Research article

On more general inequalities for weighted generalized proportional Hadamard fractional integral operator with applications

Shuang-Shuang Zhou¹, Saima Rashid²*, Erhan Set³, Abdulaziz Garba Ahmad⁴ and Y. S. Hamed⁵

¹ School of Science, Hunan City University, Yiyang 413000, China
² Department of Mathematics, Government College University, Faisalabad, Pakistan
³ Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu, Turkey
⁴ Department of Mathematics Programme, National Mathematical Centre, Abuja, Nigeria
⁵ Department of Mathematics, Faculty of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia

* Correspondence: Email: saimarashid@gcuf.edu.pk.

Abstract: Fractional calculus has been the target of the work of many mathematicians for more than a century. Some of these investigations are of inequalities and fractional integral operators. In this article, a novel fractional operator which is known as weighted generalized proportional Hadamard fractional operator with unknown attribute weight is proposed. First, a fractional formulation is constructed, which covers a subjective list of operators. With the aid of the above mentioned operators, numerous notable versions of Pólya-Szegö, Chebyshev and certain related variants are established. Meanwhile, new outcomes are introduced and new theorems are exhibited. Taking into account the novel generalizations, our consequences have a potential association with the previous results. Furthermore, we demonstrate the applications of new operator with numerous integral inequalities by inducing assumptions on weight function \( \varpi \) and proportionality index \( \varphi \). It is hoped that this research demonstrates that the suggested technique is efficient, computationally, very user-friendly and accurate.

Keywords: weighted generalized proportional Hadamard fractional integrals; weighted Chebyshev inequality; Pólya-Szegö type inequality; Cauchy Schwartz inequality

Mathematics Subject Classification: Primary: 47H09, 47H10, 90C39, 45D05, 34A12; Secondary: 54H25
1. Introduction

In recent years, a useful extension has been proposed from the classical calculus by permitting derivatives and integrals of arbitrary orders is known as fractional calculus. It emerged from a celebrated logical conversation between Leibniz and L’Hopital in 1695 and was enhanced by different scientists like Laplace, Abel, Euler, Riemann, and Liouville [1]. Approaches based on the fractional calculus and fractional differential equations has been widely applied in diffusion equation, polymer physics, medical sciences, bioengineering mathematics, turbulence, fluid flow through porous media and in the model problems of nanoscale flow [2–12]. The concept of this new calculus was applied in several distinguished areas previously with excellent developments in the frame of novel approaches and posted scholarly papers, see [13–26].

Various notable generalized fractional integral operators such as the Riemann-Liouville, Hadamard, Caputo, Marichev-Saigo-Maeda, Riez, the Gaussian hypergeometric operators and so on, are helpful for researchers to recognize real world phenomena. Therefore, the Caputo, Riemann-Liouville and Hadamard were the most used fractional operators having singular kernels. It is remarkable that all the above mentioned operators are the particular cases of the operators investigated by Jarad et al. [27]. The utilities are currently working on weighted generalized fractional operators. Inspired by the consequences in the above mentioned papers, we introduce a new weighted framework of generalized proportional Hadamard fractional integral operator. Also, some new characteristics of the aforesaid operator are apprehended to explore new ideas, amplify the fractional operators and acquire fractional integral inequalities via generalized fractional operators (see Remark 2 below).

Recently, by employing the fractional integral operators, several researchers have established a bulk of fractional integral inequalities and their variant forms with fertile applications (see [28–34]). These sorts of speculations have remarkable use in fractional differential/difference equations and fractional Schrödinger equations [35].

Our intention is to establish a more general form for the most appealing and noteworthy Pólya-Szegő-Chebyshev type inequalities [36, 37] and certain related variants via weighted generalized proportional Hadamard fractional integral operator. Also, more appropriate than the existing ones.

In 1882, Chebyshev pondered the noted result [36]:

\[
P(\tilde{f}, \tilde{g}) := \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \tilde{f}(x)\tilde{g}(x)dx - \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \tilde{f}(x)dx \left( \frac{1}{\eta_2 - \eta_1} \int_{\eta_1}^{\eta_2} \tilde{g}(x)dx \right),
\]

for integrable functions \( \tilde{f} \) and \( \tilde{g} \) on \([\eta_1, \eta_2]\) and both the functions are simultaneously increasing or decreasing for the same values of \( x \) in \([\eta_1, \eta_2]\), that is,

\[
(\tilde{f}(x) - \tilde{f}(y))(\tilde{g}(x) - \tilde{g}(y)) \geq 0
\]

for any \( x, y \in [\eta_1, \eta_2] \).

Butt et al. [38], Rashid et al. [39] and Set et al. [40] established the fractional integral inequalities via generalized fractional integral operator having Raina’s function, generalized \( K \)-fractional integral and Katugampola fractional integral inequalities similar to the variant (1.1). For more recent literature, (see [41–51]).
The intensively studied Grüss inequality [52] for two integrable functions \( \tilde{f} \) and \( \tilde{g} \) on \([\eta_1, \eta_2]\) is presented as follows:

\[
P(\tilde{f}, \tilde{g}) \leq \frac{(Q - q)(S - s)}{r},
\]

where the integrable functions \( \tilde{f} \) and \( \tilde{g} \) satisfy \( q \leq \tilde{f} \leq Q \) and \( s \leq \tilde{g} \leq S \) for all \( x \in [\eta_1, \eta_2] \) and for some \( q, s, Q, S \in \mathbb{R} \).

The Pólya-Szegö type inequality [37] can be stated as follows:

\[
\frac{\int_{\eta_1}^{\eta_2} \tilde{f}^2(x)dx \int_{\eta_1}^{\eta_2} \tilde{g}^2(x)dx}{\left( \int_{\eta_1}^{\eta_2} \tilde{f}(x)\tilde{g}(x)dx \right)^2} \leq \frac{1}{4\left( \sqrt{QS} \right)^2 + \sqrt{qs}} \frac{\eta_2 - \eta_1}{4} \sqrt{qsQS}.
\]

The constant \( \frac{1}{4} \) is best feasible in (1.3) make the experience it cannot get replaced by a smaller constant. With the aid of the Pólya-Szegö inequality, Dragomir and Diamond [53] derived the inequality

\[
|P(\tilde{f}, \tilde{g})| \leq \frac{(Q - q)(S - s)}{4(\eta_2 - \eta_1)^2 \sqrt{qsQS}} \int_{\eta_1}^{\eta_2} \tilde{f}(x)dx \int_{\eta_1}^{\eta_2} \tilde{g}(x)dx
\]

holds for all \( x \in [\eta_1, \eta_2] \) if the mappings \( \tilde{f} \) and \( \tilde{g} \) defined on \([\eta_1, \eta_2]\) satisfies \( q \leq \tilde{f}(x) \leq Q \) and \( s \leq \tilde{g}(x) \leq S \). Here we should emphasize that, inequalities (1.1) and (1.3) are a remarkable instrument for reconnoitering plentiful scientific regions of investigation encompassing probability theory, statistical analysis, physics, meteorology, chaos and henceforth. Nisar et al. [54] proposed the weighted fractional integral inequalities of (1.1) and (1.3) within the weighted generalized fractional integral operator. Shen et al. [55] introduced the time scale version similar to (1.1) and (1.3), respectively. Ntouyas et al. [42] are the ones who contemplated the fractional version of (1.1) and (1.3) via Riemann-Liouville fractional integral operator. For more recent literature, we refer to the readers [56–63] and the references cited therein.

The motivation for this paper is twofold. First, we introduce a novel framework named weighted generalized proportional Hadamard fractional integral operator, then current operator employed to on the Pólya-Szegö-Chebyshev and certain related inequalities for exploring the analogous versions of (1.1) and (1.3). The study is enriched by giving remarkable cases of our results which are not computed yet. Interestingly, particular cases are designed for Hadamard fractional integral, generalized proportional Hadamard fractional integral and weighted Hadamard fractional integral inequalities. It is worth mentioning that these operators have the ability to recapture several generalizations in the literature by considering suitable assumptions of \( \varpi \) and \( \varphi \).

2. Prelude

This section demonstrates some essential preliminaries, definitions and fractional operators which will be utilized in this paper.
**Definition 2.1.** (\([27]\)) Let \(\sigma \neq 0\) be a mapping defined on \([\eta_1, \eta_2]\), \(\tilde{g}\) is a differentiable strictly increasing function on \([\eta_1, \eta_2]\). The space \(\chi_\sigma^p(\eta_1, \eta_2)\), \(1 \leq p < \infty\) is the space of all Lebesgue measurable functions \(\tilde{f}\) defined on \([\eta_1, \eta_2]\) for which \(\|\tilde{f}\|_{l_\sigma^p}\), where

\[
\|\tilde{f}\|_{l_\sigma^p} = \left( \int_{\eta_1}^{\eta_2} |\sigma(x)\tilde{f}(x)|^p \tilde{g}'(x) \, dx \right)^{\frac{1}{p}}, \quad 1 < p < \infty
\]  

and

\[
\|\tilde{f}\|_{l_\sigma^p} = \text{ess sup}_{x \in \eta_1, x \in \eta_2} \left| \sigma(x)\tilde{f}(x) \right| < \infty.
\]

**Remark 1.** Clearly we see that \(\tilde{f} \in \chi_\sigma^p(\eta_1, \eta_2) \implies \sigma(x)\tilde{f}(x)(\tilde{g}'(x))^{1/p} \in L_\nu(\eta_1, \eta_2)\) for \(1 \leq p < \infty\) and \(\tilde{f} \in \chi_\sigma^p(\eta_1, \eta_2) \implies \sigma(x)\tilde{f}(x) \in L_\nu(\eta_1, \eta_2)\).

Now, we show a novel fractional integral operator which is known as the weighted generalized proportional Hadamard fractional integral operator as follows.

**Definition 2.2.** (\([29]\)) Let \(\tilde{f} \in \chi_\sigma^p[1, \infty)\) and \(\sigma \neq 0\) be a function on \([1, \infty)\). Then the left and right-sided weighted generalized proportional Hadamard fractional integral operator of order \(\varphi > 0\) are described as:

\[
\mathcal{H}_{\chi_\sigma^p}^\varphi \tilde{f}(x) = \frac{1}{\varphi \Gamma(\varphi)} \int_{\eta_1}^{\eta_2} \frac{\exp\left[\frac{\sigma(x)}{\varphi}(\ln \frac{x}{\varphi})\right] \tilde{f}(\phi) \sigma(\phi)}{\phi} \, d\phi, \quad \eta_1 < x
\]  

and

\[
\mathcal{H}_{\chi_\sigma^p}^\varphi \tilde{f}(x) = \frac{1}{\varphi \Gamma(\varphi)} \int_{x}^{\eta_2} \frac{\exp\left[\frac{\sigma(x)}{\varphi}(\ln \frac{x}{\varphi})\right] \tilde{f}(\phi) \sigma(\phi)}{\phi} \, d\phi, \quad x < \eta_2,
\]

where \(\varphi \in (0, 1)\) is the proportionality index, \(\varphi \in C, \Re(\varphi) > 0\) and \(\Gamma(x) = \int_{0}^{\infty} \phi^{x-1} e^{-\phi} \, d\phi\) is the Gamma function.

**Remark 2.** Some particular fractional operators are the special cases of (2.5) and (2.6).

I. Setting \(\sigma(x) = 1\) in Definition 2.2, then we get the generalized proportional Hadamard fractional operator introduced by Rahman et al. \([62]\) stated as follows:

\[
\mathcal{H}_{\chi_\sigma^p}^\varphi \tilde{f}(x) = \frac{1}{\varphi \Gamma(\varphi)} \int_{\eta_1}^{\eta_2} \frac{\exp\left[\frac{\sigma(x)}{\varphi}(\ln \frac{x}{\varphi})\right] \tilde{f}(\phi)}{\phi} \, d\phi, \quad \eta_1 < x
\]  

and

\[
\mathcal{H}_{\chi_\sigma^p}^\varphi \tilde{f}(x) = \frac{1}{\varphi \Gamma(\varphi)} \int_{x}^{\eta_2} \frac{\exp\left[\frac{\sigma(x)}{\varphi}(\ln \frac{x}{\varphi})\right] \tilde{f}(\phi)}{\phi} \, d\phi, \quad x < \eta_2.
\]
II. Setting \( \varphi = 1 \) in Definition 2.2, then we get the weighted Hadamard fractional operators stated as follows:

\[
\mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_1}, \varphi \gamma_{\eta_1}} \tilde{f}(x) = \frac{\varphi^{-1}(x)}{\Gamma(\varphi)} \int_{\eta_1}^{x} \tilde{f}(\phi) \varphi(\phi) d\phi, \quad \eta_1 < x
\]  
(2.7) and

\[
\mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_2}, \varphi \gamma_{\eta_2}} \tilde{f}(x) = \frac{\varphi^{-1}(x)}{\Gamma(\varphi)} \int_{x}^{\eta_2} \tilde{f}(\phi) \varphi(\phi) d\phi, \quad \eta_2 < x.
\]  
(2.8)

II. Setting \( \varphi(\chi) = 1 \) and \( \varphi = 1 \) in Definition 2.2, then we get the Hadamard fractional operator proposed by Samko et al. [18] and Kilbas et al. [19], respectively, stated as follows:

\[
\mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_1}, \varphi \gamma_{\eta_1}} \tilde{f}(x) = \frac{1}{\Gamma(\varphi)} \int_{\eta_1}^{x} \tilde{f}(\phi) d\phi, \quad \eta_1 < x
\]  
(2.9) and

\[
\mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_2}, \varphi \gamma_{\eta_2}} \tilde{f}(x) = \frac{1}{\Gamma(\varphi)} \int_{x}^{\eta_2} \tilde{f}(\phi) d\phi, \quad \eta_2 < x.
\]  
(2.10)

Remark 3. (Semi-group property) For \( \varphi, \psi > 0, \varphi \in (0, 1] \) with \( 1 \leq p < \infty \) and let \( \tilde{f} \in \mathcal{H}^{p}_{\varphi}(\eta_1, \eta_2) \). Then

\[
\left( \mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_1}, \varphi \gamma_{\eta_1}} \mathcal{H}^{\psi}_{\varphi \gamma_{\eta_1}, \varphi \gamma_{\eta_1}} \right) \tilde{f} = \left( \mathcal{H}^{\varphi}_{\varphi \gamma_{\eta_1}, \varphi \gamma_{\eta_1}} \right) \tilde{f}.
\]  
(2.11)

This section consists of some novel Pólya-Szegö type inequalities regarding the generalized proportional Hadamard fractional operators which are also utilized to obtain Chebyshev type integral inequalities and related variants. Throughout the present investigation, for the consequences related to (1.1) and (1.3), it is assumed that all functions are integrable in the Riemann sense.

Theorem 2.3. Let two positive integrable functions \( \tilde{f} \) and \( \tilde{g} \) defined on \([\eta_1, \infty)\). Assume that there exist four integrable functions \( v_1, v_2, v_3 \) and \( v_4 \) defined on \([\eta_1, \infty)\) such that

\[
(A) \quad 0 < v_1(\ell) \leq \tilde{f}(\ell) \leq v_2(\ell) \quad \text{and} \quad 0 < v_3(\ell) \leq \tilde{g}(\ell) \leq v_4(\ell),
\]  
(2.12)

for all \( \ell \in [\eta_1, x] \) \((x > \eta_1), \eta_1 \in \mathbb{R}^+_0\). Then, the inequality

\[
\frac{\mathcal{H}^{\varphi \gamma_{\varphi \eta_1}}_{\varphi \gamma_{\varphi \eta_1}}[v_3 v_4 \tilde{f}^2](x) \mathcal{H}^{\varphi \gamma_{\varphi \eta_1}}_{\varphi \gamma_{\varphi \eta_1}}[v_1 v_2 \tilde{g}^2](x)}{\mathcal{H}^{\varphi \gamma_{\varphi \eta_1}}_{\varphi \gamma_{\varphi \eta_1}}[(v_1 v_3 + v_2 v_4) \tilde{f} \tilde{g}](x)^2} \leq \frac{1}{4}
\]  
(2.13)

holds for all \( \varphi \in (0, 1] \), \( \varphi \in \mathcal{C} \) and \( \Re(\varphi) > 0 \) with \( \varphi > 0 \).
Proof. By means of given hypothesis, we obtain

\[
\frac{\tilde{f}(\ell)}{\tilde{g}(\ell)} \leq \frac{v_2(\ell)}{v_3(\ell)} \quad \text{and} \quad \frac{u_1(\ell)}{u_4(\ell)} \leq \frac{\tilde{f}(\ell)}{\tilde{g}(\ell)} \quad (\ell \in [\eta_1, x] \ (x > \eta_1)).
\]  

(2.14)

Thus, we have

\[
\left( \frac{v_2(\ell)}{v_3(\ell)} + \frac{u_1(\ell)}{u_4(\ell)} \right) \tilde{f}(\ell) \tilde{g}(\ell) \geq \frac{\tilde{f}^2(\ell)}{\tilde{g}^2(\ell)} + \frac{v_1(\ell)u_2(\ell)}{v_3(\ell)u_4(\ell)}.
\]

(2.15)

which imply that

\[
(v_1(\ell)v_3(\ell) + v_2(\ell)v_4(\ell))\tilde{f}(\ell)\tilde{g}(\ell) \geq v_3(\ell)v_4(\ell)\tilde{f}^2(\ell) + v_1(\ell)v_2(\ell)\tilde{g}^2(\ell).
\]

(2.16)

Here, taking product each side of the above inequality by the following term \(\frac{1}{\varphi^\Gamma(\varphi)} \left( \frac{\exp[-\left(\frac{\ln \varphi}{\varphi}\right)]}{\ell} \right)\) \((\ell \in [\eta_1, x]) \ (x > \eta_1)\) and integrating the resulting inequality with respect to \(\ell\) on \([\eta_1, x]\), we have

\[
\frac{1}{\varphi^\Gamma(\varphi)} \int_{\eta_1}^{\infty} \frac{\exp[-\left(\frac{\ln \varphi}{\varphi}\right)]}{(\ln \varphi)^{1-\varphi}} \frac{\varphi(\ell)(v_1(\ell)v_3(\ell) + v_2(\ell)v_4(\ell))\tilde{f}(\ell)\tilde{g}(\ell)}{\ell} d\ell \geq \frac{1}{\varphi^\Gamma(\varphi)} \int_{\eta_1}^{\infty} \frac{\exp[-\left(\frac{\ln \varphi}{\varphi}\right)]}{(\ln \varphi)^{1-\varphi}} \frac{\varphi(\ell)v_3(\ell)v_4(\ell)\tilde{f}^2(\ell)}{\ell} d\ell + \frac{1}{\varphi^\Gamma(\varphi)} \int_{\eta_1}^{\infty} \frac{\exp[-\left(\frac{\ln \varphi}{\varphi}\right)]}{(\ln \varphi)^{1-\varphi}} \frac{\varphi(\ell)v_1(\ell)v_2(\ell)\tilde{g}^2(\ell)}{\ell} d\ell.
\]

(2.17)

Multiplying both sides of the above equation by \(\varphi^{-1}(x)\) and employing Definition 2.2, we have

\[
\mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{(v_1v_3 + v_2v_4)\tilde{f}\tilde{g}\}(x) \geq \mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{v_3v_4\tilde{f}^2\}(x) + \mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{v_1v_2\tilde{g}^2\}(x).
\]

(2.18)

Taking into account the arithmetic-geometric mean inequality, we have

\[
\mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{(v_1v_3 + v_2v_4)\tilde{f}\tilde{g}\}(x) \geq 2 \sqrt{\mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{v_3v_4\tilde{f}^2\}(x) \mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{v_1v_2\tilde{g}^2\}(x)}.
\]

(2.19)

which leads to the inequality (2.13). This completes the proof. \(\square\)

Corollary 1. Let two positive integrable functions \(\tilde{f}\) and \(\tilde{g}\) defined on \([\eta_1, \infty)\) such that

\[
0 < q \leq \tilde{f}(\ell) \leq Q \quad \text{and} \quad 0 < s \leq \tilde{g}(\ell) \leq S,
\]

(2.20)

for all \(\ell \in [\eta_1, x] \ (x > \eta_1), \eta_1 \in \mathbb{R}_0^+\) with \(\varphi > 0\). Then, the inequality holds:

\[
\frac{\mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{\tilde{f}^2\}(x) + \mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{\tilde{g}^2\}(x)}{(\mathcal{H}_\varphi \mathcal{Y}_{\eta_1}^\varphi \{\tilde{f}\}(x))^2} \leq \frac{1}{4} \left( \sqrt{\frac{sq}{QS}} + \sqrt{\frac{QS}{sq}} \right)^2.
\]

(2.21)

Remark 4. Under all assumptions of Theorem 2.3 and Corollary 1:

(1) If we take \(\varphi(x) = \varphi = 1\), then we get the result similar to Lemma 2.1 (by taking \(\kappa = 1\)) of [64].

(2) If we take \(\varphi(x) = \varphi = 1\), then we get the result similar to Corollary 1 (by taking \(\kappa = 1\)) of [64].
Theorem 2.4. Let two positive integrable functions $\tilde{f}$ and $\tilde{g}$ defined on $[\eta_1, \infty)$ such that the assumption (A) satisfying (2.12). Then, for all $\ell, \phi \in [\eta_1, x] (x > \eta_1)$, $\eta_1 \in \mathbb{R}^+_0$, the inequality

$$\frac{\mathcal{H}_\eta \mathcal{S}_\eta \{v_1 v_4\}}{\mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{f}^2\}} + \frac{\mathcal{H}_\eta \mathcal{S}_\eta \{v_2 v_3\}}{\mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}^2\}} \leq \frac{1}{4}$$

holds for all $\varphi \in (0, 1], \phi, \psi \in C$ and $\Re(\varphi), \Re(\psi) > 0$ with $\varpi > 0$.

Proof. By means of assumption (2.12), we have

$$\left(\frac{v_2(\ell)}{v_5(\phi)} - \frac{\tilde{f}(\ell)}{\tilde{g}(\phi)}\right) \geq 0 \quad \text{and} \quad \left(\frac{\tilde{f}(\ell)}{\tilde{g}(\phi)} - \frac{v_1(\ell)}{v_4(\phi)}\right) \geq 0 \quad (\ell, \phi \in [\eta_1, x] (x > \eta_1), \quad (2.23)$$

which imply that

$$\left(\frac{v_1(\ell)}{v_4(\phi)} + \frac{v_2(\ell)}{v_3(\phi)}\right) \tilde{f}(\ell) \tilde{g}(\phi) \geq \tilde{f}(\ell) \tilde{g}(\phi) + \frac{v_1(\ell)v_2(\ell)}{v_3(\phi)v_4(\phi)}.$$

Conducting product each side of the inequality (2.24) by $v_1(\phi)v_2(\phi)\tilde{g}^2(\phi)$, we get

$$v_1(\ell)\tilde{f}(\ell)v_3(\phi)\tilde{g}(\phi) + v_2(\ell)\tilde{f}(\ell)v_4(\phi)\tilde{g}(\phi) \geq v_3(\phi)v_4(\phi)\tilde{f}(\ell) + v_1(\ell)v_2(\ell)\tilde{g}^2(\phi). \quad (2.25)$$

Here, taking product each side of the above inequality by the following term

$$\frac{1}{\varphi^{1+\phi}} \frac{\Gamma(\varphi)\Gamma(\psi)}{(\ln \frac{\varphi}{\psi})^\psi (\ln \frac{\varphi}{\psi})^{1-\phi}} \ell \phi \quad (\ell, \phi \in [\eta_1, x], \ x > \eta_1)$$

and integrating the resulting inequality with respect to $\ell$ and $\phi$ on $[\eta_1, x]$. Then, multiplying both sides of the inequality by $\varpi^{-2}(x)$ and employing Definition 2.2, we obtain

$$\mathcal{H}_\eta \mathcal{S}_\eta \{v_1 \tilde{f}\} \mathcal{H}_\eta \mathcal{S}_\eta \{v_3 \tilde{g}\} + \mathcal{H}_\eta \mathcal{S}_\eta \{v_2 \tilde{f}\} \mathcal{H}_\eta \mathcal{S}_\eta \{v_4 \tilde{g}\} \geq \mathcal{H}_\eta \mathcal{S}_\eta \{v_3 v_4\} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{f}^2\} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}^2\} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}^2\}. \quad (2.26)$$

By employing the arithmetic-geometric mean inequality, we have

$$\mathcal{H}_\eta \mathcal{S}_\eta \{v_1 \tilde{f}\} \mathcal{H}_\eta \mathcal{S}_\eta \{v_3 \tilde{g}\} \mathcal{H}_\eta \mathcal{S}_\eta \{v_2 \tilde{f}\} \mathcal{H}_\eta \mathcal{S}_\eta \{v_4 \tilde{g}\} \geq 2 \sqrt{\mathcal{H}_\eta \mathcal{S}_\eta \{v_3 v_4\} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{f}^2\} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}^2\}} \mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}^2\}, \quad (2.27)$$

which leads to the desired inequality (2.22). Hence the proof is complete. \qed

Corollary 2. Let two positive integrable functions $\tilde{f}$ and $\tilde{g}$ defined on $[\eta_1, \infty)$ satisfying (2.20). Then, for all $\ell, \phi \in [\eta_1, x] (x > \eta_1)$, $\eta_1 \in \mathbb{R}^+_0$, the inequality

$$\frac{\mathcal{H}_\eta \mathcal{S}_\eta \{I\}}{\mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{f}\}} \left(\frac{\mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{f}\}}{\mathcal{H}_\eta \mathcal{S}_\eta \{\tilde{g}\}}\right)^2 \leq \frac{1}{4} \left(\sqrt{\frac{q_s}{\mathcal{Q}_S}} + \sqrt{\frac{\mathcal{Q}_S}{q_s}}\right)^2 \quad (2.28)$$

holds for all $\varphi \in (0, 1], \phi, \psi \in C, \Re(\varphi), \Re(\psi) > 0$ with $\varpi > 0$ and $I(x)$ is the identity mapping.
**Remark 5.** Under all assumptions of Theorem 2.4 and Corollary 2:

1. If we take \( \sigma(x) = \varphi = 1 \), then we get the result similar to Lemma 2.2 (by taking \( \kappa = 1 \)) of [64].
2. If we take \( \sigma(x) = \varphi = 1 \), then we get the result similar to Corollary 2 (by taking \( \kappa = 1 \)) of [64].

**Theorem 2.5.** Under the assumption of Theorem 2.4, then for all \( x > \eta_1 \) and \( \ell, \phi \in [\eta_1, x] \). Then, the inequality

\[
\mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ f^2 \}}(x) \leq \mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ \tilde{f}^2 \}}(x)
\]  

(2.29)

holds for all \( \varphi \in (0, 1], \psi, \psi \in C \) and \( \mathcal{R}(\varphi), \mathcal{R}(\psi) > 0 \) with \( \sigma > 0 \).

**Proof.** By means of assumption (2.12), we have

\[
\frac{1}{\varphi^\psi \Gamma(\psi)} \int_{\eta_1}^{x} \frac{\exp[\varphi^{-1} (\ln \psi)]}{(\ln \psi)^{-\psi}} \frac{\varphi(x)}{\ell} d\ell
\]

\[
\leq \frac{1}{\varphi^\psi \Gamma(\psi)} \int_{\eta_1}^{x} \frac{\exp[\varphi^{-1} (\ln \psi)]}{(\ln \psi)^{-\psi}} \frac{\varphi(x)}{\ell} d\ell.
\]

(2.30)

Multiplying both sides of the above equation by \( \sigma^{-1}(x) \) and employing of Definition 2.2, we have

\[
\mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ f^2 \}}(x) \leq \mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ \tilde{f}^2 \}}(x).
\]

(2.31)

By similar argument, we have

\[
\frac{1}{\varphi^\psi \Gamma(\psi)} \int_{\eta_1}^{x} \frac{\exp[\varphi^{-1} (\ln \psi)]}{(\ln \psi)^{-\psi}} \frac{\varphi(x)}{\ell} d\ell
\]

\[
\leq \frac{1}{\varphi^\psi \Gamma(\psi)} \int_{\eta_1}^{x} \frac{\exp[\varphi^{-1} (\ln \psi)]}{(\ln \psi)^{-\psi}} \frac{\varphi(x)}{\ell} d\ell.
\]

(2.32)

Multiplying both sides of the above equation by \( \sigma^{-1}(x) \) and employing of Definition 2.2, we have

\[
\mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ \tilde{f}^2 \}}(x) \leq \mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ \tilde{g}^2 \}}(x).
\]

(2.33)

Taking product of the inequalities (2.31) and (2.33) side by side, then we obtain the desired inequality (2.29).

\( \square \)

**Corollary 3.** Let two positive integrable functions \( \tilde{f} \) and \( \tilde{g} \) defined on \([\eta_1, \infty)\) satisfying (2.20). Then for all \( \ell, \phi \in [\eta_1, x] (x > \eta_1) \), \( \eta_1 \in \mathbb{R}^+_0 \), the inequality

\[
\frac{\mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ f^2 \}}(x)}{(\mathcal{H}_{\sigma}^{\gamma_{\psi}; \{ \tilde{f}^2 \}}(x))^2} \leq \frac{Qs}{qs}
\]

(2.34)

holds for all \( \varphi \in (0, 1], \psi, \psi \in C \) and \( \mathcal{R}(\varphi), \mathcal{R}(\psi) > 0 \) with \( \sigma > 0 \).
Remark 6. Under all assumptions of Theorem 2.5 and Corollary 3:
(1) If we take \( \varpi(x) = \varphi = 1 \), then we get the result similar to Lemma 2.3 (by taking \( \kappa = 1 \)) of [64].
(2) If we take \( \varpi(x) = \varphi = 1 \), then we get the result similar to Corollary 3 (by taking \( \kappa = 1 \)) of [64].

Our next result is the Chebyshev type integral inequality within the weighted generalized proportional Hadamard fractional integral operator defined in (2.3), with the aid of Pólya-Szego type inequality established in Theorem 2.3.

**Theorem 2.6.** Let two positive integrable functions \( f \) and \( g \) defined on \([\eta_1, \infty)\) such that the assumption (A) satisfying (2.12). Then, for all \( \ell, \phi \in [\eta_1, \infty) \), \( \eta_1 \in \mathbb{R}^+ \), the inequality

\[
\left| \frac{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(f)(x) + \mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(g)(x)}{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(f)(x) - \mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(g)(x)} \right| \leq \left| \Upsilon_1(f, u_1, u_2)(x) + \Upsilon_2(f, u_1, u_2)(x) \right|^2
\]

where

\[
\Upsilon_1(f, u_1, u_2)(x) := \frac{(\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(u_1 + u_2) f^2)(x)}{4 \mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(u_1 u_2)(x)} - \frac{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(f)(x)}{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(g)(x)}
\]

and

\[
\Upsilon_2(f, u_1, u_2)(x) := \frac{(\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(u_1 + u_2) f^2)(x)}{4 \mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(u_1 u_2)(x)} - \frac{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(f)(x)}{\mathcal{H}_{\varpi} \gamma_{\varphi, \psi} \mathcal{I}(g)(x)}
\]

holds for all \( \varphi \in (0, 1], \varphi, \psi \in C \) and \( \Re(\varphi), \Re(\psi) > 0 \) with \( \varpi > 0 \).

**Proof.** For \( \ell, \phi \in [\eta_1, \infty) \) with \( \ell > \eta_1 \), we define \( \Delta(\ell, \phi) \) as

\[
\Delta(\ell, \phi) := (\tilde{f}(\ell) - \tilde{f}(\phi))(\tilde{g}(\ell) - \tilde{g}(\phi)),
\]

or, equivalently,

\[
\Delta(\ell, \phi) = \tilde{f}(\ell)\tilde{g}(\ell) + \tilde{f}(\phi)\tilde{g}(\phi) - \tilde{f}(\ell)\tilde{g}(\phi) - \tilde{f}(\phi)\tilde{g}(\ell).
\]

Taking product each side of the above inequality by the following term

\[
\frac{1}{\varphi^{\psi+\Upsilon(\varphi, \psi)}} \exp\left[ \frac{\varphi - 1}{\varphi} \ln \left( \frac{\varphi}{\psi} \right) \right] \exp\left[ \frac{\varphi - 1}{\varphi} \ln \left( \frac{\varphi}{\psi} \right) \right] \varpi(\ell) \varpi(\phi) (\ell, \phi \in [\eta_1, \infty), \ell > \eta_1)
\]

and integrating the resulting inequality with respect to \( \ell \) and \( \phi \) on \([\eta_1, \infty)\), then we have

\[
\frac{1}{\varphi^{\psi+\Upsilon(\varphi, \psi)}} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \exp\left[ \frac{\varphi - 1}{\varphi} \ln \left( \frac{\varphi}{\psi} \right) \right] \exp\left[ \frac{\varphi - 1}{\varphi} \ln \left( \frac{\varphi}{\psi} \right) \right] \varpi(\ell) \varpi(\phi) (\ell, \phi \in [\eta_1, \infty), \ell > \eta_1)
\]
\[ \begin{align*}
&= \frac{1}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} f(\ell) g(\ell) d\ell d\phi \\
&+ \frac{1}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} f(\ell) g(\ell) d\ell d\phi \\
&- \frac{1}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} f(\ell) g(\ell) d\ell d\phi \\
&- \frac{1}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} f(\ell) g(\ell) d\ell d\phi. 
\end{align*} \]

Multiplying both sides of the above equation by \( \sigma^{-2}(x) \) and employing Definition 2.2, we have

\[ \begin{align*}
\frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} & \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \Delta(\ell, \phi) d\ell d\phi \\
&= \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ \tilde{f} \tilde{g} [\mathcal{M}] \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ I \} [\mathcal{M}] \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ I \} [\mathcal{M}] \\
&- \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ \tilde{f} \tilde{g} [\mathcal{M}] \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ I \} [\mathcal{M}] \\
&- \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ \tilde{f} \tilde{g} [\mathcal{M}] \mathcal{H}_{\varphi \varphi} \mathcal{S}_{\eta_1} \{ I \} [\mathcal{M}] \}.
\end{align*} \]

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals in (2.41), we can write that

\[ \begin{align*}
&\left[ \frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \Delta(\ell, \phi) d\ell d\phi \right]^{1/2} \\
&\leq \left[ \frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} f^2(\ell) d\ell d\phi \\
&+ \frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \tilde{f}^2(\ell) d\ell d\phi \\
&- \frac{2 \sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \tilde{f}(\ell) \tilde{f}(\ell) d\ell d\phi \\
&\times \left[ \frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \tilde{g}^2(\ell) d\ell d\phi \
&+ \frac{\sigma^{-2}(x)}{\varphi^{\nu} \Gamma(\varphi) \Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp[\frac{-1}{\nu}(\ln \xi)] \exp[\frac{-1}{\nu}(\ln \xi)]}{(\ln \xi)^{1-\nu} (\ln \xi)^{1-\psi}} \frac{\sigma(\ell) \sigma(\phi)}{\ell \phi} \tilde{g}^2(\ell) d\ell d\phi \right].
\end{align*} \]
In view of Definition 2.2, we get

\[
- \frac{2\sigma^{-2}(x)}{v^{\sigma^v}\Gamma(\sigma)\Gamma(\psi)} \int_{\eta_1}^{\infty} \int_{\eta_1}^{\infty} \frac{\exp\left[\frac{1}{x}(\ln \frac{\nu}{x})\right] \exp\left[\frac{1}{x}(\ln \frac{\psi}{x})\right]}{\sigma(\ell)\sigma(\phi) \ell} g(\ell)g(\phi)d\ell d\phi \right]^{\frac{1}{2}}. \tag{2.42}
\]

Applying Theorem 2.3 and setting \(\nu_3(x) = \nu_4(x) = \tilde{g}(x) = 1\), we find

\[
\mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}^2\}(x) \leq \frac{(\mathcal{H}_{\nu_3}^{\nu_4}\{(u_1 + u_2)^2\})}{4 \mathcal{H}_{\nu_3}^{\nu_4}\{u_1 u_2\}(x)}. \tag{2.44}
\]

This implies that

\[
\mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}^2\}(x) - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}(x)\} \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}\}(x)
\leq \frac{(\mathcal{H}_{\nu_3}^{\nu_4}\{(u_1 + u_2)^2\})}{4 \mathcal{H}_{\nu_3}^{\nu_4}\{u_1 u_2\}(x)} - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}(x)\} \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{f}\}(x)
= \Upsilon_1(\tilde{f}, u_1, u_2)(x) \tag{2.45}
\]

and

\[
\mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}^2\}(x) - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x) \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x)
\leq \frac{(\mathcal{H}_{\nu_3}^{\nu_4}\{(u_1 + u_2)^2\})}{4 \mathcal{H}_{\nu_3}^{\nu_4}\{u_1 u_2\}(x)} - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}(x)\} \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x)
= \Upsilon_2(\tilde{f}, u_1, u_2)(x). \tag{2.46}
\]

Analogously, setting \(u_1(x) = u_2(x) = \tilde{g}(x) = 1\), we find

\[
\mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}^2\}(x) - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x) \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x) \leq \Upsilon_1(\tilde{g}, u_3, u_4)(x) \tag{2.47}
\]

and

\[
\mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}^2\}(x) - \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x) \mathcal{H}_{\nu_3}^{\nu_4}\{\tilde{g}\}(x) \leq \Upsilon_2(\tilde{g}, u_3, u_4)(x). \tag{2.48}
\]

A combination of (2.42)–(2.48), we get the immediate consequence (2.35). This completes the proof of (2.35). \qed
Remark 7. If we take \( \varpi(x) = \varphi = 1 \) in Theorem 2.6, then we get the result similar to Theorem 1 (by taking \( \kappa = 1 \)) of [64].

The following lemma play a vital role for generating new outcomes by employing weighted generalized proportional Hadamard fractional integral operator

Lemma 2.7. (\cite{65}) Let \( \sigma \geq 0, \theta_1 \geq \theta_2 \geq 0 \) and \( \theta \neq 0 \). Then

\[
\sigma^{\theta_2 / \theta_1} \leq \left( \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} + \frac{\theta_1 - \theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} \right), \quad \text{for any } \kappa > 0.
\]

Theorem 2.8. For \( \theta_1 \geq \theta_2 \geq 0, \theta_1 \neq 0 \) and let an integrable function \( \tilde{f} \) defined on \([\eta_1, \infty)\). Moreover, assume that there exist two integrable functions \( v_1, v_2 \) defined on \([\eta_1, \infty)\) such that

\[
v_1(\ell) \leq \tilde{f}(\ell) \leq v_2(\ell), \quad \forall \eta_1 \in \mathbb{R}^+.
\] (2.49)

Then, the inequality

\[
(c_1) \quad H_{\varphi, \eta_1}^{\varphi, \eta_1}(v_2 - \tilde{f})_+^{\eta_1}(x) + \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(\tilde{f})(x)
\]

\[
\leq \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(v_2)(x) + \frac{\theta_1 - \theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(I)(x),
\]

\[
(c_2) \quad H_{\varphi, \eta_1}^{\varphi, \eta_1}(\tilde{f} - v_1)_+^{\eta_1}(x) + \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(v_2)(x)
\]

\[
\leq \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(\tilde{f})(x) + \frac{\theta_1 - \theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} H_{\varphi, \eta_1}^{\varphi, \eta_1}(I)(x)
\] (2.50)

holds for all \( \varphi \in (0, 1), \varphi \in \mathbb{C} \) and \( \Re(\varphi) > 0 \) with \( \varpi > 0 \).

Proof. By means of Lemma 2.7 and utilizing the assumption (2.49), for \( \theta_1 \geq \theta_2 \geq 0, \theta_1 \neq 0 \), for any \( \kappa > 0 \), it follows that

\[
(v_2(\ell) - \tilde{f}(\ell))_+^{\eta_1} \leq \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} (v_2(\ell) - \tilde{f}(\ell)) + \frac{\theta_1 - \theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1}.
\] (2.51)

Taking product each side of the above inequality by the following non-negative term

\[
\frac{1}{\varphi^\Gamma(\varphi)} \int_{\eta_1}^x \frac{\exp\left[\frac{1}{\varphi}(\ln \frac{x}{\ell})\right] \varphi(\varphi - \tilde{f}(\ell))_+^{\eta_1}}{(\ln \frac{x}{\ell})^{1-\varphi}} d\ell
\]

\[
\leq \frac{\theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} \int_{\eta_1}^x \frac{\exp\left[\frac{1}{\varphi}(\ln \frac{x}{\ell})\right] \varphi(\varphi - \tilde{f}(\ell))_+^{\eta_1}}{(\ln \frac{x}{\ell})^{1-\varphi}} d\ell + \frac{\theta_1 - \theta_2}{\theta_1} K_\varphi^{\theta_2 / \theta_1} \int_{\eta_1}^x \exp\left[\frac{1}{\varphi}(\ln \frac{\eta_1}{\ell})\right] \varphi(\varphi - \tilde{f}(\ell))_+^{\eta_1} d\ell,
\] (2.52)
Multiplying both sides of the above equation by $\varpi^{-1}(x)$ and employing of Definition 2.2, we have
\begin{align}
\mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(u_2 - \tilde{f})^i(\varpi - \tilde{g})^j\}(x) \\
\leq \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(u_2 - \tilde{f})^i\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{\tilde{f}\}(x) + \frac{\theta_1 - \theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{I\}(x),
\end{align}
which leads to inequality (c1). Inequality (c2) can be proved by similar argument. \hfill \square

**Theorem 2.9.** For $\theta_1 \geq \theta_2 \geq 0$, $\theta_1 \neq 0$ and under the assumptions of Theorem 2.3. Then, the inequalities
\begin{align}
(c_3) & \quad \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(u_2 - \tilde{f})^i(\varpi - \tilde{g})^j\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{u_2\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{\tilde{f}\}(x) + \frac{\theta_1 - \theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{I\}(x), \\
(c_4) & \quad \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(u_2 - \tilde{f})^i(\varpi - \tilde{g})^j\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{u_2\}(x) + \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{\tilde{f}\}(x) + \frac{\theta_1 - \theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{I\}(x), \\
(c_5) & \quad \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(\tilde{f} - u_1)^i(\varpi - \tilde{g})^j\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{u_2\}(x) + \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{\tilde{f}\}(x) + \frac{\theta_1 - \theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{I\}(x), \\
(c_6) & \quad \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{(\tilde{f} - u_1)^i(\varpi - \tilde{g})^j\}(x) + \frac{\theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{u_2\}(x) + \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{\tilde{f}\}(x) + \frac{\theta_1 - \theta_2}{\theta_1} \mathcal{H}_n^\varphi\mathcal{H}_n^\psi\{I\}(x),
\end{align}
holds for all $\varphi \in (0, 1]$, $\varphi, \psi \in \mathcal{C}$ and $\mathcal{R}(\varphi), \mathcal{R}(\psi) > 0$ with $\varpi > 0$.

**Proof.** The inequalities (c3) – (c6) can be deduced by utilizing Lemma 2.7 and the following assumptions:
\begin{align}
(c_3) & \quad \sigma = (u_2(l) - \tilde{f}(l))(\varpi(l) - \tilde{g}(l)), \\
(c_4) & \quad \sigma = (u_2(l) - \tilde{f}(l))(\varpi(\phi) - \tilde{g}(\phi)), \\
(c_5) & \quad \sigma = (\tilde{f}(l) - u_1(l))(\varpi(l) - \varpi(\phi)), \\
(c_6) & \quad \sigma = (\tilde{f}(l) - u_1(l))(\varpi(l) - \varpi(\phi)).
\end{align}
\hfill \square
**Remark 8.** Under all assumptions of Theorem 2.8 and Theorem 2.9:
(1) If we take $\varphi(x) = \varphi = 1$, then we get the result similar to Theorem 15 of [66].
(2) If we take $\varphi(x) = \varphi = 1$, then we get the result similar to Theorem 22 of [66].

### 3. Particular cases

In the sequel, we derive the certain novel estimates via weighted generalized proportional Hadamard fractional and generalized Hadamard fractional integral operator as follows.

**I.** Setting $\varphi = \psi$ and considering Theorem 2.6, then we get a new result for weighted generalized proportional Hadamard fractional integral operator.

**Corollary 4.** Let two positive integrable functions $\tilde{f}$ and $\tilde{g}$ defined on $[\eta_1, \infty)$ such that the assumption (A) satisfying (2.12). Then, for all $\ell \in [\eta_1, x] (x > \eta_1), \eta_1 \in \mathbb{R}^+_+$, the inequality

$$
\left| \mathcal{H}_{\frac{\varphi}{\varphi}} \mathcal{N}_{\eta_1} \mathcal{J} \mathcal{I}_{\varphi}(\tilde{f}, \tilde{g})(x) \right| \leq \left| \mathcal{Y}(\tilde{f}, \tilde{g}, \tilde{u}_1, \tilde{u}_2)(x) \right|^{\frac{1}{2}},
$$

where

$$
\mathcal{Y}(\tilde{f}, \tilde{u}_1, \tilde{u}_2)(x) := \left( \mathcal{H}_{\frac{\varphi}{\varphi}} \mathcal{N}_{\eta_1} \mathcal{J} \mathcal{I}_{\varphi}(\tilde{f})(x) \right)^2 - \left( \mathcal{H}_{\frac{\varphi}{\varphi}} \mathcal{N}_{\eta_1} \mathcal{J} \mathcal{I}_{\varphi}(x) \right)^2
$$

holds for all $\varphi \in (0, 1], \varphi \in C$ and $\Re(\varphi) > 0$ with $\varphi > 0$.

**II.** Considering the assertion (2.20) and Theorem 2.6, then we get a new result for weighted generalized proportional Hadamard fractional integral operator.

**Corollary 5.** Let two positive integrable functions $\tilde{f}$ and $\tilde{g}$ defined on $[\eta_1, \infty)$. Then, the inequality

$$
\left| \mathcal{H}_{\frac{\varphi}{\varphi}} \mathcal{N}_{\eta_1} \mathcal{J} \mathcal{I}_{\varphi}(\tilde{f}, \tilde{g})(x) \right| \leq \left( \frac{S - J}{4\sqrt{Q}} \right) \mathcal{H}_{\frac{\varphi}{\varphi}} \mathcal{N}_{\eta_1} \mathcal{J} \mathcal{I}_{\varphi}(\tilde{f})(x),
$$

holds for all $\varphi \in (0, 1], \varphi \in C$ and $\Re(\varphi) > 0$ with $\varphi > 0$.

**II.** Setting $\varphi = 1$, then we get a new result for generalized proportional Hadamard fractional integral operator.

**Corollary 6.** Suppose all assumptions of Theorem 2.8 be satisfied. Then, the inequality

$$
\left( \begin{array}{l}
(d_1) \\
(d_2)
\end{array} \right)
$$

holds for all $\varphi \in C$ and $\Re(\varphi) > 0$ with $\varphi > 0$. 

**AIMS Mathematics**

Volume 6, Issue 9, 9154–9176.
IV. Setting $\varphi = 1$, then we get a new result for generalized proportional Hadamard fractional integral operator.

**Corollary 7.** For $\theta_1 \geq \theta_2 \geq 0$, $\theta_1 \neq 0$ and under the assumptions of Theorem 2.3. Then, the inequalities

\[
\begin{align*}
& (c_3) \quad \mathcal{H}^{\varphi} |(x_2 - f)\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) \\
& \leq \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_1 - \theta_2}{\theta_1} k^{\theta_4} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x),
\end{align*}
\]

\[
\begin{align*}
& (c_4) \quad \mathcal{H}^{\varphi} |(x_2 - f)\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) \\
& \leq \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_1 - \theta_2}{\theta_1} k^{\theta_4} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x),
\end{align*}
\]

\[
\begin{align*}
& (c_5) \quad \mathcal{H}^{\varphi} |(x_2 - f)\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) \\
& \leq \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_1 - \theta_2}{\theta_1} k^{\theta_4} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x),
\end{align*}
\]

\[
\begin{align*}
& (c_6) \quad \mathcal{H}^{\varphi} |(x_2 - f)\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) \\
& \leq \frac{\theta_2}{\theta_1} k^{(0_2 - \theta_2)/\theta_1} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x) + \frac{\theta_1 - \theta_2}{\theta_1} k^{\theta_4} \mathcal{H}^{\varphi} |\mathcal{H}^{\psi}((x_4 - g)\mathcal{H}^{\varphi})|^{\frac{\theta_2}{\theta_1}}(x),
\end{align*}
\]

holds for all $\varphi \in (0, 1], \varphi, \psi \in C$ and $\mathcal{R}(\varphi), \mathcal{R}(\psi) > 0$ with $\varphi > 0$.

**Example 3.1.** Let $\eta_1 > 1, \varphi, \psi > 0, p_1, q_1 > 1$ having $p_1^{-1} + q_1^{-1} = 1$, and $\varpi \neq 0$ be a function defined on $[0, \infty)$. Let $f$ be an integrable function defined on $[1, \infty)$ and $\mathcal{H}^{\varphi} f$ be the weighted generalized proportional Hadamard fractional integral operator. Then we have

\[
\begin{align*}
\mathcal{H}^{\varphi} f(x) & \leq \Xi \left| (f \circ \varphi) \phi \right|_{L^1(1, x)}, \\
\Xi & = \frac{\varpi^{-1}(x)}{\varphi \Gamma(\varphi)} \left( (\varphi + p_1) - 2p_1 \varphi \right)^{1/p_1} \\
& \times \Gamma^{1/p_1} \left( (\varphi - 1) p_1 + 1, (p_1 + \varphi - 2p_1 \varphi) \ln x \right) \left| (f \circ \varphi) \phi \right|_{L^1(1, x)}
\end{align*}
\]

and $\Gamma^{1/p_1}(1, x) = \int_0^x e^{-y^p} dy$ is the incomplete gamma function [68].
Proof. In view of Definition 2.2 and applying modulus property that

\[ \left| \mathcal{H}^{\varphi \psi} f(x) \right| \leq \varphi^{-1}(x) \int_1^x \exp \left[ \frac{\varphi^{-1} \left( \ln \varphi \right)}{\varphi \Gamma(\varphi)} \right] \frac{f(\phi)}{\phi} d\phi, \]

for \( \phi > 1. \)

By the virtue of the noted Hölder inequality, we have

\[ \left| \mathcal{H}^{\varphi \psi} f(x) \right| \leq \varphi^{-1}(x) \left( \int_1^x \exp \left[ \frac{\varphi^{-1} \left( \ln \varphi \right)}{\varphi \Gamma(\varphi)} \right] \phi^{p_1} (\ln \varphi)^{p_1(1-\varphi)} d\phi \right)^{1/p_1} \left\| \tilde{f} \circ \sigma \varphi \right\|_{L_1(1,\alpha)}. \]

Substituting \( \nu = \ln \left( \frac{x}{\varphi} \right). \) Then elaborated computations represents

\[ \left| \mathcal{H}^{\varphi \psi} f(x) \right| \leq \varphi^{-1}(x) \left( \frac{\varphi x^{1-p_1}}{\varphi \Gamma(\varphi)} \right)^{1/p_1} \times (\varphi - 1)p_1 + 1, (p_1 + \varphi - 2p_1 \varphi \ln x) \left\| \tilde{f} \circ \sigma \varphi \right\|_{L_1(1,\alpha)}. \]

\[ \square \]

4. Applications

In the sequel we demonstrate a new methodology for establishing the four bounded mappings and employ them to show certain bounds of Chebyshev type weighted generalized proportional Hadamard fractional integral inequalities of two unknown mappings.

Consider a unit step function \( \chi \) be defined as

\[ \chi(x) = \begin{cases} 1, & x > 0, \\ 0, & x \geq 0. \end{cases} \]

and assuming a Heaviside unit step function \( \chi_{\eta}(x) \) defined by

\[ \chi_{\eta}(x) = \chi(x - \eta) = \begin{cases} 1, & x \geq \eta, \\ 0, & x < \eta. \end{cases} \]

The main characteristic of the unit step function are its frequent use in the differential equations and piece-wise continuous functions when sum of pieces defined by the series of functions. Assume that a piece-wise continuous function \( v_1(\ell) \) defined on \([\eta_1, T]\) can be presented a follows:

\[
\begin{align*}
v_1(x) &= h_1(s_{x_0}(x) - s_{x_1}(x)) + h_2(s_{x_1}(x) - s_{x_2}(x)) + h_3(s_{x_2}(x) - s_{x_3}(x)) + \ldots + h_{q+1}s_{x_q}(x) \\
&= h_1s_{x_0} + (h_2 - h_1)s_{x_1}(x) + (h_3 - h_2)s_{x_2}(x) + \ldots + (h_{q+1} - h_q)s_{x_q}(x)
\end{align*}
\]
The weighted generalized proportional Hadamard fractional integral of \( \tilde{\varphi} \) follows:

\[
\sum_{i=0}^{q} (h_{i+1} - h_i) s_x(x),
\]

(4.1)

where \( h_0, h_j \in \mathbb{R} (j = 0, 1, ..., q) \) and \( \eta_1 = x_0 < x_1 < x_2 < ... < x_i < x_q = T \). Analogously, we define the mappings \( \nu_2, \nu_3 \) and \( \nu_4 \) as follows

\[
\begin{align*}
\nu_2(x) &= \sum_{i=0}^{q} (H_{i+1} - H_i) s_x(x), \\
\nu_3(x) &= \sum_{i=0}^{q} (r_{i+1} - r_i) s_x(x), \\
\nu_4(x) &= \sum_{i=0}^{q} (R_{i+1} - R_i) s_x(x),
\end{align*}
\]

(4.2)

where \( r_0 = R_0 = H_0 = 0 \) and \( r_j, R_j, H_j \in \mathbb{R} (j = 0, 1, ..., q) \).

Suppose an integrable function \( \tilde{f} \) defined on \([\eta_1, T]\) satisfying assumption (2.12), (4.1) and (4.2), respectively, then we have \( h_{i+1} \leq \tilde{f}(x) \leq H_{i+1} \) for every \( x \in (x_i, x_{i+1}) \) \( (l = 0, 1, ..., q). \) Specifically, \( q = 4 \), the time theory of \( \tilde{f} \) presented in (4.1).

The weighted generalized proportional Hadamard fractional integral of \( \tilde{f} \) on \([\eta_1, T]\) can be described as follows:

\[
\mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f} \} (T) = \sum_{l=0}^{q} \mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f} \} (x),
\]

(4.3)

where

\[
\mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f} \} (x) := \frac{\varpi^{-1}(x) \int_{x_i}^{x} \exp \left[ \frac{1}{\varphi} \ln \left( \frac{3}{\varphi} \right) \right] \varphi (\phi) \tilde{f}(\phi) d\phi}{(\ln(3))^1} \quad (l = 0, 1, 2, ..., i)
\]

(4.4)

**Proposition 1.** Let two integrable functions \( \tilde{f} \) and \( \tilde{g} \) defined on \([\eta_1, T]\) satisfying the assumptions (2.12), (4.1) and (4.2), respectively. Then, the inequality

\[
\left( \sum_{l=0}^{q} R_{l+1} H_{l+1} \mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f}^2 \} (T) \right) \left( \sum_{l=0}^{q} h_{l+1} H_{l+1} \mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{g}^2 \} (T) \right)
\]

\[
\leq \frac{1}{4} \sum_{l=0}^{q} \left( R_{l+1} + H_{l+1} \right) \left( \mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f} \} (T) \right)^2
\]

(4.5)

holds for all \( \varphi \in (0, 1), \varphi \in C \) and \( \Re(\varphi) > 0 \) with \( \varpi > 0 \).

**Proof.** By employing Definition 2.2, we have

\[
\mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \nu_3 \nu_4 \tilde{f}^2 \} (T) = \sum_{l=0}^{q} R_{l+1} H_{l+1} \mathcal{H} \mathcal{L}^{\phi}_x \mathcal{L}^{\psi}_u \{ \tilde{f}^2 \} (T)
\]
\[
\mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \nu_1 \nu_2 g^2 \}(T) = \sum_{l=0}^{q} h_{i+1}^l H_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ g^2 \}_{\nu}(T) \quad (4.6)
\]

and

\[
\mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{(\nu_1 \nu_3 + \nu_2 \nu_4) \tilde{g}^2 \}(T) = \sum_{l=0}^{q} (r_{i+1}^l h_{i+1}^l + R_{i+1}^l H_{i+1}^l) \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{g}^2 \}(T). \quad (4.7)
\]

Using the fact of Lemma 2.3, inequalities (4.6) and (4.7), the desired inequality (4.5) is established.

**Proposition 2.** Let two positive integrable functions \( \tilde{f} \) and \( \tilde{g} \) defined on \([\eta_1, T]\) such that the assumption (A) satisfying (2.12). Then, the inequality

\[
\frac{\sum_{l=0}^{q} r_{i+1}^l H_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{f} \}^2 \}(T) + \sum_{l=0}^{q} h_{i+1}^l H_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{g}^2 \}(T)}{\left( \sum_{l=0}^{q} h_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{f} \}(T) + \sum_{l=0}^{q} r_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{g} \}(T) + \sum_{l=1}^{q} H_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{f} \}(T) + \sum_{l=1}^{q} R_{i+1}^l \mathcal{H}_\varphi \mathcal{J}_\eta^\psi \{ \tilde{g} \}(T) \right)^2} \leq \frac{1}{4} \quad (4.8)
\]

holds for all \( \varphi \in (0, 1], \varphi, \psi \in C \) and \( \Re(\varphi), \Re(\psi) > 0 \).

**Proof.** The proof is simple by following (4.1), (4.2) and Theorem 2.4.

\[\square\]

**Remark 9.** The accuracy of the approximated estimates (4.5) and (4.8) depends on the value of \( q \in \mathbb{N} \).

**5. Conclusions**

This paper proposes a new generalized fractional integral operator. The novel investigation is used to generate novel weighted fractional operators in the Hadamard and generalized proportional Hadamard fractional operator, which effectively alleviates the adverse effect of weight function \( \varpi \) and proportionality index \( \varphi \). Utilizing the weighted generalized proportional Hadamard fractional operator technique, we derived the analogous versions of the weighted Pólya-Szegő-Chebyshev and certain associated type inequalities that improve the accuracy and efficiency of the proposed technique. Contemplating the Remark 2, several existing results can be identified in the literature. It is important to note that our generalizations are refinements of the results obtained by [69]. Some innovative particular cases constructed by this method are tested and analyzed for statistical theory, fractional Schrödinger equation [35]. The results show that the method proposed in this paper can stably and efficiently generate integral inequalities for convexity with better operators’ performance, thus providing a reliable guarantee for its application in control theory [67].

**Acknowledgments**

The authors would like to express their sincere thanks to referees for improving the article and also thanks to Natural Science Foundation of China (Grant Nos. 61673169) for providing financial assistance to support this research. The authors would like to express their sincere thanks to the support.
of Taif University Researchers Supporting Project Number (TURSP-2020/155), Taif University, Taif, Saudi Arabia.

The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11601140) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 16B047).

Competing interests

The authors declare that they have no competing interests.

References

1. I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
2. R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
3. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Application of Fractional Differential Equations, North Holland Mathematics Studies, 204, 2006.
4. R. L. Magin, Fractional Calculus in Bioengineering, Begell House Publishers, 2006.
5. S. Patnaik, F. Semperlotti, A generalized fractional order elastodynamic theory for non local attenuating media, P. Roy. Soc. A, 476 (2020), 20200200.
6. S. Patnaik, S. Sidhardh, F. Semperlotti, Towards a unified approach to nonlocal elasticity via fractional order mechanics, Inter. J. Mechanical Scis, 189 (2021), 105992.
7. G. Alotta, M. D. Paola, G. F. Francesco, P. Pinnola, On the dynamics of non local fractional viscoelastic beams under stochastic agencies, Compos. Part B: Eng., 137 (2018), 102–110.
8. G. Alotta, O. Barrera, A. C. F. Cocks, M. D. Paola, On the behavior of a three dimensional fractional viscoelastic constitutive model, Meccanica, 52 (2017), 2127–2142.
9. G. Failla, M. Zingale, Advanced materials modelling via fractional calculus: Challenges and perspectives, 2020, 20200050.
10. S. Patnaik, J. P. Hollkamp, F. Semperlotti, Applications of variable order fractional operators: A review, P. Roy. Soc. A 476 (2020), 20190498.
11. S. Patnaik, F. Semperlotti, Variable order fracture mechanics and its application to dynamic fracture, J. Comput. Materials, 7 (2021), 18.
12. M. D. Paola, G. Alotta, A. Burlon, G. Failla, A novel approach to nonlinear variable order fractional viscoelasticity, Philos. T. Roy. Soc. A, 378 (2020), 20190296.
13. E. K. Akgül, A. Akgül, M. Yavuz, New illustrative applications of integral transforms to financial models with different fractional derivatives, Chaos, Solitons Fract., 146 (2021), 110877.
14. M. Yavuz, N. Sene, Fundamental calculus of the fractional derivative defined with Rabotnov exponential kernel and application to nonlinear dispersive wave model, J. Ocean. Engin. Sci., 6 (2021), 196–205.
15. M. Yavuz, European option pricing models described by fractional operators with classical and generalized Mittag-Leffler kernels, *Numer. Meth. Partial Diff. Equus*, (2021). doi:10.1002/num.22645.

16. M. Yavuz, T. A. Sulaiman, F. Usta, H. Bulut, Analysis and numerical computations of the fractional regularized long-wave equation with damping term, *Math. Meth. Appl. Sci.*, 44 (2021), 7538–7555.

17. M. Yavuz, T. A. Sulaiman, A. Yusuf, T. Abdeljawad, The Schrödinger-KdV equation of fractional order with Mittag-Leffler nonsingular kernel, *Alexandria Eng. J.*, 60 (2021), 2715–2724.

18. S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, *Gordon and Breach, Yverdon*, 1993.

19. A. A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, 38 (2001), 1191–1204.

20. F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivative, *Adv. Diff. Equ.*, 2012 (2012), Article ID: 142.

21. U. N. Katugampola, New approach to generalized fractional integral, *Appl. Math. Comput.*, 218 (2014), 860–865.

22. F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Special Topics*, 226 (2017), 3457–3471.

23. O. M. Agrawal, Generalized multi-parameters fractional variational calculus, *Int. J. Differ. Equ.*, 2012 (2012), doi:10.1155/2012/521750.

24. O. M. Agrawal, Some generalized fractional calculus operators and their applications in integral equations, *Frac. Cal. Appl. Anal.*, 15 (2012), 700–711.

25. M. Al-Refai, A. M. Jarrah, Fundamental results on weighted Caputo-Fabrizio fractional derivative, *Chaos Soliton Fract.*, 126 (2019), 7–11.

26. M. Al-Refai, On weighted Atangana-Baleanu fractional operators, *Adv. Differ. Equ.*, 2020 (2020), Article ID: 3.

27. F. Jarad, T. Abdeljawad, K. Shah, On the weighted fractional operators of a function with respect to another function, *Fractals*, (2020), doi: 10.1142/S0218348X20400113.

28. Y. M. Chu, S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, More new results on integral inequalities for generalized K-fractional conformable integral operators, *Discrete, Cont. Dyn. Syss. Series S*, (2021), DOI: 10.3934/dcdss.2021063.

29. S. S. Zhou, S. Rashid, A. Rauf, F. Jarad, Y. S. Hamed, K. M. Abualnaja, Efficient computations for weighted generalized proportional fractional operators with respect to a monotone function, *AIMS Math.*, 6 (2021), 8001–8029.

30. S. Rashid, S. Sultana, F. Jarad, H. Jafari, Y. S. Hamed, More efficient estimates via h-discrete fractional calculus theory and applications, *Chaos Solitons Fract.*, 147 (2021), 110981.

31. H. G. Jile, S. Rashid, F. B. Farooq, S. Sultana, Some inequalities for a new class of convex functions with applications via local fractional integral, *J. Func. Spaces*, 2021 (2021).

32. S. Rashid, S. Parveen, H. Ahmad, Y. M. Chu, New quantum integral inequalities for some new classes of generalized ψ-convex functions and their scope in physical systems, *Open Physics*, 19 (2021), DOI: 10.1515/phys-2021-0001.
33. A. A. El-Deeb, S. Rashid, On some new double dynamic inequalities associated with Leibniz integral rule on time scales, *Adv. Differ. Equas.*, 2021 (2021), DOI: 10.1186/s13662-021-03282-3

34. S. S. Zhou, S. Rashid, S. Parveen, A. O. Akdemir, Z. Hammouch, New computations for extended weighted functionals within the Hilfer generalized proportional fractional integral operator, *AIMS Math.*, 6 (2021). DOI: 10.3934/math.2021267.

35. Y. Zhang, X. Liu, M. R. Belic, W. Zhong, Y. P. Zhang, M. Xiao, Propagation dynamics of a light beam in a fractional Schrödinger equation, *Phys. Rev. Lett.*, 115 (2015), 180403.

36. P. L. Čebyšev, Sur les expressions approximatives des intégrales par les auters prises entre les mêmes limites, *Proc. Math. Soc. Charkov*, 2 (1882), 93–98.

37. G. Pólya, Szegö, Aufgaben und Lehrsätze aus der Analysis i, *Springer*, New York, 1964.

38. S. I. Butt, A. O. Akdemir, M. Y. Bhatti, M. Nadeem, New refinements of Chebyshev-Pólya-Szegö-type inequalities via generalized fractional integral operators, *J. Inequal. Appl.*, 2020 (2020), Article ID: 157.

39. S. Rashid, F. Jarad, H. Kalsoom, Y. M. Chu, On Pólya-Szegö and Čebysev type inequalities via generalized k-fractional integrals, *Adv. Differ. Equas*, 2020 (2020), Article number: 125.

40. E. Set, Z. Dahmani, İ Mumcu, New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Pólya-Szegö inequality, *An Inter. J. Optim. Cont. Theories Appl.*, 8 (2018), 137–144. doi.org/10.11121/ijocta.01.2018.00541.

41. E. Deniz, A. O. Akdemir, E. Yüksel, New extensions of Chebyshev-Pólya-Szegö type inequalities via conformable integrals, *AIMS Math.*, 5 (2020), 956–965.

42. S. Ntouyas, P. Agarwal, J. Tariboon, On Pólya-Szegö and Čebyšev types inequalities involving the Riemann-Liouville fractional integral operators, *J. Math. Inequal.*, 10 (2016), 491–504.

43. S. B. Chen, S. Rashid, M. A. Noor, R. Ashraf, Y. M. Chu, A new approach on fractional calculus and probability density function, *AIMS Math.*, 5 (2020), 7041–7054.

44. M. Al-Qurashi, S. Rashid, S. Sultana, H. Ahmad, K. A. Gepreeifl, New formulation for discrete dynamical type inequalities via $h$-discrete fractional operator pertaining to nonsingular kernel, *Math. Bioscis. Eng.*, 18 (2021), 1794–1812. DOI: 10.3934/mbe.2021093.

45. Y. M. Chu, S. Rashid, J. Singh, A novel comprehensive analysis on generalized harmonically $\Psi$-convex with respect to Raina’s function on fractal set with applications *Math. Meth. Appl. Scis.*, (2021), DOI: 10.1002/mma.7346.

46. S. Rashid, F. Jarad, Z. Hammouch, Some new bounds analogous to generalized proportional fractional integral operator with respect to another function, *Discrete. Conti. Dyn. Syss-Series S*, (2021).

47. S. Rashid, S. I. Butt, S. Kanwal, H. Ahmad, M. K. Wang, Quantum integral inequalities with respect to Raina’s function via coordinated generalized $\psi$-convex functions with applications, *J. Fun. Spaces*, 2021 (2021). DOI: 10.1155/2021/6631474.

48. S. Rashid, Y. M. Chu, J. Singh, D. Kumar, A unifying computational framework for novel estimates involving discrete fractional calculus approaches, *Alexandria Eng. J.*, 60 (2021), DOI: 10.1016/j.aej.2021.01.003.
49. M. Al Qurashi, S. Rashid, Y. Karaca, Z. Hammouch, D. Baleanu, Y. M. Chu, Achieving more precise bounds based on double and triple integral as proposed by generalized proportional fractional operators in the Hilfer sense, *Fractals*, (2021). DOI: 10.1142/S0218348X21400272.

50. M. K. Wang, S. Rashid, Y. Karaca, Z. Hammouch, D. Baleanu, Y. M. Chu, New multi-functional approach for kth-order differentiability governed by fractional calculus via approximately generalized $(\psi, h)$-convex functions in Hilbert space, *Fractals*, (2021). DOI: 10.1142/S0218348X21400193.

51. M. Al Qurashi, S. Rashid, A. Khalid, Y. Karaca, Y. M. Chu, New computations of Ostrowski type inequality pertaining to fractal style with applications, *Fractals*, (2021), DOI: 10.1142/S0218348X21400260.

52. G. Grüss, Über das Maximum des absoluten Betrages von \( \frac{1}{\eta_2-\eta_1} \int_{\eta_1}^{\eta_2} f(x)g(x)dx \leq \left( \frac{1}{\eta_2-\eta_1} \right)^2 \int_{\eta_1}^{\eta_2} f(x)dx \int_{\eta_1}^{\eta_2} g(x)dx, \) *Math. Z.*, 39 (1935), 215–226.

53. S. S. Dragomir, N. T. Diamond, Integral inequalities of Grüss type via Polya-Szego and Shisha-Mond results, *East Asian Math. J.*, 19 (2003), 27–39.

54. K. S. Nisar, G. Rahman, D. Baleanu, M. Samraiz, S. Iqbal, On the weighted fractional Pólya-Szegö and Chebyshev-type integral inequalities concerning another function, *Adv. Differ. Eqs.*, 2020 (2020), 623.

55. J. M. Shen, S. Rashid, M. A. Noor, R. Ashraf, Y. M. Chu, Certain novel estimates within fractional calculus theory on time scales. *AIMS Math.*, 5 (2020), 6073–6086, doi: 10.3934/math.2020390.

56. S. Rashid, T. Abdeljawad, F. Jarad, M. A. Noor, Some estimates for generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications, *Math*, 7 (2019), 807.

57. S. B. Chen, S. Rashid, M. A. Noor, Z. Hammouch, Y. M. Chu, New fractional approaches for n-polynomial p-convexity with applications in special function theory, *Adv. Differ. Eqs.*, 2020 (2020), Article ID: 543.

58. T. Abdeljawad, S. Rashid, Z. Hammouch, İ. İşcan, Y. M. Chu, Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications, *Adv. Differ. Eqs.*, 2020 (2020), Article ID: 496.

59. F. Jarad, T. Abdeljawad, S. Rashid, Z. Hammouch, More properties of the proportional fractional integrals and derivatives of a function with respect to another function, *Adv. Differ. Eqs.*, 2020 (2020), Article ID: 303.

60. S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom, Y. M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics*, 7 (2020), 1225.

61. F. Jarad, T. Abdeljawad, J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.*, 226 (2017), 3457–3471.

62. G. Rahman, T. Abdeljawad, F. Jarad, A. Khan, K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Diff. Eqs.*, 2019 (2019), Article ID: 454.
63. T. U. Khan, M. Adil Khan, Generalized conformable fractional operators, *J. Comput. Appl. Math.*, **346** (2019), 378–389.

64. M. Tomar, S. Mubeen, J. Choi, Certain inequalities associated with Hadamard $k$-fractional integral operators, *J. Inequal. Appl.*, **2016** (2016), 234.

65. F. Jiang, F. Meng, Explicit bounds on some nonlinear integral inequalities with delay, *J. Comput. Appl. Math.*, **205** (2007), 479–486.

66. W. Sudsutad, S. K. Ntouyas, J. Tariboon, Fractional integral inequalities via Hadamard’s fractional integral, *Abstract. Appl. Anal.*, **2014** (2014), Article ID: 563096.

67. D. R. Anderson, D. J. Ulness, Newly Defined Conformable Derivatives, *Adv. Dyn. Syst. Appl.*, **10** (2015), 109–137.

68. N. N. Lebedev, Special functions and their applications Prentice-Hall, *INC. Englewood Cliffs*, 1965.

69. E. Set, A. Kashuri, I. Mumcu, Chebyshev type inequalities by using generalized proportional Hadamard fractional integrals via Pólya-Szegö inequality with applications, *Chaos. Solitions Fract.*, **146** (2021), 110860.

© 2021 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)