Transition Operators

J. Alcock-Zeilinger\(^1\) and H. Weigert\(^1\)

\(^1\)University of Cape Town; Dept. of Physics, Private Bag X3, Rondebosch 7701, South Africa

Abstract: In this paper, we give a generic algorithm of the transition operators between Hermitian Young projection operators corresponding to equivalent irreducible representations of SU\((N)\), using the compact expressions of Hermitian Young projection operators derived in [1]. We show that the Hermitian Young projection operators together with their transition operators constitute a fully orthogonal basis for the algebra of invariants of \(V^\otimes m\) that exhibits a systematically simplified multiplication table. We discuss the full algebra of invariants over \(V^\otimes 3\) and \(V^\otimes 4\) as explicit examples. In our presentation we make use of various standard concepts such as Young projection operators, Clebsch-Gordan operators, and invariants (in birdtrack notation). We tie these perspectives together and use them to shed light on each other.

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1 Introduction

Applications of representation theory generally are concerned with irreducible representations of the group under scrutiny. Physics applications in particular are generally aimed at finding all irreducible representations in an \( m \)-particle Fock space. The textbook example here is of course angular momentum and spin with the group \( SU(2) \) and the construction of the periodic table in quantum mechanics via the irreducible multiplets for \( m \)-electrons in an atom with \( m \) protons that classify its orbital configuration, its spectral and chemical properties. In quantum chromodynamics we meet flavor symmetry (flavor \( SU_f(2) \) or \( SU_f(3) \)) as an approximate symmetry that guide us through interpreting the mesons and baryons as members of irreducible representation of the flavor group in the eight-fold way \([2, 3]\). Gauge invariance and confinement force the same particles into singlets of the color gauge group \( SU_c(3) \). The latter are of particular interest in the color glass condensate which dominates QCD in high energy applications, i.e. in modern collider experiments. In this set of applications Wilson line correlators appear whose color structures are of central importance and the presently existing techniques are limited to explicit calculations at a given order of complexity. In \([1]\), we have established an efficient algorithm to construct a full set of Hermitian projection operators for the decomposition of a product of \( m \) fundamental representations of \( SU(N) \) as a subset of the associated algebra of invariants. Here we aim to find a complete basis for the algebra of invariants that is fully shaped by irreducible representations these operators represent and identify the missing basis elements as transition operators between equivalent representation contained in the product. In a future paper \([4]\), this information will be used to give a full account of all singlets (i.e. all one dimensional representations that remain invariant under the group action) constrained in a product of \( m \) fundamental representations with \( m' \) anti-fundamental representations. In physics parlance, this gives access to the gauge invariant part of the Fock space of \( m \) particles and \( m' \) anti-particles.

There are, of course, several different technologies on the market to address these questions, the most familiar to the practising physicist being the construction of Clebsch-Gordan coefficients \([5]\) (see \([6–8]\) for textbook introductions), Elie Cartan’s method of roots and weights \([9]\) and Alfred Young’s combinatorial method of classifying the idempotents on the algebra of permutations \([10]\). The Schur-Weyl duality, \([11]\) relates these idempotents to the irreducible representations of compact, semi-simple Lie groups and thus to \( SU(N) \). This duality is based on the theory of invariants, \([11, 12]\), which exploits the invariants (in particular the primitive invariants) of a Lie group \( G \) and constructs projection operators corresponding to the irreducible representations of \( G \) from the invariants of that group. It is this method that allows us to carry \( N \) as a parameter throughout, which has important advantages in applications in QCD we are ultimately interested in. The core part of our discussion will deal with a product representations of \( SU(N) \) constructed from the fundamental representation of a Lie group \( G \) on a given vector space \( V \) with \( \dim(V) = N \), whose action will simply be denoted by \( v \mapsto Uv \) for all \( U \in SU(N) \) and \( v \in V \). Choosing a basis \( \{ e_i \} \) \( i = 1, \ldots, \dim(V) \)
such that \( v = v^i e_{(i)} \) this becomes \( v^i \mapsto U^i_j v^j \). This immediately induces product representations of SU(N), representations on \( V^\otimes m \), if one uses this basis of \( V \) to induce a basis on \( V^\otimes m \) so that a general element \( \mathbf{v} \in V^\otimes m \) takes the form \( \mathbf{v} = v^{i_1 \ldots i_m} e_{(i_1)} \otimes \ldots \otimes e_{(i_m)} \):

\[
U \circ \mathbf{v} = U \circ v^{i_1 \ldots i_m} e_{(i_1)} \otimes \ldots \otimes e_{(i_m)} := U^{i_1}_{j_1} \ldots U^{i_m}_{j_m} v^{j_1 \ldots j_m} e_{(i_1)} \otimes \ldots \otimes e_{(i_m)} \tag{1}
\]

Since all the factors in \( V^\otimes m \) are identical, the notion of permuting the factors is a natural one and leads to a linear map on \( V^\otimes m \) according to

\[
\rho \circ \mathbf{v} = \rho \circ v^{i_1 \ldots i_m} e_{(i_1)} \otimes \ldots \otimes e_{(i_m)} := v^{\rho(i_1) \ldots \rho(i_m)} e_{(i_1)} \otimes \ldots \otimes e_{(i_m)} \tag{2}
\]

where \( \rho \) is an element of \( S_m \), the group of permutations of \( m \) objects.\(^1\) From the definitions (1) and (2) one immediately infers that the product representation commutes with all permutations on any \( \mathbf{v} \in V^\otimes m \):

\[
U \circ \rho \circ \mathbf{v} = \rho \circ U \circ \mathbf{v} . \tag{3}
\]

In other words, any such permutation \( \rho \) is an invariant of SU(N) (or in fact any Lie group \( G \) acting on \( V \)):

\[
U \circ \rho \circ U^{-1} = \rho . \tag{4}
\]

It can further be shown that these permutations in fact span the space of all linear invariants of SU(N) over \( V^\otimes m \) \(^{[12]}\). The permutations are thus referred to as the primitive invariants of SU(N) over \( V^\otimes m \). The full space of linear invariants is then given by

\[
\text{API}(\text{SU}(N), V^\otimes m) := \left\{ \sum_{\sigma \in S_m} \alpha_{\sigma} \sigma \mid \alpha_{\sigma} \in \mathbb{R}, \sigma \in S_m \right\} \subset \text{Lin} \left( V^\otimes m \right) . \tag{5}
\]

As defined in (5), \( \text{API}(\text{SU}(N), V^\otimes m) \) is a real vector space and becomes a real algebra with the product induced by the product of permutations. It will become obvious from our presentation that this space is large enough to encompass all group-theoretically interesting objects, namely

1. Hermitian projectors onto irreducible representations (see \([1, 13]\)), and
2. any transition operators associated with equivalent representations.

We will show that these operators do not only fit into it, they in fact span the whole space and form an orthogonal basis for \( \text{API}(\text{SU}(N), V^\otimes m) \). We will call this the projector basis for \( \text{API}(\text{SU}(N), V^\otimes m) \) in the remainder of this paper and discuss in detail its unique structures and the freedom of choice still left open.

Naively, since the number of permutations in \( S_m \) is \( m! \), one might expect the dimension of \( \text{API}(\text{SU}(N), V^\otimes m) \) to be simply \( m! \) and indeed this is the maximal dimension of the algebra. However, this is realized only if \( N = \text{dim}(V) \geq m \). Failing that, not all permutations remain linearly independent (as elements of the vector space \( \text{Lin}(V^\otimes m) \)). In the projector basis we construct this is particularly clearly exhibited: a number of clearly distinguished basis elements will explicitly become null operators, all others will remain non-zero and thus form a basis for the now smaller space of invariants (see appendix A). It is this feature that allows us to use \( N \) as a parameter in calculations.

We begin with a short presentation of some background material needed to fully appreciate the arguments made in this paper, sec. 2. This section begins by introducing the birdtrack formalism \(^{[12]}\), which is particularly suited for dealing with the objects discussed in this paper. We then proceed by summarizing

\[^1\]Permuting the basis vectors instead involves \( \rho^{-1}: v^{\rho(i_1) \ldots \rho(i_m)} e_{(\rho(i_1))} \otimes \ldots \otimes e_{(\rho(i_m))} = v^{i_1 \ldots i_m} e_{(\rho^{-1}(i_1))} \otimes \ldots \otimes e_{(\rho^{-1}(i_m))} \)
classic textbook material on Young tableaux and their corresponding projection operators, see for example \cite{6, 7, 12, 14, 15}. Lastly, we state some cancellation rules for birdtrack operators \cite{16}.

Section 3 provides the first new results: we show that the Young projection operators can be augmented by what we choose to call transition operators to give an alternative basis for the algebra of invariants $\mathcal{API}(\text{SU}(N), V^\otimes m)$ for $m \leq 4$, and proceed to give the full basis for $\mathcal{API}(\text{SU}(N), V^\otimes 3)$ (a diagram depicting the full basis up to $m = 4$ is given in Figure 2). Since orthogonality of Young projection operators breaks down beyond $m = 4$, the Young basis cannot be generalized to larger $m$. This motivates a basis in terms of Hermitian projection operators and their unitary transition operators:

Section 4 discusses such a basis through Clebsch-Gordan operators for all $m$. As it turns out, Hermiticity and unitarity of these operators automatically guarantee mutual orthogonality of the basis elements with respect to the inner product $\langle A, B \rangle := \text{tr}(A^\dagger B)$. Since this method requires the construction of $N^m$ normalized states to find $m!$ basis elements for $\mathcal{API}(\text{SU}(N), V^\otimes m)$ we then proceed to present a more efficient algorithm to reach this goal. Our method is based on streamlined methods to construct Hermitian Young projection operators \cite{1} (themselves based on earlier work by Keppeler and Sjödahl \cite{13}). These Hermitian Young projection operators are complemented with unitary transition operators to provide a full basis for $\mathcal{API}(\text{SU}(N), V^\otimes m)$ for all $m$ in section 5, Theorem 4. This construction algorithm for transition operators serves as a starting point for a more efficient method, Theorem 5, yielding much shorter expressions of the transition operators. The proof of Theorem 5 can be found in appendix D.

We close with some examples: We give the basis of $\mathcal{API}(\text{SU}(N), V^\otimes m)$ in terms of Hermitian Young projection operators and unitary transition operators for both $m = 3$ and $m = 4$ in section 6. Figures 2 and 3 summarize the most important aspects of Young and Hermitian Young decompositions of $\mathcal{API}(\text{SU}(N), V^\otimes m)$ for all $m \leq 4$.

## 2 Background: Birdtracks, Young tableaux, notations and conventions

### 2.1 Birdtracks, scalar products, and Hermiticity

In the 1970’s Penrose devised a graphical method of dealing with primitive invariants of Lie groups including Young projection operators, \cite{17, 18}, which was subsequently applied in a collaboration with MacCallum, \cite{19}. This graphical method, now dubbed the birdtrack formalism, was modernized and further developed by Cvitanović, \cite{12}, in recent years. The immense benefit of the birdtrack formalism is that it makes the actions of the operators visually accessible and thus more intuitive. For illustration, we give as an example the permutations of $S_3$ written both in their cycle notation (see \cite{6} for a textbook introduction) as well as birdtracks:

\[
S_3 = \{ \text{id}, (12), (13), (23), (123), (132) \}.
\]

The action of each of the above permutations on a tensor product $v_1 \otimes v_2 \otimes v_3$ is clear, for example

\[
(123)(v_1 \otimes v_2 \otimes v_3) = v_3 \otimes v_1 \otimes v_2.
\]

In the birdtrack formalism, this equation is written as

\[
\frac{v_3}{\otimes v_1} = \frac{v_3}{\otimes v_2}.
\]
where each term in the product $v_1 \otimes v_2 \otimes v_3$ (written as a tower $v_1^\uparrow v_2^\uparrow v_3^\uparrow$) can be thought of as being moved along the lines of $\uparrow\uparrow\uparrow$. Birdtracks are thus naturally read from right to left as is also indicated by the arrows on the legs.

The graphical structure faithfully represents the multiplication table of $S_m$ by implementing a “glue and follow the lines prescription” along the lines of

$$S_{a_1,\ldots,a_n} := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma a_1,\ldots,a_n \tag{10}$$

where $\sigma a_1,\ldots,a_n$ denotes a permutation in $S_n$ over (any subset of) the letters $a_1,\ldots,a_n$, and antisymmetrizers

$$A_{a_1,\ldots,a_n} := \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma a_1,\ldots,a_n \tag{11}$$

These may act on subsets of $n$ factors in $V^\otimes m$. Both symmetrizers and antisymmetrizers are by definition idempotent,

$$S_{a_1,\ldots,a_n}^2 = S_{a_1,\ldots,a_n} \quad \text{and} \quad A_{a_1,\ldots,a_n}^2 = A_{a_1,\ldots,a_n} \tag{12}$$

they are projection operators.

All of these have birdtrack representations in which symmetrizers (resp. antisymmetrizers) are shown as unfilled (filled) boxes covering the lines to be symmetrized (resp. antisymmetrized). Take for example $S_{134} \in \text{API} (SU(N), V^\otimes 5)$ and $A_{35} \in \text{API} (SU(N), V^\otimes 5)$, which take the form

$$S_{134} = \quad \in \text{API} (SU(N), V^\otimes 5) \quad \text{and} \quad A_{35} = \quad \in \text{API} (SU(N), V^\otimes 5) \tag{13}$$

We note in passing that Hermitian conjugation for birdtracks (in the sense of linear maps on $V^\otimes m$ with the scalar product inherited from $V$) is achieved by reflection around a vertical axis, followed by a reversal of the arrows, [12]. As an example take

$$\begin{align*}
\text{reflect} & \quad \rightarrow \quad \text{rev. arr} \quad i.e \quad (\uparrow\uparrow\uparrow)^\dagger = \uparrow\uparrow\uparrow.
\end{align*} \tag{14}$$

This implies that all symmetrizers and antisymmetrizers as defined above are Hermitian

$$S_{a_1,\ldots,a_n}^\dagger = S_{a_1,\ldots,a_n} \quad \text{and} \quad A_{a_1,\ldots,a_n}^\dagger = A_{a_1,\ldots,a_n} \tag{15}$$

and that all permutations are unitary:

$$\sigma^{-1} = \sigma^\dagger \quad \text{for all} \ \sigma \in S_m. \tag{16}$$
The direction of arrows on the legs also allows us to account for complex conjugation, *c.f.* [12]. In this paper, we will exclusively be working with real operators and thus suppress the direction of the arrows, for example

\[ \begin{array}{c}
\leftarrow \rightarrow \\
\rightarrow \leftarrow 
\end{array} \]

will refer to \[ \begin{array}{c}
\rightarrow \rightarrow \\
\leftarrow \leftarrow 
\end{array} \].

Lastly, if a *Hermitian* projection operator \( A \) projects onto a subspace completely contained in the image of a projection operator \( B \), then we denote this as \( A \subset B \), transferring the familiar notation of sets to the associated projection operators. In particular, \( A \subset B \) if and only if

\[ A \cdot B = B \cdot A = A \tag{18} \]

for the following reason: If the subspaces obtained by consecutively applying the operators \( A \) and \( B \) in any order is the same as that obtained by merely applying \( A \), then not only need the subspaces that \( A \) and \( B \) project onto overlap (as otherwise \( A \cdot B = B \cdot A = 0 \)), but the subspace corresponding to \( A \) must be completely contained in the subspace of \( B \) - otherwise the last equality of (18) would not hold. Hermiticity is crucial for these statements – they thus do not apply to most Young projection operators on \( V^\otimes m \) if \( m \geq 3 \).

A familiar example for this situation is the relation between symmetrizers of different length: a symmetrizer \( S_\mathcal{N} \) can be absorbed into a symmetrizer \( S_\mathcal{N}' \), as long as the index set \( \mathcal{N} \) is a subset of \( \mathcal{N}' \), and the same statement holds for antisymmetrizers, [12]. For example,

\[ S_\mathcal{N} \subset S_\mathcal{N}' \tag{20} \]

Thus, by the above notation, \( S_\mathcal{N}' \subset S_\mathcal{N} \) if \( \mathcal{N} \subset \mathcal{N}' \). Or, as in our example,

\[ \begin{array}{c}
\big[ & \big]\subset \big[ & \big] \\
\leftarrow & \rightarrow \\
\rightarrow & \leftarrow 
\end{array} = \big[ & \big] = \big[ & \big] \\
\big] \subset \big[ & \big] \tag{19} \]

API \((SU(N), V^\otimes m)\) itself is equipped with a scalar product for linear maps that is consistent with the scalar product on \( V^\otimes m \) and simply given by a trace

\[ \langle \, , \, \rangle : \text{API} (SU(N), V^\otimes m) \otimes \text{API} (SU(N), V^\otimes m) \to \mathbb{R}, \quad \langle A, B \rangle := \text{tr}(A^\dagger B) . \tag{21} \]

In birdtrack notation, the trace \( \text{tr} \) merely connects each line exiting \( A^\dagger B \) on the left with the line entering \( A^\dagger B \) on the right that is on the same level,

\[ \text{tr} (A^\dagger B) = \begin{array}{c}
\big[ & \big]\langle A^\dagger B \rangle \\
\big[ & \big]\end{array} \tag{22} \]

For example,

\[ \text{tr} \left( \left( \begin{array}{c}
\big[ & \big]\end{array} \right)^\dagger \left( \begin{array}{c}
\big[ & \big]\end{array} \right) \right) = \text{tr} \left( \left( \begin{array}{c}
\big[ & \big]\end{array} \right)^\dagger \left( \begin{array}{c}
\big[ & \big]\end{array} \right) \right) = \text{tr} \left( \big[ & \big] \right) = \big[ & \big] = N^2, \tag{23} \]

where we have drawn the lines originating from the trace in red for visual clarity. Each closed loop in the trace yields a factor of \( N \) (the dimension of the fundamental representation), so that the scalar product (21)

\[ \text{As can be explicitly verified by an example.} \]
will always yield a polynomial of $N$. In particular, it is clear that this polynomial will not be identically 0 if $A, B \in S_m$ (and hence $A^\dagger \in S_m$), since $S_m$ is a group and thus $A^\dagger B \in S_m$. The cyclic property of the trace becomes very apparent in birdtrack notation, as the operator $A_1$ in

$$\text{tr}(A_1A_2\ldots A_n)$$

(24)
can just be “pulled” to the right of $A_n$ along the lines induced by the trace.

### 2.2 The hierarchical nature of Young tableaux

A Young tableau is a conglomerate of numbered boxes with shape and ordering restrictions imposed. The shape and ordering restrictions automatically emerge if we construct these tableaux iteratively, starting from a single box $\begin{array}{c} 1 \end{array}$ in a process governed by branching rules (see for example [14, 15]). The second box $\begin{array}{c} 2 \end{array}$ and all further boxes are attached (in order) to the right of or below an existing box in all possible ways that lead to a set of boxes in which no row is longer than that above it and no column is longer than the one left of it. The process results in a tree whose first few branchings are displayed in Fig. 1.

![Branching tree of Young tableaux from its root to the 4th generation](image)

Figure 1: Branching tree of Young tableaux from its root to the 4th generation

We denote the set of Young tableaux with $n$ boxes by $\mathcal{Y}_n$ and note that in each branching step every Young tableau $\Theta \in \mathcal{Y}_{n-1}$ creates a whole set of descendant tableaux in $\mathcal{Y}_n$. We will refer to this set by $\{\Theta \otimes \pi\}$ and notice that it has no overlap with the descendants of any other element of $\mathcal{Y}_{n-1}$. Any $\mathcal{Y}_n$ is the disjunct union of descendant sets: For example,

$$\mathcal{Y}_3 := \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array}, \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array}, \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} = \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\} \cup \left\{ \begin{array}{c} 1 \end{array} \begin{array}{c} 2 \end{array} \begin{array}{c} 3 \end{array} \right\}. \quad (25)$$

Traversing the tree of Fig. 1 downwards is a branching operation, in which each descendant has a well defined ancestry chain: Starting at a Young tableau and taking away the box with the highest entry is a map in the mathematical sense, it yields a unique tableau. We call this map the parent map and denote it by $\pi$. $\pi$ can
then repeatedly be applied to the resulting tableau generating the ancestry chain for a given tableau. An example for part of such a chain is

\[ \ldots \xrightarrow{\pi} \begin{array}{ccc} 1 & 3 & 6 \\ 2 & 5 & 4 \end{array} \xrightarrow{\pi} \begin{array}{ccc} 1 & 3 \\ 2 & 5 & 4 \end{array} \xrightarrow{\pi} \begin{array}{ccc} 1 & 3 \\ 2 & 4 \end{array} \xrightarrow{\pi} \begin{array}{ccc} 1 \end{array} \xrightarrow{\pi} \ldots . \] (26)

This idea can obviously be formalized in a way that provides some useful notation:

**Definition 1 (parent map and ancestor tableaux)** Let \( \Theta \in Y_n \) be a Young tableau. We define its parent tableau \( \Theta^{(1)} \in Y_{n-1} \) to be the tableau obtained from \( \Theta \) by removing the box \( n \) of \( \Theta \). Furthermore, we will define a parent map \( \pi \) from \( Y_n \) to \( Y_{n-1} \), for a particular \( n \),

\[ \pi : Y_n \rightarrow Y_{n-1}, \] (27)

which acts on \( \Theta \) by removing the box \( n \) from \( \Theta \), \(^3\)

\[ \pi : \Theta \mapsto \Theta^{(1)}. \] (28)

In general, we define the successive action of the parent map \( \pi \) by

\[ Y_n \xrightarrow{\pi} Y_{n-1} \xrightarrow{\pi} Y_{n-2} \xrightarrow{\pi} \ldots \xrightarrow{\pi} Y_{n-m}, \] (29)

and denote it by \( \pi^m \),

\[ \pi^m : Y_n \rightarrow Y_{n-m}, \quad \pi^m := Y_n \xrightarrow{\pi} Y_{n-1} \xrightarrow{\pi} Y_{n-2} \xrightarrow{\pi} \ldots \xrightarrow{\pi} Y_{n-m} \] (30)

We will further denote the tableau obtained from \( \Theta \) by applying the map \( \pi \) \( m \) times, \( \pi^m(\Theta) \), by \( \Theta^{(m)} \), and refer to it as the ancestor tableau of \( \Theta \) \( m \) generations back. Applying the map \( \pi^m \) to a Young tableau \( \Theta \) then yields the unique tableau \( \Theta^{(m)} \),

\[ \pi^m : \Theta \mapsto \Theta^{(m)}. \] (31)

We now reverse direction again and return to thinking about adding boxes. As we keep adding more and more boxes we encounter more and more tableaux that share their overall shape, they only differ by the ordering of entries. The shape (represented by the boxes with the entries deleted) is commonly referred to as a Young diagram. The reordering required to relate two tableaux of the same shape defines a tableau permutation:

**Definition 2 (tableau permutation)** Consider two Young tableaux \( \Theta, \Theta' \in Y_n \) with the same shape. Then, \( \Theta' \) can be obtained from \( \Theta \) by permuting the numbers of \( \Theta \); clearly, the permutation needed to obtain \( \Theta' \) from \( \Theta \) is unique. We denote this permutation mapping \( \Theta' \) into \( \Theta \) by \( \rho_{\Theta' \Theta} \),

\[ \Theta = \rho_{\Theta' \Theta}(\Theta') \iff \Theta' = \rho_{\Theta' \Theta}^{-1}(\Theta) = \rho_{\Theta \Theta'}(\Theta). \] (32)

### 2.3 From Young tableaux to Young operators

The literature (see for example [6]) provides a standard manner in which each Young tableau is associated with a Young projection operator that is constructed from symmetrizers and antisymmetrizers – symmetrizers

\(^3\)We note that the tableau \( \Theta^{(1)} \) is always a Young tableau if \( \Theta \) was a Young tableau, since removing the box with the highest entry cannot possibly destroy the properties of \( \Theta \) (and thus \( \Theta^{(1)} \)) that make it a Young tableau.
for the rows and antisymmetrizers for the columns. (For completeness we assign the identity permutation for rows or columns of length one. If all rows or columns are exactly of length one, we refer to this as the symmetrizers or antisymmetrizers becoming trivial.) As such, the Young projection operators corresponding to tableaux in \( Y_m \) are elements of the (real) algebra of invariants \( API(SU(N), V^\otimes m) \).

Take, for example

\[
\Theta = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 \end{pmatrix}.
\]  

(33)

The Young projection operator corresponding to this tableau, \( Y_\Theta \), is given by

\[
Y_\Theta = 2 \cdot S_{134} S_{25} A_{12} A_{35},
\]

(34)

where the constant 2 ensures idempotency of \( Y_\Theta \). c.f. eq. (38). (This is a textbook topic. For a reminder on how to construct Young projection operators from Young tableaux, readers are referred to [6, 14, 15].)

All the symmetrizers of a tableau commute with each other and so do the antisymmetrizers, since no number appears more than once in any tableau. Thus, when constructing the birdtrack corresponding to \( Y_\Theta \), we are able to draw the two symmetrizers appearing in it underneath each other (since they are disjoint), and similarly for the two antisymmetrizers,

\[
Y_\Theta = 2 \cdot \begin{array}{c} \includegraphics[width=0.2\textwidth]{birdtrack.png} \end{array}.
\]

(35)

We denote the set (or product – it does not really matter as they mutually commute) of symmetrizers associated with the tableau \( \Theta \) by \( S_\Theta \) and the set (or product) of antisymmetrizers by \( A_\Theta \). However, the symmetrizers of a tableau do not commute with its antisymmetrizers (unless one or both are trivial):

\[
[S_\Theta, A_\Theta] \neq 0.
\]

(36)

Therefore their relative order matters in the general definition of a Young projector

\[
Y_\Theta := \alpha_\Theta S_\Theta A_\Theta,
\]

(37)

where \( \alpha_\Theta \in \mathbb{R} \) is defined as

\[
\alpha_\Theta := \frac{H_\Theta}{\prod_R |\text{length}(R)|! \prod_C |\text{length}(C)|!};
\]

(38)

the products in the denominator run over every row \( R \) respectively over every column \( C \) of \( \Theta \) [12] and \( H_\Theta \) is the hook length of the tableau \( \Theta \) [14, 15]: For a particular Young tableau \( \Theta \), form its underlying Young diagram \( Y_\Theta \) by deleting all entries and then re-fill each box \( c \) in \( Y_\Theta \) with the integer counting all boxes to the right and below it (including itself), called the hook of \( c \) \( H_c \), for example

\[
\Theta = \begin{pmatrix} 1 & 3 & 5 & 8 \\ 2 & 6 & 7 \\ 4 \end{pmatrix} \quad \text{delete entries}\quad \begin{array}{c} \includegraphics[width=0.2\textwidth]{young_diagram.png} \end{array} \quad \text{hooks} \quad H_c = \begin{pmatrix} 6 & 4 & 3 & 1 \\ 3 & 2 & 1 \\ 1 \end{pmatrix}.
\]

(39)

\(^4\text{Placing the antisymmetrizers to the right of the symmetrizers is a choice of convention – the reverse order leads to equivalent structural results, it only matters to stay consistent.}\)
The hook length of $\Theta$, $H_\Theta$, (equivalently the hook length of $Y_\Theta$, $H_{Y_\Theta}$) is defined to be the product of all the hooks $H_c$,

$$
H_\Theta = H_{Y_\Theta} := \prod_{c \in \Theta} H_c;
$$

for the tableau in (39), the hook length is $H_\Theta = 6 \cdot 4^2 \cdot 3 \cdot 2 = 576$.

The Young projection operators are nonzero precisely if all their columns are at most of length $N$, otherwise they vanish identically – we refer to this as being dimensionally zero (see app. A).

From (37) one infers that $Y_\Theta$ is not Hermitian (unless at least one of the sets is trivial):

$$
Y_\Theta^\dagger = \alpha_\Theta (S_\Theta A_\Theta)^\dagger = \alpha_\Theta A_\Theta S_\Theta \neq Y_\Theta.
$$

Hermiticity (or the lack thereof) in birdtrack notation is best judged after expanding in primitive invariants, for example

$$
Y_{\begin{array}{c} 1 \\ 3 \\ 2 \\ 5 \\ 4 \end{array}} = \frac{4}{3} \cdot \begin{array}{c} = \alpha_\Theta (S_\Theta A_\Theta)^\dagger = \alpha_\Theta A_\Theta S_\Theta \neq Y_\Theta \end{array}
$$

The definition of $Y_{\begin{array}{c} 1 \\ 3 \\ 2 \\ 5 \\ 4 \end{array}}$ in (34) speaks of a linear map on $V^{\otimes 5}$, i.e. an element of $\text{Lin} (V^{\otimes 5})$, or with equal validity of a linear map on a larger space $V^{\otimes m}$, $m \geq 5$, in which the factors beyond the first five remain unaffected. We speak of this case as the canonical embedding of $\text{Lin} (V^{\otimes n}) \hookrightarrow \text{Lin} (V^{\otimes m})$ (with $m \geq n$). For a given tableau $\Theta \in \mathcal{Y}_n$ we will, in a slight abuse of notation, employ the same notation, $Y_\Theta$, to talk both about the original case or any of the embeddings. The idea of an embedding in birdtrack terms requires to explicitly draw the “unaffected lines”, for example the operator $\bar{Y}_{\begin{array}{c} 1 \\ 3 \\ 2 \end{array}}$ is canonically embedded into $\text{Lin} (V^{\otimes 5})$ as

$$
\begin{array}{c} \Rightarrow \end{array} \bar{Y}_{\begin{array}{c} 1 \\ 3 \\ 2 \end{array}}
$$

Furthermore, for any operator $O$, the symbol $\hat{O}$ will refer to a product of symmetrizers and antisymmetrizers without any additional scalar factors. For example,

$$
Y_\Theta := \frac{4}{3} \cdot \begin{array}{c} = \alpha_\Theta (S_\Theta A_\Theta)^\dagger = \alpha_\Theta A_\Theta S_\Theta \neq Y_\Theta \end{array}
$$

The benefit of this notation is that it allows us to ignore all additional scalar factors; in particular, for a non-zero scalar $\omega$,

$$
\omega \cdot O \neq O \quad \text{but} \quad \omega \cdot \hat{O} = \hat{O}.
$$

Tableau permutations can be represented as birdtracks. A graphical procedure is probably the most efficient mean to obtain this representation:
**Definition 3 (tableau permutations as birdtracks)** To construct the birtrack form of the tableau permutation \( \rho_{\Theta \Phi} \) between tableaux of the same shape (c.f. Def. 2) explicitly, write the Young tableau \( \Theta \) and \( \Phi \) next to each other, such that \( \Theta \) is to the left of \( \Phi \) and then connect the boxes in the corresponding position of the two diagrams, such as

\[
\Theta \rightarrow \begin{array}{c}
\text{boxes}
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{boxes}
\end{array} \quad \leftarrow \Phi.
\]

Then, write two columns of numbers from 1 to \( n \) next to each other in descending order; the left column represents the entries of \( \Theta \) and the right column represents the entries of \( \Phi \). Lastly, connect the entries in the left and the right column in correspondence to (46). The resulting tangle of lines is the birtrack corresponding to \( \rho_{\Theta \Phi} \) and thus determines the permutation.

As a birdtrack, \( \rho_{\Theta \Phi} \) immediately becomes a linear map in \( \text{API}(V^\otimes m) \) and as such directly relates the associated Young projectors:

\[
Y_\Theta = \rho_{\Theta \Phi} Y_\Phi \rho_{\Theta \Phi}^{-1} = \rho_{\Theta \Phi} Y_\Phi \rho_{\Phi \Theta}.
\] (47)

This property is in fact part and parcel of the very definition of Young projectors in [6, def. 5.4]. Eq. (47) demonstrates that tableaux of the same shape correspond to equivalent representations and \( \rho_{\Theta \Phi} \) is the isomorphism that seals the equivalence. For the Hermitian projection operators (c.f. sec. 5.1), eq. (47) breaks down, as is exemplified in appendix B.

It may help to illustrate this with an example: Take the equivalence pair corresponding to the Young tableaux

\[
\Theta := \begin{pmatrix} 1 & 2 \\
3 & \end{pmatrix} \quad \text{and} \quad \Phi := \begin{pmatrix} 1 & 3 \\
2 & \end{pmatrix}
\]

with

\[
Y_\Theta = \begin{array}{c}
\text{birdtrack}
\end{array} \quad \text{and} \quad Y_\Phi = \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

To find the permutation \( \rho_{\Theta \Phi} \), we connect boxes between \( \Theta \) and \( \Phi \)

\[
\begin{array}{c}
\text{boxes}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\text{boxes}
\end{array} \quad \leftarrow \begin{array}{c}
\text{boxes}
\end{array}
\]

and identify \( \rho_{\Theta \Phi} \) as \( \begin{array}{c}
\text{birdtrack}
\end{array} \). Evidently eq. (47) holds, since

\[
\begin{array}{c}
\text{birdtrack}
\end{array} \neq \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

\[
Y_\Theta = \begin{array}{c}
\text{birdtrack}
\end{array} \quad \text{and} \quad Y_\Phi = \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

\[
\begin{array}{c}
\text{birdtrack}
\end{array} \neq \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

\[
Y_\Theta = \begin{array}{c}
\text{birdtrack}
\end{array} \quad \text{and} \quad Y_\Phi = \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

\[
\begin{array}{c}
\text{birdtrack}
\end{array} \neq \begin{array}{c}
\text{birdtrack}
\end{array}.
\]

### 2.4 Cancellation rules

In [16], we established various rules designed to easily manipulate birdtrack operators comprised of symmetrizers and antisymmetrizers. Since all operators considered in this paper are of this form, the simplification rules of [16] are immediately applicable here. One of them plays a crucial role throughout this paper so we recall the result without repeating the proof:
Theorem 1 (cancellation of parts of the operator [16]) Let \( \Theta \in \mathcal{Y}_n \) be a Young tableau and \( M \in \text{API}(\text{SU}(N), V^\otimes n) \) be an algebra element. Then, there exists a (possibly vanishing) constant \( \lambda \) such that

\[
O := S_{\Theta} M A_{\Theta} = \lambda \cdot Y_{\Theta} .
\]

(52)

Note that if the operator \( O \) is non-zero, then \( \lambda \neq 0 \).

To benefit of this statement we provide some crucial criteria that allow us to identify particularly important cases of nonzero \( O \) in \( \text{API}(\text{SU}(N), V^\otimes n) \subset \text{Lin}(V^\otimes n) \):

1. Let \( A_{\Phi_i} \supset A_{\Theta} \) and \( S_{\Phi_j} \supset S_{\Theta} \) be (anti-) symmetrizers that can be absorbed into \( A_{\Theta} \) and \( S_{\Theta} \) for every \( i \in \{1, 3, \ldots, k - 1\} \) and for every \( j \in \{2, 4, \ldots, k\} \). If \( M \) in (52) is of the form

\[
M = A_{\Phi_1} S_{\Phi_2} A_{\Phi_3} S_{\Phi_4} \cdots A_{\Phi_{k-1}} S_{\Phi_k} ,
\]

then \( O \) is non-zero unless \( Y_{\Theta} \) is dimensionally zero.

2. Let \( \Theta, \Phi \in \mathcal{Y}_n \) be two Young tableaux with the same shape and construct the permutations \( \rho_{\Theta\Phi} \) and \( \rho_{\Phi\Theta} \) between the two tableaux according to Definition 3. Furthermore, let \( D_{\Phi} \) be a product of symmetrizers and antisymmetrizers which can be absorbed into \( S_{\Phi} \) and \( A_{\Phi} \) respectively. If \( M \) in (52) is of the form

\[
M = \rho_{\Theta\Phi} D_{\Phi} \rho_{\Phi\Theta} ,
\]

then \( O \) is non-zero unless \( Y_{\Theta} \) is dimensionally zero.

3. If \( M \) is a product of expressions of the form of (53) and (54), then \( O \) is non-zero unless \( Y_{\Theta} \) is dimensionally zero.

The general proof of these statements can again be found in [16], but it is apparent that under the conditions listed for the ingredients of (53) and (54), any dimensional zero of \( O \) manifests itself as a dimensional zero of \( Y_{\Theta} \) since \( A_{\Theta} \) automatically contains the longest antisymmetrizer in \( O \).

As an example, consider the operator

\[
O := \begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 & 2
\end{array} = \{S_{125}, S_{34}\} \cdot \{A_{13}\} \cdot \{S_{12}, S_{34}\} \cdot \{A_{13}, A_{24}\} .
\]

(55)

This operator meets all conditions of the above Theorem 1: the sets \( \{S_{125}, S_{34}\} \) and \( \{A_{13}, A_{24}\} \) together constitute the birdtrack of a Young projection operator \( \bar{Y}_{\Theta} \) corresponding to the tableau

\[
\Theta := \begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 & & \end{array} .
\]

(56)

The set \( \{A_{13}\} \) corresponds to the ancestor tableau \( \Theta_{(2)} \), and the set \( \{S_{12}, S_{34}\} \) corresponds to the ancestor tableau \( \Theta_{(1)} \) and thus can be absorbed into \( A_{\Theta} \) resp. \( S_{\Theta} \), c.f. eq. (19). Hence \( O \) can be written as

\[
O = S_{\Theta} A_{\Theta_{(2)}} S_{\Theta_{(1)}} A_{\Theta} .
\]

(57)

According to the Cancellation-Theorem 1, we may cancel the wedged ancestor sets \( A_{\Theta_{(2)}} \) and \( S_{\Theta_{(1)}} \) at the cost of a non-zero constant \( \lambda \),

\[
O = \lambda \cdot \begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 & & \end{array} = \bar{Y}_{\Theta} ,
\]

(58)

which is proportional to \( Y_{\Theta} \).
3 Young projection and transition operators over $V^\otimes m$ for small $m$: an inspiration for a multiplet adapted basis for $\text{API}(\text{SU}(N), V^\otimes m)$

The group theoretical interest in Young operators is that they project onto irreducible representations. They satisfy the following three properties:

1. Young projection operators are idempotent, that is they satisfy\(^5\)

$$Y_{\Theta} \cdot Y_{\Theta} = Y_{\Theta}. \quad (59a)$$

They are mutually orthogonal as projectors: if $\Theta$ and $\Phi$ are two distinct Young tableaux in $Y_m$, then

$$Y_{\Theta} \cdot Y_{\Phi} = 0 \quad \text{for } m = 1, 2, 3, 4 \quad (59b)$$

and for all $m$ if $\Theta$ and $\Phi$ have a different shape.

2. The set of Young projection operators for $\text{SU}(N)$ over $V^\otimes m$ sum up to the identity element of $V^\otimes m$:

$$\sum_{\Theta \in Y_m} Y_{\Theta} = \text{id}_{V^\otimes m} \quad \text{for } m = 1, 2, 3, 4. \quad (59c)$$

In physics parlance this constitutes a completeness relation.

The Young tableaux underlying the projection operators fully classify the irreducible representations of $\text{SU}(N)$ over $V^\otimes m$ for any $m$. If $m \leq 4$, the Young projectors split the space $V^\otimes m$ into mutually orthogonal subspaces, which can be shown to be irreducible [6]. For $m \geq 4$, generalizations of Young projectors take over this role. Such generalizations include subtracted operators [20, 21] and Hermitian Young projection operators [1, 13]. In this paper, we will focus on the latter.

Besides Young projection operators not being pairwise orthogonal for $m \geq 5$ as linear maps, they are not orthogonal with respect to the scalar product on $\text{API}(V^\otimes m)$ even for smaller $m$, for example

$$\text{tr}\left(Y_{\Theta}^+ Y_{\Phi}\right) \neq 0 \quad (60)$$

as emerges quickly from an explicit calculation:

$$\text{tr}\left(Y_{\Theta}^+ Y_{\Phi}\right) = \text{Tr}\left(\left(\frac{4}{3}\right)^2 \begin{array}{cccc} \Theta & \Xi & \Xi & \Xi \\ \Xi & -2 \cdot \Xi & \Xi & \Xi \\ \Xi & \Xi & -2 \cdot \Xi & \Xi \\ \Xi & \Xi & \Xi & -2 \cdot \Xi \end{array}\right)$$

$$= \frac{1}{9} \text{Tr}\left(\begin{array}{cccc} \Xi & -2 \cdot \Xi & \Xi & \Xi \\ -2 \cdot \Xi & \Xi & \Xi & \Xi \\ \Xi & \Xi & -2 \cdot \Xi & \Xi \\ \Xi & \Xi & \Xi & -2 \cdot \Xi \end{array}\right)$$

$$= -\frac{N_c^3}{9} + \frac{N_c}{9} \neq 0.$$

This mishap is only possible since the Young projectors are not Hermitian – otherwise their orthogonality as projectors would create a zero automatically at least for $m \leq 4$.

\(^5\)This is surprisingly hard to demonstrate unless you have access to the cancellation rule Theorem 1.

\(^6\)This is truly an issue with Hermiticity, not a consequence of choosing an unsuitable scalar product: The sets of left and right eigenvectors differ if $Y_{\Theta}$ is not Hermitian.
Nevertheless the trace of $Y_\Theta$ corresponding to $\Theta \in \mathcal{Y}_m$, normalized as a projector, uncovers the dimension of the associated irreducible representation (see [12, app. B4.] for a textbook exposition):
\[
\text{tr}(Y_\Theta) = \text{dim}(\Theta) \quad \text{for all } m.
\] (61)

As is evident from Fig. 1, $|\mathcal{Y}_m|$, the number of Young tableaux in $\mathcal{Y}_m$, is smaller than the dimension of $\text{API}(\text{SU}(N), V^{\otimes m})$, which is $m!$ (up to dimensional zeros).

### 3.1 Transition operators for Young projectors over $V^{\otimes m}$ for $m \leq 4$

Generally, the established goal of representation theory is to find a set of operators that satisfy idempotency, orhthogonality and decomposition of unity, nothing more, nothing less, and for $m \leq 4$, Young projection operators do just that.

We step beyond this point by noting that there is an additional set of linearly independent operators in $\text{API}(\text{SU}(N), V^{\otimes m})$ for $m \leq 4$ that are closely related to the set of Young projectors and that complete it to a basis of the full algebra in a transparent way. Recall that for any pair of equivalent representations corresponding to tableaux $\Theta$ and $\Phi$ in $\mathcal{Y}_m$, there exist a unique tableau permutations $\rho_{\Theta\Phi}$ such that $Y_\Theta = \rho_{\Theta\Phi} Y_\Phi \rho_{\Phi\Theta}$ (c.f. eq. (47)). From this define transition operators $T_{\Theta\Phi} := \rho_{\Theta\Phi} Y_\Phi = Y_\Theta \rho_{\Theta\Phi} Y_\Phi$ for $m = 1, 2, 3, 4$ (62) and observe that they seamlessly extend the multiplication table of the Young projectors:

\[
\begin{align*}
Y_\Theta T_{\Theta\Phi} &= T_{\Theta\Phi} = T_{\Phi\Theta} Y_\Phi & (63a) \\
T_{\Theta\Phi} T_{\Phi\Theta} &= Y_\Theta & (63b) \\
T_{\Phi\Theta} T_{\Theta\Phi} &= Y_\Phi & (63c)
\end{align*}
\]

We see that $T_{\Theta\Phi}$ maps the image $Y_\Phi(V^{\otimes m})$ bijectively onto $Y_\Theta(V^{\otimes m})$ ($m \leq 4$). The inverse on the images is $T_{\Phi\Theta}$. The $T_{\Theta\Phi}$ are transition operators between the irreducible representations. An example of all Young projection and transition operators over $V^{\otimes 3}$ is given in section 3.3.

Note that in general the transition operators between Young projectors are not unitary on the subspaces, $$(T_{\Theta\Phi})^\dagger = Y_\Phi^\dagger \rho_{\Phi\Theta} Y_\Theta^\dagger \neq T_{\Phi\Theta},$$ (64) since the Young projection operators are not Hermitian (for an explicit example see app. C).

### 3.2 A multiplet adapted basis for $\text{API}(\text{SU}(N), V^{\otimes m})$

In this section we prove that the set of all mutually orthogonal projection operators corresponding to irreducible representations of $\text{SU}(N)$ over $V^{\otimes m}$ and their transition operators – call this set $\mathcal{S}_m$ – spans the algebra of invariants $\text{API}(\text{SU}(N), V^{\otimes m})$. This proof holds for all $m$ allowing us to construct $\mathcal{S}_m$: for Young projection operators, this means that $m \leq 4$. Later on (section 4), we see that $\mathcal{S}_m$ can be constructed for all $m$ if Hermitian Young projection operators are used.

The projection operators corresponding to irreducible representations of $\text{SU}(N)$ over $V^{\otimes m}$ project onto equivalent irreducible representations if and only if the corresponding Young tableaux have the same shape, [6, 12], and thus correspond to the same underlying Young diagram. Suppose now that a particular Young diagram

14
Young tableaux. Then, the set of all projection operators corresponding to these $l$ tableaux and all transition operators between them – let us denote this set by $\mathcal{S}_Y$ – will be of size $l^2$,

$$|\mathcal{S}_Y| = l^2 ,$$

(65)

since one may always arrange the elements of $\mathcal{S}_Y$ into an $l \times l$ matrix which has the projection operators on the diagonal and each off-diagonal element in position $ij$ is the transition operator between the diagonal elements $ii$ and $jj$. Fortunately, there is a way of counting how many Young tableaux can be obtained from a Young diagram with a particular shape, namely via the hook length $H_Y$ [14, 15], c.f. eq. (40): If $Y$ is a particular Young diagram, then the set $\mathcal{S}_Y$ has size $(m!/H_Y)^2$ [22],

$$|\mathcal{S}_Y| = \left( \frac{m!}{H_Y} \right)^2 .$$

(66)

If we sum the $|\mathcal{S}_Y|$ over all Young diagrams $Y$ consisting of $m$ boxes, we obtain the aggregate number of all projection and transition operators associated with $\text{SU}(N)$ over $V^\otimes m$ $|\mathcal{S}_m|$,.

$$|\mathcal{S}_m| = \sum_Y |\mathcal{S}_Y| = \sum_Y \left( \frac{m!}{H_Y} \right)^2 .$$

(67)

To proceed further, we need to use some well established facts of the representation theory of the permutation group of $m$ elements, $S_m$, which can be found in many standard textbooks such as [23]. To this end, let us briefly recapitulate: Each irreducible representation of $S_m$ corresponds to a Young diagram $Y$ (not a Young tableau!), and the multiplicity of each representation in the regular representation of the symmetric group is given by $m!/H_Y$. From representation theory of finite groups (such as $S_m$), it is further known that the sum of the square of the multiplicities of all irreducible representations of a finite group $G$ is equal to the size of the group. In particular, for the finite group $S_m$, this means that

$$|S_m| = \sum_Y \left( \frac{m!}{H_Y} \right)^2 ,$$

(68)

where we sum over all Young tableaux $Y$ consisting of $m$ boxes. (For a bijective proof of eq. (68) see [15].) However, (67) tells us that the sum on the right hand side of equation (68) also represents the aggregate number of all Hermitian Young projection and transition operators of $\text{SU}(N)$ over $V^\otimes m$, so that

$$|S_m| = |\mathcal{S}_m| .$$

(69)

Provided that $N \geq m$ so that dimensional zeroes are absent, the projection and transition operators in $\mathcal{S}_m$ are all linearly independent (see app. A for the general case). It follows that these operators span the algebra of invariants over $V^\otimes m$, and thus constitute an alternative basis of this algebra,

$$\text{API} (\text{SU}(N), V^\otimes m) = \left\{ \alpha_k g_k | \alpha_k \in \mathbb{R}, g_k \in \mathcal{S}_m \right\} .$$

(70)

### 3.3 An example: the full algebra over $V^\otimes 3$ in a Young projector basis

$\text{API} (\text{SU}(N), V^\otimes 3)$ is spanned by the primitive invariants

$$\{ \Xi, \Xi, \Xi, \Xi, \Xi, \Xi \} \subset \text{Lin} (V^\otimes 3) .$$

(71)

Note that one often finds the statement that “the number of tableaux corresponding to a diagram is given by the hook length.” It would be less misleading to state that it is a function of the hook length.
If $N \geq 3$, its dimension is $3! = 6$. There are 3 Young diagrams consisting of 3 boxes, which give rise to a total of 4 Young tableaux,

\[
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
1 \ 2
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\begin{array}{ccc}
\begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\] (72)

Indeed, we find that the sum of the squares of $3! / \mathcal{H}_Y$, corresponding to each diagram $Y_i$ is equal to the size of the group $S_3$,

\[
3! = |S_3| = \sum_{Y_i} \left( \frac{3!}{\mathcal{H}_Y} \right)^2 = 1^2 + 2^2 + 1^2.
\] (73)

The first and last Young tableaux have a unique shape and thus project onto unique irreducible representations of SU($N$). The corresponding Young projection operators are given by

\[
Y_1 = \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array}
\text{ and } Y_4 = \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array},
\] (74)

where $Y_i$ corresponds to the $i^{th}$ tableau (read from left to right) in (72). The central two tableaux of (72) stem from the same Young diagram. Thus, their corresponding projection operators project onto equivalent irreducible representations; there must therefore exist two transition operators between them. The projection operators $Y_2$ and $Y_3$ are

\[
Y_2 = \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array}
\text{ and } Y_3 = \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array},
\] (75)

and the transition operators $T_{ij}$ between $Y_i$ and $Y_j$ are

\[
T_{23} = Y_2 \rho_{23} = \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array}
\text{ and } T_{32} = Y_3 \rho_{32} = \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array},
\] (76)

in accordance with eq. (62). Arranging all projection and transition operators in a matrix $\mathcal{M}$ where the diagonal elements $m_{ii}$ are projection operators and each off-diagonal element $m_{ij}$ is the transition operator between $m_{ii}$ and $m_{ij}$, one obtains the following matrix of operators,

\[
\mathcal{M} = \begin{pmatrix}
0 & \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array} & 0 & 0 \\
0 & \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array} & 0 \\
0 & \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array} & 0 & \begin{array}{c}
\begin{array}{c}
1 \ 3
\end{array}
\end{array} \\
0 & 0 & 0 & \begin{array}{c}
\begin{array}{c}
1 \ 2 \ 3
\end{array}
\end{array}
\end{pmatrix},
\] (77)

where all projection operators are highlighted in blue.

Notice that the operator in the $1 \times 1$-block in the bottom right corner contains an antisymmetrizer of length 3. Thus, for $N \leq 2$, this operator is a null-operator and only the remaining two blocks are non-trivial in the above matrix, c.f. app. A. If $N \leq 1$, also the central $2 \times 2$-block vanishes as it contains antisymmetrizers of length 2.
4 Orthogonal projector bases

As we have seen in section 3, the Young projection and transition operators provide a basis for the algebra of invariants of SU(N) over $V^\otimes m$ provided $m \leq 4$. Due to a lack of pairwise orthogonality and completeness of the Young projection operators beyond this point, c.f. eqs. (59b) and (59c), the Young basis cannot be generalized to larger $m$. This motivates a basis in terms of Hermitian Young projection operators, since these are orthogonal and complete for all values of $m$ [13].

We re-state the most important aspects of Hermitian Young projection operators in section 4.1, before discussing transition operators between Hermitian projections in sections 4.2 (in terms of Clebsch-Gordan operators) and 5 (between Hermitian Young projection operators).

Section 4.3 discusses the multiplication table of the basis of the algebra of invariants of SU(N) over $V^\otimes m$ in terms of Hermitian projectors and their corresponding transition operators.

4.1 Hermitean projection operators

If we replace the Young projectors $Y_\Theta$ by their (more complicated) Hermitian counterparts $P_\Theta$, either following Keppeler and Sjödahl [13, 24, 25] or our own improved versions thereof [1, 16], the group theoretically important features of Young projection operators now apply for all values of $m$ [13]:

1. The Hermitian Young projection operators are idempotent and mutually orthogonal as projectors: for any two Young tableaux $\Theta$ and $\Phi$ in $\mathcal{Y}_m$ they satisfy

$$ P_\Theta \cdot P_\Theta = \delta_{\Theta\Phi} P_\Theta \quad \text{for all } m . $$

2. They provide a complete decomposition of unity on $V^\otimes m$

$$ \sum_{\Theta \in \mathcal{Y}_m} P_\Theta = \text{id}_{V^\otimes m} \quad \text{for all } m $$

into irreducible representations.

3. Unlike their Young counterparts, they are Hermitian

$$ P_\Theta^\dagger = P_\Theta . $$

Due to this new property (Hermiticity) several new features appear [1]:

4. The projectors in the descendant set $\{ \Theta \otimes \underline{m} \}$ of any $\Theta \in \mathcal{Y}_{m-1}$ sum to the parent projector thus augmenting the single identity (78b) by a whole nested set of inclusion sums or partial completeness statements [1]:

$$ \sum_{\Phi \in \{ \Theta \otimes \underline{m} \}} P_\Phi = P_\Theta . $$

This can be generalized further,

$$ \sum_{\Phi \in \{ \Theta \otimes \underline{k} \otimes \cdots \otimes \underline{m} \}} P_\Phi = P_\Theta \quad \text{for } \Theta \in \mathcal{Y}_{k-1}, \Phi \in \mathcal{Y}_m \text{ and } k < m . $$

The constructions used in this paper are summarized in section 5.1.
5. Unlike their conventional Young counterparts $Y_\Theta$ (or the corrected Littlewood-Young operators $L_\Theta$, see [1, 20]), the Hermitian Young projectors $P_\Theta$ are automatically orthogonal with respect to the scalar product on $\text{API}(\text{SU}(N), V^\otimes m)$

\[
\langle P_\Theta, P_\Phi \rangle = \text{tr} \left( P_\Theta^\dagger P_\Phi \right) = \text{tr} (P_\Theta P_\Phi) = 0 \quad \text{for } \Theta \neq \Phi
\] (78c)

Both of these properties hinge crucially on Hermiticity – standard (Littlewood-) Young projectors do not share them.

The most direct way to achieve this is in terms of Clebsch-Gordan operators, and this will immediately ensure that operators satisfying (78) exist. This method remains computationally expensive and keeping $N$ a parameter appears a hopeless task, sec. 4.2. In section 5 we give an effective construction of transition operators for Hermitian Young projection operators.

4.2 A full orthogonal basis for $\text{API}(\text{SU}(N), V^\otimes m)$ via Clebsch-Gordan operators

To make this explicit, consider a general Clebsch-Gordan operator $C_{\lambda; j_1, j_2, \ldots, j_n}$ that implements the projection and basis change from a product of irreducible representations labelled by $j_1, \ldots, j_n$ (with states labeled by $m_1, \ldots, m_n$) into an irreducible representation labelled by $\lambda$ (where, of course, $\lambda$ stands in for a tableau $\Theta$ and with states labeled by $\kappa$) [6]

\[
l_{\kappa; j_1, j_2, \ldots, j_n} = \lambda, \kappa\langle j_1, m_1 | j_2, m_2 | \ldots | j_n, m_n \rangle =: C_{\lambda; j_1, j_2, \ldots, j_n}
\] (79)

the part marked as $C_{\lambda; j_1, j_2, \ldots, j_n}$ is the usual Clebsch-Gordan coefficient and the diagram on the right is the birdtrack representation of this operator (c.f. [12]). Since we are interested only in products of the fundamental representation acting on $V^\otimes n$ (so that the $j_i$ all refer to this one representation) we can suppress the corresponding label, but we must retain $\lambda$ to reference a specific irreducible representation contained in this product. Accordingly, we simplify notation in the birdtrack spirit by removing the redundant indices according to

\[
C_{\lambda; j_1, j_2, \ldots, j_n} = _{\lambda, \kappa\langle j_1, m_1 | j_2, m_2 | \ldots | j_n, m_n \rangle =: C_{\lambda, n} := \lambda
\] (80)

By its very nature, the Clebsch-Gordan operator translates a product representation into its irreducible sub-blocks labeled by $\lambda$, i.e.

\[
\lambda \rightarrow U^\dagger_{(\lambda)} \rightarrow \lambda
\] (81)

for all $U \in \text{SU}(N)$ and $U_{(\lambda)}$ in the $\lambda$ representation of SU(N). Orthonormality of the new states,

\[
\kappa \rightarrow \lambda \rightarrow \kappa' = \delta_{\lambda, \lambda'} \delta_{\kappa, \kappa'}
\] (82)
allows us to cast projection operators in the form

\[ P_\lambda := \sum_\kappa |\lambda, \kappa\rangle \langle \lambda, \kappa| = C^\dagger_{\lambda, n} \cdot C_{\lambda, n} \]

which are clearly mutually orthogonal,

\[ P_\lambda P_{\lambda'} = \delta_{\lambda, \lambda'} P_\lambda \]  \hspace{1cm} (84)

Equation (83) also introduces the birdtrack notation of \( C^\dagger_{\lambda, n} \), the Hermitian conjugate of \( C_{\lambda, n} \) in Eq. (79). The \( P_\lambda \) are mutually orthogonal elements of the algebra of primitive invariants \( \text{API}(\text{SU}(N), V \otimes n) \) due to eq. (81):

\[ U^\dagger U = I \]  \hspace{1cm} (85)

and general theory assures us that these yield projectors on all irreducible subspaces contained in \( V \otimes n \) [6].

From the perspective of Clebsch-Gordan operators there are obvious candidates for transition operators: When two equivalent representations \( \lambda \) and \( \lambda' \) are isomorphic, one can choose the states \( |\lambda, \kappa\rangle \) and \( |\lambda', \kappa\rangle \) such that the representation matrices are identical, \( U_{(\lambda')} = U_{(\lambda)} \). This allows us to identify the transition operators

\[ T_{\lambda' \lambda} := \sum_\kappa |\lambda', \kappa\rangle \langle \lambda, \kappa| = C^\dagger_{\lambda', n} \cdot C_{\lambda, n} \]

as additional algebra elements, since, with this particular basis choice, they also are invariant:

\[ T_{\lambda' \lambda} = T_{\lambda \lambda'} \]  \hspace{1cm} (87)

Unlike the projection operators \( P_\lambda \) the transition operators \( T_{\lambda' \lambda} \) are clearly not Hermitian. From their definition in terms of states (86) it however follows that they are unitary (on the subspaces corresponding to \( \lambda \) and \( \lambda' \)),

\[ (T_{\lambda' \lambda})^\dagger = T_{\lambda \lambda'} \]  \hspace{1cm} (88)

These operators in fact define the isomorphisms that in the standard perspective allow us to claim equivalence between the two representations in the first place. Our point here is that these isomorphisms are elements of \( \text{API}(\text{SU}(N), V \otimes n) \).

The totality of all projection and transition operators (83) and (86) obviously exhausts the algebra of invariants due to the completeness of Clebsch-Gordan operators; the transition operators (86) provide all the missing basis elements, which fixes their total number. This matches with the counting arguments of section 3 and seamlessly fits into the multiplication pattern of closed subalgebras discussed in sec. 4.3.

Note that this procedure leads to a particular realization of projectors and associated transition operators, any other equivalent construction may produce results that differ by a similarity transformation as discussed
in section 4.3.\textsuperscript{9} As a means to find a basis in a practical calculation this procedure is exceedingly inefficient: It relies on finding a total of $N^n$ normalized states in $V^\otimes n$ as a stepping stone to produce $n!$ basis states for $API(SU(N), V^\otimes n)$ while keeping $N \geq n$ to avoid dimensional zeroes.\textsuperscript{10} It clearly is not the most efficient option to achieve this goal, in particular if one aims to keep $N$ as a parameter. Therefore we use the Clebsch-Gordan method as a proof of concept and abstract the main features of the resulting basis as the goalposts for a more efficient construction to be presented in sec. 5.

We observe:

1. $T_{\Theta\Phi}$, as a map from $V^\otimes n$ to $V^\otimes n$, projects onto the image of $P_\Phi$ and maps that surjectively onto the image of $P_\Theta$,

$$T_{\Theta\Phi} P_\Phi = T_{\Theta\Phi} = P_\Theta T_{\Theta\Phi}, \quad (89)$$

It thus can be considered a map from the image of $P_\Phi$, $P_\Phi(V^\otimes n)$, to the image of $P_\Theta$, $P_\Theta(V^\otimes n)$.

2. $T_{\Theta\Phi}^\dagger$ is the right inverse of $T_{\Theta\Phi}$ on $P_\Theta(V^\otimes n)$,

$$T_{\Theta\Phi} T_{\Theta\Phi}^\dagger = P_\Theta. \quad (90)$$

3. $T_{\Theta\Phi}^\dagger$ is the left inverse of $T_{\Theta\Phi}$ on $P_\Phi(V^\otimes n)$,

$$T_{\Theta\Phi}^\dagger T_{\Theta\Phi} = P_\Phi. \quad (91)$$

We see that $T_{\Theta\Phi}$ maps the image $P_\Phi(V^\otimes n)$ bijectively onto $P_\Theta(V^\otimes n)$ – the inverse is $T_{\Phi\Theta} = T_{\Theta\Phi}^\dagger$. The $T_{\Theta\Phi}$ are unitary transition operators between the irreducible representations.

These properties are sufficient to uniquely characterize the $T_{\Theta\Phi}$. The argument for uniqueness follows a similar pattern as the uniqueness proof for inverses in a group.

Alternatively one may cast the statements 2 and 3 as

$$2'. \quad T_{\Theta\Phi}^\dagger = T_{\Phi\Theta}$$

$$3'. \quad T_{\Theta\Phi} T_{\Phi\Theta} = P_\Theta$$

This is the form we use in the definition:

**Definition 4 (unitary transition operators)** Let $\Theta, \Phi \in Y_n$ be two Young tableaux with the same underlying Young diagram, and let $P_\Theta$ and $P_\Phi$ be their respective Hermitian Young projection operators. Then, the operator $T_{\Theta\Phi}$ satisfying the following three properties is called the transition operator between $P_\Theta$ and $P_\Phi$.

$$T_{\Theta\Phi} P_\Phi = T_{\Theta\Phi} = P_\Theta T_{\Theta\Phi} \quad (92a)$$

$$T_{\Theta\Phi}^\dagger = T_{\Phi\Theta} \quad (92b)$$

$$T_{\Theta\Phi} T_{\Phi\Theta} = P_\Theta \quad (92c)$$

\textsuperscript{9}We will see that this similarity transform leaves the block-structure of the associated matrix $M$ invariant, sec. 4.3 and refined below in eq. (100).

\textsuperscript{10}This forces us into the domain where $N^n \geq n^n > n!$. There will always be more states than multiplets.
4.3 The multiplication table of the algebra of invariants

If we look at a given Young diagram $Y_i$, then the set of all projection and transition operators corresponding to tableaux with shape $Y_i$, $\Theta_i$, forms a closed subalgebra of $\text{API}(\text{SU}(N), V^\otimes m)$. Its multiplication table is given by eqns. (84) and (92) and evidently decouples from the rest of the algebra. A simple relabelling allows to condense these equations into a single one (see eq. (94b) below). To do so, form a matrix pattern in which the projection operators are placed on the diagonals, such that

$$m_{ii} = P_{\Theta_i} \quad \text{for all } \Theta_i \in \mathcal{Y}_m \text{ with underlying diagram } Y_i$$  \hspace{1cm} (93a)

and populate the off-diagonal sites with the transition operators, such that

$$m_{ij} = T_{\Theta_i \Theta_j} \quad \text{for all } \Theta_i, \Theta_j \in \mathcal{Y}_m \text{ with underlying diagram } Y_i.$$  \hspace{1cm} (93b)

Calling the matrix of elements for this subalgebra $\mathfrak{M}_{Y_i}$, we can then assemble all such blocks into a matrix pattern of independent closed subalgebras by placing the blocks along the diagonal while filling the remainder with zeroes (this makes sense since the “transition operators” between projectors belonging to different blocks should be zero). Schematically we obtain

$$\mathfrak{M} = \begin{pmatrix}
\mathfrak{M}_{Y_1} & & \\
& \mathfrak{M}_{Y_2} & \\
& & \mathfrak{M}_{Y_i} \\
& & \\
& & \mathfrak{M}_{Y_k}
\end{pmatrix}.$$  \hspace{1cm} (94a)

This matrix once again illustrates the fact that the sum of all projection operators and transition operators of $\text{SU}(N)$ must be a sum of squares (c.f. eq. (67)), as $\mathfrak{M}$ is block diagonal. It is clear that the matrix elements $m_{ij}$ of $\mathfrak{M}$ in (94a) satisfy the property

$$m_{ij} m_{kl} = \delta_{jk} m_{il},$$ \hspace{1cm} (94b)

Eq. (94b) (together with the statement which of the $m_{ij}$ are zero) is probably the most compact form to encode both eqns. (84) and (92) simultaneously. In the new notation, we have

$$\text{API}(\text{SU}(N), V^\otimes m) = \left\{ \alpha_{ij} m_{ij} | \alpha_{ij} \in \mathbb{R}, m_{ij} \in \mathcal{S}_m \right\}$$ \hspace{1cm} (94c)

where the sum is over all $i, j \in \{1, \ldots, m\}$ and $\mathcal{S}_m$ once again denotes the set of all projection and transition operators corresponding to the irreducible representations of $\text{SU}(N)$ over $V^\otimes m$. This basis has the advantage that dimensional zeroes manifest themselves directly as zeroes of the basis elements: if a dimensional zero appears at $N < m$ it affects whole equivalence blocks. All operators in the block turn into null-operators simultaneously, c.f. appendix A or section 6 for an explicit example. No additional dimensional zeroes can arise from linear combinations of the remaining basis elements.

In particular, the product (94b) yields a non-zero result only if either $i = j = k = l$ (squaring a projection operator), or if we multiply a diagonal element with an off-diagonal element of the same block in the correct
order. In fact, (94b) is the structure of the multiplication table of the multiplet basis for API(SU(N), V⊗m), and even for API(SU(N), V⊗m ⊗ (V*)⊗m′).

The multiplication table (94b) is clearly more structured than that of the primitive invariant basis of API(SU(N), V⊗m), which is directly the multiplication table of S_m:

\[ \rho_i \rho_j = A_{ij}^k \rho_k . \]  

This has consequences: The simpler structure of (94b) also gives access to the uniqueness of the operators m_{ij} appearing in it. While the types and equivalence patterns (the block structure) of irreducible representations contained in API(SU(N), V⊗m), or API(SU(N), V⊗m ⊗ (V*)⊗m′) are uniquely determined by N, m and m′ the operators themselves are not uniquely determined by the multiplication table and decomposition of unity representations alone, if the size of the block it falls into is greater than one.

This can be seen as follows: Since the block structure is fixed, two realizations \( M \) and \( \tilde{M} \) of bases with the same block structure must satisfy

\[ m_{ij} m_{kl} = \delta_{jk} m_{il} \quad \text{and} \quad \tilde{m}_{ij} \tilde{m}_{kl} = \delta_{jk} \tilde{m}_{il} \]  \hspace{1cm} (96)

and be related by a general real linear transformation as

\[ \tilde{m}_{ij} := a_{\alpha\beta} m_{\alpha\beta} b_{\beta j} \]  \hspace{1cm} (97)

This implies that (note that the \( m \) are operators, while the \( a_{ij} \) and \( b_{ij} \) are real coefficients that commute with the \( m \))

\[ \tilde{m}_{ij} \tilde{m}_{kl} = a_{\alpha\beta} m_{\alpha\beta} b_{\beta j} a_{\gamma\lambda} m_{\gamma\lambda} b_{\lambda i} = a_{\alpha\beta} m_{\alpha\beta} b_{\lambda i} \left( a_{\kappa\gamma} m_{\kappa\gamma} b_{\gamma j} \right) = \delta_{kj} \tilde{m}_{il}, \quad \text{i.e.} \quad b = a^{-1} . \]  \hspace{1cm} (98)

With this constraint \( a \) must have the same block structure as both \( M \) and \( \tilde{M} \).

Everything we said up to this point also holds for a basis of Young projection and transition operators over API(SU(N), V⊗m) provided \( m \leq 4 \). The main disadvantage (besides being restricted to small \( m \)) of this basis remains that it is not orthogonal under the standard scalar product on API(SU(N), V⊗m) provided by \( \langle A, B \rangle = \text{tr} (A^\dagger B) \), as is exemplified in (60).

If we populate the subalgebra listing \( M \) with Hermitian projectors and their unitary transition operators results in an orthogonal basis:

\[ (m_{ij}, m_{kl}) = \text{tr} \left( m_{ij}^\dagger m_{kl} \right) \overset{(92b)}{=} \text{tr} (m_{ji} m_{kl}) \overset{(94b)}{=} \delta_{ik} \text{tr} (m_{jl}) = \delta_{ik} \delta_{jl} \dim(\Theta_j) , \]  \hspace{1cm} (99)

where \( \Theta_j \) labels the representation corresponding to the projection operator \( m_{ij} \). Note that this is a general statement based purely on the multiplication table, Hermiticity and unitarity without any reference to a specific realization of the basis elements and thus automatically also applies to the basis we construct in sec. 5.

Hermiticity and unitarity also restrict the freedom to change the \( m_{ij} \) beyond what we had seen in eq. (98): Using the Hermiticity properties (\( m_{ij}^\dagger = m_{ji} \), \( \tilde{m}_{ij}^\dagger = \tilde{m}_{ji} \) and \( a_{ij} \in \mathbb{R} \) since the algebra is real) then lead to

\[ \tilde{m}_{ij}^\dagger = a_{\alpha\beta} m_{\alpha\beta} a_{\beta j}^{-1} = a_{\alpha\beta} m_{\beta\alpha} a_{\alpha l}^{-1} = (a^{-1})_j^\dagger m_{\beta\alpha} a_{\alpha l}^\dagger = \tilde{m}_{ji} , \quad \text{i.e.} \quad a^{-1} = a^\dagger , \]  \hspace{1cm} (100)

the freedom is restricted to (blockwise!) orthogonal transformations of the \( m_{ij} \).

\footnote{We have not provided transition operators for the Littlewood-Young operators, so that at this point we need to switch to Hermitian operators as soon as \( m \geq 5 \).}
5 Unitary transition operators for Hermitian Young projectors

Like the (non-unitary) transition operators for Young projectors over \(V^\otimes m\) \((m \leq 4)\) introduced in sec. 3, the unitary transition operators associated with Hermitian Young projectors are based on the projectors themselves. In the present case, the building blocks will be a set \(\{P_\Theta|\Theta \in Y_m\}\) (with the full list of properties listed in sec. 4.1), where we need not put a restriction on \(m\). We first recapitulate their ingredients in sec. 5.1 before we use them and the tableau permutations of Definition 3 to construct the transition operators in sec. 5.2.

5.1 Construction methods of Hermitian Young projection operators

At the present time, there exist three ways of constructing (completely equivalent) Hermitian Young projection operators. The first is an iterative method that goes back to Keppeler and Sjödahl (KS) and is discussed in [13]. The second method is based on the KS-algorithm but produces substantially shorter operators [1]. The third method exploits the structure of Young tableaux and their “lexical ordering” [1]. Since the second and third method are used in this paper, we summarize these two construction algorithms in the present section.

In the first method, Hermitian Young projection operators are constructed by forming a product of consecutively “older” Young projection operators:

**Theorem 2 (staircase form of Hermitian Young projectors [1])** Let \(\Theta \in Y_n\) be a Young tableau. Then, the corresponding Hermitian Young projection operator \(P_\Theta\) is given by

\[
P_\Theta = Y_{\Theta_{n-2}} Y_{\Theta_{n-3}} Y_{\Theta_{n-4}} \ldots Y_{\Theta_{(1)}} Y_{\Theta_{(2)}} Y_{\Theta_{(3)}} \ldots Y_{\Theta_{(n-4)}} Y_{\Theta_{(n-3)}} Y_{\Theta_{(n-2)}}.
\]

(101)

In the second method, one takes into account the lexical ordering of the Young tableau. In order to accomplish this we require a few more definitions.

**Definition 5 (column- & row-words and lexical ordering)** Let \(\Theta \in Y_n\) be a Young tableau. We define the column-word of \(\Theta\), \(C_\Theta\), to be the column vector whose entries are the entries of \(\Theta\) as read column-wise from left to right. Similarly, the row-word of \(\Theta\), \(R_\Theta\), is defined to be the row vector whose entries are those of \(\Theta\) read row-wise from top to bottom.

We say that a tableau \(\Theta\) is (lexically) ordered if either its row-word or its column-word (or both) is in lexical order. If we want to be more specific, we might call \(\Theta\) row-ordered resp. column-ordered.

For example, the tableau

\[
\Phi := \begin{pmatrix}
1 & 5 & 7 & 9 \\
2 & 6 & 8 \\
3 \\
4
\end{pmatrix}
\]

(102)

has a column-word

\[
C_\Phi = (1, 2, 3, 4, 5, 6, 7, 8, 9)^t,
\]

(103)

and a row-word

\[
R_\Phi = (1, 5, 7, 9, 2, 6, 8, 3, 4).
\]

(104)
Since $C_{\Phi}$ is lexically ordered, we say that $\Phi$ is a (column-) ordered tableau.

It should be noted that the above definition of the row-word is different to the definition given in the standard literature such as [14, 15] (there, the row word is read from bottom to top rather than from top to bottom). However, for the purposes of this paper, the above given definition is more useful than the standard definition.

**Definition 6 (measure of lexical disorder (MOLD))** Let $\Theta \in \mathcal{Y}_n$ be a Young tableau. We define its measure of lexical disorder (MOLD) to be the smallest natural number $M(\Theta) \in \mathbb{N}$ such that

$$\Theta(M(\Theta)) = \pi^{M(\Theta)}(\Theta)$$

is a lexically ordered tableau. (Recall from Definition 1 that $\pi^{M(\Theta)}$ refers to $M(\Theta)$ consecutive applications of the parent map $\pi$ to the tableau $\Theta$.)

We note that the MOLD of a Young tableau is a well-defined quantity, since one will always eventually arrive at a lexically ordered tableau, as, for example, all tableaux in $\mathcal{Y}_3$ are lexically ordered. This then implies that the MOLD of a tableau $\Theta \in \mathcal{Y}_n$ has an upper bound,

$$M(\Theta) \leq n - 3,$$

making it a well-defined quantity. As an example, consider the tableau $\Phi := \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 \end{pmatrix}$.

The MOLD of the above tableau is $M(\Theta) = 2$, one needs to apply $\pi$ twice to arrive at the first lexically ordered ancestor, which in this case is row ordered:

$$\Phi \mapsto \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 3 \end{pmatrix}.$$

The following construction algorithm of Hermitian Young projection operators uses the MOLD of the corresponding Young tableau.

**Theorem 3 (MOLD operators [1])** Consider a Young tableau $\Theta \in \mathcal{Y}_n$ with MOLD $M(\Theta) = m$. Furthermore, suppose that $\Theta^{(m)}$ has a lexically ordered row-word. Then, the Hermitian Young projection operator corresponding to $\Theta$, $P_{\Theta}$, is, for $m$ even,

$$P_{\Theta} = \beta_{\Theta} \cdot S_{\Theta^{(m)}} A_{\Theta^{(m-1)}} S_{\Theta^{(m-2)}} \cdots S_{\Theta^{(2)}} A_{\Theta^{(1)}} S_{\Theta^{(1)}} Y_{\Theta}^\dagger Y_{\Theta}$$

and, for $m$ odd,

$$P_{\Theta} = \beta_{\Theta} \cdot S_{\Theta^{(m)}} A_{\Theta^{(m-1)}} S_{\Theta^{(m-2)}} \cdots A_{\Theta^{(2)}} S_{\Theta^{(1)}} Y_{\Theta}^\dagger Y_{\Theta}$$

Similarly, if $\Theta^{(m)}$ has a lexically ordered column-word, $P_{\Theta}$ is given by, for $m$ even,

$$P_{\Theta} = \beta_{\Theta} \cdot A_{\Theta^{(m)}} S_{\Theta^{(m-1)}} A_{\Theta^{(m-2)}} \cdots A_{\Theta^{(2)}} S_{\Theta^{(1)}} Y_{\Theta}^\dagger Y_{\Theta}$$

and, for $m$ odd,

$$P_{\Theta} = \beta_{\Theta} \cdot A_{\Theta^{(m)}} S_{\Theta^{(m-1)}} A_{\Theta^{(m-2)}} \cdots A_{\Theta^{(2)}} A_{\Theta^{(1)}} Y_{\Theta}^\dagger Y_{\Theta}$$

In the above, all symmetrizers and antisymmetrizers are understood to be canonically embedded into the algebra over $V^\otimes n$; $\beta_{\Theta}$ is a non-zero constant chosen such that $P_{\Theta}$ is idempotent.

This construction seems complicated at first glance as four cases need to be considered. In [1] we discuss why this is necessary and how the structure of Theorem 3 can be understood.
5.2 Unitary transition operators for Hermitian Young projectors

For the standard Young projection operators, the tableau permutation \( \rho_{\Theta \Phi} \), viewed as an element of \( \text{Lin} (V \otimes m) \), directly relates any associated Young projectors:

\[
Y_{\Theta} = \rho_{\Theta \Phi} Y_{\Phi} \rho_{\Theta \Phi}^{-1}.
\]

c.f. eq. (47). For Hermitian Young projection operators, this is no longer true in general: There exist tableaux \( \Theta \) and \( \Phi \) such that

\[
P_{\Theta} \neq \rho_{\Theta \Phi} P_{\Phi} \rho_{\Theta \Phi}^{-1}.
\]

The simplest example for such a mismatch is probably the equivalence pair corresponding to the Young tableaux from eq. (48)

\[
\Theta := \begin{array}{c}
1 \\
3 \\
2
\end{array} \quad \text{and} \quad \Phi := \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

with

\[
Y_{\Theta} = \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array} \quad \text{and} \quad Y_{\Phi} = \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array}
\]

and

\[
P_{\Theta} = \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array} \quad \text{and} \quad P_{\Phi} = \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array}
\]

respectively. We recall the associated tableau permutation from eq. (50): \( \rho_{\Theta \Phi} = \Phi \Theta \). Evidently

\[
\frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array} = \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array} \quad \text{while} \quad \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array} \neq \frac{4}{3} \cdot \begin{array}{c}
\begin{array}{c}
\ast & \ast & \\
& \\
&
\end{array}
\end{array}
\]

as claimed.

However, what remains true is that

\[
P_{\Theta} \cdot \rho_{\Theta \Phi} P_{\Phi} \rho_{\Theta \Phi}^{-1} \neq 0,
\]

since all symmetrizers and anti-symmetrizers in \( \rho_{\Theta \Phi} P_{\Phi} \rho_{\Theta \Phi}^{-1} \) can be absorbed into \( S_{\Theta} \) and \( A_{\Theta} \) respectively, see part 2 (eq. (54)) in section 2.4. The fact that \( P_{\Theta} \cdot \rho_{\Theta \Phi} P_{\Phi} \rho_{\Theta \Phi}^{-1} \neq 0 \) in eq. (116) is the main ingredient that guarantees that the transition operators constructed below fulfill all necessary criteria. A more involved example illustrating the action of \( \rho_{\Theta \Phi} \) on \( P_{\Phi} \) is given in appendix B.

Here is a first version of the construction algorithm for transition operators:

**Theorem 4 (unitary transition operators)** Let \( \Theta, \Phi \in Y_n \) be two Young tableaux with the same underlying Young diagram, and let \( P_{\Theta} \) and \( P_{\Phi} \) be their respective Hermitian Young projection operators and \( T_{\Theta \Phi} \) the transition operator between them. Then, \( T_{\Theta \Phi} \) is given by

\[
T_{\Theta \Phi} = \tau \cdot P_{\Theta} \rho_{\Theta \Phi} P_{\Phi},
\]

where \( \tau \) is a non-zero constant and \( \rho_{\Theta \Phi} \in S_n \) is the permutation constructed according to Definition 3. The constant \( \tau \) is constrained by (92c) and can be determined through explicit calculation (c.f eq. (129)).

That the operator (117) defined in Theorem 4 satisfies all conditions (92) is readily seen:
\textbf{Property (92a)}, \( T_{\Theta\Phi} P_\Phi = T_{\Theta\Phi} = P_\Theta T_{\Theta\Phi} \), is easily shown: Let
\[ T_{\Theta\Phi} := \tau \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi \quad \text{with} \quad \tau \in \mathbb{R} \setminus \{0\} . \tag{118} \]
Then,
\[ T_{\Theta\Phi} \cdot P_\Phi := \tau \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi \cdot P_\Phi = \tau \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi , \tag{119} \]
since \( P_\Phi \) is a projection operator. Similarly
\[ P_\Theta \cdot T_{\Theta\Phi} := \tau \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi \cdot P_\Phi = \tau \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi . \tag{120} \]

\textbf{Property (92b)}, \( T_{\Theta\Phi}^\dagger = T_{\Phi\Theta} \):
\[ T_{\Theta\Phi}^\dagger = (P_\Theta \rho_{\Theta\Phi} P_\Phi)^\dagger = P_\Phi \rho_{\Phi\Theta}^\dagger P_\Theta = T_{\Phi\Theta} \tag{121} \]
where the last equality holds since \( \rho_{\Phi\Theta}^\dagger = \rho_{\Theta\Phi} \) is the inverse permutation of \( \rho_{\Theta\Phi} \), c.f. Definition 3.

\textbf{Property (92c)}, \( T_{\Theta\Phi} T_{\Phi\Theta} = P_\Theta \): We unpack
\[ T_{\Theta\Phi} T_{\Phi\Theta} = \tau^2 \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi \cdot P_\Phi = \tau^2 \cdot P_\Theta \rho_{\Theta\Phi} P_\Phi \rho_{\Theta\Phi}^\dagger P_\Theta . \tag{122} \]
writing \( \rho_{\Theta\Phi} \) as \( \rho_{\Theta\Phi}^\dagger \) for clarity in the steps to follow. Of the equivalent ways to express the projectors \( P_\Theta \) and \( P_\Phi \) [1, 13], we choose \( P_\Theta \) and \( P_\Phi \) to be constructed according to Theorem 2 (sec. 5.1):
\[ \frac{T_{\Theta\Phi} T_{\Phi\Theta}}{\tau^2} = \underbrace{Y_{\Theta_{(n-2)}} \cdots Y_{\Theta_{(n-2)}}}_{P_\Theta} \cdot \underbrace{Y_{\Phi_{(n-2)}} \cdots Y_{\Phi_{(n-2)}}}_{P_\Phi} \rho_{\Phi\Theta} \rho_{\Theta\Phi}^\dagger \cdot \underbrace{Y_{\Theta_{(n-2)}} \cdots Y_{\Theta_{(n-2)}}}_{P_\Theta} . \tag{123} \]
Writing each Young projection operator as a product of symmetrizers and antisymmetrizers, \( Y_{\Xi} = \alpha_{\Xi} S_{\Xi} A_{\Xi} \), eq. (123) becomes
\[ \frac{T_{\Theta\Phi} T_{\Phi\Theta}}{\tau^2 \beta^2_{\Theta\Theta} \beta^2_{\Phi}} = \underbrace{S_{\Theta_{(n-2)}} \cdots S_{\Theta_{(1)}} A_{\Theta_{(n-2)}} \cdots A_{\Theta_{(1)}}}_{P_\Theta} \cdot \underbrace{S_{\Phi_{(n-2)}} \cdots S_{\Phi_{(2)}} A_{\Phi_{(n-2)}} \cdots A_{\Phi_{(2)}}}_{P_\Phi} \cdot \underbrace{S_{\Theta_{(n-2)}} \cdots S_{\Theta_{(1)}} A_{\Theta_{(n-2)}} \cdots A_{\Theta_{(1)}}}_{P_\Theta} , \tag{124} \]
where the constants \( \beta_{\Theta} \) and \( \beta_{\Phi} \) lump together all the constants \( \alpha_{\Xi} \) appearing in \( P_\Theta \) and \( P_\Phi \) respectively. Let us now take a closer look the part of \( T_{\Theta\Phi} T_{\Phi\Theta} \) that is enclosed in a green box in (124): We notice that this part is of the form
\[ O := S_{\Theta} M^{(1)} M^{(2)} M^{(1)} A_{\Theta} , \tag{125} \]
where the $M^{(i)}$ are defined in (124). According to the Cancellation-Theorem 1, there exists a constant $\lambda$ such that

$$O = \lambda Y_\Theta .$$

(126)

Furthermore, we know that $\lambda \neq 0$, if the operator $O$ itself is non-zero. In section 2.4, we gave two conditions under which $O$ is guaranteed to be non-zero. From the definition of the $M^{(i)}$ (124), it is clear that $M^{(1)}$ satisfies the first such condition (condition 1), while $M^{(2)}$ satisfies the second condition (condition 2). Thus, a combination of the two conditions hold and $O$ is non-zero (condition 3). This implies that (126) holds for a non-zero constant $\lambda$. We may therefore simplify (124) as

$$T \Theta \Phi T = \lambda \cdot A_{\Theta \Theta(1)} A_{\Phi} S_{\Theta \Theta(1)} A_{\Theta \Theta(1)} A_{\Phi} \ldots A_{\Theta \Theta(1)} .$$

(127)

Once again writing the sets of symmetrizers and antisymmetrizers as Young projection operators, $Y_\Xi = \alpha_\Xi A_\Xi$ (where the $\alpha_\Xi$ are encoded in the constants $\beta$), the product $T_{\Theta \Phi} T_{\Phi \Theta}$ becomes

$$T_{\Theta \Phi} T_{\Phi \Theta} = \left( \tau^2 \beta_{\Theta \Phi} \lambda \right) \cdot Y_{\Theta \Theta(1)} Y_{\Theta \Theta(1)} \ldots Y_{\Theta \Theta(1)} .$$

(128)

Thus, for

$$\tau = \frac{1}{\sqrt{\beta_{\Theta \Phi} \lambda}} ,$$

(129)

the transition operator $T_{\Theta \Phi}$ also satisfies Property 3 of Definition 4.

Since $T_{\Theta \Phi}$ does indeed satisfy all properties laid out in eqs. (92), we conclude that it is the transition operator between the Hermitian Young projection operators $P_\Theta$ and $P_\Phi$.

Due to the length of the operator expressions Theorem 4 becomes inefficient very easily. We will build on this result to refine our methods in Theorem 5, which provides a more efficient way of constructing the transition operators.

Returning to our example of eq. (114) we obtain

$$T_{\Theta \Phi} = \tau \cdot A_{\Theta \Theta(1)} A_{\Phi} S_{\Theta \Theta(1)} A_{\Theta \Theta(1)} A_{\Phi} .$$

(130)

Using Theorem 1, this can be simplified to

$$T_{\Theta \Phi} = \sqrt{\frac{2}{3}} .$$

(131)

The constant $\sqrt{\frac{2}{3}}$ is determined via implementing eq. (92c). In fact, one can incorporate this simplification step directly in the construction, arriving at a general efficient algorithm.

Our description of the algorithm is based on a specific graphical convention for the birdtracks used to represent the projection operators: For any birdtrack operator we will align all sets of symmetrizers and antisymmetrizer at the top. If a particular set of symmetrizers $S_{\Theta}$ contains several symmetrizers such that each $S_i \in S_{\Theta}$ corresponds to the $i^{th}$ row of $\Theta$, then we draw $S_i$ above $S_j$ if $i < j$. A similar convention is used for antisymmetrizers corresponding to the columns of $\Theta$. 

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For birdtrack operators containing 3 index lines, we have neglected to follow this convention in two cases for consistency with the literature, for example [12]. This is remedied using the two identities

$$\begin{align*}
\begin{array}{c}
\text{\ (132)}
\end{array}
\end{align*}$$

**Theorem 5 (compact transition operators)** Let $\Theta$ and $\Phi$ be two Young tableaux of equivalent representations of $SU(N)$. They therefore have the same shape and sets of antisymmetrizers $A_\Theta$ and $A_\Phi$ are in one to one correspondence: For each element of $A_\Theta$ there exists a counterpart in $A_\Phi$ with the same length (this is important for the graphical matching described below). Let $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ be the birdtracks of two Hermitian Young projection operators constructed according to the MOLD-Theorem 3, drawn using the conventions listed in the previous paragraph. Then $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ contain $A_\Theta$ and $A_\Phi$ at least once, but at most twice. This determines how to proceed:

1. If both $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ each contain exactly one set of $A_\Theta$ respectively $A_\Phi$, then pick this set in each operator.

2. If one of $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ contains one copy of $A_\Theta$ respectively $A_\Phi$, the other contains two, then pick the left-most set $A_\Theta$ in $\tilde{P}_\Theta$ and the right-most set $A_\Phi$ in $\tilde{P}_\Phi$.

3. If both $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ each contain two sets of $A_\Theta$ respectively $A_\Phi$, then pick either the left-most set or the right-most set in both operators. (It does not matter which one, but it needs to be the same in both operators.)

Now split $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ by vertically cutting through the tower of antisymmetrizers chosen according to these rules. The next step discards everything to the right of the cut in $\tilde{P}_\Theta$ and everything to the left of the cut in $\tilde{P}_\Phi$ and glues the remaining pieces together at the cut. The resulting birdtrack is $\tilde{T}_{\Theta\Phi}$.

The proof of this Theorem is rather lengthy and thus deferred to Appendix D. This proof will also shed light on the three distinctions 1, 2 and 3 we had to make in the Theorem.

To forestall any misunderstanding about the cut, discard and glue procedures (the significance of which is discussed in appendix D.1), we will now clarify them with an example: Consider the two Hermitian Young projection operators

$$\begin{align*}
P_\Theta &= \frac{3}{2} \cdot \begin{array}{c} \text{\ (133)} \end{array} \\
P_\Phi &= 2 \cdot \begin{array}{c} \text{\ (133)} \end{array}
\end{align*}$$

corresponding to the Young tableaux

$$\begin{align*}
\Theta &= \begin{array}{c} \text{\ (134)} \end{array} \\
\Phi &= \begin{array}{c} \text{\ (134)} \end{array}
\end{align*}$$

respectively. We construct $\tilde{T}_{\Theta\Phi}$ according to the compact Theorem 5: we first split the left-most antisymmetrizer $A_{123}$ of $\tilde{P}_\Theta$ and discard everything to the right of it,

$$\begin{align*}
\tilde{P}_\Theta &= \begin{array}{c} \text{\ (135)} \end{array} \\
\quad \Rightarrow \begin{array}{c} \text{\ (135)} \end{array}
\end{align*}$$

It should be noted that this gluing can always be done, since the two Young tableaux $\Theta$ and $\Phi$ have the same shape, thus do their sets of antisymmetrizers $A_\Theta$ and $A_\Phi$, and the two sets are top-aligned.
Similarly,

\[ P_{\Phi} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}.
\end{array} \tag{136} \]

Gluing the remaining pieces together at the cut then yields

\[ \tilde{T}_{\Theta \Phi} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \; \tag{137} \]

and indeed, the transition operator \( T_{\Theta \Phi} = \sqrt{2} \tilde{T}_{\Theta \Phi} \), as can be easily checked by direct calculation.

Readers should note that one can replace antisymmetrizer sets (\( A_\Theta \) respectively \( A_\Phi \)) by symmetrizer set (\( S_\Theta \) respectively \( S_\Phi \)) in all the steps outlined in Theorem 5. This leads to the same birdtrack \( \tilde{T}_{\Theta \Theta} \) as becomes evident in the proof. Basing the procedure on antisymmetrizers however makes the discussion on “vanishing representations” in appendix A clearer.

To obtain \( T_{\Theta \Phi} = \tau \tilde{T}_{\Theta \Phi} \) one still needs to find the normalization constant \( \tau \) from direct calculation by requiring eq. (92c) to hold. The relatively compact expression are well suited for automated treatment.

\section{Examples}

\subsection*{6.1 API (SU(N), V^{\otimes 3}) – the full algebra of 3 quarks}

Revisiting the Young tableaux in \( Y_3 \) (eq. (72)),

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}.
\end{array} \tag{138} \]

Denote the Hermitian projection operator corresponding to the \( i^{th} \) tableau in (138) (read from left to right) by \( P_i \). The Hermitian projection operators corresponding to the first and last tableau in (138) are equal to the Young projection operators (74)

\[ P_1 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = Y_1 \quad \text{and} \quad P_4 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} = Y_4 , \tag{139} \]

since the Young projectors are Hermitian to begin with. The Hermitian projection operators corresponding to the central two tableaux are different from their Young counterparts (75)

\[ P_2 = \frac{4}{3} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \neq Y_2 \quad \text{and} \quad P_3 = \frac{4}{3} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \neq Y_3 , \tag{140} \]

and similarly for their transition operators \( T_{ij} \) between \( P_i \) and \( P_j \),

\[ T_{23} = \sqrt{\frac{4}{3}} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \quad \text{and} \quad T_{32} = \sqrt{\frac{4}{3}} \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} . \tag{141} \]
The birdtracks of the $\tilde{T}_{ij}$ were constructed using Theorem 5, and the constants were determined to match eq (92c). Arranging all projection operators and transition operators in a matrix $\mathfrak{M}$ as in (94a), one obtains

$$
\mathfrak{M} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

(142)

where all projection operators are highlighted in blue. The Hermitian Young projection operators in (142) were already known \cite{12}, the transition operators are a new result.

As is the case for the Young projector matrix (77), the operator in the bottom right corner in (142) becomes a null-operator for $N \leq 2$, and so does the $2 \times 2$-block for $N \leq 1$.

6.2 API ($\text{SU}(N), V^{\otimes 4}$) – the full algebra of 4 quarks

The general pattern analyzed in sec. 4.3 and once again observed in sec. 6.1 must reappear if $m$ increases. The case $m = 4$ provides additional illustration.

All Young tableaux of $\mathfrak{Y}_4$ and the Young diagrams from which they originate are

(143)

The first and last tableau each stem from a unique Young diagram and their corresponding representations thus are not equivalent to any other irreducible representation of $\text{SU}(N)$. Tableaux 2, 3 and 4 (as counted from the left) all have the same shape and therefore correspond to equivalent irreducible representations. Similarly for tableaux 5 and 6 and tableaux 7, 8 and 9.

If we arrange the Young projection operators corresponding to the tableaux in (143), as well as the transition operators in a block-diagonal matrix as was done in (94a) (with projection operators on the diagonal and transition operators on the off-diagonal), the resulting block diagonal matrix will be of the form

$$
\mathfrak{M} = \begin{pmatrix}
1 & 3 \\
3 & 2 \\
2 & 3 \\
3 & 1 \\
1 & 2 \\
\end{pmatrix},
$$

(144)
where the number in each block gives the size of the block. Indeed, we again find that

\[ 4! = |S_4| = \sum_{Y_i} \left( \frac{4!}{H_{Y_i}} \right)^2 = 1^2 + 3^2 + 2^2 + 3^2 + 1^2. \]  

(145)

Since the matrix (144) would be rather large, we will now give each block separately. The first block, consisting only of one Hermitian Young projection operator is

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix},
\]  

(146)

and corresponds to an irreducible representation of SU(N) with dimension \( d = \frac{N(N+1)(N+2)(N+3)}{24} \). The second block, a \( 3 \times 3 \)-block, is

\[
\begin{pmatrix}
\begin{array}{ccc}
\frac{1}{\sqrt{6}} & \sqrt{2} & 0 \\
\sqrt{2} & 0 & \sqrt{2} \\
\sqrt{2} & \sqrt{2} & 0
\end{array}
\end{pmatrix},
\]  

(147)

All Hermitian Young projection operators on the diagonal of this block correspond to equivalent irreducible representations of SU(N) with dimension \( d = \frac{N(N+2)(N^2-1)}{8} \). The following \( 2 \times 2 \)-block has projection operators on its diagonal that correspond to equivalent irreducible representations of dimension \( d = \frac{N(N^2-1)}{12} \),

\[
\begin{pmatrix}
\begin{array}{cc}
\frac{1}{\sqrt{3}} & \sqrt{2} \\
\sqrt{2} & -1
\end{array}
\end{pmatrix},
\]  

(148)

The next \( 3 \times 3 \) block is given by

\[
\begin{pmatrix}
\begin{array}{ccc}
\frac{1}{\sqrt{3}} & \sqrt{2} & 0 \\
\sqrt{2} & 0 & \sqrt{2} \\
\sqrt{2} & \sqrt{2} & 0
\end{array}
\end{pmatrix},
\]  

(149)

here the projection operators each correspond to irreducible representations of dimension \( d = \frac{N(N-2)(N^2-1)}{8} \). What remains is a \( 1 \times 1 \)-block:

\[
\begin{pmatrix}
1
\end{pmatrix},
\]  

(150)
This operator corresponds to an irreducible representation of dimension \(d = \frac{N(N-1)(N-2)(N-3)}{24}\).

All the projection operators given above have previously been known \([12]\). The transition operators again are a new result.

Similarly to what we observed for the 3-quark algebra, we find that the blocks described above give null-operators from bottom right to top left as we incrementally decrease \(N\) below 4: For \(N = 3\), only the last \(1 \times 1\)-block turns into a null-operator. For \(N = 2\), the last \(1 \times 1\)-block as well as the second-to-last \(3 \times 3\)-block consists of null-operators. All but the top-most \(1 \times 1\) block give null-operators for \(N = 1\). The entire matrix will (trivially) consist of null-operators if we decrease \(N\) to 0. In fact, we can read off which operators will be null-operators from their dimension formula, as \(d = 0\) for a null-operator.

Figs. 2 and 3 expand on \([12, \text{figs. 9.1 and 9.2}]\): The figures collect the hierarchy of Young tableaux and the associated nested Hermitian projector decompositions (in the sense of embeddings into \(\text{API}(\text{SU}(N), V^{\otimes 4})\)) and adds the transition operators we have derived in this paper (recall that for \(m \leq 4\) the construction algorithm for transition operators between Young projectors over \(V^{\otimes m}\) is well-defined). We would like to draw attention to the fact that only the leftmost and rightmost branches in each tree, consisting solely of a single symmetrizer or antisymmetrizer, are fully unique as they are not connected by transition operators listed on the right.

7 Conclusion & outlook

The representation theory of \(\text{SU}(N)\) is an old theory with many successful applications in physics. Yet some of the tools remain awkward and only applicable in specific situations, like the general theory of angular momentum or the construction of Young projection operators that lack Hermiticity. Newer tools like the birdtrack formalism remain only partially connected with these time honored results. We have a very specific interest in applications to QCD in the JIMWLK context, in jet physics, in energy loss and generalized parton distributions, so we have aimed at creating a set of tools that we know will aid in these applications and, in the process have pointed out where the existing tools fall short of our needs.

Building on the previously found Hermitian Young projection operators \([1, 13]\), the main result of this paper is the inclusion of transition operators that complement the set of multiplet projectors to a basis for the full algebra of invariants \(\text{API}(\text{SU}(N), V^{\otimes m})\); for Young projectors and their associated transition operators this is only possible up to \(m = 4\). Any subset of projectors encoding mutually equivalent representations together with their transition operators form closed subalgebras. Relabeling the set of basis operators as \(m_{ij}\) (with double indices according to (93)) so that

\[
\text{API}(\text{SU}(N), V^{\otimes m}) = \left\{ \alpha_{ij}m_{ij} | \alpha_{ij} \in \mathbb{R}, m_{ij} \in \mathfrak{S}_m \right\}
\]

leads to a simplified multiplication table for the new basis elements

\[
m_{ij}m_{kl} = \delta_{jk}m_{il},
\]

significantly simpler than the standard basis of primitive invariants \(\rho_i \in S_m, \rho_i\rho_j = A_{ij}^{kl}\rho_k\).

The transition operators obtained from Hermitian projectors are automatically unitary, causing the basis elements to be mutually orthogonal and normalized to match the dimension of the irreducible representations

\[
\langle m_{ij}, m_{kl} \rangle = \delta_{ik}\delta_{jl}\dim(\Theta_j).
\]
Figure 2: Hierarchy of Young tableaux and the associated (non-nested) Young projector decompositions over $V^\otimes m$ for $m = 1, 2, 3, 4$ (in the sense of embeddings into $\text{API}(\text{SU}(N), V^\otimes 4)$): The lines indicate ancestry. The associated transition operators for groups of equivalent representations are listed to the right. Note that this tree cannot be extended beyond $m = 4$ due to the failure of the corresponding Young projectors to be orthogonal or complete.
\[ d = N \]

\[ d = \frac{N(N+1)}{2} \]

\[ d = \frac{N(N-1)}{2} \]

\[ d = \frac{N(N^2-1)}{2} \]

\[ d = \frac{N(N+1)(N+2)}{24} \]

\[ d = \frac{N(N^2-1)}{12} \]

\[ d = \frac{N(N-1)(N^2-1)}{8} \]

\[ d = \frac{N(N-2)(N^2-1)}{8} \]

\[ d = \frac{N(N-3)(N^2-1)}{24} \]

\[ d = \frac{N(N+2)(N^2-1)}{8} \]

\[ d = \frac{N(N-1)(N^2-1)}{12} \]

\[ d = \frac{N(N-2)(N^2-1)}{8} \]

\[ d = \frac{N(N-3)(N^2-1)}{24} \]

\[ P_1 = \]

\[ P_1^2 = \]

\[ P_1^3 = \]

\[ P_1^{34} = \]

**Figure 3:** Hierarchy of Young tableaux and the associated nested Hermitian Young projector decompositions over \( V^\otimes m \) for \( m \leq 4 \) (in the sense of embeddings into \( \text{API}(\text{SU}(N), V^\otimes 4) \)): The arrows indicate which operators sum to which ancestors (see [1] – this does not apply to their standard Young counterparts shown in Fig. 2). The associated transition operators for groups of equivalent representations are listed to the right (recall that these transition operators are unitary on the image of the projection operators, \( T_{\Theta\Phi} = T_{\Phi\Theta}^\dagger \)). This tree can be extended to arbitrary \( m \) using the construction algorithm of Hermitian Young projectors [1] and that of transition operators given in this paper.
This is an essential prerequisite for a future publication that aims at constructing an orthonormal basis for the space of all global color singlet states for a given Fock-space configuration; it is important to note that (153) does not hold for the standard Young projectors and their associated transition operators.

We have used the new form of the multiplication table (152) to show that the projection operators in $\text{API}(\text{SU}(N),V^\otimes m)$ are only uniquely determined if the representation occurs precisely once in the decomposition. All Hermitian projectors onto equivalent representations and their associated transition operators are only unique up to orthogonal rotations as described in sec. 4.3 and 5. Figs. 2 and 3 collect all the examples worked out in this paper displaying all their relationships in a compact form for reference.

Our own list of future applications for the tools and insights presented in this paper are QCD centric: Global singlet state projections of Wilson-line operators that appear in a myriad of applications due to factorization of hard and soft contributions help analyzing the physics content in all of them. We hope that our presentation is suitable to unify perspectives provided by the various approaches to representation theory of $\text{SU}(N)$ and that the results prove useful beyond these immediate applications.

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A Dimensional zeroes

For small enough values of $N$ (and we will define what we mean by “small enough” shortly), some of the irreducible representation of $\text{SU}(N)$ over $V^\otimes m$ vanish. The reason for this is simple: An antisymmetrizer over $p$ legs is, viewed as a linear map on $V^\otimes p$, a null-operator, if $\dim(V) < p$.

$$\begin{array}{c}
\begin{array}{ccc}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\end{array}
: V^\otimes p \rightarrow 0 \quad \text{if } \dim(V) < p. \quad (154)
$$

Thus, if an operator in $\text{Lin}(V^\otimes m)$ with $\dim(V) = N$ contains an antisymmetrizer of length $> N$, the operator will be a null-operator on the space $V^\otimes m$. For example, the antisymmetrizer

$$\begin{array}{c}
\begin{array}{ccc}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\end{array}
= \frac{1}{3!} (\ldots - \Xi - \Xi - \Xi + \Xi + \Xi)
$$

acts as a null-operator on the space $V^\otimes 3 = V \otimes V \otimes V$ if the dimension of $V$ is $\leq 2$. The reason for this is that the primitive invariants constituting the antisymmetrizer $A_{123}$ as elements of $\text{Lin}(V^\otimes m)$ are not linearly independent if the vector space $V$ has dimension $\leq 2$. In this situation, the identity permutation (for example) can be expressed as a linear combination of the remaining primitive invariants,

$$\begin{array}{c}
\begin{array}{ccc}
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
= \frac{\dim(V) \leq 2}{3!} \Xi + \Xi + \Xi - \Xi - \Xi. \quad (155)
$$

We discussed previously that each irreducible representation of $\text{SU}(N)$ over $V^\otimes m$ corresponds to a particular Young tableau in $\mathcal{Y}_m$. From the construction Theorems of (Hermitian) Young projection operators (eq. (37) for Youngs and sec. 5.1 for Hermitian) and their transition operators (sec. 3 for Youngs up to $m = 4$ and sec. 5.2 for unitary operators for all $m$), it is evident that the longest antisymmetrizer present in such an
operator corresponds to the longest column of the corresponding Young tableau. In particular, if \( N < m \), there will be at least one Young tableau containing a column which is longer than \( N \), namely

\[
\begin{array}{c}
1 \\
2 \\
\vdots \\
m
\end{array}
\]

(157)

There may be more tableaux with columns longer than \( N \), depending by how much \( N \) differs from \( m \). If two (non-) Hermitian Young projection operators \( P_\Theta \) and \( P_\Phi \) (resp. \( Y_\Theta \) and \( Y_\Phi \)) correspond to equivalent irreducible representations of \( SU(N) \), they both will contain antisymmetrizers of equal length (\( A_\Theta \) or \( A_\Phi \) respectively) and so will their transition operators by construction (see eq. (62) for Young operators up to \( m = 4 \) resp. Theorem 5 for the unitary operators for all values of \( m \)). They will therefore all vanish simultaneously if \( N \) is too small.

To summarize, we see that all multiplets and transition operators are only present in \( API(SU(N), V^{\otimes m}) \) if \( N \geq m \). If \( N \) is smaller than \( m \), some of them become null-operators. These can explicitly be identified by their corresponding Young tableau \( \Theta \) or directly by \( A_\Theta \) in the birdtrack notation.

B Illustrating the action of \( \rho_{\Theta\Phi} \) on Hermitian Young projection operators: an example

In this section, we illustrate why eq. (116),

\[
P_{\Theta} \cdot \rho_{\Theta\Phi} P_{\Phi} \rho_{\Phi\Theta} \neq 0 ,
\]

(158)

holds by means of an example. In the process, we will show that eq. (47) (saying that \( Y_\Theta = \rho_{\Theta\Phi} Y_\Phi \rho_{\Phi\Theta} \)) breaks down for Hermitian projection operators,

\[
P_{\Theta} \neq \rho_{\Theta\Phi} P_{\Phi} \rho_{\Phi\Theta} .
\]

(159)

Consider two Young tableaux

\[
\Theta = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array}
\quad \text{and} \quad
\Phi = \begin{array}{ccc}
1 & 2 & 6 \\
3 & 5 \\
4
\end{array}
\]

(160)

The permutation \( \rho_{\Theta\Phi} \) as defined in Definition 3 is given by

\[
\rho_{\Theta\Phi} = \begin{array}{cccc}
1 & 3 & 5 & 2 \\
2 & 4 & 6 & 3 \\
3 & 5 & 6 & 4 \\
4 & 6 & 3 & 2
\end{array}
\]

(161)

Let us now construct the MOLD-operators (c.f. Theorem 3) corresponding to \( \Theta \) and \( \Phi \). To do so, we need to construct their MOLD-ancestries (c.f. Definition 6 for the MOLD of a tableau),

\[
\Theta = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc}
1 & 3 \\
2 & 4
\end{array}
\]

(162)
and

\[
\Phi = \begin{array}{ccc}
1 & 2 & 6 \\
3 & 5 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 6 \\
3 & 5 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 \\
3 & 4 \\
\end{array}
\]

(163)

The MOLD-projectors \( P_{\Theta} \) and \( P_{\Phi} \) are thus determined by

\[
P_{\Theta} = \begin{array}{ccc}
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
\end{array}
\]

\[
P_{\Phi} = \begin{array}{ccc}
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
\end{array}
\]

where we simplified \( \bar{P}_{\Theta} \) according to Theorem 1 in the second step. The full projection operators \( P_{\Theta} \) and \( P_{\Phi} \) require additional constants \( \beta_{\Theta} \) and \( \beta_{\Phi} \) respectively to ensure their idempotency. From the differing lengths of \( P_{\Theta} \) and \( P_{\Phi} \) (due to the different MOLD of the tableaux \( \Theta \) and \( \Phi \)) it is abundantly clear that \( P_{\Theta} \neq \rho_{\Theta \Phi} P_{\Phi} \rho_{\Phi \Theta} \), confirming eq. (159). Let us however take a closer look at \( \rho_{\Theta \Phi} P_{\Phi} \rho_{\Phi \Theta} \),

\[
\rho_{\Theta \Phi} P_{\Phi} \rho_{\Phi \Theta} = \begin{array}{ccc}
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
& & \\
\hline
& & \\
\end{array}
\]

(166)

By transforming \( P_{\Phi} \) with the permutation \( \rho_{\Theta \Phi} \), we have transformed each set of (anti-)symmetrizers into a different set of the same shape. In particular, the (anti-)symmetrizers of the ancestor tableaux of \( \Phi \) have been transformed into the (anti-)symmetrizers of tableaux obtained from \( \Theta \) by deleting the corresponding boxes,

\[
\Phi \rightarrow \Phi_{(1)} \rightarrow \Phi_{(2)} \rightarrow \Phi_{(3)}
\]

(167a)

\[
\Theta \rightarrow \Theta_{(\Phi,1)} \rightarrow \Theta_{(\Phi,2)} \rightarrow \Theta_{(\Phi,3)}
\]

(167b)

Each tableau \( \Theta_{(\Phi,k)} \) in (167b) was obtained from the predecessor \( \Theta_{(\Phi,k-1)} \) by removing the box which is in the same position as the box with the highest number in \( \Phi_{(k-1)} \). We shall refer to the tableaux in (167b) as the \( \Phi \)-MOLD ancestry of \( \Theta \). It should be noted however, that most of the tableaux in the \( \Phi \)-MOLD ancestry of \( \Theta \) are not the ancestor tableaux of \( \Theta \), in fact, most of them are not even Young tableaux. The \( \Theta_{(\Phi,k)} \) emerge by superimposing the \( \Phi_{(i)} \) in cookie cutter fashion over \( \Theta \) and thus intrinsically differ from the ancestry of \( \Theta \) itself – compare

\[
\Theta \rightarrow \Theta_{(1)} \rightarrow \Theta_{(2)} \rightarrow \Theta_{(3)}
\]

(168)
with eq. (167b).

We now see that the symmetrizers and antisymmetrizers in the operator (166) are exactly those corresponding to the tableaux in the $\Phi$-MOLD ancestry of $\Theta$ eq. (167b). This means that, the (anti-)symmetrizers in (166) can be obtained from $S_\Theta$ and $A_\Theta$ by removing index legs. Thus, all symmetrizers (resp. antisymmetrizers) of (166) are contained in $S_\Theta$ (resp. $A_\Theta$), yielding the product $P_\Theta \cdot \rho_{\Theta \Phi} P_\Phi \rho_{\Phi \Theta}$ to be non-zero as claimed in (158).

C Consequences of non-Hermiticity – an example

In this appendix, we illustrate the non-unitarity of transition operators between Young projection operators as given in eq. (64),

$$(T_{\Theta \Phi})^\dagger = Y_\Phi^\dagger \rho_{\Phi \Theta} Y_\Theta^\dagger \neq T_{\Phi \Theta},$$

by means of an example. Consider the two Young tableaux

$$\Theta := \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \text{and} \quad \Phi := \begin{array}{c} 1 \\ 3 \\ 2 \end{array}. \quad (170)$$

In eq. (50) we found that $\rho_{\Theta \Phi} = \infty$. Using eq. (62) we construct $T_{\Theta \Phi}$

$$
\begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
\quad T_{\Theta \Phi}
\quad \begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
\quad Y_\Phi
=\begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
\quad T_{\Theta \Phi}
\quad \begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
\quad Y_\Theta.
\end{array}
\quad (171)$$

and $T_{\Phi \Theta}$

$$
\begin{array}{c}
\begin{array}{c}
\rho_{\Phi \Theta}
\end{array}
\end{array}
\quad T_{\Phi \Theta}
\quad \begin{array}{c}
\begin{array}{c}
\rho_{\Phi \Theta}
\end{array}
\end{array}
\quad Y_\Theta
=\begin{array}{c}
\begin{array}{c}
\rho_{\Phi \Theta}
\end{array}
\end{array}
\quad T_{\Phi \Theta}
\quad \begin{array}{c}
\begin{array}{c}
\rho_{\Phi \Theta}
\end{array}
\end{array}
\quad Y_\Phi.
\end{array}
\quad (172)$$

From this example, it is immediately clear that $(T_{\Theta \Phi})^\dagger \neq T_{\Phi \Theta},$

$$(T_{\Theta \Phi})^\dagger = \begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\rho_{\Theta \Phi}
\end{array}
\end{array}
\neq T_{\Phi \Theta},$$

and vice versa

$$(T_{\Phi \Theta})^\dagger \neq T_{\Theta \Phi}.$$
D Proof of Theorem 5 “compact transition operators”

D.1 The significance of the cutting-and-gluing procedure

Before we present the proof of Theorem 5, we need to make some observations: Let \( I \) be any set of symmetrizers or antisymmetrizers, and let \( \rho \) be a permutation. Then, using the fact that \( \rho^\dagger = \rho^{-1} \) for any permutation,\(^{13}\) we have that

\[
\rho I = \rho I \rho^\dagger \rho = \rho I \rho^\dagger \rho = I' \rho, \tag{175}
\]

where \( I' \) is now a set of symmetrizers, respectively antisymmetrizers, over a different set of indices.\(^{14}\)

In the proof of Theorem 5, we will come across a particular such case, namely where \( \rho \) is the permutation \( \rho = \rho \Theta \Phi \) as defined in Definition 3. The simplest case we encounter are the products \( \rho \Theta \Phi S \Phi \) and \( \rho \Theta \Phi A \Phi \). By its very definition \( \rho \Theta \Phi \) explicitly relates \( \Theta \) and \( \Phi \) such that

\[
\rho \Theta \Phi S \Phi = S \Theta \rho \Theta \Phi S \Phi = S \Theta \rho \Theta \Phi S \Phi, \tag{177a}
\]

\[
\rho \Theta \Phi A \Phi = A \Theta \rho \Theta \Phi A \Phi = A \Theta \rho \Theta \Phi A \Phi, \tag{177b}
\]

where the last equality follows from the fact that each (anti-) symmetrizer individually is idempotent (12). Recognizing the parallel between eq. (177) and transition operators eq. (117) (between Hermitian projectors, such as symmetrizers \( S \Xi \) and antisymmetrizers \( A \Xi \)), the objects (177a) and (177b) can be viewed as transition operators between individual sets of (anti-) symmetrizers. This observation establishes the connection to the graphical cutting-and-gluing procedure discussed in Theorem 5: cutting antisymmetrizers \( A \Theta \) and \( A \Phi \) vertically and gluing them as suggested by the Theorem is equivalent to forming the product \( A \Theta \rho \Theta \Phi A \Phi \) (and similarly for symmetrizers). This is illustrated in the following example: For the Young tableaux

\[
\Theta = \begin{array}{c} 1 \ 3 \\ 2 \\ 4 \end{array} \quad \text{and} \quad \Phi = \begin{array}{c} 1 \ 2 \\ 3 \\ 4 \end{array}, \tag{178}
\]

we have

\[
\begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta \\
\rho \Theta \Phi
\end{array}
\end{array}\quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Phi
\ \ \ \ \ \ \ \ A \Theta
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\ \ \ \ \ \ \ \ A \Phi
\end{array}
\end{array}. \tag{179}
\]

The feature observed in this example is fully general: \( \rho \Theta \Phi \) is defined to translate the ordering of the left legs on \( A \Phi \) into the ordering of the right legs on \( A \Theta \) – this is precisely what the cutting and gluing procedure achieves graphically:

\[
A \Theta \rightarrow \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\end{array}
\end{array} \quad \text{and} \quad A \Phi \rightarrow \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Phi
\ \ \ \ \ \ \ \ A \Theta
\end{array}
\end{array} \rightarrow \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\ \ \ \ \ \ \ \ A \Phi
\end{array}
\end{array}. \tag{180}
\]

\(^{13}\)This becomes evident in the birdtrack formalism, where the inverse of a permutation \( \rho \) is obtained by flipping \( \rho \) about its vertical axis [12], which is incidentally also the process for Hermitian conjugation of a birdtrack [12].

\(^{14}\)We consider this to be self evident, but an example may help diffuse anxiety:

\[
\begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\ \ \ \ \ \ \ \ A \Phi
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\ \ \ \ \ \ \ \ A \Phi
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\ \ \ \ \ \ \ \ A \Theta
\ \ \ \ \ \ \ \ A \Phi
\end{array}
\end{array}. \tag{176}
\]

where we had \( I = \{ S_{123}, S_{45} \} \) and \( I' = \{ S_{124}, S_{35} \} \).
Both procedures lead to the same result (this is a consequence of relation (177)). Thus, we will refer to the algebraic construct (177b) as the cut-antisymmetrizer and denote it by

\[ K_{\Theta \Phi} := A_{\Theta \rho \Theta \Phi} A_{\Phi} = A_{\Theta \rho \Theta \Phi} = \rho_{\Theta \Phi} A_{\Phi} , \]

and similarly for the cut-symmetrizer \( S_{\Theta \Phi} := S_{\Theta \rho \Theta \Phi} S_{\Phi} \). For the proof of Theorem 5, we will only concern ourselves with cut-antisymmetrizers, as we already did in the Theorem. However, all the following arguments hold equally well if we consider cut-symmetrizers instead.

Before we dive into the proof, we need to notice that eq. (177) does not hold for the ancestor sets \( S_{\Phi}(k) \) and \( A_{\Phi}(l) \) of \( S_{\Phi} \) and \( A_{\Phi} \), however such ancestor sets will be transformed (upon commutation with the permutation \( \rho_{\Theta \Phi} \)) into sets of the same shape that can be obtained from \( S_{\Theta} \) resp. \( A_{\Theta} \) by dropping lines. Thus, the resulting (anti-) symmetrizers can be absorbed into \( S_{\Theta} \) and \( A_{\Theta} \) respectively,

\[ \rho_{\Theta \Phi} S_{\Phi}(k) = S_{\Theta}(\Phi, k) \rho_{\Theta \Phi} \quad \text{for} \quad S_{\Theta}(\Phi, k) \supset S_{\Theta} \]
\[ \rho_{\Theta \Phi} A_{\Phi}(l) = A_{\Theta}(\Phi, l) \rho_{\Theta \Phi} \quad \text{for} \quad A_{\Theta}(\Phi, l) \supset A_{\Theta} , \]

the (anti-) symmetrizers \( S_{\Theta}(\Phi, k) \) and \( A_{\Theta}(\Phi, l) \) correspond to tableaux in the \( \Phi \)-MOLD ancestry of \( \Theta \), c.f. eq. (167) in app. B. For further clarification, we refer the reader to appendix B for an explicit example.

We will now present a proof for the short-hand graphical construction of the birdtracks of transition operators, Theorem 5.

D.2 Proof of Theorem 5

Let \( \Theta, \Phi \in Y_n \) be two Young tableaux with the same shape, thus corresponding to equivalent irreducible representations of \( SU(N) \), and let the corresponding Hermitian Young projection operators \( P_{\Theta} \) and \( P_{\Phi} \) be constructed according to the MOLD-Theorem 3. Furthermore, let \( I \) denote either a set of symmetrizers or antisymmetrizers, and \( B \) denote the other set (that is, if \( I \) denotes a set of symmetrizers then \( B \) denotes a set of antisymmetrizers and vice versa): we use these generalized sets rather than the concrete sets \( A \) and \( S \) in order to discuss all possible forms of \( P_{\Theta} \) and \( P_{\Phi} \) in one go. We then have that \( \bar{P}_{\Theta} \) is given by

\[ \bar{P}_{\Theta} = C_{\Theta} I_{\Theta} B_{\Theta} I_{\Theta} C_{\Theta}^\dagger , \]

where \( C_{\Theta} \) consists of ancestor sets of (anti-) symmetrizers of \( \Theta \), and the exact structure of \( C_{\Theta} \) is determined by the MOLD of \( \Theta \), \( \mathcal{M}(\Theta) \), and the parity of \( \mathcal{M}(\Theta) \). Similarly, \( \bar{P}_{\Phi} \) is of the form

\[ \bar{P}_{\Phi} = D_{\Phi} I_{\Phi} B_{\Phi} I_{\Phi} D_{\Phi}^\dagger \quad \text{or} \quad \bar{P}_{\Phi} = D_{\Phi} B_{\Phi} I_{\Phi} B_{\Phi} D_{\Phi}^\dagger , \]

where, like \( C_{\Theta} \), \( D_{\Phi} \) consists of ancestor sets of (anti-) symmetrizers of \( \Phi \): in equation (184), we have taken into account that the central part of \( P_{\Phi} \) can either have the same form as \( P_{\Theta} \) (which is IBI), or it may have symmetrizers and antisymmetrizers exchanged from \( P_{\Theta} \). It should be noted that the set \( D_{\Phi} \) will be different whether the central part of \( P_{\Phi} \) is \( I_{\Phi} B_{\Phi} I_{\Phi} \) or \( B_{\Phi} I_{\Phi} B_{\Phi} \), but in both cases it will consist of ancestor sets of symmetrizers and antisymmetrizers of \( \Theta \). Understanding this, we have chosen not to introduce different symbols for the set \( D_{\Phi} \) in order to introduce the following compact notation for \( \bar{P}_{\Phi} \),

\[ \bar{P}_{\Phi} := D_{\Phi} \left\{ I_{\Phi} B_{\Phi} I_{\Phi} \right\} D_{\Phi}^\dagger , \]
which says that the central part of $\tilde{P}_\phi$ is either given by the top row, or by the bottom row in the curly bracket.\(^{15}\) According to Theorem 4, the birdtrack of the transition operator $T_{\Theta \Phi}$ is given by

$$
\tilde{T}_{\Theta \Phi} = C_\Theta \left[ \begin{array}{c}
I_\Theta B_\Theta I_\Theta \cr c_\Theta
\end{array} \right] \rho_{\Theta \Phi} \left[ \begin{array}{c}
I_\Phi B_\Phi \cr c_\Phi
\end{array} \right] D_\Phi \left\{ \begin{array}{c}
I_\Phi B_\Phi I_\Phi \cr B_\Phi I_\Phi B_\Phi
\end{array} \right\} D_\Phi^T.
$$

(186)

As was discussed in sec. D.1, the permutation $\rho_{\Theta \Phi}$ can be commuted with $D_\Phi$, in accordance with relations (182). Furthermore, equations (177) tell us that $\rho_{\Theta \Phi} I_\Phi = I_\Theta \rho_{\Theta \Phi}$ and $\rho_{\Theta \Phi} B_\Phi = B_\Theta \rho_{\Theta \Phi}$.

In commuting the $\rho_{\Theta \Phi}$ through the sets $I_\Phi$ and $B_\Phi$, it will be convenient to stop the commutation in a different place in the top row than the bottom row of $\tilde{T}_{\Theta \Phi}$,

$$
\tilde{T}_{\Theta \Phi} = C_\Theta I_\Theta B_\Theta I_\Theta c_\Theta D_\Theta \left\{ \begin{array}{c}
I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi \cr B_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi}
\end{array} \right\} D_\Phi^T.
$$

(187)

this choice may seem arbitrary at this point, but the position of $\rho_{\Theta \Phi}$ in (187) will turn out to specify the position of the cut in the cutting-and-gluing procedure, c.f. sec. D.1.

We may apply the Cancellation-Theorem 1 to the operator (187) to simplify $\tilde{T}_{\Theta \Phi}$ as

$$
\tilde{T}_{\Theta \Phi} \xrightarrow{\text{Thm. } 1} C_\Theta I_\Theta B_\Theta I_\Theta \left\{ \begin{array}{c}
I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi \cr B_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi}
\end{array} \right\} D_\Phi^T = C_\Theta \left\{ \begin{array}{c}
I_\Theta B_\Theta I_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi \cr I_\Theta B_\Theta I_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi}
\end{array} \right\} D_\Phi^T.
$$

(188)

Let us now look at the central part of $\tilde{T}_{\Theta \Phi}$ (the part in the curly brackets) in more detail: Since $I_\Theta$ denotes either $A_\Theta$ or $B_\Theta$, and $B_\Theta$ denotes the other set, then the product $I_\Theta B_\Theta$ is proportional to either a Young projection operator or the Hermitian conjugate thereof, $I_\Theta B_\Theta = Y_\Theta^{(1)}$. Thus, if the central part of $\tilde{T}_{\Theta \Phi}$ is given by the top option (implementing that $I_\Theta I_\Theta = I_\Theta$), we can use the fact that $Y_\Theta^{(1)}$ is quasi-idempotent to obtain

$$
\underbrace{I_\Theta B_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi}_{Y_\Theta^{(1)}} \propto \underbrace{I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi}_{Y_\Theta^{(1)}}.
$$

(189)

Similarly, if the central part of $\tilde{T}_{\Theta \Phi}$ is given by the bottom option of (188), we may reduce it to

$$
\underbrace{I_\Theta B_\Theta I_\Theta B_\Theta \rho_{\Theta \Phi}}_{Y_\Theta^{(1)}} \propto \underbrace{I_\Theta B_\Theta \rho_{\Theta \Phi}}_{Y_\Theta^{(1)}}.
$$

(190)

This turns (188) into (using the bar-notation introduced in eq. (45) to retain equality)

$$
\tilde{T}_{\Theta \Phi} = C_\Theta \left\{ \begin{array}{c}
I_\Theta B_\Theta \rho_{\Theta \Phi} I_\Phi \cr I_\Theta B_\Theta \rho_{\Theta \Phi}
\end{array} \right\} D_\Phi^T.
$$

(191)

In Theorem 5, we discussed three different cutting-and-gluing procedures, depending on the exact structure of the projection operators $P_\Theta$ and $P_\Phi$.

1. Option 1 requires both operators $\tilde{P}_\Theta$ and $\tilde{P}_\Phi$ to contain exactly one set of antisymmetrizers $A_\Theta$ and $A_\Phi$ respectively. This occurs if we choose the top option of $T_{\Theta \Phi}$ as given in (186) (and hence the top option of $\tilde{T}_{\Theta \Phi}$).\(^{15}\) This notation is convenient, as it will allow us to discuss both cases simultaneously.
line in (191)) and if $B$ denotes the set of antisymmetrizers and thus $I$ denotes the set of symmetrizers,

\begin{equation}
\bar{T}_{\Theta\Phi} = C_{\Theta} I_{\Theta} B_{\Theta} \rho_{\Theta\Phi} I_{\Phi} D_{\Phi}^{\dagger} \xrightarrow{B=A, I=S} C_{\Theta} S_{\Theta} \underbrace{A_{\Theta} \rho_{\Theta\Phi}}_{=A_{\Theta}} S_{\Phi} D_{\Phi}^{\dagger},
\end{equation}

where we marked the cut-antisymmetrizer $\mathcal{K}_{\Theta\Phi}$ (see eq. (181)) in the above. Clearly, (192) coincides with the cutting-and-gluing prescription of Theorem 5 if each projector $P_{\Theta}$ and $P_{\Phi}$ contains exactly one set $A_{\Theta}$ and $A_{\Phi}$ respectively.  

2. Option 2 of Theorem 5 requires $\bar{P}_{\Theta}$ and $\bar{P}_{\Phi}$ to have a different number of $A_{\Theta}$ and $A_{\Phi}$. The bottom option of operator (186) (and hence operator (191)) corresponds to this case, and it does not matter whether $B$ denotes the set of antisymmetrizers and $I$ the set of symmetrizers or the other way around: If $B$ denotes the set of antisymmetrizers, we have

\begin{equation}
\bar{T}_{\Theta\Phi} = C_{\Theta} I_{\Theta} B_{\Theta} \rho_{\Theta\Phi} I_{\Phi} D_{\Phi}^{\dagger} \xrightarrow{B=A, I=S} C_{\Theta} S_{\Theta} \underbrace{A_{\Theta} \rho_{\Theta\Phi}}_{=A_{\Theta}} D_{\Phi}^{\dagger}.
\end{equation}

The operator (193a) is the same operator that would have resulted from cutting $\bar{P}_{\Theta}$ at its left-most set $A_{\Theta}$ and $\bar{P}_{\Phi}$ at its right-most set $A_{\Phi}$, and gluing the pieces in the appropriate manner as described by the Theorem 5. Similarly, if $I$ denotes the set of antisymmetrizers, then

\begin{equation}
\bar{T}_{\Theta\Phi} \xrightarrow{\text{eq. (177a)}} C_{\Theta} A_{\Theta} S_{\Theta} \rho_{\Theta\Phi} D_{\Phi}^{\dagger},
\end{equation}

where we used the commutation relation (177a) to commute $S_{\Theta}$ and $\rho_{\Theta\Phi}$. This again yields the same result as the cutting-and-gluing procedure of Theorem 5.

3. Lastly, suppose that both $\bar{P}_{\Theta}$ and $\bar{P}_{\Phi}$ each contain two sets of antisymmetrizers $A_{\Theta}$ and $A_{\Phi}$ respectively. Then, we once again need to look at the top option of the operator $\bar{T}_{\Theta\Phi}$ as given in (186) (and hence (191)), but this time we require that $I$ denotes the set of antisymmetrizers. Then,

\begin{equation}
\bar{T}_{\Theta\Phi} = C_{\Theta} I_{\Theta} B_{\Theta} \rho_{\Theta\Phi} I_{\Phi} D_{\Phi}^{\dagger} \xrightarrow{\text{eq. (177a)}} C_{\Theta} \underbrace{A_{\Theta} \rho_{\Theta\Phi}}_{=A_{\Theta}} A_{\Phi} D_{\Phi}^{\dagger}.
\end{equation}

Equivalently,

\begin{equation}
\bar{T}_{\Theta\Phi} = C_{\Theta} A_{\Theta} S_{\Theta} \rho_{\Theta\Phi} A_{\Phi} D_{\Phi}^{\dagger} \xrightarrow{\text{eq. (177a)}} C_{\Theta} \underbrace{A_{\Theta} \rho_{\Theta\Phi}}_{=A_{\Theta}} S_{\Phi} A_{\Phi} D_{\Phi}^{\dagger};
\end{equation}

eq. (194a) corresponds to cutting-and-gluing at the right-most sets of antisymmetrizers $A_{\Theta}$ and $A_{\Phi}$ (respectively) in both $\bar{P}_{\Theta}$ and $\bar{P}_{\Phi}$, while eq. (194b) corresponds to cutting-and-gluing the left-most sets of antisymmetrizers $A_{\Theta}$ and $A_{\Phi}$ in both $\bar{P}_{\Theta}$ and $\bar{P}_{\Phi}$. Thus, we have shown that $\bar{T}_{\Theta\Phi}$ can indeed be obtained by the graphical cutting-and-gluing prescription given in the Theorem 5, concluding the proof.
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