3-extra connectivity of 3-ary $n$-cube networks

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Abstract

Let $G$ be a connected graph and $S$ be a set of vertices. The $h$-extra connectivity of $G$ is the cardinality of a minimum set $S$ such that $G - S$ is disconnected and each component of $G - S$ has at least $h + 1$ vertices. The $h$-extra connectivity is an important parameter to measure the reliability and fault tolerance ability of large interconnection networks. The $h$-extra connectivity for $h = 1, 2$ of $k$-ary $n$-cube are gotten by Hsieh et al. in [Theoretical Computer Science, 443 (2012) 63-69] for $k \geq 4$ and Zhu et al. in [Theory of Computing Systems, arxiv.org/pdf/1105.0991v1 [cs.DM] 5 May 2011] for $k = 3$. In this paper, we show that the $h$-extra connectivity of the 3-ary $n$-cube networks for $h = 3$ is equal to $8n - 12$, where $n \geq 3$.

Keywords: Interconnection networks, 3-ary $n$-cube networks, extra connectivity, conditional connectivity

1. Introduction

It is well known that a topological structure of an interconnection network can be modeled by a loopless undirected graph $G = (V, E)$, where the vertex set $V$ represents the processors and the edge set $E$ represents the communication links. In this paper, we use graphs and networks interchangeably.

Let $G$ be a simple undirected graph. Two vertices $v_1, v_2$ in $V(G)$ are said to be adjacent if and only if $(v_1, v_2) \in E(G)$. The neighborhood of a vertex $u$ in $G$ is the set of all vertices adjacent to $u$ in $V(G)$, denoted by $N_G(u)$. The cardinality $|N_G(u)|$ represents the degree of $u$ in $G$, denoted by $d_G(u)$ (or simply $d(u)$), $\delta(G)$ the minimum degree of $G$. For a vertex subset

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The neighborhood of $S$ in $G$ is $N_G(S) = (\bigcup_{u \in S} N_G(u)) - S$. A subgraph of $G = (V, E)$ is a graph $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. For a subgraph $H$ of $G$, $N_G(V(H))$ can be simplified as $G[H]$. For a subset $S$ of $V(G)$, the induced subgraph of $S$, written by $G[S]$, is a subgraph of $G$, whose vertex set is $S$ and an edge $e \in G[S]$ if and only if both end vertices of $e$ are in $S$. $N[S]$ is also used to denote the induced subgraph of $N_G(S) \cup S$. A subset $S \subseteq V(G)$ is a vertex cut if $G - S$ is disconnected. The components of $G$ are its maximal connected subgraphs.

A path $P_k = (v_1, v_2, \ldots, v_k)$ for $k \geq 2$ in a graph $G$ is a sequence of distinct vertices such that any two consecutive vertices are adjacent, and $v_1$ and $v_k$ are the end-vertices of the path. For convenience, use $P_t$ to denote a path of $t$ vertices. A path of $G$ of length $n$ will be called an $n$-path. A cycle $C_k = (v_1, v_2, \ldots, v_k, v_1)$ for $k \geq 3$ is a sequence of vertices in which any two consecutive vertices are adjacent, where $v_1, v_2, \ldots, v_k$ are all distinct. A cycle of $G$ of length $n$ will be called an $n$-cycle. A complete graph of $n$ vertices, denoted by $K_n$, is a simple graph whose vertices are pairwise adjacent.

Let $G$ and $H$ be two graphs. $G$ and $H$ are distinct if their vertex sets are different, and disjoint if they have no common vertices. An isomorphism from a graph $G$ to a graph $H$ is a bijection function $\pi : V(G) \rightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(\pi(u), \pi(v)) \in E(H)$. We write $G \cong H$ if there is an isomorphism from $G$ to $H$.

The connectivity $\kappa(G)$ of a connected graph $G$ is the minimum number of vertices removed to get the graph disconnected or trivial. A graph $G$ is said to be super connected, or simply super-$\kappa$, if every minimum vertex cut creates exactly two components, one of which is a singleton. Connectivity as a measure of reliability underestimates the fault tolerance ability of these multiprocessor systems.

Conditional connectivity introduced by Harary [15] can be used to better measure the reliability of multiprocessor systems. If any component of $G - S$ has some property $P$, where $S$ is a vertex cut of $G$, then $S$ is called a $P$-vertex cut. The $P$-conditional connectivity of $G$ is defined to be the minimum over all cardinalities of $P$-vertex cuts. J. Fàbrega and M.A. Fiol [10] introduced the extra connectivity of interconnection networks as follows. A vertex set $S \subseteq V(G)$ is called to be an $h$-extra vertex cut if $G - S$ is disconnected and every component of $G - S$ has at least $h + 1$ vertices. The $h$-extra connectivity of $G$, denoted by $\kappa_h(G)$, is defined as the cardinality of a minimum $h$-extra vertex cut, if exist. An $(h + 1)$-extra vertex cut of a graph $G$ is clearly an
It is obvious that $\kappa_0(G) = \kappa(G)$ for any graph $G$ that is not a complete graph. In particular, the 1-extra vertex cut is called as the extra vertex cut and the 1-extra connectivity is called as the extra connectivity. The problem of determining the $h$-extra connectivity of numerous networks has received a great deal of attention in recent years. Interested readers may refer to [1],[9],[21] or others for further details.

The $k$-ary $n$-cube $Q^k_n$, proposed by Scott and Goodman [19], is one of the most popular interconnection networks. Some properties of the $k$-ary $n$-cube network have been investigated, for example, fault diameter [7], pan-connectivity [17] etc. Moreover, many interconnection networks can be viewed as the subclasses of $Q^k_n$, including the cycle, the torus and the hypercube. The $h$-extra connectivity for $h = 1, 2$ of $k$-ary $n$-cube are gotten by Hsieh et al. in [16] for $k \geq 4$ and Zhu et al. in [22] for $k = 3$. In this paper, we show that the 3-extra connectivity of the 3-ary $n$-cube network is $8n - 12$ for $n \geq 3$.

Definitions which not been given here are referred to [3] and [20]. The remainder of this paper is organized as follows. In Section 2, the $k$-ary $n$-cube and its properties will be given. Section 3 discusses the 3-extra connectivity of the 3-ary $n$-cube. Section 4 concludes the paper. Last is acknowledgements.

2. The $k$-ary $n$-cube and its properties

The $k$-ary $n$-cube, denoted by $Q^k_n$, where $k \geq 2$ and $n \geq 1$ are integers, is a graph consisting of $k^n$ vertices. Each of these vertices has the form $u = u_{n-1}u_{n-2}\cdots u_0$ where $u_i \in \{0, 1, 2, \cdots, k-1\}$ for $0 \leq i \leq n-1$. Two vertices $u = u_{n-1}u_{n-2}\cdots u_0$ and $v = v_{n-1}v_{n-2}\cdots v_0$ in $Q^k_n$ are adjacent if and only if there exists an integer $j$, where $0 \leq j \leq n-1$, such that $u_j = v_j \pm 1(\text{mod } k)$, and $u_i = v_i$ for every $i \in \{0, 1, 2, \cdots, j-1, j+1, \cdots, n-1\}$. In this case, $(u, v)$ is a $j$-dimensional edge. For clarity of presentation, “(mod $k$)” does not appear in similar expressions in the remainder of the paper. Obviously, $Q^1_n$ is a cycle of length $k$, $Q^2_n$ is an $n$-dimensional hypercube, $Q^3_2$ is a $k \times k$ wrap-around mesh. This study considers 3-ary $n$-cube, $Q^3_2$ and $Q^3_3$ are illustrated in Fig.1.

It is possible to partition $Q^k_n$ over $j$-dimension, for a $j \in \{0, 1, 2, \cdots, n-1\}$, into $k$ disjoint subcubes, denoted by $Q^k_{n-1}[0], Q^k_{n-1}[1], \cdots, Q^k_{n-1}[k-1]$ by deleting all the $j$-dimensional edges from $Q^k_n$. For convenience, abbreviate
these as $Q[0], Q[1], \ldots, Q[k-1]$ if there is no ambiguity. Moreover, $Q[i]$ for $0 \leq i \leq k-1$ is isomorphic to $k$-ary $(n-1)$-cube and there are $kn^{n-1}$ edges between $Q[i]$ and $Q[i+1]$. For each vertex $u \in V(Q[i])$, the right neighbor (respectively, left neighbor) of $u$, denoted by $u_R$ (respectively, $u_L$), is the outer neighbor of $u$ in $Q[i+1]$ (respectively, $Q[i-1]$).

The following useful properties of $Q_k^n$ which will be used later on can be found in [4], [12], [19], [20].

**Lemma 2.1.** ([19]) For $n \geq 1$, $Q_k^n$ is $n$-regular and has $nk^{n-1}$ edges when $k = 2$; $Q_k^n$ is $2n$-regular and has $nk^n$ edges when $k \geq 3$.

**Lemma 2.2.** ([12, 20]) For $n \geq 2$, $\kappa(Q_k^n) = \delta(Q_k^n) = 2n$ when $k \geq 3$; $\kappa(Q_k^n) = \delta(Q_k^n) = n$ when $k = 2$. Moreover, $Q_k^n$ is super-connected for $n \geq 2$.

**Lemma 2.3.** ([12]) For $k \geq 2, n \geq 2$, $Q_k^n$ is vertex transitive and edge transitive.

**Lemma 2.4.** ([4]) For $k \geq 3, n \geq 2$, $Q_k^n$ can be divided into $k$ disjoint subgraphs, each subgraph is isomorphic to $Q_{n-1}^k$. The two outer neighbors of every vertex in $Q_k^n$ are in different subgraphs which contained in $\{Q[i] : 0 \leq i \leq k-1\}$. 
3. Main result

The $h$-extra connectivity for $h = 1, 2$ of $k$-ary $n$-cube are gotten by Hsieh et al. in [16] for $k \geq 4$ and Zhu et al. in [22] for $k = 3$. Nevertheless, the $h$-extra for $h \geq 3$ connectivity of 3-ary $n$-cube has not been obtained yet. In this section, the 3-extra connectivity of the 3-ary $n$-cube will be determined.

**Lemma 3.1.** ([22]) Any two adjacent vertices in $Q^3_n$ have exactly one common neighbor for $n \geq 1$; If any two nonadjacent vertices in $Q^3_n$ have common neighbors, they have exactly two common neighbors for $n \geq 2$.

**Lemma 3.2.** ([22]) $\kappa_1(Q^3_n) = 4n - 3$ for $n \geq 2$ and $\kappa_2(Q^3_n) = 6n - 7$ for $n \geq 3$.

**Lemma 3.3.** Suppose that $F \subseteq V(Q^3_n)$ with $|F| \leq 4n - 4$ is a vertex cut of $Q^3_n$ for $n \geq 2$, then $Q^3_n - F$ has two components, one of which is a singleton.

**Proof.** This lemma can be proved by using induction on $n$.

For $n = 2$, the graph is $Q^3_2$, shown in Fig.1, in this case, $|F| \leq 4n - 4 = 4$; Since $F$ is a vertex cut of $Q^3_2$, then $|F| \geq \kappa(Q^3_2) = 2n = 4$. Thus $|F| = 4$, $F$ is the minimum vertex cut. By Lemma 2.2, $Q^3_2 - F$ has two components, one of which is a singleton.

Assume now $n \geq 3$ and the lemma is true for $Q^3_{n-1}$. Recall that $Q[i]$ is $2(n-1)$-regular and is isomorphic to $Q^3_{n-1}$ for any $i \in \{0, 1, 2\}$. Let $F_i = F \cap V(Q[i])$, so $\sum_{i=0}^{2} |F_i| \leq 4n - 4$. Then at least one of $|F_0|, |F_1|$ and $|F_2|$ is strictly less than $2(n-1)$ since $F_0 \cap F_1 \cap F_2 = \emptyset$ and $|F_0| + |F_1| + |F_2| = |F| \leq 4n - 4 < 6n - 6$ for $n \geq 2$. Without loss of generality, suppose $|F_0| \leq 2(n - 1) - 1$. We consider the following cases.

**Case 1.** For any $i \in \{1, 2\}$, $|F_i| \leq 2(n - 1) - 1$.

Since $\kappa(Q^3_{n-1}) = 2(n-1)$, then $Q[i] - F_i$ is connected. There are $3^{n-1}$ edges between $Q[i]$ and $Q[j]$ for $0 \leq i \neq j \leq 2$, and $3^{n-1} > 2(n - 1) - 1 + 2(n - 1) - 1 = 4n - 6$ for $n \geq 2$, then $Q[i] - F_i$ is connected to $Q[j] - F_j$. By the arbitrary of $i$ and $j$, $Q^3_n - F$ is connected which is a contradiction.

**Case 2.** $|F_1| \leq 2(n - 1) - 1$ and $2(n - 1) \leq |F_2| \leq 4(n - 1) - 4$ (The case of $2(n - 1) \leq |F_1| \leq 4(n - 1) - 4, |F_2| \leq 2(n - 1) - 1$ is the similar discussion).

By the similar argument to Case 1, $Q[0] - F_0$ is connected to $Q[1] - F_1$. $Q[i] - F_i$ for $i \in \{0, 1\}$ belong to a same component of $Q^3_n - F$, denoted by $C$. If $F_2$ is not a vertex cut of $Q[2]$, then $Q[2] - F_2$ is connected, there are at
least \(3^{n-1} - (4n - 8) - (2n - 3) \geq 2\) edges between \(Q[i] - F_i\) for each \(i \in \{0, 1\}\) and \(Q[2] - F_2\) for \(n \geq 3\), thus \(Q[2] - F_2\) is contained in \(C\), it implies that \(Q_n^3 - F\) is connected which is a contradiction. So \(F_2\) is a vertex cut of \(Q[2]\). By inductive hypothesis, \(Q[2] - F_2\) has two components, one of which is a singleton. Let \(B\) be the largest component of \(Q[2] - F_2\). We will show that \(B\) is connected to \(C\). Note that \(B\) has at least \(3^{n-1} - (4n - 8) - 1\) vertices, and \(2[3^{n-1} - (4n - 8) - 1]\) different outer neighbors. Since there are at most \(4n - 4 - (2n - 2) = 2n - 2\) vertices in \(F - F_2\), and \(2[3^{n-1} - (4n - 8) - 1] > 2n - 2\) for \(n \geq 3\), there must be an edge between \(B\) and \(C\), then \(B\) is contained in \(C\). Thus \(Q_n^3 - F\) has two components, one of which is a singleton.

**Case 3.** For some \(i \in \{1, 2\}\), \(|F_i| \geq 4n - 7\).

Since \(2(4n - 7) > 4n - 4\) for \(n \geq 3\), there is only one \(i \in \{1, 2\}\) such that \(|F_i| \geq 4n - 7\). Without loss of generality, let \(i = 1\). Since \(|F| \leq 4n - 4\), then \(\sum_{j \neq 1} |F_j| \leq 3\). By the the similar argument as Case 1, \(Q[j] - F_j\) for \(j \in \{0, 2\}\) belong to a same component of \(Q_n^3 - F\), denoted by \(C\). If a component, denoted by \(B\), of \(Q[1] - F_1\) has an edge, then its endpoints have exactly four distinct outer neighbors, so \(B\) is contained in \(C\). Thus only singletons of \(Q[1] - F_1\) may not be contained in \(C\). Since each singleton of \(Q[1] - F_1\) has two outer neighbors, which are all different, there can be only one such singleton. Hence \(Q_n^3 - F\) has two components, one of which is a singleton.

**Case 4.** For any \(i \in \{1, 2\}\), \(4n - 7 \geq |F_i| \geq 2(n - 1)\).

Since \(|F| \leq 4n - 4\), then \(|F_0| \leq 0\), that is \(|F_0| = 0\). So \(Q[0] - F_0\) is connected, denoted it by \(C\). Every vertex of \(Q[i] - F_i\) has one neighbor outside \(Q[1] \cup Q[2]\) for each \(i \in \{1, 2\}\), so any component of \(Q[i] - F_i\) is connected to \(C\). Then \(Q_n^3 - F\) is connected which is a contradiction.

The lemma is completed. \(\square\)

**Lemma 3.4.** Suppose that \(F \subseteq V(Q_n^3)\) with \(|F| \leq 6n - 8\) is a vertex cut of \(Q_n^3\) for \(n \geq 3\), then \(Q_n^3 - F\) either has two components, one of which is a singleton, or an edge; or has three components, two of which are singletons.

**Proof.** Let \(F_i = V(Q[i]) \cap F\) for \(i \in \{0, 1, 2\}\), so \(\sum_{i=0}^2 |F_i| = |F| \leq 6n - 8\). Since \(F_0 \cap F_1 \cap F_2 = \emptyset\) and \(|F_0| + |F_1| + |F_2| = |F| \leq 6n - 8 < 6n - 6\) for \(n \geq 3\), then at least one of \(|F_0|, |F_1|\) and \(|F_2|\) is strictly less than \(2(n - 1)\). Without loss of generality, suppose \(|F_0| \leq 2(n - 1) - 1\). We will prove the lemma by using induction on \(n\).

**Case 1.** We prove the result for \(Q_n^3\) with \(n = 3\).
The graph is $Q_3^n$, shown in Fig.1. Let $F \subseteq V(Q_3^n)$ with $|F| \leq 6n - 8 = 10$ be a vertex cut of $Q_3^n$. We consider the following subcases.

**Subcase 1.1.** For any $i \in \{1, 2\}$, $|F_i| \leq 3$.

Since $\kappa(Q_2^3) = 4$, then $Q[i] - F_i$ is connected. There are $3^2 - |F_i| - |F_j| \geq 3^2 - 2 \times 3 = 3$ edges between $Q[i] - F_i$ and $Q[j] - F_j$ for $0 \leq i \neq j \leq 2$, thus $Q[i] - F_i$ is connected to $Q[j] - F_j$. By the arbitrary of $i$ and $j$, $Q_3^n - F$ is connected which is a contradiction.

**Subcase 1.2.** $|F_1| \leq 3$ and $4 = 2(n - 1) \leq |F_2| \leq 6(n - 1) - 8 = 4$, that is $|F_2| = 4$ (The case of $|F_1| = 4$ and $|F_2| \leq 3$ is the similar discussion).

By the similar argument to Subcase 1.1, $Q[0] - F_0$ is connected to $Q[1] - F_1$. $Q[i] - F_i$ for $i \in \{0, 1\}$ belong to a same component of $Q_3^n - F$, denoted by $C$. If $Q[2] - F_2$ is connected, since there are at least $3^2 - |F_i| - |F_2| \geq 3^2 - 4 - 3 = 2$ edges between $Q[i] - F_i$ and $Q[2] - F_2$ for $i \in \{0, 1\}$, then $Q[2] - F_2$ is contained in $C$, thus $Q_3^n - F$ is connected which is a contradiction. Hence $Q[2] - F_2$ is disconnected, since $|F_2| = \kappa(Q_2^3) = 4$, $F_2$ is a minimum vertex cut of $Q[2]$, by Lemma 2.2, $Q[2] - F_2$ has two components, one of which is a singleton. Let $B$ be the largest component of $Q[2] - F_2$. Note that $B$ has 4 vertices, which has 8 different outer neighbors. Since there are at most $10 - 4 = 6$ vertices in $F - F_2$, there must be two edges between $B$ and $C$, then $B$ is contained in $C$. Thus $Q_3^n - F$ has two components, one of which is a singleton.

**Subcase 1.3.** For some $i \in \{1, 2\}$, $|F_i| \geq 6n - 13 = 5$.

Without loss of generality, suppose $|F_2| \geq 5$. Since $|F| \leq 10$ and $|F_0| \leq 3$, then $\sum_{j \neq 2} |F_j| \leq 5$, $|F_1| \leq 2$. Since $\kappa(Q_2^3) = 4$, then $Q[i] - F_i$ is connected for $i \in \{0, 1\}$. By the similar argument as Subcase 1.1, $Q[0] - F_0$ is connected to $Q[1] - F_1$. $Q[i] - F_i$ for $i \in \{0, 1\}$ belong to a same component of $Q_3^n - F$, denoted by $C$. Every vertex of $Q[2] - F_2$ has two outer neighbors, any component with more than two vertices of $Q[2] - F_2$ is contained in $C$. Thus $Q_3^n - F$ either has two components, one of which is a singleton, or an edge; or has three components, two of which are singletons.

**Subcase 1.4.** $|F_1| \geq 4, |F_2| \geq 4$.

Since $|F| \leq 10$, then $|F_0| = |F| - |F_1| - |F_2| \leq 2$. So $Q[0] - F_0$ is connected. Let $C$ be a component of $Q_3^n - F$ which contains $Q[0] - F_0$. Every vertex of $Q[i] - F_i$ has one neighbor outside $Q[1] \cup Q[2]$ for $i \in \{1, 2\}$, so any component with more than two vertices of $Q[i] - F_i$ is contained in $C$. Thus $Q_3^n - F$ either has two components, one of which is a singleton, or an edge; or has three components, two of which are singletons.

Then the result holds for $n = 3$. In what follows, assume that $n \geq 4$ and the result holds for $Q_3^{n-1}$.
Case 2. We prove the result for $Q^3_n$ and $n \geq 4$. We consider the following subcases.

Subcase 2.1. For any $i \in \{1, 2\}$, $|F_i| \leq 2(n-1) - 1$.
Since $\kappa(Q^3_n) = 2(n-1)$, then $Q[i] - F_i$ is connected. There are $3^{n-1}$ edges between $Q[i]$ and $Q[j]$ for $0 \leq i \neq j \leq 2$, and $3^{n-1} > 2(n-1) - 1 + 2(n-1) - 1 = 4n - 6$ for $n \geq 3$, then $Q[i] - F_i$ is connected to $Q[j] - F_j$. By the arbitrary of $i$ and $j$, $Q^3_n - F$ is connected which is a contradiction.

Subcase 2.2. $|F_1| \leq 2(n-1) - 1$ and $2(n-1) \leq |F_2| \leq 6n - 14$ (The case of $2(n-1) \leq |F_1| \leq 6n - 14$ and $|F_2| \leq 2(n-1) - 1$ is the similar discussion).

By the similar argument as Subcase 2.1, $Q[0] - F_0$ is connected to $Q[1] - F_1$. The component which contains $Q[i] - F_i$ for $i \in \{0, 1\}$ of $Q^3_n - F$ is denoted by $C$. If $Q[2] - F_2$ is connected, there are at least $3^{n-1} - (6n-14) - (2n-3) \geq 12$ edges between $Q[i] - F_i$ (for $i \in \{0, 1\}$) and $Q[2] - F_2$ for $n \geq 4$, thus $Q[2] - F_2$ is connected to $C$. It implies that $Q^3_n - F$ is connected which is a contradiction. Hence $F_2$ is a vertex cut of $Q[2]$. By inductive hypothesis, there are at most three components in $Q[2] - F_2$, with two of them having at most two vertices in total. Let $D$ be the largest component of $Q[2] - F_2$. $D$ has at least $3^{n-1} - (6n-14) - 2$ vertices, by Lemma 2.4, $D$ has at least $2[3^{n-1} - (6n-14) - 2]$ distinct outer neighbors. Since $|F_0| + |F_1| = |F| - |F_2| \leq 6n - 8 - (2n-2) = 4n - 6$ and $2[3^{n-1} - (6n-14) - 2] > 4n - 6$ for $n \geq 4$, then there exists at least one edge between $D$ and $C$. Hence $D$ is contained in $C$. Thus the smallest component at most contains two vertices, the result is proved in this case.

Subcase 2.3. For some $i \in \{1, 2\}$, $|F_i| \geq 6n - 13$.
Without loss of generality, suppose $|F_2| \geq 6n - 13$. Then $\sum_{j \neq 2} |F_j| = |F| - |F_2| \leq 5 \leq 2(n-1) - 1$ for $n \geq 4$. Since $\kappa(Q^3_n) = 2(n-1) \geq 6$ for $n \geq 4$, then $Q[i] - F_i$ is connected for each $i \in \{0, 1\}$. By the similar argument as Subcase 2.1, $Q[0] - F_0$ is connected to $Q[1] - F_1$. Let $C$ be the component of $Q^3_n - F$ which contains $Q[i] - F_i$ for $i \in \{0, 1\}$. Every vertex has two outer neighbors, any component of $Q[2] - F_2$ with more than two vertices is contained in $C$. Thus $Q^3_n - F$ either has two components, one of which is a singleton, or an edge; or has three components, two of which are singletons.

Next we only need to consider the case that $F_i$ is greater than $2(n-1) - 1$ for each $i \in \{1, 2\}$.
Subcase 2.4. $|F_1| \geq |F_2| \geq 2(n - 1)$ (The case of $|F_2| \geq |F_1| \geq 2(n - 1)$ is the similar discussion).

Clearly, $|F_2| \leq |F_1| \leq 6n - 8 - 2(n - 1) = 4n - 6$. If $|F_1| = 4n - 6$, then $|F_2| = 2n - 2$, $|F_0| = 0$. So $Q[0] - F_0$ is connected, denoted it by $C$. Every vertex of $Q[i] - F_i$ has one neighbor outside $Q[1] \cup Q[2]$ for each $i \in \{1, 2\}$, so any component of $Q[i] - F_i$ is contained in $C$. Then $Q^3_n - F$ is connected which is a contradiction.

Now we consider $|F_1| \leq 4n - 7$. First consider $|F_1| = 4n - 7$, if $|F_2| = 2n - 1$, we have done by the previous argument. So the left case is $|F_2| = 2n - 2$, $|F_0| = 1$. Thus $Q[0] - F_0$ is connected, denoted it by $C$. Since every vertex in $Q[1]$ and $Q[2]$ has an outer neighbors in $Q[0]$ and $|F_0| = 1$, then at most one vertex can be disconnected from $C$ in $Q^3_n - F$, hence $Q^3_n - F$ has two components, one of which is a singleton.

Finally, we consider those cases where $|F_2| \leq |F_1| \leq 4n - 8 = 4(n - 1) - 4$. This case is divided into three subcases.

Subcase 2.4.1. Both $Q[1] - F_1$ and $Q[2] - F_2$ are connected.

In this case, there are at least $3^{n-1} - |F_1| - |F_0| \geq 3^{n-1} - (4n - 8) - (2n - 3) \geq 14$ edges between $Q[0] - F_0$ and $Q[i] - F_i$ for each $i \in \{1, 2\}$ and $n \geq 4$, thus $Q[i] - F_i$ is connected to $Q[0] - F_0$. Hence $Q^3_n - F$ is connected which is a contradiction.

Subcase 2.4.2. Only one of $Q[1] - F_1$ and $Q[2] - F_2$ is connected.

Without loss of generality, assume that $Q[1] - F_1$ is connected and $Q[2] - F_2$ is disconnected. By the similar argument as Subcase 2.4.1, $Q[1] - F_1$ is connected to $Q[0] - F_0$. Let $C$ be the component of $Q^3_n - F$ which contains $Q[0] - F_0$ and $Q[1] - F_1$. Since $Q[2] - F_2$ is disconnected and $|F_2| \leq 4n - 8 = 4(n - 1) - 4$, by Lemma 3.3, $Q[2] - F_2$ has two components, one of which is a singleton. Let $D$ be the largest component of $Q[2] - F_2$. Note that $D$ has at least $3^{n-1} - (4n - 8) - 1$ vertices, and has at least $3^{n-1} - (4n - 8) - 1$ neighbors in $Q[0]$, since $3^{n-1} - (4n - 8) - 1 > 2(n - 1) - 1 \geq |F_0|$ for $n \geq 4$, thus $D$ is contained in $C$. Hence $Q^3_n - F$ has two components, one of which is a singleton.

Subcase 2.4.3. Both $Q[1] - F_1$ and $Q[2] - F_2$ are disconnected.

By Lemma 3.3, $Q[i] - F_i$ for each $i \in \{1, 2\}$ has two components, one of which is a singleton, denoted by $x_i$. Since $|F_0| \leq 2(n - 1) - 1$, then $Q[0] - F_0$ is connected. Let $B_i$ be the largest component of $Q[i] - F_i$ for each $i \in \{1, 2\}$. Note that $B_i$ has at least $3^{n-1} - (4n - 8) - 1$ vertices, and has at least $3^{n-1} - (4n - 8) - 1$ outer neighbors in $Q[0]$, since $3^{n-1} - (4n - 8) - 1 > 2(n - 1) - 1$ for $n \geq 4$, thus $B_i$ is connected to $Q[0] - F_0$. Let $C$ be the component of
$Q_n^3 - F$ which contains $B_i$ and $Q[0] - F_0$.

If both $x_1$ and $x_2$ are contained in $C$, then $Q_n^3 - F$ is connected which is a contradiction. If only one of $x_i$ is contained in $C$, then $Q_n^3 - F$ has two components, one of which is a singleton. Besides, the two singletons in $Q[1] - F_1$ and $Q[2] - F_2$ may either remain singleton components in $Q_n^3 - F$ or they could belong to one component of $Q_n^3 - F$, forming a $K_2$. Hence $Q_n^3 - F$ either has two components, one of which is an edge; or has three components, two of which are singletons. The result holds in this case.

The proof of the lemma is finished. □

**Theorem 3.5.** For $n \geq 3$, $\kappa_3(Q_n^3) \leq 8n - 12$.

**Proof.** Let $u = (0, 0, 0, 0, \ldots, 0)$, $v = (0, 1, 0, 0, \ldots, 0)$, $w = (0, 1, 1, 0, \ldots, 0)$ and $t = (0, 0, 1, 0, \ldots, 0)$ be four vertices in $Q_n^3$, $P_4 = uvwt \in Q_n^3$ be a path of length three. Let $F = N(P_4)$, obviously, $Q_n^3 - F$ is disconnected. Note that $(u, v, w, t, u)$ is a cycle of length four. By Lemma 3.1 and the structure of $Q_n^3$, $u$ has a neighbor set $X_1$ with the order $2n - 2$ in $V(Q_n^3) - V(P_4)$; Since $u$ and $v$ have one common neighbor $x_1 = (0, 2, 0, 0, \ldots, 0)$, $v$ has a neighbor set $X_2$ with the order $2n - 2 - 1 = 2n - 3$ in $V(Q_n^3) - V(P_4) - X_1$; Since $w$ and $u$ have two common neighbors $v$ and $t$, $w$ and $v$ have one common neighbor $x_2 = (0, 1, 2, 0, \ldots, 0)$, $w$ has a neighbor set $X_3$ with the order $2n - 2 - 1 = 2n - 3$ in $V(Q_n^3) - V(P_4) - X_1 - X_2$; Since $t$ and $u$ have one common neighbor $x_3 = (0, 0, 2, 0, \ldots, 0)$, $t$ and $v$ have two common neighbors $w$ and $u$, $t$ and $w$ have one common neighbor $x_4 = (0, 2, 1, 0, \ldots, 0)$, $t$ has a neighbor set $X_4$ with the order $2n - 2 - 1 = 2n - 3$ in $V(Q_n^3) - V(P_4) - X_1 - X_2 - X_3$. Thus

$$|F| = |X_1| + |X_2| + |X_3| + |X_4| = (2n - 2) + (2n - 3) + (2n - 3) + (2n - 3) = 8n - 12.$$ We will show that $F$ is a 3-extra vertex cut of $Q_n^3$ for $n \geq 3$.

For $n = 3$, from Fig.1, it is easy to see that $F$ is a 3-extra vertex cut of $Q_3^3$. We assume that $n \geq 4$ in the following. Recall that $N[P_4] = N(P_4) \cup P_4$, we will prove that $Q_n^3 - N[P_4]$, connected for $n \geq 4$.

Without loss of generality, we partition $Q_n^k$ over 0-dimension. Let $F_i = N(P_4) \cap Q[i]$, where $i \in \{0, 1, 2\}$. Note that $P_4 = uvwt \in Q[0]$, the two outer neighbors of every vertex in $P_4$ are in different subgraph $Q[j]$, $j \neq i$, thus $|F_k| = 4$, $k = 1, 2$. By Lemma 2.2, $\kappa(Q_{n-1}^3) = 2(n - 1) \geq 6 > 4$, then $Q[k] - F_k$ ($k = 1, 2$) is connected. Since there are $3^n - 1 \geq 9$ edges between $Q[i]$ and $Q[j]$ for $0 \leq i \neq j \leq 2$ and $n \geq 3$. Thus $Q_n^3 - Q[0] - N[P_4]$ is connected, denoted by $C$.

Now we consider $Q[0] - N[P_4]$, for any $x \in Q[0] - N[P_4]$, $x$ has two outer neighbors $x_L$ and $x_R$, obviously, $x_L(x_R)$ is not in $N[P_4] \cap (Q_n^3 - Q[0])$. Hence
x is connected to C. By the arbitrary of x, \( C \cup (Q[0] - N[P_4]) = Q^3_n - N[P_4] \) is connected.

Thus \( Q^3_n - F \) has two components, \( Q^3_n - N[P_4] \) and \( P_4 \). Then \( F \) is a 3-extra vertex cut of \( Q^3_n \) for \( n \geq 3 \), thus \( \kappa_3(Q^3_n) \leq |F| = 8n - 12 \). The theorem is completed. \( \square \)

In the following, suppose \( F \subseteq V(Q^3_n) \) is a faulty vertex set of \( Q^3_n \). For convenience, let \( F_i = F \cap V(Q[i]) \), \( I = \{ i \mid Q[i] - F_i \) is disconnected for \( i \in \{0, 1, 2\} \}, J = \{0, 1, 2\} \setminus I \). \( F_i = \bigcup_{i \in I} F_i, F_J = \bigcup_{j \in J} F_j, Q[I] = \bigcup_{i \in I} Q[i], Q[J] = \bigcup_{j \in J} Q[j] \).

**Lemma 3.6.** Let \( F \subseteq V(Q^3_n) \) with \( |F| \leq 8n - 13 \) be a faulty vertex set of \( Q^3_n \). Then, \( Q[J] - F_J \) is connected when \( |I| \leq 2 \) and \( n \geq 4 \). Furthermore, let \( H \) be a component of \( Q^3_n - F \) and \( H \cap (Q[J] - F_J) = \emptyset \), then \( N_{Q[I]}(H) \subseteq F_I, N_{Q[J]}(H) \subseteq F_J \).

**Proof.** Clearly, \( |I| \leq 3, |F_i| \geq 2n - 2 \) for any \( i \in I \). By the definition of \( J \), for any \( j \in J \), \( Q[j] - F_j \) is connected. We consider the following three cases.

**Case 1.** \( |I| = 0 \).

For any \( j \in J \), \( Q[j] - F_j \) is connected. Since \( 3^{n-1} - (8n - 13) \geq 8 \) for \( n \geq 4 \), there must be an edge between \( Q[j] - F_j \) and \( Q[k] - F_k \) for \( j \neq k \) and \( j, k \in \{0, 1, 2\} \). \( Q[J] - F_J \) is connected for \( n \geq 4 \).

**Case 2.** \( |I| = 1 \).

Without loss of generality, we assume that \( I = \{0\} \), then \( |F_0| \geq \kappa(Q^3_{n-1}) = 2(n - 1) \), both \( Q[1] - F_1 \) and \( Q[2] - F_2 \) are connected. There are at least \( 3^{n-1} - (|F| - |F_0|) \geq 3^n - 8n + 13 - (2n - 2) \geq 3^n - (19 - 6) = 14 \) edges between \( Q[1] - F_1 \) and \( Q[2] - F_2 \) for \( n \geq 4 \), thus \( Q[J] - F_J \) is connected.

**Case 3.** \( |I| = 2 \).

Without loss of generality, we assume that \( I = \{0, 1\} \). Then \( Q[J] - F_J = Q[2] - F_2 \) is also connected.

Hence \( Q[J] - F_J \) is connected for \( |I| \leq 2 \) and \( n \geq 4 \).

Suppose there exists a vertex \( u \in N_{Q[I]}(H) \), \( u \notin F_I \). Then \( u \) is connected to \( H \), hence \( u \) belongs to \( H \), which leads to a contradiction. Thus \( N_{Q[I]}(H) \subseteq F_I \). If there exists a vertex \( v \in N_{Q[J]}(H), v \notin F_J \), then \( v \) is connected to \( H, v \) is contained in \( H \), then \( H \cap (Q[J] - F_J) = \{v\} \), which is contradict to \( H \cap (Q[J] - F_J) = \emptyset \). Hence, \( N_{Q[J]}(H) \subseteq F_J \). The lemma is completed. \( \square \)

**Theorem 3.7.** Suppose that \( F \subseteq V(Q^3_n) \) with \( |F| \leq 8n - 13 \) is a vertex cut of \( Q^3_n \) for \( n \geq 3 \), then \( Q^3_n - F \) has one of the following conditions:
(1) two components, one of which is a singleton or an edge or a 2-path or a 3-cycle.
(2) three components, two of which are singletons.
(3) three components, two of which are a singleton and an edge, respectively.
(4) four components, three of which are singletons.

Proof. This theorem can be proved by using induction on \( n \). 

Case 1. Suppose that \( n = 3 \) and \( F \subseteq V(Q^3_3) \) with \( |F| \leq 8n - 13 = 11 \), is a vertex cut of \( Q^3_3 \). We consider the following four subcases.

Subcase 1.1. \( |I| = 0 \).

Since \( J = \{0, 1, 2\}\backslash I = \{0, 1, 2\} \).

If for any \( j \in J \), \( |F_j| < \kappa(Q^3_{n-1}) = 2(n-1) = 4 \). Since \( 3^{3-1} - |F_j| - |F_k| \geq 3^{3-1} - 4(3 - 1) = 1 \), there must be an edge between \( Q[j] - F_j \) and \( Q[k] - F_k \) for \( j \neq k \) and \( j, k \in \{0, 1, 2\} \). In this case \( Q[\{j\}] - F_j \) is connected which is a contradiction.

If there exists only one \( j \in J \), \( |F_j| \geq \kappa(Q^3_{n-1}) = 2(n-1) = 4 \), suppose \( j = 0 \). \( Q[1] - F_1 \) is connected to \( Q[2] - F_2 \). Since \( |V(Q[0])| = 3^2 = 9 \), then \( 4 \leq |F_0| \leq 9 \). If \( Q[0] - F_0 \) is empty, then \( \sum_{j=1}^{2} |F_j| \leq 2 \), there are at least \( 9 - 2 = 7 \) edges between \( Q[1] - F_1 \) and \( Q[2] - F_2 \), thus \( Q^3_3 - F \) is connected.

If \( Q[0] - F_0 \) has one vertex, then \( \sum_{j=1}^{2} |F_j| \leq 3 \). By Lemma 2.4, \( Q^3_3 - F \) is connected or has two components, one of which is a singleton. If \( Q[0] - F_0 \) has two vertices, then \( \sum_{j=1}^{2} |F_j| \leq 4 \). In this case, \( Q^3_3 - F \) is connected or has two components, one of which is an edge. If \( Q[0] - F_0 \) has three vertices, then \( \sum_{j=1}^{2} |F_j| \leq 5 \). The three vertices have six outer neighbors, \( Q^3_3 - F \) is connected in this case. If \( Q[0] - F_0 \) has four or five vertices, \( Q^3_3 - F \) is connected by the similar reason.

If there exists only two \( j \in J \), \( |F_j| \geq \kappa(Q^3_{n-1}) = 2(n-1) = 4 \), suppose \( j \in \{0, 1\} \). Then \( |F_2| \leq 3 \). Clearly, \( |F_0| \leq |F| - |F_1| \leq 7 \), \( |F_1| \leq |F| - |F_0| \leq 7 \).

Suppose \( |F_0| \leq |F_1| \). If \( |F_1| = 7 \), then \( |F_0| = 4 \), \( |F_2| = 0 \), every vertex of \( Q[j] - F[j] \) for \( j \in \{0, 1\} \) has an outer neighbor in \( Q[2] \), then \( Q^3_3 - F \) is connected. Now we consider \( |F_1| = 6 \), if \( |F_0| = 5 \), as the similar reason, \( Q^3_3 - F \) is connected. The left case is \( |F_0| = 4 \), \( |F_2| \leq 1 \), note that \( |V(Q[0] - F[0])| = 5 \) and \( |V(Q[1] - F[1])| = 3 \), every vertex of \( Q[j] - F[j] \) for each \( j \in \{0, 1\} \) has an outer neighbor in \( Q[2] \). Thus there is an edge between \( Q[j] - F[j] \) and \( Q[2] - F[2] \), then \( Q^3_3 - F \) is connected. Next consider \( |F_1| = 5 \), if \( |F_0| = 5 \), then \( |F_2| \leq 1 \), \( Q^3_3 - F \) is connected; if \( |F_0| = 4 \), then \( |F_2| \leq 2 \), \( Q^3_3 - F \) is also connected.
If for any $j \in J$, $|F_j| \geq \kappa(Q_{n-1}^3) = 2(n - 1) = 4$, then $|F| \geq 12$, this is contradict to $|F| \leq 8n - 13 = 11$.

In summary, $Q_n^3 - F$ has two components, one of which is a singleton or an edge. The result holds.

**Subcase 1.2.** $|I| = 1.$

Without loss of generality, we assume that $I = \{0\}$. Then $Q[0] - F_0$ is disconnected. By Lemma 2.2, $|F_0| \geq 2n - 2 = 4$. Thus $|F| - |F_0| \leq 11 - 4 = 7$. There are at least $3^{2-1} - (|F| - |F_0|) \geq 3^2 - (11 - 4) \geq 2$ edges between $Q[1] - F_1$ and $Q[2] - F_2$, then $Q[J] - F_J$ is connected.

Suppose $W$ is the union of all components of $Q_n^3 - F$ and has no vertices in $Q[J] - F_J$. Since $Q_n^3 - F$ is disconnected, then $W$ exists. Every vertex in $W$ has two outer neighbors, so $2|W| \leq |F| - |F_0| \leq 7$, that means $|W| \leq 3$, the result holds.

**Subcase 1.3.** $|I| = 2.$

Without loss of generality, we assume that $I = \{0, 1\}$. Then $Q[0] - F_0$ and $Q[1] - F_1$ are disconnected, $Q[2] - F_2$ is connected. By Lemma 2.2, $|F_0| \geq 2n - 2 = 4$ and $|F_1| \geq 2n - 2 = 4$, so $|F_2| = |F| - |F_0| - |F_1| \leq 11 - 4 - 4 = 3$. Suppose $W$ is the union of all components of $Q_n^3 - F$ and has no vertices in $Q[2] - F_2$. Since $Q_n^3 - F$ is disconnected, then $W$ exists. Every vertex in $W$ has one outer neighbor in $Q[2]$, so $|W| \leq |F_2| \leq 3$. The desired result.

**Subcase 1.4.** $|I| = 3.$

In this case, $I = \{0, 1, 2\}$. For any $i \in I$, $|F_i| \geq 2n - 2 = 4$, then $|F| = |F_0| + |F_1| + |F_2| \geq 12$, this is contradict to $|F| \leq 11$.

In summary, we have proved the result holds for $Q_n^3$.

In what follows, assume that $n \geq 4$ and the result holds for $Q_{n-1}^3$.

**Case 2.** We prove the result for $Q_n^3$ and $n \geq 4$. We divide the proof into the following two subcases.

**Subcase 2.1.** For any $i \in \{0, 1, 2\}$, $|F_i| \leq 6n - 14$.

Recall that $|F| \leq 8n - 13$, by Lemma 3.6, $Q[J] - F_J$ is connected when $|I| \leq 2$. We consider the following four subcases.

**Subcase 2.1.1.** $|I| = 0.$

Since $J = \{0, 1, 2\} \setminus I = \{0, 1, 2\}$, by Lemma 3.6, $Q[J] - F_J = Q_n^3 - F$ is connected which is a contradiction.

**Subcase 2.1.2.** $|I| = 1.$

Without loss of generality, we assume that $I = \{0\}$. By Lemma 3.6, $Q[J] - F_J$ is connected. Since $Q[0] - F_0$ is disconnected, and $|F_i| \leq 6n - 14 = 6(n - 1) - 8$, by Lemma 3.4, $Q[0] - F_0$ either has two components, one of
which is a singleton or an edge, denoted by $X_0$; or has three components, two of which are singletons, denoted by $X_0 = \{u, v\}$. Let $B$ be the largest component of $Q[0] - F_0$. Next we show $B$ is connected to $Q[J] - F_J$. Note that $B$ has at least $3^{n-1} - (6n - 14) - 2$ vertices, and has at least $2[3^{n-1} - (6n - 14) - 2]$ outer neighbors in $Q[J]$. Since $2[3^{n-1} - (6n - 14) - 2] > 8n - 13 - (2n - 2) = 6n - 11$ for $n \geq 4$, then $B$ is connected to $Q[J] - F_J$. Thus $Q_n^3 - F$ either has two components, one of which is a singleton or an edge; or has three components, two of which are singletons. The result holds.

**Subcase 2.1.3.** $|I| = 2$.

Without loss of generality, we assume that $I = \{0, 1\}$. By Lemma 3.6, $Q[2] - F_2$ is connected. For $i \in I$, since $|F_i| \leq 6n - 14 = 6(n - 1) - 8$, by Lemma 3.4, $Q[i] - F_i$ either has two components, one of which is a singleton or an edge, denoted by $X_i$; or has three components, two of which are singletons, denoted by $X_i = \{u_i, v_i\}$. Let $B_i$ be the largest component of $Q[i] - F_i$, then $X_i = Q[i] - F_i - B_i$. We claim that $B_i$ is connected to $Q[2] - F_2$. In fact, $B_i$ has at least $3^{n-1} - (6n - 14) - 2$ vertices, and $3^{n-1} - (6n - 14) - 2$ outer neighbors in $Q[2]$, since $|F_2| = |F| - |F_0| - |F_1| \leq 8n - 13 - 2(2n - 2) = 4n - 9$ and $3^{n-1} - (6n - 14) - 2 > 4n - 9$ for $n \geq 4$, then $B_i$ is connected to $Q[2] - F_2$. Let $C$ be the component of $Q_n^3 - F$ which contains $B_i$ and $Q[2] - F_2$. Next we consider the following three subcases.

**Subcase 2.1.3a.** Both $Q[0] - F_0$ and $Q[1] - F_1$ have two components, one of which is a singleton or an edge.

If both $Q[0] - F_0$ and $Q[1] - F_1$ have two components, one of which is a singleton. By the similar argument as Subcase 2.4.3 of Lemma 3.4, $Q_n^3 - F$ either has three components, two of which are singletons; or has two components, one of which is a singleton or an edge. The result holds.

If only one of $Q[0] - F_0$ and $Q[1] - F_1$ has two components, one of which is a singleton. Without loss of generality, assume that $Q[0] - F_0$ has two components, and one of which is a singleton which is denoted by $x_0$ and $Q[1] - F_1$ has two components, and one of which is an edge which is denoted by $X_1 = u_1v_1$. If both $x_0$ and $X_1$ are contained in $C$, then $Q_n^3 - F$ is connected which is a contradiction. If $x_0$ is contained in $C$, $X_1$ is not contained in $C$, then $Q_n^3 - F$ has two components, one of which is an edge. If $x_0$ is not contained in $C$ and $X_1$ is contained in $C$, then $Q_n^3 - F$ has two components, one of which is a singleton. Besides, the singleton $x_0$ in $Q[0] - F_0$ and the isolated edge $X_1$ in $Q[1] - F_1$ may either remain singleton and isolated edge in $Q_n^3 - F$; or they could belong to one component of $Q_n^3 - F$, forming a 2-path. Thus $Q_n^3 - F$ either has three components, two of which are a singleton and
an edge, respectively; or has two components, one of which is a 2-path. The result holds in this case.

If both $Q[0] - F_0$ and $Q[1] - F_1$ have two components, one of which is an edge, denoted by $X_i$ for $i \in I$. By Lemma 3.3, $|F_i| \geq 4(n-1) - 3 = 4n - 7$, $|F_2| = |F| - |F_0| - |F_1| \leq 8n - 13 - 2(4n - 7) = 1$. Every vertex in $X_i$ has one outer neighbor in $Q[2]$. By Lemma 3.6, $N_{Q[2]}(X_i) \subseteq F_2$, then $2 = |N_{Q[2]}(X_i)| \leq |F_2| \leq 1$ which is a contradiction.

**Subcase 2.1.3b.** Only one of $Q[0] - F_0$ and $Q[1] - F_1$ has two components.

Without loss of generality, assume that $Q[0] - F_0$ has two components and $Q[1] - F_1$ has three components.

If $Q[0] - F_0$ has two components, one of which is a singleton, denoted by $x_0$. $Q[1] - F_1$ has three components, two of which are singletons, denoted by $u_1, v_1$. If $x_0$, $u_1$, and $v_1$ are contained in $C$, then $Q^3_n - F$ is connected which is a contradiction. If $x_0$ is contained in $C$, only one of $u_1$ and $v_1$ is contained in $C$, then $Q^3_n - F$ has two components, one of which is a singleton. If $x_0$ is contained in $C$, both $u_1$ and $v_1$ are not contained in $C$, then $Q^3_n - F$ has three components, two of which are singletons. If $x_0$ is not contained in $C$, both $u_1$ and $v_1$ are contained in $C$, $Q^3_n - F$ has two components, one of which is a singleton. If $x_0$ is not contained in $C$, only one of $u_1$ and $v_1$ is contained $C$. In this case, $Q^3_n - F$ has two components, one of which is an edge; or $Q^3_n - F$ has three components, two of which are singletons. Besides, the singletons in $Q[1] - F_1$ and $Q[0] - F_0$ may remain singleton components in $Q^3_n - F$; or they could belong to two components of $Q^3_n - F$, $K_2$ and a singleton. Thus $Q^3_n - F$ either has four components, three of which are singletons; or has three components, two of which are a singleton and an edge, respectively. The result holds.

If $Q[0] - F_0$ have two components, one of which is an edge, denoted by $X_0$. By Lemma 3.3, $|F_0| \geq 4(n-1) - 3 = 4n - 7$ and $|F_1| \geq 4(n-1) - 3 = 4n - 7$, by the similar argument as Subcase 2.1.3a, 2 = $|N_{Q[2]}(X_0)| \leq |F_2| \leq 1$ which is a contradiction.

**Subcase 2.1.3c.** Both $Q[0] - F_0$ and $Q[1] - F_1$ have three components, two of which are singletons, denoted by $X_i = \{u_i, v_i\}$ for $i \in I$.

By Lemma 3.3, $|F_0| \geq 4(n-1) - 3 = 4n - 7$ and $|F_1| \geq 4(n-1) - 3 = 4n - 7$. Then $|F_2| = |F| - |F_0| - |F_1| \leq 8n - 13 - 2(4n - 7) = 1$. By Lemma 3.6, $N_{Q[2]}(u_i) \subseteq F_2$, and $N_{Q[2]}(v_i) \subseteq F_2$. Then $2 = |N_{Q[2]}(X_i)| \leq |F_2| \leq 1$ which is a contradiction.

**Subcase 2.1.4.** $|I| = 3$. 

15
We have $I = \{0, 1, 2\}$, by Lemma 2.2, for any $i \in I$, $|F_i| \geq 2(n-1) = 2n - 2$. Since $|F| \leq 8n - 13$, then $|F_i| \leq 8n - 13 - 2(n-2) = 4n - 9 \leq 4(n-1) - 4$. By Lemma 3.3, $Q[i] - F_i$ has two components, one of which is a singleton $x_i$. Let $B_i$ be the largest component of $Q[i] - F_i$ for $i \in I$. Note that $B_i$ has at least $3^{n-1} - (4n - 9) - 1$ vertices, and has at least $3^{n-1} - (4n - 9) - 1$ neighbors in $Q[k]$ for $0 \leq k \neq i \leq 2$. $B_i$ is connected to $B_k$ since $|F_k| \leq 4n - 9$ and $3^{n-1} - (4n - 9) - 1 > 4n - 9$ for $n \geq 4$. Let $C$ be the component of $Q^3_n - F$ which contains $B_i$ for $i \in I$. We consider the following four subcases.

**Subcase 2.1.4a.** For any $i \in I$, $x_i$ is contained in $C$.

In this case, $Q^3_n - F$ is connected which is a contradiction.

**Subcase 2.1.4b.** There exists only two $i \in I$, such that $x_i$ is contained in $C$.

In this case, $Q^3_n - F$ has two components, one of which is a singleton.

**Subcase 2.1.4c.** Only one $i \in I$, such that $x_i$ is contained in $C$.

In this case, $Q^3_n - F$ either has three components, two of which are singletons; or has two components, one of which is an edge. The result holds.

**Subcase 2.1.4d.** For any $i \in I$, $x_i$ is not contained in $C$.

In this case, the three singletons in $Q[i] - F_i$ may remain singleton components of $Q^3_n - F$; or they could belong to two components, a singleton and an edge; or they could belong to one component forming a 3-cycle or a 2-path. Thus $Q^3_n - F$ either has four components, three of which are singletons; or has three components, two of which are a singleton and an edge, respectively; or has two components, one of which is a 2-path, or a 3-cycle. Then the result holds.

**Subcase 2.2.** There exists some $i \in \{0, 1, 2\}$, such that $|F_i| \geq 6n - 13$.

Without loss of generality, we assume that $|F_0| \geq 6n - 13$. For $j \in \{1, 2\}$, $|F_j| \leq 8n - 13 - (6n - 13) = 2n < 4(n-1) - 3 = 4n - 7$ for any $n \geq 4$. We consider the following two subcases.

**Subcase 2.2.1.** For $j \in \{1, 2\}$, $Q[j] - F_j$ is connected.

Since there are at least $3^{n-1} - |F_1| - |F_2| \geq 3^{n-1} - 2n - 2n > 11$ edges between $Q[1] - F_1$ and $Q[2] - F_2$ for $n \geq 4$, then $Q[1] - F_1$ is connected to $Q[2] - F_2$. The component which contains $Q[j] - F_j$ for $j \in \{1, 2\}$ of $Q^3_n - F$ is denoted by $C$.

If $|F_0| \leq 8(n-1) - 13 = 8n - 21$, suppose $Q[0] - F_0$ is connected. There are at least $3^{n-1} - (8n - 21) - (2n - 2) \geq 10$ edges between $Q[j] - F_j$ and $Q[0] - F_0$ for $n \geq 4$, then $Q[0] - F_0$ is contained in $C$. It implies that $Q^3_n - F$ is connected which is a contradiction. Hence $F_0$ is a vertex cut of $Q[0]$. By induction on $n$, $Q[0] - F_0$ satisfies one of the conditions (1)-(4). Let $D$ be the
Suppose $|F_0| > 8n - 21$, then $|F| - |F_0| < 8n - 13 - (8n - 21) = 8$. Suppose $W$ is the union of the components of $Q^3_n - F$ and has no vertices in $C$. Since $Q^3_n - F$ is disconnected, then $W$ exists. By Lemma 3.6, the outer neighbors of $W$ is in $F - F_0$. By Lemma 2.4, $2|W| \leq |F| - |F_0| < 8$, then $|W| < 4$. Hence $|W| \leq 3$, the desired result.

**Subcase 2.2.2.** There exists $j \in \{1, 2\}$, $Q[j] - F_j$ is disconnected.

Without loss of generality, we assume that $j = 1$ and $Q[1] - F_1$ is disconnected. By Lemma 2.2, $|F_1| \geq \kappa(Q^3_{n-1}) = 2n - 2$, $|F_2| = |F| - |F_0| - |F_1| \leq 8n - 13 - (6n - 13) -(2n - 2) = 2$, then $Q[2] - F_2$ is connected. Furthermore, $|F_0| \leq 8n - 13 - (2n - 2) = 6n - 11$, $|F_1| \leq 8n - 13 - (6n - 13) = 2n \leq 4(n - 1) - 4$ for $n \geq 4$. By Lemma 3.3, $Q[1] - F_1$ has two components, one of which is a singleton $x_1$. Let $D$ be the largest component of $Q[1] - F_1$, then $D$ is connected to $Q[2] - F_2$. In fact, $D$ has at least $3^{n-1} - 2n - 1$ vertices and $3^{n-1} - 2n - 1$ neighbors in $Q[2]$, since $|F_2| \leq 2$ and $3^{n-1} - 2n - 1 - 2 > 0$ for $n \geq 4$, then $D$ is connected to $Q[2] - F_2$. The component which contains $D$ and $Q[2] - F_2$ of $Q^3_n - F$ is denoted by $C$.

If $Q[0] - F_0$ is connected, since $|F_2| \leq 2$, $|F_0| \geq 6n - 13$ and $3^{n-1} - (6n - 13) \geq 17 > |F_2|$ for $n \geq 4$, there is at least one edge between $Q[0] - F_0$ and $Q[2] - F_2$. Then $Q[0] - F_0$ is connected to $Q[2] - F_2$, $Q[0] - F_0$ is contained in $C$. Thus $Q^3_n - F$ has two components, one of which is a singleton. Then the result holds.

If $Q[0] - F_0$ is disconnected, suppose $X$ is the union of the components of $Q[0] - F_0$ and has no neighbors in $C$. We will show $|X| \leq 2$. Suppose by the way of contradiction that $|X| \geq 3$. Every vertex in $X$ has one outer neighbor in $Q[2] \cap F$, by Lemma 3.6 and $|F_2| \leq 2$, thus $|X| \leq 2$ which is a contradiction. Let $E$ be the largest component of $Q[0] - F_0$, then $E$ is connected to $Q[2] - F_2$. In fact, there must be an edge between $Q[0] - F_0$ and $Q[2] - F_2$ since $|F_2| \leq 2$ and $3^{n-1} - (2n - 2) - 2 > 2 \geq |F_2|$ for $n \geq 4$. Thus $E$ is contained in $C$.

Suppose $W$ is the union of the components of $Q^3_n - F$ and has no vertices in $C$. Since $Q^3_n - F$ is disconnected, then $W$ exists. Obviously, $W \subseteq \{x_1\} \cup X$. Since $|X| \leq 2$, then $|W| \leq 3$. The desired result.

This covers all possibilities and the proof of the theorem is complete. □
The following theorem about the 3-extra connectivity of the 3-ary $n$-cube network follows from Theorem 3.5 and Theorem 3.7.

**Theorem 3.8.** For $n \geq 3$, $\kappa_3(Q_n^3) = 8n - 12$.

**Proof.** By Theorem 3.7, $\kappa_3(Q_n^3) \geq 8n - 12$ for $n \geq 3$. On the other hand, by Theorem 3.5, $\kappa_3(Q_n^3) \leq 8n - 12$ for $n \geq 3$. Hence, $\kappa_3(Q_n^3) = 8n - 12$ for $n \geq 3$. The proof of the theorem is complete. \hfill \qed

4. Conclusion

In this paper, the 3-extra connectivity of the 3-ary $n$-cube networks is gotten. The result shows that at least $8n - 12$ vertices must be moved to disconnect the 3-ary $n$-cube for $n \geq 3$, provided that the removal of these vertices does not isolate either a singleton, an edge, a 2-path, or a 3-cycle. We will further study 3-extra connectivity of the $k$-ary $n$-cube networks for $k \geq 4$, $h$-extra connectivity of the $k$-ary $n$-cube networks for $h \geq 4$ and $h$-extra connectivity of other interconnection networks. Determining the $h$-extra connectivity of various multiprocessor systems requires further research efforts.

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