A sharp estimate for Neumann eigenvalues of the Laplace–Beltrami operator for domains in a hemisphere

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Here, we prove an isoperimetric inequality for the harmonic mean of the first $N-1$ non-trivial Neumann eigenvalues of the Laplace–Beltrami operator for domains contained in a hemisphere of $S^N$.

Keywords: Neumann eigenvalues; Laplace–Beltrami operator; sphere; isoperimetric inequalities.

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and let us consider the eigenvalues of the classical Neumann–Laplacian in $\Omega$,

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots$$

arranged in a non-decreasing sequence where each eigenvalue is repeated according to its multiplicity. Isoperimetric inequalities for the $\mu_i$’s go back to the classical theorem of Szegő [17] and Weinberger [19]: the ball maximizes $\mu_1(\Omega)$ among all bounded smooth domains $\Omega$ in $\mathbb{R}^N$ having the same measure. Szegő, using conformal
maps, proved it for simply connected domains in $\mathbb{R}^2$, while Weinberger introduced a method that allowed him to get this result in full generality in $\mathbb{R}^N$. His technique has been adapted in different contexts to establish isoperimetric results for combination of eigenvalues of the Laplacian with Dirichlet or Neumann boundary conditions (see e.g. [2] [5] [6] [9] [11] [12] [16]). For further references see, e.g. the monographs [10] [14] [15] and the survey paper [1]. Actually, as well-known, the conformal map technique used by Szegő allows to prove the stronger inequality

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \geq \frac{2}{\mu_1(\Omega^*)},$$

(1)

again for simply connected domains in $\mathbb{R}^2$. Here and in the sequel, $\Omega^*$ will denote the disk, or, more in general, the ball in $\mathbb{R}^N$ having the same measure as $\Omega$. Inequality (1) is sharp since equality is achieved if and only if $\Omega$ is a disk. Later, in [3], the assumption of simply connectedness was removed. In the same paper, the authors conjectured that an inequality analogous to (1) holds true in $\mathbb{R}^N$ ($N \geq 1$), namely

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_N(\Omega)} \geq \frac{N}{\mu_1(\Omega^*)}.$$  

Very recently, in [18] the authors made an important step toward the proof of this conjecture, by showing the following inequality

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_{N-1}(\Omega)} \geq \frac{N-1}{\mu_1(\Omega^*)},$$

for $N \geq 2$.

The aim of this paper is to prove an analogous result for the Laplace–Beltrami operator with Neumann boundary conditions. Precisely, we deal with nontrivial Neumann eigenvalues of an arbitrary domain $\Omega$ contained in a hemisphere of $\mathbb{S}^N$, defined by the following boundary value problem:

$$\begin{cases}
-\Delta_{\mathbb{S}^N} u = \mu u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}$$

(2)

where $\nu$ is the unit normal to $\partial \Omega$. We still denote with $\{\mu_i(\Omega)\}_i$, the non-decreasing sequence of eigenvalues of (2), where each eigenvalue is repeated according to its multiplicity, that is

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots .$$

Let us denote by $u_i$ an eigenfunction corresponding to $\mu_i(\Omega)$, with $i \in \mathbb{N}_0$. The following variational characterization holds true

$$\mu_i(\Omega) = \min \left\{ \frac{\int_{\Omega} |\nabla \phi|^2 \, d\omega}{\int_{\Omega} \phi^2 \, d\omega} : \phi \in H^1(\Omega) \setminus \{0\}, \phi \in \text{span}\{u_0, u_1, \ldots, u_{i-1}\}^\perp \right\}.$$  

(3)
The analogous of the Szegő–Weinberger result is already known and was proved in [4]. Our main result is the following.

**Theorem 1.1.** With the notation as above,

\[
\sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \geq \sum_{i=1}^{N-1} \frac{1}{\mu_i(D_\gamma)} = \frac{N-1}{\mu_1(D_\gamma)},
\]

where \(D_\gamma\) is a geodesic ball contained in a hemisphere of \(S^N\) having the same \(N\)-volume as \(\Omega\), and \(\gamma\) is its radius. More precisely, \(\gamma\) is determined by

\[
|\Omega| = N\omega_N \int_0^\gamma \sin^{N-1} t dt,
\]

where \(\omega_N\) denotes the volume of the unit ball in \(\mathbb{R}^N\). Equality sign holds in (4) if and only if \(\Omega\) is a geodesic ball.

2. Properties of the Neumann Eigenvalues and Eigenfunctions of a Geodesic Ball

Let \(D_\gamma\) be a geodesic ball on \(S^N\) having radius \(\gamma\). We think of this geodesic ball as the set of points of \(S^N\) with angle from the positive \(x_{N+1}\)-axis less than \(\gamma\), that is a polar cap. By the standard separation of variables technique, we find that the eigenvalues of (2), with \(\Omega = D_\gamma\), are the eigenvalues of the following one-dimensional problems

\[
\begin{cases}
-\frac{1}{\sin^{N-1} \theta} \frac{d}{d\theta} \left( \sin^{N-1} \theta \frac{d}{d\theta} y \right) + \frac{l(l + N - 2)}{\sin^2 \theta} y = \mu_{l,k} y & \text{in } (0, \gamma), \\
y(0) \text{ finite, } y'(\gamma) = 0
\end{cases}
\]

with \(l \in \mathbb{N}_0, k \in \mathbb{N}\). Clearly, \(\mu_1(D_\gamma) = \min\{\mu_{0,2}, \mu_{1,1}\}\). In [4] the authors show that \(\mu_1(D_\gamma) = \mu_{1,1}\) at least if \(\gamma \leq \frac{\pi}{2}\). Hence, an eigenfunction \(g\) (assumed positive) associated to \(\mu_{1,1} = \mu_1(D_\gamma)\) satisfies

\[
\begin{cases}
-\frac{g''}{g} - (N-1) \cot \theta g' + \frac{N-1}{\sin^2 \theta} g = \mu_1(D_\gamma) g & \text{in } (0, \gamma), \\
g(0) = g'(\gamma) = 0
\end{cases}
\]

(5)

Multiplying the equation in (5) by \(g\) and then integrating on \(D_\gamma\) yields

\[
\mu_1(D_\gamma) = \frac{\int_{D_\gamma} \left[ g'(\theta)^2 + (N-1) \frac{g(\theta)^2}{\sin^2 \theta} \right] d\omega}{\int_{D_\gamma} g(\theta)^2 d\omega}.
\]

(6)

The following properties are also proved in [4]:

(i) If \(0 < \gamma \leq \frac{\pi}{2}\), then \(g' > 0\) in \([0, \gamma]\), thus \(g\) is strictly increasing in \([0, \gamma]\),
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(ii) $\mu_1(D_\gamma)$ is a strictly decreasing function of $\gamma$ for $0 < \gamma \leq \frac{\pi}{2}$, 
(iii) $\mu_1(D_\gamma) > N = \mu_1(D_{\pi/2})$ for $0 < \gamma < \frac{\pi}{2}$.

We also recall that $\mu_1(D_\gamma)$ is $N$-fold degenerate, that is 
$\mu_1(D_\gamma) = \mu_2(D_\gamma) = \cdots = \mu_N(D_\gamma)$.

Now, define $G : [0, \frac{\pi}{2}] \rightarrow [0, +\infty)$ by

$$G(\theta) = \begin{cases} g(\theta) & \theta \leq \gamma, \\ g(\gamma) & \theta > \gamma. \end{cases} \quad (7)$$

Then, we have the following.

**Lemma 2.1.** The function $\frac{G(\theta)}{\sin \theta}$ is strictly decreasing in $[0, \frac{\pi}{2}]$.

**Proof.** By Taylor–Frobenius expansion we have $G(\theta) = \theta - a \theta^3 + o(\theta^3)$, where

$$a = \frac{\mu_1(D_\gamma) - \frac{2}{3}(N-1)}{2N + 4} > 0.$$ 

We explicitly observe that we are assuming $G'(0) = 1$. In order to get the claim it is enough to prove that $W(\theta) := G'(\theta) - G(\theta) \cot \theta < 0$.

Using the behavior of $G(\theta)$ near $\theta = 0$, we have

$$W(\theta) = \left(\frac{1}{3} - 2a\right) \theta^2 + o(\theta^2) = \left(\frac{N - \mu_1(D_\gamma)}{N + 2}\right) \theta^2 + o(\theta^2).$$ 

Property (iii) implies that $W(\theta) < 0$ is close to 0. We also know that $W(\gamma) < 0$. Assume by contradiction that $W(\theta)$ attained a positive maximum at a point $\tilde{\theta} \in (0, \gamma)$. Hence

$$W(\tilde{\theta}) > 0, \quad W'(\tilde{\theta}) = G''(\tilde{\theta}) - G'(\tilde{\theta}) \cot \tilde{\theta} + \frac{G(\tilde{\theta})}{\sin^2 \tilde{\theta}} = 0.$$

Using this last identity in $\mathbb{M}$, we obtain

$$N \left[ G'(\tilde{\theta}) \cot \tilde{\theta} - \frac{G(\tilde{\theta})}{\sin^2 \tilde{\theta}} \right] = -\mu_1(D_\gamma) G(\tilde{\theta}),$$

that is

$$N[W(\tilde{\theta}) \cot \tilde{\theta} - G(\tilde{\theta})] = -\mu_1(D_\gamma) G(\tilde{\theta}).$$

Since we are assuming that $W(\tilde{\theta}) > 0$, property (3) immediately gives a contradiction. \(\square\)

**Remark 2.2.** (i) Notice that $\sin \theta \cdot W(\theta)$ is the Wronskian between the eigenfunction corresponding to the first nontrivial Neumann eigenvalue of $D_{\pi/2}$, whose derivative is equal to 1 at 0, and the extension of the eigenfunction corresponding to the first nontrivial Neumann eigenvalue of $D_\gamma$, $0 < \gamma < \pi / 2$, introduced in $(7)$. 

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(ii) Using the same method as above we can prove that $G(\theta)$ is concave and, in fact, that $G'(\theta)$ is decreasing. This, together with the result of Lemma 2.1, provides an alternative proof of the fact that the function $B(\theta)$ in [4] Theorem 4.1 is a decreasing function of $\theta$.

3. Some Mathematical Tools Needed for the Proof of Theorem 1.1

For the proof of our main result, Theorem 1.1, it is convenient to parametrize the points of $\Omega$ in terms of the coordinates of their stereographic projection (see, for example, [7, 13]). For a point $P \in \Omega$, we denote by $P'$ its stereographic projection from the south pole $S$ onto the “equator” (as illustrated in Fig. 1).

For $P'$ we use Cartesian coordinates $(x_1, x_2, \ldots, x_N, 0)$. We also use $s = \sqrt{\sum_{i=1}^{N} x_i^2}$, the Euclidean distance from $P'$ to the origin $O$. As usual we denote by $\theta$ the azimuthal angle, i.e. the angle between $ON$ and $OP$, where $N$ stands for the north pole. Moreover, we denote by $\varphi$ the angle between $SN$ and $SP$. It is clear that $\theta = 2\varphi$ and $\tan \varphi = s$. Hence,

$$\theta = 2 \arctan s \quad (8)$$

from which we immediately get

$$\frac{d\theta}{ds} = \frac{2}{1 + s^2} = p(s), \quad (9)$$

the conformal factor associated to the differential structure on $S^N$. In terms of the conformal factor $p$ we can write

$$\nabla_{S^N} = \frac{1}{p} \nabla_{\mathbb{R}^N},$$

Fig. 1. Stereographic coordinates.
where $\nabla_{R^N}$ is the standard gradient on the equator. We also have

$$-\Delta_{S^N} = - p^{-N} \text{div}(p^{N-2} \nabla_{R^N} u).$$

Finally, from the figure (or directly from (8) and (9)) we also have that

$$\sin \theta = p \cdot s.$$  (10)

In the sequel we also need to compute $\theta, i := \frac{\partial \theta}{\partial x_i}$. Using (9), the definition of $s$ and the chain rule we have

$$\theta, i = \frac{\partial \theta}{\partial s} \cdot s, i = 1, \ldots, N,$$

and

$$\sum_{i=1}^N \theta, i^2 = p^2.$$  (11)

With the notation introduced above, we define

$$\Phi_i(x) = G(\theta) \frac{x_i}{s}, \quad i = 1, \ldots, N,$$  (12)

where $G(\theta)$ is defined in (7). In order to use $\Phi_i$ as test function in (3), we need the following orthogonality conditions

$$\int_{\Omega} \Phi_i u_j d\omega = 0, \quad i = 1, \ldots, N, \quad j = 0, \ldots, i - 1,$$  (13)

where, as we said, $u_j$ is an eigenfunction corresponding to $\mu_j(\Omega)$. To fulfill these conditions, we need a special “orientation” of the sphere $S^N$. When $j = 0$, conditions (13) can be immediately deduced from [4, Theorem 2.1] via the following identity:

$$\int_{\Omega} \Phi_i d\omega = \int_{\Omega} G(\theta) \frac{x_i}{s} d\omega = \int_{\Omega} \frac{G(\theta)}{\sin \theta} y_i d\omega,$$

choosing $\hat{G}(\theta) = \frac{G(\theta)}{\sin \theta}$. When $j > 0$, conditions (13) can be proved arguing in an analogous way as in the proof of [3 Theorem 2.1].

4. Proof of Theorem 1.1

Recalling the definition of $\Phi_i$ given in (12), we get

$$(\nabla \Phi_i)_j \equiv \Phi_{i,j} = G'(\theta) p \frac{x_i x_j}{s^2} + G(\theta) \frac{x_i x_j}{s^2} = G(\theta) \frac{x_i x_j}{s^2}, \quad j = 1, \ldots, N.$$  (14)

Using (11), the definition of $s$ and (12), we have

$$\frac{1}{p^2} |\nabla \Phi_i|^2 = \frac{x_i^2}{s^2} + \frac{G(\theta)^2}{s^2} \frac{1}{p^2} - \frac{G(\theta)^2 x_i^2}{p^2 s^4}. $$  (15)
Hence, from (10) and (15),

$$\sum_{i=1}^{N} |\nabla_{\Sigma} \Phi_i|^2 = \frac{1}{p^2} \sum_{i=1}^{N} |\nabla \Phi_i|^2 = G'(\theta)^2 + G(\theta)^2 \frac{N-1}{s^2 p^2} = G'\theta)^2 + G(\theta)^2 \frac{N-1}{\sin^2 \theta}.$$  

Taking into account the orthogonality conditions (13), we use $\Phi_i$ as test function in the variational characterization (3) of $\mu_i(\Omega)$, and we get

$$\int_{\Omega} \Phi_i^2 d\omega \leq \frac{1}{\mu_i(\Omega)} \int_{\Omega} G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega$$

$$= \frac{1}{\mu_i(\Omega)} \int_{\Omega\cap D_\omega} G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega$$

$$\leq \frac{1}{\mu_i(\Omega)} \int_{D_\omega} G'(\theta)^2 \frac{x_i^2}{s^2} d\omega + \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega$$

$$= \frac{1}{N\mu_i(\Omega)} \int_{D_\omega} G'(\theta)^2 d\omega + \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega. \quad (16)$$

Summing over $i = 1, \ldots, N$ we get

$$\int_{\Omega} G(\theta)^2 d\omega \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\omega} G'(\theta)^2 d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} \left(1 - \frac{x_i^2}{s^2}\right) d\omega.$$  

Now notice that

$$\sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \left(1 - \frac{x_i^2}{s^2}\right) - \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} = \frac{1}{\mu_N(\Omega)} - \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \frac{x_i^2}{s^2} \leq 0,$$

which follows from $\mu_i(\Omega) \leq \mu_N(\Omega)$ for all $i = 1, \ldots, N - 1$ and the definition of $s$. Hence,

$$\int_{\Omega} G(\theta)^2 d\omega \leq \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\omega} G'(\theta)^2 d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} d\omega. \quad (17)$$

By Lemma 2.1 we know that the function $\frac{G(\theta)}{\sin \theta}$ is decreasing in $\langle 0, \gamma \rangle$. Recalling that $|\Omega| = |D_\gamma|$, we get

$$\int_{\Omega} \frac{G(\theta)^2}{\sin^2 \theta} d\omega = \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2 \theta} d\omega + \int_{\Omega \setminus D_\gamma} \frac{G(\theta)^2}{\sin^2 \theta} d\omega$$

$$\leq \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2 \theta} d\omega + \int_{\Omega \setminus D_\gamma} \frac{G(\gamma)^2}{\sin^2 \gamma} d\omega$$

$$= \int_{\Omega \setminus D_\gamma} \frac{G(\gamma)^2}{\sin^2 \gamma} |\Omega \setminus D_\gamma|$$

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\[ \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2 \theta} d\omega + \frac{G(\gamma)^2}{\sin^2 \gamma} |D_\gamma \setminus \Omega| \leq \int_{\Omega \cap D_\gamma} \frac{G(\theta)^2}{\sin^2 \theta} d\omega + \int_{D_\gamma \setminus \Omega} \frac{G(\theta)^2}{\sin^2 \theta} d\omega \]
\[ = \int_{D_\gamma} g(\theta)^2 d\omega. \quad (18) \]

On the other side, since \( G(\theta) \) is non-decreasing in \((0, \frac{\pi}{2})\), we have
\[
\int_{\Omega} G(\theta)^2 d\omega = \int_{\Omega \cap D_\gamma} G(\theta)^2 d\omega + \int_{\Omega \setminus D_\gamma} G(\theta)^2 d\omega \\
\geq \int_{\Omega \cap D_\gamma} G(\theta)^2 d\omega + G(\gamma)^2 |\Omega \setminus D_\gamma| \\
= \int_{\Omega \cap D_\gamma} G(\theta)^2 d\omega + G(\gamma)^2 |D_\gamma \setminus \Omega| \\
\geq \int_{\Omega \cap D_\gamma} G(\theta)^2 d\omega + \int_{D_\gamma \setminus \Omega} g(\theta)^2 d\omega \\
= \int_{D_\gamma} g(\theta)^2 d\omega. \quad (19) \]

Using (17)–(19) and the monotonicity of the sequence \( \{\mu_i(\Omega)\}_i \), we have
\[
\int_{D_\gamma} g(\theta)^2 d\omega \leq \frac{1}{N-1} \sum_{i=1}^{N} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} g'(\theta)^2 d\omega + \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \frac{g(\theta)^2}{\sin^2 \theta} d\omega \\
\leq \frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \int_{D_\gamma} \left[ g'(\theta)^2 + (N-1) \frac{g(\theta)^2}{\sin^2 \theta} \right] d\omega.
\]

Finally, from (6) we conclude
\[
\frac{1}{N-1} \sum_{i=1}^{N-1} \frac{1}{\mu_i(\Omega)} \geq \frac{1}{\mu_1(D_\gamma)}. \quad (20)
\]

Since \( G(\theta)/\sin \theta \) is strictly decreasing in \((0, \gamma)\), when \( \gamma < \pi/2 \), the equality sign holds in (20) if and only if \( \Omega \) is a geodesic ball.

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