The Influence of Percolation in the generalized Chalker–Coddington Model

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We numerically investigate the influence of classical percolation on the quantum Hall localization-delocalization transition. This is accomplished within the framework of the generalized Chalker–Coddington network model which allows us to control the number of classical saddle points by setting the width $W$ of the saddle point distribution. It is found that increasing this width causes a new microscopic length scale to appear which depends on $W$ and scales with the exponent $X \approx 1.36$ which indicates a close connection to the classical percolation length $\xi$ and its exponent $\nu_p = 4/3$. Furthermore, the influence of an increase in $W$ on the spectral statistics of the quasieenergies of the network model is investigated. An effect similar to the increase of the potential correlation length in the Landau model is seen.

KEYWORDS: Chalker-Coddington model, level spacing distribution, scaling, quantum Hall systems, multifractality

§1. Introduction

The Chalker-Coddington network\cite{1} is a model for quantum Hall systems with long ranged disorder potentials. It represents a system of two-dimensional (2D) electrons in a strong magnetic field and smooth disorder potential. As a member of the quantum Hall universality class\cite{2}, it has been used to determine various critical quantities at the localization-delocalization (LD) transition point between the quantized plateaus of the Hall conductance.\cite{3, 4, 5}

The model is based on the semi-classical time evolution picture of 2D electrons moving along the equipotential contours of a smooth disorder potential under the influence of a strong magnetic field. The electronic states are defined by the amplitudes on the network links representing the equipotential contours and the time evolution is determined by unitary scattering matrices at the nodes of the network corresponding to the saddle points of the random potential where tunneling between contours occurs. The scattering strength is determined by the electron energy and the energy of the saddle point. In contrast to the original network model introduced by Chalker and Coddington, who explicitly excluded any percolation effects by setting all saddle point energies to zero, the generalized version of the model allows percolation effects by introducing an energy range $[-W, W]$ for the distribution of saddle point energies\cite{6}. While this generalization of the model does not change the critical behavior as long as the investigated systems are large enough, it introduces a microscopic length scale $a$ that depends on $W/E_t$, where $E_t$ is the tunneling energy at the saddle points\cite{7}. Since the motion of electrons with energy $E$ at saddle points with energies $u_k$ that obey $|u_k - E| \gg E_t$ follows the classical path, it was concluded in refs.\cite{8} and\cite{9} that this length scale must be connected to the classical percolation length, meaning that $a$ scales

with the classical percolation exponent $\nu_p = 4/3$, i.e.

$$a \left( \frac{W}{E_t} \right) \propto \left( \frac{W}{E_t} \right)^{\nu_p}. \quad (1)$$

As long as the system size $L$ is much larger than this length scale the generalized model will show the same critical properties as the original.

In the following we will first give a short description of the generalized Chalker–Coddington model followed by a review of the arguments given in refs.\cite{5} and\cite{7} for the influence of percolation effects. We will then give numerical evidence that the microscopic length scale indeed shows the scaling behavior we expected. After that we take a look at the spectral properties of the network model and their dependence on $W$. We will show that the shape of the level spacing distribution function at criticality changes more and more to Poissonian behavior with increasing $W$. This is also the case for the number variance and consequently the spectral compressibility. In spite of these changes the scaling exponent $\nu$ seems to be unaffected. The change of shape in the level spacing distribution is similar to that observed by Ono et al.\cite{21} when increasing the correlation length of the disorder potential.

§2. The Network Model

As we have already mentioned the Chalker–Coddington model is based on the semi-classical picture of electrons in two dimensions moving under the influence of a strong perpendicular magnetic field $B$ in a long-ranged disorder potential $V$, i.e. the correlation length $l_V$ of $V$ is large compared to the magnetic length $l_c = \sqrt{\hbar c/eB}$. Starting from this picture the following network model was developed.\cite{10}

The network consists of a 2D regular lattice (see Fig. 1) whose links are unidirectional channels and whose nodes are scattering centers represented by $2 \times 2$ unitary scattering matrices $S_k$, where $k$ is the node index. The ma-
The correlation length exponent \( \alpha \) for other critical exponents like multifractal analysis of critical eigenfunctions.

5) \( \psi \) approaches the critical energy \( E \) in a magnetic field and random potential \( L/E \). It also connects the two limiting cases of the network model directly to the relevant parameter of the network \( \text{size} \times \text{critical energy} \).

The network operator \( U \) also functions as a time evolution operator for the network states, i.e. \( \Psi(t+\tau) = U\Psi(t) \), where \( \tau \) is the characteristic scattering time. Equation (5) states that a wave function which is an eigenfunction of \( U(E) \) with eigenvalue 1 is an eigenfunction of the modeled system. Such eigenfunctions will only occur at discrete values \( E_n = E \) forming the eigenenergy spectrum of the system. These eigenenergies are not easily determined, but it was found in ref. [12] that the eigenvalues \( \omega_n \) defined by the equation

\[
U(E)\Psi = \Psi, \tag{3}
\]

for a fixed value of \( E \) show the same statistics as the eigenenergies \( E_n \) close to \( E \). These so called quasienergies \( \omega_n(E) \) are much easier to determine. This and the fact that one can choose the exact point on the energy scale where we want to investigate the statistics make them the ideal tool for spectral investigations.

We can set \( E = E_c \) and use all the critical quasienergies \( \omega_n(E_c) \), i.e. the eigenphases of \( U(E_c) \) to determine spectral properties at criticality. This is an enormous advantage compared to other methods where only a small fraction of the spectrum is critical. We can even improve this method if we use the fact that for every eigenphase \( \omega_n \) of the network operator the phase \( \omega_n + \pi \) (or in the case of double periodic boundary conditions the phases \( \omega_n + \pi/2, \omega_n + \pi \) and \( \omega_n + 3\pi/2 \)) is also an eigenphase (see Appendix). This leads to a reduction of the matrix size increasing the speed of the numerical determination and also the attainable system sizes. Additionally, every eigenstate determined for a unitary operator \( U(E_c) \) at the critical energy \( E_c \) is critical and can be used for multifractal analysis.

§3. Percolation and Multifractality

Although percolation, i.e. \( W > 0 \), does not seem to have an influence on scaling and thereby on the result of the multifractal analysis, there are nevertheless some effects which have to be investigated. They are very obvious if one looks at the wave functions themselves (see Fig. 2), but also show some influence in the scaling analysis.

The eigenstates in Fig. 2 obtained in the generalized network model at \( E = E_c \) show the typical self-similar shape of critical wave functions which lead to the use of multifractal analysis on critical systems. In this context the scale invariance of the eigenfunctions is...
investigated by obtaining the following quantities:

\[ m(l_b) = \int r_0 \, d|\Psi(\mathbf{r} + \mathbf{r}_0)|^2, \]

which denote the probability of a particle to be in a box of linear size \( l_b \) centered at \( \mathbf{r}_0 \), respectively. The disorder averaged \( q \)-moments of \( m(l_b) \), \( m_q(l_b) = \langle m^q(l_b) \rangle \), scale over a wide range of box sizes with definite exponents,

\[ m_q(\lambda) \propto \lambda^{d+\tau(q)}, \]

where \( \lambda = l_b/L \) is the quotient of box size \( l_b \) to system size \( L \). The exponents \( \tau(q) \) depend non-linearly on \( q \) and characterize the universality class of the system. Often the single exponent \( \alpha_0 = d\tau(q)/dq(q = 0) \) is used instead of the entire \( \tau(q) \) spectrum, because it describes the scaling of the typical value of the squared amplitude with the system size, \( \exp(\ln|\Psi|^2) \propto L^{-\alpha_0} \). The results of numerical investigations of these exponent show that they do not depend on the size of \( W \) (i.e. \( W/E_t \)), we will set \( E_t = 1 \) in the following. However, we can observe that the range of box sizes for which scaling behavior is seen does change. For \( W < 1 \) the box probabilities scale over the entire range \( a < l_b < L \), where \( a \) is the lattice constant of the network. If we increase \( W \), scaling deteriorates for small box sizes, i.e. small \( \lambda \)'s. In Fig. 3 we show the \( \ln[m^q(l_b)] - \ln \lambda \)-plots for four different values of \( W \) at \( q = -0.5 \). We see that the deterioration of scaling shows a significant dependence on \( W \) indicating that the valid range of lengths for scaling follows a law of the kind

\[ a \left( \frac{W}{E_t} \right) < l_b < L, \]

where \( a(x) \) is a function we have to determine.

If we take the values of \( \ln \lambda \) for each \( W \) at which the linear approximation becomes invalid and plot them against \( \ln W \), a linear dependence becomes apparent (see Fig. 4). A linear fit of the data yields a slope of \( X = 1.36 \pm 0.06 \). Consequently, the minimal length where scaling can be seen follows the following relation

\[ a \propto \left( \frac{W}{E_t} \right)^x. \]

Another effect of the increase of \( W \) are the increasing sample to sample fluctuations and the rapid increase of the statistical error in the \( f(x) \) data, especially for large values of \( |q| \). This all seems to be accompanied by a visible change of the characteristics of the wave function, as seen in Fig. 2. It is obvious that for \( W = 150 \) the wave function shows large areas, where the square amplitude changes only slightly, whereas for \( W = 0 \) such areas of constant amplitude are much smaller. In fact one can see a steady increase of the size of these areas if one gradually increases \( W \).

If we take all these effects into account, we have to conclude that, apparently, the introduction of variable saddle point energies, i.e. \( W > 0 \), leads to a new length scale in our system.

In order to understand how classical percolation can alter the shape of the quantum mechanical wave functions and influence their scaling behavior we will repeat the discussion published previously in literature. Take a look at those saddle points whose energy \( w_k \) differs more than \( E_t \) from the electron energy \( E = 0 \). Of course, such saddle points occur only for \( W > E_t \). The transmission coefficients \( T_k \) at these saddle points are exponentially small and the amplitude of an incoming link is essentially transmitted to a single outgoing link (classical behavior). Therefore, on a path avoiding sad-
Fig. 5. Along the classical percolation path where the saddle point energies are large, |\(u_k|\) > \(E_t\), the amplitude of the wave functions remain mainly constant. The resulting critical clusters form the links of the rescaled network.

In the metallic regime, nor strongly localized. The direct connection between level distribution and multifractal wave functions. This equation could be verified numerically for the Chalker–Coddington network, and shows that we are again between the metallic (\(\chi = 1\)) behavior. If our arguments concerning the influence of classical percolation are valid, we should obtain the same kind of results for the critical spectra regardless of the value of \(W\). On the other hand, we have seen that increasing \(W\) causes finite size effects to appear faster which should also be visible in the spectral properties.

We start by taking a look at the \(P(s)\) distribution for different values of \(W\). In Fig. 6 we can see \(P(s)\) for \(W = 0\) to \(W = 40\). In this case we chose the system size \(L = 80\), but we obtain the same \(P(s)\) for other system sizes. The distribution is still system size independent at least for the range of system sizes we are able to investigate. This rules out finite size effects which could be expected due to the decrease of effective system size observed for the wave functions. Nevertheless, the distribution seems to become more and more Poissonian for larger \(W\). This effect has already been seen by Ono et al. who used the Landau model for the quantum Hall effect. In this case an increase of the potential’s correlation length, which is
1. According to eq. (3) comparable to an increase of $W$, led to a similar result. In their publication Ono et al. fitted the $P(s)$ distribution to the formula:

$$P(s) = A s^\beta \exp(-B s^\alpha),$$

(10)

where $A$ and $B$ are determined by the normalization conditions $\langle s \rangle = 1$ and $\alpha$ and $\beta$ are the fitting parameters. This way of fitting was prompted by the results of Kravtsov et al. [3, 13] which predicted that $\beta = 2$ as for the Gaussian unitary ensemble (GUE), where $P(s) = \frac{12}{\pi^2} s^2 \exp(-4/\pi s^2)$, and $\alpha = 1 + 1/(d \nu) \approx 1.21$. Although the numerical values for $\alpha$ do not fit that prediction, we can still use eq. (10) to compare our data with that in ref. 21. The result of such a fit to the distributions in Fig. 6 is shown in Table I. We can see that $W \approx 30$

![Image of Fig. 6: The $P(s)$ spectrum for $W = 0, 10, 20, 30, 40$. The smooth curves show the Poisson and GUE distributions. The inset shows the same on a logarithmic scale.](image)

2. For both models it is very clear that the level repulsion for small values of $s$ decreases with increasing $W$ and correlation length, respectively. This is indicated by the increase of $P(s = 0)$ and the decrease of the fit parameter $\beta$.

3. In ref. 13 it was found that the tail of $P(s)$ shows an exponential decay of the form $P(s) \propto \exp(-\kappa s)$, which was predicted by Altshuler et al. [3] with $\kappa = 1/(2\chi)$ and confirmed for QHE systems by the numerical results in refs. 3 and 24. In Fig. 7 we show the behavior of the tail region of $P(s)$ for different $W$. Although the tail seems to remain an exponential the factor $\kappa$ decreases with increasing $W$ getting closer to the Poissonian behavior, where $\kappa = 1$. This would indicate that the compressibility $\chi$ also changes with $W$. In Fig. 8 we show the level number variance $\Sigma_2(N)$ from which we obtain $\chi$ by determining its slope. We can see that the slope increases with increasing $W$ which would correspond to the decrease of $\kappa$, but a look at Table II shows that the relation $\kappa = 1/(2\chi)$ no longer holds in the case $W > 0$.

![Image of Fig. 7: The tail of the $P(s)$ spectrum for $W = 0, 10, 20, 30, 40$. The smooth curves are the linear fits for respective curves.](image)

![Image of Table I: The results for the fit parameters $\alpha$ and $\beta$ for different values of $W$. The last two entries give the values for different potential ranges determined by Ono et al.](image)

| $W$ | $\alpha$ | $\beta$ |
|-----|---------|---------|
| 0   | 1.54    | ±0.02  |
| 10  | 1.71    | ±0.04  |
| 20  | 1.79    | ±0.02  |
| 30  | 1.94    | ±0.03  |
| 40  | 2.04    | ±0.03  |

| $W$ | $\kappa$ | $\chi$ | $1/(2\chi)$ |
|-----|---------|--------|-------------|
| 0   | 4.09    | ±0.10  | 0.127 ±0.002 |
| 10  | 3.72    | ±0.06  | 0.190 ±0.003 |
| 20  | 3.26    | ±0.05  | 0.254 ±0.001 |
| 30  | 2.83    | ±0.04  | 0.291 ±0.002 |
| 40  | 2.68    | ±0.04  | 0.313 ±0.002 |

Table II. The results for $\kappa$ and $\chi$ for different values of $W$.

This is also true for the relation $\chi = \frac{4D_2}{2s}$ connecting the compressibility $\chi$ to the multifractal exponent $D_2$. From the investigation of the critical wave functions we know that $D_2$ does not change with $W$.

Finally, observing that increasing $W$ has a strong effect on the shape of $P(s)$ the question arises whether a function like

$$J_0(L, E) = \frac{1}{2} \langle s^2 \rangle_{L, E},$$

(11)

corresponds to the results obtained by Ono et al. for a potential correlation length of $2l_c$, whereas a correlation length of $l_c$ did not produce significant changes from the uncorrelated results (i.e. $W = 0$). We would expect that, because the relevant microscopic length in the Landau model is $l_c$ so that the potential correlation length has to be larger to show an appreciable effect. This corresponds to the fact that $W \approx E_t$ also has no effect since the tunnel energy $E_t$ sets the energy scale for our system and $W$ has to be larger in order to obtain classical saddle points.
which depends on as well as describes the shape of $P(s)$ still shows the same one-parameter scaling for $W > 0$ as for $W = 0$. The insets in Fig. 9 shows $J_0$ for $W = 5, 15$ as a function of $E$ for different system sizes. At the critical point $E_c = 0$ all curves come together at a single size independent point which varies with $W$ reflecting the dependence of the shape of $P(s)$ on $W$. If one-parameter scaling still holds then a scaling function $f$ of the following form should exist:

$$J_0(E, L) = f(L/\xi), \quad \xi = \epsilon_0|E - E_c|^{-\nu}. \quad (12)$$

We determined $f(x)$ by fitting a fourth degree polynomial in $x = \ln(L/\epsilon_0) + \nu \ln E$ to our data using $\epsilon_0$ and $\nu$ as fitting parameters. The values we found for $\nu$ and the critical values of $J_0$ are given in Table III. Although the critical values $J_0(W)$ vary systematically with $W$ the scaling exponent $\nu$ shows no systematic dependence on $W$ and is consistent within previous results for $\nu$ obtained by various studies. [10, 11, 12] within the error bars.

| $W$ | $\nu$ | $J_0^c$ |
|-----|-------|---------|
| 0   | 2.1   | 0.600   | $\pm 0.005$ |
| 2   | 2.1   | 0.601   | $\pm 0.01$  |
| 5   | 2.2   | 0.604   | $\pm 0.01$  |
| 10  | 2.1   | 0.611   | $\pm 0.01$  |
| 15  | 2.3   | 0.623   | $\pm 0.01$  |
| 30  | 2.3   | 0.664   | $\pm 0.01$  |
| 40  | 2.2   | 0.685   | $\pm 0.01$  |

Table III. The results for the scaling exponent $\nu$ and the critical value of $J_0$ for different values of $W$.

5. Conclusion

In this paper we have discussed the influence of percolation effects on critical wave functions and the critical quasienegy spectrum of the Chalker–Coddington network model. Classical percolation effects appear in the generalized version of the model when the range $W$ of the saddle point energies, i.e. node energies of the network, is increased from $W = 0$ in the original model to

$W > 1$.

We found that multifractal analysis of the critical wave functions results in the same exponents regardless of the value of $W$. Nevertheless, we can see the effect of classical percolation in the appearance of a new microscopic length scale which increases with $W$ and scales with an exponent $X = 1.36 \pm 0.06$ consistent with the exponent $\nu_p = 4/3$ of the classical correlation length. This confirms the connection of the correlation length to that new length scale, which sets the minimum length for scaling and therefore reduces the effective size of the system, thereby causing increasing fluctuations when $W$ grows.

In the case of the spectral statistics of the quasienegy spectrum of the network the influence of an increasing $W$ is more profound. An increasing shift from the critical statistics towards Poissonian behavior is observed for all the spectral quantities we have investigated. Although this shift does not seem to have an influence on the scaling exponent $\nu$ it is clearly visible. Since the effect is not changing with system size, it can not be ruled as a mere finite size effect resulting from the decrease of effective system size observed for the critical wave functions. Furthermore, we observed that the changes that come with an increase of $W$ are very similar to those connected with
the increase of the potential correlation length observed by Ono et al.\textsuperscript{[11]}

It seems, although the multifractal behavior of the critical wave functions is well understood, that more elaborate analytical studies are indicated to understand the level statistics at the LD transition.

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Appendix: Degeneracy of Eigenvalues

The chiral structure of the Chalker-Coddington network leads to a pseudo degeneracy of the eigenvalues of its network operator $U$. For each eigenvector $\Psi_\alpha$ with eigenvalue $e^{i\omega_\alpha}$ there exits an eigenvector $\tilde{\Psi}_\alpha$ with eigenvalue $e^{i(\omega_\alpha + \pi)}$.

In order to prove this, let us first point out the fact that the operator $U$ maps vertical links only to horizontal links and vice versa. This means that $U^2$ maps vertical to vertical and horizontal to horizontal links, thereby creating two orthogonal $U^2$-invariant subspaces.

We can therefore define the projection operators $P_v$ and $P_h$ which will project onto the two subspaces, respectively.

Let $\Psi_\alpha$ be an eigenvector of $U$ with eigenvalue $e^{i\omega_\alpha}$. Then $\tilde{\Psi}_\alpha$ is also an eigenvector of $U^2$ with eigenvalue $e^{2i\omega_\alpha}$. We can write $\Psi_\alpha$ as a linear combination of the two projections $\tilde{\Psi}_\alpha = P_v\Psi_\alpha$ and $\tilde{\Psi}_\alpha = P_h\Psi_\alpha$:

$$
\Psi_\alpha = \Psi_\alpha^v + \Psi_\alpha^h = v\Psi_\alpha^v + h\Psi_\alpha^h,
$$

where $v = |\Psi_\alpha^v|$, $h = |\Psi_\alpha^h|$, $\Psi_\alpha^v = v^{-1}\Psi_\alpha^v$ and $\Psi_\alpha^h = h^{-1}\Psi_\alpha^h$. Since $\Psi_\alpha^v$ and $\Psi_\alpha^h$ are orthogonal to each other they are both also eigenvectors of $U^2$ with eigenvalue $e^{2i\omega_\alpha}$. This means that this eigenvalue is degenerate.

Let us now construct the following eigenvector of $U^2$:

$$
\tilde{\Psi}_\alpha = h\Psi_\alpha^v - v\Psi_\alpha^h.
$$

$\tilde{\Psi}_\alpha$ is obviously orthogonal to $\Psi_\alpha$. In order to see what will happen if we use $U$ on $\tilde{\Psi}_\alpha$, let us first use it on $\Psi_\alpha$ and keep in mind that $U\Psi_\alpha = e^{i\omega_\alpha}\Psi_\alpha$ and $U\Psi_\alpha^v = e^{i\omega_\alpha}\Psi_\alpha^v$, where $e^{2i\omega_\alpha} = e^{2i\omega_\alpha - i\omega_\alpha}$.

For eqs. (A.2) and (A.3) it follows that $v = h = e^{i(\omega_\alpha - \omega_\alpha)}$ and consequently

$$
U\tilde{\Psi}_\alpha = h\Psi_\alpha^v - v\Psi_\alpha^h
= e^{2i(\omega_\alpha - \omega_\alpha)}\Psi_\alpha^v e^{i\omega_\alpha} - e^{2i(\omega_\alpha - \omega_\alpha)}\Psi_\alpha^v e^{i\omega_\alpha} = e^{i\omega_\alpha} - e^{2i(\omega_\alpha - \omega_\alpha)}\Psi_\alpha^v e^{i\omega_\alpha}
= -e^{i\omega_\alpha} - e^{i\omega_\alpha} \tilde{\Psi}_\alpha.
$$

This means $\tilde{\Psi}_\alpha$ is an eigenvector of $U$ with eigenvalue $e^{i(\omega_\alpha + \pi)}$.

The same kind of argument leads to a further pseudo-degeneracy when we impose double periodic boundary conditions (torus) with system size $L$ a multiple of 2. In that case each eigenphase $\omega_\alpha$ is accompanied by three other eigenphases $\omega_\alpha + \pi/2$, $\omega_\alpha + \pi$ and $\omega_\alpha + 3\pi/2$. This is caused by the chiral symmetry above and the boundary conditions which lead to the formation of two more (in this case $U^2$) invariant subspaces within each of the $U^2$-invariant subspaces of vertical and horizontal lines. These subspaces consist of the sets combining every second vertical row or every second horizontal column, respectively. Without the boundary conditions those subspaces mix.

References

[1] J. T. Chalker, P. D. Coddington, J. Phys. C 21, 2665 (1988).
[2] M. Janssen, O. Viehweger, U. Fastenrath and J. Hajdu: Introduction to the Theory of the Integer Quantum Hall Effect (VCH, Weinheim; New York; Basel; Cambridge; Tokyo, 1994).
[3] J. F. G. Eastmond, Ph. D. thesis, Oxford University (1992).
[4] D. H. Lee, Z. Wang and S. Kivelson: Phys. Rev. Lett. 70(1993) 4130.
[5] R. Kliesse and M. Metzler, Europhys. Lett. 32(1995) 229.
[6] We read the data directly from plots like in Fig. 3 with error bars corresponding to the possible errors made by doing so.
[7] D. G. Polyakov and M. E. Raikis: Phys. Rev. Lett. 75(1995) 1368.
[8] H. A. Fertig: Phys. Rev. B 38(1988) 996.
[9] R. Kliesse and M. Metzler, to be published.
[10] R. Kliesse, Ph.D. Thesis, Universitat zu Kolin, (AWOS-Verlag, Erfurt) 1996.
[11] B. Huckestein and R. Kliesse: Phys. Rev. B 55(1997) R7303.
[12] R. Kliesse and M. Metzler: Phys. Rev. Lett. 79(1997) 721.
[13] M. Metzler and I. Varga, J. Phys. Soc. Jpn. 67, 1856 (1998).
[14] M. Metzler, Ph.D. Thesis, Universitat zu Kolin 1996.
[15] M. Janssen: Int. J. Mod. Phys. 8(1994) 943.
[16] M. B. Shicheno, Rev. Mod. Phys. 64, 961 (1992).
[17] D. Stauffer: Introduction to Percolation Theory, Taylor & Francis, 1985.
[18] J. T. Chalker, V. E. Kravtsov and I. V. Lerner: JETP Lett. 64(1996) 386.
[19] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler and A. G. Aronov: Phys. Rev. Lett. 72(1994) 888.
[20] A. G. Aronov, V. E. Kravtsov and I. V. Lerner: JETP Lett. 59(1994) 39.
[21] Y. Ono, T. Ohhtsuki and B. Kramer: J. Phys. Soc. Jpn. 65(1996) 1734.
[22] M. L. Mehta: Random Matrices, 2nd ed. (Academic Press, New York, 1991).
[23] B. L. Altshuler, I. Kh. Zarekesherov, S. A. Kotochigova and B. Shklovskii: Sov. Phys.-JETP 67(1988) 625.
[24] M. Batsch and L. Schweitzer: High Magnetic Fields in the Physics of Semiconductors II: Proceedings of the International Conference, Würzburg 1996, edited by G. Landwehr and W. Ossau (World Scientific Publishers Co., Singapore, 1997), pp. 47-50.
[25] Both relations do not hold for the localized regime where $\chi = 1 \neq 1/2$. This is to be expected since the assumptions made for their derivation are not valid in the localized regime. (See also ref. [18].)
[26] H. Aoki and T. Ando: Phys. Rev. Lett. 54(1985) 831.
[27] B. Huckestein and B. Kramer: Phys. Rev. Lett. 64(1990) 1473.
[28] Y. Hsu and R. N. Bhatt: Phys. Rev. Lett. 68(1992) 1375.
[29] Dongzi Liu and S. Das Sarma: Phys. Rev. B 49(1994) 2677.
[30] T. Ohhtsuki and Y. Ono: J. Phys. Soc. Jpn. 64(1995) 4088.