Remarks on the Gauge Dependence of the RI/MOM Renormalization Procedure

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Abstract

The RI/MOM non-perturbative renormalization scheme is studied on the lattice in SU(3) quenched QCD with Wilson fermions. The gauge dependence of some fermion bilinear renormalization constants is discussed by comparing data which have been gauge-fixed in two different realizations of the Landau gauge and in a generic covariant gauge. The very good agreement between the various sets of results and the theory indicates that the numerical uncertainty induced by the lattice gauge-fixing procedure is moderate and below the statistical errors.

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1 Introduction

Non-perturbative renormalization techniques \[1, 2\] have become a crucial ingredient in lattice computations of fundamental QCD parameters and hadronic matrix elements. They allow the limitations of lattice perturbation theory to be overcome in the determinations of the relevant renormalization constants by removing the uncertainties due to unknown higher order terms.

The RI/MOM technique proposed in Refs. \[1, 3\] has been successfully applied to compute renormalization constants of composite fermion operators in many lattice regularizations: Wilson fermions \[1, 3–10\], Kogut–Susskind fermions \[11\], domain-wall fermions \[12\] and overlap fermions \[13\]. The method imposes RI/MOM renormalization conditions on conveniently defined amputated Green functions computed non-perturbatively between off-shell quark states at large virtuality in a fixed gauge. In practical computations the renormalization scale $\mu^2$, determined from the virtuality $p^2$ of the external states, must satisfy the condition $\Lambda_{\text{QCD}} \ll \mu \ll O(1/a)$ to ensure the reliability of the results \[1\].

The RI/MOM renormalization constants depend in general on the external states and therefore on the gauge in which the RI/MOM renormalization conditions have been imposed. For gauge-invariant operators the coefficients needed to match in a given gauge-invariant scheme will cancel these dependences up to higher orders in the continuum perturbative expansion and up to discretization errors. A necessary assumption in this procedure is that, at large virtualities, the continuum limit of the renormalized gauge-dependent matrix elements, computed by imposing the non-perturbative Landau lattice gauge fixing, lead to the corresponding values computed in perturbation theory up to higher order \[14\].

The existence of both continuum \[15\] and lattice Gribov copies \[16, 17\] and the numerical noise that they can generate \[18\] (for a recent review see Ref. \[19\]) is an unsolved problem of the lattice non-perturbative gauge fixing. The real concern is, of course, the influence that these phenomena may have on the values of the renormalization constants, when computed using non-gauge-invariant quantities as in the RI/MOM scheme.

The aim of this paper is to study the systematics induced by the gauge-fixing procedure on the RI/MOM determinations of the renormalization constants. We have computed the quark field $(Z_\psi)$, the axial-vector $(Z_A)$ and scalar $(Z_S)$ renormalization constants, by imposing the standard lattice Landau gauge and another realization of the Landau condition that, in the following, will be called Landau1 gauge. Whereas the two realizations impose the same gauge-fixing condition in the naive continuum limit, they are affected by different Gribov ambiguities. By comparing the different sets of results for the renormalization constants, we have found negligible differences within our statistical errors.

As will be discussed in section 2, a very interesting feature of the Landau1 gauge is that it can be generalized to impose a generic covariant gauge on the lattice, as proposed in Ref. \[13\]. By exploiting this opportunity, we have extended
the study of the gauge dependence to off-shell Green functions measured in a
generic covariant gauge.

2 Gauge-Fixing Conditions

In this section we briefly describe the continuum analogous of the non-perturbative
gauge-fixing conditions whose discretized forms have been used to obtain the nu-
merical results presented below.

In the standard Landau gauge the expectation value of a gauge-dependent
operator is given by

$$\langle O_F \rangle = \int \delta A_\mu O e^{-S(A)} \Delta_F[A] \delta(\partial_\mu A_\mu),$$

(1)

where the Faddeev–Popov factor is defined as

$$\Delta_F[A] = \det(\partial_\mu D_\mu)$$

(2)

and the gauge-fixing condition

$$\partial_\mu A_\mu^G = 0 \text{ with periodic boundary conditions},$$

(3)

is enforced non-perturbatively by minimizing the Gribov functional

$$F_A[G] \equiv ||A^G||^2 = \int d^4x \text{Tr} \left[A^G_\mu A^G_\mu\right].$$

(4)

Since the functional in Eq. (4) can have multiple minima in the non-perturbative
regime, the corresponding gauge-fixing condition is incomplete, a fact that is
known as the Gribov copy problem.

In order to study the effects induced by Gribov’s ambiguities on gauge-
dependent matrix elements, the Landau gauge can also be fixed by minimizing a
different functional

$$H_A[G] \equiv \int d^4x \text{Tr} \left[(\partial_\mu A_\mu^G)^2\right],$$

(5)

which reaches its minima when

$$D_\nu \partial_\nu \partial_\mu A_\mu^G = 0,$$

(6)

where $D_\nu$ is the covariant derivative. Equation (5), supplemented with periodic
boundary conditions, defines a new gauge which we call Landau1 in the following.

In this case the expectation value of a gauge-dependent operator is given by

$$\langle O_H \rangle = \int \delta A_\mu O e^{-S(A)} \Delta_H[A] \delta(D_\nu \partial_\nu \partial_\mu A_\mu).$$

(7)
where $\Delta_H[A]$ is defined analogously as in Eq. (3). By comparing Eqs. (1) and (7) it is straightforward to show that the two procedures lead to the same values of gauge-dependent matrix elements in the perturbative regime.

The solutions of Eq. (6) are all absolute minima of $H_A[G]$, at variance with the copies associated with the standard functional in Eq. (4). Therefore we expect a different distribution of Gribov’s copies for the two procedures, which could generate different effects in off-shell matrix elements. It is clear that Eq. (1) could also have “spurious” solutions, which correspond to zero modes of the operator $D_\nu \partial_\nu$ and do not satisfy the gauge condition in Eq. (3). The numerical results of the exploratory study presented in Ref. [14], and, a posteriori, the new results presented below, indicate that such a possibility can be discarded.

The functional $H_A[G]$ has been proposed in the literature because it can be easily generalized to the following form

$$H_{(A,\Lambda)}[G] \equiv \int d^4 x \text{Tr} \left[ (\partial_\mu A^G_\mu - \Lambda)(\partial_\nu A^G_\nu - \Lambda) \right],$$

where $\Lambda(x)$ are matrices belonging to the Lie algebra of the group and are generated according to a Gaussian distribution with the variance $\xi$ fixed at a given value. It is the value of this parameter that determines the corresponding covariant gauge. The $H_{(A,\Lambda)}[G]$ minima satisfy the gauge condition

$$D_\nu \partial_\nu (\partial_\mu A^G_\mu - \Lambda) = 0,$$

with periodic boundary conditions.

The expectation value of a given operator is obtained by

$$\langle O_H \rangle_\xi = \int \delta A e^{-\frac{1}{\xi} \int d^4 x [A^2]} \int \delta A_\mu \ O e^{-S(A)} \Delta_H[A] \delta (\partial_\mu A_\mu - \Lambda).$$

For a complete description of the method, see Ref. [14]. In the following we perform a further check of the gauge-dependence comparing the results obtained in the Landau gauge and in one of the lattice covariant gauges.

### 3 Numerical Results and Discussion

We have used a sample of 80 SU(3) gauge configurations retrieved from the repository at the “Gauge Connection” (http://qcd.nersc.gov/), which were generated with the Wilson gluonic action at $\beta = 6.0$ and $V \times T = 16^3 \times 32$. By using the discretized version (see [14] for details) of the functionals in Eqs. (4), (5) and (8) we have rotated each configuration in the Landau gauge, in the Landau1 gauge and in the covariant gauge with the bare value of the gauge fixing parameter $\xi = 8$ respectively.

The various gauges have been enforced with a quality factor $\theta < 10^{-6}$ (see [14] for details) with the Origin2000 computers at Boston University. The inversion of
Table 1: Main parameters of our simulations.

| \( \beta \)     | 6.0 |
|------------------|-----|
| Action           | Wilson |
| Lattice Size     | \( 16^3 \times 32 \) |
| # Configurations | 80 |
| Gauges           | Landau |
|                  | Landau1 |
| \( \xi = 8 \)    |     |
| \( k \)          | 0.1530 |
|                  | 0.1535 |
|                  | 0.1540 |
|                  | 0.1545 |
|                  | 0.1550 |
|                  | 0.1555 |
| \( k_c \)        | 0.15703(2) |
| \( a^{-1} \)     | 2.26(5) GeV |

the Wilson–Dirac operator has been performed on an INFN APE100 machine for masses corresponding to the values of the \( k \) parameter shown in Table 1. Once the quark propagator \( S(x, 0) = \langle \psi(x) \bar{\psi}(0) \rangle \) has been computed for each mass and gauge and Fourier-transformed, we have determined the inverse Euclidean bare-quark propagator \( S^{-1}(pa) \). We have computed the amputated Green functions of the local quark bilinears \( O_\Gamma(x) = \bar{\psi}(x) \Gamma \psi(x) \) in the Fourier space

\[
A_\Gamma(pa) = S^{-1}(pa)G_\Gamma(pa)S^{-1}(pa),
\]

where

\[
G_\Gamma(p) = \int d^4x d^4ye^{-ip(x-y)}\langle \psi(x)O_\Gamma(0)\bar{\psi}(y) \rangle
\]

and \( \Gamma \) is a generic Dirac matrix corresponding to \( \Gamma = \{ A, S \} = \{ \gamma_\mu \gamma_5, I \} \).

In the RI/MOM scheme, the wave-function renormalization constant \( Z_\psi \) is defined as

\[
Z_\psi(\mu a) = \lim_{m \to 0} -i \frac{1}{12} \text{Tr} \left[ \frac{\partial S^{-1}(pa)}{\partial \mu} \right]_{p^2=\mu^2}.
\]

In order to avoid numerical derivatives, it is convenient to use

\[
Z'_\psi(\mu a) = \lim_{m \to 0} -i \frac{1}{12} \text{Tr} \left[ \frac{\sum_{\mu=1,4} \gamma_\mu \sin(p_\mu a)S^{-1}(pa)}{4 \sum_{\mu=1,4} \sin^2(p_\mu a)} \right]_{p^2=\mu^2},
\]

which differs from \( Z_\psi \) by a finite term of order \( \alpha_s \). The matching between the two renormalization constants can be computed in continuum perturbation theory.

\[2\] In what follows we adopt the conventions of Ref. [5].
Since this correction is negligible at the level of the numerical precision of our calculation (see below), we will not take it into account in the following. Therefore throughout this paper we assume $Z_\psi = Z'_\psi$.

The renormalization constant $Z_R^{\Gamma}(\mu, g_0)$, which defines the renormalized operator $\hat{O}_R^{\Gamma} = Z_R^{\Gamma} \hat{O}_\Gamma$, is fixed by imposing in the chiral limit the renormalization condition

$$Z_R^{\Gamma}(\mu)Z^{-1}_\psi(\mu)\text{Tr} P_\Gamma \Lambda_\Gamma(pa)|_{p^2=\mu^2}=1,$$

(15)

where $P_\Gamma$ is a suitable projector on the tree-level amputated Green function [1].

Even though $O_A$ and $O_S$ are gauge-independent operators, their matrix elements between quark states acquire a gauge dependence. In Fig. 1 are shown the numerical results for $Z_\psi$, $Z_A$ and $Z_S$ calculated in the Landau and in the Landau1 gauge, as a function of the square lattice momenta $(ap^\mu_L)^2 = 4 \sin^2\left(\frac{ap^{\mu}}{2}\right)$.

In the perturbative region, $Z_\psi(a\mu)$ has a flat dependence on the renormalization scale. This is expected since in the Landau gauge it has zero anomalous dimension at the leading order. Also $Z_A(a\mu)$ shows a flat behaviour due to the compatibility of the RI/MOM renormalization conditions with the axial Ward identities. The value of $Z_S(a\mu)$ increases with a logarithmic behaviour, as expected in perturbation theory (see also Fig. 2). Our data are in very good agreement with the results reported in the literature [1].

The results for all correlators, corresponding to Landau and Landau1 gauges, coincide within the statistical errors. As a consequence we can conclude that the data do not show any residual effect due to the presence of lattice Gribov’s copies in the statistical sample of configurations generated and for the lattice used. This represents one of the main results of this paper.

To show the amount of gauge dependence that can be found in our correlators and therefore our sensitiveness to the gauge, in the last part of this section we compare the results obtained in the Landau and in the covariant gauge with the bare gauge parameter fixed to $\xi = 8$. The gauge dependence of the primary quantities $Z_\psi$, $\Lambda_A$ and $\Lambda_S$ measured on the lattice is clearly visible in Fig. 3, where they are plotted for the Landau and the covariant gauge as a function of the momenta $(ap)^2$. All three quantities show a statistically significant gauge dependence while the renormalization constants $Z_A$, $Z_S$ shown in Fig. 3, obtained by computing the ratios as indicated in Eq. (15), do not exhibit any visible gauge dependence within the statistical precision reached in our simulation. For these quantities, the fluctuations of the simulation hide the weak gauge dependence that is expected in perturbation theory from the next-to-leading order terms in $\alpha_s$. These results indicate an upper limit to the numerical troubles that can be expected in the RI/MOM renormalization constants because of the lattice gauge fixing.
Figure 1: The $Z_A$ and $Z_S$ renormalization constants are shown as a function of $(a\mu)^2$ for the Landau and Landau1 gauges, as discussed in the text. The data have been slightly displaced in the $x$-direction and the error bars for one set only have been shown in order to help the eye. The error bars for the other sets are comparable. The errors are jackknife.
Figure 2: The behaviour of $Z_\psi$, $\Lambda_A$ and $\Lambda_S$ is shown as a function of $(ap)^2$ for the Landau (black dots) and the covariant gauge (crosses) data. The errors are jackknife.
Figure 3: The behaviour of $Z_A$ and $Z_S$ is shown as a function of $(a\mu)^2$ for the Landau (black dots) and the covariant gauge (crosses) data. The errors are jackknife.
4 Conclusions

We have proposed a method to study the effects of Gribov’s copies on lattice gauge-dependent matrix elements. We have implemented it numerically in quenched QCD with Wilson fermions and we have computed the matrix elements needed to fix the quark field, the axial-vector and scalar renormalization constants in the RI/MOM non-perturbative renormalization technique. Our numerical results do not show any significant differences between data coming from the standard Landau and the Landau1 gauge. We do not find any relevant effect that can be traced back to the incompleteness of the gauge fixing for a number of gauge-field configurations and for lattice sizes of the order of those used in this calculation. The main indication of our paper is therefore that, at the level of the precision that we have reached, the systematics due to the numerical gauge-fixing machinery has negligible effect on the matrix elements needed for implementing the RI/MOM scheme.

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