Structure in 6D and 4D $\mathcal{N} = 1$ supergravity theories from F-theory

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Abstract: We explore some aspects of 4D supergravity theories and F-theory vacua that are parallel to structures in the space of 6D theories. The spectrum and topological terms in 4D supergravity theories correspond to topological data of F-theory geometry, just as in six dimensions. In particular, topological axion-curvature squared couplings appear in 4D theories; these couplings are characterized by vectors in the dual to the lattice of axion shift symmetries associated with string charges. These terms are analogous to the Green-Schwarz terms of 6D supergravity theories, though in 4D the terms are not generally linked with anomalies. We outline the correspondence between F-theory topology and data of the corresponding 4D supergravity theories. The correspondence of geometry with structure in the low-energy action illuminates topological aspects of heterotic-F-theory duality in 4D as well as in 6D. The existence of an F-theory realization also places geometrical constraints on the 4D supergravity theory in the large-volume limit.
1. Introduction

F-theory [1, 2, 3] provides a very general approach to constructing string vacua in even-dimensional space-times. In particular, F-theory gives a nonperturbative description of a wide range of string compactifications. F-theory describes structures such as gauge groups, matter fields, and Yukawa couplings in a simple geometric framework that is amenable to the use of powerful mathematical tools from algebraic geometry. F-theory as it is currently understood is incomplete as a physical theory. In its elemental geometric formulation there is no action principle or complete characterization of the fundamental degrees of freedom. The clearest definition of F-theory is as a limit of M-theory. M-theory itself, however, is also not a completely well defined theory, and some of the mathematical simplicity of F-theory is less apparent in the M-theory framework. Nonetheless, even with its current limitations, F-theory has proven to be a powerful tool for exploring both the large-scale structure of the landscape of string vacua and detailed aspects of semi-realistic phenomenology.

In eight and six dimensions, the set of F-theory compactifications includes vacua with spectra matching those of most or all supergravity theories that can be realized using other
known string theory constructions (for a review of 8D and 6D supergravity/F-theory models and many further references, see [4]). Given a six-dimensional supergravity theory, the spectrum and action of the theory provide data that can be used to identify the geometry of a corresponding F-theory construction, when one exists [5, 6, 7, 8]. The F-theory geometry in turn imposes certain constraints on the spectrum and action of the low-energy theory [9, 7, 10, 11]. Some, but not all, of these constraints are understood from macroscopic/low-energy consistency conditions such as anomaly cancellation. The set of 6D F-theory vacua forms a complicated moduli space with many components associated with different F-theory “base” geometries connected through extremal tensionless string transitions [12, 3]. Recent work has begun to systematically classify the set of 6D F-theory compactifications, using connections between the F-theory geometry and corresponding structure in the low-energy supergravity theory [13, 14, 15, 16, 17, 18].

In four dimensions F-theory gives rise to an even broader and richer class of vacua than in higher dimensions. Recent efforts have focused on compactifications relevant for semi-realistic GUT phenomenology [19, 20] including constructions of compact fourfolds for global models [21, 22, 23, 24]. In four dimensions, however, with only one supersymmetry, the space of string solutions is complicated by various perturbative and nonperturbative effects such as fluxes that remove massless moduli and produce a “landscape” containing isolated distinct vacua connected through regions of off-shell string physics (for reviews of flux compactifications and related developments see e.g. [25, 26, 27]). In this context, the limitations of F-theory in its current form become more apparent, and using this nonperturbative approach to study the global space of solutions becomes more challenging. Recent work has focused on incorporating more directly into F-theory degrees of freedom such as fluxes on the world-volume of 7-branes [28, 30, 31, 32, 33, 34, 35], and the related transverse scalar fields on multiple branes that can carry noncommuting structures such as “T-branes” [36]. While these features are present in the M-theory description of F-theory, the 4D physics described by F-theory is only reached in a singular limit that is as yet not fully understood. In four dimensions, there also appear to be many types of string solutions that are not easily described in the F-theory framework, such as $G_2$ compactifications of M-theory [37, 38, 39], heterotic compactifications on Calabi-Yau manifolds that (unlike K3) have no elliptic fibration in their moduli space, and other more exotic possibilities that may include a vast range of asymmetric orbifolds [40] and/or non-geometric flux vacua [11, 12, 13, 14, 15, 16] (that may also have asymmetric orbifold descriptions [17]). Despite these limitations, it can be argued that at this stage F-theory provides the broadest perspective on the range of possible phenomena that may emerge from string theory in 4D supergravity theories. In this work we address some global questions regarding the structure of F-theory vacua within the existing framework. For many practical questions we use the definition of F-theory as a limit of M-theory as recently studied in the 4D context in [27, 48].

In six dimensions, the key to reconstructing the geometry of an F-theory compactification from the data of the supergravity theory lies in the Green-Schwarz terms of the form $BR^2$ and $BF^2$, and in the related lattice of dyonic string charges. While the original
understanding of the Green-Schwarz terms arose through the anomaly cancellation mechanism, it seems that there may be deeper reasons underlying the existence and structure of these terms. In four dimensions similar topological couplings arise between axions $\rho$ and gauge and gravitational curvature-squared terms, of the form

$$ (a \cdot \rho) R \wedge R + \sum_A (b_A \cdot \rho) F^A \wedge F^A. \quad (1.1) $$

While in some cases these terms are connected with a generalized Green-Schwarz mechanism for cancellation of gauge and mixed abelian-gravitational anomalies $[49, 50]$, these terms are not uniquely determined by this condition; for example, the $\rho R \wedge R$ terms appear even in theories without massless gauge fields. A detailed discussion of the generalized Green-Schwarz mechanism in related weakly coupled Type IIB scenarios can be found in $[51]$. Terms of the form (1.1), and the associated integral lattice of axionic string/instanton charges (containing $a$ and $b_A$ in (1.1)), relate 4D supergravity theories to F-theory geometry in a parallel fashion to the six-dimensional story. In particular, $a$ contains geometric information about the F-theory compactification manifold (the canonical class of the threefold base), while $b_A$ captures information about the geometric structure giving rise to the simple factors in the gauge group (the locations of the 7-branes supporting the gauge group factors). As in six dimensions, this information, along with other structure in the 4D supergravity theory, can be used in a “bottom-up” fashion to identify the F-theory geometry needed for a UV completion of the theory. We describe in this paper how the terms (1.1) arise from F-theory, and match with dual heterotic constructions. This connection gives a simple perspective on the topological structure of heterotic-F-theory duality that is valid for $SO(32)$ as well as $E_8 \times E_8$ heterotic vacua with F-theory duals. More generally, these couplings and the structure of the related string charge lattice may provide a useful tool for addressing global questions about the space of string vacua and related duality symmetries in 4D just as they have done in 6D.

Some previous progress towards relating the degrees of freedom and action of 4D $\mathcal{N} = 1$ supergravity theories to the data used in an F-theory construction via M-theory was presented in $[48, 52, 53, 54]$. In four dimensions, the structure of F-theory compactifications is complicated by the necessary presence of fluxes that produce a superpotential or D-terms that lift some moduli of the theory. In this paper, we assume that the theory is in a regime where these moduli are light, corresponding to a large-volume F-theory compactification. In this regime, F-theory geometry places certain constraints on the spectrum and action of the associated 4D supergravity theory. An important direction for further extension of the work in this paper is to develop an understanding the role of the structure and constraints presented here away from the large-volume F-theory limit.

Six-dimensional supergravity theories and F-theory vacua are described in Section 2. The spectrum and relevant terms in the action of 4D theories are described in Section 3. This section also contains an analysis of axion–curvature squared couplings in heterotic theories, and uses these terms to determine topological aspects of the general heterotic/F-theory duality correspondence for 4D theories. Section 4 contains a brief description of some
structures and constraints on 4D theories associated with large-volume F-theory compactifications that are close analogues of similar structures and constraints in six dimensions. Section 5 contains concluding remarks.

2. Six-dimensional supergravity theories and F-theory vacua

In this section we summarize some key features of 6D supergravity theories and F-theory vacua. We outline the correspondence between data in the supergravity theory and geometric structures in F-theory. Most of this material is known and is described in earlier papers, but here we consolidate together a variety of results on 6D theories into a coherent picture for comparison with the 4D story. We also add a few new observations on some aspects of 6D theories that help to clarify both the 6D and 4D stories. The material in this section is also used in Section 4 to characterize constraints on consistent 6D F-theory vacua in terms of data in the supergravity theory.

2.1 F-theory vacua and 6D spectra

A review of 6D supergravity theories and associated F-theory constructions appears in [4]; another review of 6D string vacuum constructions from a variety of approaches including F-theory is given in [55]. We summarize the basics here, beginning with supergravity and then describing F-theory models.

We begin with some generalities on 6D supergravity theories with $\mathcal{N} = (1, 0)$ supersymmetry, corresponding to eight supercharges. The spectrum of such a theory contains:

- One gravity multiplet,
- $T$ tensor multiplets,
- $V$ vector multiplets in a general (nonabelian $\times$ abelian) gauge group,
- $H$ hypermultiplets spanning a quaternionic Kähler manifold.

Each hypermultiplet contains four real scalars as bosonic components. Note that the bosonic components of the gravity multiplet contain, in addition to the metric, a two-form field with self-dual field strength. The tensor multiplets each contain an anti-self-dual two-form field as well as a real scalar. The field content, couplings, and equations of motion of 6D supergravity theories were studied in [56, 57, 58, 59, 60]. Dyonic strings in the theory carry charges under the self-dual and anti-self-dual two-form fields in the gravity and tensor multiplets. These strings should appear in any quantized theory of 6D supergravity as quantum excitations charged under the two-form fields, independent of whether the UV completion of the gravity theory involves a conventional formulation of string theory. The charges of the quantized dyonic strings lie in a lattice $\Gamma$ that must be unimodular [10].

In six dimensions, anomaly cancellation [57, 58, 61] provides a powerful set of constraints on the set of possible theories, as well as a useful tool for analyzing supergravity
theories. For example, the numbers of multiplets $H, V, T$ introduced above are not independent, but rather linked through the gravitational anomaly relation

$$H - V = 273 - 29T.$$  \hspace{1cm} (2.1)

We briefly review the complete set of 6D anomaly cancellation conditions in Appendix B.

We now turn to F-theory constructions of 6D $\mathcal{N} = (1,0)$ supergravity theories. Such models arise from compactification of F-theory on an elliptically fibered Calabi-Yau threefold $X$ over a complex surface base $B$. A detailed description of 6D compactifications of F-theory is given in \cite{2, 3}; we briefly review the structure of these vacua, emphasizing the correspondence between the topology of $B, X$ and the field content of the low-energy theory.

In the type IIB picture, an F-theory construction is given by a set of 7-branes wrapped on the space $B$. The nonabelian part of the gauge group arises from coincident 7-branes on $B$, which give singularities in the elliptic fibration $X$ associated with codimension one loci (divisors) on the base. The nonabelian gauge group factor on such a divisor can be determined from the Kodaira/Tate classification of the local codimension one singularity \cite{2, 3, 24, 31}. The elliptically fibered threefold $X$ is given in the Weierstrass description by

$$y^2 = x^3 + fx + g,$$  \hspace{1cm} (2.2)

where $f,g$ are sections of $-4K, -6K$, with $K$ being the canonical bundle of the base $B$. In the type IIB picture, the 7-branes are wrapped on the two-cycles in the base where the elliptic fibration degenerates. This degeneration locus is given by the vanishing of the discriminant $\Delta = 4f^3 + 27g^2$. The Kodaira condition that the total space of the elliptic fibration be Calabi-Yau states that

$$-12 [K] = [\Delta] = \sum_A \nu_A [S_A] + [Y]$$  \hspace{1cm} (2.3)

where $[K] = -c_1(B)$ is the canonical class of the base, $[\Delta]$ is the total class of the singularity locus, $[S_A]$ are the classes of the irreducible effective divisors carrying simple gauge group factors $G_A$, and $Y$ is the residual discriminant locus, which does not give rise to nonabelian gauge symmetries. The divisors $S_A$ carrying nonabelian gauge group factors are associated with singularities in the fibration characterized by integer multiplicities $\nu_A$ depending on the group $G_A$. (e.g., $\nu = N$ for $SU(N)$, $\nu = 10$ for $E_8$, etc.) We use Poincaré duality to move freely between divisor classes in $H_2(B)$ and elements of $H^{1,1}(B)$. Two viewpoints on such compactifications will be useful. Either we can consider the base $B$ supplemented by additional data for the 7-branes or we can study the complete singular threefold $X$.

We can now describe how the 6D spectrum is related to the F-theory geometry. The number of tensor multiplets is related to the topology of the F-theory base through

$$T = h^{1,1}(B) - 1.$$  \hspace{1cm} (2.4)
There is a unimodular lattice $\Gamma = H_2(B, \mathbb{Z})$ of dyonic string charges of signature $(1, T)$ associated with type IIB D3-branes wrapped on the 2-cycles of $B$. These strings are charged under the self-dual and anti-self-dual two-form fields in the theory.

The number of vector fields depends not only on the topology of the F-theory base but also on the singularity structure of the fibration encoded in $\Delta$. These singularities will generically render the total space $X$ of the elliptic fibration singular. One can, however, canonically blow up the singularities at each codimension, producing a smooth Calabi-Yau space $\hat{X}$. The rank of the gauge group of the 6D theory is then given in terms of the topologies of $\hat{X}$ and $B$ by

$$ r = h^{1,1}(\hat{X}) - h^{1,1}(B) - 1. $$

This rank can include a number of abelian vector fields in addition to the nonabelian gauge fields. Abelian vector fields are associated with extra sections of the fibration that increase the rank of the Mordell-Weil group $[3]$; the treatment of such abelian factors is rather subtle.

Finally, the number of uncharged scalar fields is

$$ H_{\text{neutral}} = h^{2,1}(\hat{X}) + 1. $$

These fields come from the complex structure moduli on $X$, with one modulus for the overall Kähler class of the base $B$; equivalently, these fields correspond to physical moduli in the Weierstrass description (2.3) of the F-theory model1. (More precisely, the fields $H_{\text{neutral}}$ are quaternionic, with four real degrees of freedom in each field; half of the degrees of freedom in each neutral field other than the overall Kähler modulus come from complex structure/Weierstrass moduli, the other half come from degrees of freedom on the 7-branes and the bulk form fields.) In general, charged matter fields arise from codimension 2 singularities in the elliptic fibration $[62, 64, 3, 17, 11]$, with some matter fields such as adjoint representations arising nonlocally on divisors $S_A$ of higher genus $[65, 66]$. Throughout this work we will primarily focus on 6D theories without matter fields for simplicity; in many theories there is a phase in which all matter fields are Higgsed and there is no massless charged matter $[17]$.

Note that the identifications (2.5), (2.4) and (2.6) allow us to give a simple expression for the Euler character of the resolved threefold for theories without charged matter

$$ \chi(\hat{X}) = 2(h^{1,1}(X) - h^{2,1}(X)) = 2(r + T - H_{\text{neutral}} + 3) $$

This equation provides the simplest link between the topology of $\hat{X}$ and the 6D spectrum. As we discuss later, $\chi(\hat{X})$ can also be related to the constant coefficients determining the 6D Green-Schwarz terms. This connection between the topology of the F-theory compactification space and the structure of the supergravity spectrum and action provides a constraint on 6D supergravity theories that, as we discuss further in Section 4, matches with 6D anomaly cancellation conditions.

1A detailed counting of physical vs. non-physical degrees of freedom in the Weierstrass coefficients of 6D F-theory models appears in $[48]$. 

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2.2 F-theory geometry and terms in the 6D supergravity action

In the previous section we described the correspondence between topology of the F-theory compactification space and the spectrum of the 6D theory. We now consider the connection between further geometric structures of the F-theory compactification and terms in the supergravity action. In particular, some terms in the supergravity action carry discrete geometric information about the F-theory picture. Through understanding this correspondence we can construct a map from data in the supergravity spectrum and action to data of the F-theory geometry, allowing us to identify which specific F-theory vacuum should correspond to any given supergravity theory. As we describe in Section 2.4, this gives a simple way of describing dualities such as heterotic/F-theory duality at the level of topology. Using the correspondence in the opposite direction, we can interpret constraints associated with F-theory geometry as necessary conditions for a supergravity theory to admit an F-theory realization, as we discuss in Section 4.

2.2.1 Couplings in the 6D supergravity action

As discussed above, from the spectrum of a given 6D theory one can already infer some core topological data of the base $B$ and total space $X$ of the F-theory elliptic fibration using (2.4), (2.5), and (2.6). More precise data from the supergravity theory is required to construct information about specific divisor classes on the base $B$, such as the canonical class $K$ of $B$, and the divisor classes $S_A$ carrying the nonabelian gauge group factors. This information is carried in the structure of the Green-Schwarz terms of the 6D supergravity action, which take the schematic form

$$(K \cdot B) \wedge R \wedge R, \quad (S_A \cdot B) \wedge F^A \wedge F^A$$

(2.8)

(where $B$ is the vector of two-forms). We now describe these terms in further detail. The analogous structure in four dimensions is one of the main focal points of this paper.

The study of the action for 6D effective supergravity theories is complicated by the fact that one has to deal with self-dual and anti-self-dual two-forms $B_2^\alpha$. This problem can be overcome, however, by working with a pseudoaction, where the duality constraints are imposed by hand after determining the equations of motion [56, 60], as done for F-theory in [8]. We focus here on the terms that are needed to identify the internal geometry. We take the Einstein-Hilbert term to have the canonical normalization $S_{EH} = - \int \frac{1}{2} R_s * 1$. The terms in the action in which we are particularly interested are the quadratic terms in
the space-time, nonabelian, and two-form curvatures ²

\[ S^{(6)} = - \frac{1}{2} \left( j^\alpha \Omega_{\alpha \beta} a^\beta \right) \tr R \wedge R + \frac{2}{\lambda_A} \left( j^\alpha \Omega_{\alpha \beta} b^\beta_A \right) \tr F^A \wedge *F^A \]

\[ + \frac{1}{4} \left( B^\alpha \Omega_{\alpha \beta} a^\beta \right) \wedge \tr R \wedge R + \frac{1}{\lambda_A} \left( B^\alpha \Omega_{\alpha \beta} b^\beta_A \right) \wedge \tr F^A \wedge F^A \]

\[ + \frac{1}{4} G_{\alpha \beta} H_3^\alpha \wedge *H_3^\beta + \frac{1}{2} G_{\alpha \beta} d j^\alpha \wedge *d j^\beta . \]

Here \( R = \frac{1}{2} R_{\mu \nu} dx^\mu \wedge dx^\nu \) is the \( SO(1,5) \)-valued curvature two-form. We have introduced the field strengths \( H_3^\alpha \) of the \( B_2^\alpha \) that are given by

\[ H_3^\alpha = dB_2^\alpha + \frac{1}{2} a^\alpha w_{CS}(R) + 2 \frac{b^\alpha_A}{\lambda_A} w_{CS}^A(F) , \]

where the Chern-Simons forms are given by

\[ w_{CS}(R) = \tr \left( \hat{\omega} \wedge d\hat{\omega} + \frac{3}{2} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right) , \]

\[ w_{CS}^B(F) = \tr \left( A^B \wedge dA^B + \frac{3}{2} A^B \wedge A^B \wedge A^B \right) , \]

with \( \hat{\omega} \) being the spin-connection one-form. The field \( j^\alpha \) and the coefficients \( a^\alpha, b^\alpha_A \) transform as vectors in the space \( \mathbb{R}^{1,T} \), which carries a symmetric inner product \( \Omega_{\alpha \beta} \) of signature \((1,T)\). Note that the self- and anti-self duality conditions for \( H_3^\alpha \) must be imposed by hand on the level of the equations of motion by demanding

\[ \Omega_{\alpha \beta} * H_3^\beta = G_{\alpha \beta} H_3^\beta . \]

The field \( j^\alpha \) contains the scalars in the \( T \) tensor multiplets. The additional degree of freedom in \( j^\alpha \) is fixed by the condition \( j^\alpha \Omega_{\alpha \beta} j^\beta = 1 \). By convention, in \((2.9) \) “tr” of \((F^B)^2\) denotes the trace in the fundamental representation, and \( \lambda_B \) are normalization constants depending on the type of each simple group factor. These constants are related to the dual Coxeter numbers \( c_{GB} \) of the gauge group factors \( G_B \) and trace normalization factors \( A_{\text{adjoint}}^{(B)} \) through \( \lambda_B = 2c_{GB} / A_{\text{adjoint}}^{(B)} \), with coefficients discussed in Appendix B.

The first two terms on the second line of \((2.9) \) can be written as

\[ S_{GS}^{(6)} = - \frac{1}{2} \int \Omega_{\alpha \beta} B_2^\alpha \wedge X_4^\beta \]

\[ \text{where} \]

\[ X_4^\alpha = \frac{1}{2} a^\alpha \tr R \wedge R + \sum_A b^\alpha_A \left( \frac{2}{\lambda_A} \tr F^A \wedge F^A \right) . \]

For a 6D supergravity theory arising from F-theory, the \( T + 1 \) two-form fields \( B_2^\alpha \) arise by expanding the R-R four-form \( C_4 \) of Type IIB into a basis \( \omega_\alpha \) of \( h^{1,1}(B) \) two-forms

²In this action we have included a term \( \tr R \wedge *R \) in analogy to \( \tr F^A \wedge *F^A \). The precise form of this term can be altered by a field redefinition involving the metric. In F-theory these higher derivative terms can be determined via 5D M-theory compactifications generalizing [8]. In five dimensions the supersymmetric completion of the curvature squared terms is known [8].
spanning $H^2(B)$ as $C_4 = B_2^2 \wedge \omega_\alpha$. The 6D tensors satisfy the duality condition (2.13) due to the 10D self-duality of the field strength $F_5$ of $C_4$. Due to the varying dilaton, however, this decomposition cannot in general be described in a weakly coupled supergravity limit. The 6D action can be derived in a more precise fashion via a 5D M-theory compactification [8].

We now discuss the various terms appearing in (2.9) and comment on the topological information of the F-theory compactification space that we can extract from these terms.

### 2.2.2 Topological couplings in the Green-Schwarz term

We now discuss the terms from (2.9) that appear in (2.14). Equation (2.14) describes the 6D Green-Schwarz terms that are needed for anomaly cancellation [57, 59, 61]. In 6D there are gauge, gravitational, and mixed gauge-gravitational anomalies. These anomalies are captured by an 8-form anomaly polynomial $I_8(R, F)$ that is a function of the curvature tensor $R$ and the gauge field strengths $F_A$ of all gauge groups. If this polynomial factorizes as $I_8 = \frac{1}{2} \Omega_{\alpha\beta} X_\alpha^a X_\beta^b$, then the anomaly can be cancelled using the Green-Schwarz counterterm, as described in detail in Appendix B.

For 6D supergravity theories arising from an F-theory compactification, the $SO(1, T)$ vectors $a^\alpha, b^\alpha_A$ appearing in the 6D Green-Schwarz terms carry topological information about the F-theory geometry. These vectors correspond to the canonical class and divisor classes carrying the nonabelian gauge group factors in the F-theory picture. Specifically,

$$a^\alpha = K^\alpha, \quad b^\alpha_A = C^\alpha_A,$$

(2.16)

where the coefficients $K^\alpha$ and $C^\alpha_A$ arise in the two-form expansions

$$[K] = K^\alpha \omega_\alpha, \quad [S_A] = C^\alpha_A \omega_\alpha,$$

(2.17)

of the canonical class of $B$ and the 7-brane classes in (2.3).

There are several different ways in which the correspondence given by (2.16) can be derived and/or confirmed. The anomaly cancellation conditions provide a consistency check on this identification; the intersection products between the vectors $a, b_A$ are given in terms the matter content of the theory through the anomaly equations, and match with the intersection products between $K, S_A$ in the F-theory geometry [1, 3, 9]. When $K, S_A$ span the entire cohomology lattice $H_2(B, \mathbb{Z})$ then this correspondence suffices to prove (2.16). The correspondence (2.16) can also be confirmed directly from the dual M-theory picture; details of this computation are given in [3]. In cases where the F-theory model has a heterotic dual, it is furthermore possible to directly derive the coefficients in the Green-Schwarz term by dimensional reduction of the heterotic 10D theory and to confirm (2.16) [69]; we describe this connection to 6D heterotic theories in Section 2.4.

A fourth approach to deriving the correspondence (2.16) arises from the expansion of the curvature-corrected Chern-Simons action of the 7-branes. This approach, originally taken by Sadov [3], is somewhat heuristic as the perturbative 7-brane action is extrapolated to the nonperturbative regime. This argument is, however, the easiest approach to
generalize to the analogous 4D context, so we focus on this method here. The Dirac-Born-Infeld world-volume action of the 7-branes [70] (reviewed in [26]) contains Chern-Simons type couplings that can be written in the schematic form

\[ \int_{M_6 \times S_A} C_4 \wedge \left( \text{tr}(\hat{F}^A)^2 - \frac{1}{48} \text{tr}\hat{R}^2 \right) , \]

where \( \hat{F}^A \) is the 7-brane field strength, \( \hat{R} \) is the curvature two-form restricted to the 7-brane world volume, and \( S_A \) are the divisors in \( B \) wrapped by the 7-branes. After dimensional reduction to 6D, each stack of branes on a divisor \( S_A \) associated with a nonabelian gauge group factor produces the term of the form \( B \cdot S_A \text{tr}(F^A)^2 \) in (2.14). All 7-branes, including those that do not carry nonabelian gauge group factors, should in principle carry \( R^2 \) terms. From the Kodaira condition (2.3), the sum over these branes gives precisely the class \([-12K]\), so that the sum over all branes of the \( R^2 \) terms reproduces the term of the form \( B \cdot K/4 \text{tr}R^2 \) in (2.14). This derivation can be understood clearly in the limit of F-theory discussed by Sen [71, 72] where the 7-branes not carrying nonabelian gauge groups combine into orientifold planes. We give a more detailed description of the analogous analysis in the 4D case in Section 3.

We see then from the correspondence (2.16) that the canonical class of the base and the divisors carrying the nonabelian gauge group factors can be read off directly from the topological couplings in the supergravity action. To understand this relationship better it is helpful to discuss the inner product structure on \( SO(1,T) \) vectors somewhat further. As discussed above, the inner product \( \Omega_{\alpha\beta} \) has signature \((1,T)\). For convenience we use a shorthand notation

\[ x \cdot y = x^\alpha \Omega_{\alpha\beta} y^\beta . \]  

(2.19)

The vectors \( a, b^A \) are associated with charges of dyonic strings given by gravitational and gauge theory instantons. These vectors lie in an integral lattice. The integrality of the inner products \( a \cdot a, a \cdot b^A, b^A \cdot b_B \) follows simply from the absence of anomalies in any 6D supergravity theory, independent of consideration of quantized string charges [6]. Furthermore, in any consistent theory these vectors must lie in a signature \((1,T)\) lattice \( \Gamma \) that is self-dual (unimodular) [10]. In a theory with an F-theory realization, this lattice corresponds to the second cohomology lattice of the F-theory base

\[ \Gamma = H_2(B, \mathbb{Z}) . \]

(2.20)

The intersection product on this lattice corresponds to the inner product given by \( \Omega_{\alpha\beta} \) in the supergravity theory. Furthermore, any charge \( x \in \Gamma \) with \( j \cdot x > 0 \) for all \( j \) in the Kähler cone corresponds to an effective divisor in \( B \). Thus, knowledge of the spectrum of charged string excitations in the theory provides a complete picture of the cohomology and effective divisors (Mori cone) of \( B \). The lattice spanned by \( a, b^A \) is in general a sublattice of the full lattice \( \Gamma \).
2.2.3 Kinetic terms

We next consider the first two terms in (2.9). These two terms are related by supersymmetry to the terms in (2.14) \[59\]. Consider first the kinetic terms of the 6D vectors with field strengths $F^A$. Independent of the supersymmetry relating these terms to the topological $BF^2$ terms, one can compare the general form of the kinetic term of the 6D vectors with the kinetic term of the vectors arising in an F-theory reduction. This is done either by an M-theory lift as in \[8\], or by a direct evaluation of the Dirac-Born-Infeld action for D7-branes. In the latter route, by analogy to (2.18) this term is given by

$$
\int_{M_6 \times S_A} \text{Tr} (\hat{F}^A \wedge *_8 \hat{F}^A) = \int_{M_6} \text{Tr} (F^A \wedge * F^A) \cdot \frac{\int_{S_A} J_0}{V_b^{1/2}}, \tag{2.21}
$$

where $*_8$ is the Hodge-star on the 7-brane world-volume, and $J_0$ is the Kähler form of the base $B$. Note that the factor of the base volume $V_b = \frac{1}{2} \int_B J_0 \wedge J_0$ in (2.21) arises from the Weyl rescaling of the metric to bring the Einstein-Hilbert term to standard form.\[3\] Expanding the base Kähler form as $J_0 = v_0^b \omega_\alpha$, and comparing (2.21) with (2.9) one infers

$$
J_\alpha = \frac{v_0^b \omega_\alpha}{(2V_b)^{1/2}}, \quad J_\alpha \Omega_{\alpha\beta} J_\beta = 1, \tag{2.22}
$$

where the latter condition is automatically satisfied. Similarly, one can in principle evaluate higher curvature terms in the Dirac-Born-Infeld action of a D7-brane to fix the first term in (2.9). In contrast to the kinetic terms of the vectors $F^A$ this term contains the contraction of the form $K^\alpha \Omega_{\alpha\beta} J^\beta$, where $K^\alpha$ is canonical class of the base as in (2.17).

Finally, we discuss the kinetic term of the two-forms $B_2^\alpha$ that is the remaining term in (2.9). It contains the metric $G_{\alpha\beta}$, which due to supersymmetry can be given as a simple expression in terms of the real scalars $j_\alpha$. The main purpose of including this term is to contrast it with its four-dimensional analogue (in Section 3) where such strong supersymmetry constraints do not apply. One notes, however, that $G_{\alpha\beta}$ also admits a small $j^\alpha$ expansion that is valid for large two-cycle volumes $v_0^b$ in the base $B$. Explicitly one finds

$$
G_{\alpha\beta} = -\Omega_{\alpha\beta} + O(j^2), \tag{2.23}
$$

as discussed in more detail in \[8\]. Hence, in this large-volume limit the kinetic term of the $B_2^\alpha$ allows us to infer the intersection matrix $\Omega_{\alpha\beta}$ from the low-energy effective action. In six dimensions, this matrix is always equivalent under a linear field redefinition to the matrix $	ext{diag}(+1, -1, -1, \ldots)$; in four dimensions, however, the analogous structure is more complex.

We close our discussion by noting that in 6D one can in many cases use the discrete data $T$ and the anomaly lattice to uniquely identify the F-theory base and topological data of the discriminant locus from the data of the low-energy theory \[3 \ 7\]. When augmented with information about the dyonic string lattice of the low-energy theory this data is always sufficient to uniquely determine the topology of the F-theory base, including the precise structure of effective divisors, i.e. the Mori cone.

\[3\] One has to perform the rescaling of the 6D metric $g_{\mu\nu} \rightarrow V_b^{1/2} g_{\mu\nu}$.
2.3 Examples of 6D F-theory models

We give a few brief examples of 6D F-theory models to illustrate some of the points just reviewed.

2.3.1 $T = 0$

The simplest F-theory base for a 6D supergravity model is $\mathbb{P}^2$, with $h^{1,1} = 1$ so $T = 0$. 6D supergravity theories with $T = 0$ were analyzed extensively in [14] from the point of view of supergravity constraints, and in [15, 16] from the point of view of F-theory. In all $T = 0$ models, $\Gamma = \mathbb{Z}$, $-a = 3$, since $K = -3H$ where $H$ is the hyperplane generating $H_2(\mathbb{P}^2, \mathbb{Z})$ with $H \cdot H = 1$, and $b_A$ is an integer for each gauge group where $S_A = b_A H$.

2.3.2 $T = 1$

The F-theory bases with $T = 1$ are the Hirzebruch surfaces $\mathbb{F}_m$, $m \leq 12$ [3]. These are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$. A basis for $H_2(\mathbb{F}_m, \mathbb{Z})$ is $\Sigma, F$, with $\Sigma$ a section and $F$ a fiber, and intersection numbers $\Sigma \cdot \Sigma = -m, \Sigma \cdot F = 1, F \cdot F = 0$. The irreducible effective divisors in this basis are $\Sigma, F$, and $q \Sigma + p F$ with $q > 0, p \geq m q$. The generic Weierstrass model over $\mathbb{F}_m$ for $m = 0, 1, 2$ has no gauge group or matter, and for $m = 3, 4, 5, 6, 8, 12$ has a gauge group $SU(3), SO(8), F_4, E_6, E_7, E_8$ with no charged matter. We focus here on the structure of the Green-Schwarz terms for these models. In the following section these terms are related to the dual heterotic picture.

There is a natural linear basis for $H^2(\mathbb{F}_m, \mathbb{Z})$ given by

$$\omega_f = [\Sigma + (m/2) F], \quad \omega_b = [F], \quad (2.24)$$

where the brackets indicate that we consider the Poincaré dual two-forms. In this basis the inner product is given by

$$\Omega_{\alpha\beta} = \int_{B_2} \omega_\alpha \wedge \omega_\beta, \quad \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.25)$$

While this basis is not an integral basis for the lattice for $m$ odd, it will be useful in matching to the heterotic theory.

The 6D two-forms in Type IIB on $\mathbb{F}_m$ arise from the $C_4$ R-R field via the decomposition $C_4 = B_f \wedge \omega_f + B_b \wedge \omega_b$ into the two-forms $\omega_f, \omega_b$ introduced in (2.24). To evaluate the 6D Green-Schwarz terms for $B_f$ and $B_b$ we first determine $a$ for this geometry. In the $\omega_f, \omega_b$ basis we have

$$-[K] = [2 \Sigma + (2 + m) F] = 2 \omega_f + 2 \omega_b, \quad a = (K^\alpha) = (-2, -2), \quad (2.26)$$

where $[K]$ is the canonical class of $B_2$. The vector $b$ is determined by the wrapping of the 7-brane. For a gauge group factor wrapped on a divisor $S = p \Sigma + q F$, one has

$$[S] = p \omega_f + (q - p m/2) \omega_b, \quad b = (C^\alpha) = (p, q - p m/2). \quad (2.27)$$
The Green-Schwarz terms are then obtained by inserting (2.26), (2.27), and (2.25) into the general expression (2.14) such that
\[
S_{\text{GS}}^{(6)} = \frac{1}{2} \int (B_f + B_b) \wedge \text{tr} R^2 - (p B_b + (q - p m/2)B_f) \wedge \frac{2}{\lambda} \text{tr} F^2. \tag{2.28}
\]

2.4 Six-dimensional heterotic models

The \( T = 1 \) 6D models discussed above are also well understood in a dual heterotic picture [3]. Generic Weierstrass models over \( \mathbb{F}_m \) correspond to heterotic \( E_8 \times E_8 \) compactifications on K3 with \( 12 \pm m \) instantons in each \( E_8 \) factor, or for \( \mathbb{F}_4 \) to heterotic \( SO(32) \) compactification on K3. The 6D Green-Schwarz terms for these theories can be derived directly from the heterotic 10D action. This can be seen on the one hand as a method for confirming the form of the Green-Schwarz terms. On the other hand, this can be seen as a simple way of determining the F-theory dual of the heterotic theories: by finding the low-energy data associated with a given heterotic model and constructing from this the F-theory data we can directly determine the F-theory dual of a given heterotic theory. This discussion is intended as a warmup for the 4D case discussed in the following section, where similar statements hold.

2.4.1 Heterotic in 10D

We first recall the 10D heterotic supergravity action with gauge groups \( SO(32) \) and \( E_8 \times E_8 \). Since we want to determine a 6D action of the form (2.9) with the duality constraint (2.13) imposed on tensors, it will be convenient to start with a pseudo action in 10D. This action depends on the heterotic B-field \( \hat{B} \) and its dual six-form field \( \hat{B}_6 \). Throughout this section we use hats (as in \( \hat{B} \)) to denote 10D quantities; fields without hats refer to 6D quantities. A well-known global constraint on 6D heterotic compactifications arises from the Bianchi identity of the modified heterotic three-form field strength \( \hat{H} \). Due to the Chern-Simons connections in \( \hat{H} \) it satisfies
\[
d\hat{H} = \frac{2}{\lambda} \text{Tr} \hat{F}^2 - \text{tr} \hat{R}^2,
\]
with the duality \( *\hat{H} = e^{2\phi} \hat{H}_7 \) imposed on the level of the equations of motion. Note that the equations of motion of (2.30) supplemented by the duality constraint precisely reproduce the equations of motion of the heterotic action. The last two terms in (2.30) are the 10D Green-Schwarz terms with
\[
\hat{X}_8 = \frac{1}{24} \text{Tr} \hat{F}^4 - \frac{1}{7200} (\text{Tr} \hat{F}^2)^2 - \frac{1}{240} \text{Tr} \hat{F}^2 \text{tr} \hat{R}^2 + \frac{1}{8} (\text{tr} \hat{F}^4) + \frac{1}{32} (\text{tr} \hat{R}^2)^2, \tag{2.31}
\]
\[
\hat{X}_4 = \frac{2}{\lambda} \text{tr} \hat{F}^2 - \lambda \text{tr} \hat{R}^2, \tag{2.32}
\]
\[\text{We have used in this action and in (2.24) a normalization of the B-field convenient for heterotic/F-theory duality discussed below.}\]
where again $\lambda = 2$ for $SO(32)$, and $\lambda = 60$ for $E_8 \times E_8$. Note that the equations of motion of $\hat{B}_6$ determined from (2.30) are precisely the Bianchi identity (2.29) via the duality of the field strengths.

### 2.4.2 Green-Schwarz terms and duality to F-theory

We now describe in detail the derivation of the Green-Schwarz terms for 6D heterotic compactifications and the connection through duality to F-theory. As we describe in the following section, a very similar analysis holds in four dimensions. Green-Schwarz anomaly cancellation in 6D heterotic compactifications on K3 was first analyzed in [57]. The derivation of the 6D Green-Schwarz terms from the heterotic theory was done by Honecker in [69], and the determination of the structure of the 6D terms from anomalies was worked out by Erler in [61]. The trace factors needed for this computation are given in [61]. Note that the conventions of [61, 6, 7] differ from those of [57, 69]. We follow the former conventions here.

Consider the heterotic theory compactified on K3, described as a $T^2$ fibration over $\mathbb{P}^1$. There is one 6D tensor that we shall call $B_0$ coming from the 10D B-field $\hat{B}$ in non-compact directions. This $B_0$ is not chiral but a linear combination of the self-dual and anti-self-dual tensors that are part of the 6D gravity and tensor multiplets respectively. In the action formulation (2.30), with duality condition $*\hat{H} = e^{2\phi} \hat{H}_7$ imposed on the level of the equations of motion, one also gets a second 6D tensor from $\hat{B}_6$. We denote this 6D tensor arising from $\hat{B}_6$ wrapped on the wrapped on the K3 by $B_1$. The 10D duality of the $B$ field to $B_6$ reduces to the 6D duality of $B_0, B_1$ with the inner product matrix (2.25). The contribution of $B_1$ to the 6D action comes the last term in (2.30) and yields

$$S_{GS}^{(6)}(B_1) = \frac{1}{2} \int B_1 \wedge (\text{tr} R^2 - \frac{2}{\lambda} \text{tr} F^2).$$  

(2.33)

The contribution of $B_0$ to the 6D action comes from the dimensional reduction of the 10D Green-Schwarz term involving $\hat{X}_8$. To get the 6D action we replace half of the indices in $\hat{X}_8$ with internal (compact) indices; we denote curvatures in the compact directions by $\mathcal{R}, \mathcal{F}$.

In 6D compactifications on K3 the Bianchi identity (2.29) implies\footnote{Here we have fixed the normalization of $\mathcal{F}, \mathcal{R}$ such that $(2/\lambda) \int_{K3} \text{tr} \mathcal{F}^2, \int_{K3} \text{tr} \mathcal{R}^2$ are integers.}

$$\frac{2}{\lambda} \int_{K3} \text{tr} \mathcal{F}^2 = \int_{K3} \text{tr} \mathcal{R}^2 = 24.$$

(2.34)

This implies a fixed total instanton number for $SO(32)$, but allows for the distribution $12 \pm k$ of instantons between the two gauge group factors in the $E_8 \times E_8$ case.

We now consider separately the heterotic $SO(32)$ and $E_8 \times E_8$ theories. For the $SO(32)$ theory in a generic instanton background, we replace [61]

$$\text{Tr} \hat{F}^2 = 30 \text{tr} \hat{F}^2, \quad \text{Tr} \hat{F}^4 = 24 \text{tr} \hat{F}^4 + 3(\text{tr} \hat{F}^2)^2.$$

(2.35)
The $tr\hat{R}^4, tr\hat{F}^4$ terms in the fundamental representation from (2.31) do not contribute in 6D since the curvature in the compact directions is associated with different indices from the 6D curvatures. Thus, (2.31) gives

$$X^S_{SO(32)} = \int_{K3} \left( -\frac{1}{8} tr F^2 tr R^2 - \frac{1}{8} tr R^2 tr F^2 + \frac{1}{16} tr R^2 tr R^2 \right).$$

(2.36)

We thus have the Green-Schwarz couplings

$$S^{(6)}_{SO(32)} = \frac{1}{2} \int (B_1 + B_0) \wedge tr R^2 - (B_1 - 2B_0) \wedge tr F^2.$$  

(2.37)

Comparing this Green-Schwarz term with the general expression (2.14), we read off in the basis $(B_0, B_1)$ with intersection product (2.25) the vectors

$$a = (-2, -2), \quad b = (1, -2).$$

(2.38)

Note that the first entry of $a$ and $b$ is easy to infer by comparing the modified heterotic field strength (2.29) with the general 6D expression (2.10). These agree with the F-theory picture (2.28) under the identifications

$$(B_f, B_b) \leftrightarrow (B_0, B_1).$$

(2.39)

The vector $a$ in (2.38) agrees with the F-theory expression (2.26). In the $SO(32)$ case the vector $b = (1, -2)$ shows that in F-theory the compactification manifold must be $F_4$ and the remaining gauge group ($SO(8)$ when maximally broken) must arise from a 7-brane wrapping the divisor $\Sigma$ on $F_4$. To see this, we use the fact that the vector $b = (1, -2)$ only encodes an irreducible effective divisor on $F_m$ for $F_4$. For $m < 4$ the corresponding divisor is not effective and for $m > 4$ it is not irreducible. This reproduces the standard picture of heterotic/F-theory duality in this case [3]

For the $E_8 \times E_8$ case, we have a similar analysis. Now there are $12 \pm k$ instantons in the two $E_8$ factors. Using the $E_8$ trace normalization and relation [61]

$$\text{Tr}\hat{F}^2 = tr\hat{F}^2, \quad \text{Tr}\hat{F}^4 = \frac{1}{100} tr\hat{F}^4$$

(2.40)

and inserting into (2.31) we get

$$X^S_{E_8} = \int_{K3} \left( \frac{1}{3600} \left[ 2 tr F_1^2 tr F_1^2 + 2 tr F_2^2 tr F_2^2 - tr F_1^2 tr F_1^2 - tr F_2^2 tr F_2^2 \right] - \frac{1}{240} \left[ tr F_1^2 tr R^2 + tr R^2 tr F_1^2 + tr F_2^2 tr R^2 + tr R^2 tr F_2^2 \right] + \frac{1}{16} \left[ tr R^2 tr R^2 \right] \right).$$

(2.41)

Inserting $\int tr R^2 = 24, \int tr F_{1,2}^2 = 30(12 \pm k)$ gives the Green-Schwarz terms

$$S^{(6)}_{E_8} = \frac{1}{2} \int (B_1 + B_0) \wedge tr R^2 - \frac{1}{30} \left( B_1 - \frac{k}{2} B_0 \right) \wedge tr F_1^2 - \frac{1}{30} \left( B_1 + \frac{k}{2} B_0 \right) \wedge tr F_2^2.$$

(2.42)
Comparing this with the general 6D expression \((2.14)\) gives

\[ a = (-2, -2), \quad b_1 = (1, -k/2), \quad b_2 = (1, k/2). \tag{2.43} \]

This is in agreement with the F-theory picture with the identification \((2.39)\) where the remaining components of the two \(E_8\) groups wrap \(\Sigma, \Sigma + kF\) on \(\mathbb{F}_k\). We see that the heterotic/F-theory correspondence of these terms immediately determines the space for the F-theory dual of each choice of instanton distribution in the heterotic theory as well as the locus on which the branes carrying the two gauge group factors are wrapped. For generic instanton configurations, only one of these gauge groups remains unbroken; by convention this is taken to be the gauge group associated with the divisor \(\Sigma\) on \(\mathbb{F}_k\).

We thus see that by computing the Green-Schwarz terms on the heterotic side, we can immediately determine the topology and divisor classes of the base manifold and 7-branes carrying gauge groups for a dual F-theory model. Note that on the heterotic side, it possible to have a K3 that is not elliptically fibered. In this case there is no clear F-theory dual. The determination of the F-theory dual through the Green-Schwarz terms is only topological, however. Because the non-elliptically fibered K3’s are in the same moduli space as elliptically fibered K3 surfaces, they can be reached by a continuous deformation from models admitting F-theory duals. It would be interesting to understand better how this works in the dual F-theory picture.

3. Four-dimensional supergravity theories and F-theory vacua

We now carry out a similar analysis for 4D supergravity theories and F-theory constructions. This section is structured in a parallel fashion to the 6D story in the previous section, though some of the technical and conceptual aspects are more complicated. As in 6D, the supergravity spectrum and topological terms correspond closely to the topological structure of 4D F-theory vacua, at least for large-volume compactifications where the moduli can be clearly identified from the low-energy theory. Section \([3.1]\) contains some simple observations on the connection of 4D spectra with F-theory geometry. We describe the general structure of axion–curvature squared terms in the 4D action in Section \([3.2]\).

The topological nature of these terms in 4D encodes much of the relevant structure of the F-theory compactification geometry, just as the Green-Schwarz \(BF^2\) and \(BR^2\) terms in 6D encode key aspects of the topology of the corresponding elliptically fibered F-theory threefold. This story is complicated in four dimensions, however, by the appearance of similar terms associated with additional axion fields, for example at weak string coupling from the 10D axiodilaton. In Section \([3.3]\) we describe as examples F-theory compactifications on bases that are complex threefolds with the structure of a \(\mathbb{P}^1\) fibration. These are dual to 4D heterotic compactifications over elliptically fibered threefolds; we describe these models in Section \([3.4]\) and show how the axionic–curvature squared terms can be derived from the heterotic theory and used to identify the topology of the F-theory dual.

Note that while in six-dimensional supergravity theories the spectrum of the theory is massless, and the structures visible from F-theory geometry are clearly apparent through-
out the moduli space, the story is more complicated in four dimensions. Perturbative and nonperturbative effects, including the fluxes needed for D3-brane tadpole cancellation, lift some moduli of the F-theory geometry. Structures in the action and constraints that are apparent in the large volume F-theory limit are not protected against perturbative and nonperturbative corrections, and may be lost or modified in the full off-shell configuration space of the theory. In discussing the spectrum and terms in the 4D supergravity action, we are working in a limit where the compactification volume is large, and where the spectrum of light fields, while possibly lifted by fluxes, is still related to the geometry of the F-theory compactification. As we discuss at the end of the paper, going beyond this limit and exploring the implication of these structures and constraints on the broader off-shell configuration space is an interesting open problem for further research.

3.1 F-theory vacua and 4D spectra

We begin by summarizing the field content of 4D $\mathcal{N} = 1$ supergravity theories, and describing the spectrum that will appear in any F-theory compactification. Much of this correspondence is known [27, 48], but we add some further observations here. The spectrum of a general $\mathcal{N} = 1$ theory contains a single gravity multiplet and a number of chiral multiplets $C$, as well as a number of vector multiplets $V$. A chiral $\mathcal{N} = 1$ multiplet contains a single complex scalar comprising one real scalar and one real pseudoscalar degree of freedom [73] while a vector multiplet contains a vector as bosonic components. The standard form of the $\mathcal{N} = 1$ supergravity action is well-known and can be found, e.g. in [74].

3.1.1 Scalar spectrum and couplings

In an F-theory construction of a 4D $\mathcal{N} = 1$ theory, we have a Calabi-Yau fourfold $X$ that is elliptically fibered over a complex threefold base $B_3$. The origin of the various fields in the 4D theory is described in [27, 48] from the point of view of F-theory as a limit of M-theory. We focus here on neutral scalar fields; charged fields are discussed in later sections. Since a non-Abelian gauge group on the 7-branes generically renders the fourfold $X$ singular, as in 6D we use the resolved fourfold $\hat{X}$ to determine the spectrum of the theory. In the F-theory picture, there are $h^{3,1}(\hat{X})$ neutral chiral multiplets associated with the complex structure moduli $z^k$ of $\hat{X}$, or equivalently with the physical moduli in the Weierstrass model describing the 7-brane configuration. Counting $h^{3,1}(\hat{X})$ corresponds to considering deformations that preserve the 7-brane gauge group singularities that are smoothed in the resolution from $X$ to $\hat{X}$. Other chiral multiplets arise from a basis $\omega_\alpha$ of $H^{1,1}(B_3)$ when expanding the Kähler form $J_b$ of $B_3$ and R-R 4-form $C_4$ as

$$J_b = v^a_b \omega_\alpha, \quad C_4 = B^a_2 \wedge \omega_\alpha.$$  \hspace{1cm} (3.1)

In 4D we can dualize the resulting two-forms to give axions $\rho_\alpha$ that complexify the fields from the Kähler class into complex moduli $T_\alpha$. In contrast to the 6D compactifications discussed in the previous section, there is yet another class of chiral multiplets, associated
to the third non-trivial Hodge number of a Calabi-Yau fourfold. In general, there will be $h^{2,1}(\hat{X}) - h^{2,1}(B_3)$ multiplets of this type, which arise in the M-theory picture as complex scalars in the expansion of the three-form potential into the respective three-form basis of $X$. At weak string coupling these fields correspond to modes of the Type IIB R-R and NS-NS two-forms and the Wilson line modes on the 7-branes. We decompose the chiral multiplets into different types, where the numbers of the different types of fields are related to the Hodge numbers of the F-theory compactification through

$$C_{cs} = h^{3,1}(\hat{X}) - 1,$$

$$C_{sa} = h^{1,1}(B_3) + 1,$$

$$C_{21} = h^{2,1}(\hat{X}) - h^{2,1}(B_3).$$

While the total number of complex scalars is $C = C_{cs} + C_{sa} + C_{21}$, these different types of scalars have distinct properties and couplings in the large-volume F-theory limit. In particular, the scalars $C_{sa} + C_{21}$ are distinguished from the scalars $C_{cs}$ by the fact that the $C_{sa} + C_{21}$ can immediately be identified as having pseudoscalar components with an axionic shift symmetry. At weak string coupling this is apparent from the fact that the Kähler moduli, the B-field moduli and the dilaton are complexified by real scalars arising in the expansion of the Ramond-Ramond forms with discrete shift symmetries [75].

To further distinguish the types of scalar fields, we can study their couplings in the effective action. As we discuss in more detail in Section 3.2, the $C_{sa}$ scalars generically contribute to axion-curvature squared terms of the form $\rho R \wedge R, \rho F \wedge F$, where $\rho$ is the pseudoscalar component. It is less clear, however, whether the $C_{21}$ scalars have couplings of this form. The absence of such couplings for these axions might be linked with the fact that the scalars $C_{21}$ have an additional discrete symmetry, as we discuss next. We recall that at large volume the definition of the real part of the $h^{1,1}(B_3)$ Kähler moduli $T_{\alpha}$ contains divisor volumes in the base $B_3$. The imaginary part of $T_{\alpha}$ are the axionic scalars dual to the two-form fields obtained from reducing $C_4$. One observes that the $C_{21}$ complex scalars $N^a$ appear quadratically in $T_{\alpha}$, with a coupling function $d_{aab}(z, \bar{z})$ determined by a holomorphic functions of the complex structure moduli $z^k$ of $X$. Using the corresponding M-theory reduction [77, 78] one explicitly finds [48]

$$T_{\alpha} = \frac{1}{2} \kappa_{\alpha\beta\gamma} v^\beta_b v^\gamma_b + \frac{1}{4} d_{aab} (N + \bar{N})^a(N + \bar{N})^b + i \rho_{\alpha},$$

where $v^a_b$ are the base two-cycle volumes introduced in the expansion (3.1), and $\kappa_{\alpha\beta\gamma}$ is the triple intersection number on $B_3$. The leading classical Kähler potential $K$ determining the kinetic terms of the scalars is given as a function of the base volume $V_b$ and the $h^{3,1}(\hat{X})$ complex structure moduli of $\hat{X}$. It must be evaluated as a function of the complex moduli $T_{\alpha}, N^a$ and $z^k$ by solving (3.2) for $v^a_b$ and inserting the result into $V_b \propto \kappa_{\alpha\beta\gamma} v^\beta_b v^\gamma_b$. This

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6This is equally true for the 7-brane Wilson line moduli also contained in $C_{21}$ [76].
implies that $K$ is only a function of $T_\alpha + \bar{T}_\alpha$ and $N^a + \bar{N}^a$. Given these expressions we note that the kinetic terms of the action have the classical shift symmetries

$$N^a \rightarrow N^a + i\Lambda^a, \quad T_\alpha \rightarrow T_\alpha + i\Lambda_\alpha.$$  \hfill (3.6)

We expect that generally these symmetries will be broken to discrete shifts by quantum states coupling to $T_\alpha, N^a$. Further, observe that one has the symmetry $\pi : N^a \rightarrow -N^a$ due to the quadratic appearance of $N^a$ in (3.5). It is tempting to conjecture that this symmetry $\pi$ is preserved at the quantum level, and allows one to distinguish the $C_{21}$ scalars $N^a$ from the others. Such a symmetry can also potentially forbid curvature-squared couplings.

The main structure that we focus on in this section is the axion-curvature squared terms mentioned above that couple the pseudoscalar components of the fields $C_{sa}$ to the gravitational and gauge curvatures. The existence of such couplings is connected with the set of quantum string states in the theory that are magnetically charged under the axion fields. In general, each axion field obeys a discrete shift symmetry where the shift of the field lies in the lattice of possible axionic string charges. Note that, just as in six dimensions, the lattice of quantized string states should arise in four dimensions in any quantum theory of gravity containing axions under which the strings can carry charges, independent of the UV completion of the theory. Each of the fields of the type $C_{sa}$ contains a pseudoscalar axion with such a shift symmetry as its imaginary part, and in the next section we compute the couplings of these axions to curvature squared terms in the action. While chiral multiplets in general include pseudoscalar components as their imaginary parts, which may also act as axions under which string excitations of the 4D theory are magnetically charged, it is less clear how this works for the other types of scalar fields $C_{cs}, C_{21}$. There is no apparent axionic shift symmetry for generic complex structure moduli $h^{3,1}(\hat{X})$ in the F-theory construction using an elliptic fourfold. Nevertheless one can find couplings of the scalars $C_{cs}$ to certain $U(1)$-curvature squared terms as we discuss in (3.10). It would be very interesting to investigate the set of couplings for these scalars in more detail. It is possible that away from special limits in the F-theory complex structure moduli space, such as the weak string coupling, or the heterotic limit, all scalars $h^{3,1}(\hat{X})$ mix with other moduli and correct the curvature-squared couplings. For curvature-squared couplings involving the 7-brane field strength this was also found in [54]. Indeed, in the context of mirror symmetry for $\mathcal{N} = 2$ theories [79], complex structure and Kähler moduli are related through a duality symmetry, suggesting that generically both types of moduli may admit shift symmetries and engage in couplings to curvature-squared terms.

### 3.1.2 Vector spectrum and gauge kinetic functions

Vector fields in the 4D theory come from two sources. The first class of vector fields arises in complete analogy to the 6D compactifications. The nonabelian gauge symmetries arise from the codimension one singularities of the elliptic fibration of $X$ over the base $B_3$. Physically these singularities signal the presence of space-time filling 7-branes. The rank of the gauge group can be determined by resolving $X$ to $\hat{X}$. The total rank $r_v$ of the gauge
group is
\[ r_v = h^{1,1}(\hat{X}) - h^{1,1}(\mathcal{B}_3) - 1 \] (3.7)
as in (2.5). As discussed in section 2.1 this general expression also counts massless \( U(1) \) factors. These are obtained when the elliptic fibration \( X \) has more than one section.

Let us review the form of the gauge coupling function for a non-Abelian gauge group on a stack of 7-branes wrapped on divisors \( S_A \). This coupling can be computed by using an M-theory dual description [48], or from the 7-brane action at weak coupling as discussed below, and is given at leading order by
\[ f_A = \frac{1}{2} C^\alpha_A T_\alpha, \quad [S_A] = C^\alpha_A \omega_\alpha, \] (3.8)
where \( \omega_\alpha \) is a basis of two-forms of \( \mathcal{B}_3 \), and the \( T_\alpha \) have been given in (3.5). This expression for \( f_A \) is well-known for D7-branes [76]. From the weak coupling analysis, however, one expects additional classical corrections to \( f_A \). These can be induced by fluxes, or by a non-trivial curvature on the brane as we discuss below. In the F-theory context such corrections have not been studied in full detail. This is due to the fact that their M-theory origin is more involved, as recently shown in [54].

In contrast to 6D compactifications one finds in addition \( h^{2,1}(\mathcal{B}_3) \) \( U(1) \) vector fields that arise from expanding \( C_4 \) into harmonic three-forms of the base \( \mathcal{B}_3 \). The rank of this abelian part is denoted by
\[ r_{21} = h^{2,1}(\mathcal{B}_3), \] (3.9)
which is equal to the number of such \( U(1) \) factors. The gauge coupling functions for these \( r_{21} \) vectors are given at leading order by [48]
\[ \tau_{\kappa\lambda}(z) = \frac{i}{2} \left( \int_B \beta^\mu \wedge \bar{\psi}^\kappa \right)^{-1} \int_B \alpha_\lambda \wedge \bar{\psi}^\mu. \] (3.10)
Here \( (\alpha_\kappa, \beta^\kappa) \) is a real symplectic basis on \( \mathcal{B}_3 \), while \( \psi^\kappa \) is a basis of \( (2,1) \) forms on \( \mathcal{B}_3 \) varying with the complex structure moduli \( z^k \). In other words, at this leading order \( \tau_{\kappa\lambda} \) only depends on the complex structure moduli \( z^k \). The imaginary part of \( \tau_{\kappa\lambda} \) thus couples to \( F^\kappa \wedge F^\lambda \) inducing a coupling of type axion-curvature squared to the \( h^{2,1}(\mathcal{B}_3) \) Ramond-Ramond \( U(1) \) vectors. Similarly one expects subleading corrections to (3.8) depending on the complex structure moduli \( z^k \).

### 3.1.3 Fluxes, D3-brane tadpole and chiral spectrum

A key difference from the situation for 6D F-theory compactifications is the fact that 4D vacua allow for a non-trivial background flux. Including such fluxes in four-dimensional F-theory constructions is the subject of substantial current work [27]-[34]. In fact, such fluxes are often necessary for tadpole cancellation and have to be present in a consistent vacuum. This leads to an intriguing interplay of geometric data and flux data. It will be a far reaching task to unify both into a common framework. Here we make some basic observations that will be useful in the analysis below of the 4D effective action.
To begin with, we note that there are three types of background fluxes in F-theory: R-R and NS-NS three-form flux in the bulk, and two-form fluxes on the 7-branes. While an individual description of these fluxes can be difficult to integrate with the Weierstrass description of an F-theory model, there is a natural lift of these fluxes into a single type of four-form flux $G_4$ that can be interpreted as an actual four-form on a smooth geometry $\hat{X}$ in the dual M-theory compactification, where $G_4$ is the field strength of the M-theory three-form. A general $G_4$ induces a 4D superpotential as well as a D-term. The superpotential is given by $W(z) = \int G_4 \wedge \Omega$ \cite{80}, and depends holomorphically on the $h^{3,1}(\hat{X})$ complex structure moduli of $\hat{X}$. The large volume D-term depends on the Kähler moduli via the Kähler form on $\hat{X}$. It will be useful to introduce the the matrix

$$\Theta_{\Sigma \Lambda} = \int_{\hat{X}} \omega_{\Sigma} \wedge \omega_{\Lambda} \wedge G_4 . \quad (3.11)$$

where $\omega_{\Sigma}$ is a basis of two-forms of $H^2(\hat{X}, \mathbb{Z})$ on the resolved fourfold, including all new classes $\omega_{iB}$ obtained after resolution of gauge group $G_B$ singularities for non-Abelian 7-branes. Components of $\Theta_{\Sigma \Lambda}$ determine the D-terms. The D-terms arise from gaugings of the shift symmetries \cite{36} of the imaginary part of the scalars $T_\alpha$ given in \cite{35}. In the M-theory dual Coulomb branch description the gauge-invariant derivative is given by

$$DT_\alpha = dT_\alpha + i \Theta_{\alpha iB} A^{iB} \quad (3.12)$$

where $i_B$ labels the forms $\omega_{iB}$ arising from resolving the gauge group singularities for $G_B$. In the 4D F-theory compactification one has to replace $\Theta_{\alpha iB}$ with an adjoint valued matrix $\Theta_{\alpha B}$ and the invariant derivative takes the form

$$DT_\alpha = dT_\alpha + i \text{Tr}(\Theta_{\alpha B} A^B) \quad (3.13)$$

Note that $\Theta_{\alpha B}$ corresponds to a non-Abelian flux background on the $B$th 7-branes and thus will break the gauge group $G_B$.

Let us stress here, that both the superpotential as well as the D-terms give mass to some of the $C_{cs} + C_{sa}$ moduli. This complicates the identification of light states in the 4D effective theory. Since, however, the masses both from the superpotential and the D-terms are suppressed by a volume factor of higher power than for the masses of the KK-modes, one can identify light fields at large volume.

It is crucial to note that $G_4$ has to obey various constraints. First, it has to be quantized appropriately, since $G_4 + c_2(\hat{X})/2$ has to be an integral class \cite{81}. This condition has recently been analyzed systematically for F-theory geometries in \cite{82, 83}. Secondly, certain components of $\Theta_{\Sigma \Lambda}$ have to vanish in order that $G_4$ lifts to an F-theory flux and preserves 4D Poincaré invariance \cite{27, 31, 53}. Fluxes are crucial to induce a 4D chiral matter spectrum as recently studied in \cite{84, 85, 86, 30-34}. In general the components

$$\Theta_{iAJB} = \int_{\hat{X}} \omega_{iA} \wedge \omega_{jB} \wedge G_4 , \quad (3.14)$$

with $\omega_{iA}, \omega_{jB}$ resolving the gauge group singularities for $G_A, G_B$ on one or more 7-branes, can be non-zero. Physically these components of $\Theta_{\Sigma \Lambda}$ carry the information about the 4D chiral matter spectrum integrated out at one loop in the M-theory compactification \cite{83}. 

---
In 4D compactifications the fluxes and geometry are linked via the well-known global consistency condition ensuring cancellation of 3-brane tadpoles. In the M-theory language, this tadpole constraint is

\[ \frac{\chi(\hat{X})}{24} = \frac{1}{2} \int G_4 \wedge G_4 + N_3 , \]  

where \( \chi(\hat{X}) \) is the Euler character of the resolved Calabi-Yau fourfold. Here \( N_3 \) is the number of 3-branes, which are point-like objects in \( B_3 \). The number of independent components of \( G_4 \) depends upon the Hodge number \( h^{2,2}(X) \). A linear relation between the Hodge numbers on a Calabi-Yau fourfold \[ h^{2,2}(X) = 44 + 4h^{1,1}(X) + 2h^{2,1}(X) - 4h^{3,1}(X) \] shows that the Euler character can then be written as

\[ \chi(\hat{X}) = 6(8 + h^{1,1}(\hat{X}) + h^{3,1}(\hat{X}) - h^{2,1}(\hat{X})) \]

\[ = 6(9 + C_{sa} + C_{cs} + r_v - (C_{21} + r_{21})) . \]  

This links the light spectrum with the fluxes via (3.15). Note, however, that fluxes also make it hard to identify massless moduli since, as mentioned above, the flux-induced D-term will generate a potential for the fields introduced in section 3.1.1. In general, there will be many discrete choices of flux associated with a given geometric F-theory background. This can be either achieved by using \( G_4 \) fluxes or introducing a number \( N_3 \) of 3-branes.

Note that the presence of D3-branes leads to additional light degrees of freedom corresponding to the moduli of the D3-branes. For \( N_3 \) separated D3-branes they shift the number of fields

\[ \Delta C_{cs} = 3N_3 , \quad \Delta r_v = N_3 , \]  

where the scalars arise from the three complex positions of the D3-brane in \( B_3 \). Note that each D3-brane comes with an additional four-dimensional \( U(1) \) gauge symmetry which is generically unbroken. This leads to a link \[ (3.18) \] of the number of \( U(1) \)'s with the number of massless deformations. This should be contrasted with the case of 7-branes, or even D7-branes, where the brane \( U(1) \)'s can be massive at the Kaluza-Klein scale due to a geometric St{"u}ckelberg term \[ (76, 52) \]. This is true both in 4D and 6D since in both cases the 7-brane embedding into the base \( B_3 \) can have a non-trivial topology. We do not work with theories here that have separate D3-brane degrees of freedom.

### 3.2 Couplings in the 4D supergravity action from F-theory

In this section we discuss some specific terms in the action of the 4D \( \mathcal{N} = 1 \) effective supergravity theories that arise by compactifying F-theory on the elliptically fibered fourfold \( X \). As in the 6D discussion \[ (2.4) \] we focus on terms in the action that contain information about the topological data of the compactification manifold \( X \). It is important to remember that 4\( D, \mathcal{N} = 1 \) is much less protected against corrections than its 6D counterpart due to the smaller number of supersymmetries and space-time dimensions. It is therefore more
challenging to extract the geometric data from a low-energy 4D supergravity theory than in 6D. As discussed above, this connection can be made most clearly in the large-volume limit. We find it useful in this discussion to frame part of the analysis involving scalar-axion fields in terms of a dual 4D picture where these fields are analogs of 6D tensor multiplets.

3.2.1 Topological couplings in the F-theory effective action

We now wish to focus on particular terms in the $\mathcal{N} = 1$ effective action involving the scalar-axion fields $C_{sa}$. In particular, we consider terms in the theory that are analogous to the couplings (2.14), and have the form

$$S^{(4)}_{ax} = \frac{1}{8} \int \frac{1}{2} a^A \rho_A \tr R \wedge R + \frac{2}{\lambda_A} b^A_A \rho_A \tr F^A \wedge F^A.$$  (3.19)

In these couplings, $\rho_A$ are the axions appearing as the imaginary parts of the $C_{sa}$ scalar fields. The index $A$ is taken to be 0 for the axion associated with the axiodilaton at weak coupling, and $\alpha = 1, \ldots, h^{1,1}(B_3)$ are the remaining $C_{sa}$ fields. As in 6D, $a^\alpha$ describes the canonical class of the F-theory base, while $b^\alpha_A$ contains information about the divisor classes on which the nonabelian gauge group factors are wrapped. Just as in six dimensions, the terms (3.19) can play a role in anomaly cancellation through the generalized Green-Schwarz mechanism [49, 50, 51]. These terms appear, however, independent of the need for anomaly cancellation, and are present even in theories without unbroken gauge groups. The appearance in these terms of the canonical class of the F-theory base and the classes carrying the 7-branes associated with gauge groups plays a key role in the correspondence between the low-energy theory and the F-theory construction, just as in six dimensions. The 4D story is more complicated, however, due to the existence of extra axions that do not tie directly into the divisor geometry in F-theory. For example, at weak string coupling the dimensionally reduced 10D axiodilaton will appear in the curvature-squared couplings, despite the fact that it does not admit a two-form interpretation in 6D. We discuss such additional axions in section 3.2.2. As we describe in more detail below in section 3.2.3, the couplings (3.19) admit a dual description when the axions are dualized to two-forms and lead to corrected field strengths of the form (2.10).

We now describe how the terms appearing in the action (3.19) are determined in an F-theory reduction on a Calabi-Yau fourfold. We concentrate first on the terms involving the $h^{1,1}(B_3)$ axions in $T_\alpha$. To determine $b^\alpha_A$ we note that the $F^2$ coupling is given by the imaginary part of the gauge coupling function $f_A$. For a 7-brane these can be extracted at weak string coupling using an argument analogous to (2.18), as we show below. Alternatively, as discussed above, one can use the duality between M-theory and F-theory to derive $f_A$ [48]. Either approach gives a leading gauge coupling function (3.8) that is linear in $T_\alpha$. Comparing this with (3.19) gives

$$b^\alpha_A = C^\alpha_A, \quad [S_A] = C^\alpha_A \omega_\alpha,$$  (3.20)

where $S_A$ are the divisors in $B_3$ wrapped by the 7-branes. The weak-coupling 7-brane analysis below also describes the higher curvature terms $R^2$ with coupling $a^\alpha$. The upshot
is that, just as in 6D, \(a^\alpha\) corresponds to the canonical class of the F-theory base manifold through
\[
a^\alpha = K^\alpha, \quad c_1(B_3) = -K^\alpha \omega_\alpha.
\]

One expects that the expression for \(a^\alpha\) can also be determined via an M-theory reduction as in the 6D/5D reduction \[8\]. This is more involved, however, than in the 6D case, due to the fact that the \(R \wedge R\) term couples to the axion \(\rho_\alpha\) rather then the two-form \(B_3^\alpha\) that is the 4D dual of \(\rho_\alpha\).

We now give the weak coupling 7-brane analysis showing that the identifications (3.20) and (3.21) are indeed correct, and fix the numerical factors in the latter coupling. This analysis proceeds in analogy to the similar analysis for 6D compactifications. We consider the Chern-Simons couplings of the D7-branes and O7-planes. The branes will admit SU\((N^\hat{A}_{D7})\) gauge groups, such that \(\lambda^\hat{A} = 1\) in (3.19). We focus here particularly on the terms coupling to the R-R four-form \(C_4\) which admits the expansion
\[
C_4 = \rho_\alpha \tilde{\omega}_\alpha,
\]
where \(\tilde{\omega}_\alpha\) are the four-forms on \(B_3\) dual to \(\omega_\alpha\). Inserting this expansion into the Chern-Simons actions and integrating over the compact directions one obtains
\[
S^{CS}_{D7}(R, F) = -\frac{1}{2} \int_{M^{3,1}} C^\alpha A^\rho_\alpha \left( \frac{1}{96} \text{tr}(R \wedge R) N^\hat{A}_{D7} + \frac{1}{2} \text{tr}(F^\hat{A} \wedge F^\hat{A}) \right) + \ldots ,
\]
\[
S^{CS}_{O7}(R) = +2 \int_{M^{3,1}} \tilde{C}^\alpha \rho_\alpha \left( -\frac{1}{192} \text{tr}(R \wedge R) \right) + \ldots ,
\]
where \(C^\alpha \) is the restriction to the \(\hat{A}\)th D7-brane stack, and \(N^\hat{A}_{D7}\) is the number of D7-branes on the \(\hat{A}\)th stack. Here \(\hat{A}\) will run only over indices labeling the D7-branes. The restriction to the O7-plane is denoted by \(\tilde{C}^\alpha\). Note that in F-theory, both \(C^\alpha \) and \(\tilde{C}^\alpha\) are combined into \(C^\alpha \). The fact that in F-theory the O7-plane is split into two 7-branes is captured by the relative factor of 2 when comparing the \(\text{tr}(R^2)\) for the D7-brane and O7-planes after using \(\mu_{O7} = -4 \mu_{D7}\). From the \(F \wedge F\) terms we confirm (3.20). We are now also in a position to show (3.21). At weak coupling the discriminant in F-theory splits through
\[
-12[K] = [\Delta] = 2[S_{O7}] + N^\hat{A}_{D7}[S^\hat{A}] = (2\tilde{C}^\alpha + N^\hat{A}_{D7} C^\alpha) \omega_\alpha ,
\]
where the first equality is the Kodaira constraint (2.3). Hence, one infers 
\(12K^\alpha = -(2\tilde{C}^\alpha + N^\hat{A}_{D7} C^\alpha)\), which inserted into the sum of the actions (3.22) yields the identification (3.21) for \(f_A\).

### 3.2.2 An additional axion

As discussed above, we have included an additional scalar-axion field contributing to \(C_{sa}\) thereby treating it on a similar footing to the \(h^{1,1}(B_3)\) scalars \(T_\alpha\); we denote this field by \(T_0\). This can be motivated by the fact that there are two limits in the complex structure moduli space of \(\hat{X}\), where one scalar-axion field is singled out.

Firstly, at weak string-coupling \(T_0\) corresponds to the dimensionally reduced axiodilation \(\tau_{\text{IB}}\) setting
\[
T_0 = -i\tau_{\text{IB}} = e^{-\phi} - iC_0 ,
\]
\[\text{(3.24)}\]
In this weak coupling limit $\text{Re} T_0$ is large, which is the analog to the $\text{Re} T_\alpha$ in the large-volume limit of $\mathcal{B}_3$. Using an expansion of the Chern-Simons actions of D7-branes and O7-branes one can derive the couplings $a^0, b^0_A$ of $\text{Im} T_0$ to $R^2$ and $(F^A)^2$. However, away from the weak coupling limit in a generic F-theory compactification the axiodilaton $\tau_{\mathcal{H}B_3}$ is no longer a well-defined 4D field, since it admits $SL(2,\mathbb{Z})$ monodromies around the 7-branes. The appropriate coordinates are now the $h^{3,1}(\hat{X})$ complex structure deformations that contain this weak-coupling degree of freedom. We thus expect couplings of the form

$$\frac{1}{2} \tilde{a}(z) \text{tr} R \wedge R + \frac{2}{\lambda_A} \tilde{b}_A(z) \text{tr} F^A \wedge F^A,$$

(3.25)

where $\tilde{a}(z), \tilde{b}_A(z)$ admit appropriate monodromy properties for a given Calabi-Yau geometry. At weak coupling the functions $\tilde{a}, \tilde{b}_A$ can be expanded into a term linear in $\text{Im} T_0$ with exponentially suppressed corrections. We do not have a clear derivation of these terms in a general F-theory setup. In principle it should be possible to compute these couplings by gluing together contributions from all 7-brane actions focusing on the axiodilaton coupling. Note that the described situation is very similar to the complex structure dependent couplings (3.10) for the Ramond-Ramond $U(1)$ vectors, and hence to the analogous $\mathcal{N} = 2$ story for Calabi-Yau threefold compactifications. $\tilde{a}(z), \tilde{b}_A(z)$ are expected to be complicated functions of the complex structure moduli that depend on the point in moduli space and have some characteristic expansion with coefficients determined by the topological data of the base $\mathcal{B}_3$ and gauge bundles on the 7-branes.

A second limit in which the couplings $\tilde{a}(z), \tilde{b}_A(z)$ can be expanded into a term linear in a single axion with exponentially suppressed corrections can be accessed for F-theory geometries with a heterotic dual. We will make this precise in Section 3.4, and derive these couplings through the duality to heterotic theory.

Note that, as mentioned in the introduction to this section, giving up the the large-volume limit for $\mathcal{B}_3$ one expects to also lose the linearity in the $T_\alpha$, and all axions may mix non-trivially. It is not clear from the F-theory point of view how these couplings can be computed, however. We leave further investigation of this question to future work.

### 3.2.3 Remarks on the triple intersection numbers and identification of F-theory geometry

We have seen so far how the Hodge numbers, canonical class, and 7-brane divisors of an F-theory compactification are encoded in the corresponding 4D supergravity theory. As in 6D, a key part of the topological structure of the F-theory compactification geometry lies in the intersection form of the F-theory base. In 4D, this intersection form is the triple intersection product $\kappa_{\alpha\beta\gamma}$. In contrast to the 6D story, however, the triple intersection product is not immediately visible in a general 4D $\mathcal{N} = 1$ supergravity theory. Due to perturbative and non-perturbative corrections, the kinetic terms of the scalars are not protected. Only at leading order in the large-volume limit in an F-theory compactification is the triple product visible.
The situation is relatively clear when the scalars $T_\alpha$ can be replaced by linear multiplets containing as bosonic components a real scalar $L^\alpha$, which is the Legendre dual $\text{Re} T_\alpha$ with respect to the Kähler potential, and a two-form $B^\alpha$ dual to $\text{Im} T_\alpha$. This can be done if $\text{Im} T_\alpha$ possesses a shift symmetry (3.6). Then, as in 6D in (2.10), the field-strength of $B^\alpha$ is given by

$$H_3^\alpha = d B_2^\alpha + \frac{1}{2} a^\alpha w_{\text{CS}}(R) + 2 \frac{b^\alpha}{\lambda A} w_A w_{\text{CS}}(F),$$

(3.26)

For small $L^\alpha$ one can then expand the metric $\tilde{G}_{\alpha\beta}$ for $L^\alpha$ and $H_3^\alpha$ as

$$\tilde{G}_{\alpha\beta} \propto \kappa_{\alpha\beta\gamma} L^\gamma + \ldots ,$$

(3.27)

which is the analogue to (2.23) in the 6D action. At large volume the 4D couplings then contain the triple intersections $\kappa_{\alpha\beta\gamma}$ with the third index contracted with $L^\alpha$. It remains to be shown how much of this structure survives quantum corrections. In particular, away from the large-volume limit where $L^\alpha$ can be large, the structure (3.27) is not expected to be preserved. Nonetheless, in the large-volume limit the triple intersection coefficients are contained in the leading order term in the expansion (3.27).

We have thus outlined a way in which much of the topological structure of the F-theory base $B_3$ and fourfold $X$ can be identified from the 4D supergravity action, at least in the large-volume limit where the spectrum remains light and corrections are small. We have identified the Hodge numbers, canonical class, and intersection form of the F-theory base. From Wall’s theorem [91], the homotopy type of a compact complex 3-manifold can be identified from the Hodge numbers, triple intersection form, and first Pontryagin class $p_1(B_3)$. In principle, information about $p_1$ will be contained in higher-order terms in the string action. In particular, at weak string coupling one can use the higher-curvature corrections in the D7-brane Chern-Simons action to find couplings of the axiodilaton to the $p_1$ restricted to the branes. A more complete understanding of the allowed couplings away from special limits in the F-theory complex structure field space might thus yield the desired information. In principle then, enough data to reconstruct the topology of the F-theory compactification space is contained in the structure of the 4D supergravity theory at large volume. While providing additional information, the couplings of additional axions can also complicate the reconstruction of the geometry. To reconstruct the topology of the F-theory base it is necessary to know which axions correspond to the complex structure moduli. While in some cases this is clear in the large-volume limit, in general this may require further information. Knowing the topology of the F-theory base is also not sufficient to completely determine the geometry. It is also necessary to know the complex structure on the base to fully identify the theory. As in 6D, the quantum spectrum of supersymmetric charged string solitons encodes the Mori cone of the threefold base in a 4D compactification; this is discussed further in Section 4.2.4. In the specific examples we discuss below for heterotic/F-theory duality, the data given in the 4D theory, coupled with an identification of axions, is sufficient to uniquely determine the F-theory geometry.
3.3 4D F-theory examples with \( \mathbb{P}^1 \) fibered base

We now discuss a general class of examples of F-theory compactifications, where the base \( B_3 \) of the Calabi-Yau fourfold is a \( \mathbb{P}^1 \) fibration over some complex surface \( B_2 \). This is the class of 4D F-theory models that admit a duality to heterotic compactifications \([92, 93, 94, 95]\). We describe the general topological structure of these fibrations in section 3.3.1, and discuss explicit examples with \( B_2 = \mathbb{P}^2 \) in section 3.3.2. Duality to the heterotic string is described in Section 3.4.

3.3.1 On the geometry of general \( \mathbb{P}^1 \) fibrations

We begin by reviewing some generalities regarding \( \mathbb{P}^1 \) fibered bases \( B_3 \), following \([92, 95]\). We consider \( \mathbb{P}^1 \) fibrations with a section \( \Sigma \). Such fibrations can be characterized by two two-forms \( r, t \) that are obtained as follows. First, consider the sum \( \mathcal{O} \oplus \mathcal{L} \) of two complex line bundles \( \mathcal{O} \) and \( \mathcal{L} \). The base \( B_3 \) is the projectivization of this vector bundle, \( i.e. B_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \). There are two distinguished two-forms \( r, t \) on \( B_3 \), which are given by the first Chern classes

\[
\begin{align*}
    r &= c_1(\mathcal{O}(1)) = [\Sigma] , \\
    t &= c_1(\mathcal{L}) .
\end{align*}
\]

(3.28)

Here \( \mathcal{O}(1) \) is a line bundle on \( B_3 \) that restricts to the typical line bundle of each \( \mathbb{P}^1 \) fiber of \( B_3 \), and \( r \) restricts to the \( \mathbb{P}^1 \) hyperplane class. The classes \( r, t \) satisfy

\[
r(r + t) = 0 .
\]

(3.29)

In the evaluation of the low-energy couplings we need the characteristic classes \(^7\)

\[
\begin{align*}
    c_1(B_3) &= c_1(B_2) + (2r + t) , \\
    c_2(B_3) &= c_2(B_2) + c_1(B_2)(2r + t) ,
\end{align*}
\]

(3.30)

as can be shown by using the adjunction formulas.

In order to perform the F-theory reduction we introduce the basis of two-forms \( \omega_\alpha = (\omega_f, \omega_i) \) on \( B_3 \), and expand \( C_4 = B_2^2 \wedge \omega_\alpha \). The four-dimensional two-forms \( B_2^2 \) are dual to the axions \( \rho_\alpha \). The internal two-forms \( \omega_i \) are pulled back from two-forms of \( B_2 \). In summary we introduce the basis

\[
\omega_f = r + \frac{t}{2} , \quad \omega_i .
\]

(3.31)

Using (3.29) one infers the triple intersections

\[
\kappa_{ijk} = 0 , \quad \kappa_{fij} = \kappa_{ij} , \quad \kappa_{fji} = 0 , \quad \kappa_{fff} = \frac{1}{4} t^i t^j \kappa_{ij} .
\]

(3.32)

with intersection form \( \kappa_{ij} = \int_{B_2} \omega_i \wedge \omega_j \), and \( t^i \) appearing in the expansion \( t = t^i \omega_i \). This is the higher-dimensional analog of (2.24), (2.25). In this basis we use (3.30) to determine the vector \( K^\alpha \) as

\[
(a^\alpha_F) \equiv (K^\alpha) = (-2, K^i) ,
\]

(3.33)

\(^7\)Note that \( \int_{B_3} c_1(B_3)c_2(B_3) = 2 \int_{B_2} c_1(B_2)c_2(B_2) = 24 \), by using the fact that \( \chi_0(B_3) = 1 \).
where \( K^i \) are the coefficients of the canonical class of \( B_2 \) in the basis \( \omega_i \). In general, 7-branes carrying gauge groups can be wrapped on an arbitrary effective irreducible four-cycle in \( B_3 \) with Poincaré-dual class

\[
[S] = p\omega_f + \sum_i q_i\omega_i,
\]

similar to the 6D case (2.27).

### 3.3.2 Examples: \( \mathbb{P}^1 \) fibrations over \( \mathbb{P}^2 \)

A simple example for \( \mathbb{P}^1 \) fibered base spaces are threefolds that are described by \( \mathbb{P}^1 \) bundles over \( B_2 = \mathbb{P}^2 \). These manifolds, which we denote by \( \tilde{F}_k \) are close relatives of the Hirzebruch surfaces \( F_m \). They are studied in the physics context, for example, in [88, 109, 96]. These models have a simple toric description\(^8\), but can be equally specified using the construction of section 3.3.1. We denote by \( H \) the hyperplane class of the two-fold base \( \mathbb{P}^2 \) pulled back to \( B_3 \), and \( \Sigma \) the class of a section corresponding to \( r \). Furthermore, we identify \( t \) specifying the \( \mathbb{P}^1 \) bundle as

\[
t = k[H].
\]

Clearly, using (3.30) and the fact that \( c_1(\mathbb{P}^2) = 3[H] \) and \( c_2(\mathbb{P}^2) = 3[H]^2 \) we find

\[
c_1(\tilde{F}_k) = 2[\Sigma] + (3 + k)[H], \quad \quad c_2(\tilde{F}_k) = 6[\Sigma \cdot H] + (3 + 3k)[H]^2.
\]

The triple intersections are simply given by

\[
\Sigma H^2 = 1, \quad H^3 = 0, \quad \Sigma^2 H = -k, \quad \Sigma^3 = k^2.
\]

The first condition is inherited from the base \( B_2 = \mathbb{P}^2 \), while the second corresponds to the fact that three elements of the base cannot intersect for a fibration. The last two are a trivial consequence of the general fact that \( r(r + t) = 0 \) for \( \tilde{F}_k \) implies that \( \Sigma^2 = -k\Sigma \cdot H \).

The threefolds \( \tilde{F}_k, k = 0, 1, 2, 3 \) are bases for generic elliptically fibered Weierstrass models without codimension one singularities that would impose a gauge group on the 4D theory. For higher values of \( k \), the divisor \( \Sigma \) is rigid and \( f, g \) and \( \Delta \) must vanish to a degree that mandates the appearance of a nonabelian gauge group over that divisor. For example, \( \tilde{F}_4 \) carries a minimal gauge group \( SU(2) \), and \( \tilde{F}_{18} \) carries an \( E_8 \) over \( \Sigma \). \( \tilde{F}_k \) cannot be a good F-theory base for \( k > 18 \) [96]. F-theory compactifications on the threefolds \( \tilde{F}_k \) can be dual to heterotic compactifications on Calabi-Yau threefolds that are elliptically fibered over \( \mathbb{P}^2 \). We can use the analysis of Section 3.2.1 to read off which divisor classes in \( \tilde{F}_k \) must support the gauge group, and hence identify which of the surfaces \( \tilde{F}_k \) is needed for the F-theory dual from knowledge of the bundle structure on the heterotic side. We discuss the 4D heterotic models next.

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\(^8\)The toric 1D cones for \( \tilde{F}_k \) are generated by \( e_1, e_2, e_3, -e_3 \) and \( -e_1 - e_2 - k e_3 \), where \( e_i \) are the unit vectors of \( \mathbb{R}^3 \).
3.4 Four-dimensional heterotic models and heterotic/F-theory duality

The Green-Schwarz two-form–curvature squared terms in 6D theories of the form $BF^2, BR^2$ provide an illuminating connection between heterotic and F-theory models, as discussed in section 2.4. A similar relationship holds in four dimensions, which we describe in this section. The relationship between 4D heterotic and F-theory compactifications was studied in detail in the seminal work by Friedman, Morgan, and Witten [92]. They showed that many heterotic bundles admit a “stable degeneration limit” in which duality to F-theory can be clearly understood. This work, and subsequent developments following [94] have led to an extensive study of this duality; for a review of some of this work see [95]. As in the 6D heterotic/F-theory duality, the 4D duality relates the heterotic theory on a Calabi-Yau manifold that is elliptically fibered over a base $\mathcal{B}_2$ to an F-theory model on a $\mathbb{P}^1$ fibration over $\mathcal{B}_2$. Friedman, Morgan, and Witten relate the bundle structure on the heterotic side to the twisting of the $\mathbb{P}^1$ fibration on the F-theory side for $E_8 \times E_8$ heterotic theory. We find here that this identification of bundle structure with twisting follows naturally from the structure of the axion–curvature-squared terms in the 4D action, and that in many cases the F-theory geometry dual to a given heterotic model is uniquely determined by the structure of these terms. The locus of the 7-branes carrying the gauge group action on the F-theory side is also uniquely determined by the axion–curvature-squared terms arising from a given bundle structure on the heterotic side. These considerations are independent of the type of bundle construction, and give a topological picture of heterotic/F-theory duality that is valid for the $SO(32)$ theory as well as for the $E_8 \times E_8$ theory.

As described in Section 2.2.2, for a 6D F-theory compactification the coefficients of the $BR^2$ term are components of a vector in the string charge lattice characterizing the canonical class of the F-theory base, while the coefficients of the $BF^2$ term characterize the divisor class on which the 7-branes giving each nonabelian gauge group factor are wrapped. The corresponding coefficients can be computed directly in the heterotic theory, as described in Section 2.4, by reduction of the 10D $H^2$ and Green-Schwarz terms. For any 6D supergravity that has dual descriptions in terms of heterotic and F-theory compactifications, this correspondence in the 6D Green-Schwarz terms provides a direct topological description of the duality. In particular, given any heterotic compactification with an F-theory dual, by computing the 6D Green-Schwarz terms we can read off the canonical class of the F-theory base and the divisor classes on which the 7-branes are wrapped in the dual F-theory model. Essentially the same story holds in four dimensions.

In this section we compute the axion–curvature squared terms of the form $\rho R^2$ and $\rho F^2$ for a general heterotic compactification on an elliptically fibered threefold with section. This not only serves as a check on the structure of these terms as described for F-theory models, but also provides a direct means for identifying the structure of the dual F-theory model. In Section 3.4.1 we describe the general class of heterotic models and compute the axion–curvature-squared terms in this general context. Section 3.4.2 describes some explicit examples of this approach to understanding heterotic/F-theory duality.
3.4.1 General heterotic models with F-theory duals

We consider a general 4D $\mathcal{N} = 1$ supergravity model that has both a weakly coupled large-volume heterotic description and a large-volume F-theory description in appropriate regimes. On the heterotic side, the 10D $SO(32)$ or $E_8 \times E_8$ heterotic theory is compactified on a Calabi-Yau threefold $Z_3$ that has the form of an elliptic fibration over a base manifold $\mathcal{B}_2$ that is a complex surface. We assume that the elliptic fibration has one (but not more than one) section, so that $h^{1,1}(Z_3) = h^{1,1}(\mathcal{B}_2) + 1$. On the F-theory side, the compactification manifold is an elliptic fibration over a complex threefold $\mathcal{B}_3$ that is a $\mathbb{P}^1$ fibration over $\mathcal{B}_2$.

The axions on the heterotic side consist of an axion $\chi_0$ arising from wrapping the 10D six-form $\hat{B}_6$ (dual to the two-form $\hat{B}$) on the compactification space $Z_3$, and $h^{1,1}(Z_3)$ axions $\chi^I = (\chi^i, \chi_f)$ from wrapping the 10D two-form $\hat{B}$ on the two-cycles of $Z_3$. One thus expands

$$\hat{B} = b_2 + \chi^I \omega_I = b_2 + \chi_f \omega_B + \sum_i \chi^i \omega_i, \quad (3.38)$$

where $\omega_B$ is the two-form Poincaré dual to the base $\mathcal{B}_2$ of $Z_3$, and $\omega_i$ are dual to the two-cycles in $H_2(\mathcal{B}_2)$. The four-dimensional two-form $b_2$ is dual to $\chi_0$. We can now follow essentially the same analysis as in Section 2.4 to determine the 4D axion–curvature squared couplings from this compactification.

Before doing that, first note that the global consistency conditions arising from the Bianchi identity (2.29) now split into $h^{1,1}(Z_3)$ conditions

$$c_I - \lambda_I = 0, \quad (3.39)$$

where we have defined

$$c_I = \int_{Z_3} \omega_I \wedge \text{tr} R^2 = \int_{Z_3} \omega_I \wedge c_2(Z_3), \quad \lambda_I = \frac{2}{\lambda} \int_{Z_3} \omega_I \wedge \text{Tr} F^2, \quad (3.40)$$

with $\lambda = 2$ for $SO(32)$, and $\lambda = 60$ for $E_8 \times E_8$ as above. For an elliptically fibered threefold $Z_3$ we can determine $c_I$ more explicitly, by using [12]

$$c_2(Z_3) = 11c_1(\mathcal{B}_2)^2 + c_2(\mathcal{B}_2) + 12c_1(\mathcal{B}_2) \wedge \omega_B, \quad (3.41)$$

which gives $c_2(Z_3)$ in terms of classes of $\mathcal{B}_2$, and we have suppressed the pullback to $Z_3$. Evaluated on a basis one finds

$$c_i = -12 \kappa_{ij} K^j, \quad (3.42)$$

$$c_f = \int_{\mathcal{B}_2} c_2(Z_3) = \int_{\mathcal{B}_2} c_2(\mathcal{B}_2) - c_1(\mathcal{B}_2)^2 = -8 + 2h^{1,1}(\mathcal{B}_2),$$

where $\kappa_{ij}$ is the intersection form of the $\omega_i$ on $\mathcal{B}_2$, and $K$ is again the canonical class of the base $\mathcal{B}_2$.

We can now determine the four-dimensional couplings of the axions. The coupling of $\chi_0$ to $F^2$ and $R^2$ comes from the kinetic term of $\hat{H}$ in ten dimensions by using $*\hat{H} = \hat{H}_7$. 


It is given by
\[ \chi_0 \left( \text{tr} R^2 - \frac{2}{\lambda} \text{tr} F^2 \right) . \] (3.43)
We can then analyze the contributions from \( B \wedge \hat{X}_8 \) separately in the \( SO(32) \) and \( E_8 \times E_8 \) theories as in six dimensions; the algebra follows in a practically identical fashion in both cases to the six-dimensional analysis.

For the \( SO(32) \) theory, the couplings are
\[ S^{(4)}_{SO(32)} = \frac{1}{2} \int \left( \chi_0 + \frac{1}{24} \chi^j c_f \right) \text{tr} R^2 - \left( \chi_0 - \frac{1}{12} \chi^j c_f \right) \text{tr} F^2 \] (3.44)
\[ = \frac{1}{2} \int \left( \chi_0 - \frac{1}{2} \chi^i \kappa_{ij} K^j + \frac{1}{24} \chi_f c_f \right) \text{tr} R^2 \]
\[ - \left( \chi_0 + \chi^i \kappa_{ij} K^j - \frac{1}{12} \chi_f c_f \right) \text{tr} F^2 . \]

In order to determine the vectors \( a \) and \( b \) we compare (3.44) with the general form (3.19). We first define
\[ \tilde{\chi}_0 = 8 \chi_0 , \quad \tilde{\chi}_i = 8 \kappa_{ij} \chi^j , \quad \tilde{\chi}_f = 8 \chi_f . \] (3.45)
In the basis \( (\tilde{\chi}_0, \tilde{\chi}_i, \tilde{\chi}_f) \) we read off, using \( \lambda = 2 \) for \( SO(32) \), vectors \(^9\)
\[ a = \left( -2, K^i, -\frac{1}{12} c_f \right) , \quad b = \left( 1, K^i, -\frac{1}{12} c_f \right) , \] (3.46)
with \( c_f = -8 + 2 h^{1,1}(B_2) \) as shown in (3.42).

Let us now turn to the discussion of the \( E_8 \times E_8 \) theory. In this case we have to specify two bundles \( V_1 \oplus V_2 \). We introduce the general split of the curvature four-forms
\[ \frac{1}{30} \text{tr} F^2_i = \eta_i \wedge \omega_B + \zeta_i , \] (3.47)
where \( \eta_i \) is a two-form, and \( \zeta_i \) is a four-form inherited from \( B_2 \). The Bianchi identity (2.23) implies in cohomology
\[ \text{tr} F^2_i + \text{tr} F^2_j = 30 \text{tr} R^2 . \] (3.48)
Using the second Chern class of \( Z_3 \) as given in (3.41) we have
\[ \eta_1 + \eta_2 = 12 c_1(B_2) , \] (3.49)
\[ \zeta_1 + \zeta_2 = 11 c_1(B_2)^2 + c_2(B_2) \equiv C_2 . \] (3.50)
To satisfy these two conditions we can make the general Ansatz
\[ \eta_1 = 6 c_1(B_2) + \tilde{\ell} , \quad \eta_2 = 6 c_1(B_2) - \tilde{\ell} , \] (3.51)
\[ \zeta_1 = \frac{1}{2} C_2 + \Phi , \quad \zeta_2 = \frac{1}{2} C_2 - \Phi . \]
where \( \tilde{\ell}, \Phi \) are a two-form and a four-form inherited from \( B_2 \).

\(^9\)Note that we can formally obtain the 6D result (2.38) for a twofold base by setting \( K_{p1} = -2 \) and dropping the last entry in (3.46).
For the $E_8 \times E_8$ theory, the couplings are

$$S_{E_8}^{(4)} = \frac{1}{2} \int (\chi_0 + \frac{1}{12} \chi^I c_I) \text{tr} R^2 - (\chi_0 - \frac{1}{12} \chi^I \tilde{c}_I) \text{tr} F_1^2 - (\chi_0 + \frac{1}{12} \chi^I \tilde{c}_I) \text{tr} F_2^2 , \quad (3.52)$$

where we can treat the vector of axions $\chi^I = (\chi^i, \chi_f)$. In this expression we have defined

$$\tilde{c}_I = \int_{Z_3} \omega_I \wedge (i \wedge \omega_B + \Phi) = (-1)^i \int_{Z_3} \omega_I \wedge \left( \frac{1}{2} \text{tr} R^2 - \frac{1}{30} \text{tr} F_i \right) . \quad (3.53)$$

where $i = 1, 2$, and the second identity is a trivial consequence of (3.47), (3.49) and (3.51). We can evaluate $\tilde{c}_I$ for the basis $\omega_i, \omega_B$ to find

$$\tilde{c}_i = \kappa_{ij} \tilde{t}_j , \quad \tilde{c}_f = \int_{B_2} \Phi - \tilde{t}^2 . \quad (3.54)$$

Using these expressions and comparing (3.52) with the general action (3.19) and $\lambda = 60$, we find in the basis $(\tilde{\chi}_0, \tilde{\chi}_i, \tilde{\chi}_f)$ defined in (3.45) the vectors

$$a = \left( -2, K^i, -\frac{1}{12} c_f \right) , \quad b_1 = \left( 1, -\frac{1}{2} \tilde{t}_i, -\frac{1}{12} \tilde{c}_f \right) , \quad b_2 = \left( 1, \frac{1}{2} \tilde{t}_i, \frac{1}{12} \tilde{c}_f \right) . \quad (3.55)$$

From the results (3.44) and (3.52) we can directly read off topological information about the dual F-theory model. The F-theory axions $\rho_0, \rho_\alpha$ must be related to the heterotic axions through

$$\rho_0 \leftrightarrow \tilde{\chi}_f \quad (3.56)$$

$$\rho_b \leftrightarrow \tilde{\chi}_0 \quad (3.57)$$

$$\rho_i \leftrightarrow \tilde{\chi}_i , \quad (3.58)$$

where $\rho_0$ is the F-theory axion from the 10D IIB axiodilaton, $\rho_b$ comes from $C_4$ integrated over the base $B_2$, and $\rho_i$ come from $C_4$ integrated over four-cycles in $B_3$ obtained from curves in $B_2$. Comparing the expressions for the vector $a$ from (3.55) and (3.46) to (3.33), we see that this identification of axions gives a clear match between the canonical class of the bases $B_2$ used in the heterotic and F-theory constructions.

Considering the axion–$F \wedge F$ terms, comparing the vectors $b$ from (3.53) and (3.46) to (3.34) we can directly read off the divisor classes on which the 7-branes supporting the gauge group factors are wrapped on the F-theory side from the information about the heterotic bundle, just as in 6D. From the $\chi_0$ terms we see that every brane wraps the base cycle $\Sigma$ on the F-theory side precisely once, again as in the 6D story. While many F-theory models could be constructed on $B_3$ with gauge groups wrapping other cycles that do not wrap the base once, these models will have no perturbative heterotic dual and require the introduction of heterotic five-branes. For the $SO(32)$ theory the brane also wraps the classes on the base with twice the multiplicity of the canonical divisor $K$. For $E_8 \times E_8$ theories, the divisor classes on which the branes giving the gauge group factors are
wrapped depends upon the class of the difference $\tilde{t}$ appearing in the splitting of the bundle into two components. The fact that these divisors must be effective and irreducible places constraints on the line bundle $t$ describing the $\mathbb{P}^1$ bundle over $\mathcal{B}_2$. This leads naturally to the identification

$$t^i \leftrightarrow \tilde{t}^i, \quad (3.59)$$

corresponding to the association between the heterotic bundle and F-theory $\mathbb{P}^1$ fibration found in [92]. In some situations this identification is the unique possibility that satisfies the constraint on the divisor classes carrying the gauge group factors. We give below a simple class of examples where this identification is uniquely determined, in analogy with the 6D heterotic/F-theory duality story.

It is interesting to note that with the identification (3.59) and the heterotic axion–curvature-squared terms (3.46), the heterotic $SO(32)$ theory is always dual to F-theory on the same space that carries the dual to the $E_8 \times E_8$ theory where the decomposition (3.51) is

$$\eta_1 = 8c_1(\mathcal{B}_2), \quad \eta_2 = 4c_1(\mathcal{B}_2). \quad (3.60)$$

This matches nicely with the observation that on the F-theory side, in the weak coupling orientifold limit described by Sen [71, 72], the O7-planes carry precisely 1/3 of the total Kodaira bound of $-12K$. Thus the orientifold limit fits naturally with the $SO(32)$ heterotic string, as also suggested by the gauge groups and representations that appear in that limit.

Note that the analysis here does not depend upon knowing anything about the construction of the specific bundles on the heterotic side, only on the decomposition of the bundle in a way that satisfies the Bianchi identity. Various constructions of bundles are known for the heterotic theory, including spectral cover [92, 93, 94, 97, 95] and monad [98] (see e.g. [99] and references therein for a recent overview) constructions. The topological information about the duality found here should be valid and agree with all of these constructions in the appropriate limits.

Finally, we turn to the terms of the form $\chi_{\text{tr}} R^2, \chi_{\text{tr}} F^2$ in the heterotic picture. These correspond to couplings proportional to $\rho_0$ in the F-theory picture. As discussed earlier, we do not have a way to directly compute these couplings in F-theory. Heterotic/F-theory duality gives us the answer for this computation in the weak-coupling large-volume heterotic limit. We leave as an open problem the connection of this result to a more general computation in the F-theory context.

### 3.4.2 Heterotic/F-theory duality: examples

We conclude the heterotic/F-theory discussion with a brief description of how the general duality dictionary described above applies for the examples $\tilde{\mathcal{B}}_k$ introduced in section 3.3.2 and make some general statements about a broader class of examples.

We begin with the $E_8 \times E_8$ heterotic string on a Calabi-Yau that is elliptically fibered over $\mathcal{B}_2 = \mathbb{P}^2$. From (3.49) the total instanton number is 36, so we distribute the instantons $18 \pm m$ to the two bundles, with $\tilde{t} = m[H]$. The F-theory dual will have a base $\mathcal{B}_3$ that
is a $\mathbb{P}^1$ fibration over $\mathbb{P}^2$, and thus is the threefold $\tilde{F}_k$, with $t = k[H]/2$. This matches with the results of [26]. From (3.55) we see that the divisors carrying the gauge group in the F-theory picture are $\Sigma + (k/2 \pm m/2)H$. These are only irreducible effective divisor classes if $k = m$. Thus, the topological heterotic/F-theory determined by the vectors $a, b$ controlling the axion-curvature squared terms uniquely determines the F-theory manifold and gauge brane divisor classes for any given topological class of bundle on the heterotic side.

A similar story holds for the $SO(32)$ theory as in the $E_8 \times E_8$ case and as in 6D. For $SO(32)$, from (3.46) the contribution of $b$ in the base is $-3$ since $K = -3H$, so we see that the branes on the F-theory side are wrapped on $\Sigma + (k/2 - 3)H$. It follows that the $\mathbb{P}^1$ bundle on the F-theory side is always $\tilde{F}_6$, matching with the fact that the unbroken gauge group on this base is generically $SO(8)$.

A much broader class of examples of 4D heterotic/F-theory duality can be found in the recently produced list of 61,539 toric bases $B_2$ that can be used for an elliptic fibration with section of a Calabi-Yau threefold [18]. In principle, any of these spaces can be used to construct an elliptically fibered threefold, and the dual F-theory model will be on a base $B_3$ that is a $\mathbb{P}^1$ fibration over $B_2$. The simplest case to consider is one where the heterotic theory is an $E_8 \times E_8$ theory with the bundle evenly split, so $\tilde{t} = 0$. This will be dual to an F-theory model on the trivial bundle $\mathbb{P}^1 \times B_2$. In general, the toric bases in this list have many “non-Higgsable clusters” [17] giving rise to multiple copies of gauge groups such as $E_8$ and $F_4$ with no charged matter. These should correspond on the heterotic side to singular Calabi-Yau geometries, where enhanced gauge groups appear at the singular loci. Understanding the heterotic moduli that may smooth these singularities on the F-theory side, where fluxes may be involved, is an interesting direction for further work.

4. Geometrical constraints in 6D and 4D

Not all classical supergravity theories can be realized in F-theory. The geometric structure of F-theory places specific constraints on the spectrum and action of supergravity theories that admit an F-theory construction. In six dimensions, some geometrical F-theory constraints correspond to anomaly cancellation conditions or other known consistency conditions for a quantum low-energy supergravity theory. In other cases, it is not known whether the geometrical constraints from F-theory are necessary for consistency of any quantum 6D supergravity theory. In this section we review the structure of these constraints in 6D and describe related constraints on F-theory models in four dimensions. Some constraints on 4D supergravity theories arising from F-theory are closely analogous to 6D F-theory constraints that can be understood in terms of anomalies in 6D. In 4D, however, these constraints cannot be understood from gravitational anomaly cancellation or other known consistency conditions. Other constraints in 4D are analogous to the F-theory constraints on 6D theories for which there is as yet no macroscopic explanation in terms of consistency of supergravity theories; these constraints also lack macroscopic 4D interpretations. In general, the constraints that we find on 4D theories can only be clearly
formulated in the large-volume F-theory limit where the lifted moduli of the theory are still light. Unlike $\mathcal{N} = 1$ theories in six dimensions that have 8 supercharges and are quite constrained, $\mathcal{N} = 1$ theories in four dimensions with only 4 supercharges are relatively unconstrained. Away from the large-volume F-theory limit the fields become massive and couplings mix, and it is difficult to identify global constraints. Nonetheless, the F-theory constraints we describe here may provide a useful window on some aspects of the general structure of 4D $\mathcal{N} = 1$ string vacua.

We begin in Section 4.1 with a description of the constraints provided by geometry for 6D F-theory compactifications. We then describe the analogous 4D structures in Section 4.2 and discuss the nature and range of validity of the 4D constraints. The underlying geometric formulae we use here are not new; in particular, the connection between the Euler character of an elliptically fibered Calabi-Yau manifold and the geometry of the base manifold was described in detail in [88, 97]. The emphasis here, however, is on framing these geometric relations in terms of the spectrum and terms in the action of the low-energy supergravity theory, where they become constraints on theories admitting an F-theory realization.

4.1 Geometrical constraints on 6D effective supergravity theories

In Section 2 we described the correspondence between data in the spectrum and action of a 6D supergravity theory and topological aspects of the F-theory compactification geometry giving rise to the 6D supergravity theory. We now turn to the question of what constraints are imposed by F-theory on this 6D supergravity data.

We focus here on constraints for the restricted class of theories that contain no charged matter fields, since these are the simplest constraints to understand geometrically. This restriction still gives us insight into a wide range of F-theory compactification spaces, since for many (but not all) 6D supergravity theories, all matter fields can be Higgsed so that the generic model over the given base has no charged matter [17, 18]. There is also a deep relationship for 6D theories between F-theory geometry and anomaly constraints on theories with charged matter [3, 3, 15, 11]. While this structure is substantially richer than for theories without matter, the description of matter in four dimensions makes the 4D analogue of these constraints more complicated, and we do not attempt to systematically understand constraints on 4D theories with matter in this paper.

4.1.1 Constraints on theories without gauge groups

Let us start by considering 6D theories with no gauge group (or where the gauge group has been completely Higgsed). In this case the number of vector fields vanishes, $V = 0$, and there are no vectors $b_A$ appearing in the action. As shown in [17], these theories have $T \leq 9$, and correspond to F-theory compactifications on base surfaces that are generalized del Pezzo surfaces, containing no effective irreducible curves of self-intersection $-3$ or less. There are two constraints that are imposed by F-theory on the numbers of scalar and
tensor fields $H, T$ in the spectrum and the gravitational Chern-Simons vector $a$

$$a \cdot a = 9 - T$$  \hspace{1cm} (4.1)

$$273 = 29T + H$$  \hspace{1cm} (4.2)

These constraints are also quantum consistency conditions for the supergravity theory based on gravitational anomaly cancellation. Thus, in this case the F-theory constraints correspond to known consistency conditions on the low-energy theory. As we will see, F-theory imposes an analogous constraint on 4D theories, though in that case the constraint is on light rather than massless fields and is only clearly formulated in the large-volume F-theory limit. There is no known anomaly condition in the low-energy theory associated with the corresponding 4D constraint.

**Derivation of constraints from F-theory**

For completeness, and for comparison with the analogous 4D story, we now review explicitly how the constraints (4.1), (4.2) follow from the geometry of F-theory. Related arguments appear in [3, 100, 9, 11, 101]. In terms of the F-theory geometry, using the correspondences (2.4) and (2.16), (4.1) is the condition

$$K \cdot K = 10 - h^{1,1}(B).$$  \hspace{1cm} (4.3)

This can be proven geometrically as follows: The holomorphic Euler characteristic of the base is [102]

$$\chi_0(B) = \frac{1}{12} \int_B (c_1^2 + c_2)$$  \hspace{1cm} (4.4)

where $c_i$ are the Chern classes of $B$. The left-hand side of (4.4) is just $\chi_0(B) = 1$ since $h^{0,0} = 1, h^{0,i} = 0, i > 0$. Hence, $12 = \int c_1^2 + \chi(B)$, where

$$\chi(B) = \int_B c_2 = \sum_i (-1)^i b^i = 2 + h^{1,1},$$  \hspace{1cm} (4.5)

with $b^i$ the Betti numbers of $B$. It follows that $10 - h^{1,1}(B) = \int c_1^2 = K \cdot K$.

Now consider the constraint (4.2). In the absence of a nonabelian gauge group, the F-theory compactification is described by a smooth Calabi-Yau threefold $X$ with an elliptic fibration over the base $B$. In order to completely specify $X$ one has to give the global properties of the elliptic fiber. The number of massless $U(1)$'s in the 6D theory is given by

$$n_{U(1)} = h^{1,1}(X) - h^{1,1}(B) - 1,$$  \hspace{1cm} (4.6)

which coincides with the rank $r$ given in (2.3). The number $n_{U(1)}$ can be also determined by counting the number of sections of the elliptic fibration (Mordell-Weil group). With no abelian gauge group, $n_{U(1)} = 0$, and $r = 0$ in (2.3). For spaces with $n_{U(1)} = 0$, there is a Weierstrass model of the form (2.2), and the Euler character of $X$ is related to the topology of the base by [88]

$$\chi(X) = -60 \int_B c_1(B)^2 = -60 \Omega_{\alpha\beta} K^\alpha K^\beta = 2(T - H + 3),$$  \hspace{1cm} (4.7)
where we have used (2.7), and all hypermultiplets are neutral. Combining this with (4.1), we have
\[ 30(9 - T) = H - T - 3 \quad \Rightarrow \quad 273 = H + 29T, \quad (4.8) \]
and we have shown that the constraint (4.2) follows from F-theory geometry.

**Transitions between F-theory bases**

Before including nonabelian gauge groups in the discussion, we now briefly discuss the transitions among vacua with different numbers of tensor multiplets. All F-theory bases for 6D supergravity theories are connected through transitions associated with blowing down \(-1\) curves until a minimal model surface is reached \([103, 18]\). The constraints (4.1) and (4.2) must hold globally on the space of theories without gauge groups and can be characterized by their invariance under transitions that preserve this property.

As is well known \([12, 3]\), the tensionless string transitions in 6D theories associated with blowing down curves to points in the base \(B\) produce a change in field content of the 6D theory where a tensor field is exchanged for 29 scalar fields. From the 6D supergravity point of view, the number of fields replacing the tensor multiplet must be 29 to satisfy the anomaly condition (2.1). There are also several ways in which this change in field content can be understood from the F-theory geometry. Starting from the Weierstrass model on the blown-down base it can be seen that 29 scalars corresponding to Weierstrass coefficients must be tuned to blow up a point in the base \([66]\). Equivalently, using the expression (4.7) for the Euler character of \(X\) one can infer this change from purely geometric arguments.

To extract the change in the Hodge numbers using (4.7) we continue to restrict to the case of having no nonabelian gauge symmetries and no massless \(U(1)\)'s. We consider a modification of the base \(B\) by blowing up a generic point into an exceptional curve \(E\) in a new base \(B'\), denoting the blow-down map by
\[ \pi : B' \rightarrow B \quad (4.9) \]
The first Chern class changes according to
\[ c_1(B') = \pi^*c_1(B) - [E]. \quad (4.10) \]
Using the fact that \([E] \wedge \pi^*[c_1(B)] = 0\), as well as \(E^2 = -1\) we have
\[ \chi(X') = \chi(X) - 60. \quad (4.11) \]
Since one new Kähler class is gained in the blow-up, we infer using \(\chi = 2(h^{1,1} - h^{2,1})\) that 29 elements of \(h^{2,1}\) are lost. This implies by (2.4) and (2.6) that the spectrum changes as
\[ \Delta T = 1, \quad \Delta H_{\text{neutral}} = -29, \quad (4.12) \]
Hence, in the blow-up transition one tensor multiplet is added to the spectrum, while 29 neutral hypermultiplets are lost.
From the point of view of the complete Calabi-Yau threefold, this transition corresponds to blowing up a singular point into a del Pezzo 8 surface. After performing a flop transition this becomes a del Pezzo 9 (half K3), which is an elliptic fibration over the exceptional curve $E = \mathbb{P}^1$ in $B' \setminus \mathbb{P}^1$.

### 4.1.2 Constraints on theories without charged matter

We now generalize the discussion further and allow for nonabelian gauge groups in the 6D effective theory. This leads us to consider a broader class of F-theory compactifications on singular elliptic fibrations. We continue to restrict attention to models without charged matter fields, as discussed above.

We state briefly some additional constraints that arise from F-theory geometry for 6D theories without matter. For theories with nonabelian gauge groups, these constraints involve the vectors $b_A$ associated with each gauge group factor, and the total number of vector multiplets $V = \sum_A \dim G_A$. In theories without charged matter there are no abelian gauge group factors. The constraints from F-theory geometry are

\begin{align}
(-12a - \nu_A b_A) \cdot b_A &= 0 \quad (4.13) \\
b_A \cdot b_B &= 0, \quad A \neq B \quad (4.14) \\
29T - 273 &= V - H \quad (4.15)
\end{align}

Like the constraints (4.1) and (4.2), these F-theory constraints can be understood in the supergravity theory from anomaly cancellation; similar constraints in four dimensions, however, have no analogous understanding in terms of the low-energy theory.

We now briefly describe the constraints just listed from the point of view of F-theory geometry. We begin with (4.13), (4.14). Matter fields in F-theory arise either from codimension two singularities in the discriminant locus or from higher genus topology of divisor classes $S_A$. Codimension 2 singularities can either occur when different components of the discriminant locus intersect, or when a single component acquires a singularity. For a non-abelian gauge group factor $G_A$ that carries no charged matter, it must be the case that the corresponding divisor class $S_A$ has no intersection with the rest of the discriminant locus $\Delta - \nu_A S_A$. (4.13) and (4.14) are simply the conditions in the 6D supergravity theory that the divisor carrying a gauge group $G_A$ in the F-theory picture have no intersection either with the remainder of the discriminant locus or with any other particular divisor carrying a gauge group. Note that (4.14) is both necessary and sufficient for the absence of charged matter under a given pair of gauge group factors $G_A, G_B$, while (4.13) is a necessary but not sufficient condition for the absence of matter charged under a single gauge group factor $G_A$. Matter charged under a single gauge group may appear for example as adjoint matter when $S_A$ is a higher-genus surface, or from singularities within $S_A$ itself, as studied in [3, 13]. From the point of view of the low-energy theory, the relations (4.13) and (4.14) are apparently nontrivial constraints on the topological terms (2.14) appearing in the action, although as mentioned above they follow from anomaly cancellation conditions in six dimensions.
Now we consider the constraint (4.15), for which we consider a 4D analogue in Section 4. In the F-theory picture any gauge group factor that carries no charged matter is associated with a codimension one singularity on a divisor with topology $\mathbb{P}^1$. (Gauge groups on divisors with higher genus topology always carry adjoint or equivalent matter.) We can use this fact to generalize the argument in the previous subsection for the F-theory constraint associated with the Euler character. The correction to the Euler character of the total resolved Calabi-Yau space $X$ when multiple gauge group factors $G_A$ arise on codimension one loci $S_A$ in the base surface $B$ gives

$$ \chi(\hat{X}) = -60 \int_B c_1^2(B) - \sum_A r_{G_A} c_{G_A} (2 - 2g_A), $$

(4.16)

where $r_{G_A}, c_{G_A}, g_A$ are the rank and dual Coxeter number of $G_A$, and $g_A$ is the genus of $S_A$. Since as noted above, $S_A$ is a genus 0 curve when there is no matter charged under $G_A$, and $c_{G_A}r_{G_A} = \dim_{G_A} - r_{G_A}$, the modification of the Euler character is just twice the difference between the rank and the dimension of $G$. The Euler character is then

$$ \chi(X) = 60T - 540 + 2 \sum_A (r_{G_A} - \dim_{G_A}) $$

(4.17)

where we have used (4.7) for the unmodified $\int c_1^2$ in (4.16). Comparing (4.17) with the general expression (2.7) one finds

$$ 29T - 273 = \sum_A \dim_{G_A} - H, $$

(4.18)

thus producing from the F-theory geometry the constraint (4.15), equivalent to the anomaly constraint (2.1) for a theory with no charged matter fields.

Clearly, one can also perform the geometric transitions discussed in (4.9) in an F-theory configuration with divisors carrying nonabelian gauge groups. Using (4.16) and (4.10), the change in the spectrum is identical to (4.12) as long as the point blown up does not live in a divisor carrying a gauge group factor. When the blown-up point lives on a divisor carrying a gauge group factor, the gauge group generally changes, but this always occurs in a fashion compatible with (4.18).

### 4.1.3 Sign constraints and the Kodaira condition

In addition to the constraints on the spectrum that we have already discussed, F-theory imposes a set of positivity conditions on the vectors $a$ and $b_A$

$$ j \cdot (-a) > 0 $$

(4.19)

$$ j \cdot b > 0 $$

(4.20)

$$ j \cdot (-12a - \sum_A \nu_A b_A) > 0. $$

(4.21)

The geometric statement of these conditions in F-theory is that the anti-canonical class $-K$, all divisors $S_A$ carrying gauge group factors, and the residual divisor locus $Y$ defined through (2.3) are all effective divisors.
The condition (4.20) has a simple interpretation in terms of the 6D supergravity theory; it states that the kinetic term $F \wedge *F$ for each gauge group factor has the proper sign [59]. The other two conditions do not have known interpretations in terms of the low-energy theory. The condition (4.19) states that the quadratic term in the curvature of the form $R \wedge *R$ must have a specific sign. As discussed in [7], this condition may follow from causality, following an argument analogous to that of [104]. We discuss the analogous 4D constraint in the following section.

The Kodaira condition (2.3) from which constraint (4.21) follows is of crucial importance in F-theory compactifications. The Kodaira condition expresses the geometric condition that the total space $X$ is Calabi-Yau and hence preserves supersymmetry in the dimensionally reduced theory. In the weak coupling limit this condition simply corresponds to the well-known fact that the 7-brane tadpoles have to globally cancel. Written in cohomology (2.3) can be evaluated on a basis and amounts to

$$-12K^\alpha - \sum_A \nu_A C_A^\alpha - Y^\alpha = 0.$$  \hspace{1cm} (4.22)

The geometric condition that the residual divisor locus $Y$ be effective corresponds to the constraint (4.21). It would be very interesting to achieve some understanding of this constraint from the point of view of the supergravity theory. In particular, this inequality plays a key role in bounding the set of possible F-theory compactifications when the number of tensors $T$ becomes large [6]; understanding this constraint as a consistency condition on low-energy theories would be one of the final steps needed in matching low-energy consistency conditions to consistency conditions from string theory for 6D theories [105].

4.1.4 Lattice structure for dyonic string charges

As discussed earlier, in any supergravity theory arising from an F-theory compactification, the lattice of dyonic string charges takes the form $\Gamma = H_2(B, \mathbb{Z})$. By Poincaré duality this lattice is self-dual/unimodular. Thus, F-theory imposes the constraint that the dyonic string charge lattice is unimodular. It was shown in [10] that this condition is also necessary for any consistent 6D supergravity theory. Thus, this is an example of a consistency condition arising from F-theory that is also a quantum consistency constraint, where the understanding of the F-theory picture motivated the identification of the macroscopic consistency condition. It may be that other F-theory constraints will eventually be understood in this fashion from low-energy/macroscopic considerations.

To understand the corresponding structure arising in four dimensions, it may be helpful to review the nature of the inner product structure on the lattice $\Gamma$. In the F-theory picture this is just the intersection form on $H_2(B, \mathbb{Z})$. From the point of view of the supergravity theory, the elements $x \in \Gamma$ are charges for dyonic strings that couple to the self-dual and anti-self-dual two-forms $B^\alpha$. The inner product $x \cdot y$ of the charges between two such dyonic strings must be an integer by the generalization of the Dirac quantization condition to six dimensions [106].
4.2 Geometrical constraints from F-theory on 4D supergravity theories

In this section we investigate various simple constraints that the geometry of F-theory imposes on 4D supergravity theories. While in 6D the constraints from F-theory are clear discrete constraints on massless spectra that are satisfied across all continuous branches of the moduli space, the constraints from F-theory on 4D theories are more subtle. In particular, as discussed above, in a general 4D F-theory compactification many of the continuous geometric moduli are lifted. F-theory constraints on the spectrum become less clear from the low-energy action when the fields become sufficiently massive that the F-theory moduli are no longer clearly distinguishable from other massive fields in the theory. In the discussion here of constraints on 4D supergravity theories, we assume that we are working in a large volume compactification where fields coming from F-theory geometry are all light and can be identified in the spectrum. Understanding how these constraints and other structures extend further into the moduli space beyond the large volume approximation represents a challenge for future work.

Another significant limitation in treating F-theory constraints on 4D theories is the absence of a systematic formalism for treating the degrees of freedom on the 7-brane world volumes in a way that is naturally compatible with Weierstrass models of F-theory. In particular, while recently there has been progress in understanding certain classes of “G-flux” configurations in F-theory [28]-[35], and in principle fluxes can be integrated with F-theory from the point of view of M-theory [27, 33], there is no unified synthesis of gauge fluxes on 7-branes, or the related adjoint scalars that appear in “T-brane” constructions [36] with F-theory geometry. This poses an obstacle to a systematic treatment of constraints, particularly when matter fields are involved. There is a close interplay between fluxes, geometry and matter in 4D theories; in particular, fluxes can change the matter content on a 7-brane world volume or at intersections between 7-branes. We focus therefore on simple constraints from F-theory geometry that do not depend critically on the detailed structure of matter. A clear direction for further extension of this work would be a more careful treatment of matter, fluxes, and codimension two and three singularities in 4D F-theory constructions.

4.2.1 Constraints on theories without gauge groups

We begin our discussion as in 6D, by focusing on effective theories that have no gauge group. Such theories arise, for example, for generic F-theory Weierstrass models over Fano threefold bases such as $\mathbb{P}^3$ where the geometry does not impose any gauge group on the theory by requiring vanishing of $f, g$ on any particular divisor locus. In this case, the associated elliptically fibered Calabi-Yau fourfold has no singularities. Moreover, the elliptic fibration has just a single section, so that there are no massless $U(1)$ symmetries $r_v = 0$, and we have $r_{21} = 0$. In such situations there is a constraint on the spectrum analogous to (4.8). The Euler character of the elliptically fibered fourfold $X$ is related to
the topology of the base $B_3$ by \cite{78, 88} \footnote{Note that the base $B_3$ of an elliptically fibered Calabi-Yau fourfold always has $24\chi_0(B_3) = \int_{B_3} c_1(B_3) c_2(B_3) = 24$. This follows from the fact that $\chi_0(B_3) = \sum (-1)^n h^{0,n}(B_3) = 1$, for a base of an elliptically fibered Calabi-Yau manifold since $h^{1,0} = h^{2,0} = h^{3,0} = 0$ for $X$ and $B_3$.}

$$\chi(X) = 288 + 360 \int c_1^3(B_3).$$ (4.23)

Using (3.17) together with $r_v = 0$ this gives the constraint

$$39 - 60\kappa_{\alpha\beta\gamma} K^\alpha K^\beta K^\gamma = 39 - 60 \langle \langle a, a, a \rangle \rangle = C_{sa} + C_{cs} - C_{21},$$ (4.24)

on the numbers of the different types of scalar fields. To streamline equations and to clarify the analogy to 6D, we use here a shorthand notation for the triple intersection product of three vectors under $\kappa_{\alpha\beta\gamma}$, the intersection form on the base $B_3$ of a 4D F-theory compactification.

$$\langle \langle x, y, z \rangle \rangle \cong \kappa_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma.$$ (4.25)

Note that this intersection product is computed using only the components of $a^a$ and not the component $a^0$ related to the axiodilaton. The constraint (4.24) should be satisfied by any 4D $\mathcal{N} = 1$ supergravity theory arising from an F-theory compactification in a phase with no unbroken gauge group. A simple consequence of (4.24) depends only upon the light spectrum of the theory and not upon the details of $\kappa_{\alpha\beta\gamma}$ or $K^\alpha$

$$C_{sa} + C_{cs} - C_{21} \equiv 39 \pmod{60}.$$ (4.26)

As a simple example of a 4D F-theory model satisfying (4.24), consider a generic Weierstrass model over the base $\mathbb{P}^3$, with $C_{sa} = h^{1,1}(B_3) + 1 = 2$. In this case, the cubic coupling of $a^a$ is characterized by a single integer

$$\kappa = \kappa_{HHH} = 1.$$ (4.27)

With $-K = 4H$, in terms of the hyperplane section $H$, the constraint (4.24) then becomes

$$39 - 60\kappa a^3 = 3879 = 2 + C_{cs} - (C_{21} + r_{21}).$$ (4.28)

This is satisfied for a generic F-theory Weierstrass model on $\mathbb{P}^3$, which has $C_{cs} = h^{3,1}(X) - 1 = 3877$ scalar degrees of freedom, and $C_{21} = r_{21} = 0$. The quantity $h^{3,1}(X)$ can be computed directly for any toric base from the number of monomials in the global Weierstrass model, minus the number of automorphisms. The number of automorphisms can be determined from the “polar polytope” \cite{107} (for example see \cite{27, 18}).

As another example of the constraint (4.24), consider the base $\mathbb{F}_2$. Over this base there is no gauge group required by vanishing of $f, g$ on any divisor. From the form of $-K = 2\Sigma + 5F$ and the triple intersection products given in (3.37), we have

$$k = 2 : \quad 60 \langle \langle -K, -K, -K \rangle \rangle = 3720,$$ (4.29)
and, using the fact that for the base $\tilde{F}_2$, $h^{3,1}(X) = 3757$,

$$C_{sa} + C_{cs} - (C_{21} + r_{21}) = 3 + 3756 - (0 + 0) = 3759 = 39 - 60 \langle \langle a, a, a \rangle \rangle.$$  \hspace{1cm} (4.30)

So (4.24) is again satisfied.

The base $\tilde{F}_3$ is an interesting case. While $f, g$ are not required to vanish on $\Sigma$ in this case, and there is no gauge group, there is only a one-parameter family of constant functions for each of $f$ and $g$ that do not vanish on $\Sigma$. In parallel with the 6D case, where there is a $-2$ curve on $F_2$ associated with a complex degree of freedom that has been tuned and is not visible in the Weierstrass parameterization of $[18]$, there is an extra degree of freedom of type $C_{21}$ on $\tilde{F}_3$. This combines with $C_{sa} = 4357$ to give

$$C_{sa} + C_{cs} - C_{21} = 4359,$$  \hspace{1cm} (4.31)

again matching with (4.24).

The constraint (4.24) is reminiscent of the analogous 6D constraint (4.1) on the number of scalars arising from the gravitational anomaly, although there is no known pure gravitational anomaly in four dimensions. As we have emphasized repeatedly, in contrast to the 6D situation, constraints such as (4.24) and (4.26) are only clearly formulated in the regime where the geometric moduli of the F-theory compactification are light compared to other massive fields in the theory, so that the numbers of scalar fields of each type can be distinguished, and for (4.24) so that the canonical class $K$ and triple intersection form can be extracted from couplings in the action as discussed in Section 3. To extend these constraints away from the class of large-volume F-theory compactifications, it would be necessary to have a definitive way of identifying $C_{cs}, C_{sa},$ and $C_{21}$ from the point of view of the low-energy theory in a general context, where some of the fields may become very massive. In the context of general $\mathcal{N} = 1$ supergravity theories, however, it is unclear how to make sense of these moduli fields. Not only do they mix with other massive fields such as Kaluza-Klein modes, but they also can in principle mix with one another, so that we do not have a definitive way of distinguishing the fields $C_{21}$ from $C_{cs}$ and $C_{sa}$ away from the large-volume F-theory limit. These considerations suggest that it may be difficult to identify clear constraints that are valid for general 4D supergravity theories not associated with a specific type of string compactification. Nonetheless, if any such global constraint on $\mathcal{N} = 1$ theories does exist, constraints such as (4.24) that hold in specific contexts such as F-theory should provide a helpful window and guide to understanding the more general constraints.

Note that in six dimensions, there are two constraints on the spectrum for theories without gauge groups; in addition to (4.1), there is a second constraint (4.2). It is natural to wonder whether there is an analogous second constraint on the spectrum for 4D F-theory vacua without unbroken gauge groups in four dimensions. We believe that there is no such second constraint, even if we allow the number $r_{21}$ to enter the constraint, as suggested by the structure of geometric transitions discussed below. We briefly outline the argument for this conclusion. Aside from the spectrum, the only objects available for a constraint are
\( \kappa_{\alpha\beta\gamma} \) and \( K^\alpha \). The only invariant that can be formed is the triple intersection \( \langle \langle K, K, K \rangle \rangle \).

If there is a second linear constraint then there must be one linear combination that only contains the numbers \( C_s, r_{21} \) in the spectrum. If there were such a linear constraint on the spectrum then the existence of a pair of compactifications that differ only in one number in the spectrum would indicate that that number could not appear in the linear combination. As we discuss below, there exist such pairs, indicating that \( C_{cs}, C_{sa}, r_{21} \) cannot be in the linear combination. Since there are solutions with different values of \( C_{21} \), we conclude that there cannot be a further linear constraint on the spectrum and triple intersection of \( K \) for 4D theories arising from F-theory compactifications.

**Transitions between F-theory threefold bases**

Just as in the 6D story, transitions associated with blowing down divisors in a threefold base connect different branches of the geometric moduli space of elliptically fibered Calabi-Yau fourfolds that can be used to produce a 4D supergravity theory from F-theory. Such transitions must respect the F-theory constraints on the supergravity data such as the spectrum constraint described in the previous section. While in the physics of 4D F-theory models some geometric moduli are lifted by fluxes and the superpotential, these geometric moduli still underlie the configuration space of the theory and describe the off-shell geometry of the theory. It is in this sense of the underlying off-shell geometry that we can systematically describe transitions between different F-theory geometries as connecting components of the continuous geometric moduli space, even though the physical moduli space is more constrained. Extending these off-shell parameters outside the F-theory framework presents an interesting challenge for developing a deeper understanding of the theory.

We can use invariance under these transitions as an aid in understanding constraints on the physical spectrum of the theory. The constraints may also shed light on the physics of the transitions. For 4D theories, there is a much richer set of transitions than in 6D, corresponding to different kinds of blow-up and blown-down processes. The network of transitions for a particularly simple class of (Fano) F-theory bases is explored in [108, 109, 10]. The geometry of threefolds, however, is much more complicated than that of surfaces. The mathematics of the Mori program is aimed at understanding the connections between complex varieties for dimensions 3 and higher analogous to minimal surface theory in complex dimension 2, and classifying the types of singularities that may arise [110]. A full exploration of the physics associated with this story will be a substantial research endeavor. Here, we focus on the simplest class of transitions, where a point or a curve in a smooth base is blown up into a divisor in another smooth base, and where neither base requires the presence of a gauge group.

Considering the blow-up of a single point in a smooth base we can derive the change in the \( \mathcal{N} = 1 \) spectrum by using the formula (4.23). Note that in case of a point blow-up the exceptional divisor is \( \mathbb{P}^2 \), which we will refer to as \( E \). Using the formulae of appendix A together with (4.23), the Euler characteristic obtained for an elliptic fibration over the
new base is
\[ \chi(X') = \chi(X) - 2880. \] (4.32)
We can infer the change in the number of chiral multiplets by using this equation. Note that \( h^{1,1}(X) \) increases by one due to the new exceptional divisor \( E \). Since \( E \) has no three-forms the Hodge number \( h^{2,1}(X) \) will not change. Hence, from (3.17) the number of chiral multiplets changes as
\[ \Delta C_{cs} = -481, \quad \Delta C_{sa} = +1, \quad \Delta C_{21} = \Delta r_{21} = 0. \] (4.33)
From the point of view of the elliptic fibration the transition (4.33) can be viewed as a tuning of 481 moduli to enforce a singularity over the point in the base, which is then resolved. This number of moduli can also be derived directly by counting degrees of freedom in the Weierstrass model, as discussed below. Note that the congruency condition (4.26) is invariant under this transition, since \( 1 - 481 = -480 \equiv 0 \pmod{60} \). This transition must be possible from the point of view of F-theory geometry, but is not understood at this point physically from the point of view of the low-energy theory.

As a simple example of a transition of this type, consider \( \tilde{F}_1 \), which can be realized as a blow-up of a point on \( \mathbb{P}^3 \) just as \( F_1 \) is given by blowing up a point on \( \mathbb{P}^2 \). It is easy to check that under this blow up, the numbers in the spectrum change through (4.33). Indeed, in this case the change in the number of complex structure moduli \( h^{3,1}(X) \) can be computed directly along the lines of the argument in [66]. Starting from the F-theory base \( B'_3 = \tilde{F}_1 \), the divisor \( \Sigma \) can be blown down to give \( B_3 = \mathbb{P}^3 \). We can describe \( \tilde{F}_1 \) as a \( \mathbb{P}^1 \) bundle over \( \mathbb{P}^2 \) in terms of coordinates \((x_1, x_2, x_3, x_4, x_5)\) subject to the relations
\[ (x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda \mu x_2, \lambda x_3, \mu x_4, \lambda x_5). \] (4.34)
The functions \( f, g \) are of degrees \((16, 8)\) and \((24, 12)\) in \( \lambda, \mu \). Blowing down \( \Sigma \) gives \( \mathbb{P}^3 \), where \( \tilde{f}, \tilde{g} \) are of degree 16 and 24, and descend from \( f, g \) as in the case of \( F_1 \rightarrow \mathbb{P}^2 \). We can then directly count the number of degrees of freedom that must be tuned in \( \tilde{f}, \tilde{g} \). The power of \( x_2 \) in \( f \) is at most 8. This requires tuning \( 1 + 3 + 6 + \cdots + 36 = 120 \) coefficients. Similarly, \( g \) has a power of \( x_2 \) that is at most 12, requiring the tuning of \( 1 + 3 + \cdots + 78 = 364 \) coefficients. Thus 484 coefficients must be tuned, with a three-parameter space of points where the tuning may be done, giving a total of 481 moduli that are removed in the transition. From the point of view of the theory on the blown-down base \( B_3 \), the tuning just described corresponds to arranging coefficients in the Weierstrass model so that there is a codimension 3 point where \( f, g \) vanish to degrees 8, 12. This tuning is local in the vicinity of the singularity that must be blown up. It is straightforward to verify by counting monomials in the dual polytope (and subtracting automorphisms in the polar polytope as in [18]) that for any blow-up of a point in a toric base, the number 481 of degrees of freedom that must be tuned to give a transition of this type will be the same, as long as no additional singularity (such as would give a gauge group or matter field) is required on either threefold. Note that a codimension 3 point where \( f, g \) vanish to degrees 4 and 6 is singular but cannot be blown up to a divisor, suggesting a pathology of such theories that is not yet well understood [67].
In a three-dimensional base one can also blow up a smooth curve $\mathcal{C}$ to a divisor in the threefold base. This is the 4D analogue of the tensionless string transition in 6D theories. In the F-theory picture a 3-brane wrapped on a $\mathbb{P}^1$ fiber over the curve $\mathcal{C}$ in the blown up space becomes a tensionless string in the limit as the fiber shrinks. In this case one finds a different pattern in the change of the Euler character and spectrum. The changes in the Chern classes and the intersection numbers of the blown-up base are summarized in appendix A. Using (4.23) gives

$$\chi(X') = \chi(X) - 1440 \left( 2 - 2g_\mathcal{C} + \int_\mathcal{C} c_1(B_3) \right),$$

where $g_\mathcal{C}$ is the genus of the curve $\mathcal{C}$. In contrast to the blow-up of a point, the blow-up divisor $E$ is a $\mathbb{P}^1$-bundle over $\mathcal{C}$ and hence has a more non-trivial topology. In particular, it has $g_\mathcal{C}$ new $(2,1)$-forms that are obtained by wedging the two-form of the $\mathbb{P}^1$-fiber with a $(1,0)$-form on $\mathcal{C}$. This implies that the number of $(2,1)$-forms of the base $B_3$ and the fourfold $X$ will also change in this transition. Translated into the change of the four-dimensional spectrum one finds

$$\Delta C_{cs} = -481(1 - g_\mathcal{C}) - 240 \int_\mathcal{C} c_1(B_3), \quad \Delta C_{sa} = +1,$$

$$\Delta r_{21} = g_\mathcal{C}, \quad \Delta C_{21} = 0.$$  

Note that this change in the spectrum will be modified if the blow-up changes the structure so that a divisor has $f,g$ vanishing, requiring a change in the gauge group. Even in the marginal case where on some divisor $f,g$ are constant, as mentioned above for the example $\tilde{\mathbb{P}}_3$, additional scalars $C_{cs}, C_{21}$ can arise in such a transition.

The change of spectrum in this transition includes a change in $r_{21}$. This motivates us to generalize the constraint (4.1), (4.2) to include models with $r_{21}$; including this term from (3.17) gives

$$39 - 60 \kappa_{\alpha\beta\gamma} K^\alpha K^\beta K^\gamma = 39 - 60 \langle \langle a,a,a \rangle \rangle = C_{sa} + C_{cs} - C_{21} - r_{21}.$$  

Just as the change in spectrum under a tensionless string transition is compatible with the constraints (4.1), (4.2) we expect that the changes of spectrum (4.33), (4.36) are compatible with the 4D constraint (4.37). A detailed check of this would require explicit computation of the triple intersection numbers on both sides of the transition; this can be done in any particular case from the geometry. More generally, however, we can easily confirm that the associated congruence

$$C_{sa} + C_{cs} - C_{21} - r_{21} \equiv 39 \pmod{60}.$$  

is invariant under both these transitions. This serves as a check that this constraint is indeed valid for general F-theory models. Further transitions would be needed to span the space of connected F-theory bases, however, and — unlike in 6D — in 4D not all F-theory bases can be connected by the transitions associated with Mori theory: blowing up, blowing down, flips and flops.
We return now to complete the proof of the statement made in the previous section that there is only one linear constraint involving \( \langle K, K, K \rangle \) on the spectrum of fields of 4D F-theory models with \( r_v = 0 \). Note that if we blow up a curve of genus 0, the change in the spectrum is identical to that for blowing up a point, except for the last term in \( \Delta C_{cs} \). Since this term can be nonzero, as discussed above there can be two compactifications that only differ in this number in the spectrum, so \( C_{cs} \) cannot appear in any linear constraint. Given this, from (4.33) it follows that \( C_{sa} \) also cannot appear. But then \( r_{21} \) also cannot appear since we can blow up a curve of nonzero genus without changing \( r_{21} \). Since there are models with different values of \( C_{21} \) there cannot be any further linear constraints beyond (4.24).

### 4.2.2 Constraints on 4D theories without charged matter

We now relax the condition that there is no gauge group in the 4D theory, and generalize the 4D constraint derived above to include theories with gauge groups. Including charged matter in 4D is significantly more involved compared to 6D. Unlike in 6D, where codimension two loci in the base giving matter are pointlike, in 4D codimension two singularities are themselves surfaces, whose Euler character affects the matter content of the theory. Fluxes \( G_4 \) alter the spectrum by modifying the equations for the massless matter eigenstates on the worldvolume and intersections of the 7-branes. In this paper we only comment briefly on the complications associated with chiral matter and fluxes, and focus on topological constraints on \( B_3 \) and \( X \). There are presumably more complicated constraints involving charged matter fields when fluxes are correctly included, perhaps related to the relation (3.16).

As in six dimensions, a simple condition that implies the absence of any matter charged under two groups \( G_A \) and \( G_B \) associated with divisors \( S_A, S_B \) is that the intersection between the two divisors vanish identically

\[
\kappa_{\alpha\beta\gamma} S_A^\alpha S_B^\beta = 0, \quad \forall \gamma.
\]

(4.39)

This is parallel to the 6D constraint (4.14), though in the 4D case this is only a sufficient condition for the absence of multiply charged matter while in 6D the condition is also necessary, since in 6D every intersection between divisors is a pointlike codimension two singularity that carries matter degrees of freedom. On the other hand, the 6D condition (4.13) stating that the divisor carrying a gauge group is orthogonal to the residual divisor locus \( Y \) is necessary for theories without matter, but not sufficient since a gauge group can carry non-local matter such as an adjoint. In 4D the analogous condition is neither necessary or sufficient, for the same reasons stated above.

The constraints on the absence of 4D chiral matter have a closer analogy to the 6D constraints of section 4.1.2. Physically this is due to the fact that 4D chiral matter induces non-Abelian anomalies, just as a general matter spectrum does in 6D. We have recalled in (3.14) that the chiral spectrum of a 4D F-theory compactification on the singular space \( X \) can be derived from the \( G_4 \) flux on \( \hat{X} \) by studying the constant couplings \( \Theta_{i\alpha i\beta} \). These
couplings capture the 4D chiral spectrum as 3D loop corrections in the Coulomb branch. Thus, absence of 4D chiral matter simply implies

$$\Theta_{i_Ai_B} = \int_X \omega_{i_A} \wedge \omega_{i_B} \wedge G_4 = 0 , \quad (4.40)$$

where $\omega_{i_A}, \omega_{i_B}$ are the resolution divisors for the gauge groups $G_A, G_B$ in the M-theory picture. We note that it is possible to have $A = B$ in (4.40), which captures information about the chiral matter of the intersection of $S_A$ with the rest of the discriminant. This implies that each F-theory compactification without chiral matter has to admit a special non-trivial $G_4$ satisfying (4.40) together with the D3-tadpole cancellation condition (3.15). The 6D analog of (4.40) is a constraint on the triple intersection numbers of the resolved Calabi-Yau threefold with three indices labeling exceptional resolution divisors for the gauge groups. In order to promote (4.40) to a constraint on the low-energy data, just as in 6D one has to find other couplings that involve the same flux data. It will be interesting to find such couplings in 4D compactifications. The immediate analog to 6D appears if a 4D Green-Schwarz coupling is required to cancel anomalies.

For theories in which there is no charged matter and there are no codimension two singularities on the gauge group divisor loci, the argument leading to (4.24) can be generalized to include theories with gauge group factors. A set of identities known as Plücker identities can be used to show that the Euler character of the Calabi-Yau fourfold that is elliptically fibered over a threefold base with a homogeneous degeneration over divisors carrying a pure gauge group with no codimension two singularities is given by [88, 97]

$$\chi(X) = 288 + 360 \int_B c_1(B)^3 - \sum_A r_{G_A} c_{G_A}(c_{G_A} + 1) \int_{S_A} c_1(S_A)^2 . \quad (4.41)$$

This generalizes the constraint (4.24) for a theory with pure gauge group factors and no matter to

$$39 - 60 \langle\langle a, a, a \rangle\rangle = C_{sa} + C_{cs} + r_v - (C_{21} + r_{21}) + \frac{1}{6} \sum_A r_{G_A} c_{G_A}(c_{G_A} + 1) \langle\langle a + b_A, a + b_A, b_A \rangle\rangle . \quad (4.42)$$

As an example, consider $\tilde{F}_4$, which carries a gauge group $SU(2)$ on the divisor class $\Sigma$, as discussed above. In this case we have $-K = 2\Sigma + 7F$, $h^{3,1} = 5187$, and the group has rank $r_v = 1$ so

$$39 - 60\langle\langle a, a, a \rangle\rangle = 5199 = 3 + 5186 + 1 + \frac{1}{6}(1 \cdot 2 \cdot 3 \cdot 9) , \quad (4.43)$$

confirming (4.42).

4.2.3 Sign conditions and Kodaira condition

Just as in six dimensions, the anticanonical class $-K$ and the divisors $S_A$ carrying any non-abelian gauge group must be effective. This imposes, in particular, positivity constraints on the volumes of these divisors, so that in the 4D theory we must have

$$\langle\langle -K, v, v \rangle\rangle = \kappa_{\alpha\beta\gamma} (-a^\alpha) v^\beta v^\gamma > 0 \quad (4.44)$$
and
\[ \langle \langle S_A, v, v \rangle \rangle = \kappa_{\alpha\beta\gamma} b_A^\alpha v^\beta v^\gamma > 0. \] (4.45)
The analogous constraint in 6D to (4.45) simply corresponds in the supergravity theory to the constraint that the kinetic term for the gauge group factor \( G_A \) has the proper sign. In 4D this is complicated by the appearance of the additional real second term in (3.5) and additional axions discussed in section 3.2.2, which as noted below (4.25) are not included in the intersection product \( \langle \langle \cdot, \cdot, \cdot \rangle \rangle \).

As discussed in the 6D context, the Kodaira constraint (2.3) is another constraint imposed by F-theory that at the present time is not clearly understood from the low-energy point of view. The Kodaira constraint sharpens the inequalities (4.44) and (4.45) to
\[ 12 \langle \langle -K, v, v \rangle \rangle \geq \sum_A \nu_A \langle \langle S_A, v, v \rangle \rangle > 0. \] (4.46)

As an example of the Kodaira constraint, consider again F-theory compactifications on the base \( \mathbb{P}^3 \), with a gauge group \( SU(N) \). In this case, as in (4.27), there is only a single axion \( \rho_\kappa \) other than \( \rho_0 \), and \( -a = 4H \). The \( SU(N) \) gauge group lives on a divisor that can be identified from the 4D Green-Schwarz-like couplings to be \( b = mH \), with \( m > 0 \) an integer. The Kodaira constraint in this case is
\[ 48 \geq mN. \] (4.47)
In (15) a more careful analysis of such theories with \( m = 1 \) \( (b = H) \) showed that in this case \( N \leq 32 \).

We do not have any clear understanding at present of how the inequalities (4.44) and (4.46) should be understood in terms of the low-energy theory. We note in passing, however, that some possibly related constraints on Gauss-Bonnet couplings have been discussed using AdS/CFT e.g. in [111].

### 4.2.4 Lattice structure for string states

In 6D supergravity theories, the charges \( a, b_i \) appearing in the \( BR^2, BF^2 \) topological couplings live in a sublattice \( \Lambda \) of the lattice of dyonic charges \( \Gamma \) associated with strings in space-time. As discussed in the 6D section, there is an inner product on \( \Gamma \) associated with Dirac quantization, under which \( \Gamma \) takes the structure of an integral and self-dual (unimodular) lattice. This lattice plays an important role in the relationship with F-theory; the charges on this lattice characterize elements of \( H_2(B, \mathbb{Z}) \).

There is a similar lattice structure in 4D, though the absence of an inner product makes the story less transparent. In 4D, the classical continuous axionic shift symmetries are broken to a discrete lattice \( L \), so that we have an invariance under
\[ \rho_\alpha \rightarrow \rho_\alpha + l_\alpha, \quad l \in L. \] (4.48)
This breaking of the continuous shift symmetries arises from nonperturbative terms. This can also be understood in terms of quantized strings that carry magnetic axion charges \( q \).
in the lattice $L$. A quantum string gives rise to an axionic charge measured along a loop around the string where $\int d\rho_\alpha = l_\alpha$ is an element of $L$. The charges $a, b_A$ that parameterize the 4D topological couplings (3.13) must lie in the dual lattice

$$a, b_A \in L^*, \quad (4.49)$$

since a shift of the axions under (4.48) must leave the action invariant up to an overall additive constant $2\pi k, k \in \mathbb{Z}$, and the gauge and gravitational instanton numbers are integrally quantized. From the F-theory point of view, this characterization follows from the fact that axions are associated with components of $C_4$ that are 4-forms in $B_3$, while $a, b_A$ are associated with divisor classes in $H_4(B, \mathbb{Z})$.

For a 4D F-theory compactification, the lattice $L$ characterizes the charges of fundamental strings in space-time that are charged under the axion $\rho_0$ as well as axionic strings that arise from D3-branes wrapped on cycles in $H_2(B, \mathbb{Z})$. These strings are electrically charged under the two-form fields $B^A$ and magnetically charged under the axions $\rho_A$. Whereas in 6D, the intersection product between dyonic strings has a clear physical interpretation in terms of the phase appearing in the Dirac quantization condition, we do not have an analogous simple physical interpretation of any structure associated with a set of 3 strings or instantons in four dimensions. The structure of the triple intersection product suggests that there is a natural triple product between sets of 3 axionically charged strings in the 4D theory. This may be more naturally formulated in the language of instantons. Euclidean D3-branes wrapped on 4-cycles $H_4(B, \mathbb{Z})$ correspond to instantons in the 4D theory that couple to the axions $\rho_\alpha, \alpha > 0$. The triple intersection product $\kappa_{\alpha\beta\gamma}$ naturally associates an integer with any set of 3 such instantons. It would be interesting to identify a natural physical interpretation of this product in four dimensions.

In principle, the set of supersymmetric axionically-charged string excitations of a 4D theory describes the Mori cone of effective curves on the F-theory compactification threefold $B_3$. This characterizes the complex structure of the compactification space. The geometric F-theory structure of axions and axionically charged strings/instantons in 4D is complicated, however, by extra 4D axions as discussed above. In parallel to the additional axion from the axiodilaton, additional instantons appear in four dimensions associated with pointlike D(-1)-branes; these instantons couple to $\rho_0$ just as Euclidean D3-branes couple to the other axions $\rho_\alpha, \alpha > 0$. As discussed in Section 3.4, the 4D axions have a different geometric interpretation in heterotic compactifications, and the axion $\rho_0$ ties into the geometric structure of the compactification manifold. Away from the large-volume F-theory limit, many of the axion fields become massive and their identity becomes difficult to distinguish. It seems likely that in general 4D supergravity theories, the axion-instanton couplings can be completely general; gauge and gravitational instantons may couple to any axions in the theory, including those in $C_2_1$ and $C_{cs}$ as well. In this case there will be massive string excitations associated with each of the axions. Many of these theories will not have large-volume F-theory or heterotic interpretations. A better understanding of additional structure such as the triple intersection product on the axionic string/instanton charge lattice may be of value in developing this part of the story further.
5. Conclusions

We have found that, just as in six dimensions, much of the geometric data associated with an F-theory compactification to four dimensions is encoded in the spectrum and action of the 4D theory. In particular, for large volume compactifications the spectrum of light fields directly encodes the Hodge numbers of the F-theory compactification manifold, and the canonical class and 7-brane divisor classes on the F-theory base are encoded in topological terms coupling 4D axions to curvature squared terms in the action. Strings carrying magnetic axion charges and couplings in the supergravity action further characterize the full second homology and triple intersection product of the F-theory compactification manifold. In 6D this correspondence makes it possible to read off the F-theory geometry directly from the structure of the low-energy theory. In 4D, this correspondence is only transparent in the large-volume limit where the geometric moduli are light. F-theory geometry in 4D is obscured in the low-energy theory in the bulk of the moduli space due to various corrections, lifting of moduli by the superpotential generated by fluxes, additional axions, and various other complications. Further analysis of how the results of this work can be relevant outside the large-volume F-theory limit is an interesting direction for future work.

A particular element that has played a key role in understanding the space of 6D supergravities, and that seems to play a related structural role in 4D supergravities, is the set of couplings between two-form/axion fields and curvature squared terms in the action. In 4D, axions have a discrete shift symmetry associated with magnetic charges of stringlike excitations of the theory. These axions couple to $F \wedge F$ and $R \wedge R$ in the 4D action in a way that captures topological aspects of the compactification geometry in the case of models that arise from an F-theory compactification. These coupling terms illuminate the structure of 4D supergravity theories just as Green-Schwarz terms illuminate 6D supergravity theories, even though in 4D these terms are not strongly restricted by anomalies. One specific application of these terms that we have explored in this paper is to heterotic/F-theory duality. We have computed the axion–curvature squared terms explicitly in a general class of 4D heterotic compactifications with F-theory duals, providing a check on the general structure presented here, and giving a simple dictionary indicating which $\mathbb{P}^1$ bundle acts as the F-theory base and which divisors in an F-theory compactification carry the gauge group factors in any case of heterotic/F-theory duality. These results provide a complementary perspective to other methods such as the spectral cover method and the stable degeneration limit previously used for understanding heterotic/F-theory duality in four dimensions, and suggest that further insight into this duality may follow from further considerations along the lines pursued in this paper. For example, the axion–curvature squared terms provide information about heterotic/F-theory duality for the $SO(32)$ theory that is less amenable to analysis through the stable degeneration limit. Further development of heterotic/F-theory and other dualities through the structure of the low-energy theory and topological axion-curvature squared couplings may help to clarify how the various string constructions of 4D supergravity theories are related and to chart the space of 4D $\mathcal{N} = 1$ string compactifications.
We have identified some constraints on 4D supergravity theories that hold in the large-volume F-theory limit where the geometric moduli of the compactification remain light. The simplest of these constraints is a condition on a linear combination of the numbers of different types of fields \( (3.2) \) in the theory

\[
C_{sa} + C_{cs} - C_{21} \equiv 39 \pmod{60},
\]

(5.1)

for any theory with a completely broken gauge group. This constraint is an analogue of the gravitational anomaly constraints appearing in 6D supergravity theories, though there is no known gravitational anomaly in 4D that would give rise to constraints of this form. A more precise version of this constraint involves the canonical class \( K \) of the F-theory base, which is encoded in the axion-curvature squared terms. Note that a consequence of (5.1) is that any large-volume F-theory model must have at least 39 light scalar fields, independent of the distribution of fields between the various types. Both in 6D and in 4D there are also constraints from F-theory on the signs of curvature-squared terms proportional to \( R \wedge R \) and \( F \wedge F \). In the latter case this constraint simply follows in the low-energy theory from the condition that the gauge kinetic term have the standard (negative) sign for stability of the theory. In the case of the metric curvature terms, no low-energy reason for a sign constraint is known.

A key issue that should be incorporated better into the considerations of this paper regarding 4D F-theory vacua is the role of fluxes, and more generally world-volume fields on the 7-branes. While F-theory provides a simple and beautiful context for nonperturbative exploration of a large region of the space of possible string compactifications, this formulation of string theory is still incomplete. In particular, the description in terms of a Weierstrass model is not coupled to a natural description of gauge fields on the world volume, or fields such as the adjoint scalars in the Higgs bundle on 7-branes. Fluxes, however, play a key role in determining the physics of \( \mathcal{N} = 1 \) 4D string vacua. While these fluxes are conceptually clear in the M-theory picture, this framework loses the geometric picture of the Weierstrass model in the F-theory context. As a result, the tools for working simultaneously with F-theory Weierstrass models and fluxes are still at an early stage of development. To understand the kinds of constraints we have discussed here better, the incorporation of fluxes is clearly crucial. In particular, fluxes play a key role in determining the structure of matter in 4D theories. We leave the further integration of fluxes into the story begun here as a challenging open problem for future research.

In six dimensions, the lattice of dyonic string charges plays an important role in the structure of an \( \mathcal{N} = 1 \) supergravity theory. The structure of axion-curvature squared terms in 4D supergravity theories arising from F-theory suggests that the analogous lattice of axionic string charges should play a similar role in four-dimensional theories. Away from the large-volume F-theory limit, it seems that all axions in the theory can mix, but some integral structure will still be supplied by the underlying massive axionic string lattice. In particular, for any massive string state the associated magnetic charge will correspond to an axionic shift symmetry. Even when supersymmetric string states are very massive, they still play a role in the basic symmetry structure of the theory. A better physical
understanding of the axionic string lattice in low-energy $\mathcal{N} = 1$ supergravity theories may be a key to applying the methods and results of this paper to deeper issues in the structure of 4D theories. It may be that further consideration of the world-volume theory on the charged strings will shed light on constraints and/or structure in general 4D supergravity theories.

A central question in the study of string compactifications is the extent to which the geometry of the compactification can be uniquely identified from data in the supergravity theory. In six dimensions, the story in this regard is quite clear. In many cases, knowledge of the spectrum and Green-Schwarz coefficients of the low-energy supergravity theory is sufficient to uniquely determine the geometry of a corresponding F-theory construction, when one exists. In all cases, further knowledge of the spectrum and Dirac-quantized charge products of supersymmetric dyonic string states fixes the intersection product and Mori cone of the base, uniquely determining the F-theory geometry when it exists. In four dimensions, the story is more complicated. Only in the large-volume regime of F-theory are the geometric moduli of the F-theory compactification clearly distinguishable from other massive modes in the theory. In this regime the couplings between axions and curvature-squared terms play a similar role to the Green-Schwarz terms in six dimensions. Combining this information with the string charge lattice, triple intersection product, and other information from the supergravity spectrum and action it is in principle possible to determine a corresponding F-theory geometry. Beyond the lifting of moduli by fluxes and other effects, however, there are also additional fields, such as the axion that is associated with the axiodilaton in the weak coupling limit, which must be disentangled in order to identify the F-theory geometry. While in the case of heterotic/F-theory duality we were able to explicitly identify the correspondence between axions to determine the map between heterotic and F-theory geometry, it is not clear how this can be done in general. It can be, for example, that there are dual F-theory compactifications on distinct threefold bases that give equivalent 4D physics, but with a different distribution of axions between complex structure moduli and other fields. For example, this may occur if a Calabi-Yau fourfold admits two distinct elliptic fibrations with different bases. Situations of this kind in the dual heterotic setting have been discussed, for example, in [112, 113]. Further analysis of the extent to which 4D supergravity determines compactification geometry promises to lead to new insights into the global structure of the space of string vacua, for which the tools and methods developed in this paper may be of some use.

Another direction in which the methods of this paper may be applied is towards the systematic understanding of the space of elliptically fibered fourfolds underlying the space of 4D F-theory models. Just as for elliptically fibered threefolds, elliptically fibered fourfolds form a complicated moduli space, with continuous branches connected by non-Higgs type phase transitions such as tensionless string transitions. Recently, a global exploration of the space of threefold bases for 6D F-theory vacua has been initiated [17, 18], following the minimal model approach [103]. The structure of the axion-curvature squared terms and constraints described in this work may provide useful tools in exploring the space of 4D F-theory vacua, and a better understanding of the connections between the branches.
of the theory associated with different bases and the exotic transitions connecting these theories.

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A. Blowing up curves and points in a smooth threefold base

In this appendix we summarize the necessary equations to discuss the blow-ups of points and smooth curves in a threefold $B$. We denote the blown-up space by $B'$ and name the blow-down map $\pi: B' \to B$. The exceptional divisor obtained after blow-up is denoted by $E$.

We first consider the blow-up of a point in $B$. The exceptional divisor $E$ in this blow-up is simply a $\mathbb{P}^2$. The Chern classes of the threefold change as

$$
c_1(B') = \pi^*c_1(B) - 2[E], \quad c_2(B') = \pi^*(c_2(B)), \quad \text{(A.1)}
$$

We will also need the intersection numbers after blow-up. One finds that

$$
E^2 = f, \quad E \cdot f = -1, \quad E \cdot \pi^*D = E \cdot \pi^*\tilde{C} = f \cdot \pi^*D = 0, \quad \text{(A.2)}
$$

for all divisors $D$ and curves $\tilde{C}$ in $B$.

Let us now turn to the blow-up of a smooth curve $C$. The exceptional divisor is then given by the projectivisation of the normal bundle $N_BC$ of the curve in $B$, i.e. $E = \mathbb{P}(N_BC)$. The Chern classes of $B$ now change as

$$
c_1(B') = \pi^*c_1(B) - [E], \quad \text{(A.3)}
$$

$$
c_2(B') = \pi^*(c_2(B) + [C]) - [E] \wedge \pi^*c_1(B), \quad \text{(A.4)}
$$

The blow-up space has the following intersection numbers

$$
E^2 = -\pi^*C + \text{deg}(N_BC)f, \quad E \cdot f = -1, \quad \text{(A.5)}
$$

$$
E \cdot \pi^*D = (C \cdot D)f, \quad f \cdot \pi^*D = 0, \quad E \cdot \pi^*\tilde{C} = 0, \quad \text{(A.6)}
$$

for all divisors $D$ and curves $\tilde{C}$ in $B$. The degree of $N_BC$ can be written as

$$
\text{deg}(N_BC) = -\chi(C) + \int_C c_1(B). \quad \text{(A.7)}
$$
B. Anomalies in 6D supergravity

The anomaly cancellation condition can be written in terms of the 8-form anomaly polynomial as \[ I_8(R, F) = \frac{1}{2} \Omega_{\alpha\beta} X_4^\alpha X_4^\beta. \] (B.1)

Here

\[ X_4^\alpha = \frac{1}{2} a^\alpha \text{tr} R^2 + \sum_i b_i^\alpha \left( \frac{2}{\lambda_i} \text{tr} F_i^2 \right) \] (B.2)

with \( a^\alpha, b_i^\alpha \) transforming as vectors in the space \( \mathbb{R}^{1,T} \) with symmetric inner product \( \Omega_{\alpha\beta} \); “tr” of \( F_i^2 \) denotes the trace in the fundamental representation, and \( \lambda_i \) are normalization constants depending on the type of each simple group factor. Cancellation of the individual terms in (B.1) gives

\[ H - V = 273 - 29T \] (B.3)

\[ 0 = B_\text{adj}^i - \sum_R X_R^i B_R^i \] (B.4)

\[ a \cdot a = 9 - T \] (B.5)

\[ -a \cdot b_i = \frac{1}{6} \lambda_i \left( \sum_R x_R^i A_R^i - A_\text{adj}^i \right) \] (B.6)

\[ b_i \cdot b_i = \frac{1}{3} \lambda_i^2 \left( \sum_R x_R^i C_R^i - C_\text{adj}^i \right) \] (B.7)

\[ b_i \cdot b_j = \lambda_i \lambda_j \sum_{RS} x_{RS}^{ij} A_R^i A_S^j \] (B.8)

where \( A_R, B_R, C_R \) are group theory coefficients defined through

\[ \text{tr}_R F^2 = A_R \text{tr} F^2, \quad \text{tr}_R F^4 = B_R \text{tr} F^4 + C_R (\text{tr} F^2)^2, \] (B.9)

and where \( x_R^i \) and \( x_{RS}^{ij} \) denote the number of matter fields that transform in the irreducible representation \( R \) of gauge group factor \( G_i \), and \((R, S)\) of \( G_i \otimes G_j \) respectively. Note that for groups such as \( SU(2) \) and \( SU(3) \), which lack a fourth order invariant, \( B_R = 0 \) and there is no condition (B.4).

It is shown in [7] using elementary group theory that the inner products on the LHS of conditions (B.5-B.8) are all integral as a consequence of global and local anomaly cancellation. This gives an integral anomaly lattice \( \Lambda \) formed from vectors \( a, b_i \in \mathbb{R}^{1,T} \). The vector \( a \) is associated with a coupling \( a \cdot B \text{ tr} R^2 \) of the \( B \) fields to space-time curvature, while the vectors \( b_i \) are associated with couplings \( b_i \cdot B \text{ tr} F_i^2 \) of the \( B \) fields to the field strengths \( F_i \) of the various factors in the gauge group; together these terms form the Green-Schwarz counterterm

\[ B \cdot X_4 = B^\alpha \Omega_{\alpha\beta} \left[ \frac{1}{2} a^\beta \text{tr} R^2 + \sum_i b_i^\beta \left( \frac{2}{\lambda_i} \text{tr} F_i^2 \right) \right] \] (B.10)

The lattice \( \Lambda \) is a sublattice of the complete unimodular lattice \( \Gamma \) of dyonic string charges for the 6D theory.
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