Statistical duality of the Laplace distribution

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Abstract

The statistical duality of distributions is a powerful tool for statistical inferences. In the paper the statistical duality of Laplace distribution is discussed. As shown the confidence density of the parameter of this distribution is uniquely determined.

Key words: Uncertainty, Statistical duality, Measurement, Estimation
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1 Introduction

In ref.[1] is introduced the notion of statistical duality and is shown that several pairs of distributions (Poisson and Gamma, normal and normal, Cauchy and Cauchy) are statistically dual distributions. These distributions allow to exchange the parameter and the random variable, conserving the same formula for the distribution of probabilities. The interrelation between the statistically dual distributions and conjugate families is considered in ref.[2]. It allows to use the statistical duality for estimation of the distribution parameter.

In the paper we show that Laplace distributions are statistically dual distributions.

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2 Statistical duality and estimation of the parameter of distribution

The probability density of the Laplace distribution is

\[ L(x; a, b) = \frac{1}{2b} e^{-\frac{|x-a|}{b}}, \]  

(1)

where \( x \) is a real variable, \( a \) and \( b > 0 \) are real parameters. Here we can exchange the parameter \( a \) and variable \( x \) with conserving of the same formula for new density. This new density can be named as a confidence density of the parameter \( a \) (for example, [3])

\[ \tilde{L}(a; x, b) = \frac{1}{2b} e^{-\frac{|x-a|}{b}}, \]  

(2)

where \( a \) is a real variable, \( x \) and \( b > 0 \) are real parameters.

By this means the Laplace distributions are a statistically self-dual distributions [1].

Let \( \hat{x} \) be a result of single observation of random variable \( x \). If we define the confidence interval \((a_1, a_2)\) (see, also, [1,4]) as

\[ P(a_1 \leq a \leq a_2 | \hat{x}) = P(x \leq \hat{x}|a_1) - P(x \leq \hat{x}|a_2), \]  

(3)

where \( a_1 \leq a_2 \) are real values and \( P(x \leq \hat{x}|a) = \int_{-\infty}^{\hat{x}} L(x; a, b)dx \), we can reconstruct the confidence density of the parameter \( a \). This is due to the fact that the identity

\[ \int_{\hat{x}}^{\infty} L(x; a_1, b)dx + \int_{a_1}^{a_2} \tilde{L}(a; \hat{x}, b)da + \int_{-\infty}^{\hat{x}} L(x; a_2, b)dx = 1 \]  

(4)

takes place for any real values \( b > 0 \) and \( a_1 \leq a_2 \).

Let us suppose that \( \tilde{L}(a; \hat{x}, b) \) is the confidence density of parameter of the Laplace distribution if observed value of random variable \( x \) is equal to \( \hat{x} \). It is a conditional probability density. As it follows from formulae (Eqs.2,4), the \( \tilde{L}(a; \hat{x}, b) \) is the density of Laplace distribution by definition.

\(^1\) Note, in this case the given definition coincides with definition of fiducial interval [5].
On the other hand: if $\tilde{L}(a; \hat{x}, b)$ is not equal to this confidence density and the confidence density of the Laplace parameter is the other function $h(a; \hat{x}, b)$ then there takes place another identity

$$\int_{\hat{x}}^{\infty} L(x; a_1, b) dx + \int_{a_1}^{a_2} h(a; \hat{x}, b) da + \int_{-\infty}^{\hat{x}} L(x; a_2, b) dx = 1$$

(5)

This identity is correct for any real $a_1 \leq a_2$ and $\hat{x}$ too. The first and third terms in the left part of this identity determine the boundary conditions on the confidence interval.

If we subtract Eq.5 from Eq.4 then we have

$$\int_{a_1}^{a_2} \tilde{L}(a; \hat{x}, b) - h(a; \hat{x}, b) da = 0.$$  (6)

We can choose the $a_1$ and $a_2$ by the arbitrary way. Let us make this choice so that $\tilde{L}(a; \hat{x}, b)$ is not equal $h(a; \hat{x}, b)$ in the interval $(a_1, a_2)$ and, for example, $\tilde{L}(a; \hat{x}, b) > h(a; \hat{x}, b)$ and $a_2 > a_1$. In this case we have

$$\int_{a_1}^{a_2} \tilde{L}(a; \hat{x}, b) - h(a; \hat{x}, b) da > 0.$$  (7)

and we have contradiction. Hence $\tilde{L}(a; \hat{x}, b) = h(a; \hat{x}, b)$ everywhere except, may be, a finite set of points.

As a consequence, the reconstruction of the confidence density of the Laplace distribution parameter $a$ is an unique, i.e. the confidence density of the parameter $a$ is the probability density $\tilde{L}(a; \hat{x}, b)$.

So, the statistical duality allows to connect the estimation of the parameter with the measurement of the random variable of the Laplace distribution.

3 Conclusion

In the paper we show that the Laplace distribution is a statistically self-dual distributions. The confidence density of the Laplace distribution parameter is determined in case of single observation $\hat{x}$ of random variable $x$. It allows to construct the confidence intervals for parameter $a$ of the distribution by the easy way.
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References

[1] S.I. Bityukov, V.A. Taperechkina, V.V. Smirnova, Statistically dual distributions and estimation of the parameters. e-Print: math.ST/0411462, 2004.

S.I. Bityukov, N.V. Krasnikov, V.A. Taperechkina, V.V. Smirnova, Statistically dual distributions in statistical inference, in preparing.

[2] S.I. Bityukov, N.V. Krasnikov, Statistically dual distributions and conjugate families, 25th Internat. Workshop on Bayesian Inference and Maximum Entropy Methods in Science and Engineering (MaxEnt 2005), San Jose State University, San Jose CA USA Aug 7-12, 2005.

[3] B. Efron, R.A. Fisher in the 21st Century. Stat.Sci. 13 (1998) 95-122.

[4] S.I. Bityukov, N.V. Krasnikov, V.A. Taperechkina, Confidence intervals for Poisson distribution parameter. Preprint IFVE 2000-61, Protvino, 2000; e-Print: hep-ex/0108020, 2001.

[5] R.A. Fisher, Inverse probability. Proc. of the Cambridge Philosophical Society 26 (1930) 528-535.