ON THE PRODUCT IN NEGATIVE TATE COHOMOLOGY FOR
FINITE GROUPS

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Abstract. Our aim in this paper is to give a geometric description of the cup
product in negative degrees of Tate cohomology of a finite group with integral
coefficients. By duality it corresponds to a product in the integral homology
of
\[ H_n(BG, \mathbb{Z}) \otimes H_m(BG, \mathbb{Z}) \rightarrow H_{n+m+1}(BG, \mathbb{Z}) \]
for \( n, m > 0 \). We describe this product as join of cycles, which explains the shift
in dimensions. Our motivation came from the product defined by Kreck using
stratifold homology. We then prove that for finite groups the cup product in
negative Tate cohomology and the Kreck product coincide. The Kreck product
also applies to the case where \( G \) is a compact Lie group (with an additional
dimension shift).

Introduction

For a finite group \( G \) one defines Tate cohomology with coefficients in a \( \mathbb{Z}[G] \)
module \( M \), denoted by \( \hat{H}^*(G, M) \). This is a multiplicative theory:
\[ \hat{H}^n(G, M) \otimes \hat{H}^m(G, M') \rightarrow \hat{H}^{n+m}(G, M \otimes M') \]
and the product is called cup product. For \( n > 0 \) there is a natural isomorphism
\( H^n(G, M) \rightarrow \hat{H}^n(G, M) \), and for \( n < -1 \) there is a natural isomorphism
\( \hat{H}^n(G, M) \rightarrow H_{-n-1}(G, M) \). We restrict ourselves to coefficients in the trivial
module \( \mathbb{Z} \). In this case, \( \hat{H}^*(G, \mathbb{Z}) \) is a graded ring. Also, in this case the
group cohomology and homology are actually the cohomology and homology of
a topological space, namely \( BG \), the classifying space of principal \( G \) bundles -
\( H^n(G, \mathbb{Z}) \cong H^n(BG, \mathbb{Z}) \) and \( H_n(G, \mathbb{Z}) \cong H_n(BG, \mathbb{Z}) \). Combining this with the
isomorphism we had before \( \hat{H}^n(G, \mathbb{Z}) \rightarrow H_{-n-1}(G, \mathbb{Z}) \) for \( n < -1 \) we get a product
\( H_n(BG, \mathbb{Z}) \otimes H_m(BG, \mathbb{Z}) \rightarrow H_{n+m+1}(BG, \mathbb{Z}) \) for \( n, m > 0 \). Note the dimension
shift. This product, with coefficients in a field of characteristics \( p \) rather than \( \mathbb{Z} \),
was studied in \[ B-C \]. Our aim in this paper is to give a geometric description of this
product. We give a rather concrete description in singular homology that involves
the join of cycles, and that explains the shift in dimension.

Our motivation came from a geometric description of \( H_*(G, \mathbb{Z}) \) which appears in
\[ K2 \] and the product defined by Kreck using stratifold homology. We then prove
that the cup product in negative Tate cohomology and the Kreck product coincide.
An advantage in Kreck theory is that it holds also for compact Lie groups giving a product:
\[ H_n(BG, \mathbb{Z}) \otimes H_m(BG, \mathbb{Z}) \rightarrow H_{n+m+1+\dim(G)}(BG, \mathbb{Z}) \]
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Tate cohomology

Remark. In this paper $R$ is assumed to be a ring with unit, not necessarily commutative, and all modules are assumed to be left $R$-modules unless stated otherwise. The group $G$ is assumed to be finite unless stated otherwise.

We start by defining Tate cohomology and the cup product as appears in [C]. To do so we introduce the language taken from the stable module category. We will not get into details, for a formal treatment the reader is referred to the appendix.

Let $M, N$ be two $R$-modules, denote by $\text{Hom}_R(M, N)$ the quotient of $\text{Hom}_R(M, N)$ by the maps that factor through some projective module.

**Definition 1.** Given an $R$-module $M$, denote by $\Omega^k M$ the following module:

Take any partial projective resolution of $M$, $P_{k-1} \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} P_0 \rightarrow M$ then $\Omega^k M = \ker(P_{k-1} \xrightarrow{d_{k-1}} P_{k-2} \ldots P_0 \rightarrow M)$. If $k = 1$ we simply denote it by $\Omega M$. This module clearly depends on the choice of the resolution. Nevertheless, as proved in the appendix, the modules $\text{Hom}_R(\Omega^k M, \Omega^j N)$ do not depend on the choice of resolutions i.e., they are well defined up to canonical isomorphisms. If we would like to stress the dependency on $P$ we would use the notation $\Omega^k_P M$.

Note that there is a natural map $\Psi : \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(\Omega M, \Omega N)$.

**Definition 2.** The Tate cohomology of $G$ with coefficients in a $\mathbb{Z}[G]$ module $M$ is given by:

$\widehat{H}^n(G, M) = \text{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M) = \lim_{m} \text{Hom}_{\mathbb{Z}[G]}(\Omega^{n+m} \mathbb{Z}, \Omega^m M)$

where $\mathbb{Z}$ is the trivial $\mathbb{Z}[G]$-module (if $n < 0$ we start this sequence from $m = -n$).

In our case, where $G$ is finite, we have the following proposition which is proved in the appendix:

**Proposition 3.** If $G$ is a finite group and $M$ is a $\mathbb{Z}[G]$-module which is projective as a $\mathbb{Z}$-module then the homomorphism $\Psi : \text{Hom}_{\mathbb{Z}[G]}(M, N) \rightarrow \text{Hom}_{\mathbb{Z}[G]}(\Omega M, \Omega N)$ is an isomorphism.

Therefore, since $\mathbb{Z}$ and $\Omega^k \mathbb{Z}$ are projective as $\mathbb{Z}$-modules this limit equals to

$\widehat{H}^n(G, M) = \text{Hom}_{\mathbb{Z}[G]}(\Omega^n \mathbb{Z}, M)$

if $n \geq 0$ or

$\widehat{H}^n(G, M) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{-n} M)$

if $n < 0$. Our main interest will be the second case, especially when $M = \mathbb{Z}$.

**Example 4.** $\widehat{H}^{-1}(G, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega \mathbb{Z})$. Take the following exact sequence

$0 \rightarrow I \rightarrow \mathbb{Z}[G] \xrightarrow{f} \mathbb{Z} \rightarrow 0$ where the map $f$ is the augmentation map and $I$ is the augmentation ideal, so $I = \Omega \mathbb{Z}$. Therefore $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, I) = \{0\}$

so $\widehat{H}^{-1}(G, \mathbb{Z}) = \{0\}$.

Let $G$ be a finite group. We construct a natural isomorphism $\widehat{H}^{-n-1}(G, \mathbb{Z}) \rightarrow H_n(G, \mathbb{Z})$ for $n \geq 1$. Before that we prove a small lemma.
Lemma 5. Let $G$ be a finite group and $P$ a projective $\mathbb{Z}[G]$-module, then for every element $x \in P$ we have:

1) $x \in P^G \iff \exists y \in P, x = Ny$

2) $y \otimes 1 = y' \otimes 1 \in P \otimes_{\mathbb{Z}[G]} \mathbb{Z} \iff Ny = Ny'$

Where $P^G$ are the invariants of $P$ under the action of $G$, $N$ is the norm homomorphism defined by multiplication by the element $N = \sum g \in \mathbb{Z}[G]$.

Proof. For every $\mathbb{Z}[G]$-module $M$ the following sequence is exact:

$$0 \to \tilde{H}^{-1}(G, M) \to H_0(G, M) \to H^0(G, M) \to \tilde{H}^0(G, M) \to 0$$

where the map $H_0(G, M) \to H^0(G, M)$ is the norm map $N : M \otimes \mathbb{Z} \to M^G$ given by $(N(x \otimes k) = kNx)$ ([1], VI.4). If $M$ is projective then $\tilde{H}^m(G, M) = 0$ for all $m \in \mathbb{Z}$, hence $N$ is an isomorphism. We conclude:

1) Surjectivity of $N$ implies that $x \in P^G \iff \exists y \in P, x = Ny$.
2) Injectivity of $N$ implies that $y \otimes 1 = y' \otimes 1 \iff Ny = Ny'$ for all $y, y' \in P$.

□

Proposition 6. Let $G$ be a finite group then there is an isomorphism between $\tilde{H}^{-n-1}(G, \mathbb{Z})$ and $H_n(G, \mathbb{Z})$ for $n \geq 1$.

Proof. Take a projective resolution of $\mathbb{Z} \to \cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to \mathbb{Z}$, tensoring it with $\mathbb{Z}$ gives us the chain complex for the homology of $G$ which we denote by $C_*(G)$. We define a map from $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}\mathbb{Z})$ to $C_n(G)$ the following way: Given a homomorphism $f : \mathbb{Z} \to \Omega^{n+1}\mathbb{Z}$, $f(1) = x$ is an invariant element in $P_n$. By the lemma, since $P_n$ is projective and $x$ is invariant then some $y \in P_n$ such that $x = Ny$. We define $\Phi(f) = y \otimes 1$. This doesn’t depend on the choice of $y$ since $Ny = Ny' \iff y \otimes 1 = y' \otimes 1$ by the lemma above. We know that $Nd_n(y) = d_n(Ny) = d_n(x) = 0$ and by the lemma this implies that $d_n(y) \otimes 1 = 0$ ($P_{n-1}$ is projective and here we use the fact that $n \geq 1$). We deduce that $y \otimes 1 \in Z_n(G)$. The map described now $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}\mathbb{Z}) \to Z_n(G)$ is surjective since given an element $y \otimes 1 \in C_n(G)$ such that $d_n(y) \otimes 1 = 0$ this implies that $Nd_n(y) = 0$, so we define $f(k) = kNy$, this is well defined since $Ny$ is invariant and in the kernel of $d_n$.

We now have a surjective homomorphism $\Phi : \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}\mathbb{Z}) \to H_n(G, \mathbb{Z})$. If $f \in \text{ker}(\Phi)$ then there exist $s \in P_{n+1}$ such that $\Phi(f) = y \otimes 1 = d_{n+1}(s) \otimes 1$ then the map $f : \mathbb{Z} \to \Omega^{n+1}\mathbb{Z}$ factors through $P_{n+1}$ which is projective by $1 \to Ns$. On the other hand if $f$ factors through a projective module, w.l.o.g. $P_{n+1}$, then $Ny = f(1) = d_{n+1}(Ns)$ (every invariant element in $P_{n+1}$ is of the form $Ns$ by the lemma). This implies that $Nd_{n+1}(s) = Ny \iff d_{n+1}(s \otimes 1) = d_{n+1}(s) \otimes 1 = y \otimes 1$.

We conclude that the induced map:

$$\Phi : \tilde{H}^{-n-1}(G, \mathbb{Z}) \xrightarrow{\Phi} \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}\mathbb{Z}) \to H_n(G, \mathbb{Z})$$

is an isomorphism for all $n \geq 1$.

□

Remark. Since $\tilde{H}^{-1}(G, \mathbb{Z}) = \{0\}$ we conclude that $\tilde{H}^{-n-1}(G, \mathbb{Z}) \cong \check{H}_n(G, \mathbb{Z})$ for $n \geq 0$ where $\check{H}_n(G, \mathbb{Z})$ is the reduced homology.
The product structure

The cup product in Tate cohomology
\( \hat{H}^{-n}(G, \mathbb{Z}) \otimes \hat{H}^{-m}(G, \mathbb{Z}) \to \hat{H}^{-n-m}(G, \mathbb{Z}) \)

is given by composition (this is also called the Yoneda composition product). Given:

\[
[f] \in \hat{H}^{-n}(G, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n \mathbb{Z})
\]

\[
[g] \in \hat{H}^{-m}(G, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[G]}(\Omega^n \mathbb{Z}, \Omega^{n+m} \mathbb{Z})
\]

we compose them to get a map:

\[
[f] \cup [g] = [g \circ f] \in \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+m} \mathbb{Z})
\]

Since for \( n, m \geq 2 \) we have \( \hat{H}^{-n}(G, \mathbb{Z}) \cong H_{n-1}(G, \mathbb{Z}) \), \( \hat{H}^{-m}(G, \mathbb{Z}) \cong H_{m-1}(G, \mathbb{Z}) \) we have a product \( H_{n-1}(G, \mathbb{Z}) \otimes H_{m-1}(G, \mathbb{Z}) \to H_{n+m-1}(G, \mathbb{Z}) \). We would like to have a description of the isomorphism \( \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[G]}(\Omega^n \mathbb{Z}, \Omega^{n+m} \mathbb{Z}) \) which is concrete. To do so we use the following construction:

**The join of augmented chain complexes.**

Let \( G \) be a finite group and let \( P \) and \( Q \) be the following augmented chain complexes over \( \mathbb{Z}[G] \) - \( \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \) and \( \cdots \to Q_2 \to Q_1 \to Q_0 \to \mathbb{Z} \).

We define the join of those two chain complexes to be \( P \ast Q = \Sigma(P \otimes Q) \) that is the suspension of the tensor product over \( \mathbb{Z} \) (with a diagonal \( G \) action). To be more specific \( (P \ast Q)_n = \bigoplus_{0 \leq k \leq n+1} P_{k-1} \otimes_{\mathbb{Z}} Q_{n-k} \):

\[
\cdots \to P_1 \otimes_{\mathbb{Z}} \mathbb{Z} \oplus P_0 \otimes_{\mathbb{Z}} Q_0 \oplus \mathbb{Z} \oplus Q_1 \to P_0 \otimes_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Z} \oplus Q_0 \to \mathbb{Z} \oplus \mathbb{Z} \]

\( P \ast Q \) is an augmented \( \mathbb{Z}[G] \) chain complex in a natural way.

**Lemma 7.** If both \( P \) and \( Q \) are projective and acyclic augmented \( \mathbb{Z}[G] \) chain complexes then \( P \ast Q \) is a projective and acyclic augmented \( \mathbb{Z}[G] \) chain complex.

**Proof.** \( P \) and \( Q \) are projective acyclic chain complexes over \( \mathbb{Z} \) so the same is true for their tensor product, by the Künneth formula. \( (P \ast Q)_n \) is projective over \( \mathbb{Z}[G] \) for \( n \geq 0 \) since each of the modules \( P_{k-1} \otimes_{\mathbb{Z}} Q_{n-k} \) is projective. \( \square \)

**Lemma 8.** Let \( P \) and \( Q \) be two resolutions of \( \mathbb{Z} \) over \( \mathbb{Z}[G] \), and let \( s \in Q_{n-1} \) be an element, \( n > 1 \). Define a map \( s_* : P_{k-1} \to (P \ast Q)_{k+n-1} \) by \( s_*(x) = x \otimes s \) called the join with \( s \). Then we have:

1) \( s_* \) is a group homomorphism.
2) If \( s \) is \( G \)-invariant then \( s_* \) is a homomorphism over \( \mathbb{Z}[G] \).
3) If \( s \in \ker(Q_{n-1} \to Q_{n-2}) \) then \( s_* \) is a chain map of degree \( n \).

**Proof.** 1) Follows from the properties of the tensor product.
2) For every \( g \in G \) we have: \( g(s_*(x)) = g(x \otimes s) = g(x) \otimes g(s) = g(x) \otimes s = s_*(g(x)) \)
3) \( \partial(s_*(x)) = \partial(x \otimes s) = \partial(x) \otimes s + (-1)^{|x|+1} x \otimes \partial s = \partial(x) \otimes s = s_*(\partial(x)) \) \( \square \)

This implies the following:

**Theorem 9.** Let \( n, m > 0 \), the product \( \hat{H}^{-n}(G, \mathbb{Z}) \otimes \hat{H}^{-m}(G, \mathbb{Z}) \to \hat{H}^{-n-m}(G, \mathbb{Z}) \) is given by \( [f] \cup [g] = [f \ast g] \) where \( (f \ast g)(k) = k \cdot f(1) \otimes g(1) \in \Omega^{m+n} \mathbb{Z} \) and \( k \in \mathbb{Z} \).

**Proof.** Take a projective resolution \( P \) for \( \mathbb{Z} \) over \( \mathbb{Z}[G] \). Let \( [f] \in \hat{H}^{-n}(G, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^n \mathbb{Z}) \), \( [g] \in \hat{H}^{-m}(G, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^m \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[G]}(\Omega^n \mathbb{Z}, \Omega^{n+m} \mathbb{Z}) \)

Choose representatives \( f, g \) and define a degree \( m \) map \( P \to P \ast P \) by \( x \mapsto x \otimes g(1) \).

Since \( g(1) \) is invariant and in the kernel this map is a chain map of \( \mathbb{Z}[G] \) chain...
complexes of degree $m$. This gives us a concrete construction of the isomorphism $\text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}^m, \Omega^n \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[G]}(\Omega^n \mathbb{Z}, \Omega^{n+m} \mathbb{Z})$. The composition is therefore $g \circ f(1) = f(1) \otimes g(1)$. 

**A description of the product by joins of cycles.**

We now consider resolutions which come from singular chains of spaces. Let $G$ be a finite group, recall that a contractible $G - CW$ complex with a free $G$ action is denoted by $EG$, the quotient space $EG/G$ is the classifying space of principal $G$ bundles and is denoted by $BG$.

We consider now the augmented singular chain complex of $EG$ denoted by $C_*(EG)$. The action on $EG$ is free so $C_*(EG)$ is projective ($n \geq 0$) and $EG$ is contractible so $C_*(EG)$ is acyclic.

As we saw before (prop. D) every element of $H_n(G, \mathbb{Z})$ can be considered as an invariant cycle in $C_n(EG)$ (modulo invariant boundary), we will show that the product can be considered as the join of the two such cycles, which is naturally an invariant cycle in $C_*(EG \ast EG)$ where $EG \ast EG$ is the join of two copies of $EG$. Note that since the join of contractible spaces is contractible, $EG \ast EG$ is contractible, and it has a natural $G$ action, given by $g(x, y, t) = (gx, gy, t)$, which is free since it is free on both copies of $EG$. This implies that its augmented singular chain complex is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$.

We now associate the join of chain complexes to the join of spaces.

**Lemma 10.** Let $A$ and $B$ be two spaces and let $C_*(A)$ and $C_*(B)$ be their augmented (!) singular chain complexes, then there is a natural chain map:

$$h : C_*(A) \ast C_*(B) \to C_*(A \ast B)$$

If $G$ acts on $A$ and $B$ then it also acts on $A \ast B$ and the chain complexes are complexes over $\mathbb{Z}[G]$ and $h$ is a map of $\mathbb{Z}[G]$ chain complexes.

**Proof.** We first note that for $n, m \geq 0$, for every two singular simplices $\sigma \in C_n(A)$ and $\tau \in C_m(B)$ there is a canonical singular chain $\sigma \ast \tau \in C_{n+m+1}(A \ast B)$ and this definition can be extended in a bilinear way to chains. Define $h$ the following way: Given an element $s \otimes t \in C_n(A) \otimes C_m(B)$, if $n, m \geq 0$ then $h(s \otimes t) = s \ast t$, else $n = -1$ (or $m = -1$) then $s$ is an integer, denote it by $k$ then $h(s \otimes t) = h(k \otimes t) = k \cdot t$ where $t$ is the chain induced by the inclusion of $B$ in $A \ast B$ (and similarly for $m = -1$).

We have to show that $h$ is a chain map. For two simplices of positive (!) dimension we have the formula $\partial(\sigma \ast \tau) = \partial(\sigma) \ast \tau + (-1)^{\dim(\sigma)+1} \sigma \ast \partial(\tau)$. The formula extends to chains, so we have:

$$\partial h(s \otimes t) = \partial(s \ast t) = \partial(s) \ast t + (-1)^{|s|+1}s \ast \partial(t) = h(\partial(s) \otimes t + (-1)^{|s|+1}s \otimes \partial(t)) = h(\partial(s \otimes t)).$$

For $\sigma$, a simplex of dimension $0$ (a point), $\sigma \ast \tau$ is the cone over $\tau$ and its boundary is given by $\partial(\sigma \ast \tau) = \tau + (-1)^{\dim(\sigma)+1} \sigma \ast \partial(\tau)$. Since the boundary map $C_0(A) \to \mathbb{Z}$ is the augmentation map we see indeed that also in this case $h$ commutes with the boundary (with respect to the way we have defined $h(k \otimes t)$).

The boundary formula is not (!) true when one of the simplices is zero dimensional due to the non symmetric way we define the faces of a zero simplex (the $n$ simplex has $n + 1$ faces while the zero simplex has no faces). If we wanted to be consistent with the boundaries of the higher simplices we should have used only augmented chain complexes. More in this direction appears in [F].
When there is a $G$ action on both spaces then clearly all the complexes are complexes over $\mathbb{Z}[G]$. $h$ is a $\mathbb{Z}[G]$ chain map since for every $g \in G$ we have:

$$h(g(s \otimes t)) = h(gs \otimes gt)) = gs \ast gt = g(s \ast t) = g(h(s \otimes t))$$

$\square$

**Theorem 11.** The cup product in negative Tate cohomology gives a product

$$H_n(G, \mathbb{Z}) \otimes H_m(G, \mathbb{Z}) \rightarrow H_{n+m+1}(G, \mathbb{Z})$$

$(n, m > 0)$. Each homology class in $H_n(G, \mathbb{Z})$ is represented by an invariant cycle in $EG$. The product of two classes is given by the join of those cycles, which is an invariant cycle in $EG \ast EG$.

*Proof.* We already saw that the product can be described by the join of resolutions. By the proposition above there is a degree zero chain map $C_*(EG) \ast C_*(EG) \rightarrow C_*(EG \ast EG)$). The image of $f(1) \otimes g(1)$ under this map is the join of $f(1)$ with $g(1)$. This gives a more concrete model where the cycles are actual invariant singular cycles of the space $EG \ast EG$. $\square$

**Corollary 12.** The product in $\hat{H}_*(G, \mathbb{Z})$ comes from the chain map:

$$(P \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \otimes_{\mathbb{Z}} (P \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \rightarrow (P \ast P) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

given by: $(x \otimes 1) \otimes (y \otimes 1) \rightarrow ((Nx) \otimes y) \otimes 1$, where $P$ is an augmented projective resolution.

This map is equal to the composition of two maps. The first one:

$$(P \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \otimes_{\mathbb{Z}} (P \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \rightarrow (P \ast P) \otimes_{\mathbb{Z}[G \times G]} \mathbb{Z}$$

is given by $(x \otimes 1) \otimes (y \otimes 1) \rightarrow (x \otimes y) \otimes 1$. This is an exterior product, which is injective (in homology) by the *Künneth* theorem. Note that the homology of $(P \ast P) \otimes_{\mathbb{Z}[G \times G]} \mathbb{Z}$ need not be equal to $\hat{H}_*(G \times G, \mathbb{Z})$ since $P \ast P$ is not projective over $\mathbb{Z}[G \times G]$. $(P \ast P) \otimes_{\mathbb{Z}[G \times G]} \mathbb{Z}$ is the chain complex of $BG \ast BG$, and this is the join product:

$$\hat{H}_n(BG, \mathbb{Z}) \otimes \hat{H}_m(BG, \mathbb{Z}) \rightarrow \hat{H}_{n+m+1}(BG \ast BG, \mathbb{Z})$$

The second map, $(P \ast P) \otimes_{\mathbb{Z}[G \times G]} \mathbb{Z} \rightarrow (P \ast P) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is given by $(x \otimes y) \otimes 1 \rightarrow ((Nx) \otimes y) \otimes 1$, which is a transfer map.

**Comparing Kreck’s product and the cup product in Tate cohomology**

The Kreck product is defined using stratifolds and stratifold homology. Stratifolds are generalization of manifolds. They were introduced by Kreck [K1] and used in order to define a bordism theory, denoted by $SH_*$, which is naturally isomorphic to singular homology. We will use them to describe group homology with integral coefficients and the Kreck product.
Stratifolds.

Kreck defined stratifolds as spaces with a sheaf of functions, called the smooth functions, fulfilling certain properties but for our purpose the following definition is enough (these stratifolds are also called p-stratifolds):

A stratifold is a pair consisting of a topological space and a subsheaf of the sheaf of real continuous functions, which is constructed inductively in a similar way to the way we construct CW complexes. We start with a discrete set of points denoted by \( X^0 \) and define inductively the set of smooth functions which in the case of \( X^0 \) are all real functions.

Suppose \( X^{k-1} \) together with a smooth set of functions is given. Let \( W \) be a \( n \) dimensional smooth manifold “the \( n \) strata” with boundary and a collar \( c \), and \( f \) a continuous map from the boundary of \( W \) to \( X^{n-1} \). We require that \( f \) is smooth, which means that its composition with every smooth map from \( X^{n-1} \) is smooth. Define \( X^n = X^{n-1} \cup_f W \). The smooth maps on \( X^n \) are defined to be those maps \( g : X^n \to \mathbb{R} \) which are smooth when restricted to \( X^{n-1} \) and to \( W \) and such that for some \( 0 < \delta \) we have \( gc(x,t) = gf(x) \) for all \( x \in \partial W \) and \( t < \delta \).

Among the examples of stratifolds are manifolds, real and complex algebraic varieties \([G]\), and the one point compactification of a smooth manifold. The cone over a stratifold and the product of two stratifolds are again stratifolds.

We can also define stratifolds with boundary which are analogous to manifolds with boundary. A main difference is that every stratifold is the boundary of its cone, which is a stratifold with boundary.

Given two stratifolds with boundary \((T',S')\) and \((T'',S'')\) and an isomorphism \( f : S' \to S'' \) there is a well defined stratifold structure on the space \( T' \cup_f T'' \) which is called the gluing. On the other hand, given a smooth map \( g : T \to \mathbb{R} \) such that there is a neighborhood of 0 which consists only of regular values then the preimages \( g^{-1}((-\infty,0]) = T' \) and \( g^{-1}([0,\infty)) = T'' \) are stratifolds with boundary and \( T \) is isomorphic to the gluing \( T' \cup_{Id} T'' \).

To obtain singular homology we specialize our stratifolds in the following way: We use compact stratifolds, require that their top stratum will be oriented and the codimension one stratum will be empty.

Remark. Regarding regularity, a condition often required, see \([K3]\).

Stratifold homology.

Stratifold homology was defined by Kreck in \([K1]\). We will describe here a variant of this theory called parametrized stratifold homology, which is naturally isomorphic to it for CW complexes. In this paper we will refer to parametrized stratifold homology just as stratifold homology and use the same notation for it.

Stratifold homology is a homology theory, denoted by \( SH_* \). It is naturally isomorphic to integral homology and gives a new geometric point of view on it.

Definition 13. Let \( X \) be a topological space and \( n \geq 0 \), define \( SH_n(X) \) to be \( \{g : S \to X\} / \sim \) i.e., bordism classes of maps \( g : S \to X \) where \( S \) is a compact oriented stratifold of dimension \( n \) and \( g \) is a continuous map. We often denote the class \([g : S \to X]\) by \([S,g]\) or by \([S \to X]\). \( SH_n(X) \) has a natural structure of an Abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If \( f : X \to Y \) is a continuous map than we can define an induced map by composition \( f_* : SH_n(X) \to SH_n(Y) \).

One constructs a boundary operator and prove the following:
Theorem 14. (Mayer-Vietoris) The following sequence is exact:
\[ \cdots \to SH_n(U \cap V) \to SH_n(U) \oplus SH_n(V) \to SH_n(U \cup V) \overset{\partial}{\to} SH_{n-1}(U \cap V) \to \cdots \]
where, as usual, the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

\( SH_* \) with the boundary operator is a homology theory.

Theorem 15. There is a natural isomorphism of homology theories \( \Phi : SH_* \to H_* \)

Proof. See for example [T]. \( \Phi \) is given by \( \Phi_n([S,f]) = f_*([S]) \) where \( [S] \in H_n(S,\mathbb{Z}) \) is the fundamental class of \( S \).

Stratifold group homology.

One defines the group homology of a group \( G \) with coefficients in a \( \mathbb{Z}[G] \) module \( M \) to be \( H_*(BG,M) \) where \( M \) is considered as a local coefficients system. Our main interest is when \( M = \mathbb{Z} \) with the trivial action, then this reduces to the integral homology \( H_*(BG,\mathbb{Z}) \). These groups are naturally isomorphic to the groups \( SH_*(BG,\mathbb{Z}) \) by the theorem above.

Let \( G \) be a compact Lie group of dimension \( d \). Denote by \( SH_n(G,\mathbb{Z}) \) the set of compact oriented stratifolds of dimension \( n \) with a free and orientation preserving \( G \) action modulo \( G \)-cobordism, i.e. a cobordism with a free \( G \) action extending the given action on the boundary (all actions on the stratifolds are assumed to be smooth). We denote the class of the stratifold and the action by \( [S,\rho] \).

The following lemma and proposition are an easy exercise:

Lemma 16. 1) Let \( S \) be a compact oriented stratifold of dimension \( n \) and \( \tilde{S} \to S \) a covering space then \( \tilde{S} \) can be given a unique structure of an oriented stratifold such that the covering map is an orientation preserving local isomorphism. If \( S \) is compact and the fibers are finite then \( \tilde{S} \) is compact.

2) Let \( S \) be a compact oriented stratifold of dimension \( n \) with an orientation preserving free \( G \) action then \( S/G \) can be given a unique structure of a compact oriented stratifold such that the projection will be an orientation preserving local isomorphism.

Proposition 17. Let \( G \) be a finite group, the map \( \Psi : SH_n(G,\mathbb{Z}) \to SH_n(BG,\mathbb{Z}) \) given by \( [S,\rho] \mapsto [S/G \xrightarrow{f_*} BG] \), where \( f \) is the classifying map, is an isomorphism. The map \( \Psi^{-1} : SH_n(BG,\mathbb{Z}) \to SH_n(G,\mathbb{Z}) \) is given by \([S \xrightarrow{f} BG] \mapsto [\tilde{S},\rho]\) where \( \tilde{S} \to S \) is the pull back of the universal bundle \( EG \to BG \) and \( \rho \) is the induced action.

Remark. Similarly, an isomorphism \( \Psi : SH_{n+d}(G,\mathbb{Z}) \to SH_n(BG,\mathbb{Z}) \) can be constructed for a compact Lie group \( G \) of dimension \( d \).

There is a natural product structure \( SH_n(G,\mathbb{Z}) \otimes SH_m(G,\mathbb{Z}) \to SH_{n+m}(G,\mathbb{Z}) \) given by the Cartesian product with the diagonal action:

\( [S,\rho] \otimes [S',\rho'] \mapsto [S \times S',\Delta] \)

This product vanishes whenever \( n,m > 0 \) since it is the boundary of \([CS \times S',\hat{\rho}]\)
where \( \hat{\rho} \) is the obvious extension of the action \( \Delta \), but it is also the boundary of \([S \times CS',\hat{\rho}]\) where \( \hat{\rho} \) is the obvious extension of the action \( \Delta \).
The Kreck product is a secondary product defined by gluing \((CS \times S', \hat{\rho})\) and \((S \times CS', \hat{\rho})\) along their common boundary \((S \times S', \Delta)\)

\[
[S, \rho] \otimes [S', \rho'] \rightarrow [S \ast S', \rho \ast \rho']
\]

(note that after the gluing what we get is the join of the two stratifolds).

The product \(SH_n(G, \mathbb{Z}) \otimes SH_m(G, \mathbb{Z}) \rightarrow SH_{n+m+1}(G, \mathbb{Z})\) does not vanish in general. When \(G\) is finite cyclic then \(SH_n(G, \mathbb{Z})\) is infinite cyclic when \(n = 0\), zero when \(n\) is even and isomorphic to \(G\) when \(n\) is odd. The generators can be taken to be odd dimensional spheres with the action induced by the complex multiplication, when the sphere is considered as the unit sphere in a complex space. In this case the product of generators is again a generator. A similar construction will hold for \(G = S^1\) and \(S^3\). This implies that the product is non trivial for every group with a free and orientation preserving smooth action on a sphere.

There is an isomorphism \(\Psi : SH_n(G, \mathbb{Z}) \rightarrow \tilde{H}^{-n-1}(G, \mathbb{Z})\) for \(n > 0\) given by the composition:

\[
SH_n(G, \mathbb{Z}) \rightarrow SH_n(BG, \mathbb{Z}) \rightarrow H_n(BG, \mathbb{Z}) \rightarrow \tilde{H}^{-n-1}(G, \mathbb{Z})
\]

One might show that this isomorphism is given the following way: Take some model for \(EG\). Its singular chain complex \(C_\ast(EG)\) is a projective resolution for \(\mathbb{Z}\) over \(Z[G]\). Let \([(S, \rho)]\) be an element in \(SH_n(G, \mathbb{Z})\). There is a map \(f : S \rightarrow EG\) that commutes with the action of \(G\). This map is unique up to \(G\) homotopy (\(f\) is called the classifying map), any two such maps are \(G\) homotopic. Since \(f\) commute with the action of \(G\) it induces a map of the singular chain complexes which are complexes of \(Z[G]\) modules - \(C(S) \xrightarrow{f} C(EG)\). As shown in [T], \(S\) has a fundamental class, we take some representative of it which is \(G\) invariant (we can do that by lifting a fundamental cycle of \(S/G\) and denote it by \(s\). We get an element \(f_\ast(s) \in C(EG)_n\) which is both invariant and a cycle thus we get an element in \(\text{Hom}(\mathbb{Z}, \Omega^{
+1})\). As before different choices of \(S\) and \(f\) will give elements that differ by a map which factors through a projective (the fundamental class of the cobordism is mapped into \(C(EG)_{n+1}\) which is projective), hence gives a homomorphism \(SH_n(G, \mathbb{Z}) \rightarrow \tilde{\text{Hom}}_{\mathbb{Z}[G]}(\mathbb{Z}, \Omega^{n+1}_\mathbb{Z}) = \tilde{H}^{-n-1}(G, \mathbb{Z})\) which is exactly the isomorphism above.

Now we would like to show that the Kreck product is the same as the cup product. We show that the join of two fundamental classes is equal to the fundamental class of their join.

**Lemma 18.** Let \(S\) and \(S'\) be two compact oriented stratifolds of dimension \(n\) and \(m\) \((m, n > 0)\) respectively. Denote the fundamental classes of \(S, S', S \ast S'\) by \(a_S, a_{S'}, a_{S \ast S'}\). then \(a_S \ast a_{S'} = a_{S \ast S'}\).

**Proof.** Let \(U = \{(s, s', t) \in S \ast S' \mid t < 1\}\), \(V = \{(s, s', t) \in S \ast S' \mid 0 < t\}\) then \(U \simeq S, V \simeq S', U \cup V \simeq S \times S', U \cup V = S \ast S'\). By Mayer-Vietoris the boundary map \(\partial : H_{n+m+1}(S \ast S', \mathbb{Z}) \rightarrow H_{n+m}(S \times S', \mathbb{Z})\) is injective (an isomorphism actually) and by the definition of the boundary we have \(\partial(a_{S \ast S'}) = a_{S \times S'}\). It will be enough to show that \(\partial(a_S \ast a_{S'}) = a_{S \ast S'}\). We do know that \(\partial(a_S \ast a_{S'}) = a_S \ast a_{S'}\), this follows from the definition of the boundary after taking the suitable representative for \(a_S \ast a_{S'}\). So we reduced the problem to proving that \(a_S \ast a_{S'} = a_{S \times S'}\). This fact follows from the following commutative diagram:
Let $G$ be a finite group, then there is a natural isomorphism between $SH_n(G, \mathbb{Z})$ and $H^{-n-1}(G, \mathbb{Z})$ and this isomorphism respects the product.

In other words, the product in group homology defined by Kreck using stratifold homology and the join agrees with the cup product in negative Tate cohomology.

Appendix - The stable module category

In this appendix we give the background needed for the construction we used for Tate cohomology.

Again $R$ is a ring with unit, not necessarily commutative, and all modules are assumed to be left $R$-modules.

The stable category $St - mod(R)$.

Let $M$ and $N$ be two $R$-modules, denote by $P\text{Hom}_R(M, N)$ the set of $R$-homomorphisms $M \xrightarrow{f} N$ that factors through a projective $R$-module, i.e. there exists a projective $R$-module $P$ and two maps $M \xrightarrow{f_1} P \xrightarrow{f_2} N$ such that $f = f_2 \circ f_1$. The following proposition is left as an easy exercise:

**Proposition 20.** $P\text{Hom}_R(M, N)$ is a sub module of $\text{Hom}_R(M, N)$ and the composition of two homomorphisms such that one of them factors through a projective module also factors through a projective module.

By the proposition above we can define $\text{Hom}_R(M, N) = \text{Hom}_R(M, N)/P\text{Hom}_R(M, N)$ which is an $R$-module, and a composition $\text{Hom}_R(N, K) \times \text{Hom}_R(M, N) \to \text{Hom}_R(M, K)$ which is $R$-bilinear.

**Definition 21.** Let $R$ be a ring, denote by $St - mod(R)$ the category whose objects are all $R$-modules and the morphisms between each $M$ and $N$ are $\text{Hom}_R(M, N)$. This category is called the stable module category.
The functor $\Omega$.

For every $R$-module $M$ choose (once and for all) a projective cover, that is a surjective map $\pi_M : P_M \to M$ where $P_M$ is a projective $R$-module (for example the canonical free cover).

Define a functor $\Omega : St - mod(R) \to St - mod(R)$ the following way: For an object $M$ define $\Omega(M) = Ker(\pi_M)$. For a morphism $[f] \in \text{Hom}_R(M, N)$ choose some representative $f : M \to N$, use the fact that $P_M$ is projective and $\pi_M$ is surjective to define a map $\tilde{f} : P_M \to P_N$ such that the following diagram become commutative:

\[
\begin{array}{ccccccccc}
0 & \to & \Omega(M) & \to & P_M & \to & M & \to & 0 \\
\downarrow & & \downarrow \tilde{f}|_{\Omega(M)} & & \downarrow \tilde{f} & & \downarrow f & & \\
0 & \to & \Omega(N) & \to & P_N & \to & N & \to & 0
\end{array}
\]

Now take $\Omega(f)$ to be the class of the induced map $\tilde{f}|_{\Omega(M)} : \Omega(M) \to \Omega(N)$. This is well defined by the following lemma:

**Lemma 22.** 1) In the previous notations, if $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of $f \circ \pi_M$ then $\tilde{f}_1|_{\Omega(M)}$ and $\tilde{f}_2|_{\Omega(M)}$ represent the same element in $\text{Hom}_R(\Omega M, \Omega N)$.
2) The map $\text{Hom}_R(M, N) \to \text{Hom}_R(\Omega M, \Omega N)$ is a homomorphism.
3) If $f$ factors through a projective then also $\tilde{f}|_{\Omega(M)}$ does, thus we get a homomorphism $\text{Hom}_R(M, N) \to \text{Hom}_R(\Omega M, \Omega N)$.

**Proof.** 1) Assume we have two such lifts $\tilde{f}_1$ and $\tilde{f}_2$ then the following diagram is commutative (where $h = \tilde{f}_1|_{\Omega(M)} - \tilde{f}_2|_{\Omega(M)}$):

\[
\begin{array}{ccccccccc}
0 & \to & \Omega(M) & \to & P_M & \to & M & \to & 0 \\
\downarrow & & \downarrow h & & \downarrow \tilde{f}_1 - \tilde{f}_2 & & \downarrow 0 & & \\
0 & \to & \Omega(N) & \to & P_N & \to & N & \to & 0
\end{array}
\]

It will be enough to show that $h$ factors through $P_M$ which is projective. This follows from the fact that the image of the map $\tilde{f}_1 - \tilde{f}_2$ is contained in $\Omega(N)$ by the commutativity of the diagram.

2) Choose the lifting of $a \cdot f + b \cdot g$ to be $a \cdot \tilde{f} + b \cdot \tilde{g}$.
3) Assume $f$ factors through a projective module $P$. We have the following diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \Omega(M) & \to & P_M & \to & M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & P & \to & P & \to & 0 \\
\downarrow & & \downarrow & & \downarrow s & & \downarrow & & \\
0 & \to & \Omega(N) & \to & P_N & \to & N & \to & 0
\end{array}
\]

The map $s : P \to P_N$ can be defined using the fact that $P$ is projective and the map $P_N \to N$ is surjective. We get that the induced map $\Omega(M) \to \Omega(N)$ is the zero map.

The following is important for the definition of Tate cohomology:

**Proposition 23.** Let $G$ be a finite group and $R = \mathbb{Z}[G]$. If $M$ is a $\mathbb{Z}[G]$ module which is projective as an Abelian group then the map $\text{Hom}_R(M, N) \to \text{Hom}_R(\Omega M, \Omega N)$ is an isomorphism.

**Proof.** Before we start recall ([Br] VI 2) that a $\mathbb{Z}[G]$-module $Q$ is called relatively injective if for every injection $A \hookrightarrow B$ of $\mathbb{Z}[G]$-modules which splits as an injection
of Abelian groups and every \( \mathbb{Z}[G] \) homomorphism \( A \to Q \) there exists an extension to a \( \mathbb{Z}[G] \) homomorphism \( B \to Q \), and that if \( G \) is a finite group every projective module is relatively injective.

We construct an inverse to this map. Given a map \( f : \Omega M \to \Omega N \). We have the following diagram:

\[
\begin{array}{cccccc}
0 & \to & \Omega(M) & \to & P_M & \to & M & \to & 0 \\
\downarrow & & \downarrow f & & \downarrow & & \downarrow & & \\
0 & \to & \Omega(N) & \to & P_N & \to & N & \to & 0
\end{array}
\]

Since \( M \) is projective as an Abelian group the upper row splits as Abelian groups. This means that \( \Omega(M) \to P_M \) is a split injection as Abelian groups. \( P_N \) is projective and hence relatively injective therefore we can extend the homomorphism \( \Omega(M) \to P_N \) to a homomorphism \( f : P_M \to P_N \) such that the diagram will commute. This induces a homomorphism \( \Omega f : M \to N \). Of course \( \Omega f \) depends on the choice of \( \Omega f \). Suppose that \( \tilde{f}_1, \tilde{f}_2 \) are two extensions then \( \tilde{f}_1 - \tilde{f}_2 \) vanishes on \( \Omega(M) \) hence the map \( \tilde{f}_1 - \tilde{f}_2 : M \to N \) factors through \( P_N \) which is projective. This gives a well defined homomorphism \( \text{Hom}_{R}(\Omega M, \Omega N) \to \text{Hom}_{R}(M, N) \). Assume \( f : \Omega M \to \Omega N \) factors through a projective \( P \) then we can choose \( \tilde{f} \) to factor through \( P \) again since it is relatively injective and get that \( \tilde{f} \) is the zero map:

\[
\begin{array}{cccccc}
0 & \to & \Omega(M) & \to & P_M & \to & M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & P & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow s & & \downarrow & & \\
0 & \to & \Omega(N) & \to & P_N & \to & N & \to & 0
\end{array}
\]

Hence we get a homomorphism \( \text{Hom}_{R}(\Omega M, \Omega N) \to \text{Hom}_{R}(M, N) \) which is easily seen to be the inverse of the homomorphism \( \text{Hom}_{R}(M, N) \to \text{Hom}_{R}(\Omega M, \Omega N) \).

We have defined the endofunctor \( \Omega \). We define \( \Omega^n \) by induction: \( \Omega^0 = \text{Id} \) and \( \Omega^n = \Omega \circ \Omega^{n-1} \).

**Proposition 24.** Let \( M \) be an \( R \)-module and let \( \ldots \to Q_{n-1} \to \ldots \to Q_0 \to M \) be any projective resolution of \( M \), then \( \Omega^n(M) \) can be identified with \( \text{Ker}(Q_{n-1} \to Q_{n-2}) \), that is there is a canonical map \( \text{Ker}(Q_{n-1} \to Q_{n-2}) \to \Omega^n(M) \) which is an isomorphism in the category \( \text{St} - \text{mod}(R) \).

**Proof.** Given an \( R \)-module \( M \) we construct a canonical projective resolution of it using the projective covers we have chosen before. We do it by induction where \( P_n \) is defined to be the projective cover of \( \text{Ker}(P_{n-1} \to P_{n-2}) \) with the induced map \( P_n \to P_{n-1} \), which clearly make this into a projective resolution. Notice that by the definition of \( \Omega \) we have \( \Omega^n(M) = \text{Ker}(P_{n-1} \to P_{n-2}) \), and for a map \( f : M \to N \) the map \( \Omega^n(f) \) can be be constructed by extending the map \( f \) to a chain map between the two resolutions. In order to prove the proposition it will suffice to show that given two projective resolutions of \( M \to Q_{n-1} \to \ldots \to Q_0 \to M \) and \( \ldots \to P_{n-1} \to \ldots \to P_0 \to M \) there is a canonical isomorphism \( \text{Ker}(Q_{n-1} \to Q_{n-2}) \to \text{Ker}(P_{n-1} \to P_{n-2}) \). This follows directly by induction from what we have already showed in the case of a the projective cover of \( M \).
Remark. By similar reasons we can compute the induced maps $\Omega^n(f)$ for any map $f: M \to N$ by taking any two resolutions for $M$ and for $N$ and extending $f$ into a chain map between the two resolutions.

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