Existence of Homoclinic Solutions for Difference Equations on Integers Via Variational Method

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ABSTRACT: In this paper, we study the elastic membrane equation with dynamic boundary conditions, source term and a nonlinear weak damping localized on a part of the boundary and past history. Under some appropriate assumptions on the relaxation function the general decay for the energy have been established using the perturbed Lyapunov functions and some properties of convex functions.

Key Words: Discrete boundary value problem, Nonlinear difference equation, Critical point, Lipschitz condition, Variational methods.

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1. Introduction

The aim of this paper is to establish the existence of homoclinic solutions for the following discrete boundary value problem

\[
\begin{aligned}
&-\Delta (a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k, u(k)) + \mu g(k, u(k)) + h(u(k)), \quad k \in \mathbb{Z}, \\
&u(k) \to 0
\end{aligned}
\]

where \(1 < p < +\infty\), \(\lambda > 0\), \(\mu \geq 0\), \(\phi_p(t) = |t|^{p-2}t\) for all \(t \in \mathbb{R}\), \(a, b : \mathbb{Z} \to (0, \infty)\), \(f, g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}\) are two continuous functions in the second variables, \(h : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function of order \(p-1\) with Lipschitzian constant \(L \geq 0\) such that \(h(0) = 0\), \(\Delta u(k) = u(k+1) - u(k)\) is the forward difference operator. A solution \(u = u(k)\) of \((P_{\lambda,\mu}^{f,g,h})\) is homoclinic if \(\lim_{|k| \to \infty} u(k) = 0\). Also we define the following conditions:

\(A_1\) \(b(k) \geq \beta > 0\) for all \(k \in \mathbb{Z}\), \(b(k) \to +\infty\) as \(|k| \to +\infty\).

There are many papers about existence of solutions to boundary value problems for finite difference equations with \(p\)-Laplacian operator which branching out in many fields such as biologic, economic, farm and other areas. There are various methods such fixed point, variational methods, critical point theory, Morse theory and the mountain-pass theorem. For background and recent results, we refer the reader to \[1,2,3,4,5,6,7,8,12,13,14,19,20,22,23,24,26,27,28,29,30,32,33\]. For example, Henderson and Thompson investigated existence multiple solutions for second order discrete boundary value problems in [19]. Wong and Xie proved three symmetric solutions of lidstone boundary value problem for partial equation in [33]. In [12] Cabada and Tersian studied the existence of homoclinic solutions for semilinear \(p\)-Laplacian difference equations with periodic coefficients based on the Brezis-Nirenberg’s mountain pass theorem. In [13] Candito and D’Agui studied discrete nonlinear Neumann problems to find three solutions. In addition, Iannizzotto and Tersian in [20] via critical point theory investigated the existence of multiple homoclinic solutions for the discrete \(p\)-Laplacian problem

\[
\begin{aligned}
&-\Delta (\phi_p(\Delta u(k-1))) + a(k)\phi_p(u(k)) = \lambda f(k, u(k)), \quad \forall k \in \mathbb{Z}, \\
&u(k) \to 0 \quad \text{as} \quad |k| \to \infty
\end{aligned}
\]
where $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$ is a positive and coercive weight function and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function in the second variable. In [23] Kong studied the problem $(P_{\lambda, \mu}^{f,g,h})$, in the case $\mu = 0$ and $h \equiv 0$ and proved the problem has infinitely many homoclinic solutions. Also, a variant of the fountain theorem was utilized. Kong in [22] studied the following higher order difference equation defined on $\mathbb{Z}$ with $p$-Laplacian

$$
\begin{aligned}
&\{ (-1)^n \Delta^n (a(k-n) \phi_p (\Delta^n u(k-n)) + b(k) \phi_p (u(k))) \\
&= \lambda f(k,u(k+1),u(k),u(k-1)), \quad \forall k \in \mathbb{Z} \\
&u(k) \to 0 \quad \text{as} \quad |k| \to \infty
\end{aligned}
$$

(1.2)

where $n > 1$ is an integer, $\lambda > 0$ is a parameter, $\phi_p = |t|^{p-2}t$ for $t \in \mathbb{R}$, $\Delta$ is the forward difference operator defined by $u(k) = u(k+1) - u(k)$ for $k \in \mathbb{Z}$, $\Delta^i u(k) = \Delta(\Delta^{i-1}u(k))$ for $i \in \mathbb{N}$, $a,b : \mathbb{Z} \to (0, \infty)$, and $f : \mathbb{Z} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous in the second, third, and fourth variables. By using the critical point theory, sufficient conditions were obtained for the existence of infinitely many homoclinic solutions of the problem (1.2), based on the fountain theorem in combination with the variational technique. Štegliński in [28] determined a concrete interval of positive parameter $\lambda$, for prove the existence of homoclinic solutions for the problem (1.1), while in [30] dealt with the problem $(P_{\lambda, \mu}^{f,g,h})$, in the case $\mu = 0$ and $h \equiv 0$, and using both the general variational principle of Ricceri and the direct method introduced by Faraci and Kristály proved the existence of infinitely many homoclinic solutions for the problem where the nonlinear term $f$ has an appropriate oscillatory behavior at zero. In [4], sufficient conditions for the existence of at least one homoclinic solution for a nonlinear second-order difference equation with $p$-Laplacian were presented.

Inspired by the above results, in the present paper, we obtain the existence of at least three distinct nonnegative solutions for the problem $(P_{\lambda, \mu}^{f,g,h})$, in which two parameters are involved. Estimation of these two parameters $\lambda$ and $\mu$ will be given. In particular, in Theorem 3.1 we establish the existence of at least three distinct nonnegative solutions for the problem $(P_{\lambda, \mu}^{f,g,h})$. Theorem 3.4 is a consequence of Theorem 3.1. The Examples 3.3 and 3.5 help us to illustrate our main results. In Example 3.3 the hypotheses of Theorem 3.1 are fulfilled and in Example 3.5 the condition of Theorem 3.4 are satisfied. As a special case of Theorem 3.4, we obtain Theorem 3.6 which under suitable conditions on $f$ at zero and at infinity, ensures two positive solutions for the autonomous case of the problem. Finally, by the way of example, we point out Theorem 3.7, as simple consequence of Theorem 3.6.

2. Preliminaries

In this paper $X$ denotes a finite dimensional real Banach space

$$
X = \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{k \in \mathbb{Z}} [a(k) |\Delta u(k-1)|^p + b(k) |u(k)|^p] < \infty \right\}
$$

with the norm

$$
\|u\| = \left( \sum_{k \in \mathbb{Z}} [a(k) |\Delta u(k-1)|^p + b(k) |u(k)|^p] \right)^{\frac{1}{p}},
$$

and as is shown in [20] $(X, \|\cdot\|)$ is a reflexive Banach space and the embedding $X \hookrightarrow \ell^p$ is compact and $I_\lambda : X \to \mathbb{R}$ is a functional satisfying the following structure hypothesis: $I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$ for all $u \in X$ where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class $C^1$ on $X$ with $\Phi$ coercive, i.e. $\lim_{\|u\| \to \infty} \Phi(u) = +\infty$, and $\lambda$ is a positive real parameter.

In this framework a finite dimensional variant of [8, Theorem 3.3 ] (see also Corollary 3.1 and Remark 3.9 of [8]) is the following:

Let $X$ be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r_1$, $r_2$, $r_3$, with $r_2 > r_1$ and $r_2 > \inf_X \Phi$, and all $r_3 > 0$, we define

$$
\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty,r)} \frac{\sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(u)}{r - \Phi(u)},
$$

Theorem 2.1 ([8, Theorem 3.3]). Assume that

(B1) $\Phi$ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;

(B2) for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s \in [0,1]} \Psi(s u_1 + (1-s) u_2) \geq 0.$$ 

Assume that there are three positive constants $r_1, r_2, r_3$ with $r_1 < r_2$, such that

(B3) $\varphi(r_1) < \beta(r_1, r_2)$;

(B4) $\varphi(r_2) < \beta(r_1, r_2)$;

(B5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in ]\frac{1}{\varphi(r_1, r_2)}, \frac{1}{\alpha(r_1, r_2, r_3)}[$ the functional $\Phi - \lambda \Psi$ admits three distinct critical points $u_1, u_2, u_3$ such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

We refer the interested reader to the papers [11,14,15,16,17,18,21] in which Theorem 2.1 has been successfully employed to get the existence of at least three solutions for boundary value problems.

For Banach space $X$ and the norm explained above the following inequality is obvious

$$\|u\|_{\infty} \leq \|u\|_p \leq \beta^{\frac{1}{p-1}} \|u\| \quad \forall u \in X,$$  \hspace{1cm} (2.1)$$

where $(A_1)$ is satisfied. Let two functions $f, g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be two continuous functions in the second variables and $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function of order $p - 1$ with Lipschitzian constant $L \geq 0$, i.e,

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|^{p-1}$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$. Suppose the constant $L \geq 0$ satisfies $L \beta^{-\frac{1}{p-1}} < 1$. Corresponding to the function $f, g$ and $h$, we introduce $F, G : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$, respectively, as follows

$$F(k, t) := \int_0^t f(k, \xi) d\xi \quad \forall (k, t) \in \mathbb{Z} \times \mathbb{R},$$

$$G(k, t) := \int_0^t g(k, \xi) d\xi \quad \forall (k, t) \in \mathbb{Z} \times \mathbb{R}$$

and

$$H(t) := \int_0^t h(\xi) d\xi \quad \forall t \in \mathbb{R}. $$
3. Main results

Set
$$G^\theta := \sum_{k \in \mathbb{Z}} \max_{|\xi| \leq \theta} G(k, \xi) \quad \forall \theta > 0$$
and
$$G_\eta := \sum_{k \in \mathbb{Z}} \inf_{[0,\eta]} G(k, t) \quad \forall \eta > 0.$$ 

If $g$ is sign-changing, then $G^\theta \geq 0$ and $G_\eta \leq 0$.

We formulate our main result as follows:

Fixing four positive constants $\theta_1, \theta_2, \theta_3, \eta$ and the integer $k_0$, put
$$\delta_{\lambda,G} := \min \left\{ \frac{1}{p} \min \left\{ \frac{(1 - L \beta^{-1}) \beta \theta_1^p - \lambda p \sum_{k \in \mathbb{Z}} F(k, \theta_1)}{G^{\theta_1}}, \frac{(1 - L \beta^{-1}) \beta \theta_2^p - \lambda p \sum_{k \in \mathbb{Z}} F(k, \theta_2)}{G^{\theta_2}}, \frac{(1 - L \beta^{-1}) \beta (\theta_3^p - \theta_2^p) - \lambda p \sum_{k \in \mathbb{Z}} F(k, \theta_3)}{G^{\theta_3}}, \frac{1}{p} \left(1 + L \beta^{-1}\right) \left(a(k_0 + 1) + a(k_0) + b(k_0)\right) \eta^p - \lambda \sum_{k \in \mathbb{Z}} F(k, \theta_1) \right\} \right\}.$$ 

Theorem 3.1. Assume that there exist positive constants $\theta_1, \theta_2, \theta_3, \eta$ and the integer $k_0$ with
$$\theta_1 < \beta^\frac{1}{p} \left(1 + L \beta^{-1}\right) \left(a(k_0 + 1) + a(k_0) + b(k_0)\right) \eta^p,$$
and $\theta_2 > \theta_3$ such that

(A2) $f(k, t) \geq 0 \quad \forall (k, t) \in \mathbb{Z} \times [0, \theta_3]$;

(A3) $\max \left\{ \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}{\theta_1^p}, \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_2)}{\theta_2^p}, \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_3)}{\theta_3^p} \right\} < \frac{(1 - L \beta^{-1}) \beta}{1 + L \beta^{-1}\left(a(k_0 + 1) + a(k_0) + b(k_0)\right) \eta^p} \left(F(k_0, \eta) - \sum_{k \in \mathbb{Z}} F(k, \theta_1)\right).$

Then, for every
$$\lambda \in \Lambda := \left\{ \frac{1}{p} \left(1 + L \beta^{-1}\right) \left(a(k_0 + 1) + a(k_0) + b(k_0)\right) \eta^p, \frac{(1 - L \beta^{-1}) \beta}{p} \min \left\{ \frac{\theta_1^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}, \frac{\theta_2^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_2)}, \frac{\theta_3^p - \theta_2^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_3)} \right\} \right\}$$
and for every nonnegative function $g : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$, there is $\delta_{\lambda,G}$ given by (3.1) such that for each $\mu \in [0, \delta_{\lambda,G})$, the problem $(P_{\lambda,\mu})$ has at least three nonnegative solutions $u_1, u_2, u_3$ such that $\max_{k \in \mathbb{Z}} |u_1(k)| < \theta_1, \max_{k \in \mathbb{Z}} |u_2(k)| < \theta_2$ and $\max_{k \in \mathbb{Z}} |u_3(k)| < \theta_3$.

Proof. Our goal is to apply Theorem 2.1 to the problem $(P_{\lambda,\mu})$. We consider the auxiliary problem
$$(P_{\lambda,\mu})$$
$$\begin{cases} -\Delta(a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) = \lambda f(k, u(k)) \\
+\mu g(k, u(k)) + h(u(k)), \\
u(k) \rightarrow 0 \quad \text{as} \quad |k| \rightarrow \infty
\end{cases}$$

in $\mathbb{Z}$. The rest of the proof is similar to the proof of Theorem 2.1.
where \( \hat{f} : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \) is a continuous function defined putting

\[
\hat{f}(k, \xi) = \begin{cases} 
  f(k, 0) & \text{if } \xi < 0, \\
  f(k, \xi) & \text{if } 0 \leq \xi \leq \theta_3, \\
  f(k, \theta_3) & \text{if } \xi > \theta_3.
\end{cases}
\] (3.2)

From (A1) any solution of the problem \((P_{\lambda, \mu}^{f,g,h})\) is nonnegative. In addition, if it satisfies also the condition \(0 \leq u(k) \leq \theta_3\), and for every \(k \in [1, T]\), clearly it turns to be also a nonnegative solution of \((P_{\lambda, \mu}^{f,g,h})\). Therefore, for our goal, it is enough to show that our conclusion holds for \((P_{\lambda, \mu}^{f,g,h})\). Let the functions \(\Phi, \Psi\) for every \(u \in X\), defined by

\[
\Phi(u) = \left\| u \right\|^p - \sum_{k \in \mathbb{Z}} H(u(k)),
\] (3.3)

and

\[
\Psi(u) = \sum_{k \in \mathbb{Z}} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))].
\] (3.4)

The function \(\Psi\) is differentiable at \(u \in X\)

\[
\Psi'(u)(v) = \sum_{k \in \mathbb{Z}} [f(k, u(k)) + \frac{\mu}{\lambda} g(k, u(k))] v(k) \quad \forall v \in X.
\]

From the definition of \((p - 1)\)-Lipschitz continues and \(h(0) = 0\), so we deduce

\[
\frac{1 - L\beta^{-1}}{p} \left\| u \right\|^p \leq \Phi(u) \leq \frac{1 + L\beta^{-1}}{p} \left\| u \right\|^p,
\] (3.5)

which the condition \(L(1 + L\beta^{-1}) < 1\) deduce that \(\Phi\) is coercive. In addition \(\Phi\) is continuously differentiable whose differential at the point \(u \in X\) is

\[
\Phi'(u)(v) = \sum_{k \in \mathbb{Z}} \left[ a(k) \phi_p(\Delta u(k - 1)) \Delta v(k - 1) + b(k) \phi_p(u(k)) v(k) - h(u(k)) v(k) \right]
\]

for every \(v \in X\). Furthermore, \(\Phi\) is sequentially weakly lower semicontinuous. Indeed, since \(\sum_{k \in \mathbb{Z}} H(u(k))\) continuous on \(X\), one has

\[
\lim_{n \to \infty} \inf \Phi(u_n) = \lim_{n \to \infty} \inf \frac{\|u_n\|^p}{p} - \lim_{n \to \infty} \sum_{k \in \mathbb{Z}} H(u_n(k)) \geq \frac{\|u\|^p}{p} - \sum_{k \in \mathbb{Z}} H(u(k)) = \Phi(u).
\]

Therefore, the assumptions on \(\Phi\) and \(\Psi\), as requested in Theorem 2.1 are verified. We know the critical points of the function \(\Phi - \lambda \Psi\) are the solutions for problem \((P_{\lambda, \mu}^{f,g,h})\). Define \(w(k_0) = \eta\) and \(w(k) = 0\) for every \(k \in \mathbb{Z} \setminus \{k_0\}\). By using (3.3) and (3.5) we have

\[
\frac{1 - L\beta^{-1}}{p} \left( (a(k_0 + 1) + a(k_0) + b(k_0)) \eta^p \right) \leq \Phi(w)
\]

\[
\leq \frac{1 + L\beta^{-1}}{p} \left( (a(k_0 + 1) + a(k_0) + b(k_0)) \right) \eta^p.
\] (3.6)

Setting \(r_1 = \frac{1 - L\beta^{-1}}{p} \theta_1^p\), \(r_2 = \frac{1 - L\beta^{-1}}{p} \theta_2^p\) and \(r_3 = \frac{1 - L\beta^{-1}}{p} (\theta_3^p - \theta_2^p)\), so from (3.6) and from the conditions \(\theta_1 < \beta \eta \frac{1}{p} (a(k_0 + 1) + a(k_0) + b(k_0))^{\frac{1}{p}} \eta\), \(\beta \eta \frac{1}{p} (1 + L\beta^{-1}) \left( (a(k_0 + 1) + a(k_0) + b(k_0))^{\frac{1}{p}} \right) \eta < \theta_2\) and \(\theta_2 < \theta_3\) we deduce

\[
r_1 < \Phi(w) < r_2.
\]
Now the estimation \( \Phi(u) \leq r_1 \) follows that
\[
|u(k)|^p \leq \|u\|_\infty^p \leq \beta^{-1}\|u\|^p \leq \frac{p}{(1-L\beta^{-1})\beta}\Phi(u) \\
\leq \frac{p}{(1-L\beta^{-1})\beta}r_1 = \theta_1^p, \quad \forall k_0 \in \mathbb{Z}.
\]
From the definition of \( r_1 \), it follows that
\[
\Phi^{-1}(\infty, r_1) = \{ u \in X; \Phi(u) < r_1 \} \subseteq \{ u \in X; |u| \leq \theta_1 \},
\]
Hence, by using the assumption \((A_2)\), one has
\[
\sup_{u \in \Phi^{-1}(\infty, r_1)} \sum_{k \in \mathbb{Z}} F(k, u(k)) \leq \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_1} F(k, t) \leq \sum_{k \in \mathbb{Z}} F(k, \theta_1).
\]
Similarity, we have
\[
\sup_{u \in \Phi^{-1}(\infty, r_2)} \sum_{k \in \mathbb{Z}} F(k, u(k)) \leq \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_2} F(k, t) \leq \sum_{k \in \mathbb{Z}} F(k, \theta_2)
\]
and
\[
\sup_{u \in \Phi^{-1}(\infty, r_3)} \sum_{k \in \mathbb{Z}} F(k, u(k)) \leq \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_3} F(k, t) \leq \sum_{k \in \mathbb{Z}} F(k, \theta_3)
\]
Therefore, since \( 0 \in \Phi^{-1}(\infty, r_1) \) and \( \Phi(0) = \Psi(0) = 0 \), one has
\[
\varphi(r_1) \leq \frac{\sup_{u \in \Phi^{-1}(\infty, r_1)} \Psi(u) - \Psi(u)}{r_1 - \Phi(u)} \\
= \frac{\sup_{u \in \Phi^{-1}(\infty, r_1)} \sum_{k \in \mathbb{Z}} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))] - \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_1} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))]}{r_1} \\
\leq \frac{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_1} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))] - \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_1} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))]}{r_1}
\]
By replacing \( r_1 = \frac{(1-L\beta^{-1})\beta}{p} \theta_1^p \), we conclude
\[
\varphi(r_1) \leq \frac{p}{(1-L\beta^{-1})\beta} \left( \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}{\theta_1^p} + \frac{\mu}{\lambda} \frac{G^{\theta_1}}{\theta_1^p} \right)
\]
and
\[
\varphi(r_2) \leq \frac{\sup_{u \in \Phi^{-1}(\infty, r_2)} \Psi(u)}{r_2} \\
= \frac{\sup_{u \in \Phi^{-1}(\infty, r_2)} \sum_{k \in \mathbb{Z}} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))] - \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_2} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))]}{r_2} \\
\leq \frac{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_2} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))] - \sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_2} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))]}{r_2}
\]
Then by replacing \( r_2 = \frac{(1-L\beta^{-1})\beta}{p} \theta_2^p \), we deduce
\[
\varphi(r_2) \leq \frac{p}{(1-L\beta^{-1})\beta} \left( \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_2)}{\theta_2^p} + \frac{\mu}{\lambda} \frac{G^{\theta_2}}{\theta_2^p} \right)
\]
and
\[
\gamma(r_2, r_3) \leq \sup_{u \in \Phi^{-1}(\infty, r_2 + r_3)} \frac{\Psi(u)}{r_3} = \sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \frac{\sum_{k \in \mathbb{Z}} [F(k, u(k)) + \frac{\mu}{\lambda} G(k, u(k))]}{r_3} \\
\leq \frac{p}{(1 - L\beta^{-1})\beta} \left( \frac{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \theta_1} [F(k, t) + \frac{\mu}{\lambda} G(k, t)]}{\theta_3^p - \theta_2^p} \right).
\]

Now for \( u \in \Phi^{-1}(-\infty, r_1) \) we have
\[
\beta(r_1, r_2) \geq \frac{F(k_0, \eta) - \sum_{k \in \mathbb{Z}} F(k, \theta_1) + \frac{\mu}{\lambda} (G(\eta) - G(\theta_1))}{\Phi(u) - \Phi(a)}
\]
\[
\geq \frac{F(k_0, \eta) - \sum_{k \in \mathbb{Z}} F(k, \theta_1) + \frac{\mu}{\lambda} (G(\eta) - G(\theta_1))}{1 + L\beta^{-1}} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right) \eta^p.
\]

Due to (A_3) we get
\[
\alpha(r_1, r_1, r_1) < \beta(r_1, r_2).
\]

Therefore, the assumptions (B_1) and (B_2) of Theorem 2.1 are verified. We verify \( \Phi - \lambda \Psi \) admits three distinct critical points. Let two local minima \( u_2 \) and \( u_3 \) for \( \Phi - \lambda \Psi \) are solutions for the problem \( (F_{\lambda, \mu, h}) \) and we want to prove that they are nonnegative. Let \( u_1 \) be a nontrivial solution of the problem \( (P_{\lambda, \mu, h}) \).

Arguing by a contradiction, assume that the set \( A = \{ k \in \mathbb{Z} : u_1(k) < 0 \} \) is nonempty and of positive measure. Put \( \bar{v}(k) = \min \{0, u_1 \} \) for all \( k \in \mathbb{Z} \) and we have
\[
\sum_{k \in \mathbb{Z}} \left[ a(k) \phi_p(\Delta u_1(k - 1)) \Delta \bar{v}(k - 1) \right.
\]
\[
+ b(k) \phi_p(u_1(k)) \bar{v}(k) - h(u_1(k)) \bar{v}(k) \]
\[
- \lambda \sum_{k \in \mathbb{Z}} [f(k, u_1(k)) + \mu g(k, u_1(k))] \bar{v}(k) = 0 \quad \forall \bar{v}(k) \in \mathbb{X}.
\]

Thus, from our sign assumptions on the data one has
\[
0 \leq (1 - L\beta^{-1}) \left( \sum_{k \in A} [a(k)|\Delta u(k - 1)|^p + b(k)|u(k)|^p] \right)
\]
\[
\leq \sum_{k \in A} \left[ a(k) \phi_p(\Delta u_1(k - 1)) \Delta u_1(k - 1) \right.
\]
\[
+ b(k) \phi_p(u_1(k)) u_1(k) - h(u_1(k)) u_1(k) \left. \right] \leq 0.
\]

Hence, since \( 1 > L\beta^{-1}, u_1 = 0 \) and this is absurd. Then, we deduce \( u_2 \geq 0 \) and \( u_3 \geq 0 \). Thus, it follows that \( su_2 + (1 - s)u_3 \geq 0 \) for all \( s \in [0, 1] \), and that \( (\lambda f + \mu g)(k, su_2 + (1 - s)u_3) \geq 0 \), and consequently, \( \Psi(su_2 + (1 - s)u_3) \geq 0 \), for all \( s \in [0, 1] \). Hence from Theorem 2.1 for every
\[
\lambda \in \left[ \frac{1 + L\beta^{-1}}{p} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right) \eta^p \right.
\]
\[
\left. \frac{(1 - L\beta^{-1})\beta}{p} \times \min \left\{ \frac{\theta_1^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}, \frac{\theta_2^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_2)}, \frac{\theta_3^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_3)} \right\} \right].
\]
and $\mu \in [0, \delta_{\lambda, G})$, the functional $\Phi - \lambda \Phi$ has three critical points $u_1$, $u_2$ and $u_3$ in $X$ in order to $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$ and $\Phi(u_3) < r_3 + r_2$, that is, $\max_{k \in \mathbb{Z}} |u_1(k)| < \theta_1$, $\max_{k \in \mathbb{Z}} |u_2(k)| < \theta_2$, and $\max_{k \in \mathbb{Z}} |u_3(k)| < \theta_3$.

**Remark 3.2.** If either $f(k,t) \neq 0$ for some $k \in \mathbb{Z}$ or $g(k,t) \neq 0$ for some $k \in \mathbb{Z}$, or both hold true the solutions of Theorem 3.1 are not trivial.

Now, we present the following example in which the hypotheses of Theorem 3.1 are satisfied.

**Example 3.3.** Consider the following problem

$$
\begin{align*}
-\Delta(a(k)\Delta u(k-1) + b(k)u(k)) &= \lambda f(k,u(k)) + \mu g(k,u(k)) + h(u(k)), \\
u(k) &\to 0
\end{align*}
$$

for each $k \in \mathbb{Z}$ and $\theta = r < \max_{k \in \mathbb{Z}} |u(k)| = 1$, $\lambda \neq 0$, $\mu \neq 0$, $\theta_1 < 1 + 10\delta$ and $\theta_2 < 1 + 10\delta$.

We define $b(k)$ as

$$b(k) = \begin{cases} 10^{-4k}, & k \in \mathbb{Z}^+, \\ 10^{4k}, & k \in \mathbb{Z}^- \end{cases}$$

and

$$f(k,t) = \begin{cases} 2 \times 5(\frac{1}{2})^{|k|+1}t^4, & t < 1, \\ 2 \times (\frac{1}{2})^{|k|+1}5, & t \geq 1, \end{cases}$$

also, we define $h(t) = \frac{1}{10^3}(1 - \cos t)$ for $t \in \mathbb{R}$. So, by the definition of $F$ and $H$ it's obvious

$$F(k,t) = \begin{cases} 2 \times 5(\frac{1}{2})^{|k|+1}t^4, & t < 1, \\ 2 \times (\frac{1}{2})^{|k|+1}(1 + 5 \ln(t)), & t \geq 1, \end{cases}$$

and $H(t) = \frac{1}{10^3}(t - \sin t)$ for every $t \in \mathbb{R}$. We suppose $\theta_1 = 10^{-4}$, $\theta_2 = 10^2$, $\theta_3 = 10^4$ and $\eta = 1$, in addition by selecting $\beta = 10^8$, $L = 10^{-4}$ and fixed $k_0 = 1$, since the hypotheses of Theorem 3.1 are fulfilled since $\theta_1 < 10^{-4}(\frac{3}{2} + 10^{-4})$, $\frac{10^{-4}(1 + 10^{-12})}{1 - 10^{-12}} < 10^{-12}$, $\frac{3}{2} + 10^{-12} < \theta_2$ and $\theta_2 < 10^{-12}$, and $f(k,t) \geq 0$ for each $(k,t) \in \mathbb{Z} \setminus \{0\} \times [0,10^4]$, and taking into account that

$$\max \left\{ \frac{\sum_{k \in \mathbb{Z}} F(k,10^{-4})}{10^{-8}}, \frac{\sum_{k \in \mathbb{Z}} F(k,10^2)}{10^8}, \frac{\sum_{k \in \mathbb{Z}} F(k,10^4)}{10^{10}} \right\} = \frac{1 + 10 \ln(10)}{10^4}$$

and

$$\frac{(1 - 10^{-12})10^8}{1 + 10^{-12}(\frac{3}{2} + 10^{-4})} \left( F(1,1) - \sum_{k \in \mathbb{Z}} F(k,10^{-4}) \right) \approx \frac{(1 - 10^{-12})10^8}{2} \frac{\frac{1}{2} - 10^{-20}}{1 + 10^{-12}(\frac{3}{2} + 10^{-4})},$$

so the condition $(A_3)$ of Theorem 3.1 is verified because

$$\frac{1 + 10 \ln(10)}{10^4} < \frac{(1 - 10^{-12})10^8}{2} \frac{\frac{1}{2} - 10^{-20}}{1 + 10^{-12}(\frac{3}{2} + 10^{-4})}.$$
Then, for every
\[
\lambda \in \Lambda := \left( \frac{1 + 10^{-12}(1.5 + 10^{-12})}{3 - 10^{-20}}, \frac{1 - 10^{-12}(10^8)}{3} \times 10^4 \right)
\]
we set nonnegative function \( g(t) = |\sin t| \), and constant \( \lambda = 1 \). Since
\[
\min \left\{ \frac{(1 - 10^{-12})10^{-8} - 3 \times 10^{-20}}{0.99}, \frac{(1 - 10^{-12})10^{16} - 3(1 + 10 \ln 10)}{0.17}, \frac{(1 - 10^{-12})(10^{20} - 10^{16}) - 3(1 + 10 \ln 10)}{0.17} \right\} = \frac{(1 - 10^{-12})10^{-8} - 3 \times 10^{-20}}{0.99}
\]
and
\[
\min \left\{ \frac{(1 - 10^{-12})10^{-8} - 3 \times 10^{-20}}{0.99}, \frac{(1 + 10^{-12})\frac{3}{2} + 10^{-4} - (1 - 10^{-20})}{1 - 0.99} \right\} = \frac{(1 - 10^{-12})10^{-8} - 3 \times 10^{-20}}{0.99}.
\]
So, there is \( \delta_{\lambda,G} = \frac{(1-10^{-12})10^{-8} - 3 \times 10^{-20}}{0.99} \) such that for each \( \mu \in [0, \delta_{\lambda,G}) \), the problem (3.7) has at least three nonnegative solutions \( u_1, u_2, u_3 \) such that \( \max_{k \in \mathbb{Z}} |u_1(k)| < 10^{-4} \), \( \max_{k \in \mathbb{Z}} |u_2(k)| < 10^2 \) and \( \max_{k \in \mathbb{Z}} |u_3(k)| < 10^4 \).

For positive constants \( \theta_1, \theta_2 \) and \( \eta \), put
\[
\delta_{\lambda,G} := \min \left\{ \frac{1}{p} \min \left\{ (1 - L\beta^{-1})\beta \theta_1 - p\lambda \sum_{k \in \mathbb{Z}} F(k, \theta_1), \frac{(1 - L\beta^{-1})\beta \theta_4 - 2p\lambda \sum_{k \in \mathbb{Z}} F(k, \frac{1}{2p} \theta_4)}{2G^{\beta_4}}, \frac{(1 - L\beta^{-1})\beta \theta_4 - 2p\lambda \sum_{k \in \mathbb{Z}} F(k, \theta_4)}{2G^{\beta_4}}, \frac{(1 + L\beta^{-1})}{p} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right) \eta^p G_\eta - G_\eta^{\beta_1}, \frac{\lambda \left( F(k_0, \eta) - \sum_{k \in \mathbb{Z}} F(k, \theta_1) \right)}{G_\eta - G_\eta^{\beta_1}} \right\} \right\}
\]
(3.8)

We, now deduce the following consequence of Theorem 3.1.

**Theorem 3.4.** Assume that there exist three positive constants \( \theta_1, \theta_4, \eta \) and the integer \( k_0 \) which we put
\[
\theta_1 < \min \left\{ \eta, \beta \frac{1}{p} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right)^{1/p} \eta \right\}
\]
and
\[
\beta \frac{1}{p} \frac{\sqrt{2}(1 + L\beta^{-1})}{2(1 - L\beta^{-1})} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right)^{1/2} \eta < \theta_4
\]
such that
\[
(A_4) \ f(k,t) \geq 0 \ for \ each \ (k_0, t) \in \mathbb{Z} \times [0, \theta_4],
\]
\[(A_5) \max \left\{ \frac{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}{\theta_4^2}, \frac{2 \sum_{k \in \mathbb{Z}} F(k, \theta_4)}{\theta_4^p} \right\} \lesssim \frac{1 - L\beta^{-1}}{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1}) (a(k_0 + 1) + a(k_0) + b(k_0))} \times \frac{F(k_0, \eta)}{\eta^p}. \]

Then, for every
\[
\lambda \in \left( \frac{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1}) (a(k_0 + 1) + a(k_0) + b(k_0))\eta^p}{\max \left\{ \frac{\theta_1^p}{p \beta^{-1} F(k_0, \eta)}, \frac{1 - L\beta^{-1}}{p} \min \left\{ \frac{\theta_1^p}{\sum_{k \in \mathbb{Z}} F(k, \theta_1)}, \frac{\theta_4^p}{2 \sum_{k \in \mathbb{Z}} F(k, \theta_4)} \right\} \right\}}. \]

and for every nonnegative function \( g : \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \), there is \( \delta'_{L,G} > 0 \) given by (3.8) such that for each \( \mu \in [0, \delta'_{L,G}) \), the problem \((P_{\lambda,\mu}^{f,g,h})\) has at least three nonnegative solutions \( u_1, u_2, u_3 \) such that
\[
\max_{k \in \mathbb{Z}} |u_1(k)| < \theta_1, \max_{k \in \mathbb{Z}} |u_2(k)| < \frac{1}{\sqrt{2}} \theta_1, \max_{k \in \mathbb{Z}} |u_3(k)| < \theta_4.
\]

Proof. We put \( \theta_2 = \frac{1}{\sqrt{2}} \theta_4 \) and \( \theta_3 = \theta_4 \). So from \((A_5)\) we get
\[
\sum_{k \in \mathbb{Z}} F(k, \theta_2) = \frac{2 \sum_{k \in \mathbb{Z}} F(k, \frac{1}{\sqrt{2}} \theta_4)}{\theta_4^p} \leq \frac{2 \sum_{k \in \mathbb{Z}} F(k, \theta_4)}{\theta_4^p} \times \frac{1 - L\beta^{-1}}{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1}) (a(k_0 + 1) + a(k_0) + b(k_0))} \times \frac{F(k_0, \eta)}{\eta^p} = (3.9)
\]

and
\[
\sum_{k \in \mathbb{Z}} F(k, \theta_3) = \frac{2 \sum_{k \in \mathbb{Z}} F(k, \theta_4)}{\theta_3 - \theta_2} \leq \frac{1 - L\beta^{-1}}{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1}) (a(k_0 + 1) + a(k_0) + b(k_0))} \times \frac{F(k_0, \eta)}{\eta^p}. = (3.10)
\]

From \((A_5)\) and taking into account \( \theta_1 < \eta \) we have
\[
\frac{(1 - L\beta^{-1})\beta}{1 + L\beta^{-1} (a(k_0 + 1) + a(k_0) + b(k_0))} (F(k_0, \eta) - \sum_{k \in \mathbb{Z}} F(k, \theta_1)) \geq \frac{(1 - L\beta^{-1})\beta}{1 + L\beta^{-1} (a(k_0 + 1) + a(k_0) + b(k_0))} (F(k_0, \eta)) - \frac{(1 - L\beta^{-1})\beta}{1 + L\beta^{-1} (a(k_0 + 1) + a(k_0) + b(k_0))} (\sum_{k \in \mathbb{Z}} F(k, \theta_1)) \geq \frac{1 - L\beta^{-1}}{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1}) (a(k_0 + 1) + a(k_0) + b(k_0))} \times \frac{F(k_0, \eta)}{\eta^p}. = (3.11)
\]

Hence, from (3.9), (3.10), and (A5), the assumption \((A_4)\) of Theorem 3.1 is satisfied, and since the critical points of function \( \phi - \lambda \psi \) are the solutions of the problem \((P_{\lambda,\mu}^{f,g,h})\) we have the conclusion. \( \square \)

Now, we present the following example in which the conditions of Theorem 3.1 are satisfied.
Example 3.5. Consider the problem

\[
\begin{cases}
-\Delta(a(k)\Delta u(k-1) + b(k)u(k)) = \lambda f(k, u(k)) \\
+\mu g(k, u(k)) + h(u(k)), \\
u(k) \to 0
\end{cases}
\]

where \(a(k) = 10^{-k}\) for every \(k \in \mathbb{Z}\) and \(b(k) = |\cos k\pi|\) and

\[
f(k, t) = \begin{cases} 
\frac{2}{3} \times 3(\frac{1}{2})^{k+1}t^2 & t < 1, \\
\frac{2}{3} \times (\frac{1}{2})^{k+1}t & t \geq 1,
\end{cases}
\]

also \(h(t) = \frac{1}{1 + t^2} - 1\) for \(t \in \mathbb{R}\), so by the definitions of \(F\) and \(H\) it is obvious and

\[
F(k, t) = \begin{cases} 
\frac{2}{3} \times 3(\frac{1}{2})^{k+1}t^3 & t < 1, \\
\frac{2}{3} \times (\frac{1}{2})^{k+1}(1 + 3 \ln(t)) & t \geq 1
\end{cases}
\]

and \(H(t) = \arctan(t - t)\) for every \(t \in \mathbb{R}\). We suppose that \(\theta_1 = 10^{-3}, \theta_4 = 10^3\) and \(\eta = 1\), in addition by choosing \(\beta = 10^4, L = 10^{-3}\) and fixed \(k_0 = 1\), since the assumptions of Theorem 3.1 are satisfied since

\[
\theta_1 < 10^{-2}(10^{-1} + 10^{-2})^{\frac{1}{2}} \quad \text{and} \quad 10^{-2} \frac{(\sqrt{2})(1 + 10^{-7})}{2(1 - 10^{-7})}(10^{-1} + 10^{-2})^{\frac{1}{2}} < \theta_4,
\]

and \(f(k, t) \geq 0\) for each \((k, t) \in \mathbb{Z} \times [0, 10^3]\), and

Since

\[
\max \left\{ \frac{\sum_{k \in \mathbb{Z}} F(k, 10^{-3})}{10^{-6}}, \frac{2\sum_{k \in \mathbb{Z}} F(k, 10^3)}{10^6} \right\} = \max \left\{ \frac{10^{-9}}{10^{-6}}, \frac{2(1 + 9 \ln 10)}{10^6} \right\} = \frac{10^{-9}}{10^{-6}}
\]

and

\[
\frac{1 - 10^{-7}}{1 - 10^{-7} + 10^{-4}(1 + 10^{-7})(10^{-1} + 10^{-2} + 1)} \times \frac{F(1, 1)}{1} = \frac{1 - 10^{-7}}{1 - 10^{-7} + 10^{-4}(1 + 10^{-7})(10^{-1} + 10^{-2} + 1)} \times \frac{1}{6}
\]

so the condition \((A_5)\) of Theorem 3.4 is verified because

\[
\frac{10^{-9}}{10^{-6}} < \frac{1 - 10^{-7}}{1 - 10^{-7} + 10^{-4}(1 + 10^{-7})(10^{-1} + 10^{-2} + 1)} \times \frac{1}{6}.
\]

Then, for every

\[
\lambda \in \Lambda \simeq \left( \frac{1 - 10^{-7} + 10^{-4}(1 + 10^{-7})(10^{-1} + 10^{-2} + 1)}{2 \times 10^{-4} \times \frac{1}{6}}, \frac{1 - 10^{-7}}{2} \times \frac{10^{-6}}{10^{-9}} \right)
\]

and for every nonnegative function \(g = |\sin t|\) and constant \(\lambda = 1\) since

\[
\min \left\{ \frac{(1 - 10^{-7})10^{-2} - 2 \times 10^{-9}}{1}, \frac{(1 - 10^{-7})10^{10} - 4(\frac{10^{-9}}{10^{-6}})}{2}, \frac{(1 - 10^{-7})10^{10} - 4(\frac{10^{-9}}{10^{-6}})}{2 \times 0.17} \right\} = \frac{(1 - 10^{-7})10^{-2} - 2 \times 10^{-9}}{1}
\]
and
\[
\min \left\{ \frac{(1 - 10^{-7})10^{-2} - 2 \times 10^{-9}}{1}, \frac{(1 - 10^{-7})}{3}(10^{-1} + 10^{-2} + 1) + 10^{-9} \right\}
\]
\[
= \frac{(1 - 10^{-7})10^{-2} - 2 \times 10^{-9}}{1}.
\]
So, there is \( \delta'_{\lambda,G} = \frac{(1 - 10^{-7})10^{-2} - 2 \times 10^{-9}}{1} \) such that for each \( \mu \in [0, \delta'_{\lambda,G}) \), the problem (3.12) has at least three nonnegative solutions \( u_1, u_2, u_3 \) such that \( \max_{k \in \mathbb{Z}} |u_1(k)| < 10^{-3}, \max_{k \in \mathbb{Z}} |u_2(k)| < \frac{10^3}{\sqrt{2}} \) and \( \max_{k \in \mathbb{Z}} |u_3(k)| < 10^3 \).

Now, we deduce the following straightforward consequence of Theorem 3.4.

**Theorem 3.6.** Let \( f \) be a nonnegative continuous and nonzero function such that
\[
\lim_{t \to 0^+} \frac{f(t)}{|t|^{p-1}} = \lim_{t \to +\infty} \frac{f(t)}{|t|^{p-1}} = 0
\]
for every \( \lambda > \lambda^* \) where
\[
\lambda^* = \inf \left\{ \frac{1 - \lambda \beta^{-1} + \beta^{-1}(1 + \lambda \beta^{-1})(a(k_0 + 1) + a(k_0) + b(k_0))}{p\beta^{-1}F(\eta)} : \eta > 0, F(\eta) > 0 \right\}
\]
there exists
\[
\bar{\mu} := \min \left\{ \frac{1}{p} \min \left\{ (1 - L\beta^{-1})\beta\theta^p - p\lambda F(\theta_1), (1 - L\beta^{-1})\beta\theta^p, (1 - L\beta^{-1})\beta\theta^p - 2p\lambda F(k, \theta_4) \right\}, \frac{1 + \lambda \beta^{-1}}{2G_{\beta^{-1}}} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right) \right\}
\]
where \( \theta_1, \theta_4 \) and \( \eta \) are positive constants, so the problem
\[
\begin{cases}
-\Delta(a(k)\phi_k(\Delta u(k-1))) + b(k)\phi_k(u(k)) = \lambda f(u(k)) \\
+\mu g(u(k)) + h(u(k)) \\
u(k) \to 0
\end{cases}
\]
for all \( k \in \mathbb{Z}, |k| \to \infty, \) \( \eta > 0 \) is nonnegative continuous and nonzero function, for each \( \mu \in [0, \bar{\mu}) \) has at least two distinct positive solutions.

**Proof.** Fix \( \lambda > \lambda^* \) which
\[
\lambda > \frac{1 - \lambda \beta^{-1} + \beta^{-1}(1 + \lambda \beta^{-1})(a(k_0 + 1) + a(k_0) + b(k_0))}{p\beta^{-1}F(\eta)}
\]
where \( F(t) = \int_0^t f(\xi)d\xi \) for all \( t \in \mathbb{R} \) and \( \eta > 0, F(\eta) > 0 \).

From (3.13) there is \( \theta_1 > 0 \) such that \( \theta_1 < \min\{\eta, \beta^{\frac{1}{p}}(a(k_0 + 1) + a(k_0) + b(k_0))^{\frac{1}{p}}\} \) and
\[
\beta^{\frac{1}{p}}\sqrt{2(1 + L\beta^{-1})} \left( a(k_0 + 1) + a(k_0) + b(k_0) \right)^{\frac{1}{p}} \eta < \theta_4
\]
such that \( \frac{F(\theta_1)}{\theta_1} < \frac{1 - L\beta^{-1}}{2p\beta^{-1}\lambda} \) and \( \frac{F(\theta_4)}{\theta_4} < \frac{1 - L\beta^{-1}}{2p\beta^{-1}\lambda} \).

Finally, we point out the following simple consequence of Theorem 3.4 when \( \mu = 0 \) and \( h \equiv 0 \).

**Theorem 3.7.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function such that \( tf(t) > 0 \) for all \( t \neq 0 \) and

\[
\lim_{t \to 0} \frac{f(t)}{t^{p-1}} = \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = 0. \tag{3.15}
\]

Then, for every \( \lambda > \lambda^{**} \) where

\[
\lambda^{**} = \frac{1 - L\beta^{-1} + \beta^{-1}(1 + L\beta^{-1})(a(k_0 + 1) + a(k_0) + b(k_0))p^{-1}}{p\beta^{-1}F(\eta)} \times \max \left\{ \inf_{\eta > 0} \frac{\eta^p}{F(\eta)}, \inf_{\eta < 0} \frac{-\eta^p}{F(\eta)} \right\}
\]

the problem (3.14) in the case \( \mu = 0 \) and \( h \equiv 0 \) possesses at least four distinct nontrivial solutions.

**Proof.** Setting

\[
f_1(t) = \begin{cases} 0 & t < 0, \\ f(t) & t \geq 0 \end{cases}
\]

and

\[
f_2(t) = \begin{cases} 0 & t < 0, \\ -f(-t) & t \geq 0 \end{cases}
\]

and employing Theorem 3.6 to \( f_1 \) and \( f_2 \) we have the result. \( \square \)

**References**

1. R.P. Agarwal, K. Perera, D. O’Regan, *Multiple positive solutions of singular discrete p-Laplacian problems via variational methods*, Adv. Differ. Equ. **2005** (2005), 93–99.
2. C. Bereanu, P. Jebelean, C. Şerban, *Periodic and Neumann problems for discrete p-Laplacian*, J. Math. Anal. Appl. **399** (2013), 75–87.
3. L.H. Bian, H.R. Sun, Q.G. Zhang, *Solutions for discrete p-Laplacian periodic boundary value problems via critical point theory*, J. Differ. Equ. Appl. **18** (2012), 345–355.
4. M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, *Existence of at least one homoclinic solution for a nonlinear second-order difference equation*, Int. J. Nonlinear Sci. Numerical Simul. **20** (2019), 433–439.
5. G. Bonanno, *A critical points theorem and nonlinear differential problems*, J. Global Optim. **28** (2004), 249–258.
6. G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012), 2992–3007.
7. G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal. **54** (2003), 651–665.
8. G. Bonanno, P. Candito, *Infinitely many solutions for a class of discrete nonlinear boundary value problems*, Appl. Anal. **884** (2009), 605–616.
9. G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differ. Equ. **244** (2008), 3031–3059.
10. G. Bonanno, P. Candito, *Nonlinear difference equations investigated via critical point methods*, Nonlinear Anal. **70** (2009), 3180-3186.
11. G. Bonanno, B. Di Bella, *A boundary value problem for fourth-order elastic beam equations*, J. Math. Anal. Appl. **343** (2008), 1166–1176.
12. A. Cabada, C. Li, S. Tersian, *On homoclinic solutions of a semilinear p-Laplacian difference equation with periodic coefficients*, Adv. Differ. Equ. **2010** (2010), 1–17.
13. P. Candito, G. D’Aguì, *Three solutions to a perturbed nonlinear discrete Dirichlet problem*, J. Math. Anal. Appl. **375** (2011), 594–601.
14. S. Heidarkhani, G.A. Afrouzi, G. Caristi, J. Henderson, S. Moradi, *A variational approach to difference equations*, J. Differ. Equ. Appl. **22** (2017), 1761–1776.
15. S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, Existence of three solutions for multi-point boundary value problems, J. Nonlinear Funct. Anal. 2017 (2017), Article ID 47.

16. S. Heidarkhani, A. Cabada, G.A. Afrouzi, S. Moradi, G. Caristi, A variational approach to perturbed impulsive fractional differential equations, J. Comput. Appl. Math. 341 (2018), 42–60.

17. S. Heidarkhani, A.L.A. De Araujo, G.A. Afrouzi, S. Moradi, Multiple solutions for Kirchhoff- type problems with variable exponent and nonhomogeneous Neumann conditions, Math. Nachr. 291 (2018), 326–342.

18. S. Heidarkhani, S. Moradi, S.A. Tersian, Three solutions for second-order boundary-value problems with variable exponents, Electron. J. Qual. Theory Differ. Equ. 33 (2018), 1–19.

19. J. Henderson, H.B. Thompson, Existence of multiple solutions for second order discrete boundary value problems, Comput. Math. Appl. 43 (2002), 1239–1248.

20. A. Iannizzotto, S. Tersian, Multiple homoclinic solutions for the discrete p-Laplacian via critical point theory, J. Math. Anal. Appl. 403 (2013), 173–182.

21. L. Kong, Existence of solutions to boundary value problems arising from the fractional advection dispersion equation, Electron. J. Diff. Equ. 2013 (2013), pp. 1–15.

22. L. Kong, Homoclinic solutions for a higher order difference equation with p-Laplacian, Indag. Math. 27 (2016), no.1, 124–146.

23. L. Kong, Homoclinic solutions for a second order difference equation with p-Laplacian, Appl. Math. Comput. 247 (2014), 1103–1121.

24. H. Liang, P. Weng, Existence and multiple solutions for a second-order difference boundary value problem via critical point, J. Math. Anal. Appl. 326 (2007), 511–520.

25. B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000), 401–410.

26. B. Ricceri, On a three critical points theorem, Arch. Math. 75 (2000), 220–226.

27. R. Stegliński, On homoclinic solutions for a second order difference equation with p-Laplacian, Discrete Continuous Dyn. Syst.-Ser. B 23 (2018), 487–492.

28. R. Stegliński, On sequences of large homoclinic solutions for a difference equations on integers, Adv. Differ. Equ. 2016 (2016), 1–11.

29. R. Stegliński, On sequences of large solutions for discrete anisotropic equations, Electronic J. Qual. Theo. Differ. Equ. 2015, No. 25, 1–10.

30. R. Stegliński, Sequences of small homoclinic solutions for difference equations on integers, Electron. J. Differential Equations, Vol. 2017 (2017), No. 228, pp. 1–12.

31. L. Zhilong, Existence of positive solutions of superlinear second-order Neumann boundary value problem, Nonlinear Anal. 72 (2010), 3216–3221.

32. D.B. Wang, W. Guan, Three positive solutions of boundary value problems for p-Laplacian difference equations, Comput. Math. Appl. 55 (2008), 1943–1949.

33. P.J.Y. Wong, L. Xie, Three symmetric solutions of Lidstone boundary value problems for difference and partial difference equations, Comput. Math. Appl. 45 (2003), 1445–1460.

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