Mutually connected component of network of networks with replica nodes

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We describe the emergence of the giant mutually connected component in networks of networks in which each node has a single replica node in any layer and can be interdependent only on its replica nodes in the interdependent layers. We prove that if in these networks, all the nodes of one network (layer) are interdependent on the nodes of the same other interconnected layer, then, remarkably, the mutually connected component does not depend on the topology of the network of networks. This component coincides with the mutual component of the fully connected network of networks constructed from the same set of layers, i.e., a multiplex network.

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I. INTRODUCTION

Complex networks structures strongly affect cooperative and critical phenomena in them 1 2. Despite the huge interest in the topic only recently it has become clear that in order to characterize the function and the dynamics of the majority of complex systems, it is necessary to make a step further and consider networks of networks 3 4. For example, if we want to understand the robustness of critical infrastructures 5 it is necessary to characterize the complex interdependencies between them, or if we aim at characterizing the function of a cell, we need to scale up the analysis of single cellular networks such as a protein interaction network and a metabolic network and study also interactions between different cellular networks. Critical phenomena in a network of networks and multilayer structures 3 4 6 show surprising new features 5 7 24. In particular it has been recently shown 5 7 9 that when we consider several interdependent networks, the system as a whole might be much more fragile than single networks taken in isolation, and that the interdependencies between different networks can trigger cascading failure events of dramatic impact on the networks of networks.

After the seminal work 5 it has become commonly accepted that the robustness of interdependent networks can be evaluated by considering the size of the mutually connected component of the interdependent networks. While the emergence of the giant mutually connected component in multiplex networks (graphs with nodes of one kind and different types of links) is already well understood 5 7 11, few works focused on the mutually connected component in a network of networks in which nodes in each individual network (layer) are interdependent with nodes in some other layers 10–22. In the interdependent networks, a mutually connected component is introduced as a subgraph remaining after the cascade of failures spreading back-and-forth through intralinks within layers and through interlinks—interdependencies—from layer to layer. For a sufficiently large number of layers in the network of networks 15, say, greater than 3, the complex structure of interconnections makes the problem of the mutually connected component principally richer than for a pair of interdependent networks 5 7 8. The key question is how far are complex networks of networks from the multiplex networks in respect of their mutual components? In a series of publications 16–18 a network of networks in which each node can be linked to a random node in an interdependent layer (i.e., there are no replica nodes) was considered. It was demonstrated that if this type of network of networks is a proper tree (in respect of interlinks), then the size of its mutual component is determined by the number of layers and does not depend on the structure of this tree; but it was found, contrastingly, that for a network of networks with loops, its global structure matters 20. In the works 16 18 20 it was found that if the supernetwork of interdependencies between the layers is a tree, then the mutually connected component of such a network of networks follows the same equations as for a multiplex network formed by these layers. In contrast, it was shown in Ref. 20 that the supernetwork of interconnections in a network of networks of this class (each node in a layer is interconnected with a random node in a respective interdependent layer) contains loops, then the mutually connected component does not satisfy these equations. More recently, it has been found in Ref. 24 that the difference between these topologies can be dramatic. While in the case of a tree supernetwork, all the layers percolate at once resulting in a single discontinuous phase transition, in the case of a loopy supernetwork with heterogeneous degrees of the nodes, layers with different number of interdependencies fail one after the other as
FIG. 1: (Color online) Schematic view of a typical network of networks with replica nodes considered in this paper. Interdependencies (interlinks between nodes from different levels) are shown by the black dashed lines. Intralinks between nodes within layers are shown as solid red lines.

the initial damage inflicted to the network increases \[^{23}\] . This results in a chain of phase transitions.

Here we will show that the way the nodes are connected between the layers affects very significantly the robustness properties of the network of networks. In fact if each node has a single replica nodes in the other layers and the interconnections are organized in such a way that all the nodes of a layer are interdependent on the nodes of the same other layer (see Fig. 1), then, remarkably, the mutually connected component does not depend on topology of the network of networks. We emphasize that it can be tree or it can be loopy, in both cases, apart from natural dependence on the structures of individual layers, only the size of the network of networks (the number of layers) matters. Thus we show that the problem of a wide range of networks of networks with replica nodes is actually reduced to the well studied problem of multiplex networks. We will arrive at this unexpected conclusion by applying the convenient message-passing technique \[^{8,11,12,24,26}\]. In the case of locally tree-like networks, this approach, which is also called the cavity method, believe propagation, message-passing equations, etc., leads to the same equations as the more traditional technique \[^{27}\] used in Refs. \[^{5,8,9,16,18,20}\] and in numerous works for other complex networks \[^{2}\]. On the other hand, when a network has finite loops, the message passing technique provides an approximate solution, which was reported to be reasonably precise in investigated loopy networks \[^{26}\]. In addition, this powerful techniques is particularly convenient for formalization and unified consideration of the class of problems under consideration, and so it constitutes a useful framework for analysis of a wide range of these complex networks.

II. NETWORK OF NETWORKS WITH REPLICA NODES

A network of networks in this work is formed by \(M\) networks \(\alpha = 1, 2, \ldots, M\) each of \(N\) nodes \(i = 1, 2, \ldots, N\). We assume that \(M\) is finite and \(N\) is infinite. Every node \((i, \alpha)\) can be connected to nodes \((j, \alpha)\) in the same network or with its “replica nodes” \((i, \beta)\) in other networks. If network \(\alpha\) is interdependent with network \(\beta\), each node \((i, \alpha)\) of network \(\alpha\) is interdependent on node \((i, \beta)\) of network \(\beta\) and vice versa (see Fig. 1). We define the network of networks with a super-adjacency matrix of elements \(a_{i\alpha,j\beta} = 1\) if there is a link between node \((i, \alpha)\) and node \((j, \beta)\) and zero otherwise. In these specific networks, \(a_{i\alpha,j\beta} = 0\) if both \(i \neq j\) and \(\alpha \neq \beta\). We call the graph of interdependencies with an adjacency matrix \(A_{\alpha\beta}\), such that for every \(i\) \(A_{\alpha\beta} = a_{i\alpha,i\beta}\), the supernetwork, \(\Gamma\), of the network of networks.

The mutually connected component can be defined as follows. Each node \((i, \alpha)\) is in the mutually connected component if it has at least one neighbor \((j, \alpha)\) which belongs to the mutually connected component and if all the linked nodes \((i, \beta)\) in the interdependent networks are also in the mutually connected component. From this definition, it follows that if a node \(i\) in one layer of a connected (in terms of interlinks) network of networks is in a mutual component, then all its “replica” nodes in all layers are in this mutual component. Then, for infinite \(N\), the same set of nodes \(\{i\}\) in each level belong to the giant mutually connected component. Moreover, it is clear from the same definition that the set of replicas \(\{i\}\) belonging to the mutual component will not change if we rearrange connections in the supernetwork retaining it connected. A cascade of failures in the networks of networks can be initiated by deleting a fraction of nodes (as is natural, this removal keeps all interdependencies functioning). After such a removal of a node, all its replica nodes necessarily should fall apart from the mutual component, so our conclusions do not change. In this paper we will prove strictly that these heuristic arguments are correct.

For a given network of networks it is easy to construct a message passing algorithm that allow us to determine if node \((i, \alpha)\) is in the mutually connected component. We denote by \(\sigma_{i\alpha \rightarrow j\alpha} = 1\) the message in within a layer, from node \((i, \alpha)\) to node \((j, \alpha)\) and indicating \(\sigma_{i\alpha \rightarrow j\alpha} = 1\) if node \((i, \alpha)\) is in the mutually connected component when we consider the cavity graph by removing the link \((i, j)\) in network \(\alpha\). Furthermore, let us denote by \(S^{\alpha}_{i\alpha \rightarrow i\beta} = 0, 1\) the message between the “replicas” \((i, \alpha)\) and \((i, \beta)\) of node \(i\) in layers \(\alpha\) and \(\beta\). The message \(S^{\alpha}_{i\alpha \rightarrow i\beta} = 1\) indicates if the node \((i, \alpha)\) is in the mutually connected component when we consider the cavity graph by removing the link between node \((i, \alpha)\) and node \((i, \beta)\). In addition, we indicate with \(s_{i\alpha} = 0\) a node that is removed from the network as an effect of the damage inflicted to the network, otherwise \(s_{i\alpha} = 1\).

The message passing equations for these messages are
of the following forms:
\[
\sigma_{i\alpha\to j\alpha} = s_{i\alpha} \prod_{\beta \in \mathcal{C}(\alpha)\setminus \alpha} \left[ 1 - \prod_{\ell \in N_{\alpha}(i) \setminus j} (1 - \sigma_{\ell\alpha \to i\alpha}) \right],
\]
\[
S'_{i\alpha\to i\beta} = s_{i\alpha} \prod_{\gamma \in \mathcal{N}(\alpha)\setminus \beta} \left[ 1 - \prod_{\ell \in N_{\alpha}(i)} (1 - \sigma_{\ell\alpha \to i\alpha}) \right],
\]
where \(N_{\alpha}(i)\) indicates the set of nodes \((\ell, \alpha)\) which are neighbors of node \(i\) in network \(\alpha\), and \(\mathcal{N}(\alpha)\) indicates the layers that are interdependent on network \(\alpha\). Using simple properties of these messages, the expression for the messages \(\sigma_{i\alpha\to i\beta}\) can be simplified giving
\[
\sigma_{i\alpha\to j\alpha} = S'_{i\alpha\to i\beta} S'_{i\beta\to i\alpha} \left[ 1 - \prod_{\ell \in N_{\alpha}(i) \setminus j} (1 - \sigma_{\ell\alpha \to i\alpha}) \right], \tag{2}
\]
(see Appendix \(A\) for the detailed derivation). Finally \(S_{i\alpha}\) indicates if a node \((i, \alpha)\) is in the mutually connected component or not \((S_{i\alpha} = 1, 0)\), namely
\[
S_{i\alpha} = s_{i\alpha} \prod_{\beta \in \mathcal{C}(\alpha)} \left[ 1 - \prod_{\ell \in N_{\alpha}(i)} (1 - \sigma_{\ell\alpha \to i\alpha}) \right], \tag{3}
\]

III. GENERAL SOLUTION OF THE ALGORITHM

In this paragraph we will show that the general solution of the message passing algorithm in Eqs. \(1\) in a network of networks with replica nodes, where the supernetwork may consist of an arbitrary number of connected components, is given by
\[
\sigma_{i\alpha\to j\alpha} = \prod_{\beta \in \mathcal{C}(\alpha)\setminus \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_{\beta}(i)} (1 - \sigma_{\ell\beta \to i\beta}) \right] \right\} \times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_{\alpha}(i)} (1 - \sigma_{\ell\alpha \to i\alpha}) \right], \tag{4}
\]
\[
S_{i\alpha} = \prod_{\beta \in \mathcal{C}(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_{\beta}(i)} (1 - \sigma_{\ell\beta \to i\beta}) \right] \right\}. \tag{5}
\]
Here and in the following \(\mathcal{C}(\alpha)\) is the connected component in the supernetwork \(\mathcal{G}\) to which layer \(\alpha\) belongs. In particular, if the supernetwork contains only a single connected component, then \(\mathcal{C}(\alpha) = \mathcal{G}\) for every layer \(\alpha\) and the cardinality of this component (its size) \(|\mathcal{C}(\alpha)|\) coincides with \(\mathcal{M}\), the total number of layer of the network of networks.

In the Appendix \(B\) we show that indeed this formula is valid for simple examples of supernetworks, such as a tree, a forest and a single loop.

Here in the following we want to show that Eqs. \(1\) and \(5\) are valid for every supernetwork topology. If we consider a connected supernetwork, Eq. \(1\) for the messages \(S'_{i\alpha\to i\beta}\) can be written as
\[
S'_{i\alpha\to i\beta} = \prod_{P_{\alpha\beta}(\gamma)} \prod_{\ell, \gamma, \gamma'} \left\{ s_{i\alpha} \left[ 1 - \prod_{\ell \in N_{\gamma}(i)} (1 - \sigma_{\ell\alpha \to i\alpha}) \right] \right\} \times \prod_{\zeta \in P_{\alpha\beta}(\gamma)} \left\{ s_{i\zeta} \left[ 1 - \prod_{\ell \in N_{\zeta}(i)} (1 - \sigma_{\ell\zeta \to i\zeta}) \right] \right\}, \tag{6}
\]
where \(P_{\alpha\beta}(\gamma)\) are all the directed paths of the supernetwork that can be drawn from any node \(\gamma\) and that, starting from the superlink \((\gamma, \gamma')\), arrive at node \(\alpha\) from nodes different from \(\beta\).

Let us first consider the case of a tree supernetwork (see Fig. \(2\)). Since the product in Eq. \(6\) involves only zeros and ones, we can consider only the paths starting from the leaves of the supernetwork. If \(\gamma\) is a leaf of the supernetwork, it follows from Eq. \(1\) that
\[
S'_{i\alpha\to i\gamma'} = s_{i\gamma} \left[ 1 - \prod_{\ell \in N_{\gamma}(i)} (1 - \sigma_{\ell\gamma \to i\gamma}) \right]. \tag{7}
\]
Therefore we obtain
\[
S'_{i\alpha\to i\beta} = \prod_{\ell \in T_{\alpha\beta}} \left\{ s_{i\ell} \left[ 1 - \prod_{\ell \in N_{\ell}(i)} (1 - \sigma_{\ell\ell \to i\ell}) \right] \right\}, \tag{7}
\]
where \(T_{\alpha\beta}\) is the subtree in the supernetwork that has the root given by the layer \(\alpha\) and branching departing from every link of layer \(\alpha\) in the supernetwork except the link to layer \(\beta\). Using the expression given by Eq. \(2\) we find for this tree supernetwork that the messages \(\sigma_{i\alpha\to j\alpha}\) are given by
\[
\sigma_{i\alpha\to j\alpha} = \prod_{\beta \in \mathcal{C}(\alpha)\setminus \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_{\beta}(i)} (1 - \sigma_{\ell\beta \to i\beta}) \right] \right\} \times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_{\alpha}(i) \setminus j} (1 - \sigma_{\ell\alpha \to i\alpha}) \right], \tag{8}
\]
where we refer the reader to Appendix \(B\) for more details. Finally, using Eq. \(5\), it can be shown that \(S_{i\alpha}\) can be expressed in terms of the messages as
\[
S_{i\alpha} = \prod_{\beta \in \mathcal{G}} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_{\beta}(i)} (1 - \sigma_{\ell\beta \to i\beta}) \right] \right\}. \tag{9}
\]
We note here that Eqs. \(8\)–\(9\) are equivalent to Eqs. \(4\)–\(5\) on a connected supernetwork where \(\mathcal{C}(\alpha) = \mathcal{G}\) for every \(\alpha\).

Let us prove that Eqs. \(6\) imply that Eqs. \(8\) are actually valid for every connected supernetwork topology. We
call the set of layers connected to a layer $\alpha$ in the supernetwork at least by two non-overlapping paths the loopy component of layer $\alpha$. We call each connected component formed by the layers connected in the supernetwork to layer $\alpha$, but not belonging to the loopy components, the dangling components of layer $\alpha$. The dangling components might be trees or might contain loops (see Fig. 2).

Let us assume for simplicity that the supernetwork coincides with its loopy component and prove that Eq. (5) remains valid in this supernetwork topology, which principally differs from trees. Since in Eq. (6) every path $P_{\alpha\beta}(\gamma)$ contributes either by a zero or a one to the product, in order to evaluate the messages $S''_{i\alpha\beta}$, we can consider only the paths $P_{\alpha\beta}(\alpha)$ that can be drawn from layer $\alpha$ and that starting form the interlink $(\alpha, \gamma')$ are returning to layer $\alpha$ through links coming from layers different from $\beta$. Then we have

$$S''_{i\alpha\beta} = \prod_{P_{\alpha\beta}(\alpha)} \prod_{\gamma' \in P_{\alpha\beta}(\alpha)} S'_{i\alpha\gamma'} \times \prod_{\xi \in P_{\alpha\beta}(\alpha)} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\xi(i)} (1 - \sigma_{i\ell \rightarrow i\xi}) \right] \right\}. \quad (10)$$

This expression is valid for every node $i$ and every pair of interdependent layers $\alpha, \beta$. Since every layer of the loopy component can be reached at least by two non-overlapping paths in the supernetwork, the product over the layers $\xi$ in Eq. (10) includes all layers of the supernetwork. Therefore, all the messages $S''_{i\alpha\beta}$ of the supernetwork $G$ are all equal to one, $S''_{i\alpha\beta} = 1$, if and only if

$$\prod_{\xi \in G} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\xi(i)} (1 - \sigma_{i\ell \rightarrow i\xi}) \right] \right\} = 1. \quad (11)$$

Therefore the messages $S''_{i\alpha \rightarrow j\alpha}$ can be expressed as

$$S''_{i\alpha \rightarrow j\alpha} = \prod_{\xi \in G} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\xi(i)} (1 - \sigma_{i\ell \rightarrow i\xi}) \right] \right\}. \quad (12)$$

Using this result and Eqs. (2) and summarizing the results obtained for a supernetwork formed by a loopy component we can express the messages $\sigma_{i\alpha \rightarrow j\alpha}$ in within each layer as Eqs. (4).

In particular this result holds for the special case of a supernetwork formed by a single loop (see Appendix B for details). Proceeding in a similar way, the result obtained for trees and for loopy components can be directly extended to a connected supernetwork with general topology finding Eqs. (8). Indeed, this network is actually formed by a combination of loopy and dangling components, which allows for convenient iterative decomposition and complete analysis of all possible paths of messages (see Appendix C for further details). Here

$S_{i\alpha}$ are given by the same Eq. (9). Thus the mutually connected component depends only on the structure of the layers. Finally, since every connected component of supernetwork is independent, Eqs. (8)–(10) can be generalized for supernetwork topologies with several connected components giving Eqs. (4)–(5).

IV. AVERAGE OVER AN ENSEMBLE OF NETWORK OF NETWORKS WITH REPLICA NODES

Let us assume that the supernetwork is given (fixed by its adjacency matrix $A_{\alpha\beta}$) and connected and that each layer $\alpha$ is generated independently from a configuration model with a degree distribution $P_\alpha(k)$. That is, in our network of networks, the supernetwork is not random, while the layers are infinite uncorrelated random networks. Each layer $\alpha$ has a degree sequence $\{k_\alpha^\alpha\}$, and the degrees of the replica nodes in the different layers are uncorrelated. Furthermore, we assume that nodes $(i, \alpha)$ are removed from layers with probability $1 - p_\alpha$ (instigators of cascading failures). Consequently, the probability $P(\{s_{i\alpha}\})$ of the variables $s_{i\alpha}$ is given by

$$P(\{s_{i\alpha}\}) = \prod_{\alpha=1}^{M} \prod_{i=1}^{N} p_{i\alpha}^s (1 - p_\alpha)^{1 - s_{i\alpha}}. \quad (13)$$

In order to evaluate the expected size of the mutually connected component, we average the messages over this ensemble of the network of networks. We indicate with $\sigma_\alpha$ the average message in within a layer $\langle \sigma_{i\alpha \rightarrow j\alpha} \rangle = \sigma_\alpha$. The equations for the average messages in within a layer are given in terms of the parameters $p_\beta = \langle s_{i\beta} \rangle$ and the generating functions $G_0^\beta(z)$ and $G_1^\beta(z)$ defined as $G_0^\beta = \sum_k P_\beta(k) z^k$ and $G_1^\beta = \sum_k k P_\beta(k) / \langle k \rangle_\beta z^{k-1}$,
respectively. Using Eq. (8) we find
\[
\sigma_\alpha = p_\alpha \prod_{\beta \in \mathcal{V}_\alpha} \left\{ \beta [1 - G_0^\beta (1 - \sigma_\beta)] \right\} [1 - G_0^\alpha (1 - \sigma_\alpha)],
\]
(14)

Moreover, using Eq. (9) we can derive the probability for \( S_\alpha = \langle S_\alpha \rangle \) that a randomly chosen node in layer \( \alpha \) belongs to the mutually connected component:
\[
S_\alpha = \prod_{\beta \in \mathcal{V}} \left\{ p_\beta [1 - G_0^\beta (1 - \sigma_\beta)] \right\}.
\]
(15)

In particular, if all layers \( \beta \) have the same topology and equal probabilities \( p_\beta = p \), then the average messages in within different layers are all equal \( \sigma_\beta = \sigma \) and are given by
\[
\sigma = p^M [1 - G_0 (1 - \sigma)]^{M-1} [1 - G_1 (1 - \sigma)],
\]
(16)

where \( M \) is the cardinality (number of nodes) of the supernetwork. Furthermore, the probability \( S = \langle S_\alpha \rangle \) that a node in a given layer is in the mutually connected component is given by
\[
S = \left\{ p (1 - G_0 (1 - \sigma)) \right\}^M.
\]
(17)

Thus, as long as the supernetwork is connected, the resulting mutual component is determined only by the size of the supernetwork, and not by its structure. In particular, this supernetwork can be a fully connected graph. Therefore characterization of this model can be reduced to the problem of finding the mutually connected component of a multiplex network \([5, 9]\). Equations (14) and (15) or Eqs. (16) and (17) coincide with those for a multiplex network with \( M \) layers \([5, 8]\). We emphasize that these equations are identical to, respectively, Eqs. (29) and (30) or Eqs. (34) and (35) from Ref. \([17]\) obtained for a network of networks with random matching of the nodes in interdependent layers (a node in a layer is interconnected with a single random node in an interdependent layer) having a tree supernetwork. Clearly, this network is identical to the particular case of our network of networks with replica nodes and the same tree supernetwork. Indeed, since interlinks in this network do not form loops, one can relabel nodes in individual layers ascribing the same label to each of interlinked nodes, so that the interlinked nodes are actually replicas. This is, however, a particular network of considered in the present work. Importantly, we have shown that Eqs. (14) and (15) or Eqs. (16) and (17) are valid to a much wider class of networks with replica nodes having an arbitrary connected supernetwork with any loops. Note that if \( M \to \infty \), then \( \sigma = 0 \) or \( 1 \), where the solution \( \sigma = 1 \) only can be found in the case of \( p = 1 \).

V. DISCUSSION AND CONCLUSIONS

Our findings show that a mutually connected component is independent on the structure of a supernetwork for any networks of networks with replica nodes and a fixed supernetwork. We proved that in these networks the phenomenon of the independence of the topology is actually not about the presence or absence of loops in the networks of networks but rather about correlations in the set of interdependencies between nodes in pairs of layers. In particular, the results obtained above are valid for a ring of interdependent networks, i.e., for a supernetwork having a form of a simple loop.

Interestingly, our results are still valid even for more general networks of networks than were considered in this paper. Let us return to the direct consequences of the definition of the mutual component, which we discussed introducing the model of a network of networks. Consider again a connected network of networks of this type (see Fig. 1). Let us rearrange separately and independently interconnections between nodes in each individual set, that consists of node \( i \) and all its replicas, retaining each of this sets connected. Clearly, this rearrangement preserves the set of nodes belonging to the mutually connected component. On the other hand, after this rearrangement, different nodes in the same layer of the resulting network of networks already can be interconnected with different sets of layers, which differs principally from Fig. 1.

The form of our final results, Eqs. (14) and (15) or Eqs. (16) and (17), assumes that each of the individual layers is an uncorrelated network. We presented however simple arguments allowing us to suggest that our qualitative conclusion, namely, the independence on the topology of the supernetwork, is actually valid for arbitrary structures of the layers.

Finally, we should indicate an alternative class of networks of networks in which organization of interconnections really matters.

For example, let the interlinks connecting nodes in different layers be completely random similarly to the configuration model. In other words, the interlinks of the nodes of each layer are assumed to be uncorrelated in contrast to Fig. 1. The detailed general analysis of this class of networks of networks show that percolation in these networks of networks is irreducible to multiplex networks and that these networks of networks display multiple percolation transitions \([22]\).

In summary we have shown that in a surprisingly wide class of networks of networks, a mutually connected component does not depend on their global topology. The mutual component problem for these networks of networks is resolved by reduction to that for multiplex networks.

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Appendix A: Derivation of the simplified expression for the messages $\sigma_{i\alpha \rightarrow j\alpha}$ given by Eq. (2)

In this appendix, from the definition in Eqs. (1) that we rewrite here for convenience,

$$\sigma_{i\alpha \rightarrow j\alpha} = s_{i\alpha} \prod_{\beta \in N(\alpha)} S'_{\beta \rightarrow i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S'_{i\alpha \rightarrow i\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right], \quad (A1)$$

we derive the simplified expression for the messages $\sigma_{i\alpha \rightarrow j\alpha}$ given by Eq. (2), i.e.,

$$\sigma_{i\alpha \rightarrow j\alpha} = S'_{i\alpha \rightarrow i\beta} S'_{i\beta \rightarrow j\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right], \quad (A2)$$

Let us rewrite Eqs. (A1), introducing the auxiliary message $y_{i\alpha \rightarrow j\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow i\alpha}$. Then we have

$$\sigma_{i\alpha \rightarrow j\alpha} = y_{i\alpha \rightarrow i\beta} S'_{i\beta \rightarrow j\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S'_{i\alpha \rightarrow i\beta} = y_{i\alpha \rightarrow i\beta} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right]. \quad (A3)$$

We notice that $\sigma_{i\alpha \rightarrow j\alpha} = 1$ only if also $\left( 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$. The condition $\left( 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$ implies that automatically $\left( 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$.

We summarize these considerations writing the following system of equations for the messages

$$\sigma_{i\alpha \rightarrow j\alpha} = S'_{i\alpha \rightarrow i\beta} S'_{i\beta \rightarrow j\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S'_{i\alpha \rightarrow i\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right] \quad (A4)$$

that must be satisfied for every connected pair of layers $(\alpha, \beta)$.

Appendix B: The application of the message passing equations to simple networks

In this appendix we provide simple examples for which we verify the validity of the general relations given by Eqs. (4) and (5). For the sake of convenience, we rewrite here these relations,

$$\sigma_{i\alpha \rightarrow j\alpha} = \prod_{\beta \in C(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i\beta}) \right] \right\},$$

$$\times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S_{i\alpha} = \prod_{\beta \in C(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i\beta}) \right] \right\}, \quad (B1)$$

where $C(\alpha)$ is the connected component of the supernetwork to which the layer $\alpha$ belongs. In particular, if the supernetwork contains only a single connected component, $C(\alpha) = G$ for every layer $\alpha$, and the cardinality of this component (its size) $|C(\alpha)|$ coincides with $M$.

a. Message passing equations on a forest supernetwork

Let us consider the message passing Eqs. (1) on a forest supernetwork, i.e., a supernetwork formed by several components, in which each component is a tree. Starting from the message passing equations

$$\sigma_{i\alpha \rightarrow j\alpha} = s_{i\alpha} \prod_{\beta \in N(\alpha)} S'_{i\beta \rightarrow i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S'_{i\alpha \rightarrow i\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right]. \quad (B2)$$

We notice that the messages can be expressed as

$$\sigma_{i\alpha \rightarrow j\alpha} = y_{i\alpha \rightarrow i\beta} S'_{i\beta \rightarrow j\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right],$$

$$S'_{i\alpha \rightarrow i\beta} = y_{i\alpha \rightarrow i\beta} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i,\alpha}) \right],$$

$$y_{i\alpha \rightarrow i\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow i\alpha}. \quad (B3)$$

We observe that $\sigma_{i\alpha \rightarrow j\alpha} = 1$ only if also $\left( 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$. The condition $\left( 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$ implies that automatically $\left( 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i\alpha}) \right) = 1$. 
We can summarize these considerations writing the following equations for the messages

\[
\sigma_{i\alpha \rightarrow ja} = S'_{i\alpha \rightarrow i\beta} S'_{i\beta \rightarrow ia} \\
\times \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right],
\]

\[
S'_{i\alpha \rightarrow i\beta} = s_{i\alpha} \prod_{\gamma \in N(\alpha) \setminus \beta} S'_{\gamma \rightarrow ia} \\
\times \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] \quad \text{(B4)}
\]

that must be satisfied for every connected pair of networks \((\alpha, \beta)\). If we consider a connected network, the last equation for the messages \(S'_{i\alpha \rightarrow i\beta}\) has the form

\[
S'_{i\alpha \rightarrow i\beta} = \prod_{P_{\alpha\beta}(\gamma)} \prod_{\xi \in P_{\alpha\beta}(\gamma)} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\xi(i)} (1 - \sigma_{\ell\xi \rightarrow i\xi}) \right] \right\},
\]

\[
\text{(B5)}
\]

where \(P_{\alpha\beta}(\gamma)\) are all the directed paths that can be drawn from node \(\alpha\) from nodes different from node \(\beta\). Therefore the entire set of messages \(S'_{i\alpha \rightarrow i\beta}\) of the connected component \(C(\alpha)\) including node (layer) \(\alpha\) of a supernetwork, are all equal to one, \(S'_{i\alpha \rightarrow i\beta} = 1\), if and only if

\[
\prod_{\beta \in C(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell\beta \rightarrow i\beta}) \right] \right\} = 1. \quad \text{(B6)}
\]

Summarizing the results of this section we can say that the messages \(\sigma_{i\alpha \rightarrow ja}\) in within each layer is given by

\[
\sigma_{i\alpha \rightarrow ja} = \prod_{\beta \in C(\alpha) \setminus \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell\beta \rightarrow i\beta}) \right] \right\} \\
\times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right]. \quad \text{(B7)}
\]

Finally \(S_{i\alpha}\) indicates if a node \((i, \alpha)\) is in the mutually interdependent component \((S_{i\alpha} = 1, 0)\), and this indicator function can be expressed in terms of the messages as

\[
S_{i\alpha} = s_{i\alpha} \prod_{\beta \in C(\alpha) \setminus \alpha} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell\beta \rightarrow i\beta}) \right] \\
\times \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right]. \quad \text{(B8)}
\]

\[\text{b. Message passing equations (1) on a single loop supernetwork}\]

In this subsection we consider the message passing equations on a supernetwork formed by a loop and find similar results. We indicate the layers as \(\alpha = 1, 2, \ldots, M\), where each layer \(\alpha\) is linked to the layers \(\alpha + 1\) and \(\alpha - 1\). Here we identify layer \(M + 1\) with layer 1 and layer 0 with layer \(M\). The original message passing equations applied to this simple supernetwork topology are given by

\[
\sigma_{i\alpha \rightarrow ja} = s_{i\alpha} S'_{i\alpha \rightarrow j+1} S'_{i\alpha \rightarrow j-1} \\
\times \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right],
\]

\[
S'_{i\alpha \rightarrow i\alpha \pm 1} = s_{i\alpha} S'_{i\alpha \rightarrow i\alpha \pm 1} \\
\times \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right]. \quad \text{(B9)}
\]

Solving recursively the equations for \(S'_{i\alpha \rightarrow i\alpha \pm 1}\) we get

\[
S'_{i\alpha \rightarrow i\alpha \pm 1} = S'_{i\alpha \rightarrow i\alpha \pm 1} \prod_{\alpha} \left\{ s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] \right\}
\]

which yields the solution \(S' = S'_{i\alpha \rightarrow i\alpha \pm 1} = 1\) only if

\[
\prod_{\beta = 1, M} s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell\beta \rightarrow i\beta}) \right] = 1, \quad \text{(B10)}
\]

i.e., only if all the “replica” nodes \((i, \alpha)\) of node \(i\) are in the mutually connected component. Therefore Eqs. \(\text{(B9)}\) reduce to

\[
\sigma_{i\alpha \rightarrow ja} = \prod_{\beta = 1, M} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell\beta \rightarrow i\beta}) \right] \right\} \\
\times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right]. \quad \text{(B11)}
\]

Now we notice that

\[
\left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] = \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] \quad \text{(B12)}
\]

because if node \((i, \alpha)\) has at least one neighbor \((\ell, \alpha) \neq (j, \alpha)\) in the mutually connected component, then we have automatically

\[
\left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell\alpha \rightarrow ia}) \right] = 1. \quad \text{(B13)}
\]
Therefore the equations for the loop supernetwork are given by

\[ \sigma_{\alpha \rightarrow j \beta} = \prod_{\beta \neq \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \right\} \times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i \alpha}) \right]. \]  

(B14)

Finally, \( S_{i\alpha} \) indicates if a node \((i, \alpha)\) is in the mutually connected component \((S_{i\alpha} = 1, 0)\) and can be expressed in terms of the messages as

\[ S_{i\alpha} = s_{i\alpha} \prod_{\beta \neq \alpha} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \times \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i \alpha}) \right]. \]  

(B15)

Appendix C: Derivation of Eqs. (8) and (9) for a network of networks with replica nodes and connected supernetwork.

Given a node (layer) \( \alpha \) of the supernetwork, the set of layers connected to the layer \( \alpha \) at least by two non-overlapping paths, is called the loopy component of layer \( \alpha \) in the supernetwork. We call each connected component formed by the layers connected in the supernetwork to layer \( \alpha \), but not belonging to the loopy components, the dangling components of layer \( \alpha \). The dangling components might be trees or might contain loops (see Fig. 2). In Sec. III we have shown that for a tree supernetwork topology the messages \( S'_{i\alpha \rightarrow j \beta} \) are given by Eq. (7) in the main text, i.e.,

\[ S'_{i\alpha \rightarrow j \beta} = \prod_{\xi \in T_{\alpha \beta}} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i \xi}) \right] \right\}, \]  

(C1)

where \( T_{\alpha \beta} \) is the subtree in the supernetwork that has the root given by the layer \( \alpha \) and branching departing from every link of layer \( \alpha \) in the supernetwork except the link connected to layer \( \beta \). For a supernetwork topology formed by a single loopy component the messages, instead the messages \( S'_{i\alpha \rightarrow j \beta} \) have been found to follow Eq. (12), i.e.

\[ S'_{i\alpha \rightarrow j \beta} = \prod_{\xi \in C_{\alpha \beta}} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\alpha(i)} (1 - \sigma_{\ell \alpha \rightarrow i \xi}) \right] \right\}. \]  

(C2)

These two expressions [Eqs. (C1) and (C2)] are both equivalent to the following one

\[ S'_{i\alpha \rightarrow j \beta} = \prod_{\xi \in C_{\alpha \beta}} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in N_\xi(i)} (1 - \sigma_{\ell \xi \rightarrow i \xi}) \right] \right\}, \]  

(C3)

where \( C_{\alpha \beta} \) is the connected component of the supernetwork that can be reached from layer \( \alpha \) departing from links pointing to layers different from \( \beta \). In fact in the case of a tree supernetwork \( C_{\alpha \beta} = T_{\alpha \beta} \) and in the case of a loopy component \( C_{\alpha \beta} = C(\alpha) \). From these expressions it follows that for the tree supernetwork and for the supernetwork formed by a single loopy component, the messages \( \sigma_{i\alpha \rightarrow j \beta} \) and the indicator function \( S_{i\alpha} \) are given by

\[ \sigma_{i\alpha \rightarrow j \beta} = \prod_{\beta \in C(\alpha) \setminus \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \right\} \times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i \alpha}) \right], \]  

(C4)

\[ S_{i\alpha} = \prod_{\beta \in C(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \right\}. \]  

(C5)

Here we show that these equations are valid for every supernetwork topology. Let us consider a completely general supernetwork and a layer \( \alpha \) in the supernetwork. The supernetwork can be decomposed into the loopy component of layer \( \alpha \) and the dangling components of layer \( \alpha \). Given a generic layer \( \gamma \) belonging to one of such dangling components we can further decompose this dangling component into the loopy component of layer \( \gamma \) and the dangling components of layer \( \gamma \). Continuing this iteration we will arrive at the end of the iteration in which we will have dangling components with a tree topology. Given this possible decomposition of the supernetwork, here we prove the validity of Eqs. (C3), (C4) and Eqs. (C5) using an iterative argument. In particular we want to prove that if we assume that the messages coming from every dangling components of layer \( \alpha \) follow Eqs. (C3), then the messages in the loopy component of layer \( \alpha \) also follow Eqs. (C3) and therefore Eqs. (C4) and (C5). Starting from the message passing Eqs. (1) valid for \( S'_{i\alpha \rightarrow j \beta} \), we can follow the messages coming to node \((i, \alpha)\) from other layers different from layer \( \beta \) backwards. The paths starting from the layer \( \alpha \) and departing from the links of the supernetwork connecting layers different from layer \( \beta \), can be distinguished between paths belonging only to the loopy component of layer \( \alpha \), that are reaching again layer \( \alpha \), closing a loop, or paths that reach the dangling components of layer \( \alpha \). Therefore, if we indicate with \( M_\alpha \) the neighbors of layer \( \alpha \) in the loopy component, by \( D_\alpha \) the layers belonging to the dangling components of layer \( \alpha \) linked to the loopy component of layer \( \alpha \), and finally if we denote by \( L_\alpha \) the loopy component of layer \( \alpha \) we obtain the following expression for the messages

\[ \sigma_{i\alpha \rightarrow j \beta} = \prod_{\beta \in C(\alpha) \setminus \alpha} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \right\} \times s_{i\alpha} \left[ 1 - \prod_{\ell \in N_\alpha(i) \setminus j} (1 - \sigma_{\ell \alpha \rightarrow i \alpha}) \right]. \]  

(C4)

\[ S_{i\alpha} = \prod_{\beta \in C(\alpha)} \left\{ s_{i\beta} \left[ 1 - \prod_{\ell \in N_\beta(i)} (1 - \sigma_{\ell \beta \rightarrow i \beta}) \right] \right\}. \]  

(C5)
\( S'_{\alpha \rightarrow \beta} \) departing from layer \( \alpha \),

\[
S'_{\alpha \rightarrow \beta} = \prod_{\mathcal{P}_{\alpha \beta}(\alpha)} \prod_{\gamma \in \mathcal{P}_{\alpha \beta}(\alpha)} S_{\alpha \rightarrow i\gamma} \prod_{\psi \in \mathcal{D}(\alpha) \setminus \beta} S'_{i\psi \rightarrow i\phi}
\]

\[
\times \prod_{\xi \in \mathcal{L}_\alpha} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in \mathcal{N}_{i}(i)} (1 - \sigma_{i\xi \rightarrow i\ell}) \right] \right\}, \tag{C6}
\]

where \( \gamma \in \mathcal{M}_\alpha \) and \( \mathcal{P}_{\alpha \beta}(\alpha) \) are all the directed paths of the loopy component of the supernetwork that can be drawn from node \( \alpha \) and that starting from the interlink \((\alpha, \gamma)\) arrive at layer \( \alpha \) from layers different from \( \beta \). The expression \( \text{(C6)} \) is valid either if \( \beta \in \mathcal{M}(\alpha) \) or if \( \beta \in \mathcal{D}_\alpha \).

Now, assuming that all the messages \( S'_{i\psi \rightarrow i\phi} \) coming from the dangling components satisfy Eq. \( \text{(C3)} \), we find that all the messages \( S'_{\alpha \rightarrow i\beta} \) have the same value, and that these messages are given by 1 only if

\[
1 = \prod_{\xi \in \mathcal{L}_\alpha} \left\{ s_{i\xi} \left[ 1 - \prod_{\ell \in \mathcal{N}_{i}(i)} (1 - \sigma_{i\xi \rightarrow i\ell}) \right] \right\}, \tag{C7}
\]

where \( \mathcal{C}_{\alpha \beta} \) is the connected component of the supernetwork that can be reached from layer \( \alpha \) departing from links pointing to layers different from \( \beta \). This implies that Eq. \( \text{(C3)} \) remains valid in this case. This result concludes our proof by iteration that Eq. \( \text{(C3)} \) is valid from every topology of the supernetwork. Finally using Eq. \( \text{(A2)} \) it can be shown that this result implies that also Eqs. \( \text{(C4)} \) and \( \text{(C5)} \) identical to Eqs. \( \text{(8)} \) and \( \text{(9)} \) are valid for every connected topology of the supernetwork.