Derivative of symmetric square \( p \)-adic \( L \)-functions via pull-back formula

Giovanni Rosso *

January 8, 2014

In this paper we recall the method of Greenberg and Stevens to calculate derivatives of \( p \)-adic \( L \)-functions using deformations of Galois representation and we apply it to the symmetric square of a modular form Steinberg at \( p \). Under certain hypotheses on the conductor and the Nebentypus, this prove a conjecture of Greenberg and Benois on trivial zeros.

Contents

1 Introduction 1
2 The method of Greenberg and Stevens 3
3 Eisenstein measures 7
  3.1 Siegel modular forms 7
  3.2 Eisenstein series 10
  3.3 Families for \( \text{GL}_2 \times \text{GL}_2 \) 12
4 \( p \)-adic \( L \)-functions 16

1 Introduction

Let \( M \) be a motive over \( \mathbb{Q} \) and suppose that it is pure of weight 0 and irreducible. We suppose also that \( s = 0 \) is a critical integer à la Deligne.

Fix a prime number \( p \) and let \( V \) be the \( p \)-adic representation associated to \( M \). We fix once for all an isomorphism \( \mathbb{C} \cong \mathbb{C}_p \). If \( V \) is semistable, it is conjectured that for each regular submodule \( D \) \[\text{Ben11, §0.2}\] there exists a \( p \)-adic \( L \)-function \( L_p(V, D, s) \). It is supposed to interpolate the special values of the \( L \)-function of \( M \) twisted by finite-order character of \( 1 + p\mathbb{Z}_p \) \[\text{PR95}\]. In particular, we expect the following interpolation formula at \( s = 0 \):

\[
L_p(V, D, s) = E(V, D) \frac{L(V, 0)}{\Omega(V)},
\]

*PhD Fellowship of Fund for Scientific Research - Flanders, partially supported by a JUMO grant from KU Leuven (Jumo/12/032), a ANR grant (ANR-10-BLANC 0114 ArShiFo) and a NSF grant (FRG DMS 0854964).
for $\Omega(V)$ a complex period and $E(V, D)$ some Euler type factors which conjecturally have to be removed in order to allow $p$-adic interpolation (see [Ben11 §2.3.2] for the case when $V$ is crystalline). It may happen that certain of these Euler factors vanish. In this case the connection with what we were interested in, the special values of the $L$-function, is lost. Motivated by the seminal work of Mazur-Tate-Teitelbaum [MTT86], Greenberg, in the ordinary case [Gre94], and Benois [Ben11] have conjectured the following;

**Conjecture 1.1.** [Trivial zeros conjecture] Let $e$ be the number of Euler-type factors of $E_p(V, D)$ which vanish. Then the order of zeros at $s = 0$ of $L_p(V, D, s)$ is $e$ and

$$
\lim_{s \to 0} \frac{L_p(V, D, s)}{s^e} = \mathcal{L}(V, D) E^*(V, D) \frac{L(V, 0)}{\Omega(V)}
$$

(1.2)

for $E^*(V, D)$ the non-vanishing factors of $E(V, D)$ and $\mathcal{L}(V, D)$ a non-zero number called the $\mathcal{L}$-invariant.

There are many different ways in which the $\mathcal{L}$-invariant can be defined; the first definition one could think is the one of an analytic $\mathcal{L}$-invariant

$$
\mathcal{L}^{an}(V, D) = \lim_{s \to 0} \frac{L_p(V, D, s)}{s^e} \frac{L(V, 0)}{\Omega(V)}.
$$

Clearly, with this definition, the above conjecture reduces to the statement on the order of $L_p(V, D, s)$ at $s = 0$ and the non-vanishing of the $\mathcal{L}$-invariant.

In [MTT86] the authors give a more arithmetic definition of the $\mathcal{L}$-invariant for an elliptic curve, in term of an extended regulator on the extended Mordell-Weil group. The search for an intrinsic, Galois theoretic interpretation of this error factor led to the definition of the arithmetic $\mathcal{L}$-invariant $\mathcal{L}^{ar}(V, D)$ given by Greenberg [Gre94] (resp. Benois [Ben11]) in the ordinary case (resp. semistable case) using Galois cohomology (resp. cohomology of $(\varphi, \Gamma)$-module).

For 2-dimensional Galois representations many more definitions have been proposed and we refer to [Col05] for a detailed exposition. When the $p$-adic $L$-function can be constructed using an Iwasawa cohomology class and the big exponential [PR95], one can use the machinery developed in [Ben12 §2.2] to prove formula (1.2) with $\mathcal{L} = \mathcal{L}^{an}$. Unluckily, it is a very hard problem to construct classes in cohomology which are related to special values. Kato’s Euler system has been used in this way in [Ben12] to prove many instances of Conjecture 1.1 for modular forms. It possible that the construction of Lei-Loeffler-Zerbes [LLZ12] of Euler system for Rankin product could produce such an Iwasawa classes for other Galois representation; in particular, for $V = \text{Sym}^2(V_f)(1)$, where $V_f$ is the Galois representation associated to a weight 2 modular form (see also [PR98]).

We would like to present in this paper a different method which has already been extensively used in many cases and which we think to be more at hand at the current state; the method of Greenberg and Stevens [Gre94]. Under certain hypotheses which we shall state in the next section, it allows us to calculate the derivative of $L_p(V, D, s)$. The main ingredient of their method is the fact that $V$ can be $p$-adically deformed in a one dimensional family; as an example, modular forms can be deformed in a Hida-Coleman family. We have decided to present this method because the
recent developments on families of automorphic forms \cite{alp12, bra13} have open the door for the construction of families of $p$-adic $L$-function in many different settings. For example, we refer to the ongoing PhD thesis of L. Zheng on $p$-adic $L$-functions for Siegel modular forms. Consequently, we expect that one could prove many new instances of Conjecture \ref{conjecture}.

In Section \ref{section:examples} we shall apply this method to the case of the symmetric square of a modular form which is Steinberg at $p$. The theorem which will be proved is the following;

**Theorem 1.3.** Let $f$ be a modular form of trivial Nebentypus, weight $k_0$ and conductor $Np$, $N$ squarefree and prime to $p$. Then Conjecture \ref{conjecture} (up to the non-vanishing of the $L$-invariant) is true for $L_p(s, \text{Sym}^2(f))$.

In this case there is only one choice for the regular submodule $D$, and the trivial zero appears at $s = k_0 - 1$. This theorem generalizes \cite[Theorem 1.3]{ros13b} in two ways: we allow $p = 2$ and, when $p \neq 2$, we do not require $N$ to be even. In particular, we cover the case of the elliptic curve $X_0(11)$ (for $p = 11$).

In the ordinary setting, the same theorem (but with the hypothesis at 2) has been proved by Greenberg and Tilouine (unpublished). The importance of this formula for the proof of the Greenberg-Iwasawa Main Conjecture \cite{urb06} has been put in evidence in \cite{htu97}.

The improvement from \cite{ros13b} is achieved using a different construction of the $p$-adic $L$-function, namely the one of \cite{bs00}. We express the complex $L$-function using Eisenstein series for $GSp_4$, a pullback formula to the Igusa divisor and a double Petersson product. We are grateful to É. Urban for suggesting us this approach. In Section \ref{section:overview} we recall briefly the theory of Siegel modular forms and develop a theory of $p$-adic modular forms for $GL_2 \times GL_2$ necessary for the construction of $p$-adic families of Eisenstein series.

This paper is part of the author PhD thesis and the author is very grateful to his director J. Tilouine for the constant guidance and the attention given. We would like to thank É. Urban for inviting the author at Columbia University (where this work has seen the day) and for his generosity with which he shared with us his ideas and insights. We would like thank R. Casalis for interesting discussions on Kähler differentials.

\section{The method of Greenberg and Stevens}

The aim of this section is to recall the method of Greenberg and Stevens \cite{gs93} to calculate analytic $L$-invariant. This method has been used successfully many other times \cite{mok09, ros13a, ros13b}.

It is very robust and easily adaptable to many situations in which the expected order of the trivial zero is one. We shall describe also some obstacles which occur while trying to apply this method to higher order zeros, and we shall discuss in detail derivation along the “weight variable”.

We let $K$ be a $p$-adic local field, $\mathcal{O}$ its valuation ring and $\Lambda$ the Iwasawa algebra $\mathcal{O}[[T]]$. Let $V$ be a $p$-adic Galois representation as before.

We suppose that we can construct a $p$-adic $L$-function for $L_p(s, V, D)$ and that it presents a single trivial zero.

We suppose also that $V$ can be deformed in a $p$-adic family $V(\kappa)$. Let us be more precise; we denote by $\mathcal{W}$ the rigid analytic space whose $\mathbb{C}_p$-points are $\text{Hom}_{\text{cont}}(\mathbb{Z}_p, \mathbb{C}_p)$. We have a map $\mathbb{Z} \to \mathcal{W}$ defined by $k \mapsto [k] : z \mapsto z^k$. Let us fix once for all, if $p \neq 2$ (resp. $p = 2$) a generator $u$ of $1 + p\mathbb{Z}_p$ (resp.
1 + 4\mathbb{Z}_2) and a decomposition \( \mathbb{Z}_p^* = \mu_1 \times 1 + p\mathbb{Z}_p \) (resp \( \mathbb{Z}_2^* = \mu_2 \times 1 + 4\mathbb{Z}_2 \)). Here \( \mu = \mathbb{GL}_m^\text{tors}(\mathbb{Z}_p) \). As rigid space we have the following isomorphism

\[
\mathcal{W} \cong \mathbb{GL}_m^\text{tors}(\mathbb{Z}_p) \wedge B(1, 1^-) \quad \kappa \mapsto (\kappa_\mu, \kappa(\mu)),
\]

where the first set has the discrete topology and the second is the rigid open unit ball around 1. Let \( 0 < r < \infty, r \in \mathbb{R} \), we define

\[
\mathcal{W}(r) = \{ (\zeta, z) | \zeta \in \mathbb{GL}_m^\text{tors}(\mathbb{Z}_p), |z - 1| \leq p^{-r} \}.
\]

We fix an integer \( h \) and we denote by \( \mathcal{H}_h \) the algebra of \( h \)-admissible distributions over \( 1 + p\mathbb{Z}_p \) (or \( 1 + 4\mathbb{Z}_2 \) if \( p = 2 \)) with values in \( K \). Here we take the definition of admissibility as in [Pan03, §3], so measures are 1-admissible, i.e. \( \Lambda \otimes \mathbb{Q}_p \cong \mathbb{H}_1 \). The Mellin transform gives us a map

\[
\mathcal{H}_h \to \text{An}(0),
\]

where \( \text{An}(0) \) stands for the algebra of \( \mathbb{Q}_p \)-analytic, locally convergent functions around 0. If we see \( \mathcal{H}_h \) as a subalgebra of the ring of formal series, this amounts to \( T \mapsto u^s - 1 \).

We suppose that we are given an affinoid \( \mathcal{U} \), finite over \( \mathcal{W}(r) \). Let us write \( \pi : \mathcal{U} \to \mathcal{W}(r) \). Let us denote by \( \mathfrak{I} \) the Tate algebra corresponding to \( \mathcal{U} \); we shall suppose that \( \mathfrak{I} \) is integrally closed. We suppose that we have a big Galois representation \( V(\kappa) \) with values in \( \mathfrak{I} \) and a point \( \kappa_0 \in \mathcal{U} \) such that \( V = V(\kappa_0) \). We define \( \mathcal{U}^{cl} \) to be the set of \( \kappa \in \mathcal{U} \) satisfying the followings conditions:

- \( \pi(\kappa) = [k] \), with \( k \in \mathbb{Z} \),
- \( V(\kappa) \) is motivic,
- \( V(\kappa) \) is semistable as \( G_{\mathbb{Q}_p} \)-representation,
- \( s = 0 \) is a critical integer for \( V(\kappa) \).

We make also the following assumption on \( \mathcal{U}^{cl} \):

(CI) for every \( n > 0 \), there are infinitely many \( \kappa \in \mathcal{U}^{cl} \) and \( s \in \mathbb{Z} \) such that:

- \( |\kappa - \kappa_0|_\mathcal{U} < p^{-n} \)
- \( s \) critical for \( V(\kappa) \),
- \( s \equiv 0 \mod p^n \).

This amounts to ask that the couples \((\kappa, [s])\) in \( \mathcal{U} \times \mathcal{W} \) with \( s \) critical for \( V(\kappa) \) accumulate at \((\kappa_0, 0)\).

We suppose that there is a global triangulation \( D(\kappa) \) of the \( (\varphi, \Gamma) \)-module associated to \( V(\kappa) \) and that this induces the regular submodule used to construct \( L_p(s, V, D) \). Under these hypotheses, it is natural to conjecture the existence of a two variables \( p \)-adic \( L \)-function (depending on \( D(\kappa) \)) \( L_p(\kappa, s) \in \mathfrak{I} \otimes \mathcal{H}_h \) interpolating the \( p \)-adic \( L \)-functions of \( V(\kappa) \), for all \( \kappa \) in \( \mathcal{U}^{cl} \). Conjecturally [Pan94, Pot13], \( h \) should be defined solely in term of the \( p \)-adic Hodge theory of \( V(\kappa) \) (and \( D(\kappa) \)).

The two key hypotheses that we make on this \( p \)-adic \( L \)-function are:

4
i) there exists a subspace of dimension one \((\kappa, s(\kappa))\) containing \((\kappa_0, 0)\) over which \(L_p(\kappa, s(\kappa))\) vanishes identically;

ii) there exists an improved \(p\)-adic \(L\)-function \(L^*_p(\kappa)\) in \(\mathcal{I}\) such that \(L_p(\kappa, 0) = E(\kappa)L^*_p(\kappa)\), for \(E(\kappa)\) a non-zero element which vanishes at \(\kappa_0\).

The idea is that \(i)\) allows us to express the derivative we are interested in in term of the “derivative with respect to \(\kappa\)”. The latter can be calculated using \(ii)\). In general, we expect that \(s(\kappa)\) is a simple function of \(\pi(\kappa)\). We let 
\[
\log_p(z) = -\sum_{n=1}^{\infty} \frac{(1-z)^n}{n}, \text{ for } |z - 1|_p < 1
\]

\[
\text{Log}_p(\kappa) = \frac{\log_p(\kappa(u'))}{\log_p(u')}
\]

for \(r\) any integer big enough.

For example, in [Gre94, Mok09] we have \(s(\kappa) = \frac{1}{2}\log_p(\pi(\kappa))\) and in [Ros13a, Ros13b] we have \(s(\kappa) = \log_p(\pi(\kappa)) - 1\). In the first case the line corresponds to the vanishing on the central critical line, which is a consequence of the constancy of the \(\varepsilon\)-factor in the family. In the second case, the vanishing is due to a line of trivial zeros, as all the motivic specializations present a trivial zero.

The idea behind \(ii)\) is the Euler factor which brings the trivial zero for \(V\) varies analytically along \(V(\kappa)\) once one fixes the cyclotomic variable. This is often the case with one dimensional deformations. If we allow deformations of \(V\) in more than one variable, it is unlikely that the removed Euler factors define \(p\)-analytic functions, due to the fact that eigenvalues of the crystalline Frobenius do not vary \(p\)-adically or equivalently, that the Hodge-Tate weights are not constant.

We now give the example of families of Hilbert modular forms. For simplicity of notation, we consider a totally real field \(F\) of degree \(d\) where \(p\) is split. Let \(\mathbf{f}\) be a Hilbert modular form of weight \((k_1, \ldots, k_d)\) such that the parity of \(k_i\) does not depend on \(i\). We define \(m = \max(k_i - 1)\) and \(v_i = \frac{m+1-k_i}{2}\). We suppose that \(\mathbf{f}\) is nearly-ordinary [Hid89, Ros13a, Ros13b] we have \(s(\kappa) = \log_p(\pi(\kappa)) - 1\). In the first case the line corresponds to the vanishing on the central critical line, which is a consequence of the constancy of the \(\varepsilon\)-factor in the family. In the second case, the vanishing is due to a line of trivial zeros, as all the motivic specializations present a trivial zero.

The idea behind \(ii)\) is the Euler factor which brings the trivial zero for \(V\) varies analytically along \(V(\kappa)\) once one fixes the cyclotomic variable. This is often the case with one dimensional deformations. If we allow deformations of \(V\) in more than one variable, it is unlikely that the removed Euler factors define \(p\)-analytic functions, due to the fact that eigenvalues of the crystalline Frobenius do not vary \(p\)-adically or equivalently, that the Hodge-Tate weights are not constant.

It may happen that the Euler factor which brings the trivial zero for \(V\) is (locally) identically zero on the whole family; this is the case for the symmetric square of a modular form of prime-to-\(p\) conductor, and more generally, for the standard \(L\)-function of parallel weight Siegel modular forms of prime-to-\(p\) level. That’s why in [Mok09, Ros13a] the authors deal only with forms of parallel weight. It seems quite hard to generalize the method of Greenberg and Stevens to higher order derivatives without new ideas.

We have seen in the examples above that \(s(\kappa)\) is a linear function of the weight. Consequently, one needs to evaluate the \(p\)-adic \(L\)-function \(L_p(V(\kappa), s)\) at \(s\) which are big for the archimedean norm. If \(s\) is not critical, it is quite an hard problem to evaluate the \(p\)-adic \(L\)-function. That is why we have supposed (CI). It is not an hypothesis to be taken for granted; one example is the spinor \(L\)-function for genus 2 Siegel modular forms of any weight, which has only one critical integer.
The improved \( p \)-adic \( L \)-function is said so because \( L_p'(\kappa_0) \) is supposed to be the special values we are interested in.

The rest of the section will be devoted to make precise the expression “derive with respect to \( \kappa \).”

Under the above hypotheses, we can define then a new analytic \( d \)-invariant (which a priori depends on the deformation \( V(\kappa) \) and \( d \)) by

\[
\mathcal{L}_d^{an}(V) := -\log_p(u) \cdot d(s(\kappa))^{-1} d(E(\kappa)) \bigg|_{\kappa=\kappa_0} = -d(s(\kappa))^{-1} \frac{d(L_p(0,\kappa))}{L_p(\kappa)} \bigg|_{\kappa=\kappa_0}.
\]

We remark that with the notation above we have \( \log_p(u) \frac{d}{ds} = \frac{d}{ds} \). We apply \( d \) to \( L_p(\kappa, s(\kappa)) = 0 \) to obtain

\[
0 = d(L_p(\kappa, s(\kappa))) = \log_p(u) \cdot \frac{d}{ds} L_p(\kappa, s) \bigg|_{s=s(\kappa)} d(s(\kappa)) + d(L_p(\kappa, s)) \bigg|_{s=s(\kappa)}.
\]
Evaluating at $\kappa = \kappa_0$ we deduce
\[
\frac{d}{ds} L_p(V, s) \bigg|_{s=0} = L^an_d(V) L^\kappa_p(\kappa_0)
\]
and consequently
\[
L^an(V) = L^an_d(V).
\]
In the cases in which $L^\mathrm{al}$ has been calculated, namely symmetric powers of Hilbert modular forms, it is expressed in term of the logarithmic derivative of Hecke eigenvalues at $p$ of certain finite slope families [Ben10, HJ13, Hid06, Mok12]. Consequently, the above formula should allows us to prove the equality $L^\mathrm{an} = L^\mathrm{al}$.
Moreover, the fact that the $L$-invariant is a derivative of a non constant function shows that $L^\mathrm{an} \neq 0$ except in finitely many cases. In this direction, positive results for a given $V$ have been obtained only in the case of an elliptic curve with $p$-adic uniformization, using a deep theorem of transcendent number theory [BSDGP96].

3 Eisenstein measures

In this section we first fix the notation concerning genus 2 Siegel forms. We recall then a normalization of certain Eisenstein series for $GL_2 \times GL_2$ and develop a theory of $p$-adic families of modular forms (of parallel weight) on $GL_2 \times GL_2$. Finally we construct two Eisenstein measures which will be used in the next section to construct two $p$-adic $L$-functions.

3.1 Siegel modular forms

We follow closely the notation of [BS00] and we refer to the first section of loc. cit. for more details. Let us denote by $\mathbb{H}_1$ the complex upper half-plane and by $\mathbb{H}_2$ the Siegel space for $\text{GSp}_4$. We have explicitly
\[
\mathbb{H}_2 = \left\{ Z = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right| Z^t = Z \text{ and } \text{Im}(Z) > 0 \right\}.
\]
It has a natural action of $\text{GSp}_4(\mathbb{R})$ via fractional linear transformation; for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\text{GSp}_4^+(\mathbb{R})$ and $Z$ in $\mathbb{H}_2$ we define
\[
M(Z) = (AZ + B)(CZ + D)^{-1}.
\]
Let $\Gamma = \Gamma_0^{(2)}(N)$ be the congruence subgroup of $\text{Sp}_4(\mathbb{Z})$ of matrices whose lower block $C$ is congruent to 0 modulo $N$. We consider the space $M_k^{(2)}(N, \phi)$ of scalar Siegel forms of weight $k$ and Nebentypus $\phi$:
\[
\left\{ F : \mathbb{H}_2 \to \mathbb{C} \left| F(M(Z))(CZ + D)^{-k} = \phi(M)F(Z) \quad \forall M \in \Gamma, f \text{ holomorphic} \right\}.
\]
Each \( F \) in \( M_k(\Gamma, \phi) \) admits a Fourier expansion

\[
F(Z) = \sum_T a(T)e^{2\pi i \text{tr}(TZ)},
\]

where \( T = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_4 \end{pmatrix} \) ranges over all matrices \( T \) positive and semi-defined, with \( T_1, T_4 \) integer and \( T_2 \) half-integer.

We have two embeddings (of algebraic groups) of \( \text{SL}_2 \) in \( \text{Sp}_4 \):

\[
\text{SL}_2^\uparrow(R) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(R) \right\},
\]

\[
\text{SL}_2^\downarrow(R) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(R) \right\}.
\]

We can embed \( \mathbb{H}_1 \times \mathbb{H}_1 \) in \( \mathbb{H}_2 \) in the following way

\((z_1, z_4) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} \).

If \( \gamma \) belongs to \( \text{SL}_2(\mathbb{R}) \), we have

\[
\gamma^\uparrow \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} \gamma(z_1) & 0 \\ 0 & z_4 \end{pmatrix},
\]

\[
\gamma^\downarrow \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & \gamma(z_4) \end{pmatrix}.
\]

Consequently, evaluation at \( z_2 = 0 \) gives us a map

\[
M_k^{(2)}(N, \phi) \hookrightarrow M_k(N, \phi) \otimes \mathbb{C} M_k(N, \phi),
\]

where \( M_k(N, \phi) \) denotes the space of elliptic modular forms of weight \( k \), level \( N \) and Nebentypus \( \phi \).

This also induces a closed embedding of two copies of the modular curve in the Siegel threefold. We shall call its image the Igusa divisor. On points, it corresponds to abelian surfaces which decompose as product of two elliptic curves.

We consider the following differential operators on \( \mathbb{H}_2 \):

\[
\partial_1 = \frac{\partial}{\partial z_1}, \quad \partial_2 = \frac{1}{2} \frac{\partial}{\partial z_2}, \quad \partial_4 = \frac{\partial}{\partial z_4}.
\]
We define
\[
D_l = z_2 \left( \partial_1 \partial_2 - \partial_2^2 \right) - \left( l - \frac{1}{2} \right) \partial_2,
\]
\[
D^*_l = D_{l+s-1} \circ \ldots \circ D_l,
\]
\[
D^*_l = D^*_l|_{z_2=0}.
\]

The importance of \( \hat{D}^*_l \) is that it preserves holomorphicity.

Let \( I = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_4 \end{pmatrix} \); we define \( b^*_l(I) \) to be the only homogeneous polynomial in the indeterminates \( T_1, T_2, T_4 \) of degree \( s \) such that
\[
\hat{D}^*_l e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4} = b^*_l(I) e^{T_1 z_1 + T_4 z_4}.
\]

We need to know a little bit more about the polynomial \( b^*_l(I) \). Let us write
\[
\hat{D}^*_l e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4} = P^*_l(z_2, I) e^{T_1 z_1 + T_4 z_4},
\]
we have
\[
\hat{D}^*_l e^{T_1 z_1 + 2T_2 z_2 + T_4 z_4} = \begin{pmatrix} P^*_l(z_2, I) \end{pmatrix} e^{\text{tr}(ZI)}. \tag{3.1}
\]

We obtain easily
\[
P^*_l(z_2, I) = \begin{pmatrix} P_1^l(z_2, I) \\ \text{det}(I) \end{pmatrix} - \left( l - \frac{1}{2} \right) T_2,
\]
\[
b^*_l(I) = P^*_l(0, I). \tag{3.2}
\]

We can write [BS00], page 1381, that
\[
\hat{D}^*_l = \sum_{J=\{j_1, j_2, j_4\} \mid j_1 + j_2 + j_4 = s} c^s_{j_1} \partial^{j_1}_{1} \partial^{j_2}_{2} \partial^{j_4}_{4}.
\]

We have easily
\[
\partial^{j_1}_{1} \partial^{j_2}_{2} \partial^{j_4}_{4} (z_1^{j_1} z_2^{j_2} z_4^{j_4}) = j_1! j_2! j_4! 2^{-j_2},
\]
\[
\partial^{j_1}_{1} \partial^{j_2}_{2} \partial^{j_4}_{4}(e^{\text{tr}(ZI)}) = T_1^{j_1} T_2^{j_2} T_4^{j_4}(e^{\text{tr}(ZI)}).
\]

We pose
\[
c^s_j := c^s_{j,0,0},
\]
\[
(-1)^s c^s_j = \prod_{i=1}^{s} \left( l - 1 + s - \frac{i}{2} \right) = 2^{-s} \frac{(2l - 2 + 2s - 1)!}{(2l - 2 + s - 1)!}.
\]

When \( L \mid T_1, T_4 \) we obtain \( 2s \hat{D}^*_l \equiv 2s c^s_j \partial^s_2 \mod L \).
3.2 Eisenstein series

The aim of this section is to recall certain Eisenstein series which can be used to construct the $p$-adic $L$-functions, as in [BS00]. In loc. cit. the authors consider certain Eisenstein series for $\text{GSp}_{4g}$ whose pullback to the Igusa divisor is a holomorphic Siegel modular form. Fix now a (parallel weight) Siegel modular form $f$ for $\text{GSp}_{2g}$. We can write the standard $L$-function of $f$ as a double Petersson product between $f$ and these Eisenstein series (see Proposition 3.4). When $g = 1$, the standard $L$-function of $f$ coincides, up to a twist, with the symmetric square $L$-function of $f$ we are interested in.

The main advantage of this method is that the Eisenstein series we need are holomorphic on the Igusa divisor, so no theory of nearly overconvergent Siegel modular form (nor of overconvergent projector) is needed to construct the $p$-adic $L$-function.

In general, for an algebraic group bigger than $\text{GL}_2$ it is quite hard to find the normalization of Eisenstein series which maximizes, in a suitable sense, the $p$-adic behavior of its Fourier coefficients. In [BS00] §2 the authors develop a twisting method which allow them to define Eisenstein series whose Fourier coefficients satisfy Kummer’s congruences when the character associated with the Eisenstein series varies $p$-adically. This is the key for their construction of the one variable (cyclotomic) $p$-adic $L$-function and of our two variables $p$-adic $L$-function.

When the character is trivial modulo $p$, we dispose of a simple relation between the twisted and the not-twisted Eisenstein series [BS00, §6 Appendix]. To construct the improved $p$-adic $L$-function, we shall simply interpolate the not-twisted Eisenstein series. Let us now recall these Fourier developments.

We fix a weight $k$, an integer $N$ prime to $p$ and a Nebentypus $\phi$. Let $f$ be an eigenform in $M_k(Np, \phi)$, of finite slope for the Hecke operator $U_p$. We write $N = N_{ss}N_{nss}$, where $N_{ss}$ (resp. $N_{nss}$) is divisible by all primes $q \mid N$ such that $U_q f = 0$ (resp. $U_q f \neq 0$). We fix an integer coprime with $N$ and $p$ and $N_1$ a positive integer such that $N_{ss} \mid N_1 \mid N$. We fix a Dirichlet character $\chi$ modulo $N_1Rp$ which we write as $\chi \chi' \varepsilon_1$, with $\chi_1$ defined modulo $N_1$, $\chi'$ primitive modulo $R$ and $\varepsilon_1$ defined modulo $p$. We shall explain after Proposition 3.4 why we introduce $\chi_1$.

Let $t \geq 1$ be an integer and $\mathbb{F}^{t+1}(w, z, R^2N^2p^{2n}, \phi, u)^{(\chi)}$ be the twisted Eisenstein series of [BS00 (5.3)]. We define

$$H_{L, \chi}(w, z) := L(t + 2s, \phi \chi) \mathcal{D}_{t+1}^s \left( \mathbb{F}^{t+1}(w, z, R^2N^2p^{2n}, \phi, u)^{(\chi)} \right)^2 U_L, U_L$$

for $s$ a non-negative integer and $p^n \mid L$, with $L$ a $p$-power. It is a form for $\Gamma_0(N^2R^2p) \times \Gamma_0(N^2R^2p)$ of weight $t + 1 + s$.

We shall sometimes chose $L = 1$ and in this case the level is $N^2R^2p^{2n}$.

For any prime number $q$ and matrices $I$ as in the previous section, let $B_q(X, I)$ be the polynomial of degree at most 1 of [BS00 Proposition 5.1]. We pose

$$B(t) = (-1)^{t+1} \frac{2^{1+2t}}{\Gamma(3/2)} \pi^{\frac{5}{2}}.$$

We deduce easily from [BS00 Theorem 7.1] the following theorem.
Theorem 3.3. The Eisenstein series defined above has the following Fourier development:

\[
\mathcal{H}'_{L, \chi}(z, w)|_{u=\frac{1}{2}-t} = B(t)(2\pi i)^t G(\chi) \sum_{T_1 \geq 0} \sum_{T_2 \geq 0} \left( \sum_I b^w_{i+1}(I)(\chi)^{-1}(2T_2) \sum_{G \in \text{GL}_2(\mathbb{Z}) : D(I)} (\phi \chi)^2(\text{det}(G))|\text{det}(G)|^{2t-1} \right.
\]

\[
L(1-t, \sigma_{\text{det}(2I)} \phi \chi) \prod_{q|\text{det}(2G^{-1}IG^{-1})} B_q \left( \chi \phi(q)q^{t-2}, G^{-1}IG^{-1} \right) e^{2\pi i(T_1 z + T_2 w)},
\]

where the sum over \( I \) runs along the matrices \( \left( \begin{array}{cc} L^2T_1 & T_2 \\ T_2 & L^2T_4 \end{array} \right) \) positive definite and with \( 2T_2 \in \mathbb{Z} \), and

\[
\mathcal{D}(I) = \{ G \in M_2(\mathbb{Z})|G^{-1}IG^{-1} \text{ is a half-integral symmetric matrix} \}.
\]

Proof. The only difference from loc. cit. is that we do not apply \( \left( \begin{array}{cc} 1 & 0 \\ 0 & N^2S \end{array} \right) \). \( \square \)

In fact, contrary to [BS00], we prefer to work with \( \Gamma_0(N^2S) \) and not with the opposite congruence subgroup.

In particular each sum over \( I \) is finite because \( I \) must have positive determinant; moreover, we can rewrite as a sum over \( T_2 \), with \( (2T_2, p) = 1 \) and \( T_1 - T_2^2 > 0 \).

It is proved in [BS00] Theorem 8.5 that (small modifications of) these functions \( \mathcal{H}'_{L, \chi}(z, w) \) satisfy Kummer’s congruences. The key fact is what they call the twisting method [BS00] (2.18); the Eisenstein series \( \mathcal{F}^{t+1} \left( w, z, R^2N^2p^{2n}, \phi, u \right)^{(\chi)} \) are obtained weighting \( \mathcal{F}^{t+1} \left( w, z, R^2N^2p^{2n}, \phi, u \right) \) with respect to \( \chi \) over integral matrices modulo \( NRp^n \). To ensure these Kummer’s congruences, even when \( p \) does not divide the conductor of \( \chi \), the authors are forced to consider \( \chi \) of level divisible by \( p \). Using nothing more than Tamagawa’s rationality theorem for \( \text{GL}_2 \), they find the relation [BS00] (7.13’)

\[
\mathcal{F}^{t+1} \left( w, z, R^2N^2p^{2n}, \phi, u \right)^{(\chi)} = \mathcal{F}^{t+1} \left( w, z, R^2N^2p^{2n}, \phi, u \right)^{(\chi \chi')} \left( \text{id} - p \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right).
\]

So the Eisenstein series we want to interpolate to construct the improved \( p \)-adic \( L \)-function is

\[
\mathcal{H}_{L, \chi'}(z, w) := L(t+1+2s, \phi \chi) \hat{\mathcal{D}}^{t+1}_{\chi'} \left( \mathcal{F}^{t+1} \left( w, z, R^2N^2p, \phi, u \right)^{(\chi \chi')} \right).
\]

In what follows, we shall specialize \( t = k - k_0 + 1 \) (for \( k_0 \) the weight of the form in the theorem of the introduction) to construct the improved one variable \( p \)-adic \( L \)-function.

For each prime \( q \), let us denote by \( \alpha_q \) and \( \beta_q \) the roots of the Hecke polynomial at \( q \) associated to \( f \). We define

\[
D_q(X) := (1 - \alpha_q^2 X)(1 - \alpha_q \beta_q X)(1 - \beta_q^2 X).
\]
For each Dirichlet character \( \chi \) we define
\[
\mathcal{L}(s, \text{Sym}^2(f), \chi) := \prod_q D_q(\chi(q)q^{-s})^{-1}.
\]

This \( L \)-function differs from \( L(s, \text{Sym}^2(p_f) \otimes \chi) \) by a finite number of Euler factors at prime dividing \( N \). We conclude with the integral formulation of \( \mathcal{L}(s, \text{Sym}^2(f), \chi) \) [BS00, Theorem 3.1, Proposition 7.1 (7.13)].

**Proposition 3.4.** Let \( f \) be a form of weight \( k \), Nebentypus \( \phi \). We put \( t + s = k - 1 \), \( s_1 = \frac{1}{2} - t \) and \( \mathcal{H}' = \mathcal{H}'_{1,\chi}(z, w) \mid_{u=s_1} ; \) we have
\[
\left\langle f(w) \left| \begin{array}{cc} 0 & -1 \\ N^2p^{2n}R^2 & 0 \end{array} \right. \right\rangle_{N^2p^{2n}R^2} = \frac{\Omega_{k,s}(s_1)p_{s_1}(t + 1)}{\chi(-1)d_{s_1}(t + 1)} \left( R N_1 p^n \right)^{s_1+3-k} \left( \frac{N}{N_1} \right)^{2-k} \times
\]
\[
\times \mathcal{L}(s + 1, \text{Sym}^2(f), \chi^{-1}) f(z) |_{U_{N^2}/N_1^2}
\]
for
\[
p_{s_1}(t + 1) = c_{t+1} \quad \frac{d_{s_1}(t + 1)}{c_{t+1}^2} = \frac{\prod_{i=1}^s (s + t - \frac{i}{2})}{\prod_{i=1}^s (s - \frac{i-1}{2})},
\]
\[
\Omega_{k,s} \left( \frac{1}{2} - t \right) = 2^{t}(1) \frac{\Gamma(k-t) \Gamma(k-t-\frac{1}{2})}{\Gamma(\frac{3}{2})}.
\]

**Proof.** With the notation of [BS00, Theorem 3.1] we have \( M = R^2N^2p^{2n} \) and \( N = N_1R p^n \). We have that \( \mathcal{H}'_{1,\chi}(z, w) \) is the holomorphic projection of the Eisenstein series of [BS00, Theorem 3.1] (see [BS00] (1.30),(2.1),(2.25)). The final remark to make is the following relation between the standard (or adjoint) \( L \)-function of \( f \) and the symmetric square one:
\[
\mathcal{L}(1 - t, \text{Ad}(f) \otimes \phi, \chi) = \mathcal{L}(k - t, \text{Sym}^2(f), \chi).
\]

The authors of [BS00] prefer to work with an auxiliary character modulo \( N \) to remove all the Euler factors at bad primes of \( f \), but we do not want to do this. Still, we have to make some assumptions on the level \( \chi \). Suppose that there is a prime \( q \) dividing \( N \) such that \( q \mid N_1 \) and \( U_q f = 0 \), then the above formula would give us zero. That is why we introduce the character \( \chi_1 \) defined modulo a multiple of \( N_{ss} \).

At the level of \( L \)-function this does not change anything as for \( q \mid N_{ss} \) we have \( D_q(X) = 1 \).

For \( f \) as in Theorem 1.13 we can take \( N_1 = 1 \).

### 3.3 Families for \( \text{GL}_2 \times \text{GL}_2 \)

The aim of this section is the construction of families of modular forms on two copies of the modular curves. Let us fix a tame level \( N \), and let us denote by \( X \) the compactified modular curve of level \( \Gamma_1(N) \cap \Gamma_0(p) \). For \( n \geq 2 \), we shall denote by \( X(p^n) \) the modular curve of level \( \Gamma_1(N) \cap \Gamma_0(p^n) \).
where \( \eta \) from [AIS12, Pil13] that, for \( v \) and \( r \) suitable, there exists an invertible sheaf \( \omega^\kappa \) on \( X(v) \times_K \mathcal{W}(r) \). This allows us to define families of overconvergent forms as

\[
M_\kappa(N) := \lim_{v \to 0} H^0(X(v) \times_K \mathcal{W}(r), \omega^\kappa).
\]

We denote by \( \omega^{\kappa, (2)} \) the sheaf on \( X(v) \times_K X(v) \times_K \mathcal{W}(r) \) obtained by base change over \( \mathcal{W}(r) \). We define

\[
\begin{align*}
M_{\kappa,v}^{(2)}(N) &:= H^0(X(v) \times_K X(v) \times_K \mathcal{W}(r), \omega^{\kappa, (2)}) \\
&= H^0(X(v) \times \mathcal{W}(r), \omega^\kappa) \hat{}_{\mathcal{O}(\mathcal{W}(r))} H^0(X(v) \times \mathcal{W}(r), \omega^\kappa); \\
M_\kappa^{(2)}(N) &:= \lim_{v \to 0} M_{\kappa,v}^{(2)}(N).
\end{align*}
\]

We believe that this space should correspond to families of Siegel modular forms [AIP12] of parallel weight restricted on the Igusa divisor, but for shortness of exposition we do not examine this now. We have a correspondence \( C_p \) above \( X(v) \) defined as in [Pil13, §4.2]. We define by fiber product the correspondence \( C_p^2 \) on \( X(v) \times_K X(v) \) which we extend to \( X(v) \times_K X(v) \times_K \mathcal{W}(r) \). This correspondence induces a Hecke operator \( U_p^{\otimes 2} \) on \( H^0(X(v) \times_K X(v) \times_K \mathcal{W}(r), \omega^{\kappa, (2)}) \) which corresponds to \( U_p \otimes U_p \). These are potentially orthonormalizable \( \mathcal{O}(\mathcal{W}(r)) \)-modules [Pil13, §5.2] and \( U_p^{\otimes 2} \) acts on these spaces as a completely continuous operator (or compact, in the terminology of [Buz07, §1]) and this allows us to write

\[
M_\kappa^{(2)}(N) \leq \alpha = \bigoplus_{\alpha_1 + \alpha_2 = \alpha} M_\kappa(N) \leq \alpha_1 \hat{}_{\mathcal{O}(\mathcal{W}(r))} \lim \alpha_2 \leq M_\kappa(N) \leq \alpha_2.
\]

(3.5)

Here and in what follows, for \( A \) a Banach ring, \( M \) a Banach \( A \)-module and \( U \) a completely continuous operator on \( M \), we write \( M \leq \alpha \) for the finite dimensional submodule of generalized eigenspaces associated to the eigenvalues of \( U \) of valuation smaller or equal than \( \alpha \). We write \( \Pr \leq \alpha \) for the corresponding projection.

We remark [Urb11, Lemma 2.3.13] that there exists \( v > 0 \) such that

\[
M_\kappa^{(2)}(N) \leq \alpha = M_{\kappa,v}^{(2)}(N) \leq \alpha
\]

for all \( 0 < v' < v \). We define similarly \( M_\kappa^{(2)}(NP^n) \).

We now use the above Eisenstein series to give an example of a family. Let us fix \( \chi = \chi_1 \chi' \varepsilon_1 \) as before. We suppose moreover \( \chi \) even. We recall the Kubota-Leopoldt \( p \)-adic \( L \)-function;

**Theorem 3.6.** Let \( \eta \) be a even Dirichlet character. There exists a \( p \)-adic \( L \)-function \( L_p(\kappa, \eta) \) satisfying for any integer \( t \geq 1 \), and finite-order character \( \varepsilon \) of \( 1 + p\mathbb{Z}_p \)

\[
L_p(\varepsilon(u)[t], \eta) = (1 - (\varepsilon \omega^{-t} \eta_0(p))L(1 - t, \varepsilon \omega^{-t} \eta),
\]

where \( \eta_0 \) stands for the primitive character associated to \( \eta \). If \( \eta \) is not trivial, then \( L_p([t], \eta) \) is holomorphic. Otherwise, it a simple pole at \([0] \).
We can consequently define a $p$-adic analytic function interpolating the Fourier coefficients of the Eisenstein series defined in the previous section; for any $z$ in $\mathbb{Z}_p^s$, we define $l_z = \frac{\log_p(z)}{\log_p(u)}$ and

$$a_{T_1,T_4,L}(\kappa,\kappa') = \left( \sum_I \kappa(u_2T_2)\chi^{-1}(2T_2) \sum_{G \in \text{GL}_2(\mathbb{Z})} (\phi \chi)^2(\det(G))|\det(G)|^{-1}\kappa'^2(u_{2|\det(G)}) \right)$$

$$L_p(\kappa', \sigma_{-\det(2I)}\phi \chi) \prod_{q|\det(2\text{det}(G))} B_q \left( (\phi \chi')^{2}(u_{2,t}q^2,G^{-t}IG^{-1}) \right)$$

If $p = 2$, then $\chi^{-1}(2T_2)$ vanishes when $T_2$ is an integer, so the above above sum is only on half-integral $T_2$ and $\kappa(u_{2T_2})$ is a well-defined 2-analytic function.

We recall that if $p^N \mid T_1, T_4$ we have \textbf{BS00} (1.21, 1.34), for $s = k - t - 1$,

$$4^s \mathcal{B}_{t+1}(I) \equiv (-1)^s 4^s c_{t+1}T_2^s \mod p^N.$$  Consequently, if we define

$$\mathcal{H}_L(\kappa,\kappa') = \sum_{T_1 \geq 0} \sum_{T_2 \geq 0} a_{T_1,T_4,L}(\kappa[-1]\kappa^{-1},\kappa') \frac{T_1}{q_1^{\frac{T_1}{2}}} \frac{T_4}{q_2^{\frac{T_4}{2}}}$$

we have,

$$(-1)^s 2^s c_{t+1} A \mathcal{H}_L([k],\varepsilon[i]) \equiv 2^s \mathcal{H}_{L,\chi_0\varepsilon^{-\frac{1}{2}}}(z, w) \mod L,$$

with

$$A = A(t, k, \varepsilon) = B(t)(2\pi i)^s G \left( \chi \varepsilon^{\omega^{-s}} \right).$$

We have exactly as in \textbf{Pan03} Definition 1.7.

**Lemma 3.7.** There exists a projector

$$\Pr_{\infty}^\leq \alpha : \bigcup_n M_\kappa^{(2)}(Np^n) \to M_\kappa^{(2)}(N)^{\leq \alpha}$$

which on $M_\kappa^{(2)}(Np^n)$ is $(U_p^\leq \alpha)^{\frac{1}{i} \text{Pr}^\leq \alpha (U_p^\leq \alpha)^{\frac{1}{i}}}$, independent of $i \geq n$.

We shall now construct the improved Eisenstein family. Fix $k_0 \geq 2$ and $s_0 = k_0 - 2$. It is easy to see from 3.2 and 3.1 that when $k$ varies $p$-adically the value $\mathcal{B}_L^{(s_0)}(I)$ varies $p$-adically analytic too. We define $\mathcal{B}_L^{(s_0)}(I)$ to be the only polynomial in $O(W(r))[T_i]$, homogeneous of degree $s_0$ such that $\mathcal{B}_L^{(s_0)}(I) = \mathcal{B}_L^{(s_0)}(I)$. Its coefficients are products of $\text{Log}_p(\kappa[i])$. We let $\chi = \chi_1 \chi_0^{k_0 - 2}$ and we define

$$a_{T_1,T_4}(\kappa) = \left( \sum_I \mathcal{B}_L^{(s_0)}(I) \chi^{-1}(2T_2) \sum_{G \in \text{GL}_2(\mathbb{Z})} (\phi \chi_1)^2(\det(G))|\det(G)|^{-1}\kappa(u_{2|\det(G)}) \right)$$

$$\times L_p(\kappa, \sigma_{-\det(2I)}\phi \chi_1) \prod_{q|\det(2\text{det}(G))} B_q \left( (\phi \chi_1(q)\kappa(u_{2,t}q^2,G^{-t}IG^{-1}) \right)$$
We can now construct another $p$-adic family of Eisenstein series:
\[ \mathcal{H}^*(\kappa) = \Pr^{\leq\alpha}(-1)^{k_0} \sum_{T_1 \geq 0} \sum_{T_2 \geq 0} \alpha_{T_1,T_2}^*(\kappa[1-k_0])q_1^{T_1}q_2^{T_2}. \]

**Proposition 3.8.** We have a $p$-adic family $\mathcal{H}(\kappa,\kappa') \in M^{(2)}_\kappa(N^2R^2)^{\leq\alpha}$ such that
\[ \mathcal{H}([k],\varepsilon[t]) = \frac{1}{(1)^s c_{l+1}^s A} (U_p^\otimes 2)^{-2l} \Pr^{\leq\alpha} \mathcal{H}'_{p^l,\chi^\varepsilon \omega^{-s}}(z,w). \]

We also have $\mathcal{H}^*(\kappa) \in M^{(2)}_\kappa(N^2R^2)^{\leq\alpha}$ such that
\[ \mathcal{H}^*(\kappa) = A^{s-1}(-1)^{k_0} \Pr^{\leq\alpha} \mathcal{H}_{1,\chi^s}(z,w)|_{w=\frac{1}{2}-k+k_0-1}, \]
for
\[ A^s = B(k-k_0+1)(2\pi i)^{k_0-2}G(\chi'\chi_1). \]

**Proof.** From its own definition we have $(U_p^\otimes 2)^{2j} \mathcal{H}_{1,\chi}^s(z,w) = \mathcal{H}'_{p^j,\chi}(z,w)$. We define
\[ \mathcal{H}(\kappa,\kappa') = \lim_{j} (U_p^\otimes 2)^{-2j} \Pr^{\leq\alpha} \mathcal{H}_{p^j,\chi}(\kappa,\kappa'). \]

With Lemma 3.7 and the previous remark we obtain
\[ 4^s (U_p^\otimes 2)^{-2l} \Pr^{\leq\alpha} \mathcal{H}_{p^l,\chi^\varepsilon \omega^{-s}}(z,w) = 4^s (U_p^\otimes 2)^{-2j} \Pr^{\leq\alpha} \mathcal{H}'_{p^j,\chi^\varepsilon \omega^{-s}}(z,w) \equiv A(-1)^s 4^s c_{l+1}^s (U_p^\otimes 2)^{-2l} \Pr^{\leq\alpha} \mathcal{H}_{p^l}(\kappa,\varepsilon[t]) \mod p^j. \]

The punctual limit is then a well-defined classical, finite slope form. By the same method of proof of [Urb13, Corollary 3.4.7] or [Ros13b, Proposition 2.17] (in particular, recall that when the slope is bounded the ray of overconvergence can be fixed), we see that this $q$-expansion defines a family of finite slope forms.

For the second family, we remark that $\log_p(\kappa)$ is bounded on $W(r)$, for any $r$ and we reason as above. \qed

We want to explain briefly why the construction above works.

It is slightly complicated to calculate explicitly the polynomial $b_{i+1}^s(I)$. But we know that $\tilde{D}_i^s$ is an homogeneous polynomial in $\partial_i$ of degree $s$. In particular, there is a single monomial which does not involve $\partial_1$ and $\partial_4$, namely $c_{i+1}^s \tilde{D}_i^s$. Consequently, in $b_{i+1}^s(I)$ there is a single monomial without $T_1$ and $T_4$.

When the entries on the diagonal of $I$ are divisible by $p^j$, $4^s b_{i+1}^s(I)$ reduces to $(-1)^s 4^s c_{i+1}^s$ modulo $p^j$. We apply $U_p^\otimes 2$ many times to ensure that $T_1$ and $T_4$ are very divisible by $p$. Speaking $p$-adically, we approximate $\tilde{D}_i^s$ by $\partial_2$ (multiplied by a constant). The more times we apply $U_p^\otimes 2$, the better we can approximate $p$-adically $\tilde{D}_i^s$ by $\partial_2$. At the limit, we obtain equality. This should remind the reader of the fact that on $p$-adic forms the Maass-Shimura operator and Dwork $\Theta$-operator coincide [Urb13, §3.2] (but here the hypothesis that the forms are of finite slope is necessary).
\section{p-adic L-functions}

We now construct two p-adic L-functions: the two-variable one and the improved one. Necessary for the construction is a p-adic Petersson product \cite{Pan03} §6 which we now recall. We fix a family $F = F(\kappa)$ of finite slope modular forms which we suppose primitive, i.e. all its classical specialization are primitive forms, of prime-to-$p$ conductor $N$. We consider characters $\chi, \chi'$ and $\chi_1$ as in Section \ref{sec:eigenvarieties}. We keep the same decomposition for $N$ (because the local behavior at $q$ is constant along the family). We shall write $N_0$ for the conductor of $\chi_1$.

Let $\mathcal{C}_F$ be the corresponding irreducible component of the Coleman-Mazur eigencurve. It is finite flat over $\mathcal{W}(r)$, for a certain $r$. Let $\mathcal{I}$ be the coefficients ring of $F$ and $\mathcal{K}$ its field of fraction. For a classical form $f$, let us denote by $f^c$ the complex conjugated form. We shall denote by $\tau_N$ the Atkin-Lehner involution of level $N$ normalized as in \cite{Hid90} h4. When the level will be clear from the context, we shall simply write $\tau$.

Standard linear algebra allows us to define a $\mathcal{K}$-linear form $l_F$ on $M_\kappa(N)_{\leq \alpha} \otimes_\mathcal{I} \mathcal{K}$ with the following property \cite{Pan03, Proposition 6.7}:

\begin{proposition}
For all $G(\kappa)$ in $M_\kappa(N)_{\leq \alpha} \otimes_\mathcal{I} \mathcal{K}$ and any $\kappa_0$ classical point, we have

$$ l_F(G(\kappa))|_{\kappa = \kappa_0} = \frac{\langle F(\kappa_0)^c | \tau, G(\kappa_0) \rangle_{Np}}{\langle F(\kappa_0)^c | \tau, F(\kappa_0) \rangle_{Np}}. $$

We can find $H_F(\kappa)$ in $\mathcal{I}$ such that $H_F(\kappa)l_F$ is defined over $\mathcal{I}$. We shall refer sometimes to $H_F(\kappa)$ as the denominator of $l_F$.

We define consequently a $\mathcal{K}$-linear form on $M_\kappa(2)(N)_{\leq \alpha} \otimes_\mathcal{I} \mathcal{K}$ by

$$ l_{F \times F} := l_F \otimes l_F $$

under the decomposition in \textit{\ref{sec:eigenvarieties}}.

Before defining the p-adic L-functions, we need an operator to lower the level of the Eisenstein series constructed before. We follow \cite{Hid88} §1 VI]. Fix a prime-to-$p$ integer $L$, with $N|L$. We define for classical weights $k$:

$$ T_{L/N,k} : M_k(Lp, A) \rightarrow M_k(Np, A) \quad \text{with} \quad f \mapsto (L/N)^{k/2} \sum_{[\gamma] \in \Gamma(N)/\Gamma(N,L/N)} f|_k \left( \begin{array}{cc} 1 & 0 \\ 0 & L/N \end{array} \right) |_{k\gamma}. $$

As $L$ is prime to $p$, it is clear that $T_{L/N,k}$ commutes with $U_p$. It extends uniquely to a linear map

$$ T_{L/N} : M_\kappa(L) \rightarrow M_\kappa(N) $$

which in weight $k$ specializes to $T_{L/N,k}$.

We have a map $M_\kappa(N)_{\leq \alpha} \rightarrow M_{\kappa}(N^2R_2)^{\leq \alpha}$; we define $1_{N^2R_2/N}$ to be one left inverse. We define

$$ L_p(\kappa, \kappa') = \frac{N_0}{N^2R_2N_1} l_F \otimes l_F \left( (U_{N^2/N_2}^{-1} \circ 1_{N^2R_2/N}) \otimes T_{N^2R_2/N}(\mathcal{H}(\kappa, \kappa')) \right), $$

$$ L_0(\kappa) = \frac{N_0}{N^2R_2N_1} l_F \otimes l_F \left( (U_{N^2/N_2}^{-1} \circ 1_{N^2R_2/N}) \otimes T_{N^2R_2/N}(\mathcal{H}^*(\kappa)) \right). $$
We will see in the proof of the following theorem that it is independent of the left inverse \(1_{N^2R^2/N}\) which we have chosen.

We fix some notations. For a Dirichlet character \(\eta\), we denote by \(\eta_0\) the associated primitive character. Let \(\lambda_p(\kappa) \in \mathcal{I}\) be the \(U_p\)-eigenvalue of \(F\). We say that \((\kappa, \kappa') \in \mathcal{C}_F \times \mathcal{W}\) is of type \((k; t, \varepsilon)\) if \(\kappa|_{W(\varepsilon)} = [k]\) with \(k \geq 2\), and \(\kappa' = \varepsilon[t]\) with \(1 \leq t \leq k - 1\), and \(\varepsilon\) modulo \(p^n\), \(n \geq 1\). We let as before \(s = k - t - 1\). We define

\[
E_1(\kappa, \kappa') = \lambda_p(\kappa)^{-2n_0}(1 - (\chi\varepsilon\omega^{-s})\eta_0(p)\lambda_p(\kappa)^{-2p^s})
\]

where \(n_0 = 0\) (resp. \(n_0 = n\)) if \(\chi\varepsilon\omega^{-s}\) is (resp. is not) trivial at \(p\).

If \(F(\kappa)\) is primitive at \(p\) we define \(E_2(\kappa, \kappa') = 1\), otherwise

\[
E_2(\kappa, \kappa') = (1 - (\chi^{-1}e^{-1}\omega^s)\eta_0(p)p^{k-2-s}) \times (1 - (\chi^{-1}e^{-1}\omega^s)\eta_0(p)\lambda_p(\kappa)^{-2p^{2k-3-s}}).
\]

We shall denote by \(F^\circ(\kappa)\) the primitive form associated to \(F(\kappa)\). We shall write \(W'(F(\kappa))\) for the prime-to-\(p\) part of the root number of \(F^\circ(\kappa)\). If \(F(\kappa)\) is not \(p\)-primitive we pose

\[
S(F(\kappa)) = (-1)^k \left(1 - \frac{\phi_0(p)p^{k-1}}{\lambda_p(\kappa)^2} \right) \left(1 - \frac{\phi_0(p)p^{k-2}}{\lambda_p(\kappa)^2} \right),
\]

and \(S(F(\kappa)) = (-1)^k\) otherwise. We pose

\[
C_{\kappa, \kappa'} = is!G\left((\chi\varepsilon\omega^{-s})^{-1}\right)(\chi\varepsilon\omega^{-s})\eta_0(p)p^{n-n_0}(N_1Rp^{n_0})^sN^{-k/2}2^{-3s-t},
\]

\[
C_\kappa = C_{\kappa, [k-k_0+1]},
\]

\[
\Omega(F(\kappa), s) = W'(F(\kappa))(2\pi i)^{s+1}(F^\circ(\kappa), F^\circ(\kappa)).
\]

We have the following theorem, which will be proven at the end of the section.

**Theorem 4.2.**  

i) The function \(L_p(\kappa, \kappa')\) is defined on \(\mathcal{C}_F \times \mathcal{W}\), it is meromorphic in the first variable and of logarithmic growth \(h = [2a] + 2\) in the second variable (i.e., as function of \(t\), \(L_p(\kappa, [t])\prod_{i=0}^{h-1} \log(p^{t-i})\) is holomorphic on the open unit ball). For all classical points \((\kappa, \kappa')\) of type \((k; t, \varepsilon)\) with \(k \geq 2\), \(1 \leq t \leq k - 1\), we have the following interpolation formula

\[
L_p(\kappa, \kappa') = C_{\kappa, \kappa'}E_1(\kappa, \kappa')E_2(\kappa, \kappa') \frac{\mathcal{L}(s + 1, \text{Sym}^2(F(\kappa)), \chi^{-1}\varepsilon^{-1}\omega^s)}{S(F(\kappa))\Omega(F(\kappa), s)}.
\]

ii) The function \(L^*_p(\kappa)\) is meromorphic on \(\mathcal{C}_F\). For \(\kappa\) of type \(k\) with \(k \geq k_0\), we have the following interpolation formula

\[
L^*_p(\kappa) = C_\kappa E_2(\kappa, [k-k_0+1]) \frac{\mathcal{L}(k_0 - 1, \text{Sym}^2(F(\kappa)), \chi'^{-1}\chi^{-1})}{S(F(\kappa))\Omega(F(\kappa), k_0 - 2)}.
\]

We can deduced the fundamental corollary which allows us to apply the method of Greenberg and Stevens:
Corollary 4.3. We have the following factorization of locally analytic functions around $\kappa_0$ in $C_F$:

$$L_p(\kappa, [k-k_0+1]) = (1 - \chi' \chi_1(p) \lambda_p(\kappa)^{-2} p^{k_0-2}) L_p^*(\kappa).$$

So can prove the main theorem of the paper;

**Proof of Theorem 4.3.** Let $f$ be as in the statement of the theorem; we take $\chi' = \chi_1 = 1$. We define $L_p(\text{Sym}^2 f, s) = C_{\kappa_0}^{-1}[1] L_p(\kappa_0, [k_0-s])$. The two variables $p$-adic $L$-function vanishes on $\kappa' = [1]$. As $f$ is Steinberg at $p$, we have $\lambda_p(\kappa_0)^2 = p^{k_0-2}$.

Consequently, the following formula is a straightforward consequence of Section 2 and Corollary 4.3.

$$\lim_{s \to 0} \frac{L_p(\text{Sym}^2 f, s)}{s} = -2 \frac{d \log \lambda_p(\kappa)}{d\kappa} |_{\kappa = \kappa_0} \frac{L(\text{Sym}^2 f, k_0 - 1)}{S(F(\kappa)) \Omega(f, k_0 - 2)}.$$

From [Ben10, Mok12] we obtain

$$L_{\text{alg}}(\text{Sym}^2 f) = -2 \frac{d \log \lambda_p(\kappa)}{d\kappa} |_{\kappa = \kappa_0}.$$

Under the hypothesis of the theorem, $f$ is Steinberg at all primes of bad reduction and we see from [Ros13b, §3.3] that

$$L(\text{Sym}^2 f, k_0 - 1) = L(\text{Sym}^2 f, k_0 - 1)$$

and we are done. □

**Proof of Theorem 4.2.** We point out that most of the calculations we need in this proof and have not already been quoted can be found in [Hid88, Pan03].

If $\epsilon$ is not trivial at $p$, we shall write $p^a$ for the conductor of $\epsilon$. If $\epsilon$ is trivial, then we let $n = 1$.

We recall that $s = k - t - 1$; we have

$$L_p(\kappa, \kappa') = \frac{N_0 \left\langle F(\kappa)^t | \tau_{Np}, U_p^{-1} N^2 \right\rangle \left\langle F(\kappa)^t | \tau_{Np}, T_{N^2 R^2} \right\rangle \mathcal{H}(k, \epsilon[t])}{N^2 R^2 N_1 \left\langle F(\kappa)^t | \tau, F(\kappa) \right\rangle^2}.$$

We have as in [Pan03, (7.11)]

$$\left\langle F(\kappa)^t | \tau_{Np}, U_p^{-2n+1} P_{r^{-2n} U_p^{-2n-1} g} \right\rangle = \lambda_p(\kappa)^{1-2n} p^{(2n-1)(k-1)} \left\langle F(\kappa)^t | \tau_{Np}, | p^{2n-1} \right\rangle,$$  \hspace{1cm} (4.4)

where $f)[| p^{2n-1}|(z) = f[p^{2n-1}z]$. We recall the well-known formulae [Hid88 page 79]:

\begin{align*}
\left\langle f \right| p^{2n-1}, T_{N^2 R^2} \right\rangle &= (NR^{2k}) \left\langle f \right| p^{2n R^{2k}} \right\rangle, \\
\tau_{Np} \left| p^{2n-1} R^2 \right| &= \left(N R^{2k} 2^{2n-1} \right)^{k/2} \tau_{N^2 R^2 p^{2n}}, \\
\left\langle F(\kappa)^t | \tau_{Np}, F(\kappa) \right\rangle &= (-1)^k W' F(\kappa)^t p^{(2-k)/2} \lambda_p(\kappa) \times \left(1 - \frac{\phi_0(p)}{\lambda_p(\kappa)^2} \right) \left(1 - \frac{\phi_0(p) p^{k-2}}{\lambda_p(\kappa)^2} \right).
\end{align*}
So we are left to calculate

\[
\frac{N_0(NR^2)^{k/2} \lambda_p(\kappa)^{1-2n}p^{(2n-1)}(\frac{k}{2} - 1)}{A(\frac{k}{2}) c_{n+1}N_1 N_k N^2 R^2} \left\langle F(\kappa)^c | \tau_{NP}, U_{N_k/N_1}^{-1} \left\langle F(\kappa)^c | \tau_{N^2 R^2 p^n}, \mathcal{H}_{1, \delta \omega^{-1}}(z, w)\right|_s = \frac{r}{s} \right\rangle.
\]

We use Proposition 3.3 and [BS00 (3.29)] (with the notation of loc. cit. \( \beta_1 = \frac{\lambda_1(\kappa)}{p^{s-1}} \)) to obtain that the interior Petersson product is a scalar multiple of

\[
\mathcal{L}(\text{Sym}^2(f), s + 1) \bar{E}(\kappa, k) E_2(\kappa, k) F(\kappa)|_{U_{N_k/N_1}^2}.
\]

Here

\[
\bar{E}(\kappa, k') = -p^{-1}(1 - \chi \varepsilon^{-1}(p) \lambda_p(\kappa)^{2} t s + 1 - \kappa)
\]

if \( \chi \varepsilon \omega^{-s} \) is trivial modulo \( p \), and 1 otherwise. The factor \( E_2(\kappa, k') \) appears because \( F(\kappa) \) could not be primitive. Clearly it is independent of \( 1_{N_k/N_1} \) and we have \( l_F(F(\kappa)) = 1 \). We explicit the constant which multiplies (4.5):

\[
(-1)^s \frac{N_0(NR^2)^{k/2} \lambda_p(\kappa)^{1-2n}p^{(2n-1)}(\frac{k}{2} - 1) N_1 + 1 (R \omega^s)^{s+3-k} N^2 - k \omega^{k-t-1} (-1)^{\Omega} p s_1 (t + 1)}{B(t)(2\pi)^s c_{n+1} G ((\chi \varepsilon \omega^{-s})) N^2 R^2 N_1} \frac{1}{d s_1 (t + 1)}
\]

\[
= \frac{N^2 \frac{k}{2} N_0 N_1^4 R \omega^s \lambda_p(\kappa)^{1-2n}p^{(2n-1)}(2s)!2^{-2s}2^{s+1} 3/2}{(2\pi)^s G ((\chi \varepsilon \omega^{-s})) 2^{s+1} 3 \omega^{-s}}
\]

If \( \varepsilon \omega^{-s} \) is not trivial we obtain

\[
\frac{N^2 \frac{k}{2} N_0 N_1^4 R \omega^s \lambda_p(\kappa)^{1-2n}p^{(2n-1)}(2s)!2^{-2s}2^{s+1} 3/2}{(2\pi)^s G ((\chi \varepsilon \omega^{-s})) 2^{s+1} 3 \omega^{-s}}
\]

otherwise

\[
\frac{N^2 \frac{k}{2} N_0 N_1^4 R \omega^s \lambda_p(\kappa)^{1-2n}p^{(2n-1)}(2s)!2^{-2s}2^{s+1} 3/2}{(2\pi)^s G ((\chi \varepsilon \omega^{-s})) 2^{s+1} 3 \omega^{-s}}
\]

The calculations for \( L_p^*(\kappa) \) are similar. We have to calculate

\[
\frac{N_0(NR^2)^{k/2}}{A^* N^2 R^2 N_1} \left\langle F(\kappa)^c | \tau_{NP}, U_{N_k/N_1}^{-1} \left\langle F(\kappa)^c | \tau_{N^2 R^2 p^n}, (-1)^{\Omega} \mathcal{H}_{1, \delta \omega^{-1}}(z, w)\right|_s = 1 \right\rangle,
\]

where \( s_1 = \frac{k}{2} - k_0 - 1 \). The interior Petersson product equals (see [BS00 Theorem 3.1] with \( M, N \) of loc. cit. as follows: \( M = R^2 N^2 p, N = N_k R \))

\[
\frac{R^{k_0+1-k} N^2 p}{d_{s_1}(k - k_0 + 2)} \langle F(\kappa)^c, F(\kappa)^c \rangle W'(F(\kappa)) p^{(2-k)/2} \lambda_1(\kappa) S(F(\kappa)) F(\kappa)|_{U_{N_k/N_1}^2}.
\]
so we have
\[ L_p^*(\kappa) = i \frac{N^{k_0-2} R^{k_0-2} N^{k/2}(k_0 - 2)! E_2(\kappa, [k - k_0 + 1]] L(k_0 - 1, \text{Sym}^2(F(\kappa)), (\chi_1 \chi')^{-1})}{(2\pi i)^{k_0-1} G(\chi_1 \chi') 2^{2k_0-5+k} G(F(\kappa))(F(\kappa)^n, F(\kappa)^n)}. \]

To prove the bound on the logarithmic growth, we note that the measure \( \mathcal{H}(\kappa, \kappa') \) is bounded. We use (4.4) and we have, as a function of \( n \),
\[ |L_p(\kappa, [t] \varepsilon)| = O(|\lambda_p(\kappa)^{-2n}|); \]
the estimate follows from [Pan03, Definition 3.2 c), Theorem 3.4].

We give here some concluding remarks. Unless \( F(\kappa) \) has complex multiplication by \( \chi \) we can see exactly as in [Hid90, Proposition 5.2] that \( H_F(\kappa)L_p(\kappa, \kappa') \) is holomorphic along \( \kappa' = 0 \) (which is the pole of the Kubota Leopoldt \( p \)-adic \( L \)-function). If \( p \neq 2 \), this \( p \)-adic \( L \) coincides, up to an Euler factor at 2, with the one of [Ros13b, Theorem 4.14]. Thanks to [Ros13b, Appendix], it can be divided by Iwasawa functions interpolating the missing Euler factors to obtain a holomorphic function interpolating the primitive complex \( L \)-function. We are confident that the same method should also apply for \( p = 2 \).

Note also that our choice of periods is not optimal [Ros13b, §6].

References

[AIP12] Fabrizio Andreatta, Adrian Iovita, and Vincent Pilloni. \( p \)-adic families of Siegel modular cuspforms. \textit{to appear Ann. of Math.}, 2012.

[AIS12] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens. Overconvergent modular sheaves and modular forms for \( GL_2/F \). \textit{to appear Israel J. Math.}, 2012.

[Bel12] Joël Bellaïche. \( p \)-adic \( L \)-functions of critical CM forms. \textit{preprint available at http://people.brandeis.edu/~jbellaic/preprint/CML-functions4.pdf}, 2012.

[Ben10] Denis Benois. Infinitesimal deformations and the \( \ell \)-invariant. \textit{Doc. Math.}, (Extra volume: Andrei A. Suslin sixtieth birthday):5–31, 2010.

[Ben11] Denis Benois. A generalization of Greenberg’s \( \mathcal{L} \)-invariant. \textit{Amer. J. Math.}, 133(6):1573–1632, 2011.

[Ben12] Denis Benois. Trivial zeros of \( p \)-adic \( L \)-functions at near central points. \textit{to appear J. Inst. Math. Jussieu http://dx.doi.org/10.1017/S1474748013000261}, 2012.

[Bra13] Riccardo Brasca. Eigenvarieties for cuspforms over PEL type Shimura varieties with dense ordinary locus. \textit{in preparation}, 2013.

[BS00] S. Böcherer and C.-G. Schmidt. \( p \)-adic measures attached to Siegel modular forms. \textit{Ann. Inst. Fourier (Grenoble)}, 50(5):1375–1443, 2000.
[BSDGP96] Katia Barré-Sirieix, Guy Diaz, François Gramain, and Georges Philibert. Une preuve de la conjecture de Mahler-Manin. *Invent. Math.*, 124(1-3):1–9, 1996.

[Buz07] Kevin Buzzard. Eigenvarieties. In *L-functions and Galois representations*, volume 320 of *London Math. Soc. Lecture Note Ser.*, pages 59–120. Cambridge Univ. Press, Cambridge, 2007.

[Col05] Pierre Colmez. Zéros supplémentaires de fonctions $L$ $p$-adiques de formes modulaires. In *Algebra and number theory*, pages 193–210. Hindustan Book Agency, Delhi, 2005.

[DG12] Mladen Dimitrov and Eknath Ghate. On classical weight one forms in Hida families. *J. Théor. Nombres Bordeaux*, 24(3):669–690, 2012.

[Dim13] Mladen Dimitrov. On the local structure of ordinary hecke algebras at classical weight one points. *to appear in Automorphic Forms and Galois Representations, proceedings of the LMS Symposium, 2011*, to appear, 2013.

[Gre94] Ralph Greenberg. Trivial zeros of $p$-adic $L$-functions. In *$p$-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991)*, volume 165 of *Contemp. Math.*, pages 149–174. Amer. Math. Soc., Providence, RI, 1994.

[GS93] Ralph Greenberg and Glenn Stevens. $p$-adic $L$-functions and $p$-adic periods of modular forms. *Invent. Math.*, 111(2):407–447, 1993.

[Hid88] Haruzo Hida. A $p$-adic measure attached to the zeta functions associated with two elliptic modular forms. II. *Ann. Inst. Fourier (Grenoble)*, 38(3):1–83, 1988.

[Hid89] Haruzo Hida. On nearly ordinary Hecke algebras for GL(2) over totally real fields. In *Algebraic number theory*, volume 17 of *Adv. Stud. Pure Math.*, pages 139–169. Academic Press, Boston, MA, 1989.

[Hid90] Haruzo Hida. $p$-adic $L$-functions for base change lifts of GL$_2$ to GL$_3$. In *Automorphic forms, Shimura varieties, and $L$-functions, Vol. II (Ann Arbor, MI, 1988)*, volume 11 of *Perspect. Math.*, pages 93–142. Academic Press, Boston, MA, 1990.

[Hid06] Haruzo Hida. *Hilbert modular forms and Iwasawa theory*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006.

[HJ13] Robert Harron and Andrei Jorza. On symmetric power $L$-invariants of Iwahori level Hilbert modular forms. *preprint available at www.its.caltech.edu/~ajorza/papers/harron-jorza-non-cm.pdf*, 2013.

[HTU97] Haruzo Hida, Jacques Tilouine, and Eric Urban. Adjoint modular Galois representations and their Selmer groups. *Proc. Nat. Acad. Sci. U.S.A.*, 94(21):11121–11124, 1997. Elliptic curves and modular forms (Washington, DC, 1996).

[Liu13] Ruochuan Liu. Triangulation of refined families. *preprint available at http://arxiv.org/abs/1202.2188*, 2013.
Éric Urban. Nearly overconvergent modular forms. To appear in the Proceedings of conference IWASAWA 2012, 2013.