Linear representation of categorical values*

Arnaud Berny

June 15, 2021

Abstract

We propose a binary representation of categorical values using a linear map. This linear representation preserves the neighborhood structure of categorical values. In the context of evolutionary algorithms, it means that every categorical value can be reached in a single mutation. The linear representation is embedded into standard metaheuristics, applied to the problem of Sudoku puzzles, and compared to the more traditional direct binary encoding. It shows promising results in fixed-budget experiments and empirical cumulative distribution functions with high dimension instances, and also in fixed-target experiments with small dimension instances.

Keywords: Combinatorial optimization, categorical values, binary representation, linear representation, Sudoku

1 Introduction

Representation is an important topic for evolutionary algorithms [12] and other metaheuristics, especially when applied to combinatorial optimization. It directly influences the range of problems which can be addressed by metaheuristics and the quality of their solutions. Many evolutionary algorithms have been designed with binary domains in mind. Although most of them can be adapted more or less easily to other domains, it is still desirable to be able to represent values from non binary domains in binary domains so as to leverage theoretical and practical knowledge of evolutionary algorithms.
in binary domains along with their implementations. In this paper, we are concerned with the binary representation of categorical values.

Often, categorical values are represented by means of a direct binary encoding. As an example, let us address the problem of representing the four nucleobases $A$, $T$, $C$, and $G$ found in DNA. Using 2 bits, we can arbitrarily decide that $A = 00$, $T = 01$, $C = 10$, and $G = 11$. In the neighborhood system defined by 1-bit flips, it appears that each nucleobase has 2 neighbors. Hence it is not possible to go from $A$ to $G$ in a single bit flip. The so called unary representation is also able to represent categorical values. The idea is to assign categorical values to bit strings based on their Hamming weights. In the case of DNA, we can arbitrarily decide that $A$, $T$, $C$, and $G$ are represented by 3-bit strings of Hamming weights 0, 1, 2, and 3 respectively. Consequently, $A$ is represented by only one bit string (000) whereas $C$ is represented by 3 bit strings (011, 101, and 110). Just as for direct encoding, the neighborhood system of 1-bit flips is not complete in the sense that $A$ and $G$ have only one neighbor as compared to $T$ and $C$ which have two neighbors. Moreover, going from $A$ to $G$ requires 3 bit flips.

Those representations are inappropriate because the resulting neighborhood systems among categorical values are not complete. In a set of categorical values, every element is the neighbor of every other element. In other words, categorical values are the vertices of a complete graph. In this paper, we propose a binary representation of categorical values which is based on a linear map and which satisfies this requirement.

The paper is organised as follows. In Sect. 2 we define binary representations for categorical values. In Sect. 3 we propose a linear representation for categorical values. In Sect. 4 we apply the linear representation to Sudoku puzzles seen as optimization problems. Sec. 5 concludes the paper.

### 2 Representation

Let $V = \{v_1, v_2, \ldots, v_N\}$ be a set of $N$ categorical values, where $N \in \mathbb{N}$ and $N \geq 2$. We want to represent those values in a binary domain $\{0,1\}^n$ of dimension $n \in \mathbb{N}$. A binary representation of $V$ is a surjective map $\phi : \{0,1\}^n \rightarrow V$, that is, for all $v \in V$, there exists $x \in \{0,1\}^n$ such that $\phi(x) = v$. The binary vector $x$ is called a representative of $v$ which might have more than one representative. Such binary representations can be used, for example, to apply metaheuristics designed for binary spaces to the optimization of functions defined on categorical values.

Let $(e_1, e_2, \ldots, e_n)$ be the canonical basis of $\{0,1\}^n$. For example, in $\{0,1\}^3$, $e_1 = (1,0,0)^t$, where $t$ denotes transpose (we use column vectors).
For all $x \in \{0, 1\}^n$, let $B(x, 1)$ be the Hamming ball of radius 1 centered at $x$, that is $B(x, 1) = \{x\} \cup \{x + e_i \mid i \in [1..n]\}$. Throughout this paper, we identify the set $\{0, 1\}$ as the finite field $\mathbb{F}_2$. Thus, addition on $\{0, 1\}$ or $\{0, 1\}^n$ must be understood modulo 2 and is equivalent to the exclusive-or operator. For example, if $x = (1, 1, 1)^t$ then $x + e_1 = (0, 1, 1)^t$.

With mutation based metaheuristics or local search in mind, we would like to be able to reach any categorical value in a single bit mutation. We say that $\phi$ is locally bijective if, for all $x \in \{0, 1\}^n$, its restriction $\phi : B(x, 1) \to V$ is bijective. In this case, necessarily, $n + 1 = N$. We can restate this property in the language of graph theory by saying that the hypercube $\{0, 1\}^n$ is a covering graph of the complete graph $K_N$ and $\phi$ a covering map from $\{0, 1\}^n$ to $K_N$. The map $\phi$ is also an $N$-coloring of $\{0, 1\}^n$.

### 3 Linear representation

We propose a linear representation which is locally bijective. We suppose for now that $N = 2^k$, where $k \in \mathbb{N}$. The categorical values are first identified with $k$-bit binary vectors in an arbitrary manner. We are looking for a surjective linear representation, that is a $k \times n$ binary matrix of rank $k$. Let $x \in \{0, 1\}^n$ be the current search point and $y = Ax \in \{0, 1\}^k$ its corresponding categorical value. The neighbors of $y$ are $A(x + e_i) = Ax + Ae_i = y + Ae_i$, where $i \in [1..n]$. Let $y' \in \{0, 1\}^k$ be any categorical value but $y$. Then, $A(x + e_i) = y' \iff Ae_i = y + y'$. The last equation has a unique solution if and only if the set $\{Ae_i \mid i \in [1..n]\}$ is the set $\{0, 1\}^k \setminus \{0\}$ and $n = N - 1 = 2^k - 1$, which means that the columns of $A$ are made of all the vectors of $\{0, 1\}^k$ but $0$.

We observe that $A$ is precisely the parity-check matrix of the binary Hamming code $[4, 11]$. It is remarkable that a requirement in the context of local search leads to a well known object of coding theory. We want to point out that we use $A$ differently, though. The Hamming code is defined as the set of vectors $x$ of $\{0, 1\}^n$ such that $Ax = 0$, that is, as the kernel (null space) of $A$. We use $A$ more as a generator matrix. The matrix $A$ is also the generator matrix of the simplex code which is the dual code of the Hamming code. It is defined as the set $\{y^tA \mid y \in \{0, 1\}^k\}$, where we have used left-multiplication as is traditional in coding theory literature. Again, we use $A$ differently.

As an example, let us consider again the case of DNA where $N = 4$ categories, $k = 2$, $n = 3$, and

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$
The order of the columns does not matter. Fig. 1 shows a geometric representation of $A$. For example, let $x = (1, 0, 1)^t$, that is the bottom right vertex on the front face of the cube. Then

$$Ax = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (\text{mod} 2),$$

which gives the bit string 01 and the nucleobase $C$. Let us compute the neighbors of $x$:

$$A(x + e_1) = Ax + Ae_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = T$$

$$A(x + e_2) = Ax + Ae_2$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A$$

$$A(x + e_3) = Ax + Ae_3$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = G$$

It is clear that we can reach every categorical value (nucleobase) other than $C$ in a single mutation.

Following [12], we can say that the linear representation has high locality; is uniformly redundant (each categorical value has exactly $2^{n-k}$ representatives); and is non synonymously redundant (for each categorical value, its representatives are spread all over the hypercube).

If $N$ is not a power of 2 then we let $k$ be the smallest natural such that $N < 2^k$ and repeat the construction of $A$ with $n = 2^k - 1$. It is then necessary to map the output of $A$ to $V$ as in

$$x \mapsto Ax \mapsto i \mapsto j = 1 + (i \mod N) \mapsto v_j,$$

where $Ax$ is the binary representation of integer $i$. The resulting binary representation $\phi$ of $V$ is still locally surjective but not locally bijective.

From our observations, it seems that the linear representation is isotropic, a property which generalizes the fact that it is locally bijective. Intuitively speaking, it means that, from any search point, a uniform mutation of exactly $r$ bits yields a quasi uniform distribution over the categorical values. For
Figure 1: Geometrical representation of a linear representation in the case of $N = 4$ categories, $k = 2$, and $n = 3$. Each vertex is labeled with a 2-bit string which, as a binary vector, is the image of its coordinates under the matrix $A$. For illustration purpose, each 2-bit string is also arbitrarily identified as one of the four nucleobases found in DNA.
example, with \( N = 16 \) categories, \( n = 15 \) bits, and \( r = 3 \) bits, from any search point \( x \) and its category \( y = Ax \), there are 35 3-bit mutations yielding the same category \( y \) and, for each category \( z \neq y \), there are 28 3-bit mutations yielding category \( z \). In total, there are \( 35 + 15 \times 28 = 455 = \binom{15}{3} \) 3-bit mutations.

For all \( x \in \{0, 1\}^n \) and all \( r \in [0..n] \), let \( S(x, r) \) be the Hamming sphere of radius \( r \) centered at \( x \), that is \( S(x, r) = \{ x' \in \{0, 1\}^n \mid d_H(x, x') = r \} \), where \( d_H \) is the Hamming distance. We say that a representation \( \phi \) is isotropic if, for all \( x \in \{0, 1\}^n \) and \( r \in [0..n] \), the map \( \phi : S(x, r) \setminus \ker(A) \to \{0, 1\}^k \setminus \{0\} \) balanced. The question of whether the linear representation is isotropic remains open.

4 Experiments

4.1 Problem

We apply linear representation to Sudoku puzzles. A Sudoku puzzle consists in filling a 9 by 9 board using digits and satisfying a set of constraints. The board is divided into 3 by 3 blocks. Each row, column, and block must contain all 1 to 9 digits. We turn a Sudoku puzzle into an optimization problem by counting the number of unsatisfied constraints. More precisely, the objective is to minimize the function \( f : V^{81} \to \mathbb{N} \) defined by

\[
f(x) = \sum_{i \in V} \sum_{k \in V} \sum_{j \in V} [x_{ij} = k] - 1 \quad \text{(rows)}
+ \sum_{j \in V} \sum_{k \in V} \sum_{i \in V} [x_{ij} = k] - 1 \quad \text{(columns)}
+ \sum_{(i,j) \in \{1,4,7\}^2} \sum_{k \in V} \sum_{(k,l) \in \{0,1,2\}^2} [x_{i+k,l+k} = k] - 1 \quad \text{(blocks)},
\]

where \( V = \{1, 2, ..., 9\} \) and \([P] = 1\) if the statement \( P \) is true, 0 otherwise (Iverson bracket). Usually, in a Sudoku puzzle, some of the \( x_{ij} \)'s are given. Solving the puzzle is equivalent to minimizing the function \( g \) defined by \( g(x_U) = f(x_U, x_K) \), where \( U \subset V \times V \) is the set of locations of unknowns and \( K = (V \times V) \setminus U \) is the set of locations of known digits. The puzzle is solvable if and only if the minimum of \( g \) is zero.

From the point of view of linear representation, Sudoku is a worst-case scenario since \( N = 9 \) is one past a power of 2. Direct representation requires
\( n = 4 \) bits whereas linear representation requires \( n = 15 \) bits. For example, instance sudoku-intermediate-54 has 54 unknowns and the size of its (binary) search space is 216 bits with direct representation and 810 bits with linear representation.

### 4.2 Fixed-budget experiments

We have generated Sudoku instances of varying difficulty using an online generator (which is also open source) \([10]\). It should be noted that the difficulty grade (simple, easy, intermediate, expert) is relevant to algorithms relying on constraint programming or backtracking rather than metaheuristics. We have generated two instances per difficulty grade. We have also included two 17-hint uniquely completable instances \([13]\).

The study includes the following metaheuristics: random local search with restart (RLS), hill climbing with restart (HC), simulated annealing (SA) \([8]\), genetic algorithm (GA) \([7]\), \((1 + 1)\) evolutionary algorithm (EA), \((10 + 1)\) evolutionary algorithm, population-based incremental learning (PBIL) \([1]\), mutual information maximization for input clustering (MIMIC) \([2]\), univariate marginal distribution algorithm (UMDA) \([9]\), linkage tree genetic algorithm (LTGA) \([14]\), parameter-less population pyramid (P3) \([3]\). All metaheuristics have been applied to the same set of instances with direct and linear representations. They have been given the same arbitrary budget (300,000 function evaluations) and run 20 times. All experiments have been produced with the HNCO framework \([6]\).

Fig. 2 shows the mean values of solutions found by metaheuristics on a particular instance (sudoku-intermediate-54). Despite increased dimension of the search space, all metaheuristics but MIMIC consistently improve the quality of their solutions with linear representation.

Tab. 1 shows the summary statistics of solutions on instance sudoku-easy-56. Metaheuristics are ranked according to their median, first and third quartiles, minimum and maximum, in that order. On this particular instance, only MIMIC does not take advantage of linear representation.

Tab. 2 shows the summary statistics of ranks on the set of all ten instances. Metaheuristics are globally ranked. On this particular set of instances and within the given budget, for all metaheuristics, linear representation dominates direct representation.

### 4.3 Empirical cumulative distribution functions

To account for the dynamical behavior of metaheuristics with respect to representation, we have studied their empirical cumulative distribution functions
| Algo. | rep. | Value |  |  |  |
|-------|------|-------|---|---|---|
| sa    | linear | 4 | 4.00 | 4.0 | 6.00 | 8 |
| sa    | direct | 4 | 8.00 | 8.0 | 12.50 | 14 |
| ga    | linear | 4 | 8.00 | 10.0 | 12.00 | 16 |
| ea-10p1 | linear | 6 | 9.50 | 10.0 | 12.00 | 18 |
| ga    | direct | 4 | 9.50 | 12.0 | 14.00 | 18 |
| p3    | linear | 10 | 13.50 | 14.0 | 16.00 | 18 |
| ea-1p1 | linear | 8 | 13.50 | 17.0 | 18.00 | 24 |
| rls   | linear | 12 | 18.00 | 19.0 | 20.00 | 24 |
| p3    | direct | 14 | 18.00 | 21.0 | 22.00 | 28 |
| ea-10p1 | direct | 10 | 18.00 | 22.0 | 24.50 | 30 |
| umda  | linear | 16 | 20.00 | 22.0 | 26.00 | 28 |
| pbil  | linear | 14 | 20.00 | 23.0 | 26.00 | 32 |
| ea-1p1 | direct | 12 | 21.50 | 24.0 | 26.50 | 32 |
| mimic | direct | 16 | 22.00 | 24.0 | 26.00 | 34 |
| ltga  | linear | 22 | 24.00 | 25.0 | 26.50 | 30 |
| mimic | linear | 20 | 25.50 | 28.0 | 28.50 | 38 |
| pbil  | direct | 26 | 31.00 | 32.0 | 36.00 | 38 |
| umda  | direct | 26 | 28.00 | 33.0 | 36.00 | 42 |
| hc    | linear | 28 | 31.50 | 36.0 | 36.50 | 42 |
| rls   | direct | 28 | 34.00 | 36.0 | 36.00 | 38 |
| ltga  | direct | 30 | 35.50 | 36.0 | 38.00 | 40 |
| hc    | direct | 42 | 49.50 | 52.0 | 52.00 | 56 |

Table 1: Value statistics in fixed-budget experiments (lower is better, instance sudoku-easy-56, 300,000 function evaluations, 20 runs).
Table 2: Rank statistics in fixed-budget experiments (value based, all ten instances).

| Algo.  | rep. | Rank | min | $Q_1$ | med. | $Q_3$ | max |
|--------|------|------|-----|-------|------|-------|-----|
| sa     | linear | 1    | 1.00 | 1.0   | 1.00 | 1     |
| ga     | linear | 2    | 2.00 | 2.0   | 3.00 | 3     |
| sa     | direct | 2    | 2.00 | 2.5   | 3.00 | 3     |
| ea-10p1| linear | 4    | 4.00 | 4.0   | 4.75 | 5     |
| ga     | direct | 4    | 4.25 | 5.0   | 5.00 | 6     |
| ea-1p1 | linear | 5    | 6.00 | 6.0   | 7.00 | 7     |
| p3     | linear | 6    | 6.00 | 7.0   | 7.00 | 7     |
| ea-10p1| direct | 8    | 8.00 | 8.5   | 9.75 | 10    |
| rls    | linear | 8    | 8.00 | 9.0   | 9.00 | 10    |
| p3     | direct | 8    | 9.00 | 9.5   | 10.75| 15    |
| ea-1p1 | direct | 9    | 10.25| 11.0  | 11.75| 13    |
| pbil   | linear | 11   | 11.25| 12.0  | 12.00| 15    |
| mimic  | linear | 12   | 13.00| 13.5  | 14.75| 16    |
| umda   | linear | 11   | 13.00| 14.0  | 14.00| 15    |
| mimic  | direct | 11   | 13.25| 14.0  | 15.00| 15    |
| ltga   | linear | 15   | 16.00| 16.0  | 16.00| 16    |
| pbil   | direct | 17   | 17.00| 17.0  | 17.00| 18    |
| umda   | direct | 17   | 18.00| 18.5  | 19.00| 20    |
| rls    | direct | 18   | 19.00| 19.0  | 20.00| 21    |
| hc     | linear | 18   | 18.25| 20.0  | 20.00| 21    |
| ltga   | direct | 19   | 20.00| 21.0  | 21.00| 21    |
| hc     | direct | 22   | 22.00| 22.0  | 22.00| 22    |
Figure 2: Mean value of solutions found by metaheuristics with direct and linear representations (lower is better, instance sudoku-intermediate-54, 300,000 function evaluations, 20 runs).

(ECDF) [5]. For each instance, every metaheuristic has been run 20 times with a budget of $10^7$ evaluations per run. For each run, every improvement has been recorded. Then, for each instance, the range of function values has been evenly divided into 50 targets. Finally, for each metaheuristic, the mean proportion of targets reached at each number of evaluations has been computed.

The metaheuristics and instances considered in this section are the same as in the previous one. Result are shown in Fig. 3-13. It should be noted that representations, linear or direct, are ranked in the keys according to their final scores. If a curve stops before $10^7$ evaluations, it means that the metaheuristic-representation pair did not make any further progress. Almost all curves have the shape of a sigmoid. We can identify the following patterns:

- The curves gradually diverge and linear representation dominates. This is the case for RLS (Fig. 3), SA (Fig. 5), (1 + 1) EA (Fig. 6), (10 + 1) EA (Fig. 7), and GA (Fig. 8).

- The curve for linear representation eventually crosses from below the one for direct representation after a delayed and sharp transition. This is the case for HC (Fig. 4), PBIL (Fig. 9), UMDA (Fig. 10), and, by a small margin, MIMIC (Fig. 11).

In the case of P3 (Fig. 13), the curves have a smooth staircase shape before a few thousands evaluations but can still be considered as sigmoids on a larger
scale. Linear representation dominates in the range $[10^3, 1.5 \cdot 10^6]$ and direct representation overtakes it afterward. In the case of LTGA (Fig. 12), the curves do not have the shape of a sigmoid. Linear representation dominates in the range $[924, 4.7 \cdot 10^5]$ and direct representation overtakes it afterward. However, LTGA with linear representation makes progress until the end, on the contrary to LTGA with direct representation, and almost catches up with it.

4.4 Fixed-target experiments

As we have seen in fixed-budget experiments, metaheuristics almost never find an optimal solution, that is a solution which satisfies every constraint. As a consequence, we have generated easy instances, starting from complete boards and erasing a small number $r \in [1..10]$ of digits. For each dimension, 4 instances have been generated. For each instance, every metaheuristic has been run 100 times. A run is successful if an optimal solution has been found before $10^6$ evaluations. In this case, the runtime is the number of evaluations needed to find it. For each instance, metaheuristics are ranked according to median runtime. Finally, they are ranked on a given set of instances according to success rate then rank statistics. It should be noted that all runs until $r = 6$ erased digits are successful. Hence, the full set of instances has been further divided into low dimension instances, for which $r \in [1..5]$, and medium dimension instances, for which $r \in [6..10]$. Tab. 3 gives the results for a particular small dimension instance (small-dimension-3-a) with $r = 3$ .
Figure 4: ECDF’s of HC with direct and linear representations (20 runs).

Figure 5: ECDF’s of SA with direct and linear representations (20 runs).
Figure 6: ECDF’s of $(1 + 1)$ EA with direct and linear representations (20 runs).

Figure 7: ECDF’s of $(10 + 1)$ EA with direct and linear representations (20 runs).
Figure 8: ECDF’s of GA with direct and linear representations (20 runs).

Figure 9: ECDF’s of PBIL with direct and linear representations (20 runs).
Figure 10: ECDF’s of UMDA with direct and linear representations (20 runs).

Figure 11: ECDF’s of MIMIC with direct and linear representations (20 runs).
Figure 12: ECDF’s of LTGA with direct and linear representations (20 runs).

Figure 13: ECDF’s of P3 with direct and linear representations (20 runs).
Table 3: Runtime statistics in fixed-target experiments (instance small-dimension-3-a, 100 runs).

| Algorithm         | Number of evaluations | Success |
|-------------------|-----------------------|---------|
|                   | min | $Q_1$ | med. | $Q_3$ | max |     |
| rls-linear        | 6   | 18.75 | 30.0 | 50.25 | 269 | 100.0 % |
| rls-direct        | 1   | 20.00 | 38.0 | 64.50 | 600 | 100.0 % |
| ea-1p1-linear     | 4   | 29.00 | 45.0 | 71.00 | 192 | 100.0 % |
| ea-1p1-direct     | 3   | 28.50 | 47.5 | 72.75 | 336 | 100.0 % |
| p3-linear         | 5   | 56.75 | 87.0 | 134.25 | 316 | 100.0 % |
| sa-linear         | 3   | 46.25 | 93.5 | 216.50 | 1,171 | 100.0 % |
| hc-linear         | 1   | 79.50 | 97.5 | 113.00 | 128 | 100.0 % |
| p3-direct         | 11  | 67.50 | 107.0 | 157.50 | 357 | 100.0 % |
| hc-direct         | 1   | 59.25 | 115.0 | 248.00 | 1,155 | 100.0 % |
| umda-direct       | 3   | 118.25 | 156.5 | 207.50 | 348 | 100.0 % |
| ga-direct         | 2   | 109.00 | 166.0 | 217.75 | 415 | 100.0 % |
| ga-linear         | 4   | 111.75 | 181.5 | 238.50 | 419 | 100.0 % |
| pbil-direct       | 2   | 127.00 | 186.5 | 265.00 | 461 | 100.0 % |
| mimic-direct      | 1   | 136.75 | 209.5 | 244.25 | 418 | 100.0 % |
| ea-10p1-direct    | 2   | 107.00 | 228.0 | 365.50 | 1,151 | 100.0 % |
| ea-10p1-linear    | 4   | 107.00 | 269.0 | 430.00 | 1,791 | 100.0 % |
| pbil-linear       | 13  | 209.00 | 307.0 | 658.75 | 1,858 | 100.0 % |
| umda-linear       | 1   | 104.25 | 348.0 | 715.00 | 1,566 | 100.0 % |
| ltga-linear       | 2   | 135.00 | 352.5 | 524.50 | 610 | 100.0 % |
| mimic-linear      | 2   | 130.50 | 387.5 | 686.25 | 1,715 | 100.0 % |
| ltga-direct       | 5   | 140.75 | 443.5 | 555.25 | 840 | 100.0 % |
| sa-direct         | 13  | 39.50 | 102.0 | 243.75 | 845 | 99.0 % |

missing digits. Tab. 4 gives the results for a particular medium dimension instance (small-dimension-8-a) with $r = 8$ missing digits. Representation can have a significant impact on maximum runtime, as exemplified in this table by UMDA, MIMIC, HC, and (1 + 1) EA.

Tab. 5 shows the rank distributions of metaheuristics on all instances. All unsuccessful metaheuristic-representation pairs but one use direct representation. Only SA has been unable to succeed with either representation. For all metaheuristics but GA, linear representation dominates direct representation. Tab. 6 shows the rank distributions of metaheuristics on low dimension instances. Only SA with direct representation has been unable to succeed. For all metaheuristics but GA, PBIL, UMDA, and MIMIC, linear represen-
| Algorithm       | Number of evaluations | Success |
|-----------------|-----------------------|---------|
|                 | min       | $Q_1$     | med.    | $Q_3$     | max       |         |
| rls-linear      | 46        | 154.00    | 259.0   | 815.50    | 4,217     | 100.0%  |
| ea-1p1-linear   | 50        | 169.00    | 266.0   | 536.75    | 19,447    | 100.0%  |
| rls-direct      | 38        | 162.75    | 403.0   | 834.25    | 3,920     | 100.0%  |
| p3-linear       | 141       | 473.00    | 709.5   | 992.25    | 1,979     | 100.0%  |
| p3-direct       | 275       | 513.50    | 782.5   | 1,092.25  | 2,095     | 100.0%  |
| mimic-direct    | 432       | 801.25    | 861.5   | 1,033.25  | 16,391    | 100.0%  |
| hc-linear       | 532       | 757.00    | 876.0   | 1,657.50  | 4,481     | 100.0%  |
| ltga-linear     | 567       | 641.75    | 920.0   | 1,112.00  | 2,218     | 100.0%  |
| ga-direct       | 302       | 813.25    | 994.0   | 1,277.25  | 2,010     | 100.0%  |
| ga-linear       | 510       | 962.25    | 1,228.0 | 1,451.75  | 2,956     | 100.0%  |
| ea-10p1-linear  | 432       | 955.75    | 1,569.0 | 2,383.00  | 5,572     | 100.0%  |
| ea-10p1-direct  | 422       | 1,078.00  | 1,696.5 | 2,554.00  | 8,081     | 100.0%  |
| umda-linear     | 1,351     | 1,770.50  | 1,948.5 | 2,171.00  | 20,398    | 100.0%  |
| mimic-linear    | 2,519     | 3,663.50  | 4,139.5 | 4,701.25  | 25,444    | 100.0%  |
| ltga-direct     | 608       | 2,047.75  | 4,151.5 | 6,835.50  | 21,157    | 100.0%  |
| pbil-linear     | 3,567     | 4,823.25  | 5,160.5 | 5,647.25  | 9,616     | 100.0%  |
| hc-direct       | 920       | 9,232.75  | 22,709.5| 39,932.50 | 166,010   | 100.0%  |
| umda-direct     | 213       | 519.75    | 574.5   | 655.25    | 922,799   | 98.0%   |
| sa-direct       | 167       | 4,675.75  | 6,833.5 | 8,244.50  | 12,039    | 98.0%   |
| sa-linear       | 101       | 4,082.50  | 6,878.0 | 8,872.00  | 13,342    | 98.0%   |
| pbil-direct     | 746       | 905.50    | 1,029.0 | 1,174.75  | 3,315     | 94.0%   |
| ea-1p1-direct   | 55        | 188.25    | 317.0   | 715.75    | 949,772   | 90.0%   |

Table 4: Runtime statistics in fixed-target experiments (instance small-dimension-8-a, 100 runs).
| Algorithm        | Rank | Success |
|------------------|------|---------|
| rls-linear       | 1    | 100.0%  |
| ea-1p1-linear    | 1    | 100.0%  |
| rls-direct       | 1    | 100.0%  |
| p3-linear        | 4    | 100.0%  |
| hc-linear        | 1    | 100.0%  |
| p3-direct        | 4    | 100.0%  |
| ltga-linear      | 3    | 100.0%  |
| ga-direct        | 4    | 100.0%  |
| ga-linear        | 3    | 100.0%  |
| ea-10p1-linear   | 10   | 100.0%  |
| ea-10p1-direct   | 3    | 100.0%  |
| umda-linear      | 10   | 100.0%  |
| ltga-direct      | 3    | 100.0%  |
| mimic-linear     | 7    | 100.0%  |
| pbil-linear      | 8    | 100.0%  |
| mimic-direct     | 4    | 99.0%   |
| sa-linear        | 2    | 98.0%   |
| hc-direct        | 2    | 98.0%   |
| pbil-direct      | 5    | 98.0%   |
| sa-direct        | 1    | 97.0%   |
| umda-direct      | 5    | 97.0%   |
| ea-1p1-direct    | 2    | 96.0%   |

Table 5: Rank statistics (runtime based) of metaheuristics on all instances.

...tation dominates direct representation. Tab. 7 shows the rank distributions of metaheuristics on medium instances. The unsuccessful metaheuristics are almost the same as for the full set of instances. The difference with the full set of instances is that, in this experiment, MIMIC has been successful. For all metaheuristics but GA and MIMIC, linear representation dominates direct representation.

5 Conclusion

We have proposed a linear representation for categorical values in binary domains. It mostly targets evolutionary algorithms and other metaheuristics expressed in terms of binary domains. It preserves the neighborhood rela-
| Algorithm          | Rank | Q₁    | med. | Q₃    | max | Success |
|--------------------|------|-------|------|-------|-----|---------|
| rls-linear         | 1    | 1.00  | 1.0  | 2.50  | 19  | 100.0%  |
| ea-1p1-linear      | 2    | 3.00  | 3.0  | 4.00  | 17  | 100.0%  |
| ea-1p1-direct      | 1    | 3.00  | 4.0  | 4.00  | 21  | 100.0%  |
| hc-linear          | 1    | 5.00  | 5.5  | 7.00  | 12  | 100.0%  |
| p3-linear          | 5    | 5.75  | 6.5  | 14.50 | 22  | 100.0%  |
| p3-direct          | 6    | 7.00  | 8.0  | 10.00 | 22  | 100.0%  |
| umda-direct        | 4    | 7.00  | 8.0  | 12.00 | 19  | 100.0%  |
| mimic-direct       | 9    | 9.00  | 11.0 | 14.25 | 20  | 100.0%  |
| ga-direct          | 9    | 10.00 | 11.0 | 14.50 | 21  | 100.0%  |
| sa-linear          | 2    | 6.00  | 11.5 | 18.00 | 22  | 100.0%  |
| pbil-direct        | 3    | 11.00 | 12.0 | 12.25 | 15  | 100.0%  |
| ga-linear          | 7    | 11.00 | 12.0 | 13.00 | 19  | 100.0%  |
| hc-direct          | 2    | 8.75  | 13.0 | 15.50 | 22  | 100.0%  |
| ea-10p1-linear     | 2    | 13.00 | 14.0 | 15.25 | 21  | 100.0%  |
| ea-10p1-direct     | 3    | 14.00 | 16.0 | 17.00 | 21  | 100.0%  |
| ltga-linear        | 7    | 12.75 | 16.5 | 18.00 | 20  | 100.0%  |
| ltga-direct        | 1    | 15.25 | 17.0 | 18.00 | 22  | 100.0%  |
| umda-linear        | 8    | 15.25 | 18.0 | 20.00 | 21  | 100.0%  |
| mimic-linear       | 6    | 17.25 | 20.0 | 21.00 | 22  | 100.0%  |
| pbil-linear        | 1    | 14.25 | 20.5 | 21.00 | 22  | 100.0%  |
| sa-direct          | 5    | 15.75 | 19.0 | 22.00 | 22  | 99.0%   |

Table 6: Rank statistics (runtime based) of metaheuristics on low dimension instances.
| Algorithm      | Rank | min  | Q₁ | med. | Q₃ | max  | Success |
|----------------|------|------|----|------|----|------|---------|
| rls-linear     | 1    | 1.00 | 1.0| 1.00 | 2  | 100.0% |
| ea-1p1-linear  | 1    | 2.00 | 2.0| 2.00 | 3  | 100.0% |
| rls-direct     | 2    | 3.00 | 3.0| 4.00 | 9  | 100.0% |
| p3-linear      | 4    | 5.00 | 5.0| 6.00 | 10 | 100.0% |
| ltga-linear    | 3    | 5.00 | 6.0| 7.25 | 12 | 100.0% |
| p3-direct      | 4    | 5.00 | 6.5| 8.00 | 8  | 100.0% |
| hc-linear      | 3    | 6.00 | 8.0| 9.25 | 12 | 100.0% |
| mimic-direct   | 4    | 7.00 | 8.0| 9.00 | 11 | 100.0% |
| ga-direct      | 6    | 9.00 | 9.0| 10.00| 12 | 100.0% |
| ga-linear      | 8    | 10.00| 10.0| 13.00| 13 | 100.0% |
| ea-10p1-linear | 11   | 11.00| 12.0| 14.25| 15 | 100.0% |
| umda-linear    | 10   | 11.75| 13.0| 15.25| 17 | 100.0% |
| ea-10p1-direct | 11   | 13.00| 13.0| 14.00| 16 | 100.0% |
| mimic-linear   | 14   | 14.00| 15.0| 17.00| 19 | 100.0% |
| ltga-direct    | 14   | 15.00| 16.0| 16.00| 19 | 100.0% |
| pbil-linear    | 15   | 15.00| 16.0| 18.00| 20 | 100.0% |
| sa-linear      | 17   | 18.00| 19.0| 20.25| 22 | 98.0%  |
| hc-direct      | 17   | 17.00| 19.5| 21.00| 22 | 96.0%  |
| sa-direct      | 18   | 18.75| 21.0| 21.25| 22 | 96.0%  |
| pbil-direct    | 9    | 12.00| 20.0| 20.00| 22 | 96.0%  |
| umda-direct    | 4    | 6.75 | 19.0| 21.00| 21 | 95.0%  |
| ea-1p1-direct  | 3    | 4.00 | 18.5| 22.00| 22 | 92.0%  |

Table 7: Rank statistics (runtime based) of metaheuristics on medium dimension instances.
tions between categorical values. Every value can be reached with a single mutation. This requirement has, in turn, lead to an unexpected connexion with coding theory.

Linear representation has been paired with 11 standard metaheuristics and applied to Sudoku puzzles. In fixed-budget experiments and empirical cumulative distribution functions, high dimension instances have been used to rank metaheuristic-representation pairs according to the quality of their solutions. Linear representation has shown a clear advantage over direct representation with all metaheuristics but MIMIC, LTGA, and P3. Only in the case of P3 has direct representation overtaken linear representation by a significant margin within the considered budget. In fixed-target experiments, small dimension instances have been used to rank metaheuristic-representation pairs according to runtime. Linear representation has surpassed direct representation with all metaheuristics but GA and MIMIC.

One drawback of linear representation is its size, which is linear in the number of categorical values but exponential in the size of direct representation. This could explain some of the negative experimental results as an increased size of the search space usually implies a degraded performance in terms of runtime or quality of solutions.

The results in this paper have to be confirmed in the context of other problems, preferably real-world ones. The influence of the number of categories and the number of categorical variables on the performance of metaheuristic-representation pairs are of particular interest.

References

[1] Shumeet Baluja and Rich Caruana. Removing the genetics from the standard genetic algorithm. In Armand Prieditis and Stuart Russell, editors, *Proc. of the 12th Annual Conf. on Machine Learning*, pages 38–46. Morgan Kaufmann, 1995.

[2] J. S. De Bonet, C. L. Isbell, and P. Viola. MIMIC: finding optima by estimating probability densities. In *Advances in Neural Information Processing Systems*, volume 9. MIT Press, Denver, 1996.

[3] Brian W. Goldman and William F. Punch. Fast and efficient black box optimization using the parameter-less population pyramid. *Evolutionary Computation*, 23(3):451–479, 2015.

[4] R. W. Hamming. Error detecting and error correcting codes. *The Bell System Technical Journal*, 29(2):147–160, 1950.
[5] Nikolaus Hansen, Anne Auger, Dimo Brockhoff, Dejan Tusar, and Tea Tusar. COCO: performance assessment. *CoRR*, abs/1605.03560, 2016.

[6] HNCO. [https://github.com/courros/hnco](https://github.com/courros/hnco) v0.16.

[7] John H. Holland. *Adaptation in natural and artificial systems*. University of Michigan Press, Ann Arbor, 1975.

[8] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.

[9] Heinz Mühlenbein. The equation for response to selection and its use for prediction. *Evolutionary Computation*, 5(3):303–346, 1997.

[10] Stephen Ostermiller. Qqwing. [https://qqwing.com/](https://qqwing.com/) Accessed: 2020-10-20.

[11] Vera Pless. *Introduction to the Theory of Error-Correcting Codes*. John Wiley & Sons, Ltd, 1989.

[12] Franz Rothlauf. *Representations for genetic and evolutionary algorithms*. Springer, 2006.

[13] Gordon Royle. Minimum sudoku. [https://web.archive.org/web/20131019184812if_/http://school.maths.uwa.edu.au/~gordon/sudokumin.php](https://web.archive.org/web/20131019184812if_/http://school.maths.uwa.edu.au/~gordon/sudokumin.php) Accessed: 2020-10-20.

[14] Dirk Thierens. The linkage tree genetic algorithm. In Robert Schaefer, Carlos Cotta, Joanna Kołodziej, and Günter Rudolph, editors, *Parallel Problem Solving from Nature, PPSN XI*, pages 264–273, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.