Measuring Coherence of Quantum Measurements

Valeria Cimini, Ilaria Gianani, Marco Sbroscia, Jan Sperling, and Marco Barbieri

1 Dipartimento di Scienze, Università degli studi Roma Tre, Via della vasca Navale 84, 00146 Rome, Italy
2 Integrated Quantum Optics Group, Applied Physics, University of Paderborn, 33098 Paderborn, Germany
3 Consiglio Nazionale delle Ricerche, Largo E. Fermi 6, 50125 Florence, Italy

(Dated: March 5, 2019)

The superposition of quantum states lies at the heart of physics and has been recently found to serve as a versatile resource for quantum information protocols, defining the notion of quantum coherence. In this contribution, we report on the implementation of its complementary concept, coherence from quantum measurements. By devising an accessible criterion which holds true in any classical statistical theory, we demonstrate that noncommutative quantum measurements violate this constraint, rendering it possible to perform an operational assessment of the measurement-induced quantum coherence. In particular, we verify that polarization measurements of a single photonic qubit, an essential carrier of one unit of quantum information, are already incompatible with classical, i.e., incoherent, models of a measurement apparatus. Thus, we realize a method which enables us to quantitatively certify that quantum measurements follow fundamentally different statistical laws than expected from classical theories and, at the same time, quantify the usefulness within the modern framework of resources for quantum information technology.

I. INTRODUCTION

Quantum interference phenomena are a key property that enable us discern classical physics from the quantum realm [1–5]. Different forms of quantum coherence constitute the basis for a variety of notions of nonclassicality, such as entanglement that is a result of nonlocal superpositions [6–10]. The application of quantum coherence as a resource for realizations of quantum information protocols recently gained a lot of attention (see Refs. [11–13] for introductions) because it connects fundamental question about our physical nature with practical aspects of the implementations of upcoming quantum technologies.

In order to show how classical expectations are superseded by quantum physics, a number of measurable criteria have been proposed. Most prominently, Bell’s inequality [14] enables us to show that local hidden-variable models do not sufficiently describe general correlations between quantum systems. More generally, the concept of contextuality provides a broadly applicable approach which demonstrates the superiority of quantum-mechanical joint probabilities over their classical counterparts [15, 16]. The underlying constraints for both examples provide criteria which were derived in a classical framework; secondly, we relate them to the manipulation and preparation of quantum states [34–37], conditional quantum correlations [38, 39], as well as questions concerning the collapse of the wave function [27, 40–43]. The other way around, Heisenberg’s seminal uncertainty relation [44] poses a fundamental precision limitation to quantum measurements of multiple observables [45–47], which is not the case in classical models. Thus, an experimentally accessible distinction between classical and quantum statistics, bases on the outcomes of a measurements, is vital for many applications. While some measurements have been performed, for example, to confirm the noncommutativity of certain observables [48, 49], a general connection between the quantumness of measurements and the state-based notion of quantum coherence, together its experimental certification, is still missing.

In this contribution, we close this gap between the theory of quantum coherence of states and experiments with incompatible quantum measurements. To derive our experimentally accessible and generally applicable criteria, we firstly perform a derivation in a purely classical framework; secondly, we relate our findings to a theoretical framework of quantum coherence of measurements. Then, we directly apply our technique to data obtained in our experiment of polarization measurements of photons, detecting one qubit of information. Our results not only experimentally verify with high statistical significance if and when a classical interpretation of a measurement ultimately fails in quantum systems, but it also provides a quantifier of the measurement-based quantum coherence in the system under study. Thus, we provide and implement a practical
tool to study the fundamentals and application-oriented properties of quantum measurements.

II. CLASSICAL LAW OF TOTAL PROBABILITIES

Similarly to the approach by Bell and others, let us formulate our classical constraints solemnly based on universally valid properties in classical statistics. For this reason, we consider a probability distribution \( P \), where \( P(x) \) and \( P(y) \) are the probabilities to measure the outcomes \( x \) and \( y \) for two random variables. Further, the probability to measure \( y \) after a measurement of \( x \) is given by the conditional probability \( P(y|x) = P(y,x)/P(x) \), where \( P(y,x) \) is the joint probability for the given outcomes. Consequently, the probability to detect \( y \) regardless of the prior outcome \( x \) is given by \( P'(y) = \sum_x P(y|x)P(x) \). According to the law of total probability \[50, 51\], we have

\[
P'(y) = \sum_x P(y|x)P(x),
\]

for any classical system. It is worth emphasizing that the law of total probability applies to any classical model even if the measurement is not an ideal one.

Using the classical identity (1), we can now formulate a variance-based constraint for classical statistics,

\[
\mathbb{V}_{P'(y)}[y] = \mathbb{V}_{P(y)}[y],
\]

where \( \mathbb{V} \) denotes the variance. This classical relation is known as the law of total variance \[50, 51\] and follows from the decomposition \( \mathbb{V}_{P'(y)}[y] = \mathbb{E}_{P(x)}[\mathbb{V}_{P(y|x)}[y]] + \mathbb{V}_{P(x)}[\mathbb{E}_{P(y|x)}[y]] \), which is based on the construction of \( P' \) via conditional probabilities and where \( \mathbb{E} \) denotes the mean value. A violation of the classically universal law in Eq. (2) certifies the incompatibility of the measurement with classical statistics. It is worth emphasizing that beyond second-order criterion (2), a generalization to higher-order moments is possible, using Eq. (1).

III. RELATION TO QUANTUM COHERENCE

Let us now establish the relation of the above criterion to the notion of quantum coherence. For this reason, we identify an observable, represented through the operator \( \hat{x} \), to serve as our incoherent gauge when compared to a second, general observable \( \hat{y} \). The decomposition of those observables reads \( \hat{x} = \sum_x \hat{x}_x \) and \( \hat{y} = \sum_y \hat{y}_y \), using the positive operator-valued measures \( \{\hat{x}_x\} \) and \( \{\hat{y}_y\} \).

Measuring the outcome \( x \) is achieved with the probability \( P(x) = \text{tr}(\rho \hat{x}_x) = \langle \hat{x}_x | \rho \rangle \) and leaves us with a post-measurement state \( \hat{\rho}_x = \hat{x}_x^{1/2}\rho \hat{x}_x^{1/2}/P(x) \). In analogy to the classical case, we now ignore the first outcome, resulting in

\[
\hat{\rho}' = \sum_x P(x)\hat{\rho}_x = \sum_x \hat{x}_x^{1/2}\hat{\rho}_x\hat{x}_x^{1/2}.
\]

For our purpose, it is now convenient to define that a state is incoherent if the map in Eq. (3) leaves the state unchanged, i.e., \( \hat{\rho} \rightarrow \hat{\rho}' = \hat{\rho} \). Conversely, quantum coherence is given by \( \hat{\rho}' \neq \hat{\rho} \). It is worth mentioning that \( \hat{\rho} \rightarrow \hat{\rho}' \) is a so-called strictly incoherent operation \[2, 52\]. In general, assessing coherence demands a choice of a preferred basis on grounds of physical considerations. Here, it is motivated through a detection of \( \hat{x} \) because, in itself, it does have a completely classical model in terms of the measured statistics \( P(x) \).

For comparing the two cases, the measurement of \( \hat{y} \) without and with a prior measurement of \( \hat{x} \) yields

\[
\mathbb{V}_{P'(y)}[y] = \langle (\Delta \hat{y})^2 \rangle \hat{\rho} \quad \text{and} \quad \mathbb{V}_{P(y)}[y] = \langle (\Delta \hat{y})^2 \rangle \hat{\rho}',
\]

respectively, corresponding to the variances for the previously discussed classical case. Here, however, the classical law of total variances does not apply, and we can find \( \langle (\Delta \hat{y})^2 \rangle \hat{\rho} \neq \langle (\Delta \hat{y})^2 \rangle \hat{\rho}' \) in the presence of quantum coherence, \( \hat{\rho} \neq \hat{\rho}' \).

It is worth emphasizing that our approach does indeed measure the incompatibility of the performed measurements because \( \{\hat{x}, \hat{y}\} = \{\forall x, y\} \) implies \( P(y) = P'(y) \), regardless of the coherence initial state \( \hat{\rho} \) \[53\]. Therefore, when the classical constraint (2) is violated, we can directly infer that the quantum measurement \( \hat{y} \) exhibits quantum coherence with respect to the detection of \( \hat{x} \). In an ideal scenario, where the measurements are represented through orthonormal bases, i.e., \( \hat{x}_x = |x\rangle\langle x| \) and \( \hat{y}_y = |y\rangle\langle y| \), this means that the measurement of \( \hat{y} \) is not described through incoherent mixtures (\( \{\hat{y}_{y}, \hat{y}_{y'}\} \neq \sum_q q_k |x_{y', y}\rangle \langle y_{y', y}| \)); rather, it requires quantum superpositions \( |y\rangle = \sum_q c_{y, y} |x_{y, y}\rangle \), i.e., quantum coherence. More specifically, our intermediate definition of the coherence of the states, based on the measurement of \( \hat{x} \) [cf. Eq. (3)], actually serves as a proxy to infer the quantum coherence of the second measurement \( \hat{y} \) when compared to \( \hat{x} \). Consequently, we have formulated an observable criterion that assesses the quantum coherence of measurements.

IV. IMPLEMENTATION

We explore the previously devised concepts for qubits. While our approach works to arbitrary system, qubits are fundamental quantum objects as they represent the basic unit of quantum information science \[32, 54, 55\]. Our qubits are encoded in the polarization of single photons. The preferred basis is given by the horizontal (\( H \)) and vertical (\( V \)) polarization, also defining the reference measurement \( \hat{x} = -|H\rangle\langle H| + |V\rangle\langle V| = [1 \, 1 \, 0 \, 1] \). The states we prepare take the general form

\[
\hat{\rho} = \begin{bmatrix} 1 - p & \sqrt{p(1-p)} \gamma \\ \sqrt{p(1-p)} \gamma & p \end{bmatrix},
\]

where \( p \in [0, 1] \) indicates the population unbalance between the two levels, and the parameter \( \gamma \in [0, 1] \) determines the coherence in the state when compared to their incoherent counterparts, \( \hat{\rho}' = (1 - p)|H\rangle\langle H| + p |V\rangle\langle V| \), cf. Eq. (3). Note that \( \gamma \) can be additionally equipped with a complex phase factor, \( e^{i\phi} \), to account for the most general case of a qubit; this, however, does not lead to any conceptional advantage and is, therefore, fixed to one \( (e^{i\phi} = 1) \) in our treatment. The qubit
is subjected to two consecutive measurements. The first one measures the Pauli-$z$ operator (here, denoted as $\hat{x}$); the second one measures an arbitrary observable

$$
\hat{y} = \cos \theta \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
$$

parametrized through the angle $\theta$.

In our experiment, we prepare linear polarization states from $H$-polarized photons by means of a half-wave plate (HWP) at an angle $\alpha$—hence, $p = \sin^2(2\alpha)$. The statistics for mixed states is obtained by inputting states corresponding to the setting $\alpha$ and $-\alpha$ with the respective weights $w_+, w_- \geq 0$. These are chosen in such a way that $w_+ + w_- = 1$ and $w_+ - w_- = \gamma$ hold true.

In order to implement the $\hat{x}$ measurement without destroying the signal photon, we couple the photon with an ancillary meter by means of a controlled sign gate [56–58]. This is constituted by a beam splitter with polarization-dependent transmittivities, $T_H = 1$ and $T_V = 1/3$. Two-photon nonclassical interference occurs selectively on the beam splitter only for the vertical components of the signal and the meter photon polarizations. These consequently acquire a $\pi$ phase shift with respect to the other three terms, when post-selecting those events in which the two photons emerge on distinct outputs of the beam splitter—this then demands to register only coincidence events between the arms. Two other identical beam splitters, rotated by 90 degrees, are used to balance polarization-dependent loss induced by the first one [59].

The action of the described gate can be switched on and off by controlling the polarization of the meter [60–62]. Consider a pure signal state state ($\gamma = 1$) arriving at the gate. If a $H$-polarized meter is injected, no coupling can occur; hence, the joint state remains separable, $(\sqrt{1-p}|H\rangle_s + \sqrt{p}|V\rangle_s) \otimes |H\rangle_m$. No information can be inferred from the meter about the signal. If the meter is, however, injected in a diagonal polarization, $|m\rangle_m = (|H\rangle_m + |V\rangle_m)/\sqrt{2}$, the output two-photon polarization state is entangled, $\sqrt{1-p}|H\rangle_s \otimes |+\rangle_m + \sqrt{p}|V\rangle_s \otimes |-\rangle_m$, due to the phase shift imparted on the gate. By measuring the meter in the diagonal basis, one can extract the full information about the $\hat{x}$ measurement of the original signal state.

The observable $\hat{y}$ is measured conventionally by a HWP at $\beta = \theta/4$ and a polarization beam splitter.

The experiment is carried out in two steps: firstly, we measure the unperturbed variance, predicted to be $\langle (\Delta\hat{y})^2 \rangle_\rho = 1 - (2p - 1) \cos \theta + 2\sqrt{p(1-p)}\gamma \sin \theta$; secondly, we measure the variance resulting from a prior measurement of $\hat{x}$, expected to follow $\langle (\Delta\hat{y})^2 \rangle_{\rho'} = 1 - (1 - 2p)^2 \cos^2 \theta$. In order to account for experimental artifacts, both measurements are performed with the photons passing through the gate, with the polarization of the meter set accordingly, and registering coincidences counts. In both cases, the polarization of the meter is not analyzed since we are ignoring the outcome $x$, as expressed in Eq. (3).

![FIG. 1. Variance difference between the two measurement configurations [Eq. (7)]. (a) The surface depicts the expected theoretical behavior when varying $p$ and $\theta$; the points show the experimental data. (b) Cut of the plot (a) for $p = 0.165$. (c) Cut of the plot (a) graph for $p = 0.552$.](image)

V. RESULTS

To assess the amount of measurement-induced quantum coherence when compared to the classical constraint (2), it is convenient to consider the following difference of variances:

$$
\Delta V = \mathbb{V}_{\rho^{(y)}}[y] - \mathbb{V}_{\rho^{(x)}}[y] = \langle (\Delta\hat{y})^2 \rangle_\rho - \langle (\Delta\hat{y})^2 \rangle_{\rho'}.
$$

Because of Eq. (2), a significant deviation from $\Delta V = 0$ is our figure of merit to quantify the amount of coherence.

Figure 1(a) shows the measured deviation of $\Delta V$ as a function of $\theta$ and $p$ for a measurement of $\hat{y}$ [Eq. (6)] for pure states [$\gamma = 1$ in Eq. (5)]. As one can observe, our data are in good agreement with the quantum-mechanical model of the detection processes. We also confirm that for $\theta = 0$, i.e., $[\hat{x}, \hat{y}] = 0$ [Eq. (6)], no coherence can be observed ($\Delta V = 0$) regardless of the input state as predicted in the theory part. Moreover, we can rule out that our system can be mimicked by any classical model of a measurement as $\Delta V$ significantly deviates from zero in almost all other cases. The maximally positive and negative deviation from the classical bound is shown in Figs. 1(b) and 1(c). In particular, Fig. 1(c) certifies that the absolute maximal violation is obtained for $\theta \approx \pi/2$, which corresponds to a measurement $\hat{y}$ that is a Pauli-$x$ measurement, thus maximally incompatible with the reference measurement $\hat{x}$ of the Pauli-$z$ operator.

In addition, we explore mixed states to probe the quantum coherence between the measurements in Fig. 2(a). There, we fix the value of $p = \sin^2(2\alpha)$ at $\alpha = 12^\circ$ and study the difference of the variances as a function of $\gamma$ [Eq. (5)]. We can observe that, in general, the highest purity ($\gamma \to 1$) yields the most significant verification of quantum coherence, $\Delta V \neq 0$, which also represents the scenario with the highest coherence.
of the probe state $\hat{\rho}$. Again, Figs. 2(b) and 2(c) show the cuts with the optimal deviations from the classical bound zero.

Finally, we can also measure the maximal deviation for the state $\hat{\rho}$ prior to the measurement $\hat{x}$ and the state $\hat{\rho}'$ after the detection took place [63,64], cf. Eq. (3). The result is shown in Fig. 3 as functions of $p$ and $\gamma$, which define the prepared state in Eq. (5). The shown results enable us to quantify the measurement-induced decoherence because one can straightforwardly prove [65] that the trace distance between $\hat{\rho}$ and $\hat{\rho}'$ is identical to $\Delta V^2$ for $\theta = \pi / 2$. For instance, we can verify from Figs. 3(a) and 3(b) that the maximal decoherence occurs for $p \approx 1/2$ and $\gamma \approx 1$, corresponding to a maximally coherent input state, $\hat{\rho} = |\psi\rangle\langle\psi|$ with $|\psi\rangle = (|H\rangle + |V\rangle) / \sqrt{2}$, being converted into a maximally incoherent one, $\hat{\rho}' = (|H\rangle\langle H| + |V\rangle\langle V|) / 2$, through the detection of $\hat{x}$.

VI. DISCUSSION

In summary, we formulated and implemented a method that enables us to certify quantum coherence between two measurements. We applied the law of total probabilities (and variances) to formulate conditions that apply to all classical measurements. The translation to the quantum domain enabled us to violate these classical requirements, and, thereby, we revealed a connection to the notion of quantum coherence between measurements. We confirmed our theory by probing the quantumness of different and essential qubit measurements, encoded in the polarization of photons. This allowed us to experimentally verify the fundamental incompatibility of quantum measurements with classical statistical models on a quantitative basis. Furthermore, we were able to assess the measurement-induced decoherence which occurs when a measurement is performed on a quantum system.

Our studies reveal fundamental and application-oriented properties genuine to the quantum description of measurements. First, we confirmed—with an easily accessible, alternative approach and high significance—that the quantum statistics of measurements has fundamentally different properties than expected from any classical perspective. Second, we were able to connect the resource-theoretic notion of quantum coherence of quantum states to the coherence between two measurement scenarios. Specifically, one measurement defines a classical reference, the incompatibility of this reference with the employed state and the second measurement then leads to quantum effects beyond classical physics. In this scenario, the coherence of the state serves as a medium to prove that the description of the second measurement requires quantum superpositions since for commuting observables, any quantum coherence of the state become meaningless. This further demonstrates that, in quantum physics, it makes a profound difference if one measures a second observable in the context of preceding one or not—even if one is ignorant to the outcome of the first detection event. Let us also comment on the fact that the role of the first and second measurement is fixed by reasons of experimental practicality, however, it can be exchanged in our treatment without affecting any of the general observations for noncommuting observables.

Moreover, our approach enables us to quantify the loss of coherence as a result of the alteration of a state after a quantum-measurement process took place. This also relates to the collapse of the wave function. In particular, we show that intervening with a measurement has a disruptive action on the quantum information carried by the state’s coherence. Indeed, a prior measurement cancels the presence of coherence in the state, affecting a subsequent measurement, which is also the basis of our quantumness criteria. Furthermore, our approach applies to general quantum systems beyond qubits, and second-order criteria can be straightforwardly extended, outlining possible future generalizations of our method.

It is also worth emphasizing that measurement-based quantum protocols rely on realizing measurements which are incompatible. Here, we were able to assess the quantum coherence between such measurements to quantify this resource of incompatibility, analogously to the requirements on quantum protocols which exploit the coherence of the state. Therefore, we additionally provide a useful tool to quantify the coherence of measurements for practical purposes.
ACKNOWLEDGMENTS

M. S. acknowledges support from the ADAMO project of Distretto Tecnologico Beni e Attività Culturali, Regione Lazio. The Integrated Quantum Optics group acknowledges financial support from the Gottfried Wilhelm Leibniz-Preis (Grant No. SI1115/3-1).

[1] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. 113, 140401 (2014).
[2] A. Winter and D. Yang, Operational Resource Theory of Coherence, Phys. Rev. Lett. 116, 120404 (2016).
[3] T. Theurer, N. Killoran, D. Egloff, and M. B. Plenio, Resource Theory of Superposition, Phys. Rev. Lett. 119, 230401 (2017).
[4] T. Biswas, M. G. Diaz, and A. Winter, Interferometric visibility and coherence, Proc. Roy. Soc. London A 473, 20170170 (2017).
[5] Y.-T. Wang, J.-S. Tang, Z.-Y. Wei, S. Yu, Z.-J. Ke, X.-Y. Xu, C.-F. Li, and G.-C. Guo, Directly Measuring the Degree of Quantum Coherence using Interference Fringes, Phys. Rev. Lett. 118, 020403 (2017).
[6] W. Vogel and J. Sperling, Unified quantification of nonclassicality and entanglement, Phys. Rev. A 89, 052302 (2014).
[7] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Measuring Quantum Coherence with Entanglement, Phys. Rev. Lett. 115, 020403 (2015).
[8] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Converting Nonclassicality into Entanglement, Phys. Rev. Lett. 116, 080402 (2016).
[9] E. Chitambar and M.-H. Hsieh, Relating the Resource Theories of Entanglement and Quantum Coherence, Phys. Rev. Lett. 117, 020402 (2016).
[10] L.-F. Qiao et al., Entanglement activation from quantum coherence and superposition, Phys. Rev. A 98, 052351 (2018).
[11] G. Adesso, T. R. Bromley, and M. Cianciaruso, Measures and applications of quantum correlations, J. Phys. A: Math. Theor. 49, 473001 (2016).
[12] A. Streltsov, G. Adesso, and M. B. Plenio, Quantum coherence as a resource, Rev. Mod. Phys. 89, 041003 (2017).
[13] E. Chitambar and G. Gour, Quantum Resource Theories, arXiv:1806.06107.
[14] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics 1, 195 (1964).
[15] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Contextuality supplies the ’magic’ for quantum computation, Nature (London) 510, 351 (2014).
[16] R. Raussendorf, Contextuality in measurement-based quantum computation, Phys. Rev. A 88, 022322 (2013).
[17] A. Zhang, H. Xu, J. Xie, H. Zhang, B. J. Smith, M. S. Kim, and L. Zhang, Experimental Test of Contextuality in Quantum and Classical Systems, Phys. Rev. Lett. 122, 080401 (2019).
[18] J. Åberg, Quantifying Superposition, arXiv:quant-ph/0612146.
[19] J. Åberg, Catalytic Coherence, Phys. Rev. Lett. 113, 150402 (2014).
[20] J. Sperling and W. Vogel, Convex ordering and quantification of quantumness, Phys. Scr. 90, 074024 (2015).
[21] R. J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. 131, 2766 (1963).
[22] U. M. Titulaer and R. J. Glauber, Correlation functions for coherent fields, Phys. Rev. 140, B676 (1965).
[23] L. Mandel, Non-classical states of the electromagnetic field, Phys. Scr. T 1986, 34 (1986).
[24] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, (Cambridge University Press, Cambridge, 1995).
[25] W. Vogel and D.-G. Welsch, Quantum Optics (Wiley-VCH, Weinheim, 2006).
[26] D. Walls and G. J. Milburn, Quantum Coherence and Measurement Theory, In: D. Walls and G. J. Milburn, Quantum Optics (Springer, Berlin, Heidelberg, 2008).
[27] Yao Yao, G. H. Dong, Xing Xiao, Mo Li, and C. P. Sun, Interpreting quantum coherence through a quantum measurement process, Phys. Rev. A 96, 052322 (2017).
[28] F. Bischof, H. Kampermann, and D. Bruß, Resource theory of coherence based on positive-operator-valued measures, arXiv:1812.00018.
[29] P. Skrzypczyk and N. Linden, Robustness of Measurement, discrimination games and accessible information, arXiv:1809.02570.
[30] P. Skrzypczyk, I. Šupić, and D. Cavalcanti, All sets of incompatible measurements give an advantage in quantum state discrimination, arXiv:1901.00816.
[31] T. Guff, N. A. McMahon, Y. R. Sanders, and A. Gilchrist, A Resource Theory of Quantum Measurements, arXiv:1902.08490.
[32] M. Van den Nest, A. Miyake, W. Dür, and H. J. Briegel, Universal Resources for Measurement-Based Quantum Computation, Phys. Rev. Lett. 97, 150504 (2006).
[33] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, Measurement-based quantum computation, Nat. Phys. 5, 19 (2009).
[34] X. Hu and H. Fan, Extracting quantum coherence via steering, Sci. Rep. 6, 34380 (2016).
[35] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, Assisted Distillation of Quantum Coherence, Phys. Rev. Lett. 116, 070402 (2016).
[36] T. Ma, M.-J. Zhao, S.-M. Fei, and G.-L. Long, Remote creation of quantum coherence, Phys. Rev. A 94, 042312 (2016).
[37] D. Girolami, How Difficult is it to Prepare a Quantum State?, Phys. Rev. Lett. 122, 010505 (2019).
[38] J. Sperling, T. J. Bartley, G. Donati, M. Barbieri, X.-M. Jin, A. Datta, W. Vogel, and I. A. Walmsley, Quantum Correlations from the Conditional Statistics of Incomplete Data, Phys. Rev. Lett. 117, 083601 (2016).
[39] E. Agudelo, J. Sperling, L. S. Costanzo, M. Bellini, A. Zavatta, and W. Vogel, Conditional Hybrid Nonclassicality, Phys. Rev. Lett. 119, 120403 (2017).
[40] M. Fuwa, S. Takeda, M. Zwierz, H. M. Wiseman, and A. Furusawa, Experimental proof of nonlocal wavefunction collapse for a single particle using homodyne measurements, Nat. Commun. 6, 6665 (2015).
[41] G. C. Knee, K. Kakuyanagi, M.-C. Yeh, Y. Matsuzaki, H. Toida, H. Yamaguchi, S. Saito, A. J. Leggett, and W. J. Munro, A strict experimental test of macroscopic realism in a superconducting...
Continuity of measurement outcomes

X.-Y. Xu et al., Measurements of nonlocal variables and demonstration of the failure of the product rule for a pre- and postselected pair of photons, arXiv:1902.08473.

J. Sperling, Continuity of measurement outcomes, arXiv:1805.12404.

W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Physik 43, 172 (1927).

U. Singh, A. K. Pati, and M. N. Bera, Uncertainty Relations for Quantum Coherence, Mathematics 4, 47 (2017).

X. Yuan, G. Bai, T. Peng, and X. Ma, Quantum uncertainty relation using coherence, Phys. Rev. A 96, 032313 (2017).

H. Dolatkhah, S. Haseli, S. Salimi, and A. S. Khorashad, Tightening the entropic uncertainty relations for multiple measurements and applying it to quantum coherence, Quantum Inf. Process. 18, 13 (2019).

V. Parigi, A. Zavatta, M. Kim, M. Bellini, Probing Quantum Commutation Rules by Addition and Subtraction of Single Photons to/from a Light Field, Science 317, 1890 (2007).

A. Zavatta, V. Parigi, M. S. Kim, H. Jeong, and M. Bellini, Experimental Demonstration of the Bosonic Commutation Relation via Superpositions of Quantum Operations on Thermal Light Fields, Phys. Rev. Lett. 103, 140406 (2009).

D. R. Brillinger, The calculation of cumulants via conditioning, Ann. Inst. Stat. Math. 21, 215 (1969).

M. J. Schervish, Theory of Statistics (Springer, New York, NY, 1995).

B. Yadim, J. Ma, D. Girolami, M. Gu, and V. Vedral, Quantum Processes Which Do Not Use Coherence, Phys. Rev. X 6, 041028 (2016).

The relations $[\hat{X}_x, \hat{A}_y] = 0$ and Eq. (3) imply $P(y) = \langle \hat{A}_y \rangle_{\rho'} = \sum_{\gamma} \text{tr}(\hat{\Sigma}_x^{1/2} \hat{\rho} \hat{\Sigma}_x^{1/2} \hat{\Pi}_y) = \text{tr}(\hat{\rho} \hat{\Pi}_y \sum_{\gamma} \hat{\Sigma}_x) = P(y)$, using the identities $\text{tr}(\hat{a} \hat{b} \hat{c}) = \text{tr}(\hat{c} \hat{a} \hat{b})$ (\forall \hat{a}, \hat{b}, \hat{c}) and $1 = \sum_{\gamma} \hat{\Sigma}_x$.

E. Knill and R. Laflamme, Power of One Bit of Quantum Information, Phys. Rev. Lett. 81, 5672 (1998).

M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).

N. K. Langford et al., Demonstration of a simple entangling optical gate and its use in Bell-state analysis, Phys. Rev. Lett. 95, 210504 (2005).

N. Kiesel, C. Schmid, U. Weber, R. Ursin, and H. Weinfurter, Linear optics controlled-phase gate made simple, Phys. Rev. Lett. 95, 210505 (2005).

K. Okamoto, H. F. Hofmann, S. Takeuchi, and K. Sasaki, Demonstration of an optical quantum Controlled-NOT gate without path interference, Phys. Rev. Lett. 95, 210506 (2005).

E. Roccia, I. Gianani, L. Mancino, M. Sbroschia, F. Somma, M. G. Genoni, and M. Barbieri, Entangling Measurements for Multiparameter Estimation with Two Quubits, Quantum Sci. Technol. 3, 01LT01 (2018).

G. J. Pryde, J. L. O’Brien, A. G. White, S. D. Bartlett, and T. C. Ralph, Measuring a Photonic Qubit without Destroying It, Phys. Rev. Lett. 92, 190402 (2004).

L. Mancino, M. Sbroschia, E. Roccia, I. Gianani, F. Somma, P. Mataloni, M. Paternostro, and M. Barbieri, The Entropic Cost of Quantum Generalized Measurements, npj Quantum Inf. 4, 20 (2018).

L. Mancino, M. Sbroschia, E. Roccia, I. Gianani, V. Cimini, M. Paternostro, and M. Barbieri, Information-reality complementarity in photonic weak measurements, Phys. Rev. A 97, 062108 (2018).

M. Sawerwain and J. Wiśniewska, Quantum Coherence Measures for Quantum Switch, arXiv:1803.03321.

A. Venugopalan, S. Mishra, and T. Qureshi, Monitoring Decoherence via Measurement of Quantum Coherence, Physica A 516, 308 (2019).

The trace distance between $\hat{\rho}$ [Eq. (5)] and $\hat{\rho}'$ [Eq. (5) for $\gamma = 0$] reads $\|\hat{\rho} - \hat{\rho}'\| = 2\gamma \sqrt{p(1-p)}$. For $\theta = \pi/2$ in Eq. (6), we directly get $\Delta V = 4\gamma^2 p(1-p)$. 
