LOCAL STABILITY OF A FRACTIONAL ORDER SIS EPIDEMIC MODEL WITH SPECIFIC NONLINEAR INCIDENCE RATE AND TIME DELAY

MOUHCINE NAIM¹, GHASSANE BENRHMAC⁰¹, FOUAD LAHMIDI¹, ABDELWAHED NAMIR²

¹Laboratory of Analysis, Modeling and Simulation, Faculty of Sciences Ben M’sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco
²Laboratory of Information Technology and Modeling, Faculty of Sciences Ben M’sik, Hassan II University, P.O. Box 7955, Sidi Othman, Casablanca, Morocco

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Abstract. In this paper, we study the stability of a fractional order SIS epidemic model with specific functional response and time delay, where the fractional derivative is defined in the Caputo sense. Using the theory of stability of differential equations of delayed fractional order systems, we prove that the disease-free equilibrium is locally asymptotically stable when the basic reproduction number $R_0 < 1$. Also, we show that if $R_0 > 1$, the endemic equilibrium is locally asymptotically stable. Numerical simulations are presented to illustrate the theoretical results of this work.

Keywords: fractional derivative; SIS epidemic model; time delay; stability.

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1. INTRODUCTION

Epidemiology is the study of the spread of diseases in human populations and the factors that are responsible for or contribute to their occurrence. Consequently, it has been investigated...
by several researchers through study the dynamical behavior of infectious diseases by mathematical models (see, e.g., [1, 2, 3, 4]). Particularly, the SIS (susceptible-infected-susceptible) epidemic model is often used to model the dynamics of the diseases such as the bacterial diseases and some sexually transmitted diseases where infection with the disease does not confer permanent immunity against re-infection so that those who survived the infection revert to the class of wholly-susceptible individuals [5].

Fractional calculus is the field of mathematical analysis aiming at the investigation of integrals and derivatives of arbitrary (non integer) orders. The main advantage of fractional order derivative in comparison in integer order is that fractional order derivative can be describe the memory and hereditary effects in various substances. Therefore, many applied researchers have treat many real processes using the fractional derivative such as botanical electrical impedances [6], viscoelasticity of cancellous bone [7], human root dentin [8], financial processes [9], \( PI^\lambda D^\mu \) controller [10], and so on.

Due to the memory effects which is has an important role on the spread of an infectious disease, many investigators have started to study the fractional order epidemic models, see, e.g., [11, 12, 13, 14]. In 2014, El-Saka in [14] introduced a fractional order SIS model with variable population size where the author study the stability of equilibrium points. Our aim in this present work is to extend the model presented in [14] to a model with specific functional response and time delay. In this way, we propose the following fractional order SIS epidemic model

\[
\begin{align*}
D^\alpha S(t) &= \Lambda - \mu S(t) - \frac{\beta S(t)I(t-\tau)}{1 + \alpha_1 S(t) + \alpha_2 I(t-\tau) + \alpha_3 S(t)I(t-\tau)} + rI(t), \\
D^\alpha I(t) &= \frac{\beta S(t)I(t-\tau)}{1 + \alpha_1 S(t) + \alpha_2 I(t-\tau) + \alpha_3 S(t)I(t-\tau)} - (\mu + a + r)I(t),
\end{align*}
\]

where \( \alpha \in (0, 1] \) is the order of the fractional derivative, \( S(t) \) is the proportion of susceptible individuals at time \( t \), \( I(t) \) is the proportion of infected individuals at time \( t \), \( \Lambda \) is the recruitment rate of the susceptible, \( \mu \) is the natural death rate of the population, \( a \) is the death rate due to disease, \( r \) is the recovery rate of infective individuals, \( \beta > 0 \) is the contact transmission coefficient, which measures the infection force of the disease and \( \alpha_1, \alpha_2, \alpha_3 \geq 0 \) are the saturation factors measuring the psychological or inhibitory effect. The constant \( \tau \geq 0 \) is the time delay.
in which the infectious individuals develop in the vector and it is only after that time that the infected vector can infect a susceptible individual (see, e.g., [15], [16]).

The fractional order derivative used in model (1) is in the sense of Caputo definition, which is a modification of the Riemann-Liouville integral definition, and has the advantage that the initial values for fractional differential equations with Caputo derivatives take the same form as that for integer order differential equations. Also, another advantage of this definition is that the Caputo derivative of a constant is zero.

The Riemann-Liouville fractional integral and Caputo fractional derivative are defined respectively as follows [17, 18].

**Definition 1.** The Riemann-Liouville integral of order \( \alpha > 0 \) for an integrable function \( f : \mathbb{R}_+ \mapsto \mathbb{R} \) is defined as

\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,
\]

where \( \Gamma \) is the Gamma function defined by the integral

\[
\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.
\]

**Definition 2.** The Caputo fractional derivative of order \( \alpha > 0 \) for a function \( f \in C^n(\mathbb{R}_+, \mathbb{R}) \) is defined as

\[
D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,
\]

where \( n \) is a positive integer such that \( \alpha \in (n-1, n] \). In particular, when \( \alpha \in (0, 1] \), one has

\[
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds.
\]

The rest of this paper is organized as follows. In the next section, we investigate the existence and the local stability of equilibria. In Section 3, we present the numerical simulation to illustrate our results, and finally we gave our conclusion in Section 4.

2. **Stability Analysis**

In this section, we discuss the existence and the local stability of the equilibria of system (1). In this sense, we define the basic reproduction number of model (1) as follows

\[
R_0 = \frac{\beta \Lambda}{(\mu + \alpha_1 \Lambda)(\mu + a + r)}.
\]
From biological point of view, $R_0$ represents the average number of secondary infections that occur when one infectious individual is introduced into a completely susceptible population [19].

The equilibrium of model (1) is obtained by setting $D^\alpha S = D^\alpha I = 0$. Then, system (1) always has a disease-free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$. Further, if $R_0 > 1$, then system (1) has a unique endemic equilibrium $E^* = (S^*, I^*)$, where

$$S^* = \frac{\Lambda - (\mu + a)I^*}{\mu},$$

$$I^* = \frac{2\sigma(\mu + \alpha_1 \Lambda)(R_0 - 1)}{(\mu + a)(\beta - \alpha_1 \sigma) + \sigma(\alpha_2 \mu + \alpha_3 \Lambda) + \sqrt{\Delta}},$$

with $\sigma = a + \mu + r$ and

$$\Delta = \left[(\mu + a)(\beta - \alpha_1 \sigma) + \sigma(\alpha_2 \mu + \alpha_3 \Lambda)\right]^2 - 4\alpha_3(\mu + a)\sigma \left[\beta \Lambda - (\mu + \alpha_1 \Lambda)\sigma\right]$$

$$= \left[(\mu + a)(\beta - \alpha_1 \sigma) + \sigma(\alpha_2 \mu - \alpha_3 \Lambda)\right]^2 + 4\alpha_3 \mu \sigma^2 (\mu + a + \alpha_2 \Lambda).$$

Consider the following linear delayed fractional differential system

$$D^\alpha x(t) = Ax(t) + Bx(t - \tau), \quad t \geq 0,$$

where $\alpha \in (0, 1], x(t) \in \mathbb{R}^n, A, B \in \mathbb{R}^{n \times n}$ and $\tau \geq 0$. The characteristic equation of system (2) is

$$\Delta(s) = \det(s^\alpha I_n - A - Be^{-s\tau}) = 0.$$

If $\tau = 0$, system (2) can be expressed as

$$D^\alpha x(t) = Mx(t),$$

where the coefficient matrix $M = A + B$.

In the case of $A = 0$, Deng et al. in [20] obtained the following two stability results.

**Lemma 1.** If all the roots of the characteristic equation $\Delta(s) = 0$ have negative real parts, then the zero solution of system (2) is Lyapunov globally asymptotically stable.
Lemma 2. If all the eigenvalues $\lambda$ of $B$ satisfy $|\arg(\lambda)| > \frac{\alpha \pi}{2}$, and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system (2) is Lyapunov globally asymptotically stable.

If $A \neq 0$, according to [21], we have the following conclusion.

Lemma 3. If all the eigenvalues $\lambda$ of $M$ satisfy $|\arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system (2) is Lyapunov globally asymptotically stable.

Remark 1. If $A \neq 0$, the stability of system (2) is not guaranteed under conditions that the eigenvalues of $M$ are satisfied $|\arg(\lambda)| > \frac{\alpha \pi}{2}$. In fact, when the eigenvalues of $M$ are satisfied $\frac{\alpha \pi}{2} < |\arg(\lambda)| \leq \frac{\pi}{2}$, and the characteristic equation $\Delta(s) = 0$ has no purely imaginary roots for any $\tau > 0$, the zero solution has unstable situation (see Section 5 in [21]).

Remark 2. To study the local asymptotic stability of equilibria of nonlinear fractional order systems, we investigate the stability of the linearized systems of such nonlinear systems around these equilibria based on the previous lemmas.

2.1. Stability of the disease-free equilibrium. This subsection is devoted to studying the stability of the diseases-free equilibrium $E_0$ of system (1). For this, let $x(t) = S(t) - \frac{A}{\mu}$ and $y(t) = I(t)$. Then the linearized system of (1) around $E_0$ takes the following form

$$
\begin{align*}
D^\alpha x(t) &= -\mu x(t) - \frac{\beta A}{\mu + \alpha_1 \lambda} y(t - \tau) + ry(t), \\
D^\alpha y(t) &= \frac{\beta A}{\mu + \alpha_1 \lambda} y(t - \tau) - (\mu + a + r)y(t).
\end{align*}
$$

(3)

The associated characteristic equation of system (3) can be described as

$$
\Delta(s) = \det \begin{pmatrix} s^\alpha + \mu & \frac{\beta A}{\mu + \alpha_1 \lambda} e^{-s \tau} - r \\
0 & s^\alpha - \frac{\beta A}{\mu + \alpha_1 \lambda} e^{-s \tau} + \mu + a + r \end{pmatrix} = 0,
$$

which leads to

$$
\Delta(s) = (s^\alpha + \mu) [s^\alpha + (\mu + a + r)(1 - R_0 e^{-s \tau})] = 0.
$$

(4)
Theorem 1. If \( R_0 < 1 \), then the disease-free equilibrium \( E_0 \) is asymptotically stable for all \( \tau \geq 0 \) and \( \alpha \in (0, 1] \). \( E_0 \) is unstable if \( R_0 > 1 \).

Proof. When \( \tau = 0 \), the coefficient matrix \( M \) of system (3) satisfies
\[
M = \begin{pmatrix}
-\mu & -\frac{\beta \Lambda}{\mu + \alpha_i \Lambda} + r \\
0 & \frac{\beta \Lambda}{\mu + \alpha_i \Lambda} - (\mu + \alpha + r)
\end{pmatrix}.
\]

The eigenvalues of the coefficient matrix \( M \) are \( \lambda_1 = -\mu < 0 \) and \( \lambda_2 = (\mu + \alpha + r)(R_0 - 1) < 0 \) if \( R_0 < 1 \). Whence \( |\arg \lambda_i| = \pi > \frac{\pi}{2} \) \( (i = 1, 2) \), so that all the eigenvalues \( \lambda \) of \( M \) satisfy \( |\arg(\lambda)| > \frac{\pi}{2} \) if \( R_0 < 1 \). If \( R_0 > 1 \), then \( \lambda_2 > 0 \) and consequently \( E_0 \) is unstable [22].

Now, we research the circumstance of delay \( \tau > 0 \). We only need to analyze the second factor of (4) as it contains \( \tau \), so substituting \( s = iw = w(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \) in the second factor of (4), with \( w > 0 \). Then
\[
w^\alpha (\cos \frac{\alpha \pi}{2} + i \sin \frac{\alpha \pi}{2}) + (\mu + a + r) [1 - R_0 (\cos w \tau - i \sin w \tau)] = 0.
\]

Separating real and imaginary parts gives
\[
\begin{align*}
(\mu + a + r)R_0 \cos w \tau & = w^\alpha \cos \frac{\alpha \pi}{2} + (\mu + a + r), \\
(\mu + a + r)R_0 \sin w \tau & = -w^\alpha \sin \frac{\alpha \pi}{2}.
\end{align*}
\]

Thus
\[
w^{2\alpha} + 2w^\alpha (\mu + a + r) \cos \frac{\alpha \pi}{2} + (\mu + a + r)^2 (1 - R_0^2) = 0.
\]

Obviously, since \( (\mu + a + r) \cos \frac{\alpha \pi}{2} \geq 0 \) for \( \alpha \in (0, 1] \) and our assumption that \( R_0 < 1 \), then the Eq. (5) has no positive roots. Which ensures that Eq. (4) has no purely imaginary roots. According to Lemma 3, the equilibrium \( E_0 \) is asymptotically stable for any delay \( \tau \geq 0 \) and \( \alpha \in (0, 1] \) if \( R_0 < 1 \). The proof is completed. \( \square \)

2.2. Stability of the endemic equilibrium. In this subsection, we analyse the stability of the endemic equilibrium of the system (1). To begin with, we linearise the system about the endemic equilibrium \( E^* \). Let the transformation \( x(t) = S(t) - S^* \) and \( y(t) = I(t) - I^* \). Then by linearizing system (1) around \( E^* = (S^*, I^*) \), we get the following system
\[
\begin{cases}
D^\alpha x(t) = -m_1 x(t) - m_2 y(t - \tau) + r y(t), \\
D^\alpha y(t) = m_3 x(t) + m_2 y(t - \tau) - m_4 y(t),
\end{cases}
\]

where

\[
\begin{align*}
  m_1 &= \mu + \frac{\beta I^* (1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\
  m_2 &= \frac{\beta S^* (1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\
  m_3 &= \frac{\beta I^* (1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2} > 0, \\
  m_4 &= \mu + a + r > 0.
\end{align*}
\]

Characteristic equation which is associated with system (6) is given by

\[
\Delta(s) = \det \begin{pmatrix}
  s^\alpha + m_1 & m_2 e^{-s\tau} - r \\
  -m_3 & s^\alpha - m_2 e^{-s\tau} + m_4
\end{pmatrix} = 0.
\]

Hence, the above equation can be rewritten equivalently as

\[
\Delta(s) = s^{2\alpha} + a_1 s^\alpha + a_2 - (a_3 s^\alpha + a_4) e^{-s\tau} = 0,
\]

where

\[
\begin{align*}
  a_1 &= m_1 + m_4 > 0, \\
  a_2 &= m_1 m_4 - r m_3 = (\mu + a)m_3 + \mu m_4 > 0, \\
  a_3 &= m_2 > 0, \\
  a_4 &= \mu m_2 > 0.
\end{align*}
\]

**Theorem 2.** If \( R_0 > 1 \), then the endemic equilibrium \( E^* \) is asymptotically stable for all \( \tau \geq 0 \) and \( \alpha \in (0, 1] \).
Proof. When \( \tau = 0 \), the characteristic equation of the coefficient matrix \( M \) of system (6) is

\[
\lambda^2 + (a_1 - a_3)\lambda + (a_2 - a_4) = 0.
\]

Since

\[
m_4 - m_2 = (\mu + a + r) - \frac{\beta S^* (1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2}
\]

\[
= \frac{\beta S^*}{1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*} - \frac{\beta S^* (1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2}
\]

\[
\geq 0,
\]

then \( a_1 - a_3 = m_1 + (m_4 - m_2) > 0 \) and \( a_2 - a_4 = (\mu + a)m_3 + \mu(m_4 - m_2) > 0 \). Hence the two roots \( \lambda_i \ (i = 1, 2) \) of the Eq. (8) have negative real parts, so that all the eigenvalues of \( M \) of system (6) satisfy \( |\arg(\lambda)| > \frac{\pi}{2} \) if \( R_0 > 1 \).

For \( \tau > 0 \), let \( s = iw \) is a root of Eq. (7), with \( w > 0 \). Substituting \( s \) into (7) gives

\[
w^{2\alpha}(\cos(\alpha \pi) + i \sin(\alpha \pi)) + w^\alpha a_1(\cos(\frac{\alpha \pi}{2} + i \sin(\frac{\alpha \pi}{2}))) + a_2 - (w^\alpha a_3 \cos(\frac{\alpha \pi}{2}) + a_4
\]

\[
+ iw^\alpha a_3 \sin(\frac{\alpha \pi}{2}) \cos w\tau - i \sin w\tau) = 0.
\]

We separate the real and imaginary parts to have

\[
\begin{cases}
w^{2\alpha} \cos(\alpha \pi) + w^\alpha a_1 \cos(\frac{\alpha \pi}{2}) + a_2 = w^\alpha a_3 \cos(\frac{\alpha \pi}{2} - w\tau) + a_4 \cos w\tau, \\
w^{2\alpha} \sin(\alpha \pi) + w^\alpha a_1 \sin(\frac{\alpha \pi}{2}) = w^\alpha a_3 \sin(\frac{\alpha \pi}{2} - w\tau) - a_4 \sin w\tau.
\end{cases}
\]

Squaring and adding the two equations in (10), we obtain

\[
w^{4\alpha} + \eta_1 w^{3\alpha} + \eta_2 w^{2\alpha} + \eta_3 w^\alpha + \eta_4 = 0,
\]

where

\[
\begin{align*}
\eta_1 &= 2a_1 \cos(\frac{\alpha \pi}{2}), \\
\eta_2 &= a_1^2 - a_3^2 + 2a_2 \cos(\alpha \pi), \\
\eta_3 &= 2(a_1 a_2 - a_3 a_4) \cos(\frac{\alpha \pi}{2}), \\
\eta_4 &= a_2^2 - a_4^2.
\end{align*}
\]
Since $\alpha \in (0, 1]$ and $a_1 > 0$, then $\eta_1 \geq 0$. In addition, we have
\[a_2^2 - a_4^2 = (a_2 - a_4)(a_2 + a_4) > 0,\]
and
\[a_1a_2 - a_3a_4 = a_1(a_2 - a_4) + a_4(a_1 - a_3) > 0,\]
since $a_2 - a_4 > 0$, $a_1 - a_3 > 0$ and $a_1, a_2, a_4 > 0$. Then $\eta_3 \geq 0$ and $\eta_4 > 0$. And since $a_2 > 0$, we have
\[\eta_2 = a_1^2 - a_3^2 + 2a_2 \cos \alpha \pi \geq a_1^2 - a_3^2 - 2a_2 = (m_1 + m_4)^2 - m_2^2 - 2(m_1m_4 - rm_3) = m_1^2 + 2rm_3 + m_4^2 - m_2^2.\]

From (9), we have $m_4^2 - m_2^2 \geq 0$ since $m_4 + m_2 > 0$. Hence $\eta_2 > 0$. Therefore the Eq. (11) has no positive real roots, which implies that Eq. (7) has no purely imaginary roots. Thus, according to Lemma 3, the equilibrium point $E^*$ is asymptotically stable for delay $\tau \geq 0$ and $\alpha \in (0, 1]$. This concludes the proof.

\[\square\]

3. Numerical Simulations

In this section, we give some numerical simulations in order to illustrate our theoretical results.

Consider the following parameters $\Lambda = 0.95$, $\beta = 0.1$, $\mu = 0.2$, $a = 0.03$, $r = 0.3$, $\alpha_1 = 0.1$, $\alpha_2 = 0.03$, $\alpha_3 = 0.05$. By calculation, we obtain $R_0 = 0.63 < 1$, then, by Theorem 1, $E_0$ is asymptotically stable for different values of $\tau \geq 0$ and $\alpha \in (0, 1]$ (see Figs. 1, 2 and 3).
Figure 1. Stability of the disease-free equilibrium $E_0$ and $\tau = 2$

Figure 2. Stability of the disease-free equilibrium $E_0$ and $\tau = 5$

Figure 3. Stability of the disease-free equilibrium $E_0$ and $\tau = 8$

Now, we keep all the parameter values except that $\beta$ is increased to 0.3 from 0.1. In this case, we have $R_0 = 1.78 > 1$. Hence, we can conclude, by Theorem 2, that $E^*$ is asymptotically stable for different values of $\tau \geq 0$ and $\alpha \in (0, 1]$ (see Figs. 4, 5 and 6).
Figure 4. Stability of the endemic equilibrium $E^*$ and $\tau = 2$

Figure 5. Stability of the endemic equilibrium $E^*$ and $\tau = 5$

Figure 6. Stability of the endemic equilibrium $E^*$ and $\tau = 8$
4. Conclusion

In this paper, we have presented a fractional order SIS epidemic model with the Caputo fractional derivative and a specific functional response with delay given by

\[ \beta S(t) I(t-\tau) \over 1 + \alpha_1 S(t) + \alpha_2 I(t-\tau) + \alpha_3 S(t) I(t-\tau) \].

We show that if the basic reproduction number \( R_0 \), is less than one, the disease-free equilibrium is locally asymptotically stable for all \( \tau \geq 0 \) and \( 0 < \alpha \leq 1 \), which means that the disease will go to extinction. Moreover, we prove that if \( R_0 > 1 \), the endemic equilibrium is locally asymptotically stable, so the disease will be persistent at the unique endemic equilibrium. In the end some numerical simulations are given to illustrate the results. From, our theoretical and numerical analysis, we can observe that the different values of \( \alpha \) and \( \tau \) have no effect on the stability of both equilibria but affect the time to reach the steady states.

Conflict of interests

The author(s) declare that there is no conflict of interests.

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