Suppression of oscillatory congestion via trunk link bandwidth and control gain in star network*

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Abstract The time delay-induced instability in an Internet congestion control model is investigated. The star topology is considered, and the link bandwidth ratio and the control gain are selected as the tunable parameters for congestion suppression. The stability switch boundary is obtained by the eigenvalue analysis for the linearized system around the equilibrium. To investigate the oscillatory congestion when the equilibrium becomes unstable, the center manifold reduction and the normal form theory are used to study the periodic oscillation induced by the delay. The theoretical analysis and numerical simulation show that the ratio between bandwidths of the trunk link and the regular link, rather than these bandwidths themselves, is crucial for the stability of the congestion control system. The present results demonstrate that it is not always effective to increase the link bandwidth ratio for stabilizing the system, and for some certain delays, adjusting the control gain is more efficient.

Key words Internet congestion control, Hopf bifurcation, delayed differential equation, stability, normal form

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1 Introduction

The smooth functioning of the Internet mainly relies on the transmission control protocol/Internet protocol (TCP/IP). Congestion control is a central issue of TCP which aims at improving the quality of Internet services. The congestion control algorithm regulates the end user’s window size, which is defined as the maximum number of data packets during data transmission, as well as the queue size at the link or in the buffer of the router to avoid an overload in the network system[1–2]. TCP-based congestion control has achieved admirable success since its early implementation[3–5] and has evolved into various TCP congestion control schemes and active queue management (AQM) policies, such as TCP-Tahoe, TCP-Reno, TCP-NewReno for

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TCP congestion control, random early detection (RED), random exponential marking (REM), and active virtual queue (AVQ) for AQM policies[6–12]. Generally, these congestion control schemes feature a moderate increase in the window size for low-level congestion and strong window control when the congestion indicator becomes remarkably significant, i.e., the additive increasing and multiplicative decreasing (AIMD) characteristic.

Hence, TCP-based congestion control plays a crucial role in current applications involving the Internet. However, the current congestion control scheme is not the ideal solution to the congestion problem. For example, some TCP congestion control schemes have long been criticized for the oscillatory feature embedded in them[13]. According to the interpretation for the TCP-based congestion control from the operational research perspective[14], congestion will not occur if data packets are transferred at a constant rate, pre-determined using the method of optimization. In other words, the stable steady state of the congestion control system rejects the occurrence of congestion. Conversely, the instability brings the risk of congestion, especially for the synchronous oscillation case in which the network may collapse at the peak of oscillation of data transmission rates[15]. Besides, oscillatory rates of data transmission are harmful to the quality of service for specific Internet applications, such as video games, online chatting, and online video watching. Therefore, significant effort has been made to study the stability of the congestion control system, especially the factors that potentially lead to the oscillatory rates of data transmission[16–17].

The round trip time (RTT), or time delay, is among the most important parameters of TCP-based congestion control systems. Unlike physical and algorithmic parameters such as link bandwidth and control gain, the time delay is determined by neither hardware nor software. Therefore, significant uncertainty is introduced to the congestion control problem. Previous studies have shown that the time delay may induce the oscillatory rates of data transmission, and consequently degrade the performance of the network[18–20]. Another key parameter is the bandwidth of the link, especially the trunk link, namely, the link at which data packet from multi-sources stack. Such link is paramount in the investigation of congestion control. Before the data packets are injected into the trunk link, congestion is not likely to occur because the network resources serving these packets are distributive and thus are sufficient for packets transmission. However, after these packets enter the trunk link, packets from various users will have to compete for finite network resources, namely, the trunk link bandwidth. Therefore, the congestion may arise. As it is natural to ascribe the Internet congestion to the limited bandwidth of the trunk link, one common intervention for frequently congested network is to increase the trunk link bandwidth. However, such treatment may induce new problems. For example, it was argued that the excessively large buffer size, which acts similar to the link bandwidth in the network, may cause unnecessary latency and consequently cause harm to the network performance. A recent research has reported that the Braess’ paradox, which states that increasing the network bandwidth may unexpectedly induce congestion, may also be observed in the communication network[21]. Therefore, a careful investigation is necessary to uncover the relation between the link bandwidth and the network performance for congestion control. Moreover, the Internet congestion control system is present in the form of a feedback control system. It is well known that the control gain has significant influence on the stability of such system[22–23]. Therefore, it is rational to seek the possibility of stabilizing the congestion control system by selecting proper control gains.

In this work, we study the oscillatory congestion induced by the time delay, as well as the congestion suppression by adjusting the trunk link bandwidth and the control gain for a network with the star topology. The star network is a typical in the Internet. It is characterized by the existence of a central hub that connects various hosts. Owing to such features, congestion may occur in the star network especially at the link that connects the central hub (typically, a router) with the outer Internet as such link is in fact a trunk link where traffic can be heavily congested. However, to our best knowledge, investigations related to the theoretical study of the congestion
control problem in the star network are scarce. To investigate such problems both qualitatively and quantitatively, a mathematical model that describes the temporal evolution of congestion control system is necessary. In Ref. [24], Kelly proposed a model for which the equilibrium is proved to be the global optimum of the corresponding source allocation problem over the network. Hollot et al. [25] and Misra [26] formulated the mathematical model of TCP/AQM using stochastic differential equations and studied the stability of the equilibrium of the model. Such a model includes the maximum amount of the technical details of the congestion control algorithm, and therefore attracts much attention from the researchers [27–31]. However, due to the same reason, the model is highly complicated for further analysis from the theoretical perspective. In this work, we use the model proposed by Kelly as the basic model since it captures the AIMD feature of TCP and meanwhile is well simplified.

To our best knowledge, few works are related to the joint influence of the delay, the trunk link bandwidth, and the control gain on the stability and dynamics of the congestion control model. In this paper, we study the impact of time delay on the congestion control model as well as the principle of selecting tunable parameters such as the link bandwidth ratio and control gain. The remainder of the paper is organized as follows. The mathematical model of the Internet congestion control system and the location of the equilibrium are discussed in Section 2. The stability of the equilibrium is investigated in Section 3. Based on the center manifold reduction together with the normal form theory, the formulae for determining the properties of the Hopf bifurcating periodic solutions are derived in Section 4. In Section 5, we discuss the suppression of oscillatory congestion via tuning the trunk link bandwidth ratio and control gain. Moreover, numerical results are provided to verify our theoretical analysis. Conclusions are presented in Section 6.

2 Congestion control model and its equilibrium

In this section, we formulate the mathematical model of the congestion control algorithm for the star network and study the equilibrium of the model.

2.1 Mathematical model

In this paper, we focus on the Internet congestion control model with two sources and three links, as shown in Fig. 1.

![Fig. 1 Topology of network under investigation](image)

According to Fig. 1, the Internet congestion control model is described by

\[
\begin{align*}
\dot{x}_1(t) &= k_1 \left( w_1 - x_1 \left( p_1(x_1 + x_2) + \tilde{p}(x_1 + x_2) \right) \right), \\
\dot{x}_2(t) &= k_2 \left( w_2 - x_2 \left( p_2(x_2) + \tilde{p}(x_1 + x_2) \right) \right),
\end{align*}
\]

where \(x_1(t)\) and \(x_2(t)\) represent the sending rates of sources 1 and 2 at time \(t\), respectively, and \(k_1\) and \(k_2\) are the positive control gains. \(w_1\) and \(w_2\) are the parameters of adjusting the equilibria, \(p_1(\cdot), p_2(\cdot),\) and \(\tilde{p}(\cdot)\) are the congestion indication functions or marking functions, which are increasing, nonnegative, and not identically zero. It is noteworthy that, if congestion does not occur in the Internet, \(\dot{x}_i(t) = k_i w_i (i = 1, 2)\), implying that the rate of data transmission is
increasing unboundedly. When the Internet congestion occurs, the source receives a signal to reduce the data transmission rate. In (1), \( x_i = x_i(t - \tau) \) (\( i = 1, 2 \)), where \( \tau \) represents the time delay. We assume for simplicity that the time lags in the delayed terms in \( p_1(\cdot), p_2(\cdot), \) and \( \bar{p}(\cdot) \) are identical.

As in area networks, the network equipment and Internet speed are nearly the same for all users. Herein, we consider the special case in which \( k_1 = k_2 = k \), and \( w_1 = w_2 = w \). The congestion indication functions are of the following form:

\[
p_1(x) = p_2(x) = p(x) = \frac{\theta x}{\theta x + 2(C_1 - x)}, \quad \bar{p}(x) = \frac{\theta x}{\theta x + 2(C_2 - x)},
\]

where \( C_1 \) is the bandwidth of the regular link, namely, the link that is used by each source alone, and \( C_2 \) is the bandwidth of the trunk link. According to the physical meaning of \( C_1 \) and \( C_2 \), the sending rates of the two sources should satisfy the following condition:

\[
0 < x_1(t) \leq C_1, \quad 0 < x_2(t) \leq C_1, \quad 0 < x_1(t) + x_2(t) \leq C_2.
\]

In the remainder of this paper, we assume \( \theta x = 0.5 \) and \( w = 1 \) as in Ref. [22]. It is noteworthy that (1) describes the evolution of the congestion control system for the simplest star network because the router on the right-hand side in Fig. 1 can be viewed as the hub.

To simplify the congestion control model and reduce the number of the system parameters, we introduce the following transformations that will not change the system qualitatively:

\[
y_i(t) = \frac{x_i(t)}{C_1}, \quad W = \frac{w}{C_1}, \quad C = \frac{C_1}{C_2}, \quad i = 1, 2.
\]

Then, (1) is transformed into

\[
\begin{cases}
y_1(t) = k(W - y_{1\tau}(p(y_{1\tau}) + \bar{p}(y_{1\tau} + y_{2\tau}))), \\
y_2(t) = k(W - y_{2\tau}(p(y_{2\tau}) + \bar{p}(y_{1\tau} + y_{2\tau}))),
\end{cases}
\]

where \( p(y) = \frac{\theta y}{\theta y + 2w} \), and \( \bar{p}(y) = \frac{\theta y}{\theta y + 2w - 3w} \). In the following theoretical analysis, we consider (4) instead of (1).

The model investigated in the present work originates from the congestion control model of a single user with a single link constructed by Kelly [24] as follows:

\[
\dot{x}(t) = k(w - x_{\tau} p(x_{\tau})),
\]

where \( x(t) \), \( k \), and \( w \) have the same meanings as their counterparts in the previous models. Again, \( x_{\tau} \) denotes the rate of data transmission delayed by \( \tau \). The congestion indicator, or equivalently, the penalty function \( p(\cdot) \), is given by [24]:

\[
p(x) = \frac{\theta x}{\theta x + 2(C - x)},
\]

where \( \theta x \) can be viewed as a parameter that regulates the strength of the punitive part in the congestion control algorithm, and \( C \) represents the link bandwidth. Figure 2 shows the configuration of \( p(\cdot) \) for different values of \( \theta x \) when \( C = 5 \). It can be seen that for a small rate of link occupancy, the network encourages the user to send packets because the penalty function \( p(\cdot) \) assumes a small value. When the link occupancy rate becomes large, the value of \( p(\cdot) \) approaches 1, implying the full punishment on the user. It should be noted that although this model is derived based on the nonlinear optimization theory [24], it is usually interpreted as a mathematical abstraction of the aforementioned AIMD mechanism of the Internet congestion control. More specifically, the right-hand side of the model above shows the balance between the “additive increasing” and “multiplicative decreasing” of the TCP algorithm.
First, we present a lemma to illustrate the symmetric property of the equilibrium.

Lemma 1 Assume that the conditions guaranteeing \( 0 < y_1^* \leq 1, 0 < y_2^* \leq 1, \) and \( y_1^* + y_2^* \leq C \) are satisfied. Then, \( y_1^* = y_2^* \).

Proof From (5), we obtain
\[
\frac{(y_1^* - y_2^*) (4y_1^* + y_2^*) - 3y_1^*y_2^*}{(4 - 3y_1^*)(4 - 3y_2^*)} = \frac{(y_1^* - y_2^*) (y_1^* + y_2^*)}{4C - 3(y_1^* + y_2^*)}.
\]

Now, we show \( y_1^* = y_2^* \). Conversely,
\[
\frac{4(y_1^* + y_2^*) - 3y_1^*y_2^*}{(4 - 3y_1^*)(4 - 3y_2^*)} = \frac{y_1^* + y_2^*}{4C - 3(y_1^* + y_2^*)}.
\]

From (2) and (3), we have \( 0 < y_1^* \leq 1, 0 < y_2^* \leq 1, \) and \( y_1^* + y_2^* \leq C \). It follows
\[
\frac{4(y_1^* + y_2^*) - 3y_1^*y_2^*}{(4 - 3y_1^*)(4 - 3y_2^*)} = \frac{y_1^* (1 - \frac{3}{4}y_2^*) + y_2^* (1 - \frac{3}{4}y_1^*)}{4 (1 - \frac{3}{4}y_1^*) (1 - \frac{3}{4}y_2^*)} > 0.
\]

It is obvious that the right-hand side of (6) is negative, which brings a contradiction. Thus, we obtain \( y_1^* = y_2^* \).

Let \( y^* = y_1^* = y_2^* \), and \( y^* \) be the solution to the following equation:
\[
6y^3 + (9W - 4 - 2C)y^2 - 6W(C + 2)y + 8WC = 0.
\]

Clearly, \( 0 < y^* \leq 1, \) and \( 2y^* \leq C \).

According to Ref. [32], we can obtain the discriminant of the roots of (7) as \( H(W) = H_1W^4 + H_2W^3 + H_3W^2 + H_4W \), where \( H_1 = -8748C^2 + 34992C - 34992, H_2 = -11664C^3 + 23328C^2 + 46656C - 93312, H_3 = -432C^4 - 24192C^3 + 93312C^2 - 96768C - 6912, \) and \( H_4 = -768C(C + 2)^3 \). Then, the following four cases are possible regarding the solutions to (7).

Case 1 If \( W = 0 \), and \( C = -2 \), then (7) has a triple root \( y = 0 \), which is contrary with the fact that \( y \) is positive.
Case 2 If $H(W) > 0$, then (7) has one real root and one pair of conjugate complex roots.

Case 3 If $H(W) < 0$, then (7) has three different real roots.

Case 4 If $H(W) = 0$, then (7) has three real roots, two of which are repeated roots.

$H_1$, $H_2$, $H_3$, and $H_4$ are plotted in Fig. 3. From Fig. 3, we can find that $H_1$ and $H_2$ are not greater than zero (equal to zero when $C = 2$), while $H_3$ and $H_4$ are always less than zero. For $W > 0$, we conclude that $H(W) < 0$ always holds.

Based on the analysis above, we can conclude that only Case 3 may occur. Moreover, the two roots of (7) are positive, and the third one is negative. However, it is difficult to derive a concise expression for the solution to (7) in terms of $W$ and $C$. Therefore, the numerical method is introduced to study the relation between the equilibrium and parameters qualitatively.

According to the implicit function theorem, we have

$$
\frac{\partial y}{\partial C}\bigg|_{y=y^*} = \frac{(y^*)^2 + 3Wy^* - 4W}{9(y^*)^2 + (9W - 4 - 2C)y^* - 3W(C + 2)}
$$

(8)

Through a simple deduction, we conclude that the numerator of (8) cannot be zero. The same results can be obtained for the denominator. That is to say, the sign of (8) is positive or negative. We use $W = \frac{1}{4}$, i.e., $C_1 = 3$ as an example to verify the sign of $\frac{\partial y}{\partial C}$. It is clear from Fig. 4 that $\frac{\partial y}{\partial C}\bigg|_{y=y^*} > 0$ for any $C$, i.e., $y^*$ increases with $C$. For convenience in the following discussion, we define $y_{\infty}^* = \lim_{C \to \infty} y^*$ and $\tau_{\infty}^*$ as the critical value of the delay for the stability switch when $C \to \infty$. To obtain the limits of $y^*$, we set the right-hand side of (8) to be zero.
Then, we have \( y_\infty^* = \sqrt{9W^2 + 16W - 3W^2} \). This implies that if the bandwidth of the link used by each source alone is incomparable to that of the trunk link, regardless of the size of \( C_1 \), the equilibrium will not change significantly.

![Fig. 4 Derivative of equilibrium with respect to \( C \)](image)

Up to now, we have analyzed the number and location of the equilibria of the congestion control model under consideration. Moreover, we have verified that the coordinates of the equilibrium will increase with the link bandwidth ratio \( C \) and approach their limits when \( C \) approaches infinity.

Next, we study the stability of the equilibrium as well as the nonlinear dynamics that arises when the equilibrium becomes unstable.

### 3 Existence of Hopf bifurcation in Internet congestion control model

In this section, we consider the existence of bifurcating periodic solutions in the Internet congestion control model.

Setting \( u_i(t) = y_i(t) - y^* \) (\( i = 1, 2 \)) and \( u(t) = (u_1(t), u_2(t))^T \), we obtain

\[
\dot{u}(t) = Lu(t) + H(u(t)) + \text{h.o.t.},
\]

where

\[
Lu(t) = \begin{bmatrix} a_1u_1 + b_1u_2 \\ a_2u_1 + b_2u_2 \end{bmatrix},
\]

\[
H(u(t)) = \begin{bmatrix} d_{11}u_1^2 + d_{12}u_1u_2 + d_{13}u_2^2 + e_{11}u_1^3 + e_{12}u_1^2u_2 + e_{13}u_1u_2^2 + e_{14}u_2^3 \\ d_{21}u_1^2 + d_{22}u_1u_2 + d_{23}u_2^2 + e_{21}u_1^3 + e_{22}u_1^2u_2 + e_{23}u_1u_2^2 + e_{24}u_2^3 \end{bmatrix},
\]

and h.o.t. stands for higher order terms. The coefficients \( a_i, b_i, d_{ij}, e_{ik} \) \((i = 1, 2; j = 1, 2, 3; k = 1, 2, 3, 4)\) and the derivatives of the congestion indication functions are listed in Appendix A. Due to the symmetry of the system, some of the coefficients are identical. For example, it should be noted that \( a_1 = b_2 \), and \( a_2 = b_1 \). Further, \( 0 < y^* < 1, 2y^* < C \), and \( k > 0 \). Through a simple calculation, we can verify that all the coefficients in (9) are negative.

The linearized equation of (4) at the equilibrium \((y^*, y^*)^T\) is

\[
\begin{align*}
\dot{u}_1(t) &= a_1u_1 + a_2u_2, \\
\dot{u}_2(t) &= a_2u_1 + a_1u_2,
\end{align*}
\]

and the corresponding characteristic equation of (10) is given by

\[
\lambda^2 - 2a_1\lambda e^{-\lambda\tau} + (a_1^2 - a_2^2)e^{-2\lambda\tau} = 0.
\]
To obtain the explicit expression of the critical stability boundary, we assume that \( \lambda = i \omega \) is the root of (11), where \( \omega > 0 \) is a real number. Then, substituting \( \lambda = i \omega \) into (11) and equating the real and imaginary parts to zero yield

\[
\begin{align*}
-2a_1 \omega \sin(\omega \tau) + (a_1^2 - a_2^2) \cos(2\omega \tau) &= \omega^2, \\
-2a_1 \omega \cos(\omega \tau) - (a_1^2 - a_2^2) \sin(2\omega \tau) &= 0,
\end{align*}
\]

(12)

which can be rewritten as the following two equations:

\[
\begin{align*}
-2a_1 \omega \sin(\omega \tau) + (a_1^2 - a_2^2) \cos(2\omega \tau) &= \omega^2, \\
\cos(\omega \tau) &= 0,
\end{align*}
\]

or

\[
\begin{align*}
-2a_1 \omega \sin(\omega \tau) + (a_1^2 - a_2^2) \cos(2\omega \tau) &= \omega^2, \\
\sin(\omega \tau) &= -\frac{a_1 \omega}{a_1^2 - a_2^2}.
\end{align*}
\]

(14)

The following discussion will be divided into two cases.

Case 1 From \( \cos(\omega \tau) = 0 \), it follows that \( \cos(2\omega \tau) = 2 \cos(\omega \tau)^2 - 1 = -1 \). If \( \sin(\omega \tau) = -1 \), substituting \( \sin(\omega \tau) = \cos(2\omega \tau) = -1 \) into the first equation of (13) yields \( \omega^2 - 2a_1 \omega + (a_1^2 - a_2^2) = 0 \). It can be checked that \( a_1^2 - a_2^2 > 0 \). Obviously, the equation contains two negative roots \( \omega_1 = a_1 + a_2 \) and \( \omega_2 = a_1 - a_2 \), which contracts the assumption that \( \omega > 0 \). Likewise, for the case \( \sin(\omega \tau) = 1 \), we obtain

\[
\omega^2 + 2a_1 \omega + (a_1^2 - a_2^2) = 0.
\]

(15)

The roots of (15) are \( \omega_1 = -a_1 + a_2 \) and \( \omega_2 = -a_1 - a_2 \). According to \( \sin(\omega \tau) = 1 \) and \( \cos(\omega \tau) = 0 \), we obtain \( \omega \tau = \frac{(4n + 1)\pi}{2\omega l} \) \((n = 0, 1, 2, \cdots)\). Thus, one has

\[
\tau_{nl} = \frac{(4n + 1)\pi}{2\omega l}, \quad n = 0, 1, 2, \cdots, \quad l = 1, 2.
\]

Case 2 From \( \sin(\omega \tau) = -\frac{a_1 \omega}{a_1^2 - a_2^2} \), we have

\[
\cos(2\omega \tau) = 1 - 2 \sin(\omega \tau)^2 = \frac{(a_1^2 - a_2^2)^2 - 2a_1^2 \omega^2}{(a_1^2 - a_2^2)^2}.
\]

Substituting the formula above into the first equation of (14) yields \( \omega = \sqrt{a_1^2 - a_2^2} \). Because \( \sin(\omega \tau) \in [-1, 1] \), we can deduce \( -a_2^2 \geq 0 \) from \( \sin(\omega \tau) = -\frac{a_1 \omega}{a_1^2 - a_2^2} \in [-1, 1] \), which is impossible for any nonzero \( a_2 \). Hence, this case cannot occur.

Based on the discussion above, one can draw the conclusion that for the case \( \sin(\omega \tau) = 1 \) and \( \cos(\omega \tau) = 0 \), (11) contains a pair of purely imaginary roots \( \pm i \omega_l \) \((l = 1, 2)\), where \( \omega_1 = -a_1 + a_2, \omega_2 = -a_1 - a_2 \), and \( \tau_{nl} = \frac{(4n + 1)\pi}{2\omega l} \) \((n = 0, 1, 2, \cdots, l = 1, 2)\).

Now, we verify the second condition for the occurrence of Hopf bifurcation, i.e.,

\[
\left. \frac{dR(\lambda)}{d\tau} \right|_{\tau = \tau_{nl}} \neq 0.
\]

The proof is detailed in Appendix A.

It is obvious that the roots of (11) depend continuously on the parameter \( \tau \). When \( \tau = 0 \), we know that (11) contains two roots \( \lambda_1 \) and \( \lambda_2 \), which satisfy

\[
\lambda_1 + \lambda_2 = 2a_1 < 0, \quad \lambda_1 \lambda_2 = a_1^2 - a_2^2 > 0.
\]

Therefore, both \( \lambda_1 \) and \( \lambda_2 \) contain negative real parts. Suppose that \( \tau_{01} \) is the smallest positive value so that (11) has a pair of purely imaginary roots. Because the roots of (11)
continuously depend on $\tau$, it is clear that the roots of (11) contain negative real parts for $\tau \in [0, \tau_0]$. It can be proved that (11) contains a pair of purely imaginary roots $\pm i\omega_1$ and all the other roots have strictly negative real parts when $\tau = \tau_0$. Assume the opposite, namely, $\lambda = u + iv$, where $u > 0$ is a root of (11) when $\tau = \tau_0$. Since the roots of (11) continuously depend on $\tau$, there exists $\tau' \in (0, \tau_0)$ so that (11) has purely imaginary roots at $\tau = \tau'$, which contradicts the assumption that $\tau_0$ is the smallest such $\tau$. Moreover, it follows from $\frac{d\lambda}{d\tau} > 0$ that (11) has at least a pair of roots with positive real parts when $\tau > \tau_0$.

Up to now, we have studied the critical condition for the occurrence of Hopf bifurcation. Figure 5 shows the stability switch boundary curve of the equilibrium of system (4) with $k = 1$, $w = 1$, and $C_1 = 3$.

![Figure 5](image-url)

**Fig. 5** Stability switch boundary of equilibrium of (4)

Based on Fig. 5, we claim that the system is free of congestion near the equilibrium for the parameters selected from the shaded region. However, the mechanism that induces the instability and the type of congestion that may arise after the stability switch remains unclear. The nonlinear analysis will be employed to study these issues in the next section.

4 Calculation of normal form on center manifold and stability of bifurcating periodic orbit

Let $C = C([-\tau, 0], \mathbb{R}^2)$ and $u_t(\theta) = u(t + \theta)$, where $-\tau \leq \theta \leq 0$, and $\tau = \tau^* + \mu$. Then, (9) becomes

$$\dot{u}(t) = L_\mu(u_t) + F(\mu, u_t),$$

(16)

where

$$L_\mu(u_t) = \begin{bmatrix} a_1 u_{1\tau} + a_2 u_{2\tau} \\ a_2 u_{1\tau} + a_1 u_{2\tau} \end{bmatrix},$$

$$F(\mu, u_t) = \begin{bmatrix} d_{11} u_{1\tau}^2 + d_{12} u_{1\tau} u_{2\tau} + \ldots + e_{11} u_{1\tau}^3 + e_{12} u_{1\tau}^2 u_{2\tau} + \ldots + e_{14} u_{2\tau}^3 \\ d_{21} u_{2\tau}^2 + d_{22} u_{1\tau} u_{2\tau} + \ldots + e_{21} u_{2\tau}^3 + e_{22} u_{1\tau}^2 u_{2\tau} + \ldots + e_{24} u_{2\tau}^3 \end{bmatrix}.$$  

The expressions of the coefficients above are listed in (A1).

According to the Riesz representation theorem, the function of bounded variation $\eta(\theta)$ exists with $\theta \in [-\tau, 0]$ such that

$$L_\mu(\phi) = \int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta).$$

$\$
Define $L_0 = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}$. Then, $\eta(\theta, \mu) = L_0 \delta(\theta + \tau)$, where $\delta$ represents the Dirac delta function.

For $\phi \in C^1([-\tau, 0], \mathbb{R}^2)$, we define
\[
A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-\tau, 0), \\
\int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0,
\end{cases}
\]
\[
R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\
F(\mu, \theta), & \theta = 0.
\end{cases}
\]

Then, (16) can be rewritten as
\[
\dot{u}_t = A(\mu)u_t + R(\mu)u_t. 
\tag{17}
\]

For $\psi \in C^*([0, \tau], \mathbb{R}^2)$, the adjoint operator $A^*$ of $A$ is defined as
\[
A^*(\mu)\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tau], \\
\int_{-\tau}^{0} d\eta^T(s, \mu)\psi(-s), & s = 0.
\end{cases}
\]

For $\psi \in C^*([0, \tau], \mathbb{R}^2)$, and $\phi \in C([-\tau, 0], \mathbb{R}^2)$, we define a bilinear form
\[
\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \bar{\psi}^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,
\tag{18}
\]
where $d\eta(\theta) = d\eta(\theta, 0)$, and $C^*$ is the dual space of $C$.

Let $q(\theta)$ and $q^*(\theta)$ be such that $A(0)q(\theta) = i\omega q(\theta)$, and $A^*(0)q^*(\theta) = -i\omega q^*(\theta)$. In other words, $q(\theta)$ and $q^*(\theta)$ are the eigenvectors of $A(0)$ and $A^*(0)$ associated with $i\omega$ and $-i\omega$, respectively.

Obviously, $q(\theta)$ and $q^*(\theta)$ are of the form
\[
q(\theta) = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} e^{i\omega \theta}, \quad q^*(s) = N \begin{bmatrix} 1 \\ \beta \end{bmatrix} e^{i\omega s},
\]
where $\alpha, \beta$, and $N$ are given in Appendix B.

### 4.1 Center manifold reduction

According to the work of Faria and Magalhaes\cite{33}, the phase space $C$ can be decomposed by $\Lambda = \{i\omega, -i\omega\}$ as $C = P \oplus Q$, where $P$ is the generalized eigenspace associated with $\Lambda$. Let $\Phi$ and $\Psi$ be the bases for $P$ and $P^*$ associated with the eigenvalues $i\omega$ and $-i\omega$, respectively. Define $m = \dim(P)$, and assume that $B$ is an $m \times m$ matrix with the point spectrum $\sigma(B) = \Lambda$. For the Internet congestion control model considered in this paper, we have $m = 2, B = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix}$, and
\[
\Phi = (\phi_1, \phi_2), \quad \phi_1 = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} e^{i\omega \theta}, \quad \phi_2 = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} e^{-i\omega \theta},
\]
\[
\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad \psi_1 = (N, N\beta) e^{-i\omega s}, \quad \psi_2 = (N, N\beta) e^{i\omega s},
\]
where $N$ is the conjugation of $N$. 
4.2 Normal form of (4)

Recall \( \mu = \tau - \tau^* \). Therefore, the Hopf bifurcation occurs at \( \mu = 0 \). Then, (17) can be rewritten as

\[
\dot{u}(t) = L_{\tau^*}(u_t) + \bar{F}(u_t, \mu),
\]

where \( \bar{F}(u_t, \mu) \) is the nonlinear term in (17). Enlarge the phase space \( C \) to the Banach space \( BC = C \times \mathbb{R}^2 \), i.e., the space of uniformly continuous functions from \( [-\tau, 0] \) to \( \mathbb{R}^2 \) with a jump discontinuity at zero. Using the decomposition \( u_t = \Phi x(t) + y \), with \( x(t) \in \mathbb{C}^2 \) and \( y \in Q^1 = Q \cap C \) leading to the decomposition \( BC = P \oplus F\pi, Q \subseteq F\pi \), where \( F \) denotes the kernel function. Then, (17) is decomposed as

\[
\begin{aligned}
\dot{x} &= Bx + \Psi(0) \bar{F}(\Phi x + y, \mu), \\
\dot{y} &= A_{Q^1} y + (I - \pi)X_0 \bar{F}(\Phi x + y, \mu),
\end{aligned}
\]

where \( A_{Q^1} : Q^1 \subseteq F\pi \) is defined as

\[
A_{Q^1} \phi = \dot{\phi} + X_0(L_{\tau^*} \phi - \dot{\phi}(0))
\]

with

\[
X_0(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\
I, & \theta = 0 \end{cases}
\]

and

\[
\pi : BC \to P, \quad \pi(\phi + X_0 \gamma) = \Phi((\Psi, \phi) + \Psi(0) \gamma).
\]

Expand the nonlinear terms of (20) into the Taylor series as follows:

\[
\begin{aligned}
\Psi(0) \bar{F}(\Phi x + y, \mu) &= \frac{1}{2!} f^1_2(x, y, \mu) + \frac{1}{3!} f^1_3(x, y, \mu) + \cdots, \\
(I - \pi)X_0 \bar{F}(\Phi x + y, \mu) &= \frac{1}{2!} f^2_2(x, y, \mu) + \frac{1}{3!} f^3_2(x, y, \mu) + \cdots,
\end{aligned}
\]

where \( f^1_j(x, y, \mu) \) and \( f^2_j(x, y, \mu) \) represent homogeneous polynomials in \( (x, y, \mu) \) of degree \( j \) with coefficients in \( \mathbb{C}^2 \) and \( F\pi \), respectively. Therefore, (20) can be written as

\[
\begin{aligned}
\dot{x} &= Bx + \sum_{j \geq 2} \frac{1}{j!} f^1_j(x, y, \mu), \\
\dot{y} &= A_{Q^1} y + \sum_{j \geq 2} \frac{1}{j!} f^2_j(x, y, \mu).
\end{aligned}
\]

The normal form of (17) on the center manifold at \( \mu = 0 \) is written as

\[
\dot{x} = Bx + \frac{1}{2!} g^1_2(x, 0, \mu) + \frac{1}{3!} g^1_3(x, 0, \mu) + \text{h.o.t.},
\]

where \( g^1_2(x, 0, \mu) \) and \( g^1_3(x, 0, \mu) \) are the second-order and third-order terms in \( (x, \mu) \), respectively.

\[
\begin{aligned}
g^1_2(x, y, \mu) &= f^1_2(x, y, \mu) - (B, U^1_2)(x), \\
g^1_3(x, y, \mu) &= f^2_3(x, y, \mu) - (D_x U^3_2(x) Bx - A_{Q^1}(U^2_2(x)))
\end{aligned}
\]
with \((B, U_j^1)(x) = D_x U_j^1(x)Bx - BU_j^1(x)\). Here, \(U_j^1 \in V_j^3(C^2)\), and \(U_j^2 \in V_j^3(Q^1)\), where \(V_j^3(C^2)\) represents the homogeneous polynomials of degree \(j\) in variables \(x_1, x_2,\) and \(\mu\), and \(Q^1 = Q \cap C\).

Let \(M_j^1\) be the operator from \(V_j^3(C^2)\) to itself, which is defined by

\[
\begin{cases}
(M_j^1 h)(x, \mu) = D_x h(x, \mu)Bx - Bh(x, \mu), \\
V_j^3(C^2) = G(M_j^1) \oplus F(M_j^1), \quad g_j^1(x, 0, \mu) \in F(M_j^1),
\end{cases}
\]  

and

\[
V_j^2(C^2) = \text{span}\left\{\left(\begin{array}{c}
x_1^2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_1x_2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_2^2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_1\mu \\
0
\end{array}\right), \left(\begin{array}{c}
x_2\mu \\
0
\end{array}\right), \left(\begin{array}{c}
\mu^2 \\
0
\end{array}\right), \\
\left(\begin{array}{c}
0 \\
x_1^2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_1x_2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2^2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_1\mu
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2\mu
\end{array}\right), \left(\begin{array}{c}
0
\end{array}\right)
\right\}.
\]

Through a simple calculation, we have

\[
G(M_j^1) = \text{span}\left\{\left(\begin{array}{c}
x_1^2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_1x_2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_2^2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_1\mu \\
0
\end{array}\right), \left(\begin{array}{c}
x_2\mu \\
0
\end{array}\right), \left(\begin{array}{c}
\mu^2 \\
0
\end{array}\right), \\
\left(\begin{array}{c}
0 \\
x_1^2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_1x_2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2^2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_1\mu
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2\mu
\end{array}\right), \left(\begin{array}{c}
0
\end{array}\right)
\right\}.
\]

Then, we have

\[
F(M_j^1) = \text{span}\left\{\left(\begin{array}{c}
x_1\mu \\
0
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2\mu
\end{array}\right)\right\}.
\]

Similarly,

\[
F(M_j^1) = \text{span}\left\{\left(\begin{array}{c}
x_1x_2 \\
0
\end{array}\right), \left(\begin{array}{c}
x_1\mu^2 \\
0
\end{array}\right), \left(\begin{array}{c}
0 \\
x_1x_2
\end{array}\right), \left(\begin{array}{c}
0 \\
x_2\mu^2
\end{array}\right)\right\}.
\]

From the above results, we obtain

\[
\frac{1}{2!} f_2^1(x, 0, \mu) = \Psi(0)L_\mu(\Phi x)
\]

\[
= \left[\begin{array}{c}
N \\
N\beta
\end{array}\right] (L_\mu(\phi_1)x_1 + L_\mu(\phi_2)x_2).
\]

The second term on the right-hand side of the normal form is given by

\[
\frac{1}{2!} g_2^1(x, 0, \mu) = \frac{1}{2!} P_{F(M_j^1)} f_2^1(x, 0, \mu)
\]

\[
= P_{F(M_j^1)} \left[\begin{array}{c}
N \\
N\beta
\end{array}\right] (L_\mu(\phi_1)x_1 + L_\mu(\phi_2)x_2),
\]

where

\[
L_\mu(\phi_1) = i\mu\omega \left[\begin{array}{c}
1 \\
\alpha
\end{array}\right], \quad L_\mu(\phi_2) = -i\mu\omega \left[\begin{array}{c}
1 \\
\alpha
\end{array}\right].
\]

Thus, we obtain

\[
\frac{1}{2!} g_2^1(x, 0, \mu) = \left[\begin{array}{c}
Lx_1\mu \\
Lx_2\mu
\end{array}\right].
\]
Similarly, we have

\[
\frac{1}{3!} f_1 = \frac{1}{3!} f_3(x, 0, 0) = \Psi(0)F_3(\Phi x, \tau^*) = \left[ \begin{array}{c} N \\ N \beta \end{array} \right] \times \left[ \begin{array}{c} h_1 \\ h_2 \end{array} \right],
\]

\[
\frac{1}{3!} g_1(x, 0, 0) = \frac{1}{3!} F_3(\Phi x, \tau^*) = \left[ \begin{array}{c} K x_1^2 x_2 \\ K x_1 x_2^2 \end{array} \right].
\]

where

\[
\begin{align*}
h_1 &= e_{11}(x_1 + x_2)^3 + e_{12}(x_1 + x_2)^2(x_1\alpha + x_2\alpha) \\
&\quad + e_{13}(x_1 + x_2)(x_1\alpha + x_2\alpha)^2 + e_{14}(x_1\alpha + x_2\alpha)^3,
\end{align*}
\]

\[
\begin{align*}
h_2 &= e_{14}(x_1 + x_2)^3 + e_{13}(x_1 + x_2)^2(x_1\alpha + x_2\alpha) \\
&\quad + e_{12}(x_1 + x_2)(x_1\alpha + x_2\alpha)^2 + e_{11}(x_1\alpha + x_2\alpha)^3,
\end{align*}
\]

\[S = \text{span} \left\{ \left( x_1^2 x_2, 0 \right), \left( 0, x_1 x_2^2 \right) \right\}. \]

L and K are shown in Appendix B.

Thus, the normal form of (17) is given by

\[
\dot{x} = Bx + \left[ \begin{array}{c} L x_1 \mu \\ T x_2 \mu \end{array} \right] + \left[ \begin{array}{c} K x_1^2 x_2 \\ K x_1 x_2^2 \end{array} \right] + O(|x|^2 + |x|^4).
\]

(25)

Introduce the following coordinate transformation:

\[
x_1 = p_1 - ip_2, \quad x_2 = p_1 + ip_2, \quad p_1 = \rho \cos(\xi), \quad p_2 = \rho \sin(\xi).
\]

Then, (25) can be rewritten as

\[
\dot{\rho} = r \mu \rho + s \rho^3 + O(\mu^2 \rho + |(\rho, \mu)|^4), \quad \dot{\xi} = -\omega + O(|(\rho, \mu)|),
\]

(26)

where \( r = \mathcal{R}(L) \), and \( s = \mathcal{R}(K) \).

We should note that \( rs \) determines the direction of the Hopf bifurcation, i.e., if \( rs < 0 \) (\( > 0 \)), the Hopf bifurcation is supercritical (subcritical). \( s \) determines the stability of the bifurcating periodic solutions, i.e., the solutions are orbitally stable (unstable) if \( s < 0 \) (\( > 0 \)). Now, we use an example to verify our theoretical results.

### 4.3 Numerical simulation

In the following numerical simulation, we choose \( k = w = 1, C_1 = 3, \) and \( C_2 = 5 \) in the original equation (1), resulting in \( W = \frac{1}{3} \) and \( C = \frac{5}{3} \). According to (7), the equation about the equilibrium \((y^*, y^*)^T\) is

\[
6y^3 - \frac{13}{3} y^2 - \frac{22}{3} y + \frac{40}{9} = 0.
\]

(27)

Through a direct calculation, the equilibrium is obtained as \((y^*, y^*)^T = (0.565, 0, 0.565, 0)^T\). Substituting \( W = \frac{1}{3} \) and \( C = \frac{5}{3} \) into (A1), we have \( a_1 = -1.366, 2, a_2 = -0.350, 8 \), and thus
$\omega^* = 1.7170$, $\tau^* = \frac{\pi}{2\omega^*} = 0.9148$. From the theoretical results above, the normal form of (4) is given by

$$\dot{\rho} = 0.7778\mu\rho - 5.4739\rho^3 + O(|(\rho,\mu)|^4), \quad \dot{\xi} = -1.7170 + O(|(\rho,\mu)|).$$  \hspace{1cm} (28)

According to the previous analysis, the equilibrium is stable if $\tau < 0.9148$ and unstable if $\tau > 0.9148$. Therefore, a stable periodic solution bifurcates from the unstable equilibrium owing to a supercritical Hopf bifurcation. The amplitude of the periodic solution to (28) is calculated as

$$\rho = 0.3770\sqrt{\mu},$$

which will be compared with the numerical results in the following discussion. In the numerical simulations, the values of the variables are listed as in Table 1 unless otherwise stated.

| Variable | $k$ | $w$ | $C_1$ | $C_2$ | $x_1(0)$ | $x_2(0)$ |
|----------|-----|-----|-------|-------|----------|----------|
| Value    | 1   | 1   | 3     | 5     | 0.3      | 0.15     |

From Figs. 6 and 7, we can find that the equilibrium is stable when $\tau < \tau^*$ and unstable if $\tau > \tau^*$. A stable periodic orbit bifurcates from the unstable equilibrium as predicted by the theoretical analysis. The bifurcation diagram of (1) is shown in Fig. 8. As can be seen from Figs. 6 and 7 that (1) makes no qualitative difference with (4). Thus, we only plot the bifurcation diagram of (1) for simplicity.

Fig. 6 Time histories and phase portraits of Internet congestion control model with $\tau = 0.8$ for (1) ((a) and (b)) and (4) ((c) and (d))
Suppression of oscillatory congestion via trunk link bandwidth and control gain

As previously stated, the delay may deteriorate the system performance and even make the system unstable. This section is devoted to the discussion on the selection of parameters, especially the suppression of the delay-induced oscillatory congestion via adjusting the tunable parameters, i.e., the link bandwidth ratio and the control gain. The idea is to select these parameters appropriately so that the equilibrium is stabilized based on the study of Section 2.

5 Suppression of oscillatory congestion via link bandwidth ratio and control gain

As previously stated, the delay may deteriorate the system performance and even make the system unstable. This section is devoted to the discussion on the selection of parameters, especially the suppression of the delay-induced oscillatory congestion via adjusting the tunable parameters, i.e., the link bandwidth ratio and the control gain. The idea is to select these parameters appropriately so that the equilibrium is stabilized based on the study of Section 2.

5.1 Stabilization of equilibrium through adjusting link bandwidth ratio \( C \)

From Section 2, the relationship between \( C \) and the critical delay \( \tau^* \) is given by

\[
\tau^* = \frac{1}{C}
\]

Fig. 7  Time histories and phase portraits of Internet congestion control model with \( \tau = 0.95 \) for (1) ((a) and (b)) and (4) ((c) and (d))

Fig. 8  Bifurcation diagram of (1)
\[ \tau^* = \frac{\pi (4 - 3y^*)^2 (2C - 3y^*)^2}{-4ky^* (27(y^*)^3 - 36(C + 2)(y^*)^2 + 6(C^2 + 16C + 4)y^* - 16C(C + 2))}. \quad (29) \]

We have obtained that the limit equilibrium is \( y^*_\infty = \sqrt{\frac{3W^2 + 16W - 3W}{2}} \) and the corresponding critical delay \( \tau^*_\infty = \frac{(3\sqrt{3W^2 + 16W - 9W - 8})^2 \pi}{2k(\sqrt{3W^2 + 16W - 3W})(9W + 16 - 3\sqrt{3W^2 + 16W})} \). Recall that we study only the cases of \( k = w = 1 \) \([22]\) and \( C_1 = 3 \), implying that \( W = \frac{1}{3}, y^*_\infty = 0.7583 \), and \( \tau^*_\infty = 1.0767 \).

The stability switch boundary curve can be obtained based on the analysis in Section 2, as shown in Fig. 9. It can be seen from Fig. 9 that there is a vertical asymptote as \( C \) grows along the boundary curve, corresponding to \( \tau^*_\infty = 1.0767 \). Based on Fig. 9, several observations that are of practical importance can be obtained. First, increasing the link bandwidth ratio always benefits the stability of the equilibrium for small delays. For example, when \( \tau = 0.9 \) (the dashed green curve in Fig. 9(a)), as \( C \) increases, the system parameters enter the region for which the equilibrium is stable. The green dashed line crosses the stability switch boundary for only once, implying that the equilibrium remains stable as \( C \) increases further. Next, for medium \( \tau \), there exists an interval on which the equilibrium is stable. In other words, an excessively large or small \( C \) deteriorates the system stability for such cases. For instance, the dashed black line intersects the stability switch boundary twice, implying that the equilibrium switches to being unstable when \( C \) becomes sufficiently large, as shown in Fig. 9(c). Finally, as \( \tau > 1.1265 \) (the horizontal coordinate of the rightmost point on the boundary curve), the equilibrium is always unstable regardless of \( C \). In other words, when \( \tau \) exceeds some threshold, it is impossible to stabilize the congestion control system by adjusting the link bandwidth ratio, as revealed in Fig. 9(d).

![Fig. 9](image-url)  
Stabilization of (1) via adjusting link bandwidth ratio \( C \) based on stability switch boundary in (a) for (b) \( \tau = 0.9 \), (c) \( \tau = 1.1 \), and (d) \( \tau = 1.2 \) (color online)
5.2 Optimum $C$ when equilibrium is stable

When the equilibrium is stable, the optimal $C$ should be obtained so that the rate of data transmission moves towards the equilibrium at the fastest speed. Hence, three examples are provided as an illustration of the idea by plotting $R(\hat{\lambda})$ versus $C$, where $\hat{\lambda}$ represents the eigenvalue with the largest real part among all eigenvalues.

Example 1 $\tau = 1.1 > \tau^*_\infty$. The equilibrium is stable for $C \in (2.807, 13.962)$. As shown in Fig. 10(a), the optimal value of $C$ is 4.41.

Example 2 $\tau = 1.0767 > \tau^*_\infty$. The equilibrium is stable only if $C > 2.485$. According to Fig. 10(b), $C = 4.41$ is the optimal value in the whole stable interval.

Example 3 $\tau = 0.9 < \tau^*_\infty$. The smallest $C$ such that the equilibrium is stable is 1.625. As shown in Fig. 10(c), the optimal $C$ is 4.41.

Based on the observations above, one can postulate that the optimal value of $C$ is independent of $\tau$ when the equilibrium is stable. This assertion is validated by Figs. 11 and 12, in which $R(\hat{\lambda}) - R(\tilde{\lambda}_\tau)$ instead of $R(\hat{\lambda})$ itself, is plotted versus the $C\tau$-plane, where $\tilde{\lambda}_\tau$ is the eigenvalue of (10) evaluated at $C = 4.41$ for a given $\tau$. It can be seen from Fig. 11 that the strongest "damping" is achieved at $C = 4.41$ for any $C$ provided that the equilibrium is stable.

Figures 11 and 12 are plotted using DDE-BIFTOOL[34] and the integral evaluation method[35–37], respectively. First, we calculate the largest real part of eigenvalue for a specific point $(C, \tau) = (3, 0.5)$ as $-1.113$ 501 6 based on the method in Ref. [36]. The characteristic equation is given by $f(\lambda) = \lambda^2 - 2a_1\lambda e^{-\lambda\tau} + 2(a_1^2 - a_2^2)e^{-2\lambda\tau}$, where $a_1$ and $a_2$ are shown in
Appendix A. Through a direct calculation, we obtain

$$\int_{0}^{50} \mathcal{R} \left( \frac{f'(\lambda) - 1 + i\omega}{f(-1 + i\omega)} \right) d\omega \approx 3.0974,$$

(30)

$$\int_{0}^{50} \mathcal{R} \left( \frac{f''(-2 + i\omega)}{f(-2 + i\omega)} \right) d\omega \approx -9.4834.$$  

(31)

This implies that the largest real part of all eigenvalues is $\sigma \in (-2, -1)$. Repeating the procedure above, we can narrow the interval in which $\sigma$ exists as $(-1.2, -1)$. Then, the initial estimate of $\sigma$ is chosen as $-1.1$. Substituting $\lambda = -1.1 + i\omega$ into $f(\lambda)$, and equating the real and imaginary parts of $f(\lambda)$ with zero, respectively, $\sigma$ can be solved by using the Newton-Raphson iteration method. Thus, we obtain the rightmost eigenvalue as $-1.11350166 + 2.20709411i$. Suppose that the step lengths of $C$ and $\tau$ are both 0.01. Then, the largest real part of the eigenvalues for different values of $C$ and $\tau$ can be obtained as shown in Fig. 12. Obviously, Fig. 12 agrees well with Fig. 11.

5.3 Stabilization of equilibrium through adjusting $k$

Figure 9(d) shows that it is impossible to stabilize the equilibrium by adjusting $C$ for $\tau > 1.1265$. In this part, we seek the possibility of stabilizing the equilibrium by adjusting the control gain $k$. The stability switch boundary in the space of $C$, $\tau$, and $k$ is plotted in Fig. 13, in which the equilibrium of the system is stable for the parameters selected from the region...
below the boundary surface. Figure 13 shows that the smaller the control gain $k$, the larger the critical delay $\tau^*$. In other words, lowering the control gain delays the onset of the Hopf bifurcation. For example, the equilibrium is unstable for $k = w = 1$, $C_1 = 3$, $C_2 = 5$, and $\tau = 0.95 > \tau^* = 0.9148$. If $k$ is decreased to 0.8 while the other parameters remain unchanged, the equilibrium of (1) becomes stable, as shown in Fig. 14.

![Fig. 13](image1.png)

**Fig. 13** (a) Stability switch boundary in space of $C$, $\tau$, and $k$, where equilibrium of system is stable for parameters selected from region below boundary surface, and (b) corresponding contour plot in $Ck$-plane

![Fig. 14](image2.png)

**Fig. 14** Time histories of (1) for (a) $k = 1$ and (b) $k = 0.8$

### 6 Conclusions

In this paper, we study the oscillatory congestion induced by the time delay in an Internet congestion control model, which describes the evolution of the congestion control algorithm for a star network. The link bandwidth ratio, defined as the ratio of bandwidth of the trunk link and that of the common link, and the control gain are chosen as tunable parameters to suppress the delay-induced oscillatory congestion. The stability switch boundary in the parameter space is obtained through the stability analysis for the equilibrium. The center manifold reduction and normal form theory are adopted to investigate the mechanism which leads to the oscillatory
congestion. Our analysis shows that, for the fixed control gain, the stability switch boundary curve is not monotonic in the plane consisting of link bandwidth ratio and delay. Consequently, a complex relation is found between the delay and the interval of the link bandwidth ratio on which the oscillatory congestion can be suppressed. The length of such interval depends on the delay. If the delay exceeds some threshold, the equilibrium cannot be stabilized through adjusting the link bandwidth ratio. Besides, when the equilibrium is stable, there exists an optimal value of the link bandwidth ratio such that the transient behavior of the congestion control system dies out rapidly. Moreover, re-selecting the control gain may help stabilize the equilibrium which cannot be stabilized by adjusting the link bandwidth ratio alone. The numerical simulations are in good agreement with the theoretical analysis.

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Appendix A

In this appendix, we want to give the proof process of transversality condition and some formulae in Section 4. First, we give the coefficients in (9),

\[
\begin{align*}
    a_1 &= -k(p(y^*) + y^*p'(y^*) + \bar{p}(2y^*) + y^*\bar{p}'(2y^*)), \\
    b_1 &= -ky^*\bar{p}'(2y^*), \\
    a_1 &= b_2, \quad a_2 = b_1, \\
    d_{11} &= -k\left(p'(y^*) + \frac{p''(y^*)}{2}y^* + \bar{p}'(2y^*) + \frac{\bar{p}''(2y^*)}{2}y^*\right), \\
    d_{12} &= -k\left(p'(2y^*) + y^*\bar{p}''(2y^*)\right), \\
    d_{13} &= -\frac{k\bar{p}''(2y^*)}{2}y^*, \\
    d_{21} &= d_{13}, \quad d_{22} = d_{12}, \quad d_{23} = d_{11}, \\
    e_{11} &= -k\left(\frac{p''(y^*)}{2} + \frac{p'''(y^*)}{6}y^* + \frac{\bar{p}''(2y^*)}{2} + \frac{\bar{p}'''(2y^*)}{6}y^*\right), \\
    e_{12} &= -k\left(\bar{p}''(2y^*) + \frac{\bar{p}'''(2y^*)}{2}\right), \\
    e_{13} &= -k\left(\bar{p}''(2y^*) + \frac{\bar{p}'''(2y^*)}{2}y^*\right), \\
    e_{14} &= -\frac{\bar{p}'''(2y^*)}{6}y^*, \\
    e_{21} &= e_{14}, \quad e_{22} = e_{13}, \quad e_{23} = e_{12}, \quad e_{24} = e_{11}, \\
\end{align*}
\]

(A1)

\[
\begin{align*}
    p'(y) &= \frac{4}{(4-3y)^2}, \quad p''(y) = \frac{24}{(4-3y)^3}, \quad p'''(y) = \frac{216}{(4-3y)^4}, \\
    \bar{p}'(y) &= \frac{4C}{(4C-3y)^2}, \quad \bar{p}''(y) = \frac{24C}{(4C-3y)^3}, \quad \bar{p}'''(y) = \frac{216C}{(4C-3y)^4}.
\end{align*}
\]

(A2)

The proof process of transversality condition is presented as follows.

Proof Taking the derivative of \( \lambda \) with respect to \( \tau \) in (11), we get

\[
\frac{d\lambda}{d\tau} = \frac{\lambda(a_1^2 - a_2^2)e^{-2\lambda\tau} - a_1\lambda^2e^{-\lambda\tau}}{\lambda - a_1e^{-\lambda\tau} + a_1\lambda e^{-\lambda\tau} - (a_1^2 - a_2^2)e^{-2\lambda\tau}}.
\]

For the sake of simplicity, \( \omega \) and \( \tau \) are denoted by \( \omega_i \) and \( \tau_{ni} \), respectively. Then,

\[
\frac{d\lambda}{d\tau} = \frac{\omega(a_1^2 - a_2^2)e^{-2i\omega_i\tau} + a_1\omega^2e^{-i\omega_i\tau}}{\omega_i - a_1e^{-i\omega_i\tau} + a_1\omega e^{-i\omega_i\tau} - (a_1^2 - a_2^2)e^{-2i\omega_i\tau}} \triangleq \frac{A_1 + B_1i}{A_2 + B_2i},
\]

where

\[
\begin{align*}
    A_1 &= \omega(a_1^2 - a_2^2)\sin(2\omega\tau) + a_1\omega^2\cos(\omega\tau), \\
    B_1 &= \omega(a_1^2 - a_2^2)\cos(2\omega\tau) - a_1\omega^2\sin(\omega\tau), \\
    A_2 &= -a_1\cos(\omega\tau) + a_1\omega\tau\sin(\omega\tau) - \tau(a_1^2 - a_2^2)\cos(2\omega\tau), \\
    B_2 &= \omega + a_1\sin(\omega\tau) + a_1\omega\tau\cos(\omega\tau) + \tau(a_1^2 - a_2^2)\sin(2\omega\tau).
\end{align*}
\]
Let
\[ Q = A_2^2 + B_2^2 \]
\[ = (-a_1 \cos(\omega \tau) + a_1 \omega \tau \sin(\omega \tau) - \tau(a_1^2 - a_2^2) \cos(2\omega \tau))^2 \]
\[ + (\omega + a_1 \sin(\omega \tau) + a_1 \omega \tau \cos(\omega \tau) + \tau(a_1^2 - a_2^2) \sin(2\omega \tau))^2. \]

Then,
\[
 QR \left( \frac{d\lambda}{d\tau} \right) = \text{Re}((A_1 + B_1i)(A_2 - B_2i)) \\
 = A_1A_2 + B_1B_2 \\
 = (\omega(a_1^2 - a_2^2) \sin(2\omega \tau) + a_1 \omega^2 \cos(\omega \tau))(\omega \sin(\omega \tau) + a_1 \omega \cos(\omega \tau) - a_1 \omega^2 \sin(\omega \tau)) \\
 \cdot (\omega + a_1 \sin(\omega \tau) + a_1 \omega \tau \cos(\omega \tau) + \tau(a_1^2 - a_2^2) \sin(2\omega \tau)).
\]

Thus, we have
\[
 QR \left( \frac{d\lambda}{d\tau} \right) \bigg|_{\tau=n\omega} = (-\omega(a_1^2 - a_2^2) - a_1 \omega^2)(\omega + a_1) \\
 = -\omega((a_1^2 - a_2^2)\omega + (a_1^2 - a_2^2)a_1 + a_1 \omega^2 + a_2^2) \\
 = -\omega((a_1^2 - a_2^2)\omega + (a_1^2 - a_2^2)a_1 - 2a_1 \omega - a_1(a_1^2 - a_2^2) + a_2^2) \\
 = a_2^2 \omega^2 \\
 > 0.
\]

This completes the proof.

**Appendix B**

According to the definition of \( q(\theta) \) at \( \theta = 0 \), one obtains
\[
\int_{-\tau}^{0} \left[ \begin{array}{cc} a_1 & a_2 \\ a_2 & a_1 \end{array} \right] \delta(\theta + \tau) \left[ \begin{array}{c} 1 \\ \alpha \end{array} \right] e^{i\omega \theta} d\theta = i\omega \left[ \begin{array}{c} 1 \\ \alpha \end{array} \right] \\
\Rightarrow \left[ \begin{array}{cc} a_1 & a_2 \\ a_2 & a_1 \end{array} \right] \left[ e^{-i\omega \tau} \begin{array}{c} 1 \\ ae^{-i\omega \tau} \end{array} \right] = i\omega \left[ \begin{array}{c} 1 \\ \alpha \end{array} \right].
\]

Thus, we get the expression of \( \alpha \),
\[ \alpha = \frac{a_2}{i\omega e^{i\omega \tau} - a_1}. \]

Similarly,
\[
\int_{-\tau}^{0} \left[ \begin{array}{cc} a_1 & a_2 \\ a_2 & a_1 \end{array} \right] \delta(s + \tau) \left[ \begin{array}{c} 1 \\ \beta \end{array} \right] e^{i\omega s} ds = -i\omega \left[ \begin{array}{c} 1 \\ \beta \end{array} \right] \\
\Rightarrow \left[ \begin{array}{cc} a_1 & a_2 \\ a_2 & a_1 \end{array} \right] \left[ e^{i\omega \tau} \begin{array}{c} 1 \\ \beta e^{i\omega \tau} \end{array} \right] = -i\omega \left[ \begin{array}{c} 1 \\ \beta \end{array} \right],
\]

and
\[ \beta = \frac{-a_2 e^{i\omega \tau}}{a_1 e^{i\omega \tau}}. \]
In order to obtain \( N \), let \( \langle q^*, q \rangle = 1 \).

\[
\langle q^*, q \rangle = (q^*)^T(0)q(0) - \int_{-\tau}^{0} \int_{0}^{\theta} (q^*)^T(\xi - \theta) d\eta(\theta) q(\xi) d\xi \\
= N(1 + \alpha \beta) - \int_{-\tau}^{0} \int_{0}^{\theta} N(1, \beta)e^{-i\omega(\xi - \theta)} \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix} \frac{\delta(\theta + \tau)}{\alpha} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} e^{i\omega \xi} d\xi \\
= N(1 + \alpha \beta) - \int_{-\tau}^{0} \int_{0}^{\theta} \frac{N(1, \beta)}{\alpha} \begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix} \frac{\delta(\theta + \tau)}{\alpha} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} d\theta d\xi \\
= N(1 + \alpha \beta) + N \tau e^{-i\omega \tau} (a_1 + a_2 \alpha + a_2 \beta + a_1 \alpha \beta).
\]

Then,

\[
N = \frac{1}{1 + a_2 \beta + \tau e^{-i\omega \tau}(a_1 + a_2 \alpha + a_2 \beta + a_1 \alpha \beta)}.
\]

\( L \) and \( K \) in Section 4 are

\[
L = i\omega N(1 + a_2 \beta),
\]

\[
K = \frac{N}{1 + a_2 \beta + \tau e^{-i\omega \tau}(a_1 + a_2 \alpha + a_2 \beta + a_1 \alpha \beta)} \left( e_{12} + e_{13} \beta (2\alpha + \alpha \beta) + (e_{13} + e_{12} \beta) (\alpha^2 + 2\alpha \beta)
+ 3(e_{14} + e_{11} \beta) \alpha^2 \alpha \beta + 3(e_{11} + e_{14} \beta) \right).
\]