BRST Invariance of Non-local Charges and Monodromy Matrix of Bosonic String on $AdS_5 \times S^5$

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ABSTRACT: Using the generalized Hamiltonian method of Batalin, Fradkin and Vilkovsky we develop the BRST formalism for the bosonic string on $AdS_5 \times S^5$ formulated as principal chiral model. Then we show that the monodromy matrix and non-local charges are BRST invariant.

KEYWORDS: string theory.

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1. Introduction and Summary

One of the most remarkable developments of the string theory in recent years is the celebrated duality between string theory on an $AdS_5 \times S^5$ background and $N = 4$ supersymmetric Yang-Mills theory living on the boundary of this space $\cite{1, 2, 3}$. These theories have many properties in common, as for example global symmetries. On the other hand the direct comparison of general states in these two theories is difficult by the weak/strong nature of the duality and for a long time could only be applied to the special states that are protected by supersymmetry.

One of the most remarkable tools for checking this duality is the discussion of integrability on both sides of this duality. This program began when Minahan and Zarembo demonstrated that the one loop anomalous dimension operator, acting on single trace scalar operators, could be interpreted as the Hamiltonian of an integrable spin chain $\cite{6}$. Therefore, the anomalous dimensions could be found using Bethe ansatz. At present no gauge theory calculation of these single trace operators has contracted integrability.

In string theory the investigation of the integrability began with the discovery of complete set of classically conserved non-local charges in $\cite{11}$. In summary, the evidence of the integrability on both sides of the duality is a compelling new argument in favor of the duality.

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1 For review, see for example $\cite{4, 5}$.
2 For review and extensive list of references, see $\cite{7, 8, 9, 10}$.
3 For some of the related works, see $\cite{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29}$.
The next step would be to extend the context of the integrability on the quantum level. It turns out that this is much more difficult problem, for careful discussion, see \cite{30}. In the quantum treatment one should certainly consider the Green-Schwarz superstring on the $AdS_5 \times S^5$. On the other hand there is no covariant quantization scheme for Green-Schwarz superstring. Even if the light-cone treatment of the Green-Schwarz string is very useful and gives many interesting results \cite{31,32} it would be certainly desirable to have covariant description of the superstring. Such a formulation has been developed in last few years by Berkovits \cite{33,34,35,36}. Pure spinor string in $AdS_5 \times S^5$ has been studied in \cite{43,44,42,47,48,49,50} and the aspects of the classical integrability have been discussed in \cite{51,46}. On the other hand the explicit solution of the pure spinor string in $AdS_5 \times S^5$ background is still lacking.

Since the BRST treatment of the string theory and its relation to the integrability is not well known we mean that it would be useful to study its properties in simpler dynamical system. One example of such a system is a bosonic string theory on $AdS_5 \times S^5$ formulated as principal sigma model coupled to two dimensional world-sheet gravity. Due to the diffeomorphism invariance this system possesses explicit gauge invariance and can be treated with the Hamiltonian method of Batalin, Fradkin and Vilkovsky (BFV) \cite{52}. We will develop this formalisms for the principal sigma model coupled to two dimensional gravity. We start with the Hamiltonian formalism for corresponding principal model. We introduce the canonical variables using the method developed in \cite{53}. Then we introduce the extended phase space following \cite{57}. We determine the action of the BRST charge on the phase space variables using the canonical Poisson brackets. We obtain, by definition, expressions that hold for fields that are off-shell. On the other hand it is well known that the monodromy matrix is defined using the Lax connection that is constructed from currents that define given principal model. In other words the monodromy matrix is defined in the configuration space. For that reason we use the field redefinitions given in \cite{57} that map the ghosts given in BFV formalism to ones that appear in the geometrical treatment. It turns out however that even in this case the BRST transformations of the currents do not correspond to the geometrical one. We will argue that the original BRST transformations take the geometrical form on condition that the matter currents obey appropriate form of the equations of motion. This fact however implies that we have to work in the specific gauge in order to define Hamiltonian that determines the time evolution in the extended phase space.

We then show that the non-local charges, defined either by iteration procedure given in \cite{58} or by monodromy matrix are BRST invariant. We define these non-local charges using the prescription given recently in \cite{59} that allows us to introduce infinite number of non-local charges even on string world-sheet with finite spatial extend.

Using again the prescription given in \cite{58} we define the monodromy matrix for bosonic string on $AdS_5 \times S^5$. We again show that this monodromy matrix is conserved and BRST invariant.

\footnote{For review of pure spinor formalism in superstring theory, see \cite{37,38,39,40,41}.}

\footnote{For recent application of this method in string theory and quantum gravity, see \cite{24,19,48,54,55,56}.}
In conclusion, we hope that the BRST treatment of the bosonic string theory on $AdS_5 \times S^5$ background brings new insight on an existence and importance of the infinite number of non-local and local conserved charges. The fact that they are BRST invariant emphasize their physical importance for the description of given state of the string and consequently serves as further support of the integrability of the bosonic string on $AdS_5 \times S^5$. Of course one can expect that in order to properly include quantum effects one should take fermions into account. For that reason we hope that better understanding of the relation between BRST symmetry and integrability could be useful for further development of the pure spinor string in $AdS_5 \times S^5$ background.

This paper is organized as follows. In next section (2) we formulate the bosonic string on $AdS_5 \times S^5$ as principal model. We perform its Hamiltonian analysis and determine all constrains that given theory have to obey. In section (3) we introduce extended phase space and determine BRST charge and its action on fundamental fields. Then we perform field redefinitions that map the original BRST transformations to ones that have the properties of the geometrical BRST transformations. In section (4) we review the iterative definition of the non-local charges that holds for general world-sheet metric. We will argue that these non-local charges can be defined on the string world-sheet as well and that they are conserved. Then we show that these charges are BRST invariant as well. Finally in section (5) we introduce the monodromy matrix and review its main properties. We carefully review the calculation of the Poisson brackets between the monodromy matrix and any function defined on extended phase space. Then we will show that this monodromy matrix has vanishing Poisson brackets with the Hamiltonian and with the BRST charge. In other words we show that the monodromy matrix is time independent and BRST invariant.

2. Bosonic string on $AdS_5 \times S^5$ as principal model

In this section we formulate the bosonic string on the $AdS_5 \times S^5$ as principal chiral model, following the notation given in [26]. The five sphere $S^5$ is parameterized by five variables: coordinates $y^i, i = 1, \ldots, 4$ and the angle variable $\phi$. In terms of six real embedding coordinates $Y^A, A = 1, \ldots, 6$ obeying the condition $Y_A Y^A = 1$ the parametrisation reads

\[
\begin{align*}
\mathcal{Y}_1 &= Y_1 + i Y_2 = \frac{y_1 + iy_2}{1 + \frac{y^2}{4}}, \\
\mathcal{Y}_2 &= Y_3 + i Y_4 = \frac{y_3 + iy_4}{1 + \frac{y^2}{4}}, \\
\mathcal{Y}_3 &= Y_5 + i Y_6 = \frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}} \exp(i\phi).
\end{align*}
\]

(2.1)

The metric induced on five sphere $S^5$ from the flat induced metric is

\[
dY_A dY_A = \left(\frac{1 - \frac{y^2}{4}}{1 + \frac{y^2}{4}}\right)^2 d\phi^2 + \frac{1}{\left(1 + \frac{y^2}{4}\right)^2} dy_i dy_i,
\]

(2.2)
where \( y^2 \equiv y_i y_i \). In the same way we describe the \( AdS_5 \) space when we introduce four coordinates \( z_i \) and \( t \). The embedding coordinates \( Z_A \) that obey \( Z_A Z_B \eta^{AB} = -1 \) with the metric \( \eta^{AB} = (-1, 1, 1, 1, -1) \) is now parameterized as

\[
\begin{align*}
Z_1 &= Z_1 + iZ_2 = -\frac{z_1 + iz_2}{1 - \frac{z_2^2}{4}}, \quad Z_2 = Z_3 + iZ_4 = -\frac{z_3 + iz_4}{1 - \frac{z_4^2}{4}}, \\
Z_3 &= Z_0 + iZ_5 = \frac{1 + \frac{z_2^2}{4}}{1 - \frac{z_5^2}{4}} \exp(it)
\end{align*}
\]

so that the induced metric takes the form

\[
\eta_{AB} dZ^A dZ^B = \frac{(1 + \frac{z_i^2}{4})^2}{(1 - \frac{z_i^2}{4})^2} dt^2 + \frac{1}{(1 - \frac{z_i^2}{4})^2} dz_i dz_i , 
\]

where again \( z^2 \equiv z_i z_i \). We presume closed string and hence all fields \( Y_A, Z_A \) are periodic functions of the world-sheet variable \( \sigma \in (0, 2\pi) \). It is remarkable fact that the bosonic string on the \( AdS_5 \times S^5 \) can be formulated as principal chiral model with the action

\[
S = -\frac{\sqrt{\lambda}}{4\pi} \int dt d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} \text{Tr}(g^{-1}\partial_\alpha gg^{-1}\partial_\beta g) ,
\]

where \( \sqrt{\lambda} \) is related to the radius \( R \) of \( S^5 \) (\( AdS_5 \)) and the slope \( \alpha' \) of the Regge trajectory as \( \sqrt{\lambda} = \frac{R^2}{\alpha'} \) and where \( \gamma^{\alpha\beta}, \alpha = \tau, \sigma \) is world-sheet metric. In (2.3) the matrix \( g \) takes the form

\[
g = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix} .
\]

Here \( g_a \) and \( g_s \) are the following \( 4 \times 4 \) matrices

\[
g_a = \begin{pmatrix} 0 & Z_3 & -Z_2 & Z_1^* \\ -Z_3 & 0 & Z_1 & Z_2^* \\ Z_2 & Z_1 & 0 & -Z_3^* \\ -Z_1^* & Z_2^* & Z_3^* & 0 \end{pmatrix}, \quad g_s = \begin{pmatrix} 0 & \mathcal{Y}_1 & -\mathcal{Y}_2 & \mathcal{Y}_3^* \\ -\mathcal{Y}_1 & 0 & \mathcal{Y}_3 & \mathcal{Y}_2^* \\ \mathcal{Y}_2 & \mathcal{Y}_3 & 0 & \mathcal{Y}_1^* \\ -\mathcal{Y}_3^* & -\mathcal{Y}_2^* & -\mathcal{Y}_1^* & 0 \end{pmatrix},
\]

where \( Z_k, k = 1, 2, 3 \) are the complex embedding coordinates for \( AdS_5 \) defined in (2.3) and \( \mathcal{Y}_k, k = 1, 2, 3 \) are the complex embedding coordinates for sphere defined in (2.1). The matrix \( g_a \) is an element of the group \( SU(2, 2) \) since it can be shown that

\[
g_a^\dagger E g_a = E , \quad E = \text{diag}(-1, -1, 1, 1)
\]

provided the following condition is satisfied

\[
Z_1^* Z_1 + Z_2^* Z_2 - Z_3^* Z_3 = -1 .
\]

In fact \( g_a \) describes embedding of an element of the coset space \( SO(4, 2)/SO(5, 1) \) into group \( SU(2, 2) \) that is locally isomorphic to \( SO(4, 2) \). We use this isometry to work with
4 × 4 matrices rather with 6 × 6 ones. Note that due to the explicit choice of the coset representative above there is not any gauge symmetry left. Quite analogously \( g_s \) is unitary

\[ g_s g_s^\dagger = 1 \]  

(2.10)
on condition that \( \mathcal{Y}_1^* \mathcal{Y}_1 + \mathcal{Y}_2^* \mathcal{Y}_2 + \mathcal{Y}_3^* \mathcal{Y}_3 = 1 \). The matrix \( g_s \) describes an embedding of an element of the coset \( SO(6)/SO(5) \) into \( SU(4) \) being isomorphic to \( SO(6) \). In what follows we use the abstract formulation of the action

\[ S = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{\alpha\beta} J^A_\alpha \gamma^B_\beta K_{AB} \]  

(2.11)
where

\[ J = g^{-1} dg = J^A T_A \]  

(2.12)
and where \( T_A \) are generators of the algebra \( g \) with the metric

\[ K_{AB} = \text{Tr}(T_A T_B) \]  

(2.13)
In order to develop the canonical formalisms we follow [53]. Using the flatness of the current \( J^A \)

\[ \partial_\tau J^A_\sigma - \partial_\sigma J^A_\tau + J^B_\tau J^C_\sigma f^A_{BC} = 0 \]  

(2.14)
we can express \( J^A_\tau \) as

\[ \partial_\tau J^A_\sigma = (\partial_\sigma \delta^A_C + J^B_\tau f^A_{BC}) J^C_\tau \equiv D^A_C J^C_\tau \]  

(2.15)
that implies

\[ J^A_\tau = (D^{-1})^A_B \partial_\tau J^B_\sigma \]  

(2.16)
We see that it is natural to interpret \( J^A_\sigma \) as a canonical variable with the corresponding conjugate momentum \( \Pi_A \). These canonical variables have following Poisson bracket

\[ \{ J^A_\sigma (\sigma_1), \Pi_B (\sigma_2) \} = \delta^A_B \delta (\sigma_1 - \sigma_2) . \]  

(2.17)
In order to find the corresponding momentum \( \Pi_A \) we insert (2.16) into the action (2.11) and perform the variation with respect to \( \partial_\tau J^A_\sigma \). After some calculations we obtain

\[ \Pi_A = \frac{\delta S}{\delta \partial_\tau J^A_\sigma} = \frac{\sqrt{\lambda}}{2\pi} K_{AB} (D^{-1})^B_C (\sqrt{-\gamma} \gamma^{\tau\tau} (D^{-1})^C_D (\partial_\tau J^D_\sigma) + \sqrt{-\gamma} \gamma^{\tau\sigma} J^C_\sigma) . \]  

(2.18)
Using this relation we can express \( J^A_\tau \) as a function of \( J^A_\sigma \) and \( \Pi_A \)

\[ J^A_\tau = \frac{1}{\sqrt{-\gamma} \gamma^{\tau\tau}} \left( \frac{2\pi}{\sqrt{\lambda}} D^A - \sqrt{-\gamma} \gamma^{\tau\sigma} J^A_\sigma \right) , \]  

(2.19)
where we have defined

\[ D^A \equiv D^A_B \Pi^B \]  

(2.20)
As the next step we define the Hamiltonian as

\[
H_0 = \int d\sigma \mathcal{H}_0 = \int d\sigma (\Pi_A \partial_\sigma J^A_\sigma - \mathcal{L}) = \\
= \int d\sigma \left[ \frac{\sqrt{-\gamma}}{\gamma_{\sigma\sigma}} \left( \frac{\pi}{\sqrt{\lambda}} \mathcal{D}^C \mathcal{D}^D K_{CD} + \frac{\sqrt{\lambda}}{4\pi} J^C_\sigma J^D_\sigma K_{CD} + \frac{\gamma_{\tau\sigma}}{\gamma_{\sigma\sigma}} J^C_\sigma K_{CD} \mathcal{D}^D \right) \right].
\]

(2.21)

We choose the parametrisation of the metric \(\gamma_{\alpha\beta}\) as \([57]\)

\[
\lambda^\pm = \frac{\sqrt{\gamma + \gamma_{\tau\sigma}}}{\gamma_{\sigma\sigma}}, \quad \xi = \ln \gamma_{\sigma\sigma},
\]

(2.22)

where \(\lambda^\pm\) are manifestly invariant under Weyl transformation

\[
g'_{\alpha\beta}(\sigma, \tau) = e^{\phi(\sigma, \tau)} g_{\alpha\beta}(\sigma, \tau)
\]

(2.23)

while \(\xi\) transforms as \(\xi'(\sigma, \tau) = \xi(\sigma, \tau) + \phi(\sigma, \tau)\). Since the action (2.11) does not contain time-derivative of \(\gamma_{\alpha\beta}\) it follows that the momenta conjugate to \(\lambda^\pm, \xi\) are zero:

\[
\pi^\pm = \frac{\delta S}{\delta \partial_\tau \lambda^\pm} = 0, \quad \pi_\xi = \frac{\delta S}{\delta \partial_\tau \xi} = 0.
\]

(2.24)

These conditions consist primary constraints of the theory. Using (2.22) the Hamiltonian density (2.21) takes the form

\[
\mathcal{H}_0 = \frac{\sqrt{-\gamma}}{\gamma_{\sigma\sigma}} \left( \frac{\pi}{\sqrt{\lambda}} \mathcal{D}^C \mathcal{D}^D K_{CD} + \frac{\sqrt{\lambda}}{4\pi} J^C_\sigma J^D_\sigma K_{CD} + \frac{\gamma_{\tau\sigma}}{\gamma_{\sigma\sigma}} J^C_\sigma K_{CD} \mathcal{D}^D \right) = \\
= \frac{\sqrt{-\gamma}}{\gamma_{\sigma\sigma}} T_0 + \frac{\gamma_{\tau\sigma}}{\gamma_{\sigma\sigma}} T_1 = \frac{\lambda^+}{2} (T_0 + T_1) + \frac{\lambda^-}{2} (T_0 - T_1).
\]

(2.25)

We see an advantage of the parametrisation (2.22) since the Hamiltonian density \(\mathcal{H}_0\) contains variables \(\lambda^\pm\) only and hence it is Weyl invariant. According to the general analysis the time evolution of the primary constraints \(\pi^\lambda_{\pm}\) imply an existence of the secondary constraints

\[
T_\pm = \frac{1}{2} (T_0 \pm T_1) \approx 0.
\]

(2.26)

As the next step we determine some fundamental Poisson brackets. Recall that \(\mathcal{D}^A\) is equal to

\[
\mathcal{D}^A = \partial_\sigma \Pi^A + J^B_\sigma \Pi^C f^A_{BC}.
\]

(2.27)

Then using the canonical Poisson bracket (2.17) it is easy to see that

\[
\{ J^A_\sigma(\sigma), \mathcal{D}^B(\sigma') \} = -\partial_\sigma \delta(\sigma - \sigma') K^{AB} - J^C_\sigma(\sigma) f^A_{CD} K^{DB} \delta(\sigma - \sigma'),
\]

\[
\{ \Pi^A(\sigma), \mathcal{D}^B(\sigma') \} = -K^{AC} f^B_{CD} \Pi^D(\sigma) \delta(\sigma - \sigma'),
\]

\[
\{ \mathcal{D}^A(\sigma), \mathcal{D}^B(\sigma') \} = -K^{AC} f^B_{CD} \mathcal{D}^D(\sigma) \delta(\sigma - \sigma').
\]

(2.28)
As the next step we determine the Poisson brackets between \( J_\sigma, D \) and \( T_{0,1} \)

\[
\begin{align*}
\{J^A_\sigma(\sigma), T_0(\sigma')\} &= \partial_\sigma \delta(\sigma - \sigma') \sqrt{-\gamma} \gamma^{\alpha \tau} J^A_\sigma(\sigma') - \sqrt{-\gamma} \gamma^{\alpha \tau} J^B_\sigma J^C_\tau (\sigma') \partial^A_B \partial(\sigma - \sigma') , \\
\{J^A_\sigma(\sigma), T_1(\sigma')\} &= \partial_\sigma \delta(\sigma - \sigma') J^A_\sigma(\sigma') , \\
\{D^A(\sigma), T_1(\sigma')\} &= \partial^{\sigma'} \delta(\sigma - \sigma') D^A(\sigma') , \\
\{D^A(\sigma), T_0(\sigma')\} &= \frac{\sqrt{x}}{2\pi} \partial_\sigma \delta(\sigma - \sigma') J^A_\sigma(\sigma') .
\end{align*}
\]

(2.29)

Using these Poisson brackets we easily determine

\[
\begin{align*}
\{T_1(\sigma), T_1(\sigma')\} &= -2T_1(\sigma) \partial_\sigma \delta(\sigma - \sigma') - \partial_\sigma T_1(\sigma) \delta(\sigma - \sigma') , \\
\{T_0(\sigma), T_1(\sigma')\} &= -2T_0(\sigma) \partial_\sigma \delta(\sigma - \sigma') - \partial_\sigma T_0(\sigma) \delta(\sigma - \sigma') , \\
\{T_0(\sigma), T_0(\sigma')\} &= -2T_1(\sigma) \partial_\sigma \delta(\sigma - \sigma') - \partial_\sigma T_1(\sigma) \delta(\sigma - \sigma') .
\end{align*}
\]

(2.30)

It is convenient to define following combinations

\[
T_+ = \frac{1}{2}(T_0 + T_1) , \quad T_- = \frac{1}{2}(T_0 - T_1)
\]

(2.31)

that have following Poisson brackets

\[
\begin{align*}
\{T_+(\sigma), T_+(\sigma')\} &= -2T_+(\sigma) \partial_\sigma \delta(\sigma - \sigma') - \partial_\sigma T_+(\sigma) \delta(\sigma - \sigma') , \\
\{T_-(\sigma), T_-(\sigma')\} &= 2T_-(\sigma) \delta(\sigma - \sigma') + \partial_\sigma T_-(\sigma) \delta(\sigma - \sigma') , \\
\{T_+(\sigma), T_-(\sigma')\} &= 0 .
\end{align*}
\]

(2.32)

After determination of these Poisson brackets we can proceed to the BRST analysis of the principal model.

### 3. Extended phase space for principal model

We introduce the extended phase space following the approach presented in [57]. The extended phase space of the BFV theory is defined as including to the classical phase space the ghost- auxiliary field sector

\[
(C^A, \overline{\xi}_A), (P^A, \overline{C}_A), (N^A, B_A) ,
\]

(3.1)

where \( A = (\lambda^\pm, \xi, \pm) \) label the first class constraints \( \pi_\pm = \pi_\xi = 0 \) and \( T_\pm = 0 \). \( C^A \) and \( P^A \) are the BFV ghosts fields carrying one unit of the ghost number while \( \overline{\xi}_A \) and \( \overline{C}_A \) are their canonical momenta. The last canonical pairs are auxiliary fields that carry zero ghost number. Note that these fields have following Poisson brackets

\[
\begin{align*}
\{C^A(\sigma), \overline{P}_B(\sigma')\} &= \{\overline{P}_B(\sigma'), C^A(\sigma)\} = -\delta^A_B \delta(\sigma - \sigma') , \\
\{P^A(\sigma), \overline{C}_B(\sigma')\} &= \{\overline{C}_B(\sigma'), P^A(\sigma)\} = -\delta^A_B \delta(\sigma - \sigma') , \\
\{N^A(\sigma), B_B(\sigma')\} &= -\{B_B(\sigma'), N^A(\sigma)\} = \delta^A_B \delta(\sigma - \sigma') .
\end{align*}
\]

(3.2)
Given the constraints (2.24) and (2.26) we can construct the BRST charge in standard way

\[ Q = \int d\sigma [C^+ H^\lambda + C^- H^- + C^+ (T_+ + P_+ \partial_\sigma C^+) + 
+ C^- (T_- - P_- \partial_\sigma C^-) + B_A P_A]. \]

(3.3)

Using the Poisson brackets given in (3.2) it is straightforward to determine the BRST transformations of the ghost fields

\[
\begin{align*}
\delta \lambda^\pm &= \{ \lambda^\pm, Q \} = C^\pm, \\
\delta \xi &= \{ \xi, Q \} = C^\xi, \\
\delta \pi^\lambda &= \delta \pi^\xi = 0, \quad \delta C^\lambda = \delta C^\xi = 0, \\
\delta P^\pm &= \{ P^\pm, Q \} = -\pi^\pm, \\
\delta B_A &= \{ B_A, Q \} = -B_A, \quad \delta B_A = 0.
\end{align*}
\]

(3.4)

In order to determine the BRST transformation of the current \( J^A_\sigma \) we use the Poisson brackets (2.29) together with (3.2) and we get

\[
\begin{align*}
\delta_Q J^A_\sigma(\sigma) &= \{ J^A_\sigma(\sigma), Q \} = -\frac{(C^+ + C^-)}{2}(\sqrt{-\gamma} \gamma_\tau^r \partial_\sigma J^A_\sigma + \sqrt{-\gamma} \gamma_\tau^r \partial_\tau J^A_\sigma) - \\
&\quad - \frac{(C^+ - C^-)}{2} \partial_\sigma J^A_\sigma - \partial_\sigma [(C^+ + C^-) \sqrt{-\gamma} \gamma_\tau^r \partial_\tau J^A_\sigma] - \frac{1}{2} \partial_\sigma (C^+ - C^-) J^A_\sigma,
\end{align*}
\]

(3.5)

where \( \alpha = \tau, \sigma \). In principle the metric components \( \gamma_{\alpha \beta} \) can be expressed using the variables \( \lambda^\pm, \xi \) but it will not be necessary. In order to find the BRST transformation of \( J^A_\tau \) we have to use the relation (2.19) and then the Poisson brackets given in (2.29), (3.2). After some calculations we obtain

\[
\begin{align*}
\delta_Q J^A_\tau(\sigma) &= \{ J^A_\tau(\sigma), Q \} = \frac{(C^+ + C^-)}{2}(\sqrt{-\gamma} \gamma_\sigma^r \partial_\sigma J^A_\sigma + \sqrt{-\gamma} \gamma_\sigma^r \partial_\tau J^A_\sigma) - \\
&\quad - \frac{1}{2\sqrt{-\gamma}} \partial_\sigma (C^+ - C^-) J^A_\sigma + \\
&\quad + \frac{\gamma_\tau^r}{\gamma_{\sigma \tau}} \partial_\sigma [(C^+ + C^-) \sqrt{-\gamma} \gamma_\tau^r \partial_\tau J^A_\sigma] - \frac{1}{2} \partial_\sigma (C^+ - C^-) J^A_\tau - \\
&\quad - \frac{1}{2} \partial_\sigma [(C^+ - C^-) \sqrt{-\gamma} \gamma_\tau^r \partial_\tau J^A_\sigma] + \frac{\gamma_\tau^r}{2\gamma_{\tau \tau}} \partial_\sigma (C^+ - C^-) J^A_\tau - \\
&\quad - \frac{1}{2} \frac{(C^+ + C^-)}{\sqrt{\lambda}} D^A + \frac{(C^+ - C^-)}{2}. \frac{\pi}{\sqrt{\lambda}} D^A_\lambda + (C^+ - C^-) J^A_\tau.
\end{align*}
\]

(3.6)

We see that the BRST transformations expressed using the BFV ghosts is rather complicated. On the other hand the Lax connection and monodromy matrix are defined using
the covariant currents $J_\alpha$ so that it was necessary to find the BRST transformation of $J_\tau$. However we will argue below that it is possible to express the BRST transformation of the currents in more geometrical form.

Before we proceed to the study of this question we focus our attention on gauge fixed action for fields defined in the extended phase space. The gauge fixed action in the BFV formalism is defined as

\[
S = \int d\tau d\sigma \left( \dot{J}^A \Pi_A + \dot{\lambda}^\pi \pi^\pi + \dot{\lambda}^\pi \pi^\xi + \dot{\pi}^\xi - \mathcal{H} \right) = \\
= \int d\tau d\sigma \left[ \frac{2}{\lambda^+ + \lambda^-} \sqrt{\lambda} \dot{J}^A K_{AB} J^B + \frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-} \dot{J}^A K_{AB} J^B + \dot{\mathcal{C}}_A \mathcal{P}_A + \dot{\lambda}^\pi \pi^\pi + \dot{\lambda}^\pi \pi^\xi + \dot{\pi}^\xi - \mathcal{H} \right],
\]

where $\dot{X} \equiv \partial_\tau (X)$ and where

\[
H \equiv \int d\sigma \mathcal{H} = H_0 - \{ Q, \Psi \}.
\]

$\Psi$ given above is the gauge fermion. Note that we have used the fact that

\[
\mathcal{C}_A \mathcal{P}_A + B_A N^A = - \left\{ \mathcal{C}_A N^A, Q \right\}
\]

and hence this term can be absorbed into $\Psi$ by redefinition of $\chi^A$.

As we have seen $H_0$ is a sum of the constraints $T_\pm$. Then according to the general principles of the BRST Hamiltonian formalism we say that $H_0 = 0$.

Clearly we can not go further without any assumption on $\Psi$ so that we consider it in the standard form

\[
\Psi = \int d\sigma [\mathcal{C}_A \chi^A + \mathcal{P}_A N^A],
\]

where $\chi_A$ are gauge-fixing functions. With this choice of $\Psi$ we obtain the Hamiltonian in the form

\[
H = - \{ \Psi, Q \} = \int d\sigma (H_{CL} + H_{GF} + H_{FP}),
\]

where

\[
H_{CL} = N^+ T^+ + N^- T^- + \pi^\lambda_\lambda N^+ + \pi^\lambda_\lambda N^- + \pi^\xi N^\xi = \\
= \frac{1}{2} (N^+ + N^-) T_0 + \frac{1}{2} (N^+ - N^-) T_1 + \pi^\lambda_\lambda N^+ + \pi^\lambda_\lambda N^- + \pi^\xi N^\xi,
\]

\[
H_{GF} = B_A \chi^A,
\]

\[
H_{FP} = - \mathcal{C}_A \{ \chi^A, Q \} + \mathcal{P}_A \mathcal{P}_A + \\
+ [2 \mathcal{P}_+ \partial_\sigma C^+ + \partial_\sigma \mathcal{P}_+ C^+] N^- - [2 \mathcal{P}_- \partial_\sigma C^- + \partial_\sigma \mathcal{P}_- C^-] N^-.
\]

Comparing $H_{CL}$ with $H_0$ suggests that $N^\pm$ can be related to two of the metric variables as

\[
N^0 = \frac{1}{2} (N^+ + N^-) = \frac{1}{\sqrt{-\gamma^{\tau\tau}}} = \frac{1}{2} (\lambda^+ + \lambda^-),
\]

\[
N^1 = \frac{1}{2} (N^+ - N^-) = \frac{\gamma^{\tau\sigma}}{\gamma_{\tau\sigma}} = \frac{1}{2} (\lambda^+ - \lambda^-).
\]

(3.12)
Then in order to recover the original form of the metric we impose following gauge conditions
\[ \chi^+ = \lambda^+ - N^+, \quad \chi^- = \lambda^- - N^- \] (3.13)
without changing the relation \( \xi = \ln g_{11} \). Then the solving the equation of motion with respect to \( B^\lambda_{+, -} \) we obtain that
\[ \chi^+_\lambda = 0, \quad \chi^-_\lambda = 0 . \] (3.14)
On the other hand if we calculate the Poisson brackets between \( \chi^\pm_\lambda \) given in (3.13) and \( Q \) we obtain
\[ - \{ \chi^+_\lambda, Q \} = - C^+_\lambda + P^+, \quad - \{ \chi^-_\lambda, Q \} = - C^-_\lambda + P^- . \] (3.15)
Then the gauge fixed action takes the form (after integration out \( B^\lambda_{\pm} \))
\[ S = S_{\text{matter}} + S_{\text{ghosts}} + S_{\text{FG}}, \]
\[ S_{\text{matter}} = \int d\sigma d\tau [\dot{\lambda}^+ \pi^\lambda_+ + \dot{\lambda}^- \pi^-_\lambda + \dot{\lambda}^\xi \pi_\xi - \frac{\sqrt{\lambda}}{4\pi} \sqrt{-\gamma} \alpha^\beta \gamma_{\alpha} J^K_{\beta} K_{KL} - \pi^\lambda_+ N^+_\lambda - \pi^-_\lambda N^-_\lambda - \pi^\xi N^\xi_\lambda], \]
\[ S_{\text{ghosts}} = \int d\sigma d\tau [\bar{\mathcal{C}} A \mathcal{F} A - \mathcal{H}_{FP}], \]
\[ \mathcal{H}_{FP} = - \bar{\mathcal{C}}^+ \{ \chi^+, Q \} - \bar{\mathcal{C}}^- \{ \chi^-, Q \} - \bar{\mathcal{C}}^\xi \{ \chi^\xi, Q \} - \bar{\mathcal{C}}^+_\lambda c^+_\lambda + \bar{\mathcal{C}}^-_\lambda p^+ - \bar{\mathcal{C}}^\lambda p^- + \bar{\mathcal{C}} A p^A + [2\bar{\mathcal{C}}^+ \partial_\sigma c^+ + \partial_\sigma \bar{\mathcal{C}}^+ c^+] \lambda^+ - [2\bar{\mathcal{C}}^- \partial_\sigma c^- + \partial_\sigma \bar{\mathcal{C}}^- c^-] \lambda^- , \]
\[ S_{\text{FG}} = - \int d\sigma d\tau (B^+_\lambda \chi^+_\lambda + B^-_\lambda \chi^-_\lambda + B^\xi \chi^\xi), \] (3.16)
where we have used
\[ \frac{1}{\sqrt{-\gamma}} T_0(x) + \frac{\gamma^{\tau\sigma}}{\gamma_{\sigma\sigma}} T_1(x) = \frac{\sqrt{\lambda}}{4\pi} (\gamma^{\tau\tau} J^A_{\tau} \gamma^B J^K_{AB} K_{AB} - \gamma^{\sigma\sigma} J^A_{\sigma} J^K_{AB} K_{AB}) \] (3.17)
and the relation (2.19).

After introduction of the basic object of the BFV formalism we come to the question how to simplify the BRST transformations of the currents. It turns out that the natural way how to do this is to introduce more familiar geometrical BRST transformations. In fact, the primary constraints \( \pi^\lambda_{\pm} \) and \( \pi_\xi \) are responsible for the fact that the variables \( \lambda^\pm \) and \( \xi \) have lost their original meanings.
3.1 Review of the geometrical BRST transformations

In this subsection we review the geometrical BRST transformations. These BRST transformations have their origin in the transformations of the world-sheet modes under the diffeomorphism and Weyl transformations.

Let us consider the following diffeomorphism transformations

\[ x'^\mu = x^\mu + \omega^\mu(x), \quad \partial_{x'^\mu} \equiv \partial_{x^\mu} x'^\alpha = \delta_\alpha^\mu + \partial_{x^\mu} \omega^\alpha, \quad (3.18) \]

where \( x^\mu \equiv (x^0 = \tau, x^1 = \sigma) \). Since currents are one-form on the world-sheet their transformation properties can be easily determined from

\[ J'_\mu(x') = J_\mu(x) \Rightarrow \delta J_\mu(x) = J'_\mu(x) - J_\mu(x) = -\partial_\alpha J_\mu \omega^\alpha - J_\alpha(x) \partial_\mu \omega^\alpha. \quad (3.19) \]

Analogously we determine the variation of the metric under diffeomorphism transformations

\[ \delta \gamma_{\mu\nu} = -\partial_\alpha \gamma_{\mu\nu} \omega^\alpha - \partial_\mu \omega^\alpha \gamma_{\alpha\nu} \quad (3.20) \]

while its variation under Weyl transformation takes the form

\[ \delta W \gamma_{\alpha\beta} = -\omega W \gamma_{\alpha\beta}. \quad (3.21) \]

For letter purposes we also determine the variation of \( \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \) where

\[ \epsilon^{\mu\nu} = \epsilon^{\mu\nu}, \quad \epsilon^{01} = -e^{10} = 1. \quad (3.22) \]

Since \((\gamma \epsilon)^{\gamma}_{\alpha}\) is an object with one lower and one upper index it is clear that its variation takes the form

\[ \delta(\gamma_{\alpha\beta} \epsilon^{\beta\gamma}) = -\omega^{\delta} \partial_6 (\gamma_{\alpha\beta} \epsilon^{\beta\gamma}) - \partial_6 \omega^{\delta} \gamma_{\delta\gamma} \epsilon^{\beta\gamma} + \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_6 \omega^\gamma \quad (3.23) \]

while it is invariant under Weyl transformations. Finally, using the fact that

\[ \sqrt{-\gamma'(x')} = \sqrt{-\gamma(x)} \left| \det(\partial_\alpha x'^\mu)^{-1} \right| \quad (3.24) \]

we get

\[ \delta \sqrt{-\gamma(x)} = -\partial_\alpha \left[ \sqrt{-\gamma} \right] \omega^\alpha - \partial_\alpha \omega^\alpha \sqrt{-\gamma} \quad (3.25) \]

while its variation under Weyl transformation takes the form

\[ \delta W \sqrt{-\gamma} = \omega W \sqrt{-\gamma}. \quad (3.26) \]

These transformations appear as the BRST transformations in the extended configuration space. The extended configuration space contains three independent metric components \( \gamma_{\alpha\beta} \), currents \( J_\alpha \) with two reparametrisation ghosts \( C^\alpha \) and corresponding anti-ghosts \( \overline{C}_\alpha \).
For letter purposes we also determine following BRST variations of the flat connection (3.28) where \( \Lambda \) is spectral parameter. Using (3.27) it is easy to determine the BRST transformation of the flat connection (3.28)

\[
\delta_Q \gamma_{\mu\nu} = \partial_\alpha \gamma_{\mu\nu} C^\alpha + \gamma_{\mu\nu} \partial_\alpha C^\alpha + \partial_\mu C^\alpha \gamma_{\alpha\nu} + C_W \gamma_{\mu\nu},
\]

\[
\delta_Q (\gamma_{\alpha\beta} \epsilon^{\beta\gamma}) = C^\delta \partial_\delta (\gamma_{\alpha\beta} \epsilon^{\beta\gamma}) + \partial_\alpha C^\delta \gamma_{\delta\beta} \epsilon^{\beta\gamma} - \gamma_{\alpha\beta} \epsilon^{\beta\delta} \partial_\delta C^\gamma,
\]

\[
\delta_Q J_\mu = \partial_\alpha J_\mu C^\alpha + J_\mu \partial_\alpha C^\alpha,
\]

\[
\delta_Q J_\mu = \partial_\alpha J_\mu C^\alpha - J_\mu \partial_\alpha C^\alpha,
\]

\[
\delta_Q C^\alpha = C^\beta \partial_\beta C^\alpha,
\]

\[
\delta_Q C_W = C^\alpha \partial_\alpha C_W,
\]

\[
\delta_Q \sqrt{-\gamma} = \partial_\alpha [\sqrt{-\gamma} C^\alpha + \partial_\alpha C^\alpha \sqrt{-\gamma}].
\]

Let us now return to the principal chiral model and consider the Lax connection

\[
\hat{J}_\alpha^A (\Lambda) = \frac{1}{1 - \Lambda^2} (J_\alpha^A - \Lambda \gamma_{\alpha\beta} \epsilon^{\beta\gamma} J_\gamma^A),
\]

where \( \Lambda \) is spectral parameter. Using (3.27) it is easy to determine the BRST transformation of the flat connection (3.28)

\[
\delta_Q J_\alpha^A = C^\beta \partial_\beta [\frac{1}{1 - \Lambda^2} (J_\alpha^A - \Lambda \gamma_{\alpha\gamma} \epsilon^{\gamma\delta} J_\delta^A)] + \frac{1}{1 - \Lambda^2} J_\beta^A \partial_\alpha C^\beta - \frac{\Lambda}{1 - \Lambda^2} \partial_\alpha [\gamma_{\alpha\beta} \epsilon^{\beta\gamma} C^\gamma] + \delta_Q \sqrt{-\gamma} = \partial_\alpha \sqrt{-\gamma} C^\alpha + \partial_\alpha \sqrt{-\gamma} C^\alpha.
\]

For letter purposes we also determine following BRST variations

\[
\delta_Q (\sqrt{-\gamma} \gamma^{\alpha\tau} J_\alpha) = \partial_\tau (\sqrt{-\gamma} J_\alpha C^\tau) + \partial_\alpha (\sqrt{-\gamma} J_\alpha C^\tau) - \sqrt{-\gamma} J_\alpha C^\tau,
\]

\[
\delta_Q (\sqrt{-\gamma} \gamma^{\sigma\alpha} J_\alpha) = \partial_\sigma (\sqrt{-\gamma} J_\alpha C^\sigma) + \partial_\alpha (\sqrt{-\gamma} J_\alpha C^\sigma) - \sqrt{-\gamma} J_\alpha C^\sigma.
\]

After this review of the geometrical BRST transformations we return to the question how the BFV ghosts are related to the geometrical ghosts \( C^\alpha, C_W \). This problem was carefully studied in [57]. It was shown here that in order to find such a relation we have to use the equations of motion for BFV ghosts and consequently we have to specify remaining functions \( \chi^\pm, \chi^\xi \) in the gauge fermion. It was argued that it is sufficient to presume that these functions do not depend on \( \bar{F}_A, \pi^A_\pm \) and \( \pi^\xi \) but are arbitrary otherwise. Then it was shown [57] that the geometrical ghosts are related to the BFV ghosts as \(^6\)

\[
C^\tau = \frac{C^\tau}{\chi^\times}, \quad C^\sigma = -C^\sigma + \frac{\chi^\sigma}{\chi^\times} C^\tau,
\]

\(^6\)It was shown in [57] that in the same way we can find the relation between \( C_W \) and BFV ghosts. Since these formulas will not be important in what follows we do not include them here.
where $C^\pm = C^\tau \pm C^\sigma$. In order to express the BRST transformations (3.5) and (3.6) using the ghosts $C^\pm$ we have to invert the relation (3.31) and we get

$$
C^\tau = \frac{1}{2}(C^+ + C^-) = \lambda^\tau C^\tau = \sqrt{\gamma}^\tau C^\tau,
$$

$$
C^\sigma = \frac{1}{2}(C^+ - C^-) = -C^\sigma + \frac{1}{2}(\lambda^+ - \lambda^-)C^\tau = -C^\sigma + \gamma^\tau_\tau C^\tau.
$$

(3.32)

With the help of these results we obtain that (3.5) takes the form

$$
\{ J^A_\sigma, Q \} = C^\tau \partial^\tau J^A_\sigma + C^\sigma \partial_\sigma J^A_\sigma + \partial_\sigma C^\tau J^A_\sigma + \partial^\sigma C^\tau J^A_\sigma
$$

(3.33)

that coincides with the variation of $J_\sigma$ given in (3.27). On the other hand if we use (3.32) in (3.6) we obtain more complicated result

$$
\{ J^A_\tau, Q \} = \partial_\tau C^\tau J^A_\tau + C^\sigma \partial_\sigma J^A_\tau + +C^\sigma \partial_\sigma J^A_\tau +
$$

$$+ C^\tau \left[ -\frac{1}{\sqrt{-\gamma}} \gamma^\tau_\tau \partial_\tau \sqrt{-\gamma} \gamma^\tau_\tau J^A_\sigma \right] - \frac{1}{\sqrt{-\gamma}} \gamma^\tau_\tau \partial_\sigma \sqrt{-\gamma} \gamma^\tau_\tau J^A_\tau -
$$

$$
- \frac{1}{\gamma^\tau_\tau} \partial_\tau \sqrt{-\gamma} \gamma^\tau_\tau J^A_\tau - \frac{1}{\gamma^\tau_\tau} \partial_\sigma \sqrt{-\gamma} \gamma^\tau_\tau J^A_\tau ,
$$

(3.34)

where we have also used the equations of motion for $\alpha^\pm$ in order to express them as functions of $C^\pm$ and $\lambda^\pm$. We see that this expression can be written in the standard form on condition that currents obey the equation of motion

$$
\partial_\alpha \left[ \sqrt{-\gamma} \gamma^{\alpha \beta} J^A_\beta \right] = 0
$$

(3.35)

that follow from the variation of the action (3.16) on condition that the gauge fixing functions $\chi^\pm, \chi^\xi$ do not depend on $J$. Then if we demand that currents obey (3.35) we can rewrite (3.34) into the form

$$
\{ J^A_\tau, Q \} = C^\alpha \partial_\alpha J^A_\tau + J^A_\alpha \partial_\tau C^\alpha
$$

(3.36)

that coincides with the variation of $J_\tau$ given in (3.27).

4. BRST invariance of the non-local conserved charges

4.1 Definition of the non-local charges for general world-sheet metric

It is easy to see that the currents $J = g^{-1}dg$ and consequently the action (3.16) are invariant under the left multiplication of the group element $g$ by constant matrix $h$ from $G$. Then for $h = 1 + \epsilon$, where $\epsilon = \epsilon^A T_A$ belongs to $g$ the algebra of $G$ the variation of the current is equal to

$$
\delta J = g^{-1}dg
$$

(4.1)
or in components ($\delta J = \delta J^A T_A, \epsilon = \epsilon^A T_A$)

$$\delta J^A = d e^B C_{BC} K^{CA}, \quad C_{BC} = \text{Tr}(g^{-1} T_B g T_C).$$

Then we obtain that the variation of the action (3.16) takes the form (We again presume that $\chi^\pm, \chi^\xi$ do not depend on $J$)

$$\delta S_{\text{matter}} = \frac{-\sqrt{\lambda}}{2\pi} \int d\sigma^2 \sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha \epsilon^A C_{AB} K^{BC} K_{CD} J^D.$$

Then the standard arguments imply an existence of the current

$$j_{\alpha\beta} = \frac{\sqrt{\lambda}}{2\pi} C_{AC} J^C_{\beta}$$

(4.4)

that is conserved

$$\partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha\beta} j_{\alpha\beta}] = 0$$

(4.5)

on condition that all fields obey the equations of motion. Note also that the current (4.4) $j = j^A T_A$ can be written in the following form

$$j = \frac{\sqrt{\lambda}}{2\pi} d g g^{-1} = \frac{\sqrt{\lambda}}{2\pi} g J g^{-1}.$$  

Then from (4.6) we obtain important identity

$$dj + j \wedge j = 0$$

(4.7)

or explicitly

$$\partial_\alpha J^A_{\beta} - \partial_\beta J^A_{\alpha} + j^B_{\alpha} J^C_{\beta} f^A_{BC} = 0.$$  

(4.8)

The equation (4.8) together with (4.5) allows us to construct non-local conserved charges using the iteration procedure presented in [58].

To begin with we define the operator $(D_\alpha)_B^A$ as

$$(D_\alpha)_B^A = \delta_B^A \partial_\alpha + j^C_\alpha f^A_{CB}.$$  

(4.9)

This operator acts on an element $X^A$ as

$$(D_\alpha)_B^A X^B = \partial_\alpha X^A + j^C_\alpha f^A_{CB} X^B.$$  

(4.10)

In the same way we introduce the operator

$$(D_\alpha)_B^A = \delta_B^A \partial_\alpha.$$  

(4.11)

Then it is easy to see that

$$[(\partial_\alpha)_B^A (\sqrt{-\gamma} \gamma^{\alpha\beta} D^B_{\beta C}) - D^A_{\alpha B} (\sqrt{-\gamma} \gamma^{\alpha\beta} (\partial_\beta)_C)] X^C = \partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha\beta} j^B_{\beta} f^A_{BC} X^C = 0$$

(4.12)

due to the fact that $j$ obeys the equation (4.5).
In the same way we can show that
\[
[D_\alpha, D_\beta] A X^B = [\partial_\alpha j^{C}_{\beta} - \partial_\beta j^{C}_{\alpha} + j^{D}_{\alpha} D^{E}_{\beta} f^{C}_{DE}] f^{A}_{CB} X^B = 0
\]
(4.13)
due to the equation (1.8). Now we return to the iterative procedure given in [58] generalised to the case of general world-sheet metric. Let us consider conserved current \( j^{(n)}_{\alpha A} \) that obeys the equation
\[
\partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha \beta} j^{(n)}_{\beta A}] = 0 .
\]
(4.14)
Then we can define the function \( \xi^{(n)}_A \) in the following way
\[
j^{(n)}_{\alpha A} = \gamma_{\alpha \delta} \epsilon^{\beta \delta} \partial_\beta \xi^{(n)}_A
\]
(4.15)
since then (4.14) is trivially satisfied as follows from
\[
\partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha \beta} j^{(n)}_{\beta A}] = \partial_\alpha [\sqrt{-\gamma} \epsilon^{\alpha \beta} \partial_\beta \xi^{(n)}_A] = \epsilon^{\alpha \beta} \partial_\beta \partial_\alpha \xi^{(n)}_A = 0 ,
\]
(4.16)
where we have again used the fact that
\[\epsilon^{\alpha \beta} = \frac{\epsilon^{\alpha \beta}}{\sqrt{-\gamma}}, \epsilon^{01} = -\epsilon^{10} = 1 .\]
(4.17)
Note also that the equation (4.13) implies
\[
\partial_\alpha \xi^{(n)A} = K^{AB} \epsilon^{\beta \gamma} \gamma_{\beta \gamma} j^{(n)}_{\gamma B} = K^{AB} \gamma_{\alpha \beta} \epsilon^{\beta \gamma} j^{(n)}_{\gamma B}
\]
(4.18)
using the fact that \( \gamma_{\alpha \beta} \epsilon^{\beta \delta} = \epsilon_{\alpha \beta} \gamma^{\beta \delta} \). Then we define current
\[
j^{(n+1)}_{\alpha A} = K_{AB} (D_{\alpha} C^{B} C^{(n)A}) = \partial_\alpha \xi^{(n)}_A + K_{AB} j^{D}_{\alpha} C^{(n)C} f^{B}_{DC} .
\]
(4.19)
This current is also conserved as follows from
\[
\partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha \beta} j^{(n+1)}_{\beta A}] = K_{AB} \partial_\alpha [\sqrt{-\gamma} \gamma^{\alpha \beta} (D_{\beta} C^{B} C^{(n)C})] = K_{AB} (D_{\alpha} C^{B} C^{(n)C}) = K_{AB} (D_{\alpha} C^{B} C^{(n+1)}) = K_{AB} (D_{\beta} C^{(n-1)D}) = 0 ,
\]
(4.20)
where we have used (4.12) and (4.13). The fact that \( j^{(n+1)} \) obeys (4.20) implies that this iterative procedure generates infinite number of non-local currents with initial condition
\[
j^{(0)}_{\alpha A} = j_{\alpha A} .
\]
(4.21)
As an example we determine the first order non-local current. Firstly from (4.18) we obtain
\[
\xi^{(0)}_A (\sigma, \tau) = - \int_{0}^{\tau} \, d\tau' \sqrt{-\gamma} \gamma^{\sigma \delta} j_{\delta A}(\sigma, \tau') + \int_{-\infty}^{\sigma} \, d\sigma' \sqrt{-\gamma} \gamma^{\sigma \delta} j_{\delta A}(\tau', 0) ,
\]
(4.22)
where we have used the initial condition $\lim_{\tau \to -\infty} j(\tau) = 0$. Then we obtain

$$\begin{align*}
j_{\sigma A}^{(1)}(\sigma, \tau) &= \epsilon_{\sigma\alpha\gamma}^{} j_{\beta A}(\sigma, \tau) + K_{AB} j_{\sigma}^D \xi^{(0)C}(\tau, \sigma) f_{BC}^D, \\
j_{\tau A}^{(1)}(\tau, \sigma) &= \epsilon_{\tau\alpha\gamma}^{} j_{\beta A}(\sigma, \tau) + K_{AB} j_{\tau}^D \xi^{(0)C}(\tau, \sigma) f_{BC}^D.
\end{align*}$$

(4.23)

In order to define the time-independent conserved charge we have to carefully specify the integration domain due to the finite size of the string world-sheet. To do this we use the proposal given recently in [59] and define the conserved charge $Q^{(n)}$ as

$$Q_A^{(n)} = - \int_{-\infty}^{\tau} d\tau' \sqrt{-\gamma} \gamma^{\sigma A} j_A^{(n)}(\tau, 0) + \int_0^{2\pi} d\sigma' \sqrt{-\gamma} \gamma^{\tau A} j_A^{(n)}(\sigma', \tau) + \int_{-\infty}^{\tau} d\tau' \sqrt{-\gamma} \gamma^{\sigma A} j_A^{(n)}(\tau, 2\pi).$$

(4.24)

Then using the conservation of $j_A^{(n)}$ and integration by parts we can show that the charge $Q^{(n)}$ defined in (4.24) is conserved

$$\frac{dQ_A^{(n)}}{d\tau} = - \frac{d}{d\tau} \sqrt{-\gamma} \gamma^{\sigma A} j_A^{(n)}(\tau, 0) - \int_0^{2\pi} d\sigma' \partial_{\sigma'} [\sqrt{-\gamma} \gamma^{\sigma A} j_A^{(n)}(\sigma', \tau)] + \sqrt{-\gamma} \gamma^{\sigma A} j_A^{(n)}(\tau, 2\pi) = 0.$$  

(4.25)

We mean that it is very satisfactory result that we can define non-local conserved charges in case of finite domain as well. This fact could be useful for further study of the integrability of the string theory on $AdS_5 \times S^5$.

4.2 BRST invariance of the conserved non-local charges

Now we are going to show that the non-local charges given above are BRST invariant as well. To do this we have to determine the BRST variation of $\xi_A^{(n)}$. It is easy to see that this is equal to

$$\delta Q \xi_A^{(n)} = C^\beta \partial_\beta \xi_A^{(n)}$$

(4.26)

since then

$$\delta Q j_A^{(n)} = \delta Q (\gamma_{\alpha\beta} \delta_\beta \xi_A^{(n)}) + \gamma_{\alpha\beta} \delta_\beta \delta Q (\xi_A^{(n)}) = C^\beta \partial_\beta j_A^{(n)} + j_A^{(n)} \partial_\alpha C^\beta$$

(4.27)

that is a correct form of the BRST transformation of the current $j_A$ as follows from (3.27).

Then the BRST variation of the current $j_A^{(n+1)}$ is equal to

$$\delta Q j_A^{(n+1)} = \partial_\alpha \delta Q (\xi_A^{(n)}) + K_{AB} \delta Q (j_A^D) \xi_A^{(n)C} f_{DC}^B + K_{AB} j_A^D \delta Q (\xi_A^{(n)}C) f_{DC}^B = \partial_\alpha C^\gamma (\partial_\gamma \xi_A^{(n)}) + K_{AB} j_A^D \delta Q (\xi_A^{(n)}C) f_{DC}^B = \partial_\alpha C^\beta j_A^{(n+1)} + \partial_\beta j_A^{(n+1)} C^\beta.$$  

(4.28)
Then using (3.30) we easily get
\[
\delta Q(\sqrt{-\gamma} j^{(n)\alpha}_{A(\alpha)}(\tau, 0)) = \partial_\tau (\sqrt{-\gamma} j^{(n)\alpha}_{A(\alpha)}(\tau, 0)) + \partial_\sigma (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0)) - \sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0) \Delta_\sigma C^\tau,
\]
\[
\delta Q(\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0)) = \partial_\sigma (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0)) + \partial_\tau (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0)) - \sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)}(\tau, 0) \Delta_\tau C^\sigma.
\]
(4.29)

Let us now calculate following Poisson bracket
\[
\{Q^{(n)}_A, Q\} = - \int_{-\infty}^\tau d\tau' \delta Q(\sqrt{-\gamma} j^{(n)\alpha}_{A(\alpha)}(\tau', 0)) + \int_0^{2\pi} d\sigma' \delta Q(\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\sigma', \tau)) + \int_{-\infty}^\tau d\tau' \delta Q(\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau', 2\pi)).
\]
(4.30)

Using (4.29) the first expression above can be calculated as
\[
- \int_{-\infty}^\tau d\tau' \delta Q(\sqrt{-\gamma} j^{(n)\alpha}_{A(\alpha)}(\tau', 0)) = - (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0)) + (\sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)} C^\sigma)(\tau, 0).
\]
(4.31)

In the same way we proceed with the third term in (4.30) and we obtain the result
\[
\int_{-\infty}^\tau d\tau' \delta Q(\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau', 2\pi)) = (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 2\pi)) - (\sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)} C^\sigma)(\tau, 2\pi).
\]
(4.32)

Finally we will calculate the second expression in (4.30)
\[
\int_0^{2\pi} d\sigma' \delta Q(\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\sigma', \tau)) = \\
\int_0^{2\pi} d\sigma' (\partial_\tau (\sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)} C^\sigma) + \partial_\sigma (\sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)} C^\tau) - \sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0) \Delta_\sigma C^\tau) = \\
- \sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 2\pi) + \sqrt{-\gamma} j^{(n)\sigma}_{A(\alpha)}(\tau, 0) + \sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)}(\tau, 2\pi) - \sqrt{-\gamma} j^{(n)\tau}_{A(\alpha)}(\tau, 0).
\]
(4.33)

Then collecting (4.31), (4.32) and (4.33) we obtain the final result
\[
\{Q^{(n)}_A, Q\} = 0.
\]
(4.34)

In other words the non-local conserved charges are BRST invariant.
5. BRST invariance of the monodromy matrix

In this section we show that monodromy matrix of the principal model is BRST invariant as well. We mean that this result could be useful for further study of the properties of the bosonic string on $\text{AdS}_5 \times S^5$. For example, it is well known that the trace of the monodromy matrix is a generator of the local conserved charges. Then the BRST invariance of the monodromy matrix immediately implies BRST invariance of these charges as well. Moreover, it is well known that the monodromy matrix is the generator of non-local charges and hence these charges are BRST invariant. We mean that the fact that local and non-local charges are BRST invariant support their physical importance for the description of the state of the bosonic string on $\text{AdS}_5 \times S^5$.

Before we proceed to the study of the BRST properties of the monodromy matrix we review its properties with respect to phase space and canonical Poisson brackets.

5.1 Hamiltonian analysis of the monodromy matrix

In this section we review the main properties of the monodromy matrix. We closely follow [53, 60, 61]. We also construct, following [59], the monodromy matrix on the string world-sheet that has finite spatial extend.

The monodromy matrix is defined as

$$ T(\gamma, \Lambda) = P \exp \left( \int_{\gamma} J_\alpha(\Lambda) \frac{dx^\alpha}{d\xi} \right), $$

(5.1)

where $\gamma = (x^\alpha(\xi) \equiv (\tau(\xi), \sigma'(\xi))$ is a curve on two dimensional cylinder, $J_\alpha \frac{dx^\alpha}{d\xi}$ is an embedding of the Lax connection with spectral parameter $\Lambda$ on the curve $\gamma$ and $\xi$ is a parameter that labels points on given curve. Finally $P$ in (5.1) means path ordering.

In order to define the time-independent monodromy matrix on string world-sheet we consider following form of the curve $\gamma$

$$
\begin{align*}
\gamma &= \gamma_1(-\infty, 0, \tau, 0) + \gamma_2(\tau, 0, \tau, 2\pi) + \gamma_3(\tau, 2\pi, -\infty, 2\pi), \\
\gamma_1 &= (\tau' = \xi, \sigma' = 0, \xi \in (-\infty, \tau)), \\
\gamma_2 &= (\tau' = \tau, \sigma' = \xi, \xi \in (0, 2\pi)), \\
\gamma_3 &= (\tau' = \xi, \sigma' = 2\pi, \xi \in (\tau, -\infty)).
\end{align*}
$$

(5.2)

Then for the curve (5.2) the monodromy matrix $T$ takes the form

$$ T(\Lambda) = P \exp\left( \int_{-\infty}^{\tau} d\tau' J_\tau(\tau', 0) + \int_{0}^{2\pi} d\sigma' J_\sigma(\tau, \sigma') + \int_{\tau}^{-\infty} d\tau' J_\tau(\tau', 2\pi) \right). $$

(5.3)

Let us now calculate the Poisson bracket between any local quantity (for example $H$ or $Q$) and the monodromy matrix for general form of the curve $\gamma$. Using the definition of Poisson bracket we obtain

$$ \{X, T_{ij}(\gamma, \Lambda)\} = \int_{0}^{2\pi} d\sigma \left( \frac{\delta X}{\delta J_\sigma^A(\sigma)} \frac{\delta T_{ij}(\gamma, \Lambda)}{\delta \Pi_A(\sigma)} - \frac{\delta X}{\delta \Pi_A(\sigma)} \frac{\delta T_{ij}(\gamma, \Lambda)}{\delta J_\sigma^A(\sigma)} \right). $$

(5.4)
where \((ij)\) label the matrix indices of \(\mathcal{T}\). To proceed we use the fact that (For simplicity we omit the symbol \(\Lambda\))

\[
\frac{\delta T_{ij}(\gamma)}{\delta J^A_k(\sigma)} = \int_\gamma d\xi \frac{\delta J^A_k(x(\xi))}{\delta J^B_\sigma(\sigma)} \frac{dx^\alpha}{d\xi} \frac{\delta T_{ij}(\gamma)}{\delta J^A(x(\xi))},
\]

\[
\frac{\delta T_{ij}(\gamma)}{\delta \Pi_B(\sigma)} = \int_\gamma d\xi \frac{\delta J^A_k(x(\xi))}{\delta \Pi_B(\sigma)} \frac{dx^\alpha}{d\xi} \frac{\delta T_{ij}(\gamma)}{\delta J^A(x(\xi))}.
\]

(5.5)

Then we can rewrite (5.4) into the form

\[
\{X, T_{ij}(\gamma)\} = \int_\gamma d\xi \frac{\delta T_{ij}(\gamma)}{\delta J^A_k(x(\xi))} \frac{dx^\alpha}{d\xi} \int_0^{2\pi} d\sigma \left( \frac{\delta X}{\delta J^B_\sigma(\sigma)} \frac{\delta J^A_k(x(\xi))}{\delta J^B_\sigma(\sigma)} - \frac{\delta X}{\delta \Pi_B(\sigma)} \frac{\delta J^A_k(x(\xi))}{\delta \Pi_B(\sigma)} \right) = \]  

\[
= \int_\gamma d\xi \frac{\delta T_{ij}(\gamma)}{\delta J^A_k(x(\xi))} \frac{dx^\alpha}{d\xi} \left\{ X, J^A_k(x(\xi)) \right\}.
\]

(5.6)

As the next step we determine the variation of \(T_{ij}\) with respect to \(\tilde{J}\). If we generalize the arguments given in [60, 61] to the case of arbitrary curve \(\gamma\) we obtain

\[
\delta \mathcal{T}(\gamma) = \int_\gamma d\xi \mathcal{T}(\gamma' < \gamma(\xi)) \delta \tilde{J}_\alpha(\xi) \frac{dx^\alpha}{d\xi} \mathcal{T}(\gamma' > \gamma(\xi)),
\]

(5.7)

where we have introduced following notation. \(\gamma' > \gamma(\xi)\) means the part of the curve \(\gamma\) that starts at the point \(\gamma(\xi)\) and ends at the final point of curve \(\gamma(\xi_f)\). In the same way \(\gamma' < \gamma(\xi)\) denotes the curve that starts at the initial point of \(\gamma(\xi_i)\) and ends at the point \(\gamma(\xi)\). Finally \(\mathcal{T}(\gamma' > \gamma(\xi))\), \(\mathcal{T}(\gamma' < \gamma(\xi))\) denote monodromy matrices evaluated on the curves \(\gamma' > \gamma(\xi)\), \(\gamma' < \gamma(\xi)\) respectively. Then if we use the fact that \(\tilde{J} = \tilde{J}^A T_A\) we can rewrite (5.7) as

\[
\delta T_{ij}(\gamma) = \int_\gamma d\xi \mathcal{T}(\gamma' < \gamma(\xi)) \delta J^A(\gamma) \left( \frac{dx^\alpha}{d\xi} T_{ij}(\gamma' > \gamma(\xi)) = \right) = \]  

\[
= \int_\gamma d\xi \delta J^A(\gamma(\xi)) \left( \frac{dx^\alpha}{d\xi} T_{ik}(\gamma' < \gamma(\xi)) \right) \frac{dx^\alpha}{d\xi} T_{ij}(\gamma' > \gamma(\xi))
\]

(5.8)

and finally

\[
\frac{\delta T_{ij}(\gamma)}{\delta J^A(\gamma(\xi))} = T_{ik}(\gamma' < \gamma(\xi)) (T_A)_{kl} T_{lj}(\gamma' > \gamma(\xi)),
\]

(5.9)

where \(\tilde{J}^A(\xi) = \tilde{J}^A(x(\xi)) \frac{dx^\alpha}{d\xi}\). Then using (5.9) the Poisson bracket (5.6) takes the final form

\[
\{X, T_{ij}(\gamma)\} = \int_\gamma d\xi T_{ik}(\gamma' < \gamma(\xi)) (T_A)_{kl} T_{lj}(\gamma' > \gamma(\xi)) \frac{dx^\alpha}{d\xi} \left\{ X, J^A(\gamma(\xi)) \right\}
\]

(5.10)
or in matrix notation

\[
\{X, \mathcal{T}(\gamma)\} = \int_{\gamma} d\xi T(\gamma' < \gamma(\xi)) \left\{X, \dot{J}_\alpha(\gamma(\xi))\right\} \frac{dx^\alpha}{d\xi} \mathcal{T}(\gamma' > \gamma(\xi)) .
\] (5.11)

With the help of this formula we show that \(\mathcal{T}\) evaluated on the curve (5.2) commutes with Hamiltonian on condition that \(\dot{J}\) is Lax pair. To do this we use the fact that the time evolution of any component of the Lax connection is governed by the equation

\[
\partial_\tau \dot{J}_\alpha(\sigma, \tau) = - \left\{H, \dot{J}_\alpha(\sigma, \tau)\right\} .
\] (5.12)

Then the equation (5.11) for \(X = H\) implies

\[
- \{\mathcal{T}, H\} = \int_{\gamma} d\xi T(\gamma' < \gamma(\xi)) \partial_\tau \dot{J}_\alpha(\gamma(\xi)) \frac{dx^\alpha}{d\xi} \mathcal{T}(\gamma' > \gamma(\xi)) =
\]

\[
= \int^\tau_{-\infty} d\tau' T(-\infty, 0, \tau', 0) \partial_{\tau'} \dot{J}_\tau(\tau', 0) \mathcal{T}(\tau', 0, -\infty, 2\pi) +
\]

\[
+ \int_{0}^{2\pi} d\sigma T(-\infty, 0, \tau, \sigma) \partial_\sigma \dot{J}_\sigma(\tau, \sigma) \mathcal{T}(\tau, \sigma, -\infty, 2\pi) +
\]

\[
+ \int_{\tau}^{-\infty} d\tau' T(-\infty, 0, \tau', 2\pi) \partial_{\tau'} \dot{J}_\tau(\tau', 2\pi) \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) .
\] (5.13)

The first term in (5.13) can evaluated using the integration by parts and we obtain

\[
\int^\tau_{-\infty} d\tau' T(-\infty, 0, \tau', 0) \partial_{\tau'} \dot{J}_\tau(\tau', 0) \mathcal{T}(\tau', 0, -\infty, 2\pi) =
\]

\[
= \mathcal{T}(-\infty, 0, \tau, 0) \dot{J}_\tau(\tau, 0) \mathcal{T}(\tau, 0, -\infty, 2\pi) ,
\] (5.14)

where we have used the fact that

\[
\frac{\partial \mathcal{T}(\gamma)}{\partial Y^\alpha} = \mathcal{T}(\gamma) \dot{J}_\alpha(Y) , \quad Y \equiv \gamma(\xi_f) ,
\]

\[
\frac{\partial \mathcal{T}(\gamma)}{\partial X^\alpha} = - \dot{J}_\alpha(X) \mathcal{T}(\gamma) , \quad X \equiv \gamma(\xi_i) ,
\] (5.15)

where \(X^\alpha\) is the initial point of the curve \(\gamma\) and \(Y^\alpha\) is the final point of the curve. In the same way we can show that

\[
\int^{\infty}_{\tau} d\tau' T(-\infty, 0, \tau', 2\pi) \partial_{\tau'} \dot{J}_\tau(\tau', 2\pi) \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) =
\]

\[
= - \mathcal{T}(-\infty, 0, \tau, 2\pi) \dot{J}_\tau(\tau, 2\pi) \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) .
\] (5.16)
Finally we evaluate the second expression in (5.13) using the fact that $\hat{J}$ is Lax connection. Explicitly we get
\[
\int_0^{2\pi} d\sigma \mathcal{T}(-\infty, 0, \tau, \sigma) \partial_\tau \hat{J}_\sigma(\tau, \sigma) \mathcal{T}(\tau, \sigma, -\infty, 2\pi) = \\
= \int_0^{2\pi} d\sigma \mathcal{T}(-\infty, 0, \tau, \sigma)(\partial_\sigma \hat{J}_\tau(\tau, \sigma) - [\hat{J}_\tau, \hat{J}_\sigma](\tau, \sigma)) \mathcal{T}(\tau, \sigma, -\infty, 2\pi) \\
= \mathcal{T}(-\infty, 0, \tau, 2\pi) \hat{J}_\tau(\tau, 2\pi) \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) - \mathcal{T}(-\infty, 0, \tau, 0) \hat{J}_\tau(\tau, 0) \mathcal{T}(\tau, 0, -\infty, 2\pi) ,
\]
where we have performed integration by parts and used (5.15). If we now combine (5.17) together with (5.14) and (5.16) we obtain that
\[
\{\mathcal{T}(\Lambda), H\} = 0 .
\]
In other words the monodromy matrix defined in (5.3) is time independent and hence it can be considered as generator of infinite number of local and non-local charges.

Finally we show that the monodromy matrix (5.3) is BRST invariant. Recall that the BRST transformation of the Lax connection takes the form
\[
\delta_Q \hat{J}_\alpha = \{\hat{J}_\alpha, Q\} = \partial_\alpha C^\beta \hat{J}_\beta + \partial_\alpha \hat{J}_\beta C^\beta .
\]
Now using (5.11) we obtain
\[
\{\mathcal{T}(\gamma), Q\} = \int_\gamma^\tau d\xi \mathcal{T}(\gamma' < \gamma(\xi)) \{\hat{J}_\alpha(x(\xi)), Q\} \frac{dx_\alpha}{d\xi} \mathcal{T}(\gamma' > \gamma(\xi)) = \\
= \int_{-\infty}^\tau d\tau' \mathcal{T}(-\infty, 0, \tau', 0) \{\hat{J}_\tau(\tau', 0), Q\} \mathcal{T}(\tau', 0, -\infty, 2\pi) + \\
+ \int_0^{2\pi} d\sigma \mathcal{T}(-\infty, 0, \tau, \sigma) \{\hat{J}_\sigma(\tau, \sigma), Q\} \mathcal{T}(\tau, \sigma, -\infty, 2\pi) + \\
+ \int_{\tau}^{-\infty} d\tau' \mathcal{T}(-\infty, 0, \tau', 2\pi) \{\hat{J}_\tau(\tau', 2\pi), Q\} \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) .
\]
Using (5.13) we can calculate the first term in (5.20) as
\[
\int_{-\infty}^\tau d\tau' \mathcal{T}(-\infty, 0, \tau', 0) \{\hat{J}_\tau(\tau', 0), Q\} \mathcal{T}(\tau', 0, -\infty, 2\pi) = \\
= \mathcal{T}(-\infty, 0, \tau, 0) C^\alpha \hat{J}_\alpha \mathcal{T}(\tau, 0, -\infty, 0) ,
\]
where we have performed the integration by parts and used (5.13) and also the fact that Lax connection is flat. In the same way we obtain
\[
\int_{\tau}^{-\infty} d\tau' \mathcal{T}(-\infty, 0, \tau', 2\pi) \{\hat{J}_\tau(\tau', 2\pi), Q\} \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) = \\
= -\mathcal{T}(-\infty, 2\pi, \tau, 2\pi) C^\alpha \hat{J}_\alpha \mathcal{T}(\tau, 2\pi, -\infty, 2\pi) .
\]
Finally we will calculate second term in (5.20) and we get

\[
\int_0^{2\pi} d\sigma \mathcal{T}(-\infty, 0, \tau, \sigma) \left\{ \hat{J}_\alpha(\tau, \sigma), Q \right\} \mathcal{T}(\tau, \sigma, -\infty, 2\pi) = \\
\mathcal{T}(-\infty, 0, \tau, 2\pi) C^\alpha \hat{J}_\alpha(\tau, 2\pi) \mathcal{T}(\tau, \sigma, \infty, 2\pi) - \\
\mathcal{T}(-\infty, 0, \tau, 0) C^\alpha \hat{J}_\alpha(\tau, 0) \mathcal{T}(\tau, 0, \infty, 2\pi).
\]

(5.23)

Collecting (5.21), (5.22) and (5.23) together we obtain

\[
\{ \mathcal{T}, Q \} = 0.
\]

(5.24)

In other words, the monodromy matrix is BRST invariant.

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**References**

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [arXiv:hep-th/9711200].

[2] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183 [arXiv:hep-th/9905111].

[5] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” arXiv:hep-th/0201253.

[6] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for N = 4 super Yang-Mills,” JHEP 0303 (2003) 013 [arXiv:hep-th/0212208].

[7] J. A. Minahan, “A Brief Introduction To The Bethe Ansatz In N=4 Super-Yang-Mills,” J. Phys. A 39 (2006) 12657.

[8] J. Plefka, “Spinning strings and integrable spin chains in the AdS/CFT correspondence,” arXiv:hep-th/0507136.
[9] N. Beisert, “The dilatation operator of N = 4 super Yang-Mills theory and integrability,” Phys. Rept. 405 (2005) 1 [arXiv:hep-th/0407277].

[10] A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” arXiv:hep-th/0311139.

[11] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69, 046002 (2004) [arXiv:hep-th/0305116].

[12] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[13] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69, 046002 (2004) [arXiv:hep-th/0305116].

[14] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[15] I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69, 046002 (2004) [arXiv:hep-th/0305116].

[16] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[17] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[18] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[19] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[20] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[21] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[22] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[23] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[24] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.

[25] N. Mann and S. E. Vazquez, “Classical Open String Integrability,” arXiv:hep-th/0612038.
[29] L. F. Alday, “Non-local charges on $AdS(5) \times S^5$ and pp-waves,” JHEP 0312 (2003) 033 [arXiv:hep-th/0310146].

[30] N. Mann and J. Polchinski, “Bethe ansatz for a quantum supercoset sigma model,” Phys. Rev. D 72 (2005) 086002 [arXiv:hep-th/0508232].

[31] R. R. Metsaev, C. B. Thorn and A. A. Tseytlin, “Light-cone superstring in $AdS$ space-time,” Nucl. Phys. B 596 (2001) 151 [arXiv:hep-th/0009171].

[32] R. R. Metsaev and A. A. Tseytlin, “Superstring action in $AdS(5) \times S(5)$: kappa-symmetry light cone gauge,” Phys. Rev. D 63 (2001) 046002 [arXiv:hep-th/0007036].

[33] N. Berkovits, “Super-Poincare covariant quantization of the superstring,” JHEP 0004, 018 (2000) [arXiv:hep-th/0001035].

[34] N. Berkovits and B. C. Vallilo, ‘Consistency of super-Poincare covariant superstring tree amplitudes,” JHEP 0007, 015 (2000) [arXiv:hep-th/0004171].

[35] N. Berkovits, “Cohomology in the pure spinor formalism for the superstring,” JHEP 0009, 046 (2000) [arXiv:hep-th/0006003].

[36] N. Berkovits, “Relating the RNS and pure spinor formalisms for the superstring,” JHEP 0108, 026 (2001) [arXiv:hep-th/0104247].

[37] N. Berkovits, “ICTP lectures on covariant quantization of the superstring,” arXiv:hep-th/0209059.

[38] P. A. Grassi, “$N = 2$ superparticles, RR fields and noncommutative structures of (super)-spacetime,” arXiv:hep-th/0511015.

[39] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “An introduction to the covariant quantization of superstrings,” Class. Quant. Grav. 20 (2003) S395 [arXiv:hep-th/0302147].

[40] P. A. Grassi, G. Policastro and P. van Nieuwenhuizen, “Yang-Mills theory as an illustration of the covariant quantization of superstrings,” arXiv:hep-th/0211095.

[41] N. A. Nekrasov, “Lectures on curved beta-gamma systems, pure spinors, and anomalies,” arXiv:hep-th/0511008.

[42] N. Berkovits and O. Chandia, “Superstring vertex operators in an $AdS(5) \times S(5)$ background,” Nucl. Phys. B 596, 185 (2001) [arXiv:hep-th/0009168].

[43] N. Berkovits, “Quantum consistency of the superstring in $AdS(5) \times S^{**5}$ background,” JHEP 0503, 041 (2005) [arXiv:hep-th/0411170].

[44] B. C. Vallilo, “One loop conformal invariance of the superstring in an $AdS(5) \times S(5)$ background,” JHEP 0212 (2002) 042 [arXiv:hep-th/0210064].

[45] N. Berkovits and O. Chandia, “Massive superstring vertex operator in $D = 10$ superspace,” JHEP 0208, 040 (2002) [arXiv:hep-th/0204121].

[46] N. Berkovits, “BRST cohomology and nonlocal conserved charges,” JHEP 0502 (2005) 060 [arXiv:hep-th/0409159].

[47] J. Kluson, “Note about classical dynamics of pure spinor string on $AdS(5) \times S(5)$ background,” arXiv:hep-th/0603228.

[48] M. Bianchi and J. Kluson, “Current algebra of the pure spinor superstring in $AdS(5) \times S(5)$,” JHEP 0608 (2006) 030 [arXiv:hep-th/0606188].
[49] V. G. M. Puletti, “Operator product expansion for pure spinor superstring on AdS(5) x S**5,” arXiv:hep-th/0607076.

[50] P. A. Grassi and J. Kluson, “Pure spinor strings in TsT deformed background,” arXiv:hep-th/0611151.

[51] B. C. Vallilo, “Flat currents in the classical AdS(5) x S**5 pure spinor superstring,” JHEP 0403 (2004) 037 [arXiv:hep-th/0307018].

[52] I. A. Batalin and G. A. Vilkovisky, “Relativistic S Matrix Of Dynamical Systems With Boson And Fermion Constraints,” Phys. Lett. B 69 (1977) 309.

[53] L. D. Faddeev and L. A. Takhtajan, “HAMILTONIAN METHODS IN THE THEORY OF SOLITONS,”

[54] D. Korotkin and H. Samtleben, “Yangian symmetry in integrable quantum gravity,” Nucl. Phys. B 527 (1998) 657 [arXiv:hep-th/9710210].

[55] B. H. Miller, “Conserved charges in the principal chiral model on a supergroup,” JHEP 0608 (2006) 010 [arXiv:hep-th/0602006].

[56] J. Kluson, ‘Hamiltonian analysis of non-linear sigma model on supercoset target,” JHEP 0610 (2006) 046 [arXiv:hep-th/0608146].

[57] T. Fujiwara, Y. Igarashi, J. Kubo and K. Maeda, “A New Insight Into Bst Anomalies In String Theory,” Nucl. Phys. B 391 (1993) 211 [arXiv:hep-th/9210038].

[58] E. Brezin, C. Itzykson, J. Zinn-Justin and J. B. Zuber, “Remarks About The Existence Of Nonlocal Charges In Two-Dimensional Models,” Phys. Lett. B 82 (1979) 442.

[59] M. Hatsuda and S. Mizoguchi, “Nonlocal charges of T-dual strings,” JHEP 0607, 029 (2006) [arXiv:hep-th/0603097].

[60] H. J. de Vega, H. Eichenherr and J. M. Maillot, “Classical And Quantum Algebras Of Nonlocal Charges In Sigma Models,” Commun. Math. Phys. 92 (1984) 507.

[61] A. G. Izergin and V. E. Korepin, “The Inverse Scattering Method Approach To The Quantum Shabat-Mikhailov Model,” Commun. Math. Phys. 79 (1981) 303.