Noncommutative Dipole Field Theories

K. Dasgupta\textsuperscript{1}, M. M. Sheikh-Jabbari\textsuperscript{2}

Department of Physics, Stanford University
382 Via Pueblo Mall, Stanford CA 94305-4060, USA

Abstract

Assigning an intrinsic constant dipole moment to any field, we present a new kind of associative star product, the dipole star product, which was first introduced in [hep-th/0008030]. We develop the mathematics necessary to study the corresponding noncommutative dipole field theories. These theories are sensible non-local field theories with no IR/UV mixing. In addition we discuss that the Lorentz symmetry in these theories is “softly” broken and in some particular cases the CP (and even CPT) violation in these theories may become observable. We show that a non-trivial dipole extension of $\mathcal{N} = 4$, $D = 4$ gauge theories can only be obtained if we break the $SU(4)$ $R$ (and hence super)-symmetry. Such noncommutative dipole extensions, which in the maximal supersymmetric cases are $\mathcal{N} = 2$ gauge theories with matter, can be embedded in string theory as the theories on D3-branes probing a smooth Taub-NUT space with three form fluxes turned on or alternatively by probing a space with R-symmetry twists. We show the equivalences between the two approaches and also discuss the M-theory realization.

December 2001

\textsuperscript{1} keshav@itp.stanford.edu
\textsuperscript{2} jabbari@itp.stanford.edu
1. Introduction

In the last three years a great amount of work have been devoted to the field theories on the Moyal plane, usually called noncommutative field theories. The noncommutative Moyal plane defined by the coordinate operators with

\[ [x^\mu, x^{\nu}] = i \theta^{\mu\nu}, \]  

where \( \theta \) is a constant, can be realized in string theory as the world volume of D-branes in a constant background \( B_{\mu\nu} \) field, probed by the open strings \[1,2,3\]. In the absence of D-branes, a constant B-field can be gauged away completely. However, in the presence of D-branes constant \( B_{\mu\nu} \) fields with both \( \mu \) and \( \nu \) directions along the brane cannot be gauged away \[4\] and in fact such components lead to the noncommutativity on the brane world volume. The effective low energy world volume field theories on the brane turn out to be noncommutative (supersymmetric) gauge theories. The very characteristic of noncommutative gauge theories, in general, is that they are non-local theories with the fields effectively describing the dynamics of “dipole” like objects \[5,6\]. The dipole moments of these particles are proportional to their momentum

\[ d_\mu \sim \theta_{\mu\nu} p_\nu. \]  

This dipole nature is the familiar effect of the motion of particles in an external magnetic field \[7\]. From field theory point of view the momentum dependence of the dipole moments (and the corresponding non-locality) shows up in the loop expansion of the noncommutative field theories as the IR/UV mixing \[8\].

The components of the B-field not parallel to the D-branes however can be gauged away if the B-field is a constant. This can be waived if in some way we stabilize the B-field along the transverse directions to the D-brane to support a non-constant B-flux. This is possible, for example, if we compactify the transverse direction to a brane with a varying size of the compact circle. Putting one leg of the B-field along that direction effectively gives us a non-zero three form \( H_{NS} \) field. The case that we want to study further in this paper, however, is to stabilize the B-field when it has one leg along the brane and the other transverse to it. The possibility of such configuration was discussed earlier in \[9,10,11\], where the “twisted” compactification were introduced. We shall review this in section five. As discussed in \[12\] performing the “twisted” compactification leads us to introduce a new type of star product between the fields at the level of effective field theories. As a result of the twisted compactification we find the possibility of associating a constant
dipole length to any field. This dipole length is proportional to the “winding” number of the fields in the twisted directions. We shall review this in section five. Then, at the level of the effective field theory we obtain some sort of noncommutative field theory which we call “noncommutative dipole field theory” (NCDFT) to distinguish it from the theories on the Moyal plane. We would like to stress that unlike the Moyal case, the origin of the noncommutativity in noncommutative dipole field theories is not the noncommutativity in space-time, but an inherent property of each field and we can also have fields with zero dipole length.

Besides the string theoretic appeal, the noncommutative dipole field theories, NCDFT’s, are also interesting by themselves. As we will discuss, considering some definite noncommutative dipole gauge theories, there is the chance of finding a CP (and even CPT) violating theory. We also discuss the renormalizability of noncommutative dipole scalar and gauge field theories and argue that these theories are renormalizable in the sense that adding finite numbers of non-local counterterms at each loop level will remove the divergences. We show that the β-function of the noncommutative dipole QED is not affected by the dipole nature of the fields. Furthermore we argue that, in general, there is no IR/UV mixing effect in the noncommutative dipole theories. We show that in the noncommutative gauge theory with a fundamental matter fields, unlike the Moyal case, we can have $SU(N)$ gauge theories. However, for the adjoint matter field, we will argue that $SU(N)$ is not possible and one should take $U(N)$.

We then study the possible supersymmetric extension of the noncommutative dipole theories. We show that unlike the Moyal noncommutative gauge theories, the maximal SUSY noncommutative dipole gauge theory is the $\mathcal{N} = 2$ (eight supercharges).

The plan of this paper is as follows. In section 2, first we present the explicit form of the dipole star product and study some of its properties, such as associativity. We show that the usual integral over the space-time can provide a natural $Tr$ over the $C^*$-algebra of functions with the dipole star products. Then in section 3, we study noncommutative dipole field theories and their renormalizability. In section 4, we discuss the SUSY extension of noncommutative dipole gauge theories. In section 5, the string theory realization of NCDFT’s is discussed (the first two parts of this section is an expanded review of Ref.[12]). We also discuss how the NCDFT’s can be understood as the effective field theories on D3-branes probing a Taub-NUT space with a B-field switched on. Section 6 is devoted to the M theory realizations. In section 7 we study another kind of star-product which uses both the dipole nature of open-strings and the noncommutativity of the underlying space-time. We end with a discussion and outlook.
2. Mathematical Preliminaries

To formulate any noncommutative field theory, one should start by defining the proper associative star product. Then, the fields are members of the $C^*$-algebra of functions with respect to that star product. To any element of the $C^*$-algebra, $\phi_i$, we assign a constant space-like dipole length $\vec{L}_i$ and define the “dipole star product” as

$$ (\phi_i * \phi_j)(x) \equiv \phi_i(x - \frac{1}{2}L_j) \phi_j(x + \frac{1}{2}L_i) . \quad (2.1) $$

Furthermore we need to identify the dipole moment of $(\phi_i * \phi_j)(x)$. Demanding our algebra to be associative, it is straightforward to show that the dipole moment of (star) product of two functions should be sum of their dipole moments:

$$ (\phi_i * \phi_j) * \phi_k = (\phi_i(x - \frac{1}{2}L_j) \phi_j(x + \frac{1}{2}L_i)) * \phi_k $$

$$ = \phi_i(x - \frac{L_j + L_k}{2}) \phi_j(x + \frac{L_i - L_k}{2}) \phi_k(x + \frac{L_i + L_j}{2}) . \quad (2.2) $$

We would like to mention that the above star product cannot be expressed in terms of a commutation relation among the space-time coordinates. In other words, the noncommutativity in the dipole case is not a property of space-time, but originating from the dipole length associated to each field.

Now we should introduce a suitable $Tr$ on the algebra. The natural choice for the $Tr$, similar to the Moyal case, is the integral over the space-time. However naively taking the integral over star products of arbitrary functions, one can easily check that they do not enjoy the necessary cyclicity condition. This problem is removed if we restrict the integrand to have a total zero dipole length. More explicitly, the integral serves as the proper $Tr$, over the functions of zero dipole length:

$$ \int \phi_1 * \phi_2 * \cdots * \phi_n = \int \phi_n * \phi_1 * \cdots \phi_{n-1} , \quad (2.3) $$

with the condition that $\sum_{i=1}^n \vec{L}_i = 0$, where $L_i$ are the dipole lengths for $\phi_i$. In the field theory actions this condition is translated into the fact that any term in the proposed action should have a total vanishing dipole length (as well as the usual hermiticity conditions which guarantees the electric charge conservation). Therefore, maintaining with the translational invariance, in each vertex both the sum of the external momenta and the total dipole length should vanish.
Next, we should define complex conjugate of a field and the behaviour of the star product under complex conjugation. Demanding \((\phi^\dagger \star \phi)\) to be real valued, i.e.

\[
\phi^\dagger \star \phi = (\phi^\dagger \star \phi)^\dagger,
\]

(2.4)

fixes the dipole length of \(\phi^\dagger\) to be the same as that of \(\phi\) though with a minus sign. Therefore the dipole length of any real (hermitian) field, and in particular the gauge fields, is zero. Then, one can show that

\[
(\phi_i \star \phi_j)^\dagger = \phi_j^\dagger \star \phi_i^\dagger
\]

(2.5)

To write down the field theory action we also need the derivative operators with respect to the star product. It is easy to check that the usual derivative does satisfy the Leibniz rule, i.e.

\[
\partial_\mu (\phi_1 \star \phi_2) = (\partial_\mu \phi_1) \star \phi_2 + \phi_1 \star (\partial_\mu \phi_2),
\]

(2.6)

i.e. they can be used as the proper derivatives for writing down the kinetic terms of the NCDFT’s.

3. Noncommutative Dipole Field Theories

Equipped with the above mathematics we are ready to formulate noncommutative dipole field theories (NCDFT’s). In general, to obtain the NCDFT’s actions it is enough to replace the product of fields in the commutative actions with the dipole star product Eq. (2.1) . However, one should insert the proper dipole lengths for all fields. We start with a simple scalar field theory and then formulate fermions and gauge theories.

3.1. Scalar noncommutative dipole theory

Along our general recipe, the action proposed for the scalar noncommutative dipole theory is

\[
S = \int \partial_\mu \phi^\dagger \star \partial_\mu \phi - V_\star (\phi^\dagger \star \phi),
\]

(3.1)

where \(V_\star\) is the potential with the products replaced by star products Eq. (2.1) . We note that, written as a function of \(\phi^\dagger \star \phi\), it is guaranteed that the terms in the above action satisfy the necessary zero-dipole-length condition. Using the cyclicity condition, we can drop the star product in the quadratic part of the action(s) (much like the Moyal-star product case) and the effects of dipole moments appear only through the interaction
terms. Therefore the propagator of the above noncommutative dipole theory (and also any NCDFT) is the same as the commutative case. Moreover, since

\[ (\phi^\dagger \star \phi)(x) = (\phi^\dagger \phi)(x - \frac{1}{2}L), \]

then

\[
\int (\phi^\dagger \star \phi)_{*}^{n} = \int \left( (\phi^\dagger \phi)(x - \frac{1}{2}L) \right)^{n} = \int \left( (\phi^\dagger \phi)(x) \right)^{n}.
\]

Hence, even in the potential term, being only a function of \( \phi^\dagger \star \phi \), all the dipole dependence is removed. As a result the scalar noncommutative dipole theory introduced by the action Eq. (3.1) is exactly the same as its commutative counter-part.

Noting that \( \phi \star \phi^\dagger \) is also of zero dipole length, there is another possibility for potential: to be a function of both \( \phi \star \phi^\dagger \) and \( \phi^\dagger \star \phi \). Hence, potential for the noncommutative dipole version of the \( \phi^4 \) theory in the most general case is

\[
V_{*} = \lambda_{0}(\phi^\dagger \star \phi) \star (\phi^\dagger \star \phi) + \lambda_{1}(\phi^\dagger \star \phi) \star (\phi \star \phi^\dagger),
\]

and therefore

\[
\int V_{*} = \int \lambda_{0}(\phi^\dagger \phi)(x)(\phi^\dagger \phi)(x) + \lambda_{1}(\phi^\dagger \phi)(x - \frac{L}{2})(\phi^\dagger \phi)(x + \frac{L}{2}).
\]

Performing loop calculations, one can show that at the level of one loop, two point functions will show the same kind of divergences as the commutative theory. Furthermore, we have both planar and non-planar diagrams (where the exponential phases involving the dipole length and the loop momenta appear in the loop integrals). The non-planar diagrams are finite and we do not face IR/UV mixing, because the dipole length, unlike the Moyal noncommutative case, is a constant. However, considering the one loop four point function, something interesting happens now. We have planar and non-planar one loop four point functions and again non-planar diagrams are finite. As for the planar diagrams, we face two type of divergences: those which can be cancelled with the usual counter-terms if \( \lambda_{0} = \lambda_{1} \), and those which cannot be absorbed in the terms already present in the action. Therefore, the noncommutative dipole theory with potential Eq. (3.3) is not renormalizable in the usual sense used for local field theories. However, these extra divergences can be absorbed in a term like

\[
\int (\phi^\dagger \phi)(x - \frac{2L}{2})(\phi^\dagger \phi)(x + \frac{2L}{2}).
\]
This procedure should be continued to the \( m \) loops order. In the loops of order \( m \) the divergences in this theory can be cancelled by the following choice of counter-terms:

\[
\sum_{n=0}^{m+1} \lambda_n \int (\phi^{\dagger} \phi)(x - \frac{nL}{2})(\phi^{\dagger} \phi)(x + \frac{nL}{2}).
\]  

(3.5)

Although the above noncommutative dipole theory is not renormalizable in the usual sense, we see that the counter-terms are under strict control — in fact are of the same form as the original Lagrangian — and we do not require any extra degrees of freedom at the UV. In particular if instead of the potential Eq. (3.3) we start with Eq. (3.5) (with the sum going to infinity) with all the coefficient \( \lambda_n \) to be equal, the theory, though non-local, would be renormalizable in the usual sense. In fact, we will show in section 4 that, this special case is what one finds from string theory.

3.2. Noncommutative dipole gauge theory

Here we restrict ourselves to the \( SU(N) \) gauge theories and the other gauge groups can be studied in the same spirit. As we discussed the gauge fields, being hermitian, should have a zero dipole length. So, the pure gauge theory is defined exactly in the same way as the commutative gauge theory. However, the gauge fields coupled to the matter fields may uncover the dipole structure of the charged matter fields. In order to define the fundamental matter coupled to gauge field theory, we need the covariant derivative of a dipole field. This is given as:

\[
D_{\mu} \psi \equiv \partial_{\mu} \psi + i g A_{\mu} \ast \psi
= \partial_{\mu} \psi + i g A_{\mu}(x - \frac{L}{2})\psi(x).
\]  

(3.6)

Then it is straightforward to check that the action

\[
S = \int \overline{\psi} \gamma^{\mu} D_{\mu} \psi,
\]  

(3.7)

is invariant under gauge transformations:

\[
\psi \rightarrow U \ast \psi
A_{\mu} \rightarrow U \ast A_{\mu} \ast U^{-1} + \frac{i}{g} \partial_{\mu} U \ast U^{-1} = U A_{\mu} U^{-1} + \frac{i}{g} \partial_{\mu} U U^{-1},
\]  

(3.8)

\[\text{1 We would like to comment that for the Moyal noncommutative case again we have two possibilities for strict renormalizabilty in the usual sense. Besides } \lambda_1 = 0, \text{ the } \lambda_0 = \lambda_1 \text{ case is also renormalizable [13] with } \lambda_i = 0 \text{ for } i \geq 2.\]
where $U \in SU(N)$ (and of course the dipole length assigned to $U$ is zero). Note that all the fermions in the $N$-vector of $SU(N)$ fundamental representation should have the same dipole length. We would like to comment that, unlike the noncommutative Moyal QED, in the noncommutative dipole QED we can have particles with arbitrary electric charge. Expanding Eq. (3.6) in powers of $L$ the first order terms will give the dipole interactions, where the dipole moment is

$$d = \frac{1}{2} g \langle \bar{\psi} \gamma^0 \psi \rangle \vec{L}.$$  \hspace{1cm} (3.9)

It is worth noting that the noncommutative dipole gauge theory under the parity is not invariant and as expected the matter field with dipole length $\vec{L}$ is mapped into the theory with dipole length $-\vec{L}$, while under charge conjugation and also time reversal the theory remains invariant. So, the dipole theory, with the dipole lengths $\vec{L}_i$ under CP (as well as CPT) is mapped to another dipole theory with $-\vec{L}_i$. However, we will argue momentarily that this CP and CPT violation is only observable in very particular cases. We would also like to comment that the dipole moment Eq. (3.9) under both parity and charge conjugation change sign and hence invariant under CP.

Along our previous discussions the propagator for fermions and the gauge fields in the dipole case is the same as the commutative case. However, as it is seen from Eq. (3.6) the interaction vertices of the noncommutative dipole gauge theory (with “fundamental” matter) compared to the commutative case, involve an extra dipole dependent phase factor, $e^{-\frac{i}{2} \vec{p} \cdot \vec{L}}$ (where $p$ is the in-going gauge field momentum). Performing loop calculations, since only the momentum of the gauge field appears in the noncommutative phase factors, the noncommutative dipole phase factors never contain the momentum running in the loop. Therefore, we do not face any non-planar diagram in our noncommutative dipole theory.

---

2 Similar to the Moyal noncommutative case, we can have another type of fermion (or covariant derivative):

\[ D_\mu \psi' = \partial_\mu \psi' + ig \psi' A_\mu = \partial_\mu \psi' + ig A_\mu (x + \vec{L}/2) \psi'(x). \]

The $\psi$ and $\psi'$ type fermions are related by parity transformation, while the two types of fermions in the Moyal NCQED are related by charge conjugation. The gauge transformation for the $\psi'$ type fermion is $\psi' \rightarrow \psi' \ast U^{-1}$.

3 The parity, charge conjugation and time reversal are defined in the same way as the usual commutative theories.

4 Also note that the in the noncommutative dipole theories Lorentz symmetry is “softly” broken.
coupled to the fundamental matter and hence, there is no IR/UV mixing phenomenon. In
particular, the fermion and gauge particle one loop two point functions are not altered by
the dipole length at all. As for the fermion-gluon vertex, the noncommutative dipole phase
factor will appear just in front of the loop integrals. Consequently, all the loop integrals are
the same as the commutative case and hence the \( \beta \)-function of the field theory is the same
as the commutative case. Furthermore, the dipole vector \( \vec{L} \) do not receive any quantum
corrections.

It is straightforward to check that the noncommutative dipole gauge theory is invariant under BRST symmetry. The corresponding BRST transformations are obtained by inserting the dipole star product Eq. (2.1) into the commutative expressions. The ghost field and the BRST generators have zero dipole lengths. Therefore, the above one loop argument ensures the renormalizability of the theory.

One may wonder now that the dipole length \( L \) is not appearing in the loop dynamics
of the gauge theory, it may be removed by a field re-definition as:

\[
A_\mu(x - \frac{L}{2}) = \tilde{A}_\mu(x). \tag{3.10}
\]

Then it is easy to check that the theory written in terms of \( \tilde{A} \) is exactly the usual gauge
theory. This field re-definition can be understood in a more intuitive way. As we know
[16], by a translation in the origin of the coordinate system, any point like \( 2^n \)-pole looks
as a bunch of \( 2^m \)-poles \((m \geq n)\). In particular our charged-dipole like particle can be
viewed as a pure charge in a translated frame. This is just equivalent to the above field
re-definition.

Note that the above field re-definition can remove the dipole length \( L \) when we have
only one kind of particle. If we allow particles of different dipole lengths, this simple field
re-definition will not work. However, it is not hard to see that, if we have a gauge theory of a
simple group the CP (and CPT) violating phase factor, even if we have fundamental matter
fields with different dipole lengths, is not observable at tree level. As an explicit example
consider the “noncommutative dipole” QED with electron and muon which have dipole
lengths \( L_e \) and \( L_\mu \) respectively. It is straightforward to check that our renormalizability
argument is not affected. Now consider the \( e - \mu \) scattering, the scattering amplitude
picks up a phase factor of \( e^{ip(L_e - L_\mu)} \) where \( p \) is the momentum of the exchanged photon.
Although we have a non-trivial dipole phase factor, it will not appear in the cross sections,
i.e. having different dipole lengths is not enough to make the dipole effects traceable. In order to observe noncommutative dipole phase (for fundamental matter) in the tree level cross sections, we need to fulfill two other conditions: (1) We should take a semi-simple group, and (2) we should allow particles to have different dipole lengths for different simple group factors.

These conditions can easily be satisfied in a noncommutative dipole version of the usual Standard Model, and then the CP and CPT violating phase factor could in principle be observable in the $e - \mu$ scattering.

3.3. Adjoint noncommutative dipole matter fields

Besides the matter fields in the fundamental representation one can introduce the matter field in the adjoint representation, with the covariant derivative defined as

$$D_\mu \phi = \partial_\mu \phi + ig(A_\mu * \phi - \phi * A_\mu) = \partial_\mu \phi + ig \left[ A_\mu (x - \frac{L}{2}) \phi(x) - \phi(x) A_\mu (x + \frac{L}{2}) \right], \quad (3.11)$$

and as usual the $\phi$ field under gauge transformations should transform as

$$\phi \rightarrow U * \phi * U^{-1}. \quad (3.12)$$

As it is seen from Eq. (3.11) if we start with $SU(N)$ gauge group, i.e. $\phi, A_\mu \in su(N)$, in the interaction vertices we will have both the completely symmetric and anti-symmetric $su(N)$ tensors (usually denoted by $d^{abc}$ and $f^{abc}$ respectively). The appearance of $d^{abc}$ factors, to make the theory “renormalizable”, eventually will force us to include the central $U(1)$ factor. In other words, in the noncommutative dipole theory with the adjoint matter, it is not possible to have a $SU(N)$ theory while $U(N)$ is possible. We would like to note that this case is actually what we find from string theory.

Hereafter we restrict ourselves to the $U(1)$ case. The dipole moment of the adjoint matter with dipole length $L$, as we expect, is twice bigger than the fundamental matter. It is worth noting that in this case, since the dipole is lowest pole present (there is no pure charge), the field redefinition through simple translation does not work. One can check

5 We should mention that this is not the case if we consider loop effects. In particular consider the noncommutative dipole version of the Hydrogen atom with $L_e$ and $L_P$ for electron and proton dipoles, respectively. The $L_e - L_P$ will appear in the potential term (which can be thought of summing the whole ladder) in the corresponding Schrödinger equation and this will change the spectrum.
that re-defining $A_\mu(x - \frac{L}{2}) - A_\mu(x + \frac{L}{2})$ as the new gauge field, we will lose the simple
gauge transformations rule for the re-defined gauge field. (Or equivalently the gauge theory
action will not look as simple as $F_{\mu\nu}F^{\mu\nu}$.)

The fermion-photon vertex in this case is obtained by replacing the exponential non-
commutative dipole factor of the fundamental matter, by $2i \sin \frac{1}{2} p \cdot L$. Here we present the
results of loop calculations and the renormalizability for the $U(1)$ case, and the full and
detailed study of the noncommutative dipole theories with adjoint matter is postponed to
future works [17]. One should note that for the noncommutative dipole QED (with adjoint
matter), the corresponding commutative theory is trivial, it is just an uncharged particle
plus a pure $U(1)$ theory.

One loop photon propagator

Upon insertion of the dipole sine factor, the photon self-energy diagram is multiplied by
the factor of $-4 \sin^2(\frac{1}{2} p \cdot L)$ (times the usual QED result coming from the fermion running
in the loop). In other words the noncommutative dipole factor will not enter into the loop
integral. So, the divergent part of this diagram will show the same tensorial structure as
the usual QED and hence the gauge invariance is guaranteed. However, because of this sine
factor now the renormalization factor for the photon field, $Z_A$, is multiplied by a factor of
$-4 \sin^2 \frac{1}{2} p \cdot L$. In other words, to absorb the divergences, besides the usual $F_{\mu\nu}F^{\mu\nu}$ type
term we should add $F_{\mu\nu}(x - \frac{L}{2})F^{\mu\nu}(x + \frac{L}{2})$ type counter-terms. At higher loop level still
the counter-terms needed, can be expressed in terms of $F_{\mu\nu}$, but in a more non-local way
(similar to the Eq. (3.5) ).

One loop fermion propagator

In the noncommutative dipole QED (with adjoint matter), there is only one diagram which
contributes to the one loop fermion two point function. Here the noncommutative dipole
factor will enter into the loop integrals, i.e. we have planar and non-planar diagrams. The
planar part is essentially the same as usual QED, while the non-planar part is finite and
there is no IR/UV mixing.

One loop fermion-photon vertex

The only diagram that contributes to the fermion-photon vertex contains both planar and
non-planar diagrams. Again (up to some numeric factors) the divergent part (coming from
planar diagram) is the same as usual QED, and the non-planar part is finite.

All together, the above mentioned theory is a sensible theory (in the same spirit as
discussed in sub-section 3.1), however the $\beta$-function is different from the usual QED.

10
The theory is in fact asymptotically free (note that there is an extra factor of $i$ in the expression for the vertex). Moreover, unlike the Moyal noncommutative gauge theory where the noncommutativity parameter $\theta$ do not appear in the $\beta$-function explicitly, in the noncommutative dipole case, dipole length $L$ will enter into the expression for the $\beta$-function. However, the dipole length itself is not receiving any quantum corrections.

4. Supersymmetric Noncommutative Dipole Gauge Theory

Having defined noncommutative dipole gauge theories, one can check that it admits a supersymmetric extension, though the maximal supersymmetric case is now $\mathcal{N} = 2$ in four dimensions (8 supercharges). To see this we recall that, as discussed earlier, the gauge fields should appear with zero dipole lengths and supersymmetry requires that all the fields in the vector multiplet to have the same dipole length. Therefore the $D = 4$, $\mathcal{N} = 4$ case which only contains the vector multiplet does not admit a non-trivial dipole extension. The $\mathcal{N} = 4$ vector multiplet, in the $\mathcal{N} = 2$ language can be decomposed as a vector multiplet + a hyper multiplet. So, the first possibility arises when we allow the $\mathcal{N} = 2$ hyper multiplet, which is of course in the adjoint representation of the gauge group, to have a non-zero dipole length (while the vector multiplets still have zero dipole lengths). From the $\mathcal{N} = 2$ point of view, the dipole length of the hypermultiplets can be understood as a new (dimensionful) moduli of the theory. In other words, the dipole version of $\mathcal{N} = 4$ can be understood as a perturbation of the commutative $\mathcal{N} = 4$ case by dimension five (and higher dimensional) operators. Such a perturbation will break the $SU(4)$ R-symmetry. For the case that the two chiral multiplets in the $\mathcal{N} = 2$ hyper multiplet have the same dipole length, the R-symmetry is broken to $SU(2)_R \times SU(2)_L \times U(1)$, i.e. an $\mathcal{N} = 2$ theory. We note that the specific $\mathcal{N} = 2$ theory described above, in the zero dipole length limit goes back to a $\mathcal{N} = 4$ theory and therefore, this theory may be understood as a noncommutative dipole extension of $\mathcal{N} = 4$ gauge theory. In fact, in the next section we will present the brane configuration which exactly leads to this noncommutative dipole SYM theories.

Of course one can consider the theories with lower SUSY, by further breaking of the R-symmetry. This is possible by assigning different dipole lengths to the chiral matter fields.

---

6 We would like to comment that this is not the case with the Moyal noncommutative gauge theory, which admits 16 SUSY extension. In that case using the Seiberg-Witten map, the noncommutative theory, can be understood as the commutative theory perturbed by a dimension six (and higher dimensional) operators which all preserve $SU(4)$ R-symmetry. [13].
5. String Theory Realization of Noncommutative Dipole Theory

In this section we will study how to embed the noncommutative dipole theories in string theory. This issue has been discussed earlier in [12, 11]. When we orient the background $B_{NS}$ field so that it has only one leg parallel to the brane then the low energy theory on the world volume is a noncommutative dipole theory.

First we show how the dipole star product Eq. (2.1) arises in the twisted compactified string theory. Then we present the supergravity solutions corresponding to T-dual of D2-brane with a twisted compactification [12]. Here we will concentrate on a special twist which preserves eight supercharges. Taking the near horizon limit allows us to study the UV and IR behaviour of the large $N$ limit of the supersymmetric noncommutative dipole gauge theories. As an alternative way, we study a D3-brane probing a Taub-NUT with a particular $B_{NS}$ background [11] and discuss the similarities and differences between the two approaches.

5.1. Dipole star product from twist

Consider a D2-brane along 012 directions with the following “twisted” compactification

$$(x_{012}, x_3, x_{3+a}) \rightarrow (x_{012}, x_3 + 2\pi R, \sum_{b=1}^{6} O_{ba} x_{3+a}),$$  \hspace{1cm} (5.1)

where $a = 1, \ldots, 6$. The explicit form of $O$ is given by $O = e^{2\pi i R M/\alpha'}$ where $M$ is a finite matrix of the Lie algebra $so(6) \equiv su(4)$ with dimensions of length. Under the action of $O$

$$\delta x_a = \Omega_{ab} x_b dx_3,$$  \hspace{1cm} (5.2)

where $\Omega = 2\pi i M/\alpha'$. The rank of $\Omega$ determines the number of supersymmetry preserved. In [12] $\Omega$ was of rank 4 and therefore the model had no supersymmetry. We shall take $\Omega$ with rank 2 preserving $\mathcal{N} = 2$ SUSY. The low energy effective field theory of the above brane configuration, after a T-duality in third direction, is a noncommutative dipole theory. To see this, we review and expand the arguments of [12] showing how the dipole product can be derived from the above twist analysis. First we note that the dipole product of Eq. (2.1) can be equivalently written as

$$(\phi_1 \ast \phi_2)(x) \equiv \exp \left( \frac{1}{2} \left( L_1^\mu \frac{\partial}{\partial x_2^\mu} - L_2^\nu \frac{\partial}{\partial x_1^\nu} \right) \right) \phi_1(x_1) \phi_2(x_2) \bigg|_{x_1 = x_2 = x}$$ \hspace{1cm} (5.3)

7 The Melvin twisted compactification of Matrix Models have also been discussed in [19].
which tells us that inserting a phase
\[ e^{i \sum_{1 \leq i < j \leq n} p_i \cdot L_j} \]
with the requirement \( \sum_{i=1}^{n} L_i = 0 = \sum_{j=1}^{n} p_j \), will generate the noncommutative dipole theory from a given commutative theory.

To show how the phase factor Eq. (5.4) arise from the twist analysis, we consider D2-branes probing the twisted background Eq. (5.1). For simplicity, we can assume the twist acting on only two coordinates \( Z = x_6 + ix_7 \) as \( Z \to e^{i\alpha} Z \), in terms of our previous notation, \( \Omega, \alpha = \Omega R \). In the presence of the twist, going round the compactified circle, open strings can gain a non-trivial “winding” number along the \( Z \) directions. This winding is then an integer multiple of \( \alpha \), the twist angle. Following [12], we call this winding as \( Z \)-charge. Now, let us consider \( n \) interacting open strings with winding numbers \( w_1, ..., w_n \) (around the \( x_3 \) direction) and with \( Z \)-charges \( q_1, ..., q_n \). The scattering amplitude for these \( n \) open strings is equal to the zero twist case, but, now we should also insert the total phase of
\[ e^{i \sum_{1 \leq i < j \leq n} w_i q_j} \]  

(5.6)

We would like to stress that to have a meaningful twisted compactification, we should confine the scattering amplitudes to those which respect the symmetries need for twisted compactification of Eq. (5.1). This means that in our twisted string theory only the scattering processes with the vanishing total windings are allowed. In other words, besides inserting Eq. (5.6) phase factor we should also restrict them to
\[ \sum_{i=1}^{n} w_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} q_i = 0 . \]  

(5.7)

Now, we make a T-duality in \( x_3 \) direction according which, we should replace \( w_i \) with the momenta in third direction and the \( Z \)-charges with the dipole length \( L_i \) (more precisely, \( L_i = \tilde{R} q_i \), where \( \tilde{R} = \frac{\alpha'}{R} \) is the compactification radius after T-duality). So upon the T-duality, Eq. (5.6) will produce the looked-for phase factor of Eq. (5.4). It is worth noting that, all the dipole vectors obtained in this way are along the T-duality direction, \( x^3 \); moreover the ratio of the dipole length for different fields can only be a rational number.

---

8 The necessity of these requirements from the noncommutative dipole field theories side have been discussed earlier.
We would like to point out that in order to find finite dipole lengths in the $\tilde{R} \rightarrow \infty$ (decompactification) limit one should send the twist angle, $\alpha$ to zero. However, this is compatible with the gravity decoupling limit, which we will discuss in the next part.

However, one should note that in the low energy effective field theory of the above twisted open strings, all the possible $q_i$ should be considered, i.e. if the lowest dipole lengths for a twisted string is denoted by $L$, a field can have all the integer multiples of this dipole length. More precisely, this low energy theory is a special case of noncommutative dipole theory we introduced and studied in previous sections. For example, in the $\phi^4$ case, in fact the potential that we find from the above analysis is

$$\lambda \sum_{Q=0}^{\infty} (\phi^\dagger \phi)(x - \frac{QL}{2})(\phi^\dagger \phi)(x + \frac{QL}{2}). \quad (5.8)$$

If we just ignore the higher $Z$-charges, and start with the lowest dipole lengths, these terms will be generated through loop effects. From the field theory point of view this means that in order to regularize the theory, although we do not require new degrees of freedom, we need to add some non-local counter-terms; i.e. if we start with a dipole length corresponding to $q_i = \alpha$ as the lowest dipole length in the above notation, then the non-local counter-terms would involve all the integer multiples of $\alpha$. Note that a twisted string with multiple windings should not be treated as a new degree of freedom (a new field).

5.2. Supergravity background

The supergravity solution which shows a D3-brane probing the above twisted geometry can be obtained by starting with a D2-brane and making a T-duality in the third direction \[12\]. Noting the definition of the twist, the “genuine” distance scale is $dx_a - \delta x_a$ which gives rise to the following background metric

$$ds^2 = f^{-1/2}\left(dx_{012}^2 - \frac{dx_3^2}{1 + (\Omega x)^2}\right) - f^{1/2}\left(dx_a dx_a - \frac{(dx_a \Omega x)^2}{1 + (\Omega x)^2}\right), \quad (5.9)$$

In the above we have made $x_3$ dimensionless by putting $R = l_s = 1$ where $R$ is the radius of the compact $x_3$ direction. The twist $\Omega$ also generates a $B_{NS}$ field given by

$$B_{3a} dx_a = -\frac{\Omega x dx}{1 + (\Omega x)^2}. \quad (5.10)$$

It is worth noting that under the four dimensional parity transformation the above B-field will change sign. (This is not the case with the Moyal case where $B$ has two legs along
the brane and hence does not change sign under parity.) So, the noncommutative dipole theory that we obtain in presence of the above B-field is parity violating.

Since the noncommutative dipole theory is a decoupled theory the near horizon geometry determines the gravity dual of the theory. In fact the decoupling limit is obtained by $\alpha' \to 0$ while keeping $\Omega$ fixed. In terms of $\Omega$ the dipole lengths $L \sim \alpha' \Omega Q$, where $Q$ is an integer determining the winding along the twisted direction. Therefore, the non-trivial dipole effects only appear from the large $Z$-charges, or in the Eq. (5.8) through the terms with large $Q$ ($\alpha'Q = \text{fixed}$). Writing $x_a = r n_a$ with $\|n_a\|^2 = 1$, the near horizon metric is

$$ds^2 = \frac{1}{u^2} \left( dx_{012}^2 - du^2 - \frac{u^2}{u^2 + (\Omega n)^2} dx_3^2 \right) - \left( dn^2 - \frac{(\Omega n.dn)^2}{u^2 + (\Omega n)^2} \right), \quad (5.11)$$

where $u = 1/r$. Observe that the metric is a deformed AdS and a deformed $S^5$. Deformed AdS shows that the theory is no longer conformal which is expected because there is an inherent scale – the dipole length – in the theory. Deformed sphere tells us that the R-symmetry is no longer $SO(6)$. In fact from the metric its clear that, taking a rank 2 twist, the R-symmetry is $SU(2) \times SU(2) \times U(1)$. The behaviour of the NSNS two form is given by

$$\sum_a B_{3a} dn_a = -\frac{\Omega n.dn}{u^2 + (\Omega n)^2}. \quad (5.12)$$

In the near horizon this field breaks Lorentz invariance explicitly on the world volume theory. The dilaton behaves as

$$e^{2(\phi - \phi_0)} = \frac{u^2}{u^2 + (\Omega n)^2}. \quad (5.13)$$

(a) **IR physics**

The parameter $u$ determines the RG scale of the theory. Therefore the IR corresponds to large $u$. For large $u$ the background is $AdS_5 \times S^5$ and hence the noncommutative dipole theories are determined from SYM by perturbing it by a dimension 5 operator $[10,11]$

$$\frac{i}{g_{YM}^2} \left( tr[F_{\mu\nu} \phi^I D_\nu \phi^J] + \sum_K (D_\mu) \phi^K \phi^I \phi^J + \text{fermions} \right), \quad (5.14)$$

where $I, J = 1, ..., 6$ are R-symmetry indices, $\phi^I$ are the scalars, $D_\mu$ is the covariant derivative with respect to gauge fields $A_\mu$ of field strength $F_{\mu\nu}$ and $[...,]$ is complete antisymmetrization. The operator Eq. (5.14) transforms in the $15$ of the R-symmetry group $SU(4)$. As expected, the dual of this operator in supergravity is our rank two field $B_{NS}$.
transforming as 15. The other correspondences from supergravity for the field theory at far IR have been worked out in [12].

(b) UV physics

The UV physics appears on the gravity side when we go near the boundary or equivalently when $u \to 0$. Something interesting happens now. From the form of the metric we see that some of the components vanish. In fact the fibered circle of the deformed $S^5$ — which is a $S^1$ fibered over a base $CP^2$ — shrinks to zero size. T-dualising along that direction we obtain the metric

$$ds^2 = \frac{1}{u^2}[ds^2_{012u} - (dx_3 + Ld\gamma)^2] - d\gamma^2 - ds^2_{CP^2}, \quad (5.15)$$

where $\gamma$ is the circle coordinate. The dilaton is a constant and there are non-trivial $H_{NS}$ and four-form backgrounds.

The above form of metric makes explicit the non-local nature of the underlying field theory:

(i) The proper distance between two points $(x_3, \gamma)$ and $(x_3 + 2\pi L, \gamma)$ is $2\pi$ which is of stringy scale (we have taken $R = l_s = 1$).

(ii) The proper distance between two points $(x_3, \gamma)$ and $(x_3 + \Delta, \gamma)$ when $\Delta$ is not an integer multiple of $L$ is $\frac{1}{u} \to \infty$.

This should be compared with the similar case that we observe for the noncommutative Moyal theory. The metric along the noncommutativity directions $x_2, x_3$ goes as [20,21]

$$ds^2 = \frac{u^2}{u^4 + \theta^4}(dx_2^2 + dx_3^2), \quad (5.16)$$

where $\theta$ is the noncommutativity parameter. At the UV, i.e when $u \to 0$, the metric shrinks to zero. Assuming $x_{2,3}$ forming a torus, \[ we T-dualise the above metric to get

$$ds^2 = \frac{1}{u^2}(\theta dx_2 + dx_3)^2, \quad (5.17)$$

which is of the same form as Eq. (5.15). Also observe that from Eq. (5.15) we actually restore the 4 dimensional superconformal invariance. This was also seen from some related field theory loop calculations in the earlier sections. Before we end we note that the problem regarding T-duality and fermions which appeared in [12] because of the non-SUSY background, will not appear here.

9 This will avoid the IR problems also.
5.3. Noncommutative dipole theories from Taub-NUT background

In [11] another way of studying noncommutative dipole theory was developed. It was shown that when we place a D3-brane near the origin of a Taub-NUT space and switch on a $B_{NS}$ background which has one component along the D3-brane and another component along the shrinking cycle of the Taub-NUT the low energy theory on the world volume of D3-brane is a noncommutative dipole theory. Below we mention the steps.

Consider a D3-brane oriented along $x_0, x_1, x_2, x_3$ and orthogonal to a Taub-NUT space along $x_6, ..., x_9$. In the absence of a $B$ field the metric of a Taub-NUT space probed by a D3-brane is non-singular in a good coordinate system $u'$, and is given by

$$ds^2 = f_2^{-1/2}ds_{0123}^2 + f_2^{1/2}[ds_{45}^2 + du'^2 + u'^2d\Omega^2 + u'^2(dx_6 + B_{6i}dx_i)^2]$$ (5.18)

with $f_2^{-1/2} = u'^2$ in the near horizon limit.

Let us now switch on a $B_{NS}$ field with one leg oriented along the brane. If the asymptotic value of the $B_{NS}$ field is $b$ and

$$h^{-1} = \sin^2\theta + f_1\cos^2\theta$$ (5.19)

then the classical SUGRA background for the system is given by

$$ds^2 = f_2^{-1/2}[ds_{012}^2 + hf_1dx_3^2] + f_2^{1/2}[ds_{45}^2 + du'^2 + u'^2d\Omega^2 + h(dx_6 + B_{6i}dx_i)^2]$$ (5.20)

where $f_1 = u'^{-2}$ in the near horizon limit. The above metric goes to a flat one asymptotically. Using the good coordinate system $u'$, the D3 probes a smooth Taub-NUT space with a $B_{NS}$ field. Therefore from the above analysis we expect that the metric component $g_{33}$ in Eq. (5.20) is given, for small values of $b$, by

$$g_{33} = hf_1f_2^{-1/2} = \frac{u'^2}{1 + u'^2b^2}$$ (5.21)

To compare this to Eq. (5.11) we have to identify $u = 1/u'$. Under this identification the $g_{33}$ components look similar if we replace $| \Omega n |$ with $b$. To see whether this is a generic phenomena with our background we have to identify the other components of the metric and the NSNS field. The background $B'_{NS}$ field is given by

$$B' = h \tan\theta dx_3 \wedge (dx_6 + B_{6i}dx_i) = \frac{b}{b^2 + u'^{-2}} dx_3 \wedge (dx_6 + B_{6i}dx_i)$$ (5.22)
This is the same as Eq. (5.12) if we replace $|\Omega n|$ with $b$. Now let us check for the metric along the sphere direction. The components $g_{66}$ and $g_{77}$ (and also $g_{67}$) have coefficients

$$h f_2^{1/2} = \frac{1}{1 + w^2 b^2} = \frac{u^2}{u^2 + b^2}$$

which is again same under the above replacement. Finally its easy to check that the dilaton also behaves in the expected way.

One last thing to identify is the nature of the dipoles in the noncommutative dipole theory. This can be argued in two ways. First is directly from D3 probing the Taub-NUT with $B_{NS}$. In [11] the dipoles in these theories were identified with rotating arched strings. These strings have angular momentum along (say) $x_6, x_7$ directions and they rotate in the background three form field $H_{367}$. The D3-brane, as usual, is stretched along $x_{0123}$. The system is stable because the tension of the string is balanced by the outward pull due to the rotating string in $H$ background. The identification is now clear from the fact that R-symmetry corresponds to the simultaneous rotations of the $6-7$ and $8-9$ planes by the same angle. The dipole length in this case will be determined by the angular momentum. The fact that interactions of these strings should preserve the internal angular momentum implies that $\sum_i L_i = 0$. However this simple classical picture is valid in the limit of large $b$. Otherwise we have to take into account the radiative corrections for the metric and the $B$-fields radiations.

The second is from the T-dual version of the above model. As we know, the T-dual picture is a NS5-brane oriented along $x^{0,1,...,5}$ and a D4-brane along $x^{0,1,2,3,6}$ where $x^6$ is the compact direction. The directions $x^3, x^4$ are on a slanted torus and therefore the D4 comes back to itself with a shift along $x^3$. The strings connecting the D4 across the NS5-brane are expected to form dipoles and are charged under $(N, \overline{N})$ of the gauge group $U(N)$. Therefore they transform as an adjoint. Its also clear from the brane construction that the vectors do not pick up a dipole length. Recall that this is what we need from our earlier discussions on adjoint matters. However if the matter is in the fundamental then we expect product gauge groups to have a non-trivial noncommutative dipole theory. This also can be easily seen from multiple parallel NS5-branes on the $x^6$ circle with D4-branes along that circle. In the original model this corresponds to a multi Taub-NUT probed by D3-branes. The non-conformal extension of this is to probe the background with integer and fractional D3-branes. Existence of fractional D3 branes gives rise to a scale in the theory. And therefore there would be logarithmic RG flow. Switching on further $B_{NS}$ fields will generate dipole lengths of the fundamental matters on the D3-brane with product gauge groups. This would then be a realization of the noncommutative dipole theory with a RG flow.

18
Therefore from the above detailed analysis we can conclude that the noncommutative dipole theory generated using a Taub-NUT background is an approximation of the one generated using a twist (when we take the twist metric to have a rank two). This is encouraging because the Taub-NUT background is useful to do BPS analysis and now under this identification it can therefore be extended to the twisted case too. However this simple identification doesn’t extend (as far as it seems) to higher rank twists.

6. M-Theory Realization

As discussed in the introduction, the oriented $B_{NS}$ fields give various new theories. These theories can also be realized from M-theory by keeping a M5 brane near a Taub-NUT singularity and switching on an appropriate $C_{\mu\nu\rho}$ field. The M5-brane is oriented along $x^{0,1,2,3,4,5}$ and is orthogonal to a Taub-NUT space along $x^{7,8,9,10}$ where $x^{7}$ is the Taub-NUT circle. The noncommutative dipole theory can be generated from a M5 brane with a $C$ field having two components $\mu, \nu = 4, 5$ along the M5-brane and the other component $\rho$ along the Taub-NUT circle $x^{7}$.

When the external parameters are carefully chosen this leads to a decoupled noncommutative dipole theory in six dimensions. The limits of the external parameters are:

$$C \to \epsilon, \quad R_{7} \to \epsilon, \quad M_{p} \to \epsilon^{-\beta}, \quad \beta > 1 \quad (6.1)$$

In this limit the energy scale of the excitations of the M5-brane is kept finite whereas the other scales in the problem are set to infinity. This decoupling is kinematical. For a different scaling of external parameters

$$C \to finite, \quad R_{7} \to finite, \quad M_{p} \to \infty, \quad R_{10} \to 0 \quad (6.2)$$

we get a dynamical decoupling. This decoupling is in the same spirit as the little string theory.

However in both the cases above we have kept the value of $C$ very low. An interesting case is when $C \to \infty$ and we remove the M5-brane from the picture. It turns out that if we consider the following limits:

$$C \to \infty, \quad M_{p} \to \infty, \quad M_{p}^{3}C^{-1} \to fixed \quad (6.3)$$

with the identification that the M-theory circle is now $x^{7}$ we get a $6 + 1$ dimensional noncommutative YM theory whose coupling constant

$$g_{YM}^{2} = M_{p}^{-3}C = fixed \quad (6.4)$$
This limit is consistent with (and in fact it’s the same as) the limit studied by Seiberg-Witten. The exact limits which give us a noncommutative 6 + 1 dimensional theory is:

\[ C \rightarrow \epsilon^{-1/2}, \quad M_p \rightarrow \epsilon^{-1/6}, \quad R_7 \rightarrow \text{constant}, \quad g^{M}_{\mu\nu} \rightarrow \epsilon^{2/3} \quad (6.5) \]

where \( g^{M}_{\mu\nu} \) is the dimensionless M-theory metric. However, as discussed in \cite{22}, unlike what is naively expected, the theory is not decoupled from gravity.

7. Other Kinds of *-Products

In the above sections we discussed about the new kind of *-product coming from noncommutative dipole theory. This *-product, as opposed to noncommutative theory, doesn’t affect the underlying space-time. But as seen from the supergravity point of view, these theories share the non-local behaviour at the UV region.

An interesting extension is the idea of nonabelian geometry \cite{23}. The derivation of the nonabelian *-product takes into account both the noncommutativity of space-time and the dipole nature of the open strings. The two ends of these dipoles lie on noncommutative spaces with different fluxes on them. Interaction of these dipoles can be thought of as interactions between the center-of-mass of these dipoles.

The end points of the dipoles are labelled by \( x_1 \) and \( x_2 \) respectively and each of the end points see different noncommutativity parameter \( \Omega_i \) and \( \Omega_j \). These \( \Omega \)'s can be transformed as canonical forms

\[ \Omega_i = T_i J T_i^\top \quad (7.1) \]

where \( J \) is a canonical matrix \cite{23}. From the above decomposition the center-of-mass of our dipole turns out to be:

\[ x_c \equiv (T_i^{-1} + T_j^{-1})^{-1}(T_i^{-1} x_1 + T_j^{-1} x_2) \quad (7.2) \]

This choice of center-of-mass changes the interaction matrices and therefore affects the *-product for this theory as:

\[ (\Psi^i_j \ast_{ijk} \Phi^j_k)(x) = \exp \left( \frac{i}{2} \frac{\partial}{\partial x'^\mu} \Omega_{ij;jk} \frac{\partial}{\partial x''^\nu} \right) \Psi^i_j(x') \Phi^j_k(x'') \bigg|_{x'=S^{ik}_{ij}, x''=S^{jk}_{ik}} \quad (7.3) \]

where \( S \) is defined in \cite{23} and \( \Omega_{ij;jk} \) can be given in terms of \( T_i \) as:

\[ \Omega_{ij;jk} = \left( \frac{T_i^{-1} + T_j^{-1}}{2} \right)^{-1} J \left( \frac{T_j^{-1} + T_k^{-1}}{2} \right)^\top (-1) \quad (7.4) \]
The above *-product Eq. (7.3) can be interpreted as the star product for multiple branes having different noncommutativity parameter on them. The quantity $J$ is the usual non-commutativity\footnote{Multiple noncommutativity on the branes were first discussed in \cite{24}, in connection with conifold geometry, and in \cite{25}, for parallel branes in a slowly varying background field.} (due to $B$ field) and $T_i$’s are due to the non abelian nature. Eq. (7.4) therefore encodes this intertwining clearly and there is no way to separate them\footnote{Multiple noncommutativity on the branes were first discussed in \cite{24}, in connection with conifold geometry, and in \cite{25}, for parallel branes in a slowly varying background field.}.

7.1. Embedding in String Theory

From the above constructions of the nonabelian theory it would seem difficult to have supersymmetric brane configurations with different fluxes on their world volumes. However there are two interesting configurations which require different fluxes on their world-volumes to be stable:

(i) A pair of $D5 - \overline{D5}$ wrapped on two cycle of a conifold. The absence of tachyons in this system require different fluxes $b_i$ on their world-volumes. The zero point energy of a string connecting them is

$$E = -\frac{1}{2} \left( |\nu - \frac{1}{2}| + \frac{1}{2} \right)$$  \hspace{1cm} (7.5)

with $\nu$ being the shift in the mode number given by

$$e^{2\pi i\nu} = \frac{(1 - ib_1)(1 + ib_2)}{(1 + ib_1)(1 - ib_2)}$$  \hspace{1cm} (7.6)

Using the right GSO projection one can show that the tachyon in this system becomes massless. For this system one can show that the *-product simplifies as

$$\psi_{ij} *_{ijk} \phi_{jk} \approx \psi_{ij} *_j \phi_{jk}$$  \hspace{1cm} (7.7)

where $*_j$ is defined in terms of $\Omega_j \equiv \Omega_{jj,ijj}$.

(ii) A pair of $Dp - \overline{Dp}$ in flat space with world-volume magnetic and electric fields turned on. This system is $\frac{1}{4}$ BPS because in the presence of these fields the SUSY preserved by both branes and anti-branes are same as long as the magnetic fields come with opposite sign. This construction has been discussed in details in \cite{26}.

8. Discussions and Open Problems

Motivated by string theory brane configuration, where a D3-brane is probing the “twisted” geometry of Eq. (5.1) or a D3-brane probing a Taub-NUT space with a B-field turned on, we studied the noncommutative dipole field theories (NCDFT’s) in more
detail. Besides the string theory, there are some strong motivations from the usual particle physics: charged leptons, neutrinos and neutron and proton have non-zero electric dipole moments [27]. Furthermore the existence of an inherent electric dipole moment is a sign of CP violation. In fact, as we discussed, the noncommutative dipole gauge theories we have studied here in some particular cases show signs of CP and even CPT violation.

With the above motivations we introduced the \( C^* \)-algebra of functions with the appropriate dipole star product Eq. (2.1). Then, studying the renormalizability of noncommutative dipole theories, we showed that there is no IR/UV mixing in this case.

As the first open question to mention here, it would be very interesting to build up a full dipole version of the Standard Model. In this case, unlike the Moyal case [28], it is possible to have \( SU(N) \) noncommutative dipole theories with the fundamental matter fields, and building such a model is more straightforward. Furthermore, we do not face charge quantization problem of the Moyal noncommutative QED [14,28]. In the dipole version, although the Lorentz symmetry is broken, we have the advantage that the theory is still under control and we have a sensible field theory. In particular, we would like to mention the possibility to consider the neutrinos as an adjoint matter field under the dipole version of QED.

The other interesting problem we would like to address here is solitonic and topological solutions of the NCDFT’s. As we discussed, the dipole star product will not appear in the dipole- scalar and pure gauge theories and hence here we do not find the solutions of Moyal noncommutative theories discussed in Ref. [29]. However, the noncommutative dipole version of the ’t Hooft-Polyakov monopole solutions will keep traces of the dipole deformation. It is straightforward to show that the BPS monopole equations in the dipole case reads as

\[
\vec{B}^a(x - \frac{L}{2}) = \pm (\vec{D} \Phi^a)(x),
\]

where \( B_i^a = \epsilon_{ijk} F^{ija} \), \( a \) stands for the SU(2) indices and \( \pm \) corresponds to monopole (or anti-monopole) solutions. Then one can check that the solution to the above equation is obtained by making the shift, \( x \rightarrow x + \frac{L}{2} \) in the argument of the gauge field. It is readily seen that the monopole charge for this solution (\( \int_{S^2} \Phi^a B^a \cdot dS \)) would give the same value as the commutative case. However, this monopole solution besides the monopole charge, also carries (magnetic) dipole moment.

From the string theory point of view the noncommutative dipole theories fall under the general framework of studying various theories on the world-volume of D3-branes with \( B_{NS} \) fields oriented in different ways on the D3-branes. As it is studied in detail here and also in [11,12] when we have a \( B_{NS} \) field with one leg along the D3 brane and other leg orthogonal to it, the low energy theory on it is a NCDFT. However when we orient the
$B_{NS}$ field completely orthogonal to the D3-brane the world volume theory is the *pinned brane* theory \[9\]. Here the D3 has minimum tension at some point in space (origin of a Taub-NUT singularity). The hypermultiplet in these theories are massive. Finally, as is well known, when we have both the legs of the $B_{NS}$ field along the brane the world volume theory is noncommutative geometry.

**Acknowledgements**

We would like to thank Savas Dimopoulos, Ori Ganor, David Kaplan, Mark Van Raamsdonk and Lenny Susskind for many helpful discussions, M. Alishahiha for comments on the manuscript and the Stanford theory group for critical comments. The research of K.D. is supported in part by David and Lucile Packard Foundation Fellowship 2000-13856. The research of M. M. Sh-J. is supported in part by NSF grant PHY-9870115 and in part by funds from the Stanford Institute for Theoretical Physics.
References

[1] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, “Dirac Quantization of Open Strings and Noncommutativity in Branes,” Nucl. Phys. B576 (2000) 578, hep-th/9906161.
M.M. Sheikh-Jabbari and A. Shirzad, “Boundary Conditions as Dirac Constraints,” Eur.Phys.J. C19 (2001) 383, hep-th/9907055.

[2] C.-S. Chu and P.-M. Ho, “Constrained Quantization of Open String in Background B-Field and Noncommutative D-Brane,” Nucl. Phys. B568 (2000) 447, hep-th/9906192.

[3] N. Seiberg and E. Witten, “String Theory and Noncommutative Geometry,” JHEP 9909 (1999) 032, hep-th/9908142.

[4] E. Witten, “Bound States Of Strings And p-Branes,” Nucl.Phys. B460 (1996) 335, hep-th/9510135.

[5] M.M. Sheikh-Jabbari, “Open Strings in a B-field Background as Electric Dipoles,” Phys. Lett. B455 (1999) 129, hep-th/9901080.

[6] M.M. Sheikh-Jabbari, “One Loop Renormalizability of Supersymmetric Yang-Mills Theories on Noncommutative Two-Torus,” JHEP 9906 (1999) 015, hep-th/9903107.

[7] D. Bigatti, L. Susskind, “Magnetic Fields, Branes and Noncommutative Geometry,” Phys.Rev. D62 (2000) 066004, hep-th/9908056.

[8] S. Minwalla, M. Van Raamsdonk, N. Seiberg “Noncommutative Perturbative Dynamics,” JHEP 0002 (2000) 020, hep-th/9912072.

[9] S. Chakravarty, K. Dasgupta, O.J. Ganor and G. Rajesh, “Pinned Branes and New Non-Lorentz Invariant Theories,” Nucl.Phys. B587 (2000) 228, hep-th/0002173.

[10] A. Bergman and O.J. Ganor, “Dipoles, Twists and Noncommutative Gauge Theory,” JHEP 0010, 018 (2000), hep-th/0008030.

[11] K. Dasgupta, O. J. Ganor and G. Rajesh, “Vector Deformations of N = 4 Yang-Mills Theory, Pinned Branes and Arched Strings,” hep-th/0010072.

[12] A. Bergman, K.Dasgupta, O. J. Ganor, J. L. Karczmarek, G. Rajesh, “Non-local Field Theories and Their Gravity Duals,” hep-th/0103090.

[13] I. Ya. Aref’eva, D. M. Belov, A. S. Koshelev, “A Note on UV/IR for Noncommutative Complex Scalar Field,” hep-th/0001215.

[14] M. Hayakawa, “Perturbative Analysis on Infrared and Ultraviolet Aspects of Noncommutative QED on R^4,” hep-th/9912116.

[15] M.M. Sheikh-Jabbari, “Discrete Symmetries (C,P,T) in Noncommutative Field Theories,” Phys.Rev. Lett. 84 (2000) 5265, hep-th/0001167.

[16] J. D. Jackson “Classical Electro-Dynamics,” New York : Wiley, c1999.

[17] K. Dasgupta and M.M. Sheikh-Jabbari, “Work in Progress”.

[18] K. Intriligator, “Maximally Supersymmetric RG Flows and AdS Duality,” Nucl.Phys. B580 (2000) 99, hep-th/9909082.

[19] L. Motl, “Melvin Matrix Models,” hep-th/0107002.
[20] A. Hashimoto and N. Itzhaki, “Noncommutative Yang-Mills and the AdS/CFT Correspondence,” Phys. Lett. B 465, 142 (1999), hep-th/9907166.

[21] J. M. Maldacena and J. G. Russo, “Large N Limit of Noncommutative Gauge Theories,” JHEP 9909, 025 (1999), hep-th/9908134.

[22] M. Alishahiha, H. Ita, Y. Oz, “Graviton Scattering on D6 Branes with B Fields,” JHEP 0006 (2000) 002, hep-th/0004011.

[23] K. Dasgupta and Z. Yin, “Non-Abelian Geometry,” hep-th/0011034.

[24] R. Tatar, “A Note on Noncommutative Field Theory and Stability of Brane-Antibrane Systems,” hep-th/0009213.

[25] L. Dolan and C. Nappi, “A Scaling Limit with Many Noncommutativity Parameters,” hep-th/0009223.

[26] D. S. Bak and A. Karch, “Supersymmetric brane-antibrane configurations,” hep-th/0110039.

[27] D. E. Groom, et. al., “The Review of Particle Physics,” Eur. Phys. J C15 (2000) 1.

[28] M. Chaichian, P. Presnajder, M. M. Sheikh-Jabbari and A. Tureanu, “Noncommutative standard model: Model building,” hep-th/0107055.

[29] R. Gopakumar, S. Minwalla, A. Strominger, “Noncommutative Solitons,” JHEP 0005 (2000) 020, hep-th/0003160.