Zero-density estimates for $L$-functions attached to cusp forms

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Abstract

Let $S_k$ be the space of holomorphic cusp forms of weight $k$ with respect to $SL_2(\mathbb{Z})$. Let $f \in S_k$ be a normalized Hecke eigenform, $L_f(s)$ the $L$-function attached to the form $f$. In this paper we consider the distribution of zeros of $L_f(s)$ in the strip $\sigma \leq \text{Re} \, s \leq 1$ for fixed $\sigma > 1/2$ with respect to the imaginary part. We study estimates of

$$N_f(\sigma, T) = \# \{ \rho \in \mathbb{C} \mid L_f(\rho) = 0, \, \sigma \leq \text{Re} \, \rho \leq 1, \, 0 < \text{Im} \, \rho \leq T \}$$

for $1/2 \leq \sigma \leq 1$ and large $T > 0$. Using the methods of Karatsuba and Voronin we shall give another proof for Ivić’s method.

1 Introduction

It is conjectured that non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\text{Re} \, s = 1/2$ (in short RH). For $1/2 \leq \sigma \leq 1$ and large $T > 0$, let $N(\sigma, T)$ be the number of zeros of $\zeta(s)$ in the region $\sigma \leq \text{Re} \, s \leq 1$ and $0 < \text{Im} \, s \leq T$. If RH is true then $N(\sigma, T) = 0$ for $1/2 < \sigma \leq 1$. The purpose of the zero-density theory for $\zeta(s)$ is to estimate $N(\sigma, T)$ as

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\end{footnotesize}
small as possible to support RH. For this problem Bohr and Landau \[2\] began to study zero-density for \(\zeta(s)\) and proved \(N(\sigma, T) \ll T\) uniformly for \(1/2 \leq \sigma \leq 1\). Ingham \[9\] improved their result to \(N(\sigma, T) \ll T^{3/(2\sigma - 1)}(\log T)^{5/4} (1/2 \leq \sigma \leq 1)\). In this paper we study the distribution of zeros of \(L\)-functions associated to holomorphic cusp forms. Let \(S_k\) be the space of cusp forms of weight \(k \in \mathbb{Z}_{\geq 12}\) with respect to the full modular group \(SL_2(\mathbb{Z})\). Let \(f \in S_k\) be a normalized Hecke eigenform, and \(a_f(n)\) the \(n\)-th Fourier coefficient of \(f\). It is known that all \(a_f(n)\)’s are real numbers (see \([1, \text{Chapter 6.14}]\)) and estimated as \(|a_f(n)| \leq d(n)n^{k-1/2}\) by Deligne \[3\], where \(d(n)\) is the divisor function defined by \(d(n) = \sum_{m|n} 1\). The \(L\)-function attached to \(f\) is defined by

\[
L_f(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p: \text{prime}} \frac{1}{1 - \lambda_f(p)p^{-s} + p^{-2s}} \quad (\text{Re } s > 1),
\]

where \(\lambda_f(n) = a_f(n)n^{-k-1/2}\). Hecke \[7\] proved that \(L_f(s)\) has an analytic continuation to the whole \(s\)-plane and the completed \(L\)-function

\[
\Lambda_f(s) = (2\pi)^{-s-k-1/2} \Gamma(s + k-1/2)L_f(s) = \int_0^{\infty} f(iy)y^{s+k-1/2-1}dy
\]

satisfies \(\Lambda_f(s) = (-1)^{k/2}\Lambda_f(1 - s)\), namely,

\[
L_f(s) = \chi_f(s)\Lambda_f(1 - s)
\]
for all \( s \in \mathbb{C} \), where \( \chi_f(s) \) is defined by
\[
\chi_f(s) = (-1)^{k/2} (2\pi)^{2s-1} \frac{\Gamma(1-s + \frac{k-1}{2})}{\Gamma(s + \frac{k-1}{2})}.
\]
By (1.4) and (1.6), \( L_f(s) \) has no zeros in \( \text{Re } s > 1 \) and \( \text{Re } s < 0 \) except \( s = -n + 1/2 \) (\( n \in \mathbb{Z}_{\geq k/2} \)). The zeros of \( L_f(s) \) in the critical strip \( 0 \leq \text{Re } s \leq 1 \) are called non-trivial zeros. Moreover, by (1.5) and (1.6), the non-trivial zeros are located symmetrically with respect to \( \text{Im } s = 0 \) and \( \text{Re } s = 1/2 \).

It is opened that all of non-trivial zeros of \( L_f(s) \) lie on \( \text{Re } s = 1/2 \), which is called the Generalized Riemann Hypothesis (in short GRH). Let \( N_f(\sigma, T) \) be the function defined by
\[
N_f(\sigma, T) = \# \{ \rho \in \mathbb{C} \mid L_f(\rho) = 0, \ \sigma \leq \text{Re } \rho < 1, \ 0 < \text{Im } \rho \leq T \}.
\]
(1.7)

As in the case of the Riemann zeta function, it is important to study the behavior of \( N_f(\sigma, T) \). It is Ivić [10] who first proved the non-trivial estimates of \( N_f(\sigma, T) \).

The aim of this paper is to give an alternative proof of Ivić’s estimate, namely,

**Theorem 1.1.** Let \( f \in S_k \) be a normalized Hecke eigenform. For any large \( T \) we have
\[
N_f(\sigma, T) \ll \begin{cases} 
T^{\frac{1}{2}(1-\sigma)+\varepsilon}, & 1/2 \leq \sigma \leq 3/4, \\
T^{\frac{3}{4}(1-\sigma)+\varepsilon}, & 3/4 \leq \sigma \leq 1
\end{cases}
\]
uniformly \( 1/2 \leq \sigma \leq 1 \). Here and later \( \varepsilon \) denotes arbitrarily positive small constant.

Ivić [10] proof is based on the second and the sixth power moments of \( L_f(s) \) due to Good [5, Theorem] and Jutila [11, (4.4.2)]. In this paper, instead of using the higher power moments of \( L_f(s) \), we follow Karatsuba and Voronin’s approach and use only the approximate functional equation of \( L_f(s) \) (see Lemma 2.2) and the well-known estimates of exponential sum (see Lemma 2.6, 2.7).

It is important that we can construct a set \( \mathcal{E} \) of zeros of \( L_f(s) \) such that the estimate of \( R = N_f(\sigma, T_1) - N_f(\sigma, T_1/2) \) is reduced to that of \( S(\rho) \), where \( 1/2 \leq \sigma \leq 1, \ 2 \leq T_1 \leq T \) and \( S(\rho) \) is a function obtained by multiplying
the approximate functional equation of $L_f(s)$ by $1/L_f(s)$. The existence of $E$ works to obtain an estimate

$$R \ll (\log T_1)^{4\alpha+3} \sum_{\rho \in E} |S(\rho)|^{2\alpha}$$

where $\alpha$ is any fixed positive integer. The upper bound of sum of (1.9) is obtained by the technique of Karatsuba and Voronin's calculating, in which power moment of $L_f(s)$ is not needed.

## 2 Preliminary Lemmas

To prove Theorem 1.1 we need Lemmas 2.1–2.7. First Lemmas 2.1–2.3 are required to show Proposition 3.1 (see Section 3), which is required for the proof of Theorem 1.1.

**Lemma 2.1.** If we write

$$\frac{1}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\mu_f(n)}{n^s} \quad (\text{Re } s > 1),$$

then we see that $\mu_f(n)$ is multiplicative and given by

$$\mu_f(p^r) = \begin{cases} 
1, & r = 0, \\
-\lambda_f(p), & r = 1, \\
1, & r = 2, \\
0, & r \in \mathbb{Z}_{\geq 3}
\end{cases}$$

for any prime number $p$. In addition we have $|\mu_f(n)| \leq d(n)$ for $n \in \mathbb{Z}_{\geq 1}$.

**Proof.** By expanding the right-hand side of (1.4) and using Deligne’s result, we can obtain the assertion of this lemma. \qed

**Lemma 2.2** (The approximate functional equation of $L_f(s)$, [4, KOROL-LAR 2]). There exist $\alpha \in (0, 1/2)$ and $\beta \in \mathbb{R}_{>0}$ such that

$$\sum_{x \leq n \leq x(1 + x^{-\alpha})} |a_f(n)|^2 \ll x^{k-\beta}$$
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and

$$L_f(s - \frac{k-1}{2}) = \sum_{n \leq y} \frac{a_f(n)}{n^s} + (2\pi)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \sum_{n \leq y} \frac{a_f(n)}{n^{k-s}} + O(|t|^\frac{k-1}{2} - \sigma - \frac{\alpha + \beta}{2})$$

uniformly for $(k-1)/2 \leq \sigma \leq (k+1)/2$ where $s = \sigma + it$ and $y = |t|/(2\pi)$. Moreover we have

$$L_f(s) = \sum_{n \leq y} \mathcal{L}_f(n) + \mathcal{X}_f(s) \sum_{n \leq y} \mathcal{L}_f(n) + O(|t|^{1/2 - \sigma + \varepsilon})$$

where $\mathcal{X}_f(s)$ is given by (1.6).

Proof. We shall show (2.2) by using (2.1). Since $|a_f(n)| \ll n^{k-1/2 + \varepsilon}$ for any $n \in \mathbb{Z}_{\geq 1}$ from Deligne’s result, it follows that

$$\sum_{x \leq n \leq (1+x-\alpha)} |a_f(n)|^2 \ll x^{k-\alpha + \varepsilon}$$

for $\alpha \in \mathbb{R}_{>0}$. If we put $\alpha = 1/2 - \varepsilon$ and $\beta = \alpha - \varepsilon$, then the $O$-term in (2.1) becomes $O(|t|^{1/2 - \sigma + \varepsilon})$. Replacing $s$ by $s + (k-1)/2$ in (2.1), we obtain the formula (2.2).

Lemma 2.3 (Moreno [14, Theorem 3.5]). Let $N_f(T)$ be the number of zeros of $L_f(s)$ in the region $0 \leq \text{Re} \, \rho \leq 1$ and $0 \leq \text{Im} \, \rho \leq T$. For large $T > 0$ we have

$$N_f(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T).$$

Furthermore, we have $N_f(\sigma, T+1) - N_f(\sigma, T) \ll \log T$ uniformly for $1/2 \leq \sigma \leq 1$.

When we estimate $\sum_{r=0}^{k} |S(t_r)|^{2} - \frac{1}{d} \int_{t_0}^{t_k} |S(t_k)|^2 dt + 2 \sqrt{\int_{t_0}^{t_k} |S(t)|^2 dt} \sqrt{\int_{t_0}^{t_k} |S'(t)|^2 dt}.$

Lemma 2.4 ([13] Lemma IV.1.1]). Let $t_0 < t_1 < \cdots < t_{k-1} < t_k$, $S(t)$ be a complex-valued $C^1$-class function on $[t_0, t_k]$. Set $d = \min_{0 \leq r \leq k-1} |t_{r+1} - t_r|$. Then we have

$$\sum_{r=0}^{k} |S(t_r)|^2 \leq \frac{1}{d} \int_{t_0}^{t_k} |S(t)|^2 dt + 2 \sqrt{\int_{t_0}^{t_k} |S(t)|^2 dt} \sqrt{\int_{t_0}^{t_k} |S'(t)|^2 dt}. $$
Lemma 2.5 ([13] Lemma IV.1.2). Let \( a(n) \) be an arithmetic function. The parameters \( X, X_1, N \) and \( N_1 \) satisfy \( 0 < X < X_1 \leq 2X \) and \( 3 \leq N < N_1 \leq 2N \). Then we have

\[
\int_X^{X_1} \left| \sum_{N<n \leq N_1} a(n)n^it \right|^2 dt \ll (X + N \log N) \sum_{N<n \leq N_1} |a(n)|^2.
\]

When we calculate \( \sum_{a<x \leq b} \varphi(x)e^{2\pi if(x)} \) where \( \varphi(x) \) and \( f(x) \) are real-valued \( C^\infty \)-class functions on \([a, b]\), we use Lemmas 2.6 or 2.7.

Lemma 2.6 ([13] Corollary 1 of Lemma V.2.1). Let \( \varphi \) and \( f \) be real-valued continuous functions on \([a, b]\). Suppose the following conditions are satisfied:

(C1) The functions \( \varphi^{(2)}(x) \) and \( f^{(4)}(x) \) are continuous.

(C2) The function \( f^{(2)}(x) \) satisfies \( 0 < f^{(2)}(x) \ll 1 \).

(C3) There exist the parameters \( H, U > 0 \) such that \( 0 < b - a \leq U \), \( \varphi(x) \ll H \), \( \varphi^{(1)}(x) \ll H/U \), \( \varphi^{(2)}(x) \ll H/U^2 \), \( f^{(2)}(x) \gg 1/A \), \( f^{(3)}(x) \ll 1/AU \), \( f^{(4)}(x) \ll 1/AU^2 \).

Then we have

\[
\sum_{a<x \leq b} \varphi(x)e^{2\pi if(x)} = \int_a^b \varphi(x)e^{2\pi if(x)} dx + O(H)
\]

where the constant in the \( O \)-term depends on \( C \).

Lemma 2.7 ([13] Theorem II.3.1]). Let \( \varphi \) and \( f \) be real-valued continuous functions on \([a, b]\), suppose the following conditions:

(C1) The functions \( \varphi^{(2)}(x) \) and \( f^{(4)}(x) \) are continuous.

(C2) The parameters \( H, A, U > 0 \) satisfy \( 1 \ll A \ll U \), \( 0 < b - a \leq U \) and

\[
\varphi(x) \ll H, \quad \varphi^{(1)}(x) \ll H/U, \quad \varphi^{(2)}(x) \ll H/U^2, \quad f^{(2)}(x) \gg 1/A, \quad f^{(3)}(x) \ll 1/AU, \quad f^{(4)}(x) \ll 1/AU^2.
\]
Suppose that the numbers \( x_n \) are determined from the equation \( f(1)(x_n) = n \). Then we have

\[
\sum_{a < x \leq b} \varphi(x)e^{2\pi if(x)} = \sum_{f(1)(a) \leq n \leq f(1)(b)} c(n)Z(n) + R
\]

where

\[
c(n) = \begin{cases} 
1, & n \in (f(1)(a), f(1)(b)), \\
1/2, & n = f(1)(a) \text{ or } n = f(1)(b),
\end{cases}
\]

\[
Z(n) = e^{\frac{2\pi i}{f(2)(x_n)}} e^{2\pi if(x_n) - nx_n},
\]

\[
R = O \left( H \left( T(a) + T(b) + \log(f(1)(b) - f(1)(a) + 2) \right) \right),
\]

\[
T(x) = \begin{cases} 
0, & f(1)(x) \in \mathbb{Z}, \\
\min(1/\|f(1)(x)\|, \sqrt{A}), & f(1)(x) \not\in \mathbb{Z},
\end{cases}
\]

\[
\|f(1)(x)\| = \min(\{f(1)(x)\}, 1 - \{f(1)(x)\}),
\]

and \( \{X\} \) is the fractional part of \( X \).

3 Proof of the Theorem 1.1

In this section we shall prove Theorem 1.1. By a standard argument, it is enough to get upper bound \( R = N_f(\sigma, T_1) - N_f(\sigma, T_1/2) \) for \( 2 \leq T_1 \leq T \) and \( \sigma > 1/2 \). Let \( X = T_1^{3/2} \) where \( \delta \) is a positive integer which is chosen later. Let \( M_X(s) \) be a function defined by

\[
M_X(s) = \sum_{m \leq X} \frac{\mu_f(m)}{m^s}.
\]

We multiply both sides of (2.2) by \( M_X(s) \) and obtain

\[
L_f(s)M_X(s) = 1 + \sum_{X < t \leq Xy} \frac{c_f(t)}{t^s} + \chi_f(s) \sum_{m \leq X} \frac{\mu_f(m)}{m^s} \sum_{n \leq y} \frac{\lambda_f(n)}{n^{1-s}} + O(\delta^{3/2 - \sigma}|M_X(s)|)
\]
where

\[ c_f(l) = \sum_{l=mn, m \leq X, n \leq y} \mu_f(m) \lambda_f(n) = \begin{cases} 1, & l = 1, \\ 0, & 2 \leq l \leq X. \end{cases} \]  

(Note that \(|c_f(l)| \leq d_4(l) \ll l^\varepsilon\). By a trivial estimate \(M_X(s) \ll X^{1-\sigma+\varepsilon}\), we have

\[ |t|^{\frac{1}{2} - \sigma} |M_X(s)| \ll T_1^{\left(1 + \frac{1}{2}\right)(1-\sigma) - \frac{1}{4} + \varepsilon}. \]

If we choose \(\delta\) sufficiently large, then we see that the exponent of \(|t|\) in the error term of (3.1) becomes negative, that is, there exists a positive constant \(c\) such that \(|t|^{\frac{1}{2} - \sigma} |M_X(s)| \ll T_1^{-c}\). Therefore taking \(s = \rho \in U\) in (3.1) where \(U\) is a set of zeros of \(L_f(s)\) in \(\sigma \leq \text{Im } s \leq 1\) and \(T_1/2 < \text{Im } s \leq T_1\), we get

\[ 1/2 \leq 1 + O(T_1^{-c}) \]

\[ \sum_{l \leq X} \frac{c(l)}{l^\rho} \bigg| \sum_{m \leq X} \frac{\mu_f(m)}{m^\rho} \bigg| \bigg| \sum_{n \leq y} \frac{\lambda_f(n)}{n^{1-\rho}} \bigg| \]

for sufficiently large \(T_1\).

Dividing the intervals of summations over \(l, m, n\) into subintervals of the form \((Z, 2Z]\) (the last subintervals are of the form \((Z, Z_0]\) where \(Z < Z_0 \leq 2Z\) and \(Z_0 = Xy, X, y\) respectively), we can write (3.3) as

\[ \frac{1}{2} \leq \sum_{\nu=1}^{D} |S_\nu(\rho)| \]

where

\[ S_\nu(\rho) = \sum_{L_\nu < l \leq L_\nu'} \frac{c(l)}{l^\rho}, \]

\[ S_\nu(\rho) = \chi_f(\rho) \sum_{M_\nu < m \leq M_\nu'} \frac{\mu_f(m)}{m^\rho} \sum_{N_\nu < n \leq N_\nu'} \frac{\lambda_f(n)}{n^{1-\rho}}. \]

(Note that the number of summands of (3.3) is \(\ll (\log T_1)^2\), namely \(D \ll (\log T_1)^2\).)

Now following Karatsuba and Voronin, we shall show the existence of a set \(E\) of zeros of \(L_f(s)\) with playing an important role later.
Proposition 3.1. There exists a set $E$ of zeros of $L_f(s)$ such that

\begin{align}
(3.6a) & \quad |S(\rho)| \geq \frac{1}{2D} \quad (\rho \in E), \\
(3.6b) & \quad \#E \gg \frac{R}{D \log T_1}, \\
(3.6c) & \quad |\text{Im} \rho - \text{Im} \rho'| \geq 1 \quad (\rho \neq \rho' \in E),
\end{align}

where $S(\rho) = S_{\nu_0}(\rho)$ with some number $\nu_0 \in \{1, \ldots, D\}$.

Proof. From (3.4) it is clear that $U = \bigcup_{1 \leq \nu \leq D} A_{\nu}$ where $A_{\nu} = \{\rho \in U \mid |S_\nu(\rho)| \geq 1/(2D)\}$. Then we see that there exists $\nu_0$ such that $|S(\rho)| \geq 1/(2D)$ for $\rho \in A$ and $\#A \geq R/D$ where $A = A_{\nu_0}$ and $S(\rho) = S_{\nu_0}(\rho)$.

Let $\rho_{m,n}$ be $\rho \in A$ such that $\text{Im} \rho$ is the $n$-th minimum number in $(T_1/2 + m, T_1/2 + m + 1]$. By using Lemma 2.3 we can write

$$
A = \bigcup_{j=0,1} \bigcup_{1 \leq n \leq C \log T_1} E_{n,j}, \quad E_{n,j} = \{\rho_{j,n}, \rho_{2+j,n}, \rho_{4+j,n}, \ldots, \rho_{2\lfloor T_1/4\rfloor+j,n}\}
$$

where $C$ is a positive constant. Then there exist $n_0 \in \{1, 2, \ldots, [C \log T_1]\}$ and $j_0 \in \{0, 1\}$ such that $\#A \leq C(\log T_1) \sum_{j=0,1} \#E_{n_0,j} \leq 2C(\log T_1)\#E$ where $E = E_{n_0,j_0}$. Since $E \subset A$ and $\#A \geq R/D$, it follows that (3.6a) and (3.6b) are shown. And (3.6c) is shown because $|\text{Im} \rho_{2l+j_0,n_0} - \text{Im} \rho_{2l'+j_0,n_0}| \geq 1$ for $l \neq l' \in \{0, 1, \ldots, [T_1/4]\}$. \qed

From (3.6a) and (3.6b) we can reduce the estimate of $R$ to that of $S(\rho)$, that is, by taking $2\alpha$-th power of both sides of (3.6a) we get

\begin{equation}
R \ll D(\log T_1)\#E = D(\log T_1) \sum_{\rho \in E} 1^{2\alpha} \ll (\log T_1)^{4\alpha+3} \sum_{\rho \in E} |S_\rho(\rho)|^2
\end{equation}

where $\alpha$ is any fixed positive integer.

First we consider the case that $S(\rho)$ is of the form (3.5a). We shall give a preliminary upper bound of $R$ as

Proposition 3.2. Let $S(\rho)$ in (3.7) has the form

\begin{equation}
S(\rho) = \sum_{L < \xi \leq L'} \frac{c_f(l)}{l^\rho}
\end{equation}

with $L < L' \leq 2L$. For any positive integer $\alpha$ we have

\begin{equation}
R \ll L^{(1-2\alpha)}(T_1 + L^\alpha)T_1^\varepsilon.
\end{equation}
Proof. From (3.8) the $\alpha$-th power of $S(\rho)$ has the form

$$S^\alpha(\rho) = \sum_{L^\alpha < l \leq L'^\alpha} \frac{A_\alpha(l)}{l^\beta} = \sum_{L^\alpha < l \leq L'^\alpha} \frac{A_\alpha(l)}{l^{\beta+i\gamma}},$$

where

$$A_\alpha(l) = \sum_{l = l_1 \cdots l_\alpha, L' < l_1, \ldots, l_\alpha \leq L} c_f(l_1) \cdots c_f(l_\alpha).$$

(Note that $|A_\alpha(l)| \leq d_{\alpha}(l) \ll \varepsilon$.) If we put $C(t) = \sum_{L^\alpha < l \leq t} A_\alpha(l) l^{-i\gamma}$, then by partial summation formula and Cauchy’s inequality we have

$$|S^\alpha(\rho)|^2 = \left| \frac{C(L^\alpha)}{(L^\alpha)^{\beta}} - \beta \int_{L^\alpha}^{L'^\alpha} \frac{C(t)}{t^{\beta+1}} dt \right|^2 \ll \frac{|C(L^\alpha)|^2}{L^{2\alpha\sigma}} + \frac{1}{L^{2\alpha(\sigma+1)}} \left( \int_{L^\alpha}^{L'^\alpha} |C(t)| dt \right)^2$$

(3.10)

Hence from (3.7) and (3.10) we obtain

$$R \ll \frac{(\log T_1)^{4\alpha+3}}{L^{2\alpha\sigma}} \sum_{\rho \in \mathcal{E}} |C(L_0)|^2.$$  

(3.11)

where $L_0$ is chosen such that $\sum_{\rho \in \mathcal{E}} |C(L_0)|^2$ is the maximal value. To estimate this maximal value, we divide again the interval $(L^\alpha, L_0]$ into the subintervals of the form $(Z, Z']$ where $Z < Z' \leq 2Z$ and let $L_1 \in (L^\alpha, L_0]$ be chosen so that $\sum_{\rho \in \mathcal{E}} |\sum_{L_1 < l \leq L_1'} A_\alpha(l) l^{-i\gamma}|$ is maximal. (Note that the number of divided interval is at most $\alpha$ because $L_0 \leq 2^\alpha L^\alpha$.) Then we have

$$\sum_{\rho \in \mathcal{E}} |C(L_0)|^2 \ll \sum_{\rho \in \mathcal{E}} \left( \sum_{L_1 < l \leq L_1'} A_\alpha(l) l^{-i\gamma} \right)^2.$$  

(3.12)

Now we apply Lemma 2.4 to the right hand side of the above formula:

$$\sum_{\rho \in \mathcal{E}} \left( \sum_{L_1 < l \leq L_1'} A_\alpha(l) l^{-i\gamma} \right)^2 \ll I_1 + \sqrt{I_1 I_2},$$  

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where

\[
I_1 = \int_{T_1}^{T_1} \left| \sum_{K_0 < l \leq K'_{0}} A_\alpha(l) l^{i\gamma} \right|^2 d\gamma, \quad I_2 = \int_{T_1}^{T_1} \left| \sum_{K_0 < l \leq K'_{0}} A_\alpha(l) l^{i\gamma} \log l \right|^2 d\gamma.
\]

By using Lemma 2.5 and noting \(|A_\alpha(l)| \ll l^\varepsilon\), we see that

\[
I_1 \ll L(T_1 + L)T_1^\varepsilon, \quad I_2 \ll L(T_1 + L)T_1^\varepsilon.
\]

(3.14)

Combining (3.11)–(3.14) we obtain the assertion of Proposition 3.2.

We divide the interval \((X, Xy]\) into subintervals of the form:

\[
\mathcal{F}_r = \begin{cases} 
(T_1, T_1^{1+\frac{\delta}{2}}], & r = 1, \\
(T_1^{\frac{1}{r}}, T_1^{\frac{1}{r}-1}], & r \in \{2, \ldots, \delta\}.
\end{cases}
\]

(3.15)

We see that there exists \(r \in \{1, 2, \ldots, \delta\}\) such that \(L \in \mathcal{F}_r\). If we use (3.9) under some conditions of \(L\), then we can obtain the following upper bounds of \(R\):

**Proposition 3.3.** We have

\[
R \ll \begin{cases} 
T_1^{2(1-\sigma)+\varepsilon}, & L \in \mathcal{F}_1, \\
T_1^{\frac{1}{2\varepsilon}(1-\sigma)+\varepsilon}, & L \in \bigcup_{2 \leq r \leq \delta} \mathcal{F}_r.
\end{cases}
\]

Proof. First in the case of \(L \in \mathcal{F}_1\), taking \(\alpha = 1\) in (3.9) and choosing \(\varepsilon \geq 2/\delta\), we obtain

\[
R \ll L^{2(1-\sigma)}T_1^\varepsilon \ll T_1^{2(1-\sigma)+\varepsilon} \ll T_1^{2(1-\sigma)+\varepsilon}.
\]

(3.16)

Next we consider the upper bound of \(R\) in the case of \(L \in \bigcup_{2 \leq r \leq \delta} \mathcal{F}_r\). Let \(2 \leq A \leq 4\). When \(r \in \{2, \ldots, \lceil A/(A-2) \rceil\}\), that is, \(A \leq 2r/(r-1)\), we can divide \(\mathcal{F}_r\) to \((T_1^{\frac{1}{r}}, T_1^{\frac{1}{r}-1}\]\) and \((T_1^{\frac{1}{r}}, T_1^{\frac{1}{r}-1}\]\). Hence \(L \in (T_1^{\frac{1}{r}}, T_1^{\frac{1}{r}}]\), taking \(\alpha = r\) in (3.9) we have

\[
R \ll L^{2r(1-\sigma)}T_1^\varepsilon \ll T_1^{A(1-\sigma)+\varepsilon}.
\]

(3.17)
When \( r \in \{\lceil A/(A-2) \rceil + 1, \ldots, \delta \} \) and \( L \in \mathcal{F}_r \), we see that \( A > 2r/(r-1) \) and

\[
R \ll L^{2r(1-\sigma)}T_1^\varepsilon \ll T'_1 \left( A^{(1-\sigma)+\varepsilon} \right) \ll T'_1 A^{(1-\sigma)+\varepsilon}.
\]

(3.18)

On the other hand, in the case of \( L \in \bigcup_{2 \leq r < A/(A-2)} (T_1 \sqrt{r}, T_1 \sqrt{r+1}) \), taking \( \alpha = r-1 \) in (3.9) we obtain

\[
R \ll r^{(r-1)(1-2\sigma)}T_1^{1+\varepsilon} \ll T_1^{1+\frac{1}{2r} A^{(1-2\sigma)+\varepsilon}}.
\]

(3.19)

Here we suppose \( 1 + A(1-2\sigma)(r-1)/2r \leq A(1-\sigma) \), that is, \( A \geq (2\sigma-1)(r-1)/2r + (1-\sigma)^{-1} \). If we put \( A = \max_{r \geq 2} (2\sigma-1)(r-1)/2r + (1-\sigma)^{-1} = 4/(3-2\sigma) \), then we obtain

\[
R \ll T_1^{A(1-\sigma)+\varepsilon} \ll T_1^{A^{(1-\sigma)+\varepsilon}}.
\]

(3.20)

From the results (3.16)–(3.19), the proof of Proposition 3.2 is completed. \( \square \)

We consider an improvement of estimate of \( R \) when \( L \in \bigcup_{2 \leq r \leq \delta} \mathcal{F}_r \). We replace \( 2\alpha \) by \( r-1 \) in (3.5a), then

\[
R \ll (\log T_1)^{2r+1} \sum_{\rho \in \mathcal{E}} |S^{r-1}(\rho)|.
\]

(3.20)

By writing \( |S^{r-1}(\rho)| = S^{r-1}(\rho)e^{-i\theta(\rho)} \) and using Cauchy’s inequality, (3.20) becomes

\[
R \ll (\log T_1)^{2r+1} \sum_{\rho \in \mathcal{E}} e^{-i\theta(\rho)} \sum_{L^{r-1} < l \leq L^r-1} \frac{A_{r-1}(l)}{l^{\gamma}} \ll \sqrt{W} \sqrt{W'} T_1^\varepsilon
\]

(3.21)

where \( \theta(\rho) = \arg S(\rho) \), and

\[
W = \sum_{L^{r-1} < l \leq L^r-1} \left| \sum_{\rho \in \mathcal{E}} e^{-i\theta(\rho)}/l^{\rho} \right|^2, \quad W' = \sum_{L^{r-1} < l \leq L^r-1} |A_{r-1}(l)|^2.
\]

By noting \( |A_{r-1}(l)| \ll l^{\varepsilon} \), it is obvious that

\[
W' \ll L^{r-1} T_1^\varepsilon.
\]

(3.22)
From (3.6c) we see that $\rho = \rho'$ when $\gamma = \gamma'$, and obtain

$$W = \sum_{L^{r-1} < l \leq L'} \sum_{\rho, \rho' \in E} e^{-i(\theta(\rho) - \theta(\rho'))} \frac{l^{i(\gamma' - \gamma)}}{l^{\beta + \beta'}}.$$  

(3.23)

We shall calculate two terms of (3.23). Since $\#E \leq R$, it follows that

$$\sum_{\rho \in E} \sum_{L^{r-1} < l \leq L'} \frac{1}{l^{2\beta}} \ll RL^{(r-1)(1-2\sigma)}.$$  

(3.24)

Using partial summation formula and putting $C_{\gamma, \gamma'}(t) = \sum_{L^{r-1} < l \leq t} l^{i(\gamma' - \gamma)}$, we see that

$$\sum_{\gamma \neq \gamma'} \sum_{L^{r-1} < l \leq L'} \frac{l^{i(\gamma' - \gamma)}}{l^{\beta + \beta'}} \ll L^{-2(r-1)\sigma} \sum_{\gamma \neq \gamma'} |C_{\gamma, \gamma'}(L_0)|$$  

(3.25)

where $L_0$ is chosen such that $\sum_{\gamma \neq \gamma'} |\sum_{L^{r-1} < l \leq L_0} l^{i(\gamma' - \gamma)}|$ is the maximal value. In order to estimate the maximal sum, we shall divide the above sum as

$$\sum = \sum_{1 < |\gamma - \gamma'| \leq 2} + \sum_{2 < |\gamma - \gamma'| \leq 4} + \cdots + \sum_{2^D_1 < |\gamma - \gamma'| \leq T_1/2}.$$  

(Note that the number of divided sums is $\ll \log T_1$, that is, $D_1 \ll \log T_1$.) Then we see that

$$\sum_{\gamma \neq \gamma'} |C_{\gamma, \gamma'}(L_0)| \ll (\log T_1) \sum_{V < |\gamma - \gamma'| \leq V'} |C_{\gamma, \gamma'}(L_0)|$$  

(3.26)

where $V \in [1, T_1/2]$ and $V < V' \leq 2V$. By fixing $\gamma' = \gamma_1 = \min_{\rho \in E} \gamma$, we get

$$\sum_{V < |\gamma - \gamma'| \leq V'} |C_{\gamma, \gamma'}(L_0)| \ll R \sum_{V < \gamma - \gamma_1 \leq V'} |C_{\gamma, \gamma_1}(L_0)|.$$  

(3.27)
Combining the above (3.23)–(3.27), we have

\begin{equation}
W \ll RL^{(r-1)(1-2\sigma)} + RL^{-2(r-1)\sigma} T_1^{\varepsilon} \sum_{V<\gamma-\gamma_1 \leq V'} |C_{\gamma,\gamma_1}(L_0)|.
\end{equation}

Taking squares in both sides of (3.21), from (3.22) and (3.28) we obtain

\begin{equation}
R \ll L^{2(r-1)(1-\sigma)} T_1^{\varepsilon} + L^{(r-1)(1-2\sigma)} T_1^{\varepsilon} \sum_{V<\gamma-\gamma_1 \leq V'} |C_{\gamma,\gamma_1}(L_0)|.
\end{equation}

By estimating the sum of the second term of (3.29) in the case of \(2V \leq \pi L^{-1}\) or \(2V > \pi L^{-1}\), we can give upper bounds of \(R\) as Proposition 3.4.

We have

\begin{equation}
R \ll \begin{cases} T_1^{2(1-\sigma)+\varepsilon}, & L \in \bigcup_{2 \leq r \leq \delta} F_r \text{ and } 2V \leq \pi L^{-1}, \\ T_1^{2(1-\sigma)+\varepsilon}, & L \in \bigcup_{2 \leq r \leq \delta} F_r, \ 2V > \pi L^{-1} \text{ and } 2/3 \leq \sigma \leq 1. \end{cases}
\end{equation}

Proof. First we consider the upper bound of sum of (3.29) in the case of \(2V \leq \pi L^{-1}\), namely, \(1 \ll L^{-1}/V\). By using Lemma 2.6 we see that

\[ C_{\gamma,\gamma_1}(L_0) = \int_{L_0}^{L} u^{-i(\gamma-\gamma_1)} du + O(1) \ll \frac{L^{-1}}{\gamma - \gamma_1} + 1 \ll \frac{L^{-1}}{V}. \]

This formula and (3.29) imply that

\begin{equation}
R \ll L^{2(r-1)(1-\sigma)} T_1^{\varepsilon} \left( 1 + \sum_{V<\gamma-\gamma_1 \leq V'} \frac{1}{V} \right) \ll T_1^{2(1-\sigma)+\varepsilon}
\end{equation}

for \(L \in F_r \ (2 \leq r \leq \delta)\).

Next in the case of \(2V > \pi L^{-1}\) we apply the estimate of \(C_{\gamma,\gamma_1}(L_0)\) to Lemma 2.7 then

\begin{equation}
C_{\gamma,\gamma_1}(L_0) = e^{i\left(\frac{\pi}{2} - (\gamma-\gamma_1) \log \frac{\gamma - \gamma_1}{\pi e}\right) \sqrt{\frac{\gamma - \gamma_1}{2\pi}}} \left( \sum_{N_1 \leq n \leq N_2} \frac{n^{i(\gamma-\gamma_1)}}{n} + O\left(\frac{L^{-1}}{\sqrt{V}}\right) \right)
\end{equation}
where \(N_1 = (\gamma - \gamma_1)/(2\pi L_0)\) and \(N_2 = (\gamma - \gamma_1)/(2\pi L^{r-1})\). We shall calculate the sum on the right hand side of (3.31). Since

\[
\sum_{N_1 \leq n \leq N_2} \frac{n^i(\gamma - \gamma_1)}{n} = \frac{1}{B} \sum_{b \neq 0, -\frac{B}{2} + 1 \leq b \leq \frac{B}{2}} \sum_{N_3 \leq n \leq N_4} \frac{n^i(\gamma - \gamma_1)e^{2\pi ibn}}{n} \sum_{N_1 \leq m \leq N_2} e^{-\frac{2\pi ibm}{B}} + \\
+ \frac{1}{B} \sum_{N_3 \leq n \leq N_4} \frac{n^i(\gamma - \gamma_1)}{n} \sum_{N_1 \leq m \leq N_2} 1
\]

and

\[
\left| \sum_{N_1 \leq m \leq N_2} e^{-\frac{2\pi ibm}{B}} \right| \leq \frac{2}{|1 - e^{-\frac{2\pi ib}{B}}|} \leq \frac{B}{|b|} \quad (b \neq 0),
\]

where \(N_3 = V/(2\pi L_0)\), \(N_4 = V/(\pi L^{r-1})\), \(B = 2[V/L^{r-1}] \) (note that \(N_2 - N_1 < N_4 - N_3 < B\), \(N_3 \ll VL^{-(r-1)}\) and \(N_4 \ll VL^{-(r-1)}\)), it follows that

(3.32) \[
\sum_{N_1 \leq n \leq N_2} \frac{n^i(\gamma - \gamma_1)}{n} \ll \sum_{-\frac{B}{2} + 1 \leq b \leq \frac{B}{2}} \frac{1}{|b|} \left| \sum_{N_3 \leq n \leq N_4} \frac{n^i(\gamma - \gamma_1) e^{2\pi ibn}}{n} \right| + \sum_{N_3 \leq n \leq N_4} \frac{n^i(\gamma - \gamma_1)}{n}.
\]

Combining the results (3.31) and (3.32) we have

(3.33) \[
\sum_{V<\gamma-\gamma_1 \leq V'} |C_{\gamma, \gamma_1}(L_0)| \ll \sqrt{VT_1'} \sum_{V<\gamma-\gamma_1 \leq V'} X \left( \left| \sum_{N_3 \leq n \leq N_4} \frac{n^i(\gamma - \gamma_1) e^{2\pi ibn}}{n} \right| + \frac{L^{r-1}}{V} \right)
\]

where \(b_0\) is chosen such that \(\sum_{V<\gamma-\gamma_1 \leq 2V} \sum_{N_3 \leq n \leq N_4} n^i(\gamma - \gamma_1) - 1 e^{2\pi ibn} \) is the maximal value. Here we apply (3.6a) to \(X\) in (3.33), that is,

(3.34) \[
X = 1 \ll T_1' \sum_{L^{r-1} \leq \xi \leq L^{r-1}} \left| \frac{A_{r-1}(l)b^\gamma}{l^{\beta}} \right|.
\]
From (3.29), (3.33) and (3.34) we obtain

\[ (3.35) \quad R \ll L^{2(r-1)(1-\sigma)} T_1^c + \]
\[ + L^{(r-1)(1-2\sigma)} \sqrt{V T_1^c} \sum_{V < \gamma - \gamma_1 \leq V'} \left| \sum_{L^{r-1} < l \leq L^{r-1}} \frac{A_{r-1}(l)^{l^\gamma}}{l^3} \right| \times \]
\[ \times \left( \left| \sum_{N_3 \leq n \leq N_4} \frac{n^{i(\gamma-\gamma_1)e^{2\pi i n \theta_1}/n}}{n} \right| + \frac{L^{r-1}}{V} \right). \]

Put \( C_1(t) = \sum_{L^{r-1} < l \leq L^{r-1}} A_{r-1}(l)^{l^\gamma} \) and \( C_2(u) = \sum_{N_3 \leq n \leq u} n^{i(\gamma-\gamma_1)e^{2\pi i n \theta_1}/n} \). Using partial summation formula, and Cauchy’s inequality, we have

\[ (3.36) \quad \sum_{V < \gamma - \gamma_1 \leq V'} \left| \sum_{L^{r-1} < l \leq L^{r-1}} \frac{A_{r-1}(l)^{l^\gamma}}{l^3} \right| \left( \left| \sum_{N_3 \leq n \leq N_4} \frac{n^{i(\gamma-\gamma_1)e^{2\pi i n \theta_1}/n}}{n} \right| + \frac{L^{r-1}}{V} \right) \]
\[ \ll L^{(r-1)(1-3\sigma)} V^{-\frac{1}{4}} T_1^c (W_1 + W_2) \]

where

\[ W_1 = \sum_{V < \gamma - \gamma_1 \leq V'} |C_1(L_1)|, \quad W_2 = \sum_{V < \gamma - \gamma_1 \leq V'} |C_1(L_2)||C_2(N_3)|, \]

and \( L_1, L_2, N_3 \) are chosen such that \( W_1, W_2 \) are the maximal values.

We shall calculate \( W_1 \) and \( W_2 \). Writing

\[ |C_1(L_1)| = C_1(L_1)e^{-i\theta_1(\gamma)}, \quad |C_1(L_2)||C_2(N_3)| = C_1(L_2)C_2(N_3)e^{-i\theta_2(\gamma)} \]

and using Cauchy’s inequality we obtain

\[ (3.37) \quad W_1 \ll \sqrt{S_{1,1}} \sqrt{S_{1,2}}, \quad W_2 \ll \sqrt{S_{2,1}} \sqrt{S_{2,2}} \]

where

\[ S_{1,1} = \sum_{L^{r-1} < l \leq L_1} |A_{r-1}(l)|^2, \quad S_{1,2} = \sum_{L^{r-1} < l \leq L_1} \left| \sum_{V < \gamma - \gamma_1 \leq V'} e^{-i\theta_1(\gamma)p(\gamma - \gamma_1)} \right|^2, \]
\[ S_{2,1} = \sum_{M_1 < m \leq M_2} B(m)^2, \quad S_{2,2} = \sum_{M_1 < m \leq M_2} \left| \sum_{V < \gamma - \gamma_1 \leq V'} e^{-i\theta_2(\gamma)m(\gamma - \gamma_1)} \right|^2, \]
Here we shall calculate the above sums as

\[ B(m) = \sum_{l|m, \, L^{-1}r \leq l \leq L, \, N_3 < m/l \leq N_5} |A_{r-1}(l)|, \]

By using Lemma 2.4 we get

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\[ \theta_1(\gamma) = \text{arg} C_1(L_1), \quad \theta_2(\gamma) = \text{arg} C_1(L_2)C_2(N_3), \quad M_1 = V/2^{r+2}, \quad M_2 = 2^{r-2}V. \]

(Note that \( (L^{-1}N_3, L_2N_5) \subset (M_1, M_2). \) It is trivial that

\[ (3.38) \quad S_{1,1} \ll VT_1^\varepsilon, \quad S_{2,1} \ll VT_1^\varepsilon. \]

By using Lemma 2.4 we get

\[ (3.39) \quad S_{1,2} \ll I_{1,2} + \sqrt{I_{1,2}J_{1,2}}, \quad S_{2,2} \ll I_{2,2} + \sqrt{I_{2,2}J_{2,2}} \]

where

\[ I_{1,2} = \int_{L^{-1}}^{L_1} \left| \sum_{V < \gamma < V'} e^{-i\theta_1(\gamma)}u^{i(\gamma - \gamma_1)} \right|^2 \, du, \]

\[ J_{1,2} = \int_{L^{-1}}^{L_1} \left| \sum_{V < \gamma < V'} e^{-i\theta_1(\gamma)}u^{i(\gamma - \gamma_1)-1}i(\gamma - \gamma_1) \right|^2 \, du, \]

\[ I_{2,2} = \int_{M_1}^{M_2} \left| \sum_{V < \gamma < V'} e^{-i\theta_2(\gamma)}u^{i(\gamma - \gamma_1)} \right|^2 \, du, \]

\[ J_{2,2} = \int_{M_1}^{M_2} \left| \sum_{V < \gamma < V'} e^{-i\theta_2(\gamma)}u^{i(\gamma - \gamma_1)-1}i(\gamma - \gamma_1) \right|^2 \, du. \]

Here we shall calculate the above sums as

\[ \int_I \left| \sum_{\gamma} f_\gamma(u) \right|^2 \, du = \sum_{\gamma=\gamma'} \int_I f_\gamma(u)\overline{f_{\gamma'}(u)}du + \sum_{\gamma\neq\gamma'} \int_I f_\gamma(u)\overline{f_{\gamma'}(u)}du, \]

and use the trivial estimates \( |\gamma - \gamma_1| \ll V, \ |L_1 - L^{-1}| \ll L^{-1}, \ M_1 \gg V, \ |M_2 - M_1| \ll V \) and \( \#E \ll R. \) Then we can obtain

\[ (3.40) \quad I_{1,2} \ll RL^{r-1}T_1^\varepsilon, \quad J_{1,2} \ll RV^2L^{1-r}T_1^\varepsilon, \]

\[ I_{2,2} \ll RVT_1^\varepsilon, \quad J_{2,2} \ll RVT_1^\varepsilon. \]

Hence from \( (3.37) \) - \( (3.40) \) and the condition \( 2V > \pi L^{-1} \) we have

\[ (3.41) \quad W_1 \ll \sqrt{RVT_1^\varepsilon}, \quad W_2 \ll \sqrt{RVT_1^\varepsilon}. \]
Combining the results (3.35), (3.36) and (3.41) we obtain

\[ R \ll L^{2(r-1)(1-\sigma)}T_1^\varepsilon + L^{(r-1)(1-3\sigma)}\sqrt{RV} T_1^\varepsilon \]
\[ \ll L^{2(r-1)(1-\sigma)}(1 + \sqrt{RV}L^{-(r-1)\sigma})T_1^\varepsilon. \]  

Moreover we shall calculate the right hand side of (3.42). In the case of \( 1 \gg \sqrt{RV}L^{-(r-1)\sigma} \) and \( r \in \{2, \ldots, \delta\} \), (3.42) becomes

\[ R \ll L^{2(r-1)(1-\sigma)}T_1^\varepsilon \ll T_1^{2(1-\sigma)+\varepsilon} \]

for \( L \in \mathcal{F}_r \). On the other hand, in the case of \( 1 \ll \sqrt{RV}L^{-(r-1)\sigma} \) and \( r \in \{2, \ldots, \delta\} \), (3.42) becomes \( R \ll L^{(r-1)(2-3\sigma)}\sqrt{RV}T_1^\varepsilon \), that is,

\[ R \ll VL^{2(r-1)(2-3\sigma)}T_1^\varepsilon \ll T_1^{1+\frac{r-1}{2(2-3\sigma)}A^{(2-3\sigma)+\varepsilon}} \]

for \( \sigma \geq 2/3 \) and \( L \in (T_1^{\frac{4}{2}}, T_1^{\frac{1}{r-1}}] \). Here we suppose \( 1 + A(2-3\sigma)(r-1)/r \leq A(1-\sigma) \), that is, \( A \geq (1-\sigma + (3\sigma-2)(1-1/r))^{-1} \). If we put \( A = \max_{r>2}(1-\sigma + (3\sigma-2)(1-1/r))^{-1} = 2/\sigma \) where \( 2/3 \leq \sigma \leq 1 \), then from (3.44) we obtain

\[ R \ll T_1^2(1-\sigma)+\varepsilon \quad (2/3 \leq \sigma \leq 1, \ L \in \bigcup_{2 \leq r \leq \delta} (T_1^{\frac{4}{2}}, T_1^{\frac{1}{r-1}}]). \]

Combining the results (3.30), (3.43) and (3.45), the proof of Proposition 3.4 is completed.

Finally we consider in the case of \( S(\rho) = S_\nu(\rho) \) (see (3.5b)). We shall give an upper bound of \( R \):

**Proposition 3.5.** Let \( S(\rho) \) in (3.7) has the form

\[ S(\rho) = \chi_f(\rho) \sum_{M < m \leq M'} \frac{\mu_f(m)}{m^\rho} \sum_{N < n \leq N'} \frac{\lambda_f(n)}{n^{1-\rho}} \]

with \( M < M' \leq 2M \) and \( N < N' \leq 2N \). When \( \alpha = 1 \), we have

\[ R \ll T_1^{2(1-\sigma)+\varepsilon} \quad (1/2 < \sigma \leq 1). \]
Proof. If we take $\alpha = 1$ in (3.7), then we have

$$R \ll (\log T_1)^7 \sum_{\rho \in E} |\chi_f(\rho)|^2 \left| \sum_{M < m \leq M'} \frac{\mu_f(m)}{m^\rho} \sum_{N < n \leq N'} \frac{\lambda_f(n)}{n^{1-\rho}} \right|.$$  

Here using Stirling’s formula with the condition $T_1/2 < \gamma \leq T_1$ we have

$$|\chi_f(\rho)| = (2\pi)^{2\beta-1} \frac{\gamma^{1-\beta+\frac{1}{4}\lambda_1} e^{-\frac{\pi}{2}\sqrt{2\pi}(1+O(1/\gamma))}}{\gamma^{\beta+\frac{1}{2}\lambda} e^{-\frac{\pi}{2}\sqrt{2\pi}(1+O(1/\gamma))}} \ll T_1^{1-2\beta}.$$  

We put $C_3(t) = \sum_{M < m \leq t} \mu_f(m)m^{-i\gamma}$ and $C_4(u) = \sum_{N < n \leq u} \lambda_f(n)n^{-i\gamma}$. Using partial summation formula and Cauchy’s inequality, we have

$$\left| \sum_{M < m \leq M'} \frac{\mu_f(m)}{m^\rho} \sum_{N < n \leq N'} \frac{\lambda_f(n)}{n^{1-\rho}} \right|^2 \ll \frac{1}{M^{2\beta}} \left( |C_3(M')|^2 + \frac{1}{M} \int_M^{M'} |C_3(t)|^2 dt \right) \times \frac{1}{N^{2(1-\beta)}} \left( |C_4(N')|^2 + \frac{1}{N} \int_N^{N'} |C_4(u)|^2 du \right).$$

From (3.46)–(3.48) and the condition $N \leq y$ (i.e. $NT_1^{-1} \ll 1$), we have

$$R \ll T^\varepsilon \sum_{\rho \in E} T_1^{1-2\beta} M^{-2\beta} N^{2(\beta-1)} |C_3(M_0)|^2 |C_4(N_0)|^2$$

$$\ll T_1^{1-2\alpha + \varepsilon} M^{-2\alpha_0} N^{-1} \sum_{\rho \in E} |C_3(N_0)|^2 |C_4(M_0)|^2.$$  

where $M_0, N_0$ are chosen such that the sum of (3.49) is the maximal value.

We shall calculate the sum of (3.50). Multiplying $C_3(M_0)$ by $C_4(M_0)$, dividing the interval $(MN, M_0N_0]$ into the form $(P_\nu, P'_\nu]$ where $P_\nu < P'_\nu \leq 2P_\nu$, $\nu \in \{1, \ldots, D_5\}$ (note that $D_5 \leq 2$ because $M_0N_0 \leq 4MN$), we can calculate the above sum as

$$\sum_{\rho \in E} |C_3(N_0)|^2 |C_4(M_0)|^2 = \sum_{\rho \in E} \left| \sum_{\nu=1}^{D_5} \sum_{P_\nu < l \leq P'_\nu} \frac{B(l)}{l^{i\gamma}} \right|^2$$
where
\[ B(l) = \sum_{n \mid l, M < m \leq M_0, N < l/m \leq N_0} \mu_f(m) \lambda_f \left( \frac{l}{m} \right). \]

(Note that \(|B(l)| \leq d_4(l) \ll l^\varepsilon\).) If we choose \(\nu_0\) such that
\[ \sum_{\rho \in \mathcal{E}} \left| \sum_{P_0 < l \leq P} B(l) l^{-i\gamma} \right| \text{is the maximal value and apply Lemma 2.4} \]
then we have
\[ \sum_{\rho \in \mathcal{E}} \left| \sum_{P_0 < l \leq P} B(l) l^{-i\gamma} \right|^2 \ll \sum_{\rho \in \mathcal{E}} \left| \sum_{P_0 < l \leq P} B(l) l^{-i\gamma} \right|^2 \ll I_1 + \sqrt{I_1 I_2} \]
where
\[ I_1 = \int_{\frac{T_1}{2}}^{T_1} \left| \sum_{P_0 < l \leq P' \nu_0} B(l) l^{-i\gamma} \right|^2 d\gamma, \quad I_2 = \int_{\frac{T_1}{2}}^{T_1} \left| \sum_{P_0 < l \leq P' \nu_0} B(l) l^{-i\gamma} \log l \right|^2 d\gamma. \]

Using Lemma 2.5 we see that
\[ I_1 \ll MN(T_1 + MN) \varepsilon, \quad I_2 \ll MN(T_1 + MN) \varepsilon. \]

Finally from (3.50)–(3.53) we obtain the desired estimate:
\[ R \ll M^{1-2\sigma} (T_1 + MN) T_1^{1-2\sigma + \varepsilon} \]
\[ \ll (M^{1-2\sigma} T_1^{2(1-\sigma)} + M^{2(1-\sigma)} (NT_1^{-1}) T_1^{2-2\sigma}) T_1^{\varepsilon} \]
\[ \ll (T_1^{2(1-\sigma)} + T_1^{2(1-\sigma)} T_1^{2(1-\sigma)}) T_1^{\varepsilon} \ll T_1^{2(1-\sigma) + \varepsilon} \]
where \(\varepsilon\) is chosen such that \(\varepsilon \geq 2/\delta\).

By Propositions 3.3–3.5 the proof of Theorem 1.1 is completed.

References

[1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, second edition, Springer-Verlag, 1990.
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[2] H. Bohr and E. Landau, *Sur les zéros de la fonction ζ(s) de Riemann*, Comp. Rend. Acad. Sci. Paris **158** (1914), 106–110.

[3] P. Deligne, *La conjecture de Weil. I*, Publ. Math. Inst. Hautes Études Sci. **43** (1974), 273–307.

[4] A. Good, *Approximative Funktionalgleichungen und Mittelwertsätze für Dirichletreihen, die Spitzenformen assoziiert sind*, Comment. Math. Helv. **50** (1975), 327–361.

[5] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika, **29** (1982), 278–295.

[6] G. H. Hardy and J. E. Littlewood, *The approximate functional equation for ζ(s) and ζ(s)^2*, Proc. London Math. Soc. (2) **29** (1929), 81–97.

[7] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I*, Math. Ann. **114** (1937), 1–28.

[8] M. N. Huxley, *On the difference between consecutive primes*, Invent. math. **15** (1972), 164–170.

[9] A. Ingham, *On the estimation of N(σ, T)*, Quart. J. Math. Oxford **11** (1940), 291–292.

[10] A. Ivić, *On zeta-functions associated with Fourier coefficients of cusp forms*, in Proceedings of the Amalfi Conference on Analytic Number Theory, E. Bombieri et al. (eds.), Università di Salerno, 1992, 231–246.

[11] M. Jutila, *Lectures on a Method in the Theory of Exponential Sums*, Tata Inst. Fund. Res. Lectures on Math. and Phys. **80**, Springer, Berlin, 1987.

[12] A. A. Karatsuba, *The distribution of prime numbers*, Russ. Math. Surv. **45** (1990), 99–171.

[13] A. A. Karatsuba and S. M. Voronin, *The Riemann Zeta-Function*, Walter de Gruyter de Gruyter Explosions in Mathematics 5, 1992.

[14] C. J. Moreno, “Explicit formulas in the theory of automorphic forms” *in Number Theory Day*, Lecture Notes in Math. **626**, Springer-Verlag, Berlin (1977), 73–216.