Analysis of Reciprocity Breakdown in Nonlinear Systems

Graeme Manson, Keith Worden and Michael Wood
Department of Mechanical Engineering, University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK
E-mail: graeme.manson@sheffield.ac.uk

Abstract. The breakdown of the principle of reciprocity is a well-known phenomenon of nonlinear systems. A structure or system is said to exhibit reciprocity when the response at some point \( j \) to an input at some point \( i \) is identical to the response at point \( i \) when the same input is applied at point \( j \). This paper seeks to explain this phenomenon by adopting a functional series representation which describes the input-output relationship. The frequency-domain Volterra Series representation utilises Higher-Order Frequency Response Functions (HFRFs) (generalisations of the linear FRF) to explain the behaviour of nonlinear systems. This breakdown in reciprocity may be observed through a breakdown in symmetry of HFRFs.

1. Introduction
There exists a number of techniques for identifying whether a system is linear or nonlinear. The most well-known of these techniques are the principle of superposition, and its more restricted forms of harmonic distortion and homogeneity, and the coherence function [1]. The subject of the current paper is the property of reciprocity which, for a linear system, states that the response at some point \( j \) to an input at some point \( i \) is identical to the response at point \( i \) when the same input is applied at point \( j \). An alternative way of expressing this behaviour is that the frequency response function measured at response point \( j \) to an input at point \( i \), \( H^{(j:i)}(\omega) \), is equal to \( H^{(i:j)}(\omega) \). Although the breakdown of reciprocity is often used to detect the presence of nonlinearity, such as in the study of Farrar et al. [2] where the reciprocity breakdown in a reinforced-concrete bridge column was observed due to the presence of multiple fatigue cracks, there has been little investigation to explain this behaviour or attempt to exploit it.

The current work adopts a Volterra series approach to attempt to better understand the breakdown of reciprocity in nonlinear systems and is the latest in a series of papers [3, 4, 5] which extend the understanding of structural dynamics observables using the Volterra functional series approximation of nonlinear system behaviour.

It is assumed throughout this work that the Volterra series representation exists for the systems. Sufficient conditions for this requirement are known [6] [7]. Essentially, it is required that the system nonlinearity be analytic. Another restriction is that the nonlinear system must have finite memory. Essentially this means that the system will reach a steady-state condition not governed by the initial conditions. This requirement precludes the use of the Volterra series for hysteretic systems and systems with limit cycles. Of greater concern is the issue of
some point this breakdown can occur regardless of the nature of the forcing input. Illustrate the breakdown of reciprocity in nonlinear systems but, it should be made clear that for the purposes of the current work, a sinusoidal input force will be employed in order to does not breakdown despite the presence of nonlinearity.

2. Volterra Series Representation of Nonlinear System Response

The Volterra series representation [9] [10] of the output response of a nonlinear system, \( y(t) \), to an input, \( x(t) \), is a generalisation of the input-output relation for linear systems and takes the form of an infinite series:

\[
y(t) = y_1(t) + y_2(t) + y_3(t) + \ldots
\]

where

\[
y_n(t) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\tau_1 \ldots d\tau_n h_n(\tau_1, \ldots, \tau_n)x(t - \tau_1) \ldots x(t - \tau_n)
\]

The functions \( h_n(\tau_1, \ldots, \tau_n) \) are known as the Volterra kernels of the system.

As in the linear case, there exists a dual frequency-domain representation for nonlinear systems. The higher order FRFs (HFRFs) or Volterra kernel transforms \( H_n(\omega_1, \ldots, \omega_n) \), \( n = 1, \ldots, \infty \) are defined as the multi-dimensional Fourier transforms of the kernels, i.e.

\[
H_n(\omega_1, \ldots, \omega_n) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\tau_1 \ldots d\tau_n h_n(\tau_1, \ldots, \tau_n)e^{-i(\omega_1 \tau_1 + \ldots + \omega_n \tau_n)}
\]

which leads to the frequency-domain dual of equation (1):

\[
Y(\omega) = Y_1(\omega) + Y_2(\omega) + Y_3(\omega) + \ldots
\]

where

\[
Y_n(\omega) = \frac{1}{(2\pi)^n-1} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\omega_1 \ldots d\omega_{n-1} H_n(\omega_1, \ldots, \omega - \omega_1 - \ldots - \omega_{n-1})
\]

\[
\times X(\omega_1) \ldots X(\omega - \omega_1 - \ldots - \omega_{n-1})
\]

3. Volterra Series Representation of Reciprocity Breakdown

In this section the Volterra series will be used to demonstrate the breakdown of reciprocity in nonlinear systems under sinusoidal forcing and investigate the conditions under which reciprocity does not breakdown despite the presence of nonlinearity.

3.1. Volterra Series Representation of Response to Sinusoidal Input

For the purposes of the current work, a sinusoidal input force will be employed in order to illustrate the breakdown of reciprocity in nonlinear systems but, it should be made clear that this breakdown can occur regardless of the nature of the forcing input.

If an input is applied at some point \( i \), the Volterra series representation of the response at some point \( j \) will be:

\[
y^{(ji)}(t) = y_1^{(ji)}(t) + y_2^{(ji)}(t) + y_3^{(ji)}(t) + \ldots
\]

Note that this is simply equation (1) with the addition of superscripts to reflect that this response is at some point \( j \) to an input at some point \( i \). If a cosinusoidal input, \( x'(t) = X \cos(\Omega t) \), is applied at point \( i \), it may be shown [1] that the individual response components will be:

\[
y_1^{(ji)}(t) = X \left| H_1^{(ji)}(\Omega) \right| \cos \left( \Omega t + \angle H_1^{(ji)}(\Omega) \right)
\]

\[
y_2^{(ji)}(t) = \frac{X^2}{2} \left\{ \left| H_2^{(ji)}(\Omega, \Omega) \right| \cos \left( 2\Omega t + \angle H_2^{(ji)}(\Omega, \Omega) \right) + H_2^{(ji)}(\Omega, -\Omega) \right\}
\]
The form of these HFRFs for a particular multi-degree-of-freedom system will now be considered. Forcing will be necessary that all of the HFRFs (or at least their diagonal terms in the case of sinusoidal forcing) display symmetry, i.e.

\[ y^{(j:i)}(t) = \frac{X^3}{4} \left\{ \left[ H^{(j:i)}_3(\Omega, \Omega, \Omega) \right] \cos \left( 3\Omega t + \angle H^{(j:i)}_3(\Omega, \Omega, \Omega) \right) + 3 \left[ H^{(j:i)}_3(\Omega, \Omega, -\Omega) \right] \cos \left( \Omega t + \angle H^{(j:i)}_3(\Omega, \Omega, -\Omega) \right) \right\} \]

etc. Note that the addition of the superscripts on the HFRFs also represent the response point \( j \) and the input point \( i \).

Examination of the previous components shows that each odd-order HFRF contributes a component to each odd harmonic of that order or less and each even-order HFRF contributes a component to each even harmonic of that order or less: this includes a DC component.

The FRF of a linear system can be thought of as the magnitude and phase relationship between an input sinusoid and the output sinusoid of the same frequency and it is possible to adopt this representation for nonlinear systems to construct describing functions which the authors have termed composite frequency response functions [1]. The first sinusoidal composite FRF for response point \( j \) to an input at point \( i \), \( \Lambda^{(j:i)}_{s1}(\Omega) \), may be defined as:

\[ \Lambda^{(j:i)}_{s1}(\Omega) = \frac{Y^{(j:i)}(\Omega)}{X^{(i)}(\Omega)} = H^{(j:i)}_1(\Omega) + \frac{3X^2}{4} H^{(j:i)}_3(\Omega, \Omega, -\Omega) + \frac{5X^4}{8} H^{(j:i)}_5(\Omega, \Omega, -\Omega, -\Omega) + \ldots \] (8)

The \( s \) subscript is to emphasise that this composite FRF is for a nonlinear system under sinusoidal forcing as opposed to other forcing, such as random in which case an alternative composite FRF can be defined [3]. By examining the response at other harmonics it is possible to obtain the higher-order composite FRFs. e.g.

\[ \Lambda^{(j:i)}_{s2}(\Omega, \Omega, \Omega) = \frac{2Y^{(j:i)}(2\Omega)}{X^{(i)}(\Omega)^2} = H^{(j:i)}_2(\Omega, \Omega) + X^2H^{(j:i)}_4(\Omega, \Omega, -\Omega) + \frac{15X^4}{16} H^{(j:i)}_6(\Omega, \Omega, -\Omega, -\Omega) + \ldots \] (9)

and

\[ \Lambda^{(j:i)}_{s3}(\Omega, \Omega, \Omega, \Omega, \Omega) = \frac{4Y^{(j:i)}(3\Omega)}{X^{(i)}(\Omega)^3} = H^{(j:i)}_3(\Omega, \Omega, \Omega) + \frac{5X^2}{4} H^{(j:i)}_5(\Omega, \Omega, \Omega, -\Omega) + \frac{21X^4}{16} H^{(j:i)}_7(\Omega, \Omega, \Omega, -\Omega, -\Omega) + \ldots \] (10)

3.2. Conditions for Reciprocity in a Nonlinear System

In order for reciprocity to hold between two points \( i \) and \( j \) in a nonlinear system it is necessary that the response at point \( j \) to an input at point \( i \) be identical to the response at point \( i \) to the same input applied at point \( j \), i.e.

\[ y^{(j:i)}(t) = y^{(i:j)}(t) \] (11)

Examination of equations 6 to 10 shows that, for this to be the case, it is necessary that each of the composite FRFs must display reciprocity, i.e.

\[ \Lambda^{(j:i)}_{11}(\Omega) = \Lambda^{(i:j)}_{11}(\Omega) \; ; \; \Lambda^{(j:i)}_{12}(\Omega, \Omega) = \Lambda^{(i:j)}_{12}(\Omega, \Omega) \; ; \; \Lambda^{(j:i)}_{13}(\Omega, \Omega, \Omega) = \Lambda^{(i:j)}_{13}(\Omega, \Omega, \Omega) \ldots \] (12)

For this to be the case for all levels of forcing, \( X \), (validity of the Volterra series permitting), it will be necessary that all of the HFRFs (or at least their diagonal terms in the case of sinusoidal forcing) display symmetry, i.e.

\[ H^{(j:i)}_n(\Omega, \Omega, \ldots, -\Omega, -\Omega, -\Omega) = H^{(i:j)}_n(\Omega, \Omega, \ldots, -\Omega, -\Omega, -\Omega) \] (13)

The form of these HFRFs for a particular multi-degree-of-freedom system will now be considered.
3.3. HFRF Calculation via Harmonic Probing

In order to determine the analytical form of the HFRFs, the method of harmonic probing was introduced by Bedrosian and Rice in [11] specifically for systems with continuous-time equations of motion. The procedure essentially consists of setting the input to be one or more pure harmonics (a single harmonic is used to calculate the linear FRF expression and multiple harmonics are used to calculate the higher-order FRFs), calculating the relevant terms of the Volterra response to this harmonic input then substituting the input and response expressions into the equations of motion and equating terms at frequencies of interest. This is best illustrated by considering a particular system.

A 2 degree-of-freedom system with the possibility of quadratic and cubic stiffness and damping elements at each location is shown in Figure 1. The order of the nonlinear elements is given by the subscript to the left of the $c$ or $k$ and the connectivity of the elements are indicated by the subscripts to the right of the $c$ or $k$ (where 11, 22 and 12 indicate a grounded element connected to mass $m_1$, a grounded element connected to mass $m_2$ and an internal element between masses $m_1$ and $m_2$ respectively).

This system has the following equations of motion:

$$
m_1\ddot{y}^{(1)}(t) + \sum_{p=1}^{3} \left\{ p \ c_{11}\dot{y}^{(1)}(t)^p + p \ c_{12} (\dot{y}^{(1)}(t) - \dot{y}^{(2)}(t))^p \right\} = x^{(1)}(t)
$$

$$
m_2\ddot{y}^{(2)}(t) + \sum_{p=1}^{3} \left\{ p \ c_{22}\dot{y}^{(2)}(t)^p - p \ c_{12} (\dot{y}^{(1)}(t) - \dot{y}^{(2)}(t))^p \right\} = x^{(2)}(t)
$$

(14)

In order to calculate an expression for the linear HFRF matrix, a periodic excitation composed of a single harmonic is applied at mass 1 (the subscript $p_1$ denotes that this is the first probing input):

$$x^{(1)}_{p_1}(t) = e^{i\Omega t}$$

(15)

Substituting this expression into equation (2) and forming the total response as in equation (1) gives an expression for the responses at mass $j$ where $j = 1, 2$ due to the input at mass 1:

$$y^{(j:1)}(t) = H^{(j:1)}(\Omega)e^{i\Omega t} + H^{(j:1)}(\Omega, \Omega)e^{2i\Omega t} + H^{(j:1)}(\Omega, \Omega, \Omega)e^{3i\Omega t} + \ldots$$

(16)

It transpires that only the first term in the above expression is required for calculating the $H_1$ matrix and therefore the probing expression for the output responses will be:

$$y^{(j:1)}_{p_1}(t) = H^{(j:1)}(\Omega)e^{i\Omega t}$$

(17)
Substituting the expressions for the first probing input and first probing responses of equations (15) and (17) respectively into the equations of motion (14) and equation coefficients of $e^{i\Omega t}$ leads to:

$$\begin{bmatrix} H_1^{(1:1)}(\Omega) \\ H_1^{(2:1)}(\Omega) \end{bmatrix} = \begin{bmatrix} -\Omega^2 m_1 + (i\Omega)(1c_{11} + 1c_{12}) + (k_{11} + k_{12}) & -(i\Omega)1c_{12} - k_{12} \\ -(i\Omega)1c_{12} + k_{12} & -\Omega^2 m_2 + (i\Omega)(1c_{12} + 1c_{22}) + (k_{12} + k_{22}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (18)

Repeating this procedure but applying the single harmonic at mass 2 and combining with the above result leads to the familiar expression for the $H_1$ matrix for a linear 2 degree-of-freedom system:

$$\begin{bmatrix} H_1^{(1:1)}(\Omega) & H_1^{(1:2)}(\Omega) \\ H_1^{(2:1)}(\Omega) & H_1^{(2:2)}(\Omega) \end{bmatrix} = \begin{bmatrix} -\Omega^2 m_1 + (i\Omega)(1c_{11} + 1c_{12}) + (k_{11} + k_{12}) & -(i\Omega)1c_{12} - k_{12} \\ -(i\Omega)1c_{12} + k_{12} & -\Omega^2 m_2 + (i\Omega)(1c_{12} + 1c_{22}) + (k_{12} + k_{22}) \end{bmatrix}^{-1}$$ (19)

Reciprocity up to this order is confirmed by the fact that $H_1^{(1:2)}(\Omega) = H_1^{(2:1)}(\Omega)$ for all $\Omega$.

In order to obtain an expression for the $H_2$ matrix, the input probing expression takes the form of a two harmonic input, applied first at mass 1 and then at mass 2:

$$x_p^{(i)}(t) = e^{i\Omega_1 t} + e^{i\Omega_2 t}$$ (20)

and the resulting probing response expressions are of the form

$$y_p^{(j:i)}(t) = H_2^{(j:i)}(\Omega) e^{i\Omega_1 t} + H_2^{(j:i)}(\Omega) e^{i\Omega_2 t} + 2 H_2^{(j:i)}(\Omega_1, \Omega_2) e^{i(\Omega_1 + \Omega_2) t}$$ (21)

Substituting the above two probing expressions into the equations of motion and equating coefficients of $e^{i(\Omega_1 + \Omega_2) t}$ leads to the following expression for the $H_2$ matrix:

$$\begin{bmatrix} H_2^{(1:1)}(\Omega_1, \Omega_2) & H_2^{(1:2)}(\Omega_1, \Omega_2) \\ H_2^{(2:1)}(\Omega_1, \Omega_2) & H_2^{(2:2)}(\Omega_1, \Omega_2) \end{bmatrix} = \begin{bmatrix} H_1^{(1:1)}(\Omega_1 + \Omega_2) & H_1^{(1:2)}(\Omega_1 + \Omega_2) \\ H_1^{(2:1)}(\Omega_1 + \Omega_2) & H_1^{(2:2)}(\Omega_1 + \Omega_2) \end{bmatrix} \begin{bmatrix} N_2^{(1:1)} & N_2^{(1:2)} \\ N_2^{(2:1)} & N_2^{(2:2)} \end{bmatrix}$$ (22)

where $N_2$ is the matrix which depends upon the quadratic nonlinearities present in the system under consideration. The rows of $N_2$ are given by:

$$N_2^{(1:i)} = -\{(i\Omega_1)(i\Omega_2)c_{11} + k_{11}\} H_1^{(1:i)}(\Omega_1) H_1^{(1:i)}(\Omega_2)$$

$$-\{(i\Omega_1)(i\Omega_2)c_{12} + k_{12}\} H_1^{(2:i)}(\Omega_1) H_1^{(2:i)}(\Omega_2)$$

$$N_2^{(2:i)} = -\{(i\Omega_1)(i\Omega_2)c_{22} + k_{22}\} H_1^{(2:i)}(\Omega_1) H_1^{(2:i)}(\Omega_2)$$

$$+\{(i\Omega_1)(i\Omega_2)c_{21} + k_{12}\} H_1^{(1:i)}(\Omega_1) H_1^{(1:i)}(\Omega_2)$$

for $i = 1, 2$. Whilst it is not relevant for the current work, it is also possible to calculate the expression for the $H_2$ cross-kernel transform, which describes the interaction due to inputs at multiple points, using an extension to the harmonic probing procedure [12].

A similar procedure to that above may be followed to obtain the $H_3$ matrix but, due to space considerations (each element of the $N_3$ matrix is composed of eight components), it will not be included here.
3.4. Conditions for HFRF Symmetry and therefore Reciprocity

Expansion of equation (22) gives the following expressions for $H_2^{(1:2)}(\Omega_1, \Omega_2)$ and $H_2^{(2:1)}(\Omega_1, \Omega_2)$:

\[
H_2^{(1:2)}(\Omega_1, \Omega_2) = H_1^{(1:1)}(\Omega_1 + \Omega_2) N_2^{(1:2)} + H_1^{(1:2)}(\Omega_1 + \Omega_2) N_2^{(2:2)} \\
H_2^{(2:1)}(\Omega_1, \Omega_2) = H_1^{(2:1)}(\Omega_1 + \Omega_2) N_2^{(1:1)} + H_1^{(2:2)}(\Omega_1 + \Omega_2) N_2^{(2:1)}
\]  

(24)

Examination of the above equations show that each of the $H_2(\Omega_1, \Omega_2)$ expressions are made up of a summation of two complex products of a linear $H_1$ term and a nonlinear $N_2$ term which is some function of the nonlinear parameters and $H_1$ (in an $m$-dof system there will be $m$ complex products for each HFRF expression). The only two terms which are guaranteed to be equal are the $H_1^{(1:2)}(\Omega_1 + \Omega_2)$ and $H_1^{(2:1)}(\Omega_1 + \Omega_2)$ terms, due to linear reciprocity as shown in equation (19). It appears that the only situation which will result in the above two expressions being non-zero. So, the final condition for HFRF symmetry for this 2 dof system is $c_{12} = k_{12} = 0$. The reason for this final condition is the manner in which the quadratic stiffness and damping forces from the elements between two masses are defined. Under the current definition they always act to push the masses apart for positive values of $2c_{12}$ and $2k_{12}$.

Although the construction of the $H_3$ matrix was not included above, it may still be examined to see under what conditions the HFRFs $H_3^{(1:2)}(\Omega_1, \Omega_2, \Omega_3)$ and $H_3^{(2:1)}(\Omega_1, \Omega_2, \Omega_3)$ will be equal. It transpires that this is simply an extension of the conditions for $H_2$: in addition to those conditions, it is also necessary that $c_{111} = c_{221}$ and $k_{111} = k_{221}$, i.e. confirming symmetry of the parameters. An interesting departure is that the two $H_3$ terms will be equal, assuming that the previous conditions are met, even if the cubic nonlinearities between the two masses, $3c_{12}$ and $3k_{12}$, are non-zero.

4. Illustrative Examples

In the previous section it was shown that the HFRFs $H_2^{(1:2)}(\Omega_1, \Omega_2)$ and $H_2^{(2:1)}(\Omega_1, \Omega_2)$ will be equal and $H_3^{(1:2)}(\Omega_1, \Omega_2, \Omega_3)$ and $H_3^{(2:1)}(\Omega_1, \Omega_2, \Omega_3)$ will be equal if the underlying linear system were symmetric and the nonlinear parameters were symmetric as long as the internal nonlinearity were of odd order (i.e. has no quadratic component). This will be shown to be the case by considering a numerical example. Consider the system shown in Figure 1 and let the linear parameters be $m_1 = m_2 = 1$ kg, $c_{111} = c_{12} = c_{22} = 20$ N/(m/s) and $k_{111} = k_{12} = k_{22} = 1 \times 10^4$ N/m. The nonlinear parameters are set to $c_{211} = c_{222} = 500$ N/(m/s)$^2$, $k_{211} = k_{222} = 1 \times 10^7$ N/m$^2$ and $3k_{11} = k_{12} = k_{22} = 5 \times 10^9$ N/m$^3$ thereby satisfying the conditions outlined above.

The system was constructed in Simulink and simulated using 4th-order Runge-Kutta. First, an input was applied at mass $m_1$ given by $x^{(1)}(t) = 10 \sin(100t)$ and the response $y^{(2:1)}(t)$ was plotted then the same input was applied at mass $m_2$ ($x^{(2)}(t) = 10 \sin(100t)$) was applied and the response $y^{(1:2)}(t)$ was plotted. The resulting responses for the first 0.5 seconds are shown in Figure 2 and it may be seen that they are identical, as expected. Figure 3 shows the two responses when a quadratic stiffness of $2k_{12} = 1 \times 10^7$ N/m$^2$ is added to the previous nonlinear system. It may be seen that, as predicted by the calculations in the previous subsection, this addition has resulted in a breakdown in reciprocity with the two responses no longer being the same.

It should be stated that, whilst a Volterra series approach was used to identify which nonlinear systems displayed reciprocity, there is no necessity that the Volterra series representation be
Figure 2. Displacement responses from symmetric nonlinear system with no quadratic nonlinearity between masses 1 and 2.

Figure 3. Displacement responses from same nonlinear system as Figure 2 with addition of quadratic stiffness between masses 1 and 2.

convergent for the forcing levels being applied in the example: this would only be necessary if the representation were being used to construct the responses.

5. Discussion and Conclusions
This paper has attempted to illuminate the subject of reciprocity breakdown in nonlinear systems by adopting a Volterra series approach. It should be emphasized that the conclusions are somewhat limited to low-order nonlinearities and that going higher-order may be computationally prohibitive. To a certain extent, the finding that nonlinear systems which are symmetrical in nature are capable of maintaining reciprocity may seem somewhat facile with the possible exception of the demonstration that even-ordered nonlinear elements between masses causes reciprocity breakdown (irrespective of symmetry). That said, it is the authors’ opinion that a functional series approach is capable of providing valuable insight into the behaviour of nonlinear systems which may be exploited in such areas as system identification: this will be the subject of forthcoming work.

6. References
[1] Worden K and Tomlinson G R 2001 Nonlinearity in Structural Dynamics: Detection, Identification and Modelling (Bristol: Institute of Physics Publishing)
[2] Farrar C R and Worden K 2012 Structural Health Monitoring: A Machine Learning Perspective (John Wiley & Sons)
[3] Worden K and Manson G 1998 Random vibrations of a Duffing oscillator using the Volterra series J. Sound Vibrat. 217 781–89
[4] Worden K and Manson G 1999 Random vibrations of a a multi-degree-of-freedom nonlinear system using the Volterra series J. Sound Vibrat. 226 397–405
[5] Worden K and Manson G 2005 A Volterra series approximation to the coherence of the Duffing oscillator J. Sound Vibrat. 286 529–47
[6] Palm G and Poggio T 1977 The Volterra representation and the Wiener expansion: validity and pitfalls SIAM J. App. Math. 33 (2) 195–216
[7] Rugh W J 1981 Nonlinear System Theory - The Volterra/Wiener Approach (John Hopkins University Press)
[8] Barrett J F 1965 The use of Volterra series to find region of stability of a non-linear differential equation Int. J. Control 1 (3) 209–16
[9] Volterra V 1959 Theory of Functionals and Integral equations (New York: Dover Publications)
[10] Schetzen M 1980 The Volterra and Wiener Theories of Nonlinear Systems (New York: John Wiley Interscience Publication)
[11] Bedrosian E and Rice S O 1971 The output properties of Volterra systems driven by harmonic and Gaussian inputs P. IEEE 59 1688–707
[12] Worden K, Manson G and Tomlinson G R 1997 A harmonic probing algorithm for the multi-input Volterra series J. Sound Vibrat. 201 67–84