Odd Colorings of Sparse Graphs

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Abstract

A proper coloring of a graph is called odd if every non-isolated vertex has some color that appears an odd number of times on its neighborhood. The smallest number of colors that admits an odd coloring of a graph $G$ is denoted $\chi_o(G)$. This notion was introduced by Petruševski and Škrekovski, who proved that if $G$ is planar then $\chi_o(G) \leq 9$; they also conjectured that $\chi_o(G) \leq 5$.

For a positive real number $\alpha$, we consider the maximum value of $\chi_o(G)$ over all graphs $G$ with maximum average degree less than $\alpha$; we denote this value by $\chi_o(G_\alpha)$. We note that $\chi_o(G_\alpha)$ is undefined for all $\alpha \geq 4$. In contrast, for each $\alpha \in [0, 4)$, we give a (nearly sharp) upper bound on $\chi_o(G_\alpha)$. Finally, we prove $\chi_o(G_{20/7}) = 5$ and $\chi_o(G_3) = 6$. Both of these results are sharp.

1 Introduction

A proper coloring of a graph is odd if every non-isolated vertex has some color that appears an odd number of times on its neighborhood. The smallest number of colors that admits an odd coloring of a graph $G$ is denoted $\chi_o(G)$. Clearly, $\chi_o(G) \leq |V(G)|$, since we can simply color each vertex with its own color. This notion was introduced by Petruševski and Škrekovski [13], who proved that if $G$ is planar then $\chi_o(G) \leq 9$; they also conjectured that $\chi_o(G) \leq 5$.

Odd coloring is motivated by various types of hypergraph coloring. A hypergraph $H$ consists of a set $V$ of vertices and a set $E$ of (hyper)edges, each of which consists of an arbitrary set of vertices in $V$. Most varieties of hypergraph coloring assign colors (integers in $\{1, \ldots, k\}$) to the elements of $V$ subject to certain constraints. Standard hypergraph coloring requires only that no edge in $E$ is monochromatic. Even et al. [6] introduced conflict-free coloring, which requires that each edge in $E$ has some color that appears exactly once on its vertices. This topic has been widely studied [5, 8, 9, 12, 14]. Cheilaris et al. [4] studied odd coloring of hypergraphs, which requires that each edge in $E$ has some color that appears an odd number of times on its vertices. Aspects of this problem have been studied in [1, 2, 7]. For graphs, Cheilaris [3] studied conflict-free colorings with respect to open neighborhoods. That is, for each vertex $v$ some color appears exactly once on $N(v)$. Finally, Petruševski and Škrekovski [13] studied odd colorings of graphs, which are proper colorings where each vertex $v$ has some color that appears an odd number of times on $N(v)$. It is this parameter that we consider in the present short note.

The average degree of a graph $H$ is $2|E(H)|/|V(H)|$. The maximum average degree of a graph $G$, denoted mad$(G)$, is the maximum, over all non-empty subgraphs $H$ of $G$, of the average degree of $H$. That is, mad$(G) := \max_{H \subseteq G} 2|E(H)|/|V(H)|$. For each positive real number $\alpha$, let $G_\alpha$ denote the family of graphs $G$ with mad$(G) < \alpha$. We denote by $\chi_o(G_\alpha)$ the maximum value of $\chi_o(G)$ over all $G \in G_\alpha$. The focus of this paper is bounding $\chi_o(G_\alpha)$ for $\alpha \in [0, 4)$.
various values of $\alpha$. We observe that $\chi_o(G_o)$ is undefined for all $\alpha \geq 4$. That is, there exists a sequence of graphs $G_n$ such that $\chi_o(G_n) = n$ and $\text{mad}(G_n) < 4$ for all $n$. In contrast, for each $\alpha \in [0, 4)$, we give a (nearly sharp) upper bound on $\chi_o(G_o)$. We have two main results.

**Theorem 1.** Fix $\epsilon$ such that $0 < \epsilon \leq 8/5$. If $\text{mad}(G) \leq 4 - \epsilon$, then $\chi_o(G) \leq [8/\epsilon] + 2$. As $\epsilon \to 0$, infinitely often there exists $G_\epsilon$ such that $\text{mad}(G_\epsilon) = 4 - \epsilon$ and $\chi_o(G_\epsilon) = [8/\epsilon] - 1$.

When $\epsilon \in \{1, 8/7\}$, we prove sharper upper bounds on $\chi_o(G_{4-\epsilon})$.

**Theorem 2.** Fix a graph $G$. (a) If $\text{mad}(G) < 3$, then $\chi_o(G) \leq 6$; and (b) if $\text{mad}(G) < 20/7$, then $\chi_o(G) \leq 5$. Furthermore, neither of these upper bounds on $\chi_o(G)$ can be decreased, and neither of the inequalities can be weakened to allow equality.

For a proper coloring $\varphi$ of some subgraph $H$ of $G$, let $\varphi_o(v)$ denote the unique color that appears an odd number of times in $N_H(v)$ if such a color exists; otherwise, $\varphi_o(v)$ is undefined. Most of the rest of our notation and definitions are standard. But for the reader’s convenience we highlight a few terms. The **girth** of a graph $G$ is the length of a shortest cycle in $G$ (the girth of an acyclic graph is infinite). A $k$-vertex is a vertex of degree $k$. A $k^+$-vertex (resp. $k^-'$-vertex) is a vertex of degree at least (resp. at most) $k$. A $k$-neighbor of a vertex $v$ is a neighbor of $v$ that is a $k$-vertex. Both $k^+$-neighbor and $k^-'$-neighbor are defined analogously.

To close this introduction, we prove three easy results about graphs $G$ with $\chi_o(G) \leq 4$.

**Proposition 1.** A graph $G$ has $\chi_o(G) = 1$ if and only if $G$ has vertices but no edges. And $G$ has $\chi_o(G) = 2$ if and only if $G$ is bipartite and the degree of each vertex in $G$ is either 0 or odd.

**Proof.** The first statement is obvious, so now we prove the second. If $G$ is bipartite, then $G$ has a proper 2-coloring $\varphi$. If the degree of each vertex is either 0 or odd, then $\varphi$ is also an odd 2-coloring of $G$. If $G$ is not bipartite, then $G$ has no proper 2-coloring, so clearly $\chi_o(G) > 2$.

Suppose instead $G$ is bipartite, but some vertex $v$ has positive even degree. Now the component of $G$ containing $v$ has only a single 2-coloring (up to permuting colors), but this 2-coloring is not odd since $N(v)$ has only a single color, which appears an even number of times.

**Proposition 2.** If $3 \mid n$, then $\chi_o(C_n) = 3$. If $n = 5$, then $\chi_o(C_n) = 5$. Otherwise, $\chi_o(C_n) = 4$.

**Proof.** If $3 \mid n$, then we can repeat the colors $1, 2, 3, 1, 2, 3, \ldots$ around $C_n$ to get an odd 3-coloring. Suppose instead that $3 \nmid n$. By the Pigeonhole principle, in every proper 3-coloring $\varphi$, some color appears twice on the neighborhood of some vertex $v$, so $\varphi$ is not odd. Thus, $\chi_o(C_n) \geq 4$.

If $3 \nmid n$ and $n \neq 5$, then we can begin with $1, 2, 3, 4$ or $1, 3, 2, 4, 1, 2, 3, 4$ and continue with $1, 2, 3, 1, 2, 3, \ldots$. Finally, it is easy to check that each proper 4-coloring $\varphi$ of $C_5$ uses the same color on both neighbors of some vertex, so $\varphi$ is not an odd coloring. Thus, $\chi_o(C_5) = 5$.

It is interesting to note that the class of graphs $G$ with $\text{mad}(G) = 2$ and $\chi_o(G) = 4$ is richer than simply cycles $C_n$ with $3 \mid n$. Denote the vertices of $C_n$ by $v_1, \ldots, v_n$. Starting from such a graph, we can add arbitrarily many leaves at each $v_i$ with $3 \mid i$. It is straightforward to check that the resulting graph $G'$ also has $\text{mad}(G') = 2$ and $\chi_o(G') = 4$. More generally, we can identify these $v_i$ with vertices of an arbitrary graph and $\chi_o$ will not decrease.

**Proposition 3.** Every tree $T$ has $\chi_o(T) \leq 3$. Thus, if $\text{mad}(T) < 2$, then $\chi_o(T) \leq 3$. This bound on $\text{mad}(T)$ is sharp.

**Proof.** Let $T$ be a tree. We use induction on $|T|$. The case $|T| = 1$ is trivial. Now suppose that $|T| \geq 2$. Let $v$ be a leaf of $T$. Let $T' := T - v$ and note that $T'$ is a tree. By hypothesis, $T'$ has an odd 3-coloring $\varphi$. Let $w$ be the neighbor of $v$ in $T$. To extend $\varphi$ to an odd 3-coloring of $T$, we color $v$ with a color outside $\{\varphi(w), \varphi_o(w)\}$. This gives the desired odd 3-coloring of $T$.

If $\text{mad}(G) < 2$, then $G$ is a forest. So the second statement follows from the first. Finally, the third statement follows from Proposition 2.
The following proposition is folklore. But we include a proof for completeness.

**Proposition 4.** If $G$ is a planar graph with girth at least $g$, then $\text{mad}(G) < 2g/(g - 2)$.

**Proof.** Fix a plane embedding of a planar graph $G$ with girth at least $g$. Since each subgraph of $G$ also has girth at least $g$, it suffices to show that $2|E(G)|/|V(G)| < 2g/(g - 2)$. Summing the lengths of all facial walks gives $2|E(G)| \geq g|F|$, where $F$ is the set of all faces. To prove the proposition, we substitute this inequality into Euler’s formula and solve for $2|E(G)|/|V(G)|$. \qed

2 \hspace{1cm} \text{mad}(G) < 4

Recall, for each $\alpha > 0$, that $\mathcal{G}_\alpha$ denotes the set of all graphs $G$ with $\text{mad}(G) < \alpha$; and we write $\chi_\alpha(\mathcal{G}_\alpha)$ to denote the maximum value of $\chi_\alpha(G)$ over all $G \in \mathcal{G}_\alpha$. Proposition 3 shows that $\chi_\alpha(\mathcal{G}_2) = 3$. In this section, we determine the set of all values $\alpha$ such that $\chi_\alpha(\mathcal{G}_\alpha)$ is defined; when it is defined, we prove a (nearly sharp) upper bound on its value.

We denote by $K_n^*$ the graph formed from $K_n$ by subdividing each edge once.

**Lemma 5.** We have $\chi_\alpha(K_n^*) = n$ and $\text{mad}(K_n^*) = 4 - 8/(n + 1)$.

**Proof.** For each $n \geq 1$, let $\epsilon_n := 8/(n + 1)$. To show that $\text{mad}(K_n^*) = 4 - \epsilon_n$, we give a fractional orientation of $K_n^*$ where each vertex has indegree exactly $2 - \epsilon_n/2$. We orient each edge with fraction $1 - \epsilon_n/4$ towards its endpoint of degree 2 and fraction $\epsilon_n/4$ toward its endpoint of degree $n$. Each 2-vertex has indegree $2(1 - \epsilon_n/4) = 2 - \epsilon_n/2$. Each $(n - 1)$-vertex has indegree $(n - 1)(\epsilon_n/4) = (n - 1)(2/(n + 1)) = (2n - 2)/(n + 1) = 2 - 4/(n + 1) = 2 - \epsilon_n/2$. Thus, $\text{mad}(G) = 4 - \epsilon_n$, as desired.

In an odd coloring of $K_n^*$, all $(n - 1)$-vertices must get distinct colors. So $\chi_\alpha(K_n^*) \geq n$. Given any coloring where all $(n - 1)$-vertices get distinct colors, it is easy to extend to an odd $n$-coloring of $K_n^*$. Thus, as claimed, $\chi_\alpha(K_n^*) = n$. \qed

**Corollary 6.** There exists a sequence $\epsilon_1, \epsilon_2, \ldots$ such that $\epsilon_n > 0$ for all $n \geq 1$ and $\lim_{n \to \infty} \epsilon_n = 0$ and for each $n \geq 1$ there exists a graph $G_n$ with $\text{mad}(G_n) = 4 - \epsilon_n$ and $\chi_\alpha(G_n) = 8/\epsilon_n - 1$.

**Proof.** Let $\epsilon_n := 8/(n + 1)$ and $G_n := K_n^*$. Now $\chi_\alpha(G_n) = n = (n + 1) - 1 = 8/\epsilon_n - 1$. \qed

**Corollary 7.** $\chi_\alpha(\mathcal{G}_\alpha)$ is undefined whenever $\alpha \geq 4$.

In Lemma 5, we considered $K_n^*$ which is formed by subdividing each edge of $K_n$. Applying the same construction to any $n$-chromatic graph $H$ yields a graph $H'$ with $\chi_\alpha(H') = n$ and $\text{mad}(H') < 4$. Since there exist graphs $H$ with both chromatic number and girth arbitrarily large, there also exist graphs $H'$ with $\chi_\alpha(H')$ and girth arbitrarily large, and with $\text{mad}(H') < 4$. However, every $n$-chromatic graph $H$ that does not contain $K_n$ as a subgraph gives an upper bound on $4 - \epsilon$ that is worse (larger) than that in Corollary 6. Consider an $n$-critical (sub)graph $H$ with $a$ vertices and $b$ edges. Recall that $\delta(H) \geq n - 1$. But $H$ is not $(n - 1)$-regular, by Brooks’ Theorem. Thus, $b = |E(H)| > (n - 1)|V(H)|/2 = (n - 1)a/2$. Subdividing each edge of $H$ gives $H'$ with $|V(H')| = a + b$ and $|E(H')| = 2b > a(n - 1)$. Thus, $\text{mad}(H') \geq 2|E(H')|/|V(H')| = 4b/(a + b) > 2a(n - 1)/(a(n - 1) + 2a) = 2a(n - 1)/(a(n + 1)/2) = 4(n - 1)/(n + 1) = 4 - 8/(n + 1)$.

We next show that the construction $K_n^*$ in Lemma 5 is nearly sharp.

**Theorem 1.** Fix $\epsilon$ such that $0 < \epsilon \leq 8/5$. If $\text{mad}(G) \leq 4 - \epsilon$, then $\chi_\alpha(G) \leq \lfloor 8/\epsilon \rfloor + 2$. As $\epsilon \to 0$, infinitely often there exists $G_\epsilon$ such that $\text{mad}(G_\epsilon) = 4 - \epsilon$ and $\chi_\alpha(G_\epsilon) = \lfloor 8/\epsilon \rfloor - 1$.

**Proof.** The second statement follows directly from Corollary 6.

For the first statement, let $k := \lfloor 8/\epsilon \rfloor + 2$ and note that $k \geq 7$. Our proof is by induction on $|V(G)|$. The base case is when $|V(G)| = 1$, so $\chi_\alpha(G) = 1$. Instead assume $|V(G)| \geq 2$. If $G$
contains a 1-vertex v with neighbor w, then G − v has an odd k-coloring ϕ and we can extend ϕ to v by coloring v to avoid ϕ(w) and ϕ_o(w). If G contains a 3-vertex v, then denote N(v) by \{w_1, w_2, w_3\}. Now G − v has an odd k-coloring ϕ and we can extend ϕ to v by coloring v to avoid \{ϕ(w_1), ϕ(w_2), ϕ(w_3), ϕ_o(w_1), ϕ_o(w_2), ϕ_o(w_3)\}. So we assume that G has neither 1-vertices nor 3-vertices. Suppose that G has adjacent 2-vertices v_1 and v_2, and denote the remaining neighbors of v_1 and v_2, respectively, by v_0 and v_3. Now G − \{v_1, v_2\} has an odd k-coloring ϕ. To extend this to v_1 and v_2, we first color v_1 to avoid \{ϕ(v_0), ϕ_o(v_0), ϕ(v_3)\} and then color v_2 to avoid the new color on v_1 as well as \{ϕ(v_0), ϕ(v_1), ϕ_o(v_3)\}.

Since \text{mad}(G) < 4, we know that δ(G) = 2. We use discharging to show that some vertex v with d(v) ≤ 8/ε − 2 has “many” 2-neighbors. Similar to the case of adjacent 2-vertices above, we will delete v and its 2-neighbors, find an odd k-coloring ϕ for this smaller graph, and extend ϕ to all of G. Let x := 1 − ε/2. We give each vertex v initial charge d(v) and use the following single discharging rule: Each 2-vertex takes charge x from each neighbor. Since G does not have adjacent 2-vertices, each 2-vertex finishes with charge 2 + 2x = 4 − ε. For each 4^+-vertex v, let d_2(v) denote the number of 2-neighbors of v. Now v finishes with charge d(v) − xd_2(v). If d(v) − d_2(v) > 2 + 2x = 4 − ε for each v, then we contradict the assumption \text{mad}(G) ≤ 4 − ε. So there exists v such that d(v) − xd_2(v) ≤ 2 + 2x.

Form \( G' \) from G by deleting v and all of its 2-neighbors. By induction, \( G' \) has an odd k-coloring ϕ. To extend ϕ to G, we first color v to avoid the colors on the colored neighbors (in G) of its deleted 2-neighbors (in G') as well as, for each 4^-neighbor w, to avoid ϕ(w) and ϕ_o(w). For convenience, we denote this color used on v by ϕ(v). Then we color each 2-neighbor x of v, with other neighbor y, to avoid \{ϕ(v), ϕ_o(v), ϕ(y), ϕ_o(y)\}. This gives an odd coloring ϕ' of G. We must only show that ϕ' uses at most k colors. In total, v must avoid at most 2d(v) − d_2(v) colors. So it will suffice to show that k ≥ 1 + (2d(v) − d_2(v)). Note that 1 + 2d(v) − d_2(v) < 1 + d(v) + (d(v) − xd_2(v)) ≤ 1 + d(v) + 2 + 2x < 1 + d(v) + 4. Since 1 + 2d(v) − d_2(v) is an integer, 1 + 2d(v) − d_2(v) ≤ d(v) + 4. So now we must bound d(v). Clearly, d(v) − d(v)x ≤ d(v) − d_2(v)x ≤ 2 + 2x. So d(v) ≤ (2 + 2x)/(1 − x) = (4 − ε)/(ε/2) = 8/ε − 2. Since d(v) is an integer, d(v) ≤ [8/ε] − 2. Thus, d(v) + 4 ≤ ([8/ε] − 2) + 4 = [8/ε] + 2 = k. □

**Remark 8.** In the previous proof we handled adjacent 2-vertices as follows. “Suppose that G has adjacent 2-vertices v_1 and v_2, and denote the remaining neighbors of v_1 and v_2, respectively, by v_0 and v_3. Now G − \{v_1, v_2\} has an odd k-coloring ϕ. To extend this to v_1 and v_2, we first color v_1 to avoid \{ϕ(v_0), ϕ_o(v_0), ϕ(v_3)\} and then color v_2 to avoid the new color on v_1 as well as \{ϕ(v_0), ϕ(v_3), ϕ_o(v_3)\}.” In this argument we implicitly assumed that v_3 = v_0. Otherwise, our choice for color on v_1 might create a problem for v_3. Specifically, suppose that v_0 is adjacent to 2-vertices v_1 and v_2, which are also adjacent to each other, and that colors 1 and 2 each appear once on N(v) under (and no other color appears an odd number of times); this is possible when v_3 = v_0. Now ϕ_o(v_0) is undefined. But if we are careless and color v_1 with 1 and color v_2 with 2, then we create a problem for v_0. This obstacle has an easy work-around. Rather than defining ϕ_o(v_0) relative to the coloring ϕ, we define it relative to the current coloring, including any vertices that were deleted but are now already colored. For brevity, we omit mention of this issue throughout the paper, since the solution above always works.

It is natural to consider a list-coloring analogue of odd coloring (although we omit a formal definition). We note that, with minor modifications, the proof of Theorem \( \text{III} \) works equally well in the context of odd list-coloring and even in the context of odd correspondence coloring. The same is true for the proofs of Theorem 2(a) and Theorem 2(b).

We suspect that the construction in Lemma \( \text{E} \) is sharp.

**Conjecture 1.** Fix ε such that 0 < ε ≤ 8/5. If \text{mad}(G) ≤ 4 − ε, then \( \chi_o(G) ≤ [8/ε] − 1 \).

Conjecture \( \text{II} \) cannot be extended to allow ε = 2, since \text{mad}(C_{3s+1}) = 2, but \( \chi_o(C_{3s+1}) = 4 \), for each positive integer s. In contrast, when ε = 1 and ε = 8/7, we prove the conjecture in a
stronger form. We show that if \( \text{mad}(G) \leq 4 - \epsilon \), then \( \chi_o(G) \leq \lfloor 8/\epsilon \rfloor - 2 \) unless \( G \) contains a subgraph \( K^*_s \), where \( s := \lfloor 8/\epsilon \rfloor - 1 \). We present these proofs in Sections 3 and 4.

Recall that a graph \( H \) is \( k \)-critical if \( \chi(H) = k \) and \( \chi(H - e) < k \) for every edge \( e \in E(H) \). Analogously, a graph \( H \) is odd-\( k \)-critical if \( \chi_o(H) = k \) and \( \chi_o(H - e) < k \) for every edge \( e \in E(H) \). The latter notion is more subtle, since possibly \( H' \) is a subgraph of \( H \), but \( \chi_o(H') > \chi_o(H) \). More concretely, if we form \( H \) from \( K^*_n \) with \( n \geq 3 \) by adding a leaf adjacent to each vertex, then \( \chi_o(H) \leq 3 \), even though \( \chi_o(K^*_n) = n \). It is easy to check that subdividing every edge of a \( k \)-critical graph, with \( k \geq 6 \), yields an odd-\( k \)-critical graph. Kostochka and Yancey [10, 11] showed that the sparsest \( k \)-critical graphs are so-called \( k \)-Ore graphs. We then show that if \( \text{mad}(\cdot) \geq k \) for every \( k \geq 4 \), by subdividing all edges in \( k \)-Ore graphs, we get an infinite family of odd-\( k \)-critical graphs with maximum average degrees slightly larger than \( 4 - 8/(k + 3) \). We expect that in the limit no infinite family of odd-\( k \)-critical graphs has smaller maximum average degree.

3 Odd 6-Colorings

The goal of this section is to prove the following result.

**Theorem 2(a).** If \( \text{mad}(G) < 3 \), then \( \chi_o(G) \leq 6 \). This includes all planar \( G \) of girth at least 6.

Note that Theorem 3 is sharp in two senses. First, \( \text{mad}(K^*_3) = 4 - 8/(6 + 1) = 2 + 6/7 \) and \( \chi_o(K^*_3) = 6 \), so the upper bound of 6 cannot be decreased. Second, \( \text{mad}(K^*_4) = 4 - 8/(7 + 1) = 3 \) and \( \chi_o(K^*_4) = 7 \), so the hypothesis \( \text{mad}(G) < 3 \) cannot be weakened at all.

**Proof.** The second statement follows from the first by Proposition 4. So we prove the first. Assume the statement is false, and \( G \) is a counterexample with as few vertices as possible. As in the proof of Theorem 3, we assume \( G \) has no 1-vertex and \( G \) has no adjacent 2-vertices. Assume \( G \) has a 3-vertex \( v \) with neighbors \( w_1, w_2, w_3 \), where \( d(w_1) = 2 \). Denote the other neighbors of \( w_1 \) by \( x_1 \). Let \( G' := G - \{v, w_1\} \). By minimality, \( G' \) has an odd 6-coloring \( \varphi \). To extend \( \varphi \) to \( G \), we color \( v \) to avoid \( \{\varphi(w_1), \varphi(w_2), \varphi(w_3), \varphi_o(w_2), \varphi_o(w_3)\} \) and then color \( w_1 \) to avoid the new color on \( v \) and also to avoid \( \{\varphi(x_1), \varphi_o(x_1)\} \). Thus, \( G \) has a 2-neighbor.

Suppose \( G \) has a 4-vertex with neighbors \( w_1, w_2, w_3, w_4 \), where \( d(w_1) = d(w_2) = d(w_3) = d(w_4) = 2 \). For each \( i \in \{1, 2, 3\} \), denote the other neighbor of \( w_i \) by \( x_i \). Let \( G' := G - \{v, w_1, w_2, w_3\} \). By minimality, \( G' \) has an odd 6-coloring \( \varphi \). To extend \( \varphi \) to \( G \), color \( v \) to avoid \( \{\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi_o(w_4), \varphi_o(w_3)\} \) and then color each \( w_i \) to avoid the new color on \( v \) and also avoid \( \{\varphi(x_1), \varphi_o(x_1), \varphi_o(x_1)\} \). Thus, \( G \) has 3- or more 2-neighbors.

Suppose \( G \) has a 5-vertex \( v \) with five 2-neighbors \( w_1, w_2, w_3, w_4, w_5 \). For each \( i \in \{1, 2, 3, 4, 5\} \), denote the other neighbor of \( w_i \) by \( x_i \). Let \( G' := G - \{v, w_1, w_2, w_3, w_4, w_5\} \). By minimality, \( G' \) has an odd 6-coloring \( \varphi \). To extend \( \varphi \) to \( G \), we color \( v \) to avoid \( \{\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4), \varphi(x_5)\} \). Then we color each \( w_i \) to avoid the new color on \( v \), as well as \( \{\varphi(x_i), \varphi_o(x_i), \varphi(v)\} \). Thus, \( G \) has 5-vertex with five 2-neighbors.

Now we use discharging to reach a contradiction, which will finish the proof. We give each vertex \( v \) initial charge \( d(v) \) and use the following single discharging rule: Each 2-vertex takes 1/2 from each neighbor. We show that each vertex finishes with charge at least 3, which contradicts the hypothesis \( \text{mad}(G) < 3 \). Recall that \( \delta(G) \geq 2 \).

If \( d(v) = 2 \), then \( v \) finishes with \( 2 + 2(1/2) = 3 \). If \( d(v) = 3 \), then \( v \) has no 2-neighbors, so \( v \) finishes with 3. If \( d(v) = 4 \), then \( v \) has at most two 2-neighbors, so \( v \) finishes with at least \( 4 - 2(1/2) = 3 \). If \( d(v) = 5 \), then \( v \) has at most four 2-neighbors, so \( v \) finishes with at least \( 5 - 4(1/2) = 3 \). If \( d(v) \geq 6 \), then \( v \) finishes with at least \( d(v) - d(v)/2 = d(v)/2 \geq 3 \), since \( d(v) \geq 6 \). This finishes the proof.

With a bit more analysis, we can prove the following stronger result.

\footnote{An interesting example is \( C_5 \), since \( \chi_o(C_5) = 5 \), but \( C_5 \) has no odd-4-critical subgraph.}
Corollary 9. If \( \text{mad}(G) \leq 3 \) and \( G \) does not contain \( K_7^- \) as a subgraph, then \( \chi_o(G) \leq 6 \).

Proof. In the proof of Theorem 3 to contradict the hypothesis \( \text{mad}(G) < 3 \), we showed that each vertex finished with charge at least 3. To contradict the weaker hypothesis \( \text{mad}(G) \leq 3 \), it suffices to show that also some vertex finishes with charge more than 3. Suppose the contrary.

Now \( G \) has no \( 7^- \)-vertex and each 6-vertex has six 2-neighbors. Each 5-vertex has exactly four 2-neighbors and each 4-vertex has exactly two 2-neighbors. Now it is straightforward to check that \( G \) cannot have any two adjacent \( 3^- \)-vertices \( v \) and \( w \); in each case, we delete \( v, w \), and their 2-neighbors, color the smaller graph and extend. Thus, \( G \) is bipartite, where one part consists of 2-vertices and the other consists of 6-vertices. Form \( G' \) from \( G \) by contracting one edge incident to each 2-vertex. Now \( G' \) is 6-regular, but no component is \( K_7^- \), so \( \chi(G') \leq 6 \) by Brooks’ Theorem. A 6-coloring of \( G' \) induces a 6-coloring of the 6-vertices in \( G \), which we can easily extend to an odd 6-coloring of \( G \).

4 Odd 5-Colorings

The goal of this section is to prove the following result.

Theorem 2(b). If \( \text{mad}(G) < 20/7 \), then \( \chi_o(G) \leq 5 \). This includes all planar \( G \) of girth at least 7.

Proof. The second statement follows from the first by Proposition 4. So we prove the first. Assume the statement is false, and \( G \) is a counterexample with as few vertices as possible. As in the proof of Theorem 1 we assume \( G \) has no 1-vertex and \( G \) has no adjacent 2-vertices. Suppose \( G \) has a 3-vertex \( v \) with neighbors \( w_1, w_2, w_3 \), where \( d(w_1) = d(w_2) = 2 \). Let the other neighbors of \( w_1 \) and \( w_2 \) be, respectively, \( x_1 \) and \( x_2 \). Let \( G' := G - \{v, w_1, w_2\} \). By minimality, \( G' \) has an odd 5-coloring \( \varphi \). To extend \( \varphi \) to \( G \), we color \( v \) to avoid \( \{\varphi(w_1), \varphi(w_2), \varphi(w_3), \varphi(x_1), \varphi(x_2)\} \) and color each \( x_i \) to avoid the new color on \( v \) and avoid \( \varphi(x_1), \varphi(x_2) \). Thus, each 3-vertex has at most one 2-neighbor. More generally, each 3-vertex has at most one 3-neighbor.

If not, then we delete \( v \) and any 2-neighbor. We color \( v \) to avoid the color on each 3-neighbor, the color on the colored neighbor of each (deleted) 2-neighbor, as well as \( \{\varphi(w), \varphi_o(w)\} \), where \( w \) is the third neighbor of \( v \). Thus, each 3-vertex has at most one 3-neighbor.

Suppose \( G \) has a 4-vertex with four 2-neighbors \( w_1, w_2, w_3, w_4 \). For each \( i \in \{1, 2, 3, 4\} \), denote the other neighbor of \( w_i \) by \( x_i \). Let \( G' := G - \{v, w_1, w_2, w_3, w_4\} \). By minimality, \( G' \) has an odd 5-coloring \( \varphi \). To extend \( \varphi \) to \( G \), color \( v \) to avoid \( \{\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)\} \) and then color each \( w_i \), in succession, to avoid the new color on \( v \) and also avoid \( \{\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)\} \) and then color each \( w_i \), in succession, to avoid the new color on \( v \) and also avoid \( \{\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(x_4)\} \). Thus, no 4-vertex in \( G \) has four 2-neighbors. Similarly, if each \( w_i \) has degree at most 3 and at least one \( w_i \) is a 2-vertex, then we form \( G'' \) from \( G \) by deleting \( v \) and all its 2-neighbors. Again \( G'' \) has an odd 5-coloring \( \varphi \). We extend \( \varphi \) to \( v \) by avoiding the color on each 3-neighbor and the color on the other neighbor of each 2-neighbor. Finally, we can extend the coloring to all 2-neighbors to get an odd 5-coloring of \( G \). Thus, \( G \) has no 4-vertex with at least one 2-neighbor and all 3-neighbors. Similarly, \( G \) does not contain adjacent 4-vertices, \( v \) and \( w \), each with three 2-neighbors. If so, then \( G - (N[v] \cup N[w]) \) has an odd 5-coloring. We color \( v \) and \( w \) with distinct colors that each differ from the colors on the three colored vertices at distance two in \( G \). Again, we can extend this coloring to an odd 5-coloring of \( G \).

Now we use discharging to reach a contradiction, which will finish the proof. We give each vertex \( v \) initial charge \( d(v) \) and use the following two discharging rules.

(R1) Each 2-vertex takes \( 3/7 \) from each neighbor.

(R2) Each 3-vertex with a 2-neighbor and each 4-vertex with three 2-neighbors takes \( 1/7 \) from each 3^-neighbor.

We show that each vertex finishes with charge at least \( 20/7 \), which contradicts the hypothesis \( \text{mad}(G) < 20/7 \). Recall that \( \delta(G) \geq 2 \). If \( d(v) = 2 \), then \( v \) finishes with \( 2 + 2(3/7) = 20/7 \). If
$d(v) = 3$, then $v$ has at most one 2-neighbor. Further, $v$ does not give away charge by (R2), since no 3-vertex has both a 2-neighbor and a 3-neighbor, and also no 4-vertex has both three 2-neighbors and a 3-neighbor. So, if $v$ has a 2-neighbor, then it has two 4$^+$-neighbors and receives charge $1/7$ from each. Thus, $v$ finishes with at least $3 - 3/7 + 2(1/7) = 20/7$. If $v$ has no 2-neighbor, then $v$ starts and finishes with 3.

Let $v$ be a 4-vertex, and recall that $v$ has at most three 2-neighbors. If $v$ has at most two 2-neighbors, then $v$ finishes with at least $4 - 2(3/7) - 2(1/7) = 20/7$. If $v$ has three 2-neighbors, then its fourth neighbor is a 4$^+$-neighbor that does not receive charge from $v$ by (R2) but rather gives $v$ charge $1/7$ by (R2). So $v$ finishes with $4 - 3(3/7) + 1/7 = 20/7$.

If $v$ is a 5$^+$-vertex, then $v$ finishes with at least $d(v) - 3d(v)/7 = 4d(v)/7 \geq 20/7$, since $d(v) \geq 5$. This finishes the proof.

With a bit more analysis, we can prove the following stronger result. The proof is similar to that of Corollary 9, so we just provide a sketch.

**Corollary 10.** If $\text{mad}(G) \leq 20/7$ and $G$ does not contain $K^*_6$ as a subgraph, then $\chi_o(G) \leq 5$.

**Proof Sketch.** Assume $G$ is a counterexample. So $G$ has no 6$^+$-vertices and each 5-vertex in $G$ has five 2-neighbors. Each 3-vertex has a single 2-neighbor. Each 4-vertex either has exactly three 2-neighbors or has exactly two 2-neighbors and gives charge $1/7$ to each of its 3$^+$-neighbors. In this case, we delete $v$, its 4-neighbors and the 2-neighbors of all deleted 4-vertices. This smaller graph has an odd 5-coloring $\varphi$, and it is straightforward to check that we can extend $\varphi$ to $G$. So $G$ has no 4-vertices. This implies, in turn, that $G$ has no 3-vertices, since $G$ has no 3-vertex with two 3$^-$-neighbors. So $G$ is bipartite with vertices in one part of degree 2 and those in the other part of degree 5. We contract one edge incident to each 2-vertex, and 5-color the resulting graph by Brooks’ Theorem. Finally, we extend this 5-coloring to an odd 5-coloring of $G$.

**References**

[1] D. P. Bunde, K. Milans, D. B. West, and H. Wu. Parity and strong parity edge-coloring of graphs. In Proceedings of the Thirty-Eighth Southeastern International Conference on Combinatorics, Graph Theory and Computing, volume 187, pages 193–213, 2007.

[2] D. P. Bunde, K. Milans, D. B. West, and H. Wu. Optimal strong parity edge-coloring of complete graphs. *Combinatorica*, 28(6):625–632, 2008.

[3] P. Cheilaris. *Conflict-Free Coloring*. 2009. Thesis (Ph.D.)–City University of New York.

[4] P. Cheilaris, B. Keszegh, and D. Pálvölgyi. Unique-maximum and conflict-free coloring for hypergraphs and tree graphs. *SIAM J. Discrete Math.*, 27(4):1775–1787, 2013.

[5] P. Cheilaris and G. Tóth. Graph unique-maximum and conflict-free colorings. *J. Discrete Algorithms*, 9(3):241–251, 2011.

[6] G. Even, Z. Lotker, D. Ron, and S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comput.*, 33(1):94–136, 2003.

[7] I. Fabrici and F. Göring. Unique-maximum coloring of plane graphs. *Discuss. Math. Graph Theory*, 36(1):95–102, 2016.

[8] R. Glebov, T. Szabó, and G. Tardos. Conflict-free colouring of graphs. *Combin. Probab. Comput.*, 23(3):434–448, 2014.

[9] A. Kostochka, M. Kumbhat, and T. Łuczak. Conflict-free colourings of uniform hypergraphs with few edges. *Combin. Probab. Comput.*, 21(4):611–622, 2012.

[10] A. Kostochka and M. Yancey. Ore’s conjecture on color-critical graphs is almost true. *J. Combin. Theory Ser. B*, 109:73–101, 2014.
[11] A. Kostochka and M. Yancey. A Brooks-type result for sparse critical graphs. *Combinatorica*, 38(4):887–934, 2018.

[12] J. Pach and G. Tardos. Conflict-free colourings of graphs and hypergraphs. *Combin. Probab. Comput.*, 18(5):819–834, 2009.

[13] M. Petruševski and R. Škrekovski. Colorings with neighborhood parity condition. Dec. 2021, [arXiv:2112.13710](https://arxiv.org/abs/2112.13710).

[14] S. Smorodinsky. Conflict-free coloring and its applications. In *Geometry—intuitive, discrete, and convex*, volume 24 of *Bolyai Soc. Math. Stud.*, pages 331–389. János Bolyai Math. Soc., Budapest, 2013.