GENERALIZED RELATIVISTIC MAGNETOHYDRODYNAMIC EQUATIONS FOR PAIR AND ELECTRON–ION PLASMAS

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ABSTRACT

We derived one-fluid equations based on a relativistic two-fluid approximation of $e^\pm$ pair plasma and electron–ion plasma to reveal the specific relativistic nature of their behavior. Assuming simple condition on the relativistic one-fluid equations, we propose generalized relativistic magnetohydrodynamic (RMHD) equations which satisfy causality. We show the linear analyses of these equations regarding various plasma waves to show the validity of the generalized RMHD equations derived here and to reveal the distinct properties of the pair plasma and electron–ion plasma. The distinct properties relate to (1) the inertia effect of electric charge, (2) the momentum of electric current, (3) the relativistic Hall effect, (4) the thermal electromotive force, and (5) the thermalized energy exchange between the two fluids. Using the generalized RMHD equations, we also clarify the condition that we can use standard RMHD equations and that we need the distinct RMHD equations of pair and electron–ion plasmas. The standard RMHD is available only when the relative velocity of the two fluids is nonrelativistic, a difference of the enthalpy densities of the two fluids is much smaller than the total enthalpy density, and the above distinct properties of the pair/electron–ion plasma are negligible. We discuss a general relativistic version of the equations applicable to the pair and electron–ion plasmas in black hole magnetospheres. We find the effective resistivity due to the shear of frame dragging around a rotating black hole.

Key words: galaxies: active – galaxies: jets – galaxies: magnetic fields – galaxies: nuclei – methods: analytical – plasmas – relativity

1. INTRODUCTION

The most powerful engines in the universe are located in the centers of active galactic nuclei (AGNs), quasars (QSOs), micro quasars ($\mu$QSOs), and gamma-ray bursts (GRBs). Observations have shown that all of them eject relativistic jets (Pearson & Zensus 1987; Biretta et al. 1999; Junor et al. 1999; Mirabel & Rodriguez 1994, 1998; Kulkarni 1999). In spite of the drastic difference of their characteristic scales and powers, it is believed that the activities of these objects, such as the relativistic jet ejection, are supported commonly by the drastic phenomena of accretion disks around black holes (Mirabel & Rodriguez 1998). However, the distinct mechanism of the activity is not confirmed yet. Models with interaction between the plasma and magnetic field in the very strong gravity of the black hole are thought to be most promising. The region where the plasma and magnetic field interact each other near a black hole is called a black hole magnetosphere. A black hole magnetosphere consists of an accretion disk and a corona around a black hole and it is thought to be composed of various kinds of plasmas. In the case of an active galactic nucleus (AGN), it is suggested that the disk is made of electron–ion plasma (Ford et al. 1994), the core of the relativistic jet is mainly of electron–positron (pair) plasma (Wardle et al. 1998), and the corona is by both of them, while the actual components of such plasmas have not been confirmed observationally yet. It is a natural idea that a corona near a black hole consists mainly of pair plasma, and near its disk, mainly of electron–ion plasmas. Such a difference of the kinds of plasmas may influence the dynamics of a black hole magnetosphere, but none of the works has been done to investigate it.

Black hole magnetosphere has been studied on the basis of the nonrelativistic magnetohydrodynamics (MHD) with pseudo-Newtonian potential (Paczynski & Wiita 1980) and ideal general relativistic MHD (GRMHD; e.g., Takahashi et al. 1990; Tomimatsu 1994). The relativistic MHD (RMHD) or GRMHD is a one-fluid approximation of the plasma and is based on the relativistic conservation laws of particle number, momentum, and energy, Maxwell equations, and the simple Ohm’s law in the plasma rest frame. The simple Ohm’s law means that the electric field $E'$ is proportional to the current density $J'$ in the plasma rest frame: $E' = \eta J'$, where $\eta$ is the electric resistivity. We call such RMHD (GRMHD) equations the “standard” RMHD (GRMHD) equations in this paper. When we set $\eta$ zero, the standard RMHD (GRMHD) becomes the “ideal” RMHD (GRMHD). Recently, numerical simulations of the ideal GRMHD have become prevailing and yielded interesting, important results with respect to relativistic outflow/jet formation (Koide 2004; McKinney 2006) and extraction of black hole rotational energy by magnetic field (Koide et al. 2002; Koide 2003; Komissarov 2005). However, since they used the standard GRMHD and thus the one-fluid approximation, they ignored the unique properties of the pair/electron–ion plasma. To begin with, we have to check the validity of the standard RMHD and clarify its application conditions for plasmas around black holes.

To clarify these points, we reconsider the RMHD equations. Such a task was first performed by Ardavan (1976) using the Vlasov–Boltzmann equation for a pulsar magnetosphere. It yielded a relativistic version of the generalized Ohm’s law and a new condition for the validity of the MHD approximation for a pulsar magnetosphere (where the Lorentz factor of plasma is much larger than unity). A more generalized treatment, which included annihilation of electrons and positrons, radiation, Compton scattering, and pair photoproduction was formulated by Blackman & Field (1993) and Gedalin (1996). Reconsideration of ideal MHD in a neutral cold plasma based on two-fluid approximation was presented by Melatos & Melrose (1996), who investigated the conditions under which the MHD approximation breaks down. To investigate black hole magneto-
spheres, Khanna (1998) formulated a general relativistic version of the two-fluid approximation in the Kerr metric. A more generalized version in a time-varying spacetime was derived by Meier (2004) from the general relativistic Vlasov–Boltzmann equation.

To investigate the distinct properties of the pair and electron–ion plasmas and the applicability of the standard RMHD, we derived one-fluid equations using a relativistic two-fluid approximation of a plasma consisting of positively charged particles, each has charge $e$ and mass $m_+,$ and the other fluid of negatively charged particles, each with charge $-e$ and mass $m_-.$

Throughout this paper, we assume the charge of ions is $e,$ while in a pair plasma case, they are equal. With respect to the one-fluid equations with finite resistivity larger than a certain value, causality seems to be broken mathematically (Koide 2008). We consider the dispersion relation of the electromagnetic waves and show the mathematical condition where the equations satisfy causality.

On the basis of the above results, we propose the “generalized RMHD equations” for the plasmas, which are introduced in this field for the first time by the present work. These equations can reveal the distinct properties of the electron–ion and pair plasmas. The differences between the generalized RMHD equations and the standard RMHD equations concern (1) the inertia effect of charge, (2) the momentum of current, (3) the relativistic Hall effect, (4) the thermal electromotive force, and (5) the thermalized energy exchange rate. Mainly using linear analyses, we investigate the influence of such distinct properties to plasma dynamics which cannot be clarified from the standard RMHD equations. The conditions that the distinct properties of the electron–ion and pair plasmas are negligible and the premise conditions of the generalized RMHD clarify the application condition of the standard RMHD equations.

In Section 2, we derive one-fluid equations from the relativistic two-fluid equations. In Section 3, dispersion relation of the electromagnetic waves and show the mathematical condition where the equations satisfy causality.

To derive one-fluid equations of the plasma, we define the average and difference variables with respect to plasma dynamics which cannot be clarified from the standard RMHD equations. In Section 4, we propose a set of generalized RMHD equations with any mass ratio $m_+/m_-$ from the one-fluid equations. To show the reasonable property of the generalized RMHD equations, in Section 5, we investigate the linear modes of various plasma waves derived by the generalized RMHD equations and show the unique nature of the pair and electron–ion plasmas. Section 6 shows the unique properties of the electron–ion and pair plasmas and estimations of them. In Section 7, we discuss the breakdown condition of the generalized RMHD equations and the nonlinear distinct nature of the electron–ion and pair plasmas in the jet-forming region around the black hole. In Section 8, we suggest the distinct general relativistic effects shown by the generalized GRMHD equations. Here, we indicate that the shear of frame dragging induces effective resistivity near a rotating black hole. In Section 9, the results are summarized.

2. TWO-FLUID EQUATIONS AND ONE-FLUID EQUATIONS

We employ relativistic two-fluid equations of plasma consisting of two fluids. One fluid is composed of positively charged particles, and the other is composed of the negatively charged particles. We derive a set of one-fluid equations equivalent to the two-fluid equations in the Minkowski spacetime $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ where the line element is given by

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$
We find the following relations between the variables with respect to the enthalpy density:

\[
\Delta h^1 = n^2 \left( \frac{h_+ 2m_-}{n_+^2 m} - \frac{h_- 2m_+}{n_-^2 m} \right),
\]

(11)

\[
h^1 = n^2 \left[ \frac{h_+ (2m_-)^2}{n_+^2 m} + \frac{h_- (2m_+)^2}{n_-^2 m} \right],
\]

(12)

\[
\Delta h^1 = n^2 \left( \frac{h_+ 2m_-}{n_+^2 m} - \frac{h_- 2m_+}{n_-^2 m} \right),
\]

(13)

\[
\Delta h^2 = n^2 \left[ \frac{h_+ (2m_-)^3}{n_+^2 m} - \frac{h_- (2m_+)^3}{n_-^2 m} \right].
\]

(14)

We find the following relations between the variables with respect to the average and difference variables of the two fluids as

\[
h^1 = 4\mu h^1 - \Delta \mu \Delta h^1,
\]

(15)

\[
\Delta h^1 = 2\Delta \mu h^1 + \Delta h^1,
\]

(16)

\[
\Delta h^2 = -8\mu \Delta \mu h^1 + 4(1 - 3\mu)\Delta h^1,
\]

(17)

where \( \mu = m_+ m_- / m^2 \) and \( \Delta \mu = (m_+ - m_-) / m \). It is noted that we have the relation \( \mu = [1 - (\Delta \mu^2)] / 4 \). We also find the relations with respect to the average and difference variables of the two fluids as

\[
m_+ n_+ U^\mu_+ + m_- n_- U^\mu_- = \rho U^\mu - \frac{1}{e} (m_+ - m_-) J^\mu,
\]

(18)

\[
h_+ U^\mu_+ U^\nu_+ + h_- U^\mu_- U^\nu_- = h^1 U^\mu U^\nu + \frac{h^1}{(2ne)^2} J^\mu J^\nu
\]

\[+ \frac{\Delta h^1}{2ne} (U^\mu J^\nu + J^\mu U^\nu),
\]

(19)

\[
m_- h_+ U^\mu_+ U^\nu_+ - m_+ h_- U^\mu_- U^\nu_-
\]

\[= \frac{h^1}{4n^2 e} \rho (U^\mu J^\nu + J^\mu U^\nu) + \frac{\Delta h^1}{2mn^2} \rho^2 U^\mu U^\nu + \frac{m}{8n^2 e^2} \Delta h^2 J^\mu J^\nu.
\]

(20)

Using the above variables and their relations, we derive the one-fluid equations from the two-fluid Equations (1), (2) as

\[
\partial_t (\rho U^\nu) = 0,
\]

(21)

\[
\partial_t \left[ \frac{h^1}{2ne} (U^\mu J^\nu + J^\mu U^\nu) + \frac{\Delta h^1}{2ne} (U^\mu J^\nu + J^\mu U^\nu) \right]
\]

\[= -\partial^\mu p + J^\nu F^\mu _\nu,
\]

(22)

\[
\frac{1}{2ne} \partial_t \left[ \frac{h^1}{2ne} (U^\mu J^\nu + J^\mu U^\nu) + \Delta h^1 U^\mu U^\nu + \frac{\Delta h^2}{(2ne)^2} J^\mu J^\nu \right]
\]

\[= \frac{1}{2ne} \partial^\mu (\Delta \mu p - \Delta \rho) + \left( U^\nu - \frac{\Delta \mu}{ne} J^\nu \right) F^\mu _\nu + \frac{R^\mu}{ne},
\]

(23)

where \( p = p_+ + p_- \), \( \Delta \rho = p_+ - p_- \), and

\[
R^\mu = -\eta ne \sqrt{M} \left[ J^\mu - \frac{Q}{M} (1 + \Theta) U^\mu \right].
\]

(24)

Here, \( M = U^\nu U_\nu \), \( Q = U^\nu J_\nu \), and \( \Theta \) is the thermal energy exchange rate from negatively charged fluid to the positive fluid (see Appendix A). In the electron–ion plasma case, we usually assume that the energy exchange due to the friction between the two fluids is not significant: \( \Theta = 0 \). On the other hand, in the pair plasma case, since the energy of electron fluid and positron fluid is exchanged frequently, we set

\[
\Theta = \frac{2m^2}{e^2} \frac{Q^2 - MW}{(2mnM)^2 - (mQ/e)^2} \theta,
\]

(25)

where \( W = J_\nu J^\nu \) and \( \theta \) is the redistribution coefficient of the thermalized energy due to the friction of the two fluids (see Appendix A).

### 3. PROPAGATION OF ELECTROMAGNETIC WAVES AND CAUSALITY

With respect to the one-fluid equations of a pair plasma, Koide (2008) indicated that causality with the electromagnetic wave propagation breaks down if the electric resistivity of the pair plasma exceeds a critical value. To clarify the causal condition on the one-fluid equations, we investigate electromagnetic waves in unmagnetized, uniform plasma using the linear analysis of the one-fluid Equations (21)–(23) and the Maxwell Equations (3) and (4). We write the variables of the uniform, rest plasma and uniform magnetic field as \( \rho = \bar{\rho} \), \( p = \bar{p} \), \( h^1 = \bar{h} \), \( U = 0, B = 0 \), and \( E = 0 \). When the perturbations of the variables are written by \( \tilde{\rho} = \rho - \bar{\rho} \), \( \tilde{p} = p - \bar{p} \), \( \tilde{h} = h^1 - \bar{h} \), \( \tilde{U} = U, \tilde{B} = B \), and \( \tilde{E} = E \), we have the linearized equations in the 3-vector form,

\[
\frac{\partial}{\partial t} \tilde{\rho} + \bar{\rho} \nabla \cdot \tilde{U} = 0,
\]

(26)

\[
\frac{\partial}{\partial t} \tilde{h} - \bar{h} \nabla \cdot \tilde{U} = -\bar{h} \nabla \cdot \tilde{U},
\]

(27)

\[
\frac{\partial}{\partial t} \tilde{E} - \bar{E} = -\bar{E} \cdot \tilde{J},
\]

(29)

\[
\nabla \cdot \tilde{B} = 0,
\]

(31)
where \( q = ne \) is the “latent” charge density, \( \vec{K} = \vec{J}/(ne) \), \( \tilde{q} = \tilde{n}e \), and \( \vec{K} = \vec{J}/(\tilde{n}e) \). Hereafter, we note an equilibrium variable by a bar, and perturbation by a tilde. We assume the resistivity is uniform and constant. We also assume that the perturbation of any variables \( \Delta \) is proportional to \( \exp(i \mathbf{k} \cdot \mathbf{r} - i \omega t) = \exp(i \eta \mu \omega \mathbf{x}^\nu) \), where \( k^\mu = (\omega, \mathbf{k}) \) is the constant covariant vector called the wave number 4-vector. Furthermore, for simplicity, we investigate the transverse modes of the electromagnetic waves in unmagnetized plasma, thus we set \( \tilde{\mathbf{E}} \perp \mathbf{k} \).

Then we have the following linearized equations:

\[
\frac{\partial}{\partial t} \tilde{\mathbf{B}} = -\nabla \times \tilde{\mathbf{E}}, \tag{32}
\]

\[
\tilde{\mathbf{J}} + \frac{\partial}{\partial t} \tilde{\mathbf{E}} = \nabla \times \tilde{\mathbf{B}}, \tag{33}
\]

where \( \eta = \eta(\tilde{q}^2/\mu_\tilde{h}) \) is the “latent” charge density, \( \mu_\tilde{h} = \mu h \). This notation is mathematically identical to the dispersion relation of electromagnetic wave in the resistive pair plasma (Koide 2008). The detailed analysis of the dispersion relation (Equation (42)) performed by Koide (2008) shows that the group velocity of the electromagnetic wave is smaller than the speed of light when \( h < 2 \), while it is larger than the light speed (unphysical) when \( h > 3 \). That is, \( h > 3 \) is forbidden, while the details in the range \( 2 < h < 3 \) are shown in Koide (2008). Consequently, the condition of the causal one-fluid equations is given by \( h < 2 \), that is,

\[
\eta < \frac{\sqrt{2}}{\omega_p}. \tag{43}
\]

Then we obtain the dispersion relation

\[
\omega = \Omega_h = \frac{\tilde{q}^2}{k h} + i \omega , \tag{41}
\]

where \( H = (\eta \tilde{q}^2)/\mu \tilde{h} \). This equation is mathematically identical to the dispersion relation of electromagnetic wave in the resistive pair plasma (Koide 2008). The detailed analysis of the dispersion relation (Equation (42)) performed by Koide (2008) shows that the group velocity of the electromagnetic wave is smaller than the speed of light when \( H < 2 \), while it is larger than the light speed (unphysical) when \( H > 3 \). That is, \( H > 3 \) is forbidden, while the details in the range \( 2 < H < 3 \) are shown in Koide (2008). Consequently, the condition of the causal one-fluid equations is given by \( H < 2 \), that is,

\[
\eta < \frac{\sqrt{2}}{\omega_p}. \tag{43}
\]
The generalized RMHD equations for a pair plasma are given by

\[ \partial_t (\rho U^\nu) = 0, \]  
\[ \partial_\nu \left[ \frac{1}{\rho} \left( U^\mu J^\nu + J^\mu U^\nu \right) \right] = -\partial^\mu p + J^\nu F^\mu_v, \]  
\[ \frac{1}{2q^2} \partial_\nu \left[ \frac{1}{q} (U^\mu J^\nu + J^\mu U^\nu) \right] = - \frac{1}{2} \partial^\mu \Delta p + U^\nu F^\mu_v - \eta [J^\mu + Q(1 + \Theta)U^\mu]. \]  

In this pair plasma case, the Hall term disappears, while the current inertia effect is found in the equation of motion (Equation (56)). The thermal electromotive force is different from that of the electron–ion plasma. It is also noted that \( \Theta \) is not negligible.

In this section, we proposed a set of generalized RMHD Equations (46)–(48), the electron–ion plasma RMHD Equations (49)–(51), and the pair plasma RMHD Equations (55)–(57). The comparison between these sets of equations and the standard RMHD Equations (52)–(54) reveals the unique properties of the electron–ion and pair plasmas. We examine these unique properties in the following sections.

5. LINEAR ANALYSES OF RMHD WAVES

We newly proposed a set of generalized RMHD Equations (46)–(48) which are applicable to electron–ion and pair plasmas in the previous section. In this section, we show the linear analyses of these equations concerning various plasma waves to show the reasonable property of the generalized RMHD equations and to clarify the distinct properties of the electron–ion and pair plasmas. First, we derive the general dispersion relations with respect to the generalized RMHD equations, and next we reveal the unique nature of the electron–ion and pair plasmas, respectively.

We investigate waves propagating in a uniform, rest plasma and a uniform magnetic field. Linearized equations of perturbations, \( \hat{\rho} = \rho - \bar{\rho}, \hat{p} = p - \bar{p}, \hat{h} = h^i - \bar{h}, \hat{U} = U, \hat{B} = B - \bar{B}, \) and \( \hat{E} = E \), are given by

\[ \frac{\partial}{\partial t} \hat{\rho} + \hat{\rho} \nabla \cdot \hat{U} = 0, \]  
\[ \hat{h} \frac{\partial}{\partial t} \hat{U} = -\nabla \hat{p} + \hat{J} \times \hat{B}, \]  
\[ \hat{\mu} \frac{\partial}{\partial t} \hat{\mathbf{K}} = \frac{1}{2q} \nabla \cdot (\Delta \mu \hat{p} - \Delta \hat{p}) \]  
\[ + (\hat{U} - \Delta \mu \hat{\mathbf{K}}) \hat{B} + \hat{E} - \eta \hat{J}, \]  
\[ \nabla \cdot \hat{E} = \hat{\rho}_e, \]  
\[ \nabla \cdot \hat{B} = 0, \]  
\[ \frac{\partial}{\partial t} \hat{B} = -\nabla \times \hat{E}, \]  
\[ \hat{J} + \frac{\partial}{\partial t} \hat{E} = \nabla \times \hat{B}. \]  

These equations are closed with the equation of state (EoS), \( h = h(\rho, p) = \rho \hat{h}(p/\rho) \). The adiabatic EoS for single-component relativistic fluids, which are in thermal equilibrium, has been known, and is given by

\[ \frac{\hat{h}_\pm}{\rho_\pm} = \frac{K_3(\rho_\pm/p_\pm)}{K_2(\rho_\pm/p_\pm)} \equiv \hat{h}(p_\pm/\rho_\pm) \]  

(Chandrasekhar 1938; Synge 1957). Here, \( K_2 \) and \( K_3 \) are the modified Bessel functions of the second kind of order 2 and 3, respectively. Useful approximation of the EoS

\[ \hat{h}_{\text{RC}}(\hat{p}) = 1 + \hat{p} \left( 1 + \frac{3}{3}\frac{\hat{\rho} + 3}{\hat{\rho} + 2} \right) \]  

was proposed by Ryu et al. (2006), where \( \hat{\rho} = p/\rho \). This expression has the functional dependence similar to \( \hat{h}(\hat{p}) \) of Equation (65) and replaces complex calculations of modified Bessel functions by algebraic calculations. When we consider the adiabatic one-component fluid in the rest frame, Equations (1) and (2) reduce to

\[ \partial_t (n_\pm U^\nu_{\pm}) = 0, \]  
\[ \partial_\nu (h_{\pm} U^\mu_{\pm} U^\nu_{\pm}) = -\partial^\mu p_{\pm}. \]  

The linearized equations of the perturbations, \( \tilde{n}_\pm = n_\pm - \bar{n}_\pm, \tilde{\rho}_\pm = p_\pm - \bar{\rho}_\pm, \tilde{h}_\pm = h_\pm - \bar{h}_\pm, \) and \( \tilde{U}_\pm = U^\mu_{\pm} - \bar{U}^\mu_{\pm} \) where \( U^\mu_{\pm} = (1, 0, 0, 0) \) are

\[ \tilde{Q}_0 \tilde{n}_\pm + \tilde{Q}_1 \sum_i \tilde{Q}_i \tilde{U}^i_{\pm} = 0, \]  
\[ \tilde{Q}_0 \tilde{h}_\pm + \tilde{Q}_1 \sum_i \tilde{Q}_i \tilde{U}^i_{\pm} = -\tilde{\rho}_\pm. \]  

These equations yield

\[ \frac{\tilde{n}_\pm}{\tilde{h}_\pm} = \frac{\tilde{h}_\pm - \tilde{\rho}_\pm}{\tilde{h}_\pm}. \]  

From Equation (65), we find

\[ \frac{\tilde{\rho}_\pm}{\rho_\pm} \frac{\tilde{\rho}_\pm}{\rho_\pm} = \frac{\hat{h}'(\tilde{\rho}_\pm)}{\hat{h}'(\tilde{\rho}_\pm)} \equiv (\tilde{\rho}_\pm). \]
In general, $\Gamma$ is not constant and is a function of $\tilde{\rho}/\bar{\rho}$. $\Gamma$ is called the effective adiabatic index. When $\Gamma(\tilde{\rho}/\bar{\rho}) \approx \Gamma(\tilde{\rho}/\bar{\rho})$ which is satisfied only if $\tilde{\rho}/\bar{\rho} \ll 1$ or $\tilde{\rho}/\bar{\rho} \gg 1$, we have $(1/\bar{T})\tilde{\rho} = \Gamma\bar{\rho}$, that is,

$$\frac{\dot{\tilde{\rho}}}{\tilde{\rho}} = \frac{\Gamma}{\bar{\rho}} \left( \frac{d\bar{p}}{d\bar{\rho}} \right). \tag{74}$$

In general, $(\tilde{\rho}/\bar{\rho})(\tilde{\rho}/\bar{\rho}) = \Gamma(\tilde{\rho}, \bar{\rho})$ depends on both $\tilde{\rho}$ and $\bar{\rho}$.

### 5.1. Plasma Oscillation

To show the inclusion of plasma oscillation as a fundamental mode of the generalized RMHD Equations (46)–(48) with the Maxwell Equations (3) and (4), we simply assume

$$\eta = 0, \quad \bar{B} = 0, \quad \tilde{\rho} = 0. \tag{75}$$

Because $\tilde{\rho} = 0$, according to Equation (73), we have $\tilde{\rho} = \Delta \tilde{\rho} = 0$. The linearized Equations (58)–(64) become

$$\frac{\tilde{\Delta} \tilde{\rho}}{\bar{q}} \frac{\partial}{\partial t} \tilde{E}^k = \tilde{E}^k, \tag{76}$$

$$\tilde{J} + \frac{\partial}{\partial t} \tilde{E}^j = 0. \tag{77}$$

Immediately, we have the dispersion relation

$$\omega^2 = \omega_p^2, \tag{78}$$

where $\omega_p^2 = \bar{q}^2/(\mu \bar{\rho})$. This mode is recognized as the plasma oscillation, because $\omega_p$ is identical to the plasma frequency $\omega_p = \sqrt{4\pi e^2/\mu m}$ when $\tilde{\rho} = 0$; for the electron–ion plasma, the plasma frequency is $\omega_p = \sqrt{4\pi e^2/\mu m}$, and in the pair plasma case, $\omega_p = \sqrt{2\pi e^2/\mu m}$. As we will show in the following subsection, $\omega_p$ is the plasma frequency when the pressure is relativistically high.

### 5.2. Compressional Modes

In this subsection, we derive a dispersion relation of longitudinal oscillation modes ($\tilde{U} \parallel k$) in an unmagnetized, rest plasma with a uniform, finite pressure $\bar{\rho}$. For simplicity, we assume the temperatures of the two fluids are the same: $\bar{T} = \bar{T}_e = \bar{T}_i$. Using Equation (5) and zeroth component of Equation (8), we have $\gamma n_{\pm} = \gamma (\gamma \rho \pm m_\pm r_\pm e)/m$ when $\gamma = \gamma_e \approx \gamma_i$. Using these equations, we have

$$\frac{\partial \gamma n}{\partial t} = -\frac{2}{m} \nabla \cdot (\rho \tilde{U}) + \frac{\mu}{e} \left( \frac{\nabla \cdot \tilde{J}}{\rho} \right). \tag{79}$$

In the present nonrelativistic case, setting $\gamma = 1$, we have

$$\frac{\partial n}{\partial t} = -\frac{2}{m} \nabla \cdot (\rho \tilde{U}) + \frac{\mu}{e} \left( \frac{\nabla \cdot \tilde{J}}{\rho} \right). \tag{80}$$

If $\Gamma(\tilde{\rho}/\bar{\rho}) = \Gamma(\tilde{\rho}/\bar{\rho}) = \Gamma$ is uniform and constant, we obtain

$$\frac{\partial \tilde{\rho}}{\partial t} = -\frac{2\Gamma}{m} \left[ \nabla \cdot (\tilde{\rho} \tilde{U}) - \frac{\Delta \mu}{2e} (\nabla \cdot \tilde{J}) \right]. \tag{81}$$

These equations yield

$$\frac{\mu \tilde{h}}{q^2} \frac{\partial^2}{\partial t^2} \tilde{J} = -\frac{\Delta \mu \Gamma T}{e} \nabla (\nabla \cdot \tilde{U})$$

$$+ \frac{\Gamma T}{2q^2} (1 + (\Delta \mu)^2) \nabla (\nabla \cdot \tilde{J}) - \tilde{J}, \tag{82}$$

$$\frac{\partial^2}{\partial t^2} \tilde{U} = \frac{2\Gamma}{m} \tilde{\rho} \frac{\nabla \cdot \tilde{U}}{\Gamma} - \frac{\Delta \mu \Gamma T}{e} \nabla (\nabla \cdot \tilde{J}). \tag{83}$$

Because $\tilde{J} \parallel k$ and $\tilde{U} \parallel k$ in the longitudinal modes, we obtain

$$-\omega^2 \frac{\mu \tilde{h}}{q^2} \tilde{J}_l = \frac{\Delta \mu \Gamma T}{e} k^2 \tilde{U}_l - \frac{\Gamma T}{2q^2} (1 + (\Delta \mu)^2) k^2 \tilde{J}_l - \tilde{J}_l, \tag{84}$$

$$-\omega^2 \tilde{h} \tilde{U}_l = -\frac{2\Gamma}{m} \tilde{\rho} \frac{k^2 \tilde{U}_l}{\Gamma} + \frac{\Delta \mu \Gamma T}{e} k^2 \tilde{J}_l. \tag{85}$$

where $\tilde{J}_l \equiv (\tilde{J} \cdot k)/k$ and $\tilde{U}_l \equiv (\tilde{U} \cdot k)/k$, $k \neq 0$. Then we get the following dispersion relation:

$$\left[ \omega^2 - \frac{c_s^2}{2\mu} (1 + (\Delta \mu)^2) k^2 - \omega_p^2 \right] \left( \omega^2 - 2c_s^2 k^2 \right) = \frac{(\Delta \mu)^2 (c_s k)^4}{\mu}, \tag{86}$$

where $c_s^2 = \Gamma \tilde{p}/\bar{h} = (\tilde{p}/\bar{h})(dp/d\rho)$.

In the case of an electron–ion plasma ($\Delta \mu \approx 1$, $\mu = m_e/m \ll 1$), when $\omega \approx c_s k/\sqrt{\mu}$, the dispersion relation becomes

$$\omega^2 = \frac{c_s^2}{2\mu} k^2 + \omega_p^2. \tag{87}$$

This expression shows the dispersion relation of the plasma oscillation for the plasma with finite pressure. In the case that $\omega^2 \ll \omega_p^2/\mu$, we have the dispersion relation of sound waves

$$\omega^2 = 2c_s^2 k^2. \tag{88}$$

In the case of a pair plasma ($\Delta \mu = 0$, $\mu = 1/4$), we have two modes

$$\omega^2 = \omega_p^2 + 2c_s^2 k^2, \tag{89}$$

and

$$\omega^2 = 2c_s^2 k^2. \tag{90}$$
The former is the dispersion relation of plasma oscillation and the latter is that of the sound wave.

We investigate the compressional wave of the magnetized plasma. When \( k \parallel \vec{B} \) and \( \vec{U} \parallel \vec{B} \), the dispersion relation is the same as that of the unmagnetized plasma wave. Then, we investigate the case that \( k \perp \vec{B} \), \( k \perp \vec{U} \), and \( \rho_e = 0 \) in this paragraph. Because \( \rho_e = 0 \), we have \( \Delta \rho_p = 0 \). In the case that \( k \ll \omega_p \) and \( \omega \ll \omega_c \), the left-hand side, first term, and the Hall term on the right-hand side of the Ohm’s law (Equation (60)) are negligible. The validity of this assumption is checked below. Here, we defined a characteristic frequency of the charged particle with mass \( m \), charge \( e \) in the magnetic field \( \vec{B} \), \( \omega_c = e \vec{B} / m \). Using linearized Equations (58)–(64), we have the dispersion relation

\[
\omega^2 = v^2_k k^2, \quad (96)
\]

where \( v^2_k = (\Gamma \vec{p} + \vec{B}^2) / (\bar{\mu} + \vec{B}^2) \). This shows the dispersion relation of the fast wave. It is also noted that \( v_f < 1 \). Below, we check the validity of the assumption \( k \ll \omega_p \) and \( \omega \ll \omega_c \) that we used. Using the Ohm’s law (Equation (60)) with this assumption and the dispersion relation (Equation (96)), the left-hand side of the Ohm’s law (Equation (60)) becomes

\[
\frac{i \omega \mu \vec{h}}{q} \vec{J} = \left( \frac{k}{\omega_p} \right)^2 (v^2_k - 1) \vec{U} \times \vec{B}, \quad (97)
\]

This clearly shows that the term on the left-hand side can be ignored in Equation (60) when \( k \ll \omega_p \). The absolute value of the summation of the thermal electromotive force term and the Hall term of the Ohm’s law (Equation (60)) is estimated by

\[
\left| \frac{1}{q} i k \Delta \mu \vec{p} - \Delta \mu \vec{K} \times \vec{B} \right| = \left| \frac{\Delta \mu}{q} \omega \vec{U} \left( \left( \frac{\Gamma \vec{p}}{2} + \vec{B}^2 \right) \frac{1}{v_f} - \vec{B}^2 \right) \right| = \left| \frac{\Delta \mu}{q} \omega h \vec{U} \right|. \quad (98)
\]

This is negligible compared to \( |\vec{U} \times \vec{B}| \sim \vec{U} \vec{B} \) because the ratio of the two terms is \( \sim \Delta \mu \omega / \omega_c \) when \( \omega \ll \omega_c \).

5.3. Transverse Wave Propagating Along the Magnetic Field

We investigate transverse waves propagating through the ideal MHD plasma along the magnetic field lines

\[
\vec{B} \parallel k, \quad \vec{E}, \vec{B}, \vec{U} \parallel k, \quad \eta = 0. \quad (100)
\]

We assume any perturbation \( \vec{A} \) is proportional to \( \exp(i \vec{k} \cdot \vec{r}) \vec{A}(t) \). The linearized equations become

\[
\frac{\hbar}{\omega} \frac{\partial}{\partial t} \vec{U} = \vec{J} \times \vec{B}, \quad (101)
\]

\[
\frac{\mu \hbar}{q} \frac{\partial}{\partial t} \vec{J} = (q \vec{U} - \Delta \mu \vec{J}) \times \vec{B} + q \vec{E}, \quad (102)
\]

\[
i k \cdot \vec{E} = 0, \quad (103)
\]

\[
i k \cdot \vec{B} = 0, \quad (104)
\]

\[
\frac{\partial}{\partial t} \vec{B} = -i k \times \vec{E}, \quad (105)
\]

\[
\vec{J} + \frac{\partial}{\partial t} \vec{E} = i k \times \vec{B}. \quad (106)
\]

From these linearized equations, we have

\[
\frac{\mu \hbar}{q} \frac{\partial^2}{\partial t^2} \vec{J} = -\frac{q}{\hbar} (\vec{\square} \vec{J}) \vec{B}^2 - \Delta \mu \frac{\partial}{\partial t} (\vec{\square} \vec{J}) \times \vec{B} + \frac{q}{\hbar} \frac{\partial^2}{\partial t^2} \vec{J}, \quad (107)
\]

where \( \vec{\square} = \partial_r \partial^r \). When we set

\[
\frac{\partial \vec{J}}{\partial t} = i \omega \vec{J} + \Omega \times \vec{J}, \quad (108)
\]

we obtain the following dispersion relations:

\[
\frac{\hat{q} \vec{B}^2}{h} k^2 = \hat{q} \omega^2 + \frac{\mu \hbar}{q} (\omega^2 k^2 - 4 \omega^2 \Omega^2) - \Delta \mu (k^2 - 2 \omega^2) \Omega, \quad (109)
\]

\[
2 \Omega \left[ \frac{\mu \hbar}{q} (k^2 - \omega^2) + \hat{q} + \frac{\hat{q} \vec{B}^2}{h} \right] = \Delta \mu (k^2 - 2 \Omega^2) \vec{B}, \quad (110)
\]

where \( k^2 = k^2 - \omega^2 - \Omega^2 \) and \( \omega = \omega^2 + \Omega^2 \).

Let us consider the case of pair plasma \( (\mu = 1/4, \Delta \mu = 0) \). In this case, we have \( \Omega = 0, \omega^2 = \omega^2, k^2 = k^2 - \omega^2 \), and the dispersion relation becomes

\[
\left[ \frac{\hat{B}^2}{h} + \left( \frac{\omega}{\omega_p} \right)^2 \right] (k^2 - \omega^2) = \omega^2. \quad (111)
\]

When \( \omega_p^2 \gg \omega^2 \), we have

\[
\omega^2 = \frac{u_A^2 k^2}{1 + u_A^2}, \quad (112)
\]

where \( u_A = \sqrt{B^2 / h} \) is the Alfvén 4-velocity. This shows the dispersion relation of the relativistic Alfvén wave.

For the electron–ion plasma case \( (\mu \approx m_e / m \ll 1, \Delta \mu \approx 1) \), we arrange the dispersion Equations (109) and (110) as

\[
\frac{u_A^2 k^2}{1 + u_A^2} = \omega^2 - (k^2 - 2 \omega^2) \frac{\omega_c}{\omega_p^2} \Omega, \quad (113)
\]
When \( \omega \) Alfvén wave in the electron–ion plasma case

Equations (116) and (118) yield the dispersion relation of the becomes nearly vacuum, and we assume

When \( \Omega \lesssim k, \omega \ll \omega_{\text{eq}} \), then \( \Omega \parallel \ll k \) and \( k^2 = \omega^2 \), thus we have

Substituting this equation into Equation (113), we obtain

When \( \omega_{\text{eq}} \ll \omega_{\text{p}}, \omega \ll \omega_{\text{p}}, \) we finally have

Equations (116) and (118) yield the dispersion relation of the Alfvén wave in the electron–ion plasma case

This implicitly contains two ratios:

The differences between the standard RMHD equations and generalized RMHD equations of electron–ion plasmas and pair plasmas are only following five points. To find out these points, we should just consider the terms with \( \mu \) and \( \Delta \mu \) in Equations (46)–(48).

1. The second term on the left-hand side of equation of motion (Equation (47)): this term is due to the momentum of current. In the pair plasma case, it is significant, while it is negligible in an electron–ion plasma.
2. The time-derivative term of the Ohm’s law (Equation (48)): this is due to the inertia effect of the current. We can ignore it only when the electric resistivity is zero in an electron–ion plasma. When the resistivity is finite, we cannot ignore it because of causality.
3. The second term with respect to \( J^\mu \) on the right-hand side of the Ohm’s law (Equation (48)): this is the Hall term. In the pair plasma case, it disappears.
4. The term with respect to pressure gradient on the right-hand side of the Ohm’s law (Equation (48)): this is the term of the thermal electromotive force. In a pair plasma, it influences through \( \Delta \rho \), while in an electron–ion plasma through \( p_{\parallel} \).
5. The term proportional to \( \Theta \) in the Ohm’s law (Equation (48)): this is due to the difference of the thermalized energy exchange between the electron–positron pair and the electron–ion pair during friction.

The item (3) was indicated by many authors (e.g., Blackman & Field 1993; Meier 2004) and the other items are explicitly described in this paper for the first time. As shown above, it is clear that most of the differences come from the Ohm’s law (Equation (48)). The exception is the effect of current moment in the equation of motion shown in the distinct property (1). The influence of the distinct property (1) is estimated by comparison between \( \sqrt{\mu J^\mu/(ne)} \) and \( U^\mu \), which are on the order of the square root of two terms on the left-hand side of Equation (47). This implicitly contains two ratios:

To evaluate the latter ratio, we introduce the transformation between the two frames \( t, r \) and \( t', r' \) whose relative velocity is \( v_0 \), like the Galilean transformation,

For any 3-vector \( A \), the vector length \( |A'| \) is equal to or smaller than the length of the vector transformed by the Lorentz transformation, \( |A'| \). This is because

The finite \( \Omega_1 \) shows the Faraday rotation of the electromagnetic wave along the magnetic field line in the electron–ion plasma. In the pair plasma case (\( \Delta \mu = 0 \)), no rotation of polarization is found.
where the vector with the subscript \(\perp\) (perpendicular) component to \(\mathbf{v}_0\) and \(\gamma_0 = (1 - v_0^2)^{-1/2}\). When we take the velocity of the center of mass of the two fluid locally as \(\mathbf{v}_0\), that is, \(\mathbf{v}_0 = \mathbf{U}\), we have

\[
\Delta \mathbf{U}^0 \equiv \mathbf{U}_+^0 - \mathbf{U}_-^0 = \frac{1}{\gamma_f} (\gamma_f \mathbf{U}_+ - \gamma_f \mathbf{U}_-) = \frac{\gamma_f}{\gamma_f} (\mathbf{V}_+ - \mathbf{V}_-),
\]

and

\[
|\Delta \mathbf{U}^0| \leq |\mathbf{U}_+ - \mathbf{U}_-|.
\]

We can write

\[
\mathbf{U}_\pm = \frac{1}{\rho_f} (\gamma_f \rho_f \mathbf{U} \pm \gamma_f \rho \Delta \mathbf{U}^0),
\]

(130)

\[
K = \frac{\rho_f}{\rho} \mathbf{V} + \frac{n_s n_m}{n^2} \Delta \mathbf{U}^0.
\]

(131)

In general, since \(\Delta \mu \leq 1\), the condition of the neglect of the current momentum is

\[
\rho_c \ll \frac{n_e}{\sqrt{\mu}}, |\mathbf{U}_+ - \mathbf{U}_-| \ll |\mathbf{U}| \sqrt{\mu}.
\]

(132)

This is because

\[
\left| \frac{\mathbf{J}}{n\mathbf{e}} \right| = |\mathbf{K}| = \frac{\rho_f}{\rho} |\mathbf{V} + \frac{n_s n_m}{n^2} \Delta \mathbf{U}^0| \ll \frac{\rho_f}{n_e} |\mathbf{V}| + \frac{n_s n_m}{n^2} |\Delta \mathbf{U}^0| < \frac{\rho_c}{n_e} |\mathbf{U}| + |\mathbf{U}_+ - \mathbf{U}_-| \ll |\mathbf{U}|.
\]

The influence of the unique nature (3) is negligible in the pair plasma case (\(\Delta \mu = 0\)), the Hall effect disappears without any restriction.

The distinct property (2), the time-derivative term of the Ohm’s law, is evaluated by the ratio

\[
\frac{\mu}{c^2 (n\mathbf{e})^2} U J : U^0 F_{\perp} \sim \frac{\mu}{c^2 (n\mathbf{e})^2} \frac{h}{J : B}.
\]

(134)

The ratio should be evaluated in a more realistic case because it depends on the characteristic timescale \(\tau\). An example on black hole magnetospheres of AGNs is discussed in the following section. With respect to the unique property (4) of the thermal electromotive force, the influence is estimated by the ratio

\[
\frac{\Delta \mu \rho - \Delta \rho}{2 n e l} : U^0 F_{\perp},
\]

(135)

where \(l\) is the characteristic scale length. As for the unique nature (5), the term is negligible in the electron–ion plasma case in a short timescale, because of the poor energy exchange between the electron fluid and ion fluid. However, we cannot ignore it in the pair plasma case. In this paper, we do not discuss the details of the last two distinct properties.

7. VALIDITY OF GENERALIZED AND STANDARD RMHD EQUATIONS

In this section, we show the applicability of the generalized RMHD equations and evaluate the significance of the pair and electron–ion plasma distinct properties in a global, astrophysical situation. The standard RMHD equations are applicable only when the conditions of the generalized RMHD are satisfied and the unique properties of pair and electron–ion plasmas can be ignored (see Section 6).

7.1. Breakdown Condition of Generalized RMHD Equations

We consider the breakdown condition of the generalized RMHD Equations (46)–(48) with the Maxwell Equations (3) and (4) on the basis of the two-fluid approximation and the average procedure (Equations (5)–(8)). To derive the generalized RMHD equations from the one-fluid equations, we used the assumptions \(\mathbf{M} = U_\nu \mathbf{U} = -1\) and \(\Delta \mathbf{h} / \mathbf{h} \ll \mathbf{h}\). It is noted that the one-fluid Equations (21)–(23) and the two-fluid Equations (1)–(2) are consistent within the averaging procedure (Equations (5)–(8)). The condition \(\mathbf{M} = -1\) is necessary so that \(U_\nu \mathbf{U}\) has meaning of the 4-velocity, and that means the averaged Lorentz factor of the fluid of positively charged particles and the other fluid of negatively charged particles is nonrelativistic in the center-of-mass frame because \(\mathbf{M} = U_\nu \mathbf{U} = U_\nu \mathbf{U} = U_\nu \mathbf{U}^0 = -(\mathbf{J}_\parallel)^2 = -1\). Here, the prime indicates a variable observed by the local center-of-mass frame of the two fluids. This condition is consistent with the assumption of the current density independency of the resistivity (see Appendix A). When the relative velocity of the two fluids is relativistic, the frictional force of the two fluids is not proportional to the relative velocity, and then the resistivity depends on the current density. This condition restricts the net charge density and current density as follows. Using Equations (7) and (8), we get

\[
U_\nu \mathbf{U} = -1 - \frac{m_s m_m}{\rho^2} \left[ (n_s - n_m)^2 + \frac{1}{e^2} J_{\parallel} J_{\perp} \right],
\]

(136)

by some algebraic calculations. When \(\mathbf{M} = U_\nu \mathbf{U} \approx 1\), we find

\[
\frac{m_s m_m}{\rho^2} \left[ (n_s - n_m)^2 + \frac{1}{e^2} J_{\parallel} J_{\perp} \right] \ll 1.
\]

Using this condition and \(J_{\parallel} J_{\perp} = -\rho_c^2 + |J|^2\), we find

\[
(n_s - n_m)^2 + \frac{1}{e^2} |J|^2 \ll \frac{\rho_c^2}{e^2} + \frac{\rho^2}{m_s m_m},
\]

then we have

\[
|J|^2 \ll \rho_c^2 + \frac{(e n)^2}{\mu}.
\]

(137)

This condition shows the current must be much smaller than en\(\sqrt{\mu}\) or \(\rho_c\).

For an electron–ion plasma, if \(\gamma_s \ll m_s / m_m\) and \(n_s \approx n_m\), we have the condition \(\gamma' \approx 1\). This is because

\[
\gamma' \approx \gamma_s' + \frac{m_m}{m_s} \gamma_m' \approx 1,
\]

since

\[
|U'| = \frac{m_m}{m_s} n_s U'_m \leq \frac{m_m}{m_s} \gamma_s' \approx 1.
\]

For a pair plasma, we need \(\gamma_s' \approx 1\) to keep the condition of \(\gamma' \approx 1\) because \(\gamma_s' \approx \gamma'_s\) and \(\gamma' \approx (\gamma'_s + \gamma'_p) / 2 \approx \gamma'_p\) if \(n_s \approx n_m\).
With respect to the condition $\Delta h^i \ll h^i$, in the case of the nonrelativistic pressure limit $p_\pm \ll \rho_\pm$, we find

$$2 \mu m \left( \frac{1}{n_+} - \frac{1}{n_-} \right) \ll \frac{m_+}{n_+} + \frac{m_-}{n_-}.$$  

This condition requires $n_+ \approx n_-$. It is noted that this does not mean the charge neutrality because net charge density is given by $\rho_e = e(n_+ y_+ - n_- y_-) \neq e(n_+ - n_-)$. We call the condition $n_+ \approx n_-$ the “proper charge neutrality.” In the relativistic pressure case, we find this condition yields

$$\frac{2}{m} \left( h^1_m - \frac{h^1_m}{n_+^2} \right) \ll \frac{h^1_m}{n_+^2} + \frac{h^1_m}{n_-^2},$$

and then using specific enthalpy (Equation (65)), which is approximated by Equation (66), we have

$$\frac{1}{n^1_m} \left( \frac{p_\pm}{\rho_\pm} \right) \approx \frac{1}{n^1_m} \left( \frac{p_\pm}{\rho_\pm} \right). \quad (138)$$

This condition is satisfied when $n_+ \approx n_-$ and $p_\pm/\rho_\pm \approx p_\pm/\rho_\pm$ except for a special case.

In the case of an electron–ion plasma with proper charge neutrality, the condition $p_\pm/\rho_\pm \approx p_\pm/\rho_\pm$ is implausible when the ion temperature $T_i$ is not much larger than the electron temperature $T_e$. Because it requires $T_i = p_\pm/\rho_\pm \approx (n_+/n_-) T_e$ and $n_+ \approx n_-$. Then, the application condition of the generalized RMHD equations is $p_\pm \ll \rho_\pm$. In the case of a pair plasma with proper charge neutrality, the application condition is $p_\pm/\rho_\pm \approx p_\pm/\rho_\pm$. In general (both cases of the electron–ion and pair plasmas), the generalized RMHD equations are applicable to the plasmas with the proper charge neutrality $n_+ \approx n_-$, the nonrelativistic thermal energy $p_\pm \ll \rho_\pm$, and the nonrelativistic (nonsuperrelativistic for electron–ion plasmas) relative velocity of the two fluids $U_\perp \ll (m_+/m_-) \Delta U_\perp$ if $m_+ \gg m_-$. The term $\rho_\perp$ is the plasma around the black hole has zero resistivity, which is a typical assumption for jet formation models (see Figure 1; Blandford & Payne 1982; Shibata & Uchida 1986; Kudoh et al. 1998; Koide et al. 1998, 1999, 2000). The disk twists the magnetic field lines rapidly near the black hole to increase the magnetic pressure. The magnetic pressure (and tension) blows off the plasma near the disk. The outflow of the plasma is collimated by the magnetic tension to form a jet. Here we make a rough estimation of the distinct properties of the electron–ion and pair plasmas through the process around the black hole. For simplicity, we ignore the general and special relativistic effects such as the frame-dragging effect of the rotating black hole and the inertia effect of the magnetic field. The former effects are discussed in Section 8. In this section, we use the MKSA units.

As shown in Figure 1, in such a situation, current is induced along the rotation axis of the disk, and the return current is formed. The generalized RMHD equations suggest that the current carries the momentum to influence the jet formation. We evaluate the influence as follows. We assume that the velocity of the disk rotation is the Keplerian velocity $v_K = \sqrt{GM_{BH}/r}$, where $G$ is the gravitational constant, $M_{BH}$ is the mass of the black hole, and $r$ is the distance from the center of the black hole to the jet-forming region. The azimuthal component of the twisted magnetic field $B_\phi$ is estimated by

$$\frac{B_\phi^2}{2\mu_0 \rho} = \frac{1}{2} \chi r^2,$$

where $\rho$ is the mass density of the coronal plasma around the black hole. On the other hand, the Ampere’s law yields $\mu_0 J \sim B_\phi/r$. As shown in Section 6 (Equation (128)), we evaluate the influence of current momentum and the Hall effect

![Figure 1. Schematic picture of a relativistic jet-forming region in a black hole magnitosphere of an AGN. The disk is thought to be mainly composed of an electron–ion plasma, while the corona near the black hole and the core of the jet would be mainly of pair plasma.](image-url)
by the ratio \( |\mu|/(ne)/|U| \) and \( \mu_\rho/(ne\gamma) \). The former ratio is estimated as

\[
\frac{1}{\sqrt{ne}} \mu \sim \frac{c}{\sqrt{ne} \gamma^2} = \frac{c_\mu}{\tau_{\rho}}
\]

In the pair plasma case, the Hall effect disappears. In the electron–ion plasma case, it is negligible when

\[
\frac{c}{\tau_{\rho}} \Delta \mu \approx \frac{c}{\tau_{\rho}} \ll 1.
\]  
(139)

The critical particle number density, with which the current momentum or the Hall effect is significant, \( n_{\text{crit}} \), near the black hole is estimated by

\[
\frac{c}{r_S} \frac{\varepsilon}{n_{\text{crit}} e^2} \max(\mu, \Delta \mu) \sim 1,
\]

and using \( \max(\mu, \Delta \mu) \leq 1 \), we have

\[
n_{\text{crit}} \sim \frac{c^2 \varepsilon}{r_S^2} \frac{\varepsilon}{e^2} (\max(\mu, \Delta \mu))^2 \leq \frac{c^6}{(2GM_{\odot})^2} \frac{\varepsilon}{e^2} = 3.2 \times 10^{6} \left(\frac{M_{\odot}}{M_{\odot}}\right)^2 \text{m}^{-3}.
\]

Here, we set \( r = r_S \) where \( r_S = 2GM_{\odot}/c^2 \) is the Schwarzschild radius. In the case of an AGN, we calculate the critical number density as \( n_{\text{crit}} \approx 3 \times 10^{-2} \text{ m}^{-3} \) when \( M_{\odot} \approx 10^8 M_{\odot} \). This critical number density \( n_{\text{crit}} \) is much less than the actual value of the plasma around the black hole and this evaluation shows insignificance of the current momentum and Hall effect.

We evaluate the inertia effect of the current in the Ohm’s law (Equation (48)). As shown in Section 6 (Equation (134)), the influence is estimated by the ratio

\[
\frac{\mu}{c} \frac{m n^2 J}{c t' n_e B} \sim \frac{\mu}{\tau_\rho r_\rho} \sim \left(\frac{c}{r_\rho}\right)^2 \frac{v_K}{c},
\]

where \( \tau \) is the characteristic timescale of the jet formation \( \tau \sim r/v_K \). This shows that the current momentum and the Hall effect are negligible, we can neglect the current inertia effect because of Equation (139) and the condition \( v_K < c \). As a result, the distinct properties of the pair and electron–ion plasmas in the jet-forming region near the black hole are not significant globally in the AGN case. Inversely, the condition that the current momentum is significant is

\[
r \lesssim \frac{c}{\omega_p} \mu.
\]

The Hall effect is not negligible when

\[
r \lesssim \frac{c}{\omega_p} \Delta \mu
\]

in the electron–ion plasma case, while it disappears in the pair plasma case. The inertia of the current is significant in the case that

\[
\tau \sim \frac{1}{\omega_p} \left(\frac{c \omega_p^{-1}}{r}\right).
\]

Here, we neglect the thermal electromotive force and the thermalized energy exchange between the two fluids. These conditions can be satisfied only in the small-scale phenomena. The global plasmas of black hole magnetospheres may be influenced by these distinct effects through the magnetic reconnection around the black holes (Koide & Arai 2008). This is discussed in Section 9 briefly.

### 8. Generalized GRMHD Equations for Electron–Ion/Pair Plasmas Around Black Holes

Around a footpoint of a jet near a black hole shown in Figure 1, the general relativistic effect becomes significant because of strong gravity of the black hole. In this section, we discuss the general relativistic effects of the plasma dynamics near black holes, while in the previous sections we investigated the RMHD equations in the special relativistic framework. To treat the plasmas near black holes appropriately, we have to use the general relativistic equations. We get the equations by transforming the generalized RMHD Equations (46)–(48) with the Maxwell Equations (3) and (4) to the covariant form of the general relativity

\[
\nabla_v (\rho U^\nu) = 0,
\]

\[
\nabla_v \left[ h \left( \frac{U^\nu U^\mu + \mu}{q^2} J^\mu J^\nu \right) \right] = -\nabla^\nu p + J^\nu F^\mu_\nu, \quad (141)
\]

\[
\frac{1}{q} \nabla_v \left[ \frac{\mu h}{q} \left( U^\mu J^\nu + J^\mu U^\nu - \frac{2\mu \Delta \mu h}{q^2} J^\mu J^\nu \right) \right] = \frac{1}{2q} \nabla^\nu (\Delta \mu p - \Delta p) + \left( U^\nu - \frac{\Delta \mu}{q} J^\nu \right) F^\mu_\nu
\]

\[
- \eta [J^\mu + Q(1 + \Theta)U^\mu], \quad (142)
\]

\[
\nabla_v F^\mu_\nu = 0, \quad (143)
\]

\[
\nabla_v F^\nu_\mu = J^\nu, \quad (144)
\]

where \( \nabla_v \) is the covariant derivative (Misner et al. 1970; Weinberg 1972). Using the equation derived by the Maxwell equations

\[
(\nabla_v F_{\mu\nu}) F^{\nu\sigma} = \frac{1}{4} g_{\mu\nu} \nabla^\nu (F^{\kappa\lambda} F_{\kappa\lambda}),
\]

and Equation (144), the equation of motion (Equation (141)) becomes

\[
\nabla_v T^{\mu\nu} = 0, \quad (145)
\]

where

\[
T^{\mu\nu} = p g^{\mu\nu} + h \left( U^\mu U^\nu + \frac{\mu}{q^2} J^\mu J^\nu \right) + F^{\mu\sigma} F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} (F^{\kappa\lambda} F_{\kappa\lambda}).
\]

This equation corresponds to the equation of motion in the standard GRMHD, for example, Equation (A2) in Appendix A of Koide et al. (2006). The newly additional term is only that of the current momentum density \( \mu J^\mu J^\nu/q^2 \). The Ohm’s law becomes

\[
\nabla_v \left[ \frac{\mu h}{q^2} (U^\mu J^\nu + J^\mu U^\nu - \frac{2\mu \Delta \mu h}{q^2} J^\mu J^\nu) \right] = \frac{1}{2q} \nabla^\nu (\Delta \mu p - \Delta p) + \left( U^\nu - \frac{\Delta \mu}{q} J^\nu \right) F^\mu_\nu
\]

\[
- \eta [J^\mu + Q(1 + \Theta)U^\mu], \quad (146)
\]

when we ignore the nonuniformity and time variation of \( q \).

As shown in Section 3, \( H = (\eta q)^2/\mu h \) should be smaller than several (almost unity) because of causality. Equation (146)
with \( H \ll 1 \) is called the equation of the “generalized general-relativistic Ohm’s law” and Equations (140), (141), (146), (143), and (144) with \( H \ll 1 \) are called the “generalized GRMHD equations.” Equation (146) is drastically different from the ideal MHD condition \( U^\mu F^\nu_{\mu} = 0 \), which is used customarily in present GRMHD simulations. To show the plasma dynamics governed by the generalized GRMHD equations, we discuss them using the 3+1 formalism shown by Koide et al. (2006; see Appendix A of the paper; definition of variables therein). The variable observed by the “zero angular momentum observer (ZAMO)” frame is denoted by a hat. The energy density observed by the ZAMO frame is

\[
\epsilon + \gamma p = \hat{T}^{00} = h \gamma y^2 + \frac{\mu}{q^2} \rho e^2 - p + \frac{\hat{B}_i^2}{2} + \frac{\hat{E}_i^2}{2},
\]

where \( \epsilon + \gamma p \) is the total energy density. This shows “electric charge” has the rest mass energy, which is disregarded in the standard GRMHD. The three components of the momentum density,

\[
\hat{J}_i = \hat{T}^{i0} = h \left( \gamma \hat{U}_i + \frac{\mu}{q^2} \rho \hat{E}_i \right) + (\hat{E} \times \hat{B})_i,
\]

show that the current has the momentum. The stress tensor,

\[
\hat{T}_{ij} = p \delta_{ij} + \left( \hat{U}_i \hat{U}_j + \frac{\mu}{q^2} \hat{J}_i \hat{J}_j \right) + \left( \frac{\hat{B}_i^2}{2} + \frac{\hat{E}_i^2}{2} \right) \delta_{ij} - \hat{B}_i \hat{B}_j - \hat{E}_i \hat{E}_j,
\]

indicates that the current also carries the momentum. Summarizing the above results, we find that the difference of equation of motion between the generalized GRMHD and standard GRMHD equations is only with respect to the inertia and momentum of the charge and current, which are found in the special relativistic MHD equations.

We next examine the Ohm’s law. Defining

\[
T^{\mu \nu}_{\text{ohm}} = \frac{\mu h}{q^2} (U^\mu J^\nu + J^\mu U^\nu) - \frac{2\mu \Delta \rho h}{q^3} J^\mu J^\nu,
\]

we have another formalism of the general relativistic generalized Ohm’s law

\[
\nabla_{\nu} T^{\mu \nu}_{\text{ohm}} = \frac{1}{2q} \nabla^\mu (\Delta \rho p - \Delta p) + \left( U^\mu - \frac{\Delta \rho h}{q} J^\mu \right) F^\mu_\nu - \eta [J^\mu + Q(1 + \Theta) U^\mu]. \tag{147}
\]

The \( \delta \)th component of Equation (147) yields the 3+1 form

\[
\frac{\partial}{\partial t} \left( \frac{\mu h}{q^2} \hat{J}^\delta \right) = -\frac{1}{h_i h_j h_3} \sum_j \partial_j \left[ \frac{\mu h h_j}{h_i} \hat{T}^{i0}_{\text{ohm}} + \mu h \hat{J}^i \right] - \frac{\mu h}{q^2} \frac{1}{h_i} \frac{\partial \alpha}{\partial x^i} \hat{J}^\delta_{\text{ohm}} - \frac{\mu h}{q^2} \sigma_{\delta j} \hat{J}^j + \sum_j \frac{\mu h}{q^2} \alpha \beta j \left( G_{ij} \hat{J}^j - G_{ji} \hat{J}^j \right) + \frac{1}{2q} \frac{1}{h_i} \frac{\partial}{\partial x^i} (\Delta \rho p - \Delta p)
\]

\[
+ \gamma [E^i + (V \times B)_i - \frac{\Delta \rho}{q} (\mu h E^i + (J \times B)_i)] - \eta [J^\delta + Q(1 + \Theta) U^\delta], \tag{148}
\]

where \( h_i^2, h_j^2, h_3^2 \) are spacial diagonal elements of the metrics, \( \beta^1, \beta^2, \beta^3 \) are the shift vectors, \( \alpha \) is the lapse function, \( \sigma_{ij} \) is the shear of the frame dragging \( \sigma_{ij} = \left( 1/h_i h_j \right) \left[ \delta^a b_{ij} \partial_\alpha (a b^\mu \partial^\alpha) \right] \), \( f_{\text{ohm}} = \sum_j (G_{ij} \hat{J}^j_{\text{ohm}} - G_{ji} \hat{J}_{\text{ohm}}) \), and \( G_{ij} = -1/(h_i h_j) (\partial h_i / \partial x^j) \).

Here, \( \rho \equiv (q^2/\mu h) \hat{T}_{00} \) or \( \rho \equiv (q^2/\mu h) \hat{T}_{00} \) are on the order of the charge density and the current density, respectively, and \( \alpha \) is regarded as the gravitational potential. With respect to the term \( -(\mu h/q^2)/(1/h_i h_j) \partial h_i / \partial x^j \), it clearly shows the gravity influences the Ohm’s law and behaves like electromotive force as Khanna (1998) pointed out. Near the black hole where the gravitational acceleration is very strong, such “gravitational electromotive force” becomes significant when the charge separation of the plasma occurs. This effect is also found in the Newtonian two-fluid model when we consider the gravitation term in the equation of motion of the two fluids. The fourth and fifth terms of the right-hand side of Equation (148)

\[
- \sum_j \frac{\mu h}{q^2} \sigma_{ji} \hat{J}^j + \frac{1}{2q} \frac{1}{h_i} \frac{\partial}{\partial x^i} (\Delta \rho p - \Delta p)
\]

are also regarded as the effective electromotive force. In the electromotive force, the shear of the frame dragging around black holes, \( (\mu h/q^2) \sigma_{ji} \) and \( (\mu h/q^2) \alpha \beta j \), corresponds to the electric resistivity. This effective resistivity due to the frame-drudging shear is a pure general relativistic effect, which may cause the dynamo effect and magnetic reconnection near black holes (Khanna & Camenzind 1994, 1996; Koide & Arai 2008).

9. SUMMARY AND DISCUSSION

In this paper, we derived the one-fluid equations of the two-component plasma from the relativistic two-fluid equations. On the basis of the new one-fluid equations and the dispersion relation analysis of the electromagnetic wave, we proposed a set of generalized RMHD Equations (46)–(48) with Maxwell Equations (3) and (4) which are applicable to both pair and electron–ion plasmas. The generalized RMHD equations yielded the RMHD equations for the electron–ion plasma (Equations (49)–(51)) and the pair plasma (Equations (55)–(57)). The comparison between these sets of equations and the standard Equations (52)–(54) clearly show the distinct properties of the electron–ion and pair plasmas. We investigated the linear modes propagating in the pair and electron–ion plasmas using the generalized RMHD equations. In the analysis, we found the effect of polarity rotation of Alfven wave in the electron–ion plasma, while no rotation in the pair plasma. This effect is similar to the Faraday rotation of the electromagnetic wave propagating along magnetic field lines in a plasma, but not identical. We also found other plausible unique properties of the electron–ion and pair plasmas, such as the plasma oscillation, fast wave, and electromagnetic wave, which also show the validity of the generalized RMHD equations. With respect to the nonlinear effects, we evaluated the distinct properties of the pair and electron–ion plasmas with the significance of each distinct term of the generalized RMHD equations and with the special case of black hole magnetospheres where jets are formed. It confirmed the validity of the standard RMHD Equations (52)–(54), (3), and (4) on the global phenomena in black hole magnetospheres of AGNs. We also discussed the generalized GRMHD Equations (140)–(144) and revealed the unique effects of the plasmas around black holes. As indicated by Khanna (1998), we confirmed the appearance of the “gravitational electromotive force” due to the
strong gravity of a black hole when the plasma has charge separation. We newly found the effective resistivity due to the shear of frame dragging around a rotating black hole. To investigate the nonlinear phenomena qualitatively, numerical method is required and should be developed in the near future.

According to the results in Section 7.1, if the relative velocity of the two fluids and the thermal energy of the fluids are nonrelativistic and their proper particle number densities are approximately equal, \( y' \approx 1, \rho_+ \ll \rho_\pm, n_+ \approx n_\pm \), the generalized RMHD equations are applicable. Otherwise, the premise condition (A5) of the generalized RMHD equations require \( \rho_+ / \rho_\pm \approx p_+ / p_\pm \), that is, \( T_+ = (m_+ / m_\pm)T_\pm \). This condition is implausible for the electron–ion plasma when the ion temperature is not much larger than the electron temperature. In the pair plasma case, it yields \( p_+ / n_+ = T_+ \approx T_\pm = p_\pm / n_\pm \). The special condition of the electron–ion plasma, \( T_\pm = (m_\pm / m_\mp)T_\mp \) may be possible for optically thin advection-dominated accretion flows (ADAFs) in the electron–ion plasma of the accretion disk around black holes (e.g., Kato et al. 1998, see Table 10.1). If these premise conditions are not satisfied, the generalized RMHD equations, which are reducible to the standard RMHD equations, are not applicable to the plasma. That is, a relativistically hot plasma \( (\rho_\pm \gtrsim \rho_\pm) \) cannot be treated by RMHD if \( p_\pm \approx p_\mp (m_\pm / m_\pm) \) and \( n_\pm \approx n_\mp \) are not satisfied. In such a case, the generalized RMHD approximation breaks down. Then, the generalized RMHD equations should be replaced by more appropriate equations for the plasma with relativistic internal energy (relativistic high temperature, relativistically fast relative velocity of the two fluids). The standard RMHD equations, which have been utilized indiscriminately in astrophysics, are applicable to the astrophysical objects when the premise of the generalized RMHD equations is satisfied and the unique properties of the electron–ion and pair plasma are negligible. These conditions indicated in this paper depend on the averaging procedure of the variables (Equation (10)). The averaging of the variables (10) enables us to derive the one-fluid Equations (21)–(23) from the two-fluid Equations (1) and (2) relatively easily. However, we have to check the validity of the averaging method. To confirm it, the Vlasov–Boltzmann equation of the plasma should be utilized. This task is beyond the scope of this paper, while it is interesting and important. It should be investigated in the near future.

As shown in Section 7.2, distinct properties of the electron–ion and pair plasmas do not influence the global dynamics of the plasmas in black hole magnetospheres of AGNs because they are large-scale RMHD phenomena. On the other hand, local phenomena such as magnetic reconnection may change the influence to the global dynamics drastically, because they are possible to change the magnetic configuration globally, although the reconnection region is much smaller than the global scale of magnetospheres. In such a small region, the distinct properties of the electron–ion and pair plasmas may become significant. For example, the inertia effect of the current plays a role to cause the magnetic diffusion instead of the electric resistivity. This distinct magnetic reconnection may be important in the pair plasma, which is thought to locate at the coronal regions near black holes of AGNs and GRBs.

With respect to the corona between a black hole and its disk, in generally speaking, the plasmas would be composed of electrons, ions, and positrons, while the amount of positrons in the plasmas around black holes has not been confirmed observationally yet. This is regarded as a mixture of an electron–ion plasma and pair plasma. The simplest treatment of the mixed plasma is given by \( m_- = m_e \) and \( m_+ = m_e + (m_i - m_e) \xi \), where \( m_i \) is the mass of the ion and \( \xi = n_i / n_e \) is the relative proportion of ions \((n_i \text{ and } n_e \text{ are the particle number densities of ions and electrons, respectively})\). Here, we assumed the local (proper) charge neutrality, \( n_e = n_+ + n_\mp \), where \( n_\mp \) is the number density of the positrons. In this treatment, the relative proportion of ions \( \xi \) should be given by other condition. For example, we may be able to trace the plasma element where \( \xi \) is constant. However, when the relative velocity between the positron fluid and the ion fluid is significant, \( \xi \) on a certain plasma element changes. In such a case, the treatment becomes difficult. The advanced formalism of the mixed plasma is a forthcoming and important subject in the relativistic plasma physics. Additionally, other effects such as pair creation/annihilation, radiation, atomic processes, and more general EoS should be considered. Furthermore, as we mentioned above, for the relativistically hot plasma and relativistically strong current plasma, the premise of the generalized RMHD equations breaks down. In plasmas at black hole magnetospheres, it is not surprising that the temperature and current density are relativistically high and strong. To deal with such relativistic plasmas, we have to develop a new scheme beyond the present generalized RMHD equations.

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APPENDIX A

FRictional 4-force Density of Two Fluids

We assume that a relative velocity of two fluids is nonrelativistic so that \( M = U_r U'_r = -1 \). Then, when we observe the fluids from the local center-of-mass frame \( S' \), the fluids can be treated as nonrelativistic fluids. The 4-force density of the friction from the negatively charged fluid to the positively charged fluids is

\[
f'_+ = -m_+ \sigma_{-+} v_{n_+ n_+} (V^{0'}_+ - V^{0'}_+). \tag{A1}
\]

where \( \sigma_{-+} \) is the cross-section of the collision between the positive particle and negative particle, \( v_i \) is the average relative velocity between the two fluids including the thermal velocity. Similarly, the frictional 4-force density of the positively charged fluid to the negatively charged fluid is

\[
f'_- = -m_- \sigma_{+-} v_{n_- n_-} (V^{0'}_- - V^{0'}_-). \tag{A2}
\]

According to the action–reaction principle, we have

\[
m_+ \sigma_{-+} = m_- \sigma_{+-}. \tag{A3}
\]

Then we define the following variables:

\[
A = \eta e^2 = m_+ \sigma_{-+} v_1 = m_- \sigma_{+-} v_2. \tag{A4}
\]

In the center-of-mass frame \( S' \), we have

\[
m_+ n_+ U^{i'}_+ + m_- n_- U^{i'}_- = 0. \tag{A5}
\]

When we consider the Lorentz transformation \( x^\mu' = b^\nu x^{\nu'} \), we find

\[
b^{\alpha 0} = \frac{U^\mu}{\gamma'.} \tag{A6}
\]
We calculate the frictional 4-force density by
\[ f_+^\mu = b_+^\mu v_{+}^\nu = \frac{U^\mu}{\gamma'} f_+^\nu + b_+^\mu f_+^\nu, \]
\[ = \frac{U^\mu}{\gamma'} f_+^\nu - \frac{\Delta p}{\rho e} \sqrt{-\Delta} \left[ \frac{J^\mu - \frac{Q}{M} U^\mu}{\Delta M} \right], \]
where \( M = U_v U^v = -\gamma'^2 \) and \( Q = U_v J^v = -\gamma' J^0 \). We also note that
\[ f_+^\nu v_{+i} = -\frac{A\rho m_e}{m_+ n_+ \gamma'_+ \gamma'^2} (Q^2 - MW) < 0, \]
where \( W = J_v J^v \) and
\[ f_+^\nu v_{-i} = -\frac{A\rho m}{m_- n_- \gamma' \gamma'^2} (Q^2 - MW). \]

We consider the thermal energy gain rate of the positive charged fluid,
\[ f_+^\nu = -f_+^\nu v_{+i} \theta_+ - f_+^\nu v_{-i} \theta_- - \frac{n_+ \gamma_+ - f_+^\nu v_{+i} (1 - \theta_+)}{\gamma'_+ n_+ + \gamma'_- n_-} \]
\[ = \frac{1}{\gamma'_+ n_+ + \gamma'_- n_-} \left( \theta_+ n_+ \gamma_+ f_+^\nu v_{+i} - \theta_- n_- \gamma'_- f_+^\nu v_{+i} \right) \]
\[ = \frac{A\rho}{m_e} \frac{\theta_+ m_- \left[ \rho M - \frac{m_+}{e} \right] \left[ \theta_- m_+ \left[ \rho M + \frac{m_-}{e} \right] \right]}{(2\rho M + \frac{m_-}{e} Q) \left( \rho M - \frac{m_+}{e} Q \right) \left( \rho M + \frac{m_-}{e} Q \right)} \times (Q^2 - MW), \]
where \( \theta_+ \), \( \theta_- \) are the redistribution coefficient of the thermalized energy to the positively charged fluid and negative one with the equipartition principle, respectively. When \( \theta_+ = \theta_- = \theta \), we obtain
\[ f_+^\nu = -\frac{A\rho}{m_e} \Theta Q \Theta \leq 0, \]
\[ = \frac{\Delta p}{\rho e} \sqrt{-\Delta} \left( J^\mu - \frac{Q}{M} U^\mu \right). \]

Using \( \Theta \), we finally obtain the frictional 4-force density
\[ f_+^\nu = -\frac{A\rho}{m_e} \sqrt{-\Delta} \left( J^\mu - \frac{Q}{M} (1 + \Theta) U^\mu \right). \]

Because \( R^\mu = f_+^\mu \), we obtain the Ohm’s law.

Here, we recognize \( \eta \) is the resistivity.

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ERRATUM: GENERALIZED RELATIVISTIC MAGNETOHYDRODYNAMIC EQUATIONS FOR PAIR AND ELECTRON–ION PLASMAS (2009, API, 696, 2220)

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As the result of an error at the publisher, Equation (60) was printed without the vector operation between the two vectors of the second term of the right-hand side of the equation. This equation is reprinted below in its correct form:

\[
\frac{\hbar}{q} \frac{\partial}{\partial t} \vec{K} = \frac{1}{2q} \nabla \cdot (\Delta \mu \vec{p} - \Delta \vec{p}) + (\vec{U} - \Delta \mu \vec{K}) \times \vec{B} + \vec{E} - \eta \vec{J}.
\]  

(60)