Fekete-Szegö problem for generalized bi-subordinate functions of complex order

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Abstract

In this paper, we obtain the Fekete-Szegö inequality for the generalized bi-subordinate functions of complex order. The various results, which are presented in this paper, would generalize those in related works of several earlier authors.

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1. Introduction

Let \( A \) be the class of analytic functions in the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and let \( S \) be the class of functions \( f \) that are analytic, univalent in \( \mathbb{D} \) and are of the form

\[
 f(z) = z + \sum_{k=2}^{\infty} a_k z^k. 
\]  

The Koebe one-quarter theorem assures that the image of unit disk \( \mathbb{D} \) under every univalent function \( f \in A \) contains a disk of radius \( 1/4 \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying

\[
 f^{-1}(f(z)) = z \quad (z \in \mathbb{D}) \quad \text{and} \quad f(f^{-1}(w)) = w, \quad (|w| < r_0, \ r_0 \geq 1/4).
\]

Furthermore, the Taylor-Maclaurin series of \( f^{-1} \) is given by

\[
 f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots .
\]

A function \( f \in A \) is said to be bi-univalent in \( \mathbb{D} \) if \( f \) is univalent and \( f^{-1} \) has univalent analytic continuation, which we denote by \( g \), to the unit disk \( \mathbb{D} \). Let \( \sigma \) denote the class of bi-univalent functions defined in the unit disk \( \mathbb{D} \). Coefficient problem for bi-univalent functions were recently investigated by several authors \[1,4–8,15–17,19,20\]. A function \( f \in A \) is said to be subordinate to a function \( h \in A \), denoted by \( f \prec h \), if there exists an analytic function \( w \in B_0 \), where \( B_0 := \{ w : w(0) = 0, \ |w(z)| < 1, \ z \in \mathbb{D} \} \) such that \( f(z) = h(w(z)) \). We let \( S^* \) consist of starlike functions \( f \in A \), that is, \( \Re\{zf'(z)/f(z)\} > 0 \) in \( \mathbb{D} \) and \( C \) consist of convex functions \( f \in A \), that is, \( 1 + \Re\{zf''(z)/f'(z)\} > 0 \) in \( \mathbb{D} \). Ma and

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Minda [12] unified various subclasses of starlike and convex functions for which either of the quantity $z f'(z)/f(z)$ or $1 + z f''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function $\varphi$ with positive real part in the unit disk $\mathbb{D}$ and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $z f'(z)/f(z) \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + z f''(z)/f'(z) \prec \varphi(z)$. Extensions of the above two classes (see [14]) are

$$S^*(\gamma; \varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \varphi(z), \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}$$

and

$$C(\gamma; \varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma} \left( \frac{zf''(z)}{f'(z)} - 1 \right) \prec \varphi(z), \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}.$$  

In literature, the functions belonging to these classes are called Ma-Minda starlike and convex of complex order $\gamma$ (where $\gamma \in \mathbb{C} \setminus \{0\}$), respectively. A function $f$ is bi-starlike of Ma-Minda type of complex order $\gamma$ (where $\gamma \in \mathbb{C} \setminus \{0\}$) and bi-convex of Ma-Minda type of complex order $\gamma$ (where $\gamma \in \mathbb{C} \setminus \{0\}$). The classes consisting of bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type of complex order $\gamma$ (where $\gamma \in \mathbb{C} \setminus \{0\}$) are denoted by $S^*_\sigma(\gamma; \varphi)$ and $C_\sigma(\gamma; \varphi)$, respectively. As a special case $\gamma = 1$ the classes $S^*_\sigma$ and $C_\sigma$ reduce to bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are denoted by $S^*_\sigma(\varphi)$ and $C_\sigma(\varphi)$, respectively.

In this paper, we consider more general class $S_\sigma(\lambda, \gamma; \varphi)$ for $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C} \setminus \{0\}$ which was investigated by Deniz [5] wherein he obtained the bounds for $a_2$ and $a_3$. This motivated us to study the Fekete-Szegő inequality for the class $S_\sigma(\lambda, \gamma; \varphi)$. Recently, some authors have investigated the Fekete-Szegő problem for various subclasses of $\sigma$ (see [3,9,13,21,22]).

### 2. Coefficient estimates

Throughout this paper $\varphi$ denotes an analytic univalent function in $\mathbb{D}$ with positive real part and normalized by $\varphi(0) = 1$, $\varphi'(0) > 0$. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \quad (B_1 > 0).$$

**Definition 2.1.** For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$, the class $S(\lambda, \gamma; \varphi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda) f(z) + \lambda z f'(z)} - 1 \right) \prec \varphi(z) \quad (z \in \mathbb{D}).$$

The class $S_\sigma(\lambda, \gamma; \varphi)$ consists of functions $f \in \sigma$ such that $f, g \in S(\lambda, \gamma; \varphi)$ where $g$ is the analytic continuation of $f^{-1}$ to the unit disk $\mathbb{D}$.

The class $S(\lambda, \gamma; \varphi)$ was introduced by [18]. Motivated by this class the second author [5] defined and studied the class $S_\sigma(\lambda, \gamma; \varphi)$, which is called the class of generalized bi-subordinate functions of complex order $\gamma$ and type $\lambda$. As special cases of the class $S_\sigma(\lambda, \gamma; \varphi)$, we have $S_\sigma(0, \gamma; \varphi) \equiv S^*_\sigma(\gamma; \varphi)$ and $S_\sigma(1, \gamma; \varphi) \equiv C_\sigma(\gamma; \varphi)$.

The class $S_\sigma(\lambda, \gamma; \varphi)$ includes many earlier classes, which are mentioned below:

- $S_\sigma(0, 1; \varphi) \equiv S^*_\sigma(\varphi)$ and $S_\sigma(1, 1; \varphi) \equiv C_\sigma(\varphi)$, are classes of Ma-Minda bi-starlike and Ma-Minda bi-convex functions, respectively, introduced and studied in [11].
- $S_\sigma((0, 1; (1 + A z)/(1 + B z)) \equiv S_\sigma(A, B)$ and $S_\sigma((1, 1; (1 + A z)/(1 + B z)) \equiv C_\sigma(A, B)$ ($-1 < B < A \leq 1$) are, respectively, the classes of Janowski bi-starlike and bi-convex functions. Additionally, for $0 \leq \beta < 1$, $S_\sigma(1 - 2\beta, 1) \equiv S_\sigma(\beta)$ and $C_\sigma(1 - 2\beta, 1) \equiv C_\sigma(\beta)$ are, respectively, the classes of bi-starlike and bi-convex functions of order $\beta$ introduced and studied in [2].
For $0 < \beta \leq 1$, $S_\sigma \left( 0, 1; \left( \frac{1+z}{1-z} \right)^\beta \right) \equiv SS_\sigma^* (\beta)$ and $S_\sigma \left( 1, 1; \left( \frac{1+z}{1-z} \right)^\beta \right) \equiv SC_\sigma^* (\beta)$ are, respectively, classes of strongly bi-starlike and strongly bi-convex functions of order $\beta$ introduced and studied in [2].

For $\gamma \in \mathbb{C} \setminus \{ 0 \}$, $S_\sigma (0, \gamma; (1+z)/(1-z)) \equiv S_\sigma^* [\gamma]$ and $S_\sigma (1, \gamma; (1+z)/(1-z)) \equiv C_\sigma [\gamma]$ are classes of bi-starlike and bi-convex functions of complex order, respectively.

To prove our next theorems, we shall need the following well-known lemma (see [10]).

**Lemma 2.2** ([10]). Let the function $w \in B_0$ be given by

$$w(z) = c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{D}),$$

then for every complex number $s$,

$$|c_2 - sc_1^2| \leq 1 + (|s| - 1)|c_1|^2.$$

In the following theorem, we consider functional $|a_3 - \mu a_2^2|$ for $\gamma$ nonzero complex number and $\mu \in \mathbb{C}$.

**Theorem 2.3.** Let the function $f$ given by (1.1) be in the $S_\sigma (\lambda, \gamma; \varphi)$. For $\gamma \in \mathbb{C} \setminus \{ 0 \}$ and $\mu \in \mathbb{C}$, we have

$$|a_2| \leq \frac{|\gamma| B_1}{1+\lambda}, \quad (2.2)$$

$$|a_3| \leq \frac{|\gamma| B_1}{4(1+2\lambda)} \max \{ 2, (|s| + |t|) \} \quad (2.3)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1|\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L} \leq 2 \\ \frac{B_1|\gamma|}{4(1+2\lambda)} \mathcal{L} & \text{if } \mathcal{L} > 2 \end{cases} \quad (2.4)$$

where $s = \frac{B_2}{B_1} - \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$, $t = \frac{B_2}{B_1}$ and $\mathcal{L} = \frac{B_2}{B_1} + (1 - \mu) \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2} + \frac{|B_2|}{B_1^2}$.

**Proof.** Since $f \in S_\sigma (\lambda, \gamma; \varphi)$, there exists two analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = 0 = v(0)$, such that

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right) = \varphi (u(z)) \quad (z \in \mathbb{D}) \quad (2.5)$$

and

$$1 + \frac{1}{\gamma} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{(1-\lambda)g(w) + \lambda wg'(w)} - 1 \right) = \varphi (v(w)). \quad (2.6)$$

Define the functions $u$ and $v$ by

$$u(z) = c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad v(w) = d_1 w + d_2 w^2 + \cdots. \quad (2.7)$$

Using (2.1) with (2.7), it is evident that

$$\varphi (u(z)) = 1 + (B_1c_1)z + (B_1c_2 + B_2c_1^2)z^2 + \cdots \quad (2.8)$$

and

$$\varphi (v(w)) = 1 + (B_1d_1)w + (B_1d_2 + B_2d_1^2)w^2 + \cdots. \quad (2.9)$$

Also, using (1.1), we get

$$1 + \frac{1}{\gamma} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right) = 1 + \frac{(1+\lambda) a_2}{\gamma} z + \left[ \frac{2(1+2\lambda) a_3 - (1+\lambda)^2 a_2^2}{\gamma} \right] z^2 + \cdots \quad (2.10)$$

and using (1.2), we get
\[
1 + \frac{1}{\gamma} \left( \frac{wg'(w) + \lambda w^2 g''(w)}{(1 - \lambda) g(w) + \lambda wg'(w)} - 1 \right)
= 1 - \frac{(1 + \lambda) a_2}{\gamma} w \left[ -2 \frac{(1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} \right] w^2 + \cdots . \tag{2.11}
\]

Equating coefficients of right sides of equations (2.8) with (2.10) and (2.9) with (2.11) yield
\[
\frac{(1 + \lambda) a_2}{\gamma} = B_1 c_1, \quad \frac{2 (1 + 2\lambda) a_3 - (1 + \lambda)^2 a_2^2}{\gamma} = B_1 c_2 + B_2 c_1^2 \tag{2.12}
\]
and
\[
\frac{-(1 + \lambda) a_2}{\gamma} = B_1 d_1, \quad \frac{-2 (1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} = B_1 d_2 + B_2 d_1^2 \tag{2.13}
\]
so that, on account of (2.12) and (2.13)
\[
c_1 = -d_1, \tag{2.14}
\]
\[
a_2 = \frac{\gamma B_1}{1 + \lambda} c_1 \tag{2.15}
\]
and
\[
a_3 = a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right]. \tag{2.16}
\]

Taking into account (2.14), (2.15), (2.16) and the well known estimate $|c_1| \leq 1$ of the Schwarz lemma, we get
\[
|a_2| = \left| \frac{\gamma B_1}{1 + \lambda} c_1 \right| \leq \left| \frac{\gamma}{1 + \lambda} \right| \tag{2.17}
\]
and from Lemma 2.2,
\[
|a_3| = \left| a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right] \right|
= \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} c_1^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ \left( B_1 c_2 - B_2 c_1^2 \right) - \left( B_1 d_2 - B_2 d_1^2 \right) \right]
= \frac{\gamma B_1}{4(1 + 2\lambda)} \left\{ \left[ c_2 - \frac{B_2}{B_1} \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} c_1^2 \right] - \left[ d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\}
\leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ \left| c_2 - \frac{B_2}{B_1} \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} c_1^2 \right| + \left| d_2 - \frac{B_2}{B_1} d_1^2 \right| \right\}
\leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ 1 + (|s| - 1) \left| c_1^2 \right| + 1 + (|t| - 1) \left| c_1^2 \right| \right\}
= \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ 2 + (|s| + |t| - 2) \left| c_1^2 \right| \right\}.
\]
Thus, using $|c_1| \leq 1$ we have the desired estimate for $|a_3|:
\[
|a_3| \leq \frac{|\gamma| |B_1|}{4(1 + 2\lambda)} \max \{ 2, (|s| + |t|) \},
\]
where $s = \frac{B_2}{B_1} - \frac{4B_1 \gamma (1 + 2\lambda)}{(1 + \lambda)^2}$ and $t = \frac{B_2}{B_1}$.

To find an estimate for $|a_3 - \mu a_2^2|$, we express $a_3 - \mu a_2^2$ in terms of $c_i$ and $d_i$. Using the equality (2.16), we have
\[
a_3 - \mu a_2^2 = (1 - \mu) a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right].
\]
Therefore from Lemma 2.2, we obtain
\[
|a_3 - \mu a_2^2| = \left| (1 - \mu) a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} \left[ B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2 \right] \right|
\]
\[
= \left| \frac{\gamma B_1}{4(1 + 2\lambda)} \left( \left| c_2 - \left( B_2 - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) c_1^2 \right| - \left| d_2 - \frac{B_2}{B_1} d_1^2 \right| \right) \right|
\]
\[
\leq \left| \frac{\gamma |B_1|}{4(1 + 2\lambda)} \left\{ 2 + \left( \left| B_2 - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right| + \left| B_1 \right| - 2 \right) \right| c_1^2 \right\} \tag{2.18}
\]
As a result of this, from \(|c_1| \leq 1\) we obtain
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{B_1 |c_1|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \leq 1 \\
\frac{\gamma |B_1|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu > 1
\end{array} \right.
\]
where \(\mathcal{L} = \left| \frac{B_2}{B_1} + (1 - \mu) \frac{4B_1 \gamma (1 + 2\lambda)}{(1 + \lambda)^2} \right| + \left| \frac{B_2}{B_1} \right| \).
Thus the proof is completed. \(\square\)

We next consider the cases \(\gamma\) and \(\mu\) are real.

**Theorem 2.4.** Let the function \(f\) given by (1.1) be in the \(S_{\sigma}(\lambda, \gamma; \varphi)\). For \(\gamma > 0\) and \(\mu \in \mathbb{R}\), we have

1. If \(|B_2| \geq B_1\), then
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{\gamma |B_1|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \leq 1 \\
\frac{\gamma |B_1|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu > 1
\end{array} \right.
\]

2. If \(|B_2| < B_1\), then
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{\gamma |B_1|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } \mu \leq 1 - \mathcal{T} \\
\frac{\gamma |B_1|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} & \text{if } 1 - \mathcal{T} \leq \mu < 1 + \mathcal{T}
\end{array} \right.
\]
where \(\mathcal{T} = \frac{(1 + \lambda)^2 |B_1| - |B_2|}{2 \gamma B_1^2 (1 + 2\lambda)}\).

**Proof.** Using (2.18) and Lemma 2.2, we obtain
\[
|a_3 - \mu a_2^2| = \left| \frac{\gamma B_1}{4(1 + 2\lambda)} \left( \left| c_2 - \left( B_2 - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right) c_1^2 \right| - \left| d_2 - \frac{B_2}{B_1} d_1^2 \right| \right) \right|
\]
\[
\leq \left| \frac{\gamma B_1}{4(1 + 2\lambda)} \left\{ 2 + \left( \left| B_2 - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right| + \left| B_1 \right| - 2 \right) \right| c_1^2 \right\}
\]
\[
\leq \left| \frac{\gamma B_1}{2(1 + 2\lambda)} + \left( \frac{\gamma |B_2| - B_1}{2(1 + 2\lambda)} + |\mu - 1| \frac{\gamma^2 B_1^2}{(1 + \lambda)^2} \right) \right| c_1^2 \right| \right|. \tag{2.19}
\]
Now, the proof will be presented in two cases:
Firstly, we consider the case $|B_2| \geq B_1$. If $\mu \leq 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\}
$$

$$
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2}.
$$

If $\mu > 1$, then using (2.19) and $|c_1| \leq 1$, we obtain

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\}
$$

$$
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2}.
$$

Finally, we consider the case $|B_2| < B_1$. By using (2.19) and $|c_1| \leq 1$, we obtain the following results according to the cases of $\mu$ and $\mathcal{F}$.

For $\mu \leq 1 - \mathcal{F}$, we have

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\}
$$

$$
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} - (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2},
$$

and for $1 - \mathcal{F} < \mu \leq 1$, we yield

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (1 - \mu) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)}.
$$

Similarly for $1 < \mu < 1 + \mathcal{F}$, we obtain

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)}.
$$

Finally for $\mu \geq 1 + \mathcal{F}$, we have

$$
|a_3 - \mu a^2_2| \leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\} |c_1^2|
$$

$$
\leq \frac{\gamma B_1}{2(1 + 2\lambda)} + \left\{ \frac{\gamma (|B_2| - B_1)}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2} \right\}
$$

$$
= \frac{\gamma |B_2|}{2(1 + 2\lambda)} + (\mu - 1) \frac{\gamma^2 B^2_1}{(1 + \lambda)^2}.
$$
Thus the proof is completed.

Finally, we consider the cases of $\gamma$ nonzero complex number and $\mu \in \mathbb{R}$.

**Theorem 2.5.** Let the function $f$ given by (1.1) be in the $S_\sigma(\lambda, \gamma; \varphi)$. For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$, we have

1. If $\frac{|1+|\sin \theta||B_2|}{2B_1} \geq 1$, then
   \[
   |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
   \frac{|\gamma|^2 B_2^2}{(1 + \lambda)} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2||(1 + |\sin \theta|)}{4(1 + 2\lambda)} & \text{if } \mu \leq 1 - \Re(k_1) \\
   \frac{|\gamma||B_2||(1 + |\sin \theta|)}{4(1 + 2\lambda)} - \frac{|\gamma|^2 B_2^2}{(1 + \lambda)} (1 - \mu - \Re(k_1)) & \text{if } \mu > 1 - \Re(k_1).
   \end{array} \right.
   \]

2. If $\frac{|1+|\sin \theta||B_2|}{2B_1} < 1$, then
   \[
   |a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
   \frac{|\gamma|^2 B_2^2}{(1 + \lambda)} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2||(1 + |\sin \theta|)}{4(1 + 2\lambda)} & \text{if } 1 - \Re(k_1) + N < \mu < 1 - \Re(k_1) - N \\
   \frac{|\gamma||B_2||(1 + |\sin \theta|)}{4(1 + 2\lambda)} - \frac{|\gamma|^2 B_2^2}{(1 + \lambda)} (1 - \mu - \Re(k_1)) & \text{if } \mu \geq 1 - \Re(k_1) - N
   \end{array} \right.
   \]

where $k_1 = \frac{B_2(1+\lambda)e^{i\theta}}{4B_1(1+(1+2\lambda)\gamma)}$, $|\gamma| = |\gamma|e^{i\theta}$ and $N = \frac{(1+\lambda)^2||B_2||(1 + |\sin \theta|) - 2B_1}{4B_1(1+(1+2\lambda)\gamma)}$.

**Proof.** Let $f \in S_\sigma(\lambda, \gamma; \varphi)$. By using (2.18) and Lemma 2.2, then we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{4(1 + 2\lambda)} \left\{ 2 + \left|\frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1 (1 + 2\lambda)}{(1 + \lambda)^2} \right| + \left|\frac{B_2}{B_1} - 2\right| \right\} |c_1|^2.
\]

Taking $|\gamma| = |\gamma|e^{i\theta}$, $k_1 = \frac{B_2(1+\lambda)e^{i\theta}}{4B_1(1+(1+2\lambda)\gamma)}$ and $l_1 = \frac{(|B_2| - 2B_1)(1 + \lambda)^2}{4B_1(1+(1+2\lambda)\gamma)}$, for $B_1, B_2 \in \mathbb{R}$ and $B_1 > 0$, we rewrite

\[
|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} (1 - \mu - k_1 + l_1) |c_1|^2 \tag{2.20}
\]

Firstly, we consider the case $\frac{1+|\sin \theta||B_2|}{2B_1} \geq 1$.

Let $\mu \leq 1 - \Re(k_1)$. Then from (2.20) and $|c_1| \leq 1$, we obtain

\[
|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left|\frac{|\gamma|^2 B_2^2}{(1 + \lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2||(1 + |\sin \theta|) - 2B_1|}{4(1 + 2\lambda)} \right| |c_1|^2.
\]

Thus the proof is completed. \qed
Let $\mu > 1 - \Re (k_1)$. Then from (2.20) and $|c_1| \leq 1$, we yield

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left[ \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} |1 - \mu - \Re (k_1)| + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)} \right]|c_1|$$

$$\leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (\mu + \Re (k_1) - 1) + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)}$$

$$= \frac{|\gamma| B_2(1 + |\sin \theta|)}{4(1 + 2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)).$$

Finally, we want to consider the case with $\frac{(1 + |\sin \theta|)B_2}{2B_1} < 1$. By using (2.20) and $|c_1| \leq 1$, we obtain the following results according to the cases of $\mu, k_1$ and $N$.

For $\mu \leq 1 - \Re (k_1) + N$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left[ \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} |1 - \mu - \Re (k_1)| + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)} \right]|c_1|$$

$$\leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} (1 - \mu - \Re (k_1)) + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)}$$

and for $1 - \Re (k_1) + N \leq \mu \leq 1 - \Re (k_1)$, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left[ \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} |1 - \mu - \Re (k_1)| + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)} \right]|c_1|$$

Similarly, for $1 - \Re (k_1) < \mu < 1 - \Re (k_1) - N$, we yield

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left[ \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} |1 - \mu - \Re (k_1)| + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)} \right]|c_1|$$

and finally, for $\mu \geq 1 - \Re (k_1) - N$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1 + 2\lambda)} + \left[ \frac{|\gamma|^2 B_1^2}{(1 + \lambda)^2} |1 - \mu - \Re (k_1)| + \frac{|\gamma| [B_2(1 + |\sin \theta|) - 2B_1]}{4(1 + 2\lambda)} \right]|c_1|$$

Thus the proof is completed.

Taking $\gamma = 1$, $\lambda = 0$ and $\varphi(z) = (1 + Az)/(1 + Bz)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

**Corollary 2.6.** If $f \in A$ is given by (1.1) belongs to the class $S_\sigma [A, B]$, then

1. For $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \frac{A - B}{2} \frac{|B| + |4(1 - \mu)(A - B) - B|}{|B| + |4(1 - \mu)(A - B) - B|} \frac{|a_2|}{|a_2|} \frac{1}{4} \frac{\lambda}{\mu}$$

if $|B| + |4(1 - \mu)(A - B) - B| < 2$.

This completes the proof.
(2) For $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|B(A-B)|}{2} - (\mu - 1) (A-B)^2 & \text{if } \mu \leq 1 - \frac{1-|B|}{2(A-B)} \\
\frac{A-B}{2} & \text{if } 1 - \frac{1-|B|}{2(A-B)} < \mu < 1 + \frac{1-|B|}{2(A-B)} \\
\frac{|B|(A-B)}{2} + (\mu - 1) (A-B)^2 & \text{if } \mu \geq 1 + \frac{1-|B|}{2(A-B)} 
\end{cases}$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{(A-B)}{2} \left[ (A-B) (1 - \mu) + \frac{|B| + B}{4} \right] & \text{if } \mu \leq 1 + \frac{|B| + B - 2}{4(A-B)} \\
\frac{A-B}{2} & \text{if } 1 + \frac{|B| + B - 2}{4(A-B)} < \mu < 1 - \frac{|B| - B - 2}{4(A-B)} \\
(A-B) \left[ (A-B) (1 - \mu) + \frac{|B| - B}{4} \right] & \text{if } \mu \geq 1 - \frac{|B| - B - 2}{4(A-B)} 
\end{cases}$$

Taking $\gamma = 1$, $\lambda = 1$ and $\varphi(z) = (1 + A z)/(1 + B z)$ $(-1 \leq B < A \leq 1)$ in Theorems 2.3, 2.4 and 2.5, we have the following corollary.

**Corollary 2.7.** If $f \in A$ is given by (1.1) belongs to the class $C_\sigma [A, B]$, then

1. For $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{A-B}{6} & \text{if } |B| + |3 (1 - \mu) (A-B) - B| < 2 \\
\frac{|B| + 3 (1 - \mu) (A-B) - B|}{2} & \text{if } |B| + |3 (1 - \mu) (A-B) - B| \geq 2 
\end{cases}$$

2. For $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|B|(A-B)}{6} - (\mu - 1) (A-B)^2 & \text{if } \mu \leq 1 - \frac{2(1-|B|)}{3(A-B)} \\
\frac{A-B}{6} & \text{if } 1 - \frac{2(1-|B|)}{3(A-B)} < \mu < 1 + \frac{2(1-|B|)}{3(A-B)} \\
\frac{|B|(A-B)}{6} + (\mu - 1) (A-B)^2 & \text{if } \mu \geq 1 + \frac{2(1-|B|)}{3(A-B)} 
\end{cases}$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{A-B}{12} \left[ |3 (A-B) (1 - \mu) + |B| + B| \right] & \text{if } \mu \leq 1 + \frac{2|B| + 2B - 1}{6(A-B)} \\
\frac{A-B}{6} & \text{if } 1 + \frac{2|B| + 2B - 1}{6(A-B)} < \mu < 1 + \frac{2|B| - 2B - 1}{6(A-B)} \\
\frac{A-B}{12} \left[ |3 (A-B) (\mu - 1) + |B| - B| \right] & \text{if } \mu \geq 1 - \frac{2|B| - 2B - 1}{6(A-B)} 
\end{cases}$$

Taking $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda = 0$ and $\varphi(z) = (1 + z)/(1 - z)$ in Theorems 2.3, 2.4 and 2.5, then we have the following corollary.

**Corollary 2.8.** If $f \in A$ is given by (1.1) belongs to the class $S^\ast [\gamma]$], then

1. For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{\gamma}{2} |1 + (1 - \mu) 8\gamma| + 1 & \text{if } |1 + (1 - \mu) 8\gamma| < 1 \\
\frac{\gamma}{2} & \text{if } |1 + (1 - \mu) 8\gamma| \geq 1 
\end{cases}$$

2. For $\gamma > 0$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\gamma - 4 (\mu - 1) \gamma^2 & \text{if } \mu \leq 1 \\
\gamma + 4 (\mu - 1) \gamma^2 & \text{if } \mu > 1 
\end{cases}$$

3. For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{4 |\gamma|^2 |1 - \mu| + |\gamma(1+|\sin \theta| - |\cos \theta|)|}{2} & \text{if } \mu \leq 1 + \chi_1 (\gamma, \theta) \\
\frac{|\gamma| (1+|\sin \theta| - |\cos \theta|)}{2} - 4 |\gamma|^2 (1 - \mu) & \text{if } 1 + \chi_1 (\gamma, \theta) < \mu < 1 - \chi_2 (\gamma, \theta) \\
\frac{|\gamma| (1+|\sin \theta| + |\cos \theta| - 1)}{8|\gamma|} & \text{if } \mu \geq 1 - \chi_2 (\gamma, \theta) 
\end{cases}$$

where $\chi_1 (\gamma, \theta) = \frac{|\sin \theta - |\cos \theta| - 1|}{8|\gamma|}$ and $\chi_2 (\gamma, \theta) = \frac{|\sin \theta + |\cos \theta| - 1|}{8|\gamma|}$. 
Taking $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda = 1$ and $\varphi(z) = (1 + z)/(1 - z)$ in Theorems 2.3, 2.4 and 2.5, we obtain the following corollary.

**Corollary 2.9.** If $f \in \mathcal{A}$ is given by (1.1) belongs to the class $\mathcal{C}_\sigma[\gamma]$, then

(i) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C},$

$$|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\left| \gamma \right| \frac{1}{2} & \text{if } |1 + (1 - \mu) 6\gamma| < 1 \\
\left| \gamma \right| & \text{if } |1 + (1 - \mu) 6\gamma| \geq 1
\end{array} \right.$$

(ii) For $\gamma > 0$ and $\mu \in \mathbb{R},$

$$|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\frac{7}{3} - (\mu - 1) \gamma^2 & \text{if } \mu \leq 1 \\
\frac{3}{3} + (\mu - 1) \gamma^2 & \text{if } \mu > 1.
\end{array} \right.$$

(iii) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R},$

$$|a_3 - \mu a_2| \leq \left\{ \begin{array}{ll}
\frac{\left| \gamma \right|^2 (1 - \mu) + |\gamma| |1 + |\sin \theta| - |\cos \theta|)}{\sqrt{6}} & \text{if } \mu \leq 1 + \varphi_1(\gamma, \theta) \\
\frac{\left| \gamma \right|^2 (1 + |\sin \theta| - |\cos \theta|)}{\sqrt{6}} - \left| \gamma \right|^2 (1 - \mu) & \text{if } 1 + \varphi_1(\gamma, \theta) < \mu < 1 - \varphi_2(\gamma, \theta)
\end{array} \right.$$

where $\varphi_1(\gamma, \theta) = \left| \frac{|\sin \theta| - |\cos \theta| - 1}{6\gamma} \right|$ and $\varphi_2(\gamma, \theta) = \left| \frac{|\sin \theta| + |\cos \theta| - 1}{6\gamma} \right|.$

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