Abstract

This paper proposes a method for estimating the norms of a system in a pure data-driven fashion based on their identified Impulse Response (IR) coefficients using only a single batch of data (one-shot). The calculation of norms is briefly reviewed and the main expressions for the IR-based estimations are presented. As a case study, the $H_1$, $H_2$, and $H_\infty$ norms of the sensitivity transfer function of five different discrete-time closed-loop systems are estimated for a Signal-to-Noise-Ratio (SNR) of 10 dB, achieving low percent error values if compared to the real value. To verify the influence of the noise amplitude, norms are estimated considering a wide range of SNR values, for a specific system, presenting low Mean Percent Error (MPE) if compared to the real norms. The proposed technique is also compared to an existing state-space-based method in terms of $H_\infty$, through Monte Carlo experiments, showing a reduction of approximately 48% in the MPE for a wide range of SNR values.

Key words: Norm estimation; Data-driven; Impulse Response.

1 Introduction

A system norm is a single number that contains information regarding gain, energy, robustness, among other possible physical interpretations, which can be used as a tool for system analysis and controller design [1]. In a case where the model of a process is not available or is too complex to be obtained, an alternative data-driven method can be used to estimate system norms.

The most common data-driven approach is to obtain the norms via frequency response [2], but it usually requires more than one experiment to acquire the necessary data. Iterative methods as Power Iterations and Weighted Thompson Sampling [3] are also a viable solution at the expense of high computational cost. A time domain approach for $H\infty$ estimation with the Toeplitz matrix of the system’s Markov parameters is presented in [4], and a different computation to the same approach is shown in [5], based on the system’s subspace (state-space) identification.

This work, inspired by [5], presents a time domain data-driven and one-shot method for estimating the norms of a Single-Input Single-Output (SISO) system solely relying on its estimated Impulse Response (IR) coefficients. In order to achieve a lower estimation error than state-space-based methods with corrupted data, the IR is identified through regularized least squares [6]. Solutions are presented for norms $H_1$, $H_2$, and $H\infty$.

2 Norm estimation by impulse response

The output signal $y(k)$ of a linear discrete-time causal and SISO system $G$ is given by

$$G : y(k) = g(k) * u(k) = \sum_{n=0}^{\infty} g(k-n)u(n),$$

which is the convolution between the input signal $u(k)$ and the signal that represents the impulse response of the system, $g(k)$. In other words, knowing $g(k)$ is enough to characterize any input-output relation of a linear system $G$. Other important definitions are the norms of signals and systems, widely used to characterize their measurements. According to [1 A.5], the norms $\mathcal{L}_p$, $\mathcal{L}_\infty$, used for signals, can be defined as
where \( x(k) \) is any time domain signal, and \( p \) can assume different values, typically 1 or 2. The \( H_1, H_2 \) and \( H_\infty \) norms used for SISO systems, also from [1, A.5], are defined as

\[
\begin{align*}
\mathcal{H}_1 : ||G||_1 & = \sum_{k=0}^{\infty} |g(k)| = \max_{u(k) \neq 0} \frac{||g(k) \ast u(k)||_\infty}{||u(k)||_\infty}, \\
\mathcal{H}_2 : ||G||_2 & = \sqrt{\sum_{k=0}^{\infty} |g(k)|^2} = ||g(k)||_2, \\
\mathcal{H}_\infty : ||G||_\infty & = \max_{u(k) \neq 0} \frac{||g(k) \ast u(k)||_2}{||u(k)||_2}.
\end{align*}
\]

In the case of a stable system, it is known that \( \lim_{k \to \infty} g(k) = 0 \). In order to use limited data for estimating system norms, the IR terms with an order greater than \( M \) are assumed to be negligible. Therefore, the convolution in \( (4) \) can be truncated at \( M \) terms, resulting in the approximation

\[
G : y(k) = \sum_{n=0}^{\infty} g(k-n)u(n) \approx \sum_{n=0}^{M} g(k-n)u(n) \quad \text{as} \quad |g(M+1)| < \epsilon, \quad \text{with} \quad \epsilon \to 0^+.
\]

Notice that only the \( M \) first elements of the IR are considered to be sufficient to characterize the input-output relation of a system. In terms of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) norms, from \( (4) \) and \( (5) \), considering the truncated convolution in \( (7) \), the following approximations can be obtained:

\[
\begin{align*}
||G||_1 & \approx \sum_{k=0}^{M} |g(k)|, \\
||G||_2 & \approx \sqrt{\sum_{k=0}^{M} |g(k)|^2}.
\end{align*}
\]

For the \( \mathcal{H}_\infty \) norm, which represents the maximum value of convolution considering all sets of input signals, expression \( (6) \) cannot be directly used, since it requires to consider all possible input signals in its evaluation. An alternative strategy is to obtain a matrix relation for \( g(k) \), allowing for the use of induced norm properties. Expanding the relation given in \( (7) \) to the \( M \) first terms,

\[
\begin{align*}
y(0) & = g(0)u(0) \\
y(1) & = g(1)u(0) + g(0)u(1) \\
\cdots \\
y(M) & = g(M)u(0) + \cdots + g(0)u(M),
\end{align*}
\]

the following matrix relation is obtained:

\[
\begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(M)
\end{bmatrix} =
\begin{bmatrix}
g(0) & 0 & \cdots & 0 \\
g(1) & g(0) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
g(M) & g(M-1) & \cdots & g(0)
\end{bmatrix}
\begin{bmatrix}
u(0) \\
u(1) \\
\vdots \\
u(M)
\end{bmatrix}.
\]

It should be noted that, in [11], matrix \( G_M \) represents the terms of the system’s impulse response, in the form of a Toeplitz matrix, and that its multiplication with the input vector \( U_M \), truncated at \( M \) elements, results in the output vector \( Y_M \). From the assumption that \( M \) is sufficiently high, it can be said that matrix \( G_M \) characterizes the IR \( g(k) \) and, consequently, system \( G \).

The characterisation of system \( G \) in a matrix form, represented by \( G_M \), allows the use of some matrix properties, as it is the case of the induced norm, [11, A.5]:

\[
||G_M||_p = \max_{U_M \neq 0} \frac{||G_M U_M||_p}{||U_M||_p},
\]

where the subscript \( i_p \) stands for induced. In short, \( ||G_M||_p \) is a form of representing the gain of the system \( G \) considering a set of possible input signals \( U_M \). Among the possible induced norms, the induced-2 norm can be highlighted:

\[
||G_M||_2 = \sigma(G_M) = \sqrt{\lambda_max(G_M^t G_M)},
\]

where \( \sigma \) and \( \lambda_{max} \) stands for largest singular value and largest eigenvalue, respectively. Note that \( (13) \) is evaluated directly with \( G_M \), which can be obtained with a single input signal. Comparing \( (13) \) to the \( \mathcal{H}_\infty \) norm definition from \( (6) \), knowing that convolution \( (7) \) is equivalent to the matrix expression in \( (11) \), it is observed that

\[
||G||_\infty \approx \max_{U_M \neq 0} \frac{||G_M U_M||_2}{||U_M||_2} = ||G_M||_2.
\]

Consequently, the calculation of the \( \mathcal{H}_\infty \) norm can be approximated by

\[
||G||_\infty \approx \sigma(G_M) = \sqrt{\lambda_max(G_M^t G_M)}.
\]

Note that in \( (8), (9), \) and \( (15) \), only the knowledge of the \( M \) first elements of the impulse response is sufficient to estimate the system norms. Using this idea, one-shot algorithms for the IR estimation can be used to estimate the coefficients of \( g(k) \), and consequently, the norms of \( G \). An option for a more precise estimation of the IR is to use regularization, for the following reasons: i) the
The objective of this case study is to obtain $H_1$, $H_2$ and $H_\infty$ norms of the sensitivity transfer function $S(z)$ of a closed-loop discrete-time system, where $z$ is the discrete time-shift operator. In practice, the norm $||S(z)||_\infty$ is used as a measure of robustness [1]. In a data-driven case, for example, $S(z)$ cannot be obtained with models since they are not available. Assuming an experiment where a given reference signal $r(k)$ is applied to the closed-loop system, generating an output $y(k)$, being

$$y(k) = T(z)r(k) + S(z)n(k),$$

where

$$T(z) = \frac{C(z)G(z)}{1 + C(z)G(z)} \quad S(z) + T(z) = 1,$$

and $n(k)$ is the output noise. It follows that $y(k) = [1 - S(z)]r(k) + S(z)n(k) = r(k) + S(z)[n(k) - r(k)]$. Consequently $y(k) - r(k) = S(z)[n(k) - r(k)]$, resulting in $r(k) - y(k) = C^{-1}(z)u(k) = S(z)[r(k) - n(k)]$. Henceforth, the input-output set $\{r(k), C^{-1}(z)u(k), k = 1...N\}$ can be acquired from the process and used for the estimation of the impulse response of $S(z)$, $s(k)$. Note that $n(k)$ is not considered in the data set since it cannot be acquired. The norms $||S(z)||_1$ and $||S(z)||_2$ can be directly obtained by expressions [3] and [4], respectively. In order to estimate the $H_\infty$ norm of $S(z)$, a Toeplitz matrix $S_M$ can be built with its IR, as in [5]. Finally, from [15], $||S(z)||_\infty \approx ||S_M||_2$, with $M$ sufficiently large.

3.1 Examples

To illustrate the proposed method, five closed-loop systems - plant $G(z)$ and controller $C(z)$ - are used as subjects, since they represent the structure of systems that are commonly found in engineering problems, which are presented in Table 1. Table 2 shows the $H_1$, $H_2$ and $H_\infty$ norms of $S(z)$ calculated by model, assumed here as the real values, and estimated by the proposed data-driven method. The reference signal is a Pseudo-Random Binary Signal (PRBS) with $N = 2000$ samples, and the length of the IR was arbitrarily set as $M = 100$. The IR is estimated in a regularized fashion with the Tuned-Correlated (TC) kernel [8]. Additive white Gaussian noise with zero mean and a Signal-to-Noise-Ratio (SNR) of 10 dB was included in the process output $y(k)$ and fed back to the system. The percent error between the real and estimated value is also shown in Table 2 and indicates good performance of the proposed technique for all cases, since the obtained error values are low.

Now, in order to verify the influence of the noise amplitude, $H_1$, $H_2$, and $H_\infty$ norms of $S(z)$ for System 2 are estimated with an SNR value varying from 0.1 to 50 dB with a step of 0.1 dB for the noise at the output, as it is shown in Figure 1. The estimations are done using the same $r(k)$ as aforementioned. The Mean Percent Error (MPE) between estimated and real norm is used as a measure of precision, given as the average of the errors found for each experiment for all SNR values. The estimation of $H_1$ for the whole tested SNR range presents an MPE of 2.4287 %, whilst for the norm $H_2$ the MPE is 0.0903 %, and for $H_\infty$ the MPE achieves 0.3729 %.

For the sake of comparison, Figure 2 presents a Monte Carlo experiment of 100 different noise realizations for each SNR value of the estimated $H_\infty$ norm of $S(z)$ for System 1, considering $M = 100$ for all cases, by the technique proposed in this paper in a regularized fashion, and by the state-space-based technique proposed in [5]. The approach proposed in this work obtained an MPE for the Monte Carlo experiment of 0.6351 %, whilst the state-space approach achieved 1.2187 % of MPE, showing a reduction of 47.8871 % in MPE.

4 Conclusion

This paper presented a one-shot method for estimating the $H_1$, $H_2$, and $H_\infty$ norms of a system, using time domain signals, relying solely on its estimated impulse response coefficients. A closed loop case study with the objective of estimating the $H_1$, $H_2$, and $H_\infty$ norms of $S(z)$ is presented for five different systems. All data-driven estimated norms were close to the real norms, showing a maximum error of 5.2485 % - for System 2, norm $H_1$, and $M = 100$. For a wide range of SNR, norms were estimated for System 2, presenting a low MPE for all cases. In a Monte Carlo comparison to a state-space-based technique, for the $H_\infty$ norm, the proposed method has shown to reduce the MPE by 47.8871 %. As for future related research subjects, are suggested: finding bounds for the estimation error; automatic choice of the estimated IR length $M$.

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Table 1
System’s transfer functions $G(z)$ and controllers $C(z)$ used as examples.

| System | $G(z)$ | $C(z)$ |
|--------|--------|--------|
| 1      | $\frac{0.5}{(z-0.9)}$ | $0.3979(z-0.9)$ |
| 2      | $\frac{-0.1(z-0.5)}{(z-0.9)(z-0.8)}$ | $1-1.1600(z-0.9719)$ |
| 3      | $\frac{-0.05(z+0.2)}{(z^2+1.8z+0.82)}$ | $-3.7144(z-0.9351)(z-0.4210)$ |
| 4      | $\frac{-0.05(z-1.4)}{(z-0.9)(z-0.8)}$ | $\frac{-4.7942(z-0.9)(z-0.8)}{z(z-1)}$ |
| 5      | $\frac{2.05(z+0.55)(z^2-1.62z+0.6586)}{(z^2-1.84z+0.8561)(z^2-1.26z+0.4069)}$ | $0.0519(z-0.8977)$ |

Table 2
Each system’s $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ norm of $S(z)$ calculated by model (Real), estimated via data (Data), and its percent error.

| System | Real | Data | Error (%) | Real | Data | Error (%) | Real | Data | Error (%) |
|--------|------|------|-----------|------|------|-----------|------|------|-----------|
| 1      | 2.0000 | 2.0076 | 0.3825 | 1.0511 | 1.0520 | 0.0784 | 1.1049 | 1.1210 | 1.4635 |
| 2      | 2.0544 | 1.9466 | 5.2485 | 1.0441 | 1.0428 | 0.1293 | 1.1619 | 1.1521 | 0.8408 |
| 3      | 2.1656 | 2.1496 | 0.7358 | 1.0542 | 1.0509 | 0.3166 | 1.1272 | 1.1331 | 0.5186 |
| 4      | 2.4794 | 2.4778 | 0.0649 | 1.0811 | 1.0846 | 0.3175 | 1.5348 | 1.5375 | 0.1767 |
| 5      | 2.0307 | 1.9970 | 1.6578 | 1.0523 | 1.0543 | 0.1889 | 1.1006 | 1.1224 | 1.9781 |

Fig. 1. Estimated $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of $S(z)$, for System 2, with its real values in dashed lines, for a wide SNR range.

Fig. 2. Mean value of 100 Monte Carlo runs for the proposed and a state-space method [5], for estimating $\|S(z)\|_\infty$ of System 1, as well as its box plot representation.

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