Minimal paths between communities induced by geographical networks

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Abstract. In this work we investigate the betweenness centrality in geographical networks and its relationship with network communities. We show that vertices with large betweenness define what we call characteristic betweenness paths in both modeled and real-world geographical networks. We define a geographical network model that possesses a simple topology while still being able to present such betweenness paths. Using this model, we show that such paths represent pathways between entry and exit points of highly connected regions, or communities, of geographical networks. By defining a new network, containing information about community adjacencies in the original network, we describe a means to characterize the mesoscale connectivity provided by such characteristic betweenness paths.

Keywords: random graphs, networks
1. Introduction

As a consequence of their potential to represent real systems, complex networks have become the subject of growing interest in recent years [1]. Some examples of systems that have been studied using complex networks are electric power grids [2], airline routes [3], the World-Wide Web [4], among others [5]. Networks whose vertices are embedded in a space with well-defined coordinates and their connectivity is related to the distance between the vertices are named geographical networks [6]. Examples of geographical networks include street, airline, and neural networks. Models to generate artificial networks with geographic dependency have been created [7] in order to better understand the real systems.

The spatial influence on the connectivity gives rise to many interesting characteristics [8]. In a previous work about the influence of gene expression on neuronal characteristics [9], we computed the betweenness centrality [10] of neuronal networks. While visualizing those neuronal networks we observed that the vertices having the highest betweenness centrality values were connected together creating chains of vertices, which here we call betweenness paths. The presence of betweenness paths in the neuronal network suggests a two-level hierarchy of the system structure, where the betweenness paths form the main pathways for information flow and the remaining paths distribute this information locally. Such hierarchical organization has also been observed for road networks [11–15]. In [16] its argued that road networks should be planned in different scales, according to their purpose, function or management policies. Such scales range from the entire city level down to local streets used for property access or pedestrian movement. According to [17, 18] the traffic in road networks, modeled by applying the betweenness centrality, and the movement of people in a city have also been shown to be power-law distributed, meaning that a few paths concentrate most of the city traffic.
Given that a hierarchical structure of pathways has been observed in a range of systems, it is interesting to study the reasons why they are formed, their importance for the correct operation of the system, their influence in the network flow, and in which kind of geographical networks the betweenness paths can be found. To better understand the hierarchical structures, it is necessary to study how the network topology is inducing those paths. Furthermore, the knowledge about betweenness paths and their characteristics can be used for better planning the system.

In this work, we study geographical networks composed by two hierarchical levels: a main structure forming the backbone of the network, and secondary connections supported by such backbone. We considered networks generated from geographic models and real-world street networks. From these networks, we identified connected vertices with high betweenness centrality. We observed that these vertices provide a good covering of the topology, in the sense that any network vertex is always near at least one or more characteristic betweenness paths. We found that such paths tend to be related with the community structure [19] of the network. So, to better understand the characteristic betweenness paths and its relationship with communities, we created a random geographic model and used community detection to divide the space in well-defined regions. In order to study the relationship between such communities and their geographical positions, we created a sub-summed community network, where each vertex is a different community and the connection weights are based on the amount of shortest paths crossing the community border.

This paper is organized as follows. Section 2 presents the basic network theory concepts that are used in this paper. In section 3 we discuss betweenness paths in network models and real cases. In section 4 we create a network model to better understand the betweenness paths. In section 5, we describe the relationship between geographic and topological measurements.

2. Basic concepts

There are many ways to compute network centralities, each being developed to evaluate different network characteristics [7].

In order to locally quantify the network centrality, it is possible to measure the degree $k_i$ for each vertex $i$. Furthermore, in weighted networks, we can consider the edges to define a measurement called strength [8], which is computed for each vertex $i$ as

$$S_i = \sum_{j \in r_i} w_{ij},$$

where $r_i$ is the set of neighbors of vertex $i$ and $w_{ij}$ is the weight of the edge connecting $i$ to $j$.

The network centrality can also be computed globally, that is, considering the whole network structure. One important global centrality measure is the stress centrality ($SC_u$) [20]. This measure is defined for each vertex $u$ as

$$SC_u = \sum_{ij} \sigma(i, u, j),$$

where $\sigma(i, u, j)$ is the number of shortest paths between vertices $i$ and $j$ passing through vertex $u$.
where $\sigma(i, u, j)$ represents the number of shortest paths between the vertices $i$ and $j$ crossing the vertex $u$. If one vertex is removed from the network it can affect the stress of all other vertices.

Another important global measurement is the betweenness centrality, which also considers the amount of shortest paths crossing a vertex [10]. This measure is defined as

$$B_u = \sum_{ij} \frac{\sigma(i, u, j)}{\sigma(i, j)},$$

where $\sigma(i, u, j)$ represents the number of shortest paths between the vertices $i$ and $j$ crossing the vertex $u$ and $\sigma(i, j)$ is the total amount of shortest paths between $i$ and $j$. The betweenness centrality can be implemented using a fast algorithm [21, 22], where the betweenness centrality is computed and the result is used to compute the stress centrality.

Betweenness centrality is frequently used to study flows in networks [23–25] and can be used in several applications. In social networks it can be described in terms of its message communications, considering that each message take the shortest path [26]. In power grid networks, this measure can quantify the load of each vertex [27]. It is also used in the cascading-failure dynamics [28].

In addition to the centrality measures, another important concept studied in complex networks theory is the community structure of the system. Where a community is defined as a set of vertices densely interconnected, and with few connections to vertices outside this set. This property can be found in many real networks such as social networks [29], metabolic networks [30], and in the structure of cities [31]. In city networks, those communities are shown to be subdivisions of urban structure emerging urban elements and their interactions [31].

A common approach to quantify how well-defined are the communities in the network is the modularity measure [19]. The modularity is represented as

$$Q = \frac{1}{2m} \sum_{i,j} \left( A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j),$$

where $m$ is the number of network edges, $A$ is the network adjacency matrix, $k_i$ and $k_j$ are the vertex degree $i$ and $j$, respectively, $c_i$ and $c_j$ are the communities of vertex $i$ and $j$, respectively, and $\delta(c_i, c_j)$ is 0 if $c_i \neq c_j$ or 1 if $c_i = c_j$.

There are several possibilities to find network communities [32]. For example using methods based on high betweenness centrality or low clustering for inter-community edges [33, 34], but in this study we applied a method that detect communities through the optimization of modularity. For a comprehensive review of modularity optimization methods, please refer to [32]. In this work we use the fast greedy [35] algorithm to detect the communities.

3. Characteristic betweenness paths

A characteristic betweenness network is defined as a sub-network composed by the vertices from the original network having the highest betweenness centrality values. In order to find such a sub-network, we compute the betweenness centrality of the vertices.
in the original network and divide them into two groups. Vertices having betweenness larger than a threshold $T$ are called characteristic betweenness vertices, while the rest of the vertices in the network are called secondary vertices. The characteristic betweenness vertices together with all edges connecting pairs of characteristic betweenness vertices define the characteristic betweenness network. We will show that such a network is mainly composed by chains of vertices with some occasional crossing between chains. Such chains are henceforth called characteristic betweenness paths.

In the following we use four networks to study the characteristic betweenness paths. Two of these networks are artificially generated, and the other two are street networks. In the first model, known as Waxman [36], the vertices are organized randomly in a plane. For each vertex, the probability to connect with all other vertices follows an exponential decay of the distance between them. In the second model, known as random geometric graph [37], the vertices are randomly organized and the connections are created only when the geographical distance is lower than a fixed value. We compared the network models with two real street networks obtained from San Joaquin County in USA and Oldenburg city in Germany [38].

In order to find a reasonable value for the betweenness threshold, we first computed the fraction of network vertices that become characteristic betweenness vertices when setting different values for the threshold. This fraction is found by dividing the number of characteristic betweenness vertices by the size of the network. We performed these tests using the two street networks. The obtained values for the fraction of characteristic betweenness vertices can be seen in figure 1(a). Moreover, we also plot in figure 1(a) the fraction of characteristic betweenness vertices when considering only the largest connected component (giant component) of the characteristic betweenness network. The results show that the majority of the characteristic vertices are included in the giant component for all different threshold values. Furthermore, the curves are similar for both cities.

In order to better characterize the emergence of characteristic betweenness paths at different threshold values, we also calculated the fraction of characteristic betweenness vertices that have degree two when considering the characteristic betweenness network. Note that, in this case, the ratio is calculated as the number of characteristic betweenness vertices having degree two divided by the size of the characteristic betweenness network. If all characteristic betweenness vertices form a single path, this ratio is equal to one. Figure 1(b) shows the evolution of the proposed degree-two ratio for different threshold values. The plot reveals that the fraction of degree-two characteristic betweenness vertices tends to increases with the threshold. This means that the characteristic betweenness network is neither breaking apart into isolated vertices nor forming a core at the center of the original network, but defining paths along the 2D space. Such paths only disappear when the threshold is close to the maximum betweenness value.

It is clear that there is no indication of a proper threshold for defining the characteristic betweenness vertices. If the threshold is too small, the paths do not appear in the resulting network. If it is too large, a small characteristic betweenness network is generated and, as a result, this network spans only a fraction of the original 2D space. Therefore, we consider that a good compromise between path emergence and coverage is when the threshold is in the range $[10^3, 10^7]$. Thus, we selected a threshold for
which 20% of the network vertices become characteristic vertices, which corresponds to $T \approx 10^6$. If we consider the subnetwork formed by such vertices and their respective connections, it is possible to show that for each analyzed network an interesting chain-like structure is obtained. Such structure contains most of the flow in the network, represented by the betweenness, and can be understood as the backbone of the original network. The secondary vertices rely upon this backbone for proper communication with the rest of the system. The characteristic betweenness paths were also found in other works, but their origins were not analyzed [12, 13, 17].

In order to illustrate the characteristic betweenness paths, we generated single realizations of the Waxman and random geometric models. The Waxman network has 1707 vertices and average degree 2.20 (the network is shown in figure 2(a)). The random geometric graph has 4883 vertices and average degree 6.18 (shown in figure 2(b)). We also show the street network from Oldenburg, which has 2873 vertices and average degree 2.57 (figure 2(c)). For the Oldenburg network, the node colors represent the detected communities. The San Joaquin network (14 503 vertices and average degree 2.75) is not shown in figure 2 because the network is very large, which hinders a proper visualization of its paths and communities.

For each network we computed the betweenness centrality and highlighted the 20% of vertices with largest betweenness values, which defines the characteristic betweenness paths. By visual inspection, in each network the characteristic vertices seem to define a backbone structure having many two-degree vertices. Such backbone is formed by what we call the characteristic betweenness paths. In street networks this result can indicate a tendency that the volume of cars is bigger in some streets than in others if we consider that all drivers are using the shortest paths for the formation of high flow streets.

In every example we could observe the characteristic betweenness paths and it was possible to identify more than one connected component with high betweenness centrality.
Figure 2. Geographical networks, where black circles and highlighted edges show the characteristic betweenness paths. Item (a) represents the network of the Waxman model, item (b) represents the network of the random geometric model, and item (c) represents the network of Oldenburg streets, where each color represents a different community.
centrality in the same network. The component sizes were different for each network. In the random geometric model, the giant component represents 85% of all vertices in the characteristic betweenness network. In the other networks the giant component represents more than 98% of all vertices in the paths.

We also analyze the coverage of the characteristic betweenness paths. In order to do so, we measured the distance $d_i$ between each secondary vertex $i$ and the closest characteristic vertex, because the shorter the distances are, the better are the paths coverage. In order to provide a fair comparison between different networks, the calculated distances are normalized by the average path length, $l$, of each network. The aim of normalizing by the average path length is to measure the distances between characteristic and secondary vertices relative to what one would typically expect from a randomly selected pair of vertices. Therefore, we define the normalized shortest path distance, $\Theta_i$, as

$$\Theta_i = \frac{d_i}{l}.$$ 

We generate 60 samples from each network model. In this case, we adjust the model variables in order to compare the results with the Oldenburg network (modularity 0.92). The ensemble of networks generated by the Waxman model have, typically, $2976.70 \pm 16.61$ vertices, average degree $2.71 \pm 0.04$, and modularity $0.79 \pm 0.01$. The samples of random geographic model have $2892.33 \pm 58.46$ vertices, average degree $4.50 \pm 0.09$, and modularity $0.97 \pm 0.001$. In order to do a comparison considering the statistical variations of the models, these results are shown as histograms in figure 3. We observed that the fraction of vertices with small distances (first bin in the histogram) is larger in the random geometric model (figure 3(a)) than in the Waxman model (figure 3(b)). Comparing the two models, we observe that secondary vertices tend to be closer to characteristic betweenness paths in the random geometric graph model. Furthermore, the standard deviation of frequencies, shown as blue lines, are small in both models, which indicates that this effect is not a statistical artifact caused by fluctuation. The shortest distances were also analyzed for the street networks (as can be seen in figures 3(c) and (d)). The dashed lines represent the average values of $\Theta_i$ which we call the coverage of the characteristic betweenness paths. The coverage of Oldenburg is larger than for San Joaquin, which may be a consequence of the higher complexity of Oldenburg’s topology [38].

4. Characteristic paths model

In order to better understand the emergence of characteristic betweenness paths in geographic networks, we define a model able to generate networks with a simple lattice-like topology, but still containing a well-defined backbone structure. The initial structure of the network is a 2D regular lattice, where the vertices are organized in lines and equally spaced columns. First we defined the size, $L$, of the network (number of lines and columns) and two variables, the maximum distance to connect two vertices as $d_{MAX}$ and a probability $x$ to remove randomly the edges for each vertex. The algorithm follows
three steps: (1) the vertices are organized as a square grid with toroidal boundary condition; (2) for each pair of vertices \( i \) and \( j \) when the distance \( d_{ij} \) is less or equal than \( d_{\text{MAX}} \) an edge connecting \( i \) to \( j \) is created; (3) each edge is removed with probability \( x \).

In order to visualize the betweenness paths in the model presented in this section, we created a network using the parameter \( L = 200 \), so that the network has \( N = 40,000 \). The connectivity range was set to \( d_{\text{MAX}} = \sqrt{2} \). Therefore, each vertex of the network has degree 8. We randomly remove the connections with probability \( x = 0.65 \).
We measured the betweenness centrality for each network vertex and the results can be seen in figure 4(a). We can observe vertices with higher values (characteristic vertices) generating preferential paths in the network, in a similar manner as main streets. Our hypothesis is that the network communities are inducing the formation of betweenness paths through connections between them. In figure 4(b) we show the network communities and characteristic betweenness paths (indicated in black). The result suggests that the paths are going through communities.

To compare the characteristic betweenness paths displayed by our network model and real networks, we executed the same tests as in the previous section. In order to infer the statistical variation of the model, we created 60 network samples. We used the parameter $L = 56$ and we randomly removed the connections with probability $x = 0.7$. These parameters were set so that the generated networks have a size and average degree as close as possible to compare with the Oldenburg network. The network measurements are: number of vertices $2829.95 \pm 21.34$, average degree $2.59 \pm 0.03$, and modularity $0.94 \pm 0.002$.

Our model has similarities and differences from the street networks and the other models. The giant component of the characteristic betweenness network represents $(85 \pm 8)\%$ of all characteristic vertices. Regarding the shortest distance between secondary vertices and characteristic betweenness paths (figure 5), the secondary vertices tend to be more distant from the backbone than in the Oldenburg and Waxman networks.

We compared the average of shortest distance between the secondary vertices and the preferential betweenness paths in network models and the Oldenburg network. For each generated network we calculate coverage, which we represent by $\Theta$, and in figure 6 we show the distribution of $\Theta$ for the generated networks. The dashed line
shows the coverage for Oldenburg. The coverage for Oldenburg is closest to the measured values for the characteristic path model. Besides, the standard deviation is low, which confirms that the betweenness paths have a similar structure in all generated networks. Therefore, in the following we further analyze the betweenness paths using a single realization of our model.

To quantify our network communities in terms of their respective degree and betweenness centrality, we considered two types of vertices, the geographical border community vertices and the internal community vertices. The first type represents vertices that have at least one neighbor in the grid belonging to a different community, no matter if they are connected or not. The second type represents a vertex that is not a geographical border. In order to define each vertex type, we considered all eight geographical neighbors, of the vertices. In some analyses we also considered the topological border community vertices, which are the vertices connected with one or more vertices from different community. We generated a network with $L = 200$ and average degree 2.92 according to our model. Using this network we computed three different comparisons. In the first, we compare the average degree between internal vertices and border community vertices. In this comparison it is possible to observe that the average degree in the internal community vertices is higher than the border community vertices. Thereby, we can affirm that the vertex degree is influencing how the community is formed.

In order to describe how the vertices are connecting communities, we measured the average degree in the geographical community border and in the topological community border. In 98% of the communities, the average degree of vertices belonging to the geographical border is lower than for vertices contained in the topological border.
We made a similar comparison, between geographic and topological borders, but now measuring the betweenness centrality instead of the degree. This analysis can show if connections between communities (topological border) are generating a characteristic betweenness paths. We observed that the average geographical border betweenness is lower than the topological border betweenness for all communities. Thus, in average, the vertices where the paths are crossing the community borders have higher betweenness centrality and degree than the geographical borders. Some vertices of characteristic betweenness paths are part of the topological border. In other words, they are connecting the communities.

The results indicate that the paths represent routes of efficient communication between the communities. This idea is illustrated in figure 7. Vertices included in the highlighted gray region belong to the same community. The community has three vertices on its topological border and the shortest paths connecting them, shown in red, are creating the preferential paths with higher betweenness centrality.

In order to validate these results, we created an artificial geographic network with fixed communities. In this network, each community is a regular lattice, where the vertices are connected with their eight nearest neighbors, and the communities are interconnected through some selected vertices. Figure 8 shows the network and highlight the highest values of betweenness centrality (the characteristic vertices). In this test, it is possible to observe that the characteristic betweenness paths are connecting the communities. Besides, through this model, it is possible to note that the paths are not created at random, that is, their emergence have a well-defined mechanism.

It is also interesting to analyze if the number of shortest paths connecting pairs of communities are related to the sizes of the communities geographical border. In order to do so, we plot the perimeter of adjacency between communities against the sum of the vertex stress connecting such communities, which can be seen in figure 9. Besides, we calculated the Pearson correlation coefficient between the measurements, obtaining a value of 0.23. Thus, these measurements are uncorrelated.
5. Community adjacency network

In order to observe the relationship between communities and their importance, considering geographical characteristics, we created a network based on the topological adjacency between communities of our characteristic paths model. Through this model it is possible to compare the geographical characteristics (e.g. the community area) with topological characteristics, such as the degree of a vertex, which represents the number...
of communities connected with a given community. In this network, each vertex represents a community. A weight equal to \( SC_i + SC_j \) (defined in equation (2)) is associated to each edge connecting communities \( i \) and \( j \). The resulting network is undirected. In figure 10 we show a typical realization of this network. It is possible to observe small communities having strong connections, indicating that betweenness paths are also present in such small communities. On the other hand, there are big communities with weak connections, showing that few preferential paths are crossing them.

We compared local topology characteristics of the community adjacency network with geometrical characteristics (the scatter plots can be seen in figure 11). Figures 11(a)–(c) show the degree as a function of area, perimeter and diameter, respectively and figures 11(d)–(f) show the same measures using the vertex strength instead of the degree. The degree is highly correlated with the geographical characteristics of communities. On the other hand, the strength has little relationship with the communities geometry. This happens because the strength is uncorrelated with the geographical characteristics of communities.

6. Conclusion

The structure of a geographical network has a direct impact on how the system can be navigated. Our analysis revealed that betweenness paths are a common characteristic of geographical networks, since they can be found in geographical models as well as real world networks. We characterized the characteristic betweenness paths in terms of the distance between each secondary vertex and the closest characteristic vertex. Such
measurement can be related to the covering provided by the paths to reach the entire network. We showed that in both the geographical models and the street networks the betweenness paths provide a good covering of the system. In order to better understand the establishment of the characteristic betweenness paths, we proposed a model to generate geographical networks having a simple topology but that can still originate such paths. It was found that the characteristic betweenness paths have a strong relationship with the presence of well-defined communities in the networks. We compared the vertices degrees and betweenness centrality among geographical and topological borders of the communities, as well as the internal vertices of the communities. This analysis revealed that the characteristic betweenness paths are usually induced by the communities, in the sense that such paths provide a communication between entry and exit points of the communities.

The number of shortest paths passing through the topological border of adjacent communities seems to be independent of the geometry of the interface between the communities. This was revealed by the absence of correlation between the total stress of vertices connecting adjacent communities and the perimeter of the adjacency. In order to better understand the relationship between communities and characteristic betweenness paths, we defined a new structure based on the vertices betweenness and the community organization of the network being studied, which we called community network. By characterizing this meta structure, we concluded that geographical characteristics can influence the connectivity between communities, which takes place along characteristic betweenness paths.

Figure 11. Scatter plots comparing geographical and topological measurements in the community networks. The items (a)–(c) represent the unweighted degree against geographical characteristics and the items (d)–(f) represent the strength against geographical characteristics. The Pearson correlation, \( \rho \), is shown inside each plot.
It remains an open question if other, non-geographical, network models can also present such characteristic betweenness paths. Also, the proper impact that such paths may have on a dynamics taking place on the system is also an interesting aspect for future studies.

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