Electronic properties of graphene quantum ring with wedge disclination

Abdelhadi Belouad\textsuperscript{1}, Ahmed Jellal\textsuperscript{1,2,a}, and Hocine Bahlouli\textsuperscript{3}

\textsuperscript{1} Laboratory of Theoretical Physics, Faculty of Sciences, Chouaïb Doukkali University, PO Box 20, 24000 El Jadida, Morocco
\textsuperscript{2} Canadian Quantum Research Center, 204-3002 32 Ave, Vernon, BC V1T 2L7, Canada
\textsuperscript{3} Physics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

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Abstract. We study the energy spectrum and persistent current of charge carriers confined in a graphene quantum ring geometry of radius $R$ and width $w$ subject to a magnetic flux. We consider the case where the crystal symmetry is locally modified through dislocations created by replacing the original carbon hexagon by a pentagon, square, heptagon or octagon. To model this type of defect, we include appropriate boundary conditions for the angular coordinate. The electrons are then confined to a finite width strip in the radial direction by setting an infinite mass boundary conditions at the edges of the strip. The solutions are expressed in terms of Hankel functions and their asymptotic behavior allows to derive quantized energy levels in the presence of an energy gap. We also investigate the persistent currents that appear in the quantum ring in the presence of a quantum flux at the center of the ring and how wedge disclination influences different quantum transport quantities.

1 Introduction

Graphene is a two-dimensional lattice of carbon atoms that are arranged in a honeycomb pattern forming a hexagonal lattice \cite{1}. After its experimental discovery, graphene has attracted a lot of attention, which is, in part, due to its exotic electronic properties. Some of these include high electron mobility, excellent conductivity, peculiar tunneling phenomena and other interesting transport and structural properties, which would be too many to list \cite{2}. Graphene is interesting from a fundamental research perspective, as well as for future potential technological applications \cite{3}, such as in the study of liquid crystals \cite{4}, solar cells \cite{5} and high-frequency electronic devices \cite{6}. Particles in pristine graphene near high symmetry $K$ points are described by an effective low energy model, the Dirac–Weyl Hamiltonian for massless charged particles. One of its striking features is the linear gapless energy dispersion relation represented by conic conduction and valence bands \cite{1}.

Recently, graphene-based quantum rings produced by lithographic techniques have been experimentally investigated \cite{7}. These systems have been studied theoretically using a tight-binding model, which does not provide simple analytical expressions for the eigenstates and eigenvalues \cite{8,9}. It has also been shown \cite{10} that it is possible to realize quantum rings of finite width using electric fields to confine electrons in this geometry. The influence of an applied magnetic field has drawn a lot of attention \cite{11} in the context of the important and interesting physical properties of quantum rings in graphene. In this respect, one has to mention several recent theoretical studies related to different properties of quasiparticles confined in nanostructures such as quantum dots \cite{12,13,14} and quantum rings \cite{15,16}. However, some of the most interesting effects on quasiparticles in graphene are due to the presence of a disclination defect, which can significantly alter the electronic structure, magnetic and transport properties \cite{2}. The desire to employ graphene for studying curvature effects is motivated by the simplicity of the Hamiltonian and the important potential applications of graphene in nanoelectronics and potential future quantum computing devices \cite{2}. An individual dislocation in free-standing graphene layers has been imaged using transmission electron microscopy \cite{17}. Topological defects resulting from either kinetic factors or substrate imperfections have also been reported for epitaxially grown graphene on SiC \cite{18}, Ir (111) \cite{19} and polycrystalline Ni surfaces \cite{20}.

In this paper, we follow a similar strategy and consider a quantum ring of graphene with a wedge dislocation that can be understood from Volterra's cut-and-glue constructions \cite{21}. The calculations are performed in the continuum approximation limit in the vicinity of the Dirac points. After a general model description,
we solve the model taking into account the radial and angular degrees of freedom. We find the eigenspinors in terms of the Hankel functions showing quantized energy levels in the asymptotic limit. These results allow us to end up with a gap opening separating the conduction and valence bands. We also find analytical expressions for the persistent currents as a function of ring radius, total momentum, magnetic field, width of quantum ring and an integer index \( n \). Such an index is induced by disclination defect and quantifies the curvature of our geometry. To investigate the behavior of our system, we provide numerical studies for a suitable selection of the physical parameters characterizing our system.

The manuscript is organized as follows. In Sect. 2, we present our theoretical model based on the Dirac Hamiltonian to describe the new geometry obtained via Volterra construction. Then, we introduce a magnetic flux and obtain the analytic solutions for the eigenvectors and quantized eigenenergies in Sect. 3. We then explicitly determine the corresponding persistent currents generated in our system in Sect. 4. In Sect. 5, we discuss our numerical results related to the energy spectrum and the persistent currents. Section 6 contains a summary of the main results and conclusions.

2 Theoretical model

We consider a graphene quantum ring of radius \( R \) and width \( w \) in the presence of a magnetic flux as shown in Fig. 1a and study its electronic properties.

We use the Volterra construction [21] to model the disclination defect in our system bounded by the regularized rings of radii \( R_1 \) and \( R_2 \) around the apex and the removed wedge disclination as shown in Fig. 1b.

Particles in graphene have an electronic band structure with low energy band crossings at two inequivalent high symmetry points, \( K \) and \( K' \), at low energies one can neglect contributions away from Dirac points.

Due to translational invariance the Hamiltonians at the two different Dirac points can be described independently by

\[
H = v_F [\tau_z \sigma_x p_x + \sigma_y p_y] + \Delta(r) \tau_z \sigma_z,
\]

where \( \sigma_i, \tau_i \) are Pauli matrices denoting the sublattice and valley degrees of freedom, respectively, \( v_F \) is the Fermi velocity and \( \Delta(r) \) is the confining potential of the ring as shown in Fig. 1, which is defined by

\[
\Delta(r) = \begin{cases} 
0, & R_1 \leq r \leq R_2 \\
\infty, & \text{otherwise}.
\end{cases}
\]

The Hamiltonian (1) acts on the two component spinor

\[
\Psi^\tau(r, \varphi) = \begin{pmatrix} \Psi_A(r, \varphi) \\
\Psi_B(r, \varphi) \end{pmatrix},
\]

where \( \Psi_{A/B}(r, \varphi) \) is the spinor component describing either of the two graphene sublattices \( A \) and \( B \). It is worth mentioning that in reality our ring has also a finite thickness along the z-direction, parallel to the ring axis, however, under normal growth conditions the vertical confinement is much stronger than the lateral one and hence one can always assume that the electron is frozen in the lowest energy state of the quantum well potential that confines the ring in the z-direction. It is known that deformations of the honeycomb lattice enter the continuum description via fictitious gauge fields [22]. As summarized below, topological point defects manifest themselves through spatially well-localized fluxes of the fictitious fields [23].

Now, we discuss mathematically the new geometry obtained via Volterra construction. Indeed, in the non-rotated system, the eigenvalue equation \( H \psi(0) = \)

![Fig. 1](image-url) **Fig. 1** a The conical ring after Volterra construction. b Unfolded plane of lattice where a wedge of angle \( n\pi/3 \) is removed (\( n = 1 \) here). A potential (sketch on the left part of the ring) confines the electrons in the lowest radial mode on a ring of radius \( R \), avoiding the singularity at the origin. We rescaled the angle \( \varphi \) of the unfolded plane to the new angle \( \alpha = \varphi/(1 - \frac{2}{3}) \). The carbon atoms of the removed sector are denoted by open symbols, those which remain after the cut are represented by solid symbols.
\(E\psi(0)\) implies that the new spinor
\[
\psi(\varphi) = e^{i\varphi\sigma_z\tau_z/2}\psi(0)
\] (4)
fulfils \(H(R_z(\varphi))\psi(\varphi) = E\psi(\varphi)\) after rotation where \(R_z(\varphi)\) is a rotation matrix. This is because Hamiltonian (1) transforms as
\[
H(R_z(\varphi)) = e^{i\varphi\sigma_z\tau_z/2}He^{i\varphi\sigma_z\tau_z/2}
\] (5)
which can easily be checked. Let us remove a sector from the graphene sheet (e.g., with a pentagon replacement) and then glue the sheet back together at the edge where we cut it as shown in Fig. 1b. Now, one of the cuts can be chosen by convention to be at angle 0 and the other at angle \(-n\pi/3\) where \(n\) is the curvature index that quantifies disclination defects. After gluing both of these angles become the same point and are related by a rotation of angle \(-n\pi/3\). Then, the wave function at this point has to be single valued which requires directly the boundary condition
\[
\Psi(r, \alpha = 0) = 0
\] (6)
where we rescale the angle \(\varphi\) of the unfolded plane to the new angle \(\alpha = \varphi / (1 - \frac{\pi}{6})\), \(\alpha\) vary from 0 to \(2\pi\). This, however, is not the complete story and one has to look slightly beyond the continuum model to find the full story. Particularly, let us go to momentum space and cut a wedge in the reciprocal lattice and then glue the lattice back together. We then find that for angles \(2n\pi/3\) equivalent \(K\) points get glued onto each other. For angles \((2n+1)\pi/3\), however, \(K\) points get glued onto inequivalent \(K'\) points (a \(60^\circ\) rotation connects inequivalent \(K\) points). Now one can switch the \(K\) and \(K'\) blocks in the Hamiltonian by the unitary transformation \(e^{i\pi\sigma_y\tau_y/2}\). Keeping track of both observations one finds the boundary condition
\[
\Psi(r, \alpha = 2\pi) = -e^{i2\pi[0-(n6)]\sigma_z\tau_z/2}\Psi(r, \alpha = 0),
\] (7)
For a general index \(n\), we introduce polar coordinates \((r, \varphi)\) defined in the unfolded plane according to Fig. 1b and perform two singular transformations
\[
\Psi(r, \alpha = 2\pi) = \lambda(\varphi)\mu_n(\alpha)\Psi(r, \alpha = 0)
\] (8)
where \(\lambda(\varphi) = e^{i\varphi\sigma_z\tau_z/2}\) and \(\mu_n(\alpha) = e^{in\sigma_y\tau_y/4}\). The first one, \(\lambda(\varphi)\), transforms \(\Psi\) to a spinor that is expressed in the local frame \((e_r, e_y)\) with unit vectors along the radial and azimuthal directions, effectively replacing \(\partial_\varphi\) by \(\partial_\varphi + 1/(2r)\) in the Hamiltonian. The second \(\mu_n(\alpha)\) introduces a matrix-valued gauge field into the Hamiltonian, effectively replacing \(\partial_\alpha\) by \(\partial_\alpha + i\frac{\pi}{2}\sigma_y\tau_y\). One should note that this transformation is performed to simplify the boundary conditions [24].

### 3 Magnetic flux and energy levels

At this stage, we introduce an additional magnetic flux through the center of the ring. In the effective Hamiltonian, a magnetic flux \(\Phi\) crossing through the origin is accounted for by replacing momentum \(p\) by the conical momentum \(p + eA\) with a vector potential defined by
\[
A = \Phi \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} / (\Omega_n \Phi_0 r),
\] (9)
where \(\Omega_n = 1 - \frac{n}{6}\) and \(\Phi_0 = \frac{e}{\hbar}\) is the magnetic flux quantum. We write Hamiltonian (1) in polar coordinate and apply the transformations through \(\lambda(\varphi)\) and \(\mu_n(\alpha)\), namely
\[
\hat{H}(r, \alpha) = \lambda^\dagger n\mu_n^\dagger \hat{H}\mu_n^\alpha \lambda
\] (10)
to obtain the transformed Hamiltonian
\[
\hat{H}(r, \alpha) = \hbar v_F \left( k_r - \frac{i}{2r} \right) \sigma_x + \hbar v_F \left( k_\alpha + \frac{\Phi}{\Omega_n \Phi_0 r} + \frac{n}{4\Omega_n r} \tau_z \right) \sigma_y + \Delta \tau_z \sigma_z
\] (11)
we have set \(k_r = -i\partial / \partial r\) and \(k_\alpha = -i\partial / \partial \alpha\).

Since \(\hat{H}(r, \alpha)\) commutes with the total angular momentum \(J_z = L_z + S_z\), then we can split the eigen-spinors into angular and radial parts
\[
\Psi(r, \alpha) = e^{ij\alpha} \begin{pmatrix} \chi_A(r) \\ i\chi_B(r) \end{pmatrix}
\] (12)
such that \(j = m + 1/2\) are the eigenvalues of \(J\) and \(m \in \mathbb{Z}\).

Inside the quantum ring, we use \(H\Psi = E\Psi\) and show that the radial components satisfy
\[
\begin{align*}
\frac{\partial}{\partial r} + &\frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r\Omega_n} \left( j + \beta + \frac{n\tau}{4} \right) \chi_B(r) = \frac{E}{\hbar v_F} \chi_A(r) \\
\frac{\partial}{\partial r} + &\frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r\Omega_n} \left( j + \beta + \frac{n\tau}{4} \right) \chi_A(r) = -\frac{E}{\hbar v_F} \chi_B(r)
\end{align*}
\] (13) (14)
where we have defined the quantum quantity \(\nu = \frac{1}{\hbar}(m + \frac{1}{2} + \beta + \frac{n\tau}{4})\) and \(\beta = \frac{\Phi}{\Omega_n}\) is the dimensionless flux. Decoupling the above equations, we arrive at the Hankel differential equation for \(\chi_A(r)\)
\[
\left[ r^2 \frac{\partial^2}{\partial r^2} + r \frac{\partial}{\partial r} + \left( \frac{E r}{\hbar v_F} \right)^2 - \left( \nu - \frac{1}{2} \right)^2 \right] \chi_A(r) = 0
\] (15)
In graphene systems, the persistent current (PC) can be generated as follows. We wrap up graphene ribbon to form a tube or ring with a magnetic flux passing through its central part, then the helical edge state provide a robust channel for PC. On the other hand, the existence of PC in a normal metal ring was first proposed by Buttiker, Imry and Landauer [25]. Recently, with the advent of nano-fabrication techniques, several experimental investigations have been made to confirm the existence [26–28] and the periodicity [29,30] of PC in semiconductor quantum rings. As for graphene rings, there are several studies of PC with various geometrical shapes (without spin-orbit interaction). For example, flat rings (similar to Corbino-disks) with various types of boundaries [8,31,32] or a tube with zigzag edges [33] and folding a graphene ribbon into a ring [34]. The total equilibrium persistent current of a ring can be calculated using the following formula [35]:

\[ I = -\sum_i \frac{\partial E_i}{\partial \phi} \]  

with the magnetic flux \( \Phi \) and \( i \) stands for a set of quantum numbers used to label the corresponding eigenstates.

The above formula just states that the current is the partial derivative of the total energy of the system with respect to the magnetic flux. Adopting the above approach to our situation, we calculate the persistent current, carried by a given electron state, for our model of a graphene quantum ring with disclination. Then, as far as our system is concerned we have the relation

\[ I = -\frac{1}{\Phi_0} \sum_m \frac{\partial E_{mn}}{\partial \beta} = \sum_m I_{mn}, \]

where we limit ourselves to zero temperature and hence summation is realized over occupied states. At finite temperature, \( T \neq 0 \), we need to multiply each energy level contribution by the Fermi function \( f_m = 1/[\exp(E_{mn} - \mu/K_BT)] + 1 \) which accounts for the probability of occupancy of the single energy level under
consideration. After some algebra, we end up with

$$I(\tau, \beta, n) = \mp I_0 \sum_m \frac{1}{\Omega^2_m} \frac{m + \frac{1}{2} + \frac{n}{4} + \beta}{\left( \frac{m}{\Omega^2_m} \right)^2 + \left( m + \frac{1}{2} + \frac{n}{4} + \beta \right) \left( \frac{w}{R^2} \right)^2}$$

(26)

with $I_0 = \frac{E_0}{\Phi_0} \left( \frac{w}{R} \right)^2$. This result will be investigated numerically for a suitable choice of physical parameters.

5 Numerical results

We consider a graphene quantum ring, with inner and outer radii $R_1$ and $R_2$, respectively, of width $w = R_2 - R_1$ and a confining potential $\Delta(r)$ for the ring defined in (2) as shown in Fig. 1. Here, we assume that an applied magnetic flux $\Phi$ only passes through a disk region enclosed by the inner circle (i.e., $r \leq R_1$), allowing us to explore the effect of Aharonov–Bohm phase due to the magnetic flux. We will analyze the energy levels and the persistent currents to extract useful information relevant our system.

In Fig. 2, we present the results for the energy levels as a function of the angular quantum number $m$ in a graphene layer. We show in panel (a) the case of a pentagon defect $n = 1$, in panel (b) the case of a square defect $n = 2$, in panel (c) the case of a heptagon defect $n = -1$ and in panel (d) the case of an octagon defect $n = -2$. We consider different valleys $K$ ($\tau = 1$) and three values of $\beta = -0.5$ (green curve), $\beta = 0$ (blue curve) and $\beta = 0.5$ (red curve). From (22), one may see that the spectrum exhibits an approximate minimum if the angular quantum number takes the following values:

$$m = -\left( \frac{1}{2} + \beta + \frac{n\tau}{4} \right)$$

(27)

for a given value for $\beta$ and is independent of $w$ and $R$ [37]. Comparing our results with those in [38], we notice the creation of an induced pseudo-field $\beta_1 = \frac{n\tau}{4}$ due to the presence of the disclination effect. Such field is measured by a non-zero integer index $n$ that quantifies the curvature and allows the energy spectrum to be displaced in opposite directions at the two Dirac points $K$ and $K'$. As a consequence, the valley the dependence of the energy, for both the conduction and valence bands, on $m$ is symmetric with respect to change in sign $\pm m$

$$E(\tau, m, n, \beta) = E(\tau, -m, n, -\beta).$$

(28)

However, the energy also has valley-index $\tau$-dependent symmetries

$$E(\tau, m, n, \beta) = -E(-\tau, m, n, \beta),$$

$$E(\tau, m, n, \beta) = -E(-\tau, -m, n, -\beta).$$

(29)

Furthermore, in Fig. 2a–d we see that there is an energy gap between the conduction and valence bands, which is given by

$$\Delta E = 2E_0 \sqrt{\left( \frac{\tau\pi}{2} \right)^2 + \nu^2 \left( \frac{w}{R} \right)^2}$$

(30)
Energy levels with $m = 0$ (blue), $m = 1, 2, 3$ (red) and $m = -1, -2, -3$ (green) as function of magnetic flux $\beta$ with $R = 30 \text{ nm}$, $w = 5 \text{ nm}$ for pentagon ($n = 1$) panels (a, e), heptagon ($n = -1$) panels (b, f), square ($n = 2$) panels (c, g), octagon ($n = -2$) panels (d, h). The left panels are for $\tau = 1$ and the right ones are for $\tau = -1$.

It depends on different values of the physical parameters such as quantum angular moment $m$, curvature index $\beta_n = \beta + \frac{\pi \tau}{4}$, quantum ring parameters $w$, $R$ and valley index $\tau$.

Figure 3 shows the energy levels as a function of the magnetic flux $\beta$ for different valleys $K$ ($\tau = 1$) and $K'$ ($\tau = -1$) while the disclination is defined by the curvature index ($n = -2, -1, 0, 1, 2$). In the absence of magnetic flux ($\beta = 0$), the states with angular quantum number $m$ in the valleys $K$ and $K'$ have different energy for the valence and conduction bands

$$E(\tau, m, \beta, n) = E(\tau, -m, -\beta, n) \quad (31)$$

that is because of the deformation of graphene. For $\beta \neq 0$, the states with an opposite $m$ at $K$ and $K'$ have the same energies.

$$E(\tau, m, \beta, n) = E(\tau, -m, -\beta, n). \quad (32)$$

However, it is interesting to notice that the degeneracy may still possibly persist for a non-zero flux, which is not the case for graphene with no defect ($n = 0$) [8,38]. This comes from the periodic dependence of the energy spectrum on magnetic flux $\beta_n = \beta + \frac{\pi \tau}{4}$. Physically, the periodic dependence of the energy spectrum on $\beta_n$ originates from the combined effect of graphene defect and the presence of a magnetic flux, which affects the
phase of the wave function as predicted in Aharonov–Bohm effect.

The energy levels $E/E_0$ as function of ring radius $R$ are shown in Fig. 4 for valleys $K (\tau = 1)$ and $K' (\tau = -1)$ with $\beta = 1/2$, $w = 5$ nm, $m = 0$ (black line), $m = 1$ (blue dashed), $m = -1$ (red line) such that panel (a): pentagon defect ($n = 1$), panel (b): square defect ($n = 2$), panel (c): heptagon defect ($n = -1$), panel (d): octagon defect ($n = -2$). For small radii, $E/E_0$ have branches converging as $1/R$ with an energy gap between the conduction and valence bands, which depends on the curvature index $n$. This behavior quantitatively differs from that found in standard quantum rings in graphene ($n = 0$) in the absence and presence of magnetic flux [38]. In the limit of large radii $R$, we find that there is a constant energy gap $\Delta E = \frac{2E_0\pi}{R}$, which does not depend on the index $n$. As a consequence, the symmetry

$$E(\tau, m = 0, n) = -E(-\tau, m = -1, n)$$  \hspace{1cm} (33)

holds for all $n$. We also find that the symmetry $E(\tau, n, m) = E(-\tau, -n, m)$ is broken.

To better understand the dependence of the energy levels $E/E_0$ on the confinement of the graphene quantum ring in the presence of a disclination, we present in Fig. 5 $E/E_0$ as a function of the ring radius $R$, at constant flux $\beta = 0.5$ and $n = 2$ (square) for several values of ring width $w$. In the regime $R < 0.1 \text{nm}$, $E(\tau) = E(-\tau) = 0$, for $R = 0.1 \text{nm}$, the energy spectrum has a maximum at $E(\tau = 1) = 8800 \text{ meV}$ for the valley $K$ and $E(\tau = -1) = 3000 \text{ meV}$ for the valley $K'$. We find that the energy spectrum decreases with the increase in $R$. We observe that for small values of $w$, we reach large values of $E$, such effect is due to the strong confinement. By increasing $R$ to higher values, we notice that the energy spectrum changes only slightly and converges towards constant values. It is clearly seen that the energy spectrum exhibits an asymmetry $E(\tau) \neq E(-\tau)$.

In Fig. 6, we show the behavior of the persistent current $I$ as function of magnetic flux $\beta$ for disclination index values $n = -2, -1, 1, 2$ and fixed ratio $\frac{w}{R} = 0.5$. Note that solid and dashed lines correspond to the valleys $K (\tau = 1)$ and $K' (\tau = -1)$, respectively. We observe that there is a non-zero persistent current with
with disclination defect as a function of for the indices in Fig. 7. The red, green, blue and magenta curves are curvature index \( n \). Moreover, the amplitude of the current depends on the angular momentum \( m \) and \( k \) for different values of the curvature index \( n \) and the angular momentum \( m \). We observe that \( I_{mn} \) exhibits the same behavior as the total current \( I \), but for the limit \( w/R \to 0 \) we have \( I_{mn}(\tau) \neq I_{mn}(-\tau) \). When \( w/R \to \infty \), it is clearly seen that \( I_{mn} \) remains constant and tends towards a non-zero value while the total current vanishes in the same limit \( I \to 0 \).

### 6 Conclusion

We have studied a graphene system in the shape of a quantum ring of inner radius \( R_1 \), outer radius \( R_2 \) and width \( w \) subject to a magnetic flux in its central region which induces an extra Aharonov–Bohm phase in the electronic wavefunction. We have considered a disclination defect using a procedure known as the Volterra process [21]. This transformation can be looked at as a cut-and-glue process where we cut and remove a sector in the quantum ring of a graphene layer. Due to the sixfold rotational symmetry of graphene honeycomb lattice the removed angular sector must be a multiple of \( \pi/6 \). We have included a topological defect in a graphene layer using a fictitious gauge field following the approach used in [22] where a gauge field is introduced in the Dirac equation to reproduce the already known effect of the disclination on the behavior of the spinor. We have obtained analytical expressions for the energy levels and the corresponding eigenspinors as well as the persistent current using the infinite mass boundary condition [8,14]. Our numerical results were exposed as a function of variations in the ring radius \( R \), total moment, magnetic field, ring width and integer-valued curvature index \( n \).

In particular, we have shown that the disclination modifies the energy spectrum and shifts the magnetic flux in opposite directions at the valleys \( K \) and \( K' \). Furthermore, the charge current is affected by the pseudo-magnetic field produced by the disclination having opposite signs at the two Dirac points. We have found zero flux at the two valleys and for any value of the index \( n \). This shows that the degeneracy of the valley, \( I(\tau) = I(-\tau) \) holds in the ranges \( \beta \leq -4 \) and \( \beta \geq 4 \), but is broken in the interval \( -4 < \beta < 4 \). The persistent current oscillates as a function of magnetic flux with the same period as in the well-known Aharonov–Bohm oscillations [8]. The persistent current displays the following symmetries:

\[
I(\tau, \beta, n) = I(-\tau, -\beta, n), \quad I(\tau, \beta, n) = -I(\tau, -\beta, n). \tag{34}
\]

Moreover, the amplitude of the current depends on the curvature index \( n \).

Results for the persistent current \( I \) of a graphene ring with disclination defect as a function of the ratio \( w/R \) are shown in Fig. 7. The red, green, blue and magenta curves are for the indices \( n = 1, 2, -1, -2 \), respectively, while the solid curves and dashed curves correspond to the valleys \( k \) and \( k' \). These results show that the persistent current has a maximum when \( w/R \to 0 \) as shown in panel (a), and verify the following symmetry:

\[
I(n, \tau) = -I(-n, -\tau). \tag{35}
\]

In addition, we can clearly see that when \( w/R \) increases the persistent current \( I \) decreases until \( w/R \simeq 1 \). The degeneration of the \( k \) and \( k' \) valleys is lifted, i.e., \( I(n, \tau) = -I(-n, -\tau) \). In the \( w/R \to \infty \) regime, the persistent current \( I \) becomes almost constant and converges to zero, see panel (b).

Figure 8 shows the numerical results for the partial current (partial contribution) \( I_{mn} \) as a function of the ratio \( w/R \) for the valley \( K \) and \( K' \) with \( \beta = 0.5 \) for several values of the index \( n \). These results show that the persistent current \( I \) changes as a function of magnetic flux \( \beta \) and \( n \). The degeneracy of the valley, \( K \), for \( n = 0 \) and \( n = 1 \), \( K' \) for \( n = 2 \) is lifted, i.e., \( I(\tau) = 0 \) for \( \tau = 0 \) and \( \tau = 1 \). This shows that the degeneracy of the valley \( K \) for \( n = 0 \) and \( n = 1 \) and valley \( K' \) for \( n = 2 \) is lifted, i.e., \( I(\tau) = 0 \) for \( \tau = 0 \) and \( \tau = 1 \). The degeneracy of the valley \( K \) for \( n = 0 \) and \( n = 1 \), \( K' \) for \( n = 2 \) is lifted, i.e., \( I(\tau) = 0 \) for \( \tau = 0 \) and \( \tau = 1 \). In the \( w/R \to \infty \) regime, the persistent current \( I \) becomes almost constant and converges to zero, see panel (b).

We have studied a graphene system in the shape of a quantum ring of inner radius \( R_1 \), outer radius \( R_2 \) and width \( w \) subject to a magnetic flux in its central region which induces an extra Aharonov–Bohm phase in the electronic wavefunction. We have considered a disclination defect using a procedure known as the Volterra process [21]. This transformation can be looked at as a cut-and-glue process where we cut and remove a sector in the quantum ring of a graphene layer. Due to the sixfold rotational symmetry of graphene honeycomb lattice the removed angular sector must be a multiple of \( \pi/6 \). We have included a topological defect in a graphene layer using a fictitious gauge field following the approach used in [22] where a gauge field is introduced in the Dirac equation to reproduce the already known effect of the disclination on the behavior of the spinor. We have obtained analytical expressions for the energy levels and the corresponding eigenspinors as well as the persistent current using the infinite mass boundary condition [8,14]. Our numerical results were exposed as a function of variations in the ring radius \( R \), total moment, magnetic field, ring width and integer-valued curvature index \( n \).

In particular, we have shown that the disclination modifies the energy spectrum and shifts the magnetic flux in opposite directions at the valleys \( K \) and \( K' \). Furthermore, the charge current is affected by the pseudo-magnetic field produced by the disclination having opposite signs at the two Dirac points. We have found zero flux at the two valleys and for any value of the index \( n \). This shows that the degeneracy of the valley, \( I(\tau) = I(-\tau) \) holds in the ranges \( \beta \leq -4 \) and \( \beta \geq 4 \), but is broken in the interval \( -4 < \beta < 4 \). The persistent current oscillates as a function of magnetic flux with the same period as in the well-known Aharonov–Bohm oscillations [8]. The persistent current displays the following symmetries:

\[
I(\tau, \beta, n) = I(-\tau, -\beta, n), \quad I(\tau, \beta, n) = -I(\tau, -\beta, n). \tag{34}
\]

Moreover, the amplitude of the current depends on the curvature index \( n \).

Results for the persistent current \( I \) of a graphene ring with disclination defect as a function of the ratio \( w/R \) are shown in Fig. 7. The red, green, blue and magenta curves are for the indices \( n = 1, 2, -1, -2 \), respectively, while the solid curves and dashed curves correspond to the valleys \( k \) and \( k' \). These results show that the persistent current has a maximum when \( w/R \to 0 \) as shown in panel (a), and verify the following symmetry:

\[
I(n, \tau) = -I(-n, -\tau). \tag{35}
\]

In addition, we can clearly see that when \( w/R \) increases the persistent current \( I \) decreases until \( w/R \simeq 1 \). The degeneration of the \( k \) and \( k' \) valleys is lifted, i.e., \( I(n, \tau) = -I(-n, -\tau) \). In the \( w/R \to \infty \) regime, the persistent current \( I \) becomes almost constant and converges to zero, see panel (b).

Figure 8 shows the numerical results for the partial current (partial contribution) \( I_{mn} \) as a function of the ratio \( w/R \) for the valley \( K \) and \( K' \) with \( \beta = 0.5 \) for different values of the curvature index \( n \) and the angular momentum \( m \). We observe that \( I_{mn} \) exhibits the same behavior as the total current \( I \), but for the limit \( w/R \to 0 \) we have \( I_{mn}(\tau) \neq I_{mn}(-\tau) \). When \( w/R \to \infty \), it is clearly seen that \( I_{mn} \) remains constant and tends towards a non-zero value while the total current vanishes in the same limit \( I \to 0 \).
Fig. 8  Partial current (partial contribution) $I_{mn}$ as function of the ratio $w/R$ for the valley $k$ ($\tau =1$) and $k'$ with $\beta =0.5$. The red, green, blue, and magenta curves for $m=1$, $2$, $-1$, $-2$, respectively, correspond to the curvature index with panel a: $n = 1$, panel b: $n = 2$, panel c: $n = -1$, panel d: $n = -2$

an interesting behavior in the presence of a induced pseudo-magnetic field, which has no analog in quantum rings with no defect ($n=0$). Indeed, for a graphene quantum ring with a defect ($n \neq 0$), we have generated a gap opening in the energy spectrum between the conduction and valence bands that depends on the index $n$. We also have found that the persistent current has oscillations as a function of the magnetic flux with an analogous period to the famous Aharonov–Bohm oscillations [8] and obey the following symmetries $I(\tau, \beta, n) = I(-\tau, -\beta, n)$ and $I(\tau, \beta, n) = -I(\tau, -\beta, n)$. Moreover, we have shown that the amplitude of the energy spectrum and the current depends on the index $n$ of the defects in graphene and on the width $w$ of the quantum ring. We need to stress that all calculations performed in this work were done at zero temperature, $T = 0$, at finite temperature we expect different type of scattering process to take over and in particular electron-phonon interaction might cause the decay of the generated persistent current. Thus we expect the persistent current to decay above room temperature due to the fact that electron-phonon scattering becomes stronger and breaks electron–hole symmetry. However, the computations of the lifetime of such persistent current is a very debatable issue and is still an open “Million Dollar” question even at the level of normal metal ring threaded by a magnetic flux [28].

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