A Refined Inertial DCA for DC Programming

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Abstract We consider the difference-of-convex (DC) programming problems whose objective function is level-bounded. The classical DC algorithm (DCA) is well-known for solving this kind of problems, which returns a critical point. Recently, de Oliveira and Tcheo incorporated the inertial-force procedure into DCA (InDCA) for potential acceleration and preventing the algorithm from converging to a critical point which is not d(directional)-stationary. In this paper, based on InDCA, we propose two refined inertial DCA (RInDCA) with enlarged inertial step-sizes for better acceleration. We demonstrate the subsequential convergence of our refined versions to a critical point. In addition, by assuming the Kurdyka-Łojasiewicz (KL) property of the objective function, we establish the sequential convergence of RInDCA. Numerical simulations on image restoration problem show the benefit of enlarged step-size.

Keywords Difference-of-convex programming · Refined Inertial DCA · Enlarged inertial step-size · Kurdyka-Łojasiewicz property · Image restoration

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1 Introduction

Difference-of-convex (DC) programming, referring to the problems of minimizing a function which is the difference of two convex functions, forms an important class of nonconvex programming and has been studied extensively.
for decades, e.g., [11, 26, 31, 12, 25, 16]. In the paper, we consider the standard DC program in form of
\[
\min\{f(x) := f_1(x) - f_2(x) : x \in \mathbb{R}^n\},
\]
where \(f_1\) and \(f_2\) are proper closed and convex functions. Such a function \(f\) is called a DC function; \(f_1 - f_2\) is a DC decomposition of \(f\); \(f_1\) and \(f_2\) are DC components of \(f\). Throughout the paper, we make the mild Assumption 1 for problem \((P)\).

**Assumption 1**
(a) \(\text{dom } f_1 \subseteq \Omega \subseteq \text{dom } f_2\), where \(\Omega \subseteq \mathbb{R}^n\) is an open and convex set;
(b) \(f\) is level-bounded.

Note that (a) and (b) imply the finiteness of the optimal value of \((P)\).

DC Algorithm (DCA) [28, 14, 26, 27] is a well-known algorithm for solving DC program \((P)\), which has been introduced by Pham Dinh Tao in 1985 and extensively developed by Le Thi Hoai An and Pham Dinh Tao since 1994. Specifically, at iteration \(k\), DCA obtains the next iteration point \(x^{k+1}\) by solving the convex subproblem
\[
x^{k+1} \in \arg\min\{f_1(x) - \langle y^k, x \rangle : x \in \mathbb{R}^n\},
\]
where \(y^k \in \partial f_2(x^k)\). Then it has been proved in [27] that any limit point \(\bar{x}\) of the generated sequence \(\{x^k\}\) is a critical point of problem \((P)\), i.e., the necessary local optimality condition \(\partial f_1(\bar{x}) \cap \partial f_2(\bar{x}) \neq \emptyset\) is verified.

Recently, several accelerated algorithms for DC programming have been studied. Actacho et al. proposed in [2] a boosted DC algorithm for unconstrained smooth DC program where both \(f_1\) and \(f_2\) are smooth. It was verified there that the direction \(x^{k+1} - x^k\), determined by the consecutive iterations of DCA, is a descent direction of \(f\) at \(x^{k+1}\) when \(f_2\) is strongly convex, thus a line-search procedure can be conducted along it to obtain a better candidate with lower objective value. Meanwhile, Niu et al. [21] also developed a boosting DCA by incorporating line-search procedure for general DC program (both smooth and nonsmooth cases) with convex constraints. Later, Actacho et al. investigated the line-search idea in unconstrained nonsmooth DC program [3] and nonsmooth DC program with linear constraint [1], both of them could be considered as special cases of the one proposed in [21]. Different from the boosted approaches above, another type of acceleration is the momentum methods. There are two renowned momentum methods: the Polyak’s heavy-ball method [34] and the Nesterov’s acceleration technique [20, 19]. Indeed, Nesterov’s acceleration belongs to the heavy ball family. These two momentum methods have been successfully applied for various nonconvex optimizations problems (see, e.g., [22, 29, 18]). In DC programming, Oliveira et al. [24] introduced the heavy-ball inertial-force into DCA and proposed an inertial DC algorithm (InDCA) for problem \((P)\) in the premise that \(f_2\) is strongly convex; while Wen et al. [32] incorporated the Nesterov’s extrapolation technique into the proximal DCA by considering a special class of DC program where \(f_1\) is the
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sum of a convex function and a smooth convex function. Note that criticality is a weaker optimality condition than \(d\)-stationarity (see, e.g., \([25, 23]\)). However, methods with inertial-forces often help prevent algorithms from converging to critical points that are not \(d\)-stationary. Next, we introduce some works about obtaining the \(d\)-stationary solution of DC program (\(P\)). By exploring the structure of \(f_2\), i.e., \(f_2\) is the supremum of finitely many convex smooth functions, Pang et al. [25] proposed a novel enhanced DC algorithm (EDCA) to obtain a \(d\)-stationary solution. Later, in order to make EDCA cheaper at each iteration, Lu et al. [16] developed an inexact variant of EDCA, the solution of which has the same property as that of EDCA; furthermore, considering additionally the structure of \(f_1\), i.e., \(f_1\) consists of a convex function \(f_n\) plus a smooth convex function \(f_s\), then a proximal version of EDCA was developed with a low cost in each iteration when the proximal operator of \(f_n\) is easy to compute; besides, this algorithm incorporated the Nesterov’s extrapolation for a possible acceleration.

In this paper, based on the inertial DC algorithm (InDCA) [24], we propose two refined versions (RInDCA) equipped with larger inertial step-sizes: the refined exact one (RInDCA\(_e\)) and the refined inexact one (RInDCA\(_n\)). As an example, RInDCA\(_e\) obtains at \(k\)-th iteration the next trial point \(x^{k+1}\) by solving the convex subproblem

\[
x^{k+1} \in \text{argmin}\{f_1(x) - \langle y^k + \gamma(x^k - x^{k-1}), x \rangle : x \in \mathbb{R}^n\},
\]

where \(y^k \in \partial f_2(x^k)\). Our analysis shows that the inertial step-size \(\gamma \in [0, (\sigma_1 + \sigma_2)/2)\) is adequate for guaranteeing convergence where \(\sigma_1\) and \(\sigma_2\) are the strong convex parameters of \(f_1\) and \(f_2\). Particularly, when \(\sigma_1 > 0\), this enlarges the range of \(\gamma\) compared with \([0, \sigma_2/2)\) in InDCA. A larger inertial step-size will potentially accelerate the convergence. Moreover, in some practical applications, due to the lack of strong convexity in \(f_1\) and/or \(f_2\), InDCA may not be applicable. We often encounter two cases:

- \(f_1\) is strongly convex, while \(f_2\) is not;
- both \(f_1\) and \(f_2\) are not strongly convex.

For the first case, InDCA can not be applied directly, but RInDCA can; for the second case, a strongly convex function, e.g., \(\rho\|\cdot\|^2/2\) with \(\rho > 0\), is usually added to \(f_1\) and \(f_2\) for strong convexity of DC components, so that both InDCA and RInDCA are applicable. More importantly, when \(f_1\) and \(f_2\) are both strongly convex, then RInDCA has larger inertial step-sizes which potentially yield more acceleration.

Our contributions are: (1) propose two refined versions of InDCA, each of which is equipped with larger inertial step-size compared with InDCA; besides, the relation between the inertial-type DCA and the classical DCA is pointed out; (2) establish the sequential convergence of our refined versions by assuming the Kurdyka-Lojasiewicz (KL) property of the objective function.

The rest of the paper is organized as follows: In Section 2, we recall some notations and preliminaries in convex and variational analysis. Then we introduce in Section 3 our two refined versions of InDCA, followed respectively by
their subsequential convergence analysis. Next in Section 4, by assuming the KL property of the objective function, we prove the sequential convergence of our refined versions. Numerical results summarized in Section 5 on image restoration problem demonstrate the benefit of enlarged inertial step-size by comparing with InDCA, classical DCA with fixed DC decomposition, and successive DCA. Some concluding remarks are discussed in the final section.

2 Notations and preliminaries

Let $\mathbb{R}^n$ denote the finite dimensional vector space equipped with the canonical inner product $\langle \cdot, \cdot \rangle$, and the induced norm $\|\cdot\|$, i.e., $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. The entry of a vector $x$ is denoted as $x_i$, and the entry of a matrix $A$ is denoted as $A_{i,j}$.

For an extended real-valued function $h : \mathbb{R}^n \to (-\infty, \infty]$, the set $\text{dom} h := \{x \in \mathbb{R}^n : h(x) < \infty\}$ denotes its effective domain. If $\text{dom} h \neq \emptyset$, and $h$ does not attain the value $-\infty$, then $h$ is called a proper function. The set $\text{epi} h := \{(x, t) : h(x) \leq t, x \in \mathbb{R}^n, t \in \mathbb{R}\}$ denotes the epigraph of $h$, and $h$ is closed (resp. convex) if $\text{epi} h$ is closed (resp. convex). A proper closed function $h$ is said to be level-bounded if the lower level set $\{x \in \mathbb{R}^n : h(x) \leq r\}$ is bounded for any $r \in \mathbb{R}$.

Given a proper closed function $h : \mathbb{R}^n \to (-\infty, \infty]$, the Fréchet subdifferential of $h$ at $x \in \text{dom} h$ is given by

$$\partial F h(x) := \left\{ y \in \mathbb{R}^n : \liminf_{z \to x} \frac{h(z) - h(x) - \langle y, z - x \rangle}{\|z - x\|} \geq 0 \right\},$$

while for $x \notin \text{dom} h$, $\partial F h(x) = \emptyset$. The (limiting) subdifferential of $f$ at $x \in \text{dom} h$ is defined as

$$\partial h(x) := \{y \in \mathbb{R}^n : \exists \{x^k \to x, h(x^k) \to h(x), y^k \in \partial F h(x^k)\} \text{ such that } y^k \to y\},$$

and $\partial h(x) = \emptyset$ if $x \notin \text{dom} h$. Note that if $h$ is also convex, then the Fréchet subdifferential and the limiting subdifferential will coincide with the convex subdifferential, that is,

$$\partial F h(x) = \partial h(x) = \{y \in \mathbb{R}^n : h(z) \geq h(x) + \langle y, z - x \rangle \text{ for all } z \in \mathbb{R}^n\}.$$

Given a nonempty set $C \subseteq \mathbb{R}^n$, the distance from a point $x \in \mathbb{R}^n$ to $C$ is denoted as $\text{dist}(x, C) := \inf\{\|x - z\| : z \in C\}$. We now recall the Kurdyka-Lojasiewicz (KL) property $[9, 4, 5]$. For $\eta \in (0, \infty]$, we denote by $\Xi_\eta$ the set of all concave continuous functions $\varphi : [0, \eta) \to [0, \infty)$ that are continuously differentiable over $(0, \eta)$ with positive derivatives and satisfy $\varphi(0) = 0$. 
Definition 2.1 (KL property) A proper closed function $h$ is said to satisfy the KL property at $\bar{x} \in \text{dom}\partial h := \{x \in \mathbb{R}^n : \partial h(x) \neq \emptyset\}$ if there exist $\eta \in (0, \infty]$, a neighborhood $U$ of $x$, and a function $\varphi \in \Xi_\eta$ such that for all $x$ in the intersection

$$U \cap \{x \in \mathbb{R}^n : h(\bar{x}) < h(x) < h(\bar{x}) + \eta\},$$

it holds that

$$\varphi'(h(\bar{x}) - h(x))\text{dist}(0, \partial h(x)) \geq 1.$$

If $h$ satisfies the KL property at any point of dom$\partial h$, then $h$ is called a KL function.

The following uniformized KL property plays an important role in our sequential convergence analysis:

Lemma 2.1 (Uniformized KL property, see [10]) Let $\Omega \subseteq \mathbb{R}^n$ be a compact set and let $h : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be a proper closed function. If $h$ is constant on $\Omega$ and satisfies the KL property at each point of $\Omega$, then there exist $\varepsilon, \eta > 0$ and $\varphi \in \Xi_\eta$ such that

$$\varphi'(h(\bar{x}) - h(x))\text{dist}(0, \partial h(x)) \geq 1$$

for any $x \in \Omega$ and any $x$ satisfying $\text{dist}(x, \Omega) < \varepsilon$ and $h(\bar{x}) < h(x) < h(\bar{x}) + \eta$.

Now, let $h : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be a proper closed and convex function, and let $\varepsilon > 0$, the set

$$\partial_\varepsilon h(x) := \{y \in \mathbb{R}^n : h(y) \geq h(x) + \langle y, z - x \rangle - \varepsilon \text{ for all } z \in \mathbb{R}^n\}$$

denotes the $\varepsilon$-subdifferential of $h$ at $x$, and any point in $\partial_\varepsilon h(x)$ is called a $\varepsilon$-subgradient of $h$ at $x$. Clearly, $\partial_\varepsilon h(x) \neq \emptyset$ implies that $x \in \text{dom}h$. A proper closed function $h$ is called $\sigma$-strongly convex (cf. $\sigma$-convex) with $\sigma \geq 0$ if for any $x, z \in \text{dom}h$ and $\lambda \in [0, 1]$, it holds that

$$h(\lambda x + (1 - \lambda)z) \leq \lambda h(x) + (1 - \lambda)h(z) - \frac{\sigma}{2} \lambda(1 - \lambda)\|x - z\|^2.$$

Moreover, let $h$ be a $\sigma$-convex function. Then it is known [7] that for any $x \in \text{dom}h$, $y \in \partial h(x)$ and $z \in \text{dom}h$, we have

$$h(z) \geq h(x) + \langle y, z - x \rangle + \frac{\sigma}{2}\|z - x\|^2.$$

Finally, we give a result related with strong convexity and $\varepsilon$-subdifferential, which will be used in analyzing our refined inexact DC algorithm.

Lemma 2.2 Let $h : \mathbb{R}^n \rightarrow (-\infty, \infty]$ be a $\sigma$-convex function, and let $\varepsilon \geq 0$, $t \in (0, 1]$. Then for any $x \in \text{dom}h$, $z \in \text{dom}h$ and $y \in \partial h(x)$, we have

$$h(z) \geq h(x) + \langle y, z - x \rangle + \frac{\sigma(1 - t)}{2}\|z - x\|^2 - \frac{\varepsilon}{t}.$$  \hspace{1cm} (1)
Proof For any $x \in \text{dom} h$, $z \in \text{dom} h$ and $t \in (0, 1]$, it follows from the definition of $\sigma$-strong convexity of $h$ that
\[ h(x + t(z - x)) \leq (1 - t)h(x) + th(z) - \frac{\sigma}{2}t(1 - t)\|x - z\|^2. \] (2)

On the other hand, $y \in \partial \varepsilon h(x)$ implies that
\[ h(x + t(z - x)) \geq h(x) + \langle y, t(z - x) \rangle - \varepsilon. \] (3)
Combining (2) and (3), then we obtain (1).

3 A refined inertial DCA for DC programming

In this section, we will focus on two refined versions (RInDCA) of the inertial DC algorithm (InDCA) [24] equipped with larger inertial step-size compared with InDCA for problem ($P$).

Firstly, we will briefly review the basic idea of InDCA. Suppose that the objective function $f$ has a DC decomposition $f = f_1 - f_2$ where $f_2$ is $\sigma_2$-convex ($\sigma_2 > 0$). The original InDCA is described in Algorithm 1.

**Algorithm 1: InDCA**

| Input: $x^0 \in \text{dom} f_1$; $\lambda \in [0, 1)$; $\gamma \in [0, (1 - \lambda)\sigma_2/2)$; $x^{-1} = x^0$. |
| 1 for $k = 0, 1, 2, \ldots$ do |
| 2 find $x^{k+1} \in \mathbb{R}^n$ such that |
| $\partial \varepsilon h_k f_1(x^{k+1}) \cap (\partial f_2(x^k) + \gamma(x^k - x^{k-1})) \neq \emptyset$ (4) |
| with $0 \leq \varepsilon^{k+1} \leq \lambda \frac{\sigma_2}{2} \|x^{k+1} - x^k\|^2$. |
| 3 end |

Specifically, if $\lambda = 0$, then $\varepsilon^{k+1} = 0$ and thus $\partial \varepsilon h_k f_1(x^{k+1}) = \partial f_1(x^{k+1})$ in InDCA. In this case, the iteration point $x^{k+1}$ in (4) could be obtained by solving the next convex subproblem:
\[ x^{k+1} \in \arg\min \{f_1(x) - \langle y^k + \gamma(x^k - x^{k-1}), x \rangle : x \in \mathbb{R}^n \}, \] (5)
where $y^k \in \partial f_2(x^k)$. We will particularly refer this case as the exact version of InDCA (namely, InDCA$_e$). Note that if $\gamma = 0$, then InDCA$_e$ reduces to the classical DCA applied to the DC decomposition $f = f_1 - f_2$. On the other hand, if $\lambda \in (0, 1)$, then the corresponding InDCA is referred as the inexact version of InDCA (namely, InDCA$_n$). In this case, the iteration point $x^{k+1}$ could be computed by bundle methods [24]. It has been shown in [24] that any limit point of the generated sequence by InDCA is a critical point.

Note that, for applying InDCA, the strong convexity parameter of the second DC component $f_2$ requires to be greater than 0. However, in some practical applications, $f_2$ may not be strongly convex, but we can still use
InDCA by introducing a strong convex function on both DC components of the initial DC components. Here are two examples in signal processing:

**Example 3.1 (1D signals recovery by $\ell_1-\ell_2$)** Consider the nonconvex sparse signal recovery problem for 1D signals:

$$
\min \{ f(x) = \frac{1}{2} \|Ax - b\|^2 + \lambda \|x\|_1 - \lambda \|x\|_2 : x \in \mathbb{R}^n \},
$$

where $\lambda > 0$ is a trade-off constant.

This model has a DC decomposition $f = f_1 - f_2$ with $f_1(x) = \|Ax - b\|^2/2 + \lambda \|x\|_1$ and $f_2(x) = \lambda \|x\|_2$ where both $f_1$ and $f_2$ are not supposed to be strongly convex. In this case, it is usually suggested to add the strongly convex term $\psi : x \mapsto \rho \|x\|^2/2$ into both $f_1$ and $f_2$, yielding the new DC decomposition $(f_1 + \psi) - (f_2 + \psi)$. Then InDCA can be applied with $\gamma \in [0, \rho/2)$ to solve the problem in Example 3.1.

**Example 3.2 (1D signals denoising)** Consider the nonconvex sparse signal recovery problem for 1D signals:

$$
\min \{ \frac{\mu}{2} \|x - b\|^2 + \sum_{i=1}^{n-1} \phi(|x_{i+1} - x_i|) : x \in \mathbb{R}^n \},
$$

where $\mu > 0$ is a trade-off constant, and $\phi$ is a concave function for inducing sparsity, e.g., $\phi(r) := \log(1 + 2r)/r$.

In this example, taking $\phi(r) = \log(1 + 2r)/r$, we have a DC decomposition as $f_1 - f_2$ with $f_1(x) = \frac{\mu}{2} \|x - b\|^2 + \sum_{i=1}^{n-1} |x_{i+1} - x_i|$ and $f_2 = \sum_{i=1}^{n-1} |x_{i+1} - x_i| - \sum_{i=1}^{n-1} \phi(|x_{i+1} - x_i|)$ where $f_1$ is strongly convex and $f_2$ is not supposed to be strongly convex. In [24], a similar trick as adding $\psi : x \mapsto \|x\|^2$ into both $f_1$ and $f_2$ is used to make the second DC component 2-strong convex, and then InDCA can be employed to solve the resulting problem.

As shown in Examples 3.1 and 3.2, in order to apply InDCA, the first DC component will be equipped with strong convexity. However, InDCA only considers the strong convexity of the second DC component to determine the range of the inertial step-size $\gamma$ for guaranteeing convergence, while the strong convexity of the first DC component is not taken into account yet. In the next two subsections, we will involve the strong convexity of both DC components to yield two refined versions of InDCA with enlarged inertial step-sizes. Moreover, the subsequential convergence are also established.

### 3.1 Refined exact version of InDCA

Assumption 2 Suppose that the DC components $f_1$ is $\sigma_1$-convex ($\sigma_1 \geq 0$) and $f_2$ is $\sigma_2$-convex ($\sigma_2 \geq 0$) with $\sigma_1 + \sigma_2 > 0$. 


Algorithm 2: RInDCA

Input: $x^0 \in \text{dom} f_1$; $\gamma \in [0,(\sigma_1 + \sigma_2)/2)$; $x^{-1} = x^0$.

1. for $k = 0, 1, 2, \cdots$ do
2. find $x^{k+1} \in \mathbb{R}^n$ such that
   $\partial f_1(x^{k+1}) \cap (\partial f_2(x^k) + \gamma(x^k - x^{k-1})) \neq \emptyset$. 
3. end

Now, we describe our refined exact algorithm RInDCA in Algorithm 2, where the range of $\gamma$ is enlarged from $[0,\sigma_2/2]$ in InDCA to $[0,(\sigma_1 + \sigma_2)/2]$ in RInDCA.

Remark 3.1 Note that RInDCA requires that $\sigma_1 + \sigma_2 > 0$. Particularly, it holds for the case where $\sigma_2 = 0$ and $\sigma_1 > 0$ as in Example 3.2, without requiring additional strong convexity of $f_2$.

Next, we will focus on the convergence theorem of RInDCA.

**Lemma 3.1** Let $\{x^k\}$ be the sequence generated by RInDCA, $\sigma_1 + \sigma_2 > 0$ and $\gamma \in [0,(\sigma_1 + \sigma_2)/2)$. Then the sequence $\{f(x^k) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^k - x^{k-1}\|^2\}$ is nonincreasing and for all $k \geq 0$,

$$f(x^{k+1}) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^{k+1} - x^k\|^2 \leq f(x^k) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^k - x^{k-1}\|^2 - \frac{\sigma_1 + \sigma_2 - 2\gamma}{2}\|x^k - x^{k-1}\|^2. \quad (7)$$

**Proof** Because $x^{k+1}$ satisfies (6), and let $y^k \in \partial f_2(x^k)$ such that $y^k + \gamma(x^k - x^{k-1}) \in \partial f_1(x^{k+1})$, then by $\sigma_1$-convexity of $f_1$, we have

$$f_1(x^k) \geq f_1(x^{k+1}) + \langle y^k + \gamma(x^k - x^{k-1}), x^k - x^{k+1} \rangle + \frac{\sigma_1}{2}\|x^{k+1} - x^k\|^2. \quad (8)$$

On the other hand, it follows from $y^k \in \partial f_2(x^k)$ and $\sigma_2$-convexity of $f_2$ that

$$f_2(x^{k+1}) \geq f_2(x^k) + \langle y^k, x^{k+1} - x^k \rangle + \frac{\sigma_2}{2}\|x^{k+1} - x^k\|^2. \quad (9)$$

Summing (8) to (9), and reshuffling the terms, we derive that

$$f(x^k) \geq f(x^{k+1}) + \gamma\|x^k - x^{k-1}, x^k - x^{k+1}\| + \frac{\sigma_1 + \sigma_2}{2}\|x^{k+1} - x^k\|^2. \quad (10)$$

By applying $\langle x^k - x^{k-1}, x^k - x^{k+1} \rangle \geq -\left(\|x^k - x^{k-1}\|^2 + \|x^k - x^{k+1}\|^2\right)/2$ to (10), we obtain

$$f(x^{k+1}) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^{k+1} - x^k\|^2 \leq f(x^k) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^k - x^{k-1}\|^2 - \frac{\sigma_1 + \sigma_2 - 2\gamma}{2}\|x^k - x^{k-1}\|^2. \quad (11)$$

Moreover, we get from $\gamma < (\sigma_1 + \sigma_2)/2$ that $(\sigma_1 + \sigma_2 - 2\gamma)/2 > 0$, thus the sequence $\{f(x^k) + \frac{\sigma_1 + \sigma_2 - \gamma}{2}\|x^k - x^{k-1}\|^2\}$ is nonincreasing. \[\Box\]
Theorem 3.1 Let \( \{x^k\} \) and \( \{y^k\} \) be the sequences generated by RInDCA, \( \sigma_1 + \sigma_2 > 0 \) and \( \gamma \in [0, (\sigma_1 + \sigma_2)/2) \). Then the following statements hold:

(i) The sequences \( \{x^k\} \) and \( \{y^k\} \) are bounded;
(ii) \( \lim_{k \to \infty} \|x^k - x^{k-1}\| \to 0; \)
(iii) Any limit point \( \bar{x} \) of \( \{x^k\} \) is a critical point of problem (P).

Proof (i) Let us denote \( \eta_1 := (\sigma_1 + \sigma_2 - \gamma)/2 \) and \( \eta_2 := (\sigma_1 + \sigma_2 - 2\gamma)/2 \). Clearly, \( \eta_1 > 0, \eta_2 > 0 \), and the inequality (7) reads as

\[
 f(x^{k+1}) + \eta_1 \|x^{k+1} - x^k\|^2 \leq f(x^k) + \eta_1 \|x^k - x^{k-1}\|^2 - \eta_2 \|x^k - x^{k-1}\|^2. \tag{11}
\]

Then, for all \( k \geq 0 \), it follows from \( x^0 = x^{-1} \) and (11) that

\[
 f(x^k) \leq f(x^k) + \eta_1 \|x^k - x^{k-1}\|^2 \leq f(x^0).
\]

Recall that \( f \) is assumed to be level-bounded in Assumption 1, then the boundedness of \( \{x^k\} \) is an immediate consequence of the level-boundedness of \( f \), and the boundedness of \( \{y^k\} \) is derived from the boundedness of \( \{x^k\} \) and \( \{x^k\} \subseteq \Omega \subseteq \text{dom} f_2 \). Thus statement (i) holds.

(ii) By summing (11) from \( k = 0 \) to \( n \), we obtain that

\[
 \eta_2 \sum_{k=0}^n \|x^k - x^{k-1}\|^2 \leq f(x^0) - (f(x^{n+1}) + \eta_1 \|x^{n+1} - x^n\|^2). \tag{12}
\]

Recalling that the optimal value of \( f \), denoted as \( f^* \), is finite, we have

\[
 f(x^{n+1}) + \eta_1 \|x^{n+1} - x^n\|^2 \geq f^*,
\]

then

\[
 f(x^0) - (f(x^{n+1}) + \eta_1 \|x^{n+1} - x^n\|^2) \leq f(x^0) - f^*. \tag{13}
\]

Therefore, we get from (12) and (13) that

\[
 \sum_{k=0}^n \|x^k - x^{k-1}\|^2 \leq \frac{f(x^0) - f^*}{\eta_2}, \quad \forall n \in \mathbb{N}.
\]

Thus, \( \sum_{k=0}^\infty \|x^k - x^{k-1}\|^2 < \infty \) which implies \( \lim_{k \to \infty} \|x^k - x^{k-1}\| = 0 \).

(iii) Let \( \bar{x} \) be any limit point of \( \{x^k\} \). Then there exists a convergent subsequence such that \( \lim_{i \to \infty} x^{k_i} = \bar{x} \). Moreover, since \( \{y^{k_i}\} \) is bounded in \( \mathbb{R}^p \), there exists a convergent subsequence of \( \{y^{k_i}\} \). Without loss of generality, we can assume that the sequence \( \{y^{k_i}\} \) is convergent and \( \lim_{i \to \infty} y^{k_i} = \bar{y} \). Then taking into account that \( y^{k_i} \in \partial f_2(x^{k_i}) \) for all \( i \), \( y^{k_i} \to \bar{y}, \ x^{k_i} \to \bar{x} \), and the closedness of the graph \( \partial f_2 \) (see Theorem 24.4 in [30]), we obtain

\[
 \bar{y} \in \partial f_2(\bar{x}). \tag{14}
\]

On the other hand, the next relation holds for all \( i \)

\[
 y^{k_i} + \gamma (x^{k_i} - x^{k_i-1}) \in \partial f_1(x^{k_i+1}).
\]
By the fact that \( \lim_{i \to \infty} x^{k_i+1} = \bar{x} \) since \( \lim_{i \to \infty} \|x^{k_i} - x^{k_i-1}\| \to 0 \), then combining \( y^{k_i} \to \tilde{y}, \gamma(x^{k_i} - x^{k_i-1}) \to 0, x^{k_i+1} \to \bar{x} \) and the closedness of the graph \( \partial f_1 \), we have
\[
\tilde{y} \in \partial f_1(\bar{x}).
\] (15)

We conclude from (14) and (15) that \( \tilde{y} \in \partial f_1(\bar{x}) \cap \partial f_2(\bar{x}) \neq \emptyset \), thus \( \bar{x} \) is a critical point of problem \( (P) \). \( \square \)

3.2 Refined inexact version of InDCA

In some practical problems, obtaining \( x^{k+1} \) satisfying (6) requires exactly solving a hard convex subproblem, whose cost may be expensive. Thus, the inexact algorithms come into playing their importance. Next, we describe in Algorithm 3 our refined inexact algorithm RInDCA\(_n\) with larger inertial step-size compared with InDCA\(_n\), followed by its convergence analysis.

**Algorithm 3: RInDCA\(_n\)**

**Input:** \( x^0 \in \text{dom} f_1; \lambda \in (0, 1); t \in (0, 1]; \gamma \in [0, (\sigma_1(1-t) + \sigma_2)/2 - \lambda \sigma_2/2t]; \)
\( x^{-1} = x^0 \).

1. for \( k = 0, 1, 2, \ldots \) do
2. find \( x^{k+1} \in \mathbb{R}^n \) such that
\[
\partial f_1(x^{k+1}) \cap (\partial f_2(x^k) + \gamma(x^k - x^{k-1})) \neq \emptyset
\] with \( 0 \leq \varepsilon^{k+1} \leq \lambda \frac{\sigma}{2} \|x^{k+1} - x^k\|^2 \).
3. end

Note that the only difference between InDCA\(_n\) and RInDCA\(_n\) is the distinct ranges of \( \gamma \). Next, we will focus on showing the enlarged inertial step-size provided by our refined inexact version RInDCA\(_n\).

**Lemma 3.2** Let \( \{x^k\} \) be the sequence generated by RInDCA\(_n\), \( \lambda \in (0, 1), t \in (0, 1] \) and \( \sigma_2 > 0 \) such that \( \sigma_t = \sigma_1(1-t) + \sigma_2 - \lambda \sigma_2/t > 0 \) and \( \gamma < \sigma_t/2 \). Then the sequence \( \{f(x^k) + \frac{\sigma_t - \gamma}{2} \|x^k - x^{k-1}\|^2\} \) is nonincreasing and for all \( k \geq 0 \), we have
\[
f(x^{k+1}) + \frac{\sigma_t - \gamma}{2} \|x^{k+1} - x^k\|^2 \leq f(x^k) + \frac{\sigma_t - \gamma}{2} \|x^k - x^{k-1}\|^2
\] minus
\[
- \frac{\sigma_t - 2\gamma}{2} \|x^k - x^{k-1}\|^2.
\] (17)
Proof Because $x^{k+1}$ satisfies (16), and let $y^k \in \partial f_2(x^k)$ such that $y^k + \gamma(x^k - x^{k-1}) \in \partial_{x^{k+1}} f_1(x^{k+1})$, we have

$$f_1(x^k) \geq f_1(x^{k+1}) + \langle y^k + \gamma(x^k - x^{k-1}), x^k - x^{k+1} \rangle + \frac{\sigma_1 (1 - t)}{2} \|x^{k+1} - x^k\|^2 - \varepsilon_{k+1}$$

$$\geq f_1(x^{k+1}) + \langle y^k + \gamma(x^k - x^{k-1}), x^k - x^{k+1} \rangle + \frac{\sigma_1 (1 - t) - \lambda \sigma_2 / t}{2} \|x^{k+1} - x^k\|^2,$$

where the second inequality is implied by the fact $\varepsilon_{k+1} \leq \lambda \sigma_2 / t \|x^{k+1} - x^k\|^2$. On the other hand, $y^k \in \partial f_2(x^k)$ and $\sigma_2$-convexity of $f_2$ imply that

$$f_2(x^{k+1}) \geq f_2(x^k) + \langle y^k, x^{k+1} - x^k \rangle + \frac{\sigma_2}{2} \|x^{k+1} - x^k\|^2. \quad (19)$$

Summing (18) to (19), and reshuffling the terms, we obtain that

$$f(x^k) \geq f(x^{k+1}) + \gamma \langle x^k - x^{k-1}, x^k - x^{k+1} \rangle + \frac{\bar{\sigma}_t}{2} \|x^{k+1} - x^k\|^2. \quad (20)$$

Moreover, by applying $\langle x^k - x^{k-1}, x^k - x^{k+1} \rangle \geq -((\|x^k - x^{k-1}\|^2 + \|x^k - x^{k+1}\|^2)/2)$ to (20), we obtain (17). 

The proof of the next theorem is omitted since it follows from similar arguments as in Theorem 3.1.

**Theorem 3.2** Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by RInDCA$_n$. Then the following statements hold:

(i) The sequences $\{x^k\}$ and $\{y^k\}$ are bounded;

(ii) $\lim_{k \to \infty} \|x^k - x^{k-1}\| \to 0$;

(iii) Any limit point $\bar{x}$ of $\{x^k\}$ is a critical point of problem $(P)$.

**Remark 3.2** In RInDCA$_n$, the range of $\gamma$ has three different cases:

(a) If $\sigma_1 = 0$, then one can select $t = 1$ such that

$$\frac{\bar{\sigma}_t}{2} = \frac{(1 - \lambda) \sigma_2}{2},$$

in this case, the range of $\gamma$ coincides with the one in InDCA$_n$;

(b) If $\sigma_1 > 0$ and $\sqrt{\lambda \sigma_2 / \sigma_1} \geq 1$, then one can also choose $t = 1$ such that

$$\frac{\bar{\sigma}_t}{2} = \frac{(1 - \lambda) \sigma_2}{2},$$

which gives the same range of $\gamma$ as in InDCA$_n$;
(c) If $\sigma_1 > 0$ and $\sqrt{\lambda \sigma_2 / \sigma_1} < 1$, then one can pick $t = \sqrt{\lambda \sigma_2 / \sigma_1}$ such that
\[
\frac{\sigma_t}{2} > \frac{(1 - \lambda) \sigma_2}{2}.
\]
In this case, the range of $\gamma$ is larger than that given in InDCA$_n$, which shows that our refined inexact version RInDCA$_n$ takes a larger inertial step-size. In our previous Examples 3.1 and 3.2, the case (c) holds exactly since $\sigma_1 \geq \sigma_2 > 0$ and for all $\lambda \in (0, 1)$, we have $\sqrt{\lambda \sigma_2 / \sigma_1} < 1$, so that RInDCA$_n$ is applicable to these examples.

Comparing InDCA$_n$ and RInDCA$_n$, we can visualize in Fig. 1 the upper bound of $\gamma$ with respect to $\lambda$ in the case (c). Here the supremum of $\gamma$ for RInDCA$_n$ is $H_1(\lambda) := (\sigma_1 + \sigma_2)/2 - \sqrt{\lambda \sigma_1 \sigma_2}$ obtained at $t = \sqrt{\lambda \sigma_2 / \sigma_1}$; while for InDCA$_n$, the supremum of $\gamma$ is $H_2(\lambda) := (1 - \lambda) \sigma_2 / 2$. We plot $H_1$ and $H_2$ over $\lambda \in (0, 1)$ with respect to $(\sigma_1, \sigma_2) \in \{(1, 1), (2, 1), (3, 1), (4, 1)\}$ respectively. It can be observed that, when fixing $\lambda \in (0, 1)$ and $\sigma_2 = 1$,

![Fig. 1 H1 and H2 over λ ∈ (0, 1) with (σ1, σ2) ∈ {(1, 1), (2, 1), (3, 1), (4, 1)} respectively.](image)

then by increasing the value of $\sigma_1$, the difference between $H_1(\lambda)$ and $H_2(\lambda)$ is enlarged. This observation implies that taking into consideration the strong convexity of $f_1$ may enlarge the inertial step-size, which could potentially result in acceleration.
3.3 Relation between the inertial-type DCA and the classical DCA

It is worth noting that the inertial-type DCA (both InDCA and RInDCA) is related to the successive DCA, namely SDCA (see [14]). We will take the exact version of InDCA for illustration. Suppose that \( f \) has a basic DC decomposition \( f_1 - f_2 \), we take at \( k \)-th iteration of DCA the DC decomposition \( f = (f_1 + \varphi^k) - (f_2 + \varphi^k) \) where \( \varphi^k(x) = \frac{\gamma}{2} \|x - x^{k-1}\|^2 \), then DCA applied to this special DC decomposition yields

\[
x^{k+1} = \arg\min_{x \in \mathbb{R}^n} \{ f_1(x) + \varphi^k(x) - \langle z^k, x \rangle : x \in \mathbb{R}^n \},
\]

(21)

where \( z^k \in \partial(f_2 + \varphi^k)(x^k) \). Therefore, we can find \( y^k \in \partial f_2(x^k) \) such that

\[
z^k = y^k + \nabla \varphi^k(x^k) = y^k + \gamma (x^k - x^{k-1}).
\]

Then (21) reads as

\[
x^{k+1} = \arg\min_{x \in \mathbb{R}^n} \{ f_1(x) + \varphi^k(x) - \langle y^k + \gamma (x^k - x^{k-1}), x \rangle : x \in \mathbb{R}^n \},
\]

(22)

where \( y^k \in \partial f_2(x^k) \).

Comparing SDCA described in (22) and InDCA\( _e \) in (5), we observe that the only difference is the presence of the strong convex term \( \varphi^k(x) \) in (22). This term plays as a regularizer which can be interpreted as finding a point \( x^{k+1} \) close to \( x^{k-1} \) when trying to minimize the convex function \( x \mapsto f_1(x) - \langle y^k + \gamma (x^k - x^{k-1}), x \rangle \). This is the basic idea of proximal point method, thus this SDCA is often called proximal DCA as well. The absence of \( \varphi^k \) in InDCA\( _e \) means that the iteration point \( x^{k+1} \) is not supposed to be close to \( x^{k-1} \), which may lead to potential acceleration. As a conclusion, the term \( \varphi^k \) plays a key role to make difference between InDCA\( _e \) and DCA, and potentially yields acceleration in InDCA\( _e \).

Remark 3.3 Indeed, the convex problem (22) is irrelevant to \( x^{k-1} \) since

\[
\arg\min_{x \in \mathbb{R}^n} \{ f_1(x) + \varphi^k(x) - \langle y^k + \gamma (x^k - x^{k-1}), x \rangle : x \in \mathbb{R}^n \}
\]

\[
= \arg\min_{x \in \mathbb{R}^n} \{ f_1(x) + \frac{\gamma}{2} \|x - x^{k-1}\|^2 - \langle y^k + \gamma x^k, x \rangle + \langle \gamma x^{k-1}, x \rangle : x \in \mathbb{R}^n \}
\]

\[
= \arg\min_{x \in \mathbb{R}^n} \{ f_1(x) + \frac{\gamma}{2} \|x\|^2 - \langle y^k + \gamma x^k, x \rangle : x \in \mathbb{R}^n \},
\]

which is exactly the subproblem of DCA with the fixed DC decomposition

\[
f = (f_1 + \frac{\gamma}{2} \|\cdot\|^2) - (f_2 + \frac{\gamma}{2} \|\cdot\|^2).
\]

3.4 Some results on the sequences obtained by RInDCA

In this subsection, we will prove some important results on the sequences obtained by RInDCA\( _e \) and RInDCA\( _n \). We denote \( \Omega_1 \) (resp. \( \Omega_2 \)) as the set of limit points of the sequence \( \{x^k\} \) generated by RInDCA\( _e \) (resp. RInDCA\( _n \)). Clearly, both \( \Omega_1 \) and \( \Omega_2 \) are nonempty.
Proposition 3.1 Suppose that the assumption in Theorem 3.1 holds. Let \( \{x^k\} \) be the sequence generated by \( \text{RInDCA}_e \). Then the following statements hold:

(i) \( \lim_{k \to \infty} f(x^k) := \zeta \) exists.

(ii) \( f(x) \equiv \zeta \) on \( \Omega_1 \).

Proof (i) Lemma 3.1 implies that the sequence \( \{f(x^k) + \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1}{2} \cdot \frac{\sigma_2}{2} \cdot ||x^k - x^{k-1}||^2\} \) is nonincreasing, clearly, this sequence is bounded below, thus it converges. On the other hand, (ii) of Theorem 3.1 implies that \( \lim_{k \to \infty} ||x^k - x^{k-1}||^2 = 0 \).

Therefore, \( \lim_{k \to \infty} f(x^k) := \zeta \) exists.

(ii) Because \( \Omega_1 \) is nonempty, thus given any \( \bar{x} \in \Omega_1 \), there exists a subsequence \( \{x^{k_i}\} \) such that \( \lim_{i \to \infty} x^{k_i} = \bar{x} \). Then, we get from \( y^{k_i} \in \partial f_2(x^{k_i}) \) and \( \sigma_2 \)-convexity of \( f_2 \) that

\[
\begin{align*}
  f_2(x^{k_i+1}) & \geq f_2(x^{k_i}) + \langle y^{k_i}, x^{k_i+1} - x^{k_i} \rangle + \frac{\sigma_2}{2} ||x^{k_i+1} - x^{k_i}||^2. \\
& \quad \text{(23)}
\end{align*}
\]

Similarly, we obtain as well from \( y^{k_i} + \gamma(x^{k_i} - x^{k_i-1}) \in \partial f_1(x^{k_i+1}) \) and \( \sigma_1 \)-convexity of \( f_1 \) that

\[
\begin{align*}
  f_1(x) & \geq f_1(x^{k_i+1}) + \langle y^{k_i} + \gamma(x^{k_i} - x^{k_i-1}), x - x^{k_i+1} \rangle + \frac{\sigma_1}{2} ||x - x^{k_i+1}||^2. \\
& \quad \text{(24)}
\end{align*}
\]

Then we have

\[
\begin{align*}
\zeta = & \lim_{k \to \infty} f(x^k) \\
= & \lim_{k \to \infty} f_1(x^{k+1}) - f_2(x^{k+1}) \\
\overset{(23)}{\leq} & \limsup_{i \to \infty} f_1(x^{k_i+1}) - \{f_2(x^{k_i}) + \langle y^{k_i}, x^{k_i+1} - x^{k_i} \rangle + \frac{\sigma_2}{2} ||x^{k_i+1} - x^{k_i}||^2\} \\
\overset{(24)}{\leq} & \limsup_{i \to \infty} f_1(x) - \langle y^{k_i} + \gamma(x^{k_i} - x^{k_i-1}), x - x^{k_i+1} \rangle - \frac{\sigma_1}{2} ||x^{k_i+1} - x^{k_i}||^2 \\
& - f_2(x^{k_i}) - \langle y^{k_i}, x^{k_i+1} - x^{k_i} \rangle - \frac{\sigma_2}{2} ||x^{k_i+1} - x^{k_i}||^2 \\
= & f(\bar{x}),
\end{align*}
\]

where the last equality follows that \( \lim_{i \to \infty} ||x^{k_i} - x^{k_i-1}|| = 0 \), \( f_2 \) is continuous over \( \text{dom} f_1 \), and \( \{y^{k_i}\} \) is bounded. On the other hand, because \( f \) is a closed function, then we have

\[
\begin{align*}
f(\bar{x}) & = \lim_{i \to \infty} f_1(x^{k_i}) - f_2(x^{k_i}) \\
& \leq \liminf_{i \to \infty} f_1(x^{k_i}) - f_2(x^{k_i}) \\
& = \liminf_{i \to \infty} f(x^{k_i}) = \zeta.
\end{align*}
\]

Thus, we obtain that \( f(\bar{x}) = \zeta \) and conclude that \( f \equiv \zeta \) on \( \Omega_1 \). \( \Box \)
We can obtain a similar proposition for RInDCA\(_n\) described as follows whose proof will be omitted.

**Proposition 3.2** Suppose that the assumption in Theorem 3.2 holds. Let \(\{x^k\}\) be the sequence generated by RInDCA\(_n\). Then the following statements hold:

(i) \(\lim_{k \to \infty} f(x^k) := \zeta\) exists.
(ii) \(f(x) \equiv \tilde{\zeta}\) on \(\Omega_2\).

### 4 Sequential Convergence of RInDCA

In this section, we establish the sequential convergence of RInDCA\(_n\) and RInDCA\(_m\) by assuming the KL property of the following auxiliary functions:

\[
E(x, y, z) = f_1(x) - \langle x, y \rangle + f_2^*(y) + \frac{\sigma_1 - \gamma}{2} \|x - z\|^2, \quad (25)
\]

\[
\tilde{E}(x, y, z) = f_1(x) - \langle x, y \rangle + f_2^*(y) + \frac{\sigma_2 - \gamma}{2} \|x - z\|^2, \quad (26)
\]

where \(f_1\) and \(f_2\) are defined in problem \((P)\); \(\sigma_1\) and \(\sigma_2\) are the strong convexity parameters of \(f_1\) and \(f_2\); \(\tilde{\sigma}_t\) coincides with that in Lemma 3.2; \(f_2^*\) is the conjugate function of \(f_2\), defined by

\[
f_2^*(y) = \sup\{\langle y, x \rangle - f_2(x) : x \in \mathbb{R}^n\}, \quad y \in \mathbb{R}^n.
\]

Note that our construction of such \(E\) and \(\tilde{E}\) is motivated by [6, 15]. Moreover, if \(f = f_1 - f_2\) is a KL function, then so are \(E\) and \(\tilde{E}\) (see [15]).

Next, we will concentrate on creating the sequential convergence of RInDCA\(_n\) in the premise that \(E\) is a KL function. The details about establishing that of RInDCA\(_m\) are omitted which can be easily developed based on the KL property of \(E\), Proposition 3.2 and the corresponding Proposition 4.1 for \(\tilde{E}\).

**Proposition 4.1** Suppose that the assumption in Theorem 3.1 holds. Let \(E\) be defined as in (25), and \(\{x^k\}\) be the sequence generated by RInDCA\(_n\). Then the following statements hold:

(i) For any \(k \geq 1\),

\[
E(x^{k+1}, y^k, x^k) \leq E(x^k, y^{k-1}, x^{k-1}) - \frac{\sigma_1 + \sigma_2 - 2\gamma}{2} \|x^k - x^{k-1}\|^2; \quad (27)
\]

(ii) The set of limit points of the sequence \(\{(x^k, y^{k-1}, x^{k-1})\}\) denoted by \(T\), is a nonempty compact set, and \((\bar{x}, \bar{y}, \bar{z}) \in T\) implies \(\bar{x} = \bar{z} \in \Omega_1\);

(iii) \(\lim_{k \to \infty} E(x^k, y^{k-1}, x^{k-1}) := \zeta\) exists, and \(E \equiv \zeta\) on \(T\).

**Proof** (i) For any \(k \geq 1\), \(y^{k-1} \in \partial f_2(x^{k-1})\) together with \(\sigma_2\)-convexity of \(f_2\) implies that

\[
f_2(x^k) \geq f_2(x^{k-1}) + \langle y^{k-1}, x^k - x^{k-1} \rangle - \frac{\sigma_2}{2} \|x^k - x^{k-1}\|^2. \quad (28)
\]
Thus we have
\[
(y^{k-1}, x^k) - f_2(x^k) + \frac{\sigma_2}{2} \|x^k - x^{k-1}\|^2 \leq (y^{k-1}, x^{k-1}) - f_2(x^{k-1}) = f_2(y^{k-1}),
\]
where the equality is implied by \( y^{k-1} \in \partial f_2(x^{k-1}) \), and thus \((y^{k-1}, x^{k-1}) = f_2(x^{k-1}) + f_2^*(y^{k-1})\) (see Theorem 4.20 in [7]). Moreover, \( y^k + \gamma(x^k - x^{k-1}) \in \partial f_1(x^{k+1}) \) and \( \sigma_1 \)-convexity of \( f_1 \) imply that
\[
f_1(x^k) \geq f_1(x^{k+1}) + \langle y^k + \gamma(x^k - x^{k-1}), x^k - x^{k+1} \rangle + \frac{\sigma_1}{2} \|x^k - x^{k+1}\|^2. \quad (30)
\]
Next, we have
\[
E(x^{k+1}, y^k, x^k)
= f_1(x^{k+1}) - (x^{k+1}, y^k) + f_2^*(y^k) + \frac{\sigma_1 - \gamma}{2} \|x^{k+1} - x^k\|^2
\leq f_1(x^k) - (y^k + \gamma(x^k - x^{k-1}), x^k - x^{k+1}) - \frac{\sigma_1}{2} \|x^{k+1} - x^k\|^2
- (x^{k+1}, y^k) + f_2(y^k) + \frac{\sigma_1 - \gamma}{2} \|x^{k+1} - x^k\|^2
= f_1(x^k) - f_2(x^k) - (\gamma(x^k - x^{k-1}), x^k - x^{k+1}) - \frac{\gamma}{2} \|x^{k+1} - x^k\|^2
\leq f_1(x^k) - f_2(x^k) + \frac{\gamma}{2} \|x^k - x^{k+1}\|^2
\leq f_1(x^k) - (x^k, y^{k-1}) + f_2^*(y^{k-1}) + \frac{\gamma - \sigma_2}{2} \|x^k - x^{k-1}\|^2
= E(x^k, y^{k-1}, x^{k-1}) - \frac{\sigma_1 + \sigma_2 - 2\gamma}{2} \|x^k - x^{k-1}\|^2,
\]
where the second equality is implied by \( y^k \in \partial f_2(x^k) \), and thus \( f_2(x^k) + f_2^*(y^k) = (x^k, y^k) \); the second inequality follows from \( (x^k - x^{k-1}, x^k - x^{k+1}) \geq -\|x^k - x^{k-1}\|^2 - \|x^k - x^{k+1}\|^2 \)/2.

(ii) The boundedness of \( \{x^k\} \) and \( \{y^{k-1}\} \) follows from (i) of Theorem 3.1. Thus, the sequence \( \{(x^k, y^{k-1}, x^{k-1})\} \) is bounded, yielding that \( \mathcal{T} \) is a nonempty compact set. Moreover, it is easy to see that \((x, y, z) \in \mathcal{T} \) implies \( x = z \in \Omega_1 \).

(iii) For any \( k \geq 1 \), we have
\[
E(x^k, y^{k-1}, x^{k-1}) = f_1(x^k) - (x^k, y^{k-1}) + f_2^*(y^{k-1}) + \frac{\sigma_1 - \gamma}{2} \|x^k - x^{k-1}\|^2
\geq f_1(x^k) - f_2(x^k) + \frac{\sigma_1 - \gamma}{2} \|x^k - x^{k-1}\|^2
\geq f^* + \frac{\sigma_1 - \gamma}{2} \|x^k - x^{k-1}\|^2,
\]
where the first inequality follows from \( f_2(x^k) + f_2(y^{k-1}) \geq \langle x^k, y^{k-1} \rangle \) and \( f^* \) is the finite optimal value of \( f \). Thus, \( \liminf_{k \to \infty} E(x^k, y^{k-1}, x^{k-1}) \geq f^* \). Then, combining the fact that the sequence \( \{E(x^k, y^{k-1}, x^{k-1})\} \) is nonincreasing (statement (i)), we obtain the existence of \( \lim_{k \to \infty} E(x^k, y^{k-1}, x^{k-1}) \). Next, we prove the last part of (iii). Given any \((\bar{x}, \bar{y}, \bar{x}) \in \mathcal{T}\), there exists a subsequence \((x^{k_i}, y^{k_i-1}, x^{k_i-1})\) such that

\[
\lim_{i \to \infty} \|(x^{k_i}, y^{k_i-1}, x^{k_i-1}) - (\bar{x}, \bar{y}, \bar{x})\| = 0.
\]

Then we have

\[
\begin{align*}
\lim_{k \to \infty} E(x^k, y^{k-1}, x^{k-1}) &= \lim_{i \to \infty} E(x^{k_i+1}, y^{k_i}, x^{k_i}) \\
&= \lim_{i \to \infty} f_1(x^{k_i+1}) - \langle x^{k_i+1}, y^{k_i} \rangle + f_2(y^{k_i}) + \frac{\sigma_1 - \gamma}{2} \|x^{k_i+1} - x^{k_i}\|^2 \\
&= \lim_{i \to \infty} f_1(x^{k_i+1}) - \langle x^{k_i+1}, y^{k_i} \rangle - f_2(x^{k_i}) + \langle y^{k_i}, x^{k_i} \rangle + \frac{\sigma_1 - \gamma}{2} \|x^{k_i+1} - x^{k_i}\|^2 \\
&\leq \limsup_{i \to \infty} f_1(x^{k_i}) - \langle y^{k_i} + \gamma(x^{k_i} - x^{k_i-1}), x^{k_i} - x^{k_i+1} \rangle - \frac{\gamma}{2} \|x^{k_i+1} - x^{k_i}\|^2 \\
&\quad - \langle x^{k_i+1}, y^{k_i} \rangle - f_2(x^{k_i}) + \langle y^{k_i}, x^{k_i} \rangle + \frac{\sigma_1 - \gamma}{2} \|x^{k_i+1} - x^{k_i}\|^2 \\
&= \lim_{i \to \infty} f(x^{k_i}) - \gamma \langle x^{k_i} - x^{k_i-1}, x^{k_i} - x^{k_i+1} \rangle - \frac{\gamma}{2} \|x^{k_i+1} - x^{k_i}\|^2 \\
&= \lim_{i \to \infty} f(x^{k_i}) \\
&= \zeta = f(\bar{x}) \leq E(\bar{x}, \bar{y}, \bar{x}),
\end{align*}
\]

where the third equality follows from \( y^{k_i} \in \partial f_2(x^{k_i}) \) and thus \( f_2(x^{k_i}) + f_2(y^{k_i}) = \langle y^{k_i}, x^{k_i} \rangle \); the first inequality is implied by \( y^{k_i} + \gamma (x^{k_i} - x^{k_i-1}) \in \partial f_1(x^{k_i+1}) \) and \( \sigma_1 \)-convexity of \( f_1 \); the fifth equality follows from \( \lim_{i \to \infty} \|x^{k_i} - x^{k_i-1}\| = 0 \). On the other hand, the closedness of \( E \) implies that \( E(\bar{x}, \bar{y}, \bar{x}) \leq \zeta \). Thus, we obtain that \( E(\bar{x}, \bar{y}, \bar{x}) = \zeta \) and conclude that \( E \equiv \zeta \) on \( \mathcal{T} \).

**Theorem 4.1** Suppose that the assumption in Theorem 3.1. Let \( E \) defined in \((25)\) be a KL function, and \( \{x^k\} \) be the sequence generated by \( R\text{InDCA}_\omega \). Then the following statements hold:

(i) There exists \( D > 0 \) such that \( \forall k \geq 1, \)

\[
\text{dist}(0, \partial E(x^k, y^{k-1}, x^{k-1})) \leq D(||x^k - x^{k-1}|| + ||x^{k-1} - x^{k-2}||).
\]

(ii) The sequence \( \{x^k\} \) converges to a critical point of problem \((P)\).
Proof (i) Note that the subdifferential of $E$ at $w^k := (x^k, y^{k-1}, x^{k-1})$ is:

$$
\partial E(w^k) = \begin{bmatrix}
\partial f_1(x^k) - y^{k-1} + \frac{\gamma - \sigma_2}{2} (x^k - x^{k-1}) \\
-x^k + \partial f_2(y^{k-1}) \\
-\frac{\gamma - \sigma_2}{2} (x^k - x^{k-1})
\end{bmatrix}.
$$

Because $y^{k-1} \in \partial f_2(x^{k-1})$, thus $x^{k-1} \in \partial f_2(y^{k-1})$; besides, we have $y^{k-1} + \gamma(x^{k-1} - x^{k-2}) \in \partial f_1(x^k)$. Combining these relations, we have

$$
\begin{bmatrix}
\gamma(x^{k-1} - x^{k-2}) + \frac{\gamma - \sigma_2}{2} (x^k - x^{k-1}) \\
-x^k + x^{k-1} \\
-\frac{\gamma - \sigma_2}{2} (x^k - x^{k-1})
\end{bmatrix} \in \partial E(w^k).
$$

Thus, it is easy to see that there exists $D > 0$ such that $\forall k \geq 1$,

$$
\text{dist}(0, \partial E(x^k, y^{k-1}, x^{k-1})) \leq D(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\|).
$$

(ii) It is sufficient to prove that $\sum_{k=1}^{\infty} \|x^k - x^{k-1}\| < \infty$. If there exists $N_0 \geq 1$ such that $E(w^{N_0}) = \zeta_1$, then the inequality (27) implies that $x^k = x^{N_0}, \forall k \geq N_0$; naturally, the sequence $\{x^k\}$ is convergent. Thus, we can assume that

$$
E(w^k) > \zeta, \forall k \geq 1.
$$

Then, it follows from $T$ is a compact set, $T \subseteq \text{dom} \partial E$ and $E \equiv \zeta$ on $T$ that there exists $\varepsilon > 0$, $\eta > 0$ and $\varphi \in \Xi_\eta$ such that

$$
\varphi'(E(x, y, z) - \zeta)\text{dist}(0, \partial E(x, y, z)) \geq 1
$$

for all $(x, y, z) \in U$ with

$$
U = \{(x, y, z) : \text{dist}((x, y, z), T) < \varepsilon \} \cap \{(x, y, z) : \zeta < E(x, y, z) < \zeta + \eta\}.
$$

Moreover, it is easy to derive from $\lim_{k \to \infty} \text{dist}(w^k, T) = 0$ and $\lim_{k \to \infty} E(w^k) = \zeta$ that there exists $N \geq 1$ such that $w^k \in U, \forall k \geq N$. Thus,

$$
\varphi'(E(w^k) - \zeta)\text{dist}(0, \partial E(w^k)) \geq 1, \ \forall k \geq N.
$$

Using the concavity of $\varphi$, we see that $\forall k \geq N$,

$$
\begin{align*}
&[\varphi(E(w^k) - \zeta) - \varphi(E(w^{k+1}) - \zeta)]\text{dist}(0, \partial E(w^k)) \\
\geq & \varphi'(E(w^k) - \zeta)\text{dist}(0, \partial E(w^k))(E(w^k) - E(w^{k+1})) \\
\geq & E(w^k) - E(w^{k+1}).
\end{align*}
$$

(32)

Let $\Delta_k := \varphi(E(w^k) - \zeta) - \varphi(E(w^{k+1}) - \zeta)$ and $C := \frac{\gamma + \sigma_2 - \sigma_1}{2}\|x^k - x^{k-1}\|^2$. Then combining (32), (31) and (i) of Proposition 3.2, we have

$$
\|x^k - x^{k-1}\|^2 \leq \frac{D}{C} \Delta_k(\|x^k - x^{k-1}\| + \|x^{k-1} + x^{k-2}\|).
$$
Thus, taking square roots on both sides, we obtain
\[ \|x^k - x^{k-1}\| \leq \sqrt{\frac{2D}{C} \Delta_k \left( \|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| \right)} \]
\[ \leq \frac{D}{C} \Delta_k + \frac{1}{4} \|x^k - x^{k-1}\| + \frac{1}{4} \|x^{k-1} - x^{k-2}\|, \]
this yields
\[ \frac{1}{2} \|x^k - x^{k-1}\| \leq \frac{D}{C} \Delta_k + \frac{1}{4} \|x^{k-1} - x^{k-2}\| - \frac{1}{4} \|x^k - x^{k-1}\|. \]
Considering also that \( \sum_{k=1}^{\infty} \Delta_k < \infty \), thus the above inequality implies that \( \sum_{k=1}^{\infty} \|x^k - x^{k-1}\| < \infty \), which indicates that \( \{x^k\} \) is a Cauchy sequence, and thus convergent. The proof is completed.

5 Application to image restoration

5.1 Image restoration problem

Consider a gray-scale image of \( m \times n \) pixels and with entries in \([0, 1]\), where 0 represents pure black and 1 represents pure white. The original image \( X \in \mathbb{R}^{m \times n} \) is assumed to be first blurred by a linear operator \( \mathcal{L} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n} \), and then corrupted with a noise \( B \). Thus, we obtain the observed image \( \tilde{X} = \mathcal{L}X + B \), which is known to us. Then, we aim to recover the original image \( X \) by the next nonconvex optimization problem
\[
\min_{X} \left\{ \frac{\mu}{2} \| \mathcal{L}X - \tilde{X} \|_F^2 + \text{TV}_\phi(X) : X \in \mathbb{R}^{m \times n} \right\},
\]
where \( \mu > 0 \) is a trade-off constant; \( \tilde{X} \) is the observed image; \( \| \cdot \|_F \) represents the Frobenius norm; \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R} \) is a concave function and \( \text{TV}_\phi \) is defined as:
\[
\text{TV}_\phi(X) = \sum_{i,j} \phi(\|\nabla X_{i,j}\|)
\]
with
\[
\nabla X_{i,j} = \begin{cases} 
X_{i,j} - X_{i,j+1} & i = m, j \neq n \\
X_{i,j} - X_{i+1,j} & i \neq m, j = n \\
0 & i = m, j = n \\
(X_{i,j} - X_{i+1,j}, X_{i,j} - X_{i,j+1}) & \text{otherwise},
\end{cases}
\]
and
\[
\|\nabla X_{i,j}\| = \begin{cases} 
|X_{i,j} - X_{i,j+1}| & i = m, j \neq n \\
|X_{i,j} - X_{i+1,j}| & i \neq m, j = n \\
0 & i = m, j = n \\
\sqrt{(X_{i,j} - X_{i+1,j})^2 + (X_{i,j} - X_{i,j+1})^2} & \text{otherwise}.
\end{cases}
\]
If $\mathcal{L}$ is the identity operator, then (33) refers to image denoising problem, and (33) is image denoising and deblurring problem when $\mathcal{L}$ is a blur operator.

Denote $\text{TV} := \text{TV}_\phi$ when $\phi \equiv 1$. For some specific $\phi$ (see Table 1, introduced in [24]), $\text{TV}_\phi(X)$ has a DC decomposition as

$$\text{TV}_\phi(X) = \text{TV}(X) - (\text{TV}(X) - \text{TV}_\phi(X)).$$

| $\phi_a(r)$ | $\phi'_a(r)$ |
|-------------|-------------|
| $\log(1+ar)$ | $\frac{1}{1+ar}$ |
| $\frac{x}{1+ar/2}$ | $\frac{1}{a^2r^2+2ar+4}$ |
| $\frac{\tan((1+ar)/\sqrt{3})-n/6}{a^\sqrt{3}/4}$ | $\frac{1}{\exp(ar)}$ |
| $\frac{1-exp(-ar)}{a}$ | $1$ |

When $\mathcal{L}$ is the identity operator, we get from (33) the next DC program

$$\min\left\{ \frac{\mu}{2}\|X - \tilde{X}\|_F^2 + \text{TV}(X) - (\text{TV}(X) - \text{TV}_\phi(X)) : X \in \mathbb{R}^{m \times n} \right\}.$$  \hspace{1cm} (34)

When $\mathcal{L}$ is a blur operator, one obtains from (33) the DC program

$$\min\left\{ \tilde{f}_1(X) - \tilde{f}_2(X) : X \in \mathbb{R}^{m \times n} \right\},$$  \hspace{1cm} (35)

where $\tilde{f}_1(X) = \frac{t}{2}\|X - \tilde{X}\|_F^2 + \text{TV}(X)$ and $\tilde{f}_2(X) = \text{TV}(X) - \text{TV}_\phi(X) + \frac{t}{2}\|X - \tilde{X}\|_F^2$. With $t \geq \mu \|\mathcal{L}\|^2$ and $\|\mathcal{L}\|$ being the spectral norm of $\mathcal{L}$.

Note that DCA, SDCA and RInDCAe can be applied for problem (34), but this is not the case for InDCAe because $f_2$ is not strongly convex; Whereas, for problem (35) and taking $t > \mu \|\mathcal{L}\|^2$, all methods DCA, SDCA, RInDCAe and InDCAe are applicable, and all of them require solving convex subproblems in form of

$$\min\left\{ \tilde{\ell}_\phi(X) - \tilde{\ell}_\phi(X) : X \in \mathbb{R}^{m \times n} \right\},$$

where $\tilde{\ell} > 0$ and $Z \in \mathbb{R}^{m \times n}$, which can be efficiently solved by the algorithm of FISTA type proposed in [8].

5.2 Numerical simulations

In this part, we conduct numerical simulations on image restoration problem. All experiments are implemented in Mathab 2019a on a 64-bit PC with an Intel(R) Core(TM) i5-6200U CPU (2.30GHz) and 8GB of RAM.

The original image (with $512 \times 512$ pixels) is taken from SIPI Image database\(^1\), and each pixel is re-scaled by dividing 255. For image denoising

\(^1\) http://sipi.usc.edu/database/database.php?volume=textures&image=64
problem, white noise with standard deviation 80/255 is added to the scaled image to produce the observed image in (34); while for the image denoising and deblurring problem, the scaled image is first blurred by convolving with the Disk kernel of radius 3 (a $7 \times 7$ matrix generated by the Matlab command `fspecial('disk',3)`), and then white noise with standard deviation 80/255 is added, yielding the observed image in (35). In this case, based on the linearity of the convolution operator and the fact that the entries of the Disk kernel is a positive convex combination, we can evaluate the spectral norm of the corresponding $L$ in problem (35) as $\|L\| \leq 1$.

To measure the quality of reconstruction images, we use the Structural SIMilarity (SSIM) index (see [17]). In general, the closer to 1 of SSIM indicates the better reconstruction quality. For all the involved algorithms DCA, RInDCAe and SDCA for problem (34) together with DCA, InDCAe and RInDCAe for problem (33), the observed images are initialized for the corresponding algorithms, whose convex subproblems are all solved by FISTA. Moreover, the stopping condition for FISTA is that the distance between two consecutive iterations is smaller than $10^{-4}$.

**Image denoising** We test DCA, SDCA and RInDCAe for problem (34) with $\mu$ ranging from 0.55 to 1.25, and $\phi = \phi_{\text{atan}}$ with $a = 6$. For SDCA, we select $\gamma \in \{1, 0.5\}$; while for RInDCAe, we choose $\gamma = 0.5 \times \mu \times 99\% = 0.495\mu$. The trends of the function values of $f$ and SSIM within 20 iterations when $\mu = 0.95$ and $\phi = \phi_{\text{atan}}$ with $a = 6$ are shown in Fig 2. Moreover, the recovered images after 20 iterations together with the original and corrupted images are demonstrated in Fig 3. In Table 2, we show in detail the function values of $f$ and SSIM after 20 iterations with $\mu$ ranging from 0.95 to 1.25 and $\phi = \phi_{\text{atan}}$ with $a = 6$.

![Fig. 2](image-url) The trends of the function values of $f$ and SSIM within 20 iterations when $\mu = 0.95$ and $\phi = \phi_{\text{atan}}$ with $a = 6$.
Fig. 3 Original image, noisy image and reconstructions. (a) Original image. (b) Noisy image. (c), (d) is respectively the reconstruction image by DCA and RInDCA after 20 iterations when \( \mu = 0.95 \) and \( \phi = \phi_{\text{atan}} \) with \( a = 6 \); while (e), (f) is respectively the reconstruction image by SDCA with \( \gamma \in \{1, 0.5\} \) after 20 iterations when \( \mu = 0.95 \), \( \phi = \phi_{\text{atan}} \) with \( a = 6 \).

**Image denoising and deblurring** We test DCA, InDCA and RInDCA for problem (35) with \( t = \mu + 2 \) where \( \mu \) varies from 0.95 to 1.65 and \( \phi = \phi_{\text{atan}} \) with \( a \in \{4, 8\} \). Clearly, \( f_1 \) is \( t \)-convex and \( f_2 \) is 2-convex. We set \( \gamma \) in InDCA as 0.5 \times 2 \times 99\% = 0.99, and \( \gamma \) in RInDCA as 0.5 \times (\mu + 4) \times 99\% = 0.495\mu + 1.98. The trends of the function values of \( f \) and SSIM of DCA, InDCA and RInDCA within 20 iterations when \( \mu = 1.35 \) and \( \phi = \phi_{\text{atan}} \) with \( a \in \{4, 8\} \) are shown in Fig. 4. In addition, the recovered images after 20 iterations together with the original and corrupted images are demonstrated in Fig. 5. In Table 3, we show in detail the function values of \( f \) and SSIM after 20 iterations with \( \mu \) ranging from 0.95 to 1.65 and \( \phi = \phi_{\text{atan}} \) with \( a \in \{4, 8\} \).

We conclude that RInDCA performs better than InDCA and DCA (with fixed DC decomposition given in (34) and (35)), while InDCA performs better than DCA and SDCA (with fixed parameter \( \gamma \), and SDCA is the worst. This indicates the benefit of enlarged inertial step-size.

6 Conclusion and perspective

In this paper, based on the inertial algorithm (InDCA) [24] for DC programming, we propose two refined versions: the refined exact one and the refined inexact one, each of which is equipped with larger inertial step-size for better
Table 2 The values of $f$ and SSIM obtained by DCA, RInDCA, and SDCA after 20 iterations with $\mu$ taking from 0.55 to 1.25, and $\phi = \phi_{\text{atan}}$ with $a = 6$

| Algorithm | $\mu$ | $f$       | SSIM  |
|-----------|-------|-----------|-------|
| DCA       |       | 3.4356e+04 | 0.9505 |
| RInDCA $\epsilon$ | 0.55  | 3.435e+04   | 0.9513 |
| SDCA, $\gamma = 1$ |       | 3.4355e+04 | 0.9505 |
| SDCA, $\gamma = 0.5$ | | 3.4358e+04 | 0.9500 |
| DCA       | 0.65  | 3.5656e+04 | 0.9539 |
| RInDCA $\epsilon$ |       | 3.5652e+04 | 0.9533 |
| SDCA, $\gamma = 1$ |       | 3.5664e+04 | 0.9504 |
| SDCA, $\gamma = 0.5$ | | 3.5662e+04 | 0.9511 |
| DCA       | 0.75  | 3.6956e+04 | 0.9517 |
| RInDCA $\epsilon$ |       | 3.6952e+04 | 0.9533 |
| SDCA, $\gamma = 1$ |       | 3.6970e+04 | 0.9477 |
| SDCA, $\gamma = 0.5$ | | 3.6963e+04 | 0.9502 |
| DCA       | 0.85  | 3.8261e+04 | 0.9479 |
| RInDCA $\epsilon$ |       | 3.8248e+04 | 0.9505 |
| SDCA, $\gamma = 1$ |       | 3.8290e+04 | 0.9422 |
| SDCA, $\gamma = 0.5$ | | 3.8270e+04 | 0.9451 |
| DCA       | 0.95  | 3.9567e+04 | 0.9448 |
| RInDCA $\epsilon$ |       | 3.9550e+04 | 0.9459 |
| SDCA, $\gamma = 1$ |       | 3.9589e+04 | 0.9345 |
| SDCA, $\gamma = 0.5$ | | 3.9577e+04 | 0.9380 |
| DCA       | 1.05  | 4.088e+04  | 0.9288 |
| RInDCA $\epsilon$ |       | 4.0849e+04 | 0.9394 |
| SDCA, $\gamma = 1$ |       | 4.0914e+04 | 0.9190 |
| SDCA, $\gamma = 0.5$ | | 4.0896e+04 | 0.9274 |
| DCA       | 1.15  | 4.2194e+04 | 0.9140 |
| RInDCA $\epsilon$ |       | 4.2159e+04 | 0.9256 |
| SDCA, $\gamma = 1$ |       | 4.2236e+04 | 0.9022 |
| SDCA, $\gamma = 0.5$ | | 4.2211e+04 | 0.9089 |
| DCA       | 1.25  | 4.3511e+04 | 0.8942 |
| RInDCA $\epsilon$ |       | 4.3464e+04 | 0.9115 |
| SDCA, $\gamma = 1$ |       | 4.3563e+04 | 0.8797 |
| SDCA, $\gamma = 0.5$ | | 4.3535e+04 | 0.8874 |

acceleration compared with InDCA. Numerical simulations on image restoration problem show the benefit of larger step-size.

Note that SDCA could be improved by decaying the parameter $\gamma$ during iterations whose performance deserves more attention in the future. Moreover, the inertial-force procedure can be extended to partial DC programming [13] in which the objective function $f$ depends on two variables $x$ and $y$ where $f(x,.)$ and $f(.,y)$ are both DC functions.

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Fig. 4 a, b indicate the trends of the function values of $f$ and SSIM within 20 iterations when $\mu = 1.35$ and $\phi = \phi_{\text{atan}}$ with $a = 4$; while c, d mean that when $\mu = 1.35$ and $\phi = \phi_{\text{atan}}$ with $a = 8$

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Fig. 5 Original image, blurry and noisy image and reconstructions. 

(a) Original image. 
(b) Blur and noisy image. 
(c, d, e) is respectively the reconstruction image by DCA, InDCA, and RInDCA after 20 iterations when \( \mu = 1.35 \) and \( \phi = \phi_{\text{atan}} \) with \( a = 4 \); while (f, g, h) is respectively the reconstruction image by DCA, InDCA, and RInDCA after 20 iterations when \( \mu = 1.35 \) and \( \phi = \phi_{\text{atan}} \) with \( a = 8 \).

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Table 3 The values of $f$ and SSIM obtained by DCA, InDCA, and RInDCA after 20 iterations with $\mu$ taking from 0.95 to 1.65, and $\phi = \phi_{atan}$ with $a \in \{4, 8\}$

| Algorithm | $\mu$ | $\alpha$ | $f$         | SSIM   |
|-----------|-------|----------|-------------|--------|
| DCA       | 0.95  | 4        | 5.2870e+04  | 0.9357 |
| InDCA$_e$ |       |          | 5.2867e+04  | 0.9360 |
| RhnDCA$_e$|       |          | 5.2859e+04  | 0.9353 |
| DCA       | 0.95  | 8        | 3.2898e+04  | 0.9252 |
| InDCA$_e$ |       |          | 3.2878e+04  | 0.9269 |
| RhnDCA$_e$|       |          | 3.2865e+04  | 0.9285 |
| DCA       | 1.05  | 4        | 5.4165e+04  | 0.9362 |
| InDCA$_e$ |       |          | 5.4162e+04  | 0.9368 |
| RhnDCA$_e$|       |          | 5.4155e+04  | 0.9366 |
| DCA       | 1.15  | 4        | 5.5459e+04  | 0.9370 |
| InDCA$_e$ |       |          | 5.5455e+04  | 0.9375 |
| RhnDCA$_e$|       |          | 5.5446e+04  | 0.9382 |
| DCA       | 1.15  | 8        | 3.4198e+04  | 0.9248 |
| InDCA$_e$ |       |          | 3.4177e+04  | 0.9269 |
| RhnDCA$_e$|       |          | 3.4157e+04  | 0.9298 |
| DCA       | 1.25  | 4        | 5.6753e+04  | 0.9370 |
| InDCA$_e$ |       |          | 5.6749e+04  | 0.9375 |
| RhnDCA$_e$|       |          | 5.6739e+04  | 0.9389 |
| DCA       | 1.25  | 8        | 3.6743e+04  | 0.9293 |
| InDCA$_e$ |       |          | 3.6767e+04  | 0.9264 |
| RhnDCA$_e$|       |          | 3.6743e+04  | 0.9293 |
| DCA       | 1.35  | 4        | 5.8046e+04  | 0.9370 |
| InDCA$_e$ |       |          | 5.8042e+04  | 0.9377 |
| RhnDCA$_e$|       |          | 5.8030e+04  | 0.9395 |
| DCA       | 1.35  | 8        | 3.8038e+04  | 0.9254 |
| InDCA$_e$ |       |          | 3.8065e+04  | 0.9254 |
| RhnDCA$_e$|       |          | 3.8038e+04  | 0.9284 |
| DCA       | 1.45  | 4        | 5.9341e+04  | 0.9372 |
| InDCA$_e$ |       |          | 5.9345e+04  | 0.9382 |
| RhnDCA$_e$|       |          | 5.9321e+04  | 0.9391 |
| DCA       | 1.45  | 8        | 3.9429e+04  | 0.9204 |
| InDCA$_e$ |       |          | 3.9364e+04  | 0.9248 |
| RhnDCA$_e$|       |          | 3.9331e+04  | 0.9288 |
| DCA       | 1.55  | 4        | 6.0625e+04  | 0.9374 |
| InDCA$_e$ |       |          | 6.0627e+04  | 0.9380 |
| RhnDCA$_e$|       |          | 6.0613e+04  | 0.9394 |
| DCA       | 1.55  | 8        | 4.0623e+04  | 0.9288 |
| InDCA$_e$ |       |          | 4.0666e+04  | 0.9233 |
| RhnDCA$_e$|       |          | 4.0623e+04  | 0.9288 |
| DCA       | 1.65  | 4        | 6.1926e+04  | 0.9375 |
| InDCA$_e$ |       |          | 6.1921e+04  | 0.9376 |
| RhnDCA$_e$|       |          | 6.1904e+04  | 0.9398 |
| DCA       | 1.65  | 8        | 4.1960e+04  | 0.9223 |
| InDCA$_e$ |       |          | 4.1960e+04  | 0.9223 |
| RhnDCA$_e$|       |          | 4.1915e+04  | 0.9287 |
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