Isometric immersions of RCD spaces

Shouhei Honda *

May 5, 2020

Abstract

We prove that if an RCD space has a regular isometric immersion in a Euclidean space, then the immersion is a locally bi-Lipschitz embedding map. This result leads us to prove that if a compact non-collapsed RCD space has an isometric immersion in a Euclidean space via an eigenmap, then the eigenmap is a locally bi-Lipschitz embedding map to a sphere, which generalizes a fundamental theorem of Takahashi in submanifold theory to a non-smooth setting. Applications of these results include a topological sphere theorem and topological finiteness theorems, which are new even for closed Riemannian manifolds.

Contents

1 Introduction ........................................... 2
1.1 Almost isometric embedding and isometric imbedding via eigenmap ...... 2
1.2 Non-smooth spaces: RCD(K, N) spaces .................................. 3
1.3 Main results ............................................ 4
1.4 Outline of the proofs ....................................... 7
1.5 Organization ........................................... 8

2 Preliminaries .......................................... 8
2.1 RCD space .............................................. 8
2.2 Structure of RCD(K, N) space .................................... 9
2.3 Non-collapsed RCD space ..................................... 10
2.4 Second order calculus ..................................... 12
2.5 Convergence of tensors ..................................... 15
2.6 Splitting theorem via splitting map ................................ 17

3 Isometric immersion ..................................... 20

4 Eigenmap ................................................ 22
4.1 Definition ............................................... 22
4.2 Compactness for eigenmaps ................................... 22

5 Isometric immersion via eigenmap ..................... 24

6 Sphere theorem ......................................... 26
6.1 Convergence result for eigenmaps .................................. 27
6.2 Proof of Theorems 1.3 and 1.4 .................................. 28

*Tohoku University, shouhei.honda.e4@tohoku.ac.jp
1 Introduction

Nash’s smooth embedding theorem states that any finite dimensional (not necessarily complete) Riemannian manifold \((M^n, g)\) can be embedded isometrically into a Euclidean space \(\mathbb{R}^N\):

\[
\Phi : M^n \hookrightarrow \mathbb{R}^N
\]

as Riemannian manifolds, that is, the pull-back metric \(\Phi^*g_{\mathbb{R}^N}\) of the flat Riemannian metric \(g_{\mathbb{R}^N}\) of the Euclidean space coincides with the original Riemannian metric \(g: \Phi^*g_{\mathbb{R}^N} = g\).

It is natural to ask when such an embedding map can be chosen by a canonical one. One of the main purposes of the paper is to give an answer to this question in a non-smooth setting by adopting eigenmaps as canonical one, with giving applications to the smooth setting.

1.1 Almost isometric embedding and isometric imbedding via eigenmap

Let \((M^n, g)\) be an \(n\)-dimensional closed, that is, compact without boundary, Riemannian manifold. A map \(\Phi = (\varphi_1, \ldots, \varphi_k) : M^n \rightarrow \mathbb{R}^k\) is said to be a \(k\)-dimensional eigenmap if each \(\varphi_i\) satisfies \(\Delta \varphi_i + \mu_i \varphi_i = 0\) for some \(\mu_i \in \mathbb{R}\), that is, it is an eigenfunction of \(-\Delta\) on \((M^n, g)\), or vanishes everywhere. Moreover we say that \(\Phi\) is irreducible if each \(\varphi_i\) is not a constant function.

Bérrard-Besson-Gallot proved in [BBG94] that for any \(t \in (0, \infty)\), a map \(\Phi_{t, \infty} : M^n \rightarrow \ell^2\) defined by

\[
\Phi_{t, \infty}(x) := \left(c(n)t^{(n+2)/2}e^{-\lambda_i t} \psi_i(x)\right)_{i=1}^k
\]

is a smooth embedding map with the following asymptotic formula:

\[
\Phi_{t, \infty} g_{\ell^2} = g - \frac{t}{3} \left(\operatorname{Ric}^n_{M^n} - \frac{1}{2} \operatorname{Scal}^n_{M^n} g\right) + O(t^2), \quad t \rightarrow 0^+,
\]

where \(c(n)\) is a positive constant depending only on \(n\) defined by

\[
c(n) := (4\pi)^n \left(\int_{\mathbb{R}^n} |\partial_x(e^{-\frac{|x|^2}{4}})|^2 \, dx\right)^{-1},
\]

\(g_{\ell^2}\) is the standard flat Riemannian metric of \(\ell^2\), \(\lambda_i\) denotes the \(i\)-th eigenvalue (\(\lambda_0 = 0\)) of \(-\Delta\) counted with multiplicities, and \(\psi_i\) is a corresponding eigenfunction with the standard normalization:

\[
\frac{1}{\operatorname{vol} M^n} \int_{M^n} |\psi_i|^2 \, d\operatorname{vol}_g = 1.
\]

Thus (1.3) says that \(\Phi_{t, \infty}\) is an almost isometric embedding when \(t\) is small.

Portegies established a quantitative finite dimensional reduction of this result in [P16], that is, he proved: for all \(\epsilon, \tau, d \in (0, \infty)\) and any \(K \in \mathbb{R}\) there exists \(t_0 := t_0(n, K, \epsilon, \tau, d) \in (0, 1)\) such that for any \(t \in (0, t_0)\) there exists \(N_0 := N_0(n, K, \epsilon, \tau, d, t) \in \mathbb{N}\) such that if \((M^n, g)\) satisfies \(\text{diam}(M^n, d_g) \leq d, \text{Ric}^n_{M^n} \geq K\) and \(\text{inj}^g_{M^n} \geq \tau\), where \(\text{diam}(M^n, d_g)\) and \(\text{inj}^g_{M^n}\) denote the diameter and the injectivity radius, respectively, then for any \(m \in \mathbb{N},\) an irreducible eigenmap \(\Phi_{t, m} : M^n \rightarrow \mathbb{R}^m\) defined by

\[
\Phi_{t, m}(x) := \left(c(n)t^{(n+2)/2}e^{-\lambda_i t} \psi_i(x)\right)_{i=1}^m
\]
is a smooth embedding map with
\[ \|\Phi^*_{t,m}g_{R^m} - g\|_{L^\infty} < \epsilon. \] (1.7)

Thus (1.7) says that for any closed Riemannian manifold, the Riemannian metric can be approximated by the pull-back metric of an eigenmap. Let us emphasize that for a fixed \( i \in \mathbb{N} \), we have \( \|c(n)t^{(n+2)/2}e^{-\lambda t}\psi_i\|_{L^\infty} \to 0 \) as \( t \to 0^+ \).

Next we discuss the equality case, that is, let us assume that there exists an eigenmap \( \Phi: M^n \to \mathbb{R}^{k+1} \) (which is not necessary an embedding map) with
\[ \Phi^*g_{\mathbb{R}^{k+1}} = g. \] (1.8)

A fundamental theorem in submanifold theory proved in [T66] by Takahashi states that (1.8) implies that \( \Phi \) is a minimal immersion in a sphere \( S^k(a) := \{x \in \mathbb{R}^{k+1}; |x|_{\mathbb{R}^{k+1}} = a\} \) for some \( a \in (0, \infty) \). Conversely any minimal immersion of \((M^n, g) \) in \( S^k(a) \) can be obtained by an eigenmap along this way. Next let us discuss the above for non-smooth spaces, so-called \( \text{RCD}(K, N) \) spaces whose introduction is given in the next subsection.

### 1.2 Non-smooth spaces: \( \text{RCD}(K, N) \) spaces

A triple \((X, d, m)\) is said to be a \textit{metric measure space} if \((X, d)\) is a complete separable metric space and \(m\) is a Borel measure on \(X\) with full support. The \textit{curvature-dimension condition} \( \text{CD}(K, N) \) introduced in [LV09] by Lott-Villani and [St06a, St06b] by Sturm, independently, for metric measure space \((X, d, m)\) catches the lower bound \(K \in \mathbb{R}\) on the Ricci curvature and the upper bound \(N \in [1, \infty)\) of the dimension of \((X, d, m)\) in a synthetic sense. Then adding a Riemannian structure to a metric measure space satisfying the \( \text{CD}(K, N) \) condition, \( \text{RCD}(K, N) \) \textit{spaces} are introduced in [AGS14b] by Ambrosio-Gigli-Savaré, [AMS19] by Ambrosio-Mondino-Savaré, [G15, G13] by Gigli, and [EKS15] by Erbar-Kuwada-Sturm. Roughly speaking, \((X, d, m)\) is said to be an \( \text{RCD}(K, N) \) space if the following holds:

- \( \text{Ric}(X, d, m) \geq K, \dim(X, d, m) \leq N \), and the Sobolev space \(H^{1,2}(X, d, m)\) is a Hilbert space.

See subsection 2.1 for the precise definition.

Thanks to quick developments on the study of \( \text{RCD}(K, N) \) spaces, we have nice structure results on \( \text{RCD}(K, N) \) spaces. For example it is proved in [BS18] by Brué-Semola that for any \( \text{RCD}(K, N) \) space \((X, d, m)\) with \(N < \infty\), the \textit{essential dimension}, denoted by \( \dim_{d,m}(X) \in \mathbb{N} \cap [1, N)\), is well-defined by the unique \(n \in \mathbb{N}\) satisfying that the \(n\)-dimensional regular set has positive \(m\)-measure. This gave a generalization of a result proved in [CN12] by Colding-Naber to \( \text{RCD}(K, N) \) spaces.

Typical examples of \( \text{RCD}(K, N) \) spaces can be found in weighted Riemannian manifolds \((M^n, d_g, e^{-f} \text{vol}_g)\), where \(f \in C^\infty(M^n)\). That is, \((M^n, d_g, e^{-f} \text{vol}_g)\) is an \( \text{RCD}(K, N) \) space if and only if \(N \geq n\) holds, and the \(N\)-Bakry-Émery Ricci curvature (the LHS of (1.9)) is bounded from below by \(K\):
\[ \text{Ric}_g^n + \text{Hess}_{d_g}^q - \frac{df \otimes df}{N-n} \geq Kg. \] (1.9)

Let us fix a compact \( \text{RCD}(K, N) \) space \((X, d, m)\) with \(N < \infty\), that is, it is an \( \text{RCD}(K, N) \) space, and \((X, d)\) is a compact metric space. Then as in the case of closed Riemannian manifolds, the spectrum of \(-\Delta\) on \((X, d, m)\) is discrete and unbounded. In
particular an eigenmap \( \Phi : X \to \mathbb{R}^k \) on \((X, d, m)\) makes sense. Note that for an weighted Riemannian manifold \((M^n, d_g, e^{-f} \vol_g)\), the corresponding Laplacian \(\Delta_f\) as an \(\text{RCD}(K, N)\) space coincides with the Witten Laplacian: \(\Delta_f h = \text{tr}(\text{Hess}_h) - g(\nabla f, \nabla h)\).

Then it is natural to ask whether the Riemannian metric \(g_X\), which is also well-defined (Proposition 2.13), can be approximated by the pull-back metric of an eigenmap or not. A positive answer to this question was obtained in [AHPT18] by Ambrosio-Portegies-Tewodrose and the author for \(\text{non-collapsed} \ \text{RCD}(K, N)\) spaces whose definition is introduced in [DePhG18] by De Philippis-Gigli as a synthetic counterpart of non-collapsed Ricci limit spaces (Definition 2.7). Note that non-collapsed \(\text{RCD}(K, N)\) spaces have nicer properties rather than that of general \(\text{RCD}(K, N)\) spaces (Theorem 2.8).

A main result of [AHPT18] is stated as follows: for any \(p \in [1, \infty)\) and all \(\epsilon, \tau, d \in (0, \infty)\) and any \(K \in \mathbb{R}\) there exists \(t_0 = t_0(n, K, \epsilon, \tau, d, p) \in (0, 1)\) such that for any \(t \in (0, t_0)\) there exists \(N_0 := N_0(n, K, \epsilon, \tau, d, p, t) \in \mathbb{N}\) such that if a non-collapsed \(\text{RCD}(K, N)\) space \((X, d, \mathcal{H}^n)\), where \(\mathcal{H}^n\) denotes the \(n\)-dimensional Hausdorff measure, satisfies \(\text{diam}(X, d) \leq d\) and \(\mathcal{H}^n(X) \geq \tau\), then for any \(m \in \mathbb{N}_{\geq N_0}\), an eigenmap \(\Phi_{t, m} : X \to \mathbb{R}^m\) defined by (1.6) with the standard normalization of eigenfunctions \(\psi_i\) similarly satisfies

\[
\|\Phi^*_{t, m} g_{\mathbb{R}^m} - g_X\|_{L^p} < \epsilon. \tag{1.10}
\]

Let us remark that we do not know whether \(\Phi_{t, m}\) is actually a topological embedding or not. It is also worth pointing out that this was new even for closed Riemannian manifolds and that the finiteness of \(p, p < \infty\), is sharp which is a different point from the case of closed Riemannian manifolds. For example, any \(n\)-dimensional closed ball in \(\mathbb{R}^n\) with \(\mathcal{H}^n\) is a non-collapsed \(\text{RCD}(0, n)\) space, but it does not satisfy the \(L^\infty\)-version of (1.10).

Compare with (1.7). Therefore (1.10) says that for any compact non-collapsed \(\text{RCD}(K, n)\) space, the Riemannian metric can be approximated by the pull-back metric of an eigenmap in the \(L^p\)-sense for any \(p \in [1, \infty)\).

### 1.3 Main results

A map \(\Phi = (\varphi_1, \ldots, \varphi_k) : U \to \mathbb{R}^k\) from an open subset \(U\) of an \(\text{RCD}(K, N)\) space \((X, d, m)\) to \(\mathbb{R}^k\) is said to be an isometric immersion if it is locally Lipschitz and

\[
\Phi^* g_{\mathbb{R}^k} := \sum_{i=1}^k d\varphi_i \otimes d\varphi_i = g_X \tag{1.11}
\]

holds as \(L^\infty_{\text{loc}}\)-tensors. We say that \(\Phi\) is regular if each \(\varphi_i\) is included in the domain of the (local) Laplacian with \(\Delta \varphi_i \in L^\infty(U, m)\). Thanks to a regularity result proved in [J14] by Jiang, any regular map is locally Lipschitz. Our first main result is stated as follows.

**Theorem 1.1 (Isometric immersion implies locally bi-Lipschitz embedding).** Let \((X, d, m)\) be an \(\text{RCD}(K, N)\) space with \(N < \infty\) and \(n = \dim_{d, m}(X)\), let \(U\) be an open subset of \(X\) and let \(\Phi : U \to \mathbb{R}^k\) be a regular isometric immersion. Then \(\Phi\) is a locally bi-Lipschitz embedding. Moreover we have the following:

1. For any \(x \in U\) and any \(\epsilon \in (0, 1)\) there exists \(r \in (0, 1)\) such that \(\Phi|_{B_r(x)}\) is a bi-Lipschitz embedding and that \(\Phi|_{B_r(x)}\) and \((\Phi|_{B_r(x)})^{-1}\) are \((1 + \epsilon)\)-Lipschitz.

2. \(U\) is locally Reifenberg flat, that is, for any \(x \in U\) and any \(\epsilon \in (0, 1)\) there exists \(r_0 \in (0, 1)\) such that

\[
d_{\text{GH}}(B_r(y), B_r(0_n)) < \epsilon r, \quad \forall y \in B_{r_0}(x), \forall r \in (0, r_0) \tag{1.12}
\]

holds, where \(d_{\text{GH}}\) denotes the Gromov-Hausdorff distance.
In particular $U$ is homeomorphic to an $n$-dimensional topological manifold without boundary.

It is worth pointing out that the regularity assumption of $\Phi$ in Theorem 1.1 is essential. In order to check this, let us give two simple examples. The first one is to consider a map $\Phi : \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x) = |x|$ which is globally Lipschitz with $\Phi^* g_\mathbb{R} = g_\mathbb{R}$ as $L^\infty$-tensors. Thus $\Phi$ is an isometric immersion. However $\Phi$ is not locally bi-Lipschitz around the origin. Note that $\Phi$ is not regular, in fact, $\Phi$ is not included in the domain of the Laplacian.

The second one is to consider a canonical inclusion map $i = (x_1, \ldots, x_n)$ from the closed unit ball $\overline{B}_1(0_n) := \{ x \in \mathbb{R}^n; |x|_{\mathbb{R}^n} = 1 \}$ to $\mathbb{R}^n$ (recall that $(\overline{B}_1(0_n), d_{\mathbb{R}^n}, H^n)$ is an isometric immersion). However the map is not regular because the corresponding Laplacian of $(\overline{B}_1(0_n), d_{\mathbb{R}^n}, H^n)$ as an RCD$(0, n)$ space coincides with the Neumann Laplacian. Since each coordinate function $x_i$ does not satisfy the Neumann boundary condition, $x_i$ is not included in the domain of the Laplacian. More precise description of these observations can be found in Remark 2.31.

Our second main result is stated as follows, which gives a generalization of Takahashi’s theorem to RCD$(K, N)$ spaces except for the minimality parts. Recall again that thanks to regularity results obtained in [J14] and in [JLZ16] by Jiang-Li-Zhang, any eigenmap is a regular Lipschitz map.

**Theorem 1.2 (Isometric immersion via eigenmap).** Let $(X, d, m)$ be a compact RCD$(K, N)$ space with $N < \infty$ and $n = \dim_{d, m}(X)$ and let $\Phi : X \to \mathbb{R}^{k+1}$ be an eigenmap. Assume that $\Phi$ is an isometric immersion. Then the following two conditions are equivalent:

(a) $|\Phi|$ is a constant function.

(b) $(X, d, H^n)$ is a non-collapsed RCD$(K, n)$ space with

$$m = \frac{m(X)}{H^n(X)} H^n. \quad (1.13)$$

Furthermore if (a) (or (b), equivalently) holds, then the following holds.

1. $\Phi : (X, d) \to (S^k(|\Phi|), d_{S^k(|\Phi|)})$ is $1$-Lipschitz.

2. $\Phi : (X, d) \to (S^k(|\Phi|), d_{S^k(|\Phi|)})$ is a locally bi-Lipschitz embedding map. More precisely for any $x \in X$ and any $\epsilon \in (0, 1)$ there exists $r > 0$ such that $\Phi|_{B_r(x)}$ is a bi-Lipschitz embedding and that $(\Phi|_{B_r(x)})^{-1}$ is $(1 + \epsilon)$-Lipschitz.

3. If $k = n$, then $\Phi$ gives an isometry from $(X, d)$ to $(S^n(|\Phi|), d_{S^n(|\Phi|)})$.

Note that it is a direct consequence of Theorem 1.1 that $k \geq n$ holds in Theorem 1.2 because Theorem 1.1 yields that $X$ is homeomorphic to an $n$-dimensional closed topological manifold, and any $n$-dimensional closed topological manifold cannot have a local homeomorphism into $\mathbb{R}^l$ for $l \leq n$. It is worth pointing out that this observation leads us to get a gap theorem (Theorem 7.3).

Let us introduce applications of Theorem 1.2. Before that, let us define a non-negative number $L(\Phi) \in [0, \infty)$ of an eigenmap $\Phi = (\varphi_1, \ldots, \varphi_{k+1})$ from a compact RCD$(K, N)$ space $(X, d, m)$ to $\mathbb{R}^{k+1}$ by

$$L(\Phi) := \min_{1 \leq i \leq k+1} \frac{1}{m(X)} \int_X |\varphi_i|^2 \, dm. \quad (1.14)$$

Note that the author does not know a suitable definition of minimal immersions from RCD$(K, N)$ spaces to Riemannian manifolds.
We will also establish convergence/compactness results for eigenmaps with respect to the measured Gromov-Hausdorff convergence (Proposition 4.4 and Corollary 4.6). Since compactness results for sequences of RCD\((K,N)\) spaces are already known (Theorems 2.10 and 2.20), combining these compactness results for eigenmaps and for sequences of RCD\((K,N)\) spaces with (5) of Theorem 1.2 gives us the following result which is new even for closed Riemannian manifolds.

**Theorem 1.3** (Almost characterization of sphere via eigenmap). We have the following:

1. For any \(K \in \mathbb{R}\), any \(n \in \mathbb{N}\) and all \(\epsilon, \tau, d \in (0, \infty)\) there exists \(\delta := \delta(n, K, \epsilon, \tau, d) \in (0, 1)\) such that if a compact non-collapsed RCD\((K,n)\) space \((X, d, \mathcal{H}^n)\) satisfies \(\text{diam}(X, d) \leq d\) and

   \[
   \frac{1}{\mathcal{H}^n(X)} \int_X |\Phi^* g_{\mathbb{R}^{n+1}} - g_X| \, d\mathcal{H}^n < \delta \tag{1.15}
   \]

   for some irreducible \((n+1)\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^{n+1}\) with \(L(\Phi) \geq \tau\), then \(d_{\text{GH}}(X, S^n(a)) < \epsilon\) holds, where

   \[
   a^2 = \frac{1}{\mathcal{H}^n(X)} \int_X |\Phi|^2_{g_{\mathbb{R}^{n+1}}} \, d\mathcal{H}^n. \tag{1.16}
   \]

2. For any \(K \in \mathbb{R}\), any \(n \in \mathbb{N}\) and all \(\epsilon, d \in (0, \infty)\) there exists \(\delta := \delta(n, K, \epsilon, d) \in (0, 1)\) such that if a compact non-collapsed RCD\((K,n)\) space \((X, d, \mathcal{H}^n)\) satisfies \(\text{diam}(X, d) \leq d\) and \(d_{\text{GH}}(X, S^n(a)) < \delta\) for some \(a \in [\tau, d/\pi]\), then there exists an irreducible \((n+1)\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^{n+1}\) with \(|L(\Phi) - a^2(n+1)^{-1}| < \epsilon\) such that

   \[
   \frac{1}{\mathcal{H}^n(X)} \int_X |\Phi^* g_{\mathbb{R}^{n+1}} - g_X| \, d\mathcal{H}^n < \epsilon \tag{1.17}
   \]

holds.

As a corollary of Theorem 1.3, we obtain the following topological sphere theorem, which is also new even for closed Riemannian manifolds.

**Theorem 1.4** (Topological sphere theorem via eigenmap). For any \(K \in \mathbb{R}\), any \(n \in \mathbb{N}\) and all \(d, \tau \in (0, \infty)\), there exists \(\delta := \delta(n, K, d, \tau) \in (0, 1)\) such that if a compact non-collapsed RCD\((K,n)\) space \((X, d, \mathcal{H}^n)\) with \(\text{diam}(X, d) \leq d\) satisfies that there exists an irreducible \((n+1)\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^{n+1}\) with \(L(\Phi) \geq \tau\) such that

   \[
   \frac{1}{\mathcal{H}^n(X)} \int_X |\Phi^* g_{\mathbb{R}^{n+1}} - g_X| \, d\mathcal{H}^n < \delta \tag{1.18}
   \]

holds, then \(X\) is homeomorphic to \(S^n := S^1(1)\). Moreover if \((X, d)\) is isometric to a Riemannian manifold, then the homeomorphism can be improved to a diffeomorphism.

Let us emphasize that in Theorems 1.3 and 1.4 we do not necessary to assume the positive lower bound on Ricci curvature and the positive lower bound on the volume \(\mathcal{H}^n(X)\), which are different from previous sphere theorems, for instance [C96a, C96b] by Colding, [KM19] by Kapovitch-Mondino, and [HM19] by Mondello and the author.

By (3) of Theorem 1.2 we know that the sphere is the only non-collapsed RCD\((K,n)\) space whose Riemannian metric can be realized by the pull-back metric of an \((n+1)\)-dimensional eigenmap. Therefore it is natural to ask:
For fixed $k$ how many non-collapsed $\text{RCD}(K,n)$ space $(X,d,H^n)$ satisfy (1.11) for an eigenmap $\Phi$?

In order to give an answer to this question, we will provide two topological finiteness theorems in more general setting. Roughly speaking, they tell us that from the point of view of topology, only finite spaces realize this condition. See Theorems 7.4 and 7.5.

Let us explain how to achieve these results in the next subsection.

### 1.4 Outline of the proofs

Let us start a simple observation. A (globally) Lipschitz map $\Phi := (\varphi_1, \ldots, \varphi_k) : \mathbb{R}^n \to \mathbb{R}^k$ is an isometric embedding as metric spaces (that is, $\Phi$ preserves the distance) if and only if each $\varphi_i$ is a harmonic function on $\mathbb{R}^n$ with

$$\Phi^* g_{\mathbb{R}^k} = g_{\mathbb{R}^n}. \quad (1.19)$$

The harmonicity of $\varphi_i$ is essential in this observation because, for example, as already observed, the map $\Phi : \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x) = |x|$ is globally Lipschitz and satisfies $\Phi^* g_{\mathbb{R}}$ as $L^\infty$-tensors, however it is not an isometric embedding as metric spaces.

The first step to prove Theorem 1.1 is to establish a quantitative version of this observation, that is, roughly speaking, if $\Phi^* g_{\mathbb{R}^k}$ is close to $g_X$ around a point $x$ of an $\text{RCD}(K,N)$ space $(X,d,m)$, then $\Phi$ gives a Gromov-Hausdorff approximation to the image on a neighbourhood of $x$, and the point $x$ is almost $n$-dimensional regular, where $n = \dim_{\mathbb{R}^d,m}(X)$ (Theorem 2.30). This is done by a blow-up argument with (1.11) and stability results of Sobolev functions with respect to the measured Gromov-Hausdorff convergence established in [AH17, AH18] by Ambrosio and the author. Let us emphasize that the regularity assumption of $\Phi$ is essentially used here. The second step is to prove, by using the first step, that a regular almost isometric immersion implies a local bi-Lipschitz embedding and an almost Reifenberg flatness of the space (Theorem 3.4), which leads us to get Theorem 1.1.

For Theorem 1.2, we will divide the proof into several lemmas in Section 5. In order to show the equivalence between (a) and (b) in Theorem 1.2, we prove the following formula for any eigenmap $\Phi : X \to \mathbb{R}^{k+1}$ (Theorem 4.2):

$$\nabla^* (\Phi^* g_{\mathbb{R}^{k+1}}) = -\frac{1}{4} d\Delta |\Phi|^2_{\mathbb{R}^{k+1}}. \quad (1.20)$$

By using (1.20), we know that under assuming (1.11), $|\Phi|$ is a constant function if and only if $\nabla^* g_X = 0$ holds. Moreover thanks to a development on the second order differential calculus in [G18] by Gigli, $\nabla^* g_X = 0$ holds if and only if

$$\Delta f = \text{tr}(\text{Hess}_f), \quad \forall f \quad (1.21)$$

holds.

Let us assume that (a) holds. Thus we have (1.21). Then we can apply a result proved in [BS18], which confirmed a conjecture raised in [DePhG18], to show that $(X,d,m)$ is a weakly non-collapsed $\text{RCD}(K,n)$ space. Since any compact weakly non-collapsed $\text{RCD}(K,n)$ space is actually a non-collapsed $\text{RCD}(K,n)$ space up to multiplying a positive constant to the measure, which is proved in [H19] by the author, we have (b). The converse implication, from (b) to (a), is justified along the same line with a result proved in [Han18] by Han (Lemma 5.1).

Under assuming (a), Theorem 1.1 allows us to prove (1) and (2) of Theorem 1.2 (Lemmas 5.3 and 5.4). Moreover we know that if $k = n$, then $\Phi$ is a local isometry.
In particular \((X, d, m)\) is isometric to a closed Riemannian manifold whose sectional curvature is equal to 1. Since we can check that the first positive eigenvalue \(\lambda_1\) of \((X, d, m)\) is at most \(n/|\Phi|^2\), the final statement, (3) of Theorem 1.2, follows from Obata’s theorem [O62] which states that if a closed \(n\)-dimensional Riemannian manifold \((M^n, g)\) satisfies \(\text{Ric}_{M^n} \geq n - 1\) and \(\lambda_1 \leq n\), then \((M^n, g)\) is isometric to \((S^n, g_{S^n})\) (Lemma 5.6).

The remaining results, Theorems 1.3 and 1.4, are justified by contradiction after combining Theorem 1.2 with compactness results for eigenmaps established in Section 4. Roughly speaking, we will prove the following in Section 4:

- The set of all compact RCD(K, N) spaces \((X, d, m, \Phi)\) with irreducible eigenmaps \(\Phi : X \to \mathbb{R}^k\) satisfying that \(\text{diam}(X, d) \leq d\) and \(L(\Phi) \geq \tau\), is compact with respect to the joint convergences of measured Gromov-Hausdorff convergence of \((X, d, m)\) and of irreducible eigenmaps \(\Phi\).

See Theorem 4.4 and Corollary 4.6.

1.5 Organization

In the next section we will introduce knowledges on the RCD theory minimally. In Section 3 we will discuss isometric immersions of RCD(K, N) spaces, in particular, we will prove Theorem 1.1. We will study eigenmaps on RCD(K, N) spaces in Section 4. Section 5 is devoted to the proof of Theorem 1.2. Finally, we will prove remaining statements introduced above in Sections 6 and 7.2.

Acknowledgement. The author would like to thank Shin Nayatani and Toshihiro Shoda for valuable suggestions. He acknowledges the supports of the Grant-in-Aid for Scientific Research (B) of 20H01799 and the Grant-in-Aid for Scientific Research (B) of 18H01118.

2 Preliminaries

In this section we give a quick introduction on the RCD theory in order to understand the paper minimally under assuming a bit of knowledges of metric measure geometry. Therefore we will sometimes refer only suitable references for the details. For example we omit the definitions of Gromov-Hausdorff (GH) convergence, measured Gromov-Hausdorff (mGH) convergence, and pointed measured Gromov-Hausdorff (pmGH) convergence which are metrizable by \(d_{GH}, d_{mGH}\), and \(d_{pmGH}\), respectively. See [BBI01, GMS13, LV09, St06a, St06b, Vi09].

2.1 RCD space

We refer [A19, AGMR15, AGS14a, AGS14b, AMS19, CM16, EKS15, LV09, St06a, St06b] as references in this subsection.

We recall the definition of metric measure spaces again: a triple \((X, d, m)\) is said to be a metric measure space if \((X, d)\) is a complete separable metric space and \(m\) is a Borel measure on \(X\) with full support. We fix a metric measure space \((X, d, m)\) below.

Define the \textit{Cheeger energy} \(\text{Ch}: L^2(X, m) \to [0, \infty]\) by

\[
\text{Ch}(f) := \inf \left\{ \liminf_{i \to \infty} \int_X \text{lip}^2 f_i \, dm : f_i \in \text{Lip}(X, d) \cap (L^2 \cap L^\infty)(X, m), \|f_i - f\|_{L^2} \to 0 \right\},
\]  

(2.1)

\[2\text{More precisely, we should use } \mathbf{D}_{pG_w} \text{ instead of using } d_{mGH}, d_{pmGH} \text{ due to [GMS13, St06a]. However in order to keep our notation simply we will use these notations.} \]
where \( \text{lip}_f(x) \) denotes the local slope of \( f \) at \( x \), and \( \text{Lip}(X,d) \) is the set of all Lipschitz functions on \( X \). Then the Sobolev space \( H^{1,2} = H^{1,2}(X,d,m) \) is defined as the finiteness domain of \( \text{Ch} \) in \( L^2(X,m) \) and it is a Banach space equipped with the norm 
\[
\| f \|_{H^{1,2}} = \sqrt{\| f \|_{L^2}^2 + \text{Ch}(f)}.
\]
We are now in a position to introduce the definition of RCD\((K,N)\) space:

**Definition 2.1 (RCD\((K,N)\) space).** \((X,d,m)\) is said to be an RCD\((K,N)\) space for some \( K \in \mathbb{R} \) and some \( N \in [1, \infty) \) if the following four conditions hold:

- (Volume growth condition) There exist \( C \in (0, \infty) \) and \( x \in X \) such that \( m(B_r(y)) \leq Ce^{Cd(x,y)^2} \) holds for any \( y \in X \) and any \( r \in (0, \infty) \).
- (Infinitesimally Hilbertian condition) \( H^{1,2} \) is a Hilbert space. In particular for all \( f_i \in H^{1,2} (i = 1, 2) \),
\[
\langle \nabla f_1, \nabla f_2 \rangle := \lim_{t \to 0} \frac{|\nabla (f_1 + tf_2)|^2 - |\nabla f_1|^2|}{2t} \in L^1(X,m) \tag{2.2}
\]
is well-defined, where \( |\nabla f_i| \) denotes the minimal relaxed slope of \( f_i \).
- (Sobolev-to-Lipschitz property) Any function \( f \in H^{1,2} \) satisfying \( |\nabla f|(y) \leq 1 \) for \( m \)-a.e. \( y \in X \) has 1-Lipschitz representative.
- (Bochner inequality) For any \( f \in D(\Delta) \) with \( \Delta f \in H^{1,2} \), we have
\[
\frac{1}{2} \int_X |\Delta \phi||\nabla f|^2 \, dm \geq \int_X \phi \left( \frac{(|\Delta f|^2)}{N} + \langle \nabla \Delta f, \nabla f \rangle + K|\nabla f|^2 \right) \, dm \tag{2.3}
\]
for any \( \phi \in D(\Delta) \cap L^\infty(X,m) \) with \( \Delta \phi \in L^\infty(X,m) \) and \( \phi \geq 0 \), where
\[
D(\Delta) := \left\{ f \in H^{1,2}; \exists h =: \Delta f \in L^2, \text{s.t. } \int_X \langle \nabla f, \nabla \psi \rangle \, dm = -\int_X h\psi \, dm, \forall \psi \in H^{1,2} \right\}. \tag{2.4}
\]

Throughout the paper the parameters \( K \in \mathbb{R} \) and \( N \in [1, \infty) \) will be kept fixed.

### 2.2 Structure of RCD\((K,N)\) space

Let \((X,d,m)\) be an RCD\((K,N)\) space with \( \text{diam}(X,d) > 0 \). It is known that \((X,d)\) is a proper geodesic space (see [GRS16, Cor.1.4] for more information on geodesics).

**Definition 2.2 (Regular set \( \mathcal{R}_k \)).** For any \( k \geq 1 \), we denote by \( \mathcal{R}_k \) the \( k \)-dimensional regular set of \((X,d,m)\), namely, the set of all points \( x \in X \) such that \((X,r^{-1}d,m(B_r(x))^{-1}m,x)\) pmGH converge to \((\mathbb{R}^k, d_{\mathbb{R}^k}, \omega_k^{-1}\mathcal{H}^k,0_k)\) as \( r \to 0^+ \), where \( \omega_k := \mathcal{H}^k(B_1(0_k)) \).

The following result is proved in [BS18, Thm.0.1] after [MN19] which gives a generalization of [CN12, Thm.1.12] to RCD\((K,N)\) spaces.

**Theorem 2.3 (Essential dimension of RCD\((K,N)\) spaces).** Let \((X,d,m)\) be an RCD\((K,N)\) space. Then, there exists a unique integer \( n \in [1,N] \cap \mathbb{N} \), denoted by \( \dim_{d,m}(X) \), such that
\[
m(X \setminus \mathcal{R}_n) = 0 \tag{2.5}
\]
The following is a direct consequence of the Bishop-Gromov inequality [LV09, Thm.5.31], [St06b, Thm.2.3] (see [Vi09, Thm.30.11]) and the Poincaré inequality [Raj12, Thm.1] with [HK00, Thm.5.1].

**Theorem 2.4** (Rellich compactness). If \((X, d)\) is compact (or equivalently, \(\text{diam}(X, d) < \infty\) by properness), then the canonical inclusion map:

\[
i : H^{1,2}(X, d, m) \hookrightarrow L^2(X, m)
\]

is a compact operator.

**Definition 2.5** (Eigenfunction). A function \(f \in L^2(X, m)\) is said to be an eigenfunction (of \(-\Delta\)) on \((X, d, m)\) if \(f \in D(\Delta)\) holds with \(f \neq 0\) and \(\Delta f + \lambda f \equiv 0\) for some \(\lambda \in \mathbb{R}\), where \(\lambda\) is called the eigenvalue of \(f\).

A direct consequence of Theorem 2.4 is that if \((X, d)\) satisfies \(\text{diam}(X, d) \leq d < \infty\), then \(-\Delta\) admits a discrete non-negative spectrum:

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,
\]

where \(\lambda_i := \lambda_i(X, d, m)\) is the \(i\)-th eigenvalue of \(-\Delta\) counted with multiplicities. We denote the corresponding eigenfunctions by \(\psi_0, \psi_1, \ldots\) with the standard normalization:

\[
\frac{1}{m(X)} \int_X |\psi_i|^2 \, dm = 1.
\]

It is known that \(\psi_i\) is Lipschitz, in fact, it holds that

\[
\|\psi_i\|_{L^\infty} \leq C \lambda_i^{N/4}, \quad \|\nabla \psi_i\|_{L^\infty} \leq C \lambda_i^{(N+2)/4}, \quad \lambda_i \geq C^{-1} i^{2/N},
\]

where \(C := C(d, K, N) > 0\). See [J14, JLZ16] (see also [AHPT18]).

Let us recall the (local) Sobolev space \(H^{1,2}(U, d, m)\) for an open subset \(U\) of \(X\). A function \(f \in L^2(U, m)\) belongs to \(H^{1,2}(U, d, m)\) if and only if \(\varphi f \in H^{1,2}(X, d, m)\) holds for any \(\varphi \in \text{Lip}(X, d)\) with compact support in \(U\), and \(|\nabla f| \in L^2(U, m)\) is satisfied. See also [Ch99, Sh00]. Then we denote by \(D(\Delta, U)\) the set of all \(f \in H^{1,2}(U, d, m)\) satisfying that there exists a unique \(h \in L^2(U, m)\), denoted by \(\Delta_U f\) (or \(\Delta f\) for short), such that

\[
\int_U \langle \nabla f, \nabla \varphi \rangle \, dm = -\int_U h \varphi \, dm
\]

holds for any \(\varphi \in \text{Lip}(X, d)\) with compact support in \(U\).

Finally let us end this subsection by introducing (global) harmonic functions.

**Definition 2.6** (Harmonic function). A function \(f : X \to \mathbb{R}\) is said to be harmonic if \(f|_U \in D(\Delta, U)\) holds with \(\Delta f = 0\) for any bounded open subset \(U\) of \(X\).

See also [AH18, AHPT18].

### 2.3 Non-collapsed RCD space

Let us recall a special class of RCD\((K, N)\) spaces, so-called non-collapsed space, introduced in [DePhG18] in order to give a synthetic counterpart of non-collapsed Ricci limit spaces whose study was developed in [CC97, CC00a]. Note that a similar notion of non-collapsed space is provided in [K19], which is a-priori weaker than that of [DePhG18] (however it is conjectured in [DePhG18] that they are essentially equivalent each other as discussed below).
Definition 2.7 (Non-collapsed RCD\((K, N)\) space). An RCD\((K, N)\) space \((X, d, m)\) is said to be non-collapsed if \(m = \mathcal{H}^N\) holds.

Non-collapsed RCD\((K, N)\) spaces have nicer properties rather than that of general RCD\((K, N)\) spaces. Let us introduce some of them:

Theorem 2.8. Let \((X, d, \mathcal{H}^N)\) be a non-collapsed RCD\((K, N)\) space. Then the following holds.

1. We have \(\dim_{d, m}(X) = N\).
2. It holds that for any \(x \in X\),
   \[
   \lim_{r \to 0^+} \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} \leq 1. \tag{2.11}
   \]
   Moreover the equality in (2.11) is satisfied if and only if \(x \in \mathcal{R}_N\) holds.

The inequality (2.11) is sometimes refered as the Bishop inequality. See [DePhG18, Thm.1.3 and 1.6]. It is worth pointing out that a quantitative version of the rigidity part of the Bishop inequality is also satisfied. In order to explain it, let us use a standard notation in the convergence theory:

\[
\Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_l; c_1, c_2, \ldots, c_m) \tag{2.12}
\]

denotes a function \(\Psi : (\mathbb{R}_{>0})^l \times \mathbb{R}^m \to (0, \infty)\) satisfying

\[
\lim_{(\epsilon_1, \ldots, \epsilon_l) \to 0} \Psi(\epsilon_1, \epsilon_2, \ldots, \epsilon_l; c_1, c_2, \ldots, c_m) = 0, \quad \forall c_i. \tag{2.13}
\]

Then the quantitative rigidity of (2.11) is stated as follows:

Theorem 2.9 (Almost rigidity of Bishop inequality). Let \((X, d, \mathcal{H}^N)\) be a non-collapsed RCD\((K, N)\) space and let \(x \in X\). If

\[
\left| \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} - 1 \right| < \epsilon. \tag{2.14}
\]

holds for some \(r \in (0, 1)\), then we see that

\[
d_{\text{GH}}(B_{r/2}(x), B_{r/2}(0_N)) < \Psi(\epsilon, r; K, N)r \tag{2.15}
\]

holds and that for any \(t \in (0, 1)\)

\[
d_{\text{pmGH}}\left((X, t^{-1}d, m(B_t(x))^{-1}m), (\mathbb{R}^N, d_{\mathbb{R}^N}, 0_N, \omega_N^{-1}\mathcal{H}^N)\right) < \Psi(\epsilon, t/r, r; K, N) \tag{2.16}
\]

holds. Conversely if

\[
d_{\text{GH}}(B_r(x), B_r(0_N)) < \epsilon r \tag{2.17}
\]

holds for some \(r \in (0, 1)\), then

\[
\left| \frac{\mathcal{H}^N(B_r(x))}{\omega_N r^N} - 1 \right| < \Psi(\epsilon, r; K, N) \tag{2.18}
\]

is satisfied.
See [DePhG18, Thm.1.3 and 1.6] (see also [AHPT18, Prop.6.5]).

In connection with (1) of Theorem 2.8 it is conjectured in [DePhG18, Rem.1.11] that the converse implication is true up to multiplying a positive constant to the measure, that is, if a \( \text{RCD}(K,N) \) space \((X,d,m)\) satisfies \( \dim_{d,m}(X) = N \), then \( m = a\mathcal{H}^N \) holds for some \( a \in (0,\infty) \). This conjecture was proved in [H19, Cor.1.3] when \((X,d)\) is compact by using \( L^2 \)-embedding results via the heat kernel obtained in [AHPT18]. Namely:

**Theorem 2.10.** Let \((X,d,m)\) be a compact \( \text{RCD}(K,N) \) space. Then \( \dim_{d,m}(X) = N \) holds if and only if \( m = a\mathcal{H}^N \) holds for some \( a \in (0,\infty) \).

Finally let us end this subsection by giving the following convergence result proved in [DePhG18, Thm.1.2].

**Theorem 2.11** (GH implies mGH). Let \((X_i,d_i,x_i,\mathcal{H}^N)\) be a sequence of pointed non-collapsed \( \text{RCD}(K,N) \) spaces. If \((X_i,d_i,x_i)\) \text{pGH}-converge to a pointed complete metric space \((X,d,x)\), then

\[
\mathcal{H}^N(B_r(z_i)) \to \mathcal{H}^N(B_r(z))
\]

holds for any \( r \in (0,\infty) \) and any \( z_i \in X_i \to z \in X \).

### 2.4 Second order calculus

Let \((X,d,m)\) be an \( \text{RCD}(K,\infty) \) space. The main purpose of this subsection is to recall the second order differential calculus developed in [G18] minimally. To keep short presentation in the paper we omit several fundamental notions, for instance,

- the space of all \( L^2 \)-vector fields, \( L^2 \)-one forms and \( L^2 \)-tensors of type \((0,2)\), on \( A \subset X \), denoted by \( L^2(T(A,d,m)), L^2(T^*(A,d,m)) \) and \( L^2((T^*)^\otimes 2(A,d,m)) \), respectively;
- the gradient operator \( \nabla : H^{1,2}(U,d,m) \to L^2(T(U,d,m)) \) for an open subset \( U \) of \( X \), and the exterior derivative \( d \).

We denote the pointwise Hilbert-Schmit norm and the pointwise scalar product by \(|T|_{\text{HS}}\) (or \(|T|_1\) for short) and \(\langle S,T \rangle\), respectively. See [G18, Subsect.3.2] (see also [AH17, Sect.10]). Put

\[
\text{TestF}(X,d,m) := \{ f \in D(\Delta) \cap \text{Lip}(X,d) \cap L^\infty(X,m); \Delta f \in H^{1,2}(X,d,m) \} \tag{2.20}
\]

and recall that \(\text{TestF}(X,d,m)\) is an algebra with \( |\nabla f|^2 \in H^{1,2}(X,d,m) \) for any \( f \in \text{TestF}(X,d,m) \). We need the following important notion, the Hessian of a function:

**Theorem 2.12** (Hessian). For any \( f \in \text{TestF}(X,d,m) \) there exists a unique \( T \in L^2((T^*)^\otimes 2(X,d,m)) \), called the Hessian of \( f \), denoted by \( \text{Hess} f \), such that for all \( f_i \in \text{TestF}(X,d,m) \),

\[
\langle T, df_1 \otimes df_2 \rangle = \frac{1}{2} (\langle \nabla f_1, \nabla (\nabla f_2, \nabla f) \rangle + \langle \nabla f_2, \nabla (\nabla f_1, \nabla f) \rangle - \langle f, \nabla (\nabla f_1, \nabla f_2) \rangle) \tag{2.21}
\]

holds for \( m \)-a.e. \( x \in X \).

See [G18, Thm.3.3.8]. Moreover it is proved in [G18, Thm.3.3.8 and Cor.3.3.9] that the Hessian is well-defined for any \( f \in D(\Delta) \) by satisfying (2.21), and that the Bochner inequality involving the Hessian term:

\[
\frac{1}{2} \int_X |\nabla f|^2 \Delta \varphi \, dm \geq \int_X \varphi \left( |\text{Hess} f|^2 + \langle \nabla \Delta f, \nabla f \rangle + K |\nabla f|^2 \right) \, dm \tag{2.22}
\]

holds for any \( f \in \text{TestF}(X,d,m) \) and \( \varphi \in D(\Delta) \) with \( \varphi \geq 0 \) and \( \varphi, \Delta \varphi \in L^\infty(X,m) \).

Let us define the Riemannian metric as follows. See [AHPT18, Prop.3.2] and [GP16, Thm.5.1] for the proof.
Proposition 2.13 (Riemannian metric). There exists a unique $g_X \in L^\infty((T^*X)^{\otimes 2}(X,d,m))$ such that for any $f_i \in \text{Test}F(X,d,m)$ we have

$$\langle g_X, df_1 \otimes df_2 \rangle = \langle \nabla f_1, \nabla f_2 \rangle, \quad \text{for } m\text{-a.e. } x \in X.$$  

(2.23)

We call $g_X$ the Riemannian metric of $(X,d,m)$. Moreover if $(X,d,m)$ is an $\text{RCD}(K,N)$ space with $n = \text{dim}_d m(X)$, then

$$|g_X| = \sqrt{m}, \quad \text{for } m\text{-a.e. } x \in X. \quad (2.24)$$

Let us introduce a relationship between $\Delta$, $g_X$ and $\text{Hess}_f$.

Theorem 2.14 (Laplacian is trace of Hessian under maximal dimension). Assume that $(X,d,m)$ is an $\text{RCD}(K,N)$ space with $\text{dim}_d m(X) = N$. Then for all $f \in D(\Delta)$ we see that

$$\Delta f = \text{tr}(\text{Hess}_f)(:= \langle \text{Hess}_f, g_X \rangle) \quad \text{for } m\text{-a.e. } x \in X. \quad (2.25)$$

See [Han18, Prop.3.2] for the proof.

Definition 2.15 (Adjoint operator $\delta$). Let us denote by $D(\delta)$ the set of all $\omega \in L^2(T^*X,d,m)$ such that there exists a unique $f \in L^2(X,m)$, denoted by $\delta \omega$, such that

$$\int_X \langle \omega, dh \rangle \, dm = \int_X fh \, dm, \quad \forall h \in H^{1,2}(X,d,m) \quad (2.26)$$

holds.

Let us recall the space of all test 1-forms:

$$\text{Test}T^*(X,d,m) := \left\{ \sum_{i=1}^l f_{0,i} df_{1,i}; l \in \mathbb{N}, f_{j,i} \in \text{Test}F(X,d,m) \right\} \quad (2.27)$$

which is a dense subspace of $L^2(T^*(X,d,m))$. It is proved in [G18, Prop.3.5.12] that $\text{Test}T^*(X,d,m) \subset D(\delta)$ holds with

$$\delta (f_1 df_2) = -\langle \nabla f_1, \nabla f_2 \rangle - f_1 \Delta f_2, \quad \forall f_i \in \text{Test}F(X,d,m). \quad (2.28)$$

Definition 2.16 (Sobolev spaces $W_C^{1,2}$). Let us denote by $W_C^{1,2}(T^*(X,d,m))$ the set of all $\omega \in L^2(T^*(X,d,m))$ such that there exists a unique $T \in L^2((T^*)^{\otimes 2}(X,d,m))$, denoted by $\nabla \omega$, such that for all $f_i \in \text{Test}F(X,d,m) (i = 1,2)$ we have

$$\int_X \langle T, f_0 df_1 \otimes df_2 \rangle \, dm = \int_X \langle (\omega, df_2) \delta (f_0 df_1) - f_0 \langle \text{Hess}_{f_2}, \omega \otimes df_1 \rangle \rangle \, dm. \quad (2.29)$$

See [G18, Def.3.4.1] (note that although the definition of [G18, Def.3.4.1] is stated for vector fields, it is equivalent to the above under the canonical isometry: $L^2(T^*(X,d,m)) \cong L^2(T(X,d,m))$). It is proved in [G18, Thm.3.4.2] that $\text{Test}T^*(X,d,m) \subset W_C^{1,2}(T^*(X,d,m))$ holds with

$$\nabla (f_1 df_2) = df_1 \otimes df_2 + f_1 \text{Hess}_{f_2}, \quad \forall f_i \in \text{Test}F(X,d,m). \quad (2.30)$$

Definition 2.17 (Adjoint operator $\nabla^*$). Let us denote by $D(\nabla^*)$ the set of all $T \in L^2((T^*)^{\otimes 2}(X,d,m))$ such that there exists $\alpha \in L^2(T^*(X,d,m))$, denoted by $\nabla^* T$, such that

$$\int_X \langle T, \nabla \omega \rangle \, dm = \int_X \langle \alpha, \omega \rangle \, dm, \quad \forall \eta \in \text{Test}T^*(X,d,m) \quad (2.31)$$

holds.
Note that since the space of all test forms of type \((0,2)\):

\[
\text{Test}(T^*)^\otimes 2(X,d,m) := \left\{ \sum_{i=1}^{l} f_{0,i}d f_{1,i} \otimes d f_{2,i}; l \in \mathbb{N}, f_{j,i} \in \text{Test}F(X,d,m) \right\}
\]

is also dense in \(L^2((T^*)^\otimes 2(X,d,m))\), the existence of \(\alpha\) in Definition 2.17 implies the uniqueness.

**Proposition 2.18.** For any \(\varphi \in \text{Test}(X,d,m)\) we have \(d\varphi \otimes d\varphi \in D(\nabla^*)\) with

\[
\nabla^*(d\varphi \otimes d\varphi) = -\Delta \varphi d\varphi - \frac{1}{2} d|d\varphi|^2.
\]

**Proof.** For all \(f_1, f_2 \in \text{Test}F(X,d,m)\) we have

\[
\int_X (d\varphi \otimes d\varphi, \nabla(f_1d f_2)) \, dm
\]

\[
= \int_X \left( (\langle d\varphi, df_1 \rangle, \langle df_2, d\varphi \rangle) + f_1(\text{Hess}_{f_2}, d\varphi \otimes d\varphi) \right) \, dm
\]

\[
= \int_X \left( (\langle d\varphi, df_2 \rangle, \langle df_2, d\varphi \rangle) + \frac{f_1}{2} \left( 2\langle d\varphi, d(df_2, d\varphi) \rangle - \langle df_2, d|d\varphi|^2 \rangle \right) \right) \, dm
\]

\[
= \int_X \left( (\delta(\langle d\varphi, df_2 \rangle) f_1 + f_1 \langle d\varphi, d(df_2, d\varphi) \rangle - \frac{f_1}{2} \langle df_2, d|d\varphi|^2 \rangle \right) \, dm
\]

\[
= \int_X \left( -f_1 \langle d\varphi, d(df_2, d\varphi) \rangle - f_1 \langle d\varphi, df_2 \rangle \Delta \varphi + f_1 \langle d\varphi, d(df_2, d\varphi) \rangle - \frac{f_1}{2} \langle df_2, d|d\varphi|^2 \rangle \right) \, dm
\]

which completes the proof. \(\square\)

Finally let us prove a characterization of compact non-collapsed \(\text{RCD}(K,N)\) spaces:

**Theorem 2.19.** Let \((X,d,m)\) be a compact \(\text{RCD}(K,N)\) space with \(n = \text{dim}_{d,m}(X)\). Then the following two conditions are equivalent:

1. \(g_X \in D(\nabla^*)\) holds with \(\nabla^* g_X = 0\).
2. It holds that

\[
m = \frac{m(X)}{\mathcal{H}^n(X)} \mathcal{H}^n
\]

and that \((X,d,\mathcal{H}^n)\) is a non-collapsed \(\text{RCD}(K,n)\) space.

**Proof.** Let us first prove the implication from (1) to (2). Assume that (1) holds. Then for any \(f_i \in \text{Test}F(X,d,m)\) \((i = 1, 2)\) we have

\[
\int_X \langle g_X, \nabla(f_1d f_2) \rangle \, dm = 0,
\]

thus by (2.30)

\[
\int_X \langle df_1, df_2 \rangle \, dm = -\int_X f_1 \text{tr}(\text{Hess}_{f_2}) \, dm.
\]

Therefore we have \(\Delta f_2 = \text{tr}(\text{Hess}_{f_2})\). Then applying [BS18, Thm.3.12] with [G18, Thm.1.12] yields \(N = \text{dim}_{d,m}(X)\). Thus we have (2) because of Theorem 2.10.

The converse implication is a direct consequence of (2.30) and Theorem 2.14. Thus we conclude. \(\square\)
2.5 Convergence of tensors

Let us recall the following well-known compactness result for a sequence of RCD(K,N) spaces (see [AGS14a, Thm.6.11], [EKS15, Thm.5.3.22], [GMS13, Thm.7.2], [LV09, Thm.5.19], and [Sto6a, Thm.4.20]).

**Theorem 2.20** (Compactness). Let \((X, d, x, m)\) be a sequence of pointed RCD(K,N) spaces with

\[
0 < \inf_i m_i(B_1(x_i)) \leq \sup_i m_i(B_1(x_i)) < \infty. \tag{2.38}
\]

Then after passing to a subsequence, there exists a pointed RCD(K,N) space \((X, d, x, m)\) such that \((X, d_i, x_i, m_i)\) pmGH-converge to \((X, d, x, m)\).

Let us fix \(R \in (0, \infty)\) and a pmGH convergent sequence of pointed RCD(K,N) spaces:

\[
(X, d_i, x_i, m_i) \xrightarrow{pmGH} (X, d, x, m). \tag{2.39}
\]

In this setting it is well-defined that a sequence \(f_i \in L^p(B_R(x_i), m_i)\) \(L^p\)-strongly/weakly converge to \(f \in L^p(B_R(x), m)\) on \(B_R(x)\) for \(p \in [1, \infty)\). Note that \(B_R(x) = X\) when \(R = \infty\). We skip the precise definition (because we omitted the pmGH convergence. A typical example is that \(1_{B_r(y)}\) \(L^p\)-strongly converge to \(1_{B_r(y)}\) on \(B_R(x)\) for any \(r \in (0, \infty)\) and any \(y_i \in X_i \to y \in X\). See [AH17, AL18, AST16, D02, GMS13, H15, KS03].

Instead of giving that, we give the definition of \(L^2\)-convergence of tensors as follows. Let us denote by \(\nabla_i, \Delta_i\) and etc, the gradient operator, Laplacian etc for \((X, d_i, m_i)\) (such notations will be used later immediately).

**Definition 2.21** (Convergence of tensors). We say that a sequence \(T_i \in L^2((T^*) \otimes (\mathcal{B}(B_R(x_i), d_i, m_i)))\) \(L^2\)-weakly converge to \(T \in L^2((T^*) \otimes (\mathcal{B}(B_R(x), d, m)))\) on \(B_R(x)\) if the following two conditions are satisfied.

1. \(\sup_i \|T_i\|_{L^2(B_R(x_i))} < \infty\) holds.

2. We see that \(\langle T_i, df_{1,i} \otimes df_{2,i}\rangle\) \(L^2\)-weakly converge to \(\langle T, df_{1} \otimes df_{2}\rangle\) on \(B_R(x)\) whenever \(f_{j,i} \in \text{Test}(X_i, d_i, m_i)\) \(L^2\)-strongly converge to \(f_j \in \text{Test}(X, d, m)\) with\[
\sup_{i,j} \left(\|f_{j,i}\|_{L^\infty(X_i)} + \|\nabla_i f_{j,i}\|_{L^\infty(X_i)} + \|\Delta_i f_{j,i}\|_{L^2(X_i)}\right) < \infty. \tag{2.40}
\]

Moreover we say that \(T_i\) \(L^2\)-strongly converge to \(T\) on \(B_R(x)\) if it is \(L^2\)-weak convergent sequence on \(B_R(x)\) with \(\limsup_{i \to \infty} \|T_i\|_{L^2(B_R(x_i))} \leq \|T\|_{L^2(B_R(x))}\) holds.

Compare with [AHPT18, Def.5.18 and Lem.6.4] (see also [AH17, Thm.10.3] and [H15, Prop.3.64]). Next let us recall the definition of \(H^{1,2}\)-strong convergence:

**Definition 2.22** \((H^{1,2})\)-strong convergence). We say that a sequence of \(f_i \in H^{1,2}(B_R(x_i), d_i, m_i)\) \(H^{1,2}\)-strongly converge to \(f \in H^{1,2}(B_R(x), d, m)\) on \(B_R(x)\) if \(f_i\) \(L^2\)-strongly converge to \(f\) on \(B_R(x)\) with \(\lim_{i \to \infty} \|\nabla f_i\|_{L^2(B_R(x_i))} = \|\nabla f\|_{L^2(B_R(x))}\).

In connection with Definition 2.22, we introduce a Rellich type compactness result with respect to mGH-convergence (see [GMS13, Thm.6.3] and [AH17, Thm.7.4]).

**Theorem 2.23** (Convergence of gradient operators). If a sequence \(f_i \in H^{1,2}(B_R(x_i), d_i, m_i)\) satisfies \(\sup_i \|f_i\|_{H^{1,2}} < \infty\), then after passing to a subsequence, there exists \(f \in H^{1,2}(B_R(x), d, m)\) such that

\[
\lim_{i \to \infty} \|\nabla f_i\|_{L^2(B_R(x_i))} \geq \|\nabla f\|_{L^2(B_R(x))} \tag{2.41}
\]
holds and that $f_i$ $L^2$-strongly converge to $f$ on $B_r(x)$ for any $r \in (0, R)$. Moreover if in addition $f_i H^{1,2}$-strongly converge to $f$ on $B_r(x)$ for some $r \in (0, R]$, then $|\nabla f_i|^2$ $L^1$-strongly converge to $|\nabla f|^2$ on $B_r(x)$.

Note that in Theorem 2.23 if $R < \infty$, then $L^2$-strong convergence of $f_i$ to $f$ is satisfied on $B_R(x)$, which is justified by using the Sobolev embedding theorem $H^{1,2} \hookrightarrow L^{2N/(N-2)}$. See [AH18, Thm.4.2]. Based on Theorem 2.23, we can easily check the following by an argument similar to the proof of [AH17, Thm.10.3] (see also [AHPT18, Def.5.18 and Lem.6.4] and [H15, Prop.6.44]).

**Proposition 2.24** (Lower semicontinuity of $L^2$-norms). A sequence $T_i \in L^2((T^*)^\otimes 2(B_R(x), d_i, m_i))$ $L^2$-weakly converge to $T \in L^2((T^*)^\otimes 2(B_R(x), d, m))$ on $B_R(x)$, then it holds that

$$\liminf_{i \to \infty} \|T_i\|_{L^2(B_R(x))} \geq \|T\|_{L^2(B_R(x))}. \quad (2.42)$$

The convergence of the heat flows with respect to (2.39) is discussed in [GMS13, Thm.5.7] (more precisely they discussed it in more general setting, for CD($K, \infty$) spaces under pmG-convergence). As a corollary, it is proved in [GMS13, Thm.7.8] that the following spectral convergence result holds, which will play a key role later (see [CC00b, Thm.7.3 and 7.9] for Ricci limit spaces. Compare with [AH18, Prop.3.3]).

**Theorem 2.25** (Spectral convergence). If $(X, d)$ is compact, then

$$\lambda_j(X_i, d_i, m_i) \to \lambda_j(X, d, m), \quad \forall j. \quad (2.43)$$

Moreover for any $\varphi_j \in D(\Delta)$ with $\Delta \varphi_j + \lambda_j(X, d, m)\varphi_j = 0$, there exists a sequence of $\varphi_{j,i} \in D(\Delta_i)$ such that $\Delta_i \varphi_{j,i} + \lambda_j(X_i, d_i, m_i)\varphi_{j,i} = 0$ holds and that $\varphi_{j,i}$ $H^{1,2}$-strongly converge to $\varphi_j$ on $X$.

Let us recall the following stability results proved in [AH18, Thm.4.4].

**Theorem 2.26** (Stability of Laplacian on balls). Let $f_i \in D(\Delta_i, B_R(x_i))$ satisfy

$$\sup_i (\|f_i\|_{H^{1,2}(B_R(x_i))} + \|\Delta_i f_i\|_{L^2(B_R(x_i))}) < \infty,$$

and let us assume that $f_i$ $L^2$-strongly convergent to $f \in L^2(B_R(x), m)$ on $B_R(x)$ (so that, by Theorem 2.23, $f \in H^{1,2}(B_R(x), d, m)$). Then we have the following.

1. $f \in D(\Delta, B_R(x))$.
2. $\Delta_i f_i$ $L^2$-weakly converge to $\Delta f$ on $B_R(x)$.
3. $f_i H^{1,2}$-strongly converge to $f$ on $B_r(x)$ for any $r < R$.

Note that in Theorem 2.26 if $R = \infty$, then the $H^{1,2}$-strong convergence of $f_i$ to $f$ is satisfied on $B_R(x) = X$. See [AH17, Cor.10.4].

Finally let us mention the following result, where (2.44) is already proved in [K19, Thm.1.5] by a different way (see also [AHPT18, Rem.5.20] and [H15, Prop.3.78]).

**Proposition 2.27** ($L^2_{loc}$-weak convergence of Riemannian metrics). We see that $g_{X_i}$ $L^2$-weakly converge to $g_X$ on $B_R(x)$. In particular we see that

$$\liminf_{i \to \infty} \text{dim}_{d_i, m_i}(X_i) \geq \text{dim}_{d, m}(X) \quad (2.44)$$

holds and that $g_{X_i}$ $L^2$-strongly converge to $g_X$ on $B_R(x)$ if and only if $\text{dim}_{d_i, m_i}(X_i) = \text{dim}_{d, m}(X)$ holds for any sufficiently large $i$. 

16
Proof. The desired $L^2$-weak convergence is a direct consequence of Theorems 2.23 and 2.26. Moreover Propositions 2.13 and 2.24 yield
\[
\liminf_{i \to \infty} \dim_{d_i, m_i}(X_i) = \liminf_{i \to \infty} \frac{1}{m_i(B_R(x_i))} \int_{B_R(x_i)} |g_{X_i}|^2 \, dm_i \geq \frac{1}{m(B_R(x))} \int_{B_R(x)} |g_X|^2 \, dm = \dim_{d, m}(X). \tag{2.45}
\]
The remaining statement is trivial from this observation. \hfill \square

2.6 Splitting theorem via splitting map

We say that a map $\gamma$ from $\mathbb{R}$ to a metric space $(Z, d_Z)$ is a line if it is an isometric embedding as metric spaces, that is, $d_Z(\gamma(s), \gamma(t)) = |s - t|$ holds for all $s, t \in \mathbb{R}$. Then the Buseman function of $\gamma$, $b_\gamma : Z \to \mathbb{R}$ is defined by
\[
b_\gamma(x) := \lim_{t \to \infty} (t - d_Z(\gamma(t), x)). \tag{2.46}
\]
Let us introduce an important result on the RCD theory, so-called the splitting theorem, proved in [G13, Thm.1.4] (see also [ABS19, Lem.1.21]):

Theorem 2.28 (Splitting theorem). Let $(X, d, m)$ be an RCD$(0, N)$ space and let $x \in X$. Assume that the following (1) or (2) holds.

1. There exist lines $\gamma_i : \mathbb{R} \to X(i = 1, 2, \ldots, k)$ such that $\gamma_i(0) = x$ and
\[
\int_{B_1(x)} b_{\gamma_i} b_{\gamma_j} \, dm = 0, \quad \forall i \neq j \tag{2.47}
\]
are satisfied.

2. There exist harmonic functions $f_i : X \to \mathbb{R}(i = 1, 2, \ldots, k)$ such that $f_i(x) = 0$ and $(df_i, df_j) \equiv \delta_{ij}$ are satisfied.

Let us put $\varphi_i := b_{\gamma_i}$ if (1) holds, $\varphi_i := f_i$ if (2) holds. Then there exist a pointed RCD$(K, N - k)$ space $(Y, d_Y, y, m_Y)$ and an isometry
\[
\Phi : (X, d, x, m) \to \left( \mathbb{R}^k \times Y, \sqrt{d_{\mathbb{R}^k}^2 + d_{Y}^2}, (0_k, y), \mathcal{H}^k \otimes m_Y \right) \tag{2.48}
\]
such that $\varphi_i \equiv \pi_i \circ \Phi$ holds, where $\pi_i : \mathbb{R}^k \times Y \to \mathbb{R}$ is the projection to the $i$-th $\mathbb{R}$ of the Euclidean factor $\mathbb{R}^k$.

Note that quantitative versions of Theorem 2.28 are also justified. See [G13, Thm.1.5] and [BPS19, Prop.3.7 and 3.9] (see also [BPS20, Prop.1.4 and 1.5]). Let us give a variant of that after introducing the following well-known result. We give a proof for reader’s convenience.

Proposition 2.29. Let $(X, d, m)$ be an RCD$(0, N)$ space and let $f$ be a harmonic function on $(X, d, m)$ with $\|\nabla f\|_{L^\infty} < \infty$. Then $|\nabla f|$ is a constant function. In particular $f$ is Lipschitz on $X$ by Theorem 2.28.
Proof. Applying the existence of good cut-off functions constructed in [MN19, Lem.3.1] for rescaled RCD(0, N) space \((X, R^{-1} d, m)\), we see that for any \(R \in [1, \infty)\) and any \(x \in X\) there exists \(\varphi_R \in D(\Delta)\) such that \(0 \leq \varphi_R \leq 1\), \(\text{supp} \varphi_R \subset B_{2R}(x)\), \(\varphi_R|_{B_R(x)} \equiv 1\), \(|\nabla \varphi_R| \leq C(N) R^{-1}\), and \(|\Delta \varphi_R| \leq C(N) R^{-2}\) are satisfied. Then applying the Bochner inequality \((2.22)\) as \((\varphi, K) = (\varphi_R, 0)\) and then letting \(R \to \infty\) yield \(\text{Hess}_f = 0\). In particular

\[|\nabla|\nabla f|^2| \leq 2|\nabla f| : \text{Hess}_f = 0. \tag{2.49}\]

Since \(|\nabla f|^2|_U \in H^{1,2}(U, d, m)\) holds for any bounded open subset \(U\) of \(X\), \((2.49)\) yields that \(|\nabla f|^2\) is a constant function.

Let us end this subsection by giving the following result which is a variant of quantitative versions of Theorem 2.28, which will be used later.

**Theorem 2.30.** Let \(\delta \in (0, 1)\), let \(C, L, \tau \in (0, \infty)\), let \(k \in \mathbb{N}\), let \((X, d, x, m)\) be a pointed RCD\((-\delta, N)\) space with \(n = \dim_d m(X)\). Assume that there exist \(\varphi_i \in D(\Delta, B_L(x))(i = 1, 2, \ldots, k)\) such that \(|\nabla \varphi_i| \leq C\) on \(B_L(x)\),

\[
\frac{1}{m(B_1(x))} \int_{B_1(x)} (\Delta \varphi_i)^2\, dm < \delta, \tag{2.50}
\]

and

\[
\frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \sum_{i=1}^k d_i \varphi_i \otimes d \varphi_i - g_X \right|\, dm < \delta \tag{2.51}
\]

are satisfied. Then we see that \((X, d, x, m(B_1(x))^{-1} m)\) is \(\Psi(\delta, L^{-1}; C, N, \tau)\)-pmGH close to \((\mathbb{R}^n, d_{\mathbb{R}^n}, 0_n, \omega_n^{-1} \mathcal{H}^n)\) and that \(\Phi = (\varphi_1, \ldots, \varphi_k) : B_1(x) \to \mathbb{R}^k\) gives a \(\Psi(\delta, L^{-1}; C, N, \tau)\)-GH approximation map to the image.

*Proof.* The proof is done by contradiction. If not, then there exist \(\epsilon_0 \in (0, 1)\), a sequence of \(\delta_i \to 0^+\), a sequence of \(L_i \to \infty\), a sequence of pointed RCD\((-\delta_i, N)\) spaces \((X_i, d_i, x_i, m_i)\) with \(m_i(B_1(x_i)) = 1\), a sequence of functions \(\varphi_{j,i} \in D(\Delta, B_{L_i}(x_i))(j = 1, 2, \ldots, k)\) with \(|\nabla \varphi_{j,i}| \leq C\) on \(B_{L_i}(x_i)\) such that the following holds.

1. We have

\[
\int_{B_{r}(x_i)} \left| \sum_{j=1}^k d_i \varphi_{j,i} \otimes d \varphi_{j,i} - g_{X_i} \right|\, dm_i + \int_{B_{L_i}(x_i)} (\Delta_i \varphi_{j,i})^2\, dm_i \to 0. \tag{2.52}
\]

2. For any \(i \in \mathbb{N}\), one of the following two conditions (a) and (b) is satisfied:

(a) \((X_i, d_i, x_i, m_i)\) is not \(\epsilon_0\)-pmGH close to \((\mathbb{R}^n, d_{\mathbb{R}^n}, 0_n, \omega_n^{-1} \mathcal{H}^n)\), where \(n_i = \dim_{d_i,m_i}(X_i)\).

(b) \(\Phi_i = (\varphi_{1,i}, \ldots, \varphi_{k,i}) : B_1(x_i) \to \mathbb{R}^k\) is not an \(\epsilon_0\)-GH-approximation to the image.

By Theorems 2.20, 2.23 and 2.26, with no loss of generality we can assume that \(\dim_{d_i,m_i}(X_i)\) does not depend on \(i\) (thus we denote it by \(n\)) and that there exist an RCD\((0, N)\) space \((X, d, m)\) and harmonic functions \(\varphi_j\) on \((X, d, m)\) such that \(|\nabla \varphi_j|_{L^\infty} < \infty\) holds, that \((X_i, d_i, x_i, m_i)\) pmGH-converge to \((X, d, x, m)\) and that \(\varphi_{j,i} \in H^{1,2}\)-strongly converge to \(\varphi_j\) on \(B_r(x)\) for any \(r \in (0, \infty)\). Note that \((2.52)\) implies

\[
\int_{B_{r}(x)} \left| \sum_{j=1}^k d \varphi_{j,i} \otimes d \varphi_{j,i} - g_{X_i} \right|\, dm_i \to 0. \tag{2.53}
\]
because of $|\nabla_i \varphi_{j,i}| \leq C$. In particular Propositions 2.24 and 2.27 yield
\[
\int_{B_r(x)} \left| \sum_{j=1}^k d\varphi_j \otimes d\varphi_j - g_X \right|^2 \, dm \leq \liminf_{i \to \infty} \int_{B_r(x_i)} \left| \sum_{i=1}^k d\varphi_{j,i} \otimes d\varphi_{i,j} - g_X \right|^2 \, dm_i = 0.
\]
(2.54)
Thus we have
\[
\sum_{j=1}^k d\varphi_j \otimes d\varphi_j = g_X
\]
on $B_r(x)$.

On the other hand since Proposition 2.29 yields that $(\langle \nabla \varphi_j, \nabla \varphi_i \rangle)_{ij}$ is a constant matrix on $X$, applying Theorems 2.28 for $\varphi_j$ shows that (2.55) holds on $X$, that $(X, d, m)$ is isometric to $(\mathbb{R}^l, d_{\mathbb{R}^l}, \omega^{-1}_l \mathcal{H}^l)$ for some $l \in \mathbb{N}_{\leq k}$ (because the (LHS) of (2.55) consists of the Riemannian metric of the Euclidean factor coming from $\varphi_j$), and that the map $\Phi = (\varphi_1, \ldots, \varphi_k) : X \to \mathbb{R}^k$ is an isometric embedding as metric spaces (because each $\varphi_j$ is a linear function). Since $\varphi_{i,j}$ converge uniformly to $\varphi_j$ on $B_r(0_k)$ for any $r \in (0, \infty)$, for any $\epsilon \in (0, \infty)$, $\Phi_i : B_1(x_i) \to \mathbb{R}^k$ gives an $\epsilon$-GH-approximation to the image for any sufficiently large $i$. Therefore it is enough to prove $l = n$ in order to get a contradiction.

Combining Proposition 2.27 with (2.53) yields that $g_X, L^2$-strongly converge to $g_{\mathbb{R}^l}$ on $B_r(0_l)$. Thus we have
\[
l = \frac{1}{\mathcal{H}^l(B_r(0_l))} \int_{B_r(0_l)} |g_{\mathbb{R}^l}|^2 \, d\mathcal{H}^l = \lim_{i \to \infty} \frac{1}{m_l(B_r(x_i))} \int_{B_r(x_i)} |g_X|^2 \, dm_i = n.
\]
(2.56)
\[\square\]

Remark 2.31. As an immediately consequence of Theorem 2.30 (although it may be independent of our interest in the paper), it is easy to see that the following holds:

- Let $U$ be an open subset of an RCD($K, N$) space $(X, d, m)$ with $n = \text{dim}_{\text{d,dm}}(X)$, let $x \in U$, let $\Phi = (\varphi_1, \ldots, \varphi_k) : U \to \mathbb{R}^k$ be a locally Lipschitz map. Assume that $x$ is a harmonic point of each $\varphi_i$ and that
\[
\lim_{r \to 0^+} \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \sum_{i=1}^k d\varphi_i \otimes d\varphi_i - g_X \right| \, dm = 0
\]
holds. Then $x \in \mathcal{R}_n$.

See [AHPT18, Def.5.2] for the definition of harmonic points of $H^{1,2}$ functions.

Instead of giving the precise definition, let us emphasize that this notion is closely related to the differentiability of the function at a given point and that for any $H^{1,2}$ function, the set of all harmonic points of the function has full measure. For instance in the Riemannian case $(X, d, m) = (\mathbb{M}^n, d_g, \text{vol}_g)$ with $f \in C^1(\mathbb{M}^n)$, every point $x \in \mathbb{M}^n$ is a harmonic point of $f$. On the other hand if $f(x) = |x|$ on $\mathbb{R}^n$, then $0_n$ is not a harmonic point of $f$.

Let us consider the canonical inclusion $\iota : \overline{B}_1(0_n) \hookrightarrow \mathbb{R}^n$ with $e_n = (0, 0, \ldots, 0, 1) \in \overline{B}_1(0_n)$. Note that $(\overline{B}_1(0_n), d_{\mathbb{R}^n}, \mathcal{H}^n)$ is a compact non-collapsed RCD($0, n$) space with
\[
\sum_{i=1}^n dx_i \otimes dx_i = g_{\overline{B}_1(0_n)}.
\]
(2.58)
Then it is worth pointing out that $e_n$ is a harmonic point of the coordinate function $x_i$ for any $i \in \mathbb{N}_{\leq n-1}$. However $e_n$ is not a harmonic point of $x_n$. 

19
3 Isometric immersion

Let us fix an RCD($K$, $N$) space with $n = \dim d_m(X)$.

**Definition 3.1** (Pull-back metric). Let $A$ be a Borel subset of $X$ with $m(A) > 0$, and let $\Phi = (\varphi_1, \varphi_2, \ldots, \varphi_k) : A \to \mathbb{R}^k$ be a locally Lipschitz map, that is, for any $x \in A$ there exists $r \in (0, 1)$ such that $\Phi |_{A \cap B_r(x)}$ is Lipschitz. Define the pull-back (Riemannian semi) metric $\Phi^* g_{\mathbb{R}^k}$ by

$$\Phi^* g_{\mathbb{R}^k} := \sum_{i=1}^{k} d\varphi_i \otimes d\varphi_i \in L^0((T^*)^{\otimes 2}(A, d, m)), \quad (3.1)$$

where $d\varphi_i$ is the restriction to $A$ of the exterior derivative $d\varphi_i$ of a Lipschitz extension $\varphi_i$ of $\varphi_i |_{A \cap B_r(x)}$ to $X$ (that is, $d\varphi_i = d\varphi_i |_A$) which is well-defined because of the locality of the minimal relaxed slope. Then we say that $\Phi$ is an isometric immersion if $\Phi^* g_{\mathbb{R}^k} = g_X$ holds in $L^0((T^*)^{\otimes 2}(A, d, m))$.

This definition is compatible with [AHPT18, Prop.4.9] which gives the definition of pull-back (Riemannian semi) metric of a general Lipschitz map into a real separable Hilbert space. Note that if $\Phi$ is $C$-Lipschitz, then $\|\Phi^* g_{\mathbb{R}^k}\|_{L^\infty(A)} \leq C^2n$.

**Definition 3.2** (Regular map). Let $U$ be an open subset of $X$. Then a map $\Phi := (\varphi_1, \ldots, \varphi_k) : U \to \mathbb{R}^k$ is said to be regular on $U$ if each $\varphi_i$ is in $D(\Delta, U)$ with $\Delta \varphi_i \in L^\infty(U, m)$.

Note that thanks to a regularity result proved in [J14, Thm.3.1], any regular map $\Phi : U \to \mathbb{R}$ is locally Lipschitz.

**Definition 3.3** (Locally uniformly $\delta$-isometric immersion). Let $U$ be an open subset of $X$, let $\delta \in (0, \infty)$, let $\Phi : U \to \mathbb{R}^k$ be a locally Lipschitz map and let $A$ be a Borel subset of $U$. We say that $\Phi$ is locally uniformly $\delta$-isometric immersion on $A$ if for any $x \in A$ there exists $r_0 \in (0, 1)$ such that $B_{r_0}(x) \subset U$ and

$$\frac{1}{m(B_r(y))} \int_{B_r(y)} |\Phi^* g_{\mathbb{R}^k} - g_X| \, dm < \delta, \quad \forall y \in B_{r_0}(x) \cap A, \forall r \in (0, r_0) \quad (3.2)$$

are satisfied. Moreover $\Phi$ is said to be a locally uniformly isometric immersion on $A$ if it is a locally uniformly $\delta$-isometric immersion on $A$ for any $\delta \in (0, 1)$.

It is trivial that for a locally Lipschitz map $\Phi : U \to \mathbb{R}^k$ on open subset $U$ of $X$, $\Phi$ is a locally uniformly isometric immersion on $U$ if and only if $\Phi$ is an isometric immersion.

**Theorem 3.4**. Let $U$ be an open subset of $X$, let $A$ be a Borel subset of $U$, and let $\Phi = (\varphi_1, \ldots, \varphi_k) : U \to \mathbb{R}^k$ be a regular map with

$$\|\nabla \varphi_i\|_{L^\infty(U)} + \|\Delta \varphi_i\|_{L^\infty(U)} \leq C. \quad (3.3)$$

Assume that $\Phi$ is locally uniformly $\delta$-isometric immersion on $A$. Then we have the following:

1. The set $A$ is locally $\Psi(\delta; K, N, C)$-Reifenberg flat, that is, for any $x \in A$ there exists $r_1 \in (0, 1)$ such that

$$d_{GH}(B_r(y), B_r(0_n)) < \Psi(\delta; K, N, C)r, \quad \forall r \in (0, r_1], \forall y \in B_{r_1}(x) \cap A \quad (3.4)$$

holds.
2. \( \Phi|_A \) gives a locally bi-Lipschitz embedding map from \( A \) to \( \mathbb{R}^k \) whenever \( \delta \) is sufficiently small depending only on \( K, N \) and \( C \). More precisely, for any \( x \in A \), there exists \( r_2 \in (0,1) \) such that \( \Phi_{B_{r_2}(x) \cap A} \) and \( (\Phi|_{B_{r_2}(x) \cap A})^{-1} \) are \((1 + \Psi(\delta; K, N, C))\)-Lipschitz maps.

**Proof.** Let \( x \in A \), let \( r_0 \) be as in Definition 3.3, and let \( y \in B_{r_0}(x) \cap A \). For \( r \in (0, r_0] \) let us denote by \( \nabla_r, \Delta_r \) etc denote the gradient operator, the Laplacian, etc of the rescaled space

\[
(X_r, d_r, m_r) := (X, r^{-1}d, m(B_r(y))^{-1}m),
\]

which is an \( \operatorname{RCD}(r^2 K, N) \) space. Note that the rescaled functions \( \varphi_{r,i} := r^{-1} \varphi_i \) satisfy

\[
|\nabla_r \varphi_{r,i}| = |\nabla \varphi| \leq C \text{ and}
\]

\[
\int_{B_L^r(y)} |\Delta_r \varphi_{r,i}|^2 \, dm_r = \int_{B_L^r(y)} r^2 |\Delta \varphi_i| \, dm_r \\
\leq C r^2 m_r(B_L^r(y)) \\
\leq C r^2 \int_0^L \sinh^{N-1} t \, dt \cdot \left( \int_0^1 \sinh^{N-1} t \, dt \right)^{-1} \\
\leq C r \left( \int_0^1 \sinh^{N-1} t \, dt \right)^{-1},
\]

(3.6)

where \( L \) is defined by satisfying

\[
r^{-1} = \int_0^L \sinh^{N-1} t \, dt
\]

(3.7)

and we used the Bishop-Gromov inequality as an \( \operatorname{RCD}(-(N-1), N) \) space (because \( r^2 K \geq -(N-1) \) when \( r \) is small). Then applying Theorem 2.30 for \( (X_r, d_r, y, m_r) \) and \( \varphi_{r,i} \) yields that \( (X, r^{-1}d, y, m(B_r(y))^{-1}m) \) is \( \Psi(\delta; r; K, N, C)\)-pmGH close to \( (\mathbb{R}^n, d_{\mathbb{R}^n}, 0, \omega_n^{-1} \mathcal{H}^n) \) and that \( \Phi : B_r(y) \to \mathbb{R}^k \) gives a \( \Psi(\delta, r; K, N, C)r\)-GH-approximation to the image. Thus we have (1) and

\[
||\Phi(z) - \Phi(w)||_{\mathbb{R}^k} - (d(z, w)) \leq \Psi(\delta, r; K, N, C)r, \quad \forall z, w \in B_r(y).
\]

(3.8)

In particular for all \( z, w \in B_{r_0/4}(x) \) letting \( r := d(z, w) \) and \( w = y \) in (3.8) shows

\[
||\Phi(z) - \Phi(w)||_{\mathbb{R}^k} - (d(z, w)) \leq \Psi(\delta, r; K, N, C)d(z, w),
\]

(3.9)

that is,

\[
(1 - \Psi(\delta, r; K, N, C))d(z, w) \leq ||\Phi(z) - \Phi(w)||_{\mathbb{R}^k} \leq (1 + \Psi(\delta, r; K, N, C))d(z, w).
\]

(3.10)

Therefore \( \Phi|_{B_{r_0/4}(x)} \) is a \((1 + \Psi(\delta, r; K, N, C))\)-Lipschitz embedding satisfying that \((\Phi|_{B_{r_0/4}(x)})^{-1}\) is \((1 - \Psi(\delta, r; K, N, C))^{-1}\)-Lipschitz, whenever \( \delta, r \) are sufficiently small depending only on \( K, N, \) and \( C \). Thus we conclude. \(\square\)

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.**

Applying Theorem 3.4 as \( A = U \) and arbitrary \( \delta \in (0,1) \) yields (1) and (2) of Theorem 1.1. The remaining part, that \( U \) is homeomorphic to an \( n \)-dimensional topological manifold without boundary, is justified by the Reifenberg flatness with the intrinsic Reifenberg theorem proved in [CC97, Thm.A.1.2]. \(\square\)
4 Eigenmap

In this section we discuss eigenmaps on compact RCD(\(K,N\)) spaces.

4.1 Definition

Let us fix a compact RCD(\(K,N\)) space \((X,d,m)\).

**Definition 4.1 (Eigenmap).** We say that a map \(\Phi = (\varphi_1,\ldots,\varphi_k): X \to \mathbb{R}^k\) is a \(k\)-dimensional eigenmap if each \(\varphi_i\) is an eigenfunction of \(-\Delta\) for any \(i\) or \(\varphi_i \equiv 0\), that is, \(\Delta \varphi_i + \lambda_i \varphi_i \equiv 0\) for some \(\lambda_i \in [0,\infty)\). Put \(A(\Phi) := \{i; |\nabla \varphi_i| \not\equiv 0\}\), \(\Lambda(\Phi) := \max_{i \in A(\Phi)} \lambda_i\), \(L(\Phi) := \min_{1 \leq i \leq k} \frac{1}{m(X)} \int_X \varphi_i^2 dm\).

Moreover \(\Phi\) is said to be irreducible if each \(\varphi_i\) is not a constant function.

**Theorem 4.2.** Any \(k\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^k\) satisfies
\[
\nabla^* \Phi^* g_{\mathbb{R}^k} = -\frac{1}{4} d \Delta |\Phi|^2_{\mathbb{R}^k}.
\]

**Proof.** It is enough to check \((4.2)\) under assuming \(k = 1\). Then since \(\Delta |\Phi|^2 = 2\Phi \Delta \Phi + 2|d\Phi|^2\), Proposition 2.18 shows
\[
d\Delta |\Phi|^2 = 2\Delta \Phi d\Phi + 2d \Delta \Phi + 2|d\Phi|^2
\]
\[
= -4\lambda \Phi d\Phi + 2|d\Phi|^2
\]
\[
= 4\Phi d\Delta \Phi + 2|d\Phi|^2
\]
\[
= -4\nabla^* (d\Phi \otimes d\Phi),
\]
where \(\lambda\) denotes the eigenvalue of \(\Phi\).

4.2 Compactness for eigenmaps

Throughout this subsection let us fix an mGH-convergent sequence of compact RCD(\(K,N\)) spaces
\[(X_i,d_i,m_i) \xrightarrow{mGH} (X,d,m)\].

Let us discuss a convergence of eigenmaps in the following sense.

**Definition 4.3 (Convergence of eigenmaps).** We say that a sequence of \(k\)-dimensional eigenmaps \(\Phi_i : X_i \to \mathbb{R}^k\) converge to a \(k\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^k\) if \(\pi_j \circ \Phi_i \to \pi_j \circ \Phi\) \(L^2\)-strongly on \(X\) for any \(j\), where \(\pi_j : \mathbb{R}^k \to \mathbb{R}\) is the projection to the \(j\)-th \(\mathbb{R}\), and \(\Phi_i^* g_{\mathbb{R}^k}\) \(L^2\)-strongly converges to \(\Phi^* g_{\mathbb{R}^k}\) on \(X\).

Let us fix \(k \in \mathbb{N}\) and \(k\)-dimensional eigenmaps \(\Phi_i = (\varphi_{1,i},\ldots,\varphi_{k,i}) : X_i \to \mathbb{R}^k\) below.

**Proposition 4.4 (Compactness for general eigenmaps).** Assume
\[
\sup_i (\|\Phi_i\|_{L^2} + \|\Phi_i^* g_{\mathbb{R}^k}\|_{L^1} + \Lambda(\Phi_i)) < \infty.
\]
Then after passing to a subsequence, there exists a \(k\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^k\) such that \(\Phi_i\) converge to \(\Phi\) with
\[
\lim_{i \to \infty} \Lambda(\Phi_i) \geq \Lambda(\Phi), \quad \limsup_{i \to \infty} L(\Phi_i) \leq L(\Phi).
\]
Proof. By Theorems 2.23, 2.25 and 2.26 with (2.9) and (4.5), after passing to a subsequence, with no loss of generality we can assume that there exist $\lambda_j \in \mathbb{R}$ and $\varphi_j \in D(\Delta)$ such that $\Delta \varphi_j + \lambda_j \varphi_j = 0$ holds, that $\sup_i \| \nabla \varphi_{j,i} \|_{L^\infty} < \infty$ holds and that $\varphi_{j,i} \in H^{1,2}$ strongly converge to $\varphi_j$. Then it is easy to check that $\Phi_i$ converge to a $k$-dimensional eigenmap $\Phi := (\varphi_1, \ldots, \varphi_k) : X \to \mathbb{R}^k$ and that (4.6) holds. \qed

Remark 4.5. In Theorem 4.4 it is essential to assume an uniform upper bound on $\Lambda(\Phi)$. In order to check this, let us consider a collapsing tori:

\[(X_r, d_r, m_r) := \left( S^1 \times S^1(r), \sqrt{d_{S^1}^2 + d_{S^1(r)}^2}, \frac{1}{4\pi r^2} H^2 \right) \to \left( S^1, d_{S^1}, \frac{1}{2\pi} H^1 \right) := (X, d, m) \tag{4.7} \]

as $r \to 0^+$. Then for any $r \in (0, \infty)$, the canonical inclusion $\Phi_r : X_r \to \mathbb{R}^4$ is an irreducible 4-dimensional eigenmap with $\Phi^*_r g_{S^4} = g_{X_r}$. Note that $\Lambda(\Phi_r) \to \infty$ holds, that $L(\Phi_r) \to 0$ holds, and that $g_{X_r}$ $L^2$-weakly converge to $g_X$, but it is not an $L^2$-strong convergence because of

\[\int_{X_r} |g_{X_r}|^2 \, dm_r \equiv 2 > 1 \equiv \int_X |g_X|^2 \, dm. \tag{4.8} \]

In particular $\Phi^*_r g_{S^4}$ has no $L^2$-strong convergent subsequence as $r \to 0^+$.

Corollary 4.6 (Compactness for irreducible eigenmaps). Assume that each $\Phi_i$ is irreducible with

\[\sup_i \| \Phi^*_i g_{\mathbb{R}^k} \|_{L^1} < \infty \tag{4.9} \]

and

\[\inf_i L(\Phi_i) > 0. \tag{4.10} \]

Then it holds that (4.5) is satisfied and that after passing to a subsequence, there exists a $k$-dimensional irreducible eigenmap $\Phi : X \to \mathbb{R}^k$ such that $\Phi_i$ converge to $\Phi$

\[L(\Phi_i) \to L(\Phi), \quad \Lambda(\Phi_i) \to \Lambda(\Phi). \tag{4.11} \]

Proof. Since

\[
\int_{X_i} |\varphi_{j,i}|^2 \, dm_i \leq \lambda_1(X_i, d_i, m_i)^{-1} \int_{X_i} |d\varphi_{j,i}|^2 \, dm_i = \lambda_1(X_i, d_i, m_i)^{-1} \int_{X_i} \langle \Phi^*_i g_{\mathbb{R}^k}, g_{X_i} \rangle \, dm_i \\
\leq \lambda_1(X_i, d_i, m_i)^{-1} \sqrt{\dim_{d^*,m_i}(X_i)} \int_{X_i} |\Phi^*_i g_{\mathbb{R}^k}| \, dm_i, \tag{4.12}
\]

we have $\sup_i \| \Phi_i \|_{L^2} < \infty$. Let us denote by $\lambda_{j,i}$ the eigenvalue of $\varphi_{j,i}$. Then since

\[
\lambda_{j,i} = \left( \int_{X_i} |d\varphi_{j,i}|^2 \, dm_i \right) \cdot \left( \int_{X_i} |\varphi_{j,i}|^2 \, dm_i \right)^{-1} \leq \sqrt{\dim_{d^*,m_i}(X_i)} \int_{X_i} |\Phi^*_i g_{\mathbb{R}^k}| \, dm_i \cdot \left( \inf_i L(\Phi_i) \right)^{-1}, \tag{4.13}
\]

we have $\sup_i \lambda(\Phi_i) < \infty$. Thus applying Proposition 4.4 yields that after passing to a subsequence, there exists a $k$-dimensional eigenmap $\Phi : X \to \mathbb{R}^k$ such that $\Phi_i$ converge to $\Phi$. The irreducibility of $\Phi$ comes from a fact that $\lambda_1(X_i, d_i, m_i) \to \lambda_1(X, d, m)$ by Theorem 2.25. Then it is easy to see (4.11). \qed
Finally let us mention the following approximation result.

**Proposition 4.7** (Approximation). Let \( \hat{\Phi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_k) : X \to \mathbb{R}^k \) be a \( k \)-dimensional (irreducible, respectively) eigenmap. Then there exists a sequence of \( k \)-dimensional (irreducible, respectively) eigenmaps \( \hat{\Phi}_i : X_i \to \mathbb{R}^k \) such that \( \hat{\Phi}_i \) converge to \( \hat{\Phi} \), that \( \sup_i \| \hat{\Phi}_i^* g_{\mathbb{R}^k} \|_{L^\infty} < \infty \) and

\[
\Lambda(\hat{\Phi}_i) \to \Lambda(\hat{\Phi}), \quad L(\hat{\Phi}_i) \to L(\hat{\Phi}) \quad (4.14)
\]

are satisfied.

**Proof.** Let us find \( \lambda_j \in [0, \infty) \) with \( \Delta \hat{\varphi}_j + \lambda_j \hat{\varphi}_j \equiv 0 \). By Theorem 2.25, for any \( j \), we can find a convergent sequence \( \lambda_{j,i} \to \lambda_j \) and sequence of \( \hat{\varphi}_{j,i} \in D(\Delta^i) \) with \( \Delta^i \hat{\varphi}_{j,i} + \lambda_{j,i} \hat{\varphi}_{j,i} \equiv 0 \) such that \( \hat{\varphi}_{j,i} \) \( H^{1,2} \)-strongly converge to \( \hat{\varphi}_j \). Then eigenmaps \( \hat{\Phi}_i := (\hat{\varphi}_{1,i}, \ldots, \hat{\varphi}_{k,i}) : X_i \to \mathbb{R}^k \) satisfy the desired properties. \( \square \)

## 5 Isometric immersion via eigenmap

The main purpose of this section is to prove Theorem 1.2. We divide the proof into several lemmas as follows. Throughout the section we fix a compact RCD \((K,N)\) space \((X,d,m)\) with \( n := \dim_d m(X) \) and a \((k+1)\)-dimensional eigenmap \( \Phi = (\varphi_1, \ldots, \varphi_{k+1}) : X \to \mathbb{R}^{k+1} \) with

\[
\Phi^* g_{\mathbb{R}^{k+1}} = g_X. \quad (5.1)
\]

Thanks to (2.9), \( \Phi \) is a \( C_1 \)-Lipschitz map from \( X \) to \( \mathbb{R}^{k+1} \) for some \( C_1 \in (0, \infty) \).

**Lemma 5.1.** The following two conditions are equivalent:

1. \( |\Phi|_{\mathbb{R}^{k+1}} \) is a constant function.

2. \((X,d,H^n)\) is a non-collapsed RCD\((K,n)\) space with

\[
m = \frac{m(X)}{\mathcal{H}^n(X)} \mathcal{H}^n. \quad (5.2)
\]

**Proof.** Let us first prove the implication from (1) to (2). Assume that \( |\Phi|_{\mathbb{R}^{k+1}} \) is a constant function. Then Corollary 4.2 shows

\[
\nabla^* \Phi^* g_{\mathbb{R}^{k+1}} = -\frac{1}{4} \Delta |\Phi|^2_{\mathbb{R}^{k+1}} = 0. \quad (5.3)
\]

Thus \( g \in D(\nabla^*) \) with \( \nabla^* g = 0 \) which completes the proof of (2) because of Theorem 2.19.

Next we prove the remaining implication. Assume that \((X,d,H^n)\) is a non-collapsed RCD\((K,n)\) space. Then applying Theorem 2.19 again yields \( \Delta |\Phi|^2_{\mathbb{R}^{k+1}} = 0 \). Since \( \Delta |\Phi|^2_{\mathbb{R}^{k+1}} \in H^{1,2}(X,d,H^n) \), \( |\Phi|^2_{\mathbb{R}^{k+1}} \) is a constant function. In particular for any \( f \in D(\Delta) \) with \( \Delta f + \lambda f \equiv 0 \) for some \( \lambda \in (0, \infty) \) we have

\[
\int_X f \Delta |\Phi|^2_{\mathbb{R}^{k+1}} \, dm = 0 \quad (5.4)
\]

which implies

\[
\int_X f |\Phi|^2_{\mathbb{R}^{k+1}} \, dm = 0. \quad (5.5)
\]

Since \( f \) is arbitrary, \( |\Phi|^2_{\mathbb{R}^{k+1}} \) is also a constant function. \( \square \)
From now on we assume that $|\Phi|$ is a constant function. By rescaling, with no loss of generality we can assume $|\Phi| \equiv 1$.

**Lemma 5.2.** We have $\min_i \lambda_i \leq n$.

**Proof.** (5.1) yields

$$\langle \Phi^* g_{\mathbb{R}^{k+1}}, g \rangle = \langle g, g \rangle = n.$$  \hfill (5.6)

Thus integrating (5.6) over $X$ shows

$$\min_i \lambda_i \leq \frac{k+1}{\mathcal{H}^n(X)} \int_X |d\varphi_i|^2 \, d\mathcal{H}^n = n.$$ \hfill (5.7)

\[ \square \]

**Lemma 5.3.** For any $x \in X$ and any $\epsilon \in (0, 1)$ there exists $r \in (0, 1)$ such that $\Phi|_{B_r(x)} : B_r(x) \to (\Phi(B_r(x)), d_{S^k})$ is a bi-Lipschitz map, that $\Phi|_{B_r(x)}$ is $(1 + \epsilon)$-Lipschitz, and that $(\Phi|_{B_r(x)})^{-1} : (\Phi(B_r(x)), d_{S^k}) \to (X, d)$ is $(1 + \epsilon)$-Lipschitz.

**Proof.** Fix a sufficiently small $\epsilon \in (0, 1)$. Since $O(k + 1)$ acts on $S^k$ transitively, with no loss of generality we can assume that $\varphi_{k+1}(x) = 1$. Note that

$$\sum_{j=1}^{k} \left(1 + \frac{\varphi_j^2}{1 - \sum_{j=1}^{k} \varphi_j^2}\right) d\varphi_j \otimes d\varphi_j + \sum_{j \neq l} \frac{\varphi_j \varphi_l}{1 - \sum_{j=1}^{k} \varphi_j^2} d\varphi_j \otimes d\varphi_l = g_X \hfill (5.8)$$

holds on a neighbourhood of $x$. Thus we can find $r_0 \in (0, \epsilon)$ satisfying that

$$\left| \sum_{j=1}^{k} d\varphi_j \otimes d\varphi_j - g_X \right| (y) < \epsilon \hfill (5.9)$$

for $\mathcal{H}^n$-a.e. $y \in B_{2r_0}(x)$. In particular for any $x \in B_{r_0}(y)$ and any $r \in (0, r_0]$ we have

$$\frac{1}{m(B_r(y))} \int_{B_r(y)} \left| \sum_{j=1}^{k} d\varphi_j \otimes d\varphi_j - g_X \right| dm < \epsilon. \hfill (5.10)$$

Define a map $\hat{\Phi}$ from $X$ to $\mathbb{R}^k$ by

$$\hat{\Phi} := (\varphi_1, \varphi_2, \ldots, \varphi_k). \hfill (5.11)$$

Then (5.10) says that $\hat{\Phi}$ is a locally uniformly $\epsilon$-isometric immersion on $B_{r_0}(x)$. Thus applying Theorem 3.4 gives that there exists $r_1 \in (0, r_0)$ such that $\hat{\Phi}|_{B_{r_1}(x)}$ is a $(1 + \Psi(\epsilon; K, n, C_1))$-Lipschitz embedding map and that $(\hat{\Phi}|_{B_{r_1}(x)})^{-1}$ is $(1 - \Psi(\epsilon; K, n, C_1))^{-1}$-Lipschitz.

On the other hand recall that a map $\pi := \pi_r$ from $(B_r(\Phi(x)), d_{S^k})$ to $(\mathbb{R}^k, d_{\mathbb{R}^k})$ defined by

$$\pi(x_1, x_2, \ldots, x_{k+1}) := (x_1, x_2, \ldots, x_k) \hfill (5.12)$$

is a 1-Lipschitz embedding satisfying that $(\pi|_{B_r(\Phi(x))})^{-1}$ is $(1 + \Psi(r; k))$-Lipschitz. Therefore since $\Phi = \pi^{-1} \circ \hat{\Phi}$ holds on $B_{r_1}(x)$, we have the desired statement because $\epsilon$ is arbitrary. \[ \square \]

**Lemma 5.4.** $\Phi : (X, d) \to (S^k, d_{S^k})$ is 1-Lipschitz.
Proof. By Lemma 5.3, for any 1-Lipschitz function \( F \) on \( S^k \) and any \( \epsilon \in (0, 1) \) we see that \( \text{lip}(F \circ \Phi)(x) \leq 1 + \epsilon \) holds for any \( x \in X \). Therefore \( F \circ \Phi \) is also 1-Lipschitz because \( \epsilon \) is arbitrary. In particular for fixed \( x \in X \), taking \( F(z) = d_{S^k}(\Phi(x), z) \) yields

\[
d_{S^k}(F(\Phi(x)), F(\Phi(y))) = |F(\Phi(x)) - F(\Phi(y))| \leq d(x, y)
\]

which completes the proof. \( \square \)

From now on we assume \( k = n \).

Lemma 5.5. \( \Phi : (X, d) \rightarrow (S^n, d_{S^n}) \) is a local isometry.

Proof. Let us remark that \( \Phi(X) \) is an open subset of \( S^n \) because \( X \) is homeomorphic to an \( n \)-dimensional topological closed manifold by Theorem 1.1, where we used a fact that if \( F : M^n \rightarrow N^n \) is an injective continuous map between two \( n \)-dimensional topological manifolds \( M^n \) and \( N^n \), then \( F \) is a local homeomorphism to \( N^n \). Moreover since \( \Phi(X) \) is compact, in particular, it is closed. Thus we see that \( \Phi \) is a surjective locally bi-Lipschitz map to \( S^n \).

Using the compactness of \( X \) with Lemma 5.3, find \( r \in (0, 1) \) satisfying that \( \Phi|_{B_{2r}(x)} \) is a bi-Lipschitz map to the image including \( B_r(\Phi(x)) \) for any \( x \in X \). Fix \( x \in X \) and let us define \( f \in \text{Lip}(B_r(\Phi(x)), d_{S^n}) \) by

\[
f(y) := d \left( x, (\Phi|_{B_{2r}(x)})^{-1}(y) \right).
\]

Then Lemma 5.3 yields

\[
\text{lip} f(y) \leq (1 + \epsilon) \text{lip}(F \circ \Phi)((\Phi|_{B_{2r}(x)})^{-1}(y)) \leq 1 + \epsilon
\]

holds for any \( y \in B_r(\Phi(x)) \) and any \( \epsilon \in (0, 1) \). Since \( B_r(\Phi(x)) \) is a convex subset of \( S^n \), (5.15) shows that \( f \) is 1-Lipschitz on \( B_r(\Phi(x)) \) because \( \epsilon \) is arbitrary. In particular we have for any \( y \in B_r(\Phi(x)) \)

\[
d(x, y) = |f(\Phi(x)) - f(\Phi(y))| \leq d_{S^n}(\Phi(x), \Phi(y))
\]

which completes the proof because of Lemma 5.4. \( \square \)

The following lemma completes the proof of Theorem 1.2:

Lemma 5.6. \( \Phi : (X, d) \rightarrow (S^n, d_{S^n}) \) is an isometry.

Proof. Lemma 5.5 shows that \( (X, d) \) is isometric to an \( n \)-dimensional closed Riemannian manifold whose sectional curvature is equal to 1. Then Lemma 5.2 with Obata’s theorem yields that \( (X, d) \) is isometric to \( (S^n, d_{S^n}) \). In particular \( \Phi : (X, d) \rightarrow (S^n, d_{S^n}) \) is an isometry because it is easy to see that if an eigenmap \( \tilde{\Phi} : S^n \rightarrow \mathbb{R}^{n+1} \) satisfies \( \tilde{\Phi}^*g_{\mathbb{R}^{n+1}} = g_{S^n} \), then \( \tilde{\Phi} \) coincides with the canonical inclusion \( \iota : S^n \hookrightarrow \mathbb{R}^{n+1} \) up to multiplying an element of \( O(n + 1) \). \( \square \)

6 Sphere theorem

The main purpose of this section is to prove Theorems 1.3 and 1.4.
6.1 Convergence result for eigenmaps

The main result of this subsection is the following, which will play a key role in the proofs of Theorems 1.3 and 1.4.

**Theorem 6.1.** Let \((X_i, d_i, \mathcal{H}^n)\) be a sequence of compact non-collapsed RCD\((K, n)\) spaces, let \(k \in \mathbb{N}\) and let \(\Phi_i : X_i \to \mathbb{R}^k\) be a \(k\)-dimensional eigenmap. Assume that

\[
\frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |\Phi_i^*g_{\mathbb{R}^k} - g_{X_i}| \, d\mathcal{H}^n \to 0 \tag{6.1}
\]

and

\[
\sup_i (\text{diam}(X_i, d_i) + \|\Phi_i\|_{L^2} + \Lambda(\Phi_i)) < \infty \tag{6.2}
\]

are satisfied. Then after passing to a subsequence, there exist a compact non-collapsed RCD\((K, n)\) space \((X, d, \mathcal{H}^n)\) and a \(k\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^k\) such that the following holds.

1. \((X_i, d_i, \mathcal{H}^n)\) mGH-converge to \((X, d, \mathcal{H}^n)\).
2. \(X_i\) is homeomorphic to \(X\) for any sufficiently large \(i\).
3. \(\Phi_i\) converge to \(\Phi\).
4. \(\Phi_i^* g_{\mathbb{R}^k} = g_X\).

**Proof.** By Theorem 2.20, with no loss of generality we can assume that there exists a compact RCD\((K, n)\) space \((X, d, m)\) such that

\[
\left( X_i, d_i, \mathcal{H}^n(X_i)^{-1} \mathcal{H}^n \right) \xrightarrow{\text{mGH}} (X, d, m) \tag{6.3}
\]

holds.

Applying Proposition 4.4, after passing a subsequence, there exists a \(k\)-dimensional eigenmap \(\Phi : X \to \mathbb{R}^k\) such that \(\Phi_i\) converge to \(\Phi\). By Proposition 2.27 with (6.1), we see that \(g_{X_i} L^2\)-strongly converge to \(g_X\). In particular we have

\[
\int_X |\Phi_i^* g_{\mathbb{R}^k} - g_X|^2 \, dm = \lim_{i \to \infty} \frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |\Phi_i^* g_{\mathbb{R}^k} - g_{X_i}|^2 \, d\mathcal{H}^n = 0, \tag{6.4}
\]

(thus (4) of Theorem 6.1 holds) and

\[
\dim_{d, m}(X) = \frac{1}{m(X)} \int_X |g_X|^2 \, dm = \lim_{i \to \infty} \frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |g_{X_i}|^2 \, d\mathcal{H}^n = \lim_{i \to \infty} \dim_{d_i, m_i}(X_i) = n. \tag{6.5}
\]

Therefore Theorem 2.10 yields that \((X, d, \mathcal{H}^n)\) is a non-collapsed RCD\((K, n)\) space with \(\mathcal{H}^n(X) m = m(X) \mathcal{H}^n\). Then we see that the sequence \(\{(X_i, d_i)\}_i\) is uniformly Reifenberg flat, that is:

\(\heartsuit\) For any \(\epsilon \in (0, 1)\) there exist \(i_0 \in \mathbb{N}\) and \(r_0 \in (0, \epsilon)\) such that for any \(i \in \mathbb{N}_{\geq i_0}\), any \(x_i \in X_i\) and any \(r \in (0, r_0)\) we have \(d_{GH}(B_r(x_i), B_r(0_n)) < \epsilon r\).

Although the proof of \(\heartsuit\) is well-known, for reader’s convenience, let us give a proof (see also [HM19, Rem.2.4]).

Fix \(\epsilon \in (0, 1)\). Thanks to Theorem 1.1, \(X\) is Reifenberg flat. Thus there exists \(r_0 \in (0, \epsilon)\) such that

\[
d_{GH}(B_r(x), B_r(0_n)) < \epsilon r, \quad \forall r \in (0, 2r_0), \forall x \in X \tag{6.6}
\]
holds. In particular there exists $i_0 \in \mathbb{N}$ such that for any $i \in \mathbb{N}_{>i_0}$ and any $x_i \in X_i$ we have
\begin{equation}
\text{d}_{GH}(B_{2r_0}(x_i), B_{2r_0}(0_n)) < 4e r_0. \tag{6.7}
\end{equation}
Fix $i \in \mathbb{N}_{>i_0}$ and $x_i \in X_i$. Therefore Theorem 2.9 yields
\begin{equation}
1 - \Psi(\epsilon; K, n) \leq \frac{\mathcal{H}^n(B_{2r_0}(x_i))}{\mathcal{H}^n(B_{2r_0}(0_n))} \leq 1 + \Psi(\epsilon; K, n).
\end{equation}

The Bishop-Gromov and Bishop inequalities (Theorem 2.8) yield that
\begin{equation}
1 - \Psi(\epsilon; K, n) \leq \frac{\mathcal{H}^n(B_{2r}(x_i))}{\mathcal{H}^n(B_{2r}(0_n))} \leq 1 + \Psi(\epsilon; K, n), \quad \forall r \in (0, r_0]
\end{equation}
holds. Thus applying Theorem 2.9 again shows
\begin{equation}
\text{d}_{GH}(B_{r}(x_i), B_{r}(0_n)) < \Psi(\epsilon; K, n)r, \quad \forall r \in (0, r_0]
\end{equation}
which completes the proof of (♠) because $\epsilon$ is arbitrary.

Then the remaining statement, that $X_i$ is homeomorphic to $X$ for any sufficiently large $i$, follows from (♠) with the topological stability theorem proved in [CC97, Thm A.1.2 and A.1.3]. \hfill \Box

### 6.2 Proof of Theorems 1.3 and 1.4

Let us finish the proofs of Theorems 1.3 and 1.4.

**Proof of Theorem 1.3.**

Let us prove (1) of Theorem 1.3. The proof is done by contradiction. If not, there exist $\epsilon_0 \in (0, 1)$, a sequence of compact non-collapsed RCD$(K, n)$ spaces $(X_i, d_i, \mathcal{H}^n)$ with $\text{diam}(X_i, d_i) \leq d$, and a sequence of irreducible $(n + 1)$ dimensional eigenmaps $\Phi_i : X_i \to \mathbb{R}^{n+1}$ with $L(\Phi_i) \geq \tau$ such that
\begin{equation}
\frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |\Phi_i^* g_{\mathbb{R}^{n+1}} - g_{X_i}|^2 d\mathcal{H}^n \to 0 \tag{6.11}
\end{equation}
and
\begin{equation}
\text{d}_{GH}(X_i, S^n(a_i)) \geq \epsilon_0 \tag{6.12}
\end{equation}
are satisfied, where
\begin{equation}
a_i^2 = \frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |\Phi_i|^2 d\mathcal{H}^n. \tag{6.13}
\end{equation}

Applying Theorem 6.1 with Corollary 4.6 shows that after passing to a subsequence, there exist a compact non-collapsed RCD$(K, n)$ space and an irreducible $(n + 1)$ dimensional eigenmap $\Phi : X \to \mathbb{R}^{n+1}$ such that $(X_i, d_i, \mathcal{H}^n)$ mGH-converge to $(X, d, \mathcal{H}^n)$, that $\Phi_i$ converge to $\Phi$ and that $\Phi_i^* g_{\mathbb{R}^{n+1}} = g_X$. Then Theorem 1.2 yields that $|\Phi|$ is a constant function and that $(X, d)$ is isometric to $(S^n(|\Phi|), d_{S^n(|\Phi|)})$ which contradicts (6.12) because of $a_i^2 \to |\Phi|^2$.

The proof of (2) of Theorems 1.3 is similarly done by contradiction after establishing the following result:

(\checkmark) If a sequence of non-collapsed RCD$(K, n)$ spaces $(X_i, d, \mathcal{H}^n)$ satisfies $\text{d}_{GH}(X_i, S^n(a)) \to 0$ for some $a \in (0, \infty)$, then there exists a sequence of irreducible $(n + 1)$ dimensional eigenmaps $\Phi_i : X_i \to \mathbb{R}^{n+1}$ such that $\Phi_i$ converge to the canonical inclusion $\iota : S^n(a) \to \mathbb{R}^{n+1}$ with $L(\Phi_i) \to L(\iota)$.
However (◦) is a direct consequence of Proposition 4.7 with Theorem 2.11. Thus we conclude. □

**Proof of Theorem 1.4.**

The proof is also done by contradiction. If not, then we can find a sequence of compact non-collapsed RCD\((K,n)\) spaces \((X_i, d_i, \mathcal{H}^n)\) with \(\text{diam}(X_i, d_i) \leq d\), and a sequence of irreducible \((n + 1)\)-dimensional eigenmaps \(\Phi_i : X_i \rightarrow \mathbb{R}^{n+1}\) with \(L(\Phi_i) \geq \tau\) such that \(X_i\) is homeomorphic to \(\mathbb{S}^n\) and that

\[
\frac{1}{\mathcal{H}^n(X_i)} \int_{X_i} |\Phi_i^* g_{\mathbb{R}^{n+1}} - g_{X_i}| \, d\mathcal{H}^n \rightarrow 0
\]

(6.14)

holds. Then applying Theorems 1.3 and 6.1 with Corollary 4.6, after passing to a subsequence, we see that \((X_i, d_i)\) GH-converge to \((\mathbb{S}^n(a), d_{\mathbb{S}^n(a)})\) for some \(a \in (0, \infty)\) and that \(X_i\) is homeomorphic to \(\mathbb{S}^n(a)\) for any sufficiently large \(i\). This is a contradiction. □

7 Topological finiteness theorems

In this section we establish topological finiteness theorems for almost isometric immersions.

7.1 Convergence result for regular maps

Let us fix \(k \in \mathbb{N}\) and a mGH-convergent sequence of compact RCD\((K,N)\) spaces:

\[(X_i, d_i, m_i) \xrightarrow{\text{mGH}} (X, d, m).\] (7.1)

Note that if a sequence \(\varphi_i \in D(\Delta_i)\) satisfies \(\sup_i \| \Delta_i \varphi_i \|_{L^\infty} < \infty\), then \(\sup_i \| \nabla_i \varphi_i \|_{L^\infty} < \infty\) holds, which is a direct consequence of a regularity result proved in [J14, Thm.3.1]. We will use this fact immediately below. As in Definition 4.3 for eigenmaps, let us define the convergence of Lipschitz (not necessary eigen) maps as follows.

**Definition 7.1** (Convergence of Lipschitz maps). We say that a sequence of Lipschitz maps \(\Phi_i : X_i \rightarrow \mathbb{R}^k\) converge to a Lipschitz map \(\Phi : X \rightarrow \mathbb{R}^k\) if \(\pi_j \circ \Phi_i \rightarrow \Phi\) \(L^2\)-strongly converges to \(\pi_j \circ \Phi\) for any \(j\), where \(\pi_j : \mathbb{R}^k \rightarrow \mathbb{R}\) is the projection to the \(j\)-th \(\mathbb{R}\), and \(\Phi_i^* g_{\mathbb{R}^k}\) \(L^2\)-strongly converge to \(\Phi^* g_{\mathbb{R}^k}\) on \(X\).

By an argument similar to the proof of Proposition 4.4 we have the following:

**Proposition 7.2** (Compactness for regular maps). Let \(\Phi_i = (\varphi_{1,i}, \ldots, \varphi_{k,i}) : X_i \rightarrow \mathbb{R}^k\) be a sequence of regular maps with

\[
\sup_{j,i} (\| \varphi_{j,i} \|_{L^2} + \| \Delta_i \varphi_{j,i} \|_{L^\infty}) < \infty.
\]

(7.2)

Then after passing to a subsequence there exists a regular map \(\Phi : X \rightarrow \mathbb{R}^k\) such that \(\Phi_i\) converge to \(\Phi\).

Let us introduce the following “gap theorem”:

**Theorem 7.3** (Gap theorem). For any \(K \in \mathbb{R}\), any \(N \in [1, \infty)\) and all \(d, \lambda \in (0, \infty)\) there exists \(\epsilon_0 := \epsilon_0(K, N, d, \lambda) \in (0, 1)\) such that if a compact RCD\((K,N)\) space \((X, d, m)\) with \(n = \text{dim}_{d,m}(X)\) satisfies \(\text{diam}(X, d) \leq d\), then

\[
\frac{1}{m(X)} \int_X |\Phi^* g_{\mathbb{R}^k} - g_X| \, dm \geq \epsilon_0
\]

(7.3)

holds for any \(k \in \mathbb{N}_{\leq n}\) and any regular map \(\Phi = (\varphi_1, \ldots, \varphi_k) : X \rightarrow \mathbb{R}^k\) with \(\| \Delta \varphi_i \|_{L^\infty} \leq \lambda\).
Proof. The proof is done by contradiction. If not, there exist $n \in \mathbb{N}$, $k \in \mathbb{N}_{\leq n}$, a sequence of compact RCD($K,N$) spaces $(X_i,d_i,m_i)$, and a sequence of regular maps $\Phi_i = (\varphi_{1,i}, \ldots, \varphi_{k,i}) : X_i \to \mathbb{R}^k$ such that $\|\Delta_i \varphi_{j,i}\|_{L^\infty} \leq \lambda$, $\sup_{j,i} \|\varphi_{j,i}\|_{L^2} < \infty$, $\dim(X_i,d_i) \leq d$, $m_i(X_i) = 1$, $\dim_{d,m_i}(X_i) = n$ and

$$\int_X |\Phi_i^* g_{\mathbb{R}^k} - g_{X_i}| \, dm_i \to 0 \quad (7.4)$$

are satisfied. Thus Proposition 7.2 yields that after passing to a subsequence, there exists a regular map $\Phi : X \to \mathbb{R}^k$ such that $\Phi_i$ converge to $\Phi$. Therefore (7.4) with Proposition 2.27 shows that $\Phi^* g_{\mathbb{R}^k} = g_X$ holds and that $g_X$, $L^2$-strongly converge to $g_X$ on $X$. In particular

$$\dim_{d,m}(X) = \int_X |g_X|^2 \, dm = \lim_{i \to \infty} \int_{X_i} |g_{X_i}|^2 \, dm_i = \lim_{i \to \infty} \dim_{d_i,m_i}(X_i) = n. \quad (7.5)$$

Therefore (2) of Theorem 1.1 yields that $X$ is homeomorphic to an $n$-dimensional closed manifold. In particular $X$ cannot have a local homeomorphism into $\mathbb{R}^k$, which contradicts (1) of Theorem 1.1.

\[\square\]

7.2 Topological finiteness theorems

We are now in a position to give two topological finiteness theorems. The first one is stated for any compact (not necessary non-collapsed) RCD($K,N$) spaces as follows:

**Theorem 7.4 (Topological finiteness theorem I).** For any $K \in \mathbb{R}$, any $k \in \mathbb{N}$, any $N \in [1, \infty)$ and all $d, \lambda \in (0, \infty)$, there exists $\delta := \delta(K,k,N,d,\lambda) \in (0,1)$ such that the following holds. Let us denote by $\mathcal{M} := \mathcal{M}(K,k,N,d,\lambda)$ the set of (isometry classes of) compact RCD($K,N$) space $(X,d,m)$ satisfying that $\dim(X,d) \leq d$ and that there exists a regular map $\Phi = (\varphi_1, \ldots, \varphi_k) : X \to \mathbb{R}^k$ such that $\|\Delta \varphi_i\|_{L^\infty} \leq \lambda$ and

$$\|\Phi^* g_{\mathbb{R}^k} - g_X\|_{L^\infty} \leq \delta \quad (7.6)$$

are satisfied. Then for any $(X,d,m) \in \mathcal{M}$, $X$ is homeomorphic to an $n$-dimensional topological closed manifold. Moreover $\mathcal{M}$ has finitely many members up to homeomorphism.

Proof. Since the proof of the first statement is quite similar to that of Theorems 1.3 and 1.4, we give a proof only for the topological finiteness statement.

If not, then we can find a sequence of compact RCD($K,N$) spaces $(X_i,d_i,m_i)$ and a sequence of regular maps $\Phi_i = (\varphi_{1,i}, \ldots, \varphi_{k,i}) : X_i \to \mathbb{R}^k$ such that $\dim(X_i,d_i) \leq d$ holds, that $m_i(X_i) = 1$ holds, that $\|\Delta_i \varphi_{j,i}\|_{L^\infty} \leq \lambda$ holds, that $\sup_{j,i} \|\varphi_{j,i}\|_{L^2} < \infty$ holds, that $X_i$ is not homeomorphic to $X_j$ for all $i \neq j$, and that

$$\|\Phi_i^* g_{\mathbb{R}^k} - g_{X_i}\|_{L^\infty} \to 0 \quad (7.7)$$

holds. Applying Theorem 2.20 with Proposition 7.2 shows that after passing to a subsequence, there exist a compact RCD($K,N$) space $(X,d,m)$ and a regular Lipschitz map $\Phi : X \to \mathbb{R}^k$ such that $(X_i,d_i,m_i)$ mGH-converge to $(X,d,m)$ and that $\Phi_i$ converge to $\Phi$. In particular (7.7) with Proposition 2.27 yields that $g_{X_i}$ $L^2$-strongly converge to $g_X$ on $X$ and that

$$\Phi^* g_{\mathbb{R}^k} = g_X \quad (7.8)$$
holds. Thus Theorem 1.1 yields that \( X \) is Reifenberg flat. With no loss of generality we can assume that \( \dim_{d_i,m_i}(X_i) \) does not depend on \( i \), thus we denote it by \( n \). Then we have

\[
\dim_{d,m}(X) = \int_X |g_X|^2 \, dm = \lim_{i \to \infty} \int_{X_i} |g_{X_i}|^2 \, dm_i = \lim_{i \to \infty} \dim_{d_i,m_i}(X_i) = n. \tag{7.9}
\]

On the other hand applying Theorem 2.30 with (7.7) (see also the proof of Theorem 3.4) shows that the sequence of \( \{(X_i,d_i)\}_i \) is uniformly Reifenberg flat. Therefore applying the topological stability theorem proved in [CC97, Thm.A.1.2 and A.1.3] with the Reifenberg flatness of \( X \) and (7.9) yields that \( X \) is homeomorphic to \( X_i \) for any sufficiently large \( i \), which is a contradiction.

The second one is stated for non-collapsed RCD\((K,n)\) spaces, however the assumption (7.10) is weaker than (7.6):

**Theorem 7.5** (Topological finiteness theorem II). For any \( K \in \mathbb{R} \), any \( k,n \in \mathbb{N} \) and all \( d, \lambda \in (0, \infty) \), there exists \( \delta := \delta(K,k,n,d,\lambda) \in (0,1) \) such that the following holds. Let us denote by \( M := M(K,k,n,d,\lambda) \) the set of (isometry classes of) compact non-collapsed RCD\((K,n)\) space \( (X,d,H^n) \) satisfying that \( \text{diam}(X,d) \leq d \) and that there exists a regular map \( \Phi = (\varphi_1, \ldots, \varphi_k) : X \to \mathbb{R}^k \) such that \( \|\Delta \varphi_i\|_{L^\infty} \leq \lambda \) and

\[
\frac{1}{H^n(X)} \int_X |\Phi^* g_{\mathbb{R}^k} - g_X| \, dH^n \leq \delta \tag{7.10}
\]

are satisfied. Then for any \( (X,d,H^n) \in M \), \( X \) is homeomorphic to an \( n \)-dimensional topological closed manifold. Moreover \( M \) has finitely many members up to homeomorphism.

**Proof.** The proof is quite similar to that of Theorem 7.4. The only different part is in the proof of the uniform Reifenberg flatness. However this is justified by the standard way using the non-collapsed condition as already done in the proof of Theorem 6.1. Thus we omit it.

**References**

[A19] L. Ambrosio: *Calculus, heat flow and curvature-dimension bounds in metric measure spaces*, Proceedings of the ICM 2018, Vol. 1, World Scientific, Singapore, (2019), 301–340.

[AGMR15] L. Ambrosio, N. Gigli, A. Mondino, T. Rajala: *Riemannian Ricci curvature lower bounds in metric measure spaces with \( \sigma \)-finite measure*. Trans. Amer. Math. Soc. 367 (2015), no. 7, 4661–4701.

[AGS14a] L. Ambrosio, N. Gigli, G. Savaré: *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math. 195 (2014), 289–391.

[AGS14b] L. Ambrosio, N. Gigli, G. Savaré: *Metric measure spaces with Riemannian Ricci curvature bounded from below*, Duke Math. J. 163 (2014), 1405–1490.

[AH17] L. Ambrosio, S. Honda: *New stability results for sequences of metric measure spaces with uniform Ricci bounds from below*, in Measure Theory in Non-Smooth Spaces, 1–51, De Gruyter Open, Warsaw, 2017.
[AH18] L. Ambrosio, S. Honda: Local spectral convergence in $\text{RCD}^\ast(K,N)$ spaces, Nonlinear Anal. 177 Part A (2018), 1–23.

[AHPT18] L. Ambrosio, S. Honda, J. W. Portegies, D. Tewodrose: Embedding of $\text{RCD}^\ast(K,N)$-spaces in $L^2$ via eigenfunctions, arXiv:1812.03712.

[AMS19] L. Ambrosio, A. Mondino, G. Savaré: Nonlinear diffusion equations and curvature conditions in metric measure spaces, Mem. Amer. Math. Soc. 262 (2019), no. 1270, v+121 pp.

[AST16] L. Ambrosio, F. Stra, D. Trevisan: Weak and strong convergence of derivations and stability of flows with respect to MGH convergence, J. Funct. Anal. 272 (2017), 1182–1229.

[ABS19] G. Antonelli, E. Bruè, D. Semola: Volume bounds for the quantitative singular strata of non collapsed $\text{RCD}$ metric measure spaces, Anal. Geom. Metr. Spaces 7 (2019), no. 1, 158–178.

[BBG94] P. Bérard, G. Besson, S. Gallot: Embedding Riemannian manifolds by their heat kernel, Geom. Funct. Anal. 4(4) (1994), 373–398.

[BPS19] E. Brué, E. Pasqualetto, D. Semola: Rectifiability of the reduced boundary for sets of finite perimeter over $\text{RCD}(K,N)$ spaces, arXiv:1909.00381.

[BPS20] E. Brué, E. Pasqualetto, D. Semola: Rectifiability of $\text{RCD}(K,N)$ spaces via $\delta$-splitting maps, arXiv:2001.07911.

[BS18] E. Brué, D. Semola: Constancy of dimension for $\text{RCD}^\ast(K,N)$ spaces via regularity of Lagrangian flows. arXiv:1803.04387 (2018), to appear in Comm. Pure and Appl. Math.

[BBI01] D. Burago, Y. Burago, S. Ivanov A course in metric geometry, Graduate Studies in Mathematics, 33. American Mathematical Society, Providence, RI, 2001. xiv+415 pp.

[CM16] F. Cavalletti, E. Milman: The Globalization Theorem for the Curvature Dimension Condition. arXiv:1612.07623.

[Ch99] J. Cheeger: Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999), 428–517.

[CC97] J. Cheeger, T. H. Colding: On the structure of spaces with Ricci curvature bounded below. I, J. Differential Geom. 46 (1997), 406–480.

[CC00a] J. Cheeger, T. H. Colding: On the structure of spaces with Ricci curvature bounded below. II, J. Differential Geom. 54 (2000), 13–35.

[CC00b] J. Cheeger, T. H. Colding: On the structure of spaces with Ricci curvature bounded below. III, J. Differential Geom. 54 (2000), 37–74.

[C96a] T. H. Colding: Shape of manifolds with positive Ricci curvature, Invent. Math. 124 (1996), 175–191.

[C96b] T. H. Colding: Large manifolds with positive Ricci curvature, Invent. Math. 124 (1996), 193–214.
[CN12] T. H. Colding, A. Naber: Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications, Annals of Math. 176 (2012), 1173-1229.

[DePhG18] G. De Philippis, N. Gigli: Non-collapsed spaces with Ricci curvature bounded below, Journal de l’École polytechnique, 5 (2018), 613-650.

[D02] Y. Ding: Heat kernels and Green’s functions on limit spaces, Comm. Anal. and Geom. 10(3) (2002), 475-514.

[EKS15] M. Erbar, K. Kuwada, K.-T. Sturm: On the equivalence of the entropic curvature-dimension condition and Bochner’s inequality on metric measure spaces, Invent. Math. 201 (2015), 993–1071.

[G13] N. Gigli: The splitting theorem in non-smooth context, arXiv:1302.5555 (2013).

[G15] N. Gigli: On the differential structure of metric measure spaces and applications, Mem. Amer. Math. Soc. 236 (2015), no. 1113.

[G18] N. Gigli: Nonsmooth differential geometry – An approach tailored for spaces with Ricci curvature bounded from below, Mem. Amer. Math. Soc. 251 (2018), no. 1196.

[GMS13] N. Gigli, A. Mondino, G. Savaré: Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows, Proceedings of the London Mathematical Society 111 (2015), 1071–1129.

[GRS16] N. Gigli, T. Rajala, K.-T. Sturm: Optimal maps and exponentiation on finite dimensional spaces with Ricci curvature bounded from below, J. Geom. Anal. 26 (2016), 2914–2929.

[GP16] N. Gigli, E. Pasqualetto: Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces, arXiv: 1611.09645 (2016).

[HK00] P. Hajlasz, P. Koskela: Sobolev met Poincaré. Mem. Amer. Math. Soc., 145 (2000), 1–101.

[Han18] B.-X. Han, Ricci tensor on RCD*(K,N) spaces. J. Geom. Anal 28 (2018), 1295–1314.

[H15] S. Honda: Ricci curvature and L^p-convergence, J. Reine Angew Math. 705 (2015), 85–154.

[H19] S. Honda: New differential operator and non-collapsed RCD spaces, arXiv:190500123v2, to appear in Geom. Topol.

[HM19] S. Honda, I. Mondello: Sphere theorems for RCD and stratified spaces, arXiv:1907.03482, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5).

[J14] R. Jiang: Cheeger-harmonic functions in metric measure spaces revisited, J. Funct. Anal. 266 (2014), 1373–1394.

[JLZ16] R. Jiang, H. Li, and H.-C. Zhang: Heat Kernel Bounds on Metric Measure Spaces and Some Applications, Potent. Anal. 44 (2016), 601–627.

[KM19] V. Kapovitch, A. Mondino: On the topology and the boundary of N-dimensional RCD(K,N) spaces, arXiv:1907.02614, to appear in Geom. Topol.
[K17] Y. Kitabeppu: A Bishop-type inequality on metric measure spaces with Ricci curvature bounded below, Proc. Amer. Math. Soc. 145 (2017), 3137–3151.

[K19] Y. Kitabeppu: A sufficient condition to a regular set of positive measure on RCD spaces, Potential Anal. 51 (2019), 179–196.

[KS03] K. Kuwae, T. Shioya: Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry, Commun. Anal. Geom. 11 (2003), 599-673.

[LV09] J. Lott, C. Villani: Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. 169 (2009), 903–991.

[MN19] A. Mondino, A. Naber: Structure theory of metric measure spaces with lower Ricci curvature bounds, J. Eur. Math. Soc. 21 (2019), 1809–1854.

[O62] M. Obata: Certain Conditions for a Riemannian Manifold to be Isometric with a Sphere, J. Math. Soc. Japan 14 (1962), 333—340.

[P16] J. W. Portegies: Embeddings of Riemannian manifolds with heat kernels and eigenfunctions, Comm. Pure Appl. Math. 69 (3) (2016), 478–518.

[Raj12] T. Rajala: Local Poincaré inequalities from stable curvature conditions on metric spaces, Calc. Var. Partial Differential Equations 44(3) (2012), 477–494.

[Sh00] N. Shanmugalingam: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoamericana 16 (2000), 243–279.

[St06a] K.-T. Sturm: On the geometry of metric measure spaces, I. Acta Math. 196 (2006), 65–131.

[St06b] K.-T. Sturm: On the geometry of metric measure spaces, II. Acta Math. 196 (2006), 133–177.

[T66] T. Takahashi: Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan 18 (1966), 380-385.

[Vi09] C. Villani: Optimal transport. Old and new, vol. 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2009.