DAHA AND SKEIN ALGEBRA OF SURFACE: DOUBLE-TORUS KNOTS

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ABSTRACT. We study a topological aspect of rank-1 double affine Hecke algebra (DAHA). Clarified is a relationship between the DAHA of $A_1$-type (resp. $C^\vee C_1$-type) and the skein algebra on a once-punctured torus (resp. a 4-punctured sphere), and the $SL(2;\mathbb{Z})$ actions of DAHAs are identified with the Dehn twists on the surfaces. Combining these two types of DAHA, we construct the DAHA representation for the skein algebra on a genus-two surface, and we propose a DAHA polynomial for a double-torus knot, which is a simple closed curve on a genus two Heegaard surface in $S^3$. Discussed is a relationship between the DAHA polynomial and the colored Jones polynomial.

1. INTRODUCTION

The double affine Hecke algebra (DAHA) was introduced by Cherednik, and it is a powerful tool in studies of the Macdonald polynomials associated with root systems (see, e.g., [13, 35]). The Macdonald polynomial is ubiquitous in mathematics and physics, and an interpretation as a $q$-deformation of a wave-function of the quantum Hall effect suggest an importance of a topological structure of the DAHA in studies of topological orders. The DAHA was recently applied to quantum topology. Proposed was a DAHA polynomial invariant [14, 15], and discussed was a relationship with the refined Chern–Simons invariant and the Khovanov homology [2]. The construction of the DAHA polynomial [14] is purely algebraic, but the DAHA polynomial is limited only to torus knots and their descendants, i.e., all are non-hyperbolic. An attempt [3] was made towards DAHA for a genus-two surface generalizing DAHA of $A_1$-type, but a relationship with the known quantum polynomial invariants is unclear.

A purpose of this article is to combine two rank-1 DAHAs of $A_1$-type and $C^\vee C_1$-type to construct the DAHA representation for double-torus knots. The double-torus knot [28, 29] is a simple closed curve on a genus two Heegaard surface in $S^3$, and a large family of knots such as twist knots belong to this type. Originally the DAHAs of $A_1$-type and $C^\vee C_1$-type are for the Rogers polynomial (or the $q$-ultraspherical polynomial) and the Askey–Wilson polynomial respectively. The Askey–Wilson polynomial [4] is on top of the Askey scheme of classification of orthogonal polynomials of hypergeometric-type, and its algebraic structure receives recent active interests (see [32, 33, 45, 46]). Here we pay attentions to a relationship between the DAHA and the Kauffman bracket skein algebra on surfaces. It is known that the DAHA of $C^\vee C_1$-type represents a quantization of the affine cubic surface which is the character variety of a 4-punctured sphere [41], while the DAHA of $A_1$-type is related to the character
variety of a once-punctured torus. Based on the fact [9, 42] that the coordinate ring of the character varieties is a specialization of the Kauffman bracket skein algebra, discussed also is a relationship with the skein algebra on the 4-punctured sphere and the once-punctured torus [6, 7].

For each simple closed curve on the genus-two surface, we assign a DAHA operator which represents the skein algebra on the surface. A benefit of our method combining two types of rank-1 DAHAs is that we can make use of their well-known automorphisms. Due to the relationship between the DAHA and the skein algebra, the DAHA automorphisms are regarded as the mapping class group [8, 19], the group of isotopy classes of orientation-preserving diffeomorphisms of surface. As the mapping class group is generated by the Dehn twists, we can clarify the $SL(2; \mathbb{Z})$ actions of the DAHA of $A_1$-type and $C^\vee C_1$-type as the Dehn twists about curves on each surface. The $q$-difference DAHA operator is indeed constructed by use of the automorphisms of DAHA as in the case of torus knots by Cherednik [14]. Using the DAHA operator assigned to a simple closed curve $c$ on the surface, we propose a DAHA polynomial for $c$. We compute explicitly the DAHA polynomial for double-twist knots, and discuss a relationship with the colored Jones polynomial.

This paper is organized as follows. In Section 2, we study the once-punctured torus $\Sigma_{1,1}$. We recall properties of the DAHA of $A_1$-type, and establish a relationship with the skein algebra on $\Sigma_{1,1}$. The DAHA polynomial proposed by Cherednik is also reviewed. Section 3 is for the 4-punctured sphere $\Sigma_{0,4}$. We recall both the DAHA of $C^\vee C_1$-type and the skein algebra on $\Sigma_{0,4}$. Based on the correspondence, we associate a DAHA operator for a simple closed curve with a rational slope. In these sections, the $SL(2; \mathbb{Z})$-actions on the rank-1 DAHAs are interpreted as the Dehn twists about certain curves on the surfaces $\Sigma_{1,1}$ and $\Sigma_{0,4}$. In Section 4 as a prototype toward the genus-two surface, we study the skein algebra on a twice-punctured torus $\Sigma_{1,2}$. We “glue” two types of the rank-1 DAHAs, $A_1$-type and $C^\vee C_1$-type, using a quantum dilogarithm function, and give the DAHA representation of the skein algebra on $\Sigma_{1,2}$. Section 5 is for the genus-two surface. We propose the DAHA polynomial for a simple closed curve on the surface, and study a relationship with the colored Jones polynomial. In the rest of this section, we collect our notations such as special functions.

1.1. Preliminaries. The Kauffman bracket skein module $\text{KBS}_\lambda(M)$ of a 3-manifold $M$ is defined by

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{vertically above} \\
\end{array}
\end{array} &= A \quad \begin{array}{c}
\begin{array}{c}
\text{vertically above} \\
\end{array}
\end{array}, \quad (1.1)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{vertically above} \\
\end{array}
\end{array} &= -A^2 - A^{-2}.
\end{align*}
\]

When $M = \Sigma \times [0, 1]$ with an oriented surface $\Sigma$, we write $\text{KBS}_\lambda(\Sigma)$. Here, a multiplication $x \cdot y$ of curves $x$ and $y$ means that $x$ is vertically above $y$,

\[
x \cdot y = \begin{array}{c}
\begin{array}{c}
\text{vertically above} \\
\end{array}
\end{array} \quad (1.2)
\]
Throughout this paper, we use for simplicity a variant of hyperbolic functions
\[ sh(x) = x - x^{-1}, \quad \text{ch}(x) = x + x^{-1}. \tag{1.3} \]

We recall the standard notations of \( q \)-calculus. We use the \( q \)-Pochhammer symbol defined by
\[ (x; q)_n = \prod_{j=1}^{n} (1 - x q^{j-1}), \tag{1.4} \]
and
\[ (x_1, x_2, \cdots; q)_n = (x_1; q)_n (x_2; q)_n \cdots. \]

Here we mean \( (x; q)_0 = 1 \), and for negative integers
\[ (x; q)_{-n} = \frac{(x; q)_\infty}{(x q^{-n}; q)_\infty} = \frac{1}{(x q^{-n}; q)_n}. \tag{1.5} \]

We also use the \( q \)-hypergeometric series
\[ \phi_r \left[ \frac{a_1, \cdots, a_r}{b_1, \cdots, b_s}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} \left( (-1)^n q^2 n(n-1) \right)^{1+s-r} z^n. \tag{1.6} \]

See, e.g., [22] for properties of the hypergeometric functions.

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2. Once-Punctured Torus

2.1. Skein Algebra. We study the skein module on a once-punctured torus \( \Sigma_{1,1} \). We set simple closed curves \( x, y, z, \) and \( b \) as in Fig. 1. See that \( b \) denotes the boundary circle of the puncture.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{torus.png}
\caption{Depicted are simple closed curves on the once-puncture torus \( \Sigma_{1,1} \).}
\end{figure}

**Proposition 2.1 ([10]).** The KBSA(\( \Sigma_{1,1} \)) is generated by \( x, y, \) and \( z, \) satisfying
\[ A x y - A^{-1} y x = (A^2 - A^{-2}) z, \]
\[ A y z - A^{-1} z y = (A^2 - A^{-2}) x, \]
\[ A z x - A^{-1} x z = (A^2 - A^{-2}) y. \tag{2.1} \]
It is noted that the boundary circle $\mathfrak{b}$ is generated by
\[ \mathfrak{b} = A x y z - A^2 x^2 - A^{-2} y^2 - A^2 z^2 + A^2 + A^{-2}. \] (2.2)

2.2. **DAHA of $A_1$-type.** We collect several properties of DAHA of $A_1$-type. Essential references are [13, 35].

**Definition 2.2.** The DAHA of $A_1$-type $\mathcal{H}_{q,t}$ is $\mathbb{C}(q,t)$-algebra generated by $Y^\pm 1, X^\pm 1, T^\pm 1$ satisfying
\[
(T + t) (T - t^{-1}) = 0, \\
TXT = X^{-1}, \\
T^{-1}YT^{-1} = Y^{-1}, \\
XY = q^{-1} YXT^2. \] (2.3)

We use an idempotent
\[ e = \frac{1}{t + t^{-1}} (t + T), \] (2.4)
which satisfies
\[ e^2 = e, \\
eT = T e = t^{-1} e. \] (2.5)

**Definition 2.3.** The spherical DAHA $\mathcal{S}H_{q,t}$ is
\[ \mathcal{S}H_{q,t} = e \mathcal{H}_{q,t} e. \]

The automorphisms of $A_1$-DAHA are listed in the following [13].

**Lemma 2.4.**
- An automorphism $\epsilon : \mathcal{H}_{q,t} \rightarrow \mathcal{H}_{q^{-1},t^{-1}}$;
\[
\epsilon : \begin{pmatrix} X \\ Y \\ T \end{pmatrix} \mapsto \begin{pmatrix} Y \\ X \\ T^{-1} \end{pmatrix}. \] (2.6)
- An anti-automorphism $\epsilon' : \mathcal{H}_{q,t} \rightarrow \mathcal{H}_{q,t}$;
\[
\epsilon' : \begin{pmatrix} X \\ Y \\ T \end{pmatrix} \mapsto \begin{pmatrix} Y^{-1} \\ X^{-1} \\ T \end{pmatrix}. \] (2.7)

**Lemma 2.5 ([13]).** The $\text{SL}(2; \mathbb{Z})$ action on $\mathcal{H}_{q,t}$ is generated by
\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \tau_R, \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \mapsto \tau_L, \] (2.8)
where $\tau_{\bullet} : \mathcal{H}_{q,t} \rightarrow \mathcal{H}_{q,t}$ is
\[
\tau_R : \begin{pmatrix} T \\ Y \\ X \end{pmatrix} \mapsto \begin{pmatrix} T \\ q^\frac{1}{2} XY \\ X \end{pmatrix}, \\
\tau_L : \begin{pmatrix} T \\ Y \\ X \end{pmatrix} \mapsto \begin{pmatrix} T \\ Y \\ q^{-\frac{1}{2}} YX \end{pmatrix}. \] (2.9)
We note that, with (2.6), we have
\[ \tau_L = \epsilon \tau_R \epsilon. \]  
(2.10)

2.3. Polynomial Representation and Macdonald Polynomial. We recall a representation on ring of the Laurent polynomials \( \mathbb{C}[x^{\pm 1}] \) [13]. We use an involution \( s \) and a \( q \)-difference operator \( \delta \) respectively defined by
\[
 s f(x) = f(x^{-1}), \quad \delta f(x) = f(q x),
\]
(2.11)
where \( f \in \mathbb{C}[x^{\pm 1}] \).

**Proposition 2.6.** A polynomial representation in \( \mathbb{C}[x^{\pm 1}] \) is given by
\[
 T \mapsto t^{-1} s + (t^{-1} - t) \frac{1}{x^2 - 1} (s - 1), \\
 X \mapsto x, \\
 Y \mapsto \delta s T.
\]
(2.12)

Here \( Y \) is called the Dunkl–Cherednik operator. We see that \( T f = t^{-1} f \) for the symmetric Laurent polynomials \( f \in \mathbb{C}[x + x^{-1}] \), and that \( e \mathbb{C}[x] = \mathbb{C}[x + x^{-1}] \). Thus \( \mathcal{H}_{q,t} \) preserves a symmetric space \( \mathbb{C}[x + x^{-1}] \). As a \( q \)-difference operator of \( \mathcal{H}_{q,t} \), we have the following expression,
\[
 Y + Y^{-1} \big|_{\text{sym}} \mapsto t x - t^{-1} x^{-1} - \frac{x - x^{-1}}{\delta} + \frac{t^{-1} x - t x^{-1}}{x - x^{-1}} - \delta^{-1},
\]
(2.13)
where \( h \rvert_{\text{sym}} \) means that \( h \in \mathcal{H}_{q,t} \) acts on the symmetric Laurent polynomial space \( \mathbb{C}[x + x^{-1}] \). This operator is known as the Macdonald operator (see, e.g., [34]). One also finds that
\[
 q^{1/2} X Y + q^{-1/2} Y^{-1} X^{-1} \big|_{\text{sym}} \mapsto q^{1/2} x \frac{t x - t^{-1} x^{-1}}{x - x^{-1}} - \frac{x - x^{-1}}{\delta} + q^{-1/2} x^{-1} \frac{t^{-1} x - t x^{-1}}{x - x^{-1}} - \delta^{-1}.
\]
(2.14)

We have the non-symmetric Macdonald polynomials \( E_m(x; q, t) \) as eigenfunctions of \( Y \),
\[
 Y E_m(x; q, t) = t^{-1} q^{-m} E_m(x; q, t), \\
 Y E_m(x; q, t) = t q^m E_m(x; q, t).
\]
(2.15)

Here \( m > 0 \), and the Laurent polynomials \( E_m(x; q, t) \) have forms of
\[
 E_{-m}(x; q, t) = x^{-m} + \frac{(t - t^{-1})}{t q^m - t^{-1} q^{-m}} x^m + \cdots, \\
 E_m(x; q, t) = x^m + \cdots,
\]
(2.16)
where \( \cdots \) means Laurent polynomials \( x^k \) with \( |k| < m \). It is noted that \( E_0(x; q, t) = 1 \) and that \( Y E_0(x; q, t) = t^{-1} E_0(x; q, t) \).

Symmetric eigenfunctions of (2.13) are the Macdonald polynomials of \( A_1 \)-type (also known as the \( q \)-ultraspherical polynomial, or the Rogers polynomial [35]). Explicitly, we have
\[
 (Y + Y^{-1}) M_n(x; q, t) = (t q^n + t^{-1} q^{-n}) M_n(x; q, t),
\]
(2.17)
where
\[ M_n(x; q, t) = x^n \cdot 2\phi_1 \left[ \frac{t^2}{1 - q^{2n}} : q^2, t^2 q^{2n} x^{-2} \right] \]  
(2.18) 
\[ = \frac{(q^2; q^2)_n}{(t^2; q^2)_n} \sum_{j+k=n} \frac{(t^2; q^2)(t^2; q^2)_k}{(q^2; q^2)(q^2; q^2)_k} x^{j-k}. \]

Here the polynomials are normalized to be \( M_0(x; q, t) = 1 \) and
\[ M_n(x; q, t) = (x^n + x^{-n}) + \cdots, \]
for \( n > 0 \). The polynomials \( M_n(x; q, t) \) span the symmetric Laurent polynomial space \( \mathbb{C}[x + x^{-1}] \). In terms of the non-symmetric polynomials (2.15), we have
\[ M_m(x; q, t) = E_m(x; q, t) + \frac{q^m - q^{-m}}{t^2 q^m - q^{-m}} E_m(x; q, t) \]  
(2.19) 
\[ = t^{-1} (T + t) E_m(x; q, t). \]

Some of them are explicitly written as follows;
\[ M_0(x; q, t) = 1, \]
\[ M_1(x; q, t) = x + x^{-1}, \]  
(2.20) 
\[ M_2(x; q, t) = x^2 + x^{-2} + \frac{(1 + q^2)(1 - t^2)}{1 - q^2 t^2}, \]
\[ M_3(x; q, t) = x^3 + x^{-3} + \frac{(1 - q^6)(1 - t^2)}{(1 - q^2)(1 - q^4 t^2)} (x + x^{-1}). \]

Note that the generating function of the \( A_1 \)-type Macdonald polynomials is
\[ \sum_{n=0}^{\infty} M_n(x; q, t) \frac{(t^2; q^2)_n}{(q^2; q^2)_n} z^n = \frac{(t^2 x z, t^2 x^{-1} z; q^2)_\infty}{(x z, x^{-1} z; q^2)_\infty}. \]  
(2.21) 

One sees that the Macdonald polynomial (2.18) reduces at \( q = t \) to
\[ M_n(x; q, q) = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}} = S_n(x + x^{-1}). \]  
(2.22) 

Here \( S_n(z) \) is the Chebyshev polynomial of the second kind, which is also defined recursively by
\[ z S_n(z) = S_{n+1}(z) + S_{n-1}(z), \]  
(2.23) 
with \( S_0(z) = 1, S_1(z) = z. \)

We list some identities of the \( A_1 \)-type Macdonald polynomials. As a typical property of orthogonal polynomials, we have the three-term recurrence relation,
\[ (X + X^{-1}) M_n(X; q, t) = M_{n+1}(X; q, t) + \frac{(1 - q^{2n})(1 - q^{2n-2} t^4)}{(1 - q^{2n-2} t^2)(1 - q^{2n} t^2)} M_{n-1}(X; q, t). \]  
(2.24) 
We also have
\[ \left( q^{-\frac{1}{2}} Y X + q^{\frac{1}{2}} X^{-1} Y^{-1} \right) M_n(X; q, t) \]
\[ = t q^{n+\frac{1}{2}} M_{n+1}(X; q, t) + t^{-1} q^{-n+\frac{1}{2}} \frac{(1 - q^{2n})(1 - q^{2n-2} t^4)}{(1 - q^{2n-2} t^2)(1 - q^{2n} t^2)} M_{n-1}(X; q, t). \]  
(2.25) 

For our later computations, we introduce the raising and lowering operators of \( M_n(x; q, t) \).
Proposition 2.7. We have the raising operator with a parameter shift,

$$
\left\{ \frac{(1-t^2x^2) (1-q^2t^2x^2)}{q \ t^2x (x^2-1)} - \delta - \frac{(t^2-x^2) (t^2q^2-x^2)}{q \ t^2x (x^2-1)} \delta^{-1} \right\} M_m(x; q, q \ t) = (q^{m+1} t^2 - q^{-m-1} t^{-2}) M_{m+1}(x; q, t). \tag{2.26}
$$

The lowering operator with a parameter shift is given by

$$
\frac{x}{x^2-1} (\delta - \delta^{-1}) M_m(x; q, t) = (q^m - q^{-m}) M_{m-1}(x; q, q \ t). \tag{2.27}
$$

Proof. It can be proved by calculating actions on the generating function (2.21). See also [30, 31].

These raising and lowering operators, which preserve the symmetric Laurent polynomial space $\mathbb{C}[x + x^{-1}]$, can be rewritten using the generators of DAHA. For brevity, we denote the raising and lowering operators in Prop. 2.7 as $K^{(+)}$ and $K^{(-)}$ respectively. By use of (2.12), we have

$$
t \ Y - t^{-1} Y^{-1}\big|_{\text{sym}} = t^{-1} (t^{-1} - t) \delta\big|_{\text{sym}} = t^{-1} \frac{t^{-1}x^2 - t}{x^2 - 1} (1 - s) \delta\big|_{\text{sym}},
$$

which proves [13, 33] that the lowering operator (2.27) is written as

$$
K^{(-)}\big|_{\text{sym}} = \frac{t}{t^{-1}X - t X^{-1}} \left( t \ Y - t^{-1} Y^{-1} \right)\big|_{\text{sym}}. 
$$

Note that $K^{(-)}$ does not depend on $t$ as operators on $\mathbb{C}[x + x^{-1}]$.

Combining the identities (2.26) and (2.27), we have

$$
K^{(+)} K^{(-)} M_m(x; q, t) = \left\{ (Y + Y^{-1})^2 - (t + t^{-1})^2 \right\} M_m(x; q, t).
$$

Using the above expression for $K^{(-)}$ and the fact that the Macdonald polynomials $M_m(x; q, t)$ are bases of $\mathbb{C}[x + x^{-1}]$, we find

$$
K^{(+)}\big|_{\text{sym}} = t^{-1} \left( t^{-1} Y - t Y^{-1} \right) \left( t^{-1}X - t X^{-1} \right)\big|_{\text{sym}}.
$$

To conclude, we have the following. We recall that $\text{sh}(x)$ is defined in (1.3).

Proposition 2.8. Both the raising and lowering operators preserve the symmetric Laurent polynomial space $\mathbb{C}[x + x^{-1}]$, and they are written as

$$
t^{-1} \ \text{sh} \left( t^{-1} Y \right) \ \text{sh} \left( t^{-1}X \right) M_m(x; q, q \ t) = \left( q^{m+1} t^2 - q^{-m-1} t^{-2} \right) M_{m+1}(x; q, t),
$$

$$
\frac{t}{\text{sh} \left( t^{-1} X \right)} \ \text{sh} \left( t \ Y \right) M_m(x; q, q \ t) = \left( q^m - q^{-m} \right) M_{m-1}(x; q, q \ t). \tag{2.28}
$$

2.4. Automorphisms as Conjugation. We revisit the $SL(2; \mathbb{Z})$ action (2.9) in the polynomial representation of DAHA. As a completion of DAHA [14], we introduce a function

$$
U_R = \exp \left( \frac{(\log X)^2}{2 \log q} \right).
$$

As $U_R$ is symmetric in $X \leftrightarrow X^{-1}$ and $s \ U_R = U_R \ s$, it commutes with $T$. One easily sees that $\delta \ U_R = q^{\frac{1}{2}} X \ U_R \ \delta$, and we obtain

$$
Y \ U_R = \delta \ s \ T \ U_R = q^{\frac{1}{2}} X \ U_R \ \delta \ s \ T = U_R q^{\frac{1}{2}} X \ Y.
$$
Trivial is a commutativity between $U_R$ and $X$, and thus, the automorphism $\tau_R$ (2.9) is identified with a conjugation by $U_R$. For $\tau_L$ (2.9), we recall (2.10) to find the following $U_L$.

**Proposition 2.9.** The $SL(2; \mathbb{Z})$ action on $SH_{q,t}$ is given as conjugation. In particular, the automorphisms $\tau_*$ (2.8) are written as conjugations

$$
\tau_* : h \mapsto U_*^{-1} h U_*,
$$

where

$$
U_R = \exp \left( \frac{(\log X)^2}{2 \log q} \right), \quad U_L = \exp \left( -\frac{(\log Y)^2}{2 \log q} \right).
$$

See [18] where the operators $U_*$ were introduced for DAHA of $A_n$-type.

2.5. **Shift Operator.** As seen from the raising (2.26) and the lowering operators (2.27), it is useful to introduce a parameter shift operator $\delta_t$ in $SH_{q,t}$ satisfying

$$
\delta_t t = q t \delta_t.
$$

The $SL(2; \mathbb{Z})$ actions $\tau_*$ (2.8) on $t$ are trivial, but we have the following action on $\delta_t$.

**Proposition 2.10.** We have $\tau_R : \delta_t \mapsto \delta_t$, and

$$
\tau_L : \delta_t \mapsto \frac{1}{\text{sh}(t^{-1} q^{-\frac{1}{2}} Y X)} \text{sh}(t^{-1} X) \delta_t.
$$

**Proof.** The conjugation (2.29) shows an invariance of $\delta_t$ under $\tau_R$. For $\tau_L$, we use the fact that the Macdonald polynomials span the symmetric polynomial space $\mathbb{C}[x + x^{-1}]$, and we compute actions on $M_m(x; q, t)$. Recalling that $M_n(x; q, t)$ is a sum of the non-symmetric polynomials (2.19) and that the operator $U_L$ is symmetric in $Y \leftrightarrow Y^{-1}$, we have the following equalities;

$$
U_L^{-1} \delta_t U_L M_m(x; q, t)
$$

$$
= U_L^{-1} \delta_t e^{-\frac{(\log(qm))}{2 \log q}} M_m(x; q, t) = U_L^{-1} e^{-\frac{(\log(qm+1))}{2 \log q}} M_m(x; q, q, t)
$$

$$
e^{-\frac{(\log(qm+1))}{2 \log q}} U_L^{-1} t \sh(t^{-1} X) \frac{1}{\sh(t^{-1} Y)} \left( q^{m+1} t^2 - q^{-m-1} t^{-2} \right) M_{m+1}(x; q, t)
$$

$$
= e^{-\frac{(\log(qm+1))}{2 \log q}} t \sh(t^{-1} Y) U_L^{-1} \left( q^{m+1} t^2 - q^{-m-1} t^{-2} \right) M_{m+1}(x; q, t)
$$

$$
= \frac{1}{\sh(t^{-1} q^{-\frac{1}{2}} Y X)} \sh(t^{-1} X) M_m(x; q, q, t).
$$

This proves the statement. \qed
2.6. Algebra Embedding. We shall give a DAHA representation for $\text{KBS}_A(\Sigma_{1,1})$ defined in (2.1). For a simple closed curve $c_{(r,s)}$ with slope $s/r$ on $\Sigma_{1,1}$, we assign DAHA operators as

$$c_{(r,s)} \mapsto m_{(r,s)} = \text{ch}(G_{(r,s)}) \in \text{SH}_{q,t},$$

(2.33)

where we recall that $\text{ch}(x)$ is defined in (1.3). The curves in Fig. 1 are identified as $c_{(1,0)} = \infty$, $c_{(0,1)} = y$, and $c_{(1,1)} = z$. The algebra $\text{KBS}_A(\Sigma_{1,1})$ was studied in detail in [20] (see also [5]), and known is a “product-to-sum formula” for $c_{(r,s)}$. For example, we can check easily the following skein algebra;

$$c_{(1,0)} c_{(0,1)} = A c_{(1,1)} + A^{-1} c_{(1,-1)},$$

$$c_{(0,1)} c_{(1,1)} = A c_{(0,0)} + A^{-1} c_{(1,2)},$$

$$c_{(1,1)} c_{(0,0)} = A c_{(0,1)} + A^{-1} c_{(2,1)}.$$  

(2.34)

Here we put

$$m_{(1,0)} = \text{ch}(X),$$

$$m_{(0,1)} = \text{ch}(Y),$$

$$m_{(1,1)} = \text{ch}\left(q^{\frac{1}{2}}XY\right) = \text{ch}\left(q^{-\frac{1}{2}}XY\right).$$  

(2.35)

Recall that $m_{(0,1)}$ is the Macdonald operator of $\text{SH}_{q,t}$, and see (2.13) and (2.14) for explicit forms of the $q$-difference operators on the symmetric Laurent polynomial space $\mathbb{C}[x + x^{-1}]$. By direct computations, we get a representation for (2.34)

$$m_{(1,0)} m_{(0,1)} = q^{-\frac{1}{2}} m_{(1,1)} + q^{\frac{1}{2}} m_{(1,-1)},$$

$$m_{(0,1)} m_{(1,1)} = q^{-\frac{1}{2}} m_{(0,0)} + q^{\frac{1}{2}} m_{(1,2)},$$

$$m_{(1,1)} m_{(1,0)} = q^{-\frac{1}{2}} m_{(0,1)} + q^{\frac{1}{2}} m_{(2,1)},$$

(2.36)

where we have defined

$$m_{(1,-1)} = \text{ch}\left(q^{\frac{1}{2}}X^{-1}Y\right),$$

$$m_{(1,0)} = \text{ch}(YXY),$$

$$m_{(2,1)} = \text{ch}(XYX).$$

(2.37)

Using these relations, we can check (2.1) in Prop. 2.1 and we obtain the following.

Theorem 2.11. We have an algebra embedding $\text{KBS}_A(\Sigma_{1,1}) \to \text{SH}_{q,t}$ with $A = q^{\frac{1}{2}}$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} m_{(1,0)} \\ m_{(0,1)} \\ m_{(1,1)} \end{pmatrix}.$$  

(2.38)

We can also check that

$$m_{(1,-1)} m_{(1,1)} = q^{-1} m_{(2,0)} + q m_{(0,2)} + q + q^{-1} - (t^2 q^{-1} + t^{-2} q),$$

(2.39)

where we follow the notation of [20],

$$m_{(2,0)} = T_2(m_{(1,0)}) = X^2 + X^{-2},$$

$$m_{(0,2)} = T_2(m_{(0,1)}) = Y^2 + Y^{-2}.$$
which satisfies the same recurrence equation (2.23) with the second kind polynomial. From
the viewpoint of the skein algebra, the last term in (2.39) denotes the boundary circle \( \mathcal{b} \) in Fig. 1

\[
\mathcal{b} \mapsto -t^2 q^{-1} - t^{-2} q.
\] (2.41)

Indeed the identities (2.36) and (2.39) give

\[
q^{-1} m_{(1,0)} m_{(0,1)} m_{(1,1)} = q^{-1} m_{(1,0)}^2 + q m_{(0,1)}^2 + q^{-1} m_{(1,1)}^2 + (-q - q^{-1}) + (-t^2 q^{-1} - t^{-2} q).
\] (2.42)

Recalling the skein algebra relation (2.2), we see that the embedding (2.38) induces (2.41).

Actions of \( m_{(r,s)} \) on the Macdonald polynomial \( M_\alpha(x; q, t) \) as a basis of the symmetric Laurent polynomials \( \mathbb{C}[x + x^{-1}] \) can be computed explicitly in principle. The Macdonald polynomials are eigenpolynomials of \( m_{(1,0)} = Y + Y^{-1} \) as in (2.17), and the three-term recurrence relations (2.24) and (2.25) denote respectively the actions of \( m_{(1,0)} \) and \( m_{(1,1)} \). These relations reduce to results of representation of \( KBS_\alpha(\Sigma_{1,1}) \) \([16, 36]\) when \( q \) is a root of unity.

Other operators \( m_{(r,s)} \) can be given explicitly using the \( SL(2; \mathbb{Z}) \)-action of \( \text{SH}_{q,t} \). For instance, when we apply the automorphisms (2.9) to (2.36), we get

\[
m_{(n,1)} m_{(1,0)} = q^{-1} m_{(n-1,1)} + q \frac{1}{X} m_{(n+1,1)},
\]

\[
m_{(1,1)} m_{(1,n)} = q^{-1} m_{(1,n-1)} + q \frac{1}{Y} m_{(1,n+1)},
\] (2.43)

where for \( n \geq 1 \)

\[
m_{(n,1)} = \text{ch} \left( q^{-\frac{n}{2}} x^{n-1} Y X \right),
\]

\[
m_{(1,n)} = \text{ch} \left( q^{\frac{n}{2}} y^{n-1} X Y \right).
\] (2.44)

With our algebra embedding, the \( SL(2; \mathbb{Z}) \) actions (2.9) of DAHA naturally induce the mapping class group \( \text{Mod}(\Sigma_{1,1}) \equiv SL(2; \mathbb{Z}) \) (see, e.g., \([8, 19]\)), which is generated by the Dehn twists \( T_x \) (resp. \( T_y \)) about the curve \( x \) (resp. \( y \)). As seen from the fact that \( T_R : c_{(1,0)} \mapsto c_{(1,1)} \), \( T_L : c_{(1,0)} \mapsto c_{(1,0)} \), and that the boundary circle \( \mathcal{b} \) is fixed, the automorphism \( T_R \) (2.8) denotes the right Dehn twist \( T_x^{-1} \) about \( x = c_{(1,0)} \). In the same manner, as we have \( T_L : c_{(1,0)} \mapsto c_{(0,1)} \) and \( T_L : c_{(1,0)} \mapsto c_{(1,1)} \), we can identify the automorphism \( T_L \) (2.8) with the left Dehn twist \( T_y \) about \( y = c_{(0,1)} \). For our later use, we summarize the actions of the Dehn twists as follows.

\[
\mathcal{J}_y \mapsto T_L, \quad T_L^{\pm 1} : \begin{pmatrix} T \\ X \\ Y \\ \delta_t \end{pmatrix} \mapsto \begin{pmatrix} T \\ q^{\mp \frac{1}{2}} Y^{\pm 1} X \\ Y \\ \frac{1}{\text{ch}(t^{-1} q^{\pm \frac{1}{2}} Y^{\pm 1} X)} \text{sh}(t^{-1} X) \delta_t \end{pmatrix},
\]

\[
\mathcal{J}_x \mapsto T_R^{-1}, \quad T_R^{\pm 1} : \begin{pmatrix} T \\ X \\ Y \\ \delta_t \end{pmatrix} \mapsto \begin{pmatrix} T \\ X \\ q^{\pm \frac{1}{2}} X^{\pm 1} Y \\ \delta_t \end{pmatrix}.
\] (2.45)
2.7. **DAHA Polynomial.** The $N$-colored Jones polynomial of knot $K$ is a linear combination of the Kauffman bracket polynomials for $k$ parallel copies of knot $K$ ($1 \leq k \leq N - 1$), so that the polynomial for an unknot is $(−1)^{N−1}A^{N−1}$. We should recall that this is the Chebyshev polynomial of the second kind. A case of $N = 2$ is the Jones polynomial. As a simple closed curve $c_{(r,s)}$ with coprime integers $(r, s)$ on the genus-one Heegaard surface in $S^3$ denotes a torus knot, the DAHA operator $M_{(r,s)}$ associated to $c_{(r,s)}$ is expected to be related with the (colored) Jones polynomial of the torus knot. Following Cherednik [14,15], we define

$$P_n(x, q, t; c_{(r,s)}) = M_{n-1}(\mathcal{O}_{(r,s)}; q, t)(1).$$

(2.46)

Here $\mathcal{O}_{(r,s)} = \tau(X)$ where the $SL(2; Z)$-action is $\mathcal{T} \mapsto \tau$ when the curve is $c_{(r,s)} = \mathcal{T}(c_{(1,0)})$. Note that the case of $n = 2$ is simply given from (2.20) as

$$P_2(x, q, t; c_{(r,s)}) = M_{(r,s)}(1)$$

$$= \text{ch}(\mathcal{O}_{(r,s)})(1).$$

(2.47)

For instance, we have

$$P_n(x, q, t; c_{(1,0)}) = M_{n-1}(x; q, t),$$

$$P_n(x, q, t; c_{(0,1)}) = M_{n-1}(t^{-1}; q, t).$$

(2.48)

As the Macdonald polynomial reduces to the Chebyshev polynomial of the second kind at a specific setting $q = t$, both polynomials $P_n(x, q, t; c_{(r,s)})$ for $(r, s) = (0, 1), (1, 0)$ are regarded as a deformation of the $n$-colored Jones polynomial for unknot.

We show an explicit result for the curve $c_{(2k+1,2)} = \left(\mathcal{T}_x^{-k} \circ \mathcal{T}_y^2\right)(c_{(1,0)})$. We have

$$\mathcal{O}_{(2k+1,2)} = \left(\tau_R^k \circ \tau_I^2\right)(X) = q^{k-1}(X^2 Y^2) X,$$

(2.49)

and the DAHA operator associated to $c_{(2k+1,2)}$ is given by

$$M_{(2k+1,2)} = \text{ch}\left(q^{k-1}(X^2 Y^2) X\right).$$

This can be written as the operator on the symmetric polynomials $C[x + x^{-1}]$ as

$$M_{(2k+1,2)} \mapsto \tilde{M}_{(2k+1,2)}(x; q, t)$$

$$= (qx)^{2k+1} \left(\frac{1 - t^2 x^2}{t^2 (1 - x^2)} \frac{1 - q^2 t^2 x^2}{1 - q^2 x^2}\right) \delta^2 + \left(q x^{-1}\right)^{2k+1} \left(\frac{t^2 - x^2}{t^2 (1 - x^2)} \frac{q^2 t^2 - x^2}{q^2 x^2}\right) \delta^{-2}$$

$$- q t^{-2} \left(q^2 - t^2\right) \left(1 - t^2\right) \frac{x (1 + x^2)}{(q^2 - x^2) (1 - q^2 x^2)}. \quad (2.50)$$

From this expression we obtain

$$P_2(-q, q; c_{(2k+1,2)}) = \left.\tilde{M}_{(2k+1,2)}(x; q, t)(1)\right|_{x = \pm 1}$$

$$= 1 - q^{4k} - q^{4k+2} - q^{4k+4}$$

$$= -q^{6k+3} (q + q^{-1}) J_2(q^2; T_{(2k+1,2)}),$$

(2.51)
where \( J_N(q; T_{(s,t)}) \) is the colored Jones polynomial for torus knot \(^{[38]}\) normalized to be \( J_N(q; \text{unknot}) = 1 \),

\[
J_N(q; T_{(s,t)}) = \frac{q^{1/2} (1 - N^2)}{q^2 - q^{-2}} \sum_{r=\frac{N-1}{2}}^{N-1} \left( q^{str^2-(s+t)r+\frac{1}{2}} - q^{str^2-(s-t)r-\frac{1}{2}} \right). \tag{2.52}
\]

We note that the three-term recurrence relation (2.24) for the Macdonald polynomial gives a recursion relation for the DAHA polynomial,

\[
P_{n+1}(x, q, t; c(2k+1,2)) = \tilde{M}_{(2k+1,2)}^{(0)}(x; q, t) \left( P_n(x, q, t; c(2k+1,2)) \right) - \frac{(1 - q^{2n-2}) (1 - q^{2n-4} t^4)}{(1 - q^{2n-4} t^2) (1 - q^{2n-2} t^2)} P_{n-1}(x, q, t; c(2k+1,2)). \tag{2.53}
\]

We find that this agrees with the colored Jones polynomial up to framing factor at a specific point,

\[
P_m(-q, q, -q; c(2k+1,2)) = q^{2(k+1)(m^2 - 1)} \frac{q^m - q^{-m}}{q - q^{-1}} J_m(q; T(2k+1,2)). \tag{2.54}
\]

See \(^{[14, 15]}\) for further computations of the \( A_q \) DAHA polynomials.

### 3. 4-Punctured Sphere

#### 3.1. Skein Algebra

We set simple closed curves \( x, y, z \), and \( b_j \) on a 4-punctured sphere \( \Sigma_{0,4} \) as in Fig. 2. The boundary circles \( b_j \) of the punctures are central.

![4-punctured sphere](image)

**Figure 2.** Depicted are simple closed curves on the 4-punctured sphere \( \Sigma_{0,4} \).

**Proposition 3.1 (\(^{[10]}\)).** The Kauffman bracket skein module \( \text{KBS}_A(\Sigma_{0,4}) \) is generated by \( x, y, z \), and \( b_j \) satisfying

\[
\begin{align*}
A^2 x y - A^{-2} y x &= (A^4 - A^{-4}) z + (A^2 - A^{-2}) (b_2 b_3 + b_1 b_4), \\
A^2 y z - A^{-2} z y &= (A^4 - A^{-4}) x + (A^2 - A^{-2}) (b_1 b_2 + b_3 b_4), \\
A^2 z x - A^{-2} x z &= (A^4 - A^{-4}) y + (A^2 - A^{-2}) (b_1 b_3 + b_2 b_4), \tag{3.1}
\end{align*}
\]

with

\[
\begin{align*}
A^2 x y z &= A^4 x^2 + A^{-4} y^2 + A^4 z^2 \\
&\quad + A^2 (b_1 b_2 + b_3 b_4) x + A^{-2} (b_1 b_3 + b_2 b_4) y + A^2 (b_1 b_4 + b_2 b_3) z \\
&\quad + b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_1 b_2 b_3 b_4 - (A^2 + A^{-2})^2. \tag{3.2}
\end{align*}
\]
Essential simple closed curves on $\Sigma_{0,4}$ are parameterized by a slope $Q \cup \{\infty\}$. Curves $x, y, \text{and} z$ are identified respectively with curves $c_{(1,0)}, c_{(0,1)}, \text{and} c_{(1,1)}$, where $c_{(r,s)}$ with coprime integers $r$ and $s$ denotes a simple closed curve with a slope $s/r$ (see Fig. 3). The multiplicative structure of these curves were investigated in detail [5], and a “product-to-sum formula” [20] was given. For example, one finds the following skein relation:

$$c_{(1,0)} c_{(0,1)} = A^2 c_{(1,1)} + A^{-2} c_{(1,-1)} + (b_1 b_4 + b_2 b_3),$$

$$c_{(1,0)} c_{(1,1)} = A^2 c_{(2,1)} + A^{-2} c_{(0,1)} + (b_1 b_3 + b_2 b_4),$$

$$c_{(1,1)} c_{(0,1)} = A^2 c_{(1,2)} + A^{-2} c_{(1,0)} + (b_1 b_2 + b_3 b_4). \quad (3.3)$$

### 3.2. $C^V C_1$-DAHA

A generalization of the rank-1 DAHA is known as DAHA of $C^V C_1$-type (or the Askey–Wilson type), which has four parameters besides $q$.

**Definition 3.2.** The DAHA of $C^V C_1$-type $\mathcal{H}_{q,t_0,t_1,t_2,t_3}$ is generated by $T_0^{\pm 1}, T_1^{\pm 1}, T_0^\vee^{\pm 1}$, and $T_1^\vee^{\pm 1}$, satisfying

$$\begin{align*}
(T_0 - t_0^{-1}) (T_0 + t_0) &= 0, \\
(T_1 - t_1^{-1}) (T_1 + t_1) &= 0, \\
(T_0^\vee - t_2^{-1}) (T_0^\vee + t_2) &= 0, \\
(T_1^\vee - t_3^{-1}) (T_1^\vee + t_3) &= 0,
\end{align*} \quad (3.4)$$

and

$$T_1^\vee T_1 T_0 T_0^\vee = q^{-1}. \quad (3.5)$$

We use an idempotent

$$e = \frac{1}{t_1 + t_1^{-1}} (t_1 + T_1). \quad (3.6)$$

satisfying

$$\begin{align*}
e^2 &= e, \\
e T_1 &= T_1 e = t_1^{-1} e.
\end{align*} \quad (3.7)$$

Hereafter we denote $t = (t_0, t_1, t_2, t_3)$.

**Definition 3.3.** The spherical DAHA $SH_{q,t}$ of $C^V C_1$-type is defined by $SH_{q,t} = e \mathcal{H}_{q,t} e$.

We recall some of the known automorphisms of DAHA. See, e.g., [40, 41].
Lemma 3.4. We have an involutive anti-automorphism $\epsilon'$ defined by

$$
\epsilon' : \begin{pmatrix} T_0 \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} T_0^\vee \\ T_1 \\ T_0 \\ T_1^\vee \end{pmatrix}, \quad \begin{pmatrix} t_0 \\ t_1 \\ t_0^\vee \\ t_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} t_3 \\ t_1 \\ t_2 \\ t_0 \end{pmatrix}.
$$

(3.8)

Lemma 3.5. The $SL(2; \mathbb{Z})$ action on $\mathcal{H}_{q,t}$ is generated by

$$
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto \sigma_R, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \sigma_L,
$$

(3.9)

where the automorphisms are

$$
\sigma_R : \begin{pmatrix} T_0 \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} T_0 T_0^{-1} \\ T_1 \\ T_0 \\ T_1^\vee \end{pmatrix}, \quad \begin{pmatrix} t_0 \\ t_1 \\ t_0^\vee \\ t_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} t_2 \\ t_1 \\ t_2 \\ t_0 \end{pmatrix},
$$

$$
\sigma_L : \begin{pmatrix} T_0 \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} T_0^{-1} \\ T_1 \\ T_0^\vee \\ T_1^\vee \end{pmatrix}, \quad \begin{pmatrix} t_0 \\ t_1 \\ t_0^\vee \\ t_1^\vee \end{pmatrix} \mapsto \begin{pmatrix} t_0 \\ t_1 \\ t_3 \\ t_2 \end{pmatrix}.
$$

(3.10)

We note that

$$
\sigma_R = \epsilon' \sigma_L \epsilon',
$$

(3.11)

where $\epsilon'$ is an involutive anti-automorphism (3.8).

3.3. Polynomial Representation and Askey–Wilson Polynomial. We review the representation on the Laurent polynomials $\mathbb{C}[x^\pm 1]$. See, e.g., [39, 40, 43].

Proposition 3.6. A polynomial representation is given by

$$
T_0 \mapsto t_0^{-1} s \delta - q^{-1} (t_0^{-1} - t_0) x^2 + q^{-1/2} (t_2^{-1} - t_2) x (1 - s \delta),
$$

$$
T_1 \mapsto t_1^{-1} s + (t_1^{-1} - t_1) + (t_3^{-1} - t_3) x (s - 1),
$$

$$
T_0^\vee \mapsto q^{-1/2} T_0^{-1} x,
$$

$$
T_1^\vee \mapsto x^{-1} T_1^{-1}.
$$

(3.12)

Based on this representation, we define

$$
Y = T_1 T_0,
$$

$$
X = (T_1^\vee T_1)^{-1},
$$

(3.13)

where $Y$ is the Dunkl–Cherednik operator for the Askey–Wilson polynomial.

As in the case of $A_1$-type, we have $T_1 f = t_1^{-1} f$ for a symmetric Laurent polynomial $f \in \mathbb{C}[x + x^{-1}]$. We see that the projection $e$ is $\mathbb{C}[x] \to \mathbb{C}[x + x^{-1}]$, and $SH_{q,t}$ preserves $\mathbb{C}[x + x^{-1}]$. On this symmetric polynomial space, the so-called Askey–Wilson operator is explicitly written as

$$
Y + Y^{-1}_{\text{sym}} \mapsto A(x; t) (\delta - 1) + A(x^{-1}; t) (\delta^{-1} - 1) + t_0 t_1 + (t_0 t_1)^{-1},
$$

(3.14)
where
\[ A(x; t) = t_0 t_1 \frac{\left(1 - \frac{1}{t_0 t_1} x\right) \left(1 + \frac{t_1}{t_0} x\right) \left(1 - \frac{q^{\frac{1}{2}}}{t_0 t_2} x\right) \left(1 + \frac{q^{\frac{1}{2}} t_2}{t_0} x\right)}{(1 - x^2) (1 - q x^2)}. \]  

(3.15)

Eigenfunctions of \(Y\) \((3.13)\) are called the non-symmetric Askey–Wilson polynomial,
\[ Y E_m(x; q, t) = (t_0 t_1)^{-1} q^m E_m(x; q, t), \]
\[ Y E_{-m}(x; q, t) = t_0 t_1 q^{-m} E_{-m}(x; q, t). \]  

(3.16)

Here \(m > 0\) and
\[ E_m(x; q, t) = x^m + \frac{(t_0^2 - 1) t_1^2 + q^m (t_1^2 - 1)}{(t_0 t_1)^2 q^m - q^m} x^{-m} + \cdots, \]
\[ E_{-m}(x; q, t) = x^{-m} + \cdots, \]  

(3.17)

where \(\cdots\) denote the Laurent polynomials \(x^k\) with \(|k| < m\). We note that \(E_0(x; q, t) = 1\) and that \(YE_0(x; q, t) = (t_0 t_1)^{-1} E_0(x; q, t)\).

The eigenfunctions of \((3.14)\) are the symmetric Askey–Wilson polynomials \([4]\). We have
\[ (Y + Y^{-1}) P_m(x; q, t) = \left((t_0 t_1)^{-1} q^m + t_0 t_1 q^{-m}\right) P_m(x; q, t). \]  

(3.18)

Here we have
\[ P_m(x; q, t) = \frac{(a b, a c, a d; q)_m}{a^m (a b c d q^{-m-1}; q)_m} 4\phi_3 \left[ q^{-m}, q^{m-1} a b c d, a x, a x^{-1}; q, q \right], \]  

(3.19)

where
\[ a = \frac{1}{t_1 t_3}, \quad b = -\frac{t_3}{t_1}, \quad c = \frac{q^{\frac{1}{2}}}{t_0 t_2}, \quad d = -\frac{q^{\frac{1}{2}} t_2}{t_0}. \]  

(3.20)

Note that we have normalized the polynomials so that
\[ P_m(x; q, t) = (x^m + x^{-m}) + \cdots, \]
and \(P_0(x; q, t) = 1\). These are written in terms of the non-symmetric polynomials as
\[ P_m(x; q, t) = E_m(x; q, t) + \frac{(q^m - 1) \left(t_0^2 + q^m\right) t_1^2}{q^{2m} - (t_0 t_1)^2} E_{-m}(x; q, t) \]
\[ = t_1 (T_1 + t_1) E_{-m}(x; q, t). \]  

(3.21)

Some of them are explicitly written as
\[ P_0(x; q, t) = 1, \]
\[ P_1(x; q, t) = x + x^{-1} + \frac{q^{\frac{1}{2}} t_0 \left(1 + t_1^2\right) \left(1 - t_2^2\right) t_3 + (q + t_0^2) t_1 t_2 \left(1 - t_2^2\right)}{(q - t_0^2 t_2^2) t_2 t_3}. \]  

(3.22)

Higher order polynomials are generated from the three-term recurrence relation. It is read as (see, e.g., \([22]\))
\[ (X + X^{-1}) P_m(x; q, t) = P_{m+1}(x; q, t) + B_m P_m(x; q, t) + C_m P_{m-1}(x; q, t). \]  

(3.23)
Here using (3.20) we have
\[ C_n = \frac{1 - abcdq^{n-2}}{(1 - abcdq^{2n-3})(1 - abcdq^{2n-2})} \times \frac{1 - q^n}{(1 - acdqn^{-1})(1 - abdqn^{-1})}, \]
\[ B_n = a + a^{-1} \frac{1 - abcdq^{n-1}}{(1 - abcdq^{2n-3})(1 - abcdq^{2n-2})} \times \frac{1 - q^n}{(1 - acdqn^{-1})(1 - abdqn^{-1})}. \]

It is noted that we also have
\[ \left( T_1 T_0' + (T_1 T_0')^{-1} \right) P_m(x; q, t) = q^{m+\frac{1}{2}}(t_0 t_1)^{-1} P_{m+1}(x; q, t) \]
\[ + \left( B_m = q^{-m-\frac{1}{2}} \frac{(t_0 - q^{\frac{1}{2}}t_1)}{(1 + q^{\frac{1}{2}}t_0 t_1)} \frac{(1 + t_2 t_3)}{(1 + t_0 t_2 t_3)} \right) P_m(x; q, t) \]
\[ + t_0 t_1 q^{-m+\frac{1}{2}} C_m P_{m-1}(x; q, t). \] (3.25)

3.4. Automorphisms as Conjugation. We study the \( SL(2; \mathbb{Z}) \) action (3.10) of \( S_{q,t} \) under the polynomial representation (3.12). We introduce
\[ V_R = \exp \left( -\frac{(\log X)^2}{2 \log q} \right). \]
This function is symmetric in \( X \leftrightarrow X^{-1} \), thus \( s V_R = V_R s \), and commutes with \( T_1 \) and \( T_0' \). As we have \( s \delta V_R = q^{-\frac{1}{2}}X V_R s \delta \), we get
\[ V_R T_0 = \left( q^{\frac{1}{2}}x^{-1}a(x) s \delta + b(x) \right) V_R, \]
where we have used \( T_0 = a(x) s \delta + b(x) \) in (3.12) for brevity. We see that the expression in the parenthesis coincides with
\[ T_0' = q^{-\frac{1}{2}} T_0^{-1} x = q^{\frac{1}{2}}x^{-1}a(x) s \delta + q^{-\frac{1}{2}}x \left( b(x) + t_0 - t_0^{-1} \right), \]
when
\[ (t_0 - t_2)(1 + t_0 t_2) = 0. \] (3.26)
So assuming (3.26), we have \( V_R^{-1} T_0' V_R = T_0' \), and also
\[ T_0 V_R T_0 = T_0 T_0' V_R = q^{-\frac{1}{2}} x V_R = V_R q^{-\frac{1}{2}} x = V_R T_0 T_0', \]
which proves \( V_R^{-1} T_0 V_R = T_0 T_0' T_0^{-1} \). The automorphism \( \sigma_R (3.10) \) is thus realized by conjugation of \( V_R \).

For \( \sigma_L (3.10) \), we recall (3.11) where the anti-involution \( \epsilon' (3.8) \) sends \( X \leftrightarrow Y^{-1} \) and \( Y \leftrightarrow X^{-1} \). As in the case of \( A_1 \)-type, the automorphism \( \sigma_L \) is also realized by conjugation of \( V_L \). To conclude, we have the following.

**Proposition 3.7.** Under the condition (3.26), the \( SL(2; \mathbb{Z}) \) actions (3.10) are conjugations
\[ \sigma_* : h \mapsto V_*^{-1} h V_* \]
where

\[ V_R = \exp \left( -\frac{(\log X)^2}{2 \log q} \right), \quad V_L = \exp \left( -\frac{(\log Y)^2}{2 \log q} \right). \]  

(3.27)

3.5. **Algebra Embedding.** We shall define \( \mathcal{A}_{(r,s)} \in \text{SH}_{q,t} \) associated to a simple closed curve \( c_{(r,s)} \) on the sphere \( \Sigma_{0,4} \) with a slope \( s/r \),

\[ c_{(r,s)} \mapsto \mathcal{A}_{(r,s)} = \text{ch} \left( \mathfrak{O}_{(r,s)} \right). \]  

(3.28)

Amongst them, we put

\[ \mathcal{A}_{(1,0)} = X + X^{-1} = \text{ch} \left( T_1^\vee T_1 \right) = \text{ch} \left( q^{\frac{1}{2}} T_0 T_0^\vee \right), \]

\[ \mathcal{A}_{(0,1)} = Y + Y^{-1} = \text{ch} \left( T_1 T_0 \right), \]

\[ \mathcal{A}_{(1,1)} = \text{ch} \left( T_1 T_0^\vee \right) = \text{ch} \left( q^{-\frac{1}{2}} T_1 T_0^{-1} \right). \]  

(3.29)

A tedious but straightforward computation proves

\[ \mathcal{A}_{(1,0)\mathcal{A}_{(0,1)}} = q^{-\frac{1}{2}} \mathcal{A}_{(1,1)} + q^{\frac{1}{2}} \mathcal{A}_{(1,-1)} - t_{03,12}, \]  

(3.30)

\[ \mathcal{A}_{(1,1)\mathcal{A}_{(0,1)}} = q^{-\frac{1}{2}} \mathcal{A}_{(1,2)} + q^{\frac{1}{2}} \mathcal{A}_{(1,0)} - t_{02,13}, \]

\[ \mathcal{A}_{(1,0)\mathcal{A}_{(1,1)}} = q^{-\frac{1}{2}} \mathcal{A}_{(2,1)} + q^{\frac{1}{2}} \mathcal{A}_{(0,1)} - t_{01,23}. \]

Here we have

\[ \mathcal{A}_{(1,-1)} = \text{ch} \left( q^{\frac{1}{2}} T_0 T_1^\vee \right), \]

\[ \mathcal{A}_{(1,2)} = \text{ch} \left( T_1 T_0^\vee T_1 \left( T_0^\vee \right)^{-1} \right) = \sigma_L^{-1}(\mathcal{A}_{(1,1)}), \]

\[ \mathcal{A}_{(2,1)} = \text{ch} \left( T_1 \left( T_0^\vee \right)^{-1} T_0 T_0^\vee \right) = \sigma_R^{-1}(\mathcal{A}_{(1,1)}). \]  

(3.31)

and

\[ t_{03,12} = \left( q^{\frac{1}{2}} t_1 - q^{-\frac{1}{2}} t_1^{-1} \right) \left( t_2 - t_2^{-1} \right) + \left( t_0 - t_0^{-1} \right) \left( t_3 - t_3^{-1} \right), \]

\[ t_{02,13} = \left( q^{\frac{1}{2}} t_1 - q^{-\frac{1}{2}} t_1^{-1} \right) \left( t_3 - t_3^{-1} \right) + \left( t_0 - t_0^{-1} \right) \left( t_2 - t_2^{-1} \right), \]

\[ t_{01,23} = \left( t_2 - t_2^{-1} \right) \left( t_3 - t_3^{-1} \right) + \left( t_0 - t_0^{-1} \right) \left( q^{\frac{1}{2}} t_1 - q^{-\frac{1}{2}} t_1^{-1} \right). \]  

(3.32)

Furthermore, we can check that

\[ \mathcal{A}_{(1,-1)\mathcal{A}_{(1,1)}} = q^{-1} \mathcal{A}_{(2,0)r} + q \mathcal{A}_{(0,2)r} - q^{-\frac{1}{2}} t_{02,13} \mathcal{A}_{(1,0)} - q^{\frac{1}{2}} t_{01,23} \mathcal{A}_{(0,1)} + \left( q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right)^2 \]

\[ - \left( t_0 - t_0^{-1} \right)^2 \left( q^{\frac{1}{2}} t_1 - q^{-\frac{1}{2}} t_1^{-1} \right)^2 - \left( t_2 - t_2^{-1} \right)^2 \left( t_3 - t_3^{-1} \right)^2 \]

\[ + \left( t_0 - t_0^{-1} \right) \left( q^{\frac{1}{2}} t_1 - q^{-\frac{1}{2}} t_1^{-1} \right) \left( t_2 - t_2^{-1} \right) \left( t_3 - t_3^{-1} \right). \]  

(3.33)

where we have in terms of the Chebyshev polynomial \([2.40]\)

\[ \mathcal{A}_{(2,0)r} = T_2(\mathcal{A}_{(1,0)}) = (T_1^\vee T_1)^2 + (T_1^\vee T_1)^{-2}, \]

\[ \mathcal{A}_{(0,2)r} = T_2(\mathcal{A}_{(0,1)}) = (T_1 T_0)^2 + (T_1 T_0)^{-2}. \]  

(3.34)

Combining (3.30) with (3.33), we find that \( \mathcal{A}_{(1,0)}, \mathcal{A}_{(0,1)}, \) and \( \mathcal{A}_{(1,1)} \) fulfill the cubic relation (3.2). See also [41][45]. As a result, we have the algebra embedding of the skein algebra (3.1) and (3.2) as follows.
Theorem 3.8. We have an algebra embedding $K_{S(A)}(\Sigma_{0,4}) \to \text{SH}_{q,t}$ with $A^2 = q^{-\frac{1}{2}}$ by

$$
\begin{pmatrix}
\chi \\
y \\
z
\end{pmatrix} \mapsto \begin{pmatrix}
\mathcal{A}_{(1,0)} \\
\mathcal{B}_1 \\
\mathcal{B}_2 \\
\mathcal{B}_3 \\
\mathcal{B}_4
\end{pmatrix}, \\
\begin{pmatrix}
\mathcal{B}_1 \\
\mathcal{B}_2 \\
\mathcal{B}_3 \\
\mathcal{B}_4
\end{pmatrix} \mapsto \pm \begin{pmatrix}
i(t_0 - t_0^{-1}) \\
i(t_2 - t_2^{-1}) \\
i(q^\frac{1}{2}t_1 - q^{-\frac{1}{2}}t_1^{-1}) \\
i(t_3 - t_3^{-1})
\end{pmatrix}.
$$

(3.35)

With the embedding, the three term relations (3.23) and (3.25) give the representation of $x$ and $z$ in terms of the Askey–Wilson polynomials as eigenfunctions of $y$ (3.18). As in the case of the skein algebra on the once-punctured torus [16], we obtain a finite-dimensional representation when we set $q$ to be a root of unity.

It should be noted that a torus is a two-fold branched cover of a sphere over four-points. This may correspond to the fact [25] that the DAHA of $C^\vee C_1$–type is constructed using the $A_1$–type with the reflection equation [44].

The other DAHA operators $A_{(r,s)}$ associated to $c_{(r,s)}$ can be given using the $SL(2;\mathbb{Z})$ actions on $\text{SH}_{q,t}$. For instance, when we apply the automorphisms (3.10) to (3.30), we get

$$
A_{(1,n)}A_{(0,1)} = q^{-\frac{1}{2}}A_{(1,n+1)} + q^{\frac{1}{2}}A_{(1,n-1)} - \begin{cases}
t_{02,13}, & \text{for odd } n, \\
t_{03,12}, & \text{for even } n,
\end{cases}
$$

$$
A_{(1,0)}A_{(n,1)} = q^{-\frac{1}{2}}A_{(n+1,1)} + q^{\frac{1}{2}}A_{(n-1,1)} - \begin{cases}
t_{01,23}, & \text{for odd } n, \\
t_{03,12}, & \text{for even } n,
\end{cases}
$$

(3.36)

where

$$
A_{(1,2k)} = \text{ch} \left( Y^{-k}X^{-1}T_1^{-1}Y^kT_1 \right),
$$

$$
A_{(1,2k+1)} = \text{ch} \left( q^{-\frac{1}{2}}Y^{-k-1}T_1XY^kT_1 \right),
$$

$$
A_{(2k,1)} = \text{ch} \left( T_1X^{-k}T_1^{-1}YX^k \right),
$$

$$
A_{(2k+1,1)} = \text{ch} \left( q^{-\frac{1}{2}}T_1X^{-k}Y^{-1}T_1X^{k+1} \right).
$$

(3.37)

See [5] for the algorithms to obtain the curve $c_{(r,s)}$ from the viewpoint of the skein algebra on the sphere $\Sigma_{0,4}$.

3.6. Automorphisms and Braiding. The relationship between DAHA of $C^\vee C_1$–type and the skein algebra $K_{S(A)}(\Sigma_{0,4})$ gives an interpretation of the $SL(2;\mathbb{Z})$ action (3.10) of DAHA. The mapping class group $\text{Mod}(\Sigma_{0,4})$ is generated by half Dehn twists, and it is known that the $SL(2;\mathbb{Z})$ action corresponds to the Artin braid group $B_3$, which denotes the subgroup of $\text{Mod}(\Sigma_{0,4})$ fixing one puncture [8][19]. As seen from the fact that

$$
\sigma_R : A_{(0,1)} \mapsto A_{(1,-1)}, \quad A_{(2,1)} \mapsto A_{(1,1)},
$$

$$
\sigma_L : A_{(1,0)} \mapsto A_{(1,-1)}, \quad A_{(1,2)} \mapsto A_{(1,1)},
$$

the automorphism $\sigma_R$ (resp. $\sigma_L$) is identified with a braiding of punctures $b_1$ and $b_2$ (resp. $b_2$ and $b_4$) as in Fig. 4 and it denotes the Dehn twist about $x$ (resp. $y$).
3.7. DAHA polynomials for $c_{(r,s)}$. For the simple closed curve $c_{(r,s)}$ with slope $s/r$ on $\Sigma_{0,4}$, we have assigned the DAHA operator $\mathcal{A}_{(r,s)} \in \text{SH}_q,t$ \eqref{eq:DAHA}. The DAHA polynomial associated to the curve $c_{(r,s)}$ is defined by

$$P_n(x, q, t; c_{(r,s)}) = M_{n-1}(\mathcal{O}_{(r,s)}; q, q)(1).$$

\label{eq:DAHA_poly}

Especially

$$P_2(x, q, t; c_{(r,s)}) = \mathcal{A}_{(r,s)}(1).$$

\label{eq:DAHA_poly_2}

This is why we use the $A_1$-type Macdonald polynomial in \eqref{eq:DAHA_poly} rather than the Askey–Wilson polynomial, $P_1(x; q, t) \neq x + x^{-1}$, but we may introduce a new parameter as a $t$-parameter of the $A_1$-Macdonald polynomial \eqref{eq:DAHA_poly}. Indeed a different definition was used in \cite{15} as the $C^\vee C_1$ DAHA polynomial.

We give some explicit forms in the following. We have

$$P_n(x, q, t; c_{(1,0)}) = M_{n-1}(x; q, q),$$

$$P_n(x, q, t; c_{(0,1)}) = M_{n-1}((t_0 t_1)^{-1}; q, q).$$

Using \eqref{eq:f1} and \eqref{eq:f3} we have

$$P_2(x, q, t; c_{(1,1)}) = q^{1/2}(t_0 t_1)^{-1} (x + x^{-1}) - q^{1/2}t_0^{-1} (t_3 - t_5^{-1}) - t_4^{-1} (t_2 - t_2^{-1}),$$

$$P_2(x, q, t; c_{(1,-1)}) = q^{-1/2}x^{-1} A(x; t) + q^{1/2}x A(x^{-1}; t) - A(x; t) - A(x^{-1}; t) + t_1 t_2 + (t_1 t_2)^{-1},$$

where $\tilde{t} = (t_2, t_1, t_0, t_3)$.

4. Twice-punctured Torus

4.1. Skein Algebra. The skein algebra $KBS_A(\Sigma_{1,2})$ on a twice-punctured torus was studied in \cite{10}. We define simple closed curves $x, y, x_u, y_u, b_3, b_4$ as in Fig. 5 where $b_3$ and $b_4$ are the boundary circles. We regard $\Sigma_{1,2}$ as the surface constructed by gluing an annulus $S^1 \times [0, 1]$ with $\Sigma_{0,4}$ in Fig. 2 $\Sigma_{1,2} = \Sigma_{0,4} \cup \mathcal{S}^1_1 \times [0, 1]$, where both $\mathcal{S}^1_1 \times \{0\} \approx b_1$ and $\mathcal{S}^1_1 \times \{1\} \approx b_2$ are isotopic to $x_u$. Then we have the skein algebra of $\Sigma_{0,4}$-type, \eqref{eq:KBS} and \eqref{eq:KBS_2} with $b_1 \approx b_2 \approx x_u$. Here $z$ is generated by $x$ and $y$ from the first identity of \eqref{eq:KBS}. On the other hand, $\Sigma_{1,2}$ is regarded as the surface given by gluing a once-punctured torus with a thrice-punctured sphere, $\Sigma_{1,2} = \Sigma_{1,1} \cup \Sigma_{0,3}$, where the boundary circle of $\Sigma_{1,1}$ is $x$ and the three boundary circles of $\Sigma_{0,3}$ are isotopic to $x, b_3, b_4$. Then we have the algebra \eqref{eq:KBS_1} of $\Sigma_{1,1}$ for $x_u, y_u$, and $z_u$. Here $z_u$ is generated by $x_u$ and $y_u$ by the first identity of \eqref{eq:KBS_1}. We see that $x$, which is isotopic to the boundary circle of $\Sigma_{1,1}$, is generated by \eqref{eq:KBS_2}. In addition, we need the consistency condition \eqref{eq:consistency} for $y$ and $y_u$ as the skein algebra for $\Sigma_{1,2}$, which can be checked directly.
Proposition 4.1. The skein algebra $KBS_A(\Sigma_{1,2})$ is as follows:

- $\Sigma_{0,4}$-type,
  \[
  A^2 x y - A^{-2} y x = (A^4 - A^{-4}) z + (A^2 - A^{-2}) (b_3 + b_4) x_u, \\
  A^2 y z - A^{-2} z y = (A^4 - A^{-4}) x + (A^2 - A^{-2}) (x_u^2 + b_3 b_4), \\
  A^2 z x - A^{-2} x z = (A^4 - A^{-4}) y + (A^2 - A^{-2}) (b_3 + b_4) x_u, 
  \]
  with
  \[
  A^2 x y z = A^4 x^2 + A^{-4} y^2 + A^4 z^2 \\
  + A^2 (x_u^2 + b_3 b_4) x + A^{-2} (b_3 + b_4) x_u y + A^2 (b_4 + b_3) x_u z \\
  + 2 x_u^2 + b_3^2 + b_4^2 + x_u^2 b_3 b_4 - (A^2 + A^{-2})^2, 
  \]

- $\Sigma_{1,1}$-type,
  \[
  A x_u y_u - A^{-1} y_u x_u = (A^2 - A^{-2}) z_u, \\
  A y_u z_u - A^{-1} z_u y_u = (A^2 - A^{-2}) x_u, \\
  A z_u x_u - A^{-1} x_u z_u = (A^2 - A^{-2}) y_u, 
  \]
  with
  \[
  x = A x_u y_u z_u - A^2 x_u^2 - A^{-2} y_u^2 - A^2 z_u^2 + A^2 + A^{-2}, 
  \]
  consistency,
  \[
  -y^2 y_u + (A^2 + A^{-2}) y y_u y - y_u y^2 = (A^2 - A^{-2})^2 y_u, \\
  -y_u^2 y + (A^2 + A^{-2}) y u y y - y y_u^2 = (A^2 - A^{-2})^2 y, 
  \]
  with
  \[
  x y_u = y u x, \\
  x_u y = y x_u. 
  \]

It is noted that the boundary circles $b_3$ and $b_4$ are central.

We note that the $\Sigma_{1,1}$-type relations (4.3) are redundant, and we have

\[
- y_u^2 x_u + (A^2 + A^{-2}) y_u x_u y_u - x_u y_u^2 = (A^2 - A^{-2})^2 x_u, \\
- x_u^2 y_u + (A^2 + A^{-2}) x_u y_u x_u - y_u x_u^2 = (A^2 - A^{-2})^2 y_u. 
\]
4.2. Construction of DAHA. To construct the DAHA representation for $KBS_A(\Sigma_{1,2})$ in Prop. 4.1 we shall first make use of the DAHA of $C^\vee C_1$-type which represents $KBS_A(\Sigma_{0,4})$ in Fig. 2. Due to the fact that the curves $b_1$ and $b_2$ are set to be isotopic, $b_1 \approx b_2 \approx x_u$, and that we have the embedding (3.35) for $KBS_A(\Sigma_{0,4})$, we put $t_0 = t_2 = i x_u$. Namely the parameters of $C^\vee C_1$-DAHA $\mathcal{H}_{q,t}$ (3.14) are set to be

$$(t_0, t_1, t_2) = \left( i x_u, i q^{-\frac{1}{2}} x_r, i x_u, i x_r \right) = t_2.$$ (4.8)

In the spherical $C^\vee C_1$-DAHA $\mathcal{H}_{q,t}$ with $t_1$ (4.8), we assign $X + X^{-1}$ and $Y + Y^{-1}$ for the curves $\chi$ and $\gamma$ respectively. Explicitly, we have the following representation on $\mathbb{C}[x + x^{-1}]$ satisfying the $\Sigma_{0,4}$-type skein relations (4.1),

$$\begin{align*}
\chi &\mapsto x + x^{-1}, \\
\gamma &\mapsto -\beta(x, x_u, x_r, x_r) \delta - \beta(x^{-1}, x_u, x_r, x_r) \delta^{-1} - \varphi(x, x_u, x_r, x_r), \\
x_u &\mapsto x_u + x_u^{-1}, \\
b_3 &\mapsto x_r + x_r^{-1}, \\
b_4 &\mapsto x_r + x_r^{-1}.
\end{align*}$$ (4.9)

where

$$\begin{align*}
\beta(x, x_u, x_r, x_r) &= \frac{\left( x_r + q^{\frac{1}{2}} x x_u \right) \left( q^{\frac{1}{2}} x + x_r x_x \right) \left( q^{\frac{1}{2}} x + x_u^2 \right)}{q^{\frac{1}{2}} \left( 1 - q^{\frac{1}{2}} x \right) \left( 1 - x^2 \right) x_r x_x x_u}, \\
\varphi(x, x_u, x_r, x_r) &= \frac{x (x_r + x_x) (1 + x_r x_x) (1 + x_u^2)}{\left( 1 - q^{-\frac{1}{2}} x \right) \left( 1 - q^{\frac{1}{2}} x \right) x_r x_x x_u}. \tag{4.10}
\end{align*}$$

To give representations for $x_u$ and $y_u$ satisfying (4.3) and (4.4), we use the $A_1$-type DAHA $\text{SH}_{q,u,t}$. The skein algebra embeddings in Theorems 2.11 and 3.8 suggest to set

$$q_u = q^{\frac{1}{2}}.$$ (4.11)

Recalling that the boundary circle $b$ in Fig. 2 is generated by (2.2) and that the DAHA representation gives (2.41), we see that (4.4) is satisfied by

$$t = i q_u^{\frac{1}{2}} x_u^{\frac{1}{2}}. \tag{4.12}$$

For the consistency conditions (4.5), we may take a conjugation of $\text{SH}_{q,u,t}$ by use of a “gluing function” $G(x, x_u)$ to be determined as

$$\begin{align*}
x_u &\mapsto G(x, x_u)^{-1} (x_u + x_u^{-1}) G(x, x_u) = x_u + x_u^{-1}, \\
y_u &\mapsto G(x, x_u)^{-1} (y(x, x_u) \delta_u + y(x, x_u^{-1}) \delta_u^{-1}) G(x, x_u).
\end{align*}$$ (4.13)

Here $\delta_u$ is a difference operator for $x_u$,

$$\delta_u x_u = q_u x_u \delta_u, \tag{4.14}$$

and $y(x, x_u)$ is for the Macdonald operator (2.13) of $\text{SH}_{q_u i q_u^{\frac{1}{2}} x_u^{\frac{1}{2}}}$

$$y(x, x_u) = i q_u^{\frac{1}{2}} x_u^{-\frac{1}{2}} x_u - \left( i q_u^{\frac{1}{2}} x_u^{\frac{1}{2}} \right)^{-1} x_u^{-1}. \tag{4.15}$$
Under this setting, the commutativity (4.6) is trivial, and we have the $\Sigma_{1,1}$-type skein relation (4.3) because of the conjugation of the $A_1$ DAHA representation. To check the remaining consistency condition (4.5), we assume that

$$y_u \mapsto y_1(x_u) = y_2(x, x_u) = -1,$$  \hfill (4.16)

By brute force computations, we find that the consistency conditions (4.5) are fulfilled when

$$y_1(x_u) = i q^\frac{1}{2} \frac{-1}{1-x_u^2},$$
$$y_2(x, x_u) = i q^\frac{1}{4} \left(1 + q^{-\frac{1}{2}} x^{-1} x_u^2\right) \left(1 + q^{-\frac{1}{2}} x x_u^2\right).$$  \hfill (4.17)

The representation (4.16) is indeed the form of (4.13), when the gluing function is defined in terms of the quantum dilogarithm function by

$$G(x, x_u) = e^{\log x \log x_u (\log q)} (-q^{\frac{1}{2}} x x_u^2; q).$$  \hfill (4.18)

We should note that the gluing function $G(x, x_u)$ satisfies the following $q$-difference equations,

$$G(q x, x_u) = \frac{x_u}{1 + q^2 x x_u^2},$$
$$G(x, q^2 x u) = \frac{x^2}{1 + q^2 x u^2}.$$  \hfill (4.19)

As a result, we obtain the following.

**Theorem 4.2.** We have an algebra embedding (4.9), (4.13), (4.18) of $\text{KBS}_{A}^{\Sigma_{1,2}} \to \text{SH}_{q,t_{\natural}}$ with $A^2 = q^{-\frac{1}{2}}$ as operators on symmetric polynomial $C[x + x^{-1}]$.

It should be stressed that the above representation preserves the symmetric Laurent polynomial space $C[x + x^{-1}]$. Due to conjugation by the gluing function $G(x, x_u)$ (4.18), broken is a symmetry $x_u \leftrightarrow x_u^{-1}$. We can recover the symmetry of $x_u$ when we discard the symmetry $x \leftrightarrow x^{-1}$, by taking an inverse conjugation, $h \mapsto G(x, x_u) h G(x, x_u)^{-1}$, for the representations in Theorem 4.2. We use $t$ by

$$x = -\frac{t^2}{q_u},$$  \hfill (4.20)

as in (4.12), and a $q$-difference operator for $t$ in $\text{SH}_{q,t_{\natural}}$ is

$$\delta_t \frac{t}{q_u} \delta_t.$$  \hfill (4.21)

We obtain the following representation which acts on $C[x_u + x_u^{-1}]$. 
Corollary 4.3. We have an algebra embedding, $\text{KBS}_A(\Sigma_{1,2}) \to \text{SH}_{q,t}$, with $A^2 = q_u^{-1}$,

\[
\begin{align*}
x_u &\mapsto x_u + X_u^{-1}, \\
y_u &\mapsto Y_u + Y_u^{-1}, \\
b_3 &\mapsto x_t + X_t^{-1}, \\
b_4 &\mapsto x_t + X_t^{-1}, \\
y &\mapsto -\frac{q_u t^2 (x_t - x_t t^2) (x_t x_r - t^2)}{(1 + t^2) (q_u^2 - t^4) x_t x_r} (X_u^{-1} t^{-1} - X_u) (X_u t^{-1} - X_u^{-1} t) \delta_t, \\
&\quad - \frac{q_u (q_u^2 x_r - x_t t^2) (q_u^2 - x_t x_r t^2)}{(1 + t^2) (q_u^2 - t^4) X_x x_r} \delta_t^{-1} - \frac{q_u t^2 (x_t + x_r) (1 + x_t x_r)}{(1 + t^2) (q_u^2) x_t x_r} (X_u + X_u^{-1}).
\end{align*}
\]

(4.22)

Here $X_u$ and $Y_u$ defined by $\text{(2.12)}$ acting on the Laurent polynomial of $x_u$ constitute the DAHA $\text{SH}_{q,t}$. The representation (4.22) preserves the symmetric polynomials $\mathbb{C}[x_u + x_u^{-1}]$, and they are explicitly written as follows;

\[
\begin{align*}
x_u &\mapsto x_u + X_u^{-1}, \\
y_u &\mapsto t x_u - t^{-1} x_u^{-1} \delta_u + t^{-1} x_u - t x_u^{-1} \delta_u^{-1}, \\
b_3 &\mapsto x_t + X_t^{-1}, \\
b_4 &\mapsto x_t + X_t^{-1}, \\
y &\mapsto -\frac{q_u (x_t - x_r t^2) (x_t x_r - t^2)}{(1 + t^2) (q_u^2 - t^4) (x_t x_r - t^2)} (x_u^2 - t^2) \delta_t, \\
&\quad - \frac{q_u (q_u^2 x_r - x_t t^2) (q_u^2 - x_t x_r t^2)}{(1 + t^2) (q_u^2 - t^4) x_t x_r} \delta_t^{-1} - \frac{q_u t^2 (x_t + x_r) (1 + x_t x_r)}{(1 + t^2) (q_u^2) x_t x_r x_u}.
\end{align*}
\]

(4.23)

4.3. DAHA Polynomial of Simple Closed Curves on $\Sigma_{1,2}$. The representation in Theorem 4.2 is on $\mathbb{C}[x + x^{-1}]$, while the representation [4.22] is on $\mathbb{C}[x_u + x_u^{-1}]$. Using Theorem 4.2, we can assign an operator $\mathcal{A}$ acting on $\mathbb{C}[x + x^{-1}]$ for a simple closed curve $c$ on $\Sigma_{1,2},$

\[
c \mapsto \mathcal{A} = \text{ch}(\Theta).
\]

(4.24)

When $c$ is given by Dehn twists from $y$, $c = \mathcal{T}(y)$, we have $\Theta = y(\gamma)$ due to that $y \mapsto Y + Y^{-1} \in \text{SH}_{q,t}$. The automorphism $y$ is induced from $\mathcal{T}$, and it is written as the conjugation by $U_\gamma$ and $V_\gamma$. We then define the DAHA polynomial for $c$ by

\[
Q_n(x, x_u, x_t, x_r, q; c) = M_{n-1}(\Theta; q, q)(1).
\]

(4.25)

Especially we have

\[
Q_2(x, x_u, x_t, x_r, q; c) = \mathcal{A}(1).
\]

For example, we have

\[
Q_2(x, x_u, x_t, x_r, q; y_u) = i q^{-\frac{1}{2}} x_u^2 \frac{x_u^2}{1 - x_u^2} (x + x^{-1}) - i q^{-\frac{1}{2}} q(1 - q) x_u^4 \frac{1}{1 - x_u^2},
\]

\[
Q_2(x, x_u, x_t, x_r, q; y) = -q^{-\frac{1}{2}} x_u x_t - q^2 x_u^{-1} x^{-1}.
\]

These are in $\mathbb{C}(q, x_u, x_t, x_r)[x + x^{-1}]$, and we do not know a relationship with the previously known quantum polynomial invariants at this stage.
We shall pay attention to the representation \((4.22)\) which acts on \(\mathbb{C}[x_u + x_u^{-1}]\). We assume that a simple closed curve \(c\) on \(\Sigma_{1,2}\) is given from either \(x_u\) or \(c_{(r,s)}\) on subsurface \(\Sigma_{0,4} \subset \Sigma_{1,2}\) by the Dehn twist \(\mathcal{T}\), where \(\mathcal{T}\) is generated by \(\mathcal{T}_{x_u}\) and \(\mathcal{T}_{y_u}\). Then we have

\[
\mathcal{c} \mapsto \begin{cases} 
\text{ch}(\gamma(X_u)), & \text{when } \mathcal{c} = \mathcal{T}(x_u), \\
\text{ch}(\gamma(O_{(r,s)})), & \text{when } \mathcal{c} = \mathcal{T}(c_{(r,s)}), 
\end{cases}
\]

where \(\gamma\) denotes automorphisms of DAHA \(\mathcal{H}_{q,t}\) induced from \(\mathcal{T}\). We define

\[
P_n(x_u, x_r, x_r, q_u, t; c) = \begin{cases} 
M_{n-1}(\gamma(X_u); q_u, t)(1), & \text{when } \mathcal{c} = \mathcal{T}(x_u), \\
M_{n-1}(\gamma(O_{(r,s)}); q_u, t)(1), & \text{when } \mathcal{c} = \mathcal{T}(c_{(r,s)}). 
\end{cases}
\]

We show some concrete examples. In the following \(\tau_{\bullet(u)}\) denotes the automorphisms \((2.8)\) for \(X_u\) and \(Y_u\) of \(A_1\)-DAHA, which correspond to the Dehn twists about the curve \(x_u\) and \(y_u\) respectively. The first example is

\[
c'_{(2k+1,2)} = \left( \mathcal{T}_{x_u}^{-k} \circ \mathcal{T}_{y_u}^{2} \right)(x_u),
\]

In the DAHA representation we have the automorphism \(\tau_{R(u)}^{k} \circ \tau_{L(u)}^{2}\) on \(X_u\) \((2.49)\) to obtain the same results for torus knots \(c_{(2k+1,2)}\) on the once-punctured torus,

\[
c'_{(2k+1,2)} \mapsto \tilde{M}_{(2k+1,2)}(x_u; q_u, t),
\]

where \(\tilde{M}_{(2k+1,2)}(x; q, t)\) is defined in \((2.50)\). We thus have

\[
P_2(x_u, x_l, x_r, q_u, t; c'_{(2k+1,2)}) = \tilde{M}_{(2k+1,2)}(x_u; q_u, t)(1),
\]

which reduces to

\[
P_2 \left( x_u = -q, x_l = -q^{-1}, x_r = -q^{-1}, q, t = q; c'_{(2k+1,2)} \right) = 1 - q^{4k} - q^{4k+2} - q^{4k+4},
\]

which is the Jones polynomial for torus knot \(T_{(2k+1,2)}\) as in \((2.51)\).

As the second example, we treat

\[
c''_{(2k+1,2)} = \left( \mathcal{T}_{x_u}^{-k} \circ \mathcal{T}_{y_u}^{2} \right)(y),
\]

We need the action \(\tau_{R(u)}^{k} \circ \tau_{L(u)}^{2}\) on the representation \((4.22)\) of \(y\) arising from the DAHA \(\mathcal{H}_{q,t}\) of \(C^*C_1\) type. Recalling \((2.45)\), we have

\[
c''_{(2k+1,2)} \mapsto \frac{q_u t^2 \left( x_l - x_r t^2 \right) \left( x_l x_r - t^2 \right)}{(1 + t^2) \left( q_u^2 - t^4 \right) x_l x_r} \text{sh} \left( t q_u k^{-1}(X_u Y_u)^2 X_u \right) \text{sh}(t^{-1}X_u) \delta_l - \frac{q_u \left( q_u^2 x_r - x_r t^2 \right) \left( q_u^2 - x_r t^2 \right)}{(t^2 + q_u^2) \left( q_u^2 - t^4 \right) x_l x_r} \delta_{l-1} \frac{1}{\text{sh}(t^{-1}X_u)} \text{sh} \left( t^{-1} q_u k^{-1}(X_u Y_u)^2 X_u \right) - \frac{q_u t^2 \left( x_l + x_r \right) (1 + x_l x_r)}{(1 + t^2) \left( t^2 + q_u^2 \right) x_l x_r} \text{ch} \left( q_u k^{-1}(X_u Y_u)^2 X_u \right).
\]
This can be written as an operator on $\mathbb{C}[x_u + x_u^{-1}]$ as

$$ c''_{(2k+1,2)} \mapsto -\frac{q_u t^2 (x_\ell - x, t^2) (x_\ell x_r - t^2)}{(1 + t^2) (q_u^2 - t^4) x_\ell x_r} \hat{M}^{(+)\phantom{(-)}}_{(2k+1,2)}(x_u; q_u, t) \delta_t $$

$$ -\frac{q_u (q_u^2 x_r - x_\ell t^2) (q_u^2 - x_\ell x_r t^2)}{(t^2 + q_u^2) (q_u^2 - t^4) x_\ell x_r} \delta_t^{-1} \hat{M}^{(-)}_{(2k+1,2)}(x_u; q_u, t) $$

where we use (2.50) and the operators $\hat{M}^{(\pm)}_{(2k+1,2)}(x; q, t)$ are given by

$$ \hat{M}^{(+)\phantom{(-)}}_{(2k+1,2)}(x; q, t) = \left( (q x)^{2k} \frac{1 - t^2 x^2}{q t^4 (1 - x^2) (1 - q^2 t^2 x^2)} \right) \delta^2 + \{ x \to x^{-1} \} \delta^{-2} $$

$$ + \frac{q (1 + q^2) (1 - t^2) (t^2 - x^2) (1 - t^2 x^2)}{t^4 (q^2 - x^2) (1 - q^2 x^2)} $$

$$ \hat{M}^{(-\phantom{+})}_{(2k+1,2)}(x; q, t) = \left( (q x)^{2k} \frac{q x^2 (1 - t^2 x^2)}{(1 - x^2) (1 - q^2 x^2)} \right) \delta^2 + \{ x \to x^{-1} \} \delta^{-2} + \frac{q (1 + q^2) (1 - t^2) x^2}{(q^2 - x^2) (1 - q^2 x^2)} $$

Here $\delta$ is a $q$-difference operator for $x$, and $\{ x \to x^{-1} \}$ in the second term denotes the coefficient $\{ \cdots \}$ of $\delta^2$ in the first term replacing $x$ by $x^{-1}$ so that the operators $\hat{M}^{(\pm)}(x; q, t)$ preserve the symmetric Laurent polynomial space $\mathbb{C}[x + x^{-1}]$. Then we obtain the DAHA polynomial for $c''_{(2k+1,2)}$ by acting (4.30) on 1. This is indeed a deformation of the Jones polynomial for torus knot $T_{(2k+1,2)}$; it reduces to the Jones polynomial for $c_{(2k+1,2)}$ up to framing when we take specific values for deformation parameters,

$$ P_2(x_u = -q, x_\ell = -q^{-1}, x_r = -q^{-1}, q, t = q; c''_{(2k+1,2)}) = 1 - q^{4k} - q^{4k+2} - q^{4k+4}. $$

5. Genus-Two Torus

5.1. Skein Algebra. We shall construct the skein algebra $\text{KBS}_A(\Sigma_{2,0})$ based on the previous sections. We define simple closed curves on $\Sigma_{2,0}$ as in Fig. 6. We regard the two-punctured torus $\Sigma_{1,2}$ in Fig. 5 as a subsurface of $\Sigma_{2,0}$, and we can construct $\Sigma_{2,0} = \Sigma_{1,2} \cup S \times [0, 1]$ where both $S \times \{ 0 \} \approx \mathbb{H}$ and $S \times \{ 1 \} \approx \mathbb{H}$ are isotopic to $x_d$. As a reduction of the $C^\ell C_\ell$-type DAHA (3.1), the simple closed curves $x, y, z$ constitute the following skein algebra (5.1) of $\Sigma_{0,4}$-type. Here $z$ is generated by $x$ and $y$ as in (5.1). On the other hand, we can regard $\Sigma_{2,0}$ as a union of two once-punctured tori, $\Sigma_{2,0} = \Sigma_{u,1} \cup \Sigma^{d}_{1,1}$, where $\Sigma^{u}_{1,1} \cap \Sigma^{d}_{1,1} \approx x$, and the curves $x_u$ and $y_u$ (resp. $x_d$ and $y_d$) satisfy the skein algebra (5.3) on the once-puncture torus $\Sigma^{u}_{1,1}$ (resp. $\Sigma^{d}_{1,1}$). Two sets of curves $\{ x_\emptyset, y_\emptyset, z_\emptyset \}$ for $\emptyset \in \{ u, d \}$ fulfill the skein algebra of $\Sigma_{1,1}$ whose boundary circle is isotopic to $x$. Further we need skein relations for $\bar{y}$ in Fig. 5 and they can be written in (5.7).

In summary, we have the following skein algebra. See also [3].

**Proposition 5.1.** The skein algebra $\text{KBS}_A(\Sigma_{2,0})$ is
\( \Sigma_{0,4}\)-type,
\[
A^2 x y - A^{-2} y x = (A^4 - A^{-4}) \, z + 2 \left(A^2 - A^{-2}\right) \, x_d x_u, \tag{5.1}
\]
\[
A^2 y z - A^{-2} z y = (A^4 - A^{-4}) \, x + \left(A^2 - A^{-2}\right) \left(x_u^2 + x_d^2\right),
\]
\[
A^2 z x - A^{-2} x z = (A^4 - A^{-4}) \, y + 2 \left(A^2 - A^{-2}\right) \, x_d x_u,
\]
with
\[
A^2 x y z = A^4 x^2 + A^{-4} y^2 + A^4 z^2 + A^2 \left(x_u^2 + x_d^2\right) \, x + 2 A^{-2} x_d x_u y + 2 A^2 x_d x_u z + 2 x_u^2 + 2 x_d^2 + x_u^2 x_d^2 - \left(A^2 + A^{-2}\right)^2. \tag{5.2}
\]

\( \Sigma_{1,1}\)-type,
\[
A x_\otimes y_\otimes - A^{-1} y_\otimes x_\otimes = \left(A^2 - A^{-2}\right) \, z_\otimes, \tag{5.3}
\]
\[
A y_\otimes z_\otimes - A^{-1} z_\otimes y_\otimes = \left(A^2 - A^{-2}\right) \, x_\otimes,
\]
\[
A z_\otimes x_\otimes - A^{-1} x_\otimes z_\otimes = \left(A^2 - A^{-2}\right) \, y_\otimes,
\]
with
\[
x = A x_\otimes y_\otimes z_\otimes - A^2 x_\otimes^2 - A^{-2} y_\otimes^2 - A^2 z_\otimes^2 + A^2 + A^{-2}, \tag{5.4}
\]
where \( \otimes \in \{u, d\} \),

consistency,
\[
- y_\otimes^2 y_\otimes + \left(A^2 + A^{-2}\right) y_\otimes y - y_\otimes y_\otimes^2 = \left(A^2 - A^{-2}\right)^2 y_\otimes, \tag{5.5}
\]
\[
- y_\otimes^2 y + \left(A^2 + A^{-2}\right) y_\otimes y_\otimes y - y_\otimes y_\otimes^2 = \left(A^2 - A^{-2}\right)^2 y,
\]
with
\[
x y_\otimes = y_\otimes x, \quad x_\otimes y = y x_\otimes, \tag{5.6}
\]
where \( \otimes \in \{u, d\} \), and
\[
x_u x_d = x_d x_u, \quad x_u x_d = x_d x_u,
\]
\[
y_u x_d = x_d y_u, \quad y_u x_d = x_d y_u,
\]
skein relations for \( \bar{y} \),
\[
\bar{y} y_\otimes = y_\otimes \bar{y}, \tag{5.7}
\]
\[
- \bar{y}_\otimes^2 x_\otimes + \left(A^2 + A^{-2}\right) \bar{y}_\otimes x_\otimes \bar{y} - x_\otimes \bar{y}_\otimes^2 = \left(A^2 - A^{-2}\right)^2 x_\otimes,
\]
\[
-x_\otimes^2 \bar{y} + \left(A^2 + A^{-2}\right) x_\otimes \bar{y} x_\otimes - \bar{y} x_\otimes^2 = \left(A^2 - A^{-2}\right)^2 \bar{y}.
\]
where $\otimes \in \{u, d\}$, and
\[
\begin{align*}
\widetilde{y} x_u y_u - y_u x_u \widetilde{y} &= y y_d x_d - x_d y_d y, \\
\widetilde{y} y &= y \widetilde{y}.
\end{align*}
\] (5.8)

Note that the above $\Sigma_{1,1}$-type relations \([5.3]\) are redundant, and we have
\[
\begin{align*}
-y^2_{\otimes}x_{\otimes} + (A^2 + A^{-2}) y_{\otimes} x_{\otimes} y_{\otimes} - x_{\otimes} y^2_{\otimes} &= (A^2 - A^{-2})^2 x_{\otimes}, \\
-x^2_{\otimes} y_{\otimes} + (A^2 + A^{-2}) x_{\otimes} y_{\otimes} y_{\otimes} - y_{\otimes} x^2_{\otimes} &= (A^2 - A^{-2})^2 y_{\otimes}.
\end{align*}
\] (5.9)

5.2. **Polynomial Representation.** To give a representation, we make use of the DAHA representation for the twice-punctured torus in the previous section. We glue the two punctures of $\Sigma_{1,2}$ in Fig. 5 by setting
\[x_t = x_r = x_d.\] (5.10)

It should be remarked that four parameters $t$ of DAHA of $C^1C_1$-type SH$_q,t$ (see Fig. 2) are now set to be
\[(t_0, t_1, t_2, t_3) = (ix_u, iq^{-\frac{1}{2}}x_d, ix_u, ix_d),\] (5.11)
which means that we have glued the punctures, $\mathfrak{b}_1$ with $\mathfrak{b}_2$, and $\mathfrak{b}_3$ with $\mathfrak{b}_4$, together. In gluing $\mathfrak{b}_1$ with $\mathfrak{b}_2$ in the previous section, we have employed DAHA of $A_1$-type $SH_{q_1, iq_1^{-\frac{1}{2}}x_1^{-\frac{1}{2}}}$ so that $x$ is generated from $x_u$ and $y_d$ as the boundary circle (2.2) of $\Sigma^u_{1,1}$. As we have $\Sigma^u_{1,1} \cap \Sigma^d_{1,1} \approx x$, another DAHA $SH_{q_2, iq_2^{-\frac{1}{2}}x_2^{-\frac{1}{2}}}$ plays the role of $\Sigma^d_{1,1}$ so that $x$ is also generated from $x_d$ and $y_d$ as the boundary circle (2.2) of $\Sigma^d_{1,1}$. Thus it is natural to use the gluing function (4.18) for $y_d$, and to put as follows;

\[
x \mapsto x + x^{-1},
\]
\[
y \mapsto -\beta(x, x_u, x_d) \delta - \beta(x^{-1}, x_u, x_d) \delta^{-1} - \varphi(x, x_u, x_d),
\]
\[
x_u \mapsto x_u + x_u^{-1},
\]
\[
y_u \mapsto G(x, x_u)^{-1} \left( y(x, x_u) \delta_u + y(x, x_u^{-1}) \delta_u^{-1} \right) G(x, x_u)
\]
\[
= \gamma_1(x_u) \delta_u + \gamma_2(x, x_d) \delta_u^{-1},
\]
\[
x_d \mapsto x_d + x_d^{-1},
\]
\[
y_d \mapsto G(x, x_d)^{-1} \left( y(x, x_d) \delta_d + y(x, x_d^{-1}) \delta_d^{-1} \right) G(x, x_d)
\]
\[
= \gamma_1(x_d) \delta_d + \gamma_2(x, x_d) \delta_d^{-1},
\] (5.12)

where $\gamma(x, x_u)$, $\gamma_1(x_u)$, and $\gamma_2(x, x_u)$ are defined in (4.15) and (4.16). The functions $\beta(x, x_u, x_d)$ and $\varphi(x, x_u, x_d)$ come from the Askey–Wilson operator (3.14) with $t_*$ (5.11),

\[
\begin{align*}
\beta(x, x_u, x_d) &= \frac{q^{-\frac{1}{2}} + x}{(1 - q^2 x)(1 - x^2)} \left( x_u + q^{\frac{1}{2}} x x_u^{-1} \right) \left( x_d + q^{\frac{1}{2}} x x_d^{-1} \right), \\
\varphi(x, x_u, x_d) &= \frac{2}{(q^{-\frac{1}{2}} - x^{-1})(1 - q^{\frac{1}{2}} x)} \left( x_u + x_u^{-1} \right) \left( x_d + x_d^{-1} \right).
\end{align*}
\]

We should note that $y$ is represented by the Askey–Wilson difference operator (3.14), while $y_u$ and $y_d$ are the (conjugated) $A_1$-type Macdonald difference operators (2.13). The $q$-difference
operators $\delta_u$ and $\delta_d$ are for $x_u$ and $x_d$ respectively

$$\delta_u \delta_d = \delta_d \delta_u,$$

$$\delta_u f(x, x_u, x_d) = f(x, q_u x_u, x_d),$$

$$\delta_d f(x, x_u, x_d) = f(x, x_u, q_d x_d),$$

where we mean $q_u = q_d = q^{1/2} (4.11)$. Note that the representation for $y$ is symmetric $x_u \leftrightarrow x_d$, and that the associated Askey–Wilson polynomial for $t_*$ (5.11) is written in a symmetric form by use of the Sears’ transformation formula [22] to (5.19) as

$$P_m(x; q, x_u, x_d) = (-1)^m q^{-\frac{m-1}{2}} \left(\frac{q}{x_u}, \frac{q}{x_d}; q \right)_m \phi_3 \left[ q^{-m}, q^{m+1} x, -q^{\frac{1}{2}} x, -q^{\frac{1}{2}} x^{-1}; q, q \right].$$  \hspace{1cm} (5.13)

By construction, both skein relations of the $\Sigma_{0,4}$-type (5.1) and the $\Sigma_{1,1}$-type (5.3) are fulfilled by the above representation (5.12). Moreover the above Askey–Wilson operator for $y$ is symmetric in $x_u \leftrightarrow x_d$, and the consistency conditions (5.5) are satisfied thanks to the results for the skein algebra on $\Sigma_{1,2}$ in the previous section. Hence a non-trivial is for the simple closed curve $\bar{y}$ in Fig. 5. To give a representation, we suppose that

$$\bar{y} \mapsto \sum_{\varepsilon \in \{-, +\}} \sum_{\varepsilon_u, \varepsilon_d = \pm 1} a_{\varepsilon, \varepsilon_u, \varepsilon_d}(x, x_u, x_d) \delta_u^\varepsilon \delta_d^\varepsilon,$$  \hspace{1cm} (5.14)

and consider a condition for the first identity of (5.8). Equating each coefficient of $\delta \delta_u^2 \delta_d^{-1}$, $\delta \delta_u^{-2} \delta_d^{-1}$, $\delta \delta_d^{-1}$, we have the following functional equations for $a_{++-}(x, x_u, x_d)$:

$$a_{++-}(x, q_u x_u, x_d) = \gamma_1(q_u x_u),$$

$$a_{++-}(x, x_u, x_d) = \gamma_1(x_u),$$

$$a_{+-+}(x, q_u^{-1} x_u, x_d) = \gamma_2(q x, q_u^{-1} x_u),$$

$$a_{+-+}(x, x_u, x_d) = \gamma_2(x_u),$$

$$(q_u^{-1} x_u + q_u x_u^{-1}) \left\{ \gamma_1(q_u^{-1} x_u) a_{+-+}(x, x_u, x_d) - \gamma_2(x_u) a_{++-}(x, q_u^{-1} x_u, x_d) \right\}$$

$$- (q_u x_u + q_u^{-1} x_u^{-1}) \left\{ \gamma_1(x_u) a_{+-+}(x, q_u x_u, x_d) - \gamma_2(q x, q_u x_u) a_{++-}(x, x_u, x_d) \right\}$$

$$= 1 - q \left( 1 + q^{1/2} x \right) \left( q^{1/2} x + x_d^2 \right) \left( q^{1/2} x + x_u^2 \right).$$

The first two are solved to be

$$a_{+-+}(x, x_u, x_d) = \tilde{a}_{+-+}(x, x_d) \frac{q^{1/2} x + x_u^2}{1 - x_u^2},$$

$$a_{++-}(x, x_u, x_d) = \tilde{a}_{++-}(x, x_d) \frac{1}{1 - x_u^2}.$$

Due to that functions $\tilde{a}_{++-}$ and $\tilde{a}_{+-+}$ do not depend on $x_u$, we can solve them from the above third equations,

$$\tilde{a}_{+-+}(x, x_d) = \frac{(1 + q^{1/2} x) \left( q^{1/2} x + x_u^2 \right) \left( q^{1/2} x + x_d^2 \right)}{q^{1/2} x \left( 1 - x_u^2 \right) \left( 1 - q^{1/2} x \right) \left( 1 - x_d^2 \right)},$$

$$\tilde{a}_{++-}(x, x_d) = -q x \tilde{a}_{+-+}(x, x_d).$$
In this manner, we get $a_{x,u,x_d}(x, x_u, x_d)$.

For $a_{0,±,±}(x, x_u, x_d)$, we see that a sum for $\varepsilon \neq 0$ in (5.14) commute with both $y_u$ and $y_d$. Consequently we can suppose that a sum for $\varepsilon = 0$ in (5.14) has a form of

$$\psi(x) \left( y_1(x_d) \delta_d + y_2(x, x_d) \delta_d^{-1} \right) \left( y_1(x_u) \delta_u + y_2(x, x_u) \delta_u^{-1} \right).$$

A commutativity between $\tilde{y}$ and $\gamma$ solves $\psi(x)$, and as a result we obtain

$$\tilde{y} \mapsto (\kappa(x_d) \delta_d + \lambda(x, x_d) \delta_d^{-1}) \left( \kappa(x_u) \delta_u + \lambda(x, x_u) \delta_u^{-1} \right) \omega(x) \delta$$

$$+ (\kappa(x_d) \delta_d + \lambda(x^{-1}, x_d) \delta_d^{-1}) \left( \kappa(x_u) \delta_u + \lambda(x^{-1}, x_u) \delta_u^{-1} \right) \omega(x^{-1}) \delta^{-1}$$

$$+ \psi(x) \left( y_1(x_d) \delta_d + y_2(x, x_d) \delta_d^{-1} \right) \left( y_1(x_u) \delta_u + y_2(x, x_u) \delta_u^{-1} \right), \quad (5.15)$$

where

$$\omega(x) = \frac{x \left( 1 + q^2 x \right)}{q^2 x \left( 1 - x^2 \right) \left( 1 - q^2 x \right)}, \quad \psi(x) = \frac{2 x}{\left( 1 - q^{-2} x \right) \left( 1 - q^2 x \right)}, \quad (5.16)$$

$$\lambda(x, x_u) = \frac{q^2 x + x_u^2}{q x \left( 1 - x_u^2 \right)}, \quad \kappa(x_u) = \frac{-1}{1 - x_u^2}.$$  

With the setting (5.15), we can check the relations (5.7) by tedious computations.

As a result, we have obtained the following theorem.

**Theorem 5.2.** We have a representation of $KBS_A(\Sigma_{2,0})$ by (5.12), (5.15) with $A^2 = q^{-\frac{1}{2}}$.

This representation denotes the difference operators on $C[x + x^{-1}]$. As representation on $C[x_u + x_u^{-1}, x_d + x_d^{-1}]$, we take a conjugation $h \mapsto G h G^{-1}$ with

$$G = G(x, x_u) G(x, x_d). \quad (5.17)$$

Using (4.20), we have the following representation.

**Corollary 5.3.** We have a representation of $KBS_A(\Sigma_{2,0})$ with $A^2 = q_u^{-1}$ by

$$x_\otimes \mapsto X_\otimes + X_\otimes^{-1},$$

$$y_\otimes \mapsto Y_\otimes + Y_\otimes^{-1}, \quad \text{for } \otimes \in \{u, d\}, \quad (5.18)$$

$$y \mapsto -\frac{q_u t^4 \left( 1 - t^2 \right)}{(1 + t^2) \left( q_u^2 - t^4 \right)} \left\{ \prod_{\otimes \in \{u, d\}} \sh(t X_\otimes) \sh(t^{-1} X_\otimes) \right\} \delta_t$$

$$- \frac{q_u^3 \left( q_u^2 - t^2 \right)}{(q_u^2 + t^2) \left( q_u^2 - t^4 \right)} \delta_t^{-1} - \frac{2 q_u t^2}{(t^2 + q_u^2) (t^2 + 1)} \left\{ \prod_{\otimes \in \{u, d\}} \ch(X_\otimes) \right\}. \quad (5.19)$$
\[
\tilde{y} \mapsto \frac{q_u t^4 (1 - t^2)}{(1 + t^2) (q_u^2 - t^4)} \left\{ \prod_{\otimes \in \{u,d\}} \left( t^{-1} \text{sh} (t^{-1} Y_\otimes) \text{sh} (t^{-1} X_\otimes) \right) \right\} \delta_t
\]
\[
+ \frac{q_u^3 (q_u^2 - t^2)}{(q_u^2 + t^2) (q_u^2 - t^4)} \left\{ \prod_{\otimes \in \{u,d\}} \left( \frac{t}{\text{sh} (t^{-1} X_\otimes)} \text{sh} (t Y_\otimes) \right) \right\} \delta_t^{-1}
\]
\[- \frac{2 q_u t^2}{(q_u^2 + t^2) (1 + t^2)} \left\{ \prod_{\otimes \in \{u,d\}} \left( \text{ch} (Y_\otimes) \right) \right\} .
\]

(5.20)

Here \(\{X_\otimes, Y_\otimes, T_\otimes\}\) are generators of \(\mathcal{H}_{q_u,t}\), and these representations preserve symmetric space \(\mathbb{C}[x_u + x_u^{-1}, x_d + x_d^{-1}]\). Recalling (2.28), these are explicitly written as operators on the symmetric polynomial space,

\[
x_\otimes \mapsto x_\otimes + x_\otimes^{-1},
\]
\[
y_\otimes \mapsto \frac{t x_\otimes - t^{-1} x_\otimes^{-1}}{x_\otimes - x_\otimes^{-1}} \delta_\otimes + \frac{t^{-1} x_\otimes - t x_\otimes^{-1}}{x_\otimes - x_\otimes^{-1}} \delta_\otimes^{-1}, \quad \text{for } \otimes \in \{u,d\},
\]
\[
x \mapsto -q_u t^{-2} - q_u^{-1} t^2,
\]
\[
y \mapsto -\frac{q_u t^4 (1 - t^2)}{x_\otimes - x_\otimes^{-1}} \delta_\otimes + \frac{t^{-1} x_\otimes - t x_\otimes^{-1}}{x_\otimes - x_\otimes^{-1}} \delta_\otimes^{-1}, \quad \text{for } \otimes \in \{u,d\},
\]
\[
(5.21)
\]

We should note that the representation (5.20) of \(\tilde{y}\) can be recovered by use of the automorphisms of \(\mathcal{H}_{q_u,t}\). As shown in Fig. 7, we have \(\tilde{y} = \mathcal{T}_{x_u} \mathcal{T}_{x_d}^{-1} \mathcal{T}_{y_u} \mathcal{T}_{y_d}^{-1} (y)\), and the DAHA operator for \(y = c_{(0,1)}\) is \(c_{(0,1)} \mapsto \mathcal{A}_{(0,1)} = \text{ch}(Y)\) in (5.19). Using (2.45), an action \(\tau_{\mathcal{T}_{(u)} \mathcal{T}_{(d)} \mathcal{T}_{(u)} \mathcal{T}_{(d)}}^{-1}\) is

\[
\begin{pmatrix} x_u \\ x_d \\ \delta_t \end{pmatrix} \mapsto \begin{pmatrix} q^{-1} x_u^{-1} Y_u X_u \\ \frac{1}{\text{sh} (r^{-1} X_u) \text{sh} (r^{-1} Y_u)} \frac{1}{\text{sh} (r^{-1} X_d) \text{sh} (r^{-1} Y_d)} \delta_t \end{pmatrix} .
\]

We then recover (5.20) from (5.19) as operators on \(\text{SH}_{q_u,t} \times \text{SH}_{q_u,t}\).

As depicted in Fig. 7, the simple closed curve \(\tilde{y}\) is also given from the curve \(c_{(1,1)}\) by \(\tilde{y} = \mathcal{T}_{x_u} \mathcal{T}_{x_d} \mathcal{T}_{y_u} \mathcal{T}_{y_d} (c_{(1,1)})\). The associated operator \(c_{(1,1)} \mapsto \mathcal{A}_{(1,1)} = \text{ch}(T_1 T_0)\) in \(\text{SH}_{q_\ast,t}\).
is explicitly written as the operator on $\mathbb{C}[x + x^{-1}]$,
\[
\mathcal{A}_{(1,1)} \bigg|_{\text{sym}} = -\frac{x \left(1 + q^\frac{1}{2} x\right) \left(q^\frac{1}{2} x u^{-1} + x u\right) \left(q^\frac{1}{2} x d^{-1} + x d\right)}{(1 - x^2) \left(1 - q^\frac{1}{2} x\right)} \delta^{-1} + \frac{2q^\frac{1}{2} x \left(x u + u^{-1}\right) \left(x d + d^{-1}\right)}{(q^\frac{1}{2} - x) \left(1 - q^2 x\right)},
\]
which follows from (3.29) with $t_*$ (5.11). We take a conjugation $\mathcal{A}_{(1,1)} \mapsto G \mathcal{A}_{(1,1)} G^{-1}$ with the gluing function (5.17), to have an operator on $\mathbb{C}[x u + x^{-1} u, x d + x^{-1} d]$. Making a change of variables (4.20), we get
\[
G \mathcal{A}_{(1,1)} \bigg|_{\text{sym}} G^{-1} = \frac{q u t^5 - t^2}{(q u^2 - t^4) (q u^2 + t^2)} \prod_{\ominus \in \{u,d\}} \sh (t X_\ominus) \sh (t^{-1} X_\ominus) \delta_t + \frac{2 q u t^2}{(q u^2 + t^2) (1 + t^2)} \prod_{\ominus \in \{u,d\}} \ch (X_\ominus). \tag{5.24}
\]
Applying the automorphism $\tau^{-1}_{R(u)} \tau^{-1}_{R(d)} \tau_L(u) \tau_L(d)$, we obtain (5.20) as well.

As a simple closed curve $\tilde{y}$ does not intersect with both $y_u$ and $y_d$, the Dehn twists $\mathcal{F}_{y_u}$ and $\mathcal{F}_{y_d}$ have trivial actions on $\tilde{y}$, $\tilde{y} = \mathcal{F}_{y_u}(\tilde{y}) = \mathcal{F}_{y_d}(\tilde{y})$. Our representation for $\tilde{y}$ is indeed invariant under $\tau_L(\ominus)$ due to the following proposition.

**Proposition 5.4.** The operators of $\text{SH}_{q,t}$,
\[
t^{-1} \sh(t^{-1} Y) \sh(t^{-1} X) \delta_t,
\]
and
\[
\frac{t}{\sh(t^{-1} X)} \sh(t Y) \delta_t^{-1},
\]
are invariant under $\tau_L$. 

![Figure 7. A simple closed curve $\tilde{y}$ is given from curves with slope-1/0 and 1/1 by Dehn twists.](image)
Proof. It is straightforward to see the invariance of the first operator using (2.8) and (2.32).

For the second, we recall that it denotes the lowering operator $K^{(-)}$ which does not depend on $t$, and that it commutes with $\delta_t$. We then have

$$
\delta_t^{-1} \frac{t}{\text{sh}(t^{-1}X)} \text{sh}(tY) \leftrightarrow \delta_t^{-1} \frac{1}{\text{sh}(t^{-1}X)} \text{sh}(t^{-1}q^{\frac{i}{2}}YX) \frac{t}{\text{sh}(t^{-1}q^{\frac{i}{2}}YX)} \text{sh}(tY).
$$

which proves the invariance under $\tau_L$. \hfill \qed

5.3. Double-Torus Knots. A simple closed curve on a genus two Heegaard surface in $S^3$ is called a double-torus knot. A construction of a non-trivial knot was studied in [28,29], but a classification of the double-torus knots is far from complete.

We shall propose a DAHA polynomial for the double-torus knot. We assign a difference operator for the simple closed curve $c$ as follows. We suppose that $c$ is given from $c_{(r,s)}$ by Dehn twists $\mathcal{T}$ which is generated by $\mathcal{T}_{x_u}, \mathcal{T}_{y_u}, \mathcal{T}_{x_d}$, and $\mathcal{T}_{y_d}$,

$$
c = \mathcal{T}(c_{(r,s)}),
$$

and that a curve $c_{(r,s)}$ does not intersect with $x_u, x_d$. Here we mean that $c_{(r,s)}$ is a simple closed curve on $\Sigma_{0,4} \subset \Sigma_{2,0}$ with slope $s/r$, when we regard $\Sigma_{2,0} = \Sigma_{0,4} \cup S^1 \times [0,1] \cup S^1 \times [0,1]$ and the boundary circles in Fig.2 are $b_1 \approx b_2 \approx x_u$ and $b_3 \approx b_4 \approx x_d$. Using the $C^*C_1$-type DAHA $\text{SH}_{q,t_*}$ with $t_*(5.11)$, we have the $q$-difference operator

$$
c_{(r,s)} \mapsto \mathcal{A}_{(r,s)} = \text{ch}(O_{(r,s)}) \in \text{SH}_{q,t_*}.
$$

We then take a conjugation $G \mathcal{A}_{(r,s)} G^{-1}$ by the gluing function (5.17) and make a change of variables (4.20). Applying the automorphism $\gamma$ associated to the Dehn twist $\mathcal{T}$, we have the difference operator

$$
c \mapsto \gamma(G \mathcal{A}_{(r,s)} G^{-1}).
$$

We then define the DAHA polynomial of $c$ by

$$
P_n(t, q_u, x_u, x_d; c) = \gamma(\mathcal{A}_{n-1}(\mathcal{O}_{(r,s)}; q, q) G^{-1}) (1)
$$

$$
= \gamma(\mathcal{A}_{n-1}(\mathcal{A}_{(r,s)} G^{-1}) (1).
$$

It is noted that we can further take a conjugation by $\Phi$ to be determined,

$$
c \mapsto \Phi(q_u, t) \gamma(G \mathcal{A}_{(r,s)} G^{-1}) \Phi(q_u, t)^{-1},
$$

and can modify the definition of the DAHA polynomial as

$$
P_n(t, q_u, x_u, x_d; \Phi; c) = \gamma(\Phi(q_u, t) G \mathcal{A}_{n-1}(\mathcal{A}_{(r,s)} G^{-1} \Phi(q_u, t)^{-1} (1).
$$

We expect that, by suitably choosing the function $\Phi$, there may exist a relationship between our DAHA polynomial and a Poincaré polynomial of knot homology (see, e.g., [21]), but we do not know at this stage.

To see a relationship with the Jones polynomial, we pay attention to the constant term $\delta_t^0$ in the operators $S_{n-1}(\mathcal{A}_{(r,s)})$. This extraction is realized in (5.26) by putting $t = q_u$ with a condition

$$
\Phi(q_u, q_u) = 0.
$$
For simplicity, we write the reduced DAHA polynomial as the constant term $\delta^0$ of $S_{n-1}(\mathcal{A}_{(r,s)})$ as

$$
\overline{P}_n(q_u, x_u, x_d; c) = \gamma \left( \Phi(q_u, t) G M_{n-1}(\Theta_{(r,s)}; q, q) G^{-1} \Phi(q_u, t) \bigg|_{t=q_u} \right) (1)
$$

(5.28)

where $\text{Const}(\bullet)$ is the $\delta^0$ term of $\bullet$. We do not need to take a conjugation by the gluing function $G$ to pick up the constant term.

**Conjecture 5.5.** The DAHA invariant $\overline{P}_n(q_u, x_u, x_d; c)$ for a simple closed curve $c$ on $\Sigma_{2,0}$ coincides with the $n$-colored Jones polynomial for the double-torus knot $c$ up to framing when $x_u = x_d = q_u$.

![Figure 8](image.png)

**Figure 8.** Shown is a curve $c_{(1,2),(1,-1)}$ as an example of a Dehn twist on a simple closed curve $c_{(1,2)}$ with slope 2. In this case, we get the figure-eight knot $4_1$ in $S^3$.

We give explicit examples. When $\mathcal{T}$ is generated by $\mathcal{T}_{x_u}, \mathcal{T}_{y_u}, \mathcal{T}_{x_d}, \mathcal{T}_{y_d}$, the Dehn twist $\mathcal{T}(c_{(1,1)})$ is a connected sum of torus knots. To have a hyperbolic knot, we consider a simple closed curve $c = \mathcal{T}(c_{(1,2)})$ given from the slope-2/1 curve. Let $c_{(1,2),(k,\ell)}$ be a simple curve on $\Sigma_{2,0}$ given by $\mathcal{T}_{y_u}^k \mathcal{T}_{y_d}^\ell (c_{(1,2)})$. See Fig. 8 for a case of $(k, \ell) = (1, -1)$. For small $k$ and $\ell$, the simple closed curve $c_{(1,2),(k,\ell)}$ is identified with a knot of Rolfsen’s notation as in Table 1. The curve $c_{(1,2),(1,p)}$ is the twist knot $K_p$ in $S^3$.

| knot | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|------|---|---|---|---|---|---|---|----|----|----|----|
| $(k, \ell)$ | $(1, 1)$ | $(1, -1)$ | $(1, 2)$ | $(-1, 2)$ | $(1, 3)$ | $(-2, -2)$ | $(-1, 3)$ | $(-2, 2)$ | $(1, 4)$ | $(2, 3)$ | $(-1, 4)$ | $(-2, 3)$ |

**Table 1.** Examples of double-torus knots $c_{(1,2),(k,\ell)}$. The so-called twist knot corresponds to the case of $k = 1$.

The operator associated to $c_{(1,2)}$ on $\Sigma_{0,4}$, $c_{(1,2)} \mapsto \mathcal{A}_{(1,2)} \in \text{SH}_{q,t}$, is given in (3.31), which preserves $\mathbb{C}[x + x^{-1}]$. At $t_*$ (5.11) it is written as

$$
\mathcal{A}_{(1,2)} \big|_{\text{sym}} = A_{(1,2)}^{[0]}(q, x, x_u, x_d) + \sum_{j=1}^2 \sum_{\ell=1} A_{(1,2)}^{[j]}(q, x, x_u, x_d) \delta^{\ell j},
$$

(5.29)
where
\[
A_{(1,2)}^2(q,x,x_u,x_d) = \frac{x \left( 1 + q^2 x \right) \left( 1 + q^3 x \right)}{(1 - x^2) (1 - q^2 x^2) \left( 1 - q^3 x \right) \left( 1 - q^2 x \right)} \prod_{\sigma \in \{u,d\}} \left( x_{\sigma} + q^{1/2} x x_{\sigma}^{-1} \right) \left( x_{\sigma} + q^{3/2} x x_{\sigma}^{-1} \right),
\]
\[
A_{(1,2)}^1(q,x,x_u,x_d) = -\frac{2q^2 x \left( 1 + q^4 x \right)^2}{(1 - x^2) \left( q^2 - x \right) \left( 1 - q^2 x \right) \left( 1 - q^2 x \right)} \prod_{\sigma \in \{u,d\}} \left( x_{\sigma} + x^{-1} \right) \left( x_{\sigma} + q^{1/2} x x_{\sigma}^{-1} \right),
\]
\[
A_{(1,2)}^0(q,x,x_u,x_d) = \frac{q^2 x \left( q \left( 1 + x^2 \right)^2 + 4 \left( 1 + q^2 \right) x^2 + 3 q^{1/2} \left( 1 + q \right) \left( 1 + x^2 \right) x \right)}{(q^2 - x^2) (1 - q^2 x^2) \left( 1 - q^2 x^2 \right)} \left( x_u + x_u^{-1} \right)^2 \left( x_d + x_d^{-1} \right)^2 + q \left( 1 - q \right) x \left( 1 + x^2 \right) \left( q^2 - x^2 \right) (1 - q^2 x^2) \right) \right\}
\]

We take a conjugation by the gluing function \( G \) (5.17), and then replace \( x \) with \( t \) (4.20) to obtain an operator on \( SH_{q_u,t} \times SH_{q_u,t} \) as
\[
G \cdot A_{(1,2)} = A_{(1,2)}^2(q_u,t) \left[ \sh(t^{-1}X_u) \sh(t X_u) \sh(t^{-1}X_d) \sh(t X_d) \delta_t \right]^2 + A_{(1,2)}^1(q_u,t) \ch(X_u) \ch(X_d) \left[ \sh(t^{-1}X_u) \sh(t X_u) \sh(t^{-1}X_d) \sh(t X_d) \delta_t \right] + A_{(1,2)}^0(q_u,t,X_u,X_d) + A_{(1,2)}^1(q_u,t_u^{-1}) \ch(X_u) \ch(X_d) \delta_t^{-1} + A_{(1,2)}^2(q_u,q_u t^{-1}) \delta_t^{-2}. \tag{5.30}
\]

Here we have
\[
A_{(1,2)}^2(q_u,t) = \frac{-q_u^5 t^{10} \left( 1 - t^2 \right) \left( 1 - q_u^2 t^2 \right)}{(q_u^6 - t^4) \left( 1 + t^2 \right) \left( 1 - q_u^2 t^4 \right) \left( 1 + q_u^2 t^2 \right)},
\]
\[
A_{(1,2)}^1(q_u,t) = \frac{2 q_u^3 t^6 \left( 1 - t^2 \right)^2}{1 + t^2 \left( q_u^6 - t^4 \right) \left( q_u^2 + t^2 \right) \left( 1 + q_u^2 t^2 \right)},
\]
and
\[
A_{(1,2)}^0(q_u,t,X_u,X_d) = \frac{-q_u^3 t^2 \left( t^4 \left( 4 - 3 t^2 + t^4 \right) + q_u^4 \left( 1 - 3 t^2 + 4 t^4 \right) + q_u^2 t^2 \left( -3 + 2 t^2 - 3 t^4 \right) \right)}{(q_u^6 - t^4) \left( 1 - q_u^2 t^4 \right) \left( 1 + t^2 \right) \left( q_u^2 + t^2 \right)} \ch^2(X_u) \ch^2(X_d) + \frac{q_u t^2 \left\{ t^2 (1 + q_u^6) + q_u^4 (t^2 - 2) + q_u^2 t^2 (1 - 2 t^2) \right\}}{(q_u^6 - t^4) \left( 1 - q_u^2 t^4 \right)} \right\}
\]
\[
\ch^2(X_u) + \ch^2(X_d) + \frac{-q_u t^2 \left( 1 - q_u^2 \right)^2 q_u^2 + t^4}{(q_u^6 - t^4) \left( 1 - q_u^2 t^4 \right)} \right). \tag{5.31}
\]

The reduced DAHA polynomial \( \overline{P}_k \) for the simple closed curve \( c_{(1,2)(k,l)} \) is given by applying automorphism \( \tau_{R(u)}^k \tau_{R(d)}^l \) to (5.30). At \( t = q_u \), the constant term \( \delta^0 \) of \( A_{(1,2)} \) reduces to a
symmetric bilinear form,

\[
a^{[0]}_{(1,2)}(q_u, q_u, X_u, X_d) = -\frac{q_u (1-q_u^2)}{1-q_u^d} \left( \frac{(1-q_u^2)^2}{q_u^2 (1-q_u^d)^2} - \frac{1-q_u^2}{q_u^2 (1-q_u^d)} \right) \prod_{\varnothing \in \{u,d\}} (1-q_u^2 X_{\varnothing}^2) \left( 1-q_u^2 X_{\varnothing}^{-2} \right)
\]

where we have used the Chebyshev polynomial of the second kind \(T_2\). As we have for symmetric bilinear form,\(\) \(t\) \(\cdots\) \(\text{sym}(\ldots)\) \(\text{sym}(\ldots)\)

\[
\text{We have checked that } \mathbf{P}_2(q_u, X_u, X_d; \mathbf{c}_{(1,2),(k,\ell)}) = a^{[0]}_{(1,2)}(q_u, q_u, \frac{\mathbf{c}}{2} Y_u X_u, q_u \frac{\mathbf{c}}{2} Y_u X_u)(1)
\]

\[
= -\frac{q_u (1-q_u^2)}{1-q_u^d} \left( 1, q_u^k \left( 1, S_2(\mathbf{ch} X_u) \right) \left( 1, q_u^k \left( \frac{1}{1-q_u^2} \right) \left( 1, \left( \frac{1}{1-q_u^2} \right) \right) \right) \right).
\]

We have checked that \(\mathbf{P}_2(q_u, q_u, q_u; \mathbf{c}_{(1,2),(k,\ell)})\) for \((k, \ell)\) in Table 1 agrees with the Jones polynomial up to framing (see, e.g., \(\text{[11]}\)).

We compute the colored reduced DAHA polynomial \(\overline{P}_{n>2}\) \(\text{[5.28]}\). For \(n = 3\), we need the difference operator \(S_2(\mathbf{A}_{(1,2)}) = \mathbf{A}_{(1,2)}^2 - 1\). Using \(\text{[5.30]}\), we find that the operator which survives at \(t = q_u\) is a term \(\delta^0\) given by

\[
a^{[2]}_{(1,2)}(q_u, q_u) a^{[2]}_{(1,2)}(q_u, q_u^2) \prod_{\varnothing \in \{u,d\}} \text{sh}(q_u^{-1} X_{\varnothing}) \text{sh}(q_u^{-1} X_{\varnothing}^{-1}) \text{sh}(q_u X_{\varnothing}) \text{sh}(q_u X_{\varnothing}^{-1})
\]

\[
+ a^{[1]}_{(1,2)}(q_u, q_u) a^{[1]}_{(1,2)}(q_u, q_u^{-1}) \prod_{\varnothing \in \{u,d\}} \text{ch}(X_{\varnothing}) \text{sh}(q_u^{-1} X_{\varnothing}) \text{sh}(q_u X_{\varnothing}) + \left( a^{[0]}_{(1,2)}(q_u, q_u, X_u, X_d) \right)^2 - 1
\]

\[
= \frac{q (1-q)}{1-q^3} \left( \frac{(1-q^3)^2}{q^2 (1-q^2)} - \frac{1-q^d}{q^2 (1-q^d)} \prod_{\varnothing \in \{u,d\}} (1-q X_{\varnothing}^2) (1-q X_{\varnothing}^{-2}) \right)
\]

\[
+ \frac{(1-q) (1-q^2)}{q^3 (1-q^d)} \prod_{\varnothing \in \{u,d\}} (1-q X_{\varnothing}^2) (1-q X_{\varnothing}^{-2}) (1-q^2 X_{\varnothing}^2) (1-q^2 X_{\varnothing}^{-2}) \right),
\]

where \(q = q_u^2\). The 3-colored polynomial for \(\mathbf{c}_{(1,2),(k,\ell)}\) is given by replacing \(X_u\) (resp. \(X_d\)) by \(\frac{\mathbf{k}}{2} Y_u X_u\) (resp. \(\frac{\mathbf{k}}{2} Y_d X_d\)). The DAHA \(\mathbf{SH}_{q,t}\) proves

\[
M_j(q^{-\mathbf{k}/2} Y^k X; q, t)(1) = \left( q^{\mathbf{k}/2} t^{\mathbf{k}} \right)^k M_j(x; q, t),
\]

\(\text{[5.34]}\)
which gives

$$\bar{P}_s(q_u, x_u, x_d; c_{(1,2)(k,\ell)}) = \frac{q_u^N (1 - q_u^N)}{1 - q_u^N}$$

\[
\left(1, q_u^{4k} S_2(ch x_u), q_u^{12k} S_4(ch x_u)\right) = \left(1, \frac{1}{(1-q_u^N)(1-q_u)} - \frac{1}{1-q_u^N} \right) \left(1, \frac{1}{1-q_u^N} \right) \left(1, q_u^{12k} S_4(ch x_d)\right). \tag{5.35}
\]

We have checked that the results for knots in Table 1 coincide with the known $N = 3$ colored Jones polynomial in [11].

The DAHA polynomial for $n = 4$ is given similarly from the difference operator $S_3(\mathcal{A}(1,2)) = \mathcal{A}(3) - 2 \mathcal{A}(1,2)$. Based on these explicit computations, we have the following observation for the $N$-colored reduced DAHA polynomial $\bar{P}_N$. We define

$$s_N(x) = \left(S_2(x + x^{-1})\right)_{0 \leq j \leq N-1}, \tag{5.36}$$

$$v_N(q, x) = \left(q x^2, q x^{-2}; q\right)_{0 \leq j \leq N-1} \tag{5.37}.$$  

These are bases of the symmetric Laurent polynomial space $\mathbb{C}[x + x^{-1}]$ of even power, and we have

$$v_N(q, x) = s_N(x) B_N(q), \tag{5.38}$$

where the $N \times N$ triangular matrix $B_N(q)$ is defined by

$$B_N(q))_{j,k} = \begin{cases} (-1)^j q^{\frac{1}{2}j(j+1)} \frac{(q; q)_{2k+1}}{(q; q)_{k-j} (q; q)_{k+j+1}}, & \text{for } 0 \leq j \leq k \leq N-1, \\ 0, & \text{otherwise.} \end{cases}$$

One sees that these bases were used in [24, 37] for a cyclotomic expansion of the colored Jones polynomial.

**Conjecture 5.6.** The constant term $\delta^0$ of $S_{N-1}(\mathcal{A}(1,2))$ at $t = q_u = q^\frac{1}{2}$ is given by

$$\text{Const} \left(S_{N-1}(\mathcal{A}(1,2))_{1 = q_u}\right) = (-1)^{N-1} q^{\frac{1}{2}(N-1)} \frac{1}{1 - qN} v_N(q, x_u) T_N(q; c_{(1,2)}) v_N(q, x_d)^T, \tag{5.39}$$

where the diagonal matrix $T_N$ is

$$T_N(q; c_{(1,2)}) = \text{diag} \left((-1)^{k-1} q^{\frac{1}{2}k(k+1)-N} \left(q^{2k}; q\right)_{N+1-2k} \left(q^{2k}; q\right)_{N-k} \right)_{1 \leq k \leq N}. \tag{5.40}$$

For computations of the reduced DAHA polynomial of $\mathcal{Y}^k_{y_u, y_d}(c_{(1,2)})$, we need $v_N(q, t^2_u(X_\circ))(1)$ in (5.39). By changing bases using (5.38) to the Chebyshev polynomial, we have $s_N(t^2_u(X_\circ))(1) = s_N(q_u^{-\frac{1}{2}} Y_\circ X_\circ)(1)$ at $t = q_u$, which can be computed by $\text{SH}_{q_u t}$ as (5.34). To conclude, Conjecture 5.5 of the reduced DAHA polynomial for $c_{(1,2)(k,\ell)}$ is read under Conjecture 5.6 as follows.

**Conjecture 5.7.** The $N$-colored Jones polynomial for $c_{(1,2)(k,\ell)} = \mathcal{Y}^k_{y_u, y_d}(c_{(1,2)})$ coincides up to framing with the reduced DAHA polynomial $\bar{P}_N(q_u, x_u, x_d; c)$ at $x_u = x_d = q_u = q^\frac{1}{2}$, where we
have
\[ P_N(q,u,x_u,x_d;c_{(1,2)}(k,\ell)) = (-1)^{N-1} q^{\frac{j}{2}(N-1)} \frac{1-q}{1-q^N} \]
\[ \times s_N(x_u) \text{ diag} \left( q^{(j+1)k} \right)_{0 \leq j \leq N-1} B_N(q) \left( q^\ell \right) \text{ diag} \left( q^{(j+1)\ell} \right)_{0 \leq j \leq N-1} s_N(x_d)^\top. \]
\[ (5.41) \]

One can check the conjecture for small \( N \) and the knots in Table 1. Furthermore, as an explicit form of the \( N \)-colored Jones polynomial for twist knot \( K_p \) is given in [37], we have checked the equality for small \( p \) and \( N \),
\[ P_N(q,u,x_u,x_d;c_{(1,2)}(1_p)) = (-1)^{N-1} q^{\frac{N}{2}} \frac{q^{-\frac{N}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}} J_N(q;K_p). \]
\[ (5.42) \]
Here the colored Jones polynomial is read as
\[ J_N(q;K_p) = \sum_{n=0}^{\infty} q^n (q^{1-N}, q^{1+N}, q_n) \sum_{j=0}^{n} (-1)^j q^{(j+1)p+\frac{j}{2}(j-1)} \left( 1-q^{2j+1} \right) \frac{(q; q)_n}{(q; q)_{n+j+1}(q; q)_{n-j}}, \]
which is normalized such that \( J_N(q; \text{unknot}) = 1 \). See also [23] where a similar expression with (5.41) was given for the Poincaré polynomial for knot homology \( c_{(1,2)}(k,\ell) \).

6. Concluding Remarks

We have clarified topological aspects of rank-1 DAHAs, \( A_1 \)-type and \( C^\vee_1 \)-type. Motivated by the DAHA–Jones polynomial for torus knots by Cherednik [14], we have proposed to combine these two rank-1 DAHAs as a representation of the skein algebra on the genus-two surface \( \Sigma_{2,0} \). We have constructed the reduced DAHA polynomial \( P_n(q,u,x_u,x_d;c) \) for a simple closed curve \( c \) on \( \Sigma_{2,0} \), and have clarified the relationship with the colored Jones polynomial for double-torus knot. In this paper, we have mainly studied double twist knots \( \mathcal{F}_y^k \mathcal{F}_y^\ell(c_{(1,2)}) \), and in Appendix A we give some results on other simple closed curves given from \( c_{(1,2)} \).

In Appendix B we give some results concerning simple closed curves derived from \( c_{(1,3)} \), and discuss a relationship with the colored Jones polynomial. These results indicate an importance of the Askey–Wilson operator at \( t_r(5.11) \) associated to a simple closed curve \( c_{(r,s)} \) in studies of quantum invariants of knot. The symmetric bilinear form (5.41) for those knot families seems to be promising for other quantum polynomial invariants of knots. Also studies of the (non-reduced) DAHA polynomial \( P_n(t,q,u,x_u,x_d;\Phi,c) \), higher rank cases, and skein algebra on higher genus surfaces will reveal a fruitful structure [27]. Also the cluster algebraic construction of the skein algebra [26] will be useful in DAHA.

We have studied the reduced DAHA polynomial for a simple curve \( c = \mathcal{F}(c_{(r,s)}) \) where the Dehn twist \( \mathcal{F} \) is generated by \( \mathcal{F}_u, \mathcal{F}_y, \mathcal{F}_d \), and \( \mathcal{F}_y \). In this case, we need only to apply the \( SL(2; \mathbb{Z}) \) actions of \( A_1 \)-type DAHA to the \( q \)-difference operator which is defined in terms of the \( C^\vee_1 \)-type DAHA. Even in the case that we further have the Dehn twist \( \mathcal{T}_y \) about \( y \), it is possible to define the DAHA polynomial using \( V_L \) corresponding to the Dehn twist about \( y \).
as we have constructed the conjugation (3.27) for the \( C^\omega C_1 \)-type DAHA. Unfortunately such computations are much involved, and it remains for future studies.

**APPENDIX A. OTHER SIMPLE CLOSED CURVES DERIVED FROM \( c_{(1,2)} \)**

The Dehn twist \( \mathcal{T}_{x_\circ}^{-k} \mathcal{T}_{y_\circ} \) induces the automorphism of \( S_{-k} \mathcal{T}_{x_\circ}^{-1} \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ} = q^{\frac{j}{2}(k-1)}X^kYX \). At \( t = q \), we have

\[
S_{2j}(\text{ch}(q^{\frac{j}{2}(k-1)}X^kY))(1) = \sum_{i=-j}^{j} S_{2(k+1)i}(\text{ch} x) q^{2i((k+1)+1)}.
\]

(A.1)

Here as we have \( S_n(x + x^{-1}) = \frac{x^{n+1} + x^{-n-1}}{x-x^{-1}} \) for \( n \geq 0 \), the Chebyshev polynomial for negative integers means

\[
S_{-n}(ch x) = -S_{n-1}(ch x).
\]

The Dehn twist \( \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ} \mathcal{T}_{y_\circ} \) induces the SH \( q,t \) automorphism, \( t^k \mathcal{T}_{x_\circ} \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ} = q^{-k}Y^kXY^{k+1}X \). At \( t = q \), we have

\[
S_{2j}(\text{ch}(q^{-k}Y^kXY^{k+1}X))(1) = \sum_{i=0}^{2j} (-1)^i q^{\frac{i}{2}(k+1)(i+1)} S_{2i}(ch x).
\]

(A.2)

As we have discussed, the reduced DAHA polynomials can be given from the constant term (5.39) by applying the automorphisms. For instance, when a simple closed curve \( c \) is given from \( c_{(1,2)} \) as \( \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ}^{-1} \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ} \), the reduced DAHA polynomial is then given by

\[
\bar{P}_N(q, u, x, d; \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ}^{-1} \mathcal{T}_{y_\circ} \mathcal{T}_{x_\circ}(c_{(1,2)})) = (-1)^N \frac{q^{\frac{j}{2}(N-1)}}{1 - q^{\frac{j}{2}}} s_{2N-1}(x_u) \left((-1)^i q^{\frac{j}{2}(2k+1)(i+1)}\right)_{0 \leq i \leq 2j \leq 2(N-1)} \times B_N(q) T_{N}(q; c_{(1,2)}) B_N(q)^\top \text{ diag}(q^{(j-1)\ell})_{1 \leq j \leq N} s_N(x_d)^\top.
\]

(A.3)

We have checked the polynomial at \( x_u = x_d = q_u \) for \( (k, \ell) = (-1, 1) \) (resp. \( (-1, -1) \)) coincidences with the \( N \)-colored Jones polynomial for \( \delta_1 \) (resp. \( \delta_2 \)). We can give the formula for the curves obtained by the above Dehn twists for \( X_\circ \) by applying (A.1) and (A.2)

**APPENDIX B. DOUBLE-TORUS KNOTS FROM \( c_{(1,3)} \)**

We give some results on the reduced DAHA polynomial for double-torus knots, which are given from the simple closed curve \( c_{(1,3)} \) on \( \Sigma_{2,0} \). The difference operator \( \mathcal{A}_{(1,3)} \in \text{SH}_{q,t} \) associated to the curve \( c_{(1,3)} \) is given in (3.37). At \( t_* \) (5.11), we can compute it explicitly as
follows.

\[
G \mathcal{A}_{(1,3)} \big|_{\text{sym}} G^{-1} = a_{(1,3)}^3(q_u, t) \left[ \text{sh}(t^{-1}X_u) \text{sh}(X_u) \text{sh}(tX_d) \text{sh}(tX_d) \delta_t \right]^3 \\
+ a_{(1,3)}^2(q_u, t) \text{ch}(X_u) \text{ch}(X_d) \left[ \text{sh}(t^{-1}X_u) \text{sh}(tX_u) \text{sh}(t^{-1}X_d) \text{sh}(tX_d) \delta_t \right]^2 \\
+ a_{(1,3)}^1(q_u, t, X_u, X_d) \left[ \text{sh}(t^{-1}X_u) \text{sh}(tX_u) \text{sh}(t^{-1}X_d) \text{sh}(tX_d) \delta_t \right] \\
+ a_{(1,3)}^0(q_u, t, X_u, X_d) + a_{(1,3)}^1(q_u, q_u t^{-1}, X_u, X_d) \delta_t^{-1} \\
+ a_{(1,3)}^2(q_u, q_u t^{-1}) \text{ch}(X_u) \text{ch}(X_d) \delta_t^{-2} + a_{(1,3)}^3(q_u, q_u t^{-1}) \delta_t^{-3},
\] (B.1)

where \( G \) is the gluing function (5.17), and

\[
a_{(1,3)}^3(q_u, t) = \frac{q_u^{13}t^{14}(1-t^2)(1-q_u^2t^2)(1-q_u^4t^2)}{(1+t^2)(1+q_u^2t^2)(1+q_u^4t^2)(1-q_u^2t^4)(1-q_u^6t^4)},
\]

\[
a_{(1,3)}^2(q_u, t) = \frac{-2q_u^2t^{10}(1-t^2)(1-q_u^2t^2)(1+q_u^2t^2)(1+q_u^4t^2)(1-q_u^2t^4)}{(1+t^2)(1+q_u^2t^2)(1+q_u^4t^2)(1-q_u^2t^4)},
\]

\[
a_{(1,3)}^1(q_u, t, X_u, X_d) = \frac{q_u^2t^6(1-t^2)}{(1+t^2)\left(q_u^2+q_u^4t^2\right)\left(q_u^6+q_u^4t^2\right)\left(q_u^6-t^4\right)\left(1-q_u^2t^4\right)} \\
\times \left\{-q_u^2(1+q_u^8)(-4+3t^2)-q_u^2t^2(1+q_u^4)(-3+5t^2-4t^4+6^4) \\
+ q_u^4(1-4t^2-5t^4+6t^6-4t^8)\right\} \text{ch}^2(X_u) \text{ch}^2(X_d) \\
- (q_u^2+t^2)(1+q_u^2t^2)\left(t^2(1+q_u^8)+q_u^2(1+q_u^4t^2)(1-2t^2)+q_u^4(-2+t^2+t^6)\right) \left\{ \text{ch}^2(X_u) + \text{ch}^2(X_d) \right\} \\
+ (q_u^2+t^2)(1+q_u^2t^2)(-q_u^2+q_u^2t^2)t^4+q_u^2+q_u^4t^4) \right\},
\]

\[
a_{(1,3)}^0(q_u, t, X_u, X_d) = \frac{2q_u t^2}{(1+t^2)\left(q_u^2+q_u^4t^2\right)\left(q_u^4+q_u^4t^2\right)\left(q_u^6+q_u^4t^2\right)\left(q_u^6-t^4\right)\left(1-q_u^2t^4\right)} \text{ch}(X_u) \text{ch}(X_d) \\
\times \left\{q_u^4t^2\left(t^4(3-3t^2+t^4)+q_u^2t^2(-3+5t^2-4t^4+t^6)+q_u^4(1-4t^2+5t^4-3t^6) \\
+ q_u^6(1-3t^2+3t^4)\right) \text{ch}^2(X_u) \text{ch}^2(X_d) \\
- q_u^2\left(t^6+q_u^2t^6(3-t^2-2t^4+t^6)+q_u^4t^4(-1+t^4-2t^4)+q_u^8t^2(-2+t^2-t^6) \\
+ q_u^8(1-2t^2-3t^6)\right) \text{ch}^2(X_u) + \text{ch}^2(X_d) \right\} \\
+ \left\{t^6(1+t^2)+q_u^2t^4(1+2t^2-2t^4-t^8)-q_u^4t^4(-2-5t^2+2t^4+3t^6+t^8) \\
- q_u^4t^4(2+t^2-t^4+3t^6+t^8)-q_u^8(1+3t^2-t^4+t^6+2t^8)+q_u^{10}t^6(-1-3t^2-2t^4+5t^6+2t^8) \\
+ q_u^{12}(-1+2t^4+t^6+t^8)+q_u^{14}t^8(1+t^2)\right\} \right\}.
\]

The constant term \( \delta^0 \) in \( \mathcal{A}_{(1,3)} \) at \( t = q_u \) is written as

\[
a_{(1,3)}^0(q_u, q_u, X_u, X_d)
\]
automorphisms to (B.2), the reduced DAHA polynomial is given by

\[
P_2(q_u, x_u, x_d; c,(1,3)) = q_{(1,3)}^{[0]} (q_u, q_u, q_u^{-1}) Y^\ell Y_d X_d(1).
\]

As \(X_0\) and \(Y_0\) are generators of \(SH_{q_u,t}\), we can make use of (5.34) of \(SH_{q,t}\) in the computation. As a result, we obtain a symmetric bilinear form for the reduced DAHA polynomial

\[
P_2(q_u, x_u, x_d; c,(1,3)) = -q_u \left(1 - q_u^2\right) \frac{1}{1 - q_u^4} \times \left(q_u^{-k} S_1(ch x_u), q_u^{-k} S_3(ch x_u)\right) \left(1 + \left(1 - q_u^2\right) \frac{1 - q_u^4}{1 - q_u^2} \right) \left(\frac{q_u^2}{1 - q_u^2} \right) \left(q_u^2 S_1(ch x_d), q_u^2 S_3(ch x_d)\right).
\]

We see that the curve \(c,(1,3)\) with \((k, \ell) = (1, -1)\) denotes a connected sum of trefoils \(3_1 \# \overline{3_1}\) (the square knot). Also, SnapPy \cite{17} tells us that the curves \(c,(1,3)\) for \((k, \ell) = (1, 1)\) and \((1, 2)\) are \(9_{46}\) and \(k7_{125}\) respectively. We have checked that the reduced DAHA polynomial \(\overline{P}_2\) (B.4) for these closed curves coincide with the Jones polynomials for \(3_1 \# \overline{3_1}\), \(9_{46}\), and \(k7_{125}\) in [11][12] at \(x_u = x_d = q_u\).

For the \(n = 3\) colored reduced DAHA polynomial \(\overline{P}_3\), the constant term of \(S_2(\overline{A}_{(1,3)}) = \overline{A}_{(1,3)}^2 - 1\) at \(t = q_u = q_u^\frac{1}{2}\) is computed as

\[
ap_{(1,3)}^{[3]}(q_u, q_u) a_{(1,3)}^{[3]}(q_u, q_u^{-3}) \prod_{\in \{u,d\}} \left(\sum_{j=1}^{3} \text{sh}(q_u^{-j}X_0) \text{sh}(q_u^jX_0)\right)
\]

\[
+ a_{(1,3)}^{[2]}(q_u, q_u) a_{(1,3)}^{[2]}(q_u, q_u^{-2}) \prod_{\in \{u,d\}} \left(\sum_{j=1}^{2} \text{ch}^2(X_0) \text{sh}(q_u^{-j}X_0) \text{sh}(q_u^jX_0)\right)
\]

\[
+ a_{(1,3)}^{[1]}(q_u, q_u, X_u, X_d) a_{(1,3)}^{[1]}(q_u, q_u^{-1}, X_u, X_d) \prod_{\in \{u,d\}} \left(\text{sh}(q_u^{-1}X_0) \text{sh}(q_uX_0)\right)
\]

\[
+ a_{(1,3)}^{[0]}(q_u, q_u, X_u, X_d)^2 - 1
\]

\[
= \frac{q (1 - q^2)(1 - q^2) \prod_{\in \{u,d\}} (q X_0^2, q X_0^{-2}, q X_0^2, q X_0^{-2}) \left(\text{ch}(X_0^2) + 1 + \frac{q}{1 - q^2}\right) + q (1 - q) \prod_{\in \{u,d\}} \left(\text{ch}(X_0^2) + 1\right)}{(1 - q^2)(1 - q^2)(1 - q^4)(1 - q^4)}
\]

\[
+ \frac{q (1 - q)(1 - q^2)^2 \prod_{\in \{u,d\}} (1 - q X_0^2, 1 - q X_0^{-2}) \text{ch}^2(X_0^2) + q (1 - q) \prod_{\in \{u,d\}} \left(\text{ch}(X_0^2) + 1\right)}{1 - q^2}.
\]
Applying the $SL(2;\mathbb{Z})$ actions on $X_{\emptyset}$, we obtain a symmetric bilinear form for the reduced DAHA polynomial of $c_{(1,3);(k,\ell)}$ as

$$\overline{P}_3(q_u, x_u, x_d; c_{(1,3);(k,\ell)}) = \frac{q(1-q)}{1-q^2} s_4(x_u) \text{ diag} \left( 1, q^{2k}, q^{6k}, q^{12k} \right)$$

\begin{equation}
\times \begin{pmatrix}
1 & \frac{1-q(1-q^3)}{1-q(1-q^2)} & \frac{-q(1-q^2)}{1-q(1-q^3)} & 0 \\
\frac{(1-q)(1-q^3)}{(1-q)(1-q^2)} & \frac{1-q(1-q^3)}{(1-q)(1-q^2)} & \frac{1-q(1-q^2)}{(1-q)(1-q^3)} & \frac{-q(1-q^2)}{(1-q)(1-q^3)} \\
\frac{-q(1-q^2)}{1-q(1-q^2)} & \frac{1-q(1-q^3)}{(1-q)(1-q^2)} & \frac{1-q(1-q^2)}{1-q(1-q^3)} & \frac{q(1-q^2)}{(1-q)(1-q^3)} \\
0 & \frac{1-q(1-q^3)}{1-q(1-q^2)} & \frac{1-q(1-q^2)}{1-q(1-q^3)} & \frac{-q(1-q^2)}{(1-q)(1-q^3)}
\end{pmatrix} \times \text{ diag} \left( 1, q^{2\ell}, q^{6\ell}, q^{12\ell} \right) s_4(x_d)^T,
\end{equation}

where $s_N(x)$ is defined in [5.36]. We have checked that the results for $(k, \ell) = (1, -1)$ and $(1, 1)$ agree up to framing factor with the $n = 3$ colored Jones polynomials for the square knot $3_1\#3_1$ and $9_{46}$ in [11] respectively when $x_u = x_d = q_u = q^2$ as expected.

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