Response to the Comment by G. Emch on Projective Group Representations in Quaternionic Hilbert Space

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Abstract

We discuss the differing definitions of complex and quaternionic projective group representations employed by us and by Emch. The definition of Emch (termed here a strong projective representation) is too restrictive to accommodate quaternionic Hilbert space embeddings of complex projective representations. Our definition (termed here a weak projective representation) encompasses such embeddings, and leads to a detailed theory of quaternionic, as well as complex, projective group representations.

I. PRELIMINARIES NOT INVOLVING GROUP STRUCTURE

Before turning to a discussion of what is an appropriate definition of a quaternionic projective group representation, we first address several issues that do not involve the notion of a group of symmetries. We follow throughout the Dirac notation used in our recent book [1], in which linear operators in Hilbert space act on ket states from the left and on bra states from the right, as in $\mathcal{O}|f\rangle$ and $\langle f|\mathcal{O}$, while quaternionic scalars in Hilbert space act on ket states from the right and on bra states from the left, as in $|f\rangle\omega$ and $\omega\langle f|$.
We begin by recalling the statement (see Sec. 2.3 of Ref. [1]) of the quaternionic extension of Wigner’s theorem, which gives the Hilbert space representation of an individual symmetry in quantum mechanics. Physical states in quaternionic quantum mechanics are in one-to-one correspondence with unit rays of the form \( |\mathbf{f}\rangle = \{|\mathbf{f}\rangle \omega \} \), with \( |\mathbf{f}\rangle \) a unit normalized Hilbert space vector and \( \omega \) a quaternionic phase of unit magnitude. A symmetry operation \( S \) is a mapping of the unit rays \( |\mathbf{f}\rangle \) onto images \( |\mathbf{f}'\rangle \), which preserves all transition probabilities,

\[
S|\mathbf{f}\rangle = |\mathbf{f}'\rangle \\
|\langle \mathbf{f}'|\mathbf{g}'\rangle| = |\langle \mathbf{f}|\mathbf{g}\rangle|.
\]

(1)

Wigner’s theorem, as extended to quaternionic Hilbert space, asserts that by an appropriate \( S \)-dependent choice of ray representatives for the states, the mapping \( S \) can always be represented (in Hilbert spaces of dimension greater than 2) by a unitary transformation \( U_S \) on the state vectors, so that

\[
|\mathbf{f}'\rangle = U_S|\mathbf{f}\rangle.
\]

(2)

Conversely, any unitary transformation of the form of Eq. (2) clearly implies the preservation of transition probabilities, as in Eq. (1). When only one symmetry transformation is involved, the issue of projective representations does not enter, since Wigner’s theorem asserts that this transformation can be given a unitary representation on appropriate ray representative states in Hilbert space. The issue of projective representations arises only when we are dealing with two (or more) symmetry transformations, in which case the ray representative choices which reduce the first symmetry transformation to unitary form may not be compatible with the ray representative choices which reduces a second symmetry transformation to unitary form. Thus we disagree with Emch’s statement, in the semifinal paragraph of his Comment, that Wigner’s theorem (which he notes is a form of the first fundamental theorem of projective geometry) may be dependent on the definition adopted for quaternionic projective group representations.

In the first section of his Comment, Emch proves a Proposition stating that if an operator \( \mathcal{O} \) commutes with all of the projectors \( |\mathbf{f}\rangle\langle \mathbf{f}| \) of a quaternionic Hilbert space of dimension 2
or greater, then $\mathcal{O}$ must be a real multiple of the unit operator 1 in Hilbert space. When $\mathcal{O}$ is further restricted to be a unitary operator (as obtained from a symmetry transformation via the Wigner theorem), the real multiple is further restricted to be $\pm 1$. Since we will refer to this result in the next section, let us give an alternative proof, based on the spectral representation of a general unitary operator $U$ in quaternionic Hilbert space,

$$U = \sum_{\ell} |u_{\ell}\rangle e^{i\theta_{\ell}} \langle u_{\ell}| , \quad 0 \leq \theta_{\ell} \leq \pi , \quad (3)$$

in which the sum over $\ell$ spans a complete set of orthonormal eigenstates of $U$. Let us focus on a two state subspace spanned by $|u_1\rangle$ and $|u_2\rangle$, and construct the projector $P = |\Phi\rangle\langle\Phi|$, with

$$|\Phi\rangle = |u_1\rangle + |u_2\rangle \omega ,$$

$$\omega = -\omega , \quad \omega = \omega_\alpha + j\omega_\beta , \quad \omega_\alpha \omega_\beta \neq 0 , \quad (4)$$

where $\omega_{\alpha,\beta}$ are symplectic components lying in the complex subalgebra of the quaternions spanned by 1 and $i$. Then the projector $P$ is given by

$$P = |u_1\rangle\langle u_1| + |u_2\rangle\langle u_2| + |u_2\rangle\omega \langle u_1| - |u_1\rangle \omega \langle u_2| , \quad (5a)$$

and the part of $U$ lying in the $|u_{1,2}\rangle$ subspace is

$$U_{1,2} = |u_1\rangle e^{i\theta_1} \langle u_1| + |u_2\rangle e^{i\theta_2} \langle u_2| . \quad (5b)$$

The commutator of $U$ and $P$ is then given by

$$[U, P] = [U_{1,2}, P] = |u_2\rangle (e^{i\theta_2} \omega - \omega e^{i\theta_1}) \langle u_1| - |u_1\rangle (e^{i\theta_1} \omega - \omega e^{i\theta_2}) \langle u_2| , \quad (6)$$

which vanishes only if $e^{i\theta_1} = e^{i\theta_2}$ (from equating to zero the coefficient of $\omega_\alpha$) and $e^{i\theta_1} = e^{-i\theta_2}$ (from equating to zero the coefficient of $\omega_\beta$). Since $0 \leq \theta_{1,2} \leq \pi$, this requires either $\theta_1 = \theta_2 = 0$ or $\theta_1 = \theta_2 = \pi$. Repeating the argument for each dimension 2 subspace in turn, we learn that $U = \pm 1$. Note that in a complex Hilbert space, the analogous argument shows only that $e^{i\theta_1} = e^{i\theta_2}$, from which we conclude (again by repeating the argument for each
dimension 2 subspace in turn) that $U = e^{i\theta}$, which commutes with all projectors because any complex number is a $c$-number in complex Hilbert space.

Clearly, the argument just given involves only elementary properties of the projectors in Hilbert space, and makes no reference to the notion of a group of symmetries. The same is true of the proposition given in Sec. I of Emch’s Comment. Since Schur’s Lemma ordinarily describes the restrictions on an operator that commutes with the representation matrices of an irreducible group representation, and since the projectors in Hilbert space do not form a group (they are not invertible and the product of two different projectors is not a projector), it is a misnomer to describe Emch’s Proposition, or the corollary given here, as a “quaternionic Schur’s lemma”. In addition to disagreeing with Emch’s terminology, we also disagree with his statement, in the second paragraph of Sec. III of his Comment, that the analysis leading to his Proposition is dependent on the definition adopted for quaternionic projective group representations; in fact, the notion of a group of symmetries does not enter into either his analysis, or the corollary for unitary matrices proved here.

II. HOW SHOULD ONE DEFINE QUATERNIONIC PROJECTIVE GROUP REPRESENTATIONS?

Let us now address the central question of how one should generalize to quaternionic Hilbert space the notion of a projective group representation. We begin by reviewing how projective group representations arise in complex Hilbert space. Let $\mathcal{G}$ be a symmetry group composed of abstract elements $a$ with group multiplication $ab$. By Wigner’s theorem, each group element is represented, after an $a$-dependent choice of ray representatives, by a unitary operator $U_a$ acting on the states of Hilbert space. In the simplest case, in which the $U_a$ are said to form a vector representation, the $U$’s obey a multiplication law isomorphic to that of the corresponding abstract group elements,

$$U_a U_b = U_{ab} .$$  (7)
However, when the complex rephasings of the states used in Wigner’s theorem are taken into account, there exists the more general possibility that for any state $|f\rangle$, the states $U_a U_b |f\rangle$ and $U_{ab} |f\rangle$ are not equal, but rather differ from one another by a change of ray representative, i.e.,

$$U_a U_b |f\rangle = U_{ab} |f\rangle e^{i\phi(a,b,f)} .$$  \hspace{1cm} (8)

Corresponding to Eq. (8), there are two possible definitions of a projective representation in complex Hilbert space:

\begin{itemize}
  \item **Definition (1)** In a weak projective representation, the multiplication law of the $U$’s obeys Eq. (8) on one complete set of states $\{|f\rangle\}$. This suffices, by superposition, to determine the multiplication law of the $U$’s on all states. \\
  \\
  \item **Definition (2)** In a strong projective representation, the multiplication law of the $U$’s obeys Eq. (8) on all states in Hilbert space. In this case, we can easily prove that the phases $\phi(a,b;f)$ are independent of the state label $f$. To see this, let us define $V_{ab} = U_{ab}^{-1} U_a U_b$; then Eq. (8) implies that

$$V_{ab} |f\rangle = |f\rangle e^{i\phi(a,b,f)} ,$$  \hspace{1cm} (9)

which immediately implies that $V_{ab}$ commutes with the projector $|f\rangle \langle f|$, for all states $|f\rangle$ in Hilbert space. But invoking the complex Hilbert space specialization of the result of the preceding section, we learn that $V_{ab}$ must be a c-number, $V_{ab} = e^{i\phi(a,b)}$. This is the customary definition of a projective representation in complex Hilbert space, and is well known to have nontrivial realizations.

Let us now turn to the question of how to define projective representations in quaternionic Hilbert space. Emch chooses as his generalization the strong definition given above, which by the reasoning following Eq.(9), and the quaternionic result of Sec. 1, implies that $V_{ab} = (-1)^{n_{a,b}}$, with $n_{a,b}$ an integer that can depend in general on $a$ and $b$. In other words, the only strong quaternionic projective representations are real projective representations.
The problem with adopting the strong definition, however, is that it excludes from consideration as a quaternionic projective representation the embedding into quaternionic Hilbert space of a nontrivial complex projective representation realized on a complex Hilbert space. Thus, potentially interesting structure is lost. To avoid this problem, Ref. [1] adopts as the quaternionic generalization of the notion of a projective representation the weak definition given above, which in quaternionic Hilbert space states that

$$U_a U_b |f⟩ = U_{ab} |f⟩ \omega_{a,b} , \quad |\omega_{a,b}| = 1 \quad (10)$$

for one particular complete set of states \(\{|f⟩\}\). As discussed in Ref. 1, Eq. (10) can also be rewritten in the operator form

$$U_a U_b = U_{ab} \Omega(a, b) , \quad (11a)$$

with

$$\Omega(a, b) = \sum_f |f⟩ \omega(a, b; f) ⟨f|. \quad (11b)$$

Since the operator \(\Omega\) depends on the particular complete set of states on which the projective phases are given, a more complete notation (not employed in Ref. 1) would in fact be \(\Omega(a, b; \{|f⟩\})\). Using the result of an analysis [2] of the associativity condition for weak quaternionic projective representations, Tao and Millard [3] have recently given a beautiful complete structural classification theorem for weak quaternionic projective representations. The complex specialization of their Corollary 2, incidentally, states that in a complex Hilbert space, the weak definition of a projective representation implies the strong one.

Can the weak definition of a quaternionic projective representation be weakened even further, by using a different complete set of states \(\{|f⟩\}\) to specify the projective phases for each pair of group elements \(a\) and \(b\) [4]? In this case, the operator \(\Omega\) takes the form \(\Omega(a, b; \{|f⟩\}_{a,b})\). However, since any unitary operator is diagonalizable on some complete set of states, this further weakening allows an arbitrary specification of \(\Omega\) for each \(a, b\), and any relationship of the unitary representation to the underlying group structure is lost.
III. DISCUSSION

We conclude that the difference between our analysis and that of Emch is traceable to what I have here termed the difference between a strong and a weak definition of projective representation. The strong definition is the customary one in complex Hilbert space, but it excludes potentially interesting structure when applied to quaternionic Hilbert space. Since the weak definition leads to a detailed theory [1, 2, 3] of projective group representations in quaternionic Hilbert space, and since it implies [3] the strong definition in complex Hilbert space, the weak definition is in fact the more appropriate one in both complex and quaternionic Hilbert spaces.

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[4] T. Tao, private communication.