Almost orthogonality and Hausdorff interval topologies of atomic lattice effect algebras

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Abstract

We prove that the interval topology of an Archimedean atomic lattice effect algebra $E$ is Hausdorff whenever the set of all atoms of $E$ is almost orthogonal. In such a case $E$ is order continuous. If moreover $E$ is complete then order convergence of nets of elements of $E$ is topological and hence it coincides with convergence in the order topology and this topology is compact Hausdorff compatible with a uniformity induced by a separating function family on $E$ corresponding to compact and cocompact elements. For block-finite Archimedean atomic lattice effect algebras the equivalence of almost orthogonality and s-compact generation is shown. As the main application we obtain the state smearing theorem for these effect algebras, as well as the continuity of $\oplus$-operation in the order and interval topologies on them.

Key words: Non-classical logics, D-posets, effect algebras, MV-algebras, interval and order topology, states

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1. Introduction, basic definitions and facts

In the study of effect algebras (or more general, quantum structures) as carriers of states and probability measures, an important tool is the study of
topologies on them. We can say that topology is practically equivalent with the concept of convergence. From the probability point of view the convergence of nets is the main tool in spite of that convergence of filters is easier to handle and preferred in the modern topology. It is because states or probabilities are mappings (functions) defined on elements but not on subsets of quantum structures. Note also, that connections between order convergence of filters and nets are not trivial. For instance, if a filter order converges to some point of a poset then the associated net need not order converge (see e.g. [12]).

On the other hand certain topological properties of studied structures characterize also their certain algebraic properties and conversely. For instance a known fact is that a Boolean algebra \( B \) is atomic iff the interval topology \( \tau_i \) on \( B \) is Hausdorff (see [20, Corollary 3.4]). This is not more valid for lattice effect algebras (even MV-algebras). By Frink’s Theorem the interval topology \( \tau_i \) on \( B \) (more generally on any lattice \( L \)) is compact iff it is a complete lattice [5]. In [14] it was proved that if a lattice effect algebra \( E \) (more generally any basic algebra) is compactly generated then \( E \) is atomic.

We are going to prove that on an Archimedean atomic lattice effect algebra \( E \) the interval topology \( \tau_i \) is Hausdorff and \( E \) is \((o)\)-continuous if and only if \( E \) is almost orthogonal. Moreover, if \( E \) is complete then \( \tau_i \) is compact and coincides with the order topology \( \tau_o \) on \( E \) and this compact topology \( \tau_i = \tau_o \) is compatible with a uniformity on \( E \) induced by a separating function family on \( E \) corresponding to compact and cocompact elements of \( E \).

As the main corollary of that we obtain that every Archimedean atomic block-finite lattice effect algebra \( E \) has Hausdorff interval topology and hence both topologies \( \tau_i \) and \( \tau_o \) are Hausdorff and they coincide. In this case almost orthogonality of \( E \) and \( s \)-compact generation by finite elements of \( E \) are equivalent. As an application the state smearing theorem for these effect algebras is formulated. Moreover, the continuity of \( \oplus \)-operation in \( \tau_i \) and \( \tau_o \) on them is shown.

**Definition 1.1.** A partial algebra \((E; \oplus, 0, 1)\) is called an effect algebra if 0, 1 are two distinct elements and \( \oplus \) is a partially defined binary operation on \( E \) which satisfy the following conditions for any \( a, b, c \in E \): 

- (Ei) \( b \oplus a = a \oplus b \) if \( a \oplus b \) is defined,
- (Eii) \( (a \oplus b) \oplus c = a \oplus (b \oplus c) \) if one side is defined,
- (Eiii) for every \( a \in E \) there exists a unique \( b \in E \) such that \( a \oplus b = 1 \) (we put \( a' = b \)),
- (Eiv) if \( 1 \oplus a \) is defined then \( a = 0 \).

We often denote the effect algebra \( (E; \oplus, 0, 1) \) briefly by \( E \). In every effect algebra \( E \) we can define the partial order \( \leq \) by putting \( a \leq b \) and \( b \oplus a = c \) iff \( a \oplus c \) is defined and \( a \oplus c = b \), we set \( c = b \oplus a \).

If \( E \) with the defined partial order is a lattice (a complete lattice) then \((E; \oplus, 0, 1)\) is called a lattice effect algebra (a complete lattice effect algebra).
Recall that a set $Q \subseteq E$ is called a sub-effect algebra of the effect algebra $E$ if

(i) $1 \in Q$

(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in $Q$, then $a, b, c \in Q$.

If $Q$ is simultaneously a sublattice of $E$ then $Q$ is called a sub-lattice effect algebra of $E$.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-orthogonal if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$) exists in $E$. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus a_k$ exists and $\bigoplus a_k \leq a'_n$. An arbitrary system $G = (a_k)_{k \in H}$ of not necessarily different elements of $E$ is $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_k)_{k \in H}$ the element $\bigoplus G$ exists iff $\cup \{\bigoplus K \mid K \subseteq G, K \text{ is finite} \}$ exists in $E$ and then we put $\bigoplus G = \cup \{\bigoplus K \mid K \subseteq G\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_k)_{k \in H_1}$).

Recall that elements $x$ and $y$ of a lattice effect algebra are called compatible (written $x \leftrightarrow y$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$ [13]. For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements are compatible then $E$ is called an MV-effect algebra. In fact, every MV-effect algebra can be organized into an MV-algebra (see [2]) if we extend the partial $\oplus$ to a total operation by setting $x \oplus y = x \oplus (y \ominus (x \wedge y))$ for all $x, y \in E$ (also conversely, restricting a total $\oplus$ into partial $\oplus$ for only $x, y \in E$ with $x \leq y'$ we obtain a MV-effect algebra).

Moreover, in [23] it was proved that every lattice effect algebra is a set-theoretical union of MV-effect algebras called blocks. Blocks are maximal sub-sets of pairwise compatible elements of $E$, under which every subset of pairwise compatible elements is by Zorn’s Lemma contained in a maximal one. Further, blocks are sub-lattices and sub-effect algebras of $E$ and hence maximal sub-MV-effect algebras of $E$. A lattice effect algebra is called block-finite if it has only finitely many blocks.

Finally note that lattice effect algebras generalize orthomodular lattices [10] (including Boolean algebras) if we assume existence of unsharp elements $x \in E$, meaning that $x \wedge x' \neq 0$. On the other hand the set $S(E) = \{x \in E \mid x \wedge x' = 0\}$ of all sharp elements of a lattice effect algebra $E$ is an orthomodular lattice [8]. In this sense a lattice effect algebra is a “smeared” orthomodular lattice, while an MV-effect algebra is a “smeared” Boolean algebra. An orthomodular lattice $L$ can be organized into a lattice effect algebra by setting $a \oplus b = a \lor b$ for every pair $a, b \in L$ such that $a \leq b^\perp$.

For an element $x$ of an effect algebra $E$ we write $\text{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer $n$ and we write $\text{ord}(x) = n_x$ if $n_x x$ exists in $E$. An effect algebra $E$ is Archimedean if $\text{ord}(x) < \infty$ for all $x \in E$. We can show that every complete effect algebra is Archimedean (see [22]).
An element $a$ of an effect algebra $E$ is an atom if $0 \leq b < a$ implies $b = 0$ and $E$ is called atomic if for every nonzero element $x \in E$ there is an atom $a$ of $E$ with $a \leq x$. If $u \in E$ and either $u = 0$ or $u = p_1 \oplus p_2 \oplus \cdots \oplus p_n$ for some not necessarily different atoms $p_1, p_2, \ldots, p_n \in E$ then $u \in E$ is called finite and $u' \in E$ is called cofinite. If $E$ is a lattice effect algebra then for $x \in E$ and an atom $a$ of $E$ we have $a \leftarrow x$ iff $a \leq x$ or $a \leq x'$. It follows that if $a$ is an atom of a block $M$ of $E$ then $a$ is also an atom of $E$. On the other hand if $E$ is atomic then, in general, every block in $E$ need not be atomic (even for orthomodular lattices [1]).

The following theorem is well known.

**Theorem 1.2.** [25, Theorem 3.3] Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then to every nonzero element $x \in E$ there are mutually distinct atoms $a_\alpha \in E$, $\alpha \in \mathcal{E}$ and positive integers $k_\alpha$ such that

$$x = \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee \{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$$

under which $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \text{ord}(a_\alpha)$ for all $\alpha \in \mathcal{E}$.

**Definition 1.3.** (1) An element $a$ of a lattice $L$ is called compact iff, for any $D \subseteq L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice $L$ is called compactly generated iff every element of $L$ is a join of compact elements.

The notions of cocompact element and compactly generated lattice can be defined dually. Note that compact elements are important in computer science in the semantic approach called domain theory, where they are considered as a kind of primitive elements.

2. Characterizations of interval topologies on bounded lattices

The order convergence of nets (($o$)-convergence), interval topology $\tau_i$ and order-topology $\tau_o$ (($o$)-topology) can be defined on any poset. In our observations we will consider only bounded lattices and we will give a characterization of interval topologies on them.

**Definition 2.1.** Let $L$ be a bounded lattice. Let $\mathcal{H} = \{[a, b] \subseteq L \mid a, b \in L$ with $a \leq b\}$ and let $\mathcal{G} = \{\bigcup_{k=1}^n[a_k, b_k] | [a_k, b_k] \in \mathcal{H}, k = 1, 2, \ldots, n\}$. The interval topology $\tau_i$ of $L$ is the topology of $L$ with $\mathcal{G}$ as a closed basis, hence with $\mathcal{H}$ as a closed subbasis.

From definition of $\tau_i$ we obtain that $U \in \tau_i$ iff for each $x \in U$ there is $F \in \mathcal{G}$ such that $x \in L \setminus F \subseteq U$.

**Definition 2.2.** Let $L$ be a poset.

(i) A net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of $L$ order converges (($o$)-converges, for short) to a point $x \in L$ if there exist nets $(u_\alpha)_{\alpha \in \mathcal{E}}$ and $(v_\alpha)_{\alpha \in \mathcal{E}}$ of elements of $L$ such that

$$x \uparrow u_\alpha \leq x_\alpha \leq v_\alpha \downarrow x, \alpha \in \mathcal{E}$$
where \( x \upharpoonright u_\alpha \) means that \( u_\alpha \leq u_{\alpha_2} \) for every \( \alpha_1 \leq \alpha_2 \) and \( x = \bigvee \{ x_\alpha \mid \alpha \in \mathcal{E} \} \). The meaning of \( v_\alpha \downarrow \) is dual.

We write \( x_\alpha \xrightarrow{(o)} x, \alpha \in \mathcal{E} \) in \( L \).

(ii) A topology \( \tau_0 \) on \( L \) is called the order topology on \( L \) iff

(a) for any net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) of elements of \( L \) and \( x \in L \): \( x_\alpha \xrightarrow{(o)} a \) in \( L \) \( \Rightarrow \)

\( x_\alpha \xrightarrow{\tau} x, \alpha \in \mathcal{E} \), where \( x_\alpha \xrightarrow{\tau} x \) denotes that \( (x_\alpha)_{\alpha \in \mathcal{E}} \) converges to \( x \)

in the topological space \( (L, \tau_0) \),

(b) if \( \tau \) is a topology on \( L \) with property (a) then \( \tau \subseteq \tau_0 \).

Hence \( \tau_0 \) is the strongest (finest, biggest) topology on \( L \) with property (a).

Recall that, for a directed set \( (\mathcal{E}, \leq) \), a subset \( \mathcal{E}' \subseteq \mathcal{E} \) is called cofinal in \( \mathcal{E} \) iff for every \( \alpha \in \mathcal{E} \) there is \( \beta \in \mathcal{E}' \) such that \( \alpha \leq \beta \). A special kind of a subnet of a net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) is net \( (x_\beta)_{\beta \in \mathcal{E}'} \) where \( \mathcal{E}' \) is a cofinal subset of \( \mathcal{E} \). This kind of subnets works in many cases of our considerations.

In what follows we often use the following useful characterization of topological convergence of nets:

**Lemma 2.3.** For a net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) of elements of a topological space \( (X, \tau) \) and \( x \in X \):

\[ x_\alpha \xrightarrow{\tau} x, \alpha \in \mathcal{E} \text{ iff for all } \mathcal{E}' \subseteq \mathcal{E}, \text{ where } \mathcal{E}' \text{ is cofinal in } \mathcal{E} \text{ there exist } \mathcal{E}'' \subseteq \mathcal{E}', \mathcal{E}'' \text{ cofinal in } \mathcal{E}' \text{ such that } x_\gamma \xrightarrow{\tau} x, \gamma \in \mathcal{E}''. \]

**Proof.** \( \Rightarrow \): It is trivial.

\( \Leftarrow \): Let for every \( \mathcal{E}' \subseteq \mathcal{E} \), where \( \mathcal{E}' \) is cofinal in \( \mathcal{E} \) there exist \( \mathcal{E}'' \subseteq \mathcal{E}' \), \( \mathcal{E}'' \) cofinal in \( \mathcal{E}' \) and \( x_\alpha \xrightarrow{\tau} x, \gamma \in \mathcal{E}'' \), and let \( x_\beta \nrightarrow x, \alpha \in \mathcal{E} \). Then there exist \( \mathcal{E}' \subseteq \mathcal{E} \) such that for all \( \alpha \in \mathcal{E} \) there exist \( \beta_\alpha \in \mathcal{E} \) with \( \beta_\alpha \geq \alpha \) and \( x_{\beta_\alpha} \notin U(x) \). Let \( \mathcal{E}' = \{ \beta_\alpha \mid \alpha \in \mathcal{E}, \beta_\alpha \geq \alpha, x_{\beta_\alpha} \notin U(x) \} \) then \( x_{\beta_\alpha} \xrightarrow{\tau} x, \beta_\alpha \in \mathcal{E}' \) and for all cofinal \( \mathcal{E}'' \subseteq \mathcal{E}' \): \( x_\gamma \xrightarrow{\tau} x, \gamma \in \mathcal{E}'' \). Hence there exists \( \mathcal{E}' \subseteq \mathcal{E} \) cofinal in \( \mathcal{E} \) and for all \( \mathcal{E}'' \subseteq \mathcal{E}' \), \( \mathcal{E}'' \) cofinal in \( \mathcal{E}' \): \( x_\gamma \xrightarrow{\tau} x, \gamma \in \mathcal{E}'' \) a contradiction. \( \Box \)

Further, let us recall the following well known facts:

**Lemma 2.4.** Let \( L \) be a bounded lattice. Then

(i) \( F \subseteq L \) is \( \tau_0 \)-closed iff for every net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) of elements of \( L \) and \( x \in L \):

\( x_\alpha \xrightarrow{(o)} x, \alpha \in \mathcal{E} \Rightarrow x \in F \).

(ii) For every \( a, b \in L \) with \( a \leq b \) the interval \([a, b]\) is \( \tau_0 \)-closed.

(iii) \( \tau_i \subseteq \tau_0 \).

(iv) For any net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) of elements of \( L \) and \( x \in L \):

\[ x_\alpha \xrightarrow{(o)} x, \alpha \in \mathcal{E} \Rightarrow x_\alpha \xrightarrow{\tau} x, \alpha \in \mathcal{E}. \]

(v) If \( \tau_i \) is Hausdorff then \( \tau_0 = \tau_i \) (see [3]).
Conversely, since \( F \) for \( c, d \) evidently \( H \cap \tau(a) \): Let \( L \) is a complete sub-lattice of \( L \).

Let Theorem 2.5.

The interval topology of a complete lattice \( L \) is compact iff \( L \) is a complete lattice (see [2]).

Finally, let us note that compact Hausdorff topological space is always normal. Thus separation axiom \( T_2, T_3 \) and \( T_4 \) are trivially equivalent for the interval topology of a complete lattice \( L \).

**Theorem 2.5.** Let \( L \) be a complete lattice with interval topology \( \tau_i \). If \( F \subseteq L \) is a complete sub-lattice of \( L \) then

(a) \( \tau_i^F = \tau_i \cap F \) is the interval topology of \( F \),
(b) for any net \( (x_\alpha)_{\alpha \in \mathcal{E}} \) of elements of \( F \) and \( x \in F \):

\[
x_\alpha \xrightarrow{\tau_i^F} x, \alpha \in \mathcal{E} \iff x_\alpha \xrightarrow{\tau_i} x, \alpha \in \mathcal{E}.
\]

**Proof.** (a): Let \( \mathcal{H} \) and \( \mathcal{H}_F \) be a closed subbasis of \( \tau_i \) and \( \tau_i^F \) respectively. Then evidently \( \mathcal{H} \cap F = \{ [a, b] \cap F | [a, b] \in \mathcal{H} \} \) is a closed subbasis of \( \tau_i \cap F \). Further for \( [c, d]_F \in \mathcal{H}_F \) we have \( [c, d]_F = \{ x \in F | c \leq x \leq d \} = [c, d] \cap F \in \mathcal{H} \cap F \). Conversely, since \( F \) is a complete sub-lattice of \( L \), if \( [a, b] \in \mathcal{H} \) then \( [a, b] \cap F = \{ x \in F | a \leq x \leq b \} \) and either \( [a, b] \cap F = 0 \) or there exist \( c = \land \{ x \in F | a \leq x \leq b \} \) and \( d = \lor \{ x \in F | a \leq x \leq d \} \) and \( [a, b] \cap F = [c, d]_F \in \mathcal{H}_F \). This proves that \( \tau_i^F = \tau_i \cap F \).

(b): This is an easy consequence of (a).

\[\square\]

3. Hausdorff interval topology of almost orthogonal Archimedean atomic lattice effect algebras and their order continuity

The atomicity of Boolean algebra \( B \) is equivalent with Hausdorffness of interval topology on \( B \) (see [11], [29] and [20, Corollary 3.4]). This is not more valid for lattice effect algebras, even also for MV-algebras.

**Example 3.1.** Let \( M = [0, 1] \subseteq \mathbb{R} \) be a standard MV-effect algebra, i.e., we define \( a \oplus b = a + b \) iff \( a + b \leq 1 \), \( a, b \in M \). Then \( M \) is a complete \((o)\)-continuous lattice with \( \tau_i = \tau_o \) being Hausdorff and with \((o)\)-convergence of nets coinciding with \( \tau_o \)-convergence. Nevertheless, \( M \) is not atomic.

We have proved in [16] that a complete lattice effect algebra is atomic and \((o)\)-continuous lattice iff \( E \) is compactly generated. Nevertheless, in such a case, the interval topology on \( E \) need not be Hausdorff.

**Example 3.2.** Let \( E \) be a horizontal sum of infinitely many finite chains \((P_i, \bigoplus, 0_i, 1_i)\) with at least 3 elements, \( i = 1, 2, \ldots, n, \ldots \) (i.e., for \( i = 1, 2, \ldots, n, \ldots \), we identify all 0\(_i\) and all 1\(_i\) as well, \( \bigoplus \), on \( P_i \) are preserved and any \( a \in P_i \setminus \{0_i, 1_i\} \), \( b \in P_j \setminus \{0_j, 1_j\} \) for \( i \neq j \) are noncomparable). Then \( E \) is an atomic complete lattice effect algebra, \( E \) is not block-finite and the interval topology \( \tau_i \) on \( E \) is compact. Nevertheless, \( \tau_i \) is not Hausdorff because e.g., for \( a \in P_i, b \in P_j, i \neq j, a, b \) noncomparable, we have \( [a, 1] \cap [0, b] = 0 \) and there is no finite family \( \mathcal{I} \) of closed intervals in \( E \) separating \( [a, 1], [0, b] \) (i.e., the lattice
Let $E$ be a lattice effect algebra, $x, y \in E$. Then $x \land y = 0$ and $x \leq y'$ iff $kx \land ly = 0$ and $kx \leq (ly)'$, whenever $kx$ and $ly$ exist in $E$.

Proof. Let $x \leq y'$, $x \land y = 0$ and $2y$ exists in $E$. Then $x \oplus y = (x \lor y) \oplus (x \land y) = x \lor y \leq y'$ and hence there exists $x \oplus 2y = (x \lor y) \oplus y = (x \oplus y) \lor 2y = x \lor y \lor 2y = x \lor 2y$, which gives that $x \leq (2y)'$ and $x \land 2y = 0$. By induction, if $ly$ exists then $x \land ly = x \lor ly$ and hence $x \leq (ly)'$ and $x \land ly = 0$.

Now, $x \leq (ly)'$ iff $ly \leq x'$ and because $x \land ly = 0$, we obtain by the same argument as above that $ly \oplus kx = ly \lor kx$, hence $kx \leq (ly)'$ and $ly \lor kx = 0$ whenever $kx$ exists in $E$.

Conversely, $kx \land ly = 0$ implies that $x \land y = 0$ and $kx \leq (ly)'$ implies $x \leq kx \leq (ly)' \leq y'$.

In the next we will use the statement of Lemma 3.3 in the following form: For any $x, y \in E$ with $x \land y = 0$, $x \not\leq y'$ iff $kx \not\leq (ly)'$, whenever $kx$ and $ly$ exist in $E$.

**Definition 3.4.** Let $E$ be an atomic lattice effect algebra. $E$ is said to be almost orthogonal if the set $\{ b \in E \mid b \not\leq a', b \text{ is an atom} \}$ is finite for every atom $a \in E$.

Note that our definition of almost orthogonality coincides with the usual definition for orthomodular lattices (see e.g. [14, 15]).

**Theorem 3.5.** Let $E$ be an Archimedean atomic lattice effect algebra. Then $E$ is almost orthogonal if and only if for any atom $a \in E$ and any integer $l$, $1 \leq l \leq n_a$, there are finitely many atoms $c_1, \ldots, c_m$ and integers $j_1, \ldots, j_m$, $1 \leq j_1 \leq n_{c_1}, \ldots, 1 \leq j_m \leq n_{c_m}$ such that $j_k c_k \not\leq (la)'$ for all $k \in \{1, \ldots, m\}$ and, for all $x \in E$, $x \not\leq (la)'$ implies $j_{k_0} c_{k_0} \leq x$ for some $k_0 \in \{1, \ldots, m\}$.

Proof. $\Longrightarrow$: Assume that $E$ is almost orthogonal. Let $a \in E$ be an atom, $1 \leq l \leq n_a$. We shall denote $A_a = \{ b \in E \mid b \text{ is an atom}, b \not\leq a' \}$. Clearly, $A_a$ is finite i.e. $A_a = \{ b_1, \ldots, b_n \}$ for suitable atoms $b_1, \ldots, b_n$ from $E$.

Let $b \in E$ be an atom, $1 \leq k \leq n_b$ and $kb \not\leq (la)'$. Either $b = a$ or $b \not= a$ and in this case we have by Lemma 3.3(iv) that $b \not\leq a'$. Hence either $b = a$ or $b \in A_a$. Let us put $\{ c_1, \ldots, c_m \} = \{ A_a \}$ if $a \in S(E)$; otherwise . In both cases we have that $a \in \{ c_1, \ldots, c_m \}$.

Now, let $x \in E$ and $x \not\leq (la)'$. By Theorem 1.2 there is an atom $c \in E$ and an integer $1 \leq j \leq n_c$ such that $jc \leq x$ and $jc \not\leq (la)'$. Either $c = a$ or $c \not= a$. In
the first case we have that \( j \geq (n_a - l + 1) \), i.e. \( x \geq (n_a - l + 1)a \). In the second case we get that \( c \leq a' \), i.e. \( c \in A_a \) and \( x \geq b_i \) for suitable \( i \in \{1, \ldots, n\} \). Hence it is enough to put \( j_k = 1 \) if \( c_k \in A_a \) and \( j_k = (n_a - l + 1) \) if \( c_k = a \).

\( \equiv \) Conversely, let \( a \in E \) be an atom. Then there are finitely many atoms \( c_1, \ldots, c_m \) and integers \( j_1, \ldots, j_m \), \( 1 \leq j_1 \leq n_{c_1}, \ldots, 1 \leq j_m \leq n_{c_m} \), such that \( j_k c_k \not\leq a' \) for all \( k \in \{1, \ldots, m\} \) and, for all \( x \in E \), \( x \not\leq a' \) implies \( j_k c_k \leq x \) for some \( k_0 \in \{1, \ldots, m\} \). Let us check that \( A_a \subseteq \{c_1, \ldots, c_m\} \). Let \( b \in A_a \). Then \( b \geq j_k c_k \geq c_k \) for some \( k_0 \in \{1, \ldots, m\} \). Hence \( b = c_{k_0} \). This yields \( A_a \) is finite. 

\( \square \)

**Lemma 3.6.** Let \( E \) be an almost orthogonal Archimedean atomic lattice effect algebra. Then, for any atom \( a \in E \) and any integer \( l \), \( 1 \leq l \leq n_a \) there are finitely many atoms \( b_1, \ldots, b_n \) and integers \( j_1, \ldots, j_n \), \( 1 \leq j_1 \leq n_{b_1}, \ldots, 1 \leq j_n \leq n_{b_n} \) such that

\[
E = [0, (la)^\gamma] \cup (\bigcup_{k=1}^n [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1])
\]

and

\[
[0, (la)^\gamma] \cap (\bigcup_{k=1}^n [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1]) = \emptyset.
\]

Hence \([0, (la)^\gamma]\) is a clopen subset in the interval topology.

**Proof.** Let \( a \in E \) be an atom, \( 1 \leq l \leq n_a \). By Definition \ref{def:3.5}, let \( \{j_1 b_1, \ldots, j_n b_n\} \) be the finite set of non-orthogonal finite elements to \( la \) of the form \( j_k b_k \), \( 1 \leq j_k \leq n_{b_k} \), \( b_1, \ldots, b_n \) are atoms different from \( a \). We put \( D = [0, (la)^\gamma] \cup (\bigcup_{k=1}^n [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1]) \). Let us check that \( D = E \). Clearly, \( D \subseteq E \). Now, let \( z \in E \). Then by Theorem \ref{thm:3.2} there are mutually distinct atoms \( c_\gamma \in E \), \( \gamma \in \mathcal{E} \) and integers \( t_\gamma \) such that

\[
z = \bigoplus \{t_\gamma c_\gamma | \gamma \in \mathcal{E}\} = \sqrt[\gamma]{t_\gamma c_\gamma | \gamma \in \mathcal{E}}.
\]

Either \( t_\gamma c_\gamma \leq (la)^\gamma \) for all \( \gamma \in \mathcal{E} \) and hence \( z \in [0, (la)^\gamma] \) or there exists \( \gamma_0 \in \mathcal{E} \) such that \( t_{\gamma_0} c_{\gamma_0} \leq (la)^\gamma \). Hence, by almost orthogonality, either \( j_k c_k \leq t_{\gamma_0} c_{\gamma_0} \leq z \) for some \( k_0 \in \{1, \ldots, n\} \) or \( (n_a + 1 - l)a \leq t_{\gamma_0} c_{\gamma_0} \leq z \). In both cases we get that \( z \in D \).

Now, assume that \( y \in [0, (la)^\gamma] \cap (\bigcup_{k=1}^n [j_k b_k, 1] \cup [(n_a + 1 - l)a, 1]) \). Then \( (n_a + 1 - l)a \leq y \leq (la)^\gamma \) or \( j_k b_k \leq y \leq (la)^\gamma \) for some \( k \in \{1, \ldots, n\} \). In any case we have a contradiction. 

\( \square \)

**Proposition 3.7.** Let \( E \) be an almost orthogonal Archimedean atomic lattice effect algebra. Then, for any not necessarily different atoms \( a, b \in E \) and any integers \( l, k; 1 \leq l \leq n_a, 1 \leq k \leq n_b \), the interval \([kb, (la)^\gamma]\) is clopen in the interval topology.

**Proof.** From Lemma \ref{lem:3.6} we have that \([0, (la)^\gamma]\) is a clopen subset. Since a dual of an almost orthogonal Archimedean atomic lattice effect algebra is an almost orthogonal Archimedean atomic lattice effect algebra as well, we have that \([kb, 1]\) is again clopen in the interval topology. Hence also \([kb, (la)^\gamma]\) is clopen in the interval topology. 

\( \square \)
**Theorem 3.8.** Let $E$ be an almost orthogonal Archimedean atomic lattice effect algebra. Then the interval topology $τ_1$ on $E$ is Hausdorff.

**Proof.** Let $x, y ∈ E$ and $x ≠ y$. Then (without loss of generality) we may assume that $x ∭ y$. Then by [25, Theorem 3.3] there exists an atom $b$ from $E$ and an integer $k$, $1 ≤ k ≤ n_a$ such that $kb ≤ x$ and $kb ∭ y$. Applying the dual of [25, Theorem 3.3] there exists an atom $a$ from $E$ and an integer $l$, $1 ≤ l ≤ n_a$ such that $y ≤ (la)'$ and $kb ∭ (la)'$. Clearly, $x ∈ [kb, 1)$, $y ∈ [0, (la)']$.

Assume that there is an element $z ∈ E$ such that $z ∈ [kb, 1] ∩ [0, (la)']$. Then $kb ≤ z ≤ (la)'$, a contradiction. Hence by Proposition 3.7, $[kb, 1]$ and $[0, (la)']$ are disjoint open subsets separating $x$ and $y$. □

**Theorem 3.9.** Let $E$ be an almost orthogonal Archimedean atomic lattice effect algebra. Then $E$ is compactly generated and therefore (o)-continuous.

**Proof.** It is enough to check that, for any atom $a ∈ E$ and any integer $l$, $1 ≤ l ≤ n_a$ the element $la$ is compact in $E$ since any element of $E$ is a join of such elements (see Theorem 3.2 resp. [25, Theorem 3.3]).

Let $x = \bigvee_{x_a ∈ E} x_a$ for some net $(x_a)_{a ∈ E}$ in $E$, $la ≤ x$, i.e., $(la)' ≥ x' \downarrow x_a$.

By Lemma 3.6 we have $E = [0, (la)'] ∪ (∪_{l=1}^{n_a}[j_k b_k], 1] ∪ [(n_a + 1 - l)a, 1]$, $[0, (la)'] ∩ (∪_{k=1}^{n_a}[b_k, 1] ∪ [(n_a + 1 - l)a, 1]) = ∅$, $b_1, \ldots, b_n$ are atoms of $E$, $1 ≤ j_k ≤ n_k$, $1 ≤ k ≤ n$.

Since $E$ is directed upwards, there exists a cofinal subset $E' ⊆ E$ such that $x_{β} ∈ [0, (la)']$ for all $β ∈ E'$ or there exists $k_0 ∈ \{1, 2, ..., n\}$ such that $x_{β} ∈ [j_{k_0} b_{k_0}, 1]$ for all $β ∈ E'$ or $x_{β} ∈ [(n_a + 1 - l)a, 1]$ for all $β ∈ E'$. If $x_{β} ∈ [0, (la)']$ for all $β ∈ E'$ then clearly $la ≤ x_{β}$ for all $β ∈ E'$. If there exists $k_0 ∈ \{1, 2, ..., n\}$ such that $x_{β} ∈ [j_{k_0} b_{k_0}, 1]$ for all $β ∈ E'$ or $x_{β} ∈ [(n_a + 1 - l)a, 1]$ for all $β ∈ E'$ we obtain that $x' ∈ [j_{k_0} b_{k_0}, 1]$ for all $β ∈ E'$ or $x' ∈ [(n_a + 1 - l)a, 1]$ which is a contradiction with $x' ∈ [0, (la)']$. □

Let $E$ be an Archimedean atomic lattice effect algebra. We put $U = \{x ∈ E \mid x = ∪_{i=1}^{n} a_i, a_i, \ldots, a_n$ are atoms of $E$ $1 ≤ l ≤ n_a$, $1 ≤ i ≤ n$, $n$ natural number $\}$ and $V = \{x ∈ E \mid x' ∈ U\}$. Then by [25, Theorem 3.3], for every $x ∈ L$, we have that

$$x = \bigvee \{u ∈ U \mid u ≤ x\} = \bigwedge \{v ∈ V \mid x ≤ v\}.$$

Consider the function family $Φ = \{f_u \mid u ∈ U\} ∪ \{g_v \mid v ∈ V\}$, where $f_u, g_v : L → \{0, 1\}$, $u ∈ U, v ∈ V$ are defined by putting $f_u(x) = \begin{cases} 1 & \text{iff } u ≤ x \\ 0 & \text{iff } u ≥ x \end{cases}$ and $g_v(y) = \begin{cases} 1 & \text{iff } x ≤ v \\ 0 & \text{iff } x ≥ v \end{cases}$ for all $x, y ∈ L$.

Further, consider the family of pseudometrics on $L$: $Σ_Φ = \{ρ_u \mid u ∈ U\} ∪ \{π_v \mid v ∈ V\}$, where $ρ_u(a, b) = |f_u(a) - f_u(b)|$ and $π_v(a, b) = |g_v(a) - g_v(b)|$ for all $a, b ∈ L$.
Let us denote by $\mathcal{U}_0$ the uniformity on $L$ induced by the family of pseudo-metrics $\Sigma_\Phi$ (see e.g. [3]). Further denote by $\tau_\Phi$ the topology compatible with the uniformity $\mathcal{U}_0$.

Then for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of $L$

$$x_\alpha \xrightarrow{\tau_\Phi} x \iff \varphi(x_\alpha) \to \varphi(x) \text{ for any } \varphi \in \Phi.$$  

This implies, since $f_u, u \in \mathcal{U}$, and $g_v, v \in \mathcal{V}$, is a separating function family on $L$, that the topology $\tau_\Phi$ is Hausdorff. Moreover, the intervals $[u, v] = [u, 1] \cap [0, v] = f_u^{-1}(\{1\}) \cap g_v^{-1}(\{1\})$ are clopen sets in $\tau_\Phi$.

**Definition 3.10.** Let $E$ be an Archimedean atomic lattice effect algebra. Let $\Phi$ be a separating function family on $E$ defined above. We will denote by $\tau_\Phi$ the uniform topology on $E$ defined by this function family, that means for every net $(x_\alpha)_{\alpha \in \mathcal{E}}$ of elements of $L$

$$x_\alpha \xrightarrow{\tau_\Phi} x \iff \varphi(x_\alpha) \to \varphi(x) \text{ for any } \varphi \in \Phi.$$ 

**Theorem 3.11.** Let $E$ be an almost orthogonal Archimedean atomic lattice effect algebra. Then $\tau_1 = \tau_0 = \tau_\Phi$.

**Proof.** Since by Theorem 3.8 $\tau_1$ is Hausdorff we obtain by [4] that $\tau_1 = \tau_0$. Further if $O \in \tau_0$ and $x \in O$ then by Theorem 1.2 we have $x = \bigvee \{u \in \mathcal{U} \mid u \leq x\} = \bigwedge \{v \in \mathcal{V} \mid x \leq v\}$, which by [12] implies that there exist finite sets $F \subseteq \mathcal{U}$, $G \subseteq \mathcal{V}$ such that $x \in \bigvee F \cap \bigwedge G \subseteq O$. Hence $\tau_o \subseteq \tau_\Phi$. By Theorem 3.9 and [10, Theorem 1] we obtain $\tau_o = \tau_\Phi$. \qed

**Theorem 3.12.** Let $E$ be an Archimedean atomic block-finite lattice effect algebra. Then $\tau_1 = \tau_0$ is a Hausdorff topology.

**Proof.** As in [18], it suffices to show that for every $x, y \in E$, $x \not\leq y$ there are finitely many intervals, none of which contains both $x$ and $y$ and the union of which covers $E$.

By [13], $E$ is a union of finitely many atomic blocks $M_i$, $i = 1, 2, \ldots, n$. Choose $i \in \{1, 2, \ldots, n\}$. If $x, y \in M_i$ then there is an atom $a_i \in M_i$ and an integer $l_i$, $1 \leq l_i \leq n_{a_i}$ such that $l_i a_i \leq x$, $l_i a_i \not\leq y$. Let us put $k_i = n - l_i + 1$. Since $M_i$ is almost orthogonal (the only possible non-orthogonal $kb$ to $la$ for an atom $a$, $1 \leq l \leq n_a$ is that $a = b$) we have by Lemma 3.5 that $M_i = ([0, (k_i a_i)] \cap M_i) \cup ([n_{a_i} + 1 - k_i) a_i, 1) \cap M_i)$. Hence $M_i \subseteq [0, (k_i a_i)] \cup [(n_{a_i} + 1 - k_i) a_i, 1]$. Let us check that $[0, (k_i a_i)] \cap [(n_{a_i} + 1 - k_i) a_i, 1] = \emptyset$. Assume that $(n_{a_i} + 1 - k_i) a_i \leq z \leq (k_i a_i)$. Then $(n_{a_i} + 1 - k_i) a_i \leq (k_i a_i)$, a contradiction. Put $J_i = [0, (k_i a_i)]$, $K_i = [(n_{a_i} + 1 - k_i) a_i, 1]$. This yields $x \in K_i$, $y \in J_i$, $M_i \subseteq J_i \cup K_i$ and $J_i \cap K_i = \emptyset$. Let $x \not\in M_i$. Then there exists an atom $a_i \in M_i$ that is not compatible with $x$. Let us check that $x \not\in [0, (a_i)] \cup [n_{a_i} a_i, 1]$. Assume that $x \in [0, (a_i)]$ or $x \in [n_{a_i} a_i, 1]$. Then $x \leq (a_i)$ or $a_i \leq n_{a_i} a_i \leq x$, i.e., in both cases we get that $x \leftrightarrow a_i$, a contradiction. Let us put $J_i = [0, (a_i)]$, $K_i = [n_{a_i} a_i, 1]$. As above, $M_i \subseteq J_i \cup K_i$, $J_i \cap K_i = \emptyset$ and moreover $x \not\in J_i \cup K_i$. The remaining case $y \not\in M_i$ can be checked by similar considerations. We obtain $E = \bigcup_{i=1}^n M_i \subseteq \bigcup_{i=1}^n (J_i \cup K_i) \subseteq E$ and none of the intervals $J_i, K_i$, $i = 1, 2, \ldots, n$ contains both $x$ and $y$. \qed
4. Order and interval topologies of complete atomic block-finite lattice effect algebras

We are going to show that on every complete atomic block-finite lattice effect algebra \( E \) the interval topology is Hausdorff. Hence both topologies \( \tau_i \) and \( \tau_o \) are in this case compact Hausdorff and they coincide. Moreover, a necessary and sufficient condition for a complete atomic lattice algebra \( E \) to be almost orthogonal is given.

For the proof of Theorems 4.2 and 4.3 we will use the following statement, firstly proved in the equivalent setting of D-posets in [19].

**Theorem 4.1.** [19, Theorem 1.7] Suppose that \( (E; \oplus, 0, 1) \) is a complete lattice effect algebra. Let \( \emptyset \neq D \subseteq E \) be a sub-lattice effect algebra. The following conditions are equivalent:

1. For all nets \( (x_\alpha)_{\alpha \in \mathcal{E}} \) such that \( x_\alpha \in D \) for all \( \alpha \in \mathcal{E} \)
   \[ x_\alpha \stackrel{(o)}{\longrightarrow} x \text{ in } E \text{ if and only if } x_\alpha \stackrel{(o)}{\longrightarrow} x \text{ in } D. \]
2. For every \( M \subseteq D \) with \( \bigvee M = x \) in \( E \) it holds \( x \in D \).
3. For every \( Q \subseteq D \) with \( \bigwedge Q = y \) in \( E \) it holds \( y \in D \).
4. \( D \) is a complete sub-lattice of \( E \).
5. \( D \) is a closed set in order topology \( \tau_o \) on \( E \).

Each of these conditions implies that \( \tau_o^D = \tau_o^E \cap D \), where \( \tau_o^D \) is an order topology on \( D \).

Important sub-lattice effect algebras are blocks, \( S(E), B(E) = \bigcap \{ M \subseteq E \mid M \text{ block of } E \} \) and \( C(E) = B(E) \cap S(E) \) (see [6, 7, 13, 21, 23]).

**Theorem 4.2.** Let \( E \) be a complete lattice effect algebra. Then for every \( D \in \{ S(E), C(E), B(E) \} \) or \( D = M \), where \( M \) is a block of \( E \), we have:

1. \( x_\alpha \stackrel{\tau_i}{\rightarrow} x \iff x_\alpha \stackrel{\tau_o}{\rightarrow} x \), for all nets \( (x_\alpha)_{\alpha \in \mathcal{E}} \) in \( D \) and all \( x \in E \).
2. If \( \tau_i \) is Hausdorff then
   \[ x_\alpha \stackrel{\tau_i}{\rightarrow} x \iff x_\alpha \stackrel{\tau_o}{\rightarrow} x \], for all nets \( (x_\alpha)_{\alpha \in \mathcal{E}} \) in \( D \) and all \( x \in E \).

Proof. The first part of the statement follows by Theorem 2.5 and the fact that if \( E \) is a complete lattice effect algebra then \( M, S(E), C(E) \) and \( B(E) \) are complete sub-lattices of \( E \) (see [6, 24]). The second part follows by [4] since \( \tau_i \) is Hausdorff implies \( \tau_i = \tau_o \) and by Theorem 4.1.

**Theorem 4.3.** (i) The interval topology \( \tau_i \) on every Archimedean atomic MV-effect algebra \( M \) is Hausdorff and \( \tau_i = \tau_o = \tau_\Phi \).
(ii) For every complete atomic MV-effect algebra \( M \) and for any net \( (x_\alpha) \) of \( M \) and any \( x \in M \),
\[ x_\alpha \stackrel{\tau_o}{\rightarrow} x \text{ if and only if } x_\alpha \stackrel{(o)}{\rightarrow} x \ (\text{briefly } \tau_o \equiv (o)). \]
Moreover, \( \tau_o \) is a uniform compact Hausdorff topology on \( M \).

(iii) For every atomic block-finite lattice effect algebra \( E \), \( E \) is a complete lattice iff \( \tau_i = \tau_o \) is a compact Hausdorff topology.

Proof. (i), (ii): This follows from the fact that every pair of elements of \( M \) is compatible, hence every pair of atoms is orthogonal. Thus for (i) we can apply Theorem 3.11 and for (ii) we can use (i) and [16, Theorem 2] since \( M \) is compactly generated by finite elements and \( \tau_i \) is compact.

(iii) From Theorem 3.12 we know that \( \tau_i = \tau_o \) is a Hausdorff topology. By Lemma 2.1 (vi) the interval topology \( \tau_i \) on \( E \) is compact iff \( E \) is a complete lattice.

In what follows we will need Corollary 4.5 of Lemma 4.4.

**Lemma 4.4.** Let \( E \) be an Archimedean atomic lattice effect algebra. Then

(i) If \( c, d \in E \) are compact elements with \( c \leq d' \) then \( c \oplus d \) is compact.

(ii) If \( u = \bigoplus G \), where \( G \) is a \( \oplus \)-orthogonal system of atoms of \( E \), and \( u \) is compact then \( G \) is finite.

Proof. (i) Let \( c \oplus d \leq \bigvee D \). Let \( \mathcal{E} = \{ F \subseteq D : F \text{ is finite}\} \) be directed by set inclusion and let for every \( F \in \mathcal{E} \) be \( x_F = \bigvee F \). Then \( x_F \uparrow x = \bigvee D \). Since \( c \leq \bigvee D \) and \( d \leq \bigvee D \) there is a finite subset \( F_1 \subseteq D \) such that \( c \vee d \leq \bigvee F_1 \). Therefore, for \( F \geq F_1 \), \( x_F \oplus c \uparrow x \oplus c, d \leq x \oplus c \). Then there is a finite subset \( F_2 \subseteq D \), \( F_1 \leq F_2 \) such that \( d \leq x_{F_2} \oplus c \). Hence \( c \oplus d \leq x_{F_2} \).

(ii) Let \( u \in E \), \( u = \bigoplus G = \bigvee \{ \bigoplus K \mid K \subseteq G \text{ is finite}\} \) where \( G = (a_\varkappa)_{\varkappa \in H} \) is a \( \oplus \)-orthogonal system of atoms. Clearly if \( K_1, K_2 \subseteq G \) are finite such that \( K_1 \subseteq K_2 \) then \( \bigoplus K_1 \leq \bigoplus K_2 \).

Assume that \( u \) is compact. Hence there are finite \( K_1, K_2, \ldots, K_n \subseteq G \) such that \( u \leq \bigvee \{ \bigoplus K_i \mid i = 1, 2, \ldots, n\} \). Let \( K_0 = \bigcup \{ K_i \mid i = 1, 2, \ldots, n\} \), then \( K_0 \subseteq G \). \( K_0 \) is finite and \( \bigoplus K \leq \bigoplus K_0 \), \( i = 1, 2, \ldots, n \), which gives that \( \bigvee \{ \bigoplus K_i \mid i = 1, 2, \ldots, n\} \leq \bigoplus K_0 \). It follows that \( u \leq \bigoplus K_0 \leq u = \bigvee \{ \bigoplus K \mid K \subseteq G \text{ is finite}\} \). Hence \( u = \bigoplus K_0 \), \( K_0 \subseteq G \) is finite. Further, for every finite \( K \subseteq G \setminus K_0 \) we have \( \bigoplus K_0 \subseteq \bigoplus (K_0 \cup K) = \bigoplus K_0 \oplus \bigoplus K \leq u = \bigoplus K_0 \), which gives that \( \bigoplus K = 0 \). Hence \( K = \emptyset \) and thus \( G \setminus K_0 = \emptyset \) which gives that \( K_0 = G \).

**Corollary 4.5.** Let \( E \) be an o-continuous Archimedean atomic lattice effect algebra. Then every finite element of \( E \) is compact.

Proof. Clearly, by [16, Theorem 7] we know that \( E \) is compactly generated. Therefore, any atom of \( E \) is compact. The compactness of every finite element follows by an easy induction.

**Theorem 4.6.** Let \( E \) be an Archimedean atomic lattice effect algebra. Then the following conditions are equivalent:

(i) \( \tau_i = \tau_o = \tau_\emptyset \).

(ii) \( E \) is o-continuous and \( \tau_i \) is Hausdorff.
(iii) $E$ is almost orthogonal.

Proof. (i) $\implies$ (ii): Since $\tau_o = \tau_\Phi$ we have by [16, Theorem 1] that $E$ is compactly generated and hence o-continuous. The condition $\tau_i = \tau_\Phi$ implies that $\tau_i$ is Hausdorff because $\tau_\Phi$ is Hausdorff.

(ii) $\implies$ (i), (iii): Since $\tau_i$ is Hausdorff we obtain $\tau_i = \tau_o$ by [4]. Moreover, from [16, Theorem 7] and Corollary 4.5 the (o)-continuity of $E$ implies that $E$ is compactly generated by the elements from $U$. This gives $\tau_o = \tau_\Phi$ from [16, Theorem 1].

Let $a \in E$ be an atom, $1 \leq l \leq n_a$. Then the interval $[0, (la)']$ is a clopen set in the order topology $\tau_o = \tau_\Phi = \tau_i$. Hence there is a finite set of intervals in $E$ such that $0 \in E \setminus \bigcup_{i=1}^{n_a} [u_i, v_i] \subseteq [0, (la)']$. Thus $E \subseteq [0, (la)'] \cup \bigcup_{i=1}^{n_a} [u_i, v_i] \subseteq [0, (la)'] \cup \bigcup_{i=1}^{n_a} [k_ib_i, 1]$, where $b_i \in E$ are atoms such that $k_ib_i \leq u_i$, $1 \leq k_i \leq n_i$, $i = 1, \ldots, n$. This yields that $E$ is almost orthogonal.

(iii) $\implies$ (ii): From Theorems 3.8 and 3.9 we have that $\tau_i$ is Hausdorff and $E$ is compactly generated, hence (o)-continuous.

Corollary 4.7. Let $E$ be a complete atomic lattice effect algebra. Then the following conditions are equivalent:

(i) $E$ is almost orthogonal.

(ii) $\tau_i = \tau_o = \tau_\Phi \equiv (o)$.

(iii) $E$ is (o)-continuous and $\tau_i$ is Hausdorff.

Proof. It follows from Theorems 1.6 and the fact that by (o)-continuity of $E$ [27, Theorem 8] we have $\tau_o \equiv (o)$.

The next example shows that a complete block-finite atomic lattice effect algebra need not be (o)-continuous and almost orthogonal in spite of that $\tau_i = \tau_o$ is a compact Hausdorff topology.

Example 4.8. Let $E$ be a horizontal sum of finitely many infinite complete atomic Boolean algebras $(B_i, \bigoplus_i, 0_i, 1_i)$, $i = 1, 2, \ldots, n$. Then $E$ is an atomic complete lattice effect algebra, $E$ is not almost orthogonal, $E$ is not compactly generated by finite elements (hence $\tau_o \neq \tau_\Phi$), $E$ is block-finite, $\tau_i = \tau_o$ is Hausdorff by Theorem 3.12 and the interval topology $\tau_i$ on $E$ is compact.

5. Applications

Theorem 5.1. Let $E$ be a block-finite complete lattice effect algebra. Then the following conditions are equivalent:

(i) $E$ is almost orthogonal.

(ii) $E$ is compactly generated.

(iii) $E$ is (o)-continuous.

(iv) $\tau_i = \tau_o = \tau_\Phi \equiv (o)$.
Proof. By Theorem 3.12, \( \tau_i \) is a Hausdorff topology. This by 16, Theorem 7 gives that (ii) \( \iff \) (iii) and by Corollary 4.7 we obtain that (i) \( \iff \) (iii) \( \iff \) (iv).

In Theorem 5.1 the assumption that \( E \) is atomic can not be omitted. For instance, every non-atomic complete Boolean algebra is \((o)\)-continuous but it is not compactly generated, because in such a case \( E \) must be atomic by 16, Theorem 6.

Remark 5.2. If a \( \oplus \)-operation on a lattice effect algebra \( E \) is continuous with respect to its interval topology \( \tau_i \) meaning that \( x_{\alpha} \leq y_{\alpha}, x_{\alpha} \buildrel \tau_i \over \rightarrow x, y_{\alpha} \buildrel \tau_i \over \rightarrow y, \alpha \in \mathcal{E} \) implies \( x_{\alpha} \oplus y_{\alpha} \buildrel \tau_i \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \) then \( \tau_i \) is Hausdorff (see [14]). Hence \( \oplus \)-operation on complete \((o)\)-continuous atomic lattice effect algebras which are not almost orthogonal cannot be \( \tau_i \)-continuous, by [14] and Corollary 4.7.

Theorem 5.3. Let \( E \) be a block-finite complete atomic lattice effect algebra.
Let \((x_{\alpha})_{\alpha \in \mathcal{E}}\) and \((y_{\alpha})_{\alpha \in \mathcal{E}}\) be nets of elements of \( E \) such that \( x_{\alpha} \leq y_{\alpha} \) for all \( \alpha \in \mathcal{E} \).

If \( x_{\alpha} \buildrel \tau_i \over \rightarrow x, y_{\alpha} \buildrel \tau_i \over \rightarrow y, \alpha \in \mathcal{E} \) then \( x \leq y \) and \( x_{\alpha} \oplus y_{\alpha} \buildrel \tau_i \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \). Moreover, \( \tau_i = \tau_o \).

Proof. Since, by Theorem 6.12, \( \tau_i \) is Hausdorff, we obtain that \( \tau_i = \tau_o \) by [4]. Let \( \{M_1, \ldots, M_n\} \) be the set of all blocks of \( E \). Further, for every \( \alpha \in \mathcal{E} \), elements of the set \( \{x_{\alpha}, y_{\alpha}, x_{\alpha} \oplus y_{\alpha}\} \) are pairwise compatible. It follows that for every \( \alpha \in \mathcal{E} \) there exists a block \( M_{k_{\alpha}} \) of \( E \), \( k_{\alpha} \in \{1, \ldots, n\} \) such that \( \{x_{\alpha}, y_{\alpha}, x_{\alpha} \oplus y_{\alpha}\} \subseteq M_{k_{\alpha}} \).

Let \( \mathcal{E}' \) be any cofinal subset of \( \mathcal{E} \). Since \( \mathcal{E}' \) is directed upwards, there exists a block \( M_{k_{\alpha}} \) of \( E \) and a cofinal subset \( \mathcal{E}'' \) of \( \mathcal{E}' \) such that \( \{x_{\beta}, y_{\alpha}, x_{\beta} \oplus y_{\beta}\} \subseteq M_{k_{\beta}} \) for all \( \beta \in \mathcal{E}'' \). Otherwise we obtain a contradiction with the finiteness of the set \( \{M_1, \ldots, M_n\} \). Further, by Theorem 2.5 we obtain that \( \tau_{i,M_{k_{\alpha}}} = \tau_i \cap M_{k_{\alpha}} \) as \( M_{k_{\alpha}} \) is a complete sublattice of \( E \) (see Theorem 4.2). It follows that the interval topology \( \tau_{i,M_{k_{\alpha}}} \) on the complete MV-effect algebra \( M_{k_{\alpha}} \) is Hausdorff.

The last by [14, Theorem 3.6] gives that \( x_{\beta} \oplus y_{\beta} \buildrel \tau_{i,M_{k_{\alpha}}} \over \rightarrow x \oplus y, \beta \in \mathcal{E}'' \) and hence \( x_{\beta} \oplus y_{\beta} \buildrel \tau_{i,M_{k_{\alpha}}} \over \rightarrow x \oplus y, \beta \in \mathcal{E}'' \), as \( \tau_{i,M_{k_{\alpha}}} = \tau_i \cap M_{k_{\alpha}} \). It follows that \( x_{\alpha} \oplus y_{\alpha} \buildrel \tau_i \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \) by Lemma 2.3.

In [22, Theorem 4.5] it was proved that a block-finite lattice effect algebra \( (E; \oplus, 0, 1) \) has a MacNeille completion which is a complete effect algebra \((MC(E); \oplus, 0, 1)\) containing \( E \) as a (join-dense and meet-dense) sub-lattice effect algebra iff \( E \) is Archimedean. In what follows we put \( \hat{E} = MC(E) \).

Corollary 5.4. Let \( E \) be a block-finite Archimedean atomic lattice effect algebra.
Then for any nets \((x_{\alpha})_{\alpha \in \mathcal{E}}\) and \((y_{\alpha})_{\alpha \in \mathcal{E}}\) of elements of \( E \) with \( x_{\alpha} \leq y_{\alpha} \), \( \alpha \in \mathcal{E} \):
\[ x_{\alpha} \buildrel \tau_i \over \rightarrow x, y_{\alpha} \buildrel \tau_i \over \rightarrow y, \alpha \in \mathcal{E} \implies x_{\alpha} \oplus y_{\alpha} \buildrel \tau_i \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \].

Proof. By [20, Lemma 1.1], for interval topologies \( \hat{\tau_i} \) on \( \hat{E} \) and \( \tau_i \) on \( E \), we have \( \hat{\tau_i} \cap E = \tau_i \). Thus for \( x_{\alpha}, y_{\alpha}, x, y \in E \) we obtain \( x_{\alpha} \oplus y_{\alpha} \buildrel \hat{\tau_i} \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \) which gives \( x_{\alpha} \oplus y_{\alpha} \buildrel \tau_i \over \rightarrow x \oplus y, \alpha \in \mathcal{E} \) by the fact that \( \hat{\tau_i} \cap E = \tau_i \).
**Definition 5.5.** Let $E$ be a lattice. Then

(i) An element $u$ of $E$ is called strongly compact (briefly $s$-compact) iff, for any $D \subseteq E$: $u \leq c \in E$ for all $c \geq D$ implies $u \leq \bigvee F$ for some finite $F \subseteq D$.

(ii) $E$ is called $s$-compactly generated iff every element of $E$ is a join of $s$-compact elements.

**Theorem 5.6.** Let $E$ be a block-finite Archimedean atomic lattice effect algebra. Then the following conditions are equivalent:

(i) $E$ is almost orthogonal.

(ii) $\hat{E} = MC(E)$ is almost orthogonal.

(iii) $\hat{E} = MC(E)$ is compactly generated.

(iv) $E$ is $s$-compactly generated.

**Proof.** By J. Schmidt [30], a MacNeille completion $\hat{E}$ of $E$ is (up to isomorphism) a complete lattice such that for every element $x \in \hat{E}$ there exist $P, Q \subseteq E$ such that $x = \bigvee \hat{E} P = \bigwedge \hat{E} Q$ (taken in $\hat{E}$). Here we identify $E$ with $\varphi(E)$, where $\varphi : E \to \hat{E}$ is the embedding (meaning that $E$ and $\varphi(E)$ are isomorphic lattice effect algebras). It follows that $E$ and $\hat{E}$ have the same set of all atoms and coatoms and hence also the same set of all finite and cofinite elements, which implies that (i) $\iff$ (ii).

Moreover, for any $A \subseteq E$ and $u \in E$, we have ($d \in E, A \leq d$ implies $u \leq d$) iff $u \leq \bigvee \hat{E} A$. Then $u$ is $s$-compact in $E$ iff $u$ is compact in $\hat{E}$, which gives (iii) $\iff$ (iv).

Finally (ii) $\iff$ (iii) by Theorem 5.1. \qed

**Definition 5.7.** Let $E$ be an effect algebra. A map $\omega : E \to [0, 1]$ is called a state on $E$ if $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ exists in $E$.

**Theorem 5.8.** (State smearing theorem for almost orthogonal block-finite Archimedean atomic lattice effect algebras) Let $(E; \oplus, 0, 1)$ be a block-finite Archimedean atomic lattice effect algebra. If $E$ is almost orthogonal then:

(i) $E_1 = \{ x \in E \mid x$ or $x'$ is finite $\}$ is a sub-lattice effect algebra of $E$.

(ii) If there exists an $(o)$-continuous state $\omega$ on $E_1$ (or on $S(E_1) = S(E) \cap E_1$, or on $S(E)$) then there exists an $(o)$-continuous state $\hat{\omega}$ on $E$ extending $\omega$ and an $(o)$-continuous state $\tilde{\omega}$ on $\hat{E} = MC(E) = MC(E_1)$ extending $\hat{\omega}$.

**Proof.** (i) By Theorem 5.6, $E$ is $s$-compactly generated and thus by [28, Theorem 2.7] $E_1$ is a sub-lattice effect algebra of $E$.

(ii) Since $E$ is $s$-compactly generated, we obtain the existence of $(o)$-continuous extensions $\tilde{\omega}$ on $E$ and $\hat{\omega}$ on $\hat{E}$ by [28, Theorem 4.2]. \qed
References

[1] E.G. Beltrametti, G. Cassinelli, The Logic of Quantum Mechanics, Addison-Wesley, Reading, MA, 1981.

[2] C.C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.

[3] A. Császár, General Topology, Akadémiai Kiadó, Budapest (1978).

[4] M. Erné, S. Weck, Order convergence in lattices, Rocky Mountain J. Math. 10 (1980), 805–818.

[5] O. Frink, Topology in lattices, Trans. Amer. Math. Soc. 51 (1942), 569–582.

[6] R.J. Greechie, D.J. Foulis, S. Pulmannová, The center of an effect algebra, Order 12 (1995), 91–106.

[7] S. P. Gudder, Sharply dominating effect algebras, Tatra Mt. Math. Publ. 15 (1998), 23–30.

[8] G. Jenča, Z. Riečanová, On sharp elements in lattice ordered effect algebras, BUSEFAL 80 (1999) 24–29.

[9] G. Jenča, Z. Riečanová, A Survey on Sharp Elements in Unsharp Quantum Logics, Journal of Electrical Engineering, 52 (No 7-8) (2001), 237-239.

[10] G. Kalmbach, Orthomodular Lattices, Kluwer Academic Publ. Dordrecht, 1998.

[11] M. Katětov, Remarks on Boolean algebras, Colloq. Math. 11 (1951), 229–235.

[12] H. Kirchheimová, Z., Riečanová, Note on order convergence and order topology, Appendix B, in the book Riečan, B., Neubrunn, T., Measure, Integral and Order, Ister Science (Bratislava) and Kluwer Academic Publishers (Dordrecht-Boston-London), 1997.

[13] F. Kôpka, Compatibility in D-posets, Inter. J. Theor. Phys. 34 (1995), 1525–1531.

[14] Lei Qiang, Wu Junde and Li Ronglu, Interval topology of lattice effect algebras, Applied Math. Letters. 22 (2009), 1003–1006.

[15] K. Mosná, Atomic lattice effect algebras and their sub-lattice effect algebras, J. Electrical Engineering 58 (No 7/8) (2007), 3–6.

[16] J. Paseka, Z. Riečanová, Compactly generated de Morgan lattices, basic algebras and effect algebras, International J. Theoret. Phys. (2009), doi:10.1007/s10773-009-0011-4.
[17] S. Pulmannová, Z. Riečanová, Compact topological orthomodular lattices, Contributions to General Algebra 7, Verlag Hölder - Pichler - Tempsky, Wien, Verlag B.G. Teubner, Stuttgart (1991), 277–282.

[18] S. Pulmannová, Z. Riečanová, Blok finite atomic orthomodular lattices, Journal Pure and Applied Algebra 89 (1993), 295–304.

[19] Z. Riečanová, On Order Continuity of Quantum Structures and Their Homomorphisms, Demonstratio Mathematica 29 (1996), 433–443.

[20] Z. Riečanová, Lattices and Quantum Logics with Separated Intervals, Atomicity, International J. Theoret. Phys. 37 (1998), 191–197.

[21] Z. Riečanová, Compatibility and central elements in effect algebras, Tatra Mountains Math. Publ. 16 (1999), 151–158.

[22] Z. Riečanová, Archimedean and block-finite lattice effect algebras, Demonstratio Mathematica 33 (2000), 443–452.

[23] Z. Riečanová, Generalization of blocks for D-lattices and lattice-ordered effect algebras, International Journal of Theoretical Physics 39 (2000), 231–237.

[24] Z. Riečanová, Orthogonal Sets in Effect Algebras, Demonstratio Mathematica, 34 (2001), 525–532.

[25] Z. Riečanová, Smearings of states defined on sharp elements onto effect algebras, International Journal of Theoretical Physics 41 (2002), 1511–1524.

[26] Z. Riečanová, Continuous Lattice Effect Algebras Admitting Order-Continuous States, Fuzzy Sets and Systems 136 (2003), 41–54.

[27] Z. Riečanová, Order-topological lattice effect algebras, Contributions to General Algebra 15, Proceedings of the Klagenfurt Workshop 2003 on General Algebra, Klagenfurt, Austria, June 19-22, (2003), 151–160.

[28] Z. Riečanová, J. Paseka, State smearing theorems and the existence of states on some atomic lattice effect algebras, Journal of Logic and Computation, Advance Access, published on March 13, 2009, doi:10.1093/logcom/exp018.

[29] T.A. Sarymsakov, S.A. Ajupov, Z. Chadzhiev, V.J. Chilin, Ordered algebras, FAN, Tashkent, (in Russian), 1983.

[30] J. Schmidt, Zur Kennzeichnung der Dedekind-Mac Neileschen Hülle einer Geordneten Menge, Archiv d. Math. 7 (1956), 241–249.