On Colorings of Graph Powers

Hossein Hajiabolhassan
Department of Mathematical Sciences
Shahid Beheshti University
P.O. Box 19834, Tehran, Iran
hhaji@sbu.ac.ir

Abstract

In this paper, some results concerning the colorings of graph powers are presented. The notion of helical graphs is introduced. We show that such graphs are hom-universal with respect to high odd-girth graphs whose $(2t+1)$st power is bounded by a Kneser graph. Also, we consider the problem of existence of homomorphism to odd cycles. We prove that such homomorphism to a $(2k+1)$-cycle exists if and only if the chromatic number of the $(2k+1)$st power of $S_2(G)$ is less than or equal to 3, where $S_2(G)$ is the 2-subdivision of $G$. We also consider Nešetřil’s Pentagon problem. This problem is about the existence of high girth cubic graphs which are not homomorphic to the cycle of size five. Several problems which are closely related to Nešetřil’s problem are introduced and their relations are presented.

Keywords: graph homomorphism, graph coloring, circular coloring.

Subject classification: 05C

1 Introduction

Throughout this paper we only consider finite graphs. A homomorphism $f : G \rightarrow H$ from a graph $G$ to a graph $H$ is a map $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. The existence of a homomorphism is indicated by the symbol $G \rightarrow H$. Two graphs $G$ and $H$ are homomorphically equivalent if $G \rightarrow H$ and $H \rightarrow G$. Also, the symbol Hom$(G, H)$ is used to denote the set of all homomorphisms from $G$ to $H$ (for more on graph homomorphisms see [2, 3, 8, 13]).

If $n$ and $d$ are positive integers with $n \geq 2d$, then the circular complete graph $K_{(n,d)}$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$ in which $v_i$ is connected to $v_j$ if and only if $d \leq |i - j| \leq n - d$. A graph $G$ is said to be $(n,d)$-colorable if $G$ admits a homomorphism to $K_{(n,d)}$. The circular chromatic number (also known as the star chromatic number [31]) $\chi_c(G)$ of a graph $G$ is the minimum of those ratios $\frac{n}{d}$ for which $\gcd(n,d) = 1$ and such that $G$ admits a homomorphism to $K_{(n,d)}$. It can be shown that one may only consider onto-vertex homomorphisms [33]. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all $n$-subsets of $[m]$. For a given subset $A \subseteq [m]$, the complement of $A$ in $[m]$ is denoted by $\overline{A}$. The Kneser graph $KG(m,n)$ is the graph with vertex set $\binom{[m]}{n}$, in which $A$ is connected to $B$ if and only if $A \cap B = \emptyset$. It was conjectured by Kneser [16] in 1955, and proved by Lovász [20] in 1978, that $\chi(KG(m,n)) = m - 2n + 2$. A subset $S$ of $[m]$ is called 2-stable if $2 \leq |x - y| \leq m - 2$ for all distinct elements $x$ and $y$ of $S$. The Schrijver

---

1This research was partially supported by Shahid Beheshti University.
2Correspondence should be addressed to hhaji@sbu.ac.ir.
graph $SG(m, n)$ is the subgraph of $KG(m, n)$ induced by all 2-stable $n$-subsets of $[m]$. It was proved by Schrijver [27] that $\chi(SG(m, n)) = \chi(KG(m, n))$ and that every proper subgraph of $SG(m, n)$ has a chromatic number smaller than that of $SG(m, n)$. The fractional chromatic number, $\chi_f(G)$, of a graph $G$ is defined as

$$\chi_f(G) \overset{\text{def}}{=} \inf \{ \frac{m}{n} \mid \text{Hom}(G, KG(m, n)) \neq \emptyset \}.$$ 

For more about the fractional coloring see [26]. The local chromatic number of a graph was defined in [6] as the minimum number of colors that must appear within distance 1 of a vertex. Here is the formal definition.

**Definition 1.** The local chromatic number $\psi(G)$ of a graph $G$ is

$$\psi(G) \overset{\text{def}}{=} \min \max_{c} \{ |\{c(v) : u \in N(v)\}| + 1 \},$$

where the minimum is taken over all proper colorings $c$ of $G$ and $N(v) = N_G(v)$ denotes the neighborhood of a vertex $v$ in a graph $G$.

It is easy to verify that for any graph $G$, $\psi(G) \leq \chi(G)$. Also, it was shown in [17] that $\chi_f(G) \leq \psi(G)$ holds for any graph $G$.

For a graph $G$, let $G^{(k)}$ be the $k$th power of $G$, which is obtained on the vertex set $V(G)$, by connecting any two vertices $u$ and $v$ for which there exists a walk of length $k$ between $u$ and $v$ in $G$. Note that the $k$th power of a simple graph is not necessarily a simple graph itself. For instance, the $k$th power may have loops on its vertices provided that $k$ is an even integer. For a given graph $G$ with $og(G) \geq 7$, the chromatic number of $G^{(5)}$ provides an upper bound for the local chromatic number of $G$. In [25], it was proved $\psi(G) \leq \lceil \frac{m}{2} \rceil + 2$ whenever $\chi(G^{(5)}) \leq m$.

The following simple and useful lemma was proved and used independently in [5, 25, 30].

**Lemma A.** Let $G$ and $H$ be two simple graphs such that $\text{Hom}(G, H) \neq \emptyset$. Then for any positive integer $k$, $\text{Hom}(G^{(k)}, H^{(k)}) \neq \emptyset$.

Note that Lemma A trivially holds whenever $H^{(k)}$ contains a loop, e.g., when $k = 2$. As immediate consequences of Lemma A we obtain $\chi_c(P) = \chi(P)$ and $\text{Hom}(C, C_5) = \emptyset$, where $P$ and $C$ are the Petersen and the Coxeter graphs, respectively, see [5].

In what follows we are concerned with some results concerning the colorings of graph powers. First, the notion of helical graphs is introduced. We show that such graphs are hom-universal with respect to high odd-girth graphs whose $(2t + 1)$st power is bounded by a Kneser graph. Then, we consider the problem of existence of homomorphism to odd cycles. We prove that such homomorphism to a $(2k+1)$-cycle exists if and only if the chromatic number of the $(2k + 1)$st power of $S_2(G)$ is less than or equal to 3, where $S_2(G)$ is the 2-subdivision of $G$. We also consider Nešetřil’s Pentagon problem. This problem is about the existence of high girth cubic graphs which are not homomorphic to the cycle of size five. Several problems which are closely related to Nešetřil’s problem are introduced and their relations are presented.
2 Helical Graphs

For a given class \( \mathcal{C} \) of graphs, a graph \( U \) is called hom-universal with respect to \( \mathcal{C} \) if for any \( G \in \mathcal{C} \), \( \text{Hom}(G,U) \neq \emptyset \), in which case the class \( \mathcal{C} \) is said to be bounded by the graph \( U \). The problem of the existence of a bound with some special properties, for a given class of graphs, has been a subject of study in graph homomorphism. In the following definition, we introduce a new family of hom-universal graphs, namely the family \( H(m,n,k) \) of the helical graphs.

**Definition 2.** Let \( m, n, \) and \( k \) be positive integers with \( m \geq 2n \). Set \( H(m,n,k) \) to be the helical graph whose vertex set contains all \( k \)-tuples \( (A_1,\ldots,A_k) \) such that for any \( 1 \leq r \leq k \), \( A_r \subseteq [m] \), \(|A_1| = n_i\), \(|A_r| \geq n\) and for any \( s \leq k - 1 \) and \( t \leq k - 2 \), \( A_s \cap A_{s+1} = \emptyset \). Also, two vertices \((A_1,\ldots,A_k)\) and \((B_1,\ldots,B_k)\) of \( H(m,n,k) \) are adjacent if for any \( 1 \leq i,j \leq k \), \( A_i \cap B_i = \emptyset \), \( A_j \subseteq B_j+1 \), and \( B_j \subseteq A_j+1 \).

Note that \( H(m,1,1) \) is the complete graph \( K_m \) and \( H(m,n,1) \) is the Kneser graph \( KG(m,n) \). It is easy to verify that if \( m > 2n \), then the odd-girth of \( H(m,n,k) \) is greater than or equal to \( 2k+1 \).

For a given graph \( G \) and \( v \in V(G) \), set

\[
N_i(v) \overset{\text{def}}{=} \{ u | \text{there is a walk of length } i \text{ joining } u \text{ and } v \}.
\]

Also, for a coloring \( c : V(G) \rightarrow \binom{[m]}{n} \), define

\[
c(N_i(v)) \overset{\text{def}}{=} \bigcup_{u \in N_i(v)} c(u).
\]

The chromatic number of graph powers has been studied in the literature (see \([1, 5, 7, 23, 28, 30]\)). In the theorem below, we show that the helical graphs are hom-universal graphs with respect to the family of high odd-girth graphs whose \((2k-1)\)st power is bounded by a Kneser graph.

**Theorem 1.** Let \( G \) be a non-empty graph with odd-girth at least \( 2k+1 \). Then we have \( \text{Hom}(G^{(2k-1)}, KG(m,n)) \neq \emptyset \) if and only if \( \text{Hom}(G, H(m,n,k)) \neq \emptyset \).

**Proof.** First, let \( c \in \text{Hom}(G^{(2k-1)}, KG(m,n)) \). If \( v \) is an isolated vertex of \( G \), then consider an arbitrary vertex, say \( f(v) \), of \( H(m,n,k) \). For any non-isolated vertex \( v \in V(G) \), define

\[
f(v) \overset{\text{def}}{=} (c(v), c(N_1(v)), c(N_2(v)), \ldots, c(N_{k-1}(v))).
\]

If \( i \leq j \) and \( i \equiv j \mod 2 \), we have \( N_i(v) \subseteq N_j(v) \), implying that \( c(N_i(v)) \subseteq c(N_j(v)) \). Also, since \( c \) is a homomorphism from \( G^{(2k-1)} \) to \( KG(m,n) \), for any \( i \leq j \leq k-1 \) and \( i \not\equiv j \mod 2 \), we obtain \( c(N_i(v)) \cap c(N_j(v)) = \emptyset \). Hence, for any vertex \( v \in V(G) \), \( f(v) \in V(H(m,n,k)) \). Moreover, for any \( 0 \leq i,j+1 \leq k-1 \), we have \( N_i(v) \cap N_i(u) = \emptyset \), \( N_j(v) \subseteq N_{j+1}(u) \), and \( N_j(u) \subseteq N_{j+1}(v) \) provided that \( u \) is adjacent to \( v \). Hence, \( f \) is a graph homomorphism from \( G \) to \( H(m,n,k) \).
Next, let $\text{Hom}(G, H(m, n, k)) \neq \emptyset$ and $f : G \rightarrow H(m, n, k)$. Assume $v \in V(G)$ and $f(v) = (A_1, A_2, \ldots, A_k)$. Define, $c(v) \overset{\text{def}}{=} A_1$. Assume further that $u, v \in V(G)$ such that there is a walk of length $2t + 1$ ($t \leq k - 1$) between $u$ and $v$ in $G$, i.e., $uv \in E(G^{(2k-1)})$. Consider adjacent vertices $u'$ and $v'$ such that $u' \in N_t(u)$ and $v' \in N_t(v)$. Also, let $f(v) = (A_1, A_2, \ldots, A_k)$, $f(u) = (B_1, B_2, \ldots, B_k)$, $f(v') = (A'_1, A'_2, \ldots, A'_k)$, and $f(u') = (B'_1, B'_2, \ldots, B'_k)$. In view of the definition of the helical graph, we obtain $A_1 \subseteq A'_{t+1}$ and $B_1 \subseteq B'_{t+1}$. On the other hand, $A'_{t+1} \cap B'_{t+1} = \emptyset$, which yields $c(v) \cap c(u) = \emptyset$. Thus, $\text{Hom}(G^{(2k-1)}, KG(m, n)) \neq \emptyset$, as desired. 

It was conjectured in [21] that a class $C$ of graphs is bounded by a graph $H$ whose odd-girth is at least $2k + 1$ provided that the set $\{\chi(G^{(2k-1)}) | G \in C\}$ is bounded and that all graphs in $C$ have odd-girth at least $2k + 1$. It is worth noting that Theorem 1 shows the above conjecture is true. This conjecture however was proved by Tardif recently (personal communication, see [21]).

It was proved by Schrijver [27] that $SG(m, n)$ is the vertex-critical subgraph of $KG(m, n)$. Motivated by the construction of the Schrijver graphs, we introduce a family of subgraphs of the helical graphs.

**Definition 3.** Let $m, n, k$ be positive integers with $m \geq 2n$. Define $SG(m, n, k)$ to be the induced subgraph of $H(m, n, k)$ whose vertex set contains all $k$-tuples $(A_1, \ldots, A_k) \in V(H(m, n, k))$ such that for any $1 \leq r \leq k$, $A_r = \cup_s B_s$, where every $B_s$ is a 2-stable $n$-subset of $[m]$. ♠

One can deduce the following theorem whose proof is almost identical to that of Theorem 1 and the proof is omitted for the sake of brevity.

**Theorem 2.** Let $G$ be a non-empty graph with odd-girth at least $2k + 1$. Then, $\text{Hom}(G^{(2k-1)}, SG(m, n)) \neq \emptyset$ if and only if $\text{Hom}(G, SG(m, n, k)) \neq \emptyset$.

In [4], it was proved that $\chi(H(m, 1, 2)) = m$. Later in [1, 28], it was shown $\chi(H(m, 1, k)) = m$. We would like to remark that the graph $H(m, 1, k)$ is defined in a completely different way in [1, 28]. Simonyi and Tardos [28] showed that $\chi(H(m, 1, k)) = m$ by proving the existence of homomorphism from $SG(a, b)$ to $H(m, 1, k)$, where $a - 2b + 2 = m$ and $a$ is sufficiently large. Similarly, one can show that $\chi(H(m, n, k)) = \chi(SG(m, n, k)) = m - 2n + 2$, where $m \geq 2n$.

**Lemma B.** [28] Let $u, v \subset [a]$ be two vertices of $SG(a, b)$. If there is a walk of length $2s$ between $u$ and $v$ in $SG(a, b)$, then $|u \setminus v| \leq s(a - 2b + 2)$.

**Theorem 3.** Let $m, n, k$ be positive integers with $m \geq 2n$. The chromatic number of the helical graph $H(m, n, k)$ is equal to $m - 2n + 2$. Moreover, $\chi(SG(m, n, k)) = m - 2n + 2$.

**Proof.** For a given vertex $v = (A_1, A_2, \ldots, A_k) \in V(H(m, n, k))$, define $f(v) \overset{\text{def}}{=} A_1$. It is easy to check that $f$ is a graph homomorphism from $H(m, n, k)$ to $KG(m, n)$. It follows that $\chi(SG(m, n, k)) \leq \chi(H(m, n, k)) \leq m - 2n + 2$. Now, we prove that $m - 2n + 2$ is a lower bound for the chromatic number of $SG(m, n, k)$.
To this end, it suffices to show, first, that for \( a \overset{\text{def}}{=} 2(k - 1)m(m - 2n + 2) + m \) and \( b \overset{\text{def}}{=} (k - 1)m(m - 2n + 2) + n \), we have \( \text{Hom}(SG(a, b)^{(2k - 1)}, SG(m, n)) \neq \emptyset \). Then, Theorem 2 applies, and hence the assertion follows. Now, let \([a]\) be partitioned into \( m \) sets, each of which contains \( 2(k - 1)(m - 2n + 2) + 1 \) consecutive elements of \([a]\). In other words, \([a]\) is partitioned into \( m \) disjoint sets \( D_1, \ldots, D_m \), where each \( D_i \) contains consecutive elements and \( |D_i| = 2(k - 1)(m - 2n + 2) + 1 \). Note that \( b = (k - 1)m(m - 2n + 2) + n \) and \( \sum_{i=1}^{m} \frac{|D_i| - 1}{2} = (k - 1)m(m - 2n + 2) \). Therefore, for every \( 2 \)-stable subset \( u \) of \([a]\) of size \( b \), there are at least \( n \) indices \( i_1, \ldots, i_n \) such that \( u \) contains \((k - 1)(m - 2n + 2) + 1\) elements of \( D_{i_j} \), \( 1 \leq j \leq n \). Note also that \( D_i \) contains a unique subset of cardinality \((k - 1)(m - 2n + 2) + 1\) which does not contain any two consecutive elements. Use \( E_i \) to denote this unique subset of \( D_i \), which is readily seen to consist of the smallest elements of \( D_i \), the third smallest elements of \( D_i \), and so on and so forth. For any vertex \( u \in SG(a, b) \), we define a coloring \( c \) by choosing \( n \) indices \( i_j \) (\( 1 \leq j \leq n \)) such that \( E_{i_j} \subseteq u \) and we set \( c(u) \overset{\text{def}}{=} \{i_1, \ldots, i_n\} \). Since \( u \) is a \( 2 \)-stable subset of \([a]\), it is easy to verify that \( c(u) \) is a \( 2 \)-stable subset of \([m]\) too. One needs to show that for any two vertices \( u \) and \( v \) for which there is a walk of length \( 2r - 1 \) between them, where \( 1 \leq r \leq k \), we have \( c(u) \cap c(v) = \emptyset \). To prove this, suppose that \( i \in c(v) \) and \( v = v_0, v_1, \ldots, v_{2r-1} = u \) be a walk between \( u \) and \( v \), where \( 1 \leq r \leq k \). By Lemma 1, \( |v \setminus v_{2r-2}| \leq (k - 1)(m - 2n + 2) \). In particular, \( v_{2r-2} \) contains all but at most \((k - 1)(m - 2n + 2) + 1\) elements of \( E_i \). As \( |E_i| = (k - 1)(m - 2n + 2) + 1 \), we see that \( v_{2r-2} \cap E_i \neq \emptyset \). Thus, the set \( u \), which is disjoint from \( v_{2r-2} \), cannot contain all elements of \( E_i \), showing that \( i \notin c(u) \). This proves that \( c(u) \cap c(v) = \emptyset \). Therefore, Theorem 2 applies, finishing the proof. \( \blacksquare \)

For a given graph \( G \), if \( u \) and \( v \) are distinct vertices of \( G \) and the neighborhood of \( u \) is a subset of that of \( v \), then the graph \( G \) is certainly not a vertex-critical graph. Note that in the graph \( SG(7, 2, 2) \), the neighborhood of the vertex \( \{\{1, 3\}, \{4, 5, 6, 7\}\} \) is a subset of that of the vertex \( \{\{1, 3\}, \{2, 4, 5, 6, 7\}\} \). Hence, the graph \( SG(m, n, k) \) in general is not a vertex-critical graph. This motivates us to present the following definition.

**Definition 4.** Let \( m, n, \) and \( k \) be positive integers with \( m \geq 2n \). Define \( SH(m, n, k) \) to be the induced subgraph of \( H(m, n, k) \) whose vertex set contains all \( k \)-tuples \((A_1, \ldots, A_k) \in V(H(m, n, k))\) such that for any \( 1 \leq r \leq k \), \( A_r = \cup_s B_s \) and \( \overline{A_r} = \cup_s C_s \), where \( B_s \)’s and \( C_s \)’s are all \( 2 \)-stable \( n \)-subsets of \([m]\). 

One can check that \( SH(m, n, k) \) has the property that for any two distinct vertices \( u, v \in V(SH(m, n, k)), N(u) \nsubseteq N(v) \) and \( N(v) \nsubseteq N(u) \). Also, it is straightforward to see that \( SH(m, n, k) \) is the maximal subgraph of \( SG(m, n, k) \) with the aforementioned property. To prove this, we modify the graph \( SG(m, n, k) \) by performing the following **WHILE-loop**.

**WHILE** there exist two distinct vertices \( u = (A_1, \ldots, A_k) \) and \( v = (B_1, \ldots, B_k) \), where \( N(u) \subseteq N(v) \), then **DO** the following: remove the vertex \( u \).

We claim that in the **WHILE-loop** algorithm when the input is the graph \( SG(m, n, k) \) with \( m \geq 2n \), then the output is the graph \( SH(m, n, k) \). To show
this, note that in the **WHILE-loop** each time we search in the new graph for the bad vertex \(u\). So a vertex \(u\) may be good at the beginning, and become bad later. Suppose that **WHILE-loop** is not completed yet. In the last graph obtained from the **WHILE-loop** algorithm, let \(i\) be the greatest positive integer for which there exists at least a vertex \(u = (A_1, \ldots, A_k) \in V(SG(m, n, k))\) such that \(\overline{A}_i\) is not a union of 2-stable \(n\)-subsets of \([m]\). Note that as \(|A_1| = n\), it is easy to verify that \(\overline{A}_1\) is a union of 2-stable \(n\)-subsets of \([m]\), and hence \(i \geq 2\). Also, by the assumption, for any \(i < j\), \(\overline{A}_j\) is a union of 2-stable \(n\)-subsets of \([m]\). Set 

\[ v \overset{\text{def}}{=} (A_1, \ldots, A_{i-1}, A_i \cup B, A_{i+1}, \ldots, A_k), \]

where

\[ B \overset{\text{def}}{=} \{j \mid j \in \overline{A}_i \text{ and } j \text{ does not appear in any } 2\text{-stable } n\text{-subsets of } \overline{A}_i\}. \]

For any \(j \in B\), since \(A_{i-1} \subseteq \overline{A}_i\) and that \(A_{i-1}\) is a union of 2-stable \(n\)-subsets of \([m]\), it is easy to show that \(\{j-1, j+1\} \subseteq A_{i-1} \subseteq \overline{A}_i\) (mod \(m\)). Therefore, \(A_i \cup B\) is a union of 2-stable \(n\)-subsets of \([m]\). Also, by considering the assumption, we should have \(B \subseteq A_{i+2}\). Thus, \(v \in V(SG(m, n, k))\) and also \(N(u) \subseteq N(v)\). Consequently, when the **WHILE-loop** is completed, we obtain the graph \(SH(m, n, k)\). Also, this shows that \(SH(m, n, k)\) and \(SG(m, n, k)\) are homomorphically equivalent. In view of the above observation, we suggest the following question.

**Question 1.** Let \(m, n, \) and \(k\) be positive integers with \(m \geq 2n\). Is it true that the graph \(SH(m, n, k)\) is a vertex-critical graph?

The problem whether the circular chromatic number and the chromatic number of the Kneser graphs and the Schrijver graphs are equal has received attention and has been studied in several papers \([4, 9, 15, 19, 22, 28]\). Johnson, Holroyd, and Stahl \([15]\) proved that \(\chi_c(KG(m, n)) = \chi(KG(m, n))\) if \(m \leq 2n + 2\) or \(n = 2\). They also conjectured that the equality holds for all Kneser graphs.

**Conjecture 1.** \([15]\) For all \(m \geq 2n + 1\), \(\chi_c(KG(m, n)) = \chi(KG(m, n))\).

It was shown in \([9]\) that if \(m \geq 2n^2(n - 1)\), then the circular chromatic number of \(KG(m, n)\) is equal to its chromatic number. Later, it was proved independently in \([22, 28]\) that \(\chi(KG(m, n)) = \chi_c(KG(m, n)) = m - 2n + 2\) whenever \(m\) is an even natural number. Also in \([1, 28]\), it was shown that \(\chi(H(m, 1, k)) = m\). Simonyi and Tardos \([28]\) used the fact that \(\text{Hom}(SG(a,b),H(m,1,k)) \neq \emptyset\), where \(a + b - 2 = m\), and hence \(m - 1\) is a lower bound for the co-index of the box complex of \(H(m, 1, k)\). For definition of the box complex and more about this concept refer to \([28]\).

**Theorem A.** \([22, 28]\) If coind\((B_0(G))\) is odd for a graph \(G\), then \(\chi_c(G) \geq \text{coind}(B_0(G)) + 1\).

It was shown in \([28]\) that circular chromatic number and chromatic number of \(H(m, 1, k)\) are equal.

**Theorem 4.** Let \(m, n, \) and \(k\) be positive integers, where \(m \geq 2n\) and \(m\) is an even positive integer. Then, \(\chi_c(SG(m, n, k)) = \chi_c(H(m, n, k)) = m - 2n + 2\). Furthermore, \(\chi_c(SH(m, n, k)) = m - 2n + 2\).
Proof. As proved in Theorem\textsuperscript{3} if \( a - 2b = m - 2n \) and \( a = 2(k - 1)m(m - 2n + 2) + m \), then \( \text{Hom}(SG(a, b), SG(m, n, k)) \neq \emptyset \). This implies \( \text{coind}(B_o(SG(a, b)) \leq \text{coind}(B_o(SH(m, n, k))) \). Also, it is well known that \( \text{coind}(B_o(SG(a, b)) = a - 2b + 1 \). Thus, by Theorem\textsuperscript{A} we have \( \chi_c(SG(m, n, k)) = \chi_c(H(m, n, k)) = m - 2n + 2 \). Also, two graphs \( SH(m, n, k) \) and \( SG(m, n, k) \) are homomorphically equivalent. Thus, \( \chi_c(SH(m, n, k)) = m - 2n + 2 \).

In \textsuperscript{22, 28}, the authors made use of Theorem\textsuperscript{A} to prove that \( \chi_c((SG(a, b)) = \chi_c((SG(a, b)) \) provided that \( a \) is an even positive integer. In view of \( \chi_c((SG(a, b)) = \chi_c((SG(a, b)), \) where \( a \) is an even integer number, one can present an alternate proof of Theorem\textsuperscript{3}. However, note that the equality \( \text{coind}(B_o(H(m, n, k)) = m - 2n + 1 \) provides more information about the colorings of the helical graph \( H(m, n, k) \) (see \textsuperscript{28, 29}).

It was conjectured in \textsuperscript{19} and proved in \textsuperscript{9}, that for every fixed \( n \), there is a threshold \( t(n) \) such that \( \chi_c(SG(m, n)) = \chi(SG(m, n)) \) for all \( m \geq t(n) \). Note that \( H(3, 1, 2) \) is the nine cycle and that \( \chi_c(H(3, 1, 2)) = \frac{9}{7} \). Hence, the following question arises naturally.

**Question 2.** Given positive integers \( n \) and \( k \), does there exist a number \( t(n, k) \) such that the equality \( \chi_c(SH(m, n, k)) = \chi_c(H(m, n, k)) = \chi(H(m, n, k)) = m - 2n + 2 \) holds for all \( m \geq t(n, k) \)?

### 3 Homomorphism to Odd Cycles

In this section, we investigate the problem of existence of homomorphisms to odd cycles. A graph \( H \) is said to be a subdivision of a graph \( G \) if \( H \) is obtained from \( G \) by subdividing some of the edges. The graph \( S_t(G) \) is said to be the \( t \)-subdivision of a graph \( G \) if \( S_t(G) \) is obtained from \( G \) by replacing each edge by a path with exactly \( t \) inner vertices. Note that \( S_0(G) \) is isomorphic to \( G \). In the following theorem, we prove that a homomorphism to \((2k + 1)\)cycle exists if and only if the chromatic number of \((2k+1)\)st power of \( S_2(G) \) is less than or equal to \( 3 \).

**Theorem 5.** Let \( G \) be a graph with odd-girth at least \( 2k+1 \). Then, \( \chi(S_2(G)^{(2k+1)}) \leq 3 \) if and only if \( \text{Hom}(G, C_{2k+1}) \neq \emptyset \).

**Proof.** First, if there exists a homomorphism from \( G \) to \( C_{2k+1} \), then it is obvious to see that there is a homomorphism from \( S_2(G) \) to \( C_{6k+3} = H(3, 1, k + 1) \). In view of Theorem\textsuperscript{1} we have \( \chi(S_2(G)^{(2k+1)}) \leq 3 \).

Next, if \( \chi(S_2(G)^{(2k+1)}) \leq 3 \), then \( \text{Hom}(S_2(G), C_{6k+3}) \neq \emptyset \). Consequently, \( \text{Hom}(S_2(G)^{(3)}, C_{6k+3}^{(3)}) \neq \emptyset \). Also, it is easy to verify that \( G \) is a subgraph of \( S_2(G)^{(3)} \) and that there is a homomorphism from \( C_{6k+3}^{(3)} \) to \( C_{2k+1} \). Therefore, we have \( \text{Hom}(G, C_{2k+1}) \neq \emptyset \).

Considering Theorem\textsuperscript{5} it is worth to study the following question.

**Question 3.** Let \( G \) be a non-bipartite graph. What is the value of

\[
\sup\left\{ \frac{2k + 1}{2t + 1} | \chi(S_{2t}(G)^{(2k+1)}) = \chi(G), \frac{2k + 1}{2t + 1} < \text{og}(G) \right\}.
\]
In [24], Nešetřil posed the Pentagon problem.

**Problem 1.** Nešetřil's Pentagon Problem [24]

If $G$ is a cubic graph of sufficiently large girth, then $\text{Hom}(G, C_5) \neq \emptyset$.

It should be noted that if in the problem $C_5$ is replaced by $C_3$, then the problem holds; and in fact it is a quick consequence of Brook’s theorem. On the other hand, the problem is known to be false if one replaces $C_5$ by $C_{11}, C_9$ or $C_7$ [10, 18, 32].

In view of Theorem 5, it is possible to rephrase the Pentagon Problem as follows.

**Question 4.** Let $G$ be a cubic graph of sufficiently large girth, is it true that $\chi(S_2(G)^{(5)}) \leq 3$?

If the Pentagon problem holds, then it follows from Lemma A that there exists a number $g_0$ with the property that the chromatic number of the third power of any cubic graph with girth larger than $g_0$ is less than six.

**Question 5.** [5] Is it true that for any natural number $g_0$, there exists a cubic graph $G$ whose girth is larger than $g_0$ and $\chi(G^{(3)}) \geq 6$?

It is interesting to find $\max_{g(G) \geq g} \chi(G^{(3)})$, where maximum is taken over all cubic graphs with girth at least $g$. It should be noted that by Brook’s theorem this maximum is less than or equal to 16. In view of Theorem 1 the following question is equivalent to question 5.

**Question 6.** Is it true that for any natural number $g_0$, there exists a cubic graph $G$ whose girth is larger than $g_0$ and $\text{Hom}(G, H(5, 1, 2)) = \emptyset$?

Note that $H(3, 1, 2)$ is the nine cycle. It was proved in [32] that the above question has an affirmative answer when $H(5, 1, 2)$ is replaced by $H(3, 1, 2)$. This motivates us to suggest the following question.

**Question 7.** Is it true that for any natural number $g_0$, there exists a cubic graph $G$ whose girth is larger than $g_0$ and $\text{Hom}(G, H(4, 1, 2)) = \emptyset$?

The fractional chromatic number of graphs with odd-girth greater than 3 has been studied in several papers [11, 12]. Heckman and Thomas [12] posed the following conjecture.

**Conjecture 2.** [12] Every triangle free graph with maximum degree at most 3 has the fractional chromatic number at most $\frac{14}{5}$.

The helical graphs bound high girth graphs. Thus, it may be interesting to compute their fractional chromatic number and their local chromatic number.

**Question 8.** Let $m, n,$ and $k$ be positive integers with $m \geq 2n$. What are the values of $\chi_f(H(m, n, k))$ and $\psi(H(m, n, k))$?
Let $\mathcal{P}_{2k+1}$ be the class of planar graphs of odd-girth at least $2k+1$. Naserasr \cite{23} posed an upper bound for the chromatic number of planar graph powers as follows.

**Conjecture 3.** \cite{23} For every $G \in \mathcal{P}_{2k+1}$ we have $\chi(G^{(2k-1)}) \leq 2^{2k}$.

Again in view of Theorem \cite{1}, one can rephrase Naserasr’s conjecture in terms of the helical graphs. The following conjecture is Jaeger’s modular orientation conjecture restricted to planar graphs.

**Conjecture 4.** Jaeger’s Conjecture \cite{14}

Every planar graph with girth at least $4k$ has a homomorphism to $C_{2k+1}$.

Considering Theorem \cite{5} one can reformulate Jaeger’s conjecture as follows.

**Conjecture 5.** Let $P$ be a planar graph with girth at least $4k$. Then, we have $\chi(S_2(P)^{(2k+1)}) \leq 3$.

**Acknowledgement:** This paper was written during the sabbatical leave of the author in Zurich University. He wishes to thank J. Rosenthal for his hospitality. Also, the author wishes to thank an anonymous referee, G. Simonyi, C. Tarif, G. Tardos, B.R. Yahaghi, and X. Zhu who drew the author’s attention to the references \cite{1}, \cite{7} and \cite{12} and for their useful comments.

**References**

\cite{1} S. Baum and M. Stiebitz, *Coloring of graphs without short odd paths between vertices of the same color class*, manuscript 2005.

\cite{2} A. Daneshgar and H. Hajiabolhassan, *Graph homomorphisms through random walks*, J. Graph Theory, 44 (2003), 15–38.

\cite{3} A. Daneshgar and H. Hajiabolhassan, *Graph homomorphisms and nodal domains*, Linear Algebra and Its Applications, 418 (2006), 44–52.

\cite{4} A. Daneshgar and H. Hajiabolhassan, *Circular colouring and algebraic no-homomorphism theorems*, European J. Combinatorics, 28 (2007), 1843–1853.

\cite{5} A. Daneshgar and H. Hajiabolhassan, *Density and power graphs in graph homomorphism problem*, Discrete Mathematics, to appear.

\cite{6} P. Erdös, Z. Füredi, A. Hajnal, P. Komjath, V. Rödl, and A. Seress, *Coloring graphs with locally few colors*, Discrete Mathematics, 59 (1986), 21-34.

\cite{7} A. Gyárfás, T. Jensen, and M. Stiebitz , *On graphs with strongly independent color classes*, Journal of Graph Theory, 46(2004), 1–14.

\cite{8} G. Hahn and C. Tarif, *Graph homomorphisms: structure and symmetry*, in Graph Symmetry, G. Hahn and G. Sabidussi, eds., no. 497 in NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer, Dordrecht, 1997, 107–167.
[9] H. Hajiabolhassan and X. Zhu, Circular chromatic number of Kneser graphs, J. Combinatorial Theory Ser. B, 88 (2003), 299-303.

[10] H. Hatami, Random cubic graphs are not homomorphic to the cycle of size 7, J. Combin. Theory Ser. B, 93 (2005), 319–325.

[11] H. Hatami and X. Zhu, The fractional chromatic number of graphs of maximum degree at most 3, manuscript 2006.

[12] C.C. Heckman and R. Thomas, A new proof of the independence ratio of triangle-free cubic graphs, Discrete Mathematics, 233 2001, 233–237.

[13] P. Hell and J. Nešetřil, Graphs and Homomorphisms, Oxford Lecture Series in Mathematics and its Applications, 28, Oxford University press, Oxford (2004).

[14] F. Jaeger, On circular flows in graphs in Finite and Infinite Sets, Colloquia Mathematica Societatis Janos Bolyai, edited by A. Hajnal, L. Lovasz, and V.T. Sos., North-Holland, 37 (1981), 391–402.

[15] A. Johnson, F. C. Holroyd, and S. Stahl, Multichromatic numbers, star chromatic numbers and Kneser graphs, J. Graph Theory, 26 (1997), 137–145.

[16] M. Kneser, Aufgabe 300, Jber. Deutsch. Math.-Verein., 58 (1955), 27.

[17] J. Körner, C. Pilotto, and G. Simonyi, Local chromatic number and Sperner capacity, Journal of Combinatorial Theory, Ser. B, 95 (2005), 101–117.

[18] A. Kostochka, J. Nešetřil, and P. Smolikova, Colorings and homomorphisms of degenerate and bounded degree graphs, Discrete Mathematics, 233 (2001), 257–266.

[19] K.W. Lih and D.F. Liu, Circular chromatic numbers of some reduced Kneser graphs, J. Graph Theory, 41 (2002), 62–68.

[20] L. Lovász, Kneser’s conjecture, chromatic number, and homotopy, J. Combinatorial Theory Ser. A, 25 (1978), 319–324.

[21] T. Marshal, R. Naserasr, and J. Nešetřil, On homomorphism bounded classes of graphs, European J. Combinatorics, 27 (2006), 592–600.

[22] F. Meunier, A topological lower bound for the circular chromatic number of Schrijver graphs, J. Graph Theory, 49 (2005), 257-261.

[23] R. Naserasr, Homomorphisms and edge-colourings of planar graphs, J. Combinatorial Theory Ser. B, 97 (2007), 394-400.

[24] J. Nešetřil, Aspects of structural combinatorics (graph homomorphisms and their use), Taiwanese J. Math., 3 (1999), 381-423.

[25] J. Nešetřil and P. Ossona De Mendez, Colorings and Homomorphisms of Minor Closed Classes, Discrete and Computational Geometry: The Goodman–Pollack Festschrift (ed. B. Aronov, S. Basu, J. Pach, M. Sharir), Springer Verlag, 2003, 651–664.
[26] E. R. Scheinerman and D. H. Ullman, *Fractional graph theory*, John Wiley & Sons, Inc., New York, 1997.

[27] A. Schrijver, *Vertex-critical subgraphs of Kneser graphs*, Nieuw Arch. Wiskd., III. Ser. **26** (1978), 454–461.

[28] G. Simonyi and G. Tardos, *Local chromatic number, Ky Fan's theorem, and circular colorings*, Combinatorica, **26** (2006), 587–626.

[29] G. Simonyi and G. Tardos, *Colorful subgraphs in Kneser-like graphs*, manuscript 2006.

[30] C. Tardif, *Multiplicative graphs and semi-lattice endomorphisms in the category of graphs*, J. Combinatorial Theory, Ser. B, **95** (2005), 338–345.

[31] A. Vince, *Star chromatic number*, J. Graph Theory, **12** (1988), 551–559.

[32] I. M. Wanless and N. C. Wormald, *Regular graphs with no homomorphisms onto cycles*, J. Combinatorial Theory, Ser. B, **82** (2001), 155–160.

[33] X. Zhu, *Circular chromatic number: a survey*, Discrete Math., **229** (2001), 371–410.