On the Universality and Extremality of Graphs with a Distance Constrained Colouring

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Abstract. A lambda colouring (or $L(2, 1)$-colouring) of a graph is an assignment of non-negative integers (with minimum assignment 0) to its vertices such that the adjacent vertices must receive integers at least two apart and the vertices at distance two must receive distinct integers. The lambda chromatic number (or the $\lambda$ number) of a graph $G$ is the least non-negative integer among all the maximum assigned non-negative integer over all possible lambda colouring of the graph $G$. Here we have shown that every edge maximum graph with lambda chromatic number $t \geq 5$, admits an equitable $L(2, 1)$ colouring. Further, we have established a classification result, identifying all possible $n$-vertex graphs with lambda chromatic number $t \geq 3$, where $n \geq t + 1$, which contain the maximum number of edges. Such classification provides a complete solution to a problem posed more than two decades ago in 1996 by John P. Georges and David W. Mauro.

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1. Introduction

The channel assignment problem is the task of assigning frequencies to radio transmitters of a communication network. In this problem, there is a trade-off between deploying the minimum number of frequencies (or channels) and avoiding interference due to the proximity of transmitters. This distance restriction necessitates a separation of frequencies among nearby transmitters to mitigate the interference. The usable spectrum of frequencies is a scarce and a costly resource. For this reason, an efficient assignment of frequencies is
desirable. The frequency assignment to the transmitters, constrained by distance related parameters, can be mapped to varieties of \textit{distance constrained colouring} problems of a graph. Hale [9] modelled these problems as several generalised versions of graph colouring problem. In one of such models, the transmitters are considered as vertices and edges correspond to the unordered pairs of interfering transmitters. The assignment of frequencies (represented by non-negative integers) is done in such a manner that “close” transmitters (i.e. vertices at distance two) are assigned different frequencies and “very close” transmitters (i.e. adjacent vertices) are assigned frequencies in a difference of at least two. This channel assignment problem is translated to a colouring (or labelling) problem of graphs. Griggs and Yeh [8] had referred to this colouring as the non-negative integer \(\min V_c\) problem as \(L(2,1)\)-colouring problem of graphs. We refer to it as the lambda colouring problem of graphs. From the complexity point of view, the lambda colouring problem of graphs is an \(NP\)-hard problem [9, Theorem 57]. Survey articles on this well-investigated problem can be found in [1,16,18].

Throughout this article, the set of all non-negative integers is denoted by \(\mathbb{N}\). All the graphs are simple and their vertex sets are non-empty and finite. For a graph \(G\), the vertex set and the edge set are denoted by \(V(G)\) and \(E(G)\) respectively. The subset \(N_u = \{v \in V(G) : \{u, v\} \in E(G)\}\) of the vertex set of a graph \(G\) is called the neighbour set or simply the neighbour of the vertex \(u\). A (vertex) colouring of a graph \(G\) is a mapping \(c : V(G) \rightarrow \mathbb{N}\). The lambda colouring of a graph \(G\) is a mapping \(c : V(G) \rightarrow \mathbb{N}\) such that for each \(u, v \in V(G)\), \(|c(u) - c(v)| + d_G(u, v) \geq 3\). Here \(d_G(u, v)\) denotes the distance between vertices \(u\) and \(v\), i.e. the minimum number of edges connecting the vertices \(u\) and \(v\) through a path. If there is no path connecting the vertices \(u\) and \(v\), then we may put \(d_G(u, v) = \infty\). By well ordering property of \(\mathbb{N}\), the range of the mapping \(c\) attains \(\min_{u \in V(G)} c(u)\). Therefore, without loss of generality, we assume \(\min_{u \in V(G)} c(u) = 0\). The lambda chromatic number of the graph \(G\) is the non-negative integer \(\min \{\max_{u \in V(G)} c(u) : c\) is a lambda colouring of \(G\}\). A lambda colouring \(c\) of \(G\) is said to be \textit{optimal} if \(\max_{u \in V(G)} c(u)\) equals to the lambda chromatic number. The \textit{coloured partition} of the vertex set \(V(G)\) with respect to a lambda colouring \(c\) is \(C_0, C_1, \ldots, C_m, \ldots, C_T\), where \(C_m := \{u \in V(G) : c(u) = m\}\) and \(T = \max_{u \in V(G)} c(u)\). Such coloured partition is said to be \textit{equitable} if for all integers \(i\) and \(j\) with \(0 \leq i, j \leq T\), \(|C_i| - |C_j| \leq 1\). With respect to a lambda colouring \(c\) of the graph \(G\), if the coloured partition of \(V(G)\) is an equitable partition, then we call such lambda colouring is \textit{equitable}. Fu and Xie [4] have studied equitable lambda colouring for Sierpiński graphs. A positive integer \(h\) is said to be a \textit{hole} of the lambda colouring \(c\), if for each vertex \(u\), \(c(u) \neq h\) but there exist at least one vertex \(v\) such that \(h < c(v)\). Note that a hole corresponds to an empty colour class. (Caution: The lambda colouring may not be an onto mapping, contrary to usual colouring of graphs.)

Fishburn and Roberts [2,3] extensively studied the possible graphs admitting at least one optimal but onto lambda colouring. Such graphs are known as \textit{full colourable} graphs. However, in [12], Lu et al. concentrated specifically on onto lambda colourings of any graph irrespective of being full colourable. They
studied the associated optimal value in terms of its bounds. In fact, they had obtained their results in the case where these bounds are attained. These results were expressed in terms of the number of edges, diameters and number of connected components. On the other hand, the extremal nature of the graphs, from the viewpoint of inter-relation between lambda chromatic number and the minimum number of holes, was studied in [13] by Lu and Zhai.

In [7], Gonçalves has shown lambda chromatic number of any graph cannot be more than $\Delta^2 + \Delta - 2$, $\Delta$ being the maximum degree of the graph. So far, the best known upper bound of the lambda chromatic number of an arbitrary graph is due to Havet et al. [10]. There, authors proved that for the graphs with sufficiently large maximum degree $\Delta$, the lambda chromatic number cannot be more than $\Delta^2$. It settles the Griggs–Yeh conjecture (which states that lambda chromatic number of a graph is at most $\Delta^2$) for the graphs with sufficiently large $\Delta$. This clearly implies that the lambda chromatic number of any graph cannot be more than $\Delta^2 + C$, for some constant $C$.

Among recent developments in the study of lambda colouring, Junosza-Szaniawski et al. [11] provided a general framework for designing online algorithms for lambda coloring of unit disk intersection graphs with better competitive ratio. Chybowski-Sokół et al. [17] have shown that $\frac{4}{9}\Delta^2 + 25\Delta + 20$ is an upper bound for disk graphs. Moreover they settled the Griggs–Yeh conjecture for disk graphs with $\Delta \geq 126$ and for unit disk graphs with $\Delta \geq 11$. In [15], Mandal and Panigrahi studied lambda colouring for direct product of two graphs with emphasis on the product of a complete graph and a path; a path and a cycle as well as a cycle and a path. In [14], the authors have studied the lambda chromatic number of power graph of some finite groups.

**An Open Problem of Extremal Graph Theory**

In the study of lambda colouring, one of the most fundamental problem lying open for more than two decades, is the basic extremal question of characterizing graphs with maximum number of edges, when the number of vertices and the lambda chromatic number are given. Local restrictions and less independence in edge distribution distinguish this problem from other varieties of extremal issues associated with a graph colouring problem.

Let $G$ be an $n$-vertex graph with lambda chromatic number $t$. In 1996, Georges and Mauro in their seminal paper [6] obtained formulae for the minimum and, with an algorithmic approach, the maximum numbers of edges in an $n$-vertex graph with lambda chromatic number $t$. Also they characterized all such graphs with minimum edges. Though they were able to describe the graphs with maximum number of edges when $n \leq t + 1$, the question for $n > t + 1$ was posed open [6, page 55]. No progress is known so far towards solving this problem, even partially, up to the best of our knowledge.
Our Contribution

In this article, we give a complete answer to the above extremal question. We concentrated mainly on the edge distribution of an $n$-vertex graph (i.e. a graph with $n$ many vertices) with lambda chromatic number $t$. In fact, this article has a two-way orientation, namely universality and extremality of family of graphs with lambda chromatic number $t$.

**Question.** Our main focus is to answer the following two questions. Finding the answer to Question (b) is the most prevailing part of this article. Answer to Question (a) is interesting on its own and it is also an essential component in the investigation of Question (b).

(a) Can we find a graph $\Omega$ with lambda chromatic number $t$ such that any graph $G$ with lambda chromatic number $t$ is a subgraph of $\Omega$?

(b) By means of explicit constructions, can we classify all the $n$-vertex graphs, with lambda chromatic number $t$, which contain the maximum number of edges?

Regarding the Question (b), for $n \leq t + 1$ we refer to [6]. However, a reader may realise that the answer of this case is incurred within Proposition 3.3. Here we focus mainly for $n \geq t + 1$.

A Brief Discussion on Our Approach

To answer the first question, we have constructed a family (set) of graphs with lambda chromatic number $t$. We call it $G(t, l)$, where $l$ is a positive integer (see Construction 2.2). We have concluded (in Theorem 2.7), that any graph $G$ with lambda chromatic number $t$, is a subgraph of a member graph of $G(t, l)$ for some positive integer $l$.

The basic idea used to answer the second question is analogous to finding the global maximum value of a smooth function over a compact domain. We first obtain the stationary points, where the local maximum values are attained. Further analysis rules out the stationary points not leading to the global maximum value of the function. It also ensures that the remaining stationary points of the function achieve the global maximum value. We translate this idea into our work by dynamically altering a partition of the vertex set and the edge distribution of a graph.

A lambda colouring, in essence, induces a coloured partition of the vertex set of a graph with some local restrictions. Due to distance constrained conditions of the lambda colouring, between any two vertices of a colour class, there cannot exist any path of length at most two. In addition, there is no edge between two consecutively assigned colour classes. Such locally forbidden structures reduce the independence (i.e. increase the dependence) for distributing edges between different colour classes. From this point of view, we believe that our approach has far-reaching consequences. In fact, our method has been meticulously handy for solving this extremal graph colouring problem which is typically riddled with local restrictions and having less independence in
edge distribution. Many such problems can be modelled from the $\mathcal{NP}$-hard frequency assignment problems proposed in [9].

Let $u$, $v$ and $w$ be three vertices of a graph $G$ with lambda chromatic number $t \geq 3$. With respect to an optimal colouring $c : V(G) \to \{0, 1, \ldots, t\}$, suppose $c(u) + 2 \leq c(v)$ and $c(v) + 2 \leq c(w)$. Also suppose $\{u, v\}$ and $\{v, w\}$ are edges of $G$. Note that there is no certainty that $\{u, w\}$ is an edge of $G$. Instead, there may exist other vertices $u'$ and $w'$ with $c(u) = c(u')$ and $c(w) = c(w')$ such that $\{u, w'\}$ and $\{u', w\}$ are edges of $G$. We need to deal with such unpredictability of edge distribution among the different colour classes along a path.

To address this issue, we have developed a technique called the edge standardisation technique. This technique transforms graphs into ones with deliberately altered edge distribution such that $u$ becomes identified with $u'$ and $w$ with $w'$. Moreover, the transformed $n$-vertex graph becomes a subgraph of a member graph of $G(t, l)$ for some integer $l$ (depends on $n$). The technique keeps the vertex set, the coloured partition and the lambda chromatic number invariant (Proposition 3.4). Thus the edge standardisation technique cleans up uncontrolled situations as well as unnecessary conditions, without changing the graph parameters into consideration. Most importantly, this technique ensures that the transformed graphs keep the same number of edges (under a process of deletion and insertion of vertices, explained later) before hitting certain necessary conditions. These necessary conditions will act as stationary conditions. Such stationary conditions narrow down our search for an $n$-vertex graph $G$ with lambda chromatic number $t$ and having maximum number of edges (see Propositions 3.11, 3.12, 3.13 and Sect. 4). At this point, we rule out those graphs not containing the maximum number of edges through further analysis. As a consequence, we establish the classification results (Theorem 4.9 and Corollary 4.10) which solve Question (b). With this classification, an open problem is solved. This problem was posed more than two decades ago in 1996 [6, Page 55, Last Paragraph].

2. Two Examples and Their Universal Properties

If $G$ is a graph with lambda chromatic number 2, then $G$ is a disjoint union of some edges. But if $G$ is a graph with lambda chromatic number $t \geq 3$, then the problems (described in Question (a) and (b)) become non-trivial. Here we begin this section with (a) a sequence of graphs $\{G_n\}_{n=3}^{\infty}$ and (b) a doubly sequence of family (set) of graphs $\{G(t, l) : t \geq 3, l \geq 1\}$ and study their lambda chromatic number and other properties.

Construction 2.1. Let $\{G_n : n \in \mathbb{N}, n \geq 3\}$ be a sequence of simple graphs defined via a recursive rule as follows: $G_3$ be the graph with vertex set $\{v_0, v_1, v_2, v_3\}$ and edge set $\{(v_0, v_2), (v_0, v_3), (v_1, v_3)\}$. For $n \geq 4$, the graph $G_n$ has vertex set $\{v_i : 0 \leq i \leq n\}$. The edges of $G_n$ are all the edges of $G_{n-1}$ and the edge of the form $\{v_i, v_n\}$, where $i$ is an integer with $0 \leq i \leq n - 2$. In total, for each integer $t \geq 3$, $G_t$ has exactly $(t + 1)$ many vertices and $\binom{t}{2}$ many edges.
It will be shown later that each graph can be modified through edge standardisation into a disjoint union of some $G_t$'s with or without some deleted vertices.

Construction 2.2. Let $t \geq 3$ be an integer and $V_i$, where $i$ is an integer with $0 \leq i \leq t$, be mutually disjoint sets of size $l$. Let $G(t,l)$ be the family of graphs with vertex set $\bigcup_{i=0}^{t} V_i$. Each graph $G \in G(t,l)$ satisfies the following two properties.

- If $x \in V_m$, then there exists a unique $y \in V_p$ such that $\{x,y\}$ is an edge of $G$, where $m$ and $p$ are integers with $0 \leq m \leq t$, $0 \leq p \leq t$ and $|m-p| \geq 2$.
- Whenever $u, v \in V_m$, where $0 \leq m \leq t$, $\{u, v\}$ is not an edge of $G$.

In total, each $G \in G(t,l)$ has exactly $(t+1)l$ many vertices and $(\binom{t}{2})l$ many edges.

There is a link between the above two constructions. Precisely for each $n \geq 3$, the graph $G_n$ is the only member of $G(n,1)$. In the following result, we obtain an optimal lambda colouring of $G_n$, where $n \geq 3$, given by $v_i \mapsto i$ from $V(G_n) = \{v_i : 0 \leq i \leq n\}$ to $\mathbb{N}$. We found that such optimal colouring is an onto mapping. This implies that under such colouring of $G_n$ there is no hole, for each $n \geq 3$.

Theorem 2.3. For each $n \geq 3$, the mapping $v_i \mapsto i$, from $V(G_n)$ to $\mathbb{N}$, is a lambda colouring and the lambda chromatic number of $G_n$ is $n$.

Proof. Suppose the result is true for $n = m \geq 3$. Therefore, $\bar{c} : V(G_m) \to \{0, 1, \ldots, m\}$ defined by $\bar{c}(v_i) = i$ is a lambda colouring. Now the mapping $c : V(G_{m+1}) \to \mathbb{N}$ defined by

$$c(v_i) = \begin{cases} 
\bar{c}(v_i) & \text{if } 0 \leq i \leq m \\
 m+1 & \text{if } i = m+1
\end{cases}$$

is a colouring of $G_{m+1}$.

Claim: $c$ is a lambda colouring of $G_{m+1}$.

Proof of Claim. We note that for each $u, v \in V(G_{m+1})$ $|c(u) - c(v)| \geq 1$, so if $d_{G_{m+1}}(u, v) = 2$, then $|c(u) - c(v)| + d_{G_{m+1}}(u, v) \geq 3$. Now if $d_{G_{m+1}}(u, v) = 1$ for some $u, v \in V(G_m)$, then

$$|c(u) - c(v)| = |\bar{c}(u) - \bar{c}(v)| \geq 2,$$

since by assumption, $\bar{c}$ is a lambda colouring of $G_m$. As any two vertices are at most 2 distance apart, the only case left where $d_{G_{m+1}}(v_{m+1}, u) = 1$ with $u \in V(G_m)$. Here we explicitly have $u = v_i$, where $0 \leq i \leq m - 1$ and consequently $\bar{c}(u) = \bar{c}(v_i) = i$, it implies that

$$|c(v_{m+1}) - c(u)| = |m + 1 - \bar{c}(u)| = |m + 1 - i| \geq 2.$$

Hence, the claim is established.
By the above claim, it implies that the lambda chromatic number of $G_{m+1}$ is at most $m + 1$. Let $c : V(G_{m+1}) \to \mathbb{N}$ be a lambda colouring. Since for each $u, v \in V(G_{m+1})$, $1 \leq d_{G_{m+1}}(u, v) \leq 2$; therefore, every vertex must receive distinct colours (non-negative integers). Hence, we need at least $|V(G_{m+1})| = m + 2$ many colours to colour the vertices of $G_{m+1}$. Consequently,

$$\max\{c(u) : u \in V(G_{m+1})\} \geq \max\{0, 1, \ldots, m + 1\} = m + 1.$$ 

Since $c : V(G_{m+1}) \to \mathbb{N}$ is an arbitrary lambda colouring, the lambda chromatic number of $G_{m+1}$ is at least $m + 1$. This means the result is true for $n = m + 1$. It can be verified directly that the result is true for $n = 3$. Hence, the result follows by induction on $n$.

The following two results give us a no-hole optimal lambda colouring of each of the member graphs of $G(t, l)$.

**Theorem 2.4.** For positive integers $t \geq 3$ and $l$, the colouring map $v_m \mapsto m$, where $v_m \in V_m$ and $0 \leq m \leq t$ is a lambda colouring of each member graph $G \in G(t, l)$.

**Proof.** We denote the aforementioned mapping by $c : \sqcup_{i=0}^{t} V_i \to \mathbb{N}$. Let $G$ be a member graph of $G(t, l)$ and $u, v$ are two vertices of $G$. If $d_G(u, v) \geq 3$, then $|c(u) - c(v)| + d_G(u, v) \geq 3$. Therefore, we have to check only the following two cases.

**Case I:** $d_G(u, v) = 2$. To show $|c(u) - c(v)| \geq 1$.

Suppose $c(u) = c(v)$. Since $d_G(u, v) = 2$, it means there exists $w \in V(c(u))$ such that $\{u, w\}$ and $\{v, w\}$ are edges of $G$. A contradiction to the definition of $G$ as $u, v \in V(c(u))$. Hence, $|c(u) - c(v)| \geq 1$.

**Case II:** $d_G(u, v) = 1$, i.e. if $\{u, v\}$ is an edge of $G$. To show $|c(u) - c(v)| \geq 2$.

Let $u \in V_m$, where $1 \leq m \leq t - 1$, then $v \in V_i$, where $i$ is an integer with $0 \leq i \leq t$ but $i \neq m - 1, m, m + 1$. Hence, $|c(u) - c(v)| = |m - i| \geq 2$. If $u \in V_0$, then $v \in V_i$, where $i$ is an integer with $2 \leq i \leq t$. Hence, $|c(u) - c(v)| = |0 - i| \geq 2$. Also if $u \in V_i$, then $v \in V_i$, where $i$ is an integer with $0 \leq i \leq t - 2$. Hence, $|c(u) - c(v)| = |t - i| \geq 2$.

**Theorem 2.5.** For positive integers $t \geq 3$ and $l$, the lambda chromatic number of each member graph $G \in G(t, l)$ is $t$.

**Proof.** Let $c : V(G) \to \mathbb{N}$ be a lambda colouring of $G$. Fix a $v_0 \in V_0$, and let $v_{0i}$, where $2 \leq i \leq t$, denote the unique neighbour of $v_0$ in $V_i$. Hence, $v_0$ has $t - 1$ neighbours in $G$. Since $d_G(v_{0i}, v_{0j}) \leq 2$, for $2 \leq i, j \leq t$, each of the $v_{0i}$ must receive distinct non-negative integers (colours) $c(v_{02}), \ldots, c(v_{0t})$. Moreover, $v_0$ is adjacent to each of $v_{0i}$; therefore, $|c(v_0) - c(v_{0i})| \geq 2$, for each integer $i$ with $2 \leq i \leq t$. It implies that $\max\{c(v) : v \in V(G)\} \geq \max\{c(v_0), c(v_{02}), \ldots, c(v_{0t})\} \geq t$.

Since $c$ is an arbitrary lambda colouring, we have the lambda chromatic number of such graph $G$ is at least $t$. From Theorem 2.4, we have the lambda chromatic number of such graph $G$ is at most $t$ and the result follows.
Let $G$ be a graph and $c : V(G) \to \mathbb{N}$ be a lambda colouring. We note that the relation $\sim$ defined on $V(G)$ by $u \sim v$ if and only if $c(u) = c(v)$. Such relation is an equivalence relation on $V(G)$. Let $C_0, \ldots, C_t$ denote the equivalence classes under the relation $\sim$. We note that for each $m \in \{0, \ldots, t\}$, $C_m = \{u \in V(G) : c(u) = m\}$. (The reader is cautioned here, for some $m \in \{0, \ldots, t\}$, $C_m$ may be an empty set. And if $C_m$ is an empty set, then the colour $m$ is defined as hole of the lambda colouring $c$.)

**Lemma 2.6.** Let $G$ be a graph, $c : V(G) \to \mathbb{N}$ be a lambda colouring and for some $r, s \in \mathbb{N}$ let $C_r, C_s$ be two different non-empty colour classes. Then for each $x \in C_r$ there exists at most one vertex $y \in C_s$ such that $\{x, y\}$ is an edge of $G$.

**Proof.** Suppose there exist $y_1, y_2 \in C_s$ such that $\{x, y_1\}$ and $\{x, y_2\}$ are edges. Since $c(y_1) = c(y_2) = s$, we have $y_1$ and $y_2$ are not adjacent, hence, $d_G(y_1, y_2) = 2$, which leads to a contradiction.

The above lemma is pretty obvious and it implies that between any two distinct colour classes $A$ and $B$, the subgraph $\{\{x, z\}, \{y, z\}\}$, where $x, y \in A$ and $z \in B$, is the forbidden subgraph in the graph $G$. The following result is one of the main theorems of this article. It asserts the affirmative answer of Question (a) posed in the introduction.

**Theorem 2.7.** Let $G$ be a graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to an optimal lambda colouring $c : V(G) \to \{0, \ldots, t\}$. Then there exists a graph $G^* \in G(t, l)$ such that $G$ is a subgraph of $G^*$, where $l = \max\{|C_0|, \ldots, |C_t|\}$.

**Proof.** For each integer $m$ with $0 \leq m \leq l$, $0 \leq |C_m| \leq l$. Here we adjoin a disjoint set $Y_m$ with $C_m$ such that $|C_m| + |Y_m| = l$. (Such $Y_m$’s, $0 \leq m \leq l$, are also mutually disjoint.) So let $Y_m := \{x^m_i : |C_m| + 1 \leq i \leq l\}$ and $V_m := C_m \sqcup Y_m$. For $0 \leq m, p \leq l$ and $m \neq p$, let

$Z_{m,p} = \{u \in V_m : u$ has no neighbour in $V_p\}$.

Using Lemma 2.6, we have for $0 \leq m \leq p-2 \leq t-2$ and for each $x \in V_m \setminus Z_{m,p}$ there exists exactly one vertex $y \in V_p \setminus Z_{p,m}$ such that $\{x, y\}$ is an edge in $G$. Conversely, if $\{u, v\}$ is an edge in $G$, then due to the fact $c$ is a lambda colouring, there exist integers $m$ and $p$, with $0 \leq m \leq p-2 \leq t-2$, such that $u \in C_m$ and $v \in C_p$. From Lemma 2.6, we conclude that $v$ is the only neighbour of $u$ in $C_p$ and vice versa.

We construct a graph $G^*$ with vertex set $\sqcup_{m=0}^t V_m$. The edges of $G$ are edges of $G^*$. We note that $|Z_{m,p}| = |Z_{p,m}|$ and for each $u \in Z_{m,p}$, we associate a unique $v \in Z_{p,m}$ and construct an edge $\{u, v\}$ of $G^*$, where $0 \leq m \leq p-2 \leq t-2$. Hence, $G^*$ is the required member of $G(t, l)$.

### 3. Maximum Number of Edges and Equitable Partition

Let $G$ be a graph with lambda chromatic number $t$ and $c : V(G) \to \{0, \ldots, t\}$ be a lambda colouring. Also let $C_0, \ldots, C_t$ be the coloured partition, where for
Theorem 3.1. Let $c$ be a lambda colouring of the graph $G$, we denote

$$
M_c(G) := \{C_M : |C_M| = \max_{0 \leq j \leq t} |C_j|\},
$$

$$
m_c(G) := \{C_m : |C_m| = \min_{0 \leq j \leq t} |C_j|\} \text{ and }
$$

$$
\nabla_c(G) := \max_{0 \leq j \leq t} |C_j| - \min_{0 \leq j \leq t} |C_j|.
$$

We also fix two more notations here. Let

$$
M(C_0, \ldots, C_t) := \sum_{i=0}^{t-2} \sum_{j=i+2}^{t} \min\{|C_i|, |C_j|\},
$$

and $e_G(X, Y)$ denote the number of edges of the form $\{x, y\}$, where $x \in X$ and $y \in Y$, where $X$ and $Y$ are two subsets of the vertex set of the graph $G$.

**Theorem 3.1.** Let $G$ be a graph with lambda chromatic number $t$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to an optimal lambda colouring $c$. Then

(a) $G$ has at most $M(C_0, \ldots, C_t)$ many edges.

(b) $G$ has exactly $M(C_0, \ldots, C_t)$ many edges if and only if for each $x \in C_i$ there exists exactly one vertex $y \in C_j$ such that $\{x, y\}$ is an edge of $G$, where $0 < |C_i| \leq |C_j|$ with $0 \leq i, j \leq t$ and $|i - j| \geq 2$.

**Proof.** Since $c$ is a lambda colouring, we have $e_G(C_p, C_{p+1}) = 0$ for each integer $p$, with $0 \leq p \leq t - 1$. Using Lemma 2.6, we have $e_G(C_i, C_j) \leq \min\{|C_i|, |C_j|\}$, where $i$ and $j$ are integers with $0 \leq i \leq j - 2 \leq t - 2$. Hence, the result (a) follows:

If for all integers $p$ and $q$, with $0 \leq p \leq q - 2 \leq t - 2$, and for each $x \in C_p$ there exists exactly one vertex $y \in C_q$ such that $\{x, y\}$ is an edge of $G$, then $0 < |C_p| \leq |C_q|$ and $e_G(C_p, C_q) = |C_p| = \min\{|C_p|, |C_q|\}$. Hence, the number of edges is $M(C_0, \ldots, C_t)$. Conversely, if for some integers $p$ and $q$, with $0 \leq p \leq q - 2 \leq t - 2$, there exists $x \in C_p$ such that for all $y \in C_q$, $\{x, y\}$ is not an edge of $G$, then using the argument from part (a), we have $e_G(C_p, C_q) < \min\{|C_p|, |C_q|\}$. It implies that the number of edges is strictly less than $M(C_0, \ldots, C_t)$ and the result (b) follows.

The above theorem informs us that the quest for a graph $G$ with lambda chromatic number $t$, which contains maximum number of edges, is boiled down to the search for a coloured partition $C_0, \ldots, C_t$, originating from an optimal lambda colouring $c : V(G) \to \{0, \ldots, t\}$, where $M(C_0, \ldots, C_t)$ is maximum. Henceforth, our main focus is to search for such coloured partitions.

**Proposition 3.2.** Let $G'$ be a subgraph of the graph $G$. If the lambda chromatic numbers of $G'$ and $G$ are respectively $t'$ and $t$, then $t' \leq t$. 
Proof. Let $\lambda : V(G) \to \mathbb{N}$ be a lambda colouring of $G$ and $\lambda' := \lambda/\mathbb{N}(G')$. Suppose $u, v \in V(G')$, then $d_{G'}(u, v) \geq d_G(u, v)$. Consequently, $|\lambda'(u) - \lambda'(v)| + d_{G'}(u, v) \geq |\lambda(u) - \lambda(v)| + d_G(u, v)$, which means $\lambda' : V(G') \to \mathbb{N}$ is a lambda colouring of $G'$. Therefore,

$$t' = \min \left\{ \max_{u \in V(G')} c(u) : c \text{ is a lambda colouring of } G' \right\} \leq \max_{u \in V(G')} \lambda'(u) \leq \max_{u \in V(G)} \lambda(u).$$

The result follows since the right-hand side of the above inequality is true for any lambda colouring $\lambda$ of $G$.

Definition. Let $G$ be a graph with lambda chromatic number $t$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set $V(G)$ with respect to the lambda colouring $c$, where for each such $m$, $C_m = \{u_i^m : 1 \leq i \leq |C_m|\}$. The graph with vertex set $V(G)$ and edges of the form $\{u_i^m, u_i^{m+1}\}$, where $1 \leq i \leq \min\{|C_m|, |C_{m+1}|\}$, $C_m$ and $C_{m+1}$ are non-empty and $0 \leq m \leq p-2 \leq t$, is called the edge standardised graph of $G$ with respect to the lambda colouring $c$ and denoted as $\mathcal{S}_c[G]$. Such edge standardised graph of $G$ contains $M(C_0, \ldots, C_t)$ many edges.

Under the edge standardisation, the vertex set and its coloured partition remain invariant. We consider the subgraph of $\mathcal{S}_c[G]$ induced by the set of vertices $\{u_i^m : 0 \leq m \leq t, C_m \neq \emptyset\}$ of the graph $G$. We refer to such subgraph as $\mathbb{T}[G]$.

Definition. Let $G$ be an edge standardised graph and with respect to the underlying lambda colouring $c$, let $C_M \in \mathbb{M}_c(G)$ and $C_m \in \mathbb{m}_c(G)$. The subgraph obtained by deleting the vertex $u_{|C_M|}^M$ and all the edges through that vertex of the graph $G$, is called the edge deleted graph. We denote such graph as $\mathcal{D}_{C_M}[G]$. Similarly, the graph obtained by adding the vertex $u_{|C_m|+1}^{M}$ and all possible edges of the form $\{u_{|C_m|+1}^{M}, u_{|C_m|+1}^{P}\}$, where $0 \leq m \leq t$, $p \in \{0, \ldots, t\} \setminus \{m - 1, m, m + 1\}$ and $|C_p| \geq |C_m| + 1$, with the graph $G$, is called the edge inserted graph. We denote such graph as $\mathcal{I}_{C_m}[G]$.

Proposition 3.3. Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. Then the lambda chromatic number of $\mathcal{I}_{C_m}[G]$ is $t$. If $\nabla_c(G) \geq 2$, then the lambda chromatic number of $\mathcal{D}_{C_M}[G]$ is $t$.

Proof. Since $C_0$ and $C_t$ are non-empty sets and $t \geq 3$, therefore, $\{u_0^0, u_1^0\}$ is an edge of $\mathbb{T}[G]$.

Claim: The lambda chromatic number of the graph $\mathbb{T}[G]$ is $t$.

Before we prove this claim, we look into Theorem 1.1 from [5]. A path of length $k$, where $k$ is a positive integer, in a graph $G$ is a sequence $\{u_i\}_{i=1}^{k+1}$ of distinct vertices such that for $1 \leq i \leq k$, $\{u_i, u_{i+1}\}$ is an edge of $G$. The vertices $u_1$ and $u_{k+1}$ are called the initial and terminal vertices, respectively. A path covering of $G$, denoted as $\mathcal{C}(G)$, is a collection of vertex disjoint paths in $G$.
such that for each vertex \( u \in V(G) \) there exists a (unique) \( C \in \mathcal{C}(G) \) such that \( u \in C \). A minimum path covering of \( G \) is a path covering of \( G \) with minimum cardinality and the path covering number \( \tau_p(G) \) of \( G \) is the cardinality of a minimum path covering of \( G \). By the Theorem 1.1 of [5], the lambda chromatic number of an \( n \)-vertex graph \( G \) is linked to the path covering number of its complement graph \( \overline{G} \), in the following way.

- \( \tau_p(\overline{G}) = 1 \) if and only if the lambda chromatic number of \( G \) is less or equals to \( n - 1 \).
- \( \tau_p(\overline{G}) \geq 2 \) if and only if the lambda chromatic number of \( G \) is \( n + \tau_p(\overline{G}) - 2 \).

**Proof of Claim.** If \( |V(\mathbb{T}[G])| = t + 1 \), then for each integer \( m \), with \( 0 \leq m \leq t \), \( C_m \) is a non-empty set. Hence, the graph \( \mathbb{T}[G] \) is isomorphic to the graph \( \mathcal{G}_t \).

From Theorem 2.3, we conclude that the lambda chromatic number of \( \mathbb{T}[G] \) is \( t \). Hence, we are done this case. Now if \( |V(\mathbb{T}[G])| = t + 1 - r \) for some positive integer \( r \), then among the \( t + 1 \) coloured classes \( C_0, \ldots, C_t \) exactly \( r \) coloured classes are empty. We note that \( C_0 \) and \( C_t \) can never be empty. Since \( C \) is an optimal lambda colouring of \( G \), we have that if \( C_m \) is empty for some integer \( m \), with \( 1 \leq m \leq t - 1 \), then both \( C_{m-1} \) and \( C_{m+1} \) are non-empty. This implies that the complement graph \( \overline{\mathbb{T}[G]} \) of \( \mathbb{T}[G] \) contains a path covering of size \( r + 1 \), consisting of vertex disjoint paths (path graphs) \( P_0, \ldots, P_r \). We note that such paths \( P_0, \ldots, P_r \) form a path covering of \( \overline{\mathbb{T}[G]} \).

Hence, \( \tau_p(\overline{\mathbb{T}[G]}) \leq r + 1 \). Now we note that for any two paths \( P \) and \( Q \) from \( P_0, \ldots, P_r \), there does not exist any edge of the form \( \{u, v\} \), where \( u, v \) are vertices of the paths \( P \) and \( Q \), respectively, in \( \overline{\mathbb{T}[G]} \). With such property, it implies that \( P_0, \ldots, P_r \) is the only path covering of \( \overline{\mathbb{T}[G]} \) with cardinality less or equals to \( r + 1 \). Hence, \( \tau_p(\overline{\mathbb{T}[G]}) \geq r + 1 \). Consequently, \( \tau_p(\overline{\mathbb{T}[G]}) = r + 1 \).

Now using Theorem 1.1 of [5], we have the lambda chromatic number of \( \mathbb{T}[G] \) is \( t + 1 - r + (r + 1) - 2 = t \). Hence, the claim is established.

Since \( \nabla_c(G) \geq 2 \), we have that the graph \( \mathcal{D}_{C_M}[G] \) contains \( \mathbb{T}[G] \) as its induced subgraph. Using similar arguments as in the above claim and Proposition 3.2, we have the lambda chromatic number of such graph is at least \( t \). The graph \( \mathcal{D}_{C_M}[G] \) is a subgraph of \( G \) and the lambda chromatic number of \( G \) is \( t \). Hence, again using Proposition 3.2, the lambda chromatic number of \( \mathcal{D}_{C_M}[G] \) is at most \( t \), which concludes the lambda chromatic number of \( \mathcal{D}_{C_M}[G] \) is \( t \).

Since \( G \) is a subgraph of the graph \( \mathcal{I}_{C_m}[G] \), using Proposition 3.2 we conclude that the lambda chromatic number of such graph is at least \( t \). We associate \( u^m_{|C_m|+1} \leftrightarrow m \) and for each \( u^p_i \leftrightarrow p \), where \( 0 \leq p \leq t \) and \( 1 \leq i \leq |C_p| \).

We call this map as \( \Lambda : V(\mathcal{I}_{C_m}[G]) \rightarrow \mathbb{N} \) and claim the following.

**Claim:** The map \( \Lambda \) is a lambda colouring of \( \mathcal{I}_{C_m}[G] \).

**Proof of Claim.** Let \( u^p_i \) and \( u^q_j \) be two distinct vertices of \( G' := \mathcal{I}_{C_m}[G] \), where \( 0 \leq p \leq t \), \( 0 \leq q \leq t \), \( i \geq 1 \) and \( j \geq 1 \). We note that if \( i \neq j \), then
Thus, using Proposition 3.2, we conclude that the lambda chromatic number of \( S \) similarly as in a part of the proof of Proposition 3.3, we conclude that \( \Theta \) is an optimal lambda colouring and for each \( c \) of the graph \( S \), we have \(|\Lambda(u) - \Lambda(v)| + d_{G'}(u, v) \geq 3\), which establishes the claim.

We note that \( \max_{u \in V(\mathcal{C}_m[G])} \Lambda(u) = t \). Hence, from the above claim the lambda chromatic number of such graph is at most \( t \), which concludes the lambda chromatic number of \( \mathcal{C}_m[G] \) is \( t \).

The following proposition ensures us that this technique does not reduce the number of edges.

**Proposition 3.4.** Let \( G \) be a graph with lambda chromatic number \( t \geq 3 \) and \( C_0, \ldots, C_t \) be the coloured partition of the vertex set of \( G \) with respect to an optimal lambda colouring \( c \). Then such \( c \) is also an optimal lambda colouring of \( \mathcal{C}_c[G] \). Moreover, \(|E(\mathcal{C}_c[G])| \geq |E(G)| \) and \( \nabla_c(\mathcal{C}_c[G]) = \nabla_c(G) \).

**Proof.** As a part of the argument of the proof of Proposition 3.3, we have shown that \( \mathcal{C}_c[G] \) contains a subgraph \( T[G] \), whose lambda chromatic number is \( t \). Thus, using Proposition 3.2, we conclude that the lambda chromatic number of \( \mathcal{C}_c[G] \) is at least \( t \). To get an upper bound, we associate for each \( u_i^p \mapsto p \), where \( 0 \leq p \leq t \) and \( 1 \leq i \leq |C_p| \). We call this map as \( \Theta : V(\mathcal{C}_c[G]) \to \mathbb{N} \) and arguing similarly as in a part of the proof of Proposition 3.3, we conclude that \( \Theta \) is a lambda colouring of the graph \( \mathcal{C}_c[G] \). We note that \( \max_{u \in V(\mathcal{C}_c[G])} \Theta(u) = t \).

Hence, the lambda chromatic number of such graph is at most \( t \). It implies the lambda chromatic number of \( \mathcal{C}_c[G] \) is \( t \). It also implies that \( \Theta \) is an optimal lambda colouring and for each \( u \in V(G) \), \( \Theta(u) = c(u) \). Hence, \( c \) is an optimal lambda colouring of \( \mathcal{C}_c[G] \) and \( \nabla_c(\mathcal{C}_c[G]) = \nabla_c(G) \).

We note that for \( 0 \leq p \leq q - 2 \leq t - 2 \), \( e_G(C_p, C_q) \leq \min\{|C_p|, |C_q|\} \) and for \( 0 \leq p \leq t - 1 \) \( e_G(C_p, C_{p+1}) = 0 \). Since \( \mathcal{C}_c[G] \) has exactly \( M(C_0, \ldots, C_t) \) many edges; therefore, by Theorem 3.1, we have for \( 0 \leq p \leq q - 2 \leq t - 2 \), \( e_{\mathcal{C}_c[G]}(C_p, C_q) = \min\{|C_p|, |C_q|\} \) and for \( 0 \leq p \leq t - 1 \), \( e_{\mathcal{C}_c[G]}(C_p, C_{p+1}) = 0 \). Hence, \(|E(\mathcal{C}_c[G])| \geq |E(G)| \).

**Remark.** It is obvious that an optimal lambda colouring \( c : V(G) \to \{0, \ldots, t\} \) induces the coloured partition \( C_0, \ldots, C_t \) of the vertex set of a graph \( G \). In addition, when \( G \) is an edge standardised graph, then both the graphs \( \mathcal{C}_m[G] \) and \( \mathcal{C}_m[G] \), where \( C_M \in \mathcal{M}_c(G) \) and \( C_m \in \mathcal{M}_c(G) \), are edge standardised graphs. Indeed \( \mathcal{C}_m[G] \) and \( \mathcal{C}_m[G] \) are defined only for the edge standardised graph \( G \). Also, both have lambda chromatic number \( t \). Therefore, it is completely legitimate to study the edge standardised graph \( \mathcal{C}_m[D_M[G]] := \mathcal{C}_m[\mathcal{C}_M[G]] \). Such graph is obtained after deletion of the vertex \( u_{M[G]}^M \).
along with, all the edges through that vertex of the edge standardised graph $G$ and then inserting the vertex $u_{|G|+1}^m$ as well as, all possible edges of the form $\{u_{|G|+1}^m, u_{|G|+1}^p\}$, where $0 \leq m \leq t$, $p \in \{0, \ldots, t\} \setminus \{m-1, m, m+1\}$ and $|C_p| \geq |C_m| + 1$.

We start with the coloured partition $C_0, \ldots, C_m, \ldots, C_M, \ldots, C_t$ of $G$. Such coloured partition has changed to $C_0, \ldots, C_m, \ldots, C_M \setminus \{u_{|G|+1}^M\}, \ldots, C_t$ in the graph $\mathcal{D}_C[G]$ and such coloured partition has changed to $C_0, \ldots, C_m \cup \{u_{|G|+1}^m\}, \ldots, C_M, \ldots, C_t$ in the graph $\mathcal{I}_C[G]$. Therefore, using Proposition 3.3 repeatedly, we say that during the edge deletion and insertion procedure, the coloured partition $C_0, \ldots, C_m, \ldots, C_M, \ldots, C_t$ of $G$ has changed in the graph $\mathcal{I}_C \mathcal{D}_C[G]$, which is

$$C_0, \ldots, C_m \cup \{u_{|G|+1}^m\}, \ldots, C_M \setminus \{u_{|G|+1}^M\}, \ldots, C_t,$$

as long as deleting $u_{|G|+1}^M$ does not produce two consecutive empty classes (i.e. holes) in $\mathcal{D}_C[G]$. For the sake of simplicity, we refer to each of the lambda colouring of $\mathcal{D}_C[G], \mathcal{I}_C[G]$ and $\mathcal{I}_C \mathcal{D}_C[G]$ inducing the aforementioned vertex partition as $c$. Though the lambda chromatic number remains invariant during the edge deletion and insertion procedure, but $\mathcal{M}_c(G)$ and $\mathcal{m}_c(G)$ change to $\mathcal{M}_c(\mathcal{D}_C[G])$ and $\mathcal{m}_c(\mathcal{I}_C[G])$, respectively.

**Definition.** Let $G$ be a graph with lambda chromatic number $t$ and let an optimal lambda colouring $c : V(G) \rightarrow \{0, \ldots, t\}$ induce the coloured partition $C_0, \ldots, C_t$ of the vertex set of $G$. Then

$$P_G(C_m) = \begin{cases} \{C_1\} & \text{if } m = 0 \\ \{C_{t-1}\} & \text{if } m = t \\ \{C_{m-1}, C_{m+1}\} & \text{if } 1 \leq m \leq t-1 \end{cases}$$

denotes the prohibited zone of the colour class $C_m$ in the graph $G$. We use the term “prohibited” due to the property that for each $x \in C \in P_G(C_m)$ there does not exist $y \in C_m$ such that $\{x, y\}$ is an edge of $G$.

The solution of Question (b) relies on the following three lemmas.

**Lemma 3.5.** Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$, $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$ and $\nabla_c(G) \geq 2$. Then, for $C_M \in \mathcal{M}_c(G)$ the lambda chromatic number of $\mathcal{D}_C[G]$ is $t$ and the following formula holds:

$$|E(G)| = |E(\mathcal{D}_C[G])| + |\mathcal{M}_c(G)| - 1 - |\mathcal{M}_c(G) \cap P_G(C_M)|.$$

**Proof.** By Proposition 3.3, we conclude that the lambda chromatic number of $\mathcal{D}_C[G]$ is $t$. Since $G$ is an edge standardised graph, we have the following:

$$|E(G)| - |E(\mathcal{D}_C[G])| = \begin{cases} |\mathcal{M}_c(G)| - 1 & \text{if } |\mathcal{M}_c(G) \cap P_G(C_M)| = 0 \\ |\mathcal{M}_c(G)| - 2 & \text{if } |\mathcal{M}_c(G) \cap P_G(C_M)| = 1 \\ |\mathcal{M}_c(G)| - 3 & \text{if } |\mathcal{M}_c(G) \cap P_G(C_M)| = 2. \end{cases}$$

Hence, the formula holds.
Lemma 3.6. Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. Then, for $C_m \in m_c(G)$, the following formula holds:

$$|E(\mathcal{I}_{C_m}[G])| = |E(G)| + t - 1 - |m_c(G) \cup P_G(C_m)|.$$  

Proof. The result holds since there are exactly $t - 1 - |m_c(G) \cup P_G(C_m)|$ many edges which are edges of $\mathcal{I}_{C_m}[G]$ but not edges of $G$. The counting is as follows. Since $G$ is an edge standardised graph, the vertex $u^m_{|C_m|+1}$ is adjacent with the vertex of the form $u^p_{|C_m|+1}$, where $0 \leq p \leq t$ and $|C_p| \geq |C_m| + 1$. Hence, there are at most $t + 1$ many such vertices. But if $C_q \in m_c(G) \cup P_G(C_m)$, then there is no such vertex of the form $u^q_{|C_m|+1} \in C_q$ such that $\{u^m_{|C_m|+1}, u^q_{|C_m|+1}\}$ is an edge of $\mathcal{I}_{C_m}[G]$ and vice versa.

Corollary 3.7. Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $\nabla_c(G) \geq 2$, $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$, then the following formula holds:

$$|E(\mathcal{I}_{B \mathcal{R}_A[G]})| - |E(G)| = t + 2 + |\mathcal{M}_c(G) \cap P_G(A)| + |m_c(\mathcal{I}_A[G]) \cap P_{\mathcal{R}_A[G]}(B)| - (|\mathcal{M}_c(G)| + |m_c(\mathcal{R}_A[G])| + |P_{\mathcal{R}_A[G]}(B)|).$$  

(*)

Proof. The formula directly follows by combining Lemma 3.5 and Lemma 3.6.

During the transformation of the edge standardised graph $G$ into $\mathcal{I}_B \mathcal{R}_A[G]$, we track down the difference in the number of edges. This justifies to refer to (*) as the difference formula. Such formula provides us with the opportunity to understand the role of the intermediate graph $\mathcal{R}_A[G]$. In the subsequent development, we focus on the cases, when such “difference” equals $-1$ or $0$ or is at least $1$.

The development of this section and the following one is all about searching an $n$-vertex graph with lambda chromatic number $t \geq 3$ containing maximum number of edges. In this regard, we now focus on the following question.

Question. Let $G$ be an $n$-vertex graph $G$ with lambda chromatic number $t \geq 3$. If $\nabla_c(G) \geq 2$ for some optimal lambda colouring $c$ of $G$, is it possible that $G$ contains maximum number of edges?

We answer the above question affirmatively. More precisely, we answer that for $t = 3$ and $t = 4$ we can find such families of graphs. However, we cannot find any such graph when $t \geq 5$. Formally, we explain it in the following results.

Lemma 3.8. Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $G$ contains maximum number of edges, then $\nabla_c(G) \leq 1$ or all but at most one $C_i$ are members of $\mathcal{M}_c(G) \cup m_c(G)$ (where $0 \leq i \leq t$).
Proof. Suppose it does not hold that $\nabla_c(G) \leq 1$ or all but at most one $C_i$ are members of $\mathcal{M}_c(G) \cup m_c(G)$. This implies $\nabla_c(G) \geq 2$ and $2 \leq |\mathcal{M}_c(G)| + |m_c(G)| \leq t - 1$. We choose $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$ and construct two edge standardised graphs $\mathcal{D}_A[G]$ and $\mathcal{J}_B \mathcal{D}_A[G]$. We note that $\nabla_c(G) \geq 2$ implies $m_c(\mathcal{D}_A[G]) = m_c(G)$. Hence, $B \in m_c(\mathcal{D}_A[G])$. Also $|P\mathcal{D}_A[G](B)| = |P_G(B)|$.

Now using the difference formula $(\star)$ of Corollary 3.7, and the inequality
\[
3 \leq |\mathcal{M}_c(G)| + |m_c(G)| + |P_G(B)| = |\mathcal{M}_c(G)| + |m_c(\mathcal{D}_A[G])| + |P\mathcal{D}_A[G](B)| \leq t + 1
\]
we get,
\[
|E(\mathcal{J}_B \mathcal{D}_A[G])| - |E(G)| \geq 1 + |\mathcal{M}_c(G) \cap P_G(A)| + |m_c(\mathcal{D}_A[G]) \cap P\mathcal{D}_A[G](B)|.
\]
Both the graphs $\mathcal{J}_B \mathcal{D}_A[G]$ and $G$ are edge standardised graphs and $|E(\mathcal{J}_B \mathcal{D}_A[G])| \geq |E(G)| + 1$. Hence, $G$ does not contain maximum number of edges, a contradiction arises and the result follows.

Proposition 3.9. Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $\nabla_c(G) \geq 2$ and $G$ contains maximum number of edges, then exactly one of the following holds.

(a) For all $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$, $|\mathcal{M}_c(G) \cap P_G(A)| = 0$, $|m_c(G) \cap P_G(B)| = 0$, whenever $|m_c(G)| + |\mathcal{M}_c(G)| = t$.

(b) For all $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$,
\[
1 + |\mathcal{M}_c(G) \cap P_G(A)| + |m_c(G) \cap P_G(B)| \leq |P_G(B)|,
\]
whenever $|m_c(G)| + |\mathcal{M}_c(G)| = t + 1$.

Proof. Let $|m_c(G)| + |\mathcal{M}_c(G)| = t$. Suppose for some $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$ we have
\[
|\mathcal{M}_c(G) \cap P_G(A)| + |m_c(G) \cap P_G(B)| \geq 1.
\]
Then we construct two edge standardised graphs $\mathcal{D}_A[G]$ and $\mathcal{J}_B \mathcal{D}_A[G]$. We note that $\nabla_c(G) \geq 2$ implies $m_c(\mathcal{D}_A[G]) = m_c(G)$. Hence, $B \in m_c(\mathcal{D}_A[G])$ and $|m_c(\mathcal{D}_A[G]) \cap P\mathcal{D}_A[G](B)| = |m_c(G) \cap P_G(B)|$. Now using the difference formula $(\star)$ of Corollary 3.7 and the inequality
\[
|\mathcal{M}_c(G)| + |m_c(G)| + |P_G(B)| = |\mathcal{M}_c(G)| + |m_c(\mathcal{D}_A[G])| + |P\mathcal{D}_A[G](B)| \leq t + 2
\]
we get,
\[
|E(\mathcal{J}_B \mathcal{D}_A[G])| - |E(G)| \geq |\mathcal{M}_c(G) \cap P_G(A)| + |m_c(\mathcal{D}_A[G]) \cap P\mathcal{D}_A[G](B)|
\]
\[
= |\mathcal{M}_c(G) \cap P_G(A)| + |m_c(G) \cap P_G(B)| \geq 1.
\]
Both the graphs $\mathcal{J}_B \mathcal{D}_A[G]$ and $G$ are edge standardised graphs and $|E(\mathcal{J}_B \mathcal{D}_A[G])| \geq |E(G)| + 1$. Hence, $G$ does not contain maximum number of edges, a contradiction arises. This concludes (a).

Let $|m_c(G)| + |\mathcal{M}_c(G)| = t + 1$. Suppose for some $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$ we have
\[
|\mathcal{M}_c(G) \cap P_G(A)| + |m_c(G) \cap P_G(B)| \geq |P_G(B)|.
\]
Then we construct two edge standardised graphs $\mathcal{D}_A[G]$ and $\mathcal{J}_B \mathcal{D}_A[G]$. We note that $\nabla_c(G) \geq 2$ implies $m_c(\mathcal{D}_A[G]) = m_c(G)$. Hence, $B \in m_c(\mathcal{D}_A[G])$ and $|m_c(\mathcal{D}_A[G]) \cap P_{\mathcal{D}_A[G]}(B)| = |m_c(G) \cap P_G(B)|$. Also $|P_{\mathcal{D}_A[G]}(B)| = |P_G(B)|$. Now using the difference formula $(*)$ of Corollary 3.7, we get

$$|E(\mathcal{J}_B \mathcal{D}_A[G])| - |E(G)| = 1 + |\mathcal{M}_c(G) \cap P_G(A)|$$
$$+ |m_c(\mathcal{D}_A[G]) \cap P_{\mathcal{D}_A[G]}(B)| - |P_{\mathcal{D}_A[G]}(B)|$$
$$= 1 + |\mathcal{M}_c(G) \cap P_G(A)|$$
$$+ |m_c(G) \cap P_G(B)| - |P_G(B)| \geq 1.$$

Both the graphs $\mathcal{J}_B \mathcal{D}_A[G]$ and $G$ are edge standardised graphs and $|E(\mathcal{J}_B \mathcal{D}_A[G])| \geq |E(G)| + 1$. Hence, $G$ does not contain maximum number of edges, a contradiction arises. This concludes (b).

The (***) condition of Proposition 3.9 expounds the following corollary.

**Corollary 3.10.** With the same assumptions to conclude (***) (in Proposition 3.9), the followings hold.

(a) Any three consecutive members (i.e. members of the form $C_i, C_{i+1}, C_{i+2}$, where $0 \leq i \leq t-2$) of $C_0, \ldots, C_t$ must contain at least one member of $\mathcal{M}_c(G)$ and at least one member of $m_c(G)$.
(b) Occurrence of two consecutive members (i.e. members of the form $C_i, C_{i+1}$, where $0 \leq i \leq t-1$) in $\mathcal{M}_c(G)$ forbids occurrence of two consecutive members in $m_c(G)$ and vice versa.

**Remark.** Suppose $G$ is an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ is the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c : V(G) \rightarrow \{0, \ldots, t\}$. Also let, $\nabla_c(G) \geq 2$. Following the result (a) of Proposition 3.9 and the difference formula $(*)$ of Corollary 3.7, if $G$ has maximum number of edges and $|\mathcal{M}_c(G)| + |m_c(G)| = t$, then $|E(\mathcal{J}_B \mathcal{D}_A[G])| = |E(G)|$ for all $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$. But if $G$ has maximum number of edges and $|\mathcal{M}_c(G)| + |m_c(G)| = t + 1$, then using (b) of Proposition 3.9, either $|E(\mathcal{J}_B \mathcal{D}_A[G])| = |E(G)|$ or $|E(\mathcal{J}_B \mathcal{D}_A[G])| = |E(G)| - 1$ holds, where $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$.

Here we derive the necessary conditions to prohibit the increase in the number of edges. The first two results (Propositions 3.11 and 3.12) are about the transformations that keep the number of edges invariant. However, the third one (Proposition 3.13) is about the transformations when the number of edges reduces. Eventually, these lead to stationary conditions (see the definition of the stationary graph) in the subsequent development.

**Proposition 3.11.** Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $\nabla_c(G) \geq 2$ and $G$ contains maximum number of edges with $|m_c(G)| + |\mathcal{M}_c(G)| = t$, then $t = 3$ or $t = 4$; moreover $|m_c(G)| = 1$ (and consequently), $\nabla_c(G) = 2$ and $m_c(G) = \{C_i\}$, where $1 \leq i \leq t - 1$. 
Proof. Since $\nabla_c(G) > 0$, there exists $A \in \mathcal{M}_c(G)$. We assume that for some $B \in \mathfrak{m}_c(G)$, $|P_G(B)| = 1$. Now both the graphs $\mathcal{I}_B \mathcal{D}_A[G]$ and $G$ are edge standardised graphs. By the difference formula $(\star)$ of Corollary 3.7, we have $|E(\mathcal{I}_B \mathcal{D}_A[G])| \geq |E(G)| + 1$. Hence, $G$ does not contain maximum number of edges. A contradiction arises. Hence, for each $B \in \mathfrak{m}_c(G)$, $|P_G(B)| = 2$. Consequently, $C_i \in \mathfrak{m}_c(G)$ implies $1 \leq i \leq t - 1$.

We now prove the following claim.

Claim: With these conditions, $|\mathcal{M}_c(G)| \geq 2$.

Proof of Claim. Suppose $|\mathfrak{m}_c(G)| \leq 1$, then $|\mathfrak{m}_c(G)| = 1$. Say $Y$ is the unique member of $\mathfrak{m}_c(G)$. Since $\nabla_c(G) \geq 2$, we have $E(\mathcal{D}_Y[G]) = E(G)$ and for each $X \in \mathfrak{m}_c(G)$, $|Y| - 1 \geq |X| + 1$, which produces an additional edge in the graph $\mathcal{I}_X \mathcal{D}_Y[G]$. This implies, $|E(\mathcal{I}_X \mathcal{D}_Y[G])| \geq |E(G)| + 1$ for each $X \in \mathfrak{m}_c(G)$. Since $G$ and $\mathcal{I}_X \mathcal{D}_Y[G]$ both are edge standardised graphs, we have $G$ does not contain maximum number of edges, a contradiction arises. Hence, the claim is established.

Suppose with these conditions, we have $|\mathfrak{m}_c(G)| \geq 2$. Using the above claim, we choose $A, A' \in \mathfrak{m}_c(G)$ and $B, B' \in \mathfrak{m}_c(G)$. We construct the graphs $\mathcal{D}_A[G], \mathcal{D}_A' \mathcal{D}_A[G], \mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]$, and $\mathcal{I}_B' \mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]$. Now $\nabla_c(G) \geq 2$. Also $|\mathfrak{m}_c(G)| \geq 2$; therefore, $\nabla_c(\mathcal{D}_A[G]) \geq 2$ and by Lemma 3.5, lambda chromatic number of $\mathcal{D}_A' \mathcal{D}_A[G]$ is $t$. Then using Proposition 3.3, $\mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]$ and $\mathcal{I}_B' \mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]$ are of lambda chromatic number $t$. We note the following:

$$|\mathfrak{m}_c(\mathcal{D}_A[G])| = |\mathfrak{m}_c(G)| - 1,$$

$$|\mathfrak{m}_c(\mathcal{D}_A' \mathcal{D}_A[G])| = |\mathfrak{m}_c(G)| \quad \text{and}$$

$$|\mathfrak{m}_c(\mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G])| = |\mathfrak{m}_c(G)| - 1.$$

Applying Lemma 3.5, we have

$$|E(G)| = |E(\mathcal{D}_A' \mathcal{D}_A[G])| + 2|\mathfrak{m}_c(G)| - 3 - |\mathfrak{m}_c(G) \cap P_G(A)| \quad (\otimes)$$

$$+ |\mathfrak{m}_c(\mathcal{D}_A[G]) \cap P_{\mathcal{D}_A[G]}(A')|.$$

Now applying $(\otimes)$ and Lemma 3.6, we have

$$|E(\mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G])| = |E(G)| + t + 4 + |\mathfrak{m}_c(\mathcal{D}_A[G]) \cap P_{\mathcal{D}_A[G]}(A')| \quad (\otimes\otimes)$$

$$+ |\mathfrak{m}_c(G) \cap P_G(A)| + |\mathfrak{m}_c(\mathcal{D}_A' \mathcal{D}_A[G]) \cap P_{\mathcal{D}_A' \mathcal{D}_A[G]}(B)|$$

$$- 2|\mathfrak{m}_c(G)| - |\mathfrak{m}_c(G)| - |P_{\mathcal{D}_A' \mathcal{D}_A[G]}(B)|.$$

Again applying $(\otimes\otimes)$ and Lemma 3.6, we have

$$|E(\mathcal{I}_B' \mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G])| - |E(G)| = 2t + 6 + |\mathfrak{m}_c(\mathcal{D}_A[G]) \cap P_{\mathcal{D}_A[G]}(A')|$$

$$+ |\mathfrak{m}_c(G) \cap P_G(A)|$$

$$+ |\mathfrak{m}_c(\mathcal{D}_A' \mathcal{D}_A[G]) \cap P_{\mathcal{D}_A' \mathcal{D}_A[G]}(B)|$$

$$+ |\mathfrak{m}_c(\mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]) \cap P_{\mathcal{I}_B \mathcal{D}_A' \mathcal{D}_A[G]}(B')|$$
As we have

\[I: \text{Suppose Case } t \]

If \( |D_A(D_B(D_A(G)))| \geq 2t + 6 + |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_G(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[≥ 2 + |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_G(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[+ |M_c(D_A[G]) \cap P_D(A)| \]
\[≥ 2. \]

Both the graphs \( D_B[D_A(D_A(G))] \) and \( G \) are edge standardised graphs. Also \( |E(D_B[D_A(D_A(G))]| \geq |E(G)| + 2 \). Hence, \( G \) does not contain maximum number of edges. A contradiction arises. This proves \( |m_c(G)| = 1 \).

Let \( B \) be the unique member of \( m_c(G) \). Using the aforementioned arguments we have \( |P_G(B)| = 2 \). Also, we have \( |M_c(G)| + |m_c(G)| = t \) and hence, using (a) of Proposition 3.9, we have for all \( A \in M_c(G), |M_c(G) \cap P_G(A)| = 0 \) and \( |m_c(G) \cap P_G(B)| = 0 \). These together imply \( 2 \leq |M_c(G)| \leq 3 \). Hence, \( 4 \leq t + 1 \leq 5 \), i.e. \( 3 \leq t \leq 4 \).

Let \( A \in M_c(G) \) and \( C \) denote the unique coloured class with \( |B| + 1 \leq |C| \leq |A| - 1 \). We consider the edge standardised graph \( G' := D_B(D_A(G)) \). Suppose \( \nabla_c(G) \geq 3 \), then we conclude \( |C| - |B| \geq 2 \) or \( |A| - |C| \geq 2 \).

Case I: Suppose \( |C| - |B| \geq 2 \).

As we have \( |M_c(G)| = t - 1 \geq 2 \). Consequently \( |m_c(G')| = 1 \) and \( |M_c(G')| + |m_c(G')| = t - 1 \). Hence, we conclude by Lemma 3.8, \( G' \) can not contain maximum number of edges and by the difference formula (\( \ast \)) of Corollary 3.7, \( |E(G')| \geq |E(G)| \). But \( G \) contains maximum number of edges, which leads to a contradiction.

Case II: Suppose \( |A| - |C| \geq 2 \).

If \( |C| - |B| \geq 2 \), then \( |M_c(G)| = t - 1 \geq 2 \) and consequently \( |M_c(G')| + |m_c(G')| = t - 1 \). With a similar argument as in Case I, we have a contradiction.

If \( |C| - |B| = 1 \), then \( |M_c(G')| + |m_c(G')| = t \) but \( |m_c(G')| = 2 \). Hence, by the difference formula (\( \ast \)) of Corollary 3.7, we conclude \( |E(G')| \geq |E(G)| \). Since \( G' \) is an edge standardised graph and \( G \) contains maximum number of edges, we conclude that \( G' \) contains maximum number of edges. A contradiction arises, as for such edge standardised graph with maximum number of edges has the property \( |m_c(G')| = 1 \).

This shows for each of the cases leads to a contradiction. Hence, \( |C| = 1 + |B| \) and \( |A| = 1 + |C| \), i.e. \( \nabla_c(G) = 2 \).

**Proposition 3.12.** Let \( G \) be an edge standardised graph with lambda chromatic number \( t \geq 3 \) and \( C_0, \ldots, C_t \) be the coloured partition of the vertex set of \( G \).
with respect to the underlying optimal lambda colouring $c$. If $\nabla_{c}(G) \geq 2$, $G$ contains maximum number of edges with $|m_{c}(G)| + |M_{c}(G)| = t + 1$ and for some $A \in M_{c}(G)$ and $B \in m_{c}(G)$, $|E(\mathcal{I}_{B} \mathcal{D}_{A}[G])| = |E(G)|$, then $|m_{c}(G)| = 1$. Furthermore, the following results hold.

(a) $3 \leq |m_{c}(G)| \leq 4$ and $3 \leq t \leq 4$.
(b) $m_{c}(G) = \{C_{i}\}$, where $1 \leq i \leq t - 1$.
(c) If $t = 4$, then $\nabla_{c}(G) = 2$.
(d) If $t = 3$, then $2 \leq \nabla_{c}(G) \leq 3$.

**Proof.** If $M_{c}(G)$ contains a unique element $X$, then due to the assumption $\nabla_{c}(G) \geq 2$, we have for each $Y \in m_{c}(G)$, $|E(\mathcal{I}_{Y} \mathcal{D}_{X}[G])| \geq 1 + |E(G)|$. Since both the graphs $\mathcal{I}_{Y} \mathcal{D}_{X}[G]$ and $G$ are edge standardised graphs, this leads to a contradiction to the assumption that $G$ contains maximum number of edges. Hence, $|M_{c}(G)| \geq 2$. Let for some $A \in M_{c}(G)$ and $B \in m_{c}(G)$, $|E(\mathcal{I}_{B} \mathcal{D}_{A}[G])| = |E(G)|$ holds. Suppose $|m_{c}(G)| \geq 2$, then for the edge standardised graph $G' := \mathcal{I}_{B} \mathcal{D}_{A}[G]$ we have $|M_{c}(G')| + |m_{c}(G')| = t - 1$. Hence, we conclude by Lemma 3.8, $G'$ cannot contain maximum number of edges. But $|E(G')| = |E(G)|$. This implies $G$ can not contain maximum number of edges. A contradiction arises. Hence, $|m_{c}(G)| = 1$ and the first result follows.

Suppose $|M_{c}(G)| \geq 5$. This, together with $|m_{c}(G)| = 1$ and $|M_{c}(G)| + |M_{c}(G)| = t + 1$, yields the existence of an integer $i$, with $0 \leq i \leq t - 2$ such that $C_{i}$, $C_{i+1}$, $C_{i+2}$ (i.e. consecutive three members) are members of $M_{c}(G)$, this contradicts (a) of Corollary 3.10, therefore, $|M_{c}(G)| \leq 4$. Since $|M_{c}(G)| + |m_{c}(G)| = t + 1 \geq 4$, we have $|M_{c}(G)| \geq 3$. Hence, the result (a) holds.

Suppose $C_{0}$ is the unique member of $m_{c}(G)$. This, together with (a), yields the existence of an integer $i$, with $1 \leq i \leq t - 2$ such that $C_{i}$, $C_{i+1}$, $C_{i+2}$ (i.e. consecutive three members) are members of $M_{c}(G)$, this contradicts (a) of Corollary 3.10. Similarly our assumption, $C_{i}$ is the unique member of $m_{c}(G)$ leads to a contradiction. This implies the result (b).

To show (c), first we note that since $t = 4$, using (b) of Proposition 3.9, $\{C_{0}, C_{1}, C_{3}, C_{4}\}$ is the complete list of members of $M_{c}(G)$ and $m_{c}(G) = \{C_{2}\}$. Suppose $\nabla_{c}(G) \geq 3$. We note that for this case $|M_{c}(G')| = 3$ and $m_{c}(G')$ contains unique member say $B'$. Therefore, there exists $A' \in M_{c}(G')$ such that $|P_{G'}(A') \cap m_{c}(G')| = 1$. Also $|P_{G'}(B')| = 2$. Hence, using the difference formula (c) of Corollary 3.7, we have $|E(\mathcal{I}_{B'} \mathcal{D}_{A'}[G'])| \geq 1 + |E(G')| = 1 + |E(G)|$. This is a contradiction to the assumption that $G$ contains maximum number of edges. Hence, $\nabla_{c}(G) = 2$.

To show (d), first we note that since $t = 3$, using (b) of Proposition 3.9, either $\{C_{0}, C_{2}, C_{3}\}$ is the complete list of members of $M_{c}(G)$ and $m_{c}(G) = \{C_{1}\}$ or $\{C_{0}, C_{1}, C_{3}\}$ is the complete list of members of $M_{c}(G)$ and $m_{c}(G) = \{C_{2}\}$. Note that for the first choice and second choice of the coloured partitions, we have $|E(G')| = |E(G)|$ implies $A \neq C_{0}$ and $A \neq C_{3}$ for the respective choices. Suppose $\nabla_{c}(G) \geq 4$. We note that for this case $|M_{c}(G')| = 2$ and $m_{c}(G')$ contains unique member say $B'$. Let $A' \in M_{c}(G')$ and we construct $G'' := \mathcal{I}_{B'} \mathcal{D}_{A'}[G']$. Here $|M_{c}(G'\prime\prime)| = 1 = |m_{c}(G'\prime\prime)|$. We assume $M_{c}(G'\prime\prime) = ...$
\{X\} and \(m_c(G'') = \{Y\}\), then \(|M_c(G'')| + |m_c(G'')| = 2 = t - 1\). Hence, we conclude by Lemma 3.8, \(G''\) cannot contain maximum number of edges. But \(|E(G'')| = |E(G')| = |E(G)|\). This contradicts the assumption that \(G\) contains maximum number of edges. Hence, \(2 \leq \nabla_c(G) \leq 3\).

**Proposition 3.13.** Let \(G\) be an edge standardised graph with lambda chromatic number \(t \geq 3\) and \(C_0, \ldots, C_t\) be the coloured partition of the vertex set of \(G\) with respect to the underlying optimal lambda colouring \(c\). If \(\nabla_c(G) \geq 2\), \(G\) contains maximum number of edges and \(|E(\mathcal{I}_B \mathcal{D}_A[G])| = |E(G)| - 1\) for all \(A \in M_c(G)\) and \(B \in m_c(G)\), then \(|m_c(G)| = 2\), \(|M_c(G)| = 3\), \(t = 4\) and \(\nabla_c(G) = 2\). Moreover, if \(C_i \in m_c(G)\), then \(1 \leq i \leq t - 1\).

**Proof.** Here we observe from the difference formula (*) of Corollary 3.7, that for all \(A \in M_c(G)\) and \(B \in m_c(G)\), \(|E(\mathcal{I}_B \mathcal{D}_A[G])| = |E(G)| - 1\) if and only if \(|M_c(G) \cap P_G(A)| = 0\), \(|m_c(G) \cap P_G(B)| = 0\) for all \(A \in M_c(G)\) and \(B \in m_c(G)\), \(|m_c(G)| = |m_c(G)| + 1 \geq 4\). We choose \(X \in M_c(G)\) and \(Y \in m_c(G)\) and construct the graph \(G' := \mathcal{I}_Y \mathcal{D}_X[G]\), then by assumption \(|E(G')| = |E(G)| - 1\). Also \(|m_c(G')| = |m_c(G)| - 1 \geq 2\) and \(|M_c(G')| = |M_c(G)| - 1 \geq 3\). Therefore, \(|M_c(G')| + |m_c(G')| = t - 1\). We note that \(|M_c(G') \cap P_G(A')| = 0\), \(|m_c(G') \cap P_G(B')| = 0\) and \(|P_G(B')| = 2\), for all \(A' \in M_c(G')\) and \(B' \in m_c(G')\). Now we choose \(X' \in M_c(G')\) and \(Y' \in m_c(G')\) and construct the graph \(G'' := \mathcal{I}_Y \mathcal{D}_X[G']\). Therefore, \(|M_c(G'')| = |M_c(G')| - 1 \geq 2\), \(|m_c(G'')| = |m_c(G')| - 1 \geq 1\) and \(|M_c(G'')| + |m_c(G'')| = t - 3\). Hence, we conclude by Lemma 3.8, \(G''\) cannot contain maximum number of edges. But using the difference formula (*) of Corollary 3.7, we have \(|E(G'')| = |E(G')| + 1 = E(G)|. This contradicts the assumption that \(G\) contains maximum number of edges. Hence, \(|m_c(G)| \leq 2\). Now suppose \(|m_c(G)| = 1\). Since \(|M_c(G) \cap P_G(A)| = 0\), \(|m_c(G) \cap P_G(B)| = 0\) and \(|P_G(B)| = 2\) for all \(A \in M_c(G)\) and \(B \in m_c(G)\); therefore, we have \(|M_c(G)| = 2\), i.e. \(t = 2\). A contradiction arises since \(t \geq 3\). Hence, \(|m_c(G)| = 2\). Consequently, \(|M_c(G)| = 3\) and \(t = 4\).

Suppose \(\nabla_c(G) \geq 3\). We choose \(X \in M_c(G)\) and \(Y \in m_c(G)\) and construct the graph \(G' := \mathcal{I}_Y \mathcal{D}_X[G]\), then by assumption \(|E(G')| = |E(G)| - 1\). We note that, here \(|m_c(G')| = |m_c(G)| - 1 = 1\) and \(|M_c(G')| = |M_c(G)| - 1 = 2\). We choose \(X' \in M_c(G')\) and \(Y' \in m_c(G')\) and construct the graph \(G'' := \mathcal{I}_Y \mathcal{D}_X[G']\). Therefore, \(|M_c(G'')| = |M_c(G')| - 1 = 1\), \(|m_c(G'')| = |m_c(G')| + 1 = 2\) and \(|M_c(G'')| + |m_c(G'')| = 3 = t - 1\). Hence, we conclude by Lemma 3.8, \(G''\) cannot contain maximum number of edges. Since \(|M_c(G') \cap P_G(A')| = 0\), \(|m_c(G') \cap P_G(B')| = 0\) and \(|P_G(B')| = 2\), for all \(A' \in M_c(G')\) and \(B' \in m_c(G')\), using the difference formula (*) of Corollary 3.7, we have \(|E(G'')| = |E(G')| + 1 = E(G)|. This contradicts the assumption that \(G\) contains maximum number of edges. Hence, \(\nabla_c(G) = 2\).
4. The Final Arc: From the Stationary Results to the Classification Results

In this section, our main aim is to establish a classification result. We classify all $n$-vertex graphs with lambda chromatic number $t \geq 3$ and $n \geq t + 1$.

**Proposition 4.1.** Let $G$ be a graph with lambda chromatic number $t$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to an optimal lambda colouring $c$. Then the mapping $\bar{c} : V(G) \rightarrow \{0, \ldots, t\}$ defined by $u \mapsto t - c(u)$ is an optimal lambda colouring of $G$ and $\bar{C}_0, \ldots, \bar{C}_t$, where $\bar{C}_i = C_{t-i}$ for each integer $i$ with $0 \leq i \leq t$, is the coloured partition of the vertex set of $G$ with respect to $\bar{c}$.

**Proof.** The result follows immediately since $|c(u) - c(v)| + d_G(u, v) = |\bar{c}(u) - \bar{c}(v)| + d_G(u, v)$ for all $u, v \in V(G)$.

**Definition.** The aforementioned $\bar{c} : V(G) \rightarrow \{0, \ldots, t\}$ is called the dual of the optimal lambda colouring $c$. The coloured partition $C_0, \ldots, C_t$ is called the dual coloured partition of $C_0, \ldots, C_t$.

The following five results are the most important ingredients to prove the classification results (Theorem 4.9 and Corollary 4.10). Here we fix the notation

$$K_G(U_0, \ldots, U_t) := \{|(i, i + 1) : |U_i| = |U_{i+1}| = \max\{|U_n| : 0 \leq n \leq t\}\},$$

for a partition $U_0, \ldots, U_t$ of the vertex set of $G$. Therefore, $K_G(U_0, \ldots, U_t)$ counts the number pairs of the form $(U_i, U_{i+1})$, where $0 \leq i \leq t - 1$ (i.e. the consecutive pairs) and $|U_i| = |U_{i+1}| = \max_{0 \leq n \leq t}|U_n|$.

**Theorem 4.2.** Let $G$ be an edge standardised graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $G$ contains $n$ many vertices and $\nabla_c(G) = 1$, then $n = |B|(t + 1) + |M_c(G)|$ and

$$|E(G)| = |B| \left(\frac{t}{2}\right) + \left(\frac{|M_c(G)|}{2}\right) - K_G(C_0, \ldots, C_t),$$

where $B \in m_c(G)$. Moreover, $|B| = \left\lfloor\frac{n}{t+1}\right\rfloor$ and $|M_c(G)| = n - (t+1)\left\lfloor\frac{n}{t+1}\right\rfloor$. Also $G$ contains maximum number of edges if and only if the value $K_G(C_0, \ldots, C_t)$ is minimum over any equitable partition into (unequally sized) $t+1$ parts (subsets) of the vertex sets of all possible $n$-vertex graphs with lambda chromatic number $t$.

**Proof.** Let $B \in m_c(G)$. Here the vertex set of the edge standardised graph $G$ admits a partition into two parts (subsets) $U_1$ and $U_2$ such that $e_G(U_1, U_2) = 0$, where

$$U_1 := \left(\bigcup_{m=0}^{t} \{u_i^m : 1 \leq i \leq |B|\}\right) \quad \text{and} \quad U_2 := \left(\bigcup_{C_m \in M_c(G)} \{u_i^m : i = |B|+1\}\right).$$

Hence, the formula for $n$ holds. And to count the edges, it is enough to count the edges of the subgraphs induced by the subsets $U_1$ and $U_2$. Since $G$ is an edge standardised graph, the subgraph induced by the subset $U_1$ of
$V(G)$ is $|B|$ disjoint copies of $G_t$. Therefore, such subset yields $|B|\binom{t}{2}$ edges. To calculate the remaining edges we note that $G$ is an edge standardised graph. So, the subgraph induced by the subset $U_2$ of $V(G)$ is the almost complete graph on $|\mathcal{M}_c(G)|$ many vertices i.e. complete graph on $|\mathcal{M}_c(G)|$ many vertices with $K_G(C_0,\ldots,C_t)$ many deleted edges. Therefore, there are $\left[|\mathcal{M}_c(G)| - K_G(C_0,\ldots,C_t)\right]$ many additional edges.

For the next part we note that $\nabla_c(G) = 1$ implies $1 \leq |\mathcal{M}_c(G)| \leq t \Leftrightarrow \frac{1}{t+1} \leq \frac{n-1}{t+1}$. Consequently, $|B| = \left\lfloor\frac{n}{t+1}\right\rfloor$ and $|\mathcal{M}_c(G)| = n - (t + 1)\left\lfloor\frac{n}{t+1}\right\rfloor$. The remaining part follows immediately from the edge formula.

**Theorem 4.3.** Let $G$ be an edge standardised graph with lambda chromatic number $3$ and $C_0,C_1,C_2,C_3$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. Then $G$ contains maximum number of edges and $\nabla_c(G) \geq 2$ if and only if the coloured partition or its respective dual partition satisfies one of the following four types of properties.

(a) $\mathcal{M}_c(G) = \{C_0,C_3\}$, $m_c(G) = \{C_2\}$ and $|C_0| = |C_1| + 1 = |C_2| + 2 = |C_3|$. 
(b) $\mathcal{M}_c(G) = \{C_1,C_3\}$, $m_c(G) = \{C_2\}$ and $|C_0| + 1 = |C_1| = |C_2| + 2 = |C_3|$. 
(c) $\mathcal{M}_c(G) = \{C_0,C_2,C_3\}$, $m_c(G) = \{C_1\}$ and $|C_0| = |C_1| + 2 = |C_2| = |C_3|$. 
(d) $\mathcal{M}_c(G) = \{C_0,C_2,C_3\}$, $m_c(G) = \{C_1\}$ and $|C_0| + 1 = |C_2| + 3 = |C_2| = |C_3|$.

**Proof.** If $G$ contains maximum number of edges, $|\mathcal{M}_c(G)| + |m_c(G)| = t = 3$ and $\nabla_c(G) \geq 2$, then conclusions (a) and (b) directly follow from Propositions 3.11 and 3.9. For the converse part of (a), let $G$ be the same as mentioned in the statement with the additional property that (a) holds. Then we construct, $G' := \mathcal{I}_{C_2}\mathcal{D}_{C_3}[G]$. Here $\nabla_c(G') = 1$ and $|\mathcal{M}_c(G')| = 1$. Hence, $K_{G'}(C_0',C_1',C_2',C_3') = 0$, where $C_0',C_1',C_2',C_3'$ is the transformed (coloured) partition of $C_0,C_1,C_2,C_3$. Therefore, using Theorem 4.2, we have $G'$ contains maximum number of edges. We note that during the transformation, only one edge between $C_0$ and $C_3$ is deleted, and only one edge between $C_0$ and $C_2$ is inserted, which results $|E(G)| = |E(G')|$. Hence, $G$ contains maximum number of edges. For the converse part of (b), we assume $G$ be the same as mentioned in the statement with the additional property that (b) holds. Then with a same construction and similar argument, we have such $G$ contains maximum number of edges.

Similarly, if $G$ contains maximum number of edges, $|\mathcal{M}_c(G)| + |m_c(G)| = t + 1 = 4$ and $\nabla_c(G) \geq 2$, then for $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$ we have either $|E(\mathcal{I}_B\mathcal{D}_A[G])| = |E(G)|$ or $|E(\mathcal{I}_B\mathcal{D}_A[G])| = |E(G)| - 1$ holds. Since $t = 3$, we have by Proposition 3.13, there exist $A \in \mathcal{M}_c(G)$ and $B \in m_c(G)$ such that $|E(\mathcal{I}_B\mathcal{D}_A[G])| = |E(G)|$. Hence, conclusions (c) and (d) follow from Proposition 3.12. Conversely, let $G'$ be the same as mentioned in the statement with the additional property that (c) holds. Then $\nabla_c(G') = 2$ and we let $G' := \mathcal{I}_{C_1}\mathcal{D}_{C_3}[G]$. Using Theorem 4.2 and a similar argument as before, we have $G$ contains maximum number of edges. Conversely, let $G$ be the same as mentioned in the statement with the additional property that (d) holds. Then $\nabla_c(G) = 3$ and we let $G' := \mathcal{I}_{C_1}\mathcal{D}_{C_2}\mathcal{I}_{C_1}\mathcal{D}_{C_3}[G]$. Using Theorem 4.2 and a similar argument as before, we have $G$ contains maximum number of edges.
**Theorem 4.4.** Let $G$ be an edge standardised graph with lambda chromatic number 4 and $C_0, \ldots, C_4$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If $G$ contains maximum number of edges and $\nabla_c(G) \geq 2$, then there are only following two types of coloured partitions or their respective dual coloured partitions.

(a) $\mathcal{M}_c(G) = \{C_0, C_2, C_4\}$, $m_c(G) = \{C_1\}$ and $|C_0| = |C_1| + 2 = |C_2| = |C_3| + 1 = |C_4|$. 

(b) $\mathcal{M}_c(G) = \{C_0, C_2, C_4\}$, $m_c(G) = \{C_1, C_3\}$ and $|C_0| = |C_1| + 2 = |C_2| = |C_3| + 2 = |C_4|$. 

**Proof.** If $G$ contains maximum number of edges, $|\mathcal{M}_c(G)| + |m_c(G)| = t = 4$ and $\nabla_c(G) \geq 2$, then the conclusion (a) directly follows from Propositions 3.9 and 3.11.

If $G$ contains maximum number of edges, $|\mathcal{M}_c(G)| + |m_c(G)| = t + 1 = 5$ and $\nabla_c(G) \geq 2$, then the conclusion (b) follows from Proposition 3.13. However, following Propositions 3.9 and 3.12, we have $\mathcal{M}_c(G) = \{C_0, C_1, C_3, C_4\}$ and $m_c(G) = \{C_2\}$, where $|C_0| = |C_1| = |C_2| = |C_3| = |C_4|$. But such a graph $G$ satisfies the property that for each $X \in \mathcal{M}_c(G)$, $G' := \mathcal{I}_{C_2} \mathcal{I}_X [G]$ implies $|\mathcal{M}_c(G')| = 3$ and there exists exactly one integer $i$, with $i = 0$ or $i = 3$, such that $\{C_i', C_{i+1}'\} \subset \mathcal{M}_c(G')$. Hence, $K_{G'}(C_0', \ldots, C_4') = 1$. Suppose $H$ is a graph with lambda chromatic number $t = 4$, $\nabla_c(H) = 1$, $|\mathcal{M}_c(H)| = 3$ with respect to the underlying optimal lambda colouring $c : V(H) \rightarrow \{0, \ldots, 4\}$. Then using Theorem 4.2, such $H$ contains maximum number of edges if and only if the value of $K_H(C_0, \ldots, C_4)$ is 0. Here $G'$ satisfies $t = 4$, $\nabla_c(G') = 1$, $|\mathcal{M}_c(G')| = 3$ but $K_{G'}(C_0', \ldots, C_4') = 1$. Therefore, such $G'$ can not contain maximum number of edges. We note that $|E(G)| = |E(G')|$. Hence, such a graph $G$ cannot contain maximum number of edges.

**Theorem 4.5.** Let $G$ be an edge standardised graph with lambda chromatic number 4 and $C_0, \ldots, C_4$ be the coloured partition of the vertex set of $G$ with respect to the underlying optimal lambda colouring $c$. If the coloured partition or its respective dual coloured partition satisfies one of the following two properties, then $G$ contains maximum number of edges.

(a) $\mathcal{M}_c(G) = \{C_0, C_2, C_4\}$, $m_c(G) = \{C_1\}$ and $|C_0| = |C_1| + 2 = |C_2| = |C_3| + 1 = |C_4|$. 

(b) $\mathcal{M}_c(G) = \{C_0, C_2, C_4\}$, $m_c(G) = \{C_1, C_3\}$ and $|C_0| = |C_1| + 2 = |C_2| = |C_3| + 2 = |C_4|$. 

**Proof.** We assume (a) holds. Let $G' := \mathcal{I}_{C_1} \mathcal{I}_{C_4} [G]$. Here $\nabla_c(G') = 1$ and $|\mathcal{M}_c(G')| = 2$. Also $K_{G'}(C_0', \ldots, C_4') = 0$, where $C_0', \ldots, C_4'$ is the transformed (coloured) partition of $C_0, \ldots, C_4$. Therefore, using Theorem 4.2, we have $G'$ contains maximum number of edges. We note that $|E(G)| = |E(G')|$. Hence, such $G$ contains maximum number of edges, and the result follows for this case.

We assume (b) holds. Let $G' := \mathcal{I}_{C_3} \mathcal{I}_{C_2} \mathcal{I}_{C_1} \mathcal{I}_{C_4} [G]$. Here $\nabla_c(G') = 1$ and $|\mathcal{M}_c(G')| = 1$. Hence, $K_{G'}(C_0', \ldots, C_4') = 0$, where $C_0', \ldots, C_4'$ is the transformed (coloured) partition of $C_0, \ldots, C_4$. Therefore, using Theorem 4.2, we
have $G'$ contains maximum number of edges. We note that $|E(G)| = |E(G')|$. Hence, such $G$ contains maximum number of edges, and the result follows for this case.

To propose the classification results, we cannot restrict ourselves only to edge standardised graphs. So we need to get rid of the “edge distribution”-related restriction that makes a graph edge standardised. Henceforth, the graphs are not necessarily edge standardised.

**Theorem 4.6.** Let $G$ be a graph with lambda chromatic number $t \geq 3$ and $C_0, \ldots, C_t$ be the coloured partition of the vertex set of $G$ with respect to an optimal lambda colouring $c$. If $G$ contains $n \geq t + 1$ many vertices, where $n$ is a multiple of $t + 1$, then $G$ contains maximum number of edges if and only if $G$ is a member graph of $G(t, \frac{n}{t+1})$.

**Proof.** Let $G$ be a member graph of $G(t, \frac{n}{t+1})$. Then $G$ has $n$ many vertices and its lambda chromatic number is $t$. It follows from Theorem 2.7, that members of $G(t, l)$, for some integer $l$, has maximum number of edges among the graphs with lambda chromatic number $t$ and at most $l(t+1)$ many vertices. Hence, for $l = \frac{n}{t+1}$, $G$ has maximum number of edges.

Conversely, suppose $G$ has maximum number of edges and $c : V(G) \rightarrow \{0, \ldots, t\}$ is an optimal lambda colouring of $G$. If $\nabla_c(G) = 1$, then $(t+1)$ does not divide $n$. A contradiction arises. Now if $\nabla_c(G) \geq 2$, then we construct $\mathcal{S}_c[G]$. From the definition of edge standardised graph, we have $V(\mathcal{S}_c[G]) = V(G)$. Using Proposition 3.4, we have the lambda chromatic number of $\mathcal{S}_c[G]$ is $t$. Since $G$ contains maximum number of edges, again using Proposition 3.4 we have $|E(\mathcal{S}_c[G])| = |E(G)|$. This means the edge standardised graph $\mathcal{S}_c[G]$ contains maximum number of edges. Therefore, using Propositions 3.11, 3.12 and 3.13, we have $t = 3$ or $t = 4$. Consequently, using Theorems 4.3 and 4.5, we have $\nabla_c(G) = \nabla_c(\mathcal{S}_c[G]) = 2$ or $3$. If $\nabla_c(G) = \nabla_c(\mathcal{S}_c[G]) = 2$ or $3$, then it follows from Theorems 4.3 and 4.5 that $n = |V(\mathcal{S}_c[G])| = |V(G)|$ is of the form $s(t + 1) + r$, where $s \geq 0$ and $r \geq 1$ are integers. Here $r$ equals $5$ mod $4$, $5$ mod $4$, $6$ mod $4$, $9$ mod $4$, $7$ mod $5$ and $6$ mod $5$ for the respective cases. Hence, we conclude $(t + 1)$ does not divides $n$. This leads to a contradiction. Therefore, $\nabla_c(G) = 0$ and consequently $G$ is a member graph of $G(t, \frac{n}{t+1})$.

We now define a stationary graph. Such graphs behave in conformity with the necessary conditions, developed in Theorems 4.2, 4.3, 4.4 and 4.6. The intrinsic local restrictions of an optimal lambda colouring are also maintained.

**Definition.** An $n$-vertex graph $G$ is said to be a stationary graph if the vertex set is partitioned into $t + 1$ subsets $V_0, \ldots, V_t$, where $n \geq t + 1 \geq 4$, and the edge distribution follows the following four properties.

- $V_0$ and $V_t$ are non-empty and if for some integer $i$, with $1 \leq i \leq t - 1$, $V_i$ is empty then both $V_{i-1}$ and $V_{i+1}$ are non-empty.
- $e_c(V_i, V_j) = 0$ for each integer $i$, with $0 \leq i \leq t$.
- $e_G(V_i, V_{i+1}) = 0$ for each integer $i$, with $0 \leq i \leq t - 1$. 


\textbf{Lemma 4.7.} Let $G$ be an $n$-vertex stationary graph. Then lambda chromatic number of $G$ is at most $t$. Moreover, for each integer $i$, with $0 \leq i \leq t$, and for each $v \in V_i$, $v \mapsto i$ is a lambda colouring of $G$.

\textbf{Proof.} To show the lambda chromatic number of $G$ is at most $t$, it is enough to establish the mapping mentioned in the statement (say) $c$ is a lambda colouring of $G$.

If $d_G(u, v) \geq 3$, then $|c(u) - c(v)| + d_G(u, v) \geq 3$. If for some $u \in V_i$ and $v \in V_j$, where $0 \leq i, j \leq t$, suppose $d_G(u, v) = 1$, then $j \neq i$ since $e_G(V_i, V_j) = 0$ for each integer $i$, with $0 \leq i \leq t$. Also $e_G(V_i, V_{i+1}) = 0$ for each integer $i$, with $0 \leq i \leq t - 1$. Hence, $|c(u) - c(v)| = |i - j| \geq 2$. Consequently, $|c(u) - c(v)| + d_G(u, v) \geq 3$ whenever $d_G(u, v) = 1$ for some $u, v \in V(G)$. If for some $u, v \in V(G)$, suppose $d_G(u, v) = 2$, then $u$ and $v$ can not belong to same $V_i$, where $0 \leq i \leq t$. Otherwise there would exist $w \in V_j$, where $j \neq i$, such that $\{u, w\}$ and $\{v, w\}$ are edges. This is a contradiction. Hence, $c(u) \neq c(v)$, i.e. $|c(u) - c(v)| + d_G(u, v) \geq 3$ whenever $d_G(u, v) = 2$ for some $u, v \in V(G)$.

\textbf{Lemma 4.8.} Let $G$ be an $n$-vertex graph with lambda chromatic number $t \geq 3$ and $n \geq t + 1$. If $G$ contains maximum number of edges, then $G$ is a stationary graph.

\textbf{Proof.} Let $c : V(G) \to \{0, \ldots, t\}$ be an optimal lambda colouring of $G$ and $C_0, \ldots, C_t$ be the corresponding coloured partition of $V(G)$. This partition of $V(G)$ satisfies the edge distribution related properties in the definition of a stationary graph. By Proposition 3.4, $c$ is also an optimal lambda colouring of $\mathcal{S}_c[G]$. We note that if $G$ contains maximum number of edges then $|E(G)| = |E(\mathcal{S}_c[G])|$. From the definition of edge standardised graph, we have $V(\mathcal{S}_c[G]) = V(G)$. Moreover, using Proposition 3.4, we have lambda chromatic number of $\mathcal{S}_c[G]$ is $t$. Hence, $\mathcal{S}_c[G]$ contains maximum number of edges. Also note that $C_i = \{u \in V(\mathcal{S}_c[G]) : c(u) = i\}$ for each integer $i$, with $0 \leq i \leq t$.

Now $\nabla_c(\mathcal{S}_c[G]) = \nabla_c(G)$. If $\nabla_c(G) \geq 2$, then using Theorems 4.3 and 4.4, the (coloured) partition $C_0, \ldots, C_t$ of the vertex set of $\mathcal{S}_c[G]$ (and hence, $G$) follows exactly one of the conditions stated from (b) to (g), in the definition of
stationary graph. If $\nabla_c(G) \leq 1$, then the vertex partition $C_0, \ldots, C_t$ of vertex set follows condition (a) in the definition of the stationary graph.

Now we are in a position to establish our final classification results. This concludes our article.

**Theorem 4.9.** Let $G$ be an $n$-vertex graph with lambda chromatic number $t \geq 3$ and $n \geq t + 1$. Then $G$ contains maximum number of edges if and only if exactly one of the following holds.

(i) $G$ is isomorphic to an $n$-vertex member graph $G^*$ of $G(t, \frac{n}{t + 1})$, where $n \equiv 0 \mod t + 1$.

(ii) $G$ is isomorphic to an $n$-vertex stationary graph satisfying the property (a) (mentioned in the definition), such that the value $K_G(V_0, \ldots, V_t)$ is minimum over any equitable partition into $t + 1$ unequally sized parts (subsets) of the vertex sets of all possible $n$-vertex graphs, where $n \not\equiv 0 \mod t + 1$.

(iii) $G$ is isomorphic to exactly one of the $n$-vertex stationary graph satisfying the property (b) to (g) (mentioned in the definition), where $n \not\equiv 0 \mod t + 1$.

**Proof.** If $n \equiv 0 \mod t + 1$, then from Theorem 4.6, $G$ has maximum number of edges if and only if (i) holds.

Suppose $n \not\equiv 0 \mod t + 1$. Let (ii) hold. Then $V_0, \ldots, V_t$ is a partition of $V(G)$ and there exist integers $i, j$, with $0 \leq i, j \leq t$, such that $|V_i| = 1 + |V_j|$. Since the graph $G$ is a stationary graph, the colouring $c : V(G) \rightarrow \{0, \ldots, t\}$ defined by $c(v) = i$, where $v \in V_i$, $0 \leq i \leq t$, is an optimal lambda colouring of $G$. Clearly, $V_0, \ldots, V_t$ is the underlying coloured partition of the edge standardised graph $\mathcal{S}_c[G]$. Now $e_G(V_i, V_j) = e_{\mathcal{S}_c[G]}(V_i, V_j)$, $0 \leq i, j \leq t$. Hence, $|E(G)| = |E(\mathcal{S}_c[G])|$. Therefore, using Theorem 4.2, $\mathcal{S}_c[G]$ and hence, $G$ has maximum number of edges.

Let (iii) hold. Then using a similar argument as above, Theorems 4.3 and 4.5, $\mathcal{S}_c[G]$ and hence, $G$ has maximum number of edges.

Conversely, suppose $G$ has maximum number of edges. Then by Lemma 4.8, $G$ is a stationary graph. Therefore, $G$ satisfies exactly one condition from (a) to (g) stated in the definition of the stationary graph. Further $n \not\equiv 0 \mod t + 1$ excludes the fact (i) of this hypothesis. Hence, facts of (ii) and (iii) of hypothesis follow.

The following two results are natural consequences of the above classification theorem.

**Corollary 4.10.** Let $G$ be an $n$-vertex graph with lambda chromatic number $t \geq 5$ and $n \geq t + 1$. If $G$ contains maximum number of edges, then $G$ admits an optimal equitable partition.

The converse of the above corollary is not true. However, if an $n$-vertex stationary graph $G$ with equitable partition $V_0, \ldots, V_t$ satisfies the property that the value $K_G(V_0, \ldots, V_t)$ is minimum over any equitable partition into $t + 1$ parts (subsets) of the vertex sets of all possible $n$-vertex graphs, then $G$ contains maximum number of edges.
Corollary 4.11. Let \( G \) be an \( n \)-vertex graph with lambda chromatic number \( t \geq 3 \) and \( n \geq t + 1 \). If \( G \) contains maximum number of edges then there exist a member graph \( G^* \) of \( G(t, \lfloor \frac{n}{t+1} \rfloor) \) and a member graph \( G^{**} \) of \( G(t, \lfloor \frac{n}{t+1} \rfloor + 3) \) such that \( G^* \) is subgraph of \( G \) and \( G \) is subgraph of \( G^{**} \).

The above corollary connotes an approximation result. Roughly, an \( n \)-vertex graph with lambda chromatic number \( t \geq 3 \), where \( n \geq t + 1 \), and having maximum number of edges can be approximated by an “inner” graph \( G^* \) and an “outer” graph \( G^{**} \).

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