HOMOTOPY BOTT–TAUBES INTEGRALS AND THE TAYLOR TOWER FOR THE SPACE OF KNOTS

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Abstract. This work continues the study of a homotopy-theoretic construction of the author [7] inspired by the Bott–Taubes integrals. Bott and Taubes [2] constructed knot invariants by integrating differential forms along the fiber of a bundle over the space of knots $\text{Emb}(S^1, \mathbb{R}^3)$. Their techniques were later used to construct real cohomology classes in the space $\text{Emb}(S^1, \mathbb{R}^d)$ of knots in $\mathbb{R}^d$, $d \geq 4$ [3]. By doing this integration via a Pontrjagin–Thom construction, the author constructed cohomology classes in the knot space with arbitrary coefficients [7]. Here we carry out the construction over the stages of the Taylor tower for the knot space, which arises from the Goodwillie–Weiss embedding calculus. We work both over the totalization of Sinha’s cosimplicial model [12] and directly over the stages of the Taylor tower.

1. Introduction and background

The subject of this paper is the space $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ of long knots in $\mathbb{R}^d$, that is, embeddings of $\mathbb{R}$ into $\mathbb{R}^d$ with prescribed behavior outside of the interval $[-1, 1] \subset \mathbb{R}$. This space is closely related to the space of closed knots $\text{Emb}(S^1, \mathbb{R}^d)$. For example, for $d = 3$, the connected components of either space correspond to isotopy classes of knots, the subject of classical knot theory. When $d \geq 4$, $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ is connected but is far from being topologically trivial (see for example [3, 8]). Our main theorems are in the case $d \geq 4$, though some of the results apply to $d = 3$ as well. We build on previous work [7], in which the author produced cohomology classes with arbitrary coefficients in these spaces via a construction inspired by the configuration space integrals of Bott and Taubes. The purpose of this paper and ongoing research is to study these classes in more detail. In particular, we show two ways of essentially carrying out the construction over the stages of the Taylor tower of the knot space.

1.1. Bott–Taubes integrals. In [2], Bott and Taubes constructed knot invariants by considering a bundle

$$
\begin{array}{ccc}
F_{q,t} & \longrightarrow & E_{q,t} \\
\downarrow & & \downarrow \\
\text{Emb}(S^1, \mathbb{R}^3)
\end{array}
$$

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over the space of knots in $\mathbb{R}^3$. The fiber $F_{q,t}$ is a compactification of a configuration space of $q + t$ points in $\mathbb{R}^3$, $q$ of which lie on the knot. Another way of saying this is that the total space $E = E_{q,t}$ is the pullback in the square below:

$$
\begin{array}{ccc}
E_{q,t} & \longrightarrow & C_{q+t}[\mathbb{R}^3] \\
\downarrow & & \downarrow \\
\text{Emb}(S^1, \mathbb{R}^3) \times C_q[S^1] & \longrightarrow & C_q[\mathbb{R}^3]
\end{array}
$$

Here $C_q[M]$ denotes the Axelrod–Singer/Fulton–Macpherson compactification of the space $C_q(M)$ of configurations of $q$ points on a manifold $M$. The right-hand map is projection to the first $q$ points, while the bottom map comes from the fact that an embedding induces a map on configuration spaces.

Bott and Taubes considered differential forms coming from the cohomology of configuration spaces, integrated them along the fiber of the bundle, and showed that the result is a zero-dimensional closed form, which thus represents a knot invariant. Their methods were used by D. Thurston to construct all finite-type knot invariants from functionals on trivalent diagrams $[13, 16]$. The square (1) above still makes sense if one replaces $\mathbb{R}^3$ by $\mathbb{R}^d$, $d \geq 4$. Cattaneo, Cotta-Ramusino, and Longoni constructed nontrivial real cohomology classes in $\text{Emb}(S^1, \mathbb{R}^d)$ in this way $[3]$. In this paper, we will work with long knots instead of closed knots, but the Bott–Taubes construction can be carried out equally well in this case; one just replaces $S^1$ by $\mathbb{R}$ in (1). In fact, we will actually work with embeddings of $I$ into $I^d$, so we replace $S^1$ and $\mathbb{R}^3$ by $I$ and $I^d$ respectively.

1.2. A related homotopy-theoretic construction. The author considered the Bott–Taubes bundle described above, and carried out “integration along the fiber” homotopy-theoretically $[7]$, which we briefly review now. First, neatly embed the total space $E = E_{q,t}$ by a map $e_N$ into a product of the base $\mathcal{K}$ and Euclidean space with corners. Then collapse by the complement of a tubular neighborhood of $E$. Quotienting by boundary subspaces gives a map

$$
\tau : \Sigma^N \mathcal{K}_+ \to E^{\nu_N} / (\partial E^{\nu_N}).
$$

from the $N$-fold suspension of the base space $\mathcal{K}$ (union a disjoint basepoint) to the Thom space of the normal bundle $\nu_N$ of $e_N$, modulo its boundary. In cohomology this gives a map corresponding to integration along the fiber.

By letting $N$ in $e_N$ approach $\infty$, we then get a map from the suspension spectrum of $\mathcal{K}$ to the Thom spectrum of the normal bundle to the total space, which induces in cohomology a map similar to the Bott–Taubes integration along the fiber. Furthermore, the obvious space-level connect-sum of long knots $\mu : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ lifts to a multiplication

$$
\mu_E : E_{q,t} / \partial E_{q,t} \times E_{r,s} / \partial E_{r,s} \to E_{q+r,t+s} / \partial E_{q+r,t+s}
$$
on the total space of our bundle modulo its boundary. In summary, there is the following result:

**Theorem 1.1** ([7]). Let $\mathcal{K} = \text{Emb}(\mathbb{R}, \mathbb{R}^d)$, $d \geq 3$ or $\text{Emb}(S^1, \mathbb{R}^d)$, $d \geq 3$. The total spaces $E_{q,t}$ of the bundles over the knot space have a multiplication $E_{q,t}/\partial E_{q,t} \times E_{r,s}/\partial E_{r,s} \to E_{q+r,t+s}/\partial E_{q+r,t+s}$ which makes the wedge of Thom spectra $\bigvee_{q,t \in \mathbb{N}} E_{q,t}^\nu/\partial E_{q,t}^\nu$ into a ring spectrum. The Thom collapse maps for various $q, t$ induce a map of ring spectra

$$\bigvee_{q,t} \tau_{q,t} : \bigvee_{q,t} \Sigma^\infty \mathcal{K}_+ \to \bigvee_{q,t} E_{q,t}^\nu/\partial E_{q,t}^\nu$$

where the multiplication in $\bigvee \Sigma^\infty \mathcal{K}_+$ comes from the space-level connect-sum. Using the Thom isomorphism, this induces an “integration along the fiber” map in cohomology with arbitrary coefficients, producing classes in $H^* \mathcal{K}$.

The compatibility of the multiplication on $\bigvee_{q,t} E_{q,t}^\nu/\partial E_{q,t}^\nu$ with connect-sum on $\mathcal{K}$ leads to the other main result of [7], a product formula in the case of long knots, which roughly computes the evaluations of “Bott–Taubes-type” classes on a “connect-sum” of knot homology classes in terms of evaluations of smaller Bott–Taubes-type classes on the respective homology classes.

1.3. **The embedding calculus applied to the knot space.** In this paper, we adapt the homotopy-theoretic Bott–Taubes construction to the cosimplicial model for the space of knots coming from the Goodwillie–Weiss embedding calculus. We now review the main ideas of the embedding calculus as they apply to knot spaces. From now on, let $\mathcal{K}$ denote the space of long knots in $\mathbb{R}^d$ where $d \geq 3$.

The embedding calculus gives a “Taylor tower” for $\mathcal{K}$

$$\ldots \to T_n \mathcal{K} \to T_{n-1} \mathcal{K} \to \ldots \to T_1 \mathcal{K}$$

with maps $\mathcal{K} \to T_n \mathcal{K}$, where each $T_n \mathcal{K}$ is roughly the space of $n$-times punctured knots. For $d > 3$, work of Goodwillie and Weiss implies that the map from $\mathcal{K}$ to the $n$th stage of Taylor tower is $(n - 1)(d - 3)$-connected [5]. So in this case the tower converges to the original embedding functor, i.e., the homotopy limit is weakly equivalent to $\mathcal{K}$. Although convergence of the tower to the knot space not known for $d = 3$, Volić showed that all finite-type invariants factor through the Taylor tower in this case [15].

Closely related to the Taylor tower is a cosimplicial model $X^\bullet$ for $\mathcal{K}$ developed by Sinha in [12] (see also [11] and [15]). This model is analogous to the cosimplicial model for the loop space, and its levels $X^k$ are roughly configuration spaces of points in $\mathbb{R}^d$. Sinha showed that the homotopy-invariant partial totalization $\widetilde{\text{Tot}}^n X^\bullet$ is homotopy equivalent to $T_n \mathcal{K}$. This implies that for $d > 3$ the full totalization $\widetilde{\text{Tot}} X^\bullet$ is weakly equivalent to $\mathcal{K}$. 
1.4. Main results, organization, and future directions. In section 2 we first review Sinha’s cosimplicial model for the knot space (section 2.1) and then define a “cosimplicial” version $E^n$ of the total space $E$ in the Bott–Taubes construction (section 2.2). The $E^n$ themselves are not cosimplicial spaces, but they fiber over the $\tilde{\text{Tot}}^n X^\bullet$. We establish in Proposition 2.5 that there is a tower of fibrations

\[
\cdots \to E^{n+1} \to E^n \to E^{n-1} \to \cdots
\]

and the map from $E$ to the limit of the $E^n$ is an equivalence in the case of knots in $\mathbb{R}^d$ for $d > 3$.

Despite the lack of smoothness, a Pontrjagin–Thom-type construction can be carried out for the fibration $E^n \to \tilde{\text{Tot}}^n X^\bullet$ by quotienting by the complement of a neighborhood of an embedding into $\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N$. Section 3 provides the details of this construction. In section 3.1 we show that the quotients by neighborhood complements form a spectrum $\Theta E^n$, which is like a Thom spectrum in this non-smooth setting. In section 3.2 we show that the Pontrjagin–Thom collapse maps for the $E^n$ are compatible with the one for $E$. This compatibility together with the fact that the limit of the $E^n$ is $E$ provide an analogue of the Thom isomorphism (section 3.3). This establishes the main result of section 3, which appears later as Theorem 3.10:

**Theorem 1.2.** Let $\mathcal{K} = \text{Emb}(\mathbb{R}, \mathbb{R}^d)$, $d \geq 4$. For sufficiently large $n$, the collapse maps for the embedding of $E^n$ in $\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N$ for various $N$ induces a map of spectra compatible with the pre-transfer map in the smooth setting:

\[
\Sigma^\infty \mathcal{K}_+ \to E^n/\partial E^n \to \Theta E^n/\partial \Theta E^n
\]

This induces in cohomology a commutative square

\[
H^{* - \dim F} \mathcal{K} \leftarrow H^*(E, \partial E) \leftarrow H^*(E^n, \partial E^n)
\]

thus extending the homotopy-theoretic Bott–Taubes construction to the cosimplicial setting.

In the last section, we observe that we can also carry out the construction over the space $T_n \mathcal{K}$ itself instead of $\tilde{\text{Tot}}^n X^\bullet$. We prove Theorem 4.3 a result completely analogous to the
Theorem above. This is in fact easier than working with the totalizations of the cosimplicial model because we can work entirely with spaces which are smooth manifolds with corners (manifolds with faces, even).

The reason we work with the cosimplicial model (and present this approach first) is that we would like to exploit a product structure, just as we had a product formula in [7], for the purpose of calculations. It is not obvious how to construct a product on the stages $T^nK$ (say, a map $T^mK \times T^nK \rightarrow T^{m+n}K$). On the other hand, it is very likely that there is a map $\tilde{\text{Tot}}^mX^\bullet \times \tilde{\text{Tot}}^nX^\bullet \rightarrow \tilde{\text{Tot}}^{m+n}X^\bullet$ which via the evaluation maps from the knot space to $\tilde{\text{Tot}}^nX^\bullet$ agrees with connect-sum of knots. Then such a product structure and a formula incorporating it would be the correct analogue of connect-sum and the product formula from [7]. Nonetheless, we include Theorem 4.3 in the last section, as this result may also be of interest.

The cosimplicial model also has the advantage that each level $X^n$, as a configuration space, is well understood. Lambrechts, Turchin and Volić showed using the Bousfield–Kan spectral sequence that the rational homology of a space closely related to $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ for $d \geq 4$ is in some sense built out of the homology of the $X^n$ [8]. Our eventual goal is to use the Bousfield–Kan spectral sequence to make explicit computations involving the classes we have constructed. Theorem 3.10 brings us a step closer to doing this.

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2. Cosimplicial Bott-Taubes

2.1. Review of the cosimplicial model for the knot space. This subsection provides some details of Sinha’s cosimplicial model for the space $\mathcal{K}$ of long knots in $\mathbb{R}^d$. These can be found in Sinha’s papers [12] [11] [10], but we recall them here for the convenience of the reader.

First recall that $\Delta$ is the category whose objects are finite ordered sets $[n] = \{0, 1, \ldots, n\}$ and whose morphisms are order-preserving maps. There are morphisms $\delta^i : [n-1] \rightarrow [n], i = 0, \ldots, n$ which “skip $i$” and $\sigma^i : [n+1] \rightarrow [n], i = 0, \ldots, n$ which “merge $i$ and $i+1$”. Any morphism can be written as a composition of the $\delta^i$ and $\sigma^i$. A cosimplicial object is a covariant functor $X$ from $\Delta$; in particular, a cosimplicial space is a covariant functor from $\Delta$ to Spaces. A cosimplicial object is thus defined by its levels $X^n = X([n])$, the coface maps $d^i = X(\delta^i)$, and codegeneracy maps $s^i = X(\sigma^i)$. The standard cosimplicial space $\Delta^\bullet$ has
levels $\Delta^n$ and standard inclusions and projections of simplices as the coface and codegeneracy maps.

For the cosimplicial model for the knot space, it is convenient to consider the space of long knots as the space of embeddings of $I = [-1, 1]$ into $I^d$ which send $\pm 1$ to $(0, ..., 0, \pm 1)$ with unit tangent vector $(0, ..., 0, \pm 1)$. This space is clearly equivalent to the space $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ described in the introduction. Some results below will only hold for $d \geq 4$, but unless stated otherwise, we will generally assume only that $d \geq 3$.

To define the cosimplicial model $X^\bullet$ for the knot space, we need to discuss configuration spaces in some detail. The configuration space $C_n(M)$, the space of $n$ distinct points in a compact manifold $M$, is not compact. The Axelrod–Singer/Fulton–MacPherson compactification (sometimes called the “canonical compactification”) $C_n[M]$ is a space that was originally defined in terms of blow-ups and has the advantage of being a smooth manifold with corners $[1, 4]$. However, neither this compactification nor the original “open” configuration space give rise to a cosimplicial space. Instead, another compactification is needed. Although the next definition can be made for arbitrary compact $M$, we restrict to the case $M = I^d$, which suffices for our purposes.

**Definition 2.1** (Sinha [12, 10]). Consider the embedding

$$C_n(I^d) \hookrightarrow (I^d)^n \times (S^{d-1})^\binom{n}{2}$$

where the map to the $\{i > j\}$ factor $S^{d-1}$ is given by $(x_1, ..., x_n) \mapsto (x_i - x_j) / |x_i - x_j|$. Define the “simplicial compactification” $C_n(I^d)$ as the closure of the image of the map above. □

Unlike $C_n[I^d]$, $C_n(I^d)$ is not a smooth manifold with corners. However, $C_n(I^d)$ is a quotient of $C_n[I^d]$. In fact, Sinha showed that $C_n[I^d]$ can be defined as the closure of the image of

$$C_n(I^d) \hookrightarrow (I^d)^n \times (S^{d-1})^\binom{n}{2} \times [0, \infty)^\binom{n}{3}$$

where the map to the $\{i > j > k\}$ factor $[0, \infty]$ is given by $|x_i - x_j| / |x_j - x_k|$. See [10] for a proof, as well as more details on both compactifications.

For the tangential data, define $C_n'(I^d)$ as the pullback

$$\begin{array}{ccc}
C_n'(I^d) & \longrightarrow & (ST(I^d))^n \\
\downarrow & & \downarrow \\
C_n(I^d) & \longrightarrow & (I^d)^n
\end{array}$$

where $ST(I^d)$ is the unit sphere tangent bundle of $I^d$. Identifying $ST(I^d)$ with $S^{d-1}$ we can identify $C_n'(I^d)$ as a subspace of points $((x_1, ..., x_n), (v_{ij})_{1 \leq i \leq j \leq n}) \in (I^d)^n \times \prod_{i \leq j} S^{d-1}$, where the $\{i, i\}$ factors correspond to factors of $ST(I^d)$. 

To account for the prescribed behavior of the knots on the boundary, we need to define $C'_n(I^d, \partial)$ as the subspace of points $((x_0, \ldots, x_{n+1}), (v_{ij})_{0 \leq i, j \leq n+1}) \in C'_{n+2}(I^d)$ such that $x_0 = v_{00} = (0, \ldots, 0, -1), x_{n+1} = v_{n+1,n+1} = (0, \ldots, 0, 1)$.

**Definition 2.2** (Sinha [12]; see also Volić [15]). Let $X^\bullet$ be the cosimplicial space with $n^{th}$ level $X^n = C'_n(I^d, \partial)$. The codegeneracies $s^i$ are given by simply forgetting $x_i$. The coface maps $d^i$ are as follows:

$$d_i \begin{pmatrix} (x_0, \ldots, x_{n+1}), & \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1,n+1} \\ v_{22} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ & & & v_{n+1,n+1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} (x_0, \ldots, x_i, x_i, x_{n+1}), & \begin{pmatrix} v_{11} & v_{12} & \cdots & \cdots & v_{1,n+1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ v_{ii} & v_{ii} & \cdots & \cdots & v_{ii} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ v_{n+1,n+1} & v_{n+1,n+1} & \cdots & \cdots & v_{n+1,n+1} \end{pmatrix} \end{pmatrix}$$

The totalization of any cosimplicial space $Y^\bullet$ is $\text{Tot}Y^\bullet = \text{Hom}_\Delta(\Delta^\bullet, Y^\bullet)$ and the partial totalizations are defined similarly, but with $\Delta^\bullet$ replaced by its $n^{th}$ coskeleton: $\text{Tot}^nY^\bullet = \text{Hom}_\Delta(\Delta[n]^\bullet, Y^\bullet)$. We will need to use the *homotopy-invariant* totalization $\widetilde{\text{Tot}}X^\bullet$ of $X^\bullet$ since this is the space that is weakly equivalent to $K$. Similarly we consider the partial totalizations $\widetilde{\text{Tot}}^nX^\bullet$, which are homotopy-equivalent to the stages $T_nK$ of the Taylor tower for $K$ [12]. These can be defined by taking a cofibrant replacement for $\Delta^\bullet$ in the model category structure on cosimplicial spaces where all objects are fibrant. More explicitly define $\widetilde{\text{Tot}}X^\bullet := \text{Hom}_\Delta(\tilde{\Delta}^\bullet, X^\bullet)$, where $\tilde{\Delta}^\bullet$ is given by $\tilde{\Delta}^k = B(\Delta \downarrow [k])$. By definition, this is the same as $\text{holim}_\Delta X^\bullet$.

We want to construct maps between $\text{Tot}^nX^\bullet$ and $\widetilde{\text{Tot}}^nX^\bullet$. Note that $\Delta^k \cong BP_0(k)$ where $P_0(k)$ is the category of nonempty subsets of $[k]$ and inclusions. There is a functor $(\Delta \downarrow [k]) \to P_0(k)$ given by sending $f : [n] \to [k]$ to the image of $f$; there is also a functor $P_0(k) \to (\Delta \downarrow [k])$ given by sending a subset of $\ell$ elements to the corresponding morphism $[\ell - 1] \to [k]$. Note that the composite $P_0(k) \to (\Delta \downarrow [k]) \to P_0(k)$ is the identity. These induce maps on classifying spaces $\tilde{\Delta}^k \to \Delta^k$ and $\Delta^k \to \tilde{\Delta}^k$ and hence maps $\text{Tot}X^\bullet \to \widetilde{\text{Tot}}X^\bullet$ and $\widetilde{\text{Tot}}X^\bullet \to \text{Tot}X^\bullet$. Note that the composite $\text{Tot}X^\bullet \to \widetilde{\text{Tot}}X^\bullet \to \text{Tot}X^\bullet$ is the identity.

2.2. “Cosimplicial” Bott–Taubes bundle. We will define a fibration $E^n \to \widetilde{\text{Tot}}^nX^\bullet$ as follows. Throughout, we only consider $n \geq q$. 

**Definition 2.3.** Define the space \( E^n := E^n_{q,t} \) as the pullback below:

\[
\begin{array}{ccc}
E^n & \longrightarrow & C_{q+t}(I^d) \\
\downarrow & & \downarrow \\
\tilde{\text{Tot}}^n X^* \times C_q(I) & \longrightarrow & \text{Tot}^n X^* \times C_q(I) & \longrightarrow & C_q(I^d)
\end{array}
\]

In this square, the right-hand vertical map is just the projection to the first \( q \) points. The left-hand map in the bottom row comes from the map \( \tilde{\text{Tot}} X^* \to \text{Tot} X^* \) just explained above. The right-hand map in the bottom row is slightly more subtle. To understand it, first recall that

\[
\text{Tot}^n X^* \subset \prod_{i=0}^n \text{Map}(\Delta^i, C'_i(I^d, \partial))
\]

so since \( n \geq q \), we can project onto the \( i = q \) factor:

\[
\text{Tot}^n X^* \to \text{Map}(\Delta^q, C'_q(I^d, \partial))
\]

Note that \( C_q(I) \cong \Delta^q \). So the lower right-hand map is obtained by the above projection followed by evaluation on the \( q \)-simplex \( C_q(I) \) to give an element of \( C'_q(I^d, \partial) \); forgetting the two boundary points \( x_0 \) and \( x_{n+1} \) and the tangential data \( \{v_{ij}\} \) gives an element of \( C_q(I^d) \).

Thus \( E^n \) is a subset of \( \tilde{\text{Tot}}^n X^* \times C_q(I) \times C_{q+t}(I^d) \). We stress that \( E^n \) always depends on two natural numbers \( q, t \), though we will typically suppress these for ease of notation. Unlike in the original Bott-Taubes pullback square (1), the data in the upper right \( C_{q+t}(I^d) \) in the square (4) does not determine the data in the lower left \( C_q(I) \) because the map \( \Delta^q \to C'_q(I^d) \) coming from the projection \( \text{Tot}^n X^* \to \text{Map}(\Delta^q, C'_q(I^d)) \) need not be injective.

The map \( E^n \to \tilde{\text{Tot}}^n X^* \) is the left-hand map in (4) followed by projection to the first factor.

**Lemma 2.4.** The map \( E^n \to \tilde{\text{Tot}}^n X^* \) is a fibration with fiber \( F^n \) homotopy equivalent to \( F \), the fiber of the original bundle in the Bott-Taubes construction.

**Proof.** The projection \( C_{q+t}(I^d) \to C_q(I^d) \) is a fibration. Thus the left-hand map in (4) is a fibration with fiber \( F' := \text{fib}(C_{q+t}(I^d) \to C_q(I^d)) \). The inclusion of the open configuration space into either the simplicial or the canonical compactification of configuration space is a homotopy-equivalence. Thus considering the maps of the fibration \( C_{q+t}(I^d) \to C_q(I^d) \) into the corresponding fibrations of compactified configuration spaces, we see that this fiber \( F' \) is equivalent to the fiber \( F \) from the original Bott-Taubes construction. Thus the map \( E^n \to \tilde{\text{Tot}} X^n \) is a fibration whose fiber is homotopy-equivalent to \( F \times C_q(I) \simeq F \) since \( C_q(I) \cong \Delta^q \) is contractible. \( \square \)
There is a map $\mathcal{K} \to \text{Tot}^n X^\bullet$ given by evaluating a knot at $n$ points. More precisely, the map to each factor in (3) is given by $K \mapsto ((t_1, ..., t_i) \mapsto (K(t_1), ..., K(t_i))$. This induces a map $\mathcal{K} \to \tilde{\text{Tot}}^n X^\bullet$ (using the map $\text{Tot}^n X^\bullet \to \tilde{\text{Tot}}^n X^\bullet$). Sinha showed that the space $\tilde{\text{Tot}}^n X^\bullet$ is homotopy-equivalent to the $n$th stage $\tilde{\text{Tot}}^n X^\bullet$ of the Taylor tower for the space of knots, and the map $\mathcal{K} \to \tilde{\text{Tot}}^n X^\bullet$ agrees with the canonical map $\mathcal{K} \to \tilde{\text{Tot}}^n X^\bullet$ [12] (also [11]). This together with work of Goodwillie and Weiss [5] implies that for $d > 3$ the evaluation map $\mathcal{K} \to \tilde{\text{Tot}}^n X^\bullet$ is $(n-1)(d-3)$-connected.

We have a map of fibrations as below:

(6) $$
\begin{array}{ccc}
\mathcal{K} & \to & \text{Tot}^n X^\bullet \\
\downarrow & & \downarrow \\
\tilde{\text{Tot}}^n X^\bullet & & \text{Tot}^n X^\bullet \\
\end{array}
$$

Recall $E \subset \mathcal{K} \times C_q[I] \times C_{q+t}[I^d]$ and $E^n \subset \text{Tot}^n X^\bullet \times C_q[I] \times C_{q+t}[I^d]$. There is a map

$$
\mathcal{K} \times C_q[I] \times C_{q+t}[I^d] \to \tilde{\text{Tot}}^n X^\bullet \times C_q[I] \times C_{q+t}[I^d]
$$

given by evaluation in the first factor and by quotient maps on the compactified configuration spaces; this gives the map $E \to E^n$. Since for $d \geq 4$, the map on the base spaces above is $(n-1)(d-3)$-connected, the map $E \to E^n$ is also $(n-1)(d-3)$-connected in this case. There are fibrations $\tilde{\text{Tot}}^{n+1} X^\bullet \to \tilde{\text{Tot}}^n X^\bullet$ which induce fibrations $E^{n+1} \to E^n$. Recalling that $n \geq q$, it is easy to see that the fibers of these two maps are the same. Thus we have the following:

**Proposition 2.5.** There is a tower of fibrations

$$
\begin{array}{ccc}
\vdots & \to & \vdots \\
\downarrow & & \downarrow \\
E^{n+1} & \to & E^n \\
\downarrow & & \downarrow \\
E^n & \to & E^{n-1} \\
\vdots & & \vdots
\end{array}
$$
and the map from $E$ to the limit of the $E^n$ is an equivalence in the case of knots in $\mathbb{R}^d$ for $d \geq 4$. □

3. Pre-transfer map in the cosimplicial setting

3.1. Analogue of a Thom spectrum. We now construct a pre-transfer map (i.e., integration along the fiber) for the fibration over $\widetilde{\text{Tot}}^n X^\bullet$. The construction of this map by quotienting by complements of appropriate neighborhoods is the main result of this section. These quotients fit together into a spectrum analogous to a Thom spectrum.

The first main step is the following:

**Proposition 3.1.** For sufficiently large $N$, there exists an open neighborhood

$$\eta(E) = \eta_N(E) \subset \widetilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N$$

which deformation retracts onto $E^n$. Moreover, $U \subset \widetilde{\text{Tot}}^n X^\bullet \times B$ where $B$ is a bounded subset of $\mathbb{R}^N$.

In section 3.3 we will need such neighborhoods in defining our analogue of a Thom spectrum in order for this spectrum to have the correct cohomology.

We need two lemmas and a definition to prove Proposition 3.1. The first lemma is a straightforward observation that just streamlines the proof.

**Lemma 3.2.** Let $A \subset B \subset X$ and suppose there exists $V_A \subset B$ which is open in $B$ and deformation retracts onto $A$. Suppose there also exists open $V_B \subset X$ which deformation retracts onto $B$. Then there exists open $U \subset X$ which deformation retracts onto $A$.

**Proof.** Let $h_t : V_B \to X, t \in [0,1]$, be the deformation retract onto $B$, which gives a map

$$h_1 : V_B \to B.$$  

The preimage $U := h_1^{-1}(V_A)$ is an open subset of $V_B$, hence an open subset of $X$. Let $g_t : V_A \to B, t \in [0,1]$ be the deformation retract onto $A$. Then $h_{2t}|U$ followed by $g_{2t-1}$ gives a deformation retract from $U$ onto $A$. □

**Lemma 3.3.** $C_r(I^d)$ is a Euclidean neighborhood retract (ENR).

**Proof.** It suffices to show that the space in question is locally contractible in the weak sense, that is, for any $x$ and any open $U$ containing $x$ there is an open $V \subset U$ containing $x$ such that the inclusion $V \to U$ is nullhomotopic. (See for example Theorem A.7 (p.525) of [6].) We know that $C_r(I^d)$ is locally contractible since it is an open subset of $I^d$; thus it suffices to consider only the case where $x$ is in the boundary.

Suppose given $x \in \partial C_r(I^d)$ and $U \ni x$. The inclusion $i : C_r(I^d) \hookrightarrow C_r(I^d)$ is a homotopy equivalence. Let $f$ be its homotopy inverse and let $h : C_r(I^d) \times [0,1] \to C_r(I^d)$ be the
homotopy from the identity map to \( i \circ f \). We claim we can choose \( f \) and \( h \) so that \( h_t(x) := h(x, t) \) lies in \( U \) for all \( t \in [0, 1] \). In fact, \( C_r(I^d) \) is diffeomorphic to the complement of any neighborhood of the boundary of \( C_r(I^d) \), so it is clear that we can arrange for \( f(\partial C(I^d)) \subset C(I^d) \subset C(I^d) \) to be arbitrarily close to \( \partial C(I^d) \subset C(I^d) \).

Since \( C_r(I^d) \) is locally contractible and \( i^{-1}U \) is open, we can find open \( W \subset i^{-1}U \subset U \) such that \( f(x) \in W \) and so that \( W \) is contractible by a homotopy \( g_t : W \to U, t \in [0, 1] \).

By definition \( 2.1 \) the simplicial compactification \( C_r(I^d) \) can be viewed as a subset of Euclidean space. This gives rise to a metric on it. Let \( \varepsilon \) be the distance from the set \( \{ h_t(x) | t \in I \} \) to the complement of \( U \); by compactness of the former, \( \varepsilon > 0 \). By compactness of \( C(I^d) \), \( h \) is uniformly continuous, so find \( \delta > 0 \) such that \( |(y, t) - (z, s)| < \delta \) implies \( |h_t(y) - h_s(z)| := |h(y, t) - h(z, s)| < \varepsilon \). Then for all \( t \in [0, 1] \) the image \( h_t(B_\delta(x)) \) of a \( \delta \)-neighborhood of \( x \) is contained in \( B_\varepsilon(h_t(x)) \), which is contained in \( U \).

Let \( V = f^{-1}W \cap B_\delta(x) \). Then we can define a nullhomotopy of \( V \) which stays in \( U \) by \( h_{2t}|_V, t \in [0, 1/2]; g_{2t-1}|_V, t \in [1/2, 1] \).

Finally, we define a neighborhood \( \nu(E) \subset \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d) \) as follows. A point in \( \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d) \) maps to some point

\[
(\varphi_1, \ldots, \varphi_n, t, x) \in \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d)
\]

where we think of each \( \varphi_i \) as a map \( \Delta^i(\cong C_i(I)) \to C_i^d(I^d, \partial) \to C_i(I^d) \), with the latter map forgetting the boundary points and tangential data. Let \( \pi : \tilde{C}_{q+t}(I^d) \to \tilde{C}_q(I^d) \) denote the projection to the first \( q \) points. If such a point is in \( E^n \), then \( \varphi_q(t) = \pi(x) \). By definition \( 2.1 \) the simplicial compactifications of configuration spaces can be viewed as subsets of Euclidean space. This gives rise to a metric on these spaces.

**Definition 3.4.** Define \( \nu(E^n) = \nu_{\varepsilon}(E^n) \) as the space of all points in \( \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d) \) such that \( |\varphi_q(t) - \pi(x)| < \varepsilon \).

Thus \( \nu(E^n) \) is an open set in \( \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d) \) containing \( E^n \).

**Proof of Proposition 3.1.** We will apply Lemma 3.2 with \( X = \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N, B = \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \tilde{C}_{q+t}(I^d), \) and \( A = E^n \), and \( \eta(E) \) will be the resulting \( U \).

We claim that \( V_A \) can be taken to be \( \nu(E^n) = \nu_{\varepsilon}(E^n) \) defined above (for sufficiently small \( \varepsilon \)). First define a deformation retraction from \( \nu(E^n) \cap \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \times \text{int} \tilde{C}_{q+t}(I^d) \) to \( E^n \) as follows. This deformation retraction will be constant on the factor \( \tilde{\text{Tot}}^n X^\bullet \times \tilde{C}_q(I) \). A point in the domain maps to a point \( (\varphi, t, x) \) as in (17), but since \( x \) is the interior, we can actually write \( x = (x_1, \ldots, x_{q+t}) \in (I^d)^{q+t} \). Define the deformation retraction by straightline homotopies from \( (x_1, \ldots, x_q) \) to \( \varphi_q(t) \). Then extend the map by continuity to the whole compactification \( \tilde{C}_{q+t}(I^d) \), and this gives a deformation retraction from \( \nu(E^n) \) to \( E^n \).
Now clearly the proof of Lemma 3.3 applies to show that $C_q(I) \times C_{q+t}(I^d)$ is also an ENR, so there exists $W = W_N \subset \mathbb{R}^N$ which deformation retracts onto $C_q(I) \times C_{q+t}(I^d)$; by compactness of the latter space, we can take $W$ to lie in some bounded $B \subset \mathbb{R}^N$. Then there exists a deformation retract $\tilde{\text{Tot}}^n X^\bullet \times W \times I \to \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N$ onto $\tilde{\text{Tot}}^n X^\bullet \times C_q(I) \times C_{q+t}(I^d)$ which is just the identity on the $\tilde{\text{Tot}}^n X^\bullet$ factor. Thus $\tilde{\text{Tot}}^n X^\bullet \times W$ plays the role of $V_B$ in Lemma 3.2 which completes the proof of the Proposition.

\[\square\]

We can choose the inclusions $i_N$ so that the square below commutes for all sufficiently large $N$:

\[
\begin{array}{ccc}
E^n & \xrightarrow{i_N} & \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \\
\downarrow & & \downarrow \text{id \times 0} \\
E^n & \xrightarrow{i_{N+1}} & \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \times \mathbb{R}
\end{array}
\]

Consider the “Thom collapse map”

\[\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \to \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(E^n)).\]

Since $\eta_N(E^n)$ is in the product of $\tilde{\text{Tot}}^n X^\bullet$ with a bounded set, this map factors through

\[\Sigma^N (\tilde{\text{Tot}}^n X^\bullet)_{+} \to \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(E^n)).\]

We can choose the $\eta_N(E^n)$ so that $\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(E^n)$ maps into $\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^{N+1} - \eta_{N+1}(E^n)$ (e.g., take $\eta_{N+1}(E^n) = \eta_N(E^n) \times (-1, 1)$). Using that the projection of $\eta_N(E^n)$ to $\mathbb{R}^N$ is bounded, we get a map

\[\Sigma \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(E^n)) \to \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^{N+1} / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^{N+1} - \eta_{N+1}(E^n))\]
Notice also that as $n$ increases, $E_n \subset \tilde{\text{Tot}}^n X^\bullet \times C_q(I) \times C_{q+t}(I^d)$ only varies in the $\tilde{\text{Tot}}^n X^\bullet$ factor, so we have a commutative square for all large enough $N$

$$
\begin{array}{c}
E^n & \overset{i_N}{\longrightarrow} & \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \\
\downarrow & & \downarrow \\
E^{n-1} & \overset{i_N}{\longrightarrow} & \tilde{\text{Tot}}^{n-1} X^\bullet \times \mathbb{R}^N 
\end{array}
$$

where both vertical arrows come from the fibration at the $n$th level in the tower:

$$
\ldots \rightarrow \tilde{\text{Tot}}^n X^\bullet \rightarrow \tilde{\text{Tot}}^{n-1} X^\bullet \rightarrow \ldots
$$

We have now proven the following:

**Proposition 3.5.** There is a spectrum $\Theta E^n$ whose $N^\text{th}$ space is the quotient

$$
\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(E^n))
$$

The Thom collapse maps give a map of spectra

$$
\Sigma^\infty(\tilde{\text{Tot}}^n X^\bullet)_+ \rightarrow \Theta E^n
$$

These collapse maps are compatible as $n$ varies with the maps in the tower

$$
\ldots \rightarrow \tilde{\text{Tot}}^n X^\bullet \rightarrow \tilde{\text{Tot}}^{n-1} X^\bullet \rightarrow \ldots
$$

The notation $\Theta E^n$ is supposed to indicate that this space is like a Thom space for the embedding of $E^n$ in $\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N$.

### 3.2. Comparison to the smooth construction.

We now would like to show compatibility of these collapse maps with those in the smooth setting covered in [7]. This will also help us relate the cohomology of $\Theta E^n$ to that of $E^n$ in the next section.

For ease of notation we denote $\mathbb{R}^{(L),N} := [0, \infty) \times \mathbb{R}^N$. Recall that the collapse maps are constructed by considering (for each sufficiently large $N$) a neat embedding which is a composite

$$
E \overset{\subset}{\longrightarrow} \mathcal{K} \times C_q[I] \times C_{q+t}[I^d] \overset{\subset}{\longrightarrow} \mathcal{K} \times \mathbb{R}^{(L),N}
$$

and taking the quotient by the complement of a tubular neighborhood. The quotient is the Thom space of the normal bundle of the embedding, and the collapse map induces an “integration along the fiber” map in cohomology.

The construction in [7] dealt with “thickened long knots”, i.e., $\mathcal{K}$ was actually the space of embeddings $\mathbb{R} \times D^2 \hookrightarrow \mathbb{R} \times D^2$ (or $\mathbb{R} \times D^{d-1} \hookrightarrow \mathbb{R} \times D^{d-1}$), but clearly the construction works just as easily (perhaps even more easily) with $\mathcal{K}$ as the space of ordinary long knots. The construction in [7] also used $\text{Emb}(\mathbb{R}, \mathbb{R}^d)$ rather than $\text{Emb}(I, I^d)$ but it is easy to see
that one can indeed use $I$ and $I^d$ instead, just by using large enough $L$ to account for the corners in $I^d$.

**Lemma 3.6.** For all sufficiently large $L, N$, there exist neat embeddings $C_q[I] \times C_{q+t}[I^d] \hookrightarrow \mathbb{R}^{(L),N}$ and inclusions $C_q(I) \times C_{q+t}(I^d) \hookrightarrow \mathbb{R}^{L+N}$ so that the square below commutes on subspaces of the left-hand spaces away from their boundaries:

$$
\begin{array}{ccc}
C_q[I] \times C_{q+t}[I^d] & \hookrightarrow & \mathbb{R}^{(L),N} \\
\downarrow & & \downarrow \\
C_q(I) \times C_{q+t}(I^d) & \hookrightarrow & \mathbb{R}^{L+N}
\end{array}
$$

**Proof.** Start with a neat embedding $C_q[I] \times C_{q+t}[I^d] \hookrightarrow \mathbb{R}^{(L),N}$, as in section 3.2 of [7]. Fix some small $\delta > 0$ and let $A_\delta$ be the complement in $C_q[I] \times C_{q+t}[I^d]$ of a $\delta$-neighborhood of the boundary. Note that $A_\delta$ is homeomorphic to $C_q(I) \times C_{q+t}(I^d)$. Also $A_\delta$ is homeomorphic to the complement in $C_q(I) \times C_{q+t}(I^d)$ of a $\delta$-neighborhood of the boundary; let $B_\delta$ denote the closure of the $\delta$-neighborhood of the boundary in $C_q(I) \times C_{q+t}(I^d)$. Now $A_\delta$ is smooth and $B_\delta$ embeds into some Euclidean space via the map (2). So by increasing $N$ and using bump functions, we can construct an embedding of $C_q(I) \times C_{q+t}(I^d)$ into $\mathbb{R}^{L+N}$ which agrees on $A_\delta$ with the neat embedding of the smooth compactification (up to taking a product with an extra factor of some $\mathbb{R}^M$).

Thus we have a commutative diagram as below. The horizontal arrows on the left come from the definition of the pullback. The vertical arrows come from maps $E \to E^n$ and $\mathcal{K} \to \text{Tot}^n X^\bullet$ mentioned earlier. The arrow on the right is dashed to indicate that the right-hand square will only commute when the maps are restricted away from the boundaries of the compactified configuration spaces.

$$
\begin{array}{ccc}
E & \hookrightarrow & \mathcal{K} \times C_q[I] \times C_{q+t}[I^d] \\
\downarrow & & \downarrow \\
E^n & \hookrightarrow & \text{Tot}^n X^\bullet \times C_q(I) \times C_{q+t}(I^d)
\end{array}
\quad
\begin{array}{ccc}
\mathcal{K} \times \mathbb{R}^{(L),N} & \hookrightarrow & \mathcal{K} \times \mathbb{R}^{L+N} \\
\downarrow & & \downarrow \\
\text{Tot}^n X^\bullet \times \mathbb{R}^{L+N}
\end{array}
$$

We must now quotient by boundary subspaces. We consider a subspectrum $\partial \Theta E^n$ defined similarly to $\Theta E^n$. In more detail, first define

$$
\partial E^n := E^n \cap (\text{Tot}^n X^\bullet \times \partial(C_q(I) \times C_{q+t}(I^d))).
$$

We will use the next lemma in defining the boundary subspectrum, even though from a logical standpoint it is strictly necessary only in the next subsection.

**Lemma 3.7.** There is an open $\eta(\partial E^n) = \eta_N(\partial E^n) \subset \text{Tot}^n X^\bullet \times \mathbb{R}^N$ which deformation retracts onto $\partial E^n$. 

Proof. For any \( d \geq 1 \) and any \( r \) \( C_q[I^d] \) is a manifold with corners, so contains an open set \( U \) that deformation retracts by some \( h_t \) onto its boundary. Note that the quotient map \( \pi : C_q[I^d] \to C_q(I^d) \) is a homeomorphism away from the boundary. Thus the image \( q(U) \) of the open set \( U \) under this map is open (since it is the preimage of its image). On non-boundary points \( g_t := q \circ h_t \circ q^{-1} \) gives a map to the boundary which can be extended continuously to the boundary by the identity map. Thus there is a neighborhood in \( \tilde{\text{Tot}}^n X^\bullet \times C_q(I) \times C_{q+t}(I^d) \) retracting onto \( \tilde{\text{Tot}}^n X^\bullet \times \partial(C_q(I) \times C_{q+t}(I^d)) \). Then intersection with \( \nu(E^n) \) gives an open set \( \nu(\partial E^n) \) with a deformation retraction onto \( \partial E^n \). Taking the preimage of \( \nu(\partial E^n) \) under the deformation retraction from the neighborhood \( \eta_N(E^n) \subset \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \) to \( E^n \) gives \( \eta_N(\partial E^n) \subset \eta_N(E^n) \) which is open in \( \tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N \) and which deformation retracts onto \( \partial E^n \). \qedhere

Define the boundary subspectrum \( \partial \Theta E^n \) as the spectrum whose \( N \)th space is

\[
\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N / (\tilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^N - \eta_N(\partial E^n))
\]

Following the Thom collapse map

\[
\Sigma^\infty(\tilde{\text{Tot}}^n X^\bullet)_{+} \to \Theta E^n
\]

by a quotient map by the boundary subspectrum, we have

\[
\Sigma^\infty(\tilde{\text{Tot}}^n X^\bullet)_{+} \to \Theta E^n / \partial \Theta E^n.
\]

In the smooth setting, there is the Thom spectrum of the normal bundle of \( E \hookrightarrow K \times \mathbb{R}^{(L)_N} \) (where \( N \to \infty \) indexes the spaces in the spectrum) which we denote \( E^\nu \). From section 3.4 of \cite{7}, we have the pre-transfer map (Pontrjagin–Thom collapse map)

\[
\Sigma^\infty K_{+} \to E^\nu / \partial E^\nu.
\]

We want to choose the tubular neighborhood \( \eta(E) \) of \( E \hookrightarrow K \times \mathbb{R}^{(L)_N} \) so as to assure compatibility with the construction involving \( E^n \). First take a neighborhood \( v_\varepsilon(E) \subset K \times C_q[I] \times C_{q+t}[I^d] \) defined similarly to the neighborhood \( v_\varepsilon(E^n) \) of Definition \ref{definition_3.4}. Explicitly, a point in \( v_\varepsilon(E) \) is a point \( (K, t, x) \in K \times C_q[I] \times C_{q+t}[I^d] \) such that \( |K(t) - \pi(x)| < \varepsilon \), where \( \pi : C_{q+t}[I^d] \to C_q[I] \) is the projection to the first \( q \) points, and where distances on the canonical compactifications of configuration spaces come from embedding them in Euclidean space via the map \cite{3}. The map

\[
K \times C_q[I] \times C_{q+t}[I^d] \to \tilde{\text{Tot}}^n X^\bullet \times C_q(I) \times C_{q+t}(I^d)
\]

takes the complement of \( v_\varepsilon(E) \) to the complement of \( v_\varepsilon(E^n) \) away from the boundary subspaces. (This is not true on the boundary because the map forgets relative rates of approach of points.)
By Lemma 3.6, we can arrange for the complement of $\eta(E)$ to be taken to the complement of $\eta(E^n)$. Let $\text{int}_\delta E$ (resp. $\text{int}_\delta E^n$) denote the subspace of $E$ (resp. $E^n$) of points of distance at least $\delta$ from $\partial E$ (resp. $\partial E^n$). We have homeomorphisms
\begin{equation}
\frac{\text{int}_\delta E}{\partial (\text{int}_\delta E)} \simeq E/\partial E \\
\frac{\text{int}_\delta E^n}{\partial (\text{int}_\delta E^n)} \simeq E^n/\partial E^n.
\end{equation}
Taking these into account, diagram (10) gives the following:

**Proposition 3.8.** There is a commutative square of spectra:

\[
\begin{array}{ccc}
\Sigma^\infty K_+ & \longrightarrow & E^n/\partial E^n \\
\downarrow & & \downarrow \\
\Sigma^\infty (\widetilde{\text{Tot}}^n X^\bullet)_+ & \longrightarrow & \Theta E^n/\partial \Theta E^n
\end{array}
\]

\[\square\]

### 3.3. Analogue of the Thom isomorphism.

We still need to relate the cohomology of $\Theta E^n/\partial \Theta E^n$ to that of $E^n/\partial E^n$. This will occupy the rest of the section. The results of this section will require that $d \geq 4$.

Let $\eta(E)$ denote a neighborhood of $E$ in $K \times \mathbb{R}^{(L),N}$ which deformation retracts onto $E$, as defined at the end of the previous section. We note that by compactness of the canonical compactification of configuration space, we can take $\eta(E)$ to be the subspace of points $(K, r) \in K \times \mathbb{R}^{(L),N}$ such that there is a point $(K, s) \in E$ with $|r - q| \leq \varepsilon$ for $\varepsilon$ fixed (even as $K$ varies). Define $\eta(\partial E) = \eta_\varepsilon(\partial E)$ as an $\varepsilon$-neighborhood of $\partial E$ in the same way, but requiring that the point $(K, s)$ lies in $\partial E$. By choosing $\varepsilon$ sufficiently small we can arrange for $\eta(\partial E)$ to deformation retract onto $\partial E$, since the boundary of a manifold with corners is a manifold with corners.

Let $\eta(E^n) = \eta_\varepsilon(E^n)$ and $\eta(\partial E^n) = \eta_\varepsilon(\partial E^n)$ be neighborhoods in $\widetilde{\text{Tot}}^n X^\bullet \times \mathbb{R}^{L+N}$ as defined previously. These neighborhoods deformation retract onto $E^n$ and $\partial E^n$ respectively, by Lemmas 3.1 and 3.7. By compactness of the simplical compactification of configuration space, we can take these to be $\varepsilon$-neighborhoods for some fixed $\varepsilon$.

Let $\partial \eta(E)$ be the subspace of points in $K \times \mathbb{R}^{(L),N}$ whose distance from $E$ is exactly $\varepsilon$; similarly define $\partial \eta(E^n)$. (Beware that "$\partial$" and "$\eta$" do not commute!)

We claim that we can arrange for the right-hand vertical map in (10) on the ambient spaces to take $\eta(E)$ to $\eta(E^n)$, $\partial \eta(E)$ to $\partial \eta(E^n)$, and $\eta(\partial E)$ to $\eta(\partial E^n)$. In the first two cases, these statements are clearly true away from the boundary of $E$. Around the boundary, this can be accomplished by taking sufficiently small neighborhoods of $E$ and $\partial E$.

Part (e) of the lemma below is needed for our analogue of the Thom isomorphism. The first four parts are needed to prove part (e).
Lemma 3.9. Let $K$ be the space of long knots in $\mathbb{R}^d$ with $d \geq 4$. The following maps induce isomorphisms in (co)homology for a range of degrees that increases with $n$:

(a) $\eta(E)/\eta(\partial E) \to \eta(E^n)/\eta(\partial E^n)$
(b) $\eta(\partial E) \to \eta(\partial E^n)$
(c) $\partial \eta(E) \to \partial \eta(E^n)$
(d) $\eta(\partial E) \cup \partial \eta(E) \to \eta(\partial E^n) \cup \partial \eta(E^n)$
(e) $\eta(E)/(\eta(\partial E) \cup \partial \eta(E)) \to \eta(E^n)/(\eta(\partial E^n) \cup \partial \eta(E^n))$

Proof. For part (a), we have the following map of fibrations:

$$
\begin{array}{ccc}
F/\partial F & \xrightarrow{\sim} & F^n/\partial F^n \\
\downarrow & & \downarrow \\
E/\partial E & \longrightarrow & E^n/\partial E^n \\
\downarrow & & \downarrow \\
K & \longrightarrow & \tilde{\text{Tot}}^n X^* \\
\end{array}
$$

The map on fibers is a homotopy equivalence because either fiber is the subspace of $(I^d)^{q+t}$ such that the first $q$ points lie on the knot (or approximation of the knot), modulo the intersection of this subspace with the fat diagonal in $(I^d)^{q+t}$. Since $d \geq 4$, the results of Goodwillie and Weiss [5] imply that the middle map is $(n-1)(d-3)$-connected.

We have a map of pairs $(E, \partial E) \to (\eta(E), \eta(\partial E))$. The two maps are both homotopy equivalences, so by the long exact sequences of these pairs and the five-lemma, the map on quotients $E/\partial E \to \eta(E)/\eta(\partial E)$ is a homology isomorphism. The same argument with $E$ replaced by $E^n$ shows that the map $E^n/\partial E^n \to \eta(E^n)/\eta(\partial E^n)$ is a homology isomorphism. Finally consider the square below:

$$
\begin{array}{ccc}
E/\partial E & \xrightarrow{\sim} & \eta(E)/\eta(\partial E) \\
\downarrow & & \downarrow \\
E^n/\partial E^n & \xrightarrow{\sim} & \eta(E^n)/\eta(\partial E^n) \\
\end{array}
$$

We showed that the left-hand map is $(n-1)(d-3)$-connected, hence the right-hand map induces homology isomorphisms in a range of degrees, proving part (a).

Part (b) now just follows from the map on long exact sequences from the map of pairs $(\eta(E), \eta(\partial E)) \to (\eta(E^n), \eta(\partial E^n))$, part (a), and the five-lemma.

For (c), we will use the fact that we defined $\eta(E)$ and $\eta(E^n)$ as $\varepsilon$-neighborhoods of $E \subset K \times \mathbb{R}^{(L)}$ and $E^n \subset K \times \mathbb{R}^{L+N}$ respectively. We can consider the boundaries as the fibers
over $\varepsilon \in [0, \varepsilon]$ in the fibrations below:

\[
\begin{array}{ccc}
\partial \eta(E) & \to & \partial \eta(E^n) \\
\downarrow & & \downarrow \\
\eta(E) & \to & \eta(E^n) \\
\downarrow & & \downarrow \\
[0, \varepsilon] & \to & [0, \varepsilon]
\end{array}
\]

Since $E \to E^n$ is $(n-1)(d-3)$-connected, so is the middle map of neighborhood retracts above. Hence so is the top map. Thus it induces isomorphisms in homology in a range of degrees.

For part (d), we can use a map of Mayer-Vietoris sequences as follows. Let $\eta(\partial \eta(E))$ be an open set in our ambient space which deformation retracts onto $\partial \eta(E)$. (For example, one could take the preimage under a map like the lower-left map in (12) of $(\varepsilon-\delta, \varepsilon+\delta)$ for small $\delta$.) This gives a deformation retract from $\eta(\partial E) \cup \eta(\partial \eta(E))$ onto $\eta(\partial E) \cup \partial \eta(E)$. Similarly find an open set $\eta(\partial \eta(E^n))$ such that $\eta(\partial E^n) \cup \eta(\partial \eta(E^n))$ deformation retracts onto $\eta(\partial E^n) \cup \partial \eta(E^n)$; this can clearly be done so that these deformation retracts are compatible with maps induced by the evaluation $E \to E^n$. In either case, we now have a union of open sets, each of which we have established homology isomorphisms for. To see that the map on the intersections induces an isomorphism in homology, we can use a diagram similar to diagram (12). Then the Mayer-Vietoris sequence together with the five-lemma gives homology isomorphisms (in a range) for the union. Since the deformation retracts are compatible with the evaluation maps, this proves part (d).

For part (e), we have maps of the long exact sequences of the pairs $(\eta(E), \eta(\partial E) \cup \partial \eta(E))$ and $(\eta(E^n), \eta(\partial E^n) \cup \partial \eta(E^n))$. We have homology isomorphisms in a range induced by the maps between the first spaces in the pairs and by part (e, ) the maps between the second spaces in the pairs. The five-lemma then gives the desired result.

Let $\mathcal{R} = \eta(E)/(\eta(\partial E) \cup \partial \eta(E))$ and $\mathcal{R}^n = \eta(E^n)/(\eta(\partial E^n) \cup \partial \eta(E^n))$ ($\mathcal{R}$ for “relative”). By the part (e) of the previous lemma, the maps $H_*(\mathcal{R}) \to H_*(\mathcal{R}_n)$ and $H^*(\mathcal{R}_n) \to H^*(\mathcal{R})$ are isomorphisms in a range of degrees which is (at least roughly) $* < (n-1)(d-3)$; what is important is that this upper bound approaches $\infty$ as $n \to \infty$.

In the smooth setting, we have the Thom isomorphism

\[ H^*(E, \partial E) \cong H^*(\eta(E), \eta(\partial E)) \xrightarrow{\cong} H^{*+k}(\eta(E), \partial \eta(E) \cup \eta(\partial E)) \]

given by cup product with the Thom class $u \in H^k(\eta(E), \partial \eta(E) \cup \eta(\partial E)) \cong \tilde{H}^k(\mathcal{R})$. Since $H^*(\mathcal{R}_n) \to H^*(\mathcal{R})$ is an isomorphism in a range of degrees, there must be a class $u_n \in H^k(\mathcal{R}_n)$ that maps to $u$ for each sufficiently large $n$. Since the composition $E \to E^{n+1} \to E^n$
is the map $E \to E^n$ (and the same for $\mathcal{R}$, $\mathcal{R}^{n+1}$, and $\mathcal{R}^n$), $u_n$ maps to $u_{n+1}$. Using the diagram

$$
\begin{array}{ccc}
H^*(\eta(E), \eta(\partial E)) & \xrightarrow{\cup u_n} & H^{*+k}(\eta(E), \eta(\partial E) \cup \partial \eta(E)) \\
\uparrow & & \uparrow \\
H^*(\eta(E^n), \eta(\partial E^n)) & \xrightarrow{\cup u_n} & H^{*+k}(\eta(E^n), \eta(\partial E^n) \cup \partial \eta(E^n))
\end{array}
$$

together with the fact that the vertical maps are isomorphisms for a range of degrees $\ast$, we see that the map $\cup u_n$ is an isomorphism for $\ast$ in a range which increases with $n$. Thus we have a Thom isomorphism for sufficiently large $n$, despite the fact we are not dealing with bundles over smooth manifolds. Note that $\eta_N(E^n)/(\eta_N(\partial E^n) \cup \partial \eta_N(E^n))$ is the $N$th space of the spectrum $\Theta E^n/\partial \Theta E^n$. We now see that the map

$$
\tau : \Sigma^{\infty}(\widetilde{\text{Tot}}^n X^\bullet)_+ \to \Theta E^n/\partial \Theta E^n.
$$

induces in cohomology

$$
\begin{array}{ccc}
H^*(\eta(E^n), \eta(\partial E^n)) & \xrightarrow{\cup u_n} & H^{*+k}(\eta(E^n), \eta(\partial E^n) \cup \partial \eta(E^n)) \\
\uparrow & & \uparrow \\
H^*(E^n, \partial E^n) & \xrightarrow{\tau^*} & H^{*+k-N}(\widetilde{\text{Tot}}^n X^\bullet)
\end{array}
$$

As in the smooth case, the codimension $k$ of the embeddings used for the Thom collapse maps is $N - \dim F$ so the map above lowers degree by the dimension of the fiber $F$. Combining this with the compatibility of these collapse maps with those in the smooth setting, as shown in Proposition 3.8, we deduce the main result of this section:

**Theorem 3.10.** For sufficiently large $n$, the commutative square of spectra

$$
\begin{array}{ccc}
\Sigma^{\infty} \mathcal{K} & \longrightarrow & E^n/\partial E^n \\
\downarrow & & \downarrow \\
\Sigma^{\infty}(\widetilde{\text{Tot}}^n X^\bullet)_+ & \longrightarrow & \Theta E^n/\partial \Theta E^n
\end{array}
$$

induces a commutative square in cohomology with arbitrary coefficients

$$
\begin{array}{ccc}
H^{*-\dim F} \mathcal{K} & \xleftarrow{\tau} & H^*(E, \partial E) \\
\uparrow & & \uparrow \\
H^{*-\dim F}(\widetilde{\text{Tot}}^n X^\bullet) & \xleftarrow{\tau} & H^*(E^n, \partial E^n)
\end{array}
$$

thus extending the homotopy-theoretic Bott–Taubes construction to the cosimplicial setting. \qed
4. Homotopy Bott–Taubes on the Taylor tower

In \cite{Volic2014}, Volić showed that the Bott–Taubes integrals can be done (with differential forms) on the stages of the Taylor tower for the knot space $\mathcal{K} = \text{Emb}(I, I^d)$. In this section we observe that the homotopy-theoretic construction can also be carried out on these spaces.

4.1. Review of the Taylor tower and Bott–Taubes integrals over the stages of the tower. We begin by briefly reviewing the Taylor tower for the knot space, which is also described in papers of Volić \cite{Volic2015, Volic2014} and Sinha \cite{Sinha2013a, Sinha2013b}. It consists of spaces $T_n\mathcal{K}$ with maps $T_n\mathcal{K} \to T_{n-1}\mathcal{K}$ and compatible evaluation maps $\mathcal{K} \to T_n\mathcal{K}$. To describe the space $T_n\mathcal{K}$, fix some closed disjoint subintervals $A_0, ..., A_n$ of $I$. Let $\mathcal{P}(n)$ be the category of subsets of $\{0, 1, ..., n\}$. Let $\mathcal{P}_0(n)$ be the full subcategory of $\mathcal{P}(n)$ consisting of nonempty subsets.

**Definition 4.1.** Let $EC_n$ be the functor from $\mathcal{P}(n)$ to spaces given by

$$S \mapsto E_S := \text{Emb}(I - \bigcup_{i \in S} A_i, I^d).$$

Thus $EC_n$ is an $(n+1)$-cubical diagram. Define $T_n\mathcal{K}$ as the homotopy limit of the restriction of $EC_n$ to $\mathcal{P}_0(n)$. In other words, $T_n\mathcal{K}$ is the holim of the associated subcubical diagram. □

More concretely, this homotopy limit can be described as the subspace of

$$\prod_{\emptyset \neq S \subseteq \{0, ..., n\}} \text{Map}(\Delta^{[S]}, E_S)$$

such that all squares constructed from these maps and inclusions $S \hookrightarrow T$ of subsets of $\{0, 1, ..., n\}$ commute.

Since $\mathcal{K}$ is the initial object in $EC_n$ (and hence the homotopy limit of that diagram), the map $\mathcal{K} \to T_n\mathcal{K}$ is just the map from the holim of the whole cube to the holim of the restriction to $\mathcal{P}_0(n)$. (Also note that for $n \geq 2$, $\mathcal{K}$ is the limit of the diagram $EC_n|\mathcal{P}_0(n)$.) The maps $T_n\mathcal{K} \to T_{n-1}\mathcal{K}$ are the restrictions induced by the inclusions of categories $\mathcal{P}(n-1) \hookrightarrow \mathcal{P}(n)$.

By taking $T_n\mathcal{K}$ to consist of only smooth maps, one can make $T_n\mathcal{K}$ a smooth infinite-dimensional manifold. One would like to define a bundle over $T_n\mathcal{K}$ by replacing $\mathcal{K}$ by $T_n\mathcal{K}$ in the original Bott–Taubes pullback square. Note that $T_n\mathcal{K}$ sits inside a product of spaces $\text{Map}(\Delta^{[S]}, E_S)$ and that the quotient $C_q[\mathcal{K}]$ of $C_q[I]$ is diffeomorphic as a manifold with corners to $\Delta^q$. Thus including $C_q[I] \hookrightarrow C_q[I] \times C_q[I] \cong C_q[I] \times \Delta^q$ as the graph of the quotient map almost defines a map

$$T_q\mathcal{K} \times C_q[I] \to \prod_S \text{Map}(\Delta^{[S]}, E_S) \times C_q[I] \to C_q[I^d].$$

The only problem is that each $E_S$ is a space of punctured knots, and a point in $C_q[I]$ may lie in one of the punctures. Volić resolves this difficulty in \cite{Volic2014} by constructing spaces $\Gamma_q$ and
HOMOTOPY BOTT–TAUBES AND THE TAYLOR TOWER FOR THE SPACE OF KNOTS

$\Gamma_{q,t}$ which fit into the pullback square

\[
\begin{array}{ccc}
\Gamma_{q,t} & \rightarrow & C_{q+t}[\mathbb{R}^d] \\
\downarrow & & \downarrow \\
\Gamma_q \times T_nK & \rightarrow & C_q[\mathbb{R}^d]
\end{array}
\]

where $n \geq q$.

The space $\Gamma_q$ is the graph of a map

$$f : C_q[I] \rightarrow C_q\langle I \rangle \cong \Delta^q \rightarrow \Delta^q$$

such that the image of $(x_1, ..., x_q) \in C_q[I]$ in $\Delta^q$ lies in $\bigcap_i \partial_i \Delta^q$, where $\partial_i \Delta^q$ is the face of $\Delta^q$ where the $i$th barycentric coordinate is 0, and where the intersection is taken over all $i$ such that some $x_j \in A_i$. This ensures that for every $x = (x_1, ..., x_q) \in C_q[I]$, pairing $\Delta^{[S]}$ with an element in $\text{Map}(\Delta^{[S]}, E_S)$ gives a punctured knot $e_S \in E_S$ which comes from some $e_T \in E_T$ (with $T \subseteq S$) such that no $x_j$ lies in any of the punctures in the domain of $e_T$. This allows the bottom arrow in the square above to be defined. The ingredients for Volić’s construction of the map $f$ are a partition of unity and induction on the dimension of the simplex. Projecting to $T_nK$ gives a bundle $\Gamma_{q,t} \rightarrow T_nK$ for any $n \geq q$, whose fiber is a smooth manifold with corners. This allows one to integrate along the fiber just as in the original Bott–Taubes construction.

4.2. The homotopy-theoretic version. In [7], we used a categorical description due to Laures [9] of certain manifolds with corners, namely manifolds with faces, which when equipped with $L$ codimension-one strata satisfying certain conditions, are called $\langle L \rangle$-manifolds. These manifolds with corners are precisely the ones which for some $N$ can be embedded neatly in $\mathbb{R}^{(L),N} := [0, \infty)^L \times \mathbb{R}^N$, i.e., in a way that preserves corner structure. We will use the relevant terminology in this section, but we refer the reader to [9] or [7] for precise definitions and more details.

We note that $\Delta^q$ is a $\langle q + 1 \rangle$-manifold, with the $q + 1$ faces being the usual codimension one faces. We showed in [7] (Lemma 3.2.1) that $C_q[\mathbb{R}^3]$ has a $\langle k \rangle$-manifold structure for some $k$; a similar proof shows that the same is true of $C_q[I]$ (with a larger $k$). Thus $C_q[I] \times \Delta^q$ is a $\langle L \rangle$-manifold for some $L \in \mathbb{N}$, and hence so is $\Gamma_q \subset C_q[I] \times \Delta^q$.

This allows for a neat embedding which is the composite

$$\Gamma_{q,t} \hookrightarrow T_nK \times \Gamma_q \times C_{q+t}[\mathbb{R}^d] \rightarrow T_nK \times \mathbb{R}^{(L),N}$$

(13)

for some $L, N$. As in our original construction with the knot space itself, there is a well defined normal bundle $\nu_N$ of this embedding and resulting Thom collapse maps

$$T_nK \times \mathbb{R}^{(L),N} \rightarrow \Gamma_{q,t}$$
where \( \Gamma_{q,t}^{\nu_N} \) is the Thom space of the normal bundle \( \nu_N \to \Gamma_{q,t} \). The bundle \( \nu_N \) has \((L + N)\)-dimensional fibers; we choose the indexing merely to be consistent with that of the original construction in [7].

Taking the quotient by all the boundary subspaces gives
\[
\Sigma^{L+N}(T_nK) \to \Gamma_{q,t}^{\nu_N}/\partial(\Gamma_{q,t}^{\nu_N})
\]
and a map of spectra
\[
\Sigma^\infty(T_nK) \to \Gamma_{q,t}^{\nu_N}/\partial\Gamma_{q,t}^{\nu_N}
\]
where \( \Gamma_{q,t}^{\nu_N} \) is the Thom spectrum with \( N \)th space \( \Gamma_{q,t}^{\nu_N} \).

Notice that the fiber of \( \Gamma_{q,t} \to T_nK \) is a (possibly twisted) product of \( \text{fib}(C_{q+t}[I^d] \to C_q[I^d]) \)
and \( \Gamma_q \). Since \( \Gamma_q \) is homeomorphic to \( C_q[I] \), its dimension is the same as that of \( C_q[I] \).\(^1\) Thus the dimension of the fiber is the same as that of the fiber \( F_{q,t} \) of \( E_{q,t} \to K \). So applying the suspension isomorphism and the Thom isomorphism in cohomology, we have shown the following:

**Proposition 4.2.** There is a map of spectra
\[
\Sigma^\infty(T_nK) \to \Gamma_{q,t}^{\nu_N}/\partial\Gamma_{q,t}^{\nu_N}
\]
inducing a map in cohomology with arbitrary coefficients:
\[
H^*(\Gamma_{q,t}, \partial\Gamma_{q,t}) \to H^{*-\text{dim } F_{q,t}}(T_nK).
\]

We now show that this map is compatible with the original “homotopy-theoretic Bott–Taubes construction”. Consider the two pullback squares:

\[
\begin{array}{ccc}
E_{q,t} & \longrightarrow & C_{q+t}[I^d] \\
\downarrow & & \downarrow \pi \\
K \times C_q[I] & \longrightarrow & C_q[I^d]
\end{array}
\quad
\begin{array}{ccc}
\Gamma_{q,t} & \longrightarrow & C_{q+t}[I^d] \\
\downarrow & & \downarrow \pi \\
T_nK \times \Gamma_q & \longrightarrow & C_q[I^d]
\end{array}
\]

There is map from each entry of the left-hand square to the corresponding entry in the right-hand square. For the two pairs of right-hand entries, these maps are the identity. For the lower-left entries, we use the evaluation \( K \to T_nK \) and the canonical diffeomorphism \( C_q[I] \cong \Gamma_q \). The diagram

\[
\begin{array}{ccc}
K \times C_q[I] & \longrightarrow & C_q[I^d] \\
\downarrow & & \downarrow \cong \\
T_nK \times \Gamma_q & \longrightarrow & C_q[I^d]
\end{array}
\]

\(^1\)The two are not diffeomorphic as manifold with corners, as \( \Gamma_q \) may have corners of higher codimension than any of those in \( C_q[I] \); however, this does not pose us any problems.
commutes because \( \mathcal{K} \) maps to \( T_n\mathcal{K} \) as the subspace where all the homotopies (given by maps of simplices) are constant. In other words, we can essentially ignore the map \( f: C_q[I] \to \Delta^n \). Thus there is a map \( E_{q,t} \to \Gamma_{q,t} \) as claimed.

In the original homotopy-theoretic Bott–Taubes construction, we embedded \( E_{q,t} \to \mathcal{K} \times \mathbb{R}^{(L),N} \) for some \( L \in \mathbb{N} \). Although we could take this \( L \) to be less than the smallest possible \( L \) in equation (13), we may as well choose one sufficiently large \( L \), since we do not need either number to be minimal here. We can construct a map of neat embeddings:

\[
E_{q,t} \hookrightarrow \mathcal{K} \times C_q[I] \times C_{q+t}[I^d] \hookrightarrow \mathcal{K} \times \mathbb{R}^{(L),N}
\]
\[
\Gamma_{q,t} \hookrightarrow T_n\mathcal{K} \times \mathcal{K} \times C_{q+t}[I^d] \hookrightarrow T_n\mathcal{K} \times \mathbb{R}^{(L),N}
\]

It is clear from the above that the obvious maps make the left-hand square commute. As for the right hand square, commutativity is just a matter of choosing large enough \( L \) and \( N \) so that we can neatly embed \( \Gamma_q \subset C_q[I] \times \Delta^n \) by embedding \( C_q[I] \times \Delta^n \hookrightarrow \mathbb{R}^{(L_1),N_1} \times \mathbb{R}^{(L_2),N_2} = \mathbb{R}^{(L),N} \) with the first and second factors in the domain mapping respectively to the first and second factors in the codomain.

We note that all the maps above are maps of \( \langle L \rangle \)-manifolds. Thus we can restrict to each filtration indexed by some \( a \in 2^L \), and proceed with the Thom collapse maps we describe below on such subspaces, just as we remarked in the original construction in [7].

For the collapse maps, we need neighborhoods of \( E_{q,t} \) and \( \Gamma_{q,t} \) in their respective ambient spaces. We define \( v_\varepsilon(E_{q,t}) \) as we did at the end of section 3.2 namely as the set of all \((K, t, x) \in \mathcal{K} \times C_q[I] \times C_{q+t}[I^d]\) such that \(|K(t) - \pi(x)| < \varepsilon\) where \( \pi: C_{q+t}[I^d] \to C_q[I^d] \) is the projection to the first \( q \) points.

A point in \( \Gamma_{q,t} \subset T_n\mathcal{K} \times \mathcal{K} \times C_{q+t}[I^d] \) can be thought of as \(((K_s)_{s \in \Delta^n}, t \in C_q[I], x)\) such that \( K_f(t) = \pi(x); \) here \( f: C_q[I] \to \Delta^n \) is the “partition of unity map” constructed by Volić, and the inclusion \( \Delta^n \hookrightarrow \Delta^n \) is the same one used to define the projection \( T_n\mathcal{K} \to T_q\mathcal{K} \). So define \( v_\varepsilon(\Gamma_{q,t}) \) as the set of points where \(|K_f(t)(x) - \pi(x)| < \varepsilon\).

Now it is clear that that in the diagram (15), the vertical maps take the complement of \( v_\varepsilon(E_{q,t}) \) to the complement of \( v_\varepsilon(\Gamma_{q,t}) \). It is also clear that by choosing the appropriate maps as above for the right-hand square in (15), the complement of a tubular neighborhood \( \eta(E_{q,t}) \subset \mathcal{K} \times \mathbb{R}^{(L),N} \) is sent to the complement of a tubular neighborhood \( \eta(\Gamma_{q,t}) \subset T_n\mathcal{K} \times \mathbb{R}^{(L),N} \).
Thus we get for every sufficiently large $N$ a diagram of collapse maps $^3$:

\[ \mathcal{K} \times R^{(L),N} \longrightarrow E^\nu_{q,t} \]

\[ T_n \mathcal{K} \times R^{(L),N} \longrightarrow \Gamma^\nu_{q,t} \]

This commutative diagram clearly descends to the quotients by boundaries. Furthermore the cube of squares incorporating indices $N$ and $N + 1$ commutes. Hence we have the following:

**Theorem 4.3.** For $n \geq q$, there is a commutative square of spectra

\[ \Sigma^\infty \mathcal{K} \longrightarrow E^\nu_{q,t} / \partial E^\nu_{q,t} \]

\[ \Sigma^\infty (T_n \mathcal{K}) \longrightarrow \Gamma^\nu_{q,t} / \partial \Gamma^\nu_{q,t} \]

inducing the following commutative square in cohomology with arbitrary coefficients

\[ H^{* - \dim F_{q,t}} (T_n \mathcal{K}) \longrightarrow H^* (\Gamma_{q,t}, \partial \Gamma_{q,t}) \]

\[ H^{* - \dim F} (\mathcal{K}) \longrightarrow H^* (E, \partial E) \]

thus extending the homotopy-theoretic Bott–Taubes construction to stages of the Taylor tower.

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$^3$Here and below, the symbol $\nu_N$ itself abusively stands for two different bundles, but we only use it together with the base space, so this should cause no confusion. The same applies to the symbol $\nu$ in the Theorem.
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