Dedicated to Eric Opdam
on the occasion of his 60th birthday

INTEGRAL FORMULAS FOR
DAHA INNER PRODUCTS

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Abstract. The main aim is to obtain integral formulas for DAHA coinvariants and the corresponding inner products for any values of the DAHA parameters. In the compact case, our approach is similar to the procedure of “picking up residues” due to Arthur, Heckman, Opdam and others; the resulting formula is a sum of integrals over double affine residual subtori. A single real integral provides the required formula in the noncompact case. As $q$ tends to 0, our integral formulas result in the trace formulas for the corresponding AHA, which calculate the Plancherel measures for the spherical parts of the regular AHA modules. The paper contains a systematic theory of DAHA coinvariants, including various results on the affine symmetrizers and induced DAHA modules.

Key words: Affine Hecke algebras; spherical functions; residual subtori; Plancherel measure; trace formulas; nonsymmetric Macdonald polynomials.
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Contents

1. Introduction
2. Affine root systems and AHA
3. Basic DAHA theory
4. Polynomial representation
5. Induced modules
6. Residues and closed subsystems
7. Residual subtori and points
8. Integral presentations
9. Meromorphic continuations
10. Pole expansion for $A_n$
11. Integral formulas for $A_2$
References

1. Introduction

This paper is partially based on the talk by the first author at the conference “From E6 to double affine E60” in the honor of Eric Opdam’s 60th birthday. The main aim is to obtain integral formulas for

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DAHA coinvariants and the corresponding inner products for any values of the DAHA parameter $t = q^k$, where $0 < q < 1$. As $q \to 0$ and upon the restriction to symmetric functions, our integral formula results in the trace formula for the corresponding Affine Hecke Algebra, AHA for short. This formula calculates the Plancherel measure for the decomposition of the spherical part of the regular representation of AHA in terms of irreducible unitary modules. The standard AHA trace is the limit $q \to 0$ of the DAHA coinvariant for the anti-involution $\diamond$ associated with the basic inner product in the polynomial representation.

There are two directions of this paper: algebraic theory of DAHA coinvariants, including the affine symmetrizers and norm-formulas, and integral formulas for DAHA coinvariants and inner products in the compact and noncompact settings. The corresponding identities are DAHA trace formulas. The integral formulas in the compact case are obtained in a way similar to “picking up residues” due to Arthur, Heckman, Opdam and others (can be traced back to Hermann Weyl); they are sums over double affine residual subtori. The DAHA-invariance of our formulas is an important new tool, but the combinatorial aspects are involved so far in the $q,t$-theory. However, a single real integral provides the required meromorphic continuation in the noncompact case.

A challenge here is to upgrade this approach to global fields: with the $c$-functions expressed via the completed Dedekind zeta-function: Kazhdan-Okounkov [KO] and De Martino-Heiermann-Opdam [DHO]. The trace formula becomes Langlands’ formula for the inner product of two pseudo-Eisenstein series, $(\theta_\phi, \theta_\psi)$ in the notation from [DHO]; see there for the definitions and justifications. One of the key points of these two papers is that Dedekind’s zeta can be replaced by other functions satisfying the functional equation (to ensure the cancelation of the “unwanted” residues). We expect that the adelic product of DAHA trace formulas can serve global fields, where adding $q$ provides new and interesting deformations of Langlands’ formulas.

Our starting point is that the integral formulas for the level-zero and level-one coinvariants are relatively straightforward for $|t| < 1$ ($\Re k > 0$). They generalize the Macdonald formula in the AHA theory for $|t| > 1$ outlined in Section 2; the DAHA $t$-parameter used throughout this paper corresponds to $1/t$ in the standard AHA setting.

These integrals are essentially $\int f(x) \mu(q^x; q, t) dx$ for suitable spaces of functions integrated over $i\mathbb{R}$ in the compact case and $\mathbb{R}$ in the noncompact case, where $\mu$ is the measure-function in DAHA theory defined in (4.18). The problem is to extend them to $|t| > 1$.

All basic DAHA facts and references we need can be found in this paper. We frequently adjust them, generalize and develop. See [Ch1, Ch2, ChM, ChD, Ch3] for the main features of DAHA inner products.
and coinvariants. One of the changes vs. [Ch1] is that we use the anti-involution $\diamond$ that does not involve the conjugation of $q, t$.

Beyond the DAHA theory, only “$q$-calculus” and standard theory of residues is really necessary to obtain our integral formulas, though they appeared involved. This is similar to [HO1, O1].

**Meromorphic continuations.** In this paper, we mostly consider the spaces of Laurent polynomials or Laurent series $f(q^x)$, which are in terms of $q^{x\alpha_i}$ for $x_{\alpha_i} = (x, \alpha_i)$ for simple roots $\alpha_i$. The integration is mostly over the (imaginary) periods of $q^{x\alpha_i}$; however, the full imaginary integration, real and Jackson integrations play a significant role too.

Generally, the problem is to find the meromorphic continuation of the imaginary integrals to $\Re k \leq 0$ ($|t| > 1$). Interestingly, a single integral provides the required meromorphic function for all $k$ (with sufficiently small $\Im k$) in the noncompact case: for the integrations in the real directions. This is Theorem 8.1; “picking up residues” is not needed there. We note that the reciprocals of theta-functions and their expansions occur naturally in this case when the Gaussians are added to the space of functions. See e.g. [Car]. The corresponding $q, t$-Gauss integrals, noncompact variants of difference Macdonald-Mehta formulas from [Ch2], involve Appell functions and similar ones.

In the compact case, the integrals are analytic in terms of $k$ with $\Re k$ from a disconnected union of segments between the consecutive singularities of $\mu$. The problem is to use these integrals to obtain the meromorphic continuation of the initial integral, the one serving $\Re k > 0$, to all $0 \geq \Re k > -\infty$.

The resulting formula is a linear combination of finitely many integrals over double affine residual subtori, where their number depends on $\Re k$. See Theorem 9.1. The contribution of double affine residual points is very interesting; the corresponding residues generalize formal degrees of the AHA discrete series. Concerning the latter, let us mention here (at least) Kazhdan, Lusztig, Reeder, Shoji, Opdam, Ciubotaru, S.Kato; also, see some references below.

Similar to the AHA theory, the leading term of the resulting integral formula is $\int_{R^n} f(x)\mu(q^x)dx$, where $\Re k$ is arbitrary negative. This functional is AHA-invariant but not DAHA-invariant; so “corrections” are needed, which are integrals over residual subtori. It is expected that the DAHA-invariance of our formula is sufficient to fix uniquely the corresponding “measures” of residual subtori and those in the AHA limit $q \to 0$. So the action of DAHA is a major “hidden symmetry” of the AHA Plancherel formula, which is of conceptual importance.

The meromorphic continuation is basically by shifting the contours of integration in the real directions followed by “picking up the residues”. We need the analyticity of $f(x)$ to ensure that the contours can be
moved and the integrability in the imaginary directions. When $k \to 0_{-}$, the link to the procedure from [HO1] is discussed in (a) from “Concluding remarks” after Theorem 9.1. We note that the pole decomposition is the key to our approach in some contrast to AHA.

**DAHA aspects.** Our integral formulas are not directly related to the reducibility of the polynomial $\mathcal{H}$-module $\mathcal{X}$, where $\mathcal{H}$ denotes DAHA. The reducibility is for singular $t$, some special $t$ satisfying $|t| > 1$; our formulas are for any $|t| > 1$. However, there is an important connection. When the coefficients (the residues) in our formula have singularities in terms of $k$, our integral formulas for the inner product result in a certain Jantzen-type filtration of $\mathcal{X}$ in terms of $\mathcal{H}$-modules. Namely, the largest submodule is the radical of the leading term of the inner product, the 2nd is the radical of the restriction of our integral formula to the 1st and so on. For $A_n$, this filtration is essentially sufficient to decompose the polynomial representation (at least for small $n$). See e.g. [En, Ch4] about the so-called Kasatani decomposition.

Moreover, the subquotients here are naturally supplied with inner products, given by some integrals, that can be unitary even if $\mathcal{X}$ is not unitary; see [Ch3], Corollary 6.3 for an example. This is always very interesting. Generally, the problem of unitary dual is one of the keys in harmonic analysis; see e.g. [ES] for the case of rational DAHA.

In contrast to the trace of AHA, it is not immediate to see that the DAHA coinvariants are meromorphic functions in terms of $t$. This fact can be proven via (a) the theory of nonsymmetric Macdonald polynomials, (b) the theory of basic anti-involutions or (c) the theory of affine symmetrizers.

The existence of the affine symmetrizer $\widehat{P}_+(f)$ and its proportionality to $\widehat{F}_+(f)$ from Theorem 4.5 seem the most fundamental here. The origin is in the $p$-adic theory of spherical functions. Basically, we generalize the fact that Matsumoto spherical functions can be identified with nonsymmetric Hall polynomials in the AHA theory.

We extend in this paper the theory of basic anti-involutions and coinvariants to $Y$-induced DAHA modules $\mathcal{L}_\xi$, where $\xi \in \mathbb{C}^n$ is considered as a character of the $Y$-subalgebra of $\mathcal{H}$. For instance, Theorem 5.1 gives the norm-formulas for such representations and simultaneously proves the uniqueness of the corresponding coinvariant up to proportionality for sufficiently general $q, t, \xi$.

Modules $\mathcal{L}_\xi$ are important in this paper because of several reasons. First, they are related to residual points $\xi$; the irreducible quotients of $\mathcal{L}_\xi$ for “non-Steinberg” $\xi$ are interesting analogs of $\mathcal{X}$. Second, generic $\mathcal{L}_\xi$ can be naturally identified with the full regular representation of AHA, the main subject of the AHA harmonic analysis. One can define
the integration and obtain integral formulas for $I_\xi$, but this is beyond the present paper. This is related to Jackson integrals $J_\xi$ and global hypergeometric functions from [Ch2]; see also [Sto, SSV].

Some perspectives. See also “Concluding remarks” after Theorem 9.1. The decomposition of the regular AHA representation in terms of irreducible modules involves deep geometric methods (Kazhdan-Lusztig and others) and a lot of functional analysis (Opdam and others). Our approach potentially allows finding the formal degrees of AHA discrete series via DAHA without any geometry. Paper [O1] does this within the AHA theory. The DAHA approach is expected to be analytically more transparent and with additional rich symmetries, which are not present in AHA. The $q,t$-generalization of the discrete series remains to be discovered. Actually, the whole $X$ behaves as such for sufficiently large $\Re k < 0$; the affine symmetrizer $\hat{\mathcal{P}}$ acts there, which is an important feature of AHA representations of discrete series.

As we already discussed, $X$ is the spherical quotient of the regular AHA representation supplied with the structure of $\mathcal{H}$-module, $I_\xi$ are those for the whole regular representation. The classical AHA trace becomes the basic $\mathcal{H}$-coinvariant. The presentation of the trace as some integral over unitary dual is reduced to some combinatorial calculations for $\mathcal{H}$. They are far from simple but no DAHA unitary dual is needed.

In the case of $A_n$, we calculate explicitly in Theorem 10.1 the required meromorphic continuation to $|t| > 1$ as the pole decomposition of the “slightly shifted” initial integral. This can be generalized to any root systems and any orderings of iterated integrations, but the combinatorics of the resulting formulas requires further analysis. Moreover, non-Steinberg-type residual points occur beyond $A_n$.

The pole decompositions we obtain converge at any $\Re k < 0$, but only for relatively small spaces of $f(x)$ depending on $\Re k$. Such $f$ are basically Laurent polynomials of degrees bounded by $\text{const}[-\Re k]$ or the corresponding Paley-Wiener functions. Practically arbitrary analytic functions $f(x)$ can be considered when we switch to finite sums of integrals over certain double affine residual subtori.

This passage is a combinatorial problem, but not a simple one. Essentially, we need to combine the poles into families corresponding to proper residual subtori. We provide the final finite integral formulas only for $A_2$ in Section 11; see [Ch3] for the case of $A_1$. For arbitrary root systems, the calculations are involved even in the simplest interval $0 > \Re k > -\frac{1}{h}$ for the Coxeter number $h$, where the combinatorics of residual subtori is similar to that from [HO1, O1].

A natural challenge here is the case of nonequal parameters $t_{\text{sht}}$ and $t_{\text{ling}}$ for the root systems $BCFG$, i.e. for generic $k_{\text{sht}}$ and $k_{\text{ling}}$. All
main results in this paper are for any sufficiently general \( k \). For instance, the pole decomposition is obtained for any \( \Re k_{\text{sht}} < 0 \) and \( \Re k_{\text{lng}} < 0 \). However, the explicit combinatorial description of residual points is provided only when \( k_{\text{lng}} = \kappa k_{\text{sht}} \), where \( \kappa = 1 \) or \( \kappa = (\alpha_{\text{lng}}, \alpha_{\text{lng}})/(\alpha_{\text{sht}}, \alpha_{\text{sht}}) \). See Theorems 6.1 and 7.2 in terms of the closed root subsystems of maximal rank in affine root systems.

The harmonic analysis and unitary dual for DAHA are open projects by now. However, there are quite a few special theories, where this paper can be used as such. They are (a) the AHA limit as \( q \to 0 \) (the starting point for us), (b) the Kac-Moody limit as \( t \to \infty (0 < q < 1, k \to -\infty) \), (c) the quantum groups as \( t = q \), (d) level-one Demazure characters as \( t = 0 \), and (e) the Heckman-Opdam limit \([\text{HO2}]\): \( q \to 1, t = q^k \).

Case (d) and the limit \( t \to \infty \) correspond to the theory of nil-DAHA; see \([\text{ChO}, \text{ChK}]\). In the case of (e), the variables \( X_b = q^{x_b} \) for \( b \in P \) are considered unchanged in the limit (they occur as torus coordinates). For (c), there are actually two quantum group limits in the twisted case: when \( t_{\text{sht}} = q \) and \( t_{\text{lng}} = t_{\text{sht}}^\kappa \) for \( \kappa \) as above.

The simplest “special theory” is actually for \( t = 1 \); then DAHA becomes the Weyl algebra. It generalizes the main feature of the latter, the projective action of \( PSL_2(\mathbb{Z}) \). It is the key feature of DAHA theory, which collapses in the limits above unless in the following two cases.

First, this action exists for the reduced category in case (c) when \( q \) is a root of unity and, equivalently (due to Kazhdan-Lusztig and Finkelberg), for the category of integrable Kac-Moody modules in case (b). The Grothendieck ring of the reduced category becomes then the perfect representation of DAHA at \( t = q \).

The second case is the action of \( PSL_2(\mathbb{C}) \) in the rational Heckman-Opdam theory (with the Calogero operators instead of the Sutherland ones in physics literature). This is the limit \( q \to 1, t = q^k \), where \( x_b \) above are taken as the variables. The Fourier transform, which is the action of \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) becomes the non-symmetric Hankel transform (due to Dunkl for any root systems and due to Hermite for \( A_1 \)). When \( k = 0 \), we arrive at the Heisenberg algebra.

We note that the usage of Lie groups only is generally insufficient to incorporate the Fourier transform; one needs the Heisenberg-Weyl algebras and DAHA, their (flat) deformations. Similar to the classical polynomial representations for Heisenberg-Weyl algebras, DAHA provides nonsymmetric theories, which were new even for \( A_1 \). The nonsymmetric Macdonald polynomials generalize the characters and spherical functions in the Lie theory, which are symmetric (unless for Demazure characters). Our paper is “nonsymmetric”.
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2. Affine root systems and AHA

Let $R \subset \mathbb{R}^n$ be a reduced irreducible (indecomposable) root system, $Q = \oplus_{i=1}^{n} \mathbb{Z} \alpha_i, P = \oplus_{i=1}^{n} \mathbb{Z} \omega_i$, where $\alpha_i$ are simple roots and $\{\omega_i\}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the coroots $\alpha_j^\vee = 2\alpha_j / (\alpha_j, \alpha_j)$. Replacing $\mathbb{Z}$ by $\mathbb{Z}^\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$, we obtain $P^\pm$ and $Q^\pm$. See e.g., [B] or [Ch1]. The normalization will be twisted throughout this paper: $(\alpha_{sht}, \alpha_{sht}) = 2$ for short roots. Accordingly, $\vartheta = \vartheta(R_+)$ will denote the maximal short root in $R_+$, the set of positive roots. When necessary, we use the notation $\theta = \theta(R_+)$ for the maximal (long) root. One has $\vartheta(R_+^\vee) = \theta(R_+^\vee)$ due to our normalization of $(\cdot, \cdot)$, which means that $\vartheta$ becomes the maximal root in $R_+^\vee = \{\alpha^\vee, \alpha \in R\}$.

Setting $\nu_{\alpha} = (\alpha, \alpha)/2$, the vectors $\tilde{\alpha} = [\alpha, \nu_{\alpha}] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the twisted affine root system $\tilde{R} \supset R$, where $\alpha \in R$ are considered as $[\alpha, 0]$. We will sometimes use the notation $\nu_{sht}$ and $\nu_{lng}$ for short and long roots.

The inner product $(\tilde{\alpha}, \tilde{\beta})$ is that from $\mathbb{R}^n$, i.e. the affine components are ignored. However, somewhat abusing the notation, we set $(\tilde{\alpha}, z) = (\alpha, z) + \nu_{\alpha}j$, when the pairing is between $\mathbb{R}^n \ni z$ and $\tilde{\alpha}$ is considered, which will be obvious from the context. In [Ch1], we used the notation $(\tilde{\alpha}, z + d)$ for this pairing.

We add $\alpha_0 \overset{\text{def}}{=} [-\vartheta, 1]$ to the simple roots. The corresponding set $\tilde{R}_+$ of positive roots is $R_+ \cup \{[\alpha, \nu_{\alpha}j], \alpha \in R, j > 0\}$. The corresponding affine (extended) Dynkin diagram will be the usual extended one for $R_+^\vee$ where all arrows are reversed.

Note that $P \subset P^\vee$ and $Q \subset Q^\vee$ for $P^\vee, Q^\vee$ defined for $R_+^\vee$. The minuscule weights are $\omega_r$ such that $(\omega_r, \alpha^\vee) \leq 1$ for any $\alpha \in R_+$. Equivalently, $\nu_r n_r = 1$ where $\vartheta = \sum_{i=1}^{n} n_i \alpha_i$. The usage of the name “twisted” is not as in Kac-Moody theory, but there is a direct connection for the systems $B, C, F, G$.

The twisted setup is convenient for us because it is “self-dual” with respect to the DAHA Fourier transform. Also, the “level-one theory” for the $C$-type in the untwisted setting is actually “level-two”, much more difficult than “level-one” is supposed to be. There are other advantages, but the untwisted root systems are generally equally important and quite standard in Kac-Moody theory.

The set of the indices of the images of $\alpha_0$ under the action of automorphisms of the affine Dynkin diagram will be denoted by $O$. Let $O' \overset{\text{def}}{=} \{r \in O, r \neq 0\}$. The minuscule $\omega_r$ are those for $r \in O'$. We set
\[ \omega_0 = 0 \] for the sake of uniformity. All fundamental weights are minuscule for \( A_n \). There are no minuscule weights and \( O' = \emptyset \) for \( E_7,8, F_4, G_2 \).

**Affine Weyl group.** Given \( \tilde{\alpha} = [\alpha, \nu, \eta] \in \tilde{R}, \ b \in P \), let
\[ s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \ \ b'(\tilde{z}) = [z, \zeta - (z, b)] \]
for \( \tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1} \).

The **affine Weyl group** \( \hat{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle \) is the semidirect product \( W = \hat{R} \rtimes Q \) of its subgroups \( W = \langle s_{\alpha}, \alpha \in R_+ \rangle \) and \( Q \), where \( \alpha \in R \subset Q \) are identified with the following elements from \( \hat{W} \):
\[ R \ni \alpha \mapsto s_{\alpha} s_{\alpha_\nu} s_{\alpha} = s_{[-\alpha, \nu_\alpha]} s_{\alpha} \in \hat{W}. \]

The **extended Weyl group** \( \hat{W} \) is \( W \rtimes P \), which can be defined via its action in \( \mathbb{R}^{n+1} \) extending that of \( \hat{W} \) in \( \hat{R} \):
\[ (wb)([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for} \quad w \in \hat{W}, b \in P, z \in \mathbb{C}^n. \]

Notice the minus-sign of \( -(z, b) \).

We need the action of \( \hat{W} \ni \hat{w} \) on the functions \( X_{[a, \lambda]} \) defined as the **action on the indices**: \( \hat{w}(X_{[a, \lambda]}) = X_{\hat{w}[a, \lambda]} \). Generally, \( \hat{w}(f(x)) = f(\hat{w}^{-1}(x)) \) for any function of \( x \). This action is dual to the following **affine action** on vectors \( z \in \mathbb{C}^n \): \( \hat{w}(z) = w(z) + b \) for \( \hat{w} = bw \). The corresponding extension of the pairing \( \langle \cdot, \cdot \rangle \) is \( \langle z, [a, \lambda] \rangle = (z, a) + \lambda \). Namely, one has:
\[ wb(x_a) = (x, wb(a)) = (x, [w(a), -(b, a)]) = (x, w(a)) - (b, a) = (w^{-1}(x) - b, a) = (b^{-1}w^{-1}(x), a) = wb(x_a), \]
where the former \( wb(x_a) \) is the action on indices, the latter \( wb(x_a) \) is the action on functions of \( x \). We will use the notation \( \langle \cdot, \cdot \rangle \) only when some confusion is likely; almost always, \( \hat{w}(\cdot) \) will denote either the action on \( [a, \lambda] \in \mathbb{R}^{n+1} \) or on \( z \in \mathbb{C}^n \) depending on the context. Throughout the paper: \( X_a = q^{x_a}, X_{[a, \lambda]} = q^\lambda X_a \), and we set \( X_a(q^b) = q^{(a, b)} \) for \( X_a \) and other functions of \( \{X_a\} \).

The **Gaussian** \( q^{x^2/2} \) is defined for \( x^2 = \sum x_{\alpha_i} x_{\omega_i} \); it is \( W \)-invariant, and \( b(q^{x^2/2}) = q^{b_2/2}X_b^{-1}q^{x^2/2} \). It is sometimes used as a symbol, when only the action of \( \hat{W} \) on it is of importance. However, \( q^{\pm x^2/2} \) will be considered functions for real and imaginary integrations.

The group \( \hat{W} \) is isomorphic to \( \hat{W} \rtimes \Pi \) for \( \Pi \) defined \( P/Q \). The latter group consists of \( \pi_0 = \text{id} \) and the images \( \pi_r \) of minuscule \( \omega_r \) in \( P/Q \); also, see (2.4). We note that \( \pi^{-1} = \pi_\varsigma \), where \( \varsigma \) is the standard involution (sometimes trivial) of the **nonaffine** Dynkin diagram induced by \( \alpha_i \mapsto -w_0(\alpha_i) \), where \( w_0 \) is the longest element in \( W \). Generally \( \varsigma(b) = -w_0(b) = b^\varsigma \); we set \( X_b^\varsigma = X_{b^\varsigma} \).
The group $\Pi$ is naturally identified with the subgroup of $\widehat{W}$ of the elements of zero length; the length is defined as follows:

\[(2.3) \quad l(\hat{w}) = |\Lambda(\hat{w})| \quad \text{for} \quad \Lambda(\hat{w}) \overset{\text{def}}{=} \tilde{R}_+ \cap \hat{w}^{-1}(-\tilde{R}_+).\]

I.e. $\Lambda(\hat{w})$ is the set of positive affine roots that become negative upon the application of $\hat{w}$. Similarly, let $l_\nu$ be the number of $\alpha$ in $\Lambda(\hat{w})$ with $\nu_\alpha = \nu$. Setting $\hat{w} = \pi, \hat{w} \in \widehat{W}$ for $\pi_r \in \Pi, \hat{w} \in \widehat{W}$, $l(\hat{w})$ coincides with the length of any reduced decomposition of $\hat{w}$ in terms of the simple reflections $s_i, 0 \leq i \leq n$. Respectively, let $l_\nu$ count the number of $s_i$ for short and long $\alpha_i (i \geq 0)$.

For the sake of completeness, we mention that the equivalence of these two definitions is based on the key property of $\Lambda$-sets:

\[\Lambda(\hat{w}^\nu) = \hat{w}^{-1}(\Lambda(\hat{w})) \cup \Lambda(\hat{w}), \quad \Lambda(\hat{w}^{-1}) = -\hat{w}(\Lambda(\hat{w})).\]

The union here is disjoint if $l(\hat{w}^\nu) = l(\hat{w}) + l(\hat{u})$; generally, the cancelation of any pairs $\{\hat{\alpha}, -\hat{\alpha}\}$ must be performed if they occur in the union. See e.g. [Ch4]. Also, $l(b) = 2(\rho^\vee, b)$ for $b \in P_+$. Here and below $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \rho_{\text{shl}} + \rho_{\text{lng}}$, $\rho^\vee = \frac{1}{2} \sum_{\alpha > 0} \alpha^\vee = \sum_\nu \frac{\rho_\nu}{\nu_\rho} = \rho_{\text{shl}} + \frac{\rho_{\text{lng}}}{\nu_{\rho}}$.

For $b = \omega_i$, $l(\omega_i)$ gives the number of $\alpha \in R_+$ that contain $\alpha_i$.

One has $\omega_r = \pi_r u_r$ for $r \in O'$, where $u_r$ is the element $u \in W$ of minimal length such that $u(\omega_r) \in P_-$, equivalently, $w = w_0 u$ is of maximal length such that $w(\omega_r) \in P_+$. The elements $u_r$ are very explicit. Let $w_0^r$ be the longest element in the subgroup $W_0^r \subset W$ of the elements preserving $\omega_r$; this subgroup is generated by $s_i$ for $0 < i \neq r$. One has:

\[(2.4) \quad u_r = w_0 w_0^r \quad \text{and} \quad (u_r)^{-1} = w_0^r w_0 = u_{r'}, \quad \text{for} \quad r \in O'.\]

For instance, $\omega_1 = \pi_1 s_3 s_2 s_1$, $\omega_2 = \pi_2 s_2 s_1 s_3 s_2$, $\omega_3 = \pi_3 s_1 s_2 s_3$ for $A_3$.

For $B_n$: $\alpha_n$ is a unique short simple root, $\omega_n = \alpha_{n-1} + 2\alpha_n$ is a unique minuscule weight and $\vartheta = \alpha_1 + \cdots + \alpha_n$. Also, $\omega_n = \pi_n u_n$, where $u_n$ sends $\alpha_i \mapsto -\alpha_{n-i}$ for $1 \leq i \leq n - 1$ and $\alpha_n \mapsto -\vartheta$.

The extended Affine Hecke Algebra for $\tilde{R}$, AHA for short, is defined as the span $H \overset{\text{def}}{=} \langle \Pi, T_i (0 \leq i \leq n) \rangle$ for the generators subject to the standard homogeneous Coxeter relations for $T_i$ and the quadratic relations $(T_i - t_i^\frac{1}{2})(T_i + t_i^{-\frac{1}{2}}) = 0$ for $1 \leq i \leq n$, where $t_i$ depends only on the length of $\alpha_i$, i.e. on $\nu_i = \nu_{\alpha_i}$. The ring of coefficients will be $\mathbb{Z}[t_i^{\pm 1/2}]$ or $\mathbb{C}$ if $t_i$ are considered in $\mathbb{C}^*$. Concerning $\Pi$, the following relations are imposed: $\pi_r T_i \pi_r^{-1} = T_j$ if $\pi_r(\alpha_i) = \alpha_j$ for $r \in O', 0 \leq i \leq n$.

In the standard $p$-adic setting, $t = p^\ell$, where $p^\ell$ is the cardinality of the corresponding residue field $\mathbb{F}$; different $t_{\text{shl}}, t_{\text{lng}}$ are in the so-called case of unequal parameters. The DAHA $t$ is actually $p^{-\ell}$ (below).
We set $T_{\hat{w}} = \pi T_{i_1} \cdots T_{i_l}$ for any reduced decomposition $\hat{w} = \pi s_{i_l} \cdots s_{i_1} \in \hat{W}$, i.e. where $l = l(\hat{w})$. Considering $P$ as a subgroup in $\hat{W}$ we obtain that $Y_b = Y_{\hat{w}}$ for $b \in P_+$ (for dominant weights) are pairwise commutative. Then we extend it to any $b \in P$ using $Y_{b-c} = Y_b Y_c^{-1}$ for $b, c \in P_+$. This is due to Bernstein-Zelevinsky and Lusztig. The defining relations of $\mathcal{H}$ in terms of $Y_b$ are: $T_i Y_b^{-1} = Y_b^{-1} T_i^{-1}$ for $(b, \alpha_i^\vee) = 1, 0 \leq i \leq n$, and $T_i Y_b = Y_b T_i$ for $(b, \alpha_i^\vee) = 0$, where $0 \leq i \leq n$.

The canonical anti-involution, trace and unitary scalar product are:
$$T_{\hat{w}}^\ast \overset{\text{def}}{=} T_{\hat{w}^{-1}}, \quad \langle T_{\hat{w}} \rangle_{\text{reg}} = \delta_{\hat{w}, \hat{w}}, \quad \langle f, g \rangle_{\text{reg}} \overset{\text{def}}{=} \langle f^\ast g \rangle_{\text{reg}} = \sum_{w \in \hat{W}} c_{\hat{w}} d_{\hat{w}},$$
where $f = \sum c_{\hat{w}} T_{\hat{w}}$, $g = \sum d_{\hat{w}} T_{\hat{w}} \in L^2(\mathcal{H}) = \{ f \mid \sum c_{\hat{w}}^2 < \infty \}$. We assume that $t_i$ are real and add the complex conjugation to the definition of $\ast$, which results in $\bar{c}_{\hat{w}}$. The complex conjugation, which is necessary for unitarity, will be omitted in the DAHA theory below.

In the spherical case, we consider $\mathcal{HP}_+$ for the symmetrizer $\mathcal{P}_+ = \sum_{w \in \hat{W}} t^{-l(w)/2} T_w^{-1}$. By definition, $t^l(\hat{w})/2 = \prod_{\nu} t^l_{\nu}(\hat{w})/2 = t^l_{\text{shf}}(\hat{w})/2 t^l_{\text{lng}}(\hat{w})/2$.

This space has a natural left action of $\mathcal{H}$. We have $\mathcal{HP}_+ = \mathbb{C}[Y_b] \mathcal{P}_+$, for the algebra of Laurent polynomials $\mathbb{C}[Y_b, b \in P_+]$, which identification is the key in the theory of non-symmetric Matsumoto spherical functions; see [Mat, O2], [Ch1] (Section 2.11.2) and [Ion, ChM]. For instance, the formulas for $\langle P(Y') \rangle$, where $P(Y') \in \mathbb{C}[Y_b, b \in P_+]$, are sufficient to recover the trace for any $T_{\hat{w}}$.

According to Dixmier, $\langle f, g \rangle_{\text{reg}} = \int_{\pi \in \mathcal{H}^\vee} \text{Tr}(\pi(f^\ast g)) d\eta(\pi)$ for some non-negative measure $d\eta$ in the space $\mathcal{H}^\vee$ of irreducible unitary $h$-modules $\pi$, the unitary dual of $\mathcal{H}$. We omit here some analytic details concerning the classes of functions. In the spherical case (referred to as “sph” below), one takes $f, g \in \mathcal{P}_+ \mathcal{HP}_+$, for $Y_b$ in $\mathcal{P}_+ \mathcal{HP}_+$. It terms of $Y_b$, we consider the symmetric ($W$-invariant) Laurent polynomials, which is based on the so called Bernstein Lemma. The measure reduces correspondingly.

Macdonald found that $\eta_{\text{sph}}(\pi)$ as $t > 1$ (in the case of one $t$) is proportional to $\frac{ds}{c(s,t) c(s^{-1}, t)}$ in terms of the corresponding $c$-function, where $s \in \exp(\mathfrak{h}^{\mathbb{R}^n})$. Its meromorphic extension to $0 < t < 1$ can be obtained by “picking up residues” due to Arthur, Heckman, Opdam and others [CKK, HO1, O1, OS]. The final (spherical) formula reads:
$$\int \{ \cdot \} d\eta^\text{fin}_{\text{sph}}(\pi) = \sum C_{s_0 S} \int \{ \cdot \} d\eta_{s_0 S},$$
summed over affine residual subtori $s_0 S$, where $s_0 S = \exp(\alpha_0 + \mathfrak{a})$ for some $\alpha_0 \in \mathbb{R}^{\mathfrak{a}} \supset \mathfrak{a}$. See [HO1]. Residual points are when $\dim \mathfrak{a} = 0$; they correspond to square integrable irreducible modules: their characters $\chi_\pi$ extend to $L^2(\mathcal{H})$. This formula involves deep geometric representation theory; see [KL, Lu1, Lu2].
The key point for us is that 
\( c(s,t)c(s^{-1},t) \) is a limit \( q \to 0 \) of the corresponding symmetric Macdonald’s measure-function \( \delta(s; q, t) \) upon \( t \mapsto 1/t \). This measure makes the symmetric Macdonald polynomials pairwise orthogonal. We switch to its nonsymmetric variant \( \mu \) in this paper, the measure-function that makes the nonsymmetric Macdonald polynomials pairwise orthogonal.

In the DAHA approach, the problem is to find meromorphic continuations of the DAHA inner products by presenting them as integrals over double affine residual subtori. The main claims are as follows.

The \( q,t \)-generalization of the picking up residues is a presentation of the standard (\( \otimes \)-invariant) inner product in the DAHA polynomial representation as a finite linear combination of integrals over double affine residual subtori, where the measure-function \( \mu \) reduces naturally. This formula provides the meromorphic continuation of the integral formula for this inner product from \(|t| < 1\) for any \(|t| \geq 1\). Its DAHA-invariance and some assumptions about the structure are expected to determine the corresponding coefficients uniquely. Upon taking the limit \( q \to 0 \), we obtain an alternative tool for finding the \( C_{s,S} \)-coefficients for AHA including the formal degrees (for the residual points).

### 3. Basic DAHA theory

Let \( m \) be the least natural number such that \((P, P) = (1/m)\mathbb{Z}\). Thus \( m = |\Pi| \) unless \( m = 2 \) for \( D_{2k} \) and \( m = 1 \) for \( B_{2k}, C_k \).

The double affine Hecke algebra, DAHA, depends on the parameters \( q, t, (\nu \in \{\nu_\alpha, \alpha \in R\}) \) and is defined over the ring \( \mathbb{Z}_{q,t} \) formed by polynomials in terms of \( q^{\pm 1/m}, t^{\pm 1/2} \) and \( \{t_\nu^{\pm 1/2}\} \).

It will be convenient to use \( t_\nu = q^{k_\nu} = q^{\nu k_\nu} \) for \( q_\nu = q' \).

For \( \tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R} \), \( 0 \leq i \leq n \), we set

\[
\tag{3.5}
\tilde{t}_{\tilde{\alpha}} = t_{\nu_\alpha} = q^{k_\alpha}, q_{\tilde{\alpha}} = q^{\nu_\alpha}, t_i = t_{\alpha_i} = q_i^{k_i}, \rho_k = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha.
\]

Using \( \rho_\nu \overset{\text{def}}{=} \frac{1}{2} \sum_{\nu_\alpha=\nu} \alpha \), we have: \( \rho_k = \sum_\nu k_\nu \rho_\nu = k_{sht} \rho_{sht} + k_{lng} \rho_{lng} \).

The standard argument based on the application of \( s_i \) for \( i > 0 \) to \( \rho_\nu \) gives that \((\rho_\nu, \alpha_i^\vee) = 1\) for \( \nu_i = \nu \) and 0 otherwise for \( i > 0 \). We obtain that \( \rho_k = \sum_i k_i \omega_i \).

For pairwise commutative \( X_{\omega_1}, \ldots, X_{\omega_n} \), let

\[
\tag{3.6}
X_{\tilde{b}} \overset{\text{def}}{=} q^2 \prod_{i=1}^n X_{\tilde{t}_{\omega_i}}^{l_{\omega_i}} \text{ if } \tilde{b} = [b, j], \tilde{w}(X_{\tilde{b}}) = X_{\tilde{w}(\tilde{b})},
\]

where \( b = \sum_{i=1}^n l_{\omega_i} \in P, j \in \frac{1}{m} \mathbb{Z} \) and \( \tilde{w} \in \hat{W} \).
Technically, \( X_b = q^{(x,b)} \) and \( X_{\omega_i} = q^{(x,\omega_i)} \). Also, \( X_{\alpha_0} = qX_{\alpha_0}^{-1} \).

Recall that \( \omega_r = \pi_r u_r \) for \( r \in O' \) (see above) and \( \pi_r^{-1} = \pi_{\varsigma(r)} \), where \( \varsigma \) is the action of \(-w_0\) on roots and weights; we set \( X_b^* = X_{b'} \).

**Definition 3.1.** The double affine Hecke algebra \( \hat{\mathcal{H}} \) is generated over \( \mathbb{Z}_{q,t} \) by \( \mathcal{H} = \langle \Pi, T_i, 0 \leq i \leq n \rangle \), subject to the homogeneous Coxeter relations and the quadratic relations \((T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0\), and by pairwise commutative \( \{X_b, b \in P\} \) satisfying (3.6). The following “cross-relations” are imposed:

(i) \( T_iX_b = X_bT_i^{-1} \) if \( (b, \alpha_i^\vee) = 1, 0 \leq i \leq n \);

(ii) \( T_iX_b = X_bT_i \) if \( (b, \alpha_i^\vee) = 0 \) for \( 0 \leq i \leq n \);

(iii) \( \pi_rX_b\pi_r^{-1} = X_{\pi_r(b)} = X_{w_r^{-1}(b)}q^{(\omega_r, b)} \) for \( r \in O' \).

The action of \( \hat{W} \) in \( \mathbb{R}^{n+1} \) is used in (iii). Namely: \( \pi_r(b) = \omega_r u_r^{-1}(b) = [u_r^{-1}(b), -(\omega_r, u_r^{-1}(b))] \), where \(-\omega_r, u_r^{-1}(b)) = (b, u_r(\omega_r)) = (b, \omega_r(\varsigma(r)))\) and \( u_r^{-1} = u_{\varsigma(r)} \). Recall that \( u_r(\omega_r) = w_0(\omega_r) = -\omega_r \).

The pairwide commutative elements \( Y_b \) are as above:

\[
(3.7) \quad Y_b \overset{\text{def}}{=} \prod_{i=1}^n Y_i^{l_i} \quad \text{if} \quad b = \sum_{i=1}^n l_i \omega_i \in P, \quad Y_i \overset{\text{def}}{=} T_{\omega_i}, \ b \in P.
\]

When acting in the polynomial representation (see below), they are called difference Dunkl operators. We arrive at the presentation \( \hat{\mathcal{H}} = \langle X_b, T_w, Y_b, q^{1/2m}, t_w^{1/2} \rangle, b \in P, w \in W \). The relations for \( \{Y_b\} \) result from those for \( T_0 \) and the relations in \( \mathcal{H}_X \overset{\text{def}}{=} \langle T_iX_b \rangle \), where \( 1 \leq i \leq n, b \in P \). The algebra \( \mathcal{H}_X \) is isomorphic to \( \mathcal{H} = \mathcal{H}_Y \) under \( X_b \mapsto Y_b^{-1}, T_w \mapsto T_w \).

**Automorphisms and anti-involutions.** The following maps can be (uniquely) extended to automorphisms of \( \hat{\mathcal{H}} \), where \( q^{1/(2m)} \) must be added to \( \mathbb{Z}_{q,t} \) (see [Ch1], (3.2.10)–(3.2.15)):

\[
(3.8) \quad \tau_+ : \ X_b \mapsto X_b, \ T_i \mapsto T_i (i > 0), \ Y_{\omega_r} \mapsto X_{\omega_r} Y_{\omega_r} q^{(\omega_r, \omega_r)/2},
\]

\[
\tau_+ : \ 0 \mapsto q^{-1} X_0 T_0^{-1}, \ \pi_r \mapsto q^{(\omega_r, \omega_r)/2} X_{\omega_r} \pi_r (r \in O'),
\]

\[
(3.9) \quad \tau_- : \ Y_b \mapsto Y_b, \ T_i \mapsto T_i (i \geq 0), \ X_{\omega_r} \mapsto X_{\omega_r} X_{\omega_r} q^{(\omega_r, \omega_r)/2},
\]

\[
\tau_-(X_0) = q T_0 X_0^{-1} T_0^{-1}; \quad \sigma \overset{\text{def}}{=} \tau_+ \tau_0^{-1} \tau_+ = \tau_0^{-1} \tau_+ \tau_0^{-1},
\]

\[
(3.10) \quad \sigma(X_b) = Y_b^{-1}, \ \sigma(Y_b) = T_{w_0}^{-1} X_b^{-1} T_{w_0}, \ \sigma(T_i) = T_i (i > 0).
\]

Formally, \( \tau_+^{l} (H) = q^{1/2} H(q^{-l}x^{2/2}) \) for any \( H \in \hat{\mathcal{H}} \); this is in the polynomial representation, which is faithful for generic \( q \). For instance, \( q^{1/2} Y_b q^{-l}x^{2/2} = q^{l/2-1/2(x-b)^2} Y_b = q^{-l}x^{2/2} X_b Y_b \) for minuscule \( b \), which is direct from the formula for \( Y_b \) (the Gaussian is \( W \)-invariant).
In particular, \(\sigma(Y_r) = \tau^{-1}_+ \tau^{-1}_-(Y_r) = q^{-(\omega_r, \omega_r)} Y_r X_r Y_r\) for \(r \in O'\).

Also, \(\sigma(\pi_r) = T^{-1}_{ur} X^{-1}_{(r)}\), which gives that \(\sigma(Y_r) = T^{-1}_{ur} X^{-1}_{(r)} T_{ur}\). See formulas (3.2.16) and (3.2.22) from [Ch1].

The justification is as follows. First, \(\sigma(\pi_r) = T^{-1}_{w_0} X^{-1}_{(r)} T_{w_0} T^{-1}_{ur} = T^{-1}_{ur} X^{-1}_{(r)}\). Second, we represent \(w_0 = vu_r\) where \(v = w_0^{\omega(r)}\). Then, \(T_{w_0} = T_v T_{ur}\), where \(v(\omega_r) = \omega_r\), which gives that \(T_v\) commutes with \(X_{(r)}\). See (2.4) and formula (3.15) below.

We note that \(T^{-1}_{w_0} T_i T_{w_0} = T_{\zeta(i)}\) for \(i > 0\), \(T^{-1}_{w_0} T_0 T_{w_0} = T_0\) and \(T^{-1}_{w_0} \pi_i T_{w_0} = \pi_\zeta(r)\). Generally, \(\zeta^2(H) = T^{-1}_{w_0} \zeta(H) T_{w_0}\), where the involution \(\zeta\) is naturally extended to an automorphism of \(\mathcal{H} \ni H\):

\[
X_b \mapsto X_{b'}, \ Y_b \mapsto Y_{b'}, \ T_i \mapsto T_{\zeta(i)}, \ \pi_r \mapsto \pi_{\zeta(r)}, \ b \in P, \ i \geq 0, \ r \in O'.
\]

We obtain that the projective \(PSL_2(\mathbb{Z})\) due to Steinberg acts in \(\mathcal{H}\); it generated by \(\tau_{\pm}\) subject to the relations \(\tau_+ \tau_- = \tau_- \tau_+\). This group is isomorphic to the braid group \(B_3\). We note the relation \(\sigma \tau_{\pm} = \tau_{\pm}^{-1} \sigma\). The automorphism \(\sigma^{-1}\) is the DAHA-Fourier transform.

All these automorphisms fix \(t_u, q\) and their fractional powers, as well as the anti-involution \(\varphi\):

\[
\varphi: \ X_b \mapsto Y^{-1}_b, \ Y_b \mapsto X^{-1}_b, \ T_w \mapsto T^{-1}_{w-1} (w \in W),
\]

also sending \(\pi_r \mapsto T^{-1}_{ur} X^{-1}_r, \ T_0 \mapsto (X_0 T_{s_0})^{-1}\).

One has for \(b \in P:\)

\[
\varphi \tau_+ \varphi = \tau_-, \ \varphi \sigma = \sigma^{-1} \varphi, \ \varphi \sigma^{-1}(Y_b) = Y_b, \ \varphi(\tau^{-1}_+(Y_b)) = \tau^{-1}_+(Y_b),
\]

which is direct from the definitions. Also, for \(i \geq 0\) and \(r \in O:\)

\[
\varphi(\tau_+(T_i)) = \tau_+(T_i), \ \varphi(\tau_+(\pi_r)) = (\tau_+(\pi_r))^{-1} = \tau_+(\pi_{\zeta(r)}).
\]

For the sake of completeness, let us justify (3.13). We need to check the first one only for \(i = 0\), where \(\tau_+(T_0) = q^{-1} X_0 T_{s_0} Y_0^{-1}\) is obviously \(\varphi\)-invariant. For the 2nd: \(\pi_r = Y_r T^{-1}_{ur} = \pi_{\zeta(r)}^{-1} = T_{ur} Y^{-1}_r\). Applying \(\varphi\), we obtain the identities \(T^{-1}_{ur} X^{-1}_r = X_r T_{ur}\), \(X_r T_{ur} = T^{-1}_{ur} X^{-1}_r\) and

\[
\tau_+(\pi_r) = q^{-(\omega_r, \omega_r)} X_r \pi_r = q^{-(\omega_r, \omega_r)} X_r Y_r T^{-1}_{ur} =
\]

\[
= q^{-(\omega_r, \omega_r)} \pi_r X_r^{-1} = q^{-(\omega_r, \omega_r)} Y_r T^{-1}_{ur} X^{-1}_r = q^{-(\omega_r, \omega_r)} Y_r X_r T_{ur}.
\]

Therefore \(\varphi(X_r Y_r T^{-1}_{ur}) = T^{-1}_{ur} X^{-1}_r Y^{-1}_r\) and we obtain the required. See formula (3.2.12) in [Ch1].

The following anti-involution \(\ast\) results directly from the group nature of the DAHA relations. Let \(H^* = H^{-1}\) for \(H \in \{T_w, X_b, Y_b, \pi_r, q, t_u\}\). To be exact, it is naturally extended to the fractional powers of \(q, t\):

\[
\ast: \ t \mapsto t^{-\frac{1}{2}}, \ q^{\pm m} \mapsto q^{-\frac{1}{2}}.
\]
It commutes with any (anti-)isomorphisms of \( \mathcal{H} \). This anti-involution serves the standard inner product in the theory of the DAHA polynomial representation \( \mathcal{X} \), but we will use \( \ominus \) instead. For \( l \in \mathbb{Z} \), the anti-involutions \( \ominus_l \) preserve \( q, t_u \) and send:

\[
\ominus_l = q^{lx^2/2} \circ \circ \circ q^{-lx^2/2} : X_a \mapsto X_a^\ominus, \quad Y_b \mapsto q^{lx^2/2} Y_b q^{-lx^2/2} = \tau^l_+(Y_b),
\]

where \( b \in P, w \in W, r \in O \). Here, formally \( \ominus(q^{lx^2/2}) = q^{lx^2/2} \), we use that \( x^2 \) is \( W \)-invariant and \( \varsigma \)-invariant. Thus, \( \ominus_l \) is the composition \( \tau^l_+ \circ \ominus \). We note that \( \ominus = \varphi \sigma^{-1} \ominus \circ \varsigma \ominus \sigma^{-1} \circ \ominus \).

\( \ominus \) Chapter 3 of [Ch1] is actually the theory of \( \varphi, \star, \ominus^{-1} \) and the corresponding symmetric forms in the polynomial representation and its Fourier-dual, which is the space generated by delta-functions at the points \( \pi_b(-\rho_k) = b - u_b^{-1}(\rho_k) \) for \( b \in P \).

Let us provide the counterpart of the symmetries from (3.13) for \( \ominus \):

\[
\ominus(\sigma(T_i)) = \sigma(T_i)(i \geq 0), \quad \ominus(\sigma(\pi_r)) = (\sigma(\pi_r))^{-1} = \sigma(\pi_r).
\]

The first relation is not immediate only for \( \sigma(T_0) = T_{s_{q^{-1}}}X_{\vartheta^{-1}} \). One has: \( \ominus(\sigma(T_0)) = T_{s_{q^{-1}}}X_{\vartheta^{-1}}T_{w_0}T_{s_{q^{-1}}}X_{\vartheta}T_{s_{q^{-1}}}X_{\vartheta^{-1}}T_{s_{q^{-1}}}X_{\vartheta}T_{s_{q^{-1}}}X_{\vartheta^{-1}} \). We use that \( T_{w_0}X_{\vartheta}X_{\vartheta}X_{\vartheta} = T_{s_{q^{-1}}}X_{\vartheta}X_{\vartheta}X_{\vartheta}T_{s_{q^{-1}}}X_{\vartheta}X_{\vartheta}X_{\vartheta} \), which follows from (3.22) in [Ch1], and can be check directly using that \( w_0 = us_\vartheta \) for \( u \) such that \( u(\vartheta) = \vartheta \); indeed, \( w_0(\vartheta) = -\vartheta = s_\vartheta(\vartheta) \). We obtain that \( T_{w_0}^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta} = T_{s_{q^{-1}}}^{-1}T_{u}^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta}T_{s_{q^{-1}}}X_{\vartheta}X_{\vartheta}X_{\vartheta} \), where \( T_u \) commutes with any polynomial of \( X_\vartheta \).

The second equality is justified as follows. One has: \( \ominus(\sigma(\pi_r)) = \ominus(T_u^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta}) = T_u^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta}T_u^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta}^{-1} \) due to \( T_{w_0}^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta} = T_{s_{q^{-1}}}^{-1}T_{u}^{-1}X_{\vartheta}X_{\vartheta}X_{\vartheta}T_{s_{q^{-1}}}X_{\vartheta}X_{\vartheta}X_{\vartheta} \). Alternatively, one can use here and above that \( \ominus = \varphi \sigma^{-1} \).

4. Polynomial representation

Its theory is based on the PBW Theorem (actually, there are 6 of them for different orderings of \( X, T, Y \)):

**Theorem 4.1 (PBW for DAHA).** Every element in \( \mathcal{H} \) can be uniquely written in the form

\[
\sum_{a, w, b} C_{a, w, b} X_a T_w Y_b \quad \text{for} \quad C_{a, w, b} \in \mathbb{C}_{q,t}, \quad a \in P, \ w \in W, \ b \in P^Y, \quad \text{where} \quad \mathbb{C}_{q,t} = \mathbb{C}[q^\pm 1/m, t^{\pm 1/2}] ; \quad \text{actually,} \quad \mathbb{Z}_{q,t} \text{is sufficient.} \quad \Box
\]

The theorem readily results in the definition of the polynomial representation of \( \mathcal{H} \) in \( \mathcal{X} \) \( \mathcal{H} \mathcal{X} \). Using Theorem 4.1, we can identify \( \mathcal{X} \) with the induced representation \( \text{Ind}_{\mathcal{H}} \mathcal{H} \mathcal{X} \mathbb{C}_+ \), where \( \mathbb{C}_+ \) is
the one-dimensional module of $H$ such that $T_w^\omega \mapsto t^{l(\tilde{\omega})/2}$, with $t^{l(\tilde{\omega})/2} \overset{\text{def}}{=} \prod_i t_i^{l_i(\tilde{\omega})/2}$.

We note that $t^{l(\tilde{\omega})/2} = \prod_i t_i^{l_i(\tilde{\omega})/2} = \prod_i q^{\frac{\nu_i(\tilde{\omega})}{2}} = q^{\frac{\nu_i(\tilde{\omega})}{2}}$ for $b \in P_+$.

The generators $X_b$ act by multiplication; $T_i(i \geq 0)$ and $\pi_r(r \in O^*)$ act in $\mathcal{X}$ as follows:

\begin{equation}
(4.17) \quad \pi_r \mapsto \pi_r, \quad T_i \mapsto t_i^{1/2} s_i + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i} - 1} (s_i - 1) \quad \text{for} \ t_i = t_{\alpha_i}.
\end{equation}

Recall that $s_0(X_b) = X_b X_{-}^{(b,\vartheta)} q^{(b,\vartheta)}$. The images of $T_i$ for $i > 0$ are Demazure-Lusztig operators.

**DAHA coinvariants.** Generally, they can be defined for any anti-involutions of $\mathcal{H}$ and $\mathcal{H}$-modules; $\mathcal{X}$ will be considered here.

**Definition 4.2.** (i) The Shapovalov anti-involution $\varpi$ of $\mathcal{H}$ for $Y$ is such that $T_w^\varpi = T_w^{-1}$ and the following property holds: for any $H \in \mathcal{H}$, the decomposition $H = \sum c_{ab} Y_a^x T_w Y_b$ exists and is unique.

(ii) Given $\varpi$, the corresponding coinvariant $\varrho$ is a functional (a linear map to $\mathbb{C}$) on $\mathcal{H}$ such that $\varrho(H^\varpi) = \varrho(H)$. Then $\{A, B\}_\varrho \overset{\text{def}}{=} \varrho(A^x B) = \{B, A\}_\varrho$ and $\{HA, B\}_\varrho = \{A, H^x B\}_\varrho$.

(iii) A anti-involution $\varpi$ is called basic if $\varrho$ is unique up to proportionality among the functionals acting via the projection $\mathcal{H} \ni H \mapsto H(1)$ onto $\mathcal{X}$. The Shapovalov ones are basic.

For Shapovalov $\varpi$, the coinvariant $\varrho$ normalized by the relation $\varrho(1) = 1$ is unique: $\varrho(H) = \sum c_{ab} \varrho(Y_a) \varrho(T_w) \varrho(Y_b)$, where $H$ is expanded as in (i). Here $\varrho$ is the character of $\mathcal{H}$ sending $T_i \mapsto t_i^{1/2}$ for $i \geq 0$ and $\pi_r \mapsto 1$. This formula for $\varrho$ works for arbitrary $q, t$.

An anti-involutions $\varpi$ is basic if and only if $\text{dim}(\mathcal{H}/(\mathcal{J} + \mathcal{J}^\varpi)) = 1$, where $\mathcal{X} = \mathcal{H}/\mathcal{J}$ for the left ideal $\mathcal{J} = \{H \in \mathcal{H} \mid H(1) = 0\}$, where $1 \in \mathcal{X}$ and $H(\cdots)$ is the action of $H$ in $\mathcal{X}$.

The anti-involution $\varphi$ from (3.11) is a Shapovalov one due to “PBW”. The corresponding evaluation pairing provides the duality and evaluation conjectures practically without calculations; see [Ch4]. We will use sometimes the notation $\{\cdot\}$ or $\{\cdot\}_{\rho_k}$ for it. The corresponding form $\{A, B\}$ and its restriction to $\mathcal{X}$ are well defined for any $q, t$ and the study of its radical is an important tool in the theory of the polynomial representation of DAHA.

The anti-involution $\star$, sending $g \mapsto g^{-1}$ for $g = X_a, Y_b, T_w, q, t$, is basic for generic $q, t$ but not a Shapovalov one. It is proven in [Ch1] that the corresponding inner product in $\mathcal{X}$ is unique up to proportionality for generic $q, t$.

Similarly, the anti-involution $\diamond$ from (3.14) is basic for generic $q, t$ but not a Shapovalov one. It governs the inner product in $\mathcal{X}$ making the nonsymmetric Macdonald polynomials (below) pairwise orthogonal.
and fixing \( q, t_\nu \). The corresponding bilinear form is the key in the DAHA harmonic analysis, including the Plancherel formula for \( \mathcal{X} \) and its Fourier image, the representation of \( \mathcal{H} \) in delta-functions. The notation \( \langle \cdot \rangle \) will be used below for the corresponding coinvariant.

The conjugations \( \diamond \pm_1 \) of \( \diamond \) by \( q^{\pm x^2/2} \) are Shapovalov ones. So the corresponding symmetric form is well-defined for any \( q, t_\nu \); the notation will be \( \langle \cdot \rangle_{\pm_1} \). The radical of the pairing for \( \diamond_1 \) is closely related to that for \( \varphi \); they coincide in the rational theory. The anti-involutions \( \diamond \pm_1 \) are the key in the difference Mehta-Macdonald formulas and are used to calculate the Fourier transform of the DAHA modules \( \mathcal{X} q^{\pm x^2/2} \).

**Mu-functions.** We set

\[
\mu(X; q, t_\nu) = \prod_{\alpha > 0} \frac{1 - X_\alpha}{1 - t_\alpha X_\alpha}, \quad \tilde{\mu}(X; q, t_\nu) = \prod_{\alpha > 0} \frac{1 - t_\alpha^{-1} X_\alpha}{1 - X_\alpha}.
\]

Recall that \( \Lambda(\tilde{w}) \overset{\text{def}}{=} \tilde{R}_+ \cap \tilde{w}^{-1}(\tilde{R}_-) = \{ \tilde{\alpha} > 0 \mid \tilde{w}(\tilde{\alpha}) < 0 \} \) for \( \tilde{w} \in \tilde{W} \); this set consists of \( l(\tilde{w}) \) positive roots. The following are the key relations for the functions \( \mu, \tilde{\mu} \):

\[
\frac{\tilde{w}^{-1}(\mu)}{\mu} = \frac{\tilde{w}^{-1}(\tilde{\mu})}{\tilde{\mu}} = \prod_{\tilde{\alpha} \in \Lambda(\tilde{w})} \frac{1 - t_\alpha^{-1} X_\alpha^{-1}}{1 - X_\alpha^{-1}} \cdot \frac{1 - X_\alpha}{1 - t_\alpha^{-1} X_\alpha} = \prod_{\tilde{\alpha} \in \Lambda(\tilde{w})} \frac{t_\alpha^{-1} - X_\alpha}{1 - t_\alpha^{-1} X_\alpha}.
\]

We see that \( \mu/\tilde{\mu} \) is (formally) a \( \tilde{W} \)-invariant function. Note that both functions, \( \mu \) and \( \tilde{\mu} \), are invariant under the action of \( \Pi = \{ \pi_r, r \in O \} \) and under the automorphisms of the non-affine Dynkin diagram. Also, \( \tilde{w}^{-1}(\mu) \) is invariant under the “conjugation” \( q \mapsto q^{-1}, t_\nu \mapsto t_\nu^{-1} \), which sends \( X_b \mapsto X_b^{-1} \) and \( X_\alpha \mapsto X_\alpha^{-1} \) (in \( \mu \)).

The action on functions here and generally is \( \tilde{w}(f(x)) = f(\tilde{w}^{-1}(x)); \) notice \( \tilde{w}^{-1} \). This results in the action of \( \tilde{w} \) (without \( \{ \cdot \}^{-1} \)) on the indices of \( X_\alpha \). For instance, \( w(X_a) = q^{(w^{-1}(x), a)} X_{w(a)} = X_{w(a)} \), \( b(X_a) = b(q^{(x,a)}) = q^{(x-b(x), a)} X_{q^{(x-b,a)} X_a} = X_{a-(b,a)} \).

The \( W \)-symmetrization of \( \mu \) is essentially the Macdonald’s function:

\[
\delta \overset{\text{def}}{=} \mu \prod_{\alpha > 0} \frac{1 - X_\alpha^{-1}}{1 - t_\alpha X_\alpha^{-1}} : \sum_{w \in W} w^{-1}(\delta) = \sum_{w \in W} w \left( \prod_{\alpha > 0} \frac{1 - t_\alpha X_\alpha^{-1}}{1 - X_\alpha^{-1}} \right)
\]

\[
(4.20) \quad = \sum_{w \in W} (-1)^{l(w)} \prod_{\alpha > 0} \frac{X_{w(a)}^{1/2} - t_\alpha X_{w(a)}^{-1/2}}{X_\alpha^{1/2} - X_\alpha^{-1/2}} = \sum_{w \in W} t^{l(w)} = P(t_\nu),
\]

where the latter is the Poincare polynomial of \( W \).
For the sake of completeness, let us provide the formula for the constant term \( \text{ct}(t_\nu) \) of \( \mu \) (the coefficient of \( X^0 \)):

\[
\text{ct}(t_\nu) \overset{\text{def}}{=} \text{CT}(\mu) = \prod_{\alpha \in R_+} \prod_{i=1}^{\infty} \frac{(1 - q^{(\alpha, \rho_k) + i\nu_\alpha})^2}{(1 - t_\alpha q^{(\alpha, \rho_k) + i\nu_\alpha})(1 - t_\alpha^{-1} q^{(\alpha, \rho_k) + i\nu_\alpha})}.
\]

To define \( \text{ct}(\mu) \) we expand \( \mu \) in terms of \( t_\nu \) and \( X_\alpha \) for \( \tilde{\alpha} > 0 \). Then \( \text{ct}(t_\nu) \) is an element in \( \mathbb{Z}[t_\nu][[q]] \). We will use this formula mainly for \( t_\nu^{-1} \) instead of \( t_\nu \). Here, as above, \( q^{(\alpha, \rho_k) + i\nu_\alpha} = q_\alpha^{(\alpha, \rho_k) + i} \).

**Jackson integrals.** We mostly follow here [Ch5] and [Ch1]. Let us fix \( \xi \in \mathbb{C}^n \) and set \( X_\alpha(bw) \overset{\text{def}}{=} q^{0 + w(\xi), a} \) for \( bw \in \tilde{W} \). For instance, \( \mu(0) = \mu(q^\xi) \) and \( (\tilde{w}^{-1}(\mu)/\mu)(0) = \mu(\tilde{w})/\mu(0) \).

Provided the convergence, the Jackson integral is defined as \( J_\xi(f) = J(f; \xi) \overset{\text{def}}{=} \sum_{\tilde{w} \in \tilde{W}} f(\tilde{w}) \mu(\tilde{w})/\mu(0) \). It is a sum, but can be expressed as a difference of some integrals (see below). From formula (4.19):

\[
(4.21) \quad \mu(\tilde{w})/\mu(0) = \prod_{[\alpha, j\nu_\alpha] \in \Lambda(\tilde{w})} \frac{t_\alpha^{-1} - q_\alpha^{(\alpha, \xi) + j}}{1 - t_\alpha^{-1} q_\alpha^{(\alpha, \xi) + j}}.
\]

Recall that \( q^{(\alpha, \xi) + j\nu_\alpha} = q_\alpha^{(\alpha, \xi) + j} \). The key property of these “integrals” is that \( J(f; \xi) \) does not depend on \( \xi \) up to a coefficient of proportionality (serving all \( f \)) for the spaces \( \mathcal{B}^\xi \) and \( \mathcal{Q} q^\xi/2 \). The coefficient of proportionality is formula (3.5.14) from [Ch1]. This is due to the uniqueness of coinvariants for basic \( \xi \).

Also, \( J(f; \xi) = 0 \) in these spaces if \( (\tilde{\alpha}, \xi) = 0 \) for some \( \tilde{\alpha} \), where the pairing is affine: \( (\alpha, j, \xi) = (\alpha, \xi) + j \). More exactly, we have the following lemma, which will be used later.

**Lemma 4.3.** For generic \( t \) and sufficiently general \( \xi \): \( J(f; \xi) = 0 \) in any spaces of functions if \( (\tilde{\alpha}, \xi) = 0 \) for at least one \( \tilde{\alpha} \in \tilde{R} \).

**Proof.** Applying a proper \( \tilde{u} \in \tilde{W} \) to \( \xi \) (for the affine action) we can assume that such \( \tilde{\alpha} \) form a root subsystem with simple roots \( \alpha_i' \) for \( i' \) from a subdiagram of the affine Dynkin diagram of \( \tilde{R} \). One has:

\[
\Lambda(\tilde{w}s_i) = \Lambda(\tilde{w}) \cup \{\alpha_i\} \quad \text{for this } i, \quad \text{and } \frac{\mu(\tilde{w}s_i)}{\mu(0)} = \frac{\mu(\tilde{w})}{\mu(0)} t_i^{-1} = -\frac{\mu(\tilde{w})}{\mu(0)}.\]

Recall that \( bw \) is considered here as the point \( q^{w(\xi) + b} \). Thus, the Jackson summation is identically zero upon the restriction to any right coset \( \{\tilde{w}W'\} \) for the Weyl group \( W' \) generated by \( s_i' \).

The following modification of \( J(f; \xi) \) is needed for \( \xi = -\rho_k \): we set \( J(f; -\rho_k) \overset{\text{def}}{=} \sum_{\pi_b} f(\tilde{w}) \mu(\pi_b)/\mu(0) \), where \( b \in P \). For an explicit formula, let \( \Lambda'(\tilde{w}) = \{[-\alpha, j\nu_\alpha] \mid \tilde{\alpha} = [\alpha, j\nu_\alpha] \in \Lambda(\tilde{w})\} \). We follow Section 4 of [Ch5]. Recall that \( b = \pi_b u_b \) for minimal \( u_b \) such that
\( b_- = u_0(b), \ b_+ = w_0(b_-) \) and \( -b_- = b_+^\xi \). From (3.1.17) in [Ch1]:

\[
\Lambda'(\pi_b) = \left\{ \left[ \alpha, j\nu_\alpha \right] \text{ s.t. } \alpha \in R_+,-(b_-,\alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_- \right\}. \tag{4.22}
\]

Then \( \mu(0)/\mu(\hat{w}) \) is well-defined for any \( \hat{w} \) and it is non-zero if and only if \( \hat{w} = \pi_b \) for \( b \in P \), which follows from

\[
\mu(\hat{w}) = t^{-(\pi_b)} \prod_{[\alpha, j\nu_\alpha] \in \Lambda'(\hat{w})} \frac{1 - t_{\alpha} q_\alpha^{(\alpha^\vee, \rho_k) + j}}{1 - t_{\alpha}^{-1} q_\alpha^{(\alpha^\vee, \rho_k) + j}} t^{l(\hat{w})} \overset{\text{def}}{=} \prod_{\nu} t_{\nu}^{l_{\nu}(\hat{w})}. \tag{4.23}
\]

As an application, we obtain the following Jantzen-type filtration in \( \mathcal{H} \)-modules \( \mathcal{F}_\xi \) linearly generated by the characteristic functions at points \( \hat{w} \). The action of \( \check{T}_i, \pi_\nu \) is dual to that in terms of Demazure-Lusztig operators; see [Ch1] and Theorem 5.2 below.

Here \( \xi \) can be arbitrary. It is deformation \( \xi_\epsilon = -(1 + \epsilon)\xi \) becomes generic for small \( \epsilon \) and we can define \( J(f; \xi_\epsilon) \) and find \( \ell_1 > 0 \) such that \( J_1(f) = e^{\ell_1} J(f; \xi_\epsilon) \) is non-singular and nonzero identically. The first term of this filtration will be then the \( \mathcal{H} \)-submodule of \( \mathcal{F}_\xi \) generated by the characteristic functions where \( J_1 \) vanishes. Inside this module, consider \( J_2(f) = e^{\ell_2} J(f; \xi_\epsilon) \) for some \( \ell_2 < \ell_1 \) that is nonzero; the second module will be the span of characteristic functions where \( J_2 \) vanishes. Continue by induction.

Continue by induction.

As an example, let \( \xi = -\rho_k \). Then for generic \( q, t_\nu \) the last submodule will be the Fourier transform of \( \mathcal{X} \). This procedure can be applicable to any \( q, t_\nu \); then \( \mathcal{X} \) will be decomposed further.

**Affine symmetrizers.** We continue to assume that \( 0 < q < 1 \) and use the notation \( t_\alpha = q_\alpha^{l_\alpha} \). Let

\[
\check{P}_+(f) \overset{\text{def}}{=} \sum_{\hat{w} \in \check{W}} t^{-l(\hat{w})/2} T_{\hat{w}}^{-1} f, \quad \check{J}_+(f) \overset{\text{def}}{=} \sum_{\hat{w} \in \check{W}} \hat{w}(\check{\mu} f).
\]

Also, the affine Poincaré series, is defined as \( \check{P}(t_\nu) = \sum_{\hat{w} \in \check{W}} t^{l(\hat{w})} \); it is \( \prod \frac{1}{(1-t_{d_i})} \prod_{i=1}^n \frac{1 - q_{d_i} t_{d_i}^{-1}}{1 - t_{d_i}^{-1}} \) in terms of the degrees \( d_i \) in the simply-laced case. See Theorem 2.8 from [ChM]. In any module over \( \mathcal{H} = \langle T_w, Y_b \rangle \), the operator \( \check{P}_+ / \hat{P}(t_\nu^{-1}) \) is a projection onto the space of spherical vectors defined as follows: \( \{ v \mid t_{\hat{w}} v = t^{l(\hat{w})/2} v \} \). This is provided the convergence of \( \check{P}_+ \) and when \( \hat{P}(t_\nu^{-1}) \neq 0 \).

All constructions below can be extended to the minus-symmetrizers (generally, to arbitrary characters of \( \mathcal{H} \)), but we will stick to \( \check{P}_+ \). Recall that \( 0 < q < 1 \).

**Theorem 4.4.** Let us assume that \( \mathcal{X} \) has a nonzero symmetric form \( \langle f, g \rangle \) with the anti-involution \( \check{\diamond} \) normalized by \( \langle 1, 1 \rangle = 1 \). Given any
$f, g \in \mathcal{H}$, $\langle f, g \rangle$ is a rational function in terms of $q, t$. Provided that $\Re k_\nu < 0$ and $|\Re k_\nu|$ are sufficiently large (depending on $f, g$),

$$
\langle f, g \rangle = t^{-l(w_0)/2} \tilde{\mathcal{P}}_+(f T_{w_0}(g^c))/\tilde{P}(t^{-1}_\nu),
$$

(4.24)

where $\tilde{\mathcal{P}}_+(f)$ is a constant for $f \in \mathcal{H}$ assuming the convergence. Thus, formula (4.24) supplies any $\mathbb{H}$-quotient of $\mathcal{H}$ with a partially defined (when converges!) bilinear symmetric form associated with $\Diamond$, which satisfies the normalization condition $\langle 1, 1 \rangle = 1$. □

This is Theorem 2.16 from [ChM]. The following Theorem is an adjustment of some of the claims from Theorems 2.5, 2.6, 2.11 there.

**Theorem 4.5.** (i) We assume that $\Re k_\nu < 0$ for all $\nu$. Given $a_+ \in P_+$, the sums $\tilde{\mathcal{J}}_+(X_a)$ absolutely converge for any $a \in W(a_+)$ if and only if $|t^{-l(a_+)}q^{-(a_+, \omega_i)}| < 1$ for all $i = 1, \ldots, n$, where $l(a_+) = 2(p^{\nu}, a)$. Equivalently, the conditions become $\Re(2\rho_k + a_+, \omega_i) < 0$ in terms of $k_\nu$.

(ii) Employing the formulas from (4.17) for $T_{\bar{w}}$, the expansion $\tilde{\mathcal{P}}_+ = \sum_{\bar{w} \in \tilde{W}} a_{\bar{w}} \bar{w}$ is with $t$-meromorphic functions $a_{\bar{w}}$. This holds by construction for $\tilde{\mathcal{J}}_+$. As formal series and as operators acting in $\mathcal{H}$: $\tilde{\mathcal{P}}_+ = \mathcal{C}(t^{-1}_\nu).\tilde{\mathcal{J}}_+$, where $\mathcal{C}(t^{-1}_\nu)$ is the constant term and the conditions from (i) are imposed if these operators act in $\mathcal{H}$.

(iii) Let $l > 0$. The operators $\tilde{\mathcal{J}}_+$ and $\tilde{\mathcal{P}}_+$ converge absolutely at any given $f \in \mathcal{H}q^{12z/2}$ for any $k$ for $\tilde{\mathcal{J}}_+$ and under the constraints $\Re(h_k^{\text{log}}, \Re(h_k^{\text{sht}}) < 1$ for the operator $\tilde{\mathcal{P}}_+$. Here $h_k^{\text{sht}} = (\rho_k, \theta) + k_\text{sht}$, $h_k^{\text{log}} = (\rho_k, \theta^\prime) + k_\text{log}$ and $\theta$ is the maximal root in $R_+$. For instance, this constraint is $\Re k < 1/k$ for $h_k = kh$ in the simply-laced case, where $h$ is the Coxeter number. Then $\tilde{\mathcal{P}}_+ = \mathcal{C}(t^{-1}_\nu).\tilde{\mathcal{J}}_+$. □

The convergence conditions in (i) follow directly from (4.19). We note that $\tilde{\mathcal{P}}_+(f)$ is regular by construction for $f \in \mathcal{H}q^{12z/2}$ but is well-defined only for $\Re k < 1/h$; $\tilde{\mathcal{J}}_+(f)$ is well-defined for any $k$ but has poles. For instance, their proportionality gives that the latter has no poles for $\Re k < 1/h$, which is far from obvious from its definition.

The adelic version of this argument is expected to provide an alternative approach to the fact that the Langlands formula for the inner product of pseudo-Eisenstein series has no singularities due to the Dedekind zeta-functions. See [KO, DHO].

**E-polynomials.** One of the key results in the DAHA theory is that the norms of nonsymmetric Macdonald polynomials under the spherical normalization are $\frac{\mu_0}{\mu(w)}$. For generic $q, t$, they are defined as follows:

$$
E_b^{\text{def}} = E_b/E_b(q^{-\rho_k}), \quad Y_a(E_b) = q^{(a, -\pi_k(\rho_k))}E_b = q^{(a, -b + \mu_0(\rho_k))}E_b, \quad b \in P.
$$
The normalization of \( E_b = X_b + (\text{lower terms}) \). The following formula is based on the technique of intertwiners and relations (3.13), (3.15) above. For generic \( q,t \) and \( b,c \in P \):

\[
(4.25) \quad t^{-l(w_0)/2} \text{ct} (E_b T_{w_0} (E_c) \mu(X; q, t)) / \text{ct}(t) = \delta_{b,c} \mu(q^{-\rho_k}) / \mu(\pi_b).
\]

This is essentially Corollary 3.4.1 from [Ch1], where the anti-involution \( \diamond \) occur there in formula (3.4.22).

Using (4.25), we obtain a direct demonstration of the fact that the coinvariant \( \langle \cdot \rangle \) associated with \( \diamond \) is a meromorphic function for any \( k \); see (4.2). Indeed, \( \langle f \rangle \) must be proportional to \( \text{ct}(f \mu) \) for any Laurent polynomial \( f \) due to the uniqueness of the coinvariant for \( \diamond \). The coefficient of proportionality is explicit. Then we express \( f \) via \{\( E_b \}\) and use that \( \langle E_b \rangle = 0 \) for \( b \neq 0 \).

Actually, the proof of (4.25) contains the justification of the uniqueness of \( \langle \cdot \rangle \). Let us extend this formula and its proof to general \( Y \)-induced representations.

\section{5. Induced modules}

The technique of intertwiners and the theory of basic coinvariants can be naturally extended to \( Y \)-induced \( \mathcal{H} \)-modules. We mostly follow [Ch1, ChM].

Given \( \xi \in \mathbb{C}^n \), the induced representation \( \mathcal{I}_\xi \) is defined as a unique (up to isomorphisms) \( \mathcal{H} \)-module induced from the character \( \xi \) of the algebra \( \mathbb{C}[Y_b, b \in P] \) defined as follows: \( \tilde{\xi}(Y_b) = q^{-\xi(b)} \). In the main examples, \( \xi \) depend of \( q \) and \( t_\nu \), which are considered as nonzero numbers or as formal parameters.

As a vector space, \( \mathcal{I}_\xi \) is naturally isomorphic to the affine Hecke algebra \( \mathcal{H}_X = (T_w, X_b) \). It is \( Y \)-semisimple with simple \( Y \)-spectrum if and only if \( q^{\tilde{\alpha}(\rho_k)} \neq q^{\rho_k} \) for any id \( \tilde{\alpha} \in \tilde{W} \).

The module \( \mathcal{I}_\xi \) is a canonical quotient of \( \mathcal{I}_\xi \) for \( \xi = -\rho_k \). We will mostly assume that \( 0 < q < 1 \) and it is generic with respect to \( t_\nu \): \( q^a \notin t_\nu^2 \) for \( m \geq 1 \). Then \( \mathcal{I}_{-\rho_k} \) is semisimple when and only when \( w(\xi) \neq \xi \) modulo \( 2\pi a \) for any \( w \in W \).

The \( Y \)-spectrum of \( \mathcal{I}_\xi \) for any \( \xi \) is \( \{q^{w(\xi)+a}\} \), where \( a \in P, w \in W \): the spaces of pure eigenvectors are \( \{v \mid Y_b(v) = q^{-(b,a+w(\xi))} v, \ b \in P \} \). They are nonzero for any \( a, w \) and the corresponding generalized spaces of eigenvectors linearly generate \( \mathcal{I}_\xi \) for any \( \xi \). This module is irreducible if and only if \( q^{\tilde{\alpha}(\xi)} \notin \{t_\nu q_\nu^\xi \} \) for any \( \alpha \in R \) and \( \nu = \nu_\alpha \).

Given Shapovalov \( x \) and \( \xi \in \mathbb{C}^n \), the coinvariants \( \varphi \) are defined by the relations \( g(H^a) = g(H) \), \( g(HY_b) = \tilde{\xi}(Y_b) g(H) \) and \( g(T_w) = \tau(T_w) \), where \( \tau : \mathcal{H} \to \mathbb{C} \) is an arbitrary linear map satisfying the relation \( \tau(T_w) = \tau(T_{w^{-1}}) \). The simplest choice is \( \tau(T_w) = t^{l(w)/2} \) for \( w \in W \).
One has then: \( \phi((Y^\alpha) T_w Y_b) = \tilde{\xi}(Y_{a+b}) \tau(T_w) = q^{(\xi,a+b)} \tau(T_w) \). We see that given Shapovalov \( \pi, \tau \) and an arbitrary \( \xi \), there exists a unique coinvariant up to proportionality.

The anti-involutions \( *, \diamond \) do not require a choice of \( \tau \) for their definition and the uniqueness. They are basic for generic \( q, t, \xi \), i.e. the corresponding coinvariant is unique under the normalization \( \phi(\text{vac}) = 1 \). We will prove this below in process of obtaining the norm-formula in \( \mathcal{I}_\xi \) for generic \( \xi \). This will be based on the technique of intertwiners.

The notation for the coinvariants with \( \xi \) for \( \diamond_l \) will be \( \langle \cdot \rangle_{I, \xi} \); we write \( \langle \cdot \rangle_\xi \) for \( \langle \cdot \rangle_{l, \xi} \) for the polynomial representation.

**The norm-formula.** We follow Theorem 3.6.1 from [Ch1] and (3.6.23). It was stated there for the anti-involution \( * \); we adjust it accordingly and change the proof. The next theorem includes the uniqueness of \( \langle \cdot \rangle_{I, \xi} \) above for \( \diamond \) and for generic parameters. We set:

\[
(5.26) \quad \Phi_i = T_i + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i} - 1}, \quad \phi_i = t_i^{1/2} + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i} - 1}
\]

where \( 0 \leq i \leq n, \hat{\omega} = \pi_r s_{i_1} \cdots s_{i_l} \); recall that \( X_{\alpha_0} = q X_{\alpha_1} \). The decomposition of \( \hat{\omega} \in \hat{W} \) is not necessarily reduced here because \( S_i^2 = 1 = G_i^2 \) for \( 0 \leq i \leq n \). This relation and the fact that \( S, G \) do not depend on the choice of the reduced decomposition follow from the symmetries \( S_{\hat{\omega}} X_b = X_{\hat{\omega}(b)} S_{\hat{\omega}} \) and \( G_{\hat{\omega}} X_b = X_{\hat{\omega}(b)} G_{\hat{\omega}} \) for \( \hat{\omega} \in \hat{W} \). We obtain that \( S_i^2 \) is a rational function in terms of \( X_b \) and \( S_i^2(1) = 1 \) in \( \mathcal{K} \), which gives that \( S_i^2 = 1 \) and \( G_i^2 = \phi_i S_i^2 \phi_i^{-1} = 1 \).

We will need actually \( \hat{S}_{\hat{\omega}} \overset{\text{def}}{=} \sigma(S_{\hat{\omega}}), \hat{G}_{\hat{\omega}} \overset{\text{def}}{=} \sigma(G_{\hat{\omega}}) \). One has: \( \hat{S}_{\hat{\omega}} Y_b = Y_{\hat{\omega}(b)} \hat{S}_{\hat{\omega}} \) for \( \hat{\omega} \in \hat{W} \), and the same symmetry holds for \( \hat{G} \).

Accordingly, we set \( f_{\hat{\omega}} \overset{\text{def}}{=} \hat{S}_{\hat{\omega}}(v), e_{\hat{\omega}} \overset{\text{def}}{=} \hat{G}_{\hat{\omega}}(v) \), where \( v = \text{vac} \) is the cyclic generator of \( \mathcal{I}_\xi \), \( \hat{\omega} \in \hat{W} \). To obtain explicit formulas for \( f_{\hat{\omega}}, e_{\hat{\omega}} \) in terms of \( \pi_r, T_i, \) let

\[
S_i(c) = \frac{T_i + (t_i^{1/2} - t_i^{-1/2})/(X_{\alpha_i}(q^c) - 1)}{t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})/(X_{\alpha_i}(q^{-c}) - 1)}, \quad G_i(c) = \frac{T_i + t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i}(q^c) - 1}.
\]

Here \( c \in \mathbb{C}^n \). Using the affine action \( bw((z)) = w(z) + b \):

\[
 f_{\hat{\omega}} = \sigma(\pi_r S_{i_1}(c_1) \cdots S_{i_l}(c_1))(v), \quad e_{\hat{\omega}} = \sigma(\pi_r G_{i_1}(c_1) \cdots G_{i_l}(c_1))(v)
\]

\[
(5.27) \quad \text{for } \hat{\omega} = \pi_r s_{i_1} \cdots s_{i_l}, \quad c_1 = \xi, \quad c_2 = s_{i_1}(c_1), \ldots, c_l = s_{i_{l-1}}(c_{l-1}).
\]

These formulas justify that \( \{e_{\hat{\omega}}, f_{\hat{\omega}}\} \) are well-defined and nonzero for generic \( q, t \in \mathbb{C}^* \). One has for \( \hat{w} = bw, b \in P, w \in W \):

\[
 Y_a(f_{\hat{w}}) = q^{-(a,b+w(\xi))} f_{\hat{w}} \quad \text{and} \quad \mathcal{I}_\xi = \oplus_{\hat{\omega} \in \hat{W}} \mathbb{C} f_{\hat{\omega}}.
\]
The same relations hold for \( \{ c_{\tilde{w}} \} \).

For \( \xi = -\rho_k \) and generic \( q, t_{\nu} \), the module \( \mathcal{I}_\xi \) has a canonical quotient obtained by imposing additional relations \( T_w(vac) = t^{(w)/2} vac \) for \( w \in W \), which is \( \mathcal{X} \). The elements \( e_{\tau_{b}}, f_{\pi_{b}} \) and their images in \( \mathcal{X} \) are well-defined generic \( q, t \). We note that we used the following normalization in formulas (3.3.42), (3.3.44) from \([\text{Ch1}]:\)

\[
\tilde{E}_b = \tau_+ (\pi_\tau G_{i_\epsilon}(c_\ell) \cdots G_{i_1}(c_1)) (1) \quad \text{for} \quad b \in P.
\]

The relation to spherical polynomials is: \( \mathcal{E}_b = q^{(\rho_{k} + b_{+}, b_{-})} \tilde{E}_{\pi_{b}} \) for \( b \in P \), which results from formula (3.13).

The following norm formula in \( \mathcal{I}_\xi \) is actually the fundamental fact that the DAHA-Fourier transform of \( \mathcal{I}_\xi \) is the corresponding Delta-representation. The Fourier-images of \( \tilde{\tau} \) are considered as \( W \)-representation. The Fourier-images of \( \tau \) and \( \delta \)-function at \( T \) obtained by imposing additional relations acting via \( E \) for any \( \xi \), \( \eta \).

Concerning the spherical normalization \( \{ \cdot \}_\xi = 1 \) for the evaluation coinvariants \( \{ \cdot \}_\xi \), one needs to calculate its change under the action of \( \tau_+(S_{\tilde{w}}) \), which follows Proposition 6.6 from \([\text{Ch4}]\). The simplest choice of \( \tau \) is \( \tau(T_w) = t^{(w)/2} \) for \( w \in W \); however \( \tau(T_0) \) will then depend on \( \xi \), which makes the final formula somewhat more involved than that for \( \xi = -\rho_k \) with \( \{ \cdot \} \) acting via \( \mathcal{X} \).

The next theorem is the calculation of change of the norms is mostly parallel to Theorem 3.6.1 from \([\text{Ch1}] \) and (3.6.23). They were for the anti-involution \( \ast \); we will do this for \( \diamond \) and with some improvements.

**Theorem 5.1.** For generic \( \xi, q, t_\nu \) let \( \langle f \rangle_\xi \) be the coinvariant for \( \diamond \) acting via \( \mathcal{I}_\xi \) normalization by the condition \( \langle vac \rangle_\xi = 1 \). Then:

\[
\langle f_{\tilde{u}}^\diamond f_{\tilde{w}} \rangle_\xi = \delta_{\tilde{u}, \tilde{w}} \mu(\tilde{w})/\mu(0), \quad \langle e_{\tilde{u}}^\diamond e_{\tilde{w}} \rangle_\xi = \delta_{\tilde{u}, \tilde{w}} \mu(0)/\mu(\tilde{w}),
\]

for any \( \tilde{u}, \tilde{w} \in \tilde{W} \), where \( \delta \) is the Kronecker delta and \( 0 = \text{id} \in \tilde{W} \) is considered as \( q^k \). In particular, such \( \langle H \rangle_\xi \) is unique and its values at \( H = X_\alpha T_\nu Y_\beta \in \mathcal{H} \) are rational in terms of \( q^{(\xi, \alpha)} \) for \( \alpha \in R \) and fractional powers of \( q, t_\nu \).

**Proof.** It is based on the formulas in (3.15) coupled with the identity \( \mathcal{S}_i^2 = 1 \) for \( 0 \leq i \leq n \). We set \( \psi_i = \sigma(\phi_i) = t_i^{1/2} + t_i^{1/2 - t_i^{-1/2}} \) \( = t_i^{1/2 Y_{a_i}^{-1} - t_i^{-1/2}} \), where \( Y_{a_i} = q^{-1}Y_{a_i}^{-1} \). Then \( \mathcal{S}_i = \psi_i^{-1}(\sigma(T_i) + t_i^{1/2 - t_i^{-1/2}} Y_{a_i}^{-1} - 1) \),

\[
(5.28) \quad \mathcal{S}_i^\diamond = \psi_i \mathcal{S}_i \psi_i^{-1} = t_i^{1/2 Y_{a_i}^{-1} - t_i^{-1/2}} \left( t_i^{1/2 Y_{a_i}^{-1} - t_i^{-1/2}} \right)^{-1} \mathcal{S}_i = t_i^{1/2 - t_i^{-1/2}} \mathcal{S}_i.
\]

\[
(5.29) \quad \mathcal{S}_i = \frac{t_i^{1/2} - t_i^{-1/2} Y_{a_i} \mathcal{S}_i}{t_i^{1/2} - t_i^{-1/2}} = \frac{1 - t_i^{-1} Y_{a_i} \mathcal{S}_i}{1 - t_i^{-1} Y_{a_i}} \mathcal{S}_i = t_i^{-1} - Y_{a_i} \mathcal{S}_i.
\]
Also, \((\sigma(\pi_r))^\circ = \sigma(\pi_{\zeta(r)})\) for \(r \in O'\). We arrive at the relations

\[
\langle (\hat{S}_i f_{\hat{u}}) \hat{S}_i f_{\hat{w}} \rangle_{\xi} = \langle f_{\hat{u}}^\circ (\hat{S}_i^\circ ) f_{\hat{w}} \rangle_{\xi} = \frac{t_i^{-1} - Y_{\alpha_i}^{-1}(Y_{\alpha_i} \circ q^{-1}(\xi))}{1 - t_i^{-1} Y_{\alpha_i}^{-1}(Y_{\alpha_i} \circ q^{-1}(\xi))} \langle f_{\hat{u}}^\circ f_{\hat{w}} \rangle_{\xi},
\]

where \([\alpha, j \nu_\alpha, z] = (\alpha, z) + j \nu_\alpha\), which is needed here for \(\alpha_0 = [-\vartheta, 1]\). Similarly, \(\langle (\hat{G}_t e_{\hat{u}}) \hat{G}_t e_{\hat{w}} \rangle_{\xi} = \frac{1 - t_i^{-1} q(\alpha_i, \hat{w}(\xi))}{1 - t_i^{-1} q(\alpha_i, \hat{w}(\xi))} \langle e_{\hat{u}}^\circ e_{\hat{w}} \rangle_{\xi}\). Using \(\Lambda(\hat{w}) = \{ \alpha_i, s_i (s_i^2(\alpha_{i+1})), \ldots, \hat{w}^{-1}s_i(\alpha_{i+1}) \}\) for a reduced decomposition \(\hat{w} = \pi_r s_i \cdots s_i\) (formula (3.1.10) from [Ch1]), we obtain:

\[
(5.30) \quad \langle f_{\hat{u}}^\circ f_{\hat{w}} \rangle_{\xi} = \delta_{\hat{u},\hat{w}} \prod_{[\alpha, j \nu_\alpha] \in \Lambda(\hat{w})} \frac{t_i^{-1} - q^{(\alpha', \xi)}}{1 - t_i^{-1} q^{(\alpha', \xi)}} = \frac{\mu(\hat{w})}{\mu(0)},
\]

and its reciprocal for \(\langle e_{\hat{u}}^\circ e_{\hat{w}} \rangle_{\xi}\).

We will interpret this theorem as the Plancherel formula for the DAHA-Fourier transform of \(I_\xi\). Let \(\mathcal{F}_\xi = \bigoplus_{\hat{w} \in \hat{W}} \mathbb{C} \chi_{\hat{w}}\) for the characteristic functions \(\chi_{\hat{w}}\) at \(\hat{w} = bw\) considered as points \(q^{w(\xi)+b}\). It is a module over the smash product of \(\mathbb{C}[X_\alpha, a \in P]\) and the group algebra \(\hat{W}\). The action is \(S_{\hat{w}}(\chi_{\hat{w}}) = \chi_{\hat{w}}\) and \(X_a(\chi_{\hat{w}}) = X_a(\hat{w})\chi_{\hat{w}}\) for \(\hat{u}, \hat{w} \in \hat{W}\) and \(a \in P\). Here, as above, \(X_a(\hat{w}) = q^{(a, w(\xi)+b)}\).

The action of \(\mathcal{H}\) in \(\mathcal{F}_\xi\) is obtained when we use the action of \(S_{\hat{w}}\) to define that of \(T_{\hat{w}}\), namely the formulas \(T_i = \phi_i S_i - \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i}^{-1}}, S_{\pi_r} = \pi_r\). The resulting action will involve the denominators in terms of \(X\), so we need to assume that \(\xi, q, t_i\) are in a general position when applying them to \(\chi_{\hat{w}}\). See formula (3.4.10) from [Ch1].

**Theorem 5.2.** For generic \(\xi, q, t_i\) and any \(\hat{w} \in \hat{W}, \) the \(\mathbb{C}\)-linear map \(F : f \mapsto \hat{f}\) sending \(f_{\hat{w}} \mapsto \chi_{\hat{w}}\) induces the automorphism \(\sigma^{-1}\) for \(H \in \mathcal{H}\): \(FH = \sigma^{-1}(H)F\). The inner product in \(I_\xi\) given by the formula \(\langle f, g \rangle = \sum_{\hat{w} \in \hat{W}} f(\hat{w})g(\hat{w})\mu(\hat{w})/\mu(0)\) corresponds to the anti-involution \(\sigma^{-1} \circ \sigma = \sigma \circ \sigma^2\). Here \(f(bw) = f(q^{w(\xi)+b})\) etc. For any \(f, g \in I_\xi\), we have the Plancherel formula: \(\langle f^\circ g \rangle_{\xi} = \langle \hat{f}, \hat{g} \rangle\).

**Proof.** By construction: \(\sigma^{-1}(\hat{S}_{\hat{w}}) = S_{\hat{w}}\). This gives \(FH = \sigma^{-1}(H)F\). Then we use that the pairing \(\sum_{\hat{w} \in \hat{W}} f(\hat{w}) T_{\nu_0}(g)(\hat{w})\mu(\hat{w})\) corresponds to the anti-involution \(\hat{\diamond}\) and that \(\sigma^2(H) = T_{\nu_0}H T_{\nu_0}^{-1}\) for \(H \in \mathcal{H}\). See Corollary 3.4.3 from [Ch1]. This is a general fact for any kind of \(\hat{W}\)-invariant integration with the measure function \(\mu\); the Jackson integration \(\sum_{\hat{w} \in \hat{W}} f(\hat{w})\mu(\hat{w})\) is taken here as such.

Recall that the numerator of \(\mu\) is nonzero at \(q^i\) if and only if the corresponding \(I_\xi\) is \(Y\)-semisimple with simple spectrum; the denominator of \(\mu(q^x)\mu(q^{-x})\) is nonzero at \(x = \xi\) if and only if \(I_\xi\) is irreducible.
Equivalently, $\mathcal{I}_\xi$ is irreducible if and only if all binomials in the numerators and denominators of (5.30) for all $\hat{w}$ are nonzero.

The values $\mu(\hat{w})$ are naturally some residues, which will be used to obtain meromorphic continuations of the integral formulas for the inner products. Thus, we interpreted these values as norms of $Y$-eigenvectors $f_{\hat{w}} \in \mathcal{I}_\xi$. For $\xi = -\rho_b$, the elements $e_{\pi_b}, f_{\pi_b}$ become special normalizations of Macdonald’s polynomials in $\mathcal{A}$, the quotient of $\mathcal{I}_\xi$.

6. RESIDUES AND CLOSED SUBSYSTEMS

We will use the definition of the residues from [GH], Ch.5. Generically, $\text{Res}_0 \left( \frac{h(x)}{f_1(x)^{\cdot n_1} f_2(x)^{\cdot n_2}} d x_1 \wedge \cdots \wedge d x_n \right) = h(0)/\det \left( \frac{\partial f_i}{\partial x_j} \right)(0)$, where the orientation of the integration domain $\{ x = (x_i) \in \mathbb{C}^n \mid |f_i(x)| < \epsilon \}$ is by the inequality $d(\arg(f_1)) \wedge \cdots \wedge d(\arg(f_n)) \geq 0$. The assumptions here are that $h(x)$ is regular at $x = 0$, $f_i(0) = 0$ and the determinant is nonzero, i.e. 0 is a nondegenerate singularity.

We will fix below the orientation to ensure that

$$
(6.31) \quad \text{Res}(\mu, 0) = \text{Res}_0(\mu) = \prod_{\alpha > 0} \prod_{i=1}^\infty \frac{1 - q^{i+\langle \xi, \alpha \rangle}}{1 - q^{i-k_\alpha+\langle \xi, \alpha \rangle}} \cdot \frac{1 - q^{-i+(\xi, \alpha)\nu}}{1 - q^{i-k_\alpha-(\xi, \alpha)\nu}}.
$$

We have here $f_i = (1-t_i X_{\alpha_i})$. If the variable $x_{\alpha_i}$, $1 \leq i \leq n$ are naturally ordered, then the orientation is clockwise for the loops around $f_i = 0$. Permuting $\{x_{\alpha_i}\}$ will not change the residue, because the orientation will change too. The orientation and the corresponding wedge forms will be extended below to points $\hat{w}$ using the action of $\tilde{W}$.

We note that the residue of any Laurent series in terms of $X_i$ is its constant term and it does not change if $X_i$ are changed to variables $X'_i = \prod_{i=1}^n X_i^{c_{i,j}}$ for $(c_{i,j}) \in GL(n, \mathbb{Z})$. We will use this below. However, the presentation of a function as a Laurent series depends on the domain where the function is considered.

Generally, the residues can be complicated to calculate algebraically. Analytically, they are integrals of some top wedge forms $\omega$ over $\Gamma = \{ x \in \mathbb{C}^n \mid |f_i(x)| = \epsilon, 1 \leq i \leq n \}$ and depend only on the (middle) homology class of $\Gamma$ in $H_n(\{ x \mid \prod_{i=1}^n f_i(x) \neq 0 \})$ and the class of the form $\omega$ in the corresponding cohomology. See [GH].

Closed subsystems. We will need closed root subsystems $R' \subset R$ ("closed subroot systems" is used too) or those in $\tilde{R}$ of the same rank as $R$. By definition, it is required that $\tilde{\alpha} + \tilde{\beta} \in R'$ for any roots $\tilde{\alpha}, \tilde{\beta}$ in $R'$ if this sum belongs to $\tilde{R}$. Also, we will consider full affine extensions $\tilde{R}'$ of $R'$, which are with all $[\alpha, \nu_0 \mathbb{Z}]$ if $[\alpha, \cdots] \in R'$. The positivity there will be induced from that $\tilde{R}$ unless stated otherwise. We will actually use the notation $R'$ for subsystems in $R$; otherwise (in $\tilde{R}$), the notation $R'^{\dagger}$ will be used.
Let $\widetilde{R}_{\text{lng}}$ and $\widetilde{R}_{\text{slt}}$ be the root subsystems formed by long and short roots in $\widetilde{R}$ (similarly, for $R$); they are of rank $n$. The sum $\tilde{\alpha} + \tilde{\beta} \in \widetilde{R}$ of 2 long roots $\tilde{\alpha}$ and $\tilde{\beta}$ can be only long, so $\widetilde{R}_{\text{lng}}$ is a closed root subsystem of rank $n$. Indeed, $(\tilde{\alpha}, \tilde{\beta}) < 0$ in this case; otherwise, $|\tilde{\alpha} + \tilde{\beta}|^2/2 > |\tilde{\alpha}|^2/2 = \nu_{\text{lng}}$, which is impossible. Thus, $\tilde{\alpha} + \tilde{\beta} = s_{\tilde{\alpha}}(\tilde{\beta})$, i.e. it is long. Recall that $(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta)$ for the non-affine components for $\tilde{\alpha}, \tilde{\beta}$.

Similarly, $(\tilde{\alpha}, \tilde{\beta}) < 0$ for short $\tilde{\alpha}$ and long $\tilde{\beta}$ if $\tilde{\alpha} + \tilde{\beta} \in \widetilde{R}$. Thus, $\tilde{\beta} + \nu_{\text{lng}} \tilde{\alpha} = s_{\tilde{\alpha}}(\tilde{\beta})$ is long and $\tilde{\beta} + \tilde{\alpha} = s_{\tilde{\beta}}(\tilde{\alpha})$ is short in this case.

Similarly, $\tilde{\alpha} + \tilde{\beta}$ can be a long root for short $\tilde{\alpha}$ and $\tilde{\beta}$ only if $(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta) = 0$ unless for $G_2$. In this case, $s_{\tilde{\alpha}}(\tilde{\alpha}) + \tilde{\beta} = \tilde{\beta} - \tilde{\alpha}$ is a long root too. For $G_2$, $\tilde{\beta} - \tilde{\alpha}$ will be long if $(\alpha, \beta) = 0$ for short $\tilde{\alpha}$ and $\tilde{\beta}$.

In the finite case, the list of closed maximal subsystems $R' \subset R$ of rank $n$ is essentially due to Borel- de Siebenthal; there are no such subsystems for $A_n$ and they are always reducible unless for $B_n, E_{7,8}, F_4, G_2$. Setting $\theta = \sum n_i \alpha_i$, the key step is that any $\alpha_i$ with $n_i > 1$ (assumed prime for the maximality) can be replaced by $-\theta$ to generate such an $R'$, possibly reducible.

We note that the usage of $\vartheta$ here instead of $\theta$ leads to root subsystems of rank $n$ in $R$, but they can be non-closed. For instance, $B_m \oplus B_{n-m} \subset B_n$ for $2 \leq m \leq n - 2$ can occur in this way. It is of rank $n$ but non-closed: $\varepsilon_m + \varepsilon_{m+1}$ in the standard notation is a root, but not in this subsystem. Here $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$, $\alpha_n = \varepsilon_n$ and $\vartheta = \varepsilon_1$.

The Dynkin diagram of $B_n$ extended by $\alpha_0 = [-\vartheta, 1]$ is that for the usual extended diagram of $C_n$ with the reversed arrows. So the examples above are when it splits into two connected segments.

In the affine case, the description of maximal closed subsystems $R^\dagger \subset \tilde{R}$ is quite similar; see Theorem 5.6 from [FRT] and [RV] for the maximal ones. The affine classification is basically the nonaffine one with the list of $p_\alpha > 0$ such that affine roots $[\alpha, \nu_{\alpha,j}] \in \tilde{R}^\vee$ are those for $\{j\} = \{j_0 + p_\alpha \mathbb{Z}\}$; such $p_\alpha$ always exist. If the maximal closed ones in $\tilde{R}$ are known, then all closed root subsystems of the same rank as $R$ can be found by induction. Basically, the tables of maximal closed root subsystems in $R$ are sufficient for this.

We will need below the affine root subsystems $\tilde{R}' = \{[\alpha', \nu'_\alpha,j], \alpha' \in R'\}$, where $\nu'_\alpha = \nu_{\alpha'}$ is taken from $R$, and other affine definitions for reduced (=decomposable) $R'$. The corresponding $\vartheta'$ and $\alpha'_0$ are not unique then. They must be defined for each connected component of the Dynkin diagram of $R'$. The affine Weyl group $\widehat{W}'$ becomes the direct products of those from the connected components; the corresponding $P$-lattice $P'$ and the extended affine Weyl group $\widehat{W}'$ are the products of those for the connected components.
Theorem 6.1. Let $0 < q < 1$ be generic: $q^n \neq t_{\nu}^l$ for any integer $l, m \neq 0$ and $\nu$. Assume that the numerator of $\mu$ is nonzero at $q^\xi$, which condition does not depend on the choice of the positivity in $R$. Equivalently, $\mathcal{I}_\xi$ is semisimple with the simple spectrum.

(i) For $\xi = -\rho_k$, assume that $t_{\text{sht}}^{\text{ling}} \neq 1$; also, let $t_{\text{sht}}^{\text{ling}} \neq -1$ for any $1 \leq j < n$ in the case of $C_n$ and $t_{\text{sht}}^{\text{ling}} \neq 1$ for $F_4$. Then there are exactly $n$ binomials in the denominator of $\mu$ vanishing at $\hat{w} = bw$, i.e. at $q^{\hat{w}(\rho_k)}$, if and only if $\hat{w} = \pi_b$ for $b \in P$, i.e. when $w = u_b^{-1}$. Given $\pi_b$, these binomials are $(1 - t_iX_{\pi_b(\alpha_i)})$ for $1 \leq i \leq n$. For other $\hat{w}$, the number of such binomials vanishing at $\hat{w}$ is smaller than $n$ and $\mu(\hat{w})/\mu(0) = 0$; see formula (4.23).

(ii) Let $t_{\text{sht}} = t_{\text{ling}}$ or $t_{\text{sht}}^{\text{ling}} = t_{\text{ling}}$. We continue to assume that $\xi$ is such that the numerator of $\mu$ is nonzero at $q^\xi$ and assume now that its denominator has exactly $n$ binomials $(1 - t X_{\tilde{\beta}_i})$ that vanish at $q^\xi$. Let $\tilde{\beta}_i = [\beta_i, \cdots]$. Then $\{\tilde{\beta}_i, 1 \leq i \leq n\}$ is a set of simple roots in the closed root subsystem $R^\dagger = \hat{R} \cap \bigoplus_{i=1}^n \mathbb{Z}\tilde{\beta}_i$. Unless there exist short $\tilde{\beta}_i, \tilde{\beta}_j$ for the systems $BCFG$ such that $\tilde{\beta}_i - \tilde{\beta}_j = [\beta, m]$ for long $\beta \in R$, where $m$ is not divisible by $\nu_{\text{ling}}$, the set $\{\beta_i, 1 \leq i \leq n\}$ is a set of simple roots in $R^\dagger = R \cap \bigoplus_{i=1}^n \mathbb{Z}\beta_i$.

(iii) Continuing (ii), let $\hat{R}' = \{[\alpha, \nu_{\alpha} j] | \alpha \in R', j \in \mathbb{Z}\} \subset \hat{R}$, $|t_{\text{sht}}| > 1$ and $q$ is such that $q < t_{\text{ling}}^{-h}$, where $h$ is the maximum of Coxeter numbers of the irreducible components of $R^\dagger$. Then $\{(\hat{w}')^{-1}(\tilde{\beta}_i)\}$ become simple roots of $\hat{R}'$ for the positivity induced from $R_+$ and some $\hat{w}' \in \hat{W}' \subset \hat{W} \subset \hat{W}$, where $\hat{W}'$ is defined for $\hat{R}'$. More exactly, for every connected component of $R'$, exactly one simple root $\alpha'_i$ for $i^\circ$ from the corresponding twisted-affine Dynkin diagram is not in $\{(\hat{w}')^{-1}(\tilde{\beta}_i)\}$.

(iv) For $\xi = -\rho_k$ as in (i), $R_s(\mu, \pi_b) = \mu(\pi_b) \mu(0) \tilde{\mu}(\mu, 0)$, where the ratio is calculated in (4.23) and the residues are as above. Explicitly,

$$
\text{Res}(\mu, 0) = \prod_{i=1}^n (1 - t_i X_{\alpha_i}) \prod_{\tilde{\alpha} > 0} \frac{1 - X_{\tilde{\alpha}}}{1 - t_{\alpha} X_{\tilde{\alpha}}}. \tag{6.32}
$$

For $\xi$ in the setting of (ii – iii), the formulas are as follows. The corresponding residues must be calculated for $\hat{R}', \mu'$ as for (i) and then multiplied by $\mu/\mu'(q^\xi)$, which is assumed nonzero.

Proof. (i). The binomials $\alpha_i(1 \leq i \leq n)$ are such for $\hat{w} = 0$, i.e. at the point $q^{-\rho_k}$. Then $(1 - t_i X_{\hat{w}(\alpha_i)})$ for $1 \leq i \leq n$ belong to the denominator of $\mu$ if and only if $\hat{w}(\alpha_i) \in \hat{R}_+$ for $1 \leq i \leq n$ and $\Lambda(\hat{w})$ does not contain roots from $R_+$. This is the defining property of elements $\pi_b$; see (4.22). Thus $\hat{w} = \pi_b$ for some $b \in P$ and the ratio $\mu(\hat{w})/\mu(0)$ is then nonzero due to (4.23). Thus, it suffices to check that the binomials from the denominator of $\mu$ vanishing at $0$ are exactly those for $\{\alpha_i, 1 \leq i \leq n\}$.
Next, if \((1 - t_\alpha X_{\bar{\alpha}})(q^{-\rho_k}) = 0\) for \(\bar{\alpha} = [\alpha, \nu, \alpha]\) > 0, then \(j = 0\) because \(q\) is generic, i.e. \(\bar{\alpha}\) is nonaffine and \(\alpha > 0\); let \(\nu = \nu_\alpha\).

For this \(\alpha\) and any \(i > 0\) such that \(\nu_i = \nu\), one has: \((\alpha, \alpha_i) \leq 0\). Otherwise, there exists \(\alpha_i\) such that \(\beta = \alpha - \alpha_i\) is a positive root in \(R\) satisfying \((1 - X_\beta)(q^{-\rho_k}) = 0\). However, the assumption is that this is impossible for any \(\beta\) (positive or negative). These inequalities give that \(\alpha\) and the roots \(\alpha_i\) such that \(\nu_i = \nu\) are linearly independent, which is impossible in the case of \(A, D, E\).

Let us consider now \(B, C, F, G\). Then such \(\{a_l\}\) are simple roots in the root subsystem \(R_\alpha\) formed by all roots \(\beta \in R\) such that \(\nu_\beta = \nu\), but possibly not all simple roots there. The positivity in \(R_\nu\) is that induced from \(R\); \(\alpha\) remains positive in \(R_\nu\). Let us check that \(\alpha\) is simple in \(R_\nu\). We will use the notation from the tables of \([B]\).

For any non-simple positive short root \(\alpha\), there exists \(\alpha_i\) of the same length such that \(\beta = \alpha - \alpha_i \in R\). This gives that \(X_\beta(q^{-\rho_k}) = 1\), which contradicts our condition for the numerator of \(\mu\). We conclude that \(\alpha\) can be only long if it is non-simple.

The same claim (the existence of \(\alpha_i\)) holds for long \(\alpha\) in \(R_{\text{long}}\) unless \(\alpha\) is simple in \(R_{\text{long}}\) with one reservation. In the case of \(C_n\), there is no such \(\alpha_i\) for \(\alpha = \varepsilon_j\) for \(j < n\) in the notation from \([B]\). For such \(\alpha\), \(\beta = (\alpha - \alpha_n)/2 = \varepsilon_j - \varepsilon_n\) is a short root in \(R\) and \(X_\beta(q^{-\rho_k}) = \pm 1 = t_{\text{short}}^{-\varepsilon_j}\). The latter relation is excluded and we can omit \(C_n\) in the next considerations.

Let us consider now \(B, G\). Since long \(\alpha\) is linearly independent with \(\alpha_i\), the dimension of the space generated by \(\alpha_i\) is \((n - 1)\). Thus \(\alpha\) must be the unique simple root of \(R_{\text{long}}\) that is not one of \(\alpha_i\). Recall that \(\{\alpha_i\}\) are simple in \(R\) and remain simple in \(R_{\text{long}}\), but the latter system contains other simple roots (unless for \(ADE\)). We obtain that \((\alpha - \alpha_m)/\nu_{\text{long}}\) is a short root in \(R\) for some \(\alpha_i\), which contradicts the condition \(t_{\text{short}}^{\nu_{\text{long}}} \neq 1\).

So the simplicity of long \(\alpha\) remains to be checked only in \(R_{\nu}\) for \(F_4\). Then \((R_{\text{long}})_+ = \{\varepsilon_i - \varepsilon_j\}\) for \(i < j\) and \(\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4\) for \(F_4\) in the notation from Plate 8 of \([B]\). Then for any non-simple root \(\beta > 0\) in \((R_{\text{long}})_+\) either \(\beta - \alpha_i\) or \((\beta - \alpha_i)/2\) belongs to \(R\), where \(i = 1\) or \(i = 2\). We come to a contradiction.

Thus, we obtain that \(\alpha\) must be simple in \(R_\nu\) (but not simple in the whole \(R\)). One has: \(\alpha = \alpha_0 + \nu_{\text{long}}\alpha_0\) and \(\alpha = \varepsilon_1 - \varepsilon_2\) unless possibly for \(F_4\). Thus, \(t_{\text{short}}^\nu = 1\) or \(\alpha = 2\alpha_4 + \alpha_2 + 2\alpha_3\) for \(F_4\), which results in \(t_{\text{short}}^\nu = 1\). These two relations were excluded. They can really occur as well as the relation \(t_{\text{short}}^{\nu_{\text{long}} - j} = \pm 1\) in the case of \(C_n\).

We note that, actually, the classification is not strictly necessary for the last step. One can use that there exists at least one short \(\alpha_i\) such that \((\alpha, \alpha_j) > 0\). Indeed, the rank of \(R\) would be \(> n\) otherwise. Thus,
\( \alpha - \nu \alpha_i \) is a long positive root. It can be only simple in \( R \) if \( \alpha_i \) is neighboring to long simple roots in the Dynkin diagram. The case of \( \alpha_i = \alpha_4 \) for \( F_4 \) is exceptional and must be considered separately.

(ii). Similarly to the considerations above, one has: \((\beta_i, \beta_j) \leq 0 \) for \( 1 \leq i < j \leq n \). Indeed, \( \tilde{\beta}_i - \tilde{\beta}_j \in \tilde{R} \) otherwise and, additionally, \( \tilde{\beta}_i - \nu_{\text{ng}} \tilde{\beta}_j \in \tilde{R} \) if \( \nu_j < \nu_i \). This gives that one of these differences will make the numerator of \( \mu \) vanishing, which is impossible. Then the required claims result from the following lemma.

**Lemma 6.2.** Let \((\beta_i, \beta_j) \leq 0 \) for \( \tilde{\beta}_i = [\beta_i, \cdots] \in \tilde{R} \) and \( 1 \leq i < j \leq n \). Then \( \beta_i \) can assumed in \( R_+ \) upon the action of some \( w \in W \). Provided this, assume that \( \beta = \sum_{i=1}^n m_i \beta_i \in R \) with \( m_i \in \mathbb{Z} \) such that \( m_i m_j < 0 \) for at least one pair \((i, j)\). Then there exist \( i, j \) such that \( \beta_i - \beta_j \in R \) and, additionally, \( \beta_i - \nu_{\text{ng}} \beta_j \in R \) if \( \nu_j < \nu_i \). Moreover, \( \tilde{\beta}_i - \nu_{\text{ng}} \tilde{\beta}_j \in \tilde{R} \), including the ADE systems. For BCFG, \( \tilde{\beta}_i - \tilde{\beta}_j \in \tilde{R} \) unless \( \beta_i, \beta_j \) are short and \( \beta_i - \beta_j \) is long or (always) if \( \tilde{\beta} = \sum_{i=1}^n m_i \tilde{\beta}_i \in \tilde{R} \).

Proof. The positivity condition making \( \{\beta_i\} \) positive is \( (\eta, \beta) > 0 \) for \( \eta = -\sum_{i=1}^n \eta_i \beta_i \) for sufficiently general \( \eta_i > 0 \) and they are linearly independent, which is standard. We set \( x = \beta - \sum_{m_j < 0} m_j \beta_j = \sum_{m_i > 0} m_i \beta_i \). Then \( (x, x) > 0 \) and \( (\beta, \sum_{m_i > 0} m_i \beta_i) > 0 \). Therefore, \( (\beta, \beta_i) > 0 \) for at least one \( \beta_i \) with \( m_i > 0 \), and \( \beta' = \beta - \beta_i \in \tilde{R} \) has the corresponding \( \sum_{m_i' > 0} m_i' \) smaller by 1 than that for \( \beta \). Similarly, we can diminish \( -\sum_{m_j < 0} m_j \) by 1 and continue diminishing the sums \( \sum_i \) or \( \sum_j \) until we obtain \( \beta_i - \beta_j \in R \). If \( \nu_j < \nu_i \) here, then \( (\beta_i, \beta_j) > 0 \); otherwise, \( |\beta_i - \beta_j| > |\beta_j| \). This results in \( \beta_i - \nu_{\text{ng}} \beta_j = s_{\beta_j}(\beta_i) \in \tilde{R} \).

Moreover, then \( \tilde{\beta}_i - \nu_{\text{ng}} \tilde{\beta}_j - s_{\tilde{\beta}_j}(\tilde{\beta}_i) \in \tilde{R} \).

The argument above used for the relation \( \tilde{\beta} = \sum_{i=1}^n m_i \tilde{\beta}_i \in \tilde{R} \) provides (formally) that \( \tilde{\beta}_i - \tilde{\beta}_j \in \tilde{R} \). Generally, \( \beta_i - \beta_j \) does not imply \( \tilde{\beta}_i - \tilde{\beta}_j \in \tilde{R} \) only if \( \beta_i, \beta_j \) are short, \( \tilde{\beta}_i - \tilde{\beta}_j = [\beta, m] \) is long and \( m \) is not divisible by \( \nu_{\text{ng}} \). This proves the last claim. \( \square \)

(iii). The roots \( \tilde{\beta}_i \) are positive with respect to the following positivity condition \( |X_{\tilde{\beta}}(q^\xi)| < 1 \) due to the inequalities \( |t_\nu| > 1 \). Recall that \( X_{\tilde{\beta}} = X_\beta q^{\nu_{\beta j}} \) for \( \tilde{\beta} = [\beta, \nu \beta j] \) and \( X_{\tilde{\beta}}(q^\xi) = t_{\tilde{\beta}}^{-1} \) for \( \tilde{\beta} = \tilde{\beta}_i \). It is possible that \( |X_{\tilde{\beta}}| = 1 \) for some \( \tilde{\alpha} \in \tilde{R} \), so we may need to deform \( \xi \) a little to ensure that this is really some positivity in \( \tilde{R} \). For \( \tilde{R}' \), it suffices to assume that \( q \) is sufficiently small, which will be checked together with the simplicity of \( \tilde{\beta}_i \).

For \( \tilde{\beta} \in \tilde{R}' \), the range of the values \( |X_{\tilde{\beta}}(q^\xi)| \) is a union of \( V_j = \{q^j | t_{\text{sh}}^m \} \), where \( j \in \mathbb{Z} \) and \(-C < m < C\) for some constant \( C \) calculated in terms of the Coxeter numbers of the irreducible components.
of $R^\dagger$. One can assume that $V_i \cap V_j = \emptyset$ for $i \neq j$ for sufficiently small $q$. Then $V_0 = \{ |X_\beta(q^\xi) | \text{ s.t. } \beta \in R^\dagger \}$ by construction.

If $|X_\beta(q^\xi)| < 1$ is not a positivity condition for the root system $\tilde{R}'$ or if $\{\tilde{\beta}_i\}$ are not simple for this positivity, then there exists $\tilde{\beta} \in \tilde{R}'$ such that $|X_\beta(q^\xi)| \in V_0$, which can be only if $\tilde{\beta} \in R^\dagger$. Using Lemma 6.2, we obtain that then there exists $\hat{\beta} = \beta_i - \tilde{\beta}_j \in \tilde{R}'$ such that $X_\beta(q^\xi) = q^{\pm m}$ for $m > 0$. However, this is impossible for sufficiently small $q$. A more exact analysis shows that the inequality for $q$ from (iii) is sufficient here. Alternatively, one can use (ii), which states that $\tilde{\beta}_i$ are simple in $R^\dagger$ for some positivity.

Then we find $\tilde{\omega}'$ in the affine Weyl group $\widetilde{W}' \subset \widetilde{W}$ of $\tilde{R}'$ transforming the standard affine Weyl chamber for $\tilde{R}'$ to that for the positivity above. The roots $(\tilde{\omega}')^{-1}(\tilde{\beta}_i)$ then become those described in (iii).

Part (iv) is actually a reformulation of (i). We set $f_i = (1 - t_i X_{\alpha_i})$ and choose the orientation clockwise. The corresponding residue is obtained from $\mu$ by the deletion of these binomials from the denominator and the evaluation of the rest at 0, which is the point $q^{-\rho_k}$. The extension to the setting of (ii – iii) is straightforward.

**Comments.** Given $t_\nu$, the inequality for $q$ in (iii) means that $-\Re k_\nu$ must be sufficiently large. Recall that $q \to 0$ is the limit to AHA. Actually, the condition for $q$ needed here is entirely algebraic. This inequality provides it, but this claim holds for generic $q$, which is similar to Lemma 6.2.

**Residues.** Let us provide a variant of formula (6.32) in (iv) for $b = 0$ when $\{\tilde{\beta}_i\} = \{\alpha_i, 0 \leq i \leq n\} \setminus \{\alpha_j\}$ for some $j \geq 0$. I.e. the formula below will be its (minor) generalization. Then $\rho_k^\dagger = \frac{1}{2} \sum_{\tilde{\alpha} \in R^+} \nu_{\tilde{\alpha}} \tilde{\alpha}$ and

$$Res(\mu, 0) = (1 - t_j X_{\alpha_j}) \prod_{i=0}^{n} (1 - X_{\alpha_i}) \prod_{\tilde{\alpha} > 0, \tilde{\alpha} \neq \alpha_i} \frac{1 - X_{\tilde{\alpha}}}{1 - t_\alpha X_{\tilde{\alpha}}} (X = q^{-\rho_k})$$

which is for a suitable choice of the orientation. The extension to arbitrary $\xi$ when the numerator of $\mu$ has no zeros and the corresponding $R'$, including the case of different $k_\nu$, is quite similar.

**Parts (ii–iii).** The conditions $t_{\text{ing}} = t_{\text{sht}}$, or $t_{\text{ing}} = t_{\text{lng}}^\rho$ there are the two cases of equal parameters in the twisted setting. Actually, the latter relation is more common; for instance, it is compatible with the usage of DAHA for quantum group invariants of links. We obtained that the classification of $\xi$ under these conditions can be reduced to that of closed finite root subsystem $R^\dagger$ of rank $n$ in a closed affine subsystem $\tilde{R}' \subset \tilde{R}$, for a closed root subsystems $R' \subset R$ of rank $n$. 
The classification of the latter up to the action of $W$ follows from the Borel - de Siebenthal theory. We note that this theory actually uses “affine tools”, so passage from AHA to DAHA seems natural from the perspective of classification the residual points.

Generally, the simplest case is $R' = R$ when $\{\tilde{w}^{-1}(\tilde{\beta}_i)\} = \{\alpha_i, i \neq i^o\}$ for some $\tilde{w} \in \hat{W}$. Furthermore, If $i^o \neq 0$, then we can assume that $\nu_{i\circ}n_{i\circ} \geq 1$ modulo the action of $\hat{W}$, where $\theta = \sum_{i=0}^{n} n_{i\circ}$. If $R' = R$ and $i^o = 0$, then we arrive at $(i): \{\tilde{w}^{-1}(\tilde{\beta}_i)\} = \{\alpha_i, 1 \leq i \leq n\}$.

The case of $A_n$. This always holds for $A_n$ because the only closed root subsystem in $R$ of rank $n$ is $R$ and all $n_i$ are 1. Thus, we can take $i^o = 0$ for $A_n$ modulo $\hat{W}$ and $\xi$ from $(ii - iii)$ are $-k\rho$ and their images under the action of $\pi_b$ for $b \in P$. This is parallel to the “orbit” of the Steinberg representation in the AHA theory.

Induced modules. Recall that the numerator of $\mu(q^\xi)$ is nonzero if and only if the $\mathcal{H}$-module $\mathcal{I}_\xi$ is $Y$-semisimple with simple spectrum. Generally, $(ii) - (iii)$ give some class of $\xi$ for generic $q,t$ where $\mathcal{I}_\xi$ are direct counterparts of $\mathcal{I}_{\rho_k}$. Their canonical irreducible $\mathcal{H}$-quotients described in Theorem 3.6.1 from [Ch1] generalize $\mathcal{X}$. In the notation there: $\Upsilon_0 = \hat{W}$ and $\Upsilon_* = \Upsilon_+$. We note some links to [VV].

Some examples. The closed subsystems $R' \nsubseteq R$ of rank $n$ from $(ii)$ can be “even” $A^n_1 = A_1 \oplus \cdots \oplus A_1$ ($n$ times) for $C_n, D_{2m \geq 4}, E_{7,8}, F_4, G_2$, the most reducible. We use the notation $X \oplus Y$ for the root system $X \cup Y$ in the direct sum of the corresponding $R$-spaces. For instance, $R' = \{\beta_i = 2\varepsilon_i, 1 \leq i \leq n\}$ is such for $C_n$. In this case, we must have $X_{\alpha_i}(q^\xi) = X_{\varepsilon_i}(q^\xi) = -1$ for $1 \leq i < n$ to ensure that the numerator of the corresponding $\mu$ is nonzero at $q^\xi$.

For $D_4$, the closed subsystem $R' = A^1_4$ is as follows: $\beta_{1,2} = \varepsilon_1 \pm \varepsilon_2, \beta_{3,4} = \varepsilon_3 \pm \varepsilon_4$. Accordingly, $\xi = -\rho'_k = -k(\varepsilon_1 + \varepsilon_3)$. One has: $X_{\bar{\beta}_i}(q^\xi) = t^{-1}$ for $1 \leq i \leq 4$ and $X_{\bar{\beta}_2-\bar{\beta}_3}(q^\xi) = -t$; notice the minus-sign. Another variant is for $\beta_1 = \varepsilon_1 \pm \varepsilon_2, \beta_2 = \varepsilon_2 \pm \varepsilon_3, \beta_{3,4} = \varepsilon_3 \pm \varepsilon_4$. i.e. for $A_1 \oplus D_3$ in $D_4$ ($D_3 = A_3$); then $X_{\varepsilon_i \pm \varepsilon_j} = t^{(4-i)\pm(4-j)}$ can be taken. These two examples can be readily extended to $D_{n-m} \oplus D_m$ in $D_n$ for any $1 \leq m \leq n, n \geq 4$. As above, the notation is from [B].

Not all closed subsystems of rank $n$ can really occur in $(ii)$; say, $A^1_1$ in $D_6$ will have zeros in the numerator of $\mu$ if we follow the above construction for $A^1_1 \subset D_4$.

Let us give an example when not all $\beta \in R'$ can be lifted to $R^\dagger$. For the root system $B_n$, we take $\tilde{\beta}_i = \alpha_i = \varepsilon_i \pm \varepsilon_i$ for $1 \leq i \leq n-2, \tilde{\beta}_{n-1} = \varepsilon_{n-1}, \tilde{\beta}_n = [-\varepsilon_n, 1]$. Then $X_{\alpha_i}(q^\xi) = q t_{sh}^i, X_{\varepsilon_i}(q^\xi) = t_{sh}^{-1}, X_{\varepsilon_{n-1}}(q^\xi) = t_{sh}^{-1}$ for $2 \leq i \leq n-1$. Thus, $X_{\varepsilon_i \pm \varepsilon_{n-1}}(q^\xi) = q$ and all other $X_{\alpha}(q^\xi)$ for $\alpha \in R$.
contain powers of $t^n$. For $|t^n| > 1$ and generic $q$, the numerator of $\mu(q^x)$ is nonzero. It is used here that $\tilde{\beta}_{n-1} - \tilde{\beta}_n = [\epsilon_{n-1} + \epsilon_{n-2}, -1]$ is not from $\tilde{R}$ because $\epsilon_{n-1} + \epsilon_{n-2}$ is long.

7. Residual subtori and points

Informally, they are those that can potentially occur in the meromorphic continuation of the functional $I^{im}(f) = \int_{\mathbb{R}^n} f(x) \mu(q^x; q, t) dx$ from $\Re k_\nu > 0$ to all complex $k_\nu$ or for $I^{ia}(f)$. If they can be obtained from each other by the action of $\hat{w} \in \hat{W}$, we say that they are in the same packet. However not all $\mu$-residual subtori and points defined below really occur in the integral formulas; finding them is a combinatorial problem, which can be involved. After they are found, the count of the corresponding coefficients, the residues for the points, is an entirely algebraic procedure. The following definition is a double affine extension of Definition 2.1 from [HO1] coupled with Theorem 2.2 to the $\mu$-function. Also, see [O3] (Theorem 7.1, Remark 7.3).

**Definition 7.1.** We continue to assume that $0 \leq q < 1$ and $t_\nu$ are sufficiently general. The double affine residual subtori, called $\mu$-residual below, are the affine tori $T$ of codimension $m$ given by the equations $1 - t_\beta X_\beta = 0$ for $1 \leq i \leq m$ and linearly independent $\tilde{\beta}_i \in \tilde{R}_+$, provided the following condition. The number $\alpha_1$ of the binomials $(1 - t_\alpha X_\alpha)$ for $\tilde{\alpha} \in \tilde{R}_+$ vanishing at $T$ must be $\geq \alpha_0 + m$ for the number $\alpha_0$ of $(1 - X_\alpha)$ for $\tilde{\alpha} \in \tilde{R}_+$ vanishing at $T$. The $\mu$-residual points are for $m = n$. □

The residual points play a key role in the $q, t$-case. They alone are sufficient to obtain the meromorphic continuation of $I^{im,ia}(f)$ for $|t_\nu| > 1$ ($\Re k_\nu < 0$) provided the integrability and the convergence of $f(x)$. The convergence conditions depend on $\Re k_\nu$ and the order of iterated integrations. Any analytic functions $f(x)$ integrable in the imaginary directions of no greater than exponential growth in the real directions can be taken here when $\Re k < 0$ is sufficiently large. We will provide a reasonably complete general description of residual points for sufficiently general $t_\nu$ in the case of “equal parameters”. The calculation of the corresponding residues is straightforward when $\alpha_1 - \alpha_0 = n$.

We note that a direct affine generalization of the AHA residual subtori from in [HO1] is more restrictive. In our context, it would be $\alpha_1 - \alpha_0 \geq m$, where $\alpha_0$ is the number of binomials $(1 - X_\alpha)$ vanishing at $0$ for $\tilde{\alpha} \in \tilde{R} \cup \tilde{R}_+$. This is basically the switch to $\delta$, the symmetrization of $\mu$, and $W$-invariant functions $f(x)$; we will not discuss this possibility in the paper.

Following the proof of part $(ii)$ of Theorem 6.1, we obtain the following claim, which reduces the description of $\mu$-residual points to some
combinatorial analysis of the corresponding root system. Any residual \(\mu\)-point \(\xi\) can be obtained by the following construction, though we do not claim that they occur in some integral formulas and that the corresponding residues are nonzero.

**Theorem 7.2.** As in (ii, iii) of Theorem 6.1, \(0 < q < 1\), let \(q^m \neq t^l\) for any integer \(l, m \neq 0\), \(|t| > 1\) for any \(v\) and either \(t_{\text{ing}} = t_{\text{shl}}\) or \(t_{\text{ing}} = t_{\text{shl}}^{\text{lng}}\). Also, we assume that \(q\) is sufficiently general by imposing the condition from (iii) there.

Given a closed root subsystem \(R^0 \subset R\) of rank \(n\), we begin with a subset \(\{\alpha_i, i \in I\} \subset \{\alpha_i^j\}\) of simple roots of \(\bar{R}_+^0 = \bar{R}_+ \cap \bar{R}_0\) such that exactly one simple root is removed from \(\{\alpha_i^j\}\) for every connected component of \(R^0\). We follow Theorem 6.1, (iii). Then we fix \(\bar{w} \in \bar{W}\). Let \(\bar{\beta}_i = \bar{w}(\alpha_i^j)\) and \(R^1\) be the (closed) root subsystem with simple roots \(\{\bar{\beta}_i = [\beta_i, \cdots] \mid i \in I\}\), which are assumed from \(\bar{R}_+\).

Next, let \(\{i \in I'\}\) be a subdiagram of the Dynkin diagram \(\{i \in I\}\) of \(R^1\) and \(R^1\) be the corresponding closed root subsystem of \(R^1\). Then we define \(\xi \in \mathbb{C}^n\) such that \(q^{\xi, \beta_i} = 1\) for \(i \in I'\) and \(q^{\xi, \bar{\beta}_i} = t_\xi^{-1}\) for \(i \in I \setminus I'\), where \(\nu_i^\ast = \nu_{\bar{\beta}_i}\), \((\xi, [\beta, j\nu]) = (\xi, \beta) + j\nu\).

Then the numerical condition for \(\mu\)-residual points \(\xi\) becomes

\[
| \{\bar{\beta} = \bar{\beta}_m + \sum_{i \in I'} c_i \bar{\beta}_i \subset R^1_+ \text{ s.t. } m \in I \setminus I', c_i \in \mathbb{Z}_+\} | - | R^1_+ | \geq n.
\]

Any \(\mu\)-residual points occur in this way for proper \(R^0, \bar{w}, R^1, R^1\). Moreover, \(\bar{w}\) can be assumed from \(\bar{W}'\) if \(\bar{R}^0 = \bar{R}'\) for the closed root subsystem \(R'\) generated by \(\beta_i\) for \(i \in I\).

**Proof.** The direct statement follows from the definition of \(\mu\)-residual points. We need to check that any \(\mu\)-residual \(\xi\) can be represented in this way. Let \(R' \subset R\) be a closed root subsystem of the same rank as \(R\) such that its standard (full) affine extension \(\bar{R}' \subset \bar{R}\) contains the set \(\mathcal{R}^1 = \{\bar{\beta} \in \bar{R}_+ \mid X_{\bar{\beta}}(q^\xi) = t^{-1}\}\) and the closed root subsystem \(\mathcal{R}^0 = \{\bar{\beta} \in \bar{R}_+ \mid X_{\bar{\beta}}(q^\xi) = 1\}\).

We take simple roots of \(\mathcal{R}^0\) and add to them primitive roots from \(\mathcal{R}^1\) defined as \(\bar{\beta}\) there such that \(\bar{\beta} \neq \bar{\beta}' + \alpha\) for \(\bar{\beta}' \in \mathcal{R}^1\) and \(\alpha \in \mathcal{R}^0\). Let this set be \(\{\bar{\beta}_i, i \in I\}\), where \(\bar{\beta}_i\) for \(i \in I'\) are all simple roots from \(\mathcal{R}_0^1\). This set linearly generates \(\mathbb{R}^n\) and satisfies the conditions \((\bar{\beta}_j, \bar{\beta}_i) \leq 0\) for \(i \in I, j \in I \setminus I'\) because \(\bar{\beta}_j\) are assumed primitive. Similarly, \((\bar{\beta}_i, \bar{\beta}_j) \leq 0\) for primitive ones, i.e. for \(i, j \in I \setminus I'\), because otherwise \(\bar{\beta}_i - \bar{\beta}_j \in \mathcal{R}^0\) and one of them cannot be primitive. Thus, \(\{\bar{\beta}_i, i \in I\}\) are linearly independent and \(|I| = n\).

Then we impose the inequality for \(q\) from Theorem 6.1, (iii) for the system \(R^1\). Following the reasoning there, we introduce the positivity
condition for $\tilde{\beta} \in \tilde{R}'$ by $|X_{\tilde{\beta}}(q^{\xi'})| < 1$, where $\xi'$ is a small deformation such that $|X_{\tilde{\beta}}(q^{\xi'})| \neq 1$ for $\tilde{\beta} \in R^0$. One has: $X_{\tilde{\beta}}(q^{\xi'}) \approx t_{\tilde{\beta}}^{-1}$, where $i \in I \setminus I'$, $\nu^*_{\tilde{\beta}} = \nu^*_{\tilde{\beta}}$, so they are still positive. This positivity may result in different simple roots in $R^0$: let us take them as $\tilde{\beta}_i$ for $i \in I$ instead of the initial ones.

Following the proof of $(iii)$, $\tilde{\beta}_i (i \in I)$ are simple roots for the positivity condition above for sufficiently small $q$ (under the inequality we imposed). Thus, they become simple in $\tilde{R}'$ upon the action of some $\tilde{w}' \in \tilde{W}'$ for the positivity condition there induced from that in $\tilde{R}_+$. We obtain that $R^0$ is $W$-conjugated to $R'$.

\section{Integral presentations}

For $\mathcal{R}k_{\nu} > 0$ the following inner products in $\mathcal{X}$ induce $\diamondsuit_{\pm i}$ for $l > 0$:

\begin{align}
\langle f, g \rangle_{l}^{im} &= \int_{\mathbb{R}^n} fT_{w_0}(g^c)q^{-lx^2/2}\mu(q^x; q, t)dx \text{ induces } \diamondsuit_{l}, \\
\langle f, g \rangle_{l}^{re} &= \int_{\mathbb{R}^n} fT_{w_0}(g^c)q^{-lx^2/2}\mu(q^x; q, t)dx \text{ induces } \diamondsuit_{-l}.
\end{align}

Here $f, g \in \mathcal{X}$, but this can be extended to any completions of $\mathcal{X}$ provided the analyticity of $f, g$ and the integrability. We use here that $q^{-lx^2/2}$ is $W$-invariant, commutes with $T_{w_0}$ (considered as an operator of multiplication), and is preserved by $\diamondsuit$; see (3.14). One has:

$$
\langle f, H(g) \rangle_{l}^{im} = \langle f, q^{-lx^2/2}H(g) \rangle_{l}^{im} = \langle f, (q^{-lx^2/2}H)(f) \rangle_{l}^{im} = \langle (q^{lx^2/2}H) \circ \cdot \cdot \cdot \circ \cdot \cdot \cdot (H) \circ \cdot \cdot \cdot \circ \cdot \cdot \cdot (H) \rangle_{l}^{im}
$$

for $H \in \mathcal{H}$. Here $q^{lx^2/2}H \circ \cdot \cdot \cdot \circ \cdot \cdot \cdot (H) = q^{lx^2/2} \circ \cdot \cdot \cdot \circ \cdot \cdot \cdot (H) = \tau_{1}(\diamondsuit(H))$. For $l = 0$, the following integral replaces (8.33):

\begin{align}
\langle f, g \rangle_{0}^{\alpha} &= \frac{1}{(2\pi i\alpha)^n} \int_{\mathbb{R}^n/2\pi i\mathbb{Z}^n} fT_{w_0}(g^c)\mu(q^x; q, t)dx = \\
&= \frac{1}{(2\pi i\alpha)^n} \int_{-\pi\alpha}^{\pi\alpha} \cdot \cdot \cdot \int_{-\pi\alpha}^{\pi\alpha} fT_{w_0}(g^c)\mu(q^x; q, t)dx_{\alpha_1} \cdot \cdot \cdot dx_{\alpha_n} \text{ for } q = e^{-1/\alpha}.
\end{align}

Here the order of integration can be arbitrary, though the meromorphic continuation to negative $\mathcal{R}k_{\nu}$ depends on this order. This integral coincides with the constant term $ct(fT_{w_0}(g^c))\mu$ for $\mathcal{R}k_{\alpha} > 0$ and provided the inequalities $|t_{\alpha}|^2 < q_\alpha$. Indeed, $\mu(q^x)$ is analytic in the annulus $t_{\alpha}q_{\alpha}^{-1} < |X_\alpha| < t_{\alpha}^{-1}$ for $\alpha \in \tilde{R}_+$. Therefore we can replace $\mu$ with the corresponding Laurent series: its expansion in terms of $q^i$ for $i \geq 0$ and $t_{\alpha}X_{\tilde{\alpha}}$ for $\tilde{\alpha} \in \tilde{R}_+$.

The fact that the imaginary integrals give the $\diamondsuit_{l}$-invariant DAHA inner products for $l \geq 0$ does require the conditions $\mathcal{R}k_{\nu} > 0$. The give that there are no singularities of $\mu$ between the initial contour of
integration and its translations by $b \in P$ when $\mathcal{R}k_{\nu}$ is sufficiently large. Then the analytic continuation to any $\mathcal{R}k_{\nu} > 0$ is used.

Indeed, the poles of $\mu$ modulo the imaginary periods are at $x_{a^\vee} = -k_\alpha - i$ and $x_{a^\vee} = k_\alpha + i + 1$ for $\alpha \in \mathbb{R}_+$, $i \geq 0$. Thus, the “gap” between $-\mathcal{R}k_\alpha - 1$ and $\mathcal{R}k_\alpha + \nu$ gives the required when $\mathcal{R}k_{\nu} \gg 0$; Stokes’ theorem is used. The integrals over $i\mathbb{R}^{\nu}$ make sense of course for any sufficiently general $k_{\nu}$ but the corresponding pairings are only $\mathcal{H}_X$-invariant (not $\mathcal{H}$-invariant for $\mathcal{R}k_{\nu} < 0$).

**Comments.** Making $g = 1$, $(f, 1)^{im}_{l_1}$ is a coinvariant of level $l$, i.e. that for $\diamondsuit_l$. For $f \in \mathcal{X}$, one can switch here from the imaginary integration to $(\cdots)^a$. Namely, we replace $q^{-lx^2/2}$ in the integrand with the sum of its translations by $2\pi a P^\vee$ and use that $\mu$ is in terms of $X_\alpha$.

Alternatively, let $(f, g)^{ia}_{l_1} \overset{\text{def}}{=} \langle f, g \Theta(q^x)^l \rangle_0^{ia}$, where we can use the theta-function $\Theta(q^x) \overset{\text{def}}{=} \sum_{bc \in P} X_b q^{bx^2/2}$ for $\tilde{R}$ instead of $q^{-x^2/2}$ because $q^{x^2/2} \Theta(q^x)$ is $\tilde{W}$-invariant. Then $(f, 1)^{ia}_{l_1} = (f, \Theta(q^x)^l)_0^{ia}$ is a coinvariant of level $l$ too. Note that for $\mathcal{R}k_{\nu} > 1$, the integral $(f, 1)^{ia}_{l_1}$ for $f \in \mathcal{X}$ is reduced to taking the corresponding constant term.

Generally, we have two different approaches, which result in the coinciding (proportional) formulas only for $l = 1$. This is because the space of coinvariants is one-dimensional for $\diamondsuit_l$ only for $l = 0, \pm 1$. For $l = 1$ the explicit connection is established via the functional equation for $\Theta(q^x)$; see, e.g., [Kac] and Lemma 4.6 from [Ch6]. Actually, this is how the functional equation for $\Theta$ can be justified.

For the sake of completeness, let us state Theorem 4.9 from [ChD] in this context. One has for $l = 1$:

\begin{equation}
(8.36) \quad (f, g)^{im}_{1} = (1, 1)^{im}_{1} (\tau^{-1}_-(f) T_{w_0} (\tau^{-1}_-(g^x)))(q^{-\rho_k}).
\end{equation}

Here we use that $\tau_-$ acts in $\mathcal{X}$; the nonsymmetric Macdonald polynomials are its eigenvectors. As above: $X_a(q^x) = q^{(a,b)}$ and for any functions here. In particular, $(f, 1)^{im}_{1} = (1, 1)^{im}_{1} (\tau^{-1}_-(f))(q^{-\rho_k})$.

**The space of coinvariants.** More generally, let us consider $(f, \Theta[l])^{ia}_{0}$ for any theta-functions $\Theta[l]$ of level $l$. They are by definition analytic in terms of $q^x$ such that $\Theta[l]/\Theta^l$ are $\tilde{W}$-invariant. These functionals are coinvariants of level $l$ for $\diamondsuit_l$. This approach actually gives that the dimension of the space of such coinvariants coincides with the number of the integrable irreducible Kac-Moody modules of level $l > 0$ for the root system $\tilde{R}$. This is an algebraic fact: Theorem 2.13 from [ChM]. The proof there was based on the deformation argument. Equivalently, this number is the dimension of the space of inner products in $\mathcal{X}$ associated with $\diamondsuit_l$; cf. Theorem 4.4.
We note that a certain $q,t$-generalization of affine Demazure characters of any level $l > 0$ was suggested in [ChM]; a connection is expected with paper [Kat] upon the limits $t \to 0, \infty$.

Given any $f \in \mathcal{R}$ and using the constant term functional, the coinvariants $\text{ct}(f \Theta[l] \mu)$ for any $l \geq 0$ and theta-functions $\Theta[l]$ of level $l$ are meromorphic functions in terms of $k_{\nu}$. The formulas are explicit for $l = 0,1$ and the nonsymmetric Macdonald polynomials taken as $f$: some products of binomials. They are the generalized difference Macdonald-Mehta identities. Also, one can use that $l$ for valov anti-involution, which provides that the coinvariants for $l = 1$ are actually analytic upon some normalization. Employing “picking up the residues” we arrive at “the DAHA trace formulas” for any $l > 0$.

**Non-compact theories.** Let us briefly discuss the real integration. Here $k$ is arbitrary complex and there is no problem with an analytic continuation to $\Re k < 0$ for $k$ sufficiently close to the real axis. The integration is $I_{\pm}^e(f) = \int_{u \in \mathbb{R}^n} f(x) \mu(q^x) dx$ for $\epsilon > 0$ and regular $\rho \in \mathbb{R}^n$; the poles of $\mu$ at $\mathbb{R}^n$ must be avoided. We can set $(f,g)_{\pm}^e = I_{\pm}^e(f T_{w_0}(g^c) q^{\imath x^2/2})$, where the Gaussian ensures the convergence.

We note that the Jackson integration $J(f; \xi)$ is related to $I_+^e - I_-^e$. In its turn, $J(f; \xi)$ is related to the imaginary integration, so we have some connection between the imaginary and real integrations via the Jackson integration. The latter is related to $\hat{\mathcal{F}}_+$. For instance, the Jackson integration of $f q^{x^2/2}$ for $l = -1$ and $\xi = -\rho_\lambda$ is basically $\hat{\mathcal{F}}_+(f q^{x^2/2})/\Theta(q^x)$, which is a constant for any Laurent polynomial $f$.

Let us provide the adjustment of the identity from (8.36) to the real integration: $(f,g)_{\pm}^e = (1,1)_{\pm}^e \left( \tau_-(f) T_{w_0}(\tau_-(g^c)) \right)(q^{-\rho_\lambda})$. The formula for $(1,1)_{\pm}^e$ is quite interesting. For $A_1$, it is in terms of Appel functions due to Etingof; see Section 2.3.5 of [Ch1]. This is an indication that we can try to replace $q^{-x^2/2}$ by $1/\Theta(q^x)$ and connect $(f,g)_{\pm}^e$ with $(f,g)_{\pm}^{im}$. The series for $1/\Theta$ is of fundamental importance; see e.g. [Car].

An important feature of the real (noncompact) theory is that $\mu(q^x)$ can be replaced by $\tilde{\mu} = \mu^{-1}(q^x; q, t_\nu^{-1})$ from (8.34). Everything in the real theory is up to quasi-constants, which are $W$-periodic functions. Using this feature, we can replace the denominator of $\tilde{\mu}$ by the Gaussian with some corrections ensuring the proper multipicators upon the action of $P$. This will “eliminate” the denominator of $\mu$ and therefore we can make $\epsilon = 0$ in the contour shift $\imath \in \mathbb{R}$ above.

**Theorem 8.1.** Let $h = (\rho^\vee, \theta) + 1$ be the dual Coxeter number, and

$$M(x) = \sin(\pi(2\rho^\vee, x)) q^{\frac{x^2}{2}} X_{\rho}^{-1} \prod_{\alpha > 0} (1 - t_\alpha^{-1} X_\alpha).$$
Then the pairing \( \int_{\mathbb{R}} f T_{w_0}(g^x) q^{tx^2/2} M(x) \, dx \) is well defined for any \( q, t \nu \) and real \( l > 0 \); it induces in \( \mathcal{X} \) \( f, g \) the anti-involution \( \vartriangleleft -l \) for \( l \in \mathbb{N} \).

Proof. We set \( x^{\nu} \overset{\text{def}}{=} x^{\alpha \nu} = (x, \alpha \nu) \); recall that \( X_a = q^{e_a} \). Let us calculate explicitly the multipliers of the functions under consideration upon the translations by \( \omega \). For \( 1 \leq j \leq n \), one has:

\[
\omega_j^{-1} \left( h x^2/2 - (x, \rho) \right) = h(x + \omega_j)^2/2 - (x + \omega_j, \rho)
\]

\[
= h x^2/2 - (x, \rho) + h(x, \omega_j) + h\omega_j^2/2 - (\omega_j, \rho).
\]

The change is \( h(x, \omega_j) + h\omega_j^2/2 - (\omega_j, \rho) \). Next, using \( l(\omega_j) = (2 \rho^\vee, \omega_j); \omega_j^{-1}(\sin(\pi(2 \rho^\vee, x))) = \sin(\pi(2 \rho^\vee, x + \omega_j)) = (-1)^{l(\omega_j)} \sin(\pi(2 \rho^\vee, x)) \).

For the denominator \( \Delta(q^x) = \prod_{a>0} (1 - X_a^x) \) of \( \hat{\mu} \), which is basically the denominator of the twisted Kac-Moody character formula, one has:

\[
\omega_j^{-1} \left( \Delta(q^x) \right) \Delta(q^x)^{-1} = \Delta(q^{2x\omega_j}) \Delta(q^x)^{-1} = \prod_{a>0} (-X_a^{-1})^{(\alpha, \omega_j)/q - \frac{\omega_j^2}{2}} \delta_j^a = (\alpha^\vee, \omega_j).
\]

It equals \((-1)^{l(\omega_j)} q^{-\sum_{a>0} (\alpha, \omega_j) x_a^\vee} \delta_j^a \delta_j^{(\alpha, \omega_j)^{-1}} \).

Then we use the standard identity: \( \sum_{a>0} (\alpha^\vee, u) \alpha = h u \), which holds for any \( u \in C^n \). For the sake of completeness, let us provide its proof. Setting \( \sum_{a>0} (\alpha^\vee, u) \alpha = \hat{u}, (\hat{u}, v) \) is a \( W \)-invariant symmetric form. We obtain that \( (\hat{u}, v) = c(u, v) \) for some constant \( c \) due to the irreducibility of \( R \), and \((\hat{\theta}, \theta^\vee) = c(\theta, \theta^\vee) = 2c \). Let us use now that \( (\alpha, \theta^\vee) = 1 \) unless \( (\alpha, \theta^\vee) = 0 \) and \( \alpha = \theta \), when it is 2. Thus, \( (\hat{\theta}, \theta^\vee) = \sum_{a>0} (\alpha, \theta^\vee) (\alpha, \theta^\vee) = (2 \rho^\vee, \theta) + (\theta^\vee, \theta) = 2(\rho^\vee, \theta) + 2 \) and \( c = h \). Using the same identity, \( \sum_{a>0} (\alpha, \omega_j) (\alpha, \omega_j)/2 = h \omega_j^2/2 \) and:

\[
\omega_j^{-1} \left( \Delta(q^x) \right) \Delta(q^x)^{-1} = (-1)^{2(\rho^\vee, \omega_j)} X_{\omega_j}^{-h} q^{-\omega_j/2} h(\rho, \omega_j).
\]

The convergence for \( l > 0 \) is the same as it was for \( \mu \). Integer levels \( l > 0 \) are needed here for \( \vartriangleleft -l \) to serve the inner product. \( \square \)

The convergence holds here for \( l = 0 \) when \( f, g \) are of sufficiently small degrees depending on \( \Re k_u < 0 \). This is exactly as in Theorem 4.5, (i). Thus, such \( f, g \) can be served by both, the imaginary and real integrations, when \( l = 0 \). The inner product will be the same up to proportionality.

The case of \( A_1 \) and \( q \)-zeta. Let \( x = x_\omega, x_\alpha = 2x_\omega \), the Gaussian is \( q^{x^2/2} \) in terms of such \( x \). We will omit 1 in \( \alpha_1, \omega_1 \). Then we obtain:

\[
M(x) = \sin(2\pi x) q^{2x^2/2} (1 - q^{2x}) \prod_{i=1}^{\infty} (1 - t^{-1} q^{2x+i})(1 - t^{-1} q^{-2x+i}).
\]

Accordingly, the pairings for \( l \geq 0 \) are \( \int_{\mathbb{R}} f(x) T(g(x)) q^{x^2} M(x) \, dx \) for any \( k \) or \( \int_{\mathbb{R}} f(x) T(g(x)) q^{-lx^2} \mu(x) \, dx \) for \( \Re k > 0 \) (subject to the meromorphic continuation to \( \Re k < 0 \)).
The integrals \( \int_{\mathbb{R}} \frac{q^{x^2}}{1 \pm q^{x^2}} M(x) \, dx \) and similar ones lead to the definition of the “real” \( q \)-zeta function and Dirichlet \( q \)-\( L \)-functions studied in [Ch7]. The imaginary integration results in their “imaginary counterparts”. Such integrands ensure the convergence for \( \int_{\mathbb{R}} \) and for \( \int_{\mathbb{R}^i} \), but there will be now poles due to their denominators.

Upon some symmetrization needed for the functional equation, the one for Dedekind’s zeta, they conjecturally satisfy the Riemann hypothesis in terms of \( s = k + \frac{1}{2} \) (Conjecture 6.3 at the end of [Ch7]). There is another version of RH there without the symmetrization: all “interesting” zeros belong to one half-plane with respect to \( \Re s = 1/2 \).

The analytic continuation to \( s < 1/2 \) is not needed for the real integration and the Jackson-type summation. In the case of imaginary integration, this continuation can be achieved using the pole decomposition and integral formulas: the ones we will do below, but with the contributions of zeros of \( 1 \pm q^{x^2} \).

The limit to the classical \( \zeta(s) \) and the corresponding \( L \)-functions \( L(s) \) times some \( \Gamma \) is when \( q \to 1 \). This is generally for any \( s \).

However, \( \int_{\mathbb{R}} \frac{q^{x^2}}{1-q^{x^2}} M(x) \, dx \) (for the minus sign here) will converge to \( \sim \tan(\pi k) \Gamma'(k)^2 \) for \( \Re k < 1/2 \) (i.e. for \( s < 1 \)). It will converge to the (modified) zeta for \( \Re s > 1 \) in this (exceptional) case. Actually, it will be like this even for \( \Re s < 1 \) unless \( a \) become very large.

The convergence is generally fast when \( q < 1 \), even for \( a \sim 1000 \) or so (for reasonably small \( \Im k \)), which is thanks to the Gaussians.

**Six major DAHA theories.** To summarize, we mainly have two theories: the one based on the imaginary integration and that for the real integration. In the Harish-Chandra theory, they are the so-called compact and noncompact cases. Totally, we have 6 major theories by now, corresponding to different choices of “integrations”: from \((i)\) to \((vi)\).

Namely, \((i)\) the usage of the constant term, \((ii)\) imaginary integrations, \((iii)\) real integrations, \((iv)\) Jackson integrations, \((v)\) the theory at roots of unity, and \((vi)\) the theory at \( |q| = 1 \) when Barnes’ Gamma functions are needed. Basically, any \( \hat{W} \)-invariant integration can be taken for the corresponding \( \mu \)-measure.

The \( \hat{W} \)-invariance of the initial integration is immediate for \((i)\) and \((iii) - (v)\). We mostly stick to the imaginary integration (case \((ii)\)) in this paper, which requires “picking up the residues” and integral formulas below for \( \Re k_\nu < 0 \). This one matches the \( p \)-adic theory and can potentially admit some adelic version.

Also, there is “DAHA-Satake theory”, which is based on the usage of the affine symmetrizers, \( \hat{P}_+ \) and \( \hat{I}_+ \). The latter operator is the summation over extended affine Weyl group “twisted” by the \( \mu \)-function, which is closely related to the Jackson integration. The former is the
affine \( t \)-symmetrizer, which does not require any integration (and
the \( \mu \)-function), and certainly has some adelic generalization.

Let us mention 2 more directions: DAHA theory over finite fields and
(related) theory when \( q \)-Gamma and the corresponding \( \mu \) are replaced
by those in terms of the \( p \)-adic Gamma.

The \( p \)-adic limit. The \( p \)-adic limit is considered in detail in [Ch1,
ChM]. Basically, \( t^{-1} \) becomes the cardinality of the residue field and
\( q \to 0 \), but we need to be more exact here.

Let us use the homomorphism \( \zeta : \mathcal{H}_X \to \mathcal{H}_Y \) sending \( X_b \mapsto Y_b \)
for \( b \in P \), \( T_i \mapsto T_i^{-1} \) for \( 1 \leq i \leq n \), and \( t_{\nu}^{1/2} \mapsto t_{\nu}^{-1/2} \). We will
extend it elements in \( \mathcal{H}_X \) and \( \mathcal{E} \) depending of \( q \) by making \( q \to 0 \); the
notation will be \( \zeta_0 \). Then \( \zeta_0(\mathcal{E}_b) = \psi_b \overset{\text{def}}{=} t^{-l(0)/2}Y_bP_+ \) for \( b \in P \), which
are Matsumoto spherical functions in \( \mathcal{H}P_+ \). Accordingly, the Satake-
Macdonald \( p \)-adic spherical functions are \( P_+ \psi_b(b \in P_-) \), the images of the symmetrizations of \( \mathcal{E}_b \).

Here, as above, \( \mathcal{E}_b \) are nonsymmetric Macdonald polynomials under
the normalization \( \zeta(q^{-\mu}) = 1 \), \( P_+ = \sum_{w \in W} t^{-l(w)/2}T_w^{-1} / \sum_{w \in W} t^{-l(w)} \).

For the AHA of type \( A_1 \),
\[
\psi_0 \overset{\text{def}}{=} t^{-\frac{\mu}{2}}T_0, \quad P_+ = (1 + t^{1/2}T)/(1 + t) \quad \text{for} \quad n \in \mathbb{Z}.
\]

Let \( \mu^0 \overset{\text{def}}{=} \lim_{q \to 0} \mu = \prod_{a > 0} \frac{1 - X_a}{1 - tX_a} \). We set \( \langle f, g \rangle^0 = \text{ct} \langle f T_{0w}(g^c)\mu^0 \rangle \),
where \( q \to 0 \). Recall that the "\( p \)-adic trace" and the standard anti-
involution in \( \mathcal{H}_Y \) are as follows: \( \langle T_w \rangle_{\text{reg}} = \delta_{id,w} \) and \( T_w^* \overset{\text{def}}{=} T_w^{-1} \). We
omit the complex conjugation of the coefficients.

We arrive at the following nonsymmetric spherical AHA-Plancherel
formula for any Laurent polynomials \( f, g \) in terms of \( X_b \):
\[
\langle f, g \rangle^0 \bigg|_{t_0^{1/2} \to t_0^{-1/2}} = \sum_{w \in W} t^{-l(m_n)/2} \left\langle \langle \zeta_0(f)P_+(\zeta_0(g)P_+) \rangle^* \right\rangle_{\text{reg}}.
\]

It includes the presentation of the Matsumoto spherical functions as
nonsymmetric Hall polynomials.

The Gaussian and the action of projective \( PSL_2(\mathbb{Z}) \) collapse as \( q \to 0 \), and the definition of the Fourier transform requires the characters of the (unitary) irreducible representations. In the \( q, t \)-setting, (8.37)
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
is direct from the action of the \( q, t \)-setting (ChM).

9. Mermoromorphic continuations

For \( 0 < q = e^{-1/a} < 1 \) as above and \( v \in \mathbb{R} \), we set:
\[
I_v^a(f) \overset{\text{def}}{=} \int_{u - i\pi a}^{u + i\pi a} \cdots \int_{v - i\pi a}^{v + i\pi a} f(q^x)\mu(q^x)dx_{\alpha_1} \cdots dx_{\alpha_n}.
\]
Here the order of simple roots \( \alpha_i \) can be arbitrary. Functions \( f(q^x) \) are assumed series in terms of
\( X_{\alpha}(a \in Q) \) convergent in sufficiently large strips \( |Re| < C \); the norm
is the standard one in \( \mathbb{R}^n \). For the sake of convenience of notations we
Theorem 9.1. (i) Let \( \mathbb{R} \) be any entire functions in sufficiently large strips \( M \) considered for proper \( \mathbb{R} \) moving the contours of integration by \( \mathbb{R} \) is considered, i.e., \( \mathbb{R} \) restrict ourselves with \( a \in Q \). If the whole polynomial representation is considered, i.e., \( X_b \) for \( b \in P \), then the integrals \( \frac{1}{M} \int_{u+im} f(u) \) must be considered for proper \( M \in 1 + Z_+ \).

For the corresponding \( I^m \) we integrate over for \( y \) can be any entire functions in sufficiently large strips \( |\mathbb{R}| < C \) provided the integrability of \( |f| \) in the imaginary directions. The notation \( I^m \) used above is for \( v = 0 \).

Theorem 9.1. (ii) We define \( \Sigma^a(f) \) as the sum of the residues of \( \mu(q^x) f(q^x) \) over \( \mu \)-residual points \( \xi \) subject to the consecutive inequalities \( \Re x_\alpha > r \) for \( 1 \leq i \leq n \) imposed when taking the iterated integrals \( \int_{u+im} f(u) \) from \( \Re k_i > 0 \) to \( \Re k_i > -\epsilon \) for some \( \epsilon > 0 \). The same holds for \( I^m(F) \) assuming that functions \( f \) are in terms of \( X_a, a \in Q \). The meromorphic continuation of \( I^m(F) \) to any \( \Re k_i < 0 \) can be presented as a finite linear combination of integrals over certain \( \mu \)-residual subtori, with the leading term \( I^m(F) \). The number of such integrals grows as \( |\Re k_i| \) increase.

Proof. The fact that \( I^m(F) \) extend \( I^m(F) \) analytically to small negative \( \Re k_i \) is straightforward. Generally, we determine the corrections when moving the contours of integration by \( \Re y \); they are iterated integrals over \( \mu \)-residual subtori of smaller dimensions. Let us take \( I^m(F) \) for the sake of concreteness. First, we replace \( \int_{u+im} f(u) \) by \( \int_{u+im} f(u) \) for every \( 1 \leq i \leq n \). The corresponding correction will be

\[
\int_{u+im} \cdots \left( \int_{u+im} - \int_{u+im} \right) \cdots \int_{u+im} f(q^x) \mu(q^x) \, dx_1 \cdots dx_n,
\]

where the difference is at place \( i \). It is a finite sums of integrals over the proper (imaginary) contours of dimension \( (n - 1) \). The integrands will be some (partial) residues for the corresponding \( x_\alpha \). Then we continue inductively: replace all remaining \( \int_{u+im} \) by \( \int_{u+im} \) in the same way. The final output will be a finite sum of integrals over certain \( \mu \)-residual subtori. It will depend on the order of integrations. The coefficients in this sum will be the corresponding (partial) residues of \( \mu \).
(ii). Taking the iterated integrals in terms of the residues in the corresponding right half-planes requires explanations. We will provide the exact algorithm for finding the set of all $\xi$ that occur in the pole decomposition of $I_{\nu}^\mu$, all of them are $\mu$-residual points, but not all will occur, which significantly depends on the order of integrations.

The description below is purely combinatorial; it suffices to assume that $1 \gg \nu \gg -k_\nu > 0$. We will set $\bar{\alpha}_i = (\alpha, z)$ for any $\alpha \in R$, where $z \in \mathbb{R}^n$. For instance, $\bar{\mu} = \mu(q^z)$. Given a pole $\xi$ of $\mu(q^z)$, let $\{\bar{\beta}_i = [\beta_i, m_i]\}$ be a sequence of consecutive binomials that result from the iterated integrations, where Generally, they can be different as (unordered) sets for different sequences and the same set can occur more than once. Here $i = 1, \ldots, n$, $m_i \in \nu(\beta_i)\mathbb{Z}_+$, $m_i > 0$ for $\beta_i < 0$. The corresponding $\xi$ modulo the periods of $X_\alpha$ will be a unique solution of the system of equations $\bar{\beta}_i + m_i + k_{\beta_i} = 0$ for $1 \leq i \leq n$.

We will treat in the following algorithm $\bar{\alpha}_j$ as undetermined variables, which will be eliminated one by one until we obtain their values at $\xi$.

One has for $i = 1$: $\beta_1 = \sum_{j=1}^n c_{ij}^1 \alpha_j$, where $c_1 \neq 0$, $\bar{\beta}_1 = \sum_{j=1}^n c_{i}^1 \bar{\alpha}_j$. We impose then the equation $k_{\beta_1} + \bar{\beta}_1 + m_1 = 0$ and obtain that $\bar{\alpha}_1 = -(m_1 + k_{\beta_1} + \sum_{j=2}^n c_{j}^1 \bar{\alpha}_j)/c_{1}^1 > 0$. Here $\Re \bar{\alpha}_j$ will become $\nu$ in the following integrations: for $j = 2, 3, \ldots, n$. Let us use that $k_\nu$ and $\nu$ are assumed very small. Then we arrive at $-(m_1 + k_{\beta_1} + C\nu)/c_{1}^1 > 0$ for some $C$ with the upper bound depending only on the root system $R$ and the terms with $k$ and $\nu$ can be disregarded. We obtain that the initial inequality holds if and only if $c_1^1 < 0$ and $m_1 > 0$. The former means that $\beta_1 < 0$ (then $m_1 > 0$ anyway). Equivalently, $(1 - t_{\beta_1} X_\bar{\beta}_1)$ belongs to the “negative half” of the denominator of $\mu$. This is so only for the 1st integration; the “positive half” of $\mu$ may contribute too.

To go to the second step, we set $\alpha_i^1 = \bar{\alpha}_1(k_\nu \mapsto 0) = \bar{\alpha}_1|_{k_\nu \mapsto 0}$ and replace $\bar{\alpha}_1$ by the relation above in all $\bar{\beta}_i$, $\bar{\alpha}_i$ for $i > 1$ and all binomials of $\mu$. The one with $t_{\beta_1} X_\bar{\beta}_1$ in the denominator of $\mu$ will be deleted and we perform the reduction of coinciding or proportional binomials in the numerator and denominator of $\bar{\mu}$. The binomials with $t_{\beta_1} X_\bar{\beta}_1$ for $i > 1$ will not be reduced by this construction since they are among the defining relations for $\xi$.

We arrive at new $\bar{\beta}_i$ and $\bar{\alpha}_i$ for $i > 1$ and $\bar{\mu}$ in terms of $\bar{\alpha}_i(i > 1)$ and $m_1$. By construction, $(\bar{\alpha}_i, \xi) = (\alpha_i, \xi)$ and $(\bar{\beta}_i, \xi) = (\beta_i, \xi)$, where $(\alpha+c, \xi) = (\alpha, \xi) + c$ here and below for $c \in \mathbb{Q}$.

Then, we represent $\bar{\beta}_2 = \sum_{j=2}^n c_{j}^2 \bar{\alpha}_j$ for new $\bar{\beta}_2$ and $\bar{\alpha}_j$ ($j \geq 2$), where $c_{2}^2 \neq 0$, and obtain: $\bar{\alpha}_2 = -(m_2 + k_{\beta_2} + \sum_{j=3}^n c_{j}^2 \bar{\alpha}_j)/c_{2}^2$. The 2nd positivity condition is: $-(m_2 + \sum_{j=3}^n c_{j}^2 \bar{\alpha}_j)/c_{2}^2 > 0$, where $(\bar{\alpha}_j)_{\alpha \mapsto 0}$ means that we delete all $\alpha$ and $k$ from $\bar{\alpha}_j$, i.e. keep only constants, which are in terms of $\{m_i\}$. Note that $c_{2}^2$ is not the coefficient
of \( \alpha_2 \) in \( \beta_2 \). Then we switch to new \( \bar{\alpha}_i, \bar{\beta}_i \) for \( i > 2 \) and \( \bar{\mu} \) as above using the formula for \( \bar{\alpha}_2 \) and continue by induction.

Finally, we obtain the complete list of substitutions \( \bar{\alpha}_i \leftrightarrow \alpha_i^* = \sum_{j>i} C_{ij} \bar{\alpha}_j + M_i + K_i \) with some \( M_i \) in terms of \( m_j \) for \( j \leq i \) and \( K_i \) in terms of \( k_\nu \). This gives the formulas for \( \{\bar{\alpha}_i\} \) in terms of \( \{m_i\} \) and \( k_\nu \), and the list of inequalities for \( \{m_i\} \). The latter are \( (\bar{\alpha}_i)^{\alpha \to 0}_{\alpha \to 0} > 0 \) for the corresponding substitution formulas. These inequalities are necessary and sufficient for \( \xi \) to occur in \( \Sigma^m.a \) for a given order of integrations. However, the corresponding residue can be 0 or there can be cancelations of the terms.

For instance, in the 2th step (the last) gives that \( \bar{\beta}_n = c_n \bar{\alpha}_n, \bar{\alpha}_n = \beta_n/c_n = -(k_\beta_n + m_n)/c_n^2 \) and the inequality is \( (\bar{\alpha}_n)^{\alpha \to 0}_{\alpha \to 0} = -m_n/c_n > 0 \). Thus, \( c_n < 0, m_n > 0 \) and we have \( (\alpha_n, \xi^*) = -m_n/c_n > 0 \), where \( \xi^* \overset{\text{def}}{=} (\xi)^{\alpha \to 0}_{\alpha \to 0} \sum_{i=1}^n Q(i) \), obtained when we solve the system above with \( \bar{\alpha}_i \leftrightarrow (\alpha_i)^* \). Actually, \( (\alpha_n, \xi) > v \) by construction, which gives \( \alpha_n, \xi^* > 0 \). Generally, we arrive at the following description of \( \xi \).

**Lemma 9.2.** Let \( \bar{\beta}_i = [\beta_i] + \langle \beta_i \rangle \), where \( [\beta_i] = \sum_{j<i} c_j^i \bar{\alpha}_j, \langle \beta_i \rangle \in \sum_{j<i} Q \bar{\beta}_j \). Accordingly, \( (\bar{\beta}_i)^* + m_i = [\beta_i] + M^i \) at step \( i \), where \( M^i = \langle \beta_i \rangle_{\beta_j \to -m_j} \) is expressed in terms of \( m_j \) for \( j \leq i \). Then \( (\xi^*, [\beta_i]) = -M^i \) and the defining inequalities for \( \xi \) become \( (\xi^*, [\beta_i]/c_i^i) = -M^i/c_i^i > 0 \) for \( i = 1, \ldots, n \).

This procedure and the inequalities for \( \xi^* \) depend on the order of integrations. These inequalities do not guarantee that such \( \xi \) occur only once and with nonzero coefficients; there can be some cancelations even for \( A_3 \) (see below).

**Concluding remarks.** (a) The \( p \)-adic limit of the sums over residual subtori from (i) for \( I^v_a(f) \) is as follows. We assume that \( 0 > k_\nu > -\epsilon \) for small \( \epsilon \), take \( f \in \mathbb{C}[X_a, a \in Q] \) and replace the integrals \( \frac{1}{2M_{\pi \alpha}} \int_{-M_{\pi \alpha}}^{M_{\pi \alpha}} \) by \( \frac{1}{2M_{\pi \alpha}} \int_{-M_{\pi \alpha}}^{M_{\pi \alpha}} \) for \( M \in 1 + Z_+ \). By the way, the usage of \( M \) here allows us to incorporate \( f = X_b \) for \( b \in P \) instead of \( X_a \) with \( a \in Q \).

Then we set \( a = \frac{1}{M}, k_\nu = c_\nu a/\nu \) for \( c_\nu < 0 \) and make \( M \to \infty \). This results in \( q \to 0, \Re k_\nu \to 0 .. t_\nu = e^{-\frac{\nu k_\nu}{2}} \to e^{c_\nu} < 1 \). We arrive at the integrals over AHA residual subtori and the formulas from [HO1]. Actually, there is one more step here: the \( W \)-symmetrization.

Recall that \( t \to 1/t \) when we go from DAHA to AHA with the standard meaning of the parameter \( t \) there, which is \( \| \tilde{f} \| \) classically. So the range \( \Re k_\nu < 0 \) or \( t_\nu > 1 \) in terms of \( t \) for DAHA corresponds to \( t_\nu < 1 \) in the standard AHA setting, when the discrete series occurs.

(b) As we already discussed, \( I^{im}(f) \) can be generally reduced to \( I^a(f) \). Namely, we replace \( f \to \sum_{b \in 2\pi \alpha P} f(z+b) \) if this sum converges. Then, the integral formulas in terms of integrals \( I^a(f) \) over residual
stori for Laurent polynomials $f$ and series coincide with $\text{ct}(f^\mu)$; to be exact, they are proportional in the corresponding range of $k_\nu$. Then, given $f \in \mathcal{X}$, the constant term $\text{ct}(f^\mu)$ is meromorphic for any $k_\nu$, which can be seen directly using the formulas for the $E$-polynomials.

There are 3 more algebraic ways to calculate $\text{ct}(f^\mu)$. One can use (i) the coinvariants for \(\diamond\), (ii) affine symmetrizers \(\hat{I}^+\) and \(\hat{P}^+\), and (iii) the Jackson integrals $J_\xi(f) = J(f; \xi)$. Here (ii – iii) require $\Re k_\nu < 0$. The Jackson integrals are the closest to $\Sigma^{a}(f)$; they are related to the formulas via real integrations. The latter provide another tool for obtaining $\text{ct}(f^\mu)$ and result in non-compact trace formulas. The existence of the affine symmetrizer in $\mathcal{X}$ for sufficiently small $\Re k < 0$ and in $\mathcal{X}_{q,x^2/2}$ for $l > 0$ and $\Re k < 1/h$ is remarkable; these modules behave as discrete series representations in AHA theory.

The problem with the usage of the series $\Sigma^{a}(f)$ is that its convergence holds for restricted classes of $f$ and heavily depends on $\Re k$. This is similar to Jackson integrals. For instance, only constants can be taken as $f$ for $\Sigma^{a}(f)$ among $W$-invariant Laurent polynomials $f \in \mathcal{X}$ when $\Re k_\nu < 0$ are close to 0; cf. [Mac]. Also, the Gaussians $q^{-lx^2/2}$ for $l > 0$ and their Laurent expansions diverge in the real directions and result in divergent $\Sigma^{a}(f)$. The $\mathcal{HH}$-modules $q^{-lx^2/2}$ are important in the DAHA theory, but $\Sigma^{im,1a}$ cannot be used for them.

By contrast, the finite sums over residual subtori from (i) can be used for practically arbitrary analytic functions provided the integrability. For the real integration, i.e. in the non-compact case, even a single integral can be used. We did not state the uniqueness of the integral formulas explicitly. The $\mathcal{HH}$-invariance is one way to fix them uniquely.

The other way is by combinatorial collecting the $\mu$-residual points from $\Sigma^{a}$ in the families corresponding to $\mu$-residual subtori, which is a canonical process. Fig. 2 for $A_2$ demonstrates this. One needs to find the pole decompositions for the integrals over the residual subtori for $\Re k_\nu < 0$ for this, starting with $I^{im,1a}_0$. Obtaining explicit integral formulas for imaginary integrations is not simple: the number of residual subtori and points that occur there grows when $|\Re k_\nu|$ increases.

(c). As an application to $\mathcal{X}$, one can consider singular $t_\nu = q^{h_{\nu}}$ such that the coefficients in the integral formulas have poles; we renormalize the integral formula making some coefficients (measures) without $k$-poles. Then functions $f$ vanishing at all subtori with singular coefficients form an $\mathcal{HH}$-submodule, and $I^{a}_0(f(q^\nu)\bar{g}(q^{-z}))$ will induce a positive definite inner product there for the complex conjugation $g \mapsto \bar{g}$ of the coefficients of $g \in \mathcal{X}$. The corresponding anti-involution of $\mathcal{HH}$ was calculated in [Ch1]; it is not $\diamond$ since $T_{w_0}$ is omitted.

It was proven under some technical conditions in [Ch4] that a certain “smallest” submodule of $\mathcal{X}$ is $Y$-semisimple for any singular $q, t_\nu$. 

The technique of intertwiners was used; the inner products were not involved. The integral formulas provide an alternative approach to this and similar facts, including generalizations to other spaces of functions.

(d). The usage of $\nu = 1/h$ for the translation $\nu \rho^\nu$ of the contour of integration in the theorem gives the greatest analyticity range of $k_\nu$ in $I_v^{im,\alpha}(f)$. Recall that arbitrarily small negative $\Re k_\nu$ in $(i)$ were used to define $\Sigma^\alpha$, but making the analyticity range “optimal” for $I_v^{im,\alpha}(f)$ is of importance. The main fact is that if $\Sigma^\alpha$ is known, where only small $\Re k_\nu < 0$ are sufficient, the corresponding series provides the required meromorphic continuation to any $\Re k_\nu < 0$ ensuring the convergence of the resulting series.

(e). The conditions from $(i)$ of Theorem 4.5 for $f = X_a$ are sufficient for the convergence in $(ii)$ of Theorem 9.1 but they are not necessary. For instance, it converges for $f = X_{ca}$ for certain $a \in Q \cap P_+$ and any $c \in \mathbb{Z}_+$. These cones are nonempty for any orders of integrations. For $A_n$, any direction $a$ can be made such for a proper order of integrations.

In the case of $I^{im}$ for $\Re k_\nu < 0$, the convergence is granted for Paley-Wiener functions (for the Laplace transform). Namely, it suffices to assume that $f(x)$ is analytic in $x \in \mathbb{C}^n$ such that for every positive $N$ there exists some $C_N > 0$ such that $|f(w(x))| \leq C_N(1 + |x|)^{-N e^{B|\Re(x)|}}$ for every $w \in W$, where $0 < B < A|\Re(\rho_k)|$ for some $A > 0$.

Let us emphasize that if the meromorphic continuation of $I_0^{im,\alpha}(f)$ from $\Re k_\nu < 0$ to $\Re k_\nu \geq 0$ exists, then it is unique and does not depend on the specific choice of variables and their order of integrations. However, the sums $\Sigma^{im,\alpha}(f)$ and the corresponding growth conditions for $f$ depend on the order of integrations. This leads to some nontrivial identities. Our integral formulas generally depend on the order of integrations too. Their uniqueness is under some assumptions.

10. Pole expansion for $A_n$

The combinatorial algorithm for finding $\Sigma^{im,\alpha}$ becomes relatively simple for $A_n$. We will provide $\Sigma^\alpha$ only for the standard order of $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ in the iterated integral, though see below an example for $A_3$ with $\alpha_1, \alpha_3, \alpha_2$. We will describe $\xi$ and $\xi^* = \xi(k \to 0)$ following the proof of Theorem 9.1.

**Theorem 10.1.** For $A_n$ and the standard sequence of $\alpha_i$ as above, let $X = \{\xi\}$ be the set of the $\mu$-residual points in $\Sigma^\alpha$. Then the relations
for \( \xi^\bullet = b \in P \) are \((b, \alpha_n + \cdots + \alpha_i) > 0 \) for \( i = 1, \ldots, n \). The corresponding \( \xi \) are \( \pi_b(-k\rho) = b-u_b^{-1}(k\rho) \), denoted simply by \( \pi_b \) in Theorem 6.1(i) for the initial \( \xi = -k\rho \). One has: \( \Sigma^a(f) = \sum_{b \in X} \text{Res}_{\pi_b}f(\pi_b) \), where \( f(bw) = f(q^{-w(k\rho)}) \). The residues here are as in Theorem 6.1(iv).

In the integral formulas from Theorem 9.1, the integrands and residues are obtained by deleting the binomials from the denominator of \( \mu \) vanishing at the corresponding \( \mu \)-residual torus followed by the evaluation of \( f(q^\theta) \) and the rest of \( \mu(q^\theta) \) at the corresponding tori and points.

**Proof.** The description of \( \{\xi\} \) follows directly from the explicit algorithm from Theorem 9.1. Due to Theorem 7.2, the \( \mu \)-residual points \( \xi \) can be potentially with zeros in the numerator of \( \mu \) for \( A_{n \geq 2} \).

For \( A_3 \), such \( \xi \) is as follows up to the action of \( \hat{W} \): \( \bar{\alpha}_2 = \bar{\xi}_2 - \bar{\xi}_3 = 0 \) and \( \bar{\xi}_i - \bar{\xi}_i + 1 = -k = \bar{\xi}_4 - \bar{\xi}_i + 1 \) for \( i = 2, 3 \). One has \( \alpha_1 - \alpha_0 = 4 - 1 = 3 = n \), so it is \( \mu \)-residual.

We claim that \( \mu \)-residual \( \xi \) with binomials in the numerator of \( \mu \) vanishing at \( \xi \) do not contribute to \( \Sigma^a \). Let us outline the justification.

Such \( \xi \) have nontrivial stabilizers \( W_{\xi} \) in \( W \); for instance, \( s_2(\xi) = \xi \) for the \( \xi \) above. The group \( W_{\xi} \) will permute the corresponding \( \beta_i \) and these permutations are non-trivial unless for \( id \in W_{\xi} \). Accordingly, such \( \xi \) will occur \( |W_{\xi}| \) times in the procedure of finding \( \Sigma^a \) from the proof of Theorem 9.1. Following Lemma 4.3, which states that the Jackson integrals \( J_{\xi}(f) \) vanish if \( |W_{\xi}| > 1 \), we check that the corresponding residues will cancel each other in this orbit.

This argument is generally applicable to other cases in Theorem 9.1, but we claim the absence of \( \xi \) with non-trivial stabilizers only for \( A_n \). Note that Lemma 4.3 cannot be applied if \( \alpha_1 - \alpha_0 > n \). However, such points will not occur for \( A_n \), which is straightforward to verify.

Concerning the inequalities for \( b, \bar{\beta}_n = c_n^a\alpha_n \) for the last step and \((b, \alpha_n) > 0 \), as it was checked in the proof of Theorem 9.1 for any root systems. One has \( c_n^a = -1 \) for \( A_n \). Similarly, \( \bar{\beta}_{n-1}^\bullet \) can be \(-\alpha_{n-1} \) or \(-\alpha_{n-1} - \alpha_n \) for the \((n - 1)\)th step, which gives \((b, \alpha_{n-1} + \alpha_n) > 0 \). Then we continue by induction using the following general lemma.

**Lemma 10.2.** For a minuscule \( \omega_r \) such that \( r \) is an endpoint of the corresponding Dynkin diagram of \( R \), let \( \alpha \mapsto \alpha' \) be the deletion of \( \alpha_r \) if it is present in \( \alpha \in R \). \( (\alpha + \beta)' = \alpha' + \beta' \) for any \( \alpha, \beta \in R \) and \( \alpha' \neq 0 \) is a root in the root system \( R' \) with simple roots \( \{\alpha_i, i \neq r\} \).

According to Lemma 9.2, we need to find \( [\beta_i] \) for \( i = 1, \ldots, n \) in the following decompositions: \( \bar{\beta}_i = [\beta_i] + \langle \beta_i \rangle \), where \( [\beta_i] = \sum_{j \geq 1} c_j^i \alpha_j \), \( \langle \beta_i \rangle \in \bigoplus_{j < i} \mathbb{Q}\beta_j \) and the relations ensuring that \( \xi^\bullet \) occurs in \( \Sigma^a \) are \((\xi^\bullet_c, [\beta_i]/c_i^j) > 0 \) and \( \bar{\beta}_i = [\beta_i, m_i] \in \hat{R}_+ \).

The following procedure is generally applicable to any roots system \( R \) with at least one minuscule weight. Namely, \( \alpha_i \) must be in the
form $\alpha_r$ from Lemma 10.2 for the system $R^{(n-i+1)}$ with simple roots $\{\alpha_1, \cdots, \alpha_n\}$. It really gives that $\xi^* = b \in P$. However, it can result in $b$ that do not actually occur as $\xi^*$ due to cancelations. Also, the conclusion that $\xi = \pi_0(-k\rho)$ is for $A_n$ only. We will check that $\xi = \pi_0(\xi^0)$ for some $\xi^0$, but the latter can be not of Steinberg type.

We obtain that $\xi^* = b \in P$ since $c_i^g = \pm 1$, which gives that $\xi = \pi_0(-k\rho)$ for $A_n$. Then all $c_i^g$ for fixed $i$ and $j \geq 0$ have the same sign (if not zero) because $\alpha_1, \cdots, \alpha_n$ are simple roots in the corresponding system $R^{(n-i+1)}$. Next, let us check that $c_1^g = -1$.

For the last step, $[\beta_n] = c_n^a\alpha_n$ and $-m_n = c_n^a(\alpha_n, b)$, where $(\alpha_n, b) > 0$. We obtain that $c_n^a < 0$, which we already know (for any root systems). For the $(n-1)$st step: $[\beta_n] = c_{n-1}^{n-1}(\alpha_{n-1} + \alpha_n)$ for $c \geq 0$ and $-m_{n-1}/c_{n-1}^{n-1} = ((\alpha_{n-1}, b) + c(\alpha_n, b)) > 0$. This gives that $c_{n-1}^{n-1} = -1$. Then we go to $c_{n-2}^{n-2}$ and so on.

Finally, we obtain for $A_n$ and the standard order of the integrations that the all possible sequences are $\{[\beta_n], [\beta_{n-1}], [\beta_{n-2}], \ldots\}$ are: $\{\alpha_n, \alpha_{n-1} + \alpha_n, \alpha_{n-2} + \alpha_n, \alpha_{n-3} + \alpha_n, \ldots\}$. They all occur and result in the statement of the theorem. \hfill $\Box$

**Examples.** (a) Let us consider the standard order of $\alpha_i$ for $A_3$. For example, let $\vec{\beta}_1 = [-\alpha_1 - \alpha_2 - \alpha_3, n_1 + 1], \vec{\beta}_2 = [\alpha_1, n_2], \vec{\beta}_3 = [\alpha_2, n_3]$, where $n_i \geq 0$. Then the consecutive substitutions for $k = 0$ are:

$\bar{\alpha}_1 \mapsto -\bar{\alpha}_2 - \bar{\alpha}_3 + n_1 + 1, \bar{\alpha}_2 \mapsto -\bar{\alpha}_3 + n_1 + n_2 + 1, \bar{\alpha}_3 = n_1 + n_2 + n_3 + 1.$

Accordingly, $b = (n_1 + n_2 + n_3 + 1)\omega_3 + (-n_3)\omega_2 + (n_2)\omega_1$ and $\xi = b + u_b^{-1}(-k\rho) = (n_1 + n_2 + n_3 + 1 + 3k)\omega_3 + (-n_3 - k)\omega_2 + (-n_2 - k)\omega_1$.

For the sequence $\vec{\beta}_1 = [-\alpha_1 - \alpha_2, n_1 + 1], \vec{\beta}_2 = [-\alpha_2 - \alpha_3, n_2 + 1], \vec{\beta}_3 = [\alpha_2, n_3]$ the substitutions (under $k = 0$) are:

$\bar{\alpha}_1 \mapsto -\bar{\alpha}_3 + n_2 + 1, \bar{\alpha}_2 \mapsto -\bar{\alpha}_3 + n_2 + 1, \bar{\alpha}_3 = n_2 + n_3 + 1.$

Finally, $b = (n_2 + n_3 + 1)\omega_3 + (-n_3)\omega_2 + (n_1 + n_3 + 1)\omega_1$ and $\xi = (n_2 + n_3 + 1 + 2k)\omega_3 + (n_1 + n_3 + 1 + 2k)\omega_1$. Recall that $\vec{\beta}_i$ are from the binomials that are taken for the corresponding integration: $dx_{\alpha_1}, dx_{\alpha_2}$ and $dx_{\alpha_3}$. Here and below $n_i \in \mathbb{Z}_+.

Let us provide the whole set of $\xi$ for $A_3$. The first three numbers in the list below give the types of the $\vec{\beta}_i(i = 1, 2, 3)$, their corresponding numbers in the following sequence of 12 types:

$[\alpha_1, m_1], [\alpha_1 + \alpha_2, m_2], [\alpha_1 + \alpha_2 + \alpha_3, m_3], [\alpha_2, m_4], [\alpha_2 + \alpha_3, m_5], [\alpha_3, m_6], [\alpha_1, m_7], [-\alpha_1 - \alpha_2, m_8], [-\alpha_1 - \alpha_2 - \alpha_3, m_9], [-\alpha_2, m_{10}], [-\alpha_2 - \alpha_3, m_{11}]$ and $[-\alpha_3, m_{12}]$ (number 12) in $\hat{R}_+$; they are from the denominator of $\mu$. Here $m_i = n_i$ for $1 \leq i \leq 6$ and $m_i = n_1 + 1$ for $i > 6$, i.e. in the “negative half” of $\mu$. The examples above correspond to $\{9, 1, 4\}$ and $\{8, 11, 4\}$. We have: $\{7, 10, 12, 1 + n_1, 1 + n_2 + n_1, 1 + n_2 + n_1 + 2, 2k + n_1n_2 + n_3\}, \{7, 11, 2, 1 + k + n_1, -1 - 2k - n_1 - n_3, 2 + 3k + n_1n_2 + n_3\}, \{8, 1, 12, -k - n_2, 1 + 2k + n_1 + n_2 \cdots \}$. 

is unique such); equivalently,

\[ \frac{u}{u} \text{ such that} \]

Recall that

\[ n \]

\[ \prod \]

union of \( \alpha \)

For the set \( \varsigma \)

one can replace \( \bar{b} \) with \( \bar{\mu} \)

to the Jackson integral

\[ \sum_{i=0}^{\infty} \pi_i(f) \]

\[ \mu(\pi_b) / \mu(0) \]

functions \( f \) invariant with respect to the (affine) action of \( \Pi = \{ \pi_i = \pi_{\omega_i} = \pi_1^i, \text{ } 0 \leq i \leq n \} \). The latter condition is not really a restriction, since one can replace \( f \mapsto \sum_{i=0}^{\infty} \pi_i(f) \) because \( \pi_i(\mu) = \mu \) and we integrate over \( \rho / h + i \mathbb{R}^n \), which is \( \Pi \)-invariant. However, the \( \Pi \)-symmetrization of \( f \) can generally worsen the convergence of \( \Sigma^\omega(f) \), as well as \( f \mapsto f + f^\varsigma \) for \( \varsigma = -w_0 \). The Jackson integral, when it exists, is \( \diamond \)-invariant.

**Lemma 10.3.** For the set \( X \subset P \) in the theorem, \( P \) is a disjoint union of \( \pi_i(X) \) for \( 0 \leq i \leq n \), where \( \pi_{w_0} = \text{id} \). I.e. \( X \) is a fundamental domain for the action of \( \Pi \) in \( P \) for the affine action \( bw(z) = b + w(z) \) in \( \mathbb{R}^n \supset \mathbb{Z} \); we need \( \pi_b(c) = u_b^{-1}(c) + b \) for \( b = \pi_i \) and \( c \in P \).

**Proof.** Let \( R^{(\nu)} \) be \( R \) for \( A_n, S_{n+1} \) the corresponding Weyl group. Recall that \( \pi_b = bw_{\pi}^{-1} \) has the smallest length in \( \{ bw, w \in W \} \) (it is unique such); equivalently, \( w \in W \) is of minimal possible length such that \( u_b(b) \in -P_+ \). Then \( v_1 = u_{\omega_1}^{-1} \) equals \( s_1 \ldots s_n \) (the Coxeter
element). It sends $\varepsilon_i \mapsto \varepsilon_2, \ldots, \varepsilon_{n+1} \mapsto \varepsilon_1$,

$$v_1: \alpha_1 \mapsto \alpha_2 \mapsto \cdots \alpha_n \mapsto -\theta = \alpha_1 + \cdots + \alpha_n \mapsto \alpha_1$$ and

$$v_1: \omega_1 \mapsto \omega_2 - \omega_1 \mapsto \cdots \omega_n - \omega_{n-1} \mapsto -\omega_n \mapsto \omega_1.$$

Let us check that:

$(i)$ $R^{(n)} = \bigcup_{m=0}^{n} B_m$ for $B_m \overset{\text{def}}{=} v_1^m (R_+^{(n)} \setminus R_+^{(n-1)})$, where the union is disjoint and $R_+^{(n-1)}$ is the root system for $\alpha_1, \ldots, \alpha_{n-1}$, and

$(ii)$ given $b \in P$ such that $(b, \alpha) \neq 0$ for any $\alpha \in R^{(n)}$, there exists a unique $v_1^m$ such that $v_1^m(b)$ has positive inner products with all $\beta \in B_0$.

The set $R_+^{(n)} \setminus R_+^{(n-1)} = \{\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n\}$ is invariant with respect to $S_n$ (for $R^{(n-1)}$) and only for such $w$. Then, $B_m = \{\varepsilon_j - \varepsilon_m, j \neq m\}$ and their union is the whole $R^{(n)}$. Explicitly: $v_1^m(\varepsilon_i - \varepsilon_{i+1}) = \varepsilon_{i+m \mod(n+1)+1} - \varepsilon_m$ for $1 \leq m \leq n$. It contains $\pm \varepsilon_n$ only for $i+m = n$.

However, the root $\varepsilon_{i+m+1} - \varepsilon_m = \varepsilon_{n+1} - \varepsilon_m$ is negative for this $i$. Thus, $v_1^m(B_0) \cap B_0 = \emptyset$ for any $1 \leq m \leq n$, which proves $(i)$. To go from $(i)$ to $(ii)$, $b = \sum c_i \varepsilon_i \in P$ belongs to $B_m$ if and only if $c_m = \min\{c_i\}$.

Let us switch from $v_1$ to $\pi_1$. Due to the inequalities $(b, \varepsilon_j - \varepsilon_{n+1}) > 0$ for $j \leq n$, we have: $X = \{b = \sum c_i \varepsilon_i \in P \mid c_{n+1} < c_i \text{ for } i \neq n+1\}$, where $c_i \in \mathbb{Z}$ and $\sum c_i = 0$. Using that $\pi_1^m = \omega_m v_1^m$ for $1 \leq m \leq n$:

$$\pi_1^m(X) = \{b = \sum c_i \varepsilon_{i+m \mod(n+1)+1} + (\varepsilon_1 + \cdots + \varepsilon_m) - \frac{m}{n+1} \sum_{i=1}^{n+1} \varepsilon_i\}$$

$$= \{b = \sum b_i \varepsilon_i \mid b_m < b_i \text{ for } i < m \text{ and } b_m \leq b_i \text{ for } i > m\}.$$

Here $b_i \in \mathbb{Z}$ and $\sum_{i=1}^{n} b_i = 0$. We see that any $b \in P$ can be represented in this form for a unique $m$: it is such that $b_m = \min\{b_i\}$ for the smallest index $i$ when this minimum is reached. □

11. Integral Formulas for $A_2$

As we see, explicit formulas for $\Sigma^{\alpha}$ can be obtained in relatively simple way for $A_n$ for the standard order of $\alpha_i$ (and the corresponding iterated integration). However, the problem of finding explicit finite sums from $(i)$ in Theorem 9.1 is subtle even in this case for arbitrary $\Re k < 0$. We will solve it only for $A_2$ and provide the answer for $A_1$.

For $A_2$, we denote $x = x_{\alpha_1}, y = x_{\alpha_2}, X = q^x, Y = q^y$. As above, the residues are obtained by deleting the binomials of $\mu$ vanishing at $\pi_b$ and evaluating the rest at $t^{-p}$. More generally, the notation $\mu_*$ will be used for this procedure at any $\xi$ when the numerator of $\mu$ has no zeros. Due to the $\Pi$-invariance of $\mu$ and the symmetry $\mu(q^x, q^y) = \mu(q^y, q^x)$:

$$\mu_*(t q^m, t q^n) = \mu_*(t q^n, t q^m) = \mu_*(t q^n, t^{-2} q^{1-m-n}) = \mu_*(t^{-2} q^{1-m-n}, t q^n).$$
We will set: \( \varpi_1(m, n) \stackrel{\text{def}}{=} \mu_\star(tq^n, tq^n) \), \( \varpi_2(m, n) \stackrel{\text{def}}{=} \mu_\star(t^{-1}q^{m-n}, t^2q^n) \). They are connected as follows:

\[
\varpi_2(m, m + n) = t^{-1} \frac{1 - t^2q^n}{1 - q^n} \varpi_1(m, n) \quad \text{for } n > 0, \quad \text{and}
\]

\[
\varpi_2(m, m) = t^{-1} \varpi_1(m, 0) \quad \text{due to } s_1(\mu) = t^{-1} \frac{1 - tX}{1 - t^{-1}X} \mu.
\]

The following are explicit formulas for \( \varpi_{1,2} \):

\[
\begin{align*}
\varpi_1(m > 0, n > 0) &= t^{3-2(m+n)} \prod_{i=1}^{m-1} \frac{(1-t^2q^{i})}{(1-q^{i})} \prod_{i=1}^{n-1} \frac{(1-t^2q^{i})}{(1-q^{i})} \prod_{i=1}^{m+n-1} \frac{(1-t^3q^{i})}{(1-tq^{i})}, \\
\varpi_2(m > 0, n \geq m) &= t^{2-2n} \prod_{i=1}^{m-1} \frac{(1-t^2q^{i})}{(1-q^{i})} \frac{(1-q^{n-i})}{(1-t^2q^{n-i})} \prod_{i=1}^{n-1} \frac{(1-t^2q^{i})}{(1-q^{i})} \frac{(1-t^3q^{i})}{(1-tq^{i})}.
\end{align*}
\]

We set

\[
\varrho_0 = \prod_{i=0}^{\infty} \frac{(1-t^{-1}q^{i})(1-tq^{i+1})}{(1-q^{i+1})(1-t^2q^{i+1})}, \quad \varrho = \mu_\star(X = t^{-1}, Y = t^{-1}) = \\
\prod_{i=0}^{\infty} \frac{(1-t^{-1}q^{i})(1-tq^{i+1})^2(1-t^{-2}q^{i})}{(1-q^{i+1})^2(1-t^2q^{i+1})(1-t^3q^{i+1})} = \varrho_0 \prod_{i=0}^{\infty} \frac{(1-tq^{i+1})(1-t^{-2}q^{i})}{(1-q^{i+1})(1-t^3q^{i+1})}.
\]

As above, \( q = \exp(-1/a), t = q^k = \exp(-k/a) \) for \( a > 0 \). Let

\[
\text{Int}_v(f) = \frac{I_v^2(f)}{(2\pi ia)^2} = \frac{1}{(2\pi ia)^2} \int_{u-\pi a}^{u+\pi a} \int_{v-\pi a}^{v+\pi a} f(q^x, q^y) \mu(q^x, q^y) \, dx \, dy.
\]

**Proposition 11.1.**

(i) Let \( \Sigma_1(f) = \varrho \sum_{m=1, n=1}^{\infty} \varpi_1(m, n) f(tq^n, tq^n) \) and \( \Sigma_2(f) = \varrho \sum_{m=1, n=m}^{\infty} \varpi_2(m, n) f(t^{-1}q^{m-n}, t^2q^n) \). Then \( \Sigma_{a}(f) = \Sigma_1(f) + \Sigma_2(f) \) provided the convergence of \( \Sigma_{1,2}(f) \).

(ii) Let \( \Re k < -m/2 \) for \( m \in \mathbb{Z}_+ \). Then \( \Sigma_{1,2}(f) \) converge absolutely for \( f = X_a \) with \( a \in \sum_{i=1}^m \alpha[i] + \mathbb{Z}_+\alpha_2 + \mathbb{Z}_+(\alpha_1 + \alpha_2) \), where \( \alpha[i] \) is either \( \pm \alpha_1 \) or \( \pm \alpha_2 \). Moreover, such \( X_a \) can be divided by any number of binomials \( (1 - X_b) \) for \( b \neq 0 \in \mathbb{Z}_+\alpha_2 + \mathbb{Z}_+(\alpha_1 + \alpha_2) \).

(iii) In particular, \( \text{ct}(f_\mu) \), which is a meromorphic function of \( t = q^k \) for any given Laurent polynomial \( f \), coincides with \( \Sigma_1(f) + \Sigma_2(f) \) for any \( \Re k < 0 \) assuming the convergence of \( \Sigma_{1,2}(f) \). This sum coincides with \( \text{Int}_{1/3}(f) \) for \(-1/3 < \Re k < 0\); here \( \text{Int}_{1/3}(f) \) extends analytically \( \text{Int}_{0}(f) \) from \( \Re k > 0 \) to \( \Re k > -1/3 \) assuming the integrability. \( \square \)

Let us provide the corresponding integral formulas for \( \text{ct}(f_\mu) \) for \( A_2 \). They are based on the pole decomposition for \( \text{Int}_0(f) \) from \( \Re k > 0 \) to \( \Re k \leq 0 \) combined with the formulas from Proposition 11.1.(i). We arrange the corresponding (infinite) sum for \( \text{Int}_{1/3}(f) - \text{Int}_0(f) \) as a sum of one-dimensional integrals and the sum of the remaining residues of \( f_\mu \). The integrands for the one-dimensional integrals will be
of

\[ \zeta_m^1(q^z) = \mu_0(t^{-1}q^{-m}, q^z) = \mu_0(t^{-1}q^{-m}) = \mu_0(tq_{-z+m+1}^{-z}, q^z), \]

\[ \zeta_m^2(q^z) = \mu_0(tq_{m+1}^{-1}, q^z) = \mu_0(q_z t, tq_{m+1}^{-1}) = \mu_0(t^{-1}q_{-z-m}, q^z). \]

One has: \( \zeta_m^2(q^z) = \frac{(1-tq_{-z}^{-1})}{(1-tq_{-z}^{-1})} \zeta_m^1(-z). \) Explicitly:

\[ \zeta_m^1(q^z) = q_0 t^{-2m} \prod_{j=1}^{m} (1 - t^2q_j^2)(1 - t^2q_{j+z}) \prod_{j=0}^{\infty} (1 - tq_{j+z}^{-1})(1 - t^2q_{j+z}^{-1}); \]

\[ \zeta_m^2(q^z) = q_0 t^{-2m-1} \prod_{j=1}^{m} (1 - t^2q_j^2)(1 - t^2q_{j+z}) \prod_{j=1}^{\infty} (1 - tq_{j+z}^{-1})(1 - t^2q_{j+z}^{-1}). \]

**Proposition 11.2.** For \( 0 > \Re k > -0.5 \) and any \( f \in \mathbb{C}[X^{\pm 1}, Y^{\pm 1}], \)

\( \text{ct}(f) = \frac{I_0^a(f)}{(2\pi a)^2} + \frac{1}{2\pi a} \int_{-\pi a}^{\pi a} (f(t^{-1}, q^{-1}) + f(q^{-1}, t^{-1})) \zeta_0^2(q^y) dy \]

\[ + \frac{1}{2\pi a} \int_{-\pi a}^{\pi a} f(t^{-1}q^{-y}, q^y) \zeta_0^2(q^y) dy + \frac{1}{2\pi a} \int_{-\pi a}^{\pi a} \phi(t^{-1}, t^{-1}). \]

For \(-0.5 > \Re k > -1, \) the term \( \phi(t^{-1}, t^{-1}) \) here must be replaced by

\[ \phi(f(t^{-1}, t^{-1})). \]

Also, the term \( \omega(1, 1) f(tq, t^{-2}q^{-1}) \) in the latter sum must be omitted when \( \Re k = -0.5. \) The functions \( f(q^z, q^y) \) here are arbitrary analytic provided the convergence of the integrals.

We note that the formula for \( \Re k < -0.5 \) contains the integrand \( f(t^{-1}q^{-y}, q^y)\zeta_0^2(q^y) \) and the term \( \omega(1, 1) f(t^2q, t^{-1}) \) for \( \Re k < -0.5 \) that are not invariant with respect to the symmetry \( \varsigma : x \leftrightarrow y. \) The meromorphic continuation must by \( \varsigma \)-invariance, i.e. the same for \( f \) and \( f^\varsigma. \) Some symmetries of \( \zeta_0^2 \) and the corresponding cancelation of residues in this range of \( k \) ensure this.

The figures below give the points \((b_1, b_2) \) such that \( b_{-k(t^{-1})} \) are the corresponding \( \mu \)-residual points that occur in \( \Sigma^a(f). \) These vectors \( b \) form the upward sector (angle) with its vertex at \( \omega_2. \) It is clearly \( 1/3\text{rd} \) of the total lattice \( P \) under the rotations by \( 2m\pi/3 \) for \( m = 0, 1, 2 \) with the center at \( \rho/3. \)

Recall that this sum (when it converges) is proportional to \( (a) \text{ct}(f), \) \( (b) \langle f \rangle_\xi \) for the coinvariant \( \langle \cdot \rangle_\xi \) for the anti-involution \( \diamond \) and \( \xi = -k\rho, \) \( (c) \) to the Jackson integral \( J_{-k}(f) \) and \( (d) \hat{\mathcal{P}}_+(f) \) for the affine symmetrizer \( \hat{\mathcal{P}}_+. \)

These figures show the set of \( b \) that occur in the pole decomposition of \( \frac{I_0^a(f)}{(2\pi a)^2} \) for the corresponding \( \Re k < 0. \) They are those belonging to the thick arrows and inside the polygon containing \((0, 0). \) This set is obtained from the sector describing \( \Sigma^a \) by removing finitely many lines and points and adding some lines and points below this sector.

The directions of the lines that are removed or added give the corresponding integrals over one-dimensional \( \mu \)-residual subtori. They are
shown by thin arrows, but the residual points due to the corresponding one-dimensional integrals are not exactly those belonging to these arrows: some must be added to the corresponding arrows.

We note that the presentation of the residual points of $\Sigma^ia$ as $I_0^ia$ plus those in one-dimensional integrals and the remaining points is unique (in this picture). We assume that the integrands are “standard”: the partial residues of $\mu$ upon the restriction to residual tori. Then the exact “thin arrows” are canonically determined by their directions.

The integral formulas from Proposition 11.2 and their generalizations to any $\Re k < 0$ result combinatorially from the description of the “support” of $I_0^ia$. Recall that the pole expansions of this integral can be
calculated for any \( \Re k \not\in -\mathbb{Z}_+ \) but the corresponding analytic functions will be not connected with each other in different strips.

The \( \mu \)-residual points (residual subtori of \( \dim = 0 \)) from the integral formula
\[
\Sigma^a \left( f \right) = \text{Int}_0 \left( f \right) + \ldots
\]
are expected to correspond to square integrable modules that occur in the regular DAHA representation, but this is a subject of some future theory.

Let us provide the integral formulas for \( -\ell - 0.5 < \Re k < -\ell \). We omit those for \( -\ell - 1 < \Re k \leq -\ell - 0.5 \). Figures 1 and 2 are for \( \ell = 1, 2 \).

**Theorem 11.3.** Let \( P = [-\pi a, \pi a] \). For \( -\ell - 0.5 < \Re k < -\ell \), \( \ell \in \mathbb{Z}_+ \):

\[
\text{ct}(f\mu) = \text{Int}_0(f) + \frac{1}{2\pi ia} \left( \sum_{m=0}^{\ell} \int_P \left( f(t^{-1}q^{-m}, q^y) + f(q^y, t^{-1}q^{-m}) \right) \zeta_m(q^y) \, dy \right)
\]

\[
+ \sum_{m=0}^{\ell} \int_P f(t^{-1}q^{-y-m}, q^y)\zeta_m^2(q^y) \, dy + \sum_{m=1}^{\ell} \int_P f(tq^{-y+m}, q^y)\zeta_{m-1}^1(q^y) \, dy
\]

\[
+ \sum_{m=1}^{\ell} \int_P (f(tq^m, q^y) + f(q^y, tq^m))\zeta_{m-1}^2(q^y) \, dy + \phi \left( \sum_{m,n=1}^{\ell} \omega_1(m,n)f(tq^m, tq^n) + \sum_{m,n=1}^{2\ell} \omega_1(m,n)f(tq^m, t^{-1}q^{-n+m}) + \sum_{m,n=1}^{2\ell} \omega_2(m,n)f(t^{-1}q^{-m+n}, t^2q^n)
\]

\[
+ \sum_{m=0}^{\ell} \omega_2(n-m, n)f(t^2q^n, t^{-1}q^{-m}) + \sum_{m=1}^{\ell} \omega_2(m,n)f(t^{-1}q^{-m+n}, t^{-1}q^{-m+1}) + \sum_{m=1}^{2\ell+1} \omega_2(n-m, n)f(t^{-2}q^{-n+1}, tq^m) \right).
\]

Vectors \( b = b_1\omega_1 + b_2\omega_2 = \xi^* \) associated with the terms in the double sums can be seen from the corresponding values of \( f \), which are \( f(t^{-\pi}q^b, t^{-\pi}q^{-b}) \). For instance, only the vector with \( m = 1 = n \) from \( \sum_{m=1}^{\ell+1} \sum_{n=m}^{m+\ell} \) occurs for \( \ell = 0 \); its contribution is \( \phi(f(t^{-1}, t^{-1})) \).

Notice that all terms in the integral formula have the coefficient 1 in this presentation. We expect this to hold for \( A_n \) and the standard order of \( \alpha_i \), but the evidence is limited beyond \( A_2 \).

**The case of** \( A_1 \). For the sake of completeness, let us provide the integral formula from [Ch3] in the case of \( A_1 \). As above, \( q = e^{-1/\alpha}, t = q^k \) and we set \( x = x_{\alpha_1} \).

**Proposition 11.4.** Let \( \{j^+, j^-\} \overset{\text{def}}{=} \{j-1, j\} \) and \( \ell \geq 0 \) be the integral part of \( -\Re k > 0 \). Then for \( \mu \) for \( A_1 \) and \( f(q^x) \in \mathbb{C}[q^{\pm x}] \):

\[
\text{ct}(f\mu) = \frac{1}{2\pi ia} \int_{-\pi a}^{+\pi a} f(q^x)\mu(q^x) \, dx
\]

\[
+ \mu^*(q^{-k}) \left( f(q^{-k}) + \sum_{\epsilon = \pm} \sum_{j=1}^{\ell} f(q^{(k+j)}t^{-j}q^{-k}) + \prod_{j=1}^{j^*} \frac{1-t^2q^j}{1-q^j} \right).
\]
Also, \[ ct(f\mu) = \mu\big(q^{-k}\big) \left( \sum_{j=1}^{\infty} f(q^{k+j}) t^{1-j} \prod_{i=1}^{j-1} \frac{1-t^{2j}q^{-i}}{1-q^{-i}} \right), \]
where \[ f(q^x) \in q^{-lx}C[q^{+x}], \]
which provides the convergence of this sum.

\[ \square \]

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