On the integration of an ODE involving the
derivative of a Preisach nonlinearity

D Flynn¹ and O Rasskazov²

¹ Dept of Civil & Environmental Engineering, University College Cork, Ireland
² Dept of Applied Mathematics, University College Cork, Ireland
E-mail: d.flynn@mars.ucc.ie, oll@phys.ucc.ie

Abstract. We offer an algorithm for numerical integration of an ODE with a Preisach
nonlinearity under the derivative on the left-hand side. Despite the exotic form, equations
of that kind are important in applied disciplines, for example in terrestrial hydrology.

1. Introduction

Consider the differential-operator equation of the form
$$\frac{dP(y(t))}{dt} = f(t, y(t)), \quad (1)$$

where $P$ is a special type of Preisach operator. Despite the exotic form, equations of that kind
are important in applied disciplines, for example in terrestrial hydrology. Numerical modelling of
such equations could shed light on processes of desert formation; for recent studies see [10, 6, 4]
and bibliographies therein. Questions of the existence and uniqueness of solutions to these
equations were orally discussed with Pavel Krejčí and A. Pokrovskii.

In this paper we offer an algorithm that allows the correct application of high-order integration
methods for obtaining numerical solutions of (1). The given algorithm can be easily modified to
compute a posteriori error estimates for the numerical solutions.

The structure of the paper is as follows. In Section 2 we briefly discuss hysteresis in hydrology,
derive a simple soil-moisture model and present the results of numerical modelling. Section 3 is
devoted to the particularities of the numerical integration, the formal definition of the Preisach
operator, its computer modelling and differentiation. The numerical integration algorithm is
given in Subsection 3.5.1.

2. Model equation

2.1. Models of flow in a porous medium

Consider the differential equation
$$\dot{\theta} = q, \quad \theta(0) = \theta_0, \quad 0 \leq \theta \leq \theta_s < 1, \quad (2)$$

where $q$ is the inflow rate; $\theta_s$ is the volumetric water-content at natural saturation and $\theta$ is
the volumetric moisture content of the medium with dimensions of $m^3$ of water per $m^3$ of soil-
matrix. When the soil matrix is perfectly dry then $\theta = 0$, when the matrix is fully saturated
with water, then $\theta = \theta_s < 1$ and finally for in-between states the matrix contains both air and water. Hence $\theta$ is bounded between 0 and $\theta_s$ as indicated in equation (2).

For $q$ we use a form of Darcy’s law [3] for the flow of a liquid phase through the homogenous porous medium, which we may write in the form

$$q = -k \nabla \Phi,$$

where $\Phi$ is the total potential, which is the sum of partial potentials such as the gravitational and matric potentials etc. Here we will only deal with the matric potential. Finally $k$ is the hydraulic conductivity of the medium.

For the purposes of this paper we will work with as simple a model as possible and consider the zero-dimensional case replacing the gradient in potential with a difference with respect to a reference potential,

$$q = -k (\psi(\theta) - \psi_{ref}(t)),$$

where $q$ is now a zero-dimensional flow rate and $\psi$ is the matric potential, $\psi_{ref}(t)$ is a time dependent reference potential. Note that all potentials dealt with here are negative i.e. $\psi < 0$ and $\psi_{ref} < 0$.

For a reference potential $\psi_{ref}(t)$ we choose a time-dependent sinusoidal driving function of the following form

$$\psi_{ref}(t) = -A \left(1 + \frac{1}{2} \left(\cos \left(\frac{2\pi t}{T}\right) + \cos \left(\frac{2\sqrt{2}\pi t}{T}\right)\right)\right),$$

where $A > 0$ is the amplitude and $T$ is the period. Note that this function satisfies the condition of always being non-positive.

If we combine equations (5) and (4) and substitute them into equation (2), we then have the following:

$$\dot{\theta}(\psi) = -k \left[\psi(\theta) + A \left(1 + \frac{1}{2} \left(\cos \left(\frac{2\pi t}{T}\right) + \cos \left(\frac{2\sqrt{2}\pi t}{T}\right)\right)\right)\right].$$

In modern models it is assumed that relationship $\theta(\psi)$ is hysteretic and that equation 6 is in the form of (1).

2.2. Soil-Moisture Hysteresis

In this paper we are interested in soil-moisture hysteresis which is important in terrestrial hydrology, agronomy, and soil-physics. As far back as 1930 [5] and 1941 [11], it was shown experimentally that there is a hysteretic effect in the relationship between water-retention and soil-moisture tension. The hysteretic effect is evident in the Soil-Water Characteristic Curve (SWCC), in that it is different depending on whether it is for wetting or drying, see Figure 1.

The origin of the hysteretic effect may be attributed to several factors [6]:

(i) Geometric non-uniformity of individual pores, resulting in the so called “ink-bottle effect”.
(ii) Different spatial connectivity of pores during drying and wetting.
(iii) Variations in the contact angle between liquid and solid.
(iv) Entrapment of air.

Important and successful mathematical descriptions of soil-moisture hysteresis have been suggested. We will just refer to the fundamental papers [1, 7, 9, 10] and the further references therein.
To describe the SWCC analytically we require an appropriate equation. The basic equation we choose for our study is the van-Genuchten equation for the Main Drying Curve (MDC), which takes the following form:

$$\theta(\psi) = \theta_r + (\theta_S - \theta_r) \left(1 + \left(\frac{\psi}{\psi_g}\right)^n\right)^{-m},$$

where $\theta_r$ is the residual water content (in most cases this is zero); $\psi_g$ is the van Genuchten pressure-head scale parameter and the two dimensionless water retention shape parameters, $m$ and $n$, are related by $m = 1 - 1/n$.

In previous work [4], the van Genuchten equation was used to formulate a Preisach model which was nicknamed the “Wedge model” in that paper. That model was used to fit empirical data and the result of the fit was an improvement when compared with the work of others and included less parameters. We also note in passing that the Main Drying Curve is the upper bound for the totality of all possible pairs $(h, \theta)$, see Figure 1.

2.3. The Wedge Model

**Proposition 2.1.** The integral of the Preisach density over any vertical line, $L(h)$ at some point $x$ must equal the derivative $MDC'(\psi)$, where $MDC(\psi)$ is the expression (7) for the Main
Drying Curve discussed above. i.e.
\[ \int_{\psi}^{u(\alpha)} \rho(\psi, \beta) \, d\beta = MDC'(\psi) \]  

This density is a one-parameter family, distributed between the line \( \beta = \alpha \) and the line \( \beta = \gamma \alpha \), where \( 0 \leq \gamma \leq 1 \) is a parameter of the model. Moreover, we suggest that the distribution is uniform within the fragment \([h, \gamma h]\) of any vertical lines of constant \( \alpha \equiv h \). Using Proposition 2.1 we conclude that this density within this range is given by:

\[ \rho_W(\alpha, \beta) = \frac{MDC'(\alpha)}{\alpha(1-\gamma)}. \]  

We define this density as

\[ \rho_W^{\gamma}(\alpha, \beta) = \frac{MDC'(\alpha)}{C(\gamma-1)} \cdot \left( \frac{1}{\alpha - \gamma \alpha} \right) \]

where we have used the same variables as in (7). We can rewrite equation (9) as

\[ \rho_W^{\gamma}(\alpha, \beta) = C \cdot \left( \frac{1}{\alpha - \gamma \alpha} \right) \cdot \left( \frac{\alpha}{h_g} \right)^{n-2} \left( 1 + \left( \frac{\alpha}{h_g} \right)^n \right)^{-m} \cdot \left( \frac{\alpha}{h_g} \right)^{n-2} \left( 1 + \left( \frac{\alpha}{h_g} \right)^n \right)^{m-1}. \]

This density is non-zero for \( \alpha < \beta < (1-\gamma)\alpha \), where \( \alpha < \beta < 0 \), and is zero otherwise.

2.4. Results

In Figure 2 (a) we have combined two plots of the evolution of the Matric Potential over time. The solid curve in this graph presents some interesting features, in regions (1) and (2) we can see examples of the discontinuities of the right-hand side of the ODE. Similarly in Figure 2 (b) we have also combined two plots of the evolution of the Volumetric Moisture content over time. In Figures 2 (c) and (d) we have separate plots for the hysteretic loops generated using the same Wedge model parameters as in Figure 2 (a). In addition to the Wedge model parameters, all of the plots used the same van Genuchten parameters, see (7), which were: \( m = 0.526, \psi_g = 19.54 \) and \( n = 4.22 \).

3. Numerical algorithm

3.1. Preisach model

Non-ideal relay The nonideal relay \( R_{\alpha, \beta} \) (with the threshold values \( \alpha < \beta \)) is the simplest hysteretic transducer. It takes an arbitrary continuous scalar function \( x(t) \) as an input and outputs either 0 or 1 depending on \( x(t) \), \( t \geq t_0 \), and on the initial state \( \eta_0 \) at the time \( t_0 \). The output of the relay at the moment \( t \) is defined by the following explicit formula:

\[ y(t) = R_{\alpha, \beta}[t_0, \eta_0]x(t) = \begin{cases} 
R_0, & \text{if } \alpha < x(\tau) < \beta \text{ for all } \tau \in [t_0, t]; \\
1, & \text{if there exists } t_1 \in [t_0, t] \text{ such that } x(t_1) \geq \beta, x(\tau) > \alpha \text{ for all } \tau \in [t_1, t]; \\
0, & \text{if there exists } t_1 \in [t_0, t] \text{ such that } x(t_1) \leq \alpha, x(\tau) < \beta \text{ for all } \tau \in [t_1, t].
\]
Definition of the Preisach nonlinearity Consider a family $\mathcal{R}$ of relays $R^\omega = R_{\alpha,\beta}$ with threshold values $\alpha, \beta \in \mathbb{R}^2$. We call such family a bundle of relays. Suppose that the set $\mathbb{R}^2 \supset \Omega = \{(\alpha, \beta) : \alpha \leq \beta\}$ is endowed with a Borel probability measure $\mu$. We suppose that the measure of $S \subset \Omega$ is given by

$$\mu(S) = \int_S \rho(\alpha, \beta) d\alpha d\beta, \quad \rho(\alpha, \beta) \geq 0, \quad \mu(\Omega) = 1.$$ 

We further suppose that $\rho(\alpha, \beta)$ is differentiable as many times as necessary and $\rho(\alpha, \alpha) \neq 0$. The two following functions will be important for our discussion

$$Q_-(a, b) = \int_a^b \rho(s, b) ds, \quad (12)$$
$$Q_1(a, b) = \int_a^b \rho(a, s) ds, \quad (13)$$

and we require that $Q_-(-\infty, b) < +\infty$ for all $b$. 

Figure 2. Numerical results using the algorithm
We call any measurable function \( \eta_0(\omega) : \Omega \to \{0, 1\} \) the initial state of the bundle \( \mathcal{R} \). For any initial state \( \eta_0 = \eta_0(\omega) \) and any continuous input \( x(t), t \geq t_0 \), we define the function
\[
y(t) = (P[t_0, \eta_0]x)(t) = \int_{\alpha<\beta} R_{\alpha,\beta}[t_0, \eta_0(\omega)]x(t) \, d\mu, \quad t \geq t_0. \tag{14}\]

We refer to this model as the Preisach nonlinearity (or when convenient, Preisach model or Preisach operator or Preisach transducer). Here \( y(t) \) is the output of the Preisach model with the initial state \( \eta_0 \) and the input \( x(t) \).

For \( t > t_0 \), it is also convenient to interpret the function
\[
\eta(\omega) : \omega \to R^*[t_0, \eta_0(\omega)]x(t) \in \{0, 1\} \tag{15}
\]
as the state of the Preisach nonlinearity at the moment \( t \).

It is convenient also to consider only admissible initial states \( \eta_0(\omega) \) that satisfy the relationships
\[
\eta_0(\omega) = \begin{cases} 
0 & \text{if } x(t_0) \leq \alpha(\omega), \\
1 & \text{if } x(t_0) \geq \beta(\omega).
\end{cases} \tag{16}
\]
Although the individual relays are discontinuous in any reasonable sense, the Preisach operators are often continuous with respect to the uniform norm and have other nice properties, see [8].

**States and outputs** For the purposes of numerical integration we consider the case when the initial and varying states \( \eta_0, \eta(t) \) are characteristic functions of sets \( S \), that is described below. On Figure 3 \( S \) is located to the left of the line \( \alpha = \beta \) and below a continuous piece-wise linear curve \( L = L_S \), with non-positive slope and links parallel to the axes. In this standard case the coordinates of the intersection of \( L \) with the diagonal coincide with the current value of the input \( x \).

In other words, such sets \( S \) admit the representation
\[
S = \{(\alpha, \beta) : \beta + \alpha \leq \psi_S(\beta - \alpha)\}
\]
where \( \psi_S(u), 0 \leq u < \infty \), is a continuous piece-wise linear function satisfying a Lipschitz condition with constant 1:
\[
|\psi_S(u) - \psi_S(v)| \leq |u - v|.
\]
The curve \( L_S \) is connected with the function \( \psi_S \) by
\[
L_S = \{(\alpha, \beta) : \beta + \alpha = \psi_S(\beta - \alpha)\}. \tag{17}
\]
We denote by \( \mathcal{E} \) the set of states \( \eta \) of the Preisach nonlinearity whose characteristic functions are described above. The set \( \mathcal{E} \) is invariant with respect to piece-wise continuous inputs.

**Derivative of the Preisach nonlinearity** Let \( y(t) \) be a differentiable function for \( t \geq t_0 \). Let us describe the derivative \( d(P[t_0, \eta_0]y)(t)/dt \) of the Preisach nonlinearity at \( t = t_1 > t_0 \). Introduce the numbers \( x_1 = x^\uparrow(t_1) \) and \( x_1 = x^\downarrow(t_1) \) as
\[
x^\uparrow_1 = \inf\{x : \forall \varepsilon, \eta_1(x, y(t_1) + \varepsilon) = 0\},
\]
\[
x_1 = \sup\{x : \forall \varepsilon, \eta_1(y(t_1) + \varepsilon, x) = 1\},
\]
where \( \eta_1 \) is a characteristic function of the Preisach nonlinearity at \( t_1 \) The derivative \( d(P[t_0, \eta_0]y)(t_1)/dt \) is given by
\[
Q(t_1) \frac{dy}{dt} \big|_{t=t_1},
\]
We note that $Q(t)$ and hence $d(P[t_0, \eta_0]y)(t)/dt$ are right-continuous.

3.2. Existence and uniqueness of the solutions

Let us consider the initial value problem

$$(P[t_0, \eta_0]x)'(t) = f(t, x), \quad x(t_0) = x_0. \quad (21)$$

Here $x \in \mathbb{R}$, $P$ is the Preisach nonlinearity with measure $\mu$ and initial state $\eta(t_0) = \eta_0$ described above. Note also that $f(t, x)$ is sufficiently smooth in $t, x$.

We call the Lipschitz function $x(t)$ the solution of (21) if it satisfies the initial condition and satisfies the differential-operator equation almost everywhere. Since Lipschitz continuity implies absolute continuity, this definition satisfies the definition of the solution of an ODE given by Caratheodory.

The equation (21) is equivalent to

$$Q[t_0, \eta_0](x(t))x'(t) = f(t, x), \quad x(t_0) = x_0. \quad (22)$$

with $Q$ being defined by (18), (19). We should mention that $Q$ is often discontinuous and is equal to 0 at the points where $x(t)$ changes direction. Thus the unique solution of the initial value problem in terms of the definition above does not necessarily exist.

As an instructive example, consider the initial value problem

$$Q(x)x' = ax, \quad x(0) = 0, \quad a \neq 0, \quad Q(x) = x. \quad (23)$$

It is easy to see that it has a family of the solutions $x_s(t)$ satisfying (23) is given by

$$x_s(t) = \begin{cases} 0 & 0 \leq t \leq s, \\ at & s < t. \end{cases}$$

Below we formulate, without proof, a proposition about existence and uniqueness of the solution which is due to P. Krejčí and A. Pokrovskii (oral discussions)
Proposition 3.1. There exists a solution \( y(t) \) to the initial value problem (21). If for every \( t_* \) such that \( y'(t_*) = 0, \frac{\partial f(t,y(t_*))}{\partial t} \neq 0 \), then \( y(t) \) is a unique solution of (21).

3.3. Algorithmic description of Preisach nonlinearity

Restricted set of states Consider the states \( \eta \) corresponding to the sets \( S \in \Pi \) bounded by the lines \( L \) with an infinite horizontal link, and with finite number of vertical and horizontal links.

Below we restate the algorithmic description of the Preisach nonlinearity given in [2].

We denote the height of the infinite horizontal link by \( w^{(1)} \). Let \( w^{(2k)}, 1 < 2k \leq K \), be the ascending sequence of the horizontal coordinates of the vertical links. That is, the vertical links belong to the straight lines \( L_{2k} = (w^{(2k)}, \beta), 1 < 2k \leq K \). Let also \( w^{(2k+1)}, 1 < 2k + 1 \leq K \), be the descending sequence of the vertical coordinates of the finite horizontal links. By definition, the numbers \( w^{(k)}, 1 \leq k \leq K \) satisfy the inequalities

\[
\begin{align*}
w^{(1)} > \ldots > w^{(2k_0-1)} > w^{(2k_0)} > \ldots > w^{(2)}, & \quad \text{if } K = 2k_0; \\
w^{(1)} > \ldots > w^{(2k_0+1)} > w^{(2k_0)} > \ldots > w^{(2)}, & \quad \text{if } K = 2k_0 + 1.
\end{align*}
\] (24)

The totality \( W \) of all states \( \eta \), described by the finite number of links as above, may be naturally identified with the family of all finite sequences

\[ w = \{w^{(k)}\}_1^K \]

satisfying the inequalities (24).

We introduce also the semi-closed intervals \( I_0 = (\infty, w^{(1)}) \), \( I_1 = (-\infty, w^{(2)}) \) and \( I_k = [w^{(k-1)}, w^{(k+1)}], 2 \leq k < K \) and the open interval \( I_k = (w^{(k)}, w^{(k-1)}) \). Clearly those intervals form a partition of the whole real line.

Transformation rule We discuss a transformation rule to compute the varying outputs of the Preisach model together with its states. Let \( \Delta(a,b) \) be the triangle with the vertices \( (a,a),(a,b),(b,b) \):

\[ \Delta(a,b) = \{(\alpha, \beta): a \leq \alpha < \beta \leq b\}. \]

Denote for \( a \leq b \)

\[ \mu(a,b) = \mu(\Delta(a,b)) = \int_{\Delta(a,b)} \rho(\alpha, \beta) \, d\alpha \, d\beta, \] (25)

and define \( \mu(b,a) = -\mu(a,b) \). In some cases there can be an explicit formula for \( \mu(a,b) \).

For a sequence \( w \in W \), with \#\( w = K \) (where the symbol ‘\#' denotes cardinality of a set), we have the corresponding array \( \mu^{(0)}, \ldots, \mu^{(K)} \) by the equalities \( \mu^{(0)} = 0, \mu^{(1)} = \mu(-\infty, w^{(1)}) \) and the recurrent relationships

\[ \mu^{(k)} = \mu^{(k-1)} + \mu(w^{(k-1)}, w^{(k)}). \]

Clearly, output of the Preisach nonlinearity coincides with \( \mu^{(K)} \).

Let the input \( x(t) \) be monotone in the interval \( \tau \leq t \leq \sigma \). Then \( w(\sigma), K(\sigma) \), and the output \( u(\sigma) \) are obtained from the state \( w(\tau) \) via the following rule

(\text{TR}*\text{ )} Let \( \ell \) be defined by \( x(\sigma) \in I_\ell \). Then

assign \( K = K(\sigma) = \#w(\sigma) := \ell + 1 \);
assign \( w^{(K)}(\sigma) := x(\sigma) \);
assign \( \mu^{(K)}(\sigma) := \mu^{(K-1)}(\tau) + \mu(w^{(K-1)}, w^{(K)}) \), with \( w^{(0)} = -\infty \);
assign \( u(\sigma) := \mu^{(K)}(\sigma) \).
Let the function \( x(t) \) have all different local extrema whose values are different from the components of \( w(\tau) \). Then \( K(t) \) jumps up only where \( x(t) \) has an extremum or is equal to \( w^{(i)}(\tau) \).

Note that if \( x(t) \) is increasing in the interval \( t \in (\tau, \sigma) \), i.e. \( x'(t) > 0 \), then \( K(t) \) is odd. Similarly, if \( x'(t) < 0 \) for \( t \in (\tau, \sigma) \), then \( K(t) \) is even. It is convenient to introduce the following notation

\[
K_1 = K_1(t) = K_1(K(t)) = K(t) + 1 - (K(t) \mod 2); \\
K_1 = K_1(t) = K_1(K(t)) = K(t) + (K(t) \mod 2).
\]

Sometimes it is also convenient to describe the future dynamics of the output in terms of the current state \( w \).

**Differentiation of the Preisach nonlinearity** At the every moment \( t \) the Preisach nonlinearity is described by an algorithmic description above, i.e., by a finite sequence \( w \) of length \( K \), satisfying (24). The derivative at the time \( t_1 \) is defined by

\[
\begin{align*}
Q_1(t_1) &= Q_1(w^{(K-1)}, w^{(K)}) & \text{if } y'(t_1) > 0, & \quad (26) \\
Q_1(t_1) &= Q_1(w^{(K)}, w^{(K-1)}) & \text{if } y'(t_1) < 0. & \quad (27)
\end{align*}
\]

**3.4. Numerical integration algorithm for (22)**

To simplify the notation we assume that state of the Preisach nonlinearity \( \eta \) is equivalent to \( w \) and we loosely use the representation

\[
\eta = w = \{w^{(1)}, \ldots, w^{(K)}\}
\]

when there is no ambiguity. We assume that the initial states of the Preisach nonlinearity \( \eta_0 \) for an initial value problems like (22) always admit analgorithmic representation.

**3.4.1. Locally equivalent systems** In order to apply high-order integration methods to (22) we introduce some locally equivalent systems.

Let \( K(t) \) denote the length of the sequence \( \{w^{(i)}(t)\} \) that describes \( \eta(t) \) and let \( K \) denote \( K(t_0) \). Let the initial conditions of (22) be coherent, i.e. \( x_0 = x(t_0) = w^{(K)}(t_0) \).

If \( f(t_0, x_0) > 0 \) or \( f(t_0, x_0) = 0 \), \( \frac{\partial f(t_0, x_0)}{\partial t} > 0 \), then the equation (22) is locally equivalent to

\[
\begin{cases}
Q(y)dy/dt = f(t, y), \\
Q(y) = \int_{w^{(K-1)}(t)} w^{(K)}(s, y)ds \\
w^{(K+1)}(t) = y(t), \\
\text{otherwise for } \, 1 \leq i < K
\end{cases}
\]

\[
(28)
\]

Denote the solution of (28), with the initial conditions \( \eta(t_0) = \eta_0 = \{w^{(1)}, \ldots, w^{(K)}\}, y(t_0) = y_0 \), at the time \( t + \tau \) by

\[
\varphi_1(t_0, y_0, \eta_0; t_0 + \tau) = (y_1(t_0 + \tau), \eta_1(t_0 + \tau)).
\]

Here \( \eta_1(t) = \{w^{(1)}, \ldots, w^{(K-1)}, y(t)(t)\} \).

The solution \( (y(t), \eta(t)) \) of (22) coincides with \( \varphi_1(t_0, y_0, \eta_0; t) \) on the interval \([t_0, t_1]\) where

\[
t_1 = \min_{t > t_0} \{t : y'(t) = 0 \text{ or } y(t) = w^{(K-2)}(t_0)\}.
\]

depends on initial conditions. We say that (28) is locally equivalent to (22) for \( t \in [t_0, t_1] \).
If \( f(t_0, x_0) < 0 \) or \( f(t_0, x_0) = 0 \), \( \frac{\partial f(t_0, x_0)}{\partial t} < 0 \) the equation is locally equivalent to the solution \( \varphi_1 \) of

\[
\begin{cases}
Q(y) dy/dt = f(t, y), \\
Q(y) = f_{w(K_1^{-1})}(y, s) ds, \\
w_{K_1}^{(t)}(t) = y(t),
\end{cases}
\tag{29}
\]

on some interval.

Both (28) and (29) are smooth and can be numerically integrated by standard methods unless \( Q(y) = 0 \). By the definition of the equivalent systems and the properties of the Preisach nonlinearity, \( Q(y_0) = 0 \) only if \( f(t_0, y_0) = 0 \).

### 3.5. Numerical integration of (28) and (29)

The equations (28) and (29) are similar, so we consider only the first in detail.

Denote a step of size \( \tau \) of the numerical integration of the equation (28) by

\[ \nu_1(t_0, y_0, \eta_0; t_0 + \tau) = (y_{\nu_1}(t_0 + \tau), \eta_{\nu_1}(t_0 + \tau)) \]

where \( \eta_{\nu_1}(t) = \{ w^{(1)}, \ldots, w^{(K_1^{-1})}, \nu_{\nu_1}(t) \} \).

By definition, \( Q(y) \) is monotone and non-negative. By the initial conditions, \( y'(t) \geq 0 \) for \( t \in [t_0, t_1] \), \( Q(y(t)) \) is strictly positive on \( [t_0, t_1] \). Thus if \( Q(y_0) \neq 0 \), the ODE in (28) can be integrated by standard methods after dividing both sides by \( Q(y) \).

Let us consider the case when \( Q(y_0) = 0 \) and \( f(t, y) = 0 \). Around \( (t_0, y_0) \) the ODE can be approximated by

\[
\rho(y_0, y_0)(y - y_0) dy/dt = \frac{\partial f(t_0, y_0)}{\partial y} (y - y_0) + \frac{\partial f(t_0, y_0)}{\partial t} (t - t_0),
\]

with the initial condition \( y(t_0) = y_0 \). After the coordinate change \( z = y - y_0 \), \( s = t - t_0 \), this equation is equivalent to

\[
z \frac{dz}{ds} = az + bs, \quad z(0) = 0,
\tag{30}
\]

with \( a = \frac{1}{\rho(y_0, y_0)} \frac{\partial f(t_0, y_0)}{\partial y} \) and \( b = \frac{\partial f(t_0, y_0)}{\partial t} \).

Solutions to (30) are given by \( z(t) = k_i t, \quad i = 1, 2 \), with \( k_i \) satisfying the quadratic equation \( k^2 - ak - b = 0 \).

Since \( \rho(y_0, y_0) > 0 \) and \( \frac{\partial f(t_0, y_0)}{\partial y} > 0 \) by the conditions of (28), then \( a > 0, k_1 > 0, \) and \( k_2 < 0 \). Since \( z(t) = k_2 t \) contradicts the requirement \( y'(t) > 0 \), which is imposed on the system (28), it is not a solution.

Thus, the approximate solution of (28) can be found by replacing the initial conditions with the new ones at time \( t_0 + h \):

\[
y(t_0 + h) = y_0 + k_1 h, \quad w_{K_1}^{(t)}(t_0 + h) = y(t_0 + h).
\tag{31}
\]

Similarly, if \( Q(y_0) = 0 \) for the system (29), the linearised equation is given by

\[
-z \frac{dz}{ds} = az + bs, \quad b < 0 \quad z(0) = 0
\tag{32}
\]

\( k \) is the solution of \( k^2 + ak + b = 0 \) and the initial condition of (29) should be replaced by

\[
y(t_0 + h) = y_0 + \frac{-a - \sqrt{a^2 - 4b}}{2} h, \quad w_{K_1}^{(t)}(t_0 + h) = y(t_0 + h).
\tag{33}
\]

Notations \( \nu_1(t_0, z_0, \eta_0; t_0 + \tau) \) and \( \varphi_{\nu_1}(t_0, z_0, \eta_0; t_0 + \tau) \) are similarly introduced for the equation (29).

Finally, let us note that the case \( Q(y) = 0, f(t, x) \neq 0 \) is impossible, by the properties of the measures of the Preisach nonlinearities considered.
3.5.1. Integration algorithm

**General description** The idea of the numerical algorithm is simple. During each cycle of the algorithm we are trying to take a time step along either (28) or (29) in such a way that there is no jump in the derivative.

If, as a result of the step, we are too close to the zone where \( f \) is zero, we check the conditions of uniqueness of the solution and, if they hold, move away from the point where \( x'(t) = 0 \) using the linear approximation to the solution.

This algorithm can be easily modified to incorporate steps allowing us to estimate the numerical integration error.

The proposed algorithm can utilise efficient high-order integration methods over the intervals where \( f(t, y)/Q(y) \) is smooth.

Let \( \nu_1(t_0, z_0, \eta_0; t_0 + \tau) \) be defined as

\[
\nu_1(\cdot) = \begin{cases} 
\nu_1(t_0, z_0, \eta_0; t_0 + \tau), & \text{if } f(t_0, x_0) > 0 \text{ or } f(t_0, x_0) = 0 \text{ and } \frac{\partial f(t_0, x_0)}{\partial t} > 0 \\
\nu_1(t_0, z_0, \eta_0; t_0 + \tau), & \text{if } f(t_0, x_0) < 0 \text{ or } f(t_0, x_0) = 0 \text{ and } \frac{\partial f(t_0, x_0)}{\partial t} < 0
\end{cases}
\]

**Integration algorithm** **INTEGRATE1()**

**Input:** Initial values \( t_0, \eta_0, y_0 \); time step range \([h_{\text{min}}, h_{\text{max}}] \); 

**Output:** Result of the integration is an array of \( \{(t_i, y_i, \eta_i, \varsigma_i)\}_{i=0}^{n_0} \)

1. \( n = 0 \)
2. \( \varsigma_0 = \text{YDIRECTION()} \)
3. \( y_{\text{jump}} = \text{GetNextQJump()} \)
4. if \( |f(t_0, y_0)| = 0 \) then **LINEARSTEP()**
5. while \( n <= n_0 \)
6. \( \tau_n = h \)
7. \( y_{n+1} = \nu_1(t_n, y_n, \eta_n; \tau_n) \)
8. if \( (y_n - y_{\text{jump}}) \cdot (y_{n+1} - y_{\text{jump}}) \leq 0 \)
9. **JUMPtoY()**
10. **GetNextQJump**
11. else
12. if \( f(t_n, y_n) \cdot f(t_{n+1}, y_{n+1}) \leq 0 \)
13. **JUMPtoYP()**
14. \( \eta_{n+1} = \text{UPDATEPREISACH}(\eta_n, y_{n+1}) \)
15. \( \varsigma_{n+1} = -\varsigma_n, \quad t_{n+1} = t_n + \tau_n, \quad n + + \)
16. **GetNextQJump**
17. **LINEARSTEP()**
18. else
19. \( \varsigma_{n+1} = \varsigma_n \)
20. \( \eta_{n+1} = \text{UPDATEPREISACH}(\eta_n, y_{n+1}) \)
21. \( t_{n+1} = t_n + \tau_n \)
22. \( n + + \)

**GetNextQJump**

**Input:** \( t_n, y_n \)

**Output:** Return value gives the position \( y_{\text{jump}} \) of first discontinuity of \( Q \) in the
direction of growth of the solution such that \( Q(y_{jump}) \neq 0 \).

(1) if \( \varsigma_n = +1 \)
(2) if \( K_\downarrow < 3 \) then return \(-\infty\)
(3) else return \( w_n^{(K_\downarrow - 2)} \)
(4) else
(5) if \( K_\uparrow < 3 \) then return \( \infty \)
(6) else return \( w_n^{(K_\uparrow - 2)} \)

YDirection()

Input: Current step \( n \)
Output: \(+1/−1\) if solution is going to increase/decrease

(1) if \( f(t_n, y_n) = 0 \) and \( \frac{\partial f(t_n, y_n)}{\partial t} = 0 \) then error
(2) if \( f(t_n, y_n) < 0 \) or \( f(t_n, y_n) = 0 \) and \( \frac{\partial f(t_n, y_n)}{\partial t} \neq 0 \) then error
(3) \(-1\)
(4) else
(5) \(+1\)

LinearStep()

Input: Current step \( n \)
Output:

(1) \( a = \frac{1}{\rho(y_n, y_n)} \cdot \frac{\partial f(t_n, y_n)}{\partial x} \), \( b = \frac{1}{\rho(y_n, y_n)} \cdot \frac{\partial f(t_n, y_n)}{\partial t} \)
(2) if \( b < 0 \) and \( \varsigma_n < 0 \)
(3) \( y_{n+1} = y_n + (-a - \sqrt{a^2 - 4b})h_{linear}/2 \)
(4) else if \( b > 0 \) and \( \varsigma_n > 0 \)
(5) \( y_{n+1} = y_n + (a + \sqrt{a^2 + 4b})h_{linear}/2 \)
(6) else
(7) error
(8) \( \tau_n = h_{linear} \)

JumpToY()

Input: Current step \( n \)
Output: Updated values of \( \tau_n, y_{n+1} \)

(1) Using bisection modify \( \tau_n \) so that new value \( y_{n+1} = \nu_1(t_n, y_n, \eta; \tau_n) \) satisfies \( |y_{n+1} - y_{jump}| < 10^{-8} \).
(2) \( y_{n+1} = y_{jump} \)

JumpToYP()

Input: Current step \( n \)
Output: Updated values of \( \tau_n, y_{n+1} \)

(1) Using bisection modify \( \tau_n \) so that new value \( y_{n+1} = \nu_1(t_n, y_n, \eta; \tau_n) \) satisfies \( |f(t_n + \tau_n, y_{n+1})| < 10^{-8} \).

UpdatePreisach(\( \eta_{old}, x \))

Input: Old state \( \eta_{old} \), new input \( x \)
Output: Updated value $\eta = \{w^{(i)}\}$ and appropriate $K$

1. $\eta = \eta_{old}$, $K = K_{old}$
2. if $|x| >= \gamma$
3. $K = 1$ $w^{(1)} = \text{sign}(x) \cdot \gamma$
4. return $\eta$
5. if $x > w^{(K_{old})}$
6. if $K \mod 2 == 0$ then $K++$
7. while $K \geq 3$ and $x > w^{(K-2)}$
8. $K = K - 2$
9. else if $x < w^{(K_{old})}$
10. if $K \mod 2 == 1$ then $K++$
11. while $K \geq 3$ and $x < w^{(K-2)}$
12. $K = K - 2$
13. if $K \geq K_{max}$ then error
14. $w^{(K)} = x$
15. return $\eta$

4. Conclusion
In this article we considered a problem of numerical integration of operator-differential equations involving derivative of a Preisach nonlinearity. Offered algorithm takes into account existence of points where right hand side of equation is not differentiable. The algorithm can be easily modified to incorporate integration error estimates.

Acknowledgments
Acknowledgement This research was partially supported by a private request to Cork University Foundation, IRCSET Embark Ireland PostDoctoral fellowship PD/2004/26 and Enterprise Ireland SC/2003/376

References
[1] R. Angulo-Jaramillo, D. Elrick, J. Parlange, P. Gerard-Marchant, and R. Haverkamp. Analysis of short-time single-ringe infiltration under falling-head conditions with gravitational effects. Hydrology Days Proceedings, pages 16–23, 2003.
[2] M. Brokate, A. Pokrovskii, D. Rachinskii, and O. Rasskazov. Science of Hysteresis (to appear), chapter Differential Equations with Hysteresis via a Canonical Example.
[3] H. Darcy. Les Fontaines Publiques de la Ville de Dijon. Dalmond, Paris, 1856.
[4] D. Flynn, H. McNamara, P. O’Kane, and A. Pokrovskii. Application of the Preisach model to soil-moisture hysteresis. BCRI Preprint 15, 2003. To be included in the book, ”The Science of Hysteresis”, edited by Giorgio Bertotti and Isaak D. Mayergoyz.
[5] W. B. Haines. Studies in the physical properties of soil. v. the hysteresis effect in capillary properties, and the modes of moisture distribution associated therewith. Journal of Agricultural Science, pages pp 97–116, 1930.
[6] D. Hillel. Introduction to Soil Physics. Academic Press Inc., New York, 1982.
[7] W. Hogarth, J. Hopmans, J.-Y. Parlange, and R. Haverkamp. Application of a simple soil-water hysteresis model. Journal of Hydrology, 9:21–29, 1988.
[8] M. A. Krasnosel’skii and A. Pokrovskii. Systems with Hysteresis. Springer-Verlag, Berlin, 1989. Translated from the Russian by Marek Niezgódka.
[9] J. Parlange, T. Steenhuis, R. Haverkamp, D. Barry, P. Culligan, W. Hogarth, P. Parlange, M.B. Ross, and F. Stagnitti. Soil Properties and Water Movement, Vadose Zone Hydrology - Cutting Across Disciplines. Oxford University Press, New York, 1999. pp. 99-129.
[10] J.-Y. Parlange. Water transport in soils. Annu. Rev. Fluid Mech., 12:77–102, 1980.
[11] L. A. Richards. Uptake and retention of water by soil as determined by distance to a water table. Jour. Amer. Soc. Agron., 33:778–786, 1941.