Operational definition of space-time in light of quantum mechanics and general relativity inevitably indicates an intrinsic imprecision in space-time structure which has to do with space-time dimension as well. The operational dimension of space-time turns out to be a scale dependent quantity slightly smaller than four at distances $\gg l_P$. Close to the Planck length the deviation of space-time dimension from four becomes appreciable. The experimental bounds on the deviation of space-time dimension from four coming from the electron $g - 2$ factor, Lamb shift in hydrogen atom and the perihelion shift in the planetary motion are still far from the theoretical predictions.

First let us summarize different approaches for operational definition of Minkowskian space-time that enables one to estimate the rate of quantum-gravitational fluctuations of the background metric. What we are interested in is to quantify to what maximal precision can we measure the space-time distance for Minkowski space. For space-time measurement an unanimously accepted method one can find in almost every textbook of general relativity consists in using clocks and light signals [16]. Let us consider a light-clock consisting of a spherical mirror inside which light is bouncing. That is, a light-clock represents inside a spherical mirror. Therefore the precision to a better accuracy than $l_P$ requires large momentum according to Heisenberg uncertainty principle, but when the momentum becomes too large its gravitational disturbance of the region under measurement becomes appreciable. So this discussion is completely in the spirit of quantum mechanical philosophy.

In this Letter following the reasoning of paper [3] and taking into account an unavoidable imprecision in space-time structure [4, 6, 7, 8, 9, 11, 12, 13, 14, 15] we estimate space-time dimension and consider some of the experimental bounds on it.

Space-time uncertainty relation

The background space-time representing a frame in which everything takes place is often taken for granted. One of the most fundamental concepts in physics is the very notion of space-time dimension. Our fundamental theories of physics usually do not predict the space-time dimension. A notable exception is provided by superstring theory the basic principles of quantum theory and gravitation inevitably indicate a finite space-time resolution. Namely, let us ask to what maximal precision can we mark a point in space by placing there a test particle. Throughout this paper we will assume $\hbar = c = 1$. In the framework of quantum field theory a particle takes up at least a volume, $\delta x^3$, defined by its Compton wavelength $\delta x \gtrsim 1/m$. Not to collapse into a black hole, general relativity insists the quantum on taking up a finite amount of room defined by its gravitational radius $\delta x \gtrsim l_P^2 m$. Combining together both quantum mechanical and general relativistic requirements one finds

$$\delta x \gtrsim \max(m^{-1}, l_P^2 m). \quad (1)$$

From this equation one sees that a particle occupies at least the volume $\sim l_P^3$. Therefore in the operational sense the point can not be marked to a better accuracy than $\sim l_P$ (this point of view was carefully analyzed through a number of Gedankenexperiments in [4]). Since our understanding of time is tightly related to the periodic motion along some length scale, this result implies in general an impossibility of space-time distance measurement to a better accuracy than $\sim l_P$. Physical meaning of this limitation consists in significant magnification of the space-time fluctuations during the observation when the measured length scale approaches the Planck one. Roughly it happens because refined length measurement requires large momentum according to Heisenberg uncertainty principle, but when the momentum becomes too large its gravitational disturbance of the region under measurement becomes appreciable. So this discussion is completely in the spirit of quantum mechanical philosophy.

Let us consider a light-clock consisting of a spherical mirror inside which light is bouncing. That is, a light-clock counts the number of reflections of a pulse of light propagating inside a spherical mirror. Therefore the precision of such a clock is set by the size of the clock. The points between which distance is measured are marked by the clocks, therefore the size of the clock $2r_c$ from the very outset manifests itself as an error in distance measurement. Let us call it a mechanical error and denote by $\delta l_{mech} \approx r_c$. Another source of error is due to quantum fluctuations of the clock. Namely denoting the mass of
the clock by \( m \) one finds that the clock is characterized with spread in velocity

\[
\delta v = \frac{\delta p}{m} \sim \frac{1}{m r_c} ,
\]

and correspondingly during the time \( t = l \) taken by the light signal to pass the distance \( l \) the clock may move the distance \( \delta t \delta v \). In what follows we will refer to it as a quantum error and denote by \( \delta l_{\text{quant}} \sim l/mr_c \). The total uncertainty in measuring the lengths scale \( l \) takes the form

\[
\delta l \gtrsim r_c + \frac{l}{m r_c} .
\]

Minimizing this expression with respect to the size of clock one finds

\[
r_c \simeq \sqrt{\frac{l}{m}} \Rightarrow \delta l \gtrsim \sqrt{\frac{l}{m}} . \tag{2}
\]

By taking the mass of the clock to be large enough the uncertainty in length measurement can be reduced but one should pay attention that simultaneously the size of the clock diminishes and its gravitational radius increases. The measurement procedure to be possible we should care the size of the clock not to become smaller than its gravitational radius to avoid the gravitational collapse into black hole. So that there is an upper bound on the clock mass

\[
r_c^{\text{min}} \simeq \sqrt{\frac{l}{m_{\text{max}}}} \simeq \frac{l^2}{m_{\text{max}}}, \Rightarrow m_{\text{max}} \simeq \frac{l^{1/3}}{l_p^{2/3}} ,
\]

which through the equation (2) determines the minimal unavoidable error in length measurement as

\[
\delta l_{\text{min}} \simeq \frac{l^{2/3}}{l_p} l^{1/3} . \tag{3}
\]

This uncertainty relation was first obtained by Károlyházy in 1966 and was subsequently analyzed by him and his collaborators in much details [8].

Let us refine our consideration by noticing that after introducing the clock the metric takes the form

\[
ds^2 = \left( 1 - \frac{2l^2 p m}{r} \right) dt^2 - \left( 1 - \frac{2l^2 p m}{r} \right)^{-1} dr^2 - r^2 d\Omega^2 .
\]

The time measured by this clock is related to the Minkowskian time as [10]

\[
t' = \left( 1 - \frac{2l^2 p m}{r_c} \right)^{1/2} t .
\]

From this expression one sees that the disturbance of the background metric to be small, the size of the clock should be much greater than its gravitational radius \( r_c \gg 2l^2 p m \). Under this assumption for gravitational disturbance in time measurement one finds

\[
t' = \left( 1 - \frac{l^2 p m}{r_c} \right) t .
\]

Thus, in addition to the mechanical and quantum errors we have the gravitational error as well. The terms contributing to the total error look like

\[
\delta l_{\text{mech}} \simeq r_c, \quad \delta l_{\text{quant}} \simeq \frac{l}{m r_c}, \quad \delta l_{\text{grav}} \simeq \frac{l^2 p m}{r_c} .
\]

The mass of the clock can not be less than \( \sim 1/r_c \), it is nothing else but the requirement the minimal size to be set by the Compton wavelength. On the other hand to avoid the gravitational collapse into black hole we should require \( r_c \gg l^2 p m \). So what we know on general grounds is that \( 1/r_c \lesssim m^{\prime} \lesssim r_c/l^2 p \). The terms \( \delta l_{\text{quant}} \) and \( \delta l_{\text{grav}} \) depend on the mass. Minimizing \( \delta l_{\text{quant}} + \delta l_{\text{grav}} \) with respect to the mass one finds an optimal value of the clock mass to be \( m \simeq m_p \). After this minimization the total error takes the form

\[
\delta l \gtrsim r_c + \frac{l^2 p}{r_c} .
\]

This expression is minimized for the size of the clock \( r_c \simeq (l p l)^{1/2} \) determining the minimal unavoidable error in length measurement as

\[
\delta l_{\text{min}} \simeq (l p l)^{1/2} . \tag{4}
\]

This uncertainty relation was discussed in [14]. Compared with Eq. (3) this expression implies more imprecision in length measurement. Notice that if we omit either \( \delta l_{\text{quant}} \) or \( \delta l_{\text{grav}} \) we will arrive at the Eq. (3), see for details [17].

**Random walk approach to the space-time measurement**

To understand the principal features of the random walk approach it suffices to consider the following one dimensional example (for a comprehensive review of the random walks one can see [17]). Assume a particle undergoes a sequence of displacements along a straight line in the form of a series of steps of equal length, \( l_s \), each of them being taken with equal probability in both directions. So that the probability of each step to be taken either in the forward or in the backward direction is 1/2 independently of the direction of all the preceding steps. After taking \( N \) such steps from the origin of axis the particle could be at any of the points

\[-l_s N, -l_s (N+1), \ldots, -l_s, 0, l_s, \ldots, l_s (N-1), l_s N .
\]

The question we are interested in is to estimate what is the probability \( W(m, N) \) that after \( N \) displacements the
particle will be at the point \( l_m \). The probability of any given sequence of \( N \) steps is \((1/2)^N\). The required probability \( W(m,N)\) is therefore \((1/2)^N\) times the number of distinct sequences of steps leading to the point \( l_m \) after \( N \) steps. In order to arrive at \( l_m \) among the \( N \) steps, we need to make \( m \) steps in the positive direction to reach this point and the remaining steps \( N - m \) should be taken in equal numbers forth and back, that is, in whole some \((N + m)/2\) steps should be taken in the positive direction and the remaining ones \((N - m)/2\) in the negative direction. The number of such distinct sequences can be easily estimated by observing that for a given sequence the permutations among \((N + m)/2\) forth and \((N - m)/2\) back elements do not produce new sequences. Thus for the number of distinct sequences one finds

\[
\frac{N!}{\left\lfloor \frac{N-m}{2} \right\rfloor ! \left\lfloor \frac{N+m}{2} \right\rfloor !}.
\]

Hence

\[
W(m,N) = \frac{N!}{\left\lfloor \frac{N-m}{2} \right\rfloor ! \left\lfloor \frac{N+m}{2} \right\rfloor !} \left(\frac{1}{2}\right)^N.
\] (5)

Now imagine we are measuring some length scale by the ruler. The ratio of the length scale under measurement to the length of the ruler determines the number of steps, \( N \), we need to perform for this measurement. Our ruler has some precision, \( l_s\), determining the uncertainty in each measurement. It is natural to assume that in making \( N \) measurements this uncertainty adds up randomly, that is, during each step the uncertainty is expected to take on \( \pm l_s\) with equal probability. Hence, under this assumption one finds that the probability to make the error \( l_m \) in length measurement by the ruler \( N \) times smaller than this length, is given by the Bernoulli distribution [17]. With respect to this distribution one can estimate \( \langle m^2 \rangle = N \frac{1}{17} \), which determines the mean square uncertainty in the measurement

\[
\text{uncertainty in the measurement} = l_s \sqrt{N}.
\]

Gravitational field is described in terms of space-time metric, that is, figuratively speaking it measures space-time distances. To measure the space-time distance gravitational field has the only intrinsic length scale \( l_P\). If we assume our ruler is just \( l_P\), that is, both its length and precision are given by the Planck length we arrive at the Eq.

\[
\delta l \simeq l_P \left(\frac{1}{l_P}\right)^{1/2} = (l_P t)^{1/2}.
\]

So it seems natural to hold that the gravitational field operating with the Planck length precision knows space-time distances with the accuracy given by Eq.

New light from an effective quantum field theory

Interestingly enough both of the equations 3 and 4 were derived in the framework of an effective quantum field theory in [18]. The Eq. 3 emerges as a relation between UV and IR scales in the framework of an effective quantum field theory satisfying the black hole entropy bound. For an effective quantum field theory in a box of size \( l \) with UV cutoff \( \Lambda \) the entropy \( S \) scales as, \( S \sim l^3 \Lambda^3 \). That is, an effective quantum field theory counts the degrees of freedom simply as the numbers of cells \( \Lambda^{-3} \) in the box \( l^3\). Nevertheless, considerations involving black holes demonstrate that the maximum entropy in a box of volume \( l^3\) grows only as the area of the box [19]

\[
S_{BH} \simeq \left(\frac{l}{l_P}\right)^2.
\] (6)

Consequently one arrives at the conclusion that the length \( l \), which serves as an IR cutoff, cannot be chosen independently of the UV cutoff, and scales as \( \Lambda^{-3} \). Rewriting this relation wholly in length terms, \( \delta l \equiv \Lambda^{-1}\), one arrives at the Eq. 3. Is it an accidental coincidence? Indeed not. The relation 4 can be simply understood from the Eq. 3. The IR scale \( l \) can not be given to a better accuracy than \( \delta l \simeq l_P^{2/3} l^{1/3} \). Therefore, one can not measure the volume \( l^3\) to a better precision than \( \delta l^3 \simeq l_P^2 l \) and correspondingly maximal number of cells inside the volume \( l^3\) that may make an operational sense is given by \( ((l/l_P)^2)\). Thus the Károlyházy relation implies the black-hole entropy bound given by Eq. 3.

From the preceding sections one may find more motivated to use the Eq. 4 instead of Eq. 3. Again the effective quantum field theory can help us to gain new insights into the problem [18]. An effective field theory that can saturate Eq. 4 may include many states with gravitational radius much larger than the box size. To see this, note that a conventional effective quantum field theory is expected to be capable of describing a system at a temperature \( T \) provided that \( T \leq \Lambda \). So long as \( T \gg 1/\Lambda\), such a system has thermal energy \( M \sim l^3 T^4\) and entropy \( S \sim l^3 T^3\). When Eq. 4 is saturated, at \( T \sim (m_p^2 l) l^{1/3}\), the corresponding gravitational radius for this system is \( l \sim (l m_P^2)^{2/3} l^{1/3} \gg l \). That is, since the maximum energy density in the effective theory is \( \Lambda^4\), the gravitational radius associated with the maximum energy of the system, \( M_{\text{max}} \sim l^3 \Lambda^4 \Rightarrow r_g \sim l_P l^3 \Lambda^4\), will be greater than the size of the system, \( l \), if UV cutoff is defined from the Eq. 4. To be on the safe side, one can impose stronger constraint requiring the size of the system to be greater than the gravitational radius associated to the maximum energy of the system [18]

\[
l_P^2 l^3 \Lambda^4 \lesssim l.
\] (7)
Here the IR cutoff scales like $\Lambda^{-2}$. This relation written in length terms ($d l \equiv \Lambda^{-1}$, $l_p \equiv m_p^{-1}$) is the Eq.(1).

So we see what is the effective quantum field theory picture behind the Eqs.(3, 4) and interplay between them.

**Space-time dimension**

The general mathematical concept of dimension was put forward long ago by Hausdorff [21]. Let us briefly recall the definition of Hausdorff dimension. We have a metric space denoted by $(\Omega, \xi)$, where $\Omega$ stands for the set of points and $\xi$ denotes the distance (metric) on it. Let $O$ be the family of all open sets in $\Omega$ and $O_\epsilon$ a subset of it such that

$$O_\epsilon = \{ U \in O \mid d(U) \leq \epsilon \},$$

where $d(U)$ is the diameter of $U$

$$d(U) = \sup \{ \xi(x, y) \mid x, y \in U \}.$$

Let us define for an arbitrary subset $E \subset \Omega$ covered by the countable number of $U_n$, the following quantity

$$\mu^\alpha(E, \epsilon) = \inf \left\{ \sum \limits_n d(U_n)^\alpha \mid U_n \in O_\epsilon, E \subset \bigcup \limits_n U_n \right\},$$

where the infimum is over all countable open covers of $E$. Then the Hausdorff dimension is defined as

$$\text{dim}(E) \equiv \alpha_H = \sup \{ \alpha \geq 0 \mid \mu^\alpha(E) = \infty \},$$

where

$$\mu^\alpha(E) = \lim \limits_{\epsilon \to 0} \mu^\alpha(E, \epsilon).$$

This definition is based on an important property that $\mu^\alpha(E) = \infty$ for $\alpha < \alpha_H$ and $\mu^\alpha(E) = 0$ for $\alpha > \alpha_H$. It is important to notice that the definition of dimension implies the limit $\epsilon \to 0$. In any real or *Gedanken* measurement one always deals with a finite resolution that naturally leads to the necessity of operational definition of the space-time dimension [3]. From the space-time uncertainty relation we infer that in measuring the dimension of space-time region with linear size $l$ the resolution can not be taken to a better accuracy than $\epsilon \geq 6l$. Since the measure is positive definite, $\mu^\alpha(E, \epsilon) \geq \mu^\alpha(E, 0)$, the operational dimension $\alpha_{\text{op}}$ satisfies

$$\alpha_{\text{op}} \leq \alpha_H.$$

In order to get a specific value for an operational dimension one has to generalize the definition of dimension to a finite resolution case. Except of some “pathological” cases that have no physical interest, the Hausdorff dimension is equivalent to the box-counting dimension introduced by Kolmogorov [21]. Assume $\Omega$ is $n$ dimensional space, where $n \geq 1$, and suppose that $N(\epsilon)$ is the minimum number of $n$-dimensional boxes of side length $\epsilon$ required to cover the set $E \subset \Omega$. Then the Kolmogorov (or box-counting) dimension is defined as:

$$\dim(E) = \lim \limits_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln (d(E)/\epsilon)},$$

where $d(E)$ is the diameter of $E$

$$d(E) = \sup \{ \xi(x, y) \mid x, y \in E \}.$$ The condition $E \subset \Omega$ immediately implies $\dim(E) \leq n$. The Kolmogorov dimension can be straightforwardly generalized to the case of a finite resolution implying small but finite number of $\epsilon$. Thus for a space-time region $l^4$, that is, a space-time box of side length $l$, the operational dimension can be defined as

$$\alpha_{\text{op}}(l^4) = \frac{\ln N(\delta l)}{\ln (l/\delta l)},$$

where we have taken into account finiteness of space-time resolution $\epsilon \geq \delta l$. Now let us notice that the deviation in $\epsilon$-box number counting can take place due to length uncertainty allowing $l \to l - \delta l$ and correspondingly

$$N(\delta l) = \left( \frac{l - \delta l}{\delta l} \right)^n.$$

Thus from the Eq.(8) one finds the operational dimension of $l^4$ as

$$\alpha_{\text{op}} = \frac{\ln \left( \frac{l}{\Delta l} - 1 \right)}{\ln \left( \frac{l}{\Delta l} \right)}.$$

For large values of distance $l \gg l_p$ the uncertainty $\delta l \ll l$ and correspondingly the deviation of space-time dimension from four takes the form

$$\delta = 4 - \alpha_{\text{op}} \approx \frac{46l}{l \ln \frac{l}{\Delta l}}.$$

Denoting $r \equiv l/l_p$ for Eqs.(3, 4) one finds

$$\delta_1 \approx \frac{6}{r^{2/3} \ln r} \quad \text{and} \quad \delta_2 \approx \frac{8}{r^{1/2} \ln r} \quad \text{respectively}.$$

Let us look at the experimental bounds for deviation of space-time dimension from four considered in [2]. The experimental bound on the deviation of space-time dimension from four coming from the electron $g - 2$ factor measurement is $\sim 10^{-7}$. At the length scale associated to this phenomenon, it is a Compton length of the electron $r \approx 10^{22}$, the above theoretical result gives $\delta_2 \approx 10^{-11}$. The Lamb shift measurement for hydrogen atom puts the limit $\sim 10^{-11}$ [22]. At the atomic scale, that is, $r \approx 10^{25}$ one gets $\delta_2 \sim 10^{-14}$. The perihelion measurement of the planet Mercury sets the bound $\sim 10^{-9}$ [22]. But it is much larger compared with the theoretical result even at the atomic scale. It would be interesting if present or near future experiments could come closer to the theoretical results. At the nuclear scale $\sim 10^{-13}$ cm, that is, $r \approx 10^{20}$ we get $\delta_2 \sim 10^{-10}$. At the scale TeV$^{-1}$ $\sim 10^{-17}$ cm it gives $\delta_2 \sim 10^{-8}$. 

Concluding remarks

Let us briefly summarize our discussion. The presented discussion of space-time uncertainty relation from different points of view strongly favors the Eq. (4). Evidently, the space-time uncertainty relation precludes us from taking the limit $\epsilon \to 0$ implied by the standard definitions of dimension. Due to positive definiteness of Hausdorff measure one naturally expects the operational dimension to be smaller than four (it is more evident in the box-counting approach to the dimension). Even on the basis of the above effective quantum field theory view of the space-time uncertainty relations implying the relation between UV and IR cutoffs one might naturally expect the space-time dimension smaller than four. Namely, for a given IR scale the presence of related UV cutoff implies the regularization of the quantum field theory divergencies which equivalently well could be done in dimensional regularisation approach. Because of space-time uncertainty relation we need to operate with a finite dimension introduced by Kolmogorov provides a straightforward way for an operational definition of dimension.

Let us notice that one could use a simple physical way for estimating of operational value of dimension. Denoting the operational dimension of space by $3 - \delta$ one finds that the modified Newtonian potential should behave as

$$V \propto r^{\delta - 1} \quad (r \equiv l/l_p).$$

But this effect of dimension reduction is the result of space-time uncertainty that in the Newtonian potential leads to the modification $r \to r - \delta r$. So we have the same effect written in different terms. By equating $r^{1-\delta} = r - \delta r$ one gets

$$\delta = 1 - \frac{\ln(r - \delta r)}{\ln r}.$$

For large values of distance $r \gg 1$ we have $\delta r \gg r$ and correspondingly this expression takes the form $\delta \approx \delta r/r \ln r$. Thus we get the same result in this less rigorous but physically motivated way.

The operational dimension of space-time turns out to be a scale dependent quantity slightly smaller than four at distances $\gg l_p$. At relatively short distances the experimental bounds considered in [3, 22] are $3 - 4$ orders of magnitude greater than the theoretical predictions but we can hope the present or near future experiments to approach the theoretical results.

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