Gaussian systems for quantum enhanced multiple phase estimation

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For a fixed average energy, the simultaneous estimation of multiple phases can provide a better total precision than estimating them individually. We show this for a multimode interferometer with a phase in each mode, using Gaussian inputs and passive elements, by calculating the covariance matrix. The quantum Cramér-Rao bound provides a lower bound to the covariance matrix via the quantum Fisher information matrix, whose elements we derive to be the covariances of the photon numbers across the modes. We find our simultaneous strategy to yield no more than a factor of two improvement in total precision. In spite of the Gaussian nature of the problem, these elements require the calculation of non-Gaussian integrals, which we accomplish analytically. We prove that this bound can be saturated. Our work shows that no modal entanglement is necessary for simultaneous quantum-enhanced estimation of multiple phases.

Introduction: Parameter estimation with quantum enhanced precision has the potential to provide substantial technological advances as well as deep insights into the fundamental workings of Nature. Originating in the quest for the increased sensitivity requirements for detecting gravitational waves using laser interferometers with squeezed light [1, 2], the field now encompasses a variety of scenarios studying the quantum limits of sensing [3–6]. Relative phase estimation in a two-mode interferometer is by far the most common, although some attention has also been cast to the simultaneous estimation of multiple parameters at the quantum limit [7–11].

A fundamental bound on the precision of an estimation is the quantum limit on the variance of the estimator. This is set by the quantum Cramér-Rao bound [12] and valuable insights into the working of quantum mechanics have been obtained by studying it in the multi-parameter scenario [13–15]. In addition to this fundamental understanding, several scenarios of practical and technological interest are intrinsically multi-parameter estimation problems, leading to new methodologies of obtaining quantum enhancements arising purely from the multidimensional nature of the problem. This includes magnetic field sensing in three dimensions [16] and imaging [10, 17–19]. These proposed schemes use a fixed number of photons in multimode entangled states, which however are not easy to prepare for increasing photon numbers.

In this Letter we show that, for a fixed average amount of energy, simultaneous estimation of a fixed number of phase parameters is better than individual estimation, as shown in Figs. 1 and 2 respectively. We do so by obtaining an analytical expression for the quantum Fisher information matrix (QFIM) as a function of the number of phases and the total average energy. The QFIM bounds the covariance matrix for multiple phase estimation, and our results are derived for Gaussian states in terms of the Husimi Q function. Gaussian states are easier to prepare in practice than fixed particle number states, and while couched in the language of optical systems, our work also applies to bosonic degrees of freedom of matter systems. We show that our bounds are attainable, and discuss some implications.

Our work can thus improve the performance of optical techniques in quantum imaging [20] and possibly gravitational wave astronomy [21], as well as opto-mechanical systems employed in fundamental studies [22, 23]. Some of these have been studied experimentally in quantum optics, where noise reduction has been observed using correlated photon pairs [24] and multimode squeezed light [25].

The problem: We study the quantum-limited estimation of d phases $\phi \in \mathbb{R}^d$ using a $d+1$-mode pure quantum probe state $|\Psi\rangle$, as shown in Fig. 1. The state $|\Psi\rangle$ picks up the phases $\phi$...
via [26]

\[ |\Phi\rangle = \hat{U}_\phi |\Psi\rangle. \]

(1)

The parameters to be estimated are encapsulated in

\[ \hat{U}_\phi = \hat{U}_\phi' \exp(-i\phi n_0) \]

\[ = \exp(i\phi_1 (n_1 - n_0) + \ldots + i\phi_d (n_d - n_0)) = \]

\[ \exp \left( i \sum_{i=1}^d \phi_i (\hat{n}_i - \hat{n}_0) \right) = \exp(i\phi \hat{g}), \]

(2)

where the unitary operator \( \hat{U}_\phi' = \exp(i\phi_0 \hat{n}_0 + i\phi_1 \hat{n}_1 + \ldots + i\phi_d \hat{n}_d) \), \( \phi = \phi_0 + \ldots + \phi_d \) captures an unmeasurable overall phase, \( \phi \equiv (\phi_1, \ldots, \phi_d)^T \) and \( \hat{g}_i = \hat{n}_i - \hat{n}_0 \) are the generators. The \( \hat{g}_i \) are traceless and hermitian, as \( SU(n) \) generators ought to be. Indeed, our problem is a special case of \( SU(n) \) interferometry, with the parameters to be estimated restricted to a diagonal subgroup. The reduction from the unitary \( \hat{U}_\phi' \) to the special unitary \( \hat{U}_\phi \) is therefore tantamount to accounting for the unmeasurable phase \( \phi \). The nuanced role of a reference mode in quantum interferometry was recently addressed [26].

Methods: The performance of any estimation process is captured by the covariance matrix \( V(\phi) \), the covariance of the estimators for unbiased estimators. This is lower bounded as

\[ V(\phi) \geq H^{-1}, \]

(3)

according to the quantum Cramér-Rao bound, where \( H \) is the quantum Fisher information matrix (QFIM) [3, 31]. This is a matrix inequality in that \( V(\phi) - H^{-1} \) is positive semidefinite, and the QFIM \( H \) is real, positive definite, symmetric matrix. For any pure state \( |\Psi\rangle \), the QFIM reads \( H_{i,j} = 4 \Re(\langle \partial_i \hat{\Phi}\rangle \partial_j \hat{\Phi}) - \langle \partial_i \hat{\Phi}\rangle \langle \partial_j \hat{\Phi} \rangle \), where \( \Re(\cdot) \) denotes the real part and \( |\partial_i \hat{\Phi}\rangle = (\partial/\partial \phi_i) |\Phi\rangle \). For \( d \) phase parameters and the corresponding phase shift generators \( \{\hat{g}_i\} \), the \( d \times d \) QFIM reduces to [16]

\[ H_{i,j} = 4(\langle \hat{g}_i \hat{g}_j \rangle - \langle \hat{g}_i \rangle \langle \hat{g}_j \rangle) = 4(h_{i,j} - h_{i,0} - h_{0,j} + h_{0,0}), \]

(4)

with \( h_{i,j} = \langle \hat{n}_i \hat{n}_j \rangle - \langle \hat{n}_i \rangle \langle \hat{n}_j \rangle \). The expectation values are calculated for the initial state \( |\Psi\rangle \). Note that for \( H_{i,j} \), the indices \( i, j \in \{1, \ldots, d\} \), while for \( h_{i,j} \), \( i, j = 0 \) should be included. Note that \( h_{i,0} = h_{0,i} = h_{0,0} \) give rise to rank one matrices, and therefore the QFIM can be inverted using the Sherman-Morrison formula.

We use Husimi Q representation to calculate the expectation values in Eq. (4). To that end, we begin with the Q representation [32] for the initial squeezed displaced states \( \prod_{k=0}^d |\beta_k; \xi_k\rangle \), which reads,

\[ Q_0(r) = \frac{1}{\pi^{d+1}} \prod_{k=0}^d \left| \langle \alpha_k | \beta_k; \xi_k \rangle \right|^2 \]

\[ = \frac{1}{\pi^{d+1}} \prod_{k=0}^d \frac{1}{\cosh |\xi_k|} \times \]

\[ \prod_{k=0}^d \exp \left[ -\frac{|\alpha_k|^2}{2} - \frac{|\beta_k|^2}{2} + \beta_k^* \alpha_k \right] \]

\[ - \frac{1}{2} \tanh |\xi_k| (|\alpha_k|^2 - |\beta_k|^2)^2, \]

(5)

where \( |\alpha_k\rangle \) is a coherent state, \( r = (\alpha, \alpha^*)^T = (\alpha_0, \ldots, \alpha_d, \alpha_0^*, \ldots, \alpha_d^*)^T \). The Q representation of the final probe state \( |\Psi\rangle \) is then given by \( Q(\alpha') = |\langle \alpha' | \Psi \rangle|^2 \), where \( \alpha' = A \alpha \). Our calculation thus exploits the simplicity of applying \( A \) on the coherent state basis rather than its conjugate on the squeezed displaced states.

Further simplification is enabled by the passive nature of the transformation \( A \) which implies \( |\alpha'|^2 = |\alpha|^2 \) and the \( \phi \)-independence of the QFIM, which can be seen in Eq. (4). The Q representation of \( |\Psi\rangle \) is thus App. 1.

\[ Q(r) = F(\beta) \exp \left( -\frac{r^\dagger M r}{2\pi} + \frac{r^\dagger r - r^\dagger r_b}{2} \right), \]

(6)

where

\[ F(\beta) = \prod_{k=0}^d \exp \left[ -\left( |\beta_k|^2 + \frac{\tanh |\xi_k|}{2} (|\beta_k|^2 + |\beta_k|^2) \right) \right], \]

with \( r_b = (b_0, \ldots, b_d, b_0^*, \ldots, b_d^*)^T \equiv (b, b^*)^T \) and

\[ b_j = \sum_{k=0}^d A_{kj}^* (\beta_k + \beta_k^* \tanh |\xi_k|). \]

(7)

The \( 2(d+1) \times 2(d+1) \) matrix \( M \) reads,

\[ M = \frac{1}{2} \left( \begin{array}{cc} 1 & N \\ \ \ I \\\n N^\dagger & I \end{array} \right) \]

(8)
with \( N = A^\dagger D A^* \), where \( D \) is a diagonal matrix with \( D_{ij} = \tanh |\xi_i| \). Note that matrix \( N \) is symmetric, i.e., \( N = N^T \), a fact to be explored later.

To calculate the QFIM in Eq. (4) using the \( Q \) representation of the probe state \( |\Psi\rangle \) at hand, we need to recast the expectation values in terms of anti-normally ordered operators. These are

\[
\langle \hat{n}_i \rangle = \langle \hat{a}_i \hat{a}_i^\dagger \rangle = 1 - (1 + \delta_{ij})\langle \hat{a}_i \hat{a}_j^\dagger \rangle,
\]

and can be obtained via a generating function \( G(\mu) \) App. II,

\[
\langle \hat{n}_i \rangle = -1 + \left( \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \lambda_j^*} \right) G(\mu]\bigg|_{\mu = 0}.
\]

The generating function, which is based on the \( Q \) representation in Eq. (6) is given by App. II,

\[
G(\mu) = \exp \left[ r_k M^{-1} \mu + \mu^\dagger M^{-1} r_k + \mu^\dagger M^{-1} \mu \right],
\]

where \( \mu = (\lambda_0, \ldots, \lambda_d, \gamma_0, \ldots, \gamma_d)^T \). Note that the derivatives required to calculate the QFIM render the relevant integrals non-Gaussian. Finally, the inverse of \( M \), obtained using Schur’s complement, is

\[
M^{-1} = 2 \left( \begin{array}{cc} E & -N^T E \\ -NE & E^T \end{array} \right) \],
\]

where \( E = A^\dagger C A \) with \( C \) a diagonal matrix whose non-zero elements read \( C_{ij} = \cosh^2 |\xi_i| \). Note that \( E^\dagger = E \).

**Results:** By virtue of Eqs. (11), (12), and (13), the elements \( h_{i,j} \) are,

\[
h_{i,j} = 4(\langle EN - \gamma \gamma^T \rangle - \langle \gamma \gamma^T \rangle)^2 - \gamma^2 (\langle \gamma \gamma^T \rangle)^2
\]

\[
+ \frac{1}{4} (E + E^\dagger) + 2 \gamma \gamma^T + 2 \gamma^2 + 2 \gamma^2 \gamma^T
\]

\[
- (E + \gamma^T I)_{ij},
\]

where \( \gamma = (2E^\dagger - E^\dagger N^* - N^* E) b/2, \delta_{i,j} \) is the Kronecker delta and \( \gamma \) denotes the Hadamard (entrywise) product.

Having obtained the general formula for the QFIM, we make two simplifying assumptions to obtain tractable analytical expressions, namely equally squeezed inputs in all the modes \( \langle |\xi_i| \rangle = |\xi| \). \( \forall i \in \{0, \ldots, d\} \) and an orthogonal interferometer \( (A^\dagger = O^T \in SO(d+1)) \). What follows in this Letter relies on these assumptions. For general displacements \( \beta_k = x_k + iy_k \), a straightforward computation leads to a diagonal plus a rank-one matrix,

\[
H_{i,j} = \delta_{i,j} h_{i,i} + h_{0,0},
\]

where \( h_{i,i} = 2 \sinh^2 2|\xi| + 4e^{-2|\xi|} x_i^2 + 4e^{2|\xi|} y_i^2 \) and \( h_{0,0} = 2 \sinh^2 2|\xi| + 4e^{-2|\xi|} x_0^2 + 4e^{2|\xi|} y_0^2 \) with \( x_i = \sum_{k=0}^d O_{i,k} x_k \) and \( y_i = \sum_{k=0}^d O_{i,k} y_k \). In matrix notation the QFIM reads,

\[
H = H' + h_{0,0} uu^T,
\]

where \( H'_{ij} = \delta_{i,j} h_{i,i} \) and \( u = (1, \ldots, 1)^T \).

We can now bound the total variance of all the parameters, given by \( \text{Tr}(N(\phi)) \). This requires the inverse of the QFIM which, obtained by the Sherman-Morrison formula, is

\[
H^{-1} = H'^{-1} - \frac{h_{0,0}}{1 + h_{0,0} uu^T} uu^T H'^{-1},
\]

leading to

\[
\text{Tr}(H^{-1}) = \sum_{i=1}^d \frac{1}{h_{i,i}} \left( \sum_{i=0}^d \frac{1}{h_{i,i}} \right)^{-1} \sum_{i=1}^d \frac{1}{h_{i,i}}. \]

**Simultaneous vs individual phase estimation:** The optimal input for estimating the relative phase in a balanced two-mode interferometer is a squeezed state \([33]\). We extend this result within the aforementioned assumptions and prove that for any \( d \), all the energy should go to squeezing for maximal precision in estimation. We do this by first showing that minimising \( \text{Tr}(H^{-1}) \) is akin to maximising each \( h_{i,i} \) independently. Note that \( h_{i,i} \) is actually a monotonic function of the fraction of the total energy in displacements, and we show that this quantity is maximum when all the energy is used in squeezing App. III. This leads to an optimal QFIM for simultaneous estimation of

\[
H_{\text{sim}} = 2(I + uu^T) \sinh^2 2|\xi| \]

The QFIM \( H_{\text{ind}} \) for individual phase estimation comes from the above equation with \( d = 1 \). We can now compare the quantum limits for the simultaneous estimation of the \( d \) phases with their individual estimation for the same expense of energy. The total energy is

\[
E = \sum_{i=0}^d (x_i^2 + y_i^2) + (d+1) \sinh^2 |\xi| = 2d \sinh^2 |\xi|,
\]

where \( \xi \) is the squeezing used for individual estimation. The ratio of the performance of the two estimation strategies is given by

\[
R = \frac{\text{Tr}(H_{\text{sim}}^{-1})}{\text{Tr}(H_{\text{ind}}^{-1})} = 1 - \frac{d-1}{2d} \tanh^2 |\xi|.
\]

In Fig. 3 the behaviour of \( R \) as a function of \( |\xi| \) and \( d \) is shown. Since \( R \leq 1 \), the simultaneous estimation strategy is superior to the individual estimation strategy. It is also easy to see that \( R \geq (1 + 1/d)/2 \). That the ratio \( R \) saturates to 1/2 is unlike the fixed photon number scenario \([10]\) where the limit goes to 0, although in both cases they fall linearly with \( d \). Possible causes for this are the restriction to Gaussian systems and our assumptions of equal squeezing and orthogonal transformations. While the limited quantum information processing
Having obtained the QFIM for multiple phase estimation, and shown that it is better than individual estimation, we now prove that this limit is attainable. A necessary and sufficient condition for the attainability of the quantum limit is \([40, 41]\),
\[
\langle \Phi | \left[ \hat{L}_i, \hat{L}_j \right] | \Phi \rangle = \text{Tr} \left( \hat{\rho}_\phi \left[ \hat{L}_i, \hat{L}_j \right] \right) = 0, \tag{23}
\]
where \(\hat{L}_i\) is the symmetric logarithmic derivative for the parameter \(\phi_i\) given via
\[
\frac{\partial \hat{\rho}_\phi}{\partial \phi_i} = \frac{\hat{L}_i \hat{\rho}_\phi + \hat{\rho}_\phi \hat{L}_i}{2}, \tag{24}
\]
we find \(\hat{L}_i = 2i \hat{\eta}_i \hat{\eta}_0 \). Using the fact that number operators for different modes commute, the cyclic permutation property of the trace and purity of the probe states \(| \Psi \rangle\), it is easy to show that the condition in Eq. (23) is satisfied.\[
\frac{\partial \hat{\rho}_\phi}{\partial \phi_i} = i \{ (\hat{\eta}_i - \hat{\eta}_0), \hat{\rho}_\phi \}. \tag{25}
\]

Conclusions: We have considered the problem of multiple phase estimation with Gaussian states and have shown that, under some assumptions, the simultaneous estimation of \(d\) phases is always superior to the optimum individual estimation strategy. A tentative cause for this improvement is that the simultaneous strategy utilises fewer reference modes, allowing more energy per mode. Our analyses have shown that the larger the variance within a mode the better the estimation. The optimal input states for individual and simultaneous strategies are product squeezed vacuum states and so the distinction boils down to the number of modes, as the simultaneous strategy uses fewer reference modes it allows a larger variance per mode and thus an improved precision. It may be for related reasons that the high energy limit of the performance ratio of the two strategies coincides with the ratio of the number of modes \(- (d + 1)/2d\).

Finally, we note that we obtain quantum enhancements from simultaneous estimation without the presence of any quantum entanglement across the modes in the system. The latter is a consequence of the two assumptions, parallel squeezings and orthogonal transformation, we made to obtain analytically tractable expressions. Nevertheless, they provide a generalisation to multimode interferometry what was known for two mode interferometers \([33, 42]\) - that modal entanglement is not a crucial resource for quantum-enhanced interferometry.

Acknowledgements: We thank Tillmann Baumgratz and George Knee for useful discussions. This work was supported by the UK EPSRC (EP/K04057X/2) and the National Quantum Technologies Programme (EP/M01326X/1, EP/M013243/1).
APPENDICES

I. Computation of the $Q$ representations

Initially we consider $d + 1$ squeezed displaced states, i.e. $\prod_{k=0}^{d} |\beta_k; \xi_k\rangle$, where $\xi_k = |\xi_k|e^{i\theta_k}$. From the definition of the $Q$ representation $Q(r) = 1/(\pi^{d+1})\langle \alpha | \hat{\beta} | \alpha \rangle$, with $r = (\alpha, \alpha^*)^T \equiv (\alpha_0, \ldots, \alpha_d, \alpha_0^*, \ldots, \alpha_d^*)^T$, one can immediately write,

$$Q_0(r) = \frac{1}{\pi^{d+1}} \prod_{k=0}^{d} |\langle \alpha_k | \beta_k; \xi_k \rangle|^2. \quad (A1)$$

The amplitude $\langle \alpha | \beta; \xi \rangle$ can be found as follows,

$$\langle \alpha | \beta; \xi \rangle = \frac{1}{\sqrt{\cosh |\xi|}} \exp \left( -\frac{|\beta|^2}{2} - \frac{\beta^* e^{i\theta} \tanh |\xi|}{2} \right) \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} H_n(\tau) \left( \frac{e^{i\theta} \tanh |\xi|}{2} \right)^{n/2} \langle \alpha | n \rangle$$

$$= \frac{1}{\sqrt{\cosh |\xi|}} \exp \left( -\frac{|\beta|^2}{2} - \frac{\beta^* e^{i\theta} \tanh |\xi|}{2} \right) \sum_{n=0}^{\infty} \frac{1}{n!} H_n(\tau) \left( \frac{e^{i\theta} \tanh |\xi|}{2} \right)^{n/2} (A2)$$

where $H_n(\tau)$ is the Hermite polynomial of the $n$-th order with $\tau = (\beta + \beta^* e^{i\theta} \tanh |\xi|)/(2e^{i\theta} \tanh |\xi|)^{1/2}$. We have also used the expansion of a squeezed state in Fock basis [43] and the Hermite polynomials generating function [44],

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(\tau) \left( \frac{u}{2} \right)^n = \exp (2\tau u - u^2). \quad (A3)$$

From Eqs. (A1) and (A2) we write,

$$Q_0(r) = \frac{1}{\pi^{d+1}} \prod_{k=0}^{d} \frac{1}{\cosh |\xi_k|} \prod_{k=0}^{d} \left| \exp \left[ -\frac{|\alpha_k|^2}{2} - \frac{|\beta_k|^2}{2} + \beta_k \alpha_k^* \frac{1}{2} e^{i\theta_k} \tanh |\xi_k|(\alpha_k^* - \beta_k)^2 \right] \right|^2$$

$$= \frac{1}{\pi^{d+1}} \prod_{k=0}^{d} \frac{1}{\cosh |\xi_k|} \times$$

$$\exp \left[ \sum_{k=0}^{d} \left( -|\alpha_k|^2 - |\beta_k|^2 + \beta_k \alpha_k^* + \beta_k^* \alpha_k \frac{1}{2} \tanh |\xi_k| (e^{i\theta_k}(\alpha_k^* - \beta_k)^2 + e^{-i\theta_k}(\alpha_k - \beta_k)^2) \right) \right]. \quad (A4)$$

The state $\prod_{k=0}^{d} |\beta_k; \xi_k\rangle$ goes through the interferometer denoted as $\hat{A}^\dagger$ and we take the state $|\Psi\rangle = \hat{A}^\dagger \prod_{k=0}^{d} |\beta_k; \xi_k\rangle$. The $Q$ representation of the $|\Psi\rangle$ state is,

$$Q(r) = \frac{1}{\pi^{d+1}} |\langle \alpha | \hat{A}^\dagger | \prod_{k=0}^{d} |\beta_k; \xi_k\rangle|^2. \quad (A5)$$

It is apparent that it is a lot easier if we act with $\hat{A}^\dagger$ on the left, i.e on $\langle \alpha |$, that is we consider the transformation $\alpha' = \Lambda \alpha$ or $\alpha'_k = \sum_{j=0}^{d} A_{k,j} \alpha_j$. Note since we consider passive transformations the total energy before and after the interferometer is
By observing Eq. (A6) we can write it in a compact form, 

\[ Q(r) = \frac{1}{\pi^{d+1}} \frac{1}{\prod_{k=0}^{d} \cosh \left| \xi_k \right|} \exp \left[ -\sum_{k=0}^{d} \left( |\beta_k|^2 + \frac{1}{2} \tanh |\xi_k| \left( e^{i\theta_k} \beta_k^2 + e^{-i\theta_k} \beta_k^2 \right) \right) \right] \times \exp \left[ -\sum_{k=0}^{d} \left( |\alpha_k|^2 + \frac{1}{2} \tanh |\xi_k| \left( e^{i\theta_k} \sum_{j=0}^{d} A_{k,j}^* \alpha_j + e^{-i\theta_k} \sum_{j=0}^{d} A_{k,j} \alpha_j \right) \right) \right] \times \exp \left[ \sum_{k=0}^{d} \beta_k \left( \sum_{j=0}^{d} A_{k,j}^* \alpha_j + e^{-i\theta_k} \sum_{j=0}^{d} A_{k,j} \alpha_j \right) \right] \times \exp \left[ -\sum_{k=0}^{d} \beta_k^* \left( \sum_{j=0}^{d} A_{k,j} \alpha_j + e^{i\theta_k} \sum_{j=0}^{d} A_{k,j}^* \alpha_j \right) \right] \right] \times \right] \]

(A6)

By observing Eq. (A6) we can write it in a compact form,

\[ Q(r) = F(\beta, \beta^*) \frac{\exp \left( -r^\dagger Mr + r^\dagger r + r^\dagger r_b \right)}{(2\pi)^{d+1} \prod_{k=0}^{d} \cosh \left| \xi_k \right|} \]

where

\[ F(\beta, \beta^*) = e^{-\sum_{k=0}^{d} \left( |\beta_k|^2 + \frac{1}{2} \tanh |\xi_k| \left( e^{i\theta_k} \beta_k^2 + e^{-i\theta_k} \beta_k^2 \right) \right)} \]

(A7)

\[ \beta = (\beta_0, \ldots, \beta_d) \]

(A8)

\[ \beta^* = (\beta_0^*, \ldots, \beta_d^*) \]

(A9)

\[ r_b = (b_0, \ldots, b_d, b_0^*, \ldots, b_d^*)^T = (b, b^*)^T \]

(A10)

with

\[ b_j = \sum_{k=0}^{d} A_{kj}^* \left( \beta_k + \beta_k^* e^{i\theta_k} \tanh |\xi_k| \right). \]

(A11)

The \(2(d+1) \times 2(d+1)\) matrix \(M\) reads,

\[ M = \frac{1}{2} \begin{pmatrix} I & N \\ N^\dagger & I \end{pmatrix} \]

(A12)

with \(N = A^\dagger DA^*\), where \(D\) is a diagonal matrix with \(D_{jj} = e^{i\theta_j} \tanh |\xi_j|\). Note that matrix \(N\) is symmetric, i.e \(N = N^\dagger\). Also the matrix \(M\) is hermitian. In what follows we will need the the matrix \(M^{-1}\), to this end we will use the Schur’s complement [45]. We write,

\[ M^{-1} = 2 \begin{pmatrix} (I - NN^\dagger)^{-1} & -N(I - NN^\dagger)^{-1} \\ -N(I - NN^\dagger)^{-1} & (I - N^\dagger N)^{-1} \end{pmatrix} \]

(A13)

From Eqs. (A12) and (A13) is easy to see that \(M^{-1} M = I\). For the hermitian matrices \(NN^\dagger\) and \(N^\dagger N\) we can readily write their diagonalization (remember that \(A\) is unitary, therefore they diagonalize hermitian matrices),

\[ NN^\dagger = A^\dagger DD^\dagger A \]

(A14)

\[ N^\dagger N = A^T DD^\dagger (A^T)^\dagger. \]

(A15)

Since \(A^\dagger A = I\) and \((A^T)^\dagger A^T = I\) we have,

\[ (I - NN^\dagger)^{-1} = A^\dagger(I - DD^\dagger)^{-1} A \]

(A16)

\[ (I - N^\dagger N)^{-1} = A^T(I - DD^\dagger)^{-1}(A^T)^\dagger. \]

(A17)
The matrix \((I - DD^\dagger)^{-1} \equiv C\). Since \((I - DD^\dagger)^{-1}\) is a diagonal matrix, the matrix \(C\) is easily found to be the diagonal matrix whose non-zero elements read \(C_{j,j} = \cosh^2 |\xi_j|\). Therefore from Eqs. (A13), (A16) and (A17) we write,

\[
M^{-1} = 2 \left( \begin{array}{cc} E & -NE^T \\ -N^T E & E^T \end{array} \right), \tag{A18}
\]

where \(E = A^\dagger CA\).

Let us now prove that the matrix \(M\) is not only hermitian but also positive semi-definite and therefore can be used in the next section as a complex covariance matrix. We will calculate the (real) eigenvalues \(\sigma\) of the (hermitian) matrix \(M\). The characteristic polynomial reads,

\[
\det(M - \sigma I) = \det \left( \left( \frac{1}{2} - \sigma \right) I - \frac{1}{2} N N^\dagger \right) = 0 \tag{A19}
\]

Since the blocks in Eq. (A19) are square and \(N^\dagger\) commutes with \(\left( \frac{1}{2} - \sigma \right) I\), from [46] we can write,

\[
\det(M - \sigma I) = \det \left( \left( \frac{1}{2} - \sigma \right)^2 I - \frac{1}{4} NN^\dagger \right) = 0. \tag{A20}
\]

By virtue of Eq. (A14), the facts that \(A\) is unitary and \(M - \sigma I\) is hermitian, and by substituting the elements of the diagonal matrices \(D\) and \(D^\dagger\), we can write,

\[
\det(M - \sigma I) = \det \left( \left( \frac{1}{2} - \sigma \right)^2 I - \frac{1}{4} DD^\dagger \right) = \prod_{i=0}^{d} \left[ \left( \frac{1}{2} - \sigma_i \right)^2 - \frac{1}{4} \tanh^2 |\xi_i| \right] = 0. \tag{A21}
\]

From Eq. (A21) we readily find,

\[
\sigma_i = \frac{1}{2} (1 + \tanh |\xi_i|) \geq 0 \tag{A22}
\]

II. Generating function and mean values

We introduce the generating function \(G(\mu)\),

\[
G(\mu) = \int dr Q(r) \exp \left( \sum_{j=0}^{d} \lambda_j \alpha_j^* + \sum_{j=0}^{d} \lambda_j^* \alpha_j \right), \tag{A23}
\]

where \(\mu = (\lambda_0, \ldots, \lambda_d, \lambda_0^*, \ldots, \lambda_d^*)^T\). The \(\lambda\)’s are the so-called sources [47], nothing else than some helping parameters when it comes to calculate somewhat difficult integrals [48]. The word sources comes from the fact that some linear terms are added into the exponential. Sometimes this is referred as Feynman’s favourite trick. It is not difficult to see that the integral in Eq. (A23) is just a Gaussian integral and therefore easy to be calculated. Also observe that when we hit Eq. (A23) with derivatives with respect to \(\lambda\)’s at \(\mu = 0\), we get expectation values of combinations of \(\hat{a}, \hat{a}^\dagger\), that justifies the name generating function. This is exactly what we need in order to calculate the QFIM for pure states. Since we use the \(Q\) representation formalism we must calculate expectation values in terms of the mean values of antinormally ordered operators, i.e. all creation operators should be on the right,

\[
\langle \hat{n}_i \rangle = \langle \hat{a}_i \hat{a}_i^\dagger \rangle - 1 \tag{A24}
\]

\[
\langle \hat{n}_i \hat{n}_j \rangle = \langle \hat{a}_i \hat{a}_j \hat{a}_i^\dagger \hat{a}_j^\dagger \rangle - \langle \hat{a}_i \hat{a}_j^\dagger \rangle - \langle \hat{a}_i^\dagger \hat{a}_j \rangle + 1 \tag{A25}
\]

where we have used \([\hat{a}, \hat{a}^\dagger] = 1\). From Eqs. (A23), (A24) and (A25) it is not difficult to see that,

\[
\langle \hat{n}_i \rangle = \left. \left( \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \lambda_i^*} \right) G(\mu) \right|_{\mu=0} - 1 \tag{A26}
\]

\[
\langle \hat{n}_i \hat{n}_j \rangle = \left. \left[ \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \lambda_j^*} \frac{\partial}{\partial \lambda_j} \frac{\partial}{\partial \lambda_i^*} - (1 + \delta_{ij}) \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \lambda_j^*} - \frac{\partial}{\partial \lambda_j} \frac{\partial}{\partial \lambda_i^*} \right] G(\mu) \right|_{\mu=0} + 1 \tag{A27}
\]
So, we have transformed the problem of calculating a non-Gaussian integral (when calculating the mean photon number for example) into one of calculating a Gaussian integral and its derivatives up to 4-th order.

In Eq. (A23) by $d\nu$ we denote integration over all $R\alpha$ and $3\alpha$. However, we find it more convenient to calculate the integral over $\alpha$ and $\alpha^*$. To this end we will need the Jacobian for the transformation $(R\alpha, 3\alpha) \rightarrow (\alpha, \alpha^*)$, which reads $1/2^{d+1}$. By doing the Gaussian integral of Eq. (A23) we find the generating function,

$$G(\mu) = \frac{\exp \left[ (r_b^1 + \mu^1)M^{-1}(r_b + \mu) \right]}{2^{d+1} \det M \prod_{k=0}^{d} \cosh |\xi_k|},$$

(A28)

where $F(\beta, \beta^*)$ was defined in Eq. (A7).

We can simplify the generating function even more by noting that $G(\mu = 0) = 1$ since this is simply the integration of the $Q$ representation over all phase space, i.e. this is just the normalization to 1 of the $Q$ quasi-probability distribution. Therefore we get,

$$G(\mu) = \exp \left[ \frac{1}{4} \left( r_b^1 M^{-1} \mu + \mu^1 M^{-1} r_b + \mu^1 M^{-1} \mu \right) \right].$$

(A29)

Now the job is straightforward, easy and boring; by carefully performing the derivatives of Eqs. (A26) and (A27) one finds the matrix elements $h_{i,j}$ and therefore the QFIM found in the main body of the text.

### III. Optimization

We have given the expression for $\text{Tr} (H^{-1})$ in terms of the elements $h_{i,j}$ under the assumptions that we have an equal squeezing in each mode and that the unitary transform is an orthogonal transform as

$$\text{Tr} (H^{-1}) = \sum_{i=1}^{d} \frac{1}{h_{i,i}} - \left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right)^{-1} \sum_{i=1}^{d} \frac{1}{h_{i,i}^2}.$$  
(A30)

We can rewrite $h_{i,i} = 2 \sinh^2 |\xi| + 4 e^{-2|\xi|} x_i^2 t + 4 e^{2|\xi|} y_i^2$ in terms of some $E_j = \sinh^2 |\xi| + x_j^2 t + y_j^2$, and $E_{\gamma j} = x_j^2 t + y_j^2$ and $\theta_{\gamma j} = \cos^{-1} \left( \frac{x_j}{\sqrt{E_{\gamma j}}} \right)$. Under this parameterisation the energy constraint becomes $\sum_{j=0}^{d} E_j = E_{\text{Tot}}[49]$. We thus write $h_{j,j} = h_{j,j}(E_j, E_{\gamma j}, \theta_{\gamma j})$ and can now extremise $\text{Tr} (H^{-1})$ over $E_{\gamma j}$ and $\theta_{\gamma j}$ without needing to construct a Lagrangian problem (as the only constraint on $E_{\gamma j}$ and $\theta_{\gamma j}$ is $0 \leq E_{\gamma j} \leq E_j$). We now consider what we need to solve in order to extremise $\text{Tr} (H^{-1})$ with respect to $m_j = E_{\gamma j}, \theta_{\gamma j}$ for $j \neq 0$[50]

$$\frac{\partial \text{Tr} (H^{-1})}{\partial m_j} = \frac{\partial h_{j,j}}{\partial m_j} \left[ -\frac{1}{h_{j,j}^2} + \frac{2}{h_{j,j}} \left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right)^{-1} \frac{1}{h_{j,j}} \left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right) - \frac{2}{h_{j,j}} \sum_{i=1}^{d} \frac{1}{h_{i,i}} \right]$$

(A31)

$$= - \frac{\partial h_{j,j}}{\partial m_j} \frac{1}{h_{j,j}^2} \left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right)^{-2} \left[ \left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right)^{-2} - \frac{2}{h_{j,j}} \sum_{i=0}^{d} \frac{1}{h_{i,i}} + \sum_{i=1}^{d} \frac{1}{h_{i,i}} \right]$$

(A32)

We first note that the terms in the square bracket can be rewritten (for $j \neq 0$) as

$$\left( \sum_{i=0}^{d} \frac{1}{h_{i,i}} \right)^{-2} + 2 \frac{h_{j,j}}{\sum_{i=0}^{d} h_{i,i}} - 2 \frac{h_{j,j}}{\sum_{i=0}^{d} h_{i,i}} + 2 \frac{h_{j,j}}{\sum_{i=1}^{d} h_{i,i}}.$$  
(A33)

The middle two terms cancel and the remaining terms are clearly positive. Thus we may freely conclude

$$\frac{\partial \text{Tr} (H^{-1})}{\partial m_j} = -\kappa \frac{\partial h_{j,j}}{\partial m_j}, \kappa > 0.$$  
(A34)

Namely, to extremise $h_{j,j}$ with respect to $m_j$ is to extremise $\text{Tr} (H^{-1})$ with respect to $m_j$, furthermore as $\kappa > 0$ if a change in $m_j$ increases $h_{j,j}$ then it necessarily decreases $\text{Tr} (H^{-1})$. We now therefore turn our attention to the maximisation of $h_{j,j}$ with respect to $E_{\gamma j}$ and $\theta_{\gamma j}$

$$h_{j,j} = A \{ -E_{\gamma j} + 2 E_j (1 + E_j - E_{\gamma j}) \} + 8 E_{\gamma j} \sqrt{(E_j - E_{\gamma j}) (1 + E_j - E_{\gamma j}) \cos 2 \theta_{\gamma j}}$$  
(A35)
No further mathematics is required to see that $h_{j,j}$ is maximised with respect to $\theta_{j\gamma}$ by $\theta_{j\gamma} = 0$, which reduces the problem to

$$h_{j,j} = 4 \left(-E_{j\gamma} + 2E_j(1 + E_j - E_{j\gamma})\right) + 8E_{j\gamma} \sqrt{(E_j - E_{j\gamma})(1 + E_j - E_{j\gamma})}$$

(A36)

$$\frac{\partial h_{j,j}}{\partial E_{j\gamma}} = 4 - 8E_j + 8 \sqrt{(E_j - E_{j\gamma})(1 + E_j - E_{j\gamma})} + 4E_{j\gamma} \frac{-1 - 2E_j + 2E_{j\gamma}}{\sqrt{(E_j - E_{j\gamma})(1 + E_j - E_{j\gamma})}} = 0$$

(A37)

We can then solve Eq. (A37) to find the solutions $\sqrt{(E_j - E_{j\gamma})(1 + E_j - E_{j\gamma})} = -E_{j\gamma}$ and $\sqrt{(E_j - E_{j\gamma})(1 + E_j - E_{j\gamma})} = E_j + E_{j\gamma} + \frac{1}{2}$. Both of these entail $E_{j\gamma}$ to lie outside of $0 \leq E_{j\gamma} \leq E_j$ (the former obviously so, the latter solutions requires the similarly unacceptable $E_{j\gamma} = -\frac{8E_j + 2E_{j\gamma}}{8E_j + 2E_{j\gamma}}$). To this end there are no extrema within the allowed values of $E_{j\gamma}$ instead $h_{j,j}$ is monotonic within those values. To this end we consider the extreme cases, $E_{j\gamma} = 0$ and $E_{j\gamma} = E_j$, which yield respectively $h_{j,j} = 8E_j(E_j + 1)$ and $h_{j,j} = 4E_j$. $E_{j\gamma} = 0$ could have been expected to yield the superior solution as $E_{j\gamma} = E_j$ corresponds to the use of a coherent state. We are now left to optimise $\text{Tr} \left( H^{-1} \right)$ over $\{E_j\}$ subject to $\sum_{j=0}^{d} E_j = E_{\text{Tot}}$, however as $E_{j\gamma} = \sinh^2|\xi_j|$ we have previously assumed $|\xi_j| = |\xi|, \forall j \in \{0, \ldots, d\}$ which takes us to $E_j = \frac{E_{\text{Tot}}}{d+1}$, this leads us to the optimal QFIM

$$H_{\text{sim}} = 2(I + uu^T) \sinh^2 2|\xi|$$

(A38)

$$= 8(I + uu^T) \frac{E_{\text{Tot}}(d + 1 + E_{\text{Tot}})}{(d + 1)^2}.$$  

(A39)

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[50] It is clear from Eq. (A30) that Tr (\( H^{-1} \)) is minimised when \( b_{0,0} \) is maximised.