Magnetic well and Mercier stability of stellarators near the magnetic axis

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We have recently demonstrated that by expanding in small distance from the magnetic axis compared to the major radius, stellarator shapes with low neoclassical transport can be generated efficiently. To extend the utility of this new design approach, here we evaluate measures of magnetohydrodynamic interchange stability within the same expansion. In particular, we evaluate magnetic well, Mercier’s criterion, and resistive interchange stability near a magnetic axis of arbitrary shape. In contrast to previous work on interchange stability near the magnetic axis, which used an expansion of the flux coordinates, here we use the ‘inverse expansion’ in which the flux coordinates are the independent variables. Reduced expressions are presented for the magnetic well and stability criterion in the case of quasisymmetry. The analytic results are shown to agree with calculations from the VMEC equilibrium code. Finally, we show that near the axis, Glasser, Greene, & Johnson’s stability criterion for resistive modes approximately coincides with Mercier’s ideal condition.

1. Introduction

The geometry of stellarators can be optimized to achieve properties such as low neoclassical transport and good magnetohydrodynamic (MHD) stability. In the design of recent experiments like W7-X (Beidler et al. 1990), HSX (Anderson et al. 1995), and NCSX (Zarnstorff et al. 2001), this optimization was done by wrapping a 3D MHD equilibrium code with a standard numerical minimization algorithm. As with any numerical calculation, this approach provides little information about the possible existence of other solutions, and it is known that the optimization algorithm can get trapped in local minima. An older approach for relating stellarator geometry to physics properties is to make an asymptotic expansion in large local aspect ratio (Mercier 1964; Solov’ev & Shafranov 1970; Mercier & Luc 1974; Lortz & Nührenberg 1976; Garren & Boozer 1991b). Such an expansion is accurate in the core of any stellarator, even those for which the aspect ratio of the boundary is low (Landreman 2019). We have recently argued (Landreman et al. 2019; Landreman & Sengupta 2019; Jorge et al. 2020a) that this asymptotic approach deserves further attention, as it complements numerical optimization. The asymptotic approach allows equilibria to be evaluated orders of magnitude faster, and it provides a practical way to generate new initial conditions for numerical optimization. In the present paper we extend our asymptotic approach, which has previously focused on neoclassical confinement, to MHD stability. Focusing on interchange modes, we show how stability can be computed directly from a solution of the reduced near-axis equations. These results enable more comprehensive design within the near-axis approximation.

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A condition for stability of radially localized ideal-MHD interchange modes, known as ‘Mercier’s criterion’, was derived by Mercier (1962, 1964) and Greene & Johnson (1961, 1962). An important related quantity is the ‘magnetic well’. In the absence of a pressure gradient, the magnetic well is \(d^2V/d\psi^2\), where \(V(\psi)\) is the volume enclosed by a flux surface and \(2\pi\psi\) is the toroidal flux. When the pressure is nonuniform, several generalized expressions for magnetic well can be defined (Greene 1998). As shown by Mercier (1964), the magnetic well is the largest term in Mercier’s criterion if one expands in large aspect ratio and makes a subsidiary expansion in \(2\mu_0p/B^2 \ll 1\). (This ratio is not assumed to be small in our calculations here.) Mercier stability and magnetic well are both commonly used in stellarator design (Anderson et al. 1995; Drevlak et al. 2019). Mercier’s criterion can be generalized to the case of nonzero plasma resistivity, giving a stricter stability condition (Glasser et al. 1975).

Already from Mercier’s original work on the ideal stability criterion, the limit of the criterion near the magnetic axis has been examined. However such previous work has generally employed the ‘direct’ expansion, in which the function \(\psi(\rho)\) is expanded, with \(\rho\) the Euclidean distance from the axis (Mercier 1964; Solov’ev & Shafranov 1970; Jorge et al. 2020). We instead follow the ‘inverse’ expansion of Garren & Boozer (1991b, a), in which the flux coordinates are the independent variables, and the position vector is expanded as a function of these variables. The direct and indirect expansions each have advantages and disadvantages. Here we focus on Garren & Boozer’s expansion because interchange stability has not previously been analyzed in this approach, and because it is convenient for obtaining configurations with omnigenity and high-order quasisymmetry (Plunk et al. 2019; Landreman & Sengupta 2019).

Following Mercier’s original work on near-axis stability, many researchers examined the stability criterion near the axis in axisymmetry (Laval et al. 1971; Küppers & Tasso 1972; Lortz & Nührenberg 1973; Mikhailovskii 1974; Weimer et al. 1975). These results for axisymmetry have been reviewed by Greene (1998) and Freidberg (2014). In nonaxisymmetric geometry, the magnetic well near the axis in vacuum was examined by Whiteman et al. (1965). Later work on stellarator Mercier stability near the axis has mostly examined special cases such as that of constant elongation (Shafranov & Yurchenko 1968), circular cross-section (Mikhailovskii & Aburdzhaniya 1979), or a planar axis (Rizk 1981).

It should be acknowledged that Mercier stability may not be critical experimentally. The LHD, W7-AS, and TJ-II stellarator experiments all have operated in Mercier-unstable regimes, without obvious strong experimental signatures when stability boundaries are crossed (Geiger et al. 2004; Watanabe et al. 2005; Weller et al. 2006; de Aguilera et al. 2015). These studies provide some indications that turbulence may increase in Mercier-unstable regimes. These observations also suggest that Mercier instabilities in stellarators may saturate at low amplitude. Nonetheless, one may still wish to include Mercier stability as a condition in new designs to minimize any possible turbulent transport associated with this class of instability.

The remainder of this paper is organized as follows. We begin the detailed calculations in the next section by defining variables and reviewing the asymptotic expansion. Then in section 3 we compute several variants of magnetic well. Mercier’s criterion is evaluated in section 4. For each of these last two sections, we present expressions for both a general stellarator and a quasisymmetric one, as a number of simplifications occur in quasisymmetry. Sections 3 and 4 also include demonstrations that our near-axis expressions agree with finite-aspect-ratio calculations using the VMEC code (Hirshman & Whitson 1983) in the appropriate limit. Then in section 5 resistive interchange stability is examined, and it is shown that the the criterion of Glasser et al. (1975) coincides
with Mercier’s condition near the axis. Previously published expressions for Mercier stability have often been derived assuming that quantities such as the toroidal flux and/or Jacobian are positive; in the appendix we show how these expressions generalize to allow other signs.

2. Notation

We will use the expansion developed by Garren & Boozer (1991) and the notation of Landreman & Sengupta (2019). The notation and expansion are summarized here for convenience. Let $\theta$ and $\phi$ denote the Boozer poloidal and toroidal angles respectively, and let $2\pi\psi$ be the toroidal flux. Then the magnetic field can be written

$$B = \nabla \psi \times \nabla \theta + \nu \nabla \phi \times \nabla \psi,$$

$$= \beta \nabla \psi + I \nabla \theta + G \nabla \phi,$$

where $I$ and $G$ are constant on flux surfaces. In case one wishes to consider quasi-helical symmetry, it is convenient to introduce a helical angle $\vartheta = \theta - N\phi$ where $N$ is a constant integer; $N$ can be set to zero if not considering quasi-helical symmetry. Defining $\iota_N = \iota - N$, then

$$B = \nabla \psi \times \nabla \vartheta + \iota_N \nabla \phi \times \nabla \psi,$$

$$= \beta \nabla \psi + I \nabla \vartheta + (G + NI) \nabla \phi.$$  

At any location in the plasma we can express the position vector $r$ as

$$r(r, \vartheta, \phi) = r_0(\phi) + X(r, \vartheta, \phi)n(\phi) + Y(r, \vartheta, \phi)b(\phi) + Z(r, \vartheta, \phi)t(\phi).$$

Here $r_0(\phi)$ is the position vector along the magnetic axis, $r(\psi)$ is an effective minor radius defined by $2\pi\psi = \pi r^2 B$, and $B$ is a constant reference field strength of the same sign as $\psi$. The Frenet-Serret frame of the axis $(t, n, b)$ is a set of orthonormal vectors satisfying $t \times n = b$ and

$$\frac{d\varphi}{d\ell} \frac{dr_0}{d\varphi} = t, \quad \frac{d\varphi}{d\ell} \frac{dt}{d\varphi} = \kappa n, \quad \frac{d\varphi}{d\ell} \frac{dn}{d\varphi} = -\kappa t + \tau b, \quad \frac{d\varphi}{d\ell} \frac{db}{d\varphi} = -\tau n.$$

Here $\ell$ is the arclength along the axis, $\kappa(\varphi)$ is the axis curvature, and $\tau(\varphi)$ is the axis torsion. (The opposite sign convention for torsion is used by Garren and Boozer.)

Let $R$ denote the scale length of the axis, i.e. $R \sim 1/\kappa \sim 1/\tau$, and let $\epsilon = r/R$. We now expand in $\epsilon \ll 1$. The coefficients $X, Y,$ and $Z$ are expanded in the following way:

$$X(r, \vartheta, \phi) = rX_1(\vartheta, \phi) + r^2X_2(\vartheta, \phi) + r^3X_3(\vartheta, \phi) + \ldots.$$  

We expand $B$ and $\beta$ in the same way but with an $r^0$ term:

$$B(r, \vartheta, \phi) = B_0(\phi) + rB_1(\vartheta, \phi) + r^2B_2(\vartheta, \phi) + r^3B_3(\vartheta, \phi) + \ldots.$$  

The radial profile functions $G(r), I(r), p(r),$ and $\iota_N(r)$ must be symmetric under $r \rightarrow -r$, so only even powers of $r$ are included in their expansions:

$$p(r) = p_0 + r^2p_2 + r^4p_4 + \ldots.$$  

The profile $I(r)$ is proportional to the toroidal current inside the surface $r$, so $I_0 = 0$. The magnetic field must be smooth, so as discussed in appendix A of Landreman & Sengupta.
that appear in the literature: et al. included as a fast-to-evaluate proxy for MHD stability (Anderson 3. Magnetic well
Garren & Boozer’s symbol
strength variation, field as discussed in the appendix. can be flipped individually by reversing the signs of the poloidal or toroidal angle, as equations are displayed in Garren & Boozer (1991 also be imposed if desired. The result is a set of equations at each order in $J$ using the dual relations. The results are substituted into the equations (2.2)
The same form applies to the expansion coefficients of $s$
quantities:
Sengupta (2019).
Negative $V$ is favorable for stability, as is positive $\hat{\eta}$
or positive $\hat{\eta}$ as a measure of the field strength variation, $B = B_0[1 + r\hat{\eta}\cos \vartheta + O(\epsilon^2)]$.

3. Magnetic well
In stellarator optimization, the magnetic well (Greene 1998; Freidberg 2014) is often included as a fast-to-evaluate proxy for MHD stability (Anderson et al. 1995; Drevlak et al. 2019). For completeness, here we will consider three expressions for magnetic well that appear in the literature: $V'' = d^2V/d\psi^2$,
\[
\hat{W} = \frac{V}{\langle B^2 \rangle} \frac{d\langle B^2 \rangle}{dV},
\]
and
\[
W = \frac{V}{\langle B^2 \rangle} \frac{d}{dV} \langle 2\mu_0 p + B^2 \rangle.
\]
Here $\langle \ldots \rangle$ denotes a flux surface average, and $V(\psi)$ is the volume enclosed by the surface $\psi$. Throughout this paper we consider $V$ to be non-negative, hence
\[
V(\psi) = s_\psi \int_0^\psi d\psi \int_0^{2\pi} d\vartheta \int_0^{2\pi} \sin \vartheta d\varphi |\sqrt{g}|,
\]
with $\sqrt{g} = (G + iI)/B^2$. For any quantity $Q$, the flux surface average is
\[
\langle Q \rangle = \frac{s_\psi |G + iI|}{V'} \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \frac{Q B^2}{B^2}
\]
where $V' = dV/d\psi = s_\psi |G + iI|^{2\pi} d\vartheta [2\pi d\varphi B^{-2}$. Using $\langle B^2 \rangle = 4\pi^2 s_\psi |G + iI|/V'$, the first two definitions of magnetic well are related by
\[
\hat{W} = -\frac{VV''}{(V')^2} + \frac{V d\ln |G + iI|}{V'}.
\]
Negative $V''$ is favorable for stability, as is positive $\hat{W}$ or positive $W$. 

(2018), the expansion coefficients must have the form
\[
X_1(\vartheta, \varphi) = X_{1s}(\varphi) \sin(\vartheta) + X_{1c}(\varphi) \cos(\vartheta),
\]
\[
X_2(\vartheta, \varphi) = X_{20}(\varphi) + X_{2s}(\varphi \sin(2\vartheta) + X_{2c}(\varphi) \cos(2\vartheta),
\]
\[
X_3(\vartheta, \varphi) = X_{3s3}(\varphi) \sin(3\vartheta) + X_{3s1}(\varphi) \sin(\vartheta) + X_{3c3}(\varphi) \cos(3\vartheta) + X_{3c1}(\varphi) \cos(\vartheta).
\]
Note that in vacuum, \( W = \hat{W} \) and the last term in (3.5) vanishes, leaving the ratio \( \hat{W}/V'' \) equal to the negative-definite quantity \( -V/(V')^2 \). Therefore the three measures of magnetic well provide equivalent information in the limit of small plasma pressure.

### 3.1. First form of magnetic well

The first quantity we consider is \( V'' \). To evaluate this quantity, we begin with (3.3). We insert the near-axis expansions and apply \( d^2/d\psi^2 \), noting \( d/d\psi = (r B)^{-1} d/dr, \) sign(\( B \)) = \( s_0 \), and eq (A50) of Landreman & Sengupta (2019). We thereby obtain

\[
V'' = 2\pi \left| \frac{G_0}{B} \right| \int_0^{2\pi} d\varphi \left[ \frac{1}{B_0^3} \left( 3 (B_{1s}^2 + B_{1c}^2) - 4B_0 B_{20} - \frac{\mu_0 \rho_2 B_0^2}{\pi} \int_0^{2\pi} \frac{d\varphi}{B_0(\varphi)^2} \right) \right] + O(\epsilon^2). \tag{3.6}
\]

Here, \( B_{1s}, B_{1c}, B_{20}, \) and \( B_0 \) are functions of \( \varphi \), except where \( B_0 \) is evaluated at \( \hat{\varphi} \) as noted. This formula applies to any toroidal plasma, not only quasisymmetric ones. Notice that the leading-order magnetic well depends on the \( O(\epsilon^2) \) variation of \( B \), so a \( O(\epsilon^1) \) solution is not accurate enough to compute the well.

In the special case of quasisymmetry, \( B_0 \) becomes independent of \( \varphi \), and we take \( \hat{B} = s_0 B_0 \). Also \( B_{1s} = 0 \), and \( B_{1c} = \tilde{\eta} B_0 \). While \( B_{20} \) is independent of \( \varphi \) in quasisymmetry, it is convenient to relax this requirement for practical construction of quasisymmetric configurations, as described in Landreman & Sengupta (2019). The elimination of the requirement \( dB_{20}/d\varphi = 0 \) makes it possible to obtain solutions for any shape of the magnetic axis. Therefore here we will not demand that \( B_{20} \) be independent of \( \varphi \). We thus obtain the following expression for the magnetic well in quasisymmetry:

\[
V'' = \frac{4\pi^2 G_0}{B_0^3} \left[ 3\tilde{\eta}^2 - \frac{4\hat{B}_{20}}{B_0} - \frac{2\mu_0 \rho_2}{B_0^2} \right] + O(\epsilon^2), \tag{3.7}
\]

where \( \hat{B}_{20} = (2\pi)^{-1} \int_0^{2\pi} d\varphi B_{20} \).

### 3.2. Second form of magnetic well

The alternative magnetic well quantity \( \hat{W} \) in (3.5) can be evaluated in the near-axis expansion by substituting the series for \( B \) into

\[
\langle B^2 \rangle = 4\pi^2 \left( \int_0^{2\pi} d\varphi \int_0^{2\pi} \frac{d\varphi}{B^2} \right)^{-1}, \tag{3.8}
\]

keeping terms through \( O(\epsilon^2) \). Then substituting the result into (3.5), one obtains

\[
\hat{W} = 2r^2 \left( \int_0^{2\pi} \frac{d\varphi}{B_0^2} \right)^{-1} \int_0^{2\pi} \frac{d\varphi}{B_0^4} \left[ -\frac{3}{4}(B_{1s}^2 + B_{1c}^2) + \hat{B}_{20} B_0 \right], \tag{3.9}
\]

for a general toroidal plasma. In the special case of quasisymmetry, this expression reduces to

\[
\hat{W} = 2r^2 \left( -\frac{3}{4}\tilde{\eta}^2 + \frac{\hat{B}_{20}}{B_0} \right). \tag{3.10}
\]

### 3.3. Third form of magnetic well

Evaluating \( (2\mu_0 V/\langle B^2 \rangle) dp/dV \) near the axis and adding the result to (3.9) gives an expression for (3.2):

\[
W = 2r^2 \left( \int_0^{2\pi} \frac{d\varphi}{B_0^2} \right)^{-1} \int_0^{2\pi} \frac{d\varphi}{B_0^4} \left[ -\frac{3}{4}(B_{1s}^2 + B_{1c}^2) + \hat{B}_{20} B_0 \right] + \frac{\mu_0 r^2 p_2}{\pi} \int_0^{2\pi} \frac{d\varphi}{B_0^2}. \tag{3.11}
\]
Figure 1. Verification of the magnetic well calculation. The five colors indicate the five quasisymmetric configurations detailed in section 5 of Landreman & Sengupta (2019). Triangles show \( \frac{d^2V}{d\psi^2} \) evaluated from (3.7). Points connected by lines show \( \frac{d^2V}{d\psi^2} \) computed on the magnetic axis of VMEC configurations constructed with various values of the boundary effective minor radius \( a \), all with mean major axis radius \( R = 1 \) m. As \( a/R \to 0 \), the VMEC results converge to (3.7), verifying the calculations.

This result applies to a general toroidal plasma. In quasisymmetry, (3.11) reduces to

\[
W = 2r^2 \left( -\frac{3}{4} \hat{\eta}^2 + \frac{B_{20}}{B_0} + \frac{\mu_0 p_2}{B_0^2} \right) .
\]

3.4. Numerical verification

The preceding near-axis approach to evaluating magnetic well can be compared to the magnetic well of finite-aspect-ratio MHD solutions. Figure 1 shows such a comparison for five families of magnetic configuration, the five cases considered in section 5 of Landreman & Sengupta (2019). This set includes both quasi-axisymmetric and quasi-helically symmetric configurations, and includes both vacuum fields and configurations with plasma pressure and current. Each family is based upon a single solution of the near-axis equations. Substituting several finite values \( a \) into the effective minor radius \( r \) then yields a set of boundary toroidal surfaces, which are each provided as input to fixed-boundary MHD solutions using the VMEC code (Hirshman & Whitson 1983). In figure 1, triangles show the magnetic well evaluated from eq (3.7) for the underlying near-axis solution, while dots show \( \frac{d^2V}{d\psi^2} \) evaluated on the magnetic axis from the corresponding VMEC solutions. For the example of section 5.5, values are divided by 100 in order to fit on the same axes. For all five examples, as the boundary minor radius \( a \) is reduced, the VMEC results converge to the near-axis results as desired. The code and data for these calculations can be obtained online (Landreman 2020a,b).

4. Mercier Criterion

The Mercier criterion (Mercier 1964; Mercier & Luc 1974) is a geometrical quantity that, in the context of ideal MHD, allows us to assess the stability of the plasma to radially localized perturbations around a rational surface. Using the present notation,
the criterion can be written as $D_{\text{Merc}} > 0$ where

$$(2\pi)^6 D_{\text{Merc}} = \left[ s_G 2\pi^2 \frac{dI}{d\psi} - \int \frac{B \cdot \Xi}{|\nabla\psi|^3} dS \right]^2 + \left[ s_\psi \mu_0 \frac{dp}{d\psi} \sqrt{2V} - \int \frac{|\Xi|^2 dS}{|\nabla\psi|^3} \right] \int \frac{B^2 dS}{|\nabla\psi|^3},$$

(4.1)

and $\Xi = \mu_0 J - I'(\psi) B$. The coefficients $s_G = \pm 1$ and $s_\psi = \pm 1$ correspond to the signs of $G$ and $\psi$, respectively, and ensure invariance under certain changes of coordinates (see Appendix A). We note that $D_{\text{Merc}}$ corresponds to Eq. (37) in Mercier & Luc (1974) multiplied by $\psi$, and the integrals are performed along a surface of constant $\psi$ such that $dS = |\nabla\psi| |\sqrt{g}| d\psi d\varphi$ with $\sqrt{g}$ the Jacobian in $(\psi, \vartheta, \varphi)$ coordinates.

Our goal is to evaluate Eq. (4.1) using the near-axis expansion based on the Garren-Boozer formalism. We focus on the leading order terms of the Mercier criterion, which in our expansion amounts to computing the $O(\epsilon^{-2})$ component for each term in Eq. (4.1). As shown below, a comparison with numerical results is made using an equivalent form of the Mercier criterion appearing in Bauer et al. (1984); Ichiguchi et al. (1993).

### 4.1. Computation of the criterion at lowest order

We start by ordering the terms appearing in Eq. (4.1). In the first brackets, the term proportional to the magnetic shear $dI/d\psi$ is $O(\epsilon^0)$. The term proportional to $B \cdot \Xi$ is also $O(\epsilon^0)$. To show this we first note from (2.3) that

$$\mu_0 J \cdot B = \left( \frac{G}{rB} \frac{dI}{dr} - \frac{I}{rB} \frac{dG}{dr} - (G + IN) \frac{\partial \beta}{\partial \vartheta} + I \frac{\partial \beta}{\partial \varphi} \right) \frac{B^2}{G + iI}. \quad (4.2)$$

Using this result we obtain

$$\int \frac{B \cdot \Xi}{|\nabla\psi|^3} dS \simeq \int \frac{\sqrt{g} d\vartheta d\varphi}{|\nabla\psi|^2} [rB_0^2 (\beta_1 \sin \vartheta - \beta_1 \cos \vartheta) + O(\epsilon^2)]. \quad (4.3)$$

To evaluate the denominator, $\nabla\psi = \sqrt{g}^{-1} \partial\psi / \partial\vartheta \times \partial\psi / \partial\varphi$ with (2.4)-(2.6) gives

$$|\nabla\psi|^2 \simeq \frac{r^2 B_0^2 V_1}{2} (1 + a \cos 2\vartheta + b \sin 2\vartheta), \quad (4.4)$$

where

$$V_1 = X_1^2 + X_2^2 + Y_1^2 + Y_2^2, \quad (4.5)$$

$$a = (X_1^2 - X_2^2 + Y_2^2 - Y_1^2)/V_1, \quad (4.6)$$

$$b = -2(X_1 X_2 Y_1 + Y_1 Y_2 X_1)/V_1. \quad (4.7)$$

Therefore $|\nabla\psi|^2 \simeq O(\epsilon^2)$, and the overall $O(\epsilon^{-1})$ contribution to (4.3) vanishes upon integration over $\vartheta$, leaving the $B \cdot \Xi$ term in Eq. (4.1) as $O(\epsilon^0)$.

Next, we evaluate the term in Eq. (4.1) proportional to $|\Xi|^2$. Using $|J|^2 = (J \cdot B)^2 / B^2 + (dp/d\psi)^2 |\nabla\psi|^2 / B^2$, we obtain

$$|\Xi|^2 = \left( \mu_0 \frac{dp}{d\psi} \right)^2 \frac{|\nabla\psi|^2}{B^2} + \frac{r^2 B_0^2}{2} \left[ \beta_1^2 + \beta_1^2 + (\beta_2^2 - \beta_1^2) \cos 2\vartheta - 2\beta_1 \beta_1 \sin 2\vartheta \right] + O(\epsilon^3). \quad (4.8)$$

Carrying out the integrations over $\vartheta$, the term in (4.1) is found to be

$$I_{\Xi^2} = \int \frac{|\Xi|^2 dS}{|\nabla\psi|^3} = 2\pi |G_0| \left( 4\mu_0^2 p_2^2 B^2 \int_0^{2\pi} \frac{d\varphi}{B_0^4} + I_\beta \right), \quad (4.9)$$

where $\beta = \beta_1 = \beta_2 = 0$. The numerical results are compared with the exact solution of the Garren-Boozer formalism for $\iota = 0$.
where

\[ I_\beta = \int_0^{2\pi} \frac{d\varphi}{B_0^2 V_1} \left( a^2 + b^2 \right) \left( (\beta_{1s}^2 + \beta_{1c}^2) + \left( \sqrt{1 - a^2 - b^2} - 1 \right) [a(\beta_{1s}^2 - \beta_{1c}^2) - 2b\beta_{1s}\beta_{1c}] \right) \frac{\sqrt{1 - a^2 - b^2}(a^2 + b^2)}{\sqrt{1 - a^2 - b^2}(a^2 + b^2)}. \]  (4.10)

Finally, the integral at the end of (4.1) is

\[ \int \frac{B^2 dS}{|\nabla \psi|^3} \approx \frac{2\pi|G_0|}{r^2|B|} \int \frac{d\varphi}{B_0} = \frac{2\pi L}{r^2|B|}, \]  (4.11)

where \( V_1 \sqrt{1 - a^2 - b^2} = 2|X_{1s} Y_{1c} - X_{1c} Y_{1s}| = 2|\vec{B}|/B_0 \) was used, and \( L = |G_0| \int_0^{2\pi} d\varphi/B_0 > 0 \) is the axis length. We thereby obtain the following form for the Mercier criterion at lowest order in \( \epsilon \):

\[ D_{\text{Merc}} = \frac{\mu_0 p_2 L}{16\pi^2 r^2 B^2} \left[ \frac{d^2 V}{d\psi^2} - \frac{2\pi|G_0|\mu_0 p_2}{|B|} \int_0^{2\pi} \frac{d\varphi}{B_0^2} - \frac{\pi|G_0| B_0 I_\beta}{\mu_0 p_2} \right]. \]  (4.12)

In this expression, \( V'' \) can be evaluated using (3.6).

Mercier (1964) and Mercier & Luc (1974) observed that the quantity in the first pair of square brackets in (4.1) is smaller than the second near the axis, consistent with our calculation here. Also, noting that \( \beta_1 \propto p_2 \) (see (A.52) in Landreman & Sengupta (2019)), (4.12) shows that as the pressure gradient becomes small (\( p_2 \rightarrow 0 \)), the magnetic well \( V'' \) becomes the dominant term in \( D_{\text{Merc}} \). If the limit of a vacuum field is taken before the near-axis limit, the result is different, \( D_{\text{Merc}} \rightarrow (d^2 V/d\psi^2)/\left(16\pi^2 \right) \).

In the case of quasisymmetry, a number of simplifications are possible. In this case, as shown in Appendix A.3 of Landreman & Sengupta (2019), \( B_0 = \text{constant}, \bar{B} = s_{\psi} B_0, |X_{1c} Y_{1s}| = 1, \beta_{1c} = 0, \) and

\[ \beta_{1s} = -\frac{4s_{\psi} \mu_0 p_2 G_0 \bar{\eta}}{\iota N_0 B_0^3}. \]  (4.13)

Therefore (4.10) reduces to

\[ I_\beta = \frac{16\mu_0^2 \mu_2^2 G_0^2 \bar{\eta}^2}{B_0^2 \iota N_0} \int_0^{2\pi} d\varphi \left( X_{1c}^2 + Y_{1c}^2 + 1 \right) \left( X_{1c}^2 + Y_{1c}^2 + Y_{1s}^2 + 2 \right) \]  (4.14)

\[ = \frac{16\mu_0^2 \mu_2^2 G_0^2 \bar{\eta}^2}{B_0^2 \iota N_0} \int_0^{2\pi} d\varphi \left( \bar{\eta}^4 + \kappa^4 \sigma^2 + \bar{\eta}^2 \kappa^2 \right) \left( \bar{\eta}^4 + \kappa^4 (1 + \sigma^2) + 2\bar{\eta}^2 \kappa^2 \right), \]

where \( \sigma = Y_{1c}/Y_{1s} \) is the quantity appearing in eq (A6) of Garren & Boozer (1991a) and (2.14) of Landreman & Sengupta (2019). Then (4.12) becomes

\[ D_{\text{Merc}} = \left| G_0 \right| \mu_0 p_2 \left[ \frac{d^2 V}{d\psi^2} - \frac{8\pi^2 |G_0| \mu_0 p_2}{B_0^5} \right] \]  (4.15)

\[ = -\frac{16\pi |G_0|^3 \mu_0 p_2 \bar{\eta}^2}{B_0^5 \iota N_0} \int_0^{2\pi} d\varphi \left( \bar{\eta}^4 + \kappa^4 (1 + \sigma^2) + 2\bar{\eta}^2 \kappa^2 \right), \]

and (3.7) can be applied.

4.2. Alternative form of the Mercier criterion

An equivalent form of the Mercier criterion in Eq. (4.1) is given in Bauer et al. (1984); Ichiguchi et al. (1993):

\[ D_{\text{Merc}} = D_{\text{Shear}} + D_{\text{Curr}} + D_{\text{Well}} + D_{\text{Geod}} > 0, \]  (4.16)
where

\[ D_{\text{Shear}} = \frac{1}{16\pi^2} \left( \frac{ds}{d\psi} \right)^2, \]  

\[ D_{\text{Curr}} = -\frac{s_G}{(2\pi)^4} \frac{ds}{d\psi} \int dS \frac{\mathbf{\Sigma} \cdot \mathbf{B}}{|\nabla \psi|^3}, \]  

\[ D_{\text{Well}} = \frac{\mu_0}{(2\pi)^6} \frac{dp}{d\psi} \left( s_\psi^2 \frac{d^2 V}{d\psi^2} - \mu_0 \frac{dp}{d\psi} \int dS \frac{B^2}{|\nabla \psi|} \right) \int dS \frac{B^2}{|\nabla \psi|^3}, \]  

\[ D_{\text{Geod}} = \frac{1}{(2\pi)^6} \left( \int dS \mu_0 |\mathbf{J} \cdot \mathbf{B}| \right)^2 \]  

\[ = \frac{1}{(2\pi)^6} \left( \int dS \mu_0 |\mathbf{J} \cdot \mathbf{B}| \right)^2 - \frac{1}{(2\pi)^6} \left( \int dS \frac{B^2}{|\nabla \psi|^3} \right) \int dS \frac{B^2}{B^2 |\nabla \psi|^3}. \]  

These same quantities are reported by VMEC \cite{HirshmanWhitson1983}. (Again, we have included factors of \( s_G \) and \( s_\psi \), as discussed in the appendix.) The equivalence between Eqs. (4.11) and (4.16) can be shown using the identity \((\mathbf{J} \cdot \mathbf{B})^2/B^2 = J^2 - \rho'(\psi)^2 |\nabla \psi|^2/B^2\).

We now evaluate each term appearing in Eq. (4.16) at lowest order in \( \epsilon \). Recall that the scaling of \( D_{\text{Curr}} \) with \( \epsilon \) was evaluated following (4.3). We note that the \( D_{\text{Shear}} \) and \( D_{\text{Curr}} \) terms are \( O(\epsilon^0) \) while the \( D_{\text{Well}} \) and \( D_{\text{Geod}} \) terms are \( O(\epsilon^{-2}) \) so we only need to evaluate the latter two. These are given by

\[ D_{\text{Well}} = \frac{\mu_0 p_2 L}{16\pi^3 r^2 B^2} \left[ \frac{d^2 V}{d\psi^2} - \frac{4\pi \mu_0 p_2 |G_0|}{B} \int_0^{2\pi} d\varphi \right] \]  

and

\[ D_{\text{Geod}} = -\frac{|G_0| LI_\beta}{16\pi^4 r^2 |B|}, \]  

with \( I_\beta \) given by (4.10). To obtain \( D_{\text{Geod}} \), it is convenient to subtract (4.21) from (4.12).

In the case of quasisymmetry, these last expressions reduce to

\[ D_{\text{Well}} = \frac{\mu_0 p_2 |G_0|}{8\pi^4 r^2 B_0^3} \left[ \frac{d^2 V}{d\psi^2} - \frac{8\pi \mu_0 p_2 |G_0|}{B_0^5} \right], \]  

\[ D_{\text{Geod}} = -\frac{2\mu_0 p_2 G_0^2}{\pi^3 r^2 B_0^5} |\xi_0^2| \int_0^{2\pi} d\varphi \frac{q^4 + q^4 |\nabla \varphi|^2 + \nabla^2 |\nabla \varphi|^2}{\nabla^2 |\nabla \varphi|^2}. \]  

4.3. Numerical Verification

We now verify the analytic results of the previous section by comparing them to computations using the VMEC equilibrium code. As with figure 1, we first compute a numerical solution of the near-axis equations to \( O(\epsilon^2) \), then use the procedure of \cite{LandremanSengupta2019} to construct a magnetic surface surrounding this axis for \( r \) equal to a small finite value \( a \). This surface is then used as the prescribed boundary for a fixed boundary VMEC calculation, with quadratic pressure profile and uniform toroidal current density. VMEC reports all the individual quantities in (4.16) with the same normalization used here) as a standard diagnostic. VMEC’s calculation of \( D_{\text{Geod}} \) converges extremely slowly with the number of radial surfaces \( NS \), so all calculations shown here use \( NS \geq 801 \), and results from the innermost VMEC grid points are dropped.

The code and data for these calculations can be obtained online \cite{Landreman2020ab}. Figure 2 shows such a comparison for the quasi-axisymmetric configuration of section 5.3 of \cite{LandremanSengupta2019}. (The field strength has been doubled to 2 T to provide test coverage for the \( B_0 \) factors in the analytic expressions.) For this figure, the boundary aspect ratio \( A = R/a \) is chosen to be 40, where \( R = 1 \) m is the mean of the axis.
major radius over the standard toroidal angle. As predicted by theory, $D_{\text{Shear}}$ and $D_{\text{Curr}}$ are far smaller than the other terms in Mercier’s criterion. Equations (4.15) and (4.23) are evaluated using (3.7). Excellent agreement is seen between the asymptotic expressions (4.15), (4.23), and (4.24) and VMEC’s finite-aspect-ratio evaluations of $D_{\text{Merc}}$, $D_{\text{Well}}$, and $D_{\text{Geod}}$.

Figure 3 shows the same comparison repeated for several values of boundary aspect ratio. Again the agreement between the asymptotic expressions and VMEC calculations is excellent, particularly as $A$ increases. The quantities in the core of the $A = 10$ configuration can be seen to not exactly overlap those from the $A = 20$ configuration. This is because the boundary surface is constructed by extrapolating out from the axis approximately, but then VMEC computes the equilibrium inside that boundary without a near-axis approximation, leading to a slightly different axis shape than the original one.

Figure 4 presents a similar comparison to figure 3 but now using the example quasi-helically symmetric configuration of section 5.5 of [Landreman & Sengupta 2019]. This configuration provides a comprehensive test covering all effects, including departure from stellarator symmetry, and nonzero quasisymmetry number $N$, pressure $p_2$, and plasma current $I_2$. (The field strength has once more been doubled to 2 T to provide test coverage for the $B_0$ factors in the analytic expressions.) This configuration is limited to quite high aspect ratio due to large $O(\epsilon^2)$ coefficients, making the expansion only accurate at quite small $r$. Again, the VMEC results are seen to converge to the expressions (4.15), (4.23), and (4.24) as the boundary aspect ratio increases.

Finally, figures 5-6 present comparisons between our analytic expressions and VMEC’s results as we scan the input parameters $p_2$ and $\tilde{\eta}$ respectively. For these figures, the VMEC results are taken from the outermost VMEC grid point. Good agreement is seen across both parameter scans, providing comprehensive verification.
Figure 3. Numerical verification of equations (4.15), (4.23), and (4.24) for the terms in Mercier’s criterion, by comparison to the VMEC code. The magnetic configuration is the quasi-axisymmetric example in section 5.3 of Landreman & Sengupta (2019). Note the $y$ axis is a two-sided log scale, with the connecting linear-scale region shown in gray.

Figure 4. Numerical verification of equations (4.15), (4.23), and (4.24) for the terms in Mercier’s criterion, by comparison to the VMEC code. The magnetic configuration is the quasi-helically symmetric example in section 5.5 of Landreman & Sengupta (2019). Note the $y$ axis is a two-sided log scale, with the connecting linear-scale region shown in gray.
5. Necessary condition for resistive MHD stability

We next consider whether the stability of radially localized interchange modes near the axis is substantially different for resistive MHD compared to ideal MHD. In Glasser et al. (1975), assuming $D_{\text{Merc}} > 0$, a necessary condition for the plasma stability under the conditions with which the resistive MHD equations are valid was derived. The criterion is given by

$$D_R = E + F + 4\pi^2 (\ell')^{-2} H^2 \leq 0$$  \hspace{1cm} (5.1)
where

\[ E = \frac{1}{(2\pi)^6} \left[ A \int \frac{\mu_0 J \cdot B dS/|\nabla \psi|}{B^2 dS/|\nabla \psi|^3} \int B^2 dS/|\nabla \psi|^3 - (s_\psi V'' \mu_0 p' + \Lambda I') \int B^2 dS \right], \tag{5.2} \]

\[ F = \frac{1}{(2\pi)^6} \left[ \int B^2 dS/|\nabla \psi|^3 \int \frac{(\mu_0 J \cdot B)^2}{B^2|\nabla \psi|^3} dS \right. \]

\[ + \mu_0^2 p' r^2 \int B^2 dS/|\nabla \psi|^3 \int \frac{dS}{B^2|\nabla \psi|} - \left( \int \frac{\mu_0 J \cdot B}{|\nabla \psi|^3} dS \right)^2 \left], \tag{5.3} \]

\[ H = \frac{A}{(2\pi)^6} \left( \int \frac{\mu_0 J \cdot B}{|\nabla \psi|^3} dS - \int B^2 dS/|\nabla \psi|^3 \int \frac{\mu_0 J \cdot B dS/|\nabla \psi|}{B^2 dS/|\nabla \psi|} \right), \tag{5.4} \]

and \( A = s_G 4\pi^2 \epsilon' \). Primes denote \( d/d\psi \).

We note that Eq. (5.1) corresponds to Eq. (16) of Glasser et al. (1975), multiplied by \( (\epsilon')^2/(4\pi^2) \).

We now show that, to lowest order in \( \epsilon \), i.e., at \( O(\epsilon^{-2}) \), \( D_R \simeq -D_{Merc} \). We first note that the criterion in Eq. (5.1) can be related to the Mercier criterion \( D_{Merc} \) by rewriting \( D_{Merc} \) as

\[ D_{Merc} = \frac{(\epsilon')^2}{16\pi^2} - E - F - H > 0. \tag{5.5} \]

The functions \( D_R \) and \( D_{Merc} \) are then related via

\[ D_R = -D_{Merc} + \frac{4\pi^2}{(\epsilon')^2} \left[ H - \frac{(\epsilon')^2}{8\pi^2} \right]^2. \tag{5.6} \]

We have already seen that \( \epsilon'(\psi) \) is \( O(\epsilon^0) \). Using (4.2), the function \( H \) turns out to have a rather compact form when written in terms of Boozer coordinates:

\[ H = \frac{\epsilon'}{(2\pi)^4} \int d\phi d\varphi \frac{I \partial \beta/\partial \varphi - (G + IN) \partial \beta/\partial \theta}{|\nabla \psi|^2}. \tag{5.7} \]

The \( \epsilon \ll 1 \) expansion has not yet been employed. As the integral over \( \theta \) of the \( O(\epsilon^{-1}) \) component of Eq. (5.7) vanishes and \( I = O(\epsilon^2) \), we conclude that \( H = O(\epsilon^0) \) and thus \( D_{Merc} \simeq -D_R + O(\epsilon^2) \). Therefore, the distinction between ideal and resistive interchange stability (for the radially localized modes of Mercier and Glasser) becomes insignificant near the axis. A special case of this result, for circular-cross-section axisymmetric equilibria, was noted in Glasser et al. (1976).

6. Discussion and conclusions

In the analysis above we have shown that the magnetic well and Mercier stability can be computed directly from a solution of the near-axis equilibrium equations in Garren & Boozer’s expansion. As demonstrated by the agreement between our analytic formulae and finite-aspect-ratio VMEC calculations in the figures, it is therefore possible to assess the Mercier stability of the constructed configurations without running a finite-aspect-ratio equilibrium code. These results contribute to the goal of carrying out stellarator design in part within the near-axis approximation, which may help resolve the problem of numerical optimizations getting stuck in local minima, as follows. In the near-axis approximation, the equilibrium equations can be solved many orders of magnitude faster than the full 3D equilibrium equations, enabling global surveys of the landscape of possible configurations that are not limited to the vicinity of a single optimum. In such a survey, the results in the present paper let us immediately exclude Mercier-unstable configurations. The most promising configurations from a high-aspect-ratio survey can
then be provided as new initial conditions for traditional local optimization with a finite-aspect-ratio 3D equilibrium code.

The key results from our analysis are as follows. Near the magnetic axis, the magnetic well is given by (3.6), (3.9), and (3.11), depending on the definition used. In the special case of quasisymmetry, these expressions simplify to (3.7), (3.10), and (3.12). In Mercier’s criterion, the terms $D_{\text{Shear}}$ and $D_{\text{Curr}}$ scale as $r^0$, while the terms $D_{\text{Well}}$ and $D_{\text{Geod}}$ scale as $r^{-2}$. Therefore $D_{\text{Shear}}$ and $D_{\text{Curr}}$ are negligible near the axis, and overall $D_{\text{Merc}} \propto r^{-2}$.

The dominant terms near the axis can be computed from (4.12), (4.21), and (4.22), with (4.10). In the case of quasisymmetry, these expressions simplify to (4.15), (4.23), and (4.24). Finally, Glasser’s criterion for resistive interchange stability coincides with Mercier’s ideal criterion to leading order near the axis.

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Appendix A. Signs in Mercier’s criterion

Typically two possible transformations exist in which the signs of certain flux coordinates are flipped while physical quantities such as $B$ are unchanged. Some expressions for Mercier stability in the literature are not invariant under these transformations, due to assumptions about signs made during their derivation. Here we show how to generalize these forms of Mercier’s criterion so they are invariant.

A.1. Parity transformations

Let us first precisely state the two coordinate transformations under which physical phenomena should be invariant. The magnetic field is represented in Boozer coordinates as

$$B = \frac{1}{2\pi} \left( \nabla \Psi \times \nabla \theta + \nabla \varphi \times \nabla \Phi \right)$$

(A1)

$$= \frac{\beta}{2\pi} \nabla \Psi + I(\Psi) \nabla \theta + G(\Psi) \nabla \varphi,$$

(A2)

where $\Psi$ and $\Phi$ are the toroidal and poloidal fluxes (not divided by $2\pi$), and the angles $\theta$ and $\varphi$ are periodic with period $2\pi$. The rotational transform is $\iota = d\Phi/d\Psi$.

In “parity transformation 1,” the signs of $\Psi$, $\theta$, $\beta$, $I$, and $\iota$ are flipped, while $\varphi$, $G$, and $\Phi$ are unchanged. This transformation leaves $B$ unchanged in both (A1) and (A2). In “parity transformation 2,” the signs of $\varphi$, $G$, $\Phi$, and $\iota$ are flipped, while $\Psi$, $\theta$, $\beta$, and $I$ are unchanged. This transformation again leaves $B$ unchanged in both (A1) and (A2).

Since $B$ is unchanged by these transformations, physical consequences such as stability should be unaltered. Expressions for Mercier stability in references such as Mercier (1964); Mercier & Luc (1974); Bauer et al. (1984) however appear to have sign changes in at least some terms. In the remainder of this section we show how these expressions for Mercier stability should be modified to handle these coordinate transformations.

A.2. Greene & Johnson form

It is convenient to begin with a form of the criterion developed by Greene & Johnson (1961, 1962b), since they explicitly state their assumption about the coordinate signs.
They use Hamada coordinates \((V, \theta_H, \varphi_H)\), with angles that are periodic with period 1, which are required to form a right-handed system:

\[
\nabla V \cdot \nabla \theta_H \times \nabla \varphi_H = +1. \tag{A3}
\]

We require the flux surface volume \(V\) to be \(\geq 0\). The magnetic field satisfies

\[
B = \nabla \Psi_H \times \nabla \theta_H + \nabla \varphi_H \times \nabla \Phi_H, \tag{A4}
\]

where we have included the \(H\) subscript to emphasize that the toroidal and poloidal fluxes \(\Psi_H\) and \(\Phi_H\) must have signs compatible with those of \((\theta_H, \varphi_H)\) in \((A4)\), which must in turn be consistent with \((A3)\). Note \(\Psi = \pm \Psi_H\) and \(\Phi = \pm \Phi_H\) are possible. Neither of the two parity transformations is allowed by itself in Greene & Johnson’s coordinates (e.g. we cannot replace \((\Psi_H, \theta_H) \rightarrow (-\Psi_H, -\theta_H)\)) since \((A3)\) would be violated. However applying both transformations simultaneously is allowed.

The stability criterion in Greene & Johnson (1961), also eq (40) in Greene & Johnson (1962b), is \(D_{GJ} > 0\) where

\[
D_{GJ} = \left[ \oint \frac{B \; d\ell}{|\nabla V|^2} \right]^{-1} \left[ \left( \frac{d\Psi_H}{dV} \frac{d^2 \Phi_H}{dV^2} - \frac{d^2 \Psi_H}{dV^2} \frac{d\Phi_H}{dV} \right) \oint \frac{d\ell}{2B} - \oint \frac{\mu_0 J \cdot B \; d\ell}{B|\nabla V|^2} \right]^2 \tag{A5}
\]

and here \(\ell\) denotes arclength along a closed field line on the rational surface. Greene & Johnson (1961), Mercier (1964), Bauer et al. (1984), Ichiguchi et al. (1993), and Glasser et al. (1975) all set \(\mu_0 = 1\); we restore \(\mu_0\) by replacing \(J \rightarrow \mu_0 J\) and \(p \rightarrow \mu_0 p\). Note the factor of \(-2\) in the last term of \((A5)\) is missing in Greene & Johnson (1962b), as noted in Greene & Johnson (1962a), but correct in Greene & Johnson (1961).

In our Boozer coordinates \((A1)-(A2)\), the quantity

\[
\nabla V \cdot \nabla \theta \times \nabla \varphi = \frac{dV}{dr} \frac{B^2}{(G + iI)rB} = s_G s_\psi \left| \frac{dV}{dr} \frac{B^2}{(G + iI)rB} \right| \tag{A6}
\]

could have either sign. Here, \(s_\psi = \text{sgn}(\Psi) = \text{sgn}(\overline{B}) = \pm 1\) and \(s_G = \text{sgn}(G) = \pm 1\), and we have used \(dV/dr > 0\). Therefore we cannot necessarily take the poloidal and toroidal fluxes from the Boozer coordinate system and insert them in \((A5)\), setting \(\Psi_H = \Psi\) and \(\Phi_H = \Phi\). This is only allowed if \(s_G s_\psi = +1\). If \(s_G s_\psi = -1\), we must perform transformation 1 or 2 (not both) before substituting the fluxes into \((A5)\). In this case, we would have either \((\Psi_H, \Phi_H) = (-\Psi, \Phi)\) or \((\Psi, -\Phi)\) This effect can be achieved by inserting a factor \(s_G s_\psi\) in Greene & Johnson’s expression:

\[
D_{GJ} = \left[ \oint \frac{B \; d\ell}{|\nabla V|^2} \right]^{-1} \left[ s_G s_\psi \left( \frac{d\Psi}{dV} \right)^2 \oint \frac{d\ell}{B} - \oint \frac{\mu_0 J \cdot B \; d\ell}{B|\nabla V|^2} \right]^2 \tag{A7}
\]

This statement of the stability condition is now in a form invariant under either transformation.

The same argument can be applied to expressions in Glasser et al. (1975), who use the same Hamada coordinates with positive Jacobian. By this reasoning, the expressions for \(E\) and \(H\) in eq (13) of Glasser et al. (1975) are made parity invariant by including a factor \(s_G s_\psi\); \(F\) does not acquire this factor. The results, multiplied by \((dt/d\psi)^2/(4\pi^2)\), give \((5.2)-(5.4)\).
Following Mercier & Luc (1974), the line integrals in (A7) can be approximately converted to area integrals by

$$
\int \frac{Q \, dl}{B} \approx \int \frac{Q \, ds}{|\nabla \Phi|}
$$

(A8)

where $Q$ is any quantity. Noting $\int ds/|\nabla \Phi| = s_\psi s_\Theta dV/d\Phi$ (which follows from (3.3)), we find the stability condition can be written as $D_M \geq 0$ where

$$
D_M = \left( \frac{sg_{G} d(1/|\iota|)}{2 \, d\Phi} + \frac{\mu_0 J \cdot B \, ds}{|\nabla \Phi|^3} \right)^2 - 2 \left[ \frac{B^2 \, ds}{|\nabla \Phi|^3} \right] \int \frac{dS \, \mu_0 J \times n \cdot (B \cdot n)}{|\nabla \Phi|^3}.
$$

(A9)

Here, Mercier’s unit vector $n = |\nabla \Phi|^{-1} \nabla \Phi$ has been introduced. The expression (A9) generalizes eq (35) on page 60 of Mercier & Luc (1974) to be properly invariant under the two parity transformations.

### A.3. Mercier’s form

To obtain the form of the stability criterion favored by Mercier, we can follow page 61 of Mercier & Luc (1974) (also the appendix of Mercier (1964)), beginning with

$$
2J \times n \cdot (B \cdot \nabla n) = \mu_0 J^2 n \cdot \nabla \left( \frac{n \cdot \nabla \chi}{|\nabla \Phi|} \right) - \frac{dp}{d\psi} \nabla \cdot \left( n \frac{n}{|\nabla \Phi|} \right),
$$

(A10)

where $\chi = (\varphi - \theta/\iota)/(2\pi)$. The derivation of this identity by Mercier & Luc (1974) in their appendix 6 makes no assumptions about signs so (A10) is already invariant. Other expressions on page 61 of Mercier & Luc (1974) however require modification to account for signs, including

$$
\int \frac{dS}{|\nabla \Phi|} \mu_0 J \cdot \nabla \left( n \frac{\nabla \chi}{|\nabla \Phi|} \right) = \left[ \frac{d(1/\iota)}{d\Phi} \right] s_\Theta \frac{dG}{\iota} \frac{dG}{d\psi}
$$

(A11)

and

$$
\int \frac{dS}{|\nabla \Phi|} \nabla \cdot \left( \frac{n}{|\nabla \Phi|} \right) = s_\psi s_\psi d^2V \frac{d^2V}{d\psi^2} + s_\psi s_\psi \left[ \frac{d(1/\iota)}{d\Phi} \right] dV \frac{dV}{d\psi},
$$

(A12)

where $s_\psi = \text{sgn}(\psi) = \pm 1$. From eq (2.7) in Landreman & Sengupta (2019) we find

$$
\frac{dG}{d\psi} + \frac{dI}{d\psi} = -\frac{s_G s_\psi \mu_0}{4\pi^2} \frac{dp}{d\psi} \frac{dV}{d\psi},
$$

(A13)

instead of the last equation on page 61 of Mercier & Luc (1974). Combining (A10)-(A13) gives

$$
2 \left( \frac{dS \mu_0 J \times n \cdot (B \cdot \nabla n)}{|\nabla \Phi|^3} \right) = \int \frac{dS \mu_0^2 J^2}{|\nabla \Phi|^3} + s_\Theta \left[ \frac{d(1/|\iota|)}{d\Phi} \right] \frac{dI}{d\psi} - \frac{s_\psi s_\psi \mu_0}{2 \iota^2} \frac{dp}{d\psi} \frac{d^2V}{d\psi^2},
$$

(A14)

which corrects the first equation on page 62 of Mercier & Luc (1974). Using this result in (A9), we obtain Mercier’s form of the stability criterion, $D_M \geq 0$ with

$$
D_M = \left( \frac{sg_{G} d(1/|\iota|)}{2 \, d\Phi} + \frac{B \cdot \Xi \, ds}{|\nabla \Phi|^3} \right)^2 - \left[ \frac{s_\psi s_\psi \mu_0}{2 \iota^2} \frac{dp}{d\Phi} \frac{d^2V}{d\psi^2} - \frac{\Xi^2}{|\nabla \Phi|^3} \right] \frac{B^2 \, ds}{|\nabla \Phi|^3}.
$$

(A15)

where $\Xi = \mu_0 J - (B dI/d\psi = \mu_0 J - (B dI_t/d\psi)$, and $I_t = 2\pi I/\mu_0$ is the toroidal current. This result, analogous to (37) of Mercier & Luc (1974), is properly invariant under either parity transformation.
In more recent publications \cite{Bauer1984} and in the VMEC code, the stability criterion is expressed in a different form, which we now derive. Using $J^2 = (\mathbf{J} \cdot \mathbf{B})^2 / B^2 + (dp/d\Psi)^2|\nabla \Psi|^2 / B^2$ (which follows from the square of $\mathbf{J} \times \mathbf{B} = \nabla p$) and defining $D_{\text{Merc}} = \delta D_M$, Eq. \ref{eq:mercier} may be expressed as $D_{\text{Merc}} \geq 0$ with

$$D_{\text{Merc}} = \frac{1}{4} \left( \frac{d \sigma}{d \Psi} \right)^2 - s_G \frac{d \sigma}{d \Psi} \int \frac{d \theta d \varphi |\sqrt{g}| \mathbf{B} \cdot \mathbf{Z}}{|\nabla \Psi|^2} + \mu_0 \frac{dp}{d\Psi} \left[ s_g \frac{d^2 V}{d\Psi^2} - \mu_0 \frac{dp}{d\Psi} \int d\theta d\varphi \right] \frac{|\sqrt{g}|B^2}{|\nabla \Psi|^2} + \left( \int d\theta d\varphi |\sqrt{g}| \mu_0 \mathbf{J} \cdot \mathbf{B} \right)^2 - \left( \int d\theta d\varphi |\sqrt{g}|B^2 \right) \left( \int d\theta d\varphi |\sqrt{g}|(\mu_0 \mathbf{J} \cdot \mathbf{B})^2 \right),$$

where $\sqrt{g} = (\nabla \Psi \cdot \nabla \varphi)^{-1}$, and $\theta$ and $\varphi$ range over $[0, 2\pi]$ in the integrals. This expression is proportional to the stability criterion stated in \cite{Bauer1984} except for the absolute values around $\sqrt{g}$ and for the $s_G$ and $s_g$ factors that appear here. Eq. \ref{eq:mercier} rigorously generalizes this form of the stability criterion in to be invariant under both parity transformations.

Finally, we note that the Mercier criterion as stated on page 1201 of \cite{Carreras1988} is parity-invariant and equivalent to \ref{eq:mercier} if one uses the following sign conventions (using notation from that paper): $s \geq 0$, $g = 1/(\nabla s \cdot \nabla \varphi \times \nabla \zeta)$ is allowed to have either sign, and $V$ is allowed to have either sign, with $V = \int_0^s ds' \int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \, g$. This definition of $V$ differs from ours by a factor $s_G s_g$.

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