D-branes in Gepner models

by

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Abstract

We discuss D-branes from a conformal field theory point of view. In this approach, branes are described by boundary states providing sources for closed string modes, independently of classical notions. The boundary states must satisfy constraints which fall into two classes: The first consists of gluing conditions between left- and right-moving Virasoro or further symmetry generators, whereas the second encompasses non-linear consistency conditions from world sheet duality, which severely restrict the allowed boundary states. We exploit these conditions to give explicit formulas for boundary states in Gepner models, thereby computing excitation spectra of brane configurations. From the boundary states, brane tensions and RR charges can also be read off directly.

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1. Introduction

D-branes were first encountered as solitonic solutions to the low-energy effective equations of motion of superstring theory, as collective excitations extending in \( p + 1 \) directions of the target ("\( p \)-branes"), well-localized in the other, carrying Ramond-Ramond (RR) charges (see [1] for a comprehensive review). Independently of these \( p \)-branes, Polchinski et al. [2] discovered that a theory of closed and open strings in a toroidal target can equivalently be described as a closed string theory with additional higher-dimensional objects which (via \( T \)-duality) "implement" the boundary conditions originally imposed on the open strings. In particular, if Neumann boundary conditions hold in \( p + 1 \) directions of the target (one of them being time) and Dirichlet conditions in the others, one obtains a Dirichlet \( p \)-brane with \( p + 1 \)-dimensional world volume to which the closed strings couple.

Later, Polchinski showed that the Dirichlet \( p \)-branes indeed provide a "microscopic" realization of the solitonic \( p \)-branes within full-fledged string theory: They carry the same RR charges and their tension scales like \( g_s^{-1} \) with the string coupling; see [3-5].

This discovery made it possible to study non-perturbative aspects of string theory and led to a completely new picture of both string theory and the quantum field theories which arise as low-energy effective theories from string theory. Among other applications, D-branes allow to test the conjecture that all the different types of string theories known before actually are related (by S-duality) to each other and to eleven-dimensional M-theory [6,7].

The appearance of the extra dimension is itself due to a Kaluza-Klein re-interpretation of the spectrum of D-brane bound states, which also are the basic ingredients of strong-weak-coupling duality tests, like of type IIB self-duality [8]. In all these considerations, the BPS property of D-branes is of crucial importance.

Many of the statements on string dualities translate into relations between different field theories, possibly living in different dimensions. One way to see this is to use the fact that the massless excitations of string-brane systems induce low-energy effective (gauge) field theories on the world volume of the D-branes; see e.g. [8-13] and references therein. It was argued in [14] that these gauge theories at the same time provide an affective description of sub-string scale physics and geometry.

Compared to the solitonic \( p \)-branes of the low-energy effective field theories, Polchinski's D-branes are "microscopic", conformal QFT objects. Nevertheless, this formulation is still phrased in terms of target properties; in the literature, D-branes are mainly viewed as submanifolds of the target space, e.g. as flat hyper-planes of toroidal targets or as supersymmetric cycles of Calabi-Yau targets [15-17] (fluctuations of the shape are induced by interactions with closed strings).

While the target point of view provides intuitive pictures and is very convenient for a discussion of low-energy effective physics (including moduli spaces), one should also try to establish a more abstract formulation in terms of world-sheet CFTs where possible. One obvious reason is that string theory after all is to include quantum gravity and thus should not be formulated in terms of classical geometry only. Apart from conceptual considerations, the algebraic approach to string theory has its merits when it comes to producing concrete data in a controllable way.
In this article, we will show how (generalized) D-branes can be treated, in a background independent fashion, within the CFT approach to string theory. We expect that the new objects cannot be entirely described in classical geometric terms, but just like the Dirichlet branes introduced by Polchinski, they correspond to non-perturbative sectors of closed string theories.

Because of the open string origin of the branes, it is not surprising that their CFT-formulation requires notions from boundary CFT: The branes bound the world sheet of the closed strings which couple to them; the boundary CFT then describes the (perturbative) physics of the system made out of closed strings and (static) branes.

Our discussion starts from ”classical” D-branes in flat toroidal targets in section 2, where we briefly review their CFT formulation in terms of coherent states associated to the Fock spaces of closed string modes. In section 3, we provide the formalism necessary to abstract from this simple situation and turn towards the concepts of boundary states and boundary CFT. We apply the general methods to CFTs which yield particularly interesting string compactifications in section 4, by constructing generalized D-branes for arbitrary Gepner models.

2. Dirichlet branes as boundary states

The CFT formulation of “classical” Dirichlet $p$-branes in a flat toroidal target was already sketched by Polchinski [3-5] and worked out in great detail in the papers [18-22]. The techniques used were more or less known from open string theory [23,24]; see [5] for further references.

Let us first look at open strings propagating in a 1-torus, i.e. at the free boson $X(t,\sigma)$ taking values in a circle of radius $r$ and with world sheet coordinates $t \in \mathbb{R}$, $0 \leq \sigma \leq \pi$. $X(t,\sigma)$ satisfies the equation of motion $\triangle X(t,\sigma) = 0$ where $\triangle$ is the two-dimensional Laplacian, but the theory is not completely fixed before we specify some boundary condition on the two boundaries of the open string world sheet $\sigma = 0, \pi$. The two obvious choices are Neumann or Dirichlet boundary conditions:

\begin{align*}
\text{N open} : & \quad \frac{\partial X(t,\sigma)}{\partial \sigma} = 0 \quad \text{for} \quad \sigma = 0, \pi \quad (2.1) \\
\text{D open} : & \quad \frac{\partial X(t,\sigma)}{\partial t} = 0 \quad \text{for} \quad \sigma = 0, \pi \quad (2.2)
\end{align*}

Note that, in string language, imposing Dirichlet conditions means fixing a constant value for the string coordinate along each boundary,

$$X(\sigma = 0, t) = x_\alpha , \quad X(\sigma = \pi, t) = x_\beta , \quad \text{for all} \ t , \quad (2.3)$$

rather than prescribing arbitrary functions $x_\alpha(t), x_\beta(t)$; otherwise, conformal symmetry of the open string would be violated (see also section 3).

The question how open string boundary conditions can be related to closed strings has been answered in [23,24]: Interacting open strings never come without closed strings; making the open string world sheet periodic in $t$ with period $2\pi \tau$, say, we obtain an open string
one-loop diagram – which can be viewed as a closed string tree-level diagram by world sheet duality.
In passing to the dual world sheet, the roles of world sheet time and space are interchanged. The space coordinate is periodic (with period $2\pi$), the boundaries now are at $t_\alpha = 0$ and $t_\beta = \pi/\tau$, and the conditions (2.1-3) become

\begin{align}
\text{N closed:} & \quad \frac{\partial X(t,\sigma)}{\partial t} = 0 \quad \text{for} \quad t = 0, \pi/\tau, \\
\text{D closed:} & \quad \frac{\partial X(t,\sigma)}{\partial \sigma} = 0 \quad \text{for} \quad t = 0, \pi/\tau, 
\end{align}

respectively $X(t_\alpha,\sigma) = x_\alpha$, $X(t_\beta,\sigma) = x_\beta$ for all $\sigma$.

Closed string cylinder diagrams like the one we arrived at usually describe a closed string propagating from a state $|v_\alpha\rangle$ at time $t_\alpha$ to a state $|v_\beta\rangle$ at time $t_\beta$ with $|v_{\alpha,\beta}\rangle$ taken from the Hilbert space $\mathcal{H}$ of perturbative closed string excitations, i.e. from the Hilbert space of the free boson CFT (defined on the plane).

It is therefore a natural idea to try and implement the conditions (2.4,5) on some “boundary states” $|B\rangle_N$ and $|B\rangle_D$ associated to $\mathcal{H}$ such that, e.g. for $t_\alpha = 0$

\begin{align}
\frac{\partial X(0,\sigma)}{\partial t} |B\rangle_N &= 0 \\
\frac{\partial X(0,\sigma)}{\partial \sigma} |B\rangle_D &= 0
\end{align}

in the Neumann case and

\begin{align}
\frac{\partial X(0,\sigma)}{\partial t} |B\rangle_D &= 0 \\
\frac{\partial X(0,\sigma)}{\partial \sigma} |B\rangle_N &= 0
\end{align}

in the Dirichlet case. (See section 3 for a more conceptual introduction of the boundary states.)

Let us try to solve (2.7) first, proceeding in analogy to [23,24]. To this end, we split the closed string coordinate into left- and right-movers $X(t,\sigma) = X_L(x^+) + X_R(x^-)$ with $x^\pm = t \pm \sigma$, and use the mode expansion

\begin{align}
X_L(x^+) &= \frac{\hat{x}}{2} + \frac{1}{2} \left( \frac{\hat{p}}{2} + r \hat{w} \right) x^+ + \frac{1}{2} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-inx^+}, \\
X_R(x^-) &= \frac{\hat{x}}{2} + \frac{1}{2} \left( \frac{\hat{p}}{2} - r \hat{w} \right) x^- + \frac{1}{2} \sum_{n \neq 0} \frac{\bar{\alpha}_n}{n} e^{-inx^-}.
\end{align}

The non-vanishing commutators among the operators $\hat{x}$, $\hat{p}$, $\hat{w}$, $\alpha_n$, $\overline{\alpha}_n$ are

\[ [\alpha_n, \alpha_m] = n \delta_{n,-m}, \quad [\overline{\alpha}_n, \overline{\alpha}_m] = n \delta_{n,-m}, \quad [\hat{x}, \hat{p}] = i, \]

and we have $\alpha_n^* = \alpha_{-n}$, $\overline{\alpha}_n^* = \overline{\alpha}_{-n}$ for the oscillator modes, while the center of mass momentum and coordinate $\hat{p}$, $\hat{x}$ and the winding number operator $\hat{w}$ are self-adjoint.

The derivatives

\[ J(x^+) := \partial_+ X_L(x^+) \quad \text{and} \quad \overline{J}(x^-) := \partial_- X_R(x^-) \]
generate two commuting copies of the $U(1)$-current algebra $\mathcal{A} \cong \mathcal{A}_L \cong \mathcal{A}_R$, and we set

$$\alpha_0 := (\hat{p}/2 + r\hat{w}) \in \mathcal{A}_L \text{ and } \overline{\alpha}_0 := (\hat{p}/2 - r\hat{w}) \in \mathcal{A}_R.$$  

The space of states of the free bosonic field $X$ is built up from tensor products $H_g \otimes H_{\overline{g}}$ of irreducible highest weight representations of $\mathcal{A}_L \otimes \mathcal{A}_R$, containing states $|w, k\rangle$ with the properties

$$\alpha_0 |w, k\rangle = \left(\frac{k}{2r} + rw\right)|w, k\rangle = g|w, k\rangle,$$

$$\overline{\alpha}_0 |w, k\rangle = \left(\frac{k}{2r} - rw\right)|w, k\rangle = \overline{g}|w, k\rangle,$$

$$\alpha_n |w, k\rangle = \overline{\alpha}_n |w, k\rangle = 0 \text{ for } n > 0.$$  

Since $X$ takes values on a circle with radius $r$, both $\hat{w}$ and $\hat{k} := r\hat{p}$ have integer spectrum. With these notations we can return to our analysis of boundary conditions. From (2.8,9), it is easy to see that condition (2.7) on the boundary state $|B\rangle_D$ implies that

$$\hat{w} |B\rangle_D = 0 \quad \text{and} \quad (\alpha_n - \overline{\alpha}_{-n}) |B\rangle_D = 0 \quad (2.10)$$

for all $n \neq 0$. Using the commutation relations above, the general solution of these equations can be constructed as a linear combination of the following coherent states

$$|0, k\rangle_D := \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \overline{\alpha}_{-n}\right) |0, k\rangle.$$  

(2.11)

The (up to normalization) unique solution $|B(x_\alpha)\rangle_D$ which satisfies the stronger requirement $X(0, \sigma) |B(x_\alpha)\rangle_D = x_\alpha |B(x_\alpha)\rangle_D$ is

$$|B(x_\alpha)\rangle_D = \frac{1}{\sqrt{2r}} \sum_{k \in \mathbb{Z}} e^{-ikx_\alpha/r} |0, k\rangle_D.$$  

(2.12)

In the literature, (2.12) is sometimes written in the form $|B(x_\alpha)\rangle_D \sim \delta(\hat{x} - x_\alpha) |(0, 0)\rangle$, expressing the fact that the string coordinate takes the fixed value $x_\alpha$ along the boundary where $|B(x_\alpha)\rangle_D$ is placed.

Analogously, one can implement Neumann boundary conditions (2.6) on a boundary state $|B\rangle_N$, see [23,24]. In terms of modes, (2.6) reads

$$\hat{p} |B\rangle_N = 0 \quad \text{and} \quad (\alpha_n + \overline{\alpha}_{-n}) |B\rangle_N = 0 \quad (2.13)$$

for all $n \neq 0$, and the general solution is a linear combination of coherent states

$$|(w, 0)\rangle_N := \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \overline{\alpha}_{-n}\right) |(w, 0)\rangle.$$  

(2.14)

Of course, Dirichlet and Neumann conditions are related by $T$-duality, accordingly one can regard the special solution

$$|B(\tilde{x}_\alpha)\rangle_N = \sqrt{r} \sum_{w \in \mathbb{Z}} e^{-2irw\tilde{x}_\alpha} |(w, 0)\rangle_N$$  

(2.15)
as an eigenstate of the “dual” coordinate $\tilde{X} = X_L - X_R$ with eigenvalue $\tilde{x}_\alpha$.

An important common property of Dirichlet and Neumann boundary states is that both preserve conformal invariance in the sense that

$$(L_n - \overline{L}_{-n}) |B\rangle_D = (L_n - \overline{L}_{-n}) |B\rangle_N = 0$$

where $L_n, \overline{L}_n$ are the left- resp. right-moving Virasoro generators of the closed string, which have the usual bilinear expressions in $\alpha_n, \overline{\alpha}_n$. This observation will be the starting point of the general considerations in the next section.

It is straightforward to form boundary states describing Dirichlet $p$-branes in a $(d+1)$-torus out of (2.12) and (2.15): Assume that Neumann conditions are prescribed in the directions $X^\mu, \mu = 0, \ldots, p$, and Dirichlet conditions in the directions $X^\nu, \nu = p+1, \ldots, d$. Then the tensor product

$$|B(p \text{-brane})\rangle = |B(\tilde{x}_\alpha^0)\rangle_N \otimes \cdots \otimes |B(\tilde{x}_\alpha^p)\rangle_N \otimes |B(x^p_{\alpha+1})\rangle_D \otimes \cdots \otimes |B(x^d_{\alpha})\rangle_D$$  \hspace{1cm} (2.16)

describes a D $p$-brane in the $(d+1)$-torus with fixed locations $x^\nu_\alpha$ in the last $d-p$ and fixed “dual locations” $\tilde{x}_\alpha^\mu$ in the first $p+1$ directions.

Generalization of the above construction from the bosonic case to boundary states describing D-branes for superstrings in a flat “super torus” is easy, too. One merely has to impose boundary conditions on the fermionic coordinates $\psi^\mu, \overline{\psi}^\mu$ as well, namely

$$\left(\psi^\mu_{r, \pm i} \overline{\psi}_{-r}^\mu\right) |B\rangle_{\psi} = 0,$$  \hspace{1cm} (2.17)

and to solve them in the fermionic Fock space; again, fermionic coherent states arise. The signs in (2.17) correspond to different choices of the spin structure, cf. [5,25], the factor $i$ will become clear in section 3.

Let us sketch how the boundary states (2.11,12) and (2.14,15) can be used to uncover properties of the string-brane system, first restricting to the one-dimensional Dirichlet case again. As mentioned above, the closed string cylinder diagram describes a transition amplitude from a boundary state $|B(x_\alpha)\rangle_D$ at time $t_\alpha$ to a boundary state $|B(x_\beta)\rangle_D$ at time $t_\beta$, with propagation driven by the closed string Hamiltonian $H_{\text{cl}} = L_0 + \overline{L}_0 - c/12$. From the open string point of view, we calculate a one-loop diagram, i.e. a partition function $Z^D_{\alpha\beta}$ of the open string Hamiltonian $H_{\text{op}} = L_0 - c/24$. (We neglect the usual offset $-1$ from normal ordering.) We have the equation

$$Z^D_{\alpha\beta}(q) \equiv \text{Tr}_{H_{op}} q^{H_{op}} = D \langle B(x_\beta) | \tilde{q}^{\frac{c}{24} H_{\text{cl}}} | B(x_\alpha) \rangle_D$$  \hspace{1cm} (2.18)

with $q = e^{2\pi i \tau}$ and $\tilde{q} = e^{-2\pi i / \tau}$. The right hand side is easily evaluated using (2.11,12) and Poisson resummation, see e.g. [26]:

$$Z^D_{\alpha\beta}(q) = \frac{1}{\eta(q)} \sum_{k \in \mathbb{Z}} q^{\frac{c}{24}[2rk+(x_\alpha-x_\beta)/\pi]^2}$$  \hspace{1cm} (2.19)
with \( \eta(q) = q^{\frac{1}{2\pi}} \prod_{n \geq 1} (1 - q^n) \). Similarly, for the case that Neumann conditions are prescribed at both ends of the open string, formulas (2.14,15) lead to

\[
Z^N_{\alpha\beta}(q) = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} q^{\frac{1}{2}[w/r+(\bar{x}_\alpha-\bar{x}_\beta)/\pi]^2} .
\]

(2.20)

Neumann conditions at one end and Dirichlet at the other can of course also be imposed; see [26] for the associated partition function.

We observe that both partition functions are linear combinations of U(1) characters with positive integer coefficients. Furthermore, during the Poisson resummation, the momentum quantum number \( k = rp \) in the Dirichlet coherent states (2.11) has acquired the role of a winding number in (2.19), whereas the winding degree of freedom \( w \) in the Neumann coherent states (2.14) has turned into a momentum degree of freedom in the partition function (2.20).

When generalized to tensor product boundary states \(|B_\alpha\rangle\) and \(|B_\beta\rangle\) as in (2.16), the partition function \( Z_{\alpha\beta}(q) = \langle B_\beta | q^{\frac{1}{2}H^{cl}} | B_\alpha \rangle \) describes the (perturbative) excitation spectrum of two \( D \) \( p \)-branes with locations \( \bar{x}_\alpha^\mu, x_\alpha^\nu \) and \( \bar{x}_\beta^\mu, x_\beta^\nu \), interchanging (tree-level) closed strings between each other. In particular, one can read off the massless states of the system from \( Z_{\alpha\beta}(q) \) – and therefore from the boundary states – by identifying the highest weight states of conformal dimension one.

We see that for large target radius \( r \), the winding modes in (2.19) become very heavy, whereas the momentum modes in (2.20) stay light: The low-energy effective field theory contains fields that depend only on the Neumann directions of the brane, i.e. on its world volume coordinates. This was explained in [4,5] and further used in [8] to show that D-branes for the superstring (with Chan-Paton factors added) support dimensionally reduced gauge theories, with additional Higgs-like degrees of freedom which can be viewed as non-commutative transverse coordinates of the brane.

The CFT partition functions above – or more generally traces with additional vertex operators inserted, cf. (3.6) below – enter string amplitudes in the form

\[
A = c \int_{\mathcal{M}} d\mu_\mathcal{M} Z_{\alpha\beta}(q) .
\]

(2.21)

Here, the integral is over the moduli of the world sheet (in the simplest case shown, over the period \( \tau \) with an appropriate invariant measure), and the constant \( c \) contains symmetry factors of the diagram and the string tension, see [5,18-22] for more details.

Some data on the low-energy effective field theory can be uncovered without performing the integral in (2.21): As indicated above, the field content of the low-energy effective field theory on the D-brane world volume follows from the partition function \( Z_{\alpha\beta}(q) \) alone, and certain couplings \( \kappa_\Psi \) in this field theory can be read off directly from the boundary states (up to universal prefactors like \( c \) above). In particular, it was shown in [19-22] that the tension of the brane and its RR charges follow simply by taking projections

\[
\kappa_\Psi \sim \langle \Psi | B(p-\text{brane}) \rangle
\]

(2.22)

where \( |\Psi\rangle \) denotes the closed string state associated to the (massless) field in question; to obtain the tension, one inserts the graviton state, for the RR charges the corresponding
anti-symmetric forms. Beyond that, Di Vecchia et al. [21] could even derive the solitonic $p$-brane solution of the low-energy effective equations of motion from the Dirichlet $p$-brane boundary state.

In none of the formulas above, the closed string coupling $g_S$ shows up. However, when we want to compare amplitudes like (2.21) to closed string diagrams describing transitions between ordinary closed string states instead of boundary states, we have to keep track of the $g_S$-powers properly. Because of the open string origin of the boundary states, one finds [3,5] that e.g. the D-brane tension, which can be expressed as a disc diagram, scales like $g_S^{-1}$ – establishing the non-perturbative nature of D-branes and boundary states.

3. Generalized D-branes, and boundary CFT

We will now discuss methods which allow to abstract from the particularly simple case above and to define ”D-branes” in CFT terms, without reference to the geometry of some target. (We will, nevertheless, stick to the familiar name for these more general non-perturbative objects of string theory.) The relevant “microscopic” formalism to discuss such situations is that of boundary CFT, which goes back to the work of Cardy [27]. In the first two subsections, we will set up this framework in a generalized form.

The central aim of the section is to explain the notion and properties of boundary states which contain the complete information about a (static) D-brane. Their explicit construction is divided into two separate steps. First, in subsection 3.3, we obtain a linear space spanned by generalized coherent or Ishibashi states, which solve linear constraints describing how the left-moving symmetry generators of the closed string are “glued” to the right-moving ones on the brane.

Arbitrary boundary states have an expansion in a basis of such Ishibashi states, and in the second step of the construction one needs to determine the expansion coefficients. Following Cardy [28], we will show in subsection 3.4 that they are severely restricted by world sheet duality (modular covariance). These non-linear conditions were, so far, not explicitly used in the D-brane literature, but did play a role in previous investigations of open string theories [29].

Further constraints and general features of boundary CFT, including the bulk-boundary operator product expansion (OPE) [30], are sketched in subsection 3.5. We conclude the section with some remarks on non-rational situations.

3.1 Boundary conformal field theory

It is well-known that tree-level closed string theory is described by a CFT on the full plane (the so-called bulk theory), whereas tree-level open string theory involves boundary CFTs on the upper half-plane (or on the strip). In the modern context, it is the D-brane which cuts the string’s world in half: The boundary CFT now describes the perturbative properties of the string-brane-system – while, of course, the presence of the D-brane, or of a world sheet boundary, is not a perturbative phenomenon when seen from the original closed string.

So let us consider a CFT in the upper half-plane $\text{Im } z \geq 0$. Occasionally, we shall write $z = \exp(t + i{\sigma})$ and think of $t$ as time and of $\sigma$ as space variable. In the interior, $\text{Im } z > 0$,
the theory behaves like a usual CFT on the full plane. More precisely, the fields of the boundary CFT are in 1-1 correspondence with fields of the associated bulk theory and locally all their structure coincides in the sense that both theories have identical equal-time commutators with chiral fields and identical operator product expansions. In particular, there exists a stress-energy tensor $T_{\mu\nu}(z,\bar{z})$ for $\text{Im } z \geq 0$ and we shall require that no energy flows across the real line, i.e., $T_{xy}(x,0) = 0$ — where the variables $x = \text{Re } z$ and $y = \text{Im } z$ have been used. This translates into the condition
\[
T(z) = \bar{T}(\bar{z}) \quad \text{for } z = \bar{z}
\]
for the standard chiral fields $T=C$ where
\[
\partial \phi \text{ obeys the usual equal time commutators in the bulk including the relation}
\]
\[
[T(z_1), \overline{T}(\bar{z}_2)] = 0 \quad \text{for } \text{Im } z_i > 0 \text{ and } |z_1| = |\bar{z}_2|,
\]
these two fields, defined on the upper half-plane only, do not suffice to construct the action of two commuting Virasoro algebras on the state space. However, due to relation (3.1), we can still construct the generators $L_n^{(H)}$ of one Virasoro algebra \cite{27} by
\[
L_n^{(H)} := \frac{1}{2\pi i} \int_C z^{n+1} T(z) dz - \frac{1}{2\pi i} \int_C \bar{z}^{n+1} \overline{T}(\bar{z}) d\bar{z}
\]
where $C$ denotes a semi-circle in the upper half-plane with ends on the real line. This implies that the space $\mathcal{H}$ of states of the boundary theory decomposes into a sum $\mathcal{H} = \bigoplus \mathcal{H}_i$ of irreducible Virasoro modules.

Primary fields $\phi(z, \bar{z})$ in the boundary theory obey the characteristic equal time commutators
\[
[T(z_1), \phi(z_2, \bar{z}_2)] = \delta(z_2 - z_1) \partial \phi(z_1, \bar{z}_1) + h \delta'(z_2 - z_1) \phi(z_1, \bar{z}_1),
\]
\[
[\overline{T}(\bar{z}_1), \phi(z_2, \bar{z}_2)] = \delta(\bar{z}_2 - \bar{z}_1) \overline{\partial} \phi(z_1, \bar{z}_1) + \bar{h} \delta'(\bar{z}_2 - \bar{z}_1) \phi(z_1, \bar{z}_1),
\]
for all $z_i$ with $\text{Im } z_1 \geq 0$, $\text{Im } z_2 > 0$, and $|z_1| = |\bar{z}_2|$. Note that the fields $\phi(z, \bar{z})$ are well-defined only in the interior of the half-plane, which is why we will sometimes refer to them as “bulk fields” (not to be confused with the fields of a bulk CFT on the full plane). From these two relations one obtains
\[
[L_n^{(H)}, \phi(z, \bar{z})] = z^n (z\partial + h(n+1)) \phi(z, \bar{z}) + \bar{z}^n (\bar{z}\overline{\partial} + \bar{h}(n+1)) \phi(z, \bar{z}).
\]
Here, $h, \bar{h}$ are the conformal dimensions of the field $\phi$. We will assume below that their difference $h - \bar{h}$ is (half-)integer for all primary fields in the theory. Using the commutation relations (3.3) we can deduce Ward identities for correlators of primary fields (see, e.g., \cite{27}). In turns out that the $N$-point functions of $N$ fields $\phi_{h_i, \bar{h}_i}(z_i, \bar{z}_i), i = 1, \ldots, N$, in the boundary CFT satisfy the same differential equations as $2N$-point functions of fields $\phi_{h_i, \bar{h}_i}(z_i, \cdot) \phi_{h_i, \bar{h}_i}(\bar{z}_i, \cdot), i = 1, \ldots, N$, with respect to the left-moving Virasoro algebra: Each field is accompanied by its “mirror charge” \cite{27}.

To conclude this subsection, let us briefly adapt the theory to cases in which there are other chiral fields $W(z)$ and $\overline{W}(\bar{z})$ of half-integer conformal dimension $h_W$ besides the two
fields obtained from the stress-energy tensor $T$. Again, the simplest way to guarantee the action of an extended chiral algebra on the state space is to assume that

$$W(z) = \overline{W}(\bar{z}) \quad \text{for} \quad z = \bar{z}. \quad (3.4)$$

But since there is no equally fundamental interpretation for this constraint as we have for the stress-energy tensor, we shall relax this condition and assume the existence of a local automorphism $\Omega$ which acts on the space of chiral fields (while respecting the equal-time commutators) such that

$$W(z) = \Omega(\overline{W})(\bar{z}) \quad \text{for} \quad z = \bar{z}. \quad (3.5)$$

The condition (3.4) is obtained when $\Omega$ is the trivial automorphism. In any case, we assume that $\Omega$ acts trivially on the Virasoro field $T$ so that the energy flow vanishes on the real line. We can now construct modes $W_n^{(H)}$ by

$$W_n^{(H)} := \frac{1}{2\pi i} \int C z^{n+h_w-1}W(z)dz - \frac{1}{2\pi i} \int C \bar{z}^{n+h_w-1}\Omega(\overline{W})(\bar{z})d\bar{z}. \quad (3.6)$$

The notions of primary fields for the extended algebra and corresponding Ward identities for correlators can be obtained as for the Virasoro algebra above. Note that the Ward identities will depend explicitly on the automorphism $\Omega$.

### 3.2 Boundary states

In this subsection we demonstrate that to each boundary CFT one may associate a boundary state which contains all the information about the boundary conditions, and thus about (static) D-branes. Roughly speaking, the latter is described by an element in an appropriate extension of the state space $H^{(P)}$ of the bulk theory. (In this subsection, we shall mark all objects of the CFT on the full plane by an upper index $P$ and those of the CFT on the half-plane by an index $H$.)

To motivate the notion of boundary states, let us investigate the (finite temperature) correlators

$$\langle \phi_1^{(H)}(z_1, \bar{z}_2) \cdots \phi_N^{(H)}(z_N, \bar{z}_N) \rangle_{\beta_0}^{\beta_0} = \text{Tr}_H(e^{-\beta_0 H^{(H)}} \phi_1^{(H)}(z_1, \bar{z}_1) \cdots \phi_N^{(H)}(z_N, \bar{z}_N)) \quad (3.6)$$

where we assume the arguments $z_i$ to be radially ordered and where $H^{(H)} = L_0^{(H)} - c/24$. If the fields $\phi^{(H)}(z, \bar{z})$ are quasi-primary so that $\phi^{(H)}(\lambda z, \lambda \bar{z}) = \lambda^{-h} \lambda^{-\bar{h}} \phi^{(H)}(z, \bar{z})$, the above correlators (3.6) are (anti)-periodic in the time variable $t = \ln |z|$ up to a scalar factor. In this sense, the underlying geometry is that of a 1-loop diagram in open string theory.

The idea now is to replace $z, \bar{z}$ by new variables $\xi, \bar{\xi}$ in terms of which the correlators (3.6) may be reinterpreted as certain correlators for a theory on the full plane. Since the latter are necessarily periodic in the space variable, we need to exchange the role of space and time in the transformation from $z$ to $\xi$. Let us first introduce the variable $w = \ln z = t + i\sigma$. Because of the periodicity properties of (3.6) the theory essentially lives on a cylinder parameterized by $\sigma \in [0, \pi]$ and $t \in [t_0, t_0 + \beta_0]$ with the two segments
at $t = t_0$ and $t_0 + \beta_0$ identified. Now we exchange the role of space and time: After an appropriate rescaling by $2\pi/\beta_0$ we obtain a cylinder which is now periodic in space with period $2\pi$ and for which the time variable runs from $0$ to $2\pi^2/\beta_0$. This interchange of space and time is again nothing but world sheet duality. With the help of the exponential mapping

$$\xi = e^{\frac{2\pi i}{\beta_0} \ln z} \quad \text{and} \quad \bar{\xi} = e^{-\frac{2\pi i}{\beta_0} \ln \bar{z}},$$

the cylinder is finally mapped onto an annulus in the full $\xi$-plane. To rewrite the original correlators in terms of these new variables we make use of the transformation behaviour

$$\phi(\xi, \bar{\xi}) = (\frac{d\xi}{dz})^H (\frac{d\bar{z}}{d\xi})^\bar{H} \phi(H)(z, \bar{z}), \quad T(\xi) = (\frac{dz}{d\xi})^2 T(H)(z) + \frac{c}{24} \{z, \xi\}$$

(3.7)

for primary fields $\phi$ and the stress-energy tensor; $\{z, \xi\}$ is the usual Schwartz derivative. It can be easily checked that the resulting correlators for the fields $\phi(\xi, \bar{\xi})$ are invariant under the substitution $\xi \mapsto \exp(2\pi i) \xi$ if $h - \bar{h}$ is an integer (for half-integer $h - \bar{h}$, the fields are anti-periodic, i.e. they live on a double cover of the annulus). This shows (at the level of correlation functions) that the fields on the lhs. of (2.7) can be consistently defined on the full plane.

The notion of a boundary state is introduced to interpret the correlators of $\phi(\xi, \bar{\xi})$ completely within the framework of CFT on the plane. By definition, the boundary state of our original boundary CFT is a “state” $|\alpha\rangle$ “in” the state space $\mathcal{H}^{(P)}$ of the bulk theory such that

$$\langle \phi_1^{(P)}(\xi_1, \bar{\xi}_1) \cdots \phi_N^{(P)}(\xi_N, \bar{\xi}_N) |\alpha\rangle^{\beta_0} = \Theta \alpha |e^{-\frac{2\pi^2}{\beta_0}} H^{(P)} \phi_1^{(P)}(\xi_1, \bar{\xi}_1) \cdots \phi_N^{(P)}(\xi_N, \bar{\xi}_N)|\alpha\rangle.$$  (3.8)

Here, $H^{(P)} = L_0 + \bar{L}_0 - c/12$ is the Hamiltonian, $\Theta$ the CPT-operator in the bulk theory and we now regard the fields $\phi(\xi, \bar{\xi})$ as living on the full plane. With the help of the rule (3.7) and the conditions (3.5) it is easy to derive that $|\alpha\rangle$ obeys the following gluing or Ishibashi conditions

$$(L_n^{(P)} - \bar{L}_{-n}^{(P)}) |\alpha\rangle_\Omega = 0 \quad \text{and} \quad (W_n^{(P)} - (-1)^{h_W} \Omega(\bar{W}_{-n}^{(P)})) |\alpha\rangle_\Omega = 0.$$  

Since these relations depend on $\Omega$, which we will also call “gluing automorphism” in the following, we mark the boundary states $|\alpha\rangle = |\alpha\rangle_\Omega$ by an extra subscript $\Omega$ throughout most of this text. The space of solutions to the above linear constraints is the subject of the next subsection.

For a given bulk CFT on the plane, one can usually find several different boundary theories, which give rise to different boundary states $|\alpha\rangle$, $|\beta\rangle$, etc. Looking at the expression (3.8) one is tempted to replace one of the boundary states $|\alpha\rangle$ by some other state $|\beta\rangle$. Indeed, such correlators

$$\langle \phi_1^{(P)}(\xi_1, \bar{\xi}_1) \cdots \phi_N^{(P)}(\xi_N, \bar{\xi}_N) |\beta\rangle^{\beta_0} = \Theta \beta |e^{-\frac{2\pi^2}{\beta_0}} H^{(P)} \phi_1^{(P)}(\xi_1, \bar{\xi}_1) \cdots \phi_N^{(P)}(\xi_N, \bar{\xi}_N)|\alpha\rangle$$  (3.9)

make perfect sense and even possess an interpretation in the original boundary theory: When we map such correlators back into the $z$-half-plane they correspond to a boundary CFT for which the boundary condition “jumps” at the point $z = 0$ (we will make this more
precise in subsection 3.5). In string theory terms, this describes a system of two different branes exchanging (tree-level) closed strings.

Most properties of a boundary CFT discussed so far hold true in these more general situations. In particular, the state space \( \mathcal{H}_{\alpha\beta} \) of the boundary CFT with discontinuous boundary condition at \( z = 0 \) carries an action of the chiral algebra generated by the modes \( W_n^{(H)} \). In comparison to the usual case there is only one essential difference: The space \( \mathcal{H}_{\alpha\beta} \) may fail to contain a vacuum state \( |0\rangle \) which is annihilated by the elements \( L_n^{(H)}, n = 0, \pm 1 \). While this prevents us from looking at vacuum correlators of “bulk fields” \( \phi^{(H)}(z, \bar{z}) \), the functions (3.6) are still well defined.

3.3 Generalized coherent states

We now want to solve the gluing conditions, i.e. find “states” \( |I\rangle \rangle \) on which left- and right-moving generators of the symmetry algebra act like

\[
(W_n - (-1)^{hw} \overline{W}_{-n}) |I\rangle \rangle = 0 ; \tag{3.10}
\]

for simplicity, we have dropped the superscript \((P)\) and restricted ourselves to the trivial gluing automorphism \( \Omega \) for the moment. Assume from now on that the symmetry generators \( W_n \) and \( \overline{W}_n \) of the bulk CFT generate identical left- and right-moving chiral algebras \( \mathcal{A}_L = \mathcal{A}_R = \mathcal{A} \), including the Virasoro algebra. For this situation, Ishibashi has shown \([31]\) that to each irreducible highest weight representation \( i \) of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H}_i \) one can associate a “state” \( |i\rangle \rangle \), which is unique up to an overall constant, such that (3.10) is satisfied. Using \( |i, N\rangle, N \in \mathbb{Z}_+ \), to denote an orthonormal basis of \( \mathcal{H}_i \), one can give the (formal) expression

\[
|i\rangle \rangle = \sum_{N=0}^{\infty} |i, N\rangle \otimes U|i, N\rangle \tag{3.11}
\]

for the Ishibashi state associated to \( i \); above, \( U \) denotes an anti-unitary operator on the total chiral Hilbert space \( \mathcal{H}_R = \bigoplus_i \mathcal{H}_i \) which satisfies the commutation relations

\[
U \overline{W}_n = (-1)^{hw} \overline{W}_{-n} U \tag{3.12}
\]

with the right-moving generators: \( U \) acts like a chiral CPT operator. Note that \( U|i, N\rangle \in \mathcal{H}_{i^+} \), the Hilbert space carrying the representation \( i^+ \) conjugate to \( i \), and we will say that \( |i\rangle \rangle \) “couples to” the Hilbert space \( \mathcal{H}_i \otimes \mathcal{H}_{i^+} \).

Proofs of the property (3.10) and of uniqueness (therefore of basis independence) can be found in \([31,28]\). Note that \( |i\rangle \rangle \) is not a state in the bulk Hilbert space, in the same way as the coherent states describing D-branes of a free boson were not ordinary states. Indeed, one can rewrite the coherent states, at least for Neumann boundary conditions, as Ishibashi states of the form (3.11), see \([31]\).

Let us remark already now that the divergences coming with objects like (3.11) will not cause any trouble later, since in all calculations to follow, \( |i\rangle \rangle \) will be accompanied with “damping operators” like \( q^{L_0 - \frac{c}{24}} \) with \( |q| < 1 \). In particular, there is an inner product between Ishibashi states that will be important below:

\[
\langle \langle j | q^{L_0 - \frac{c}{24}} | i \rangle \rangle = \delta_{i,j} \chi_i(q) \tag{3.13}
\]

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with \( \chi_i(q) = \text{Tr}_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}} \) being the conformal character of the irreducible representation \( i \); this follows from the expression (3.11).

The gluing conditions (3.10) we have considered up to now depend only on the conformal dimensions of the local fields \( W(z) \). In particular, we have \( L_n = L_{-n} \) at the boundary, as well as \( \alpha_n = -\overline{\alpha}_{-n} \) for the modes of a spin 1 current: This means that the Ishibashi states (3.11) are not suitable to incorporate Dirichlet boundary conditions on a compactified free boson.

To incorporate Dirichlet boundary conditions we now allow for non-trivial automorphisms \( \Omega \) and look for solutions \(|i\rangle \rangle_{\Omega} \)

\[
(W_n - (-1)^{hw} \Omega(\overline{W}_{-n})) |i\rangle \rangle_{\Omega} = 0.
\] (3.14)

Such “twisted” Ishibashi states \(|i\rangle \rangle_{\Omega} \) may be associated to any representation \( i \) of \( \mathcal{A} \). Given the standard Ishibashi state \(|i\rangle \rangle \equiv |i\rangle \rangle_{\text{id}} \), the twisted one \(|i\rangle \rangle_{\Omega} \) is constructed as follows:

The outer automorphism \( \Omega \) allows \( \mathcal{H}_i \) to carry another inequivalent irreducible representation where elements \( A \in \mathcal{A} \) act on states through \( A|\hbar\rangle := \Omega(A)|\hbar\rangle \). The space \( \mathcal{H}_i \) equipped with this new action of \( \mathcal{A} \) is isomorphic (as an \( \mathcal{A} \)-module) to some representation space \( \mathcal{H}_{\omega(i)} \) (this defines \( \omega(i) \)). We denote the isomorphism by

\[
V_{\Omega} : \mathcal{H}_i \longrightarrow \mathcal{H}_{\omega(i)},
\] (3.15)

and we assume that \( V_{\Omega} U = UV_{\Omega} \). Then the generalized coherent state

\[
|i\rangle \rangle_{\Omega} := (\text{id} \otimes V_{\Omega}) |i\rangle \rangle
\] (3.16)

satisfies (3.14). Note that \(|i\rangle \rangle_{\Omega} \) couples to the Hilbert space \( \mathcal{H}_i \otimes \mathcal{H}_{\omega(i)}^* \).

Formula (3.13) holds for twisted Ishibashi states as well, provided both \(|i\rangle \rangle_{\Omega} \) and \(|j\rangle \rangle_{\Omega} \) belong to the same twisting \( \Omega \). Otherwise, weighted traces of \( V_{\Omega}V_{\Omega}^* \) will appear, see section 4.2 for an example.

The first important case of an automorphism \( \Omega \) and its twisted Ishibashi states is given by the sign-flip

\[
\Omega_{\Gamma} : J(z) \mapsto -J(z)
\]

of the U(1) current algebra, implying gluing conditions \( \alpha_n = -\overline{\alpha}_n \) for the bulk modes at the boundary. The associated twisted Ishibashi states \(|i\rangle \rangle_{\Omega_{\Gamma}} \) are precisely the coherent states that implement Dirichlet conditions on a free boson on a circle.

Generalization to non-abelian current algebras is in principle straightforward: The gluing relations (3.10) with \( W_n = J_n^a \) can be twisted by outer automorphisms of the Dynkin diagram, cf. the detailed analysis by Kato and Okada in [32]. We would, however, like to point out that their interpretation of the standard gluing conditions as “Neumann conditions on the group target” is erroneous in the non-abelian case, as can be seen by expressing the currents through the group valued field. The gluing conditions determine the (world sheet) symmetry content of the brane-bulk system but in general have no simple meaning in terms of a \( \sigma \)-model target.
The other prominent example that will be important for us later on is the mirror automorphism \( \Omega_M \) of the \( N = 2 \) super Virasoro algebra

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0}, \\
[L_n, J_m] = -m J_{n+m}, \\
[L_n, G^+_r] = \left( \frac{n}{2} - r \right) G^+_{n+r}, \\
[J_n, J_m] = \frac{c}{3} n \delta_{n+m,0}, \\
[J_n, G^+_-r] = \pm G^+_{n+r}, \\
\{ G^+_r, G^-_s \} = 2L_{r+s} + (r - s) J_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0}.
\]

(3.17)

Above, \( J_n, n \in \mathbb{Z} \), are the modes of a U(1) current, \( G^\pm_r \) are spin \( 3/2 \) superpartners of the Virasoro generators \( L_n \) with \( r \in \mathbb{Z} + \frac{1}{2} \) (Neveu-Schwarz sector) or \( r \in \mathbb{Z} \) (Ramond sector). This \( N = 2 \) supersymmetric extension of the Virasoro algebra has many interesting properties (see [33] for an excellent reference), and it plays a central role in the CFT approach to superstrings.

The relations (3.17) admit the outer automorphism (the “mirror map”)

\[ \Omega_M : J_n \mapsto -J_n, \quad G^+_r \mapsto G^-_r; \]  

(3.18)

consequently, there are two possible sets of Ishibashi conditions [17], usually called **B-type conditions** for the standard gluing

\[
(L_n - \bar{L}_{-n}) |i\rangle_B = (J_n + \bar{J}_{-n}) |i\rangle_B = 0 \\
(G^+_{r} + i\eta \bar{G}^+_{-r}) |i\rangle_B = (G^-_{r} + i\eta \bar{G}^-_{-r}) |i\rangle_B = 0
\]

(3.19)

and **A-type conditions** for the twisted gluing

\[
(L_n - \bar{L}_{-n}) |i\rangle_A = (J_n - \bar{J}_{-n}) |i\rangle_A = 0 \\
(G^+_{r} + i\eta \bar{G}^+_{-r}) |i\rangle_A = (G^-_{r} + i\eta \bar{G}^-_{-r}) |i\rangle_A = 0;
\]

(3.20)

the sign freedom \( \eta = \pm 1 \) is as in eq. (2.17). In the notation used above, \( |i\rangle_B = |i\rangle |\text{id}\) and \( |i\rangle_A = |i\rangle |\Omega_M\).

In the case of bosons with values on a two- or higher-dimensional torus (with equal radii), and also in some of the examples to be studied in the next section, tensor products of identical CFTs occur. Then there exist further outer automorphisms \( \Omega_\pi \) of the chiral algebra, acting by permutation of identical component algebras. E.g., on a torus \( \mathbb{T}^d \), one can glue the \( \mu \)th left-moving current \( \partial X^\mu \) to the \( \pi(\mu) \)th right-moving current \( \bar{\partial} X^{\pi(\mu)} \) with the help of such permutation automorphisms for any \( \pi \in S_d \). Composition with the sign flip \( \Omega_\Gamma \) allows to obtain permuted Dirichlet conditions as well. Similarly, one can model target rotations of the brane (or branes carrying constant electric fields) by suitable choices of the gluing automorphisms.

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We have obtained a complete overview of the possible Ishibashi states for a bulk CFT with symmetry algebra $A \otimes A$: To each automorphism $\Omega$ of $A$ (including the identity) there is a set of gluing conditions (3.14) between the left- and right-moving generators; and to each irreducible highest weight representation $i$ of $A$ on a Hilbert space $H_i$, there is a unique (up to rescaling) Ishibashi state $|i\rangle_\Omega$ which “implements” these gluing conditions. This means that if we form a boundary state

$$|\alpha\rangle_\Omega = \sum B^i_\alpha |i\rangle_\Omega \quad (3.21)$$

as a linear combination of (twisted) Ishibashi states, the system (bulk CFT + boundary state) will have $A$ as its symmetry algebra.

However, a bulk CFT is not specified by its symmetry algebra $A_L \otimes A_R$ alone, but it comes with a given modular invariant partition function on the torus; i.e., a consistent selection of irreducible $A_L,R$-modules making up the total Hilbert space

$$H_{\text{tot}} = \bigoplus_{j,\bar{j}} H_j \otimes H_{\bar{j}} \quad (3.22)$$

$j \in I_L$, $\bar{j} \in I_R$, is prescribed, too. Therefore, not all the (twisted) Ishibashi states $|i\rangle_\Omega$ that abstractly exist can really couple to the bulk theory: The condition is that the term $H_i \otimes H_{\omega(i)+}$ occurs in (3.22).

For the Gepner models studied below, this fact implies that a different number of Ishibashi states contribute to A-type boundary states ($\Omega = \Omega_M$ in (3.21)) than to B-type boundary states ($\Omega = \text{id}$ in (3.21)). E.g., A-type boundary states receive contributions from the chiral-chiral states in $H_{\text{tot}}$, whereas the chiral-antichiral states only contribute to B-type boundary states.

The coupling conditions also explain, in CFT terms, why there exist only Dirichlet $p$-branes with $p$ even in a ten-dimensional type IIA superstring theory (with flat toroidal target) whereas type IIB superstrings couple to D $p$-branes with $p$ odd. The difference between these two theories of closed superstrings lies in the GSO projection, which eliminates different left-right-combinations of fermionic states – so that different Ishibashi states couple to the two bulk CFTs. Details can be found e.g. in [25].

### 3.4 Cardy’s conditions

We will now discuss the second class of constraints on boundary states, the Cardy constraints, which restrict the possible linear combinations (3.21) of Ishibashi states in a boundary state. While the Ishibashi states (to a given gluing condition) still form an abstract vector space, the new constraints are non-linear: The terminology boundary “states” is, therefore, even more misleading than for the generalized coherent states $|i\rangle_\Omega$, and one should rather view them as “labels” for different (non-perturbative) sectors of a bulk CFT – much in the spirit of D-brane physics.

Cardy’s derivation [28] of the conditions (see also [34] for a good exposition) starts from a specialization of the setting we have discussed in subsection 3.2: Consider a boundary CFT with symmetry algebra $A$ for which the boundary condition jumps at $z = 0$ from $\alpha$...
to \( \beta \); study the system at finite temperature, or make the “time” direction periodic with period \( \beta_0 = -2\pi i \tau \), then compute the partition function

\[
Z_{\alpha\beta}(q) = \text{Tr}_{\mathcal{H}_{\alpha\beta}} q^{H_{\alpha\beta}^{(H)}}
\]

where \( \mathcal{H}_{\alpha\beta} \) is the boundary CFT Hilbert space, where \( H_{\alpha\beta}^{(H)} = L_0^{(H)} - \frac{c}{24} \) is the Hamiltonian in the \( w = \ln z \) coordinate, and where \( q = e^{2\pi i \tau} \). This is in fact a correlator of the type (3.6), without insertions of bulk fields \( \phi(z, \bar{z}) \), and as in subsection 3.2, eq. (3.9), we can re-interpret this situation and compute the same partition function within the bulk theory. This involves the interchange of space and time familiar from world sheet duality which, in terms or the parameter \( \tau \), amounts to a modular transformation \( \tau \mapsto -\frac{1}{\tau} \). Note that, in section 2, we have used world sheet duality merely to “translate” open string Neumann or Dirichlet boundary conditions into conditions on closed string modes, following [23,24]. Cardy’s constraints, on the other hand, will arise from the concrete modular covariance properties of the partition function.

After the re-interpretation, the system is periodic in space (it lives on an annulus), and as before we assume that the boundary conditions are implemented by boundary states \( \langle \alpha \rangle \) and \( \langle \beta \rangle \), which sit at the ends of the annulus. For simplicity, let us restrict to standard gluing conditions (3.10). Let us furthermore assume that the Hilbert space of the bulk CFT on the full plane decomposes into irreducible representations of the symmetry algebra \( \mathcal{A} \otimes \mathcal{A} \) as

\[
\mathcal{H}_{\text{tot}} = \bigoplus_{j \in \mathcal{I}} \mathcal{H}_j \otimes \mathcal{H}_{j^+},
\]

(3.24)

corresponding to a so-called “charge conjugate” modular invariant partition function. For the special case under consideration, eq. (3.9) yields

\[
Z_{\alpha\beta}(q) = \langle \Theta \beta | \tilde{q}^{1/2} (L_0^{(P)} + \bar{T}_0^{(P)} - \frac{c}{24}) | \alpha \rangle
\]

(3.25)

with \( \tilde{q} = e^{-2\pi i / \tau} \). The right hand side describes free propagation from the boundary state \( |\alpha\rangle \) at “time” 0 to the boundary state \( |\Theta \beta\rangle \) at “time” \( 2\pi^2 / \beta = \pi i / \tau \), driven by the Hamiltonian \( H^{(P)} \).

Note that, deviating from Cardy, we have introduced the CPT operator \( \Theta \) in the definitions (3.8,9) because the two boundaries have opposite orientation; the same prescription was used in [24] in the context of open string theory. The boundary states of section 2 were CPT invariant, so we left out \( \Theta \) for simplicity.

As was pointed out by Cardy, the simple identity (3.25) contains severe constraints on the boundary states \( |\alpha\rangle \) and \( |\beta\rangle \): We assumed that the boundary CFT on the half-plane has \( \mathcal{A} \) as its symmetry algebra; therefore, the Hilbert space \( \mathcal{H}_{\alpha\beta} \) decomposes into irreducible representations \( \mathcal{H}_i \) of \( \mathcal{A} \), and the partition function \( Z_{\alpha\beta}(q) \) is a sum of characters

\[
Z_{\alpha\beta}(q) = \sum_i n_{\alpha\beta}^i \chi_i(q)
\]

(3.26)

with \( \chi_i(q) = \text{Tr}_{\mathcal{H}_i} q^{L_0^{(P)} - \frac{c}{24}} \) and positive integer coefficients \( n_{\alpha\beta}^i \).
On the other hand, we can compute the “bulk amplitude” with the help of the expansion (3.21) of boundary states into Ishibashi states and by applying the inner product (3.13): 

\[ \langle \Theta | \tilde{q}^{\frac{1}{2} (L_0^{(P)} + \bar{L}_0^{(P)}) - \frac{c}{24}} | \alpha \rangle = \sum_i B^i_\beta B^i_\alpha \chi_i(q) \] (3.27)

Implicitly, we have used \( \Theta B^i_j | j \rangle \rangle = B^i_j | j \rangle \rangle \) – i.e. we have picked a special normalization of \( \Theta \) – as well as the fact that the conjugate representation \( i^+ \) must occur in the index set \( I \) of (3.24) as soon as \( i \) does.

The modular transformation behind world sheet duality acts linearly on the characters,

\[ \chi_i(q) = \sum_j S_{ij} \chi_j(q) \] (3.28)

with

\[ SS^* = 1, \quad S = S^t, \quad S^2 = C \] (3.29)

where \( C_{ij} = \delta_{i,j^+} \) acts as charge conjugation. Thus, we obtain the final form of Cardy’s constraints on the coefficients \( B^i_\alpha \) of the Ishibashi states making up a full boundary state:

\[ \sum_{i,j} B^j_\beta B^i_\alpha S_{ji} \chi_j(q) = \sum_i n^i_{\alpha \beta} \chi_i(q) \] (3.30)

for some set of positive integers \( n^i_{\alpha \beta} \). Let us stress that multiplying an acceptable boundary state by an overall factor other than a positive integer in general violates (3.30).

For a rational CFT, i.e. if \( I \) is finite, Cardy has found a solution with the help of the Verlinde formula

\[ N^k_{ij} = \sum_l \frac{S_{il}S_{jl}S^*_{lk}}{S_{0l}} \] (3.31)

for the fusion rules, where 0 denotes the vacuum representation. In Cardy’s solution, the boundary states \( | a \rangle \) carry the same labels as the irreducible representations of \( A \), i.e. \( a \in I \), and their expansion into Ishibashi states is

\[ | a \rangle = \sum_i \frac{S_{ai}}{(S_{0i})^{\frac{1}{2}}} | i \rangle \] (3.32)

With (3.31) and the properties (3.29) of the modular \( S \)-matrix, it is easy to see that the partition function of the boundary CFT on the half-plane with boundary conditions described by states \( | \alpha \rangle, | \beta \rangle \) as in (3.32) is given by

\[ Z_{ab}(q) = \sum_i N^i_{a+b^+} \chi_i(q) \]

Note that, because of the CPT operator which we introduced in (3.8,9,25), we obtain an additional conjugation on the rhs. compared to [28].
Even in this simple situation, it is not generally clear that the solutions (3.32) (together with integer multiples) cover all possible boundary states that may couple to the bulk theory. However, Cardy’s solution provides sufficiently many independent states to render the coefficient matrix $B^i_\alpha$ invertible; within the analysis of sewing constraints (see [30,35]) this seems to be necessary for the notion of “completeness” proposed by Sagnotti et al. [36].

3.5 Boundary operators and bulk-boundary OPE

We have explained above that the state space $\mathcal{H}^{(H)}$ admits the action of “bulk fields” $\phi(z, \bar{z})$, defined in the interior of the half-plane, and of chiral fields $W(z), \bar{W}(\bar{z})$. It turns out, however, that one may introduce further boundary fields $\Psi(x)$ which are localized at points $x$ on the real line and are in 1-1 correspondence with the elements of $\mathcal{H}^{(H)}$.

Let us suppose for the moment that the boundary condition does not jump along the boundary so that $\mathcal{H}^{(H)}$ contains an $sl_2$-invariant vacuum state $|0\rangle$. Then, for any state $|v\rangle \in \mathcal{H}^{(H)}$, there exists a boundary operator $\Psi_v(x)$ such that

$$\Psi_v(x)|0\rangle = e^{xL_{-1}^{(H)}}|v\rangle$$

for all real $x$. In particular, the operator $\Psi_v(0)$ creates the state $|v\rangle$ from the vacuum. Note also that $L_{-1}^{(H)}$ generates translations in the $x$-direction, i.e. parallel to the boundary. If $|v\rangle$ is a primary state of conformal weight $h$, it is straightforward to derive the commutators

$$[L_n^{(H)}, \Psi_v(x)] = x^n(x \frac{d}{dx} + h(n+1)) \Psi_v(x).$$

The existence of these extra fields $\Psi(x)$ motivates us to consider more general correlation functions in which the original bulk fields $\phi(z, \bar{z})$ on the upper half-plane appear together with the boundary fields $\Psi(x)$. They obey a set of Ward identities extending those we have described in subsection 3.1 above.

Following the standard reasoning in CFT, it is easy to conclude that the primary bulk fields $\phi(z, \bar{z})$ give singular contributions to the correlation functions whenever $z$ approaches the real line. This can be seen from the fact that the Ward identities describe a mirror pair of chiral charges placed on both sides of the boundary. Therefore, the leading singularities in $\phi(z, \bar{z})$ are given by primary fields which are localized at the point $x = \text{Re} z$ of the real line, i.e. the boundary fields $\Psi_v(x)$. In other words, the observed singular behaviour of bulk fields $\phi(z, \bar{z})$ near the boundary may be expressed in terms of a bulk-boundary OPE [30]

$$\phi(z, \bar{z}) \sim \sum_k (2y)^{h_k - h - \bar{h}} C_{\phi_k} \Psi_k(x).$$

(3.34)

Here, $\Psi_k(x)$ are primary fields of conformal weight $h_k$ and $z = x + iy$ as before. We remark that the chiral bulk fields remain regular at the boundary. This was used before when we considered $W(z)$ and $\bar{W}(\bar{z})$ to be defined in the closed upper half-plane, $\text{Im} z \geq 0$.

To guarantee locality of the boundary CFT, the coefficients in (3.34) must satisfy a number of sewing constraints similar to the familiar crossing relations in CFT. Such relations have
been worked out by Cardy and Lewellen [30,35] and in particular by Sagnotti et al. [36]. Since they involve the coefficients of the bulk field OPEs, the sewing constraints are usually difficult to analyze. Nevertheless, some examples have been treated explicitly (see [36]).

In the investigation of boundary conditions for SU(2) WZW models, a particularly simple subset of sewing constraints was found: They involve only the coefficients \( C_{\phi \phi}^{0} \) from the bulk-boundary OPE (3.34) – where 0 denotes the vacuum sector of the chiral algebra –, the quantum dimensions \( d_i := (S_{0i}/S_{00})^{1/2} \), and a special combination of fusing matrices and bulk OPE coefficients. Surprisingly, the latter turns out to be given by the fusion matrices so that the sewing constraints for the normalized coefficients \( \hat{C}_{\phi \phi}^{0} := d_i C_{\phi \phi}^{0} \) become

\[
\hat{C}_{\phi \phi}^{i} \hat{C}_{\phi \phi}^{j} = \sum_k N_{ij}^{k} \hat{C}_{\phi \phi}^{k 0} .
\] (3.35)

For non-diagonal SU(2) WZW models, some extra signs appear in the sum (see [36]). Unfortunately, the computation in [36] seems to make use of very particular features of the model. Nevertheless, one may hope that the simplicity of (3.35) is not accidental [37]. We shall show below that the coefficients \( C_{\phi \phi}^{0} \), which are constrained by (3.35), are very closely related to the coefficients \( B_{i}^{\alpha} \) in the boundary states (3.21). Therefore, equations of the form (3.35) could provide further conditions for the allowed boundary states beyond those described in the previous subsection. In the examples of section 4 we will, however, not take them into account.

The relation between the coefficients \( B_{i}^{\alpha} \) and \( C_{\phi \phi}^{0} \) can be seen by analyzing the correlators (3.8) with only one primary bulk field \( \phi(i^+,\omega(i)) (\xi,\bar{\xi}) \) inserted (the subscript \( (i^+,\omega(i)) \) refers to the transformation properties of \( \phi(\xi,\bar{\xi}) \) under the action of chiral fields).

When we perform the limit \( \beta_0 \to 0 \), almost all terms in the boundary state \( \langle \Theta \alpha \rangle \) drop out of the correlation function and only the term \( B_{i}^{0} \langle 0 \rangle \) proportional to the vacuum state survives. Thus, the dominant contribution to the 1-point function for small \( \beta_0 \) is given by the correlator

\[
\langle 0 | \phi(i^+,\omega(i)) (\xi,\bar{\xi}) | \alpha \rangle = \frac{B_{i}^{0}}{(\xi \bar{\xi} - 1)^{h_{i^+} + h_{\omega(i)}}} ,
\]

where the rhs follows from invariance of both the vacuum and the boundary state under conformal transformations generated by \( L_n - \bar{T}_{-n} \).

To relate this term to the bulk-boundary OPE, we compute the same correlator in the \( z \)-plane and focus on the leading contribution when \( z \) approaches the real line – i.e. when \( \xi(z) \) approaches the unit circle. We can insert the bulk-boundary OPE (3.34) to evaluate the correlator (3.6) in this particular regime. This leaves us with a sum of traces of boundary fields \( \Psi_{i} \) multiplied by the coefficients of the bulk-boundary OPE, some factors depending on \( y = \text{Im} z \) as well as Jacobians from the coordinate transformation as in (3.7). When we let \( y \to 0 \), the term containing the trace of the identity field and the corresponding factor \( C_{(i^+,\omega(i))}^{0} \) dominates (assuming that the CFT is unitary), with the same dependence on \( \xi(z) \) as on the plane. The trace yields a factor \( (B_{0}^{0})^2 \), and comparison of the two different computations gives

\[
C_{(i^+,\omega(i))}^{0} = \frac{B_{i}^{0}}{B_{0}^{0}} .
\] (3.36)
This shows that the coefficients in front of the identity field in relation (3.34) determine the boundary state $|\alpha\rangle_{\Omega}$. On the other hand, using the sewing relations derived in [35,36], one can also reconstruct the remaining coefficients $C_{(i,\omega(i)^{+})}^{\alpha}$ from the knowledge of the boundary state $|\alpha\rangle_{\Omega}$.

Relation (3.36) allows us to make contact with an apparently different definition of boundary states which was used in [30]: Instead of mapping the upper half-plane (with coordinate $z$) to an annulus in the $\xi$-plane as above, we can map it to the complement of the unit disc in the $\zeta$-plane via

$$\zeta = \frac{1-iz}{1+iz} \quad \text{and} \quad \bar{\zeta} = \frac{1+i\bar{z}}{1-i\bar{z}};$$

given a CFT on the upper half-plane with boundary condition $\alpha$, the boundary state for the CFT on the $\zeta$-plane is defined in terms of zero-temperature correlators

$$\langle \phi_1^{(H)}(z_1,\bar{z}_2)\cdots\phi_N^{(H)}(z_N,\bar{z}_N) \rangle_\alpha = \langle 0| \phi_1^{(P)}(\zeta_1,\bar{\zeta}_1)\cdots\phi_N^{(P)}(\zeta_N,\bar{\zeta}_N)|\alpha\rangle;$$  \hspace{1cm} (3.37)

note that the vacuum state is inserted at $\zeta = \infty$ and that, again, the transformation law (3.7) is to be applied. It is easy to verify that the Ishibashi conditions arising from (3.37) coincide with the previous ones, and relation (3.36) can be derived from the bulk boundary OPE and the large $|\zeta|$ behaviour of (3.37), see [30]. Therefore, (3.37) and the procedure of section 3.2 lead to the same boundary states.

While (3.37) is certainly more convenient when dealing with sewing constraints, our previous definition contains Cardy’s conditions as a special case. Moreover, it is not obvious how to generalize the zero temperature definition to the case where the boundary condition jumps at the origin $z = 0$ from $\beta$ to $\alpha$. We therefore return to the setting of section 3.2 and conclude this section with some brief remarks on discontinuous boundary conditions.

As we have explained before, the state space $\mathcal{H}_{\alpha\beta}$ of such a boundary CFT does not contain a vacuum state so that one cannot construct vacuum correlators. Nevertheless, $\mathcal{H}_{\alpha\beta}$ does contain various primary states (with respect to the action of the modes $W_n^{(H)}$) which can be inserted as incoming and outgoing states into the correlation functions, instead of the vacuum. These correlation functions may be interpreted as correlators of bulk fields with two extra boundary fields $\Psi_{\alpha\beta}^{(x)}$ inserted at $x = 0$ and $x = \infty$. In this sense one says that non-vanishing correlators of a boundary theory with jumps in the boundary condition necessarily contain some boundary fields $\Psi_{\alpha\beta}^{(x)}$ (“boundary condition changing operators”). Operator product expansions for such more general boundary fields along with the associated sewing constraints have been described in [35,36].

Boundary CFTs have a very rich structure which in various respects seems to reach beyond that of their plane counterparts. There are many general questions which would be interesting to study. As for D-brane physics, this might lead to a deeper understanding of the non-perturbative string sectors. E.g., it is very tempting to interpret the bulk-boundary OPE (3.34) as kind of an explicit S-duality transformation expressing closed string modes – i.e. excitations of the fundamental degrees of freedom in string theory – through fields on the brane – i.e. through excitations of solitonic degrees of freedom.
3.6 Remarks on the non-rational case

The CFTs relevant for string theory are always non-rational when regarded with respect to the (super) Virasoro algebra alone, simply because the critical dimension is 10 or 26. For non-rational (boundary) CFTs, conditions (3.30) for acceptable boundary states continue to hold, now with infinite summations over the character indices \( i \) and \( j \), but it is usually very difficult to obtain general solutions because the modular transformation properties of non-rational characters are poorly understood.

If we want to use the constraints from world sheet duality to determine boundary states, we can often exploit the presence of bigger symmetry algebras. E.g., while the \( \text{U}(1)^{26} \) current algebra of bosonic string theory on a 26-torus is not big enough to make the CFT rational, its characters are sufficiently well under control so as to render Cardy's conditions tractable: In section 2, we have, without mentioning them, solved eqs. (3.30) by taking the correct linear combinations (2.12) of the Dirichlet coherent states (2.11) so as to obtain integer coefficients in the partition function (2.19). The exponential prefactors in front of the coherent states had an alternative interpretation as a \( \delta \)-function \( \delta(\hat{x}^\mu - x^\mu) \) inserted into the boundary state in order to fix the location of the D-brane in \( \mu \)-direction.

We conclude that, as is common in string theory, geometrical properties in the target follow from world sheet conditions. Moreover, \textit{D-brane moduli}, like the brane's location, can arise as free parameters in the solutions to those constraints. This situation can e.g. occur if there are different representations with the same character – as is usually the case in a non-rational theory (often with infinitely many such “degenerate” representations). Let us recall in passing that other geometric moduli, like the orientation of the brane in the target, enter the CFT description in a different way, namely as parameters (continuous or discrete ones) of the gluing automorphisms \( \Omega \) giving rise to families of Ishibashi states – cf. the remarks in section 3.3.

In other examples of string compactifications like the Gepner models, much bigger symmetry algebras are realized on the perturbative string spectrum, and with respect to these algebras the CFTs in question become indeed rational. Below, we will be content with constructing “rational boundary states” for Gepner models, i.e. boundary states which preserve the bigger symmetry. But since, from the superstring point of view, only the super conformal invariance (and not an extended symmetry) is indispensable, we have to try and justify our procedure:

First of all, the rational boundary states are also boundary states for the smaller super Virasoro algebra: Ishibashi conditions for a big symmetry algebra \( \mathcal{A} \) imply those for any subalgebra, and upon decomposing the characters of irreducible \( \mathcal{A} \)-representations into (possibly infinite) sums of characters of super Virasoro representations it is easy to see that (3.30) is satisfied also in terms of the “smaller” characters.

Second, there are indications that branes found by geometric methods typically preserve more symmetries “than necessary”. This is already true for the classical D-branes on a flat torus, which respect the \( \text{U}(1) \)s and not just the diagonal Virasoro algebra; we will encounter the same phenomenon in the simplified Gepner model example studied in section 4.2 below.

Finally, in the special situation of \( N = 2 \) superconformal models, which is most important for superstrings, certain structures are independent of whether they are looked at from an
extended symmetry or from a mere super Virasoro point of view. The prominent example is the chiral ring [33], which can e.g. be “transported” through the whole moduli space of marginal deformations without its constituents ever changing. We may expect that, in a similar fashion, at least some properties of our boundary states are independent of rationality. This view is supported by the arguments given by Ooguri, Oz and Yin in [17]. Nevertheless, constructing boundary states in a truly non-rational setting remains a challenge – not only because of the D-brane moduli mentioned above, but also in order to clarify the status of non-geometric “D-branes” (i.e. to prove or disprove their existence).

Before we conclude this section, we would like to stress the following point: Performing an explicit test of Cardy’s conditions for a set of tentative boundary states $|\alpha\rangle$ is not just a tedious exercise whose only outcome is to dismiss $|\alpha\rangle$ or not. The quantity $Z_{\alpha\beta}(q)$ computed in this process contains much of the physics of the D-brane configuration $(\alpha \beta)$, as it describes its whole (perturbative) excitation spectrum. In particular, the massless field content (in the string sense) of the low-energy effective field theory associated to the brane configuration is completely explicit. From the boundary state itself, one can immediately read off the couplings to massless closed string modes; thus, coupling constants of the low-energy effective field theory like brane tension and RR charges follow by simply “sandwiching” the boundary state with the corresponding closed string states. This was already recalled for the case of classical branes in section 2, but equally holds for arbitrary boundary states. In particular, “generalized tensions”, too, behave like $g_S^{-1}$, due to their open string tree-level origin.

4. Boundary states for Gepner models

In this section, we apply the general methods outlined above to a class of conformal field theories which are of particular interest for string theory: The Gepner models [38] provide an algebraic formulation of supersymmetric (or heterotic) string compactifications from 10 to lower dimensions, formulated in terms of rational “internal” CFTs. Since, for a long time, Gepner models were considered to be the most relevant string vacua as far as phenomenology is concerned, it is of some interest to search for D-branes in these models. In particular, we expect these non-perturbative string sectors to have consequences for dualities in gauge theories, too.

We first need to review Gepner’s construction of superstring vacua from superconformal minimal models, were it only to set up notation. In subsection 4.2, we will briefly discuss boundary states in a simplified situation, where we can in particular compare our construction to geometry-inspired results of Ooguri, Oz and Yin [17]. After that, in subsection 4.3, we present formulas for A-type and B-type boundary states in arbitrary Gepner models.

4.1 Gepner models

In [38], Gepner introduced a construction of supersymmetric string compactifications based on the minimal models of the $N = 2$ super Virasoro algebra. In this algebraic approach, the compactification is not achieved by “curling up” $10 - D$ dimensions of ten-dimensional flat space-time into a compact Calabi-Yau manifold, but rather by replacing the $10 - D$
free superfields with some “internal CFT” of central charge $15 - 3D/2$ such that certain conditions are met, most of which serve to maintain space-time supersymmetry ([38]; see also [39] for a useful discussion):

(1) The internal CFT must at least have $N = 2$ world sheet supersymmetry.
(2) The total U(1) charges must be odd integers for both left and right movers – here, total refers to internal charges plus charges of the $D - 2$ free external superfields associated to transversal uncompactified directions; this condition implements the (generalized) GSO projection.
(3) The left-moving states must be taken from the NS sectors of each subtheory (external and, in Gepner’s models, various internal sub-theories) or from the R sectors in each subtheory; analogously for the right-moving states.
(4) The torus partition function must be modular invariant.

To build concrete string compactifications, Gepner used tensor products of $N = 2$ minimal models with levels $k_j, j = 1, \ldots, r$, whose central charges sum up to the desired value of the internal CFT,

$$c_{\text{int}} = 12 - \frac{3}{2}d ;$$  \hspace{1cm} (4.1)

$12$ appears because of the light cone gauge, and we assume for later convenience that $d = D - 2$ is equal to $2$ or to $6$.

The (anti-)commutation relations of the $N = 2$ super Virasoro algebra were given in (3.17). Its minimal models have central charges

$$c = \frac{3k}{k + 2} , \quad k = 1, 2, \ldots .$$  \hspace{1cm} (4.2)

with $k = 1, 2, \ldots$, each possessing only a finite number of unitary irreducible highest weight representations. For carrying out the GSO projection onto states of definite fermion number, one needs to consider representations of the bosonic subalgebra; these are labeled by three integers $(l, m, s)$ with

$$l = 0, 1, \ldots, k , \quad m = -k - 1, -k, \ldots, k + 2 , \quad s = -1, 0, 1, 2$$  \hspace{1cm} (4.3a)

and

$$l + m + s \text{ even} .$$  \hspace{1cm} (4.3b)

Triples $(l, m, s)$ and $(k - l, m + k + 2, s + 2)$ give rise to the same representation (“field identification”).

The conformal dimension $h$ and charge $q$ of the highest weight state with labels $(l, m, s)$ are given by

$$h_{l,m,s}^l = \frac{l(l + 2) - m^2}{4(k + 2)} + \frac{s^2}{8} \hspace{1cm} (\text{mod } 1) ,$$

$$q_{l,m,s}^l = \frac{m}{k + 2} - \frac{s}{2} \hspace{1cm} (\text{mod } 2) ;$$  \hspace{1cm} (4.4)

\hspace{1cm} (4.5)
for many purposes, it is sufficient to know \( h \) (and \( q \)) up to (even) integers. The exact dimensions and charges of the highest weight state in the representation \((l, m, s)\) can be read off (4.4.5) if one first uses the field identification and the transformations \((l, m, s) \mapsto (l, m + k + 2, s)\) and \((l, m, s) \mapsto (l, m, s + 2)\) to bring \((l, m, s)\) into the standard ranges

\[
\begin{align*}
l = 0, 1, \ldots, k, & \quad |m - s| \leq l, & s = -1, 0, 1, 2, & & l + m + s \text{ even} \quad (4.6a) \\
l = 1, \ldots, k, & \quad m = -l, & s = -2. & & (4.6b)
\end{align*}
\]

see [40, 41]. Representations with an even value of \( s \) are part of the NS-sector, while those with \( s = \pm 1 \) belong to the R-sector. Within the two sectors, representations can be grouped into pairs \((l, m, s)\&(l, m, s + 2)\) which make up a full \( N = 2 \) super Virasoro module, with all states in a bosonic sub-representation having the same fermion number modulo two.

In order to write down partition functions that satisfy all the requirements listed above and therefore describe superstring compactifications, Gepner formed tensor products of \( N = 2 \) minimal models such that the central charges add up to a multiple of three, see (4.1.2), then adjoined external fermions and bosons, and finally employed an orbifold-like procedure which enforces space-time supersymmetry and modular invariance [38, 39, 40].

We need some further notation before we can state Gepner’s result. For a compactification involving \( r \) minimal models, we use

\[
\begin{align*}
\lambda := (l_1, \ldots, l_r) \quad \text{and} \quad \mu := (s_0; m_1, \ldots, m_r; s_1, \ldots, s_r) \quad (4.7)
\end{align*}
\]

to label the tensor product of representations: \( l_j, m_j, s_j \) are taken from the range (4.3), and \( s_0 = 0, 2, \pm 1 \) characterizes the irreducible representations of the \( \text{SO}(d)_1 \) current algebra that is generated by the \( d \) external fermions (the latter also contain an \( N = 2 \) algebra for each even \( d \) and, again, the NS-sector has \( s_0 \) even).

Accordingly, we write

\[
\chi^{\lambda, \mu}_\nu(q) := \chi_{s_0}(q)\chi^{l_1}_{m_1, s_1}(q) \cdots \chi^{l_r}_{m_r, s_r}(q) \quad (4.8)
\]

with \( \chi^{l, s}_{m,s}(q) = \text{Tr}_{\text{HH}_{m,s}} q^{L_0 - \frac{c}{24}} \) etc. for the conformal characters of these tensor products of internal minimal model and external fermion representations; we refer to [38] and [42] for the explicit expressions.

We introduce the special \((2r + 1)\)-dimensional vectors \( \beta_0 \) with all entries equal to 1, and \( \beta_j, j = 1, \ldots, r \), having zeroes everywhere except for the 1st and the \((r + 1 + j)\)th entry which are equal to 2.

Consider the following products of \( 2\beta_0 \) and \( \beta_i \) with a vector \( \mu \) as above:

\[
\begin{align*}
2\beta_0 \cdot \mu := & -\frac{d}{2} \frac{s_0}{2} - \sum_{j=1}^{r} \frac{s_j}{2} + \sum_{j=1}^{r} \frac{m_j}{k_j + 2}, \\
\beta_j \cdot \mu := & -\frac{d}{2} \frac{s_0}{2} - \frac{s_j}{2}. \quad (4.9)
\end{align*}
\]
It is easy to see that \( q_{\text{tot}} := 2\beta_0 \cdot \mu \) is just the total U(1) charge of the highest weight state in \( \chi_\mu^\lambda(q) \), so that the projection onto states with odd \( 2\beta_0 \cdot \mu \) will implement the GSO projection. Similarly, restricting to states with \( \beta_i \cdot \mu \) integral ensures that only states in the tensor product of \( r + 1 \) NS-sectors (or of \( r + 1 \) R-sectors) are admitted (recall that we assumed \( d = 2 \) or \( d = 6 \)).

Modular invariance of the partition function can be achieved if the above projections are accompanied by adding “twisted” sectors (in a way similar to orbifold constructions). To state Gepner’s result, we put \( K := \text{lcm}(4, 2k_j + 4) \) and let \( b_0 \in \{0, 1, \ldots, K - 1\} \), \( b_j \in \{0, 1\} \) for \( j = 1, \ldots, r \). Then the partition function of a Gepner model describing a superstring compactification to \( d + 2 \) dimensions is given by

\[
Z_G^{(r)}(\tau, \bar{\tau}) = \frac{1}{2} \frac{\text{Im} \tau}{|\eta(q)|^{2d}} \sum_{\lambda, \mu} \sum_{b_0, b_j} (-1)^{s_0} \chi_\mu^\lambda(q) \chi_\mu^{\lambda+b_0\beta_0+b_1\beta_1+\ldots+b_r\beta_r}(\bar{q})
\] (4.11)

where \( \sum^\beta \) means that we sum only over those \( \lambda, \mu \) in the range (4.3) which satisfy \( 2\beta_0 \cdot \mu \in 2\mathbb{Z} + 1 \) and \( \beta_j \cdot \mu \in \mathbb{Z} \). The summations over \( b_0, b_j \) introduce the twisted sectors corresponding to the \( \beta \)-restrictions so that, in particular, the Gepner partition function is non-diagonal. The sign is the usual one occurring in (space-time) fermion one-loop diagrams. The \( \tau \)-dependent factor in front of the sum accounts for the free bosons associated to the \( d \) transversal dimensions of flat external space-time, while the \( 1/2 \) is simply due to the field identification mentioned after (4.3). Using the modular transformation properties of the \( \text{SO}(d)_1 \) and minimal model characters, whose S-matrices are

\[
S^f_{s_0, s_0'} = \frac{1}{2} e^{-i\pi \frac{s_0 s_0'}{k+2}},
\]

\[
S^k_{(l,m,s),(l',m',s')} = \frac{1}{\sqrt{2(k+2)}} \sin \pi \frac{(l+1)(l'+1)}{k+2} e^{i\pi \frac{mm'}{k+2}} e^{-i\pi \frac{s s'}{k+2}},
\] (4.12) (4.13)

Gepner could prove that (4.11) is indeed modular invariant.

Although we will work only with the above \( (2, 2) \)-superstring compactifications, let us mention that originally one of the most interesting features of Gepner’s construction was that it is straightforward to convert (4.11) into a (modular invariant) partition function describing a heterotic string compactification, see [38]. One simply has to replace the right-moving \( \text{SO}(d)_1 \) characters by those of \( \text{SO}(d + 8) \times E_8 \) (with a suitable permutation). The massless spectrum of heterotic Gepner models then contains fermions which transform in the 27 (and \( \bar{27} \)) representations of \( E_8 \) and which may be interpreted as (anti-)generations of a supersymmetric GUT model derived as the low-energy limit of the string compactification, see e.g. [38,43,40].

In a somewhat surprising development, it turned out that Gepner’s purely algebraic construction is intimately related to geometric string compactifications on certain (complete intersection) Calabi-Yau manifolds. E.g., the number of generations and anti-generations computed from Gepner’s partition function agrees with (or can at least be related to) those found in CICY-compactifications, where they are given by the dimensions of certain Dolbault cohomologies. The connection was made more precise by the work of Greene,
Witten and other authors, see [39] for a useful review.

An important subject that arose from the connection between compactifications on Calabi-Yau manifolds and the CFT vacua constructed by Gepner is mirror symmetry. Simply using the mirror automorphism (3.18) of $N = 2$ superconformal field theory in the right-moving sector, one can revert the role of (chiral,chiral) and (chiral,anti-chiral) primaries; moving to the Calabi-Yau manifolds, however, this corresponds to a completely non-trivial map between topologically distinct manifolds. This observation, made in [44,45], see also [33] for a precursor, has had important consequences for algebraic geometry as well as for the interpretation of string theory, see e.g. [46,39] for details and further references.

### 4.2 A simplified example

We will now turn towards the construction of boundary states for the superstring compactifications reviewed above, but before we treat Gepner models in full generality, let us discuss the simplest case in a simplified variant: We want to study the tensor product of three $k = 1$ minimal models, each restricted to the NS sector, then orbifolded to integer U(1) charge. In contrast to the Gepner models, where $q_{\text{tot}}$ must be odd, we do not perform the GSO-projection and thus still deal with a true CFT with the vacuum sector in.

The reasons for starting with this simple case are two-fold: First, notations are much lighter than for general Gepner models. Second, and more importantly, this model can be written as a $\sigma$-model compactification on a 2-torus with $\mathbb{Z}_3$-symmetry; thus it has a free field description so that boundary states have a direct geometric interpretation, which was used in the treatment in [17]. Our construction does not rely on any geometric input, and in particular does not use the coherent Ishibashi states of free bosons or fermions, instead we will require our boundary states to render the resulting boundary CFT rational. We will see that the free complex fermion re-emerges naturally in the spectrum of the open string partition function determined by these “rational boundary states”, which suggests that they are the right objects to start with.

The first minimal model in the series (4.2) has $c = 1$ and can be regarded as a free boson compactified on a circle with a special radius. The irreducible representations of the (here: full) $N = 2$ superconformal algebra are specified by the conformal dimension $h$ and the charge $q$ of the highest weight vectors, in the NS sector given by

$$\left|(h, q)\right| = \left|(0, 0)\right|, \quad \left|(1/6, 1/3)\right|, \quad \left|(1/6, -1/3)\right|.$$  

We will in the following simply use the charge enumerators $a = 0, 1, 2$ (taken modulo 3) to label these representations. Note that $|0\rangle$ and $|1\rangle$ are chiral, $|0\rangle$ and $|2\rangle$ anti-chiral states.

The $N = 2$ characters in the NS sector close under the modular transformation $S : \tau \mapsto -1/\tau$, and the elements $S_{ab}$ in $\chi_a(\bar{q}) = \sum_b S_{ab} \chi_b(q)$ are given by

$$S_{ab} = \frac{1}{\sqrt{3}} \omega^{-ab} \quad (4.14)$$

with $\omega = e^{2\pi i/3}$ and $a, b = 0, 1, 2$ – as follows from (4.13), or more directly by diagonalizing the (simple current) fusion rules of the $k = 1$ model.
We take the tensor product of three $k=1$ minimal models (each with diagonal partition function, here restricted to the NS sector) and orbifold this CFT with respect to the group generated by
\[ g = \exp\{2\pi i (J_0^{\text{tot}} + \tilde{J}_0^{\text{tot}})\} \]
where $J_0 = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}$ is the total left-moving charge of the $(k=1)^3$ theory. The resulting partition function is \[ Z(q, \bar{q}) = \sum_{x=0,1,2} \sum_{\Sigma a_i \equiv 0 \, \text{(mod 3)}} \chi_a(q) \chi_{a+(x,x,x)}(\bar{q}); \quad (4.15) \]
the (non-diagonal) terms with $x \neq 0$ are the “twisted sectors”, and the restriction on the $a = (a_1, a_2, a_3)$ summation ensures that only states with integer charge occur in the orbifolded theory. Note that, since $h_L - h_R$ may be half-integer, (4.15) is only invariant under the subgroup of the modular group generated by $S$ and $T^2$ (this is already the case for the diagonal $k=1$ partition functions we started from).

The maximal chiral symmetry algebra of the orbifold theory contains the diagonal (“total”) $N=2$ super Virasoro algebra, but also the bigger algebra $\mathcal{A}^\otimes$ generated by the three super Virasoro algebras of the individual tensor factors (the $g$-action commutes with all these generators). While the simple $c=3$ orbifold theory is rational with respect to $\mathcal{A}^\otimes$, it is already non-rational with respect to the diagonal super Virasoro algebra.

If we want to construct boundary states $|\alpha\rangle$ of the orbifolded $(k=1)^3$ model, we can require that not only (one of) the diagonal $N=2$ super Virasoro algebra(s) is preserved – in the form of A-type or B-type boundary conditions, see eqs. (3.20,19) –, but that the system (bulk CFT + boundary state) in fact enjoys the full $\mathcal{A}^\otimes$-symmetry. This makes Cardy’s world sheet duality conditions tractable – without losing too much interesting structure, as comparison with the free field approach will show.

We start with boundary states satisfying A-type conditions (3.20) in each subtheory. Out of the 27 sectors of the orbifold theory (4.15) only the 9 untwisted ones ($x=0$) can contribute because of the charge constraints $q^{(i)} = \mathcal{T}^{(i)}$ for $i = 1, 2, 3$. Thus, we make the following ansatz for an A-type $\mathcal{A}^\otimes$-preserving boundary state of the orbifolded $(k=1)^3$ model in the NS sector:
\[ |\alpha\rangle_A = \frac{1}{3^{1/4}} \sum_a \omega^{-a_1 \alpha_1 - a_2 \alpha_2 - a_3 \alpha_3} |a\rangle_A \quad (4.16) \]
where $|\alpha\rangle = |(\alpha_1, \alpha_2, \alpha_3)\rangle$ is labeled by three integers defined modulo 3. Of course, this ansatz follows Cardy’s solution, but we cannot rely on his derivation because, in our case, only a subset of the terms $\chi_i(q)\chi_j(\bar{q})$ in the partition function couples to A-type Ishibashi states, in contrast to the simple situation of section 3.4. That the boundary states (4.16) nevertheless satisfy Cardy’s conditions can be checked easily:
\[ Z^A_{\alpha \tilde{\alpha}}(q) \equiv A\langle \alpha | \tilde{q}^{L_0-\tilde{\pi}} | \alpha \rangle_A \]
\[ = \frac{1}{\sqrt{3}} \sum_a \omega^{-a(\alpha-\tilde{\alpha})} \chi_a(\tilde{q}) = \frac{1}{9} \sum_c \sum_a \omega^{-a(c+\alpha-\tilde{\alpha})} \chi_c(q) \]
\[ = \sum_c \delta^{(3)}_{c_1-c_3,\alpha_3-\alpha_1-\tilde{\alpha}_3+\tilde{\alpha}_1} \delta^{(3)}_{c_2-c_3,\alpha_3-\alpha_2-\tilde{\alpha}_3+\tilde{\alpha}_2} \chi_c(q) ; \quad (4.17) \]

In the second line, we have used

\[ A\langle a' | \tilde{q}^{L_0-\tilde{\pi}} | a \rangle_A = \delta_{a',a} \chi_a(\tilde{q}) , \]

and the modular \( S \)-matrix \( (4.14) \); finally, we have carried out the two independent summations over \( a_1 \) and \( a_2 \), the third roots of unity yielding the periodic Kronecker symbols \( \delta^{(3)} \).

Note that in \((4.17)\) we did not insert the CPT operator \( \Theta \) into the “cylinder amplitude”, in contrast to our general discussion in section 3 and to our treatment of the full Gepner models below: The reason is that for simplicity we want to work with full representations of the \( N = 2 \) super Virasoro algebra and at the same time restrict ourselves to the NS sector. Since the \( N = 2 \) algebra contains fermionic generators, insertion of \( \Theta \) in front of \( | \tilde{\alpha} \rangle \) would break up the full super Virasoro representations into ones of the bosonic subalgebra, and the modular \( S \)-transformation would then force us to introduce the R sector as well. Since all that would only blur the salient points of the example, we have simply left out the CPT operator for the moment.

Not too surprisingly, the ansatz \((4.16)\) yields a sum of \( A^\otimes \)-characters with positive integer coefficients: \( A^\otimes \)-symmetry in the open string system \( Z^A_{\alpha \tilde{\alpha}}(q) \) is preserved, and Cardy’s constraints are satisfied. But the resulting boundary CFT contains unwanted states with non-integer \( U(1) \) charge unless we require

\[ \alpha_1 + \alpha_2 + \alpha_3 = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 . \quad (4.18) \]

This “compatibility condition” between two boundary states arises essentially because the terms of the partition function coupling to A-type Ishibashi states do not close under the modular \( S \)-transformation. Note, however, that \((4.18)\) holds for \( \alpha = \tilde{\alpha} \).

We have normalized the \( | \alpha \rangle_A \) such that the vacuum representation occurs precisely once in \( Z^A_{\alpha \tilde{\alpha}}(q) \), cf. [28]. To arrive at his solution \((3.32)\) for the boundary states of a rational CFT, Cardy imposed the even stronger requirement that there exists a special boundary state \( | \tilde{0} \rangle \) which has the property that only fields in the vacuum sector propagate in its presence, i.e. \( n_{\tilde{0}0} = \delta_{1,0} \).

None of our solutions \((4.16)\) meets this requirement. Instead,

\[ Z^A_{\alpha \tilde{\alpha}}(q) = \chi_0(q)^3 + \chi_1(q)^3 + \chi_2(q)^3 \quad (4.19) \]

for all the \( | \alpha \rangle_A \) in \((4.16)\), the contributions corresponding to highest weight states with total dimensions and charges \((0,0), (1/2,1), (1/2,-1)\), respectively: \( Z^A_{\alpha \tilde{\alpha}}(q) \) is the vacuum character of a free complex fermion. Furthermore, the spectrum in any configuration \((\alpha \tilde{\alpha})\) of two compatible “branes” carries a representation of this extended symmetry algebra.
This is easy to understand since the full chiral symmetry algebra of the \((k = 1)^3\) model in fact contains a free complex fermion \(\psi(z)\), we can e.g. identify highest weight states as \(|1, 1, 1\rangle = \sqrt{i} \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} |\text{vac}\rangle\), \(|2, 2, 2\rangle = \sqrt{i} \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} |\text{vac}\rangle\). With the help of the \((k = 1)^3\) fusion rules one can show that the boundary states (4.16) satisfy

\[
\psi_r |\alpha\rangle_A = -i \omega^n \overline{\psi}_{-r} |\alpha\rangle_A
\]

with \(n = \sum \alpha_i\). This means that the fermionic symmetry is indeed preserved by these boundary states. (For similar observations of extended symmetries in boundary CFT see [36]).

This remark allows us to make contact to the results of [17], where the \((c, c)\) parts of the boundary states for the \((k = 1)^3\) orbifold were given in terms of free fermion and free boson coherent states. Denote by \(\psi, \partial X\) the left-moving complex fermion resp. boson with adjoints \(\psi^*, \partial X^*\) and modes \(\psi_r, \alpha_m\) etc.; analogously for the right-movers. Then the coherent A-type boundary states of Ooguri et al. read

\[
|B_n\rangle = |B_n\rangle_X |B_n\rangle_{\psi}
\]

with

\[
|B_n\rangle_{\psi} = \exp\left\{ i \omega^n \sum_{r > 0} \psi_{-r} \overline{\psi}_{-r} + i \omega^{-n} \sum_{r > 0} \psi^*_{-r} \overline{\psi}^*_{-r} \right\} |\text{vac}\rangle ,
\]

\[
|B_n\rangle_X = \exp\left\{ -\omega^n \sum_{m > 0} \frac{1}{m} \alpha_{-m} \overline{\alpha}_{-m} - \omega^{-n} \sum_{m > 0} \frac{1}{m} \alpha^*_{-m} \overline{\alpha}^*_{-m} \right\} |\text{vac}\rangle ;
\]

they are obtained, in a slight generalization of the procedure reviewed in section 2, as solutions of the Ishibashi conditions

\[
(\psi_r + i \omega^n \overline{\psi}_{-r}) |B_n\rangle_{\psi} = 0 ,
\]

\[
(\alpha_m + \omega^n \overline{\alpha}_{-m}) |B_n\rangle_X = 0 ,
\]

for \(n = 0, 1, 2\). The conditions for \(|B_n\rangle_X\), which are just the superpartners of the ones for \(|B_n\rangle_{\psi}\), have a classical geometrical interpretation as Dirichlet and Neumann conditions on the torus target: The case \(n = 0\) describes a 1-brane (wrapped around a supersymmetric cycle) extending in the \(X = \text{real}\) direction, whereas in the \(n = 1, 2\) cases this 1-brane (or supersymmetric cycle) is rotated by \(2n\pi/3\) (corresponding to the \(\mathbb{Z}_3\)-symmetry of the torus), see [17].

Now let us compare our boundary states (4.16) to the coherent states (4.21) of Ooguri et al.: Eqs. (4.24) and (4.20) show that they obey the same Ishibashi conditions; as for the coefficients in front of the Ishibashi states, we can use the fact that the Ishibashi state associated to an irreducible representation is unique up to normalization: We only need to compare the coefficients of the \(N = 2\) highest weight states \(|1, 1, 1\rangle \otimes |1, 1, 1\rangle = i \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} \psi_{-\frac{1}{2}} |\text{vac}\rangle\) and \(|2, 2, 2\rangle \otimes |2, 2, 2\rangle = i \psi^*_{-\frac{1}{2}} \overline{\psi}^*_{-\frac{1}{2}} |\text{vac}\rangle\) in (4.16) and (4.21). In both formulas, they are given by
\( \omega^n \) resp. \( \omega^{-n} \) with \( n = \sum_i \alpha_i \), establishing agreement. The bosonic part of the boundary state is of course hidden in the \( N = 2 \) families of the fermions since \( \partial X \sim G^{-\frac{1}{2}} \psi \).

In summary, our boundary state construction, which relied on the requirement that the extended algebra \( \mathcal{A}^\otimes \) is preserved, automatically led to the coherent states which were used in the (geometric) \( \sigma \)-model approach of Ooguri et al. We take this as encouragement to construct analogous “rational” boundary states in arbitrary Gepner models, see section 4.3. In the (non-toroidal) cases beyond \( (k = 1)^3 \), comparisons to geometric results would be less direct because the \( \sigma \)-model interpretation is less explicit there.

We remark that open string theories associated to some Gepner models (toroidal orbifolds) were discussed in [48]; there, however, only the Chan-Paton structure guaranteeing tadpole cancellation was determined, which does not require explicit knowledge of the boundary states.

To complete our analysis of the simplified \( (k = 1)^3 \) model, we compute, in the same way as before, boundary states satisfying B-type conditions (3.19) for each component super Virasoro algebra. Since this requires \( q^{(i)} = -\bar{q}^{(i)} \) for \( i = 1, 2, 3 \), only 3 of the 27 sectors in (4.15) provide B-type Ishibashi states, namely those which correspond to “charge conjugate terms” \( \chi_i(q)\chi_i(q) \) in the partition function, as opposed to the diagonal terms coupling to A-type Ishibashi states; cf. the general discussion in section 3.3.

The B-type boundary states can be written as

\[
|\alpha\rangle_B = 3^{1/4} \sum_{b=(b,b,b)} \omega^{-b(\alpha_1+\alpha_2+\alpha_3)} |b\rangle_B
\]

and the same calculations as before lead to boundary CFT partition functions

\[
Z_{\alpha \bar{\alpha}}^B(q) \equiv B(\alpha|\bar{q}^{L_{\text{tot}}-\frac{c}{24}}|\alpha)_B = \sum_{c=(c_1,c_2,c_3)} \delta_{\Sigma_{c_1},\Sigma_{\bar{\alpha}_1},-\Sigma_{\alpha_1}} \chi_c(q);
\]

to obtain only states with integer U(1) charges, the same conditions (4.18) as in the A-type case have to be satisfied. The spectrum described by \( Z_{\alpha \bar{\alpha}}^B(q) \) is, however, different from that in \( Z_{\alpha \bar{\alpha}}^A(q) \).

We can also ask what the spectrum of a brane configuration consisting of an A-type and a B-type boundary state looks like. To perform the calculation, we have to realize that the “inner product” (regularized with \( \bar{q} \)-damping factors as usual) of an A-type Ishibashi state and a B-type Ishibashi state vanishes in almost all cases, simply because the left-moving basis states in an Ishibashi state (3.11,16) are tensored to different right-moving states in the A-type and B-type case. More precisely, we find for the orbifolded \( (k = 1)^3 \) model

\[
A(\langle a|\bar{q}^{L_{\text{tot}}-\frac{c}{24}}|b\rangle_B = \delta_{a,b} \delta_{b,(0,0,0)} \text{Tr}_{\mathcal{H}_{(0,0,0)}}[V_{\Omega_M} \bar{q}^{L_{\text{tot}}-\frac{c}{24}}] \]
\]

where \( V_{\Omega_M} \) is the unitary operator that implements the mirror automorphism on the vacuum Hilbert space; on Poincaré-Birkhoff-Witt vectors, \( V_{\Omega_M} \) acts as

\[
V_{\Omega_M} L_{n_1} \ldots L_{n_i} J_{m_1} \ldots J_{m_j} G^+_{r_1} \ldots G^+_{r_k} G^-_{s_1} \ldots G^-_{s_l} |0,0\rangle = L_{n_1} \ldots L_{n_i} (-J_{m_1}) \ldots (-J_{m_j}) G^-_{s_1} \ldots G^-_{s_l} G^+_{r_1} \ldots G^+_{r_k} |0,0\rangle.
\]
compare eq. (3.18) in section 3.3. Therefore, the excitation spectrum of a brane configuration consisting of an A-type and a B-type boundary state should be computed from

\[ Z_{\alpha \tilde{\alpha}}(q) \equiv A\langle \tilde{\alpha} | \tilde{q}^{L_{\text{tot}}-\frac{c}{24}} | \alpha \rangle_B = \text{Tr}_{H_{\text{vac}}} [V_{\Omega M} \tilde{q}^{L_{\text{tot}}-\frac{c}{24}}] , \]

(4.30)

independently of the choice of $|\tilde{\alpha}\rangle_A$ and $|\alpha\rangle_B$. To check whether Cardy’s conditions are satisfied for such a configuration, we would have to determine the modular transformation properties of the trace in (4.30). We do not indulge into this computation, because in the true Gepner models discussed below, the partition functions $Z_{\alpha \tilde{\alpha}}^{AB}$ vanish anyway because of space-time supersymmetry, enforced by the GSO projection. Nevertheless, it is an interesting fact that traces of automorphisms of the chiral symmetry algebra emerge naturally in the framework of boundary CFT D-branes. This should e.g. provide a general picture of supersymmetry breaking configurations.

4.3 Boundary states for arbitrary Gepner models

After this warm-up example, we discuss full-fledged Gepner models, taking into account R-sectors and GSO projection. We first have to fix the Ishibashi conditions to be imposed on the boundary states. As in the previous subsection, we want to use Cardy’s results as far as possible and thus treat the internal part of the Gepner models as rational theories, again. This means that we have to choose the boundary conditions in such a way so as to preserve an extended symmetry algebra $A$, the obvious choice being the algebra generated by all the $N=2$ Virasoro algebras of the $r$ internal component theories, together with the $SO(d)_1$ of the external fermions. (We will in the following ignore the external bosons; their coherent boundary states, describing ordinary classical D-branes, just multiply the boundary states computed below).

In the generic case, i.e. if the levels $k_j$ of the internal minimal models are pairwise different, the only way to maintain the tensor product symmetry in the presence of a boundary state is to require A-type or B-type boundary conditions as in eqs. (3.20,19) for each set of super Virasoro generators $L_n^{(j)}, J_n^{(j)}, G_r^{\pm (j)}$ separately. In special cases, when $k_{j_1} = k_{j_2}$, we could also glue the left-moving generators of subtheory $j_1$ to the right-moving generators of subtheory $j_2$, using permutation automorphisms, but we will not work out these boundary states here.

We first discuss the simpler case where A-type Ishibashi conditions (3.20) are imposed on each of the internal sub-Virasoro algebras and on the external fermions. A-type conditions imply that only left-right representations $\mathcal{H}_i \otimes \mathcal{H}_i$ with $q_i = \tilde{q}_i$ (and of course $h_i = \tilde{h}_i$) contribute Ishibashi states, in other words, we have to restrict to the diagonal part of the Hilbert space. In a Gepner model partition function (4.11), each left-character $\chi_i(q)$ is multiplied with $\chi_i(\tilde{q})$ (among others) on the right (where it is understood that $i$ labels irreducible representations of the bosonic sub-algebra of the $N=2$ algebras rather than full $N=2$ representations). Therefore, all the tensor product A-type Ishibashi states $|\lambda, \mu\rangle_A$ (in an obvious notation) can occur in the boundary state provided $(\lambda, \mu)$ occurs on the (left-moving) closed string spectrum.
We make the following ansatz for “rational” A-type boundary states in Gepner models:

$$|\alpha\rangle_A \equiv |S_0; (L_j, M_j, S_j)_{j=1}^r\rangle_A = \frac{1}{\kappa_\alpha^A} \sum_{\lambda, \mu} B^\lambda_{\alpha} B^\mu_{\bar{\alpha}} |\lambda, \mu\rangle_A$$  \hspace{1cm} (4.31)

where $S_0, L_j, M_j, S_j$ are integer labels; the summation is over states satisfying the “$\beta$-constraints” as in Gepner’s partition function (4.11), $\kappa_\alpha^A$ is some normalization constant to be determined later, and the coefficients in front of the Ishibashi states are given by

$$B^\lambda_{\alpha} = (-1)^{\frac{s^2}{2}} e^{-i\pi \frac{d}{2} s_0 s_0} \prod_{j=1}^r \sin \frac{\pi (l_j+1) (L_j+1)}{k_j+2} e^{i\pi \frac{m_j M_j}{k_j+2}} e^{-i\pi \frac{s_j S_j}{2}}. \hspace{1cm} (4.32)$$

Except for the sign $(-1)^{\frac{s^2}{2}}$, these coefficients are chosen as in Cardy’s solution (3.32) for the tensor product of minimal models and external $\text{SO}(d)_1$ factor but before orbifolding and charge projections. Moreover, after the GSO-projection we no longer deal with a genuine CFT anyway, thus we cannot simply rely on Cardy’s general arguments but rather have to verify explicitly that the boundary states (4.31) lead to acceptable open string spectra in $Z_{\alpha \bar{\alpha}}(q) = \langle \Theta \bar{\alpha} | q^{L_0 - \frac{c}{24}} | \alpha \rangle$. The computations are straightforward except for the $\beta$-constraints in the summation. We calculate

$$Z^A_{\alpha \bar{\alpha}}(q) = \frac{1}{\kappa_\alpha^A \kappa_{\bar{\alpha}}^A} \sum_{\lambda, \mu} \sum_{\lambda, \bar{\mu}} B^\lambda_{\alpha} B^\mu_{\bar{\alpha}} \langle \langle \tilde{\lambda}, -\bar{\mu} | q^{L_0 - \frac{c}{24}} | \lambda, \mu\rangle_A$$

$$= \frac{1}{\kappa_\alpha^A \kappa_{\bar{\alpha}}^A} \sum_{\lambda, \mu} \sum_{\lambda', \mu'}^\text{sym} B^\lambda_{\alpha} B^\mu_{\bar{\alpha}} S^f_{s_0, s_0'} \prod_{j=1}^r S^{(k_j)}_{l_j, m_j, s_j, (l'_j, m'_j, s'_j)} \chi^{\lambda'}_{\mu'}(q)$$

where $\sum_{\text{sym}}$ denotes summation over the full range (4.3) with $l'_j + m'_j + s'_j \in 2\mathbb{Z}$ as the only constraint, and $S^f$ and $S^{k_j}$ are the modular $S$-matrices of the external fermions resp. the $j$th minimal model, see eqs. (4.12,13).

In order to compute the prefactor of $\chi^{\lambda'}_{\mu'}(q)$ in $Z^A_{\alpha \bar{\alpha}}(q)$, we introduce Lagrange multipliers $\nu_0, \nu_j, j = 1, \ldots, r$, for the charge constraint and the $\beta_j$-conditions and rewrite

$$\sum_{\lambda, \mu}^\beta = \frac{1}{K} \sum_{\nu_0=0}^{K-1} \sum_{\nu_j=0,1} e^{i\pi \nu_0 (q_{\text{tot}} - 1)} \prod_{j=1}^r \frac{1}{2} \sum_{\nu_j=0,1} e^{i\pi \nu_j (s_0 + s_j)}$$

with $K = \text{lcm}(4, 2k_j + 4)$ and the total $\text{U}(1)$ charge $q_{\text{tot}} = 2\beta_0 + \mu$ as in eq. (4.9). Now, the summations over $s_0, l_j, m_j, s_j$ are independent of each other (except for the $l_j + m_j + s_j$ even constraint, which is easy to handle) and can be carried out directly to give

$$Z^A_{\alpha \bar{\alpha}}(q) = n^A_{\alpha \bar{\alpha}} \sum_{\lambda', \mu'}^\text{sym} \sum_{\nu_0=0}^{K-1} \sum_{\nu_1=0,1} \cdots \sum_{\nu_r=0,1} (-1)^{s'_0 + S_0 - \tilde{S}_0} \delta^{(4)}_{s'_0, 2 + \tilde{S}_0 - S_0 - \nu_0 - 2\Sigma \nu_j}$$

$$\times \prod_{j=1}^r N^j_{l_j, L_j} \delta^{(2k_j + 4)}_{m'_j, M_j - M_j - \nu_0} \delta^{(4)}_{s'_j, \tilde{S}_j - S_j - \nu_0 - 2\nu_j} \chi^{\lambda'}_{\mu'}(q) \hspace{1cm} (4.33)$$
Here, $N_{i^p}^{l^r}$ denote the fusion rules of the SU(2)$_k$ WZW model, which arise from the $l_j$ summations via the Verlinde formula. The symbol $\delta_{r,s}^{(p)}$ means that $r \equiv s \pmod{k}$. Finally, the normalization is

$$n_{\alpha\tilde{\alpha}}^A = \frac{1}{\kappa_{\alpha}^A \kappa_{\tilde{\alpha}}^A} 2^{\frac{r}{2} + 1} \frac{2(k_1 + 2) \cdots (k_r + 2)}{K}.$$ 

From eq. (4.33) we see that with the (minimal) normalization

$$\kappa_{\alpha}^A = 2\left(2^{\frac{r}{2}} \frac{(k_1 + 2) \cdots (k_r + 2)}{K}\right)^{\frac{1}{2}}$$

our boundary states (4.31,32) indeed satisfy Cardy’s conditions – suitably modified for the supersymmetric setting: The sign in (4.33) simply distinguishes space-time bosons from space-time fermions.

But there are again additional string theory requirements which impose restrictions on the integers $(S_0, (L_j, M_j, S_j))$ in the ansatz (4.32): The spin structures of the component theories in $Z_{\alpha\tilde{\alpha}}(q)$ should be coupled as in the closed string case (states in $NS^{\otimes(r+1)}$ or $R^{\otimes(r+1)}$ only), therefore we must require

$$S_0 - \tilde{S}_0 \equiv S_j - \tilde{S}_j \pmod{2}$$

for all $j = 1, \ldots, r$ so as to have $\beta_j \cdot \mu' \in \mathbb{Z}$ for all states in $Z_{\alpha\tilde{\alpha}}(q)$.

For the excitation spectrum of the brane configuration ($\alpha\tilde{\alpha}$) described by $Z_{\alpha\tilde{\alpha}}(q)$ to be supersymmetric, only states with odd total charge should be present. From the $\delta$'s in eq. (4.33) one finds that

$$2\beta_0 \cdot \mu' \equiv Q(\alpha - \tilde{\alpha}) + 1 \equiv 1 \pmod{2}$$

with

$$Q(\alpha - \tilde{\alpha}) := -\frac{d}{2} \frac{S_0 - \tilde{S}_0}{2} - \sum_{j=1}^{r} \frac{S_j - \tilde{S}_j}{2} + \sum_{j=1}^{r} \frac{M_j - \tilde{M}_j}{k_j + 2}. \quad (4.36a)$$

For $Q(\alpha - \tilde{\alpha})$ even, only states with $q_{\text{tot}} = \pm 1, \pm 3, \ldots$ contribute; since $h \geq |q|/2$ in unitary representations of the $N = 2$ super Virasoro algebra, all states in $Z_{\alpha\tilde{\alpha}}(q)$ have conformal weight $1/2$ or higher: The spectrum is tachyon-free and stable.

In particular, requirements (4.35,36) are satisfied for two identical branes $\alpha = \tilde{\alpha}$ given by A-type boundary states (4.31,32), meaning that the excitation spectrum of a single such brane is supersymmetric and stable.

On the other hand, (4.35,36) imply that not any two boundary states are “compatible” with each other. Rather, we obtain groups of mutually compatible boundary states; configurations made up of two branes from different groups lead to spectra violating supersymmetry (and stability). E.g., a “brane-antibrane system” $(\alpha \alpha^+)$ with $|\alpha^+\rangle := \Theta |\alpha\rangle$ has a tachyon in its spectrum.
Since half of the space-time supersymmetry is preserved by our boundary states, we may conclude that they describe BPS-states. A further check of this property would be to show that \( Z_{\alpha\alpha}(q) \) in fact vanishes. Since the \( \chi_{\alpha}(q) \) consist of products of theta functions, identities for the latter should in principle allow to verify \( Z_{\alpha\alpha}(q) = 0 \) in general (analogous to Polchinski’s computation for the flat case \([4,5]\), which rests on Jacobi’s “abstruse identity”). As yet, we have merely expanded \( Z_{\alpha\alpha}(q) \) for some models up to some power in \( q \), finding indeed zero.

According to the results of \([17]\), the above A-type boundary states should, via the correspondence between Gepner models and Calabi-Yau manifolds mentioned before, describe D-branes wrapping around middle-dimensional supersymmetric cycles of the CY manifold. Moreover, using topological twisting of the \( N = 2 \) theory, Ooguri et al. have argued that the coefficients \( B_{\alpha}^{\lambda,\mu_c} \) in front of Ishibashi states \(|\lambda, \mu_c\rangle_A\) corresponding to chiral primaries (fields with \( q = 2\hbar \)) are independent of the Kähler moduli and can be, in the large volume limit, expressed in geometric terms:

\[
B_{\alpha}^{\lambda,\mu_c} = \int_{\gamma_{\alpha}} \omega_{\lambda,\mu_c} \tag{4.37}
\]

where \( \omega_{\lambda,\mu_c} \) is a differential form associated to the chiral primary \( \phi_{\lambda,\mu_c} \) via topological twisting, see \([17]\) for the details. In a Gepner model, the chiral primaries correspond to irreducible representations labeled by

\[
(\lambda, \mu_c) = (\lambda, (0; \lambda; 0, \ldots, 0))
\]

and the corresponding coefficients in the boundary state \(|\alpha\rangle_A\) read

\[
B_{\alpha}^{\lambda,\mu_c} = \frac{1}{\kappa_{\alpha}^A} \prod_{j=1}^r \frac{\sin \pi \frac{(l_j+1)(L_j+1)}{k_j+2}}{\sin \frac{\pi}{k_j+2}} \frac{e^{i\pi \frac{l_j M_j}{k_j+2}}}{k_j+2} \tag{4.38}
\]

But beyond this “geometric part” of the boundary states, our formula (4.32) contains the non-chiral contributions, too.

As was recalled in section 2, eqs. (4.31,32) moreover allow to determine the tension and RR charges of the branes described by these boundary states. One merely has to project \(|\alpha\rangle_A\) onto the massless closed string state in question (and take into account universal prefactors from string amplitudes). While we could arrive at a closed formula for (rational) boundary states for Gepner model, the massless closed string spectrum does depend sensitively on the concrete model, and we will not indulge into further case studies in this article. Similar remarks apply to the excitation spectrum of brane configurations, which can be extracted from \( Z_{\alpha\alpha}(q) \). Since its massless part determines the field content of the low-energy effective field theory associated to the configuration, explicit knowledge might be interesting for the study of gauge theories in four dimensions.

Let us now discuss B-type boundary states with B-type Ishibashi conditions (3.19) imposed on each sub-theory. The U(1) charges of the left- and right-moving highest weight states
must satisfy \( q_i = -\tilde{q}_i \) (and \( h_i = \tilde{h}_i \)), and the B-type Ishibashi states couple to “charge conjugate” parts \( \mathcal{H}_i \otimes \mathcal{H}_{i+} \) of the bulk Hilbert space, as discussed in section 3.3. A little bit of calculation shows that a term \( \chi^{\lambda}_{\mu}(q) \chi^{-\lambda}_{-\mu}(\tilde{q}) \) occurs in the Gepner partition function (4.11) precisely for those \( \mu \) which satisfy

\[
m_j \equiv b \pmod{k_j + 2} \tag{4.39}
\]

for some \( b = 0, 1, \ldots, \frac{K}{2} - 1 \) and for all \( j \). It is those Ishibashi states that contribute to the sum in the ansatz for the B-type boundary states

\[
|\alpha\rangle_B \equiv |S_0; (L_j, M_j, S_j)_{j=1}^r\rangle_B = \frac{1}{\kappa_B^{\alpha}} \sum_{\lambda, \mu}^{b, b} B_{\lambda, \mu}^{\alpha} |\lambda, \mu\rangle_B \tag{4.40}
\]

with coefficients \( B_{\lambda, \mu}^{\alpha} \) as before (4.32). Note that, generically, Gepner models possess less B-type than A-type Ishibashi states; as a consequence, the excitation spectra of B-type brane configurations will typically be richer than those of A-type branes.

One can now perform the same calculation as before in order to test this ansatz; because of the restricted \( m_j \)-range (4.39), there are slight differences compared to A-type boundary conditions, but again Lagrange multipliers can be used to disentangle all summations. For the partition function describing the excitation spectrum of a configuration of two B-type branes as in eq. (4.40) one obtains

\[
Z_{\alpha\tilde{\alpha}}^B(q) = n_{\alpha\tilde{\alpha}}^B \sum^{ev}_{\lambda', \mu'} \sum_{\nu_0=0}^{K-1} \sum_{\nu_1, \ldots, \nu_r=0,1} (-1)^{s'_0 + \tilde{S}_0 - S_0} \delta^{(4)}_{s'_0, 2 + s'_{00} - S_0 - \tilde{S}_0 - \nu_0 - 2\Sigma_{\nu_j}} \times \delta^{(K')}_{\Sigma_{m'_j}, 0} \prod_{j=1}^r N_{L_j, L_j}^{i_j} \delta^{(2)}_{m'_j + M_j - \tilde{M}_j + \nu_0, 0} \delta^{(4)}_{s'_{j}, \tilde{S}_j - S_j - \tilde{S}_j - \nu_0 - 2\nu_j} \chi^{\lambda'}_{\mu'}(q) \tag{4.41a}
\]

with \( K' := \text{lcm}(2k_j + 4) \), \( \xi_j := K'/2(k_j + 4) \) and

\[
\Sigma_{m'_j} := \sum_{j=1}^r \xi_j (m'_j + M_j - \tilde{M}_j + \nu_0) ; \tag{4.41b}
\]

the overall factor is \( n_{\alpha\tilde{\alpha}}^B = 2^\tilde{r} / (\kappa_B^B \kappa_{\alpha}^{\tilde{\alpha}}) \) so that we choose the normalization in (4.40) to be

\[
\kappa_B^{\alpha} = 2^\tilde{r}. \tag{4.42}
\]

As in the A-type case, Cardy’s conditions are always satisfied for our boundary states (4.40), and the remaining string requirements for the spectrum of a brane configuration \((\alpha \tilde{\alpha})\) lead to the same conditions

\[
S_0 - \tilde{S}_0 \overset{!}{=} S_j - \tilde{S}_j \pmod{2}, \quad Q(\alpha - \tilde{\alpha}) \overset{!}{=} 0 \pmod{2}
\]

on the integer labels occurring in the ansatz.
As in the simplified \((k = 1)^3\) example treated in section 4.2, we can take a look at systems consisting of one A-type and one B-type brane. For the Gepner models, we find that A-type and B-type boundary states do not “exert static forces on each other”, or

\[
Z^{AB}_{\alpha \tilde{\alpha}}(q) = 0,
\]

simply because of the GSO projection: Again, only representations with total charge \(q = 0\) could contribute to the trace occurring as in (4.28), but the GSO projection leaves only states with odd integer charge in the spectrum.

5. Summary and outlook

We have seen that the language of boundary CFT allows for a general conceptual formulation of (static) D-branes, i.e. of non-perturbative sectors of a closed string theory. Strong constraints make it possible to arrive at concrete and complete formulas even in the relatively complicated case of Gepner models. In contrast, the geometric methods seem to yield only part of the boundary states, thus part of the excitation spectrum of brane configurations remains undetermined. In the CFT approach, we have explicit control of the excitation spectrum: Its symmetries are manifest, and so is stability. As a consequence, we discover classes of mutually compatible branes. Given the boundary state, it is straightforward to compute the coupling constants (tensions, RR charges) of the low-energy effective field theory, simply by sandwiching the boundary state with the relevant closed string states.

An obvious question that has been left open here is whether the additional sewing constraints of boundary CFT [30,35,36] eliminate some of the Gepner model boundary states we have found. There are indications that this is not the case, but one should certainly clarify this point; the general methods developed in [37], see also [49], should prove efficient in this context.

Gepner models can also be formulated starting from modular invariant partition functions that are non-diagonal in their SU(2) part. Using the methods of [36,37], our construction can be extended to this case, too. Similarly, one could discuss other string compactifications, in particular the Kazama-Suzuki models whose Ishibashi conditions were analyzed in [50].

Beyond that, one can try to make further use of the fact that the boundary state approach gives full control of the excitation spectrum of brane configurations: In particular, it would be interesting to search for configurations that break part of the supersymmetry; in the flat case, this can e.g. be achieved by letting the branes intersect at certain angles, more generally by putting boundary states with different Ishibashi conditions on both ends of an open string. In general, this gives contributions to \(Z_{\alpha \tilde{\alpha}}(q)\) like the \(\text{Tr}_{H_{inv}} V_{\Omega} q^{L_0-\frac{c}{24}}\) in section 4.2.

To study such problems, and for general reasons, it would be advantageous to have moduli in the boundary states, like the location of the classical brane in the flat target case. Even more interesting results arise from varying the moduli, e.g. from computing string amplitudes for D-branes in relative motion [51]. It seems that all this requires to consider
non-rational situations, which in general is much harder from the CFT point of view. We hope that one can make progress e.g. by deforming rational boundary states like the ones we have constructed above away from the rational point.

Another immediate question is how one can describe, in CFT terms, configurations with three or more branes. It should be possible to extend the formulation in [20] from the flat target case to our generalized D-branes. Furthermore, it appears that the boundary state formalism also allows to investigate bound states of (strings and) branes, cf. [52] and the recent work [53]; hopefully, one can use CFT methods to prove existence of the marginal bound states important for M-theory.

Definitely, one should try to establish closer contact to the geometric approach to D-branes. While the latter has certain short-comings as far as completeness of the D-brane description or calculation of concrete spectra is concerned, it is of course much more effective in controlling moduli.

We believe, however, that a true “geometric understanding” of D-branes should be sought within non-commutative geometry rather than classical geometry. The main reason was recalled already in the introduction: String theory by its very design reaches beyond classical notions of space-time, and non-commutative geometry [54] is the most general and best developed framework to discuss “quantized space-time”. Moreover, it is precisely through D-branes and their role in the matrix model proposal [55] that aspects of non-commutative geometry have appeared naturally in string theory. In the recent work [56], this interplay was exploited to study matrix theory compactifications on non-commutative tori. Indications that D-branes are related to cohomology classes of some differential calculus intrinsic to string theory were given in [57]. It is certainly worthwhile to pursue these matters further.

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