Hyperbolicity of Semigroup Algebras

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\textit{Abstract}

Let $\mathcal{A}$ be a finite dimensional $\mathbb{Q}$-algebra and $\Gamma \subset \mathcal{A}$ a $\mathbb{Z}$-order. We classify those $\mathcal{A}$ with the property that $\mathbb{Z}^2 \not\hookrightarrow \mathcal{U}(\Gamma)$. We call this last property the \textit{hyperbolic property}. We apply this in the case that $\mathcal{A} = KS$ a semigroup algebra with $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt{-d})$. In particular, when $KS$ is semi-simple and has no nilpotent elements, we prove that $S$ is an inverse semigroup which is the disjoint union of Higman groups and at most one cyclic group $C_n$ with $n \in \{5, 8, 12\}$.

\textit{Key words:} Semigroup, Semigroup Algebras, Hyperbolic Groups, Group Rings, Units

\textsuperscript{1}This article corresponds to the second chapter of the third author's PhD Thesis, see [17]

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1. Introduction

In this paper we focus on what we call the hyperbolic property. We say that a finite dimensional \( \mathbb{Q} \)-algebra \( A \) has the hyperbolic property if for every \( \mathbb{Z} \)-order \( \Gamma \subset A \) the unit group \( U(\Gamma) \) does not contain a finitely generated abelian group of rank greater than one. This terminology is suggested by the fact that hyperbolic groups have this property \([7]\).

Research in this direction goes back to Jespers, who classified those finite groups \( G \) for which \( U(\mathbb{Z}G) \) has a non-Abelian free normal complement \([9]\). More recently Juriaans-Passi-Prasad have given contributions on this topic in the integral group ring case \([11]\), and Juriaans-Passi-Souza Filho in the group ring \( RG \) when \( R \) is the ring of algebraic integers of a quadratic rational extension \([12]\).

Here we give a complete classification of the finite semigroups whose semigroup algebra \( KS \) has the hyperbolic property with \( K = \mathbb{Q} \) or \( k = \mathbb{Q}(\sqrt{-d}) \). Part of this was done by Jespers and Wang \([10]\) who classified the finite semigroups \( S \) for which the unit group \( U(\mathbb{Z}S) \) of the integral semigroup ring \( \mathbb{Z}S \) (we of course assume that this ring contains an identity) is a finite group. Firstly, we prove a structure theorem for the finite dimensional \( \mathbb{Q} \)-algebras with the hyperbolic property. We prove that the radical of such an algebra has nilpotency index at most 2 and that its Wedderburn-Malcev components consist of copies of \( \mathbb{Q} \) or quadratic fields, totally definite quaternion algebras, two-by-two matrices over \( \mathbb{Q} \) and upper-triangular matrices over \( \mathbb{Q} \). Details on the structure of these algebras are given in section 3. In section 4, we classify the finite semigroups \( S \) whose semigroup algebras \( KS \) has the hyperbolic property, with \( K = \mathbb{Q} \) or \( K \) is a quadratic extension of \( \mathbb{Q} \). In section 5 we study the idempotents of the maximal subgroups of finite semigroups \( S \) which are not semi-simple in order to obtain a best comprehension of the structure of \( S \), when \( \mathbb{Q}S \) has the hyperbolic property.

Notation is mostly standard and we refer the reader to \([3]\) and \([13]\) for the theory of semigroup and semigroup algebra. However, for the reader’s convenience, section 2 contains some basic facts on the theory of semigroups.

2. Preliminaries

A non-empty set \( S \) with an associative binary operation \( \cdot : S^2 \to S \) is a semigroup. Let \( S \) be a semigroup, the set \( S^1 = S \cup \{1\} \), such that, \( \forall s \in S, s \cdot 1 = 1 \cdot s = s \) is a monoid, that is, a semigroup with an identity element, and the set \( S^0 = S \cup \{\theta\} \), such that, \( \forall s \in S, s \cdot \theta = \theta \cdot s = \theta \), \( \theta \) called a zero element, is a semigroup with a zero element. A semigroup \( S \) with zero \( \theta \) is a null semigroup if for all \( x, y \in S \), \( x \cdot y = \theta \). An element \( e \in S \), such that, \( e^2 = e \) is an idempotent. Denote \( E(S) := \{e \in S/e^2 = e\} \) the set of idempotents of \( S \) and let \( e, f \in E(S) \); then \( e \leq f \) if \( e \cdot f = f \cdot e = e \). An idempotent \( f \in E(S) \) is primitive if \( f \neq \theta \) and if \( e \leq f \) yields \( e = \theta \) or \( e = f \). A semigroup \( S \) is simple, if it does not properly contain any two-sided ideal. A semigroup with zero \( \theta \) is 0-simple if \( S^2 \neq \{\theta\} \) and \( \{\theta\} \) is the only proper two-sided ideal of \( S \). A 0-simple semigroup \( S \) is
Let $I$ be an ideal of $S$. The semigroup of the Rees factors, denoted by $S/I$, is the set $(S \setminus I) \cup \{\emptyset\}$ subject to the operation defined by

$$s \cdot t := \begin{cases} s \cdot t, & \text{if } s \cdot t \notin I \\ \theta, & \text{if } s \cdot t \in I \end{cases}$$

A principal series of a semigroup $S$ is a chain

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

of ideals $S_i$ of $S$, such that $S_{i+1}$ is a maximal ideal of $S_i$, $1 \leq i \leq n$. The semigroups of the Rees factors $S_i / S_{i+1}$ are called factors of this principal series. It is well known that if $S$ is a finite semigroup, then the factors of $S$ are either a null semigroup with two elements which we will call null factor or a completely 0-simple semigroup.

The semigroups $S$ which are union of groups appear naturally in the context we work. Since a semigroup which is a union of groups is the disjoint union of its maximal subgroups we have the following:

**Lemma 2.1** Let $S$ be a finite semigroup whose factors are isomorphic to groups with a zero element adjoined $\theta$, that is, $S_i / S_{i+1} \cong G^{\theta}$. Then $S$ is a disjoint union of groups.

Let $G$ be a group and $I$ and $\Lambda$ arbitrary non-empty sets. By an $I \times \Lambda$ Rees matrix, we mean an $I \times \Lambda$ matrix over $G^0$ with at most a unique entry different from $\theta$. For $a \in G$, $i \in I$ and $\lambda \in \Lambda$, $(a)_{i\lambda}$ denotes an $I \times \Lambda$ Rees matrix over $G^0$, where $a$ is the entry corresponding to row $i$ and column $\lambda$ and all other entries are zero. For any $i \in I$ and $\lambda \in \Lambda$, the expression $(a)_{i\lambda}$ denotes the $I \times \Lambda$ null matrix, which is also denoted by $\theta$.

Since we are dealing with finite semigroups, it is sufficient to consider a finite number of rows and columns, $m, n$, respectively. For $1 \leq i \leq m$ and $1 \leq j \leq n$, fix $P = (p)_{ij}$ a $m \times n$ matrix over $G^0$, called a sandwich matrix, and let $\mathcal{M}^0$ be the set of the $m \times n$ Rees matrices over $G^0$. In $\mathcal{M}^0$ we define the operation $AB = A \circ P \circ B$, where $\circ$ denotes the usual matrix product, which is binary and associative and therefore the set $\{\mathcal{M}^0, \circ\}$ is a semigroup. This semigroup is denoted by $\mathcal{M}^0(G; m, n; P)$ and $G$ is called its structural group.

In a similar way we define the Munn matrices. Let $R$ be a ring and $m, n$ positive integers. Consider $\mathcal{M}(R; m, n; P)$ the set of $m \times n$ matrices over $R$. For each $A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}(R, m, n, P)$, $1 \leq i \leq m, 1 \leq j \leq n$, addition is defined by $A + B = (a_{ij} + b_{ij})$, and multiplication by $AB = A \circ P \circ B$, where $P$ is a fixed $n \times m$ matrix with entries in $R$ and $\circ$ is the usual matrix operation. The ring $\mathcal{M}(R; m, n; P)$ is called an algebra of matrix type over $R$ or a matrix algebra over $R$.

Let $\mathcal{A}$ be a finite dimensional $\mathbb{Q}$-algebra. A unitary subring $\Gamma$ of $\mathcal{A}$ is called a $Z$-order, or simply an order, if it is a finitely generated $\mathbb{Z}$-submodule such that $\mathbb{Q}\Gamma = \mathcal{A}$, (see [16, 1.4]). Remember that, by the Borel-Chandra Theorem [1], the unit group of a $Z$-order of $\mathcal{A}$ is finitely generated and hence the hyperbolicity of $\mathcal{U}(\Gamma)$ makes sense. Furthermore, if $\mathcal{U}(\Gamma)$ is a hyperbolic group then $\mathcal{U}(\Gamma_0)$ is hyperbolic for all $Z$-order $\Gamma_0 \subset \mathcal{A}$, since the unit groups of orders are commensurable. It is known, [7], that the hyperbolicity of $\mathcal{U}(\Gamma)$ implies that $\mathbb{Z}^2 \not\subset \mathcal{U}(\Gamma)$. This suggests the following definition.
Definition 2.2 Let \( A \) be a finite dimensional \( \mathbb{Q} \)-algebra and \( \Gamma \) a \( \mathbb{Z} \)-order of \( A \). We say that \( A \) has the hyperbolic property if \( \mathbb{Z}^2 \nrightarrow \mathcal{U}(\Gamma) \).

Note that, as seen above, this definition does not depend on the particular order \( \Gamma \) of \( A \), (see [1]).

Throughout the text we use the standard notation \( \text{diag}(a_1, \cdots, a_n) \) for a \( n \times n \) matrix with elements on the main diagonal set to \( a_1, \cdots, a_n \) and all the other elements set to zero. Also \( e_{ij} \) denotes the elementary matrix whose entry is 1 in the position \( i, j \) and zero otherwise. We denote by \( T_2(\mathbb{Q}) := \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & 0 \end{pmatrix} \) with the usual matrix multiplication.

3. Finite Dimensional Algebras with the Hyperbolic Property

The main result of this section is Theorem 3.1 in which we classify the finite dimensional \( \mathbb{Q} \)-algebras which have the hyperbolic property.

Theorem 3.1 Let \( A \) be a finite dimensional \( \mathbb{Q} \)-algebra, \( \mathcal{A}_i \) a Wedderburn component of \( A \) and \( \Gamma_i \subset \mathcal{A}_i \) a \( \mathbb{Z} \)-order. Then

(i) \( A \) has the hyperbolic property, is semi-simple and without nilpotent elements if, and only if,
\[
A = \bigoplus \mathcal{A}_i,
\]
and for at most one index \( i_0 \) we have that \( \mathcal{U}(\Gamma_{i_0}) \) is hyperbolic and infinite.

(ii) \( A \) has the hyperbolic property and is semi-simple with nilpotent elements if, and only if,
\[
A = (\bigoplus \mathcal{A}_i) \oplus M_2(\mathbb{Q}).
\]

(iii) \( A \) has the hyperbolic property and is non-semi-simple with central radical if, and only if,
\[
A = (\bigoplus \mathcal{A}_i) \oplus J.
\]

(iv) \( A \) has the hyperbolic property and is non-semi-simple with non central radical if, and only if,
\[
A = (\bigoplus \mathcal{A}_i) \oplus T_2(\mathbb{Q}).
\]

For each item (i) – (iv), the \( \mathcal{A}_i \)'s are either at most a quadratic imaginary extension of \( \mathbb{Q} \) or a totally definite quaternion algebra. Furthermore, in the decompositions in (i) – (iv) above the direct summands are ideals.

We will consider \( A \) a finite dimensional \( \mathbb{Q} \)-algebra with radical \( J(A) \). According to a theorem of Wedderburn-Malčev, there exists a semi-simple subalgebra \( S(A) \) of \( A \) such that
\[
A = S(A) \oplus J(A), \text{ as a direct sum of vector spaces,}
\]
with \( J(A) \) the Jacobson radical.
Lemma 3.2 Let $\mathcal{A} = S(\mathcal{A}) \oplus J(\mathcal{A})$ be a finite dimensional $\mathbb{Q}$-algebra with the hyperbolic property. Then $I^2 = 0$, $\dim_{\mathbb{Q}}(J(\mathcal{A})) \leq 1$, that is, as a vector subspace of $\mathcal{A}$, $J = J(\mathcal{A})$ has dimension at most 1. Furthermore, if $J \neq 0$, there exists $j_0 \in \mathcal{A}$ such that $j_0^2 = 0$, $J = (j_0)_{\mathbb{Q}}$, and $1 + J \cong \mathbb{Q}$ as a multiplicative group.

PROOF. Obviously $1 + J$ is a multiplicative torsion free nilpotent group. Let $G$ be any finitely generated subgroup of $1 + J$. Hence $Z(G) \neq 1$. Since $\mathbb{Z}^2 \not\subset (1 + J)$ the same holds for $G$, so $G = Z(G) \cong \mathbb{Z}$. Since $J$ is a nilpotent ideal, there exists a least positive integer $n$, $J^n = 0 = JJ^{n-1}$ thus $1 + J = Z(1 + J) = Z(1 + J^{n-1}) \cong \mathbb{Z}$. Hence $n = 2$ and $\dim_{\mathbb{Q}}J \leq 1$. If $J \neq 0$ and $\Gamma \subset A$ is a $\mathbb{Z}$-order, let $x, y \in J \cap \Gamma$. Then the group $(1 + x, 1 + y) < U(\Gamma)$, and $1 + x, 1 + y$ are units of infinite order. Since $U(\Gamma)$ is hyperbolic we have $(1 + x, 1 + y) \cong \mathbb{Z}$. Hence $(1 + x) \cap (1 + y)$ is non-trivial and there exist integers $m, n$, such that, $(1 + x)^n = (1 + y)^m$. Since $x, y$ are 2-nilpotent, we have $1 + mx = 1 + ny$, and thus $x = \frac{m}{n}y$. So $x, y$ is a $\mathbb{Q}$-linear dependent set and we conclude that $\dim_{\mathbb{Q}}(J) = 1$. Write $J = Qj_0$, so $1 + J \cong \mathbb{Q}$. Indeed, $\phi : 1 + J \to \mathbb{Q}, \phi(1 + qj_0) = q$ is an isomorphism.

If $S_1(\mathcal{A})$ and $S_2(\mathcal{A})$ are subalgebras of $\mathcal{A}$ such that $\mathcal{A} = S_1(\mathcal{A}) \oplus J(\mathcal{A})$, $l = 1, 2$, then there exists $n \in J(\mathcal{A})$ such that $S_1 = (1 - n)S_2(\mathcal{A})(1 - n)^{-1}$, hence the semi-simple subalgebra $S(\mathcal{A})$ is unique, up to isomorphism [4, Theorem 72.19]. Choose a semi-simple subalgebra $S(\mathcal{A})$ of $\mathcal{A}$ and let $E(\mathcal{A}) = \{ E_1, \cdots, E_N \}$, $N \in \mathbb{Z}^+$ be the set of orthogonal central primitive idempotents of the semi-simple subalgebra $S(\mathcal{A})$ such that $\mathcal{A}_l := S(\mathcal{A})E_l$, we have $S(\mathcal{A}) \cong \oplus \mathcal{A}_l$. The algebra $\mathcal{A}$ can be written as a finite direct sum of indecomposable two-sided ideals, [4, Theorem 55.2]. Let

$$\mathcal{A} = ( \bigoplus_{E_i \in E(\mathcal{A})} S(\mathcal{A})E_i ) \oplus J(\mathcal{A})$$

be its Wedderburn-Malcev decomposition.

Proposition 3.3 Let $\mathcal{A}$ be a finite dimensional non-semi-simple $\mathbb{Q}$-algebra with $\dim_{\mathbb{Q}}(J(\mathcal{A})) = 1$, $J(\mathcal{A}) = (j_0)$ and $N = |E(\mathcal{A})|$. The following conditions hold:

(i) For all $x \in \mathcal{A}$, there exist $\lambda_x, \mu_x \in \mathbb{Q}$, such that, $xj_0 = \lambda_xj_0$ and $j_0x = \mu_xj_0$.

(ii) If $x$ is an idempotent, then $\lambda_x, \mu_x \in \{0, 1\}$.

(iii) There exist unique $E, F \in E(\mathcal{A})$, such that $Ej_0 \neq 0$ and $j_0F \neq 0$.

(iv) If $E = F$ then $J$ is central.

(v) If $J$ is non-central then, up to an index reordering, we can suppose that $E = E_1$, and $F = E_N$ and $E_1j_0 = j_0E_N = j_0$. In particular, in this case $N \geq 2$.

PROOF. Since $J$ is a two-sided ideal of $\mathcal{A}$, for $x \in \mathcal{A}$ we have $x \cdot j_0 \in (j_0)_{\mathbb{Q}}$, hence there exists $\lambda_x \in \mathbb{Q}$, such that, $x \cdot j_0 = \lambda_xj_0$. Similarly, for some $\mu_x \in \mathbb{Q}$, $j_0 \cdot x = \mu_xj_0$.

If $x$ is an idempotent then $x \cdot j_0 = \lambda_xj_0 = x^2 \cdot j_0 = x \cdot (x \cdot j_0) = x \cdot (\lambda_xj_0) = \lambda_x^2j_0 \cdot x$ and hence $(\lambda_x^2 - \lambda_x)j_0 = 0$. It follows that $\lambda_x^2 = \lambda_x = 0$ and so $\lambda_x \in \{0, 1\}$, in a similar way $\mu_x \in \{0, 1\}$.

Since $1 = \sum_{1 \leq i \leq N} E_i$ we have that $1 \cdot j_0 = \sum_{1 \leq i \leq N}(E_i \cdot j_0) = \sum_{1 \leq i \leq N}(\lambda_i j_0) = (\sum_{1 \leq i \leq N}\lambda_i)j_0$, and so, $\sum_{1 \leq i \leq N} \lambda_i = 1$. Since each $E_i$ is an idempotent, we have that $\lambda_i \in \{0, 1\}$. This yields the existence of a unique index, $m$, say, $1 \leq m \leq N$, such that,
\(\lambda_m = 1\) and \(\lambda_i = 0\) if \(i \neq m\). Similarly there exists a unique \(k, 1 \leq k \leq N\), such that, 
\[j_0 \cdot E_k = j_0 \text{ and } j_0 \cdot E_i = 0 \text{ for } i \neq k.\]

If \(E = F, E_m \cdot j_0 = E_k \cdot j_0\) then, by uniqueness, \(m = k\) and \(j_0 \cdot E_m = E_m \cdot j_0 = j_0\). On the other hand, for \(i \neq k\), we have that \(j_0 \cdot E_i = E_i \cdot j_0 = 0\). Therefore \(J\) commutes with \(S(A)\) and thus it is central. The other statements are now also clear. \(\square\)

**Corollary 3.4** Let \(A\) be a finite dimensional non-semi-simple \(\mathbb{Q}\)-algebra with the hyperbolic property. Then \(J(A) = \langle j_0 \rangle\) is central in \(A\) if, and only if, there exists a unique \(E \in E(A)\) such that \(j_0 \cdot E = E \cdot j_0 = j_0 \cdot E = 0\).

**Remark 3.5** Denote by \(M\) the left annihilator of \(J\) in \(A_1\). Since \(\dim_{\mathbb{Q}}(J) = 1\), it follows that \(M\) is a proper ideal of \(A_1\). If \(x \in A_1\), then there exists \(\lambda_x \in \mathbb{Q}\), such that 
\[x \cdot j = \lambda_x j_0, \text{ so } x = (x - \lambda_x E_1) + \lambda_x E_1\] and thus \(A_1 = M \oplus \mathbb{Q}E_1\) which implies that \(\dim_{\mathbb{Q}}(M) + 1 = \dim_{\mathbb{Q}}(A_1)\). Since \(A\) is a simple \(\mathbb{Q}\)-algebra it must be that \(M = \{0\}\) and \(\dim_{\mathbb{Q}}(A_1) = 1\). Analogously, we obtain \(A_N \cong \mathbb{Q}\).

Let \(A\) be a rational finite dimensional algebra with the hyperbolic property, \(N = |E(A)| \geq 2\), and \(E_1, E_N \in E(A)\) the idempotents with the property \(E_1 \cdot j_0 = j_0 \cdot E_N = j_0\). We have:
\[
S(A) = (\oplus_{1 \leq i < N} S(A) \cdot E_i) \oplus (S(A) \cdot E_1 \oplus S(A) \cdot E_N) = B \oplus C.
\]

Letting \(A_i = S(A) \cdot E_i, 1 \leq i \leq N\), we have: \(A \cong B \oplus A_1 \oplus A_N \oplus J(A)\). Considering the map
\[
\varphi : A_1 \oplus A_N \oplus J(A) \to T_2(\mathbb{Q})
\]
\[
\begin{align*}
& a_1 E_1 + a_N E_N + q j_0 \mapsto \text{diag}(a_1, a_N) + q e_{12}.
\end{align*}
\]

Clearly \(\varphi\) is an algebra isomorphism. Hence \(A_1 \oplus A_N \oplus J(A) \cong T_2(\mathbb{Q})\).

Thus we proved the next theorem:

**Theorem 3.6** Let \(A\) be a finite dimensional non-semi-simple \(\mathbb{Q}\)-algebra with the hyperbolic property. If \(J(A) = \langle j_0 \rangle\) is non-central, then, up to a reordering, we have \(E_1 \cdot j_0 = j_0 \cdot E_N = j_0\), and \(E_N \cdot j_0 = j_0 \cdot E_1 = 0\). For the others idempotents \(E_i, i \notin \{1, N\}\) we have that \(E_i \cdot j_0 = j_0 \cdot E_i = 0\). Moreover, 
\[
A_1 \oplus A_N \oplus J(A) \cong T_2(\mathbb{Q})
\]
is an ideal of \(A\).

**Corollary 3.7** Let \(A\) be a finite dimensional \(\mathbb{Q}\)-algebra with the hyperbolic property, and \(|E(A)| = N\). \(J\) is non-central if and only if 
\[
A \cong B \oplus T_2(\mathbb{Q}) \cong \text{diag}(A_2, \ldots, A_{N-1}) \oplus T_2(\mathbb{Q}) \cong \text{diag}(\mathbb{Q}, A_2, \ldots, A_{N-1}, \mathbb{Q}) \oplus \mathbb{Q} e_{1N}.
\]

Moreover, for each \(1 \leq i \leq N\), \(A_i\) is at most a quadratic imaginary extension of \(\mathbb{Q}\), or a totally definite quaternion algebra.
PROOF. By the previous theorem, $B$ and $T_2(\mathbb{Q})$ are ideals whose direct sum equals $\mathcal{A}$. Consider the algebra isomorphism

$$\varphi : \text{diag}(A_2, \cdots, A_{N-1}) \oplus T_2(\mathbb{Q}) \rightarrow \text{diag}(\mathbb{Q}, A_2, \cdots, A_{N-1}, \mathbb{Q}) + \mathbb{Q}\mathfrak{e}_{1N}$$

$$\text{diag}(a_2, \cdots, a_{N-1}) \oplus \text{diag}(q_1, q_N) + q\mathfrak{e}_{12} \rightarrow \text{diag}(q_1, a_2, \cdots, a_{N-1}, q_n) + q\mathfrak{e}_{1N}$$

Let $\Gamma_0 \subset \mathcal{A}$ be a $\mathbb{Z}$-order, with $E_1 + E_2 + \cdots + E_N + j_0 = 1 + j_0 \in \mathcal{U}(\Gamma_0)$. By Lemma 3.2, $(1 + j_0) \cong \mathbb{Z}$. Suppose $\gamma_{i_0} E_{i_0} \in \Gamma_{i_0}$ is an element of infinite order and set $\gamma = E_1 + \cdots + \gamma_{i_0} E_{i_0} + \cdots + E_N, \ 1 \neq i_0 \neq N$. We have $o(\gamma) = \infty$, $(1 + j_0) \cong \mathbb{Z}$ and $(1 + j_0) \cap \langle \gamma \rangle = \{1\}$. Since $A_{i_0} \subset C_A(J)$, the centralizer of $J \subset \mathcal{A}$, we have that $(1 + j_0) \times \langle \gamma \rangle \cong \mathbb{Z}^2$ is a subgroup of $\mathcal{U}(\Gamma_0)$, a contradiction. Therefore $\mathcal{U}(\Gamma_i)$ is a torsion group and hence is finite. Obviously $|\mathcal{U}(\Gamma_i) \cong \mathcal{U}(\Gamma_N)| \leq 2$ since, by the previous theorem, $\mathcal{A}_{1} \cong \mathcal{A}_{N} \cong \mathbb{Q}$. Therefore, by Lemma 2.3 of [16], each $\mathcal{A}_i$ is at most an imaginary extension of $\mathbb{Q}$, or a totally definite quaternion algebra. To prove the converse it is enough to consider the right and left action of $J$ on the semi-simple part of $\mathcal{A}$. □

If $\mathcal{A}$ has the hyperbolic property and the radical $J \neq \{0\}$ is central, then $\mathcal{S}(\mathcal{A})$ is a direct sum of division rings: in fact, if any component of $\mathcal{S}(\mathcal{A})$ were of matrix type it would have an element of infinite order. Hence once again we could embed $\mathbb{Z}^2 \rightarrow \mathcal{U}(\Gamma)$, for some $\mathbb{Z}$-order $\Gamma$. Therefore the simple components $\mathcal{A}_i, 1 \leq i \leq N$, of $\mathcal{S}(\mathcal{A})$ are as in the corollary above.

Corollary 3.8 Let $\mathcal{A}$ be a finite dimensional $\mathbb{Q}$-algebra with the hyperbolic property, and $|\mathcal{E}(\mathcal{A})| = N \geq 2$. If $J$ is central then $\mathcal{U}(\Gamma_i)$ are finite subgroups. Moreover, if $\Gamma \subset \mathcal{A}$ is a $\mathbb{Z}$-order then $\mathcal{U}(\Gamma)$ is commensurable with $\mathbb{Z} \times C_2 \times C_2 \times \prod H_i, |H_i| < \infty$. In particular, $\mathcal{U}(\Gamma) = \Phi(\mathcal{U}(\Gamma)), \text{ the finite conjugacy center of } \mathcal{U}(\Gamma)$.

Remark 3.9 Let $\mathcal{A}$ be a ring of characteristic zero and let $\theta_1, \theta_2 \in \mathcal{A}$ be commuting 2-nilpotent elements. If $\{\theta_1, \theta_2\}$ is $\mathbb{Z}$-L.I. then $\mathbb{Z}^2$ embeds into $\mathcal{U}(\mathcal{A})$.

Now we are ready to proof the main result of this section.

PROOF. (of Theorem 3.1) Items (iii) and (iv) follow from Theorem 3.6 and its corollary.

We now prove (2): as $\mathcal{A}$ is semi-simple with nilpotent elements we have that $\mathcal{A} \cong \oplus M_{n_i}(D_i)$, where the $D_i$’s are division rings. Remark 3.9 implies that $n_i \leq 2, \forall i$. The hyperbolicity hypothesis implies that there is at most one component with $n_i = 2$ and it is isomorphic to $M_2(\mathbb{Q})$ (this follows by Remark 3.9). Let $\Gamma_i \subset \mathcal{A}_i$ be a $\mathbb{Z}$-order of $\Gamma_i$ and consider the $\mathbb{Z}$-order $\Gamma_0 = M_2(\mathbb{Z}) \oplus (\oplus_{i \neq i_0} \Gamma_i) \subset \mathcal{A}$. We have that $\mathcal{U}(\Gamma_0) \cong GL_2(\mathbb{Z}) \times (\prod \mathcal{U}(\Gamma_i))$. It follows that all $\mathcal{U}(\Gamma_i)$ are torsion groups and hence they are finite.

The converse is straightforward, since $GL_2(\mathbb{Z})$ is hyperbolic.

We now prove (i): if $\mathcal{A}$ is semi-simple with no nilpotent elements then $M_2(\mathbb{Q})$ is not a Wedderburn component of $\mathcal{A}$ and hence $\mathcal{A} \cong \oplus \mathcal{A}_i$, a direct sum of division rings. If for any $\mathbb{Z}$-order $\Gamma \subset \mathcal{A}$ it holds that $\mathcal{U}(\Gamma)$ is finite we are done. Suppose $|\mathcal{U}(\Gamma)| = \infty$. Let $\Gamma = \oplus \Gamma_i$; then $\mathcal{U}(\Gamma) \cong (\prod \mathcal{U}(\Gamma_i))$. The hyperbolicity of $\mathcal{U}(\Gamma)$ implies that there can be at most one index $i_0$ for which $\mathcal{U}(\Gamma_{i_0})$ is infinite and hence we are done. The converse is obvious. □

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Remark 3.10 Let \( A \) be a finite dimensional non-semi-simple \( \mathbb{Q} \)-algebra with the hyperbolic property and \( J \) its radical. If \( a \in A \) is a non-trivial nilpotent element then \( a \in J \).

In fact, by Theorem \( \tau \), \( A \cong B \oplus T_2(\mathbb{Q}) \) (respectively \( A \cong B \oplus J \)) if \( J \) is non-central, (respectively if \( J \) is central). Since each \( A_i, 1 < i < N \), is a division ring, hence \( a \in T_2(\mathbb{Q}) \) (respectively \( a \in J \)). It is sufficient to consider the case for \( J \) non-central. Let \( a = \text{diag}(x, z) + ye_{12}; a^2 = 0 \) yields \( x = z = 0 \), and \( y \in \mathbb{Q} \). Therefore, \( a = ye_{12} \in J \).

4. Semigroup Algebras with the Hyperbolic Property

In this section we classify the finite semigroups \( S \) for which \( \mathbb{Q}S \) has the hyperbolic property, we also classify the extensions \( K = \mathbb{Q}(\sqrt{-d}) \) with this property. First some terminology: a finite group \( G \) is called a Higman group if \( G \) is either abelian of exponent dividing 4 or 6 or a Hamiltonian 2-group. Recall that nilpotent free means the absence of nilpotent elements and \( Q_{12} \cong C_3 \times C_4 \) where \( C_4 \) acts by inversion on \( C_3 \).

Let \( K \) be a field and \( S \) a semigroup. By the semigroup algebra \( KS \) of \( S \) over \( K \) we mean an algebra \( A \) over \( K \) which contains a subset \( S \) that is a \( K \)-basis and a multiplicative semigroup of \( A \) isomorphic to \( S \). Let \( S \) be a semigroup with a zero element \( \theta \). By the contracted semigroup algebra \( K_0S \) of \( S \) we mean an algebra over \( K \) with a basis \( B \), such that, \( B \cup \{ \theta \} \) is a subsemigroup of \( K_0S \) isomorphic to \( S \). If \( S \) is a Rees matrix semigroup, \( S = M^0(G; m, n; P) \), then the contracted algebra \( K_0S \cong M(KG; m, n; P) \), [3, Lemma 5.17].

We suppose the algebra \( KS \) has a unity. By [13, Corollary 5.26], if \( S = M^0(G; m, n; P) \) is a Rees matrix semigroup then the following conditions are equivalent:

(i) \( Q_0S \) is unitary;
(ii) \( m = n \) and \( P \) is an invertible matrix in \( M_n(\mathbb{Q}G) \).

If a structural group \( G = \{1\} \) is trivial then, up to isomorphism, there exist exactly two Rees matrix semigroup \( S = M^0(\{1\}; 2, 2; P) \) with \( Q_0S \) a unitary ring. In the following remark, we exhibit these semigroups since they appear as factors of a principal series of \( S \) when \( QS \) has the hyperbolic property and contains nilpotent elements.

Remark 4.1 There are four possibilities for the Rees semigroup \( M^0(\{1\}; 2, 2; P) \), where \( P \) is invertible, whose elements are the elementary matrices \( e_{11}, e_{12}, e_{21}, e_{22} \) and the null matrix \( \theta \) with the semigroup operation \( \cdot \).

Let \( H = \langle \sigma \rangle \) be the group generated by the transposition \( \sigma = (12) \) and \( U = \sum_{i,j=1}^2 e_{ij} \).

- The semigroup \( M^\sigma = M^0(\{1\}; 2, 2; I_\sigma), \phi \in H \) such that if \( \phi \) is the identity then \( I_\phi \) is the identity matrix and if \( \phi = \sigma \) then \( I_\phi = e_{12} + e_{21} \). Note that in \( M^\sigma \), \( e_{ij} \cdot e_{kl} := e_{ij} \delta_{j(k)} \).
- The semigroup \( M^n = M^0(\{1\}; 2, 2; I^n) \) with \( I^n = U - e_{\sigma(n)n}, n = 1, 2 \). Note that in \( M^n \), \( e_{in} \cdot e_{\sigma(n)n} = e_{ij} \), and \( e_{ij} \cdot e_{kl} := e_{il} \delta_{jk} \) otherwise.

If \( \phi \) is the identity of \( H \) then we denote \( M = M^\phi \). Clearly the maps

\[
\varphi : M \rightarrow M^\sigma \quad \text{and} \quad \phi : M^1 \rightarrow M^2
\]

\[
e_{ij} \mapsto e_{i\sigma(j)} \quad \text{and} \quad e_{ij} \mapsto e_{\sigma(i)\sigma(j)}
\]
are isomorphisms, hence $M^σ ∼ M^φ ∼ M^2$ although $Q_0M ∼ Q_0M^1 ∼ M^2(Q)$. Clearly, the Rees matrix semigroup $M$ and $M^1$ are generated by nilpotent elements.

In the sequel we shall make free use of the following results:

(i) Every periodic 0-simple semigroup (in particular any finite semigroup) is completely 0-simple, [3, Corollary 2.56]. Hence, by Rees Theorem, a 0-simple semigroups is isomorphic to some Rees matrix semigroup.

(ii) Let $S$ be a finite simple semigroup, if $KS$ is semi-simple then $S$ is a group, [3, Corollary 5.24].

(iii) $QS$ is semi-simple if, and only if, $Q(S_i/S_{i+1})$ is semi-simple for each factor of $S$, [3, Theorem 5.14].

(iv) Let $QS$ be semi-simple. If $S_i/S_{i+1}$ is a factor of $S$ then $S_i/S_{i+1}$ is isomorphic to a Rees matrix semigroup.

Let $S$ be a finite semigroup, $a, b ∈ S$ are inverses if $aba = a$ and $bab = b$. An inverse semigroup is a semigroup whose non-zero elements have a unique inverse. Suppose $ZS$ has an identity, $U(ZS)$ is finite if, and only if, $S$ is an inverse semigroup which is the disjoint union of groups which are finite Abelian groups of exponent dividing 4 or 6 or 2-Hamiltonian groups [10, Theorem 6.1]. Clearly, for such semigroups the hyperbolic property holds.

We shall now start a classification of all finite semigroup whose semigroup algebra over $Q$ has the hyperbolic property. In what follows we suppose that $ZS$ has a unit. Recall that $S^θ$ is nilpotent if there exists $n ∈ Z^+$, such that, $S^n = \{θ\}$. If $s ∈ S$ and $s^m = θ$, then $s$ is called m-nilpotent element, or nilpotent. We use the expression “nilpotent free” to indicate the absence of non-trivial nilpotent elements.

**Lemma 4.2** Let $S$ be a finite semigroup. Then $QS$ is nilpotent free if, and only if, $S$ admits a principal series whose factors are isomorphic to maximal subgroups $G$, say, of $S$ and $QG$ is nilpotent free. In particular, $S$ is the disjoint union of its maximal subgroups.

**PROOF.** It is a consequence of [3, Lemma 5.17] and Lemma 2.1. □

**Theorem 4.3** The algebra $QS$ is nilpotent free and has the hyperbolic property if, and only if, $S$ admits a principal series for which every factor is isomorphic to one of the groups below:

(i) A Higman group;

(ii) One of the following cyclic groups: $C_5, C_8$ or $C_{12}$.

Furthermore, at most one of the groups of type (ii) occurs. Moreover, $S$ is an inverse semigroup and it is the disjoint union of groups of type (i) with at most one group of type (ii).

**PROOF.** Since $QS$ is nilpotent free, by the previous lemma, $S$ has a principal series $S = S_1 ⊃ S_2 ⊃ ⋯ ⊃ S_{n+1} = \emptyset$ whose factors $S_i/S_{i+1} ∼ G_i$, a group, and $S ∼ UG_i$. Thus $QS^θ = (⊕ QG_i) ∗ Qθ$ and $Γ = (Π ZG_i) × Zθ$ is an order of $QS^θ$. If $QS$ has the hyperbolic property, by Theorem 3.1 item (i), $QS ∼ ⊕ A_i$, where at most one component $A_i$ admits a $Z$-order $Γ_i$ such that the group $U(Γ_i)$ is hyperbolic infinite. Hence, by [10, Theorem 6.1], the groups $G_i, i ≠ i_0$ are Higman groups and, by [11, Theorem 3], $G_i, i_0 ∈ \{C_5, C_8, C_{12}\}$.
Obviously, $\mathcal{U}(\mathbb{Z}S^0) \cong_{\phi} (\prod \mathcal{U}(\mathbb{Z}_{0}G_i)) \times \mathcal{U}(\mathbb{Z}\theta)$, where $\phi$ is an isomorphism. If $\theta \neq x \in S$ then $x \in G$ one of the maximal subgroup of $S$, suppose $a, b \in S$ are inverses of $x$, then $xax = xbx = x \Rightarrow \phi(xax) = \phi(xbx)$ hence $a = b$ and $S$ is an inverse semigroup.

Conversely, if $S$ is a semigroup with a principal series whose factors $S_i/S_{i+1} \cong G_i$ then $Q_0S \cong \oplus Q_0(S_i/S_{i+1}) \cong \oplus Q_0G_i$. Consider the order $\Gamma$ previously defined. By hypothesis, we have at most a unique cyclic group $G_{i_0}$, say, of order 5, 8 or 12 and all other $\mathcal{U}(\mathbb{Z}G_i), i \neq i_0$ are trivial. Therefore, by Theorem 3.1 item (i), $\mathcal{Q}S$ has the hyperbolic property. □

An algebra $\mathcal{A}$ with the hyperbolic property and which has nilpotent elements may be semi-simple or not. If it is semi-simple then, by Theorem 3.1 its Wedderburn-Mal’cev decomposition has a unique component isomorphic to $M_2(\mathbb{Q})$. For any other component the unit group of every $\mathbb{Z}$-order of this component is a finite group. In the next theorem we classify the finite semigroups whose rational semigroup algebra has these properties.

**Theorem 4.4** Let $\mathcal{Q}S$ be a unitary algebra with nilpotent elements. Then $\mathcal{Q}S$ is semi-simple and has the hyperbolic property if, and only if, $S$ has a principal series with all factors, except for one, isomorphic to Higman groups. The exceptional one is isomorphic to a semigroup $K$ of the following type:

(i) $K \in \{S_3, D_4, Q_{12}, C_4 \times C_4 : C_4 \text{ acts non trivially on } C_4\}$;

(ii) $K \in \{\mathcal{M}^0(\{1\}; 2, 2; I_d) = M\quad , \quad \mathcal{M}^0(\{1\}; 2, 2; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = M^1\}$.

In particular, if $K$ is a group then $S$ is the disjoint union of Higman groups and $K$.

**PROOF.**

If $\mathcal{Q}S$ is semi-simple then $Q_0(S_i/S_{i+1}) \cong M_{n_i}(\mathbb{Q}G_i)$. If $\mathcal{Q}S$ has the hyperbolic property then, by Theorem 3.1 item (ii), $\mathcal{Q}S \cong (\oplus \mathcal{A}_i) \oplus M_2(\mathbb{Q})$. Thus, for a unique index $i_0$, either $Q_0(S_{i_0}/S_{i_0+1}) \cong M_2(\mathbb{Q})$ or $M_2(\mathbb{Q})$ is the unique matrix epimorphic image of $Q_0(S_{i_0}/S_{i_0+1})$. As a consequence, either $S_{i_0}/S_{i_0+1} \cong \mathcal{M}(\{1\}; 2, 2; P)$ or $S_{i_0}/S_{i_0+1} \cong K \in \{S_3, D_4, Q_{12}, C_4 \times C_4\}$. For each $i \neq i_0$, $n_i = 1$ and $S_i/S_{i+1} \cong G_i$ a Higman group.

Conversely, if $S$ has a principal series as described then $\mathcal{Q}S^0 \cong (\oplus \mathcal{Q}G_i) \oplus (\oplus M_{i}(\mathbb{Q})) \cong \oplus Q_0(S_{i_0}/S_{i_0+1})$, where $G_i \cong S_i/S_{i+1}$ is a Higman group for every $i \neq i_0$ and $K \cong S_{i_0}/S_{i_0+1}$ the exceptional factor. Since either $Q_0K \cong (\oplus B_i) \oplus M_2(\mathbb{Q})$ or $Q_0K \cong M_2(\mathbb{Q})$ we have

$$Q_0S \cong (\oplus QG_i) \oplus ((\oplus B_i) \oplus M_2(\mathbb{Q})) = (\oplus \mathcal{A}_i) \oplus M_2(\mathbb{Q}).$$

Clearly, if $\Gamma_i$ is a $\mathbb{Z}$-order of $\mathcal{A}_i$ then $\mathcal{U}(\Gamma_i)$ is finite. Thus by Theorem 3.1 item (ii) the algebra $\mathcal{Q}S$ has the hyperbolic property. □

**Proposition 4.5** Let $\mathcal{S}$ be a finite semigroup, such that, $J(\mathcal{Q}S) = \mathcal{Q}\langle j_0 \rangle$ for some $j_0 \in \mathcal{Q}S$, and $j_0^2 = 0$. Then for each $s \in \mathcal{S}$, there exists $\lambda_s, \rho_s \in \{-1, 0, 1\}$ such that $s \cdot j_0 = \lambda_s j_0$ and $j_0 \cdot s = \rho_s j_0$.

**PROOF.** Since $J(\mathcal{Q}S) := J = \mathcal{Q}\langle j_0 \rangle$ is an ideal, for $s \in \mathcal{S}$ there exists $\lambda_s \in \mathbb{Q}$, such that, $s \cdot j_0 = \lambda_s j_0$. For the cyclic semigroup generated by $s$ there exists $n \in \mathbb{N}$ such
that \( e := s^n \) is an idempotent \([3, \S 1.6]\). Inductively, we obtain that \( s^k \cdot j_0 = \lambda_s^k j_0 \) and by Proposition \([3.3](ii)\) it follows that \( \lambda_s \in \{-1, 0, 1\} \). In a similar way we obtain that \( \rho_s \in \{-1, 0, 1\} \). \( \square \)

If \( S \) is a finite semigroup which is non-semi-simple then, according to \([3, \text{Corollary 5.15}]\), every principal series of \( S \) admits a null factor (a null semigroup with two elements).

**Lemma 4.6** Let \( S \) be a finite non-semi-simple semigroup. If \(QS\) has the hyperbolic property then \( S \) has a principal series with a unique null factor of \( S \). Moreover, if \( S_{\lambda}/S_{\lambda+1} := \{\theta, f\} \) is the null factor then either \( f \) is a nilpotent of \( \tilde{S} := S/S_{\lambda+1} \) and \( f \) is not a nilpotent of \( S \), or \( f \) is a nilpotent of \( S \), \( \mathcal{I} := \{\theta, f\} \) is an ideal of \( S \), \( S/\mathcal{I} \) is semi-simple and \( J = \mathcal{Q}(f) \). In each case, \( f \) is the unique nilpotent element of the semigroup.

**PROOF.** Let \( S^\theta = S_1 \supset S_2 \supset \cdots \supset S_n = \{\theta\} \supset \emptyset \) be a principal series of \( S \). We have for each factor \( S_i/S_{i+1} \) and \( J(Q_0(S_i/S_{i+1})) \), the radical of \( Q_0(S_i/S_{i+1}) \), that

\[
(QS)/J(QS) \cong (Q_0(S_i/S_{i+1}))/J(Q_0(S_i/S_{i+1})).
\]

Thus, if \( S_{\lambda}/S_{\lambda+1} \) is a null factor then \( Q_0(S_{\lambda}/S_{\lambda+1}) \subseteq J(QS) \). Suppose that \( S_{\lambda}/S_{\lambda+1} \) is another null factor of \( S \), clearly if \( x_l \in S_{\lambda}/S_{\lambda+1}, l = 0, 1, \) then \( (1 + x_0, 1 + x_1) \cong \mathbb{Z}^2 \), which is contrary to hyperbolic property of \( QS \).

If \( f \) is not nilpotent in \( S \) then \( f^2 \in S_{\lambda+1} \setminus \{\theta\} \) and \( f \) is nilpotent in \( \tilde{S} := S/S_{\lambda+1} \). If \( f \) is nilpotent in \( S \), by Remark \([3.10]\), \( f \in J \) and consequently \( J = \mathcal{Q}(f) \). We claim that \( \mathcal{I} := \{\theta, f\} \) is an ideal of \( S \). In fact, if \( s \in S \) then \( sf \in J \) and hence, by the previous proposition, \( sf = \theta \) or \( sf = f \) and so \( sf \in \{\theta, f\} \). Similarly we have that \( fs \in \{\theta, f\} \). Since in each case, \( f, s \in S_{\lambda}/S_{\lambda+1} \) the unique null factor of the principal series of \( S \), clearly \( f \) is the unique nilpotent element. \( \square \)

**Proposition 4.7** Let \( S \) be a finite semigroup that admits a nilpotent element \( j_0 \in S \). \( QS \) is non-semi-simple and has the hyperbolic property if, and only if, \( \mathcal{I} := \{\theta, j_0\} \) is an ideal of \( S \) and \( S/\mathcal{I} \) has a principal series whose factors are isomorphic to Higman groups. In particular, \( S/\mathcal{I} \) is the disjoint union of its maximal subgroups.

**PROOF.** We have that \( QS \cong S(QS) \oplus J \) with non-trivial \( J \). Since \( QS \) has the hyperbolic property we have, by Theorem \([3.1]\), that \( QS \cong (\oplus A_i) \oplus X \), where \( X \in \{J, T_2(Q)\} \) depending on the centrality of \( J \). In both cases, if \( \Gamma \) is a \( \mathbb{Z}\)-order in \( QS/J \), then \( \mathcal{U}(\Gamma) \) is finite. Therefore \( QS/J \) has the hyperbolic property and is nilpotent free. By hypothesis \( j_0 \in S \) is nilpotent hence, by the previous lemma, \( \mathcal{I} := \{j_0, \theta\} \) is an ideal of \( S \) and \( J = \mathcal{Q}(j_0) \). We have that \( Q\mathcal{I} \cong Q(j_0) \) and hence \( QS/J \cong QS/Q\mathcal{I} \cong Q_0(S/\mathcal{I}) \) has the hyperbolic property and is nilpotent free. It follows, by Theorem \([1.2]\), that \( S/\mathcal{I} \) admits a series whose factors are Higman groups or the cyclic groups \( C_5, C_8 \) and \( C_{12} \). Thus the cyclic groups \( C_5, C_8 \) and \( C_{12} \) do not occur since, by the last paragraph, \( \mathcal{U}(\Gamma) \) is finite.

Conversely, \( \mathcal{I} := \{j_0, \theta\} \) is an ideal of \( S \) and \( S/\mathcal{I} \) admits a series whose factors are Higman groups then, by Lemma \([1.2]\), \( Q_0(S/\mathcal{I}) \cong \oplus_{i=1}^N QG_i \) and hence \( QS/Q\mathcal{I} \cong \oplus_{i=1}^N QG_i \). Since \( QS \cong (j_0)Q = J \), we have that the Wedderburn-Malcev decomposition is \( QS \cong \oplus QG_i \oplus (j_0)Q \), where \( S(QS) \cong \oplus QG_i \) is the semi-simple subalgebra of \( QS \). If \( J \) is non-central then, by Proposition \([3.3]\), there exist unique \( E_1, E_N \in \mathcal{E}(QS) \) such that
Let \( S \) be a finite semigroup. \( QS \) is non-semi-simple and has the hyperbolic property if, and only if, \( S \) has a principal series with a unique null factor and the other factors are isomorphic to Higman groups.

**PROOF.** By Lemma\[4.6\] if \( QS \) is non-semi-simple and has the hyperbolic property then \( S \) has a principal series with a unique null factor \( S_{i_0}/S_{i_0+1} = \{f, 0\} := \mathcal{J} \), say. If \( f \) is nilpotent in \( S \) then the result follows by the last proposition. Otherwise, since \( S_{i_0+1} \) is an ideal of \( S \) let \( \bar{S} := S/S_{i_0+1} \). Then \( \bar{Q}S \cong \bar{Q}S_{i_0+1} \oplus \bar{Q}_0\bar{S} \) is a direct sum as ideals and \( \bar{Q}S_{i_0+1} \) and \( \bar{Q}_0\bar{S} \) has the hyperbolic property. Clearly, \( \bar{S} \) has the nilpotent \( f \) and by the last proposition the factors of \( \bar{S} \), and thus the factors \( S_i/S_{i+1}, 1 < i < i_0 \), are isomorphic to Higman groups. If \( \Gamma \) is an order of \( \bar{Q}_0\bar{S} \) then \( U(\Gamma) \) is virtually cyclic hence \( \bar{Q}S_{i_0+1} \) has the hyperbolic property. Thus by Theorem\[4.3\] the factors of \( S_{i_0+1}, \) and therefore the factors \( S_i/S_{i+1}, i_0 < i, \) are Higman groups.

Conversely, on the conditions over the factors of a series of \( S \) we have that \( QS \cong QS_{i_0+1} \oplus QS_{i_0} \) is a direct sum as ideals. By Theorem\[4.3\] \( QS_{i_0+1} \) has the hyperbolic property and is nilpotent free. By the last proposition \( QS_{i_0} \) is hyperbolic. Clearly, If \( \Gamma \) is an order of \( QS \) then \( U(\Gamma) \) is hyperbolic and the result now follows. \( \square \)

In [12], [14] and [17] are classified the quadratic extensions \( K = \mathbb{Q}[\sqrt{-d}] \), where \(-d\) is a square free integer, and the finite groups \( G \) for which the group ring \( \mathfrak{o}_K[G] \) of \( G \) over the ring of integers of \( K \) has the property that the group \( U_1(\mathfrak{o}_K[G]) \) of units of augmentation 1 is hyperbolic. Therefore it is natural to classify the extensions \( K \) and the semigroups \( S \), such that the algebra \( KS \) has the hyperbolic property. By remark\[8.4\] if \( KS \) has nilpotent elements then, since the integral basis of \( \mathfrak{o}_K \) has two elements, \( KS \) does not have the hyperbolic property. Therefore a necessary condition for \( KS \) to have the hyperbolic property is that \( KS \) must be nilpotent free.

**Theorem 4.9** Let \( K = \mathbb{Q}(\sqrt{-d}) \), where \(-1 \neq d \neq 0 \), and \( S \) a finite semigroup. The algebra \( KS \) is nilpotent free and has the hyperbolic property if and only if \( KS \) is semi-simple, \( S \) admits a principal series whose factors are isomorphic to Higman groups \( G \) or at most one copy of a group \( H \), where \( G, H \) and \( d \) are determined by the following table.
Higman groups $G$ | Group $H$ | $K = \mathbb{Q}(\sqrt{-d})$
---|---|---
an elementary Abelian 2-group | | $0 < d$
| $C_3$ | $0 < d \neq 3$
| $C_4$ | $0 < d \neq 1$
| $Q_8$ | $d \equiv 7 \pmod{8}, 0 < d$
| an Abelian group of exponent dividing 4 | $C_8$ | $d = 1$
| an Abelian group of exponent dividing 6 | | $d = 3$

Moreover, $S$ is an inverse semigroup and it is a disjoint union of groups $G$ with, if it is the case, a unique group $H$.

**PROOF.** If $KS$ is nilpotent free and has the hyperbolic property then so is $QS$ and, by Theorem [4.3], $S$ has a principal series whose factors are Higman groups and at most one of the groups $C_5C_8, C_{12}$. By [12, Theorem 4.7], only $C_8$ is such that $U(\mathfrak{o}_KC_8)$ is hyperbolic and in this case $d = 1$. If a factor of $S$ is an Abelian Higman group $G$ then the free rank of $U(\mathfrak{o}_KG)$ is 0 or 1, since $U(\Gamma)$ is hyperbolic for all $\mathbb{Z}$-order $\Gamma \subset KS$. Using [12, Corollary 3.5] we determine the groups $G$ and $H$ and the extension $K$. If $G$ is non-Abelian then, by [12, Theorem 4.7], $G = Q_8$, $0 < d$ and $d \equiv 7 \pmod{8}$. Since $U(\mathfrak{o}_KQ_8)$ is an infinite hyperbolic group, the Higman group allowed as a factor is only an elementary Abelian 2-group, see [12, Corollary 3.5]. Conversely, if $\Gamma \subset KS$ is a $\mathbb{Z}$-order it follows immediately by [12] that $U(\Gamma)$ is hyperbolic. Clearly, $S$ is an inverse semigroup.

5. Idempotents of Maximal Subgroups

Let $S$ be a finite semigroup with a nilpotent element $j_0$. In this section, we investigate the idempotents of maximal subgroups of $S$. In case $QS$ is non-semi-simple and has the hyperbolic property, the study of idempotents enable us to obtain more information on the structure of $S$. In fact we prove in the last theorem that $S$ has some explicit semigroups as basic blocks of its structure which we define below as $T_2, \tilde{T}_2$ and $T'_2$.

**Definition 5.1** As a set $T_2 = \tilde{T}_2 = \{e_1, e_2, j_0, \theta\}$ and $T'_2 = T_2 \cup \{e_3\}$ are semigroups with the operation · given by the Cayley table:

| $T_2$ | $\tilde{T}_2$ | $T'_2$ |
|---|---|---|
| $e_1$ $e_1$ $e_2$ $j_0$ | $e_1$ $e_2$ $j_0$ | $e_1$ $e_2$ $e_3$ $j_0$ |
| $e_1$ $e_1$ $e_2$ $j_0$ | $e_1$ $e_1$ $j_0$ $j_0$ | $e_1$ $e_1$ $e_3$ $j_0$ |
| $e_2$ $e_2$ $e_2$ $\theta$ | $e_2$ $e_2$ $\theta$ $\theta$ | $e_2$ $e_2$ $e_3$ $\theta$ |
| $j_0$ $j_0$ $j_0$ $j_0$ | $j_0$ $j_0$ $j_0$ $j_0$ | $j_0$ $j_0$ $j_0$ $j_0$ |
| $\theta$ $\theta$ $\theta$ $\theta$ | $\theta$ $\theta$ $\theta$ $\theta$ | $\theta$ $\theta$ $\theta$ $\theta$ |
In what follows $S = \bigcup G_i \cup \{\theta, j_0\}, j_0^2 = \theta$, see Proposition \[4.7\] and $N = |E(QS)| > 2$.

If $E_l \in E(QS)$ then $1 = \sum_{1 \leq l < N} E_l + E_1 + E_N$. Let $E := E_1 + E_N$ and $e \in QS$ be any idempotent, hence $e = \sum_{1 \leq l < N} eE_l + eE$, where $(eE_l)^2 = eE_l \in A_l$ which is, by Theorem \[3.1\] a division ring, $\forall 1 < l < N$. Therefore, $eE_l \in \{E_l, 0\}$. Let $E_{el} := eE_l \neq 0$ thus $e = \sum E_{el} + eE$.

**Proposition 5.2** If $e_i$ is the group identity element of the group $G_i$ then $e_i$ has one of the following expressions:

\[
\begin{align*}
\sum E_{ii} + E_1 + \lambda j_0 \\
\sum E_{ii} + E_N + \mu j_0 \\
\sum E_{ii} + E \\
\sum E_{ii}
\end{align*}
\]

with $0 \neq E_{ii} := e_iE_i, E_i \in E(QS)$ and for some $\lambda, \mu \in \mathbb{Q}$, Moreover the last two expressions are central idempotents.

**PROOF.**

Write $e_i = \sum E_{ii} + uE_1 + vE_N + w j_0$ (recall that the $E_{ii}$ are orthogonal, central, they annihilate $j_0$, $E_1 j_0 = j_0 E_N = j_0$ and $E_N j_0 = j_0 E_1 = 0$). Hence $e_i^2 = \sum E_{ii} + u^2 E_1 + v^2 E_N + w(u + v) j_0 = e_i$ and thus $u, v \in \{1, 0\}$ and $w(u + v) = w$. If $u = v = 1$ then $w = 0$ and therefore $e_i = \sum E_{ii} + E_1 + E_N$. The others possibilities are: $u = 1, v = 0, w = 1, u = 0, v = 1, w = 1, u = v = w = 0$, resulting in the other expressions. $\square$

**Lemma 5.3** Suppose $QS \cong (\oplus A_i) \oplus \mathbb{Q} j_0$, $E_i \in E(\oplus A_i)$ and $E_1 j_0 = j_0 E_N = j_0$. Let $G, H$ be maximal subgroups of $S$. If $A_i \subseteq QG$ then $g j_0 = j_0, j_0 g = 0, \forall g \in G$ and if $A_N \subseteq QH$ then $j_0 h = j_0, h j_0 = 0, \forall h \in H$.

**PROOF.** We have that $E_1 \in A_i \subseteq QG$, and so $E_1 = \sum_{g \in G} \alpha g$, By the property of $E_1$ it holds $0 \neq j_0 = E_1 j_0 = (\sum \alpha g) j_0$ and, by Proposition \[4.5\] the $\lambda' g \in \{0, \pm 1\}, \forall g \in G$. Therefore, there exists $g_0 \in G$ such that $\lambda g_0 = 1$. If $e_1$ is the identity of $G$ then it follows that $e_1 j_0 = j_0$ and thus $g j_0 = j_0, \forall g \in G$, because $G$ is a finite group and $\{\theta, j_0\}$ is an ideal. Similarly, if $e_N$ is the identity of $H$ then $j_0 e_N = j_0$ and $j_0 h = h$ for all $h \in H$.

Since $Qj_0$ is an ideal we have that $j_0 e_1 = \rho j_0$ and $\rho \in \{0, \pm 1\}$. Suppose $0 \neq \rho = 1$, say, thus $e_1 j_0 = j_0 = j_0 e_1$; then $e_1$ centralizes $j_0$ and hence $e_1 \notin A_1$, a contradiction. In the same way we prove that $e_N j_0 = 0$. $\square$

**Proposition 5.4** Let $G$ be a maximal subgroup of $S$. Denote by $e \in G$ its identity element and suppose that $e j_0 = j_0$. If $e = \sum E_{ei} + E_1 + \lambda j_0$ then, $\forall g \in G$, $g = \sum g E_{ei} + E_1 + \lambda j_0$. Also if $e = \sum E_{ei} + E_N + \mu j_0$ then, $\forall g \in G$, $g = \sum g E_{ei} + E_N + \mu j_0$.

**PROOF.** For $g \in G$, we have that $g = g e = \sum g E_{ei} + g E_1 + g \lambda j_0$. By Lemma \[5.3\] $g j_0 = j_0$ and hence $g = \sum g E_{ei} + g E_1 + \lambda j_0$. 14
To determine \( gE_1 \), recall that \( \mathbb{Q}(E_1, E_N, j_0) \) is an ideal of \( \mathbb{Q}S \). So we may write \( gE_1 = tE_1 + sE_N + rj_0 \). There exists \( k \in \mathbb{Z} \) such that \( g^k = e \); since the orthogonality of \( E_1 \) respect \( E_1, l \neq 1 \) and \( E_1j_0 = j_0 \) we conclude that \( E_1 g^k = E_1 + \lambda j_0 \). By comparing with the equation \( (gE_1)^k = (tE_1 + sE_N + rj_0)^k = t^k E_1 + s^k E_N + r^k j_0 \) we reach: \( t^k = 0, s^k = 1 \) and \( r^k = \lambda \). Hence, \( g = \sum gE_{e_i} E_1 + \lambda j_0 \) and, multiplying on the right by \( j_0 \) and using Lemma 5.3 we obtain that \( g = \sum gE_{e_i} + E_1 + \lambda j_0 \).

For the other case: \( e = \sum E_{e_i} + E_N + \mu j_0 \), it holds that \( j_0 e = j_0 \). If \( g \in G \) then similarly \( g = \sum gE_{e_i} + E_N + \mu j_0 \). □

**Theorem 5.5** Let \( e_1 \in G_1 \) and \( e_N \in G_N \) be the group identities and suppose that \( e_1j_0 = j_0 e_N = j_0 \). Write

\[
\begin{align*}
e_1 &= \sum E_{e_1} + E_1 + \lambda j_0 \\
e_N &= \sum E_{e_N} + E_N + \mu j_0.
\end{align*}
\]

Then only one of following options holds:

(i)
\[
e_{1}e_{k} = 0 \iff e_{k}e_{1} = [\alpha]j_0 \text{ and } \lambda + \mu = \alpha, \text{ where } \{1, N\} = \{l, k\}
\]

and either \( \alpha = 0 \) and \( \{e_1, e_N, j_0, \theta\} \cong T_2 \) or \( \alpha \in \{-1, 1\} \) and \( \{e_1, e_N, j_0, \theta\} \cong \hat{T}_2 \).

(iii)
\[
\text{If } e_N e_1 \neq 0 \text{ then } e_1 e_N = e_N e_1 = : e_3 \text{ and } \lambda + \mu = 0,
\]

and \( \{e_1, e_N, e_3, j_0, \theta\} \cong T_2^e \).

**PROOF.** Since the idempotents \( E_i \in \mathbb{E}(\mathbb{Q}S) \) are orthogonal, \( j_0 e_1 = j_0 E_1 = e_N j_0 = E_N j_0 = 0 \), and \( e_1 j_0 = E_1 j_0 = j_0 e_1 = j_0 E_N = j_0 \)

\[
\begin{align*}
e_1 e_N &= \sum E_{e_1} E_{e_N} + (\lambda + \mu) j_0 \\
e_N e_1 &= \sum E_{e_N} E_{e_1} \\
e_1 e_N &= e_N e_1 + (\lambda + \mu) j_0
\end{align*}
\]

(1)

Without loss of generality suppose \( l = 1, k = N \).

If \( e_1 e_N = 0 \) then \( -(\lambda + \mu) j_0 = e_N e_1 \). In addition, if \( \lambda + \mu = 0 \) then \( e_N e_1 = 0 \), and the converse is clear. If \( \lambda + \mu \neq 0 \) then \( e_N e_1 \) is a non-trivial nilpotent element of \( S \). Thus \( e_N e_1 = j_0 \), since \( S \) has a unique nilpotent element, clearly \( \lambda + \mu = -1 = \alpha \) (for \( l = N, k = 1 \) we have \( \alpha = 1 \)). The converse is straightforward.

If \( e_1 e_N \) and \( e_N e_1 \) are non-zero elements then, by equation (1), the set \( \{e_1 e_N, e_N e_1, j_0\} \subseteq S \) is \( \mathbb{Q} \)-L.D. Since any element of this set is not zero thus \( e_1 e_N = e_N e_1 := e_3 \) is a non-trivial idempotent and \( \lambda + \mu = 0 \). The converse is clear. □

The semigroups \( T_2, T_2' \) are, in some sense, the basic building blocks of the semigroups \( S \) whose rational semigroup algebra is non-semi-simple and has the hyperbolic property.
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