Comment on "Deterministic six states protocol for quantum communication"

[Phys. Lett. A 358 (2006) 85]

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(Dated: July 25, 2011)

In [J.S. Shaari, M. Lucamarini, M.R.B. Wahiddin, Phys. Lett. A 358 (2006) 85-90] the deterministic six states protocol (6DP) for quantum communication has been developed. This protocol is based on three mutually unbiased bases and four encoding operators. Information is transmitted between the users via two qubits from different bases. Three attacks have been studied; namely intercept-resend attack (IRA), double-CNOT attack (2CNOTA) and quantum man-in-the-middle attack. In this Letter, we show that the IRA and 2CNOTA are not properly addressed. For instance, we show that the probability of detecting Eve in the control mode of the IRA is 70% instead of 50% in the previous study. Moreover, in the 2CNOTA, Eve can only obtain 50% of the data not all of it as argued earlier.

PACS numbers: 03.67.Dd, 03.67.Hk

Key words: quantum protocol, deterministic protocol, six states protocol, eavesdropper, intercept-resend attack, double-CNOT attack

In [1], the authors have developed the deterministic six states protocol (6DP) for quantum communication. This protocol can be briefly explained as follows: We have three bases, namely, $x, y, z$, which are mutually unbiased and have the forms:

$$
|z_+\rangle = |0\rangle, \quad |z_-\rangle = |1\rangle,
$$

$$
|x_\pm\rangle = (|z_+\rangle \pm |z_-\rangle)/\sqrt{2}, \quad |y_\pm\rangle = (|z_+\rangle \pm i|z_-\rangle)/\sqrt{2}.
$$

Bob chooses two qubits from (1) on condition that they are from different bases and send them to Alice. Alice upon receiving the qubits operates on both of them with one of the four operators $\hat{I}, \hat{X}, i\hat{Y}$ or $\hat{Z}$ according to the message she wants to address to Bob. Then she sends them back to Bob. He then measures the qubits in the bases, which they have been prepared earlier, to decode the key. The key extraction depends on an advance agreement between the users related to the
encoding processes, i.e. $\hat{I} \rightarrow 00, \hat{X} \rightarrow 10, i\hat{Y} \rightarrow 01, \hat{Z} \rightarrow 11$. This scenario makes the 6DP more secure than the standard BB84 protocol [2]. As we can see, no public discussion is necessary for the completion of the encoding-decoding procedure, as it happens in BB84. Moreover, the BB84 protocol has a probability of $1/2$ that a transmitted bit has to be discarded due to the wrong choice of basis on both sides. This is not the case in 6DP, where one of the users knows the measurement bases. We proceed, three types of attacks, namely, intercept-resend attack (IRA), two control-NOT gates [3] attack (2CNOTA) and quantum man-in-the-middle attack, have been discussed for 6DP in [1]. It has been shown that Eve can obtain 50% of the message by using IRA. Also, she can obtain full information from the protocol via 2CNOTA. By means of the quantum man-in-the-middle attack Eve either induces two errors in the communication, one for each path, or induces no errors at all. This is something that a "natural" noise cannot statistically give. This represents a sure signature of Eve’s presence [1].

In [1], the treatments of the IRA and the 2CNOTA have been performed when the users use two qubits from the same bases (, precisely, identical qubits). This contradicts with the notion of the protocol, which is based on two qubits from different bases. Actually, this property is necessary to establish the key in the protocol. For instance, assume that Bob sends one particle in the state, say, $|x_+\rangle$ and after a complete round he receives the particle in the same state. For him, this means that Alice used $\hat{I}$ or $\hat{X}$ as an encoder and hence the key cannot be extracted. As a result of this, we deal again with these two attacks considering two qubits from different bases, as it should be. In treating IRA, we consider the permutation of different bases in the protocol. We obtain results completely different from the previous ones [1]. For instance, we show that the probability of detecting Eve in the control mode of the IRA is 70% instead of 50% in [1]. Additionally, in contrast with the analysis given in [1], we show that 2CNOTA is not relevant to be used against this protocol. Throughout the investigation we use the same notations and some of the equations given in [1].

We start the analysis with the IRA for control mode. This can be explained as follows: Bob has to prepare two states, say, $|B_1, B_2\rangle$ from different bases and sends them to Alice, who returns them back to Bob without doing any action on them. Bob checks if Eve is on line or not by measuring these qubits using the same bases in which they have been initially prepared. In IRA, Eve measures both of the travelling qubits in the forward and in the backward paths in the same bases. Suppose that Eve performs projective measurements along the orthogonal bases [1]:

$$|\chi_j\rangle = \cos(\frac{\theta_j}{2})|0\rangle + \exp(i\varphi_j)\sin(\frac{\theta_j}{2})|1\rangle, \quad |\chi_j^\perp\rangle = \exp(-i\varphi_j)\sin(\frac{\theta_j}{2})|0\rangle - \cos(\frac{\theta_j}{2})|1\rangle, \quad j = 1, 2, \quad (2)$$
FIG. 1: The average probability of not detecting Eve during the control mode of the IRA. The angles $\theta_1$ and $\varphi_1$ define Eve’s projective measurement direction. The angles are measured in radian.

where $0 \leq \theta_j \leq \pi, 0 \leq \varphi_j \leq 2\pi$. The index $j$ indicates that Eve can use one or more different bases for the measurement. In IRA, the probability that Alice (Bob) does not detect Eve on the forward (backward) path equals the probability to obtain the states $|B_1, B_2\rangle$ prepared initially by Bob. These two partial probabilities are equal and each of which can be expressed as:

$$P_{B_j|\chi_j}^{B_1, B_2} = P_{B_j|\chi_j}^{B_1, B_2} = P_{B_1|\chi_1}P_{B_2|\chi_2} + P_{B_1|\chi_1}P_{B_2|\chi_2}^\perp,$$

where $P_{B_j|\chi_j'} = |\langle B_j|\chi_j'\rangle|^2, j, j' = 1, 2$ and the subscripts $\text{noEve}_A$ and $\text{noEve}_B$ mean that the probabilities are related to Alice (forward path) and Bob (backward path), respectively. The probability $P_{\text{noEve}}^{B_1, B_2}$ that Eve is not detected after a whole control run is the product of the two partial probabilities, i.e. $P_{\text{noEve}}^{B_1, B_2} = P_{\text{noEve}_B}^{B_1, B_2} P_{\text{noEve}_A}^{B_1, B_2}$. This quantity is quite different from Eq. (5) in [1]. Now we study the behavior of $P_{\text{noEve}}^{B_1, B_2}$ for two cases depending on whether Eve measures the travelling qubits (i.e., qubits in the transit) using the projective qubits from the same or different bases. The first case (i.e. $j = 1$ in [2]) is given for the sake of comparison with the technique used in [1]. Of course this scenario gives Eve complete information on one of the qubits and probabilistic information on the other. In spite of this, the results will be sufficient and better than the earlier ones [1]. Starting with the first case, from (1) and (2) the quantities $P_{B_j|\chi_1}$ (where $\chi_1 = \chi_2$) can be easily calculated and the average probability of not detecting Eve $P$ in the IRA
FIG. 2: The average probability of not detecting Eve during the control mode of the IRA. The angles $\theta_1$ and $\theta_2$ determine Eve’s projective measurement direction and $\phi_j = 0$. The angles are measured in radian.

It is worth mentioning that such type of terms, say, $P_{|z_\perp\pm,x_+\rangle}^{\text{noEve}}$, which include such elements $P_B^{1|x_\perp\pm,x_+\rangle,1}$, is implicitly covered in (4). Precisely, one can prove, e.g., $P_{|z_\perp\pm,x_+\rangle}^{\text{noEve}} = P_{|x_\perp\pm,z_\perp\pm,x_+\rangle}^{\text{noEve}}$. We plot the relation (4) in Fig. 1. Based on this figure one note $0.25 \leq P \leq 0.28$. The case $P = 0.25$ occurs when Eve’s projective basis reduces to one of the set given by (1) (see Fig. 1). In this respect, all elements in the right hand side of (4) tend to $1/2$, i.e. $P_{|i,j\rangle}^{\text{noEve}} = 1/2, \forall i,j \in \{x,y,z\}$ basis. This fact can be analytically checked. Additionally, one can easily realise that the maximum value $P \approx 0.28$ occurs when Eve’s basis is different from those used in the protocol, e.g., $(\theta, \phi) = (\pi/2, \pi/4)$. In other words, when Eve measures the two qubits in the same basis she will be detected at least by 72%, however, in [1] it was approximately 50%.

Next, we draw the attention to the second case, in which Eve measures the two travelling qubits in two different bases, i.e., $j = 1, 2$ in (4). Consequently, Eve could obtain more information than that extracted from the single-basis case. In this case the order of the travelling qubits should be taken into account, where $P_{|z_\perp\pm,x_+\rangle}^{\text{noEve}} \neq P_{|x_\perp\pm,z_\perp\pm,x_+\rangle}^{\text{noEve}}$. Thus, the expression (4) includes 24 terms. Information about this situation is depicted in Fig. 2 for $\varphi_j = 0$. From this figure it is obvious
that there is an improvement in Eve’s information but not too much, where \(0.25 \leq P \leq 0.3\).

This means that the probability of detecting Eve is at least 70%. The maximum value \(P = 0.3\) (minimum value \(P = 0.25\)) occurs when Eve’s bases reduce to pair (one) of the \(x, z, y\) bases.

The natural question is: Can Eve increase her information about the key by using three different bases as those used in the protocol? According to the scenario described above, the answer is no since she, in this case, may overcome difficulties in choosing which appropriate projective bases for performing the measurement. This may decrease her information compared to those obtained from the previous cases. To explain this point, assume that Bob sends two qubits in two different bases to Alice. According to above strategy, the probability that Eve measures them in the correct form in the forward as well as in the backward paths is \(1/12\), i.e. in a complete round the probability is \(1/144 \approx 0.007\). Finally, we should stress that the IRA cannot allow Eve to ascertain Alice encoding with certainty, regardless of the choice of the projection axis. This means that Alice-Eve mutual information after the IRA is less than unity. The reason is that Eve in the IRA is going to use one or two or three projective bases. In each case, as we discussed above, Eve cannot obtain full information about the message. This fact will be elaborated when we discuss the 2CNOTA below.

We conclude this part by shedding some light on the eavesdropping success probability as a function of \(P\). Assume that \(c\) is the probability of occurrence the control run. Thus the effective transmission rate reads \(r = 1 - c\), which is equal to the probability for a message transfer \([4, 5]\). The average probability of detecting Eve during a control run equals \(d = 1 - P\), where \(P\) is given by \([4]\). Thus, the probability that Eve acquires \(n\) bits of full information without being detected is \([4]\):

\[
P_n(c, d) = \left[\frac{1 - c}{1 - c(1 - d)}\right]^n.
\]

We apply the formula \([5]\) for the extreme values in Fig. 1, i.e. \(P = 0.25\) and 0.28, with \(c = 1/2\). In this case, Eve has correspondingly probabilities of about 1.1% and 1.3% to successfully eavesdrop 1 byte (i.e., 8 bits) of information. It is evident that the probabilities are very low.

Now we draw the attention to the 2CNOTA. In this attack Eve performs a first CNOT \([3]\) between the photons in transit from Bob to Alice (control qubits) and her ancillae (target qubits). The second CNOT is performed in the backward path in the same scenario. This attack can give Eve full information on the key for some protocols, e.g. \([2]\). In \([1]\) it has been argued that Eve can obtain full information about the 6DP using this attack. This is obvious where the mutual entropy between Alice and Eve as well as between Bob and Eve are unity (see (10) in \([1]\)). On
the contrary, here we prove that the 2CNOT is not an efficient attack against the 6DP. This can simply be realized as follows: Assuming that Eve uses the ancilla $|0\rangle_E$ with one of the Bob qubits in the forward path. Then in the backward path, Eve measures her ancilla. It can be either in $|0\rangle_E$ or in $|1\rangle_E$. This does not give her definite information about the encoded bits, where Alice could execute one of four operations. More illustratively, when Eve obtains $|0\rangle_E(|1\rangle_E)$ this means that Alice acted on the qubits by one of the operators $\hat{I}$ or $\hat{Z}$ ($\hat{X}$ or $\hat{Y}$). This point was not discussed in [1]. Next, we extend this conclusion when Eve uses two ancillae, say, $|a\rangle_1^E|b\rangle_2^E$ where $a, b = 0, 1$. According to the notion of the 6DP, Bob has to prepare two qubits from two different bases as:

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1|0\rangle_1 + \beta_1|1\rangle_1) \otimes (\alpha_2|0\rangle_2 + \beta_2|1\rangle_2),$$

where $\alpha_j, \beta_j$ are c-numbers, which can be arbitrarily chosen to provide the different forms of the states (1) and $|\alpha_j|^2 + |\beta_j|^2 = 1$. Eve executes her ancilla $|a\rangle_1^E|b\rangle_2^E$ with the Bob qubits and performs the first CNOT as:

$$U_{\text{cnot}}(|\psi\rangle |a\rangle_1^E|b\rangle_2^E = |\Psi_1\rangle = (\alpha_1|0\rangle_1|a\rangle_1^E + \beta_1|1\rangle_1|1 \oplus a\rangle_1^E) \otimes (\alpha_2|0\rangle_2|b\rangle_2^E + \beta_2|1\rangle_2|1 \oplus b\rangle_2^E).$$

Alice can act by any one of the four operators given above, say, $\hat{L}_j$, to encode both the qubits with the value $j = 0$ or 1. This could change the forms of the coefficients $\alpha_j, \beta_j$ to, say, $\alpha'_j, \beta'_j$. The values of $\alpha'_j, \beta'_j$ depend on which operator Alice used in the encoding process. Then we obtain:

$$\hat{L}_j|\Psi_1\rangle = |\Psi_2\rangle = (\alpha'_1|j\rangle_1|a\rangle_1^E + \beta'_1|1 \oplus j\rangle_1|1 \oplus a\rangle_1^E) \otimes (\alpha'_2|j\rangle_2|b\rangle_2^E + \beta'_2|1 \oplus j\rangle_2|1 \oplus b\rangle_2^E).$$

Upon returning on the backward path Eve performs the second CNOT as:

$$U_{\text{cnot}}|\Psi_2\rangle = (\alpha'_1|j\rangle_1 + \beta'_1|1 \oplus j\rangle_1) \otimes (\alpha'_2|j\rangle_2 + \beta'_2|1 \oplus j\rangle_2)|j \oplus a\rangle_1^E|j \oplus b\rangle_2^E.$$  

From (9) it is obvious that the ancilla have been disentangled from the qubits and they are insensitive of the initial states. Next, Eve measures her ancilla in the z basis to get some information. Actually, Eve’s information is restricted to her ancilla since she cannot access the travelling qubits to see if the initial qubits are flipped or not. This would confuse Eve since different operations can give her the same results, as we mentioned above. For instance, assume $a, b = 0, 0$ and Alice executed one of the operators $\hat{I}$ or $\hat{Z}$ ($\hat{X}$ or $\hat{Y}$). Thus Eve measurement outcome is $|0\rangle_1^E|0\rangle_2^E$ ($|1\rangle_1^E|1\rangle_2^E$). Similar conclusions can be quoted for the other values of $a, b$. Furthermore, these results are still valid even when Eve’s ancilla comprise different bases, e.g., one of the ancilla in
z basis and the other in x basis. We have checked this fact. According to this discussion Eve has a chance 50\% to get the correct result and hence the mutual entropy between different partners might be written as:

\[ I_{AE}^{2\text{CNOT}} = \frac{1}{2}, \quad I_{BE}^{2\text{CNOT}} = \frac{1}{2}, \quad I_{AB}^{2\text{CNOT}} = 1. \]  

(10)

The origin in the last relation in (10) is that the 2CNOTA does not perturb the message and hence Alice and Bob mutual information amounts always to 1. In other words, in this attack Eve’s actions do not affect the users outcomes. Comparison between the relation (10) given above with Eq. (10) in [1] is instructive.

It is worth mentioning that the analysis of the control sessions for the 2CNOTA is equivalent to that of using the identity operator \( \hat{I} \) in the encoding process by Alice in the above analysis. In this case, Eve’s ancillae do not change after a whole run. This indicates that if the legitimate users restrict the encoding operators to \( \hat{I} \) and \( \hat{Z} \) (instead of four), Eve will obtain no information about the protocol from the 2CNOTA. In this regard, the protocol is secure against this type of attack, however, it may be vulnerable against the others.

In conclusion, the key extraction in the 6DP is based on two qubits from different bases. Considering this property we treated the IRA and 2CNOTA in this Comment. The rate of security obtained from this treatment is higher than that shown in [1], in which the security analysis was evaluated to one-qubit case only.

References

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