Static Analysis for Regular Expression Exponential Runtime via Substructural Logics

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Abstract

Regular expression matching using backtracking can have exponential runtime, leading to an algorithmic complexity attack known as REDoS in the systems security literature. In this paper, we build on a recently published static analysis that detects whether a given regular expression can have exponential runtime for some inputs. We systematically construct a more accurate analysis by forming powers and products of transition relations and thereby reducing the REDoS problem to reachability. The correctness of the analysis is proved using a substructural calculus of search trees, where the branching of the tree causing exponential blowup is characterized as a form of non-linearity.

1 Introduction

Regular expressions are everywhere. Yet the backtracking virtual machines that are used to match them (in Java, .NET and other frameworks) are very different from the DFA construction used in compiling. Whereas DFAs run in linear time but may be expensive to construct, backtracking matchers have low initial cost, but may have exponential runtime for some inputs [7]. This is a problem when such matchers may be exposed to malicious input, say over a network, as an attacker could craft an input in order for the matcher to take exponential time. This problem is known as REDoS, short for Regular Expression Denial-of-Service.

For a straightforward example of an exponential blowup, consider the following regular expression:

\((a \mid b \mid ab)^* c\)
Matching this expression against input strings of the form \((ab)^n\) leads the Java virtual machine to a halt for very moderate values of \(n\) (\(~ 50\)) on a contemporary computer. Certain other backtracking matchers like the PCRE library and the matcher available in the .NET platform seem to handle this particular example well. However, the ad-hoc nature of the workarounds implemented in these frameworks are easily exposed with a slightly complicated expression / input combination:

\[(a \mid b \mid ab)^*bc\]

This expression, when matched against input strings of the form \((ab)^nac\), leads to exponential blowups on all the three matchers mentioned.

The starting point for the present paper is a recently published static analysis for regular expression denial of service vulnerabilities [20]. The analysis presented there focuses on finding so-called pumpable strings that can be used to construct malicious inputs leading to exponential blowups. While finding pumpable strings is necessary for a REDoS vulnerability to exist, they are not sufficient. The analysis also needs to construct a suitable prefix and a suffix that will be used in that attack string. Heuristics were used to find such strings. While these worked in many simple cases, they may fail in more complex ones, making the analysis unsound. Furthermore, the theoretical basis of the analysis was not developed beyond pointing out an analogy to Brzozowski’s notion of regular expression derivatives [3]. The present paper revisits the problem of REDoS analysis using tools from programming languages theory. In doing so, we repair the unsoundness of the previous analysis and aim for compositionality.

There are surprisingly tricky interactions between different parts of a regular expression that influence whether there is an exponential blowup. So our analysis presented here is considerably more complex than the earlier one. One of the challenges in designing the analysis is that it should compute strings from which a malicious input can be constructed if a REDoS vulnerability is found; but there can be infinitely many such strings, so that any attempt to generate all of them would lead to non-termination. The solution is to focus on states of the automation constructed from the regular expression, since the number of states is finite. Strings matter only in so far as they allow different states to be reached. Even so, there are subtleties that the analysis needs to keep track of: sometimes it matters whether some state can be reached, sometimes all reachable states need to be considered; sometimes the order of states matters, sometimes it does not. To manage this complexity and arrive at a clean design, we think of the analysis as a
A program analysis for REDoS is designed and implemented that corrects limitations of an earlier more naive analysis.

• The soundness and the completeness of the analysis are proved.
• The proof technique uses a substructural logic of search trees that captures the non-linearity leading to exponential blowup.

• The analysis has been implemented in OCaml and is available as an electronic appendix to this paper at http://www.cs.bham.ac.uk/~hxt/research/rxxr2/.

Outline of the paper

Section 2 presents some required background on regular expression matching in a form that will be convenient for our purpose. We then define the three phases (prefix, pumping, and suffix construction) of our REDoS analysis in Section 3 and validate it on some examples in Section 4. Section 5 and Section 6 prove the soundness and the completeness of the analysis using a substructural calculus of search trees. Section 7 presents a brief overview of the OCaml implementation of the analysis and the practical performance of our tool. We conclude with a discussion of related work in Section 8 and directions for further work in Section 9.

2 Basic constructs

This section presents some background material that will be needed for the analysis, such as non-deterministic automata. Figure 1 gives an overview of
notation. We assume that the regular expression has been converted into an automaton following one of the standard constructions.

2.1 Backtracking and the ordered NFA

The usual text-book definitions of NFAs do not impose any ordering on the transition function. For an example, a traditional NFA for the regular expression \( a(bc | bd) \) would not prioritize any of the two transitions available for character \( b \) over the other. Since backtracking matchers follow a greedy left-to-right evaluation of alternations, the alternation operator effectively becomes non-commutative in their semantics for regular expressions. Capturing this aspect in the analysis requires a specialized definition of NFAs.

If we are only concerned about acceptance, Kleene star is idempotent and alternation is commutative. If we are interested in exponential runtime, they are not. The non-commutativity of alternation is not that surprising in terms of programming language semantics, as Boolean operators like `&&` in C or `andalso` in ML have a similar semantics: first the left alternative is evaluated, and if that does not evaluate to true, the right alternative is evaluated. Since in our tool the NFA is constructed from the syntax tree, the order is already available in the data structures. The children of a NFA node have a left-to-right ordering.

**Definition 2.1 (Ordered NFA)** An ordered NFA \( \mathcal{N} \) consists of a set of states, an initial state \( p_0 \), a set of accepting states \( \mathsf{Acc} \) and for each input symbol \( a \) a transition function from states to sequences of states. We write this function as

\[
a : p \mapsto q_1 \ldots q_n
\]

For each input symbol \( a \) and current NFA state \( p \), we have a sequence of successor states \( q_i \). The order is significant, as it determines the order of backtracking.

In the textbook definition of an \( \varepsilon \)-free NFA, the NFA has a transition function \( \delta \) of type

\[
\delta : (Q \times \Sigma) \to 2^Q
\]

where \( Q \) is the set of states and \( \Sigma \) the set of input symbols. Here we have imposed an order on the sets in the image of the function, replacing \( 2^Q \) by \( Q^* \), curried the function, and swapped the order of \( Q \) and \( \Sigma \).

\[
\Sigma \to (Q \to Q^*)
\]

\[
a \mapsto p \mapsto q_1 \ldots q_n
\]
**Definition 2.2** The nondeterministic transition relation of the NFA is given by the following inference:

\[ a : p \rightarrow q_1 \ldots q_n \]

\[ a : p \rightarrow q_i \]

Note however, that we cannot recover the ordering of the successor states \( q_i \) from the non-deterministic transition relation. In this regard, the NFA on which the matcher is based has a little extra structure compared to the standard definition of NFA in automata theory. If we know that

\[ a : p \rightarrow q_1 \text{ and } a : p \rightarrow q_2 \]

we cannot decide whether the ordered transition is

\[ a : p \rightarrow q_1 q_2 \text{ or } a : p \rightarrow q_2 q_1 \]

To complement the ordered NFA, we introduce two kinds of data structures: ordered multistates \( \beta \) are finite sequences of NFA states \( p \), where the order is significant. Multistates \( \Phi \) represent sets of NFA states, so they can be represented as lists, but are identified up to reordering. Each ordered multistate \( \beta \) can be turned into a multistate given by the set of its elements. We write this set as \( \text{Set}(\beta) \). If

\[ \beta = p_1 \ldots p_n \]

then

\[ \text{Set}(\beta) = \{ p_1, \ldots, p_n \} \]

The difference between \( \beta \) and \( \text{Set}(\beta) \) may appear small, but the notion of equality for sets is less fine-grained than for sequences, which has an impact on the search space that the analysis has to explore.

### 2.2 The abstract machines

The analysis assumes exact matching semantics of regular expressions. Given regular expression \( e \) and the input string \( w \), the matcher is required to find a match of the entire string, as opposed to a sub-string. Most practical matchers search for a sub-match by default. However, such behavior can be modeled in exact matching semantics by augmenting the regular expression with match-all constructs at either end of the expression, as in \( \cdot e \cdot \). Practical implementations offer special “anchoring” constructs that allow regular expression authors to enforce exact matching semantics. For an example,
expressions of the form \( ^c e $\) require them to be matched against the entire input string.

While the theoretical formulation of our analysis assumes exact matching semantics (thus avoiding unnecessary clutter), our implementation assumes sub-match semantics, since it is more useful in practice. The translation between the two semantics is quite straightforward.

**Definition 2.3 (Backtracking abstract machine)** Given an ordered NFA, the backtracking machine is defined as follows. We assume an input string \( w \) as given. Machine transitions may depend on \( w \), but it does not change during transitions, so that we do not explicitly list it as part of the machine state. The input symbol at position \( j \) in \( w \) is written as \( w[j] \).

- States of the backtracking machine are finite sequences of the form
  \[ \sigma = (p_1, j_1) \ldots (p_n, j_n) \]
  where each of the \( p_i \) is an NFA state and each of the \( j_i \) is an index into the current input string.

- The initial state of the machine is the sequence of length 1 containing a pair of the starting state of the NFA and the string index 0:
  \[ (p_0, 0) \]

- The machine has matching transitions, which are inferred from the transition function of the ordered NFA as follows:
  \[
  w[j] = a \quad a : p \mapsto q_1 \ldots q_n \\
  w \models (p, j) \sigma \leadsto (q_1, j + 1) \ldots (q_n, j + 1) \sigma
  \]

- The machine has failing transition, of the form
  \[ (p, j) \sigma \leadsto \sigma \]
  where \( w[j] \neq a \) or \( j \) is the length of \( w \) and \( p \not\in Acc \).

- Accepting states are of the form:
  \[ w \models (p, j) \sigma \]
  where \( p \in Acc \) and \( j \) is the length of \( w \).
• Transition sequences in \( n \) steps are written as \( \sigma^n \) and inferred using the following rules:

\[
\begin{align*}
\frac{w \vdash \sigma \rightsquigarrow \sigma'}{w \vdash \sigma \rightsquigarrow 1 \rightsquigarrow \sigma'} & \quad \frac{w \vdash \sigma_1 \rightsquigarrow \sigma_2 \quad w \vdash \sigma_2 \rightsquigarrow \sigma_3}{w \vdash \sigma_1 \rightsquigarrow n \rightsquigarrow \sigma_3}
\end{align*}
\]

We write \( w \vdash \sigma_1 \rightsquigarrow^* \sigma_2 \) for \( \exists n. w \vdash (\sigma_1 \rightsquigarrow n \rightsquigarrow \sigma_2) \).

• Final states are either accepting or the empty sequence.

The state of the backtracking machine is a stack that implements failure continuations. When the state is of the form \((p, j)\sigma\), the machine is currently trying to match the symbol at position \( j \) in state \( p \). Should this match fail, it will pop the stack and proceed with the failure continuation \( \sigma \).

The backtracking machine definition leaves a lot of leeway to the implementation. Implementation details are abstracted in the ordered transition relation. The most important choice in the definition is that the machine performs a depth-first traversal of the search tree. In principle, a backtracking matcher could also use breadth-first search. In that case, our REDoS analysis would not be applicable, and such matchers may avoid exponential run-time. However, the space requirements of breadth-first search are arguably prohibitive. A more credible alternative to backtracking matchers is Thompson’s matcher [29, 7, 8], which is immune to REDoS.

**Definition 2.4 (Lockstep abstract machine)** The lockstep abstract machine, based on Thompson’s matcher [29], is defined as follows.

• The states of the lockstep matcher are of the form

\[
(\Phi, j)
\]

where \( \Phi \) is a set of NFA states and \( j \) is an index into the input string \( w \).

• The initial state is

\[
(\{p_0\}, 1)
\]

• The matching transition are inferred as follows:

\[
\begin{align*}
w[j] = a \quad a : p_1 \mapsto \beta_1 \quad \ldots \quad a : p_n \mapsto \beta_n & \quad \frac{w[j] \rightsquigarrow (\{p_1, \ldots, p_n\}, j) \rightsquigarrow (\text{Set}(\beta_1) \cup \ldots \cup \text{Set}(\beta_n), j + 1)}{(\{p_1, \ldots, p_n\}, j) \rightsquigarrow (\text{Set}(\beta_1) \cup \ldots \cup \text{Set}(\beta_n), j + 1)}
\end{align*}
\]
• An accepting state is of the form

\((\Phi, j)\)

where \(j\) is the length of \(w\) and \(\Phi \cap \text{Acc} \neq \emptyset\).

After each step, redundancy elimination is performed by taking sets rather than sequences.

The lockstep machine will not be used in the rest of the paper. It is shown here because it is instructive to see how the state space explosion that may lead to redos in the backtracking machine can be avoided.

### 2.3 The power DFA construction

Based on a construction that is standard in automata theory and compiler construction, for each NFA there is a DFA. The set of states of this DFA is the powerset of the set of states of the NFA. We refer to such sets of NFA states as multistates.

#### Definition 2.5 (Power DFA)

Given an NFA, its power DFA is constructed as follows:

• The states of the power DFA are sets \(\Phi\) of NFA states.

• The transition relation \(\Rightarrow\) is defined as

\[ a : \Phi_1 \Rightarrow \Phi_2 \]

if and only if

\[ \Phi_2 = \{p_2 \mid \exists p_1 \in \Phi_1. a : p_1 \rightarrow p_2\} \]

• The initial state of the power DFA is the singleton set \(\{p_0\}\).

• The accepting states of the power DFA are those sets \(\Phi\) for which \(\Phi \cap \text{Acc} \neq \emptyset\).

#### Definition 2.6

The transition function of the power DFA is extended from strings \(w\) to sets of strings \(W\) using the following rule

\[ W : \Phi_1 \Rightarrow \Phi_2 \]

\[ \Phi_2 = \{p_2 \mid \exists p_1 \in \Phi_1. \exists w \in W. w : p_1 \rightarrow p_2\} \]
Intuitively, we regard $W$ as the set of realizers that take us from $\Phi_1$ to $\Phi_2$. Note that it is not only the case that (elements of) $W$ will take us from (elements of) $\Phi_1$ to (elements of) $\Phi_2$. Moreover, everything in $\Phi_2$ arises this way from $\Phi_1$ and $W$. In that sense, a judgement $W : \Phi_1 \Rightarrow \Phi_2$ is stronger than realizability or pre- and post-conditions. The fact that $\Phi_2$ is uniquely determined by $\Phi_1$ and $W$ is useful for the analysis.

3 The REDoS analysis

The REDoS analysis builds on the idea of non-deterministic Kleene expressions [20]. For an example, consider matching the regular expression:

$$(a \mid b \mid ab)^*$$

against the input string $ab$. A match could be found by taking either of the two different paths through the corresponding NFA. If we repeat this string to form $abab$, now there are four different paths through the NFA; this process quickly builds up to an exponential amount of paths through the NFA as the pumpable string $ab$ is repeated.

A matcher based on DFAs would not face a difficulty in dealing with such expressions since the DFA construction eliminates such redundant paths. However, these expressions can be fatal for backtracking matchers based on NFAs, as their operation depends on performing a depth-first traversal of the entire search space.

For a given regular expression of the form $e_1e_2^*e_3$, the analysis presented in [20] attempts to derive an attack string of the form:

$$xy^n z$$

The presence of a pumpable string $y$ signals the analyser that a corresponding prefix $x$ and a suffix $z$ need to be derived in order to form the final attack string configuration. The requirements on the different segments of the attack string are as follows:

$$
\begin{align*}
x & : \ x \in L(e_1) \quad (1) \\
y & : \ y \in L(e_2^*) \quad (\text{with } b > 1 \text{ paths}) \quad (2) \\
z & : \ xy^n z \notin L(e_1e_2^*e_3) \quad (3)
\end{align*}
$$

Intuitively, the prefix $x$ leads a backtracking matcher to a point where it has to match the (vulnerable) Kleene expression $e_2^*$. At this point the matcher is presented with $n$ ($n > 0$) copies of the pumpable string $y$, increasing the
search space of the matcher to the order of $b^n$. At the end of each of the search attempts (paths through the NFA), the suffix $z$ causes the matcher to backtrack, forcing an exploration of the entire search space.

While the analyser seem to be capable of efficiently locating REDoS vulnerabilities in practice [20], we found that it also reports a considerable amount of false positives and in certain cases, generates attack strings that does not result in an exponential runtime (despite the existence of other valid attack strings). These limitations of the analyser can be attributed to certain edge cases that the original specification of the analyser has failed to capture. For an example, consider the following (idealized) expression:

\[ .* \mid (a \mid b \mid ab)^*c \]

Here the analyser generates the attack string configuration: $x = \varepsilon$, $y = ab$ and $z = \varepsilon$. It is quite clear that the expression at hand is impossible to be attacked, since a backtracking matcher would locate a match using the left alternation without having to invoke the (vulnerable) expression in the right. However, if we swap the two branches of the alternation as in:

\[ (a \mid b \mid ab)^*c \mid .* \]

Then the above attack configuration would suffice. This observation suggests that the original analysis has failed to capture the non-commutativity of the alternation operator, leading it to report false positives as a result. For a slightly more complicated example, consider the expression:

\[ (a \mid b)^* \mid c^*(a \mid b \mid ab)^*d \]

The analyser output reads: $x = \varepsilon$, $y = ab$ and $z = \varepsilon$. Again, it’s clear that any string of the form $(ab)^n$ would be matched from the left alternation rather than the right. However, a valid attack string configuration does exist in this case: $x = c$, $y = ab$ and $z = \varepsilon$. Intuitively, the original analyser has failed to see that it can avoid the left branch of the alternation by unrolling the expression $c^*$ as part of the prefix.

In summary, the ad-hoc specification of the analyser presented in [20] is not sound (as acknowledged therein). In the following sections we build on these ideas and derive a sound and complete version of the analyser.

### 3.1 The phases of the REDoS analysis

Overall, the REDoS analysis of a node $p_\ell$ (loop node) consists of three phases. The phases all work by incrementally exploring a transition relation.
Figure 2: Branching search tree with left context for $x y y z$

These relations are the power DFA transition relation $\Rightarrow$ and an ordered variant $\uparrow^\ell$.

The REDoS analysis attempts to construct a REDoS prefix $x$, a pumpable string $y$ and a REDoS suffix $z$, such that:

$$
\begin{align*}
    x & : p_0 \uparrow^\ell (\beta p_\ell \beta') \\
    y_1 & : \Phi_x \Rightarrow \Phi_{y_1} \text{ where } \Phi_x = \text{Set}(\beta p_\ell) \\
    a & : \Phi_{y_1} \Rightarrow \Phi_{y_1a} \\
    y_2 & : \Phi_{y_1a} \Rightarrow \Phi_{y_2} \text{ where } \Phi_{y_2} \subseteq \Phi_x \\
    z & : \Phi_{y_2} \Rightarrow \Phi_{\text{fail}} \text{ where } \Phi_{\text{fail}} \cap \text{Acc} = \emptyset
\end{align*}
$$

### 3.2 REDoS prefix analysis

The analysis needs to find a string that causes the matcher to reach $p_\ell$. However, due to the nondeterminism of the underlying NFA, it is not enough to check reachability. The same string $x$ could also lead to some other states before $p_\ell$ is reached by the matcher. If one of these states could lead to acceptance, the matcher will terminate successfully, and $p_\ell$ will never be reached. In this case, there is no vulnerability, regardless of any exponential blowup in the subtree under $p_\ell$. See Figure 2.

We define an operation $\gg$ for removing all but the leftmost occurrences in sequences:

$$
\beta_1 p \beta_2 p \beta_3 \gg \beta_1 p \beta_2 \beta_3
$$
We write $\beta_1 \gg \beta_2$ if the reduction is maximal, that is, $\beta_1 \gg \beta_2$ and there is no $\beta_3$ such that $\beta_2 \gg \beta_3$ and $\beta_2 \neq \beta_3$. Note that each $\gg$ step reduces the length of the sequence, ensuring termination. Moreover, in each reduced sequence each $p$ can appear at most once, so there are only finitely many sequences that will be reached in the REDoS prefix analysis. Therefore, the prefix analysis terminates.

**Definition 3.1** Let $p_{\ell}$ be the NFA state we are currently analyzing. The transition relation for ordered multistates is defined as follows:

\[
\beta_1 = (p_1 \ldots p_n) \quad a : p_1 \mapsto \gamma_i \quad (\gamma_1 \ldots \gamma_n) \gg \beta_2
\]

\[a : \beta_1 \overset{\ell}{\rightarrow} \beta_2\]

The relation is extended to strings:

\[
w : \beta_1 \overset{\ell}{\rightarrow} \beta_2 \quad a : \beta_2 \overset{\ell}{\rightarrow} \beta_3
\]

\[\beta w', (w', \beta_2) \in \mathcal{R}
\]

\[\beta \overset{\ell}{\rightarrow} \beta\]

The REDoS prefix analysis computes all ordered multistates $\beta$ reachable from $p_0$, together with a realizer $w$, using the following rules:

\[
\begin{align*}
(w, \beta_1) &\in \mathcal{R} \\
 a : \beta_1 \overset{\ell}{\rightarrow} \beta_2 &\quad \beta w', (w', \beta_2) \in \mathcal{R} \\
(w a, \beta_2) &\in \mathcal{R}
\end{align*}
\]

In the implementation, we keep a set $\mathcal{R}$. It is initialized to $(\varepsilon, p_0)$. We then repeatedly check if there is a $(w, \beta_1)$ in the set such that for some $a$ there is a transition $a : \beta_1 \overset{\ell}{\rightarrow} \beta_2$. If there is, we add $(w a, \beta_2)$ to $\mathcal{R}$ and repeat the process. We terminate when no new $\beta_2$ has been found in the last iteration. Finally, the analysis isolates $(w, \beta)$ pairs of the form $(x, \beta p_\ell \beta')$ and takes $\Phi_x$ as $\text{Set}(\beta p_\ell)$ for each such pair.

### 3.3 Pumping analysis

**Definition 3.2** A branch point is a tuple

\[
(p_N, a, \{p_{N1}, p_{N2}\})
\]

such that $p_{N1} \neq p_{N2}$, $a : p_N \rightarrow p_{N1}$ and $a : p_N \rightarrow p_{N2}$.

For example, if $p_N$ has three successor nodes $p_1$, $p_2$ and $p_3$ for the same input symbol $a$, there are three different branch points:

\[
(p_N, a, \{p_1, p_2\})
\]

\[
(p_N, a, \{p_1, p_3\})
\]

\[
(p_N, a, \{p_2, p_3\})
\]

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There can be only finitely many non-deterministic nodes in the given NFA. For each of them, we need to solve a reachability problem.

The pumping analysis can be visualized with the diagram in Figure 3. The analysis aims to find two different paths leading from \( p \) to itself. Such paths must at some point include a nondeterministic node \( p_N \) that has at least two transitions to different nodes \( p_{N1} \) and \( p_{N2} \) for the same symbol \( a \). For such a node to lie on a path from \( p \) to itself, there must be some path labeled \( y_1 \) leading from \( p \) to \( p_N \), and moreover there must be paths from the two child nodes \( p_{N1} \) and \( p_{N2} \) leading back to \( p \), such that both these paths have the label \( y_2 \). The left side of Figure 3 depicts this situation.

So far we have only considered what states may be reached. Due to the nondeterminism of the transition relation \( a : p \rightarrow q \), there may be other states that can be reached for the same strings \( y_1 \) and \( y_2 \). Therefore, we also need to perform a must analysis that keeps track of all states reachable via the strings we construct. This analysis uses the transition relation \( \Rightarrow \) of the power DFA between sets of NFA states. In Figure 3, it is shown on the right-hand side.

Intuitively, we run the two transition relations in parallel on the same input string. More formally, this involves constructing a product of two relations. Recall that for any two relations \( r \) and \( r' \) there is a product relation \( r \times r' \) defined by

\[
((p_1, p_2), (p_1', p_2')) \in (r \times r') \text{ if and only if both} \\
(p_1, p_2) \in r \text{ and} \\
(p_1', p_2') \in r'
\]

Before we reach the branching point, we run the relations \( \rightarrow \) and \( \Rightarrow \) in parallel. After the nondeterministic node \( p_N \) has produced two different successors, we need to run two copies of \( \rightarrow \) in parallel with \( \Rightarrow \). One may visualize this situation by reading the diagram in Figure 3 horizontally: above the splitting at \( p_N \), there are two arrows in parallel for \( y_1 \), whereas below that node, there are three arrows in parallel for \( a \) and \( y_2 \).

The twofold transition relation \( \rightarrow_2 \) for running \( \rightarrow \) in parallel with \( \Rightarrow \) is given by the rules in Figure 4. Analogously, the threefold product transition relation \( \rightarrow_3 \) for running two copies of \( \rightarrow \) in parallel with \( \Rightarrow \) is given by the rules in Figure 5.

In summary, the pumping analysis consists of two phases:

1. Given \( p_\ell \) and \( \Phi_x \), the analysis searches for a realizer \( y_1 \) for reaching some nondeterministic node \( p_N \):

\[
y_1 : (p_\ell, \Phi_x) \rightarrow_2 (p_N, \Phi y_1)
\]
Figure 3: Pumping analysis construction of $y_1 a y_2$: “may” on the left using $\rightarrow$, and “must” on the right using $\Rightarrow$

$$w : (p_1, \Phi_1) \rightarrow_2 (p_2, \Phi_2) \quad b : p_2 \rightarrow p_3$$

$$b : \Phi_2 \Rightarrow \Phi_3$$

$$(w b) : (p_1, \Phi_1) \rightarrow_2 (p_3, \Phi_3)$$

$$\varepsilon : (p, \Phi) \rightarrow_2 (p, \Phi)$$

Figure 4: The twofold product transition relation $\rightarrow_2$

2. Given the successor nodes $p_{N1}$ and $p_{N2}$ of some $p_N$ node, the analysis searches for a realizer $y_2$ for reaching $p_{\ell}$:

$$y_2 : (p_{N1}, p_{N2}, \Phi_{y1a}) \rightarrow_3 (p_{\ell}, p_{\ell}, \Phi_{y2})$$

Moreover, the analysis checks that the constructed state $\Phi_{y2}$ satisfies the inclusion:

$$\Phi_{y2} \subseteq \Phi_x$$

If both phases of the analysis succeed, the string $y_1 a y_2$ is returned as the pumpable string, together with the state $\Phi_{y2}$. 

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Figure 5: The threefold product transition relation $\rightarrow_3$

3.4 REDoS suffix analysis

For each $\Phi_{y2}$ constructed by the pumping analysis, the REDoS failure analysis computes all multistates $\Phi_{\text{fail}}$ such that there is a $z$ with

$$z : \Phi_{y2} \Rightarrow \Phi_{\text{fail}} \land \Phi_{\text{fail}} \cap \text{Acc} = \emptyset$$

Intuitively, $z$ fails all the states in $\Phi_{y2}$ by taking them to $\Phi_{\text{fail}}$, which does not contain any accepting states.

4 Test cases for the REDoS analysis

In order to demonstrate the behavior of our improved analyser, here we present examples that exercise the most important aspects of its operation.

4.1 Non commutativity of alternation

This aspect of the analysis can be illustrated with the following two example expressions:

$$.* | (a \mid b \mid ab)^*c$$

$$(a \mid b \mid ab)^*c \mid .*$$

Even though the two expressions correspond to the same language, only the second expression yields a successful attack. In the first expression, all the multi-states starting from $\Phi_{x1}$ ($\text{Set}(\beta p_e)$) consist of a state corresponding to the expression (.*), which implies that this expression is capable of consuming any input string thrown at it without invoking the vulnerable Kleene expression. On the other hand, $\Phi_{x1}$ calculated for the second expression
lacks a state corresponding to $(\cdot^*)$, leading to the following attack string configuration:

\[ x = \varepsilon \quad y = ab \quad z = \varepsilon \]

### 4.2 Prefix construction

Prefix construction plays one of the most crucial roles in finding an attack string. In the following example, only a certain prefix leads to a successful attack string derivation:

\[ c.\ast | (c \mid d)(a \mid b \mid ab)^*e \]

Notice that a prefix $c$ would trigger the $(\cdot^*)$ on the left due to the left-biased treatment of alternation in backtracking matchers. The prefix $d$ on the other hand forces the matcher out of this possibility. The difference between these two prefixes is captured in two different values of $(x, \Phi_x)$:

\[
(c, \{p_1, p_2\}) \quad (d, \{p_2\})
\]

Where

\[ p_1 \models \cdot^* \text{ and } p_2 \models (a \mid b \mid ab)^*e \]

Only the latter of these two leads to a successful attack string:

\[ x = d \quad y = ab \quad z = \varepsilon \]

Prefix construction may also lead to loop unrolling when necessary. For an example, consider the following regex:

\[ (a \mid b).\ast | c^*(a \mid ab \mid b)^*d \]

Without the unrolling of the Kleene expression $c^*$, any pumpable string intended for the vulnerable Kleene expression will be consumed by the alternation on the left. The analyser captures this situation again as two different values of $(x, \Phi_x)$, one for $x = c$ and the other for either $x = a$ or $x = b$. Only the former value leads to a successful attack string:

\[ x = c \quad y = ab \quad z = \varepsilon \]

The amount of loop unrolling is limited by the finite-ness of the $\Phi_x$ values. In the following example, the loop $c^*$ needs to be unrolled twice:

\[ (c \mid a \mid b)(a \mid b).\ast | c^*(a \mid b \mid ab)^*d \]

Here, unrolling $c^*$ 0 - 2 times leads to three distinct values of $\Phi_x$ due to the different matching states on the left alternation. Only one of those unrollings leads to a successful attack string:

\[ x = cc \quad y = ab \quad z = \varepsilon \]
4.3 Pumpable construction

As is the case with prefixes, the existence of an attack string may depend on the construction of an appropriate pumpable string. For an example, consider the following regex:

\[(a | a | b | b)^*(a.* | c)\]

Here the pumpable string \(a\) does not yield an attack string since it also triggers the \((.*\)) continuation. On the other hand, the pumpable string \(b\) avoids this situation and leads to the following attack string configuration:

\[x = \varepsilon \quad y = b \quad z = \varepsilon\]

Similar to the prefix analysis, pumpable analysis utilises \((y, \Phi_y)\) values to select between pumpable strings.

In some cases, the pumpable construction overlaps with prefix construction. In the example below, an attack string may be composed in two different ways:

\[d.^*|((c | d)(a | a))^*b\]

Here, choosing \(ca\) as the pumpable string leads to a successful attack string derivation:

\[x = \varepsilon \quad y = ca \quad z = \varepsilon\]

However, it is also possible to form an attack string with the following configuration:

\[x = ca \quad y = da \quad z = \varepsilon\]

The important point here is that the attack string must begin with a \(c\) instead of a \(d\) in order to avoid the obvious match on the left. The analyser is capable of finding both the configurations that meet this requirement.

Pumpable construction may also lead to loop unrolling when necessary, as demonstrated by the following example:

\[a.^*|(c^*a(b | b))^*d\]

Without unrolling the inner loop \(c^*\), the pumpable string \(ab\) would trigger the alternation on the left. A successful attack string requires the unrolling of this inner loop, as in the following configuration:

\[x = \varepsilon \quad y = cab \quad z = \varepsilon\]
As with the previous example, the unrolling of the inner loop $c^*$ may be performed as part of the prefix construction, leading to the following alternate attack string configuration:

$$x = cab \quad y = ab \quad z = \varepsilon$$

The latter configuration may be considered more desirable in that it makes the pumpable string shorter, leading to much smaller attack strings.

5 Soundness of the analysis

The backtracking machine performs a depth-first search of a search tree. Proofs about runs of the machine are thus complicated by the fact that the construction of the tree and its traversal are conflated. To make reasoning more compositional, we define a substructural calculus for constructing search trees. Machine runs correspond to paths from roots to leaves in these trees.

5.1 Search tree logic

Definition 5.1 (Search tree logic) The search tree logic has judgements of the form

$$w : \beta_1 \triangle \beta_2$$

where $w$ is an input string, and both $\beta_1$ and $\beta_2$ are sequences of NFA states. The inference rules are given in Figure 6.

Intuitively, the judgement

$$w : \beta_1 \triangle \beta_2$$

means that there is a horizontal slice of the search tree, such that the nodes at the top form the sequence $\beta_1$, the nodes at the bottom form the sequence $\beta_2$, and all paths have the same sequence of labels, forming $w$: 
Each \( w \) represents an NFA run \( w : p_1 \rightarrow p_2 \) for some \( p_1 \) that occurs in \( \beta_1 \) and some \( p_2 \) that occurs in \( \beta_2 \). The string \( w \) labels the sides of the trapezoid, since that determines the compatible boundary for parallel composition. Again we may like to think of \( w \) as a proof of reachability. Here the reachability is not in the NFA, but in the matcher based on it.

The trapezoid can be stacked on top of each other if they share a common \( \beta \) at the boundary. They can be place side-by-side if they have the same \( w \) on the inside:

\[
\begin{align*}
\frac{a : p \rightarrow \beta}{a : p \triangle \beta} \quad & \quad (\text{TRANS1}) \quad \frac{\bar{\beta}a : p \rightarrow \beta}{a : p \triangle \varepsilon} \quad (\text{TRANS2}) \\
\frac{w_1 : \beta_1 \triangle \beta_2 \quad w_2 : \beta_2 \triangle \beta_3}{(w_1 w_2) : \beta_1 \triangle \beta_3} \quad & \quad (\text{SEQCOMP}) \\
\frac{\varepsilon : \beta \triangle \beta}{(\varepsilon \text{SEQ})} \\
\frac{w : \beta_1 \triangle \beta_2 \quad w : \beta'_1 \triangle \beta'_2}{w : (\beta_1 \beta'_1) \triangle (\beta_2 \beta'_2)} \quad & \quad (\text{PARCOMP}) \\
\frac{w : \varepsilon \triangle \varepsilon}{(\varepsilon \text{PAR})}
\end{align*}
\]

Figure 6: Search tree logic

5.2 Pumpable implies exponential tree growth

We use the search tree logic to construct a tree by closely following the phases of our REDoS analysis. The exponential growth of the search tree
in response to pumping is easiest to see when thinking of horizontal slices across the search tree for each pumping of $y$. The machine computes a diagonal cut across the search tree as it moves towards the left corner. The analysis constructs horizontal cuts with all states at the same depth. It is sufficient to show that the width of the search tree grows exponentially. The width is easier to formalize than the size.

We need a series of technical lemmas connecting different transition relations.

**Lemma 5.2** The following rule is admissible:

$$ w : \Phi_1 \Rightarrow \Phi_2 \quad w : \Phi'_1 \Rightarrow \Phi'_2 \\
\quad \Rightarrow w : (\Phi_1 \cup \Phi'_1) \Rightarrow (\Phi_2 \cup \Phi'_2) $$

**Lemma 5.3** ($\Rightarrow \triangle$ simulation) If $w : \Phi_1 \Rightarrow \Phi_2$, $w : \beta_1 \triangle \beta_2$ and $\Phi_1 = \text{Set}(\beta_1)$, then $\Phi_2 = \text{Set}(\beta_2)$.

**Lemma 5.4** ($\uparrow_\ell \triangle$ simulation) If $w : \beta_1 \uparrow_\ell \beta_2$, $w : \beta'_1 \triangle \beta'_2$ and $\beta'_1 \gg \beta_1$, then $\beta'_2 \gg \beta_2$.

**Lemma 5.5** Let

$$ w : p \rightarrow q $$

Then there are sequences of states $\beta_1$ and $\beta_2$ such that

$$ w : p \triangle \beta_1 q \beta_2 $$

**Lemma 5.6** (Prefix construction soundness) Let $x$ be constructed by the prefix analysis as:

$$ x : p_0 \uparrow_\ell \beta p_\ell \beta' $$

Then there exists state sequences $\beta_L$ and $\beta_R$ such that

$$ x : p_0 \triangle \beta_L p_\ell \beta_R $$

where $\text{Set}(\beta_L) = \text{Set}(\beta)$.  

**Proof** Follows from Lemma 5.4. \qed

**Lemma 5.7** (Pumpable realizes non-linearity) Let $y$ be pumpable for some node $p_\ell$. Then there exist $\beta_1$, $\beta_2$, $\beta_3$ such that:

$$ y : p_\ell \triangle \beta_1 p_\ell \beta_2 p_\ell \beta_3 $$
**Proof** The pumpable analysis generates a string of the form:

$$ y = y_1 a y_2 $$

Where

$$ y_1 : p_\ell \rightarrow p_N $$
$$ a : p_N \rightarrow (\beta p_{N1} \beta' p_{N2} \beta'') $$
$$ y_2 : p_{N1} \rightarrow p_\ell \quad y_2 : p_{N2} \rightarrow p_\ell $$

Now, Lemma 5.5 leads to the desired result. □

**Lemma 5.8 (Pumping iteration)** Let $y$ be pumpable for the node $p_\ell$. Then for each natural number $k$ there is a sequence of states $\beta_k$ such that

$$ y^k : p_\ell \triangle \beta_k $$

where $|\beta_k| \geq 2^k$.

**Proof** By induction on $k$ and Lemma 5.7. □

**Lemma 5.9 (Stability)** Let $x, y$ be constructed from the prefix analysis and the pumpable analysis respectively. Then the following holds for any natural number $n$:

$$ \Phi_{y^n} \subseteq \Phi_{y^{n-1}} $$

Where $\Phi_{y^0} = \Phi_x$.

**Proof** By induction on $n$. □

**Lemma 5.10 (Suffix construction soundness)** Let $x, y, z$ be constructed from the analysis such that:

$$ xy : \{p_0\} \Rightarrow \Phi_y \quad z : \Phi_y \Rightarrow \Phi_{\text{fail}} \quad \Phi_{\text{fail}} \cap \text{Acc} = \emptyset $$

Then the following holds for any natural number $n$:

$$ xy^n z : \{p_0\} \Rightarrow \Phi' \Rightarrow \Phi' \cap \text{Acc} = \emptyset $$

**Proof** Follows from Lemma 5.9. □
5.3 From search tree to machine runs

The backtracking machine performs a leftmost, depth-first traversal of the search tree. We will need to address situations where some machine state \( \sigma \) consists of nodes in the search tree above some horizontal cut \( \beta \). If the input string is \( x \) of length \( n \), that means that for each state/index pair \((p_j, i_j)\) in \( \sigma \), the substring \( x[i_j..n] \) takes us from \( p \) to some \( \beta_j \), and together these \( \beta_j \) make up \( \beta \).

**Definition 5.11** For a string \( x \) of length \( n \), a machine state \( \sigma \) and a NFA state sequence \( \beta \), the judgement

\[
x : \sigma \Downarrow \beta
\]

holds under the following conditions: there are a natural number \( m \), natural numbers \( i_1, \ldots, i_m \), states \( p_1, \ldots p_m \), state sequences \( \beta_1, \ldots, \beta_m \) such that:

\[
\sigma = (p_1, i_1) \ldots (p_m, i_m)
\]

\[
\beta = \beta_1 \ldots \beta_m
\]

\[
x[i_j..n] : p_j \Delta \beta_j \quad (1 \leq j \leq m)
\]

where \( x[i_j..n] \) stands for the substring of \( x \) from position \( i_j \) to \( n \).

**Lemma 5.12** Let \( w \) be an input string, \( p \) a state and \( \beta \) a multistate such that:

\[w : p \Delta \beta \quad \text{Set}(\beta) \cap \text{Acc} = \emptyset\]

Then for any \( \sigma \), the following machine run exists:

\[w \vdash (p, 0) \sigma \leadsto^* \sigma\]

**Proof** By induction on \( |w| \). \( \square \)

**Lemma 5.13** Let \( w \) be an input string of length \( n \), \( p \) a state such that:

\[w[1 \ldots i] : p \Delta \beta_i \quad 1 \leq i \leq n\]

\[\text{Set} (\beta_n) \cap \text{Acc} = \emptyset\]

Then for any state \( q \) appearing within some \( \beta_i \) and for any machine state \( \sigma \), the following run exists:

\[w \vdash (p, 0)\sigma \leadsto^* (q, i)\sigma \leadsto^* \sigma\]
Lemma 5.14 (Tree traversal) Let $w, z$ be input strings, $\sigma_1$ a machine state and $\beta_1, \beta_2$ multi-states such that:

$$w : \sigma_1 \Downarrow \beta_1$$
$$z : \beta_1 \triangle \beta_2$$

$\text{Set}(\beta_2) \cap \text{Acc} = \emptyset$

Then for any $\sigma_2$ there exists a machine run:

$$wz \Vdash \sigma_1 \sigma_2 \overset{n}{\Rightarrow} \sigma_2$$

Where $n \geq |\beta_1|$.

Proof By induction on $|\sigma_1|$. Note that the base case follows from Lemma 5.13.

For the inductive case, we can apply the following derivation of $\sigma_1$:

$$w : (p, i) \Downarrow \beta_1 \quad (1 \leq i \leq |w|) \quad w : \sigma'_1 \Downarrow \beta_2$$

Which yields:

$$wz \Vdash (p, i)\sigma'_1\sigma_2 \overset{n_1}{\Rightarrow} \sigma'_1\sigma_2 \overset{n_2}{\Rightarrow} \sigma_2$$

Where $n_1 \geq |\beta_1|$ (base case) and $n_2 \geq |\beta_2|$ (I.H). □

In sum, we have shown that the pumped part of the search tree grows exponentially in the size of the input, and that the backtracking machine is forced to traverse all of it.

Theorem 5.15 (Redos analysis soundness) Let the strings $x$, $y$ and $z$ be constructed by the REDoS analysis. Let $k$ be an integer. Then the backtracking machine takes at least $2^k$ steps on the input string $x y^k z$.

6 Completeness of the analysis

In this section we show that for a non-pumpable NFA, the width of any search tree is bounded from above by a polynomial.

Definition 6.1 For an ordered multi-state $\beta$ and a state $p$, we define the function $[\beta]_p$ as follows:

$$[\beta]_p = \begin{cases} 1 + [\beta'']_p & \text{if } \beta = p\beta' \\ [\beta']_p & \text{if } \beta = q\beta' \land q \neq p \end{cases}$$
Definition 6.2 We define the relation $\sim$ on ordered multi-states as follows:

$$\beta \sim \beta' \iff [\beta]_p = [\beta']_p$$

It can be shown that $\sim$ is reflexive, symmetric and transitive.

Definition 6.3 The relation $\simeq$ is defined on ordered multi-states as follows:

$$\beta \simeq \beta' \iff \forall p \in Q . \beta \sim \beta'$$

It can be shown that $\simeq$ is reflexive, symmetric and transitive.

Lemma 6.4 The following properties hold with respect to $\simeq$:

$$\beta \simeq \beta' \Rightarrow \text{Set}(\beta) = \text{Set}(\beta') \land |\beta| = |\beta'|$$

$$\beta_1 \beta_2 \simeq \beta_3 \beta_4 \iff \forall \beta . \beta_1 \beta_2 \simeq \beta_3 \beta_4$$

$$\beta \simeq \beta_1 \beta_2 \land \beta' \simeq \beta'' \Rightarrow \beta \simeq \beta_1 \beta'' \beta_2$$

$$\beta_1 \simeq \beta_1' \land \beta_2 \simeq \beta_2' \Rightarrow \beta_1 \beta_2 \simeq \beta_1' \beta_2'$$

Lemma 6.5 Let $w$ be an input string, $\beta_1$, $\beta_2$ be ordered multi-states such that:

$$\beta_1 \simeq \beta_2 \wedge w : \beta_1 \Rightarrow w : \beta_2$$

Then $\beta_1 \simeq \beta_2$.

Proof The base case ($w = a$) can be established with an inner induction on the length of $\beta_1$. An induction on $|w|$ completes the proof. □

Definition 6.6 Given an NFA, a path $\gamma$ is a sequence of triples

$$(p_0, a_0, q_0) \ldots (p_n, a_n, q_n)$$

where for $1 \leq i < n$ two successive triples are compatible in the sense that $p_{i+1} = q_i$ and for each triple there is a transition $a : p_i \rightarrow q_i$ in the NFA. We write $\text{dom}(\gamma)$ for the first node $p_1$ and $\text{cod}(\gamma)$ for the last node $q_n$ in the path. The sequence of input symbols $a_1 \ldots a_n$ along the path is written as $\gamma$ and called the label of the path.

Definition 6.7 Let $\gamma$ be a path:

$$(p_0, a_0, p_1) \ldots (p_{n-1}, a_{n-1}, p_n)$$

The set of nodes $\{p_0, \ldots, p_n\}$ along $\gamma$ is written as $\text{nodes}(\gamma)$. 

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Lemma 6.8  Given a tree judgement:

\[ w : p \triangle \beta \]

For any state \( q \) such that \( \beta = \beta_1 q \beta_2 \), there exists a path \( \gamma \) with:

\[
\text{dom}(\gamma) = p \quad \text{cod}(\gamma) = q \quad \mathcal{T} = w
\]

Definition 6.9  We write

\[ w : p \Rightarrow q \]

for \( \exists p_1, p_2, w_1, w_2 \) such that

\[
p_1 \neq p_2 \quad w = w_1 w_2 \\
w_1 : p \rightarrow p_1 \quad w_1 : p \rightarrow p_2 \\
w_2 : p_1 \rightarrow q \quad w_2 : p_2 \rightarrow q
\]

6.1 Polynomial bound

Definition 6.10  Let \( \gamma \) be a path. We define the sets \( S(\gamma) \) and \( F(\gamma) \) as follows:

\[
S(\gamma) = \{ p \mid \exists \gamma_1, \gamma_2 \cdot \gamma = \gamma_1 \gamma_2 \\
\land \mathcal{T}_2 : \text{dom}(\gamma_2) \Rightarrow \text{cod}(\gamma_2) \land p = \text{dom}(\gamma_2) \}\n\]

\[
F(\gamma) = Q \setminus S(\gamma)
\]

Lemma 6.11  Suppose \( \gamma \) is a path corresponding to a non-pumpable NFA such that \( p = \text{cod}(\gamma) \). Then the following holds for any \( w \):

\[ w : p \triangle \beta \Rightarrow \text{Set}(\beta) \subseteq F(\gamma) \]

Lemma 6.11 is illustrated in Figure 7. Note that the fringes of the sibling trees rooted at the two \( p \)'s are identical (\( \triangle \) logic is deterministic), making it impossible for either of them to contain a \( q \) (\( q \) would be pumpable otherwise).

Definition 6.12  We define the reduction \( \triangleright \) on pairs of ordered multi-states according to the following rules:

\[
(q_1 \ldots q_n, \beta_1 q_2 \beta_2) \triangleright (q_1 \ldots q_i q \ldots q_n, \beta_1 \beta_2) \quad (\exists i \cdot q = q_i)
\]

\[
(q_1 \ldots q_n, \beta_1 q_2 q_3 \beta_3) \triangleright (q_1 \ldots q_n q q, \beta_1 \beta_2 \beta_3) \quad (\forall i \cdot q \neq q_i)
\]

The reduction \( \triangleright \) repeatedly groups recurring states. Given that each transition decreases the length of the second component, the reduction must terminate. We use the notation \( \triangleright \triangleright \) to denote a maximal reduction:

\[
(\alpha_1, \beta_1) \triangleright (\alpha_2, \beta_2) \Rightarrow \beta(\alpha_3, \beta_3) \cdot (\alpha_2, \beta_2) \triangleright (\alpha_3, \beta_3)
\]

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Lemma 6.13 Suppose: 
$$(\varepsilon, \beta) \triangleright (\alpha, \sigma)$$

Then the following properties hold:

$$\beta \simeq \alpha \sigma$$ (a)

$$\text{Set}(\alpha) \cup \text{Set}(\sigma) = \text{Set}(\beta)$$ (b)

$$\forall p \in \text{Set}(\alpha) . \ [\alpha]_p = |\beta|_p > 1$$ (c)

$$|\sigma| = |\text{Set}(\sigma)|$$ (d)

Definition 6.14 We define the semantics:

$$w : (\beta, \alpha, \sigma) \triangleright (\beta', \alpha', \sigma')$$

on search tree logic with the following inference rules:

$$a : \beta_1 \alpha_1 \triangle \beta_2 \quad a : \sigma_1 \triangle \beta_3 \quad (\varepsilon, \beta_3) \triangleright (\alpha_2, \sigma_2)$$

$$a : (\beta_1, \alpha_1, \sigma_1) \triangleright (\beta_2, \alpha_2, \sigma_2)$$

$$w : (\beta_1, \alpha_1, \sigma_1) \triangleright (\beta_2, \alpha_2, \sigma_2)$$

$$a : (\beta_2, \alpha_2, \sigma_2) \triangleright (\beta_3, \alpha_3, \sigma_3)$$

$$wa : (\beta_1, \alpha_1, \sigma_1) \triangleright (\beta_3, \alpha_3, \sigma_3)$$
The $\triangledown$ semantics recursively re-arranges the search tree into $\beta$, $\alpha$ and $\sigma$ components at each depth. A derivation using the $\triangledown$ semantics may be visualized as in Figure 8. Note that in this hypothetical derivation, we encounter repeated states at depth $w_1$, thus giving rise to the first non-empty $\alpha$ component ($\alpha_1$). From $w_1$ to $w_1w_2$, we have non-empty $\beta$ and $\sigma$ components. Again at depth $w_1w_2$ we can observe a non-empty $\alpha$ component, which is the result of the previous $\sigma$ component generating duplicates at this depth. The $\beta$ component can be thought of as the shadow/projection of all the previous $\alpha$ components.

**Lemma 6.15** Let $p$ be a state and $w$ an input string such that:
\[
w : p \triangledown \beta \quad w : (\varepsilon, \varepsilon, p) \triangledown (\beta', \alpha, \sigma)
\]
Then $\beta'\alpha\sigma \simeq \beta$.

**Lemma 6.16** Let $\gamma$ be a path with $p = \text{cod}(\gamma)$ and $w$ an input string such that:
\[
w : (\varepsilon, \varepsilon, p) \triangledown (\beta, \alpha, \sigma)
\]
Then the following properties hold:
\[
\begin{align*}
\text{Set}(\beta\alpha\sigma) & \subseteq \mathcal{F}(\gamma) \quad \text{(a)} \\
|\sigma| & \leq |\mathcal{F}(\gamma)| \quad \text{(b)} \\
|\alpha\sigma| & \leq |\mathcal{F}(\gamma)| \ast o \quad \text{(c)}
\end{align*}
\]

**Lemma 6.17** Let $w$ be an input string and $p$ a state. Let $i$, $k$ be indices such that:
\[
1 \leq i \leq k \leq |w|
\]
\[ w[1 \ldots i] : (\varepsilon, \varepsilon, p) \triangle (\beta_i, \alpha_i, \sigma_i) \]
\[ w[i \ldots k] : \alpha_i \bigtriangleup \alpha(i, k) \]

Then \( \beta_k = \alpha_{(1,k)} \ldots \alpha_{(k-1,k)} \)

**Proof** By induction on \( k \). \( \square \)

With reference to Figure 8, Lemma 6.17 establishes the connection between the fringe of the overall triangle and those of individual trapezoidal slices.

**Lemma 6.18** Let \( \gamma \) be a path with \( p = \text{cod}(\gamma) \) and \( w \) an input string such that:
\[ w : (\varepsilon, \varepsilon, p) \triangle (\beta, \alpha, \sigma) \]

Then for a state \( q \) such that \( \alpha = \alpha_1 q \alpha_2 \) (for some \( \alpha_1 \) and \( \alpha_2 \)), there exists a path \( \gamma' \) from \( p \) to \( q \) such that:
\[ F(\gamma \gamma') \subset F(\gamma) \]

**Proof** Follows from Lemma 6.11 (\( q \) is repeated inside \( \alpha \)). \( \square \)

**Theorem 6.19** Suppose \( \gamma \) is a path with \( p = \text{cod}(\gamma) \) and \( w \) an input string of length \( n \) such that:
\[ w : p \bigtriangleup \beta \]

Then the following holds:
\[ |\beta| < k^k + \sigma^k * n^k \]

Where \( k = |F(\gamma)| \).

**Proof** We perform an induction on \( k \).

**Base case - 1:** Suppose \( k = 0 \). Then it follows from Lemma 6.11 that \( |\beta| = 0 \), which is within the bounds of our polynomial.

**Base case - 2:** Suppose \( k = c \) (for some constant \( c \)) and:
\[ \exists \gamma' \cdot \gamma' = \gamma' \bigtriangleup \gamma'' \land F(\gamma') < c \]

This means the search tree rooted at \( \text{cod}(\gamma) \) cannot contain duplicates at any depth, for if it does, we can always find an extended path \( \gamma' \) for which \( F(\gamma') \) is less. This restriction immediately implies that the fringe of the
search tree cannot grow beyond \( c \), which is well within the bounds of our (over-estimating) polynomial \((c^c * o^c * n^c)\).

**Inductive step:** Let us assume the notation:

\[
\begin{align*}
  w[1 \ldots i] & : (\varepsilon, \varepsilon, p) \Rightarrow (\beta_i, \alpha_i, \sigma_i) \\
  w[i \ldots n] & : \alpha_i \not\prec \alpha(i,n)
\end{align*}
\]

Where \( 1 \leq i \leq n \). From Lemma \[6.17\] we deduce:

\[
\beta_n \alpha_n \sigma_n = \alpha_{(1,n)} \ldots \alpha_{(n-1,n)} \alpha_n \sigma_n
\]  

(A)

Note that it follows from Lemma \[6.18\] that we can apply the induction hypothesis to each path ending in some state within \( \alpha_i \). Therefore, we derive:

\[
\forall i \exists v < k \cdot |\alpha_{(i,n)}| < |\alpha_i| * v^v * o^v * |w[i \ldots n]|^v
\]

In terms of the illustration in Figure \[8\] this statement measures the bottom edges of the trapezoids. Now, taking into account that \( v < k \) and \( |w[i \ldots k]| < n \), we arrive at:

\[
\forall i \cdot |\alpha_{(i,n)}| < |\alpha_i| * k^k * o^k * n^k
\]

Moreover, it follows from Lemma \[6.16\] (c) that:

\[
|\alpha_i| \leq k * o
\]

Therefore, we get:

\[
\forall i \cdot |\alpha_{(i,n)}| < k^{k+1} * o^{k+1} * n^k
\]  

(B)

Now, we combine (A) and (B) to obtain:

\[
|\beta_n \alpha_n \sigma_n| < (n - 1) * k^{k+1} * o^{k+1} * n^k + |\alpha_n \sigma_n|
\]

Furthermore, it follows from Lemma \[6.16\] (c) that:

\[
|\alpha_n \sigma_n| \leq k * o < k^{k+1} * o^{k+1} * n^k
\]

Therefore, we get:

\[
|\beta_n \alpha_n \sigma_n| < k^{k+1} * o^{k+1} * n^{k+1}
\]

Since we know \( \beta \simeq \beta_n \alpha_n \sigma_n \) from Lemma \[6.15\] the inductive step holds. □
7 Implementation

We have implemented the analysis presented above in OCaml\textsuperscript{1}. It should be noted that this is a completely new implementation superseding the earlier analyser \textsuperscript{20}.

Apart from the code used for parsing regular expressions (and some other boilerplate code), the main source modules have an almost one-to-one correspondence with the concepts discussed in this paper. The following table illustrates this relationship:

| Concept (Theory) | Implementation (OCaml Module) |
|------------------|-------------------------------|
| NFA              | Nfa.mli/ml                    |
| \( \beta \)      | Beta.mli/ml                   |
| \( \Phi \)       | Phi.mli/ml                    |
| \( \rightarrow_2 \) | Product.mli/ml                |
| \( \rightarrow_3 \) | Triple.mli/ml                 |
| Prefix analysis  | XAnalyser.mli/ml              |
| Pumpable analysis (\(y_1\)) | Y1Analyser.mli/ml |
| Pumpable analysis (\(a y_2\)) | Y2Analyser.mli/ml |
| Suffix analysis  | ZAnalyser.mli/ml              |
| Overall analysis | AnalyserMain.mli/ml           |

Each module interface (.mli file) contains function definitions which directly correspond to various aspects of the analysis presented earlier. For an example, the NFA module provides the following function for querying ordered transitions:

\begin{verbatim}
val get_transitions : Nfa.t -> int ->
  ((char * char) * int) list;;
\end{verbatim}

The NFA states are represented as integers. Each symbol of the input alphabet is encoded as a pair of characters, allowing a uniform representation of character classes ([a-z]) as well as individual characters.

The NFA used in the implementation (Nfa.mli/ml) contains \(\varepsilon\) transitions, which were not part of the NFA formalization presented earlier. The reason for this deviation is that having \(\varepsilon\) transitions allows us to preserve the structure of the regular expression within the NFA representation, which in turn preserves the order of the transitions. The correctness of the implementation is unaffected as the two forms of NFA representation are isomorphic. Only a slight mental adjustment (from ordered NFAs to \(\varepsilon\)-NFAs) is required

\begin{footnote}
\url{http://www.cs.bham.ac.uk/~hxt/research/rxxr2}
\end{footnote}
to correlate the theoretical formalizations to the OCaml code. For an example, Figure 9 presents the module interface for $\beta$. The function `advance()` is utilized inside the XAnalyser.ml module to perform the closure computation (i.e. compute all $\beta$s reachable from the root node), whereas `evolve()` is a utility function used to work around the $\epsilon$ transitions. The modules (Phi / Product / Triple).ml define similar interfaces for $\Phi$, $\rightarrow_2$ and $\rightarrow_3$ constructs introduced in the analysis.

The different phases of the analysis is implemented inside the corresponding analyser modules. As an example, Figure 10 presents the Y2Analyser.ml module responsible for carrying out the analysis after the branch point ($\rightarrow_3$ simulation). The internal representation of the analyser (type `t`) holds the state of the closure computation, which is initialized with an initial triple argument through the `init()` function. We defer the interested reader to module definition (.ml) files for further details on the implementation.

### 7.1 Practical results

The new implementation was tested on the same data sets used in [20]. The most important results are as follows:
Unlike the tool presented in [20], our analyser can deliver a definitive yes/no answer for each regular expression presented (whereas in [20] it can also be “unknown”). The increase in overall analysis time can be justified given the correctness guarantee of the results.

The new tool makes every attempt to analyse a given pattern, even the ones which contain non-regular constructs like backreferences. An expression \((e_1|e_2)\) may be vulnerable due to a pumppable Kleene that occurs within \(e_1\), whereas \(e_2\) might contain a backreference. In these situations, the analyser attempts to derive an attack string which avoids the non-regular construct. If such a non-regular construct cannot be avoided, the analysis is terminated with the interrupted flag.

On certain rare occasions, search pruning is employed as an optimization. It is activated when there have been a number of unstable derivations (failing to meet \(\Phi_{y2} \subseteq \Phi_{x}\)) for a given prefix. Pruning can yield concise attack strings for certain regular expressions (longer derivations mean lengthy
attack strings). This is the case with the two instances of pruning reported above. If a pruned search does not report a vulnerability, it should be re-analysed with a higher (or infinite) prune limit in order to obtain a conclusive result.

Out of the over 12,000 patterns examined, there were two cases that failed to terminate within any reasonable amount of time. Closer inspection reveals that a pumpable Kleene expression with a vast number of states is to blame. Consider the following example (from RegExLib):

```
^((\[a-zA-Z0-9_\-\.]*)@\([a-zA-Z0-9_\-\.]\)+
  ([\.]\((\[a-zA-Z0-9_\-\.]*)@\([a-zA-Z0-9_\-\.]\)+\)
  ([a-zA-Z]\{2,5\}){1,25})+
([a-zA-Z]\{2,5\}){1,25}){1,25})+$
```

If we change the counted expressions of the form \(e^{1,25}\) into \(e^{1,5}\), the analyser returns immediately. This shows that the analysis itself can take a long time on certain inputs. However, such cases are extremely rare.

### 7.2 Comparison to fuzzers

REDoS analysers commonly used in practice are based on a brute-force approach known as fuzzing, where the runtime of a pattern is tested against a set of strings. A leading example of this approach is the Microsoft’s SDL Regex Fuzzer [23].

As is common with most brute-force approaches, the main problem with fuzzing is that it can take a considerable amount of time to detect a vulnerability. This is especially pronounced in the case of REDoS analysis as vulnerable patterns tend to take increasing amounts of time with each iteration of testing. This property alone disqualifies fuzzing based REDoS analysers from being integrated into code-analysis tools, as their operation would impose unacceptable delays. For an example, consider the following simple pattern:

```
^(a|b|ab)*c$
```

Even with a lenient fuzzer configuration (ASCII only, 100 fuzzing iterations), SDL fuzzer takes 5-10 minutes to report a vulnerability on this pattern. By comparison, our analyser can process tens of thousands of patterns in less time.

Fuzzers can also miss out on vulnerabilities. For an example, consider the following two patterns:

```
^.*|(a|b|ab)*c$
^=(a|b|ab)+c.$
```
SDL Fuzzer reports both of these patterns as being safe. However, the non-commutative property of the alternation renders the second pattern vulnerable (as explained in Section 4). Another such example is:

```
^(a|b|c|ab|bc)*a.*$
```

For this pattern, only one of the pumpable strings (bc) can lead to an attack string, and it must not end in an a. Such relationships are difficult to be caught in a heuristics-based fuzzer.

Yet another problem with fuzzers is caused by the element of randomness present in their string generating algorithms. Since fuzzers are not based on any sound theory, some form of randomness is necessary in order to increase the chance of stumbling upon a valid attack string. However, this can make the fuzzer yield inconsistent results for the same pattern. Consider the following pattern for an example:

```
(a|b)*[^c].*(c)*(a|b|ab)*d
```

The SDL fuzzer reports this pattern as being safe in most invocations, but in few cases it finds an attack string.

Finally, the ultimate purpose of using a static analyser is to detect potential vulnerabilities upfront and lead to the corresponding fixes. Our analyser pin-points the exact pumpable Kleene expression and generates a string (pumpable string) which witnesses vulnerability, making the fixing of the error a straightforward task. This is notably in contrast to the fuzzer, which outputs a random string (mostly in hex format) that does not provide any insight into the source of the problem.

8 Related work

The starting point for the present paper was the regular expression analysis RXXR [20]. While that paper was aimed at a security audience, the present paper complements it by using a programming language approach inspired by type theory and logic.

Program analysis for security is by now a well established field [6]. RENDoS is known in the literature as a special case of algorithmic complexity attacks [9, 26].

Parsing Expression Grammars (PEGs) have been proposed as an alternative to regular expressions [11] that avoid their nondeterminism. In a series of tutorials [7, 8], Cox has argued for Thompson’s lockstep matcher [29]
as a superior alternative to backtracking matchers. However, backtracking matchers vulnerable to REDoS are still widely deployed, including the matchers in the Java and .NET platforms as well as the PCRE matcher used in some intrusion detection systems. Hence the REDoS problem will remain with us for the foreseeable future.

Backtracking is a classic application of continuations, and regular expression matchers similar to the backtracking machine have been investigated in the functional programming literature [10, 15, 12]. Other recent work on regular expressions in the programming language community includes regular expression inclusion [16] and submatching [27].

Apart from some basic constructions like the power DFA covered in standard textbooks [18], we have not explicitly relied on automata theory. Instead, we regarded the matcher as an abstract machine that can be analyzed with tools from programming language research. Specifically, the techniques in this paper are inspired by substructural logics, such as Linear Logic [13, 14] and Separation Logic [19, 24]. Concerning the latter, it may be instructive to compare the sharing of $\beta$ or absence of sharing of $\beta$ in Figure 6 to the connective of Separation logic. In a conjunction, the heap $h$ is shared:

$$h \models P_1 \quad h \models P_2$$

$$h \models P_1 \land P_2$$

By contrast, in a separating conjunction, the heap is split into disjoint parts that are not shared:

$$h_1 \models P_1 \quad h_2 \models P_2 \quad h_1 \cap h_2 = \emptyset$$

$$h_1 \cup h_2 \models P_1 \ast P_2$$

Tree-shaped data structures have been one of the leading examples of separation logic and variations of it, such as Context Logic [4]. However, a difference to the search trees we have used in this paper is that the whole search tree is not actually constructed as a data structure in memory. Rather, only a diagonal cut across it is maintained at any time in the backtracking machine. The whole tree does not exist in memory, but only in space and time, so to speak. In that regard the search trees are like parse trees, which the parser only needs to construct in principle by traversing them, and not necessarily as a data structure in memory complete with details of all nodes [2, 1].

Even though the backtracking machine is sequential, parts of the analysis are reminiscent of transition systems in process algebras, particularly running two or more automata in parallel (Figures 4, 5). Seen that way, the
may and must part of the analysis are analogous to the two modalities $\langle a \rangle$ and $[a]$ in Henessy-Milner logic [17].

9 Directions for further research

At present, the analysis constructs attack strings when there is the possibility of exponential runtime. It should be possible to extend the analysis to compute a polynomial as an upper bound for the runtime when there is no REDoS vulnerability causing exponential runtime.

The efficiency of the analyser compares favorably with that of the Microsoft SDL Regex Fuzzer [23]. Given that we are computing sets of sets of states, the analysis may explore a large search space. One may take some comfort from the fact that type checking and inference for functional programming languages can have high complexity in the worst case [22, 25] that may not manifest itself in practice. Nonetheless, we aim to revisit the design of the analysis and optimize it.

Pruning the search space may lead to improvements in efficiency. An intriguing possibility is to implement the analysis on many-core graphics hardware (GPUs). Using the right data structure representation for transitions, GPUs can efficiently explore nondeterministic transitions in parallel, as demonstrated in the iNFAnt regular expression matcher [5].

The search tree logic (Figure 6) may have independent interest and possible connections to other substructural logics such as Linear Logic [13, 14], Separation Logic [19, 24], Lambek’s syntactic calculus [21], or substructural calculi for parse trees [28]. Search trees are dual to parse trees in the sense that the nodes represent a disjunction rather than a conjunction.

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