Parametrics Resonances of a Forced Modified Rayleigh-Duffing Oscillator

C. H. Miwadinou, A. V. Monwanou and J. B. Chabi Orou

May 11, 2014

Institut de Mathématiques et de Sciences Physiques, BP: 613 Porto Novo, Bénin

Abstract

We investigate in this paper the superharmonic and subharmonic resonances of forced modified Rayleigh-Duffing oscillator. We analyse this equation by method of multiple scales and we obtain superharmonic, subharmonic resonances order-two and order-three and primary resonance. We obtain also regions where steady-state subharmonic responses exist. We also use the amplitude-frequency curve for demonstrate the effect of various parameters on the response of the system. Finally, we focus our attention on chaotic motion of this oscillator by simulation and we obtain that this oscillator is chaotic for certains values for natural and excitation frequency but chaotic motion it is not the same in subharmonic and superharmonic cases.

keywords: Parametric resonance, superharmonic, subharmonic, forced modified Rayleigh-Duffing oscillator, multiple scales method, chaotic behavior.

1 Introduction

Many problems in physics, chemistry, biology, etc., are related to nonlinear self-excited oscillators [2]. For example, the self-excited oscillations in bridges and airplane wings, the beating of a heart, and the nonlinear model of a machine tool chatter [10]. A self-excited oscillator is a system which has some external source of energy upon which it can be drawn. Self-excited systems have a long history in the field of mechanics [11][12]. One of key problems in the theory of nonlinear oscillations is a search of possibilities to estimate their amplitude and period analytically. Parametric perturbations are characterised by parameters periodically in time changing and they are described by homogeneous differential equations of motion. Many works on self-excited, parametrically and externally excited are
well known and deeply investigated in the literature separately. Minorski \[14\] is one of the first authors considering the interaction between two different types of perturbations. Warminski \[16\] emphasizes the differences in modelling ideal and non-ideal systems for a chosen class of self-excited, parametric and externally excited vibrations. Many of those studies lead to the parametric excitation combined with self-excited system and subjected to an external force which quite often take the form

\[
\ddot{x} + \eta(x; \dot{x})\dot{x} + (1 - \mu \cos 2\omega t)(x + \alpha x^3) = F \cos \omega t, 
\]  

(1)

where \(\eta(x; \dot{x})\) is a nonlinear damping function. The effect of nonlinear damping on a nonlinear oscillator was investigated previously in \[15\], showing among other things how it affected the evolution of fractalisation of phase space.

Autoparametric resonance plays an important part in nonlinear engineering while posing interesting mathematical challenges. The linear dynamics is already nontrivial whereas the nonlinear dynamics of such systems is extremely rich and largely unexplored \[5\]. Tina Marie Morrison in his thesis \[4\], have investigated the dynamics of a system consisting of a simple harmonic oscillator with small nonlinearity, damping and parametric forcing in the neighborhood of 2:1 resonance near a Hopf bifurcation:

\[
\ddot{z} + \epsilon A \dot{z} + (1 + \epsilon k_1 + \epsilon B \cos 2t)z + \epsilon(\beta_1 z^3 + \beta_2 z^2 \dot{z} + \beta_3 z \dot{z}^2 + \beta_4 \dot{z}^3) = 0
\]  

(2)

Venkatanarayanan Ramakrishnan and Brian F Feeny are particulary study in \[1\] the resonances of the forced nonlinear Mathieu equation.

In the present work we consider the modified Rayleigh-Duffing oscillator modelled by following equation:

\[
\ddot{x} + \epsilon \mu(1 - x^2)\dot{x} + \epsilon \beta \dot{x}^2 + \epsilon k_1 \ddot{x} x + \epsilon k_2 \dot{x}^2 x + (\omega^2 + \epsilon \alpha \cos \Omega t)x + \epsilon \lambda x^3 = F \cos \Omega t.
\]  

(3)

Our interest in understanding the behavior of this equation is motivated by two applications. The first is a model of the El Niño Southern Oscillation (ENSO) coupled tropical ocean-atmosphere weather phenomenon \[39, 40\] in which the state variables are temperature and depth of a region of the ocean called the thermocline. The annual seasonal cycle is the parametric excitation. The model exhibits a Hopf bifurcation in the absence of parametric excitation. The second application involves a MEMS device \[41, 42\] consisting of a 30\(\mu\)m diameter silicon disk which can be made to vibrate by heating it with a laser beam resulting in a Hopf bifurcation. The parametric excitation is provided by making the laser beam intensity vary periodically in time.

We focus our attention on the study of the different resonances which can exist in the forced parametric modified Rayleigh-Duffing oscillator. We seek approximate solutions to equation \(3\) by using the method of multiple scales (MMS) and we find the peak amplitude of resonances phenomenon. We study the effects of
certain parameters of this oscillator on these different resonances. Finally, we study the chaotic motion of this oscillator by simulation in the subharmonic and superharmonic regions.

2 Resonances of the forced modified Rayleigh-Duffing oscillator

We use the method of multiple scales (MMS) to seek approximate solutions to equation (3). The analysis reveals the existence of various superharmonic and subharmonic resonances. The method of multiple scales supposed that the approximate steady solution of first order for eq.(3) in the form

\[ x(t, \epsilon) = x_0(T_0, T_1) + \epsilon x_1(T_0, T_1) + \ldots, \]

where \( T_0 = t, T_n = \epsilon^n T_0. \) Then

\[ \frac{d}{dt} = D_0 + \epsilon D_1; \]

\[ \frac{d^2}{dt^2} = D_2 + 2 \epsilon D_0 D_1 + \ldots \]

with

\[ D_m = \frac{\partial^m}{\partial T_n^m}. \]

Substituting eq.(4) into eq.(3) and equating the coefficients of the same power of small parameter \( \epsilon, \) one obtains

In order \( \epsilon^0, \)

\[ D_0^2 x_0 + \omega^2 x_0 = F \cos \Omega T_0. \]

In order \( \epsilon^1, \)

\[ D_0^2 x_1 + \omega^2 x_1 = -2 D_0 D_1 x_0 - \mu D_0 x_0 + \mu (D_0 x_0)^3 - \beta (D_0 x_0)^2 - \alpha x_0 \cos \Omega T_0 - k_1 x_0 D_0 x_0 - k_2 x_0 (D_0 x_0)^2 - \lambda x_0^3. \]

The solution for eq.(5) is

\[ x_0 = A e^{i \omega T_0} + \Lambda e^{i \Omega T_0} + cc, \]

where

\[ \Lambda = \frac{F}{2(\omega^2 - \Omega^2)}, A = \frac{1}{2} ae^{i \theta}, \]

cc stands for complex conjugate of preceding terms. Substituting the solution \( x_0 \) from eq.(7) into eq.(6), we are expanded the terms on the right hand side. We obtain

\[ D_0^2 x_1 + \omega^2 x_1 = [-2i \omega A' - i \mu \omega A + 3i \mu \omega^3 A^2 \tilde{A} + 6i \mu \omega^2 A^2 \Lambda^2 - 3 \lambda A^2 \tilde{A} - 6 \lambda \Lambda \tilde{A} - 2k_2 \omega^2 A \tilde{A} - 2k_2 \Omega^2 \Lambda \Lambda^2] e^{i \omega T_0} + [-i \mu \Lambda \Omega + 6i \mu \omega^2 \Omega \Lambda \Lambda + 3i \Omega^3 \Lambda^3 - 6 \lambda \Lambda \tilde{A} - 3 \lambda A^3 - 2k_2 \omega^2 A \Lambda \Lambda - k_2 \Lambda^3 \Omega] e^{i \Omega T_0} + [-i \mu \lambda A^3 - \lambda \Lambda^3 + k_2 \omega^2 A^3 \tilde{A} + 3i \mu \omega^2 \Omega \Lambda^3 \Lambda - 3 \lambda A^2 \Lambda + k_2 \omega^2 A \Lambda^2 - 2k_2 \omega \Omega A^2 \Lambda] e^{i(\omega + \Omega) T_0} + \ldots \]
\[-\frac{1}{2}a\bar{a} - ik_1\Omega \Lambda \bar{a} + ik_1\omega \Lambda \bar{a} - 2\beta \omega \Omega \bar{a}\Lambda e^{i(\omega + \Omega)T_0} +
\[-\frac{1}{2}a\Lambda - ik_1\Omega \Lambda^2 + \beta \Lambda^2 \Omega^2 \bar{a}^2 + 2\beta \omega^2 A \bar{a} - 2\beta \Lambda^2 \Omega^2 + cc + NST, \tag{9}\]

where NST is non resonance terms. We need to eliminate coefficients of $e^{i\omega T_0}$ that constitute the secular terms and would make the solutions unbounded. The solvability condition is thus set by equating the coefficients of $e^{i\omega T_0}$ terms to zero.

### 2.1 Superharmonic resonances

In this case, we consider first $2\Omega = \omega + \epsilon \sigma$ and after $3\Omega = \omega + \epsilon \sigma$ where $\sigma$ is a detuning parameter.

#### 2.1.1 $2\Omega = \omega + \epsilon \sigma$

If $2\Omega = \omega + \epsilon \sigma$, the condition for elimination of secular terms in eq.(9) is

\[-2i\omega A' - i\mu \omega A + 3i\mu \omega^3 A^2 \bar{a} + 6i\mu \omega^2 A \Lambda^2 - 3\lambda A^2 \bar{a} - 6\lambda \Lambda^2 - k_2 \omega^2 A^2 \bar{a} - 2k_2 \omega^2 A \Lambda^2 + (-\frac{1}{2}a\Lambda - ik_1\Omega \Lambda^2 + \beta \Lambda^2 \Omega^2) e^{i\sigma T_1} = 0, \tag{10} \]

with $T_1 = \epsilon T_0$. To this order, $A$ is considered to be a function to $T_1$ only. Then, substituting the polar form eq.(8) into eq.(10) and equating the real and imaginary parts, one gets

\[a' = (-\frac{1}{2} + 3\Omega^2 \Lambda^2) \mu a + 3\frac{\mu \omega^2 a^3}{\omega} - \frac{\Omega k_1 \Lambda^2}{\omega} \cos(\sigma T_1 - \theta) + \frac{(-\frac{1}{2}a\Lambda + \beta \Omega^2 \Lambda^2)}{\omega} \sin(\sigma T_1 - \theta), \tag{11}\]

\[a\theta' = (3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega} a + \frac{1}{8}(3\lambda + k_2 \omega^2) a^3 - \frac{\Omega k_1 \Lambda^2}{\omega} \sin(\sigma T_1 - \theta) - \frac{(-\frac{1}{2}a\Lambda + \beta \Omega^2 \Lambda^2)}{\omega} \cos(\sigma T_1 - \theta). \tag{12}\]

Letting $\gamma = \sigma T_1 - \theta$, eq.(11) and eq.(12) can be written to

\[a' = (-\frac{1}{2} + 3\Omega^2 \Lambda^2) \mu a + 3\frac{\mu \omega^2 a^3}{\omega} - \frac{\Omega k_1 \Lambda^2}{\omega} \cos \gamma + \frac{(-\frac{1}{2}a\Lambda + \beta \Omega^2 \Lambda^2)}{\omega} \sin \gamma, \tag{13}\]

\[a\gamma' = a\sigma - (3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega} a - \frac{1}{8}(3\lambda + k_2 \omega^2) a^3 + \frac{\Omega k_1 \Lambda^2}{\omega} \sin \gamma. \]

4
Putting $a' = \theta' = 0$ to find the stable period solution. We obtain

$$\left(-\frac{1}{2}\alpha \Lambda + \beta \Omega^2 \Lambda^2\right) \frac{\omega}{\mu} \cos \gamma.$$  \hspace{1cm} (14)

At steady-state the relationship between the response amplitude and the detuning parameter $\sigma$ is

$$\sigma = \left(3\lambda + k_2 \Omega^2\right) \frac{\Lambda^2}{\omega} + \frac{1}{8} \left(3\lambda + k_2 \omega^2\right) a^2.$$  \hspace{1cm} (16)

The peak amplitude would be verify the following equation:

$$\frac{\Omega^2 k_1^2 \Lambda^4}{\omega^2 a^2_p} + \left(-\frac{1}{2}\alpha \Lambda + \beta \Omega^2 \Lambda^2\right)^2 = \left(-\frac{1}{2} + 3\Omega^2 \Lambda^2 + \frac{3}{8} \omega^2 a^2_p \mu^2\right)^2.$$  \hspace{1cm} (18)

We can conclude the following:

- The peak value is independent of $k_2$ and $\lambda$.
- Parameters of modified Rayleigh-Duffing oscillator affect the peak location and as they increase, $|\sigma_p|$ increases. Now we plot the frequency-response curve from eq.(17). In Fig. 1, the frequency-response curve are plotted for fixed values of linear and nonlinear parameters. This curve shows that the amplitude of the resonance frequency when augment the external exciting force to which the order-two resonance superharmonic increases. We also note that the peak of the resonance curve becomes less sharp as the frequency increases.
Figure 1: superharmonic resonance in the space \((a, \Omega)\) for \(\mu = 0.5; \alpha = 0.5; \lambda = 0.5; \beta = 0.1; k_2 = 0.5; k_1 = 0.1; F = 0.01; \varepsilon = 0.005\).

### 2.1.2 \(3\Omega = \omega + \epsilon \sigma\)

If \(3\Omega = \omega + \epsilon \sigma\), the first term and the term which have \(3\Omega T\) as an exponential argument of the right member of eq. (9) are the secular terms. The condition for the elimination of secular terms is

\[
-2i\omega A' - i\mu\omega A + 3i\mu\omega^3 A^2 \bar{A} + 6i\mu\omega\Omega^2 A\Lambda^2 - 3\lambda A^2 \bar{A} - 6\lambda A \Lambda^2 - k_2\omega^2 A^2 \bar{A} - 2k_2\Omega^2 A\Lambda^2 + (-i\mu \Lambda^3 - \lambda \Lambda^3 + k_2\Omega^2 \Lambda^3)e^{i\sigma T} = 0,
\]

(21)

Following the analysis done in the previous section for the \(2\Omega\) superharmonic resonance, we substitute for \(A\) and separate the equation into real and imaginary parts. Using \(\gamma = \sigma T_1 - \theta\), we arrive at a homogenous set of equations in \(\gamma\) and \(a\)

\[
a' = \left(-\frac{1}{2} + 3\Omega^2 \Lambda^2\right)\mu a + \frac{3}{8}\mu \omega^2 a^3 - \mu \omega^2 \Lambda^3 \cos \gamma - \frac{(\lambda - k_2\Omega^2)}{\omega} \Lambda^3 \sin \gamma, \quad (22)
\]

and

\[
a'\gamma' = a\sigma - (3\lambda + k_2\Omega^2)\Lambda^2 \omega a - \frac{1}{8}(3\lambda + k_2\omega^2)\frac{a^3}{\omega} + \mu \omega^2 \Lambda^3 \sin \gamma - \frac{(\lambda - k_2\Omega^2)}{\omega} \Lambda^3 \sin \gamma.
\]

(23)

For steady-state solutions \(a' = \gamma' = 0\), which is satisfied if

\[
\left[\left[\sigma - (3\lambda + k_2\Omega^2)\frac{\Lambda^2}{\omega} - \frac{1}{8}(3\lambda + k_2\omega^2)\frac{a^2}{\omega}\right]^2 + \left(-\frac{1}{2} + 3\Omega^2 \Lambda^2 + \frac{3}{8}\omega^2 a^2\right)^2 \mu^2\right]a^2 = \left[\mu^2 \omega^6 + (\lambda - k_2\Omega^2)^2\right]\frac{\Lambda^6}{\omega^2}.
\]

(24)
We determine the detuning parameter $\sigma$ from eq.(24)

$$\sigma = (3\lambda + k_2\Omega^2) \frac{\Lambda^2}{\omega} + \frac{1}{8}(3\lambda + k_2\omega^2) \frac{a^2}{\omega} \pm$$

$$\left[\mu^2\omega^6 + (\lambda - k_2\Omega^2)^2\right] \frac{\Lambda^6}{\omega^2a^2_p} - \left(-\frac{1}{2} + 3\Omega^2\Lambda^2 + \frac{3}{8}\omega^2a^2_p\right) \mu^2\right]^{1/2}. \quad (25)$$

This equation is the frequency-response curve for superharmonic.

The peak amplitude verify the following equation

$$\left[\mu^2\omega^6 + (\lambda - k_2\Omega^2)^2\right] \frac{\Lambda^6}{\omega^2a^2_p} - \left(-\frac{1}{2} + 3\Omega^2\Lambda^2 + \frac{3}{8}\omega^2a^2_p\right) \mu^2 = 0, \quad (26)$$

and corresponding value of $\sigma$ is

$$\sigma_p = (3\lambda + k_2\Omega^2) \frac{\Lambda^2}{\omega} + \frac{1}{8}(3\lambda + k_2\omega^2) \frac{a^2_p}{\omega}. \quad (27)$$

Therefore, we noticed that the peak amplitude and frequency of this resonance are affected by third order non linearity parameters and by the forcing amplitude but the parametric excitation term and the coefficients of the quadratics nonlinear terms does not contribute to this resonance at first order.

We plot in Fig. 2 the frequency-response curve giving by eq.(24). This curve also prove that the amplitude of the resonance frequency when augment the external exciting force to which the order-three resonance superharmonic increases. The frequency domain where this resonance in this order appear is smaller than the case of order-two. The peak amplitude obtain at order-two superharmonic resonance is more increased than the peak amplitude of order-three for the same resonance.

Figure 2: superharmonic resonance in the space($a, \Omega$) for $\mu = 0.5; \lambda = 0.5; k_2 = 0.5; F = 0.001; \epsilon = 0.01$. 7
2.2 Subharmonic resonances

The subharmonic resonance take place if \( \Omega = 2\omega + \epsilon \sigma \) or \( \Omega = 3\omega + \epsilon \sigma \).

2.2.1 \( \Omega = 2\omega + \epsilon \sigma \)

The first term and the term with \( i(\Omega - \omega)T_0 \) in eq.(9) contribute to secular terms. The condition of the elimination of secular terms is

\[
-2i\omega A' - i\mu \omega A + 3i\mu \omega^3 A^2 \bar{A} + 6i\mu \omega^2 A\Lambda^2 - 3\lambda A^2 \bar{A} - 6\lambda A\Lambda^2 - \\
k_2\omega^2 A^2 \bar{A} - 2k_2\Omega^2 A\Lambda^2 + (-\frac{1}{2}\alpha \bar{A} - ik_1\Omega \Lambda \bar{A} + ik_1\omega \Lambda \bar{A} - \\
2\beta\omega \Lambda A)e^{i\sigma T_1} = 0,
\]

(28)

We substitute the polar notation for \( A \) in eq.(28), and equate the real and imaginary parts, and let \( \gamma = \sigma T_1 - 2\theta \)

\[
a' = (-1 + 6\Omega^2 \Lambda^2 + \frac{3}{4}\omega^2 a^2)\mu a - \frac{(\Omega + \omega)}{\omega}k_1\Lambda a \cos \gamma - \\
\left(\frac{1}{2}\alpha + 2\beta\omega \Lambda \right)\frac{a}{\omega} \sin \gamma,
\]

(29)

and

\[
a' = [\sigma - 2(3\lambda + k_2\Omega^2)\frac{A^2}{\omega} - \frac{1}{4}(3\lambda + k_2\omega^2)\frac{a^2}{\omega}]a + \frac{(\Omega + \omega)}{\omega}k_1\Lambda a \sin \gamma - \\
\left(\frac{1}{2}\alpha + 2\beta\omega \Lambda \right)\frac{a}{\omega} \cos \gamma.
\]

(30)

Seeking steady-state, we let \( a' = \gamma' = 0 \) and we eliminate \( \gamma \) dependence to get the frequency response equation as

\[
\left[\sigma - 2(3\lambda + k_2\Omega^2)\frac{A^2}{\omega} - \frac{1}{4}(3\lambda + k_2\omega^2)\frac{a^2}{\omega}\right]^2a^2 + (-1 + 6\Omega^2 \Lambda^2 + \\
\frac{3}{4}\omega^2 a^2)\mu^2 a^2 = \left[k_1^2 \Lambda^2 \frac{(\omega + \Omega)}{\omega^2} + \left(\frac{1}{2}\alpha + 2\beta\omega \Lambda \right)\right]a^2.
\]

(31)

For this equation we have the trivial solution \( a = 0 \) and another set of solutions which verify the following equation:

\[
\left[\sigma - 2(3\lambda + k_2\Omega^2)\frac{A^2}{\omega} - \frac{1}{4}(3\lambda + k_2\omega^2)\frac{a^2}{\omega}\right]^2 + \\
(-1 + 6\Omega^2 \Lambda^2 + \frac{3}{4}\omega^2 a^2)\mu^2 a^2 = \left[k_1^2 \Lambda^2 \frac{(\omega + \Omega)}{\omega^2} + \left(\frac{1}{2}\alpha + 2\beta\omega \Lambda \right)\right]^2(32)
\]

We obtain finally the non trivial solutions of the form

\[
a^2 = p_1 \pm (p_1^2 - q_1)^{\frac{1}{2}},
\]

(33)
where
\[
p_1 = \frac{4\omega(\lambda + k_2\omega^2)[\sigma - 2(3\lambda + 4k_2\omega^2)\frac{\Lambda^2}{\omega}] - 12\mu^2\omega^4(-1 + 24\omega^2\Lambda^2)}{9[\mu^2\omega^6 + 9(3\lambda + k_2\omega^2)^2]}.
\]
\[
q_1 = \frac{1}{9[\mu^2\omega^6 + 9(3\lambda + k_2\omega^2)^2]}[16\omega^2[\sigma - 2(3\lambda + 4k_2\omega^2)\frac{\Lambda^2}{\omega}]^2 + 16\mu^2\omega^2(-1 + 24\omega^2\Lambda^2)^2 - 16k_1^2\Lambda^2(\omega + \Omega)^2 - 8\alpha - 64\beta_1\omega^2\Lambda].
\] (35)

For non trivial solutions, it follows from eq. (33) that both the radical and the first term must be positive, i.e. the non trivial solutions for \(a\) are real only when \(p_1 > 0\) and \(p_1^2 \geq q_1\). Theses conditions imply that solutions will exist if
\[
4\omega(\lambda + k_2\omega^2)[\sigma - 2(3\lambda + 4k_2\omega^2)\frac{\Lambda^2}{\omega}] > 12\mu^2\omega^4(-1 + 24\omega^2\Lambda^2)
\] (36)

and
\[
\frac{[4\omega(\lambda + k_2\omega^2)[\sigma - 2(3\lambda + 4k_2\omega^2)\frac{\Lambda^2}{\omega}] - 12\mu^2\omega^4(-1 + 24\omega^2\Lambda^2)]^2}{9[\mu^2\omega^6 + 9(3\lambda + k_2\omega^2)^2]}
\] (37)

In Fig. 3 the frequency-response equation (33) is plotted. This curve shows that the subharmonic resonance in order-two apper and the maximum amplitude corresponding to resonance increases as the resonance frequency augment remaining in the field imposed by the conditions of occurrence of this resonance with the order.

![Graph showing subharmonic resonance](image)

Figure 3: subharmonic resonance in the space \((a, \Omega)\) for \(\mu = 0.5; \alpha = 3; \lambda = -1; \beta_1 = 0.1; k_2 = 0.5; k_1 = 0.1; F = 0.001; \epsilon = 0.01\).
2.2.2 $\Omega = 3\omega + \epsilon \sigma$

If we insert $\Omega = 3\omega + \epsilon \sigma$ in eq.(39), the solvability condition takes the form

$$-2i\omega A - i\mu A + 3i\mu^3 A^2 \bar{A} + 6i\mu \omega^2\Lambda A^2 - 3\lambda A^2 \bar{A} - 6\lambda \Lambda A^2 - k_2 \omega^2 A^2 \bar{A} - 2k_2 \Omega^2 \Lambda^2 + (-3i\mu \omega^2 \bar{A}^2 \Lambda - 3\lambda \Lambda^2 + k_2 \omega^2 A^2 \Lambda - 2k_2 \omega \Omega \bar{A}^2 \Lambda) e^{i\sigma T_1},$$

(38)

Reused the rule in the undo case, and let $\gamma = \sigma T_1 - 3\theta$, put $a' = \gamma' = 0$ and eliminate $\gamma$, the frequency-response equation is

$$\begin{align*}
[\sigma - 3(3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega^2} - \frac{3}{8}(3\lambda + k_2 \omega^2) \frac{a^2}{\omega^2}] &+ [(-\frac{3}{2} + 9\Omega^2) \mu + \frac{9}{8} \omega^2 a^2] \omega^2 a^2 = \\
\frac{9}{16}[9\mu^2 \omega^4 \Omega^2 - (3\lambda - k_2 \omega^2 + 2k_2 \omega \Omega)]^2 \Lambda^2 a^4 &+ \omega^2.
\end{align*}$$

(39)

The solutions of equation (39) are either $a = 0$ or

$$a^2 = p \pm (p^2 - q)^{\frac{1}{2}},$$

(40)

where

$$p = \frac{1}{9[\mu^2 \omega^6 + 9(3\lambda + k_2 \omega^2)^2]} \times$$

$$\begin{align*}
[-72\omega^4(-\frac{3}{2} + 9\Omega^2) \mu + 24\omega(3\lambda + k_2 \omega^2)[\sigma - 3(3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega}] + 18[9\mu^2 \omega^4 \Omega^2 - (3\lambda - k_2 \omega^2 + 2k_2 \omega \Omega)^2] \Lambda^2]
\end{align*}$$

(41)

$$q = \frac{64\omega^2(-\frac{3}{2} + 9\Omega^2) \mu^2 + 64\omega[\sigma - 3(3\lambda + k_2 \Omega^2 \Lambda^2)]^2}{9[\mu^2 \omega^6 + 9(3\lambda + k_2 \omega^2)^2]}.$$

(42)

Since $q$ is always positive, we need $p > 0$ and $p^2 \geq q$. This requires that

$$24\omega(3\lambda + k_2 \omega^2)[\sigma - 3(3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega}] + 18[9\mu^2 \omega^4 \Omega^2 - (3\lambda - k_2 \omega^2 + 2k_2 \omega \Omega)^2] \Lambda^2] > 72\omega^4(-\frac{3}{2} + 9\Omega^2) \mu,$$

(43)

and

$$\begin{align*}
[-72\omega^4(-\frac{3}{2} + 9\Omega^2) \mu + 24\omega(3\lambda + k_2 \omega^2)[\sigma - 3(3\lambda + k_2 \Omega^2) \frac{\Lambda^2}{\omega}] + 18[9\mu^2 \omega^4 \Omega^2 - (3\lambda - k_2 \omega^2 + 2k_2 \omega \Omega)^2] \Lambda^2] &+ 9[64\omega^2(-\frac{3}{2} + 9\Omega^2) \mu^2 + 64\omega[\sigma - 3(3\lambda + k_2 \Omega^2 \Lambda^2)]^2] \times [\mu^2 \omega^6 + 9(3\lambda + k_2 \omega^2)^2]
\end{align*}$$

(44)

We simulate eq.(40) (see Fig. 4) and we note the same comments as the case of the subharmonic of order-two but the resonance amplitudes for resonance are significant in this case i.e. the order-three.
2.3 primary resonance

In this state, we put that $F = \epsilon F$. The closeness between both internal and external frequencies is given by $\Omega = 1 + \epsilon \sigma$. In these conditions after some algebraic manipulations, we obtain:

$$D_0^2 x_1 + \omega^2 x_1 = \left[ -2i\omega A' - i\mu \omega A + 3i\mu \omega^3 A^2 \bar{A} - k_2 \omega^2 A^2 \bar{A} - 3\lambda A^2 A + \frac{1}{2} F \exp i\sigma T_1 \right] e^{i\omega T_0} + cc + NST,$$  \hspace{1cm} (45)

equating resonant terms at 0 from Eq. (45), we obtain:

$$-2i\omega A' - i\mu \omega A + 3i\mu \omega^3 A^2 \bar{A} - k_2 \omega^2 A^2 \bar{A} - 3\lambda A^2 A + \frac{1}{2} F \exp i\sigma T_1 = 0.$$ \hspace{1cm} (46)

After the same algebraic manipulations in other resonant states, the amplitude of oscillations of primary resonant states is governed by the following nonlinear algebraic equation:

$$\left( -\frac{1}{2} \mu \omega a + \frac{3}{8} \mu \omega^3 a^3 \right)^2 + \left( \omega \sigma a - \frac{1}{2} k_2 \omega^2 a^3 - \frac{3}{8} \lambda a^3 \right)^2 = \frac{F^2}{4}.$$ \hspace{1cm} (47)

3 Effects of parameters on resonant states

In this section, we study separately the effect of each parameter on each resonance of the oscillator i.e. the effect of different nonlinearity parameter and the amplitude of the exciting force on the different resonances which appear for this modified Rayleigh-Duffing oscillator.
3.1 Effects of parameters on superharmonic resonance

The first three figures respectively show the effects of the cubic nonlinear parameters for the order-three of the superharmonic resonance and the following six figures respectively show the effects of the $\alpha, \beta, k_1, \mu, \lambda$ and $k_2$ for the two levels of the superharmonic resonance. The superharmonic response of order 1/2 involves interaction between the parametric excitation and both the nonlinear parameter and the direct excitation. In fact, if the nonlinearity is not present, this resonance persists.

Figure 5 shows that for the first resonance, the parameter $\mu$ has no effect. For the two and third superharmonic resonance of order-two that appears, note that most $\mu$ parameter increases the higher the frequency of occurrence and the maximum value of the amplitude of the oscillator of this resonance decreases and the width of the curve resonance is less.

From Figure 6, we note that the same is the case of $\mu$ parameter except that here the sulfide and as $\lambda$ increases the resonance amplitude decreases, the frequency of occurrence of superharmonic resonance of order two the third time and increasing the resonance curve becomes broader.

Analysis curve of the figure 7, we find that $k_2$ and $\mu$ have virtually the same effect on the supharmonic resonance order-two.

Figure 8 shows that the amplitude of the exciting force does not have a great effect on the superharmonic resonance of order-two. Figures 9 and 10 show that the parameters $\beta$ and $k_1$ when do increase slightly increase the amplitude of the resonance at its second appearance without changing the frequency with which it entry appears.

Figures 13, 14 and 15 respectively illustrate the effects of parameters $\mu$, $\lambda$ and $k_2$ on superharmonic resonance of order-three. We note that these parameters have almost the same effects for the order of the resonance as in the case of the order-two.

Figures 11 and 12 show respectively the detuning parameter and the excitation force amplitude effects on superharmonic resonances order-two responses curves and figures 16 and 17 illustrate the effects of the parameters on superharmonic resonances order-three responses curves. Theses prove that these two parameters also affect severely the resonance response curve.
Figure 5: Effects of $\mu$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.

Figure 6: Effects of $\lambda$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.
Figure 7: Effects of $k_2$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.

Figure 8: Effects of $\alpha$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.
Figure 9: Effects of $\beta$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.

Figure 10: Effects of $k_1$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.
Figure 11: Effects of $\sigma$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.

Figure 12: Effects of $F$ on the frequency-response curves of the order-two superharmonic resonance with the parameters of figure 1.
Figure 13: Effects of $\mu$ on the frequency-response curves of the order-three superharmonic resonance with the parameters of figure 2.

Figure 14: Effects of $\lambda$ on the frequency-response curves of the order-three superharmonic resonance with the parameters of figure 2.
Figure 15: Effects of $k_2$ on the frequency-response curves of the order-three superharmonic resonance with the parameters of figure 2.

Figure 16: Effects of $\sigma$ on the frequency-response curves of the order-three superharmonic resonance with the parameters of figure 2.
3.2 Effects of parameters on subharmonic resonance

In this part, we are found the effects of the same parameter. For the subharmonic resonance of order-two $\mu$ (Figure 20) and $k_2$ (figure 19) have exactly the same effects as in the case of superharmonic resonance of the same order. $\lambda$ (Figure 18) in turn increases the value of the resonance amplitude of the oscillator whenever the two-order resonance appears by increasing its frequency of occurrence for the second time. More $\lambda$ is large, the resonance frequency is high, which makes the resonance disappear. In the case of subharmonic resonance for this oscillator, when the excitation amplitude (figure 21) and $k_1$ parameter (figure 22) increase the resonance amplitude increases to its first appearance in keeping its frequency. We notice that the beta parameter has no effect on the behavior of the oscillator in this resonance. In the case of subharmonic resonance of order-three (Figures 23, 24 and 25), the $k_2$ parameter has exactly the same effect as if the superharmonic resonance of the same order. $\lambda$ is the behavior of the system in exactly the same way as the case of the order-two of this resonance. The parameter $\mu$ is practically no effect on the resonance that order.
Figure 18: Effects of $\lambda$ on the frequency-response curves of the order-two subharmonic resonance with the parameters of figure 3.

Figure 19: Effects of $k_2$ on the frequency-response curves of the order-two subharmonic resonance with the parameters of figure 3.
Figure 20: Effects of $\mu$ on the frequency-response curves of the order-two subharmonic resonance with the parameters of figure 3.

Figure 21: Effects of $\alpha$ on the frequency-response curves of the order-two subharmonic resonance with the parameters of figure 3.
Figure 22: Effects of $k_1$ on the frequency-response curves of the order-two subharmonic resonance with the parameters of figure 3.

Figure 23: Effects of $\lambda$ on the frequency-response curves of the order-three subharmonic resonance with the parameters of figure 4.
Figure 24: Effects of $\mu$ on the frequency-response curves of the order-three sub-harmonic resonance with the parameters of figure 4.

Figure 25: Effects of $k_2$ on the frequency-response curves of the order-three sub-harmonic resonance with the parameters of figure 3.
3.3 Effects of parameters on primary resonant state

In this subsection, we found the effects of parameters $\lambda, \mu, k_2$ and $F$ on primary resonance. Figs. 26-29, show respectively the influence of $F, k_2, \lambda$ and $\mu$ on the frequency-response curves of the primary resonance in $(\sigma, a)$ space. From these figure we noticed that the jump and hysteresis phenomenon are appeared when the amplitude of external excitation and or cubic nonlinearities parameters $\lambda$ is increased. We noticed that when $F$ is increased, $a$ is increased highly but the variation of $\lambda$ or $k_2$ is not affected the peak value of amplitude of vibration in primary resonance. For the dissipation parameter $\mu$, when its is increased the peak value of $a$ is decreased.

![Figure 26: Effects of $F$ on the frequency-response curves of primary resonance with $\mu = 0.5, \lambda = 1, k_2 = 1$.](image-url)
Figure 27: Effects of $k_2$ on the frequency-response curves of primary resonance with the parameters of figure 26.

Figure 28: Effects of $k_2$ on the frequency-response curves of primary resonance with the parameters of figure 26.
Figure 29: Effects of $k_2$ on the frequency-response curves of primary resonance with the parameters of figure 26.

4 Discussion

We have shown the details of a first-order analysis of superharmonic and subharmonic resonances for the modified Rayleigh-Duffing oscillator. It appears that the curves show antiresonance and resonance peaks for some values of different parameters in presence. Through the Figures 1 and 4, it is easy to see that if we take a superharmonic resonance curve or subharmonic resonance curve the peak amplitude of resonance increases with the resonance frequency for fixed different parameters of system in the appropriate condition. We also see through the figures(5-10, 11,12) that the cubic nonlinear parameters, $\lambda, \mu, k_2$, and $\sigma$ and $F$ scales the peak response, while both the quadratic nonlinear parameter and the direct excitation level affect the frequency value of the peak response. The superharmonic response of order $\frac{1}{2}$ involves interaction between the parametric excitation and both the nonlinear parameter and the direct excitation. At order $\frac{1}{2}$ of this resonance, we also see through the figures 13-17 that the three cubic nonlinear and $\sigma$ and $F$ parameter have the same effects as the case of order $\frac{1}{2}$. For subharmonic resonance, We also see through the figures(18-25) that the cubic nonlinear parameters, $\mu, k_2$, scales the peak response, while both the quadratic nonlinear parameter and the direct excitation level affect the frequency value of the peak response respectively for two and first appearance of this resonance in order-two or order-three but $\lambda$ affect the the peak amplitude and the frequency which his corresponds when the subharmonic resonance appear. The subharmonic resonance may not be critical to modified Rayleigh-Duffing oscillator. The variation in system responses for changes in some parameters have been observed in simulations. In primary resonant state, the vibration amplitude can increasing when cubic nonlinearities parameters are increased. In thise case, the jump and hysteresis phenomenon can
be appeared. These provokes the several variation to the system. In general we note that effects due to cubic nonlinearities on the response curves have a significant from physical point of view. It is important to note that around the resonance peaks, the amplitudes and accumulate energies of the system device are higher than those received in any oscillations. In this case, this oscillator model can give more interesting applications in physical or engineering, particular when the model is used as a MEMS device, Selkov model, Brusselator etc., but the model with high energies is very dangerous since it can give rise to catastrophe damage. In the antiresonance peaks, these systems devices vibrates with small amplitude and accumulates energy. This phenomena is of particular interst when the model is used as an electromechanical vibration absorber (MEMS device consisting of a $30\mu m$ diameter silicon disk which can be made to vibrate by heating it with a laser beam resulting in a Hopf bifurcation for example). In other words, for the case of ENSO, resonance peaks correspond to a high-temperature water from the ocean. This is very serious because this would facilitate climate change especially regarding the poor living conditions of certain species of fish. This state of affairs could therefore lead to the disappearance or migration to another place these fish consequences of famine, poverty, fuck the maritime economy etc.. In the case of peaks of antiresonance, the phenomenon would be less catastrophic.
5 Chaotic vibration of system

To illustrate the chaotic vibration of the system dynamics in the resonances regions, simulations were performed for interesting values of the system parameters, using eq.(3) for the forced modified Rayleigh-Duffing oscillator. The simulations show that the model is highly sensitive to initial conditions, it can leave a quasi-periodic state for a chaotic state without changing the physical parameters. Figs. 30 show the chaotic behavior of the system in subharmonic and superharmonic case but not too close to the resonance region. For Figs. 31 and 32 represent the portraits phase of this system respectively exactly the superharmonic and subharmonic resonances. The analysis of Figures 26 shows that for given values of the parameters of the system except for the excitation frequency, the amplitude of the exciting force and the natural frequency of the system which have been selected from the slightly near resonance box, chaotic behavior is not the same. The system is less chaotic in the superharmonic case (see Fig. 30 (a), (b) and (c)) than in the subharmonic case (see Fig. 30 (d)). These figures also show the influence of the exciting force on the intensity of the chaotic behavior of the system. Figures (a) and (b) for the plotted values of frequencies and amplitudes of the exciting force at the resonance taken strictly superharmonic order-two and order-three, respectively show that the system is quasi-periodic but chaotic in the case of figures (a) and (b) obtained under the conditions corresponding to the subharmonic resonance of order-two and order-three respectively. We note that the quasi-periodicity is less enhanced in the case of order-two in the case of order-three for superharmonic resonance and the system is less chaotic in the case of order-two in the case of order-three for the subharmonic resonance. Fig. 33 show the phase portrait of the system at the superharmonic resonance for \( \omega = 2, \Omega = 1 \) (order-two) and \( \omega = 2, \Omega = 1 \) (order-three) respectively but the control parameter \( \epsilon = 0.0001 \). These Fig. 33 (a) and (b)) confirm that the system is quasi-periodic and is less enhanced in the case of order-two in the case of order-three for this resonance. Indeed by looking at their corresponding time series, one can observe a quasi-periodic and chaotic states as shown in Fig. 34-35. Finally, the system is less chaotic in the case of the superharmonic resonance in the case subharmonic resonance and in subharmonic and superharmonic case but not too close to the resonance region the system is the very chaotic in thes two cases. We notice that the chaotic behavior and the intensity of the chaotic behavior of forced modified Rayleigh-Duffing oscillator depends on the different parameters of the system which influence the different resonances region. Resonances, quasi-periodicity and chaotic is three phenomenon of this oscillator which depends between their.
Figure 30: Phase portraits of forced modified Rayleigh-Duffing oscillator for
\( \mu = 0.1; \alpha = 1; \lambda = 1; \beta = 0.01; k_2 = 0.1; k_1 = 0.1; \epsilon = 0.1; \Omega = 0.1 \).
(a) corresponding to \( \omega = 0.01 \) and \( F = 0.01 \), (b) \( \omega = 0.01 \) and \( F = 0.1 \), (c) \( \omega = 0.01 \) and \( F = 1 \), (d) \( \omega = 1 \) and \( F = 1 \).
Figure 31: Phase portraits of forced modified Rayleigh-Duffing oscillator for $\mu = 0.5; \alpha = 0.5; \lambda = 0.5; \beta = 0.1; k_2 = 0.5; k_1 = 0.1; \epsilon = 0.01; F = 0.01$. (a) corresponding to $\omega = 1$ and $\Omega = 0.5$ and (b) corresponding to $\omega = 1$ and $\Omega = 0.3125$.

Figure 32: Phase portraits of forced modified Rayleigh-Duffing oscillator for $\mu = 0.5; \alpha = 0.5; \lambda = 0.5; \beta = 0.1; k_2 = 0.5; k_1 = 0.1; \epsilon = 0.01; F = 0.01$. (a) corresponding to $\omega = 0.5$ and $\Omega = 1$ and (b) corresponding to $\omega = 0.5$ and $\Omega = 1.5$. 

30
Figure 33: Phase portraits of forced modified Rayleigh-Duffing oscillator for $\mu = 0.5; \alpha = 0.5; \lambda = 0.5; \beta = 0.1; k_2 = 0.5; k_1 = 0.1; \epsilon = 0.0001; F = 0.01$. (a) corresponding to $\omega = 2$ and $\Omega = 1$ and (b) corresponding to $\omega = 3$ and $\Omega = 1$.

Figure 34: Times series showing the quasi-periodic and chaotic states for the superharmonic and subharmonic with the parameters of figure 26.
Figure 35: Times series showing the chaotic states for the superharmonic resonance with the parameters of figure 27.

Figure 36: Times series showing the chaotic states for the subharmonic resonance with the parameters of figure 28.
Figure 37: Times series showing the chaotic states for the superharmonic resonance with the parameters of figure 29.

6 Conclusion

In this paper, superharmonic, subharmonic and primary resonant sates have been studied. Using the method of multiple scales, we obtained the primary resonance and the order-two and order-three for each type of other resonance. We found also in each case the maximum value of the amplitude of the oscillations for the system. We noted that in the case of two-order superharmonic or subharmonic resonance, this maximum value depends on all the parameters of the system but in the case of order three, only the coefficients of the terms cubic parameters affect the maximum amplitude of the resonance. It should be noted that from the simulation of different equations of the resonance curve, the amplitude of the response is higher in the case of all subharmonic resonance in the superharmonic. By fixing all the parameters of the system and varying only the amplitude of the parametric excitation above the critical value, the increasing amplitude of the parametric excitation provokes a rapid changes in the amplitude of the response to the resonances. We obtained the jump and hysteresis phenomenon in the system behaviors. The chaotic behavior have study at superharmonic and subharmonic resonances and also subharmonic and superharmonic case but not too close to the resonances regions. We found the different regions where the systems which modelled by the forced modified Rayleigh-Duffing oscillator (El Niño Southern Oscillation (ENSO) coupled tropical ocean-atmosphere weather phenomenon, MEMS device consisting of a $30\mu m$ diameter silicon disk which can be made to vibrate by heating it with a laser beam resulting in a Hopf bifurcation, the modified Selkov equations, modified abstract trimolecular chemical reaction etc.) are quasi-periodic and chaotic.
Acknowledgments

The authors thank IMSP-UAC for financial support. C.H. Miwadinou would like to thank Bernard T., Mathias H., Laurent H. and Audran K. for their help in completion of this work.

References

[1] Venkatanarayanan Ramakrishnan and Brian F Feeny, Resonances of a Forced Mathieu Equation with Reference to Wind Turbine Blades, JVA 2012

[2] Rajasekar S, Parthasarathy S, Lakshmanan M. Prediction of horseshoe chaos in BVP and DVP oscillators. Chaos, Solitons and Fractals 1992;2:271.

[3] Aubin, K., Zalalutdinov, M., Alan, T., Reichenbach, R. B., Rand, R. H., Zehnder, A., Parpia, J. and Craighead, H. G., ‘Limit Cycle Oscillations in CW Laser-Driven NEMS’, Journal of Microelectricalmechanical System 13 6:1018, 2004.

[4] Tina Marie Morrison, Three Problems in Nonlinear Dynamics With 2:1 Parametric Exitation, Ph.D. Cornell University 2006

[5] Bajaj, A. K., ‘Resonant Parametric Perturbations of the Hopf Bifurcation’, Journal of Mathematical Analysis and Applications 115:214-224, 1986.

[6] Ramakrishnan, V., and Feeny, B. F., 2011. “In-plane nonlinear dynamics of wind turbine blades”. In ASME International Design Engineering Technical Conferences, 23th Biennial Conference on Vibration and Noise, pp. no. DETC2011–48219, on CD– ROM.

[7] Batchelor, D. B, ‘Parametric Resonance of Systems with Time-Varying Dissipation’. Applied Physical Letters 29:280-281, 1976.

[8] Belhaq, M., and Houssni, M., 1999. “Quasi-periodic oscillations, chaos and suppression of chaos in a nonlinear oscillator driven by parametric and external excitations”. Nonlinear Dynamics, 18(1), June, pp. 1–24.

[9] Belhaq, M., Guennoun, K. and Houssni, M., ‘Asymptotic Solutions for a Damped Non-Linear Quasi-periodic Mathieu equation, International Journal of Non-linear Mechanics 37:445-460, 2002.

[10] Zhang W, Yu P. A study of the limit cycles associated with a generalized codimension-3 Lienard oscillator. J Sound Vib 2000;231:145.

[11] Nayfeh, A. H., and Mook D. T., 1979, Nonlinear Oscillations, John Wiley and Sons, New York.

[12] Den Hartog JP. Mechanical vibrations. New York: Dover Publishers; 1984.
[13] Hayashi, C., 1964, *Nonlinear Oscillations in Physical Systems*, McGraw-Hill, New York.

[14] Minorski N. *Mechanical vibrations*. Warsaw: WNT; 1967.

[15] Sanjuan MAF. *The effect of nonlinear damping on the universal escape oscillator*. Int J Bifurcat Chaos 1999;9:735.

[16] Warminski J, Balthazar JM. *Vibrations of a parametrically and self-excited system with ideal and non-ideal energy sources*. J Braz Soc Mech Sci Eng 2003;25:413.

[17] Warminski J. *Regular, chaotic and hyperchaotic vibrations of nonlinear systems with self, parametric and external excitations*. Mech Automat Control Robot 2003;14:891.

[18] M. Siewe Siewe, Hongjun Cao, Miguel A.F. Sanjuan, *On the occurrence of chaos in a parametrically driven extended Rayleigh oscillator with three-well potential*, Chaos, Solitons and Fractals 41 (2009) 772–782

[19] Chedjou, J. C., Fostin, H. B., and Woafo, P., 1997, "Behavior of the van der Pol Oscillator with Two External Periodic forces". Phys. Scr., 55, pp. 390-393. DEA de Physique des liquides. Paris VI- Ecole Polytechnique.

[20] Carrol, T. L., 1995, "Communicating With Use of filtered, Synchronized, Chaotic Signals." IEEE Trans, Circuits syst.,I: Fundam. Theory Appl.,42, pp. 105-110.

[21] K. Hackl, C.Y. Yang, A.H.-D. Cheng, *Stability, bifurcation and chaos of nonlinear structures with control—I. Autonomous case*, Int. J. Non-Linear Mech. 28 (1993) 441–454.

[22] A.H.D. Cheng, C.Y. Yang, K. Hackl, M.J. Chajes, *Stability, bifurcation and chaos of non-linear structures with control—II. Non autonomous case*, Int. J. Non-Linear Mech. 28 (1993) 549–565.

[23] Aubin, K., Zalalutdinov, M., Alan, T., Reichenbach, R. B., Rand, R. H., Zehnder, A., Parpia, J. and Craighead, H. G., *Limit Cycle Oscillations in CW Laser-Driven NEMS*, Journal of Micro-electrical mechanical System 13:1018-1026, 2004.

[24] Zalalutdinov, M., Olkhovets, A., Zehnder, A., Ilic, B., Czaplewski, D. and Craighead, H. G., *Optically pumped parametric amplification for micro-mechanical systems*, Applied Physics Letters 78:3142-3144, 2001.

[25] Rand, R. H., Ramani, D. V, Keith, W. L. and Cipolla, K. M., *The quadratically damped Mathieu equation and its application to submarine dynamics*, Control of Vibration and Noise: New Millennium 61:39-50, 2000.
[26] Wirkus, S., Rand, R. H. and Ruina, A., How to pump a swing, The College Mathematics Journal 29:266-275, 1998.

[27] Zhehe Y., Deqing M., Zichen C., Chatter suppression by parametric excitation: Model and experiments, Communications in Nonlinear Science and Numerical Simulation 330:2995-3005, 2011.

[28] Pandey, M., Rand, R., and Zehnder, A. T., 2007. “Frequency locking in a forced Mathieu-van der Pol-Duffing system”. Nonlinear Dynamics, 54(1-2), February, pp. 3–12.

[29] Month, L., and Rand, R., 1982. “Bifurcation of 4-1 subharmonics in the nonlinear Mathieu equation”. Mechanics Research Communication, 9(4), pp. 233–240.

[30] M. Siewe Siewe, C. Tchawoua, S. Rajasekar, Parametric Resonance in the Rayleigh–Duffing Oscillator with Time-Delayed Feedback.

[31] Darya V. Verveyko and Andrey Yu. Verisokin, Application of He’s method to the modified Rayleigh equation, Discrete and Continuous Dynamical Systems, Supplement 2011, pp. 1423–1431.

[32] Guckenheimer J., and Holmes P., Nonlinear oscillations, dynamical systems, and bifurcations of vector fields. New York: Springer-Verlag; 1983.

[33] J. C. Chedjou, L. K. Kana, L. Moussa, K. Kyamakya, A. Laurent, Dynamics of a quasiperiodically forced Rayleigh oscillator, Transactions of the ASME, 600/vol.128, September 2006.

[34] R. Tchoukuegno, B. R. Nana Nbendjo, P. Woaf, Linear feedback and parametric controls of vibrations and chaotic escape in a $\Phi^6$ potential, International Journal of Non-Linear Mechanics 38 (2003) 531–541.

[35] Nayfeh A.H., (1981), Introduction to Perturbation Technique, J.Wiley, New York.

[36] R. Yamapi, M. A. Aziz-Alaoui, Vibration analysis and bifurcations in the self-sustained electromechanical system with multiple functions, Communications in Nonlinear Science and Numerical Simulation 12 (2007) 1534-1549.

[37] H. G. Enjieu Kadji, J. B. Chabi Orou and P. Woaf, Regular and chaotic behaviors of plasma oscillations modeled by a modified Duffing equation, United Nations Educational Scientific and Cultural Organisation and International Atomic Energy Agency, ICTP IC/2005/041.

[38] H. G. Enjieu Kadji, B. R. Nana Nbendjo, J. B. Chabi Orou and P. K. Talla, Nonlinear dynamics of plasma oscillations modeled by an anharmonic oscillator, PACS numbers: 52.30.Ex, 56.65.Vv, 80.20.Wt
[39] Wang, B. and Fang, Z., ‘Chaotic Oscillations of Tropical Climate: A Dynamic System Theory for ENSO’, Journal of Atmospheric Sciences 53:2786-2802, 1996.

[40] Wang, B., Barcilon, A. and Fang, Z., ‘Stochastic Dynamics of El Niño- Southern Oscillation’, Journal of Atmospheric Sciences 56:5-23, 1999.

[41] Zalalutdinov, M., Parpia, J.M., Aubin, K.L., Craighead, H.G., T.Alan, Zehnder, A.T. and Rand, R.H., Hopf Bifurcation in a Disk-Shaped NEMS, Proceedings of the 2003 ASME Design Engineering Technical Conferences, 19th Biennial Conference on Mechanical Vibrations and Noise, Chicago, IL, Sept. 2-6, paper no.DETC2003-48516, 2003 (CD-ROM).

[42] Pandey, M., Rand, R. and Zehnder, A., ‘Perturbation Analysis of Entrainment in a Micromechanical Limit Cycle Oscillator’, Communications in Nonlinear Science and Numerical Simulation, available online, 2006.

[43] J.P. Céron et R. Washington L’ENSO El Niño et l’Oscillation Australe, Météo-France, Prévisions Saisonnière.