Long Range Scattering for the Modified Schrödinger Map in two space dimensions

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Abstract

We study the asymptotic behaviour in time of solutions and the theory of scattering for the modified Schrödinger map in two space dimensions. We solve the Cauchy problem with large finite initial time, up to infinity in time, and we determine the asymptotic behaviour in time of the solutions thereby obtained. As a by product, we obtain global existence for small data in $H^k \cap F H^k$ with $k > 1$. We also solve the Cauchy problem with infinite initial time, namely we construct solutions defined in a neighborhood of infinity in time, with prescribed asymptotic behaviour of the previous type.

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1 Introduction

This paper is devoted to the study of the asymptotic behaviour in time of solutions and to the theory of scattering for the modified Schrödinger map (MSM) system in space dimension 2. In general space dimension $n$, that system takes the form

$$i \partial_t u = -(1/2)\Delta_A u + g(u)u.$$  

(1.1)

Here $u$ is a $\mathbb{C}^n$ vector valued function defined in space time $\mathbb{R}^{n+1}$, $\Delta_A = \nabla^2_A = (\nabla - iA)^2$ is the covariant Laplacian associated with the vector potential $A$ defined by

$$A_j = 4\Delta^{-1} \partial_k \text{Im} u_k u_j,$$  

(1.2)

g($u$) is the hermitian matrix defined by

$$g_{jk}(u) = -2i \text{Im} u_k u_j - A_0 \delta_{jk},$$  

(1.3)

$$A_0 = 2\Delta^{-1} \partial_j \partial_k \text{Re} u_k u_j - |u|^2$$  

(1.4)

and summation over repeated indices is understood. The normalization and sign conventions in (1.1)-(1.4) differ from those currently used by a few signs and factors of 2 in order to allow for an easier comparison with the Maxwell-Schrödinger (MS) system. The MSM system is formally derived from the more primitive system [19]

$$i \partial_t z = -(1/2)\nabla_j \partial_j z$$  

(1.5)

where $z$ is a complex function defined in space time $\mathbb{R}^{n+1}$ and

$$\nabla_j = \partial_j - 2 \left(1 + |z|^2\right)^{-1} \overline{z} (\partial_j z).$$  

(1.6)

The system (1.5) itself is obtained through a stereographic projection from a more geometrically defined Schrödinger map (SM) system where the unknown function takes values in the unit sphere $S^2$. The latter system appears as the Landau-Lifschitz model of a ferromagnet [16]. When deriving the MSM system from the SM one, in addition to (1.1)-(1.4), one obtains a constraint satisfied by $u$. That constraint is easily seen to be formally preserved by the evolution (1.1). The MSM system can be studied with or without that constraint. In the present paper we consider it without the constraint, which makes it more general and significantly different as regards scattering (see below). It is an important question to make the correspondence between the SM and MSM systems rigorous, in order to transfer results from one
system to the other. The equivalence of the SM system to the MSM one (with the constraint) has recently been proven under mild regularity assumptions [18].

A large amount of work has been devoted to the Cauchy problem both for the system (1.5) and for the MSM system (1.1) in various space dimensions. We refer to [1]-[4], [8]-[11], [13], [14], [17]-[20], [23] and the literature therein quoted. The best available result so far for the MSM system without the constraint in space dimension 2 is local wellposedness in $H^s$ for $s > 3/4$ [11].

In the present paper we shall study the asymptotic behaviour in time of solutions and the theory of scattering for the MSM system without the constraint in space dimension 2, where it is borderline long range (see below).

Here we regard scattering theory as a method to classify the possible solutions of (1.1) by their asymptotic behaviour. That point of view leads to the following two problems.

**Problem 1.** One gives oneself a set $U_a$ of presumed asymptotic behaviours $u_a$ for the system (1.1), parametrized by some data $u_+$. For each $u_a \in U_a$, one tries to construct a solution of the system (1.1) such that $u(t) - u_a(t)$ tends to zero as $t \to +\infty$ in a suitable sense, more precisely in suitable norms. The same problem can be considered for $t \to -\infty$. We restrict our attention to the case of $t \to +\infty$. The previous problem decomposes into two steps. The first step is to construct the solution $u$ in a neighborhood of $t = +\infty$, namely in an interval $(T, \infty)$ for $T$ sufficiently large. This is the local Cauchy problem at infinity in time. The second and rather independent step consists in extending the solution to all times and reduces therefore to the global Cauchy problem at finite times. In this paper we consider only the first step and leave aside the second one.

If the previous problem can be solved for any $u_a \in U_a$, the map $u_a \to u$ thereby defined is essentially the wave operator $\Omega_+$ for positive time associated with $U_a$.

**Problem 2.** This is the converse to Problem 1. Given a generic solution $u$ of the system (1.1), one tries to find an asymptotic motion $u_a \in U_a$ such that $u(t) - u_a(t)$ tends to zero as $t \to \infty$ in a suitable sense (in suitable norms). If that problem and the same one for $t \to -\infty$ can be solved for all $u$ (in a suitable functional framework), one says that asymptotic completeness holds with respect to the set $U_a$. That property requires in particular that all possible asymptotic behaviours of the solutions of (1.1) have been identified and included in $U_a$, and is completely out
of reach in the present case. In this paper, we restrict ourselves to the construction of a set of solutions of the Cauchy problem with finite initial time, defined up to infinity in time and behaving asymptotically as functions in the set $\mathcal{U}_a$ for which we can solve the first problem. That set includes small global solutions of (1.1).

In the present case, a natural candidate for $\mathcal{U}_a$ is the set of solutions of the free Schrödinger equation, namely

$$u_a(t) = U(t) \ u_+ = \exp \left( i(t/2)\Delta \right) u_+ .$$

(1.7)

Cases where such a choice is adequate are referred to as short range cases. This requires the nonlinear interaction to decrease sufficiently fast at infinity in space and/or time. This occurs for the MSM system in space dimension $n \geq 3$ and, if the constraint is included, also in space dimension $n = 2$. This also occurs for the SM system in space dimension $n \geq 2$ [23]. The long range case is the complementary one where that set is inadequate and has to be replaced by a set of modified asymptotic behaviours. This occurs for the MSM system without the constraint in space dimension 2, the case which we treat in this paper. The modification includes the introduction of a phase in the asymptotic Schrödinger function. In that respect, the MSM system without the constraint in space dimension 2 is borderline long range and similar to the Maxwell-Schrödinger (MS) and Wave-Schrödinger (WS) systems in dimension 3, and to the Hartree equation with $|x|^{-1}$ potential in dimension $n \geq 2$.

The MSM system is also similar to the MS system in the sense that it consists of a Schrödinger equation in a magnetic field, with vector potential $A$. In contrast with the MS system however, the magnetic field does not propagate, and the vector potential is defined locally in time in terms of the Schrödinger function. This makes the problem simpler and makes it possible to apply the methods previously used for the MS system in dimension 3 to the MSM system in dimension 2.

The theory of scattering for the MS system in dimension 3 has been studied by several authors [5] [6] [7] [21] [26], following work on the WS system and on the Hartree equation. We refer to [7] for additional information and references on that matter. In this paper we study the MSM system in dimension 2 by the methods used in [5], which seem to be the most readily applicable to that system. The main results are as follows. We first solve the Cauchy problem with large finite initial time, up to infinity in time and we determine the asymptotic behaviour in time of the solutions thereby obtained. This represents our contribution to Problem 2 mentioned above,
and allows us to identify a set $\mathcal{U}_a$ of possible asymptotic behaviours. As a by product we obtain global existence for small data. We then solve the Cauchy problem with infinite initial time, namely we construct solutions defined in a neighborhood of infinity in time, with prescribed asymptotic behaviour of the previous type. The method consists in expressing the Schrödinger function $u$ in terms of a complex amplitude $v$ and a real phase $\varphi$, replacing the original system (1.1) by an auxiliary system for the pair $(v, \varphi)$, treating the corresponding problems for the latter system, and reconstructing the solutions $u$ of the original system from the solutions $(v, \varphi)$ of the auxiliary one. The detailed construction is too complicated to allow for a more precise description at this stage and will be described in heuristic terms in Section 2 below. At the end of that section, we shall also give a simplified version of the results as Propositions 2.1 and 2.2. We conclude this introduction by giving a brief outline of the contents of this paper. A more detailed description will be given at the end of Section 2. In Section 3, we collect some notation and preliminary estimates. In Section 4 we study the Cauchy problem at finite initial time both for the original system (1.1) and for the auxiliary system. In Section 5, we study the Cauchy problem with infinite initial time for the original system (1.1) and the corresponding problem for the auxiliary system.

2 Heuristics and formal computations

In this section, we perform in a formal way the algebraic computations needed to study the Cauchy problem for the MSM system (1.1) in a neighborhood of infinity in time, both for finite and infinite initial time, and we sketch the method used to solve that problem. The system (1.1) in this form is not well suited for that purpose and we perform a number of transformations leading to an auxiliary system for which that problem can be handled. The unitary group

$$U(t) = \exp(i(t/2)\Delta)$$  \hspace{1cm} (2.1)

which solves the free Schrödinger equation can be written as

$$U(t) = M(t)\ D(t)\ F\ M(t)$$  \hspace{1cm} (2.2)

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp\left(\frac{ix^2}{2t}\right),$$  \hspace{1cm} (2.3)
\( F \) is the Fourier transform and \( D(t) \) in the dilation operator defined by

\[
D(t) = (it)^{-1} D_0(t) \quad , \quad (D_0(t)f)(x) = f(x/t) .
\]

We first change variables from \( u \) to its pseudoconformal inverse \( u_c \) defined by

\[
u(t) = M(t) \ D(t) \ u_c(1/t)
\]

or equivalently

\[
\tilde{u}(t) = F \tilde{u}_c(1/t)
\]

where for any function \( f \) of space time, we define

\[
\tilde{f}(t, \cdot) = U(-t)f(t, \cdot).
\]

Correspondingly we define \( B \) by

\[
A(t) = -t^{-1}D_0(t) \ B(1/t).
\]

Substituting (2.5) (2.8) into (1.1) yields the following equation for \( u_c \):

\[
i\partial_t u_c = -\frac{1}{2}\Delta B u_c + (\bar{B}(u_c) + g(u_c)) u_c
\]

where \( B = B(u_c) \) and we have defined

\[
B(v) = B(v, v) \quad , \quad \bar{B}(v) = \bar{B}(v, v) \quad , \quad g(v) = g(v, v) ,
\]

\[
B_j(v_1, v_2) = 2\Delta^{-1} \partial_k \ \text{Im} \ (\overline{v}_{1k}v_{2j} + \overline{v}_{2k}v_{1j}) ,
\]

\[
g_{jk}(v_1, v_2) = -i \ \text{Im} \ (\overline{v}_{1k}v_{2j} + \overline{v}_{2k}v_{1j}) - B_0(v_1, v_2)\delta_{jk} ,
\]

\[
B_0(v_1, v_2) = 2\Delta^{-1} \partial_j \partial_k \ \text{Re} \ (\overline{v}_{1k}v_{2j}) - \text{Re} \ (\overline{v}_1 \cdot v_2) ,
\]

\[
\bar{B}(v_1, v_2) = 2t^{-1}x_j\Delta^{-1} \partial_k \ \text{Im} \ (\overline{v}_{1k}v_{2j} + \overline{v}_{2k}v_{1j})
\]

and more generally, for any \( IR^2 \) vector valued function of space time

\[
\tilde{f}(x, t) = t^{-1}x \cdot f(x, t).
\]

We next parametrize \( u_c \) in terms of a complex amplitude \( v \) and a real phase \( \varphi \) by

\[
u_c = v \exp(-i\varphi).
\]
Note that $u_c$ and $v$ are ($\mathbb{R}^2$) vector valued and that the phase is the same for all components, so that in particular
\begin{equation}
B(u_c) = B(v) \quad , \quad \tilde{B}(u_c) = \tilde{B}(v) \quad , \quad g(u_c) = g(v) .
\end{equation}
Substituting (2.16) into (2.9) yields the equation
\begin{equation}
i\partial_t v = -(1/2)\Delta_K v + \left(\tilde{B}(v) - \partial_t \varphi + g(v)\right) v
\end{equation}
where
\begin{equation}
K = s + B \quad , \quad s = \nabla \varphi
\end{equation}
and in the same way as before
\begin{equation}
\Delta_K = \nabla_K^2 \quad , \quad \nabla_K = \nabla - iK .
\end{equation}
We have now only one equation for two functions ($v, \varphi$). We then arbitrarily impose a second equation, namely an equation for the phase $\varphi$, thereby splitting (2.18) into a system of two equations, the other one of which is an equation for $v$. There is a large amount of freedom in the choice of the equation for the phase. The role of the phase is to cancel the long range term $\tilde{B}(v)$ in (2.18). However since that term has a relatively low regularity, it is convenient to split it into a short range and a long range part. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(\xi) = 1$ for $|\xi| \leq 1$, $\chi(\xi) = 0$ for $|\xi| \geq 2$. We define
\begin{equation}
\tilde{B}_L = F^*\chi(\cdot \cdot t^{1/2})F \tilde{B} \quad , \quad \tilde{B}_S = F^* \left((1 - \chi(\cdot \cdot t^{-1/2}))\right) F \tilde{B} .
\end{equation}
As the equation for $\varphi$, we take
\begin{equation}
\partial_t \varphi = \tilde{B}_L(v)
\end{equation}
so that the equation for $v$ becomes
\begin{equation}
i\partial_t v = H v
\end{equation}
with
\begin{equation}
H = -(1/2)\Delta_K + \tilde{B}_S(v) + g(v) .
\end{equation}
The system (2.21) (2.22) is the final form of the auxiliary system that replaces the original system (1.1). For technical reasons, it will be useful to consider also the partly linearized system for a new variable $v'$
\begin{equation}
i\partial_t v' = H v'
\end{equation}
where $H$ is still associated with $(v, \varphi)$ according to (2.23). The Cauchy problem at finite initial time $t_0 \in [1, \infty)$ for the original system (1.1) is now replaced by the Cauchy problem for the auxiliary system (2.21) (2.22) at finite initial time $\tau_0 = t_0^{-1} \in (0, 1]$. We shall solve that problem in two steps. We shall first solve the linearized system (2.24) for $v'$ with given $v$, thereby defining a map $\Gamma : v \to v'$. We shall then prove that the map $\Gamma$ has a fixed point by a contraction method. With the solution of the system (2.21) (2.22) available, the original system (1.1) can be solved by substituting the solution $(v, \varphi)$ into the formulas (2.6) (2.16).

In a similar way, the Cauchy problem at infinite initial time for the original system (1.1) is replaced by the Cauchy problem at $t = 0$ for the auxiliary system (2.21) (2.22). Since that system is singular at $t = 0$, that problem cannot be treated directly and we follow instead an indirect procedure. We choose a set of asymptotic functions $(v_a, \varphi_a)$ which are expected to be suitable asymptotic forms of $(v, \varphi)$ at $t = 0$ and we try to construct solutions of the auxiliary system (2.21) (2.22) that are asymptotic to $(v_a, \varphi_a)$ as $t \to 0$. The set $(v_a, \varphi_a)$ will be taken of the following form. For a given $v_a$, which needs not be specified at this stage, we define $\varphi_a$ by

$$
\partial_t \varphi_a = \bar{B}_L(v_a) \quad (2.25)
$$

with $\varphi_a(1) = 0$ and we define

$$
s_a = \nabla \varphi_a \quad , \quad B_a = B(v_a) \quad , \quad K_a = s_a + B_a \quad , \quad \bar{B}_a = \bar{B}(v_a) . \quad (2.26)
$$

We next define the difference variables

$$
(w, \psi) = (v - v_a, \varphi - \varphi_a) \quad , \quad (2.27)
$$

$$
G = B(v) - B_a (= B(w, 2v_a + w)) \quad , \quad (2.28)
$$

$$
\sigma = \nabla \psi \quad , \quad L = \sigma + G \quad (2.29)
$$

so that $K = K_a + L$. Substituting the definitions (2.27)-(2.29) into the system (2.21) (2.22) yields the new system for $(w, \psi)$

$$
\partial_t \psi = \bar{G}_L \left( = \bar{B}_L(w, 2v_a + w) \right) \quad (2.30)
$$

$$
i \partial_t w = H w + H_1 v_a - R \quad (2.31)
$$

where

$$
H_1 = iL \cdot \nabla_{K_a} + (i/2)\nabla \cdot \sigma + (1/2)L^2 + \bar{G}_S + g(w, 2v_a + w) \quad , \quad (2.32)
$$
\[ R = i\partial_t v_a + (1/2)\Delta_{K_a} v_a - \left(\mathring{B}_a S + g(v_a)\right) v_a. \quad (2.33) \]

Again for technical reasons, it will be useful to consider also the partly linearized system for a new variable \( w' \)
\[ i\partial_t w' = H w' + H_1 v_a - R \quad (2.34) \]
where \( H \) and \( H_1 \) are still associated with \((v, \varphi)\) (or \((w, \psi)\)).

The remainder \( R \) expresses the failure of \((v_a, \varphi_a)\) to satisfy the system \((2.21)\) \((2.22)\) and will have to tend to zero at a suitable rate in order to make it possible to solve that system.

The construction of solutions \((v, \varphi)\) of the system \((2.21)\) \((2.22)\) with prescribed asymptotic behaviour at \( t = 0 \) will be performed in two steps. The first step consists in solving the system \((2.30)\) \((2.31)\) with \((w, \psi)\) tending to zero as \( t \to 0 \) under assumptions on \((v_a, \varphi_a)\) of a general nature, the most important of which being decay assumptions on \( R \) as \( t \to 0 \). This is done by first solving the linearized system \((2.34)\) for \( w' \), for given \((w, \psi)\) tending to zero as \( t \to 0 \), with \( w' \) tending to zero as \( t \to 0 \). For that purpose, one first solves the Cauchy problem for the system \((2.34)\) with initial condition \( w'(t_0) = 0 \) for some \( t_0 > 0 \) and one takes the limit of the solution thereby obtained as \( t_0 \to 0 \). This procedure defines a map \( \Gamma : w \to w' \). One then proves by a contraction method that the map \( \Gamma \) has a fixed point in a suitable function space.

The second step of the method consists in choosing the asymptotic function \( v_a \) so as to ensure the assumptions needed for the first step, and in particular the time decay of \( R \). In the present problem, this will be simply achieved by taking \( v_a = U(t)v_+ \) for a suitably regular \( v_+ \). Substituting the previous results into the formulas \((2.6)\) \((2.16)\) will yield the corresponding results for the original system \((1.1)\). In particular the solution \( u \) thereby obtained will behave asymptotically as \( u_a \) defined by
\[ \tilde{u}_a(t) = F\tilde{u}_{ca}(1/t) \quad (2.35) \]
\[ u_{ca} = v_a \exp(-i\varphi_a) \quad (2.36) \]
in analogy with \((2.6)\) \((2.16)\).

We now give a heuristic preview of the main results of this paper, stripped from most technicalities. They will be stated in full mathematical detail in Propositions 4.2-4.5 as regards the Cauchy problem with finite initial time and in Propositions 5.6, 5.7 as regards the Cauchy problem with initial time \( t = 0 \) for \((v, \varphi)\) and \( t = \infty \).
for $u$. In order to state the results, we shall use the spaces $V^k$, $\Sigma^k$, $H^k_\geq$ and $H^\infty_\geq$ defined by (3.4) (3.9) (3.1) (3.2) below. In all those results we assume that $1 < k < 2$. The lower bound $k > 1$ plays an essential rôle, while the upper bound $k < 2$ is imposed only for convenience. It could be dispensed with at the expense of minor modifications of the proofs. The results for finite initial time can be summarized as follows.

**Proposition 2.1.** Let $1 < k < 2$.

1. Let $v_0 \in V^k$. For $\tau_0 > 0$, $\tau_0$ sufficiently small, there exists a unique solution $(v, \varphi)$ of the system (2.21) (2.22) such that $(v, \varphi)(t_0) = (v_0, 0)$, $v \in (C \cap L^\infty)(I, V^k)$, $\varphi \in C(I, H^\infty_\geq)$ where $I = (0, \tau_0]$ and $(v, \varphi)$ is estimated in those spaces. Furthermore there exists $v_+ \in V^k$ such that $v(t)$ tends to $v_+$ when $t \to 0$, and $(v, \varphi)$ behaves asymptotically as $(v_a, \varphi_a)$ when $t \to 0$, with $v_a = U(t)v_+$ and $\varphi_a$ a solution of (2.25).

2. Let $\bar{u}_0 \in FV^k$. For $t_0$ sufficiently large, there exists a unique solution $u$ of the system (1.1) such that $u(t_0) = U(t_0)\bar{u}_0$, $\bar{u} \in C(I, FV^k)$ where $I = [t_0, \infty)$, and $u$ is estimated in that space. Furthermore $u$ behaves as $u_a$ when $t \to \infty$, with $u_a$ defined by (2.35) (2.36) and $(v_a, \varphi_a)$ as in Part (1).

3. Parts (1) and (2) hold with $V^k$ replaced everywhere by $\Sigma^k$. Furthermore for $u_0 \in \Sigma^k$, $u_0$ sufficiently small, there exists a unique solution $u \in C(I, \Sigma^k)$ of the system (1.1) with $u(0) = u_0$.

We next summarize the results for zero or infinite initial time.

**Proposition 2.2.** Let $1 < k < 2$. Let $v_+ \in V^{k+1}$, let $v_a = U(t)v_+$, let $\varphi_a$ be defined by (2.25) with $\varphi_a(1) = 0$ and let $u_a$ be defined by (2.35) (2.36).

1. There exists $\tau > 0$ and there exists a unique solution $(v, \varphi)$ of the system (2.21) (2.22) such that $v \in (C \cap L^\infty)(I, V^k)$ and $\varphi \in C(I, H^{k+2}_\geq)$, where $I = (0, \tau]$, and such that $(v, \varphi)$ behaves asymptotically as $(v_a, \varphi_a)$ when $t \to 0$, in the sense that the difference $(v - v_a, \varphi - \varphi_a)$ tends to zero in suitable norms and at suitable rates when $t \to 0$.

2. There exists $T > 0$ and there exists a unique solution $u$ of the system (1.1) such that $\bar{u} \in C(I, FV^k)$, where $I = [T, \infty)$, and such that $u$ behaves asymptotically as $u_a$ when $t \to \infty$, in the sense that the difference $u - u_a$ tends to zero in suitable norms and at suitable rates when $t \to \infty$. 

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Remark 2.1. There is a loss of one derivative from the asymptotic data \( v_+ \) to the solution \( v \) in Proposition 2.2, part (1), so that Propositions 2.1 and 2.2 cannot be considered as the converse of each other. Furthermore the convergence properties of \( (v, \varphi) \) to its asymptotic form \( (v_a, \varphi_a) \) required in Proposition 2.2, part (1) to solve the system \((2.21) \) \((2.22) \) with initial time zero are stronger than those obtained for the solutions constructed in Proposition 2.1, part (1). A similar remark applies to the pair \((u, u_a) \).

We now describe the contents of the technical parts of this paper, namely Sections 3-5. In Section 3, we introduce some notation, we define the relevant function spaces and we collect a number of preliminary estimates. In Section 4 we study the Cauchy problem for finite initial time. We solve that problem for the auxiliary system \((2.21) \) \((2.22) \) (Proposition 4.2) and for the original system \((1.1) \) (Proposition 4.4). We prove in particular the existence of small global solutions for the latter. We then analyse the asymptotic behaviour of the solutions thereby obtained for the auxiliary system (Proposition 4.3) and for the original system (Proposition 4.5). In Section 5 we study the Cauchy problem with initial time zero for \((v, \varphi) \) or infinity for \( u \). We first give uniqueness results for \((v, \varphi) \) (Proposition 5.1) and for \( u \) (Proposition 5.2). We then solve the Cauchy problem with prescribed asymptotic behaviour \((v_a, \varphi_a) \) as \( t \to 0 \) for \((v, \varphi) \) (Propositions 5.3-6) and with prescribed asymptotic behaviour \( u_a \) as \( t \to \infty \) for \( u \) (Proposition 5.7).

3 Notation and preliminary estimates

In this section we introduce some notation and we collect a number of estimates which will be used throughout this paper. We denote by \( || \ ||_r \) the norm in \( L^r \equiv L^r(\mathbb{R}^n) \), to be used mostly in \( \mathbb{R}^2 \), and by \( <,> \) the scalar product in \( L^2 \). We shall use the Sobolev spaces \( H^k \equiv H^k(\mathbb{R}^n) \) defined for \( k \in \mathbb{R} \) by

\[
H^k = \left\{ u \in S'(\mathbb{R}^n) : || u; H^k || = \| < \omega >^k u \|_2 < \infty \right\},
\]

where \( < \cdot , > = (1 + | \cdot |^2)^{1/2} \), and \( \omega = (-\Delta)^{1/2} \). Besides the standard Sobolev spaces \( H^k \), we will use the associated homogeneous spaces \( \dot{H}^k \) with norm \( || u; \dot{H}^k || = \| \omega^k u \|_2 \). If \( 0 < k < n/2 \) it is understood that \( \dot{H}^k \subset L^r \) with \( k = n/2 - n/r \). In
addition we shall use the notation
\[ H^k \geq = \bigcap_{0 < \ell \leq k} \dot{H}^\ell \] (3.1)
and
\[ H^\infty = \bigcap_{\ell > 0} \dot{H}^\ell . \] (3.2)

For any Banach space \( X \subset S'(\mathbb{R}^n) \), we use the notation
\[ FX = \{ v \in S'(\mathbb{R}^n) : F^*v \in X \} . \]

For any \( k \geq 0, \ell \geq 0 \), we define the space
\[ H^{k,\ell} = \left\{ v \in S'(\mathbb{R}^n) : \| v \|_{H^{k,\ell}} = \| \dot{x}^k < \omega \dot{x} < \omega >^k u \|_2 < \infty \right\} . \] (3.3)

In particular \( H^k = H^{k,0} \) and \( FH^k = H^{0,k} \). For \( 1 < k < 2 \) we shall make extensive use of the space \( V^k \) defined by
\[ V^k = \left\{ v \in S'(\mathbb{R}^n) : \| v \|_{V^k} = \| \dot{x}^k < \omega \dot{x} < \omega >^k x v \|_2 < \infty \right\} , \] (3.4)

where for real numbers \( a \) and \( b \) we use the notation \( a \vee b = \text{Max}(a, b) \) and \( a \wedge b = \text{Min}(a, b) \). Clearly \( V^k = H^k \cap H^{k-1,1} \). More generally, for \( 0 \leq \rho \leq 1 \) we define the spaces
\[ V^{k,\rho} = H^{k-1+\rho} \cap H^{k-1,\rho} \] (3.5)
so that
\[ V^{k,0} = H^{k-1} \quad , \quad V^{k,1} = V^k . \]
The spaces \( V^{k,\rho} \) interpolate between \( H^{k-1} \) and \( V^k \).

From the commutation relation
\[ xU(-t) = U(-t)(x + it\nabla) \] (3.6)
it follows that the space \( V^k \) is invariant under the operator \( U(t) \) and that
\[ \| U(t)v; V^k \| \leq \| U(t)v; H^k \| \leq |t| \| v; H^k \| \] (3.7)
so that
\[ \| U(t)v; V^k \| \leq (1 + |t|) \| v; V^k \| . \] (3.8)

However the space \( FV^k \) is not invariant under \( U(t) \). For that and other reasons we shall also use the smaller space
\[ \Sigma^k = H^{k,0} \cap H^{0,k} \] (3.9)
where $F$ acts an an isometry. From general interpolation theory it follows that 
\[ \Sigma^k \subset V^k \] with 
\[ \| v; V^k \| \leq C \| v; \Sigma^k \| \] (3.10)

Furthermore $\Sigma^k$ is invariant under the evolution operator $U(t)$ and
\[ \| U(t)v; \Sigma^k \| = \| <\omega >^k v \|_2 \lor \| <x + it\nabla >^k v \|_2 \leq C \left( \| v; \Sigma^k \| + |t|^k \| \omega^k v \|_2 \right) \leq C(1 + |t|^k) \| v; \Sigma^k \| . \] (3.11)

For the reader’s convenience, we give a simple direct proof of those two facts and in particular of (3.10) (3.11) in the Appendix.

For any interval $I$ and for any Banach space $X$, we denote by $C(I, X)$ (resp. $C_w(I, X)$) the space of strongly (resp. weakly) continuous functions from $I$ to $X$ and by $L^\infty(I, X)$ (resp. $L^\infty_{loc}(I, X)$) the space of measurable essentially bounded (resp. locally essentially bounded) functions from $I$ to $X$. For $I$ an open interval, we denote by $D'(I, X)$ the space of vector valued distributions from $I$ to $X$. We shall say that an evolution equation has a solution in $I$ with values in $X$ if the equation is satisfied in $D'(I_0, X)$, where $I_0$ is the interior of $I$.

We shall use extensively the following Sobolev inequalities, stated here in $IR^n$, but used only in $IR^2$, and the following Leibnitz and commutator estimates.

**Lemma 3.1.** (1) Let $1 < r \leq \infty$, $1 < r_1, r_2 < \infty$ and $0 \leq j < \ell$. If $r = \infty$, assume in addition that $\ell - j > n/r_2$. Let $\sigma$ satisfy $j/\ell \leq \sigma \leq 1$ and
\[ n/r - j = (1 - \sigma)n/r_1 + \sigma(n/r_2 - \ell) . \]

Then the following estimate holds :
\[ \| \omega^j v \|_r \leq C \| v \|_{r_1}^{1-\sigma} \| \omega^\ell v \|_{r_2}^\sigma . \]

(2) Let $1 < r, r_1, r_3 < \infty$ and
\[ 1/r = 1/r_1 + 1/r_2 = 1/r_3 + 1/r_4 . \]

Then the following estimates hold :
\[ \| \omega^\ell (v_1 v_2) \|_r \leq C \left( \| \omega^\ell v_1 \|_{r_1} \| v_2 \|_{r_2} + \| \omega^\ell v_2 \|_{r_3} \| v_1 \|_{r_4} \right) \]
for $\ell \geq 0$, and
\[ \| [\omega^\ell, v_1] v_2 \|_r \leq C \left( \| \omega^\ell v_1 \|_{r_1} \| v_2 \|_{r_2} + \| \omega^{\ell-1} v_2 \|_{r_3} \| \nabla v_1 \|_{r_4} \right) \]
for $\ell \geq 1$, where $[\cdot, \cdot]$ denotes the commutator.

In particular for $n = 2$ and $0 < \ell < 1$

$$\| \omega^\ell (v_1 v_2) \|_2 \leq C \| \omega^\ell v_1 \|_2 (\| v_2 \|_\infty + \| \nabla v_2 \|_2). \quad (3.12)$$

(3) Let $0 < \ell < 1$. Then the following estimate holds:

$$\| [\omega^\ell, v_1] \nabla v_2 \|_2 \leq C \| F \nabla v_1 \|_1 \| \omega^\ell v_2 \|_2. \quad (3.13)$$

**Proof.** Part (1) follows from the Hardy-Littlewood-Sobolev (HLS) inequality [22] (from the Young inequality if $r = \infty$), from Paley-Littlewood theory and interpolation. Part (2) is proved in [12] [15] with $\omega$ replaced by $<\omega>$ and follows therefrom by a scaling argument.

Part (3) follows from the relation

$$\left(F[\omega^\ell, v_1] \nabla v_2\right)(\xi) = i \int d\eta \hat{v}_1(\xi - \eta) \left(|\xi|^{\ell} - |\eta|^{\ell}\right) \eta \hat{v}_2(\eta),$$

from the inequality

$$| |\xi|^{\ell} - |\eta|^{\ell}| |\eta| \leq |\xi - \eta| |\eta|^{\ell}$$

and from the Young inequality.

We shall also use following lemma.

**Lemma 3.2.** Let $1 < k < 2$. Then the following estimates hold:

$$\| \omega^{2k-2}(xv_1 v_2) \|_2 \leq C \| v_1; V^k \| \| v_2; V^k \|, \quad (3.14)$$

$$\| \omega^{k+1}(xv_1 v_2) \|_2 \leq C \| v_1; V^{k+1} \| \| v_2; V^{k+1} \|. \quad (3.15)$$

**Proof.** For $\ell \geq 1$ we write

$$\omega^\ell (xv_1 v_2) = [\omega^\ell, v_1] x v_2 + v_1 [\omega^\ell, x] v_2 + xv_1 \omega^\ell v_2.$$

By Lemma 3.1, part (2) we estimate

$$\| \omega^\ell (xv_1 v_2) \|_2 \leq C \left( \| \omega^\ell v_1 \|_{r_1} \| xv_2 \|_{r_2} + \| \nabla v_1 \|_{r_2} \| \omega^{\ell-1} (xv_2) \|_{r_1} \right.\left. + \| v_1 \|_{r_2} \| \omega^{\ell-1} v_2 \|_{r_1} + \| xv_1 \|_{r_2} \| \omega^\ell v_2 \|_{r_1} \right)$$
with \(1/r_1 + 1/r_2 = 1/2\), \(2 \leq r_1 < \infty\). For \(\ell = k + 1\) we take \(r_1 = 2\), \(r_2 = \infty\) which immediately implies (3.15). For \(k \geq 3/2\) and \(\ell = 2k - 2\) we take \(2/r_1 = k - 1\), \(2/r_2 = 2 - k\) and we apply Lemma 3.1, part (1) to obtain
\[
\| \omega^{2k-2}(xv_1v_2) \|_2 \leq C \left( \| \omega^k v_1 \|_2 \| \omega^{k-1}(xv_2) \|_2 + \| \omega^{k-1}(xv_1) \|_2 \| \omega^k v_2 \|_2 + \| \omega^{k-1}v_1 \|_2 \| \omega^{k-1}v_2 \|_2 \right)
\]
which implies (3.14) in that case. For \(k < 3/2\) so that \(0 < \ell = 2k - 2 < 1\) we use the fact [22] that
\[
\| \omega^\ell f \|_2^2 = C \int dy |y|^{-2-2\ell} \| \tau_y f - f \|_2^2
\]
where \((\tau_y f)(x) = f(x + y)\). Taking \(f(x) = xv_1(x)v_2(x)\) we write
\[
\tau_y(xv_1v_2) - xv_1v_2 = (\tau_y(xv_1))(\tau_yv_2 - v_2) + (\tau_yv_1 - v_1)xv_2 + y(\tau_yv_1)v_2
\]
so that from (3.11) we obtain
\[
\| \omega^\ell(xv_1v_2) \|_2^2 \leq C \left( \int dy |y|^{-2-2\ell} \| \tau_yv_1 - v_1 \|_{r_1}^2 \| xv_2 \|_{r_2}^2 + \| \tau_yv_2 - v_2 \|_{r_2}^2 \| xv_1 \|_{r_1}^2 \right) + \int dy |y|^{-2\ell} \| (\tau_yv_1)v_2 \|_2^2 \)
\]
where \(2/r_1 = k - 1\), \(2/r_2 = 2 - k\). We estimate
\[
\begin{align*}
\int dy |y|^{-2-2\ell} \| \tau_yv_1 - v_1 \|_{r_1}^2 &\leq C \int dy |y|^{-2-2\ell} \| \omega^{2-k}(\tau_yv_1 - v_1) \|_2^2 \\
&= C \| \omega^k v_1 \|_2^2
\end{align*}
\]
by Lemma 3.1, part (1), by (3.16) and the fact that \(\omega\) commutes with \(\tau_y\). We estimate \(\| xv_2 \|_{r_2}\) by Lemma 3.1, part (1), we estimate the second term in (3.17) in the same way and we estimate the last term in (3.17) by the HLS inequality and Lemma 3.1, part (1) again. This yields (3.15).

We next derive some estimates of the functions \(B\) and \(g\) defined by (2.11) (2.12) (2.13).

**Lemma 3.3.** Let \(1 < k < 2\). Then the following estimates hold:
\[
\| \omega^\ell B(v_1, v_2) \|_2 \leq C \| v_1; H^k \| \| v_2; H^k \| \quad \text{for } 0 < \ell \leq k + 1 ,
\]
\[\text{ (3.18)}\]
\[\| \omega^\ell B(v_1, v_2) \|_2 \leq C \| v_1 \|_2 \| v_2; H^k \| \quad \text{for } 0 < \ell \leq 1 , \quad (3.19)\]
\[\| \omega^\ell \tilde{B}(v_1, v_2) \|_2 \leq C t^{-1} \| v_1; V^{k} \| \| v_2; V^{k} \| \quad \text{for } 0 < \ell \leq 2k - 1 , \quad (3.20)\]
\[\| \omega^\ell \tilde{B}(v_1, v_2) \|_2 \leq C t^{-1} \| v_1; V^{k} \| \| v_2; V^{k} \| \quad \text{for } 0 < \ell \leq k - 1 , \quad (3.21)\]
\[\| \omega^\ell \tilde{B}(v_1, v_2) \|_2 \leq C t^{-1} \| v_1; V^{k} \| \| v_2; V^{k} \| \quad \text{for } 0 < \ell \leq 2(k - 1) , \quad (3.22)\]
\[\| \omega^\ell B(v) \|_2 \forall t \| \omega^\ell \tilde{B}(v) \|_2 \leq C \| v; V^{k+1} \|^2 \quad \text{for } 0 < \ell \leq k + 2 , \quad (3.23)\]
\[\| \omega^\ell g(v_1, v_2) \|_2 \leq C \| v_1; V^{k} \| \| v_2; V^{k} \| \quad \text{for } 0 \leq \ell \leq k , \quad (3.24)\]
\[\| g(v_1, v_2) \|_2 \leq C \| v_1 \|_2 \| v_2; H^k \| . \quad (3.25)\]

**Proof.** The estimates (3.18) (3.19) (3.24) (3.25) and the estimate of \( B \) in (3.23) follow from Lemma 3.1 possibly supplemented by the HLS inequality for \( B \) if \( \ell < 1 \). In order to derive the estimates for \( \tilde{B} \) we first remark that

\[\tilde{B}(v_1, v_2) = 2t^{-1} \Delta^{-1} \partial_k \text{Im}(\overline{v}_1 x \cdot v_2 + v_2 x \cdot v_1) . \quad (3.26)\]

This follows formally by commuting \( x \) with \( \Delta^{-1} \partial_k \) and can be proved by a regularization and a limiting procedure. The estimates (3.21) (3.22) use only the assumptions that \( xv_2 \in H^{k-1} \) and follow from (3.26), from Lemma 3.1 and from the HLS inequality if \( \ell < 1 \). Finally (3.20) and the estimate of \( \tilde{B} \) in (3.23) follow from Lemma 3.2.

We shall need also the following estimates.

**Lemma 3.4.** Let \( 1 < k < 2 \) and \( k \leq m \leq 2 \). Let \( v \in V^k \) and \( \nabla \varphi \in H^{m-1} \) with
\[a = \| v; V^k \| , \quad \mu = \| \nabla \varphi; H^{m-1} \| . \quad (3.27)\]

(1) The following estimates hold:
\[\| v \exp(-i\varphi) \|_2 = \| v \|_2 \leq a , \quad (3.28)\]
\[\| xv \exp(-i\varphi) \|_2 = \| xv \|_2 \leq a , \quad (3.29)\]
\[\| \omega^{k-1} xv \exp(-i\varphi) \|_2 \leq C a(1 + \mu) , \quad (3.30)\]
\[\| \omega^{k} v \exp(-i\varphi) \|_2 \leq C a(1 + \mu)^{1+(k-1)/(m-1)} . \quad (3.31)\]
(2) Let in addition \( \varphi \in L^\infty \) with \( \| \varphi \|_\infty \leq \mu \). Then
\[
\| v (\exp(-i\varphi) - 1) \|_2 \vee \| xv (\exp(-i\varphi) - 1) \|_2 \leq a\mu , \tag{3.32}
\]
\[
\| \omega^{k-1} xv (\exp(-i\varphi) - 1) \|_2 \leq C a \mu , \tag{3.33}
\]
\[
\| \omega^k v (\exp(-i\varphi) - 1) \|_2 \leq C a \mu (1 + \mu)^{(k-1)/(m-1)} . \tag{3.34}
\]

(3) Let in addition \( xv \in L^\infty \cap \dot{H}^1 \) with \( \| xv \|_\infty \vee \| \nabla xv \|_2 \leq a \). Then
\[
\| \omega^\ell xv \exp(-i\varphi) \|_2 \leq a(1 + \mu)^\ell \quad \text{for } 0 \leq \ell \leq 1 . \tag{3.35}
\]

**Proof.** Part (1). (3.28) and (3.29) are obvious. We next estimate by Lemma 3.1
\[
\| \omega^{k-1} xv \exp(-i\varphi) \|_2 \leq C \| \omega^{k-1} xv \|_2 (1 + \| \nabla \varphi \|_2) \tag{3.36}
\]
which implies (3.30), and
\[
\| \omega^k v \exp(-i\varphi) \|_2 \leq C \left( \| \omega^k v \|_2 + \| v \|_\infty \| \omega^k \exp(-i\varphi) \|_2 \right) . \tag{3.37}
\]
We then interpolate
\[
\| \omega^k \exp(-i\varphi) \|_2 \leq \| \nabla \varphi \|_2^{(m-k)/(m-1)} \| \omega^m \exp(-i\varphi) \|_2^{(k-1)/(m-1)} \tag{3.38}
\]
and we estimate by Lemma 3.1 again
\[
\| \omega^m \exp(-i\varphi) \|_2 = \| \omega^{m-1} \nabla \varphi \exp(-i\varphi) \|_2 \leq C \| \omega^m \varphi \|_2 (1 + \| \nabla \varphi \|_2) \tag{3.39}
\]
which together with (3.36) (3.37) implies (3.31).

Part (2). (3.32) is obvious. We next estimate by Lemma 3.1
\[
\| \omega^{k-1} xv (\exp(-i\varphi) - 1) \|_2 \leq C \| \omega^{k-1} xv \|_2 (\| \varphi \|_\infty + \| \nabla \varphi \|_2) \tag{3.40}
\]
which implies (3.33), and
\[
\| \omega^k v (\exp(-i\varphi) - 1) \|_2 \leq C \left( \| \omega^k v \|_2 \| \varphi \|_\infty + \| v \|_\infty \| \omega^k \exp(-i\varphi) \|_2 \right) \tag{3.41}
\]
which together with (3.37) (3.38) implies (3.34).
Part (3). (3.35) is proved by interpolation between (3.29) and

$$\| \nabla (x v \exp(-i \varphi)) \|_2 \leq \| \nabla x v \|_2 + \| x v \|_\infty \| \nabla \varphi \|_2.$$  

In order to take into account the time decay of the norms of $w$ as $t \to 0$, we introduce a function $h \in C((0,1], \mathbb{R}^+)$ such that the function $\overline{h}(t) = t^{-(3-k)/2}h(t)$ be non decreasing in $(0,1]$ and satisfy

$$\int_0^t dt' t'^{-1} \overline{h}(t') \leq C \overline{h}(t)$$ (3.39)

for some $C > 0$ and all $t \in (0,1]$. We shall use functions of the type

$$\overline{h}(t) = t^\lambda (1 - \ell n t)^\mu$$ (3.40)

with $\lambda > 0$ which clearly satisfy (3.39). Strictly speaking that function is increasing in $(0,1]$ for $\lambda \geq \mu$ only, but not for $\lambda < \mu$. In the latter case, one can remedy that fact either by restricting oneself from the start to the smaller interval $(0,\tau]$ with $\tau = \exp(1 - \mu/\lambda)$, or by replacing the previous function by

$$\overline{h}(t) = \sup_{0 < t' \leq t} t'^{\lambda}(1 - \ell n t' )^\mu$$

which also satisfies (3.39). In what follows we shall freely use (3.40) and not mention that point any more.

For any interval $I \subset (0,1]$, we define the space

$$X(I) = \left\{ v : v \in C(I,V^k) \quad \text{and} \quad \| v; X(I) \| = \sup_{t \in I} h(t)^{-1} \| v(t); V^k \| < \infty \right\}.$$ (3.41)

We finally give estimates of the short and long range parts of $\tilde{B} = \tilde{B}(v)$ for time dependent $v$, namely

$$\| \omega^m \tilde{B}_S \|_2 \leq t^{(p-m)/2} \| \omega^p \tilde{B}_S \|_2 \leq t^{(p-m)/2} \| \omega^p \tilde{B} \|_2$$ (3.42)

for $m \leq p$, and similarly

$$\| \omega^m \tilde{B}_L \|_2 \leq (2t^{-1/2})^{m-p} \| \omega^p \tilde{B}_L \|_2 \leq (2t^{-1/2})^{m-p} \| \omega^p \tilde{B} \|_2$$ (3.43)

for $m \geq p$.  

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4 The Cauchy problem with finite initial time

In this section we study the Cauchy problem with finite initial time in a neighborhood of zero for the auxiliary system (2.21) (2.22) and in a neighborhood of infinity for the original system (1.1). The main results are the existence of solutions defined down to \( t = 0 \) for \((v, \varphi)\), obtained in Proposition 4.2, and up to \( t = \infty \) for \(u\), derived therefrom in Proposition 4.4. As a by product we also obtain the existence of small global solutions for \(u\). Furthermore we derive some results on the asymptotic behaviour of \((v, \varphi)\) as \(t \to 0\) and of \(u\) as \(t \to \infty\), stated in Propositions 4.3 and 4.5 respectively. We treat the various questions in two types of function spaces. The largest convenient spaces are \(V^k\) for \(v\) and \(\tilde{v}\), and correspondingly \(FV^k\) for \(\tilde{u}\). As mentioned in Section 3, those spaces have the drawback that \(FV^k\) is not stable under the free Schrödinger evolution \(U(t)\). The largest smaller spaces where stability holds are the spaces \(\Sigma^k\) and we also state the various results specialized to those smaller spaces. One of the reasons for doing so is that the restriction to \(\Sigma^k\) is necessary when dealing with the problem of small global solutions for \(u\) (see Proposition 4.4, part (3)).

We begin this section by deriving some preliminary estimates for solutions of the partly linearized system (2.21) (2.24). We recall that \(s = \nabla \varphi\) and we use the notation \(a_+ = a \vee 0\).

**Lemma 4.1.** Let \(1 < k < 2\) and \(I \subset (0, 1]\). Let \(v \in C(I, V^k)\) and let

\[
y \equiv y(t) = \| v(t); V^k \| .
\]

(1) Let \(s \in C(I, H^{k+1})\) and let \(v' \in C(I, V^k)\) be solution of (2.24). Then \(v'\) satisfies the following estimates for all \(t \in I\):

\[
\| v'(t) \|_2 = C ,
\]

\[
|\partial_t \| xv' \|_2| \leq \| \nabla v' \|_2 + \| s \|_2 + \| v' \|_\infty + C \ y^2 \| v' \|_2 ,
\]

\[
|\partial_t \| \omega^k v' \|_2| \leq C \left\{ \left( \| \nabla s \|_\infty + \| \omega^2 s \|_2 + \| s \|_\infty \right) + y^2 \right\} \| \omega^k v' \|_2
\]

\[
+ \left( \| \omega^{k+1} s \|_2 + \| \omega^k s \|_2 + \| s \|_\infty + y^2 \right) \left( \| s \|_\infty + y^2 \right) + y^2 \| v' \|_\infty \right\} ,
\]

\[
|\partial_t \| \omega^{k-1} x v' \|_2| \leq C \left\{ \| \omega^k v' \|_2 + \| \omega^{k-1} s \|_2 (\| v' \|_\infty + \| \nabla v' \|_2)
\]

\[
+ y^2 \| \omega^{k-1} v' \|_2 + \left( \| F\nabla s \|_1 + \| \omega^2 s \|_2 + \left( \| \nabla s \|_2 + y^2 \right) \left( \| s \|_\infty + y^2 \right) \right)
\]

\[
+ y^2 \right\} .
\]

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\[ + y^2 t^{k-2} + y^4 \| \omega^{k-1} x v' \|_2 \} . \] 

Let in addition \( v' \in C(I, \Sigma^k) \). Then \( v' \) satisfies the following estimate for all \( t \in I \)

\[ \left| \partial_t \|< x >^k v' \|_2 \right| \leq k \left( \|< x >^{k-1} \nabla v' \|_2 
+ \left( \| s \|_\infty + Cy^2 \right) \|< x >^{k-1} v' \|_2 + \| v' \|_2 \right) . \] \tag{4.5}

(2) Let \( \varphi \) satisfy \((2.21)\). Then \( \varphi \) satisfies the estimate

\[ \| \omega^\ell \partial_t \varphi \|_2 \leq C \ y^2 \ t^{-\ell(2/3+1/2-k)+} \] \tag{4.6}

for all \( \ell > 0 \) and all \( t \in I \).

**Proof.** Part (1). (4.1) is obvious. (4.2) follows immediately from (2.24), from the

\[ [x, H] = \nabla K \] and from Lemma 3.3. We next estimate \( \| \omega^k v' \|_2 \). By standard energy methods followed by Lemma 3.1, we estimate

\[ \left| \partial_t \| \omega^k v' \|_2 \right| \leq \left\{ \| \omega^k s + B \nabla v' \|_2 + \| \omega^k(\nabla \cdot s) v' \|_2 + \| \omega^k(s + B)^2 v' \|_2 
+ \| \omega^k B S v' \|_2 + \| \omega^k g v' \|_2 \right\} \]

\[ \leq C \left( \| \nabla(s + B) \|_\infty + \| \omega^2(s + B) \|_2 + \| \nabla \cdot s \|_\infty + \| s + B \|_\infty^2 
+ \| B S \|_\infty + \| g \|_\infty \right) \| \omega^k v' \|_2 + \left( \| \omega^{k+1} s \|_2 + \| \omega^k(s + B) \|_2 \| s + B \|_\infty 
+ \| \omega^k B S \|_2 + \| \omega^k g \|_2 \right) \| v' \|_\infty \right\} \] \tag{4.7}

from which (4.3) follows by (3.42) and Lemma 3.3.

We next estimate \( \| \omega^{k-1} x v' \|_2 \). By standard energy methods followed by Lemma 3.1, we estimate

\[ \left| \partial_t \| \omega^{k-1} x v' \|_2 \right| \leq \| \omega^{k-1} \nabla s + B v' \|_2 + \| [\omega^{k-1}, s + B] \nabla x v' \|_2 
+ \| \omega^{k-1}(\nabla \cdot s) x v' \|_2 + \| \omega^{k-1}(s + B)^2 x v' \|_2 + \| \omega^{k-1} B S x v' \|_2 
+ \| \omega^{k-1} g x v' \|_2 \right\} \]

\[ \leq C \left( \| \omega^k v' \|_2 + \| \omega^{k-1} s \|_2 \| v' \|_\infty + \| \nabla v' \|_2 \right) + \left( \| B \|_\infty + \| \nabla B \|_2 \right) 
\times \| \omega^{k-1} v' \|_2 + \left( \| F \nabla(s + B) \|_1 + \| \nabla \cdot s \|_\infty + \| \nabla \nabla \cdot s \|_2 + \| s + B \|_\infty^2 \right) \]
\[
+ \| s+B \|_\infty \| \nabla (s+B) \|_2 + \| \tilde{B}_s \|_\infty + \| \nabla \tilde{B}_s \|_2 + \| g \|_\infty + \| \nabla g \|_2 \\
\times \| \omega^{k-1,2} v' \|_2 \}
\]

from which (4.4) follows by (3.42) and Lemma 3.3. Finally from the commutation relation

\[
[<x>^k, H] = (\nabla <x>^k) \cdot \nabla K + (1/2) (\Delta <x>^k)
\]

(4.9)

with

\[
\nabla <x>^k = k <x>^k - 2x, \quad \Delta <x>^k = k^2 <x>^k - k(k-2) <x>^{k-4},
\]

we obtain

\[
|\partial_t \| <x>^k v' \|_2 | \leq \| [<x>^k, H]v' \|_2
\]

\[
\leq k \left( \| <x>^{k-1} \nabla v' \|_2 + \| <x>^{k-1} K v' \|_2 + \| v' \|_2 \right)
\]

from which (4.5) follows.

Part (2) follows immediately from (3.43) and from Lemma 3.3.

\[\square\]

We next derive some estimates of the difference of two solutions of (2.21) (2.24) associated with two different \( v' \)'s. We shall use the following notation. Let \( f_i, i = 1, 2 \) be two functions or operators. We define \( f_\pm = (1/2)(f_1 \pm f_2) \) so that \( f_1 = f_+ + f_- \), \( f_2 = f_+ - f_- \) and \( (fg)_\pm = f_+ g_\pm + f_- g_\mp \). Let now \( v'_i, i = 1, 2 \) be a pair of solutions of (2.24) associated with a pair \((v_i, s_i), i = 1, 2\). Then \( v'_i \) satisfies the equation

\[
i \partial_t v'_i = H_+ v'_i + H_- v'_i
\]

(4.10)

where

\[
H_+ = -(1/2)\Delta_{K_+} + (1/2)K_+^2 + \tilde{B}_S + g_+,
\]

(4.11)

\[
H_- = iK_- \cdot \nabla_{K_+} + (i/2)\nabla \cdot K_- + \tilde{B}_S + g_-.
\]

(4.12)

We can now state the difference estimates of two solutions of (2.21) (2.24).

**Lemma 4.2.** Let \( 1 < k < 2 \) and \( I \subset (0, 1] \). Let \( v_i \in C(I, V^k), i = 1, 2 \) and let

\[
y = y(t) = \max_{i=1,2} \| v_i(t); V^k \|.
\]

(4.13)
(1) Let $s_i \in C(I,H^{k+1})$, $i = 1, 2$, and let $v'_i \in C(I,V^k)$, $i = 1, 2$, be solutions of (2.24) associated with $(v_i,s_i)$. Then the following estimate holds for all $t \in I$:
\[
|\partial_t v'_-|_2 \leq C \left( \| v^{2-k}s_- \|_2 + 2 \| v_- \|_2 \right) \\| v'_+ \|_2 \\
+ \left( \| s_- \|_2 \left( \| s_+ \|_\infty + y^2 \right) + 2 \| v_- \|_2 \left( \| v^{k-1}s_+ \|_2 + y^2 \right) \\
+ \| \nabla \cdot s_- \|_2 + 2 \| v_- \|_2 t^{-1+(k-1)/2} \right) \| v'_+ \|_\infty \right) .
\] (4.14)

(2) Let $\varphi_i$, $i = 1, 2$, satisfy (2.21) with $v = v_i$. Then the following estimate holds for all $\ell > 0$ and for all $t \in I$:
\[
\| \omega^\ell \partial_t \varphi_- \|_2 \leq C y \| v_- \|_2 t^{-1-(1/2)(\ell+1-k)_+} .
\] (4.15)

Proof. Part (1). From (4.10) we estimate
\[
|\partial_t v'_-|_2 \leq \| H_- v'_+ \|_2 \leq \| K_- \|_{r_1} \| v'_+ \|_{r_2} \\
+ \left( \| s_- \|_2 \| K_+ \|_\infty + \| B_- \|_{r_1} \| K_+ \|_{r_2} \\
+ \| \nabla \cdot s_- \|_2 + \| \tilde{B}_s \|_2 + \| g_- \|_2 \right) \| v'_+ \|_\infty \right) .
\] with $r_1 = 2/(k-1)$, $r_2 = 2/(2-k)$, from which (4.14) follows by (3.42) and Lemma 3.3 which implies in particular
\[
\| K_- \|_{r_1} \leq C \left( \| v^{2-k}K_- \|_2 \leq C \left( \| v^{2-k}s_- \|_2 + \| v_- \|_2 \| v_+; V^k \| \right) , \\
\| \omega^{k-1} \tilde{B}_- \|_2 \leq C t^{-1} \| v_- \|_2 \| v_+; V^k \| \right) .
\] (4.16)

Part (2). (4.15) follows immediately from (3.43) and (4.16).

We now turn to the study of the Cauchy problem with finite initial time for the auxiliary system (2.21) (2.22). The first step is to solve that problem for the linearized system (2.24).

Proposition 4.1. Let $1 < k < 2$ and $I \subset (0,1]$, let $t_0 \in I$ and $v'_0 \in V^k$. Let $v \in C(I,V^k)$ and $s \in C(I,H^{k+1})$. Then there exists a unique solution $v' \in C(I,V^k)$ of the system (2.24) with $v'(t_0) = v'_0$. That solution satisfies the estimates (4.1)-(4.4) of Lemma 4.1, part (1). The difference of two such solutions satisfies the estimate (4.14) of Lemma 4.2, part (1). Uniqueness actually holds in $L^\infty(I,V^k)$. If
in addition $v'_0 \in \Sigma^k$, then $v' \in \mathcal{C}(I, \Sigma^k)$ and $v'$ satisfies the estimate (4.17).

That proposition can be proved for instance by a parabolic regularization followed by a limiting procedure. A similar result in a more complicated context appears in Proposition 4.1 of [5].

We now turn to the main technical result of this section, namely the existence of solutions of the auxiliary system \((2.21) \ (2.22)\) with sufficiently small initial time, defined down to time zero.

**Proposition 4.2.** Let $1 < k < 2$ and let $v_0 \in V^k$ with $\|v_0; V^k\| = a$. Then

1. There exists $\tau_0$, $0 < \tau_0 \leq 1$, such that for any $\tau_0$, $0 < \tau_0 \leq \tau_0$, there exists a unique solution $(v, \varphi)$ of the system \((2.21) \ (2.22)\) with $(v, \varphi)(\tau_0) = (v_0, 0)$ and such that $v \in (\mathcal{C} \cap L^\infty)(I, V^k)$, where $I = (0, \tau_0]$. Furthermore $\varphi \in \mathcal{C}(I, H^\infty)$ and $(v, \varphi)$ satisfy

\[
\|v; L^\infty(I, V^k)\| \leq C a, \tag{4.17}
\]

\[
\|\omega^\ell \varphi\|_2 \leq C a^2 \left( t^{-\left(\ell/2+1/2-k\right)+} - \ell \ln t \right) \tag{4.18}
\]

for all $\ell > 0$ and all $t \in I$. The time $\tau_0$ depends on $a$ according to

\[
a \left( \tau_0^{(k-1)/4} \right) \leq C. \tag{4.19}
\]

In particular one can take $\tau_0 = 1$ for a sufficiently small. Uniqueness holds actually under the condition $v \in \mathcal{C}(I, V^k)$.

If in addition $v_0 \in \Sigma^k$, then $v \in (\mathcal{C} \cap L^\infty)(I, \Sigma^k)$ and $v$ satisfies the estimate

\[
\|v; L^\infty(I, \Sigma^k)\| \leq C \|v_0; \Sigma^k\|. \tag{4.20}
\]

2. The map $v_0 \to (v, \varphi)$ is continuous for fixed $\tau_0$ on the bounded sets of $V^k$, from the $L^2$-norm of $v_0$ to the norm of $(v, \varphi)$ in $L^\infty(J, H^{k'} \oplus H^\circ \Sigma)$ for any $k'$, $0 \leq k' < k$ and any $\ell > 0$, and in the weak $\star$ sense in $L^\infty(J, V^k \oplus H^\circ \Sigma)$ for any interval $J \subset I$.

If in addition $v_0 \in \Sigma^k$, the continuity of $v$ extends to norm continuity in $L^\infty(J, \Sigma^k')$ for any $k'$, $0 \leq k' < k$, and to weak $\star$ continuity in $L^\infty(J, \Sigma^k)$ for any interval $J \subset I$.

**Proof.** Part (1). The proof consists in showing that the map $\Gamma : v \to v'$ defined by Proposition 4.1 with $v'(\tau_0) = v(\tau_0) = v_0$ and $s(\tau_0) = 0$ is a contraction on a suitable
bounded set $\mathcal{R}$ of $(C \cap L^\infty)(I, V^k)$ for the norm in $L^\infty(I, L^2)$, where $I = (0, \tau_0]$. We define

$$\mathcal{R} = \{ v \in (C \cap L^\infty)(I, V^k) : v(t_0) = v_0, \| v; L^\infty(I, V^k) \| \leq Y \} \quad (4.21)$$

for some constant $Y$ to be chosen later.

We first show that $\mathcal{R}$ is mapped into itself by $\Gamma$. Integrating (4.6) between $\tau_0$ and $t$, we obtain

$$\| \omega^\ell s \|_2 \leq C Y^2 \left( t^{-(\ell/2 + 1 - k)} - \ell \ln t \right) \quad (4.22)$$

for all $\ell \geq 0$ and all $t \in I$, so that in particular

$$\| s \|_\infty \leq C Y^2 \left( t^{-(3/2 - k)} - \ell \ln t \right) ,$$

$$\| \nabla s \|_\infty \leq \| F^s \|_1 \leq C Y^2 t^{k-2} .$$

Let now

$$y' \equiv y'(t) \equiv \| v'(t); V^k \| . \quad (4.23)$$

Substituting the previous estimates into (4.2)-(4.4), we obtain

$$|\partial_t \| xv'\|_2| \leq C \left( 1 + Y^2 (1 - \ell \ln t) \right) y' , \quad (4.24)$$

$$|\partial_t \| \omega^k v'\|_2| \leq C \left( Y^2 t^{-1 + (k-1)/2} + Y^4 t^{-1 + k/2} \left( t^{-(3/2-k)} - \ell \ln t \right) \right) y' \quad (4.25)$$

$$|\partial_t \| \omega^{k-1} xv'\|_2| \leq C \left( 1 + Y^2 t^{k-2} + Y^4 \left( t^{-(3/2-k)} + (\ell \ln t)^2 \right) \right) y' . \quad (4.26)$$

Integrating (4.24)-(4.26) over time yields

$$y'(t) \leq a \exp \left\{ C \left( \tau_0 + Y^2 \tau_0^{(k-1)/2} + Y^4 \tau_0^{k/2} \left( \tau_0^{-(3/2-k)} - \ell \ln \tau_0 \right) \right) \right\} \quad (4.27)$$

so that by choosing $Y = Ca$, taking $\tau_0$ sufficiently small according to

$$a^2 \tau_0^{(k-1)/2} \leq C \quad (4.28)$$

for suitable constants $C$ and using the fact that

$$k/2 - (3/2 - k) > k - 1 \quad (4.29)$$

we obtain

$$Y' \equiv \| y'; L^\infty(I) \| \leq Y . \quad (4.30)$$

This proves that $\mathcal{R}$ defined by (4.21) is mapped into itself by $\Gamma$. 

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We next prove that $\Gamma$ is a contraction for the norm in $L^\infty(I, L^2)$ on $\mathcal{R}$. Let $v_i \in \mathcal{R}$ and $v'_i = \Gamma v_i, i = 1, 2$. We estimate the difference $v'_-$ by Lemma 4.2. From (4.15), we obtain

$$\| \omega^t s_- \|_2 \leq C \ Y_- \ t^{-1+(k-\ell)/2}$$

for all $\ell \geq 0$ and all $t \in I$, where

$$Y_- \equiv \| v_-; L^\infty(I, L^2) \| .$$

Substituting (4.31) into (4.14) and using the fact that $v'_+ \in \mathcal{R}$, we obtain

$$\left| \partial_t \| v'_- \|_2 \right| \leq C \ Y_- \left( Y^2 \ t^{-1+(k-1)/2} + Y^4 \ t^{-1+k/2} \left( t^{-2(3/2-k)+} - \ell n \ t \right) \right)$$

so that by integration over time

$$Y'_- \equiv \| v'_-; L^\infty(I, L^2) \| \leq C \ Y_- \left( Y^2 \ \tau_0^{(k-1)/2} + Y^4 \ \tau_0^{k/2} \left( \tau_0^{-(3/2-k)+} - \ell n \ \tau_0 \right) \right) .$$

Taking again $\tau_0$ sufficiently small according to (4.28) (possibly with a smaller constant) and using again (4.29), we obtain

$$Y'_- \leq (1/2) Y_-$$

which proves that $\Gamma$ is a contraction for the $L^\infty(I, L^2)$ norm on $\mathcal{R}$. Together with the fact that $\mathcal{R}$ is closed for that norm, this proves that $\Gamma$ has a unique fixed point in $\mathcal{R}$. Uniqueness for $v \in C(I, V^k)$ follows from similar estimates.

The last statement follows by integration of (4.5), using the fact that

$$\| < x >^{k-1} \nabla v \|_2 \leq C \| v; \Sigma^k \| .$$

Part (2). Let $v_i, i = 1, 2$, be two solutions of the previous type of the system (2.21) (2.22) with different initial data $v_{0i}, i = 1, 2$. We estimate $v_-$ in $L^\infty(I, L^2)$ by using again Lemma 4.2, where now $v'_i = v_i$, but $v_-(\tau_0) = v_{0-} = (1/2)(v_{01} - v_{02}) \neq 0$. By the same computation as in the contraction proof, we obtain

$$Y_- \leq 2 \| v_{0-} \|_2$$

which proves the continuity of $v$ from $L^2$ to $L^\infty(I, L^2)$. The remaining continuities follow therefrom by interpolation with boundedness of $v$ in $L^\infty(I, V^k)$ or in $L^\infty(I, \Sigma^k)$, from standard compactness arguments and from (4.15).
Remark 4.1. We have considered the Cauchy problem for the system (2.21) (2.22) with initial condition $\varphi(\tau_0) = 0$. One could easily take instead an initial condition $\varphi(\tau_0) = \varphi_0$ for some $\varphi_0 \in H^{k+2}$ satisfying (4.18) for $0 < \ell \leq k + 2$ and $t = \tau_0$. The solutions thereby obtained would exhibit continuity properties with respect to $\varphi_0$ and $\tau_0$.

We next derive asymptotic properties in time of the solutions of the auxiliary system (2.21) (2.22) obtained in Proposition 4.2. We prove in particular the existence of a limit $v_+$ of $v(t)$ when $t \to 0$ (the subscript + is used here with a different meaning from that used when comparing two solutions) and we provide two asymptotic forms of the phase $\varphi$. The first one is more accurate while the second one has a simpler form.

Proposition 4.3. Let $1 < k < 2$. Let $(v, \varphi)$ be a solution of the system (2.21) (2.22) as obtained in Proposition 4.2 and let

$$Y = \| v; L^\infty(I, V^k) \|$$

where $I = (0, \tau_0]$. Then

1. There exists $v_+ \in V^k$ such that $v(t)$ tends to $v_+$ when $t \to 0$, strongly in $V^{k, \rho}$ for $0 \leq \rho < 1$ (and in particular in $H^{k-1}$) and weakly in $V^k$. Furthermore

$$\| v_+; V^k \| \leq \liminf_{t \to 0} \| v(t); V^k \| \leq Y$$

and the following estimate holds for all $t \in I$

$$\| v(t) - v_a(t); H^{k-1} \| \leq C Y^3 (1 + Y^2) t^{k/2}$$

where

$$v_a(t) = U(t)v_+ .$$

Similar estimates hold in $V^{k, \rho}$, $0 \leq \rho < 1$, by interpolation between (4.32) (4.33) (4.34).

If in addition $v \in (C \cap L^\infty)(I, \Sigma^k)$, then $v_+ \in \Sigma^k$ and $v_+$ satisfies

$$\| v_+; \Sigma^k \| \leq \liminf_{t \to 0} \| v(t); \Sigma^k \| .$$

Furthermore $v(t)$ tends to $v_+$ when $t \to 0$ strongly in $\Sigma^{k'}$ for $0 \leq k' < k$ and weakly in $\Sigma^k$. 26
(2) Define $\varphi_1(t)$ by
\[ \partial_t \varphi_1 = \tilde{B}_L(v_a) \quad \text{and} \quad \varphi_1(1) = 0. \] (4.37)

Then there exists $\psi_{1+} \in \dot{H}^\ell$ for $0 < \ell < 3k-2$ such that $\varphi(t) - \varphi_1(t)$ tends to $\psi_{1+}$ in $\dot{H}^\ell$ for all such $\ell$ when $t \to 0$. Define $\varphi_a = \varphi_1 + \psi_{1+}$. Then the following estimates hold for all such $\ell$ and all $t \in I$:
\[ \| \omega^\ell(\varphi(t) - \varphi_a(t)) \|_2 \leq C Y^4 (1 + Y^2) t^{k/2 - (\ell/2 + 1 - k)_+}, \] (4.38)
\[ \| \omega^\ell \varphi_a(t) \|_2 \leq C Y^2 (1 + Y^2) \left( t^{-(\ell/2 + 1/2 - k)_+ - \ell \ln t} \right). \] (4.39)

(3) Define
\[ \varphi_2(t) = (\ell \ln t)x \cdot B(v_+). \] (4.40)

Then there exists $\psi_{2+} \in \dot{H}^\ell$ for $0 < \ell < 2k - 1$ such that $\varphi(t) - \varphi_2(t)$ tends to $\psi_{2+}$ in $\dot{H}^\ell$ for all such $\ell$ when $t \to 0$. Define $\varphi_b = \varphi_2 + \psi_{2+}$. Then the following estimates hold for all such $\ell$ and all $t \in I$:
\[ \| \omega^\ell(\varphi(t) - \varphi_b(t)) \|_2 \leq C Y^2 (1 + Y^2) t^{k/2 - (1/2)(\ell+1-k)_+}, \] (4.41)
\[ \| \omega^\ell \varphi_b(t) \|_2 \leq C Y^2 (1 + Y^2) \left( t^{-(\ell/2 + 1/2 - k)_+ - \ell \ln t} \right). \] (4.42)

**Proof.** Part (1). Let $\tilde{v} = U(-t)v$. Then $\tilde{v}$ satisfies the equation
\[ i\partial_t \tilde{v} = U(-t) \left\{ i(s + B) \cdot \nabla v + \left( (i/2) \nabla \cdot s + (1/2)(s + B)^2 + \tilde{B}_S + g \right) v \right\}. \] (4.43)

By Lemma 3.1, Lemma 3.3, (3.42) and (1.18), we estimate
\[
\| \omega^\ell \partial_t \tilde{v} \|_2 \leq C \left\{ \| \omega^{\ell+2-k}(s + B) \|_2 + \| \omega^{\ell+1-k}(s + B) \|_\infty \right\} \| \omega^k v \|_2 \\
+ \left( \| \omega^\ell \nabla \cdot s \|_2 + \| \omega^\ell(s + B)^2 \|_2 + \| \omega^\ell \tilde{B}_S \|_2 + \| \omega^\ell g \|_2 \right) \| v \|_\infty + \| \nabla v \|_2 \right\} \\
\leq C \left\{ Y^3 \left( t^{-(\ell/2 + 3/2 - k)_+ - \ell \ln t} \right) + Y^5 \left( t^{-(\ell/2 + 5/2 - 2k)_+ + (\ell \ln t)^2} \right) \right\} \\
\leq C Y^3 (1 + Y^2) \left( t^{-(\ell/2 + 3/2 - k)_+ + (\ell \ln t)^2} \right) \leq C Y^3 (1 + Y^2) t^{-1 + k/2} \] (4.44)

for $0 \leq \ell \leq k - 1$. Let now $0 < t_1 < t_2 \leq \tau_0$. Integrating (4.44) over time yields
\[ \| \tilde{v}(t_2) - \tilde{v}(t_1); H^{k-1} \| \leq C Y^3 (1 + Y^2) \left( t_2^{k/2} \right). \] (4.45)

From (4.45) it follows that $\tilde{v}(t)$ has a limit $v_+$ in $H^{k-1}$ when $t \to 0$ and converges to that limit according to (4.34). Together with uniform boundedness of $v$ in $V^k$, this
implies that \( v_+ \in V^k \), that \( v_+ \) satisfies (4.33), and that \( v(t) \) converges to \( v_+ \) in the convergences stated in Part (1). Similar arguments apply with \( V^k \) replaced by \( \Sigma^k \).

**Part (2).** From (4.21) (4.37) we obtain

\[
\partial_t (\varphi - \varphi_1) = \hat{B}_L (v - v_a, v + v_a).
\]  

(4.46)

Using (3.43) and Lemma 3.3, we estimate

\[
\| \omega^\ell \partial_t (\varphi - \varphi_1) \|_2 \leq C t^{-(\ell/2+1-k)+} \| \omega^{2(k-1)\wedge \ell} \partial_t (\varphi - \varphi_1) \|_2 \leq C t^{-(\ell/2+1-k)_+} \| v - v_a; H^{k-1} \| \| v + v_a; V^k \| \leq C Y^4 (1 + Y^2) t^{-1+k/2-(\ell/2+1-k)_+}
\]

(4.47)

for \( \ell > 0 \). Integrating (4.47) over time yields

\[
\| \omega^\ell (\varphi(t_2) - \varphi_1(t_2) - \varphi(t_1) + \varphi_1(t_1)) \|_2 \leq C Y^4 (1 + Y^2) t^{k/2-(\ell/2+1-k)_+}
\]

(4.48)

for \( 0 < \ell < 2k - 3 \), which implies the existence of \( \psi_{1+} \) and the estimate (4.38). The estimate (4.39) follows from (4.18) (4.38), from the inequality

\[
k/2 - (\ell/2 + 1 - k)_+ \geq (k - 1)/2 - (\ell/2 + 1/2 - k)_+\]

and from the fact that \( Y^{2\ell(k-1)/2} \leq C \) by (4.17) (4.19).

**Part (3).** From (4.37) (4.40) it follows that

\[
\partial_t (\varphi_1 - \varphi_2) = \hat{B}_L (v_a - v_+, v_a + v_+) - \hat{B}_S (v_+).
\]  

(4.49)

We estimate

\[
\| \omega^\ell \hat{B}_L (v_a - v_+, v_a + v_+) \|_2 \leq C t^{-1} \| v_a - v_+ \|_2 \| v_a + v_+; V^k \| \leq C Y^2 t^{-1+k/2} \text{ for } 0 < \ell \leq k - 1
\]

(4.50)

by Lemma 3.3, which together with (3.43) yields

\[
\| \omega^\ell \hat{B}_L (v_a - v_+, v_a + v_+) \|_2 \leq C Y^2 t^{-1+k/2-(\ell/2+1-k)_+}
\]

(4.51)

for \( \ell > 0 \). On the other hand

\[
\| \omega^\ell \hat{B}_S (v_+) \|_2 \leq C Y^2 t^{-1+k-\ell/2-1/2} \text{ for } \ell \leq 2k - 1
\]

(4.52)
by Lemma 3.3 and (3.42), so that
\[
\| \omega^\ell \partial_t (\varphi_1 - \varphi_2) \|_2 \leq C \ Y^2 \ t^{-1+k/2-(1/2)(\ell+1-k)}^+
\]  
(4.53)

for \(0 < \ell \leq 2k - 1\). Combining (4.53) with (4.47) and using the same arguments as in the proof of Part (2) yields the existence of \(\psi_2^+\) and the estimate (4.41), which together with (4.18) yields (4.42).

\[\square\]

Remark 4.2. From (4.44) one can obtain slightly better estimates of \(\| \omega^\ell (v - v_a) \|_2\) for \(0 \leq \ell < k - 1\), which imply slightly better estimates of \(\| \omega^\ell (\varphi - \varphi_a) \|_2\) for \(0 < \ell < 2(k - 1)\).

We now turn to the Cauchy problem with finite initial time for the original system (1.1) and we state the results on that problem that follow from the results for the auxiliary system contained in Propositions 4.2 and 4.3. We recall that \(\tilde{u}(t) = U(-t)u(t) = F\tilde{u}_c(1/t)\).

**Proposition 4.4.** Let \(1 < k < 2\) and let \(\tilde{u}_0 \in FV^k\) with \(\| \tilde{u}_0; FV^k \| = \tilde{a}\). Then

1. There exists \(\bar{t}_0 \geq 1\) such that for any \(t_0 \geq \bar{t}_0\) there exists a unique solution \(u\) of the system (1.1) with \(u \in C(I, FV^k)\) and \(u(t_0) = U(t_0)\tilde{u}_0\), where \(I = [t_0, \infty)\). The time \(\bar{t}_0\) depends on \(\tilde{a}\) according to

\[
\tilde{a} \leq C \ w_0^{(k-1)/4}.
\]

(4.54)

In particular one can take \(\bar{t}_0 = 1\) for small \(\tilde{a}\). The solution \(u\) satisfies the following estimate

\[
\| \tilde{u}(t); FV^k \| \leq C \ w_0 \left( 1 + \tilde{a}^2 (1 + \ell n t) \right)^{k/2}
\]

(4.55)

for all \(t \in I\). Define in addition \(\varphi\) and \(\theta\) by

\[
\varphi(1/t_0) = 0 \quad , \quad \varphi_t = \tilde{B}_L(u_c) \quad \text{for} \quad 0 < t \leq 1/t_0 \ ,
\]

(4.56)

\[
\theta(t) = -D_0(t)\varphi(1/t) \quad \text{for} \quad t \geq t_0 \ .
\]

(4.57)

Then the following estimates hold for all \(t \in I\)

\[
\| \omega^\ell \theta(t) \|_2 \leq C \ w_0^2 \ t^{-\ell} \left( t^{(\ell/2+1-k)/2} + \ell n t \right) \quad \text{for} \quad \ell > 0 
\]

(4.58)

\[
\| U(-t)u(t) \exp(i\theta(t)); L^\infty(I, FV^k) \| \leq C \ w_0 
\]

(4.59)
If in addition \( \tilde{u}_0 \in \Sigma^k \), then \( u, \tilde{u} \in \mathcal{C}(I, \Sigma^k) \) and \( u \) satisfies the estimates
\[
\| \tilde{u}(t); \Sigma^k \| \leq C \tilde{a} \left( 1 + \tilde{a}^2(1 + \ell n(1 + |t|)) \right)^{k^{3/2}} \tag{4.60}
\]
for all \( t \in I \),
\[
\| U(-t)u(t) \exp(i\theta(t)); L_1^\infty(I, \Sigma^k) \| \leq C \tilde{a} \tag{4.61}
\]
where now \( \tilde{a} = \| \tilde{u}_0; \Sigma^k \| \).

(2) The map \( \tilde{u}_0 \to \tilde{u} \) is continuous for fixed \( t_0 \) on the bounded sets of \( FV^k \) from the \( L^2 \) norm of \( \tilde{u}_0 \) to the norm of \( \tilde{u} \) in \( L^\infty(J, FV^{k'}) \) for \( 0 \leq k' < k \) and in the weak \( \star \) sense in \( L^\infty(J, FV^k) \) for any interval \( J \subset \subset I \). If in addition \( \tilde{u}_0 \in \Sigma^k \), then continuity holds on the bounded sets of \( \Sigma^k \) to the norm of \( \tilde{u} \) in \( L^\infty(J, \Sigma^k) \) for \( 0 \leq k' < k \) and in the weak \( \star \) sense in \( L^\infty(J, \Sigma^k) \).

(3) Let \( u_0 \in \Sigma^k \) with \( \tilde{a} = \| u_0; \Sigma^k \| \) sufficiently small. Then there exists a unique solution \( u \) of the system (1.1) with \( u, \tilde{u} \in \mathcal{C}(I, \Sigma^k) \) and \( u(0) = u_0 \). That solution satisfies (4.60) for all \( t \in IR \).

**Proof.** Part (1). We first prove the existence of a solution with the properties stated. Let \( \tau_0 = 1/t_0 \) and
\[
v_0 = U(\tau_0)\tilde{u}_0 = U(1/t_0)\tilde{u}_0. \tag{4.62}
\]
Let \((v, \varphi)\) be the solution of the system (2.21) (2.22) obtained in Proposition 4.2 with \((v, \varphi)(\tau_0) = (v_0, 0)\). Such a solution exists for \( \tau_0 \leq \tau_0 \) and \( \tau_0 \) satisfying (4.19). Now by (3.8)
\[
a = \| v_0; V^k \| \leq 2 \| \tilde{u}_0; V^k \| = 2 \| \tilde{u}_0; FV^k \| = 2\tilde{a} \tag{4.63}
\]
so that (4.19) follows from (4.51) (with a different constant). Define \( u \) by (2.6) (2.16). Then \( u \) solves (1.1) in \( I = [t_0, \infty) \) with \( u(t_0) = U(t_0)\tilde{u}_0 \), and \( \varphi \) satisfies (4.56) because \( \tilde{B}_L(v) = \tilde{B}_L(u_c) \), so that \( \varphi \) can actually be defined in terms of \( u \) by (4.56). Furthermore
\[
U(-t) u(t) \exp(i\theta(t)) = \tilde{F}(1/t). \tag{4.64}
\]
The regularity of \( u \) follows immediately from that of \((v, \varphi)\) through (2.6) (2.16). The estimates (4.58) (4.59) are essentially a rewriting of (4.18) (4.17). We next derive (4.55). Now by (3.8)
\[
\| \tilde{u}(t); FV^k \| = \| \tilde{u}_c(1/t); V^k \| \leq 2 \| u_c(1/t); V^k \|
\]
\[
= 2 \| (v \exp(-i\varphi))(1/t); V^k \| \tag{4.65}
\]
and we estimate the last norm by using (4.17) (4.18) (4.63) and Lemma 3.4, part (1) with 
\( m = (2k - 1) \land 2 \). This proves (4.55).

We finally prove uniqueness of \( u \) by estimating the \( L^2 \) norm of the difference of 
the pseudo conformal inverses \( u_{ci}, i = 1, 2 \), of two solutions \( u_i, i = 1, 2 \). From (2.9), 
by a simplified version of Lemma 4.2, part (1), we estimate

\[
|\partial_t ||u_{c-}||_2 | \leq C \left( t^{-1}y^2 + y^4 \right) ||u_{c-}||_2
\]  

(4.66)

where

\[
y = y(t) = \max_i \|u_{ci}(t); V^k\|
\]

from which uniqueness follows immediately.

The additional properties of \( u \) for \( \tilde{u}_0 \in \Sigma^k \) follow immediately from the last 
statement of Proposition 4.2, part (1) by similar arguments.

Part (2) follows immediately from Proposition 4.2, part (2).

Part (3). By (4.54), for \( \tilde{a} \) sufficiently small, we can take \( \tilde{T}_0 = 1 \) in Part (1) of this 
proposition. Applying that result, we obtain a solution \( u \) of the system (1.1) with 
\( u(1) = u_0 \) and \( \tilde{u} \in C([1, \infty), \Sigma^k) \) provided

\[
\tilde{a}_> = || U(-1)u_0; \Sigma^k || \leq \tilde{a}
\]

(4.67) for some \( \tilde{\pi} \) sufficiently small. Since the system (1.1) is time translation invariant, by 
translating the previous solution by \(-1\) in time, we obtain a solution \( u_> \) with 
\( u_>(0) = u_0 \) and \( U(-1)\tilde{u}> \in C([0, \infty), \Sigma^k) \), or equivalently \( \tilde{u}> \in C([0, \infty), \Sigma^k) \), satisfying the estimate (4.60) for all \( t \geq 0 \) with \( \tilde{a} \) replaced by \( \tilde{a}> \). Since the system (1.1) is also 
time reversal invariant, we can construct similarly a solution \( u_< \) with 
\( u_<(0) = u_0 \) and \( \tilde{u}_< \in C((-\infty, 0], \Sigma^k) \), satisfying the estimate (4.60) for all \( t \leq 0 \) with \( \tilde{a} \) replaced by \( \tilde{a}_< \), with

\[
\tilde{a}_< = || U(1)u_0; \Sigma^k || \leq \tilde{a}.
\]

(4.68)

Taking \( u(t) = u>_<(t) \) for \( t \geq 0 \) yields a solution \( u \) of the system (1.1) with \( u(0) = u_0 \) 
and \( \tilde{u} \in C(\mathbb{R}, \Sigma^k) \), satisfying the estimate (4.60) for all \( t \in \mathbb{R} \) with \( \tilde{a} = \tilde{a}_> \lor \tilde{a}_< \). 
Finally by (3.11), the conditions (4.67) (4.68) can both be satisfied by taking 
\( || u_0; \Sigma^k || \) sufficiently small.

\(\square\)
Remark 4.3. Proposition 4.3 allows one to take arbitrarily large $\tilde{u}_0 \in \Sigma^k$ by taking $t_0$ sufficiently large according to (4.54), thereby generating some large initial data $u_0 = U(t_0)\tilde{u}_0$ in $\Sigma^k$. However one cannot accommodate arbitrarily large $u_0 \in \Sigma^k$ since for fixed $u_0 \in \Sigma^k$ and $t_0$ large
\[
\| < x >^k \tilde{u}_0 \|_2 = \| < x + i t_0 \nabla >^k u_0 \|_2 \sim t_0^k \| u_0 ; H^k \|
\]
and taking $t_0$ large is of no help in order to fulfill (4.54).

We now turn to the asymptotic properties of the solutions obtained in Proposition 4.4 that follow from Proposition 4.3.

Proposition 4.5. Let $1 < k < 2$. Let $u$ be a solution of the system (1.1) as obtained in Proposition 4.4, let $\theta$ be defined by (4.56) (4.57) and let $\tilde{Y} = \| U(-t) \exp(i\theta(t)) ; FV^k \|$ (4.69) where $I = [t_0, \infty)$. Then

1. There exists $u_+ \in FV^k$ such that $U(-t)u(t)\exp(i\theta(t))$ tends to $u_+$ when $t \to \infty$ strongly in $FV^{k,\rho}$ for $0 \leq \rho < 1$ (and in particular in $F H^{k-1}$) and weakly in $FV^k$. Furthermore
\[
\| u_+ ; FV^k \| \leq \lim \inf_{t \to \infty} \| U(-t) \exp(i\theta(t)) ; FV^k \| \leq \tilde{Y} \quad (4.70)
\]
and the following estimate holds for all $t \in I$:
\[
\| < x >^{k-1} (U(-t) \exp(i\theta(t)) - u_+) \|_2 \leq C \tilde{Y}^2 (1 + \tilde{Y}^2) t^{-k/2} . \quad (4.71)
\]
If in addition $\tilde{u} \in C(I, \Sigma^k)$, then $u_+ \in \Sigma^k$ and $U(-t)u(t)\exp(i\theta(t))$ tends to $u_+$ when $t \to \infty$ strongly in $\Sigma^{k'}$ for $0 \leq k' < k$ and weakly in $\Sigma^k$. Furthermore
\[
\| u_+ ; \Sigma^k \| \leq \lim \inf_{t \to \infty} \| U(-t) \exp(i\theta(t)) ; \Sigma^k \| . \quad (4.72)
\]

2. Let $\varphi_\alpha$ be defined as in Proposition 4.3, part (2) with $v_+ = \overline{Fu_+}$ and define
\[
\theta_\alpha(t) = -D_0(t) \varphi_\alpha(1/t) , \quad (4.73)
\]
\[
u_\alpha(t) = \exp(-i\theta_\alpha(t))U(t) u_+ . \quad (4.74)
\]
Then $\nu_\alpha$ satisfies the estimate
\[
\| \tilde{u}_\alpha(t) ; FV^k \| \leq C \tilde{Y} \left( 1 + (\tilde{Y}^2 (1 + \tilde{Y}^2)(1 + \ell n \ t))^{k\vee 3/2} \right) . \quad (4.75)
\]
Furthermore $u$ behaves asymptotically as $u_a$ for large $t$ in the sense that $\tilde{u} - \tilde{u}_a$ tends to zero when $t \to \infty$ strongly in $FV^{k,\rho}$ for $0 \leq \rho < 1$ (and in particular in $FH^{k-1}$). The difference $\tilde{u} - \tilde{u}_a$ satisfies the estimate

$$\|x >^{k-1} (\tilde{u}(t) - \tilde{u}_a(t))\|_2 \leq C \bar{Y}^3 (1 + \bar{Y}^2)^3(1 + \ell n t) t^{-k/2}$$  \hspace{1cm} (4.76)$$

for all $t \in I$, and similar estimates in $FV^{k,\rho}$, $0 \leq \rho < 1$, obtained by interpolation between (4.76) and the estimates (4.55) (4.75) in $FV^k$. If in addition $\tilde{u} \in C(I,\Sigma^k)$, then $\tilde{u}_a$ satisfies an estimate in $\Sigma^k$ similar to (4.75) and $\tilde{u} - \tilde{u}_a$ tends to zero when $t \to \infty$ strongly in $\Sigma^k$ for $0 \leq k' < k$.

(3) Let $\varphi_b$ be defined as in Proposition 4.3, part (3) with $v_+ = \overline{Fv_+}$ and define

$$\theta_b(t) = -D_0(t) \varphi_b(1/t) = D_0(t) \left((\ell n t)x \cdot B(\overline{Fv_+}) - \psi_{2+}\right) , \hspace{1cm} (4.77)$$

$$u_b(t) = \exp (-i\theta_b(t)) U(t) u_+ . \hspace{1cm} (4.78)$$

Then $u_b$ satisfies the estimate

$$\| \tilde{u}_b(t); FV^k \| \leq C \bar{Y} \left(1 + (\bar{Y}^2(1 + \bar{Y}^2)^2(1 + \ell n t))^{k_\nu(3/2+\epsilon)}\right)$$  \hspace{1cm} (4.79)$$

for any $\epsilon > 0$. Furthermore $u$ behaves asymptotically as $u_b$ for large $t$ in the sense that $\tilde{u} - \tilde{u}_b$ tends to zero when $t \to \infty$ in $FV^{k,\rho}$, $0 \leq \rho < 1$ (and in particular in $FH^{k-1}$). The difference $\tilde{u} - \tilde{u}_b$ satisfies the estimate

$$\| < x >^{k-1} (\tilde{u}(t) - \tilde{u}_b(t)) \|_2 \leq C \bar{Y}^3 (1 + \bar{Y}^2)^3(1 + \ell n t) t^{-k/2} .$$  \hspace{1cm} (4.80)$$

for all $t \in I$, and similar estimates in $FV^{k,\rho}$, $0 \leq \rho < 1$, obtained by interpolation between (4.80) and the estimates (4.55) (4.79) in $FV^k$.

If $\tilde{u} \in C(I,\Sigma^k)$ a similar reinforcement occurs as in Part (2).

**Proof.** Part (1). The solution $u$ is obtained from a solution $(v,\varphi)$ of the system (2.21) (2.22) as in the proof of Proposition 4.4 and $u$ satisfies (4.59) with $\theta$ defined by (4.56) (4.57).

The existence of the limit $u_+$ and the convergence properties of Part (1) are a rewriting of the corresponding properties in Proposition 4.3, part (1) with $u_+ = \overline{Fv_+}$. The estimates (4.70) (4.71) (4.72) follow from (4.33) (4.34) (4.36), from the relations

$$\| U(-t) u(t) \exp (i\theta(t)) - u_+; FH^{k-1} \| = \| \tilde{v}(1/t) - v_+; H^{k-1} \| , \hspace{1cm} (4.81)$$

$$Y = \| v; L^\infty((0,\tau_0]; V^k) \| \leq 2 \| \tilde{v}; L^\infty((0,\tau_0], V^k) \| = 2\bar{Y} , \hspace{1cm} (4.82)$$
by (3.8) (4.32) (4.69), and a similar one with \( V^k \) replaced by \( \Sigma^k \).

Part (2). The definition (4.74) of \( u_a \) is actually a rewriting of

\[
\tilde{u}_a(t) = F\tilde{u}_{ca}(1/t)
\]

with

\[
u_{ca} = v_a \exp(-i\varphi_a), \quad v_a(t) = U(t)v_+,
\]

in analogy with (2.6) (2.16). Therefore

\[
\| \tilde{u}_a(t); FV^k \| \leq 2 \| (v_a \exp(-i\varphi_a))(1/t); V^k \|
\]  

(4.83)

by (3.8), which implies (4.75) by Lemma 3.4, part (1) with \( m = (2k - 1) \wedge 2, \) (4.33) (4.39) (4.82). Similarly

\[
\| \tilde{u}(t) - \tilde{u}_a(t); FH^{k-1} \| = \| (v \exp(-i\varphi) - v_a(\exp(-i\varphi_a)))(1/t); H^{k-1} \|
\]  

(4.84)

In order to prove (4.76), we write

\[
v \exp(-i\varphi) - v_a \exp(-i\varphi_a) = v_{\neq} \exp(-i\varphi),
\]

\[
v_{\neq} = v - v_a + v_a(1 - \exp(i\psi))
\]

with \( \psi = \varphi - \varphi_a \) and for \( 0 < \ell \leq k - 1 \), we estimate

\[
\| \omega^\ell v_{\neq} \exp(-i\varphi) \|_2 \leq C \| \omega^\ell v_{\neq} \|_2 (1 + \| \nabla \varphi \|_2),
\]

(4.85)

\[
\| \omega^\ell v_{\neq} \|_2 \leq C \left( \| \omega^\ell (v - v_a) \|_2 + (\| v_a \|_{\infty} + \| \nabla v_a \|_2) \| \omega^\ell \psi \|_2 \right)
\]  

(4.86)

by Lemma 3.1 and (3.16), which implies that \( \| \omega^\ell \exp(i\psi) \|_2 \leq \| \omega^\ell \psi \|_2 \). We continue (4.85) (4.86) by using (4.17) (4.18) (4.33) (4.34) (4.38) (4.82). Substituting the result into (4.84) yields (4.76), which by interpolation with (4.55) (4.75) completes the proof in the case of \( FV^k \).

The proof in the case of \( \Sigma^k \) is similar.

Part (3). The proof of (4.79) is essentially the same as that of (4.75) with the estimate (4.39) replaced by (4.42). The occurrence of \( \varepsilon > 0 \) in the exponent for \( k \leq 3/2 \) is required by the fact that (4.42) holds only for \( \ell < 2k - 1 \), the limiting case being excluded. Similarly the proof of (4.80) is essentially the same as that of
Remark 4.4. In Parts 2 and 3 of Proposition 4.5, the convergence of \( \tilde{u} - \tilde{u}_a \) to zero as \( t \to \infty \) can be extended to weak convergence in \( FH^k \) for solutions in \( \mathcal{C}(I, FV^k) \) and to weak convergence in \( \Sigma^k \) for solutions in \( \mathcal{C}(I, \Sigma^k) \). This follows from convergence in \( FH^{k-1} \) and from uniform boundedness of \( \tilde{u} - \tilde{u}_a \) in \( FH^k \) or in \( \Sigma^k \). The latter follows from uniform boundedness of \( v \neq \exp(-i\varphi) \) in \( H^k \) or in \( \Sigma^k \), which can be easily derived from the available estimates.

5 The Cauchy problem at initial time zero for \((v, \varphi)\) and at infinity for \(u\)

In this section we solve the Cauchy problem for the auxiliary system \((2.21) (2.22)\) with prescribed asymptotic behaviour at time zero and for the original system \((1.1)\) with corresponding prescribed asymptotic behaviour at infinity. The main results are contained in Proposition 5.6 for the system \((2.21) (2.22)\) and in Proposition 5.7 for the system \((1.1)\). In this section we use only the spaces \(V^k\) for \(v\) and \(FV^k\) for \(\tilde{u}\). The specialization of the results to the space \(\Sigma^k\) is straightforward and will not be considered. We start with a uniqueness result for \((v, \varphi)\).

Proposition 5.1. Let \(1 < k < 2\). Let \(0 < \tau \leq 1\) and \(I = (0, \tau]\). Let \((v_i, \varphi_i), i = 1, 2,\) be two solutions of the system \((2.21) (2.22)\) with \(v_i \in \mathcal{C}(I, V^k)\) and \(\varphi_i(t_0) \in H^{k+2}_>\) for some \(t_0 \in I\). Assume that

\[
\| v_i(t); V^k \| \leq a(1 - \ell n t)^\alpha \tag{5.1}
\]

for all \(t \in I\) and

\[
\sup_{t \in I} h_1(t)^{-1} \| v_1(t) - v_2(t) \|_2 = Y < \infty \tag{5.2}
\]

for some constants \(a, Y\) and \(\alpha \geq 0\) and for some non-decreasing function \(h_1 \in \mathcal{C}(I, \mathbb{R}^+)\) satisfying

\[
\int_0^t dt' t'^{-1-(3-k)/2}(1 - \ell n t')^\alpha h_1(t') \leq C t^{-1-(3-k)/2}(1 - \ell n t)^\alpha h_1(t) \tag{5.3}
\]

for all \(t \in I\), and

\[
\lim_{t \to 0} t^{-1-(3-k)/2}(1 - \ell n t)^\alpha h_1(t) = 0 \tag{5.4}
\]
Assume in addition that
\[ \lim_{t \to 0} (\varphi_1(t) - \varphi_2(t)) = 0. \tag{5.5} \]
Then \((v_1, \varphi_1) = (v_2, \varphi_2)\).

**Proof.** We define again \((v_\pm, \varphi_\pm) = (1/2)(v_1 \pm v_2, \varphi_1 \pm \varphi_2)\) and we estimate \((v_-, \varphi_-)\) by Lemma 4.2. We first estimate \(\varphi_-\). From (4.15) (5.1) (5.2) it follows that
\[ \| \omega^\ell \partial_t \varphi_- \|_2 \leq C a Y (1 - \ell \ln t)^\alpha t^{-1-(1/2)(\ell+1-k)+} h_1(t) \tag{5.6} \]
for \(\ell > 0\). From (5.3) (5.4) (5.6) it follows that \(\varphi_-\) has a limit in \(H^2\) when \(t \to 0\), which gives a meaning to the assumption (5.5). Furthermore
\[ \| \omega^\ell \varphi_-(t) \|_2 \leq C a Y (1 - \ell \ln t)^\alpha t^{-(1/2)(\ell+1-k)+} h_1(t) \tag{5.7} \]
for \(0 < \ell \leq 2\) and all \(t \in I\). On the other hand, by integrating (4.6) between \(t_0\) and \(t\), we obtain
\[ \| \omega^\ell \varphi_i(t) \|_2 \leq C a^2 (1 - \ell \ln t)^{2\alpha} \left( t^{-(\ell/2+1/2-k)+} - \ell \ln t \right) \tag{5.8} \]
for \(\ell > 0\). We next estimate \(v_-\) by Lemma 4.2. From (4.14) (5.1) (5.8) we obtain
\[ \text{\begin{align*}\| & \partial_t \| v_- \|_2 \|_2 \leq C a (1 - \ell \ln t)^\alpha \left\{ \| \omega^{3-k} \varphi_- \|_2 + \| \omega^2 \varphi_- \|_2 \\
& + a (1 - \ell \ln t)^\alpha t^{-(3-k)/2} \| v_- \|_2 + a^2 (1 - \ell \ln t)^{2\alpha} \left( t^{-(3/2-k)+} - \ell \ln t \right) \| \omega \varphi_- \|_2 \\
& + a^2 (1 - \ell \ln t)^{3\alpha+1} \| v_- \|_2 \right\}. \end{align*}} \tag{5.9} \]
From (5.9) (5.2) (5.7) we then obtain
\[ \text{\begin{align*}\| & \partial_t \| v_- \|_2 \|_2 \leq C a^2 (1 - \ell \ln t)^{2\alpha} t^{-(3-k)/2} Y h_1(t) \\
& \times \left( 1 + a^2 (1 - \ell \ln t)^{2\alpha} t^{1/2} \left( t^{-(3/2-k)+} - \ell \ln t \right) \right) \\
& \leq C a^2 (1 + a^2) (1 - \ell \ln t)^{2\alpha} t^{-(3-k)/2} Y h_1(t). \tag{5.10} \end{align*}} \]
Integrating (5.10) in \((0, t]\) and using (5.2) (5.3), we obtain
\[ Y \leq C a^2 (1 + a^2) (1 - \ell \ln \tau)^{2\alpha} \tau^{(k-1)/2} Y \tag{5.11} \]
which implies \(Y = 0\) by taking \(\tau\) sufficiently small and therefore \(v_- = 0\) and \(\varphi_- = 0\) by (5.7). The extension of the proof to larger \(\tau\) proceeds by standard arguments. □

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We next derive a uniqueness result for solutions of the system \((1.1)\). That result is obtained by applying Proposition 5.1 to solutions of the system \((2.21)(2.22)\) reconstructed from solutions of the system \((1.1)\).

**Proposition 5.2.** Let \(1 < k < 2\). Let \(T \geq 1\) and \(I = [T, \infty)\). Let \(u_i, i = 1, 2\) be two solutions of the system \((1.1)\) with \(\tilde{u}_i \in C(I, FV^k)\), where \(\tilde{u}(t) = U(-t)u(t)\). Assume that
\[
\| \tilde{u}_i(t); FV^k \| \leq b(1 + \ell n t)^\beta
\]
for all \(t \in I\) and
\[
\text{Sup}_{t \in I} h_2(1/t)^{-1} \| \tilde{u}_1(t) - \tilde{u}_2(t) \|_2 = Z < \infty
\]
for some constants \(b, Z\) and \(\beta \geq 0\) and for some nondecreasing function \(h_2 \in C(I_0, IR^+)\) with \(I_0 = (0, T^{-1}]\), satisfying
\[
\int_0^t dt' t'^{-(3-k)/2} (1 - \ell n t')^{\beta+2} h_2(t') \leq C t^{-(3-k)/2} (1 - \ell n t)^{\beta+2} h_2(t)
\]
for all \(t \in I_0\), and
\[
\lim_{t \to 0} t^{-(3-k)/2} (1 - \ell n t)^{\beta+2} h_2(t) = 0.
\]
Then \(u_1 = u_2\).

**Proof.** From \(u_i, i = 1, 2\), we reconstruct solutions \((v_i, \varphi_i)\) of the system \((2.21)(2.22)\). For convenience, we work with the pseudoconformal inverses \(u_{ci}, i = 1, 2\) of \(u_i\), defined by \((2.6)\). From \((3.8)\) it follows that \((5.12)(5.13)\) can be rewritten as
\[
\| u_{c_i}(t); V^k \| \leq b(1 + \ell n t)^\beta
\]
\[
\text{Sup}_{t \in I_0} h_2(t)^{-1} \| u_{c_1}(t) - u_{c_2}(t) \|_2 = Z < \infty
\]
possibly with a change of \(b\) by a factor 2. We first define the phases \(\varphi_i\). We define \(\varphi_2\) by
\[
\partial_t \varphi_2 = \tilde{B}_L(u_{c_2})
\]
with initial condition \(\varphi_2(t_0) = 0\) for some \(t_0 \in I_0\). By integrating \((4.6)\) in \([t_0, t]\) and using \((5.16)\) we obtain
\[
\| \omega^t \varphi_2(t) \|_2 \leq C b^2 (1 - \ell n t)^{2\beta} \left(t^{-(\ell/2+1/2-k)} - \ell n t\right)
\]
for $\ell > 0$ and in particular
\[ \| \omega^\ell \varphi_2(t) \|_2 \leq C b^2 (1 - \ell \ln t)^{2\beta + 1} \] for $0 < \ell \leq 2k - 1$. (5.20)

We next define $\varphi_-$ as a solution of the equation
\[ \partial_t \varphi_- = B_L(u_{c1}) - B_L(u_{c2}) = B_L(u_{c1} - u_{c2}; u_{c1} + u_{c2}) \] (5.21)
so that by (3.43), by Lemma 3.3 and by (5.16) (5.17)
\[ \| \omega^\ell \partial_t \varphi_-(t) \|_2 \leq C b Z (1 - \ell \ln t)^{\beta} t^{-1-(1/2)(\ell+1-k)} h_2(t) \] (5.22)
for all $\ell > 0$. We can then impose the initial condition $\varphi_-(0) = 0$ and we obtain by integration in $(0, t]$
\[ \| \omega^\ell \varphi_-(t) \|_2 \leq C b Z (1 - \ell \ln t)^{\beta} t^{-1-(1/2)(\ell+1-k)} h_2(t) \] for $0 < \ell \leq 2$ (5.23)
by (5.14). We next define $\varphi_1 = \varphi_2 + \varphi_-$ so that $\varphi_1$ satisfies the equation
\[ \partial_t \varphi_1 = \hat{B}_L(u_{c1}) . \] (5.24)
Furthermore $\varphi_- = \varphi_1 - \varphi_2$ tends to zero when $t \to 0$ in the norms which appear in (5.23), and it follows from (5.20) (5.23) that
\[ \| \omega^\ell \varphi_i(t) \|_2 \leq C b^2 (1 - \ell \ln t)^{2\beta + 1} \] for $0 < \ell \leq k$ (5.25)
for $i = 1, 2$, provided
\[ Z t^{-1/2} h_2(t) \leq b(1 - \ell \ln t)^{\beta+1} \] (5.26)
which can be ensured by taking $t$ sufficiently small because of (5.15). We next define
\[ v_i = u_{ci} \exp (i\varphi_i) \] (5.27)
so that $(v_1, \varphi_1)$ satisfy the system (2.21) (2.22) since $\hat{B}_L(u_{ci}) = \hat{B}_L(v_i)$. Furthermore, it follows from (5.16) (5.25) and Lemma 3.4, part (1), with $m = k$ that
\[ \| v_i(t); V^k \| \leq C b(1 - \ell \ln t)^{\beta} \left(1 + b^2(1 - \ell \ln t)^{2\beta + 1}\right)^2 \leq C b(1 + b^2)^2(1 - \ell \ln t)^{5\beta + 2} . \] (5.28)
We next estimate $\| v_1 - v_2 \|_2$. From
\[ v_1 - v_2 = (u_{c1} - u_{c2}) \exp (i\varphi_1) + u_{c2} (\exp (i\varphi_-) - 1) \exp (i\varphi_2) , \] (5.29)
from (5.16) (5.17) (5.23) it follows that
\[ \| v_1(t) - v_2(t) \|_2 \leq C Z(1 + b^2)(1 - \ln t)^{2\beta} h_2(t). \] (5.30)

Therefore \((v_i, \varphi_i)\) satisfy the assumptions of Proposition 5.1 with
\[ a = C b(1 + b^2)^2, \quad \alpha = 5\beta + 2, \quad Y = C Z(1 + b^2), \quad h_1 = h_2(1 - \ln t)^{2\beta} \]
so that the conditions (5.3) (5.4) reduce to (5.14) (5.15). It then follows from Proposition 5.1 that \((v_1, \varphi_1) = (v_2, \varphi_2)\), so that \(u_{c1} = u_{c2}\).

We now turn to the construction of solutions \((v_i, \varphi_i)\) of the system (2.21) (2.22) with prescribed asymptotic behaviour \((v_a, \varphi_a)\) as \(t \to 0\), with \(\varphi_a\) defined by (2.25) and \(\varphi_a(1) = 0\), or equivalently of solutions of the system (2.30) (2.31) with \((w, \psi)\) tending to zero in a suitable sense when \(t \to 0\). We first collect some preliminary estimates of \((G, \psi)\), of \((B_a, \varphi_a)\) and of \(H_1\).

**Lemma 5.1.**

1. Let \(v_a, w \in V^k\) and let \(\psi\) satisfy (2.30). Then
\[ \| \omega^\ell G \|_2 \leq C \| w; H^k \| \| 2v_a + w; H^k \| \quad \text{for } 0 < \ell \leq k + 1, \quad (5.31) \]
\[ \| \omega^\ell \tilde{G} \|_2 \leq C \| w; V^k \| \| 2v_a + w; V^k \| t^{-1} \quad \text{for } 0 < \ell \leq 2k - 1, \quad (5.32) \]
\[ \| \omega^\ell \partial_t \psi \|_2 \leq C \| w; V^k \| \| 2v_a + w; V^k \| t^{-1-(\ell/2+1/2-k)} \quad \text{for } \ell > 0. \quad (5.33) \]

2. Let \(v_a \in V^{k+1}\) and let \(\varphi_a\) satisfy (2.25). Then
\[ \| \omega^\ell B_a \|_2 \leq C \| v_a; V^{k+1} \|^2 \quad \text{for } 0 < \ell \leq k + 2, \quad (5.34) \]
\[ \| \omega^\ell \partial_t \varphi_a \|_2 \leq C \| v_a; V^{k+1} \|^2 t^{-1-(\ell/2+1/2-k)} \quad \text{for } \ell > 0. \quad (5.35) \]

**Proof.** The estimates (5.31) (5.32) (5.34) follow from Lemma 3.3. The estimates (5.33) (5.35) follow from (3.43) and (5.32) (5.34).

From now on we shall assume that \(v_a\) satisfies the assumption
\[ v_a \in (C \cap L^\infty)((0,1], V^{k+1}), \quad (5.36) \]
we define
\[ a = \| v_a; L^\infty((0,1], V^{k+1}) \| \quad (5.37) \]
and we assume that \( \varphi_a \) satisfies (2.25) and \( \varphi_a(1) = 0 \), so that by integration of (5.35)

\[
\| \omega^\ell \varphi_a \|_2 \leq C a^2 \left( t^{-(1/2)(\ell - 2 - k)} + \ell \ln t \right) \quad \text{for } \ell > 0 \quad (5.38)
\]

for all \( t \in (0, 1) \).

We next estimate \( H_1 v_a \), where \( H_1 \) is defined by (2.32). We rewrite \( H_1 \) as

\[
H_1 = iL \cdot \nabla + M \quad (5.39)
\]

where

\[
M = L \cdot K_a + (i/2) \nabla \cdot \sigma + (1/2) L^2 + \tilde{G}_S + g \neq , \quad (5.40)
\]

\[
g \neq = g(w, 2v_a + w) .
\]

We shall use the auxiliary norm

\[
\| f \|_* = \| f \|_\infty \vee \| \nabla f \|_2 \vee \| \omega^k f \|_2 \quad (5.41)
\]

which satisfies

\[
\| f_1 f_2 \|_* \leq \| f_1 \|_* \| f_2 \|_\infty + \| f_1 \|_\infty \| f_2 \|_*
\]

by Lemma 3.1.

**Lemma 5.2.** Let \( v_a \) satisfy (5.36) with \( a \) defined by (5.37). Let \( I \subset (0, 1) \) and let \( w \in (C \cap L^\infty)(I, V^k) \), \( \sigma \in C(I, L^\infty \cap \tilde{H}_1 \cap \tilde{H}^{k+1}) \) with

\[
\| w; L^\infty(I, V^k) \| \leq C a . \quad (5.42)
\]

Then the following estimate holds for all \( t \in I \) :

\[
\| H_1 v_a; V^k \| \leq C a \left( \| \sigma \|_* \left( 1 + \| \sigma \|_\infty + a^2(1 - \ell \ln t) \right) + \| \nabla \cdot \sigma \|_* 
+ a \| w; V^k \| \left( t^{-1+(k-1)/2} + a^2(1 - \ell \ln t) \right) \right) . \quad (5.43)
\]

**Proof.** By Lemma 3.1, we estimate

\[
\| H_1 v_a \|_2 \leq \| L \|_\infty \| \nabla v_a \|_2 + \| M \|_\infty \| v_a \|_2
\]

\[
\| xH_1 v_a \|_2 \leq \| L \|_\infty \| x \nabla v_a \|_2 + \| M \|_\infty \| xv_a \|_2
\]
\[ \| \omega^k H_1 v_a \|_2 \leq \| \omega^k L \|_2 \| \nabla v_a \|_\infty + \| L \|_\infty \| \omega^{k+1} v_a \|_2 \]
\[ + \| \omega^k M \|_2 \| v_a \|_\infty + \| M \|_\infty \| \omega^k v_a \|_2 \]
\[ \| \omega^{k-1} x H_1 v_a \|_2 \leq (\| L \|_\infty + \| \nabla L \|_2) \| \omega^{k-1} \nabla v_a \|_2 \]
\[ + (\| M \|_\infty + \| \nabla M \|_2) \| \omega^{k-1} v_a \|_2 \]

so that
\[ \| H_1 v_a; V^k \| \leq (\| L \|_* + \| M \|_* \| v_a; V^{k+1} \|) \]
\[ \leq a (\| L \|_* + \| M \|_*) . \] (5.44)

We next estimate
\[ \| L \|_* \leq \| \sigma \|_* + \| G \|_* , \] (5.45)
\[ \| M \|_* \leq \| L \|_* (\| K_a \|_* + \| L \|_\infty) + \| \nabla \cdot \sigma \|_* + \| \dot{G}_S \|_* + \| g_{\neq} \|_* . \] (5.46)

By Lemma 5.1, (3.42) and (5.37), we estimate
\[ \| G \|_* \leq C a \| w; V^k \| \leq C a^2 , \] (5.47)
\[ \| \dot{G}_S \|_* \leq C a \| w; V^k \| t^{-1+(k-1)/2} , \] (5.48)
\[ \| K_a \|_* \leq C a^2 (1 - \ell n t) . \] (5.49)

Substituting (5.47)-(5.49) into (5.45) (5.46) and substituting the result into (5.44) yields (5.43).

We next give some estimates of solutions \( w' \) of the linearized equation (2.34) associated with some \( w \in X(I) \) where \( X(I) \) is defined by (3.41) and \( I = (0, \tau] \) for some \( \tau, 0 < \tau \leq 1 \). Such a \( w \) satisfies
\[ \| w(t); V^k \| \leq Y h(t) \] (5.50)
for some \( Y > 0 \) and all \( t \in I \). The following lemma is a variant of Lemma 4.1.

**Lemma 5.3.** Let \( v_a \) satisfy (3.30) with a defined by (5.37) and let \( \varphi_a \) be defined by (2.29) and \( \varphi_a(1) = 0 \). Let \( 0 < \tau \leq 1 \) and \( I = (0, \tau] \). Let \( w \in X(I) \) satisfy (5.50) for some \( Y > 0 \) and all \( t \in I \), and let \( \tau \) be sufficiently small so that
\[ Y \mathcal{H}(\tau) \leq a . \] (5.51)
Let $\psi$ be defined by (2.30) with $\psi(0) = 0$. Then $\psi$ satisfies the estimate
\[
\| \omega^\ell \psi \|_2 \leq c a Y t^{-(\ell/2+1/2-k)_+} h(t) \quad \text{for } 0 < \ell \leq k + 2 .
\] (5.52)

Let $w' \in C(I, V^k)$ be a solution of the equation (2.34). Then the following estimates hold:
\[
| \partial_t \| w' \|_2 | \leq \| R_1 \|_2 ,
\] (5.53)
\[
| \partial_t \| x w' \|_2 | \leq C \left(1 + a^2 (1 - \ell n t)\right) \| w'; V^k \| + \| x R_1 \|_2 ,
\] (5.54)
\[
| \partial_k \| \omega^k w' \|_2 | \leq C a^2 (1 + a^2) (1 - \ell n t)^2 \| w'; V^k \| + \| \omega^k R_1 \|_2 ,
\] (5.55)
\[
| \partial_k \| \omega^{k-1} x w' \|_2 | \leq C \left(1 + a^2 (1 - \ell n t)\right)^2 \| w'; V^k \| + \| \omega^{k-1} x R_1 \|_2 ,
\] (5.56)
where $R_1 = R - H_1 v_a$.

**Proof.** It follows from (5.33) (5.50) (5.51) (3.39) that $\| \omega^\ell \partial_t \psi \|_2$ is integrable at $t = 0$ for $0 < \ell \leq k + 2$, which gives a meaning to the assumption $\psi(0) = 0$. The estimate (5.52) then follows from (5.33) by integration. Furthermore (5.51) implies
\[
\| \omega^\ell \psi \|_2 \leq c a Y \tilde{h}(\tau) \leq C a^2 \quad \text{for } 0 < \ell \leq k + 2 .
\] (5.57)

The proof of (5.53)-(5.56) is a variant of that of the estimates (4.2)-(4.4) of $v'$ in Lemma 4.1. We estimate in particular
\[
\| v; V^k \| \leq C a ,
\]
\[
\| \omega^\ell \! s \|_2 \leq \| \omega^\ell s_a \|_2 + \| \omega^\ell \sigma \|_2 \leq C a^2 (1 - \ell n t)
\] (5.58)
for $0 \leq \ell \leq k + 1$ by (5.38) (5.57). Furthermore
\[
\| \omega^\ell B \| \leq C a^2 \quad \text{for } 0 < \ell \leq k + 1 ,
\] (5.59)
\[
\| \omega^\ell g \| \leq C a^2 \quad \text{for } 0 \leq \ell \leq k
\] (5.60)
by (5.36) (5.50) (5.51) and Lemma 3.3, while
\[
\| \omega^\ell \tilde{B}_S \| \leq \| \omega^{k-1} \tilde{G} \|_2 
\]
\[
\leq C \left(a^2 t^{-1+(k+2-\ell)/2} + a Y t^{-1+k-\ell/2-1/2} h\right)
\] for $0 < \ell \leq 2k - 1$ by (4.42) (5.32) (5.34), so that by (5.51)
\[
\| \omega^\ell \tilde{B}_S \| \leq C a^2 \quad \text{for } 0 < \ell \leq k .
\] (5.61)
Substituting (5.58)-(5.61) into the analogues for \( w' \) of the estimates in the proof of Lemma 4.1, in particular (4.7) (4.8) yields (5.53)-(5.56).

\[ \square \]

We can now state the existence results of solutions of the linearized equation (2.34).

**Proposition 5.3.** Let \( 1 < k < 2 \). Let \( v_a \) satisfy (5.37) with \( a \) defined by (5.37) and let \( \varphi_a \) be defined by (2.25) and \( \varphi_a(1) = 0 \). Let \( w \in X(I) \) satisfy (5.51) for some \( Y > 0 \) and all \( t \in I \) and let \( \tau \) be sufficiently small to ensure (5.51). Let \( R \) be defined by (2.33) and satisfy

\[ \| R; L^1((0, t], V^k) \| \leq r h(t) \] (5.62)

for some \( r > 0 \) and all \( t \in I \). Then there exists a unique solution \( w' \in X(I) \) of the equation (2.34) and \( w' \) satisfies the estimate

\[ \| w'(t); V^k \| \leq \left( 1 + C(a) t(1 - \ln t)^2 \right) \left( r + C(a) Y t^{(k-1)/2} \right) h(t) \] (5.63)

for some constant \( C(a) \) depending on \( a \) and for all \( t \in I \).

**Proof.** Let \( 0 < t_0 < \tau \) and let \( w'_{t_0} \) be the solution of (2.34) in \( C(I, V^k) \) with initial condition \( w'_{t_0}(t_0) = 0 \). That solution is obtained by a minor variation of Proposition 4.1 including the inhomogeneous term \( R_1 \). We shall construct \( w' \) as the limit of \( w'_{t_0} \) when \( t_0 \to 0 \) and for that purpose we need estimates of \( w'_{t_0}(t) \) in \( V^k \) for \( t \in [t_0, \tau] \) that are uniform in \( t_0 \). From Lemma 5.2, especially (5.43) and from (5.50) (5.52), we obtain

\[ \| H_1 v_a; V^k \| \leq h(t) \equiv C \ a^2 Y \left( 1 + a^2 \ t^{1/2}(1 - \ln t) \right) t^{-(3-k)/2} \ h(t) \] (5.64)

On the other hand, from Lemma 5.3, especially (5.53)-(5.56) and from (5.62) (5.64), we obtain

\[ \| w'_{t_0}(t); V^k \| \equiv y(t) \leq \int_{t_0}^{t} f_1(t') \ y(t') \ dt' + f_2(t) \] (5.65)

where

\[ f_1(t) = C \left( 1 + a^2(1 - \ln t) \right)^2 \] (5.66)

\[ f_2(t) = r \ h(t) + \int_{0}^{t} dt' \ h_1(t') \]

\[ \leq \left( r + C \ a^2 Y \left( 1 + a^2 \ t^{1/2}(1 - \ln t) \right) t^{(k-1)/2} \right) h(t) \] (5.67)
by (3.39) (5.64). Integrating (5.65) with \( y(t_0) = 0 \) yields
\[
y(t) \leq \int_{t_0}^{t} dt' f_1(t') f_2(t') \exp \left( \int_{t'}^{t} dt'' f_1(t'') \right) + f_2(t). \tag{5.68}
\]
Substituting (5.66) (5.67) into (5.68) yields
\[
y(t) \leq \left( 1 + C \left( 1 + a^2(1 - \ln t) \right)^2 t \right) \exp \left( C \left( 1 + a^2(1 - \ln t) \right)^2 t \right)
\times \left( r + C a^2 Y \left( 1 + a^2 t^{1/2}(1 - \ln t) \right) t^{(k-1)/2} \right) h(t) \leq C(a, Y) h(t) \tag{5.69}
\]
uniformly in \( t_0 \). That estimate is of the form of (5.63). We now take the limit of \( w'_{t_0} \) when \( t_0 \to \infty \).

Let \( 0 < t_0 \leq t_1 \leq \tau \). From the conservation of the \( L^2 \) norm of the difference of two solutions of (2.34), it follows that
\[
\| w'_{t_0}(t) - w'_{t_1}(t) \|_2 = \| w'_{t_0}(t_1) \|_2 \leq C(a, Y) h(t_1) \tag{5.70}
\]
for all \( t \in [t_1, \tau] \). It follows from (5.70) that \( w'_{t_0} \) converges in \( L^\infty_{loc}(I, L^2) \) to a limit \( w' \in C(I, L^2) \). From that convergence and from the uniform estimate (5.69), it follows that \( w' \in C(I, H^{k'}) \cap (C_w \cap L^\infty)(I, V^k) \) for \( 0 \leq k' < k \) and that \( w'_{t_0} \) converges to \( w' \) in \( L^\infty_{loc}(I, H^{k'}) \) norm and weakly in \( V^k \) pointwise in time. From the previous convergences and from the uniform estimate (5.69) of \( w'_{t_0} \), it follows that \( w' \) satisfies the same estimate in \( I \). Clearly \( w' \) is a solution of (2.34). It then follows from an inhomogeneous extension of Proposition 4.1 that \( w' \in X(I) \). Finally the estimate (5.63) is a simplified version of (5.69).

\[\square\]

We can now derive the existence of solutions of the nonlinear system (2.30) (2.31).

**Proposition 5.4.** Let \( 1 < k < 2 \). Let \( v_a \) satisfy (5.36) with \( a \) defined by (5.37) and let \( \varphi_a \) be defined by (2.25) and \( \varphi_a(1) = 0 \). Let \( R \) be defined by (2.33) and satisfy (5.62) for all \( t \in (0, 1] \). Then there exists \( \tau, 0 < \tau \leq 1 \), depending on \( (a, r) \) and there exists a unique solution \( w \in X(I) \) of the equation (2.31) with \( \psi \) satisfying (2.30) and \( \psi(0) = 0 \), where \( I = (0, \tau] \). In particular \( w \) satisfies (5.50) for some \( Y \) depending on \( (a, r) \) and for all \( t \in I \) and \( \psi \) satisfies (5.52) for all \( t \in I \).

**Proof.** Let \( 0 < \tau \leq 1 \). For \( \tau \) sufficiently small, Proposition 5.3 defines a map \( \Gamma : w \to w' \) from \( X(I) \) into itself. We shall show that for \( \tau \) sufficiently small, the map \( \Gamma \) is a contraction on the subset \( R \) of \( X(I) \) defined by (5.50) for a suitable choice of \( Y \) in the norm considered in Proposition 5.1.
We first ensure that $\mathcal{R}$ is stable under $\Gamma$. From (5.63) it follows that
\[
\| w'; X(I) \| \leq \left( 1 + C(a) \tau(1 - \elln \tau)^2 \right) \left( r + C(a)Y^{(k-1)/2} \right) (5.71)
\]
and this can be made smaller than $Y$ by taking $Y = 2r$ and $\tau$ sufficiently small.

We next show that $\Gamma$ is a contraction on $\mathcal{R}$ for the $L^2$ norm of $w$. Let $w_i \in \mathcal{R}$ and $w'_i = \Gamma w_i$, $i = 1, 2$, let $w_\pm = (1/2)(w_1 \pm w_2)$ and similarly for $w'_\pm$. All those quantities belong to $\mathcal{R}$. We define the norms
\[
Y_- = \sup_{t \in I} h(t)^{-1} \| w_-(t) \|_2 \quad (5.72)
\]
\[
Y'_- = \sup_{t \in I} h(t)^{-1} \| w'_-(t) \|_2 \quad (5.73)
\]
and we estimate $Y'_-$ in terms of $Y_-$ by Lemma 4.2 with $v_- = w_-$, $v_+ = v_a + w_+$, $v'_+ = v_a + w'_+$, $s_- = \sigma_-$, $s_+ = s_a + \sigma_+$ and $y$ defined by (4.13). From (5.37) (5.50) (5.51) it follows that
\[
y \lor \| v_+ ; V^k \| \lor \| v'_+ ; V^k \| \leq C a \quad (5.74)
\]
From (5.38) (5.52) (5.51) it follows that
\[
\| \omega^\ell s_+ \|_2 \leq C a^2 (1 - \elln t) \quad \text{for } 0 \leq \ell \leq k + 1 \quad (5.75)
\]
From (4.15) it follows that
\[
\| \omega^\ell \sigma_- \|_2 \leq C a Y t^{-(\ell+2-k)/2} h(t) \quad \text{for } 0 \leq \ell \leq 1 \quad (5.76)
\]
Substituting (5.74)- (5.76) into (4.14) yields
\[
| \partial_t \| w'_- \|_2 | \leq C a^2 Y_- \left( 1 + a^2 t^{1/2}(1 - \elln t) \right) t^{(3-k)/2} h(t) \quad (5.77)
\]
and therefore by integration over time
\[
Y'_- \leq C a^2 Y_- \left( 1 + a^2 \tau^{1/2}(1 - \elln \tau) \right) \tau^{(k-1)/2} \quad (5.78)
\]
which implies the contraction property for the norm (5.72) (5.73) for $\tau$ sufficiently small. The existence result now follows from the fact that $\mathcal{R}$ is closed for that norm.

Finally the uniqueness result follows from Proposition 5.1 with $\alpha = 0$. 

\[\square\]
So far we have solved the auxiliary system (2.30) (2.31) for general \( v_a \) satisfying (5.36) and under the decay assumption (5.62) on the remainder \( R \) defined by (2.33).

We now take \( v_a = U(t)v_+ \). By (3.8), that \( v_a \) satisfies (5.36) for \( v_+ \in V^{k+1} \) with

\[
a = \| v_a ; L^\infty((0, 1], V^{k+1}) \| \leq 2 \| v_+ ; V^{k+1} \|.
\]

(5.79)

We now show that the corresponding remainder satisfies (5.62) with \( h(t) = t(1 - \elln t)^2 \).

**Proposition 5.5.** Let \( 1 < k < 2 \). Let \( v_+ \in V^{k+1} \) and \( v_a(t) = U(t)v_+ \). Let \( R \) and \( \varphi_a \) be defined by (2.33) (2.25) and \( \varphi_a(1) = 0 \). Then \( R \in C((0, 1], V^k) \) and \( R \) satisfies the estimate

\[
\| R(t) ; V^k \| \leq C a^3(1 + a^2)(1 - \elln t)^2
\]

(5.80)

for all \( t \in (0, 1] \).

**Proof.** For \( v_a = U(t)v_+ \), the remainder \( R \) takes the form

\[
R = -iK_a \cdot \nabla v_a - \left( (i/2) \nabla \cdot s_a + (1/2)K_a^2 + \tilde{B}_aS + g_a \right) v_a.
\]

(5.81)

Using the fact that the norm \( \| \cdot \|_* \) defined by (5.41) satisfies

\[
\| f_1f_2 ; V^k \| \leq C \| f_1 \|_* \| f_2 ; V^k \|
\]

(5.82)

by Lemma 3.1, we estimate

\[
\| R; V^k \| \leq C \left( \| K_a \|_* \| v_a; V^{k+1} \| + \left( \| \nabla \cdot s_a \|_* + \| K_a \|_*^2 \right) \right) \| v_a; V^k \|
\]

(5.83)

We then estimate

\[
\| B_a \|_* \leq C a^2
\]

(5.84)

\[
\| \tilde{B}_aS \|_* \leq \left( 1 + C t^{(k-1)/2} \right) \| \omega^{k+2} \tilde{B}_a \|_2 \leq C a^2
\]

(5.85)

by (5.34) (3.42),

\[
\| s_a \|_* \vee \| \nabla \cdot s_a \|_* \leq C a^2(1 - \elln t)
\]

(5.86)

by (5.38) and

\[
\| g_a \|_* \leq C a^2
\]

(5.87)
by Lemma 3.3. Substituting (5.84)-(5.87) into (5.83) yields (5.80). The continuity of $R$ in $V^k$ follows immediately from that of $v_a$ in $V^{k+1}$ and from the estimates.

Proposition 5.5 implies that $R$ satisfies the assumption (5.62) with

$$h(t) = t(1 - \ln t)^2$$

so that

$$ \overline{h}(t) = t^{(k-1)/2}(1 - \ln t)^2.$$ (5.89)

Putting together Propositions 5.4 and 5.5, we obtain the final result for the Cauchy problem at time zero for the system (2.21) (2.22) in the following form.

**Proposition 5.6.** Let $1 < k < 2$. Let $v_+ \in V^{k+1}$ with

$$a_+ = \| v_+; V^{k+1} \|.$$ (5.90)

Let $v_a = U(t)v_+$ and let $\varphi_a$ be defined by (2.25) and $\varphi_a(1) = 0$. Then there exists $\tau$, $0 < \tau \leq 1$, depending on $a_+$ and there exists a unique solution $(v, \varphi)$ of the system (2.21) (2.22) such that $v \in (C \cap L^\infty)(I, V^k)$, $\varphi \in C(I, H^{k+2})$, $v - v_a \in X(I)$ and $(\varphi - \varphi_a)(0) = 0$, where $I = (0, \tau]$ and $X(I)$ is defined by (3.41) with $h$ given by (5.88). The solution $(v, \varphi)$ satisfies the estimates

$$\| v(t) - v_a(t); V^k \| \leq Y t(1 - \ln t)^2$$

$$\| \omega^\ell (\varphi(t) - \varphi_a(t)) \|_2 \leq C a_+ Y t^{1-(\ell/2+1/2-k)+}(1 - \ln t)^2$$

for some $Y > 0$ depending on $a_+$ and for all $t \in I$.

We can now state the final result on the Cauchy problem for the system (1.1) with prescribed asymptotic behaviour at infinity.

**Proposition 5.7.** Let $1 < k < 2$. Let $u_+ \in FV^{k+1}$ with

$$a_+ = \| u_+; FV^{k+1} \|.$$ (5.93)

Let $u_a$ be defined by (2.35) (2.36) with $v_a = U(t)v_+$, $v_+ = Fu_+$ and $\varphi_a$ defined by (2.25) and $\varphi_a(1) = 0$. Then there exists $T \geq 1$ depending on $a_+$ and there exists a
unique solution \( u \) of the system \((1.1)\) such that \( \tilde{u} \in C(I, FV^k) \), where \( I = [T, \infty) \) and \( \tilde{u}(t) = U(-t)u(t) \), and such that

\[
\| \tilde{u}(t) - \tilde{u}_a(t); FV^k \| \leq C(a_+)t^{-1}(1 + \ell \ln t)^{2+k} \tag{5.94}
\]

for all \( t \in I \). More precisely, \( \tilde{u} \) satisfies the estimates

\[
\| |x|^{\ell} \tilde{u}(t) \|_2 \leq C a_+ \left(1 + a_+^2(1 + \ell \ln t)\right)^{\ell} \quad \text{for } 0 \leq \ell \leq k , \tag{5.95}
\]

\[
\| |x|^{\ell} \nabla \tilde{u}(t) \|_2 \leq C a_+ \left(1 + a_+^2(1 + \ell \ln t)\right)^{\ell} \quad \text{for } 0 \leq \ell \leq k - 1 , \tag{5.96}
\]

\[
\| \tilde{u}(t) - \tilde{u}_a(t) \|_2 \vee \| \nabla (\tilde{u}(t) - \tilde{u}_a(t)) \|_2 \leq C(a_+)t^{-1}(1 + \ell \ln t)^2 , \tag{5.97}
\]

\[
\| |x|^k (\tilde{u}(t) - \tilde{u}_a(t)) \|_2 \leq C(a_+)t^{-1}(1 + \ell \ln t)^{k+2} , \tag{5.98}
\]

\[
\| |x|^{k-1} \nabla (\tilde{u}(t) - \tilde{u}_a(t)) \|_2 \leq C(a_+)t^{-1}(1 + \ell \ln t)^3 . \tag{5.99}
\]

**Proof.** We first prove the existence of \( u \) with the properties stated. Let \((v, \varphi)\) be the solution of the system \((2.21)\) \((2.22)\) obtained in Proposition 5.6 and define \( u \) by \((2.6)\) \((2.16)\). Then \( u \) is a solution of the system \((1.1)\) defined in \( I = [T, \infty) \) with \( T = \tau^{-1} \). The properties of \( u \) follow from those of \((v, \varphi)\) and from the estimates which we now derive. By \((2.6)\) \((2.35)\) and \((3.8)\) it is sufficient to estimate \( u_c \) and \( u_c - u_{ca} \) in \( V^k \). From \((5.38)\) and from Lemma 3.4, part (1) with \( m = 2 \) and part (3), it follows that

\[
\| \omega^\ell u_{ca}(t) \|_2 \leq C a_+ \left(1 + a_+^2(1 - \ell \ln t)\right)^{\ell} \quad \text{for } 0 \leq \ell \leq k , \tag{5.100}
\]

\[
\| \omega^\ell x u_{ca}(t) \|_2 \leq C a_+ \left(1 + a_+^2(1 - \ell \ln t)\right)^{\ell} \quad \text{for } 0 \leq \ell \leq k - 1 . \tag{5.101}
\]

We next estimate the difference

\[
\begin{align*}
    u_c - u_{ca} &= v \exp(-i\varphi) - v_a \exp(-i\varphi_a) \\
    &= (v(\exp(-i\psi) - 1) + v - v_a) \exp(-i\varphi_a) \tag{5.102}
\end{align*}
\]

with \( \psi = \varphi - \varphi_a \). From \((5.91)\) \((5.92)\) and Lemma 3.4, part (2), it follows that

\[
\| v(\exp(-i\psi) - 1); V^k \| \leq C a_+ \left(a_+ Y t(1 - \ell \ln t)^2 + \left(a_+ Y t(1 - \ell \ln t)^2\right)^2\right) \\
    \leq C a_+^2 Y t(1 - \ell \ln t)^2 \tag{5.103}
\]
for $\tau$ sufficiently small so that $a_+ Y \tau (1 - \ell n \tau)^2 \leq 1$. From (5.91), (5.103) (5.38) and Lemma 3.4, part (1) with $m = 2$, it then follows that

$$\| <x>(u_c(t) - u_{ca}(t)) \|_2 \leq C \left( 1 + a_+^2 \right) Y t (1 - \ell n t)^2,$$

(5.104)

$$\| \omega^k (u_c(t) - u_{ca}(t)) \|_2 \leq C \left( 1 + a_+^2 \right)^{k+1} Y t (1 - \ell n t)^{k+2},$$

(5.105)

$$\| \omega^{k-1} x (u_c(t) - u_{ca}(t)) \|_2 \leq C \left( 1 + a_+^2 \right)^2 Y t (1 - \ell n t)^3.$$

(5.106)

The estimates (5.97)-(5.99) follow from (5.104)-(5.106) and the estimates (5.95) (5.96) follow from (5.100) (5.101) and (5.97)-(5.99).

Uniqueness of $u$ follows from Proposition 5.2 with $h_2(t) = t(1 - \ell n t)^2$ and $\beta = k$, which satisfy the conditions (5.14) (5.15).

\[\square\]

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A - Appendix. Properties of $\Sigma^k$.

We work in arbitrary space dimension $n$, although we need only the special case $n = 2$. We first show that $\Sigma^k \subset V^k$. For that purpose it suffices to prove the following lemma.

**Lemma A.1.** Let $1 < k < 2$. Then the following estimate holds

$$\| <\omega>^{k-1} x v \|_2 \leq C \left( \|<\omega>^k v\|_2 + \|<x>^k v\|_2 \right)$$

(A.1)
or equivalently

\[ \| \omega^{k-1} xv \|_2 \leq C \| \omega^k v \|_2^{1/k} \| |x|^k v \|_2^{1/k}. \quad (A.2) \]

**Proof.** From the elementary estimate

\[ \| v \|_2 \leq C \left( \| \omega^k v \|_2 + \| |x|^k v \|_2 \right) \quad (A.3) \]

it follows that \( (A.1) \) is equivalent to

\[ \| \omega^{k-1} xv \|_2 \leq C \left( \| \omega^k v \|_2 + \| |x|^k v \|_2 \right). \quad (A.4) \]

Clearly \( (A.2) \) implies \( (A.4) \). Conversely \( (A.2) \) follows from \( (A.4) \) by a dilation of \( v \) followed by an optimization of the dilation parameter.

In order to prove \( (A.1) \), we use a dyadic decomposition. Let \( \hat{\psi} \in C^\infty(\mathbb{R}^n, \mathbb{R}) \), \( 0 \leq \hat{\psi} \leq 1 \), \( \hat{\psi}(\xi) = 1 \) (resp. \( 0 \)) for \( |\xi| \leq 1 \) (resp. \( \geq 2 \)), let \( \hat{\varphi}_0 = \hat{\psi} \) and \( \hat{\varphi}_j(\xi) = \hat{\psi}(2^{-j} \xi) - \hat{\psi}(2^{-(j-1)} \xi) = \hat{\varphi}_1(2^{-(j-1)} \xi) \), so that

\[ \varphi_j(x) = 2^{(j-1)n} \varphi_1 \left( 2^{j-1} x \right) \]

and therefore

\[ \| |x|^\ell \varphi_j \|_1 = 2^{-(j-1)\ell} \| |x|^\ell \varphi_1 \|_1 \quad \text{for } \ell \geq 0 \quad (A.5) \]

where we use the notation \( \hat{\varphi} = F\varphi \). Clearly for \( \ell \geq 0 \)

\[ C^{-1} \| <\omega>^\ell v \|_2^2 \leq \sum_{j \geq 0} 2^{2j\ell} \| \varphi_j \ast v \|_2^2 \leq C \| <\omega>^\ell v \|_2^2. \quad (A.6) \]

Let now \( \ell = k - 1 \) so that \( 0 < \ell < 1 \). We estimate

\[ \| <\omega>^\ell xv \|_2^2 \leq C \sum_{j \geq 0} 2^{2j\ell} \| \varphi_j \ast xv \|_2^2. \quad (A.7) \]

Now

\[ \varphi_j \ast xv = x (\varphi_j \ast v) - (x \varphi_j) \ast v \]

so that by \( (A.5) \) and the Young inequality

\[ \| \varphi_j \ast xv \|_2 \leq \| x (\varphi_j \ast v) \|_2 + 2^{-(j-1)} \| |x|^j \varphi_1 \|_1 \| v \|_2. \quad (A.8) \]

Substituting \( (A.8) \) into \( (A.7) \) yields

\[ \| <\omega>^\ell xv \|_2^2 \leq C \sum_{j \geq 0} 2^{2j\ell} \| x (\varphi_j \ast v) \|_2^2 + C \| v \|_2^2. \quad (A.9) \]
We estimate the sum in the RHS by the Hölder inequality in \((j, x)\) as
\[
\sum_{j \geq 0} 2^{2j\ell} \| x (\varphi_j \star v) \|_2^2 \leq \left( \sum_{j \geq 0} 2^{jk} \| \varphi_j \star v \|_2^2 \right)^{\ell/k} \left( \sum_{j \geq 0} \| |x|^k (\varphi_j \star v) \|_2^2 \right)^{1/k} 
\]
\[
\leq C \| <\omega >^k v \|_2^{2\ell/k} \left( \sum_{j \geq 0} \| |x|^k (\varphi_j \star v) \|_2^2 \right)^{1/k} \tag{A.10}
\]
by (A.6). Now from the inequality
\[
||x|^k - |y|^k| \leq k|x - y|(|x - y|^{\ell} + |y|^{\ell})
\]
we obtain
\[
||x|^k (\varphi_j \star v) - \varphi_j \star |x|^k v| \leq k \left((||x|^k|\varphi_j| \star |v| + (|x| |\varphi_j|) \star |x|^\ell |v|)\right)
\]
and therefore by (A.5) and the Young inequality
\[
\| |x|^k (\varphi_j \star v) \|_2 \leq \| \varphi_j \star |x|^k v \| + k\left(2^{-(j-1)k} \| |x|^k \varphi_1 \|_1 \| v \|_2 \right.
\]
\[
+ 2^{-(j-1)} \| |x| \varphi_1 \|_1 \| |x|^\ell v \|_2 \bigg)
\tag{A.11}
\]
so that by (A.6)
\[
\sum_{j \geq 0} \| |x|^k (\varphi_j \star v) \|_2^2 \leq C \left( \| |x|^k v \|_2^2 + \| |x|^\ell v \|_2^2 + \| v \|_2^2 \right) \tag{A.12}
\]
Substituting (A.12) into (A.9) (A.10) yields (A.1).

We next prove that \(\Sigma^k\) is stable under the free Schrödinger evolution \(U(t, \cdot)\) and for that purpose we prove the estimate (3.11). We treat only the case \(1 < k < 2\), but the extension to general \(k\) is straightforward.

**Lemma A.2.** Let \(1 < k < 2\). Then
\[
\| U(t)v; \Sigma^k \| \leq C \left( \| v; \Sigma^k \| \ + \| t^k \| \omega^k v \|_2 \right) \tag{A.13}
\]

**Proof.** By (A.3), one can use for \(\Sigma^k\) the equivalent norm
\[
\| v; \Sigma^k \| = \| \omega^k v \|_2 \ + \| |x|^k v \|_2
\]
\[
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\]
and by the commutation relation (3.6), it is sufficient to prove that

$$\| x + it \nabla |^k v \| \leq C \left( \| |x|^k v \|_2 + |t|^k \| \omega^k v \|_2 \right). \quad (A.14)$$

By homogeneity (namely dilation of \( v \) by \( |t|^{1/2} \)), (A.14) is equivalent to the special case \( t = 1 \). It is then sufficient to prove that

$$\| \omega^k \exp(ix^2/2)v \|_2 \leq C \left( \| \omega^k v \|_2 + \| x \|_k v \|_2 \right) \quad (A.15)$$

We first prove (A.15) with \( k \) replaced by \( \ell \) with \( 0 < \ell < 1 \). By (3.16)

$$\| \omega^\ell \exp(ix^2/2)v \|_2^2 = C \int dy \ |y|^{-n-2\ell} \| (\tau_y - \tau_{-y}) \exp(ix^2/2)v \|_2^2 \quad (A.16)$$

Now

$$(\tau_y - \tau_{-y}) \exp(ix^2/2)v = (\tau_y \exp(ix^2/2)) (\tau_y v - v) - \tau_{-y} \exp(ix^2/2) (\tau_y v - v) + 2i \exp((i(x^2 + y^2)/2) \sin(xy)v(x)$$

so that

$$\| (\tau_y - \tau_{-y}) \exp(ix^2/2)v \|_2 \leq \| \tau_y v - v \|_2 + \| \tau_{-y} v - v \|_2 + 2 \| \sin(\cdot) v(\cdot) \|_2 \quad (A.17)$$

Substituting (A.17) into (A.16) and using again (3.16) yields

$$\| \omega^\ell \exp(ix^2/2)v \|_2^2 \leq C \left( \| \omega^\ell v \|_2^2 + \int dy \ |y|^{-n-2\ell} \| \sin(\cdot) v(\cdot) \|_2^2 \right) \quad (A.18)$$

The last integral is

$$\int dy \ dx \ |y|^{-n-2\ell} \sin^2(xy)|v(x)|^2 = C \| x \|_\ell v \|_2^2$$

which completes the proof of (A.15) with \( k \) replaced by \( \ell \).

We next take \( \ell = k - 1 \) and we estimate

$$\| x + i \nabla |^k v \|_2 = \| x + i \nabla |^\ell (x + i \nabla)v \|_2$$
$$\leq C \left( \| x |^k v \|_2 + \| \omega^\ell x v \|_2 + \| x |^\ell \nabla v \|_2 + \| \omega^k v \|_2 \right)$$
$$\leq C \| v; \Sigma_k \|$$

by (A.15) with \( k \) replaced by \( \ell \) and Lemma A.1. 
\( \square \)
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