Performance Improvement of Extremum Seeking Control using Recursive Least Square Estimation with Forgetting Factor

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Abstract: The main limitation of perturbation based extremum seeking methods is the requirement of a multiple time-scale separation between the system dynamics, the perturbation frequency, and the adaptation rate so as to avoid interactions and possible instabilities. This causes the convergence to be extremely slow. In the present work, we propose a simple modification to the perturbation-based extremum seeking control method that can be used when the system cannot be accurately approximated by a Wiener-Hammerstein model for which convergence rate acceleration schemes are available. The linear filtering used in the perturbation based extremum seeking control for estimating the objective function gradient is replaced by a recursive least square with forgetting factor estimation algorithm. It is shown that this simple modification can accelerate convergence to the optimum by removing one time scale separation.

1. INTRODUCTION

After the publication of (Krstic and Wang (2000)) in which a formal proof of convergence has been established, perturbation based extremum seeking methods (Blackman (1962)) became once more an active and popular field of research in the control community. The main limitation of this class of extremum seeking methods remains the requirement of a multiple time-scale separation between the system dynamics, the perturbation frequency, and the adaptation rate so as to avoid interactions and possible instabilities (Krstic and Wang (2000)). The perturbation frequency should be slow enough to consider the system to be a static one. This, in turn, causes the convergence to be extremely slow, i.e. two orders of magnitude slower than the system dynamics. Though this is acceptable for fast systems whose time constants range in seconds (typically found in mechanical and electrical systems) (Wang et al. (2000)), it becomes unacceptable for chemical or biochemical systems where the time constants are in hours or days (Dochain et al. (2011)). This means that one would need a month to a year to complete an optimization cycle. Two concepts have been used to improve the convergence rate of extremum-seeking control using perturbations. Firstly, control algorithms can be applied to the system in order to accelerate its dynamics and increase its bandwidth (see e.g. (Krstic (2000); Chioia et al. (2007b))). Secondly, a phase compensator can be used to correct for the phase shift introduced by the system dynamics at the perturbation frequency (Arur and Krstic (2003); Krstic (2000)). Control algorithms seek to reduce the phase shift for a wider range of frequencies but are limited by the system relative degree, the internal dynamics stability and the presence of delays. Phase compensator, on the other hand, concentrates on the given perturbation frequency and requires the knowledge of the phase shift at that frequency.

In (Atta et al. (2014)) and (Chioia et al. (2007a)), the authors follow the phase compensation idea, but instead of relying on an a priori value, the phase shift is estimated based on the available measurements. In (Atta et al. (2014)), a Kalman filter is used to estimate the phase and amplitude of the first harmonic of the system output. In (Chioia et al. (2007a)), the phase is estimated by modulating the output with the quadrature of the perturbation signal. A low frequency perturbation is added to determine the sign of the gradient which is in turn necessary for the stability of the phase adaptation loop. Applicability of the phase compensation method proposed in (Chioia et al. (2007a)) is restricted to systems that can be approximated by a Wiener-Hammerstein model.

Recently, in (Guay and Dochain (2015); Elsman and M.Guay (2014)) the authors reformulate the extremum seeking control problem as a time-varying estimation problem.
In the present work, we propose a simple modification to the perturbation-based extremum seeking control method that can be used when the system cannot be accurately approximated by a Wiener-Hammerstein model. We suggest to replace the linear filtering used in the perturbation based extremum seeking control method for the gradient estimation by a recursive least square with forgetting factor estimation and show that this leads to an acceleration of convergence to the optimum by removing one time scale separation.

The paper is organized as follows: The next section introduces the traditional perturbation method for extremum seeking. Section 3 presents the modified perturbation method with recursive least squares estimation for which a convergence analysis is provided in Section 4. A simple example is presented in Section 5 and Section 6 concludes the paper.

\[ P(\tilde{u}) = \frac{1}{2}G_0((\tilde{u}a)^2 + a^2/2) + 2G_0\tilde{u}a\sin(\omega t - \phi_0) - G_2a^2/2 \cos(2\omega t - \phi_2) \] 

(8)

Fig. 1. Extremum seeking control via perturbation method inspired from (Krístic and Wang 2000).

Convergence is established in (Krístic and Wang 2000) through the following steps:

- The exponential stability of the equilibrium point of the above averaged system ($\tilde{u}^a$, $\xi^a$, and $\eta^a$) is first proved.
- From there on, the exponentially stability of $(u, \xi, \eta)$ (non-averaged) is established using the averaging theorem (Krístic (2002)).
- This non-averaged system $(u, \xi, \eta)$ acts as the "slow" manifold, while the original system $\dot{x} = F(x, u)$ acts as the boundary layer system which is assumed to be exponentially stable. Then, singular perturbation ideas are applied to show that their interconnection is also exponentially stable (Krístic (2002)).

The key assumption is that the system is at quasi-steady state, i.e., to a second order approximation,

\[ \nu(\tilde{u}) = \frac{1}{2}\nu^a((\tilde{u}^a)^2 + a^2/2 + 2\tilde{u}^a\sin(\omega t) - \frac{a^2}{2} \cos(2\omega t)) \] 

(7)

But due to the dynamics of the system, the sinusoids of frequencies $\omega$ and $2\omega$ will have phase shifts which will affect convergence. So, instead of using $\nu(\tilde{u})$ the following dynamic operator $P(\tilde{u})$ is used:

\[ P(\tilde{u}) = \frac{1}{2}G_0((\tilde{u}^a)^2 + a^2/2) + 2G_0\tilde{u}^a\sin(\omega t - \varphi_0) - G_2\omega a^2/2 \cos(2\omega t - \varphi_2) \] 

(8)

425
where $G_{i\omega}$ and $\varphi_{i\omega}$ are the gain and the phase-shift of
the second derivative at frequency $i\omega$. The averaged
equations taking this phase shift into account are:

$$
\frac{d}{dt} \begin{bmatrix} \tilde{a}^2 \\ \tilde{a}^2 \end{bmatrix} = \begin{bmatrix} -\omega_l(\xi^a - G_{i\omega}\tilde{a}^2 \cos(\varphi_{i\omega})) \\ -\omega_h(\eta^a - G_{i\omega}(a^2 + \frac{a^2}{2})) \end{bmatrix}
$$  

(9)

3. EXTREMUM SEEKING USING RECURSIVE LEAST SQUARES WITH FORGETTING FACTOR FOR GRADIENT ESTIMATION

3.1 Phase estimation using output measurement requires a Wiener/Hammerstein structure

Wiener or Hammerstein models are widely used to represent
nonlinear dynamic systems (Wittenmark and Evans 2001). Such models have linear dynamics and a static
nonlinearity. The difference between Wiener and Hammer-
stein models comes from the order in which the linear and
nonlinear blocks are placed. The Wiener model consists of
a static nonlinearity followed by a linear dynamics, while
in the Hammerstein model, the linear dynamics are placed
first. The output of a Wiener model reads

$$
y(u, s) = P(u) = f(u)G(s)
$$  

(10)

$f(u)$ is a nonlinear static map of the input $u$ and $G(s)$ is a
linear transfer function. The first and second derivatives of
the output $y$ with respect to $u$ are respectively

$$
\frac{\partial y}{\partial u} = \frac{\partial P}{\partial u} = \frac{\partial f}{\partial u}G(s)
$$  

(11)

$$
\frac{\partial^2 y}{\partial u^2} = \frac{\partial^2 P}{\partial u^2} = \frac{\partial^2 f}{\partial u^2}G(s)
$$  

(12)

Note the phase of the system output is also the phase of
the system output first and second order derivatives and that
this property holds for a Hammerstein model. In (8), $P(\hat{u})$ is a dynamic operator and $G_{i\omega}$ and $\varphi_{i\omega}$ are
respectively the gain and phase of the second derivative
at frequency $i\omega$. For a Wiener/Hammerstein system, the
phase only depends on the term $G(s)$ in (10). Therefore,$\varphi_{i\omega}$ in (8) is also the phase of the system output and of its
first derivative at frequency $i\omega$. It follows that the phase
estimation can therefore be achieved using the system
output measurement.

This phase invariance property of Wiener-Hammerstein
system is used in (Chioa et al. 2007a) to evaluate and
compensate the system phase in order to improve conver-
gence of perturbation-based extremum seeking methods.
However, if the system cannot be approximated by a
Wiener-Hammerstein model, the phase compensation as
proposed in (Chioa et al. 2007a) would require an
evaluation of the second order derivative of the output of
a nonlinear system prohibiting therefore its use.

3.2 Convergence analysis of gradient estimation using linear filtering and recursive least squares with forgetting factor

Gradient estimation using linear filtering The gradient estimation by linear filtering used in the perturbation
based extremum seeking control method reads

$$
\dot{\xi} = -\frac{a^2}{2}\omega_l\xi + \omega_l(y - \eta)d
$$  

(13)

$$
\dot{\eta} = -\omega_h\eta + \omega_hy
$$  

(14)

where $\xi$ and $\eta$ are respectively the low filter and high filter
states (Fig.1) and $d = a \sin(\omega t)$, the perturbation signal.
The following Lyapunov function is used:

$$
V = \frac{1}{2}\eta^2 + \frac{1}{2}\xi^2
$$  

(15)

Using the two deviation variables $\bar{\xi} = \xi - \bar{\xi}^*$ and $\bar{\eta} = \eta - \bar{\eta}^*$, the state equations (13) and (14) can be rewritten as

$$
\dot{\bar{\xi}} = -\omega_l\bar{\xi}^2 + \omega_l\xi^*d + \omega_l\xi^*(d^2 - \frac{a^2}{2})
$$  

(16)

$$
\dot{\bar{\eta}} = -\omega_h\bar{\eta} + \omega_h\xi^*d
$$  

(17)

The time derivative of the Lyapunov function in (15) is:

$$
\dot{V} = -\omega_h\bar{\eta}^2 - \frac{a^2}{2}\omega_l\bar{\xi}^2 + \omega_h\bar{\eta}\bar{\xi}^*d - \omega_l\bar{\eta}\bar{\xi}^*d + \omega_l\xi^*d^2 - \frac{a^2}{2}
$$  

(18)

and one can observe that the sign of the Lyapunov function
time derivative is indefinite. Consider now the averaged
system for the two states $(\bar{\xi}, \bar{\eta})$ obtained by taking the
average of the left hand side of (13) and (14) over the
interval $[0, \frac{\pi}{2\omega}]$, the averaged states are denoted $(\bar{\cdot}, \hat{\cdot})a$.
The averaged system reads (Khalil (2002)):

$$
\dot{\bar{\xi}}^a = -\omega_l\bar{\xi}^2 + \frac{a^2}{2}\omega_l\bar{\xi}^2 + \omega_h\bar{\eta}\bar{\xi}^*d
$$  

(19)

$$
\dot{\bar{\eta}}^a = -\omega_h\bar{\eta} + \omega_h\xi^*d
$$  

(20)

the time derivative of the Lyapunov function (15), with
$\omega_l = \frac{\xi}{\omega}$ and $\omega_h = \frac{\eta}{\eta}$, reads

$$
\dot{V}^a = -(\frac{1}{2}\bar{\eta}^2 + \frac{1}{2}\bar{\xi}^2)
$$  

(21)

and $V^a \leq -kV^a \leq 0$

Using the averaging theorem (Khalil (2002)), the system is
exponentially stable for a small enough value of $k$.

Let us now examine the obtained precision on the param-
eters estimated by linear filtering, the equilibrium point
of the averaged system, (19 and 20) is $(\bar{\xi}^a, \bar{\eta}^a) = (0, 0)$.
However, depending on the choice of the cut-off frequencies
$(\omega_l, \omega_h)$ or equivalently on the choice of the gain $k$,
the estimation of $(\bar{\xi}, \bar{\eta})$ can be biased.

This bias can be explained by the averaging Theorem
(Khalil (2002)) that states that the distance between the
state of a dynamical system and the same averaged
dynamical system is bounded by a function of order $O(k)$.

To evaluate the bias resulting from a linear filtering esti-
mation, consider (17). This relation being linear, a Laplace
transform is valid and reads

$$
\hat{\eta}(s) = \frac{\omega_h\bar{\xi}^*}{d(s)}
$$  

(22)

with

$$
d(s) = \frac{a\omega}{s^2 + \omega^2}
$$  

(23)

of which the solution in time domain is

$$
\bar{\eta}(t) = \frac{a\omega_h\xi^*}{\sqrt{\omega^2 + \omega_h^2}}\sin(\omega t - \tan^{-1}(\frac{\omega}{\omega_h}))
$$  

(24)

Substituting this solution in (16), one obtains

$$
\dot{\xi} = -\omega_l\bar{\eta}^2 + \omega_lf(t)
$$  

(25)

with
\[ f(t) = \frac{a^2 \omega h \xi^*}{\sqrt{\omega^2 + \omega_h^2}} \left( \cos(2\omega t - \tan^{-1}\frac{\omega}{\omega_h}) + \omega_h \sqrt{\omega^2 + \omega_h^2} \right) \] (26)

that reads after Laplace transform

\[ \frac{\hat{\xi}(s)}{f(s)} = \frac{\omega_l}{\frac{\alpha^2}{2} \omega_h + s} \] (27)

The bias observed in average can be deduced by evaluating the average of the time domain solutions of (22) and (27). Note that those are different from the solution of the averaged system solution (19 and 20).

\[ \frac{\omega}{2\pi} \int_0^{2\pi} \hat{\xi} dt = -\frac{\omega^2 \xi^*}{\sqrt{\omega^2 + \omega_h^2}} \] (28)

\[ \frac{\omega}{2\pi} \int_0^{2\pi} \hat{\eta} dt = 0 \] (29)

This bias vanishes in the case of extremum seeking control for which the value of \( \xi^* \) (gradient of the objective function at the optimum) is zero, but could reduce the precision of the optimizing control if this used for a slope seeking control (see e.g. (Ariyur and Krstic (2004)) for a detailed description of this method).

**Gradient estimation using recursive least squares with forgetting factor** An estimation of the objective function (2) using recursive least squares with forgetting factor reads

\[ \dot{\hat{\theta}} = -R^{-1} \varphi \varepsilon \] (30)

\[ \dot{\hat{R}} = 2(\varphi \varphi^T - \lambda R) \] (31)

with \( \varepsilon = y - \hat{y} \) estimation error, \( \hat{y} = \varphi^T \hat{\theta} \) model output, linear w.r.t the parameters \( \theta \) and \( \lambda \) forgetting factor. Using this parametrization of the output \( y \), the estimation law can be rewritten as

\[ \dot{\hat{\theta}} = -R^{-1} \varphi \varphi^T \hat{\theta} \] (32)

\[ \dot{\hat{R}} = 2(\varphi \varphi^T - \lambda R) \] (33)

where \( \hat{\theta} = \theta - \hat{\theta} \) is the estimation error. Existence of the inverse of the covariance matrix \( R = \int_0^\infty e^{-\lambda t} \varphi \varphi^T d\tau + R(0)e^{-\lambda t} \) is guaranteed by the persistency of excitation condition (Anderson et al. (1986)).

Consider the Lyapunov function

\[ V = \frac{1}{2} \hat{\theta}^T R \hat{\theta} \] (34)

the time derivative of this Lyapunov function reads

\[ \dot{V} = -\lambda \hat{\theta}^T R \hat{\theta} = -2\lambda V \leq 0 \] (35)

From (35) one can conclude that contrarily to the case of a gradient estimation by linear filtering, the exponential convergence of a gradient estimation using a recursive least square with forgetting factor method does not require using the averaging theorem whose conditions of applicability impose an upper bound on the value of the estimation gain depending on the perturbation signal frequency, itself related to the system dynamics. This restriction being removed, the "only" limitation on the rate of estimation is the presence of measurement noise. To illustrate this, the following simple example compares the performance of both estimation methods for the case of a model with no structural mismatch.

**Example 1** Consider a system for which the mapping between the system input and output is \( y = a + bu \). The estimation of parameters \( a = 5 \) and \( b = 6 \) is achieved using both the linear filtering and the recursive least square with forgetting factor methods. The perturbation signal used is \( u = \sin(\omega t) \) which frequency is set to 1 h\(^{-1}\). The linear filtering estimation is tested for three values of the gain \( k = 0.3 \), \( k = 1 \) and \( k = 2 \).

Figures 2 to 4 compare the obtained results by linear filtering and recursive least squares with forgetting factor estimation for three values of the gain \( k = 0.3 \), \( k = 1 \) and \( k = 2 \). Increasing the gain value leads to an increased level of oscillation on the estimated parameter \( \hat{a} \) and linear filtering, estimation gain for linear filtering = 0.3.
Consider an adaptive extremum seeking control using a gradient descent law for the computation of the optimizing control

\[ \dot{u} = -\Gamma_o \frac{d\hat{y}}{du} \quad (36) \]

with

\[ d = a \sin(\omega t) \quad (38) \]

and \( \Gamma_o \) the adaptation gain of the control.

If the parameters estimation uses a recursive least squares with forgetting factor (31), (31) The adaptation and control laws reads

\[ \dot{\hat{u}} = -\Gamma \frac{\partial \varphi}{\partial u} \quad (39) \]

\[ \dot{\hat{R}} = 2(\varphi^T \varphi - \lambda \hat{R}) \quad (40) \]

\[ \dot{\hat{\theta}} = -R^{-1}(\varphi y - \varphi \varphi^T \hat{\theta}) \quad (41) \]

The system output to be optimized reads

\[ J = y = \varphi^T \theta \quad (42) \]

Consider the following Lyapunov function

\[ V = \frac{1}{2} \hat{\theta}^T \hat{R} \hat{\theta} + \frac{1}{2} \left( \frac{dJ}{du} \right)^2 \quad (43) \]

with \( \hat{\theta} = \theta - \hat{\theta} \)

The gradient of the objective function reads

\[ \frac{dJ}{du} = \frac{\partial \varphi}{\partial u} \theta + \frac{\varphi}{\partial u} \quad (44) \]

the time derivative of the Lyapunov function reads

\[ \dot{V} = -\hat{\theta} \left( \frac{\partial \varphi}{\partial u} \theta \right) \frac{\partial^2 J}{\partial u^2} - \frac{\partial \varphi}{\partial u} R^{-1} \frac{\partial^2 J}{\partial u \partial \theta} \varphi^T \varphi + \lambda \hat{\theta} \]

\[ -\Gamma \frac{\partial \varphi}{\partial u} \left( \frac{\partial \varphi}{\partial u} + \frac{\partial \varphi}{\partial \theta} \right)^2 \]

\[ -\left( 2\Gamma \frac{\partial \varphi}{\partial u} + R^{-1} \frac{\partial^2 J}{\partial u \partial \theta} \varphi^T \varphi \right) \hat{\theta} \frac{\partial \varphi}{\partial u} \theta + \frac{\partial \varphi}{\partial u} \quad (45) \]

using the following notations

\[ A = \lambda R > 0 \]

\[ B = \Gamma_o \frac{\partial^2 J}{\partial u^2} > 0 \]

\[ C = \Gamma_o \left( \frac{\partial \varphi}{\partial u} \right)^2 \frac{\partial^2 J}{\partial u^2} > 0 \]

\[ D = 2\Gamma_o \frac{\partial \varphi}{\partial u} + R^{-1} \frac{\partial \varphi}{\partial u} - \frac{\partial^2 J}{\partial u \partial \theta} \varphi^T \varphi \]

\[ X = \hat{\theta} \]

\[ Y = \frac{\partial \varphi}{\partial u} \theta + \frac{\partial \varphi}{\partial \theta} \frac{dJ}{du} \]

the time derivative of the Lyapunov function reads

\[ \dot{V} = -X^T (A - C) X - BY^2 + DXY \quad (52) \]

note that \( \dot{V} < 0 \) for \( D^2 < 4(A - C)B \) therefore (39-41) converges to \((X, Y) = (0, 0)\) i.e. \( \hat{\theta} \rightarrow \theta \) and \( G = \frac{\partial \varphi}{\partial u} \rightarrow 0 \)

i.e. \( J \rightarrow J_{opt} \).

5. ILLUSTRATIVE EXAMPLE

5.1 Description of the system

A simple isothermal reaction system in a continuous stirred tank reactor (CSTR) will be considered to illustrate the convergence improvement of the modified extremum control method

- Reaction system: \( A \rightarrow B \rightarrow C, 2A \rightarrow E \).
- Model equations:

\[ \frac{dC_A}{dt} = D(C_{A_{in}} - C_A) - k_1 C_A - 2k_3 C_A^2 \]

\[ \frac{dC_B}{dt} = k_1 C_A - k_2 C_B - D C_B \]

- Variables: \( C_X \), concentration of species \( X \); \( D \), Dilution rate; \( k_i \), rate constants; \( A_{in} \), inlet concentration.
- Objective: Maximize the concentration of product \( B \), \( C_B \).
- Manipulated variable: \( D \).
- Parameter values: \( k_1 = 24 \text{ h}^{-1} \); \( k_2 = 24 \text{ h}^{-1} \); \( k_3 = 0.5 \text{ mol l}^{-1} \text{ h}^{-1} \); \( C_{A_{in}} = 1 \text{ mol l}^{-1} \).

5.2 Simulation results

Comparing Figure 6 to Figure 5 shows a clear improvement of the convergence rate of the perturbation based extremum seeking control when gradient estimation is obtained using recursive least squares estimation with forgetting factor instead of linear filtering. For the system described by (53), obtaining a no biased optimizing control solution requires to limit the perturbation signal frequency to 0.2\( \pi \) \text{ h}^{-1} at the maximum and leads to a convergence time of 6300h when using linear filtering for the gradient estimation (Figure 5). On the other hand, the use of recursive least squares with forgetting factor allows reducing the convergence time to approximatively 200h for the same perturbation signal frequency of 0.2\( \pi \) \text{ h}^{-1}.
6. CONCLUSIONS

We proposed a simple modification to the classical perturbation based extremum control method that allows improving the convergence speed while guaranteeing closed loop stability in the presence of dynamics in the system under optimizing control.

The suggested method applies to a class of non linear dynamic systems that cannot be approximated by Wiener/Hammerstein model but also to those that can. It estimates the objective function gradient using a recursive least squares estimation with forgetting factor. Note that for the latter case, simulations (not reported in the present work) showed comparable performance to that of the method proposed in (Chioua et al. (2007a)).

Results on simulation of the real time optimization of a simple isothermal reaction system in a continuous stirred tank reactor show that the convergence rate is significantly increased when compared with the traditional perturbation based extremum seeking method.

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