EHRHART QUASI-PERIOD COLLAPSE IN RATIONAL POLYGONS

TYRRELL B. McALLISTER AND MATTHEW MORIARITY

Abstract. In 1976, P. R. Scott characterized the Ehrhart polynomials of convex integral polygons. We study the same question for Ehrhart polynomials and quasi-polynomials of non-integral convex polygons. Turning to the case in which the Ehrhart quasi-polynomial has nontrivial quasi-period, we determine the possible minimal periods of the coefficient functions of the Ehrhart quasi-polynomial of a rational polygon.

1. Introduction

A rational polygon $P \subseteq \mathbb{R}^2$ is the convex hull of finitely many rational points, not all contained in a line. Given a positive integer $n$, let $nP := \{nx \in \mathbb{R}^2 : x \in P\}$ be the dilation of $P$ by $n$. The 2-dimensional case of a well-known result due to Ehrhart [3] states that the lattice-point enumerator $n \mapsto |nP \cap \mathbb{Z}^2|$ for $P$ is a degree-2 quasi-polynomial function with rational coefficients. That is, there exist periodic functions $c_P,0, c_P,1, c_P,2 : \mathbb{Z} \rightarrow \mathbb{Q}$ with $c_P,2 \not\equiv 0$ such that, for all positive integers $n$,

$$L_P(n) := c_P,0(n) + c_P,1(n)n + c_P,2(n)n^2 = |nP \cap \mathbb{Z}^2|.$$  

We call $L_P$ the Ehrhart quasi-polynomial of $P$. The period sequence of $P$ is $(s_0, s_1, s_2)$, where $s_i$ is the minimum period of the coefficient function $c_P,i$ for $i = 0, 1, 2$. The quasi-period of $L_P$ (or of $P$) is lcm $\{s_0, s_1, s_2\}$. We refer the reader to [1] and [12, Chapter 4] for thorough introductions to the theory of Ehrhart quasi-polynomials.

Our goal is to examine the possible periods and values of the coefficient functions $c_P,i$. The leading coefficient function $c_P,2$ is always a constant equal to the area $A_P$ of $P$. Furthermore, when $P$ is an integral polygon (meaning that its vertices are all in $\mathbb{Z}^2$), $L_P$ is simply a polynomial with $c_P,0 = 1$ and $c_P,1 = \frac{1}{2}b_P$, where $b_P$ is the number of lattice points on the boundary of $P$. When $P$ is integral, Pick’s formula $A_P = \mathcal{I}_P + \frac{1}{2}b_P - 1$ determines $A_P$ in terms of $b_P$ and the number $\mathcal{I}_P$ of points in the interior of $P$ [9]. Hence, characterizing the Ehrhart polynomials of integral polygons amounts to determining the possible numbers of lattice points in their interiors and on their boundaries. This was accomplished by P. R. Scott in 1976:

2000 Mathematics Subject Classification. Primary 52B05; Secondary 05A15, 52B11, 52C07.

Key words and phrases. Ehrhart polynomials, Quasi-polynomials, Lattice points, Convex bodies, Rational polygons, Scott’s inequality.

First author supported by the Netherlands Organisation for Scientific Research (NWO) Mathematics Cluster DIAMANT. Second author supported by Wyoming EPSCoR.
Theorem 1.1 (Scott [11]; see also [3]). Given non-negative integers \( I \) and \( b \), there exists an integral polygon \( P \) such that \((3_P, b_P) = (I, b)\) if and only if \( b \geq 3 \) and either \( I = 0 \), \((I, b) = (1, 9)\), or \( b \leq 2I + 6 \).

In Figure 1, the small squares indicate the values of \( I \) and \( b \) that are realized as the number of interior lattice points and boundary lattice points of some convex integral polygon. After a suitable linear transformation using Pick’s Formula, these squares represent all of the Ehrhart polynomials of integral polygons.

However, not all Ehrhart polynomials of polygons come from integral polygon. Indeed, the complete characterization of Ehrhart polynomials of rational polygons, including the non-integral ones, remains open. To this end, we define a polygonal pseudo-integral polytope, or polygonal PIP, to be a rational polygon with quasi-period equal to 1. That is, polygonal PIPs are those polygons that share with integral polygons the property of having a polynomial Ehrhart quasi-polynomial. Like integral polygons, polygonal PIPs must satisfy Pick’s Theorem [7, Theorem 3.1], so, again, the problem reduces to finding the possible values of \( \mathcal{I}_P \). Determining all possible coefficient functions \( c_{P,i} \) seems out of reach at this time. However, one interesting question we can answer is, What are the possible period sequences \((s_0, s_1, s_2)\)? In [5], P. McMullen gave bounds on the \( s_i \) in terms of the indices of \( P \). Given a \( d \)-dimensional polytope \( P \) and \( i \in \{0, \ldots, d\} \), the \( i \)-index of \( P \) is the least positive integer \( j_i \) such that every \( i \)-dimensional face of the dilate \( j_i P \) contains an integer lattice point in its affine span. We call \((j_0, \ldots, j_d)\) the index sequence of \( P \).

We state McMullen’s result in the general case of \( d \)-dimensional polytopes:

Theorem 1.2 (proved on p. 4). Given integers \( I \geq 1 \) and \( b \in \{1, 2\} \), there exists a polygonal PIP \( P \) with \((3_P, b_P) = (I, b)\). However, there does not exist a polygonal PIP \( P \) with \( b_P = 0 \) or with \((3_P, b_P) \in \{(0, 1), (0, 2)\} \).

In Figure 1b the small triangles below the small squares indicate the values of \( I \) and \( b \) mentioned in Theorem 1.2. The question of whether any points \((I, b), I \geq 1 \), above the small squares are realized by PIPs remains open.

In Section 3, we construct polygonal PIPs with \( b \geq 1 \) arbitrary. This construction therefore yields infinite families of Ehrhart polynomials that are not the Ehrhart polynomials of any integral polytope. This is our first main result, which we prove in Section 3.

Theorem 1.3 (McMullen [5] Theorem 6). Let \( P \) be a \( d \)-dimensional rational polytope with period sequence \((s_0, \ldots, s_d)\) and index sequence \((j_0, \ldots, j_d)\). Then \( s_i \) divides \( j_i \), for \( 0 \leq i \leq d \). In particular, \( s_i \leq j_i \).

We will refer to the inequalities \( s_i \leq j_i \) in Theorem 1.3 as McMullen’s bounds. It is easy to see that the indices \( j_i \) of a rational polytope satisfy the divisibility relations \( j_d | j_{d-1} | \cdots | j_0 \), and hence \( j_d \leq j_{d-1} \leq \cdots \leq j_0 \). Beck, Sam, and Woods [2] showed that McMullen’s bounds are always tight in the \( i \in \{d - 1, d\} \) cases. It is also shown in [2] that, given any positive integers \( q_d | q_{d-1} | \cdots | q_0 \), there exists a polytope with \( i \)-index \( q_i \), for \( 0 \leq i \leq d \). Moreover, all of McMullen’s bounds are tight for this polytope.

Seeing this, one might hope that the coefficient-periods \( s_i \) in the period sequence are also required to satisfy some constraints. However, our second main result shows that, in the case of polygons, \( s_0 \) and \( s_1 \) may take on arbitrary values. (Of course, we always have \( s_2 = 1 \), because the leading coefficient function \( c_{P,2} \) is a constant.)
Figure 1. On the left: Small squares indicate values of \((I, b)\) corresponding to convex integral polygons (Theorem 1.1). On the right: Small triangles indicate additional values of \((I, b)\) corresponding to nonintegral PIPs (Theorem 1.2). Question marks indicate values for which the existence of corresponding PIPs remains open.

**Theorem 1.4** (proved on p. 7). Given positive integers \(r\) and \(s\), there exists a polygon \(P\) with period sequence \((r, s, 1)\). Thus, in contrast to the Beck–Sam–Woods construction, the McMullen bound \(s_0 \leq j_0\) can be arbitrarily far from tight.

We prove Theorem 1.4 in Section 4. Before giving proofs of Theorems 1.2 and 1.4 we define in Section 2 some notation and terminology that we will use in our constructions.

2. Piecewise skew unimodular transformations

Since we will be exploring the possible Ehrhart quasi-polynomials of polygons, it will be useful to have geometric tools for constructing rational polygons while controlling their Ehrhart quasi-polynomials. The main tool that we will use are piecewise affine unimodular transformations. Following [4], we call these \(p\mathbb{Z}\)-morphisms.

**Definition 2.1.** Given \(U, V \subseteq \mathbb{R}^2\) and a finite set \(\{\ell_i\}\) of lines in the plane, let \(C\) be the set of connected components of \(U \setminus \bigcup_i \ell_i\). An injective continuous map \(f: U \to V\) is a \(p\mathbb{Z}\)-morphism if, for each component \(C \in C\), \(f|_C\) is the restriction to \(C\) of an affine transformation, and \(f|_{C \cap \mathbb{Z}^2}\) is the restriction to \(C \cap \mathbb{Z}^2\) of an affine automorphism of the lattice. (That is, \(f|_C\) can be extended to an element of \(\text{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\)).

Thus, \(p\mathbb{Z}\)-morphisms are piecewise affine linear maps that map lattice points, and only lattice points, to lattice points. The key property of \(p\mathbb{Z}\)-morphisms is that they preserve the lattice and so preserve Ehrhart quasi-polynomials.
We will only need \( p \mathbb{Z} \)-morphisms that act as skew transformations on each component of their domains. It will be convenient to introduce some notation for these transformations. Given a rational vector \( r \in \mathbb{Q}^2 \), let \( r_p \) be the generator of the semigroup \((\mathbb{R}_{>0}) \cap \mathbb{Z}^2\), and define the lattice length \( \text{len}(r) \) of \( r \) by \( \text{len}(r) = \gcd(\text{lcm}(b,d)) \). Define the skew unimodular transformation \( U_r \in \text{SL}_2(\mathbb{Z}) \) by

\[
U_r(x) = x + \frac{1}{\text{len}(r)^2} \det(r,x)r,
\]

where \( \det(r,x) \) is the determinant of the matrix whose columns are \( r \) and \( x \) (in that order). Equivalently, let \( S \) be the subgroup of skew transformations in \( \text{SL}_2(\mathbb{Z}) \) that fix \( r \), and let \( U_r \) be the generator of \( S \) that translates a vector \( v \) parallel (resp. anti-parallel) to \( r \) if the angle between \( r \) and \( v \) is less than (resp. greater than) \( \pi \), measured counterclockwise about the origin.

Define the piecewise unimodular transformations \( U_r^+ \) and \( U_r^- \) by

\[
U_r^+(x) = \begin{cases} 
U_r(x) & \text{if } \det(r,x) \geq 0, \\
x & \text{else},
\end{cases}
\]

and

\[
U_r^-(x) = \begin{cases} 
x & \text{if } \det(r,x) \geq 0, \\
(U_r)^{-1}(x) & \text{else}.
\end{cases}
\]

Finally, given a lattice point \( u \in \mathbb{Z}^2 \) and a rational point \( v \in \mathbb{Q}^2 \), let \( U_{uv}^+ \) and \( U_{uv}^- \) be the affine piecewise unimodular transformations defined by

\[
U_{uv}^+(x) = U_{v-u}^{+}(x-u) + u,
\]

\[
U_{uv}^-(x) = U_{v-u}^{-}(x-u) + u.
\]

3. Constructing nonintegral PIPs

We recall our notation from the introduction. Given a polygon \( P \subseteq \mathbb{R}^2 \), let \( \mathcal{A}_P \) be the area of \( P \), and let \( \mathcal{J}_P \) and \( b_P \) be the number of lattice points in the interior and on the boundary of \( P \), respectively. We now prove Theorem 1.2 which we restate here for the convenience of the reader.

**Theorem 1.2.** Given integers \( I \geq 1 \) and \( b \in \{1, 2\} \), there exists a polygonal PIP \( P \) with \( (\mathcal{J}_P, b_P) = (I, b) \). However, there does not exist a polygonal PIP \( P \) with \( b_P = 0 \) or with \( (\mathcal{J}_P, b_P) \in \{(0,1), (0,2)\} \).

**Proof.** Let integers \( b \in \{1, 2\} \) and \( I \geq 1 \) be given. We construct a polygonal PIP \( P \) with \( (\mathcal{J}_P, b_P) = (I, b) \).

If \( b = 2 \), consider the triangle

\[
T = \text{Conv} \left\{ (0,0)^t, (I+1,0)^t, (1,1-\frac{1}{I+1})^t \right\}.
\]

It was proved in [2] that \( T \) is a PIP. Let \( P \) be the union of \( T \) and its reflection about the \( x \)-axis. Then \( \mathcal{J}_P = I \) and \( b_P = 2 \). Moreover, \( \mathcal{J}_P(n) = 2\mathcal{J}_T(n) - (I+2) \) (correcting for points double-counted on the \( x \)-axis), so \( P \) is also a PIP.

If \( b = 1 \), consider the “semi-open” triangle

\[
T_1 = \text{Conv} \left\{ (0,0)^t, (1,2I-1)^t, (-1,0)^t \right\} \setminus \{(0,0)^t, (1,2I-1)^t\}.
\]
Figure 2. The construction of a polygonal PIP with one boundary point and an arbitrary number \( I \) of interior points in the case \( I = 3 \). Black points are elements of the region. Gray line segments indicate the lines fixed by the skew transformations.

(See Figure 2a for the case with \( I = 3 \).) The Ehrhart quasi-polynomial of \( T \) is evidently a signed sum of Ehrhart polynomials of integral polytopes, so it too is a polynomial. We will apply a succession of p\( \mathbb{Z} \)-morphisms to \( T \) to produce a convex rational polygon without changing the Ehrhart polynomial.

Let \( T_2 = (U_{(0,-1),1}^+)^{2I-1}(T_1) \). (See Figure 2b. The gray line segment indicates the line fixed by the skew transformation.) Hence,

\[
T_2 = \text{Conv} \{ (1,0)^t, (0, I - 1/2)^t, (-1,0)^t \} \setminus ((0,0)^t, (1,0)^t).
\]

Now act upon the triangle below the line spanned by \((-1,-1)\) (resp. \((1,-1)\)), with \( U_{(-1,-1),1}^+ \) (resp. \( U_{(1,-1),1}^- \)). (See Figure 2c. The line segments meeting at the origin lie on the lines fixed by one of these unimodular transformations.) The result is

\[
T_3 = \text{Conv} \left\{ \left( \frac{2I - 1}{2I + 1}, \frac{2I - 1}{2I + 1} \right), \left( 0, \frac{I - 1}{2} \right), \left( -\frac{2I - 1}{2I + 1}, \frac{2I - 1}{2I + 1} \right) \right\}.
\]

At this point, we have a convex rational polygon with the desired number of interior and boundary points, so the claim is proved. However, it might be noted that we can achieve a triangle by letting \( P = (U_{(0,1),1}^-)^{2I-1}(T_3) \), yielding

\[
P = \text{Conv} \left\{ \left( \frac{0}{-1}, \frac{2I - 1}{2I + 1}, \frac{2I - 1}{2I + 1}, \frac{2I - 1}{2I + 1} \right) \right\}.
\]

\[\square\]
To prove the nonexistence claim, let a polygonal PIP $P$ be given. In [7, Theorem 3.1], it was shown that $b_{n,P} = nb_P$ for $n \in \mathbb{Z}_{>0}$. If $b_P = 0$, this implies that $b_{n,P} = 0$ for all $n \in \mathbb{Z}_{>0}$, which is impossible because, for example, some integral dilate of $P$ is integral. Hence, $b_P \geq 1$.

It was also shown in [7] that polygonal PIPs satisfy Pick’s theorem: $\mathcal{A}_P = \mathcal{I}_P + \frac{1}{2}b_P - 1$. But if $\mathcal{I}_P = 0$ and $b_P \in \{1,2\}$, this yields $\mathcal{A}_P \leq 0$. Since our polygons are not contained in a line by definition, this is impossible. Therefore, if $b_P < 3$, we must have $b_P \in \{1,2\}$ and $\mathcal{I}_P \geq 1$.

A proof of, or counterexample to, Scott’s inequality $b_P \leq 2\mathcal{I}_P + 7$ for nonintegral polygonal PIPs with interior points eludes us. (See the question marks in Figure [11]). However, it is easy to show that any counterexample $P$ cannot contain a lattice point in the interior of its integral hull $\tilde{P} := \text{Conv}(P \cap \mathbb{Z}^2)$. Indeed, the proof does not even require the hypothesis that $P$ is a PIP.

**Proposition 3.1.** If $P$ is a polygon whose integral hull contains a lattice point in its interior, then either $(\mathcal{I}_P, b_P) = (1,9)$ or $b_P \leq 2\mathcal{I}_P + 6$.

**Proof.** We are given that $\mathcal{I}_P \geq 1$. Note that $b_{\tilde{P}} \geq b_P$ and $\mathcal{I}_{\tilde{P}} \leq \mathcal{I}_P$. Since $\tilde{P}$ is an integral polygon, it obeys Scott’s inequality: $b_{\tilde{P}} \leq 2\mathcal{I}_{\tilde{P}} + 6$ unless $(\mathcal{I}_{\tilde{P}}, b_{\tilde{P}}) = (1,9)$. In the former case, we have $b_P \leq b_{\tilde{P}} \leq 2\mathcal{I}_{\tilde{P}} + 6 \leq 2\mathcal{I}_P + 6$. In the latter case, we similarly have $b_P \leq 9$ and $1 = \mathcal{I}_{\tilde{P}} \leq \mathcal{I}_P$, so either $\mathcal{I}_P = 1$ or $b_P \leq 2\mathcal{I}_P + 6$. □

4. Periods of Coefficients of Ehrhart Quasi-Polynomials

If $P$ is a rational polygon, then the coefficient of the leading term of $\mathcal{L}_P$ is the area of $P$, so the “quadratic” term in the period sequence of $P$ is always 1. However, we show below that no constraints apply to the remaining terms in the period sequence. This is our Theorem [1.4] which we restate here for the convenience of the reader.

**Theorem 1.4.** Given positive integers $r$ and $s$, there exists a polygon $P$ with period sequence $(r,s,1)$.

Before proceeding to the proof, we make some elementary observations regarding the coefficients of certain Ehrhart quasi-polynomials.

Fix a positive integer $s$, and let $\ell$ be the line segment $[0,\frac{1}{s}]$. Then we have that $\mathcal{L}_\ell(n) = \frac{1}{s}n + c_{\ell,0}(n)$, where the “constant” coefficient function $c_{\ell,0}(n) = [n/s] - n/s + 1$ has minimum period $s$. Note also that the half-open interval $h := (\frac{1}{s},1]$ satisfies $\mathcal{L}_\ell + \mathcal{L}_h = \mathcal{L}_{[0,1]}$. In particular, we have that

$$c_{\ell,0} + c_{h,0} = 1.$$ (1)

Given a positive integer $m$, it is straightforward to compute that the Ehrhart quasi-polynomial of the rectangle $\ell \times [0,m]$ is given by

$$\mathcal{L}_{\ell \times [0,m]}(n) = \frac{m}{s}n^2 + (mc_{\ell,0}(n) + \frac{1}{s})n + c_{\ell,0}(n).$$

In particular, the “linear” coefficient function has minimum period $s$, and the “constant” coefficient function is identical to that of $\mathcal{L}_\ell$. More strongly, we have the following:

**Lemma 4.1.** Suppose that a polygon $P$ is the union of $\ell \times [0,m]$ and an integral polygon $P'$ such that $P' \cap (\ell \times [0,m])$ is a lattice segment. Then $c_{P,1}$ has minimum period $s$ and $c_{P,0} = c_{\ell,0}$. 
With these elementary facts in hand, we can now prove Theorem 1.4.

**Proof of Theorem 1.4.** Any integral polygon has period sequence \((1,1,1)\), so we may suppose that either \(r \geq 2\) or \(s \geq 2\). Our strategy is to construct a polygon \(H\) with period sequence \((1,s,1)\) and a triangle \(Q\) with period sequence \((r,1,1)\). We will then be able to construct a polygon with period sequence \((r,s,1)\) for \(r,s \geq 2\) by gluing \(Q\) to \(H\) along an integral edge.

We begin by constructing a polygon with period sequence \((1,s,1)\) for an arbitrary integer \(s \geq 2\). (Figure 3 below depicts the \(s = 3\) case of our construction.) Define \(H\) to be the heptagon with vertices

\[
\begin{align*}
t_1 &= \left(-\frac{1}{s}, s(s-1)+1\right), & v_1 &= \left(1, s(s-1)\right), \\
t_2 &= \left(-\frac{1}{s}, -s(s-1)-1\right), & v_2 &= \left(1, -s(s-1)\right), \\
u_1 &= \left(0, s(s-1)+1\right), & w &= \left(s-1+\frac{1}{s}, 0\right), \\
u_2 &= \left(0, -s(s-1)-1\right).
\end{align*}
\]

To show that \(H\) has period sequence \((1,s,1)\), we subdivide \(H\) into a rectangle and three triangles as follows (see left of Figure 3):

- \(R = \text{Conv}\{t_1, t_2, u_2, u_1\}\),
- \(T_2 = \text{Conv}\{u_2, v_2, w\}\),
- \(T_1 = \text{Conv}\{u_1, v_1, u\}\),
- \(T_3 = \text{Conv}\{u_1, u_2, w\}\).

Let \(v = (s,0)\). Write \(U_1 = U_{u_1}^\uparrow\) and \(U_2 = U_{u_2}^\downarrow\). Then \(U_1(T_1) = \text{Conv}\{u_1, v, w\}\) and \(U_2(T_2) = \text{Conv}\{u_2, v, w\}\).

Let \(H' = R \cup U_1(T_1) \cup U_2(T_2) \cup T_3\) (see right of Figure 3). Though \(H'\) was formed from unimodular images of pieces of \(H\), we do not quite have \(\mathcal{L}_H = \mathcal{L}_{H'}\). This is because each point in the half-open segment \((w,v]\) has two pre-images in \(H\). Since this segment is equivalent under a unimodular transformation to \(h = (\frac{1}{s}, 1)\), the correct equation is

\[
\mathcal{L}_H = \mathcal{L}_{H'} + \mathcal{L}_h.
\]

Let \(T = U_1(T_1) \cup U_2(T_2) \cup T_3\). Then \(T\) is an integral triangle intersecting \(R\) along a lattice segment, and \(H' = R \cup T\). Hence, by Lemma 4.1, \(c_{H',1}\) has minimum period \(s\), and so, by equation (2), \(c_{H,1}\) also has minimum period \(s\).

It remains only to show that \(c_{H,0}\) has minimum period 1. Again, from equation (2), we have that

\[
c_{H,0} = c_{H',0} + c_{h,0}.
\]

From Lemma 4.1 we know that \(c_{H',0} = c_{\ell,0}\). Therefore, by equation (1), \(c_{H,0}\) is identically 1.

We now construct a triangle with period sequence \((r,1,1)\) for integral \(r \geq 2\). Let

\[
Q = u_1 + \text{Conv}\{(0,0), (1, -1), (1/r, 0)\}.
\]

McMullen’s bound (Theorem 1.3) implies that the minimum period of \(c_{Q,1}\) is 1. Hence, it suffices to show that the minimum quasi-period of \(\mathcal{L}_Q\) is \(r\). Observe that \(Q\) is equivalent to \(\text{Conv}\{(0,0), (1,0), (0,1/r)\}\) under a unimodular transformation. Hence, one easily computes that \(\sum_{k=0}^{\infty} \mathcal{L}_Q(k)\zeta^k = (1 - \zeta)^{-2} (1 - \zeta^r)^{-1}\). Note that among the poles of this rational generating function are primitive \(r^{th}\) roots of unity. It follows from the standard theory of rational generating functions that \(\mathcal{L}_Q\) has minimum quasi-period \(r\) (see, e.g., [22 Proposition 4.4.1]).
Finally, given integers $r, s \geq 2$, let $P = H \cup Q$. Note that $H$ and $Q$ have disjoint interiors, $H \cap Q$ is a lattice segment of lattice length 1, and $H \cup Q$ is convex. It follows that $P$ is a convex polygon and $\mathcal{L}_P = \mathcal{L}_H + \mathcal{L}_Q - \delta_{[0,1]}$. Therefore, $P$ has period sequence $(r, s, 1)$, as required. \qed
5. PSEUDO-REFLEXIVE POLYGONS

We conclude with some speculative remarks about the connection between PIPs, reflexive polygons, and $\mathbf{SL}_2(\mathbb{Q})$. In particular, nonintegral PIPs that contain only a single lattice point in their interior appear to be nonintegral analogues of reflexive polygons. To explore this connection, we introduce the notion of a pseudo-reflexive polygon.

Recall that an integral polytope $P \subseteq \mathbb{R}^n$ is called reflexive if the polar dual $P^\vee = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in P \}$ of $P$ is also an integral polytope. We similarly define a pseudo-reflexive polytope to be a PIP $P$ such that $P^\vee$ is integral. An example of a nonintegral pseudo-reflexive PIP is the convex hull of $\{ (0, -1), (1/3, 1/3), (-1/3, 2/3) \}$.

Poonen and Rodriguez-Villegas [10] showed that, if $P \subseteq \mathbb{R}^2$ is a reflexive polygon, then $b_P + b_P^\vee = 12$. Hille and Skarke [6] observed that this fact follows from a correspondence between reflexive polygons and words equal to the identity in a certain presentation of $\mathbf{SL}_2(\mathbb{Z})$. In particular, put $A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Then $\mathbf{SL}_2(\mathbb{Z})$ is generated by $A$ and $B$, and the relations of this presentation are

$$ABA = BAB, \quad (AB)^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (4)$$

Given a word

$$B^{b_n} A^{a_n} B^{b_{n-1}} A^{a_{n-1}} \cdots B^{b_1} A^{a_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in $\mathbf{SL}_2(\mathbb{Z})$ with $a_i, b_i$ positive integers, one reads off a closed polygonal path $v_0 v_1 \cdots v_{n-1} v_0$ with a positive winding number $w$ about the origin as follows: Put $v_0 := (1, 0)$, $d_0 := (0, 1)$, and recursively define

$$\begin{bmatrix} v_i \\ d_i \end{bmatrix} := B^{b_i} A^{a_i} \begin{bmatrix} v_{i-1} \\ d_{i-1} \end{bmatrix}, \quad \text{for } 1 \leq i < n. \quad (6)$$

If the winding number $w$ equals 1, then this path is the boundary of a reflexive polygon $P$ with $b_P = \sum_i a_i$ and $b_P^\vee = \sum_i b_i$. Moreover, every reflexive polygon can be obtained in this way, up to an automorphism of the lattice. It follows directly from the relations (4) that $b_P + b_P^\vee$ is a multiple of 12. Hille and Skarke show that, in general, $\sum_i a_i + \sum_i b_i = 12w$, from which the Poonen–Rodriguez-Villegas result follows.

Our observation is that every pseudo-reflexive polygon $P$ also corresponds to a word, this time in the infinite set of generators

$$A^r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} \quad \text{(for all } r \in \mathbb{Q}), \quad B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad (7)$$

of $\mathbf{SL}_2(\mathbb{Q})$. Let $v_0, v_1, \ldots, v_{n-1} \in \mathbb{Q}^2$ be such that the polygonal path $v_0 v_1 \cdots v_{n-1} v_0$ is the boundary of a pseudo-reflexive PIP in which each $v_i$ is a vertex. By Theorem 12, this boundary contains a lattice point. By applying an automorphism of the lattice, we may suppose that this lattice point is $v_0 = (0, 1)$. Let $a_i \in \mathbb{Q}$ be the lattice length of the segment $v_{i-1} v_i$. Set $b'_i \in \mathbb{Z}$ to be the lattice length of the edge
of \( P^v \) with outer normal \( v_i \). Put \( b_i := \text{den}(v_i)b'_i \). Then, corresponding to \( P \), we get a word

\[
B^{b_n} A^{a_n} B^{b_{n-1}} A^{a_{n-1}} \cdots B^{b_1} A^{a_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

in the generators \( \{7\} \) equal to the identity.

Conversely, given such a word with the \( a_i \in \mathbb{Q}, a_i > 0, a_1 + \cdots + a_n \in \mathbb{Z} \), and the \( b_i \) positive integers divisible by \( \left( \text{den} \left( \sum_{j=1}^n a_j \right) \right)^2 \), we have a corresponding polygonal path \( v_0v_1 \cdots v_{n-1}v_0 \) defined by the same recursive relations \( \{6\} \) used by Hille and Skarke. If the winding number of this path about the origin is 1, then the path is the boundary of a polygonal pseudo-reflexive polygon.

References

[1] M. Beck and S. Robins, *Computing the continuous discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Integer-point enumeration in polyhedra.

[2] M. Beck, S. V. Sam, and K. M. Woods, *Maximal periods of (Ehrhart) quasi-polynomials*, J. Combin. Theory Ser. A 115 (2008), no. 3, 517–525.

[3] E. Ehrhart, *Sur les polyèdres homothétiques bordés à n dimensions*, C. R. Acad. Sci. Paris 254 (1962), 988–990.

[4] P. Greenberg, *Piecewise \( SL_2 \mathbb{Z} \) geometry*, Trans. Amer. Math. Soc. 335 (1993), no. 2, 705–720.

[5] C. Haase and J. Schicho, *Lattice polygons and the number 2i + 7*, Amer. Math. Monthly 116 (2009), no. 2, 151–165.

[6] L. Hille and H. Skarke, *Reflexive polytopes in dimension 2 and certain relations in \( SL_2(\mathbb{Z}) \)*, J. Algebra Appl. 1 (2002), no. 2, 159–173.

[7] T. B. McAllister and K. M. Woods, *The minimum period of the Ehrhart quasi-polynomial of a rational polytope*, J. Combin. Theory Ser. A 109 (2005), no. 2, 345–352, arXiv:math.CO/0310255.

[8] P. McMullen, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel) 31 (1978/79), no. 5, 509–516.

[9] G. A. Pick, *Geometrisches zur zahlenlehre*, Sitzungsber. Lotos Prag 19 (1899), no. 2, 311–319.

[10] B. Poonen and F. Rodriguez-Villegas, *Lattice polygons and the number 12*, Amer. Math. Monthly 107 (2000), no. 3, 238–250.

[11] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc. 15 (1976), no. 3, 395–399.

[12] R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

(T. B. McAllister) Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA

E-mail address: tmcallis@uwyo.edu