A 1-DIMENSIONAL PEANO CONTINUUM WHICH IS NOT AN IFS ATTRACTOR

TARAS BANAKH AND MAGDALENA NOWAK

ABSTRACT. Answering an old question of M.Hata, we construct an example of a 1-dimensional Peano continuum which is not homeomorphic to an attractor of IFS.

1. Introduction

A compact metric space $X$ is called an IFS-attractor if $X = \bigcup_{i=1}^{n} f_i(X)$ for some contracting self-maps $f_1, \ldots, f_n : X \to X$. In this case the family $\{f_1, \ldots, f_n\}$ is called an iterated function system (briefly, an IFS), see [2]. We recall that a map $f : X \to X$ is contracting if its Lipschitz constant

$$\text{Lip}(f) = \sup_{x \neq y} \frac{d(f_i(x), f_i(y))}{d(x, y)}$$

is less than 1.

Attractors of IFS appear naturally in the Theory of Fractals, see [2], [3]. Topological properties of IFS-attractors were studied by M.Hata in [4]. In particular, he observed that each connected IFS-attractor $X$ is locally connected. The reason is that $X$ has property S. We recall [6, 8.2] that a metric space $X$ has property S if for every $\varepsilon > 0$ the space $X$ can be covered by finite number of connected subsets of diameter $< \varepsilon$. It is well-known [6, 8.4] that a connected compact metric space $X$ is locally connected if and only if it has property S if and only if $X$ is a Peano continuum (which means that $X$ is the continuous image of the interval $[0, 1]$). Therefore, a compact space $X$ is not homeomorphic to an IFS-attractor whenever $X$ is connected but not locally connected. Now it is natural to ask if there is a Peano continuum homeomorphic to no IFS-attractor. An easy answer is “Yes” as every IFS-attractor has finite topological dimension, see [3]. Consequently, no infinite-dimensional compact topological space is homeomorphic to an IFS-attractor. In such a way we arrive to the following question posed by M. Hata in [4].

Problem 1.1. Is each finite-dimensional Peano continuum homeomorphic to an IFS-attractor?

In this paper we shall give a negative answer to this question. Our counterexample is a rim-finite plane Peano continuum. A topological space $X$ is called rim-finite if it has a base of the topology consisting of open sets with finite boundaries. It follows that each compact rim-finite space $X$ has dimension $\dim(X) \leq 1$.

2000 Mathematics Subject Classification. Primary 28A80; 54D05; 54F50; 54F45.

Key words and phrases. Fractal, Peano continuum, Iterated Function System, IFS-attractor.

The second author was supported in part by PHD fellowships important for regional development.
Theorem 1.2. There is a rim-finite plane Peano continuum homeomorphic to no IFS-attractor.

It should be mentioned that an example of a Peano continuum $K \subset \mathbb{R}^2$, which is not isometric to an IFS-attractor was constructed by M. Kwiecieński in [5]. However the continuum of Kwiecieński is homeomorphic to an IFS-attractor, so it does not give an answer to Problem 1.1.

2. S-dimension of IFS-attractors

In order to prove Theorem 1.2 we shall observe that each connected IFS-attractor has finite $S$-dimension. This dimension was introduced and studied in [1].

The $S$-dimension $S\text{-Dim}(X)$ is defined for each metric space $X$ with property $S$. For each $\varepsilon > 0$ denote by $S_\varepsilon(X)$ the smallest number of connected subsets of diameter $< \varepsilon$ that cover the space $X$ and let

$$S\text{-Dim}(X) = \lim_{\varepsilon \to +0} -\frac{\ln S_\varepsilon(X)}{\ln \varepsilon}.$$  

For each Peano continuum $X$ we can also consider a topological invariant

$$S\text{-dim}(X) = \inf\{S\text{-Dim}(X, d) : d \text{ is a metric generating the topology of } X\}.$$  

By [1, 5.1], $S\text{-dim}(X) \geq \dim(X)$, where $\dim(X)$ stands for the covering topological dimension of $X$.

Theorem 2.1. Assume that a connected compact metric space $X$ is an attractor of an IFS $f_1, f_2, \ldots, f_n : X \to X$ with contracting constant $\lambda = \max_{i \leq n} \text{Lip}(f_i) < 1$. Then $X$ has finite $S$-dimensions

$$S\text{-dim}(X) \leq S\text{-Dim}(X) \leq -\frac{\ln(n)}{\ln(\lambda)}.$$  

Proof. The inequality $S\text{-dim}(X) \leq S\text{-Dim}(X)$ follows from the definition of the $S$-dimension $S\text{-dim}(X)$. The inequality $S\text{-Dim}(X) \leq -\frac{\ln(n)}{\ln(\lambda)}$ will follow as soon as for every $\delta > 0$ we find $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ we get

$$-\frac{\ln S_\varepsilon(X)}{\ln \varepsilon} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$  

Let $D = \text{diam}(X)$ be the diameter of the metric space $X$. Since

$$\lim_{k \to \infty} \frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} = \lim_{k \to \infty} \frac{k \ln(n)}{(k - 1) \ln(\lambda) + \ln D} = \frac{\ln(n)}{\ln(\lambda)}^+,$$

there is $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ we get

$$-\frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$  

We claim that the number $\varepsilon_0 = \lambda^{k_0-1}D$ has the required property. Indeed, given any $\varepsilon \in (0, \varepsilon_0]$ we can find $k \geq k_0$ with $\lambda^kD < \varepsilon \leq \lambda^{k-1}D$ and observe that

$$C_k = \{f_{i_1} \circ \cdots \circ f_{i_k}(X) : i_1, \ldots, i_k \in \{1, \ldots, n\}\}$$

is a cover of $X$ by compact connected subsets having diameter $\leq \lambda^kD < \varepsilon$. Then $S_\varepsilon(X) \leq |C_k| \leq n^k$ and

$$-\frac{\ln(S_\varepsilon(X))}{\ln(\varepsilon)} \leq -\frac{\ln(n^k)}{\ln(\lambda^{k-1}D)} < -\frac{\ln(n)}{\ln(\lambda)} + \delta.$$
In the next section we shall construct an example of a rim-finite plane Peano continuum $M$ with infinite $S$-dimension $S\text{-}\dim(M)$. Theorem 2.1 implies that the space $M$ is not homeomorphic to an IFS-attractor and this proves Theorem 1.2.

3. The space $M$

Our space $M$ is a partial case of the spaces constructed in \cite{1} and called "shark teeth". Consider the piecewise linear periodic function

$$\varphi(t) = \begin{cases} t - n & \text{if } t \in [n, n + \frac{1}{2}] \text{ for some } n \in \mathbb{Z}, \\ n - t & \text{if } t \in [n - \frac{1}{2}, n] \text{ for some } n \in \mathbb{Z}, \end{cases}$$

whose graph looks as follows:

For every $n \in \mathbb{N}$ consider the function

$$\varphi_n(t) = 2^{-n} \varphi(2^n t),$$

which is a homothetic copy of the function $\varphi(t)$.

Consider the non-decreasing sequence

$$n_k = \lfloor \log_2 \log_2 (k + 1) \rfloor, \quad k \in \mathbb{N},$$

where $\lfloor x \rfloor$ is the integer part of $x$. Our example is the continuum

$$M = [0, 1] \times \{0\} \cup \bigcup_{k=1}^{\infty} \{(t, \frac{1}{k} \varphi_{n_k}(t)) : t \in [0, 1]\}$$

in the plane $\mathbb{R}^2$, which looks as follows:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The continuum $M$}
\end{figure}
The following theorem describes some properties of the continuum $M$ and implies Theorem 1.2 stated in the introduction.

**Theorem 3.1.** The space $M$ has the following properties:

1. $M$ is a rim-finite plane Peano continuum;
2. $\dim(M) = 1$ and $S\dim(X) = \infty$;
3. $M$ is not homeomorphic to an IFS attractor.

**Proof.** It is easy to see that $X$ is a rim-finite plane Peano continuum. The rim-finiteness of $M$ implies that $\dim(M) = 1$. To show that $S\dim(M) = \infty$, consider the number sequence $\vec{m} = (2^{n_k})_{k=1}^\infty$ and observe that the space $M$ is homeomorphic to the “shark teeth” space $W_{\vec{m}}$ considered in [1]. Taking into account that
\[
\lim_{k \to \infty} \frac{2^{n_k}}{k^\alpha} = 0 \quad \text{for any } \alpha > 0
\]
and applying Theorem 7.3(6) of [1], we conclude that $S\dim M = S\dim W_{\vec{m}} = \infty$. By Theorem 2.1 the space $M$ is not homeomorphic to an IFS-attractor. \qed

4. **Some Open Questions**

We shall say that a compact topological space $X$ is a topological IFS-attractor if $X = \bigcup_{i=1}^n f_i(X)$ for some continuous maps $f_1, \ldots, f_n : X \to X$ such for any open cover $\mathcal{U}$ of $X$ there is $m \in \mathbb{N}$ such that for any functions $g_1, \ldots, g_m \in \{f_1, \ldots, f_n\}$ the set $g_1 \circ \cdots \circ g_m(X)$ lies in some set $U \in \mathcal{U}$. It is easy to see that each IFS-attractor is a topological IFS-attractor and each connected topological IFS-attractor is metrizable and locally connected.

**Problem 4.1.** Is each (finite-dimensional) Peano continuum a topological IFS-attractor? In particular, is the space $M$ constructed in Theorem 3.1 a topological IFS-attractor?

5. **Acknowledgment**

The authors express their sincere thanks to Wiesław Kubiś for interesting discussions and valuable comments.

**References**

1. T. Banakh, M. Tuncali, *Controlled Hahn-Mazurkiewicz Theorem and some new dimension functions of Peano continua*, Topology Appl. **154**:7 (2007), 1286–1297.
2. M. Barnsley, *Fractals everywhere*, Academic Press, Boston, 1988.
3. G. Edgar, *Measure, topology, and fractal geometry*, Springer, New York, 2008.
4. M. Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2**:2 (1985), 381–414.
5. M. Kwieciński, *A locally connected continuum which is not an IFS attractor*, Bull. Polish Acad. Sci. Math. **47**:2 (1999), 127–132.
6. S. Nadler, *Continuum theory. An introduction*, Marcel Dekker, Inc., New York, 1992.

TARAS BANAKH AND MAGDALENA NOWAK

Institute of Mathematics, National University of L'viv, Ukraine

E-mail address: t.o.banakh@gmail.com

Institute of Mathematics, Jan Kochanowski University, Kielce, Poland

Instytut Matematyki, Jagiellonian University, Krakow, Poland

E-mail address: magdalena.nowak805@gmail.com