Model-free conditional feature screening with exposure variables

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In high dimensional analysis, effects of explanatory variables on responses sometimes rely on certain exposure variables, such as time or environmental factors. In this paper, to characterize the importance of each predictor, we utilize its conditional correlation given exposure variables with the empirical distribution function of response. A model-free conditional screening method is subsequently advocated based on this idea, aiming to identify significant predictors whose effects may vary with the exposure variables. The proposed screening procedure is applicable to any model form, including that with heteroscedasticity where the variance component may also vary with exposure variables. It is also robust to extreme values or outlier. Under some mild conditions, we establish the desired sure screening and the ranking consistency properties of the screening method. The finite sample performances are illustrated by simulation studies and an application to the breast cancer dataset.

KEYWORDS AND PHRASES: Conditional screening, Feature screening, Exposure variable, Model-free, Sure screening property, Variable selection.

1. INTRODUCTION

Ultrasoundh dimensional data arise in many frontier areas, such as genetics, imaging, economics and finance. In these areas, quite often tremendous amount of explanatory variables are collected, while only a few predictors are truly important to the response. To identify these truly active predictors, a variety of variable selection methods were studied based on different models. One appealing method to select important variables and reduce the predictor dimensionality is the two-stage approach: feature screening methods are first conducted to roughly rule out the marginally unimportant predictors, and subsequent regularized regression approaches are applied to recover the final sparse models.

In the screening stage, Fan and Lv (2008) first proposed a sure independent screening procedure (SIS) based on the marginal Pearson correlation in the context of linear models. Its appealing sure screening property urges statisticians to extend the idea of SIS under different settings, including the generalized linear models (Fan and Song, 2010), semiparametric models (Li et al., 2012a), nonparametric additive models (Fan et al., 2011; He et al., 2013). Fan and Song (2010) stated that the validity of screening methods usually relies on the correct underlying-model specification, which motivates researchers to propose screening methods at a model-free basis, such as SIRS (Zhu et al., 2011), DC-SIS (Li et al., 2012b) and BCor-SIS (Pan et al., 2018). See Liu et al. (2015) for a selective overview of the screening methods.

However, the effects of predictors on the response are sometimes dependent on certain exposure variables in an unknown pattern, such as time or some environmental indices. For instance, in human genetics research, gene effects on certain phenotype, say body mass index, may rely on the current age of people. When the predictors affect the response via one or more exposure variables, the corresponding effects are often depicted by the interactions between predictors and the exposure variables in linear models, or by the nonparametric coefficient functions in varying coefficient models. In the varying coefficient model, the dependence between predictors and response can be marginally assessed by the conditional Pearson correlation given exposure variables, due to the linearity of varying coefficient models given exposure variables. Therefore, to reduce the dimensionality of such ultrahigh dimensional varying coefficient models, Liu et al. (2014) and Fan et al. (2014) studied several conditional screening methods, based on such the partial correlation and the conditional correlation learning.

In analysis of ultrahigh dimensional data, unfortunately, we are often lack of prior information on the regression structure (Zhu et al., 2011), and the aforementioned linearity can be easily violated. In addition, extreme values or outliers often take a non-negligible role when tremendous amount of data are collected, destroying the nice data structure for applying the methods to the well-designed varying coefficient models. Under some other circumstances, predictors might be responsible for the variance, rather than the mean of response and the exposure variables may also play a role in the effects on the variance component. To address these issues,
2. CONDITIONAL SURE INDEPENDENCE RANKING AND SCREENING

2.1 Some preliminaries

Suppose $Y \in \mathbb{R}^1$ is the response variable, $x = (X_1, \ldots, X_p)^T \in \mathbb{R}^p$ is the associated predictor vector and $u \in U$ is the exposure variable, where $U$ is the support of $u$. We assume $U$ to be bounded with finite constants. Given the exposure variables $u$, we define the set of active predictors without model-specification:

$A = \{k: \text{Given } u \in U, \ F(y \mid x, u) \text{ varies with } X_k \text{ for some } y \in \mathbb{R}^1\}$

where $F(y \mid x, u)$ stands for the conditional distribution function of $Y$ given $x$ and $u$. (1) indicates that the truly active predictors affect the response variable through its distribution function, which may also depend on $u$. The set of inactive predictors is denoted by $A^c$, the complementary set of $A$. A screening method aims at removing as many predictors inactive predictors $x_{A^c}$ as possible while retaining all the active predictors $x_A$. Thus, we need to adopt a reasonable metric to measure the relative importance of each predictor conditioning on the exposure variables $u$.

We briefly review the sure independent ranking and screening procedure (Zhu et al., 2011, SIRS), which identifies active predictors satisfying $F(y \mid x) = F(y \mid x_A)$ for all $y \in \mathbb{R}^1$. For easy illustration of its rationale, assume that $x$ follows standard multivariate normal distribution and each predictor is standardized. The conditional distribution $Y$ given $x$ varies with $x_A$ and stay constant with $x_A^c$. Thus it is natural to expect that $E(\partial F(y \mid X_k)/\partial X_k)$ to be non-zero for $k \in A$ and zero for $k \in A^c$. The normality assumption implies that $E(\partial F(y \mid X_k)/\partial X_k) = E(X_k Y \leq y)$, where $I(\cdot)$ is an indicator function. Thus, by defining $\rho_k(y) = E(X_k Y \leq y)$, Zhu et al. (2011) employs $E(\rho_k^2(Y))$ to rank the relative importance of predictors. The indicator function in $\rho_k(y)$ ensures the robustness of the method to extreme values and outliers.

When exposure variable $u$ is involved, however, the distribution of $Y$, as well as its association with the predictors, may vary with $u$. Under this circumstance, only considering the marginal expectations in $\rho_k(y)$ may miss important u-varying information. Instances indeed exist (Liu et al., 2014) where marginal screening procedures fail to detect those predictors with varying effects of $u$. To address this issue, we define the conditional correlation between the predictor $X_k$ for $k = 1, \ldots, p$ and the indicator function of response $Y$ conditioning on $u$ as follows.

$$\Omega_k(u, y) = \text{corr}(X_k, I(Y \leq y)\mid u)$$

$$= \frac{\text{cov}(X_k, I(Y \leq y)\mid u)}{\sqrt{\text{var}(X_k\mid u)\text{var}(I(Y \leq y)\mid u)}}$$

Then the marginal utility for screening becomes

$$\omega_k = E_y \left[ E_u \{\Omega_k(u, y)\} \right], \ k = 1, \ldots, p.$$

To estimate $\omega_k$ based on the random sample $\{(x_i, Y_i, u_i), i = 1, \ldots, n\}$, we adopt Nadaraya-Watson estimator for each conditional mean used to compute $\Omega_k(u, y)$. Specially,

$$\widehat{E}(X_k \mid u_j) = \frac{n^{-1} \sum_{i=1}^n K_h(u_i - u_j) X_{ik}}{n^{-1} \sum_{i=1}^n K_h(u_i - u_j)},$$

$$\widehat{E}(X_k^2 \mid u_j) = \frac{n^{-1} \sum_{i=1}^n K_h(u_i - u_j) X_{ik}^2}{n^{-1} \sum_{i=1}^n K_h(u_i - u_j)},$$

$$\widehat{E}\{I(Y \leq Y_i) \mid u_j\} = \frac{n^{-1} \sum_{i=1}^n K_h(u_i - u_j) I(Y_i \leq Y_i)}{n^{-1} \sum_{i=1}^n K_h(u_i - u_j)},$$

$$\widehat{E}(X_k Y \leq Y_i) \mid u_j = \frac{n^{-1} \sum_{i=1}^n K_h(u_i - u_j)X_{ik} I(Y_i \leq Y_i)}{n^{-1} \sum_{i=1}^n K_h(u_i - u_j)}.$$

where $K_h(u) = K(u/h)/h$, $K(\cdot)$ is a kernel function and $h$ is the bandwidth. Then a natural estimator of $\omega_k$ is

$$\hat{\omega}_k = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \frac{\text{cov}^2(X_k, I(Y \leq Y_i) \mid u_j)}{\text{var}(X_k \mid u_j)\text{var}(I(Y \leq Y_i) \mid u_j)}.$$
where \(\tilde{\text{cov}}(X_k, I(Y \leq Y_i) | u_j)\) can be estimated through \(\tilde{E}(X_k I(Y \leq Y_i) | u_j) - \tilde{E}(X_k | u_j)E(I(Y \leq Y_i) | u_j)\). The variance term can be estimated by similar patterns. Based on the sample estimation of \(\tilde{\omega}_t\), we conduct the screening criterion to identify the active set indexed by

\[ \tilde{A} = \{ k : \tilde{\omega}_k \text{ ranks in the top } d \} \]

among all \(\tilde{\omega}_t\)’s, for \(1 \leq k \leq p\),

where \(d\) is the user-specified threshold value. We refer the proposed conditional sure independent ranking and screening procedure as C-SIRS in the paper.

3. THEORETICAL PROPERTIES

We establish several appealing properties for our proposed screening procedure. Denote \(\lambda_{\max}(C)\) and \(\lambda_{\min}(C)\) for the largest and smallest eigenvalues of a matrix \(C\), respectively. Write \(v^2 = vv^T\) for a vector \(v\), and \(\Omega_A(u, y) = \text{corr}(x_A, I(Y \leq y) | u)\). Say \(a_n - b_n\) uniformly in \(n\) if \(\lim inf_{n \to \infty} a_n - b_n > 0\). We consider that given \(u\), \(F(y | x, u)\) depends on \(x\) only through \(x_A^T \beta_A(u)\) for some \(|A| \times K\) matrix \(\beta_A(u)\).

The following three conditions are required for Theorem 1.

(A1) \(E\{x^T \beta_A(u) | u\} = \text{cov}\{x, x_{\tilde{A}}(u)\}[\text{cov}\{x_{\tilde{A}} \beta_A(u) | u\}]^{-1} \beta_A(u) \text{ holds uniformly for } u\).

(A2) \(\min_{k \in \tilde{A}} \omega_k > E[\lambda_{\max}\{\text{cov}(x_A, x_{\tilde{A}} | u)\}\text{cov}(x_{\tilde{A}} \beta_A(u) | u)]/\lambda_{\max}\{\text{cov}(x_A, x_{\tilde{A}} | u)\}\text{ holds uniformly for } n\).

(A3) \(x\) and \(Y\) are independent conditioning on \(x_{\tilde{A}} \beta_A(u), u\).

Condition (A1) is referred to as the conditional linearity condition. Condition (A2) is the crucial assumption to guarantee the satisfactory performance of our proposal, which requires the minimal signal of the active predictors not too small. It also does not allow strong correlation between \(x_A\) and \(x_{\tilde{A}}\), or among \(x_A\) themselves given the exposure variable \(u\). Note that (A2) holds automatically if \(x_A\) and \(x_{\tilde{A}}\) are uncorrelated conditioning on \(u\). Similar conditions are assumed in Zhu et al. (2011) and Liu et al. (2014). Condition (A3) dictates that \(Y\) relies on \(x\) via the linear combinations \(x_{\tilde{A}}^T \beta_A(u)\).

**Theorem 1.** Suppose conditions (A1), (A2) and (A3) hold, then we have

\[ \liminf_{n \to \infty} \left\{ \min_{k \in \tilde{A}} \omega_k - \max_{k \in A^c} \omega_k \right\} > 0. \]

Theorem 1 illustrates that signals between the important predictors and the unimportant ones are distinguishable, which is a prerequisite for the ranking consistency property. We assume the following regularity conditions to derive the theoretical properties of C-SIRS. Define \(E(\cdot | u) = g(\cdot | u)/f(u)\).

(C1) (The Kernel Function) The kernel \(K(\cdot)\) is a density function with compact support. It is symmetric about zero and Lipschitz continuous. In addition, it satisfies

\[ \int_{-1}^{1} K(t) dt = 1, \quad \int_{-1}^{1} t^{i-1}K(t) dt = 0, 0 \leq i \leq m - 1, \]

\[ 0 \neq \int_{-1}^{1} t^{m} K(t) dt = \nu_m < \infty \]

It is bounded uniformly such that

\[ \sup_{u \in U} |K(u)| = M_k < \infty. \]

(C2) (The Density) The probability density functions of \(u\), denoted by \(f(u)\) has continuous second-order derivative on \(U\).

(C3) (The Derivatives) The \((m - 1)\)-th derivatives of both \(g(\cdot | u), f(u)\) are locally Lipschitz-continuous with respect to \(u\).

(C4) (The Bandwidth) The bandwidth \(h\) satisfies \(h = O(n^{-\theta})\), for some \(\theta\) which satisfies \((4m)^{-1} < \theta < 1/4\).

(C5) (The Moment Condition) There exists a positive constant \(s_0\) such that

\[ \sup_{u \in U} \max_{1 \leq k \leq p} E(\exp(sX_k^2 | u)) < \infty, \quad 0 < s < s_0. \]

Further assume that \(E(X_k | u)\) and \(E(X_k^2 | u)\), their first-order and second-order derivatives are finite uniformly in \(u \in U\).

**Theorem 2.** (Sure Screening Property) Under the conditions (C1)-(C5), for any \(0 < \gamma + \theta \leq 1/4\) and \(0 < \gamma \leq m\theta\), if \(p\) satisfies \(n^p \exp(-cn^{1/2-2\gamma}) \to 0\) and \(\min_{k \in \tilde{A}} \omega_k \geq 2cn^{-\gamma}\) for some \(c > 0\), then

\[ Pr(A \subset \tilde{A}) \geq 1 - O\left\{ n^{3\gamma} |A| \exp(-cn^{1/2-2\gamma}) \right\}, \]

where \(c\) is a generic constant and \(|A|\) is cardinality of \(A\).

The sure screening property (Fan and Lv, 2008) of the C-SIRS procedure ensures that all truly active predictors can be retained after screening with the probability approaching to one.

**Theorem 3.** Under conditions (C1)-(C5), in addition to conditions (A1)-(A3), if \(p\) satisfies \(n^p \exp(-cn^{1/2-2\gamma}) \to 0\) for some positive \(c\), then

\[ \liminf_{n \to \infty} \left\{ \min_{k \in \tilde{A}} \tilde{\omega}_k - \max_{k \in A^c} \tilde{\omega}_k \right\} > 0 \text{ in probability.} \]

The ranking consistency guarantees the active predictors ranked in the top, prior to the inactive ones, with an overwhelming probability.

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4. NUMERICAL STUDIES

4.1 The performance of conditional screening

In this section, we investigate the finite sample performance of our proposed screening procedure through Monte Carlo simulations, and also compare it with five screening methods including SIRS (Zhu et al., 2011), DC-SIS (Li et al., 2012b), DT-SIR (Lin et al., 2018), BCor-SIS (Pan et al., 2018) and CC-SIS (Liu et al., 2014), CDC-SIS (Wen et al., 2018). Under all model settings, we draw \( X = (X_1, X_2, \cdots, X_p)^T \) and an intermediate variable \( u^* \) from the multivariate normal distribution with mean zero and AR covariance matrix \( \Sigma = (0.5^{j-i})_{(p+1) \times (p+1)} \). Then the exposure variable \( u \) is obtained from \( u = \Phi(u^*) \), where \( \Phi() \) is cumulative distribution function of the standard normal distribution \( \mathcal{N}(0,1) \). We set the sample size \( n = 200 \) and fix \( p = 1000 \). Each experiment is repeated 1000 times. We adopt the Epanechnikov kernel \( K_n = 200 \) and fix \( \lambda = 0.5 \) for both simulations and real data analysis.

We evaluate the finite-sample performance through the following four criteria:

1. \( R_k \): The average of the ranks of each important predictor out of 1000 replications.
2. \( S \): The minimum model size to ensure that all important predictors are included after screening. We expect it to be as close to the number of truly active predictors. We report the 5%, 25%, 50%, 75% and 95% quantiles of \( S \) out of 1000 replications.
3. \( P_a \): The proportion of all active predictors selected after screening for a given model size out of 1000 replications. We consider three screened model size \( d = \nu \lfloor n^{1/3} / \log(n^{1/5}) \rfloor \) varying \( \nu = 1, 2 \) and 3. The corresponding \( d \) is 16, 32 and 48, respectively. We expect it to be as close to one as possible.
4. \( P_k \): The proportion of each active predictor selected after screening for a given model size out of 1000 replications.

Example 1. We first generate the response \( Y \) from the generalized varying coefficient models respectively:

- **Case 1**: \( \log \{ E(Y | x, u) \} = x^T \beta(u) \);
- **Case 2**: \( \log \{ E(Y | x, u) \} = x^T \beta(u) \),

where \( \beta = \{ \beta_1(u), \ldots, \beta_{1000}(u) \}^T \). \( \beta_j \) is nonzero when \( X_j \) is the active predictor and remains zero otherwise. We set the active predictors index to be \( \{2, 100, 400, 600, 1000\} \), with corresponding coefficient functions \( \beta_2(u) = 2I(u > 0.4), \beta_{100}(u) = 1 + u, \beta_{400}(u) = (2 - 3u)^2, \beta_{600}(u) = 3\sin(2\pi u), \beta_{1000}(u) = \exp\{u/(u + 1)\} \). The first model is logistic varying coefficient model while the second one is the Poisson varying coefficient model.

| Method   | \( R_2 \) | \( R_{100} \) | \( R_{400} \) | \( R_{600} \) | \( R_{1000} \) |
|----------|----------|-------------|-------------|-------------|-------------|
| SIRS     | 30.89    | 8.22        | 169.32      | 491.70      | 18.37       |
| BCor-SIS | 38.26    | 11.84       | 96.90       | 504.99      | 23.30       |
| DT-SIR   | 187.60   | 85.87       | 259.61      | 497.15      | 115.21      |
| Case 1   |          |             |             |             |             |
| DC-SIS   | 18.11    | 6.09        | 67.89       | 512.58      | 10.73       |
| CDC-SIS  | 18.25    | 4.34        | 61.82       | 457.82      | 10.26       |
| CC-SIS   | 9.85     | 9.63        | 7.63        | 2.49        | 17.68       |
| C-SIRS   | 10.41    | 7.15        | 5.69        | 3.02        | 14.34       |

| Method   | \( R_2 \) | \( R_{100} \) | \( R_{400} \) | \( R_{600} \) | \( R_{1000} \) |
|----------|----------|-------------|-------------|-------------|-------------|
| SIRS     | 7.05     | 2.44        | 123.44      | 462.06      | 3.44        |
| BCor-SIS | 11.64    | 3.34        | 38.20       | 250.44      | 6.18        |
| DT-SIR   | 89.67    | 22.00       | 158.81      | 478.59      | 37.37       |
| Case 2   |          |             |             |             |             |
| DC-SIS   | 17.53    | 3.66        | 163.17      | 164.99      | 6.36        |
| CDC-SIS  | 104.78   | 76.95       | 203.15      | 142.34      | 107.58      |
| CC-SIS   | 21.81    | 10.00       | 65.09       | 1.56        | 19.07       |
| C-SIRS   | 3.96     | 4.67        | 35.12       | 1.08        | 6.75        |

We report the simulation results of \( R_k \) and \( S \) in Table 1 and Table 2, respectively. The proposed C-SIRS method outperforms other competitors under both model settings. The rank of each predictor is on the top while the median of \( S \) is close to the number of truly active predictors, indicating that our proposal achieves a high accuracy in ranking. CC-SIS also performs satisfactorily as it is designed for the varying coefficient model. The marginal screening methods, including SIRS, BCor-SIS, DT-SIR and DC-SIS, are not able to detect the predictor \( X_{600} \). This is mainly because the expectation of \( \beta_{600}(u) \) is zero, making the predictor \( X_{600} \) marginally independent but conditional related to the response. The simulation results of selection proportions \( P_a \) and \( P_k \) are summarized in Table 3. The C-SIRS method selects all important predictors with high probability, indicating its sure screening property. The \( P_a \) of SIRS, BCor-SIS, DT-SIR, DC-SIS and CDC-SIS methods are negligible even for the largest submodel size \( d = 48 \).
Table 3. The proportions of $P_a$ and $P_k$ given the model size $d$ for Example 1

| $d$  | Method | $P_0$ | $P_{100}$ | $P_{200}$ | $P_{500}$ | $P_{1000}$ | $P_2$  |
|------|--------|-------|-----------|-----------|-----------|------------|-------|
| 16   | SIRS   | 0.71  | 0.91      | 0.38      | 0.01      | 0.81       | 0.00  |
|      | BCor-SIS | 0.67  | 0.89      | 0.44      | 0.02      | 0.78       | 0.01  |
|      | DT-SIR | 0.23  | 0.45      | 0.13      | 0.02      | 0.38       | 0.00  |
| 48   | DC-SIS | 0.81  | 0.93      | 0.52      | 0.01      | 0.87       | 0.00  |
|      | CDC-SIS | 0.82  | 0.96      | 0.56      | 0.01      | 0.90       | 0.00  |
|      | CC-SIS | 0.90  | 0.91      | 0.91      | 0.00      | 0.83       | 0.60  |
|      | C-SIRS | 0.90  | 0.93      | 0.95      | 0.00      | 0.84       | 0.65  |
| 96   | SIRS   | 0.79  | 0.95      | 0.52      | 0.03      | 0.88       | 0.01  |
|      | BCor-SIS | 0.78  | 0.94      | 0.57      | 0.03      | 0.85       | 0.02  |
|      | DT-SIR | 0.35  | 0.57      | 0.19      | 0.03      | 0.49       | 0.00  |
| 192  | DC-SIS | 0.90  | 0.97      | 0.64      | 0.03      | 0.93       | 0.01  |
|      | CDC-SIS | 0.88  | 0.98      | 0.67      | 0.02      | 0.94       | 0.01  |
|      | CC-SIS | 0.95  | 0.95      | 0.96      | 1.00      | 0.89       | 0.77  |
|      | C-SIRS | 0.96  | 0.96      | 0.98      | 1.00      | 0.91       | 0.81  |

Table 4. The mean of $R_k$ of each active predictor for Example 2

| Method | $R_2$ | $R_{100}$ | $R_{200}$ | $R_{500}$ | $R_{1000}$ | $R_2$ |
|--------|-------|-----------|-----------|-----------|------------|-------|
| SIRS   | 24.64 | 2.76      | 61.90     | 485.22    | 9.39       |       |
| BCor-SIS | 25.98 | 10.82     | 82.19     | 233.65    | 20.97      |       |
| DT-SIR | 121.18| 40.52     | 198.86    | 510.08    | 60.21      |       |
| Case1  | DC-SIS | 104.67    | 65.33     | 277.81    | 171.04     | 146.13 |
| CDC-SIS | 108.22 | 79.91     | 206.46    | 143.12    | 111.44     |       |
| CC-SIS | 188.26 | 167.32    | 131.75    | 56.50     | 169.04     |       |
| C-SIRS | 5.69  | 8.36      | 10.61     | 1.81      | 6.97       |       |
| Case2  | DC-SIS | 362.10    | 520.40    | 560.70    | 579.80     | 538.20 |
| CDC-SIS | 497.71 | 484.76    | 506.99    | 500.14    | 487.19     |       |
| CC-SIS | 231.46 | 197.64    | 143.85    | 71.32     | 188.68     |       |
| C-SIRS | 9.63  | 6.71      | 18.70     | 2.39      | 12.31      |       |
| Case3  | DC-SIRS | 20.31     | 8.49      | 52.85     | 407.09     | 31.37  |
| CDC-SIS | 18.45  | 7.61      | 50.79     | 389.78    | 29.05      |       |
| CC-SIS | 123.35 | 34.00     | 151.27    | 197.23    | 177.44     |       |
| C-SIRS | 6.49  | 1.34      | 8.67      | 19.35     | 8.22       |       |
| Case4  | DC-SIRS | 226.48    | 172.32    | 295.15    | 5.51       | 202.70 |
| CDC-SIS | 234.92 | 175.71    | 318.88    | 4.99      | 204.87     |       |
| CC-SIS | 452.48 | 465.33    | 486.95    | 74.69     | 469.96     |       |
| C-SIRS | 7.99  | 7.39      | 17.09     | 8.78      | 14.56      |       |

Example 2. We then consider following four models, in which the response $Y$ depends on the predictors nonlinearly with a given $u$:

- **Case 1:** $Y = \exp\{x^T \beta(u)\} + \varepsilon$;
- **Case 2:** $Y = \exp\{x^T \beta(u)\} + \varepsilon$;
- **Case 3:** $Y = \beta_2(u) \exp\{X_2\} + \beta_{100}(u) X_{100} + 2\beta_{400}(u) X_{400} + \beta_{1000}(u) X_{1000} + 1.5\beta_{1000}(u) X_{1000} + \varepsilon$;
- **Case 4:** $Y = \beta_2(u) X_2 + \beta_{100}(u) X_{100} + \beta_{400}(u) X_{400} + \beta_{1000}(u) X_{1000} + 2\exp\{\beta_{600}(u) X_{600}\} \varepsilon$;

where the $\beta(u)$ setting remains identical as Example 1 and the error term is independently generated from $t(1)$ for Case 1 to Case 3 and $\mathcal{N}(0,1)$ for Case 4. Notice that Case 4 demonstrates the heteroscedasticity issue, where variance component is affected by $X_{600}$, and the effect vary with $u$.

The simulation results are tabulated in Table 4 for $R_k$. Table 5 for $S$ and Table 6, 7 for $P_a$ and $P_k$. It can be clearly seen that C-SIRS method is still the obvious winner, and performs well in both ranking and selection. CC-SIS method is not able to capture the nonlinear conditional dependence, and the rankings are not accurate. The performances of DC-SIS and CDC-SIS methods are still poor, because of the heavy distribution of the response. We observe that SIRS, BCor-SIS and DT-SIR methods also fail to identify the $X_{600}$ in most cases, which is in accordance with Example 1.

Table 8 reports the average computation time, in seconds, along with the standard deviation (SD). All simulations have been carried out on a laptop computer with Intel Xeon 2.60 GHz processor. As a screening procedure involving exposure variables, we have to utilize smoothing techniques to estimate the conditional correlations, making it slower than SIRS, DC-SIS, DT-SIR, among other methods that do not need to do smoothing (and thus cannot deal with the exposure variables). However, it is still reasonably fast compared to the other methods.
Table 5. The quantiles of the minimum model size $s$ for Example 2

| Method | 5%   | 25%  | 50%  | 75%  | 95%  |
|--------|------|------|------|------|------|
| SIRS   | 68.95| 259.75| 473.50| 728.00| 933.10|
| BCor-SIS| 22.00| 92.00| 226.50| 434.50| 777.20|
| DT-SIR | 151.00| 389.00| 607.00| 805.00| 955.05|
| Case1  | DC-SIS| 82.00| 196.75| 359.00| 583.25| 844.05|
|        | CDC-SIS| 72.95| 194.00| 353.00| 550.00| 822.05|
|        | CC-SIS  | 65.95| 185.00| 328.00| 568.75| 865.00|
|        | C-SIRS   | 5.00 | 6.00 | 9.00 | 21.00 | 92.00 |
| Case2  | DC-SIS| 95.00| 263.15| 495.20| 702.00| 925.45|
|        | BCor-SIS| 31.00| 98.00| 232.50| 448.00| 796.05|
|        | DT-SIR | 183.90| 395.75| 605.00| 798.00| 952.05|
| Case3  | DC-SIS| 42.75| 126.25| 251.50| 505.20| 999.05|
|        | CDC-SIS| 563.95| 762.75| 872.00| 946.00| 989.00|
|        | CC-SIS  | 45.00| 162.00| 346.00| 578.95| 882.00|
|        | C-SIRS   | 5.00 | 7.50 | 13.25| 42.50 | 199.65|
| Case4  | DC-SIS| 43.90| 194.00| 392.00| 667.00| 920.10|
|        | BCor-SIS| 24.95| 103.75| 256.00| 509.25| 875.20|
|        | DT-SIR | 116.00| 319.00| 527.00| 729.00| 939.05|
| Case5  | DC-SIS| 50.00| 174.00| 369.50| 633.75| 911.05|
|        | CDC-SIS| 56.75| 174.00| 369.50| 633.75| 911.05|
|        | CC-SIS  | 45.00| 162.00| 346.00| 578.95| 882.00|
|        | C-SIRS   | 5.00 | 7.50 | 13.25| 42.50 | 199.65|

with other conditional screening methods according to the Table 8.

In short, thanks to the model-free start point of C-SIRS, it is applicable for both generalized varying coefficient models and any nonlinear dependence structure of the predictor effects on the exposure variables, as shown in Example 1 and Case 1 and 3 of Example 2. As it does not impose strong distributional assumptions on the error term, the heavy-tailed error distribution and outlier issues can be well addressed in Case 1, 2 and 3 of Example 2 when $\varepsilon$ is generated from $t(1)$.

Furthermore, as illustrated in Case 4 of Example 2, it is also powerful to detect significant predictor effects on the variance of response, in addition to the mean, since we utilize the entire distribution function of the response to construct the screening score.

4.2 Real data analysis

Breast cancer is the one of most common malignancy among women, with a high lethality rate. There is a urgent need for the early diagnosis of breast cancer and simultaneously monitoring the disease progression. In this section, we evaluate the performance of proposed C-SIRS procedure through the breast cancer data reported by Chin et al. (2006). The dataset, available from the R package PMA, consists of $p = 19672$ gene expressions over $n = 88$ cancer patient samples. Our interest is to detect the most influential genes to the 136 comparative genomic hybridization (CGH) measurements, as in some previous research (Witten et al., 2009; Wen et al., 2018). CGH is known as an indicator of genome copy number variation and cancer diagnosis. In addition, recent researches (McPherson et al., 2000; DeSantis et al., 2017) found that age is a risk factor that influences the incidence of breast cancer. And the effects of gene expression levels to CGH might depend on age. However, as a typical challenge of most ultrahigh dimensional problems, it
Table 7. The proportions of $P_a$ and $P_k$ given the model size $d$ for Example 2

| Method     | $P_2$ | $P_{100}$ | $P_{400}$ | $P_{600}$ | $P_{1000}$ | $P_{5000}$ |
|------------|-------|-----------|-----------|-----------|------------|------------|
| SIRS       | 0.89  | 1.00      | 0.72      | 0.01      | 0.93       | 0.01       |
| BCor-SIS   | 0.86  | 0.99      | 0.72      | 0.08      | 0.80       | 0.03       |
| DT-SIR     | 0.49  | 0.91      | 0.25      | 0.02      | 0.46       | 0.00       |
| DC-SIS     | 0.84  | 0.95      | 0.65      | 0.01      | 0.76       | 0.01       |
| CDC-SIS    | 0.85  | 0.96      | 0.65      | 0.01      | 0.77       | 0.01       |
| CC-SIS     | 0.50  | 0.84      | 0.42      | 0.23      | 0.28       | 0.03       |
| C-SIRS     | 0.93  | 1.00      | 0.92      | 0.78      | 0.93       | 0.63       |
| Case 3     |       |           |           |           |            |            |
| SIRS       | 0.94  | 1.00      | 0.81      | 0.04      | 0.96       | 0.03       |
| BCor-SIS   | 0.92  | 0.99      | 0.80      | 0.12      | 0.86       | 0.07       |
| DT-SIR     | 0.59  | 0.94      | 0.35      | 0.04      | 0.58       | 0.00       |
| DC-SIS     | 0.90  | 0.96      | 0.75      | 0.04      | 0.85       | 0.03       |
| CDC-SIS    | 0.90  | 0.97      | 0.73      | 0.05      | 0.85       | 0.03       |
| CC-SIS     | 0.58  | 0.86      | 0.51      | 0.32      | 0.40       | 0.09       |
| C-SIRS     | 0.97  | 1.00      | 0.96      | 0.88      | 0.97       | 0.79       |
| Case 4     |       |           |           |           |            |            |
| SIRS       | 0.96  | 1.00      | 0.86      | 0.07      | 0.97       | 0.06       |
| BCor-SIS   | 0.94  | 0.99      | 0.86      | 0.17      | 0.89       | 0.12       |
| DT-SIR     | 0.65  | 0.95      | 0.44      | 0.07      | 0.65       | 0.01       |
| DC-SIS     | 0.90  | 0.97      | 0.79      | 0.06      | 0.88       | 0.05       |
| CDC-SIS    | 0.93  | 0.98      | 0.79      | 0.08      | 0.87       | 0.06       |
| CC-SIS     | 0.62  | 0.88      | 0.55      | 0.38      | 0.46       | 0.12       |
| C-SIRS     | 0.98  | 1.00      | 0.97      | 0.91      | 0.98       | 0.85       |
| Case 4     |       |           |           |           |            |            |
| SIRS       | 0.78  | 0.97      | 0.58      | 0.02      | 0.93       | 0.01       |
| BCor-SIS   | 0.69  | 0.89      | 0.50      | 0.88      | 0.79       | 0.20       |
| DT-SIR     | 0.19  | 0.40      | 0.11      | 0.02      | 0.33       | 0.00       |
| DC-SIS     | 0.13  | 0.22      | 0.08      | 0.92      | 0.15       | 0.00       |
| CDC-SIS    | 0.12  | 0.20      | 0.05      | 0.95      | 0.16       | 0.00       |
| CC-SIS     | 0.03  | 0.02      | 0.02      | 0.59      | 0.02       | 0.00       |
| C-SIRS     | 0.91  | 0.95      | 0.80      | 0.87      | 0.87       | 0.50       |
| SIRS       | 0.87  | 0.98      | 0.72      | 0.04      | 0.96       | 0.02       |
| BCor-SIS   | 0.79  | 0.94      | 0.62      | 0.94      | 0.87       | 0.37       |
| DT-SIR     | 0.29  | 0.52      | 0.19      | 0.04      | 0.43       | 0.00       |
| DC-SIS     | 0.20  | 0.32      | 0.13      | 0.97      | 0.24       | 0.01       |
| CDC-SIS    | 0.20  | 0.29      | 0.10      | 0.98      | 0.23       | 0.01       |
| CC-SIS     | 0.05  | 0.05      | 0.04      | 0.68      | 0.05       | 0.00       |
| C-SIRS     | 0.95  | 0.97      | 0.88      | 0.96      | 0.92       | 0.72       |
| Case 4     |       |           |           |           |            |            |
| SIRS       | 0.90  | 0.99      | 0.78      | 0.06      | 0.97       | 0.04       |
| BCor-SIS   | 0.85  | 0.96      | 0.67      | 0.96      | 0.91       | 0.48       |
| DT-SIR     | 0.36  | 0.58      | 0.25      | 0.06      | 0.50       | 0.00       |
| DC-SIS     | 0.26  | 0.40      | 0.16      | 0.98      | 0.31       | 0.03       |
| CDC-SIS    | 0.26  | 0.35      | 0.14      | 0.99      | 0.28       | 0.01       |
| CC-SIS     | 0.07  | 0.07      | 0.06      | 0.73      | 0.06       | 0.00       |
| C-SIRS     | 0.97  | 0.97      | 0.91      | 0.99      | 0.94       | 0.81       |

Table 8. Run times (CPU seconds) for each experiment

| Method     | Mean  | SD    |
|------------|-------|-------|
| SIRS       | 0.052 | 0.005 |
| BCor-SIS   | 4.076 | 0.020 |
| DT-SIR     | 0.085 | 0.008 |
| DC-SIS     | 3.038 | 0.060 |
| CDC-SIS    | 68.498| 0.487 |
| CC-SIS     | 0.344 | 0.007 |
| C-SIRS     | 5.894 | 0.043 |

Table 9. Results for breast cancer analysis

| Method     | Size | MSPE  |
|------------|------|-------|
| SIRS       | 10   | 3.694 |
| BCor-SIS   | 15   | 4.378 |
| DT-SIR     | 12   | 4.202 |
| DC-SIS     | 9    | 4.837 |
| CDC-SIS    | 9    | 3.626 |
| CC-SIS     | 10   | 4.067 |
| C-SIRS     | 9    | 3.533 |

Figure 1. The estimated coefficient functions of the selected nine genes.
becomes most severe at about 50s, and recovers gradually thereafter. The third selected gene LOC51107 has a large positive effect at people’s early stage, but gets less important as people get older. Other functions can be interpreted in the similar fashion.

5. A BRIEF DISCUSSION

In this paper, we utilize the conditional correlation between predictors and the empirical distribution function of response given exposure variables to develop a model-free sure independence screening procedure C-SIRS. It is resistant to the heavy-tailed distribution, outliers and extreme values of the response. In addition, C-SIRS is applicable for any model involving exposure variables, including generalized varying coefficient model, and any nonlinear dependent structure. It is also powerful to detect those variables with significant effects on variance components. The ranking consistency and sure screening property of C-SIRS are rendered in both theoretical properties and simulation studies. The breast cancer dataset is systematically analyzed by C-SIRS, along with the comparison with other related methods.

6. APPENDIX: PROOF OF THEOREMS

6.1 Appendix A: Proof of Theorem 1

For easy presentation, we introduce some notations first. \( \lambda_{\max}(C) \) and \( \lambda_{\min}(C) \) denote the largest and smallest eigenvalues of a matrix \( C \). We denote \( \mathbf{v} \) by \( \mathbf{v}^T \) for a vector \( \mathbf{v} \) and \( \Omega_A(u,y) = \text{corr}(X_A, I(y \leq y)|u) \). If we say that \( a_n - b_n \) uniformly in \( n \), then it means \( \lim_{n \to \infty} a_n - b_n > 0 \).

Without loss of generality, we assume that \( E(X_k | u) = 0 \), \( \text{var}(X_k | u) = 1 \) for \( k = 1, \ldots, p \) and \( \text{var}(I(Y \leq y) | u) = 1 \). Meanwhile, \( \beta_A(u) \) satisfies \( \beta_A^T(u) \text{cov}(X_A | u) \beta_A(u) = 1 \). Then the conditional linearity condition (A1) is simplified as \( E[I(Y \leq y)|X_A, u] = \text{cov}(X_k, x_A^T \beta_A(u) | u) \beta_A^T(u) x_A \).

Following law of iterated expectations,

\[
E \{ X_k I(Y \leq y)|u, y \} = E \left[ E \left\{ X_k I(Y \leq y)|\beta_A^T(u)X_A, u, y \right\} | u, y \right] \\
= E \left[ E \left\{ X_k | \beta_A^T(u)X_A, u \right\} \right] \\
= E \left[ I(Y \leq y) | \beta_A^T(u)X_A, u, y \right] \\
= \text{cov}(X_k, x_A^T \beta_A(u) | u) E \{ \beta_A^T(u)X_A I(Y \leq y) | u, y \},
\]

where the second equation holds because of the conditional independence of \( x \) and \( Y \) given \( \beta_A^T(u)X_A \) and \( u \) in (A3) and the third equation is due to the simplified conditional linearity condition.

Thus two inequalities yield that

\[
\max_{k \in A^c} \Omega_k^2(u, y) = \max_{k \in A^c} E^2 \{ X_k I(Y \leq y) | u, y \} \\
\leq \max_{k \in A^c} \Omega_k^2(u, y) \\
\leq E \left[ \lambda_{\max} \left\{ \text{cov}(X_A, x_A^T | u) \text{cov}(X_A, x_A^T | u) \right\} \\
\cdot \lambda_{\min} \left\{ \text{cov}(X_A, x_A^T | u) \right\} \right],
\]

which completes the proof.

6.2 Appendix A: Proof of Theorem 2

Define \( E(\cdot | u) = g(\cdot | u)/f(u) \), and

\[
\hat{E}(X_k | u) = \hat{g}(X_k | u)/\hat{f}(u), \\
\hat{E}(I(Y \leq y) | u) = \hat{g}(y | u)/\hat{f}(u),
\]

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where \( \tilde{g}(X_k \mid u) = n^{-1} \sum_{i=1}^{n} K_h(u_i - u)X_k, \) \( \tilde{g}(y \mid u) = n^{-1} \sum_{i=1}^{n} K_h(u_i - u)I(Y_i \leq y), \) \( f(u) = n^{-1} \sum_{i=1}^{n} K_h(u_i - u) \) and \( K_h(u) = K(u/h)/h. \)

**Lemma 1.** (Hoeffding’s Inequality (Hoeffding, 1963)) Let \( X_1, \ldots, X_n \) be independent random variables, and \( Pr(a_i \leq X_i \leq b_i) = 1 \) for \( i = 1, \ldots, n. \) Then for any \( t > 0, \)

\[
Pr\{|X - E(X)| \geq t\} \leq 2 \exp\left\{-2n^2t^2 \sum_{i=1}^{n} (b_i - a_i)^2 \right\}.
\]

**Lemma 2.** (Liu et al., 2014) Suppose that \( X \) is a random variable with \( E(e^{a|X|}) < \infty \) for some \( a > 0. \) Then for any \( M > 0, \) there exist positive constants \( b \) and \( c \) such that

\[
Pr\{|X| \geq M\} \leq b \exp(-cM).
\]

The following lemma is a slight modified version of Theorem 3.1 of Zhu (1993) and Lemma 3.2 of Zhu and Fang (1996).

**Lemma 3.** Suppose that conditions (C1) to (C5) are fulfilled, and \( \sup_{X} |X_k| \leq M, \) then for any \( \varepsilon_n > 0, \)

\[
Pr\left\{ \sup_u |\tilde{g}(X_k \mid u) - E\tilde{g}(X_k \mid u)| \geq 8M/\delta \right\} \leq 3c(n^{1/2}/\varepsilon_n)^4 \exp\left(-\varepsilon_n^2/288\delta^2\right) + 4c\delta^{-8} \exp\left(-n\delta^2\right),
\]

where \( \delta \geq \sup\{|\text{var}(\tilde{g}(\tilde{u}))\}|^{1/2}. \)

**Lemma 4.** Suppose that conditions (C1) to (C5) are fulfilled, for any \( 0 < \gamma + \theta < 1/4 \) and \( 0 < \gamma \leq m\theta, \) then we have

\[
Pr\left\{ \sup_u |\tilde{g}(X_k \mid u) - g(X_k \mid u)| \geq cn^{-\gamma} \right\} \leq \alpha_1 n^2 \exp(-b_1 n^{1/2-2\theta-2\gamma}),
\]

where \( \alpha_1, b_1 \) and \( c \) are some positive constants.

**Proof of Lemma 4.** For any \( M > 0, \)

\[
Pr\left\{ \sup_u |\tilde{g}(X_k \mid u) - g(X_k \mid u)| \geq cn^{-\gamma} \right\} = \alpha_1 n^2 \exp(-b_1 n^{1/2-2\theta-2\gamma}) + \alpha_2 n^2 \exp(-b_2 n^{1/2-2\theta-2\gamma}) + \alpha_3 n^2 \exp(-b_3 n^{1/2-2\theta-2\gamma})
\]

where \( P_1(u) \) and \( P_2(u) \) are defined in the obvious way. We derive the first term \( P_1(u). \) The following arguments are all under the condition \( \max |X_{ik}| \leq M, \) which is omitted for notation simplicity. We expand \( g(X_k \mid u) \) in a Taylor series with the Lagrange remainder term under Condition (C3). There exists a positive constant \( c_1 \) such that

\[
\sup_u |E\tilde{g}(X_k \mid u) - g(X_k \mid u)| = \sup_u \left| \int K_h(\tilde{u} - u) \{E(X_k \mid \tilde{u}) - E(X_k \mid u)\} f(u) du \right| \leq c_1 \nu nh^m.
\]

Since the kernel function \( K \) is uniformly continuous on its compact support, we have

\[
\sup_u \left| \int K(t) \{E(X_k \mid u + ht) f(u + ht) - E(X_k \mid u) f(u) \} dt \right| \leq c_1 \nu nh^m.
\]

Following Liu et al. (2014), we take \( M = O(n^2) \), where \( 1/2 - 2\theta - 2\gamma < \sigma < 1/4. \) For large \( n, \) we can see that \( n^{1/2}/M^2 = \)
$n^{1/2-2r} > 1$, and $M n^{2r+2\gamma} n^{-1/2} = n^{\gamma+2r+2\gamma} > 1$. Now (5) becomes
\[
\Pr \left\{ \sup_u |\tilde{g}(X_u | u) - g(X_u | u)| \geq cn^{-\gamma} \right\} \\
\leq a_1 n^2 \exp(-b_1 n^{1/2-2r-2\gamma}),
\]
where $a_1$, $b_1$ and $c$ are some positive constants. The proof is complete. \hfill \Box

**Lemma 5.** Suppose that conditions (C1) to (C5) are fulfilled, for any $0 < \gamma + \theta < 1/2$ and $0 < \gamma \leq m\theta$, then we have
\[
Pr \left\{ \sup_u \left| \tilde{f}(u) - f(u) \right| \geq cn^{-\gamma} \right\} = \sup_u \left| \tilde{f}(u) - f(u) \right| = c n^{-\gamma}.
\]

Proof of Lemma 5. We use the same technique as the proof for Lemma 4. According to Theorem 3.1 of Zhu (1993) and Lemma 3.1 of Zhu and Fang (1996), for any $\varepsilon_n > 0$,
\[
(6) \quad \Pr \left\{ \sup_u \left| \tilde{f}(u) - E\tilde{f}(u) \right| \geq 8M_K h^{-1/2} \right\} \\
\leq 3A \left( \frac{n^{1/2}}{\varepsilon_n} \right)^4 \exp \left( -\frac{\varepsilon_n^2}{128\delta_1^2} \right) + 4A \delta_1^{-8} \exp \left( -n\delta_1^2 \right),
\]
where $\delta_1 \geq \sup [\var{K \left( \frac{\tilde{u} - u}{h} \right) I(Y \leq y)}]^{1/2}$ and $\delta_2 \geq \sup [\var{K \left( \frac{\tilde{u} - u}{h} \right) I(Y \leq y)}]^{1/2}$. And $A$ is a generic constant and may take different values at different places. We expand $f(u)$ and $g(y | u)$ in a Taylor series with the Lagrange remainder term under Condition (C3). There exists some positive constants $c_2$ and $c_3$ such that
\[
\sup_u \left| E\tilde{f}(u) - f(u) \right| \\
= \sup_u \left| \int K_h (\tilde{u} - u) \left\{ f(\tilde{u}) - f(u) \right\} d\tilde{u} \right| \\
= \sup_u \left| \int K_h (t) \left\{ f(u + ht) - f(u) \right\} dt \right| \\
\leq c_2 \nu_m h^m,
\]
\[
\sup_u \left| E\tilde{g}(y | u) - g(y | u) \right| \\
= \sup_u \left| \int K_h (\tilde{u} - u) \left\{ F(y | \tilde{u}) f(\tilde{u}) - F(y | u) f(u) \right\} d\tilde{u} \right| \\
\leq c_3 \nu_m h^m,
\]
\[
= \sup_u \left| \int K_h (\tilde{u} - u) \left\{ F(y | \tilde{u}) f(\tilde{u}) - F(y | u) f(u) \right\} d\tilde{u} \right| \\
\leq c_4 \nu_m h^m.
\]
Since the kernel function $K$ is uniformly continuous on its compact support, we have
\[
\sup_u \left| \int K_h (\tilde{u} - u) I(Y \leq y) \right| \\
\leq \sup_u \left| \int K_h (\tilde{u} - u) f(\tilde{u}) d\tilde{u} \right|^{1/2} \leq M_K
\]
Thus for $0 < \gamma \leq m\theta$,
\[
(8) \quad \Pr \left\{ \sup_u \left| \tilde{f}(u) - E\tilde{f}(u) \right| \geq cn^{-\gamma} \right\} \\
= \sup_u \left| \tilde{f}(u) - E\tilde{f}(u) \right| = c n^{-\gamma}.
\]
\[
(9) \quad \Pr \left\{ \sup_u \left| \tilde{g}(y | u) - E\tilde{g}(y | u) \right| \geq cn^{-\gamma} \right\} \\
= \sup_u \left| \tilde{g}(y | u) - E\tilde{g}(y | u) \right| = c n^{-\gamma}.
\]
\begin{align*}
\text{Step 1.} & \text{ We first prove that, under conditions (C1)-(C5), for any } 0 < \gamma + \theta \leq 1/4 \text{ and } 0 < \gamma \leq m\theta, \text{ there exists some positive constant } c \text{ such that} \\
\Pr \left( \left| \hat{\omega}_k - \omega_k \right| \geq cn^{-\gamma} \right) & \leq cn^3 \exp(-cn^{1/2-2\gamma-2\theta}), \\
\text{where } a_i, b_i \text{ for } i = 6, 7, 8 \text{ and } c \text{ are some positive constants.} \\
\text{Proof of Theorem 2.} & \text{ We divide the proof into two steps.} \\
\text{Step 1.} & \text{ We define } \Delta_1 \text{ and } \Delta_2 \text{ as follows,} \\
\Delta_1 & = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \text{corr}^2 \{ X_k, I(Y \leq Y_j) \mid u_i \} \\
& \quad - \text{corr}^2 \{ X_k, I(Y \leq Y_j) \mid u_i \} \right] \cdot \\
\Delta_2 & = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \text{corr}^2 \{ X_k, I(Y \leq Y_j) \mid u_i \} \right] . \\
\text{Then we have} & \\
\Pr \left( \left| \hat{\omega}_k - \omega_k \right| \geq cn^{-\gamma} \right) & \leq \Pr (\left| \Delta_1 \right| \geq cn^{-\gamma}/2) + \Pr (\left| \Delta_2 \right| \geq cn^{-\gamma}/2). \\
\text{Step 2.} & \text{ Assume the condition } \min_{k \in A} \omega_k \geq 2cn^{-1}. \text{ We prove that} \\
\Pr (A \subseteq \hat{A}) & \geq 1 - O \left( n^3 |A| \exp(-cn^{1/2-2\gamma-2\theta}) \right). \\
\text{If } A \subseteq \hat{A}, \text{ there must exist some } j \in \hat{A} \text{ such that } \hat{\omega}_j < cn^{-\gamma}. \\
\text{Under the condition } \min_{k \in A} \omega_k \geq 2cn^{-1}, \text{ we have } |\hat{\omega}_j - \omega_j| \geq cn^{-\gamma} \text{ for this particular } j, \text{ which implies} \\
\{ A \subseteq \hat{A} \} \subseteq \left\{ \left| \hat{\omega}_j - \omega_j \right| \geq cn^{-\gamma}, \text{ for some } j \in A \right\}. \\
\text{Then, it is clear that} \\
\Pr (A \subseteq \hat{A}) & \geq 1 - \Pr \left( \left| \hat{\omega}_j - \omega_j \right| \geq cn^{-\gamma}, \text{ for some } j \in A \right) \\
& \geq 1 - |A| \max_{j \in A} \Pr \left( \left| \hat{\omega}_j - \omega_j \right| \geq cn^{-\gamma} \right) \\
& \geq 1 - O \left( n^3 |A| \exp(-cn^{1/2-2\gamma-2\theta}) \right), \\
\text{which implies the desired conclusion.} \\
\end{align*}
Then by Fatou’s Lemma,

\[
\Pr \left \{ \min_{k \in A} \hat{\omega}_k - \min_{k \in A} \omega_k + \delta \leq \max_{k \in A} \hat{\omega}_k - \max_{k \in A} \omega_k \right \} \\
\leq \Pr \left \{ \left( \min_{k \in A} \hat{\omega}_k - \max_{k \in A} \hat{\omega}_k \right) - \left( \min_{k \in A} \omega_k - \max_{k \in A} \omega_k \right) \geq \delta \right \} \\
\leq \Pr \left \{ 2 \max_{1 \leq k \leq p} \left| \hat{\omega}_k - \omega_k \right| \geq \delta \right \} \\
\leq O \left \{ p n^{-3} \exp \left(-cn^{1/2-2q} \right) \right \},
\]

Then by Fatou’s Lemma,

\[
\Pr \left \{ \liminf_{n \to \infty} \left( \min_{k \in A} \hat{\omega}_k - \max_{k \in A} \hat{\omega}_k \right) \leq 0 \right \} \\
\leq \lim_{n \to \infty} \Pr \left \{ \min_{k \in A} \hat{\omega}_k - \max_{k \in A} \hat{\omega}_k \leq 0 \right \} = 0
\]

In other words, we have

\[
\Pr \left \{ \liminf_{n \to \infty} \left( \min_{k \in A} \hat{\omega}_k - \max_{k \in A} \hat{\omega}_k \right) > 0 \right \} = 1.
\]

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