THE AFFINE STRUCTURE OF GRAVITATIONAL THEORIES: SYMPLECTIC GROUPS AND GEOMETRY

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We give a geometrical description of gravitational theories from the viewpoint of symmetries and affine structure. We show how gravity, considered as a gauge theory, can be consistently achieved by the nonlinear realization of the conformal-affine group in an indirect manner: due the partial isomorphism between $\text{CA}(3,1)$ and the centrally extended $\text{Sp}(8)$, we perform a nonlinear realization of the centrally extended (CE)$\text{Sp}(8)$ in its semi-simple version. In particular, starting from the bundle structure of gravity, we derive the conformal-affine Lie algebra and then, by the non-linear realization, we define the coset field transformations, the Cartan forms and the inverse Higgs constants. Finally we discuss the geometrical Lagrangians where all the information on matter fields and their interactions can be contained.

Keywords: Affine geometry; gravity; bundle structure; conformal group.

Received (Day Month Year)
Revised (Day Month Year)

1. Introduction

The Standard Model (SM) of particles is a gauge theory, where all fields mediating interactions are represented by gauge potentials. A very debated question is to understand why the fields mediating the gravitational interaction are different from those of other fundamental forces. It is reasonable to expect that there may be a gauge theory where the gravitational fields stand on the same footing as those of the other fundamental fields. This expectation has prompted a re-examination of General Relativity (GR) from the gauge field viewpoint.

In the SM, the involved gauge groups are internal symmetries while, in GR, gauge groups are associated to external spacetime symmetries. Since GR is not a true Yang-Mills theory, we must require that it is a theory where gauge objects are not only the gauge potentials but also tetrads that relate the symmetry group to the external spacetime. For this reason, we have to take into account more complex nonlinear gauge theories.

Ashtekar \(\text{[1]}\) considered as fundamental variables the tetrad fields and the connection forms while Einstein took, as the fundamental variables, the spacetime
metric components. However gravitation may be viewed as a gauge theory \[2\] in analogy to the Yang-Mills theory \[3\]. Later, a gauge theory build with Poincaré group \(P(3,1) = T(3,1) \times SO(3,1)\) was introduced by Kibble \[4\]. Cartan \[5\] generalized the Riemann geometry to include torsion in addition to curvature. Many authors stressed that intrinsic spin may be the source of torsion of the underlying spacetime manifold \[6\\[6\] and gauge theories based on Lie groups have been widely considered \[8\\[8\].

Furthermore, it has been proposed that the Kibble gauge theory can be built considering the de Sitter group \(SO(4,1)\), i.e. one has to come into the Poincaré group by a group contraction \[15\].

Usually one adopts the fiber-bundle description to derive gauge theories on the Lie group, where the tetrads are constructed by using the affine group \(A(4,\mathbb{R}) = T(4) \ltimes GL(4,\mathbb{R})\). It is important to stress that, in metric-affine gravity, the Lagrangian can be assumed quadratic in both curvature and torsion terms in contrast to the Einstein-Hilbert Lagrangian of GR which is linear in the scalar curvature. This approach has been recently developed also for more general theories like \(f(R)\)-gravity \[16\-21\]. However, there are many attempts to formulate gravitation as a gauge theory. Despite of these approaches, no final theory can be uniquely accepted as the gauge theory of gravitation.

Besides, the non-linear approach in the context of internal symmetry groups has been considered \[23\-24\] and extended to the case of spacetime symmetries, using the nonlinear action of \(GL(4,\mathbb{R})\) \[25\-27\]. Other authors \[28\-29\] considered the simultaneous nonlinear realization of the affine and conformal groups, showing that GR can be viewed as a consequence of spontaneous symmetry breaking of the affine symmetry where the gravitons are considered as Goldstone bosons associated with the affine symmetry breaking. The nonlinear realization scheme, employing \(GL(4,\mathbb{R})\) as the principal group, has been also considered in \[13\], while the nonlinear realization induced by the spontaneous breakdown of \(SO(3,2)\) was investigated by Stelle et al. \[30\].

After, nonlinear gauge theories of the Poincaré, de Sitter, conformal and special conformal groups have been taken into account \[31\-32\] with the gravity as a spontaneously broken \(GL(4,\mathbb{R})\) gauge theory.

Finally, the nonlinear realization in the fiber bundle formalism, based on the bundle structure \(G(G/H, H)\) has been developed by several authors (see for example \[33\-34\]). In this approach, the quotient space \(G/H\) is identified with physical spacetime. Most recently, nonlinear gauge theories of gravity have been discussed on the basis of the Poincaré, affine and conformal groups \[35\-37\]. While in GR the spacetime is represented by a four-dimensional differential manifold that is assumed to be curved, in Special Relativity (SR) the manifold is represented by the flat-spacetime \(M_4\) (i.e. Minkowskian spacetime). The GR can be regarded as a gauge theory which is based on the local Lorentz group as well as, the Yang-Mills

\[\text{Here } \ltimes \text{ represents the semi-direct product.}\]
The affine structure of gravitational theories

3

gauge theory is based on the internal iso-spin gauge group. From this viewpoint, the Riemannian connection is the gravitational counterpart of the Yang-Mills gauge fields. While $SU(2)$, in the Yang-Mills theory, is an internal symmetry group, the Lorentz symmetry represents the local nature of spacetime rather than internal degrees of freedom.

On the other hand, the Einstein Equivalence Principle, asserted for GR, requires that the local spacetime structure can be identified with the Minkowski spacetime possessing the Lorentz symmetry. In order to connect the external spacetime to the local Lorentz symmetry, we need to tie the external space to the local space. The tetrad fields can be the link tools. The tetrads can be considered as objects that can be dynamically generated and the spacetime must contain torsion in order to have spinor fields. Namely, the gravitational interaction of spinning particles needs that the Riemann spacetime must be modified in a (non-Riemannian) curved spacetime with torsion. The Sciama theory fails when the tetrad fields are treated as gauge fields. However, starting from a purely gauge point of view, it is possible formulate a self-consistent gravitational theory [4,42].

In this paper we discuss how gravity can be obtained as a gauge theory achieved by the nonlinear realization of the conformal-affine group in an indirect manner: due to the partial isomorphism between $CA(3,1)$ and $Sp(8)$, we perform a nonlinear realization of the centrally extended (CE)$Sp(8)$ in its semisimple version. In order to obtain this result, we consider the structure of the $CA(3,1)$, that is $SO(4,2) \cup A(4,\mathbb{R})$. It can be described schematically as the 15 generators of the conformal group: Lorentz (6), translations (4), special conformal (4) and dilatations (1), plus the 9 traceless generators of the symmetric linear transformations (shear). Notice that the copies of the common generators between $SO(4,2)$ and $A(4,\mathbb{R})$ have been eliminated.

The relation with the centrally extended $sp(8)$ algebra is based on the observation that the nine traceless generators of the symmetric linear transformations (shear) can be factored in the eight parameter generators $F_{\rho \dot{\sigma}}$, $F_{\rho \dot{\sigma}} = (F_{\dot{\sigma} \rho})^*$ plus the axion $A$. This fact brings us to the possibility to make a nonlinear realization of the CE $Sp(8)$, where, due the partial isomorphism described before, conformal affine structure of the gravitational theories can be included [12].

A nonlinear realization is performed by the standard CE $sp(8)$ algebra: the subgroup structure is generated by $\hat{K}_{\alpha \phi}$, $\hat{Z}_{\alpha \beta}$, $X_{\alpha \alpha}$, where the Lorentz generators $L_{\alpha \lambda}$ are semisimple, in contrast with the subgroup structure generated by $K_{\alpha \alpha}$, $Z_{\alpha \beta}$, $F_{\alpha \alpha}$, $F_{\alpha \alpha}$, $F_{\alpha \alpha}$, where the Lorentz generators $L_{\alpha \lambda}$ are semidirect products. The main difference between the different structures of $sp(8)$ consist in the explicit nonlinear constraints arising after the imposition that the corresponding Cartan form of the scalar (pseudoscalar) field must be zero. This simple fact makes a sharp difference between the simplest geometrical Lagrangians (measures) obtained in each case: Yang-Mills type or Born-Infeld (Nambu-Goto) type, that is:
The layout of the paper is the following. Section 2 is devoted to a general discussion of the bundle structure of gravitation, while the conformal-affine Lie algebra is introduced in Section 3. The non-linear realization and the coset field transformations are discussed in Section 4. Relevant Cartan forms are discussed in Section 5, and the inverse Higgs constraints are derived Section 6. The affine structure of superpspace is presented in Section 7. Finally, we shown the geometrical Lagrangian of the theory and give the new symmetries and dynamics in Section 8. Conclusions and perspectives are presented in Section 9.

2. The Bundle Structure of Gravity

Let us briefly describe the bundle structure of gravity. Let $P(M, G; \pi)$ be a principal fiber bundle, where $M$ is the base space and $G$ a standard diffeomorphic fiber. It follows that the gauge transformations are characterized by the bundle isomorphisms $\lambda : P \rightarrow P$ exhausting all diffeomorphisms $\lambda_M$ on $M$. If a mapping is equivariant with respect to the action of $G$, it is called an automorphism of $P$. This amounts to restrict the action $\lambda$ of $G$ along fibers not altering the base space. As it is well known, a gauge transformation is a fiber preserving the bundle automorphisms, i.e. diffeomorphisms $\lambda$ with $\lambda_M = (id)_M$. The automorphisms $\lambda$ form a group called the automorphism group $Aut_P$ of $P$. The gauge transformations form a subgroup of $Aut_P$ called the gauge group $G(Aut_P)$ (or $G$ in short) of $P$. The map $\lambda$ is required to satisfy two conditions, namely its commutability with the right action of $G$ [the equivariance condition $\lambda(R_g(p)) = \lambda(pg) = \lambda(p)g$]

$$\lambda \circ R_g(p) = R_g(p) \circ \lambda, \ p \in P, \ g \in G,$$

according to which fibers are mapped into fibers, and the verticality condition

$$\pi \circ \lambda(u) = \pi(u),$$

where $u$ and $\lambda(u)$ belong to the same fiber. The last condition ensures that no diffeomorphisms $\lambda_M : M \rightarrow M$, given by

$$\lambda_M \circ \pi(u) = \pi \circ \lambda(u),$$

is allowed on the base space $M$. To get a gauge description of gravity, we have to gauge external transformation groups. Hence, the transformations in a space must induce corresponding transformations in the other. The usual definition of a gauge transformation, i.e. as a displacement along local fibers not affecting the base space, must be generalized to reflect this interlocking. One possible way of framing this interlocking is to employ a nonlinear realization of the gauge group $G$, provided that
a closed subgroup $H \subset G$ exists. The interlocking requirement is then transformed into the interplay between groups $G$ and one of its closed subgroups $H$ \cite{42}.

Let us denote by $G$ a Lie group with elements $\{g\}$. Let $H$ be a closed subgroup of $G$ specified by

$$H := \{h \in G | \Pi (R_h g) = \pi (g), \forall g \in G\},$$

with elements $\{h\}$ and known linear representations $\rho (h)$. Here $\Pi$ is the first of the two projection maps, and $R_h$ is the right group action. Let $M$ be a differentiable manifold with points $\{x\}$ to which $G$ and $H$ may be referred, i.e. $g = g(x)$ and $h = h(x)$. Being that $G$ and $H$ are Lie groups, they are also manifolds. The right action of $H$ on $G$ induces a complete partition of $G$ into mutually disjoint orbits $gH$. Since $g = g(x)$, all elements of $gH = \{gh_1, gh_2, gh_3, \ldots, gh_n\}$ are defined over the same $x$. Thus, each orbit $gH$ constitutes an equivalence class of point $x$, with equivalence relation $g \equiv g'$, where $g' = R_h g = gh$. By projecting each equivalence class onto a single element of the quotient space $\mathcal{M} := G/H$, the group $G$ becomes organized as a fiber bundle in the sense that $G = \bigcup_i \{g_i H\}$. In this way, the manifold $G$ is viewed as a fiber bundle $G(M, H; \Pi)$ with $H$-diffeomorphic fibers $\Pi^{-1} (\xi) : G \to \mathcal{M} = gH$ and base space $\mathcal{M}$. A composite principal fiber bundle $\mathbb{P}(M, G; \pi)$ is one whose $G$-diffeomorphic fibers possess the fibered structure $G(M, H; \Pi) \cong M \times H$ described above. The bundle $\mathbb{P}$ is then locally isomorphic to $M \times G(M, H)$. Moreover, since an element $g \in G$ is locally homeomorphic to $\mathcal{M} \times H$, the elements of $\mathbb{P}$ are - by transitivity - also locally homeomorphic to $M \times \mathcal{M} \times H \cong \Sigma \times H$ where (locally) $\Sigma \cong M \times \mathcal{M}$. Thus, an alternative view \cite{39} of $\mathbb{P}(M, G; \pi)$ is provided by the $\mathbb{P}$-associated $H$-bundle $\mathbb{P}(\Sigma, H; \bar{\pi})$. The total space $\mathbb{P}$ may be regarded as $G(M, H; \Pi)$-bundles over base space $M$ or equivalently as $H$-fibers attached to manifold $\Sigma \cong M \times \mathcal{M}$.

The nonlinear realization (NLR) technique \cite{23, 24} gives us a method to construct the transformation properties of fields defined on the quotient space $G/H$. Thanks to the Ogievetsky theorem, we can treat the NLR of $\text{Diff}(4, \mathbb{R})$, and then the algebras of $\text{Diff}(4, \mathbb{R})$ can be considerate as algebras of $SO(4, 2)$ and $A(4, \mathbb{R})$ \cite{25}. We remember that the trasformations with quadradic forms on the Minkowski spacetime are generated by the Lorentz group while infinitesimal angle-preserving transformations on the Minkowski spacetime are generated by special conformal groups. A generalization of Poincaré group is the affine group and therefore the Lorentz group is substituted by the group of general linear transformations \cite{40}. As such, with the affine group, we can generate Lorentz transformations, translations, volume preserving shear and volume changing dilation transformations.

As a consequence, the NLR of $\text{Diff}(4, \mathbb{R})/SO(3, 1)$ can be constructed by taking a simultaneous realization of the conformal group $SO(4, 2)$ and the affine group $A(4, \mathbb{R}) := \mathbb{R}^4 \times GL(4, \mathbb{R})$ on the coset spaces $A(4, \mathbb{R})/SO(3, 1)$ and $SO(4, 2)/SO(3, 1)$. Therefore, we can see that the conformal-affine group could be the subgroup of $\text{Diff}(4, \mathbb{R})$. On the other hand, a group of realization is not linear when it is subject to constraints. For example, we can choose constraints as those responsible for
the reduction of symmetry from \( \text{Diff}(4, \mathbb{R}) \) to \( SO(3, 1) \). We consider the group \( CA(3, 1) \) as the basic symmetry group \( G \). The conformal-affine group consists of the groups \( SO(4, 2) \) and \( A(4, \mathbb{R}) \). In particular, the conformal-affine group is proportional to the union \( SO(4, 2) \cup A(4, \mathbb{R}) \). We know, however, that the affine and special conformal groups have several group generators in common. These common generators reside in the intersection \( SO(4, 2) \cap A(4, \mathbb{R}) \) of the two groups, within which there are two copies of \( \Pi := D \times P(3, 1) \), where \( D \) is the group of scale transformations (dilations) and \( P(3, 1) := T(3, 1) \times SO(3, 1) \) is the Poincaré group. We define the conformal-affine group as the union of the affine and conformal groups minus one copy of the overlap \( \Pi \), i.e. \( CA(3, 1) := SO(4, 2) \cup A(4, \mathbb{R}) - \Pi \).

Being defined in this way, we recognize that \( CA(3, 1) \) is a 24 parameter Lie group representing the action of Lorentz transformations \( (6) \), translations \( (4) \), special conformal transformations \( (4) \), spacetime shears \( (9) \) and scale transformations \( (1) \).

Here, we have calculated the NLR of \( CA(3, 1) \) modulo \( SO(3, 1) \).

### 3. The Conformal-Affine Lie Algebra

In order to develop the NLR procedure, we choose to perform the partition of \( \text{Diff}(4, \mathbb{R}) \) with respect to the Lorentz group. Using the Ogievetsky theorem \([28]\), we identify representations of \( \text{Diff}(4, \mathbb{R})/SO(3, 1) \) with those of \( CA(3, 1)/SO(3, 1) \). The 20 generators of affine transformations can be decomposed into the 4 translational \( P^\mu_{\text{Aff}} \) and 16 \( GL(4, \mathbb{R}) \) transformations \( \Lambda^\alpha_\beta \). The 16 generators \( \Lambda^\alpha_\beta \) may be further decomposed into the 6 Lorentz generators \( L^\alpha_\beta \) plus the remaining 10 generators of symmetric linear transformation \( S^\alpha_\beta \), that is, \( \Lambda^\alpha_\beta = L^\alpha_\beta + S^\alpha_\beta \). The 10 parameter symmetric linear generators \( S^\alpha_\beta \) can be factored into the 9 parameter shear (the traceless part of \( S^\alpha_\beta \)) generators defined by \( ^\dagger S^\alpha_\beta = S^\alpha_\beta - \frac{1}{9} \delta^\alpha_\beta D \), and the 1 parameter dilaton generator \( D = tr (S^\alpha_\beta) \). Shear transformations generated by \( ^\dagger S^\alpha_\beta \) describe shape changing, volume preserving deformations, while the dilaton generator gives rise to volume changing transformations. The 4 diagonal elements of \( S^\alpha_\beta \) correspond to the generators of projective transformations. The 15 generators of conformal transformations are defined in terms of the set \( \{ J_{AB} \} \) where \( A = 0, 1, 2...5 \). The elements \( J_{AB} \) can be decomposed into translations \( P^\mu_{\text{Conf}} := J_{5\mu} + J_{6\mu} \), special conformal generators \( \Delta_\mu := J_{5\mu} - J_{6\mu} \), dilatons \( D := J_{56} \) and the Lorentz generators \( L^\alpha_\beta := J_{\alpha\beta} \).

The Lie algebra of \( CA(3, 1) \) is characterized by the commutation relations...
The affine structure of gravitational theories

\[ [A_{\alpha\beta}, D] = [\Delta_{\alpha}, \Delta_{\beta}] = 0, \]
\[ [P_{\alpha}, P_{\beta}] = [D, D] = 0, \]
\[ [L_{\alpha\beta}, P_\mu] = i\eta_{\mu[\alpha} P_{\beta]}, [L_{\alpha\beta}, \Delta_\gamma] = i\eta_{[\alpha\beta} \Delta_{\gamma]}, \]
\[ [A^\alpha_{\beta}, P_\mu] = i\delta^\mu_{\alpha} P_{\beta}, [A^\alpha_{\beta}, \Delta_\mu] = i\delta^\alpha_{\mu} \Delta_{\beta}, \]
\[ [S_{\alpha\beta}, P_\mu] = i\eta_{\mu[\alpha} P_{\beta]}, [P_{\alpha}, D] = -iP_{\alpha}, \]
\[ [L_{\alpha\beta}, L_{\mu\nu}] = -i (o_{\alpha[\mu} L_{\nu\beta]} - o_{\beta[\mu} L_{\nu\alpha]}), \]
\[ [S_{\alpha\beta}, S_{\mu\nu}] = i (o_{\alpha[\mu} S_{\nu\beta]} - o_{\beta[\mu} S_{\nu\alpha]}), \]
\[ [\Delta_{\alpha}, D] = i\Delta_{\alpha}, [S_{\mu\nu}, \Delta_\alpha] = i\eta_{\alpha[\mu} \Delta_{\nu]}, \]
\[ [A^\alpha_{\beta}, A^\mu_{\nu}] = i (\delta^\mu_{\alpha} A^{\nu}_{\beta} - \delta^\nu_{\beta} A^{\mu}_{\alpha}), \]
\[ [P_{\alpha}, \Delta_{\beta}] = 2i (o_{\alpha\beta} D - L_{\alpha\beta}), \]

(5)

where \( o_{\alpha\beta} = diag(-1, 1, 1, 1) \) is the Lorentz group metric. The above algebra is the core of NLR and, in some sense, of the invariance induced gravity that we are considering here. The relation with the CE \( sp(8) \) algebra is based on the observation that the 9 traceless generators of the symmetric linear transformations (shear) can be factored in the 8 parameter generators \( F_{\rho\tau}, \bar{F}_{\rho\tau} \equiv (F_{\tau\rho})^* \) plus the axion \( A \) (see below). As it can be easily seen, the standard \( sp(8) \) algebra has the commutation relations that are related with the standard algebra \( so(2, 4) \approx su(2, 2) \) that is spanned by the generators \( (L_{\alpha\beta}, T_{\alpha\beta}, P_{\alpha\beta}, K_{\alpha\beta}, D) \), that is

\[ [P_{\alpha\beta}, P_{\gamma\delta}] = [K_{\alpha\beta}, K_{\gamma\delta}] = 0, \]
\[ [P_{\alpha\beta}, K_{\rho\lambda}] = \frac{1}{2} \left( \epsilon_{\alpha\beta} \bar{L}_{\rho\lambda} - \epsilon_{\rho\lambda} L_{\alpha\beta} \right) - i\epsilon_{\alpha\rho} \epsilon_{\beta\lambda} D, \]
\[ [L_{\alpha\beta}, L_{\rho\lambda}] = \epsilon_{\alpha\rho} L_{\beta\lambda} + \epsilon_{\beta\rho} L_{\alpha\lambda} + \epsilon_{\alpha\lambda} L_{\beta\rho} + \epsilon_{\beta\lambda} L_{\alpha\rho}, \]
\[ [L_{\alpha\beta}, P_{\rho\delta}] = \epsilon_{\alpha\rho} P_{\beta\delta} + \epsilon_{\beta\rho} P_{\alpha\delta}, [L_{\alpha\beta}, K_{\rho\delta}] = \epsilon_{\alpha\rho} K_{\beta\delta} + \epsilon_{\beta\rho} K_{\alpha\delta}, \]
\[ [D, P_{\alpha\delta}] = iP_{\alpha\delta}, [D, K_{\alpha\delta}] = -iK_{\alpha\delta}. \]

(6)

The rest of non-vanishing commutators can be obtained by complex conjugation. The algebra \( sl(4, R) \) is spanned by the generators \( (L_{\alpha\beta}, T_{\alpha\beta}, A, F_{\alpha\beta}, \bar{F}_{\alpha\beta}) \). The extra generators \( A, F_{\rho\tau}, \bar{F}_{\rho\tau} \) satisfy the relations
\[ [A, F_{\alpha \beta}] = 2F_{\alpha \beta}, \]
\[ [A, F_{\alpha \beta}] = 2F_{\alpha \beta}, \]
\[ [F_{\alpha \beta}, F_{\gamma \delta}] = \frac{1}{2} \left( \epsilon_{\alpha \rho} L_{\beta \lambda} - \epsilon_{\beta \rho} L_{\alpha \lambda} \right) + \epsilon_{\alpha \rho} \epsilon_{\beta \lambda} A. \] (7)

The generalized 4D conformal algebra \( sp(8) \) is a closure of the algebras \( so(2,4) \) and \( sl(4, \mathbb{R}) \). It is obtained by adding to the generators of \( sl(4, \mathbb{R}) \) and the vectorial Abelian translation generators \( (P_{\alpha \beta}, K_{\alpha \beta}) \) the following additional 12 Abelian generators

- \( (Z_{\alpha \beta}, \tilde{Z}_{\dot{\alpha} \dot{\beta}}) \) describing 6 standard tensorial translations
- \( (\tilde{Z}_{\alpha \beta}, \tilde{Z}_{\dot{\alpha} \dot{\beta}}) \) describing 6 conformal tensorial translations.

Some of the commutation relations that they satisfy are, for example:

\[ [Z_{\alpha \beta}, \tilde{Z}_{\rho \lambda}] = \frac{1}{2} \left( \epsilon_{\alpha \rho} L_{\beta \lambda} + \epsilon_{\beta \rho} L_{\alpha \lambda} + \epsilon_{\alpha \lambda} L_{\beta \rho} + \epsilon_{\beta \lambda} L_{\alpha \rho} \right) + \epsilon_{\alpha \rho} \epsilon_{\beta \lambda} \frac{A}{2}. \]
\[ [P_{\alpha \beta}, \tilde{Z}_{\rho \lambda}] = \frac{1}{2} \left( \epsilon_{\alpha \rho} F_{\lambda \dot{\beta}} + \epsilon_{\alpha \lambda} F_{\rho \dot{\beta}} \right). \]
\[ [K_{\alpha \beta}, \tilde{Z}_{\rho \lambda}] = \frac{1}{2} \left( \epsilon_{\alpha \rho} \tilde{F}_{\lambda \dot{\beta}} + \epsilon_{\alpha \lambda} \tilde{F}_{\rho \dot{\beta}} \right). \] (8)

being the rest described in detail in [47]. Then, we have the concrete possibility to make a nonlinear realization of \( Sp(8) \), where due to the partial isomorphism described before, the conformal affine structure of the gravitational theories can be included [42]. As we have pointed out in the Introduction, the nonlinear realization of the standard \( sp(8) \) algebra described above does not lead to geometrical invariants as measures, then we need to perform the nonlinear realization with the non-standard \( CE sp(8) \) algebra. The non-standard algebra described by the new"hat" generators, defined in next Section, sets a new basis where the commu-
The affine structure of gravitational theories

The relation that differs from the standard $CE sp(8)$ description are

\[
\left[\hat{K}_{\alpha\beta}, \hat{K}_{\beta\gamma}\right] = \frac{1}{m^2} \left(\epsilon_{\alpha\beta} L_{\alpha\beta} - \epsilon_{\alpha\beta} \mathcal{T}_{\alpha\beta}\right),
\]

\[
\left[X_{\alpha\beta}, X_{\gamma\delta}\right] = -4 \left(\epsilon_{\alpha\beta} \mathcal{L}_{\beta\delta} - \epsilon_{\beta\delta} L_{\alpha\gamma}\right),
\]

\[
[D, \hat{K}] = -i \left(\hat{K} + \frac{2P}{m^2}\right),
\]

\[
[D, \hat{Z}] = -i \left(\hat{Z} + \frac{2Z}{m^2}\right),
\]

\[
[D, A] = 0,
\]

\[
[A, G] = (2iX),
\]

\[
[X, A] = 2iG,
\]

\[
\left[X_{\alpha\beta}, G_{\gamma\delta}\right] = -4i\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} A,
\]

\[
[A, Z] = 2Z,
\]

\[
[A, \hat{Z}] = -2\hat{Z},
\]

\[
\left[A, \hat{Z}\right] = -2 \left(\hat{Z} + \frac{2Z}{m^2}\right),
\]

\[
\left[\hat{Z}_{\alpha\beta}, \hat{Z}_{\rho\lambda}\right] = \frac{1}{m^2} \left(\epsilon_{\alpha\rho} L_{\beta\lambda} + \epsilon_{\beta\rho} L_{\alpha\lambda} + \epsilon_{\alpha\lambda} L_{\beta\rho} + \epsilon_{\beta\lambda} L_{\alpha\rho}\right),
\]

\[
\left[\hat{K}_{\alpha\alpha}, L_{\rho\sigma}\right] = - \left(\epsilon_{\rho\sigma} \hat{K}_{\sigma\alpha} + \epsilon_{\sigma\alpha} \hat{K}_{\rho\alpha}\right),
\]

\[
\left[\hat{K}_{\alpha\alpha}, \hat{Z}_{\rho\sigma}\right] = - \frac{1}{2m^2} \left(\epsilon_{\rho\sigma} X_{\sigma\alpha} + \epsilon_{\sigma\alpha} X_{\rho\alpha}\right),
\]

\[
\left[X_{\alpha\alpha}, \hat{K}_{\beta\beta}\right] = 4 \left(\epsilon_{\beta\alpha} \hat{Z}_{\beta\alpha} + \epsilon_{\alpha\beta} \hat{Z}_{\beta\alpha}\right),
\]
In the next Section, we will see the deep consequences that these new generators have to define the geometrical basic invariants of the theory.

4. The Non-Linear Realization approach: the coset field transformations

As it is well known, the Maurer-Cartan form for a Lie group $G$ is a distinguished differential one-form on $G$ that carries the basic infinitesimal information about the structure of $G$. As a one-form, the Maurer–Cartan form is peculiar in that it takes its values in the Lie algebra associated to the Lie group $G$. The Lie algebra is identified by the tangent space of $G$ at the identity. In the context of the NLR approach, the Cartan forms play a significant role in order to define the geometrical invariants by which the Lagrangian of the theory can be constructed. As we will see in detail soon, starting from the algebra associated to the Lie group $G$, characterizing the symmetries of the manifold, we construct (in general by exponentiation) the respective coset and, by the pullback, we extract univocally the Cartan forms associated to each generator. The Cartan forms give also a geometrical description of the field dynamics by the implementation of constraints (e.g. the "inverse Higgs") to the equations of motion. To start now with the NLR of $Sp(8)$, we proceed to define our coset. In the new basis, we introduce the following generators

\[ \hat{K}_{\alpha} = K_{\alpha} - \frac{1}{m^2} P_{\alpha}, \quad \hat{Z}_{\alpha\beta} = Z_{\alpha\beta} - \frac{1}{m^2} Z_{\alpha\beta}, \quad \hat{\tilde{Z}}_{\alpha\beta} = \tilde{Z}_{\alpha\beta} - \frac{1}{m^2} Z_{\alpha\beta} \]

\[ X_{\alpha\beta} = F_{\alpha\beta} + \tilde{F}_{\alpha\beta}, \quad G_{\alpha\beta} = i \left( F_{\alpha\beta} - \tilde{F}_{\alpha\beta} \right), \]

where the coset is rewritten as

\[ g = e^{ix} P e^{i\phi} D e^{i\alpha} A e^{i\mu} X e^{ik} \hat{K} e^{j} \hat{Z}. \]

This particular factorization of the coset is convenient in order to have non-trivial field interactions in the dynamical equations, once the Lagrangian of the theory is constructed by the Cartan forms. Then, the left covariant Cartan forms are extracted from the well known pullback equation

\[ -i g^{-1} \bar{d} g = \omega_P \cdot P + \omega_Z \cdot Z + \omega_D \cdot D + \omega_X \cdot X + \omega_K \cdot \hat{K} + \omega_{\tilde{Z}} \cdot \hat{\tilde{Z}} + \omega_A A + \omega_G \cdot G + \omega_L \cdot L. \]

Notice that:

- Generators $G_{\alpha\beta}$ besides $L_{\alpha\beta}$ ($L_{\alpha\beta}$) are in the stability group (and not only the Lorentz generators, as is in the standard NLR approach applied to gravity).
- It is useful to remark the fact that other combinations of generators of the algebra can form part of the above stability subgroup, but the parameters (fields) related to generators $X_{\alpha\beta}$ have a trivial role in the dynamics and interactions. In this sense, this non-uniqueness of the factorization (parametrization) of the group elements in the coset is a problem that can
only be solved by physical interpretation. Theoretically, this fact goes beyond the non-uniqueness (almost at the classical level) of the Lagrangian functions, and it is deeply associated to the relation between the constraint elimination of the Goldstone fields from the Cartan forms and the algebraic (non dynamical) equation of motion for the respective parameters from the corresponding Lagrangian function.

5. The Cartan Forms

Now we proceed to extract explicitly, from the pullback equation of G defined in the previous Section, the relevant Cartan forms in order to obtain the building blocks by which we can construct the possible geometrical invariants that can play the role of the Lagrangian. They are the following

\[
\omega_D = \frac{1 + \lambda^2}{1 - \frac{\lambda^2}{2}} f d\phi - \frac{d\alpha}{2} \frac{1 + \eta^2}{1 - \eta^2} f - \frac{\epsilon^\phi}{1 - \eta^2} \left( \frac{\Sigma^{\alpha\beta}\xi_{\alpha\beta}}{1 - \frac{\lambda^2}{2}} + \frac{\Sigma^{\alpha\beta}\xi_{\alpha\beta}}{1 - \frac{\lambda^2}{2}} \right)
\]

\[
- \frac{\epsilon^\phi}{1 - \eta^2} \frac{im f \lambda_{\alpha\alpha} W^{\alpha\dot{\alpha}}}{1 - \frac{\lambda^2}{2}},
\]

\[
- i\omega_A = - \frac{1 + \lambda^2}{1 - \frac{\lambda^2}{2}} f d\phi + \frac{d\alpha}{2} \frac{1 + \eta^2}{1 - \eta^2} f - \frac{\epsilon^\phi}{1 - \eta^2} \left( \frac{\Sigma^{\alpha\beta}\xi_{\alpha\beta}}{1 - \frac{\lambda^2}{2}} + \frac{\Sigma^{\alpha\beta}\xi_{\alpha\beta}}{1 - \frac{\lambda^2}{2}} \right)
\]

\[
+ \frac{\epsilon^\phi}{1 - \eta^2} \frac{im f \lambda_{\alpha\alpha} W^{\alpha\alpha}}{1 - \frac{\lambda^2}{2}},
\]

\[
\omega_{\beta\gamma} = \frac{1}{2} \left[ -2 d\phi f - \frac{2 m d\phi}{1 - \frac{\lambda^2}{2}} \lambda^{\gamma\gamma} \right] + \frac{m^2 e^\phi}{1 - \eta^2} \left( \delta^{\gamma\gamma}_{\alpha} + \frac{\lambda_{\alpha\alpha}}{1 - \frac{\lambda^2}{2}} \right) W^{\alpha\alpha} + 2 m^2 e^\phi \frac{d\phi}{1 - \frac{\lambda^2}{2}} \lambda^{\gamma\gamma}
\]

\[
\omega_{\rho\sigma} = \frac{1}{2} \left[ f \frac{d\lambda^{\gamma\gamma}}{1 - \frac{\lambda^2}{2}} - \frac{2 m d\phi}{1 - \frac{\lambda^2}{2}} \lambda^{\gamma\gamma} - \frac{m^2 e^\phi}{2 - \eta^2} \left( \delta^{\gamma\gamma}_{\alpha} + \frac{\lambda_{\alpha\alpha}}{1 - \frac{\lambda^2}{2}} \right) W^{\alpha\alpha} + \frac{2 m^2 d\phi}{1 - \frac{\lambda^2}{2}} \lambda^{\gamma\gamma} \right] - \frac{d\rho \phi}{1 - \eta^2} P^{\gamma\gamma}_{\rho \rho} 1 + \frac{\lambda^2}{2}
\]

\[
= \frac{1}{2} \left[ \frac{d\lambda^{\gamma\gamma}}{1 - \frac{\lambda^2}{2}} - \frac{2 m d\phi}{1 - \frac{\lambda^2}{2}} \lambda^{\gamma\gamma} - \frac{m^2 e^\phi}{1 - \eta^2} \left( \frac{2 m^2 e^\phi}{1 - \eta^2} \left( \delta^{\gamma\gamma}_{\alpha} + \frac{\lambda_{\alpha\alpha}}{1 - \frac{\lambda^2}{2}} \right) W^{\alpha\alpha} - \frac{d\rho \phi}{1 - \eta^2} P^{\gamma\gamma}_{\rho \rho} 1 + \frac{\lambda^2}{2} \right)
\]

\[
(13)
\]
\[ \omega_{Z}^{\alpha \beta} = \frac{1}{2} \left[ \frac{d \phi}{m} - \frac{x^{\rho} \lambda_{\alpha \rho} W^{\alpha \alpha}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} \right] \left[ \frac{\eta^{\alpha \beta} \lambda_{\alpha}^{\beta}}{1 - \xi^{2}} - \frac{\xi^{\alpha \beta} \lambda_{\alpha}^{\beta}}{(1 + \eta^{2}) (1 - \frac{\xi^{2}}{2})} \right] \\
+ 2 \phi \left[ \frac{1 + \eta^{2} \frac{i}{m} \xi^{\alpha \beta}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} + \frac{i}{1 - \eta^{2}} \left( \delta_{\gamma}^{\alpha} \delta_{\delta}^{\beta} + \xi^{\alpha \beta} \xi_{\gamma \delta} \right) \right] \xi^{\gamma \delta}, \] 

(14)

\[ \omega_{G}^{\alpha \beta} = 2m^{2} \omega_{Z}^{\alpha \beta} - \frac{d \eta^{\alpha \beta} \lambda_{\rho}^{\beta}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} + \frac{1 + \xi^{2}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} d \xi^{\alpha \beta}, \] 

(15)

\[ \omega_{X}^{\alpha \beta} = \frac{1}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} \left[ - \frac{d \lambda^{\gamma \gamma}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} + e^{\phi} \left( \frac{-2 \lambda_{\alpha \rho} \lambda_{\sigma}^{\rho} \lambda_{\alpha}^{\sigma} \eta_{\sigma}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} \right) W^{\alpha \alpha} + \frac{2d \phi}{1 - \xi^{2}} \lambda_{\alpha}^{\beta} \right] iP_{\gamma \gamma}^{\rho \rho} \\
+ f \left( \frac{d \eta^{\alpha \beta}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} - \frac{1}{1 - \xi^{2}} \left( \frac{d \xi^{\alpha \rho} \lambda_{\rho}^{\beta}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}} + \frac{d \xi^{\rho} \lambda_{\rho}^{\alpha}}{(1 - \xi^{2})} \right) \right), \] 

(17)

where we define the following important quantities:

\[ f \equiv \frac{1 + \eta^{2}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}}, \quad g \equiv \frac{\xi^{2}}{1 - \eta^{2} \frac{1 + \xi^{2}}{2}}, \] 

(18)

the 1-forms

\[ W^{\alpha \alpha} \equiv (1 + \eta^{2}) d x^{\alpha \alpha} + 2i \left( e^{-2i \alpha} d z^{\beta} \eta_{\sigma}^{\alpha} + e^{2i \alpha} d z^{\sigma} \eta_{\beta}^{\alpha} \right), \] 

(19)

\[ \Sigma^{\alpha \beta} \equiv \left[ \eta_{\alpha}^{\beta} d x^{\alpha \alpha} + ie^{-2i \alpha} (1 + \eta^{2}) d z^{\alpha \beta} \right] \] 

(20)

\[ \Sigma^{\alpha \beta} \equiv \left[ \eta_{\alpha}^{\beta} d x^{\alpha \alpha} - ie^{2i \alpha} (1 + \eta^{2}) d z^{\alpha \beta} \right] \] 

(21)

the associated vectors in the dual space fulfilling the relations

\[ \left< W^{\alpha \alpha}, (W^{-1})_{\alpha \alpha} \right> = \delta_{\beta \beta}^{\alpha \alpha}, \quad \left< \Sigma^{\alpha \beta}, (\Sigma^{-1})_{\rho \sigma} \right> = \delta_{\rho \sigma}^{\alpha \beta}, \] 

(22)

\[ \left< \Sigma^{\alpha \beta}, (\Sigma^{-1})_{\rho \sigma} \right> = \delta_{\rho \sigma}^{\alpha \beta}, \] 

(23)
\[
\langle W^{\alpha \alpha}, (\Sigma^{-1})_{\alpha \beta} \rangle = \langle W^{\beta \alpha}, (\Sigma^{-1})_{\alpha \beta} \rangle = \langle \Sigma^{\alpha \beta}, (\Sigma^{-1})_{\alpha \beta} \rangle = 0, 
\]
\[
(W^{-1})_{\alpha \alpha} = \frac{(1 + \eta^2)}{\Delta} \left[ (1 + \eta^2) \partial_{\alpha \alpha} - 2 \epsilon^{2i \alpha} \left( \eta_\alpha \beta \partial_{\alpha \beta} - \eta^\beta \alpha \partial_{\alpha \beta} \right) \right], 
\]
\[
(\Sigma^{-1})_{\alpha \beta} = -\frac{i \epsilon^{2i \alpha}}{\Delta} \left[ (1 + \eta^2)^2 - 2 \eta^2 \partial_{\alpha \beta} - (1 + \eta^2) \eta_\beta \alpha \partial_{\alpha \beta} - 2 \eta^\beta \alpha \eta_\alpha \beta \partial_{\alpha \beta} \right], 
\]
\[
(\Sigma^{-1})_{\alpha \beta} = \frac{i \epsilon^{-2i \alpha}}{\Delta} \left[ (1 + \eta^2)^2 + 2 \eta^2 \partial_{\alpha \beta} + 2 \eta^\alpha \beta \eta_\beta \alpha \partial_{\alpha \beta} - (1 + \eta^2) \eta^\alpha \beta \partial_{\alpha \beta} \right], 
\]
with
\[
\Delta = (1 + \eta^2)^{10}. 
\]
These one forms and the corresponding (dual) vectors are directly related with the gauge covariant basis in which the metric splits in several blocks corresponding to the Manifold subgroups. These subgroups represent (as we will see later) the space-time and the additional symmetries. We have also the following projector
\[
P^{\rho \gamma}_{\rho \rho} = \frac{\xi^\rho \delta^\gamma_{\rho}}{1 - \frac{\xi^2}{2}} + \frac{\xi^\gamma \delta^\rho_{\rho}}{1 - \frac{\xi^2}{2}}. 
\]
The standard transformations on the parameters in order to convert hyperbolic and trigonometrical functions in polynomial ones are
\[
\lambda^{\rho \sigma} = \frac{\tanh \sqrt{\frac{b^2}{2m^2}}}{\sqrt{\frac{b^2}{2m^2}}} \left( \frac{k^{\rho \sigma}}{m} \right), \quad \eta^{\rho \sigma} = \frac{\tanh \sqrt{\mu^2}}{\sqrt{\mu^2}} \mu^{\rho \sigma}, 
\]
\[
\xi^{\rho \sigma} = \frac{\tanh \sqrt{\frac{c^2}{m^2}}}{\sqrt{\frac{c^2}{m^2}}} \left( \frac{t^{\rho \sigma}}{m} \right), \quad \bar{\xi}^{\rho \sigma} = \frac{\tanh \sqrt{\frac{\bar{c}^2}{m^2}}}{\sqrt{\frac{\bar{c}^2}{m^2}}} \left( \frac{\bar{t}^{\rho \sigma}}{m} \right), 
\]
simplifying considerably the theoretical analysis.

6. Inverse Higgs constraints
In order to eliminate $\lambda^{\rho \sigma}$, $\xi^{\rho \sigma}$ in a covariant suitable form, we have to take into account a particular simplification. Such a simplification is based on the separability condition coming from the same structure of the $Sp(8)$ group: that is, if we impose
\[ \alpha = \alpha (\xi_{\alpha\beta}) \text{ and } \phi = \phi (\lambda_{\alpha\dot{\alpha}}), \]

the consistent inverse Higgs constraint \( \omega_D = 0 \) leads to the simple equations:

\[
d\phi = \frac{e^\phi}{1 - \eta^2} \left( \frac{m \lambda_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}}}{1 + \frac{\xi^2}{2}} \right),
\]

and

\[
-d\alpha = \frac{me^\phi}{1 + \eta^2} \left( \frac{\Sigma_{\alpha\beta} \xi_{\alpha\beta}}{1 + \frac{\xi^2}{2}} \right),
\]

where the underlying role played by the axion and dilaton with respect to the group manifold under consideration is quite evident. Then, taking into account the above conditions due to the inverse Higgs constraint, both \( \lambda_{\rho\sigma} \) and \( \xi_{\alpha\beta} \) can be easily separately eliminated. It is not so simple in the general case where \( \phi \) and \( \alpha \) depend both on \( \lambda_{\alpha\dot{\alpha}} \) and \( \xi_{\alpha\beta} \). For \( \lambda_{\rho\sigma} \), we have

\[
\overset{\equiv \chi_1}{\frac{me^\phi}{(1 - \eta^2) (1 + \frac{\lambda^2}{2})}} = (W^{-1})_{\alpha\dot{\alpha}} \phi, \quad \Rightarrow \quad \lambda_{\alpha\dot{\alpha}} = \frac{(W^{-1})_{\alpha\dot{\alpha}} \phi}{\chi_1 \left( 1 + \sqrt{1 - \frac{2(W^{-1})_{\alpha\dot{\alpha}} \phi(W^{-1})_{\alpha\dot{\alpha}} \phi}{\chi_1^2}} \right)}.
\]

Similarly, for \( \xi_{\alpha\beta} (\xi_{\alpha\beta}) \), one has

\[
\overset{\equiv \chi_2}{- \frac{d\alpha}{2}} = \frac{me^\phi}{1 - \eta^2} \left( \frac{\Sigma_{\alpha\beta} \xi_{\alpha\beta}}{1 + \frac{\xi^2}{2}} \right), \quad \Rightarrow \quad \xi_{\alpha\beta} = \frac{(\Sigma^{-1})_{\alpha\beta} \alpha}{\chi_2 \left( 1 + \sqrt{1 - \frac{2(\Sigma^{-1})_{\alpha\beta} \alpha(\Sigma^{-1})_{\alpha\beta} \alpha}{\chi_2^2}} \right)}.
\]

However, \( \xi_{\alpha\dot{\alpha}} \) is eliminated in an analog manner. On the surface of this covariant constraint, the remaining coset Cartan forms are given by the expressions

\[
\omega_A \big|_{\omega_D=0} = \frac{me^\phi}{1 - \eta^2} \left( \frac{\Sigma_{\alpha\beta} \xi_{\alpha\beta}}{1 - \frac{\xi^2}{2}} + \frac{\Sigma_{\alpha\beta} \xi_{\alpha\beta}}{1 - \frac{\lambda^2}{2}} \right),
\]

\[
\omega_\rho^\beta \big|_{\omega_D=0} = \chi \left( \delta^\rho_\alpha \delta^\beta_\dot{\alpha} - \frac{\lambda_{\alpha\dot{\alpha}} \lambda_{\rho\dot{\sigma}}}{1 + \frac{\lambda^2}{2}} \right) W^{\alpha\dot{\alpha}} = E_\rho^\alpha W^{\alpha\dot{\alpha}} = e^\phi \bar{E}_\rho^\alpha W^{\alpha\dot{\alpha}},
\]
The affine structure of gravitational theories

\[ \omega_{\rho}^{\nu} = \left. \frac{f d\lambda^\rho}{1 - \frac{\lambda^2}{2}} - \frac{B \chi m}{2 (1 - \frac{\lambda^2}{2})} \left( \delta_\alpha^{\beta} \delta^{\mu} - \frac{\lambda \alpha \lambda^\rho}{1 + \frac{\lambda^2}{2}} \right) W_\alpha^{\mu} + \frac{d\eta_{\gamma} P_{\gamma}^{\rho}}{1 - \eta^2} \right|_{\omega_D = 0} = \frac{1}{1 - \frac{\lambda^2}{2}} \left( \frac{f d\lambda^\rho}{1 - \frac{\lambda^2}{2}} - \frac{\lambda^2 + \xi^2 + \xi^2}{1 - \left( \lambda^2 + \xi^2 + \xi^2 \right)} \omega_{\rho}^{\nu} \right) + \frac{d\eta_{\gamma} P_{\gamma}^{\rho}}{1 - \eta^2}, \quad (32) \]

\[ \omega_{Z}^{\alpha\beta} \left|_{\omega_D = 0} = \frac{1}{1 - \eta^2} \left( \delta_\gamma^{\alpha} \delta_\delta^{\beta} - \frac{\xi \alpha \xi \delta}{1 + \frac{\lambda^2}{2}} \right) \Sigma_{\gamma\delta} \right., \quad (33) \]

\[ \omega_{Z}^{\alpha\beta} \left|_{\omega_D = 0} = \frac{1 + \frac{\lambda^2}{2}}{1 - \frac{\lambda^2}{2}} \frac{d\xi^{\alpha\beta}}{1 - \eta^2} - \frac{i}{1 - \eta^2} \frac{d\eta^{\alpha\rho}}{1 - \frac{\lambda^2}{2}} + 2 m^2 \omega_{Z}^{\alpha\beta} \right., \quad (34) \]

\[ \omega_{\gamma}^{\alpha\beta} \left|_{\omega_D = 0} = i m \omega_{\gamma}^{\alpha\beta} P_{\gamma}^{\rho} - i f \frac{\eta_{\gamma}}{1 + \eta^2} \left( \frac{1 + \frac{\lambda^2}{2}}{1 - \frac{\lambda^2}{2}} \right) \omega_{A} \right., \quad (35) \]

\[ \omega_{X}^{\alpha\beta} \left|_{\omega_D = 0} = - \frac{i P_{\gamma}^{\rho}}{(1 - \frac{\lambda^2}{2})} \left[ \frac{d\lambda^\gamma}{1 - \eta^2} - \lambda^2 m \omega_{\gamma}^{\alpha} \right] + f \frac{dP_{\gamma}^{\rho}}{1 - \eta^2}, \quad (36) \]

where we have defined

\[ B = \left( -2 + f \frac{1 + \lambda^2/2}{1 - \lambda^2/2} \right). \quad (37) \]

Notice that the geometrical role of \( \eta^{\rho\nu} \), namely, as it enter in the conformal Cartan forms \( \omega_{\rho}^{\nu} \left|_{\omega_D = 0} \right. \) and \( \omega_{Z}^{\alpha\beta} \left|_{\omega_D = 0} \right. \) and the action as projector-connection in \( W_{\alpha}^{\nu} \) and \( \Sigma_{\alpha\beta} \), are intrinsically related with its role into the gauge potentials as we will show in detail.

7. The G-Manifold, spacetime and affine structure

We have seen that, after the breaking of the conformal symmetry \( (\omega_D = 0) \), the original group manifold plays the role of a 10-dimensional space-time, spanned by \( W, \Sigma \) and \( \Sigma \). Once the Higgs (inverse-constraint) mechanism is established, the 10-dimensional space-time (after a suitable choice of the Lagrangian) is transformed in the arena where \( \eta, \alpha, \) and \( \phi \) are the dynamical fields of the theory. Notice that when \( \xi^{\alpha\sigma} \) and \( \eta_{\rho, a} \) are null, the well-known pure conformal case is recovered.

Previously, we have shown that there exist a basis where the structure of the Cartan forms (derived by the pullback process in the nonlinear realization of CE
$Sp(8)$ is clearly seen. The basis is given by the 1-forms

$$W^\alpha = (1 + \eta^2) dx^\alpha + 2i \left( e^{-2i\alpha} dz^\sigma \eta_\sigma^\alpha + e^{2i\alpha} d\bar{z}^\sigma \bar{\eta}_\sigma^\alpha \right),$$

$$\Sigma^\alpha \beta = \left[ \eta_\beta^\alpha dx^\beta \bar{\eta}_\beta^\alpha + i e^{-2i\alpha} (1 + \eta^2) d\bar{z}^\beta \right],$$

$$\Sigma^\alpha \beta = \left[ \eta_\alpha^\beta dx^\alpha \bar{\eta}_\alpha^\beta + i e^{+2i\alpha} (1 + \eta^2) d\bar{z}^\beta \right].$$

(38)

The associated vectors in the dual space

$$(W^{-1})_{\alpha \beta} = \frac{(1 + \eta^2)}{\Delta} \left[ (1 + \eta^2) \partial_{\alpha \beta} - 2e^{2i\alpha} \left( \eta_\alpha^\beta \partial_{\beta} - \eta_\beta^\alpha \partial_{\alpha} \right) \right],$$

$$(\Sigma^{-1})_{\alpha \beta} = \frac{-ie^{2i\alpha}}{\Delta} \left[ \left( (1 + \eta^2)^2 - 2\eta^2 \right) \partial_{\alpha \beta} - (1 + \eta^2) \eta_\beta^\alpha \partial_{\alpha \beta} - 2\eta_\alpha^\beta \eta_\alpha^\beta \partial_{\alpha \beta} \right],$$

$$(\Sigma^{-1})_{\alpha \beta} = \frac{i e^{-2i\alpha}}{\Delta} \left[ \left( (1 + \eta^2)^2 + 2\eta^2 \right) \partial_{\alpha \beta} + 2\eta_\alpha^\beta \eta_\beta^\alpha \partial_{\alpha \beta} - 2\eta_\alpha^\beta \eta_\alpha^\beta \partial_{\alpha \beta} \right],$$

(39)

with

$$\Delta = (1 + \eta^2)^{10}.$$

These bases of vectors and forms are extremely important: they are the gauge covariant bases as pointed out by Mansouri et al. [10]. In this representation, the metric tensor of the manifold is block diagonal. In the standard gauging procedure [10], the diagonal form and the basis were given, by definition, in a sharp contrast with the NLR procedure. To show clearly this statement, we give below the line element describing the 10-dimensional spacetime manifold, that is

$$ds^2 = F_1(x, z, \bar{z})W^2 + F_2(x, z, \bar{z})\Sigma^2 + F_3(x, z, \bar{z})\bar{\Sigma}^2,$$

(40)
The affine structure of gravitational theories

where

$$\left(W^{\alpha\beta}\right)^2 = (1 + \eta^2)^2 \left[dx^{\alpha\beta} + N^{\alpha\beta}_{\sigma\gamma} dz^{\sigma\gamma} + N^{\alpha\beta}_{\sigma\gamma} d\zeta^{\sigma\gamma}\right] g^{\alpha\beta}_{\alpha\beta}$$

\begin{align*}
&\times \left[dx^{\alpha\beta} + N^{\alpha\beta}_{\rho\nu} dz^{\rho\nu} + N^{\alpha\beta}_{\rho\nu} d\zeta^{\rho\nu}\right], \\
(\Sigma^{\alpha\beta})^2 &= (1 + \eta^2)^2 \left\{ie^{-2i\alpha} \left[-\frac{1}{2} N^{\alpha\beta}_{\rho\alpha} dx^{\rho\alpha} + d\zeta^{\rho\alpha}\right]\right\} g(\alpha\beta;\gamma\delta), \\
&\times \left\{ie^{-2i\alpha} \left[-\frac{1}{2} N^{\delta\gamma}_{\alpha\lambda} dx^{\alpha\lambda} + d\zeta^{\delta\gamma}\right]\right\}, \\
(\Sigma^{\alpha\beta})^2 &= (1 + \eta^2)^2 \left\{ie^{2i\alpha} \left[-\frac{1}{2} N^{\alpha\beta}_{\rho\alpha} dx^{\rho\alpha} - d\zeta^{\rho\alpha}\right]\right\} g(\alpha\beta;\gamma\delta), \\
&\times \left\{ie^{2i\alpha} \left[-\frac{1}{2} N^{\delta\gamma}_{\alpha\lambda} dx^{\alpha\lambda} - d\zeta^{\delta\gamma}\right]\right\}
\end{align*}

with

$$N^{\alpha\beta}_{\sigma\gamma} = \frac{2ie^{-2i\alpha} \eta^{\beta\alpha} \delta^{\gamma}_{\sigma}}{(1 + \eta^2)}, \quad N^{\alpha\beta}_{\sigma\gamma} = \frac{2ie^{2i\alpha} \eta^{\alpha\beta} \delta^{\gamma}_{\sigma}}{(1 + \eta^2)},$$

$$N^{\alpha\beta}_{\rho\alpha} = \frac{2ie^{2i\alpha} \eta^{\alpha\beta} \delta^{\gamma}_{\rho}}{(1 + \eta^2)}, \quad N^{\alpha\beta}_{\rho\alpha} = \frac{2ie^{-2i\alpha} \eta^{\beta\alpha} \delta^{\gamma}_{\rho}}{(1 + \eta^2)},$$

\begin{align*}
F_1(x, z, \tau) &= 1 - 2\chi^{-2} (W^{-1} \phi)^2, \\
F_2(x, z, \tau) &= 1 - 2\chi_2^{-2} (\Sigma^{-1} \alpha)^2, \\
\overline{F}_2(x, z, \tau) &= 1 - 2\chi_2^{-2} (\Sigma^{-1} \alpha)^2.
\end{align*}

In general $g^{\alpha\beta}_{\alpha\beta}$ are proportional to $\epsilon^{\alpha\beta}_{\alpha\beta}$ and $\epsilon^{\alpha\beta}_{\alpha\beta}$ (e.g. in Cartan-Killing metric, the conjugation operation is assumed in order to be consistent with the scalar product). The metric is evidently diagonal in $W, \Sigma, \overline{\Sigma}$ and generalize the Nordstrom-Kaluza construction viewed from the coordinate basis. In general, the simplest basis in the base manifold is the coordinate one and this is that is used as standard. For the bundle viewpoint, it is not convenient to work in a coordinate basis because the description of isometries in such manifolds can best be carried out in terms of an invariant basis (giving commutation relations with the Killing vectors).

It is quite evident that the field $\eta^{\alpha}_{\alpha}$ plays a central role into $N^{\alpha\beta}_{\sigma\gamma}$, the gauge potential of the fiber. In practical cases, consistently with the algebraic role of $\eta^{\alpha}_{\alpha}$, we have the freedom to select a particular forms for the gauge potentials $N^{\alpha\beta}_{\sigma\gamma}$ as the fundamental physical dynamical variables under consideration. It is also important to consider that, in general, it is quite obvious that all the geometrical quantities of interest can be splitted but weakly and without a complete reduction. It is worth noticing that supersymmetric cases are more involved in this sense.
Our task is now to define a self-consistent geometrical Lagrangian considering the above results. As stated, the field $\eta^{\alpha}_{\sigma}$ plays a central role into the gauge potential of the fiber $N^A_B$ defined now in the variables $A$, see Sect. 3. This means that the geometrical Lagrangian can be simply achieved as the wedge product of forms. One can consider the standard Einstein-Hilbert Lagrangian, linear in the Ricci scalar $R$, $f(R)$ Lagrangians or quadratic combinations of curvature invariants. For example

$$\sqrt{-G}; \sqrt{-G^\alpha} (\alpha \in \mathbb{R}); \sqrt{-G^{AB}R_{AB}}; \text{ etc.} \quad (48)$$

Furthermore, Eddington-like Lagrangians can be assumed,

$$\sqrt{\text{Det}(G_{AB} + f(R_{AB}))}; \text{ etc.} \quad (49)$$

This means that any invariant of the Riemann tensor must be into the action: if it is not the case, there are no dynamical equations for $N^A_B$ then, for $\eta^{\alpha}_{\sigma}$. In fact, although the measure $\sqrt{-G}$ contains the simplest geometrical invariant, it cannot be, from the dynamical point of view, the geometrical Lagrangian. However, it can be used as a preliminary geometrical object to be analyzed. In the simplest form, it can be defined as the wedge product of the Cartan forms

$$\omega^\rho_{\rho 1...4} \wedge ... \omega^\sigma_{\sigma 1...4} \wedge \omega^\alpha_{\alpha 5...10} \wedge ... \omega^\beta_{\beta 5...10} =$$

$$= \left( \frac{m\phi}{1 - \eta^2} \right)^{10} \left( \frac{1 - \chi^2}{2} \right)^{1 - \frac{\chi}{2}} \left( 1 - \frac{\chi}{2} \right) \left( 1 - \frac{\chi}{2} \right) \left( 1 + \frac{\chi}{2} \right) \left( 1 + \frac{\chi}{2} \right) \left( 1 - \lambda^2/2 \right) \left( 1 + \lambda^2/2 \right)$$

This wedge product can be written as a function of $\eta$ (and the scalar and pseudo-scalar $\phi$ and $\alpha$ respectively). To this end, we write the factors $b$ and $c$ of the above equation explicitly (for $a$ the dependence is clear) defining firstly

$$\left[ (W^{-1})_{\alpha \alpha} \phi \right] \left[ (W^{-1})^\alpha_{\alpha} \phi \right] \equiv (W^{-1}\phi)^2,$$

$$\left[ (\Sigma^{-1})_{\alpha \beta} \alpha \right] \left[ (\Sigma^{-1})^{\alpha \beta}_{\alpha \beta} \alpha \right] \equiv (\Sigma^{-1}\alpha)^2, \quad (50)$$

then

$$c = \sqrt{1 - 2\chi^{-2} (W^{-1}\phi)^2},$$

$$a = \sqrt{1 - 2\chi_2^{-2} (\Sigma^{-1}\alpha)^2},$$

$$b = \sqrt{1 - 2\chi_2^{-2} (\Sigma^{-1}\alpha)^2}^2. \quad (51)$$
for instance, and due the simple assumption $\xi_{\alpha\beta} = \xi_{\alpha\beta}(\alpha)$ and $\lambda_{\alpha\alpha} = \lambda_{\alpha\alpha}(\phi)$, the geometrical Lagrangian takes the form

$$L_G = \left(\frac{e^\phi (1 + \eta^2)}{1 - \eta^2}\right)^{10} \sqrt{\left[1 - 2\chi_1^{-2}(\Sigma^{-1}\alpha)^2\right]\left[1 - 2\chi_2^{-2}(\Sigma^{-1}\alpha)^2\right]\left[1 - 2\chi_3^{-2}(W^{-1}\phi)^2\right]}.$$

where the dynamics of the axion, dilaton and other fields are governed by a Born-Infeld-like action. It is evident the interplay of gravity and fields dynamics. The case is similar to the picture given by the Weyl theory where the scalar field is associated to some part of the connection. Finally, we can state that any suitable geometrical Lagrangian, constructed by suitable combination of curvature invariants in such an approach, can contain, in principle, the dynamics of gravitational field and all the information about matter fields and their interactions.

9. Conclusions and Perspectives

In this paper a nonlinearly realized representation of the local conformal-affine group has been determined. Gravity and spin are the results of such a realization. It has been found that the nonlinear Lorentz transformation law contains contributions from the linear Lorentz parameters as well as conformal and shear contributions via the nonlinear 4-boosts and symmetric $GL(4)$ parameters. We have identified the pullback of the nonlinear translational connection coefficient to a manifold $M$ as a space-time coframe. In this way, the frame fields of the theory are obtained from the (nonlinear) gauge prescription. The mixed index coframe components (tetrads) are used to convert from the Lie algebra indices to space-time indices. In this picture, the space-time metric is a secondary object constructed (induced!) from the constant $H$ group metric and the tetrads.

The problem of consistency for the determination of the form of geometrical Lagrangian (that is one of the main drawbacks of the nonlinear realization approach) is here avoided due to the exact identification of the structure of the fiber (gauge potentials). The Lagrangian must generate the corresponding dynamical equations of motion for these gauge potentials. This fact have to be pointed out because such an identification, in our knowledge, has never been achieved in a clear form.

As a concluding remark, we can say that gravity (and in general any gauge field) can be derived as the nonlinear realization of a local conformal-affine symmetry group and then gravity can be considered an interaction induced from invariance properties. In a forthcoming paper, we will study in detail the geometrical Lagrangian and the related field equations in order to point out the physical properties of such an approach.

Acknowledgements

SC and MDL acknowledge the support of INFN Sez. di Napoli (Iniziative Specifiche TEONGRAV and QGSKY). D. J. C-L. thanks JINR-BLTP (Russia) for hospitality and financial support.
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21

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