NON-KOSZUL QUADRATIC GORENSTEIN TORIC RINGS

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Abstract. Koszulness of Gorenstein quadratic algebras of small socle degree is studied. In this note, we construct non-Koszul Gorenstein quadratic toric ring such that its socle degree is more than 3 by using stable set polytopes.

Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ a polynomial ring over $K$. Let $R = S/I$ be a standard graded $K$-algebra with respect to the grading $\deg x_i = 1$ for all $1 \leq i \leq n$, where $I$ is a homogeneous ideal of $S$. Let $R_+$ denote the homogeneous maximal ideal of $R$. For an $R$-module $M$, we denote $\beta_{ij}^R(M)$ by the $(i, j)$-th graded betti number of $M$ as an $R$-module.

The Koszul algebra was originally introduced by Priddy [29]. A standard graded $K$-algebra $R$ is said to be Koszul if the residue field $K = R/R_+$ has a linear $R$-free resolution as an $R$-module, that is, $\beta_{ij}^R(K) = 0$ if $i \neq j$. Since $\beta_{2j}^R(K) = 0$ for all $j > 2$, hence Koszul algebras are quadratic, where $R = S/I$ is said to be quadratic if $I$ is generated by homogeneous elements of degree 2. Every quadratic complete intersection is Koszul by Tate’s theorem [35]. Moreover, $R = S/I$ is Koszul if $I$ has a quadratic Gröbner bases by Fröberg’s theorem [10] and the fact that $\beta_{ij}^R(K) \leq \beta_{ij}^{R'}(K)$ for all $i, j$ and for all monomial order $<$ on $S$, where $R' = S/in_<(I)$.

The notion of Koszul algebra has played an important role in the research on graded $K$-algebras, and various Koszul-like algebras have been introduced, e.g., universally Koszul [3], strongly Koszul [12], initially Koszul [2], sequentially Koszul [1], etc.

Koszulness of toric rings of integral convex polytopes is studied. Let $\mathcal{P} \subset \mathbb{R}^n$ be an integral convex polytope, i.e., a convex polytope each of whose vertices belongs to $\mathbb{Z}^n$, and let $\mathcal{P} \cap \mathbb{Z}^n = \{a_1, \ldots, a_m\}$. Assume that $\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_m = \mathbb{Z}^n$. Let $K[X^{\pm 1}, t] := K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, t]$ be the Laurent polynomial ring in $n + 1$ variables over $K$. Given an integer vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we put $X^at = x_1^{a_1} \cdots x_n^{a_n}t \in K[X^{\pm 1}, t]$. The toric ring of $\mathcal{P}$, denoted by $K[\mathcal{P}]$, is the subalgebra of $K[X^{\pm 1}, t]$ generated by $\{X^{a_1}t, \ldots, X^{a_m}t\}$ over $K$. Note that $K[\mathcal{P}]$ can be regarded as a standard graded $K$-algebra by setting $\deg X^{a_i}t = 1$. The toric ideal $I_{\mathcal{P}}$ is the kernel of a surjective ring homomorphism $\pi : K[Y] = K[y_1, \ldots, y_m] \to K[\mathcal{P}]$ defined...
by \( \pi(y_i) = X^{a_i}t \) for \( 1 \leq i \leq m \). Then \( K[\mathcal{P}] \cong K[Y]/I_\mathcal{P} \). It is known that \( I_\mathcal{P} \) is generated by homogeneous binomials.

Note that the following implications hold:

\[ \text{quadratic C.I.} \quad \Rightarrow \quad \text{quadratic Gorenstein} \quad \Rightarrow \quad \text{quadratic Cohen-Macaulay} \]

\[ \text{Koszul algebra} \quad \uparrow [10] \quad \Leftrightarrow \quad I_\mathcal{P} \text{ has a quadratic GB} \quad \Leftarrow \quad I_\mathcal{P} \text{ has a quadratic squarefree initial ideal} \]

\[ \text{sequentially Koszul} \quad \uparrow \quad \quad \Leftrightarrow \quad \text{initially Koszul} \quad \uparrow \quad \quad \quad \text{C.I.} = \text{Complete Intersection} \]

\[ \text{strongly Koszul} \quad \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \text{universally Koszul} \quad \downarrow \]

In addition, it is known the following:

(1) Conca-De Negri-Rossi posed a conjecture that the defining ideal of a strongly Koszul algebra has a quadratic Gröbner bases \([6\text{, Question 13 (1)}]\). This conjecture is true for the toric ring of edge polytope \([16]\), order polytope \([12]\), stable set polytope \([23]\) and cut polytope \([31]\).

(2) A squarefree strongly Koszul toric ring is compressed \([24\text{, Theorem 2.1]}\), where \( K[\mathcal{P}] \) is said to be compressed if \( \sqrt{\text{in}_\prec(I_\mathcal{P})} = \text{in}_\prec(I_\mathcal{P}) \) for any reverse lexicographic order \( \prec \) on \( K[Y] \). In particular, a squarefree strongly Koszul toric ring is quadratic Cohen-Macaulay.

(3) Many of toric rings associated with integral convex polytopes whose toric ideals has a quadratic Gröbner bases are constructed (e.g., \([3]\), \([13]\), \([15]\), \([17]\), \([18]\), \([19]\)). In other words, many of Koszul toric rings associated with
integral convex polytopes are constructed.

(4) Quadratic algebra is not always Koszul (see [27, Example 2.1], [30, Example 3]). Note that both of these examples are Cohen-Macaulay but are not Gorenstein.

On the other hand, Koszulness of Gorenstein quadratic algebras is studied. For a standard graded $K$-algebra $R = \oplus_{i \geq 0} R_i$ with $\dim R = d$, we denote by

$$H_R(t) = \sum_{i \geq 0} \dim_K R_i t^i = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1 - t)^d}$$

the Hilbert series of $R$, where $h_s \neq 0$, and we say that $h(R) := (h_0, h_1, \ldots, h_s)$ is the $h$-vector of $R$ and the index $s$ is the socle degree of $R$. It is known that $h_0 = 1$ and if $R$ is Gorenstein then $h_i = h_{s-i}$ for all $0 \leq i \leq \lfloor s/2 \rfloor$ ([32 Theorem 4.4]). Conca-Rossi-Valla proved that if $R$ is a quadratic Gorenstein with $h(R) = (1, n, 1)$ (in this case $n \geq 2$ since $R$ is quadratic) then $R$ is Koszul [7 Proposition 2.12].

The case for $s = 2$ is also studied. Let $R$ be a quadratic Gorenstein with $h(R) = (1, n, 1)$ (in this case $n \geq 3$ since $R$ is quadratic). If $n = 3$, then $R$ is quadratic complete intersection, hence $R$ is Koszul. Conca-Rossi-Valla proved that $R$ is Koszul if $n = 4$ [7 Theorem 6.15] and Caviglia proved that $R$ is Koszul if $n = 5$ in his unpublished master thesis. The case for $n \geq 6$ is still open.

In this note, we focus on (4). In Section 1, we remark about known result of toric rings and toric ideals of stable set polytopes, and construct non-Koszul quadratic Gorenstein toric rings by using stable set polytopes. In Section 2, we present some questions.

**Remark 0.1.** In this note, we use Macaulay2 [11] to check to be not Koszul. About checking of non-Koszulness by using Macaulay2, see [34, p. 289].

### 1. Stable set polytope and non-Koszul quadratic Gorenstein toric ring

The stable set polytope is an integral convex polytope associated with stable sets of a simple graph.

Let $G$ be a finite simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ and let $E(G)$ denote the set of edges of $G$. Recall that a finite graph is simple if it possesses no loops or multiple edges. We denote by $\overline{G}$ the complement graph of $G$.

Given a subset $W \subset [n]$, we define the $(0, 1)$-vector $\rho(W) = \sum_{i \in W} e_i \in \mathbb{R}^n$, where $e_i$ is the $i$-th unit coordinate vector of $\mathbb{R}^n$. In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^n$. 
A subset $W \subset [n]$ is said to be stable if $\{i, j\} \not\in E(G)$ for all $i, j \in W$ with $i \neq j$. Note that the empty set and each single-element subset of $[n]$ are stable. Let $S(G)$ denote the set of all stable sets of $G$. The stable set polytope of a simple graph $G$, denoted by $Q_G$, is the convex hull of $\{\rho(W) \mid W \in S(G)\}$. By definition, $Q_G$ is a $(0, 1)$-polytope and $K[Q_G] = K[t \cdot \prod_{i \in W} x_i \mid W \in S(G)] \subset K[x_1, \ldots, x_n, t]$. Note that $\dim K[Q_G] = n + 1$. Let $K[Y] = K[y_W \mid W \in S(G)]$ be the polynomial ring over $K$. Now we define a surjective ring homomorphism $\pi : K[Y] \to K[Q_G]$ by $\pi(y_W) = t \cdot \prod_{i \in W} x_i$ and let $I_{Q_G} = \ker \pi$.

To state known results of the toric ring $K[Q_G]$ and the toric ideal $I_{Q_G}$ of the stable set polytope $Q_G$ of a simple graph $G$, we introduce some classes of graphs. About terminologies for the graph theory, see [8].

A cycle graph with length $n$, denoted by $C_n$, is a connected graph which satisfies $E(C_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}, \{1, n\}\}$. An odd cycle is a cycle such that its length is odd.

A graph $G$ is said to be perfect if the chromatic number of every induced subgraph of $G$ is equal to the size of the largest clique of that subgraph. A graph $G$ is perfect if and only if both $G$ and $\overline{G}$ are $(C_{2n+3}, n \geq 1)$-free [1].

The comparability graph $G(P)$ of a partially ordered set $P = ([n], <_P)$ is the graph such that $V(G(P)) = [n]$ and $\{i, j\} \in E(G(P))$ if and only if $i <_P j$ or $j <_P i$. A graph $G$ is said to be comparability if $G$ is the comparability graph of some partially ordered set. Forbidden induced subgraphs of comparability graphs are known (see [22, p.13]).

A graph $G$ is said to be bipartite if there exist $V_1, V_2$ with $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \emptyset$ such that if $\{i, j\} \in E(G)$ then either $i \in V_1$ and $j \in V_2$ or $i \in V_2$ and $j \in V_1$. It is known that such a graph $G$ is bipartite if and only if $G$ is $(C_{2n+1}, n \geq 1)$-free.

A graph $G$ is said to be almost bipartite (see [9, p.87]) if there exists a vertex $v$ such that the induced subgraph $G_{[n] \setminus v}$ is bipartite.

**Remark 1.1.** It is known that

1. Let $G$ be a perfect graph. Then $K[Q_G]$ is Gorenstein if and only if all maximal cliques of $G$ have the same cardinality [28, Theorem 2.1(b)].

2. Let $G(P)$ be the comparability graph of a partially ordered set $P$. Then $K[Q_{G(P)}]$ is Koszul since $Q_{G(P)}$ is equal to the chain polytope of $P$ and the toric ideal of a chain polytope has a squarefree quadratic initial ideal (see [14, Corollary 3.1]).

3. If $G$ is almost bipartite, then $K[Q_G]$ is Koszul since its toric ideal $I_{Q_G}$ has a squarefree quadratic initial ideal (see [9, Theorem 8.1]).
(4) Let $G$ be a graph such that $\overline{G}$ is bipartite. Then $K[Q_G]$ is quadratic if and only if it is Koszul [25, Corollary 3.4].

Hence, if $K[Q_G]$ is quadratic but not Koszul, then $G$ is neither comparability nor almost bipartite, and $\overline{G}$ is not bipartite. From this fact and the classifications of these graphs, we have

**Proposition 1.2.** Let $G$ be a graph on $[n]$. If $K[Q_G]$ is non-Koszul quadratic Gorenstein, then $n \geq 7$.

**Proof.** First, we assume that $n \leq 5$. Then $G$ is a comparability graph if $G$ is not $C_5$. Since $C_5$ is almost bipartite, we have that $K[Q_G]$ is Koszul if $n \leq 5$.

Next, we assume that $n = 6$. If $G$ is not connected, then $G$ is a comparability graph if $G$ is not $C_5 \cup K_1$. Since $C_5 \cup K_1$ is almost bipartite, we have that $K[Q_{G(P)}]$ is Koszul.

Assume that $G$ is connected. From the classifications of comparability and almost bipartite graphs, $G$ is one of the following (see [23, p.10]):

Then we can see that

- $K[Q_{G_1}]$ is not Gorenstein since $h(K[Q_{G_1}]) = (1, 7, 10, 3)$.
- $K[Q_{G_2}]$ is Koszul since $I_{Q_{G_2}}$ has a quadratic Gröbner bases.
- $G_3$ is $C_6$, hence bipartite.
- $K[Q_{G_4}]$ is not Gorenstein since $h(K[Q_{G_4}]) = (1, 6, 8, 2)$.
- $K[Q_{G_5}]$ is Koszul since $I_{Q_{G_5}} = I_{Q_{C_5}}$ and $I_{Q_{C_5}}$ has a quadratic Gröbner bases.

Therefore we have the desired conclusion. □

For each integer $k \geq 3$, the complement of a odd cycle $C_{2k+1}$, denoted by $\overline{C_{2k+1}}$, is neither comparability nor almost bipartite. Note that $\overline{C_{2k+1}}$ is not perfect and $S(\overline{C_{2k+1}}) = \{\emptyset, \{1\}, \{2\}, \ldots, \{2k+1\}, \{1, 2\}, \{2, 3\}, \ldots, \{2k, 2k+1\}, \{1, 2k+1\}\}$.

Let $K[Y] = K[y_0, y_1, \ldots, y_{2k+1}, y_{\{1,2\}}, y_{\{2,3\}}, \ldots, y_{\{2k,2k+1\}}, y_{\{1,2k+1\}}]$. Now we study the toric ring

$$K[Q_{\overline{C_{2k+1}}}] \cong \frac{K[Y]}{I_{Q_{\overline{C_{2k+1}}}}}.$$

**Proposition 1.3.** We have the following:
(1) \(K[\mathcal{Q}_{2k+1}]\) is quadratic Cohen-Macaulay for all \(k \geq 3\).
(2) \(K[\mathcal{Q}_{2k+1}]\) is not Gorenstein for all \(k \geq 4\).
(3) \(K[\mathcal{Q}_{2k+1}]\) is Gorenstein.
(4) \(I_{\mathcal{Q}_{2k+1}}\) possesses no quadratic Gröbner bases for all \(k \geq 3\).

**Proof.**  
(1) First, by [25, Theorem 2.1], we have that \(K[\mathcal{Q}_{2k+1}]\) is normal. Hence \(K[\mathcal{Q}_{2k+1}]\) is Cohen-Macaulay. Moreover, by [25, Theorem 3.2], we have that the toric ideal \(I_{\mathcal{Q}_{2k+1}}\) is generated by the following 4\(k+2\) binomials:

- \(y(i)y(i+1) - y0y(i,i+1)\) \((1 \leq i \leq 2k)\);
- \(y(1)y(2k+1) - y0y(1,2k+1)\);
- \(y(i)y(i+1,i+2) - y(i+2)y(i,i+1)\) \((1 \leq i \leq 2k - 1)\);
- \(y(2k)y(1,2k+1) - y(1)y(2k,2k+1), y(2k+1)y(1,2) - y(2)y(1,2k+1)\).

Hence \(K[\mathcal{Q}_{2k+1}]\) is quadratic. Therefore \(K[\mathcal{Q}_{2k+1}]\) is quadratic Cohen-Macaulay.

(2) By (1), \(K[\mathcal{Q}_{2k+1}] \cong K[Y]/I_{\mathcal{Q}_{2k+1}}\) is Cohen-Macaulay with \(\dim K[\mathcal{Q}_{2k+1}] = 2k+2\). We note that \(y = y0, y(1), y(2), y(3), \ldots, y(2k-1), y(2k), y(2k+1)\) is a regular sequence of \(K[Y]/I_{\mathcal{Q}_{2k+1}}\). Then we have that

\[
\frac{K[Y]}{I_{\mathcal{Q}_{2k+1}} + (y)} \cong \frac{K[y(1), y(2), \ldots, y(2k+1)]}{I_{2k+1}}
\]

is a artinian quadratic Cohen-Macaulay ring, where \(I_{2k+1}\) is generated by the followings:

- \(y(i)y(i+1)\) \((1 \leq i \leq 2k)\);
- \(y(1)y(2k+1)\);
- \(y^2(i) - y(i-1)y(i+2)\) \((2 \leq i \leq 2k - 1)\);
- \(y^2(1) - y(3)y(2k+1), y^2(2k) - y(1)y(2k-1), y^2(2k+1) - y(2)y(2k)\).

Assume \(k \geq 4\). Then both \(y^2(2k+1) \cdot \prod_{i=1}^{k-1} y(2i)\) and

\[
\begin{align*}
\prod_{i=1}^{2k+1} y(3i) & \quad (k \equiv 1 \mod 3), \\
y(2k+1) \cdot \prod_{i=1}^{2k+1} y(3i) & \quad (k \equiv 2 \mod 3), \\
y^2(2k) \cdot \prod_{i=1}^{2k+1} y(3i) & \quad (k \equiv 0 \mod 3),
\end{align*}
\]
are socle elements of $K[y_{11}, y_{22}, \ldots, y_{(2k+1)2k+1}]/I_{2k+1}$, hence it is not Gorenstein. Therefore $K[Q_{c,2k+1}]$ is not Gorenstein for all $k \geq 4$.

(3) By the proof of (2), we have

$$\frac{K[Y]}{I_{Q_{c,7}} + (y)} \cong \frac{K[y_{11}, y_{22}, \ldots, y_{(7)7}]}{I_7}.$$

Let $<_{rev}$ be the reverse lexicographic order on $K[y_{11}, y_{22}, \ldots, y_{(7)7}]$ induced by the ordering $y_{11} < y_{22} < \cdots < y_{(7)7}$. Then the initial ideal in $<_{rev}(I_7)$ is generated by the following monomials:

$$(y_{11}y_{22}, y_{22}y_{33}, y_{33}y_{44}, y_{44}y_{55}, y_{55}y_{66}, y_{66}y_{77}, y_{11}y_{77}),
\ y_{11}^3, y_{11}^2y_{22}, y_{22}^2, y_{22}y_{33}, y_{33}^2, y_{33}y_{44}, y_{44}^2, y_{44}y_{55}, y_{55}^2, y_{55}y_{66}, y_{66}^2, y_{66}y_{77}, y_{77}^2, y_{77}y_{11}, y_{11}y_{22}, y_{22}y_{33}, y_{33}y_{44}, y_{44}y_{55}, y_{55}y_{66}, y_{66}y_{77}).$$

From this, we can compute that the Hilbert series of $\\in_{<_{rev}}(I_7)$ is $1 + 7t + 14t^2 + 7t^3 + t^4$. Hence $h(K[Q_{c,7}]) = (1, 7, 14, 7, 1)$, therefore it is Gorenstein.

(4) Assume that there exists a monomial order $<$ on $K[Y]$ such that the Gröbner bases of $I_{Q_{c,2k+1}}$ with respect to $<$ is quadratic.

We may assume that $y_{11}y_{22} < y_{33}y_{11}$. Then $y_{33}y_{44} < y_{55}y_{33}$ since $y_{55}y_{11}y_{22}y_{33} = y_{11}y_{22}y_{33}y_{55} \in I_{Q_{c,2k+1}}$ and its initial monomial is $y_{55}y_{11}y_{22}y_{33}$. Since $y_{77}y_{33}y_{44}y_{55} - y_{33}y_{44}y_{55}y_{66} \in I_{Q_{c,2k+1}}$ and its initial monomial is $y_{77}y_{33}y_{44}y_{55}$, we have $y_{55}y_{66} < y_{77}y_{55}$. By repeating this argument, we have

$$y_{11}y_{22} < y_{33}y_{11},
\ y_{33}y_{44} < y_{55}y_{33},
\ \cdots,
\ y_{(2k-2)}y_{(2k-1)}y_{(2k+1)} < y_{(2k+1)}y_{(2k-1,2k)},
\ y_{(2k+1)}y_{(1,2k)} < y_{(2k)}y_{(1,2k+1)},
\ y_{(2k)}y_{(3,4)} < y_{(4)}y_{(2,3)},
\ y_{(4)}y_{(5,6)} < y_{(6)}y_{(4,5)},
\ \cdots,
\ y_{(2k-2)}y_{(2k-1,2k)} < y_{(2k)}y_{(2k-2,2k-1)},
\ y_{(2k)}y_{(1,2k+1)} < y_{(1)}y_{(2k,2k+1)}.$$
We can construct an infinite family of non-Koszul quadratic Gorenstein toric rings by using stable set polytopes.

**Proposition 1.5.** Let $k \geq 1$ be an integer. Let $G$ be a graph on $[2k + 7]$ such that $\overline{G} = C_7 \cup K_2 \cup \cdots \cup K_2$ and the labeling of vertices is as follows:

Then we have

1. $K[Q_G]$ is quadric Gorenstein such that
   
   $$H_{K[Q_G]}(t) = (1 + 7t + 14t^2 + 7t^3 + t^4)(1 + t)^k/(1 - t)^{2k+8}.$$

2. $K[Q_G]$ is not Koszul.

**Proof.** (1) By [25, Theorem 3.2], we have that the toric ideal $I_{Q_G}$ is generated by the following binomials:

- $y(i)y(i+1) - y_0y_1, i+1$ ($1 \leq i \leq 6$);
- $y(1)y(7) - y_0y(1,7)$;
- $y(i)y(i+1,i+2) - y(i+2)y(i,i+1)$ ($1 \leq i \leq 5$);
- $y(6)y(1,7) - y(1)y(6,7), y(7)y(1,2) - y(2)y(1,7)$;
- $y(2i)y(2i+1) - y_0y(2i+1)$ ($4 \leq i \leq k+3$).

Let $K[Y] = K[y_W \mid W \in S(G)]$. Then $K[Q_G] \cong K[Y]/I_{Q_G}$. Note that

$$y = y_0, y(1) - y(2,3), y(2) - y(3,4), \ldots, y(5) - y(6,7), y(6) - y(1,7), y(7) - y(1,2),$$

$$y(8) - y(9), \ldots, y(2k+6) - y(2k+7), y(8,9), \ldots, y(2k+6,2k+7)$$

is a regular sequence of $K[Y]/I_{Q_G}$. Hence we have

$$\frac{K[Y]}{I_{Q_G} + (y)} \cong K[y_{(1)}, y_{(2)}, \ldots, y_{(7)}]/I_7 \otimes_K \frac{K[y_{(2i)} \mid 4 \leq i \leq k+3]}{(y_{(2i)}^2 \mid 4 \leq i \leq k+3)}.$$  

Thus the Hilbert series of $K[Y]/I_{Q_G} + (y)$ is $(1 + 7t + 14t^2 + 7t^3 + t^4)(1 + t)^k$. Therefore we have the desired conclusion.

(2) $K[Q_{G'}]$ is a conbinatorial pure subring (see [26]) of $K[Q_G]$. Since $K[Q_{G'}]$ is not Koszul, hence $K[Q_G]$ is not Koszul by [26, Proposition 1.3]. \hfill \square
2. Questions

As the end of this note, we present some questions.

Recall that the $h$-vector of $K[Q_{7}]$ is $(1,7,14,7,1)$. Hence the following question is interesting.

**Question 2.1.** Does exist a non-Koszul quadratic Gorenstein algebra $R$ such that $h(R) = (1,n_1,n_2,1,1)$ and $n_1 \leq 6$ ?

Note that, in this case $n_1 \geq 4$ since $R$ is quadratic. Since $n_1 = \text{embdim } R - \text{dim } R$ and $\text{embdim } K[Q_G] = \# S(G) = 1 + n + \# \{ W \in S(G) \mid \# W \geq 2 \}$ and $\text{dim } K[Q_G] = n + 1$, if $\text{embdim } K[Q_G] - \text{dim } K[Q_G] \leq 6$, then $\# \{ W \in S(G) \mid \# W \geq 2 \} \leq 6$. In particular, we have $\alpha(G) = 2$, where $\alpha(G) := \max \{ \# W \mid W \in S(G) \}$ is the stability number of $G$. Since if $G$ is perfect graph with $\alpha(G) = 2$ then $G$ is bipartite, In this case $G$ is not perfect.

Let $G$ be a graph on $[n]$ and with $E(G)$ its edge set. The edge ring of $G$, denoted by $K[G]$, is defined by

$$K[G] := K[x_ix_j \mid \{i,j\} \in E(G)] \subset K[x_1, \ldots, x_n].$$

The second question is

**Question 2.2.** Does exist a graph $G$ such that the edge ring $K[G]$ is non-Koszul quadratic Gorenstein ?

In [27, Theorem 1.2], a criterion for the edge ring $K[G]$ of $G$ to be quadratic is given. Moreover, in [20], a class of graphs with the property that the toric ideal $I_G$ of the edge ring $K[G]$ of $G$ is quadratic but $I_G$ possesses no quadratic Gröbner bases is studied. A graph $G$ is said to be $(\ast)$-minimal if $G$ satisfies the above property and every induced subgraph $H \subseteq G$ does not satisfy the property. By the computation by using Macaulay2, we have that if $G$ is $(\ast)$-minimal and the edge ring $K[G]$ is non-Koszul quadratic Gorenstein, then $n \geq 9$.

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