Abstract. In this paper we discuss a recently discovered continued fraction expansion

\[ e = 3 - \frac{1}{4 - \frac{2}{5 - \frac{3}{6 - \frac{4}{7 - \cdots}}}} \]

and its convergence properties. We show that this expansion is a particular case of a continued fraction expansion of \( e^n \), for positive integer power \( n \), and more generally, it is a special case of a continued fraction expansion of the incomplete gamma function, or equivalently, of the confluent hypergeometric function.

1. Introduction

There are several well-known continued fraction expansions for the Euler number (or Napier constant) \( e \). For instance, the following

\[ e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \cdots}}}}, \]
\[ e = 2 + \frac{2}{3 + \frac{4}{5 + \frac{4}{5 + \cdots}}}, \]
\[ e = 1 + \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \cdots}}}}. \]

can be found in the Appendix A of [4]. Recently, the expansion

\[ e = 3 - \frac{1}{4 - \frac{2}{5 - \frac{3}{6 - \frac{4}{7 - \cdots}}}} \tag{1} \]

was discovered algorithmically in [5] and proved in [8]. By comparison of the initial convergents, it is clear that the continued fraction (1) is not obtained from an equivalence transformation of any of the above expansions.

In this paper we give an alternate proof of the continued fraction expansion (1) by establishing the formula

\[ e^n = \sum_{k=0}^{n-1} \frac{n^k}{k!} + \frac{n^{n-1}}{(n-1)!} \left( 1 + n + \prod_{m=1}^{\infty} \left( 1 + \frac{-n(m+n-1)}{m+2n+1} \right) \right), \tag{2} \]

valid for \( n \in \mathbb{N} \). Here, the notation

\[ \prod_{m=1}^{\infty} \left( \frac{a_m}{b_m} \right) := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \]

is used for the continued fraction with partial numerators \( a_n \) and partial denominators \( b_n \).
The convergents of the continued fractions (2) have certain interesting properties. We illustrate this by considering the case \( n = 1 \). Denote by \( C_k = P_k/Q_k \) the \( k \)-th convergent of (1). The first convergents of (1) are given by

\[
\begin{align*}
C_0 &= 3, & C_1 &= \frac{11}{4}, & C_2 &= \frac{49}{18}, & C_3 &= \frac{87}{32}, & C_4 &= \frac{1631}{600}, & C_5 &= \frac{11743}{4320}.
\end{align*}
\]

Note that since the partial numerators of (1) are negative, the convergents are monotonically decreasing. The difference between the first successive convergents is given by

\[
C_1 - C_0 = -\frac{1}{4}, \quad C_2 - C_1 = -\frac{1}{36}, \quad C_3 - C_2 = -\frac{1}{184}, \quad C_4 - C_3 = -\frac{1}{2400}.
\]

In general (see Proposition 2.4), the absolute value of the numerators of the differences is 1, similar to the case of regular continued fractions.

The convergence of the continued fraction is faster than the regular continued fraction. In fact, in Corollary 2.5, we see that

\[
|e - C_k| = O \left( \frac{1}{k!(k+1)^2(k+2)^2} \right),
\]

for \( k \in \mathbb{N} \). In §2 we establish the corresponding properties for the continued fractions (2).

The continued fraction expansion (2) in general cannot be extended to a continued fraction for \( e^z \) where \( z \) is a complex variable (or even a real number different of \( z = n \in \mathbb{N} \)). In fact, in §3 we show that the appropriate extension of (2) to complex variable (Theorem 3.1) corresponds to a continued fraction expansion of the incomplete gamma function, or equivalently, of the confluent hypergeometric function.

Finally, we remark that the continued fraction (2) was discovered (independently of [5]) by the author in the study of orthogonal polynomials related to solutions of a certain second order differential operator used in quantum optics (the asymmetric quantum Rabi model (AQRM), see e.g. [3, 6]).

2. THE CONTINUED FRACTION EXPANSION OF \( e^n \)

In this section we prove the convergence of the continued fraction expansion (2) and provide an estimate for its convergence speed.

**Theorem 2.1.** For \( n \in \mathbb{N} \) we have

\[
e^n = \sum_{k=0}^{n-1} \frac{n^k}{k!} + \frac{n^{n-1}}{(n-1)!} \left( 1 + n + \sum_{m=1}^{\infty} \left( -n(m+n-1) \frac{m+n}{m+2n+1} \right) \right)
\]

To prove the theorem we construct a tail sequence using an associated recurrence relation and then use the Waaadeland tail theorem (see Chapter 2 of [4]) to establish the convergence. We make use of some lemmas to prove this result.

**Lemma 2.2.** For a fixed \( n \in \mathbb{N} \), the recurrence relation

\[
X_k = (k+2n+2)X_{k-1} - n(k+n)X_{k-2},
\]

has the solution

\[
X_k = (k+2)\Gamma(k+n+2),
\]

for \( k \geq 1 \).

**Proof.** We verify directly. The right-hand side equals

\[
(k+2n+2)(k+1)\Gamma(k+n+1) - n(k+n)k\Gamma(k+n)
= \Gamma(k+n+1)\left((k+2n+2)(k+1) - nk\right)
= \Gamma(k+n+1)(k+n+1)(k+2)
= (k+2)\Gamma(k+n+2),
\]

which is equal to \( X_k \), as desired. \( \Box \)
The next lemma is used for the evaluation of the hypergeometric series that appear in the computation of the limit by Waadeland’s Tail Theorem. We recall the definition of the incomplete gamma function $\gamma(s,x)$, namely

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt,$$

also, for $p, q \geq 1$, the notation $\pFq{p}{q}{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q}{x}$ is used for the hypergeometric series in the standard way (see e.g. [1]).

**Lemma 2.3.** The formula

$$2F_2(1,1;3,z+2;z) = \frac{2(z+1)}{z^2} \left( 1 + z - \frac{\gamma(z,z)}{z^2e^{-z}} \right),$$

is valid for $z$ in the cut plane $z \in \{z \in \mathbb{C} : |\text{arg}(s)| < \pi\}$. In particular, for $n \in \mathbb{N}$ we have

$$2F_2(1,1;3,n+2;n) = \frac{2(n+1)}{n^2} \left( 1 + n - \frac{(n-1)!}{n^{n-1}} \left( e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) \right).$$

**Proof.** Using the Euler’s integral transform two times to the hypergeometric series, we get

$$2F_2(1,1;3,z+2;z) = (z+1) \int_0^1 (1-t)^z_1F_1(1;3;tz) dt$$

$$= 2(z+1) \int_0^1 (1-t)^z \int_0^1 (1-s)e^{st}dsdt$$

$$= \frac{2(z+1)}{n^2} \int_0^1 (1-t)^z \left( e^{zt} - zt - 1 \right) dt,$$

then, by partial integration, we obtain

$$(3) \quad \frac{2(z+1)}{z} \int_0^1 (1-t)^{z-1} (1 + z - e^{zt}) dt = \frac{2(z+1)}{z} \left( \frac{1 + z}{z} - \int_0^1 (1-t)^{z-1}e^{zt}dt \right).$$

A change of variable $s = 1 - t$ gives the first statement of the lemma

$$\frac{2(z+1)}{z} \left( \frac{1 + z}{z} - \frac{e^z}{z} \int_0^1 s^{z-1}e^{-s}ds \right) = \frac{2(z+1)}{z} \left( \frac{1 + z}{z} - \frac{e^z}{z} \gamma(z,z) \right).$$

For the second statement, recall the expression for the residue term $R_n$ of the $n$-th order Taylor’s expansion of $f(x) = e^x$ around $x = 0$ (see 5.41 of [7]), evaluated at $n$,

$$R_n = \frac{n^n}{(n-1)!} \int_0^1 (1-t)^{n-1}e^{nt}dt = e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!}.$$  

Using this expression in (3) gives the desired expression

$$\frac{2(n+1)}{n^2} \left( 1 + n - \frac{(n-1)!}{n^{n-1}} \left( e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) \right).$$

□

Next, we present the proof of Theorem 2.1. Recall that for any $a \in \mathbb{C}$, $(a)_n$ denotes the Pochhammer symbol, that is $(a)_0 := 1$ and

$$(a)_n := a(a+1) \cdots (a+n-1),$$

for $n \in \mathbb{N}$. 
Proof of Theorem 2.1. We obtain the result by computing the value of the shifted continued fraction \( K_{m=1}^{\infty} \frac{-n(m+n)}{m+2n+1} \). By Lemma 2.2 a tail sequence \( \{t_j\}_{j=0}^{\infty} \) for the continued fraction is given by

\[
t_j = \frac{(j + n + 1)(j + 2)}{j + 1}, \quad t_0 = -2(n + 1).
\]

Setting \( b_j = j + 2n + 2 \) for \( j \in \mathbb{N} \), define

\[
\Sigma_l = \sum_{k=0}^{l} \prod_{j=1}^{k} \left( \frac{b_j + t_j}{-t_j} \right) = \sum_{k=0}^{l} \frac{(1)_k(1)_k}{(3)_k(n+2)_k} \frac{n^k}{k!},
\]

then, taking the limit and using Lemma 2.3 we obtain

\[
\Sigma_\infty = 2F_2(1; 1; n+2; n) = \frac{2(n+1)}{n^2} \left( 1 + n - \frac{(n-1)!}{n^{n-1}} \left( e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) \right).
\]

From Waadeland’s tail theorem it follows that the continued fraction \( K_{m=1}^{\infty} \frac{-n(m+n)}{m+2n+1} \) converges to a finite value \( f_1 \) given by

\[
f_1 = -2(1 + n) \left( 1 - \frac{1}{\Sigma_\infty} \right).
\]

Clearly, \( K_{m=1}^{\infty} \frac{-n(m+n-1)}{m+2n+1} \) is also convergent and

\[
\frac{K_{m=1}^{\infty} \frac{-n(m+n-1)}{m+2n+1}}{2(1+n)} = \frac{-n^2}{2(1+n) \Sigma_\infty} = \frac{(n-1)!}{n^{n-1}} \left( e^n - \sum_{k=0}^{n-1} \frac{n^k}{k!} \right) - (1 + n).
\]

The result follows immediately. \( \square \)

Next, we establish the explicit expression of the difference between successive convergents and the estimate of the rate of convergence.

Proposition 2.4. Let \( k \in \mathbb{N} \) and \( C_k = P_k/Q_k \) be the convergent of the continued fraction expansion of \( e^n \) given in Theorem 2.1. We have

\[
C_k - C_{k-1} = -\frac{n^{n+k+1}}{(n-1)!(n)_{k+1}(k+1)k}.
\]

In particular, \( n \) divides the numerator of \( C_k - C_{k-1} \).

Proof. From the Euler-Wallis identities, it holds that

\[
P_k/Q_k - \frac{P_{k-1}}{Q_{k-1}} = (-1)^{k-1} a_1 a_2 \ldots a_k \times \frac{n^{n-1}}{(n-1)!} = \frac{n^{n+k-1}(n)_k}{(n-1)!(n)_{k+1}Q_k Q_{k-1}},
\]

with \( a_k = -n(k+n-1) \). The denominators \( Q_k \) satisfy the recurrence relation \( Q_k = (k+2n+1)Q_{k-1} - n(k+n+1)Q_{k-2} \), with initial condition \( Q_0 = 0, Q_1 = 1 \), and it can easily be verified that a solution is given by \( Q_k = (1/n)(k+1)(n)_{k+1} \). Replacing this expression in (4) gives the result. \( \square \)

The curious property mentioned in the introduction on the convergents of \( \Pi \) is just a special case of the fact that in the continued fraction expansion of \( e^n \), the difference between two successive convergents is divisible by \( n \). The estimate on the rate of convergence follows immediately from the proposition.
Corollary 2.5. Let \( C_k = P_k/Q_k \) be the convergents of the continued fraction expansion of \( e^n \) given in Theorem 2.1. We have
\[
|e^n - C_k| = O \left( \frac{n^{k+1}}{(k+1)(k+2)(n)_{k+2}} \right).
\]

Remark 2.6. Here, we point out the reason for using the shifted continued fraction in the proof of Theorem 2.1. Using the original continued fraction gives a tail sequence from the associated recurrence relation (as in the proof of Proposition 2.4). However, the initial value of the tail sequence is \( t_0 = -b_1 \), hence the sequence does not satisfy the hypothesis of Waadeland’s theorem. In addition, a direct approach by finding a solution of the recurrence relation for the numerators \( P_k \) is considerably more complicated.

3. Extension to complex variable

The proof of Theorem 2.1 suggests that the continued fraction expansion of \( e^n \) is just a special case of a more general result since Lemma 2.2 still holds when we replace \( n \) with \( z \in \mathbb{C} - \{0, -1, -2, \ldots \} \).

This is indeed the case and in this section we discuss this generalization.

Theorem 3.1. For \( z \in \{ z \in \mathbb{C} : |\arg(z)| < \pi \} \), we have
\[
\frac{\gamma(z, z)}{z^2 - 1 e^{-z}} = 1 + z + \sum_{m=1}^{\infty} \frac{-z(m + z - 1)}{m + 2z + 1},
\]
pointwise and uniformly in compacts.

Proof. As mentioned above, it is clear that both lemmas and the proof of the theorem hold for \( z \) in the indicated cut plane, so it only remains to prove the uniform convergence in compacts. Suppose \( D \) is compact domain in the cut plane, next suppose \( M \in \mathbb{R}_{>0} \) is such that
\[
|z^2| \leq M,
\]
for all \( z \in D \). Let \( f_k(z) \) denote the convergents of the continued fraction (5) and \( f(z) \) the pointwise limit of \( f_k(z) \), we have
\[
|f(z) - f_k(z)| \leq M|\Sigma_{\infty}(z) - \Sigma_{k-1}(z)|,
\]
where \( \Sigma_{k-1}(z) \) and \( \Sigma_{\infty}(z) \) are defined in terms of the shifted continued fraction as in the proof of Theorem 2.1. The uniform convergence then follows from that of the hypergeometric series. □

Corollary 3.2. For \( z \in \{ z \in \mathbb{C} : |\arg(z)| < \pi \} \) and \( n \in \mathbb{N} \), it holds that
\[
_1 F_1 (1; z + 1; z) = 1 + z + \sum_{m=1}^{\infty} \frac{-z(m + z - 1)}{m + 2z + 1},
\]
\[
\int_0^1 (1 - t)^{n-1} e^{nt} dt = \frac{1}{n} \left( 1 + n + \sum_{m=1}^{\infty} \frac{-n(m + n - 1)}{m + 2n + 1} \right).
\]

It would seem at first glance that the continued fraction expansion of Theorem 2.1 could be extended to a continued fraction for arbitrary powers of \( e \). However, by the proof of Theorem 2.1 and from Theorem 3.1, we see that the expansion of \( e^n \) is accidental in the sense that it arises from special values of the incomplete gamma function \( \gamma(s, x) \).

Finally, it is interesting to compare expansion the expansion in Corollary 3.2 to the standard M-fraction representation
\[
_1 F_1 (1; b + 1; z) = \frac{b}{b - z} + \sum_{m=1}^{\infty} \frac{mz}{b + m - z},
\]
for $z \in \mathbb{C}$ and $b \in \mathbb{C} - \mathbb{Z}_{\leq 0}$ (see e.g. (16.1.17) in [2]), upon setting $b = z \in \{ z \in \mathbb{C} : \arg(z) < \pi \}$ we obtain the expansion

$$\mathbf{1}_F(1; z + 1; z) = \frac{z}{1 + \sum_{m=1}^{\infty} \left( \frac{m z}{m} \right)}.$$  

It is immediate to check that these two expansions are not related by an equivalence transformation, since the convergents are different.

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Cid Reyes-Bustos
Department of Mathematical and Computing Science, School of Computing, Tokyo Institute of Technology
2 Chome-12-1 Ookayama, Meguro, Tokyo 152-8552 JAPAN
reyes@c.titech.ac.jp