Lexicographic Composition of Choice Functions

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Abstract

Lexicographic composition is a natural way to build an aggregate choice function from component choice functions. As the name suggests, the components are ordered and choose sequentially. The sets that subsequent components select from are constrained by the choices made by earlier choice functions. The specific constraints affect whether properties like path independence are preserved. For several domains of inputs, we characterize the constraints that ensure such preservation.

1 Introduction

Given a set of options, a choice function selects a subset. Choice functions are one of the elementary building blocks of microeconomic theory. They are a way to represent individual preferences or even aggregated social preferences and priorities.\(^1\) In contexts with competing preferences and priorities, choice functions need to be aggregated and a natural way to do so is \textit{lexicographically}. That is, given a list of component choice functions, the first one selects from the full set of options, the second one selects from what is \textit{feasible} given the first choice, and so on. The final choice of the aggregated choice function is the union of what each

\(^{1}\text{See, for instance, Moulin (1985)}\)
individual choice function selects. We focus on feasibility, in terms of constraints that earlier choices place on later choices. So, we frame feasibility in terms of *exclusions*. Depending on what has previously been chosen, certain things are excluded from what can be chosen at a particular step of the lexicographic process. Specifically, an exclusion function $E$ maps every set of items to another set of items. If a set $Z$ of items has previously been chosen, then $E(Z)$ is excluded from the input to the next choice function. That is, if we start from $Y$ and $Z$ has already been chosen, then the next choice function gets to choose from $Y \setminus E(Z)$.

Here are some examples:

- Indivisible private goods: $E$ is the identity function.
- Maximizer-collectors à la *Aizerman and Malishevski* (1981): $E$ always maps to the empty set.
- Contracts as in *Roth* (1984) and *Hatfield and Milgrom* (2005): $E$ maps to the set of contracts with “doctors” named in previously chosen contracts.
- Capacity constraints: $E$ is identity if fewer items have been chosen than the capacity or the universal set otherwise.

Whatever the feasibility constraints are, lexicographic composition is equivalent to *serial dictatorship* over the competing interest. If these interests are strategic, then concerns about incentive compatibility and efficiency often narrow the possibilities for aggregation to lexicographic composition (*Svensson*, 1999; *Pápai*, 2001; *Ehlers and Klaus*, 2003; *Hatfield*, 2009; *Pycia and Ünver*, 2022). Even without strategic considerations, such composition is a common tool in the design of choice functions in the market design literature. In fact, this hierarchical structure is, in a sense, necessary for organizational choice functions to be such that stable matchings are guaranteed to exist (*Alva*, 2016).

Properties like path independence have long been studied in choice theory as a weak form of rationality (*Plott*, 1973). Path independence is specifically important in matching settings as pointed out by *Chambers and Yenmez* (2017). Our interest is in understanding what exclusion functions have the property that

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2See, *Sönmez and Switzer* (2013), *Sönmez* (2013), and *Kominers and Sönmez* (2016) among others.

3Path independence is equivalent to the combination of *substitutes* and *consistency* (also called “independence of rejected contracts” (*Aygün and Sönmez*, 2013)) (*Aizerman and Malishevski*, 1981).
the lexicographic composition of path independent choice functions is itself path independent. It is well known that for the first example above, of the identity exclusion, path independence is preserved by lexicographic composition (Alva, 2016; Chambers and Yenmez, 2017). At the other extreme, Aizerman and Malishevski (1981) show that every path independent choice function can be rationalized by the lexicographic composition of single-valued rational choice functions with the empty exclusion function. However, not all exclusion functions have this property. Famously, exclusion based on equivalence relations—like the “contracts” exclusion described above—does not (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2010).

Our main contribution is to characterize the exclusion functions such that the lexicographic composition of two responsive choice functions is path independent (Theorem 1). We then consider two variants of this result. The first is to expand the domain from responsive choice functions to arbitrary path independent choice functions (Proposition 3). The second is to consider the preservation of an additional property, size monotonicity, which is commonly studied in the matching literature (Proposition 4). In both cases, the set of surviving exclusion functions shrinks.

While our main results are stated for the composition of just two choice functions, nested applications allow one to aggregate more of them. In general, the binary operation of lexicographic composition is not associative or commutative. Consequently, there is a great richness to the ways a list of $n$ choice functions can be aggregated: they can be mapped to binary trees with $n$ leaves. Moreover, each non-terminal node in such a tree can be labeled by an exclusion function.

The aggregation of single-valued, rational choice functions is particularly salient (Kominers and Sönmez, 2016). So, we also characterize the exclusion functions that preserve path independence when one of the two inputs is single-valued (Proposition 5 and Proposition 6).

As noted above, exclusion functions based on equivalence relations violate the necessary conditions to preserve path independence. So, we consider the weaker

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4The exclusion function that maps every set to the empty set.
5A choice function is responsive if there is an ordering $\succ$ over the items and a capacity such that from any set of items, it selects the best items according to $\succ$, up to its capacity. Every responsive choice function is path independent. So, the necessary conditions from this result extend to larger domains than responsive choice functions.
6Size monotonicity is sometimes referred to as the “law of aggregate demand” (Hatfield and Milgrom, 2005).
property of path independent completability (Hatfield and Kominers, 2016). We show that the obvious adaptations of the conditions for preserving path independence are sufficient to preserve path independent completability (an application of Lemma 1).

The remainder of the paper is organized as follows. In Section 2 we set up the model. In Section 3 we give our main result. In Section 4 we examine the consequences of expanding the domain and of preserving size monotonicity along with path independence. We consider the composition of more than two choice functions in Section 5. In Section 6, we consider equivalence relation based exclusion. We end with a brief discussion in Section 7. All proofs are in the appendix.

2 Definitions

Let \( X \) be a countably infinite set of items. Let \([X] = \{Y \in [X] : |Y| < \infty\}\) be the set of finite subsets of \( X \).

A set function \( S : [X]^* \to [X] \) is a mapping from finite subsets of \( X \) to subsets of \( X \). Let \( S \) be the set of all set functions.

Two kinds of set function are particularly relevant for our analysis. First, a dilation \( D \in S \) is a set function such that, for each \( Y \in [X]^* \), \( D(Y) \supseteq Y \). Conversely, a choice function (or contraction) \( C \in S \) is a set function such that, for each \( Y \in [X]^* \), \( C(Y) \subseteq Y \). Let \( D \) and \( C \) be the set of all dilations and choice functions, respectively.

The lexicographic composition \( L_E(C_1, C_2) \in C \) of two choice functions \( C_1, C_2 \in C \) subject to the set function \( E \in S \) is defined, for each \( Y \in [X]^* \), by

\[
L_E(C_1, C_2)(Y) = C_1(Y) \cup C_2(Y \setminus E(C_1(Y))).
\]

The first choice function \( C_1 \) chooses from all of \( Y \). Given the alternatives \( C_1(Y) \) that it chooses, the set function \( E \) specifies the alternatives \( E(C_1(Y)) \) to be excluded from being chosen by the second choice function \( C_2 \). Because of the role that it plays in the definition of \( L_E \), we refer to \( E \) as an exclusion function.

Our interest is in understanding for which \( E \)'s the lexicographic composition \( L_E \) inherits certain properties of the input choice functions \( C_1 \) and \( C_2 \). For a particular property \( \pi \), let \( \mathcal{C}_\pi \subseteq C \) denote the set of choice functions that satisfy \( \pi \). Given a pair

\[\text{Danilov and Koshevoy (2009) call these “extensive operators”}\]
of properties, \( \pi \) and \( \rho \), we denote the conjunction by \( \pi \cdot \rho \) so that \( C_{\pi \cdot \rho} \) is the set of choice functions that satisfy both \( \pi \) and \( \rho \).\(^8\) Then, given a set of choice functions \( \mathcal{C}' \subseteq \mathcal{C}^{\pi} \), the lexicographic composition \( \mathcal{L}_E \) preserves \( \pi \) over \( \mathcal{C}' \) if \( \mathcal{L}_E(C_1, C_2) \in \mathcal{C}^{\pi} \) for all \( C_1, C_2 \in \mathcal{C}' \). In some instances, we consider cases where the two inputs are from different sets. In such contexts, we speak of \( \mathcal{L}_E \) preserving \( \pi \) over \( C_1 \times C_2 \) for some \( C_1, C_2 \subseteq \mathcal{C}^{\pi} \).

We do not consider the preservation of a property \( \pi \) outside of \( \mathcal{C}^{\pi} \) since the composition cannot satisfy \( \pi \) if the inputs are not ensured to. To see why, suppose that \( C_2 \) always chooses the empty set. Then the composition of any \( C_1 \) with \( C_2 \) is just \( C_1 \), no matter what \( E \) is. If \( C_1 \) violates \( \pi \), then the composition does as well. While we could impose joint conditions on \( C_1 \) and \( C_2 \) to ensure that their composition satisfies \( \pi \), we feel that it is beyond the scope of this paper.

3 Path Independence and Responsive Choice

A choice function \( C \in \mathcal{C} \) is path independent (pi) if for any set of items \( Y \in [X]^* \), the choice from \( Y \) is invariant to arbitrarily segmenting \( Y \) into several parts, choosing from each part, and then choosing again from all of the chosen items. That is, for each pair \( Y, Y' \in [X]^* \), \( C(Y \cup Y') = C(C(Y) \cup C(Y')) \) (Plott, 1973).

Any set function \( S \in \mathcal{S} \) is monotonic if, for each pair \( Z, Z' \in [X]^* \), \( Z \subseteq Z' \) implies that \( S(Z) \subseteq S(Z') \). A choice function satisfies substitutes (sub) if the rejection function associated with it is monotonic.\(^9\) In other words, \( C \in \mathcal{C} \) satisfies substitutes if, for each pair \( Y, Y' \in [X]^* \), \( Y \subseteq Y' \) implies that \( Y \setminus C(Y) \subseteq Y' \setminus C(Y') \).

A choice function \( C \in \mathcal{C} \) satisfies consistency (con), if eliminating rejected items does not affect the choice.\(^{10}\) In other words, for each pair \( Y, Y' \in [X]^* \), \( C(Y') \subseteq Y \subseteq Y' \) implies \( C(Y) = C(Y') \).

A choice function is path independent if and only if it satisfies substitutes and

\[^8\]That is, \( C_{\pi \cdot \rho} = C_{\pi} \cap C_{\rho} \).

\[^9\]This property has appeared as “Postulate 4” in Chernoff (1954) and “Property \( \alpha \)” in Sen (1971). It plays a central role in the matching literature under the name of substitutes (Kelso and Crawford, 1982).

\[^{10}\]Consistency has also been studied in the choice literature as “Postulate 6” in Chernoff (1954) and “independence of rejecting the outcast” in Aizerman and Malishevski (1981). Like substitutes, it plays an important role in the matching literature as well (Alkan and Gale, 2003; Fleiner, 2003; Aygün and Sönmez, 2013).
consistency (Aizerman and Malishevski, 1981). That is, \( C^{pl} = C^{sub-con} \).

Among path independent choice functions are those rationalized by a preference relation and capacity. A choice function \( C \in \mathcal{C} \) is responsive (res) to a linear order \( \succ \) over \( X \cup \{\emptyset\} \) and quota \( q \) if, for each \( Y \in [X]^* \), \( C(Y) \) corresponds to the \( q \) highest \( \succ \)-ranked elements in \( \{y \in Y : Y \succ \emptyset\} \).\(^{11}\) It is well known in the literature that \( C^{res} \subset C^{pl} \).

Below, we characterize the set of exclusion functions that preserve path independence over \( C^{res} \). Since \( C^{res} \subset C^{pl} \), the restriction of the domain to \( C^{res} \) strengthens the necessity results. As we will see in subsequent sections, however, expanding the domain constricts the set of exclusion functions that preserve path independence.

To ease exposition, we break up our analysis. First we separately consider “pure expansion” (in the sense of exclusion functions that are dilations) and “pure reuse” (in the sense of the exclusion function being a contraction). Then, we build on this to handle arbitrary exclusion functions.

### 3.1 Pure Expansion

As explained above, we focus in this subsection on exclusion functions that are “expansions” and therefore permit no reuse of chosen items. That is, we restrict attention to exclusion functions that are dilations.

We start by establishing some conditions on pure expansions that are necessary for \( L_E \) to preserve path independence over \( C^{res} \). The first condition is monotonicity. The connection between path independence of a choice function and monotonicity of the rejection function has been well known in the literature (Danilov and Koshevoy, 2009) and is reflected in our definition of substitutes. Even though the exclusion function is not equivalent to the rejection function of the lexicographic composition, it comes as no surprise that monotonicity is a necessary condition.

**Claim 1.** If \( E \) is a dilation and \( L_E \) preserves path independence over \( C^{res} \), then \( E \) is monotonic.

A dilation \( D \in \mathcal{D} \) is all-or-nothing if, for each \( Z \in [X]^* \), \( D(Z) \in \{Z \cup D(\emptyset), X\}.^{12} \) Since

\(^{11}\)We denote by \( \emptyset \) the option of not choosing an item. So, we define the order \( \succ \) over \( X \cup \{\emptyset\} \) rather than just \( X \) to permit the possibility that some items are unacceptable.

\(^{12}\)The “all” is in reference to the universal set \( X \). The “nothing” is in reference to the value of the
the set of items excluded by the empty set plays an important role in our analysis, we define \( K = E(\emptyset) \).

**Claim 2.** If \( E \) is a dilation and \( \mathcal{L}_E \) preserves path independence over \( \mathcal{C}^{\text{res}} \), then \( E \) is all-or-nothing.

A dilation \( D \in \mathcal{D} \) is **cardinal** if, for each pair \( Z, Z' \in [X]^* \), such that \( Z \cup D(\emptyset) \neq X \) and \( |Z| = |Z'| \), \( D(Z) = X \) implies \( D(Z') = X \).

**Claim 3.** If \( E \) is a dilation and \( \mathcal{L}_E \) preserves path independence over \( \mathcal{C}^{\text{res}} \), then \( E \) is cardinal.

The next condition is equivalent to the combination of monotonicity, all-or-nothingness, and cardinality. A dilation \( D \in \mathcal{D} \) is **threshold-linear (with threshold \( t \in \mathbb{N} \cup \{0, \infty\} \))** if, for all \( Z \in [X]^* \),

\[
D(Z) = \begin{cases} 
Z \cup K & \text{if } |Z| < t \\
X & \text{otherwise.} 
\end{cases} \tag{13}
\]

Threshold-linearity is not only necessary but also sufficient for a dilation to preserve path independence over \( \mathcal{C}^{\text{res}} \).

**Proposition 1.** If \( E \) is a dilation, then \( \mathcal{L}_E \) preserves path independence over \( \mathcal{C}^{\text{res}} \) if and only if \( E \) is threshold-linear.

### 3.2 Pure Reuse

We now turn to the case of exclusions that only limit reuse and without expanding the set. That is, we consider exclusion functions that are contractions.

Denote by \( I \) the identity exclusion function where \( I(Z) = Z \) for each \( Z \in [X]^* \) and by \( I \setminus E \) the set function that selects \( I(Z) \setminus E(Z) \) for each \( Z \). Since we are focused dilation at the empty set \( D(\emptyset) \).

\(^{13}\)We use the term *linearity* in the sense of taking the union with a constant set \( K \). For choice functions, one could use the same terminology in reference to intersections. Aizerman et al. (1977a,b) study both forms of linearity but highlight their differences in another way. On the full domain, they show that a linear dilation \( D \) is *multiplicative* in the sense that, for all \( Z, Z' \in [X]^* \), \( D(Z \cap Z') = D(Z) \cap D(Z') \). Conversely, a linear choice function \( C \) is *additive* in the sense that, for all \( Z, Z' \in [X]^* \), \( C(Z \cup Z') = C(Z) \cup C(Z') \).
on $E$ being a contraction, $I \setminus E$ is also a contraction. Specifically, $I \setminus E$ gives the items that are reusable among those that have been selected by the first choice function. We state here conditions on $I \setminus E$ rather than $E$ directly as this is useful when we consider more general exclusion functions.

The first necessary condition is that if an item is reusable in some set $Z$, then it is reusable in every superset of $Z$.

**Claim 4.** If $E$ is a choice function and $L_E$ preserves path independence over $C^{res}$, then $I \setminus E$ is monotonic.

A contraction is **cardinal** if whether an item is chosen or not depends only on the cardinality of the input set. That is, a contraction $C \in C$ is cardinal if for any $Z,Z' \in [X]^*$ such that $|Z| = |Z'|$, and any $x \in Z \cap Z'$, $x \in C(Z)$ if and only if $x \in C(Z')$.

**Claim 5.** If $E$ is a contraction and $L_E$ preserves path independence over $C^{res}$, then $I \setminus E$ is cardinal.

The next condition is equivalent to the combination of monotonicity and cardinality of $I \setminus E$. A contraction $C \in C$ is **cardinal-linear** if there is a sequence $\{T_n\}_{n=0}^{\infty}$ such that $\emptyset = T_0 \subseteq T_1 \subseteq T_2 \ldots$, and for each $Z \in [X]^*$, $C(Z) = Z \cap T|Z|$.

**Proposition 2.** If $E$ is a contraction, then $L_E$ preserves path independence over $C^{res}$ if and only if $I \setminus E$ is cardinal-linear.

### 3.3 Putting the Pieces Together

So far, we have considered two extremes of how an exclusion might behave: pure expansion and pure reuse.

An arbitrary exclusion function $E \in S$ can be decomposed into two such parts. That is, we can express it as the difference between a dilation $G_E \in D$ and a contraction $R_E \in C$ where, for each $Z \in [X]^*$

$$E(Z) = G_E(Z) \setminus R_E(Z)$$

with $G_E(Z) = E(Z) \cup Z$ and $R_E(Z) = Z \setminus E(Z)$.

These two components of the decomposition are economically meaningful. Intuitively, $R_E$ captures the scope of reuse by the second choice function $C_2$. If
$R_E(Z) \neq \emptyset$, then some items chosen by the first choice function $C_1$ can be rechosen by $C_2$. In turn, $G_E$ captures the scope of gross exclusions that cannot contribute to incremental choice by $C_2$. For each $z \in G_E(Z)$, either $z \in E(Z)$ so that $z$ is not choosable by $C_2$; or $z \in R_E(Z)$ so that $C_2$ can choose $z$ but cannot add it to the aggregate choice (as it is already chosen).

If $E$ is a contraction, then since the domain of $E$ is all finite subsets of $X$, it is never the case that $G_E(Z) = X$ for any $Z$. However, if $E$ is not a contraction, whenever $G_E(Z) = X$, $R_E$ is irrelevant. In that case, the set of items from which $C_2$ must choose is a subset of $Z$, the items already chosen by $C_1$. So, whatever scope of reuse is permitted, there is no way for $C_2$ to affect the aggregate choice. Denote the domain of sets where $R_E$ is (potentially) relevant by $\text{Dom}(R_E) = \{Z \in [X]^* : G_E(Z) \neq X\}$.

The above observation about relevance leads us to refine our necessary conditions from Section 3.2. Given an exclusion function $E \in S$, $R_E$ is monotonic on $\text{Dom}(R_E)$ if for any $Z, Z' \in \text{Dom}(R_E)$ such that $Z \subseteq Z'$, $R_E(Z) \subseteq R_E(Z')$. Similarly, $R_E$ is cardinal-linear on $\text{Dom}(R_E)$ if there is a sequence $\{T^n\}_{n=0}^\infty$ such that $\emptyset = T^0 \subseteq T^1 \subseteq T^2 \ldots$, and for each $Z \in \text{Dom}(R_E)$, $R_E(Z) = Z \cap T^{|Z|}$. The only difference from our earlier definitions is that we only require the conclusion to hold over $\text{Dom}(R_E)$ as opposed to all of $[X]^*$.

The necessary conditions in Section 3.1 and the above adaptations of those in Section 3.2 are necessary for general exclusion functions as well, except that they apply to the corresponding gross exclusion and reuse components.

**Claim 6.** If $L_E$ preserves path independence over $C^{\text{res}}$, then

1. $G_E$ is threshold-linear, and
2. $R_E$ is cardinal-linear on $\text{Dom}(R_E)$.

The conditions listed in claim 6 leave out the interaction between the gross exclusion and reuse. The next condition says that no items in $K$ can ever be reused. Given an exclusion function $E \in S$, $R_E$ is K-disjoint on $\text{Dom}(R_E)$ if for every $Z \in \text{Dom}(R_E)$, $R_E(Z) \cap K = \emptyset$.

**Claim 7.** If $L_E$ preserves path independence over $C^{\text{res}}$, then $R_E$ is K-disjoint on $\text{Dom}(R_E)$.

\[\text{If } G_E \text{ is monotonic and } Z \in \text{Dom}(R_E), \text{ then for all } Z' \subset Z, Z' \in \text{Dom}(R_E). \text{ Moreover, if } G_E \text{ is all-or-nothing, then } G_E(Z) = Z \cup K \subset X \text{ for all } Z \in \text{Dom}(R_E).\]
We now define a class of exclusion functions that satisfy all of the above mentioned necessary conditions to preserve path independence. An exclusion function $E \in S$ is **threshold-linear with cardinal reuse** if there are $t \in \mathbb{N} \cup \{0, \infty\}$, $K \subseteq X$, and $\{T^n\}_{n=0}^t$ where $\emptyset = T^0 \subseteq T^1 \subseteq \cdots \subseteq T^t \subseteq X \setminus K$, such that for each $Z \in [X]^*$, if $|Z| < t$

$$E(Z) = (Z \setminus T^{|Z|}) \cup K,$$

and otherwise $G_E(Z) = X$. It follows that $Dom(R_E) = \{Z \in [X]^*: Z \cup K \subset X \text{ and } |Z| < t\}$ and for each $Z \in Dom(R_E)$, $G_E(Z) = Z \cup K$ and $R_E(Z) = Z \cap T^{|Z|}$.

**Theorem 1.** $L_E$ preserves path independence over $C^{res}$ if and only if $E$ is threshold-linear with cardinal reuse.

Note that Theorem 1 implies both Proposition 1 and Proposition 2. For the former, we simply restrict attention to exclusion functions that are threshold-linear with cardinal reuse where $T^l = \emptyset$ for each $l$. For the latter, we restrict attention to the identity dilation, $G_E(Z) = Z$ for every $Z \in [X]^*$.

By Theorem 1, the preservation of path independence over $C^{pi}$ imposes a considerable amount of structure on the exclusion function. How much of this is due to the substitutes component of path independence as opposed to consistency? All of it: the preservation of consistency, even on the larger domain of $C^{con}$ places no constraints on the exclusion function.

**Remark 1.** For any $E \in S$, $L_E$ preserves consistency over $C^{con}$.

## 4 Broader Domains and Additional Properties

There are two pertinent ways in which we may modify the question answered by Theorem 1. The first is to extend the domain beyond $C^{res}$. By considering the preservation of path independence over $C^{pi}$, we can accommodate nested composition of more than two choice functions, which we return to in Section 5.1. In the second direction, we consider the preservation of more than just path independence.
4.1 Expanding the Domain

While threshold-linearity with cardinal reuse ensures that an exclusion function preserves path independence over $C^\text{res}$, such a guarantee does not hold for input choice functions that are path independent but not responsive.\footnote{Note that if we take inputs that are not themselves path independent, there is no hope that the composite choice function would be path independent. For this reason we stop at expanding the domain to $C^\text{pi}$.}

Since $C^\text{res} \subset C^\text{pi}$, Theorem 1 implies that it is necessary for any $E \in S$ such that $L_E$ preserves path independence over $C^\text{pi}$ to be threshold-linear with cardinal reuse. However, this is not sufficient as only some of these exclusion functions preserve path independence over the larger domain. The only threshold-linear exclusion functions with cardinal reuse that do so are ones that are almost constant (the threshold is either 0 or 1) or are linear (the threshold is $\infty$). Moreover, reuse is limited in that expanding the choice beyond two items does not induce further reusable items.

**Proposition 3.** $L_E$ preserves path independence over $C^\text{pi}$ if and only if $E$ is threshold-linear with cardinal reuse with respect to $K \subset X$, $t \in \{0, 1, \infty\}$, and $\{T^n\}_{n=0}^\ell$ such that if $t > 0$ then $\emptyset = T^0 \subseteq T^1 \subseteq T^2 = T^3 = \cdots = T^\ell \subseteq X \setminus K$.

If we further expand the domain to $C^\text{sub} \supset C^\text{pi}$, only the possibility of $t = \infty$ with $T^1 = T^2 = \cdots = T^\infty$ remains if we wish to preserve substitutes.

4.2 Preserving an Additional Property

We now see how the result changes if we are more demanding, not about the domain, but about what properties are preserved. We consider an additional property that says the choice from a set contains at least as many elements as a choice from any subset. A choice function $C \in \mathcal{C}$ is **size monotonic (sm)** if, for each pair $Y, Y' \in [X]^*$, $Y \subseteq Y'$ implies that $|C(Y)| \leq |C(Y')|$. This property arises often in the matching literature along with substitutes and the following is well known.

**Remark 2.** $C^\text{res} \subset C^\text{sub-sm} \subset C^\text{pi}$.

We now consider the preservation of not only path independence, but the conjunc-
tion of substitutes and size monotonicity over $c^{\text{res}}$. The additional requirement shrinks the set of exclusion functions, much as expanding the domain did in Section 4.1. The added exigency of preserving size monotonicity not only restricts the threshold, but also severely limits reuse.

**Proposition 4.** $L_E$ preserves substitutes and size monotonicity over $c^{\text{res}}$ if and only if $E$ is threshold-linear with cardinal reuse with respect to $K \subset X$, $t \in \mathbb{N} \cup \{0, \infty\}$, and $\{T^n\}_{n=0}^t$ such that either

1. $|X \setminus K| \leq 1$ or
2. $t = \infty$ and for each $n < t$, $T^n = \emptyset$.

5 Nested Composition

Given an exclusion function $E \in S$, $L_E$ is a binary operator on choice functions. For special cases such as the identity exclusion function, it is associative. In fact, for the empty exclusion (“full reuse” as in the representation of Aizerman and Malishevski (1981)) it is commutative as well. However, $L_E$ is not generally associative or commutative. In this section, we consider a few ways in which compositions may be nested and pay special attention to the aggregation of single valued choice functions.

5.1 Composing More Than Two Choice Functions

As mentioned above, lexicographic composition is not generally associative or commutative. Since the purpose of composition is to build up “larger” choice functions from various “parts,” there are many ways we can go about this. To illustrate what is at stake, we start with just three choice functions $C_1, C_2$, and $C_3$ in that order. In this case, there are two possibilities, which can be thought of as “left composition” and “right composition,” respectively:

$L_{E_1}(C_1, L_{E_2}(C_2, C_3))$ and $L_{E_2}(L_{E_1}(C_1, C_2), C_3)$.

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16The sufficiency part of Theorem 1 actually extends to $c^{\text{sub-sm}}$. That is, if $E \in S$ is threshold-linear with cardinal reuse, then $L_E$ preserves path independence over $c^{\text{sub-sm}}$. However, as stated in Proposition 4 threshold linearity with cardinal reuse is not sufficient to preserve size monotonicity.
Where \((E_1, E_2)\) and \((\hat{E}_1, \hat{E}_2)\) are two pairs of exclusion functions.

Neither direction of composition is more general than the other. To see this, consider what happens if \(E_1 = \hat{E}_1\) and \(E_2 = \hat{E}_2\), and we start from some set \(Y\) to choose from. In the first step, \(C_1\) selects \(Z_1 = C_1(Y)\) no matter which of the two directions we compose in. The set that \(C_2\) chooses from is then \(Y \setminus E_1(Z_1)\) in both cases. Thus, the inputs to the first two choice functions are exactly the same, so \(C_2\) selects \(Z_2 = C_2(Y \setminus E_1(Z_1))\). The difference, however, is when we get to the third choice function. The exclusion from the set that \(C_3\) chooses from, under left and right compositions, are

\[
E_2[Z_1 \cup Z_2] \text{ and } E_1(Z_1) \cup E_2(Z_2)
\]

respectively. These are not necessarily the same. As we show below, neither format is more general than the other in terms of what policies can be achieved.

Suppose \(C_1, \ldots, C_m\) are a sequence of choice functions where \(m > 2\). Consider two procedures aggregating these \(m\) choice functions.

**Procedure 1.** Fix an integer \(N\). Starting with \(C_1\) and continuing until \(C_m\), each successive choice function can choose freely from the remainder of the input set if prior choice functions selected fewer than a total of \(N\) alternatives, but it cannot choose anything otherwise.

**Procedure 2.** Fix an integer \(N\). Starting with \(C_1\) and continuing until \(C_m\), each successive choice function can choose freely from the remainder of the input set if each prior choice function individually selected fewer than \(N\) alternatives, but it cannot choose anything otherwise.

**Claim 8.**
1. Procedure 1 can be implemented via right composition but not left composition.
2. Procedure 2 can be implemented via left composition but not right composition.

These two procedures capture the crucial difference between right and left composition. Left composition cannot condition exclusion for the \(k\)th choice function on anything other than what the \((k-1)\)th choice was. In this sense, it has “no memory” of prior choices. While prior choices still affect what is available to the \(k\)th via the input set to not only the \(k\)th but also the \((k-1)\)th choice function, they do not affect it through exclusion. On the other hand, right composition necessarily conditions exclusion for the \(k\)th choice function on the union of all prior choices. In this sense,
there is an “aggregate memory” of prior choices (though the individual choices are not discernable). Both no memory and aggregate memory are restrictive. However, as we have shown above, they are restrictive in different ways.

With more than three choice functions, one can combine them by mixing right and left composition. Specifically there are $\frac{1}{m} \binom{2m-1}{m-1}$ (the $m-1$\textsuperscript{th} Catalan number) ways to combine $m$ choice functions. Below are a few examples.

**Example 1. Soft quotas.** Suppose $X$ is the universe of candidates a firm with $n$ divisions can choose to hire from. Let $C_i$ be the $i$\textsuperscript{th} division’s choice function. Moreover, suppose that each hire can only work at one division. The firm can impose a soft quota of $k$ on the number of hires as follows. For each $Z \in [X]^*$, let $E(Z)$ be $Z$ if $|Z| \leq k$ and $X$ otherwise. The aggregated choice function is then

$$L_E(L_E(...L_E(C_1,C_2),...),C_n).$$

Then, for any set of applicants $Y$, the divisions get to choose whom to hire, from the first to the $n$\textsuperscript{th}. However, the process stops as soon as at least $k$ applicants have been chosen.

If the component choice functions $C_i$ are known to be single-valued, $k$ serves as a hard constraint.

**Example 2. Nested reserves.** Let $\{C_j\}_{j=1}^n$ be a sequence of $n$ choice functions and $\{X_j\}_{j=1}^n$ be a monotonic sequence of subsets of $X$ such that $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n$. Each item in $X_j$ can only be chosen by $C_i$ such that $i \leq j$. Thinking of the choice functions as representing resources, the $j$\textsuperscript{th} resource is reserved for $X_j$. Thus, elements of $X_n$ are most favored while elements of $X_1 \setminus X_2$ are least favored.

**Example 3. Inter-district school choice.** In the school choice model, $X$ is the set of all students. Reserves like in Example 2 are a common way to implement policies like affirmative action in schools’ choice functions. The inter-district version of this model (Hafalir et al., 2018) is nested in the sense the schools’ choice functions are aggregated into a district level choice functions and students are then matched to districts.

### 5.2 Single-valued Inputs

When it comes to constructing choice functions from simpler components, single-valued, responsive choice functions (sv-res) are a salient domain.\footnote{By “single-valued” we mean that $q = 1$, so that such a choice function may select $\emptyset$.} They are
commonly studied in the matching literature and often composed using the identity exclusion function (Kominers and Sönmez, 2016). At the opposite extreme, lexicographic composition of such choice functions with the empty exclusion function spans \( C^{\text{pi}} \) (Aizerman and Malishevski, 1981). In this section, we consider the preservation of path independence when lexicographically combining a single-valued, responsive choice function with a path independent choice function.

We first consider lexicographic composition with the first choice function being in \( C^{\text{sv-res}} \) and the second in \( C^{\text{pi}} \). The only relevant parts of an exclusion function’s domain in this case are the set of singletons and the empty set.

**Proposition 5.** \( L_E \) preserves path independence over \( C^{\text{sv-res}} \times C^{\text{pi}} \) if and only if, on the set of singletons and the empty set \( E \) coincides with an exclusion function that is threshold-linear with cardinal reuse.

Next, we consider preserving path independence over \( C^{\text{pi}} \times C^{\text{sv-res}} \). The proofs of necessity building up to (and including) that of Proposition 3 do not use the full domain of \( C^{\text{res}} \) for the second choice function. They only appeal to single-valued choice functions. Sufficiency of the same conditions, of course, follows from sufficiency for the broader domain of \( C^{\text{pi}} \).

**Proposition 6.** \( L_E \) preserves path independence over \( C^{\text{pi}} \times C^{\text{sv-res}} \) if and only if \( E \) is threshold-linear with cardinal reuse with respect to \( K \subseteq X, \ t \in \{0,1,\infty\} \), and \( \{T^n\}_{n=0}^t \) such that if \( t > 0 \) then \( \emptyset = T^0 \subseteq T^1 \subseteq T^2 = T^3 \cdots = T^t \subseteq X \setminus K \).

In Appendix B, we consider the preservation of size monotonicity as well. Since applications typically involve nested composition, we restrict the domain to \( C^{\text{sv-res}} \times C^{\text{sub-sm}} \) for the analog of Proposition 5 and to \( C^{\text{sub-sm}} \times C^{\text{sv-res}} \) for the analog of Proposition 6.

## 6 Equivalence Relations and Matching With Contracts

In the canonical setting of many-to-one matching with contracts, the market participants are hospitals and doctors. While a hospital may choose multiple contracts, it cannot choose more than one contract per doctor. In this context, when the
items are contracts, there is an equivalence relation over items: Are $x$ and $y$ contracts with the same doctor? We can then consider choice functions that select no more than one element of each equivalence class.

Denote by $\sim$ the equivalence relation in question. We assume that the number of equivalence classes of $\sim$ is countable. Given $x \in X$, let $I_x$ be the equivalence class of $\sim$ that $x$ belongs to. Given a set $Z \subseteq X$, let $I_Z = \cup_{x \in Z} I_x$.

The choice function $C$ is many-to-one (mto1) if, for each $Y \in [X]^*$, $x, z \in C(Y)$ implies $x \sim z$. In this setting, the natural minimal exclusion function forces the composition of two choice functions to respect many-to-one-ness. We define it as follows. For each $Z \in [X]^*$,

$$E(Z) = I_Z.$$

Using this definition, our notion of lexicographic composition $L_E$ generalizes the slot-specific priorities model of Kominers and Sönmez (2016). They showed that $L_E$ need not preserve substitutes over $C^{res}$ but must be consistent and satisfy a weaker “bilateral” notion of substitutes first proposed by Hatfield and Kojima (2010). Subsequently, Hatfield and Kominers (2016) showed that any choice function with slot-specific priorities can be “completed” so that it satisfies substitutes and size monotonicity.

Let $[X]^*_F = \{ Y \in [X]^* : |\{ I_x \}_Y| = |Y| \}$ be the feasible sets of items that contain no more than one item from each equivalence class. Then, $\overline{C} \in C$ completes a many-to-one choice function $C \in C^{mto1}$ if, for all $Y \in [X]^*$, $\overline{C}(Y) \in [X]^*_F$ implies $\overline{C}(Y) = C(Y)$.

Analogous to earlier notation, for a given property $\pi$, denote by $C^{\pi}$ the set of many-to-one choice functions with completions that satisfy $\pi$.

To preserve many-to-one-ness of the input choice functions, it is necessary (and sufficient) that, for each set, $E$ extends $E$. Formally, an exclusion function $E \in E$ is equivalence-excluding if, for all $Z \in [X]^*_F$, $Z \cup E(Z) \supseteq E(Z)$.

**Remark 3.** $L_E$ preserves many-to-one-ness over $C^{mto1}$ if and only if $E$ is equivalence-excluding.

Inspection of the proof makes it clear that this same condition is necessary (and sufficient) to preserve many-to-one-ness for input choice functions in $C^{mto1-res}$.

For the exclusion function $E$, which often appears in the literature, while $G_E$ is monotonic, it is *not* all-or-nothing. Indeed for any equivalence-excluding exclusion
function $E$, if $G_E$ is all-or-nothing then for every nonempty $Z \in [X]^*$ $G_E(Z) = X$, implying that $\text{Dom}(R_E) = \emptyset$. This means that, to preserve path independence and many-to-oneness (over $C_{\text{pi}}$), the exclusion function $E$ must entirely shut down the second input $C_2$.

Nonetheless, simple adaptations of our earlier conditions are sufficient to preserve the possibility of a path independent completion over $C_{\text{pi}}$ and size monotonicity over $C_{\text{sub-sm}}$.

As long as the first input choice function is many-to-one, the effective domain of $E$ is the range of a many-to-one choice function. So, there is no need to impose any restrictions on $E$ beyond $[X]^*$.

With this in mind, we modify our definition of threshold-linearity with cardinal reuse. An exclusion function $E \in S$ is many-to-one threshold-linear with cardinal reuse if there are $t \in \mathbb{N} \cup \{0, \infty\}$, $K \subseteq X$, and $\{T^n\}_{n=0}^t$ where $\emptyset = T^0 \subseteq T^1 \subseteq \cdots \subseteq T^t \subseteq X \setminus K$, such that for each $Z \in [X]^*$, if $Z \cup K \subseteq X$ and $|Z| < t$, $E(Z) = (Z \setminus T^{|Z|}) \cup K$ and otherwise $G_E(Z) = X$. The only change from the definition of threshold-linearity with cardinal reuse in Section 3 is the underlined part.

Analogs to the sufficiency parts of Theorem 1 and propositions 3 and 4 are consequences of the following result.

**Lemma 1.** Suppose that $E$ is equivalence-excluding and many-to-one threshold-linear with cardinal reuse for parameters $t, K$, and $\{T^n\}$. Let $E$ be threshold-linear with cardinal reuse with the same parameters.

If $\overline{C_i}$ completes $C_i$ for $i = 1, 2$ and, in addition, $\overline{C_2}$ satisfies consistency, then $\mathcal{L}_E(\overline{C_1}, \overline{C_2})$ completes $\mathcal{L}_E(C_1, C_2)$.

It is natural to think of $E$ as “removing” items from what is available to $C_2$ as a function of what $C_1$ chooses. As stated in Remark 3, preserving many-to-oneness requires that $E$ remove all items related by $\sim$ to any item chosen by $C_1$. The idea of the exclusion functions $E$ defined in Lemma 1 is to “put back” all but the items that were actually chosen by $C_1$.

The necessary conditions, however, are not exactly analogous to those in Theorem 1 and propositions 3 and 4. In particular, the all-or-nothingness of gross exclusion is not necessary to preserve path independent completability over $C_{\text{pi}}$ (even though the standard version of this condition is necessary to preserve path independence over $C_{\pi}$).
Example 4. Let $X$ be such that $X \supseteq \{a, b, c\}$ where $a \sim c \sim b$ and $I_a = \{a, c\}$. That is, items $a$ and $c$ are related by $\sim$ and complete their equivalence class, and there is a third item $b$ that is unrelated to $a$ and $c$. Let $E$ be such that for each $Z \in [X]^*_F$,

$$E(Z) = \begin{cases} 
X \setminus \{a\} & \text{if } I_a \cap I_Z = \emptyset, \\
I_a & \text{if } I_a = I_Z, \\
X & \text{if } I_a \subsetneq I_Z.
\end{cases}$$

Note that $E$ is equivalence-excluding and $G_E$ is monotonic over $[X]^*_F$. However, it violates all-or-nothingness since, for instance, $\{b\} \cup E(\{b\}) = X \setminus \{a\}$ while $\{a, b\} \cup E(\{a, b\}) = X$. Given a pair $C_1, C_2 \in \mathcal{C}^{pl}$, let $C = L_E(C_1, C_2)$. Let $\overline{C} \in \mathcal{C}_{sub}$ be such that for each $Y \in [X]^*$,

$$\overline{C}(Y) = \begin{cases} 
C(Y) & \text{if } Y \neq \{a, c\}, \\
\{a, c\} & \text{if } a \in C(X) \text{ and } C(\{a, c\}) = \{c\}, \text{ and} \\
C(\{a, c\}) & \text{otherwise}.
\end{cases}$$

$\overline{C}$ is a path independent completion of $C$ even though $G_E$ violates all-or-nothingness on $[X]^*_F$.

Our reasoning from the proof of Theorem 1 does not hold here because the sets picked by the first choice function are not exhaustive and are limited to $[X]^*_F$. The way Example 4 deviates from it is, in a sense, the limit. In Appendix C, we weaken all-or-nothingness to allow only the violations such as in Example 4 and show that such a condition is necessary.

7 Discussion

We have considered choice functions of the type $[X]^* \to [X]$. Our formulation of feasibility as exclusion is appropriate for applications where the designer of the aggregation rule does not control the individual choice functions. In a sense, our results delineate the limits of what such a designer can achieve.

What if, instead, the designer can exert more control on the individual choice functions? Perhaps a choice function takes as input not only a set that it may choose from, but also parameter from the set $\mathcal{A}$. A choice function would then be of the type $[X]^* \times \mathcal{A} \to [X]$. An exclusion function, in this case, would also be of this type. Since the designer of the exclusion functions would have more information
to condition exclusion on, as well as an extra parameter to the individual choice functions, they would have greater influence on the aggregate choice function.

As an extreme example, the parameters could include all of the information generated by the composition process in the form of a sequence of sets: the initial set, the set chosen by the first choice function, that chosen by the second choice function, and so on. This is a strictly more general formulation than ours. In our formulation, if $C_2$ does not see an item $x$ in its input, it cannot tell whether this is because $x$ was never a possibility or because it has been excluded based on earlier choices. In the more general model, the $k^{th}$ choice function $C_k$ would have as an extra input a sequence of sets of items $\{Z_n\}_{n=0}^{k-1}$ where $Z_0$ is the initial set and $Z_1$ through $Z_{k-1}$ are the choices of the first through $k-1^{th}$ choice functions. The exclusion $E_{k-1}$ applied prior to the $k^{th}$ choice, would take the same sequence as an input and return a menu of sets that $C_k$ can choose from. The right composition of a sequence of $n$ choice functions would then produce from the input $Y$ the choice $\cup_{i=1}^n Z_i$ where $Z_0 = Y$, $Z_1 = C_1(Y, \{Z_0\})$, and for each $i = 2, \ldots, n$,

$$Z_i = C_i(Y \setminus E_i(\{Z_l\}_{l=0}^{i-1}, \{Z_l\}_{l=0}^{i-1}))$$

For such a model, the question would be "what conditions on $E$ and the way the $C_i$s depend on the parameter ensure that the resulting choice function inherits desirable properties of the individual choice functions?" A comprehensive analysis of this more general question is beyond the scope of the current paper, which is a first step towards understanding path independence of lexicographic compositions.

A tractable next step could be to focus on specific features of prior choices. For instance, contrary to our definition of $C^{res}$, where capacities are inherent to the choice functions, they might be a parameter. Exclusions could depend on the cardinalities of prior choices. Westkamp (2013) demonstrates a way in which the cardinalities along these sequences can factor into the choices so that the responsive choice functions yield a path independent choice function in the end.\textsuperscript{18}

\textsuperscript{18}Aygün and Turhan (2020) just extend this to equivalence-based setting like in Section 6.
A Proofs

A.1 Proofs From Section 3

Claims 1 to 5 are implied by Claim 6. Propositions 1 and 2 are special cases of Theorem 1. So, we omit the proofs of the results from Sections 3.1 and 3.2 and proceed to the proofs of results in Section 3.3.

Proof of Claim 6. To conclude that \( G_E \) is threshold-linear, we prove that it satisfies each of the properties defined in Section 3.1.

Monotonicity of \( G_E \): Suppose that \( G_E \) is not monotonic. Then, there are \( Z, Z' \subseteq X \) such that \( Z \subseteq Z' \) and \( G_E(Z) \not\subseteq G_E(Z') \). Let \( a \in G_E(Z) \setminus G_E(Z') \). Since \( Z \subseteq Z' \), \( a \notin Z \) and hence \( a \in E(Z) \setminus E(Z') \).

Let \( C_1 \in C^{\text{res}} \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \varnothing \} = Z' \) and \( q_1 = |Z'| \). Let \( C_2 \in C^{\text{res}} \) be induced by \( >_2 \) such that \( \{ x \in X : x >_2 \varnothing \} = \{ a \} \) and \( q_2 = 1 \).

Let \( C = L_E(C_1, C_2) \). Then, \( a \in C(Z' \cup \{ a \}) = Z' \cup \{ a \} \) but \( a \notin C(Z \cup \{ a \}) = Z \), which violates path independence. So, \( G_E \) is monotonic.

All-or-nothingness of \( G_E \): Suppose that \( G_E \) is monotonic but not all-or-nothing. Let \( \underline{Z} \in [X]^* \) be such that \( X \supseteq G_E(\underline{Z}) \supseteq \underline{Z} \cup K \) and for each \( Z \subseteq \underline{Z} \), \( G_E(Z) = Z \cup K \).

Since \( K = E(\varnothing) = G_E(\varnothing) = \emptyset \cup K \), such \( \underline{Z} \) exists. By definition of \( \underline{Z} \), there is some \( a \in X \setminus G_E(\underline{Z}) \), \( b \in \underline{Z} \), and \( c \in E(\underline{Z}) \setminus (\underline{Z} \cup K) \).

Let \( C_1 \in C^{\text{res}} \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \varnothing \} = \underline{Z} \) and \( q_1 = |\underline{Z}| \). Let \( C_2 \in C^{\text{res}} \) be induced by \( >_2 \) such that \( \{ x \in X : x >_2 \varnothing \} = \{ a, c \} \) and \( c >_2 a >_2 \varnothing \) and \( q_2 = 1 \).

Let \( C = L_E(C_1, C_2) \). Then, \( a \in C((\underline{Z} \cup \{ a, c \}) \setminus \{ b \}) = (\underline{Z} \cup \{ c \}) \setminus \{ b \} \), which violates path independence. So, \( G_E \) is all-or-nothing.

Cardinality of \( G_E \): Let \( Z, Z' \in [X]^* \) be such that \( |Z| = |Z'| \), \( Z \cup K \neq X \), and \( G_E(Z) = X \).

We consider two cases to show that this implies that \( G_E(Z') = X \).

Case 1 \((Z \cup Z' \cup K \subseteq X)\): Towards a contradiction, suppose that \( G_E(Z') \neq X \). Since \( G_E \) is all-or-nothing and monotonic, \( G_E(Z'') = Z'' \cup K \) for all \( Z'' \subseteq Z' \).

Let \( a \in X \setminus (Z \cup Z' \cup K) \).

Let \( C_1 \in C^{\text{res}} \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \varnothing \} = Z \cup Z' \) and, for each \( z' \in Z' \) and \( z \in Z \setminus Z' \), \( z' >_1 z \) and \( q_1 = |Z| \). Let \( C_2 \in C^{\text{res}} \) be induced
by $>_2$ such that \( \{x \in X : x >_2 \emptyset\} = \{a\} \) and $q_2 = 1$. Let $C = \mathcal{L}_E(C_1, C_2)$. By definition, $a \in C(Z \cup Z' \cup \{a\}) = Z' \cup \{a\}$ and $a \in C(Z \cup \{a\}) = Z$, which violates substitutes. So, $G_E(Z') = X$.

**Case 2 (\( Z \cup Z' \cap K = X \)):** Let $a \in X \setminus (Z \cup K)$ and $b \in X \setminus (Z \cup \{a\})$. Since $Z \cup Z' \cup K = X$, $a \in Z'$. Let $Z'' = (Z' \cup \{b\}) \setminus \{a\}$. Since $Z \cup Z'' \cup K \subset X$, $G_E(Z) = X$ implies $G_E(Z'') = X$ by the argument in Case 1. (This argument holds even if $b \in Z'$.) If $b \in Z'$, then $G_E(Z'') = X$ implies $G_E(Z') = X$ by the argument in Case 1. Otherwise, $b \in Z'$. Then, $G_E(Z'') = X$ implies $G_E(Z') = X$ by monotonicity of $G_E$.

Since $G_E$ is monotonic, all-or-nothing, and cardinal, it is threshold-linear. Before showing that $R_E$ is cardinal-linear on $\text{Dom}(R_E)$, we show that it is monotonic on $\text{Dom}(R_E)$.

**Monotonicity of $R_E$ on $\text{Dom}(R_E)$:** Suppose $R_E$ is not monotonic on $\text{Dom}(R_E)$. Then, there are $Z, Z' \in \text{Dom}(R_E)$ such that $Z \subset Z'$ and $R_E(Z) \not\subset R_E(Z')$. So, there is some $a \in Z$ such that $a \in R_E(Z) \setminus R_E(Z')$. Since $Z' \in \text{Dom}(R_E)$, there is some $b \in X \setminus G_E(Z')$. (Since $b \not\in E(Z')$ but $a \in E(Z')$, $b \neq a$.)

Let $C_1 \in C^{\text{res}}$ be induced by $>_1$ such that \( \{x \in X : x >_1 \emptyset\} = Z' \) and $q_1 = |Z'|$. Let $C_2 \in C^{\text{res}}$ be induced by $>_2$ such that \( \{x \in X : x >_2 \emptyset\} = \{a, b\} \) and $a >_2 b >_2 \emptyset$ and $q_2 = 1$.

Let $C = \mathcal{L}_E(C_1, C_2)$. Then, $b \in C(Z' \cup \{b\}) = Z' \cup \{b\}$ but $b \not\in C(Z \cup \{b\}) = Z$, which violates path independence. So, $R_E$ is monotonic on $\text{Dom}(R_E)$.

**Cardinal-linearity of $R_E$ on $\text{Dom}(R_E)$:** We first show that there is no pair $Z, Z' \in \text{Dom}(R_E)$ such that $|Z| = |Z'|$ for which there is $a \in Z \cap Z'$ such that $a \in R_E(Z) \setminus R_E(Z')$. If such a pair does exist, there are two cases to consider.

**Case 1 (\( K \cup Z \cup Z' \neq X \)):** Let $b \in X \setminus (K \cup Z \cup Z')$. Let $C_1 \in C^{\text{res}}$ be induced by $q_1 = |Z|$ and $>_1$ such that \( \{x \in X : x >_1 \emptyset\} = Z \cup Z' \) and for each $x' \in Z'$ and each $x \in Z \setminus Z'$, $x' >_1 x$. Then, $C_1(Z \cup \{b\}) = Z$ and $C_1(Z \cup Z' \cup \{b\}) = Z'$. Let $C_2 \in C^{\text{res}}$ be induced by $>_2$ such that \( \{x \in X : x >_2 \emptyset\} = \{a, b\} \) and $a >_2 b >_2 \emptyset$ and $q_2 = 1$.

Let $C = \mathcal{L}_E(C_1, C_2)$. Then, $b \in C(Z \cup Z' \cup \{b\}) = Z' \cup \{b\}$ but $b \not\in C(Z \cup \{b\}) = Z$, which violates path independence.

**Case 2 (\( K \cup Z \cup Z' = X \)):** Since $Z$ and $Z'$ are finite, there is some $b \in X \setminus (Z \cup Z')$. Since $Z \in \text{Dom}(R_E)$, there is some $c \in X \setminus (Z \cup K) \subset Z'$. Since $a \in Z$, $c \neq a$. 21
Let $Z'' = (Z' \setminus \{c\}) \cup \{b\}$. Since $a \in Z''$, $Z \cup Z'' \cup K = X \setminus \{c\}$ and $|Z''| = |Z'|$. By Case 1, $a \in R_E(Z)$ implies that $a \in R_E(Z'')$. Since $b \notin Z'$, $Z' \nsubseteq Z''$ and $Z' \cup Z'' \cup K = Z' \cup K \subseteq X$. So, again by Case 1, $a \in R_E(Z'')$ implies $a \in R_E(Z')$.

Thus, for each $n \in \mathbb{N} \cup \{0, \infty\}$, letting $T^n = \bigcup_{Z \in \text{dom}(R_E), |Z| = n} R_E(Z)$, we have that for each $Z \in \text{dom}(R_E)$, $R_E(Z) = Z \cap T^{|Z|}$. By monotonicity of $R_E$ on $\text{dom}(R_E)$ and since $T^0 = \emptyset$, we conclude that $\emptyset = T^0 \subseteq T^1 \subseteq T^2 \subseteq \ldots$. Thus, $R_E$ is cardinal-linear on $\text{dom}(R_E)$.

**Proof of Claim 7.** Suppose $R_E$ is not $K$-disjoint on $\text{dom}(R_E)$. Then, there is $Z \in \text{dom}(R_E)$ with some $a \in R_E(Z) \cap K$. Since $Z \in \text{dom}(R_E)$ and $G_E$ is all-or-nothing, $G_E(Z) = Z \cup K \subseteq X$. So, there is $b \in X \setminus Z \cup \{b\}$. Since $Z$ is finite, there is some $Z' \subseteq X \setminus (Z \cup \{b\})$ such that $|Z'| = |Z|$. By construction, $a \notin Z'$. Since $G_E$ is all-or-nothing, $a \not\in K \subseteq E(Z')$. Let $C_1 \in C^{\text{res}}$ be induced by $\{x \in X : x \cdot 1 \geq x\} = Z \cup Z'$ and $x \cdot 1$ for each $x \in Z'$ and $y \in Z \setminus Z'$ and $q_1 = |Z'|$. Let $C_2 \in C^{\text{res}}$ be induced by $\{x \in X : x \cdot 2 \geq x\} = \{a, b\}$ and $a \cdot 2$ and $q_2 = 1$. Let $C = L_E(C_1, C_2)$. Then, $b \in C(Z \cup Z' \cup \{b\}) = Z' \cup \{b\}$ but $b \notin C(Z \cup \{b\}) = Z$, which violates path independence. So, $R_E$ is $K$-disjoint on $\text{dom}(R_E)$.

**Proof of Theorem 1.** We first show necessity and then sufficiency.

**Necessity:** By Claim 6 and Claim 7, for $L_E$ to preserve path independence over $C^{\text{res}}$, it is necessary that

1. $G_E$ is threshold-linear.
2. $R_E$ is cardinal-linear and $K$-disjoint on $\text{dom}(R_E)$.

These together imply that $E$ is threshold-linear with cardinal reuse. This establishes the necessity part of the theorem.

**Sufficiency:** We show that if $E$ is threshold-linear with cardinal reuse, then $L_E$ preserves path independence over $C^{\text{sub-sm}}$.

Suppose $E$ is parameterized by $K \subseteq X$, $t \in \mathbb{N} \cup \{0, \infty\}$, and $(T^s)^{t_s}_{s=1}$.

Let $C = L_E(C_1, C_2)$ and fix $Y \subseteq Y' \in [X]^t$ and $x \in Y$ such that $x \in C(Y')$. We show that $x \in C(Y)$. Let $Z = C_1(Y)$ and $Z' = C_1(Y')$. If $x \in Z$ we are done since $Z \subseteq C(Y)$.

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19Note that this is stronger than what Theorem 1 claims, which is sufficiency over only $C^{\text{res}}$, a subset of $C^{\text{sub-sm}}$. 

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If \( x \in Z' \), then since \( C_1 \) is path independent, \( x \in Z \) and we are again done. It remains to consider the case where \( x \not\in Z' \) and \( x \not\in Z \). Since \( x \not\in C(Y') \), this means \( x \in C_2(Y' \setminus E(Z')) \). Then, since \( x \not\in G(E(Z')) \), \( G(E(Z')) \neq X \) and \( |Z'| < t \). So, \( E(Z') = (Z' \setminus T[Z]) \cup K \). This implies that \( x \not\in K \). Thus, since \( x \not\in Z \) and \( x \not\in K \), it follows that \( x \not\in E(Z) \) and therefore \( x \not\in Y \setminus E(Z) \).

Since \( C_2 \) is path independent, it suffices to show that \( Y \setminus E(Z) \subseteq Y' \setminus E(Z') \). Since \( C_1 \) is size monotonic and \( Y \subseteq Y' \), \( |Z| \leq |Z'| < t \). Thus, \( E(Z) = (Z \setminus T[Z]) \cup K \) and \( T[Z] \subseteq T[Z'] \).

Since \( C_1 \) is path independent, \( Z' \setminus Y \subseteq Z \), so \( (Z' \setminus Y) \setminus T[Z] \subseteq Z \setminus T[Z] \). Removing both sides from \( Y \),

\[
Y \setminus (Z \setminus T[Z]) \subseteq Y' \setminus ((Z' \setminus Y) \setminus T[Z]) \subseteq Y' \setminus ((Z' \setminus Y) \setminus T[Z]) = Y' \setminus (Z' \setminus T[Z]).
\]

Removing \( K \) from both sides,

\[
Y \setminus E(Z) = Y \setminus ((Z \setminus T[Z]) \cup K) \subseteq Y' \setminus ((Z' \setminus T[Z]) \cup K) = Y' \setminus E(Z').
\]

\( \square \)

### A.2 Proofs From Section 4.1

**Proof of Proposition 3.** We first show necessity and then sufficiency.

**Necessity:** By Theorem 1, since \( C^{\text{pi}} \supseteq C^{\text{res}} \), if \( L_E \) preserves path independence over \( C^{\text{pi}} \), it is threshold-linear with cardinal reuse with respect to some \( t \in \mathbb{N} \cup \{0, \infty\} \), \( K \subseteq X \) and \( \{T^n\}_{n=0}^\infty \).

First, we show that \( t \) is necessarily in \( [0, 1, \infty) \). Suppose otherwise, that \( 1 < t < \infty \). Let \( Z \subseteq X \) be such that \( |Z| < t \) and \( Z \cup K \not\subseteq X \) so that \( G_E(Z) = Z \cup K \neq X \). Since \( Z \) is finite, there is \( Z' \subseteq X \) such that \( Z' \not\supset Z \), \( |Z'| = t \), and \( Z' \cup Z \cup K \neq X \). Then, by threshold-linearity, \( G_E(Z) = X \).

Let \( a \in X \setminus (Z \cup Z' \cup K) \) and let \( C_1 \in C^{\text{pi}} \) be such that \( C_1(Z \cup Z' \cup \{a\}) = Z \) and \( C_1(Z' \cup \{a\}) = Z' \).

Since \( C_1 \) can be expressed as the union of maximizers of \( t \) linear orders over \( X \cup \{\emptyset\} \), by Aizerman and Malishevski (1981) it is path independent. The following is a description of specific linear orders \( \succ_i \) \( i = 1 \) to \( t \) that rationalize \( C_1 \) in this way. Let \( \{z_1, \ldots, z_l\} = Z \cap Z' \), \( \{z_{l+1}, \ldots, z_n\} = Z \setminus Z' \), and \( \{z_{l+1}' \ldots, z_{l+n}'\} = Z' \setminus Z \). Since \( |Z'| = t > |Z| \), \( t > n \). For each \( i \) between \( 1 \) and \( t \), let \( \succ_i \) be such that if \( i \leq t \) then \( \{x \in X : x \succ_i \emptyset\} = \{z_i\} \), if \( l < i < n \) then \( \{x \in X : x \succ_i \emptyset\} = \{z_i, z_i'\} \) with \( z_i \succ z_i' \), and if \( i \geq n \) then \( \{x \in X : x \succ_i \emptyset\} = \{z_n, z_n'\} \) with \( z_n \succ z_n' \).
$X : x > 2 \forall \} = \{a\}$. Let $C = \mathcal{L}_{E}(C_{1}, C_{2})$. Then, $a \in C(Z \cup Z' \cup \{a\}) = Z \cup \{a\}$ but $a \notin C(Z' \cup \{a\}) = Z'$, which violates path independence. Therefore, $t \in [0, 1, \infty]$.

To complete the proof of necessity, we show that if $t = \infty$ then, $T^{2} = T^{3} = \ldots$. Suppose, to the contrary, $t = \infty$ but for some $l \geq 2$, $T^{l} \neq T^{l+1}$. Since $T^{l+1} \supset T^{l}$, there is $a \in T^{l+1} \setminus T^{l}$. Let $Z \subseteq X$ be such that $a \in Z$ and $|Z| = l$. Since $2 \leq |Z| < \infty$ there exists $Z' \subseteq X$ such that $a \in Z \cap Z'$, $|Z'| = l + 1$, $Z \not= Z'$, and $Z \cup Z' \cup K \neq X$. Let $b \in X \setminus (Z \cup Z' \cup K)$ and let $C_{1} \in \mathcal{C}_{\text{pl}}$ be such that $C_{1}(Z' \cup \{b\}) = Z'$ and $C_{1}(Z \cup Z' \cup \{b\}) = Z'$.

Let $C_{2} \in \mathcal{C}_{\text{res}}$ be rationalized by $>_{2}$ and $q_{2} = 1$ where $\{x \in X : x >_{2} \forall \} = \{a, b\}$ and $a >_{2} b$. Let $C = \mathcal{L}_{E}(C_{1}, C_{2})$. Then, $b \in C(Z \cup Z' \cup \{b\}) = Z \cup \{b\}$ but $b \notin C(Z' \cup \{b\}) = Z'$, which violates path independence. Therefore, $T^{2} = T^{3} = \ldots$.

**Sufficiency:** Suppose $E$ is as described in the statement of Proposition 3. If $t = 0$, then we are done since $C = C_{1}$ and $C_{1}$ is path independent. So, suppose $t \neq 0$.

Let $Y, Y' \in [X]^{*}$ be such that $Y \subseteq Y'$. Let $C = \mathcal{L}_{E}(C_{1}, C_{2})$. Suppose $x \in C(Y')$ is such that $x \in Y$. It suffices to show that $x \in C(Y)$.

First consider $t = 1$. If $x \in C_{1}(Y')$, then $x \in C_{1}(Y) \subseteq C(Y)$ since $C_{1}$ is path independent. Otherwise, $x \in C_{2}(Y' \setminus E(C_{1}(Y')) \setminus C_{1}(Y'))$. Since $G_{E}(Z) = X$ for every nonempty $Z \subseteq [X]^{*}$ and $x \in Y' \setminus E(C_{1}(Y'))$, we conclude that $C_{1}(Y') = \emptyset$ and $x \notin K$. Since $C_{1}$ is consistent, $C_{1}(Y) = \emptyset$. Then, since $x \notin K$, $x \notin Y \setminus E(C_{1}(Y))$. Moreover, $Y \setminus E(C_{1}(Y)) = Y \setminus K \subseteq Y' \setminus K = Y' \setminus E(C_{1}(Y'))$. Since $C_{2}$ is path independent and $x \in C_{2}(Y' \setminus E(C_{1}(Y')))$, it then follows that $x \in C_{2}(Y \setminus E(C_{1}(Y))) \subseteq C(Y)$.

We complete the proof of sufficiency by considering the case of $t = \infty$. Let $Z = C_{1}(Y)$ and $Z = C_{1}(Y')$. If $x \in Z$, then $x \in Z \subseteq C(Y)$ directly. If $x \in Z'$, then again $x \in Z \subseteq C(Y)$ since $C_{1} \in \mathcal{C}_{\text{pl}}$. So, suppose $x \in C_{2}(Y' \setminus E(Z')) \setminus (Z \cup Z')$.

Since $C_{2} \in \mathcal{C}_{\text{pl}}$, it suffices to show that $x \in Y \setminus E(Z) \subseteq Y' \setminus E(Z')$.

Given that $x \in Y' \setminus E(Z')$ and $K \subseteq E(Z')$, we conclude that $x \notin K$. Since $E(Z) \subseteq Z \cup K$ and $x \notin Z$, it follows that $x \notin E(Z)$. Thus, since $x \notin Y$, $x \notin Y \setminus E(Z)$.

If $Y \setminus E(Z) \not\subseteq Y' \setminus E(Z')$, there is $a \in Y \setminus E(Z)$ such that $a \notin Y' \setminus E(Z')$. Then, $a \in E(Z')$.

If $a \in K$, then $a \in E(Z)$, contradicting $a \in Y \setminus E(Z)$, so $a \notin K$. Thus, since $a \in E(Z') = (Z' \setminus T[Z']) \cup K$ and $a \notin K$, $a \in Z' \setminus T[Z']$.

If $|Z'| > 1$, then regardless of $|Z|$, $T^{[Z]} \subseteq T^{[Z']}$. So, since $a \notin T^{[Z']}$, $a \notin T^{[Z]}$. Since $a \notin E(Z)$, we then conclude that $a \notin E$. However, this contradicts the path independence of $C_{1}$ since $a \in Z'$. Otherwise, $|Z'| = 1$ and since $a \in Z' \setminus T^{[Z']}, Z' = \{a\}$. Since $a \in Y \subseteq Y'$ and $C_{1}$ is path independent, $Z = Z' = \{a\}$. Then, $E(Z) = E(Z')$, 

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Footnote 21: The argument for why $C_{1} \in \mathcal{C}_{\text{pl}}$ is identical to that in Footnote 20.
Let $C | \succ Y \setminus E(Z)$. □

A.3 Proofs From Section 4.2

Proof of Proposition 4. We first show necessity and then sufficiency.

Necessity: By Theorem 1, if $L_E$ preserves path independence over $C^{res}$, it is threshold-linear with cardinal reuse with respect to some $t \in \mathbb{N} \cup \{0, \infty\}$, $K \subseteq X$ and $\{T^n\}_{n=0}^\infty$.

Suppose $|X \setminus K| > 1$. We first show that for each $n < t$, $T^n = \emptyset$. Then, we show that $t = \infty$.

Suppose, for some $n < t$, $T^n$ contains $a \in X$. Then there is $b \in X \setminus (K \cup \{a\})$. Let $Z \subseteq X \setminus \{a, b\}$ be such that $|Z| = s$. Since $T^n \neq \emptyset$, $n > 0$ and therefore there is $c \in Z$. Let $C_1 \in C^{res}$ be induced by $>_1$ such that $\{x \in X : x >_1 \emptyset\} = Z \cup \{a\}$ and, for each $z \in Z$, $a >_1 z \geq c$ and $q_1 = n$. Let $C_2 \in C^{res}$ be induced by $>_2$ such that $\{x \in X : x >_2 \emptyset\} = \{a, b\}$ and $a >_2 b$ and $q_2 = 1$. Let $C = L_E(C_1, C_2)$.

$C_1(Z \cup \{b\}) = Z$. Since $b \notin K$ and $n < t$, $b \notin (Z \cup \{b\}) \setminus E(Z)$. Since $a \notin Z$, $C_2(Z \cup \{b\}) \setminus E(Z) = \{b\}$. Thus, $C(Z \cup \{b\}) = Z \cup \{b\}$ so $|C(Z \cup \{b\})| = n + 1$.

$C_1(Z \cup \{a, b\}) = \{a\} \cup (Z \setminus \{c\})$. Since $a \in T^n$ and $\{|a\} \cup (Z \setminus \{c\})| = n$, $a \in (Z \cup \{a, b\}) \setminus E((\{a\} \cup (Z \setminus \{c\})))$, so $C_2((Z \cup \{a, b\}) \setminus E((\{a\} \cup (Z \setminus \{c\})))) = \{a\}$. Thus, $C(Z \cup \{a, b\}) = \{a\} \cup (Z \setminus \{x\})$ so $|C(Z \cup \{a, b\})| = n$. This violates size monotonicity.

We now show that $t = \infty$. Suppose, for the sake of contradiction that $t < \infty$. Let $a, b \in X \setminus K$ and $Z \subseteq X \setminus \{a, b\}$ be such that $|Z| = t$. Since $K \neq X$, $t > 0$. Thus, there is $c \in Z$.

Let $C_1 \in C^{res}$ be induced by $>_1$ such that $\{x \in X : x >_1 \emptyset\} = Z$ and $q_1 = t$. Let $C_2 \in C^{res}$ be induced by $>_2$ such that $\{x \in X : x >_2 \emptyset\} = \{a, b\}$ and $q_2 = 2$. Let $C = L_E(C_1, C_2)$.

By definition of $C_1$, $C_1(Z \cup \{a, b\}) = Z$. Since $|Z| = t$, $G_E(Z) = X$. Thus, $C(Z \cup \{a, b\}) = Z$.

By definition of $C_1$, $C_1(\{Z \setminus \{c\}\} \cup \{a, b\}) = Z \setminus \{c\}$. Since $T^{t-1} = \emptyset$ and $|Z \setminus \{c\}| = t - 1$, $E(Z \setminus \{c\}) = (Z \setminus \{c\}) \cup K$. Thus, $(\{Z \setminus \{c\}\} \cup \{a, b\}) \setminus E(Z \setminus \{c\})) = \{a, b\}$ and $C_2(\{a, b\}) = \{a, b\}$. So, $C((Z \setminus \{c\}) \cup \{a, b\}) = (Z \setminus \{c\}) \cup \{a, b\}$.

However, $|C(Z \cup \{a, b\})| = |Z| = t < t + 1 = |(Z \setminus \{c\}) \cup \{a, b\}| = |C((Z \setminus \{c\}) \cup \{a, b\})|$, in violation of size monotonicity.
Sufficiency: Suppose $E$ is as described in Proposition 4. Let $C_1, C_2 \in \mathcal{C}^{\text{sub-sm}}$ and $C = \mathcal{L}_E(C_1, C_2)$. Fix $Y \subseteq Y' \subseteq [X]^*$. We show that $|C(Y)| \leq |C(Y')|$. Let $Z = C_1(Y)$ and $Z' = C_1(Y')$. Since $C_1$ is size monotonic, $|Z| \leq |Z'|$. Since $C_1$ is path independent, $Y \setminus Z \subseteq Y \setminus Z'$.

First consider the case of $|X \setminus K| \leq 1$. If $K = X$, then $C = C_1$, so we are done. Otherwise, $X \setminus K = \{a\}$. If $a \in Y$ or if $a \in Z$, then $C(Y) = C_1(Y) = Z$. Moreover, $C(Y') \supseteq Z'$, so $|C(Y)| = |Z| \leq |Z'| \leq |C(Y')|$. If $a \in Y \setminus Z$, then $a \in Y' \setminus Z'$ so if $a \notin C_2(\{a\})$, $|C(Y)| = |Z| \leq |Z'| \leq |C(Y')|$ and if $a \in C_2(\{a\})$, $|C(Y)| = |Z| + 1 \leq |Z'| + 1 \leq |C(Y')|$.  

Now consider $|X \setminus K| > 1$. In this case, $t = \infty$ and $T^s = \emptyset$ for each $s$. Thus, $E(\bar{Z}) = K \cup \bar{Z}$ for each $\bar{Z} \in [X]^*$ so for each $\bar{Y} \in [X]^*$, $C_2(\bar{Y} \setminus E(\bar{Z})) \cap \bar{Z} = \emptyset$ meaning that $|C(\bar{Y})| = |C_1(\bar{Y})| + C_2(\bar{Y} \setminus E(\bar{Z}))$.

Since $Y \setminus (Z \cup K) \subseteq Y' \setminus (Z' \cup K)$ and $C_2$ is size monotonic, $|C_2(Y \setminus (Z \cup K))| \leq |C_2(Y \setminus (Z' \cup K))|$. Thus, $|C(Y)| = |Z| + |C_2(Y \setminus (Z \cup K))| \leq |Z'| + |C_2(Y \setminus (Z' \cup K))| = |C(Y')|$. □

A.4 Proofs From Section 5.1

Proof of Claim 8. We prove these for the case of $m = 3$. However, the proof generalizes to arbitrary $m \geq 3$.

Letting $E_1$ and $E_2$ both be threshold-linear exclusion with cardinal reuse where $t = N$, $K = \emptyset$, and $T^N = \emptyset$, right composition $(\mathcal{L}_{E_2}(\mathcal{L}_{E_1}(C_1, C_2), C_3))$ implements Procedure 1. Left composition $(\mathcal{L}_{E_1}(C_1, \mathcal{L}_{E_2}(C_2, C_3)))$ implements Procedure 2.

Next, we show that Procedure 1 cannot be implemented with left composition. To implement Procedure 1, $E_1$ is necessarily threshold-linear exclusion with cardinal reuse where $t = N$, $K = \emptyset$, and $T^N = \emptyset$. To see this, observe that if we select $C_3 \in \mathcal{C}$ to be such it always chooses $\emptyset$, then Procedure 1 is equivalent to lexicographic composition with this exclusion function. Setting $C_1 \in \mathcal{C}$ to be such that it always chooses $\emptyset$, we similarly conclude that $E_2 = E_1$. Finally, let $Z_1$ and $Z_2 \in [X]^*$ be disjoint such that $|Z_1| + |Z_2| > N$ but $|Z_1|, |Z_2| < N$. Let $z_3 \in X \setminus (Z_1 \cup Z_2)$. Let $C_1, C_2, C_3 \in \mathcal{C}$ be such that $C_1(Z_1 \cup Z_2 \cup \{z_3\}) = Z_1$, $C_2(Z_2 \cup \{z_3\}) = Z_2$, and $C_3(\{z_3\}) = z_3$. According to Procedure 1, the final choice ought to be $Z_1 \cup Z_2$. However, the left composition with $E_1$ and $E_2$ yields $Z_1 \cup Z_2 \cup \{z_3\}$. Thus, Procedure 1 cannot be implemented via left composition.

Finally, we prove that Procedure 2 cannot be implemented via right composition. Exactly as argued above, $E_1$ and $E_2$ are necessarily threshold-linear exclusion with
cardinal reuse where \( t = N, K = 0 \), and \( T^N = \emptyset \). For the same \( C_1, C_2, \) and \( C_3 \) above, according to Procedure 2, the final choice ought to be \( Z_1 \cup Z_2 \cup \{ z_3 \} \). However, the right composition with \( E_1 \) and \( E_2 \) yields \( Z_1 \cup Z_2 \). Thus, Procedure 2 cannot be implemented via right composition.

A.5 Proofs From Section 5.2

Proof of Proposition 5. We first show necessity and then sufficiency.

Necessity: We show that if \( \mathcal{L}_E \) preserves path independent over \( C^{sv-res} \times C^{sv-res} \), then setting \( K = E(\emptyset) \), either

1. for each \( x \in X, \ G_E(\{ x \}) = X, \) or
2. there is \( T \subseteq X \setminus K \) such that for each \( x \in X, \)

\[
E(\{ x \}) = \begin{cases} 
K & \text{if } x \in T \\
K \cup \{ x \} & \text{otherwise.}^{22}
\end{cases}
\]

First, we show that \( K \subseteq G_E(\{ a \}) \) for each \( a \in X \). If not, there is \( b \in K \setminus E(\{ a \}) \) such that \( b \neq a \). Let \( C_1 \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \emptyset \} = \{ a \} \). Let \( C_2 \) be induced by \( >_2 \) such that \( \{ x \in X : x >_2 \emptyset \} = \{ b \} \). Let \( C = \mathcal{L}_E(C_1, C_2) \). Then, \( C(\{ b \}) = \emptyset \) while \( C(\{ a, b \}) = \{ a, b \} \). Since this contradicts path independence, \( K \subseteq G_E(\{ a \}) \).

Second, we show that for each \( x \in X, \ G_E(\{ x \}) = \{ X, \emptyset \} \cup K \). If not, then for some \( a \in X \) there are \( b \in E(\{ a \}) \setminus (K \cup \{ a \}) \) and \( c \in X \setminus G_E(\{ a \}) \). By definition, \( a, b, \) and \( c \) are distinct. Since \( K \subseteq G_E(\{ a \}), \ c \in K. \) Let \( C_1 \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \emptyset \} = \{ a \} \). Let \( C_2 \) be induced by \( >_2 \) such that \( \{ x \in X : x >_2 \emptyset \} = \{ b, c \} \) such that \( b >_2 c. \) Let \( C = \mathcal{L}_E(C_1, C_2) \). Then, \( C(\{ a, b, c \}) = \{ a, c \} \) while \( C(\{ b, c \}) = \{ b \} \). Since this contradicts path independence, \( G_E(\{ a \}) = K \cup \{ a \} \) or \( G_E(\{ a \}) = X. \)

Third, we show that if for any pair \( a, b \in X \) such that \( \{ a \} \cup K, \{ b \} \cup K \neq X, \ G_E(\{ a \}) = X \) if and only if \( G_E(\{ b \}) = X. \) Suppose \( G_E(\{ a \}) \neq X \) but \( G_E(\{ b \}) = X. \) If there is \( c \in X \setminus (K \cup \{ a, b \}) \). Let \( C_1 \) be induced by \( >_1 \) such that \( \{ x \in X : x >_1 \emptyset \} = \{ a, b \} \) such that \( b >_1 a \). Let \( C_2 \) be induced by \( >_2 \) such that \( \{ x \in X : x >_2 \emptyset \} = \{ c \} \). Let \( C = \mathcal{L}_E(C_1, C_2) \). Then, \( C(\{ a, c \}) = \{ a \} \) while \( C(\{ b, c \}) = \{ b, c \} \). Since this contradicts path independence, \( G_E(\{ b \}) = X. \) If there is no \( c \in X \setminus (K \cup \{ a, b \}) \), then \( X = K \cup \{ a, b \}. \) Let

\(^{22}\)This is stronger than the necessity part of Proposition 5 in that the domain \( C^{sv-res} \times C^{sv-res} \) is smaller than \( C^{sv-res} \times C^{CI} \), which is the domain for Proposition 5.
$b' \in K$. Then there is $c \in X \setminus (K \cup \{a,b'\})$ and by the above argument, $G_E(\{b'\}) = X$. Since $K \cup \{b,b'\} \neq X$, we again repeat the argument to conclude that $G_E(\{b'\}) = X$ implies $G_E(\{b\}) = X$.

Fourth, we show that if there is $x \in X$ such that $G_E(\{x\}) \neq X$, then for each $x \in K$, $x \in E(\{x\})$. Suppose there is $a \in K \setminus E(\{a\})$. Since $G_E(\{x\}) \neq X$, $K \neq X$. So, there is $b \in X \setminus K$. Let $c \in X \setminus \{a,b\}$. Since $a \in K$, $a \in K \subseteq E(\{b\})$. Let $C_1$ be induced by $1$ such that $\{x \in X : x > 1 \emptyset\} = \{a,b\}$ such that $b > 1 a$. Let $C_2$ be induced by $2$ such that $\{x \in X : x > 1 \emptyset\} = \{a\}$ such that $a > 2 c$. Let $C = \mathcal{L}_E(C_1,C_2)$. Then, $C(\{a,c\}) = \{a,c\}$ while $C(\{a,b,c\}) = \{b,c\}$. Since this contradicts path independence, for each $x \in K, x \in E(\{x\})$.

To complete the proof of necessity, let $T = \{x \in X : x \notin E(\{x\}) \}$. As we have shown above, if there is $x \in X$ such that $G_E(\{x\}) \neq X$ then $T \cap K = \emptyset$. By definition, $E(\emptyset) = K$.

If there is $x \in X$ such that $\{x\} \cap K \neq X$ but $G_E(\{x\}) = X$, then by what we have shown above, for every $x \in X$, $G_E(\{x\}) = X$. Otherwise, for each $x \in X$, $G_E(\{x\}) = \{x\} \cup K$.

By definition of $T$, if $x \in T$, $x \notin E(\{x\})$ so $E(\{x\}) = K$ and if $x \notin T$, $x \in E(\{x\})$ so $E(\{x\}) = K \cup \{x\}$.

**Sufficiency:** Suppose $E \in S$ coincides on singletons and the empty set with a threshold-linear exclusion function with cardinal reuse where $K \subseteq X$, and $T \subseteq X \setminus K$.

Let $C = \mathcal{L}_E(C_1,C_2)$ and fix $Y \subseteq Y' \subseteq [X]^*$ and $x \in Y$ such that $x \in C(Y')$. We show that $x \in C(Y)$. Let $Z = C_1(Y)$ and $Z' = C_1(Y')$. If $x \in Z$ we are done since $Z \subseteq C(Y)$. If $x \in Z'$, then since $C_1$ is path independence, $x \in Z$ and we are again done. It remains to consider the case where $x \notin Z'$ and $x \notin Z$.

If $Y \setminus E(Z) \subseteq Y' \setminus E(Z')$, we are done by path independence of $C_2$. Otherwise, there is $a \in E(Z')$ such that $a \notin Z'$ but $a \notin E(Z)$. If $a \notin E(Z)$ then $a \notin K$. Since $a \notin E(Z')$ and $a \notin K$, $G_E(Z') = X$ since $a \notin Z' \cup K$. Since $G_E(Z') = X$, $C(Y') = Z'$. However, this contradicts $x \in C(Y') \setminus Z'$.

**Proof of Proposition 6.** The proofs of necessity for Theorem 1 and Proposition 3 appeal only to $C_2 \in C^{sv-res}$. Consequently, they establish the stronger result that said conditions are necessary even on the smaller domains of $C^{res} \times C^{sv-res}$ and $C^{pi} \times C^{sv-res}$ respectively. The sufficiency part of Proposition 3 implies sufficiency over the smaller domain as well.
A.6 Proofs From Section 6

Proof of Remark 3. Sufficiency of this condition is by definition. For necessity, we proceed by contradiction. Suppose that \( Z \in [X]^n \) is such that \( x \in Z \) and \( y \in E(Z) \) for \( x \sim y \) where \( y \neq x \). Let \( C_1 \in C^{\text{onto}} \) be such that \( C_1(Z \cup \{y\}) = Z \). Let \( C_2 \in C^{\text{res}} \) be induced by \( >_2 \) such that \( \{x \in X : x >_2 \emptyset\} = \{y\} \) and \( q_2 = 1 \). Let \( C = L_E(C_1, C_2) \). Then, \( C(Z \cup \{y\}) = Z \cup \{y\} \). Since \( x \sim y \), and \( x \in Z \), \( L_E \) does not preserve many-to-one.

B Size Monotonicity and Single-valued Choice

In this appendix, we consider the preservation of both path independence and size monotonicity when one of the two inputs to the lexicographic composition is single-valued. As observed in Section 4.2, compared to the corresponding results in Section 5.2, adding the requirement of preserving size monotonicity imposes severe restrictions on reuse.

For the domain \( C^{\text{sv-res}} \times C^{\text{sub-sm}} \), the analog of Proposition 5 is as follows.

Proposition 7. \( L_E \) preserves path independence and size monotonicity over \( C^{\text{sv-res}} \times C^{\text{sub-sm}} \) if and only if on the set of singletons and the empty set, \( E \) coincides with an exclusion function that is threshold-linear with cardinal reuse where if \( |X \setminus K| > 1 \), \( T^n = \emptyset \) for all \( n \).

Proof. We first show necessity and then sufficiency.

Necessity: From the proof of the necessity part of Proposition 5, since \( C^{\text{sv-res}} \subseteq C^{\text{sub-sm}} \) and \( L_E \) preserves path independence over \( C^{\text{sv-res}} \times C^{\text{sub-sm}} \), either for each \( x \in X \), \( G_E(\{x\}) = X \) or there is \( T \subseteq X \setminus K \) such that for each \( x \in X \), \( G_E(\{x\}) = \{x\} \cup K \) and \( R_E(\{x\}) = \{x\} \cap T \). In the former case, we complete the proof by noting that \( K = X \). Suppose, instead that the latter is the case. If, \( |X \setminus K| \leq 1 \), we are again done. So, supposed \( |X \setminus K| > 1 \). We show that \( T = \emptyset \). Suppose there is \( a \in T \). Since \( |X \setminus K| > 1 \), there is \( b \in X \setminus (K \cup \{a\}) \). Let \( c \in X \setminus \{a, b\} \). Let \( C_1 \) be induced by \( >_1 \) such that \( \{x \in X : x >_1 \emptyset\} = \{a, c\} \) and \( a >_1 c \). Let \( C_2 \) be induced by \( >_2 \) such that \( \{x \in X : x >_2 \emptyset\} = \{a, b\} \) and \( a >_2 b \). Let \( C = L_E(C_1, C_2) \). Then, \( C(\{a, b, c\}) = \{a\} \) while \( C(\{b, c\}) = \{b, c\} \). Since this contradicts size monotonicity, \( T = \emptyset \).

Sufficiency: This follows from the sufficiency part of Proposition 4. □
For the domain $\mathcal{C}_{\text{sub-sm}} \times \mathcal{C}_{\text{sv-res}}$, the analog of Proposition 6 is as follows.

**Proposition 8.** $L_E$ preserves path independence and size monotonicity over $\mathcal{C}_{\text{sub-sm}} \times \mathcal{C}_{\text{sv-res}}$ if and only if $E$ is threshold-linear with cardinal reuse with respect to $K \subset X$, $t \in \mathbb{N} \cup \{0, \infty\}$, and $\{T^n\}_{n=0}^\infty$ such that for each $l$, $T^l = \emptyset$ if $|X \setminus K| > 1$.

**Proof.** We first show necessity and then sufficiency.

**Necessity:** Since the proof of necessity for Theorem 1 only appeals to $C_2 \in \mathcal{C}_{\text{sv-res}}$, it follows that for $L_E$ to preserve path independence over $\mathcal{C}_{\text{sub-sm}} \times \mathcal{C}_{\text{sv-res}}$, $E$ is threshold-linear with cardinal reuse. The part of the proof of Proposition 4 where we establish that $|X \setminus K| > 1$ implies that for each $n$, $T^n = \emptyset$ also relies only on $C_2 \in \mathcal{C}_{\text{sv-res}}$. Together, these observations complete the proof of necessity.

**Sufficiency:** As noted in Footnote 19, the sufficiency part of Theorem 1 implies sufficiency for preserving path independence. So, it remains to show that for each pair $(C_1, C_2) \in \mathcal{C}_{\text{sub-sm}} \times \mathcal{C}_{\text{sv-res}}$, if $E$ is threshold-linear with cardinal reuse such that either $|X \setminus K| \leq 1$ or for all $n$, $T^n = \emptyset$, $C = L_E(C_1, C_2)$ is size monotonic.

If $|X \setminus K| \leq 1$, then the proof is exactly as in the proof of Proposition 4. So, suppose $|X \setminus K| > 1$. Then, for each $n$, $T^n = \emptyset$ so that for each $Z \in [X]^*$, $E(Z) = Z \cup K$. Fix $Y \subset Y' \in [X]^*$. Let $C_1(Y') = Z'$ and $C_1(Y) = Z$. There are two cases to consider.

**(Case 1) ($|Z| < |Z'|$):** Since $C_2$ is single-valued, regardless of $E(Z)$ and $E(Z')$, we have $|C_2(Y \setminus E(Z))| - |C_2(Y' \setminus E(Z'))| \leq 1$. Thus, $C(Y) = |Z| + |C_2(Y \setminus E(Z))| \leq |Z'| + |C_2(Y' \setminus E(Z'))| = C(Y')$.

**(Case 2) ($|Z| = |Z'|$):** Since $C_1$ is path independent, $Y \setminus Z \subseteq Y' \setminus Z'$. Removing $Z \cup K$ from both sides gives $Y \setminus (Z \cup K) \subseteq Y' \setminus (Z' \cup Z \cup K) \subseteq Y' \setminus (Z' \cup K)$. There are two subcases to consider.

(a) If $|Z| = |Z'| < t$, then $E(Z) = Z \cup K$ and $E(Z') = Z' \cup K$. Since $C_2$ is size monotonic, $|C_2(Y \setminus E(Z))| = |C_2(Y \setminus (Z \cup K))| \leq |C_2(Y' \setminus (Z' \cup K))| = |C_2(Y' \setminus E(Z'))|$. So, $|C(Y)| = |Z| + |C_2(Y \setminus E(Z))| \leq |Z'| + |C_2(Y' \setminus E(Z'))| = |C(Y')|$.

(b) If $|Z| = |Z'| \geq t$, then $E(Z) = E(Z') = X$, so $|C_2(Y \setminus E(Z))| = |C_2(Y' \setminus E(Z'))| = 0$ and $|C(Y)| = |Z| = |Z'| = |C(Y')|$.

These two cases complete the proof. □

In contrast with Proposition 6, while the added requirement of preserving size monotonicity precludes most forms of reuse, the input being size monotonic permits greater flexibility in the threshold.
C Necessity of Weak All-Or-Nothingness

In Example 4 the exclusion function jumps from $K = E(\emptyset)$ to a set $X \subset X$. This example is special in that the items in $X \setminus X$ are all in the same equivalence class. Our next result shows that this feature is necessary. A dilation $D \in D$ is weakly all-or-nothing if, for all $Z \in [X]^*_r$, $D(Z) \in (I_2 \cup K, X)$ if there are $x, y \in X \setminus (I_2 \cup K)$ such that $x \sim y$ and otherwise, $I_2 \cup K \subseteq D(Z) \subseteq X$.

Before we show necessity of weak all-or-nothingness for the preservation of path independent completability, we introduce a necessary condition akin to monotonicity. We say that the dilation $D \in D$ is many-to-one monotonic if for each $Z, Z' \in [X]^*_r$ such that $Z \subseteq Z', D(Z) \subseteq D(Z')$. As with our earlier use of the “many-to-one” modifier, it means that the condition applies only on $[X]^*_r$ as opposed to on all of $[X]^*$. Analogous to the necessity of monotonicity for the preservation of path independence, we have the following result on preserving path independent completability.

Claim 9. If $\mathcal{L}_E$ preserves many-to-oneness and path independent completability over $\mathcal{C}^{mto1\text{-res}}$, then $G_E$ is many-to-one monotonic.

Proof. By Remark 3, $E$ is equivalence-excluding. So, if $G_E$ is not many-to-one monotonic then there are $Z, Z' \in [X]^*_r$ such that $Z \subseteq Z'$ and $a \in (I_2 \cup E(Z)) \setminus (I_2 \cup E(Z'))$.

Let $C_1 \in \mathcal{C}^{mto1\text{-res}}$ be induced by $>_1$ such that $\{x \in X : x >_1 \emptyset\} = Z'$ and $q_1 = |Z'|$. Let $C_2 \in \mathcal{C}^{mto1\text{-res}}$ be induced by $>_2$ such that $\{x \in X : x >_2 \emptyset\} = \{a\}$ and $q_2 = 1$.

Since $a \notin Z'$, $C_1(Z' \cup \{a\}) = Z'$. Since $a \notin I_2 \cup E(Z')$, $a \notin (Z' \cup \{a\}) \setminus E(Z')$. Thus, $C_2((Z' \cup \{a\}) \setminus E(Z')) = \{a\}$ and therefore $C(Z' \cup \{a\}) = Z' \cup \{a\} \in [X]^*_r$.

Since $Z \subseteq Z'$, $C_1(Z \cup \{a\}) = Z$. Since $a \in E(Z)$, $a \notin (Z \cup \{a\}) \setminus E(Z)$. Thus, $C_2((Z \cup \{a\}) \setminus E(Z)) = \emptyset$ and therefore $a \notin C(Z \cup \{a\}) = Z$. This violates path independent completability of $C$. □

Finally, we show that weak all-or-nothingness is necessary.

Claim 10. If $\mathcal{L}_E$ preserves many-to-oneness and path independent completability over $\mathcal{C}^{mto1\text{-res}}$, then $G_E$ is weakly all-or-nothing.

Proof. By Remark 3 and Claim 9, if $\mathcal{L}_E$ preserves path independent completability, it is equivalence-excluding and $G_E$ is many-to-one monotonic. Suppose there is
\(Z \in [X]^*\) such that \(X \supset G_E(Z) \supset I_Z \cup K\). Let \(Z\) be such an element of \([X]^*_F\) such that for all \(Z \subset Z, G_E(Z) = I_Z \cup K\). By definition of \(Z\), there are \(a \in X \setminus G_E(Z), b \in Z,\) and \(c \in G_E(Z) \setminus (I_Z \cup K)\). Since \(c \not\in I_Z \cup K, c \not\in Z\) so \(c \not\in E(Z)\). Note that since \(a, c \not\in I_Z, (I_a \cup I_c) \cap I_Z = \emptyset\).

If \(a \sim c\), then \(L_E\) does not preserve path independent completability by the same argument given to show necessity of all-or-nothingness for Claim 6. Thus, \(a \sim c\). Since this applies for every \(a \in X \setminus G_E(Z)\) and \(c \in G_E(Z) \setminus (I_Z \cup K)\), there is some item \(d \in X\) such that \(X \setminus (I_Z \cup E(\emptyset)) = (X \setminus G_E(Z)) \cup (G_E(Z) \setminus (I_Z \cup K)) \subseteq I_d\). So, for each pair \(x, y \in X \setminus (I_Z \cup K), x \sim y\).

To complete the proof, we observe that for any \(Z \in [X]^*_F\) such that \(Z \supset Z\), since \(I_Z \supset I_Z, X \setminus (I_Z \cup K) \subseteq X \setminus (I_Z \cup K)\). So, there are items from at most one equivalence class in \(X \setminus (I_Z \cup K)\).  

In our base model we assumed that \(X\) is countably infinite in order to avoid issues that arise at the boundaries—that is, for sets of size \(|X| - 1\). While this issue looks similar to the issue with having to weaken all-or-nothingness, requiring the number of equivalence classes to be countably infinite does not change it. The weakening of all-or-nothingness is driven by the finiteness of equivalence classes in \(X \setminus K\). Since \(K\) may be countably infinite, even if \(X\) is countably infinite, \(X \setminus K\) may still be finite.

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