On a non-homogeneous version of a problem of Firey

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Abstract
We investigate the uniqueness for the Monge–Ampère type equation
\[
\det(u_{ij} + \delta_{ij} u_{i,j=1}^n = G(u)), \quad \text{on } S^{n-1},
\]
where \(u\) is the restriction of the support function on the sphere \(S^{n-1}\), of a convex body that contains the origin in its interior and \(G : (0, \infty) \to (0, \infty)\) is a continuous function. The problem was initiated by Firey (Mathematika 21(1): 1–11, 1974) who, in the case \(G(\theta) = \theta^{-1}\), asked if \(u \equiv 1\) is the unique solution to (1). Recently, Brendle et al. (Acta Math 219(1): 1–16, 2017) proved that if \(G(\theta) = \theta^{-p}, \ p > -n - 1\), then \(u\) has to be constant, providing in particular a complete solution to Firey’s problem.

Our primary goal is to obtain uniqueness (or nearly uniqueness) results for (1) for a broader family of functions \(G\). Our approach is very different than the techniques developed in Brendle et al. (2017).

1 Introduction

The primary goal of this note is to obtain uniqueness results for the Monge–Ampère type equation
\[
\det(u_{ij} + \delta_{ij} u_{i,j=1}^n = G(u)), \quad \text{on } S^{n-1},
\]
where \(G : (0, \infty) \to \mathbb{R}\) is a strictly positive continuous function and \(u\) is the restriction on \(S^{n-1}\) of a sub-linear positively homogeneous function defined on \(\mathbb{R}^n\). Here, \(S^{n-1}\) is...
the Euclidean unit sphere of \( \mathbb{R}^n \), \( u_{ij} \) denotes the covariant derivative of \( u \) with respect to a local orthonormal frame on \( \mathbb{S}^{n-1} \) and \( \delta_{ij} \) are the Kronecker symbols.

Let \( K \) be a convex body (that is, convex compact with non-empty interior) in \( \mathbb{R}^n \). The \textit{support function} \( h_K : \mathbb{R}^n \to \mathbb{R} \) of \( K \) is defined by

\[
h_K(x) = \max\{\langle x, y \rangle : y \in K\}.
\]

Recall (see e.g. [37, Chapter 1]) that a function \( u : \mathbb{R}^n \to \mathbb{R} \) is sub-linear and positively homogeneous if and only if there exists a (unique) convex body \( K \) in \( \mathbb{R}^n \), such that \( u = h_K \). Furthermore, \( h_K \) is strictly positive on \( \mathbb{R}^n \setminus \{0\} \) if and only if \( K \) contains \( 0 \) (the origin) in its interior.

The \textit{surface area measure} \( S_K \) of \( K \) is a Borel measure on \( \mathbb{S}^{n-1} \), given by

\[
S_K(\omega) = \mathcal{H}^{n-1}(\{x : x \text{ is a boundary point of } K \text{ and there exists } v \in \omega \text{ such that } \langle x, v \rangle = h_K(v)\}),
\]

for any Borel subset \( \omega \) of \( \mathbb{S}^{n-1} \). The notation \( \mathcal{H}^{n-1}(\cdot) \) stands for the \((n-1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \). It is clear that \( S_K \) is invariant under translation of \( K \). Moreover, the barycentre of \( S_K \) is always at the origin. If \( S_K \) is absolutely continuous with respect to \( \mathcal{H}^{n-1} \), its density will be denoted by \( f_K \). It is well known that (see e.g. [23])

\[
f_K(v) = \det(h_{K,ij}(v) + \delta_{ij}h_K(v)), \quad \text{for almost every } v \in \mathbb{S}^{n-1}. \tag{3}
\]

If the boundary of \( K \) is of class \( C^2 \) and its curvature \( K(K, \cdot) \) is strictly positive, then \( f_K \) is continuous (i.e. has a continuous representative) and it holds

\[
f_K(v) = \frac{1}{K(K, \eta_K^{-1}(v))}, \quad \text{for all } v \in \mathbb{S}^{n-1}. \tag{4}
\]

Here, \( \eta_K \) is the Gauss map (see Sect. 2). We refer to [37] for concepts related to (3), (4) and the definition of the surface area measure.

As (3) shows, (2) can be re-written as follows

\[
dS_K = G(h_K)d\mathcal{H}^{n-1}. \tag{5}
\]

Equation (2) and its weak form (5) have appeared in several places in literature (in the context of different areas of Mathematics, such as Convex Geometry, Differential Geometry and PDE’s). Below, we list some of them.

A result due to Simon [39] states the following.

**Theorem A** [39] Let \( u : \mathbb{S}^{n-1} \to (0, \infty) \) be the restriction of a support function on \( \mathbb{S}^{n-1} \), \( k \in \{1, \ldots, n-1\} \), \( \lambda_k(u) \) be the \( k \)-th elementary symmetric polynomial of the reciprocals of the eigenvalues of the matrix \( (u_{ij} + \delta_{ij} u)_{i,j=1}^{n-1} \) and \( G \) be a strictly positive non-decreasing function. If \( u \) satisfies
\[ \lambda_k(u) = G(u), \quad \text{on } \mathbb{S}^{n-1}, \]

then \( u \) is constant.

Theorem A, then, corresponds to (2) if \( k = n - 1 \).

Firey [16] posed the following problem: Assume that \( G(\theta) = \theta^{-1} \). Is it true that the unit ball is the only convex body \( K \) that solves (5)? He noticed that the answer to the problem is affirmative if one additionally assumes that \( b(K) = 0 \), where \( b(K) \) is the barycentre of \( K \). This follows as an almost immediate consequence of the Blaschke-Santaló inequality (see next section for notation)

\[ V(K)V(K^o) \leq V(B_2^n)^2. \]

Andrews considered a generalization of Firey’s problem by replacing \( \theta^{-1} \) with \( \theta^p \), \( p \in \mathbb{R}\setminus\{0\} \). It turns out that uniqueness fails if \( p \leq -n - 1 \) (see [2] and [3] for a complete classification of solutions in the plane). The case \( p = -n - 1 \) was settled in [32]; it was shown that, in this case, solutions are precisely the support functions of ellipsoids centered at the origin. The problem of classifying the solutions of (5) in the case \( G(\theta) = \theta^p \), \( p \in \mathbb{R}\setminus\{0\} \), is closely related to the asymptotic behaviour of the solution to the \( a \)-Gauss curvature flow (also introduced by Firey in [16] for \( a = 1 \) and by Andrews in [2] and [3] for general \( a \)), where \( a = -1/p \), and has attracted considerable attention in the past decades.

Brendle et al. [8] recently solved the problem for \( p > -n - 1 \) and, hence, gave an affirmative answer to Firey’s long standing question.

**Theorem B** (Brendle et al. [8]) If \( K \) is a convex body, containing the origin in its interior, that solves (5) for \( G(\theta) = \theta^p, \theta > 0, p > -n - 1 \), then \( K \) is a Euclidean ball centered at the origin.

Intermediate and related results to the preceding theorem include (but certainly not limited to) [1,4,13,14,18,19,21,42]. Equation (5) can be generalized as follows.

\[ dS_K = G(h_K)d\mu, \quad \text{on } \mathbb{S}^{n-1}, \quad (6) \]

where \( \mu \) is a given Borel measure on \( \mathbb{S}^{n-1} \). The case where \( G \equiv 1 \) is the classical Minkowski problem. Minkowski’s Existence and Uniqueness Theorem states that there exists a solution \( K \) to the previous equation if and only if the barycentre of \( \mu \) is the origin and \( \mu \) is not concentrated on any great sub-sphere of \( \mathbb{S}^{n-1} \) and, furthermore, any solution \( K \) is unique up to translation. The case where \( G(\theta) = \theta^{1-p}, p \in \mathbb{R} \) is the \( L^p \) Minkowski problem introduced by Lutwak [28]. While, as of the existence and uniqueness, the problem has been settled for \( p \geq 1 \) [12,28], several problems remain open for \( p \leq 1 \) (see e.g. [5,7,43] for related results and open problems and [22] for an important generalization). In particular, the question of uniqueness in the case \( p = 0 \) and \( \mu, h_K \) being even is now considered to be a major open problem in Convex Geometry (see for instance [6,26,36,40,41]).

If \( G \) is considered to be a general continuous function, (6) was introduced in [20] (under some assumptions on \( G \)). While important results concerning existence have
been obtained [20,24], very few facts are known about uniqueness. In view of (4), (5) should be viewed as the constant curvature case in the Orlicz–Minkowski problem.

Before we state our main results, we will need to agree on some notation. Given a positive integer $n$, set $A(n)$ to be the positive cone, consisting of all continuous functions $G : (0, \infty) \to (0, \infty)$, such that for some (any) antiderivative $H$ of $G$, the function
\[(0, \infty) \ni \theta \mapsto \theta G(\theta) + nH(\theta)\]
is strictly increasing. Notice that any differentiable function $G$ with $\theta G'(\theta) + (n + 1)G(\theta) > 0$, for all $\theta > 0$, belongs to $A(n)$.

**Theorem 1.1** Let $K$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior and solves (5) for some $G \in A(n)$. Then, $K$ is symmetric with respect to some straight line through the origin. In addition, $K$ is a Euclidean ball if at least one of the following hold

1. $b(K) = 0$
2. $G$ is monotone.

In particular, if $G$ is strictly monotone or $b(K) = o$, then $K$ is a Euclidean ball centered at the origin.

Clearly, the function $\theta^p$ is contained in the class $A(n)$, for $p > -n - 1$, and any continuous strictly increasing function $G : (0, \infty) \to (0, \infty)$ is also contained in $A(n)$. Hence, one immediately recovers Theorems B and A, in the case $k = n - 1$, from Theorem 1.1. Let us demonstrate that none of the assumptions of Theorem 1.1 can be removed. Indeed, if $G(\theta) = \theta^{n-1}$, then (as mentioned previously) solutions of (5) do not need to be axially symmetric even if $b(K) = o$, thus the assumption of $G$ being a member of $A(n)$ cannot be omitted. The necessity of the monotonicity of $G$ follows from the next theorem.

**Theorem 1.2** Let $a \in \mathbb{R}^n \setminus \{o\}$. There exists a centrally symmetric, non-spherical, strictly convex body $K$ in $\mathbb{R}^n$ with $C^\infty$ boundary and a function $G \in A(n)$, such that $K + a$ contains the origin in its interior and satisfies
\[f_K(v) = f_{K+a}(v) = G(h_{K+a}(v)), \quad \text{for all } v \in S^{n-1}.\]

In fact, $K$ can be taken to be arbitrarily close to a Euclidean ball and $G$ can be taken to be arbitrarily close to a constant.

We should remark that the method used to prove Theorem 1.1 is purely geometric and, therefore, quite different than the method used in [8]. More specifically, we employ a quick argument (although some preparation is needed; see Sects. 2, 3, 4) based on Steiner symmetrization (ultimately related to the Blaschke–Santaló inequality), to show that if $G \in A(n)$, $K$ is a solution of (5) and $H$ is a hyperplane through the origin that splits $K$ into two sets of equal volume, then $K$ has to be symmetric with respect to $H$, while if $b(K) = o$ then $K$ is a Euclidean ball. Based on this, we show in Sect. 5 that
$K$ is always axially symmetric. Thus, the remaining part of the theorem essentially reduces to a 2-dimensional problem (1-dimensional actually if one attempts to solve the associated ODE). The result will follow by a careful modification of a solution $K$ (Sects. 6 and 7), so that the resulting body is non-spherical, centrally symmetric and (approximately) solves (5). Theorem 1.2 will be proved in Sect. 8.

**Remark 1.3** The fact that $K$ is a Euclidean ball centered at the origin, when $G(\theta) = \theta^{-1}$, can also follow directly from the fact that $K$ is axially symmetric, by making use of a result of McCoy (see the final remark of [29]).

## 2 Preliminaries

In this section, we fix some notation and state some basic facts about convex bodies that are necessary for our purposes. As general references, we state the books of Schneider [37] and Gardner [17]. We denote the origin by $o$ and the standard (Euclidean) unit ball of $\mathbb{R}^n$ by $B^n_2$. The closure, the interior, the boundary and the volume (i.e. Lebesgue measure) of a set $A$ will be denoted by $\text{cl} \, A$, $\text{int} \, A$, $\text{bd} \, A$ and $V(A)$, respectively. The orthogonal projection of a set or a vector onto a subspace $H$ will be denoted by $\cdot | H$.

A convex body $K$ is said to be regular if the supporting hyperplane at each boundary point of $K$ is unique. It turns out that $K$ is regular if and only if its boundary is of class $C^1$. Furthermore, $K$ is of class $C^2_+$ if its boundary is of class $C^2$ and the quantity $\det(\langle h_{K,ij}(v) + \delta_{ij}h_K(v) \rangle)$ is strictly positive everywhere on $\mathbb{S}^{n-1}$ (then, $h_K$ is also of class $C^2$).

The Gauss map

$$\eta_K : \text{bd} \, K \rightarrow \mathbb{S}^{n-1}$$

takes every boundary point $x$ of the convex body $K$ to the (unique) outer unit normal vector that supports $K$ at $x$. If $K$ is strictly convex, then $\eta_K$ is invertible and $h_K$ is $C^1$. If, additionally, $K$ happens to be regular, then $\eta_K$ and $\eta_K^{-1}$ are homeomorphisms and $\eta_K^{-1}$ is given by

$$\eta_K^{-1}(v) = \nabla h_K(v), \quad \text{for all } v \in \mathbb{S}^{n-1}, \tag{7}$$

where $\nabla h_K$ is the usual gradient of $h_K$ in $\mathbb{R}^n$. We also note that the surface area measure of any strictly convex body $K$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$.

Let $L$, $M$ be convex bodies in $\mathbb{R}^n$. The first Minkowski mixed volume $V(L, M)$ is defined by

$$V(L, M) = \frac{1}{n} \frac{d}{dt} V(tL + M) \bigg|_{t=0^+}. \tag{123}$$

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Here, $A + B := \{x + y : x \in A, \ y \in B\}$ is the Minkowski sum of the sets $A$ and $B$. The following formula is well known

$$V(L, M) = \frac{1}{n} \int_{S^{n-1}} h_L(v) dS_M(v).$$

A basic inequality concerning mixed volumes is Minkowski’s first inequality, which reads as follows:

$$V(L, M) \geq V(L)^{1/n} V(M)^{(n-1)/n}. \quad (8)$$

The polar body $L^\circ$ of $L$ is defined to be the convex set

$$L^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } y \in L\}.$$

The Santaló point $s(L)$ of $L$ is defined to be the unique point $z \in \mathbb{R}^n$, such that

$$V((L - z)^\circ) = \min \{V((L - x)^\circ) : x \in \mathbb{R}^n\}.$$

Set, also, $L^* := (L - s(L))^\circ$. It is well known that $s(L) = o$ if and only if $b(L^\circ) = o$. Moreover, $L - s(L)$ contains the origin in its interior, while if $L$ contains the origin in its interior, then $L^\circ$ is also a convex body that contains the origin in its interior. In addition, it holds $(L^\circ)^\circ = L$.

The radial function $\rho_L : S^{n-1} \to \mathbb{R}$ is given by

$$\rho_L(v) = \sup \{\lambda \geq 0 : \lambda v \in L\}.$$

If $o \in \text{int } L$, then the radial function of $L$ and the support function of the polar body $L^\circ$ are related by

$$\rho_L(v) = \frac{1}{h_{L^\circ}(v)}, \quad \text{for all } v \in S^{n-1}. \quad (9)$$

Throughout this paper, $e$ will be a fixed unit vector, unless stated otherwise. Let $J$ be a subinterval of the real line. A shadow system along the direction $e$ is a family of convex bodies $\{L(t)\}_{t \in J}$ of the form

$$L(t) = \text{conv}\{x + t\beta(x)e : x \in L\}, \quad t \in J,$$

for some bounded function $\beta : L \to \mathbb{R}$. Shadow systems where introduced in [34]. It is known [34,38] (see also e.g. [11], [10]) that several functionals (such as quermass-integrals) on convex bodies are convex with respect to the parameter $t$. We are going to use in particular that $h_{L(t)}(v)$ is convex in $t$, for any $v \in S^{n-1}$. A simple consequence of this fact is the following lemma.
Lemma 2.1  Let $L$ be a convex body in $\mathbb{R}^n$, \{L(t)\}_{t \in [-1,1]} be a shadow system along direction $e$ and let $t_0 \in [-1, 1]$. Then, the function

$$(t, v) \mapsto \frac{h_{L(t)}(v) - h_{L(t_0)}(v)}{t - t_0}$$

is bounded on $([-1, 1] \setminus \{t_0\}) \times \mathbb{S}^{n-1}$.

Proof One can (trivially) extend \{L(t)\}_{t \in [-1,1]} to a shadow system \{L(t)\}_{t \in \mathbb{R}}. Since, for each $v \in \mathbb{S}^{n-1}$, $h_{L(t)}(v)$ is convex in $t$, we can write

$$-\infty < -\frac{\|h_{L(-2)}\|_{L^\infty(\mathbb{S}^{n-1})} + \|h_{L(t_0)}\|_{L^\infty(\mathbb{S}^{n-1})}}{3} \leq \frac{h_{L(-2)}(v) - h_{L(t_0)}(v)}{t - t_0} \leq \frac{h_{L(2)}(v) - h_{L(t_0)}(v)}{2 - t_0} \leq \|h_{L(2)}\|_{L^\infty(\mathbb{S}^{n-1})} + \|h_{L(t_0)}\|_{L^\infty(\mathbb{S}^{n-1})} < \infty,$$

for all $(t, v) \in ([-1, 1] \setminus \{t_0\}) \times \mathbb{S}^{n-1}$. \qed

The Steiner symmetral $S_t e(L)$ of the convex body $L$ with respect to the hyperplane $e^\perp := \{v \in \mathbb{R}^n : \langle e, v \rangle = 0\}$ is the (apparently convex body) set obtained by replacing, for each $x \in e^\perp$, the intersection of $L$ with the line that passes through $x$ and is parallel to $e$, with the line segment of the same length which is symmetric with respect to the hyperplane $e^\perp$ and passes through $x$.

A particular case of a shadow system can be constructed as follows: The convex body $L$ can be written as

$$L = \{\bar{x} + ye : \bar{x} \in L|e^\perp, z(\bar{x}) \leq y \leq w(\bar{x})\}, \quad (10)$$

where the functions $z$, $w : L|e^\perp \to \mathbb{R}$ are such that $z$ is convex, $w$ is concave and $z \leq w$. Define, then,

$$L_t := \{x - (1 - t)u(x|e^\perp)e : x \in S_t e(L)\} = \{\bar{x} + ye : \bar{x} \in L|e^\perp, z(\bar{x}) - (1 - t)u(\bar{x}) \leq y \leq w(\bar{x}) - (1 - t)u(\bar{x})\}, \quad t \in [-1, 1],$$

where $u := (z + w)/2$. One can check that $L_t$ is convex for all $t \in [-1, 1]$ and, therefore, the family \{L_t\}_{t \in [-1,1]} is indeed a shadow system along direction $e$. Moreover, $L_1 = L$, $L_{-1}$ equals the reflection of $L$ with respect to the hyperplane $e^\perp$ and $L_0 = S_t e(L)$. Thus, the shadow system \{L_t\}_{t \in [-1,1]} is a way to perform Steiner symmetrization to $L$ in a continuous way.

We will need some (relatively recent) results concerning the polar volume of a shadow system \{L(t)\}_{t \in J}. The next theorem was proved by Campi and Gronchi in the symmetric case and by Meyer and Reisner in full generality.
Theorem C (Meyer–Reisner [31]) The function $J(t) \mapsto V((L(t))^*)^{-1}$ is convex.
When we restrict our attention to the shadow system $\{L_t\}_{t \in [-1,1]}$ that corresponds to Steiner symmetrization, Theorem C easily implies the following

Corollary 2.2 (Meyer–Pajor [30]; Meyer–Reisner [31]) It holds

$$V(L^*)^{-1} \geq V((L_0)^*)^{-1}.$$  

It should be remarked that the Blaschke–Santaló inequality follows from Corollary 2.2 in a standard way.

If $L$ is a centrally symmetric non-ellipsoidal convex body, it was shown in [35] and [30] that there exists a direction, such that the Steiner symmetrization along this direction strictly increases the volume of $L$. Therefore, in view of Theorem C, this statement implies the following.

Lemma 2.3 Let $L$ be a centrally symmetric convex body in $\mathbb{R}^n$ which is not an ellipsoid. Then, there exists an orthogonal map $O : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\frac{d}{dt} V((OL_t)^*) \bigg|_{t=1^-} < 0.$$  

Denote by $(e^\perp)^+$ and $(e^\perp)^-$ the two closed half-spaces defined by the hyperplane $e^\perp$. The following fact (stated in a simplified form; it also follows from the proof of the main result in [10]) is also proved in [31, Lemma 4].

Theorem D [31] The function $J(t) \mapsto V(L(t)^* \cap (e^\perp)^\pm)^{-1}$ is convex.

As a consequence, we have the following lemma.

Lemma 2.4 Let $L$ be a convex body containing the origin in its interior. If either $b(L^o) = \emptyset$ or $V(L^o \cap (e^\perp)^+) = V(L^o \cap (e^\perp)^-)$ then

$$V(L^o) \leq V((L_t)^o),$$  

for all $t \in [-1,1]$.

Proof If $b(L^o) = \emptyset$, then due to the minimality of $V((L_t)^*)$, we have

$$V((L_t)^o)^{-1} \leq V((L_t)^*)^{-1} \leq tV((L_1)^*)^{-1} + (1-t)V((L_0)^*)^{-1} = tV((L_1)^o)^{-1} + (1-t)V((L_0)^o)^{-1},$$

where we used Theorem C. Thus, using Corollary 2.2, we obtain

$$V((L_t)^o)^{-1} - V((L_1)^o)^{-1} \leq (1-t)(V((L_0)^*)^{-1} - V((L_1)^*)^{-1}) = (1-t)(V((L_0)^o)^{-1} - V((L_1)^o)^{-1}) \leq 0.$$  

This proves our first assertion. To prove the second one, notice that if

$$V(L^o \cap (e^\perp)^+) = V(L^o \cap (e^\perp)^-),$$
Lemma 3.1 Let their proofs are simple) will be crucial for the proof of Theorem 1.1. Continuous function, such that if \( H \) is an antiderivative of \( G \), then the function \( \int_{0}^{\theta} s^{n-1} F(s) ds \), \( \theta = (\theta) \), \( \theta = (\theta) \). Then, \( H \) is \( C^1 \) and strictly increasing. Set, also, \( \overline{G} := \overline{H} \). The assumption on \( G \) implies that \( \overline{G}|_{(0, \infty)} \in \mathcal{A}(n) \). Let \( F(\theta) := \theta \overline{H}(\theta) + n \overline{H}(\theta) = \theta \overline{G}(\theta) + n \overline{H}(\theta) \) and

\[
I(\theta) := \int_{0}^{\theta} r^{n-1} F(r \theta) dr = \frac{\int_{0}^{\theta} s^{n-1} F(s) ds}{\theta^n}, \quad \theta \in (0, \infty).
\]
Notice that \( F(0) = \overline{H}(0) = n \lim_{\theta \to 0^+} I(\theta) = 0 \). Differentiating (12), we find
\[
I'(\theta) = \frac{F(\theta)}{\theta} - n \frac{I(\theta)}{\theta}
\]
and, hence,
\[
\theta I'(\theta) + n I(\theta) = \theta \overline{H}'(\theta) + n \overline{H}(\theta), \quad \text{for all } \theta \in (0, \infty).
\]
This easily implies that there exists a constant \( c \in \mathbb{R} \), such that
\[
I(\theta) = \overline{H}(\theta) + c \theta^{-n}, \quad \theta \in (0, \infty).
\]
Since \( \overline{H}(0) = \lim_{\theta \to 0^+} I(\theta) = 0 \), we conclude that \( I = \overline{H} \), hence (11) follows from (12).

**Remark 3.2** When studying (1.1), for some function \( G \in \mathcal{A}(n) \) and for some convex body \( K \) containing the origin in its interior, we are only interested in the restriction of \( G \) onto the interval \( [\min_{v \in \mathbb{S}^{n-1}} h_K(v), \max_{v \in \mathbb{S}^{n-1}} h_K(v)] \). Thus, Lemma 3.1 in particular shows that we may always assume that there exists a continuous strictly increasing function \( F : [0, \infty) \to \mathbb{R} \), satisfying \( F(0) = 0 \) and \( G(\theta) = \left( \int_0^1 r^{n-1} F(r \theta) dr \right)' \), for all \( \theta \in (0, \infty) \).

**Lemma 3.3** Let \( G \in \mathcal{A}(n) \) and let \( 0 < c_1 < c_2 \) and \( 0 < a_1 < a_2 \). If \( G(c_1) \geq a_1 \) and \( G(c_2) \leq a_2 \), then for some \( \epsilon_0 > 0 \) and for every \( \epsilon \in (0, \epsilon_0) \), there exists a function \( G_{\epsilon} \in \mathcal{A}(n) \), such that the family \( \{G_{\epsilon}\}_{\epsilon \in (0, \epsilon_0)} \) enjoys the following properties.

1. For any \( \epsilon \in (0, \epsilon_0) \), it holds \( G_{\epsilon}(\theta) = G(\theta) \), for all \( \theta \in [c_1 + \epsilon, c_2 + \epsilon] \), while \( G_{\epsilon}(c_i) = a_i \), \( i = 1, 2 \).
2. The function \( G(\epsilon, \theta) := G_{\epsilon}(\theta) \) is bounded on \((0, \epsilon_0) \times [c_1, c_2] \).

**Proof** Let \( \epsilon_0 > 0 \) be any number, such that \( c_1 + \epsilon_0 < c_2 - \epsilon_0 \) and let \( \epsilon \in (0, \epsilon_0) \). If \( G(c_1) = a_1 \) (resp. \( G(c_2) = a_2 \)), set \( c'_1 := c_1 \) (resp. \( c'_2 := c_2 \)), while if \( G(c_1) > a_1 \) (resp. \( G(c_2) < a_2 \)), fix \( c'_1 \) to be any real number in \([c_1, c_1 + \epsilon]\) (resp. fix \( c'_2 \) in \([c_2 - \epsilon, c_2]\)), such that \( G(c'_1) \geq a_1 \) (resp. \( G(c'_2) \leq a_2 \)). Define \( G_{\epsilon} : [c_1, c_2] \to (0, \infty) \) by
\[
G_{\epsilon}(\theta) = \begin{cases} 
 a_1 + (G(c'_1) - a_1) \frac{\theta - c_1}{c'_1 - c_1}, & c_1 \leq \theta \leq c'_1 \\
 G(\theta), & c'_1 < \theta < c'_2 \\
 G(c'_2) + (a_2 - G(c'_2)) \frac{\theta - c'_2}{c_2 - c'_2}, & c'_2 \leq \theta \leq c_2 
\end{cases}
\]
and set \( H_{\epsilon}(\theta) := \int_0^\theta G_{\epsilon}(r) dr, \theta > 0 \). Then, \( G_{\epsilon} \) is continuous and coincides with \( G \) on \([c_1 + \epsilon, c_2 - \epsilon]\). Moreover, \( \theta G_{\epsilon}(\theta) + n H_{\epsilon}(\theta) \) is strictly increasing on \([c_1, c'_1]\) and \([c'_2, c_2]\). Furthermore, \( H_{\epsilon}|_{[c'_1, c'_2]} \) is an antiderivative of \( G_{\epsilon}|_{[c'_1, c'_2]} = G|_{[c'_1, c'_2]} \), so the function \( \theta G_{\epsilon}(\theta) + n H_{\epsilon}(\theta) \) is also strictly increasing on \([c'_1, c'_2]\). The continuity of \( \theta G_{\epsilon}(\theta) + n H_{\epsilon}(\theta) \) shows that \( \theta G_{\epsilon}(\theta) + n H_{\epsilon}(\theta) \) is strictly increasing on \([c_1, c_2]\). Then, by Lemma 3.1, one can extend \( G_{\epsilon} \) to a function from \( \mathcal{A}(n) \). Finally, notice that \( G_{\epsilon}(\theta) \leq \max\{a_2, \max_{\theta \in [c_1, c_2]} G(\theta)\} \), for all \( \theta \in [c_1, c_2] \).
4 Differentiability properties of Steiner symmetrization

We start this section with the following simple lemma.

**Lemma 4.1** Let $L$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. For $x \in \text{bd } L$, set $N(L, x)$ to be the set of outer unit normal vectors to hyperplanes that support $L$ at $x$. Set, also,

$$m(L) := \inf \left\{ \frac{|\langle v, x \rangle|}{|x|} : x \in \text{bd } L_t, \ v \in N(L_t, x), \ t \in [-1, 1] \right\}.$$ 

Then, it holds $m(L) > 0$.

**Proof** For $t \in [-1, 1]$, one can check that $L_t$ contains the origin in its interior. Let $t \in [-1, 1], x \in \text{bd } L_t$ and $v \in N(L_t, x)$. Then, $x$ cannot be orthogonal to $v$; otherwise, the supporting hyperplane $H$ of $L_t$, whose outer unit normal vector is $v$, would be parallel to the line segment $[o, x]$ and, therefore, $[o, x]$ would be contained in $H$. This would show that the origin is not contained in the interior of $L_t$, a contradiction. Consequently, $|\langle v, x \rangle|/|x| > 0$. A continuity/compactness argument easily yields our claim. $\Box$

As a consequence, we have the following.

**Proposition 4.2** Let $L$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior and let $t_0 \in [-1, 1]$. Then, the function

$$([-1, 1] \setminus \{t_0\}) \times \mathbb{S}^{n-1} \ni (t, v) \mapsto \frac{\rho_{L_t}(v) - \rho_{L_{t_0}}(v)}{t - t_0}$$

is bounded.

**Proof** Fix $v \in \mathbb{S}^{n-1}$. Let $t_1, t_2 \in [-1, 1]$ with $t_1 \neq t_2$ and assume for instance that $\rho_{L_{t_1}}(v) > \rho_{L_{t_2}}(v)$. Set $x := \rho_{L_{t_1}}(v)v$ and $y := \rho_{L_{t_2}}(v)v$. Clearly, $x \in \text{bd } L_{t_1}$ and $y \in \text{bd } L_{t_2}$. Choose $v' \in N(L_{t_1}, x)$. Then, we have

$$h_{L_{t_1}}(v') = \langle v', x \rangle = \frac{\langle v', x \rangle}{|x|} \rho_{L_{t_1}}(v),$$

while

$$h_{L_{t_2}}(v') \geq \langle v', y \rangle = \langle v', v \rangle \rho_{L_{t_2}}(v) = \frac{\langle v', x \rangle}{|x|} \rho_{L_{t_2}}(v).$$

Consequently, it holds

$$\left| \frac{\rho_{L_{t_1}}(v) - \rho_{L_{t_2}}(v)}{t_1 - t_2} \right| \leq \left( \frac{|\langle v', x \rangle|}{|x|} \right)^{-1} \left| \frac{h_{L_{t_1}}(v') - h_{L_{t_2}}(v')}{t_1 - t_2} \right|.$$ (13)
Our claim follows immediately from Lemmas 2.1 and 4.1, after setting \( t_1 = t \) and \( t_2 = t_0 \) in (13). \( \square \)

In the next section, we will need that \( \rho_{L_t}(v) \) is differentiable from the left at \( t = 1 \). This would follow immediately if we knew that \( \rho_{L_t}(v) \) posses some concavity property (for instance, it has been mentioned that \( h_{L_t} \) is convex in \( t \)). Unfortunately, since we are not aware of any such property, we need to work a little bit harder in order to establish the desired differentiability.

**Proposition 4.3** Let \( L \) be a convex body in \( \mathbb{R}^n \), containing the origin in its interior. Then, for all \( v \in S^{n-1} \) and for all \( t_0 \in (−1, 1] \), \( \rho_{L_t}(v) \) is differentiable from the left at \( t = t_0 \).

**Proof** It is clearly enough to assume that \( n = 2 \) (just take intersections of \( L \) with 2-dimensional subspaces containing \( e \)). After a possible rotation of the coordinate system, \( L_t \) can be written as

\[
L_t = \left\{ (x, y) \in \mathbb{R}^2 : x \in [a, b], \ z(x) - (1 - t)\frac{w(x) + z(x)}{2} \leq y \leq w(x) - (1 - t)\frac{w(x) + z(x)}{2} \right\},
\]

for some \( a < 0 < b \) and some functions \( w, \ z : [a, b] \to \mathbb{R} \), such that \( z \leq w \) and \( z, -w \) are convex. Set \( u := (w + z)/2 \) and fix \( v = (v_1, v_2) \in S^1, t_0 \in [-1, 1] \). For \( t \in [-1, 1] \), set also \( x_t := \rho_{L_t}(v)v_1 \) and \( y_t := \rho_{L_t}(v)v_2 \). If \( v_1 = 0 \), it is clear that \( d\rho_{L_{t_0}}(v)/dt|_{t=t_0} = \pm u(0) \). We may, therefore, assume that \( v_1 > 0 \) (the case \( v_1 < 0 \) follows from the case \( v_1 > 0 \) by considering the reflections of \( L \) and \( v \) with respect to the \( y \)-axis). Notice, then, that \( x_t \in (0, b) \) and that \((x_t, y_t) = \rho_{L_t}(v)\) is continuous in \( t \).

Claim 1. Let \( t_0 \in (−1, 1] \). There exists \( \delta > 0 \), such that precisely one of the following holds.

(i) \( y_t = w(x_t) - (1 - t)u(x_t) \) and \( x_t \neq x_{t_0} \), for all \( t \in [t_0 - \delta, t_0] \).

(ii) \( y_t = z(x_t) - (1 - t)u(x_t) \) and \( x_t \neq x_{t_0} \), for all \( t \in [t_0 - \delta, t_0] \).

(iii) \( x_t = x_{t_0} \), for all \( t \in [t_0 - \delta, t_0] \).

To prove Claim 1, we will first show that either \( x_t \neq x_{t_0} \) in a left neighbourhood of \( t = t_0 \) or \( x_t = x_{t_0} \) in a left neighbourhood of \( t = t_0 \). If \( z(x_{t_0}) - (1 - t_0)u(x_{t_0}) < y_{t_0} < w(x_{t_0}) - (1 - t_0)u(x_{t_0}) \) or \( u(x_{t_0}) = 0 \), then it is clear that \( x_{t_0} = b \) and the point \((b, y_{t_0})\) remains a boundary point of \( L_t \), for all \( t \) in a neighbourhood of \( t_0 \). In particular, we fall in case (iii). We may, therefore, assume that \( u(x_{t_0}) \neq 0 \) and either it holds \( y_{t_0} = w(x_{t_0}) - (1 - t_0)u(x_{t_0}) \) or it holds \( y_{t_0} = z(x_{t_0}) - (1 - t_0)u(x_{t_0}) \). For \( t \in [-1, 1] \), consider the line segment \( I_t \) (which degenerates to a point if \( x_{t_0} = b \) and \( z(b) = w(b) \)), defined as the intersection of \( L_t \) with the line which is parallel to the \( x_2 \)-axis and passes through the point \((x_{t_0}, y_{t_0})\). Then, \((x_{t_0}, y_{t_0})\) is an end-point of \( I_{t_0} \) and \( I_t = I_{t_0} + (t - t_0)u(x_{t_0})(0, 1) \), \( t \in [-1, 1] \). Thus, (depending on the sign of \( u(x_{t_0}) \)) either \( I_t \) does not contain \((x_{t_0}, y_{t_0})\), for any \( t \in [-1, t_0] \) (this is always true if \( I_{t_0} \) is a singleton) or \((x_{t_0}, y_{t_0})\) is an interior point of \( I_t \), for \( t \) lying in some interval of the form \([t_0 - \delta, t_0] \). In the first case, we clearly have \( x_t < x_{t_0} \), for all \( t \in [-1, t_0] \).
Let us consider the second case. If \( x_{t_0} < b \), then \( x_t \) is an interior point of \( L_t \), for all \( t \in [t_0 - \delta, t_0) \). In particular, \( x_t > x_{t_0} \), for all \( t \in [t_0 - \delta, t_0) \). If \( x_{t_0} = b \), then the point \((b, y_{t_0})\) remains a boundary point of \( L_t \) for \( t \) close to \( t_0 \). Consequently, \( x_t = x_{t_0} = b \), for all \( t \in [t_0 - \delta, t_0) \).

To prove the remaining assertions of our claim, observe that since \((x_t, y_t)\) is continuous in \( t \) and since \( z(x) - (1 - t)u(x) < w(x) - (1 - t)u(x) \), for all \( x \in (a, b) \), it follows that if \( J \) is a subinterval of \([-1, 1] \), such that \( x_t \in (a, b) \), then either it holds \( y_t = w(x_t) - (1 - t)u(x_t) \), for all \( t \in J \) or it holds \( y_t = z(x_t) - (1 - t)u(x_t) \), for all \( t \in J \). Notice that if \( x_{t_0} < b \), then we may assume (by choosing a smaller \( \delta \) if necessary) that \( x_t < b \), for all \( t \in [t_0 - \delta, t_0) \). Thus, if \( x_t \neq x_{t_0} \) in \([t_0 - \delta, t_0) \), one can take \( J = [t_0 - \delta, t_0) \), completing the proof of Claim 1.

Next, fix \( t_0 \in (-1, 1] \). We will show that if \( x_{t_0} < b \), then \( \rho_{L_t}(v) \) is differentiable from the left at \( t = t_0 \). Since,

\[
\rho_{L_t}(v) = \sqrt{1 + \left(\frac{v_2}{v_1}\right)^2 x_t}, \quad t \in [-1, 1],
\]

it follows that case (iii) in Claim 1 can be excluded. By considering the reflections of \( L \) and \( v \) with respect to the \( x_1 \)-axis (if necessary), we may assume that case (i) occurs. For \( t \in [-1, 1] \) and \( x \in (0, b) \), set

\[
F(t, x) := \frac{w(x) - (1 - t)u(x)}{x},
\]

and notice that \( F(t, x_t) = v_2/v_1 \), for all \( t \in [t_0 - \delta, t_0) \).

First assume that \( x_t < x_{t_0} \) in \([t_0 - \delta, t_0) \). Remark that the concavity of \( w \) implies that the left derivatives \( w'_-(x) \) and \( u'_-(x) \) exist for all \( x \in (a, b) \). Using this, we easily arrive at

\[
\lim_{\substack{(t, x) \to (t_0, x_{t_0}) \\ t < t_0, x < x_{t_0}}} \frac{F(t, x) - F(t, x_{t_0})}{x - x_{t_0}} = \frac{x_1 w'_-(x_{t_0}) - w(x_{t_0})}{x_{t_0}^2}.
\]

Since

\[0 = \frac{F(t, x_t) - F(t_0, x_{t_0})}{t - t_0} = \frac{F(t, x_t) - F(t_0, x_{t_0})}{x_t - x_{t_0}} \frac{x_t - x_{t_0}}{t - t_0} + \frac{u(x_{t_0})}{x_{t_0}},\]

for \( t \in [t_0 - \delta, t_0) \) and since (15) holds, we conclude that

\[
\lim_{t \to t_0^-} \frac{x_t - x_{t_0}}{t - t_0} = -\frac{u(x_{t_0})}{x_{t_0}^2} \frac{x_{t_0}^2}{x_{t_0} w'_-(x_{t_0}) - w(x_{t_0})},
\]

as long as \( x_{t_0} w'_-(x_{t_0}) - w(x_{t_0}) \neq 0 \). But if \( x_{t_0} w'_-(x_{t_0}) - w(x_{t_0}) \) was equal to zero, then (since \( x_{t_0} u(x_{t_0}) \neq 0 \), the ratio \((x_t - x_{t_0})/(t - t_0)\) would be unbounded. This,
by (14) and Proposition 4.2 would lead us to a contradiction and, therefore (again by (14)) \((d/dt)\rho_{L_t}(v)|_{t=t_0^-}\) exists and it holds

\[
d\frac{d}{dt}\rho_{L_t}(v)\bigg|_{t=t_0^-} = -\sqrt{1 + \left(\frac{v_2}{v_1}\right)^2} \frac{x_{t_0} u(x_{t_0})}{x_{t_0} w'_-(x_{t_0}) - w(x_{t_0})}.
\] (16)

The case where \(x_t > x_{t_0}\) in a left neighbourhood of \(t = t_0\) can be treated similarly; one has to replace \(w'_-(x)\) and \(u'_-(x)\) by \(w'_+(x)\) and \(u'_+(x)\) in the previous argument.

Finally, we need to show that \((d/dt)\rho_{L_t}(v)\) is differentiable from the left at \(t = t_0\), if \(x_{t_0} = b\). As before, we may assume that there exists \(\delta > 0\), such that \(x_t < b\) and \(y_t = w(x_t) - (1 - t)u(x_t)\), for all \(t \in [t_0 - \delta, t_0]\). Then, (16) together with the concavity of \(w\) shows that \((d/dt)\rho_{L_t}(v)|_{t=s^-}\) exists for any \(s \in (t_0 - \delta, t_0)\) and the limit

\[
\lim_{s \to t_0^-} \frac{d}{dt}\rho_{L_t}(v)\bigg|_{t=s^-}
\]
also exists and is finite (notice that this is true even if \(w'_-(b) = -\infty\)). This easily shows that the left derivative \((d/dt)\rho_{L_t}(v)|_{t=t_0^-}\) exists. \(\square\)

Taking (9) into account, an immediate consequence of Proposition 4.2 and Proposition 4.3 is the following.

**Corollary 4.4** Let \(L\) be a convex body in \(\mathbb{R}^n\) that contains the origin in its interior. Then, the ratio \(\frac{h(L_t)^{\gamma}(v) - h_{t_0}^{S^n-1}(v)}{t-1}\) is bounded in \([-1, 1) \times S^{n-1}\) and the left derivative \(\frac{d}{dt}h(L_t)^{\gamma}(v)|_{t=1^-}\) exists for all \(v \in S^{n-1}\).

The next result is the infinitesimal version (its proof follows more or less the same lines) of the well known fact that the volume of the intersection of a convex body with the unit ball \(B^2_2\) increases under Steiner symmetrization.

**Proposition 4.5** Let \(L\) be a convex body in \(\mathbb{R}^n\) and \(a > 0\). Then, the function \([-1, 1] \ni t \mapsto V(B^n_2 \cap aL_t)\) is differentiable from the left at \(t = 1\). Moreover, it holds

\[
\frac{d}{dt} V(B^n_2 \cap aL_t)\bigg|_{t=1^-} \leq 0.
\] (17)

Finally, if \(L\) is not symmetric with respect to the hyperplane \(e^\perp\), then there exists an open interval \(J \subseteq (0, \infty)\), such that for any \(a \in J\), inequality (17) is strict.

**Proof** To prove the differentiability property and (17), it suffices to assume that \(a = 1\). Write \(L\) in the form (10), for some concave functions \(-z, w : L|e^\perp \to \mathbb{R}\) and set \(u := (z + w)/2\). For \(x \in e^\perp, t \in [-1, 1]\), set \(\Phi(t, x) := V(B^n_2 \cap L_t \cap (x + \mathbb{R}x))\) to be the length of the intersection of \(B^n_2\), the convex body \(L_t\) and the line which passes through \(x\) and is parallel to \(e\). Since \(\Phi(t, x) = V_1(B^n_2 \cap (x + \mathbb{R}e) \cap (L_0 \cap (x + \mathbb{R}x) + tu(x)e))\),
it is well known (and easily verified) that \( \Phi(t, x) \) is log-concave in \( t \). In fact, if \( B^n_2 \cap (x + \mathbb{R} e) = [x - ae, x + ae] \), for some \( a > 0 \), then

\[
\Phi(t, x) = \min\{w(x) - (1 - t)u(x), a\} - \max\{z(x) - (1 - t)u(x), -a\},
\]

while if \( B^n_2 \cap (x + \mathbb{R} e) = \emptyset \), then \( \Phi(t, x) \equiv 0 \). In particular, \( t \mapsto \Phi(t, x) \) is differentiable from the left at \( t = 1 \) and is also Lipschitz with Lipschitz constant bounded by \( 2 \sup_{x \in L; e^\perp} |u(x)| < \infty \). Then, Fubini’s Theorem and the Bounded Convergence Theorem yield

\[
\lim_{t \to 1^-} \frac{V(B^n_2 \cap L) - V(B^n_2 \cap L_t)}{1 - t} = \int_{e^\perp} \lim_{t \to 1^-} \frac{\Phi(1, x) - \Phi(t, x)}{1 - t} \, dt
\]

\[
= \int_{e^\perp} \frac{d}{dt} \Phi(t, x) \bigg|_{t=1^-} \, dx.
\]

Equations (18) and (20) immediately give (17).

Next, assume that the line segments \( L \cap (x + \mathbb{R} e) \), \( B^n_2 \cap (x + \mathbb{R} e) \) satisfy the following:

(a) \( L \cap (x + \mathbb{R} e) \) is not symmetric with respect to the hyperplane \( e^\perp \).

(b) \( L \cap (x + \mathbb{R} e) \not\subseteq B^n_2 \cap (x + \mathbb{R} e) \) and \( B^n_2 \cap (x + \mathbb{R} e) \not\subseteq L \cap (x + \mathbb{R} e) \).

(c) \( V_1(B^n_2 \cap L \cap (x + \mathbb{R} e)) > 0 \).

Then, assumptions (a)–(c) together with (19) and the log-concavity of \( t \mapsto \Phi(t, x) \) ensure that the function \( \Phi(t, x) \) is strictly decreasing in \( t \) on \([0, 1]\). Moreover, using again the log-concavity property together with assumption (c), we easily conclude that \( d\Phi(t, x)/dt \bigg|_{t=1^-} \) is strictly negative. This, together with (18), (20) and a continuity argument, shows that whenever there exists a point \( x \in e^\perp \) satisfying assumptions (a)–(c), inequality (17) is strict with \( a = 1 \).

To finish with our proof, simply observe that if \( L \) is not symmetric with respect to \( e^\perp \), then there exists \( a > 0 \) and a point \( x \in e^\perp \) that satisfies assumptions (a)–(c) when \( L \) is replaced with \( aL \). Another continuity argument shows that the same is true in a whole neighbourhood \( J \) of \( a \).
5 Axial symmetry of solutions

The main goal of this section is to establish the most important part of Theorem 1.1. More specifically, we prove the following.

Proposition 5.1 Let $K$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior and solves (5) for some $G \in \mathcal{A}(n)$. Then, $K$ is symmetric with respect to some straight line through the origin. Moreover, if $b(K) = 0$, then $K$ is a Euclidean ball centered at the origin.

For a convex body $L$ that contains the origin in its interior, set $L^t := ((L^o)_t)^0$, $t \in [-1, 1]$. For $v \in \mathbb{S}^{n-1}$, set also $h'_L(v)$ to be the left derivative of $h_L(v)$ at $t = 1$ (which exists always thanks to Corollary 4.4). The following two lemmas will be of great importance towards the proof of Proposition 5.1.

Lemma 5.2 Let $L$ be a convex body that contains the origin in its interior and is not symmetric with respect to the hyperplane $e^\perp$ and let $G \in \mathcal{A}(n)$. Then,

$$\int_{\mathbb{S}^{n-1}} G(h_L(v))h'_L(v)d\mathcal{H}^{n-1}(v) > 0.$$ 

Proof Let $F : [0, \infty) \to \mathbb{R}$ be the function associated with $G$ in Remark 3.2 and set $H(\theta) := \int_0^1 r^{n-1} F(r\theta)dr$. Set, also, $I(t) := \int_{\mathbb{S}^{n-1}} H(h_L(v))d\mathcal{H}^{n-1}(v)$. Corollary 4.4 and the Bounded Convergence Theorem show that

$$I'_-(1) = \int_{\mathbb{S}^{n-1}} G(h_L(v))h'_L(v)d\mathcal{H}^{n-1}(v).$$

Using polar coordinates, we obtain

$$I(t) = \int_{\mathbb{S}^{n-1}} H(h_L'(v))d\mathcal{H}^{n-1}(v) = \int_{\mathbb{S}^{n-1}} \int_0^1 r^{n-1} F(rh_L'(v))drd\mathcal{H}^{n-1}(v) = \int_{\mathbb{S}^{n-1}} \int_0^1 r^{n-1} F(h_L'(rv))drd\mathcal{H}^{n-1}(v) = \int_{B^n_2} F(h_L'(x))dx.$$ 

Thus, using the layer cake formula, the continuity and the strict monotonicity of $F$, we can write

$$I(t) = \int_0^\infty V(B^n_2 \cap \{F(h_L') \geq s\})ds = \int_0^{\sup F} V(B^n_2 \cap \{F(h_L') \geq s\})ds = \int_0^{\sup F} [V(B^n_2) - V(B^n_2 \cap \{F(h_L') < s\})]ds.$$
On a non-homogeneous version of a problem of Firey 1075

\[ = \int_0^{\sup F} [V(B_2^n) - V(B_2^n \cap \{h_{L'} < F^{-1}(s)\})] ds \]

\[ = \int_0^{\sup F} [V(B_2^n) - V(B_2^n \cap F^{-1}(s)(L^o))] ds \]

Therefore, from Proposition 4.5 and Fatou’s Lemma, we easily get

\[ I'_{-}(1) = \liminf_{t \to 1^-} \frac{I(t) - I(1)}{t - 1} \geq \int_0^{\sup F} \frac{d}{dt}[-V(B_2^n \cap F^{-1}(s)(L^o))]_{t=1^-} ds > 0, \]

where we additionally used the fact that since \( L \) is not symmetric with respect to the hyperplane \( e^\perp \), \( L^o \) also cannot be symmetric with respect to \( e^\perp \). \( \square \)

**Lemma 5.3** Let \( L \) be a convex body in \( \mathbb{R}^n \) that contains the origin in its interior.

(i) If \( b(L) = o \) or \( V(L \cap (e^\perp)^+) = V(L \cap (e^\perp)^-) \), then it holds

\[ \int_{S^{n-1}} h_L'(v) dS_L(v) \leq 0. \]

(ii) If \( L \) is centrally symmetric and not an ellipsoid, then there exists an orthogonal map \( O \), such that

\[ \int_{S^{n-1}} h_O'(L) dS_O(L) < 0. \]

**Proof** To prove (i), let \( t \in [-1, 1] \). Using Minkowski’s first inequality (8), we obtain

\[
\frac{V(L^t', L) - V(L)}{t - 1} \leq \frac{V(L^t')^{1/n} V(L)^{(n-1)/n} - V(L)}{t - 1} = V(L)^{(n-1)/n} \frac{V(L^t')^{1/n} - V(L)^{1/n}}{t - 1}.
\]

Since by Lemma 2.4, it holds \( V(L^t') \geq V(L) \), we immediately see that

\[ \liminf_{t \to 1^-} \frac{V(L^t', L) - V(L)}{t - 1} \leq 0. \]

On the other hand, using Fatou’s Lemma and Corollary 4.4, we get

\[
\liminf_{t \to 1^-} \frac{V(L^t', L) - V(L)}{t - 1} = \liminf_{t \to 1^-} \frac{1}{n} \int_{S^{n-1}} h_{L^t'}(v) - h_L(v) dS_L(v) \\
\geq \frac{1}{n} \int_{S^{n-1}} \liminf_{t \to 1^-} \frac{h_{L^t'}(v) - h_L(v)}{t - 1} dS_L(v) = \frac{1}{n} \int_{S^{n-1}} h_L'(v) dS_L(v)
\]

Thus, (i) holds.
Next assume that \( L \) and, therefore, \( L^\circ \) is centrally symmetric but not an ellipsoid. Using Lemma 2.3, we conclude that there exists an orthogonal map \( O \) such that

\[
\lim_{t \to 1^-} \frac{V(O^L)^{(n-1)/n} V((O^L)^{1/n} - V(O^L)^{1/n})}{t - 1} < 0.
\]

Using \( O^L \) in the place of \( L \) in (21) and (22), we conclude the validity of (ii). \( \square \)

The second part of Lemma 5.3 is going to be used in Sect. 7.

**Proof of Proposition 5.1** First, notice that Lemma 5.2 and Lemma 5.3 (i) imply the following: If \( b(K) = o \) or \( V(K \cap (e^\perp)^+) = V(K \cap (e^\perp)^-) \), then \( K \) is symmetric with respect to the hyperplane \( e^\perp \). In fact, if \( b(K) = o \), the previous statement shows that \( K \) is symmetric with respect to the hyperplane \( e^\perp \), for every \( e \in S^{n-1} \). This clearly shows that \( K \) is a Euclidean ball centered at the origin.

**Fact 1.** Let \( E \) be a subspace of \( \mathbb{R}^n \) of dimension at most \( n - 2 \). Then, there exists a hyperplane \( H \), containing \( E \), such that \( K \) is symmetric with respect to \( H \).

To see this, let \( W \) be any \((n-2)\)-dimensional subspace of \( \mathbb{R}^n \) that contains \( E \). Then, as in [27], a trivial continuity argument shows that there exists a hyperplane \( H \), containing \( W \), such that \( V(K \cap H^+) = V(K \cap H^-) \). Thus, according to our previous discussion, \( K \) has to be symmetric with respect to the hyperplane \( H \).

**Fact 2.** There exist \( e_1, \ldots, e_{n-1} \) mutually orthogonal unit vectors, such that \( K \) is symmetric with respect to the hyperplane \( e_i^\perp \), \( i = 1, \ldots, n - 1 \).

We will prove by induction in \( k \) that for \( 1 \leq k \leq n - 1 \), there exist \( e_1, \ldots, e_k \) mutually orthogonal unit vectors, such that \( K \) is symmetric with respect to the hyperplane \( e_i^\perp \), \( i = 1, \ldots, k \). The case \( k = 1 \) follows from Fact 1. Assume that we have found \( e_1, \ldots, e_{k-1} \) as above for some \( 2 \leq k \leq n - 1 \) and set \( H := e_1^\perp \cap \cdots \cap e_{k-1}^\perp \). The set \( \{e_1, \ldots, e_{k-1}\} \) extends to an orthonormal basis \( \{e_1, \ldots, e_{k-1}, e_k', \ldots, e_n'\} \) of \( \mathbb{R}^n \).

Thus, \( H = \text{span}\{e_k', \ldots, e_n'\} \) and \( H^\perp = \text{span}\{e_1, \ldots, e_{k-1}\} \). Since \( \dim H^\perp \leq n - 2 \), we know from Fact 1 that there exists a unit vector \( e_k' \), such that \( e_k' \supseteq H^\perp \) and \( K \) is symmetric with respect to \( e_k^\perp \). But then \( e_k \in H \), thus \( e_k \) is orthogonal to \( e_1, \ldots, e_{k-1} \) and Fact 2 is proved.

Fact 1 shows immediately that there exists a map \( G_{n,n-2} \ni W \mapsto e_W \in S^{n-1} \) (where \( G_{n,n-2} \) is the set of all \((n-2)\)-dimensional subspaces of \( \mathbb{R}^n \)), such that \( W \subseteq e_W^\perp \) and \( K \) is symmetric with respect to \( e_W^\perp \), for all \( W \in G_{n,n-2} \). Let, also, \( e_1, \ldots, e_{n-1} \) be the unit vectors constructed in Fact 2 and let \( e_n \in e_1^\perp \cap \cdots \cap e_{n-1}^\perp \cap S^{n-1} \).

**Case I.** The vector \( e_W \) is orthogonal to \( e_n \), for all \((n-2)\)-dimensional subspaces \( W \) of \( \mathbb{R}^n \).

In this case, let \( e_0 \) be any unit vector which is orthogonal to \( e_n \) and let \( W \) be any \((n-2)\)-dimensional subspace of \( e_0^\perp \) that does not contain \( e_n \). According to our assumption, \( e_n \in e_W^\perp \) and, therefore (since \( W \subseteq e_W^\perp \)), it holds \( e_W^\perp = e_0^\perp \). In other words, \( K \) is symmetric with respect to any hyperplane that contains the line \( \mathbb{R}e_n \). This shows that \( K \) is symmetric with respect to the line \( \mathbb{R}e_n \).

**Case II.** There exists an \((n-2)\)-dimensional subspace \( W \) of \( \mathbb{R}^n \), such that \( e_W \) is not orthogonal to \( e_n \).
Then, the vectors $e_1, \ldots, e_{n-1}, e_W$ are linearly independent and $K$ is symmetric with respect to the hyperplanes $e_1^\perp, \ldots, e_{n-1}^\perp, e_W^\perp$. The latter shows that

$$\int_K \langle x, e_1 \rangle dx = \ldots = \int_K \langle x, e_{n-1} \rangle dx = \int_K \langle x, e_W \rangle dx = 0.$$ 

Since $e_1, \ldots, e_{n-1}, e_W$ are linearly independent, it follows that

$$\int_K \langle x, y \rangle dx = 0, \quad \text{for all } y \in \mathbb{R}^n.$$

This shows that $b(K) = 0$ and, therefore, $K$ is a Euclidean ball centered at the origin. In particular, $K$ is symmetric with respect to some (any) straight line through the origin. \hfill \Box

The fact that any solution $K$ to (5) is axially symmetric implies a regularity property for the boundary of $K$ that will be used subsequently.

**Corollary 5.4** Let $K$ be a convex body in $\mathbb{R}^n$, that solves (5) for some $G \in \mathcal{A}(n)$. Then, $K$ is a regular strictly convex body. In particular, $h_K$ is of class $C^1$.

**Proof** We know that $S_K$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$. Moreover, if $\omega$ is a Borel subset of $S^{n-1}$, the identity

$$S_K(\omega) = \int_\omega dS_K = \int_\omega G(h_K(v))dS_K(v),$$

together with the fact that $G$ is positive, shows that

$$S_K(\omega) > 0, \quad \text{if } \mathcal{H}^{n-1}(\omega) > 0. \quad (23)$$

If $n = 2$, then a segment on the boundary of $K$ is a facet of $K$, a fact that contradicts the absolute continuity of $S_K$. If $n \geq 3$, since $K$ is a body of revolution, the strict convexity of $K$ is proved in [32, Lemma 8.7]. It remains to prove that $K$ is regular. Let $T$ be a 2-dimensional profile of the body of revolution $K$ (set $T = K$ if $n = 2$) and let $E$ be the 2-dimensional subspace of $\mathbb{R}^n$ spanned by $T$. If $K$ is not regular, then $T$ is also not regular, hence there exists a point $x \in \text{bd } T$ and a non-empty open set $N$ in $S^{n-1} \cap E$, such that $v$ supports $T$ at $x$, for all $v \in N$. If $n = 2$, this already shows that $S_K(N) = 0$, which contradicts (23). If $n \geq 3$, denote by $\overline{N}$ and $X$ the subsets of $S^{n-1}$ and $\text{bd } K$ respectively, obtained by rotating $N$ and $x$ about the axis of symmetry of $K$, respectively. Then, $\overline{N}$ is non-empty and open in $S^{n-1}$ and each vector $v \in \overline{N}$ supports $K$ at some point of $X$. This shows that $S_K(\overline{N}) = \mathcal{H}^{n-1}(X) = 0$, which is again a contradiction. \hfill \Box

It should be remarked that since $G$ is strictly positive, Corollary 5.4 can also be deduced from a general regularity theorem due to Caffarelli [9] for Monge–Ampère equations (see also [15]), without making use of Proposition 5.1. We choose to avoid using such a deep result.
6 Gluing lemmas

The main goal of this section is to establish a preparatory step (Lemma 6.2 below) towards the proof of the remaining part of Theorem 1.1 that will allow us to deduce that if $K$ is a non-spherical convex body that solves (5), then there exists a non-spherical centrally symmetric convex body $\overline{K}$ that approximately solves (5). This will be done by gluing together part of the boundary of $K$, part of the boundary of $-K$ and appropriate spherical parts. Since the barycentre of any centrally symmetric convex body is the origin, it will then not be hard to finish the proof of Theorem 1.1. Let us first fix some notation that will be used until the end of this note.

Let $e_1 = (1, 0, \ldots, 0)$ and $e_2 = (0, 1, 0, \ldots, 0)$ be the first two vectors of the standard orthonormal basis in $\mathbb{R}^n$ and $K$ be a convex body which is symmetric with respect to the $x_2$-axis (i.e. the line $\mathbb{R}e_2$). We denote by $E$, the subspace spanned by $e_1$ and $e_2$, which we identify with $\mathbb{R}^2$. We set $Q(E) \subseteq \mathbb{R}^2$ to be the planar convex body $K \cap E$.

If $T$ is a strictly convex body in $\mathbb{R}^2$ and $v \in S^1$, $p_T(v)$ will stand for the (unique) point in $\text{bd} \ T$ such that the supporting line of $T$, whose outer unit normal vector is $v$, touches $T$ at $p_T(v)$. Notice that if $T$ is also assumed to be regular, then $p_T = \eta_T^{-1}$ and, therefore, $p_T$ is a continuous map.

We will need the following observation.

**Lemma 6.1** Let $T_1, T_2$ be regular strictly convex bodies in $\mathbb{R}^2$ and let $p, q$ be two distinct intersection points of $\text{bd} \ T_1$ and $\text{bd} \ T_2$. Let $l$ be the straight line through $p$ and $q$ and let $l^+$, $l^−$ be the two closed half-planes defined by $l$. Assume, furthermore, that the tangent lines of $T_1$ and $T_2$ at $p$ and $q$ respectively, coincide. Then, the set $T := (T_1 \cap l^+) \cup (T_2 \cap l^−)$ is also a regular strictly convex body.

**Proof** Clearly, for every point $x \in \text{bd} \ T$, the tangent line $l^T_x$ of $\text{bd} \ T = ((\text{bd} \ T_1) \cap l^+) \cup (\text{bd} \ T_2 \cap l^−)$ exists (this is trivially true if $x \neq p, q$, while if $x = p$ (and similarly if $x = q$) it follows from the fact that the tangent lines of $T_1$ and $T_2$ at $p$ coincide). Thus, it suffices to show that for each point $x \in \text{bd} \ T$, it holds $l^T_x \cap \text{int} \ T = \emptyset$.

Let $x \in \text{bd} \ T$. We may assume that $x \in (\text{bd} \ T_1) \cap l^+$. Let $l_x = l^T_x$ be the tangent line of $T_1$ at $x$. Denote by $l_x^−$ the closed half-plane defined by $l_x$, that contains $T_1$ and set $l'_x := \mathbb{R}^2 \setminus l_x^−$ (the interior of the other closed half-plane). If $x = p$ or $x = q$, our assumption implies immediately that $l_x \cap \text{int} \ T = \emptyset$. Therefore, we may assume that $x \neq p, q$. Since it is trivially true that $l_x \cap \text{int} \ T_1 = \emptyset$, it suffices to show that $l_x \cap T_2 \cap l^− = \emptyset$.

Set $A$ to be the closed convex set $l^+ \cap l^p \cap l^q$. The line segment $[p, q]$ is obviously one of the three sides (possibly unbounded) of $A$. Since $l_x$ is disjoint from $[p, q]$, it follows that $A \setminus l_x$ is contained in $l^p \cap l^q$. However, $T_2 \cap l^−$ is contained in $l^p \cap l^− \subseteq (\mathbb{R}^2 \setminus A) \cup [p, q]$, thus $l_x \cap T_2 \cap l^−$ is empty, as claimed. \(\Box\)

Now we are ready to state and prove the main technical fact of this section.

**Lemma 6.2** Let $K$ be a regular strictly convex body in $\mathbb{R}^n$ with absolutely continuous surface area measure with respect to $\mathcal{H}^{n−1}$, which is symmetric with respect to the $x_2$-axis and let $B_1 := r_1 B^n_2$, $B_2 := r_2 B^n_2$, for some $0 < r_1 < r_2$. Assume, furthermore,
that for some open set \( U \subseteq \mathbb{S}^{n-1} \), \( K \) satisfies

\[
f_K(v) = G(h_K(v)), \quad \text{for almost every } v \in \mathbb{S}^{n-1} \setminus U, \tag{24}
\]

for some \( G \in \mathcal{A}(n) \). If \( T := Q(K) \), then the following are true.

(i) If \( p_T(1, 0) \) lies on the \( x_1 \)-axis, then there exists a centrally symmetric strictly convex body \( \overline{K} \) in \( \mathbb{R}^n \), with absolutely continuous surface area measure with respect to \( \mathcal{H}^{n-1} \), such that \( K \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} = \overline{K} \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} \), satisfying

\[
f_{\overline{K}}(v) = G(h_{\overline{K}}(v)), \quad \text{for almost every } v \in \mathbb{S}^{n-1} \setminus (U \cup U'), \tag{25}
\]

where \( U' \subseteq \mathbb{S}^{n-1} \) is the reflection of \( U \) with respect to the hyperplane \( e_2^\perp \).

(ii) Assume that \( U = \emptyset \) and that there exist two distinct vectors \( v_1, v_2 \in \mathbb{S}^1 \cap \mathbb{R}_{+}^2, \) such that \( p_T(v_i) = p_{Q(v_i)}(v_i), i = 1, 2 \). Assume, furthermore, that

(a) \( r_1 < h_T(v) < r_2 \), for all \( v \in \mathbb{S}^1 \) with \( \min(\langle v_1, e_2 \rangle, \langle v_2, e_2 \rangle) \leq \langle v, e_2 \rangle \leq \max(\langle v_1, e_2 \rangle, \langle v_2, e_2 \rangle) \).

(b) \( r_1^{-1} = f_{B_1}(v_1) \leq G(r_1) \) and \( r_2^{-1} = f_{B_2}(v_2) \geq G(r_2) \).

Then, there exists a family of functions \( \{ G_\varepsilon \}_{\varepsilon \in (0, \delta_0)} \subseteq \mathcal{A}(n) \), which is uniformly bounded on \( (0, \delta_0) \times [r_1, r_2] \) and a centrally symmetric strictly convex body \( \overline{K} \) in \( \mathbb{R}^n \) with absolutely continuous surface area measure with respect to \( \mathcal{H}^{n-1} \), which is not an ellipsoid, satisfying

\[
f_{\overline{K}}(v) = G_\varepsilon(h_{\overline{K}}(v)), \quad \text{for almost every } v \in \mathbb{S}^{n-1} \setminus U_\varepsilon, \tag{26}
\]

for all \( \varepsilon \in (0, \delta_0) \), where \( \{ U_\varepsilon \} \) is a family of open sets in \( \mathbb{S}^{n-1} \) such that \( \mathcal{H}^{n-1}(U_\varepsilon) \xrightarrow{\varepsilon \to 0^+} 0 \).

**Proof** (i) Since \( p_T(1, 0) \) lies on the \( x_1 \)-axis and since \( T \) is symmetric with respect to the \( x_2 \)-axis, it holds \( p_T(1, 0) = p_{-T}(0, 1) \) and \( p_T(-1, 0) = p_{-T}(-1, 0) \). Thus, by Lemma 6.1, the set \( \overline{T} : = (T \cap \{ x \in \mathbb{R}^2 : x_2 \geq 0 \}) \cup (-T \cap \{ x \in \mathbb{R}^2 : x_2 \leq 0 \}) \) is a (centrally symmetric) strictly convex body. Let \( \overline{K} \) be the centrally symmetric strictly convex body in \( \mathbb{R}^n \) obtained by rotating \( \overline{T} \) about the \( x_2 \)-axis (this is if \( n \geq 3 \), otherwise we set \( \overline{K} := \overline{T} \)). Clearly, \( S_K \) is absolutely continuous with respect to \( \mathcal{H}^{n-1} \). Then, \( K \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} = \overline{K} \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} \). Moreover, it holds

\[
h_{\overline{K}}(v) = \begin{cases} h_K(v), & \text{if } \langle v, e_2 \rangle \geq 0 \\ h_K(-v), & \text{if } \langle v, e_2 \rangle \leq 0 \end{cases},
\]

and by (3), it follows that

\[
f_{\overline{K}}(v) = \begin{cases} f_K(v), & \langle v, e_2 \rangle \geq 0 \\ f_K(-v), & \langle v, e_2 \rangle < 0 \end{cases}, \quad \text{for almost every } v.
\]
The formulae written above for $h_{K'}$ and $f_{K'}$, together with (24), immediately give (25).

(ii) Set $v_1' := v_1$, $v_2' := v_2$, $B_1' := B_1$, $B_2' := B_2$, if $(v_1, e_2) < (v_2, e_2)$ and $v_1' := v_2$, $v_2' := v_1$, $B_1' := B_2$, $B_2' := B_1$, otherwise. Let, also, $D_i := Q(B_i')$, $i = 1, 2$. Moreover, set $v_i''$ to be the reflection of $v_i'$ with respect to the $x_2$-axis, $i = 1, 2$.

Finally, let $l_i = \{x \in \mathbb{R}^2 : x_2 = t_i\}$ to be the line through the points $p_T(v_i')$ and $p_T(v_i'')$, for some $t_i \geq 0$ and let $l_i^+ := \{x \in \mathbb{R}^2 : x_2 \geq t_i\}, l_i^- := \{x \in \mathbb{R}^2 : x_2 \leq t_i\}, i = 1, 2$. Applying Lemma 6.1 twice, we conclude that the set

$$T' := (D_2 \cap l_2^+) \cup (T \cap l_2^- \cap l_1^+) \cup (D_1 \cap l_1^-)$$

is a regular strictly convex body in $\mathbb{R}^2$ that contains the origin in its interior. If $K'$ is the (regular strictly convex with absolutely continuous surface area measure with respect to $\mathcal{H}^{n-1}$) body of revolution in $\mathbb{R}^n$ (if $n = 2$, set again $K' := T'$), obtained by rotating $T'$ about the $x_2$-axis, then as in the proof of part (i), we find

$$h_{K'}(v) = \begin{cases} 
  h_{B_1'}(v), & \text{if } \langle v, e_2 \rangle \leq \langle v_1', e_2 \rangle \\
  h_{K}(v), & \text{if } \langle v_1', e_2 \rangle \leq \langle v, e_2 \rangle \leq \langle v_2', e_2 \rangle \\
  h_{B_2'}(v), & \text{if } \langle v, e_2 \rangle \geq \langle v_2', e_2 \rangle 
\end{cases}$$

and

$$f_{K'}(v) = \begin{cases} 
  f_{B_1'}(v), & \langle v, e_2 \rangle \leq \langle v_1', e_2 \rangle \\
  f_{K}(v), & \langle v_1', e_2 \rangle < \langle v, e_2 \rangle \leq \langle v_2', e_2 \rangle, \\
  f_{B_2'}(v), & \langle v, e_2 \rangle > \langle v_2', e_2 \rangle 
\end{cases}$$

for almost every $v$.

Notice that the formulae above are still valid even if $v_2' = e_2$.

Because of assumption (b), by Lemma 3.3, there exists a uniformly bounded (in $(0, \varepsilon_0) \times [r_1, r_2]$) family of functions $\{G_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0)} \subseteq A(n)$, such that $G_\varepsilon(r_i) = r_i^{n-1} - 1$, $i = 1, 2$ and for all $\varepsilon \in (0, \varepsilon_0)$, it holds $G_\varepsilon(\theta) = G(\theta)$, for all $\theta \in [r_1 + \varepsilon, r_2 - \varepsilon]$. Using assumption (a), (24) and the representation for $h_{K'}$ and $f_{K'}$ derived above, we immediately conclude that

$$f_{K'}(v) = G_\varepsilon(h_{K'}(v)), \quad \text{for almost all } v \in \mathbb{S}^{n-1} \setminus V_\varepsilon,$$

where $V_\varepsilon := \{v \in \mathbb{S}^{n-1} : r_1 < h_{K'}(v) < r_1 + \varepsilon \} \cup \{v \in \mathbb{S}^{n-1} : r_2 - \varepsilon < h_{K'}(v) < r_2\}$. Notice that $V_\varepsilon \searrow \emptyset$ and, therefore, $\mathcal{H}^{n-1}(V_\varepsilon) \xrightarrow{\varepsilon \to 0^+} 0$. Since $p_{Q(K')}((1, 0)) = p_{D_1}(1, 0)$ lies on the $x_1$-axis, the existence of a non-spherical (this follows from assumption (a)) centrally symmetric strictly convex body $\overline{K}$, with absolutely continuous surface area measure with respect to $\mathcal{H}^{n-1}$, satisfying (26) follows from part (i). Additionally, $K'$ and hence $\overline{K}$ contains a spherical part on its boundary (but is not spherical itself) and, consequently, $\overline{K}$ cannot be an ellipsoid. \(\square\)
7 Further restriction of the class of candidates and proof of Theorem 1.1

The proof of the remaining part of Theorem 1.1 will follow from the next proposition (which actually provides some extra information about solutions of (5), in the case that $G$ is not necessarily strictly monotone). Before we state it, it will be convenient to introduce a standard reparametrization of $h_T$, where $T$ is a planar convex body. More specifically, we set

$$
\overline{h}_T(\varphi) := h_T(\cos \varphi, \sin \varphi), \quad \varphi \in \mathbb{R}.
$$

**Proposition 7.1** Let $K$ be a convex body in $\mathbb{R}^n$ that contains the origin in its interior, solves (5) for some function $G \in \mathcal{A}(n)$ and is symmetric with respect to the $x_2$-axis. Then, the following hold.

(i) $\overline{h}_{\partial(K)}$ is monotone on $[-\pi/2, \pi/2]$.

(ii) If $K$ is not a Euclidean ball centered at the origin, then $\overline{h}_{\partial(K)}$ is strictly monotone in a neighbourhood of $\varphi = 0$.

**Proof of Theorem 1.1** It remains to assume that $G$ is monotone. It clearly suffices to show that $G(h_K)$ is constant on $\mathbb{S}^{n-1}$. By Proposition 5.1, we may assume that any solution $K$ to (5) is symmetric with respect to the $x_2$-axis. Then, by Proposition 7.1, $G(h_K)$ is monotone. It follows immediately that for $u = e_2$ or $u = -e_2$, it holds $G(h_K(v)) \leq G(h_K(v'))$, for any two vectors $v, v' \in \mathbb{S}^{n-1}$ with $\langle v, u \rangle \leq 0 \leq \langle v', u \rangle$. This shows in particular that

$$
\int_{\mathbb{S}^{n-1} \cap \{x_2 \leq 0\}} -\langle v, u \rangle G(h_K(v)) d\mathcal{H}^{n-1}(v) \leq \int_{\mathbb{S}^{n-1} \cap \{x_2 \geq 0\}} \langle v', u \rangle G(h_K(v)) d\mathcal{H}^{n-1}(v'),
$$

with equality if and only if $G(h_K)$ is constant on $\mathbb{S}^{n-1}$. On the other hand, since the barycentre of $\partial K$ is always at the origin, equality must hold in the previous inequality and Theorem 1.1 is proved. \hfill $\Box$

Before we prove Proposition 7.1, we will need some additional considerations.

**Proposition 7.2** Let $L, M$ be regular strictly convex bodies in $\mathbb{R}^n$, such that $S_L = f_L d\mathcal{H}^{n-1}$ and $S_M = f_M d\mathcal{H}^{n-1}$, for some continuous functions $f_L, f_M : \mathbb{S}^{n-1} \to (0, \infty)$. Assume, furthermore, that there is a point $p \in \text{bd} L \cap \text{bd} M$ and a neighbour- hood $V$ of $p$ in $L$ and $M$, such that $V$ is contained in $M$. Let $v$ be the unit normal vector of $L$ (and therefore of $M$) at $p$. If at least one of $L$ and $M$ is of class $C_2^+$, then it holds $f_L(v) \leq f_M(v)$.

Proposition 7.2 is trivial if both convex bodies $L$ and $M$ are of class $C_2^+$. However, if we assume $C_2^+$ regularity for both $L$ and $M$, then as far as we can tell, Proposition 7.1 (and therefore the final part of Theorem 1.1) can only be proved under the additional assumption that the boundary of $K$ is $C^2$. The interested reader can find a full proof of Proposition 7.2 in the Appendix of this note.
Proof of Proposition 7.2 in the case that \( L \) and \( M \) are of class \( C^2_+ \). Since both \( L \) and \( M \) are both of class \( C^2_+ \), their boundaries are of class \( C^2 \) and it is well known that the assumption of Proposition 7.2 implies that the curvature of boundary \( \partial L \) at \( p \) dominates the curvature of boundary \( \partial M \) at \( p \). Thus (since \( \eta_L(p) = \eta_M(p) = v \)), (4) yields that \( f_L(v) \leq f_M(v) \).

\[ \Box \]

Lemma 7.3 Let \( K \) be a strictly convex body, which is symmetric with respect to the \( x_2 \)-axis and contains the origin in its interior. The following are true.

(i) If for some \( \varphi_0 \in \mathbb{R} \) the function \( \overline{h}_{Q(K)} \) attains a local extremum at \( \varphi_0 \), then it holds

\[ p_{Q(K)}(\cos \varphi_0, \sin \varphi_0) = \overline{h}_{Q(K)}(\varphi_0)(\cos \varphi_0, \sin \varphi_0), \tag{27} \]

while if (27) does not hold, then \( \overline{h}_{Q(K)} \) is strictly monotone in a neighbourhood of \( \varphi_0 \).

(ii) If \( K \) solves (5) for some function \( G \in \mathcal{A}(n) \) and if \( \varphi_0 \) is a point of local minimum (resp. local maximum) for \( \overline{h}_{Q(K)} \), then we have \( G(\overline{h}_{Q(K)}(\varphi_0)) \geq \overline{h}_{Q(K)}(\varphi_0)^{n-1} \) (resp. \( G(\overline{h}_{Q(K)}(\varphi_0)) \leq \overline{h}_{Q(K)}(\varphi_0)^{n-1} \)).

\[ \]

**Proof** Notice that (27) holds if and only if the position vector of \( p_{Q(K)}(\cos \varphi_0, \sin \varphi_0) \) is orthogonal to the vector \((-\sin \varphi_0, \cos \varphi_0)\). On the other hand, (7) and the fact that \( p_{Q(K)} = \eta_K^{-1} \) give

\[ \begin{align*}
\overline{h}'_{Q(K)}(\varphi_0) &= \langle \nabla h_{Q(K)}(\cos \varphi_0, \sin \varphi_0), (-\sin \varphi_0, \cos \varphi_0) \rangle \\
&= \langle p_{Q(K)}(\cos \varphi_0, \sin \varphi_0), (-\sin \varphi_0, \cos \varphi_0) \rangle.
\end{align*} \]

This proves (i).

To prove the second part, recall that \( K \) is regular (as Corollary 5.4 shows) and set \( D := \overline{h}_{Q(K)}(\varphi_0)B_2^n \subseteq \mathbb{R}^2 \). If \( \overline{h}_{Q(K)} \) attains a local minimum at \( \varphi_0 \), then in a neighbourhood of \( \varphi_0 \) it holds \( h_{Q(K)}(\varphi) \geq h_D(\varphi) \). However, (27) shows that \( p_{Q(K)}(\cos \varphi_0, \sin \varphi_0) = p_D(\cos \varphi_0, \sin \varphi_0) \). It follows that a neighbourhood of \( p_D(\cos \varphi_0, \sin \varphi_0) \) in the boundary of \( D \) is contained in \( Q(K) \). Thus, we immediately see that if \( B := \overline{h}_{Q(K)}(\varphi_0)B_2^n \), then the boundary of \( B \) touches the boundary of \( K \) at the point \( p := (p_D(\cos \varphi_0, \sin \varphi_0), 0, \ldots, 0) \in \mathbb{R}^n \) and some neighbourhood of \( p \) in the boundary of \( B \) is contained in \( K \). Since the outer unit normal vector for both \( K \) and \( B \) at \( p \) equals \( v := (\cos \varphi_0, \sin \varphi_0, 0, \ldots, 0) \), Proposition 7.2 shows that

\[ f_K(v) \geq f_B(v) = \overline{h}_{Q(K)}(\varphi_0)^{n-1} \]

or equivalently,

\[ G(\overline{h}_K(\varphi_0)) = G(h_K(v)) \geq \overline{h}_{Q(K)}(\varphi_0)^{n-1}. \]

The remaining case (i.e. when \( \overline{h}_{Q(K)} \) attains a local maximum at \( \varphi_0 \)) can be treated similarly. \( \Box \)
**Proof of Proposition 7.1** Set \( T := Q(K) \). Assume that (ii) is not true. Then, according to Lemma 7.3, \( pt(1, 0) \) must be contained in the \( x_1 \)-axis. Then, by Lemma 6.2 (i), there exists a centrally symmetric convex body \( \overline{K} \) with \( \overline{K} \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} = K \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} \), satisfying (5). Since \( b(\overline{K}) = 0 \), Proposition 5.1 shows that \( \overline{K} = \lambda_1 B_2^n \), for some \( \lambda_1 \geq 0 \). Thus, \( K \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} = \lambda_1 B_2^n \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} \). Putting \( -K \) in the place of \( K \), we conclude that \( -K \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} = \lambda_2 B_2^n \cap \{ x \in \mathbb{R}^n : x_2 \geq 0 \} \), for some \( \lambda_2 \geq 0 \). This shows that \( K = -K = \lambda_1 B_2^n = \lambda_2 B_2^n \) and (ii) is proved.

To prove (i), it suffices to show that \( \overline{h}_T \) is monotone on \([0, \pi/2] \). Indeed, if this is true, one can replace \( K \) by \( -K \) to show that \( \overline{h}_T \) is also monotone on \([-\pi/2, 0] \).

Taking (ii) into account, this would imply that \( \overline{h}_T \) is monotone on \([-\pi/2, \pi/2] \).

Let \( \varphi_0 \) be such that the interval \([0, \varphi_0] \) is the largest interval of the form \([0, \varphi] \subseteq [0, \pi/2] \), on which \( \overline{h}_T \) is monotone. We need to show that \( \varphi_0 = \pi/2 \). Assume that this is not true. Then, \( \overline{h}_T \) is not constant on \([\varphi_0, \pi/2] \), so we can find a minimal (with respect to inclusion) interval \([\varphi_1, \varphi_2] \subseteq [\varphi_0, \pi/2] \), such that the extrema \( \min_{\varphi \in [\varphi_0, \pi/2]} \overline{h}_T (\varphi) \), \( \max_{\varphi \in [\varphi_0, \pi/2]} \overline{h}_T (\varphi) \) are attained at the points \( \varphi_1, \varphi_2 \). In particular, all values of \( \overline{h}_T \) on \([\varphi_1, \varphi_2] \) lie strictly between \( \overline{h}_T (\varphi_1) \) and \( \overline{h}_T (\varphi_2) \). Moreover, \( \varphi_1 \) and \( \varphi_2 \) are points of local extremum for \( \overline{h}_T \) (notice that since \( \overline{h}_T \) is symmetric about the point \( \varphi = \pi/2 \), this is also true even if \( \varphi_2 = \pi/2 \)). It follows from Lemma 7.3 that if \( r_i = \overline{h}_T (\varphi_i) \), then \( pt(\cos \varphi_1, \sin \varphi_1) = \overline{h}_T (\varphi_2) \), \( i = 1, 2 \). Notice, furthermore, that if \( \overline{h}_T \) attains a local minimum at \( \varphi_1 \), then \( \overline{h}_T \) attains a local maximum at \( \varphi_2 \) and vice versa. Using again Lemma 7.3, we see that in the first case we have \( r_1^{n-1} \leq G(r_1) \) and \( r_2^{n-1} \geq G(r_2) \), while in the second case the inequalities are reversed. In any case, the assumptions of Lemma 6.2 (ii) are satisfied with \( c_1 = \min \{ r_1, r_2 \} \), \( c_2 = \max \{ r_1, r_2 \} \) and \( a_i = c_i^{n-1} \), \( i = 1, 2 \). Let \( \overline{K}, \{ G_\varepsilon \}_{\varepsilon \in (0, \varepsilon_0)}, \{ U_\varepsilon \} \) be as in the statement of Lemma 6.2. Then, it holds

\[
\int_{\mathbb{R}^{n-1}} h'_{K_1}(v) dS_{K_1} < 0.
\]

Since \( \overline{K} \) is centrally symmetric but not an ellipsoid, by Lemma 5.3 (ii), there exists an orthogonal map \( O \), such that if we set \( K_1 := O \overline{K} \), it holds

\[
\int_{\mathbb{R}^{n-1}} h'_{K_1}(v) dS_{K_1} < 0.
\]

On the other hand, (28) and Lemma 5.2 show that

\[
\int_{\mathbb{R}^{n-1}} h'_{K_1}(v) dS_{K_1}(v)
\]

\[
= \int_{\mathbb{R}^{n-1}} G_\varepsilon(h_{K_1}(v))h'_{K_1}(v) dH^{n-1}(v) + \int_{U_\varepsilon} [f_{K_1}(v) - G_\varepsilon(h_{K_1}(v))] h'_{K_1}(v) dH^{n-1}(v)
\]

\[
\geq \int_{U_\varepsilon} [f_{K_1}(v) - G_\varepsilon(h_{K_1}(v))] h'_{K_1}(v) dH^{n-1}(v) \overset{\varepsilon \to 0^+}{\longrightarrow} 0,
\]

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where we used the fact that $h'_{K_1}$ is bounded (see Corollary 4.4) and, therefore, the last integral indeed exists and converges to zero. We arrived at a contradiction because we assumed that $\varphi_0 < \pi/2$. Hence, $\overline{h}_T$ is monotone on $[0, \pi/2]$, as claimed. 

\section*{8 Proof of Theorem 1.2}

We may assume that in the statement of Theorem 1.2, $a = \lambda e_2$ for some $\lambda > 0$. Fix a number $r > 2\lambda$. Then, it is clear that the ball $rB^n_2 + \lambda e_2$ contains the origin in its interior and that the function $\overline{h}_Q(rB^n_2 + \lambda e_2)(\varphi) = \overline{h}_{rB^n_2 + \lambda e_2}(\varphi) = r + \lambda \sin \varphi$, $\varphi \in \mathbb{R}$, is strictly increasing on $[-\pi/2, \pi/2]$. Let $g : [-1, 1] \to [0, \infty)$ be an even $C^\infty$ function supported on $[-1/2, 1/2]$. Since the ball $rB^n_2$ is uniformly convex, it follows that if $m \in \mathbb{N}$ is large enough, then the function $\mathbb{S}^{n-1} \ni v \mapsto r + (1/m)g(\langle v, e_2 \rangle)$ is the support function of a centrally symmetric strictly convex body $K_m$ in $\mathbb{R}^n$, which is symmetric with respect to the $x_2$-axis and has $C^\infty$ boundary. Notice that $\{h_{K_m}\}$ converges uniformly to $h_{rB^n_2} \equiv r$, while all derivatives (on the sphere) of $\{h_{K_m}\}$ converge uniformly to 0. In particular, if we define $\overline{f}_M : [-\pi/2, \pi/2] \to \mathbb{R}$ by

$$
\overline{f}_M(\varphi) := f_M(\cos \varphi, \sin \varphi, 0, \ldots, 0),
$$

where $M = K_m$ or $rB^n_2$, we conclude that

$$
\overline{f}_{K_m} \to \overline{f}_{rB^n_2} \equiv r^{n-1} \quad \text{and} \quad \overline{f}'_{K_m} \to \overline{f}'_{rB^n_2} \equiv 0, \quad (29)
$$

uniformly of $[-\pi/2, \pi/2]$. Observe, also, that

$$
\overline{f}_{K_m}(\varphi) = r, \quad \text{for all } \varphi \in [-\pi/2, -\pi/6] \cup (\pi/6, \pi/2]. \quad (30)
$$

Next, consider the function $\overline{h}_Q(K_m + \lambda e_2)(\varphi) = r + (1/m)g(\sin \varphi) + \lambda \sin \varphi$, $\varphi \in \mathbb{R}$. We have,

$$
\overline{h}'_Q(K_m + \lambda e_2)(\varphi) = \cos \varphi \left( \lambda + \frac{1}{m}g'(\sin \varphi) \right), \quad \varphi \in \mathbb{R} \quad (31)
$$

and, therefore, $\overline{h}_Q(K_m + \lambda e_2)$ is strictly increasing on $[-\pi/2, \pi/2]$, provided that $m$ is large enough. For large $m$, consider $\zeta, \zeta_m : [r - \lambda, r + \lambda] \to [-\pi/2, \pi/2]$ to be the inverses of $\overline{h}_Q(B^n_2 + \lambda e_2)\mid[-\pi/2,\pi/2]$ and $\overline{h}_Q(K_m + \lambda e_2)\mid[-\pi/2,\pi/2]$, respectively and set

$$
G_m(\theta) := \overline{f}_{K_m}(\zeta_m(\theta)), \quad \theta \in [-\pi/2, \pi/2].
$$

One can prove easily that $\{\zeta_m\}$ converges uniformly to $\zeta$ on $[r - \lambda, r + \lambda]$ and, hence, $\{G_m\}$ converges uniformly to the constant $r^{n-1}$ on $[r - \lambda, r + \lambda]$. Furthermore, as (31) shows, $\zeta_m'$ is bounded on $[r - \lambda/2, r + \lambda/2]$ (if $m$ is large), so from (29) and (30), we immediately see that $\{G_m'\}$ converges uniformly to 0 on $[r - \lambda, r + \lambda]$. It follows that the sequence $\{\theta G_m'(\theta) + (n + 1)G_m(\theta)\}$ converges uniformly to $(n + 1)r^{n-1}$.

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on \([r - \lambda, r + \lambda]\). Consequently, there exists a positive integer \(m_0\), such that for any \(m \geq m_0\), \(G_m\) is (or more precisely, can be extended to; see Remark 3.2) a member of class \(A(n)\). However, from the definition of \(G_m\), we clearly see that for \(m \geq m_0\), it holds

\[
f_{K_m + \lambda e_2}(v) = f_{K_m}(v) = G_m(h_{K_m + \lambda e_2}(v)), \quad \text{for all } v \in S^{n-1}
\]
and the proof of Theorem 1.2 is complete. \(\Box\)

**Remark 8.1** It is clear that in the construction above, for any positive integer \(k\), \(G\) can be chosen to be of class \(C^k\). Currently, we do not have an example of a non-spherical convex body \(K\) that satisfies (5) for some \(C^\infty\) function \(G \in A(n)\).

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

**A Appendix**

This Appendix is devoted to the proof of Proposition 7.2 in full generality. We will need the following fact, which we believe should be well known. Since we were unable to find an explicit reference, we provide a proof here.

**Lemma A.1** Let \(A > 0\). There exists a constant \(C = C(A, n) > 0\), such that if \(L\) is a convex body in \(\mathbb{R}^n\), with absolutely continuous surface area measure with respect to \(H^{n-1}\) and

\[
\frac{1}{A} \leq f_L(v) \leq A, \quad \text{for almost every } v \in S^{n-1}, \quad (32)
\]

then there exists \(a \in \mathbb{R}^n\), such that

\[
\frac{1}{C} B_2^n + a \subseteq L \subseteq C B_2^n + a.
\]

**Proof** Any constant \(C_1, C_2, \text{ etc.}\) that will appear in this proof will denote a positive constant that depends only on \(A\) and the dimension \(n\). For an \((n - 1)\)-dimensional subspace \(H\) of \(\mathbb{R}^n\), we write \(V_H(\cdot)\) for the volume functional in \(H\).
Recall the well known fact that if $L_1, L_2$ are convex bodies with $S_{L_1} \leq S_{L_2}$, then it holds $V(L_1) \leq V(L_2)$. Using this and (32), we obtain

$$\frac{1}{C_1} \leq V(L) \leq C_1, \tag{33}$$

for some $C_1 > 0$. Moreover, the well known formula

$$V_{e^\perp}(L|e^\perp) = \frac{1}{2} \int_{S^{n-1}} |\langle x, e \rangle| dS_L(x), \quad e \in S^{n-1}$$

immediately shows that

$$\frac{1}{C_2} \leq V_{e^\perp}(L|e^\perp) \leq C_2, \tag{34}$$

for all $e \in S^{n-1}$, where $C_2$ is another positive constant. After a suitable translation, we may assume that the maximal volume ellipsoid $E = tS B_n^2$ contained in $L$ is centered at the origin. Here, $S$ denotes a symmetric positive definite matrix of determinant $1$ and $t = (V(E)/V(B_n^2))^{1/n}$. Then, (33) together with the classical theorem of F. John [25], yields

$$\frac{1}{C_3} B_n^2 \subseteq S^{-1} L \subseteq C_3 B_n^2,$$

for some constant $C_3 > 0$. Equivalently, we may write

$$\frac{1}{C_3} S B_n^2 \subseteq L \subseteq C_3 S B_n^2. \tag{35}$$

Let $\lambda, \mu$ be the smallest and the largest eigenvalue of $S$ and $e_\lambda, e_\mu$ be the corresponding eigenvectors respectively. Then, (34) and (35) give

$$\frac{1}{C_3^{n-1}} \lambda V_{e_\lambda^\perp}(B_n^2|e_\lambda^\perp) = V_{e_\lambda^\perp}(1/C_3 S B_n^2|e_\lambda^\perp) \leq V_{e_\lambda^\perp}(L|e_\lambda^\perp) \leq C_2$$

and

$$C_3^{n-1} \mu V_{e_\mu^\perp}(B_n^2|e_\mu^\perp) = V_{e_\mu^\perp}(C_3 S B_n^2|e_\mu^\perp) \geq V_{e_\mu^\perp}(L|e_\mu^\perp) \geq \frac{1}{C_2}.$$

Consequently, if $C_4 := C_2 C_3^{n-1}$, then $1/C_4 \leq \lambda \leq \mu \leq C_4$ and hence, using again (35), we conclude

$$\frac{1}{C_3 C_4} B_n^2 \subseteq L \subseteq C_3 C_4 B_n^2.$$

This completes our proof. \qed

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Proof of Proposition 7.2 We may clearly assume that \( o \in \text{int } L \cap \text{int } M \). First, let us prove Proposition 7.2 without any regularity assumption on the boundaries of \( L \) and \( M \), but under the additional assumption that \( V \setminus \{ p \} \subseteq \text{int } M \). This, together with the fact that \( p \in \text{int } L \cap \text{int } M \) is clearly equivalent to:

(i) \( \rho_L(v_0) = \rho_M(v_0) \), where \( v_0 = p/|p| \)

(ii) \( \rho_L(v) < \rho_M(v) \), for all \( v \in U := \{ x/|x| : x \in V \} \).

Since \( f_L \) and \( f_M \) are continuous, there are sequences of strictly positive \( C^\infty \) functions \( \{ f_m \} \) and \( \{ f_m \} \), such that \( f_m \to f_L \) and \( f_m \to f_M \), uniformly on \( S^{n-1} \), while \( \int f_m dH^{n-1} = \int f_m dH^{n-1} = \rho \), for each \( m \in \mathbb{N} \). By Minkowski’s existence and Uniqueness Theorem, for \( m \in \mathbb{N} \), there exist uniquely determined up to translation convex bodies \( L_m \) and \( M_m \), such that \( S_{L_m} = f_m dH^{n-1} \) and \( S_{M_m} = f_m dH^{n-1} \). The sequences \( \{ f_m \} \) and \( \{ f_m \} \) are uniformly bounded from above and uniformly away from zero, therefore after suitable translations, as Lemma A.1 shows, the bodies \( L_m \) and \( M_m \) are contained in and contain a fixed ball. Hence, by taking subsequences, Blaschke’s Selection Theorem shows that we may assume that \( L_m \to \bar{L} \) and \( M_m \to \bar{M} \) in the Hausdorff metric, for some convex bodies \( \bar{L} \) and \( \bar{M} \). However, \( f_m \to f_L \) and \( f_m \to f_M \) uniformly and thus weakly on \( S^{n-1} \), so \( \bar{L} \) and \( \bar{M} \) are translates of \( L \) and \( M \) respectively. Finally, we may assume that \( \bar{L} = L \) and \( \bar{M} = M \). Notice, in addition, that since \( f_m \) and \( f_m \) are positive \( C^\infty \) functions, it follows (see [33]) that \( L_m \) and \( M_m \) are all of class \( C^2_+ \).

Since \( L_m \to L \) and \( M_m \to M \) (and since \( o \in \text{int } L_m \cap \text{int } M_m \) if \( m \) is large enough), we conclude that \( \rho_{L_m}/\rho_L \to 1 \) and \( \rho_{M_m}/\rho_M \to 1 \), uniformly on \( S^{n-1} \). Thus, since

\[
\min_{v \in \text{bd } V} (\rho_M(v)/\rho_L(v)) > 1,
\]

it follows that if \( m \) is large enough, then

\[
\rho_{M_m}(v) > c \rho_{L_m}(v), \quad \text{for all } v \in \text{bd } V,
\]

where \( c > 1 \) is a constant which is independent of \( m \). On the other hand, for \( 0 < \varepsilon < c - 1 \), it holds

\[
\rho_{M_m}(v_0) < (1 + \varepsilon) \rho_{L_m}(v_0) < c \rho_{L_m}(v_0).
\]

Consequently, there exists \( m_0 \in \mathbb{N} \), such that for any \( m \geq m_0 \), the minimum

\[
c_m := \min_{v \in \text{cl } U} \frac{\rho_{M_m}(v)}{\rho_{L_m}(v)}
\]

is attained inside \( U \). This shows that if \( m \geq m_0 \), there exists \( v_m \in U \), such that \( \rho_{c_mL_m}(v_m) = c_m \rho_{L_m}(v_m) = \rho_{M_m}(v_m) \), while \( \rho_{c_mL_m}(v) \leq \rho_{M_m}(v) \), for all \( v \in U \). Thus, the triple \( (c_mL_m, M_m, v_m) \) satisfies assumptions (i) and (ii) imposed previously and, therefore, satisfies the (weaker) assumptions in the statement of Proposition 7.2. Since \( c_mL_m \) and \( M_m \) are of class \( C^2_+ \), we conclude that if \( m \geq m_0 \), then

\[
f_{c_mL_m}^{-1}(v_m) = f_{c_mL_m}(v_m) \leq f_m(v_m), \quad \text{(36)}
\]
where \( \nu_m := \eta_{L_m}(p_m) \) and \( p_m := \rho_{L_m}(v_m)v_m \).

Next, set \( W := \eta_L(V) \) and let \( \{v_k\} \) be a subsequence of \( \{v_m\} \) that converges to some vector \( \nu' \in S^{n-1} \). We claim that \( \nu' \in \text{cl } W = \eta_L(\text{cl } V) \). To see this, recall that \( p_{km} \) is the unique point in \( bd L_{km} \), such that

\[
\langle p_{km}, v_{km} \rangle = h_{L_{km}}(v_{km}).
\]

By taking a subsequence, we may assume that \( p_{km} \to q \), for some point \( q \in \text{cl } V \). Thus, it holds \( \langle q, \nu' \rangle = h_L(\nu') \) and, therefore, \( \nu' = \eta_L(q) \in \eta_L(\text{cl } V) = \text{cl } W \).

Since \( f_m \) and \( \bar{f}_m \) converge uniformly on \( S^{n-1} \) and since \( c_m \to 1 \), we conclude by (36) that

\[
f_L(\nu') \leq f_M(\nu').
\]

Notice that, in the argument described above, one can replace \( V \) by any open set \( V' \subseteq V \). Having this in mind, consider a sequence \( \{V_i\} \) of open sets in \( bd L \), all contained in \( V \), such that \( V_i \searrow \{p\} \). Then, for \( l \in \mathbb{N} \), there exists a vector \( \nu_l \in \text{cl } \eta_L(V_i) \), such that \( f_L(\nu_l) \leq f_M(\nu_l) \). Since \( \text{cl } \eta_L(V_i) \searrow \eta_L(\{p\}) = \{v\} \), it follows that \( \nu_l \to v \) and consequently,

\[
f_L(v) \leq f_M(v).
\]

It remains to remove the extra assumption \( V \setminus \{p\} \subseteq \text{int } M \). Let \( L, M, V, p, v \) be as in the statement of Proposition 7.2. At this point we are going to assume that \( L \) is of class \( C^2_+ \) (the case where \( M \) is of class \( C^2_+ \) can be treated completely similarly and is left to the reader). Let \( g : S^{n-1} \to \mathbb{R} \) be a \( C^2 \) function which is strictly positive on \( S^{n-1} \setminus \{v\} \) and satisfies \( g(v) = 0 \). Then, for small positive \( t \), the function \( h_L - tg \) is also a support function of class \( C^2_+ \). Set \( L_t \) for the \( C^2_+ \) convex body whose support function equals \( h_L - tg \). Then, \( h_{L_t} \leq h_L \), thus \( L_t \) is contained in \( L \). Furthermore, it holds \( h_{L_t}(v) = h_L(v) \) and, therefore, \( L_t \) is supported by the supporting hyperplane of \( L \), whose outer unit normal vector is \( v \). But since \( L_t \) is contained in \( L \), it follows that (for small \( t \)) \( p \in bd L_t \). Moreover, \( h_{L_t}(a) < h_L(a) \), for all \( a \in W \setminus \{v\} = \eta_L(V) \setminus \{v\} \), so \( \eta_{L_t}^{-1}(W) \setminus \{p\} \subseteq \text{int } L \subseteq \text{int } M \). This, together with the fact that \( p \in bd L_t \cap bd M \), shows that

\[
f_{L_t}(v) \leq f_M(v),
\]

for small \( t > 0 \). However, since \( L \) is of class \( C^2_+ \), (3) shows that \( f_{L_t}(v) \xrightarrow{t \to 0^+} f_L(v) \) and the result follows.

\[
\square
\]

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