The rank of the walk matrix of the extended Dynkin graph $\tilde{D}_n$

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Abstract

In this paper, we provide an explicit formula for the rank of the walk matrix of the extended Dynkin graph $\tilde{D}_n$.

1 Introduction

Let $G$ be a simple graph with $n$ vertices and let $A$ be an adjacency matrix of $G$. The walk matrix $W(G)$ of $G$ is defined by

$$W(G) = [e_n \ Ae_n \ \cdots \ A^{n-1}e_n],$$

where $e_n$ is the all-one vector of length $n$. The $(i,j)$-entry of the walk matrix $W(G)$ counts the number of walks in $G$ of length $j - 1$ from the vertex $i$. A tree obtained from the path of order $n - 1$ by adding a pendant edge at the second vertex is called a Dynkin graph $D_n$ (See Figure 1) and by adding a pendant edge to the second to last vertex of $D_n$, we get the extended Dynkin graph $\tilde{D}_n$ (See Figure 2), where $n \geq 4$.

![Figure 1: Dynkin graph $D_n$](image)

Dynkin graphs and extended Dynkin graphs are widely used in the study of simple Lie algebras [3, 6, 7], representation theory [1, 5, 8, 11, 12, 14, 16], and spectral theory [2, 4, 13].

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In [15], Wang, Wang and Guo gave a formula for the rank of the walk matrix of the Dynkin graph $D_n$ ($n \geq 4$). In this paper, we find a formula for the rank of the walk matrix of the extended Dynkin graph $\tilde{D}_n$ ($n \geq 4$) which is $\text{rank } W(\tilde{D}_n) = \lfloor \frac{n}{2} \rfloor$.

For a graph $G$ with $n$ vertices, an eigenvalue of the adjacency matrix of $G$ is said to be main if its corresponding eigenvector is not orthogonal to $e_n$. In [10], Hagos showed that the rank of the walk matrix of a graph $G$ is equal to the number of its main eigenvalues. Thus our formula also provides the number of the main eigenvalues of the walk matrix of the extended Dynkin graph.

2 Rank of the extended Dynkin graph

Let $\tilde{D}_n$ ($n \geq 4$) be the extended Dynkin graph with $n + 1$ vertices. We label the vertices of $\tilde{D}_n$ as shown in Figure 2. Then the partition

$$\Pi = \{\{1, 2\}, \{3\}, \ldots, \{n - 1\}, \{n, n + 1\}\}$$

of $V(\tilde{D}_n)$ is equitable. The $(n+1) \times (n-1)$ characteristic matrix $P$ of $\Pi$ and the $(n-1) \times (n-1)$ divisor matrix $B$ of $\Pi$ are

$$P = \begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\cdots & \cdots & \cdots \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 & 1 & 2 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 1 & 1 & 2 \\
1 & 0 & 2 & 1 & 0
\end{bmatrix},$$

respectively.

For an $n \times n$ matrix $M$, the Smith normal form of $M$ is the diagonal matrix $\text{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0)$, where $r = \text{rank}(M)$ and each $d_i$ divides $d_{i+1}$ for $i = 1, \ldots, r - 1$. Two integral matrices $M_1$ and $M_2$ are integrally equivalent if there are unimodular matrices $P$ and $Q$ such that $M_2 = PM_1Q$, that is, $M_2$ can be obtained from $M_1$ by a finite sequence of elementary row operations.
operations and column operations. Two matrices are integrally equivalent if and only if they have the same Smith normal form (See [9] for details).

Let \( \hat{W}(\tilde{D}_n) \) be the \((n-1) \times (n-1)\) matrix obtained by removing the first and last rows and the last two columns from \( W(\tilde{D}_n) \). Then we define an \((n+1) \times (n+1)\) matrix \( W'(\tilde{D}_n) \) in which \( \hat{W}(\tilde{D}_n) \) is a submatrix as follows:

\[
W'(\tilde{D}_n) = \begin{bmatrix}
0 & 0 & 0 \\
\hat{W}(\tilde{D}_n) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

For an \( n \times n \) matrix \( M \), the walk matrix \( W(M) \) of \( M \) is defined by

\[
W(M) = [e_n \ M e_n \ \cdots \ M^{n-1} e_n].
\]

The following lemma provides a method for computing the rank of a walk matrix.

**Lemma 2.1.** \([13, \text{Lemma 2}]\) Let \( M \) be a real \( n \times n \) matrix which is diagonalizable over the real field \( \mathbb{R} \). Let \( v_1, v_2, \ldots, v_n \) be \( n \) linearly independent eigenvectors of \( M^T \) corresponding to eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \), respectively. Then we have

\[
\det W(M) = \frac{\prod_{1 \leq k < j \leq n} (\lambda_j - \lambda_k) \prod_{j=1}^{n} e_n^T v_j}{\det[v_1, v_2, \ldots, v_n]}.
\]

Moreover, if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are pairwise different, then

\[
\text{rank} W(M) = |\{j : 1 \leq j \leq n \text{ and } e_n^T v_j \neq 0\}|.
\]

We outline the steps for establishing our formula for the rank of the extended Dynkin graph \( \tilde{D}_n \).

1. Using Lemma 2.1 we find a formula for the rank of the walk matrix of \( B \).
2. We show that \( \hat{W}(\tilde{D}_n) = W(B) \), which implies \( \text{rank} \hat{W}(\tilde{D}_n) = \text{rank} W(B) \).
3. We prove that the Smith normal form of \( W(\tilde{D}_n) \) and \( W'(\tilde{D}_n) \) are identical.
4. We conclude that

\[
\text{rank} W(\tilde{D}_n) = \text{rank} W'(\tilde{D}_n) = \text{rank} \hat{W}(\tilde{D}_n) = \text{rank} W(B).
\]

We start by finding the eigenvalues of \( B^T \) and the corresponding eigenvectors.
Proposition 2.2. Let \( \lambda_k = 2 \cos \frac{k\pi}{n-2} \) for \( k = 0, \ldots, n-3 \) and \( \lambda_{n-2} = -2 \). Let \( v_k = \begin{bmatrix} 1 & \cos \frac{k\pi}{n-2} & \cos \frac{2k\pi}{n-2} & \cdots & \cos \frac{(n-3)k\pi}{n-2} & \cos k\pi \end{bmatrix}^T \) for \( k = 0, \ldots, n-3 \) and \( v_{n-2} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \end{bmatrix}^T \). Then \( v_k \) is an eigenvector of \( B^T \) corresponding to the eigenvalue \( \lambda_k \), that is, \( B^T v_k = \lambda_k v_k \) for \( k = 0, \ldots, n-2 \).

Proof. Since \( v_0 = e_{n-1} \) and \( \lambda_0 = 2 \), \( B^T v_0 = \lambda_0 v_0 \). It is easily check that \( B^T v_{n-2} = \lambda_{n-2} v_{n-2} \).

Now, we show that \( B^T v_k = \lambda_k v_k \) for \( k = 1, \ldots, n-3 \). Let \( \theta = k\pi/(n-2) \).

Since \( \lambda_k = 2 \cos \theta \) and \( \cos m\theta + \cos (m+2)\theta = 2 \cos \theta \cos (m+1)\theta \), we obtain

\[
\cos m\theta + \cos (m+2)\theta = \lambda_k \cos (m+1)\theta
\]

for \( m = 0, \ldots, n-4 \). We note that

\[
2 \cos(n-3)\theta = 2( \cos(n-2)\theta \cos \theta + \sin(n-2)\theta \sin \theta )
\]

Since \( \sin(n-2)\theta = \sin k\pi = 0 \) for all \( k = 1, \ldots, n-3 \), we have

\[
2 \cos(n-3)\theta = 2 \cos \theta \cos(n-2)\theta = \lambda_k \cos k\pi
\]

Hence \( B^T v_k = \lambda_k v_k \) for all \( k = 0, \ldots, n-2 \). \( \square \)

The following lemma will be used in the proof of Lemma 2.4.

Lemma 2.3. ([15] Lemma 7)

\[
\sum_{k=1}^{n} \cos(ak+b)x = \frac{1}{2\sin \frac{1}{2}ax} \left( \sin \left( \frac{n+1}{2}a+b \right)x - \sin \left( \frac{1}{2}a+b \right)x \right)
\]

for \( \sin \frac{1}{2}ax \neq 0 \).

Lemma 2.4. Let \( v_k = \begin{bmatrix} 1 & \cos \frac{k\pi}{n-2} & \cos \frac{2k\pi}{n-2} & \cdots & \cos \frac{(n-3)k\pi}{n-2} & \cos k\pi \end{bmatrix}^T \) for \( k = 0, \ldots, n-3 \) and \( v_{n-2} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots \end{bmatrix}^T \). Then

\[
e_{n-1}^Tv_k = n-1 \quad \text{and} \quad e_{n-1}^Tv_k = \begin{cases} 1, & \text{if } k \text{ is even} \\ 0, & \text{if } k \text{ is odd} \end{cases}
\]

for \( k = 1, \ldots, n-2 \).
\textbf{Proof.} It is straightforward to check that
\[ e_{n-1}^T v_0 = n - 1 \quad \text{and} \quad e_{n-1}^T v_{n-2} = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \]

Now, we give the proof for \( k = 1, \ldots, n - 3 \). Let \( \theta = k \pi / (n - 2) \). Then
\[ e_{n-1}^T v_k = 1 + \cos k \pi + \sum_{m=1}^{n-3} \cos m \theta. \]

By Lemma 2.3, we have
\[
\sum_{m=1}^{n-3} \cos m \theta = \frac{1}{2 \sin \frac{1}{2} \theta} \left( \sin \left( n - \frac{5}{2} \right) \theta - \sin \frac{1}{2} \theta \right)
= \frac{1}{\sin \frac{1}{2} \theta} \cos \frac{k}{2} \pi \sin \frac{n - 3}{2} \theta.
\]

If \( k \) is odd, then \( \cos k \pi = -1 \) and \( \cos \frac{k}{2} \pi = 0 \) and hence \( e_{n-1}^T v_k = 0 \). Suppose that \( k \) is even. Then \( \cos k \pi = 1 \) and \( \cos \frac{k}{2} \pi = \pm 1 \). Since
\[
\sin \frac{n - 3}{2} \theta = \sin \left( \frac{n - 2}{2} \theta - \frac{1}{2} \theta \right)
= \sin \frac{n - 2}{2} \theta \cos \frac{1}{2} \theta - \sin \frac{1}{2} \theta \cos \frac{n - 2}{2} \theta
= \sin \frac{k}{2} \pi \cos \frac{1}{2} \theta - \sin \frac{1}{2} \theta \cos \frac{k}{2} \pi
= - \sin \frac{1}{2} \theta \cos \frac{k}{2} \pi,
\]
we obtain
\[ \sum_{m=1}^{n-3} \cos m \theta = - \cos^2 \frac{k}{2} \pi = -1. \]

Hence \( e_{n-1}^T v_k = 1 \). \hfill \Box

We can now formulate the rank of the walk matrix of \( B \).

\textbf{Theorem 2.5.} The rank of the walk matrix of \( B \) is \( \lfloor \frac{n}{2} \rfloor \).

\textbf{Proof.} Let \( v_0, \ldots, v_{n-2} \) be the eigenvectors of \( B^T \) in Proposition 2.2. Then there are \( 1 + \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor \) eigenvectors of \( B^T \) such that \( e_{n-1} v_k \neq 0 \) by Lemma 2.4. We note that all
eigenvalues of $B^T$ are pairwise different, since the cosine function is monotonically decreasing on $[0, \pi]$. By Lemma 2.1 we have

$$\text{rank } W(B) = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

Lemma 2.6. Let $\tilde{D}_n$ be the extended Dynkin graph with $n + 1$ vertices. Then $\hat{W}(\tilde{D}_n) = W(B)$.

Proof. Since $AP = PB$, $A^kP = PB^k$ for all $k \geq 0$. Then we have $A^k e_{n+1} = P B^k e_{n-1}$ for all $k \geq 0$, because $P e_{n-1} = e_{n+1}$. It follows that

$$[e_{n+1} \ A e_{n+1} \ \cdots \ A^{n-2} e_{n+1}] = P \begin{bmatrix} e_{n-1} & B e_{n-1} & \cdots & B^{n-2} e_{n-1} \end{bmatrix}.$$ 

If we delete the first and last rows from $[e_{n+1} \ A e_{n+1} \ \cdots \ A^{n-2} e_{n+1}]$ and $P$, we get $\hat{W}(\tilde{D}_n)$ and the identity matrix, respectively. Hence $\hat{W}(\tilde{D}_n) = W(B)$. 

Remark 2.7. By Theorem 2.5 and Lemma 2.6, we have

$$\text{rank } \hat{W}(\tilde{D}_n) = \text{rank } W(B) = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

In the next proposition, we show that $W(\tilde{D}_n)$ and $W'(\tilde{D}_n)$ are integrally equivalent and hence proving that they have the same Smith normal form.

Proposition 2.8. Let $\tilde{D}_n$ be the extended Dynkin graph with $n + 1$ vertices. Then $W(\tilde{D}_n)$ and $W'(\tilde{D}_n)$ have the same Smith normal form.

Proof. Let $r = \text{rank } W(\tilde{D}_n)$. Then the first $r$ columns of $W(\tilde{D}_n)$ are linearly independent. Note that the first two rows of $W(\tilde{D}_n)$ are the same and the last two rows of $W(\tilde{D}_n)$ are the same. Since $r \leq n - 1$, the last two columns of $W(\tilde{D}_n)$ can be written as linear combinations of $e_{n+1}, A e_{n+1}, \ldots, A^{r-1} e_{n+1}$ with integer coefficients. Hence we obtain $W'(\tilde{D}_n)$ by using the elementary row and column operations on $W(\tilde{D}_n)$. Therefore, $W(\tilde{D}_n)$ and $W'(\tilde{D}_n)$ have the same Smith normal form.

Corollary 2.9. Let $\tilde{D}_n$ be the extended Dynkin graph with $n + 1$ vertices. Then

$$\text{rank } W(\tilde{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

Proof. By Proposition 2.8 and Remark 2.7,

$$\text{rank } W(\tilde{D}_n) = \text{rank } W'(\tilde{D}_n) = \text{rank } \hat{W}(\tilde{D}_n) = \text{rank } W(B) = \left\lfloor \frac{n}{2} \right\rfloor.$$ 

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We give an example of Proposition 2.8.

**Example 2.10.** Let $\tilde{D}_8$ be the extended Dynkin graph. Then

$$W(\tilde{D}_8) = \begin{bmatrix}
1 & 1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 \\
1 & 1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 \\
1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 & 256 \\
1 & 2 & 5 & 8 & 21 & 32 & 85 & 128 & 341 \\
1 & 2 & 4 & 10 & 16 & 42 & 64 & 170 & 256 \\
1 & 2 & 5 & 8 & 21 & 32 & 85 & 128 & 341 \\
1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 & 256 \\
1 & 1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 \\
1 & 1 & 3 & 4 & 11 & 16 & 43 & 64 & 171 \\
\end{bmatrix}$$

and

$$W'(\tilde{D}_8) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 3 & 4 & 11 & 16 & 43 & 64 & 0 \\
1 & 3 & 4 & 11 & 16 & 43 & 64 & 0 & 0 \\
1 & 2 & 5 & 8 & 21 & 32 & 85 & 0 & 0 \\
1 & 2 & 4 & 10 & 16 & 42 & 64 & 0 & 0 \\
1 & 2 & 5 & 8 & 21 & 32 & 85 & 0 & 0 \\
1 & 3 & 4 & 11 & 16 & 43 & 64 & 0 & 0 \\
1 & 1 & 3 & 4 & 11 & 16 & 43 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$  

The Smith normal forms of $W(\tilde{D}_8)$ and $W'(\tilde{D}_8)$ are $\text{diag}(1,1,1,7,0,0,0,0,0)$ and the rank of $W(\tilde{D}_8)$ is 4.

Since the number of main eigenvalues of $W(\tilde{D}_n)$ is equal to the rank of $W(\tilde{D}_n)$, we have the following corollary.

**Corollary 2.11.** Let $\tilde{D}_n$ be the extended Dynkin graph with $n+1$ vertices. Then the number of main eigenvalues of $\tilde{D}_n$ is $\left\lfloor \frac{n}{2} \right\rfloor$.  

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3 Future works

In this paper, we show that the rank of the walk matrix of the extended Dynkin graph $\tilde{D}_n$ is $\lfloor \frac{n}{2} \rfloor$. The next step would be to find the Smith normal form of the walk matrix of $\tilde{D}_n$. We give the following conjecture.

**Conjecture 1.** Let $\tilde{D}_n$ be the extended Dynkin graph with $n + 1$ vertices. Then the Smith normal form of $W(\tilde{D}_n)$ is

$$\begin{cases} 
\text{diag}(1, \ldots, 1, n - 1, 0, \ldots, 0), & \text{if } n \text{ is even}, \\
\text{diag}(1, \ldots, 1, \frac{n-1}{2}, 0, \ldots, 0), & \text{if } n \text{ is odd},
\end{cases}$$

where $r = \lfloor \frac{n}{2} \rfloor$.

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