EXCEPTIONAL DIRECTIONS FOR THE TEICHMÜLLER GEODESIC FLOW AND HAUSDORFF DIMENSION

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Abstract. We prove that for every flat surface \( \omega \), the Hausdorff dimension of the set of directions in which Teichmüller geodesics starting from \( \omega \) exhibit a definite amount of deviation from the correct limit in Birkhoff’s and Oseledets’ Theorems is strictly less than 1. This theorem extends a result by Chaika and Eskin where they proved that such sets have measure 0. We also prove that the Hausdorff dimension of the directions in which Teichmüller geodesics diverge on average in a stratum is bounded above by 1/2, strengthening a classical result due to Masur. Moreover, we show that the Hausdorff codimension of the set of non-weakly mixing IETs with permutation \((d, d-1, \ldots, 1)\), where \( d \) is an odd number, is exactly 1/2 and strengthen a result by Avila and Leguil.

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1. Introduction

The problem of determining the size of the set of points with non-dense orbits under a partially hyperbolic transformation has a long history. These include orbits which escape to infinity, remain confined inside a proper compact set or simply miss a given open set. In the most studied setting, the transformation preserves a natural ergodic measure and hence these non-dense orbits have measure zero. Thus, it is natural to ask whether different types of non-dense orbits are more abundant than others with respect to other notions of size among which Hausdorff dimension is the most common.

Many instances of this problem have been studied for algebraic partially hyperbolic flows on homogeneous spaces. For such flows, Margulis conjectured in his 1990 ICM address that...
orbits with closure a compact subset of a (non-compact) homogeneous space that misses a countable set of points have full Hausdorff dimension [Mar91, Conjectures A,B]. A full resolution of these conjectures was provided in subsequent papers of Kleinbock and Margulis [KM96] and Kleinbock and Weiss [KW13]. This phenomenon of abundance of non-dense orbits also takes place in the setting of hyperbolic dynamical systems. For example, Urbański showed in [Urb91] that non-dense orbits of Anosov flows on compact manifolds have full Hausdorff dimension. Then, in [Dol97], Dolgopyat studied the Hausdorff dimension of orbits of Anosov flows and diffeomorphisms which do not accumulate on certain low entropy subsets. It was shown that these trajectories have full Hausdorff dimension in many cases.

On the other hand, non-dense orbits of divergence type tend to be less abundant. In the homogeneous setting, it was shown in [KKLM17] that the divergent on average trajectories for certain flows on \( \text{SL}(n, \mathbb{R})/\text{SL}(n, \mathbb{Z}) \) do not have full Hausdorff dimension. In fact, an explicit upper bound on the Hausdorff dimension is given, generalizing earlier papers by Cheung [Che11] and Cheung and Chevallier [CC16]. In the setting of strata of quadratic differentials, Masur showed that the Hausdorff dimension of the set of non-uniquely ergodic directions for the associated translation flow is bounded above by \( 1/2 \). These correspond to divergent orbits for the Teichmüller geodesic flow.

In this article, we quantify the abundance of non-dense orbits in the setting of Teichmüller dynamics. Theorem 1.7 is the analogue of the result of [KKLM17] on the dimension of directions in which orbits of the Teichmüller geodesic flow are divergent on average. It provides a strengthening of Masur’s result mentioned above. As for non-dense orbits, we study the more general problem concerning the set of directions at a fixed basepoint in which trajectories exhibit a definite amount of deviation from the correct limit in Birkhoff’s and Oseledets’ Theorems. Theorems 1.1 and 1.4 show that the Hausdorff dimension of these sets of directions is bounded away from 1 uniformly as the basepoint varies in the complement of certain proper submanifolds of the stratum. In particular, this implies that the intersection of the set of non-dense orbits with any Teichmüller disk in the complement of these finitely many proper submanifolds has positive Hausdorff codimension (see Corollary 1.3 and Section 10).

These results generalize prior work of Chaika and Eskin [CE15] in which the aforementioned exceptional sets were shown to have measure 0. The work of Chaika and Eskin was used in [DHL14] to study the diffusion rate of billiard orbits in periodic wind-tree models. It was shown that for any choice of side lengths of the periodic rectangular obstacles, diffusion of orbits has a constant polynomial rate in almost every direction. Theorems 1.1 and 1.4 imply that the directions exhibiting different diffusion rates do not have full Hausdorff dimension. Prior to the work of Chaika and Eskin, Athreya and Forni [AF08] established a polynomial bound on the deviation of Birkhoff averages of sufficiently regular functions along orbits of translation flows on flat surfaces in almost every direction. This full measure set of directions was chosen so that the average of a certain continuous function along the Teichmüller flow orbits is close to its expected value. Theorem 1.1 can be used to show that the directions which do not satisfy this bound are of dimension strictly smaller than 1.

It is well known that Teichmüller dynamics is closely tied to interval exchange transformations. In particular, Theorem 1.7 allows us to derive a lower bound on the Hausdorff codimension of the set of non-weakly mixing IETs with permutation \((d, d - 1, \ldots, 1)\), where \(d\) is an odd number. In combination with the result of [CM18] establishing the upper bound, this allows us to compute the precise Hausdorff codimension.
Formulation of Results. Let $g \geq 1$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an integral partition of $2g - 2$. An abelian differential is a pair $(M, \omega)$, where $M$ is a Riemann surface of genus $g$ and $\omega$ is a holomorphic 1-form on $M$ whose zeroes have multiplicities $\alpha_1, \ldots, \alpha_n$. Throughout this paper, $\mathcal{H}_1(\alpha)$ will denote a stratum of Abelian differentials with area 1 with respect to the induced area form on $M$. We refer to points of $\mathcal{H}_1(\alpha)$ as translation surfaces. For the sake of brevity, we will often refer to $\omega$ itself as an element of $\mathcal{H}_1(\alpha)$.

We recall that there are well-defined local coordinates on a stratum, called period coordinates (e.g., see [FM14, Section 2.3] for details), such that all changes of coordinates are given by affine maps. In period coordinates, $\text{SL}_2(\mathbb{R})$ acts naturally on each copy of $\mathbb{C}$. Moreover, the closure of any $\text{SL}_2(\mathbb{R})$ orbit is an affine invariant manifold [EMM15], i.e., a closed subset of $\mathcal{H}_1(\alpha)$ that is invariant under the $\text{SL}_2(\mathbb{R})$ action and looks like an affine subspace in period coordinates. Therefore, it is the support of an ergodic $\text{SL}_2(\mathbb{R})$ invariant probability measure.

The action of the following one parameter subgroups of $\text{SL}_2(\mathbb{R})$ will be referred to throughout the article.

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \tilde{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$$

We recall that the actions of $g_t$, $r_\theta$, $h_s$, and $\tilde{h}_s$ correspond to the Teichmüller geodesic flow, the rotation of the flat surface by the angle $\theta$, and the expanding and contracting horocycle flows, respectively.

Birkhoff’s Ergodic Theorem. Chaika and Eskin [CE15] proved that for any translation surface $(M, \omega) \in \mathcal{H}_1(\alpha)$, and any continuous compactly supported function $f$ on $\mathcal{H}_1(\alpha)$, for almost all $\theta \in [0, 2\pi]$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(g_tr_\theta \omega) \, dt = \int_{\mathcal{M}} f \, d\nu_\mathcal{M}, \quad (1.1)$$

where $\mathcal{M} = \text{SL}_2(\mathbb{R}) \omega$ is the smallest affine invariant manifold containing $\omega$ and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$.

In this paper, we show that the Hausdorff dimension of the set of directions exhibiting a definite amount of deviation from the correct limit in (1.1) is strictly less than 1.

Theorem 1.1. Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function $f$ on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $0 < \delta < 1$, such that for all $\omega \in \mathcal{M} \setminus (\bigcup_{i=1}^k \mathcal{N}_i)$, the Hausdorff dimension of the set

$$\left\{ \theta \in [0, 2\pi] : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_tr_\theta \omega) \, dt - \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \geq \varepsilon \right\}$$

is at most $\delta$.

Remark 1.2. We note that the upper bound in Theorem 1.1 is uniform as the basepoint $\omega$ varies in the complement of finitely many proper affine invariant submanifolds in $\mathcal{M}$. This, in particular, includes points $\omega$ whose $\text{SL}(2, \mathbb{R})$ orbit is not dense in $\mathcal{M}$.

In Theorem 6.7, we obtain a version of Theorem 1.1 for discrete Birkhoff averages which is needed for later applications. It is worth noting that the exceptional sets considered in Theorem 1.1 are non-empty in most examples and can, in fact, have positive Hausdorff
dimension. By using the results in [KW04], one can find a compact set $K$ such that the Hausdorff dimension of trajectories which are contained completely in $K$ is at least $1 - \delta'$ for some $0 < \delta' < 1$. By taking $f$ to be supported in the complement of $K$ and to have $\nu_M(f) \neq 0$, these bounded trajectories will belong to the exceptional set for all $\varepsilon$ sufficiently small. A similar argument shows that directions in which geodesics diverge on average (Definition 1.6) belong to the exceptional sets of compactly supported function with non-zero average.

Using the uniform dimension estimate in Theorem 1.1, we obtain the following corollary.

**Corollary 1.3.** Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function $f$ on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $0 < \delta < 1$, such that for all $\omega \in \mathcal{M} \setminus (\bigcup_{i=1}^k \mathcal{N}_i)$, the Hausdorff dimension of the set

$$\left\{ x \in \text{SL}_2(\mathbb{R}) \cdot \omega : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t x) \, dt - \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \geq \varepsilon \right\}$$

is at most $2 + \delta$.

In particular, by a standard approximation argument, we see that for any non-empty open subset $U$ of a connected component $\mathcal{C}$ of the stratum $\mathcal{H}_1(\alpha)$, the Hausdorff dimension of the set

$$\{ x \in \text{SL}_2(\mathbb{R}) \cdot \omega : g_t x \notin U \text{ for all } t > 0 \}$$

is strictly less than the dimension of $\text{SL}_2(\mathbb{R})$, which is 3. This being true uniformly over all Teichmüller curves $\text{SL}_2(\mathbb{R}) \cdot \omega$ in the complement of finitely many lower dimensional invariant submanifolds of $\mathcal{C}$.

**Oseledets’ Theorem for the Kontsevich-Zorich Cocycle.** The next object of our study is the Lyapunov exponents of the Kontsevich-Zorich cocycle. Consider the Hodge bundle whose fiber over every point $(X, \omega) \in \mathcal{H}_1(\alpha)$ is the cohomology group $H^1(X, \mathbb{R})$. Let $\text{Mod}(X)$ be the mapping class group, i.e. the group of isotopy classes of orientation preserving homeomorphisms of $X$. Fix a fundamental domain in the Teichmüller space for the action of $\text{Mod}(X)$. Consider the cocycle $A : \text{SL}_2(\mathbb{R}) \times \mathcal{H}_1(\alpha) \to \text{Mod}(X)$, where for $x$ in the fundamental domain, $A(g, x)$ is the element of $\text{Mod}(X)$ that is needed to return the point $gx$ to fundamental domain. Then, the Kontsevich-Zorich cocycle $A(g, x)$ is defined by

$$A(g, x) = \rho(A(g, x)).$$

where $\rho : \text{Mod}(X) \to \text{Sp}(2g, \mathbb{Z})$ is given by the induced action of $\text{Mod}(X)$ on cohomology. We recall the notion of a strongly irreducible $\text{SL}_2(\mathbb{R})$ cocycle.

**Definition** (Strongly Irreducible Cocycle). Let $(X, \nu)$ be a probability space admitting an action of a locally compact group $G$ which leaves $\nu$ invariant. Let $\pi : V \to X$ be a vector bundle over $X$ on which $G$ acts fiberwise linearly. We say that $V$ admits a $\nu$-measurable almost invariant splitting if there exists $n > 1$ and for $\nu$-almost every $x$, the fiber $\pi^{-1}(x)$ splits into non-trivial subspaces $V_1(x), \ldots, V_n(x)$ satisfying $V_i(x) \cap V_j(x) = \{0\}$ for all $i \neq j$ and $gV_i(x) = V_i(gx)$ for all $i$, $\nu$-almost every $x \in X$ and for almost every $g \in G$ with respect to the (left) Haar measure on $G$. And, finally, the map $x \mapsto V_i(x)$ is required to be $\nu$-measurable for all $i$.

The $G$ action on $V$ is said to be strongly irreducible with respect to $\nu$ if the $G$-action doesn’t admit any $\nu$-measurable almost invariant splitting.
In this setting, we prove the following statement about deviations in the Lyapunov exponents of the Kontsevich-Zorich cocycle.

**Theorem 1.4.** Suppose $M \subseteq H_1(\alpha)$ is an affine invariant submanifold and $\nu_M$ is the affine measure whose support is $M$. Let $V$ be a continuous (on $M$) $SL_2(\mathbb{R})$ invariant sub-bundle of (some exterior power of) the Hodge bundle. Assume that $A_V$ is strongly irreducible with respect to $\nu_M$, where $A_V$ is the restriction of the Kontsevich-Zorich cocycle to $V$. Then, for any $\varepsilon > 0$, there exist affine invariant submanifolds $N_1, \ldots, N_k$, properly contained in $M$, and $0 < \delta < 1$, such that for all $\omega \in M \setminus (\bigcup_{i=1}^k N_i)$, the Hausdorff dimension of the set

$$\left\{ \theta \in [0, 2\pi] : \limsup_{t \to \infty} \frac{\log \| A_V(g_t, r_\theta \omega) \|}{t} \geq \lambda_V + \varepsilon \right\}$$

is at most $\delta$, where $\lambda_V$ denotes the top Lyapunov exponent for $A_V$ with respect to $\nu_M$.

This complements a result in [CE15] where they show that under the same hypotheses, for every $\omega$ and for Lebesgue almost every $\theta \in [0, 2\pi]$, the following limit exists

$$\lim_{t \to \infty} \frac{\log \| A_V(g_t, r_\theta \omega) \|}{t} = \lambda_V$$

It is shown in [EM13, Theorem A.6] that the Kontsevich-Zorich cocycle is in fact semisimple, which means that, after passing to a finite cover, the Hodge bundle splits into $\nu_M$-measurable $SL_2(\mathbb{R})$ invariant, strongly irreducible subbundles. Moreover, it is shown in [Fil16] that such subbundles can be taken to be continuous (and in fact real analytic) in period coordinates. Additionally, it is well-known that the top Lyapunov exponent of the $k$th exterior power of the cocycle is a sum of the top $k$ exponents of the cocycle itself. In this manner, we can deduce the deviation statement for all Lyapunov exponents by examining the top Lyapunov exponents of exterior powers of the cocycle. The following Corollary is the precise statement. For more details on this deduction, see the proof of Theorem 1.4 in [CE15].

**Corollary 1.5.** Suppose $(M, \omega) \in H_1(\alpha)$ and $\nu_M$ is the affine measure whose support is $M = SL_2(\mathbb{R})\omega$. Let $A$ be the Kontsevich-Zorich cocycle over $M$. Denote by $\lambda_i$ the Lyapunov exponents of $A$ (with multiplicities) with respect to $\nu_M$. For any $\theta \in [0, 2\pi]$, suppose $\psi_1(t, \theta) \leq \cdots \leq \psi_{2g}(t, \theta)$ are the eigenvalues of the matrix $A^*(g_t, r_\theta \omega)A(g_t, r_\theta \omega)$. Then, the Hausdorff dimension of the set

$$\left\{ \theta \in [0, 2\pi] : \limsup_{t \to \infty} \frac{\log \| \psi_i(t, \theta) \|}{t} \geq 2\lambda_i + \varepsilon \right\}$$

is strictly less than 1.

**Divergent Trajectories.** The study of exceptional trajectories in Birkhoff’s and Oseledets’ theorems lends itself naturally to studying trajectories which frequently miss large sets with good properties. This problem is closely connected to studying divergent geodesics, i.e. geodesics which leave every compact subset of $H_1(\alpha)$. Masur showed in [Mas92] that, for every translation surface $\omega$, the set of directions $\theta$ for which $g_t r_\theta \omega$ is divergent has Hausdorff dimension at most $1/2$. Cheung [Che03] showed that this upper bound is optimal by constructing explicit examples for which this upper bound is realized.

In this paper, we study **divergent on average geodesics**, i.e., geodesics that spend asymptotically zero percent of the time in any compact set.
Definition 1.6. A direction $\theta \in [0, 2\pi]$ corresponds to a divergent on average geodesic $g_t r_{\theta} \omega$ if for every compact set $K \subset H_1(\alpha)$,
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \chi_K(g_t r_{\theta} \omega) \, dt = 0.
\]

Note that the set of divergent on average geodesics contains the set of divergent geodesics. Therefore, Theorem 1.7 below strengthens [Mas92].

Theorem 1.7. For any translation surface the Hausdorff dimension of the directions that correspond to divergent on average geodesics is at most $1/2$.

See also Theorem 3.2 where we consider the set of directions with a prescribed divergence behavior in open strata with finitely many invariant submanifolds removed, which may be of independent interest.

Combining Theorem 1.7 with the results in [BN04], we derive the following bound on the dimension of non-weakly mixing interval exchange transformations (IETs) whose permutation is of type $W$. We refer the reader to Section 9 for detailed definitions.

Corollary 1.8. Suppose $\pi$ is a type $W$ permutation. Then, the Hausdorff codimension of the set of non-weakly mixing IETs (with respect to the Lebesgue measure) with permutation $\pi$ is at least $1/2$.

For $d \in \mathbb{N}$, we say a permutation $\pi$ on $\{1, \ldots, d\}$ is a rotation if $\pi(i + 1) = \pi(i) + 1 \mod d$ for $1 \leq i \leq d$. Avila and Forni [AF07] showed that for any irreducible permutation, which is not a rotation, Lebesgue almost every IET is weakly mixing. In [AL16], this result was extended to show that for all such permutations, non-weakly mixing IETs have positive Hausdorff codimension. Thus, Corollary 1.8 is an improvement of [AL16] in the case of type $W$ permutations. Moreover, it is shown in [CM18] that if $\pi$ is the permutation $(d, d-1, \ldots, 1)$ for $d \geq 5$, then the Hausdorff codimension of the set of non-weakly mixing IETs with permutation $\pi$ is at most $1/2$ (the case $d = 4$ was done in [AC15]). When $d$ is odd, the permutation $(d, d-1, \ldots, 1)$ is type $W$. Thus, we identify the exact Hausdorff dimension in this case.

Outline of Proofs and Paper Organization. Our general approach is to deduce the desired results (Theorems 1.1, 1.4 and 1.7) from the analogous results for horocycle arcs (Theorems 2.1, 2.2 and 2.3, respectively). The reason is that horocycles are more convenient to work with as the geodesic flow normalizes the horocycle flow in $SL_2(\mathbb{R})$. This is carried out along with the proof of Corollary 1.3 in Section 2.

The strategy for proving Theorem 2.1 on deviations of Birkhoff averages consists of three main steps. First, we show that the convergence in (1.1) holds uniformly as the basepoint $\omega$ varies over compact sets in the complement of finitely many proper affine submanifolds. Theorem 5.1 is the precise statement. This result strengthens a result in [CE15] and may be of independent interest.

Next, we show that the Hausdorff dimension of directions whose geodesics frequently miss large compact sets, chosen with the help of a height function, is bounded away from 1. This statement is made precise in Theorem 3.2 whose proof is the main content of Section 3. Using similar techniques, Theorem 2.3 is proved in Section 4.

Theorem 2.1 is proved in Section 6. The idea is to treat a long orbital average as a sum of orbital averages over shorter orbit segments. With the help of Theorem 5.1, we show
that most orbit segments which start from a suitably chosen large compact set with good properties, will have an orbital average close to the correct limit. Using Theorem 3.2, we control the dimension of those orbit segments which miss our good compact set.

A key step is to show that the sum of such averages over orbit segments behaves like a sum of weakly dependent random variables, which is achieved by Lemma 6.5. This allows us to show that the measure of badly behaved long orbit averages decays exponentially.

The proof of Theorem 2.2 treating deviations in Oseledets’ theorem spans Section 7 and Section 8. It follows the same strategy as the one outlined above. It is shown in [CE15] that Oseledets’ theorem holds uniformly in the basepoint over large open sets for random walk trajectories. Using Egorov’s and Lusin’s theorems, we translate these results into results about the Teichmüller geodesic flow. This relies on the classical fact that a random walk trajectory is tracked by a geodesic, up to sublinear error.

Finally, we show that trajectories which frequently miss such a large set with good properties exhibit deviation in the discrete Birkhoff averages of its indicator function. The dimension of those trajectories is in turn controlled by Theorem 6.7.

In Section 9 we prove Corollary 1.8. In Proposition 9.3 we relate the criterion for weak mixing of IETs with a type W permutation in [BN04] and recurrence of Teichmüller geodesics in a stratum. The combination of this relation and our Theorem 1.7 finishes the proof.

In Section 10, we describe how to modify the proof of Theorem 1.1 to show that the Hausdorff dimension of abelian differentials $\omega$ for which ergodic integrals along their Teichmüller flow exhibit a definite amount of deviation from the correct limit in Birkhoff’s theorem is strictly less than the dimension of $\text{SL}(2,\mathbb{R})\omega$.

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2. Preliminaries

2.1. Reduction to horocycles. We explain how to deduce Theorems 1.1, 1.4 and 1.7 from the analogous results for horocycle arcs Theorems 2.1, 2.2 and 2.3, respectively.

For any $\theta \in [-\pi/4, \pi/4]$, the following equality holds.

$$r_\theta = \tilde{h}_{-\tan \theta} g_{\log \cos \theta} \tilde{h}_{\tan \theta}$$

Recall that $g_t$ contracts $\tilde{h}_{-\tan \theta}$, i.e., $g_t \tilde{h}_{-\tan \theta} g_{-t} = \tilde{h}_{-\tan \theta}$, and $g_t g_{\log \cos \theta} = g_{t+\log \cos \theta}$. Therefore, we have that in each theorem formulated in the introduction $\theta$ belongs to the exceptional set if and only if $\tan \theta$ belongs to the exceptional set in the corresponding theorem formulated below.

Finally, the bounds for the Hausdorff dimensions of the corresponding sets are preserved as the map $\theta \mapsto \tan \theta$ is bi-Lipschitz on $[-\pi/4, \pi/4]$. 

Theorem 2.1 (Analogue of Theorem 1.1). Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function $f$ on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$, such that for all $\omega \in \mathcal{M} \setminus (\cup_{i=1}^k \mathcal{N}_i)$, the Hausdorff dimension of the set

$$\left\{ s \in [-1, 1] : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t h_s \omega) \, dt \geq \int_\mathcal{M} f \, d\nu_\mathcal{M} + \varepsilon \right\}$$

is at most $\delta$.

We remark that minor modifications of the proof of Theorem 2.1 also yield an upper bound on the Hausdorff dimension of the set of directions for which the lim inf is less than the correct limit by a definite amount. Moreover, the exceptional set in Theorem 1.1 can be written as

$$\left\{ \theta \in [0, 2\pi] : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega) \, dt - \nu_\mathcal{M}(f) \geq \varepsilon \right\} = \left\{ \theta \in [0, 2\pi] : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega) \, dt \geq \nu_\mathcal{M}(f) + \varepsilon \right\} \cup \left\{ \theta \in [0, 2\pi] : \liminf_{T \to \infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega) \, dt \leq \nu_\mathcal{M}(f) - \varepsilon \right\}$$

where $\nu_\mathcal{M}(f) = \int_{\mathcal{M}} f \, d\nu_\mathcal{M}$. Thus, Theorem 1.1 follows from the reduction to horocycles, Theorem 2.1, and its variant for the lim inf.

Theorem 2.2 (Analogue of Theorem 1.4). Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$ and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M} = SL_2(\mathbb{R})\omega$. Let $V$ be a continuous (on $\mathcal{M}$) $SL_2(\mathbb{R})$ invariant sub-bundle of (some exterior power of) the Hodge bundle. Assume that $A_V$ is strongly irreducible with respect to $\nu_\mathcal{M}$, where $A_V$ is the restriction of the Kontsevich-Zorich cocycle to $V$. Then, for any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$, such that for all $\omega \in \mathcal{M} \setminus (\cup_{i=1}^k \mathcal{N}_i)$, the Hausdorff dimension of the set

$$\left\{ s \in [-1, 1] : \limsup_{t \to \infty} \frac{\log \|A_V(g_t, h_s \omega)\|}{t} \geq \lambda_V + \varepsilon \right\}$$

is at most $\delta$, where $\lambda_V$ denotes the top Lyapunov exponent for $A_V$ with respect to $\nu_\mathcal{M}$.

Theorem 2.3 (Analogue of Theorem 1.7). Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$. Then, the Hausdorff dimension of the set

$$\{ s \in [-1, 1] : \text{the geodesic } g_t h_s \omega \text{ is divergent on average} \}$$

is less than or equal to $\frac{1}{2}$.

2.2. Properties of Hausdorff dimension. The exceptional sets we study in this paper are of the form $A = \limsup_{n \to \infty} A_n$, that is

$$A = \bigcap_{n \geq 1} \bigcup_{l \geq n} A_l$$

for a sequence of subsets $A_n$ of the real line.
In this section, we reduce the problem of finding an upper bound on the Hausdorff dimension of such sets to the problem of finding efficient covers of the $A_n$ (see Lemma 2.5).

First, we recall the definition of the Hausdorff dimension. Let $A$ be a subset of a metric space $X$. For any $\rho, \beta > 0$, we define

$$H^\beta_\rho(A) = \inf \left\{ \sum_{I \in \mathcal{U}} \text{diam}(I)^\beta : \mathcal{U} \text{ is a cover of } A \text{ by balls of diameter } < \rho \right\}.$$ 

Then, the $\beta$-dimensional Hausdorff measure of $A$ is defined to be

$$H^\beta(A) = \lim_{\rho \to 0} H^\beta_\rho(A).$$

**Definition 2.4.** The Hausdorff dimension of a subset $A$ of a metric space $X$ is equal to

$$\dim_H(A) = \inf \{ \beta \geq 0 : H^\beta(A) = 0 \} = \sup \{ \beta \geq 0 : H^\beta(A) = \infty \}.$$

The following lemma provides an upper bound on the Hausdorff dimension of a set for which we have efficient covers.

**Lemma 2.5.** Let $\{A_n\}_{n \geq 1}$ be a collection of subsets of $\mathbb{R}$. Suppose there exist constants $C, C', t > 0$ and $\lambda \in (0, 1)$ such that for each $n$, $A_n$ can be covered with $Ce^{2(1-\lambda)tn}$ intervals of radius $C'e^{2tn}$. Then, the Hausdorff dimension of the set $A = \limsup_{n \to \infty} A_n$ is at most $1 - \lambda$.

**Proof.** Let $\beta \in (1 - \lambda, 1)$ and $H^\beta$ denote the $\beta$-dimensional Hausdorff (outer) measure on $\mathbb{R}$. We show that $H^\beta(A) = 0$, and that implies the Lemma. For any $\rho \in (0, 1)$, let $n_0 = n_0(\rho)$ be a natural number such that $e^{-2tn} < C\rho$ for all $n \geq n_0$. Notice that $n_0$ tends to infinity as $\rho$ goes to 0. Denote by $\mathcal{U}_n$ a cover of the set $A_n$ by $Ce^{2(1-\lambda)tn}$ intervals of radius $C'e^{2tn}$. Then, $\mathcal{U} = \bigcup_{n \geq n_0} \mathcal{U}_n$ is a cover of $A$ for which the following holds.

$$\sum_{I \in \mathcal{U}} \text{diam}(I)^\beta = \sum_{n \geq n_0} \sum_{I \in \mathcal{U}_n} \text{diam}(I)^\beta = (C')^\beta \sum_{n \geq n_0} \#\mathcal{U}_n e^{-2\beta tn} \leq (C')^\beta C \sum_{n \geq n_0} e^{2(1-\lambda-\beta)tn},$$

where $\#\mathcal{U}_n$ is the number of intervals in the cover $\mathcal{U}_n$.

Thus, since $1 - \lambda - \beta < 0$, we obtain

$$H^\beta_\rho(A) \leq (C')^\beta C e^{2(1-\lambda-\beta)tn} = (C')^\beta C e^{2(1-\lambda-\beta)tn} \to 0 \quad \text{as } \rho \to 0.$$

This implies that $H^\beta(A) = 0$ for all $\beta \in (1 - \lambda, 1)$. \hfill \Box

Let us also recall some basic facts about Hausdorff dimension which will be useful for us. The first concerns the dimension of product sets.

**Proposition 2.6** (Corollary 8.11 in [Mat95]). If $A, B \subset \mathbb{R}^d$ are Borel sets, then $\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B)$. If, in addition, the upper packing dimension of $B$ is equal to its Hausdorff dimension, then

$$\dim_H(A \times B) = \dim_H(A) + \dim_H(B).$$

We remark that the lower bound on the dimension of the product is a classical fact while the upper bound can be obtained directly when $B$ is an open ball, which is the case we will be interested in.
2.3. Proof of Corollary 1.3. Using a simple approximation argument, we may assume that \( f \) is Lipschitz. By a similar argument to the one following Theorem 2.1, it suffices to prove that the following set

\[
\overline{B}(f, \varepsilon) := \left\{ x \in \mathcal{M} : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t x) \, dt \geq \int_{\mathcal{M}} f \, d\nu_{\mathcal{M}} + \varepsilon \right\}
\]

has positive Hausdorff codimension in \( \mathcal{M} \). Let \( \delta > 0 \) and \( N_1, \ldots, N_k \) be the affine invariant submanifolds properly contained in \( \mathcal{M} \) which are provided by Theorem 2.1, depending on \( f \) and \( \varepsilon \) and suppose \( \omega \in \mathcal{M} \setminus (\cup_{i=1}^k N_i) \).

Since the action of \( \text{SL}(2, \mathbb{R}) \) is locally free\(^1\), we can find a small neighborhood of identity \( \mathcal{O}_\omega \subset \text{SL}(2, \mathbb{R}) \) such that the map \( g \mapsto g\omega \) is injective on \( \mathcal{O}_\omega \). By making \( \mathcal{O}_\omega \) smaller if necessary, we may assume that \( \mathcal{O}_\omega \) is the diffeomorphic image of an open bounded neighborhood of 0 in the Lie algebra of \( \text{SL}(2, \mathbb{R}) \) under the exponential map. In particular, there are bounded neighborhoods \( \mathcal{O}_\omega^s, \mathcal{O}_\omega^c \) and \( \mathcal{O}_\omega^u \) of 0 in \( \mathbb{R} \) such that the map

\[
(z, r, s) \mapsto \tilde{h}_z g_r h_s
\]

is a diffeomorphism from \( \mathcal{O}_\omega^s \times \mathcal{O}_\omega^c \times \mathcal{O}_\omega^u \) onto \( \mathcal{O}_\omega \). Define the following set

\[
\overline{B}(f, \varepsilon)^u_\omega := \left\{ s \in \mathcal{O}_\omega^u : \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(g_t h_s \omega) \, dt \geq \int_{\mathcal{M}} f \, d\nu_{\mathcal{M}} + \varepsilon \right\}
\]

By Theorem 2.1, the Hausdorff dimension of \( \overline{B}(f, \varepsilon)^u_\omega \) is at most \( \delta \leq 1 \). Now, suppose that \( x = g\omega \in \overline{B}(f, \varepsilon) \cap \mathcal{O}_\omega \omega \) and write \( g = \tilde{h}_z g_r h_s \). Since \( g_t \) contracts \( \tilde{h}_z \) and commutes with \( g_r \), using the fact that \( f \) is Lipschitz, we see that \( s \in \overline{B}(f, \varepsilon)^u_\omega \). Conversely, for all \( s \in \overline{B}(f, \varepsilon)^u_\omega \) and all \( (z, r) \in \mathcal{O}_\omega^s \times \mathcal{O}_\omega^c \), we have that \( \tilde{h}_z g_r h_s \omega \in \overline{B}(f, \varepsilon) \cap \mathcal{O}_\omega \omega \).

In particular, we have the identification

\[
\overline{B}(f, \varepsilon) \cap \mathcal{O}_\omega \omega \cong \mathcal{O}_\omega^s \times \mathcal{O}_\omega^c \times \overline{B}(f, \varepsilon)^u_\omega
\]

under the smooth coordinate map in (2.1). Thus, by Proposition 2.6, since the upper packing dimension of an open interval in \( \mathbb{R} \) is equal to its topological and Hausdorff dimension, we get that

\[
dim_H(\overline{B}(f, \varepsilon) \cap \mathcal{O}_\omega \omega) = \dim_H \left( \mathcal{O}_\omega^s \times \mathcal{O}_\omega^c \times \overline{B}(f, \varepsilon)^u_\omega \right) \leq 2 + \delta
\]

The above argument shows that that dimension of the intersection of \( \overline{B}(f, \varepsilon) \) with any open subset of an \( \text{SL}(2, \mathbb{R}) \) orbit in the complement of \( \cup_{i=1}^k N_i \) is at most \( 2 + \delta \neq 3 \).

3. The Contraction Hypothesis and Analysis of Recurrence

In this section, we study the problem of the Hausdorff dimension of trajectories with prescribed divergence behavior. We prove an abstract result for \( \text{SL}_2(\mathbb{R}) \) actions on metric spaces which satisfy the Contraction Hypothesis (Definition 3.1) in the terminology of Benoist and Quint [BQ12, Section 2]. The results in this section closely follow the ideas in [KKLM17].

\(^1\)For every compact set \( K \subset \mathcal{M} \), there exists a bounded neighborhood of identity \( B \subset \text{SL}(2, \mathbb{R}) \) such that the map \( (g, x) \mapsto gx \) is injective from \( B \times K \) into \( \mathcal{M} \).
Let \( X \) be a manifold equipped with a smooth \( \text{SL}(2, \mathbb{R}) \) action. For \( t, \delta > 0, N \in \mathbb{N}, Q \subset X \) a (compact) set and \( x \in X \), define the following set
\[
Z_x(Q, N, t, \delta) = \left\{ s \in [-1, 1] : \frac{1}{N} \sum_{i=1}^{N} \chi_Q(g_t h_s x) \leq 1 - \delta \right\}
\] (3.1)
where \( \chi_Q \) denotes the indicator function of \( Q \).

**Definition 3.1** (The Contraction Hypothesis). Let \( Y \) be a proper \( \text{SL}(2, \mathbb{R}) \)-invariant submanifold of \( X \) \((Y = \emptyset \) is allowed). The action of \( \text{SL}_2(\mathbb{R}) \) on \( X \) is said to satisfy the contraction hypothesis with respect to \( Y \) if there exists a proper, \( \text{SO}(2) \)-invariant function \( \alpha : X \to [1, \infty] \) satisfying the following properties:

1. \( \alpha(x) = \infty \) if and only if \( x \in Y \).
2. There is constant \( \sigma > 0 \) such that for all \( x \in X \setminus Y \) and all \( t > 0 \),
   \[
e^{-\sigma t} \alpha(x) \leq \alpha(g_t x) \leq e^{\sigma t} \alpha(x)
   \] (3.2)
3. There exists a constant \( b = b(Y) > 0 \), such that for all \( a \in (0, 1) \) there exists \( t_0 = t_0(a) > 1 \) so that for all \( t > t_0 \) and all \( x \in X \setminus Y \),
   \[
   \int_0^{2\pi} \alpha(g_t r_\theta x) \ d\theta \leq a \alpha(x) + b
   \] (3.3)
4. For all \( M \geq 1 \), the sets \( \{ x \in X : \alpha(x) \leq M \} \), denoted by \( X_{\leq M} \), form a compact exhaustion of \( X \setminus Y \).

The function \( \alpha \) is called the height function.

We remark that the study of height functions as in Definition 3.1 originated in [EMM98] in the context of homogeneous spaces.

Throughout this section \( X \) is a manifold equipped with a smooth \( \text{SL}(2, \mathbb{R}) \) action and satisfies the contraction hypothesis with respect to \( Y \), which is a proper \( \text{SL}(2, \mathbb{R}) \)-invariant submanifold of \( X \).

Our goal in this section is to prove the following theorem.

**Theorem 3.2.** Given \( \delta > 0 \), there exists \( M_0 = M_0(\delta) > 0 \) and \( t_0 > 0 \) such that for all \( M \geq M_0 \) and all \( t \geq t_0 \), there exists \( \lambda = \lambda(\delta, t) \in (0, 1) \) such that for all \( x \in X \setminus Y \), the Hausdorff dimension of the set \( \limsup_{N \to \infty} Z_x(X_{\leq M}, N, t, \delta) \) is at most \( 1 - \lambda \), where \( \lambda \) tends to 0 as \( t \to \infty \).

The proof of Theorem 3.2 can be found in Section 3.3. It should be noted that the difference between this theorem and [KKLM17, Theorem 1.5] is the flexibility in the step size \( t \). As a result, the upper bound on the Hausdorff dimension of the considered set depends on \( t \). In fact, the proof of Theorem 3.2 gives an explicit value for \( \lambda \) as a function of \( t \) and \( \delta \). See also remark 3.8 for an explicit choice for \( M_0 \) depending on \( \delta \).

3.1. Estimates for integrals over horocycle orbits. In this section we obtain an integral estimate similar to (3.3) for integrals over an entire horocycle orbit.

**Lemma 3.3.** Let \( \alpha : X \to [1, \infty] \) be a height function. Then, there is a constant \( \bar{b} > 0 \), such that for all \( \bar{a} \in (0, 1) \), there exists \( t_0 = t_0(\bar{a}) > 0 \) so that for all \( t > t_0 \) and all \( x \in X \setminus Y \),
\[
\int_{-1}^{1} \alpha(g_t h_s x) \ ds \leq \bar{a} \alpha(x) + \bar{b}
\] (3.4)
Proof. Let $b = b(Y)$ and for any $a \in (0, 1)$ we have $t_0 = t_0(a)$ as in Definition 3.1. Then, by (3.3), for any $t > t_0$ and all $x \in X$,

$$
\int_{-\pi/4}^{\pi/4} \alpha(g_{t\theta}x) \, d\theta \leq a\alpha(x) + b
$$

For any $\theta \in [-\pi/4, \pi/4]$ the following equality holds.

$$
g_{t\theta} = \begin{pmatrix}
1 & 0 & \cos(\theta) & 0 \\
0 & 1 & 0 & \sec(\theta) \\
-e^{-t \tan(\theta)} & 0 & g_{h\tan(\theta)} \\
0 & g_{\log(\cos(\theta))}
\end{pmatrix}
$$

Let $K = SO(2)$. From the KAK decomposition of $SL_2(\mathbb{R})$, $K$-invariance of $\alpha$ and Property (2) in Definition 3.1, it is easy to see that there exists a positive constant $c_0 \geq 1$, that is independent of $t$, such that for all $\theta \in [-\pi/4, \pi/4]$ and all $x \in X$,

$$
\alpha(g_{h\tan(\theta)}x) \leq c_0 \alpha(g_{t\theta}x).
$$

Thus, we get that

$$
\int_{-\pi/4}^{\pi/4} \alpha(g_{h\tan(\theta)}x) \, d\theta \leq c_0(a\alpha(x) + b).
$$

Using a change of variable $s = \tan(\theta)$ and noting that the Jacobian of this change of variable is uniformly bounded on $[-\pi/4, \pi/4]$, we obtain

$$
\int_{-1}^{1} \alpha(g_{h}x) \, ds \leq 2c_0(a\alpha(x) + b).
$$

That implies the lemma with $\bar{b} = 2c_0 b$, $a = \frac{\bar{a}}{2c_0} \in (0, 1)$ and $\bar{t}_0 = t_0(a)$. \hfill \Box

Our next lemma replaces integration over a compact subinterval of the horocycle with an integral over the entire horocycle orbit against a Gaussian measure. This is a technical step needed to carry over some results from [KKLM17] to our setting.

**Lemma 3.4.** Let $\alpha : X \to [1, \infty]$ be a height function. Then, there is a constant $b > 0$, such that for all $a \in (0, 1)$, there exists $t_0 = t_0(a) > 0$ so that for all $t > t_0$ and all $x \in X \setminus Y$,

$$
\int_{\mathbb{R}} \alpha(g_{h}x) \, d\rho_1(s) \leq a\alpha(x) + b
$$

where $d\rho_1(s) = e^{-s^2}ds$ is a mean 0, variance 1 Gaussian.

**Proof.** By Property (2) in Definition 3.1, the KAK decomposition of $SL_2(\mathbb{R})$ and $K$-invariance of $\alpha$, there exist constants $\sigma > 0$ and $C > 1$, such that for all $q \in \mathbb{R}$,

$$
\alpha(h_q x) \leq C|1 + q|^\sigma.
$$

Let $\bar{b}$ and $t_0 = \bar{t}_0(\bar{a})$ for $\bar{a} \in (0, 1)$ be given by Lemma 3.3. For any $n \in \mathbb{Z}$, we define $f(n) = \min \{|n|, |n + 2|\}$.

Then, for any $t > t_0$ and $x \in X$ we have the following.
\[
\int_{\mathbb{R}} \alpha(g_th_s x) \, d\rho_1(s) = \sum_{n \in 2\mathbb{Z} - 1} \int_{[0,2]^n} \alpha(g_th_s x) e^{-s^2} \, ds
\]
\[
\leq \sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2} \int_{-1}^{1} \alpha(g_th_s h_{n+1}x) \, ds
\]
\[
\leq \sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2} (\bar{\alpha}(h_{n+1}x) + \bar{b})
\]
\[
\leq C\bar{\alpha}(x) \sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2} |n + 2|^\sigma + \bar{b} \sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2}.
\]

This completes the proof by setting
\[
b = \bar{b} \sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2} \quad \text{and} \quad \bar{a} = \min \left\{ \frac{1}{2}, C\sum_{n \in 2\mathbb{Z} - 1} e^{-(f(n))^2} |n + 2|^\sigma \right\}.
\]

3.2. Coverings and Long Excursions. In this section, we aim to find efficient coverings for the set of directions for which geodesics take long excursions outside of certain fixed compact sets. We closely follow [KKLM17, Section 5].

Throughout this section, we fix \(x \in X \setminus Y\) and use \(Z_x(M,N,t)\) to denote the set defined in (3.1). Moreover, let \(b > 0\) and \(t_0 = t_0(a) > 1\) for \(a \in (0, 1)\) be as in Lemma 3.4.

The following is the main result of this section.

**Proposition 3.5.** There exist constants \(C_1, C_2 \geq 1\) (independent of \(x\) and \(a\)) such that for all \(M > C_2 b/a\), all \(t \geq t_0\) and all \(N \in \mathbb{N}\),
\[
\int_{Z_x(M,N-1,t)} \alpha(g_N h_s x) \, ds \leq C_1 [(2a)^N \alpha(x) + (2a)^{N-1}b]
\]

(3.6)

Here we relax the restrictions on the height of \(x\) and on the dependence of \(M\) on \(t\) in comparison with [KKLM17, Proposition 5.1]

**Proof.** The proof is the same as that of [KKLM17, Proposition 5.1] with minor modifications that we discuss now. Using Property (2) in Definition 3.1, let \(C_2 \geq 1\) be such that for all \(s \in [-2,2]\),
\[
C_2^{-1} \leq \frac{\alpha(h_s x)}{\alpha(x)} \leq C_2.
\]

Consider \(M > C_2 b/a\) and \(t > t_0\). Let \(y \in X \setminus Y\) be so that \(\alpha(y) > b/a\). By Lemma 3.4, we have
\[
\int_{\mathbb{R}} \alpha(g_th_s y) \, ds \leq a\alpha(y) + b \leq 2a\alpha(y).
\]

(3.7)

Let \(N \in \mathbb{N}\) and define the following set.
\[
Z = \{(s_1, \ldots, s_N) \in \mathbb{R}^N : \alpha(g_{t_s} h_{s_1} g_{t_{s_1}} h_{s_1+1} \cdots g_{t_s} h_{s_1}) > b/a, \forall k = 1, \ldots, N-1\}
\]
Applying (3.7) repeatedly, we get
\[
\int \cdots \int_{Z} \alpha(g_{h_{s_N}} \cdots g_{h_{s_1}}x) d\rho(s_N) \cdots d\rho(s_1) \leq (2a)^{N-1} \int_{\mathbb{R}} \alpha(g_{h_{s}}x) d\rho(s) \\
\leq (2a)^N \alpha(x) + (2a)^{N-1}b \tag{3.8}
\]
On the last line, we applied Lemma 3.4, but since we don’t insist that \(\alpha(x) > b/a\), we get the extra term with \(b\). The rest of the proof is identical to that of [KKLM17, Proposition 5.1], where we take the constant \(C_1\) to be the implicit constant in that proof depending only on the Radon-Nikodym derivative of the Lebesgue measure with respect to a certain bounded variance Gaussian on \([-1, 1]\).

As a corollary, we obtain the following covering result.

**Corollary 3.6.** There exists \(b > 0\) such that for all \(a \in (0, 1)\), there exists \(t_0 > 1\) such that for all \(x \in X\), all \(M > C_2^2b/a\), all \(N \in \mathbb{N}\), and all \(t > t_0\), the set \(Z_x(M, N, t)\) can be covered by \(2C_1C_2^2a^Ne^{2tN} \max \left\{1, \frac{\alpha(x)}{M}\right\}\) intervals of radius \(e^{-2tN}\), where \(C_1, C_2 > 1\) are the absolute constants in Proposition 3.5.

**Proof.** By taking into account the different upper bound obtained in Proposition 3.5, the proof is identical to that of [KKLM17, Corollary 5.2]. \(\square\)

**Proposition 3.7** (cf. Theorem 1.5 in [KKLM17]). Suppose \(x \in X\setminus Y\). Then, for any \(\delta, a \in (0, 1)\) there exist \(M_0 > 1\) and \(t_0 > 1\), depending only on \(a\), such that for all \(M > M_0\), all \(t > t_0\) and all \(N \in \mathbb{N}\), the set \(Z_x(X_{\leq M}, N, t, \delta)\) can be covered with \(2^NC_1^N(2a)^{\deltaN}e^{2tNC}(x)\) intervals of radius \(e^{-2tN}\), where \(C_1 > 1\) is as in Corollary 3.6 and \(C(x) = \max \left\{1, \frac{\alpha(x)}{M}\right\}\).

**Proof.** We now describe the modifications needed on the proof of [KKLM17, Theorem 1.5] in order to prove the proposition. In the same notation as in Corollary 3.6, we take \(M_0 = C_2^2b/a\) and let \(M > M_0\). Then, the rest of the proof follows the same induction scheme used in [KKLM17, Theorem 1.5] with the base case being Corollary 3.6. The only modification on the scheme is to skip the steps involving enlarging \(M\) depending on the largeness of the step size \(t\) and instead work directly with the bound on covers provided by the preceding corollary.

In particular, in the second case of the inductive step in [KKLM17, Theorem 1.5], \(M\) is assumed large enough depending on \(t\) to apply their covering result [KKLM17, Corollary 5.2] which only applies to \(x \in X\) with \(\alpha(x)\) sufficiently large. Since Corollary 3.6 above works for all \(x\), such restriction on \(M\) is not needed. \(\square\)

### 3.3. Proof of Theorem 3.2

Let \(C_1\) be the constant in Proposition 3.7. Fix \(a \in (0, 1/2)\) such that \(\delta > -\ln(2C_1)/\ln(2a)\).

Let \(M_0 = M_0(a)\) and \(t_0 = t_0(a)\) be as in Proposition 3.7. Let \(M > M_0\) and \(t > t_0\). Define \(Q = X_{\leq M}, \gamma = -\ln(2a)/2t\) and \(\beta = \ln(2C_1)/2t,\) i.e., \(2C_1 = e^{2\beta t}\). Then, by Proposition 3.7, for all \(N \in \mathbb{N}\), we can cover the set \(Z_x(Q, N, t, \delta)\) with \(C' e^{2tN(1+\beta - \delta\gamma)}\) intervals of radius \(e^{-2tN}\), for some constant \(C'\) depending only on \(x\). Note that \(C'\) is finite by our assumption that \(x \in X\). \(\setminus Y\).

Therefore, by Lemma 2.5, the Hausdorff dimension of the set \(\limsup_{N \to \infty} Z_x(Q, N, t, \delta)\) is at most \(1 + \beta - \delta\gamma > 0\). By the choice of \(a\), this upper bound is strictly less than 1. Finally, we note that by definition of \(\beta\) and \(\gamma\), our upper bound is uniform over all \(x \in X\setminus Y\).
**Remark 3.8.** The proofs of Proposition 3.7 and Theorem 3.2 show that one can choose $M_0 = c''b e^{c'/\delta}$ in the conclusion of Theorem 3.2, for some positive constants $c'$ and $c''$, where $b$ is as in Definition 3.1.

4. Haussdorff Dimension of The Divergent on Average Directions

In this section we prove Theorem 2.3 which implies Theorem 1.7 (see Section 2).

**Proof of Theorem 2.3.** Consider the set

$$Z = \{ s \in [-1, 1] : g_t h_s \omega \text{ diverges on average} \}$$

Notice that for all compact sets $Q \subset \mathcal{H}_1(\alpha)$, all $0 \leq \delta \leq 1$ and all $t > 0$,

$$Z \subseteq \liminf_{N \to \infty} Z_\omega(Q, N, t, \delta) := \bigcap_{N_0=1}^{\infty} \bigcap_{N=N_0}^{\infty} Z_\omega(Q, N, t, \delta)$$

where $Z_\omega(Q, N, t, \delta)$ is defined in (3.1).

Building on earlier work of [EM01], it is shown in [Ath06] that the SL$_2(\mathbb{R})$ action on $X = \mathcal{H}_1(\alpha)$ satisfies the contraction hypothesis with respect to $Y = \emptyset$. The precise statement is the following:

**Lemma 4.1** (Lemma 2.10 in [Ath06]). For every $0 < \eta < 1$, there exists a function $\alpha_{\eta} : X \to \mathbb{R}^+$ satisfying item (1), (2) and (4) of Definition 3.1. Moreover, there are constants $c = c(\eta)$, and $t_0 = t_0(\eta) > 0$ so that for all $t > t_0$, there exists $b = b(t, \eta) > 0$ such that for all $x \in X$,

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha_{\eta}(g_t r_\theta x) \, d\theta \leq ce^{-(1-\eta)t} \alpha_{\eta}(x) + b$$

(4.1)

By using the integral estimate in (4.1) in place of the one in (3.3), we can prove the following analogue of Lemma 3.4.

**Corollary 4.2.** For every $0 < \eta < 1$, let $\alpha_{\eta} : X \to \mathbb{R}^+$ be as in Lemma 4.1. Let $\sigma$ be as in Definition 3.1. Then, there are constants $c' = c'(\eta, \sigma)$, and $t_0 = t_0(\eta) > 0$ so that for all $t > t_0$, there exists $b' = b'(t, \eta) > 0$ such that for all $x \in X$,

$$\int_{\mathbb{R}} \alpha_{\eta}(g_t h_s x) \, d\rho_1(s) \leq c' e^{-(1-\eta)t} \alpha_{\eta}(x) + b'$$

(4.2)

where $d\rho_1(s) = e^{-s^2} ds$ is a mean 0, variance 1 Gaussian. In particular, if $\alpha_{\eta}(x) > \frac{b e^{(1-\eta)t}}{c'}$, then

$$\int_{\mathbb{R}} \alpha_{\eta}(g_t h_s x) \, d\rho_1(s) \leq 2c' e^{-(1-\eta)t} \alpha_{\eta}(x)$$

(4.3)

As a result, we deduce the upper bound on $\dim_H(Z)$ from a covering result for the sets $Z_\omega(Q, N, t, \delta)$, where $Q$ will be a sublevel set of a height function $\alpha_{\eta}$ for $\eta \in (0, 1)$. Moreover, Equation (4.3) is the exact analogue of the integral estimate in [KKLM17, Corollary 4.2]. Fixing any choice of the parameter $\eta \in (0, 1)$, the rest of the proof of [KKLM17, Theorem 1.1] applies verbatim in our setting to get that $\dim_H(Z) \leq 1 - \frac{\delta(1-\eta)}{2}$. By sending $\delta$ to 1 and $\eta$ to 0, we get the theorem. □
5. Uniformity in Birkhoff’s Theorem

The purpose of this section is to prove a uniform version of [CE15, Theorem 1.1] due to Chaika and Eskin on the pointwise equidistribution of Teichmüller geodesics with respect to the Lebesgue measure on a horocycle arc. This step is crucial for our Hausdorff dimension estimates in the large deviation problems.

Throughout this section, suppose $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ is a fixed $\text{SL}_2(\mathbb{R})$ invariant affine submanifold. For an affine invariant submanifold $\mathcal{N} \subset \mathcal{H}_1(\alpha)$, we denote by $\nu_\mathcal{N}$ the unique $\text{SL}_2(\mathbb{R})$ invariant Lebesgue probability measure supported on $\mathcal{N}$. For any bounded continuous function $\phi$ on $\mathcal{H}_1(\alpha)$, let $\nu_\mathcal{H}(\phi) = \int_{\mathcal{H}_1(\alpha)} \phi \, d\nu_\mathcal{N}$. For any $T > 0$, $s \in [-1, 1]$ and $x \in \mathcal{H}_1(\alpha)$, we denote by $A_s^T(x)$ the measure defined by

$$A_s^T(x)(\varphi) := \frac{1}{T} \int_0^T \varphi(g_th_sx) \, dt$$

for any bounded continuous function $\varphi$ on $\mathcal{M}$. Similarly, for $N \in \mathbb{N}$ and $l > 0$, we define the measure $S_s^N(x)$ in the following way.

$$S_s^N(x)(\varphi) := \frac{1}{N} \sum_{n=1}^N \varphi(g_{ln}h_sx)$$

Notice that $S_s^N(x)$ depends on the step size $l$, though we do not emphasize this in the notation.

For any $h \in \text{SL}_2(\mathbb{R})$, we define $hA_s^T(x)$ and $hS_s^N(x)$ in the following way.

$$hA_s^T(x)(\varphi) := \frac{1}{T} \int_0^T \varphi(hg_th_sx) \, dt \quad \text{and} \quad hS_s^N(x)(\varphi) := \frac{1}{N} \sum_{n=1}^N \varphi(hg_{ln}h_sx)$$

Let $C_c^\infty(\mathcal{H}_1(\alpha))$ be a space of smooth compactly supported functions on $\mathcal{H}_1(\alpha)$. For any $\varphi \in C_c^\infty(\mathcal{H}_1(\alpha))$, we define a Sobolev norm $S(\varphi)$ of $\varphi$ by

$$S(\varphi) := \|\varphi\|_{L_\text{lip}} + \|\varphi\|_{\infty},$$

where $\|\phi\|_{L_\text{lip}}$ and $\|\varphi\|_\infty$ denote the Lipschitz constant and the maximum on $\mathcal{H}_1(\alpha)$ of $\varphi$, respectively. Then, for all $g \in \text{SL}_2(\mathbb{R})$ and all $x \in \mathcal{H}_1(\alpha)$, one has

$$|\varphi(gx) - \varphi(x)| \leq S(\varphi)d(g, \text{Id})$$

where $d(.,.)$ denotes some metric on $\text{SL}_2(\mathbb{R})$.

The following theorem is the main result of this section.

**Theorem 5.1.** Suppose $f$ is a bounded continuous function on $\mathcal{H}_1(\alpha)$. Then, for any $\varepsilon > 0$ there exist finitely many proper affine $\text{SL}_2(\mathbb{R})$ invariant submanifolds of $\mathcal{M}$, denoted by $\mathcal{N}_1, \ldots, \mathcal{N}_k$ such that for any compact set $F \subset \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$ and any $\kappa > 0$, there exists $T_0 = T_0(F, \kappa, \varepsilon, f) > 0$ such that for all $x \in F$ and all $T > T_0$,

$$\left| \left\{ s \in [-1, 1] : |A_s^T(x)(f) - \nu_\mathcal{M}(f)| \geq \varepsilon \right\} \right| < \kappa$$

The proof of Theorem 5.1 (see Section 5.3) is based on a combination of the techniques used to prove [CE15, Theorem 1.1] and [EMM15, Theorem 2.11], paying additional care to the unipotent invariance of limiting distributions. Following the same idea, we also prove the following discrete version of Theorem 5.1 (see Section 5.4 for the proof).
**Theorem 5.2.** Suppose \( f \) is a bounded continuous function on \( \mathcal{H}_1(\alpha) \). Then, for any \( \varepsilon > 0 \) there exist finitely many proper affine \( \text{SL}_2(\mathbb{R}) \) invariant submanifolds of \( \mathcal{M} \), denoted by \( N_1, \ldots, N_k \) such that for any compact set \( F \subset \mathcal{M} \setminus \left( \bigcup_{i=1}^k N_i \right) \), any \( \kappa > 0 \) and all \( l > 0 \), there exists \( N_0 = N_0(F, \kappa, l, \varepsilon, f) > 0 \) such that for all \( x \in F \) and all \( N > N_0 \),

\[
\left| \left\{ s \in [-1, 1] : |S_s^N(x)(f) - \nu_M(f)| \geq \varepsilon \right\} \right| < \kappa
\]  

(5.6)

Theorems 5.1 and 5.2 are in the spirit of the results of [EMM15] and [DM93].

5.1. Some Finiteness and Recurrence Results. In this section we formulate some facts that we use throughout Section 5.

The following lemma will provide us with the finite exceptional collection of invariant submanifolds in Theorem 5.1.

**Lemma 5.3** (Lemma 3.4 in [EMM15]). Given \( \varepsilon > 0 \) and \( \varphi \in C_c(\mathcal{H}_1(\alpha)) \). There exists a finite collection \( \mathcal{C} \) of proper affine invariant submanifolds of \( \mathcal{M} \) with the following property: if \( N \subset \mathcal{M} \) is an affine invariant submanifold such that \( |\nu_N(\varphi) - \nu_M(\varphi)| \geq \varepsilon \), then \( N \) is contained in some \( N' \in \mathcal{C} \).

The following proposition shows that most geodesic trajectories avoid any given finite collection of proper submanifolds of \( \mathcal{M} \).

**Proposition 5.4** (Proposition 3.8 in [EMM15]). Given \( \varepsilon > 0 \) and any (possibly empty) proper affine invariant submanifold \( N \), there exists an open neighborhood \( \Omega_{N,\varepsilon} \) of \( N \) with the following property: the complement of \( \Omega_{N,\varepsilon} \) is compact and for any compact set \( F \subset \mathcal{H}_1(\alpha) \setminus N \), there exists \( T_0 = T_0(F) > 0 \), so that for any \( T > T_0 \) and any \( x \in F \),

\[
\int_{-1}^{1} A_s^T(x)(\chi_{\Omega_{N,\varepsilon}}) \, ds < \varepsilon
\]  

(5.7)

where \( \chi_{\Omega_{N,\varepsilon}} \) denotes the indicator function of the set \( \Omega_{N,\varepsilon} \).

The following discrete version of Proposition 5.4 also holds.

**Proposition 5.5.** Given \( \varepsilon > 0 \) and any (possibly empty) proper affine invariant submanifold \( N \), there exists an open neighborhood \( \Omega_{N,\varepsilon} \) of \( N \) with the following property: the complement of \( \Omega_{N,\varepsilon} \) is compact and for any compact set \( F \subset \mathcal{H}_1(\alpha) \setminus N \) and any \( l > 0 \), there exists \( N_0 > 0 \), so that for any \( N > N_0 \) and any \( x \in F \),

\[
\int_{-1}^{1} S_s^N(x)(\chi_{\Omega_{N,\varepsilon}}) \, ds < \varepsilon
\]  

(5.8)

where \( \chi_{\Omega_{N,\varepsilon}} \) denotes the indicator function of the set \( \Omega_{N,\varepsilon} \).

The proof of Proposition 5.5 is similar to that of Proposition 5.4, i.e., it is a consequence of the contraction hypothesis (see Definition 3.1) shown in [EMM15, Proposition 2.13]. See also [EM04, Lemma 3.1].

**Proof.** By Proposition 2.13 in [EMM15], there exists a height function \( f_N \) with \( X = \mathcal{H}_1(\alpha) \) and \( Y = N \) (see Definition 3.1). Let \( m_F = \sup \{ f_N(x) : x \in F \} \). Notice that \( m_F \geq 1 \) as, by definition, \( f_N(x) \geq 1 \) for any \( x \in \mathcal{H}_1(\alpha) \). Then, by Lemma 3.3, there exists \( t_1 > 0 \) such that for all \( t > t_1 \) and all \( x \in F \),

\[
\int_{-1}^{1} f_N(g_t h_s x) \, ds \leq \frac{1}{m_F} f_N(x) + b \leq b + 1.
\]
Moreover, by Property (2) in Definition 3.1, there exists $M = M(t_1) > 0$ such that for all $0 \leq t \leq t_1$ and all $x$, 

$$f_N(g_t h_s x) \leq M f_N(x).$$

Let $L > 0$ be such that $\frac{b+2}{L} < \varepsilon$. Define a set $\Omega_{N,\varepsilon}$ in the following way.

$$\Omega_{N,\varepsilon} = \{ x \in \mathcal{H}_1(\alpha) : f_N(x) > L \}^o,$$

where $\{ \cdot \}^o$ denotes the interior of a set. Then, by Property (4) in Definition 3.1, $\Omega_{N,\varepsilon}$ is an open neighborhood of $N$ with compact complement. Let $N_0 \in \mathbb{N}$ be sufficiently large so that

$$\frac{M m_F t_1}{l N_0} < 1.$$

Then, using the above estimates, we get

$$\int_{-1}^{1} S^N_s(x)(f_N) \, ds = \frac{1}{N} \sum_{1 \leq n \leq t_1} \int_{-1}^{1} f_N(g_{ln} h_s x) \, ds + \frac{1}{N} \sum_{t_1/i < n \leq N} \int_{-1}^{1} f_N(g_{ln} h_s x) \, ds$$

$$\leq \frac{M m_F t_1}{l N} + b + 1 \leq b + 2. \tag{5.9}$$

Notice that for any $n \in \mathbb{N}$ and $s \in [-1, 1]$, we have

$$f_N(g_{ln} h_s x) \geq L \chi_{\Omega_{N,\varepsilon}}(g_{ln} h_s x). \tag{5.10}$$

Therefore, by (5.9) and (5.10), we obtain

$$\int_{-1}^{1} S^N_s(x)(\chi_{\Omega_{N,\varepsilon}}) \, ds \leq \frac{b+2}{L} < \varepsilon.$$

□

5.2. Effective Unipotent Invariance. In this section, we show a quantative version of [CE15, Proposition 3.1] (Proposition 5.7) regarding almost sure unipotent invariance of limit points of measures of the form (5.1). Also, we state an analogue of it for discrete averages (Proposition 5.8), whose proof is identical to the flow case. See [Kha17] for a generalization of this phenomenon to semisimple Lie group actions.

Suppose $x \in \mathcal{H}_1(\alpha)$, $\phi \in C^\infty_c(\mathcal{H}_1(\alpha))$ and $\beta \in \mathbb{R}$. For $t > 0$ and $s \in [-1, 1]$, we define

$$f_t(s) = \varphi(g_t h_s x) - \varphi(h_\beta g_t h_s x). \tag{5.11}$$

The following lemma formulated for horocycle arcs is an analogue of Lemma 3.3 in [CE15] which is proved for circle arcs.

**Lemma 5.6** (Analogue of Lemma 3.3 in [CE15]). There exists a constant $C_1 > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$, $\phi \in C^\infty_c(\mathcal{H}_1(\alpha))$, $\beta \in \mathbb{R}$ and all $t_1, t_2 > 0$,

$$\left| \int_{-1}^{1} f_{t_1}(s) f_{t_2}(s) \, ds \right| \leq C_1 S(\varphi)^2 e^{-2|t_1-t_2|},$$

where $S(\phi)$ and $f_t(s)$ are defined in (5.4) and (5.11), respectively.

We note that the proof of Lemma 5.6 is identical to the proof of [CE15, Lemma 3.3] and simpler if one takes into account that the group of elements $h_s$ is normalized by $g_t$. 
Proposition 5.7 (Quantitative version of Proposition 3.1 in [CE15]). Suppose \( \beta \in \mathbb{R} \). Then, there exists a constant \( C > 0 \), such that for all \( T > 0 \), all \( x \in \mathcal{H}_1(\alpha) \) and all \( \varphi \in C_c^\infty(\mathcal{H}_1(\alpha)) \), the Lebesgue measure of the set
\[
\left\{ s \in [-1, 1] : |A_s^T(x)(\varphi) - (h_\beta A_s^T(x))(\varphi)| > \frac{S(\varphi)}{T^{1/8}} \right\}
\]
is at most \( C/T^{1/4} \).

The version of Proposition 5.7 for discrete averages is the following.

Proposition 5.8. Suppose \( \beta \in \mathbb{R} \). Then, there exists a constant \( C > 0 \), such that for all \( N > 0 \), all \( x \in \mathcal{H}_1(\alpha) \) and all \( \varphi \in C_c^\infty(\mathcal{H}_1(\alpha)) \), the Lebesgue measure of the set
\[
\left\{ s \in [-1, 1] : |S_s^N(x)(\varphi) - (h_\beta S_s^N(x))(\varphi)| > \frac{S(\varphi)}{N^{1/8}} \right\}
\]
is at most \( C/N^{1/4} \).

Proof of Proposition 5.7. By Fubini’s theorem and Lemma 5.6, one has
\[
\int_{-1}^1 |A_s^T(\varphi) - (h_\beta A_s^T)(\varphi)|^2 \, ds \leq \frac{1}{T^2} \int_{[0,T]^2} \left| \int_{-1}^1 f_{t_1}(s) f_{t_2}(s) \, ds \right| \, dt_1 \, dt_2
\]
\[
= \frac{1}{T^2} \int_{|t_1 - t_2| < T^{1/2}} \left| \int_{-1}^1 f_{t_1}(s) f_{t_2}(s) \, ds \right| \, dt_1 \, dt_2 + C_1 S(\varphi)^2 e^{-2T^{1/2}}
\]
\[
\leq \frac{16||\varphi||_\infty^2}{T^{1/2}} + C_1 S(\varphi)^2 e^{-2T^{1/2}}
\]
\[
\leq 2C_2 S(\varphi)^2 T^{-1/2}
\]
where we used the facts that \( |f_t(s)| \leq 2||\varphi||_\infty \), the measure of the region \( |t_1 - t_2| < T^{1/2} \) is at most \( 2T^{3/2} \), and \( C_2 > 16C_1 \) is a constant such that for all \( T > 0 \), one has
\[
e^{-2T^{1/2}} \leq C_2 T^{-1/2}.
\]
Using the Chebyshev’s inequality, we obtain the proposition. \( \square \)

5.3. Proof of Theorem 5.1. Fix positive constants \( \varepsilon \) and \( \kappa \). Let \( \mathcal{C} \) be the finite collection of affine invariant submanifolds \( \mathcal{N}_1, \ldots, \mathcal{N}_k \) of \( \mathcal{M} \) given by Lemma 5.3 applied to the given function \( f \) and \( \varepsilon/2 \). Consider a compact subset \( F \subset \mathcal{M} \setminus \bigcup_i \mathcal{N}_i \).

Let \( \varepsilon' > 0 \) be a sufficiently small number such that \( \sqrt{\varepsilon'} < \min \left\{ \frac{\kappa}{3}, \frac{\varepsilon}{2||f||_\infty} \right\} \). By Proposition 5.4, since \( \mathcal{C} \) is a finite collection, there exists an open neighborhood \( \Omega_{\mathcal{C}, \varepsilon'} \) of \( \bigcup \mathcal{N}_i \) and \( T_0 > 0 \) depending on \( \varepsilon' \) and \( F \) such that for all \( T > T_0 \) and all \( x \in F \), we have
\[
\int_{-1}^1 A_s^T(x) \left( \chi_{\Omega_{\mathcal{C}, \varepsilon'}} \right) \, ds < \varepsilon'
\]
and hence, by Chebyshev’s inequality, we get that the measure of the set
\[
D(x, T, \varepsilon') = \left\{ s \in [-1, 1] : A_s^T(x) \left( \chi_{\Omega_{\mathcal{C}, \varepsilon'}} \right) \geq \sqrt{\varepsilon'} \right\}
\]
is at most \( \sqrt{\varepsilon'} \).

Let \( \Phi = \{ \varphi_n : n \in \mathbb{N} \} \subset C_c^\infty(\mathcal{H}_1(\alpha)) \) be a countable dense collection of functions. For each \( N \), let \( \Phi_N = \{ \varphi_n : 1 \leq n \leq N, n \in \mathbb{N} \} \subset \Phi \). Consider \( \beta_1, \beta_2 \in \mathbb{R} \) with \( \beta_1/\beta_2 \notin \mathbb{Q} \). By
Proposition 5.7, there exists a constant $C > 0$ such that for all $T > 0$ and all $x$, the measure of the sets

$$B(x, T, \Phi_N) := \bigcup_{n=1}^{N} B(x, T, \varphi_n)$$

(5.13)

is at most $CNT^{-1/4}$, where $S(\cdot)$ is a Sobolev norm (see (5.4)) and

$$B(x, T, \varphi_n) := \left\{ s \in [-1, 1] : \left| A^T_s(x) - (h_{\beta_i} A^T_s(x)) \right| > \frac{S(\varphi_n)}{T^{1/8}} \right\}.$$  

We prove the theorem by contradiction. Suppose that the conclusion of the theorem does not hold for our choice of $F$ and $\kappa$. Then, there exists a sequence $x_n \in F$ and $T_n \to \infty$ such that for each $n \in \mathbb{N}$, the measure of the set

$$Z_{x_n}(f, T_n) := \left\{ s \in [-1, 1] : \left| A^T_{s_n}(x) - \nu_M(f) \right| \geq \varepsilon \right\}$$

(5.14)

has measure at least $\kappa$.

By our estimates on the measures of the sets in (5.12), (5.13) and (5.14), and the choice of $\epsilon'$ such that $\sqrt{\epsilon'} < \kappa/3$, then the following holds. For all $n$ sufficiently large so that $CT_n^{-1/8} < \kappa/3$, we have

$$Z_{x_n}(f, T_n) \cap D(x_n, T_n, \varepsilon')^c \cap B(x_n, T_n, \Phi_{T_n^{1/8}})^c \neq \emptyset$$

(5.15)

where for a set $A \subset [-1, 1]$, we use $A^c$ to denote its complement. Therefore, for all $n$ sufficiently large we can choose a point $s_n$ that belongs to the intersection in (5.15). Since the space of Borel measures on $\mathcal{H}_1(\alpha)$ of mass at most 1 is compact in the weak-$*$ topology, after passing to a subsequence if necessary, we may assume that there is a Borel measure $\nu$ such that

$$A^T_{s_n}(x) \xrightarrow{weak-*} \nu$$

Note that a priori $\nu$ may be the 0 measure. We show that this is not the case.

We claim that $\nu$ is $SL_2(\mathbb{R})$ invariant. By Theorem 1.4 due to Eskin and Mirzakhani [EM13], it is sufficient to show that $\nu$ is invariant by $P$, the subgroup of upper triangular matrices. Clearly, $\nu$ is invariant by $g_t$ for all $t$. Moreover, by the dominated convergence theorem, it suffices to show that $\nu$ is invariant by $h_{\beta_1}$ and $h_{\beta_2}$ as they generate a dense subgroup of $U = \{ h_s : s \in \mathbb{R} \}$.

Since smooth functions are dense in the set of compactly supported continuous functions, it suffices to show that for $i = 1, 2$, $h_{\beta_i} \nu(\varphi_k) = \nu(\varphi_k)$, where $h_{\beta_i} \nu(\varphi_k) := \int_{\mathcal{H}_1(\alpha)} \phi_k(h_{\beta_i}\omega) d\nu(\omega)$ for all $\varphi_k \in \Phi$, our countable dense collection of smooth compactly supported functions.

Fix some $\varphi_k \in \Phi$. Note that for all $n$ sufficiently large, we have that $\varphi_k \in \Phi_{T_n^{1/8}}$ and, therefore, $s_n \notin B(x_n, T_n, \varphi_k)$. As a result, we have

$$\left| A^T_{s_n}(x_n) - h_{\beta_i} A^T_{s_n}(x_n) \right| \leq \frac{S(\varphi_k)}{T_n} \xrightarrow{n \to \infty} 0.$$  

Therefore, $\nu$ is $SL_2(\mathbb{R})$ invariant.

Moreover, since $s_n \in Z_{x_n}(f, T_n)$ for all $n$, we obtain that

$$|\nu(f) - \nu_M(f)| \geq \varepsilon.$$  

We show that this is not possible. By Proposition 2.16 in [EMM15], there are countably many affine invariant submanifolds in $\mathcal{H}_1(\alpha)$. Thus, since $\nu$ is $SL_2(\mathbb{R})$ invariant, it has a
countable ergodic decomposition of the form
\[ \nu = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}} \nu_{\mathcal{N}}, \]
where the sum is taken over all such proper (possibly empty) affine invariant submanifolds and \( a_{\mathcal{N}} \in [0,1] \) for all \( \mathcal{N} \). Note that since \( s_n \notin D(x_n, T_n, \varepsilon') \), we have
\[ \sum_{\mathcal{N}: \exists \mathcal{N}' \in \mathcal{C}, \mathcal{N} \subseteq \mathcal{N}'} a_{\mathcal{N}} \leq \nu(\Omega_{\mathcal{C}, \varepsilon'}) \leq \sqrt{\varepsilon'} \]

Since the complement of \( \Omega_{\mathcal{C}, \varepsilon'} \) is compact and \( A_{s_n}^{T_n}(x_n)(1 - \chi_{\Omega_{\mathcal{C}, \varepsilon'}}) \geq 1 - \sqrt{\varepsilon'} \), the total mass of \( \nu \) is at least \( 1 - \sqrt{\varepsilon'} \).

Furthermore, we have that \( |\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| < \varepsilon/2 \), for all \( \mathcal{N} \) not contained in any member of \( \mathcal{C} \), by definition of the collection \( \mathcal{C} \).

Let \( |\nu| := \sum_{\mathcal{N}} a_{\mathcal{N}} \) be the total mass of \( \nu \). Then, we have that
\[ \varepsilon \leq |\nu(f) - \nu_{\mathcal{M}}(f)| = |(1 - |\nu|)\nu_{\mathcal{M}}(f) + \sum_{\mathcal{N} \subseteq \mathcal{M}} a_{\mathcal{N}}(\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f))| \]
\[ \leq \sqrt{\varepsilon'} ||f||_{\infty} + \sum_{\mathcal{N} : \exists \mathcal{N}' \in \mathcal{C}, \mathcal{N} \subseteq \mathcal{N}'} a_{\mathcal{N}}|\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| + \sum_{\mathcal{N} : \exists \mathcal{N}' \in \mathcal{C}, \mathcal{N} \subseteq \mathcal{N}'} a_{\mathcal{N}}|\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| \]
\[ \leq \sqrt{\varepsilon'} ||f||_{\infty} + 2||f||_{\infty}\nu(\Omega_{\mathcal{C}, \varepsilon'}) + |\nu|\varepsilon/2 \]
\[ \leq 3||f||_{\infty}\sqrt{\varepsilon'} + \varepsilon/2 < \varepsilon. \]

We get the desired contradiction by our choice of \( \varepsilon' \).

5.4. **Proof of Theorem 5.2.** The proof is similar to the proof of Theorem 5.1 (the flow case), and relies on using Propositions 5.5 and 5.8 instead of Propositions 5.4 and 5.7, respectively. The proof also goes by contradiction. Assuming that the conclusion of the theorem does not hold, we construct a \( \text{SL}_2(\mathbb{R}) \) invariant measure \( \nu \). The analysis of its ergodic decomposition implies a contradiction as in Section 5.3.

The following lemma allows us to show that the constructed measure \( \nu \) is \( \text{SL}_2(\mathbb{R}) \) invariant.

**Lemma 5.9.** Let \( l > 0 \) and let \( P_l \) be the group generated by elements of the form \( g_{n, h_s} \) for \( n \in \mathbb{Z} \) and \( s \in \mathbb{R} \). Then, \( \nu \) is \( \text{SL}_2(\mathbb{R}) \) invariant if \( \nu \) is a \( P_l \) ergodic invariant probability measure on \( \mathcal{M} \).

**Proof.** Denote by \( \tilde{\nu} \) the measure defined by
\[ \tilde{\nu} := \frac{1}{l} \int_0^l (g_t)_* \nu \, dt, \quad (5.16) \]
where \((g_t)_* \nu \) is the pushforward of \( \nu \).

Then, \( \tilde{\nu} \) is invariant by the group of upper triangular matrices \( P \). Notice that for any \( t \in (0, l) \) we have \((g_t)_* \nu \) is invariant by the group \( U = \{h_s : s \in \mathbb{R}\} \) due to the fact that \( U \) is normalized by \( g_t \) and \( \nu \) is invariant by \( U \). That implies that \( \tilde{\nu} \) is invariant by \( U \) as it is a convex combination of \( U \) invariant measures. Similarly, we can show that \( \tilde{\nu} \) is invariant under \( \mathbb{Z} \) action of \( g_t \). To show the invariance under the group \( A = \{g_t : t \in \mathbb{R}\} \), we write \( t = ml + r \) for some \( m \in \mathbb{Z} \) and \( r \in [0, 1) \) and use the invariance by \( \{g_{nt} : n \in \mathbb{Z}\} \).
As a result, by [EM13, Theorem 1.4], \( \bar{\nu} \) is \( SL_2(\mathbb{R}) \)-invariant. Thus, \( \bar{\nu} \) has the following ergodic decomposition with respect to the \( SL_2(\mathbb{R}) \) action:

\[
\bar{\nu} = \sum_{\mathcal{N} \subseteq \mathcal{M}} a_\mathcal{N} \nu_\mathcal{N}, \tag{5.17}
\]

where each \( \nu_\mathcal{N} \) is ergodic under the \( SL_2(\mathbb{R}) \) action. But, by Mautner’s phenomenon, each \( \nu_\mathcal{N} \) is ergodic under the action of \( h_s \) for all \( s \neq 0 \).

On the other hand, \( (g_t)_* \nu \) is \( h_s \)-invariant for all \( t \) and \( s \). Hence, equations (5.16) and (5.17) give two decompositions of \( \bar{\nu} \) for the action of \( h_s \), one of which is a countable decomposition into ergodic measures.

Thus, by uniqueness of the ergodic decomposition, there exists a set \( A \subseteq [0, l] \) of positive Lebesgue measure \( |A| \) and an affine invariant manifold \( \mathcal{N} \) so that \( a_\mathcal{N} = |A|/l \) and

\[
\frac{1}{l} \int_{A} (g_t)_* \nu \, dt = a_\mathcal{N} \nu_\mathcal{N}.
\]

But, by ergodicity of \( \nu_\mathcal{N} \) under the action of \( h_s \), we have that \( (g_t)_* \nu = \nu_\mathcal{N} \) for almost every \( t \in A \). Since \( \nu_\mathcal{N} \) is \( SL_2(\mathbb{R}) \) invariant, then so is \( \nu \). \( \square \)

6. Dimension of Directions with Large Deviations in Birkhoff’s Theorem

The goal of this section is to prove Theorem 2.1. We also outline the modifications on the proof needed to prove Theorem 6.7 in Section 6.6.

In what follows, \( \mathcal{M} \subseteq \mathcal{H}_1(\alpha) \) is a fixed affine invariant manifold. By a simple approximation argument, it is enough to prove Theorem 2.1 when \( f \) is a Lipschitz function. We let \( S(f) \) denote the Sobolev norm (see (5.4)), and \( \nu_\mathcal{M}(f) = \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \).

Throughout this section we use the following notation. For any positive \( \varepsilon, N \in \mathbb{R}, M \in \mathbb{N} \) and a subset \( Q \subseteq \mathcal{M} \), we define the following sets.

\[
B_\omega(f, N, \varepsilon, M) := \left\{ s \in [-1, 1] : \frac{1}{MN} \int_{0}^{MN} f(g_t h_s \omega) \, dt > \nu_\mathcal{M}(f) + \varepsilon \right\}, \tag{6.1}
\]

\[
Z_\omega(Q, M, N, \varepsilon) := \left\{ s \in [-1, 1] : \# \left\{ 0 \leq i \leq M - 1 : g_{iN} h_s \omega \notin Q \right\} > \varepsilon \right\},
\]

\[
B_\omega(f, N, \varepsilon) := \limsup_{M \to \infty} B_\omega(f, N, \varepsilon, M), \quad Z_\omega(Q, N, \varepsilon, M) := \limsup_{M \to \infty} Z_\omega(Q, N, \varepsilon, M).
\]

It is straightforward to check that \( B_\omega(f, N, \varepsilon) \) is equal to the exceptional set considered in Theorem 2.1. The sets \( Z_\omega(Q, M, N, \varepsilon) \) are the same as the ones defined in (3.1).

Next, for any \( s \in [-1, 1], i \in \mathbb{N} \) and positive \( \beta, N \in \mathbb{R} \), we define the corresponding functions and sets:

\[
f_i(s) := \frac{1}{N} \int_{iN}^{(i+1)N} f(g_t h_s \omega) \, dt \tag{6.2}
\]

\[
F_i(\beta) = \{ s : f_i(s) > \nu_\mathcal{M}(f) + \beta \} \tag{6.3}
\]

Here, we drop the dependence on the basepoint \( \omega \) from the notation for simplicity.
Strategy. The strategy for proving Theorem 2.1 consists of two steps. The first step is to use Theorem 5.2 to control the measure of the sets $F_i(\beta)$. This is carried out in Lemma 6.2.

The next step is to show that the sets $F_i(\varepsilon/2)$ behave like level sets of independent random variables (Proposition 6.5). This will allow us to bound the measure of finite intersections of these sets. The proof of this independence property also yields a mechanism for controlling the number of intervals needed to cover such finite intersection using its measure (Lemma 6.6).

In order to apply Theorem 5.2, we need to insure that our trajectories land in a pre-chosen compact set. Hence, we are forced to run the above argument but restricted to the "recurrent directions". This restriction to recurrent directions is shown in Lemma 6.1. Applying Theorem 3.2, we control the Hausdorff dimension of the non-recurrent directions.

6.1. Sets and Partitions. For $N > 0$ and $i \in \mathbb{N}$, let $\mathcal{P}_i$ denote the partition of $[-1, 1]$ into intervals of radius $e^{-2iN}$. For a set $Q \subset H_1(\alpha)$, define the following sub-partitions

$$
\mathcal{R}_i(Q) = \{J \in \mathcal{P}_i : \exists s \in J, g_{iN} h_s \omega \in Q\}
$$

Let $\mathcal{D}_i(Q) = \mathcal{P}_i \setminus \mathcal{R}_i(Q)$. Here $\mathcal{R}$ signifies recurrence and $\mathcal{D}$ signifies divergence. We note that the definition of $\mathcal{R}_i$ depends on the basepoint $\omega$ but we suppress this dependence in our notation.

**Lemma 6.1.** Suppose $Q \subset H_1(\alpha)$. Then, for any $\omega \in H_1(\alpha)$, $N, \varepsilon > 0$, $0 < \delta \leq \frac{\varepsilon}{4S(f)}$, and $M \in \mathbb{N}$, we have

$$
B_\omega(f, N, \varepsilon, M) \subseteq Z_\omega(Q, M, N, \delta) \cap \bigcup_{A \subset \{0, \ldots, M-1\} \atop |A| = \lceil \delta M \rceil} \bigcap_{i \in A} F_i^{\mathcal{R}}(\varepsilon/2),
$$

where $\mathcal{R}_i := \mathcal{R}_i(Q)$ for all $i \in \mathbb{N}$ and

$$
F_i^{\mathcal{R}}(\varepsilon/2) = F_i(\varepsilon/2) \cap \bigcup_{J \in \mathcal{R}_i} J.
$$

**Proof.** First, we notice that

$$
B_\omega(f, N, \varepsilon, M) \subseteq \bigcup_{A \subset \{0, \ldots, M-1\} \atop |A| > 2\delta M} \bigcap_{i \in A} F_i(\varepsilon/2). \tag{6.4}
$$

It holds by the following inequalities.

$$
\frac{1}{MN} \int_0^{MN} f(g_t h_s \omega) \, dt = \frac{1}{M} \sum_{i=0}^{M-1} f_i(s) = \frac{1}{M} \sum_{0 \leq i \leq M-1 \atop f_i(s) \leq \nu_M(f)+\varepsilon/2} f_i(s) + \frac{1}{M} \sum_{0 \leq i \leq M-1 \atop f_i(s) > \nu_M(f)+\varepsilon/2} f_i(s) \leq \nu_M(f) + \varepsilon/2 + \frac{\|f\|_\infty}{M} \# \{i : f_i(s) > \nu_M(f) + \varepsilon/2\}
$$

$$
\leq \nu_M(f) + \varepsilon/2 + \frac{S(f)}{M} \# \{i : s \in F_i(\varepsilon/2)\}
$$

Thus, if $s \in B_\omega(f, N, \varepsilon, M)$, then we must have that

$$
\# \{i : s \in F_i(\varepsilon/2)\} > \frac{\varepsilon}{2S(f)} M \geq 2\delta M
$$

by our choice of $\delta$. 

By (6.4), it suffices to show the following to prove the lemma.

\[ Z_\omega(Q, M, N, \delta)^c \cap \bigcup_{A \subseteq \{0, \ldots, M-1\}} F_i(\varepsilon/2) \subseteq \bigcup_{A \subseteq \{0, \ldots, M-1\}} F_i^R(\varepsilon/2), \]

where for a set \( E \subseteq [-1, 1] \), \( E^c \) denotes its complement.

The set \( Z_\omega(Q, M, N, \delta) \) was defined to be the set of directions \( s \) such that \( g_{iN} h_s \omega \notin Q \) for at least \( \delta M \) natural numbers \( i < M \). Hence, we get that

\[ Z_\omega(Q, M, N, \delta)^c \subseteq \bigcup_{B \subseteq \{0, \ldots, M-1\}} \bigcup_{J \in \mathcal{R}_j} J. \]

Indeed, the right hand side describes the set of directions \( s \) which belong to \( \bigcup_{J \in \mathcal{R}_j} J \) for at least \( (1 - \delta)M \) natural numbers \( j < M \). By definition of \( \mathcal{R}_j \), this certainly contains the set of directions \( s \) for which \( g_{jN} h_s \omega \in Q \) for at least \( (1 - \delta)M \) natural numbers \( j < M \), that is the set on the left hand side.

Notice that the following inclusions hold.

\[
\begin{align*}
\left[ \bigcup_{A \subseteq \{0, \ldots, M-1\}} F_i(\varepsilon/2) \cap \bigcup_{B \subseteq \{0, \ldots, M-1\}} \bigcup_{J \in \mathcal{R}_j} J \right] & \subseteq \bigcup_{A \subseteq \{0, \ldots, M-1\}} \bigcup_{B \subseteq \{0, \ldots, M-1\}} \bigcup_{j \in \mathcal{R}_j} F_i(\varepsilon/2) \bigcap_{i \in A} \bigcap_{j \in \mathcal{R}_j} J \bigcup_{B \subseteq \{0, \ldots, M-1\}} \bigcup_{B \subseteq \{0, \ldots, M-1\}} \bigcup_{j \in \mathcal{R}_j} F_i^R(\varepsilon/2) \bigcap_{i \in A \cap B} \bigcup_{j \in \mathcal{R}_j} J \bigcup_{A \subseteq \{0, \ldots, M-1\}} \bigcup_{A \subseteq \{0, \ldots, M-1\}} F_i^R(\varepsilon/2).
\end{align*}
\]

where for the last inclusion we used the fact that for two sets \( A, B \subseteq \{0, \ldots, M-1\} \) with \( |A| > 2\delta M \) and \( |B| > (1 - \delta)M \), we have that \( |A \cap B| > \delta M \). Moreover, notice that

\[ \bigcup_{A \subseteq \{0, \ldots, M-1\}} F_i^R(\varepsilon/2) \subseteq \bigcup_{|A| = \delta M} F_i^R(\varepsilon/2). \]

This completes the proof. \( \square \)

6.2. Measure Bounds for \( F_i \). The next lemma allows us to control the measure of the proportion of a set \( F_i \) in an element of the partition \( \mathcal{R}_i(Q) \) for a suitably chosen large compact set with good properties. This will be a direct application of Theorem 5.1.

Let \( \mathcal{N}_1, \ldots, \mathcal{N}_k \) be proper affine invariant submanifolds as in Theorem 5.1 applied to \( \varepsilon \) and \( f \). By [EMM15, Proposition 2.13], for any \( i = 1, \ldots, k \) there exist height functions \( f_{\mathcal{N}_i} \) such
that for all $\ell > 0$, the sets
\[
C_\ell = \left\{ x \in \mathcal{H}(\alpha) : \sum_{i=1}^{k} f_N(x) \leq \ell \right\}
\]
are compact. The following is the main result of this section which is the form we will use Theorem 5.1 in. Recall the definition of the sets $F_i(\beta)$ in (6.3).

**Lemma 6.2.** For all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $N > T_0$, $\beta > \varepsilon$, $i \in \mathbb{N}$, all $\omega \in \mathcal{H}(\alpha)$ and all $J \in \mathcal{R}_i(C_\ell)$, we have
\[
\frac{\nu(J \cap F_i(\beta))}{\nu(J)} \leq a,
\]
where $\nu$ is the Lebesgue probability measure on $[-1, 1]$.

**Proof.** Denote by $B_1$ a neighborhood of radius 1 around identity in $SL_2(\mathbb{R})$. Fix $\ell > 0$ and $a > 0$. Let $\ell' > \ell$ be such that $B_1 C_\ell \subseteq C_{\ell'}$.

By Theorem 5.1 applied to $f$, $\varepsilon$, $a$ and the compact set $F = C_{\ell'} \subset \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$, where $\mathcal{N}_i$ are given by that theorem, there exists $T_0$ such that for all $N > T_0$ and $x \in F$, we have
\[
\left| \left\{ s \in [-1, 1] : |\frac{1}{N} \int_0^N f(g_s h_x) \, dt - \nu_M(f) \right| \geq \varepsilon \right| < a. \quad (6.5)
\]

For any $i \in \mathbb{N}$, we define $\mathcal{R}_i := \mathcal{R}_i(C_\ell)$. Fix $J \in \mathcal{R}_i$. Let $s_0 \in J$ be such that $g_i h_{s_0} \omega \in C_\ell$. By our choice of $\ell'$, we have the following holds for any $s \in J$.
\[
g_i h_{s_0} \omega = h_{e^{2iN(s-s_0)}} g_i h_{s_0} \omega \in B_1 C_\ell \subseteq C_{\ell'}
\]
In particular, the above holds for the center $c_0$ of the interval $J$. Let $s \in J - c_0$ be such that $s + c_0 \in F_i(\beta)$. Then, we get that
\[
\nu_M(f) + \beta < \frac{1}{N} \int_0^N f(g_{t+iN h_{s+c_0}}) \, dt = \frac{1}{N} \int_0^N f(g_{t+iN h_{s+c_0}} g_i h_{c_0} \omega) \, dt
\]
Thus, we obtain the following.
\[
e^{2iN} ((J \cap F_i(\beta)) - c_0) \subseteq \left\{ s \in [-1, 1] : \left| \frac{1}{N} \int_0^N f(g_{t h_s h_{iN h_{c_0} \omega}}) \, dt - \nu_M(f) \right| \geq \beta \right\}
\]
Since $g_i h_{c_0} \omega \in C_{\ell'}$, the Lemma follows from (6.5).

The following corollary is an immediate consequence of Lemma 6.2 and the fact that elements of $\mathcal{R}_i$ are disjoint.

**Corollary 6.3.** For all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $N > T_0$, $\beta > \varepsilon$, $i \in \mathbb{N}$, we have that
\[
\nu \left( F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i(C_\ell)} J \right) \leq a.
\]
6.3. Independence of the Sets $F_i$. The goal of this section is to prove that the sets $F_i(\beta)$ behave as if they are independent. More precisely, we will prove that the measure of the intersection of such sets is bounded above by the product of their measures, up to controlled error. Recall the definition of partitions $P_i$ in Section 6.1.

We start with the following simple but key observation.

**Lemma 6.4.** Suppose $i < j$, where $i$ and $j$ are natural numbers, and $\beta > 0$. Let $J \in P_j$ be such that $J \cap F_i(\beta) \neq \emptyset$. Then, $J \subseteq F_i \left( \beta - \frac{S(f)}{N} e^{2(i+1-j)N} \right)$.

**Proof.** Let $s \in J \cap F_i(\beta)$. Then, $|s - \eta| \leq e^{-2jN}$ for any $\eta \in J$. Hence, since $f$ is Lipschitz, we have that for all $t \in [iN, (i+1)N]$

$$|f(g_t h_\eta \omega) - f(g_t h_\omega)| \leq \|f\|_{Lip} d(h_{e^{-2t} (s-\eta)}, id) \leq S(f) e^{2(i-j)N},$$

where we use $d(g, h)$ to be the metric on $SL_2(\mathbb{R})$ defined by the maximum absolute value of the entries of the matrix $gh^{-1} - Id$. Averaging the above inequality in $t$, we get that

$$|f_i(\eta) - f_i(s)| \leq \frac{S(f)e^{-2jN}}{N} \int_{iN}^{(i+1)N} e^{2t} dt \leq \frac{S(f)}{N} e^{2(i+1-j)N},$$

which implies the lemma. \(\square\)

The following lemma is the main result of this section. Let the notation be the same as in Lemma 6.2.

**Lemma 6.5 (Independence Lemma).** Suppose $\varepsilon$ is given. Then, for all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $\omega \in \mathcal{H}_1(\alpha)$, $N > T_0$, $\beta > \varepsilon + \frac{S(f)}{N}$ and finite sets $A \subset \mathbb{N}$, we have

$$\nu \left( \bigcap_{i \in A} \left( F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i(C_t)} J \right) \right) \leq a^{|A|},$$

where $|A|$ is the number of elements in $A$.

**Proof.** Fix some $\ell$ and $a$. Let $T_0 > 0$ be as in Lemma 6.2, $N > T_0$, and $A \subset \mathbb{N}$ with $p = |A|$. Up to relabeling, we may assume $A = \{1, \ldots, p\}$. Finally, let $\omega \in \mathcal{H}_1(\alpha)$.

For any $\beta > \varepsilon + \frac{S(f)}{N}$ and $t \in \mathbb{N}$, we define

$$F_i^R(\beta) := F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i} J,$$

where we use $\mathcal{R}_i$ to denote $\mathcal{R}_i(C_t)$.

We proceed by induction on $p$. Since elements of $\mathcal{R}_p$ are disjoint, we have

$$\nu \left( \bigcap_{i \in A} F_i^R(\beta) \right) = \nu \left( \bigcup_{J \in \mathcal{R}_p} \left( J \cap F_p(\beta) \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \right) \right) = \sum_{J \in \mathcal{R}_p} \nu \left( J \cap F_p(\beta) \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \right).$$

Moreover,

$$\bigcap_{i=1}^{p-1} F_i^R(\beta) \subseteq \bigcap_{i=1}^{p-1} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} \right).$$

(6.6)
Let \( J \in \mathcal{R}_p \) be such that \( J \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \neq \emptyset \). Then, \( J \cap \bigcap_{i=1}^{p-1} F_i(\beta) \neq \emptyset \) and for any \( i = 1, \ldots, p - 1 \) there exists \( J' \in \mathcal{R}_i \) such that \( J \cap J' \neq \emptyset \). Hence, by Lemma 6.4,

\[
J \subseteq \bigcap_{i=1}^{p-1} F_i \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} \right).
\]

By enlarging \( N \) if necessary, we may assume that \( \mathcal{P}_j \) is a refinement of \( \mathcal{P}_i \) for \( i \leq j \). Hence, we see that

\[
J \subseteq \bigcap_{i=1}^{p-1} \bigcup_{J' \in \mathcal{R}_i} J'.
\]

In particular, we obtain the following base step in our inductive procedure.

\[
J \subseteq \bigcap_{i=1}^{p-1} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} \right) \quad (6.7)
\]

for all \( J \in \mathcal{P}_p \) satisfying \( J \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \neq \emptyset \). Therefore, it follows that

\[
\nu \left( \bigcap_{i \in A} F_i^R(\beta) \right) \leq \sum_{J \in \mathcal{R}_p \atop J \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \neq \emptyset} \nu(J \cap F_p(\beta))
\]

\[
\leq a \sum_{J \in \mathcal{R}_p \atop J \cap \bigcap_{i=1}^{p-1} F_i^R(\beta) \neq \emptyset} \nu(J) \quad \text{by Lemma 6.2}
\]

\[
\leq a \nu \left( \bigcap_{i=1}^{p-1} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} \right) \right) \quad \text{by (6.7)}.
\]

Our choice of \( \beta \) guarantees that for all \( 1 \leq k \leq p \),

\[
\beta - \frac{S(f)}{N} \sum_{j=0}^{j=k-1} e^{(i+1-(p-j))N} > \varepsilon.
\]

Note here that our assumption that \( A = \{1, \ldots, p\} \) maximizes the sum in the above inequality. In other words, our choice of \( \beta \) guarantees that the above inequality holds where the sum is taken over any set of natural numbers \( A \) of cardinality \( p \).
Hence, by induction on our base measure estimate in (6.8), via repeated application of Lemma 6.4,

\[ \nu \left( \bigcap_{i \in A} F_i^R(\beta) \right) \leq a \nu \left( \bigcap_{i=1}^{p-1} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} \right) \right) \]

\[ \leq a^2 \nu \left( \bigcap_{i=1}^{p-2} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-p)N} - \frac{S(f)}{N} e^{2(i+1-(p-1))N} \right) \right) \]

\[ \leq \ldots \]

\[ \leq a^k \nu \left( \bigcap_{i=1}^{p-k} F_i^R \left( \beta - \frac{S(f)}{N} \sum_{j=0}^{j=k-1} e^{2(i+1-(p-j))N} \right) \right) \]

\[ \leq a^p \]

as desired. \[\square\]

6.4. A Covering Lemma. As a consequence of Lemma 6.5, we obtain the following bound on the number of intervals needed to cover intersections of the recurrent parts of the sets \( F_i \).

More precisely, we obtain the following.

Lemma 6.6. Given \( \varepsilon > 0 \). Then, for all \( \ell > 0 \) and \( a > 0 \), there exists \( T_0 > 0 \) such that for all \( \omega \in \mathcal{H}_1(\alpha) \), \( N > T_0 \), all \( \beta > \varepsilon + \frac{2S(f)}{N} \), \( M \in \mathbb{N} \), and finite sets \( A \subseteq \{0, \ldots, M-1\} \), the following holds.

\[
\# \left\{ J \in \mathcal{P}_M : J \cap \bigcap_{i \in A} \left( F_i(\beta) \cap \bigcup_{J' \in \mathcal{R}_i(C_i)} J' \right) \neq \emptyset \right\} \leq e^{2MN} a^{|A|},
\]

where \( |A| \) is the number of elements in \( A \).

Proof. Fix \( \ell \) and \( a \). Let \( T_0 > 0 \) be as in Proposition 6.5, \( N > T_0 \), \( M \in \mathbb{N} \) and \( \beta > \varepsilon + \frac{2S(f)}{N} \). Suppose \( A \subset \{0, \ldots, M-1\} \). For each \( i \in \mathbb{N} \), let

\[ F_i^R(\beta) := F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i} J \]

As in the proof of Lemma 6.5, a combination of Lemma 6.4 and the fact that the partitions \( \mathcal{P}_i \) form a refining sequence of partitions (which we may assume by enlarging \( N \) slightly if necessary) shows that for all \( J \in \mathcal{P}_M \),

\[ J \cap \bigcap_{i \in A} F_i^R(\beta) \neq \emptyset \implies J \subseteq \bigcap_{i \in A} F_i^R \left( \beta - \frac{S(f)}{N} e^{2(i+1-M)N} \right). \]

In particular, for any \( J \in \mathcal{P}_M \) satisfying \( J \cap \bigcap_{i \in A} F_i^R(\beta) \neq \emptyset \), one has

\[ J \subseteq \bigcap_{i \in A} F_i^R \left( \beta - \frac{S(f)}{N} \right). \]
Therefore, by our condition on $\beta$ and Lemma 6.5, we get

$$\sum_{J \in P_M} \nu(J) \leq \nu \left( \bigcap_{i \in A} F_i^{R} \left( \beta - \frac{S(f)}{N} \right) \right) \leq a^{|A|}. \quad (6.9)$$

Recall that $P_M$ is a partition of $[-1, 1]$ into intervals of radius $e^{-2MN}$. In particular, for $J \in P_M$, $\nu(J) = e^{-2MN}$. Combined with (6.9), this implies the lemma. \qed

6.5. **Proof of Theorem 2.1.** Let us fix the following parameters so that we can apply Lemmas 6.5 and 6.6. Fix $\varepsilon > 0$. Let $\delta, a > 0$ be sufficiently small so that the following holds.

$$\delta \leq \frac{\varepsilon}{4S(f)} \quad \text{and} \quad 2 < a^{-\delta}. \quad (6.10)$$

Let $\mathcal{N}_1, \ldots, \mathcal{N}_k$ be proper affine invariant submanifolds as in Theorem 5.1 applied to $\varepsilon/50$ and $f$. By [EMM15, Proposition 2.13], for any $i = 1, \ldots, k$ there exists a height functions $f_{\mathcal{N}_i}$. For $\ell > 0$, let

$$C_\ell = \left\{ x \in \mathcal{H}_1(\alpha) : \sum_{1}^{k} f_{\mathcal{N}_i}(x) \leq \ell \right\}.$$ 

The function $\alpha = \sum_{1}^{k} f_{\mathcal{N}_i}$ satisfies all the properties in Definition 3.1 (see [EMM15, Proposition 2.13]). Suppose $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$. Thus, $\alpha(\omega) < \infty$. In particular, Theorem 3.2 applies and guarantees the existence of some $\ell = \ell(\delta)$ and $t_0 > 0$ so that for all $t > t_0$, one has

$$\dim_H(Z_{\omega}(C_\ell, t, \delta)) \leq 1. \quad (6.11)$$

where the bound is uniform over all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$.

Let $\ell > 0$ be such that (6.11) holds. Let $T_0 > 0$ be as in Lemma 6.6 applied to $f$, $\varepsilon/50$. Let $N > \max \{T_0, t_0\}$. Over the course of the proof, we will enlarge $N$ as necessary, depending only on $\varepsilon$, $a$ and $f$.

Fix some $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$. Recall the definition of the sets $F_i$ (see (6.3)), partitions $\mathcal{P}_i$ and $\mathcal{R}_i := \mathcal{R}_i(C_\ell)$ (see Section 6.1). By enlarging $N$ if necessary, we may assume that $\mathcal{P}_i$ form a refining sequence of partitions. For each $i \in \mathbb{N}$ and $\beta > 0$, define

$$F_i^{R}(\beta) = F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i} J.$$ 

By Lemma 6.1, we get that

$$B_{\omega}(f, N, \varepsilon) \subseteq Z_{\omega}(C_\ell, N, \delta) \cup \limsup_{M \to \infty} \bigcup_{A \subseteq \{0, \ldots, M-1\}, |A| = \delta M} \bigcap_{i \in A} F_i^{R}(\varepsilon/2) \quad (6.12)$$

Thus, by (6.11), it suffices to bound the Hausdorff dimension of the second set on the right hand side. Let $M \in \mathbb{N}$ and define

$$F_M^{R} = \bigcup_{A \subseteq \{0, \ldots, M-1\}, |A| = \delta M} \bigcap_{i \in A} F_i^{R}(\varepsilon/2).$$
The number of sets of the form $A$ in the above union is at most $\left\lceil \frac{M}{\delta M} \right\rceil$. Moreover, we may assume $N$ is large enough so that

$$\frac{\varepsilon}{2} > \frac{\varepsilon}{50} + \frac{2S(f)}{N}.$$  

Hence, we may apply Lemma 6.6 with $\varepsilon/50$ in place of $\varepsilon$ to get that when $N$ is large enough, we have

$$\# \{ J \in \mathcal{P}_M : J \cap F^R_M \neq \emptyset \} \leq \sum_{A \subseteq \{0, \ldots, M-1\} |A| = \left\lceil \delta M \right\rceil} \# \left\{ J \in \mathcal{P}_M : J \cap \bigcap_{i \in A} F^R_i (\varepsilon/2) \neq \emptyset \right\} \leq \left( \frac{M}{\delta M} \right) e^{2MN} a^{\delta M} \leq 2^M e^{2MN} a^{\delta M} \quad (6.13)$$

Let $\beta = \ln(2)/2N$ and $\gamma = -\frac{1}{2N} \ln \left( a^{\delta} \right)$. Then, (6.13) can be rewritten in the following way.

$$\# \{ J \in \mathcal{P}_M : J \cap F^R_M \neq \emptyset \} \leq e^{2(1+\beta-\gamma)MN}$$

By Lemma 2.5, we get that the Hausdorff dimension of $\limsup \mathcal{F}^R_M$ is at most $1 + \beta - \gamma$. This bound is strictly less than 1 if and only if $2 < a^{-\delta}$, which holds by our choice of $a$ in (6.10). Finally, we note that our upper bound depends only on $f$ and $\varepsilon$ and is uniform in the choice of $\omega$ in $\mathcal{M} \setminus (\cup_{i=1}^k \mathcal{N}_i)$. This completes the proof.

### 6.6. Deviations of Discrete Birkhoff Averages.

The same methods used in this section to prove Theorem 2.1 also imply the following analogous statement for discrete Birkhoff averages.

**Theorem 6.7.** Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function $f$ on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$, such that for all $\omega \in \mathcal{M} \setminus (\cup_{i=1}^k \mathcal{N}_i)$ and all $l > 0$, the Hausdorff dimension of the set

$$\left\{ s \in [-1, 1] : \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^N f(g_{ln} h_s \omega) - \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \right| \geq \varepsilon \right\}$$

is at most $\delta$.

We note that by modifying the definition of the functions $f_i$ in (6.2) to be

$$f_i(s) = \frac{1}{N} \sum_{k=iN}^{(i+1)N} f(g_{ik} h_s \omega)$$

the rest of the proof of Theorem 6.7 follows verbatim as in the case of flows and as such we omit it.

### 7. Random Walks and Oseledets’ Theorem

In this section, we recall some results on the growth of the Kontsevich-Zorich cocycle along random walk trajectories on $\mathcal{H}_1(\alpha)$ which were proved in [CE15]. Using the fact that a typical random walk trajectory is tracked by a geodesic up to sublinear error, we translate such results to results concerning the Teichmüller geodesic flow.
Suppose \((M, \omega) \in \mathcal{H}_1(\alpha)\) and \(\nu_M\) is the affine measure whose support is \(\mathcal{M} = \overline{\text{SL}_2(\mathbb{R})}\omega\). Let \(V\) be a continuous \(\text{SL}_2(\mathbb{R})\)-invariant subbundle over \(\mathcal{H}_1(\alpha)\) of (an exterior power of) the Hodge bundle. Denote by \(A_V : \text{SL}_2(\mathbb{R}) \times \mathcal{M} \to \text{GL}(V)\) the restriction of the Kontsevich-Zorich cocycle to \(V\). Let \(\|A_V(\cdot, \cdot)\|\) be the Hodge norm on \(V\) (see [FM14, Section 3.4]).

Denote by \(\lambda_V\) the top Lyapunov exponent of this cocycle under the Teichmüller geodesic flow with respect to \(\nu_M\). In particular, by Oseledets’ multiplicative ergodic theorem, for \(\nu_M\) almost every \(x \in \mathcal{M}\),

\[
\lim_{t \to \infty} \frac{\log \|A_V(g_t, x)\|}{t} = \lambda_V.
\]

The cocycle \(A_V\) satisfies the following (Lipschitz) property with respect to the Hodge norm: there exists a constant \(K \in \mathbb{N}\) such that for all \(x \in \mathcal{M}\) and all \(g \in \text{SL}_2(\mathbb{R})\),

\[
\|A_V(g, x)\| \leq \|g\|^K, \quad (7.1)
\]

where for \(g \in \text{SL}_2(\mathbb{R})\), we use \(\|g\|\) to denote the norm of \(g\) in its standard action on \(\mathbb{R}^2\). This follows from [For02, Lemma 2.1] (see also [FM14, Corollary 30]). We note that the power \(K\) appears since we are considering the action of an exterior power of the cocycle. Moreover, Forni’s variational formula for the derivative of the cocycle along geodesics implies \((7.1)\) for general elements of \(\text{SL}_2(\mathbb{R})\) by the \(KAK\) decomposition, the cocycle property and the fact that \(\|A(\theta g, \cdot)\| = 1\) for all \(\theta\).

Since \(A_V(id, x) = id\) for all \(x\), we see that \(A_V(g, x)^{-1} = A_V(g^{-1}, gx)\) for all \(g \in \text{SL}_2(\mathbb{R})\) and \(x \in \mathcal{M}\). Hence, by (7.1), we get

\[
\|A_V(g, x)^{-1}\| \leq \|g^{-1}\|^K. \quad (7.2)
\]

We shall need the following facts about matrix norms which follow from the \(KAK\) decomposition and the bi-invariance of \(\|\cdot\|\) under \(K\).

**Lemma 7.1.** There exist constants \(C_1 > 0\) such that for all \(g \in \text{SL}_2(\mathbb{R})\),

1. \(\log \|g\| \leq C_1 d(g, id)\).
2. \(\|g^{-1}\| = \|g\|\)

where \(d\) denotes the right invariant metric on \(\text{SL}_2(\mathbb{R})\) and \(id\) is the identity element.

### 7.1 Random Walks

In the remainder of this section and the next section, we fix a compactly supported probability measure \(\mu\) on \(\text{SL}_2(\mathbb{R})\) which is \(\text{SO}(2)\) bi-invariant and absolutely continuous with respect to the Haar measure. Let \(\text{SL}_2(\mathbb{R})^\mathbb{N}\) be the space of infinite sequences of elements in \(\text{SL}_2(\mathbb{R})\) equipped with the probability measure \(\mu^\mathbb{N}\). For each \(n\) define the random variable \(\omega_n : \text{SL}_2(\mathbb{R})^\mathbb{N} \to \text{SL}_2(\mathbb{R})\) as

\[
(g_1, g_2, \ldots, g_n, \ldots) \mapsto \omega_n = g_ng_{n-1} \cdots g_2g_1
\]

For any fixed base point \(x \in \mathcal{H}_1(\alpha)\), the orbit \(\{\omega_nx\}_{n \in \mathbb{N}}\) in \(\mathcal{H}_1(\alpha)\) is called a random walk on \(\mathcal{H}_1(\alpha)\).

A measure \(\nu\) on \(\mathcal{H}_1(\alpha)\) is called \(\mu\)-stationary if \(\mu \ast \nu = \nu\) where

\[
\mu \ast \nu = \int_{\text{SL}_2(\mathbb{R})} (g \ast \nu)d\mu(g).
\]

The measure \(\nu_M\) is an ergodic \(\mu\)-stationary measure i.e. it cannot be written as a non-trivial convex combination of other \(\mu\)-stationary measures. By a variant of Oseledets’ theorem, due
to [GM89] in the setting of random walks, there exists $\lambda^V_\nu \in \mathbb{R}$ such that for $\nu_\mathcal{M}$-almost every $x$ and for $\mu^N$ almost every $(g_1, g_2, \ldots) \in \text{SL}_2(\mathbb{R})^N$, 
$$
\lim_{n \to \infty} \frac{\log \|A_V(g_n g_{n-1} \cdots g_1, x)\|}{n} = \lambda^V_\nu
$$

The following sets were introduced in [CE15] as a way to quantify uniformity in the above limit.

**The Sets $E_{\text{good}}(\varepsilon, L)$**. Let $\varepsilon > 0$ and $L \in \mathbb{N}$. Denote by $E_{\text{good}}(\varepsilon, L)$ the set of points $y \in \mathcal{M}$ such that for all $v \in V$, there exists a set $H(v) \subseteq \text{SL}_2(\mathbb{R})^L$ such that

1. $\mu^L(H(v)) > 1 - \varepsilon$,
2. For all $(g_1, \ldots, g_L) \in H(v)$,

$$
\lambda^\mu_v - \varepsilon < \frac{\log \|A_V(g_l \cdots g_1, y)_v\|}{L} \leq \frac{\log \|A_V(g_l \cdots g_1, y)\|}{L} < \lambda^\mu_v + \varepsilon
$$

The following lemma is an important part of our proof as it is a key step in the proof of the Oseledets part of [CE15].

**Lemma 7.2** (Lemma 2.11 in [CE15]). For any fixed $\varepsilon > 0$, the sets $E_{\text{good}}(\varepsilon, L)$ are open and

$$
\lim_{L \to \infty} \nu_\mathcal{M}(E_{\text{good}}(\varepsilon, L)) = 1
$$

7.2. **From Random Walks to Flows**. Since we will be concerned with metric properties of the exceptional set, it will be important for us to translate random walk results into the language of Teichmüller geodesics. It is a classical fact that random walk trajectories induced by a stationary measure on $\text{SL}_2(\mathbb{R})$ tracks (up to sublinear error) a Teichmüller geodesic. This is made precise in the following:

**Lemma 7.3** (Lemma 4.1 in [CE15]). There exists $\lambda > 0$, depending only on $\mu$, such that there exists a measurable map $\Theta : \text{SL}_2(\mathbb{R})^N \to [-\pi/2, \pi/2]$ , defined $\mu^N$-almost everywhere, so that for $\mu^N$-a.e. $\overline{g} = (g_1, g_2, \ldots) \in \text{SL}_2(\mathbb{R})^N$,

$$
\lim_{n \to \infty} \frac{\log \|g_{\lambda_n} \Theta(\overline{g}) (g_n \cdots g_1)^{-1}\|}{n} = 0. \quad (7.3)
$$

Furthermore, $\Theta_\ast \mu^N$ coincides with the normalized Lebesgue measure. In particular, for any interval $[a, b] \subseteq [-\pi/2, \pi/2]$,

$$
\mu^N(\overline{g} : \Theta(\overline{g}) \in [a, b]) = \frac{b - a}{\pi}. \quad (7.4)
$$

**Remark 7.4.** The relationship between the Lyapunov exponent of the random walk $\lambda^V_\nu$ and the Lyapunov exponent of the Kontsevich-Zorich cocycle under the Teichmüller flow $\lambda_V$ is provided by the parameter $\lambda$ in Lemma 7.3 as follows.

$$
\lambda_V = \frac{\lambda^\mu_v}{\lambda}
$$

The following Lemma uses Lemma 7.3 to show that geodesic trajectories which start within the sets $E_{\text{good}}(\varepsilon, L)$ also exhibit good properties with respect to the cocycle.

For simplicity, throughout this section we use the notation $A := A_V$
Lemma 7.5. There exists a constant $C > 0$, depending only on the constants of the cocycle such that the following holds: for every $\varepsilon > 0$, there exists $L_0 > 0$ such that for all $L \geq L_0$, for all $y \in E_{\text{good}}(\varepsilon, L)$ and all $v \in V$, there exists $\tilde{H}(v) \subseteq [-\pi/2, \pi/2]$ such that for all $\theta \in \tilde{H}(v)$,

$$\lambda_v - C\varepsilon < \frac{\log ||A(g_{\lambda L}, r_{\theta(y)})||}{\lambda L} < \lambda_v + C\varepsilon$$

and such that $\nu(\tilde{H}(v)) > 1 - 3\varepsilon$, where $\nu$ is the normalized Lebesgue measure on $[-\pi/2, \pi/2]$.

Proof. Let $\lambda$ is as in Lemma 7.3. Using Egorov’s theorem, we can find a set $U \subseteq SL_2(\mathbb{R})^N$ with $\mu(U) > 1 - \varepsilon$ so that the convergence in (7.3) is uniform over $U$. In particular, we can choose $L \in \mathbb{N}$ sufficiently large so that for all $\tilde{\gamma} \in U$:

$$\frac{\log ||g_{\lambda L}r_{\omega(\tilde{\gamma})}(g_L \cdots g_1)^{-1}||}{L} < \varepsilon \quad (7.5)$$

Fix $y \in E_{\text{good}}(\varepsilon, L)$ and $v \in V$. Let $H(v) \subseteq SL_2(\mathbb{R})^L$ be as in the definition of $E_{\text{good}}(\varepsilon, L)$. We will regard $H(v)$ as a cylinder subset of $SL_2(\mathbb{R})^N$ in the natural way. The set $\tilde{H}(v)$ will be essentially the image of $H(v) \cap U$ under $\Theta$, except that $\Theta$ is only a measurable map. To go around this, we use Lusin’s theorem to find a compact set $K \subset SL_2(\mathbb{R})^N$, such that $\mu^N(K) > 1 - \varepsilon$ and such that the restriction of $\Theta$ to $K$ is continuous. Let $\tilde{H}(v) = \Theta(H(v) \cap U \cap K)$

Since $\Theta$ is continuous on $K$ and $\tilde{H}(v)$ is a Borel subset of $K$, we see that $\tilde{H}(v)$ is Lebesgue measurable. Moreover, by Lemma 7.3, one has

$$\nu(\tilde{H}(v)) = \mu^N(\Theta^{-1}(\tilde{H}(v))) \geq \mu^N(H(v) \cap U \cap K) > 1 - 3\varepsilon$$

To see that $\tilde{H}(v)$ satisfies the conclusion of the Lemma, let $\tilde{\gamma} \in H(v) \cap U \cap K$. For all $L$ sufficiently large so that (7.5) holds for all $\tilde{\gamma} \in U$, define $\varepsilon_L \in SL_2(\mathbb{R})$ by the following equation

$$g_{\lambda L}r_{\omega(\tilde{\gamma})} = \varepsilon_L g_L \cdots g_1$$

with $\varepsilon_L \in SL_2(\mathbb{R})$. Then, using the cocycle property, we get

$$A(g_{\lambda L}, r_{\omega(\tilde{\gamma})} \omega) = A(\varepsilon_L, g_L \cdots g_1 \omega)A(g_L \cdots g_1, \omega)$$

Hence, since $\tilde{\gamma} \in H(v)$, by definition of the set $E_{\text{good}}(\varepsilon, L)$ and by (7.1), we get

$$\frac{\log ||A(g_{\lambda L}, r_{\omega(\tilde{\gamma})} \omega)||}{L} \leq \frac{\log ||A(\varepsilon_L, g_L \cdots g_1 \omega)||}{L} + \frac{\log ||A(g_L \cdots g_1, \omega)||}{L}$$

$$\leq K \frac{\log ||\varepsilon_L||}{L} + \lambda^\mu_v + \varepsilon$$

$$\leq \lambda^\mu_v + \varepsilon(1 + K)$$

Similarly, using (7.1) and (2) of Lemma 7.1, we get

$$\frac{\log ||A(g_{\lambda L}, r_{\omega(\tilde{\gamma})} \omega)||}{L} \geq \frac{\log ||A(g_L \cdots g_1, \omega)||}{L} - \frac{\log ||A(\varepsilon_L, g_L \cdots g_1 \omega)^{-1}||}{L}$$

$$\geq \lambda^\mu_v - \varepsilon - \frac{K \log ||\varepsilon_L^{-1}||}{L} = \lambda^\mu_v - \varepsilon - \frac{K \log ||\varepsilon_L||}{L}$$

$$\geq \lambda^\mu_v - \varepsilon(1 + K)$$
Dividing both estimates by $\lambda$ and noting that by remark 7.4, $\lambda_V = \lambda_V^*/\lambda$, we get the desired conclusion.

□

As a corollary, we obtain the following statement for horocycles.

**Corollary 7.6.** There exists a constant $C_2 > 0$, depending only on the constants of the cocycle such that the following holds: for every $\varepsilon > 0$, there exists $L_0 > 0$ such that for all $L \in \mathbb{N}$ with $L \geq L_0$, for all $y \in E_{\text{good}}(\varepsilon, L)$ and all $v \in V$, there exists $G(v) \subseteq [-2, 2]$ such that for all $s \in G(v),$

$$\lambda_V - C_2\varepsilon < \frac{\log ||A(g_{\lambda L}, h_s y)||}{\lambda L} < \lambda_V + C_2\varepsilon$$

(7.6)

and such that $|G(v)| \geq 4(1 - 30\varepsilon)$, where $|\cdot|$ is the Lebesgue measure on $[-2, 2]$.

**Proof.** Fix $\varepsilon > 0$. Suppose $L \in \mathbb{N}$ is sufficiently large so that Lemma 7.5 holds, $y \in E_{\text{good}}(\varepsilon, L)$ and $v \in V$. Let $\bar{H}(v) \subseteq [-\pi/2, \pi/2]$ and $C > 0$ be as in the conclusion of Lemma 7.5. Consider

$$G(v) = \tan(\bar{H}(v)) \cap [-2, 2].$$

We verify that the corollary holds for this set. Let

$$\rho = \tan^{-1}(2).$$

For every $\theta \in \bar{H}(v) \cap [-\rho, \rho]$ we write $r_\theta = \bar{h}_{-\tan\theta \log\cos\theta}h_{\tan\theta}$. Then, using the cocycle property, we see the following.

$$A(g_{\lambda L}, r_\theta y) = A(\bar{h}_{-e^{-2\lambda L} \tan\theta \log\cos\theta}g_{\lambda L}h_{\tan\theta}y)A(g_{\lambda L}, h_{\tan\theta}y)A(\bar{h}_{-\tan\theta \log\cos\theta}h_{\tan\theta}y)^{-1}.$$

Therefore, using the Lipschitz property (7.1) and (7.2) and the fact that $\theta \in [-\rho, \rho]$, we get

$$\frac{\log ||A(g_{\lambda L}, r_\theta y)||}{\lambda L} = \frac{\log ||A(g_{\lambda L}, h_{\tan\theta}y)||}{\lambda L} + O\left(\frac{1}{L}\right).$$

Thus, for $L$ large enough, and for all $s \in [-2, 2]$ of the form $s = \tan\theta$ with $\theta \in \bar{H}(v) \cap [-\rho, \rho]$, we can find a constant $C' > 0$, independent of $L$, so that (7.6) holds.

Let $|\cdot|$ be the Lebesgue measure. Since $|\bar{H}(v)| \geq \pi(1 - 3\varepsilon)$, we get that

$$\frac{|\bar{H}(v) \cap [-\rho, \rho]|}{2\rho} \geq 1 - \frac{3\varepsilon\pi}{2\rho}.$$

Hence, since the Jacobian of the map $\theta \mapsto \tan\theta$ is bounded by $\sec^2(\rho) = 5$ on $[-\rho, \rho]$, the following holds.

$$\frac{|G(v)|}{4} \geq 1 - \frac{15\varepsilon\pi}{4} \geq 1 - 30\varepsilon$$

□
8. Large Deviations in Oseledets’ Theorem

The purpose of this section is to prove Theorem 2.2 concerning the Hausdorff dimension of directions whose geodesics exhibit deviation of the top Lyapunov exponent for the Kontsevich-Zorich cocycle. The structure of the proof is very similar to that of Theorem 2.1. The idea is to relate the directions exhibiting deviation in Oseledets theorem along a Teichmüller geodesic to the directions exhibiting deviation in Birkhoff’s theorem for the indicator function of a large open set with good properties with respect to the cocycle. The proof is written in such a way so as to mirror the proof of Theorem 2.1 on deviations in Birkhoff’s theorem.

Throughout this section we retain the notation from the previous section and also use the following. For any positive \( \varepsilon, L \in \mathbb{R} \) and \( M \in \mathbb{N} \), we define the following sets.

\[
B(A, L, \varepsilon, M) := \left\{ s \in [-1, 1] : \log \left\| A(g_{LM}, h_s \omega) \right\| > \lambda_V + \varepsilon \right\}
\]

\[
B(A, L, \varepsilon) := \limsup_{M \to \infty} B(A, L, \varepsilon, M)
\]

Using the cocycle property, it is easy to check that for any \( L > 0 \)

\[
B(A, L, \varepsilon) = \left\{ s \in [-1, 1] : \limsup_{t \to \infty} \log \left\| A(g_t, h_s \omega) \right\| \geq \lambda_V + \varepsilon \right\}
\]

Moreover, for any \( s \in [-1, 1], \beta, L > 0 \) and \( i \in \mathbb{N} \), we define the corresponding functions and sets.

\[
a_i(s) = \frac{\log \left\| A(g_{IL}, h_s \omega) \right\|}{L}
\]

\[
A_i(\beta) = \{ \theta : a_i(s) > \lambda_V + \beta \}
\]

The functions \( a_i \) and sets \( A_i \) play the role of the functions \( f_i \) (see (6.2)) and the sets \( F_i \) (see (6.3)), respectively, in the proof of large deviations in Birkhoff’s theorem.

8.1. Sets and Partitions. For \( L > 0 \) and \( i \in \mathbb{N} \), let \( \mathcal{P}_i \) denote the partition of \([-1, 1]\) into intervals of radius \( e^{-2iL} \). By enlarging \( L \) if necessary, we may assume \( e^L \in \mathbb{N} \) and that \( \mathcal{P}_{i+1} \) is a refinement of \( \mathcal{P}_i \) for all \( i \). For \( \varepsilon > 0 \), define the following sub-partitions

\[
\mathcal{E}_i(\varepsilon, L) = \{ J \in \mathcal{P}_i : \exists s \in J, g_{iL} h_s \omega \in E_{\text{good}}(\varepsilon, L) \}.
\]

Here \( \mathcal{E} \) signifies recurrence to the set \( E_{\text{good}} \).

The following Lemma is an analogue of Lemma 6.1.

**Lemma 8.1.** Let \( \varepsilon_1, \varepsilon_2, L > 0, M \in \mathbb{N} \), and \( 0 < \delta \leq \varepsilon/4K \), where \( K \in \mathbb{N} \) is the exponent in in (7.1). Then,

\[
B(A, L, \varepsilon_1, M) \subseteq Z_{\omega}(E_{\text{good}}(\varepsilon_2, L), M, L, \delta) \cup \bigcup_{B \subseteq \{1, \ldots, M\}} \bigcap_{i \in B} \left( A_i(\varepsilon_1/2) \cap \bigcup_{J \in \mathcal{E}_i(\varepsilon_2, L)} J \right),
\]

where \( Z_{\omega}(E_{\text{good}}(\varepsilon_2, L), M, L, \delta) \) is defined in (3.1).
Proof. First, we notice that for any \( \varepsilon > 0 \)
\[
B(A, L, \varepsilon, M) \subseteq \bigcup_{B \subseteq \{1, \ldots, M\}, i \in B \atop |B| > 2M} \bigcap_{i \in B} A_i(\varepsilon/2)
\]
Using the cocycle property and submultiplicativity of matrix norms, we have the following inequalities
\[
\frac{\log \|A(g_{LM}, h_{\omega\nu})\|_L}{LM} \leq \frac{1}{M} \sum_{i=1}^{M} \frac{\log \|A(g_L, g_{Li}h_{\omega\nu})\|_L}{L}
\]
From this point on, using (7.1) to bound \( \|A(g_L, \cdot)\| \), the proof is identical to that of Lemma 6.1. \( \square \)

8.2. Measure Bounds for \( A_i \). The goal of this section is to obtain a uniform bound on the measure of sets of the form \( A_i \cap J \) for any \( J \in \mathcal{E}_i \) and any \( i \). This step is analogous to Lemma 6.2.

The following is the main result of this section. The key input in the proof is Lemma 7.5.

Lemma 8.2. Let \( C_2 > 0 \) be as in Corollary 7.6. Then, for every \( \varepsilon > 0 \), there exists \( L_1 > 0 \) such that for all \( L \geq L_1 \), all \( \gamma \geq 2C_2\varepsilon \), all \( i \in \mathbb{N} \) and all \( J \in \mathcal{E}_i(\varepsilon, L) \),
\[
\frac{\nu(J \cap A_i(\gamma))}{\nu(J)} \leq 120\varepsilon,
\]
where \( \nu \) is the Lebesgue probability measure on \([-1, 1]\).

Proof. Let \( L_0 > 0 \) and \( \lambda > 0 \) be as in Corollary 7.6 and Lemma 7.3, respectively. Define \( L_1 := L_0/\lambda \). Suppose \( \gamma \in \mathbb{R} \) and \( L \in \mathbb{N} \) are such that \( \gamma \geq 2C_2\varepsilon \) and \( L \geq L_1 \).

Let \( i \in \mathbb{N} \), \( J \in \mathcal{E}_i := \mathcal{E}_i(\varepsilon, L) \), and \( s_0 \in J \) be such that \( y_0 := g_{iL}h_{s_0}\omega \in E_{\text{good}}(\varepsilon, L) \). Let \( v \in V \) and \( G(v) \subseteq [-2, 2] \) be as in Corollary 7.6. Choose \( \eta \in J - s_0 \) such that \( s_0 + \eta \in A_i(\gamma) \cap J \). Then, we have
\[
\lambda v + \gamma \leq a_i(s_0 + \eta) = \frac{\log \|A(g_L, h_{e^{2i\eta}y_0})\|}{L}
\]
Hence, by definition of \( G(v) \),
\[
e^{2iL} \eta \notin G(v) \tag{8.1}
\]
Note that \( e^{2iL}(J - s_0) \) is a subinterval of \([-2, 2]\) of length \( 2 \). In particular, we get that
\[
e^{2iL} ((A_i(\gamma) \cap J) - s_0) \subseteq [-2, 2] \setminus G(v)
\]
Thus, since the Lebesgue measure of \( G(v) \) is at least \( 4(1 - 30\varepsilon) \), we get the following measure estimate
\[
|e^{2iL} ((A_i(\gamma) \cap J) - s_0)| = \frac{\nu(J \cap A_i(\gamma))}{\nu(J)} \leq 120\varepsilon
\]
This concludes the proof in the case \( L \in \mathbb{N} \). For the \( L \geq L_1 \) with \( L \notin \mathbb{N} \), write \( L = [L] + \{L\} \) where \([L]\) is the largest natural number less than \( L \) and \( \{L\} = L - [L] \). Then, using the cocycle property, submultiplicativity of the norm and the Lipschitz property of the cocycle (7.1), we get
\[
\frac{\log \|A(g_L, \cdot)\|_L}{L} \leq \frac{\log \|A(g_{L1}, \cdot)\|_L}{L} + O\left(\frac{1}{L}\right)
\]
Thus, we see that the conclusion follows in this case from the case when $L \in \mathbb{N}$ by choosing $L_1$ sufficiently large depending on $\varepsilon$. □

8.3. Independence of the Sets $A_i$. As a consequence of the Lipschitz property of the cocycle (7.1), we are able to prove an analogue of Lemma 6.4.

**Lemma 8.3.** There exists a constant $C_3 > 0$, depending only on the constants of the cocycle $A$ so that the following holds. Suppose $i < j$, where $i$ and $j$ are natural numbers, $L > 0$, and $\gamma > 0$. Let $J \in \mathcal{P}_j$ be such that $J \cap A_i(\gamma) \neq \emptyset$. Then, $J \subseteq A_i \left( \gamma - C_3 \frac{e^{2(i+1-j)L}}{L} \right)$.

**Proof.** Let $s_0 \in J \cap A_i(\gamma)$. Then, by definition of the partition $\mathcal{P}_j$ in Section 8.1, $|s_0 - \eta| \leq e^{-2jL}$ for any $\eta \in J$.

Using the cocycle property, we have the following.

\[
A(g_L, g_i L h_{\eta} \omega) = A(g_L, h_{e^{2iL}(\eta-s_0)} g_i L h_{s_0} \omega) \\
= A(g_L, h_{e^{2iL}(\eta-s_0)} g_i L h_{s_0} \omega) A(h_{e^{2iL}(\eta-s_0)} g_i L h_{s_0} \omega)^{-1} \\
= A(h_{e^{2(i+1)L}(\eta-s_0)} g_i L h_{s_0} \omega) A(g_L, g_i L h_{s_0} \omega) A(h_{e^{2iL}(\eta-s_0)} g_i L h_{s_0} \omega)^{-1}
\]

Therefore,

\[
A(g_L, g_i L h_{s_0} \omega) = A(h_{e^{2(i+1)L}(\eta-s_0)} g_i L h_{s_0} \omega)^{-1} A(g_L, g_i L h_{\eta} \omega) A(h_{e^{2iL}(\eta-s_0)} g_i L h_{s_0} \omega).
\] (8.2)

Hence, by (7.1), (7.2) and Lemma 7.1, there exists a constant $C_1$ so that

\[
a_i(s_0) - a_i(\eta) \leq \frac{K \log ||h_{e^{2(i+1)L}(\eta-s_0)}||}{L} + \frac{K \log ||h_{e^{2iL}(\eta-s_0)}||}{L} \leq \frac{KC_1 d(h_{e^{2(i+1)L}(\eta-s_0)}, id)}{L} + \frac{KC_1 d(h_{e^{2iL}(\eta-s_0)}, id)}{L} \leq 2KC_1 e^{2(i+1-j)L}
\]

which concludes the proof. □

As a consequence, we obtain exponential decay in the measure of intersections of the sets $A_i$, similarly to Lemma 6.5.

**Lemma 8.4** (Independence Lemma for $A_i$). Let $C_3 > 0$ be as in Lemma 8.3 and $C_2 > 0$ be as in Corollary 7.6. Then, for all $\varepsilon > 0$, there exists $L_1 > 0$ such that for all $L \geq L_1$, all finite sets $B \subset \mathbb{N}$ and all $\gamma > 2C_2\varepsilon + \frac{C_3}{L}$,

\[
\nu \left( \bigcap_{i \in B} \left( A_i(\gamma) \cap \bigcup_{J \in \mathcal{E}(\varepsilon, L)} J \right) \right) \leq (120\varepsilon)^{|B|}
\]

**Proof.** The proof is identical to the proof of Lemma 6.5 which is a formal consequence of two results: Lemma 6.2 that gives an upper bound on the measure of $F_i$, and Lemma 6.4. The analogues of those two results are Lemma 8.2 and Lemma 8.3, respectively. □
8.4. A Covering Lemma. The following lemma shows existence of efficient covers for intersections of the sets $A_i$, similarly to Lemma 6.6.

**Lemma 8.5.** Let $C_3 > 0$ be as in Lemma 8.3 and $C_2 > 0$ be as in Corollary 7.6. Then, for all $\varepsilon > 0$, there exists $L_1 > 0$ such that for all $L \geq L_1$, $M \in \mathbb{N}$, sets $B \subseteq \{1, \ldots, M\}$ and $\gamma > 2C_2\varepsilon + \frac{2K_2}{L}$, we obtain the following.

$$
\# \left\{ J \in \mathcal{P}_{M+1} : J \cap \bigcap_{i \in B} A_i(\gamma) \cap \bigcup_{J' \in \mathcal{E}(\varepsilon, L)} J' \neq \emptyset \right\} \leq e^{2L(M+1)\left(120\varepsilon\right)^{|B|}},
$$

where $|B|$ is the number of element in $B$.

**Proof.** The proof is a direct consequence of Proposition 8.4 and proceeds as in the proof of Lemma 6.6.

8.5. Proof of Theorem 2.2. Fix $\varepsilon > 0$. Suppose $\varepsilon' > 0$ is a sufficiently small number (depending only on $\varepsilon$). Define $\delta := \varepsilon/4K$, where $K$ is the exponent in (7.1). By Lemma 7.2, choose $L > 0$ large enough, depending on $\varepsilon'$, so that

$$
\nu_M(E_{\text{good}}(\varepsilon', L)) > 1 - \delta/2. \tag{8.3}
$$

Let $\chi_E$ denote the indicator function of the open set $E_{\text{good}}(\varepsilon', L)$. Then, using a variant Urysohn’s lemma, we can find a Lipschitz compactly supported continuous function $f : \mathcal{M} \rightarrow [0, 1]$, satisfying $f \leq \chi_E$ and

$$
\nu_M(f) > 1 - 3\delta/4 \tag{8.4}
$$

Moreover, we have that for all $M \in \mathbb{N}$ and all $\omega \in \mathcal{M}$,

$$
Z_{\omega}(E_{\text{good}}(\varepsilon', L), M, L, \delta) \subseteq B_{\omega} \left(1 - f, L, \delta - \nu_M(1 - f), M\right)
$$

where these sets are defined in (3.1) and (6.1) (for discrete Birkhoff averages).

Note that $\delta - \nu_M(1 - f) > 0$ by (8.4). Thus, by Theorem 6.7, there exist $0 < \eta < 1$ and finitely many proper affine invariant manifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k \subset \mathcal{M}$, depending on $f$ and $\varepsilon$, so that the following holds

$$
dim_H \left(\limsup_{M} \bigcup_{\omega \in \mathcal{M} \setminus \left(\bigcup_{i=1}^{k} \mathcal{N}_i\right)} Z_{\omega}(E_{\text{good}}(\varepsilon', L), M, L, \delta)\right) \leq \eta \leq 1 \tag{8.5}
$$

uniformly for all $\omega \in \mathcal{M} \setminus \left(\bigcup_{i=1}^{k} \mathcal{N}_i\right)$. These are the affine manifolds appearing in the conclusion of Theorem 2.2. Now, fix one such $\omega$.

Recall the definition of the sets $A_i$ and partitions $\mathcal{P}_i$ in Section 8.1. By enlarging $L$ if necessary, we may assume that $\mathcal{P}_i$ form a refining sequence of partitions. For $i \in \mathbb{N}$ and $\gamma > 0$, define

$$
\mathcal{A}_i^\varepsilon(\gamma) := A_i(\gamma) \cap \bigcup_{J \in \mathcal{E}(\varepsilon', L)} J
$$

Then, by Lemma 8.1, since $\delta = \varepsilon/4K$, we get

$$
\limsup_{M} B(A, L, \varepsilon, M) \subseteq \limsup_{M} Z_{\omega}(E_{\text{good}}(\varepsilon', L), M, L, \delta) \cup \limsup_{M} \bigcup_{B \subseteq \{1, \ldots, M\}} \bigcap_{i \in B} \mathcal{A}_i(\varepsilon/2) \bigcap_{|B|=|BM|} \mathcal{A}_i(\varepsilon/2).
$$
Thus, it remains to control the Hausdorff dimension of the second set on the right side. We apply Lemma 8.5 to $\varepsilon'$ in place of $\varepsilon$ and $\gamma = \varepsilon/2$. By choosing $\varepsilon'$ to be sufficiently small and $L$ sufficiently large, we can insure that
\[
\varepsilon/2 > 2C_2\varepsilon' + \frac{2C_3}{L}
\]
where $C_2$ and $C_3$ are constants depending only on the cocycle as in the statement of Lemma 8.5.

As a result, choosing $L$ sufficiently large, we can apply Lemma 8.5 and proceed as in the proof of Theorem 2.1 to get that
\[
\dim_H \left( \limsup_M \bigcup_{B \subseteq \{1, \ldots, M\}} \bigcap_{i \in B} A_i^E(\varepsilon/2) \right) \leq 1 + \frac{\ln(2) + \delta \ln(120\varepsilon')}{2L}
\]

By choosing $\varepsilon' < 2^{-1/\delta}/120$ (thus depending only on $\varepsilon$), we get that this upper bound is strictly less than one. Moreover, observe that the parameters $\delta, \varepsilon', L$ appearing in the upper bound above are independent of $\omega$. This completes the proof.

9. Weak Mixing IETs

This section is dedicated to the proof of Corollary 1.8. We first recall some definitions and the results of [BN04] which connect weak mixing properties of IETs with the recurrence of Teichmüller geodesics in an appropriate stratum.

Throughout this section, we fix a natural number $d \geq 2$. Given a permutation $\pi$ on $d$ letters and $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}_+^d$, we define $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_d$ and an interval exchange transformation (IET) with permutation $\pi$ to be the piecewise linear map $T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ defined as follows: first we partition the interval $[0, |\lambda|]$ into $d$ ordered half open subintervals $I_i$ so that the length of $I_i$ is equal to $\lambda_i$ and $T_{\lambda,\pi}$ maps $I_i$ linearly onto $I_{\pi(i)}$ for each $1 \leq i \leq d$. More formally, for $1 \leq i \leq d$ and $x \in I_i$,
\[
T_{\lambda,\pi}(x) = x + \sum_{\pi(j) < \pi(i)} \lambda_j - \sum_{j < i} \lambda_j.
\]

An IET $T_{\lambda,\pi}$ has finitely many points of possible discontinuity
\[
\beta_0 = 0, \quad \beta_i := \sum_{j \leq i} \lambda_j, \quad 1 \leq i \leq d - 1
\]

Define (See [Vee78] and [MW14, Section 2.2]) an alternating bilinear form on $\mathbb{R}^d \times \mathbb{R}^d$ by its value on the standard basis elements $e_i$ as follows
\[
Q(e_i, e_j) = \begin{cases} 
1 & i > j, \pi(i) < \pi(j) \\
-1 & i < j, \pi(i) > \pi(j) \\
0 & \text{otherwise.}
\end{cases} \quad (9.1)
\]

for $1 \leq i, j \leq d$. Then, for each $\lambda \in \mathbb{R}_+^d$, $1 \leq i \leq d$ and $x \in [\beta_{i-1}, \beta_i)$, we have
\[
T_{\lambda,\pi}(x) - x = Q(\lambda, e_i)
\]

The cone $\mathbb{R}_+^d$ can be viewed as the space of IETs with a given permutation $\pi$ with a natural euclidean metric and Lebesgue measure. IETs preserve the Lebesgue measure on the unit
interval and we shall refer to ergodic properties (ergodicity, weak mixing, etc) of IETs with respect to it.

9.1. A criterion for weak mixing. A permutation \( \pi \) on \( d \) letters \( \{1, \ldots, d\} \) is irreducible if for every \( 1 \leq j < d \),

\[
\pi(\{1, \ldots, j\}) \neq \{1, \ldots, j\}.
\]

**Definition 9.1.** Suppose \( \pi \) is an irreducible permutation on \( d \) letters. Define inductively a finite sequence \( \{a_p\}_{p=0,1,\ldots,l} \) of natural numbers as follows.

Set \( a_0 = 1 \). If \( a_{p-1} \in \{\pi^{-1}(1), d+1\} \), then set \( l = p - 1 \) and stop. Otherwise, define \( a_p = \pi^{-1}(\pi(a_{p-1}) - 1) + 1 \). The permutation \( \pi \) is of type \( W \) if \( a_l = \pi^{-1}(1) \).

Following [BN04], we say that \( T_{\lambda,\pi} \) satisfies IDOC (the infinite distinct orbit condition) if each discontinuity point \( \beta_i \) has an infinite orbit under \( T_{\lambda,\pi} \) and for \( i \neq j \), the orbits of \( \beta_i \) and \( \beta_j \) are disjoint.

Using the orbits of the points \( \beta_i \) under the IET \( T_{\lambda,\pi} \), we define a sequence of partitions of \([0,1]\) as follows: for each \( n \geq 1 \), \( P_n \) denotes the partition into subintervals whose endpoints are the successive elements of the sets

\[
D_n = \bigcup_{0 \leq k \leq n-1} T_{\lambda,\pi}^{-k}(\{\beta_0, \ldots, \beta_{d-1}\})
\]

For each \( n \), we define \( \epsilon_n(T_{\lambda,\pi}) \) to be the length of the shortest interval in the partition \( P_n \). The following criterion of weak mixing was proved in [BN04].

**Theorem 9.2** (Theorem 5.3 in [BN04]). Suppose \( \pi \) is a type \( W \) permutation and \( T_{\lambda,\pi} \) is an ergodic IET satisfying IDOC for some \( \lambda \in \mathbb{R}^d_+ \). If \( \limsup_{n \to \infty} n \epsilon_n(T_{\lambda,\pi}) > 0 \), then \( T_{\lambda,\pi} \) is weak mixing.

Motivated by this criterion, we will say that an IET \( T_{\lambda,\pi} \) has short intervals if

\[
\lim_{n \to \infty} n \epsilon_n(T_{\lambda,\pi}) = 0 \tag{9.2}
\]

9.2. A Compactness criterion for strata. Suppose \( \mathcal{H} \) is a stratum of abelian differentials. We recall here a description of standard compact subsets of \( \mathcal{H} \). Given \( \omega \in \mathcal{H} \), denote by \( \mathcal{L}_\omega \) the set of all of its saddle connections, i.e., the set of all flat geodesic segments joining a pair of the singularities of \( \omega \). Then, we can naturally regard \( \mathcal{L}_\omega \) as a subset of vectors in \( \mathbb{C} \). Note that \( \mathcal{L}_\omega \) is a discrete set. Moreover, using the standard action of \( \text{SL}(2,\mathbb{R}) \) on \( \mathbb{C} \), for any \( g \in \text{SL}(2,\mathbb{R}) \), the set \( g \cdot \mathcal{L}_\omega \) can be identified with \( g \cdot \mathcal{L}_\omega \).

For \( v \in \mathbb{C} \), let \( \|v\| = \max\{||\text{Re}(v)||, ||\text{Im}(v)||\} \). Now, define a function \( \ell : \mathcal{H} \to \mathbb{R}_+ \) by

\[
\ell(\omega) := \min \{\|v\| : v \in \mathcal{L}_\omega\} \tag{9.3}
\]

For any \( \varepsilon > 0 \), we use the following notation

\[
\mathcal{K}_\varepsilon := \{\omega \in \mathcal{H} : \ell(\omega) \geq \varepsilon\} \tag{9.4}
\]

It is known that the sets \( \mathcal{K}_\varepsilon \) with \( \varepsilon > 0 \) are compact subsets of \( \mathcal{H} \) and that any bounded subset of \( \mathcal{H} \) is contained in \( \mathcal{K}_\varepsilon \) for some \( \varepsilon \).
9.3. Short intervals and recurrence. Given an abelian differential \( \omega \in \mathcal{H} \) on a surface \( S \), there is a well-defined vector field given by the imaginary part \( \text{Im}(\omega) \). This vector field is defined at all points in \( S \) except for the (finitely many) zeros of \( \omega \). This vector field defines a singular flow on \( S \), called the vertical flow, by moving points at linear speed in the direction \( \text{Im}(\omega) \).

By fixing a straight line segment \( I \) (a geodesic segment in the flat metric defined by \( \omega \)) which is transversal to the vertical flow lines, the first return map \( T : I \to I \) of the flow defines an IET. One can pick \( I \) parallel to the real part of \( \omega \) so that the resulting IET has \( 2g + k - 1 \) intervals, where \( g \) is the genus of \( S \) and \( k \) is the number of zeros of \( \omega \). This remains true if the angle between \( I \) and the real part is sufficiently small.

This process allows us to define a map from a neighborhood of \( \omega \) in the stratum \( \mathcal{H} \) to the space of IETs \( \mathbb{R}^{2g+k-1} \) as follows. Pick a segment \( I \) parallel to the real part of \( \omega \) as above and let \( \pi \) be the permutation associated to the IET on \( 2 \) intervals defined by the first return map of the vertical flow defined by \( \omega \) to \( I \). Then, we can find a sufficiently small open neighborhood \( U_\omega \) of \( \omega \) in \( \mathcal{H} \) so that for all \( x \in U_\omega \), the first return map of the vertical flow defined by \( x \) to the segment \( I \) is an IET with \( 2g + k - 1 \) intervals and with the same permutation \( \pi \).

This defines a Lipschitz map
\[
\mathcal{T} : U_\omega \to \mathbb{R}^{2g+k-1}
\] (9.5)
in the Teichmüller and Euclidean metrics respectively. Conversely, using Veech’s zippered rectangles construction, one can find suspension of any IET using a piecewise linear roof function to obtain an abelian differential on a compact surface. However, this construction is not unique and in general the pre-image of an IET \( T_{\lambda,\pi} \) under the map \( \mathcal{T} \) is a positive dimensional subset of \( U_\omega \).

The following proposition allows us to relate the criterion in Theorem 9.2 to the recurrence of Teichmüller geodesics in strata. A similar result was obtained in [MW14, Proposition 7.2] using a slightly different proof. We include a proof here for completeness.

**Proposition 9.3.** In the notation above, let \( \lambda \in \mathbb{R}_+^{2g+k-1} \) be in the image of the map \( \mathcal{T} \) in (9.5). Suppose that \( T_{\lambda,\pi} \) has short intervals. Then, for all \( \tilde{\omega} \in \mathcal{T}^{-1}(\lambda) \), the geodesic \( g_t\tilde{\omega} \) diverges in \( \mathcal{H}_1(\alpha) \).

**Proof of Proposition 9.3.** Suppose \( \lambda \) is as in the statement and let \( \tilde{\omega} \in \mathcal{T}^{-1}(\lambda) \). Fix some \( \varepsilon > 0 \) and let \( n_0 \geq 1 \) be such that \( n\epsilon_n(T_{\lambda,\pi}) < \varepsilon \) for all \( n \geq n_0 \). We construct a sequence of saddle connections \( v_n \) in \( \tilde{\omega} \) so that the length of \( g_{\log(n/\sqrt{\varepsilon})}v_n \) is \( \ll \sqrt{\varepsilon} \) for all \( n \geq n_0 \). Since \( \varepsilon \) is arbitrary, \( g_t\tilde{\omega} \) diverges in \( \mathcal{H}_1(\alpha) \).

For this we use an argument similar to the one found in [Bos85, Section 10]. Let \( P_1, \ldots, P_k \) be a collection of polygons in the plane representing \( \tilde{\omega} \) and let \( \tilde{I} \) be a lift of the transversal \( I \) under the covering map \( \cup P_i \to S \) which glues parallel sides by translations. We recall that \( S \) is the surface of genus \( g \geq 1 \) that supports the abelian differentials in the proposition.

For each \( n \geq n_0 \), denote by \( I_n \subset I \) be a subinterval such that
\[
|I_n| = \epsilon_n(T_{\lambda,\pi}) < \varepsilon/n
\]
We use \( \tilde{I}_n \) to denote a lift of \( I_n \) inside \( \tilde{I} \). Denote by \( \mathcal{C} \) the open cylinder consisting of the union of the vertical flow orbits of the points in the interior of \( I_n \) up to the \( n \)th time these orbits hit the transversal \( I \). By definition of the endpoints of the interval \( I_n \), the cylinder \( \mathcal{C} \) contains no zeros of \( \tilde{\omega} \).
Let \( \tilde{t} \) denote a lift of \( t \) to the complex plane which we unfold to a parallelogram in the following manner. Let \( x \) be an arbitrary point in the interior of \( \tilde{I}_n \) and denote by \( x_t := x + it \) for \( t > 0 \). Define \( t(x, n) \) to be the time \( t > 0 \) corresponding to the \( n \)th return of \( x \) to \( I \) under the vertical flow.

Next, we define a finite sequence of times \( q_i \in (0, t(x, n)) \) and polygons \( L_i \) with \( 1 \leq i \leq n \) by induction as follows. Let \( L_1 \in \{ P_1, \ldots, P_k \} \) denote the polygon containing \( x \). Define
\[
q_1 = \inf \{ 0 < t < t(x, n) : x_t \text{ meets a side of } L_1 \}
\]
As the endpoints of \( I_n \) are discontinuities of the first return IET, the set on the right-hand side is necessarily non-empty. Let \( l_1 \) denote the side of \( L_1 \) such that \( x_{q_1} \in l_1 \). Let \( r_1 \) denote the unique side of a polygon \( R_1 \in \{ P_1, \ldots, P_k \} \) which is identified to \( l_1 \) by a translation \( T_1 \) (which defines the gluing of parallel sides).

Once \( (q_j, L_j, l_j, r_j, T_j, R_j) \) have been defined for all \( 1 \leq j \leq i - 1 < n \), we define
\[
q_i = \inf \{ q_{i-1} < t < t(x, n) : x_t \text{ meets a side of } T_{i-1} \cdot R_{i-1} \}
\]
Let \( L_i = T_{i-1} \cdot R_{i-1} \), let \( l_i \) denote the side of \( L_i \) such that \( x_{q_i} \in l_i \).

Note that \( l_i \) is the image of a side \( l'_i \) of a polygon in \( \{ P_1, \ldots, P_k \} \) by a translation \( A \), i.e., \( A \) brings \( l_i \) back to a side \( l'_i \) of one of the original polygons \( \{ P_1, \ldots, P_k \} \). Denote by \( r_i \) the unique side of a polygon \( R_i \in \{ P_1, \ldots, P_k \} \) which is identified to \( l'_i \) by a translation \( B \). Define the \( i \)th translation \( T_i \) by \( T_i = A \circ B \).

Now, consider the parallelogram
\[
\mathcal{P}_n = \left\{ x_t \in \mathbb{C} : x \in \text{Int}(\tilde{I}_n), 0 \leq t \leq t(x, n) \right\}
\]
where \( \text{Int}(\tilde{I}_n) \) denotes the interior of \( \tilde{I}_n \).

By definition of the endpoints of \( I_n \), each of the two vertical sides of \( \mathcal{P}_n \) necessarily meets a vertex of one of the polygons \( L_1, \ldots, L_n \). On the other hand, the interior of \( \mathcal{P}_n \) is free from the vertices of the polygons. In particular, if we let \( v_n \) denote a straight line segment joining two of the vertices on the two vertical sides of \( \mathcal{P}_n \), we see that \( v_n \) represents a saddle connection for \( x \) which is contained entirely in \( \mathcal{P}_n \).

If we regard \( v_n \) as a vector in \( \mathbb{C} \), we see that the imaginary part \( |\text{Im}(v_n)| \) is at most the height of the parallelogram \( \mathcal{P}_n \). Thus, in particular, we get that
\[
|\text{Im}(v_n)| \asymp n \quad (9.6)
\]
where the implied constant depends only on the lengths of the sides of the polygons \( P_1, \ldots, P_k \).

Moreover, the real part \( |\text{Re}(v_n)| \) satisfies
\[
|\text{Re}(v_n)| \ll |I_n| \leq \varepsilon/n \quad (9.7)
\]
where the implied constant here depends on the angle between the segment \( \tilde{I} \) and the horizontal axis, which, in turn, depends only on the neighborhood \( \mathcal{U}_\omega \). Therefore, we see that the length of the saddle connection \( g_{\log(n/\sqrt{n})} v_n \) is \( \ll \sqrt{\varepsilon} \) as desired.

\[\square\]

9.4. Horocycles and Lines in the Space of IETs. It was shown by Minsky and Weiss in [MW14] that the image of short horocycle arcs under the map (9.5) is short line segments in \( \mathbb{R}^2 \). This result was used in the work of Athreya and Chaika in [AC15] to relate the dimension of divergent directions for the Teichmüller flow to the dimension of non-uniquely ergodic IETs. We use a similar idea to obtain the following proposition.
Proposition 9.4. Suppose \( \pi \) is an irreducible permutation on \( d \) letters. Then, the set of \( \lambda \in \mathbb{R}^d_+ \) corresponding to uniquely ergodic IETs \( T_{\lambda,\pi} \) which are IDOC and satisfy (9.2) has Hausdorff codimension at least 1/2.

The proof of Proposition 9.4 will be given in Section 9.5 after some technical preparation. We begin by recalling a result in [MW14] characterizing line segments that can arise as the image of a short horocycle segment under the map (9.5).

Proposition 9.5 (Theorem 5.3 in [MW14]). Suppose \( \lambda \in \mathbb{R}^d_+ \) is such that \( T_{\lambda,\pi} \) is uniquely ergodic and satisfies IDOC. Suppose \( b \in \mathbb{R}^d \) satisfies \( Q(\lambda, b) > 0 \). Then, there exists an \( \varepsilon > 0 \) and an open neighborhood \( \mathcal{O} \) of \((\lambda, b)\) in \( \mathbb{R}^d_+ \times \mathbb{R}^d \) and an affine homeomorphism \( q : \mathcal{O} \to H \) such that \( h_* q(\lambda, b) = q(\lambda + sb, b) \) for \( |s| < \varepsilon \). Moreover, \( T(q(\lambda, b)) = \lambda \) for all \((\lambda, b) \in \mathbb{R}^d_+ \times \mathbb{R}^d \).

We remark that Theorem 5.3 in [MW14] is not stated in the form we use here, however the statement of Proposition 9.5 follows easily from the original statement, Definition 5.1 of positive pairs and Proposition 5.2 in [MW14].

The next lemma shows that the positivity condition \( Q(\cdot, \cdot) > 0 \) in Proposition 9.5 is not restrictive.

Lemma 9.6. For Lebesgue almost every \((\lambda, b) \in \mathbb{R}^d_+ \times \mathbb{R}^d \), \( Q(\lambda, b) \neq 0 \).

Proof. Suppose \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d_+ \). We claim that \( Q(\lambda, e_1) > 0 \) and, in particular, non-zero. Note that \( Q(e_1, e_1) = 0 \) and for all \( j > 1 \), \( Q(e_j, e_1) \geq 0 \). As \( \lambda \in \mathbb{R}^d_+ \), this implies that \( Q(\lambda, e_1) > 0 \).

Now, since \( \pi \) is irreducible, we have \( \pi(1) > 1 \). Hence, we can find some \( j_0 > 1 \) so that \( \pi(j_0) = 1 < \pi(1) \). It follows that \( Q(\lambda, e_1) \geq \lambda_{j_0} Q(e_{j_0}, e_1) = \lambda_{j_0} > 0 \).

This shows that the linear form \( Q(\lambda, \cdot) \) is not identically zero. That is the kernel of \( Q(\lambda, \cdot) \) has dimension \( d - 1 \) and thus has measure 0. Hence, the lemma follows by Fubini’s theorem. \( \Box \)

9.5. Hausdorff dimension, slicing, and proof of Proposition 9.4. Denote by \( \text{Gr}(d, m) \) the Grassmanian of \( m \) dimensional subspaces in \( \mathbb{R}^d \) and let \( \gamma_{d,m} \) denote a Lebesgue class measure on \( \text{Gr}(d, m) \). The space of lines (1 dimensional affine subspaces) in \( \mathbb{R}^d \) can be naturally identified with \( \text{Gr}(d, d-1) \times \mathbb{R}^{d-1} \) and thus carries a Lebesgue class measure. The following fact about slicing Borel sets of small Hausdorff codimension with lines will be useful for us.

Proposition 9.7 (Theorem 10.8 and Corollary 8.9(3) in [Mat95]). Suppose \( A \subset \mathbb{R}^d \) is a Borel set with \( \text{dim}_H(A) > t > d - 1 \). Then, there exists a set \( B \subset \text{Gr}(d, d-1) \times \mathbb{R}^{d-1} \) of lines in \( \mathbb{R}^d \) of positive Lebesgue measure such that for each line \( \ell \in B \),

\[
\text{dim}_H(\ell \cap A) \geq t - d + 1
\]

We also recall Frostman’s lemma.

Lemma 9.8. (Theorem 8.8 in [Mat95]) Suppose \( A \) is a Borel subset of \( \mathbb{R}^d \). Let \( s > 0 \). Then, the following are equivalent:

(1) \( H^s(A) > 0 \), where \( H^s \) denotes the \( s \)-dimensional Hausdorff measure.

(2) There exists a Borel measure \( \mu \) on \( \mathbb{R}^d \) with support in \( A \) such that \( \mu(B(x, r)) \leq r^s \) for all \( x \in \mathbb{R}^d \) and \( r > 0 \), where \( B(x, r) \) is the closed ball with center \( x \) and radius \( r \).
The idea of the proof of Proposition 9.4 is the following. First, we use Proposition 9.7 to relate the dimension of the set of interest to the dimension of its intersection with line segments. Then, Proposition 9.5 allows us to relate the dimension of sets on line segments to the dimension of subsets of horocycle arcs. Finally, using Proposition 9.3, we show that the sets of interest on the horocycle arcs correspond to points with divergent $g_t$ orbits. As a result, Theorem 2.3 concludes the argument.

The suggested outline of the proof is a modified version of an argument given in [AC15, Section 6]. The main difference is the use of Lemma 9.6 to bypass the use of Rauzy induction (Lemma 6.5 in [AC15]) which we believe makes the approach more direct.

**Proof of Proposition 9.4.** Denote by $A$ the set of $\lambda \in \mathbb{R}_+^d$ such that $T_{\lambda,\pi}$ is uniquely ergodic, IDOC, and has short intervals and note that $A$ is Borel measurable. Suppose that for some $0 < c < 1$, we have

$$\text{codim}_H (A) \leq c$$

Then, by Proposition 9.7, there exists a positive measure set $\mathcal{L}$ of lines in $\mathbb{R}^d$ such that for each line $\ell \in \mathcal{L}$ a set $\ell \cap A$ has Hausdorff dimension at least $1 - c$. By Lemma 9.6, we may assume that for each line $\ell \in \mathcal{L}$ there exists some point $\lambda \in \ell$ so that $Q(\lambda, b) \neq 0$, where $b$ is a vector in $\mathbb{R}^d$ parallel to $\ell$. Let $\ell \in \mathcal{L}$ be a line such that it passes through a point $\lambda \in \mathbb{R}_+^d$ and is parallel to $b \in \mathbb{R}^d$, i.e., $\ell = \{\lambda + sb : s \in \mathbb{R}\}$, and $Q(\lambda, b) \neq 0$.

By Lemma 9.8 ((1)\(\Rightarrow\)(2)), there exists a measure $\mu$ supported on $\ell \cap A$ so that for all $x \in \ell$ and all $r > 0$, we have

$$\mu(B(x, r)) \leq r^{1-c}$$

Note that the linearity of $Q$ implies that $Q(\lambda + sb, b) \neq 0$ for all $s \neq -Q(\lambda, b)/Q(b, b)$ and for all $s \in \mathbb{R}$ if $Q(b, b) = 0$. Hence, since $\mu$ is not a Dirac mass, we can find $x \in \text{supp}\mu \subset \ell \cap A$ such that $Q(x, b) \neq 0$. In particular, $T_{x,\pi}$ is uniquely ergodic, IDOC, and has short intervals. Notice that a priori $\lambda \in \ell$ may not belong to $\text{supp}\mu$. By replacing $b$ with $-b$ if necessary, we may assume $Q(x, b) > 0$.

Hence, by Proposition 9.5, we can find $\varepsilon_0 > 0$ and a local Lipschitz inverse $q$ of the map $T$ so that $q(x + sb, b) = h_s(q(x, b))$ for $|s| < \varepsilon_0$. But, by Proposition 9.3, the forward $g_t$ orbit of the set $\{q(x + sb, b) : |s| < \varepsilon_0, x + sb \in A\}$ is divergent (on average) in the stratum $H$.

By restricting $\mu$ to the segment $x + sb$ with $|s| < \varepsilon_0$ and using Lemma 9.8 ((2)\(\Rightarrow\)(1)), we see that the Hausdorff dimension of $A \cap \{x + sb : |s| < \varepsilon_0\}$ is at least $1 - c$. Theorem 2.3, thus, implies that $1 - c \leq 1/2$.

\[\square\]

9.6. **Proof of Corollary 1.8.** Let $\pi$ be a type $W$ permutation on $d \geq 4$ letters. By Theorem 9.2, we have the following inclusion

$$\{\lambda \in \mathbb{R}_+^d : T_{\lambda,\pi} \text{ not weak mixing} \} \subseteq \{\lambda : T_{\lambda,\pi} \text{ is not IDOC} \} \cup \{\lambda : T_{\lambda,\pi} \text{ is NUE} \} \cup \{\lambda : T_{\lambda,\pi} \text{ is IDOC, and UE, and has short intervals} \}$$

(9.8)

where (N)UE denotes (non)-uniquely ergodic and having short intervals means $T_{\lambda,\pi}$ satisfies (9.2). The last two sets in the above union have codimension at least $1/2$ by [AC15, Theorem 1.6] and Proposition 9.4, respectively.

It is shown in [Kea75] that if the components of $\lambda$ are linearly independent over $\mathbb{Q}$, then $T_{\lambda,\pi}$ is IDOC. In particular, the set of IETs which are not IDOC is contained in the intersection of the simplex $\mathbb{R}_+^d$ with countably many codimension 1 subspaces of $\mathbb{R}^d$ which are defined over $\mathbb{Q}$. This implies that the set of non-IDOC IETs has Hausdorff codimension at least 1.
10. LARGE DEVIATIONS IN BIRKHOFF’S THEOREM IN STRATA - AN OUTLINE

The scheme suggested in this paper is quite flexible and can be applied to get similar results about the Hausdorff dimension in various settings. For example, using our approach for the proof of Theorem 1.1, one should be able to answer the following question affirmatively.

**Question 10.1.** Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_\mathcal{M}$ is the affine measure whose support is $\mathcal{M}$. Let $f$ be a bounded continuous function $f$ on $\mathcal{M}$ and $\varepsilon > 0$. Is the Hausdorff dimension of the set

$$\left\{ x \in \mathcal{M} : \limsup_{T \to \infty} \left| \frac{1}{T} \int_0^T f(g_t x) \, dt - \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \right| \geq \varepsilon \right\}$$

strictly less than the dimension of $\mathcal{M}$?

Note that, by a standard approximation argument, the affirmative answer to the above question implies that for any non-empty open subset $U$ of a connected component $\mathcal{C}$ of the stratum $\mathcal{H}_1(\alpha)$, the Hausdorff dimension of the set

$$\{ x \in \mathcal{C} : g_t x \notin U \text{ for all } t > 0 \}$$

is strictly less than the dimension of $\mathcal{C}$.

For clarity, we briefly outline how to apply our techniques to answer Question 10.1. The idea is to translate all the results on horocycle arcs obtained in Section 6 to results on open bounded subsets of the strong unstable manifold for the Teichmüller flow. Then, one obtains the desired result from the analogue of Theorem 1.1 for the Hausdorff dimension of the following set in the strong unstable leaf $\mathcal{W}^{su}(\omega)$ of $\omega \in \mathcal{H}_1(\alpha)$.

$$\left\{ x \in \mathcal{W}^{su}(\omega) : \limsup_{T \to \infty} \left| \frac{1}{T} \int_0^T f(g_t x) \, dt - \int_{\mathcal{M}} f \, d\nu_\mathcal{M} \right| \geq \varepsilon \right\},$$

where $\omega$ satisfies $\SL(2, \mathbb{R}) \omega = \mathcal{M}$. We recall that

$$\mathcal{W}^{su}(\omega) = \{ x \in \mathcal{H}_1(\alpha) : d_{\mathcal{H}_1(\alpha)}(g_t \omega, g_t x) \to 0 \text{ as } t \to -\infty \}$$

where $d_{\mathcal{H}_1(\alpha)}$ denotes the Teichmüller metric. Any such leaf is foliated with orbits of the horocycle flow $h_s$. In particular, we can locally find foliation charts for $h_s$-orbits within a leaf of the unstable foliation, which also provide immersed local transversals for the horocycle orbits. As a result, we obtain that a neighborhood $\mathcal{W}^{su}_{loc}(\omega)$ of a point $\omega$ inside $\mathcal{W}^{su}(\omega)$ has a product structure. This allows the disintegration of the probability measure of Lebesgue class on small bounded open sets of unstable leaves with parameter measures as conditionals along horocycles.

One can then introduce subsets of $\mathcal{W}^{su}_{loc}(\omega)$ and the corresponding functions analogous to (6.1), (6.2) and (6.3). Using Fubini’s theorem, one should be able to translate the measure bounds on exceptional subsets of horocycle arcs (see Section 6.2) into bounds for exceptional subsets of $\mathcal{W}^{su}_{loc}(\omega)$. The product structure of $\mathcal{W}^{su}_{loc}(\omega)$ can be similarly used for translating the results for horocycles in Section 3 into results for $\mathcal{W}^{su}_{loc}(\omega)$.

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