EQUIVARIANT COHOMOLOGY OF CERTAIN MODULI OF WEIGHTED POINTED RATIONAL CURVES

CHITRABHANU CHAUDHURI

1. Introduction

In [8] Hassett introduces and studies the moduli spaces of weighted pointed stable curves. A weighted pointed curve is a nodal curve with a sequence of smooth marked points, each assigned a rational number between 0 and 1. A subset of the marked points may coincide if the sum of their weights is at most 1.

The moduli spaces are connected, smooth and proper Deligne-Mumford stacks. In the special case of genus zero the moduli spaces are smooth projective varieties. Throughout this paper we work over \( \mathbb{C} \) as the base field and we always consider cohomology with \( \mathbb{C} \) coefficients.

Consider the weight data

\[
A(m, n) = \left( \begin{array}{c}
1, \ldots, 1, 1/n, \ldots, 1/n \\
\hline
m \\
\hline
n
\end{array} \right) \quad m + n \geq 3, \ m \geq 2.
\]

Let

\[
\overline{M}_{0,m|n} = \overline{M}_{0,A(m,n)}.
\]

\( \overline{M}_{0,m|n} \) parametrises nodal curves with \( m + n \) smooth marked points such that the first \( m \) marked points are distinct but any subset of the last \( n \) marked points can coincide. There is naturally an action of \( S_m \times S_n \) on \( \overline{M}_{0,m|n} \). Here \( S_m \) permutes the first \( m \) marked points and \( S_n \) permutes the last \( n \).

In this paper we study the induced action of \( S_m \times S_n \) on the cohomology of \( \overline{M}_{0,m|n} \) and calculate the equivariant Poincaré polynomial for some small values of \( m \) and \( n \).

Let \( M_{0,m|n} \) be the interior of the the moduli space, parametrizing only the smooth curves. We first derive the \( S_m \times S_n \) character on \( H^*(M_{0,m|n}) \), and write down a generating function for the characters. We then describe a recipe for calculating the generating function for the \( S_m \times S_n \) character of \( H^*(\overline{M}_{0,m|n}) \). This is achieved by analysing a spectral sequence relating the cohomology of \( M_{0,m|n} \) to that of \( \overline{M}_{0,m|n} \).

It should be noted that when \( n = 0 \), we simply get the moduli of stable rational curves with \( m \) marked points. In this case the equivariant cohomology was studied by Getzler [6].

In another direction when \( m = 2 \), the moduli spaces under consideration are the Losev-Manin spaces of [9]. The \( S_2 \times S_n \) action on the cohomology was determined in this case by Bergström and Minabe [3].

Finally Bergström and Minabe [2] give a recursive method for calculating the equivariant Poincaré polynomial of \( \overline{M}_{0,m|n} \) for all \( m \) and \( n \). However our method seems more direct. We use techniques developed by Getzler [6] and Getzler and Kapranov [7]. We adopt the notation \( \overline{M}_{0,m|n} \) from [3].

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2. Preliminaries on \( \overline{M}_{0,m|n} \)

Let \( A(m, n) \) be as in [1]. Following Hassett [8], \( \overline{M}_{0,m|n} \) is the moduli of weighted pointed stable curves of of genus zero corresponding to the weight data \( A(m, n) \).
When \( m > 3 \), \( \overline{M}_{0,m|0} \) is simply the moduli of stable rational curves with \( m \) marked points. We abbreviate it as \( \overline{M}_{0,m} \).

Denote by \( M_{0,m|n} \) the open subvariety parametrising the smooth curves.

2.1. The stable curves. An \( A(m,n) \)-stable curve \((C;p_1, \ldots, p_{m+n})\) is a nodal curve with smooth marked points \( p_i \). The marked points of \( C \) along with the nodes will be called special points. We shall call the first \( m \) marked points along with the nodes special points of type 1, whereas the last \( n \) marked points will be referred to as special points of type 2. The curve \( C \) satisfies the following,

- Arithmetic genus of \( C \) is 0.
- The points \( \{p_1, \ldots, p_m\} \) are all distinct.
- Any subset of the points \( \{p_{m+1}, \ldots, p_{m+n}\} \) can coincide, but these points are all distinct from \( \{p_1, \ldots, p_m\} \).
- Any irreducible component of \( C \) has at least 3 special points with at least 2 of type 1.

The varieties \( \overline{M}_{0,m|n} \) are smooth and projective and \( \overline{M}_{0,m|n}\backslash M_{0,m|n} \) is a divisor with normal crossings. Ceyhan \cite{Ceyhan} studies the cohomology of \( \overline{M}_{0,A} \), for any weight data \( A \). As a special case it follows that all the cohomology of \( \overline{M}_{0,m|n} \) is algebraic. This means that all the odd degree cohomology groups vanish and the even cohomology groups are isomorphic to the Chow groups

\[
H^{2i+1}(\overline{M}_{0,m|n}, \mathbb{Q}) = 0 \quad \text{and} \quad H^{2i}(\overline{M}_{0,m|n}, \mathbb{Q}) \cong A^i(\overline{M}_{0,m|n}, \mathbb{Q}).
\]

2.2. Dual graphs. A graph will be a triple \((F,V,\sigma)\). Where

1. \( F \) is the set of flags;
2. \( V \) is a partition of \( F \);
3. \( \sigma \) is an involution on \( F \).

The parts of \( V \) are the vertices of the graph. For \( v \in V \), let \( F(v) = \{ f \in F \mid f \in F \} \) be the flags incident on \( v \). The fixed points of \( \sigma \) are the leaves. The set of leaves will be denoted by \( L \), and those incident on a vertex \( v \) denoted as \( L(v) \). The two cycles of \( \sigma \) will be the edges of the graph and the set of edges denoted by \( E \).

Colouring of a graph \( G \) consists of a set \( X \) and a function \( c : F(G) \to X \) such that \( c(f) = c(\sigma f) \) for every flag \( f \). A colouring assigns a colour (an element of \( X \)) to each flag such that both flags of an edge have the same colour. It thus makes sense to talk about the colour of an edge.

Geometric realisation of a graph \( G \), denoted by \(|G|\), is a topological space. It is the quotient space of, the collection of intervals indexed by the flags of \( G \), by an equivalence relation.

\[
|G| = \frac{F(G) \times [0,1]}{\sim}
\]

Here \((f_1,0) \sim (f_2,0)\) if the flags \( f_1, f_2 \) are incident on the same vertex and \((f,1) \sim (f',1)\) if the flags \( f, f' \) are part of an edge.

A tree \( T \) is a graph such that \(|T|\) is connected and simply connected.

The dual graph of an \( A(m,n) \)-stable curve is a tree coloured by \( \{1,2\} \). The tree has one flag for each marked point and two for each node. For every irreducible component it has a vertex. The marked points correspond to the leaves and the nodes correspond to the edges. The flags corresponding to the special points of type 1 have colour 1 where as the flags corresponding to the special points of type 2 have colour 2. Further the leaves are numbered 1 to \( m+n \) according to the marked point it represents.

2.3. Strata of \( \overline{M}_{0,m|n} \). It is clear that the dual graphs of \( A(m,n) \)-stable curves have to satisfy certain constraints. Let \( T \) be such a dual graph. For any vertex \( v \in V(T) \) let \( F_1(v) \) be the flags of colour 1 and \( F_2(v) \) the flags of colour 2. Then we must have \(|F(v)| \geq 3\) and \(|F_1(v)| \geq 2\). Let us call such trees \( A(m,n) \)-stable and denote the isomorphism classes of such trees by \( T(m,n) \).
For any $T \in \mathcal{T}(m,n)$ let $M(T)$ be the subvariety of $\overline{\mathcal{M}}_{0,m|n}$ parametrising curves whose dual graphs are isomorphic to $T$. Let $\overline{M}(T)$ be the closure. It is clear that (see Ceyhan [4, Section 3])

$$ M(T) \cong \prod_{v \in V(T)} M_{0,\#F_1(v)\#F_2(v)} \quad \text{and} \quad \overline{M}(T) \cong \prod_{v \in V(T)} \overline{M}_{0,\#F_1(v)\#F_2(v)}. $$

The codimension of $\overline{M}(T)$ is equal to the number of edges $|E(T)|$ of $T$.

We have a stratification by dual graphs

$$ \overline{\mathcal{M}}_{0,m|n} = \bigsqcup_{T \in \mathcal{T}(m,n)} M(T). $$

### 3. Symmetric Group Representations

#### 3.1. Symmetric functions. For results and notation of this section we refer to Macdonald [10]. Let $\Lambda = \lim_{n \to \infty} \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ be the ring of symmetric functions. It is well known that

$$ \Lambda \otimes \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots] $$

where $p_k = \sum_{i=1}^{\infty} x_i^k$ are the power sums. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n$, which we denote by $\lambda \vdash n$; define $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$. For an $S_n$ representation $V$ we define the symmetric function

$$ ch_n(V) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}_V(\sigma)p_{\lambda(\sigma)}, $$

here $\lambda(\sigma)$ is the partition corresponding to the cycle decomposition of $\sigma$.

The irreducible representations of $S_n$ are indexed by partitions of $n$. For $\lambda \vdash n$ let $V_\lambda$ be the corresponding irreducible representation. The Schur functions also indexed by partitions of $n$ are defined as

$$ s_\lambda = ch_n(V_\lambda). $$

Schur functions $\{s_\lambda \mid \lambda \vdash n, n \geq 1\}$ form an additive basis of $\Lambda$. There are also the elementary symmetric functions $e_n = s_{1^n}$ and the complete symmetric functions $h_n = s_n$.

There is an associative product $\circ$ on $\Lambda$ called plethysm. It is characterised by the fact that

$$ ch_n \left( \text{Ind}_{S_k \wr S_n}^{S_k \wr S_n} V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_2 \right) = ch_k(V_1) \circ ch_n(V_2), $$

where $S_k \wr S_n$ is the wreath product $S_k \times (S_n)^k$, $V_1$ is a representation of $S_k$ and $V_2$ is a representation of $S_n$.

Let $\Lambda^{(2)} = \Lambda \otimes \Lambda$. We denote the symmetric functions in the first tensor factor by the superscript (1) and those in the second tensor factor by the superscript (2).

For $V$ a representation of $S_m \times S_n$ we define

$$ ch_{m|n}(V) = \frac{1}{m! \times n!} \sum_{(\sigma,\tau) \in S_m \times S_n} \text{Tr}_V(\sigma,\tau)p^{(1)}_{\lambda(\sigma)} p^{(2)}_{\lambda(\tau)} \in \Lambda^{(2)}. $$

We shall need the following result later on.

**Proposition 3.1.** Let $W$ be any representation of $S_n$ and $D$ the following differential operator on $\Lambda$

$$ D = p_1 \frac{\partial}{\partial p_1} - 1, $$

then

$$ ch_n \left( W \otimes V_{(n-1,1)} \right) = D \ ch_n(W). $$

$V_{(n-1,1)}$ is the irreducible representation corresponding to the partition $(n-1,1)$ and often referred to as the standard representation of $S_n$. 
Proof. Let \( \text{fix}(\sigma) \) denote the number of fixed points of \( \sigma \in S_n \). Recall that \( \text{Tr}_{V(n-1,1)}(\sigma) = \text{fix}(\sigma) - 1 \). Also note that \( \lambda(\sigma) = (1^{\text{fix}(\sigma)}, 2^{\text{fix}(\sigma)}, \ldots) \); so \( p_{\lambda(\sigma)} = p_1^{\text{fix}(\sigma)} p_2^{\text{fix}(\sigma)} \cdots p_n^{\text{fix}(\sigma)} \). Thus
\[
p_1 \frac{\partial p_{\lambda(\sigma)}}{\partial p_1} = \text{fix}(\sigma)p_{\lambda(\sigma)}.
\]
Hence
\[
\text{ch}_n (W \otimes V_{(n-1,1)}) = \frac{1}{n!} \sum_{\sigma \in S_n} (\text{fix}(\sigma) - 1) \text{Tr}_W(\sigma)p_{\lambda(\sigma)} \\
= \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}_W(\sigma) \left( p_1 \frac{\partial p_{\lambda(\sigma)}}{\partial p_1} - p_{\lambda(\sigma)} \right) = p_1 \frac{\partial \text{ch}_n(W)}{\partial p_1} - \text{ch}_n(W).
\]
\[\square\]

3.2. \( S \) modules. An \( S \) module (as in [6] §1) \( V \) is a sequence of graded vector spaces \( \{V(n) \mid n \in \mathbb{N}\} \) with an action of \( S_n \) on \( V(n) \). The characteristic of an \( S \) module is defined as a symmetric series in \( \Lambda[t] \)
\[
\text{ch}(V) = \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} (-t)^i \text{ch}_n(V^i(n)).
\]
Here \( V^i(n) \) is the \( i \)-th graded component of \( V(n) \).

Similarly an \( S^2 \) module \( W \) is a collection of graded vector spaces \( \{W(m,n) \mid (m,n) \in \mathbb{N}^2\} \), with an action of \( S_m \times S_n \) on \( W(m,n) \). We define the characteristic in an analogous way
\[
\text{ch}(W) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i \in \mathbb{Z}} (-t)^i \text{ch}_{m,n}(W^i(m,n)) \in \Lambda^2[\mathbb{Z}].
\]
In the case of an ungraded \( S^2 \) module \( W \) we write the characteristic as \( \text{ch}(W) \). We define the \( S^2 \) module \( \mathbb{T}W \) in the following way
\[
\mathbb{T}W(m,n) = \bigoplus_{T \in \mathbb{T}(m,n)} W(T).
\]
Here \( \mathbb{T}(m,n) \) are the isomorphism classes of \( A(m,n) \) stable trees and
\[
W(T) = \bigotimes_{v \in V(T)} W(F(v)).
\]
For a more detailed discussion see Getzler and Kapranov [7].

3.3. Partial Legendre transform. Let \( \text{rk} : \Lambda^{(2)} \rightarrow \mathbb{Q}[x,y] \) be the homomorphism such that \( \text{rk} (p_1^{(1)}) = x, \text{rk} (p_1^{(2)}) = y \) and \( \text{rk} (p_n^{(i)}) = 0 \) for \( n > 1 \) and \( i = 1, 2 \). Thus if \( V \) is a representation of \( S_n \times S_m \) then
\[
\text{rk}(\text{ch}_{m,n}(V)) = \frac{\dim V}{m!n!} x^m y^n.
\]
Let \( \mathbb{Q}[x,y]_* \) be the power series of the form \( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} x^i y^j \) where \( a_{2,0} \neq 0 \) and \( a_{i,j} = 0 \) if \( i < 2 \) or \( i + j < 3 \). Let \( \Lambda^{(2)} = \text{rk}^{-1} \mathbb{Q}[x,y]_* \).

To define the partial Legendre transform we first define a variant of plethysm; \( \circ (1) \) which is an associative product on \( \Lambda^{(2)} \):
\[
\begin{align*}
(1) & \quad f \mapsto f \circ (1) g \text{ is a homomorphism } \Lambda^{(2)} \rightarrow \Lambda^{(2)}, \text{ for any } g \in \Lambda^{(2)} \\
(2) & \quad g \mapsto p_n^{(i)} \circ (1) g \text{ is a homomorphism } \Lambda^{(2)} \rightarrow \Lambda^{(2)}, \\
(3) & \quad p_n^{(1)} \circ (1) p_k^{(i)} = p_{nk}^{(i)} \text{ and } p_n^{(2)} \circ (1) p_k^{(i)} = p_n^{(i)}.
\end{align*}
\]
Proposition 4.1. Let $W$ be an ungraded $S^2$ module such that $W(m,n) = 0$ if $m < 2$ or $m + n < 3$. $F = e^{(1)}_2 - \text{ch}(W)$ and $G = h^{(1)}_2 + \text{ch}(TW)$ are elements of $\Lambda_*(^2)$ and $G = \mathcal{L}^{(1)}F$.

The proof of this proposition is essentially the same as the proof of Theorem 7.17 of [7]. One can also look at Theorem 5.8 of [5] for a different proof.

4. Cohomology of the Interior

In this section we study the cohomology of the interior $M_{0,m|n}$. It is easy to see that

$$M_{0,m|n} \cong \left( \mathbb{P}^1 \right)^{m+n} \left( \bigcup_{i=1}^{m} \bigcup_{j=i+1}^{n} \Delta_{i,j} \right) / \text{PGL}(2, \mathbb{C}),$$

where $\text{PGL}(2, \mathbb{C})$ acts diagonally and $\Delta_{i,j} = \{(z_1, \ldots, z_{m+n}) \mid z_i = z_j\}$.

Proposition 4.1. When $m \geq 3$,

$$H^*(M_{0,m|n}) \cong H^*(M_{0,m}) \otimes H^*(P^n_m)$$

where $P_m = \mathbb{P}^1 \setminus \{1, \ldots, m\}$ is the $m$ punctured projective plane. Moreover this decomposition respects the action of $S_m \times S_n$.

The mixed Hodge structure on $H^i(M_{0,m|n})$ is pure of weight $2i$.

Proof. Consider the fiber bundle $M_{0,m|(n+1)} \to M_{0,m|n}$ with fiber $P_m$. $P_m$ is homotopic to a wedge of circles, hence a one dimensional CW complex. The fundamental group of the base acts trivially on the fibers, (see Arnold [1]), hence in the Leray spectral sequence associated to the fibration we have

$$E_2^{pq} \cong H^p(M_{0,m|n}) \otimes H^q(P_m).$$

Moreover the fiber bundle has a section given by

$$z_{m+n+1} = \frac{z_1 + \cdots + z_m}{m} + 2\left( \max_{1 \leq k,l \leq m} |z_k - z_l| \right) + 1.$$ 

It then follows that the only possible higher differential $d_2$ is trivial and we have $H^*(M_{0,m|(n+1)} \cong H^*(M_{0,m|n}) \otimes H^*(P_m)$. This completes the proof by induction on $n$.

The statement on the mixed Hodge structure of $H^i(M_{0,m|n})$ follows from the fact that $M_{0,m|n}$ is isomorphic to a complement of hyperplanes in a projective spaces. This can be seen from description [3].

From the previous proposition it follows that the Poincaré polynomial of $M_{0,m|n}$, $P_{M_{0,m|n}}(t)$, is the product of the Poincaré polynomials of $M_{0,m}$ and $P^n_m$. From Getzler [6] Section 5.6] we know that $P_{M_{0,m}}(t) = (1 - 2t)(1 - 3t) \cdots (1 - (m - 2)t)$. It is easy to see that $P_{P^n_m}(t) = (1 - (m - 1)t)^n$. Thus

$$P_{M_{0,m|n}}(t) = (1 - (m - 1)t)^n \prod_{k=2}^{m-2} (1 - kt).$$

Proposition [11] gives a clear description of the $H^*(M_{0,m|n})$ as a representation of $S_m \times S_n$.

First note that Getzler [6] determines completely the action of $S_m$ on $H^*(M_{0,m})$, whereas $S_n$ acts on it trivially.
Let $C_n$ be the vector space generated by the letters $\{x_1, \ldots, x_n\}$. $S_n$ acts on it by permuting the letters. $C_n$ is the direct sum of the standard representation and the trivial representation i.e. $C_n = V_{(n-1,1)} \oplus V_{(n)}$.

**Proposition 4.2.** We have the following description of the $S_m \times S_n$ action on the cohomology of $P^n_m$.

$$H^k(P^n_m) \cong \left( \bigotimes^k V_{(m-1,1)} \right) \boxtimes (\wedge^k C_n)$$

$$\cong \begin{cases} V_{(m)} \boxtimes V_{(n)} , & k = 0 \\ \left( \bigotimes^k V_{(m-1,1)} \right) \boxtimes \left( V_{(n-k,1^k)} \oplus V_{(n-k+1,1^{k-1})} \right) , & 0 < k < n \\ (\bigotimes^n V_{(m-1,1)}) \boxtimes V_{(1^n)} , & k = n. \end{cases}$$

**Proof.** $S_m$ acts on $P_n$ by permuting the punctures, so $H^1(P_n)$ is the standard representation $V_{m-1,1}$.

On the other hand $S_n$ acts on $P^n_m$ by permuting the factors, thus $H^1(P^n_m) \cong V_{m-1,1} \boxtimes C_n$ as a representation of $S_m \times S_n$. For $k > 1$, $H^k(P^n_m) \cong \left( \bigotimes^k H^1(P_n) \right) \boxtimes (\wedge^k C_n)$.

The second isomorphism follows from the decomposition $C_n = V_{(n-1,1)} \oplus V_{(n)}$. Thus for $k < n$ we have $\wedge^k C_n = \left( \bigotimes^k V_{(n-1,1)} \right) \oplus (V_{(n)} \oplus \wedge^k V_{(n-1,1)})$ and $\wedge^n C_n = V_{(1^n)}$. Finally it is a fact that $\wedge^k V_{(n-1,1)} = V_{(n-k,1^k)}$.

The Propositions 4.1 and 4.2 together give us the following decomposition for $m \geq 3$,

$$H^k(M_{0,m|n}) = \bigoplus_{l=0}^k H^k-l(M_{0,m}) \otimes H^l(P^n_m)$$

$$= \bigoplus_{l=0}^k \left( H^k-l(M_{0,m}) \otimes \left( \bigotimes^l V_{(n-1,1)} \right) \right) \boxtimes \left( \wedge^l C_n \right).$$

In (4) we treat $H^k-l(M_{0,m})$ as just a representation of $S_m$.

Hence we have the following relation for $m \geq 3$

$$H^*(M_{0,m|n}) = \bigoplus_{l=0}^n \left( H^*(M_{0,m}) \otimes H^l(P^n_m) \right)$$

$$= \bigoplus_{l=0}^n \left( H^*(M_{0,m}) \otimes \left( \bigotimes^l V_{(n-1,1)} \right) \right) \boxtimes \left( \wedge^l C_n \right).$$

Let $\mathcal{M}$ be the $S$ module

$$\mathcal{M}(n) = \begin{cases} H^*(M_{0,n}) , & n \geq 3, \\ 0 , & n < 3. \end{cases}$$

Let $m_n = \text{ch}_t(\mathcal{M}(n))$ and $m = \sum_{n=1}^\infty m_n = \text{ch}_t(\mathcal{M})$.

Let $\mathcal{G}$ be the $S^2$ module

$$\mathcal{G}(m, n) = \begin{cases} 0 , & m < 2 \text{ or } m + n < 3 \\ H^*(M_{0,m|n}) , & \text{otherwise} \end{cases}$$

Let $\mathcal{D}$ be the differential operator as in Proposition 3.1. From (5) it follows that when $k \geq 3$

$$\text{ch}_t(\mathcal{G}(k, n)) = \mathcal{M}_k^{(1)} s_n^{(2)} - t \mathcal{D} \mathcal{M}_k^{(1)} \left( s_n^{(2)} + s_{n-1,1}^{(2)} \right) + \cdots + (-t)^n \left( \mathcal{D}^n \mathcal{M}_k^{(1)} \right) s_1^{(2)}.$$

$\overline{M}_{0,2|n}$ are the Losev-Manin spaces and were extensively studied in [9]. Note that

$$M_{0,2|n} \cong (\mathbb{C}^\times)^n / \mathbb{C}^\times$$
where the quotient is taken under the diagonal action. It follows that $H^1(M_{0,2|n}) \cong V_{(1,1)} \otimes V_{(n-1,1)}$ (see [3 Lemma 3.3]). As before $H^k(M_{0,2|n}) \cong \wedge^k H^1(M_{0,2|n})$. Thus

$$H^k(M_{0,2|n}) \cong \begin{cases} V_{(2)} \otimes V_{(n-k,1)} & k < n \text{ even} \\ V_{(1,2)} \otimes V_{(n-k,1)} & k < n \text{ odd} \\ 0 & k \geq n. \end{cases}$$

Hence it follows that

$$\text{ch}_t(G(2, n)) = s_2^{(1)} s_n^{(2)} - t s_1^{(1)} s_{n-1,1} + \ldots = \sum_{k=0}^{n-1} (-t)^k \left(D^k s_2^{(1)} \right) s_{n-k,1}^{(2)}.$$  \hspace{1cm} (9)

Adding up $\text{ch}_t(G(k,l))$ for all $k, l$ we get

$$\text{ch}_t(G) = m^{(1)} + \left( m^{(1)} + s_2^{(1)} \right) \sum_{n=1}^{\infty} s_n^{(2)}$$

$$+ \sum_{k=1}^{\infty} (-t)^k \left( D^k m^{(1)} \sum_{n=k}^{\infty} s_{n-k+1,1}^{(2)} + D^k \left( m^{(1)} + s_2^{(1)} \right) \sum_{n=k+1}^{\infty} s_{n-k,1}^{(2)} \right).$$  \hspace{1cm} (10)

5. Cohomology of $\overline{M}_{0,m|n}$

In this section we shall determine the action of $S_m \times S_n$ on the cohomology of $\overline{M}_{0,m|n}$. To do this let us introduce the $S^2$ module $\mathcal{W}$,

$$\mathcal{H}(m,n) = \begin{cases} 0, & m < 2 \text{ or } m + n < 3 \\ H^\ast(\overline{M}_{0,m|n}), & \text{otherwise}. \end{cases}$$  \hspace{1cm} (11)

We shall derive a formula relating $\text{ch}_t(G)$ (see (9)) and $\text{ch}_t(\mathcal{H})$ using the partial Legendre transform.

Let $X$ be an algebraic variety and $\emptyset \subset X_0 \subset \ldots \subset X_n = X$ a filtration on it by closed subvarieties $X_p \subset X$. Then there is a spectral sequence in cohomology with compact support (see Petersen [11 Section 1]),

$$E_1^{p,q} = H_c^{p+q}(X_p \setminus X_{p-1}) \Longrightarrow H_c^{p+q}(X).$$

The differentials of this spectral sequence are compatible with the mixed Hodge structures. Further if a finite group $G$ acts on $X$ and keeps each $X_p$ invariant then $E_1^{p,q}$ has an action of $G$ and the differentials $d_j$ are $G$ equivariant.

In our situation let $X = \overline{M}_{0,m|n}$ and $X_p$ be the union of all strata of dimension at most $p$

$$X_p = \bigsqcup_{T \in \Gamma(m,n)} \overline{M}(T).$$

Clearly

$$X_p \setminus X_{p-1} = \bigsqcup_{T \in \Gamma(m,n)} \overline{M}(T).$$

Thus

$$E_1^{p,q} = \bigoplus_{T \in \Gamma(m,n)} H_c^{p+q}(M(T)).$$  \hspace{1cm} (12)

From Proposition 4.1 and Poincaré duality it follows that the mixed Hodge structure on $H_c^p(\overline{M}_{0,m|n})$ is pure of weight $2(i - m - n + 3)$. This implies that $E_1^{p,q}$ has a pure Hodge structure of weight $2q$. Hence the Spectral sequence collapses in the $E_2$ page

$$E_2^{p,q} \cong E_\infty^{p,q}.$$
Moreover, from the fact that all the cohomology of $\overline{M}_{0,m|n}$ is algebraic it follows that
\begin{equation}
E_2^{p,q} \cong H^{2p}(\overline{M}_{0,m|n}) \quad \text{and} \quad E_2^{p,q} = 0 \text{ if } p \neq q.
\end{equation}
Thus there is a resolution
\begin{equation}
H^{2p}(\overline{M}_{0,m|n}) \to \bigoplus_{T \in T(m,n)} H^{2p}_c(M(T)) \to \cdots \to H^{n+n-3+p}_c(M_{0,m|n}).
\end{equation}

**Theorem 5.1.** Let
\[ F = t^{-6} \text{ch}_t(G) \mid_{t^{k-2} \mapsto t^{2n+p}} , \]
then $h_2^{(1)} + \text{ch}_t(H) = \mathfrak{S}^{(1)}(e_2^{(1)} - F)$.

**Remark.** Note that $(e_2^{(1)} - F) \in \Lambda^2_*(t)$. More over $c^{(1)}$ extends to $\Lambda^2(t)$ in a natural way: $p_{\ell}^{(1)} = t^\ell$ and $p_{\ell}^{(2)} = p_{\ell}^{(2)}$. Hence $\mathfrak{S}(1)$ makes sense on $\Lambda^2(t)$.

**Proof.** As in [6, section 5.8] we shall consider (graded) $S^2$ modules $V$ with a further $\mathbb{Z}/2$-grading $V = V(0) \oplus V(1)$. In this case define
\[ \text{ch}_t(V) = \text{ch}_t(V(0)) - \text{ch}_t(V(1)) . \]
By Poincaré duality $H^k_c(M_{0,m|n}) \cong H^{2n+2n-6-k}(M_{0,m|n})^\vee \otimes C(-m - n + 3)$ where $C(-\ell)$ is the $\ell$-fold tensor power of the dual of the Tate Hodge structure. Thus $H^k_c(M_{0,m|n})$ has pure Hodge structure of weight $2(k - m - n + 3)$.

Define the $\mathbb{Z}/2$-graded $S^2$ module $V$ as follows
\[ V(m,n) = 0 \text{ if } m < 2 \text{ or } m + n < 3 , \]
otherwise
\[ V(0)(m,n) = \bigoplus_{k=0}^{\infty} H^{2k}_c(M_{0,m|n}) \]
\[ V(1)(m,n) = \bigoplus_{k=0}^{\infty} H^{2k+1}_c(M_{0,m|n}) . \]

Here we consider $H^k_c(M_{0,m|n})$ with weight grading for the mixed Hodge structure on it, that is $H^k_c(M_{0,m|n})$ is the $2(k - m - n + 3)$ graded component. Then
\[ F = \text{ch}_t(V) . \]
The construction $T$ of Section 3.2 extends naturally to $\mathbb{Z}/2$-graded $S^2$ modules (tensor product of odd and odd is even, even and even is even where as that of odd and even is odd). Proposition 3.2 generalises to the case of $\mathbb{Z}/2$-graded $S^2$ modules.

If we add up all the terms of the spectral sequence (12) placing $E_1^{p,q}$ in bi-degree $2q, (p + q) \mod 2$ we get the graded vector space $TV(m,n)$. The differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ gives a differential on $TV(m,n)$ and the resolution (14) shows that $H^*(\overline{M}_{0,m|n})$ is the homology of of the complex $(TV(m,n), d_1)$.

Hence $\text{ch}_t(\overline{M}_{0,m|n}) = \text{ch}_t(TV(m,n))$. This completes the proof. \hfill \Box

**Appendix A. Calculations**

Recall the $S^2$ modules $G$ from (8) and $H$ from (11). In this section we compute the first few terms of the characteristics of $G$ and $H$ and list them in Table 1 and Table 2 respectively.

Formula (11) gives a recipe for calculating $\text{ch}_t(G)$ from $\text{ch}_t(M)$. The $S$ module $M$ was defined in (9). The action of $S_n$ on $H^*(M_{0,n})$ was calculated in Getzler [6] (see Theorem 5.7). Let $\mu$ be the Möbius function, and let $R_n(t) = (1/n) \sum_{d \mid n} \mu(n/d) t^d$. Further let $\kappa$ be the linear
operator on \( \Lambda \) which is 0 on the 0th, 1st and 2nd graded components of \( \Lambda \) and identity on the rest. Then

\[
\text{ch}_t(\mathcal{M}) = \kappa \left( \frac{1 + tp_1}{1 - t^2} \prod_{n=1}^\infty (1 + t^n p_n) R_n(t) \right).
\]

Thus using (10) we can calculate the first few terms of \( \text{ch}_t(\mathcal{G}) \). Of course \( \text{ch}_t(\mathcal{G}(k, n)) \) starts to be interesting when \( k \geq 3 \) and \( n \geq 2 \). In Table 1 we list these terms for \( k + n \leq 6 \).

Using Theorem 5.1 we can in principle determine \( \text{ch}_t(\mathcal{H}) \). The theorem gives a fixed point formula and first few terms of \( \text{ch}_t(\mathcal{H}) \) can be obtained from \( \text{ch}_t(\mathcal{G}) \) by performing several iterations. Again the terms \( \text{ch}_t(\mathcal{H}(k, n)) \) for \( n = 1 \) can be easily computed and are uninteresting. We list the terms corresponding to \( k + n \leq 6 \) and \( n > 1 \) in Table 2.

The calculations involving symmetric functions were done using the Maple package SF [12] by Stembridge.
\[
\begin{array}{|c|c|}
\hline
(m, n) & \text{ch}_t \left( H^*(M_{0,m|n}) \right) \\
\hline
(3, 2) & s_3^{(1)} s_2^{(2)} - ts_2^{(1)} \left( s_2^{(2)} + s_3^{(2)} \right) + t^2 \left( s_3^{(1)} + s_2^{(1)} + s_3^{(1)} \right) s_1^{(2)} \\
(3, 3) & s_3^{(1)} s_3^{(2)} - ts_2^{(1)} \left( s_3^{(2)} + s_3^{(2)} \right) + t^2 \left( s_3^{(1)} + s_2^{(1)} + s_3^{(1)} \right) \left( s_2^{(1)} + s_1^{(1)} \right) - t^3 \left( s_3^{(1)} + 3s_2^{(2)} + s_1^{(1)} \right) s_1^{(2)} \\
(4, 2) & s_4^{(1)} s_2^{(2)} - t \left( s_2^{(1)} s_2^{(2)} + s_3^{(1)} \left( s_2^{(2)} + s_1^{(2)} \right) \right) + t^2 \left( s_4^{(1)} + s_2^{(1)} \right) s_1^{(2)} + \left( s_3^{(1)} + s_2^{(1)} \right) \left( s_2^{(1)} + 2s_1^{(2)} \right) - t^3 \left( s_4^{(1)} + 2s_2^{(1)} + 2s_3^{(1)} + 2s_2^{(1)} + s_1^{(1)} \right) s_1^{(2)} \\
\hline
\end{array}
\]

**Table 1.** Equivariant Poincaré polynomial for the interior

\[
\begin{array}{|c|c|}
\hline
(m, n) & \text{ch}_t \left( H^*(\overline{M}_{0,m|n}) \right) \quad \text{Poincaré Polynomial} \\
\hline
(2, 2) & (1 + t^2) s_2^{(1)} s_2^{(2)} \quad 1 + t^2 \\
(2, 3) & (1 + t^4) s_2^{(1)} s_3^{(2)} + t^2 \left( s_2^{(1)} s_3^{(2)} + s_2^{(1)} s_1^{(2)} \right) \quad 1 + 4t^2 + t^4 \\
(3, 2) & (1 + t^4) s_2^{(1)} s_2^{(2)} + t^2 \left( s_3^{(1)} s_2^{(2)} + s_3^{(1)} s_1^{(2)} \right) + s_2^{(1)} s_1^{(2)} \quad 1 + 5t^2 + t^4 \\
(2, 4) & (1 + t^6) s_2^{(1)} s_4^{(2)} + (t^2 + t^4) \left( s_2^{(1)} \left( 2s_4^{(2)} + s_3^{(2)} s_2^{(2)} + s_3^{(1)} s_2^{(2)} \right) + s_4^{(1)} \left( s_4^{(2)} + s_3^{(1)} \right) \right) \quad 1 + 11t^2 + 11t^4 + t^6 \\
(3, 3) & (1 + t^6) s_3^{(1)} s_2^{(2)} + (t^2 + t^4) \left( s_3^{(1)} \left( 3s_2^{(2)} + 2s_2^{(1)} s_2^{(2)} + s_3^{(1)} \right) + 2s_2^{(2)} + s_2^{(1)} \right) \quad 1 + 15t^2 + 15t^4 + t^6 \\
(4, 2) & (1 + t^6) s_4^{(1)} s_2^{(2)} + (t^2 + t^4) \left( s_4^{(1)} \left( 4s_2^{(2)} + s_2^{(2)} + s_3^{(1)} \right) + 2s_2^{(2)} + s_2^{(1)} + s_2^{(2)} s_2^{(2)} \right) \quad 1 + 16t^2 + 16t^4 + t^6 \\
\hline
\end{array}
\]

**Table 2.** Equivariant Poincaré polynomial of \( \overline{M}_{0,m|n} \)
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