Abstract. It was shown by Doplicher et.al. that the measurement of spacetime intervals of the order of Planck length scale is operationally impossible, as the process of measurement invariably gives rise to a black hole formation. This can be avoided by postulating non-vanishing commutation relations between the coordinates, which are now promoted to the level of operators. Formulation of quantum mechanics in these kinds of spaces through the introduction of Hilbert spaces of Hilbert-Schmidt operators is then shown to allow the construction of spectral triplets a la Connes naturally. The computation of spectral distance between pure and mixed states is then shown to exhibit a deep connection between entropy and geometry.

1. Introduction

Ever since Connes developed the mathematical basis of Noncommutative geometry (NCG) [1], it was mostly applied to the so-called “almost commutative” (AC) spaces to provide a unified framework of the Standard model of particle physics [1]. This structure is believed to be adequate to describe physics up to the GUT scale. But still at a higher energy scale, like in the vicinity of Planck scale, one has to consider more serious non-commutative spaces and adopt the prescription of Doplicher et. al [2] where the spacetime coordinates are promoted to the level of operators satisfying certain non-vanishing commutation relations like $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$ as one of the possibilities, so that the gravitational instabilities arising out of the measurement processes to localise an event can be avoided. This naturally raises the question whether the formulation of noncommutative quantum mechanics as set out in [3, 4], which entails a representation of the quantum system on the Hilbert space of Hilbert-Schmidt operators, does not naturally encode geometric information in the context of Connes NCG where the notion of a spectral triplet is introduced? Our investigation in this direction indeed shows that the answer is in the affirmative, besides giving a much simpler algorithm to compute the spectral distance in an operatorial and a basis independent approach, so that we can avoid using any star product and any ambiguities that may result there-from[5]. Furthermore, in contrast to certain other recent computations [6, 7], carried out using the Moyal star product, we do not restrict our analysis to the classical configuration space $\mathcal{H}_c$ only; we extend this to the above mentioned Hilbert space of Hilbert-Schmidt operators, called here the quantum Hilbert space $\mathcal{H}_q$, which has a natural tensorial structure $\mathcal{H}_q = \mathcal{H}_q \otimes \mathcal{H}_q$. This offers rather tantalizing possibilities in terms of modifications of the implied geometry and the possible statistical underpinnings thereof.
2. Quantum mechanics on the non-commutative plane

In two dimensions the non-commutative plane is defined through the commutation relations

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}. \tag{1} \]

One constructs standard creation and annihilation operators \( b^\dagger \) and \( b = \hat{x}_1 + i\hat{x}_2/\sqrt{2\theta} \) satisfying \([b, b^\dagger] = 1\), and view the non-commutative plane i.e. the classical configuration space \( H_c \) (counter part of \( R^2 \) in commutative quantum mechanics) as a boson Fock space spanned by the eigenstate \( |n\rangle \) of the radial operator \( b^\dagger b \):

\[ H_c = \text{span}\{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\}. \tag{2} \]

Next we introduce the quantum Hilbert space in which the physical states of the system and the non-commutative Heisenberg algebra are to be represented. This is taken to be the set of all bounded trace-class operators (the Hilbert-Schmidt operators) over \( H_c \) and we refer to it as the quantum Hilbert space, \( H_q \),

\[ H_q = \{\psi : \text{tr}_c(\psi^\dagger\psi) < \infty\}, \tag{3} \]

where the subscript \( c \) refers to tracing over \( H_c \). Elements of \( H_q \) are denoted by a round bracket \( \psi \equiv |\psi\rangle \) and the inner product is defined as \( (\phi|\psi) = \text{tr}_c(\phi^\dagger\psi) \). We reserve \( ^\dagger \) to denote hermitian conjugation on the classical Hilbert space, while \( ^\dagger_c \) denotes hermitian conjugation on the quantum Hilbert space. Note that \( H_q \) has a natural tensor product structure in that it can be viewed as the tensor product of \( H_c \) and its dual, i.e., \( H_q = H_c \otimes H_c^\ast \). We can therefore also write the elements of \( H_q \) in the form \( |\psi, \phi\rangle = |\psi\rangle\langle\phi| \). We shall use this notation quite often below and refer to \( \psi \) as the left hand and \( \phi \) as the right hand sector.

A unitary representation of the non-commutative Heisenberg algebra

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0 \tag{4} \]

is obtained by the following action:

\[ \hat{X}_i\psi = \hat{x}_i\psi, \quad \hat{P}_i\psi = \frac{1}{\theta}\epsilon_{ij}[\hat{x}_j, \psi]. \tag{5} \]

Here we use capital letters to distinguish operators acting on the quantum Hilbert space from those acting on the classical Hilbert space. It is also useful to introduce the following quantum operators \( B, B^\dagger, \hat{P}, \hat{P}^\dagger \), where

\[ B = \frac{1}{\sqrt{2\theta}} \left( \hat{X} + i\hat{Y} \right), \quad \hat{P} = \hat{P}_x + i\hat{P}_y \tag{6} \]

These operators act as follow

\[ B\psi(\hat{x}, \hat{y}) = b\psi(\hat{x}, \hat{y}), \quad B^\dagger\psi(\hat{x}, \hat{y}) = b^\dagger\psi(\hat{x}, \hat{y}), \quad P\psi(\hat{x}, \hat{y}) = -i\sqrt{\frac{\gamma}{\theta}}[b, \psi(\hat{x}, \hat{y})], \quad P^\dagger\psi(\hat{x}, \hat{y}) = i\sqrt{\frac{\gamma}{\theta}}[b^\dagger, \psi(\hat{x}, \hat{y})]. \tag{7} \]

The interpretation of this quantum system now proceeds as for a standard one. The only modification required is that position measurement must now be interpreted in the context of a weak measurement (Positive Operator Valued Measure) rather than a strong (Projective Valued Measurement). The essence of the construction is based on the minimal uncertainty
normalised coherent states on $\mathcal{H}_c$ given by $|z\rangle = e^{-z^*/2}e^{iz\theta}|0\rangle$ where $z = \frac{1}{\sqrt{2\theta}}(x + iy)$ is a dimensionless complex number. These states provide an overcomplete basis on the non-commutative configuration space. Corresponding to these states one constructs a state in quantum Hilbert space as follows

$$|z, z\rangle = |z\rangle\langle z|, \quad B|z, z\rangle = z|z, z\rangle,$$  

which leads to the natural interpretation of $(x, y)$ as the dimensionful position coordinates. These states provide an overcomplete set on the quantum Hilbert space enabling us to obtain the operators in terms of the orthonormal basis $|n\rangle$

$$\pi_z = \frac{1}{2\pi\theta}|z, z\rangle e^{\theta z\partial_z} (z, z)|z, z\rangle = \frac{1}{2\pi\theta} \sum_n |z, n\rangle\langle z, n|$$  

forming a set of complete, positive, but non-orthogonal and unnormalized projection operators $[8]$, i.e.,

$$\int dx dy \pi_z = 1_q \quad (\psi|\pi_z|\psi) \geq 0, \forall \psi, \quad \pi_z\pi_w \neq \delta(z - w), \quad \pi_z^2 \propto \pi_z.$$  

They therefore provide a Positive Operator Valued Measure (POVM) that can be used to give a consistent probability interpretation by assigning the probability of finding the outcome of a position measurement to be $(x, y)$, given that the system is described by the density matrix $\rho$, to be $P(x, y) = \text{tr}_q (\pi_z \rho)$, which is completely insensitive to the right sector.

### 3. Spectral triplets on classical configuration space

We introduce the spectral triplet $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A} = \mathcal{H}_q$, $\mathcal{H} = \mathcal{H}_c \otimes C^2$, the action of elements of $a \in \mathcal{A}$ on $\mathcal{H}$ is defined through the representation $\pi(a)(|\psi\rangle, |\phi\rangle) = (a|\psi\rangle, a|\phi\rangle)$ and the Dirac operator is defined as $D = -i\sigma^1 P_i$ with $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ being the first two Pauli matrices. Then in the commutator $[D, \pi(a)]$ the Dirac operator can effectively be identified as,

$$D = -i\sqrt{\frac{\theta}{2}} \begin{pmatrix} 0 & b^\dagger \\ b & 0 \end{pmatrix}.$$  

States $\omega$ are positive linear functionals of norm one over $\mathcal{A}$. The Connes’ spectral distance between two states is then defined by

$$d(\omega, \omega') = \sup_{a \in \mathcal{B}} |\omega(a) - \omega'(a)|, \quad \mathcal{B} = \{a \in \mathcal{A}: ||[D, \pi(a)]||_{\text{op}} \leq 1\}, \quad ||A||_{\text{op}} = \sup_{\phi \in \mathcal{H}} \frac{||A\phi||}{||\phi||}.$$  

Exploiting the close link between states and density matrices (taken to be hermitian, semipositive and normalised) $\omega$ and $\rho_\omega$ respectively, any state over $\mathcal{A}$ can be uniquely written as $[9]$ $\omega(a) = \text{tr}_c(\rho_\omega a)$. Note that $\rho_\omega \in \mathcal{A} = \mathcal{H}_q$. If $\omega$ is a pure state, so is $\rho_\omega$, i.e. it is of the form $\rho_\omega = |\psi\rangle\langle \psi|$ for some $|\psi\rangle \in \mathcal{H}_c$.

Using this we can write the Connes’ distance function as

$$d(\rho, \rho') = \sup_{a \in \mathcal{B}} |\text{tr}_c((d\rho)a)|$$  

where $d\rho = \rho - \rho'$ and we use the notation $d\rho$ to indicate that we are interested in infinitesimal changes. Using the Schwarz inequality $||[d\rho, a]|| \leq ||d\rho||_{\text{tr}}||a||_{\text{tr}}$, this allows us to write Connes’ distance function, in a more tractable form as,

$$d(\rho, \rho') = \frac{\text{tr}_c(d\rho)^2}{||[D, \pi(d\rho)]||_{\text{op}}}.$$  

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Let us apply this to the example in [6] where the distance between harmonic oscillator states is calculated. In terms of the harmonic oscillator basis $|n\rangle$, introduced in (2), we can write $a = \sum_{m,n=0} a_{m,n} |m\rangle \langle n|$. The states are taken as $\omega_m(a) = a_{m,m}$ with corresponding pure state density matrix $\rho_m = |m\rangle \langle m|$. We then take $d\rho = \rho_{m+1} - \rho_m$. The calculation is now straightforward:

$$[D, d\rho] = -i \frac{\sqrt{2}}{\theta} \begin{pmatrix} 0 & [b^\dagger, d\rho] \\ [b, d\rho] & 0 \end{pmatrix},$$

(15)

A simple calculation yields the diagonal forms of

$$||[b, d\rho]||^2_{\text{op}} = ||4(m+1)|m+1\rangle \langle m+1| + m|m\rangle \langle m| + (m+2)|m+2\rangle \langle m+2|||_{\text{op}} = 4(m+1),$$

$$||[b^\dagger, d\rho]||^2_{\text{op}} = ||(m+2)|m+1\rangle \langle m+1| + (m+1)|m+1\rangle \langle m+1| + m|m-1\rangle \langle m-1|||_{\text{op}} = 4(m+1)$$

From this we obtain $||[D, d\rho]||^2_{\text{op}} = 2\sqrt{\frac{2(m+1)}{\theta}}$ and

$$d(m+1,m) = \sqrt{\frac{\theta}{2(m+1)}},$$

(16)

which agrees precisely with the result of [6].

Next we calculate the distance between two coherent harmonic oscillator states. Here the pure state density matrix are $\rho = |z\rangle \langle z|$. Then

$$d\rho = |z + dz\rangle \langle z + dz| - |z\rangle \langle z|$$

$$= (b^\dagger - \tilde{z}) |z\rangle \langle z| dz + |z\rangle \langle z| (b - \tilde{z}) d\tilde{z} + O(dz^2, d\tilde{z}^2).$$

(17)

This can be written more simply as

$$d\rho = dz|0\rangle \langle 1| + dz|1\rangle \langle 0|$$

(18)

where we have introduced new bosons $\tilde{b} = b - \tilde{z}$, $\tilde{b}^\dagger = b^\dagger - \tilde{z}$ and the vacuum $|0\rangle = |z\rangle$ for which $\tilde{b}|0\rangle = 0$, $|1\rangle = \tilde{b}^\dagger|0\rangle$. From this

$$[b, d\rho] = dz|0\rangle \langle 0| - \sqrt{2}d\tilde{z}|0\rangle \langle 2| - dz|1\rangle \langle 1|$$

(19)

and proceeding as before, we get

$$||[b, d\rho]||^2_{\text{op}} = ||[b^\dagger, d\rho]||^2_{\text{op}} = 3dzdz$$

(20)

so that $||[D, d\rho]||^2_{\text{op}} = \sqrt{\frac{6dzdz}{\theta}}$ and

$$d(z + dz, z) = \sqrt{\frac{2\theta}{3}} \sqrt{dzdz},$$

(21)

yielding the Euclidean distance. One can quickly check that replacing the operator norm in (16) and (21) by the trace norm one recovers, up to numerical factors, the same results. In what follows we shall often replace the operator norm with the trace norm and write

$$d(\rho, \rho') = \frac{\text{tr}_c(dx)^2}{||[D, \pi(dx)]||_{\text{tr}}},$$

(22)
4. Spectral triplets on quantum Hilbert space

Unlike in the previous section, a generic state |ψ, φ⟩ = |ψ⟩⟨φ|cHq or its corresponding pure state density matrix is given by ρq = |ψ⟩⟨φ|cHq. The family of such states with fixed |ψ⟩ have many-to-one relation with the points |ψ⟩ (associated with the density matrix |ψ⟩⟨ψ|) in Hc.

On the other hand, as discussed in section 2 and further below, in non-commutative quantum mechanics a particle maximally localized at a point (x, y), without any prior knowledge of the right hand sector, is described by the mixed state density matrix ρ = ∑n |z, n⟩⟨z, n| (z = x + iy). When the right hand sector is also specified the particle is described by a pure state density matrix ρ = |z, n⟩⟨z, n|, but not that there are infinitely many choices for the right hand sector given a point (x, y) in classical configuration space. In this section, we would like to compute the various distances between pure / mixed states. To this end we introduce a further spectral triplet (A, H, D) on quantum Hilbert space where A = B(Hq), i.e., the bounded operators on Hq, H = Hq ⊗ C2, the action of elements of a ∈ A on H is defined through the representation π(a)⟨|ψ⟩, φ⟩⟩ = ⟨a|ψ⟩, a⟩ φ⟩ and the (effective) Dirac operator is defined as

\[ D = -i\sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & B^\dagger \\ B & 0 \end{pmatrix}. \]  

(23)

Note that the Dirac operator defined here acts only on the left sector as it does not involve any momentum operators.

Using the same Dirac operator we can compute the spectral distance between the states ρq(m, φ) = |m, φ⟩⟨m, φ| and ρq(m + 1, φ′) = |m + 1, φ′⟩⟨m + 1, φ′| where φ′ is different and orthogonal to φ, i.e., ⟨φ|φ′⟩ = δ0,φ′. Introducing dpq = |m + 1, φ′⟩⟨m + 1, φ′| − |m, φ⟩⟨m, φ|, we proceed as before to see that for φ = φ′ we recover the earlier result (16), as if φ is merely a spectator. But for φ ≠ φ′ we get dl(ρ(m + 1, φ′), ρ(m, φ)) = \sqrt{\frac{2\theta}{m+2}}. This shows that dl(ρ(m + 1, φ), ρ(m, φ)) < dl(ρ(m + 1, φ′), ρ(m, φ)) and demonstrates that, despite the fact that the Dirac operator only acts on the left hand sector, the distance between two states depends on the right hand sector. One may contemplate an even more general situation by taking the right hand sector to be a statistical mixture that changes from point to point, i.e., we can consider the following density matrices

\[ ρ_q(m) = \sum_n p_n(m)|m, n⟩⟨m, n|, \quad ρ_q(z, \bar{z}) = \sum_n p_n(z, \bar{z})|z, n⟩⟨z, n| \]  

(24)

subject to the respective conditions

\[ \sum_n p_n(m) = 1, \forall m, \quad \sum_n p_n(z, \bar{z}) = 1, \forall z \]  

(25)

where |m⟩ and |n⟩ are as above, while |n⟩ is an arbitrary orthonormal basis in classical configuration space and \( p_n \) a set of position dependent probabilities. By comparing the averages and fluctuations of B1 B2 and X1 in the respective states \( ρ_q(m) \) and \( ρ_q(z, \bar{z}) \), we see that \( p_q(m) \)'s are localised at a fixed radial distance \( \sqrt{\theta(2m + 1)} \) and \( ρ_q(z, \bar{z}) \) are maximally localised at (x, y).

Let us first compute the Connes’ distance function for the density matrices (24), using (22). Setting dpq = ρq(m + 1) − ρq(m) and replacing operator norm with trace norm (14)(which affects the result at most upto a constant), one obtains

\[ d(m + 1, m) = \frac{\sqrt{\theta}}{2} \sqrt{\sum_n \frac{p_n^2(m + 1) + p_n^2(m)}{(2m + 3)p_n^2(m + 1) + (2m + 1)p_n^2(m) + 2(m + 1)p_n(m + 1)p_n(m)}}. \]  

(26)

At this stage, it is not clear a priori what criterions should be used to choose the probability distributions. However, we can investigate about the following two possibilities, which present
themselves in an obvious manner. And for that we proceed one by one. Let us first ask which choice of probabilities actually minimizes the distance function. Starting with (26), the distance between two points $m_i$ and $m_f$ is given by

$$d(m_f, m_i) = \sum_{m=m_i}^{m_f-1} d(m+1, m). \tag{27}$$

Carrying out the minimization by the method of Lagrange’s multipliers, we find that all the probabilities must be equal say $(1/\Omega)$ if $\Omega$ is the cut-off for the sum over $n$ and independent from $m$ and $n$. In this case the distance function reduces to

$$d(m+1, m) = \frac{\sqrt{\theta}}{\sqrt{6\Omega}} \frac{1}{\sqrt{m+1}}, \tag{28}$$

which only differs from (16) by a probability dependent global scale factor. Note that the operators $\pi_z$ introduced in (9) are density matrices with this property.

Next we explore the geometry emerging from the second perspective. Here, we would like to fix the probabilities from the condition of local thermal equilibrium i.e. local average energy is fixed and the local entropy associated with the density matrices (24) is optimized. To do this we introduce a local entropy associated with the density matrices (24) in the usual manner

$$S(m) = -\sum_n p_n(m) \log p_n(m), \quad S(z, \bar{z}) = -\sum_n p_n(z, \bar{z}) \log p_n(z, \bar{z}). \tag{29}$$

A simple calculation shows that maximizing this local entropy leads, as above, to equally distributed probabilities. This, of course, corresponds to the $T \to \infty$ limit where only entropy plays a role and all states are assigned equal probabilities.

Associating an energy scale $\epsilon_n$ with the right hand states labeled by $|n\rangle$ and require the local average energy to be fixed on $E(m)$, maximization of the local entropy yields a local Boltzman distribution

$$p_n(m) = \frac{e^{-\beta(m)\epsilon_n}}{Z(\beta(m))}, \quad Z(\beta(m)) = \sum_n e^{-\beta(m)\epsilon_n}, \tag{30}$$

where $\beta(m)$ is the local inverse temperature. From this the distance function (26) can be computed. In particular, if we assume the local average energy and hence the temperature to be independent of $m$, we obtain

$$d(m+1, m) = \frac{\sqrt{\theta}}{\sqrt{6}} \frac{1}{Z(\beta)} \frac{1}{\sqrt{m+1}}, \tag{31}$$

which again only differs from (16) by a temperature dependent global scale factor.

This implied modified geometry for mixed states becomes even more explicit in the case of the continuous coherent state basis, which we consider next.

The Connes’ distance in the coherent state simplifies considerably with a slight redefinition of the Dirac operator. We take the Dirac operator, written in terms of $\tilde{b}$ and $|\tilde{0}\rangle = |z\rangle$ to be

$$D = -i\sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [B^\dagger, 2\pi\theta\pi_z] \\ [B, 2\pi\theta\pi_z] & 0 \end{pmatrix} = -i\sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & 0 & \sum_n |\tilde{1}, n\rangle(\tilde{0}, n| \\ \sum_n |\tilde{0}, n\rangle(\tilde{1}, n| & 0 \end{pmatrix}. \tag{32}$$

Note that the Dirac operator is still insensitive to the right hand sector.
The computation now follows the same route as before and one obtains
\[ d(z + dz, z) = \frac{\sqrt{\theta}}{2} \sqrt{2 \sum_n \left( p_n(z)^2 + \frac{\partial p_n(z)}{\partial \bar{z}} \frac{\partial p_n(z)}{\partial z} \right) d\bar{z} dz + \sum_n \left( \left( \frac{\partial p_n(z)}{\partial \bar{z}} \right)^2 d\bar{z}^2 + \left( \frac{\partial p_n(z)}{\partial z} \right)^2 dz^2 \right) }. \] (33)

This clearly exhibits a modified geometry. Assuming that \( p_n \) is just a function of the dimensionful radial coordinate \( r = \sqrt{\frac{\bar{z}z}{\theta}} \) one obtains
\[ d(z + dz, z) = \frac{1}{2\sqrt{2}} \sqrt{\sum_n \left( 2p_n(r)^2 + p_n'(r)^2 \right) \sqrt{dr^2 + r^2 d\phi^2}}, \] (34)

which is related to the Euclidean distance by a probability dependent conformal transformation. We can of course again compute the probabilities from the condition of local equilibrium. In the \( T \to \infty \) limit, the probabilities are obtained by maximizing (29), which yields equal, \( r \)-independent probabilities and subsequently, up to a global scale, an Euclidean geometry. If we constrain the local average energy to \( E(r) \), the temperature becomes position dependent and the probabilities again follow a Boltzmann distribution
\[ p_n(r) = \frac{e^{-\beta(r)\epsilon_n}}{Z(\beta(r))} , \] (35)
yielding a conformal scale factor. If the local average energy is position independent, so is the temperature, and we obtain an Euclidean geometry with a global, temperature dependent, scale factor
\[ d(z + dz, z) = \sqrt{\frac{Z(2\beta)}{2Z(\beta)}} \sqrt{dr^2 + r^2 d\phi^2}. \] (36)

As the prefactor is bounded by \( 0 < \frac{\sqrt{Z(2\beta)}}{2Z(\beta)} \leq \frac{1}{2} \), reaching its minimum at \( T \to \infty \) and its maximum at \( T \to 0 \) this suggests, quite remarkably, that distances expand as the global temperature is lowered.

5. Conclusions

The formulation of non-commutative quantum mechanics on the Hilbert space of Hilbert-Schmidt operators naturally encodes the notion of spectral triplets in the sense of Connes and thus the notion of geometry. The geometry can be made explicit on the level of the classical configuration space and the quantum Hilbert space. In the former case, as in commutative quantum mechanics, there is a one-to-one correspondence between pure state density matrices and points in the classical configuration space. In the latter case this no longer holds due to the tensor product structure of the quantum Hilbert space and the distance function depends on information encoded in the right hand sector, even though the Dirac operator only acts on the left hand sector. It then becomes natural to define a local entropy. The condition of local equilibrium then fully determines the distance function and, subsequently, the geometry. When the temperature is independent of position, Euclidean geometry results with a scale determined by the global temperature. Remarkably the scale increases as the temperature is lowered, leading to expanding distances at lower temperature.

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