A SHARP ISOPERIMETRIC PROPERTY OF THE RENORMALIZED AREA OF A MINIMAL SURFACE IN HYPERBOLIC SPACE

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Abstract. We prove an inequality bounding the renormalized area of a complete minimal surface in hyperbolic space in terms of the conformal length of its ideal boundary.

1. Introduction

Consider a two-dimensional minimal surface in $\mathbb{H}^n$, $n \geq 3$, with $C^m$-regular asymptotic boundary, $m \geq 2$, on the ideal boundary, $\partial_\infty \mathbb{H}^n$, of hyperbolic space. Following Graham and Witten [9] and Alexakis and Mazzeo [1, 2], for any fixed point $p_0 \in \mathbb{H}^n$, such a surface has an asymptotic expansion for its area of the form

$$\text{Vol}_{\mathbb{H}^n}(\Sigma \cap \overline{B}_{R_H}^{\mathbb{H}^n}(p_0)) = c_0 \cosh R + c_1 + o(1), R \to \infty.$$ (1.1)

See Appendix B for how this expansion relates to the one appearing in [9]. This expansion is a (Riemannian) analog of the entropy considered by Ryu and Takayanagi [13, 14]. Importantly, the coefficient $c_1 = \mathcal{A}(\Sigma)$ is independent of the point $p_0$. Indeed, this quantity is the renormalized area considered by Alexakis and Mazzeo in [1] who show

$$\mathcal{A}(\Sigma) = -2\pi \chi(\Sigma) - \frac{1}{2} \int_\Sigma |A_{\mathbb{H}^n}\Sigma|^2 d\text{Vol}_\Sigma.$$ (1.2)

Here $\chi(\Sigma)$ is the Euler characteristic of the surface and $A_{\mathbb{H}^n}\Sigma$ is the (normal bundle valued) second fundamental form of $\Sigma$ – see Lemma A.2. One also has

$$c_0 = \text{Vol}_{\partial_\infty \mathbb{H}^n}(\partial_\infty \Sigma, p_0)$$

which is the boundary length of the asymptotic boundary inside the ideal boundary of the appropriate (depending on $p_0$) compactification of $\mathbb{H}^n$ and depends both on $\Sigma$ and $p_0$ – see Section 2 for details. Following [3], define

$$\lambda_c[\partial_\infty \Sigma] = \sup_{p_0 \in \mathbb{H}^n} \text{Vol}_{\partial_\infty \mathbb{H}^n}(\partial_\infty \Sigma, p_0)$$

which is, essentially, the $(n - 1)$-conformal volume in the sense of Li-Yau [11] of the embedding of the boundary curve $\partial_\infty \Sigma$ into the ideal boundary of $\mathbb{H}^n$ (see also Gromov’s visual volume [10, Section 8.2]) and, unlike $c_0$, is manifestly independent of the point $p_0$.

Using ideas of Choe and Gulliver [7], we bound the renormalized area of such a minimal surface by the conformal length of its ideal boundary:

**Theorem 1.1.** Let $\Sigma$ be a two-dimensional minimal surface in $\mathbb{H}^n$, $n \geq 3$, with a $C^2$-regular asymptotic boundary, then

$$-2\pi \geq -\lambda_c[\partial_\infty \Sigma] \geq \mathcal{A}(\Sigma)$$

with equality throughout if and only if $\Sigma$ is a totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^n$.

The author was partially supported by the NSF Grant DMS-1609340 and DMS-1904674.
Moreover, if $\mathcal{A}(\Sigma) = -\lambda_c[\partial_\infty \Sigma]$, then $\Sigma$ is a totally geodesic $\mathbb{H}^2$. In particular, either $\Sigma$ is a totally geodesic $\mathbb{H}^2$ or

$$-2\pi > -\lambda_c[\partial_\infty \Sigma] > \mathcal{A}(\Sigma).$$

**Remark 1.2.** When $\Sigma$ is a topological disk, then the absolute bound $-2\pi \geq \mathcal{A}(\Sigma)$ and corresponding rigidity result follows from [12]. In fact, Alexakis and Mazzeo show [1, Section 8] that the renormalized area of $\Sigma$ is the negative of one half of the Willmore energy of a suitably doubling of $\Sigma$. As the Willmore energy of any closed surface is at least $4\pi$, this fact implies absolute bound in general. Going further, the resolution of the Willmore conjecture by Marques and Neves [12] means that if $\Sigma$ is not a disk, then $-\pi^2 \geq \mathcal{A}(\Sigma)$. As such, the significance of Theorem 1.1 is the use of the boundary geometry to refine bounds on the renormalized area spectrum – i.e., [1, (5.15)].

In [3], the author introduced a notion of entropy for submanifolds of hyperbolic space analogous to the one introduced by Colding and Minicozzi in [8] for submanifolds of Euclidean space (see also [16]). More precisely, let $H_2(t, p; t_0, p_0)$ be the heat kernel on $\mathbb{H}^2$ with singularity at $p = p_0$ at time $t = t_0$. That is, suppose it is the unique positive solution to

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} - \Delta_{\mathbb{H}^2} H_2 = 0 \quad t > 0 \\
\lim_{t \downarrow t_0} H_2 = \delta_{p_0}. \end{array} \right.$$ 

It follows from the symmetries of $\mathbb{H}^n$ that

$$H_2(t, p; t_0, p_0) = K_2(t - t_0, \text{dist}_{\mathbb{H}^2}(p, p_0))$$

where $K_2(t, \rho)$ is a positive function on $(0, \infty) \times (0, \infty)$ and $\text{dist}_{\mathbb{H}^2}(p, p_0)$ is the hyperbolic distance between $p$ and $p_0$. For $(t, p) \in (0, \infty) \times \mathbb{H}^n$ let

$$\Phi^{t_0, p_0}_2(t, p) = K_2(t_0 - t, \text{dist}_{\mathbb{H}^2}(p, p_0)).$$

In particular, $\Phi^{t_0, p_0}_2$ restricts to the backwards heat kernel that becomes singular at $(t_0, p_0)$ on any totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^n$ that goes through $p_0$.

In analogy with the Euclidean setting, for any surface $\Sigma \subset \mathbb{H}^n$, define the hyperbolic entropy of $\Sigma$ to be

$$\lambda_{\mathbb{H}}[\Sigma] = \sup_{p_0 \in \mathbb{H}^n, \tau > 0} \int_{\Sigma} \Phi^{0, p_0}_2(-\tau, p) dVol_{\Sigma}(p).$$

This quantity is monotone non-increasing along any mean curvature flow. Moreover, by [3, Theroem 1.5] if $\Sigma$ is a two-dimensional minimal surface in $\mathbb{H}^n$, $n \geq 3$, with a $C^1$-regular asymptotic boundary,

$$\lambda_c[\partial_\infty \Sigma] = 2\pi \lambda_{\mathbb{H}}[\Sigma].$$

Combining this with Theorem 1.1 yields:

**Corollary 1.3.** If $\Sigma$ is a two-dimensional minimal surface in $\mathbb{H}^n$, $n \geq 3$, with a $C^2$-regular asymptotic boundary, then

$$-2\pi \lambda_{\mathbb{H}}[\Sigma] \geq \mathcal{A}(\Sigma).$$

**Remark 1.4.** As the estimates of this paper do not depend in an essential way on embeddedness or interior regularity we can conclude that if $\Sigma$ is the image of branched minimal immersion that is not smoothly embedded, then

$$-4\pi \geq \mathcal{A}(\Sigma).$$

This also follows from the observations made in Remark 1.2.
Finally, inspired by [5], the estimate on $A(\Sigma)$ in terms of $\lambda[\Sigma]$ leads one to expect the following refinement of Theorem 1.1 in $\mathbb{H}^3$:

**Conjecture 1.5.** Suppose $\Sigma, \Sigma'$ are two-dimensional minimal surfaces in $\mathbb{H}^3$ with a $C^2$-regular asymptotic boundaries. If $\partial_\infty \Sigma = \partial_\infty \Sigma'$ and $\Sigma'$ is not a disk, then

$$-3\pi > -\frac{(2\pi)^{3/2}}{\sqrt{\varepsilon}} = -2\pi \lambda[\mathbb{S}^1] > A(\Sigma).$$

Here $\lambda[\mathbb{S}^1] \approx 1.52$ is the Colding-Minicozzi entropy of the round circle in $\mathbb{R}^2$.

**Remark 1.6.** As observed in Remark 1.2 the proof of the Willmore conjecture implies $-2\pi \lambda[\mathbb{S}^1] > -\pi^2 > A(\Sigma')$. Hence, the significance of the conjecture is that there is an improved bound for the surface $\Sigma$ of unspecified topology.

2. BACKGROUND

We use the Poincaré ball model of hyperbolic space $\mathbb{H}^n$ in order to study the asymptotic properties of minimal surfaces in $\mathbb{H}^n$. In particular, for any point $p_0 \in \mathbb{H}^n$ one obtains a corresponding compactification of hyperbolic space. The natural geometry on the ideal boundary in this compactification is one that is invariant under Möbius transformations.

Recall, the Poincaré ball model of hyperbolic space, $\mathbb{H}^n$ is the open unit ball in Euclidean space

$$\mathbb{B}^n = B_1 = \{x : |x| < 1\} \subset \mathbb{R}^n$$

together with the Poincaré metric

$$g_P = 4 \frac{dx \otimes dx}{(1 - |x|^2)^2} = 4 \frac{1}{(1 - |x|^2)^2} g_E.$$

Here $g_E$ is the Euclidean metric on $\mathbb{B}^n$. That is, for any model of hyperbolic space, $(\mathbb{H}^n, g_{\mathbb{H}^n})$, there is an isometry $i : \mathbb{H}^n \to \mathbb{B}^n$ so $i^* g_P = g_{\mathbb{B}^n}$. The isometries of $g_P$ are given by the Möbius transforms of $\mathbb{B}^n$ and so this identification is not unique. In fact, for any point $p_0 \in \mathbb{H}^n$, there is an isometry $i : \mathbb{H}^n \to \mathbb{B}^n$ so $i(p_0) = 0$. Moreover, if $i, j : \mathbb{H}^n \to \mathbb{B}^n$ satisfy $i(p_0) = j(p_0) = 0$, then $i \circ j^{-1}$ is an orthogonal transformation of $\mathbb{B}^n$. In particular, in this case $i^* g_{\mathbb{B}^n} = j^* g_{\mathbb{B}^n}$ while these metrics are different for identifications associated to distinct distinguished points.

In the remainder of this article, we will always choose a distinguished point $p_0 \in \mathbb{H}^n$ and an identification (i.e., an isometry) $i : \mathbb{H}^n \to \mathbb{B}^n$ with $i(p_0) = 0$. We use this identification to compactify $\mathbb{H}^n$ and denote the ideal boundary of $\mathbb{H}^n$ by $\partial_\infty \mathbb{H}^n$ which is identified with $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$ by extending $i : \mathbb{H}^n \to \mathbb{B}^n$ in the obvious way. This compactification is independent, as a manifold with boundary, of the choice of $p_0$ and $i$.

A complete submanifold $\Sigma \subset \mathbb{H}^n$ has $C^m$-regular asymptotic boundary for $1 \leq m \leq \infty$ if $\Sigma' = \overline{i(\Sigma)} \subset \overline{\mathbb{B}^n}$ is a $C^m$-regular manifold with boundary, $\partial \Sigma' \subset \mathbb{S}^{n-1} = \partial \mathbb{B}^n$ that meets $\mathbb{S}^{n-1}$ orthogonally. Denote by $\partial_\infty \Sigma$ the submanifold corresponding to $\partial \Sigma'$ in $\partial_\infty \mathbb{H}^n$. As Möbius transformations are smooth and conformal this is a well defined notion independent of choice of identification.

Using the identification, $i, \partial_\infty \mathbb{H}^n$ has a well defined Riemannian metric induced from $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$. While this metric depends on $p_0$, it is otherwise independent of the choice of isometry taking $p_0$ to 0. Let us denote this metric by $g^{p_0}_{\partial_\infty \mathbb{H}^n}$. Clearly, $g^{p_0}_{\partial_\infty \mathbb{H}^n}$ and $g^{q_0}_{\partial_\infty \mathbb{H}^n}$ are conformal for different choices of distinguished point $p_0$ and $q_0$ and so $\partial_\infty \mathbb{H}^n$ has a well defined conformal structure. In fact, the two metrics are related by a Möbius transform
on the sphere. Fix a \( l \)-dimensional \( C^m \) submanifold \( \Gamma \subset \partial_\infty \mathbb{H}^n \) and let \( i(\Gamma) \subset S^{n-1} \) be the corresponding submanifold of the sphere under the identification. Set

\[
Vol_{\partial_\infty \mathbb{H}^n}(\Gamma, p_0) = Vol_{\mathbb{H}^n}(i(\Gamma)) = Vol_{\mathbb{R}^n}(i(\Gamma)).
\]

If \( q_0 \) is a different choice of distinguished point, then, there is a Möbius transform, \( \psi \in \text{Mob}(S^{n-1}) \) so that

\[
Vol_{\partial_\infty \mathbb{H}^n}(\Gamma, q_0) = Vol_{\mathbb{H}^n}(\psi(i(\Gamma)))
\]

Hence, the conformal volume of \( \Gamma \subset \partial_\infty \mathbb{H}^n \) defined by

\[
\lambda_\psi[\Gamma] = \sup_{\psi \in \text{Mob}(S^{n-1})} Vol_{\mathbb{H}^n}(\psi(i(\Gamma)));
\]

is well defined independent of the choice distinguished point and of identification.

In fact, the quantity \( \lambda_\psi[\Gamma] \) is essentially the \( n \)-conformal volume of the embedding defined by Li-Yau \([11]\). Moreover, as the Möbius transformations of \( S^{n-1} \) are parameterized by \( a \in \mathbb{B}^n \) in an explicit way one has

\[
\lambda_\psi[\Gamma] = \sup_{a \in \mathbb{B}^n} \int_{\Gamma} \frac{(1 - |a|^2)^{l/2}}{(1 - a \cdot x(p))^l} dVol_{\Gamma}(p)
\]

In particular, as shown in \([6]\), this readily leads to the following elementary properties:

**Lemma 2.1.** Let \( \Sigma \subset \mathbb{H}^n \) be a \( C^1 \)-asymptotically regular \( l \)-dimensional minimal submanifold. One has

\[
\lambda_\psi[\partial_\infty \Sigma] \geq Vol_{\mathbb{H}^n}(S^{l-1})
\]

with equality if and only if \( \Sigma \) is a totally geodesic copy of \( \mathbb{H}^l \). Moreover, if

\[
\lambda_\psi[\partial_\infty \Sigma] > Vol_{\mathbb{H}^n}(S^{l-1}),
\]

then there is a \( p_0 \in \mathbb{H}^n \) so that

\[
\lambda_\psi[\partial_\infty \Sigma] = Vol_{\partial_\infty \mathbb{H}^n}(\partial_\infty \Sigma, p_0).
\]

**Proof.** As \( \partial_\infty \Sigma \) is a closed submanifold of \( S^{n-1} \), the first claim is an immediate consequence of applying \([6, \text{Proposition 1}]\) to the embedding \( \phi : \partial_\infty \Sigma \to \partial_\infty \mathbb{H}^n \). That result also gives that equality holds only when \( \partial_\infty \Sigma \) is a totally geodesic copy of \( S^{l-1} \) in \( S^{n-1} \). The rigidity of \( \Sigma \) in case of equality follows immediately from this. The final claim is, likewise, an immediate consequence of \([6, \text{Corollary 1}]\). \( \square \)

### 3. Asymptotic Expansion of Length and Area

We record here certain computations involving geometric quantities near the boundary of the compactification, \( \Sigma' \subset \mathbb{B}^n \), of an asymptotically regular minimal surface \( \Sigma \) in \( \mathbb{H}^n \).

**Proposition 3.1.** Let \( \Sigma \) be a two-dimensional minimal surface in \( \mathbb{H}^n \), \( n \geq 3 \), with a \( C^2 \)-regular asymptotic boundary. Fix a point \( p_0 \in \mathbb{H}^n \) and let \( \Sigma' \) be the compactification in \( \mathbb{B}^n \) corresponding to \( \Sigma \) and \( p_0 \).

One has,

\[
\frac{1}{4}(1 - |x|^2)H_{\Sigma'}^{ge} = x^\perp.
\]

As a consequence, for \( p \in \partial\Sigma' \subset \partial\mathbb{B}^n \),

\[
 x^\perp(p) = 0
\]

and

\[
 A_{\Sigma'}^{ge}(x, x)|_p = K_{\partial\Sigma'}^{S^{n-1}}(p).
\]
Hence,

\[
Vol_{g^n}(\Sigma' \cap \partial B_s) = sVol_{g^n}(\partial \Sigma') - \frac{(s-1)^2}{2} \int_{\partial \Sigma'} |k_{\partial \Sigma'}^{g^n-1}|^2 dVol_{g^n} + o((s-1)^2), \quad s \to 1
\]

and

\[
\int_{\Sigma' \cap \partial B_s} \frac{|x|}{|x^\perp|} dVol_{\Sigma' \cap \partial B_s} = sVol_{g^n}(\partial \Sigma') + o((s-1)^2), \quad s \to 1.
\]

Proof. By construction, \(\Sigma'\) is minimal with respect to \(g_P\) and is \(C^2\) up to \(\partial \Sigma'\). As

\[
g_P = \frac{4}{(1 - |x|^2)^2} g_E,
\]

the formula for the transformation mean curvature vector under conformal change of metric implies that, on \(\Sigma' \cap \mathbb{B}^n\),

\[
H_{\Sigma'}^{g_P} = \frac{(1 - |x|^2)^2}{4} \left( H_{\Sigma'}^{g_E} - \frac{4x^\perp}{1 - |x|^2} \right).
\]

As \(\Sigma'\) is minimal with respect to \(g_P\) this yields (3.1). As \(\Sigma'\) is \(C^2\), it follows that the left hand side of (3.1) vanishes on \(\partial \Sigma'\). This means \(x^\perp|_{\partial \Sigma'} = 0\), proving (3.2). Observe that the normal connection to \(\Sigma'\) satisfies, for any \(v\) tangent to \(\Sigma'\),

\[
\nabla_v^{\perp, \Sigma'} x^\perp = -A_{\Sigma'}^{\perp, \Sigma'}(x^\top, v).
\]

It follows that for any \(p \in \partial \Sigma'\)

\[
\lim_{q \rightarrow p} \frac{x^\perp(q)}{1 - |x(q)|} = A_{\Sigma'}^{g_E}(x, x)|_p.
\]

This together with (3.1) and the continuity of the mean curvature implies that along \(\partial \Sigma'\)

\[
\frac{1}{2} H_{\Sigma'}^{g_E} = A_{\Sigma'}^{g_E}(x, x).
\]

Clearly, along \(\partial \Sigma'\),

\[
H_{\Sigma'}^{g_E} = A_{\Sigma'}^{g_E}(x, x) + A_{\Sigma'}^{g_E}(T, T),
\]

where \(T\) is a choice of tangent vector along \(\partial \Sigma'\) with \(|T|_{g_E} = 1\). Taken together, this implies that on \(\partial \Sigma'\)

\[
A_{\Sigma'}^{g_E}(T, T) = A_{\Sigma'}^{g_E}(x, x).
\]

Finally, as \(x^\perp = 0\) on \(\partial \Sigma'\) one has

\[
k_{\partial \Sigma'}^{g^n-1} = A_{\Sigma'}^{g_E}(T, T) = A_{\Sigma'}^{g_E}(x, x).
\]

This proves (3.3).

For \(t \geq 0\), let \(\phi_t : \Sigma' \to \Sigma'\) be the flow of a \(C^1\) vector field \(V\) that satisfies

\[
V = -\frac{|x|x^\top}{|x|^2}.
\]

near \(\partial \Sigma'\). One verifies that \(V\) has been chosen so that, near \(\partial \Sigma'\), \(\nabla_V^{g_E} |x| = -1\) and so, for \(t\) sufficiently small, \(\phi_t(\partial \Sigma') \subset \partial B_{1-t}\). Clearly, (3.2) implies that for \(p \in \partial \Sigma'\),

\[
\frac{d}{dt}|_{t=0} \phi_t(p) = -x(p).
\]
One further concludes from (3.3) that
\[
\frac{d^2}{dt^2}|_{t=0} \mathbf{x}(\phi_t(p)) = \frac{d}{dt}|_{t=0} \mathbf{V}(\phi_t(p)) = \nabla^g_{\mathbf{x}} \mathbf{V}|_p \\
= A^g_{\Sigma'}(\mathbf{x}, \mathbf{x})|_p = k^g_{\Sigma'}^{n-1}(p).
\]

The fact that \( \mathbf{x}^\perp = 0 \) on \( \partial \Sigma' \) further implies that, along \( \partial \Sigma' \),
\[
k_{\partial \Sigma'} = k^g_{\Sigma'}^{n-1} - \mathbf{x}.
\]

Hence, using the first variation formula,
\[
\frac{d}{ds}|_{s=1} \text{Vol}_R(\Sigma' \cap \partial B_s) = -\frac{d}{dt}|_{t=0} \text{Vol}_R(\phi_t(\partial \Sigma')) \\
= \int_{\partial \Sigma'} k_{\partial \Sigma'} \cdot (\mathbf{x}) \ dV_{\partial \Sigma'} = \text{Vol}_R(\partial \Sigma').
\]

Likewise, the second variation formula for length (see for instance [15 Equation (9.4)]) yields
\[
\frac{d^2}{ds^2}|_{s=1} \text{Vol}_R(\Sigma' \cap \partial B_s) = \frac{d^2}{dt^2}|_{t=0} \text{Vol}_R(\Sigma' \cap \partial B_{1-t}) = \frac{d^2}{dt^2}|_{t=0} \text{Vol}_R(\phi_t(\partial \Sigma')) \\
= -\int_{\partial \Sigma'} k_{\partial \Sigma'} \cdot k^g_{\Sigma'}^{n-1} \ dV_{\partial \Sigma'} + \int_{\partial \Sigma'} (-k_{\partial \Sigma'} \cdot -\mathbf{x})^2 - (-\mathbf{x} \cdot k_{\partial \Sigma'})^2 \ dV_{\partial \Sigma'} \\
= -\int_{\partial \Sigma'} |k^g_{\partial \Sigma'}^{n-1}|^2 \ dV_{\partial \Sigma'}.
\]

Together these calculations prove (3.4).

Observe that
\[
\frac{|\mathbf{x}|}{|\mathbf{x}^\perp|} = \frac{1}{\sqrt{1 - \frac{|\mathbf{x}|^2}{|\mathbf{x}^\perp|^2}}}.
\]

As such, at any point in \( \Sigma \)
\[
\nabla^g_{\mathbf{x}^\perp} \frac{|\mathbf{x}|}{|\mathbf{x}^\perp|} = \frac{\mathbf{x}^\perp \cdot \nabla^g_{\mathbf{x}^\perp} \mathbf{x}^\perp - |\mathbf{x}^\perp|^2 |\mathbf{x}^\perp|^2}{(1 - \frac{|\mathbf{x}|^2}{|\mathbf{x}^\perp|^2})^{3/2}} = - \frac{|\mathbf{x}| \mathbf{x}^\perp \cdot A^g_{\Sigma'}(\mathbf{x}^\perp, \mathbf{x}^\perp)}{|\mathbf{x}^\perp|^3} - \frac{|\mathbf{x}^\perp|^2}{|\mathbf{x}| |\mathbf{x}^\perp|}.
\]

Hence, for \( p \in \partial \Sigma' \) and \( t \geq 0 \) small,
\[
\left. \frac{d}{dt} \frac{|\mathbf{x}(\phi_t(p))|}{|\mathbf{x}^\perp(\phi_t(p))|} \right|_{\phi_t(p)} = \nabla^g_{\mathbf{x}^\perp} \frac{|\mathbf{x}|}{|\mathbf{x}^\perp|} \bigg|_{\phi_t(p)} \\
= \left( \frac{|\mathbf{x}|^2 \mathbf{x}^\perp \cdot A^g_{\Sigma'}(\mathbf{x}^\perp, \mathbf{x}^\perp)}{|\mathbf{x}^\perp|^5} - \frac{|\mathbf{x}^\perp|^2}{|\mathbf{x}^\perp|^3} \right) \bigg|_{\phi_t(p)}.
\]

In particular, this vanishes when \( t = 0 \). Combined with the first variation formula for length this implies
\[
\frac{d}{ds}|_{s=1} \int_{\Sigma' \cap \partial B_s} \frac{|\mathbf{x}|}{|\mathbf{x}^\perp|} \ dV_{\partial \Sigma' \cap \partial B_s} = \text{Vol}_R(\partial \Sigma').
\]
Moreover, for $p \in \partial \Sigma'$ differentiating (3.7) at $t = 0$ yields
\[
\frac{d^2}{dt^2}|_{t=0} \left| \frac{\mathbf{x}(\phi_t(p))}{|\mathbf{x}^t(\phi_t(p))|} \right| = \lim_{t \to 0} \frac{1}{t} \left| \frac{|\mathbf{x}^t|^2 \cdot A_{\Sigma'}^{s} (\mathbf{x}^t, \mathbf{x}^t) - |x^t|^2}{|x^t|^3} \right| \phi_t(p)
\]
\[
= A_{\Sigma'}^{s} (\mathbf{x}^t, \mathbf{x}^t) \mid_{p} \lim_{t \to 0} \frac{\mathbf{x}^t(\phi_t(p))}{|\mathbf{x}(\phi_t(p))|} = |k_{\Sigma'}^{2s-1}|^2(p).
\]
Where we used (3.2), (3.6) and (3.3). Combining this with the second variation formula shows
\[
\frac{d^2}{ds^2}|_{s=1} \int_{\partial \Sigma' \cap \partial B_s} \frac{|\mathbf{x}|}{|\mathbf{x}^t|} dVol_{\Sigma' \cap \partial B_s} = 0.
\]
The expansion (3.5) follows immediately. \hfill \square

We also need the following geometric expansions that refine [11] and [3, Lemma 4.1] for surfaces with $C^2$-regular asymptotic boundary.

**Proposition 3.2.** Let $\Sigma$ be a two-dimensional minimal surface in $\mathbb{H}^n$, $n \geq 3$, with a $C^2$-regular asymptotic boundary. For any $p_0 \in \mathbb{H}^n$, there are constants $L_\infty, A_\infty$ and $K_\infty$ so that:

\[
L_R = Vol_{\mathbb{H}^n} (\Sigma \cap \partial B^0_R (p_0)) = L_\infty \sinh R - K_\infty e^{-R} + o(e^{-R}), \quad R \to \infty,
\]
and
\[
A_R = Vol_{\mathbb{H}^n} (\Sigma \cap \bar{B}^0_R (p_0)) = L_\infty \cosh R + A_\infty + o(e^{-R}), \quad R \to \infty.
\]
Furthermore, $L_\infty = Vol_{\partial \infty \mathbb{H}^n} (\partial \infty \Sigma, p_0)$, $A_\infty = A(\Sigma)$ and
\[
K_\infty = \int_{\partial \Sigma'} |k_{\Sigma'}^{2s-1}|^2 dVol_{\Sigma'},
\]
where $\Sigma' \subset \bar{\mathbb{H}}^n$ is the compactification of $\Sigma$ with respect to $p_0$.

**Proof.** Pick an identification $i : \mathbb{H}^n \to \mathbb{E}^n$ with $i(p_0) = 0$. As $i^*g_P = g_{\mathbb{E}^n}$, one has $i(\partial B^0_R (p_0)) = \partial B^0_R (0)$. Furthermore, as the conformal factor of $g_P$ is radial, $\partial B^0_R (0) = \partial B_R$ where $R = \ln \left( \frac{1+s}{1-s} \right)$, equivalently, $s = \frac{e^R - 1}{e^R + 1}$.

Set $\Sigma_R = \Sigma \cap \partial B^0_R (p_0)$, and let $\Sigma'_R$ be the closure of $i(\Sigma)$ be the natural compactification of $\Sigma$ relative to $p_0$. From the above, $i(\Sigma_R) = \Sigma'_R \cap \partial B_R (0) = \Sigma'_R$. Let $g_R$ be the metric on $\Sigma_R$ induced from $\mathbb{H}^n$ and $g'_R$ be the metric induced on $\Sigma'_R$ from $g_{\mathbb{E}^n}$. Clearly,
\[
i^*g'_R = \left( \frac{e^R - 1}{e^R + 1} \right)^2 \sinh^{-2}(R) g_R = \frac{1}{(1 + \cosh R)^2} g_R.
\]
In particular, as $\Sigma'_R$ is 1-dimensional,
\[
Vol_{\mathbb{H}^n} (\Sigma'_R) = \frac{Vol_{\partial \mathbb{H}^n} (\Sigma_R)}{1 + \cosh R}.
\]

Set
\[
L_\infty = Vol_{\partial \mathbb{H}^n} (\partial \Sigma') = Vol_{\partial \infty \mathbb{H}^n} (\partial \infty \Sigma, p_0).
\]
As $\Sigma$ has a $C^2$-regular asymptotic boundary, (3.4) of Proposition 3.1 ensures
\[
Vol_{\mathbb{H}^n} (\Sigma'_R) = sL_\infty - \frac{1}{2} (s - 1)^2 K_\infty + o((s - 1)^2), \quad s \to 1
\]
where
\[
K_\infty = \int_{\partial \Sigma'} |k_{\Sigma'}^{2s-1}|^2 dVol_{\partial \Sigma'}.
\]
As
\[ s = \frac{e^R - 1}{e^R + 1} = \frac{\sinh R}{1 + \cosh R} \]
and
\[ 1 - s = \frac{2}{e^R + 1} = 2e^{-R} + o(e^{-R}), \quad R \to \infty, \]
we conclude
\[ L_R = Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) = Vol_{\mathbb{H}^n}(\Sigma_R) = (1 + \cosh R)Vol_{\mathbb{H}^n}(\Sigma'_R) \]
\[ = L_\infty \sinh R - K_\infty e^{-R} + o(e^{-R}), \quad R \to \infty. \]
This gives the first expansion.
To work out the area expansions first observe that on \( B^n \) one has
\[ \frac{1}{|\nabla g_{\Sigma'}\rho|(p)} = \frac{|x(p)|^2}{2|x(p)|}X^T(p) \]
where \( \rho(p) = \text{dist}_{g_{\Sigma'}}(p, 0) \).
It follows that
\[ \frac{1}{|\nabla g_{\Sigma'}\rho|_{g_{\Sigma'}}(p)} = \frac{|x(p)|}{|x'(p)|}. \]
Hence, as \( \Sigma \) has \( C^2 \)-regular asymptotic boundary, \((3.5)\) of Proposition 3.1 implies
\[ \int_{\Sigma'_R} \frac{1}{|\nabla g_{\Sigma'}\rho|_{g_{\Sigma'}}} dVol_{\Sigma'} = L_\infty s + o((s - 1)^2), \quad s \to 1. \]
Using the identifications from before yields
\[ \int_{\Sigma_R} \frac{1}{|\nabla r|} dVol_{\Sigma_R} = (1 + \cosh R) \int_{\Sigma'_R} \frac{1}{|\nabla \rho|_{g_{\Sigma'}}} dVol_{\Sigma'_R} \]
\[ = L_\infty \sinh R + o(e^{-R}), \quad R \to \infty \]
where \( r(p) = \text{dist}_{\mathbb{H}^n}(p, p_0) \). The co-area formula ensures,
\[ \frac{d}{dR} Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) = \int_{\Sigma_R} \frac{1}{|\nabla r|} dVol_{\Sigma_R}. \]
and so
\[ \frac{d}{dR} Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) = L_\infty \sinh R + o(e^{-R}), \quad R \to \infty. \]
Hence,
\[ \frac{d}{dR} \left( Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) - L_\infty \cosh R \right) = o(e^{-R}), \quad R \to \infty. \]
As the error is integrable, we deduce
\[ A_R = Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) = L_\infty \cosh R + A_\infty + o(e^{-R}) \]
Where
\[ A_\infty = \lim_{R \to \infty} \left( Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) - L_\infty \cosh R \right) \]
\[ = \lim_{R \to \infty} \left( Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) - Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^H_{R}(p_0)) \right). \]
The definition of renormalized area ensures \( A_\infty = A(\Sigma) \) is independent of \( p_0 \) – see Lemma[5.2] \( \square \)
4. Proof of the main theorem

In order to prove the main result we need two auxiliary results inspired by [7]. First of all, given a point $p_0 \in \mathbb{H}^n$ and a curve $\gamma \subset \partial B_{R}^{\mathbb{H}^n}(p_0)$, define the cone of $\gamma$ over $p_0$ to be

$$C(\gamma; p_0) = \{ p \in \mathbb{H}^n : p \in \sigma(p_0, q) \text{ for some } q \in \gamma \}.$$  

Here $\sigma(p_0, q)$ is the minimizing geodesic segment connecting $p_0$ to $q$. If $r = \text{dist}_{\mathbb{H}^n}(\cdot, p_0)$, one readily checks that the vector field $\nabla H$ is tangent to $C = C(\gamma; p_0) \setminus \{ p_0 \}$.

When $\gamma \subset \partial B_{R}^{\mathbb{H}^n}(p_0)$, one has the following simple formula relating length and area of geodesic balls in $C(\gamma; p_0)$ centered at $p_0$.

**Lemma 4.1.** Suppose $\gamma \subset \partial B_{R}^{\mathbb{H}^n}(p_0)$ is a $C^2$ curve. For any $0 < \rho \leq R$, let

$$L^C_\rho = Vol_{\mathbb{H}^n}(C(\gamma; p_0) \cap \partial B_{\rho}^{\mathbb{H}^n}(p_0))$$

and

$$A^C_\rho = Vol_{\mathbb{H}^n}(C(\gamma; p_0) \cap \bar{B}_{\rho}^{\mathbb{H}^n}(p_0))$$

The density at $p_0$ of the cone satisfies, for all $0 < \rho \leq R$,

$$\Theta = \Theta(C(\gamma; p_0), p_0) = \frac{L^C_\rho}{2\pi \sinh \rho} = \frac{A^C_\rho}{2\pi (\cosh \rho - 1)}.$$  

Moreover, one has

$$(L^C_\rho)^2 = 4\pi \Theta A^C_\rho + (A^C_\rho)^2 = \frac{2L^C_\rho A^C_\rho}{\sinh \rho} + (A^C_\rho)^2.$$  

**Proof.** For any $C^2$ curve $\sigma \subset \partial B_{\rho}^{\mathbb{H}^n}(p_0)$ one computes

$$k_\sigma = k_\sigma^T - \coth \rho \nabla_{\mathbb{H}^n} r$$

where $k_\sigma^T$ is the component of the curvature tangent to $\partial B_{\rho}^{\mathbb{H}^n}(p_0)$ and $r$ is the radial distance in $\mathbb{H}^n$ to $p_0$. For $t \leq 0$ let

$$\phi_t : \mathbb{H}^n \setminus \bar{B}_{-t}^{\mathbb{H}^n}(p_0) \to \mathbb{H}^n \setminus \{ p_0 \}$$

be the flow of $\nabla_{\mathbb{H}^n} r$. By definition, this flow preserves cones based at $p_0$. Moreover, one has

$$\phi_{-t}(\gamma) = C(\gamma; p_0) \cap B_{R-t}^{\mathbb{H}^n}(p_0).$$

It follows from the first variation formula that

$$\frac{d}{dt}Vol_{\mathbb{H}^n}(\phi_{-t}(\gamma)) = - \coth(R - t)Vol_{\mathbb{H}^n}(\phi_{-t}(\gamma))$$

Solving this ODE gives for $t \in [0, R]$

$$Vol_{\mathbb{H}^n}(C(\gamma; p_0) \cap B_{R-t}^{\mathbb{H}^n}(p_0)) = Vol_{\mathbb{H}^n}(\phi_{-t}(\gamma)) = \sinh(R - t)Vol_{\mathbb{H}^n}(\gamma).$$

That is, for $\rho \in (0, R]$

$$L^C_\rho = Vol_{\mathbb{H}^n}(C(\gamma; p_0) \cap B_{\rho}^{\mathbb{H}^n}(p_0)) = Vol_{\mathbb{H}^n}(\gamma) \sinh \rho.$$
As $\nabla_{\mathbb{H}^n} r$ has unit length away from $p_0$ and is tangent to $C(\gamma; p_0)$ it follows from the co-area formula that for $\rho \in (0, R]$

$$A^C_\rho = \text{Vol}_{\mathbb{H}^n}(C(\gamma; p_0) \cap \mathbb{B}^n_\rho(p_0)) = \int_0^\rho \text{Vol}_{\mathbb{H}^n}(C(\gamma; p_0) \cap \mathbb{B}^n_1(p_0)) \, dt$$

$$= \int_0^\rho \text{Vol}_{\mathbb{H}^n}(\gamma) \sinh t \, dt$$

$$= (\cosh \rho - 1)\text{Vol}_{\mathbb{H}^n}(\gamma)$$

$$= \frac{\cosh \rho - 1}{\sinh \rho} L^C_\rho.$$ 

Since,

$$\lim_{\rho \to 0} \frac{\cosh \rho - 1}{\rho^2} = \frac{1}{2}$$

it follows that

$$\Theta = \lim_{\rho \to 0} \frac{A^C_\rho}{2\pi \rho^2} = \frac{L^C_\rho}{2\pi \sinh \rho} = \frac{A^C_\rho}{2\pi (\cosh \rho - 1)}.$$ 

This proves the first claim. To see the second we observe that

$$\left(L^C_\rho\right)^2 = \left(\text{Vol}_{\mathbb{H}^n}(\gamma)\right)^2 \sinh^2 \rho$$

while

$$4\pi \Theta A^C_\rho + \left(A^C_\rho\right)^2 = 2(\text{Vol}_{\mathbb{H}^n}(\gamma))^2 (\cosh \rho - 1) + (\text{Vol}_{\mathbb{H}^n}(\gamma))^2(\cosh \rho - 1)^2$$

$$= (\text{Vol}_{\mathbb{H}^n}(\gamma))^2 (\cosh^2 \rho - 1) = (\text{Vol}_{\mathbb{H}^n}(\gamma))^2 \sinh^2 \rho.$$ 

This proves the second claim. 

\[\square\]

**Proposition 4.2.** Suppose $\Sigma$ is a compact minimal surface in $\mathbb{H}^n$ with $\partial \Sigma \subset \partial \mathbb{B}^n_R(p_0)$ a $C^1$ curve. If

$$L_R = \text{Vol}_{\mathbb{H}^n}(\partial \Sigma) = \text{Vol}_{\mathbb{H}^n}(\Sigma \cap \partial \mathbb{B}^n_R(p_0))$$

and

$$A_R = \text{Vol}_{\mathbb{H}^n}(\Sigma \cap \mathbb{B}^n_R(p_0))$$

then

$$L^2_R \geq \frac{2L_R}{\sinh R} A_R + A^2_R.$$ 

**Proof.** The key fact, proved in [7, Proposition 2] is the fact that any minimal surface in $\mathbb{H}^n$ has less area than an appropriate cone competitor. That is, if $\Sigma \subset \mathbb{H}^n$ is a compact minimal $p_0 \in \mathbb{H}^n \setminus \partial \Sigma$, then

$$\text{Vol}_{\mathbb{H}^n}(C(\partial \Sigma; p_0)) \geq \text{Vol}_{\mathbb{H}^n}(\Sigma).$$

For the sake of completeness we provide a proof this fact in Lemma [A.1].

Combining this estimate with Lemma [6.1] for $\Sigma$ and $p_0$ as in the hypothesis gives

$$L^2_R = (L^C_R)^2 = \frac{2L_R A_R}{\sinh R} + A^2_R \geq \frac{2L_R A_R}{\sinh R} + A^2_R = \frac{2L_R A_R}{\sinh R} + A^2_R.$$ 

This completes the proof. 

\[\square\]

We can now prove Theorem [1.1].
Proof. Set

\[ L_R = Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^n_R (p_0)) \]

and

\[ A_R = Vol_{\mathbb{H}^n}(\Sigma \cap B^n_R (p_0)) \]

As \( \Sigma \) has \( C^2 \)-regular asymptotic boundary, the expansions of Proposition 3.2 imply

\[ L_R^2 = L^2_{\infty} \sinh^2 R - K_{\infty} + o(1), R \to \infty, \]

\[ \frac{2L_RA_R}{\sinh R} = 2L^2_{\infty} \cosh R + 2L_{\infty}A_{\infty} + o(1), R \to \infty \]

and

\[ A^2_R = L^2_{\infty} \cosh^2 R + 2L_{\infty}A_{\infty} \cosh R + A^2_{\infty} + o(1), R \to \infty. \]

Plugging these expansions into the inequality given by Proposition 4.2 yields

\[ L^2_{\infty} \sinh^2 R \geq 2L^2_{\infty} \cosh R + L^2_{\infty} \cosh^2 R \]

\[ + 2L_{\infty}A_{\infty} \cosh R + O(1), R \to \infty. \]

Rearranging this inequality and using \( \cosh^2 R = \sinh^2 R + 1 \) gives

\[ -2L^2_{\infty} \cosh R \geq 2L_{\infty}A_{\infty} \cosh R + O(1), R \to \infty. \]

As \( Vol_{\mathbb{H}^n}(\partial_{\infty} \Sigma, p_0) = L_{\infty} > 0 \) and \( A_{\infty} = A(\Sigma) \) this yields

\[ -Vol_{\mathbb{H}^n}(\partial_{\infty} \Sigma, p_0) \geq A(\Sigma) + O(e^{-R}), R \to \infty \]

Taking the limit as \( R \to \infty \) implies

\[ -Vol_{\mathbb{H}^n}(\partial_{\infty} \Sigma, p_0) \geq A(\Sigma). \]

Hence, by taking the supremum over all \( p_0 \in \mathbb{H}^n \), one obtains

\[ -\lambda_c[\partial_{\infty} \Sigma] \geq A(\Sigma). \]

As, Lemma 2.1 implies \( \lambda_c[\partial_{\infty} \Sigma] \geq 2\pi \) all the claimed estimates hold. The first rigidity result follows from the fact that if one has equality throughout, then \( \lambda_c[\Sigma] = 2\pi \) and so by Lemma 2.1 implies \( \Sigma \) is a totally geodesic copy of \( \mathbb{H}^2 \).

To complete the proof, we suppose \( \lambda_c[\partial_{\infty} \Sigma] = -A(\Sigma) \). In this case, by Lemma 2.1 either \( \lambda_c[\partial_{\infty} \Sigma] = 2\pi \) and \( \Sigma \) is a totally geodesic \( \mathbb{H}^2 \) or \( \lambda_c[\partial_{\infty} \Sigma] > 2\pi \) and there is a \( p_0 \in \mathbb{H}^n \) so

\[ \lambda_c[\partial_{\infty} \Sigma] = Vol_{\mathbb{H}^n}(\partial_{\infty} \Sigma; p_0). \]

By Proposition 3.2 this means

\[ \lambda_c[\partial_{\infty} \Sigma] = Vol_{\mathbb{H}^n}(\partial_{\infty} \Sigma; p_0) = \lim_{R \to \infty} Vol_{\mathbb{H}^n}(\Sigma \cap \partial B^n_R (p_0)) = \lim_{R \to \infty} L_R = L_{\infty}. \]

For this \( p_0 \), the rigidity hypothesis gives

\[ L_{\infty} = \lambda_c[\partial_{\infty} \Sigma] = -A(\Sigma) = -A_{\infty}. \]

Hence we may rewrite two of the expansions above as

\[ \frac{2L_RA_R}{\sinh R} = 2A^2_{\infty} \cosh R - 2A^2_{\infty} + o(1), R \to \infty \]

and, using \( \cosh^2 R = \sinh^2 R + 1 \),

\[ A^2_R = L^2_{\infty} \sinh^2 R - 2A^2_{\infty} \cosh R + 2A^2_{\infty} + o(1), R \to \infty \]

Combining these with the estimate of Proposition 4.2 and canceling terms gives

\[ -K_{\infty} \geq o(1), R \to \infty. \]
Sending \( R \to \infty \) gives,

\[
0 \geq K_{\infty} = \int_{\partial \Sigma'} |k_{\partial \Sigma'}^{2n-2}|^2 dVol_{\partial \Sigma'} \geq 0.
\]

Hence,

\[
\int_{\partial \Sigma'} |k_{\partial \Sigma'}^{2n-2}|^2 dVol_{\partial \Sigma'} = 0
\]

and so \( \partial \Sigma' \subset \partial B^n \) is a closed geodesic in \( S^{n-1} = \partial B^n \). As \( \partial \Sigma' \) has multiplicity one, this means

\[
2\pi = Vol_{\mathbb{H}^n}(\partial \Sigma') = Vol_{\partial_{\mathbb{H}^n} \mathbb{H}^n}(\partial_{\infty} \Sigma, p_0) = \lambda_c[\partial_{\infty} \Sigma] = -A(\Sigma).
\]

This returns us to the previous rigidity situation and so \( \Sigma \) must be a totally geodesic \( \mathbb{H}^2 \). \( \square \)

### Appendix A. Area Properties of Minimal Surfaces in \( \mathbb{H}^n \)

For the sake of completeness we include proofs of two facts we use in this paper about the area of minimal surfaces in \( \mathbb{H}^n \). The first is an area comparison result from \([7]\):

**Lemma A.1.** Let \( \Sigma \) be a compact minimal surface with boundary in \( \mathbb{H}^n \). For any \( p_0 \not\in \partial \Sigma \) one has

\[
Vol_{\mathbb{H}^n}(\Sigma) \leq Vol_{\mathbb{H}^n}(C(\partial \Sigma; p_0)).
\]

**Proof.** For \( p \neq p_0 \) consider the vector field on \( \mathbb{H}^n \)

\[
X(p) = \frac{\cosh r(p) - 1}{\sinh r(p)} \text{grad}_{\mathbb{H}^n} r = \frac{\cosh r(p) - 1}{\sinh^2 r(p)} \text{grad}_{\mathbb{H}^n} \cosh r
\]

where

\[
r(p) = \text{dist}_{\mathbb{H}^n}(p, p_0).
\]

The vector field \( X \) extends smoothly to \( p = p_0 \). Using

\[
\text{grad}^2_{\mathbb{H}^n} \cosh r = \cosh r \text{grad}_{\mathbb{H}^n} \cosh r,
\]

we see that for any surface \( \Gamma \subset \mathbb{H}^n \)

\[
\text{div}_{\Gamma} X = 1 + \frac{\cosh r(p) - 1}{\sinh^2 r(p)} (1 - |\text{grad}_r|^2)
\]

In particular, if \( \Gamma \) is minimal, then

\[
\text{div}_{\Gamma} X^\top = \text{div}_{\Gamma} X \geq 1.
\]

Likewise, if \( \Gamma \) is a cone over \( p_0 \), then \( \text{grad}_{\mathbb{H}^n} r \) is tangent to \( \Gamma \) and so \( |\text{grad}_r| = |\text{grad}_{\mathbb{H}^n} r| = 1 \). Hence, as \( X \) is also tangent to \( \Gamma \),

\[
\text{div}_{\Gamma} X^\top = g_{\mathbb{H}^n}(H_{\Gamma}, X) + \text{div}_{\Gamma} X = 1.
\]

If \( \nu \) is the outward normal to \( \Sigma \) along \( \partial \Sigma \) and \( \eta \) is the outward normal to \( C(\partial \Sigma; p_0) \), then, along \( \partial \Sigma \),

\[
g_{\mathbb{H}^n}(\nu, X) \leq g_{\mathbb{H}^n}(\eta, X) = 1.
\]
Hence, the first variation formula gives

\[ \text{Vol}_{\mathbb{H}^n}(\Sigma) = \int_{\Sigma} 1 \text{dVol}_\Sigma \leq \int_{\Sigma} \text{div}_\Sigma(X) \text{dVol}_\Sigma \]

\[ = \int_{\partial \Sigma} g_{\mathbb{H}^n}(\nu, X) \text{dVol}_{\partial \Sigma} \leq \int_{\partial \Sigma} g_{\mathbb{H}^n}(\eta, X) \text{dVol}_{\partial \Sigma} \]

\[ = \int_{\partial \Sigma} \text{div}_{C(\partial \Sigma, p_0)}(X) \text{dVol}_{C(\partial \Sigma, p_0)} = \int_{\partial \Sigma} 1 \text{dVol}_{C(\partial \Sigma, p_0)} = \text{Vol}_{\mathbb{H}^n}(C(\partial \Sigma; p_0)). \]

We also show that the renormalized area is defined independent of point \( p_0 \) and also verify formula (1.2) from (1):

**Lemma A.2.** Let \( \Sigma \) be a two-dimensional minimal surface in \( \mathbb{H}^n, n \geq 3 \), with a \( C^2 \)-regular asymptotic boundary. For any \( p_0 \in \mathbb{H}^n \), one has

\[ \mathcal{A}(\Sigma) = \lim_{R \to \infty} \left( \text{Vol}_{\mathbb{H}^n}(\Sigma \cap B_R^{\mathbb{H}^n}(p_0)) - \text{Vol}_{\mathbb{H}^n}(\Sigma \cap \partial B_R^{\mathbb{H}^n}(p_0)) \right) \]

is independent of \( p_0 \). Moreover,

\[ \mathcal{A}(\Sigma) = -2\pi \chi(\Sigma) - \frac{1}{2} \int_{\Sigma} |A_{\mathbb{H}^n}|^2 \text{dVol}_\Sigma. \]

**Proof.** Pick an identification \( i : \mathbb{H}^n \to \mathbb{B}^n \) so that \( i(p_0) = 0 \). As \( i^*g_{\mathbb{H}^n} = g_{\mathbb{B}^n} \), one has \( i(\partial B_R^{\mathbb{H}^n}(p_0)) = \partial B_R^{\mathbb{B}^n}(0) \). Set \( \Sigma_R = \Sigma \cap \partial B_R^{\mathbb{H}^n}(p_0) \) and let \( \Sigma' = i(\Sigma) \) be the natural compactification of \( \Sigma \) relative to \( p_0 \). Clearly, \( i(\Sigma_R) = \Sigma' \cap \partial B_s(0) = \Sigma'_s \).

Let \( k_{\Sigma'_s} \) be the mean curvature vector of \( \Sigma'_s \) with respect to \( g_P \) and \( k_{\Sigma'_s}^{E} \) be the mean curvature vector of \( \Sigma'_s \) with respect to \( g_E \). The formula for the conformal change of mean curvature vector ensures

\[ k_{\Sigma'_s}^{P} = \left( 1 - \frac{|x|^2}{4} \right) \left( k_{\Sigma'_s}^{E} - \frac{2x}{1 - |x|^2} \right). \]

Let \( \rho(p) = \text{dist}_{g_P}(p, 0) \) and set

\[ N(p) = \frac{\nabla_{g_P} \rho(p)}{\| \nabla_{g_P} \rho(p) \|} = \frac{1 - |x(p)|^2}{2|x(p)|} x^\top(p). \]

For \( s \) near 1 this is the outward unit normal to \( \Sigma'_s \) in \( \Sigma_s \). On \( \Sigma'_s \) one computes,

\[ g_P(N, k_{\Sigma'_s}^{P}) = -|x^\top| + \frac{1 - |x|^2}{2|x|^2} g_E(k_{\Sigma'_s}^{E}, x^\top). \]

Observe that for \( p \in \partial \Sigma'_s \) it follows from (3.6) that

\[ \lim_{q \to \partial \Sigma'_s} \frac{|x(q)| - |x^\top(q)|}{1 - |x(q)|} = 0. \]

Hence, on \( \Sigma'_s \),

\[ |x^\top| = s + o(1 - s), s \to 1. \]

Likewise, using (3.2) of Proposition 3.1 and fact that \( \Sigma' \) is \( C^2 \)

\[ \lim_{s \to 1} g_E(k_{\Sigma'_s}^{E}, x^\top) = g_E(k_{\partial \Sigma'_s}^{E}, x) = -1. \]
and so on $\Sigma'$,

$$\frac{1 - |x|^2}{2|x|^4} g_E(k_{\Sigma'}^E, x^T) = s - 1 + o(s - 1), s \to 1.$$ 

That is, on $\Sigma'$,

$$g_P(N, k_{\Sigma'}^E) = -1 + o(s - 1), s \to 1$$

We conclude that for $R$ large the geodesic curvature of $\Sigma_R$ in $\Sigma$ satisfies

$$\kappa_{\Sigma_R} = g^{E_n} \left( k_{\Sigma_R}, \frac{\nabla_{\Sigma}}{\nabla_{\Sigma}} \right) = 1 + o(e^{-R}), R \to \infty$$

where here $r(p) = \text{dist}_{E_n}(p, p_0)$. Hence,

$$\int_{\Sigma_R} \kappa_{\Sigma_R} dVol_{\Sigma_R} = Vol_{E_n}(\Sigma_R) + o(1), R \to \infty$$

where we used the fact proved in see [3, Lemma 4.1] or the first part of Proposition 3.2 that

$$Vol_{E_n}(\Sigma_R) = Vol_{\partial_{E_n}(\partial_{\infty} \Sigma, p_0)} \sinh R + o(e^R), R \to \infty.$$ 

Finally, by the Gauss equations, if $K_\Sigma$ is the Gauss curvature, then, as $\Sigma$ is minimal

$$K_\Sigma = -1 - \frac{1}{2} |A_\Sigma|^2$$

Hence, by the Gauss-Bonnet formula, for $R$ large,

$$Vol_{E_n}(\Sigma \cap \overline{B_{\infty}^n}(p_0)) = -\int_{\Sigma \cap \overline{B_{\infty}^n}(p_0)} K_\Sigma + \frac{1}{2} |A_\Sigma|^2 dVol_{\Sigma}$$

$$= \int_{\Sigma_R} \kappa_{\Sigma_R} dVol_{\Sigma_R} - 2\pi \chi(\Sigma \cap \overline{B_{\infty}^n}(p_0)) - \frac{1}{2} \int_{\Sigma \cap \overline{B_{\infty}^n}(p_0)} |A_\Sigma|^2 dVol_{\Sigma}$$

$$= Vol_{E_n}(\Sigma_R) - 2\pi \chi(\Sigma) - \frac{1}{2} \int_{\Sigma \cap \overline{B_{\infty}^n}(p_0)} |A_\Sigma|^2 dVol_{\Sigma} + o(1), R \to \infty.$$ 

Here the last equality used that the definition of asymptotically regular boundary means the Euler characteristic of $\Sigma \cap \overline{B_{\infty}^n}(p_0)$ stabilizes for large $R$. Both claims follow immediately from this by sending $R \to \infty$. Note that the existence of the limit, but not its independence from $p_0$, follows from the first part of Proposition 3.2 and the proof of this part does not use Lemma A.2. \qed

**APPENDIX B. GRAHAM WITTEN EXPANSION**

To connect our expansion of (1.1) with that considered by Graham and Witten [9] introduce the function

$$s = 2 \frac{1 - |x|}{1 + |x|}$$

on $\mathbb{H}^{n+k} \setminus \{0\}$. One verifies that $s$ is a boundary defining function and, in appropriate associated coordinates, the Poincaré metric has the form

$$g_P = s^{-2} \left( ds^2 + \left( 1 - \frac{s^2}{4} \right)^2 g_{\mathbb{H}^{n+k-1}} \right).$$

Hence, if $i$ is an identification of $\mathbb{H}^{n+k}$ with $\mathbb{H}^{n+k}$ sending $p_0$ to 0 and $\sigma = s \circ i$, then $\sigma^2 g_{\mathbb{H}^{n+k}}$ is a conformal compactification in the sense of [9]. Suppose $\Sigma \subset \mathbb{H}^{n+k}$ is a
$n$-dimensional minimal submanifold that has $C^\infty$-regular asymptotic boundary. Using a boundary defining function like $\sigma$, Graham and Witten [9] showed that, when $n$ is even,

$$
\text{Vol}_{\mathbb{H}^{n+k}}(\Sigma \cap B_{R(\epsilon)}^{\mathbb{H}^{n+k}}(p_0)) = \text{Vol}_{\mathbb{H}^{n+k}}(\Sigma \cap \{ \sigma > \epsilon \})
$$

$$
= c_0 \epsilon^{-n+1} + c_2 \epsilon^{-n+3} + \cdots + c_n \epsilon^{-n+1} + o(1), \epsilon \to 0
$$

while if $n$ is odd, then

$$
\text{Vol}_{\mathbb{H}^{n+k}}(\Sigma \cap B_{R(\epsilon)}^{\mathbb{H}^{n+k}}(p_0)) = \text{Vol}_{\mathbb{H}^{n+k}}(\Sigma \cap \{ \sigma > \epsilon \})
$$

$$
= c_0 \epsilon^{-n+1} + c_2 \epsilon^{-n+3} + \cdots + c_{n-3} \epsilon^{-2} + d \log \frac{1}{\epsilon} + c_{n-1} + o(1), \epsilon \to 0.
$$

Here $R(\epsilon) = -\ln \left( \frac{\epsilon}{2} \right)$. When $n = 2$, if one sets

$$
\epsilon(R) = 2e^{-R}
$$

then the the expansion is equivalent to (1.1) as in this case

$$
\cosh R(\epsilon) = e^{\epsilon} + o(1), \epsilon \to 0.
$$

Graham and Witten further showed that, when $n$ is even, $c_{n-1}$ is independent of the choice of $\sigma$ (and so independent of choice of $p_0$) and the same is true of $d$ when $n$ is odd. When $n = 2$ this quantity is precisely the renormalized area considered in [11] completing the justification of (1.1). This expansion is also the (Riemannian) analog of the entropy considered by Ryu and Takayanagi [13][14]. One observes that

$$
c_0 = \text{Vol}_{\partial_\infty \mathbb{H}^{n+k}}(\partial_\infty \Sigma, p_0)
$$

so the conformal volume is related to the leading order behavior of this expansion. Comparing with [4] and [5] it seems the renormalized area is analogous to the relative entropy.

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