Existence theory and qualitative analysis for a fully cross-diffusive predator-prey system

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Abstract

This manuscript considers a Neumann initial-boundary value problem for the predator-prey system

\[
\begin{aligned}
\frac{u_t}{D_1} &= D_1 u_{xx} - \chi_1 (uv_x)_x + u(\lambda_1 - u + a_1 v), \\
\frac{v_t}{D_2} &= D_2 v_{xx} + \chi_2 (vu_x)_x + v(\lambda_2 - v - a_2 u),
\end{aligned}
\]

(\*)

in an open bounded interval \( \Omega \) as the spatial domain, where for \( i \in \{1, 2\} \) the parameters \( D_i, a_i, \lambda_i \) and \( \chi_i \) are positive.

Due to the simultaneous appearance of two mutually interacting taxis-type cross-diffusive mechanisms, one of which even being attractive, it seems unclear how far a solution theory can be built upon classical results on parabolic evolution problems. In order to nevertheless create an analytical setup capable of providing global existence results as well as detailed information on qualitative behavior, this work pursues a strategy via parabolic regularization, in the course of which (\*) is approximated by means of certain fourth-order problems involving degenerate diffusion operators of thin film type.

During the design thereof, a major challenge is related to the ambition to retain consistency with some fundamental entropy-like structures formally associated with (\*); in particular, this will motivate the construction of an approximation scheme including two free parameters which will finally be fixed in different ways, depending on the size of \( \lambda_2 \) relative to \( a_2 \lambda_1 \).

Adequately coping with this will firstly yield a result on global existence of weak solutions for arbitrary choices of the parameters in (\*) and arbitrarily large positive initial data from \( H^1 \), and secondly allow for the conclusion that in both cases \( \lambda_2 > a_2 \lambda_1 \) and \( \lambda_2 \leq a_2 \lambda_1 \), the respectively obtained spatially homogeneous coexistence and prey-extinction states uphold their global asymptotic stability properties well-known to be present in the corresponding ODE setting, provided that both tactic sensitivities \( \chi_1 \) and \( \chi_2 \) are suitably small.

Key words: cross-diffusion, thin-film equation, large time behavior

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1 Introduction

Taxis-type migration processes have been identified as impelling mechanisms of crucial importance for the emergence of multifarious dynamics in numerous biological systems at various levels of complexity. Especially the destabilizing potential of attractive chemotaxis, and its responsibility for striking experimental findings, e.g. on paradigmatic bacterial aggregation phenomena but also on self-organization in more intricate frameworks, has received considerable interest ([17], [14]). Accordingly, a meanwhile abundant mathematical literature has been focusing on issues related to the ability of such taxis mechanisms to enforce the formation of structures, not only in contexts of simple paradigmatic Keller-Segel type chemotaxis models ([13], [6], [23], [24], [34]) but also in some more general triangular cross-diffusion systems embedding taxis processes into more complex settings ([19], [28], [11], [35], [30]).

In comparison to this, the knowledge seems much less developed in cases in which several taxis mechanisms are mutually coupled. A prototypical situation of this type is addressed in [32], where systems of the form

\[
\begin{aligned}
    u_t &= D_1 \Delta u - \chi_1 \nabla \cdot (u \nabla v) + f(u, v), \\
    v_t &= D_2 \Delta v + \chi_2 \nabla \cdot (v \nabla u) + g(u, v),
\end{aligned}
\]  

(1.1)

are proposed as refinements of classical reaction-diffusion models for predator-prey interaction, supplemented by attractive taxis of predators toward regions of increasing prey population densities, and by repulsive cross-diffusive migration of prey individuals downward population gradients of the predators (cf. also [37], [36], [33], [16] and [18] for further modeling aspects related to predator-taxis and prey-taxis). As documented in [32] and [33], formal linear analysis indicates that the indeed introduction of such taxis mechanisms enables pursuit-evasion systems of the form (1.1) to exhibit quite colorful wave-like solution behavior. From a perspective of rigorous analysis, however, passing on to such fully cross-diffusive systems, in which hence the collection of migration mechanisms can no longer be arranged in the form of a triangular diffusion operator, apparently amounts to entering quite uncharted territories. In fact, studies on cross-diffusion systems involving fully occupied diffusion matrices seem to essentially concentrate on systems of Shigesada-Kawasaki-Teramoto type, for which indeed a considerably comprehensive theory at least with regard to questions of global solvability, but partially even going beyond, could be developed ([7], [8], [9], [10], [21], [29], [22]). However, since the migration mechanisms therein do not only exhibit structures evidently different from those in (1.1), but since they moreover, and yet more drastically, are exclusively of repulsive character, such model classes can only be viewed as far relatives of (1.1), with hence quite limited potential for accessibility to similar techniques. Accordingly, the analytical literature concerned with a doubly tactic system of type (1.1) apparently reduces to the single precedent [31], in which a method is designed that in the absence of sources, that is, in the case when \( f = g = 0 \), can be used to establish results on global existence and stabilization toward spatial averages in a corresponding one-dimensional boundary value problem.

Objectives and challenges. The intention of this work is to address a system of type (1.1) in a more realistic setting of predator-prey evolution in which doubly tactic pursuit-evasion interplay is coupled to appropriate kinetics. Concentrating on classical Lotka-Volterra interaction as a paradigmatic framework therefor, we shall henceforth consider the apparently prototypical version of a fully
cross-diffusive predator-prey system given by

\[
\begin{align*}
    u_t &= D_1 u_{xx} - \chi_1 (uv_x)_x + u(\lambda_1 - u + a_1 v), \\
    v_t &= D_2 v_{xx} + \chi_2 (vu_x)_x + v(\lambda_2 - v - a_2 u),
\end{align*}
\]

in an open bounded interval as the spatial domain, where for \( i \in \{1, 2\} \) the parameters \( D_i, a_i, \lambda_i \) and \( \chi_i \) are positive.

Our particular purpose consists firstly in establishing a result on global existence of solutions within a natural weak framework, and secondly in attempting to undertake a basic step toward understanding qualitative effects of the doubly cross-diffusive mechanisms in (1.2) when accompanied by zero-order predator-prey interaction. Indeed, standard results on local existence of smooth solutions to Keller-Segel-type systems accounting for cross-diffusion exclusively in one quantity ([15], [2]) confirm that the diffusion-induced relaxing behavior known from the situation when \( \chi_1 = \chi_2 = 0 \) persists at least during suitably small initial time intervals when \( \chi_1 \neq 0 \) as long as \( \chi_2 \) yet vanishes. On the other hand, the simultaneous appearance of taxis-type cross-diffusion in both equations from (1.2) may quite drastically affect this picture in that when both taxis mechanisms are attractive in the sense that \( \chi_1 > 0 > \chi_2 \), then in general not even local solutions can be expected to exist, even in the case when the kinetic terms therein are completely disregarded and the initial data belong to \( (C^\infty(\Omega))^2 \) ([31]).

As already suggested by the outcome of the precedent work [31], such an instantaneously and thoroughly destabilizing role may not be played by doubly taxis-like interaction in cases when only one of the cross-diffusive migration processes is attractive, with the other one being repulsive. In particular, addressing the simplified variant of (1.2) given by

\[
\begin{align*}
    u_t &= D_1 u_{xx} - \chi_1 (uv_x)_x, \\
    v_t &= D_2 v_{xx} + \chi_2 (vu_x)_x,
\end{align*}
\]

with \( \chi_1 \) and \( \chi_2 \) both assumed positive, the main results in [31] assert that in an open bounded domain \( \Omega \subset \mathbb{R} \) and for all reasonably regular nonnegative initial data, a corresponding Neumann-type initial-boundary value problem possesses a globally defined nonnegative weak solution, inter alia belonging to the space \( (L^\infty((0, \infty); L \log L(\Omega)))^2 \). Moreover, this solution is shown to become bounded in the pointwise sense at least eventually, and that both of its components uniformly approach their respective temporally constant spatial mean in the large time limit. This inter alia indicates that the pattern-supporting potential of the attractive taxis mechanism therein, well-known e.g. as generating nonconstant steady states in associated Keller-Segel systems ([26], [20]), is insufficient to enforce any structure formation on large time scales when accompanied by repulsion as in (1.3).

As a fundamental technical obstacle, any analytical approach to questions concerning solvability in (1.2) needs to face the circumstance that no general theory seems available which might at least warrant local existence of some solutions. Hence forced to construct solutions either from a very basic starting point e.g. within a suitable fixed point framework, or via approximation, in this work we choose the latter type of ansatz by means of a fourth-order regularization reminiscent of the well-studied thin film equation ([5], [3], [12], [25]). Indeed, we shall see that when carefully designed, beyond providing accessibility to well-established local solution theory ([1]), this approach will in quite
a natural manner bring about the important advantage of paving the way for a qualitative analysis through the derivation of a priori estimates at the level of approximate solutions, and might thereby turn out to be more appropriate than e.g. discretization-based methods ([7], [8], [9]) which in the present context, beyond apparently unsolved problems already at the stage of solvability, seem to offer somewhat less flexibility with regard to testing procedures.

**Thin-film-type approximation.** To make our concrete strategy more precise, let us first specify the full setting in which [12] will be studied, and subsequently consider

\[
\begin{align*}
  u_t &= D_1 u_{xx} - \chi_1 (uv_x)_x + u(\lambda_1 - u + a_1 v), \quad x \in \Omega, \ t > 0, \\
  v_t &= D_2 v_{xx} + \chi_2 (vu_x)_x + v(\lambda_2 - v - a_2 u), \quad x \in \Omega, \ t > 0, \\
  u_x = v_x = 0, & \quad x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & \quad x \in \Omega,
\end{align*}
\]

(1.4)

in an open bounded interval \( \Omega \subset \mathbb{R} \), where the initial data \( u_0 \) and \( v_0 \), along with approximating families \( (u_0^\varepsilon)_{\varepsilon \in (0, 1)} \) and \( (v_0^\varepsilon)_{\varepsilon \in (0, 1)} \), will be assumed to be such that

\[
\begin{align*}
  u_0 &\in W^{1,2}(\Omega) \text{ and } v_0 \in W^{1,2}(\Omega) \text{ satisfy } u_0 > 0 \text{ and } v_0 > 0 \text{ in } \Omega, \\
  u_0^\varepsilon &\in C^5(\overline{\Omega}) \text{ and } v_0^\varepsilon \in C^5(\overline{\Omega}) \text{ for all } \varepsilon \in (0, 1) \text{ with} \\
  u_{0,xx} = u_{0,xxx} = v_{0,xx} = v_{0,xxx} = 0 \text{ on } \partial \Omega \text{ for all } \varepsilon \in (0, 1) \text{ and} \\
  \frac{1}{2} \inf_{\Omega} u_0 \leq u_0^\varepsilon \leq u_0 + 1 \text{ in } \Omega \quad \text{ and } \quad \int_\Omega u_{0,xx}^2 \leq \int_\Omega u_{0,xx}^2 + 1 \text{ for all } \varepsilon \in (0, 1), \quad \text{ that} \\
  \frac{1}{2} \inf_{\Omega} v_0 \leq v_0^\varepsilon \leq v_0 + 1 \text{ in } \Omega \quad \text{ and } \quad \int_\Omega v_{0,xx}^2 \leq \int_\Omega v_{0,xx}^2 + 1 \text{ for all } \varepsilon \in (0, 1), \quad \text{ and that} \\
  u_0^\varepsilon \to u_0 \text{ and } v_0^\varepsilon \to v_0 \text{ a.e. in } \Omega \text{ as } \varepsilon \searrow 0.
\end{align*}
\]

(1.5)

We henceforth fix any \( \alpha \in (0, \frac{1}{2}) \), and with free parameters \( n_1 > 0 \) and \( n_2 > 0 \) to be specified below we consider the regularized parabolic system

\[
\begin{align*}
  u_{\varepsilon,t} &= -\varepsilon \left( \frac{u_1^\varepsilon}{u_1^\varepsilon - n_1 - \varepsilon} u_{\varepsilon,xx} \right)_x + \varepsilon a \left( u_{\varepsilon,xx} \right)_x - D_1 u_{\varepsilon,xx} - \chi_1 \left( u_{\varepsilon,xx} \right)_x + \frac{3u_3^\varepsilon}{3u_3^\varepsilon + \varepsilon} \cdot (\lambda_1 - u_{\varepsilon} + a_1 v_{\varepsilon}), \\
  v_{\varepsilon,t} &= -\varepsilon \left( \frac{v_1^\varepsilon}{v_1^\varepsilon - n_2 - \varepsilon} v_{\varepsilon,xx} \right)_x + \varepsilon a \left( v_{\varepsilon,xx} \right)_x + D_2 v_{\varepsilon,xx} + \chi_2 \left( v_{\varepsilon,xx} \right)_x + \frac{3v_3^\varepsilon}{3v_3^\varepsilon + \varepsilon} \cdot (\lambda_2 - v_{\varepsilon} - a_2 u_{\varepsilon}),
\end{align*}
\]

(1.6)

along with homogeneous Neumann-type boundary data and under the initial conditions given by

\[
\begin{align*}
  u_{\varepsilon,x} = v_{\varepsilon,x} = u_{\varepsilon,xxx} = v_{\varepsilon,xxx} = 0, & \quad x \in \partial \Omega, \ t > 0, \\
  u_{\varepsilon}(x, 0) = u_{0,\varepsilon}(x), \ v_{\varepsilon}(x, 0) = v_{0,\varepsilon}(x), & \quad x \in \Omega,
\end{align*}
\]

(1.7)

for \( \varepsilon \in (0, 1) \). An introduction both of similar thin-film-type fourth-order diffusion operators and of corresponding second-order fast diffusion corrections has already been underlying the analysis in [31]; a difference of crucial importance in the present approach, however, consists in the circumstance that the parameters \( n_1 \) and \( n_2 \), which may be viewed as measuring a certain intermediate-scale degeneracy of the considered fourth-order diffusion mechanisms, will be allowed to attain different values here: While the development of our existence theory will merely require that \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), our
subsequent qualitative analysis will rely on the specific choices \((n_1, n_2) = (2, 2)\) and \((n_1, n_2) = (2, 1)\), respectively, depending on the parameter setting dictated by the Lotka-Volterra interaction in (1.4).

**Main results.** In fact, we shall firstly see that if we restrict respectively, depending on the parameter setting dictated by the Lotka-Volterra interaction in (1.4), the steady states of (1.4), in the case of the data.

This will bear fruit by implying global asymptotic stability of the nontrivial spatially homogeneous steady states of (1.4), and for widely arbitrary initial data:

Then there exist nonnegative functions \(u, v\) and \(v\) defined a.e. in \(\Omega \times (0, \infty)\) and satisfying

\[
\{u, v\} \subset C^0_w([0, \infty); L^1(\Omega)) \cap L^3_{\text{loc}}(\Omega \times [0, \infty)) \cap L_\text{loc}^\frac{3}{2}([0, \infty); W^{1, \frac{3}{2}}(\Omega)) \cap L^\infty((0, \infty); L \log L(\Omega)),
\]

which are such that \((u, v)\) forms a global weak solution of (1.4) in the sense of Definition 4.1 below. This solution can be obtained as the limit of solutions to (1.6)-(1.7) in that whenever \(n_1 \in [1, 2]\), \(n_2 \in [1, 2]\) and \(\alpha \in (0, \frac{1}{2})\) and \((u_{0\varepsilon})_{\varepsilon \in (0, 1)}\) and \((v_{0\varepsilon})_{\varepsilon \in (0, 1)}\) satisfy (1.7), there exists \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) such that \(\varepsilon_j \downarrow 0\) as \(j \to \infty\), and that

\[
u_{\varepsilon} \to u \quad \text{as well as} \quad v_{\varepsilon} \to v \quad \text{a.e. in} \quad \Omega \times (0, \infty) \quad \text{as} \quad \varepsilon = \varepsilon_j \downarrow 0.
\]

Beyond establishing the above existence result which with regard to its outcome essentially parallels Theorem 1.1 in [31], our analysis related to (1.8) will moreover be organized in such a way that a beneficent dependence of correspondingly gained estimates on the sensitivities \(\chi_1\) and \(\chi_2\) can be cropped out. Together with a rather straightforward observation identifying a \((\chi_1, \chi_2)\)-independent absorbing set in \((L^1(\Omega))^2\) for (1.6)-(1.7) (Lemma 2.3), the regularity information thus provided by a quasi-entropy inequality associated with (1.8) will imply eventual bounds for solutions to (1.6)-(1.7) in somewhat stronger topologies, independent of the initial data and depending on \(\chi_1\) and \(\chi_2\) in a favorably controllable manner (Lemma 5.4). In a key step to be achieved in Lemma 6.2 these ultimate bounds will further be improved so as to become manifest even in \((W^{1, 2}(\Omega))^2\), provided that \(\chi_1\) and \(\chi_2\) satisfy a smallness condition which, importantly, involves essentially no knowledge on the initial data.

This will bear fruit by implying global asymptotic stability of the nontrivial spatially homogeneous steady states of (1.4), in the case \(\lambda_2 > a_2\lambda_1\) given by \((u_*, v_*)\) with

\[
u_* := \frac{\lambda_1 + a_1\lambda_2}{1 + a_1a_2} \quad \text{and} \quad v_* := \frac{\lambda_2 - a_2\lambda_1}{1 + a_1a_2}
\]

and by \((\lambda_1, 0)\) if \(\lambda_2 \leq a_2\lambda_1\), whenever \(\chi_1\) and \(\chi_2\) are suitably small: In Section 8 addressing the former case, we shall see that then for small \(\chi_1\) and \(\chi_2\) and any choice of initial data compatible with (1.5),
a regularized variant of
\[
\int_\Omega \left( u - u^* - u^* \ln \frac{u}{u^*} \right) + \frac{a_1}{a_2} \int_\Omega \left( v - v^* - v^* \ln \frac{v}{v^*} \right)
\] (1.12)
eventually plays the role of a genuine entropy functional for (1.6)-(1.7), up to a regularization error, under a crucial additional assumption on structural consistency of our approximation, here expressed in the hypothesis that \( n_1 = n_2 = 2 \) which is fortunately in compliance with Theorem 1.1 (Lemma 8.3).

In conjunction with the compactness properties of trajectories entailed by Lemma 6.2, this further dissipative structure will lead us to the following second of our main results:

**Theorem 1.2** Let \( \Omega \subset \mathbb{R} \) be an open bounded interval. Then given \( D_1 > 0, D_2 > 0, a_1 > 0, a_2 > 0, \lambda_1 > 0 \) and \( \lambda_2 > 0 \) fulfilling
\[
\lambda_2 > a_2 \lambda_1,
\]
one can find \( \chi^{**} > 0 \) such that if \( \chi_1 \in (0, \chi^{**}) \) and \( \chi_2 \in (0, \chi^{**}) \) as well as
\[
n_1 = 2 \quad \text{and} \quad n_2 = 2,
\]
then for any choice of initial data fulfilling (1.5) the global weak solution \((u, v)\) of (1.4) from Theorem 1.1 has the properties that
\[
\{u, v\} \subset \subset C^0(\overline{\Omega} \times (T, \infty)) \cap L^\infty(\Omega \times (0, \infty)) \quad \text{for some} \quad T > 0,
\] (1.13)
and that as \( t \to \infty \),
\[
u(\cdot, t) \to v^* \quad \text{in} \quad L^\infty(\Omega),
\]
where \( u^* > 0 \) and \( v^* > 0 \) are as in (1.11).

In the opposite parameter range where \( \lambda_2 \leq a_2 \lambda_1 \), an accordingly modified analysis identifying a corresponding entropy property of a functional approximating
\[
\int_\Omega \left( u - \lambda_1 - \lambda_1 \ln \frac{u}{\lambda_1} \right) + \frac{a_1}{a_2} \int_\Omega \left( v + \frac{a_1}{2a_2 \lambda_2} \int_\Omega v^2 \right),
\] (1.14)
will finally reveal in Section 9 the following analogue of the above statement under the assumption that yet \( n_1 = 2 \) but that now \( n_2 \) satisfies the alternative consistency condition \( n_2 = 1 \), both still admissible in Theorem 1.1:

**Theorem 1.3** Whenever \( \Omega \subset \mathbb{R} \) is an open bounded interval and \( D_1 > 0, D_2 > 0, a_1 > 0, a_2 > 0, \lambda_1 > 0 \) and \( \lambda_2 > 0 \) are such that
\[
\lambda_2 \leq a_2 \lambda_1,
\]
there exists \( \chi^{**} > 0 \) such that if \( \chi_1 \in (0, \chi^{**}) \), \( \chi_2 \in (0, \chi^{**}) \),
\[
n_1 = 2 \quad \text{and} \quad n_2 = 1
\]
and if (1.5) holds, then the global weak solution \((u, v)\) of (1.4) from Theorem 1.1 satisfies (1.13) as well as
\[
u(\cdot, t) \to 0 \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad t \to \infty.
\]

6
2 Basic properties of the approximate problems

To begin with, we recall that Amann’s theory provides a basic statement on local existence and extensibility of solutions to (1.6)-(1.7) in the following sense.

Lemma 2.1 For \(i \in \{1,2\}\) let \(D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0\) and \(n_i \in (0,4)\), and suppose that (1.5) holds. Then for all \(\varepsilon \in (0,1)\) there exist \(T_{max, \varepsilon} \in (0, \infty]\) and a pair \((u_\varepsilon, v_\varepsilon)\) of functions

\[
\begin{cases}
  u_\varepsilon \in \bigcap_{i \in \{1,2\}} C^0([0, T_{max, \varepsilon}); W^{s,2}(\Omega)) \cap C^4,1(\overline{\Omega} \times (0, T_{max, \varepsilon})) \\
  v_\varepsilon \in \bigcap_{i \in \{1,2\}} C^0([0, T_{max, \varepsilon}); W^{s,2}(\Omega)) \cap C^4,1(\overline{\Omega} \times (0, T_{max, \varepsilon}))
\end{cases}
\]

satisfying \(u_\varepsilon > 0\) and \(v_\varepsilon > 0\) in \(\overline{\Omega} \times (0, T_{max, \varepsilon})\), which are such that \((u_\varepsilon, v_\varepsilon)\) solves (1.6)-(1.7) classically in \(\Omega \times (0, T_{max, \varepsilon})\), and that either \(T_{max, \varepsilon} = \infty\), or

\[
\limsup_{t \to T_{max, \varepsilon}} \left\{ \|u_\varepsilon(t)\|_{W^{2,2}(\Omega)} + \frac{1}{\varepsilon} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(t)\|_{W^{2,2}(\Omega)} + \frac{1}{\varepsilon} \|v_\varepsilon(t)\|_{L^\infty(\Omega)} \right\} = \infty.
\]

Proof. This can be seen by adapting the argument from [31 Lemma 2.1] in a straightforward manner.

2.1 Mass evolution. Absorbing sets in \(L^1(\Omega)\)

Simple integration in (1.6)-(1.7) yields a basic information on the evolution of the total mass functionals related to both solution components.

Lemma 2.2 Let \(D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0\) and \(n_i \in (0,4)\) for \(i \in \{1,2\}\), and assume (1.5). Then for all \(\varepsilon \in (0,1)\), we have

\[
\frac{d}{dt} \int_\Omega u_\varepsilon \leq \left( \lambda_1 + \frac{\sqrt{\varepsilon}}{2\sqrt{3}} \right) \int_\Omega v_\varepsilon - \int_\Omega u_\varepsilon^2 + a_1 \int_\Omega u_\varepsilon v_\varepsilon \quad \text{for all } t \in (0, T_{max, \varepsilon})
\]

(2.2)

and

\[
\frac{d}{dt} \int_\Omega v_\varepsilon \leq \left( \lambda_2 + \frac{\sqrt{\varepsilon}}{2\sqrt{3}} \right) \int_\Omega v_\varepsilon^2 - \int_\Omega v_\varepsilon^2 - a_2 \int_\Omega u_\varepsilon v_\varepsilon + \frac{a_2 \sqrt{\varepsilon}}{2\sqrt{3}} \int_\Omega u_\varepsilon \quad \text{for all } t \in (0, T_{max, \varepsilon}).
\]

(2.3)

Proof. Writing \(g_\varepsilon(s) := \frac{3s^4}{3s^4 + \varepsilon}\) for \(s \geq 0\) and \(\varepsilon \in (0,1)\), on the basis of (1.6)-(1.7) we compute

\[
\frac{d}{dt} \int_\Omega u_\varepsilon = \lambda_1 \int_\Omega g_\varepsilon(u_\varepsilon) - \int_\Omega g_\varepsilon(u_\varepsilon) u_\varepsilon + a_1 \int_\Omega g_\varepsilon(u_\varepsilon) v_\varepsilon
\]

\[
= \lambda_1 \int_\Omega u_\varepsilon - \int_\Omega u_\varepsilon^2 + a_1 \int_\Omega u_\varepsilon v_\varepsilon
\]

\[
- \lambda_1 \int_\Omega \left( u_\varepsilon - g_\varepsilon(u_\varepsilon) \right) + \int_\Omega \left( u_\varepsilon - g_\varepsilon(u_\varepsilon) \right) u_\varepsilon - a_1 \int_\Omega \left( u_\varepsilon - g_\varepsilon(u_\varepsilon) \right) v_\varepsilon
\]

(2.4)
for $t \in (0, T_{\text{max}, \varepsilon})$. Using that $[0, \infty) \ni s \mapsto s - \xi(s)$ is nonnegative and attains its maximum at $s = \frac{\sqrt{3}}{2\sqrt{3}}$ with extremal value $\frac{\sqrt{3}}{2\sqrt{3}}$, from (2.4) we directly obtain (2.2), whereas (2.3) can be derived similarly.

By taking a suitable linear combination of the latter inequalities, we obtain some genuine $L^1$ estimates which will later on play a fundamental role in our derivation of eventual bounds which do not depend on the size of the initial data.

**Lemma 2.3** For $i \in \{1, 2\}$ let $D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0$ and $n_i \in (0, 4)$, and assume that (1.5) holds. Then there exists a bounded function $m : (0, \infty) \to \mathbb{R}$ such that for all $\varepsilon \in (0, 1)$,

$$\int_{\Omega} u_\varepsilon(\cdot, t) + \int_{\Omega} v_\varepsilon(\cdot, t) \leq m(t) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}),$$

(2.5)

and such that

$$\limsup_{t \to \infty} m(t) \leq m_\infty := \frac{|\Omega|}{2} \cdot \left( \lambda_1 + \frac{1}{2\sqrt{3}} + 1 + \max\left\{ a_1^2, 1 \right\} \cdot \frac{a_2^2}{2\sqrt{3}} \right)^2 + \frac{|\Omega|}{2} \cdot \left( \lambda_2 + \frac{1}{2\sqrt{3}} + 1 \right)^2 \cdot \max\left\{ a_1^2, 1 \right\}$$

(2.6)

**Proof.** We let

$$\beta := \max \left\{ a_1^2, 1 \right\}$$

(2.7)

and combine (2.2) with (2.3) in estimating

$$\frac{d}{dt} \left\{ \int_{\Omega} u_\varepsilon + \beta \int_{\Omega} v_\varepsilon \right\} + \left\{ \int_{\Omega} u_\varepsilon + \beta \int_{\Omega} v_\varepsilon \right\} \leq \left( \lambda_1 + \frac{1}{2\sqrt{3}} + 1 + \frac{\beta a_2}{2\sqrt{3}} \right) \int_{\Omega} u_\varepsilon - \int_{\Omega} u_\varepsilon^2$$

$$+ \beta \left( \lambda_2 + \frac{1}{2\sqrt{3}} + 1 \right) \int_{\Omega} v_\varepsilon - \beta \int_{\Omega} v_\varepsilon^2$$

$$+ (a_1 - \beta a_2) \int_{\Omega} u_\varepsilon v_\varepsilon \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).$$

(2.8)

Here three applications of Young’s inequality show that

$$\left( \lambda_1 + \frac{1}{2\sqrt{3}} + 1 + \frac{\beta a_2}{2\sqrt{3}} \right) \int_{\Omega} u_\varepsilon \leq \frac{1}{2} \int_{\Omega} u_\varepsilon^2 + \frac{|\Omega|}{2} \left( \lambda_1 + \frac{1}{2\sqrt{3}} + 1 + \frac{\beta a_2}{2\sqrt{3}} \right)^2$$

for all $t \in (0, T_{\text{max}, \varepsilon})$

and

$$\beta \left( \lambda_2 + \frac{1}{2\sqrt{3}} + 1 \right) \int_{\Omega} v_\varepsilon \leq \frac{\beta}{2} \int_{\Omega} v_\varepsilon^2 + \frac{\beta |\Omega|}{2} \left( \lambda_2 + \frac{1}{2\sqrt{3}} + 1 \right)^2$$

for all $t \in (0, T_{\text{max}, \varepsilon})$

as well as

$$(a_1 - \beta a_2) \int_{\Omega} u_\varepsilon v_\varepsilon \leq a_1 \int_{\Omega} u_\varepsilon v_\varepsilon$$

$$\leq \frac{1}{2} \int_{\Omega} u_\varepsilon^2 + \frac{a_1^2}{2} \int_{\Omega} v_\varepsilon^2$$

$$\leq \frac{1}{2} \int_{\Omega} u_\varepsilon^2 + \frac{\beta}{2} \int_{\Omega} v_\varepsilon^2$$

for all $t \in (0, T_{\text{max}, \varepsilon})$. 


the latter relying on (2.7). From (2.8) we therefore obtain that
\[
\frac{d}{dt} \left\{ \int_{\Omega} u_\varepsilon + \beta \int_{\Omega} v_\varepsilon \right\} + \left\{ \int_{\Omega} u_\varepsilon + \beta \int_{\Omega} v_\varepsilon \right\} \leq c_4 := \frac{|\Omega|}{2} \left( \lambda_1 + \frac{1}{2\sqrt{3}} + 1 + \frac{\beta a_2}{2\sqrt{3}} \right)^2 + \frac{\beta |\Omega|}{2} \left( \lambda_2 + \frac{1}{2\sqrt{3}} + 1 \right)^2 \quad \text{for all } t \in (0, T_{max,\varepsilon})
\]
and that accordingly, again due to (1.5),
\[
\int_{\Omega} u_\varepsilon + \beta \int_{\Omega} v_\varepsilon \leq \left\{ \int_{\Omega} u_0 + \beta \int_{\Omega} v_0 + (\beta + 1)|\Omega| \right\} \cdot e^{-t} + c_4 \int_0^t e^{-(t-s)} ds \leq \left\{ \int_{\Omega} u_0 + \beta \int_{\Omega} v_0 + (\beta + 1)|\Omega| \right\} \cdot e^{-t} + c_4 \quad \text{for all } t \in (0, T_{max,\varepsilon}).
\]
As \( \beta \geq 1 \) by (2.7), upon an obvious choice of \( m \) this asserts both (2.5) and (2.6) in this case. \( \square \)

\section{2.2 Global extensibility in the regularized problems}

In order to show that the solutions from Lemma 2.1 are actually global in time, in view of (2.1) our goal will be to establish a priori bounds, throughout this part possibly depending on \( \varepsilon \), for \( u_\varepsilon \) and \( v_\varepsilon \), and for \( \frac{1}{u_\varepsilon} \) and \( \frac{1}{v_\varepsilon} \), in \( W^{2,2}(\Omega) \) and \( L^\infty(\Omega) \), respectively. In a first step toward this, but moreover also later on in the context of our qualitative analysis (see Lemma 6.2), we will make use of the following interpolation inequality that has extensively been exploited in the analysis of the thin film equation, albeit mostly for slightly different purposes there (11 and 3).

**Lemma 2.4** Let \( \beta \in \mathbb{R} \) be such that \( \beta \neq 1 \). Then
\[
\int_{\Omega} \varphi^{\beta-2} \varphi_x^4 \leq \frac{9}{(\beta - 1)^2} \int_{\Omega} \varphi^\beta \varphi_{xx}^2
\]
for all \( \varphi \in C^2(\overline{\Omega}) \) which are such that \( \varphi > 0 \) in \( \overline{\Omega} \) and \( \varphi_x = 0 \) on \( \partial \Omega \).

**Proof.** We integrate by parts and use the Cauchy-Schwarz inequality to see that
\[
\int_{\Omega} \varphi^{\beta-2} \varphi_x^4 = -\frac{3}{\beta - 1} \int_{\Omega} \varphi^{\beta-1} \varphi_x^2 \varphi_{xx} \leq \frac{3}{|\beta - 1|} \left\{ \int_{\Omega} \varphi^{\beta-2} \varphi_x^4 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \varphi^\beta \varphi_{xx}^2 \right\}^{\frac{1}{2}},
\]
from which (2.9) immediately follows. \( \square \)

In the present context, our first application thereof will guarantee a certain consistency of the artificial second-order fast diffusion term in (1.6)-(1.7) with regard to the evolution of the \( H^1 \) norms of solutions. Already in the following basic statement concerning this, to be immediately utilized in Lemma 2.6 but recalled also later on in the crucial Lemma 6.2 our standing assumption that \( \alpha \leq \frac{1}{2} \) plays a major role.
Lemma 2.5 For $i \in \{1, 2\}$ let $D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0$ and $n_i \in (0, 4)$, and assume (1.7). The for all $\varepsilon \in (0, 1)$ and any $t \in (0, T_{\max, \varepsilon})$,

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega v_{\varepsilon}^2 + \varepsilon \int_\Omega \frac{u_{\varepsilon}^4}{u_{\varepsilon}^{-n_1} + \varepsilon} u_{\varepsilon,xx}^2 + D_1 \int_\Omega u_{\varepsilon,xx}^2 
\leq \chi_1 \int_\Omega \left( \frac{u_{\varepsilon}^{5-n_1}}{u_{\varepsilon}^{-n_1} + \varepsilon} \right) u_{\varepsilon,xx} + 3\lambda_1 \int_\Omega u_{\varepsilon,xx}^2 + \frac{a_1^2}{2D_1} \int_\Omega u_{\varepsilon}^2 u_{\varepsilon,xx}^2
$$

(2.10)

and

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega v_{\varepsilon}^2 + \varepsilon \int_\Omega \frac{v_{\varepsilon}^4}{v_{\varepsilon}^{-n_2} + \varepsilon} v_{\varepsilon,xx}^2 + D_2 \int_\Omega v_{\varepsilon,xx}^2 
\leq -\chi_2 \int_\Omega \left( \frac{v_{\varepsilon}^{5-n_2}}{v_{\varepsilon}^{-n_2} + \varepsilon} \right) v_{\varepsilon,xx} + 3\lambda_2 \int_\Omega v_{\varepsilon,xx}^2 + \frac{a_2^2}{2D_2} \int_\Omega u_{\varepsilon}^2 v_{\varepsilon,xx}^2
$$

(2.11)

PROOF. We multiply the first equation in (1.6) by $-u_{\varepsilon,xx}$ and integrate by parts to see that

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega u_{\varepsilon,xx}^2 + \varepsilon \int_\Omega \frac{u_{\varepsilon}^4}{u_{\varepsilon}^{-n_1} + \varepsilon} u_{\varepsilon,xx}^2 + D_1 \int_\Omega u_{\varepsilon,xx}^2 
= -\varepsilon^2 \int_\Omega (u_{\varepsilon}^{-\alpha} u_{\varepsilon,xx})_x u_{\varepsilon,xx} + \chi_1 \int_\Omega \left( \frac{u_{\varepsilon}^{5-n_1}}{u_{\varepsilon}^{-n_1} + \varepsilon} \right) u_{\varepsilon,xx} 
- \int_\Omega \frac{3u_{\varepsilon}^3}{3u_{\varepsilon}^2 + \varepsilon} (\lambda_1 - u_{\varepsilon} + a_1 v_{\varepsilon}) u_{\varepsilon,xx} \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
$$

(2.12)

where another integration by parts, followed by an application of Lemma 2.4, shows that

$$
- \int_\Omega (u_{\varepsilon}^{-\alpha} u_{\varepsilon,xx})_x u_{\varepsilon,xx} = - \int_\Omega u_{\varepsilon}^{-\alpha} u_{\varepsilon,xx}^2 + \alpha \int_\Omega u_{\varepsilon}^{-\alpha - 1} u_{\varepsilon,xx}^2 u_{\varepsilon,xx}
= - \int_\Omega u_{\varepsilon}^{-\alpha} u_{\varepsilon,xx}^2 + \frac{\alpha (\alpha + 1)}{3} \int_\Omega u_{\varepsilon}^{-\alpha - 2} u_{\varepsilon,xx}^4
\leq 0 \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
$$

(2.13)

because $\frac{\alpha (\alpha + 1)}{3} \cdot \frac{9}{(\alpha + 1)^2} = \frac{3\alpha}{\alpha + 1} \leq 1$ thanks to our assumption that $\alpha \leq \frac{1}{2}$. Apart from that, observing that for all $\varepsilon \in (0, 1)$ and any $s \geq 0$ we have

$$
\frac{d}{ds} \left( \frac{3s^3}{3s^2 + \varepsilon} \right) = \frac{9s^4 + 9\varepsilon s^2}{(3s^2 + \varepsilon)^2} \leq \frac{9s^2}{3s^2 + \varepsilon} \leq 3
$$

and

$$
\frac{d}{ds} \left( \frac{3s^4}{3s^2 + \varepsilon} \right) = \frac{18s^5 + 12\varepsilon s^3}{(3s^2 + \varepsilon)^2} \geq 0,
$$

in the rightmost expression in (2.12) we may once more rely on integration by parts to estimate

$$
- \int_\Omega \frac{3u_{\varepsilon}^3}{3u_{\varepsilon}^2 + \varepsilon} (\lambda_1 - u_{\varepsilon}) u_{\varepsilon,xx} = \lambda_1 \int_\Omega \left( \frac{3u_{\varepsilon}^3}{3u_{\varepsilon}^2 + \varepsilon} \right) u_{\varepsilon,xx} - \int_\Omega \left( \frac{3u_{\varepsilon}^3}{3u_{\varepsilon}^2 + \varepsilon} \right) u_{\varepsilon,xx}
\leq 3\lambda_1 \int_\Omega u_{\varepsilon,xx}^2 \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
$$

(2.14)
As finally
\[ -a_1 \int_\Omega \frac{3u_{xx}^2}{3a_{xx}^2 + \varepsilon} v_x u_{xxx} \leq \frac{D_1}{2} \int_\Omega u_{xx}^2 + \frac{a_1^2}{2D_1} \int_\Omega \left( \frac{3u_{xx}^3}{3a_{xx}^2 + \varepsilon} \right)^2 \varepsilon^2 \]
by Young’s inequality, in view of (2.13) and (2.14) we readily infer (2.10) from (2.12). The inequality (2.11) can be derived similarly.

Under the additional requirements that \(1 \leq n_1 \leq 2\) and \(1 \leq n_2 \leq 2\), for each fixed \(\varepsilon \in (0, 1)\) the integrals in (2.10) and (2.11) originating from the cross-diffusive interaction in (1.6) can conveniently be estimated in terms of the respective higher-order dissipative contributions, thus leading to the following \(\varepsilon\)-dependent bound with respect to the norm in \(W^{1,2}(\Omega)\).

**Lemma 2.6** Let \(D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0\) and \(n_i \in [1, 2]\) for \(i \in \{1, 2\}\), and let (1.5) be valid. Then for all \(\varepsilon \in (0, 1)\) and \(T > 0\) there exists \(C(\varepsilon, T) > 0\) such that
\[
\int_\Omega u_{xx}^2(x, t)dx + \int_\Omega v_{xx}^2(x, t)dx \leq C(\varepsilon, T) \quad \text{for all } t \in (0, \hat{T}_\varepsilon) \tag{2.15}
\]
and
\[
\|u_{xx}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{xx}(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\varepsilon, T) \quad \text{for all } t \in (0, \hat{T}_\varepsilon), \tag{2.16}
\]
where \(\hat{T}_\varepsilon := \min\{T, T_{\text{max}, \varepsilon}\}\).

**Proof.** In the cross-diffusive term on the right of (2.10), we integrate by parts and use Young’s inequality to find that for all \(t \in (0, T_{\text{max}, \varepsilon})\),
\[
\chi_1 \int \left( \frac{u_{xx}^{5-n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx} \right) u_{xx} = -\chi_1 \int \frac{u_{xx}^{5-n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx} u_{xxx} \leq \varepsilon \int \frac{u_{xx}^{5-n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx}^2 + \frac{\lambda_1^2}{4\varepsilon} \int \frac{u_{xx}^{6-n_2}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx}^2, \tag{2.17}
\]
where since \(1 \leq n_1 \leq 2\), again due to Young’s inequality we obtain that
\[
\int \frac{u_{xx}^{6-2n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx}^2 \leq \int \frac{u_{xx}^{2-n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx}^2 \leq \int (u_{xx} + 1) v_{xx}^2 \leq \|v_{xx}\|_{L^\infty(\Omega)}^2 \int (u_{xx} + 1) \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}). \tag{2.18}
\]
Here we use that
\[
\sup_{t \in (0, \hat{T}_\varepsilon)} \left\{ \int_\Omega u_{xx} + \int_\Omega v_{xx} \right\} < \infty \tag{2.19}
\]
by Lemma 2.3 which in conjunction with the Gagliardo-Nirenberg inequality and Young’s inequality shows that (2.18) implies that with some \(c_1 > 0\) and \(c_2 > 0\) we have
\[
\frac{\lambda_1^2}{4\varepsilon} \int \frac{u_{xx}^{6-2n_1}}{u_{xx}^{4-n_1} + \varepsilon} v_{xx}^2 \leq c_1 \|v_{xx}\|_{L^\infty(\Omega)}^2
\]
so that applying quite a similar procedure to (2.11) we infer the existence of

On the basis of (1.6)-(1.7) and several integrations by parts, we compute

Nirenberg inequality, which thanks to (2.19) namely provide \( c_3 > 0 \) and \( c_4 > 0 \) fulfilling

\[
\frac{a_1^2}{2D_1} \int_\Omega u_{xx}^2 v_x^2 \leq \frac{a_1^2}{4D_1} \int_\Omega u_x^4 + \frac{a_1^2}{4D_1} \int_\Omega v_x^4 \\
\leq c_3 \left\{ \int_\Omega u_{xx}^2 \right\} \cdot \left\{ \int_\Omega u_x^2 \right\} + c_3 \left\{ \int_\Omega v_{xx}^2 \right\} \cdot \left\{ \int_\Omega v_x^2 \right\} + c_3 \left\{ \int_\Omega v_x^4 \right\}
\leq c_4 \int_\Omega u_{xx}^2 + c_4 \int_\Omega v_{xx}^2 \quad \text{for all } t \in (0, \hat{T}_\varepsilon).
\]

Combined with (2.17) and (2.20), this shows that (2.10) entails the inequality

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_{xx}^2 + \frac{D_1}{2} \int_\Omega u_{xx}^2 \leq \frac{D_2}{2} \int_\Omega v_{xx}^2 + (3\lambda_1 + c_4) \int_\Omega u_{xx}^2 + \left( \frac{c_2}{2D_2} + c_4 \right) \int_\Omega v_{xx}^2 \quad \text{for all } t \in (0, \hat{T}_\varepsilon),
\]

so that applying quite a similar procedure to (2.11) we infer the existence of \( c_5 > 0 \) such that

\[
\frac{d}{dt} \left\{ \int_\Omega u_{xx}^2 + \int_\Omega v_{xx}^2 \right\} \leq c_5 \cdot \left\{ \int_\Omega u_{xx}^2 + \int_\Omega v_{xx}^2 \right\} \quad \text{for all } t \in (0, \hat{T}_\varepsilon),
\]

which upon integration yields (2.15) and thereby also implies (2.16) due to the continuity of the embedding \( W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \).

The following conclusion of Lemma 2.6 parallels that of Lemma 2.3 from [31] in its statement, but its derivation proceeds in a slightly different manner.

**Lemma 2.7** Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and assume (1.5). Then given any \( \varepsilon \in (0, 1) \), for all \( T > 0 \) one can find \( C(\varepsilon, T) > 0 \) such that again writing \( \hat{T}_\varepsilon := \min\{T, T_{\max, \varepsilon}\} \) we have

\[
\int_\Omega \frac{1}{u_x^2(x,t)} \, dx + \int_\Omega \frac{1}{v_x^2(x,t)} \, dx \leq C(\varepsilon, T) \quad \text{for all } t \in (0, \hat{T}_\varepsilon).
\]  

**Proof.** On the basis of (1.6)-(1.7) and several integrations by parts, we compute

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_x^2 = \varepsilon \int_\Omega u_x \cdot \left( \frac{u_x^2}{u_x^{-n_1 - \varepsilon} u_{xxx}} \right) x - \varepsilon \frac{\partial}{\partial x} \int_\Omega u_x \cdot (u_x^{-\alpha} u_{xx}) x \\
- D_1 \int_\Omega u_x^2 u_{xxx} + \chi_1 \int_\Omega u_x^3 \cdot \left( \frac{u_x^{-n_1^{-1}}}{u_x^{-n_1} + \varepsilon} v_x \right) x \\
- 3 \int_\Omega \frac{1}{3u_x^2 + \varepsilon} \cdot (\lambda_1 - u_x + a_1 v_x) \\
= 3\varepsilon \int_\Omega \frac{1}{u_x^{-n_1 - \varepsilon} u_{xx}} \, dx - 3\varepsilon \int_\Omega \frac{u_x^2}{u_x^{-\alpha}} \\
\leq c_2 \|v_{xx}\|_{L^2(\Omega)} \|v_{xx}\|_{L^2(\Omega)} \\
\leq \frac{D_2}{2} \int_\Omega v_{xx}^2 + \frac{c_2^2}{2D_2} \int_\Omega v_{xx}^2 \quad \text{for all } t \in (0, \hat{T}_\varepsilon).
\]
where one more integration by parts, followed by an application of Young’s inequality, shows that for all $t \in (0, T_{\text{max}, \varepsilon})$,

$$
3 \varepsilon \int_{\Omega} \frac{1}{u_{\varepsilon}^{4-n_1} + \varepsilon} u_{\varepsilon x} u_{\varepsilon xx} = -3 \varepsilon \int_{\Omega} \frac{1}{u_{\varepsilon}^{4-n_1} + \varepsilon} u_{\varepsilon xx}^2 + 3(4 - n_1) \varepsilon \int_{\Omega} \frac{u_{\varepsilon}^{6-2n_1}}{(u_{\varepsilon}^{4-n_1} + \varepsilon)^2} u_{\varepsilon x}^2 u_{\varepsilon xx} \\
\leq -\frac{3 \varepsilon}{2} \int_{\Omega} \frac{1}{u_{\varepsilon}^{4-n_1} + \varepsilon} u_{\varepsilon xx}^2 + \frac{3(4 - n_1)^2 \varepsilon}{2} \int_{\Omega} \frac{u_{\varepsilon}^{6-2n_1}}{(u_{\varepsilon}^{4-n_1} + \varepsilon)^3} u_{\varepsilon x}^4. \quad (2.23)
$$

In order to estimate the rightmost summand herein appropriately, we recall that according to Lemma 2.6 there exist $c_1 = c_1(\varepsilon, T) > 0$ and $c_2 = c_2(\varepsilon, T) > 0$ such that

$$
\int_{\Omega} u_{\varepsilon x}^2 + \int_{\Omega} v_{\varepsilon x}^2 \leq c_1 \quad \text{for all } t \in (0, \hat{T}_\varepsilon) \quad (2.24)
$$

and

$$
u_{\varepsilon}(x, t) + v_{\varepsilon}(x, t) \leq c_2 \quad \text{for all } x \in \Omega \text{ and } t \in (0, \hat{T}_\varepsilon). \quad (2.25)
$$

The latter, namely ensures that

$$
\int_{\Omega} \frac{1}{u_{\varepsilon}^{4-n_1} + \varepsilon} u_{\varepsilon xx}^2 \geq \frac{1}{c_2^{4-n_1} + \varepsilon} \int_{\Omega} u_{\varepsilon xx}^2 \quad \text{for all } t \in (0, \hat{T}_\varepsilon), \quad (2.26)
$$

while trivially

$$
\int_{\Omega} \frac{u_{\varepsilon}^{6-2n_1}}{(u_{\varepsilon}^{4-n_1} + \varepsilon)^3} u_{\varepsilon x}^4 \leq \int_{\Omega} \frac{(u_{\varepsilon}^{4-n_1} + \varepsilon)^{6-2n_1}}{(u_{\varepsilon}^{4-n_1} + \varepsilon)^{4-n_1}} u_{\varepsilon x}^4 \\
= \int_{\Omega} (u_{\varepsilon}^{4-n_1} + \varepsilon)^{-6-n_1} u_{\varepsilon x}^4 \\
\leq \varepsilon^{-\frac{6-n_1}{4-n_1}} \int_{\Omega} u_{\varepsilon x}^4 \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \quad (2.27)
$$

by nonnegativity of $u_{\varepsilon}$. Since the Gagliardo-Nirenberg inequality provides $c_3 > 0$ such that

$$
\int_{\Omega} \varphi^4 \leq c_3 \|\varphi\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)}^{3/2} \quad \text{for all } \varphi \in W_0^{1,2}(\Omega),
$$

along with (2.24) this enables us to invoke Young’s inequality in making sure that

$$
\frac{3(4 - n_1)^2 \varepsilon}{2} \int_{\Omega} \frac{u_{\varepsilon}^{6-2n_1}}{(u_{\varepsilon}^{4-n_1} + \varepsilon)^3} u_{\varepsilon x}^4 \leq \frac{3(4 - n_1)^2 \varepsilon^{-\frac{2}{4-n_1}}}{2} \int_{\Omega} u_{\varepsilon x}^4 \\
\leq \frac{3(4 - n_1)^2 c_1^3 c_3^{-2}}{2} \varepsilon^{-\frac{2}{4-n_1}} \left\{ \int_{\Omega} u_{\varepsilon xx}^2 \right\}^{\frac{1}{2}} \\
\leq \frac{3 \varepsilon}{2(c_2^{4-n_1} + \varepsilon)} \int_{\Omega} u_{\varepsilon xx}^2 + c_4 \quad \text{for all } t \in (0, \hat{T}_\varepsilon) \quad (2.28)
$$
with \( c_4 = c_4(\varepsilon, T) := \frac{3}{8} (4 - n_1)^4 c_1^2 c_3^2 \left( c_2^{4-n_1} + \varepsilon \right) \cdot \varepsilon^{\frac{8-n_1}{4-n_1}} \).

Next, in the second last summand in (2.22) we use Young’s inequality to see that again thanks to Lemma 2.8, this can be obtained by verbatim copying the proof.

Again referring to a corresponding argument from [31, Lemma 2.4], we may refrain from giving details concerning the derivation of \( H^2 \) estimates of the following flavor.

\[
3 \chi_1 \int \Omega \frac{1}{u_{11}^{n-1} - (u_{11}^{4-n_1} + \varepsilon)} u_{11} u_{11x} v_{11x} \leq 3D_1 \int \frac{u_{11x}^2}{u_{11}^2} + \frac{3 \chi_1^2}{4D_1} \int \frac{u_{11x}^{6-2n_1}}{(u_{11}^{4-n_1} + \varepsilon)^2} v_{11x}^2 \leq 3D_1 \int \frac{u_{11x}^2}{u_{11}^2} + \frac{3 \chi_1^2}{4D_1} \int \frac{u_{11x}^{6-2n_1}}{(u_{11}^{4-n_1} + \varepsilon)^2} v_{11x}^2 \leq 3D_1 \int \frac{u_{11x}^2}{u_{11}^2} + c_5 \quad \text{for all } t \in (0, \hat{T}_\varepsilon)
\]

(2.29)

Finally, in the last integral in (2.22) we may use (2.25) to achieve the rough estimate

\[
|\lambda_1 - u_\varepsilon + a_1 v_\varepsilon| \leq \lambda_1 + (a_1 + 1) c_2 \quad \text{in } \Omega \times (0, \hat{T}_\varepsilon),
\]

which ensures that

\[
-3 \int \Omega \frac{1}{u_{11}^2 + \varepsilon} \cdot (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \leq c_6 = c_6(\varepsilon, T) := \frac{3(\lambda_1 + (a_1 + 1) c_1)|\Omega|}{\varepsilon} \quad \text{for all } t \in (0, \hat{T}_\varepsilon).
\]

In combination with (2.28), (2.26), (2.28) and (2.29), this shows that (2.22) entails the inequality

\[
\frac{1}{2} \frac{d}{dt} \int \Omega \frac{1}{u_{11}^2} \leq c_4 + c_5 + c_6 \quad \text{for all } t \in (0, \hat{T}_\varepsilon)
\]

and thereby yields the claimed estimate for \( u_\varepsilon \), while that for \( v_\varepsilon \) can be derived similarly. \( \square \)

As well-known from [31], the estimates from (2.6) and Lemma 2.7 in combination yield pointwise lower bounds for both solution components:

**Lemma 2.8** Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and let (1.5) hold. Then for all \( \varepsilon \in (0, 1) \) and \( T > 0 \) there exists \( C(\varepsilon, T) > 0 \) such that

\[
u_\varepsilon(x, t) \geq C(\varepsilon, T) \quad \text{and} \quad v_\varepsilon(x, t) \geq C(\varepsilon, T) \quad \text{for all } x \in \Omega \text{ and } t \in (0, \hat{T}_\varepsilon),
\]

(2.30)

where again \( \hat{T}_\varepsilon := \min\{T, T_{\max, \varepsilon}\} \).

**Proof.** Based on Lemma 2.6 with Lemma 2.7, this can be obtained by verbatim copying the argument from [31, Lemma 2.4]. \( \square \)

Again referring to a corresponding argument from [31], we may refrain from giving details concerning the derivation of \( H^2 \) estimates of the following flavor.
Lemma 2.9 Let $D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0$ and $n_i \in [1, 2]$ for $i \in \{1, 2\}$, and let (1.5) hold. Then for all $\varepsilon \in (0, 1)$ and $T > 0$ one can find $C(\varepsilon, T) > 0$ fulfilling

$$\int_{\Omega} u_{\varepsilon xx}^2(x,t)dx + \int_{\Omega} v_{\varepsilon xx}^2(x,t)dx \leq C(\varepsilon, T) \quad \text{for all } t \in (0, \hat{T}_\varepsilon),$$

(2.31)

where again $\hat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. This estimate can be derived by testing the first two equations in (1.6) against $u_{\varepsilon xxxx}$ and $v_{\varepsilon xxxx}$ and estimating all appearing ill-signed lower-order integrals in terms of the respective fourth-order dissipative summands, using thanks to Lemma 2.8 and Lemma 2.6 the latter are bounded from below by suitable positive multiples of $\int_{\Omega} u_{\varepsilon xxxx}^2$ and $\int_{\Omega} v_{\varepsilon xxxx}^2$. For a corresponding argument in a specialized setting without the kinetic terms from (1.6) we refer to [31, Lemma 2.5], and we may omit detailing the minor adaptations necessary in the present framework. □

In view of (2.1), collecting our $\varepsilon$-dependent regularity information we can now make sure that indeed all our approximate solutions actually exist globally in time.

Lemma 2.10 Let $D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0$ and $n_i \in [1, 2]$ for $i \in \{1, 2\}$, and suppose that (1.5) is satisfied. Then for all $\varepsilon \in (0, 1)$ we have $T_{max,\varepsilon} = \infty$; that is the solution $(u_{\varepsilon}, v_{\varepsilon})$ of (1.6)-(1.7) from Lemma 2.1 is global in time.

Proof. This directly results from a combination of Lemma 2.9, Lemma 2.8, Lemma 2.3 and (2.1). □

3 A quasi-entropy structure reminiscent of (1.8)

We next intend to analyze a regularization-adapted modification of the functional in (1.8). Here in contrast to the simplified setup in [31] in which the absence of kinetic terms implies a genuine Lyapunov property, in the present context the appearance of the zero-order terms in (1.6) requires substantial additional efforts. These will be prepared by the following basic observation.

Lemma 3.1 Let $\nu \in [0, 2]$. Then

$$\frac{s^\nu}{3s^2 + \varepsilon} \leq \frac{1}{2} \left(\frac{\nu}{3}\right)^{\frac{\nu}{2}} \cdot (2 - \nu)^{\frac{2-\nu}{2}} \varepsilon^{-\frac{2-\nu}{2}} \quad \text{for all } s \geq 0 \text{ and any } \varepsilon \in (0, 1).$$

(3.1)

Proof. In the case $\nu = 2$, (3.1) follows from the simple estimate

$$\frac{s^\nu}{3s^2 + \varepsilon} \leq \frac{s^\nu}{3s^2} = \frac{1}{3} \quad \text{for all } s \geq 0 \text{ and any } \varepsilon \in (0, 1).$$

When $\nu \in [0, 2)$, the function $\varphi(s) := \frac{s^\nu}{3s^2 + \varepsilon}$, $s \geq 0$, satisfies $\varphi'(s) = \frac{-3(2-\nu)s^{\nu+1} + \nu s^{\nu-1}}{(3s^2 + \varepsilon)^2}$ for all $s > 0$, whence $\varphi$ attains its maximum at $s_0 := \sqrt[\nu]{\frac{3(2-\nu)}{\nu\varepsilon}}$ with

$$\varphi(s_0) = \left(\frac{\nu}{3(2-\nu)}\right)^{\frac{\nu}{2}} \varepsilon^\frac{\nu}{2} = \frac{1}{2} \left(\frac{\nu}{3}\right)^{\frac{\nu}{2}} \cdot (2 - \nu)^{\frac{2-\nu}{2}} \varepsilon^{-\frac{2-\nu}{2}},$$

and hence

$$\frac{s^\nu}{3s^2 + \varepsilon} \leq \frac{1}{2} \left(\frac{\nu}{3}\right)^{\frac{\nu}{2}} \cdot (2 - \nu)^{\frac{2-\nu}{2}} \varepsilon^{-\frac{2-\nu}{2}} \quad \text{for all } s \geq 0 \text{ and any } \varepsilon \in (0, 1).$$

(3.1)
which verifies (3.1) also in this case.

We can now make sure that if in addition to the above we assume \( n_1 \geq 1 \) and \( n_2 \geq 1 \), then our approximation in (1.6) indeed cooperates with a fundamental structural property of (1.4) in the following sense.

**Lemma 3.2** Assume that \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \). Then there exists \( C > 0 \) such that whenever \( \chi_1 > 0 \) and \( \chi_2 > 0 \) and (1.4) holds, for all \( \varepsilon \in (0, 1) \) the functions \( F_\varepsilon \) and \( D_\varepsilon \) defined by

\[
F_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) - \int_\Omega u_\varepsilon(\cdot, t) + \frac{\varepsilon}{(3-n_1)(4-n_1)} \int_\Omega \frac{1}{u_\varepsilon^{3-n_1}(\cdot, t)}
+ \frac{\chi_1}{\chi_2} \int_\Omega v_\varepsilon(\cdot, t) \ln v_\varepsilon(\cdot, t) - \frac{\chi_1}{\chi_2} \int_\Omega v_\varepsilon(\cdot, t) + \frac{\chi_1 \varepsilon}{(3-n_2)(4-n_2) \chi_2} \int_\Omega \frac{1}{v_\varepsilon^{3-n_2}(\cdot, t)}, \quad t \geq 0, \quad (3.2)
\]

and

\[
D_\varepsilon(t) := \frac{D_1}{2} \int_\Omega u_\varepsilon(\cdot, t) + \varepsilon \int_\Omega u_\varepsilon^{n_1-1}(\cdot, t) u_\varepsilon u_\varepsilon(\cdot, t) + D_1 \varepsilon \int_\Omega \frac{u_\varepsilon^2(\cdot, t)}{u_\varepsilon^{3-n_1}(\cdot, t)}
+ \frac{\chi_1 D_2}{2 \chi_2} \int_\Omega v_\varepsilon^2(\cdot, t) + \frac{\chi_1 \varepsilon}{\chi_2} \int_\Omega v_\varepsilon^{n_2-1}(\cdot, t) v_\varepsilon^2(\cdot, t) + \frac{\chi_1 D_2 \varepsilon}{\chi_2} \int_\Omega \frac{v_\varepsilon^2(\cdot, t)}{v_\varepsilon^{3-n_2}(\cdot, t)}, \quad t > 0, \quad (3.3)
\]

satisfy

\[
\frac{d}{dt} F_\varepsilon(t) + D_\varepsilon(t) \leq C \cdot \left( 1 + \frac{\chi_1}{\chi_2} \right) \left\{ 1 + \left\{ \int_\Omega u_\varepsilon(\cdot, t) \right\}^7 + \left\{ \int_\Omega v_\varepsilon(\cdot, t) \right\}^7 \right\} \quad \text{for all } t > 0. \quad (3.4)
\]

**Proof.** For \( i \in \{1, 2\} \) abbreviating

\[
L_i(s) := s \ln s - s + \frac{\varepsilon}{(4-n_i)(3-n_i)} \cdot \frac{1}{s^{3-n_i}}, \quad s > 0,
\]

we see that

\[
L_i'(s) = \ln s - \frac{\varepsilon}{4-n_i} \cdot \frac{1}{s^{4-n_i}} \quad \text{for all } s > 0 \quad (3.5)
\]

and

\[
L_i''(s) = \frac{1}{s} + \frac{\varepsilon}{s^{3-n_i}} \quad \text{for all } s > 0, \quad (3.6)
\]

and we observe that thus, in particular,

\[
\frac{s^4}{s^{4-n_i} + \varepsilon} \cdot L_i''(s) = s^{n_i-1} \quad \text{for all } s > 0 \quad (3.7)
\]

and hence

\[
\frac{d^2}{ds^2} \left( \frac{s^4}{s^{4-n_i} + \varepsilon} L_i''(s) \right) = (n_i - 1)(n_i - 2)s^{n_i-3} \leq 0 \quad \text{for all } s > 0 \quad (3.8)
\]
according to our assumption that $n_i \in [1, 2]$. In order to make appropriate use of this, we now go back to (1.6)-(1.7) and integrate by parts in computing
\[
\frac{d}{dt} \int_{\Omega} L_1(u_\varepsilon) = \int_{\Omega} L'_1(u_\varepsilon) u_{\varepsilon t} = -\varepsilon \int_{\Omega} L'_1(u_\varepsilon) \cdot \left( \frac{u_\varepsilon^4}{u_\varepsilon^{-n_1} + \varepsilon} \right) u_{\varepsilon xxx} + \varepsilon^2 \int_{\Omega} L''_1(u_\varepsilon) \cdot (u_\varepsilon^{-\alpha} u_{\varepsilon x})_x \\
+ D_1 \int_{\Omega} L'_1(u_\varepsilon) u_{\varepsilon xx} - \chi_1 \frac{u_\varepsilon^{5-n_1}}{u_\varepsilon^{4-n_1} + \varepsilon} L''_1(u_\varepsilon) u_{\varepsilon x v_x xx} + \int_{\Omega} L'_1(u_\varepsilon) \cdot \frac{3u_\varepsilon^3}{2} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon)
\]
where (3.6) and (3.5) directly yield
\[
-\varepsilon^\alpha \int_{\Omega} u_\varepsilon^{-\alpha} L''_1(u_\varepsilon) u_{\varepsilon x}^2 \leq 0 \quad \text{for all } t > 0 \quad (3.10)
\]
and
\[
-D_1 \int_{\Omega} L''_1(u_\varepsilon) u_{\varepsilon x}^2 = -D_1 \int_{\Omega} \frac{u_\varepsilon^2}{u_\varepsilon} - D_1 \varepsilon \int_{\Omega} \frac{u_\varepsilon^2}{u_\varepsilon^{5-n_1}} \quad \text{for all } t > 0 \quad (3.11)
\]
as well as
\[
\chi_1 \int_{\Omega} \frac{u_\varepsilon^{5-n_1}}{u_\varepsilon^{4-n_1} + \varepsilon} L''_1(u_\varepsilon) u_{\varepsilon x v_x} = \chi_1 \int_{\Omega} u_{\varepsilon x v_x} \quad \text{for all } t > 0 \quad (3.12)
\]
and
\[
\int_{\Omega} L'_1(u_\varepsilon) \cdot \frac{3u_\varepsilon^3}{2} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) = 3 \int_{\Omega} \frac{u_\varepsilon^2}{3u_\varepsilon^{2} + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \\
- \frac{3\varepsilon}{4-n_1} \int_{\Omega} \frac{u_\varepsilon^{4-n_1-1}}{3u_\varepsilon^{2} + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \quad \text{for all } t > 0 \quad (3.13)
\]
In the first summand on the right of (3.9), we integrate by parts two more times to find that thanks to (3.8) and (3.7),
\[
\varepsilon \int_{\Omega} \frac{u_\varepsilon^4}{u_\varepsilon^{4-n_1} + \varepsilon} \cdot L''_1(u_\varepsilon) u_{\varepsilon xxx} = -\varepsilon \int_{\Omega} \frac{u_\varepsilon^4}{u_\varepsilon^{4-n_1} + \varepsilon} \cdot L''_1(u_\varepsilon) u_{\varepsilon xxx}
\]
so that collecting \(3.9\)–\(3.14\) we obtain

\[
\frac{d}{dt} \int_{\Omega} L_1(u_\epsilon) + \epsilon \int_{\Omega} \frac{u_\epsilon^{n_1-1}u_{\epsilon xx}}{u_\epsilon} + D_1 \int_{\Omega} \frac{u_{\epsilon x}^2}{u_\epsilon} + D_1 \epsilon \int_{\Omega} \frac{u_{\epsilon x}^2}{u_{\epsilon x}^{n_1-1}} \\
\leq \chi_1 \int_{\Omega} u_{\epsilon x} u_{\epsilon x} \\
+ 3 \int_{\Omega} \frac{u_\epsilon^3 \ln u_\epsilon}{3u_\epsilon^2 + \epsilon} (\lambda_1 - u_\epsilon + a_1 v_\epsilon) - \frac{3 \epsilon}{4 - n_1} \int_{\Omega} \frac{u_\epsilon^{n_1-1}}{3u_\epsilon^2 + \epsilon} (\lambda_1 - u_\epsilon + a_1 v_\epsilon) \quad \text{for all } t > 0.
\]

As similarly

\[
\frac{d}{dt} \int_{\Omega} L_2(v_\epsilon) + \epsilon \int_{\Omega} \frac{v_\epsilon^{n_2-1}v_{\epsilon xx}}{v_\epsilon} + D_2 \int_{\Omega} \frac{v_{\epsilon x}^2}{v_\epsilon} + D_2 \epsilon \int_{\Omega} \frac{v_{\epsilon x}^2}{v_{\epsilon x}^{n_2-1}} \\
\leq \chi_2 \int_{\Omega} u_{\epsilon x} u_{\epsilon x} \\
+ 3 \int_{\Omega} \frac{v_\epsilon^3 \ln v_\epsilon}{3v_\epsilon^2 + \epsilon} (\lambda_2 - v_\epsilon - a_2 u_\epsilon) - \frac{3 \epsilon}{4 - n_2} \int_{\Omega} \frac{v_\epsilon^{n_2-1}}{3v_\epsilon^2 + \epsilon} (\lambda_2 - v_\epsilon - a_2 u_\epsilon) \quad \text{for all } t > 0,
\]
on taking a suitable linear combination of the latter two relations we can achieve a cancellation of the respective cross-diffusive contributions and thereby obtain the inequality

\[
\frac{d}{dt} \mathcal{F}_\epsilon(t) + \mathcal{D}_\epsilon(t) = \frac{d}{dt} \left\{ \int_{\Omega} L_1(u_\epsilon) + \frac{\chi_1}{\chi_2} \int_{\Omega} L_2(v_\epsilon) \right\} + \mathcal{D}_\epsilon(t) \\
\leq -\frac{D_1}{2} \int_{\Omega} \frac{u_{\epsilon x}^2}{u_\epsilon} - \frac{\chi_1 D_2}{2 \chi_2} \int_{\Omega} \frac{v_{\epsilon x}^2}{v_\epsilon} \\
+ 3 \int_{\Omega} \frac{u_\epsilon^3 \ln u_\epsilon}{3u_\epsilon^2 + \epsilon} (\lambda_1 - u_\epsilon + a_1 v_\epsilon) - \frac{3 \epsilon}{4 - n_1} \int_{\Omega} \frac{u_\epsilon^{n_1-1}}{3u_\epsilon^2 + \epsilon} (\lambda_1 - u_\epsilon + a_1 v_\epsilon) \\
+ \frac{3 \chi_1}{\chi_2} \int_{\Omega} \frac{v_\epsilon^3 \ln v_\epsilon}{3v_\epsilon^2 + \epsilon} (\lambda_2 - v_\epsilon - a_2 u_\epsilon) - \frac{3 \chi_1 \epsilon}{4 - n_2 \chi_2} \int_{\Omega} \frac{v_\epsilon^{n_2-1}}{3v_\epsilon^2 + \epsilon} (\lambda_2 - v_\epsilon - a_2 u_\epsilon) \quad \text{(3.15)}
\]

for all \( t > 0 \). Here clearly

\[
3 \int_{\Omega} \frac{u_\epsilon^3 \ln u_\epsilon}{3u_\epsilon^2 + \epsilon} (\lambda_1 - u_\epsilon + a_1 v_\epsilon) \leq \lambda_1 \int_{\{u_\epsilon \geq 1\}} u_\epsilon \ln u_\epsilon - \int_{\{u_\epsilon \leq 1\}} u_\epsilon^2 \ln u_\epsilon + a_1 \int_{\{u_\epsilon \geq 1\}} u_\epsilon v_\epsilon \ln u_\epsilon
\]
for all $t > 0$, so that using the estimates

$$\ln \xi \leq 2\sqrt{\xi} \quad \text{for} \quad \xi \geq 1 \quad \text{and} \quad \xi^2 \ln \xi \geq -\frac{1}{2e} \quad \text{for} \quad \xi \in (0, 1), \quad (3.16)$$

by means of Young’s inequality we infer that

$$3 \int_{\Omega} u^2 \ln u (\lambda_1 - u + a_1 v) \leq 2 \lambda_1 \int_{\Omega} u^2 + \frac{|\Omega|}{2e} + 2a_1 \int_{\Omega} u^2 v$$

$$\leq 2 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \|u\|_{L^2(\Omega)}^3 + \frac{|\Omega|}{2e} \quad \text{for all} \quad t > 0. \quad (3.17)$$

Since the Gagliardo-Nirenberg inequality provides $c_1 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)}^3 \leq c_1 \|\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)} + c_1 \|\varphi\|_{L^2(\Omega)}^3 \quad \text{for all} \quad \varphi \in W^{1,2}(\Omega),$$

by several applications of Young’s inequality we can herein estimate

$$2 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \|u\|_{L^2(\Omega)}^3 \leq 2c_1 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$+ 2c_1 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$\leq 2D_1 \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\} \cdot \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$+ 2c_1 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$= \frac{D_1}{2} \int_{\Omega} u^2 + \frac{2c_1^4}{D_1^2} \cdot \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$+ 2c_1 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$\leq \frac{D_1}{2} \int_{\Omega} u^2 + \frac{2c_1^4}{D_1^2} + 2c_1 \cdot \left\{ \lambda_1^7 |\Omega| + a_1^7 \left\{ \int_{\Omega} v \right\} \right\}$$

$$+ 2c_1 \left\{ \lambda_1 |\Omega| + a_1 \int_{\Omega} v \right\} \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$\leq \frac{D_1}{2} \int_{\Omega} u^2 + \frac{2c_1^4}{D_1^2} + 2c_1 \cdot \left\{ \left( \int_{\Omega} u \right)^{\frac{3}{2}} \right\}$$

$$+ 2c_1 \left\{ \lambda_1^7 |\Omega| + a_1^7 \left\{ \int_{\Omega} v \right\} \right\} \quad (3.18)$$
for all \( t > 0 \). Next, noting that again thanks to (3.16),

\[
-\frac{v_\varepsilon^3 \ln v_\varepsilon}{3v_\varepsilon^2 + \varepsilon} \cdot v_\varepsilon \leq -\frac{1}{3} v_\varepsilon^2 \ln v_\varepsilon \leq \frac{1}{6\varepsilon} \quad \text{if } v_\varepsilon \leq 1,
\]

that

\[
\frac{v_\varepsilon^3 \ln v_\varepsilon}{3v_\varepsilon^2 + \varepsilon} (\lambda_2 - v_\varepsilon) \leq 0 \quad \text{if } v_\varepsilon \geq \max\{1, \lambda_2\}
\]

and that

\[
\frac{v_\varepsilon^3 \ln v_\varepsilon}{3v_\varepsilon^2 + \varepsilon} \leq \frac{1}{3} v_\varepsilon \ln v_\varepsilon \leq \frac{2}{3} v_\varepsilon \frac{a}{\varepsilon \chi_2} \leq \frac{2}{3} \lambda_2^+ \quad \text{if } 1 < v_\varepsilon < \lambda_2,
\]

writing \( c_2 := \max\{\frac{1}{6\varepsilon}, \frac{5}{3} \lambda_2^+\} \) we see that

\[
\frac{3\chi_2}{\varepsilon} \int_\Omega \frac{v_\varepsilon^3 \ln v_\varepsilon}{3v_\varepsilon^2 + \varepsilon} (\lambda_2 - v_\varepsilon - a_1 u_\varepsilon) \leq \frac{3\chi_2 c_2 |\Omega|}{\chi_2} \int_\Omega \frac{u_\varepsilon v_\varepsilon \ln v_\varepsilon}{\chi_2} \leq \frac{3\chi_2 c_2 |\Omega|}{\chi_2} + \frac{\chi_2 a_1}{\varepsilon \chi_2} \int_\Omega u_\varepsilon \leq \frac{3\chi_2 c_2 |\Omega|}{\chi_2} + \frac{\chi_2 a_1}{\varepsilon \chi_2} \left\{ \int_\Omega u_\varepsilon \right\}^7 + \frac{\chi_2 a_1}{\varepsilon \chi_2} \quad \text{for all } t > 0 (3.19)
\]

due to the inequality \(-\xi \ln \xi \leq \frac{1}{\xi^2}\) for \( \xi \in (0, 1) \). In the last and the third last summand in (3.15) we make use of Lemma 3.1 which when applied to \( \nu = n_1 \) entails that

\[
-\frac{3\varepsilon}{4 - n_1} \int_\Omega \frac{u_\varepsilon^{n_1-1}}{3u_\varepsilon^2 + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \leq \frac{3\varepsilon}{4 - n_1} \int_\Omega \frac{u_\varepsilon^{n_1}}{3u_\varepsilon^2 + \varepsilon} \leq \frac{3\varepsilon}{4 - n_1} \cdot \frac{1}{2} \left( \frac{n_1}{3} \right)^{\frac{n_1}{2}} (2 - n_1)^{\frac{n_1}{2}} \varepsilon^{-\frac{n_1}{2} \frac{n_1}{2} |\Omega|}
\]

\[
= \frac{[3(2 - n_1)]^{\frac{n_1}{2}} n_1^{\frac{n_1}{2}} |\Omega|}{2(4 - n_1)^{\frac{n_1}{2}}} \leq \frac{[3(2 - n_1)]^{\frac{n_1}{2}} n_1^{\frac{n_1}{2}} |\Omega|}{2(4 - n_1)^{\frac{n_1}{2}}} \quad \text{for all } t > 0 (3.20)
\]

because of our restriction that \( \varepsilon < 1 \). Twice more employing Lemma 3.1 with \( \nu = n_2 \) and \( \nu = n_2 - 1 \), respectively, furthermore shows that again due to Young’s inequality,

\[
-\frac{3\chi_2 \varepsilon}{(4 - n_2)\chi_2} \int_\Omega \frac{v_\varepsilon^{n_2-1}}{3v_\varepsilon^2 + \varepsilon} (\lambda_2 - v_\varepsilon - a_2 u_\varepsilon)
\]

\[
\leq \frac{3\chi_2 \varepsilon}{(4 - n_2)\chi_2} \int_\Omega \frac{v_\varepsilon^{n_2}}{3v_\varepsilon^2 + \varepsilon} + \frac{3\chi_2 \varepsilon}{(4 - n_2)\chi_2} \int_\Omega \frac{v_\varepsilon^{n_2-1}}{3v_\varepsilon^2 + \varepsilon} \cdot u_\varepsilon
\]

\[
\leq \frac{3\chi_2 \varepsilon}{(4 - n_2)\chi_2} \cdot \frac{1}{2} \left( \frac{n_2}{3} \right)^{\frac{n_2}{2}} (2 - n_2)^{\frac{n_2}{2}} \varepsilon^{-\frac{n_2}{2} \frac{n_2}{2} |\Omega|}
\]
Proof. Again relying on the inequality}
\[
\int_{\Omega} u_\varepsilon = 
\]

Moreover,
\[
\int_{\Omega} \chi (\cdot, t) \, u_\varepsilon + \frac{\chi_1}{\chi_2} \int_{\Omega} \chi_1 \varepsilon \int_{\Omega} \mathcal{F}_\varepsilon (t) = C \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \left\{1 + \int_{\Omega} u_\varepsilon (\cdot, t) + \int_{\Omega} v_\varepsilon (\cdot, t)\right\} \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). (3.22)
\]

Moreover,
\[
\sup_{\varepsilon \in (0, 1)} \mathcal{F}_\varepsilon (0) < \infty. (3.23)
\]

**Proof.** Again relying on the inequality $\xi \ln \xi \geq -\frac{1}{e}$ for $\xi \in (0, e]$, for all $t > 0$ and $\varepsilon \in (0, 1)$ we can estimate
\[
\int_{\Omega} u_\varepsilon \ln (u_\varepsilon + e) = \int_{\{u_\varepsilon \geq e\}} u_\varepsilon \ln (u_\varepsilon + e) + \int_{\{u_\varepsilon < e\}} u_\varepsilon \ln (u_\varepsilon + e)
\]
\[
\leq \int_{\{u_\varepsilon \geq e\}} u_\varepsilon \cdot (\ln u_\varepsilon + \ln 2) + \ln (2e) \int_{\{u_\varepsilon < e\}} u_\varepsilon
\]
\[
\leq \int_{\Omega} u_\varepsilon \ln u_\varepsilon + (\ln 2 + \ln (2e)) \int_{\Omega} u_\varepsilon + \frac{\Omega}{e},
\]

and performing the same operations on $v_\varepsilon$ we obtain that by (3.22),
\[
\int_{\Omega} u_\varepsilon \ln (u_\varepsilon + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \ln (v_\varepsilon + e)
\]

for all $t > 0$, where we have once more used that $\varepsilon < 1$, and that $n_2 \geq 1$.

In summary, we only need to insert (3.18) into (3.17), combine the latter with (3.19), (3.20) and (3.21), and to particularly observe the dependence of the obtained estimate on $\chi_1$ and $\chi_2$ exclusively through the factor $1 + \frac{\chi_1}{\chi_2}$, to infer from (3.15) that indeed (3.4) holds with an appropriate choice of $C$. □

In order to prepare appropriate integration of (3.4) for the purpose of deriving estimates essential for our existence proof (Lemma 4.1), but also later in our qualitative analysis (Lemma 5.4), let us note the following relationship between $\mathcal{F}_\varepsilon$ and the nonnegative quantities $\int_{\Omega} u_\varepsilon \ln (u_\varepsilon + e)$ and $\int_{\Omega} v_\varepsilon \ln (v_\varepsilon + e)$.

**Lemma 3.3** Let $D_i > 0, a_i > 0, \lambda_i > 0$ and $n_i \in [1, 2]$ for $i \in \{1, 2\}$. Then there exists $C > 0$ such that given any $\chi_1 > 0$ and $\chi_2 > 0$ and initial data such that (1.5) holds, with $\mathcal{F}_\varepsilon$ as in (2.2) we have
\[
\int_{\Omega} u_\varepsilon (\cdot, t) \ln (u_\varepsilon (\cdot, t) + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon (\cdot, t) \ln (v_\varepsilon (\cdot, t) + e)
\]
\[
\leq \mathcal{F}_\varepsilon (t) + C \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \left\{1 + \int_{\Omega} u_\varepsilon (\cdot, t) + \int_{\Omega} v_\varepsilon (\cdot, t)\right\} \quad \text{for all } t > 0 \quad \text{and } \varepsilon \in (0, 1). (3.22)
\]

Moreover,
\[
\sup_{\varepsilon \in (0, 1)} \mathcal{F}_\varepsilon (0) < \infty. (3.23)
\]
\[
\begin{align*}
&\leq \int_{\Omega} u_\varepsilon \ln u_\varepsilon + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \ln v_\varepsilon \\
&\quad + (\ln 2 + \ln(2e)) \cdot \left\{ \int_{\Omega} u_\varepsilon + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \right\} + \left( 1 + \frac{\chi_1}{\chi_2} \right) \cdot \frac{|\Omega|}{e} \\
&\leq \mathcal{F}_\varepsilon(t) + (\ln 2 + \ln(2e) + 1) \cdot \left\{ \int_{\Omega} u_\varepsilon + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \right\} + \left( 1 + \frac{\chi_1}{\chi_2} \right) \cdot \frac{|\Omega|}{e}
\end{align*}
\]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \), which establishes \((3.22)\).

Apart from that, since \( u_0 \) and \( v_0 \) are positive, and since \((1.5)\) requires that \( u_0, v_0 \to u_0, v_0 \) in \( L^\infty(\Omega) \) as \( \varepsilon \searrow 0 \), it is evident from \((3.2)\) that \( \mathcal{F}_\varepsilon(0) \to \int_\Omega u_0 \ln u_0 - \int_\Omega u_0 + \frac{\lambda_1}{\chi_2} \int_\Omega v_0 \ln v_0 - \frac{\lambda_1}{\chi_2} \int_\Omega v_0 \) as \( \varepsilon \searrow 0 \), which clearly entails \((3.23)\). \( \square \)

## 4 Global existence. Proof of Theorem 1.1

Before proceeding, let us now sharpen our objective by specifying our solution concept in the following natural sense.

### Definition 4.1

For \( i \in \{1, 2\} \), let \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \), and let \( u_0 \in L^1(\Omega) \) and \( v_0 \in L^1(\Omega) \) be nonnegative. Then a pair \((u, v)\) of nonnegative measurable functions defined in \( \Omega \times (0, \infty) \) and satisfying

\[
\{ u^2, v^2, u_x, v_x, uv_x, vu_x \} \subset L^1_{loc}(\overline{\Omega} \times [0, \infty))
\]

will be called a global weak solution of \((1.4)\) if

\[
-\int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = -D_1 \int_0^\infty \int_\Omega u_x \varphi_x + \chi_1 \int_0^\infty \int_\Omega uv_x \varphi_x + \int_0^\infty \int_\Omega u(\lambda_1 - u + a_1 v) \varphi
\]

and

\[
-\int_0^\infty \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = -D_2 \int_0^\infty \int_\Omega v_x \varphi_x - \chi_2 \int_0^\infty \int_\Omega vu_x \varphi_x + \int_0^\infty \int_\Omega v(\lambda_2 - v - a_1 u) \varphi
\]

for all \( \varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty)) \).

Based on Lemma 3.2 in this section we shall derive estimates for solutions to \((1.6)-(1.7)\) which allow for an appropriate subsequence extraction process, in the limit yielding a solution in the above sense. As our approach parallels that in \([31]\) in some parts, we may concentrate on the essential and novel aspects here, beyond this restricting ourselves to outlining the main steps.

### 4.1 A first integration of \((3.4)\): Global regularity properties for fixed \( \chi_i \)

Using the \( L^1 \) bounds provided by Lemma 2.3 for arbitrary but fixed cross-diffusion coefficients we obtain the following from a first integration of \((3.4)\) in a straightforward manner, emphasizing that throughout this section all appearing constants may depend on \( \chi_1 \) and \( \chi_2 \).
Lemma 4.1 For $i \in \{1, 2\}$, let $D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0$ and $n_i \in [1, 2]$, and assume (1.3). Then for all $T > 0$ there exists $C(T) > 0$ such that for any $\varepsilon \in (0, 1)$,

$$\int_{\Omega} u_\varepsilon(\cdot, t) \ln(u_\varepsilon(\cdot, t) + e) + \int_{\Omega} v_\varepsilon(\cdot, t) \ln(v_\varepsilon(\cdot, t) + e) \leq C(T) \quad \text{for all } t \in (0, T),$$

(4.4)

that with $F_\varepsilon$ as in (3.2) we have

$$F_\varepsilon(t) \leq C(T) \quad \text{for all } t \in (0, T),$$

(4.5)

and that

$$\int_0^T \int_{\Omega} \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \int_0^T \int_{\Omega} \frac{v_{\varepsilon x}^2}{v_\varepsilon} \leq C(T)$$

(4.6)

as well as

$$\varepsilon \int_0^T \int_{\Omega} u_\varepsilon^{n_1 - 1} u_{\varepsilon xx} + \varepsilon \int_0^T \int_{\Omega} v_\varepsilon^{n_2 - 1} v_{\varepsilon xx} \leq C(T).$$

(4.7)

Proof. We employ Lemma 2.3 to find $c_1 > 0$ such that

$$\int_{\Omega} u_\varepsilon + \int_{\Omega} v_\varepsilon \leq c_1 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

(4.8)

and observe that therefore Lemma 3.2 implies that with some $c_2 > 0$, the functions $F_\varepsilon$ and $D_\varepsilon$ in (3.2) and (3.3) satisfy

$$\frac{d}{dt} F_\varepsilon(t) + D_\varepsilon(t) \leq c_2 \quad \text{for all } t > 0 \text{ and any } \varepsilon \in (0, 1)$$

and hence, upon integration,

$$F_\varepsilon(t) + \int_0^t D_\varepsilon(s)ds \leq F_\varepsilon(0) + c_2 t \quad \text{for all } t > 0 \text{ and each } \varepsilon \in (0, 1).$$

(4.9)

Thus, if in accordance with Lemma 3.3 and (4.8) we pick $c_3 > 0$ and $c_4 > 0$ large enough such that

$$F_\varepsilon(0) \leq c_3 \quad \text{for all } \varepsilon \in (0, 1)$$

and

$$\int_{\Omega} u_\varepsilon \ln(u_\varepsilon + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \ln(v_\varepsilon + e) \leq F_\varepsilon(t) + c_4 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

then from (4.9) we infer that

$$\int_{\Omega} u_\varepsilon \ln(u_\varepsilon + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_\varepsilon \ln(v_\varepsilon + e) + \int_0^t D_\varepsilon(s)ds \leq F_\varepsilon(t) + c_4 + \int_0^t D_\varepsilon(s)ds \leq c_2 t + c_3 + c_4 \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

from which in view of (3.3) both (4.4) and (4.5) as well as (4.6) and (4.7) directly follow.

Further consequences can be drawn by means of the following interpolation inequality which can be found in [31, Corollary 7.6].
Lemma 4.2 There exists \( C > 0 \) such that for all \( \varphi \in W^{1,2}(\Omega) \) satisfying \( \varphi > 0 \) in \( \Omega \),
\[
\int_{\Omega} \varphi^3 \ln(\varphi + e) \leq C \cdot \left\{ \int_{\Omega} \frac{\varphi^2}{\varphi} \right\} \cdot \left\{ \int_{\Omega} \varphi \ln(\varphi + e) \right\}^2 + C \cdot \left\{ \int_{\Omega} \varphi \ln(\varphi + e) \right\}^3.
\]
Inter alia by utilizing the latter, from Lemma 4.1 we can derive three more spatio-temporal estimates, the first and the third among which will yield precompactness of the cross-diffusive fluxes from (1.6) in spatio-temporal \( L^1 \) spaces, and the second of which will play an important role in our asymptotic analysis by implying positive lower bounds for the mass functionals in Lemma 5.3.

Corollary 4.3 Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and let (4.17) be fulfilled. Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that
\[
\int_{0}^{T} \int_{\Omega} u_\varepsilon^3 \ln(u_\varepsilon + e) + \int_{0}^{T} \int_{\Omega} v_\varepsilon^3 \ln(v_\varepsilon + e) \leq C(T) \quad \text{for all} \; \varepsilon \in (0, 1) \tag{4.10}
\]
and
\[
\int_{0}^{T} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 dt + \int_{0}^{T} \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 dt \leq C(T) \quad \text{for all} \; \varepsilon \in (0, 1) \tag{4.11}
\]
as well as
\[
\int_{0}^{T} \int_{\Omega} |u_{\varepsilon x}|^2 + \int_{0}^{T} \int_{\Omega} |v_{\varepsilon x}|^2 \leq C(T) \quad \text{for all} \; \varepsilon \in (0, 1). \tag{4.12}
\]

Proof. Given \( T > 0 \), we first apply Lemma 4.1 to find \( c_1(T) > 0 \) and \( c_2(T) > 0 \) such that
\[
\int_{\Omega} u_\varepsilon \ln(u_\varepsilon + e) + \int_{\Omega} v_\varepsilon \ln(v_\varepsilon + e) \leq c_1(T) \quad \text{for all} \; t \in (0, T) \; \text{and} \; \varepsilon \in (0, 1) \tag{4.13}
\]
and
\[
\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}^2}{u_\varepsilon} + \int_{0}^{T} \int_{\Omega} \frac{v_{\varepsilon}^2}{v_\varepsilon} \leq c_2(T) \quad \text{for all} \; \varepsilon \in (0, 1). \tag{4.14}
\]
As a consequence of Lemma 4.2, combining these inequality immediately yields (4.10).

Since by the Hölder inequality,
\[
\int_{0}^{T} \int_{\Omega} |u_{\varepsilon x}|^2 \leq \left\{ \int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}^2}{u_\varepsilon} \right\}^{\frac{3}{4}} \cdot \left\{ \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^3 \right\}^{\frac{1}{4}} \quad \text{for all} \; \varepsilon \in (0, 1),
\]
using that \( \ln(u_\varepsilon + e) \geq 1 \) and arguing similarly for \( v_\varepsilon \) we thereafter obtain (4.12) from (4.10) and (4.11).

Finally, as the Gagliardo-Nirenberg inequality provides \( c_3 > 0 \) fulfilling
\[
\|\varphi\|_{L^\infty(\Omega)} \leq c_3 \|\varphi_\varepsilon\|^2_{L^2(\Omega)} \|\varphi\|^2_{L^2(\Omega)} + c_3 \|\varphi\|_{L^2(\Omega)}^4 \quad \text{for all} \; \varphi \in W^{1,2}(\Omega),
\]
once more combining (4.13) with (4.14) we infer that
\[
\int_{0}^{T} \|u_\varepsilon\|^2_{L^\infty(\Omega)} \leq c_3 \int_{0}^{T} \|u_{\varepsilon x}\|^2_{L^2(\Omega)} \|\varphi_\varepsilon\|^2_{L^2(\Omega)} + c_3 \int_{0}^{T} \|u_{\varepsilon x}\|^4_{L^2(\Omega)} \leq \frac{c_1(T)c_2(T)c_3}{4} + c_3(T)c_3 \cdot T \quad \text{for all} \; \varepsilon \in (0, 1).
and thereby conclude that also (4.11) holds.

A last preparation for our limit procedure asserts \( \varepsilon \)-independent estimates for the time derivatives of \( u_\varepsilon \) and \( v_\varepsilon \) with respect to the norm in some suitably large spaces. Beyond this, these bounds will later on once more be recalled in the course of our qualitative analysis (cf. Corollary 4.3).

**Lemma 4.4** Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and assume \((1.5)\). Then for all \( T > 0 \) one can find \( C(T) > 0 \) such that

\[
\int_0^T \| u_{\varepsilon t}(\cdot, t) \|_{(W_0^{3,2}(\Omega))'} dt + \int_0^T \| v_{\varepsilon t}(\cdot, t) \|_{(W_0^{3,2}(\Omega))'} dt \leq C(T) \quad \text{for all } \varepsilon \in (0, \varepsilon^*).
\]

**Proof.** This can be obtained by minor modification of the reasoning in \([31\text{, Lemma 3.4}]\).

Collecting the above, we arrive at the following.

**Lemma 4.5** Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and assume \((1.5)\). Then one can find \( (\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1) \) and nonnegative functions \( u \) and \( v \) defined in \( \Omega \times (0, \infty) \) such that \((1.9)\) holds, that \( \varepsilon_j \searrow 0 \) as \( j \to \infty \), and that with some null set \( N \subset (0, \infty) \) we have

\[
\begin{align*}
&u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \quad \text{a.e. in } \Omega \times (0, \infty), \quad (4.15) \\
&u_\varepsilon(\cdot, t) \to u(\cdot, t) \quad \text{and} \quad v_\varepsilon(\cdot, t) \to v(\cdot, t) \quad \text{a.e. in } \Omega \text{ for all } t \in (0, \infty) \setminus N, \quad (4.16) \\
&u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \quad \text{in } L^3_{loc}(\Omega \times [0, \infty)) \quad \text{and} \quad (4.17) \\
&u_{\varepsilon x} \to u_x \quad \text{and} \quad v_{\varepsilon x} \to v_x \quad \text{in } L^3_{loc}(\Omega \times [0, \infty)) \quad (4.18)
\end{align*}
\]

as \( \varepsilon = \varepsilon_j \searrow 0 \). Moreover, \((u, v)\) is a global weak solution of \((1.4)\) in the sense of Definition 4.1.

**Proof.** Form Corollary 4.3 we know that

\[
(u_\varepsilon)_{\varepsilon \in (0,1)} \text{ and } (v_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{are bounded in } L^3_{loc}([0, \infty); W^{1,2}(\Omega)), \quad (4.19)
\]

and that

\[
\left( u^{3}_\varepsilon \ln(u_\varepsilon + \varepsilon) \right)_{\varepsilon \in (0,1)} \text{ and } \left( v^{3}_\varepsilon \ln(v_\varepsilon + \varepsilon) \right)_{\varepsilon \in (0,1)} \quad \text{are bounded in } L^1_{loc}(\Omega \times [0, \infty)) \quad (4.20)
\]

and that hence, by the de la Vallée-Poussin theorem,

\[
(u^{2}_\varepsilon)_{\varepsilon \in (0,1)} \text{ and } (v^{3}_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{are uniformly integrable over } \Omega \times (0, T) \text{ for all } T > 0.
\]

As Lemma 4.4 warrants that

\[
(u_{\varepsilon t})_{\varepsilon \in (0,1)} \text{ and } (v_{\varepsilon t})_{\varepsilon \in (0,1)} \quad \text{are bounded in } L^1_{loc}([0, \infty); (W^{3,2}_0(\Omega))^*)
\]

the existence of \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\), nonnegative functions \( u \) and \( v \) on \( \Omega \times (0, \infty) \) and a null set \( N \subset (0, \infty) \), as well as the convergence properties \((4.15)-(4.18)\), result from straightforward arguments involving the Aubin-Lions lemma \((27)\), the Vitali convergence theorem and the Dunford-Pettis theorem. The verification of \((4.1)-(4.3)\) can thereafter be accomplished in much the same manner as demonstrated in \([31\text{, Lemma 4.1}]\), so that we may refrain from giving details here.

This in fact already contains our main result on global existence in \((1.4)\):

**Proof of Theorem 1.1** In view of Lemma 4.5, all statements are now obvious.
5 Eventual bounds I. A second integration of \((3.4)\)

Next addressing the large time behavior of the solutions gained above in the case when \(\chi_1\) and \(\chi_2\) are suitably small, since in view of Theorem 1.2 and Theorem 1.3 we intend to admit initial data of arbitrary size our first step will consist in making sure that any such solution can eventually be controlled by quantities independent of the initial data, similarly to the second conclusion from Lemma 2.3 but in more useful topological frameworks.

For this purpose, we shall once more return to the quasi-entropy inequality from Lemma 3.2, but now with the ambition to make more efficient use of the dissipation rate appearing therein. Here an obstacle toward straightforward approaches seems to be linked to the observation that according to (3.2), for each fixed \(\varepsilon \in (0, 1)\) the functional \(F_{\varepsilon}\) contains summands proportional to the integrals \(\int_{\Omega} u - n_1 \varepsilon\) and \(\int_{\Omega} v - n_2 \varepsilon\) for which our previous estimates do not provide any meaningful information.

An elementary but important interpolation lemma, formulated in a way general enough to allow for two further applications later in Lemma 8.4 and Lemma 9.3, will be of decisive support in overcoming this difficulty:

**Lemma 5.1** Let \(\varphi \in C^1(\Omega)\) be such that \(\varphi > 0\) in \(\Omega\).

i) For all \(p > 0\) and \(q > 0\),

\[
\int_{\Omega} \varphi^{-p} \leq q^\frac{2p}{q} |\Omega|^{\frac{p+q}{q}} \cdot \left\{ \int_{\Omega} \varphi^{-q-2} \varphi_x^2 \right\}^{\frac{q}{2}} + 2^\frac{2p}{q} |\Omega|^{p+1} \cdot \left\{ \int_{\Omega} \varphi \right\}^{-p}. \tag{5.1}
\]

ii) The inequality

\[
- \int_{\Omega} \ln \varphi \leq |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \varphi_x^2 \varphi^2 \right\}^{\frac{1}{2}} - |\Omega| \cdot \ln \left\{ \int_{\Omega} \varphi \right\} + |\Omega| \cdot \ln |\Omega| \tag{5.2}
\]

holds.

**Proof.**  i) We fix \(x_0 \in \Omega\) such that \(\varphi(x_0) \geq \frac{1}{|\Omega|} \int_{\Omega} \varphi\) and then use the Cauchy-Schwarz inequality to see that for all \(x \in \Omega\),

\[
\varphi^{-\frac{q}{2}}(x) = \varphi^{-\frac{q}{2}}(x_0) + \int_{x_0}^x (\varphi^{-\frac{q}{2}})_x(y)dy \leq |\Omega|^{\frac{q}{2}} \cdot \left\{ \int_{\Omega} \varphi \right\}^{-\frac{q}{2}} + |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} (\varphi^{-\frac{q}{2}})_x^2 \right\}^{\frac{1}{2}}.
\]

Thanks to Young’s inequality, this implies that

\[
\varphi^{-p}(x) = \left\{ \varphi^{-\frac{q}{2}}(x) \right\}^{\frac{2p}{q}} \leq 2^\frac{2p}{q} |\Omega|^p \cdot \left\{ \int_{\Omega} \varphi \right\}^{-p} + 2^\frac{2p}{q} |\Omega|^\frac{p}{q} \cdot \left\{ \int_{\Omega} (\varphi^{-\frac{q}{2}})_x^2 \right\}^{\frac{p}{q}} = 2^\frac{2p}{q} |\Omega|^p \cdot \left\{ \int_{\Omega} \varphi \right\}^{-p} + q^\frac{2p}{q} |\Omega|^\frac{p}{q} \cdot \left\{ \int_{\Omega} \varphi^{-q-2} \varphi_x^2 \right\}^{\frac{p}{q}} \quad \text{for all} \quad x \in \Omega
\]

and thus establishes (5.1) upon integration.
ii) Likewise,

\[- \ln \varphi(x) = - \ln \varphi(x_0) - \int_{x_0}^{x} \frac{\varphi(y)}{\varphi(y)} dy \leq - \ln \left\{ \int_{\Omega} \varphi \right\} + \ln |\Omega| + |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \frac{\varphi^2}{\varphi} \right\}^{\frac{1}{2}},\]

which after integration results in [5.2].

Indeed, an application of the first part thereof enables us to create an absorptive linear summand in [3.4] at the expense of additional expressions on its right-hand side, which however remain under control as long as the mass functionals \( \int_{\Omega} u_\varepsilon \) and \( \int_{\Omega} v_\varepsilon \) remain uniformly positive:

**Lemma 5.2** Given \( D_1 > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), one can find \( C > 0 \) such that if \( \chi_1 > 0 \) and \( \chi_2 > 0 \) and \( [1.5] \) holds, then for all \( \varepsilon \in (0, 1) \) the functions \( F_\varepsilon \) and \( D_\varepsilon \) from [6.2] and [3.3] have the property that

\[
\frac{d}{dt} F_\varepsilon(t) + \frac{1}{C} F_\varepsilon(t) + \frac{1}{2} D_\varepsilon(t) \\
\leq C \cdot \left( 1 + \frac{\chi_1}{\chi_2} \right) \cdot \left\{ 1 + \left\{ \int_{\Omega} u_\varepsilon(\cdot, t) \right\}^6 + \left\{ \int_{\Omega} v_\varepsilon(\cdot, t) \right\}^6 \\
+ \varepsilon \cdot \left\{ \int_{\Omega} u_\varepsilon(\cdot, t) \right\}^{-(3-n_1)} + \varepsilon \cdot \left\{ \int_{\Omega} v_\varepsilon(\cdot, t) \right\}^{-(3-n_2)} \right\} \]

for all \( t > 0 \).

**Proof.** In view of Lemma 3.2 and Young’s inequality, it is clearly sufficient to make sure that given \( D_1 > 0, D_2 > 0, a_1 > 0, a_2 > 0, \lambda_1 > 0 \) and \( \lambda_2 > 0 \), one can find \( c_1 > 0 \) such that whenever \( \chi_1 > 0 \) and \( \chi_2 > 0 \) and \( [1.5] \) holds, then for any \( \varepsilon \in (0, 1) \) we have

\[
F_\varepsilon(t) \leq c_1 D_\varepsilon(t) + c_1 \cdot \left( 1 + \frac{\chi_1}{\chi_2} \right) \cdot \left\{ 1 + \left\{ \int_{\Omega} u_\varepsilon \right\}^6 + \left\{ \int_{\Omega} v_\varepsilon \right\}^6 \\
+ \varepsilon \cdot \left\{ \int_{\Omega} u_\varepsilon \right\}^{-(3-n_1)} + \varepsilon \cdot \left\{ \int_{\Omega} v_\varepsilon \right\}^{-(3-n_2)} \right\} \]

for all \( t > 0 \). To achieve this, we note that e.g. by once more combining the Gagliardo-Nirenberg inequality with Youngs inequality in a straightforward manner we can find positive constants \( c_2 \) and \( c_3 \) such that for all \( \varphi \in C^1(\Omega) \) fulfilling \( \varphi > 0 \) in \( \Omega \) we have

\[
\int_{\Omega} \varphi \ln \varphi \leq \int_{\Omega} \varphi^2 \leq c_2 \| \sqrt{\varphi} \|_{L^2(\Omega)} \| \sqrt{\varphi} \|_{L^2(\Omega)}^3 + c_2 \| \sqrt{\varphi} \|_{L^2(\Omega)}^4 \leq \int_{\Omega} \frac{\varphi^2}{\varphi} + c_3 \cdot \left\{ \int_{\Omega} \varphi \right\}^6 + c_3.
\]

Therefore,

\[
F_\varepsilon(t) \leq \int_{\Omega} \frac{u_{\varepsilon x}^2}{u_\varepsilon} + c_3 \cdot \left\{ \int_{\Omega} u_\varepsilon \right\}^6 + c_3 \\
+ \frac{\chi_1}{\chi_2} \int_{\Omega} \frac{v_{\varepsilon x}^2}{v_\varepsilon} + \frac{c_3 \chi_1}{\chi_2} \cdot \left\{ \int_{\Omega} v_\varepsilon \right\}^6 + \frac{c_3 \chi_1}{\chi_2} \\
+ \frac{\varepsilon}{(3-n_1)(3-n_2)} \int_{\Omega} \frac{1}{u_\varepsilon^{3-n_1}} + \frac{\chi_1 \varepsilon}{(3-n_2)(4-n_2) \chi_2} \int_{\Omega} \frac{1}{v_\varepsilon^{3-n_2}} \text{ for all } t > 0, \tag{5.5}
\]
where the two rightmost summands can be estimated by means of Lemma 5.1 i), which when applied to \( p = q = 3 - n_i, i \in \{1, 2\} \), says that for all \( t > 0 \),

\[
\frac{\varepsilon}{(3 - n_1)(4 - n_1)} \int_{\Omega} \frac{1}{u_{3-n_1}^2} \leq \frac{(3 - n_1)|\Omega|^2}{4 - n_1} \cdot \varepsilon \int_{\Omega} \frac{u_{3-n_1}^2}{u_{3-n_1}^5} + \frac{4|\Omega|^{4-n_1}}{(3 - n_1)(4 - n_1)} \cdot \varepsilon \cdot \left\{ \int_{\Omega} u_\varepsilon \right\}^{-(3-n_1)}
\]

and

\[
\frac{\chi_1 \varepsilon}{(3 - n_2)(4 - n_2) \chi_2} \int_{\Omega} \frac{1}{v_{3-n_2}^2} \leq \frac{(3 - n_2)\chi_1|\Omega|^2}{(4 - n_2)\chi_2} \cdot \varepsilon \int_{\Omega} \frac{v_{3-n_2}^2}{v_{3-n_2}^5} + \frac{4\chi_1|\Omega|^{4-n_2}}{(3 - n_2)(4 - n_2)\chi_2} \cdot \varepsilon \cdot \left\{ \int_{\Omega} v_\varepsilon \right\}^{-(3-n_2)}
\]

In conjunction with \((5.5)\) and the definition \((8.3)\) of \( D_\varepsilon \), these inequalities readily establish \((5.3)\) with some suitably large \( C > 0 \).

As a natural next task, we are thus concerned with positive lower bounds for \( \int_{\Omega} u_\varepsilon \) and \( \int_{\Omega} v_\varepsilon \). Thanks to the second estimate provided by Corollary 4.3, however, at least within time intervals of finite length these can be obtained by simply returning to \((1.6)\)-(1.7):

**Lemma 5.3** Let \( D_i > 0, a_i > 0, \lambda_i > 0, \chi_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \), and suppose that \((1.5)\) holds. Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that whenever \( \varepsilon \in (0, 1) \),

\[
\int_{\Omega} u_\varepsilon(\cdot, t) \geq C(T) \quad \text{and} \quad \int_{\Omega} v_\varepsilon(\cdot, t) \geq C(T) \quad \text{for all} \ t \in (0, T).
\]

**Proof.** For fixed \( T > 0 \), using Corollary 4.3 and the Cauchy-Schwarz inequality we can fix \( c_1(T) > 0 \) such that

\[
\int_0^T \|u_\varepsilon\|_{L^\infty(\Omega)} + \int_0^T \|v_\varepsilon\|_{L^\infty(\Omega)} \leq c_1(T) \quad \text{for all} \ \varepsilon \in (0, 1).
\]

Thus, if going back to \((1.6)-(1.7)\) we estimate

\[
\frac{d}{dt} \int_{\Omega} u_\varepsilon = \int_{\Omega} \frac{3u_{\varepsilon}^3}{3u_{\varepsilon}^2 + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \\
\geq -\int_{\Omega} u_{\varepsilon}^2 \\
\geq -\|u_{\varepsilon}\|_{L^\infty(\Omega)} \int_{\Omega} u_\varepsilon \quad \text{for all} \ t > 0 \ \text{and} \ \varepsilon \in (0, 1),
\]

then upon integration we infer that thanks to \((5.7)\),

\[
\int_{\Omega} u_\varepsilon(\cdot, t) \geq \left\{ \int_{\Omega} u_{0\varepsilon} \right\} \cdot e^{-\int_0^t \|u_{\varepsilon}(\cdot, s)\|_{L^\infty(\Omega)} ds} \\
\geq c_2 e^{-c_1(T)} \quad \text{for all} \ t \in (0, T) \ \text{and} \ \varepsilon \in (0, 1),
\]

with \( c_2 := \inf_{\varepsilon \in (0, 1)} \int_{\Omega} u_{0\varepsilon} \) being positive due to \((1.5)\) and the positivity of \( u_0 \). Likewise, from the inequality

\[
\frac{d}{dt} \int_{\Omega} v_\varepsilon = \int_{\Omega} \frac{3v_{\varepsilon}^3}{3v_{\varepsilon}^2 + \varepsilon} (\lambda_2 - v_\varepsilon - a_2 u_\varepsilon) \\
\geq -\int_{\Omega} v_{\varepsilon}^2 - a_2 \int_{\Omega} u_\varepsilon v_\varepsilon \\
\geq -\|v_\varepsilon\|_{L^\infty(\Omega)} \int_{\Omega} v_\varepsilon - a_2 \|u_\varepsilon\|_{L^\infty(\Omega)} \int_{\Omega} v_\varepsilon \quad \text{for all} \ t > 0 \ \text{and} \ \varepsilon \in (0, 1),
\]

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through (5.7) we obtain that
\[
\int_{\Omega} u_{\varepsilon}(\cdot, t) \geq \left\{ \int_{\Omega} v_{0\varepsilon} \right\} \cdot e^{-\int_0^t \|v_{\varepsilon}(\cdot, s)\|_{L^{\infty}(\Omega)} ds} \cdot e^{-a_2 \int_0^t \|u_{\varepsilon}(\cdot, s)\|_{L^{\infty}(\Omega)} ds}
\geq c_3 e^{-c_1(T)} \cdot e^{-a_2 c_1(T)} \quad \text{for all } t \in (0, T) \text{ and any } \varepsilon \in (0, 1)
\]
with \(c_3 := \inf_{\varepsilon \in (0, 1)} \int_{\Omega} v_{0\varepsilon} > 0\).

Making use of the fact that the singular expressions on the right-hand side of (5.3) contain the small factor \(\varepsilon\), on integration thereof we infer the following data-independent eventual regularity information as the main outcome of this section.

**Lemma 5.4** For \(i \in \{1, 2\}\), let \(D_1 > 0, a_i > 0, \lambda_i > 0\) and \(n_i \in [1, 2]\). Then there exists \(K > 0\) with the following property: If \(\chi_1 > 0\) and \(\chi_2 > 0\) and (1.5) holds, one can find \(T_0 = T_0(\chi_1, \chi_2, u_0, v_0) > 0\) such that for each \(T > T_0\) there exists \(\varepsilon_0 = \varepsilon_0(T, \chi_1, \chi_2, u_0, v_0) \in (0, 1)\) such that whenever \(\varepsilon \in (0, \varepsilon_0)\),
\[
\int_{\Omega} u_{\varepsilon}(\cdot, t) \ln(u_{\varepsilon}(\cdot, t) + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_{\varepsilon}(\cdot, t) \ln(v_{\varepsilon}(\cdot, t) + e) \leq K \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \quad \text{for all } t \in (T_0, T) \quad (5.8)
\]
and
\[
\int_{t}^{t+1} \int_{\Omega} \frac{u_{\varepsilon}^2}{u_{\varepsilon}} + \frac{\chi_1}{\chi_2} \int_{t}^{t+1} \int_{\Omega} \frac{v_{\varepsilon}^2}{v_{\varepsilon}} \leq K \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \quad \text{for all } t \in (T_0, T). \quad (5.9)
\]

**Proof.** By means of Lemma 5.2 and Lemma 3.3, given any \(D_1 > 0, D_2 > 0, a_1 > 0, a_2 > 0, \lambda_1 > 0\) and \(\lambda_2 > 0\) we can find \(c_1 > 0, c_2 > 0\) and \(c_3 > 0\) with the property that whenever \(\chi_1 > 0\), \(\chi_2 > 0\) and (1.5) holds, for \(F_{\varepsilon}\) and \(D_{\varepsilon}\) taken from (3.2) and (3.3) we have
\[
\frac{d}{dt} F_{\varepsilon}(t) + c_1 F_{\varepsilon}(t) + \frac{1}{2} D_{\varepsilon}(t)
\leq c_2 \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \cdot \left\{ 1 + \left\{ \int_{\Omega} u_{\varepsilon} \right\}^7 + \left\{ \int_{\Omega} v_{\varepsilon} \right\}^7 + e \cdot \left\{ \int_{\Omega} u_{\varepsilon} \right\}^{-(3-n_1)} + e \cdot \left\{ \int_{\Omega} v_{\varepsilon} \right\}^{-(3-n_2)} \right\} \quad (5.10)
\]
and
\[
\int_{\Omega} u_{\varepsilon} \ln(u_{\varepsilon} + e) + \frac{\chi_1}{\chi_2} \int_{\Omega} v_{\varepsilon} \ln(v_{\varepsilon} + e) \leq F_{\varepsilon}(t) + c_3 \cdot \left(1 + \frac{\chi_1}{\chi_2}\right) \cdot \left\{ 1 + \int_{\Omega} u_{\varepsilon} + \int_{\Omega} v_{\varepsilon} \right\} \quad (5.11)
\]
for all \(t > 0\) and any \(\varepsilon \in (0, 1)\). Moreover, combining the inequalities (2.5) and (2.6) provided by Lemma 2.3, we see that if (1.5) is valid, then there exists \(T_1 = T_1(u_0, v_0)\) such that
\[
\int_{\Omega} u_{\varepsilon} + \int_{\Omega} v_{\varepsilon} \leq c_4 := m_\infty + 1 \quad \text{for all } t > T_1 \text{ and each } \varepsilon \in (0, 1), \quad (5.12)
\]
where \(m_\infty > 0\) is as defined in (2.6). Keeping this value of \(T_1\) fixed henceforth, we now assume that \(D_1, D_2, a_1, a_2, \lambda_1, \lambda_2, \chi_1\) and \(\chi_2\) are given positive constants, and that (1.5) holds. Then an application of Lemma 4.4 yields \(c_5 = c_5(\chi_1, \chi_2, u_0, v_0) > 0\) such that the correspondingly defined function \(F_{\varepsilon}\) in (3.2) satisfies
\[
F_{\varepsilon}(T_1) \leq c_5 \quad \text{for all } \varepsilon \in (0, 1), \quad (5.13)
\]
and we thereupon let $T_2 = T_2(\chi_1, \chi_2, u_0, v_0) > T_1$ be large enough such that
\[ c_5 e^{-(T_2 - T_1)} \leq 1. \] (5.14)

Next, fixing any $T > T_2 + 1$ we infer from Lemma 5.3 that
\[ \int_0^T u_\varepsilon \geq c_6 \quad \text{and} \quad \int_0^T v_\varepsilon \geq c_6 \quad \text{for all} \ t \in (0, T) \quad \text{and} \quad \varepsilon \in (0, 1) \] (5.15)
with some suitably small $c_6 = c_6(T, \chi_1, \chi_2, u_0, v_0) > 0$, whence it becomes possible to pick $\varepsilon_0$ such that
\[ c_6^{-(3-n_1)} \varepsilon_0 + c_6^{-(3-n_2)} \varepsilon_0 \leq 1. \]

Then on inserting (5.12) and (5.13) into (5.10) we obtain that
\[ \frac{d}{dt}F_\varepsilon(t) + c_1 F_\varepsilon(t) + \frac{1}{2} D_\varepsilon(t) \leq c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right) \int_0^t e^{-c_1(t-s)} ds \] (5.16)
where we underline that besides $c_1$, also $c_7 := c_2 \cdot (2 + 2c_4^3)$ is independent of both $\chi_1$ and $\chi_2$ as well as $u_0$ and $v_0$. After integration using (5.13) and (5.14), from (5.16) we firstly infer that
\[ F_\varepsilon(t) \leq F_\varepsilon(T_1) \cdot e^{-c_1(t-T_1)} + c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right) \int_0^t e^{-c_1(t-s)} ds \]
\[ \leq c_5 e^{c_1(t-T_1)} + c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right) \frac{1 - e^{-c_1(T_1-t)}}{c_1} \]
\[ \leq c_6 e^{-c_1(t-T_1)} + c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right) \]
\[ \leq 1 + \frac{c_7}{c_1} \left( 1 + \frac{\chi_1}{\chi_2} \right) \quad \text{for all} \ t \in (T_2, T) \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0), \] (5.17)
and that thus in view of (5.11) and (5.12), in particular,
\[ \int_\Omega u_\varepsilon \ln(u_\varepsilon + e) + \frac{\chi_1}{\chi_2} \int_\Omega v_\varepsilon \ln(v_\varepsilon + e) \leq 1 + \frac{c_7}{c_1} \left( 1 + \frac{\chi_1}{\chi_2} \right) + c_3 \left( 1 + \frac{\chi_1}{\chi_2} \right) \cdot (1 + c_4) \]
\[ \quad \text{for all} \ t \in (T_2, T) \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0). \] (5.18)

Thanks to (5.17), (5.16) secondly entails that for all $t \in (T_2, T)$ and each $\varepsilon \in (0, \varepsilon_0),$
\[ \frac{1}{2} \int_t^{t+1} D_\varepsilon(s) ds \leq F_\varepsilon(t) - F_\varepsilon(t+1) - c_1 \int_t^{t+1} F_\varepsilon(s) ds + c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right) \]
\[ \leq 1 + \frac{c_7}{c_1} \left( 1 + \frac{\chi_1}{\chi_2} \right) + c_3 (c_4 + 1) \left( 1 + \frac{\chi_1}{\chi_2} \right) + c_1 c_3 (c_4 + 1) \left( 1 + \frac{\chi_1}{\chi_2} \right) + c_7 \left( 1 + \frac{\chi_1}{\chi_2} \right), \]
because
\[ F_\varepsilon(t) \geq -c_3 \left( 1 + \frac{\chi_1}{\chi_2} \right) (1 + c_4) \quad \text{for all} \ t > T_1 \quad \text{and} \quad \varepsilon \in (0, 1) \]
according to (5.11) and (5.12). Due to the definition (3.3) of $D_\varepsilon$, the latter implies (5.9), whereas (5.8) results from (5.18).
6 Eventual bounds II. The conditional quasi-entropy \( \int_{\Omega} u_{\varepsilon x}^2 + \gamma \int_{\Omega} v_{\varepsilon x}^2 \)

Now approaching the core of our asymptotic analysis, we next intend to improve the information on eventual regularity provided by Lemma 5.4. To this end, in the key Lemma 6.2 we shall study the time evolution of \( \int_{\Omega} u_{\varepsilon x}^2 + \gamma \int_{\Omega} v_{\varepsilon x}^2 \), and more precisely we shall see there that under suitable smallness assumptions on \( \chi_1 \) and \( \chi_2 \), for carefully chosen values of the parameter \( \gamma > 0 \) depending on the ratio \( \chi_1^2 \) in a subtle manner, this functional enjoys an entropy-like property as long as its values remain small. Using Lemma 5.4 to achieve the latter at least at some large initial time, we will thereby be able to indeed conclude eventual boundedness of both solution components in \( W^{1,2}(\Omega) \).

In estimating the respective cross-diffusion terms appearing in the corresponding testing procedure adequately, we will make use of the following elementary inequalities.

Lemma 6.1 Let \( n \in [0, \frac{7}{2}] \), \( \varepsilon > 0 \) and

\[
h_\varepsilon(s) := \frac{s^5 - n}{s^4 - n + \varepsilon}, \quad s \geq 0. \tag{6.1}
\]

Then

\[
0 \leq h_\varepsilon(s) \leq s \quad \text{and} \quad 0 \leq h_\varepsilon'(s) \leq 5 - n \quad \text{for all } s \geq 0, \tag{6.2}
\]

and moreover

\[
|h_\varepsilon''(s)| \leq 2^\frac{7-2n}{2(4-n)} \cdot (4 - n)(5 - n)\varepsilon \frac{1}{2(4-n)} s^{-\frac{3}{2}} \quad \text{for all } s > 0. \tag{6.3}
\]

Proof. Computing

\[
h_\varepsilon'(s) = \frac{s^{8-2n} + (5 - n)\varepsilon s^{4-n}}{(s^4 - n + \varepsilon)^2}, \quad s \geq 0,
\]

and

\[
h_\varepsilon''(s) = \frac{-(3-n)(4-n)\varepsilon s^{7-2n} + (4-n)(5-n)\varepsilon^2 s^{3-n}}{(s^4 - n + \varepsilon)^3}, \quad s > 0, \tag{6.4}
\]

we immediately see that since \( 1 \leq 5 - n \),

\[
0 \leq h_\varepsilon'(s) \leq \frac{(5-n)s^{8-2n} + (5-n)\varepsilon s^{4-n}}{(s^4 - n + \varepsilon)^2} = (5-n)\frac{s^{4-n}}{s^4-n+\varepsilon} \leq 5-n \quad \text{for all } s \geq 0,
\]

whence (6.2) becomes evident.

Furthermore, since \( |3-n| \leq 5-n \), from (6.4) it follows that

\[
|h_\varepsilon''(s)| \leq (4 - n)(5 - n) \cdot \frac{\varepsilon s^{7-2n} + \varepsilon^2 s^{3-n}}{(s^4 - n + \varepsilon)^3} = (4 - n)(5 - n)I_\varepsilon(s) \quad \text{for all } s > 0, \tag{6.5}
\]

where clearly

\[
I_\varepsilon(s) := \frac{\varepsilon s^{3-n}}{(s^4 - n + \varepsilon)^2} \leq \frac{\varepsilon s^{3-n}}{s^{2n}} = \varepsilon s^{n-5} \quad \text{for all } s > 0,
\]

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and where on the other hand by Young’s inequality,
\[ I_\varepsilon(s) = \frac{1}{s} \cdot \frac{\varepsilon s^{4-n}}{(s^{4-n} + \varepsilon)^2} \leq \frac{1}{s} \cdot \frac{\varepsilon s^{4-2n} + \frac{1}{2}\varepsilon^2}{(s^{4-n} + \varepsilon)^2} \leq \frac{1}{2s} \quad \text{for all } s > 0. \]

An interpolation therefore shows that since \( n \leq \frac{7}{2} \),
\[ I_\varepsilon(s) = \frac{1}{2(4-n)} \cdot I_\varepsilon^{1/2} (s) \cdot I_\varepsilon^{7/2} (s) \leq \left\{ \varepsilon^{2/(4-n)} \cdot s^{2n/(4-n)} \right\} \cdot \left\{ (2s)^{-7/2n} \right\} \quad \text{for all } s > 0, \]
and that thus \((6.3)\) is a consequence of \((6.5)\).

Now the presumably most crucial step toward our derivation both of Theorem 1.2 and of Theorem 1.3 will be accomplished in the course of the following quite delicate argument. Here a considerable technical intricacy originates from the ambition to unambiguously identify smallness assumptions on \( \chi_1 \) and \( \chi_2 \), independently of \( u_0 \) and \( v_0 \), and on \( \varepsilon \), the latter possibly data-dependent, that ensure the following global absorption property considerably going beyond that from Lemma \( \text{[2.3]} \).

**Lemma 6.2** Let \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( n_i \in [1, 2] \) for \( i \in \{1, 2\} \). Then there exist \( \chi_* > 0 \) and \( C > 0 \) with the property that whenever \( \chi_1 \in (0, \chi_*) \) and \( \chi_2 \in (0, \chi_*) \) and \((1.3)\) holds, one can find \( T_\ast = T_\ast (\chi_1, \chi_2, u_0, v_0) > 0 \) and \( \varepsilon_* = \varepsilon_* (\chi_1, \chi_2, u_0, v_0) \in (0, 1) \) such that
\[ \int_\Omega u_{xx}^2 (\cdot, t) + \int_\Omega v_{xx}^2 (\cdot, t) \leq C \quad \text{for all } t > T_\ast \text{ and each } \varepsilon \in (0, \varepsilon_*) \quad (6.6) \]
and
\[ \|u_\varepsilon (\cdot, t)\|_{L^\infty (\Omega)} + \|v_\varepsilon (\cdot, t)\|_{L^\infty (\Omega)} \leq C \quad \text{for all } t > T_\ast \text{ and } \varepsilon \in (0, \varepsilon_*) \quad (6.7) \]
as well as
\[ \int_t^{t+1} \int_\Omega u_{xxx}^2 + \int_t^{t+1} \int_\Omega v_{xxx}^2 \leq C \quad \text{for all } t > T_\ast \text{ and any } \varepsilon \in (0, \varepsilon_*). \quad (6.8) \]

**PROOF.** In order to prepare our selection of \( \chi_* \), taking \( m_\infty > 0 \) from Lemma \( \text{[2.3]} \) let us set \( m := m_\infty + 1 \) and repeatedly make use of the Gagliardo-Nirenberg inequality to find positive constants \( c_1, c_2, c_3, c_4 \) and \( c_5 \) such that for any \( \varphi \in W^{2,2}(\Omega) \) fulfilling \( \|\varphi\|_{L^1 (\Omega)} \leq m \) we have
\[ \|\varphi\|_{L^\infty (\Omega)}^2 \leq c_1 \cdot \left\{ \|\varphi_{xx}\|_{L^2 (\Omega)}^\frac{2}{3} + 1 \right\} \quad (6.9) \]
and
\[ \|\varphi\|_{L^2 (\Omega)}^2 \leq c_2 \cdot \left\{ \|\varphi_{xx}\|_{L^2 (\Omega)}^\frac{2}{3} + 1 \right\} \quad (6.10) \]
and
\[ \|\varphi\|_{L^\infty (\Omega)} \leq c_3 \cdot \left\{ \|\varphi_{xx}\|_{L^2 (\Omega)}^\frac{2}{3} + 1 \right\} \quad (6.11) \]
as well as
\[ \|\varphi_{xx}\|_{L^2 (\Omega)}^2 \leq c_4 \cdot \left\{ \|\varphi_{xx}\|_{L^2 (\Omega)}^\frac{2}{3} + 1 \right\} \quad (6.12) \]
\(\|\varphi_x\|_{L^4(\Omega)} \leq c_5 \cdot \left\{ \|\varphi_{xx}\|_{L^2(\Omega)} + 1 \right\}. \) \tag{6.13}

By the same token, we can fix \(c_6 > 0\) and \(c_7 > 0\) such that
\[\|\varphi\|_{L^4(\Omega)} \leq c_6 \|\varphi_x\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \quad \text{for all } \varphi \in W^{1,2}_{0}(\Omega) \tag{6.14}\]
and
\[\|\varphi\|_{L^\infty(\Omega)} \leq c_7 \cdot \left\{ \|\varphi_x\|_{L^2(\Omega)} + 1 \right\} \quad \text{for all } \varphi \in W^{1,2}_{0}(\Omega) \text{ satisfying } \|\varphi\|_{L^2(\Omega)} \leq m. \tag{6.15}\]

We next employ Young’s inequality to choose positive numbers \(c_8, c_9, c_{10}\) and \(c_{11}\) with the properties that
\[\left( \frac{a_1^2}{2D_1} + \frac{a_2^2}{2D_2} \right) \cdot c_1 c_2 \xi \eta \leq \frac{D_1}{8} \xi^2 + c_8 \eta^\frac{5}{2} \quad \text{for all } \xi \geq 0 \text{ and } \eta \geq 0 \tag{6.16}\]
and
\[2^\frac{5}{2} c_8 \xi \eta \leq D_2 \frac{2}{16} \xi^3 + c_9 \eta^\frac{3}{2} \quad \text{for all } \xi \geq 0 \text{ and } \eta \geq 0 \tag{6.17}\]
as well as
\[3\lambda_1 c_4 \xi^\frac{4}{5} \leq \frac{D_1}{8} \xi^2 + c_{10} \quad \text{for all } \xi \geq 0 \tag{6.18}\]
and
\[3\lambda_2 c_4 \xi^\frac{4}{5} \leq \frac{D_2}{8} \xi^2 + c_{11} \quad \text{for all } \xi \geq 0, \tag{6.19}\]
and thereupon abbreviate
\[c_{12} := \frac{D_1 + D_2}{4} + c_9 \tag{6.20}\]
as well as
\[c_{13} := c_{10} + 3\lambda_1 c_4 \quad \text{and} \quad c_{14} := c_{11} + 3\lambda_2 c_4. \tag{6.21}\]

Furthermore introducing
\[c_{15} := \max_{i \in \{1, 2\}} \left\{ c_3 + (5 - n_i) c_5 c_6^\frac{8}{5} \right\} \quad \text{and} \quad c_{16} := \max_{i \in \{1, 2\}} 2^{\frac{7 - 2n_i}{3(4 - n_i)}} (4 - n_i)(5 - n_i), \tag{6.22}\]
we let
\[c_{17} := c_{12} + c_{13} + c_{14} + \frac{D_1 + D_2}{32} + 2 \quad \text{and} \quad c_{18} := c_6^4 c_{16} \tag{6.23}\]
as well as
\[c_{19} := \frac{\min\{D_1, D_2\}}{32 \cdot (3c_4)^{\frac{5}{2}}} \quad \text{and} \quad c_{20} := c_{17} + \frac{2^\frac{5}{2} \cdot \min\{D_1, D_2\}}{16}, \tag{6.24}\]
and taking \(K > 0\) from Lemma 5.4 we write
\[c_{21} := 2K \cdot \sqrt{\frac{c_7 K}{2} + c_7}, \quad c_{22} := \max\left\{ 1, 4c_{21}, 2 \cdot \left( \frac{c_{20}}{c_{19}} \right)^{\frac{3}{5}} \right\} \quad \text{and} \quad c_{23} := \min\left\{ 1, \left( \frac{c_{22}}{4c_{21}} \right)^{\frac{10}{13}} \right\} \tag{6.25}\]
and finally define
\[
\chi_* := \min \left\{ 1, \frac{\sqrt{D_1 D_2 \cdot \frac{c_2^3}{c_{23}^3}}}{\sqrt{2048 \cdot c_{15}^6 c_{22}^3}} \right\},
\]
(6.26)
noting that through the above construction, \(\chi_*\) indeed only depends on \(D_1, D_2, a_1, a_2, \lambda_1\) and \(\lambda_2\).

We now let \(\chi_1 > 0\) and \(\chi_2 > 0\) be such that
\[
\chi_1 \leq \chi_2 \leq \chi_*,
\]
(6.27)
and assuming (1.5) we introduce
\[
y_\varepsilon(t) := \int_\Omega u^2_{\varepsilon x}(\cdot, t) + \gamma \int_\Omega v^2_{\varepsilon x}(\cdot, t), \quad t > 0, \ \varepsilon \in (0, 1),
\]
(6.28)
where
\[
\gamma := c_{23} \rho^{\frac{13}{15}} \quad \text{with} \quad \rho := \frac{\chi_1}{\chi_2} \in (0, 1].
\]
(6.29)

In order to derive an appropriate upper bound for \(y_\varepsilon\), we first invoke Lemma 2.3 to see that thanks to our definition of \(m\) we can pick \(T_1 = T_1(u_0, v_0) > 0\) such that
\[
\int_\Omega u_\varepsilon + \int_\Omega v_\varepsilon \leq m \quad \text{for all} \quad t > T_1 \quad \text{and any} \quad \varepsilon \in (0, 1),
\]
(6.30)
and thereafter employ Lemma 5.4 in choosing \(T_* = T_*(\chi_1, \chi_2, u_0, v_0) > T_1 + 1\) and \(\varepsilon_1 = \varepsilon_1(\chi_1, \chi_2, u_0, v_0) \in (0, 1)\) such that
\[
\int_{T_1}^{T_*} \int_\Omega u^2_{\varepsilon x}(\cdot, t) + \rho \int_{T_1}^{T_*} \int_\Omega v^2_{\varepsilon x}(\cdot, t) \leq K \cdot (1 + \rho) \leq 2K
\]
(6.31)
according to (6.29). Then writing \(\theta := \min_{\varepsilon \in \{1, 2\}} \frac{1}{2(4-n_1)}\) we fix \(\varepsilon_* = \varepsilon_*(\chi_1, \chi_2, u_0, v_0) \in (0, \varepsilon_1)\) suitably small fulfilling
\[
\varepsilon^\theta_* \leq \min \left\{ \frac{D_1}{576 c_{16}^1 \chi_1}, \frac{\gamma D_2}{32 c_6^4 c_{16}^1 \gamma_1}, \frac{2 \gamma_{\frac{39}{10}}}{c_6^4 c_{16}^3 c_{22}^3 \chi_1} \right\}
\]
(6.32)
as well as
\[
\varepsilon^\theta_* \leq \min \left\{ \frac{D_2}{576 c_{16}^1 \chi_2}, \frac{D_1}{32 c_6^4 c_{16}^1 \gamma_2}, \frac{2}{c_6^4 c_{16}^3 c_{22}^3 \gamma_{\frac{39}{10}} \chi_2} \right\},
\]
(6.33)
and we claim that these selections ensure that
\[
y_\varepsilon(t) \leq c_{22} \gamma^{\frac{39}{10}} \quad \text{for all} \quad t > T_* \quad \text{and any} \quad \varepsilon \in (0, \varepsilon_*).
\]
(6.34)
To verify this, for \(\varepsilon \in (0, 1)\) we consider the set
\[
S_\varepsilon := \left\{ t > T_1 \mid y_\varepsilon(t) < c_{22} \gamma^{\frac{39}{10}} \right\},
\]
(6.35)
and first combine the inequalities provided by Lemma 2.5 to see upon dropping two nonnegative summands that for all \( \varepsilon \in (0, 1) \),

\[
\frac{1}{2} y_\varepsilon'(t) + \frac{D_1}{2} \int_\Omega u_{\varepsilon xx}^2 + \frac{\gamma D_2}{2} \int_\Omega v_{\varepsilon xx}^2 \\
\leq \chi_1 \int_\Omega \left( h_{1, \varepsilon}(u_\varepsilon) u_{\varepsilon xx} \right)_x u_{\varepsilon xx} - \gamma \chi_2 \int_\Omega \left( h_{2, \varepsilon}(v_\varepsilon) u_{\varepsilon xx} \right)_x v_{\varepsilon xx} \\
+ 3\lambda_1 \int_\Omega u_{\varepsilon xx}^2 + 3\gamma \lambda_2 \int_\Omega v_{\varepsilon xx}^2 + \left( \frac{a_1^2}{2D_1} + \frac{\gamma a_2^2}{2D_2} \right) \int_\Omega u_\varepsilon^2 v_\varepsilon^2 
\quad \text{for all } t > 0,
\]

(6.36)

where \( h_{i, \varepsilon}(s) := \frac{\delta_{i, \varepsilon} s}{s^2 + \varepsilon} \) for \( s \geq 0, \varepsilon \in (0, 1) \) and \( i \in \{1, 2\} \). Here since \( \gamma \leq 1 \) by (6.24) and (6.25), in view of (6.30) we may apply (6.36), (6.11), (6.16), (6.17), and Young’s inequality to estimate

\[
\left( \frac{a_1^2}{2D_1} + \frac{\gamma a_2^2}{2D_2} \right) \int_\Omega u_{\varepsilon xx}^2 \\
\leq \left( \frac{a_1^2}{2D_1} + \frac{\gamma a_2^2}{2D_2} \right) \| u_\varepsilon \|_{L^2(\Omega)}^2 \| v_\varepsilon \|_{L^2(\Omega)}^2 \\
\leq \left( \frac{a_1^2}{2D_1} + \frac{\gamma a_2^2}{2D_2} \right) \cdot c_1 c_2 \cdot \left\{ \| u_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\} \cdot \left\{ \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\} \\
\leq \frac{D_1}{8\sqrt{2}} \cdot \left\{ \| u_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\}^{\frac{3}{2}} + c_8 \cdot \left\{ \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\}^{\frac{3}{2}} \\
\leq 2^{\frac{3}{2}} \cdot \frac{D_1}{8\sqrt{2}} \cdot \left\{ \| u_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\} + 2^{\frac{3}{2}} c_8 \cdot \left\{ \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 1 \right\} \\
= \frac{D_1}{4} \int_\Omega u_{\varepsilon xx}^2 + \frac{D_1}{4} + \frac{D_2}{4} \cdot \left\{ \left( \sqrt{\gamma} \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 \right)^{\frac{3}{2}} + \gamma \right\} \\
\leq \frac{D_1}{4} \int_\Omega u_{\varepsilon xx}^2 + \frac{D_1}{4} + \frac{D_2}{4} \cdot \left\{ \gamma \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 + \gamma \right\} + c_9 \gamma^{-\frac{1}{2}} \\
\leq \frac{D_1}{4} \int_\Omega u_{\varepsilon xx}^2 + \frac{D_1}{4} + \frac{D_2}{4} \cdot \int_\Omega v_{\varepsilon xx}^2 + \frac{\gamma D_2}{4} \int_\Omega v_{\varepsilon xx}^2 + \frac{\gamma D_2}{4} + c_9 \gamma^{-\frac{1}{2}} \\
\leq \frac{D_1}{4} \int_\Omega u_{\varepsilon xx}^2 + \gamma D_2 \int_\Omega v_{\varepsilon xx}^2 + c_12 \gamma^{-\frac{1}{2}} 
\quad \text{for all } t > T_1 \text{ and } \varepsilon \in (0, 1).
\]

(6.37)

Next, due to (6.30), (6.12), (6.18) and (6.21) we have

\[
3\lambda_1 \int_\Omega u_{\varepsilon xx}^2 \leq 3\lambda_1 c_4 \| u_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 3\lambda_1 c_4 \\
\leq \frac{D_1}{8} \| u_{\varepsilon xx} \|_{L^2(\Omega)}^2 + c_{10} + 3\lambda_1 c_4 \\
= \frac{D_1}{8} \int_\Omega u_{\varepsilon xx}^2 + c_{13} \quad \text{for all } t > T_1 \text{ and } \varepsilon \in (0, 1),
\]

(6.38)

while similarly (6.30), (6.12), (6.19) and (6.21) ensure that

\[
3\gamma \lambda_2 \int_\Omega v_{\varepsilon xx}^2 \leq 3\gamma \lambda_2 c_4 \| v_{\varepsilon xx} \|_{L^2(\Omega)}^2 + 3\gamma \lambda_2 c_4
\]

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According to (6.41), (6.42) and our definition (6.22) of and here combining (6.14) with (6.13), (6.30) and Young’s inequality shows that for $t > T_1$ and $\varepsilon \in (0, 1)$,

$$
\leq \frac{\gamma D_2}{8} \|v_{\varepsilon xx}\|_{L^2(\Omega)}^2 + c_{11}\gamma + 3\gamma \lambda_2 c_4
\leq \frac{\gamma D_2}{8} \int_\Omega v_{\varepsilon xx}^2 + c_{14}
$$

for all $t > T_1$ and $\varepsilon \in (0, 1)$, \hspace{1cm} (6.39)

again because $\gamma \leq 1$.

We next intend to estimate the cross-diffusive contributions to (6.36), and to this end we use integration by parts in firstly rewriting

$$
\chi_1 \int_\Omega \left( h_{1,\varepsilon}(u_{\varepsilon})v_{\varepsilon xx} \right) u_{\varepsilon xx} = \chi_1 \int_\Omega h_{1,\varepsilon}(u_{\varepsilon})u_{\varepsilon xx} v_{\varepsilon xx} + \frac{\chi_1}{2} \int_\Omega h_{1,\varepsilon}(u_{\varepsilon})u_{\varepsilon xx}^2 v_{\varepsilon xx}
$$

$$
= \chi_1 \int_\Omega h_{1,\varepsilon}(u_{\varepsilon})u_{\varepsilon xx} v_{\varepsilon xx} - \frac{\chi_1}{2} \int_\Omega h_{1,\varepsilon}''(u_{\varepsilon})u_{\varepsilon xx}^3 v_{\varepsilon xx}
$$

for $t > 0$, where by Lemma 6.1 the Cauchy-Schwarz inequality, (6.11) and (6.30),

$$
\chi_1 \int_\Omega h_{1,\varepsilon}(u_{\varepsilon})u_{\varepsilon xx} v_{\varepsilon xx} \leq \chi_1 \int_\Omega |u_{\varepsilon xx}v_{\varepsilon xx}|
$$

$$
\leq \chi_1 \|u_{\varepsilon xx}\|_{L^2(\Omega)} \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

$$
\leq c_3 \chi_1 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 + 1 \right\} \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

for all $t > T_1$ and $\varepsilon \in (0, 1)$. Moreover, thanks to Lemma 6.1 and the Hölder inequality,

$$
\leq \frac{(5 - n_1)\chi_1}{2} \int_\Omega v_{\varepsilon xx}^2
$$

$$
\leq \frac{(5 - n_1)\chi_1}{2} \|u_{\varepsilon xx}\|_{L^4(\Omega)}^2 \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

and here combining (6.14) with (6.13), (6.30) and Young’s inequality shows that

$$
\|u_{\varepsilon xx}\|_{L^4(\Omega)}^2 = \|u_{\varepsilon xx}\|_{L^4(\Omega)}^2 \cdot \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2
$$

$$
\leq c_5^8 c_6 \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 + 1 \right\}
$$

$$
\leq 2 c_5^8 c_6 \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 + 1 \right\}
$$

for all $t > T_1$ and $\varepsilon \in (0, 1)$, \hspace{1cm} (6.41)

and (6.42)

and here combining (6.14) with (6.13), (6.30) and Young’s inequality shows that

$$
\frac{\gamma}{8} \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2
$$

According to (6.41), (6.42) and our definition (5.22) of $c_{15}$, we therefore readily obtain using Young’s inequality that since $\int_\Omega u_{\varepsilon xx}^2(\cdot, t) \leq y_\varepsilon(t)$ for all $t > 0$ by (6.28),

$$
\chi_1 \int_\Omega h_{1,\varepsilon}(u_{\varepsilon})u_{\varepsilon xx} v_{\varepsilon xx} - \frac{\chi_1}{2} \int_\Omega h_{1,\varepsilon}''(u_{\varepsilon})u_{\varepsilon xx}^3 v_{\varepsilon xx}
$$

$$
\leq \left( c_3 + (5 - n_1)c_5^8 c_6 \right) \chi_1 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)}^2 + 1 \right\} \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)} + 1 \right\} \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

$$
\leq c_{15} \chi_1 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)} + 1 \right\} \cdot \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

$$
\leq c_{15} \chi_1 \cdot \left\{ \|u_{\varepsilon xx}\|_{L^2(\Omega)} + 1 \right\} \|v_{\varepsilon xx}\|_{L^2(\Omega)}
$$

36
$$\leq \frac{D_1}{64} \cdot \left\{ \| u_{\varepsilon xx} \|_{L^2(\Omega)} + 1 \right\}^2 + \frac{16c_{15}^2 \chi_1^2}{D_1} \cdot \left\{ \frac{1}{2} y_{\varepsilon}^3(t) + 1 \right\}^2 \| v_{\varepsilon xx} \|_{L^2(\Omega)}$$

$$\leq \frac{D_1}{32} \int_\Omega u_{\varepsilon xx}^2 + \frac{D_1}{32} + \frac{16c_{15}^2 \chi_1^2}{D_1} \cdot \left\{ y_{\varepsilon}^3(t) + 1 \right\}^2 \int_\Omega v_{\varepsilon xx}^2$$

for all $t > T_1$ and $\varepsilon \in (0, 1)$. (6.44)

Here we note that since $c_{22} \geq 1$ and $\gamma \leq 1$ and thus $1 \leq c_{22}^\frac{3}{2} \gamma^{-\frac{3}{2}}$ by (6.25), in view of (6.29) and (6.26) inside the set $S_\varepsilon$ in (6.35) we may estimate

$$\frac{16c_{15}^2 \chi_1^2}{D_1} \cdot \left\{ \frac{1}{2} y_{\varepsilon}^3(t) + 1 \right\}^2 \leq \frac{512c_{15}^2 \chi_1^2}{\gamma D_1 D_2} \cdot \left\{ y_{\varepsilon}^3(t) + 1 \right\}^2$$

$$\leq \frac{512c_{15}^2 \chi_1^2}{\gamma D_1 D_2} \cdot \left( c_{22}^\frac{3}{2} \gamma^{-\frac{3}{2}} + 1 \right)^2$$

$$\leq \frac{2048c_{15}^2 c_{22}}{D_1 D_2} \cdot \chi_1^2 \gamma^{-\frac{3}{2}}$$

$$= \frac{2048c_{15}^2 c_{22}}{D_1 D_2} \cdot \chi_1^2 \gamma^{-\frac{3}{2}}$$

$$\leq \frac{2048c_{15}^2 c_{22}}{D_1 D_2} \cdot \chi_1^2 \gamma^{-\frac{3}{2}}$$

$$\leq 1$$

for all $t \in S_\varepsilon$ and each $\varepsilon \in (0, 1)$,

whence (6.44) implies that actually

$$\chi_1 \int_\Omega h_{1, \varepsilon}(u_{\varepsilon}) u_{\varepsilon xx} v_{\varepsilon xx} - \frac{\chi_1}{2} \int_\Omega h_{1, \varepsilon}'(u_{\varepsilon}) u_{\varepsilon xx}^2 v_{\varepsilon xx}$$

$$\leq \frac{D_1}{32} \int_\Omega u_{\varepsilon xx}^2 + \frac{\gamma D_2}{32} \int_\Omega v_{\varepsilon xx}^2 + \frac{D_1}{32} \int_\Omega v_{\varepsilon xx}^2$$

for all $t \in S_\varepsilon$ and any $\varepsilon \in (0, 1)$. (6.45)

In the rightmost summand in (6.40), once more relying on Lemma 6.1 enables us to combine Young’s inequality with Lemma 2.4 to see that due to (6.26) and our definition of $\theta$, and again thanks to (6.34), for all $t > 0$ and $\varepsilon \in (0, 1)$ we have

$$-\frac{\chi_1}{2} \int_\Omega h_{1, \varepsilon}'(u_{\varepsilon}) u_{\varepsilon xx}^3 v_{\varepsilon xx} \leq c_{16} \chi_1 \varepsilon^{\frac{1}{3}(4 - \gamma)} \int_\Omega \left| \frac{u_{\varepsilon xx}}{u_\varepsilon^4} \right| v_{\varepsilon xx}^3$$

$$\leq c_{16} \chi_1 \varepsilon^\theta \int_\Omega \frac{u_{\varepsilon xx}^4}{u_\varepsilon^4} + c_{16} \chi_1 \varepsilon^\theta \int_\Omega v_{\varepsilon xx}^4$$

$$\leq 9c_{16} \chi_1 \varepsilon^\theta \int_\Omega u_{\varepsilon xx}^2 + c_6^4 c_{16} \chi_1 \varepsilon^\theta \| v_{\varepsilon xx} \|_{L^2(\Omega)} \| v_{\varepsilon xx} \|_{L^2(\Omega)}^3$$

$$\leq 9c_{16} \chi_1 \varepsilon^\theta \int_\Omega u_{\varepsilon xx}^2$$

$$+ \frac{1}{2} c_6^4 c_{16} \chi_1 \varepsilon^\theta \int_\Omega v_{\varepsilon xx}^2 + \frac{1}{2} c_6^4 c_{16} \chi_1 \varepsilon^\theta \gamma^{-3} y_{\varepsilon}(t),$$

(6.46)
because \( \int_\Omega v_{ex}^2(\cdot, t) \leq \gamma^{-1} y_\varepsilon(t) \) for \( t > 0 \) by \((6.28)\). Again restricted to times belonging to \( S_\varepsilon \), due to \((6.35)\) and \((6.32)\) this implies that

\[
-\frac{\chi_1}{2} \int_\Omega h_{1,\varepsilon}(u_\varepsilon) u_{ex}^3 v_{ex} \leq 9c_1 L_1 \varepsilon^{\theta} \int_\Omega u_{ex}^2 + \frac{1}{2} 4c_0 c_{16} \chi_1 \varepsilon^{\theta} \int_\Omega v_{ex}^2 + \frac{1}{2} c_0 c_{16} c_2 \chi_1 \gamma^{-\frac{m_0}{m}} \varepsilon^{\theta} \\
\leq \frac{D_1}{64} \int_\Omega u_{ex}^2 + \frac{\gamma D_2}{64} \int_\Omega v_{ex}^2 + 1 \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_*) \quad (6.47)
\]

Now by arguments quite similar to those used in \((6.40)-(6.47)\), on splitting

\[
-\gamma \chi_2 \int_\Omega h_{2,\varepsilon}(v_\varepsilon) u_{exx} v_{exx} = -\gamma \chi_2 \int_\Omega h_{2,\varepsilon}(v_\varepsilon) u_{exx} v_{exx} + \frac{\gamma \chi_2}{2} \int_\Omega h_{2,\varepsilon}(v_\varepsilon) v_{exx}^2 u_{exx} \\
+ \frac{\gamma \chi_2}{2} \int_\Omega h_{2,\varepsilon}(v_\varepsilon) v_{exx}^2 u_{exx}, \quad t > 0, 
\]

we may first essentially copy the reasoning from \((6.41), (6.42), (6.43), (6.44), (6.45)\): Indeed, again using that \( \int_\Omega v_{exx}^2(\cdot, t) \leq \gamma^{-1} y_\varepsilon(t) \) for \( t > 0 \) we thereby obtain that

\[
-\gamma \chi_2 \int_\Omega h_{2,\varepsilon}(v_\varepsilon) u_{exx} v_{exx} + \frac{\gamma \chi_2}{2} \int_\Omega h_{2,\varepsilon}(v_\varepsilon) v_{exx}^2 u_{exx} \leq c_{15} \gamma \chi_2 \cdot \left\{ \|v_{exx}\|_{L^2(\Omega)}^{\frac{2}{3}} + 1 \right\} \|u_{exx}\|_{L^2(\Omega)} \left\{ \|v_{exx}\|_{L^2(\Omega)} + 1 \right\} \\
\leq c_{15} \gamma \chi_2 \cdot \left\{ \gamma^{-\frac{3}{2}} y_\varepsilon(t) + 1 \right\} \|u_{exx}\|_{L^2(\Omega)} \left\{ \|v_{exx}\|_{L^2(\Omega)} + 1 \right\} \\
\leq \frac{D_1}{32} \|u_{exx}\|_{L^2(\Omega)}^2 + \frac{8 c_{15}^2 \gamma^2 \chi_2^2}{D_1} \cdot \left\{ \gamma^{-\frac{3}{4}} y_\varepsilon(t) + 1 \right\} \cdot \left\{ \|v_{exx}\|_{L^2(\Omega)} + 1 \right\} \\
\leq \frac{D_1}{32} \int_\Omega u_{exx}^2 + \frac{16 c_{15}^2 \gamma^2 \chi_2^2}{D_1} \cdot \left\{ \gamma^{-\frac{3}{4}} y_\varepsilon(t) + 1 \right\} \cdot \left\{ \int_\Omega v_{exx}^2 + 1 \right\} \\
\leq \frac{D_1}{32} \int_\Omega u_{exx}^2 + \frac{\gamma D_2}{32} \int_\Omega v_{exx}^2 + \frac{\gamma D_2}{32} \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, 1), 
\]

because according to \((6.35)\) and the inequalities \( \gamma \leq 1 \) and \( c_{22}^2 \gamma^{-\frac{m_0}{m}} \geq 1 \) asserted by \((6.25)\) and \((6.29)\) we have

\[
\frac{16 c_{15}^2 \gamma^2 \chi_2^2}{D_1} \cdot \left\{ \gamma^{-\frac{3}{4}} y_\varepsilon(t) + 1 \right\}^2 = \frac{512 c_{15}^2 \gamma^2 \chi_2^2}{D_1 D_2} \cdot \left\{ \gamma^{-\frac{3}{4}} y_\varepsilon(t) + 1 \right\}^2 \\
< \frac{512 c_{15}^2 \gamma^2 \chi_2^2}{D_1 D_2} \cdot (c_{22}^2 \gamma^{-\frac{m_0}{m}} + 1)^2 \\
< \frac{2048 c_{15}^2 c_{22}^2 \gamma^2 \chi_2^2}{D_1 D_2} \\
\leq 1 \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, 1)
\]
due to (6.26) and our restriction that $1 \geq \frac{3}{23}$.

Likewise, proceeding as in (6.40) and (6.47) we find that since $\int_{\Omega} u_{xx}^2(\cdot, t) \leq v(\cdot, t) < c_{22} \gamma^{-\frac{3}{10}}$ for all $t \in S_\varepsilon$,

$$\frac{\gamma \chi^2}{2} \int_{\Omega} h'_2 v_{xx} u_{xx} v_{xx}^3 \leq 9 c_1 \gamma \chi^2 \varepsilon^3 \int_{\Omega} u_{xx}^2 + c_4^3 \gamma \chi^2 \varepsilon^3 \|u_{xxx}\| L^2(\Omega) \|u_{xx}\|^3 L^2(\Omega) \leq 9 c_1 \gamma \chi^2 \varepsilon^3 \int_{\Omega} u_{xx}^2 + \frac{1}{2} c_0^4 \gamma \chi^2 \varepsilon^3 \int_{\Omega} u_{xx}^2 + \frac{1}{2} c_0^4 c_2 \gamma \chi^2 \varepsilon^3 \leq \frac{D_1}{64} \int_{\Omega} u_{xx}^2 + \frac{\gamma D_2}{64} \int_{\Omega} v_{xx}^2 + 1 \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_*) \quad (6.50)$$

because of (6.33).

When inserted into (6.39), in view of (6.40) and (6.48) the estimates (6.37), (6.38), (6.39), (6.45), (6.47), (6.49) and (6.50) in summary show that as $\gamma \leq 1$,

$$\frac{1}{2} y'_{\varepsilon}(t) + \frac{D_1}{32} \int_{\Omega} u_{xx}^2 + \frac{\gamma D_2}{32} \int_{\Omega} v_{xx}^2 \leq c_1 \gamma^{-\frac{3}{10}} + c_2 + c_3 + c_4 + \frac{D_1}{32} + 1 + \frac{\gamma D_2}{32} + 1 \leq c_1 \gamma^{-\frac{3}{10}} \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_*) \quad (6.51)$$

according to the definition of $c_1$ in (6.23). In order to turn the two remaining integrals herein into an appropriate superlinear absorptive term, we once more make use of (6.12), (6.30) and Young’s inequality to see that again since $\gamma \leq 1$,

$$\frac{3}{2} y_{\varepsilon}^2(t) = \left\{ \int_{\Omega} u_{xx}^2 + \gamma \int_{\Omega} v_{xx}^2 \right\} \frac{3}{2} \leq c_4^3 \cdot \left\{ \|u_{xx}\|^\frac{3}{2} L^2(\Omega) + 1 + \gamma \cdot \left(\|v_{xx}\|^\frac{3}{2} L^2(\Omega) + 1\right) \right\} \frac{3}{2} \leq c_4^3 \cdot \left\{ \|u_{xx}\|^\frac{3}{2} L^2(\Omega) + \gamma \|v_{xx}\|^\frac{3}{2} L^2(\Omega) + 2 \right\} \frac{3}{2} \leq (3c_4^3) \cdot \left\{ \|u_{xx}\|^\frac{3}{2} L^2(\Omega) + \gamma \|v_{xx}\|^\frac{3}{2} L^2(\Omega) + 2 \right\} \frac{3}{2} \quad \text{for all } t > T_1 \text{ and } \varepsilon \in (0, 1),$$

so that for all $t > T_1$ and any $\varepsilon \in (0, 1)$,

$$\frac{D_1}{32} \int_{\Omega} u_{xx}^2 + \frac{\gamma D_2}{32} \int_{\Omega} v_{xx}^2 \geq \min \left\{ \frac{D_1}{32}, \frac{D_2}{32} \right\} \cdot \left\{ \int_{\Omega} u_{xx}^2 + \gamma \int_{\Omega} v_{xx}^2 \right\} \geq \frac{\min \left\{ \frac{D_1}{32}, \frac{D_2}{32} \right\} \cdot (3c_4^3) \cdot \gamma_{\varepsilon}^2(t) - \frac{2\pi}{16} \cdot \min \left\{ \frac{D_1}{32}, \frac{D_2}{32} \right\} \cdot (3c_4^3) \cdot \gamma_{\varepsilon}^2(t) \right\}$$

In light of (6.24), (6.51) therefore entails the autonomous ODI

$$\frac{1}{2} y_{\varepsilon}^2(t) + c_1 y_{\varepsilon}^2(t) \leq c_2 \gamma^{-\frac{1}{2}} \quad \text{for all } t \in S_\varepsilon \text{ and } \varepsilon \in (0, \varepsilon_*) \quad (6.52)$$

To deduce (6.33) from this, we now recall (6.31) to infer that for each $\varepsilon \in (0, 1)$ we can fix $t_\varepsilon \in (T_1, T_*)$ such that

$$\int_{\Omega} u_{xx}^2(\cdot, t_\varepsilon) + \rho \int_{\Omega} v_{xx}^2(\cdot, t_\varepsilon) \leq 2K.$$
Through (6.15) and again (6.30), this firstly entails that
\[ \| u_{\varepsilon}(\cdot, t_{\varepsilon}) \|_{L^\infty(\Omega)}^2 = \left\{ \sqrt{u_{\varepsilon}(\cdot, t_{\varepsilon})} \right\}_{x}^2 \|_{L^2(\Omega)}^2 + 1 \]
\[ = \frac{c_7}{4} \int_{\Omega} u_{\varepsilon}(\cdot, t_{\varepsilon}) + c_7 \]
\[ \leq \frac{c_7 K}{2} + c_7 \quad \text{for all } \varepsilon \in (0, 1) \]
and, similarly,
\[ \| v_{\varepsilon}(\cdot, t_{\varepsilon}) \|_{L^\infty(\Omega)}^2 \leq \frac{c_7 K}{2} + \gamma \cdot \sqrt{c_7 K} + \frac{2K}{\rho} \]
\[ \leq c_2 + c_2 \gamma \rho^{-\frac{3}{2}} \quad \text{for all } \varepsilon \in (0, 1) \] (6.54)
Therefore, (6.54) secondly implies that
\[ y_{\varepsilon}(t_{\varepsilon}) \leq \| u_{\varepsilon}(\cdot, t_{\varepsilon}) \|_{L^\infty(\Omega)} \int_{\Omega} u_{\varepsilon}(\cdot, t_{\varepsilon}) + \gamma \| v_{\varepsilon}(\cdot, t_{\varepsilon}) \|_{L^\infty(\Omega)} \int_{\Omega} v_{\varepsilon}(\cdot, t_{\varepsilon}) \]
\[ \leq \sqrt{\frac{c_7 K}{2}} + c_7 \cdot \sqrt{\frac{2K}{\rho}} \cdot \frac{2K}{\rho} \]
\[ \leq c_2 + c_2 \gamma \rho^{-\frac{3}{2}} \quad \text{for all } \varepsilon \in (0, 1) \] (6.55)
by (6.26), for we are yet assuming that \( \rho \leq 1 \).
But now the precise link between \( \gamma \) and \( \rho \) in (6.29) ensures that
\[ \frac{c_2 + c_2 \gamma \rho^{-\frac{3}{2}}}{\frac{1}{c_2} c_2 \gamma^{-\frac{3}{4}}} = \frac{4c_2 + 13\gamma \rho^{-\frac{3}{2}}}{c_2} = \frac{13}{c_2} \leq 1 \]
due to (6.25), so that since (6.25) moreover entails that
\[ \frac{c_2}{\frac{1}{c_2} c_2 \gamma^{-\frac{3}{4}}} \leq \frac{4c_2}{c_2} \leq 1, \]
from (6.54) we infer that
\[ y_{\varepsilon}(t_{\varepsilon}) \leq \frac{1}{2} c_2 \gamma^{-\frac{3}{4}} \quad \text{for all } \varepsilon \in (0, 1) \] (6.55)
This especially guarantees that \( t_{\varepsilon} \) belongs to \( S_{\varepsilon} \) and that hence
\[ T_{\varepsilon} := \sup \left\{ T > t_{\varepsilon} \mid [t_{\varepsilon}, T] \subset S_{\varepsilon} \right\} \]
is a well-defined element of \( (t_{\varepsilon}, \infty) \) for all \( \varepsilon \in (0, 1) \). According to (6.52), however, for each \( \varepsilon \in (0, \varepsilon_*) \) we actually must have \( T_{\varepsilon} = \infty \): Otherwise, namely, the definition of \( S_{\varepsilon} \) would entail that \( (t_{\varepsilon}, T_{\varepsilon}) \subset S_{\varepsilon} \) but
\[ y_{\varepsilon}(T_{\varepsilon}) = c_2 \gamma^{-\frac{3}{4}}, \] (6.56)
while since $\overline{y}(t) := \frac{1}{2}c_{22}\gamma^{-\frac{4}{3}}\cdot t$, $t \geq t_\varepsilon$, satisfies
\[
\frac{1}{2}\overline{y}(t) + c_{19}\gamma^{\frac{4}{3}}(t) - c_{20}\gamma^{-\frac{4}{3}} = \left(2^{-\frac{4}{3}}c_{19}\gamma^{\frac{4}{3}} - c_{20}\right)\cdot \gamma^{-\frac{4}{3}} \geq 0 \quad \text{for all } t > t_\varepsilon
\]
as a further consequence of (6.25), a comparison argument based on (6.55) would show that $y_\varepsilon(t) \leq \overline{y}(t)$ for all $t \in [t_\varepsilon, T_\varepsilon]$ and thereby contradict (6.56). This confirms that indeed $[t_\varepsilon, \infty) \subset S_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_*)$ and that thus (6.32) results from the fact that $t_\varepsilon < T_\varepsilon$ for any such $\varepsilon$. In view of (6.30) and e.g. (6.11), we therefore conclude that both (6.6) and (6.7) are valid whenever $0 < \chi_1 \leq \chi_2 < \chi_\ast$, whereupon going back to (6.51) we infer by integration that also (6.8) holds for all such $\chi_1$ and $\chi_2$.

According to an evident symmetry property of the presented reasoning, however, in the case when conversely $\chi_2 \leq \chi_1$ the above result can be derived in much the same manner, essentially by exchanging the roles of $u_\varepsilon$ and $v_\varepsilon$. □

By once more recalling the convergence properties obtained in Section 4 from Lemma 6.2 we readily derive the following conclusion, inter alia asserting eventual continuity and $H^1$-boundedness of the limit couple from Lemma 4.5.

**Corollary 6.3** Let $D_i > 0, a_i > 0, \lambda_i > 0$ and $n_i \in [1, 2]$ for $i \in \{1, 2\}$, and let $\chi_1 \in (0, \chi_\ast)$ and $\chi_2 \in (0, \chi_\ast)$ with $\chi_\ast > 0$ as given by Lemma 6.2. Then whenever (6.3) holds, there exists $T_\ast = T_\ast(\chi_1, \chi_2, u_0, v_0) > 0$ such that with $(\varepsilon_j)_{j \in N} \subset (0, 1)$ and $N \subset (0, \infty)$ taken from Lemma 4.5 we have
\[
u_\varepsilon(\cdot, t) \rightarrow u(\cdot, t) \quad \text{and} \quad v_\varepsilon(\cdot, t) \rightarrow v(\cdot, t) \quad \text{in } L^\infty(\Omega) \quad \text{for all } t \in (0, \infty) \setminus N \tag{6.57}
\]
as well as
\[
\int_{t_1}^{t_2} \int_{\Omega} \overline{\nu}_\varepsilon \, dx \rightarrow \int_{t_1}^{t_2} \int_{\Omega} \overline{u} \, dx \quad \text{and} \quad \int_{t_1}^{t_2} \int_{\Omega} \overline{v}_\varepsilon \, dx \rightarrow \int_{t_1}^{t_2} \int_{\Omega} \overline{u} \, dx \quad \text{for all } t_1 > T_\ast \text{ and any } t_2 > t_1 \tag{6.58}
\]
as $\varepsilon = \varepsilon_\ast \searrow 0$. Moreover, both $u$ and $v$ belong to $C^0(\overline{\Omega} \times (T_\ast, \infty))$ and satisfy $u(\cdot, t) \in W^{1,2}(\Omega)$ and $v(\cdot, t) \in W^{1,2}(\Omega)$ for all $t > T_\ast$, and one can find $C > 0$ such that
\[
\|u(\cdot, t)\|_{W^{1,2}(\Omega)} + \|v(\cdot, t)\|_{W^{1,2}(\Omega)} \leq C \quad \text{for all } t > T_\ast. \tag{6.59}
\]

**Proof.** According to Lemma 6.2 for fixed $u_0$ and $v_0$ we can find $T_\ast > 0$, $c_1 > 0$, $c_2 > 0$ and $\varepsilon_\ast \in (0, 1)$ such that for any $\varepsilon \in (0, \varepsilon_\ast)$,
\[
\|u_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \leq c_1 \quad \text{for all } t > T_\ast \tag{6.60}
\]
and
\[
\int_{T_\ast}^{T} \|u_\varepsilon(\cdot, t)\|^2_{W^{2,2}(\Omega)} dt + \int_{T_\ast}^{T} \|v_\varepsilon(\cdot, t)\|^2_{W^{2,2}(\Omega)} dt \leq c_2 \cdot (T + 1) \quad \text{for all } T > T_\ast. \tag{6.61}
\]

As (6.30) warrants relative compactness of $(u_\varepsilon(\cdot, t))_{\varepsilon \in (0, \varepsilon_\ast)}$ and of $(v_\varepsilon(\cdot, t))_{\varepsilon \in (0, \varepsilon_\ast)}$ with respect to the weak topology in $W^{1,2}(\Omega)$ for all $t > T_\ast$, it is thus clear from Lemma 4.5 that as $\varepsilon = \varepsilon_\ast \searrow 0$, $u_\varepsilon(\cdot, t) \rightarrow u(\cdot, t)$ and $v_\varepsilon(\cdot, t) \rightarrow v(\cdot, t)$ in $W^{1,2}(\Omega)$ for all $t \in (T_\ast, \infty) \setminus N$. \tag{6.62}
and thereby especially entails \((6.57)\) by compactness of the embedding \(W^{1,2}(\Omega) \to L^\infty(\Omega)\). Since thus \((u(\cdot, t))_{t \in (T_*, \infty) \setminus N} \) and \((v(\cdot, t))_{t \in (T_*, \infty) \setminus N}\) are bounded in \(W^{1,2}(\Omega)\) due to \((6.62)\) and \((6.60)\), and since from Lemma \(4.5\) we already know that both \(u\) and \(v\) belong to \(C^0_0([0, \infty); L^1(\Omega))\), it can therefore easily be verified that actually \(u\) and \(v\) are continuous throughout \((T_*, \infty)\) as \(W^{1,2}(\Omega)\)-valued functions with respect to the weak topology therein. Again through compactness of the embedding \(W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega)\), this entails that indeed \(u\) and \(v\) are contained in \(C^0((T_*, \infty); C^0(\overline{\Omega}))\) and satisfy \((6.59)\).

Apart from that, for all \(T > T_*\) \((6.61)\) ensures boundedness of the families \((u_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}\) and \((v_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}\) in \(L^2((T_*, T); W^{2,2}(\Omega))\), so that recalling the time regularity properties from Lemma \(4.4\) we may once more invoke the Aubin-Lions lemma to conclude that \(u_\varepsilon \to u\) and \(v_\varepsilon \to v\) in \(L^2((T_*, T); W^{1,2}(\Omega))\), and that in particular \((6.58)\) holds.

\[\]

### 7 Toward verifying consistency with a genuine entropy structure

Now having at hand the eventual regularity properties gained above, for establishing our main results on stabilization in the flavor of Theorem \(1.2\) and Theorem \(1.3\) it will be sufficient to make sure that the genuine gradient-like structures related to the functionals in \((1.12)\) and \((1.14)\) find some appropriate rigorous counterpart in the approximate problems \((1.6)-(1.7)\). This short section provides the fundament therefor by recording results of associated testing procedures, which will in particular foreshadow our final selections of the key parameters \(n_1\) and \(n_2\).

**Lemma 7.1** Let \(D_i > 0, a_i > 0, \lambda_i > 0\) and \(n_i \in [1, 2]\) for \(i \in \{1, 2\}\), and assume \((L.5)\).

i) If \(n_1 = 2\), then for all \(\varepsilon \in (0, 1)\),

\[
\frac{d}{dt} \left\{ -\int_\Omega \ln u_\varepsilon + \frac{\varepsilon}{6} \int_\Omega \frac{1}{u_\varepsilon^2} \right\} + \varepsilon \int_\Omega u_{\varepsilon xx}^2 + D_1 \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon} + D_1 \varepsilon \int_\Omega \frac{u_{\varepsilon xx}^2}{u_\varepsilon^2} + \varepsilon \alpha \int_\Omega \frac{1}{u_\varepsilon^{\alpha - 2}} u_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega u_{\varepsilon xx}^\alpha + \varepsilon \int_\Omega u_{\varepsilon xx}^\alpha + \varepsilon \alpha \int_\Omega \frac{1}{u_\varepsilon^{\alpha - 2}} u_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega u_{\varepsilon xx}^\alpha = \lambda_1 \int_\Omega u_\varepsilon v_\varepsilon v_{\varepsilon xx} - \lambda_1 |\Omega| + \int_\Omega u_\varepsilon - a_1 \int_\Omega v_\varepsilon \quad \text{for all } t > 0. \tag{7.1}
\]

ii) If \(n_2 = 2\), then for all \(\varepsilon \in (0, 1)\),

\[
\frac{d}{dt} \left\{ -\int_\Omega \ln v_\varepsilon + \frac{\varepsilon}{6} \int_\Omega \frac{1}{v_\varepsilon^2} \right\} + \varepsilon \int_\Omega v_{\varepsilon xx}^2 + D_2 \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} + D_2 \varepsilon \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon^2} + \varepsilon \alpha \int_\Omega \frac{1}{v_\varepsilon^{\alpha - 2}} v_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega v_{\varepsilon xx}^\alpha + \varepsilon \int_\Omega v_{\varepsilon xx}^\alpha + \varepsilon \alpha \int_\Omega \frac{1}{v_\varepsilon^{\alpha - 2}} v_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega v_{\varepsilon xx}^\alpha = -\lambda_2 \int_\Omega u_\varepsilon v_\varepsilon v_{\varepsilon xx} - \lambda_2 |\Omega| + \int_\Omega v_\varepsilon + a_2 \int_\Omega u_\varepsilon \quad \text{for all } t > 0. \tag{7.2}
\]

iii) In the case \(n_2 = 1\), for all \(\varepsilon \in (0, 1)\) we have

\[
\frac{d}{dt} \left\{ \frac{1}{2} \int_\Omega v_{\varepsilon xx}^2 + \frac{\varepsilon}{2} \int_\Omega \frac{1}{v_\varepsilon^2} \right\} + \varepsilon \int_\Omega v_{\varepsilon xx}^2 + D_2 \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon} + D_2 \varepsilon \int_\Omega \frac{v_{\varepsilon xx}^2}{v_\varepsilon^2} + \varepsilon \alpha \int_\Omega \frac{1}{v_\varepsilon^{\alpha - 2}} v_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega v_{\varepsilon xx}^\alpha + \varepsilon \int_\Omega v_{\varepsilon xx}^\alpha + \varepsilon \alpha \int_\Omega \frac{1}{v_\varepsilon^{\alpha - 2}} v_{\varepsilon xx}^2 + \varepsilon \frac{\alpha + 2}{2} \int_\Omega v_{\varepsilon xx}^\alpha
\]

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\[
\begin{align*}
&= -\chi_2 \int_{\Omega} v_\varepsilon u_\varepsilon v_\varepsilon + \lambda_2 \int_\Omega v_\varepsilon^2 - \int_{\Omega} v_\varepsilon^3 - a_2 \int_{\Omega} u_\varepsilon v_\varepsilon^2 \\
&\quad - \varepsilon \int_{\Omega} 2v_\varepsilon^2 + 3v_\varepsilon s \varepsilon (\lambda_2 - v_\varepsilon - a_2 u_\varepsilon) \quad \text{for all } t > 0. 
\end{align*}
\tag{7.3}
\]

**Proof.**

i) Writing \(\ell(s) : = - \ln s + \frac{s - \varepsilon}{6} \) for \( s > 0 \) and fixed \( \varepsilon \in (0, 1) \), we have \(\ell'(s) = - \frac{1}{s} - \frac{\varepsilon}{3s^2} = -\frac{3s^2 + \varepsilon}{3s^2} \) and \(\ell''(s) = \frac{1}{s^2} + \varepsilon = \frac{s^2 + \varepsilon}{s^2} \) for all \( s > 0 \), so that using the first equation in (1.6) we obtain

\[
\frac{d}{dt} \int_{\Omega} \ell(u_\varepsilon) = - \varepsilon \int_{\Omega} \ell'(u_\varepsilon) \cdot \left( \frac{u_\varepsilon^4}{u_\varepsilon^2 + \varepsilon} u_\varepsilon v_\varepsilon \right)_x + \frac{1}{\Omega} \int_{\Omega} \ell'(u_\varepsilon) \cdot (u_\varepsilon^{-\alpha} u_\varepsilon v_\varepsilon)_x \\
+ D_1 \int_{\Omega} \ell'(u_\varepsilon) u_\varepsilon v_\varepsilon - \chi_1 \int_{\Omega} \ell'(u_\varepsilon) \cdot \left( \frac{u_\varepsilon^3}{u_\varepsilon^2 + \varepsilon} v_\varepsilon \right)_x \\
+ \int_{\Omega} \ell'(u_\varepsilon) \cdot \frac{3u_\varepsilon^3}{3u_\varepsilon^2 + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon)
\]

\[
= \varepsilon \int_{\Omega} \ell''(u_\varepsilon) \cdot \left( \frac{u_\varepsilon^4}{u_\varepsilon^2 + \varepsilon} u_\varepsilon v_\varepsilon \right)_x - \varepsilon \int_{\Omega} \ell''(u_\varepsilon) u_\varepsilon^{-\alpha} u_\varepsilon^2 \\
- D_1 \int_{\Omega} \ell''(u_\varepsilon) u_\varepsilon^2 + \chi_1 \int_{\Omega} \ell''(u_\varepsilon) \cdot \left( \frac{u_\varepsilon^3}{u_\varepsilon^2 + \varepsilon} v_\varepsilon \right)_x \\
+ \int_{\Omega} \ell'(u_\varepsilon) \cdot \frac{3u_\varepsilon^3}{3u_\varepsilon^2 + \varepsilon} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon)
\]

\[
= \varepsilon \int_{\Omega} u_\varepsilon u_\varepsilon v_\varepsilon v_\varepsilon x - \varepsilon \int_{\Omega} \ell''(u_\varepsilon) u_\varepsilon^{-\alpha} u_\varepsilon^2 \\
- D_1 \int_{\Omega} \left( \frac{1}{u_\varepsilon^2} + \frac{\varepsilon}{u_\varepsilon^4} \right) u_\varepsilon^2 + \chi_1 \int_{\Omega} \frac{1}{u_\varepsilon} u_\varepsilon v_\varepsilon v_\varepsilon \\
- \int_{\Omega} (\lambda_1 - u_\varepsilon + a_1 v_\varepsilon) \quad \text{for all } t > 0.
\]

As clearly \(\varepsilon \int_{\Omega} u_\varepsilon u_\varepsilon v_\varepsilon v_\varepsilon x = -\varepsilon \int_{\Omega} u_\varepsilon^2 \) for all \( t > 0 \), this is equivalent to (7.1).

ii) This part analogously follows from the second equation in (1.6).

iii) We now rather let \(\ell(s) : = \frac{\varepsilon}{2} s^2 + \frac{\varepsilon^3}{2} \) for \( s > 0 \) and \( \varepsilon \in (0, 1) \), and observing that then \(\ell'(s) = s - \frac{\varepsilon}{2s^2} = \frac{2s^2 - \varepsilon}{2s^2} \) and \(\ell''(s) = 1 + \frac{\varepsilon}{2s^3} = \frac{2s^3 - \varepsilon}{2s^3} \) for all \( s > 0 \), we use (1.6)-(1.7) to compute

\[
\frac{d}{dt} \int_{\Omega} \ell(v_\varepsilon) = \varepsilon \int_{\Omega} \ell'(v_\varepsilon) \cdot \left( \frac{u_\varepsilon^4}{u_\varepsilon^2 + \varepsilon} u_\varepsilon v_\varepsilon v_\varepsilon v_\varepsilon x - \varepsilon \int_{\Omega} \ell''(v_\varepsilon) v_\varepsilon^{-\alpha} v_\varepsilon^2 \\
- D_2 \int_{\Omega} \ell'(v_\varepsilon) v_\varepsilon^2 v_\varepsilon - \chi_2 \int_{\Omega} \ell''(v_\varepsilon) \cdot \frac{u_\varepsilon^4}{u_\varepsilon^2 + \varepsilon} u_\varepsilon v_\varepsilon v_\varepsilon \\
+ \int_{\Omega} \ell'(v_\varepsilon) \cdot \frac{3u_\varepsilon^3}{3u_\varepsilon^2 + \varepsilon} (\lambda_2 - v_\varepsilon - a_2 u_\varepsilon)
\]

\[
= \varepsilon \int_{\Omega} v_\varepsilon v_\varepsilon v_\varepsilon v_\varepsilon x - \varepsilon \int_{\Omega} \ell''(v_\varepsilon) v_\varepsilon^{-\alpha} v_\varepsilon^2 \\
- D_2 \int_{\Omega} \left( 1 + \frac{\varepsilon}{v_\varepsilon^3} \right) v_\varepsilon^2 v_\varepsilon - \chi_2 \int_{\Omega} v_\varepsilon u_\varepsilon v_\varepsilon v_\varepsilon \\
\]
\[
+ \int_{\Omega} \frac{3v_\varepsilon (2v_\varepsilon^2 - \varepsilon)}{2 \cdot (3v_\varepsilon^2 + \varepsilon)} \cdot (\lambda_2 - v_\varepsilon - a_1 u_\varepsilon) \quad \text{for all } t > 0.
\]

Since
\[
\varepsilon \int_{\Omega} v_\varepsilon v_{\varepsilon x} v_{\varepsilon xxx} = -\varepsilon \int_{\Omega} v_\varepsilon v_{\varepsilon xx}^2 - \varepsilon \int_{\Omega} v_{\varepsilon xx} v_{\varepsilon xxx} = -\varepsilon \int_{\Omega} v_\varepsilon v_{\varepsilon xx}^2 \quad \text{for all } t > 0,
\]
and since
\[
\frac{3s (2s^3 - \varepsilon)}{2 \cdot (3s^2 + \varepsilon)} = s^2 - \varepsilon \cdot \frac{2s^2 + 3s}{6s^2 + 2\varepsilon} \quad \text{for all } s > 0,
\]
this yields (7.3).

**8 The case \( \lambda_2 > a_2 \lambda_1 \). Proof of Theorem 1.2**

In our convergence argument concentrating on the situation of Theorem 1.2 first, in view of (1.12) our design of an approximate variant of the functional therein will be based on parts i) and ii) of Lemma 7.1, thus suggesting to fix \( n_1 = n_2 = 2 \) henceforth in this case. For our construction here and also for later reference, let us separately state some elementary properties of the nonlinear map arising in both integrals in (1.12).

**Lemma 8.1** Let \( \xi_* > 0 \) and

\[
\phi_{\xi_*} (\xi) := \xi - \xi_* - \xi_* \ln \frac{\xi}{\xi_*}, \quad \xi > 0.
\]  

Then \( \phi_{\xi_*} \) is positive on \((0, \infty) \setminus \{ \xi_* \} \) with \( \phi_{\xi_*} (\xi_*) = 0 \), and furthermore

\[
\phi_{\xi_*} (\xi) \leq \frac{2}{\xi_*} \cdot (\xi - \xi_*)^2 \quad \text{for all } \xi \geq \frac{\xi_*}{2}.
\]  

**Proof.** Since \( \phi'_{\xi_*} (\xi) = 1 - \frac{\xi_*}{\xi} \) and \( \phi''_{\xi_*} (\xi) = \frac{\xi_*}{\xi} \) for all \( \xi > 0 \), it is clear that the zero \( \xi_* \) of \( \phi_{\xi_*} \) is a strict and global minimum point of \( \phi_{\xi_*} \), and that moreover

\[
\phi_{\xi_*} (\xi) \leq \sup_{\eta \geq \xi_*} |\phi''_{\xi_*} (\eta)| \cdot \frac{(\xi - \xi_*)^2}{2} = \frac{2}{\xi_*} \cdot (\xi - \xi_*)^2
\]

for all \( \xi \geq \frac{\xi_*}{2} \).

Then combining lemma 7.3 with (1.12) suggests to generalize the entropy structure in question as follows.

**Lemma 8.2** Let \( n_1 = n_2 = 2 \), and suppose that \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), with

\[
\lambda_2 > a_2 \lambda_1.
\]  

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Then writing
\[ A := \frac{a_1}{a_2} \] (8.4)
and taking \( u_* > 0 \) and \( v_0 > 0 \) from (7.1), for
\[ \mathcal{E}_{1,\varepsilon}(t) := \int \phi_{u_*}(u_\varepsilon(\cdot, t)) + \frac{u_*\varepsilon}{6} \int_\Omega \frac{1}{u_\varepsilon^2(\cdot, t)} + A \int \phi_{v_*}(v_\varepsilon(\cdot, t)) + \frac{A\varepsilon}{6} \int_\Omega \frac{1}{v_\varepsilon^2(\cdot, t)}, \quad t \geq 0, \ \varepsilon \in (0, 1), \]
with \( \phi_\xi \) taken from (8.1) for \( \xi > 0 \), we have
\[
\frac{d}{dt} \mathcal{E}_{1,\varepsilon}(t) + \left\{ \begin{array}{l} D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} + \frac{A\varepsilon}{2D_2} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 \| \phi_{u_*}(u_\varepsilon(\cdot, t)) \|_{L^\infty(\Omega)}^2 \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} \\
+ \frac{AD_2 v_*}{2} - \frac{\chi^2 u_*}{2D_1} \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 \int_\Omega \frac{v_\varepsilon^2}{v_\varepsilon^2} \\
+ \int_\Omega (u_\varepsilon(\cdot, t) - u_*)^2 + A \int_\Omega (v_\varepsilon(\cdot, t) - v_*)^2 \\
+ u_* \int_\Omega u^\alpha_\varepsilon v^\alpha_\varepsilon + A v^\alpha_\varepsilon v^\alpha_\varepsilon \int_\Omega \frac{v_\varepsilon^2}{v_\varepsilon^2} \\
\end{array} \right. \leq \frac{1 + a_1}{2\sqrt{3}} \cdot \sqrt{\varepsilon} \int_\Omega u_\varepsilon + \frac{A}{2\sqrt{3}} \cdot \sqrt{\varepsilon} \int_\Omega v_\varepsilon \quad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \] (8.6)

**Proof.** We take an appropriate linear combination of the inequalities and identities provided by Lemma (2.2) and Lemma (7.1) i) and ii) to see that for all \( \varepsilon \in (0, 1) \),
\[
\frac{d}{dt} \mathcal{E}_{1,\varepsilon}(t) + \left\{ \begin{array}{l} D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} + D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} \\
+ D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} + D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} \\
+ D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} + D_1 u_* \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} \\
\end{array} \right. \leq \left( \lambda_1 + \frac{\sqrt{\varepsilon}}{2\sqrt{3}} \right) \int_\Omega u_\varepsilon - \int_\Omega u_\varepsilon + A \int_\Omega v_\varepsilon + \frac{A\varepsilon}{2\sqrt{3}} \int_\Omega u_\varepsilon \\
+ \lambda_1 u_* \int_\Omega u_\varepsilon v_\varepsilon - \lambda_1 u_* |\Omega| + u_* \int_\Omega u_\varepsilon - a_1 \int_\Omega v_\varepsilon \\
\leq A \chi_2 v_* \int_\Omega \frac{1}{v_\varepsilon} u_\varepsilon v_\varepsilon - A \chi_2 v_* |\Omega| + A v_\varepsilon \int_\Omega v_\varepsilon + A v_\varepsilon \int_\Omega v_\varepsilon \quad \text{for all } t > 0. \] (8.7)

Here using Young’s inequality, we obtain that
\[
\chi_1 u_* \int_\Omega \frac{1}{v_\varepsilon} u_\varepsilon v_\varepsilon \leq \frac{D_1 u_*}{2} \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^2} + \frac{\chi^2 u_*}{2D_1} \int_\Omega v_\varepsilon^2
\]
and, similarly,
\[-A\lambda_2 v_* \int_{\Omega} \frac{1}{v_\varepsilon} u_\varepsilon v_\varepsilon \leq \frac{AD_2 v_*}{2} \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2} + \frac{A\lambda_2 v_*}{2} \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2} + \frac{A\lambda_2 v_*}{2} \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2}\]
for all \( t > 0 \). Therefore, after neglecting some signed summands and rearranging we infer from (8.7) that
\[
d\frac{d}{dt} \mathcal{E}_{1,\varepsilon}(t) + \left\{ \frac{D_1 u_*}{2} - \frac{A\lambda_2 v_*}{2} \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2} \right\} \cdot \int_{\Omega} \frac{u_\varepsilon^2}{u_\varepsilon^2} + \left\{ \frac{AD_2 v_*}{2} - \frac{A\lambda_2 v_*}{2} \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2} \right\} \cdot \int_{\Omega} \frac{v_\varepsilon^2}{v_\varepsilon^2} + u_\varepsilon \omega^2 \int_{\Omega} u_\varepsilon v_\varepsilon^2 + A v_\varepsilon \omega^2 \int_{\Omega} v_\varepsilon^2 = 0\]
\[
\leq I_\varepsilon(t) := -\int_{\Omega} u_\varepsilon^2 + (\lambda_1 + u_* + Aa_2 v_*) \int_{\Omega} u_\varepsilon - \lambda_1 u_* \int_{\Omega} u_\varepsilon - A \int_{\Omega} v_\varepsilon^2 + (A\lambda_2 - a_1 u_* + A v_*) \int_{\Omega} v_\varepsilon - A\lambda_2 v_* \int_{\Omega} v_\varepsilon + (a_1 - Aa_2) \int_{\Omega} u_\varepsilon v_\varepsilon + \frac{1 + Aa_2}{2\sqrt{3}} \cdot \varepsilon \int_{\Omega} u_\varepsilon + \frac{A}{2\sqrt{3}} \cdot \varepsilon \int_{\Omega} v_\varepsilon \quad \text{for all } t > 0. \quad (8.8)
\]
We now make use of our definition (8.4) of \( A \), which along with (1.11) namely ensures the precise identities
\[
\lambda_1 + u_* + Aa_2 v_* = \lambda_1 + u_* + a_1 v_* = 2u_* \quad \text{and} \quad A\lambda_2 - a_1 u_* + A v_* = 2Av_*
\]
as well as
\[
u_*^2 + Av_*^2 - \lambda_1 u_* - A\lambda_2 v_* = \frac{a_2 u_* (u_* - \lambda_1) + a_1 v_* (v_* - \lambda_2)}{a_2} = \frac{a_2 u_* \cdot a_1 v_* + a_1 v_* \cdot (a_2 u_*)}{a_2} = 0
\]
and clearly also \( a_1 - Aa_2 = 0 \). Accordingly, in (8.8) we may simplify
\[
I_\varepsilon(t) = -\int_{\Omega} (u_\varepsilon - u_\varepsilon)^2 - A \int_{\Omega} (v_\varepsilon - v_\varepsilon)^2 + \frac{1 + Aa_2}{2\sqrt{3}} \cdot \varepsilon \int_{\Omega} u_\varepsilon + \frac{A}{2\sqrt{3}} \cdot \varepsilon \int_{\Omega} v_\varepsilon
\]
for all \( t > 0 \) and \( \varepsilon \in (0,1) \), and thus end up with (8.6). \( \square \)

In view of the eventual bounds derived in Section 6 for small values of \( \chi_1 \) and \( \chi_2 \) but arbitrarily large \( u_0 \) and \( v_0 \) this readily entails an inequality for \( \mathcal{E}_{1,\varepsilon} \) which in the formal limit \( \varepsilon \searrow 0 \) indeed predicts eventual decrease thereof.
Lemma 8.3 Let \( n_1 = n_2 = 2 \), and for \( i \in \{1, 2\} \) let \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \) be such that \((8.3)\) holds. Then with \( \chi_* > 0 \) taken from Lemma 6.2 one can find \( \chi^{**} \in (0, \chi_*) \) and \( C > 0 \) such that if \( \chi_1 \in (0, \chi^{**}) \) and \( \chi_2 \in (0, \chi^{**}) \), and if \((1.3)\) holds, there exist \( T_0 = T_0(\chi_1, \chi_2, u_0, v_0) > 0 \) and \( \varepsilon_0 = \varepsilon_0(\chi_1, \chi_2, u_0, v_0) \in (0, 1) \) such that for all \( \varepsilon \in (0, \varepsilon_0) \), the function \( \mathcal{E}_{1, \varepsilon} \) defined in \((8.5)\) satisfies
\[
\frac{d}{dt} \mathcal{E}_{1, \varepsilon}(t) + \frac{1}{C} \mathcal{D}_{1, \varepsilon}(t) \leq C \varepsilon \sqrt{\varepsilon} \quad \text{for all } t > T_0, \tag{8.9}
\]
where for \( \varepsilon \in (0, 1),
\[
\mathcal{D}_{1, \varepsilon}(t) := \int_{\Omega} \frac{u_{\varepsilon x}^2(\cdot, t)}{u_0^2(\cdot, t)} + \int_{\Omega} \frac{v_{\varepsilon x}^2(\cdot, t)}{v_0^2(\cdot, t)} + \int_{\Omega} (u_{\varepsilon}(\cdot, t) - u_*)^2 + \int_{\Omega} (v_{\varepsilon}(\cdot, t) - v_*)^2 + \varepsilon \frac{a_2}{2} \int_{\Omega} u_0^{-\alpha-4}(\cdot, t) u_{\varepsilon x x}^2(\cdot, t) + \varepsilon \frac{a_2}{2} \int_{\Omega} v_0^{-\alpha-4}(\cdot, t) v_{\varepsilon x x}^2(\cdot, t), \quad t > 0, \tag{8.10}
\]
and where \( u_* > 0 \) and \( v_* > 0 \) are taken from \((1.14)\).

PROOF. We first invoke Lemma 6.2 to find \( c_1 > 0 \) such that if \( \chi_1 \in (0, \chi_*) \) and \( \chi_2 \in (0, \chi_*) \), then whenever \((1.3)\) holds, we can find \( T_0 = T_0(\chi_1, \chi_2, u_0, v_0) > 0 \) and \( \varepsilon_0 = \varepsilon_0(\chi_1, \chi_2, u_0, v_0) \in (0, 1) \) such that
\[
\|u_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} + \|v_{\varepsilon}(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 \quad \text{for all } t > T_0 \text{ and any } \varepsilon \in (0, \varepsilon_0), \tag{8.11}
\]
Choosing \( \chi^{**} \in (0, \chi_*) \) small enough such that with \( A > 0 \) given by \((8.4)\) we have
\[
\frac{A \chi^{**}_* v_*}{2 D_2} \cdot c_1^2 \leq \frac{D_1 u_*}{4} \quad \text{and} \quad \frac{\chi^{**}_* u_*}{2 D_1} \cdot c_1^2 \leq \frac{AD_2 v_*}{4},
\]
from Lemma 8.2 we infer that if \( \chi_1 \in (0, \chi^{**}) \), \( \chi_2 \in (0, \chi^{**}) \) and \((1.3)\) holds, then with \( T_0 \) and \( \varepsilon_0 \) as above and for all \( \varepsilon \in (0, \varepsilon_0),
\[
\frac{d}{dt} \mathcal{E}_{1, \varepsilon}(t) + \frac{D_1 u_*}{4} \int_{\Omega} \frac{u_{\varepsilon x}^2(\cdot, t)}{u_0^2(\cdot, t)} + \frac{AD_2 v_*}{4} \int_{\Omega} \frac{v_{\varepsilon x}^2(\cdot, t)}{v_0^2(\cdot, t)} + \int_{\Omega} (u_{\varepsilon}(\cdot, t) - u_*)^2 + \int_{\Omega} (v_{\varepsilon}(\cdot, t) - v_*)^2 + u_* \varepsilon \frac{a_2}{2} \int_{\Omega} u_0^{-\alpha-4}(\cdot, t) u_{\varepsilon x x}^2(\cdot, t) + A v_* \varepsilon \frac{a_2}{2} \int_{\Omega} v_0^{-\alpha-4}(\cdot, t) v_{\varepsilon x x}^2(\cdot, t) \leq \left\{ \frac{(1 + a_1) c_1 |\Omega|}{2 \sqrt{3}} + \frac{A c_1 |\Omega|}{2 \sqrt{3}} \right\} \cdot \sqrt{\varepsilon} \quad \text{for all } t > T_0,
\]
which directly yields \((8.9)\). \(\square\)

Now once more thanks to the interpolation properties from Lemma 5.1 the dissipation rate in \((8.9)\) dominates a certain superlinear power of \( \mathcal{E}_{1, \varepsilon} \) at least within bounded time intervals:

Lemma 8.4 Let \( n_1 = n_2 = 2 \), and let \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), assuming \((8.3)\). Then whenever \((1.3)\) holds, for all \( T > 0 \) there exists \( C(T) > 0 \) such that for all \( \varepsilon \in (0, 1), \) with \( \mathcal{E}_{1, \varepsilon} \) and \( \mathcal{D}_{1, \varepsilon} \) taken from \((8.5)\) and \((8.10)\) we have
\[
\mathcal{E}_{1, \varepsilon}(t) \leq C(T) \mathcal{D}_{1, \varepsilon}(t) + C(T) \quad \text{for all } t \in (0, T). \tag{8.12}
\]
Proof. According to Lemma 5.3 given $T > 0$ we can find $c_1(T) \in (0, 1)$ such that for all $\varepsilon \in (0, 1)$,

$$
\int_\Omega u_\varepsilon(\cdot, t) \geq c_1(T) \quad \text{and} \quad \int_\Omega v_\varepsilon(\cdot, t) \geq c_1(T) \quad \text{for all } t \in (0, T), \tag{8.13}
$$

whereas Lemma 2.3 provides $c_2 > 0$ such that

$$
\int_\Omega u_\varepsilon(\cdot, t) + \int_\Omega v_\varepsilon(\cdot, t) \leq c_2 \quad \text{for all } t > 0 \tag{8.14}
$$

whenever $\varepsilon \in (0, 1)$. We now recall the definition (8.5) of $E_{1, \varepsilon}$ to firstly estimate

$$
E_{1, \varepsilon}(t) \leq \int_\Omega u_\varepsilon - u_* \int_\Omega \ln u_\varepsilon + u_* \ln u_* \cdot |\Omega| + \frac{u_* \varepsilon}{6} \int_\Omega \frac{1}{u_\varepsilon^2} + \frac{1}{2} A \int_\Omega v_\varepsilon - Av_* \int_\Omega \ln v_\varepsilon + Av_* \ln v_* \cdot |\Omega| + \frac{Av_* \varepsilon}{6} \int_\Omega \frac{1}{v_\varepsilon^2} \quad \text{for all } t > 0,
$$

from which by nonnegativity of $E_{1, \varepsilon}$, as asserted by Lemma 8.1 for all $t > 0$ it follows that due to (8.14),

$$
E_{1, \varepsilon}^{\alpha+2}(t) \leq 8^{\frac{\alpha+2}{2}} \cdot \left\{ - \int_\Omega \ln u_\varepsilon \right\}^{\frac{\alpha+2}{2}} + \left( \frac{u_*}{6} \right)^{\frac{\alpha+2}{2}} \cdot \left\{ - \int_\Omega \ln u_\varepsilon \right\}^{\frac{\alpha+2}{2}} + \frac{A}{6} v_* \cdot |\Omega|^{\frac{\alpha+2}{2}} + \frac{1}{2} c_3 \tag{8.15}
$$

with $c_3 := c_2^{\frac{\alpha+2}{2}} + |u_* \ln u_* \cdot |\Omega||^{\frac{\alpha+2}{2}} + (Av_*)^{\frac{\alpha+2}{2}} + |Av_* \ln v_* \cdot |\Omega||^{\frac{\alpha+2}{2}}$. Here we use Lemma 5.1 ii) along with (8.13) and Young’s inequality to see that writing $c_4(T) := |\Omega| \cdot \ln c_4(T) + |\Omega| \cdot |\ln |\Omega||$ we have

$$
\left\{ - \int_\Omega \ln u_\varepsilon \right\}^{\frac{\alpha+2}{2}} \leq \left\{ |\Omega|^{\frac{\alpha+2}{2}} \cdot \left( \int_\Omega \frac{u_\varepsilon^{2-\frac{2}{\alpha}}}{u_*^2} \right)^{\frac{1}{2}} - |\Omega| \ln \left( \int_\Omega u_\varepsilon \right) + |\Omega| \ln |\Omega| \right\}^{\frac{\alpha+2}{2}}
$$

$$
\leq \left\{ |\Omega|^{\frac{\alpha+2}{2}} \cdot \left( \int_\Omega \frac{u_\varepsilon^{2-\frac{2}{\alpha}}}{u_*^2} \right)^{\frac{1}{2}} + c_4 \right\}^{\frac{\alpha+2}{2}}
$$

$$
\leq \left( 2 |\Omega|^{\frac{\alpha+2}{2}} \right)^{\frac{\alpha+2}{4}} \cdot \left( \int_\Omega \frac{u_\varepsilon^{2-\frac{2}{\alpha}}}{u_*^2} \right)^{\frac{1}{2}} + (2c_4)^{\frac{\alpha+2}{2}}
$$

$$
\leq \left( 2 |\Omega|^{\frac{\alpha+2}{2}} \right)^{\frac{\alpha+2}{4}} \cdot \int_\Omega \frac{u_\varepsilon^{2-\frac{2}{\alpha}}}{u_*^2} + \left( 2 |\Omega|^{\frac{\alpha+2}{2}} \right)^{\frac{\alpha+2}{4}} + (2c_4)^{\frac{\alpha+2}{2}} \quad \text{for all } t \in (0, T). \tag{8.16}
$$

Moreover, applying Lemma 5.1 i) to $p = 2$ and $q = \alpha + 2$ shows that again thanks to (8.13) and Young’s inequality,

$$
\varepsilon^{\frac{\alpha+2}{2}} \cdot \left\{ \int_\Omega \frac{u_\varepsilon^{\alpha+2}}{u_*^2} \right\}^{\frac{\alpha+2}{4}} \leq \varepsilon^{\frac{\alpha+2}{2}} \cdot \left( \alpha + 2 \right)^{\frac{\alpha+2}{4}} |\Omega|^{\alpha+2} \cdot \left\{ \int_\Omega u_\varepsilon^{\alpha+2-\frac{2}{\alpha+2}} u_*^{\alpha+2} \right\}^{\frac{2}{\alpha+2}}
$$

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For drawing conclusions from the resulting autonomous ODE for $E_1,\varepsilon$ for all $t > b$ (second solution component), this enables us to infer from (8.15) the existence of $c_5 > 0$ and $c_6(T) > 0$ such that for all $\varepsilon \in (0, 1),$

$$E_{1,\varepsilon}^{\alpha+2}(t) \leq c_5 \cdot \left\{ \int_{\Omega} u_{\varepsilon}^2 + \int_{\Omega} v_{\varepsilon}^2 + \varepsilon \frac{\alpha+2}{4} \int_{\Omega} u_{\varepsilon}^{-\alpha-4} u_{\varepsilon x}^2 + \varepsilon \frac{\alpha+2}{4} \int_{\Omega} v_{\varepsilon}^{-\alpha-4} v_{\varepsilon x}^2 \right\} + c_6(T) \quad \text{for all} \ t \in (0, T).$$

In view of the definition (8.10) of $D_{1,\varepsilon},$ this implies (8.12).

For drawing conclusions from the resulting autonomous ODE for $E_1,\varepsilon,$ let us a quantitative consequence of a simple comparison argument.

**Lemma 8.5** Let $t_0 \in \mathbb{R}, T > t_0, a > 0, b > 0$ and $\beta > 1,$ and suppose that $y \in C^0([t_0, T]) \cap C^1((t_0, T))$ is nonnegative and such that

$$y'(t) + ay^\beta(t) \leq b \quad \text{for all} \ t \in (t_0, T).$$

Then

$$y(t) \leq \left\{ (\beta - 1)a(t - t_0) \right\}^{-\frac{1}{\beta - 1}} + \left( \frac{b}{a} \right)^{\frac{1}{\beta}} \quad \text{for all} \ t \in (t_0, T).$$

**Proof.** Without loss of generality we assume that $t_0 = 0,$ and observe that $\mathcal{R}(t) := ((\beta - 1)at)^{-\frac{1}{\beta - 1}} + \left( \frac{b}{a} \right)^{\frac{1}{\beta}},$ $t > 0,$ satisfies

$$\mathcal{R}(t) + a\mathcal{R}^\beta(t) - b = (\beta - 1)^{-\frac{1}{\beta - 1}}a^{-\frac{1}{\beta - 1}} \cdot \left( -\frac{1}{\beta - 1}t^{-\frac{1}{\beta - 1}} \right) + a \cdot \left\{ (\beta - 1)at \right\}^{-\frac{1}{\beta - 1}} + \left( \frac{b}{a} \right)^{\frac{1}{\beta}} \cdot \beta^\beta - b$$

$$\geq - (\beta - 1)^{-\frac{\beta}{\beta - 1}}a^{-\frac{1}{\beta - 1}}t^{-\frac{\beta}{\beta - 1}} + a \cdot (\beta - 1)at^{-\frac{\beta}{\beta - 1}} + a \cdot \frac{b}{a} = 0$$

for all $t > 0$ due to the fact that $(\xi + \eta)^\beta \geq \xi^\beta + \eta^\beta$ for all $\xi \geq 0$ and $\eta \geq 0.$ Since $\mathcal{R}(t) \nearrow +\infty$ as $t \searrow 0,$ by continuity of $y$ at $t = 0$ this readily implies the claimed inequality by means of a comparison argument.

We can thus reap the fruit of Lemma 8.3 and Lemma 8.4 and thereby obtain, through the dissipation mechanism expressed in (8.9), the following preliminary decay information for our solution to the original problem.

**Lemma 8.6** Let $n_1 = n_2 = 2,$ let $D_i > 0, a_i > 0, \lambda_i > 0$ and $\chi_i > 0$ for $i \in \{1, 2\}$ be such that (8.3) holds, and let $\chi_1 \in (0, \chi^{**})$ and $\chi_2 \in (0, \chi^{**})$ with $\chi^{**} > 0$ as in Lemma 8.3. Then assuming (1.17), one can find $T_0 = T_0(\chi_1, \chi_2, u_0, v_0) > 0$ such that for the limit functions $u$ and $v$ obtained in Lemma 4.5, we have

$$\int_{T_0}^\infty \int_{\Omega} u_\ast^2 + \int_{T_0}^\infty \int_{\Omega} v_\ast^2 + \int_{T_0}^\infty \int_{\Omega} (u - u_\ast)^2 + \int_{T_0}^\infty \int_{\Omega} (v - v_\ast)^2 < \infty$$

(8.19)

with $u_\ast > 0$ and $v_\ast > 0$ taken from (1.71).
Proof. According to Lemma 8.3 and Lemma 6.2 supposing (1.5) to be valid we can pick \( \varepsilon_0 = \varepsilon_0(\chi_1, \chi_2, u_0, \nu_0) \in (0, 1) \), \( T_1 = T_1(\chi_1, \chi_2, u_0, \nu_0) > 0 \) and \( c_i = c_i(\chi_1, \chi_2, u_0, \nu_0) > 0 \), \( i \in \{1, 2, 3\} \), such that
\[
\frac{d}{dt} E_{1, \varepsilon}(t) + c_1 D_{1, \varepsilon}(t) \leq c_2 \sqrt{\varepsilon} \quad \text{for all } t > T_1
\] (8.20)
as well as
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_3 \quad \text{for all } t > T_1 \text{ and any } \varepsilon \in (0, \varepsilon_0),
\] (8.21)
where \( E_{1, \varepsilon} \) and \( D_{1, \varepsilon} \) are as defined in (8.3) and (8.10), respectively. Keeping this value of \( T_1 \) fixed, we thereafter invoke Lemma 8.4 to pick \( c_4 > 0 \) such that for any choice of \( \varepsilon \in (0, 1) \),
\[
E_{1, \varepsilon}^{\alpha + 2}(t) \leq c_4 D_{1, \varepsilon}(t) + c_4 \quad \text{for all } t \in (0, T_1 + 2),
\]
which when combined with (8.20) shows that
\[
\frac{d}{dt} E_{1, \varepsilon}(t) + \frac{c_1}{c_4} E_{1, \varepsilon}^{\alpha + 2}(t) \leq c_1 + c_2 \sqrt{\varepsilon} \leq c_1 + c_2 \quad \text{for all } t \in (T_1, T_1 + 2) \text{ and all } \varepsilon \in (0, \varepsilon_0).
\]
Through Lemma 8.5 the latter implies that
\[
E_{1, \varepsilon}(t) \leq \left( \frac{\alpha}{2} \frac{c_1}{c_4} (t - T_1) \right)^{-\frac{2}{\alpha}} + \left( \frac{c_1 + c_2}{c_4} \right)^{\frac{2}{\alpha + 2}} \quad \text{for all } t \in (T_1, T_1 + 2) \text{ and } \varepsilon \in (0, \varepsilon_0)
\]
and that hence, in particular,
\[
E_{1, \varepsilon}(T_1 + 1) \leq c_5 := \left( \frac{2c_4}{c_1 \alpha} \right)^{\frac{2}{\alpha}} + \left( \frac{(c_1 + c_2)c_4}{c_1} \right)^{\frac{2}{\alpha + 2}} \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
\]
Using this as \( \varepsilon \)-independent information at the initial time \( T_1 + 1 \), we now return to (8.20) to infer upon an integration therein that
\[
c_1 \int_{T_1+1}^{T} D_{1, \varepsilon}(t) dt \leq E_{1, \varepsilon}(T_1 + 1) - E_{1, \varepsilon}(T) + c_2 \sqrt{\varepsilon} \cdot (T - T_1 - 1)
\]
\[
\leq c_5 + c_2 \sqrt{\varepsilon} \cdot (T - T_1 - 1) \quad \text{for all } T > T_1 + 1 \text{ and each } \varepsilon \in (0, \varepsilon_0),
\] (8.22)
again because \( E_{1, \varepsilon} \) is nonnegative. Since (8.21) ensures that according to (8.10) we can estimate
\[
D_{1, \varepsilon}(t) \geq \frac{1}{c_3^2} \int_{T_1}^{T} u_\varepsilon^2 + \frac{1}{c_3^2} \int_{T_1}^{T} v_\varepsilon^2 + \int_{T_1}^{T} (u_\varepsilon - u_*)^2 + \int_{T_1}^{T} (v_\varepsilon - v_*)^2 \quad \text{for all } t > T_1 \text{ and } \varepsilon \in (0, \varepsilon_0),
\]
on taking \( \varepsilon = \varepsilon_j \searrow 0 \) with \( (\varepsilon_j) \subseteq (0, 1) \) as in Lemma 4.5 from (8.22) and an argument based on lower semicontinuity of norms with respect to weak convergence in Hilbert spaces we conclude that
\[
\frac{1}{c_3^2} \int_{T_1+1}^{T} u_\varepsilon^2 + \frac{1}{c_3^2} \int_{T_1+1}^{T} v_\varepsilon^2 + \int_{T_1+1}^{T} (u_\varepsilon - u_*)^2 + \int_{T_1+1}^{T} (v_\varepsilon - v_*)^2 \leq \frac{c_5}{c_1} \quad \text{for all } t > T_1 + 1,
\]
and that thus (8.19) holds with \( T_0 := T_1 + 1 \). \( \square \)
We next make use of favorable smallness properties, as implied by (8.19) for certain arbitrarily large times which we use as new starting instants, to see on going back to (8.9), but this time simply neglecting the positive summand $D_{1,\varepsilon}$ therein, that $u$ and $v$ in fact approach their expected limits in a sense much stronger than indicated in Lemma 8.6, albeit not yet identified as spatially uniform but rather in a topology associated with the entropy functional in (1.12).

**Lemma 8.7** Let $n_1 = n_2 = 2$, let $D_i > 0, a_i > 0, \lambda_i > 0$ and $\chi_i > 0$, $i \in \{1, 2\}$, satisfy (8.3), and let $\chi_1 \in (0, \chi_{**})$ and $\chi_2 \in (0, \chi_{**})$ with $\chi_{**} > 0$ taken from Lemma 8.3. Then assuming (1.5) and letting $u, v$ and $N$ be as given by Lemma 8.3 with $\phi_{u_*}$ and $\phi_{v_*}$ taken from (8.1) and (1.11) we have

$$\int_{\Omega} \phi_{u_*}(u(\cdot, t)) \, d\Omega \to 0 \quad \text{and} \quad \int_{\Omega} \phi_{v_*}(v(\cdot, t)) \, d\Omega \to 0 \quad \text{as} \quad (0, \infty) \setminus N = t \to \infty. \quad (8.23)$$

**Proof.** In order to prepare our convergence argument, let us first once more resort to Lemma 8.3 in choosing $c_1 > 0$, $T_0 = T_0(u_0, v_0) > 0$ and $\varepsilon_* = \varepsilon_*(u_0, v_0) \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_*)$,

$$\frac{d}{dt} \mathcal{E}_{1, \varepsilon}(t) \leq c_1 \sqrt{\varepsilon} \quad \text{for all} \quad t > T_0. \quad (8.24)$$

Then abbreviating $c_2 := |\Omega|^\frac{1}{2} + |\Omega|^{- \frac{1}{2}}$, given $\eta > 0$ we fix $\delta > 0$ conveniently small such that

$$2c_2 \sqrt{\delta} \leq \frac{u_*}{2} \quad \text{and} \quad 2c_2 \sqrt{\delta} \leq \frac{v_*}{2} \quad (8.25)$$

as well as

$$\delta \leq \frac{u_*^2 \eta}{64c_2^2 |\Omega|} \quad \text{and} \quad A\delta \leq \frac{v_* \eta}{64c_2^2 |\Omega|}. \quad (8.26)$$

According to Lemma 8.6 we can thereafter pick some $T_1 > T_0 + 1$ suitably large such that

$$\int_{T_{1-1}}^{\infty} \int_{\Omega} u_x^2 + \int_{T_{1-1}}^{\infty} \int_{\Omega} v_x^2 + \int_{T_{1-1}}^{\infty} \int_{\Omega} (u - u_*)^2 + \int_{T_{1-1}}^{\infty} \int_{\Omega} (v - v_*)^2 \leq \delta, \quad (8.27)$$

and we claim that then

$$\int_{\Omega} \phi_{u_*}(u(\cdot, t)) + A \int_{\Omega} \phi_{v_*}(v(\cdot, t)) \leq \eta \quad \text{for all} \quad t > T_1 \text{ such that } t \notin N. \quad (8.28)$$

To verify this, given any such $t$ we first observe that due to (8.27),

$$\int_{t-1}^{t} \left\{ \int_{\Omega} u_x^2 + \int_{\Omega} v_x^2 + \int_{\Omega} (u - u_*)^2 + \int_{\Omega} (v - v_*)^2 \right\} \leq \delta,$$

whence it is possible to find $t_* = t_*(t) \in (t - 1, t) \setminus N$ such that

$$\int_{\Omega} u_x^2(\cdot, t_*) + \int_{\Omega} v_x^2(\cdot, t_*) + \int_{\Omega} (u(v(\cdot, t_*) - u_*)^2 + \int_{\Omega} (v(\cdot, t_*) - v_*)^2 \leq \delta. \quad (8.29)$$
In particular, this entails the existence of \( x_* \in \Omega \) such that \((u(x_*, t_*)) - u_*\)^2 \( \leq \frac{\delta}{|\Omega|} \) and that hence

\[
|u(x, t_*) - u_*| \leq |u(x, t_*) - u(x_*, t_*)| + |u(x_*, t_*) - u_*| = \left| \int_{x_*}^x u_x(y, t_*)dy \right| + |u(x_*, t_*) - u_*| \\
\leq |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} u_x^2(y, t_*)dy \right\}^{\frac{1}{2}} + |u(x_*, t_*) - u_*| \\
\leq |\Omega|^{\frac{1}{2}} \delta^{\frac{1}{2}} + \left( \frac{\delta}{|\Omega|} \right)^{\frac{1}{2}} \quad \text{for all } x \in \Omega,
\]

which along with an identical argument for \( v \) shows that

\[
\|u(\cdot, t_*) - u_*\|_{L^\infty(\Omega)} \leq c_2 \sqrt{\delta} \quad \text{and} \quad \|v(\cdot, t_*) - v_*\|_{L^\infty(\Omega)} \leq c_2 \sqrt{\delta} \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_*), \tag{8.30}
\]

Now since \( t_* \in (0, \infty) \setminus N \), Corollary 6.3 applies so as to warrant that with some \( \varepsilon_{**} = \varepsilon_{**}(t_*) \in (0, \varepsilon_*) \) and \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) as in Lemma 4.5 we have

\[
\|u_{\varepsilon}(\cdot, t_*) - u_*\|_{L^\infty(\Omega)} \leq 2c_2 \sqrt{\delta} \quad \text{and} \quad \|v_{\varepsilon}(\cdot, t_*) - v_*\|_{L^\infty(\Omega)} \leq 2c_2 \sqrt{\delta} \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_*), \tag{8.31}
\]

According to our requirements on \( \delta \) in (8.25), these estimates especially entail the inequalities

\[
u_{\varepsilon}(\cdot, t_*) \geq u_* - 2c_2 \sqrt{\delta} \geq \frac{u_*}{2} \quad \text{and} \quad v_{\varepsilon}(\cdot, t_*) \geq v_* - 2c_2 \sqrt{\delta} \geq \frac{v_*}{2} \quad \text{in } \Omega \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_*), \tag{8.32}
\]

which firstly enables us to conclude from (8.2) in conjunction with (8.31) and (8.26) that for all \( \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_*),

\[
\int_{\Omega} \phi_{u_{\varepsilon}}(u_{\varepsilon}(\cdot, t_*)) \leq \frac{2}{u_*} \int_{\Omega} u_{\varepsilon}(\cdot, t_*) - u_* \leq \frac{2|\Omega|}{u_*} \|u_{\varepsilon}(\cdot, t_*) - u_*\|^2_{L^\infty(\Omega)} \leq \frac{2|\Omega|}{u_*} \cdot (2c_2 \sqrt{\delta})^2 \leq \frac{\eta}{8}, \tag{8.33}
\]

and similarly

\[
A \int_{\Omega} \phi_{v_{\varepsilon}}(v_{\varepsilon}(\cdot, t_*)) \leq \frac{\eta}{8}. \tag{8.34}
\]

Secondly, (8.32) guarantees that if we pick \( \varepsilon_{**} = \varepsilon_{**}(t_*) \in (0, \varepsilon_*) \) small enough such that

\[
\frac{2|\Omega| \varepsilon_{**}}{3u_*} \leq \frac{\eta}{8} \quad \text{and} \quad \frac{2A|\Omega| \varepsilon_{**}}{3v_*} \leq \frac{\eta}{8},
\]

then in the contributions to \( \mathcal{E}_{1,\varepsilon} \) containing the factor \( \varepsilon \) we can estimate

\[
\frac{u_* \varepsilon}{6} \int_{\Omega} \frac{1}{u_{\varepsilon}(\cdot, t_*)} \leq \frac{u_* \varepsilon}{6} \cdot \left( \frac{2}{u_*} \right)^2 |\Omega| \leq \frac{\eta}{8} \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}} \cap (0, \varepsilon_{**}).
\]
and
\[
\frac{Av_\varepsilon}{6} \int_{\Omega} \frac{1}{v^2_\varepsilon(t_*, t_*)} \leq \frac{Av_\varepsilon}{6} \cdot \left( \frac{2}{v_*} \right)^2 |\Omega| \leq \frac{\eta}{8} \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N} \cap (0, \varepsilon_{***})}.
\]

When combined with \((8.33)\) and \((8.34)\), in view of \((8.5)\) these inequalities show that
\[
E_{1, \varepsilon}(t_*) \leq 4 \cdot \eta \leq \frac{\eta}{2} \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N} \cap (0, \varepsilon_{***})},
\]
so that letting \(\varepsilon_{***} = \varepsilon_{***}(t_*) \in (0, \varepsilon_{***})\) be such that \(c_1 \sqrt{\varepsilon_{***}} \leq \frac{\eta}{2}\), on integrating \((8.24)\) we infer that at the time in question we have
\[
E_{1, \varepsilon}(t) \leq E_{1, \varepsilon}(t_*) + c_1 \sqrt{\varepsilon} \cdot (t - t_*) \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j \in \mathbb{N} \cap (0, \varepsilon_{***})},
\]

because \(t_* \geq t - 1\). Since from Lemma \((8.5)\) and our assumption that \(t \not\in N\) we know that \(u_\varepsilon(\cdot, t) \to u(\cdot, t)\) and \(v_\varepsilon(\cdot, t) \to v(\cdot, t)\) a.e. in \(\Omega\) as \(\varepsilon = \varepsilon_j \to 0\), upon an application of Fatou’s lemma we readily obtain \((8.28)\), and thus the statement of the lemma, as a consequence of \((8.35)\).

Now thanks to the eventual precompactness features implied by Corollary \((6.3)\) due to the positivity of \(\phi_{\xi_*}\) outside the point \(\xi_* > 0\) the latter readily implies the desired convergence statement.

**Lemma 8.8** Let \(n_1 = n_2 = 2\), let \(D_i > 0, a_i > 0, \lambda_i > 0\) and \(\chi_i > 0\) for \(i \in \{1, 2\}\) be such that \((8.3)\) is fulfilled, and let \(\chi_1 \in (0, \chi_{**})\) and \(\chi_2 \in (0, \chi_{**})\) with \(\chi_{**} > 0\) as in Lemma \((8.3)\). Then whenever \((1.3)\) holds, as \(t \to \infty\), the limit functions \(u, v\) obtained in Lemma \((4.3)\) satisfy
\[
u(\cdot, t) \to u_* \quad \text{in } L^\infty(\Omega) \quad \text{and} \quad v(\cdot, t) \to v_* \quad \text{in } L^\infty(\Omega),
\]
where \(u_* > 0\) and \(v_* > 0\) are as in \((1, 14)\).

**Proof.** From Corollary \((6.3)\) we know that there exists \(T_0 > 0\) such that \((u(\cdot, t))_{t > T_0}\) is bounded in \(W^{1,2}(\Omega)\) and hence relatively compact in \(C^0(\overline{\Omega})\), whence if \((8.36)\) was false, we could find \((t_k)_{k \in \mathbb{N}} \subset (T_0, \infty)\) and \(u_\infty \in C^0(\overline{\Omega})\) such that \(u_\infty \not\equiv u_*\) and that \(t_k \to \infty\) as well as \(u(\cdot, t_k) \to u_\infty\) in \(L^\infty(\Omega)\) as \(k \to \infty\). As in view of Corollary \((6.3)\) we may assume \(u\) to be continuous in \(\overline{\Omega} \times (T_0, \infty)\), by density of \([t_k, t_k + 1] \setminus N\) in \([t_k, t_k + 1]\) we can pick \(\hat{t}_k \in [t_k, t_k + 1]\) such that \(\|u(\cdot, \hat{t}_k) - u(\cdot, t_k)\|_{L^\infty(\Omega)} \leq \frac{1}{k}\), meaning that also \(u(\cdot, \hat{t}_k) \to u_\infty\) in \(L^\infty(\Omega)\) as \(k \to \infty\). As the function \(\phi_{u_*}\) from \((8.1)\) is positive in \((0, \infty) \setminus \{u_*\}\) by Lemma \((8.1)\) however, the hypothesis \(u_\infty \not\equiv u_*\) implies that \(\int_0^\infty \phi_{u_*}(u(\cdot, t_k)) \neq 0\) as \(k \to \infty\), in contradiction to Lemma \((8.7)\). Along with a similar argument for \(v\), this establishes the claim.

The proof of our main result on kinetics-driven stabilization has thereby already been accomplished:

**Proof of Theorem \((1.2)\)** We only need to combine Corollary \((6.3)\) with Lemma \((8.8)\).

**9** **The case \(\lambda_2 \leq a_2 \lambda_1\). Proof of Theorem \((1.3)\)**

In the context addressed in Theorem \((1.3)\) in view of Lemma \((7.1)\) the form of the functional in \((1.14)\) suggests to choose still \(n_1 = 2\) but now rather \(n_2 = 1\). Our analysis will then quite closely follow the lines presented in the previous section, so that here it will be sufficient to concentrate on the main modifications only.

The fundament for convergence is constituted by a natural counterpart of Lemma \((8.2)\).
Lemma 9.1 Let \( n_1 = 2 \) and \( n_2 = 1 \), and suppose that \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), and that
\[
\lambda_2 \leq a_2 \lambda_1.
\] (9.1)

Then with \( A = \frac{a_1}{a_2} \) as before and \( \phi_{\lambda_1} \) as determined by (8.1), for
\[
E_{2, \varepsilon}(t) := \int_\Omega \phi_{\lambda_1}(u_\varepsilon(\cdot,t)) + \frac{\lambda_1 \varepsilon}{6} \int_\Omega \frac{1}{u_\varepsilon^2(\cdot,t)} + A \int_\Omega v_\varepsilon(\cdot,t) + \frac{A}{2\lambda_2} \int_\Omega v_\varepsilon^2(\cdot,t) + \frac{A \varepsilon}{2\lambda_2} \int_\Omega v_\varepsilon(\cdot,t),
\] we have
\[
\frac{d}{dt} E_{2, \varepsilon}(t) + \left\{ \frac{D_1 \lambda_1}{2} - \frac{A \lambda_1^2}{2D_2 \lambda_2} \| u_\varepsilon(\cdot,t) \|^2_{L_\infty(\Omega)} \| v_\varepsilon(\cdot,t) \|^2_{L_\infty(\Omega)} \right\} \cdot \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^4}
\end{array}
+ \left\{ \frac{AD_2}{2\lambda_2} - \frac{\lambda_2^2 \lambda_1^2}{2D_1} \right\} \cdot \int_\Omega \frac{v_\varepsilon^2}{v_\varepsilon^4}
\end{array}
+ \int_\Omega (u_\varepsilon(\cdot,t) - \lambda_1)^2 + A \int_\Omega v_\varepsilon^3
\end{array}
+ \lambda_1 \varepsilon \frac{6 + 2}{12} \int_\Omega u_\varepsilon - \alpha - 4 u_\varepsilon^2 + \frac{A}{\lambda_2 \varepsilon} \frac{6 + 2}{12} \int_\Omega v_\varepsilon - \alpha - 3 v_\varepsilon^2
\end{array}
\]
\[
\leq \frac{1 + A a_2 + 2 A a_2 \lambda_2^{-1}}{2 \sqrt{3}} \cdot \int_\Omega u_\varepsilon + A + 2 A \lambda_2^{-1} \cdot \sqrt{\varepsilon} \int_\Omega v_\varepsilon
\] (9.3)
for all \( t > 0 \) and \( \varepsilon \in (0, 1) \).

**PROOF.** Combining Lemma (2.2) with Lemma (7.1) i) and iii), on dropping several nonnegative summands we see that
\[
\frac{d}{dt} E_{2, \varepsilon}(t) = D_1 \lambda_1 \int_\Omega \frac{u_\varepsilon^2}{u_\varepsilon^4} + \frac{\lambda_1 \varepsilon^{a_2 + \frac{4}{3}}}{\lambda_2} \int_\Omega u_\varepsilon - \alpha - 4 u_\varepsilon^2
\end{array}
+ \frac{AD_2}{\lambda_2} \int_\Omega v_\varepsilon^2 + \frac{A}{\lambda_2} \frac{2 \varepsilon}{12} \int_\Omega v_\varepsilon - \alpha - 3 v_\varepsilon^2
\]
\[
\leq \begin{array}{l}
\left( \lambda_1 + \frac{\sqrt{\varepsilon}}{2 \sqrt{3}} \right) \int_\Omega u_\varepsilon - \int_\Omega u_\varepsilon + a_1 \int_\Omega u_\varepsilon v_\varepsilon
\end{array}
\end{array}
+ \lambda_1 \lambda_1 \int_\Omega \frac{u_\varepsilon}{u_\varepsilon} v_\varepsilon - \lambda_1^2 |\Omega| + \lambda_1 \int_\Omega u_\varepsilon - a_1 \lambda_1 \int_\Omega v_\varepsilon
\]
+λ₁χ₁ \int_Ω \frac{u_{ex}}{u_e} v_{ex} - \frac{A\chi₂}{\lambda₂} \int_Ω v_e \frac{u_{ex}}{u_e} \\
+ \left(1 + \frac{Aa₂}{\sqrt{3}}\right)\int_Ω v_e + \frac{A\sqrt{\varepsilon}}{\sqrt{3}} \int_Ω v_e \\
- \frac{A\varepsilon}{\lambda₂} \int_Ω \frac{2v_{e}^2 + 3v_{e} \lambda₂ - v_e - a₂u_e}{6v_{e}^2 + 2\varepsilon} 
\text{for all } t > 0 \text{ and } \varepsilon \in (0, 1). \hspace{1cm} (9.4)

Here we note that according to (9.1) we have \(A\lambda₂ - a₁\lambda₁ = \frac{a₁\lambda₁}{a₂} - a₁\lambda₁ \leq 0\), and that by Young’s inequality,

\[ \lambda₁χ₁ \int_Ω \frac{u_{ex}}{u_e} v_{ex} \leq \frac{D₁\lambda₁}{2} \int_Ω \frac{u_{ex}^2}{u_e^2} + \frac{\lambda²\chi₂^2}{2D₁} \int_Ω \frac{v_{ex}^2}{u_e^2} \]

and

\[ -\frac{A\chi₂}{\lambda₂} \int_Ω v_e \frac{u_{ex}}{u_e} \leq \frac{AD₂}{2\lambda₂} \int_Ω \frac{v_{ex}^2}{u_e^2} + \frac{A\chi₂}{2D₂\lambda₂} \int_Ω \frac{v_{ex}^2}{u_e^2} \]

\[ \leq \frac{AD₂}{2\lambda₂} \int_Ω \frac{v_{ex}^2}{u_e^2} + \frac{A\chi₂}{2D₂\lambda₂} \|u_e\|_{L^∞(Ω)}^2 \|v_e\|_{L^∞(Ω)}^2 \int_Ω \frac{u_{ex}^2}{u_e^2} \]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \). As moreover maximizing \( \varphi(s) := \frac{2^2 + 3\varepsilon}{6s^2 + 2\varepsilon}, s \geq 0 \), shows that

\[ -\frac{A\varepsilon}{\lambda₂} \int_Ω \frac{2v_{e}^2 + 3v_{e} \lambda₂ - v_e - a₂u_e}{6v_{e}^2 + 2\varepsilon} \leq \frac{A\varepsilon}{\lambda₂} \|\varphi\|_{L^∞((0, \infty))} \left( \int_Ω (v_e + a₂u_e) \right) \]

\[ = \frac{A\varepsilon}{\lambda₂} \cdot \frac{1}{3} \left\{ \left( \int_Ω v_e + a₂ \int_Ω u_e \right) \right\} \]

\[ < \frac{A\lambda₂ - 1}{\sqrt{3}} \cdot \sqrt{\varepsilon} \cdot \left\{ \left( \int_Ω v_e + a₂ \int_Ω u_e \right) \right\} \]

for all \( t > 0 \) and \( \varepsilon \in (0, 1) \), from (9.4) we directly obtain (9.3). \hfill \square

This implies an inequality of the form in Lemma 8.3.

**Lemma 9.2** Let \( n₁ = 2 \) and \( n₂ = 1 \), let \( D_i > 0, a_i > 0, \lambda_i > 0 \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), and suppose that (9.1) holds. Then with \( \chiₘ > 0 \) as in Lemma 6.2 there exists \( \chiₘₗ \in (0, \chiₗ) \) and \( C > 0 \) such that if \( \chi_1 \in (0, \chiₘₗ), \chi_2 \in (0, \chiₘₗ) \) and (7.5) is valid, then there exist \( T₀ = T₀(\chi₁, \chi₂, u₀, v₀) > 0 \) and \( \varepsilon₀ = \varepsilon₀(\chi₁, \chi₂, u₀, v₀) \in (0, 1) \) such that whenever \( \varepsilon \in (0, \varepsilon₀) \), for \( E_{2, \varepsilon} \) as in (7.2) we have

\[ \frac{d}{dt} E_{2, \varepsilon}(t) + \frac{1}{C} D_{2, \varepsilon}(t) \leq C \sqrt{\varepsilon} \]

for all \( t > T₀ \), \hspace{1cm} (9.5)

where

\[ D_{2, \varepsilon}(t) := \int_Ω \frac{u_{ex}^2}{u_e^2} + \int_Ω v_{ex}^2 + \int_Ω (u_e - \lambda)^2 + \int_Ω v_{ex}^3 \\
+ \varepsilon \frac{a₂}{2} \int_Ω \frac{u_{ex}^{α - 4}}{u_e^2} + \varepsilon \frac{a₂}{2} \int_Ω v_{ex}^{α - 3} \]

\[ t > 0, \hspace{1cm} (9.6) \]

and where \( u_∗ > 0 \) and \( v_∗ > 0 \) are taken from (1.11).
Proof. Again on the basis of Lemma 8.2, this can be derived from Lemma 9.1 in much the same manner as Lemma 8.3 was deduced from Lemma 8.2.

Up to modifications in technical details, the strategy in the proof of Lemma 8.3 finds its analogue in the following.

Lemma 9.3 Let \( n_1 = 2 \) and \( n_2 = 1 \), let \( D_i > 0 \), \( a_i > 0 \), \( \lambda_i > 0 \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), and assume (9.1) as well as (1.5). Then for all \( T > 0 \) there exists \( C(T) > 0 \) such that for all \( \varepsilon \in (0, 1) \), the functions \( E_{2, \varepsilon} \) and \( D_{2, \varepsilon} \) defined in (9.2) and (9.6) satisfy

\[
E_{2, \varepsilon}^{\alpha+2}(t) \leq C(T)D_{2, \varepsilon}(t) + C(T) \quad \text{for all } t \in (0, T).
\]

Proof. We proceed in a way similar to that in Lemma 8.3 and first note that

\[
\inf_{\varepsilon \in (0, 1)} \inf_{t \in (0, T)} \int_{\Omega} u_{\varepsilon} > 0, \quad \inf_{\varepsilon \in (0, 1)} \inf_{t \in (0, T)} \int_{\Omega} v_{\varepsilon} > 0 \quad \text{and} \quad \sup_{\varepsilon \in (0, 1)} \sup_{t \in (0, T)} \left\{ \int_{\Omega} u_{\varepsilon} + \int v_{\varepsilon} \right\} < \infty, \tag{9.8}
\]

to derive the existence of \( c_1(T) > 0 \) such that for all \( t \in (0, T) \) and each \( \varepsilon \in (0, 1) \),

\[
E_{2, \varepsilon}^{\alpha+2}(t) \leq c_1(T) \cdot \left\{ - \int_{\Omega} \ln u_{\varepsilon} \right\}^{\alpha+2} + c_1(T)\varepsilon^{\alpha+2} \cdot \left\{ \int_{\Omega} \frac{1}{u_{\varepsilon}^2} \right\}^{\alpha+2} + c_1(T) \cdot \left\{ \int_{\Omega} v_{\varepsilon}^2 \right\}^{\alpha+2} + c_1(T), \tag{9.9}
\]

where again combining Lemma 5.1 with (9.8) and Young’s inequality provides \( c_2(T) > 0 \) fulfilling

\[
c_1(T) \cdot \left\{ - \int_{\Omega} \ln u_{\varepsilon} \right\}^{\alpha+2} + c_1(T)\varepsilon^{\alpha+2} \cdot \left\{ \int_{\Omega} \frac{1}{u_{\varepsilon}^2} \right\}^{\alpha+2} \leq c_2(T) \int_{\Omega} \frac{u_{\varepsilon}^2}{u_{\varepsilon}^2} + c_2(T)\varepsilon^{\alpha+2} \int_{\Omega} u_{\varepsilon}^{-\alpha+2} u_{\varepsilon}^2 + c_2(T) \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1). \tag{9.10}
\]

Another application of Lemma 5.1, now to \( p = 1 \) and \( q = \alpha + 1 \), reveals that again due to (9.8) and Young’s inequality we can moreover find \( c_3(T) > 0 \) such that

\[
c_1(T)\varepsilon^{\alpha+2} \cdot \left\{ \int_{\Omega} \frac{1}{v_{\varepsilon}} \right\}^{\alpha+2} \leq c_1(T)\varepsilon^{\alpha+2} \cdot \left\{ (\alpha + 1) \frac{2}{\alpha+1} \right\}^{\alpha+2} \cdot \left\{ \int_{\Omega} v_{\varepsilon}^{-\alpha-3} v_{\varepsilon}^2 \right\}^{\alpha+2} + 2 \cdot \frac{2}{\alpha+1} \cdot \left\{ \int_{\Omega} v_{\varepsilon}^{-\alpha-3} v_{\varepsilon}^2 \right\}^{\alpha+2} + c_3(T)\varepsilon^{\alpha+2} \leq c_3(T)\varepsilon^{\alpha+2} \int_{\Omega} v_{\varepsilon}^{-\alpha-3} v_{\varepsilon}^2 + 2c_3(T) \quad \text{for all } t \in (0, T) \text{ and } \varepsilon \in (0, 1). \tag{9.11}
\]
because \( \frac{\alpha+2}{2(\alpha+1)} \leq 1 \). Finally, the Gagliardo-Nirenberg inequality along with (9.8) and Young’s inequality ensures the existence of \( c_4 > 0 \) and \( c_5(T) > 0 \) such that for all \( t \in (0, T) \) and any \( \varepsilon \in (0, 1) \),

\[
c_1(T) \cdot \left\{ \int_{\Omega} v_x^2 \right\}^{\frac{\alpha+2}{2}} \leq c_1(T) \cdot \left\{ c_4 \|v_{\text{ex}}\|_{L^2(\Omega)}^{\frac{\alpha+2}{2}} \|v_r\|_{L^1(\Omega)}^{\frac{2(\alpha+2)}{\alpha+2}} + c_4 \|v_r\|_{L^1(\Omega)}^{\alpha+2} \right\}
\]

\[
\leq c_5(T) \|v_{\text{ex}}\|_{L^2(\Omega)}^{\frac{\alpha+2}{2}} + c_5(T)
\]

\[
\leq c_5(T) \int_{\Omega} v_{xx}^2 + 2c_5(T),
\]

for \( \frac{\alpha+2}{\alpha} \leq 2 \). Together with (9.10), (9.11) and (9.9), this immediately leads to (9.7). □

Therefore, Lemma 9.2 implies decay in a yet very weak form similar to that in Lemma 8.6.

**Lemma 9.4** Let \( n_1 = 2 \) and \( n_2 = 1 \), let \( D_i > 0, a_i > 0, \lambda_i > 0, \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \), assume (9.1), and let \( \chi_1 \in (0, \chi_{**}) \) and \( \chi_2 \in (0, \chi_{**}) \) with \( \chi_{**} > 0 \) as in Lemma 9.3. Then whenever (1.5) holds, there exists \( T_0 = T_0(\chi_1, \chi_2, u_0, v_0) > 0 \) such that \( u \) and \( v \) from Lemma 4.5 satisfy

\[
\int_{T_0}^{\infty} \int_{\Omega} u_x^2 + \int_{T_0}^{\infty} \int_{\Omega} v_x^2 + \int_{T_0}^{\infty} \int_{\Omega} (u - \lambda_1)^2 + \int_{T_0}^{\infty} \int_{\Omega} v^3 < \infty. \tag{9.12}
\]

**Proof.** Once more thanks to Lemma 6.2, this can be seen by exploiting Lemma 9.2 together with Lemma 9.3 in essentially the same way as Lemma 8.3 and Lemma 8.4 have been used in the derivation of Lemma 8.6. □

With this information returning to the inequality from Lemma 9.2 yields stabilization in a sense paralleling that of Lemma 8.7.

**Lemma 9.5** Let \( n_1 = 2 \) and \( n_2 = 1 \), let \( D_i > 0, a_i > 0, \lambda_i > 0, \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \) be such that (9.1) is valid, and let \( \chi_1 \in (0, \chi_{**}) \) and \( \chi_2 \in (0, \chi_{**}) \) with \( \chi_{**} > 0 \) as in Lemma 8.3. Then assuming (1.3) and letting \( u, v \) and \( N \) be as given by Lemma 4.5 and \( \phi_{\lambda_1} \) be as defined through (8.7), we have

\[
\int_{\Omega} \phi_{\lambda_1}(u(\cdot, t)) \rightarrow 0 \quad \text{and} \quad \int_{\Omega} v^3(\cdot, t) \rightarrow 0 \quad \text{as} \quad (0, \infty) \setminus N \ni t \rightarrow \infty. \tag{9.13}
\]

**Proof.** A verification of this can be achieved by adapting the proof of Lemma 8.7 in an obvious manner. □

Finally, by compactness the latter can be turned into uniform convergence as in Lemma 8.8.

**Lemma 9.6** Let \( n_1 = 2 \) and \( n_2 = 1 \), let \( D_i > 0, a_i > 0, \lambda_i > 0, \) and \( \chi_i > 0 \) for \( i \in \{1, 2\} \) be such that (9.1) holds, and let \( \chi_1 \in (0, \chi_{**}) \) and \( \chi_2 \in (0, \chi_{**}) \) with \( \chi_{**} > 0 \) as in Lemma 9.2. Then whenever (1.3) is satisfied, the limit functions \( u \) and \( v \) in Lemma 4.5 have the properties that

\[
u(\cdot, t) \rightarrow 0 \quad \text{in} \quad L^\infty(\Omega) \quad \text{and} \quad v(\cdot, t) \rightarrow 0 \quad \text{in} \quad L^\infty(\Omega) \]

as \( t \rightarrow \infty \).
Proof. Again relying on Corollary 6.3 and Lemma 8.1, one can readily obtain this as a consequence of Lemma 9.5 by means of an argument in the flavor of that presented in the proof of Lemma 8.8.

We have thereby established our main result on asymptotic dominance of the predator population when \( \lambda_2 \leq a_2 \lambda_1 \).

Proof of Theorem 1.3. All statements have been verified in Corollary 6.3 and Lemma 9.6.

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