Strong interconversion of non-polar phonons and Josephson plasma oscillations induced by equilibrium Josephson currents in high $T_c$ superconductors

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We analyze consequences of dynamical modulations of Josephson current by non-polar lattice mode in the Josephson junction barrier. In the high $T_c$ junctions, the effect of such modulations can be anomalously strong due to the proximity of the insulating barrier to the superconducting state. Accordingly, the interconversion of sound (as well as other non-polar phonons) and the Josephson plasma oscillations mediated by stationary Josephson currents, which may be present in the junction due to various reasons, becomes possible. We suggest that this effect can be employed for imaging of the stationary Josephson currents. Estimates of the effect are given.

I. INTRODUCTION

The nature of the superconductivity in the high $T_c$ materials continues to be a subject of hot debates. Following the suggestion [1], very significant advances have been made in revealing the dominant d-wave symmetry of the superconducting order parameter (OP) in these materials [2]. A direct consequence of this symmetry is the possibility of the $\pi$-Josephson coupling [3]. The existence of the $\pi$-type Josephson junction (JJ) and feasibility of its controlled implementation in circuits have been firmly established [2,3]. The most startling consequence of the $\pi$-type coupling is the spontaneous Josephson current which generates the magnetic flux characterized by the half of the standard unit. It should be noted that a strong interest to the unconventional JJ made of the high-$T_c$ materials is inspired, besides purely academic attention, by the very recent proposal [4] of employing such JJ as a qubit for quantum computations.

Another intriguing consequence of the non-traditional symmetry of the OP is the time-reversal symmetry breaking (TRSB) at the grain boundaries [5]. An automatic consequence of such states are spontaneous currents leading to fractional vortices, characterized by the magnetic flux which is neither the half nor the whole unit. It is worth noting that spontaneous fractional vortices have been detected in recent years [6]. Their exact nature, however, is still alluding the explanation.

Detection of single vortices is usually achieved by means of the Scanning SQUID Microscopy technique (see in Ref. [7]). It relies on measuring the total magnetic flux threading the pick up loop which scans the surface from the distance of few $\mu$m. Thus, only those spontaneous currents can be revealed which produce magnetic field normal to the surface. In fact, the resolution of the method [8] is limited by the pick up loop size (4-18$\mu$m) as well as by the distance between the loop and the surface. Furthermore, the inductive interaction between the loop and the fluxes may distort the actual current distribution. In this regard we note that it would be desirable to have a technique capable of a direct high resolution imaging of the spatial distribution of the stationary (spontaneous) Josephson currents regardless of their orientation and of the scale of their spatial variations.

In this paper, we discuss the effect which could be employed for the imaging of the Josephson currents with high resolution. This effect is based on the empirical observations of a very strong sensitivity of values of the Josephson critical current density $J_c$ in the grain boundary (GB) JJ to the misorientation angle between the grains [9]. The nature of this effect remains very controversial. Many explanations have been proposed. One of them relies on the d-wave symmetry of the OP and faceting of the boundary, so that the macroscopic (average) critical current density is much smaller than the local critical current density [10], which may change sign due to the $\pi$-type coupling. As suggested in Ref. [11], an extremely strong suppression of $J_c$ is a result of mechanical deformations concentrated at the GB. It has been suggested that superconductivity is suppressed locally as long as the local mechanical strain becomes $\geq 1\%$ [11]. In accordance with the phenomenological approach [12], the strain shifts the chemical potential of the JJ barrier, and this may strongly affect the Josephson coupling, as long as the barrier is characterized by the proximity to the metal-insulator transition. Many properties of the GB JJ have been successfully explained by this model. On the microscopic level, in Ref. [13] it has been pointed out that small changes of the local structure, existing in the vicinity of the GB, should induce large variations of the valence of the copper ions, which control the free charge carriers concentration.

Recently, the anomalous proximity effect has been induced by light in the Y-Ba-Cu-O JJ [14]. This indicates that the JJ barrier can easily be perturbed (in this case, by light) into the superconducting state. As suggested in Ref. [14], this effect can naturally be explained if the electron system in the barrier is a sort of nearly superconducting quantum critical liquid, so that any external factor can easily shift its equilibrium. In fact, such a factor could be -light; mechanical deformations; optical displacements of atoms, etc.
In this paper, we pose a general question about mechanisms by which deformations (lattice displacements, in general) affect the Josephson coupling in the GB JJ made of the high-$T_c$ materials:

Is the role of the deformations primary or secondary?

If the effect of deformations is primary with respect to the electronic system, it can be revealed by means of applying reasonable deformations externally in the superconducting state and observing subsequent strong changes of $J_c$. If it is secondary, the deformations may not affect directly the electronic system. These may predetermine the electronic properties during the formation of the GB JJ at high temperatures (way above the superconducting temperatures) through the processes of, e.g., oxygen depletion controlled by local mechanical stress and oxygen diffusion, so that no significant direct effect due to the external deformations should be expected to occur in the superconducting phase. In order to resolve this problem we note that, if the deformations play the primary role, a strong dynamical coupling should exist between sound and the Josephson phase. In contrast, if the deformations play a secondary role, no such a significant dynamical coupling should be observed.

In this paper we will assume that the deformations play the primary role in accordance with the general concepts [12][13], and will concentrate on the dynamical implications of the strong sensitivity of $J_c$ to the mechanical and, possibly, optical (non-polar) deformations of the crystal lattice. Specifically, we will analyze a coupled dynamics of the phonons and the Josephson phase. We suggest that the coupling between the phase and the phonons should be anomalously large because of the proximity of the barrier to the superconducting state. In this phenomenological analysis, no quasi-particle effects are considered. This approximation can be well justified by the requirement that the phonon frequencies under consideration are much smaller than a typical superconducting gap. In this sense, we will be employing an adiabatic approximation in which the superconducting electronic ensemble adjusts itself momentarily to Josephson phase and the phonon variable(s). Thus, the following consideration is limited by the framework of the coupled dynamical Josephson equation [14] and equation of motion for the phonon variable.

It is important to emphasize, that the first order coupling between small oscillations of the Josephson phase and sound (or optical non-polar phonons) can exist in a medium characterized by the inversion symmetry, if only some equilibrium Josephson currents are present. In other words, the linear coupling between the phase and the non-polar phonons can exist if and only if the time reversal symmetry is broken (either spontaneously or externally). This contrasts with the case of the polar phonons, which are coupled to the phase oscillations through the dynamical electric fields (due to the Josephson relation) regardless of the presence of the equilibrium currents. In this respect, we limit our consideration by the sound and the non-polar optical phonons, so that a mere presence of the linear coupling between the phase and the non-polar phonons is indicative of the stationary Josephson currents (we do not consider the case when no inversion symmetry exists). We believe that predictions following from our analysis provide an opportunity for experimental testing of the dynamical properties of the JJ barrier in the high-$T_c$ materials, and for answering fundamental questions about the nature of the Josephson effect in the high-$T_c$ materials. Furthermore, the discussed effect could be employed for the imaging of the stationary Josephson currents. This, in its turn, may shed light on the nature of the spontaneous Josephson currents and the TRSB in these materials.

In Sec.II, we show how the linear coupling between the Josephson phase and the non-polar phonons is induced by the stationary Josephson currents. Then, the implications of this coupling are considered. Specifically, in Sec.III, the effect of the conversion of the non-polar phonons into the phase oscillations are discussed. In Sec. IV, we consider the induced infra-red (IR) activity of the non-polar optical phonons in the JJ. Then, in Sec. V, the effect of the coherent generation of sound in the JJ by the incoming microwave radiation is analyzed, and the possibility of employing this effect for the imaging of the currents is addressed. The role of external magnetic field in enhancing the interconversion is discussed in Sec. VI. Finally, the discussion and conclusion are given.

II. COUPLING BETWEEN THE JOSEPHSON PHASE AND NON-POLAR PHONONS

The Josephson current density $J(\varphi)$ as a function of the phase difference $\varphi$ can be sensitive to atomic displacements in the barrier. Let us assume that $\xi(x,t)$ is some phonon variable. It can be the strain tensor $u_{ij}$, or some other variable describing optical excitations. If this variable does not correspond to a polar (IR-active) excitation, a direct coupling between $\xi$ and the phase oscillations is forbidden. In what follows, we will show how this non-polar phonon can become IR-active in the presence of the stationary Josephson currents, so that monitoring the IR absorption at the frequency of the crystal mode, which is known to be not IR-active in the absence of the static currents, would provide some information about these currents.

In our approach, the Josephson equation for the phase $\varphi(x,t)$ contains terms $\sim \xi(x,t)$. If $\xi$, which modifies somehow the tunneling matrix element between the sides of the JJ, is small, the Josephson coupling energy density (per unit area of the junction) $E_J = E_J(\varphi,\xi)$ can be expanded in $\xi$. The symmetry consideration may require that no linear term $\sim \xi$ is present in the ideal lattice. This is exactly the case for, e.g., the non-polar vector and
quadru-polar modes. If, however, $\xi$ is the strain, the scalar combination $\xi = u_{ij}$ (we employ the convention of the summation over the repeated indices) is allowed in the expansion. It should also be taken into account that the crystal structure close to the GB and at the GB can be highly distorted. For example, some stationary strain $u_{ij}^{(0)}$ is inevitably present in any GB junction just to ensure matching of the sides. This, relaxes the symmetry requirement, so that the linear term becomes allowed. Thus, the expansion has a form

$$E_J = E_J^{(0)}(\varphi)[1 + b\xi(x,t) + o(\xi^2)],$$

where $E_J^{(0)}(\varphi)$ denotes the Josephson energy density as a function of the phase in the absence of $\xi$; the coefficient $b$ is taken as some constant. In fact, the observations \cite{1} of the very strong sensitivity of the critical current to the misorientation angle, which could be due to the local deformations arising in order to insure matching of the sides, justify the values as large as $|b| \approx 10^2$. Indeed, it has been suggested that the deformations $u_0$ as small as 1% may cause the reduction of the current by almost an order of magnitude \cite{1}. Thus, we take $|b| \approx 1/|u_0| \approx 10^2$. As we mentioned above, the optical deformations of the lattice may produce a similar effect. In this case, the estimate for $b$ can be obtained from the following considerations: optical deformation $\xi$ of 1% of the lattice cell (of the size $a \approx 0.4\text{nm}$) may produce the same effect as the strain, thus $|b| \approx 10^2/a$. It should also be noted that the form \cite{1} implies that the phonons do not modify the phase dependence of the JJ energy.

The Josephson current, then, follows from eq.\cite{1} as \cite{16}

$$J(\varphi, \xi) = \frac{2e}{\hbar} \frac{\partial E_J}{\partial \varphi} = J_0(\varphi)[1 + b\xi(x,t) + o(\xi^2)],$$

where $J_0(\varphi) = \left(2e/\hbar\right)\partial E_J^{(0)}/\partial \varphi$ is the Josephson current in the absence of $\xi$, with $e > 0$ being the unit charge.

The total Josephson energy of the system is obtained as the integral $H_J = \int d^2x E_J$ over the area of the JJ. The energy $H_\xi$ of the field $\xi$ can be represented in the spirit of a general Landau expansion with respect to the smallness of $\xi$ as

$$H_\xi = \int d^3x \left[ \frac{\rho \xi^2}{2} + \frac{\rho \omega_0^2}{2} \xi^2 + \frac{\rho v_g^2}{2} (\nabla \xi)^2 + o(\xi^4) \right],$$

where $\rho$, $\omega_0$, $v_g$ are phenomenological constants, so that $\rho$ carries the meaning of some effective mass density. Accordingly, $\omega_0$ and $v_g$ determine the dispersion law of small oscillations of $\xi$.

The total energy $H$ is the sum of $H_\xi + H_J$ and of the electro-magnetic energy $H_{EM} = \int d^2x [CV^2/2 + \kappa(\nabla \varphi)^2/2]$ of the JJ \cite{14}, where $C$ stands for the electric capacitance of the JJ per unit area; $\kappa = \hbar^2 c^2/(16\pi de^2)$, with $c$, $d$ standing for the speed of light, and the magnetic thickness of the JJ (given as the sum of the geometrical thickness $d_h$ of the barrier and of the twice of the London penetration length \cite{16}), respectively. We take into account the Josephson relation

$$V = \frac{\hbar}{2e} \dot{\varphi}$$

between $\varphi(x,t)$ and the voltage $V = V(x,t)$ across the JJ, and find

$$H = H_\xi + H_{EM} + H_J = \int d^3x \left[ \frac{\rho \xi^2}{2} + \frac{\rho \omega_0^2}{2} \xi^2 + \frac{\rho v_g^2}{2} (\nabla \xi)^2 \right] +$$

$$+ \int d^2x \left[ \frac{C}{2} \left( \frac{\hbar}{2e} \right)^2 \dot{\varphi}^2 + \frac{\kappa}{2} (\nabla \varphi)^2 + E_J^{(0)}(\varphi)[1 + b\xi] \right],$$

where the integrations $\int d^3x$ and $\int d^2x$... are performed over the bulk and over the JJ barrier area, respectively. The above form yields equations of motion of the joint dynamics of $\varphi$ and $\xi$. In this paper we will concentrate on the effect of small oscillations.

The Josephson equation following from eq.\cite{1} is

$$\ddot{\varphi} + \gamma \dot{\varphi} - \tau^2 \nabla^2 \varphi + \nu J_0(\varphi)[1 + b\xi] = 0, \quad \nu = \frac{2e}{\hbar C},$$

where all the variables are taken at the JJ plane; $\tau = c/\sqrt{4\pi dC}$ stands for the effective speed of light in the JJ; the term $\gamma \dot{\varphi}$ describes the effect of dissipation in the resistively shunted JJ \cite{16}, and $\gamma = 1/(\rho_s C)$, with $\rho_s^{-1}$ being the shunt conductance per unit area of the JJ.
The equation for $\xi$ also follows from eq. (3) as

$$\dot{\xi} + \omega_0^2 \xi - \nu_g^2 \nabla^2 \xi + \frac{b}{\rho} E_j^{(0)} (\varphi) \delta(z) = 0,$$  

(7)

where we have specified the JJ barrier as the plane $z = 0$. Note that, while the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in eq. (3) acts along the JJ plane, the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}$ in eq. (7) acts in the bulk.

Equilibrium currents correspond to a non-trivial static solution of eqs. (3), (7) which in the static case become

$$-\nu_g^2 \nabla^2 \varphi_0 + \nu J_0(\varphi_0)(1 + b \xi_0(z = 0)) = 0,$$  

(8)

$$\omega_0^2 \xi_0 - \nu_g^2 \nabla^2 \xi_0 + \frac{b}{\rho} E_j^{(0)} (\varphi_0) \delta(z) = 0,$$  

(9)

where $\xi_0(z = 0)$ stands for the solution of eq. (3) taken at the JJ plane $z = 0$. In the long wave limit, the solution of eq. (3) can easily be found and substituted into eq. (7), so that it acquires the form

$$-\nu_g^2 \nabla^2 \varphi_0 + \nu J_0(\varphi_0)[1 - g E_j^{(0)} (\varphi_0)] = 0, \quad g = \frac{b^2}{2 \rho |\omega_0 \nu_g|}.$$  

(10)

It can now be seen that the interaction with phonons modifies the equilibrium Josephson current, so that the effective current becomes $\tilde{J}(\varphi) = J_0(\varphi)[1 - g E_j^{(0)} (\varphi)]$. The resulting current may significantly deviate from $J_0(\varphi)$, if the coefficient $g$ is large enough. This may occur if $b$ is anomalously large, or the phonon mode exhibits soft-mode behavior. In this paper we will not consider such possibilities, and therefore it should be clearly shown that the dynamical effect under consideration does not imply strong deviations of the current-phase relation (CPR) from the bare one $J_0(\varphi)$, even though the values of $b$ are taken large as discussed above (see below eq. (11)). To this end, we assume, for a while, that the current obeys the standard CPR $J_0 = J_c \sin \varphi$. Then, the effective current becomes $J = J_c (1 - gh J_c/(2e)) \sin \varphi + (gh J_c^2/(4e)) \sin 2 \varphi$. We require that the second term, which is $\sim 2 \varphi$, is much smaller than the main one $\sim \sin \varphi$, that is,

$$\tilde{g} = \frac{gh J_c}{4e} = \frac{b^2 h J_c}{8 \rho |\omega_0 \nu_g|} \ll 1.$$  

(11)

Later we will show that for typical values of the parameters, $\tilde{g} \approx 10^{-2} - 10^{-3}$, so that eq. (7) as well as the term $\sim \xi_0$ in eq. (8) can be safely ignored.

Here we do not specify mechanisms which may result in deviations of $J_0(\varphi)$ from the standard CPR, and, as a possible consequence, in spontaneous Josephson currents (see, e.g., [6,17]). Our goal here is studying implications of the presence of such currents on the process under consideration. In fact, the equilibrium currents may occur due to some trapped integer vortices or external magnetic field as well.

Below we will consider coupling between small oscillations of $\varphi$, which constitute small deviations from the equilibrium, and $\xi$. We introduce small deviations $\psi(x, t)$ of $\varphi$ from $\varphi_0$ as

$$\varphi(x, t) = \varphi_0(x) + \psi(x, t).$$  

(12)

The variable $\xi$ should also be represented as the sum of the static solution $\xi_0$ of eq. (3) and the oscillatory part. We will, however, ignore the static part, provided eq. (11) holds.

The form (12) can be employed in eq. (3) in order to obtain a linearized equation for $\psi$. Before, however, we will show explicitly that, as long as the spontaneous currents are present, the variables $\psi$ and $\xi$ are coupled linearly. Indeed, substituting (12) into eq. (3) and selecting the lowest order terms describing the interaction between $\psi$ and $\xi$, we find the interaction energy as

$$H_{int} = \int d^2 x \frac{h b}{2 e} J_0(\varphi_0) \psi \xi,$$  

(13)

where $\xi$ is taken at the JJ plane, and the definition of $J_0(\varphi)$ (see below eq. (2)) has been employed. Eq. (13) indicates that the spontaneous (equilibrium) currents $J_0(\varphi_0)$ induce the first order coupling between small oscillations $\psi$ and $\xi$.

It is worth noting that, the form (13) of the coupling between the Josephson phase oscillations $\psi$ and a phonon variable $\xi$ is the lowest possible coupling as long as $\xi$ describes a non-polar (IR-inactive) mode. The situation
is very much different in the case of the IR-active mode. Indeed, the IR-active phonons induce electric fields which interact with the Josephson phase directly as indicated by the Josephson relation \( \xi' \). Such a coupling will lead to the dissipation of the Josephson plasma oscillations due to transferring its energy to the IR-active phonons. This effect has been studied quite well \[18\], and it may serve as a reference for assessing the significance of the IR-absorption in the case of the non-polar phonons. In order to make a proper comparison, we introduce some polar mode. It is a vector displacement field \( \xi'(x, t) \) in the barrier, which corresponds to the electric polarization density \( ea^{-3} \xi' \) (taken per unit cell of the volume \( a^3 \)). Then, the interaction energy between the a.c. voltage \( V \) and \( \xi'(x, t) \) is simply the polarization energy

\[
H'_{\text{int}} = -\frac{e}{a^3} \int d^3 x E_z(x, y, t) \xi'_z(x, y, z, t),
\]

where \( E_z \) is the electric field in the barrier, which is taken to have only one component along the normal to the barrier and is assumed to be uniform along this direction. Then, one finds \( E_z = V/d_b \), and \( E_z \) can be related to the phase oscillations by means of the Josephson relation \( \xi' \). The integration is carried out inside the insulating barrier of the width \( d_b \). Thus, assuming that the barrier is thin, so that \( \xi'_z \) can be taken uniform along the \( z \)-axis inside the barrier, we find eq. (13) as

\[
H'_{\text{int}} = -\frac{\hbar}{2a^3} \int dx dy \psi(x, y, t) \xi'_z(x, y, t).
\]

Thus, the polar mode interacts directly with the phase oscillations regardless of the presence of the stationary currents. Later we will employ the form \( \xi' \) as a reference for assessing the strength of the IR-absorption arising in the case of the non-polar mode described by eq. (13).

We consider two types of processes initiated by the term (13): i) oscillations \( \xi \) can excite the JJ plasma waves \( \psi \) at the same frequency, in accordance with the spatial profile of \( J_0(\phi_0(x)) \); ii) the incoming electromagnetic radiation excites the phase oscillations which convert into the oscillations \( \xi \) directly at the same frequency. This constitutes the IR-absorption by the originally IR-inactive mode. The intensity of such an absorption is determined by \( J_0(\phi_0(x)) \). Accordingly, the stationary current distribution \( J_0(\phi_0(x)) \) can be inferred.

**III. DIRECT CONVERSION OF NON-POLAR PHONONS INTO THE JOSEPHSON PLASMA OSCILLATIONS**

In this section, we consider the process i). To be specific, we choose \( \xi(x, y, z, t) \) as the plane wave \( A_\omega \exp[i(\omega t - kx)] + c.c., \) characterized by the frequency \( \omega \) and by the wave vector \( \mathbf{k} = (k_x, k_y, k_z) \) as well as by some constant amplitude \( A_\omega \). The junction plane is taken at \( z = 0 \), and the insulating layer (the barrier) is taken as being of zero thickness. This approximation can be justified, if the actual barrier thickness \( d_b \) (which is about few nm) is much smaller than \( 1/k_z \), so that \( \xi \) does not vary significantly inside the barrier along the normal. Hence, in what follows we will employ the form

\[
\xi = A_\omega \exp[i(\omega t - k_x x - k_y y)] + c.c.,
\]

at the location of the barrier in eqs. (6),(13). Performing, then, the linearization of eq. (6), we obtain

\[
(-\Omega^2 + \omega^2(x)) \psi_\omega(x) - \tau^2 \nabla^2 \psi_\omega(x) + b \nu J_0(\phi_0(x)) A_\omega e^{-i(k_x x + k_y y)} = 0,
\]

where we have introduced the notation \( \psi_\omega(x) \) for the time independent amplitude of \( \psi(x, t) = \psi_\omega(x) e^{i\omega t} + c.c.; \)

\[
\Omega^2 = \omega^2 - i\gamma \omega; \quad \text{and} \quad \omega^2(x) = \nu \frac{\partial J_0(\phi_0)}{\partial \phi_0}
\]

is the square of the local Josephson plasma frequency.

Eq. (17) describes the electromagnetic linear response of the Josephson phase on the phonon field \( \xi' \). It is important to realize that the resulting a.c. Josephson voltage \( \xi' \) turns out to be given by the equilibrium Josephson currents \( J_0(\phi_0) \). This feature can be employed for the imaging of such currents. It should, however, be noted that the non-uniformity of \( \omega^2(x) \) \[18\] complicates solving eq. (17). In order to avoid this problem, we will consider the case
Taking such a limit, on one hand, allows omitting the term $\sim \omega_J^2$ from eq.(17), which simplifies the solution significantly. On the other hand, eq. (19) implies that the magnitude of the produced voltage is suppressed. A solution of eq. (17) for the uniform a.c. voltage $V(t) = V_\omega \exp(i\omega t) + c.c.$, where $V(t) = \hbar/(2eS) \int d^2x \psi$ and the integration is performed over the whole area $S = LW$ of the junction, with $L$ and $W$ standing for the length (along $x$) and the width (along $y$) of the junction, respectively, is

$$V_\omega = Z(\omega)J_{0k}, \quad J_{0k} = \int d^2x J_0(\varphi_0(x)) e^{-i(k_x x + k_y y)} = \frac{ib\omega A_\omega}{LWc_i\Omega^2},$$

(20)

where $J_{0k}$ stands for the 2D Fourier component of $J_0(\varphi_0)$ along the GB; an explicit expression for $\nu$ (see eq.(4)) has been employed. The solution (20) shows that the uniform a.c. voltage induced by the non-polar phonons incident on the JJ is proportional to the spatial Fourier component of the static Josephson current $J_0(\varphi_0(x))$ at the wave vector $(k_x, k_y)$ (along the junction plane) of the incident phonon field.

Let us estimate magnitude of the voltage (20) in the non-resonant regime, that is, when the condition (19) is satisfied. A typical extension of the current structure in the JJ is given by the Josephson penetration length $\xi = \sqrt{\hbar c^2/(8\pi e J_c)} \approx 10 \mu m$ for the critical currents $J_c \approx 10^3 A/cm^2$ and $d = 3 \cdot 10^{-5} cm$. Thus, the projection of the phonon wave vector along the junction plane (the $k_{x,y}$ components) should be $|k| \approx L^{-1}_J \approx 10^4 cm^{-1}$ in order to produce the most efficient imaging. In this case, $J_{0k} \approx J_L d J_W$ for $W \ll L_J$ and $L \gg L_J$. Then, we choose $bA_\omega = 10^{-2}$ which corresponds to the maximum relative displacement (deformation) about $10^{-4}$. We also employ the values $\tau/c \approx 10^{-2}$ and $\hbar/c \approx 10^{12} s^{-1}$. In order to satisfy the condition (19), we choose $\Omega^2 \approx 10 \omega_J^2$ which corresponds to

$$|\Omega| = \sqrt{|\omega^2 - i\gamma \omega|} \approx 3 \omega_J.$$

(21)

When dissipation is negligible (large shunt resistance, so that $\gamma \ll \omega_J$ in eq.(6)), the condition (21) can be satisfied by $\omega \gtrsim 10^{14} Hz$. While being in the range of the optical frequencies, so high $\omega$ is hardly achievable for sound. If, however, $J_c$ is in the range 1 - 10 A/cm$^2$, the plasma frequency becomes $\omega_J \approx 10^9 - 10^{10} s^{-1}$, and the condition (21) can easily be fulfilled for sound. It should also be noted that a less demanding requirement for the sound can be imposed for the over-damped JJ, where $\gamma \gg \omega_J$ in eq.(6). In this case, already $\omega_J^2/\gamma \ll \omega_J$ fulfills the condition (21). This allows one to choose $\omega$ which is much less than the above values. For example, for $\gamma = 10^{14} s^{-1}$ and $\omega_J = 10^{12} s^{-1}$, the condition (21) is satisfied by $\omega = 100 GHz$, which is on the upper limit for the hypersound techniques (8). Finally, setting the length of the junction $L \approx 10 L_J$, we obtain the value $V_\omega \approx 10^{-8} V$ in eq.(20)

Let us also justify the validity of the approximation neglecting eq.(11). As discussed above, taking eq.(11) into account may result in significant deviations of the CPR from the standard one. This, however, does not occur if the condition (11) holds. Employing the above values in eq.(11) and choosing $\omega_0 \approx \omega = 10^{12} s^{-1}$ and $v_g = 10^5 cm/s$, $\rho \approx 6 g/cm^3$, we find $\tilde{\gamma} \approx 3 \cdot 10^{-3} \ll 1$ as mentioned above.

### IV. IR-ABSORPTION INDUCED BY THE STATIC JOSEPHSON CURRENTS

Now let us consider the case ii), that is, the possibility of the coherent phonon field generation as a result of applying a uniform a.c. voltage $V_\omega \exp(i\omega t) + c.c.$ across the barrier. This voltage corresponds to the phase oscillations

$$\psi(t) = \frac{2eV_\omega}{ih\omega} e^{i\omega t} + c.c.,$$

(22)

following from eq.(4). These phase oscillations serve as a source (in eq.(6)) for the oscillations of $\xi$ in the barrier, which are subsequently emitted into the bulk. In order to describe this effect, we employ eqs.(6), (12), and retain only the part $\sim \psi$ in the last term of eq.(6). Accordingly, eq.(6) becomes

$$\ddot{\xi} + \omega_0^2 \xi - v_g^2 \nabla^2 \xi + \left( \frac{bV_\omega J_0(\varphi_0)}{i\rho \omega} e^{i\omega t} + c.c. \right) \delta(z) = 0,$$

(23)

where eq.(22) has been employed. Eq.(23) describes the linear response of $\xi$ on the external a.c. voltage. This can be interpreted as though the displacement field $\xi$ has become polar, so that it exhibits the IR-activity inside the JJ barrier. As a result, an additional dissipation of the external radiation occurs. The total power $P(\omega)$ carried
away by the phonons, which are generated in the barrier and, then, are emitted into the bulk, can be calculated by means of solving eq. (23) and substituting the result into the expression for the phonon energy flux density \(-\rho v_g^2 \xi \nabla \xi\). Averaging this flux over time and integrating over the JJ area, we find

\[
P(\omega) = R^{-1}(\omega) |V_\omega|^2, \quad R^{-1}(\omega) = \frac{b^2}{|\omega|^2g^2} \int \frac{d^2q}{(2\pi)^2} \left( \int d^2x e^{iqx} J_0(\varphi_0) \right)^2, \quad (24)
\]

where \(\omega > \omega_0\) (for \(v_g^2 > 0\)); and the integration is performed over the range \(|q| < \sqrt{v_g^{-2}(\omega^2 - \omega_0^2)}\). The above result indicates that the frequency dependence of the absorbed power is determined by the spatial profile of the stationary current distribution. Thus, measuring the IR-absorption can be employed for obtaining information about \(J_0(\varphi_0)\).

The quantity \(R^{-1}(\omega)\) carries an obvious meaning of the a.c. conductance due to the phonons.

It is useful to compare the absorption intensity \(24\), caused by the emission of the non-polar phonons, with the intensity in the case when the phonons are polar, and the coupling is represented by eq. (15). Such a comparison helps evaluating a significance of the effect under consideration. This is especially important because the technique of the IR-absorption in the JJ is well established \(18\), and therefore can be employed for detecting the effect described above. Thus, we assume that the displacement fields in the both cases are the same (\(\xi = \xi'\)). In the case \(13\), the last term in eq. (23) should be replaced by \(\epsilon V/(\rho a^3)\delta(z)\). The corresponding emission intensity \(P'(\omega)\) can be calculated similarly to how eq. (24) has been obtained. Comparing it with (24), we find

\[
P(\omega) \approx \left( \frac{b J_0 a^3}{\epsilon \omega} \right)^2, \quad (25)
\]

where we have estimated the integrals in eq. (24) as \(\int d^2q \int d^2x J_0(\omega \xi) \approx L_J W |J_0|^2\), and has selected the emission area, from which \(P'(\omega)\) is collected, as \(L_J W\). Obviously, in the case represented by eq. (24), \(L_J W\) is the effective area occupied by \(J_0(\varphi_0)\). Taking \(\omega = 10^{12}\) s\(^{-1}\), \(a = 0.4\) nm, \(b = 10^2/a\) and \(J_c = 10^6\) A/cm\(^2\), we find the ratio \((\omega)\approx 10^{-2}\). For \(\omega = 10^{11}\) s\(^{-1}\) this ratio becomes close to 1. Thus, the IR absorption per unit area occupied by the equilibrium Josephson currents can be comparable to the absorption due to the IR-active mode. This conclusion makes detecting the effect described above by means of employing the technique \(18\) quite feasible.

V. IMAGING OF THE STATIONARY JOSEPHSON CURRENTS BY SOUND

The effect discussed in Sec.III gives a principal possibility of imaging the currents \(J_0(\varphi_0(x))\) by means of detecting uniform component of the a.c. voltage produced as a result of subjecting the barrier to a coherent time dependent field \(\xi\). In this case, eq. (24) relates such a voltage to a single spatial Fourier component of \(J_0(\varphi_0(x))\). The drawback of this is that, in order to keep such a relation valid, the frequency of the incoming phonon field should be high enough. As we will discuss below, this limitation can be circumvented by employing the reverse effect discussed in Sec.IV, that is, a coherent emission of the non-polar phonons induced by external electro-magnetic radiation. In this case, measuring the phonon field \(\xi\) (in, e.g., far zone) which obeys eq. (24) would allow to achieve the same goal. We will discuss this in detail for sound, so that detecting a dynamical mechanical stress in the far zone could be employed for the imaging of \(J_0(\varphi_0(x))\).

In what follows the role of \(\xi\) is played by the strain tensor \(u_{ij}\). We employ a simplest model of the dilatation interaction between \(u_{ij}\) and the Josephson currents, so that we take \(\xi = u_{ii}\), and describe the dynamics in the bulk within the isotropic medium approximation. Thus, the energy explicitly dependent on \(u_{ii}\) is

\[
H_d\left(\int d^3x \left[ \frac{u_{ii}^2}{2} + \lambda \frac{u_{ii}^2}{2} + \mu u_{ij}^2 + \eta \psi u_{ii} \right] - \frac{\hbar b J_0(\varphi_0)}{2e} \delta(z) \right), \quad (26)
\]

where \(\lambda, \mu\) are the Lame coefficients; \(\rho\) stands for the total density; the source term \(~\eta\) is due to \(H_{int} \quad (13)\)

Let us assume that small uniform phase oscillations \(\psi(t)\) \(22\) are imposed externally by the incoming microwave radiation \(22\). We represent the deformation field as \(u_i = \nabla_i \psi\), which determines \(u_{ij} = (\nabla_i u_j + \nabla_j u_i)/2\), with \(u\) being some scalar field. We note that the transverse part of the deformations is decoupled in this simplified approach. Then, the equation for \(u\), following from eq. (26), is

\[
\ddot{u} - \nu_s^2 \nabla^2 u = \frac{2ce\eta}{\hbar \omega \rho} V \omega e^{i\omega t} + c.c., \quad (27)
\]
where \( v_s = \sqrt{(\lambda + 2\mu)/\rho} \) stands for the speed of the longitudinal sound; eq. (22) has been employed; and no dissipation of sound is considered. Solution of eq. (27) can easily be found in terms of the Fourier harmonics \( u = u_q e^{i\omega t - iqx} + c.c. \) as

\[
u_q = \frac{2\pi V_\omega}{i\hbar \omega (v_s^2 q^2 - \omega^2)} y_q,
\]

where \( y_q = \int d^3x \exp(iqx) \eta(x) \) denotes the 3D Fourier harmonic of \( \eta \) in eq. (26), and \( V_\omega \) is taken uniform along the junction. As represented, eq. (28) is valid for an arbitrary configuration of the source \( \eta \). Thus, it is not limited by the specific form (24) corresponding to the plane junction.

Let us find the \( \omega \)-harmonic of the pressure field \( \sigma = \rho v_s^2 u_{ii} = \rho v_s^2 \nabla^2 u \) far from the junction in the effective 2D geometry. That is, we assume uniformity of all the quantities in the \( y \) direction corresponding to the thickness \( W \) of a slab containing the JJ plane \( z = 0 \). The other two directions \( x, z \) are unlimited. The location of the spontaneous currents determined by the solution of eq. (8) is restricted to some region of a typical size \( L_J \) (along \( x \)) and \( W \) (along \( y \)). The distance \( r \) (along \( x, z \) directions) at which the stress is detected is considered to be much larger than \( L_J \) and \( W \). Performing the inverse Fourier transform of eq. (28) in the far zone, we find the \( \omega \)-harmonic of the pressure \( \sigma(x, z, t) = \sigma_\omega(r, \theta) \exp(i\omega t) + c.c. \) as

\[
\sigma_\omega(r, \theta) = \frac{\epsilon_\omega^{3/2} V_\omega \sqrt{e^{-iqr}}}{\sqrt{r}} y_q,
\]

where the notation \( q_\omega = \omega/v_s \) has been introduced, and \( q = q_\omega n \), with \( n \) standing for the unit vector in the direction of observation from the location of the currents, that is, \( r = (x, z) \) and \( r = r_n \), with \( n_x = \sin \theta, \ n_z = \cos \theta \) given by the angle of observation \( \theta \) with respect to the axis \( z \). Eq. (29) can be employed for imaging of the spontaneous currents by means of detecting the mechanical pressure (stress) at the frequency of the external a.c. voltage far from the junction.

For the purpose of estimating the magnitude of the stress, we assume that the equilibrium Josephson current is uniform in the \( y \)-direction. Then, after employing the explicit expression for \( \eta \) (26) in eq. (28), we find

\[
\sigma_\omega(r, \theta) = \frac{b q_\omega^{3/2} V_\omega \sqrt{e^{-iqr}}}{2\sqrt{2\pi \omega \rho v_s}} \int dx e^{i(q_\omega \sin \theta)x} J_0(\Phi_0(x)). \tag{30}
\]

This expression shows that, if the stress amplitude in the far zone is known for a sufficient range of the angles of observation \( \theta \) and frequencies \( \omega \), the stationary (spontaneous) Josephson current distribution can be restored by means of the spatial inverse Fourier transform. Later we will estimate the expected values of \( \sigma \).

Now let us obtain an expression for the total electric power dissipated due to the emission of sound (see eq. (24) for optical phonons). As we will see, this power demonstrates a universal behavior \( \sim \omega^3 \) as \( \omega \to 0 \) as long as the current distribution \( J_0 \) is characterized by some finite vorticity.

The dissipated power \( P(\omega) \) equals to the total flux of energy

\[
P(\omega) = -r W \int d\mathbf{n} \sigma \nabla \mathbf{u}
\]

carried away by the sound. Here the surface integral of the energy flux density is taken over the cylinder surface of the radius \( r \gg L_J \) and of the height \( W \). A substitution of eqs. (28), (30) yields

\[
P(\omega) = R^{-1}(\omega)|V_\omega|^2, \quad R^{-1}(\omega) = \frac{Wb^2 \omega}{4\pi \rho v_s^3} \int d\theta \left| \int dx e^{i(q_\omega \sin \theta)x} J_0(\Phi_0(x)) \right|^2. \tag{32}
\]

It is important to note that the frequency dependence of \( R^{-1}(\omega) \) turns out to be sensitive to the net vorticity. Indeed, let us consider the case when the JJ contains some vortex characterized by the phase variation \( \Phi_0(x) \) along the junction, so that the phase difference on \( x = \pm \infty \) is \( \delta \Phi_0 \neq 0 \). Then, taking into account eq. (8) in eq. (22), we obtain

\[
R^{-1}(\omega) = R_0^{-1} \omega^3, \quad R_0^{-1} = \frac{\hbar^2 e c b W |\delta \Phi_0|^2}{256 \pi^2 \rho^2 v_s^2 d^2 \rho}, \tag{33}
\]

where the wavelength \( 2\pi/q \) of the emitted sound is taken longer than the Josephson length \( L_J \) (along which the variation of the phase typically takes place), so that the Fourier transform of \( J_0 \) was taken as
\[
J_{0q} = \frac{c^2}{\nu} i_\omega \sin \theta \int_{-\infty}^{\infty} dx e^{-i(q_\omega \sin \theta)x} \frac{\partial \phi_0(x)}{\partial x} \approx \frac{c^2}{\nu} i_\omega \sin \theta \delta \phi_0 
\]

in eq. (22), and we have employed the explicit expression for \( \nu \) in eq. (1). Thus, the emission of sound induces the specific contribution \( R^{-1}(\omega) \sim \omega^3 (\delta \phi_0)^2 \) to the a.c. conductance of the JJ in the limit \( \omega \to 0 \). We note that in the absence of the vorticity \( \delta \phi = 0 \), eq. (22) yields \( J_{0q} \approx q^2 \), and eq. (23) will change to become \( R^{-1}(\omega) \sim \omega^5 \). This result indicates that a presence of the spontaneous JJ currents characterized by finite vorticity could be revealed in the frequency dependence (33), unless, of course, the low frequency quasiparticle effects dominate the JJ a.c. conductance.

Let us estimate a possible magnitude of the effect (33). For this purpose we choose the following parameters: \( \delta \phi_0 = 2\pi; W = 10^{-3}\text{cm}; v_s \approx 4 \times 10^6 \text{cm/s}; \rho = 6g/\text{cm}^3; d = 3 \times 10^{-8}\text{cm}; b = 10^2. \) This yields \( R^{-1} = 3 \times 10^{-5} \Omega^{-1}\omega^{-1}(\omega/\text{GHz})^3 \). It is important to note that the dependence (33) saturates at some value \( R^{-1}_{\text{max}} \) at frequencies larger than a frequency \( \omega_{\text{max}} \) corresponding to the wavelength of sound comparable to the Josephson length \( L_J \). For \( L_J = 1\mu\text{m} \), we find \( \omega_{\text{max}} \approx v_s/L_J = 4 \times 10^9 \text{Hz}. \) Accordingly, \( R^{-1}_{\text{max}} \approx 2 \times 10^{-3} \Omega^{-1}. \) This corresponds to \( R^{-1}_{\text{max}}/(WL_J) \approx 2 \times 10^4 \Omega^{-1}\text{cm}^{-2} \) of the inverse effective shunt resistance per unit area of the JJ.

It is worth noting that measurements of \( R^{-1}(\omega) \) do not allow restoring completely a spatial profile of the currents. As eq. (32) indicates, \( R^{-1}(\omega) \) is the quantity which is the average over the directions. Furthermore, the quasiparticle effects may dominate the a.c. conductance. In order to be able to restore \( J_0(\phi_0(x)) \), the mechanical stress amplitude (34) should be determined as a function of \( \theta \) and \( \omega \). Let us estimate its magnitude at distances \( r = 10^3 L_J \) far from the location of the currents and for \( q \approx L_J^{-1}. \) The voltage amplitude in eq. (30) should be chosen in such a way that the linearized approach is valid. Thus, \( eV_\omega \leq \hbar \omega. \) Then, the Fourier component of the current \( J_{0q} \) in eq. (30) can be evaluated as \( J_{0LJ}, \) where a typical value of the critical current can be taken as \( J_c \approx 10^5 \text{A/cm}^2. \) Then, eq. (30) yields \( |\sigma_\omega| \approx 0.1(\omega/\text{GHz})^{3/2}\text{Pa for } L_J = 1 \mu\text{m}. \)

VI. ENHANCEMENT OF THE INTERCONVERSION BY EXTERNAL MAGNETIC FIELD

It is worth noting that the discussed effects can exhibit a resonant-like behavior with respect to applied external magnetic field \( H_y \) (along the JJ width \( W \)). Indeed, an external field greater than the lower critical field \( H_{c1} = \Phi_0/(L_J d) \) (4), where \( \Phi_0 = \hbar c/2e \) is the magnetic flux quantum, leads to the periodic (along the junction length) structure of the Josephson currents due to the Josephson vortices. This periodic structure can contribute constructively to the induced voltage (20) and to the stress field (34), if the matching conditions are fulfilled. As a result, a significant enhancement of the corresponding resistances should be anticipated. Let discuss this.

In the presence of \( H_y > H_{c1}, \) the Josephson current will change its sign on the length \( L_H \approx \Phi_0/(H_y d) < L_J \) (16). Thus, \( L_H \) describes the spatial period of the main harmonic of the current. Accordingly, as follows from eqs. (23), the induced voltage will exhibit the resonant increase when the wave vector \( k_\parallel \) (along the JJ barrier ) of the incoming deformation field becomes equal to \( 2\pi/L_H, \) that is, for

\[
H_y \approx \frac{\Phi_0 k_\parallel}{2\pi d}.
\]

In this case, provided the vortex lattice is ideal, the Fourier component of the current becomes \( J_{0k} \approx J_cWL \) in eq. (24). This implies the factor of \( L/L_J \gg 1 \) increase of the voltage magnitude if compared with the case of a single vortex considered in Sec.III.

A similar situation occurs for the mechanical stress (30) generated by the incoming a.c. voltage. This stress will exhibit a resonant increase at certain angle of observation \( \theta \) given by the matching condition \( 2\pi/L_H = (\omega \sin \theta)/v_s. \) Thus, for given frequency \( \omega \) of the external radiation and the angle \( \theta, \) the stress amplitude will resonantly increase at the value

\[
H_y \approx \frac{\Phi_0 \omega \sin \theta}{2\pi v_s d}.
\]

For long JJ (16), we find from eq. (31) that \( \sigma_\omega \) will increase in proportion to the total length \( L \) of the JJ. This also constitutes the factors of \( L/L_J \gg 1 \) increase of the stress amplitude (30) in the given direction, if compared with the above estimates for the case of a single Josephson vortex.

VII. DISCUSSION AND CONCLUSION

The interconversion effect discussed above can, in principle, be expected to occur in traditional superconductors as well. Deformations should modulate the width \( d_b \) of the JJ barrier. Accordingly, the tunneling matrix element
will change. In the superconducting state, this leads to the term (13), where the coefficient $b$ is determined by the kinematics of the one electron tunneling. Assuming the exponential form $\sim \exp(-d_b/\lambda)$ for the matrix element, where $\lambda$ stands for a typical overlap length of the electronic states on the both sides of the JJ, the estimate $b \approx d_b/\lambda$ follows, if the dominant effect of the strain is the change $d_b \rightarrow d_b(1 + u_{zz})$. Obviously, in order to have $b$ as large as chosen above $b \approx 10^2$, the critical current $J_c \sim \exp(-d_b/\lambda)$ should be practically zero. Thus, in the traditional JJ, $b \approx 1 - 10$, which lowers the above estimates of (20) and (34) significantly. On the contrary, in the JJ made of the high-$T_c$ materials, the nature of the term (13) is expected to be completely different. The proximity of the barrier to the superconducting state should result in anomalously strong sensitivity of the barrier to any external factor which shifts its equilibrium. In our estimates of $b$ for the high-$T_c$ JJ, we have relied on the indirect evidence - the extremely strong sensitivity of the Josephson critical current to the misorientation angle in the GB JJ. As suggested in Ref. [1], the mechanical strain is the most plausible cause of this.

As discussed in Introduction, the mechanical strain (stress) may have no significant direct effect on the barrier transport properties, if its role is reduced to only controlling the depletion of oxygen at the GB during the preparation of the JJ at high temperatures. If so, in the superconducting state, the external stress will produce no substantial modulations of the critical current, and the value of $b$ will not be much different from the estimate for the traditional JJ. Thus, no the strong interconversion discussed in this paper is to be observed under such circumstances. Accordingly, either observing or not observing the discussed effect will help elucidate the role of the mechanical strain in suppressing the Josephson current. If the effect is observed, this will be a very strong indication favoring the concept of proximity of the barrier to the superconducting state.

If being strong enough, the dynamical effect proposed above, opens up a possibility for the high resolution imaging of the equilibrium Josephson currents in the high-$T_c$ materials. This is especially important for revealing the nature of the superconducting OP at the GB, and the mechanisms of the TRSB. The resolution is limited by the wavelength of the sound. Thus, for the frequencies higher than 10GHz, the resolution becomes better than $1\mu m$.

In summary, we have suggested that in the Josephson junction, in which the critical current is sensitive to the mechanical (optical) deformations, a direct conversion of the non-polar phonons into the electromagnetic Josephson oscillations as well as the reverse process can occur, if this JJ contains spontaneous currents. We conjecture that these effects should be especially strong in the high $T_c$-materials due to the proximity of the insulating layer of the JJ to the superconducting state. We propose that the effect of the interconversion can be employed as a crucial test for these effects should be especially strong in the high $T_c$-materials due to the proximity of the insulating layer of the JJ to the superconducting state. We conjecture that these effects should be especially strong in the high $T_c$-materials due to the proximity of the insulating layer of the JJ to the superconducting state. We propose that the effect of the interconversion can be employed as a crucial test for the role of deformations in the superconducting properties of the GB JJ made of the high-$T_c$ materials. This effect can be used for imaging of the Josephson current spatial distributions.

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