Preliminaries in Many-Particle Quantum Gravity. Einstein–Friedmann Spacetime

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(Dated: February 2, 2008)

PACS numbers: 04.60.-m, 98.80.Qc, 05.30.Jp

Keywords: Quantum Cosmology, Quantum Gravity, Boson systems, Einstein–Hilbert theory

I. INTRODUCTION

In this paper introductory results of the Fock space formulation of Quantum Gravity are discussed by example of Einstein–Friedmann Spacetime. Firstly the classical Dirac–Arnowitt–Deser–Misner Hamiltonian approach to General Relativity for the Spacetime is briefly presented, and the Wheeler–DeWitt evolution as primary quantization of constraints is formulated. Next evolution is separated on the canonical Hamilton field equations system, and consistent the von Neumann–Araki–Woods quantization in the Fock space is realized. For diagonalization the Bogoliubov–Heisenberg basis is used, and internal logarithmic conformal field theory structure in this basis is proved. By treatment of the Spacetime as the Bose–Einstein system thermodynamics of the system is computed.

II. CLASSICAL APPROACH

A. The Dirac constraints

The flat, homogenous and isotropic Spacetime investigated by Einstein [1] and Friedmann [2] is described by an interval

\[ ds^2 = a^2(\eta) \left( (d\eta)^2 - (dx^i)^2 \right), \quad \eta = N_a(x^0)dx^0, \quad (1) \]

where \( a(\eta) \) is the Friedmann conformal scale factor, \( \eta \) is the Dirac–Arnowitt–Deser–Misner conformal time [3, 4], and \( N_a \) is lapse function. The Hilbert action [5] for (1)

\[ \mathcal{A}[a] = \int dx^0 \left\{ p_a \frac{da}{dx^0} - N_a \left[ - \frac{p_a^2}{4V_0^2} + \rho(a) \right] V_0 \right\}, \quad (2) \]

where \( p_a \) is nontrivial canonical conjugate momentum, \( \rho(a) \) is energy density

\[ p_a = \frac{-2V_0}{N_a} \frac{da}{dx^0}, \quad \rho(a) = \frac{a^4}{V_0} \int d^3x \mathcal{H}_M, \quad (3) \]

\[ \mathcal{H}_M \]

and \( V_0 = \int d^3x < \infty \) is finite space volume. Variational principle with respect to \( N_d \) applied to action (2) leads to constraints

\[ \frac{\delta \mathcal{A}[a]}{\delta N_0} = 0 \Rightarrow - \frac{p_a^2}{4V_0^2} + \rho(a) = 0, \quad (4) \]

with solution given by the Hubble law

\[ \int_{a_i}^{a} \frac{da}{\sqrt{\rho(a)}} = \pm |\eta - \eta_I|, \quad (5) \]

where index \( I \) means initial data.

B. The Hamilton field equations

One can see that the primary constraints (1) simplify to a form

\[ p_a^2 - \omega^2(a) = 0, \quad \omega(a) \equiv \pm 2V_0 \sqrt{\rho(a)}, \quad (6) \]

and that direct canonical quantization

\[ i \left[ \hat{p}_a, a \right] = 1, \quad \hat{p}_a \equiv -i \frac{\partial}{\partial a}, \quad (7) \]

leads to the Wheeler–DeWitt evolution equation [2, 7]

\[ \left( \frac{\partial^2}{\partial a^2} + \omega^2(a) \right) \Psi(a) = 0. \quad (8) \]

This equation can be represented as the canonical Hamilton field equations system

\[ \begin{align*}
\frac{\partial}{\partial \Pi_\Psi} \mathcal{H}(\Pi_\Psi, \Psi) &= \frac{\partial}{\partial a} \Psi, \\
- \frac{\partial}{\partial \Psi} \mathcal{H}(\Pi_\Psi, \Psi) &= \frac{\partial}{\partial a} \Pi_\Psi,
\end{align*} \quad (9) \]

where considered Hamiltonian \( \mathcal{H}(\Pi_\Psi, \Psi) \) has a form

\[ \mathcal{H}(\Pi_\Psi, \Psi) = \frac{1}{2} \left( \Pi^2_\Psi + \omega^2(a)\Psi^2 \right). \quad (10) \]

In further text we will use compact form of (9)

\[ \frac{\partial}{\partial a} \left[ \begin{array}{c} \Psi \\ \Pi_\Psi \end{array} \right] = \left[ \begin{array}{cc}
0 & 1 \\
-\omega^2(a) & 0
\end{array} \right] \left[ \begin{array}{c} \Psi \\ \Pi_\Psi \end{array} \right]. \quad (11) \]

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III. QUANTIZATION

We focus attention on the Hamilton field equations (11).

A. Von Neumann–Araki–Woods quantization

Essence of quantization constitutes general transition between classical fields and field operators

\[
\begin{bmatrix}
\Psi(a) \\
\Pi_\Psi(a)
\end{bmatrix} \longrightarrow
\begin{bmatrix}
\Psi[a(\eta)] \\
\Pi_\Psi[a(\eta)]
\end{bmatrix},
\]

with standard Canonical Commutation Relations (CCRs)

\[
[\Pi_\Psi[a(\eta)], \Psi[a(\eta)]] = -i\delta(a(\eta) - a(\eta')).
\]

Problem of CCRs representations is not new, and was investigated in [8] and [9]. In [10] was proposed the Von Neumann–Araki–Woods quantization in the Fock space \((G, G')\) of annihilation and creation operators

\[
\begin{bmatrix}
\Psi \\
\Pi_\Psi
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{2\omega(a)}} & \frac{1}{\sqrt{2\omega(a)}} \\
-i\sqrt{\frac{\omega(a)}{2}} & i\sqrt{\frac{\omega(a)}{2}}
\end{bmatrix} \begin{bmatrix}
G[a(\eta)] \\
G'[a(\eta)]
\end{bmatrix},
\]

which is consistent with (13) by CCRs

\[
\begin{align*}
[G[a(\eta)], G'[a(\eta)]] &= \delta(a(\eta) - a(\eta')), \\
[G[a(\eta)], G[a(\eta)]] &= 0.
\end{align*}
\]

Quantization (13) applied to equations (11) gives

\[
\frac{\partial}{\partial a} \begin{bmatrix}
\Psi \\
\Pi_\Psi
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\omega^2(a) & 0
\end{bmatrix} \begin{bmatrix}
\Psi \\
\Pi_\Psi
\end{bmatrix},
\]

or in terms of the Fock space

\[
\frac{\partial}{\partial a} \begin{bmatrix}
G[a(\eta)] \\
G'[a(\eta)]
\end{bmatrix} = \begin{bmatrix}
-i\omega(a) & \Delta \\
\Delta & i\omega(a)
\end{bmatrix} \begin{bmatrix}
G[a(\eta)] \\
G'[a(\eta)]
\end{bmatrix},
\]

where \(\Delta = \partial_a \ln \left(\frac{\omega(a)}{\omega(a_1)}\right)\), and \(\omega(a_1) = \pm 4V_0\sqrt{\rho(\alpha)}\) is initial value of \(\omega(a)\).

B. The Bogoliubov–Heisenberg basis

Evolution (13) should be diagonalize in order to build correct quantum theory. We apply the Bogoliubov automorphism for Bose–Einstein systems, which lies in accordance with considered field equations (8)

\[
\begin{bmatrix}
W[a(\eta)] \\
W'[a(\eta)]
\end{bmatrix} = \begin{bmatrix}
u(a) & v(a) \\
v^*(a) & u^*(a)
\end{bmatrix} \begin{bmatrix}
G[a(\eta)] \\
G'[a(\eta)]
\end{bmatrix},
\]

This transformation preserves CCRs (15) and (16) in new basis

\[
\begin{align*}
[W[a(\eta)], W'[a(\eta)]] &= \delta(a(\eta) - a(\eta')), \\
[W[a(\eta)], W[a(\eta)]] &= 0.
\end{align*}
\]

Next we reduce evolution (18) to Heisenberg type one

\[
\frac{\partial}{\partial a} \begin{bmatrix}
v(a) \\
u(a)
\end{bmatrix} = \begin{bmatrix}
-i\omega(a) & -\Delta \\
-\Delta & i\omega(a)
\end{bmatrix} \begin{bmatrix}
v(a) \\
u(a)
\end{bmatrix},
\]

that can be solved in hyperbolic parametrization

\[
v(a) = e^{i\theta(a)} \sinh \phi(a), \quad u(a) = e^{i\theta(a)} \cosh \phi(a).
\]

Here functions–parameters

\[
\theta(a) = \int_{a_1}^a p_\alpha da, \quad \phi(a) = \ln \left| \frac{\omega(a)}{\omega(a_1)} \right|,
\]

are classical phase integral and logarithmic conformal field. In this manner we have proved that theory builded by the Von Neumann–Araki–Woods quantization in form (14) has internal logarithmic conformal field theory structure.

IV. PHYSICAL IMPLICATIONS

Now some physical results will present.

A. Density functional

We take density functional operator in standard form of occupation number consistent with quantization (13)

\[
\varrho = G' G,
\]

which in the Bogoliubov–Heisenberg basis has a following matrix representation

\[
\varrho = \begin{bmatrix}
|u|^2 & -uv \\
-\bar{u}v^* & |v|^2
\end{bmatrix}.
\]

One can see that the Von Neumann entropy for considered system has the Boltzmann one form

\[
S \equiv -\frac{\text{tr}(\varrho \ln \varrho)}{\text{tr}(\varrho)} = \ln \left\{ \frac{1}{2|u|^2 - 1} \right\}.
\]

It creates opportunity to formulate of thermodynamics for quantized the Einstein–Friedmann Spacetime represented by entropy (29).
B. Thermodynamics of Spacetime

In opposite to results of paper [10], in this text we propose use of (10) as the Hamiltonian of theory. Theory constructed by this Hamiltonian is conceptually simpler than theory presented in [10], because in (10) superfluidity term is absent. However, as it will turn out physical consequences presented below are principally the same as in [10].

The Hamiltonian (10) has a following matrix representation in the Bogoliubov–Heisenberg basis
\[
H = \begin{bmatrix}
\frac{|u|^2 + |v|^2}{2} \omega & -uv\omega \\
-\overline{u^*v}\omega & \frac{|u|^2 + |v|^2}{2} \omega
\end{bmatrix}.
\] (30)

Statistical mechanics understand internal energy as average \( U = \text{tr}(\rho H) \), which for considered case is
\[
U = \left( \frac{1}{2} + \frac{4n + 3}{2n + 1} \right) \omega(a). \] (31)

Here we introduced quantity
\[
n = \langle 0|g^\dagger g|0 \rangle = \frac{1}{4} \left| \sqrt{\frac{\omega(a)}{\omega(a)}} - \sqrt{\frac{\omega(a)}{\omega(a)}} \right|^2, \] (32)
which is number of particles produced from vacuum. Averaged number of particles is
\[
\langle n \rangle = 2n + 1. \] (33)

Chemical potential is defined as \( \mu = \frac{\partial U}{\partial n} \), in this case
\[
\mu = \left( 1 + \frac{1}{(2n + 1)^2} - \frac{1}{2n + 1} \right) \frac{4n + 1}{2n + 2} \sqrt{\frac{n}{n + 1}} \omega(a). \] (34)

Description of the system according to quantum theory principles demands using of the thermal Gibbs ensemble.

The system is characterized by CCRs (21-22). It leads to identification of function under logarithm in (29) with the Bose–Einstein partition function
\[
\frac{1}{2|u|^2 - 1} = \left\{ \exp \left[ \frac{U - \mu n}{T} \right] - 1 \right\}^{-1}. \] (35)

Using of relations (20), (31), (32), (34), and (35) give the formula for temperature of considered system
\[
T = \frac{1 + \left( \frac{2n}{2n + 1} \right)^2 + 8n^2 + 8n + 1}{4n + 2} \frac{\sqrt{n}}{2 \ln(2n + 2)} \omega(a). \] (36)

This formula describes temperature of the system as a function of Friedmann conformal scale factor \( a \).

V. SUMMARY

Above formalism points at direct way to the Quantum Cosmology understood as many-particle open quantum system. On base of this type reasoning formulation of physics is obvious and clear - in presented case description of the Einstein–Friedmann Spacetime as the quantum Bose–Einstein system leads to well-defined thermodynamics. Furthermore, for considered Spacetime open quantum system point of view gives opportunity to formulate Quantum Gravity as theory with internal structure of logarithmic conformal field theory.

In opinion of the author presented way creates substantially wider opportunities for construction of similar formalism for general gravitational fields and to research connections between geometry and physics in terms of logarithmic conformal field theory.

Acknowledgements

The author is especially thankful for valuable discussions to B.M. Barbaschov, I. Białynicki-Birula, A. Borowiec, S.J. Brodsky, G. ‘t Hooft, J. Lukierski, V.N. Pervushin, V.B. Priezzhev, and C. Rovelli.

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