AXISYMMETRIC TRAVELING FRONTS IN BALANCED BISTABLE REACTION-DIFFUSION EQUATIONS

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ABSTRACT. For a balanced bistable reaction-diffusion equation, the existence of axisymmetric traveling fronts has been studied by Chen, Guo, Ninomiya, Hamel and Roquejoffre [4]. This paper gives another proof of the existence of axisymmetric traveling fronts. Our method is as follows. We use pyramidal traveling fronts for unbalanced reaction-diffusion equations, and take the balanced limit. Then we obtain axisymmetric traveling fronts in a balanced bistable reaction-diffusion equation. Since pyramidal traveling fronts have been studied in many equations or systems, our method might be applicable to study axisymmetric traveling fronts in these equations or systems.

1. Introduction. In this paper we study a reaction-diffusion equation
\begin{align}
\frac{\partial u}{\partial t} &= \Delta u - G'(u(x, t)), \quad x \in \mathbb{R}^n, t > 0, \\
 u(x, 0) &= u_0(x), \quad x \in \mathbb{R}^n,
\end{align}

where \( n \geq 2 \) is a given integer, and given \( u_0 \in X \). Here \( X \) is the set of bounded and uniformly continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) with the norm
\[
\| u_0 \| = \sup_{x \in \mathbb{R}^n} |u_0(x)|.
\]

Now \( G \in C^2[−1, 1] \) satisfies \( G(1) = 0, G(-1) = 0, G'(1) = 0, G'(-1) = 0, G''(1) > 0 \) and \( G''(-1) > 0 \) with
\[
G(s) > 0 \quad \text{if} \quad -1 < s < 1.
\]

For \( G(u) = (1 - u^2)^2/4 \) and \( -G'(u) = u - u^3 \), (1.1) is called the Allen–Cahn equation, the scalar Ginzburg–Landau equation or the Nagumo equation.

The reaction term is called balanced when \( G(1) = G(-1) \) and is called unbalanced when \( G(1) \neq G(-1) \). When the reaction term is unbalanced multidimensional traveling fronts have been studied by [15, 16, 10, 11, 12, 17, 18, 19, 13, 23, 26, 14, 20, 21, 22] and so on. In this case, the propagation is mainly driven by the imbalance of the reaction kinetics and the curvature effect of an interface. Here a level set of a solution is often called an interface.

When the reaction term is balanced, one has no driven force caused by the reaction kinetics and the propagation is mainly driven by the curvature effect of an interface and is also driven by interaction between portions of an interface. For Equation (1.1), axisymmetric traveling fronts have been studied by Chen, Guo, Hamel, Ninomiya and Roquejoffre [4]. See del Pino, Kowalczyk and Wei [5] for a stationary solution, that is a traveling front with speed zero, related with De Giorgi’s conjecture. See [6] for a traveling wave solution with two non-planar fronts.
and for a traveling wave solution with non-convex fronts. Recently the existence of axially asymmetric traveling fronts was studied by [24] as a balanced limit of pyramidal traveling fronts in unbalanced reaction-diffusion equations. See Wang [25] for traveling waves of a mean curvature flow in $\mathbb{R}^n$. In this paper we prove the existence of an axisymmetric traveling front solution to a balanced reaction-diffusion equation (1.1) by using the method of [24]. This axisymmetric traveling front solution is monotone decreasing in the traveling axis $x_n$ and travels with any given positive speed. Since pyramidal traveling fronts have been studied in many equations or systems as in [17, 18, 19, 13, 26, 14, 1, 23], our method might be applicable to study axisymmetric traveling fronts in these equations or systems.

Let $c > 0$ be arbitrarily given. Let $x = (x', x_n) \in \mathbb{R}^n$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. We put $z = x_n - ct$ and $u(x', x_n, t) = w(x', z, t)$, and have

$$\frac{\partial w}{\partial t} - c \frac{\partial w}{\partial z} = \frac{\partial^2 w}{\partial z^2} + \sum_{j=1}^{n-1} \frac{\partial^2 w}{\partial x_j^2} - G'(w), \quad (x', z) \in \mathbb{R}^n, \ t > 0,$$

$$w(x', z, 0) = u_0(x), \quad (x', z) \in \mathbb{R}^n.$$

Now we write $z$ simply as $x_n$. Then $w(x, t)$ satisfies

$$\frac{\partial w}{\partial t} - c \frac{\partial w}{\partial x_n} = \Delta w - G'(w), \quad x \in \mathbb{R}^n, \ t > 0,$$

$$w(x, 0) = u_0(x), \quad x \in \mathbb{R}^n.$$
Then the profile equation for a traveling wave $V$ with speed $c$ is given by

$$-\Delta V - c \frac{\partial V}{\partial x_n} + G'(V) = 0, \quad x \in \mathbb{R}^n.$$  

Let $s_*$ be the largest zero point of $G'$ in $(-1,1)$, that is, $s_* \in (-1,1)$ is defined by

$$s_* = \min\{s_0 \in (-1,1) \mid -G'(s) > 0 \quad \text{if} \quad s_0 < s < 1\}.$$  

We fix $\theta_0$ with $s_* < \theta_0 < 1$ and have $-G'((\theta_0)) > 0$.

Let $r = |x'|$ and let $0' = (0, \ldots, 0) \in \mathbb{R}^{n-1}$. In Section 4, we define

$$U_0(x', x_n) = \lim_{i \to \infty} V_{k_i}^{(m_i)}(x', x_n + z_i)$$  

for all $(x', x_n)$ in any compact set in $\mathbb{R}^n$. Here $V_{k_i}^{(m_i)}(x', x_n)$ is a pyramidal traveling front given by Theorem 2. See Figure 2 for its level set. As is seen in Section 4, we can define $U(r, x_n)$ by

$$U(r, x_n) = U_0(x', x_n), \quad (x', x_n) \in \mathbb{R}^n. \quad (1.3)$$  

Then we have

$$|\nabla U_0(x', x_n)| = \sqrt{\left(\frac{\partial U}{\partial r}(r, x_n)\right)^2 + \left(\frac{\partial U}{\partial x_n}(r, x_n)\right)^2}.\quad (1.2)$$

The following is the main assertion in this paper.

**Theorem 1** (Axisymmetric traveling fronts). Let $c > 0$ be an arbitrarily given number. Let $U(r, x_n)$ be given by (1.3). Then one has $U(0, 0) = \theta_0 \in (-1,1)$ and

$$\frac{\partial^2 U}{\partial r^2} + \frac{n - 2}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial x_n^2} + c \frac{\partial U}{\partial x_n} - G'(U) = 0, \quad r > 0, x_n \in \mathbb{R},$$  

$$\frac{\partial U}{\partial x_n}(r, x_n) < 0 \quad \text{if} \quad r \geq 0, x_n \in \mathbb{R},$$  

$$\frac{\partial U}{\partial r}(r, x_n) > 0 \quad \text{if} \quad r > 0, x_n \in \mathbb{R}. \quad (1.4)$$

For every $\theta \in (-1,1)$, one has

$$\inf_{r \geq 0, x_n \in \mathbb{R}} \left\{ \sqrt{\left(\frac{\partial U}{\partial r}(r, x_n)\right)^2 + \left(\frac{\partial U}{\partial x_n}(r, x_n)\right)^2} \left| \ U(r, x_n) = \theta \right\} > 0, \quad (1.6)$$

and can define $\psi_\theta \in \mathcal{C}_1^1[0, \infty)$ by $U(r, \psi_\theta(r)) = \theta$ for all $r \geq 0$. For every $\theta \in (-1,1)$, one has

$$\psi'_\theta(0) = 0,$$

$$\psi'_\theta(r) > 0 \quad \text{for all} \ r \geq 0,$$

$$\lim_{r \to \infty} \psi'_\theta(r) = \infty.$$

Moreover, for every $\theta \in (-1,1)$, one has

$$\lim_{r_0 \to \infty} \sup_{(x,y) \in K} \left| U\left(r_0, \psi_\theta(r_0)\right) + \frac{x}{1 + \psi'_\theta(r_0)^2 \psi_\theta(r_0)}(-\psi'_\theta(r_0), 1) \right.$$

$$\left. - \frac{y}{1 + \psi'_\theta(r_0)^2 \psi_\theta(r_0)}(1, \psi'_\theta(r_0)) - \Phi\left( x + \Phi^{-1}(\theta) \right) \right| = 0$$

for any given compact set $K$ in $\mathbb{R}^2$.  

Remark 1. From Theorem 1 the following assertion follows. For every $\theta \in (-1,1)$, one has
$$ \lim_{r_0 \to \infty} \sup_{(x,y) \in K} \left| U ((r_0 - x, \psi_\theta(r_0) - y)) - \Phi (x + \Phi^{-1}(\theta)) \right| = 0 $$
for any given compact set $K$ in $\mathbb{R}^2$.

As far as the author knows, the uniqueness of an axisymmetric traveling front for (1.1) is yet to be studied. The author conjectures that an axisymmetric traveling front in Theorem 1 coincides with that of [4]. This is an open problem.

This paper is organized as follows. In Section 2, we make preparations. In Section 3, we show properties of pyramidal traveling fronts to unbalanced reaction-diffusion equations. In Section 4, we take the balanced limit of pyramidal traveling fronts, and prove Theorem 1.

2. Preliminaries. We extend $G \in C^2[-1,1]$ as a function of $C^2(\mathbb{R})$ with
$$ G(s) > 0 \quad \text{if} \quad |s| \neq 1. $$
Let
$$ \beta = \frac{1}{2} \min \{ G''(1), G''(-1) \} > 0, $$
and let $\delta_* \in (0,1/4)$ satisfy
$$ \min_{|u+1| \leq 2\delta_*} G''(u) > \beta, \quad \min_{|u-1| \leq 2\delta_*} G''(u) > \beta. $$
We put
$$ M = 1 + \max_{|u| \leq 1+2\delta_*} |G''(u)|. $$

Following [16, 4, 17, 18, 19, 24], we introduce a one-dimensional traveling front. For any $k$ with
$$ 0 < k < \sqrt{G''(-1)}, $$
let
$$ f_k(s) = -G'(s) + k \sqrt{2G(s)}, \quad s \in \mathbb{R}, $$
$$ F_k(s) = \int_{-1}^{s} f_k(\sigma) \, d\sigma. $$
Then we have
$$ f'_k(1) = -G''(1) - k \sqrt{G''(1)} < 0, $$
$$ f'_k(-1) = - \sqrt{G''(-1)} \left( \sqrt{G''(-1)} - k \right) < 0, $$
$$ -F_k(-1) = 0, \quad -F_k(1) = -k \int_{-1}^{1} \sqrt{2G(\sigma)} \, d\sigma < 0. $$
Let $k_0 \in \left(0, \sqrt{G''(-1)}\right)$ be small enough such that one has
$$ \min \{ -F_k(s) \mid s \in (-1,1), f_k(s) = 0 \} > 0 $$
for every $k \in [0,k_0)$. We define $\Phi$ by
$$ -x = \int_{0}^{\Phi(x)} \frac{ds}{\sqrt{2G(s)}}, \quad x \in \mathbb{R}. $$
Then we have $\Phi(0) = 0$ and
\[
-\Phi'(x) = \sqrt{2G(\Phi(x))}, \quad x \in \mathbb{R}, \\
\Phi''(x) = G'(\Phi(x)), \quad x \in \mathbb{R}.
\]

Thus $\Phi$ satisfies
\[
\Phi''(x) + k\Phi'(x) + f_k(\Phi(x)) = 0, \quad x \in \mathbb{R},
\]
and is a one-dimensional traveling front with speed $k \in (0, k_0)$. Now $\Phi$ also satisfies
\[
\Phi''(x) - G'(\Phi(x)) = 0, \quad x \in \mathbb{R},
\]
\[
\Phi'(x) < 0, \quad x \in \mathbb{R},
\]
\[
\Phi(-\infty) = 1, \quad \Phi(0) = 0, \quad \Phi(+\infty) = -1.
\]

Thus $\Phi$ is a planar stationary front to (1.1).

It is well-known that $(s, \phi) = (0, \Phi)$ is the unique solution to
\[
\phi''(x) + s\phi'(x) - G'(\phi(x)) = 0, \quad x \in \mathbb{R},
\]
\[-1 < \phi(x) < 1, \quad x \in \mathbb{R},
\]
\[
\phi(-\infty) = 1, \quad \phi(0) = 0, \quad \phi(+\infty) = -1.
\]

See [7, 2] for the proof of this uniqueness.

3. Properties of pyramidal traveling fronts to unbalanced reaction-diffusion equations. In this section we study properties of pyramidal traveling fronts for unbalanced reaction-diffusion equations. Two-dimensional V-form fronts and pyramidal traveling fronts in $\mathbb{R}^n$ have been studied by [15, 16, 10, 11, 12, 17, 18, 19, 13, 23, 26, 14] and so on.

Let $c > 0$ be arbitrarily given. For a given bounded and uniformly continuous function $u_0$ let $w(x, t; u_0)$ be the solution of
\[
\frac{\partial w}{\partial t} = \Delta w + c \frac{\partial w}{\partial x_n} + f_k(w), \quad (x', x_n) \in \mathbb{R}^n, t > 0,
\]
\[
w(x, 0) = u_0(x), \quad x \in \mathbb{R}^n.
\]

For any $k \in (0, \min\{k_0, c\})$, let
\[
m_* = \frac{\sqrt{c^2 - k^2}}{k}.
\]

Let $\mathbb{N}$ be the set of positive integers and and let $\bar{\mathbb{N}} = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$ with $m \geq 2$ we define $J$ as
\[
J = \{j_1 \in \bar{\mathbb{N}} \mid 0 \leq j_1 \leq 2^m - 1\} \quad \text{if } n = 3,
\]
\[
J = \{j \in \bar{\mathbb{N}}^{n-2} \mid 0 \leq j_i \leq 2^m (1 \leq i \leq n - 3), 0 \leq j_{n-2} \leq 2^m - 1\} \quad \text{if } n \geq 4.
\]

For each $j = (j_1, \ldots, j_{n-2}) \in J$, we define
\[
a_j = \begin{pmatrix}
\cos \frac{2\pi j_1}{2^m} \\
\sin \frac{2\pi j_1}{2^m}
\end{pmatrix}
\]
for $n = 3$. 
and
\[ a_j = \begin{pmatrix} \cos \left( \frac{\pi j_1}{2m} \right) \\ \sin \left( \frac{\pi j_1}{2m} \right) \cos \left( \frac{2\pi j_2}{2m} \right) \\ \sin \left( \frac{\pi j_1}{2m} \right) \sin \left( \frac{2\pi j_2}{2m} \right) \end{pmatrix} \] for \( n = 4 \),

and
\[ a_j = \begin{pmatrix} \cos \left( \frac{\pi j_1}{2m} \right) \\ \sin \left( \frac{\pi j_1}{2m} \right) \cos \left( \frac{\pi j_2}{2m} \right) \\ \vdots \\ \sin \left( \frac{\pi j_1}{2m} \right) \sin \left( \frac{\pi j_n-3}{2m} \right) \\ \sin \left( \frac{\pi j_1}{2m} \right) \sin \left( \frac{2\pi j_n-2}{2m} \right) \\ \sin \left( \frac{\pi j_1}{2m} \right) \sin \left( \frac{2\pi j_n-3}{2m} \right) \end{pmatrix} \] for \( n \geq 5 \).

Now we put \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \) and \( x = (x', x_n) = (x_1, \ldots, x_n) \in \mathbb{R}^n \) with \( |x'| = \sqrt{\sum_{i=1}^{n-1} x_i^2} \) and \( |x| = \sqrt{\sum_{i=1}^{n} x_i^2} \), respectively. For \( x' \in \mathbb{R}^{n-1} \) we set
\[ h_j(x') = m_*(a_j, x'), \quad j \in J, \]
\[ h^{(m)}(x') = \max_{j \in J} h_j(x') = m_* \max_{j \in J} (a_j, x'). \] (3.1)

Here \( (a_j, x') \) denotes the inner product of vectors \( a_j \) and \( x' \). In this paper we call \( \{(x', x_n) \in \mathbb{R}^n \mid x_n \geq h^{(m)}(x')\} \) a pyramid.

For \( j \in J \) we define
\[ \Omega_j = \{ x' \in \mathbb{R}^{n-1} \mid h^{(m)}(x') = h_j(x') \}, \]
and have
\[ \bigcup_{j \in J} \partial \Omega_j = \{ h_i(x') = h_j(x') = h^{(m)}(x') \text{ for some } i, j \in J \text{ with } a_i \neq a_j \}. \]

We call
\[ E = \left\{ (x', h^{(m)}(x')) \mid x' \in \bigcup_{j \in J} \partial \Omega_j \right\} \]
the set of edges of a pyramid. For \( \gamma > 0 \), let
\[ D(\gamma) = \{ x \in \mathbb{R}^n \mid \text{dist}(x, E) > \gamma \}. \] (3.3)

Now we define
\[ \Psi(x', x_n) = \Phi \left( \frac{k}{c} (x_n - h^{(m)}(x')) \right), \quad (x', x_n) \in \mathbb{R}^n. \] (3.4)

Pyramidal traveling fronts are stated as follows. For the proof see [15] for \( n = 2 \) and see [17, 13] for \( n \geq 3 \).
Theorem 2 ([15, 17, 13, 23]). Let $c > 0$ be an arbitrarily given number. For every $k \in (0, \min\{k_0, c\})$, let $h^{(m)}$ and $v$ be given in (3.2) and (3.4), respectively. Let $V^{(m)}_k$ be defined by

$$V^{(m)}_k(x) = \lim_{t \to \infty} w(x, t; v) \quad \text{for all } x \in \mathbb{R}^n.$$ 

Then $V^{(m)}_k$ satisfies

$$(-\Delta - cD_N)V^{(m)}_k - f_k(V^{(m)}_k) = 0 \quad \text{in } \mathbb{R}^n \quad (3.5)$$

with

$$\lim_{\gamma \to \infty} \sup_{x \in D(\gamma)} \left| V^{(m)}_k(x) - v(x) \right| = 0,$$

$$-\frac{\partial V^{(m)}_k}{\partial x_n}(x) > 0 \quad \text{for all } x \in \mathbb{R}^n,$$

$$-1 < v(x) < V^{(m)}_k(x) < 1 \quad \text{for all } x \in \mathbb{R}^n.$$

**Figure 2.** A level set $\{V_k(x', x_n) = \theta_0\}$ of a square pyramidal traveling front $V_k$.

Since $h^{(m)}$ is symmetric with respect to a plane $(x', a_j) = 0$, $V^{(m)}_k(\cdot, x_n)$ is symmetric with respect to the same plane for any fixed $x_n \in \mathbb{R}$ by the definition of $V^{(m)}_k$ in Theorem 2. Using

$$\left(\nabla h^{(m)}(x'), a_j\right) > 0 \quad \text{if } x' \in \Omega_j,$$

we have

$$\left(\nabla v(x', x_n), a_j\right) > 0 \quad \text{if } x' \in \Omega_j.$$
Then, using the definition of $V^{(m)}$, we obtain
\[
\left( \nabla V^{(m)}_k(x', x_n), a_j \right) > 0 \quad \text{if } x' \in \Omega_j, x_n \in \mathbb{R}
\] (3.6)
for every $j \in J$. For every $k \in (0, \min\{k_0, c\})$ and every $m \in \mathbb{N}$ with $m \geq 2$, we choose $z^{(m)}_k \in \mathbb{R}$ by
\[
V^{(m)}_k(0', z^{(m)}_k) = \theta_0.
\]

By the Schauder estimate [8, Theorem 9.11], there exists a positive constant $B$ such that
\[
\|V^{(m)}_k\|_{L^\infty(\mathbb{R}^n)} < B
\]
holds true for all $k \in \min\{k_0, c\}$ and all $m \geq 2$.

4. Balanced limits of pyramidal traveling fronts. In this section we study the limits of pyramidal traveling fronts for unbalanced reaction-diffusion equations as the reaction term approaches to a balanced one, and prove the existence of axisymmetric traveling fronts in balanced reaction-diffusion equations.

Let a sequence $(k_i)_{i \in \mathbb{N}}$ satisfy $\lim_{i \to \infty} k_i = 0$ and
\[
0 < k_{i+1} < k_i, \quad 0 < k_i < \min\{k_0, c\}, \quad i \in \mathbb{N}.
\]

Let a sequence $(m_i)_{i \in \mathbb{N}}$ satisfy $\lim_{i \to \infty} m_i = \infty$ with
\[
m_i \in \mathbb{N}, \quad m_i < m_{i+1}, \quad i \in \mathbb{N}.
\]

We choose $z_i \in \mathbb{R}$ with $V^{(m_i)}_{k_i}(0', z_i) = \theta_0$. Taking a subsequence if necessary, we define
\[
U_0(x', x_n) = \lim_{i \to \infty} V^{(m_i)}_{k_i}(x', x_n + z_i)
\] (4.1)
for all $(x', x_n)$ in any compact set in $\mathbb{R}^n$. Without loss of generality, we can assume that this convergence holds true for all $i \in \mathbb{N}$.

Then $U_0(x)$ satisfies the profile equation
\[
\Delta U_0 + c \frac{\partial U_0}{\partial x_n} - G'(U_0) = 0, \quad (x', x_n) \in \mathbb{R}^n
\] (4.2)
with $U_0(0) = \theta_0$ and
\[
\frac{\partial U_0}{\partial x_n}(x) \leq 0, \quad x \in \mathbb{R}^n.
\]

Since $V^{(m)}_k(\cdot, x_n)$ is symmetric with respect to a plane $(x', a_j) = 0$ for any fixed $x_n \in \mathbb{R}$, $U_0(\cdot, x_n)$ is spherically symmetric in $\mathbb{R}^{n-1}$ for any fixed $x_n \in \mathbb{R}$. Using $r = |x'|$, we define
\[
U(r, x_n) = U_0(x', x_n), \quad (x', x_n) \in \mathbb{R}^n.
\] (4.3)

Now we have $U(0, 0) = \theta_0$. Using (3.6), we obtain
\[
\frac{\partial U}{\partial r}(r, x_n) \geq 0, \quad r > 0, x_n \in \mathbb{R}.
\] (4.4)

Now we have
\[
\frac{\partial^2 U}{\partial r^2}(0, x_n) \geq 0, \quad x_n \in \mathbb{R}.
\] (4.5)

We will show the following lemma.
Lemma 1. One has
\[
\frac{\partial^2 U}{\partial r^2} + \frac{n - 2}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial x_n^2} + c \frac{\partial U}{\partial x_n} - G'(U) = 0, \quad r > 0, \; x_n \in \mathbb{R}
\]
and
\[
\frac{\partial U}{\partial x_n}(r, x_n) < 0, \quad r \geq 0, \; x_n \in \mathbb{R},
\]
\[
\frac{\partial U}{\partial r}(r, x_n) > 0, \quad r > 0, \; x_n \in \mathbb{R}.
\]

Proof. It suffices to prove the latter two inequalities. If \( \partial U/\partial x_n = 0 \) at some point in \([0, \infty) \times \mathbb{R} \), we have \((\partial U/\partial x_n)(r, x_n) = 0 \) from the maximum principle. Then \( U(r, x_n) \) is independent of \( x_n \) and is a function of \( r \geq 0 \). By (1.4) and (4.5), we have
\[
\frac{\partial^2 U}{\partial r^2} + \frac{n - 2}{r} \frac{\partial U}{\partial r} - G'(U) = -\frac{\partial^2 U}{\partial x_n^2} - c \frac{\partial U}{\partial x_n} = 0, \quad r > 0, \; x_n \in \mathbb{R}.
\]
Combining this equality and (4.5), we find \(-G'(U(0, x_n)) \leq 0 \) for all \( x_n \in \mathbb{R} \). This contradicts \( U(0, 0) = \theta_0 \) and \(-G'(\theta_0) > 0 \). Thus we have
\[
\frac{\partial U}{\partial x_n}(r, x_n) < 0, \quad r \geq 0, \; x_n \in \mathbb{R}.
\]
Next we prove the last inequality. Assume \( U_r = 0 \) at some point \((r_0, x_n^0)\) with \( r_0 > 0 \) and \( x_n^0 \in \mathbb{R} \). Here we write \( U_r(r, x_n) = (\partial U/\partial r)(r, x_n) \). Then \( U_r \) satisfies
\[
\left( \frac{\partial^2}{\partial r^2} + \frac{n - 2}{r} \frac{\partial}{\partial r} - \frac{n - 2}{r^2} + \frac{\partial^2}{\partial x_n^2} + c \frac{\partial}{\partial x_n} - G''(U) \right) U_r = 0, \quad r > 0, \; x_n \in \mathbb{R},
\]
\[
U_r \geq 0, \quad r \geq 0, \; x_n \in \mathbb{R}.
\]
Then, using the maximum principle, we get \( U_r \equiv 0 \) on \([0, \infty) \times \mathbb{R} \). Then \( U(r, x_n) \) is independent of \( r \geq 0 \) and is a function of \( x_n \), and satisfies
\[
\frac{\partial^2 U}{\partial x_n^2} + c \frac{\partial U}{\partial x_n} - G'(U) = 0, \quad x_n \in \mathbb{R}.
\]
Since the one-dimensional traveling front profile is uniquely determined and its speed is uniquely determined by [2], we obtain \( c = 0 \) and \( U(r, x_n) = \Phi(x_n) \) for all \( x_n \in \mathbb{R} \). This contradicts \( c > 0 \). Thus we obtain the last inequality. This completes the proof. □

Now the following assertion follows from [24].

Proposition 1. Let \( c > 0 \) be an arbitrarily given number. Let \( \zeta > 0 \) be arbitrarily given. Then \( U_0 \) given by (4.1) satisfies
\[
\inf_{x \in \mathbb{R}^n} \{ |\nabla U_0(x)| \mid U_0(x) = \theta \} > 0, \quad (4.6)
\]
for every \( \theta \in (-1, 1) \). Moreover one can define \( q_\theta(x') \in \mathbb{R} \) by \( U_0(x', q_\theta(x')) = \theta \) for all \( x' \in \mathbb{R}^{n-1} \). Here \( q_\theta \) belongs to \( C^1(\mathbb{R}^{n-1}) \).

Proof. The proof of this proposition can be carried out by a simplified argument as in the proof of Theorem 1 of [24]. □

Remark 2. For every \( \theta \in (-1, 1) \), a level set \( \{ x \in \mathbb{R}^n \mid U_0(x) = \theta \} \) is given by a graph of a function that is defined on the entire space \( \mathbb{R}^{n-1} \).

Now we have the following lemma.
Lemma 2. Let $U_0$ be given by (4.1). One has
\[ \|U_0\|_{C^{2,\alpha_0}(\mathbb{R}^n)} < \infty \]
for some $\alpha_0 \in (0, 1)$.

Proof. This lemma follows from general regularity theory for elliptic equations. See [8] for instance.

For every $\theta \in (-1, 1)$, we define $\psi_\theta(r) \in \mathbb{R}$ by $U(r, \psi_\theta(r)) = \theta$ for all $r \geq 0$. Then $\psi_\theta$ is of class $C^1[0, \infty)$. Then, for every $\theta \in (-1, 1)$, we have
\[ \psi_\theta'(0) = 0, \quad \psi_\theta'(r) > 0 \quad \text{for all } r > 0. \]

Now we have $\psi_0(0) = 0$.

Let $\delta_\ast$ be as in Section 2. We define
\[ \Omega_- = \{(x', x_0) \in \mathbb{R}^n | U_0(x', x_0) < -1 + \theta_*\} = \{(x', x_0) \in \mathbb{R}^n | x_0 > \psi_{-1+\theta_*}(|x'|)\}, \]
\[ \Omega_+ = \{(x', x_0) \in \mathbb{R}^n | 1 - \theta_* < U_0(x', x_0)\} = \{(x', x_0) \in \mathbb{R}^n | x_0 < \psi_{1-\theta_*}(|x'|)\}. \]

Let $\lambda > 0$ and $\lambda_1 > 0$ be positive numbers with
\[ \lambda_1 = \lambda + \frac{c}{2}, \quad \lambda_1^2 + (n-1)\lambda^2 = \frac{c^2}{4} + \beta. \]

Following [4] we show a decay estimate on $U_0$.

Lemma 3. There exists a positive constant $m_0$ such that the following assertions hold true. For every $x \in \Omega_-$ one has
\[ -1 < U_0(x) \leq -1 + m_0 e^{-\lambda p(x)}, \quad |\nabla U_0(x)| < m_0 e^{-\lambda p(x)}, \]
where
\[ p(x) = \frac{1}{\sqrt{n}} \text{dist}(x, \partial \Omega_+). \]

For every $x \in \Omega_+$ one has
\[ 1 - m_0 e^{-\lambda q(x)} \leq U_0(x) < 1, \quad |\nabla U_0(x)| < m_0 e^{-\lambda q(x)}, \]
where
\[ q(x) = \frac{1}{\sqrt{n}} \text{dist}(x, \partial \Omega_+). \]

Proof. For any given $L > 0$, we define
\[ P(x', x_n; L) = 4 \exp \left( -\lambda L - \frac{c}{2} x_n \right) \cosh(\lambda_1 x_n) \prod_{j=1}^{n-1} \cosh(\lambda x_j), \quad (x', x_n) \in \mathbb{R}^n. \]

Then $P$ satisfies
\[ -\Delta P - c \frac{\partial P}{\partial x_n} + \beta P = 0, \quad x \in \mathbb{R}^n, \]
and
\[ P(x) \geq 1 \quad \text{if } |x| \geq \sqrt{n}L. \]

We put $u_0(x) = 1 - U_0(x)$. Then $u_0$ satisfies
\[ \left( -\Delta - c \frac{\partial}{\partial x_n} - \int_0^1 G''(1 - \theta u_0) d\theta \right) u_0 = 0, \quad x \in \mathbb{R}^n. \]
Now we have 

\[ |\theta u_0(x)| \leq \delta, \quad 0 < \theta < 1, \ x \in \Omega_+. \]

For every \( y \in \Omega_+ \), let \( p(y) \) be given by 

\[ \sqrt{np(y)} = \text{dist}(y, \partial \Omega_+), \]

where \( \text{dist}(y, \partial \Omega_+) \) is the distance from \( y \) to \( \partial \Omega_+ \). Now \( \bar{u}(x) = u_0(x) - P(x - y; p(y)) \) satisfies we have

\[
\begin{align*}
-\Delta &- c \frac{\partial}{\partial x_n} - \int_0^1 G''(1 - \theta u_0)d\theta \nabla \bar{u} \leq 0, \quad x \in B(y; \sqrt{np(y)}), \\
\bar{u}(x) &\leq 0, \quad x \in \partial B(y; \sqrt{np(y)}),
\end{align*}
\]

where 

\[ B(y; \sqrt{np(y)}) = \{ x \in \mathbb{R}^n \mid |x - y| < \sqrt{np(y)} \}. \]

Then we obtain 

\[ \bar{u}(x) \leq 0 \quad x \in B(y; \sqrt{np(y)}) \]

from the maximum principle. Especially we have

\[ -1 < U_0(y) \leq -1 + 4e^{-\lambda p(y)}. \]

Then the former half of the lemma follows from the Schauder interior estimate of [8]. The latter half can be proved similarly. \( \square \)

Now we show the level set of \( U(r, x_n) \) becomes parallel to the \( x_n \)-axis as \( r \to \infty \). We prove this lemma in Section 5.

**Lemma 4.** For every \( \theta \in (-1, 1) \), one has

\[ \lim_{r \to \infty} \psi'_\theta(r) = \infty. \]

The following lemma asserts the asymptotic behavior of \( U \).

**Lemma 5.** For every \( \theta \in (-1, 1) \), one has

\[ \lim_{r_0 \to \infty} \sup_{(x,y) \in K} \left| U \left( (r_0, \psi_\theta(r_0)) + \frac{x}{\sqrt{1 + \psi'_\theta(r_0)^2}} (-\psi'_\theta(r_0), 1) - \frac{y}{\sqrt{1 + \psi'_\theta(r_0)^2}} (1, \psi'_\theta(r_0)) \right) - \Phi(x + \Phi^{-1}(\theta)) \right| = 0 \]

for any given compact set \( K \) in \( \mathbb{R}^2 \).

**Proof.** Let \( r_0 > 1 \) be given and we define

\[ u_1^{(r_0)}(x, y) = U \left( (r_0, \psi_\theta(r_0)) + \frac{x}{\sqrt{1 + \psi'_\theta(r_0)^2}} (-\psi'_\theta(r_0), 1) - \frac{y}{\sqrt{1 + \psi'_\theta(r_0)^2}} (1, \psi'_\theta(r_0)) \right) \]

for \( (x, y) \in \mathbb{R}^2 \) with

\[ r_0 - \frac{\psi'_\theta(r_0)}{\sqrt{1 + \psi'_\theta(r_0)^2}} x - \frac{y}{\sqrt{1 + \psi'_\theta(r_0)^2}} > 0. \]
Using Lemma 1, we have
\[
\frac{\partial^2 u_1^{(r_0)}}{\partial x^2}(x, y) + \frac{\partial^2 u_1^{(r_0)}}{\partial y^2}(x, y) + \frac{n - 2}{s^{(r_0)}} \left( -\frac{\psi'_0(r_0)}{\sqrt{1 + \psi'_0(r_0)^2}} \frac{\partial u_1^{(r_0)}}{\partial x}(x, y) - \frac{1}{\sqrt{1 + \psi'_0(r_0)^2}} \frac{\partial u_1^{(r_0)}}{\partial y}(x, y) \right)
\]
\[+ \frac{1}{\sqrt{1 + \psi'_0(r_0)^2}} \frac{\partial u_1^{(r_0)}}{\partial x}(x, y) - \frac{\psi'_0(r_0)}{\sqrt{1 + \psi'_0(r_0)^2}} \frac{\partial u_1^{(r_0)}}{\partial y}(x, y) - G'(u_1^{(r_0)}(x, y)) = 0
\]

for \((x, y) \in \mathbb{R}^2\) with \(s^{(r_0)} > 0\), where
\[
s^{(r_0)} = r_0 - \frac{\psi'_0(r_0)}{\sqrt{1 + \psi'_0(r_0)^2}} x - \frac{y}{\sqrt{1 + \psi'_0(r_0)^2}}.
\]

Using Lemma 4, we see that
\[u_1(x, y) = \lim_{r_0 \to \infty} u_1^{(r_0)}(x, y), \quad (x, y) \in \mathbb{R}^2\]
is a function of \(x\) and is independent of \(y\). Using (1.4) and Proposition 1, we have
\[
d^2u_1 \frac{d^2u_1}{dx^2}(x) - G'(u_1(x)) = 0, \quad x \in \mathbb{R},
\]
\[-\frac{du_1}{dx}(x) > 0, \quad -1 < u_1(x) < 1, \quad x \in \mathbb{R}.
\]
\[u_1(-\infty) = 1, \quad u_1(0) = \theta, \quad u_1(\infty) = -1.
\]

Then we obtain
\[u_1(x) = \Phi(x + a)
\]
with \(a \in \mathbb{R}\) and \(\Phi(a) = \theta\). This completes the proof.

Now Theorem 1 follows from Lemma 1, Proposition 1, Lemma 4 and Lemma 5.

5. Proof of Lemma 4. Let \(\theta \in (-1, 1)\) be arbitrarily fixed. We get a contradiction assuming \(\liminf_{r \to \infty} \psi'_0(r) < \infty\). Under this condition there exists \(\{r_j\}_{j \in \mathbb{N}}\) with \(\lim_{j \to \infty} r_j = \infty\) and \(\lim_{j \to \infty} \psi'_0(r_j) = \liminf_{r \to \infty} \psi'_0(r) < \infty\). Now we define
\[u_0^{(j)}(x, y) = U(r_j + x, \psi_0(r_j) + y)
\]
for \((x, y) \in \mathbb{R}^2\) with \(r_j + x \geq 0\). Using Lemma 1, we have
\[
\frac{\partial^2 u_0^{(j)}}{\partial x^2}(x, y) + \frac{\partial^2 u_0^{(j)}}{\partial y^2}(x, y) + \frac{n - 2}{r_j + x} \frac{\partial u_0^{(j)}}{\partial x}(x, y) + c \frac{\partial u_0^{(j)}}{\partial y}(x, y) - G'(u_0^{(j)}(x, y)) = 0
\]
for \((x, y) \in \mathbb{R}^2\) with \(r_j + x \geq 0\). Now we define
\[v(x, y) = \lim_{j \to \infty} u_0^{(j)}(x, y) \quad \text{for} \; (x, y) \in \mathbb{R}^2.
\]

Then, using Proposition 1, we have
\[
\frac{\partial^2 v}{\partial x^2}(x, y) + \frac{\partial^2 v}{\partial x^2}(x, y) + c \frac{\partial v}{\partial y}(x, y) - G'(v(x, y)) = 0, \quad (x, y) \in \mathbb{R}^2, \quad (5.1)
\]
\[-\frac{\partial v}{\partial y}(x, y) > 0, \quad \frac{\partial v}{\partial x}(x, y) \geq 0, \quad (x, y) \in \mathbb{R}^2, \quad (5.2)
\]
\[-1 < v(x, y) < 1 \quad (x, y) \in \mathbb{R}^2,
\]
\[\inf_{(x, y) \in \mathbb{R}^2} \left\{ \sqrt{v_x(x, y)^2 + v_y(x, y)^2} \bigg| v(x, t) = \theta_1 \right\} > 0 \quad \text{for any} \; \theta_1 \in (-1, 1).
\]
If we have $v_x = 0$ at some point, we get $v_x \equiv 0$, that is, $v(x,y) = \Phi(y)$, which contradicts $c > 0$. Thus we obtain
\[ \frac{\partial v}{\partial x}(x,y) > 0, \quad (x,y) \in \mathbb{R}^2. \]

Now we have
\[ \lim_{y \to -\infty} v(x,t) = -1, \quad \lim_{y \to \infty} v(x,t) = 1 \quad \text{for any fixed } x \in \mathbb{R}. \]

Now we have
\[ \| \nabla v \|_{L^\infty(\mathbb{R}^2)} < \infty. \]
See [8] for instance. We define $\gamma_\theta(x)$ by $v(x, \gamma_\theta(x)) = \theta$ for every $x \in \mathbb{R}$. Then $\gamma_\theta(x)$ is of class $C^1(\mathbb{R})$. We define
\[ D_- = \{(x,y) \in \mathbb{R}^2 \mid v(x,y) < -1 + \theta_* \}, \]
\[ D_+ = \{(x,y) \in \mathbb{R}^2 \mid 1 - \theta_* < v(x,y) \}. \]
The, by using Lemma 3, there exist constants $m_1 > 0$ and $\mu > 0$ such that we have
\[ -1 < v(x,y) \leq -1 + m_1 \exp(-\mu p_1(x,y)), \quad |\nabla v(x,y)| < m_1 \exp(-\mu p_1(x,y)), \]
if $(x,y) \in D_-$, where $p_1(x,t) = \text{dist}((x,y),\partial D_-)$. Similarly we have
\[ 1 - m_1 \exp(-\mu q_1(x,y)) \leq v(x,y) < 1, \quad |\nabla v(x,y)| < m_1 \exp(-\mu q_1(x,y)), \]
if $(x,y) \in D_+$, where $q_1(x,t) = \text{dist}((x,y),\partial D_+)$.

In the following, we will show a contradiction following Gui [9]. We introduce
\[ h(x) = - \int_{\mathbb{R}} v_x(x,y)v_y(x,y) \, dy \quad \text{for any fixed } x \in \mathbb{R}. \]
Then we have
\[ 0 < \sup_{x \in \mathbb{R}} h(x) \leq \| \nabla v \|_{L^\infty(\mathbb{R}^2)}. \]
Now we get
\[ h'(x) = - \int_{\mathbb{R}} (v_{xx}v_y + v_xv_{xy}) \, dy \]
\[ = \int_{\mathbb{R}} \left( cv_y^2 - \frac{d}{dy} \left( \frac{1}{2} v_x^2 - \frac{1}{2} v_y^2 + G(v) \right) \right) \, dy \]
\[ = \int_{\mathbb{R}} cv_y(x,y)^2 \, dy \]
for any fixed $x \in \mathbb{R}$. Integrating the both sides over $(x_1, x_2)$ with $-\infty < x_1 < x_2 < \infty$, we have
\[ c \int_{[x_1,x_2] \times \mathbb{R}} v_y(x,y)^2 \, dx \, dy = h(x_2) - h(x_1) \leq 2 \| \nabla v \|_{L^\infty(\mathbb{R}^2)}. \]
Sending $x_1 \to -\infty$ and $x_2 \to \infty$, we obtain
\[ c \int_{\mathbb{R}^2} v_y(x,y)^2 \, dx \, dy \leq 2 \| \nabla v \|_{L^\infty(\mathbb{R}^2)} < \infty. \]
Combining this inequality and (5.2), we obtain
\[ \lim_{x \to \infty} \gamma'_\theta(x) = \infty, \quad \lim_{x \to -\infty} \gamma'_\theta(x) = \infty. \]
Multiplying (5.1) by \( v_y \), we have
\[
\text{div} (v_y \nabla v) - \frac{1}{2} \frac{\partial}{\partial y} (|\nabla v|^2) + cv_y^2 - G'(v)v_y = 0, \quad (x, y) \in \mathbb{R}^2.
\]
Integrating the both sides over \( \mathbb{R} \times (y_1, y_2) \) with \(-\infty < y_1 < y_2 < \infty\), we have
\[
\int_{\mathbb{R} \times (y_1, y_2)} \left( \text{div} (v_y \nabla v) - \frac{1}{2} \frac{\partial}{\partial y} (|\nabla v|^2) + cv_y(x, y)^2 \right) \, dxdy = \int_{\mathbb{R}} (G(v(x, y_2)) - G(v(x, y_1))) \, dx.
\]
Now we have
\[
\int_{\mathbb{R} \times (y_1, y_2)} \left( \text{div} (v_y \nabla v) - \frac{1}{2} \frac{\partial}{\partial y} (|\nabla v|^2) \right) \, dxdy = \int_{\mathbb{R}} \left( -v_y(x, y_1)^2 + \frac{1}{2} |\nabla v(x, y_1)|^2 \right) \, dx.
\]
Thus we obtain
\[
\int_{\mathbb{R}} \left( \frac{1}{2} \psi'(x) \right) \, dx = \int_{\mathbb{R}} \psi'(x) \, dx = \int_{\sigma_1} \sqrt{2G(\sigma)} \, d\sigma.
\]
Using \( \lim_{\sigma \to \infty} \gamma_\theta'(\sigma) = \infty \), we have
\[
\int_{\mathbb{R}} \left( \frac{1}{2} \psi'(x) \right) \, dx = \int_{\sigma_1} \sqrt{2G(\sigma)} \, d\sigma.
\]
Then, sending \( y_1 \to -\infty \) and \( y_2 \to \infty \) in (5.3), we obtain
\[
\int_{\mathbb{R}^2} cv_y(x, y)^2 \, dxdy = 0,
\]
which contradicts (5.2). Thus we obtained \( \liminf_{r \to \infty} \psi_\theta'(r) = +\infty \). Now we complete the proof of Lemma 4.

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