

Here an original idea is suggested to prove the existence of optimal control for some types of non-linear problems. The obtained results can be considered as individual existence theorems (in some sense).

The idea of the proof consists of the following:

Any minimizing sequence is chosen. It’s clear, that this sequence is minimizing for indefinitely many other functionals. If there is strict convex among these functionals it turns out strong convergence of this sequence. (exactly the subsequence). After it under natural conditions theorems of existence are proved. The properties of minimizing sequence are used for finding such strict convex functionals.

The following problems have been considered

1. Non convex variational problems;
2. The optimal control problems described by the first and second order non-linear equations with abstract evolution;
3. The optimal control problem for the system described by the Gursa-Darbu equation;
4. The optimal control problem for the system described by the parabolic type equation.

**AT FIRST CONSIDERED NON CONVEX VARIATIONAL PROBLEMS [16]**

\[
J(x) = \int_a^b f(t, x(t), \dot{x}(t)) \, dt \to \min, \\
x_i(c_i) = d_i, i \in I.
\]

(1)

Here, \(a, b\) are the given numbers

\[
t \in [a, b] x(t) = (x_1(t), x_2(t), ..., x_n(t)),
\]

\[
\dot{x}(t) = (\dot{x}_1(t), \dot{x}_2(t), ..., \dot{x}_n(t)) d_i \in Ri \subseteq \{1, 2, ..., n\},
\]

\(c_i\) accept values \(a\) or \(b\), \(f(t, x, s)\) is the given function from \([a, b] \times R^n \times R^n\) to \(R\), \(p \geq 2\).

The set of indexes \(i\) is defined as \(I_0\) at which \(f(t, x_1, x_2, ..., x_n, \dot{x}_1, \dot{x}_2, ..., \dot{x}_n)\) doesn’t depend on a component \(x_i\). Let \(I_0 \neq \emptyset\) the following conditions also are fulfilled:

1) \(J_* = \inf J(x) > -\infty\); and \(J(x) \to +\infty\), if \(\|x\|_{W^{1,p}(a,b)} \to +\infty\);

2) \(|f_S(t, x, s)| \leq m_1 + m_2 |s|^{p-1}, \forall t \in [a,b], x \in R^n; m_1, m_2 \geq 0.\)

3) there is a non-negative function \(a(t, x, p), i \in I_0\) such, that if \(\|x\|_{W^{1,p}(a,b)} \leq \text{const}\) then \(\|a_i(\cdot, x(\cdot), \dot{x}(\cdot))\|_{L^{\infty}(a,b)} \leq \text{const}\) and the function

\[
F(t, x, s) = f(t, x, s) + \sum_{i \in I_0} a_i(t, x, s) f_{S_i}^2(t, x, s)
\]

is convex relatively \(s\) in \(R^n\).

The following is proved.

**Theorem 1.** There exists a solution of the problem (1) - (2) under the conditions 1) - 3).

The example of function \(f(t, x, p)\) satisfying all conditions of the theorem 1 which is however not convex relatively \(s\) on \(R^n\) is given.

The following multivariable problem also is considered

\[
J(x) = \int_D f(t, \dot{x}(t)) dt \to \min
\]

(3)

Here

\[
D \subset R^n t = (t_1, t_2, ..., t_n) \subset D \dot{x}(t) = (x_{t_1}(t), x_{t_2}(t), ..., x_{t_n}(t)), f : D \times R^n \to R
\]

is continuous on set of variables with the partial derivatives \(f_{S_1}, f_{S_2}, ..., f_{S_n}\).

The theorem of existence is formulated within similar conditions \((I_0 = \{1, 2, ..., n\})\).

Now we shall consider questions of resolvability of the optimal control problem for the abstract evolutionary equations of the first order ([18]).
Let $H$ be Hilbert space, $A_0$ be linear closed operator $A_0 : D(A_0) \to H$, $A(t)$ be the self-adjoint linear operator at every $t \in [0, T]$, $A(t) : D(A_0) \to H$ and, satisfying inequalities

$$m \|A_0\varphi\|_H \leq \|A(t)\varphi\|_H \leq M \|A_0\varphi\|_H \varphi \in D(A_0), m, n > 0.$$ 

Through $H_0 = D(A_0)$ we shall designate Hilbert space with norm

$$\|\varphi\|^2_{H_0} = \|\varphi\|^2_H + \|A_0\varphi\|^2_H, \varphi \in D(A_0).$$ 

Let $C$ be the linear continuous operator $C : H_0 \to H_1$, $H_1$ is Hilbert space, $V_0 \subset U$ be the closed convex bounded set in Hilbert space $U$,

$$V = \left\{ v : v \in L_2(0, T, U), v(t) \in V_0, \forall t \in (0, T) \right\}.$$ 

Operator $B(t, v, u)$ for every $t \in [0, T]$ and $v \in V_0$ continuously operates from $H_0$ in $H$; operator $K(t, v, u, p)$ for every $t \in [0, T]$ continuously operates from $V_0 \times H \times H_1$ in $R$ and at each

$$(v, u, p) \in V \times L_2(0, T; H) \times L_2(0, T; H_1), \quad K(t, v(t), u(t), p(t)) \in L_2(0, T),$$ 

$$(u)$$ is continuous functionals, defined in $H$. The Banach space of functions $u=\dot{u}(t)$ is designated by $W$, belonging to $L_2(0, T; H_0)$, strict continuous on $t$ in norm $H$, having final norm

$$\|u\|_W = \max_{0 \leq t \leq T} \|u(t)\|_H + \|A_0 u\|_{L_2(0, T, H)}.$$ 

Satisfy the subspace $W$ is designated by $W_0$, for the elements which

$$\int_0^{T-h} h^{-1} \|u(t + h) - u(t)\|^2_H dt \to 0 \text{ at } h \to 0.$$ 

The minimization problem for the functional is considered

$$J(v) = \int_0^T K(t, v(t), u(t), Cu(t)) dt + \Phi(u(T)), \quad (4)$$ 

on set $V$ at conditions

$$u_t + A^2(t) u + B(t, v, u) = 0, \quad (5)$$ 

where $\varphi \in H$. The solution of the problem (4) - (6) is understood as a function $u = u(t) \in W_0$ satisfying equality

$$(u(t), \eta(t)) + \int_0^t (- (u, \eta_t) + (A(\tau) u, A(\tau) \eta) + (B(\tau, v, u), \eta)) d\tau +$$ 

$$= (\varphi, \eta(0)), \quad \forall t \in (0, T),$$ 

for $\forall \eta = \eta(\cdot) \in L_2(0, T; H_0), \eta_t \in L_2(0, T; H), (\cdot, \cdot)$ is scalar product in $H$. It’s suggested, that the solution of the reduced problem (5) - (6) exists only and satisfies to the estimation
\[ \| u \|_W \leq N, \forall v \in V \]

Let

\[ K(t, v, u, p) = K_0(t, u, p) + K_1(t, v, u), \]
\[ B(t, v, u) = B_0(t, u) + B_1(t, v, u). \]

Operators \( K(t, v, u, p), B(t, v, u) \) have Frechet derivatives \( K_u, K_p, \Phi_z, B \); the operators \( K_1(t, v, u), B_1(t, v, u) \) have continuous Frechet derivatives on \( v \in V \). All these derivatives satisfy the Lipschitz condition in \( V \times L^2(0, T; H) \times L^2(0, T; H) \).

Introduced the Hamilton - Potryagen functional for the problem (4) - (6)

\[ H(t, v, u, \psi) = (B_1(t, v, u), \psi) - K_1(t, v, u), \quad \text{where } \psi = \psi(t) \in W_0 \]

is the solution of the adjoint problem for the (4)-(6)

\[ -\psi_t(t) + A^2(t) \psi + B_u(t, v, u) \psi = K_u(t, v, u, Cu) + C^* K_p(t, v, u, Cu) \]
\[ \psi(t) = -\Phi_u(u(T)). \]

Let the following conditions fulfill:

1) \( K_1(t, v, u), B_1(t, v, u) \) are satisfy Lipschitz condition on \( v \in U, u \in H \)
and

\[ K(t, v, u, p) \geq g(t), \quad \Phi(u) \geq \mu > -\infty, \quad g \in L^1(0, T); \]

2) For all \( v_1, v_2 \in V_0, u \in H, \forall t \in (0, T), \)

\[ \| B_1(t, \frac{v_1 + v_2}{2}, u) - \frac{1}{2} B_1(t, v_1, u) - \frac{1}{2} B_1(t, v_2, u) \|_H \leq \]
\[ \leq \frac{\chi_1}{2} v_2 - v_1 \|_U \]
\[ K_1(t, \frac{v_1 + v_2}{2}, u) - \frac{1}{2} K_1(t, v_1, u) - \frac{1}{2} K_1(t, v_2, u) \leq \]
\[ \leq -\frac{\chi_2}{4} v_2 - v_1 \|_U, \]

Where \( \chi_1 \geq 0, \chi_2 \geq 0; \)

3) \( \| \psi(t) \|_H \leq q, \forall v \in V, \forall t \in (0, T); \)

4) It is possible to choose strongly converging subsequence from the sequence \( u = u_n(t), \psi = \psi_n(t) \) in \( L^2(0, T; H) \).

Last condition is fulfilled, for example, if \( H_0 \subset H \) is compact and

\[ \| u \|_W + \| u_t \|_{L^2(0, T; H)} \leq \text{const}, \forall v \in V. \]

**Theorem 2.** Let \( \chi > q\chi_1 \). Then there exists a solution of the problem (4) - (6) and we can choose strongly converging subsequence to the solution in \( L^2(0, T, U) \) from any sequence.

It turns out the consequence from the theorem 2 which covers a wide class of nonlinear problems.
Corollary 1. Let $B_1(t, v, u)$ be linearly on $v$ and $K_1(t, v, u)$ is strict convex on $v$ on $V_0$. Then the statement of the theorem 2 is fulfilled.

The received results are applied to one optimal control problem for the parabolic equations.

Let $D \in \mathbb{R}^n$ be bounded domain with enough smooth boundary $\Gamma \Omega = D \times (0, T), S = \Gamma \times [0, T] x = (x_1, x_2, ..., x_n) \in D, V_0$ be some closed, bounded, convex set in $\mathbb{R}^n$.

Let the functional be minimized

$$J(v) = \int_{\Omega} [K_0(x, t, u(x, t), u_x(x, t))] +$$

$$+ K_1(x, t, v(x, t), u(x, t))]dxdt + \int_D \Phi(u(x, T))dx$$

(10)

on set

$$V = \{ v = v(x, t) : v = (v_1, v_2, ..., v_m) \in L^2_2(\Omega), v(x, t) \in V_0, \forall (x, t) \in \Omega \},$$

at conditions

$$u_t - \sum_{i,j=1}^{n} (a_{i,j}(x,t)u_{x_i})_{x_j} + a(x, t, v, u) + f(x, t, u, u_x) = 0, \forall (x, t) \in \Omega,$$

(11)

$$u(x, 0) = \varphi(x), x \in D,$$

(12)

$$u|_S = 0$$

Here $a_{i,j}(x,t), i, j = 1, n, a(x, t, v, u), f(x, t, v, u), \varphi(x)$ are the given measurable on $(x, t) \in \Omega$ functions, continuous on $v \in V_0 u \in \mathbb{R}, p \in \mathbb{R}^n; a_{i,j} = a_{j,i}$ and $a_{i,j}$ satisfy to a condition of uniform ellipticity; $a(x, t, v, u), f(x, t, u, p)$ and their partial derivatives relatively $v, u$ and $p$ satisfy to Lipschitz condition on $(v, u, p) \in V_0 \times R \times R^n; \varphi \in L^2(D)$.

Let the solution of the problem (11) - (14) $u = u(x, t) \in \dot{V}^{1,1/2}_2(\Omega)$ exists.

From corollary 1 it turns out

Theorem 3. Let $a(x, t, v, u)$ be linearly relatively $v$ and $K_1(x, t, v, u)$ strict convex relatively $v$ on $V_0$. Then the solution of the problem (10) - (14) exists and we can choose subsequence from any minimizing subsequence strongly converging in $L^2_2(\Omega)$ to the solution a.

In spite of the fact that $a(x, t, v, u)$ is linear relatively $v$, there is a nonlinear term in the equation $f(x, t, v, u, u_x)$ in which to take a limit, generally speaking, is impossible. It raises the importance of the Theorem 3.

If $a(x, t, v, u)$ is not linear relatively $v$ then the following conditions are put

1) $|a(x, t, \frac{v_1 + v_2}{2}, u) - \frac{1}{2}a(x, t, v_1, u) - \frac{1}{2}a(x, t, v_2, u)| \leq \frac{\chi_1}{4} \|v_2 - v_1\|^2_{\mathbb{R}^m}, \chi_1 \geq 0;$$

2) $K_1(x, t, \frac{v_1 + v_2}{2}, u) - \frac{1}{2}K(x, t, v_1, u) - \frac{1}{2}K_1(x, t, v_2, u) \leq -\frac{\chi}{4} \|v_2 - v_1\|^2_{\mathbb{R}^m}, \chi \geq 0;$$

3) $|\psi(x, t)| \leq q, q \geq 0, \forall (x, t) \in \Omega, \forall v \in V.$

It turns out the resolvability of a problem of optimal control (10) - (14) at $\chi > q\chi_1$. The optimal control problem for the abstract evolutionary equations of the second order is investigated analogically.
The optimal control problem for Goursa-Darbou system also has been considered ([14,18]). Let the functional be minimized

$$J(v) = \int_{0}^{T} \int_{0}^{l} K(x,t,u(x,t),u_{x}(x,t),u_{t}(x,t),v(x,t)) \, dx \, dt + \Phi(u(l,T))$$

at conditions

$$u_{xt}(x,t) = f(x,t,u(x,t),u_{x}(x,t),u_{t}(x,t),v(x,t)), (x,t) \in Q,$$

$$u(0,t) = \varphi_{0}(t), t \in [0,T] ; u(x,0) = \varphi_{1}(x), x \in [0,l],$$

Where $l$, $T$ is the given positive numbers $\varphi_{0}(0) = \varphi_{1}(0)$.

Let’s understand the vector function $u = u(x,t)$ as the solution of a problem [14] - [15] appropriated to control $v \in V$ which has generalized derivatives $u(x,t), u_{t}(x,t), u_{xt}(x,t) \in L^{2}(Q)$ and, satisfying the equation (14) almost everywhere in $Q$ and to conditions (6) in sense of equality of the appropriate traces $u(0, \cdot), u(\cdot, 0)$.

It is supposed, that functions $K(x,t,u,p,q,v), f^{i}(x,t,u,p,q,v), i = 1, n$, $\Phi(u)$ and their partial derivatives on $u$, $p$, $q$, $v$ are continuous on set of arguments and satisfy to Lipschitz condition on $(u, p, q, v)$.

Let

$$K(x,t,u,p,q,v) = K_{0}(x,t,u,v) + K_{1}(x,t,u,p,q)$$

$$f^{i}(x,t,u,p,q,v) = f_{0}^{i}(x,t,u,v) + f_{1}^{i}(x,t,u,p,q), i = 1, n$$

The function

$$H(x,t,u,v,\psi) = -K_{0}(x,t,u,v) + (f_{0}(x,t,u,v), \psi)$$

is introduced, where $\psi = \psi(x,t) = (\psi^{1}(x,t), \psi^{2}(x,t),...,\psi^{n}(x,t))$ is a solution of the conjugate system.

The following conditions are put:

Let $\forall (x,t) \in Q, \quad v_{i} \in D, \quad i = 1,2, \quad \lambda \in (0,1), \quad u \in R$, 

$$|f_{0}(x,t,u,\frac{v_{1}+v_{2}}{2}) - \frac{1}{2}\lambda f_{0}(x,t,u,v_{1}) - \frac{1}{2}f_{0}(x,t,u,v_{2})|_{E^{n}} \leq \frac{\frac{\alpha}{4}}{1} |v_{2} - v_{1}|_{r} ,$$

$$K_{0}(x,t,u,\frac{v_{1}+v_{2}}{2}) - \frac{1}{2}K_{0}(x,t,u,v_{1}) - \frac{1}{2}K_{0}(x,t,u,v_{2}) \leq -\frac{\frac{1}{4}}{1} |v_{2} - v_{1}|_{r} ,$$

$$|\psi(x,t)| \leq R, \quad \forall (x,t) \in Q$$
Theorem 4. The solution of the problem (13) - (15) at $\chi > \chi_1 R$ exists and we can choose strongly converging subsequence to the solution in $L_2^r(Q)$ from minimizing sequence.

From the theorem 4 is obtained the following.

Corollary 2. Let $f_i(x, t, u, v)$ be linear relatively $v$ and $K_0(x, t, u, v), \varphi(x, t) \in Q, u \in R^n$ is strict convex on $v$ in $D$. Then the statement of the theorem 4 is valid.

At last we shall note the results concerning to investigation of existence of the solution

Let the following conditions fulfill

| Condition | Description |
|-----------|-------------|
| 1) $f(x, t, v, u, p)$ almost for every $(x, t) \in \Omega$ is convex relatively $(v, u, p) \in V_0 \times R \times R^n$; |
| 2) Function $u \rightarrow f(x, t, v, u, p)$ decreases on $R$, $\forall (x, t) \in \Omega, v \in V_0, p \in R^n$; |
| 3) Function $K(x, t, v, u)$ and $(x, z)$ for $\varphi(x, t) \in \Omega$ are convex relatively $(v, u) \in V_0 \times R$ and $z \in R$; |
| 4) Functions $u \rightarrow K(x, t, v, u), z \rightarrow \Phi(x, z)$ increase on $R, \forall (x, t) \in \Omega, v \in V_0$. |

Theorem 5. The solution of the problem (20) - (22) is convex relatively $v \in V$ almost for all $(x, t) \in \Omega$, at conditions 1), 2).

Using this result, it is proved

Theorem 6. Let the conditions 1)-4) be satisfy. Then the solution of the problem (13) - (15) exists.

Introduced the Hamilton - Potryagen functional for the problem (13) - (22)

$$H(x, t, v, u, \psi) = f(x, t, v, u, u_x) \Psi - K(x, t, v, u).$$
Theorem 7. Let \( u^*(x,t) \) and \( \Psi^*(x,t) \) be the solution of the basic and adjoint problem at \( v = v^*(x,t) \in V \). Then it is sufficiency and necessary the fulfillment of condition for optimality of the control \( v^*(x,t) \)

\[
H(x,t,v^*(x,t),u^*(x,t),\psi^*(x,t)) = \max_{v \in V_0} H(x,t,v,u^*(x,t),\psi^*(x,t)), \quad \forall (x,t) \in \Omega. \tag{23}
\]

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