Development the Numerical Method to Solve the Inverse Initial Value Problem for the Thermal Conductivity Equation of Composite Materials

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Abstract. In this paper, the heat conduction equation for composite materials posed and solved. This problem is known as an inverse initial value problem for the heat conduction equation. In order to solve and formulate this inverse problem, the function spaces must be defined and represented. By studying and solving the direct problem for the heat equation in composite materials, it is possible to determine the function spaces and solve the inverse initial value problem. Scientific methods used: the separation of variables method used to solve the direct problem for the heat equation. It found that method separation of variables does not completely lead to the solution of the inverse initial value problem, since this method leads to a divergent series of solutions and rather massive errors. The heat conduction problem can be formulated as Fredholm integral first kind equations. The discretization algorithm applied to reformulated the problem as linear operator problem as matrix and vectors form. Then, Tikhonov’s regularization inversion method has been used to find an approximation solution. Finally, as shown in the numerical example the regularized approximate solutions obtained.

1. Introduction

Many applied problems formulated as inverse problems of mathematical physics belong to the class of ill-posed problems. The inverse initial value problem for the heat equation is defined as an ill-posed problem in the sense that a “small” arbitrary change in data can lead to “large” errors in the solution [1]. Many methods can solve the inverse problem under the study of the heat equation. For example, the method of regularization Tikhonov A. N. [2], the method of Lavrentiev M. M. [3, 4], and the method of quasi-solutions Ivanova V. K. [5, 6]. With the development of high-speed personal computers, it has become more convenient to use numerical methods to solve inverse problems. Theoretical concepts and computational implementation associated with the solution of inverse and ill-posed problems were investigated by AN Tikhonov, MM Lavrentiev, VK Ivanov, and their students and successive ones [7].

Many authors have discussed the theoretical concepts and computational implementation associated with the inverse problem of the heat equation for composite materials, and many methods have been described. In [8] the inverse problem of heat conduction was investigated using the Fourier series in eigenfunctions for an equation with a discontinuous coefficient. The Fourier transform was used, which made it possible to derive an inverse problem for the operator equation. Then the inverse problem of the operator equation was solved by the residual method. In [9], the author solved the problem of a moving boundary according to the Cauchy data in a one-dimensional heat equation with composite materials or a multilayer region. This problem is solved using a numerical regularization method based on the method
of fundamental solutions and the method of discrete Tikhonov regularization. An artificial neural network is used in [10] as an inverse tool for evaluating the thermal conductivity of a composite made of an aluminum core with an aluminum face sheet connected by an adhesive layer. The Picard method proposed to solve the inverse Cauchy problem to heat equation for composite materials [11] the Picard method applied in [12].

The main idea of this work is to reconstruct the source function of the heat equation by using the Tikhonov’s regularization method, and it implemented to solve the inverse initial Cauchy problem for the heat equation in composite materials.

2. Statement of the problem

The direct problem of finding the temperature in the rod that is created from composite materials is the problem of heat conduction at any moment by using the initial and boundary values of the temperature. The mathematical formulation of this problem is described [11] by a system of differential equations:

\[
\begin{align*}
\frac{\partial u_1(x,t)}{\partial t} &= a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2} ; x \in (0,x_0) ; t \in (0,T), a_1 > 0, \\
\frac{\partial u_2(x,t)}{\partial t} &= a_2^2 \frac{\partial^2 u_2(x,t)}{\partial x^2} ; x \in (x_0,1) ; t \in (0,T), a_2 > 0, \\
u_1(x,0) &= u_1(x); 0 \leq x \leq x_0, u_2(x,0) = u_2(x); x_0 \leq x \leq 1; \\
u_1(0) &= u_1(0) = u_2(1) = u_2'(1) = 0,
\end{align*}
\]

where \( u_1(x) \in (C[0,x_0] \cap C^2(0,x_0)); u_2(x) \in (C[x_0,1] \cap C^2(x_0,1)) \)

\[
\begin{align*}
\frac{\partial u_1(0,t)}{\partial x} &= 0; t \in [0,T], \\
u_2(1,t) &= 0; t \in [0,T], \\
u_1(x_0,t) &= u_2(x_0,t); t \in [0,T], \\
a_1 \frac{\partial u_1(x_0,t)}{\partial x} &= a_2 \frac{\partial u_2(x_0,t)}{\partial x}; t \in [0,T],
\end{align*}
\]

in the direct problem (1–8) we need to find the vector \( \bar{u}(x,t) \).

\[
\bar{u}(x,t) = \left[ u_1(x,t); x \in [0,x_0]; t \in [0,T] \right] \left[ u_2(x,t); x \in [x_0,1]; t \in [0,T] \right],
\]

where \( \bar{u}(x,t) \in C([0,1] \times [0,T]) \cap C^{2,1} \left( \left( [0,x_0] \cup (x_0,1) \right) \times (0,T) \right) \),

\[
\left. \bar{u}(x,t) \right|_{t \to 0} = \tilde{u}(x) 
\]

Theorem 1. For any function \( \tilde{u}(x) \) satisfying (4), \( \exists \) the unique solution of the direct problem satisfying (1–3), (5), (8, 9).

**Proof.** We will seek a formal solution to system (1–7) in the form of a series in eigenfunctions corresponding to the Sturm-Liouville problem \( \{ S_n(x) \} \)

\[
\tilde{u}(x,t) = \sum_{n=1}^{\infty} u_n e^{-\lambda_n t} S_n(x),
\]
where function $S_n(x)$ is defined by
\[ S_n(x) = \begin{cases} S_n^1 \left( \frac{x}{a_1} \right); & x \in (0,x_0), \\ S_n^2 \left( \frac{x}{a_2} \right); & x \in (x_0,1), \end{cases} \]

\[ S_n(x) = \beta_n \begin{cases} \cos \left( \frac{\lambda_n x}{a_1} \right) \sin \left( \frac{\lambda_n (1-x)}{a_2} \right); & x \in [0,x_0], \\ \sin \left( \frac{\lambda_n x}{a_2} \right) \cos \left( \frac{\lambda_n (1-x)}{a_1} \right); & x \in [x_0,1], \end{cases} \] (12)

\[ u_n = \frac{\alpha_n}{\beta_n} \begin{cases} u_n^1 = \frac{\lambda_n}{a_1} \int_0^{x_0} u(x) S_n^1 \left( \frac{x}{a_1} \right) d\left( \frac{x}{a_1} \right), \\ u_n^2 = \int_{x_0}^{1} u(x) S_n^2 \left( \frac{x}{a_2} \right) d\left( \frac{x}{a_2} \right). \end{cases} \] (13)

where $\alpha_n = \beta_n \cos \left( \frac{\lambda_n x_0}{a_2} \right)$

\[ \beta_n^2 = \frac{2a_1a_2}{a_2x_0 \sin^2 \left( \frac{\lambda_n (1-x_0)}{a_2} \right) + a_1 (1-x_0) \cos^2 \left( \frac{\lambda_n x_0}{a_1} \right)}, \] (14)

\[ \lambda_n = \frac{\pi a_2 (2n-1)}{2(a_2x_0 + a_1 (1-x_0))}, n= 1,2,3,..... \] (15)

From (9), (11) and (13) we can reduce problem as Fredholm integral first kind equation.

\[ \overline{u}(x,t) = \begin{cases} u_1(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} S_n^1(y) S_n^1(x) d\left( \frac{y}{a_1} \right), & y \in [0,x_0], \\ u_2(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} S_n^1(y) S_n^2(x) d\left( \frac{y}{a_2} \right), & y \in [x_0,1], \end{cases} \] (16)

can write (16) as following

\[ A\overline{u}(x) = \begin{cases} A_1 u_1(x) = \int_{x_0}^{x} P_1(y,x,t) u_1(y) d\left( \frac{y}{a_1} \right), & y \in [0,x_0], \\ A_2 u_2(x) = \int_{x_0}^{1} P_2(y,x,t) u_2(y) d\left( \frac{y}{a_2} \right), & y \in [x_0,1], \end{cases} = \overline{g}(x), \] (17)

where $P_1(y,x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} S_n^1(y) S_n^1(x)$, $P_2(y,x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} S_n^2(y) S_n^2(x)$

The function $\overline{u}(x)$ refers as initial temperature, see (3) and (10) $\overline{u}(x) = \begin{cases} u_1(x); & x \in [0,x_0]; \\ u_2(x); & x \in [x_0,1]; \end{cases}$

the kernels $P_1(y,x,t) \in C([0,x_0] \times [0,x_0])$, $P_2(y,x,t) \in C([x_0,1] \times [x_0,1])$, $\overline{u}(x) \in H^2_2[0,1]$ and $\overline{g}(x) \in L_2[0,1]$. The kernels of the integral operator $A$ in (17) is closed.
3. Statement of the inverse problem

In (1–8), consider the inverse initial value problem for the heat equation, that is, the temperature distribution at the time \( t = T > 0 \), which is given by the function \( \tilde{u}(x,T) \), and it is required to find the initial distribution \( \tilde{u}(x) = \left\{ \begin{array}{ll} u_i(x); & x \in [0,x_0]; \smallskip \\
 u_2(x); & x \in [x_0,1]; \end{array} \right. \); assuming that \( \tilde{u}(x) \in H^1_0[0,1] \). Suppose that we know a function \( \tilde{g}(y) \in L_2[0,1] \), which is a solution to the direct problem for \( t = T \),

\[
\tilde{u}(x,T) = \left\{ \begin{array}{ll}
 A_1 u_i(x) = \int_0^{x_0} P_i(y,x,T) u_i(x) d\left( \frac{x}{a_i} \right), & y \in [0,x_0]; \\
 A_2 u_2(x) = \int_{x_0}^1 P_2(y,x,T) u_2(x) d\left( \frac{x}{a_2} \right), & y \in [x_0,1]
\end{array} \right. = \tilde{g}(y), \quad (18)
\]

the exact value of the function \( \tilde{g}(y) = \tilde{g}_0(y) \) in (18) is unknown, and instead of it a pair \( \tilde{g}_\delta(y) \) and \( \delta \) are given, where \( \tilde{g}_\delta(y) \in L_2[0,1], \delta > 0 \),

\[
\| \tilde{g}_\delta(y) - \tilde{g}_0(y) \|_{L_2} \leq \delta. \quad (19)
\]

Required using the initial data of the problem \( \tilde{g}_\delta(y) \) and \( \delta \) to find the approximation solution \( u_\delta(x) \in L_2[0,1] \), the \( u_\delta(x) \) is converges to \( \tilde{u}(x) \) when the \( \delta \to 0 \), in \( L_2[0,1] \), as well as to obtain an estimate of the error \( \| u_\delta(x) - \tilde{u}(x) \|_{L_2} \).

4. Discretization of the integral equation of the first kind

The discretization algorithm described in [13] and implemented in [14]. For simplicity, we used the following steps to reformulate the first integral equation \( A_{1} u_i(x) = \int_0^{x_0} P_i(y,x,T) u_i(x) d\left( \frac{x}{a_i} \right), y \in [0,x_0] \) for problem (17) and we can apply same step on the second integral equation

\[
A_2 u_2(x) = \int_{x_0}^1 P_2(y,x,T) u_2(x) d\left( \frac{x}{a_2} \right), y \in [x_0,1].
\]

We considered the following integral equation for first kind without \( T \) because the is fixed number

\[
Au(x) = \int_a^b P(x,y) u(x) dx = g(y), a \leq y \leq b, \quad (20)
\]

where \( u(x) \in L_2[a,b], g(y) \in L_2[a,b] \), the function \( P(x,y) \) represent the kernel operator \( A \), where \( P(x,y), P'(x,y) \in C^{1,1}([a,b] \times [a,b]) \). We propose that for \( g(x) = g_0(x) \) there exist a true solution \( u_0(x) \) for problem (20) in the set \( M_r \)

\[
M_r = \left\{ u(x): u(x), u'(x) \in L_2[a,b], u(a) = u(b) = 0, \left\| u(x) \right\|_{L_2} \leq r^2 \right\}.
\]

The function \( g_0(x) \) is unknown instead of we have \( g_\delta \in L_2[a,b] \) and \( \delta > 0 \) such that \( \|g_\delta(y) - g_0(y)\|_{L_2} \leq \delta^2 \). In order to solve the problem (20) we need find the approximation solution \( u_\delta(x) \) by using the given information \( g_\delta(y), \delta \) and \( M_r \). Then we estimate the deviation of the approximation solution \( u_\delta(x) \) from the true solution \( u_0(x) \) in the metric of space \( L_2[a,b] \).

We need define an operator \( B: L_2[a,b] \to L_2[a,b] \) by the following formula

\[
u(x) = B\nu(x) = \int_0^1 \nu(\zeta) d\zeta, \nu(x), B\nu(x) \in L_2[a,b],
\]

(22)
There is an operator named \( C \) which can be defined by
\[
C_v(x) = ABv(x), \quad v(x) \in L_2[a,b], \quad C_f(x) \in L_2[a,b],
\]
from (23) and (24) we follow that
\[
C_f(x) = \int_a^b K(x, y) v(x) \, dy, \quad \text{where} \quad K(x, y) = -\int_a^b P(\xi, y) d\xi.
\]

The finite-dimensional operator \( C_{n,n} \) has been defined for computing the numerical solution for problem (20), the \( C \) replaced with the operator \( C_{n,n} \) and these operators satisfied the following relation where the value of the \( \eta_{n,n} \) is define by
\[
N(y) = \max_{x \in [a,b]} |P(x, y)|, \quad y \in [a,b],
\]
and
\[
N_i = \max \{ |K'_i(x, y)|; a \leq x \leq b, a \leq y \leq b \},
\]
the \( N(y) \in [a,b] \) and \( N_i \) exist because the \( P(x, y), P'_i(x, y) \in C([a,b] \times [a,b]) \).

By dividing the intervals \([a, b]\) into \( n \) equal parts where interval \([a, b]\) is divided by points \( x_i = a + \frac{i(b-a)}{n}, i = 0, 1, \ldots, n - 1 \). Now, we need to define the following functions
\[
\tilde{K}_i(y) = K(x, y),
\]
\[
K_n(x, y) = \tilde{K}_i(y); x_i \leq x \leq x_{i+1}, \quad y \in [a,b], i = 0, 1, \ldots, n - 1
\]
and
\[
K_{n,n}(x, y) = \tilde{K}_i(y); x_i \leq x \leq x_{i+1}, \quad y_j \leq y \leq y_{j+1}, i = 0, 1, \ldots, n - 1, j = 0, 1, \ldots, n - 1,
\]

By using the equations (26–28), we define the operators \( C_n \) and \( C_{n,n} \)
\[
C_n v(x) = \int_a^b K_n(x, y) v(x) \, dy, \quad y \in [a,b], \quad C_{n,n} v(x) = \int_a^b K_{n,n}(x, y) v(x) \, dy, \quad y \in [a,b],
\]

where \( C_n \) and \( C_{n,n} \) map \( L_2[a,b] \) into \( L_2[a,b] \).

Next step need to estimate the \( \|C_{n,n} - C\| \), by using the inequality relation
\[
\|C_{n,n} - C\| \leq \|C_{n,n} - C_n\| + \|C_n - C\|.
\]
Since
\[
\|K_{n,n}(x, y) - K_n(x, y)\| \leq \|\tilde{K}_i(y) - \tilde{K}_j(y)\|,
\]
for \( x_i \leq x \leq x_{i+1} \) and \( y_j \leq y \leq y_{j+1}, i = 0, 1, \ldots, n - 1, j = 0, 1, \ldots, n - 1 \), from
\[
\|\tilde{K}_i(y) - \tilde{K}_j(y)\| \leq N_i \frac{b-a}{m},
\]
finding from (30) that
\[
\|K_{n,n}(x, y) - K_n(x, y)\| \leq N_i \frac{b-a}{m};
\]
By using the equality \( \|C_{n,n} - C_n\| = \sup_{H_1} \|C_{n,n} v - C_n v\| \), we get
\[
\|C_{n,n} - C\|^2 = \sup_{H_1} \|\int_a^b [K_{n,n}(x, y) - K_n(x, y)] v(x) \, dx\|^2 dy.
\]
Driving from (31) and (32) the following
\[
\|C_{n,n} - C_n\|^2 \leq N_i^2 \left( \frac{b-a}{n} \right)^2 \int_a^b \|v(x)\|^2 \, dx \, dy.
\]
Since \( \int_a^b \|v(x)\|^2 \, dx \leq \sqrt{b-a} \|v(x)\|_L^2 \), inequality (33) implies that
\[ \|C_{n,n} - C_n\| \leq \sqrt{(b-a)(b-a)N_i} \frac{b-a}{n}. \]  

(34)

Now the term \( \|C_n - C\| \) can be estimating. Since \( C_n v(x) - C v(x) = \int_a^b (K(x,y) - K_n(x,y))v(x)dx \) and

\[ \|C_n - C\|^2 = \sup \left\{ \int_a^b \left| K_n(x,y) - K_n(x,y) \right| |v(x)|dx \right\}^2 \]  

Taking into account (24), (26) and (27) and the inequality

\[ \int_a^b |K(x,y) - K_n(x,y)| |v(x)|dx \leq \int_a^b |K(x,y) - K(x,y)| |v(x)|dx \leq \frac{b-a}{n} N(y) \int_a^b |v(x)|dx, \]

We find the following

\[ \|C_{n,n} v(x) - C_n v(x)\|^2 \leq \left( \frac{b-a}{n} \right)^2 \int_a^b \left[ |v(x)|dx \right]^2 dy. \]  

(35)

The \( |v(x)|\leq 1 \) and \( \frac{b-a}{n} \int_a^b |v(x)|dx \leq \frac{b-a}{n} \|v(x)\| \) with inequality (35) implies that

\[ \|C_n - C\| \leq \sqrt{(b-a)} \|N(y)\| \frac{b-a}{n}. \]  

(36)

Thus from (34) and (36)

\[ \eta_{n,n} = \sqrt{(b-a)(b-a)} N_i \frac{b-a}{n} + \sqrt{(b-a)} \|N(y)\| \frac{b-a}{n}. \]  

(37)

5. Finite-dimensional of the Tikhonov regularization method

This step described in [13]. We define subspace \( X_n \) of spaces \( L_2(a,b) \). Those subspaces consisting of all functions on intervals \([x_i, x_{i+1}], i = 0,1,\ldots,n-1\), and intervals \([y_j, y_{j+1}], j = 0,1,\ldots, n-1\), for space \( L_2(a,b) \). We denote by \( P_n \) the metric projection operator where \( P_n : L_2(a,b) \to X_n \).

The problem (20) reduces to linear operator problem first type.

\[ C_{n,n} v(x) = g_n^0(y), \]  

(38)

where \( g_n^0(y) = P_n g(y) \).

The approximation solution for the problem (20) can be obtained by using the generalized discrepancy method. The method reduces the problem (38) to the conditional extremum variational problem.

\[ \inf \left\{ \|v(x)\| : v(x) \in X_n, \|C_{n,n} v(x) - g_n^0(y)\| \leq \|v(x)\| \eta_{n,n} + \delta \right\}, \]  

(39)

under the condition

\[ \|g_n^0(y)\| = \|C_n v(x)\| \eta_{n,n} + \delta, \]  

(40)

The variational problem (38) has a unique solution \( v_{\delta,n_n}(x) \) such that

\[ \|C_{n,n} v_{\delta,n_n}(x) - g_n^0(y)\| = \|v_{\delta,n_n}(x)\| \eta_{n,n} + \delta. \]

The conditional problem (38) is reduced to the unconditional problem

\[ \inf \left\{ \|C_{n,n} v(x) - g_n^0(y)\| + \alpha \|v(x)\| : v(x) \in X_n \right\}, \alpha > 0, \]  

(41)

The (41) it is a finite-dimensional version of the Tikhonov regularization method [5].
There is a unique solution $v^{\alpha}_{\delta}(x)$ for problem (41). This solution should satisfy the general discrepancy principle [15].

$$\left\| C_{n,m}v^{\alpha}_{\delta}(x) - g^{\delta}_{\alpha}(y) \right\| = \left\| v^{\alpha}_{\delta}(x) \right\| \eta_{n,m} + \delta. \quad (42)$$

Under condition (40), the equation (42) has a unique solution $v^{\alpha}_{\delta}(x)$ with respect to regularization parameter $\alpha(n,n)$. That condition known in [16] and by theorem defined, there is $v_{\alpha,n}(x) = v^{\alpha(n,m)}_{\delta}(x)$ where the approximation solution is $u_{\alpha,n}(x) = Bv_{\alpha,n}(x)$.

For solving the problem (42), we get the equation

$$C_{n,m}C_{n,m}v(x) + \alpha v(x) = C_{n,m}^{T}g^{\delta}_{\alpha}(y). \quad (43)$$

In space $X_{n}$, the orthonormal bases have introduced $\{\varphi_{i}(x)\}$ by following

$$\varphi_{i}(x) = \begin{cases} \sqrt{n} \frac{1}{b-a} ; & x_{i} \leq x < x_{i+1} \\ 0 ; & x \not\in [x_{i}, x_{i+1}], i = 0,1,\ldots,n-1, \end{cases}$$

$$\varphi_{j}(y) = \begin{cases} \sqrt{n} \frac{1}{b-a} ; & y_{j} \leq y < y_{j+1} \\ 0 ; & x \not\in [y_{j}, y_{j+1}], j = 0,1,\ldots,n-1, \end{cases}$$

With these bases, we define the isometric operator $J_{s}$ where $J_{s} : R^{n} \rightarrow X_{n}$ following.

$$J_{s}[s(x)] = \sum_{i=0}^{n} s(\varphi_{i}(x), s = (s_{0}, s_{1},\ldots,s_{n-1}), \quad (44)$$

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$$\text{inf} \left\{ \left\| J_{s}^{-1}C_{n,m}v(x) - J_{s}^{-1}g^{\delta}_{\alpha}(y) \right\|^{2} + \alpha \left\| J_{s}^{-1}v(x) \right\|^{2} : J_{s}^{-1}[v(x)] \in X_{n}, \alpha > 0 \right\},$$

where

$$J_{s}^{-1}[g^{\delta}_{\alpha}(y)] = \sqrt{\frac{n}{b-a}} \int_{y_{i}}^{y_{i+1}} f(y)dy$$

$$J_{s}^{-1}[v(x)] = \sqrt{\frac{n}{b-a}} \int_{x_{i}}^{x_{i+1}} v(x)dx$$

We can rewrite the equation (44) in matrix and vector form as following

$$(\tilde{C}_{ij})^{T} \tilde{C}_{ij}v + \alpha v = (\tilde{C}_{ij})^{T} \tilde{g}_{ij}, i = 0,1,\ldots,n-1, j = 0,1,\ldots,n-1.$$

Where
\[ \bar{v}_i = J^{-1}_i[v(x)] = \left[ \frac{n}{\sqrt{b-a}} \int_a^b v(x)dx \right] \]

\[ \bar{g}_j = J^{-1}_j[g^a(y)] = \left[ \frac{n}{\sqrt{b-a}} \int_a^b g(y)dy \right] \]

\[ \bar{C}_{j,i} = \frac{b-a}{n} \sqrt{b-a} \left[ \begin{array}{cccc} \bar{K}_{i=0}(y_{j=0}) & \bar{K}_{i=1}(y_{j=1}) & \cdots & \bar{K}_{i=n-1}(y_{j=0}) \\
\bar{K}_{i=0}(y_{j=1}) & \bar{K}_{i=1}(y_{j=1}) & \cdots & \bar{K}_{i=n-1}(x_{j}) \\
\vdots & \vdots & \ddots & \vdots \\
\bar{K}_{i=0}(y_{j=n-1}) & \bar{K}_{i=1}(y_{j=n-1}) & \cdots & \bar{K}_{i=n-1}(y_{j=n-1}) \end{array} \right] \]

\[ (\bar{C}_{j,i})^T = \frac{b-a}{n} \sqrt{b-a} \left[ \begin{array}{cccc} \bar{K}_{i=0}(y_{j=0}) & \bar{K}_{i=0}(y_{j=1}) & \cdots & \bar{K}_{i=0}(y_{j=n-1}) \\
\bar{K}_{i=1}(y_{j=0}) & \bar{K}_{i=1}(y_{j=1}) & \cdots & \bar{K}_{i=1}(y_{j=n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\bar{K}_{i=n-1}(y_{j=0}) & \bar{K}_{i=n-1}(y_{j=1}) & \cdots & \bar{K}_{i=n-1}(y_{j=n-1}) \end{array} \right] \]

6. Numerical example
Considering the inverse initial value problem (1–8) for the heat equation in composed materials, we need to find the \( u(x, T) \) by using Tikhonov regularization method. The exact solution \( \tilde{u}(x) = \cos(\frac{\pi x}{2}) \) will be shown in ‘figure 1’, and the input function for inverse problem \( u(x, T) = g_0(x) \) where the \( x_0 \in (0,1) \), \( x_0 = 0.5 \). We will add some noise to \( g_0(x) \) to become \( g_\delta(x) \).
Figure 1. The exact solution for inverse problem $u_0(x)$.

We divided the intervals $[0,1]$ into $m = 100$ and the $T = 0.1$. We have applied our method with different regularization parameter value $\alpha$.

![Figure 2](image1.png)  
**Figure 2.** $u_\delta^\alpha(x)$, where $\delta = 0.02$.

From above figure 2 and figure 3, we obtain a good estimated solution compared with exact initial value $u_0(x)$.

**Figure 3.** $u_\delta^\alpha(x)$, where $\delta = 0.04$.

7. Conclusion

This work deals with the algorithm for solving the initial value problem for the heat equation in composed materials. This problem is an ill-posed problem, and a special method needs to reformulate the problem to linear operator problem. The separation of variables method used to represent the partial differential equation as an integral equation of the first kind. By using the discretization method, the integral equation converted to a system of linear equations or linear operator equation first kind. Tikhonov’s regularization method is a suitable method to obtain the approximation solution.

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