Determination of the Probability Density Function of the Mixture Signal and Additive Noise under Influence of Multiplicative Noise

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Abstract. Issues related to effect of multiplicative (modulating) (otherwise amplitude distortions) and additive noises on processed signal are considered. Analysis of statistical characteristics of probability density function (PDF) of instantaneous signal values against background of modulating interference is carried out. Expressions of the joint PDF envelope, phase and instantaneous signal values are obtained, for the case of only phase distortions, as well as the case when fluctuations in amplitude and phase are independent of each other. It is noted that in the second case, with the independence of phase and amplitude distortions, the PDF of the signal, first of all, is determined by the PDF of its instantaneous values and practically does not depend on the PDF of the signal phase. Expressions are obtained for the PDF useful signal and the most common PDF envelope when affecting the signal with modulating noise for deep phase distortions and for its uniform distribution over the interval (0, 2π). In addition, for the case of some functional relationship of phase and amplitude fluctuations, the expression for PDF instantaneous signal values is defined. The presence of such a functional connection allows calculating the PDF of the signal through the statistical characteristics of its envelope. It is obtained that PDF of mixture of signal and additive noise contains arbitrary distribution of phase and amplitude, as well as arbitrary law of distribution of envelope of processed signal. It is obtained that weight coefficients are determined by derivative of characteristic function of amplitude distortion in case when PDFs of said mixture are distributed according to normal law and functional connection exists between amplitude and phase distortions.

1. Introduction

In statistical radio engineering, radar and radio navigation applications, in the theory of communication and automatic control, information and measurement systems, methods of applied theory of random processes are widely used. Most often, scientific studies consider models in which processed signals are received as some mixture of signal and additive noise with the Gaussian nature of the distribution [1-3, etc.]. However, in real-world operation, useful signals are often exposed to multiplicative (otherwise modulating) noise [4-6, etc.]. As a result, mathematical models using the near-reality probability density function (PDF) of the mixtures of the useful signal, additive and multiplicative noise should be used to synthesize and analyze systems for processing such signals [7, 8, etc.].
The aim of the study is to obtain mathematical models that would allow you to model PDF instantaneous values of the sum of additive noise and a signal subject to modulating (multiplicative) noise.

2. Influence of multiplicative noise on PDF of instantaneous signal values

We evaluate various information parameters of the processed signal, which is simultaneously affected by additive and multiplicative non-Gaussian noise. Note that the evaluation of these parameters will be carried out in a discrete observation time.

Let some samples of the random process \( Y_h = Y(t_h) \) \( (h = 1, \ldots, H) \) be observed over time \([0, T]\). This process is a mixture of a useful signal \( s(t) \) containing information about the measured parameter and non-Gaussian multiplicative \( n(t_h) \) noise:

\[
u_n = \eta(t) \cos \Theta(t),
\]

where \( \eta(t) \geq 0 \) is a dimensionless multiplier and characterizes signal envelope changes generated by modulating noise (or amplitude distortions), \( U(t) \) is a signal envelope determined by the law of amplitude modulation of the signal, \( \alpha_0 \) is the carrier frequency of the signal, \( \phi(t) \) is represented by the law of phase modulation of a signal (at frequency modulation it is possible to write down \( \Phi = \int \Omega(t) dt \), where \( \Omega(t) \) is the law of frequency modulation), \( \phi_0 \) is an initial phase of a signal, phase changes of a signal caused by the modulating noise (phase distortions) will be written down by a symbol \( \phi(t) \).

Let the signal affected by the modulating noise in the interval \( t \in (0, T) \) be recorded as follows:

\[
s(t) = \frac{u_n(t)}{U(t)} = \eta(t) \cos \left( \alpha_0 t + \phi(t) \right),
\]

where \( \eta(t) \) and \( \phi(t) \) are, as a rule, functionally connected stochastic functions characterizing, respectively, amplitude and phase distortions, \( U(t) \) and \( \Phi = \int \Omega(t) dt \) represent laws of change of amplitude and a phase in which change the transmitted data is concluded, \( \Psi(t) \) is an instant full phase of the processed signal \( s(t) \).

Let us write, taking into account relation (1) and the properties of the \( \delta \)-function, the joint PDF of the instantaneous values of the signal \( s(t) \), the envelope \( \eta(t) \) and the phase \( \phi(t) \) in the form [1]

\[
W(s, \eta, \phi) = W(\eta, \phi) \delta(s - \eta \cos \Psi) = \frac{1}{2\pi} W(\eta, \phi) \int_{-\infty}^{\infty} \exp \left( jx(s - \eta \cos \Psi) \right) dx.
\]

We also take into account the periodic nature of the function \( \cos \Psi \) and using the expansion

\[
\exp \left( jx\eta \cos \Psi \right) = \sum_{i=-\infty}^{\infty} J_i(x\eta) \exp \left( jI\Psi \right),
\]

where \( J_i(x\eta) \) is Bessel function of real integer-order argument, we get the following expression for the required joint PDF

\[
W(s, \eta, \phi) = \frac{W(\eta, \phi)}{\pi \sqrt{\eta^2 - s^2}} \sum_{i=-\infty}^{\infty} \exp \left( jI \left( \Psi + \arcsin \frac{s}{\eta} \right) \right), \quad s \leq \eta.
\]

To calculate the PDF of instantaneous values of the signal \( W(s) \), which has amplitude and phase distortions with PDF \( W(\eta, \phi) \), expression (2) is used.

If, at least at coincident times, the amplitude and phase fluctuations are independent, then the PDF of the instantaneous signal values will be written:
applying the results 

\[ W(s) = \int_0^\infty W(s, \eta, \varphi) d\eta d\varphi = \]

\[ = \frac{1}{\pi} \int_{\sqrt{\eta^2 - s^2}}^\infty \frac{W(\eta)}{\sqrt{\eta^2 - s^2}} d\eta + \sum_{l=0}^\infty \Theta^e(l) \exp \left( j l \Psi \right) \int_{\sqrt{\eta^2 - s^2}}^\infty \frac{W(\eta)}{\sqrt{\eta^2 - s^2}} \exp \left( j l \arcsin \frac{s}{\eta} \right) d\eta, \quad |s| \leq \eta. \]  

(3)

here \( \Theta^e \) is a one-dimensional characteristic function of the signal \( s(t) \) phase distortions, \( \Psi(t) \) is the mathematical expectation is determined by the expression \( \Psi(t) = \omega_s t + \Phi(t) + \Phi_0 \), \( \Phi_0 \), in turn, is the average value of the phase distortions \( \varphi(t) \).

If the useful signal is affected only by phase distortions, which are determined by the expression \( W(\eta) = \delta(\eta - \eta_0) \), where \( \eta_0 \) is the expected value \( \eta(t) \), then the probability density of the instantaneous signal values (3) can be represented as:

\[ W(s) = \frac{1}{\pi \sqrt{\eta_0^2 - s^2}} \left[ 1 + 2 \sum_{l=0}^\infty \Theta^e(l) \cos \left( l \left( \omega_s t + \Phi(t) \right) + l \arcsin \frac{s}{\eta_0} \right) \right], \quad |s| \leq \eta_0. \]  

(4)

In this case, the first four initial moments \( m_1(s) - m_4(s) \) for the PDF (4) given in [7].

For what follows, it is necessary to know that in expressions (3) and (4) the first terms correspond to the PDF of a signal \( s(t) \) with a phase uniformly distributed in the interval \((0, 2\pi)\). Other components, more precisely, their «weight» are determined by the characteristic function of phase distortions and the condition \( \lim_{\eta \to \infty} \Theta^e(l) \to 0 \).

In addition, you should also know that with a non-uniform phase distribution on the interval \((0, 2\pi)\), PDFs (3) and (4) depend on time, and the signal is a non-stationary function.

We introduce the integral variable \( \eta = |s| \chi \), in (3) and, assuming that the amplitude and phase of the signal \( s(t) \) are independent random functions, after a series of transformations, we obtain the following expression

\[ W(s) = W_0(s) + W_\varphi(s). \]

Here the PDF of instantaneous signal values with independent amplitude-phase distortions \( W_\varphi(s) \) is determined by the expression \( W_\varphi(s) = \frac{1}{\pi} \int_0^\infty W_\varphi(|s| \chi \eta) d\eta \). In this case, it is assumed that phase distortions are uniformly distributed in the interval \((0, 2\pi)\) or obey an arbitrary distribution law if the phase variance at \( \sigma^2_{\varphi} \gg 1 \) (the first four initial moments of this distribution have the form: \( m_1(s) = 0; m_2(s) = (1/2)m_2(\eta); m_3(s) = 0; m_4(s) = (3/8)m_4(\eta) \));

\[ W_\varphi(s) = \frac{1}{\pi} \sum_{l=0}^\infty \Theta^e(l) \exp \left( j l \Psi \right) \int_{\sqrt{\eta^2 - s^2}}^\infty \frac{W(\eta)}{\sqrt{\eta^2 - s^2}} \exp \left( j l \arcsin \frac{s}{\eta} \right) d\eta = \]

\[ = \frac{1}{\pi} \sum_{l=0}^\infty \Theta^e(l) \exp \left( j l \Psi \right) \frac{1}{j l} \int_{s}^\infty \frac{d}{ds} \left[ W(\eta) \exp \left( j l \arcsin \frac{s}{\eta} \right) \right] d\eta. \]

Let us write the following signal expression for determining the PDF of instantaneous values (3), applying the results [6] and omitting a number of mathematical transformations to find \( W_\varphi(s) \):

\[ W(s) = W_0(s) + \frac{2}{\pi} \sum_{l=0}^\infty \Theta^e(l) |W_\varphi(|s|)| \sin \left( l \left( \omega_s t + \Phi(t) + \frac{\pi}{2} + \arg \Theta^e(l) \right) \right), \quad |s| \leq \eta. \]  

(5)
From expression (5) it follows that with independent amplitude and phase distortions, when \( \sigma^2_\phi >> 1 \), a \( \theta^l_i (l \geq 1) \ll 1 \), the PDF of the signal \( W(s) \) depends little on the PDF of the phase \( W_\phi(s) \) and is determined mainly only by the PDF \( W_\theta(s) \), that is, it practically depends only on the PDF of the envelope \( W(\eta) \).

Let us write an expression for a two-dimensional PDF \( W(\eta, \phi) \) under the condition that there is a functional relationship between the amplitude and phase fluctuations of the form \( \phi = f(\eta) \):

\[
W(\eta, \phi) = W(\eta, \phi) \delta\left[ \phi - f(\eta) \right].
\]  

Substitution of (6) into (2), and a number of mathematical transformations make it possible to obtain the following expression

\[
W(s) = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} \int_{\eta=-\infty}^{\eta=\infty} W(\eta, \phi) \delta\left[ \phi - f(\eta) \right] \exp\left[ \frac{j l \left( \arcsin \frac{s}{\eta} + \Psi \right)}{\sqrt{\eta^2 - s^2}} \right] d\eta d\phi = \frac{1}{\pi} W_\eta(s|x| \log) + \frac{2}{\pi} W_\eta(|\eta|) \sum_{l=1}^{\infty} \left| l \right| \sin \left[ l \left( \omega t + \Phi(t) + \frac{\pi}{2} + f(\eta) \right) \right].
\]  

You should be aware that the PDF of the useful signal \( W(s) \) can be expressed through the statistical characteristics of the envelope in the presence of a functional relationship between the amplitude and phase.

Let us present an expression for the initial moment of the \( k \)-th order of the distribution \( W(s) \) (7)

\[
m_k(s) = \frac{1}{\pi} \int_{s=-\infty}^{s=\infty} s^k W(s) ds = m_k(s)_{\rho} + m_k(s)_{\varphi},
\]  

where

\[
m_k(s)_{\rho} = \frac{1}{\pi} \int \left[ d^k d^k \theta_i^k (\theta) \right]_{\theta=0} = \frac{1}{\pi} \int \left[ d^k d^k H_0 \left[ \frac{W(\eta)}{\eta}, \theta \right] \right]_{\theta=0}
\]

is component determined by the uniform phase distribution law, and \( H_0 \left[ \frac{W(\eta)}{\eta}, \theta \right] \) is Hankel transform, \( \theta \) is positive integer;

\[
m_k(s)_{\varphi} = \frac{4}{\pi} \left[ \left[ d^k d^k \theta_i^k (\theta) \right] \right]^{-1} \int \sum_{l=1}^{\infty} \left[ d^k d^k \theta_i^k (\theta) \right]^{-1} \left[ d^k d^k \theta_i^k (\theta) \right] \exp\left[ j l \left( \omega t + \Phi(t) + f(\eta) \right) \right]
\]

is the component determined by the difference between the phase distribution law and the uniform distribution, in which \( \theta_i^k (\theta f(\eta)) \) is the characteristic function of amplitude distortions (envelope fluctuations), \( f(\eta) \) is the derivative of the characteristics \( f(\eta) \) of the amplitude distortion function.

Analysis of relation (8) shows that the nature of amplitude distortions largely determines the average value of the signal \( s(t) \) (1). In addition to the component determined by the PDF with a uniform phase distribution law on the interval \( (0, 2\pi) \), the named ratio is also determined by the component that depends on the derivative of the characteristic of the amplitude distortion function.

3. **Influence of multiplicative noise on the PDF of the sum of the signal and additive noise**

In practically significant cases, the case should be considered when the signal affected by the multiplicative noise (1) is also affected by the additive Gaussian noise \( n(t) \) with PDF

\[
W(n) = \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left( -n^2 / 2\sigma_n^2 \right),
\]

where \( \sigma_n^2 \) is variance of additive noise.
Let us determine the PDF of the mixture of the signal and the additive noise, using the obtained relations for the PDF of the instantaneous values of the signal $W_0(s)$

$$y(t) = u_n(t) + n(t), \ 0 \leq t \leq T,$$  \hspace{1cm} (9)

where from (1) it follows $u_n(t) = U(t)s(t)$.

It is known that the PDF of the sum of random variables is determined by the expression

$$W(y) = \int W(u)W(y-u)du.$$  \hspace{1cm} (2)

Changing the variable $x = \arcsin(u_n/\eta)$, we put (9) in (2) taking into account (1) and get

$$W(y, \eta, \varphi) = \frac{W(\eta, \varphi)}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2 + \eta^2 U^2}{4\sigma_n^2}\right) \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I_{2k+p} \left(\frac{\eta U^2}{\sigma_n^2}\right) I_{2k} \left(\frac{\eta^2 U^2}{4\sigma_n^2}\right) \exp(jl\Psi).$$  \hspace{1cm} (10)

where $I_0(\alpha)$ is a modified Bessel function.

Using the well-known decomposition [9] $\exp(-a \arccos x) = \sum_{n=0}^{\infty} (-1)^n I_n(\alpha) \exp(jnx)$, and a number of transformations, we write (10) in the form:

$$W(y, \eta, \varphi) = \frac{W(\eta, \varphi)}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2 + \eta^2 U^2}{4\sigma_n^2}\right) \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} I_{2k+p} \left(\frac{\eta U^2}{\sigma_n^2}\right) I_{2k} \left(\frac{\eta^2 U^2}{4\sigma_n^2}\right) \exp(jl\Psi).$$

We use the following relationship to find the PDF of the mixture of signal and additive noise

$$W(y) = \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \Theta(l) F(y, k, l) \exp\left[jl\left[\omega_0t + \Phi(t)\right]\right],$$

where $F(y, k, l) = \int_0^\infty I_{2k+p} \left(\frac{\eta U^2}{\sigma_n^2}\right) I_{2k} \left(\frac{\eta^2 U^2}{4\sigma_n^2}\right) \exp\left(-\frac{\eta^2 U^2}{4\sigma_n^2}\right) W(\eta) d\eta; \ \Psi = \omega_0t + \Phi(t) + \varphi(t).$

Let the following expression play the role of the signal-to-noise ratio (SNR) $\rho^2 = \frac{\eta^2(t)U^2(t)}{2\sigma_n^2}$.

Since the two cases of signal-to-noise ratio are of the greatest practical interest $\rho^2 >> 1$ and $\rho^2 << 1$, we will dwell on their consideration further.

Let the SNR $\rho^2 << 1$ and the amplitude and phase distortions be independent, that is, the expression $W(\eta, \varphi) = W(\eta)W(\varphi)$.

Suppose that condition $I_0(x)\exp(-x)$ is also satisfied and, restricting ourselves to the terms of the double series with number $k = 0$, we obtain

$$W(y, \eta, \varphi) = \frac{W(\eta, \varphi)}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \sum_{l=-\infty}^{\infty} I_l \left(\frac{\eta U^2}{\sigma_n^2}\right) \exp(jl\Psi).$$  \hspace{1cm} (11)

In the absence of amplitude distortions $W(\eta) = \delta(\eta-1)$ and an arbitrary phase distribution law, we obtain
$W(y) = \frac{1}{\pi\sqrt{2\pi\sigma_n^2}} \exp\left(\frac{-y^2}{2\sigma_n^2}\right) \sum_{l=-\infty}^{\infty} \theta^l(l) I_0\left(\frac{Uy}{2\sigma_n^2}\right) \exp\left\{jl\left[\omega_d t + \Phi(t)\right]\right\}, \quad \rho^2 << 1. \quad (12)$

For the cases of deep phase distortions $\sigma_n^2 >> 1$, as well as for phase distortions uniformly distributed over the interval $(0, 2\pi)$, the expression was obtained

$W(y) = \frac{1}{\pi\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) I_1\left(\frac{Uy}{\sigma_n}\right), \quad \rho^2 << 1, \quad \sigma_n^2 >> 1.$

The expression for the PDF of the total signal with an arbitrary law of the envelope distribution has the form

$W(y) = \frac{1}{\pi\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \sum_{l=0}^{\infty} \theta^l(l) F(y,l) \exp\left\{jl\left[\omega_d t + \Phi(t)\right]\right\}, \quad \rho^2 << 1,$

where $F(y,l) = \int_0^{\infty} I_0\left(\frac{\eta Uy}{\sigma_n}\right) W(\eta) d\eta$ is integral transformation of the PDF envelope $W(\eta)$ [8].

With the SNR $\rho^2 << 1$ we can assume that $I_l\left(\frac{\eta Uy}{\sigma_n}\right) \approx \frac{1}{l!} \left(\frac{\eta Uy}{2\sigma_n}\right)^l$. Considering that $I_l(x) = I_{-l}(x)$, after the necessary transformations we get:

$F(y,l) = \frac{1}{l!} \left(\frac{y U}{2\sigma_n}\right)^l m_l(\eta),$

where $m_l(\eta)$ is initial moment of the $l$-th order.

Expression (12), taking into account the latter, will take the form:

$W(y) = \frac{1}{\pi\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \left[1 + 2\sum_{l=1}^{\infty} \frac{y U}{\sigma_n} \frac{m_l(\eta)}{l!} \Re\left[\theta^l(l) \exp\left\{jl\left[\omega_d t + \Phi(t)\right]\right\}\right]\right], \quad \rho^2 << 1.$

Let's turn to the case SNR $\rho^2 >> 1$. We use the replacement $I_{2k}\left(\frac{\eta^2 U^2}{4\sigma_n^2}\right) \approx \frac{\sigma_n}{U\eta} \sqrt{\frac{\pi}{2}} \exp\left(-\frac{\eta^2 U^2}{4\sigma_n^2}\right)$ and take into account that $\sum_k I_{2k}(x) = 0,5chx, \sum_k I_{2k+1}(x) = 0,5shx$, then for (10) we obtain the following expression

$W(y,\eta,\varphi) = \frac{1}{\pi^2\eta U} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) W(\eta,\varphi) \left[\text{ch}\left(\frac{\eta U}{\sigma_n^2}\right) + \text{sh}\left(\frac{\eta U}{\sigma_n^2}\right) \cos\varphi\right], \quad \rho^2 >> 1.$

If the distribution laws of the amplitude $W(\eta)$ and phase $W(\varphi)$ are arbitrary, then we obtain:

$W(y) = \frac{1}{\pi} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \left[\Gamma_+(yU)+\Gamma_-(yU)\right] \Re\left[\theta^0(l) \exp\left\{jl\left[\omega_d t + \Phi(t)\right]\right\}\right], \quad \rho^2 >> 1. \quad (13)$

Here

$\Gamma_+(yU) = \int_0^\infty W(\eta) d\eta + \sum_{k=1}^{\infty} \frac{1}{(2k)! \eta^2 \sigma_n^{2k}} m_{2k-1}(\eta); \quad \Gamma_-(yU) = \frac{y U}{\sigma_n^2} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)! \eta^2 \sigma_n^{2k+1}} m_{2k}(\eta), \quad (14)$
where $\Gamma_c(yU)$ and $\Gamma_s(yU)$ are functions expressed in terms of the initial moments of the signal envelope (1), while $m_{2k-1}(\eta)$ and $m_{2k}(\eta)$ are the initial moment, respectively, (2k-1)-th, 2k-th of the order of the amplitude fluctuation.

By substituting (14) into (13), one can obtain the PDF expansion of the total signal $W(y)$ (8) in terms of the PDF of a normal random signal with weight coefficients (13), which, in turn, are determined by the initial moments of the signal envelope.

Let us represent relation (13), which is valid for a uniform phase distribution law or for the dispersion of phase fluctuations $\sigma_\phi^2 >> 1$:

$$W(y) = \frac{\Gamma_c(y)}{\pi^2} \exp\left(-\frac{y^2}{2\sigma_n^2}\right), \quad \rho^2 >> 1, \quad \sigma_\phi^2 >> 1.$$ 

Let us further define the PDF of the mixture of the signal and the additive noise (9) when the condition $\varphi = f(\eta)$ and relation (6) are satisfied. In other words, when the amplitude-phase distortion of the signal $u_{ad}(t)$ is functionally related to each other.

Integration of expression (11) with respect to $\eta$ and $\varphi$ at SNR $\rho^2 << 1$ leads to the expression

$$W(y) = \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \sum_{l=0}^{\infty} \exp\left\{jl[\Psi + f(0)]\right\} \int_0^{\eta U} \frac{\eta U}{\sigma_n^2} \exp\left\{j\eta f'(0)\right\} W(\eta) d\eta.$$ 

Having performed the expansion of the Bessel function in a power series and restricting ourselves to the term $I_j\left(\frac{\eta U}{\sigma_n}\right) \approx \frac{1}{l!}\left(\frac{\eta U}{\sigma_n}\right)^l$, for the SNR $\rho^2 << 1$, and also taking into account that

$$\eta \exp\{j\eta f'(0)\} = \left[\frac{d^n}{[j\eta f'(0)]^n} \exp\{j\eta f'(0)\}\right]_{n=1},$$

we give the final expression:

$$W(y) = \frac{1}{\sqrt{2\pi\sigma_n}} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{y U}{\sigma_n}\right)^l \left[\frac{d^n}{[j\eta f'(0)]^n} \exp\{j\eta f'(0)\}\right]_{n=1} \exp\{j\eta f'(0)\}, \quad \rho^2 << 1,$$

where $\Psi(t) = \alpha_0 t + \Phi(t)$.

With the SNR equal to $\rho^2 >> 1$, as well as with $\sigma_\phi^2 > 1$ and $W(\eta, \varphi) = W(\eta) \delta[\varphi - f(\eta)]$, expression (12) takes the form:

$$W(y) = \frac{1}{\pi^2} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \left[\Gamma_c(yU) + \sum_{k=1}^{\infty} C_k \cos\left[\alpha_0 t + \Phi(t) + f(0)\right]\right], \quad \rho^2 >> 1.$$ 

For the SNR $\rho^2 << 1$, the following expression is obtained

$$W(y) = \frac{1}{\pi^2} \exp\left(-\frac{y^2}{2\sigma_n^2}\right) \left[\Gamma_c(yU) + \sum_{k=1}^{\infty} C_k \cos\left[\alpha_0 t + \Phi(t) + f(0)\right]\right], \quad \rho^2 >> 1,$$

where $C_k(y) = \frac{1}{[j\eta f'(0)]^{2k+1}} \left(\frac{y U}{\sigma_n^2}\right)^{2k+1} \left(\frac{d^{2k+1}}{[j\eta f'(0)]^n} \exp\{j\eta f'(0)\}\right)_{n=1}$. 

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Analysis of expression (15) shows that, in contrast to (13), in the case of functionally coupled amplitude-phase distortions in the expansion of the PDF of the signal mixture and additive noise according to the normal law, the weight factors are determined by the derivative of the characteristic function of amplitude distortions (fluctuations envelope). The non-stationary terms in expressions (15) and (13) differ from each other only by the presence in (13) of the initial phase determined by the coupling function \( \varphi = f(\eta) \) at the zero point.

4. Conclusions
It is shown that the average value of the signal \( s(t) \) under the influence of multiplicative noise is largely determined by the nature of the amplitude distortions. The signal value is called includes two components. The first component is determined by a PDF with a uniform phase distribution law over a given interval \((0, 2\pi)\). The second component depends on the derivative of the characteristic of the amplitude distortion function.

It is also shown that when there is a functional relationship between the amplitude and phase fluctuations of the signal, the PDF can be expressed in terms of the statistical characteristics of the envelope. In addition, for functionally related amplitude-phase distortions when decomposing a mixture of PDF signals and additive noise according to the normal law, the weighting coefficients are determined by the derivative of the characteristic function of amplitude distortions.

The scientific novelty consists in the fact that static characteristics of the probability density of instantaneous signal values with the simultaneous effect of multiplicative and additive noise for a number of practically significant cases are obtained.

The practical significance is determined by the possibility of increasing the accuracy of determining the parameters of the processed signal for various radio engineering and information-measuring systems.

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