RADON TRANSFORM ON REAL, COMPLEX AND QUATERNIONIC GRASSMANNIANS

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Abstract. Let $G_{n,k}(\mathbb{K})$ be the Grassmannian manifold of $k$-dimensional $\mathbb{K}$-subspaces in $\mathbb{K}^n$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ is the field of real, complex or quaternionic numbers. For $1 \leq k \leq k' \leq n - 1$ we define the Radon transform $(Rf)(\eta)$, $\eta \in G_{n,k}(\mathbb{K})$, for functions $f(\xi)$ on $G_{n,k}(\mathbb{K})$ as an integration over all $\xi \subset \eta$. When $k + k' \leq n$ we give an inversion formula in terms of the Gårding-Gindikin fractional integration and the Cayley type differential operator on the symmetric cone of positive $k \times k$ matrices over $\mathbb{K}$. This generalizes the recent results of Grinberg-Rubin for real Grassmannians.

1. Introduction

The Radon transform on rank one symmetric spaces has been studied extensively and is related to many areas in analysis and geometry; see [7], [6] for a systematic treatment and e.g. [4], [10], [13], [12], [8] and references therein for some recent development. For higher rank symmetric spaces the theory is far from been complete and there has been comparably less progress. In a remarkable paper [3] Grinberg and Rubin find an inversion formula for the Radon transform from functions on the real Grassmannian $G_{n,k}(\mathbb{R})$ of $k$-dimensional subspaces in $\mathbb{R}^n$ to functions on $G_{n,k'}(\mathbb{R})$, with $1 \leq k \leq k' \leq n - 1$, $k + k' \leq n$, by using the Gårding-Gindikin fractional integration on the space of real symmetric $k \times k$-matrices. It is natural to ask if the corresponding results hold for the Grassmannian manifolds over the complex and quaternionic numbers, and for corresponding non-compact symmetric spaces of matrix balls. In the present paper we will answer the question and prove the results for complex and quaternionic Grassmannians. We proceed with a brief summary of our results and the technical tools needed to prove them.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex or quaternionic numbers with real dimension $d = 1, 2, 4$ and with the standard involution $x \rightarrow \bar{x}$. Let $G_{n,k}(\mathbb{K})$ be the Grassmannian manifold of $k$-dimensional subspaces over $\mathbb{K}$ in $\mathbb{K}^n$. For $1 \leq k \leq k' \leq n - 1$ the Radon transform $R : C^\infty(G_{n,k}(\mathbb{K})) \rightarrow C^\infty(G_{n,k'}(\mathbb{K}))$ is defined by

\begin{equation}
\phi(\eta) = (Rf)(\eta) = \int_{\xi \subset \eta} f(\xi)d_\eta(\xi), \quad \eta \in G_{n,k'}(\mathbb{K}),
\end{equation}

where $d_\eta(\xi)$ is a certain probability measure on the set $\{\xi \in G_{n,k}(\mathbb{K}) : \xi \subset \eta\}$ invariant with respect to the group of unitary transformations of $\eta$; it can be defined using a group-theoretic formulation, see [1,2].

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When $k = 1$ the Grassmannian $G_{n,k}(\mathbb{K})$ is the projective space of all lines $\xi$ in $\mathbb{K}^n$. The Radon transform $f(\xi) \to \phi(\eta)$ is inverted by using the integration of $\phi(\eta)$ over all subspaces $\eta$ that are at a fixed angle $\theta$ with the given $\xi$, and by a fractional integration $[6]$ with respect to the variable $\cos^2 \theta$. Motivated by that result Grinberg and Rubin $[3]$ define also a cosine of an “angle” for a $k$-plane $\xi$ and $k'$-plane in $\mathbb{R}^n$, defined as a $k \times k$ semi-positive matrix, defined up unitary $U(k, \mathbb{K})$-equivalence. To be more precise we denote $M_{n,k} := M_{n,k}(\mathbb{K})$ the space of $n \times k$-matrices over $\mathbb{K}$, and $S_{n,k} = S_{n,k}(\mathbb{K})$ the Stiefel manifold of all orthogonal $k$-frames in $\mathbb{K}^n$, identified also with the set of all isometric $\mathbb{K}$-linear transformations $x \in M_{n,k} : \mathbb{K}^k \to \mathbb{K}^n$. The Grassmannian $G_{n,k}$ will be viewed as a quotient space of $S_{n,k}$ via the mapping $x \in S_{n,k} \to \xi = \{x\} = x\mathbb{K}^k \in G_{n,k}$, and we will identify a function $f(\xi)$ on $G_{n,k}$ as a right $U(k, \mathbb{K})$-invariant function $f(x)$ on $S_{n,k}$. We define now, following $[3]$, the cosine of the “angle” $(\eta, x)$ between the element $x \in S_{n,k}$ and $\eta \in G_{n,k}'(\mathbb{K})$ to be a semi-positive $k \times k$-matrix, $\cos^2(\eta, x) = x^*P_\eta x$

where $P_\eta$ is the orthogonal projection $\mathbb{K}^n \to \eta$. (One can define an angle “$(\eta, x)$” as a $k \times k$-self adjoint matrix using the spectral calculus of $t \to \cos^2 t$, however we will only need $\cos^2(\eta, x)$.) The matrix $\cos^2(\eta, \xi)$ of the “angle” between $\eta \in G_{n,k'}$ and $\xi \in G_{n,k}$ can be defined up to the $U(k, \mathbb{K})$ equivalence as $\cos^2(\eta, x)$, with $\xi = \{x\}$.

The generalization to higher rank spaces of the integration of $\phi(\eta)$ is

$$(\mathcal{T}_r \phi)(\xi) := \int_{U(k, \mathbb{K})} du \int_{\cos^2(\eta, x) = uru^*} \phi(\eta)d\xi \eta, \quad \xi = \{x\},$$

where $d\xi \eta$ is a group-invariant measure on the set of integration.

Our main result in this paper is Theorem 4.6, it gives an inversion formula expressing $f(\xi)$ in terms the Gårding-Gindikin fractional integration $I^r$ and the Cayley type differential operator acting on the function $(\mathcal{T}_r \phi)(\xi)$ with respect to the $r$-variable. This generalizes the result of Grinberg-Rubin for real Grassmannians $[3]$. The principal technical tools in proving it are Propositions 3.3 and 3.5 giving integral formulas on matrix spaces and on Stiefel manifolds. Proposition 3.3 (ii) is proved for the real Grassmannian in $[3]$ by using rather complicated formulas involving Bessel functions proved by Herz $[9]$. Our idea is instead to find first an integral formula on the matrix space, namely Proposition 3.3 (i), and (ii) will be then a direct consequence. This then answers a question in $[3]$. A similar integral formula as our Proposition 3.5 is proved in $[3]$ Lemma 2.5] using again some interesting and however tricky computations. Here we use again integration on the matrix space. We believe that our proofs are easier, both technically and conceptually. Nevertheless it will be clear that we are very much inspired by that paper.

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2. Symmetric cones and Gårding-Gindikin Fractional Integration

In this section we fix notations and we recall some known results on the Gindikin Gamma function on symmetric cones; see [1] for a systematic treatment.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex or quaternionic numbers with real dimension
\[ d := \dim_{\mathbb{R}} \mathbb{K} = 1, 2, 4 \]
and let $x \rightarrow \bar{x}$ be the standard conjugation. Let $M_{n,k} := M_{n,k}(\mathbb{K})$ be the Euclidean space of all $n \times k$-matrices with entries $x$ in $\mathbb{K}$. For $x \in M_{n,k}$ we denote $x^* := \bar{x}^T$, where $T$ stands for the transpose. The space $M_{n,k}$ is then a Euclidean space with the metric $\text{tr}(xx^*)$ and the corresponding Lebesgue measure $dx$. Throughout this paper we will fix $1 \leq k < n$ if nothing else if specified. The vector space $\mathbb{K}^k$ will be identified as column vectors $v$ with multiplication by $c \in \mathbb{K}$ from right, $v \mapsto vc$, and all matrices $x \in M_{n,k}$ will be identified as $\mathbb{K}$-linear transformations from $\mathbb{K}^k$ to $\mathbb{K}^n$ (i.e. $x(vc) = x(v)c$, $v \in \mathbb{K}^k$, $c \in \mathbb{K}$) unless something else is specified.

The $\mathbb{R}$-subspace $A = A_k := \{a \in M_{k,k}; a^* = a\}$ of the self-adjoint $M_{k,k}$-matrices forms a formally real Jordan algebra with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the identity $I := I_k$. Its (real) dimension $N$ is given by
\[ N := N_k = \dim_{\mathbb{R}} A = k + \frac{d}{2}k(k - 1) \]
Let $\Omega = \Omega_k$ be the cone in $A$ of positive definite $k \times k \mathbb{K}$-matrices. Let $\Delta$ be the determinant function on $A$. It is a polynomial of degree $k$; for real and complex matrices it is the usual determinant, for quaternionic matrices it can be defined using the Pfaffian by identifying $A$ with a subspace of complex $2k \times 2k$ skew symmetric complex matrices. Let $GL_k(\mathbb{K})$ be the group of invertible $k \times k \mathbb{K}$-matrices. It acts on $\Omega = \Omega_k$ by $g : r \rightarrow gr^*$, and with this action the cone $\Omega$ becomes a Riemannian symmetric space, $\Omega = GL_k(\mathbb{K})/U(k, \mathbb{K})$, where $U(k, \mathbb{K})$ is the orthogonal group over $\mathbb{K}$. The $GL_k(\mathbb{K})$-invariant measure on $\Omega$ is
\[ \begin{align*}
\text{(2.1)} \\
\text{d}t(a) := \Delta(a)^{-N/k} \text{d}a.
\end{align*} \]

The Gindikin Gamma integral is defined and given by
\[ \begin{align*}
\text{(2.2)} \\
\int_{\Omega} e^{-\text{tr}(st^*)} \Delta(s)^{\lambda} \text{d}t(s) = \Gamma_{\Omega}(\lambda) \Delta(t)^{-\lambda}, & \quad t \in \Omega, \Re \lambda > \frac{N}{k} - 1
\end{align*} \]
where
\[ \Gamma_{\Omega}(\lambda) = (2\pi)^{(N-k)/2} \prod_{j=1}^{k} \Gamma(\lambda - \frac{d}{2}(j - 1)) \]
is the Gindikin Gamma function. (Note that a different normalization of the measure $\text{d}s$ is used in [3].) The corresponding Beta-integral is
\[ \begin{align*}
\text{(2.3)} \\
\int_{0}^{s} \Delta(t)^{\lambda - N/k} \Delta(s - t)^{\mu - N/k} \text{d}t = B_{\Omega}(\lambda, \mu) \Delta(s)^{\lambda + \mu - N/k}, & \quad \Re \lambda, \Re \mu > \frac{N}{k} - 1
\end{align*} \]
with
\[ B_\Omega(\lambda, \mu) = \frac{\Gamma_\Omega(\lambda) \Gamma_\Omega(\mu)}{\Gamma_\Omega(\lambda + \mu)}. \]

Let \((0, I)\) be the unit open interval \((0, I) = \{s \in \Omega; s < I\}\). The Gårding-Gindikin fractional integral for a function \(f\) on \((0, I)\) is defined by
\[
(I^\lambda f)(s) = \frac{1}{\Gamma_{\Omega}(\lambda)} \int_0^s \Delta(s - t)^{\lambda - N/k} f(t) dt, \quad \Re \lambda > \frac{N}{k},
\]
where the integration is understood as over the set \(t \in \Omega, t < s\). For our purpose later (see Lemma 4.3) we consider the space \(L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\) with \(\alpha \in \mathbb{R}\). The integral \((I^\lambda f)(s)\) is then well-defined for \(f \in L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\).

Using the previous formula it is easy to prove the following semigroup property
\[
I^\lambda(I^\mu f) = I^{\lambda + \mu} f, \quad \Re \lambda, \Re \mu > \frac{N}{k}
\]
for \(f \in L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\).

To define the operator \(I^\lambda\) for smaller \(\Re \lambda\) we need the differential operator \(\Delta(\partial)\), the so-called Cayley type differential operator, defined uniquely on \(\mathcal{A}\) by requiring that
\[
\Delta(\partial_x)e^{\text{tr}(xy^*)} = \Delta(y)e^{\text{tr}(xy^*)}.
\]

It particular it follows from the equality \([25]\) that
\[
\Delta(\partial)\Delta(s)^\lambda = \prod_{j=1}^k (\lambda + \frac{d}{2}(j - 1))\Delta(s)^{\lambda - 1}
\]
for any \(\lambda \in \mathbb{C}\), which sometimes is referred as Cayley-Capelli type identity.

We state and prove some elementary results on the integral operator \(I^\lambda\). They may have been proved in more general form in the literature.

**Lemma 2.1.**

(i) Suppose \(\Re \lambda > \frac{N}{k} - 1\) and \(\mu > \frac{N}{k} - 1\). The operator \(I^\lambda\) defines a bounded operator from \(L^1((0, I), \Delta(I - t)^{\nu + \Re \lambda - \frac{N}{k}} dt)\) into \(L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\).

(ii) Suppose \(\nu > \frac{N}{k} - 1\) and \(f \in L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\). \(I^\lambda f\) has an analytic continuation in \(\lambda \in \mathbb{C}\) as a distribution on the space \(C_0^\infty(0, I)\) of smooth functions with compact support in \((0, I)\).

(iii) Suppose \(\nu > \frac{N}{k} - 1\) and \(f \in L^1((0, I), \Delta(I - t)^{\nu - \frac{N}{k}} dt)\). Let \(m > \frac{N}{k}\) be an integer. Then
\[
\Delta(\partial)^m I^m f = f,
\]
in the sense of distributions.
Proof. (i). Let $\lambda$ and $\mu$ be as in (i). We estimate the norm of $I^\lambda f$ in $L^1((0, I), \Delta(I - t)^{\mu - \frac{N}{k}} dt)$. It is, apart from the constant $\Gamma_\Omega(\lambda)^{-1}$,

$$\int_0^I \left| \int_0^s \Delta(s - t)^{\lambda - \frac{N}{k}} f(t) dt \right| \Delta(I - s)^{\mu - \frac{N}{k}} ds \leq \int_0^I \int_0^s \Delta(s - t)^{\Re \lambda - \frac{N}{k}} \left| f(t) \right| \Delta(I - s)^{\mu - \frac{N}{k}} dt ds$$

$$= \int_0^I \left( \int_0^I \Delta(s - t)^{\Re \lambda - \frac{N}{k}} \Delta(I - s)^{\mu - \frac{N}{k}} ds \right) \left| f(t) \right| dt.$$

We compute the inner integral. Performing the change of variables $s = t + P((I - t)^{\frac{1}{2}}) \{v\}, v \in (0, I)$, where $x \to P(x)$ is the quadratic representation of the Jordan algebra $\mathcal{A}$ (see [1]). We have then $ds = \Delta(I - t)^{\frac{N}{k}} dv$, $\Delta(I - s) = \Delta(I - t) \Delta(I - v)$, $\Delta(s - t) = \Delta(I - t) \Delta(v)$. That integral is

$$\Delta(I - t)^{\mu + \Re \lambda - \frac{N}{k}} \int_0^I \Delta(v)^{\Re \lambda - \frac{N}{k}} \Delta(I - v)^{\mu - \frac{N}{k}} dv = B(\mu, \Re \lambda) \Delta(I - t)^{\mu + \Re \lambda - \frac{N}{k}}.$$

The result follows by substituting this into the previous estimate.

(ii). To define $I^\lambda$ for general $\lambda$ we note that for $\Re \lambda > \frac{N}{k}$ and a test function $\phi \in C_0^\infty(0, I)$, we have, viewing the operator $\Delta(\partial)$ as acting on distributions

$$\left( \Delta(\partial) I^\lambda f, \phi \right) = (-1)^k \left( I^\lambda f, \Delta(\partial) \phi \right) = (-1)^k \frac{1}{\Gamma_\Omega(\lambda)} \int_0^I (I^\lambda f)(s) \Delta(\partial) \phi(s) ds$$

$$= (-1)^k \frac{1}{\Gamma_\Omega(\lambda)} \int_0^I \left( \int_0^s \Delta(s - t)^{\lambda - \frac{N}{k}} f(t) dt \right) \Delta(\partial) \phi(s) ds$$

$$= \int_0^I \left( (-1)^k \frac{1}{\Gamma_\Omega(\lambda)} \int_0^I \Delta(s - t)^{\lambda - \frac{N}{k}} \Delta(\partial) \phi(s) ds \right) f(t) dt$$

To treat the inner integral (for fixed $t$) we change variables $s = t + u, u \in \mathcal{A}$ and denote $\psi(u) = \phi(s) = \phi(t + u)$. The function $\psi$ so defined is then a smooth function on $\mathcal{A}$ of compact support in the interval $(-t, I - t)$, in particular it is in the Schwartz space $S(\mathcal{A})$. Moreover, and $\Delta(\partial_u) \phi(s) = \Delta(\partial_u) \psi(u)$. The inner integral can be written as

$$(-1)^k \frac{1}{\Gamma_\Omega(\lambda)} \int_0^I \Delta(u)^{\lambda - \frac{N}{k}} \Delta(\partial) \psi(u) du = (-1)^k \frac{1}{\Gamma_\Omega(\lambda)} \int_0^\Omega \Delta(u)^{\lambda - \frac{N}{k}} \Delta(\partial) \psi(u) du.$$
(iii) By the same computation we have

$$\left(\Delta(\partial)^m I^m f, \phi\right) = \int_0^1 \left((-1)^km \frac{1}{\Gamma(m)} \int_\Omega \Delta(u)^{m-\frac{n}{k}} \Delta(\partial)^m \psi(u) du\right) f(t) dt$$

and it follows again from the first and the third equality in Theorem VII.2.2 loc. cit. that the inner integral is the delta distribution on $\psi$,

$$(-1)^km \frac{1}{\Gamma(m)} \int_\Omega \Delta(u)^{m-\frac{n}{k}} \Delta(\partial)^m \psi(u) du = \delta(\psi) = \psi(0) = \phi(t),$$

proving (iii). □

The explicit analytic continuation of $I^\lambda f$ is rather complicated; see e.g. [11] and references therein for some systematic study.

3. Polar decomposition and Bi-Stiefel decomposition

In this section we prove some integral formulas related to certain polar decompositions of matrices. Let $S_{n,k}$ be the Stiefel manifold of all orthonormal frames in $\mathbb{K}^n$. It can be realized as the manifold of all isometries $x \in M_{n,k}$.

Let $G := U(n, \mathbb{K}) = \{g \in M_{n,n}; g^* g = I_n\} = O(n), U(n), Sp(n)$

be the orthogonal, unitary, and symplectic unitary groups according to $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively. It acts transitively on $S_{n,k}$ by the defining action, and thus

$$S_{n,k} = G/U(n - k, \mathbb{K}).$$

Each $x \in S_{n,k}$ defines a $k$-dimensional subspace over $\mathbb{K}$,

$$\xi = \{x\} := x\mathbb{K}^k \subset \mathbb{K}^n \in G_{n,k}.$$

Thus $G_{n,k}$ can be identified as the space of orbits in $S_{n,k}$ under the action of the unitary group $U(k, \mathbb{K})$ on $\mathbb{K}^k$,

$$G_{n,k} = S_{n,k}/U(k, \mathbb{K}).$$

The manifold $G_{n,k}$ is therefore a compact Riemannian symmetric space

$$G_{n,k} = U(n, \mathbb{K})/U(n - k, \mathbb{K}) \times U(k, \mathbb{K});$$

see e.g. [4]. We will later specify certain reference points in $S_{n,k}$ and $G_{n,k}$ thus the respective isotropic subgroups. Throughout the paper we will identify functions $f(\xi)$ on $G_{n,k}$ with right $U(k, \mathbb{K})$-invariant functions $f(x)$ on the Stiefel manifold $S_{n,k}$.

For any compact group $K$ we let $dk$ the normalized Haar measure on $K$. Denote also $dv$ and $d\xi$ the normalized unique $G$-invariant measures on $S_{n,k}$ and on $G_{n,k}$ respectively. The following result is possibly known, for completeness we give a proof.

**Lemma 3.1.** Let $dx$ and $dv, dr$ be the measures on $M_{n,k}, S_{n,r}$ and $\Omega$ normalized as above. Almost all $x \in M_{n,k}$ can be decomposed uniquely as

$$x = vr^{\frac{1}{2}}, \quad v \in S_{n,k}, \quad r \in \Omega,$$

and under that decomposition the measure $dx$ is given by

$$dx = C_0 \Delta(r)^{\frac{2n}{k}} dv dr(r) = C_0 \Delta(r)^{\frac{2n-N/k}{k}} dv dr,$$
namely
\[ \int_{M_{n,k}} f(x) \, dx = C_0 \int_{S_{n,k}} \int_{\Omega} f(\sqrt{\frac{1}{2}}) \Delta(r) \frac{n}{2} \, dv \, dt(r), \]
where
\[ C_0 = C_0(n, k) = \frac{\sqrt{\pi} \, n \, k}{\Gamma_{\Omega}(\frac{d}{2} n)}. \]

**Proof.** The polar decomposition follows from the general polar decomposition for linear transformations and from the fact that the set of elements of full rank \( k \) (over \( \mathbb{K} \)) is an open dense subset of \( M_{n,k} \). Now by the \( G \)-invariance we see that there is a weight function depending only on \( r \), say \( W(r) \), so that
\[ \int_{M_{n,k}} f(x) \, dx = \int_{S_{n,k}} \int_{\Omega} f(\sqrt{\frac{1}{2}}) W(r) \, dt(r) \, dv, \]
in term of the invariant measure \( dt \) on \( \Omega \). To find the weight function \( W(r) \) we note that for any \( g \in GL_k(\mathbb{K}) \), the LHS is
\[ \int_{M_{n,k}} f(x) \, dx = \int_{M_{n,k}} f(xg) |\det(g)|^n \, dx = \int_{S_{n,k}} \int_{\Omega} f(\sqrt{\frac{1}{2}}) W(r) \, dt(r) \, dv, \]
where \( \det(g) \) is the real Jacobian of the \( \mathbb{R} \)-linear transformation on \( \mathbb{K}^k = \mathbb{R}^{dk} \), \( y \in \mathbb{R}^{dk} = \mathbb{K}^k \to yg \in \mathbb{K}^k = \mathbb{R}^{dk} \). We perform further the polar decomposition of \( r^{\frac{1}{2}} g \),
\[ r^{\frac{1}{2}} g = u(g^*rg)^{\frac{1}{2}}, \quad u \in U(k, \mathbb{K}), \]
so that the above integral is
\[ \int_{S_{n,k}} \int_{\Omega} f(u(g^*rg)^{\frac{1}{2}}) |\det(g)|^n W(r) \, dt(r) \, dv. \]
Performing first the change of variables \( g^*rg \to r \) on \( \Omega \) and then \( vu \to v \) we see that it is
\[ \int_{S_{n,k}} \int_{\Omega} f(vu(g^*rg)^{\frac{1}{2}}) |\det(g)|^n W((g^*)^{-1}rg^{-1}) \, dt(r) \, dv, \]
by the invariance of \( dt \) and respectively \( dv \). Thus the weight factor \( W(r) \) transforms as
\[ W(g^*rg) = |\det(g)|^n W(r), \]
from which it follows, using the fact that any \( r \in \Omega \) can be written as \( r = g^*g \), \( g \in GL_k(\mathbb{K}) \) (see e.g. [1]), that \( W(r) = c \Delta(r)^{\frac{n}{2}} \) for some constant \( c \). The constant can be evaluated by taking the function \( f(x) = e^{-\text{tr}(x^*x)} \) and using the formula (2.2). □

**Remark 3.2.** Any \( x \in M_{n,k}, n \geq k \), can be written as \( x = vr^{\frac{1}{2}} \) where \( r = x^*x \geq 0 \). The element \( v \) is generally not unique, and we can always choose \( v \) so that \( v \in S_{n,k} \). We mention also that, polar decompositions, generally speaking, are closely related to the Bessel functions, in particular Lemma 3.1 above is much related to the result in [2], p. 130.

**Proposition 3.3.** Suppose \( 1 \leq k \leq k' < n \) and \( k + k' \leq n \).
Almost all \( x \in M_{n,k} \) can be uniquely decomposed as
\[
x = \begin{bmatrix} ur^\frac{k'}{2} \\ v(I-r)^\frac{k'}{2} \end{bmatrix} s^\frac{k}{2}, \quad u \in S_{k',k}, \quad v \in S_{n-k',k}, \quad r \in (0, I), \quad s \in \Omega
\]
and under that decomposition the measure \( dx \) is given by
\[
dx = C_1 \Delta(r) \frac{4^{k'-N/k}}{\Delta(I-r)} \Delta(I-r) \frac{4^{(n-k')-N/k}}{\Delta(s)} dudvdrds
\]
with
\[
C_1 = C_1(n, k', k) = \sqrt{\pi} \frac{dnk}{B\left(\frac{d}{2}k', \frac{d}{2}n\right)} \Gamma_{\Omega}(\frac{d}{2}n)
\]
Almost all \( w \in S_{n,k} \) can be decomposed as
\[
w = \begin{bmatrix} ur^\frac{k}{2} \\ v(I-r)^\frac{k}{2} \end{bmatrix}, \quad u \in S_{k',k}, \quad v \in S_{n-k',k}, \quad r \in (0, I),
\]
and under that decomposition the measure \( dw \) is given by
\[
dw = C_2 \Delta(r) \frac{4^{k'-N/k}}{\Delta(I-r)} \Delta(I-r) \frac{4^{(n-k')-N/k}}{\Delta(s)} dudvdr
\]
with
\[
C_2 = C_2(n, k') = \frac{1}{B\left(\frac{d}{2}k', \frac{d}{2}n\right)}
\]
For the proof we need another elementary result.

**Lemma 3.4.** The mapping
\[
(r, s) \mapsto (x, y) = (s^\frac{1}{2}rs^\frac{1}{2}, s^\frac{1}{2}(I-r)s^\frac{1}{2})
\]
is a diffeomorphism from \((0, I) \times \Omega\) onto \(\Omega \times \Omega\), and its Jacobian is given by \(\Delta(s)^{N/k}\), namely
\[
dxdy = \Delta(s)^{N/k}dsdr.
\]

**Proof.** We write \( z = x + y = s^\frac{1}{2}rs^\frac{1}{2} + s^\frac{1}{2}(I-r)s^\frac{1}{2} = s \). Thus the mapping
\[
(r, s) \mapsto (x, z) = \left(s^\frac{1}{2}rs^\frac{1}{2}, s\right)
\]
maps \(\Omega \times (0, I)\) into \( \{(x, z) \in \Omega \times \Omega; z > x\} \), and its inverse is given by
\[
(x, z) \rightarrow (r, s) = (z^{-\frac{1}{2}}xz^{-\frac{1}{2}}, x).
\]
This proves the first part of the statement. For the Jacobian we have that
\[
du(z)du(x) = du(s)du(r)
\]
by the invariance. Thus, in term of the Lebesgue measure,
\[
dx = dzdx = \Delta(x)^{N/k} \Delta(z)^{N/k} du(x)du(z)
\]
\[
= \Delta(s)^{N/k} \Delta(r)^{N/k} du(x)du(z) = \Delta(s)^{N/k}drds.
\]

We prove now the Proposition.
Proof. The proof is done by a change of variables. Conceptually it is clearer to compute the integral $\int_{m_{n,k}} f(x)dx$ as in the proof of Lemma 3.1, whereas computationally it is easier to compute the measure $dx$, and we will adopt the latter. First of all we write $x$ as a $2 \times 1$-block matrix under the decomposition of $\mathbb{K}^n = \mathbb{K}^k \oplus \mathbb{K}^{n-k}$ and then perform polar decomposition

$$(3.2) \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u_1 p_1^{1/2} \\ u_2 p_2^{1/2} \end{bmatrix}$$

and we may assume that $p_1 > 0, p_2 > 0$ and $u_1 \in S_{k',k}$ and $u_2 \in S_{n-k',k}$, by the condition that $k' \geq k$, $n - k' \geq k$. Thus by Lemma 3.1,

$$(3.3) \quad dx = dx_1 dx_2, \quad dx_1 = c_1 du_1 \Delta(p_1)^{\frac{k'}{k} - \frac{n}{k}} dp_1, \quad dx_2 = c_2 du_2 \Delta(p_2)^{\frac{n-k'}{k} - \frac{n}{k}} dp_2$$

with $c_1 = C_0(k',k)$, $c_2 = C_0(n-k',k)$. On the other hand, applying Lemma 3.1 again to $x$ we can write $x$ as

$$x = w s^{\frac{1}{2}}, \quad w \in S_{n,k}, \quad s \in \Omega$$

with

$$(3.4) \quad dx = c_3 \Delta(s)^{\frac{n-1}{n-k}} dw ds, \quad c_3 = C_0(n,k).$$

We write $w \in S_{n,k}$ as a block matrix $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and perform again the polar decomposition, using the fact that $w^*w = I = I_k$,

$$(3.5) \quad w = \begin{bmatrix} ur^{\frac{1}{2}} \\ v(I-r)^{\frac{1}{2}} \end{bmatrix}$$

for some $r, 0 \leq r \leq I$ and partial isometries $u$ and $v$. We can again assume, up to a set of $dw$-measure zero, that $0 < r < I$, $u \in S_{k',k}$ and $v \in S_{n-k',k}$. Therefore $x$ has the form

$$(3.6) \quad \begin{bmatrix} u_1 p_1^{1/2} \\ u_2 p_2^{1/2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x = w s^{\frac{1}{2}} = \begin{bmatrix} ur^{\frac{1}{2}} \\ v(I-r)^{\frac{1}{2}} \end{bmatrix} s^{\frac{1}{2}} = \begin{bmatrix} ur^{\frac{1}{2}} s^{\frac{1}{2}} \\ v(I-r)^{\frac{1}{2}} s^{\frac{1}{2}} \end{bmatrix}$$

Thus

$$(3.7) \quad x_1 = ur^{\frac{1}{2}} s^{\frac{1}{2}}, \quad x_2 = v(I-r)^{\frac{1}{2}} s^{\frac{1}{2}}; \quad p_1 = s^{\frac{1}{2}} r s^{\frac{1}{2}}, \quad p_2 = s^{\frac{1}{2}} (I-r) s^{\frac{1}{2}}.$$

Now using Lemma 3.4 for the change of variables $(r, s) \rightarrow (p_1, p_2)$, we have that

$$(3.8) \quad r^{\frac{1}{2}} s^{\frac{1}{2}} = u_0 p_1^{\frac{1}{2}}, \quad (I-r)^{\frac{1}{2}} s^{\frac{1}{2}} = v_0 p_2^{\frac{1}{2}},$$

$$(3.9) \quad r^{\frac{1}{2}} s^{\frac{1}{2}} = u_0 p_1^{\frac{1}{2}}, \quad (I-r)^{\frac{1}{2}} = v_0 p_2^{\frac{1}{2}}.$$
for some \( u_0, v_0 \in U(k, \mathbb{K}) \). Thus

\[
\begin{align*}
    u_1 p_1^{\frac{1}{2}} &= x_1 = ur^{\frac{1}{2}} s^{\frac{1}{2}} = uu_0 p_1^{\frac{1}{2}} , \\
    u_1 p_1^{\frac{1}{2}} &= x_2 = v(I - r)^{\frac{1}{2}} s^{\frac{1}{2}} = vv_0 p_2^{\frac{1}{2}} ,
\end{align*}
\]

from which it follows that \( u_1 = uu_0 , \quad u_2 = vv_0 \) and

\[
du_1 du_2 = du dv,
\]

by invariance. We now obtain

\[
dx = c_1 c_2 \Delta(r)^{\frac{d}{2} - \frac{n}{2}} \Delta(I - r)^{\frac{d}{2} - \frac{n}{2}} \Delta(s)^{\frac{d}{2} - \frac{n}{2}} dv dr ds.
\]

This is our claim in (i), and (ii) follows in turn by using (3.4).

\[
\square
\]

In the next Proposition we will need another form of polar decomposition that is different from the one in Lemma 3.1. If \( 1 \leq k \leq k' \) we have that for almost all \( x \in M_{k,k} \), viewed as linear transformation \( \mathbb{K}^k \to \mathbb{K}^{k'} \to \mathbb{K}^k \) under the natural embedding and projection, that

\[
x = [0_{k,k'} - k \ s^{\frac{1}{2}}] u , \quad s \in \Omega , \quad u \in S_{k',k},
\]

by the usual polar decomposition of \( x \), where \( 0_{p,q} \) stands for the zero \( p \times q \)-matrix. However this factorization is not unique and thus no integral formula is expected as that in Lemma 3.1. Nevertheless we have the following substitute. First we shall need an integral formula, proved in [3, Lemma 2.4] for real Grassmannians, which follows easily by the invariance of the measure: For any function \( f \) defined on \( M_{k,k}(\mathbb{K}) \)

\[
(3.9) \quad \int_{S_{k',k}} f(v^* u) du = \int_{S_{k',k}} f(u^* v) du = \int_{S_{k',k}} f(v^* u) dv .
\]

**Proposition 3.5.** Let \( 1 \leq k \leq k' \). For any measurable function \( H(w) \) on \( M_{k,k}(\mathbb{K}) \), let

\[
H_1(s) = \Delta(s)^{\frac{d}{2} - \frac{n}{2}} \Delta(I - s)^{\frac{d}{2} - \frac{n}{2}} \Delta(s)^{\frac{d}{2} - \frac{n}{2}} dv , \quad s \in \Omega ,
\]

and

\[
H_2(s) = \Delta(s)^{\frac{d}{2} - \frac{n}{2}} \Delta(I - s)^{\frac{d}{2} - \frac{n}{2}} \Delta(s)^{\frac{d}{2} - \frac{n}{2}} du , \quad s \in \Omega .
\]

Then for any \( \varepsilon > -1 \),

\[
(3.10) \quad (\Gamma_\varepsilon^{N/k} H_1)(s) = C_3(I^{\varepsilon + N/k} H_2)(s),
\]

where

\[
C_3(n, k', k) = \frac{C_0(k' - k, k) C_0(k, k) \Gamma_{\Omega}(\frac{d}{2}(k' - k))}{C_0(k', k)}
\]

if \( k' - k \geq k \) and

\[
C_3(n, k', k) = \frac{C_0(k, k' - k) C_0(k, k) \Gamma_{\Omega_{k' - k}}(\frac{d}{2}k)}{C_0(k', k)}
\]

if \( k' - k < k \). Here \( \Gamma_{\Omega_{k' - k}} \) is the Gamma function associated with the symmetric cone \( \Omega_{k' - k} \) in \( M_{k' - k, k' - k} \) and \( C_0(k, k' - k) \) the constant \( C_0 \) in [3,1] with \((n,k)\) replaced by \((k,k' - k)\).
The inner integral can be computed by fixing $r \in \Omega$ and consider the integral

$$J(r) := \int_{w \in M_{k',k} : w^* w < r} H(w^* \begin{bmatrix} I_k \\ 0 \end{bmatrix}) \Delta(r - w^* v)^\varepsilon dv$$

We write the matrix $w^*$ also in block form: $w^* = \begin{bmatrix} w_1^* \\ w_2^* \end{bmatrix}$ with $w_1 \in M_{k,k}$, $w_2 \in M_{k'-k,k}$. We have

$$J(r) = \int_{w_1 \in M_{k,k} : w_1^* w_1 < r} H(w_1^*) \left( \int_{w_2 \in M_{k'-k,k} : w_2^* w_2 < r - w_1^* w_1} \Delta(r - w_1^* w_1 - w_2^* w_2)^\varepsilon dw_2 \right) dw_1.$$

The inner integral can be computed by

$$\int_{w_2 \in M_{k'-k,k} : w_2^* w_2 < r - w_1^* w_1} \Delta(r - w_1^* w_1 - w_2^* w_2)^\varepsilon dw_2 = c_\varepsilon \Delta(r - w_1^* w_1)^{\varepsilon + \frac{d}{2}(k'-k)}$$

by the changing of variables $w_2 = y(r - w_1^* w_1)^{\frac{1}{2}}$, where

$$c_\varepsilon = \int_{y \in M_{k'-k,k} : y^* y < I_k} \Delta(I - y^* y)^\varepsilon dy$$

which will be evaluated in the end of the proof. The integral $J(r)$ is then, by Lemma 3.1,

$$J(r) = c_\varepsilon \int_{w_1 \in M_{k,k} : w_1^* w_1 < r} H(w_1^*) \Delta(r - w_1^* w_1)^{\frac{d}{2}(k'-k) + \varepsilon} dw_1$$

(3.11)

$$= c_\varepsilon C_0(k,k) \int_0^r \Delta(r - s)^{\frac{d}{2}(k'-k) + \varepsilon} \Delta(s)^{\frac{d}{2} - 1} \left( \int_{u \in U(k,k')} H(u s^{\frac{1}{2}}) du \right) ds$$

$$= C_\varepsilon (I^{\frac{d}{2}(k'-k) + \varepsilon + N/k} H_2)(r)$$

with

$$C_\varepsilon = c_\varepsilon C_0(k,k) \Gamma_{k,k}(\frac{d}{2}(k'-k) + \varepsilon + N/k).$$

On the other hand, the integral $J(r)$ can be computed by Lemma 3.1, performing the polar decomposition $w = vs^{\frac{1}{2}}$,

$$J(r) = C_0(k',k) \int_0^r \Delta(s)^{\frac{d}{2}(k'- (k-1)) - 1} \Delta(r - s)^\varepsilon \left( \int_{S_{k',k'}} H(s^{\frac{1}{2}} v^* [I_k]_v) dv \right) ds$$

and the inner integral, by (3.9), is

$$\int_{S_{k',k'}} H(s^{\frac{1}{2}} [0 \ I_k]_v) dv,$$

Namely

$$J(r) = C_0(k',k) \Gamma_{k,k}(\varepsilon + N/k) (I^{\varepsilon + N/k} H_1)(r),$$

with $H_1$ given as in the statement. Comparing the two equalities (3.11) and (3) we get

$$(I^{\varepsilon + N/k} H_1)(s) = C_3 (I^{\frac{d}{2}(k'-k) + \varepsilon + N/k} H_2)(s),$$
with

$$C_3 = \frac{C_{\varepsilon}}{C_0(k', k) \Gamma_{\Omega}(\varepsilon + N/k)}.$$  

Now we evaluate $c_{\varepsilon}$ and prove that $C_3$ is as given in the Proposition (and is in fact independent of $\varepsilon$!). If $k' - k \geq k$ then by Lemma 3.1 and (2.3), $c_{\varepsilon}$ is

$$C_0(k' - k, k) \int_0^1 \Delta(I-r)^\varepsilon \Delta(r)^{\frac{d(k'-k)-N/k}{2}} dr = C_0(k'-k, k) B_{\Omega}(\varepsilon + N/k, \frac{d}{2}(k'-k)).$$

If $k' - k < k$, we have again, noticing that for any $y \in M_{k'-k, k}$, $y^* y < I_k$ is equivalent to $yy^* < I_{k'-k}$, and that $\Delta(I_k - y^* y) = \Delta(I_{k'-k} - yy^*)$, that the integral can be expressed as integration on the interval $(0, \mathcal{I}_{k'-k})$ in the symmetric cone $\Omega_{k'-k}$ of smaller rank,

$$c_{\varepsilon} = C_0(k, k' - k) \int_{0 < r < \mathcal{I}_{k'-k}} \Delta(I_{k'-k} - r)^\varepsilon \Delta(r)^{\frac{d}{2}(k'-k-1)} dr = C_0(k, k' - k) B_{\Omega_{k'-k}}(\varepsilon + 1 + \frac{d}{2}(k'-k-1), \frac{d}{2} k).$$

The constant $C_3$ can then be computed by the formula for $C_0$ and by the formula for the Gamma function, we leave the elementary and yet intricate computations to the interested reader.  

\[\square\]

4. Radon Transform and the Inverse Transform

In this section we will prove our main result, finding an inversion formula for the Radon transform.

To simplify notations we will write (with some abuse of notation) $U(l) = U(l, \mathbb{K})$ dropping the symbol $\mathbb{K}$. Let

$$x_0 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}, \quad \hat{x}_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in S_{n,k}, \quad \xi_0 = \{x_0\}, \quad \hat{\xi}_0 = \{\hat{x}_0\} \in G_{n,k},$$

be two reference points in $S_{n,k}$ and in $G_{n,k}$ respectively. Similarly, we fix

$$y_0 = \begin{bmatrix} I_{k'} \\ 0 \end{bmatrix}, \quad \hat{y}_0 = \begin{bmatrix} 0 \\ I_{k'} \end{bmatrix} \in S_{n,k'}, \quad \eta_0 = \{y_0\}, \quad \hat{\eta}_0 = \{\hat{y}_0\} \in G_{n,k'}$$

two reference points in $S_{n,k'}$ and in $G_{n,k'}$. Identifying $G_{n,k}$ with $G_{n,k} = G \cdot \hat{\xi}_0$ we have $G_{n,k} = G/K$ where

$$K = U(n-k) \times U(k) = \{k = \text{diag}(\alpha, \delta) \in G; (\alpha, \delta) \in U(n-k) \times U(k)\}$$

is the isotropic subgroup of $\hat{\xi}_0$. Correspondingly

$$S_{n,k} = G \cdot \hat{x}_0 = G/U(n-k).$$

Similarly, $G_{n,k'} = G \cdot \eta_0 = G/K'$, $S_{n,k'} = G \cdot y_0 = G/U(n-k')$, with

$$K' = U(k') \times U(n-k') = \{k = \text{diag}(\tau, \rho) \in G; (\tau, \rho) \in U(k') \times U(n-k')\}$$

the isotropic subgroup of $\eta_0$. We will henceforth fix these realizations of the groups $U(k), U(n-k), U(k')$ and $U(n-k')$ when viewed as subgroups of $G = U(n)$, if no ambiguity would arise.
For any $\eta \in G_{n,k'}$ the subset

$$S(\eta) = \{ \xi \in G_{n,k}; \xi \subset \eta \}$$

is a totally geodesic submanifold of $G_{n,k}$. It is itself a symmetric space with the induced metric. For $\eta = \eta_0$,

$$S(\eta_0) = G_{k',k} = U(k')/U(k) \times U(k' - k)$$

where $U(k) \times U(k' - k)$ (now with a different realization) consists of the elements in $G$ of the form $\text{diag}(\delta, \epsilon, I_{n-k'})$. For any $\eta \in G_{n,k'}$ let $\eta = g_\eta \eta_0$ with $g_\eta \in G$, then

$$S(\eta) = g_\eta S(\eta_0).$$

We let $d_\eta \xi$ be the unique $U(k')$ invariant measure on $S(\eta)$ via the above identification.

We define the Radon transform $\mathcal{R} : C^\infty(G_{n,k}) \to C^\infty(G_{n,k'})$ by

$$\mathcal{R}(f)(\eta) = \int_{S(\eta)} f(\xi) d_\eta \xi.$$  \hfill (4.1)

Equivalently it can be defined as

$$\mathcal{R}(f)(\eta) = \int_{U(k')} f(\eta \tau \xi_0) d\tau.$$ \hfill (4.2)

Similarly we define the Radon transform on $S_{n,k}$,

$$\mathcal{R}(f)(y) = \int_{U(k')} f(\eta \tau x_0) d\tau.$$ \hfill (4.3)

It is easy to see that $\mathcal{R}f$ is well-defined and maps $C^\infty(S_{n,k})$ to $C^\infty(S_{n,k'})$. In particular the two formulas agree when acting on $f \in C^\infty(G_{n,k})$ viewed as right $U(k)$-invariant functions on $S_{n,k}$.

As explained in the introduction, we shall define the operator $\phi(\eta) \to (\mathcal{T}_{r_{1/2}}/\phi)(\xi)$ from $C^\infty(G_{n,k'})$ to $C^\infty(G_{n,k})$, as the integration of functions $\phi$ over planes $\eta$ that are of angle $(x, \eta)$ so that $\text{Cos}^2(x, \eta) = uru^*$, $u \in U(k)$, with $x \in S_{n,k}$ a representative of $\xi$; we shall define a slight generalization, $T_a$, for a $a \in M_{k,k}$, $a^*a \leq I$. We will write the integration as one on the group $U(n-k)$. For that purpose we define the following $n \times n$ matrices, written in block matrices under the decomposition $\mathbb{K}^n = \mathbb{K}^{k'-k} \oplus \mathbb{K}^k \oplus \mathbb{K}^{n-k'-k} \oplus \mathbb{K}^k$,

$$j(a) = \begin{bmatrix} I_{k'-k} & 0 & 0 & 0 \\ 0 & (I_k - a^*a)^{1/2} & 0 & a^* \\ 0 & 0 & I_{n-k'-k} & 0 \\ 0 & -a & 0 & (I_k - aa^*)^{1/2} \end{bmatrix},$$

and

$$h(a) = \begin{bmatrix} a^* & 0 & 0 & (I_k - a^*a)^{1/2} \\ 0 & I_{n-k'-k} & 0 & 0 \\ 0 & 0 & I_{k'-k} & 0 \\ -(I_k - aa^*)^{1/2} & 0 & 0 & a \end{bmatrix},$$

where $a \in M_{k,k}$, $a^*a \leq I$.

Some simple observations about those elements are given in the next Lemma.
Lemma 4.1.  
(i) Suppose $0 \leq a^*a \leq I_k$. The elements $j(a)$ and $h(a)$ are in 
$G = U(n, \mathbb{K})$, and $j(a)^{-1} = j(a)^* = j(-a)$, $h(a)^{-1} = h(a)^* = h(a^*)$.
(ii) The projection $P_{j(a)\eta_0} = j(a)P_{\eta_0}j(a)^*$ onto the subspace $j(a)\eta_0 = \{j(a)\eta_0\} \in 
\mathbb{R}_{n,k}$ is given by

$$
P_{j(a)\eta_0} = \begin{bmatrix}
I_{k'-k} & 0 & 0 & 0 \\
0 & I_k - a^*a & 0 & (I_k - a^*a)^{\frac{1}{2}}a^* \\
0 & 0 & 0 & 0 \\
0 & a(I_k - a^*a)^{\frac{1}{2}} & 0 & aa^*
\end{bmatrix}.
$$

(iii) For any $v \in S_{k',k}$, we have

$$
j(a)^{-1} \begin{bmatrix}
v \\
0
\end{bmatrix} = \begin{bmatrix}
I_{k'-k} & 0 & 0 & (I_k - a^*a)^{\frac{1}{2}} \\
0 & (I_k - a^*a)^{\frac{1}{2}} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
v \\
0
\end{bmatrix},
$$

and

$$
h(a) \begin{bmatrix}
v \\
0
\end{bmatrix} = \begin{bmatrix}
I_{k'-k} & 0 & 0 & \eta_k \\
0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
v \\
0
\end{bmatrix}.
$$

(iv) Assume that $0 < aa^* < I_k$. For any $v \in S_{k',k}$ there exists an $l \in U(n - k)$ such 
that

$$
j(a)^{-1} \begin{bmatrix}
v \\
0_{n-k',k}
\end{bmatrix} = \text{diag}(l, I_k)h(b)\hat{x}_0
$$

where $b = [0_{k,k'-k} \ a] \ v \in M_{k,k}$.

Proof. We prove only the last statement (iv), the remaining are proved by simple 
matrix computations. First we see that $h(b)\hat{x}_0$ is the right hand side of (4.4) is, by 
(iii),

$$
h(b)\hat{x}_0 = \begin{bmatrix}
(I_k - a^*a)^{\frac{1}{2}} \\
0_{n-2k,k} & a
\end{bmatrix}.
$$

We have the left hand side of (4.4), by the first formula in (iii), is of the form

$$
j(a^{-1}) \begin{bmatrix}
v \\
0_{n-k',k}
\end{bmatrix} = \begin{bmatrix}
q \\
b
\end{bmatrix} \in S_{n,k}
$$

for some $q \in M_{n-k,k}$, $q^*q = I_k - b^*b$. By our assumption $0 < a^*a < I$ we see that 
b = [0 a]v satisfies also $b^*b < I$ and which in turn implies that $q$ is of full rank $k$. 
The polar decomposition of $q$ is then $q = u(I_k - b^*b)^{\frac{1}{2}}$, with $u \in S_{n-k,k}$, and $u$ can be 
further written as

$$
u = l \begin{bmatrix}
I_k \\
0_{n-k'-k,k}
\end{bmatrix}.$$
Then $\cos^2(4.7) T_{\omega} \text{ maps } S_{n,k}$. We give an elementary proof. □

We define, for any $b \in M_{k,k}([\mathbb{K}], b^* b \leq I$, the operator $T_b : C^\infty(S_{n,k'}) \to C^\infty(S_{n,k})$

$$T_b \phi(x) = \int_{U(n-k) \times U(k)} \phi(g_x k j(b)^{-1} y_0) dk, \quad x = g_x \hat{x}_0, \quad g_x \in G.$$  

It maps $C^\infty(G_{n,k'})$ to $C^\infty(G_{n,k})$ and has the form

$$T_b \phi(\xi) = \int_{U(n-k) \times U(k)} \phi(g_x k j(b)^{-1} \eta_0) dk.$$

The following lemma clarifies the geometric meaning of the above integral. For completeness we give an elementary proof.

**Lemma 4.2.** Let $0 \leq r \leq l = I_k$ and $\eta \in G_{n,k'}$ and $x = g_x \hat{x}_0 \in S_{n,k}$ for $g_x \in G$. Then $\cos^2(\eta, x) = \delta_0 r \delta_0^*$ for some $\delta_0 \in U(k)$ if and only if $m = g_x \rho j(r^*) \eta_0$ for some $\rho = \text{diag}(\alpha, \delta) \in K$.

**Proof.** Suppose $\rho = \text{diag}(\alpha, \delta) \in K$ and $\eta = g_x \rho j(r^*) \eta_0$. The projection $P_\eta \in M_{n,n}$ is given by

$$P_\eta g_x \rho P_{j(r^*)} \eta_0 \rho^* g_x^*$$

and

$$\cos^2(\eta, x) = x^* P_\eta x = x_0^* \rho P_{j(r^*)} \eta_0 \rho^* x_0 = \delta r \delta^*$$

by Lemma 4.1 (ii).

Conversely, suppose $\cos^2(\eta, x) = \delta_0 r \delta_0^* := s$. We let $\omega = g_x^{-1} \eta$. We prove that $\omega = \rho j(r^*)^{-1} \eta_0$ for some $\rho \in K$, which is our claim. Now $P_\eta = g_x P_\omega g_x^*$, and then by definition

$$s = \cos^2(\eta, x) = x^* P_\eta x = x_0^* P_\omega x_0.$$  

The matrix $P_\omega$ is a self-adjoint matrix on $\mathbb{K}^n$, and under the decomposition of $\mathbb{K}^n = \mathbb{K}^{n-k} \oplus \mathbb{K}^k$ it is of the block-matrix form

$$P_\omega = \begin{bmatrix} A & B \\ B^* & s \end{bmatrix}.$$  

by the previous formula, with $A$ self-adjoint and $B \in M_{n-k,k}([\mathbb{K})$. Being a projection, $P_\omega^2 = P_\omega$, that is

$$A^2 + BB^* = A, \quad AB + Bs = B, \quad B^* B + s^2 = s.$$  

The polar decomposition of $B$ is of the form, under the decomposition $\mathbb{K}^{n-k} = \mathbb{K}^{k-k'} \oplus \mathbb{K}^k \oplus \mathbb{K}^{n-k-k'}$,

$$B = \alpha_1 \begin{bmatrix} 0 \ \ s^{1/2} (I_k - s)^{1/2} \\ s^{1/2} (I_k - s)^{1/2} \ 0 \end{bmatrix}, \quad \alpha_1 \in U(n-k),$$
for $B^*B = s(I_k - s)$.

Let $A_1 = \alpha_1^{-1} A \alpha_1$. Writing $A_1$ in block matrix form under the same decomposition
and using the equation $A^2 + BB^* = A$ we see that

$$A_1 = \begin{bmatrix}
A_{11} & 0 & A_{13} \\
0 & (s)^{1/2}(I_k - s)^{1/2} & 0 \\
A_{13}^* & 0 & A_{33}
\end{bmatrix},$$

and that the matrix

$$\begin{bmatrix}
A_{11} & A_{13} \\
A_{13}^* & A_{33}
\end{bmatrix}$$

is a projection of rank $k' - k$ since $P_\omega$ is of rank $k'$. Thus

$$\begin{bmatrix}
A_{11} & A_{13} \\
A_{13}^* & A_{33}
\end{bmatrix} = \alpha_2 \begin{bmatrix}
I_{k' - k} & 0 \\
0 & 0
\end{bmatrix}$$

for some $\alpha_2 \in U(n - 2k)$. Finally we have

$$P_\omega = \rho j(\frac{1}{r^2})^{-1} P_{\eta_0j(\frac{1}{r^2})},$$

with $\rho = \text{diag}(\alpha_1 \alpha_2, \delta) \in K$, which is equivalent to

$$\omega = \rho j(\frac{1}{r^2})^{-1} \eta_0,$$

as any subspace $\omega$ is uniquely determined by the projection $P_\omega$. \hfill \Box

For $0 < r < I$ the operator $\mathcal{T}_{r^2}$ can be expressed in terms of the angle explained in
the introduction,

$$(\mathcal{T}_{r^2}\phi)(x) = \int_{y=(y): \cos^2(\eta, x) = uru^*, u \in U(k)} \phi(y) d_x y.$$  

For the real Grassmannians this is the slightly corrected version of the formulas (1.8) and (3.2) in [3] ($uru^*$ should be in place of $r$ there).

To state the next proposition we need also a mean-value operator, $\mathcal{W}_b : C^\infty(S_{n,k}) \to C^\infty(S_{n,k})$, for $b \in M_{k,k}$, $b^*b \leq I$,

$$(\mathcal{W}_b f)(x) = \int_{U(n-k)} f(gx \text{ diag}(\alpha, I_k) h(b) \tilde{x}_0) d\alpha;$$

again $\mathcal{W}_a$ is well-defined. Observe that it does not map functions on $G_{n,k}$ to functions $G_{n,k}$. If $f \in C^\infty(G_{n,k})$, the function

$$\int_{U(k)} (\mathcal{W}_{\delta a} f)(x) d\delta$$

is right $U(k)$-invariant and thus defines a functions on $G_{n,k}$. (For $k = 1$ the operator $\mathcal{W}_{\delta a}$ can roughly speaking be incorporated into the operator $\mathcal{T}$ so that it doesn’t play a role.)

First we state an integral formula, which is a direct consequence of Proposition 3.3 (ii), written in the form

$$w = \begin{bmatrix}
v(I - r)^{1/2} \\
v^* r^{1/2}
\end{bmatrix}, \quad v \in S_{n-k,k}, \quad u \in U(k), \quad r \in (0, I),$$
and of the formula \[ \text{(4.9)} \]; see also \[ \text{Lemma 3.5} \] (a different formula for the mean-value operator is defined there).

**Lemma 4.3.** Suppose \( 1 \leq p \leq \infty \) and \( f \in L^p(S_{n,k}) \). Then for any \( x \in S_{n,k} \)

\[
\int_{S_{n,k}} f(g_x w) dw = \int_{S_{n,k}} f(w) dw \\
= C_2(n, k) \int_0^{I_k} \Delta(I_k - r)^\frac{2}{2}(n-2k+1) - 1 \Delta(r)^\frac{d}{2} - 1 \left( \int_{U(k)} (W_{ur} \frac{f}{r}) (x) du \right) dr.
\]

In particular, the function

\[
\Phi_0(r) = \Phi_0(r, x) := \Delta(r)^{\frac{d}{2} - 1} \int_{U(k)} (W_{ur} \frac{f}{r}) (x) du
\]

is in \( L^1((0, I_k), \Delta(I_k - r)^\frac{d}{2}(n-2k+1) - 1 dr) \).

**Proposition 4.4.** Let \( \varepsilon \in \mathbb{C} \). Suppose \( f \in L^1(S_{n,k}) \) and \( \phi = \mathcal{R} f \). Then, in the sense of distribution,

\[
\mathcal{I}^{\varepsilon + N/k} \Phi_1 = C_3 \mathcal{I}^{\frac{d}{2}(k' - k)} \varepsilon + N/k \Phi_0,
\]

as analytic continuation in \( \varepsilon \), where

\[
\Phi_1(s) = \Phi_1(s, x) = \Delta(s)^{\frac{d}{2}(k' - (k-1)) - 1} (\mathcal{T}_{s^\frac{d}{2}} \phi)(x)
\]

and \( \Phi_0 \) is as in \( \text{(4.9)} \).

**Proof.** Let \( r \in (0, I) \). We compute \( (\mathcal{T}_{r^\frac{d}{2}} \phi)(x) = (\mathcal{T}_{r^\frac{d}{2}} \mathcal{R} f)(x) \). We prove the formula for \( \varepsilon \geq 0 \), in which case the convergence of the relevant integrals are easily justified by using Lemma \( \text{4.3} \) and the result then follows for any \( \varepsilon \) by analytic continuation.

By the definitions of \( \mathcal{T} \) and \( \mathcal{R} \) we have,

\[
(\mathcal{T}_{r^\frac{d}{2}} \phi)(x) = \int_K \int_{\tau \in U(k')} f(g_x \rho j(r^\frac{1}{2} - 1) \tau x_0) d\tau d\rho,
\]

where \( \tau = \text{diag}(\tau, I_{n-k}) \) in the formula. Introducing the variable \( v = \tau I_k \in S_{k',k} \), we have

\[
\tau x_0 = \begin{bmatrix} v \\ 0 \end{bmatrix};
\]

furthermore using Lemma 4.1 (iv) we have

\[
j(r^{1/2} - 1) \tau x_0 = \text{diag}(l, I_k) h([0, r^{1/2}] v) \hat{x}_0,
\]

for some \( l \in U(n-k) \). Now \( \rho = \text{diag}(\alpha, \delta) \),

\[
\rho j(r^{1/2} - 1) \tau x_0 = \text{diag}(\alpha, \delta) \text{diag}(l, I_k) h([0, r^{1/2}] v) \hat{x}_0 = \text{diag}(\alpha l, \delta) h([0, r^{1/2}] v) \hat{x}_0,
\]

thus changing variables \( \alpha l \rightarrow \alpha \) the previous integral becomes

\[
\int_{S_{k',k}} \int_{\alpha \in U(n-k)} \int_{\delta \in U(k)} f(g_x \text{diag}(\alpha, I_k) \text{diag}(I_{n-k}, \delta) h([0, r^{1/2}] v) \hat{x}_0)) d\delta d\alpha dv
\]

\[
= \int_{S_{k',k}} \int_{\alpha \in U(n-k)} \int_{\delta \in U(k)} f(g_x \text{diag}(\alpha, I_k) h(\delta [0, r^{1/2}] v) \hat{x}_0)) d\delta d\alpha dv.
\]
In term of the mean-value operator $W_b$ the integral can be written as

\[(4.10) \quad (\mathcal{T}_{r^2})(x) = \int_{S^k_{a',a} \cap U(k)} (W_{[0,r^2]} f)(x) d\delta dv\]

We use now Proposition 3.5 with $H(u) = \int_{\delta \in U(k)} (W_{\delta u} f)(x)$, obtaining

\[
(I_{\varepsilon+N/k} \Phi_1)(s) = C_3 (I_{\varepsilon+N/k} \Phi_0)(s), \quad s \in (0, I).
\]

The next lemma is elementary, the first two parts can be proved by the same method as that of Lemma 3.4 in [3]. The last part is proved by using the triangle inequality and the fact that $\Delta(s) \leq 1$ for $s \in (0, I)$; we omit the proof.

**Lemma 4.5.**  
(i) For any $f \in L^p(S_n,k), 1 \leq p < \infty$, we have

\[
\lim_{a \to I_{k,a,a} \ast a < I_k} W_{a} f = f.
\]

where the limit is in $L^p(S_n,k)$-sense.

(ii) If $f \in L^p(G_n,k)$ then

\[
\lim_{a \to I_{k,a,a} \ast a < I_k} \int_{U(k)} (W_{\delta a} f) d\delta = f.
\]

and the limit is taken in $L^p(G_n,k)$.

(iii) If $f \in L^p(S_n,k)$ or $L^p(G_n,k)$ then for any complex number $\zeta$,

\[
\lim_{a \to I_{k,a,a} \ast a < I_k} \Delta(a^* a)^\zeta \int_{U(k)} (W_{\delta a} f) d\delta = f.
\]

and the limit is taken in the respective spaces.

**Theorem 4.6.** Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the field of real, complex and quaternionic numbers. Let $1 \leq k < k' < n - 1, k + k' \leq n$. Let $\Delta(\partial)$ be the Cayley-type differential operator acting on the space of self-adjoint $k \times k$-matrices over $\mathbb{K}$ and $I^\lambda$ be the Gårding-Gindikin fractional integration. Suppose $1 \leq p < \infty, f \in L^p(G_n,k(\mathbb{K}))$ and $\phi = R f$. Let $m$ be such that

\[
m > \frac{d}{2}(k' - 1).
\]

Then the Radon transform $f \to \phi = R f$ is inverted by the formula

\[
f = \lim_{s \to I_k} \Delta(\partial)^m I^{m - \frac{d}{2}(k' - k)} \Phi
\]

where the limit is taken in the space $L^p(G_n,k(\mathbb{K}))$ and the differential operator acts in the sense of distribution, and where

\[
\Phi(s) = \Phi(s, \xi) = \Delta(s)^{\frac{d}{2}(k' - (k - 1)) - 1} (\mathcal{T}_{s^2} \phi)(\xi).
\]

**Proof.** Let $m$ be as in the statement and let $\varepsilon > -1$ be given so that $m = \frac{d}{2}(k' - k) + \varepsilon + N/k$. Then Proposition 4.4 can be restated as

\[
(I_m \Phi_0)(s) = (I^{m - \frac{d}{2}(k' - k)} \Phi)(s).
\]
Now Lemma 4.3 implies that $\Phi_0$ is in a $L^1$-space, and we can then use Lemma 2.1 (iii), obtaining

$$\Delta(s)^{\frac{d}{2}-1} \int_{U(k)} (W_{us/2} f)(x) d\delta = \Phi_0(s) = \Delta(\partial_s m(I^{m-d/2}(k' - k)\Phi)(s),$$

in the distributional sense. The result then follows from Lemma 4.5 (iii). □

There arise several interesting questions such as inverting Radon transform on non-compact symmetric matrix domains and finding a Plancherel formula. In a forthcoming paper we will generalize the results in this paper to that setup.

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