Spectral Functions for Regular Sturm-Liouville Problems

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Regular One-dimensional Sturm-Liouville Problems

Let $I = [0, 1] \subset \mathbb{R}$, and let $\mathcal{L}$ be the following symmetric second order differential operator

$$\mathcal{L} = -\frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + V(x),$$

with $p(x) > 0$ for $x \in I$, and $p(x)$ and $V(x)$ in $\mathcal{L}^1(I, \mathbb{R})$. For the differential operator $\mathcal{L}$ we consider the differential equation

$$\mathcal{L}\varphi_\lambda = \lambda^2 \varphi_\lambda,$$

where $\lambda \in \mathbb{C}$ and $\varphi_\lambda \in C^2(I)$.

The differential equation (1) endowed with self-adjoint boundary conditions imposed on $\varphi_\lambda$ is called a regular Sturm-Liouville problem. Furthermore, the parameter $\lambda \in \mathbb{R}$ denotes the eigenvalues of the SL problem. Self-adjoint boundary conditions can be divided in two mutually excluding classes:

- **Separated** Boundary Conditions
- **Coupled** boundary conditions.
Separated Boundary Conditions

*Separated boundary conditions* have the following general form

\[
A_1 \varphi_\lambda(0) - A_2 p(0) \varphi'_\lambda(0) = 0 ,
B_1 \varphi_\lambda(1) + B_2 p(1) \varphi'_\lambda(1) = 0 ,
\]

with \( A_1, A_2, B_1, B_2 \in \mathbb{R} \) and \( (A_1, A_2) \neq (0, 0) \), and \( (B_1, B_2) \neq (0, 0) \).

**Eigenvalues.** For each \( \lambda \) we choose a solution \( \varphi_\lambda \) satisfying the *initial conditions*

\[
\varphi_\lambda(0) = A_2 , \quad \text{and} \quad p(0) \varphi'_\lambda(0) = A_1 .
\]

The eigenfunctions of the Sturm-Liouville problem are, then, those that also satisfy the condition

\[
\Omega(\lambda) = B_1 \varphi_\lambda(1) + B_2 p(1) \varphi'_\lambda(1) = 0 ,
\]

which represents an *implicit* equation for the eigenvalues \( \lambda \).

**Remarks:** Well-known examples

- For \( A_1 = B_1 = 0 \) and \( A_2 = B_2 = 0 \) we get Neuman and Dirichlet boundary conditions, respectively.
- When \( A_1 = B_1 \) and \( A_2 = -B_2 \) we have Robin Boundary conditions.
- By setting \( A_1 = B_2 = 0 \) or \( A_2 = B_1 = 0 \) we obtain mixed or hybrid boundary conditions.
Coupled Boundary Conditions

Coupled boundary conditions can be expressed in general as

\[
\begin{pmatrix}
\varphi_\lambda(1) \\
p(1)\varphi'_\lambda(1)
\end{pmatrix} = e^{i\gamma} K \begin{pmatrix}
\varphi_\lambda(0) \\
p(0)\varphi'_\lambda(0)
\end{pmatrix},
\]

where \(-\pi < \gamma \leq 0\) or \(0 \leq \gamma < \pi\) and \(K \in \text{SL}_2(\mathbb{R})\). For \(\gamma = 0\) and \(K = I_2\) we have periodic boundary conditions.

**Eigenvalues.** We write the solution as

\[
\varphi_\lambda(x) = \alpha u_\lambda(x) + \beta v_\lambda(x),
\]

where for each \(\lambda\), \(u_\lambda(x)\) and \(v_\lambda(x)\) are defined by the initial conditions

\[
\varphi_\lambda(0) = \beta \quad \text{and} \quad p(0)\varphi'_\lambda(0) = \alpha.
\]

By imposing coupled boundary conditions and by denoting \([k_{ij}] = K\) we obtain the linear system

\[
\begin{align*}
\alpha \left[u_\lambda(1) - e^{i\gamma} k_{12}\right] + \beta \left[v_\lambda(1) - e^{i\gamma} k_{11}\right] &= 0 \\
\alpha \left[p(1)u'_\lambda(1) - e^{i\gamma} k_{22}\right] + \beta \left[p(1)v'_\lambda(1) - e^{i\gamma} k_{21}\right] &= 0,
\end{align*}
\]

which has a non-trivial solution if and only if

\[
\Delta(\lambda) = 2 \cos \gamma - [k_{22}v_\lambda(1) - k_{12}u_\lambda(1) + k_{11}p(1)u'_\lambda(1) - k_{12}p(1)v'_\lambda(1)] = 0.
\]
Spectral Zeta Function

The implicit equations for the eigenvalues provide an integral representation of the spectral zeta function valid for \( \Re(s) > 1/2 \) as

\[
\zeta\{\mathcal{S}_\mathcal{C}\}(s) = \frac{1}{2\pi i} \int_{C\{\mathcal{S}_\mathcal{C}\}} d\lambda \lambda^{-2s} \frac{\partial}{\partial \lambda} \ln \left\{ \frac{\Omega(\lambda)}{\Delta(\lambda)} \right\}.
\]

By deforming the contour to the imaginary axis and by changing variables \( i\lambda \rightarrow z \) one obtains the representation

\[
\zeta\{\mathcal{S}_\mathcal{C}\}(s) = \sin \frac{\pi s}{\pi} \int_0^\infty dzz^{-2s} \frac{\partial}{\partial z} \ln \left\{ \frac{\Omega(z)}{\Delta(z)} \right\},
\]

valid for \( 1/2 < \Re(s) < 1 \).

To perform the analytic continuation to the left of the strip \( 1/2 < \Re(s) < 1 \) we subtract and add from the integrand a suitable number of terms from the asymptotic expansion of \( \ln \Omega(z) \) and \( \ln \Delta(z) \) for \( z \rightarrow \infty \).

The desired asymptotic expansion is obtained through a *WKB analysis of the solutions of Sturm-Liouville problem.*

**Remark:**

- For a general one-dimensional Sturm-Liouville differential operator with general self-adjoint boundary conditions, \( \Omega(z) \) and \( \Delta(z) \) are *not known* explicitly.
**WKB Expansion**

In the parameter \( z \), the Sturm-Liouville differential equation reads

\[
-\frac{d}{dx} \left( p(x) \frac{d}{dx} \phi_z(x) \right) + V(x) \phi_z(x) = -z^2 \phi_z(x).
\]

By introducing the auxiliary function

\[
S(x, z) = \frac{\partial}{\partial x} \ln \phi_z(x),
\]

the equation above is equivalent to the following

\[
[p(x)S(x, z)]' = V(x) + z^2 - p(x)S^2(x, z).
\]

As \( z \to \infty \) we assume that \( S(x, z) \) has the asymptotic expansion

\[
S(x, z) \sim zS_{-1}(x) + S_0(x) + \sum_{i=1}^{\infty} \frac{S_i(x)}{z^i}.
\]

Once the asymptotic expansion of \( S(x, z) \) is known, the one for the solution \( \phi_z(x) \) will immediately follow.
WKB Expansion

By substituting the asymptotic form of $S(x, z)$ in the previous non-linear differential equation we obtain

$$S_{-1}^\pm(x) = \pm \frac{1}{\sqrt{p(x)}}$$

$$S_0^\pm(x) = -\frac{1}{2} \frac{d}{dx} \ln \left(p(x) S_{-1}^\pm(x)\right) = -\frac{p'(x)}{4p(x)},$$

$$S_1^\pm(x) = \frac{1}{2p(x) S_{-1}^\pm(x)} \left[V(x) - p(x) \left(S_0^\pm\right)^2(x) - (p(x) S_0^\pm(x))'\right],$$

and for $i \geq 1$

$$S_{i+1}^\pm(x) = -\frac{1}{2p(x) S_{-1}^\pm(x)} \left[(p(x) S_i^\pm(x))' + p(x) \sum_{m=0}^{i} S_m^\pm(x) S_{i-m}^\pm(x)\right].$$

The terms $S_i^+(x)$ and $S_i^-(x)$ provide the exponentially growing and decaying parts of the solution $\varphi_z(x)$ as

$$\varphi_z(x) = A \exp \left\{ \int_0^x S^+(t, z) dt \right\} + B \exp \left\{ \int_0^x S^-(t, z) dt \right\},$$

with $A$ and $B$ uniquely determined by the initial conditions.
Asymptotic Expansion: Separated Boundary Conditions

For separated boundary conditions the implicit equation for the eigenvalues is

\[
\ln \Omega(z) \sim \ln \left[ -A_2 p(0) S^-(0, z) + A_1 \right] + \ln \left[ B_2 p(1) S^+(1, z) + B_1 \right] \\
- \ln \left[ p(0) \left( S^+(0, z) - S^-(0, z) \right) \right] + \int_0^1 S^+(t, z) dt .
\]

By introducing the function \( \delta(x) = 1 \) for \( x = 0 \), and \( \delta(x) = 0 \) for \( x \neq 0 \) one can further expand \( \ln \Omega(z) \) to obtain

\[
\ln \Omega(z) = -\frac{1}{4} \ln p(0)p(1) + [1 - \delta(A_2)] \ln A_2 \sqrt{p(0)} + [1 - \delta(B_2)] \ln B_2 \sqrt{p(1)} \\
+ \delta(A_2) \ln A_1 + \delta(B_2) \ln B_1 - \ln 2z + [2 - \delta(A_2) - \delta(B_2)] \ln z \\
+ z \int_0^1 S^+_n(t) dt + \sum_{i=1}^{\infty} \frac{M_i}{z^i} .
\]

Remark:

- The terms \( M_i, i \geq 1 \), are expressed only in terms of \( p^{(n)}(x) \) and \( V^{(n)}(x) \) with \( n \leq i + 1 \) and their powers.
Asymptotic Expansion: Coupled Boundary Conditions

For coupled boundary conditions the implicit equation for the eigenvalues is

\[
\ln \Delta(z) \sim -\ln \left[p(0) \left(S^+(0, z) - S^-(0, z)\right)\right] + \int_0^1 S^+(t, z)\,dt \\
+ \ln \left[-k_{21} - k_{22}p(0)S^-(0, z) + k_{11}p(1)S^+(1, z) + k_{12}p(1)p(0)S^-(0, z)S^+(1, z)\right]
\]

The large-\(z\) asymptotic behavior depends on whether \(k_{12}\) vanishes or not. Both cases are described by the expression

\[
\ln \Delta(z) = -\frac{1}{4} \ln p(0)p(1) + [1 - \delta(k_{12})] \ln k_{12} \sqrt{p(0)p(1)} \\
+ \delta(k_{12}) \ln \left(k_{22} \sqrt{p(0)} + k_{11} \sqrt{p(1)}\right) \\
+ [2 - \delta(k_{12})] \ln z - \ln 2z + z \int_0^1 S^+_{-1}(t)\,dt + \sum_{i=1}^{\infty} \frac{\mathcal{N}_i}{z^i}.
\]

Remark:
- The terms \(\mathcal{N}_i, i \geq 1\), are expressed *only* in terms of \(p^{(n)}(x)\) and \(V^{(n)}(x)\) with \(n \leq i + 1\) and their powers.
Analytic Continuation of the Spectral Zeta Function

From the integral representation of $\zeta^{\{SC\}}(s)$ we add and subtract $L$ leading terms of the respective asymptotic expansions to obtain

$$\zeta^{\{SC\}}(s) = Z^{\{SC\}}(s) + \sum_{i=-1}^{L} A_i^{\{SC\}}(s),$$

with $Z^{\{SC\}}(s)$ an analytic function for $\Re s > -(L + 1)/2$, and $A_i^{\{SC\}}(s)$ meromorphic functions for $s \in \mathbb{C}$. In particular we have

$$\zeta^S(s) = Z^S(s) + \frac{\sin \pi s}{\pi} \left[ \frac{1 - \delta(A_2) - \delta(B_2)}{2s} + \frac{1}{2s - 1} \int_0^1 S_{-1}^+(t)dt - \sum_{i=1}^{L} i \frac{M_i}{2s + i} \right],$$

$$\zeta^C(s) = Z^C(s) + \frac{\sin \pi s}{\pi} \left[ \frac{1 - \delta(k_{12})}{2s} + \frac{1}{2s - 1} \int_0^1 S_{-1}^+(t)dt - \sum_{i=1}^{L} i \frac{N_i}{2s + i} \right].$$

Remarks:

• $\zeta^S(s)$ and $\zeta^C(s)$ are meromorphic functions of $s \in \mathbb{C}$ with only a simple pole at $s = 1/2$. 
Functional Determinant and Heat Kernel Coefficients

From the analytically continued expression of the spectral zeta function one can compute

- The functional determinant, \( \det(L) = \exp\{-\zeta'(0)\} \).
- The coefficients of the asymptotic expansion of \( \theta(t) = \text{Tr}_{\mathcal{L}} e^{-tL} \).

For the HKC, by using the Mellin transform one has

\[
a_{\frac{1}{2}-s} = \Gamma(s) \text{Res} \zeta(s) , \quad a_{\frac{1}{2}+n} = \frac{(-1)^n}{n!} \zeta(-n).
\]

when \( s = 1/2 \) and \( s = -(2n + 1)/2 \) with \( n \in \mathbb{N}_0 \). In our case we have

\[
a_0^S = a_0^C = \frac{1}{2\sqrt{\pi}} \int_0^1 \frac{dt}{\sqrt{p(t)}},
\]

\[
a_{\frac{1}{2}}^S = \frac{1 - \delta(A_2) - \delta(B_2)}{2}, \quad a_{2m+1}^S = -\frac{1}{(m-1)!} \mathcal{M}_{2m}, \quad a_{n+1}^S = -\frac{2^{2n}n!}{\sqrt{\pi}(2n)!} \mathcal{M}_{2n+1},
\]

\[
a_{\frac{1}{2}}^C = \frac{1 - \delta(k_{12})}{2}, \quad a_{2m+1}^C = -\frac{1}{(m-1)!} \mathcal{N}_{2m}, \quad a_{n+1}^C = -\frac{2^{2n}n!}{\sqrt{\pi}(2n)!} \mathcal{N}_{2n+1},
\]

with \( m \in \mathbb{N}^+ \) and \( n \in \mathbb{N}_0 \).
Further Research

The analysis outlined above represents the foundation for further research:

- Analysis of the Casimir energy and force for a one-dimensional piston modeled by a compact potential with separated or coupled boundary conditions. Study of the behavior of the force as the boundary conditions change.

- Generalize the technique presented here to study spectral functions for Laplace operator on manifolds of the type $I \times N$ or $I \times_f N$ with $N$ being a compact Riemannian manifold, and $I = [a, b] \subset \mathbb{R}$. These results could be applied to the analysis of the Casimir effect for potential pistons with arbitrary cross-section.

- It would be particularly interesting to develop a method similar to the one presented in this paper to obtain the analytic continuation of the spectral zeta function for one-dimensional singular Sturm-Liouville problems:
  - The functions $p(x)$ and $V(x)$ become unbounded in the neighborhood of the endpoints of $I$.
  - The interval $I = \mathbb{R}$ is unbounded and the potential $V(x) \to +\infty$, as $|x| \to \infty$, is confining.