DEGENERACY AND FINITENESS THEOREMS FOR
MEROMORPHIC MAPPINGS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. This article deals with the degeneracy and finiteness problem of meromorphic mappings sharing few hyperplanes in projective space. In this article, we prove that there are at most two meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ ($n \geq 2$) sharing $2n + 2$ hyperplanes in general position regardless of multiplicity, where all zeros with multiplicities more than certain values do not need to be counted. A degeneracy theorem for three mappings also is given. These results are improvement of previous recent results of Fujimoto, Chen-Yan, and the author.

1. Introduction

In 1926, R. Nevanlinna [3] showed that two distinct nonconstant meromorphic functions $f$ and $g$ on the complex plane $\mathbb{C}$ cannot have the same inverse images for five distinct values, and that $g$ is a special type of linear fractional transformation of $f$ if they have the same inverse images counted with multiplicities for four distinct values [3]. These results are usually called the five values and the values theorems of R. Nevanlinna.

After that, many authors extended and improved the results of Nevanlinna to the case of meromorphic mappings into complex projective spaces. The extensions of the five values theorem are usually called the uniqueness theorems, and the extensions of the four values theorem are usually called the finiteness theorems. Here we formulate some recent results on this problem.

To state some of them, first of all we recall the following.

Let $f$ be a nonconstant meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and $H$ a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Let $k$ be a positive integer or $k = \infty$. Denote by $\nu_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{Z}$ whose value $\nu_{(f,H)}(a)$ ($a \in \mathbb{C}^m$) is the intersection multiplicity of the images of $f$ and $H$ at $f(a)$. For every $z \in \mathbb{C}^m$, we set

$$
\nu_{(f,H)} \leq k(z) = \begin{cases} 
0 & \text{if } \nu_{(f,H)}(z) > k, \\
\nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \leq k, 
\end{cases}
$$

and

$$
\nu_{(f,H)} \geq k(z) = \begin{cases} 
0 & \text{if } \nu_{(f,H)}(z) < k, \\
\nu_{(f,H)}(z) & \text{if } \nu_{(f,H)}(z) \geq k, 
\end{cases}
$$

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Take a meromorphic mapping $f$ of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ which is linearly nondegenerate over $\mathbb{C}$, a positive integer $d$ and $q$ hyperplanes $H_1, \ldots, H_q$ of $\mathbb{P}^n(\mathbb{C})$ in general position with

$$\dim f^{-1}(H_i \cap H_j) \leq m - 2 \quad (1 \leq i < j \leq q)$$

and consider the set $\mathcal{F}(f, \{H_i\}_{i=1}^q, d)$ of all linearly nondegenerate over $\mathbb{C}$ meromorphic maps $g : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ satisfying the following two conditions:

(a) $\min (\nu(f, H_j), d) = \min (\nu(g, H_j), d) \quad (1 \leq j \leq q),$

(b) $f(z) = g(z)$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

We see that conditions a) and b) mean the sets of all intersecting points (counted with multiplicity to level $d$) of $f$ and $g$ with each hyperplane are the same, and two mappings $f$ and $g$ agree on these sets. If $d = 1$, we will say that $f$ and $g$ share $q$ hyperplanes $\{H_j\}_{j=1}^q$ regardless of multiplicity.

Denote by $\# S$ the cardinality of the set $S$. In 1983, L. Smiley [7] proved the following uniqueness theorem.

**Theorem A.** If $q = 3n + 2$ then $\# \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$.

In 1998, H. Fujimoto [2] proved a finiteness theorem for meromorphic mappings as follows.

**Theorem B.** If $q = 3n + 1$ then $\# \mathcal{F}(f, \{H_i\}_{i=1}^q, 2) \leq 2$.

In 2009, Z. Chen-Q. Yan [1] considered the case of $2n + 3$ hyperplanes and proved the following uniqueness theorem.

**Theorem C.** If $q = 2n + 3$ then $\# \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) = 1$.

After that, in 2011 S. D. Quang [5] improved the result of Z. Chen-Q. Yan by omitting all zeros with multiplicity more than a certain number in the conditions on sharing hyperplanes of meromorphic mappings. As far as we known, there is still no uniqueness theorem for meromorphic mappings sharing less than $2n + 3$ hyperplanes regardless of multiplicities. In 2011 Q. Yan-Z. Chen [8] also proved a degeneracy theorem as follows.

**Theorem D.** If $q = 2n + 2$ then the map $f^1 \times f^2 \times f^3$ of $\mathbb{C}^m$ into $\mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C}) \times \mathbb{P}^N(\mathbb{C})$ is linearly degenerate for every three maps $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i\}_{i=1}^q, 2)$.

The first finiteness theorem for the case of meromorphic mappings sharing $2n + 2$ hyperplanes regardless of multiplicities are seen given by S. D. Quang [6] in 2012 as follows.

**Theorem E.** If $n \geq 2$ and $q = 2n + 2$ then $\# \mathcal{F}(f, \{H_i\}_{i=1}^q, 1) \leq 2$.

However, in the above results, we see that all intersecting point of the mappings and the hyperplanes are considered. It seems to us that the technique used in the proof of the above results do not work for the case where all intersecting point with multiplicities more than a certain number are not taken to count. Our main purpose in this paper is to improve the above result by omitting all such intersecting points. In order to states the main results, we give the following.

Let $f$ be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \ldots, H_q$ be $q$ hyperplanes of $\mathbb{P}^n(\mathbb{C})$ in general position. Let $k_1, \ldots, k_q$ be $q$ positive
integers or $+\infty$. Assume that
$$\dim \{z; \nu(f,H_i), \leq k_i(z) \cdot \nu(f,H_j), \leq k_j(z) > 0\} \leq m - 2 \ (1 \leq i < j \leq q).$$
Let $d$ be an integer. We consider the set $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^q, d)$ of all meromorphic maps $g : C^m \to P^n(C)$ satisfying the conditions:

(a) $\min (\nu(f,H_i), \leq k_i, d) = \min (\nu(g,H_i), \leq k_i, d) \ (1 \leq j \leq q)$,

(b) $f(z) = g(z)$ on $\bigcup_{i=1}^q \{z; \nu(f,H_i), \leq k_i(z) > 0\}$.

Then we see that $\mathcal{F}(f, \{H_i\}_{i=1}^q, d) = \mathcal{F}(f, \{H_i, \infty\}_{i=1}^q, d)$

Firstly, we will prove the following degeneracy theorem for meromorphic mappings sharing $2n + 2$ hyperplanes regardless of multiplicities as follows.

**Theorem 1.1.** Let $f$ be a linearly nondegenerate meromorphic mapping of $C^m$ into $P^n(C)$. Let $H_1, \ldots, H_{2n+2}$ be $2n + 2$ hyperplanes of $P^n(C)$ in general position and let $k_i \geq n \ (1 \leq i \leq 2n + 2)$ be positive integers or $+\infty$ with
$$\dim \{z; \nu(f,H_i), \leq k_i(z) \cdot \nu(f,H_j), \leq k_j(z) > 0\} \leq m - 2 \ (1 \leq i < j \leq 2n + 2).$$
Assume that $n \geq 2$ and
$$\sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n} < \frac{3n + 1}{3n^2(n + 1)} + \frac{3n + 1}{3n(k_0 + 1 - n)},$$
where $k_0 = \max_{1 \leq i \leq 2n+2} k_i$. Then for three maps $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1)$ we have $f^1 \land f^2 \land f^3 = 0$.

When $k_1 = \cdots = k_{2n+2} = k$, we have the following corollary

**Corollary 1.2.** Let $f$ be a linearly nondegenerate meromorphic mapping of $C^m$ into $P^n(C)$. Let $H_1, \ldots, H_{2n+2}$ be $2n + 2$ hyperplanes of $P^n(C)$ in general position and let $k \geq 2n^2 + 4n$ be positive integers or $+\infty$ with
$$\dim \{z; \nu(f,H_i), \leq k(z) \cdot \nu(f,H_j), \leq k(z) > 0\} \leq m - 2 \ (1 \leq i < j \leq 2n + 2).$$
If $n \geq 2$ then for every three maps $f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i, k\}_{i=1}^{2n+2}, 1)$ we have $f^1 \land f^2 \land f^3 = 0$.

The last purpose of this paper is to prove the following unicity theorem.

**Theorem 1.3.** Let $f$ be a linearly nondegenerate meromorphic mapping of $C^m$ into $P^n(C)$. Let $H_1, \ldots, H_{2n+2}$ be $2n + 2$ hyperplanes of $P^n(C)$ in general position and let $k_i \geq n \ (1 \leq i \leq 2n + 2)$ be positive integers or $+\infty$ with
$$\dim \{z; \nu(f,H_i), \leq k_i(z) \cdot \nu(f,H_j), \leq k_j(z) > 0\} \leq m - 2 \ (1 \leq i < j \leq 2n + 2),$$
and
$$\sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n} < \frac{3n + 1}{3n^2(n + 1)} + \frac{3n + 1}{3n(k_0 + 1 - n)},$$
where $k_0 = \max_{1 \leq i \leq 2n+2} k_i$. We assume that

(a) if $n \geq 3$ then $\sum_{i=1}^{2n+2} \frac{1}{k_i + 1} < \frac{(n - 2)(n + 1)}{n(10n + 1)}$,

(b) if $n = 2$ then $\sum_{i=1}^{6} \frac{1}{k_i + 1} < \frac{1}{12}$. 
Then \( \#F(f, \{ H_i, k \}_{i=1}^{2n+2}, 1) \leq 2 \).

When \( k_1 = \cdots = k_{2n+2} = k \), we have the following corollary

**Corollary 1.4.** Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Let \( H_1, \ldots, H_{2n+2} \) be \( 2n+2 \) hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in general position and let \( k \geq 2n^2 + 4n \) be a positive integer or \( +\infty \) with

\[
\dim \{ z; \nu(f, H_i), \leq k(z) \cdot \nu(f, H_j), \leq k(z) > 0 \} \leq m - 2 \quad (1 \leq i < j \leq 2n + 2).
\]

We assume that

(a) if \( n \geq 3 \) then \( k > \frac{20n^2 + n + 2}{n - 2} \),
(b) if \( n = 2 \) then \( k > 71 \).

Then \( \#F(f, \{ H_i, k \}_{i=1}^{2n+2}, 1) \leq 2 \).

Then we see that in the case \( n \geq 2 \), Theorems D and E are corollaries of Theorem 1.3 when \( k_1 = \cdots = k_{2n+2} = +\infty \).

Throughout this paper, we always assume that \( n \geq 2 \).

### 2. Basic notions in Nevanlinna theory

**2.1. Counting functions of divisors.** We set \( \|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2} \) for \( z = (z_1, \ldots, z_n) \in \mathbb{C}^m \) and define

\[
B(r) := \{ z \in \mathbb{C}^m : \|z\| < r \}, \quad S(r) := \{ z \in \mathbb{C}^m : \|z\| = r \} \quad (0 < r < \infty).
\]

Define

\[
\nu_m(z) := (dd^c||z||^2)^{m-1} \quad \text{and} \quad \sigma_m(z) := d^c \log||z||^2 \wedge (dd^c \log||z||^2)^{m-1} \text{ on } \mathbb{C}^m \backslash \{0\}.
\]

We mean by a divisor divisor \( \nu \) on a domain \( \Omega \) in \( \mathbb{C}^m \) a formal sum

\[
\nu = \sum_{\lambda \in \Lambda} a_{\lambda} Z_{\lambda},
\]

where \( a_{\lambda} \in \mathbb{Z} \) and \( \{ Z_{\lambda} \}_{\lambda \in \Lambda} \) is a locally finite family of distinct irreducible hypersurfaces of \( \Omega \). Then, we may consider the divisor \( \nu \) as a function on \( \Omega \) with values in \( \mathbb{Z} \) as follows

\[
\nu(z) = \sum_{Z_{\lambda} \ni z} a_{\lambda}.
\]

The support of \( \nu \) is defined by \( \text{Supp} \nu = \bigcup_{a_{\lambda} \neq 0} Z_{\lambda} \).

For a nonzero meromorphic function \( \varphi \) on a domain \( \Omega \) in \( \mathbb{C}^m \), we denote by \( \nu_{\varphi}^0 \) (resp. \( \nu_{\varphi}^\infty \)) the divisor of zeros (resp. divisor of poles) of \( \varphi \), and denote by \( \nu_{\varphi} = \nu_{\varphi}^0 - \nu_{\varphi}^\infty \) the divisor generated by \( \varphi \).

For a divisor \( \nu \) on \( \mathbb{C}^m \) and for positive integers \( k, M \) (or \( M = \infty \)), we define the counting functions of \( \nu \) as follows. Set

\[
\nu^{(M)}(z) = \min \{ M, \nu(z) \};
\]
\[ \nu_{\leq k}(z) = \begin{cases} 0 & \text{if } \nu(z) > k, \\ \nu(z) & \text{if } \nu(z) \leq k, \end{cases} \]

\[ \nu_{> k}(z) = \begin{cases} \nu(z) & \text{if } \nu(z) > k, \\ 0 & \text{if } \nu(z) \leq k. \end{cases} \]

We define \( n(t) \) by

\[ n(t) = \begin{cases} \int_{|\nu| \cap B(t)} \nu(z)v_{n-1} & \text{if } n \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if } n = 1. \end{cases} \]

Similarly, we define \( n^{(M)}(t), n^{(M)}_{\leq k}(t), n^{(M)}_{> k}(t) \).

Define

\[ N(r, \nu) = \int_{1}^{r} \frac{n(t)}{t^{2n-1}} dt \quad (1 < r < \infty). \]

Similarly, we define \( N(r, \nu^{(M)}), N(r, \nu^{(M)}_{\leq k}), N(r, \nu^{(M)}_{> k}) \) and denote them by \( N^{(M)}(r, \nu), N^{(M)}_{\leq k}(r, \nu), N^{(M)}_{> k}(r, \nu) \) respectively.

Let \( \varphi : \mathbb{C}^m \to \mathbb{C} \) be a meromorphic function. Define

\[ N_{\varphi}(r) = N(r, \nu^0_{\varphi}), \quad N^{(M)}_{\varphi}(r) = N^{(M)}(r, \nu^0_{\varphi}), \]

\[ N^{(M)}_{\varphi, \leq k}(r) = N^{(M)}_{\leq k}(r, \nu^0_{\varphi}), \quad N^{(M)}_{\varphi, > k}(r) = N^{(M)}_{> k}(r, \nu^0_{\varphi}). \]

For brevity we will omit the superscript \( (M) \) if \( M = \infty \).

For a set \( S \subset \mathbb{C}^m \), we define the characteristic function of \( S \) by

\[ \chi_S(z) = \begin{cases} 1 & \text{if } z \in S, \\ 0 & \text{if } z \not\in S. \end{cases} \]

If the closure \( \bar{S} \) of \( S \) is a proper analytic subset of \( \mathbb{C}^m \) then we denote by \( N(r, S) \) the counting function of the reduced divisor whose support is union of all irreducible component of \( \bar{S} \) with codimension one.

### 2.2. Characteristic and Proximity functions

Let \( f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates \( (w_0 : \cdots : w_n) \) on \( \mathbb{P}^n(\mathbb{C}) \), we take a reduced representation \( f = (f_0 : \cdots : f_n) \), which means that each \( f_i \) is a holomorphic function on \( \mathbb{C}^m \) and \( f(z) = (f_0(z) : \cdots : f_n(z)) \) outside the analytic set \( \{f_0 = \cdots = f_n = 0\} \) of codimension \( \geq 2 \). Set \( \|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2} \).

The characteristic function of \( f \) is defined by

\[ T_f(r) = \int_{S(r)} \log \|f\| \sigma_m - \int_{S(1)} \log \|f\| \sigma_m. \]
Let $H$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n\}$, where $(a_0, \ldots, a_n) \neq (0, \ldots, 0)$. We set $(f, H) = \sum_{i=0}^n a_i f_i$. Then we see that the divisor $\nu_{(f, H)}$ does not depend on the reduced representation of $f$ and presentation of $H$. We define the proximity function of $H$ by

$$m_{f, H}(r) = \int_{S(r)} \log \frac{||f|| \cdot ||H||}{|(f, H)|} \sigma_m - \int_{S(1)} \log \frac{||f|| \cdot ||H||}{|(f, H)|} \sigma_m,$$

where $||H|| = (\sum_{i=0}^n |a_i|^2)^{\frac{1}{2}}$.

Let $\phi$ be a nonzero meromorphic function on $\mathbb{C}^m$, which are occasionally regarded as a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$. The proximity function of $\phi$ is defined by

$$m(r, \phi) := \int_{S(r)} \log \max (||\phi||, 1) \sigma_n.$$

As usual, by the notation “$\nu$” we mean the assertion $P$ holds for all $r \in [0, \infty)$ excluding a Borel subset $E$ of the interval $[0, \infty)$ with $\int_E dr < \infty$.

2.3. Some lemmas. The following results play essential roles in Nevanlinna theory (see [4]).

**Theorem 2.1** (The first main theorem). Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Then

$$N_{(f, H)}(r) + m_{f, H}(r) = T_f(r) \ (r > 1).$$

**Theorem 2.2** (The second main theorem). Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \ldots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Then

$$\| (q - n - 1)T_f(r) \| \leq \sum_{i=1}^q N_{(f, H_i)}^{(n)}(r) + o(T_f(r)).$$

**Lemma 2.3** (Lemma on logarithmic derivative). Let $f$ be a nonzero meromorphic function on $\mathbb{C}^m$. Then

$$\| m(r, \frac{D^\alpha(f)}{f}) = O(\log^+ T_f(r)) \ (\alpha \in \mathbb{Z}_+^m).$$

**Lemma 2.4.** Let $f$ be a linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$. Let $H$ be a hyperplanes of $\mathbb{P}^n(\mathbb{C})$, $d$ be a positive integer and $k$ is a positive integer or $+\infty$ with $k \geq d$, then

$$N^{(d)}(r, \nu_{(f, H), \leq k}) \geq \frac{k + 1}{k + 1 - d} N^{(d)}(r, \nu_{(f, H)}) - \frac{d}{k + 1 - d} T_f(r).$$
Proof. We have

\[ N^{(d)}(r, \nu_{(f,H), \leq k}) = N^{(d)}(r, \nu_{(f,H)}) - N^{(d)}(r, \nu_{(f,H), > k}) \]
\[ \geq N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1} N^{(d)}(r, \nu_{(f,H), > k}) \]
\[ = N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1} (N^{(d)}(r, \nu_{(f,H)}) - N^{(d)}(r, \nu_{(f,H), \leq k})) \]
\[ \geq N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1} \left( T_f(r) + \frac{d}{k+1} N^{(d)}(r, \nu_{(f,H), \leq k}) \right). \]

This implies that

\[ N^{(d)}(r, \nu_{(f,H), \leq k}) \geq \frac{k+1}{k+1-d} N^{(d)}(r, \nu_{(f,H)}) - \frac{d}{k+1-d} T_f(r). \]

The lemma is proved. □

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 3.1. Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Let \( H_1, \ldots, H_{2n+2} \) be \( 2n+2 \) hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) in general position and let \( k_i \geq n \) (\( 1 \leq i \leq 2n+2 \)) be positive integers or \( +\infty \) with

\[ \dim \{ z; \nu_{(f,H_i), \leq k}(z) \cdot \nu_{(f,H_j), \leq k}(z) > 0 \} \leq m - 2 \quad (1 \leq i < j \leq 2n+2). \]

Assume that

\[ \sum_{i=1}^{2n+2} \frac{1}{k_i} < \frac{n+1}{nk_0} + \frac{1}{n}, \]

where \( k_0 = \max_{1 \leq i \leq 2n+2} k_i \). Then every mapping \( g \in \mathcal{F}(f, \{ H_i, k_i \}_{i=1}^{2n+2}, 1) \) is linearly nondegenerate.

Proof. Suppose that there exists a hyperplane \( H \) satisfying \( g(\mathbb{C}^m) \subset H \). We assume that \( f \) and \( g \) have reduce representations \( f = (f_0 : \cdots : f_n) \) and \( g = (g_0 : \cdots : g_n) \) respectively. Assume that \( H = \{ (\omega_0 : \cdots : \omega_n) \mid \sum_{i=0}^{n} a_i \omega_i = 0 \} \). Since \( f \) is linearly nondegenerate, \( (f,H) \neq 0 \). On the other hand \( (f,H)(z) = (g,H)(z) = 0 \) for all \( z \in \bigcup_{i=1}^{2n+2} \{ \nu_{(f,H_i), \leq k_i} \} \), hence

\[ N_{(f,H)}(r) \geq \sum_{i=1}^{2n+2} N_{(f,H_i), \leq k_i}^{(1)}(r). \]
It yields that
\[ ||T_f(r)|| \geq N_{f,H}(r) \geq \sum_{i=1}^{2n+2} N_{f,H_i}^{(1)}(r) \geq \sum_{i=1}^{2n+2} \left( \frac{k_i + 1}{k_i} \right) N_{f,H_i}^{(1)}(r) - \frac{1}{k_i} T_f(r) \]
\[ \geq (1 + \frac{1}{k_0}) \frac{n}{n} \sum_{i=1}^{2n+2} N_{f,H_i}^{(n)}(r) - \sum_{i=1}^{2n+2} \frac{1}{k_i} T_f(r) \]
\[ \geq \left( (1 + \frac{1}{k_0}) \frac{n + 1}{n} - \sum_{i=1}^{2n+2} \frac{1}{k_i} \right) T_f(r) + o(T_f(r)). \]

Letting \( r \to +\infty \), we get
\[ \sum_{i=1}^{2n+2} \frac{1}{k_i} \geq \frac{n + 1}{nk_0} + \frac{1}{n}. \]

This is a contradiction. Hence \( g(C^n) \) can not be contained in any hyperplanes of \( P^n(C) \). Therefore \( g \) is linearly nondegenerate. \( \square \)

**Lemma 3.2.** Let \( f \) be a linearly nondegenerate meromorphic mapping of \( C^m \) into \( P^n(C) \). Let \( H_1, \ldots, H_{2n+2} \) be \( 2n+2 \) hyperplanes of \( P^n(C) \) in general position and let \( k_i \geq n \) \((1 \leq i \leq 2n+2)\) be positive integers or \( +\infty \) with
\[ \dim \{ z; \nu_{f,H_i}, \leq k(z) \cdot \nu_{f,H_j}, \leq k(z) > 0 \} \leq m - 2 \quad (1 \leq i < j \leq 2n+2). \]

Assume that
\[ \sum_{i=1}^{2n+2} \frac{1}{k_i} < \frac{n + 1}{nk_0} + \frac{1}{n}. \]

where \( k_0 = \max_{1 \leq i \leq 2n+2} k_i \). Then for every mapping \( g \in F(f, \{ H_i, k_i \}_{i=1}^{2n+2}, 1) \), we have
\[ ||T_g(r)|| = O(T_f(r)) \text{ and } ||T_f(r)|| = O(T_g(r)). \]

**Proof.** By Lemma 3.1 we have \( g \) is linearly nondegenerate. Then by the Second Main Theorem, we have
\[ ||(n + 1)T_g(r)|| \leq \sum_{i=1}^{2n+2} N_{g,H_i}^{(n)}(r) + o(T_g(r)) \]
\[ \leq \sum_{i=1}^{2n+2} n N_{g,H_i}^{(1)}(r) + o(T_g(r)) \]
\[ \leq \sum_{i=1}^{2n+2} n \left( \frac{k_i}{k_i + 1} \right) N_{g,H_i}^{(1)}(r) + \frac{1}{k_i + 1} T_g(r) + o(T_g(r)) \]
\[ \leq \sum_{i=1}^{2n+2} n \left( \frac{k_i}{k_i + 1} \right) N_{g,H_i}^{(1)}(r) + \frac{1}{k_i + 1} T_g(r) + o(T_g(r)) \]
\[ \leq \sum_{i=1}^{2n+2} n T_f(r) + \frac{1}{k_i + 1} T_g(r) + o(T_f(r) + T_g(r)). \]
Thus
\[
(n + 1 - \sum_{i=1}^{2n+2} \frac{n}{k_i + 1})T_g(r) \leq n(2n + 2)T_f(r) + o(T_f(r) + T_g(r)).
\]

We note that
\[
n + 1 - \sum_{i=1}^{2n+2} \frac{n}{k_i + 1} \geq n + 1 - \sum_{i=1}^{2n+2} \frac{n}{k_i} > n + 1 - \frac{n + 1}{k_0} - 1 \geq \frac{n + 1}{2} - 1 > 0.
\]
Hence \(|T_g(r) = O(T_f(r))\). Similarly, we get \(|T_f(r) = O(T_g(r))\).

\[\square\]

**Proof of Theorem 1.1.** From the assumption, we see that
\[
\sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n} < \frac{3n + 1}{3n^2(n + 1)} + \frac{3n + 1}{3n(k_0 + 1 - n)} \leq \frac{3n + 1}{3n^2(n + 1)} + \frac{3n + 1}{(2n + 2)3n} \sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n}.
\]
Thus
\[
\sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n} < \frac{6n + 2}{6n^2 + 9n + 1}.
\]

Then, we have
\[
\sum_{i=1}^{2n+2} \frac{1}{k_i} \leq \sum_{i=1}^{2n+2} \frac{1}{k_i + 1 - n} < \frac{6n + 2}{6n^2 + 9n + 1} < \frac{1}{n}.
\]
Hence, by Lemmas 3.1 and 3.2 we have that \(f^s\) is linearly nondegenerate and \(|T_{f^s}(r) = O(T_f(r))\) and \(|T_f(r) = O(T_{f^s})(r)\) for all \(s = 1, 2, 3\).

Suppose that \(f^1 \land f^2 \land f^3 \neq 0\). For each \(1 \leq i \leq 2n + 2\), we set
\[
N_i(r) = \sum_{u=1}^{3} N^{(u)}_{(f^u, H_i), \leq k_i} (r) - (2n + 1)N^{(1)}_{(f^1, H_i), \leq k_i} (r).
\]
We denote by \(\mathcal{I}\) the set of all permutations of the \((2n+2)\)-tuple \((1, \ldots, 2n+2)\), that means
\[
\mathcal{I} = \{ I = (i_1, \ldots, i_{2n+2}) : \{i_1, \ldots, i_{2n+2}\} = \{1, \ldots, 2n+2\}\}.
\]
For each \(I = (i_1, \ldots, i_{2n+2}) \in \mathcal{I}\) we define a subset \(E_I\) of \([1, +\infty)\) as follows
\[
E_I = \{ r \geq 1 : N_{i_1}(r) \geq \cdots \geq N_{i_{2n+2}}(r) \}.
\]
It is clear that \(\bigcup_{I \in \mathcal{I}} E_I = [1, +\infty)\). Therefore, there exists an element of \(\mathcal{I}\), for instance it is \(I_0 = (1, 2, \ldots, 2n+2)\), satisfying
\[
\int_{E_{I_0}} dr = +\infty.
\]
Then, we have \(N_1(r) \geq N_2(r) \geq \cdots \geq N_{2n+2}(r)\) for all \(r \in E_{I_0}\).

We consider \(\mathcal{M}^3\) as a vector space over the field \(\mathcal{M}\). For each \(i = 1, \ldots, 2n + 2\), we set
\[
V_i = \{(f^1, H_i), (f^2, H_i), (f^3, H_i)\} \in \mathcal{M}^3.
\]
We put

$$t = \min\{i : V_i \cap V_t \neq 0\}.$$ 

Since $f^1 \wedge f^2 \wedge f^3 \neq 0$, we have $1 < t < n + 1$. Also by again $f^1 \wedge f^2 \wedge f^3 \neq 0$, there exists an index $s \not\in \{1, t\}$ such that $V_i \cap V_t \cap V_s \neq 0$. This means that

$$P := \det(V_i, V_t, V_s) = \det\left( \begin{array}{ccc} (f^1, H_i) & (f^1, H_t) & (f^1, H_s) \\ (f^2, H_i) & (f^2, H_t) & (f^2, H_s) \\ (f^3, H_i) & (f^3, H_t) & (f^3, H_s) \end{array} \right) \neq 0. $$

Denote by $S$ the closure of $\bigcup_{1 \leq u \leq 3} I(f_u) \cup \bigcup_{1 \leq i < j \leq 2n+2} \{z ; \nu(f_i, H_i), \nu(f_j, H_j) > 0\}$. Then $S$ is an analytic subset of codimension two of $\mathbb{C}^n$.

For $z \not\in S$, we consider the following three cases:

Case 1. $z$ is a zero of $(f, H_1)$ with multiplicity at most $k_1$. We set

$$m = \min\{\nu(f^1, H_1), \nu(f^2, H_1), \nu(f^3, H_1)\}.$$ 

Then there exist a neighborhood $U$ of $z$ and holomorphic function $h$ defined on $U$ such that $\text{Zero}(h) = U \cap \text{Zero}(f, H_1)$ and $dh$ has no zero. Then the functions $\varphi_u = (f_u, H_1)/h^m$ ($1 \leq u \leq 3$) are holomorphic in a neighborhood of $z$. On the other hand, since $f_1 = f_2 = f_3$ on $\text{Zero}(f, H_1)$, we have

$$(f_u, H_1)/(f_1, H_1) = (f_u, H_s)/(f_1, H_s) \text{ on } \text{Zero}(f, H_1), \ u = 2, 3.$$

Therefore, the functions $\psi_u = ((f_u, H_1)/(f_1, H_1)) - (f_u, H_s)/(f_1, H_s))/h$ ($u = 2, 3$) are also holomorphic in a neighborhood of $z$.

We may assume that $\varphi_u$ ($1 \leq u \leq 3$), $\psi_u$ ($u = 2, 3$) are holomorphic on $U$. We rewrite $P$ on $U$ as follows

$$P = h^m \det\left( \begin{array}{ccc} \varphi_1 & (f_1, H_t) & (f_1, H_s) \\ \varphi_2 & (f_2, H_t) & (f_2, H_s) \\ \varphi_3 & (f_3, H_t) & (f_3, H_s) \end{array} \right)$$

$$= h^m (f_1, H_t)(f_1, H_s) \det\left( \begin{array}{ccc} \varphi_1 & 1 & 1 \\ \varphi_2 & (f_2, H_t) & (f_2, H_s) \\ \varphi_3 & (f_3, H_t) & (f_3, H_s) \end{array} \right)$$

$$= - h^{m+1} (f_1, H_t)(f_1, H_s) \det\left( \begin{array}{ccc} \varphi_1 & 1 & 0 \\ \varphi_2 & (f_2, H_t) & \psi_2 \\ \varphi_3 & (f_3, H_t) & \psi_3 \end{array} \right).$$

This yields that

$$\nu_P(z) \geq m + 1 = \min\{\nu(f^1, H_1), \nu(f^2, H_1), \nu(f^3, H_1)\} + 1.$$
Case 2. \( z \) is a zero of \((f, H_t)\) with multiplicity at most \( k_i \). Repeating the same argument as in Case 1, we have
\[
\nu_P(z) \geq \min \{ \nu_{(f^1, H_t), \leq k_1}(z), \nu_{(f^2, H_t), \leq k_1}(z), \nu_{(f^3, H_t), \leq k_1}(z) \} + 1.
\]

Case 3. \( z \) is a zero point of \((f, H_v)\) with multiplicity at most \( k_v \), where \( v \notin \{1, t\} \). There exist an index \( l \) such that \((f^1, H_t)(z) \neq 0\). Since \( f^1(z) = f^2(z) = f^3(z) \), we have \((f^u, H_t)(z) \neq 0 (1 \leq u \leq 3)\) and
\[
P = \det \begin{pmatrix}
(f^1, H_1) & (f^1, H_t) & (f^1, H_s) \\
(f^2, H_1) & (f^2, H_t) & (f^2, H_s) \\
(f^3, H_1) & (f^3, H_t) & (f^3, H_s)
\end{pmatrix}
\]
\[
= \prod_{u=1}^3 (f^u, H_t) \cdot \det \begin{pmatrix}
(f^1, H_1) & (f^1, H_t) & (f^1, H_s) \\
(f^2, H_1) & (f^2, H_t) & (f^2, H_s) \\
(f^3, H_1) & (f^3, H_t) & (f^3, H_s)
\end{pmatrix}
\]
\[
= \prod_{u=1}^3 (f^u, H_t) \cdot \det \begin{pmatrix}
(f^1, H_1) & (f^1, H_t) & (f^1, H_s) \\
(f^2, H_1) & (f^2, H_t) & (f^2, H_s) \\
(f^3, H_1) & (f^3, H_t) & (f^3, H_s)
\end{pmatrix}.
\]
vanishes at \( z \) with multiplicity at least two.

Thus, from the above three cases we have
\[
\nu_P(z) \geq \sum_{v=1,t}^{2n+2} \nu_{(f_1, H_v), \leq k_v}(z) - \nu_{(f_1, H_v), \leq k_v}(z) + \nu_{(f_1, H_v), \leq k_v}(z) + \nu_{(f_1, H_v), \leq k_v}(z) + \nu_{(f_1, H_v), \leq k_v}(z) + 2 \nu_{(f_1, H_v), \leq k_v}(z)
\]
for all \( z \) outside the analytic set \( S \).

Since \( \min \{a, b, c\} \geq \min \{a, n\} + \min \{b, n\} + \min \{c, n\} - 2n \) for every integers \( a, b, c \), the above inequality implies that
\[
\nu_P(z) \geq \sum_{v=1,t}^{2n+2} \left( \sum_{u=1}^3 \nu_{(f^u, H_v), \leq k_v}(z) - (2n + 1) \nu_{(f_1, H_v), \leq k_v}(z) \right) + 2 \nu_{(f_1, H_v), \leq k_v}(z),
\]
for all $z$ outside an analytic subset of co-dimension two.

Integrating both sides of the above inequality, we get

$$N_P(z) \geq \sum_{v=1,t}^3 \left( \sum_{u=1}^{3n} N_{(f^u,H_v),\leq k_v}(z) - (2n+1)N_{(f,H_v)}^{(1)}(z) \right) + 2 \sum_{v=1}^{2n+2} N_{(f,H_v),\leq k_v}(z)$$

$$= \sum_{v=1,t}^3 N_v(r) + 2 \sum_{v=1}^{2n+2} N_{(f,H_v),\leq k_v}(z).$$

Then for all $r \in E_{i_0}$, we have

$$\| N_P(r) \| \geq \frac{1}{n+1} \sum_{v=1}^{2n+2} N_v(r) + 2 \sum_{v=1}^{2n+2} N_{(f,H_v),\leq k_v}(z)\right) + 2 \sum_{v=1}^{2n+2} N_{(f,H_v),\leq k_v}(z)$$

$$\geq (1 + \frac{1}{3n}) \frac{1}{n+1} \sum_{v=1}^{2n+2} \sum_{u=1}^{3n} N_{(f^u,H_v),\leq k_v}(z)$$

$$\geq (1 + \frac{1}{3n}) \frac{1}{n+1} \sum_{u=1}^{3n} \sum_{v=1}^{2n+2} \left( \frac{k_v + 1}{k_v + 1 - n} N_{(f^u,H_v)}^{(n)}(z) - \frac{n}{k_v + 1 - n} T_{f^u}(r)\right)$$

$$\geq (1 + \frac{1}{3n}) \frac{1}{n+1} \sum_{u=1}^{3n} \sum_{v=1}^{2n+2} \left( \frac{k_v + 1}{k_v + 1 - n} N_{(f^u,H_v)}^{(n)}(z) - \frac{n}{k_v + 1 - n} T_{f^u}(r)\right)$$

$$\geq (1 + \frac{1}{3n}) \frac{1}{n+1} \sum_{u=1}^{3n} \sum_{v=1}^{2n+2} \left( \frac{(k_v + 1)(n+1)}{k_v + 1 - n} - \frac{n}{k_v + 1 - n} T_{f^u}(r)\right) + o(T(r))$$

$$= \frac{3n+1}{3n(n+1)} \left( \frac{(k_v + 1)(n+1)}{k_v + 1 - n} - \sum_{v=1}^{2n+2} \frac{n}{k_v + 1 - n} \right) T(r) + o(T(r)),$$

where $T(r) = \sum_{u=1}^{3n} T_{f^u}(r)$.

On the other hand, by Jensen’s formula and the definition of the characteristic function we have

$$N_P(r) = \int_{S(r)} \log |P| \sigma_m + O(1)$$

$$\leq \sum_{u=1}^{3} \int_{S(r)} \log \left( |(f^u,H_1)|^2 + |(f^u,H_t)|^2 + |(f^u,H_s)|^2 \right)^{\frac{1}{2}} \sigma_m + O(1)$$

$$\leq \sum_{u=1}^{3} \int_{S(r)} \log \| f^u \| \sigma_m + O(1) = T(r) + o(T(r)).$$
Thus, we have

\[ T(r) \geq \frac{3n+1}{3n(n+1)} \left( \frac{(k_0 + 1)(n + 1)}{k_0 + 1 - n} - \sum_{v=1}^{2n+2} \frac{n}{k_v + 1 - n} \right) T(r) + o(T(r)) \]

for every \( z \in E_{iu} \) outside a Borel finite measure set.

Letting \( r \to +\infty \) (\( r \in E_{iu} \)) we get

\[ 1 \geq \frac{3n+1}{3n(n+1)} \left( \frac{(k_0 + 1)(n + 1)}{k_0 + 1 - n} - \sum_{v=1}^{2n+2} \frac{n}{k_v + 1 - n} \right). \]

Thus

\[ \sum_{v=1}^{2n+2} \frac{1}{k_v + 1 - n} \geq \frac{3n+1}{3n^2(n+1)} + \frac{3n+1}{3n(k_0 + 1 - n)}. \]

This is a contradiction. Hence, \( f^1 \land f^2 \land f^3 \equiv 0 \). The theorem is proved. \( \square \)

4. PROOF OF THEOREM 1.3

Now for three mappings \( f^1, f^2, f^3 \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1) \), we define

\[ F^{ij}_k = \frac{(f^k, H_i)}{(f^k, H_j)} \quad (0 \leq k \leq 2, \ 1 \leq i < j \leq 2n + 2), \]

\[ V_i = ((f^1, H_i), (f^2, H_i), (f^3, H_i)) \in \mathcal{M}_m^3, \]

\[ T_i = \{ z \in \nu((f^i, H_i), \nu_{<k_i}(z) > 0 \}, S_i = \bigcup_{u=1}^{3}\{ z \in \nu((f_u, H_i), \nu_{>k_i}(z) > 0 \} \]

\[ R_i = \bigcap_{u=1}^{3}\{ z \in \nu((f_u, H_i), \nu_{>k_i}(z) > 0 \} \]

\[ \nu_i = \{ z \in k_i \geq \nu((f^u, H_i))(z) \geq \nu((f^v, H_i))(z) = \nu((f^t, H_i))(z) \text{ for a permutation } (u, v, t) \} \}

We write \( V_i \cong V_j \) if \( V_i \land V_j \equiv 0 \), otherwise we write \( V_i \not\cong V_j \). For \( V_i \not\cong V_j \), we write \( V_i \sim V_j \) if there exist \( 1 \leq u < v < 3 \) such that \( F^{ij}_u = F^{ij}_v \), otherwise we write \( V_i \not\sim V_j \).

Lemma 4.1. Let \( h, g \) be two elements of \( \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1) \), where \( k_i \geq n \) (\( 1 \leq i \leq 2n + 2 \)) are positive integers or \( +\infty \) with

\[ \sum_{i=1}^{2n+2} \frac{1}{k_i} < \frac{n + 1}{nk_0} + \frac{1}{n} \quad \text{and} \quad \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} < \frac{n - 1}{2n}, \]

where \( k_0 = \max_{1 \leq i \leq 2n+2} k_i \). If there exist a constant \( \lambda \) and two indices \( i, j \) such that

\[ \frac{(h, H_i)}{(h, H_j)} = \lambda \frac{(g, H_i)}{(g, H_j)} \]

then \( \lambda = 1 \).
Proof. By Lemmas 3.1 and 3.2, we see that h and g are linearly nondegenerate and have the characteristic functions of the same order with the characteristic function of f.

Setting \( H = \frac{(h, H_i)}{(h, H_j)} \) and \( G = \frac{(g, H_i)}{(g, H_j)} \) and

\[
S'_i = \{ z; \nu_{(h, H_i)} > k_i(z) > 0 \} \cup \{ z; \nu_{(g, H_i)}, > k_i(z) > 0 \},
\]

Then \( H = \lambda G \). Supposing that \( \lambda \neq 1 \), since \( H = G \) on the set \( \bigcup_{t \neq i, j} T_t \setminus (S'_i \cup S'_j) \), we have \( \bigcup_{t \neq i, j} T_t \subset S'_i \cup S'_j \). Thus

\[
0 \geq \sum_{t \neq i, j} N^{(1)}_{(f, H_i), \leq k_t}(r) - (N(r, S'_i) + N(r, S'_j))
\]

\[
\geq \frac{1}{2} \sum_{t \neq i, j} (N^{(1)}_{(h, H_i), \leq k_t}(r) + (N^{(1)}_{(g, H_i), \leq k_t}(r)) - (N(r, S'_i) + N(r, S'_j))
\]

\[
\geq \frac{1}{2} \sum_{t \neq i, j} (N^{(1)}_{(h, H_i)}(r) + N^{(1)}_{(g, H_i)}(r)) - \sum_{t=1}^{2n+2} N(r, S'_t)
\]

\[
\geq \frac{1}{2n} \sum_{t \neq i, j} (N^{(n)}_{(h, H_i)}(r) + N^{(n)}_{(g, H_i)}(r)) - \sum_{t=1}^{2n+2} (N^{(1)}_{(h, H_i), > k_t}(r) + N^{(1)}_{(g, H_i), > k_t}(r))
\]

\[
\geq \frac{n-1}{2n} (T_h(r) + T_g(r)) - \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} (T_h(r) + T_g(r)) + o(T_f(r)).
\]

Letting \( r \to +\infty \), we get

\[
\frac{n-1}{2n} \leq \sum_{t=1}^{2n+2} \frac{1}{k_t + 1}.
\]

This is a contradiction. Therefore \( \lambda = 1 \). The lemma is proved \( \square \)

Lemma 4.2. Let \( f^1, f^2, f^3 \) be three elements of \( \mathcal{F}(f, \{ H_i, k_i \}_{i=1}^{2n+2}, 1 \) \), where \( k_i \geq n (1 \leq i \leq 2n + 2) \) are positive integers or \( +\infty \). Suppose that \( f^1 \land f^2 \land f^3 \equiv 0 \) and \( V_i \sim V_j \) for some distinct indices \( i \) and \( j \). Then \( f^1, f^2, f^3 \) are not distinct.

Proof. Suppose \( f^1, f^2, f^3 \) are distinct. Since \( V_i \sim V_j \), we may suppose that \( F^{ij}_1 = F^{ij}_2 \neq F^{ij}_3 \). Since \( f^1 \land f^2 \land f^3 \equiv 0 \) and \( f^1 \neq f^2 \), there exists a meromorphic function \( \alpha \) such that

\[
F^{tj}_3 = \alpha F^{tj}_1 + (1 - \alpha) F^{tj}_2 \quad (1 \leq t \leq 2n + 2).
\]

This implies that \( F^{ij}_3 = F^{ij}_1 = F^{ij}_2 \). This is a contradiction. Hence \( f^1, f^2, f^3 \) are not distinct. The lemma is proved \( \square \)

For meromorphic functions \( F, G, H \) on \( \mathbb{C}^m \) and \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0} \), we put

\[
\Phi^\alpha(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ \mathcal{D}^\alpha(\frac{1}{F}) & \mathcal{D}^\alpha(\frac{1}{G}) & \mathcal{D}^\alpha(\frac{1}{H}) \end{vmatrix}
\]
Lemma 4.3 ([2, Proposition 3.4]). If $\Phi^\alpha(F, G, H) = 0$ and $\Phi^\alpha(F, G, H) = 0$ for all $\alpha$ with $|\alpha| \leq 1$, then one of the following assertions holds:

(i) $F = G, G = H$ or $H = F$

(ii) $F, G$ and $H$ are all constant.

Lemma 4.4. Let $f, \{H_i\}_{i=1}^{2n+2}, \{k_i\}_{i=1}^{2n+2}$ be as in Theorem [13]. Let $f^1, f^2, f^3$ be three maps in $\mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2})$. Suppose that $f^1, f^2, f^3$ are distinct and there are two indices $i, j \in \{1, 2, \ldots, 2n + 2\}$ ($i \neq j$) such that $V_i \neq V_j$

$$\Phi^\alpha := \Phi^\alpha(f_1^{ij}, f_2^{ij}, f_3^{ij}) \equiv 0$$

for every $\alpha = (\alpha_1, \ldots, \alpha_3) \in \mathbb{Z}^m_+$ with $|\alpha| = 1$. Then for every $t \in \{1, \ldots, 2n + 2\} \setminus \{i\}$, the following assertion hold:

(i) $\Phi^\alpha_t \equiv 0$ for all $|\alpha| \leq 1$, 

(ii) if $V_i \neq V_t$ then $f_1^t, f_2^t, f_3^t$ are distinct and

$$N_{(f, H_i), \leq k_i}^1(r) \geq \sum_{s \neq i, t} N_{(f, H_i), \leq k_i}^1(r) - N_{(f, H_i), \leq k_t}^1(r) - 2(N(r, S_i) + N(r, S_t))$$

$$\geq \sum_{s \neq i, t} N_{(f, H_i), \leq k_i}^1(r) - N_{(f, H_i), \leq k_t}^1(r) - 2 \sum_{u=1}^{3} \sum_{s=i, t} N_{(f, H_i), \leq k_s}^1(r).$$

Proof. By the supposition $V_i \neq V_j$, we may assume that $F_2^{ji} - F_1^{ji} \neq 0$.

(a) For all $\alpha \in \mathbb{Z}^m_+$ with $|\alpha| = 1$, we have $\Phi^\alpha_{ij} = 0$, and hence

$$D^\alpha(F_3^{ij} - F_1^{ij}) = \frac{1}{(F_2^{ij} - F_1^{ij})^2} \cdot \left( (F_2^{ij} - F_1^{ij}) \cdot D^\alpha(F_3^{ij} - F_1^{ij}) 
- (F_3^{ij} - F_1^{ij}) \cdot D^\alpha(F_2^{ij} - F_1^{ij}) \right)$$

$$= \frac{1}{(F_2^{ij} - F_1^{ij})^2} \cdot \left| \begin{array}{ccc} 1 & 1 & 1 \\ F_1^{ij} & F_2^{ij} & F_3^{ij} \\ D^\alpha(F_1^{ij}) & D^\alpha(F_2^{ij}) & D^\alpha(F_3^{ij}) \end{array} \right| = 0.$$

Since the above equality hold for all $|\alpha| = 1$, then there exists a constant $c \in C$ such that

$$\frac{F_3^{ij} - F_1^{ij}}{F_2^{ij} - F_1^{ij}} = c$$

By Theorem [11] we have $f^1 \land f^2 \land f^3 = 0$. Then for each index $t \in \{1, \ldots, 2n + 2\} \setminus \{i, j\}$ we have

$$0 = \det \left( \begin{array}{ccc} (f_1, H_i) & (f_1, H_j) & (f_1, H_t) \\ (f_2, H_i) & (f_2, H_j) & (f_2, H_t) \\ (f_3, H_i) & (f_3, H_j) & (f_3, H_t) \end{array} \right) = \prod_{u=1}^{3} (f_u, H_i) \cdot \det \left( \begin{array}{ccc} 1 & F_1^{ij} & F_1^{ti} \\ 1 & F_2^{ij} & F_2^{ti} \\ 1 & F_3^{ij} & F_3^{ti} \end{array} \right)$$

$$= \prod_{u=1}^{3} (f_u, H_i) \cdot \det \left( \begin{array}{ccc} F_2^{ij} - F_1^{ij} & F_2^{ti} - F_1^{ti} \\ F_3^{ij} - F_1^{ij} & F_3^{ti} - F_1^{ti} \end{array} \right).$$
Thus
\[(F_2^{ji} - F_1^{ji}) \cdot (F_3^{ti} - F_1^{ti}) = (F_3^{ji} - F_1^{ji}) \cdot (F_2^{ti} - F_1^{ti}).\]

If $F_2^{ti} - F_1^{ti} = 0$ then $F_3^{ti} - F_1^{ti} = 0$, and hence $\Phi_{it}^\alpha = 0$ for all $\alpha \in \mathbb{Z}_+^m$ with $|\alpha| < 1$. Otherwise, we have
\[
\frac{F_3^{ti} - F_1^{ti}}{F_2^{ti} - F_1^{ti}} = \frac{F_3^{ji} - F_1^{ji}}{F_2^{ti} - F_1^{ti}} = c.
\]

This also implies that
\[
\Phi_{it}^\alpha = F_1^{it} \cdot F_2^{ii} \cdot F_3^{it} \cdot \left| \begin{array}{ccc}
1 & 1 & 1 \\
F_1^{ti} & F_2^{ti} & F_3^{ti} \\
\mathcal{D}^\alpha (F_1^{ti}) & \mathcal{D}^\alpha (F_2^{ti}) & \mathcal{D}^\alpha (F_3^{ti})
\end{array} \right| \\
= F_1^{it} \cdot F_2^{ii} \cdot F_3^{it} \cdot \left| \begin{array}{ccc}
F_2^{ti} - F_1^{ti} & F_3^{ti} - F_1^{ti} \\
\mathcal{D}^\alpha (F_2^{ti} - F_1^{ti}) & \mathcal{D}^\alpha (F_3^{ti} - F_1^{ti})
\end{array} \right| \\
= F_1^{it} \cdot F_2^{ii} \cdot F_3^{it} \cdot \left| \begin{array}{ccc}
F_2^{ti} - F_1^{ti} & c(F_2^{ti} - F_1^{ti}) \\
\mathcal{D}^\alpha (F_2^{ti} - F_1^{ti}) & c\mathcal{D}^\alpha (F_2^{ti} - F_1^{ti})
\end{array} \right| = 0.
\]

Then one always has $\Phi_{it}^\alpha = 0$ for all $t \in \{1, \ldots, 2n + 2\} \setminus \{i\}$. The first assertion is proved.

(b) We suppose that $V_i \not\supseteq V_t$. From the above part, we have
\[cF_2^{si} + (1 - c)F_1^{si} = F_3^{si} (s \neq i).
\]

By the supposition $f^1, f^2, f^3$ are distinct, we have $c \not\in \{0, 1\}$. This implies that $F_1^{ti}, F_2^{ti}, F_3^{ti}$ are distinct.

We see that the second inequality is clear, then we prove the remain first inequality. We consider the meromorphic mapping $F^t$ of $\mathbb{C}^m$ into $\mathbb{P}^1(\mathbb{C})$ with a reduced representation
\[F^t = (F_1^{ti} h_t : F_2^{ti} h_t),\]

where $h_t$ is a meromorphic function on $\mathbb{C}^m$. We see that
\[
T_{F^t}(r) = T \left( r, \frac{F_1^{ti}}{F_2^{ti}} \right) \leq T(r, F_1^{ti}) + T \left( r, \frac{1}{F_2^{ti}} \right) + O(1) \\
\leq T(r, F_1^{ti}) + T(r, F_2^{ti}) + O(1) \leq T_{f_1}(r) + T_{f_2}(r) + O(1) = O(T_f(r)).
\]

For a point $z \not\in I(\mathbb{C}^t) \cup S_i \cup S_t$ which is a zero of some functions $F_{u}^{ti} h_t$ ($1 \leq u \leq 3$), then $z$ must be either zero of $(f, H_i)$ with multiplicity at most $k_i$ or zero of $(f, H_t)$ with multiplicity at most $k_t$, and hence
\[
\sum_{u=1}^{3} \nu_{F_{u}^{ti} h_t}^{(1)}(z) = 1 \leq \nu_{(f, H_i), \leq k_i}^{(1)}(z) + \nu_{(f, H_t), \leq k_t}^{(1)}(z).
\]

This implies that
\[
\sum_{u=1}^{3} \nu_{F_{u}^{ti} h_t}^{(1)}(z) \leq \nu_{(f, H_i), \leq k_i}^{(1)}(z) + \nu_{(f, H_t), \leq k_t}^{(1)}(z) + \chi_{S_i}(z) + \chi_{S_t}(z)
\]
outside an analytic subset of codimension two. By integrating both sides of this inequality, we get

\[(4.5) \quad \sum_{u=1}^{3} N_{F_{u}^{(1)}b_{i}}^{(1)}(r) \leq N_{(f,H_{i}),\leq k_{i}}^{(1)}(r) + N_{(f,H_{i}),\leq k_{t}}^{(1)}(r) + N(r,S_{i}) + N(r,S_{t}).\]

By the second main theorem, we also have

\[(4.6) \quad ||T_{F}(r)| \leq \sum_{u=1}^{3} N_{F_{u}^{(1)}b_{i}}^{(1)}(r) + o(T(r)).\]

On the other hand, applying the first main theorem to the map \(F_{t}\) and the hyperplane \(\{w_{0} - w_{1} = 0\}\) in \(\mathbb{P}^{1} (\mathbb{C})\), we have

\[(4.7) \quad T_{F}(r) \geq N_{(F_{1}^{(1)} - F_{2}^{(1)})b_{i}}^{(1)}(r) \geq \sum_{\substack{u=1 \\nu \neq i,t}}^{2n+2} N_{(f,H_{u}),\leq k_{u}}^{(1)}(r) - N(r,S_{i}) - N(r,S_{t}).\]

Therefore, from (4.5), (4.6) and (4.7) we have

\[ || N_{(f,H_{i}),\leq k_{i}}^{(1)}(r) \geq \sum_{\substack{u=1 \\nu \neq i,t}}^{2n+2} N_{(f,H_{u}),\leq k_{u}}^{(1)}(r) - N_{(f,H_{i}),\leq k_{t}}^{(1)}(r) - 2(N(r,S_{i}) + N(r,S_{t})) + o(T(r)). \]

The second assertion of the lemma is proved. \(\square\)

**Lemma 4.8.** Let \(f^{1}, f^{2}, f^{3}\) be three maps in \(\mathcal{F}(f, \{H_{i}, k_{i}\}_{i=1}^{2n+2}, 1)\) as in Theorem 4.3. Assume that there exist \(i, j \in \{1, 2, \ldots , 2n+2\} (i \neq j)\) and \(\alpha \in \mathbb{Z}_{+}^{n} \) with \(|\alpha| = 1\) such that \(\Phi_{ij}^{\alpha} \neq 0\). Then we have

\[
T(r) \geq \sum_{u=1}^{3} N_{(f^{u},H_{i}),\leq k_{i}}^{(n)}(r) + \sum_{k=1}^{3} N_{(f^{k},H_{j}),\leq k_{j}}^{(n)}(r) + 2 \sum_{t=1}^{2n+2} N_{(f,H_{t}),\leq k_{t}}^{(1)}(r)
- (2n+1)N_{(f,H_{i}),\leq k_{i}}^{(1)}(r) - (n+1)N_{(f,H_{j}),\leq k_{j}}^{(1)}(r) + N(r,\nu_{i})
- N(r,S_{i}) - N(r,S_{j}) - (2n-2)N(r,R_{i}) - (n-1)N(r,R_{j}) + o(T(r))
\geq \sum_{u=1}^{3} N_{(f^{u},H_{i}),\leq k_{i}}^{(n)}(r) + \sum_{k=1}^{3} N_{(f^{k},H_{j}),\leq k_{j}}^{(n)}(r) + 2 \sum_{t=1}^{2n+2} N_{(f,H_{t}),\leq k_{t}}^{(1)}(r)
- (2n+1)N_{(f,H_{i}),\leq k_{i}}^{(1)}(r) - (n+1)N_{(f,H_{j}),\leq k_{j}}^{(1)}(r) + N(r,\nu_{i})
- \sum_{u=1}^{3} ((1 + \frac{n-1}{3})N_{(f^{u},H_{i}),\leq k_{i}}^{(1)}(r) - (1 + \frac{2n-2}{3})N_{(f^{u},H_{i}),\leq k_{i}}^{(1)}(r) + o(T(r)).
\]
Proof. The second inequality is clear. We remain prove the first inequality. We have
\[
\Phi^\alpha = F^{ij} \cdot F^{ij}_2 \cdot F^{ij}_3.
\]
\[
= \begin{vmatrix}
F^{ij}_1 & F^{ij}_2 & F^{ij}_3 \\
F^{ij}_1 & F^{ij}_2 & F^{ij}_3 \\
F^{ij}_1 & F^{ij}_2 & F^{ij}_3 \\
\end{vmatrix}.
\]
Thus
\[
\Phi^\alpha_{ij} = F^{ij}_1 \left( \frac{\partial \Phi^{ji}(F^{ji}_3)}{F^{ji}_3} - \frac{\partial \Phi^{ji}(F^{ji}_2)}{F^{ji}_2} \right) + F^{ij}_2 \left( \frac{\partial \Phi^{ji}(F^{ji}_1)}{F^{ji}_1} - \frac{\partial \Phi^{ji}(F^{ji}_3)}{F^{ji}_3} \right)
\]
(4.9)
\[
+ F^{ij}_3 \left( \frac{\partial \Phi^{ji}(F^{ji}_2)}{F^{ji}_2} - \frac{\partial \Phi^{ji}(F^{ji}_1)}{F^{ji}_1} \right).
\]
By the Logarithmic Derivative Lemma, it follows that
\[
m(r, \Phi^\alpha_{ij}) \leq \sum_{u=1}^3 m(r, F^i_u) + 2 \sum_{u=1}^3 m \left( \frac{\partial \Phi^{ji}(F^i_u)}{F^i_u} \right) + O(1) \leq \sum_{u=1}^3 m(r, F^i_u) + o(T_j(r)).
\]
Therefore, we have
\[
T(r) = \sum_{u=1}^3 T(r, F^i_u) = \sum_{u=1}^3 m(r, F^i_u) + N_{1/F^i_u}(r) = m(r, \Phi^\alpha_{ij}) + \sum_{u=1}^3 N_{1/F^i_u}(r) + o(T(r))
\]
\[
\geq T(r, \Phi^\alpha_{ij}) - N_{1/F^i_u}(r) + \sum_{u=1}^3 N_{1/F^i_u}(r) + o(T(r))
\]
\[
\geq N_{\Phi^\alpha_{ij}}(r) - N_{1/F^i_u}(r) + \sum_{u=1}^3 N_{1/F^i_u}(r) + o(T(r))
\]
\[
= N(r, \nu_{\Phi^\alpha_{ij}}) + \sum_{u=1}^3 N_{1/F^i_u}(r) + o(T(r)).
\]
Then, in order to prove the lemma, it is sufficient for us to prove
\[
N(r, \nu_{\Phi^\alpha_{ij}}) \geq \sum_{u=1}^3 N^{(1)}_{(f^i_u,H_t), \leq k_i}(r) + \sum_{k=1}^{2n+2} N^{(1)}_{(f^k,H_t), \leq k_j}(r) + 2 \sum_{t=1}^3 N^{(1)}_{(f,H_t), \leq k_t}(r)
\]
(4.10)
\[
- (2n + 1)N^{(1)}_{(f,H_s), \leq k_i}(r) - (n + 1)N^{(1)}_{(f,H_s), \leq k_j}(r) - \sum_{u=1}^3 N_{1/F^i_u}(r) + N(r, \nu_j)
\]
\[
- N(r, S_i) - N(r, S_j) - (2n - 2)N(r, R_t) - (n - 1)N(r, R_t) + o(T(r)).
\]
Denote by $S$ the set of all singularities of $f^{-1}(H_t)$ ($1 \leq t \leq q$). Then $S$ is an analytic subset of codimension at least two in $C^m$. We set
\[
I = S \cup \bigcup_{s \neq t} \{ z; \nu_{(f,H_s), \leq k_s}(z) \cdot \nu_{(f,H_t), \leq k_t}(z) > 0 \}.
\]
Then $I$ is also an analytic subset of codimension at least two in $\mathbb{C}^m$.

In order to prove the inequality (4.10), it is sufficient for us to show that the following inequality

$$P : \text{Def} 3 \sum_{u=1}^{3} \nu_{(f^u, H_i), \leq k_i} + 3 \sum_{u=1}^{3} \nu_{(f^u, H_j), \leq k_j} + 2 \sum_{t=1}^{2n+2} \chi_{T_t} - (2n+1)\chi_{T_i} - (n+1)\chi_{T_j}$$

$$- 3 \sum_{u=1}^{3} \nu_{F_{ij}^u} + \chi_{T_i} - \chi_{S_i} - 2(n-1)\chi_{R_i} - (n-1)\chi_{R_j} \leq \nu_{\Phi_{ij}^\alpha}.$$ 

hold outside the set $I$.

For $z \notin I$, we distinguish the following cases

**Case 1**: $z \in T_i \setminus S_i \cup S_j$ $(t \neq i, j)$. We see that $P(z) = 2$. We write $\Phi_{ij}^\alpha$ in the form

$$\Phi_{ij}^\alpha = F_{ij}^1 \cdot F_{ij}^2 \cdot F_{ij}^3 \times \left| \frac{(F_{ij}^1 - F_{ij}^2)}{D^\alpha (F_{ij}^1 - F_{ij}^2)} \frac{(F_{ij}^1 - F_{ij}^3)}{D^\alpha (F_{ij}^1 - F_{ij}^3)} \right| .$$

Then by the assumption that $f^1, f^2, f^3$ are identify on $T_t$, we have $F_{ij}^1 = F_{ij}^2 = F_{ij}^3$ on $T_i \setminus S_i$. The property of the wronskian implies that $\nu_{\Phi_{ij}^\alpha}(z) \geq 2 = P(z)$.

**Case 2**: $z \in T_i \cap (S_i \cup S_j)$ $(t \neq i, j)$. We see that $P(z) \leq -3 \sum_{u=1}^{3} \nu_{F_{ij}^u} (z) - 1$.

From (4.9) we see that

$$\nu_{\Phi_{ij}^\alpha}(z) \geq \min \{ \nu_{F_{ij}^1}(z) - 1, \nu_{F_{ij}^2}(z) - 1, \nu_{F_{ij}^3}(z) - 1 \} \geq P(z).$$

**Case 3**: $z \in T_i \setminus S_j$. We have

$$P(z) = 3 \sum_{u=1}^{3} \nu_{(f^u, H_i), \leq k_i} (z) - (2n+1) \leq \min \{ \nu_{(f^u, H_i), \leq k_i} (z) \} - 1.$$ 

We may assume that $\nu_{(f^1, H_i)}(z) \leq \nu_{(f^2, H_i)}(z) \leq \nu_{(f^3, H_i)}(z)$. We write

$$\Phi_{ij}^\alpha = F_{ij}^1 \left[ F_{ij}^2 (F_{ij}^1 - F_{ij}^2) F_{ij}^3 D^\alpha (F_{ij}^1 - F_{ij}^3) - F_{ij}^3 (F_{ij}^1 - F_{ij}^2) F_{ij}^2 D^\alpha (F_{ij}^1 - F_{ij}^2) \right]$$

It is easy to see that $F_{ij}^2 (F_{ij}^1 - F_{ij}^2)$ and $F_{ij}^3 (F_{ij}^1 - F_{ij}^3)$ are holomorphic on a neighborhood of $z$ and

$$\nu_{F_{ij}^2 D^\alpha (F_{ij}^1 - F_{ij}^2)}(z) \leq 1$$

and

$$\nu_{F_{ij}^3 D^\alpha (F_{ij}^1 - F_{ij}^3)}(z) \leq 1.$$ 

Therefore, it implies that

$$\nu_{\Phi_{ij}^\alpha}(z) \geq \nu_{(f^1, H_i), \leq k_i} (z) - 1 \geq P(z).$$
Case 4: \( z \in T_1 \cap S_j \). The assumption that \( f^1, f^2, f^3 \) are identity on \( T_i \) yields that \( z \in R_j \). We have
\[
P(z) \leq - \sum_{u=1}^{3} \nu^{(n)}_{F^u} (z) - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) - (2n + 1) - n \leq - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) - 1.
\]
We have
\[
\nu_{\Phi^ij} (z) \geq \min \{ \nu_{F^1} (z) - 1, \nu_{F^2} (z) - 1, \nu_{F^3} (z) - 1 \} \geq - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) - 1 \geq P(z).
\]

Case 5: \( z \in T_j \setminus S_i \). Put
\[
\nu_{F^1} (z) = d_1 \geq \nu_{F^2} (z) = d_2 \geq \nu_{F^3} (z) = d_3.
\]
Choose a holomorphic function \( h \) on \( C^m \) with multiplicity 1 at \( z \) such that \( F^u_{ij} = h^{d_u} \varphi_u \) (\( 1 \leq u \leq 3 \)), where \( \varphi_u \) are meromorphic on \( C^m \) and holomorphic on a neighborhood of \( z \). Then
\[
\Phi^ij = F^1_{ij} \cdot F^2_{ij} \cdot F^3_{ij} \cdot \begin{vmatrix} F^2_{ij} - F^1_{ij} & F^3_{ij} - F^1_{ij} \\ D^a (F^2_{ij} - F^1_{ij}) & D^a (F^3_{ij} - F^1_{ij}) \end{vmatrix} = F^1_{ij} \cdot F^2_{ij} \cdot F^3_{ij} \cdot h^{d_2 + d_3} \cdot \begin{vmatrix} \varphi_2 - h^{d_1-d_2} \varphi_1 \\ D^a (h^{d_2-d_3} \varphi_2 - h^{d_1-d_3} \varphi_1) \end{vmatrix} \begin{vmatrix} \varphi_3 - h^{d_1-d_3} \varphi_1 \\ D^a (\varphi_3 - h^{d_1-d_3} \varphi_1) \end{vmatrix}.
\]
This yields that
\[
\nu_{\Phi^ij} (z) \geq \sum_{u=1}^{3} \nu_{F^u} (z) + d_2 + d_3 - \max \{0, \min \{1, d_2 - d_3\}\}.
\]

If \( z \not\in S_i \) then
\[
P(z) = - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) + \sum_{u=1}^{3} \min \{n, d_u\} - (n + 1) + \chi_{ij}.
\]
and
\[
\nu_{\Phi^ij} (z) \geq - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) + \sum_{u=1}^{3} \nu^0_{F^u} (z) + d_2 + d_3 - 1 + \chi_{ij} \geq P(z).
\]
Otherwise, if \( z \in S_i \) then \( z \in R_i \), and hence
\[
P(z) \leq - \sum_{u=1}^{3} \nu^{(n)}_{F^u} (z) - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) - 3n - 1 + \chi_{ij} \leq - \sum_{u=1}^{3} \nu^{\infty}_{F^u} (z) - 3n,
\]

and \( \nu_{F_{ij}^α}(z) \geq -\sum_{u=1}^3 \nu_{F_{ij}^u}(z) + d_2 + d_3 - 1 \)

\[ \geq -\sum_{u=1}^3 \nu_{F_{ij}^u}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 \geq P(z). \]

Case 6: \( z \in (S_1 \cup S_j) \setminus \left( \bigcup_{i=1}^{2n+2} T_i \right) \). Similarly as Case 5, we have

\[ \nu_{F_{ij}^α}(z) \geq -\sum_{u=1}^3 \nu_{F_{ij}^u}(z) + \max\{0, -d_1\} + \max\{d_2, 0\} + \max\{d_3, 0\} - 1 \]

\[ \geq -\sum_{u=1}^3 \nu_{F_{ij}^u}(z) - 1 \geq -\sum_{u=1}^3 \nu_{F_{ij}^u}(z) - \chi s_i - \chi s_j \geq P(z). \]

From the above six cases, we see that the inequality (1.11) holds. Hence the lemma is proved.

**Proof of theorem 1.3** We note that, from the assumption it easy to verify that

- \( \sum_{i=1}^{2n+2} \frac{1}{k_i} < \frac{1}{n} \) (similar as in the proof of Theorem 1.1)
- If \( n \geq 3 \) then \( \frac{(n-2)(n+1)}{n(10n+1)} \leq \frac{n-1}{3(n+2)} \leq \frac{5n-6}{14n-3} \leq \frac{n-1}{2n} \).
- If \( n = 2 \) then \( \frac{1}{12} = \frac{n-1}{3(n+2)} \leq \frac{5n-6}{14n-3} \leq \frac{n-1}{2n} \).

Suppose that there exits three distinct maps \( f_1, f_2, f_3 \in \mathcal{F}(f, \{H_i, k_i\}_{i=1}^{2n+2}, 1) \). By Theorem 1.1 we have \( f_1 \wedge f_2 \wedge f_3 \equiv 0 \). Without loss of generality, we may assume that

\[ V_{i_1} \cong \cdots \cong V_{i_{l_1}} \not\cong V_{i_{l_1}+1} \cong \cdots \cong V_{i_{l_2}} \not\cong V_{i_{l_2}+1} \cong \cdots \cong V_{i_{l_3}+1} \not\cong \cdots \not\cong V_{i_{l_s}+1}, \]

where \( l_s = 2n+2 \).

Denote by \( P \) the set of all \( i \in \{1, \ldots, 2n+2\} \) satisfying there exist \( j \in \{1, \ldots, 2n+2\} \setminus \{i\} \) such that \( V_{i} \not\cong V_{j} \) and \( \Phi_{ij}^α \equiv 0 \) for all \( α \in \mathbb{Z}_+^m \) with \( |α| \leq 1 \). We consider the following three cases.

**Case 1:** \( \#P \geq 2 \). Then \( P \) contains two elements \( i, j \). Then we have \( \Phi_{ij}^α = \Phi_{ji}^α = 0 \) for all \( α \in \mathbb{Z}_+^m \) with \( |α| \leq 1 \). By Lemma 4.3 there exist two functions, for instance they are \( F_{ij}^1 \) and \( F_{ij}^2 \), and a constant \( λ \) such that \( F_{ij}^{ij} = λF_{ij}^{ij} \). This yields that \( F_{ij}^1 = F_{ij}^2 \) (by Lemma 4.1). Then by Lemma 4.4 (ii), we easily see that \( V_{i} \cong V_{j} \), i.e., \( V_{i} \) and \( V_{j} \) belong to the same group in the above partition.

Without loss of generality, we may assum that \( i = 1 \) and \( j = 2 \). Since \( f_1, f_2, f_3 \) are supposed to be distinct, the number of each group in the above partition is less than \( n+1 \). Hence we have \( V_1 \cong V_2 \not\cong V_t \) for all \( t \in \{n+1, \ldots, 2n+2\} \). Then by Lemma 4.4 (ii), we
have

\[ N^{(1)}_{(f, H_{t}), \leq k_{t}}(r) + N^{(1)}_{(f, H_{t}), > k_{t}}(r) \geq \sum_{s \neq 1, t} N^{(1)}_{(f, H_{s}), < k_{s}}(r) - \sum_{u=1}^{3} \sum_{s=1, t} N^{(1)}_{(f^{u}, H_{s}), > k_{s}}(r), \]

and

\[ N^{(1)}_{(f, H_{2}), \leq k_{t}}(r) + N^{(1)}_{(f, H_{2}), > k_{t}}(r) \geq \sum_{s \neq 2, t} N^{(1)}_{(f, H_{s}), < k_{s}}(r) - \sum_{u=1}^{3} \sum_{s=2, t} N^{(1)}_{(f^{u}, H_{s}), > k_{s}}(r). \]

Summing-up both sides of the above two inequalities, we get

\[ 2N^{(1)}_{(f, H_{t}), \leq k_{t}}(r) \geq \sum_{s \neq 1, 2, t} N^{(1)}_{(f, H_{s}), < k_{s}}(r) - \sum_{u=1}^{3} (N^{(1)}_{(f^{u}, H_{1}), > k_{1}}(r) + N^{(1)}_{(f^{u}, H_{2}), > k_{2}}(r)) + 2N^{(1)}_{(f^{u}, H_{t}), > k_{t}}(r). \]

After summing-up both sides of the above inequalities over all \( t \in \{n + 1, 2n + 2\} \), we easily obtain

\[
\begin{align*}
|| & \sum_{u=1}^{3} ((n + 2)(N^{(1)}_{(f^{u}, H_{1}), > k_{1}}(r) + N^{(1)}_{(f^{u}, H_{2}), > k_{2}}(r)) + 2 \sum_{t=n+1}^{2n+2} N^{(1)}_{(f^{u}, H_{t}), > k_{t}}(r)) \\
& \geq (n + 2) \sum_{t=3}^{n} N^{(1)}_{(f, H_{t}), \leq k_{t}}(r) + n \sum_{t=n+1}^{2n+2} N^{(1)}_{(f, H_{t}), \leq k_{t}}(r) \\
& \geq n \sum_{t=3}^{2n+2} N^{(1)}_{(f, H_{t}), \leq k_{t}}(r) \geq \frac{n}{3} \sum_{u=1}^{3} \sum_{t=3}^{2n+2} N^{(1)}_{(f^{u}, H_{t}), \leq k_{t}}(r) \\
& \geq \frac{1}{3} \sum_{u=1}^{3} \sum_{t=3}^{2n+2} N^{(n)}_{(f^{u}, H_{t}), r}(r) - \frac{n}{3} \sum_{u=1}^{3} \sum_{t=3}^{2n+2} N^{(1)}_{(f^{u}, H_{t}), > k_{t}}(r) \\
& \geq \frac{n}{3} \sum_{u=1}^{3} \sum_{t=3}^{2n+2} N^{(1)}_{(f^{u}, H_{t}), > k_{t}}(r) + o(T(r)).
\end{align*}
\]

Therefore, we have

\[
\frac{n - 1}{3} T(r) \leq (n + 2) \sum_{u=1}^{3} \sum_{t=1}^{2n+2} \frac{1}{k_{t}+1} N^{(1)}_{(f^{u}, H_{t}), > k_{t}}(r) \leq (n + 2) \sum_{u=1}^{3} \sum_{t=1}^{2n+2} \frac{1}{k_{t}+1} N^{(1)}_{(f^{u}, H_{t})}(r)
\]

\[
\leq (n + 2) \sum_{t=1}^{2n+2} \frac{1}{k_{t}+1} T(r).
\]

Letting \( r \longrightarrow +\infty \), we get

\[
\frac{n - 1}{3(n + 2)} \leq \sum_{t=1}^{2n+2} \frac{1}{k_{t}+1}.
\]

This is a contradiction.
Case 2: \( \sharp P = 1 \). We assume that \( P = \{1\} \). We easily see that \( V_i \not\supseteq V_i \) for all \( i = 2, \ldots, 2n + 2 \) (otherwise \( i \in P \), this contradict to \( \sharp P = 1 \)). Then by Lemma 4.4 (ii), we have

\[
N_{(f, H_i), \leq k_i}(r) \geq \sum_{s \neq 1, i} N_{(f, H_s), \leq k_s}(r) - N_{(f, H_i), \leq k_i}(r) - 2 \sum_{u=1}^{3} \sum_{s=1, i} N_{(f^u, H_s), > k_s}(r) + o(T(r)).
\]

Summing-up both sides of the above inequality over all \( i = 2, \ldots, 2n + 2 \), we get

\[
(2n + 1)N_{(f, H_1), \leq k_1}(r) \geq (2n - 1) \sum_{i=2}^{2n+2} N_{(f, H_i), \leq k_i}(r) - 2 \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i), > k_i}(r)
- 2(2n + 1) \sum_{u=1}^{3} N_{(f^u, H_1), > k_1}(r) + o(T(r)).
\]

(4.12)

We also see that \( i \not\in P \) for all \( 2 \leq i \leq 2n + 2 \). We set

\[
\sigma(i) = \begin{cases} 
    i + n & \text{if } i \leq n + 2, \\
    i - n & \text{if } n + 2 < i \leq 2n + 2.
\end{cases}
\]

Then we easily see that \( i \) and \( \sigma(i) \) belong to two distinct groups, i.e, \( V_i \not\supseteq V_{\sigma(i)} \), for all \( i \in \{2, \ldots, 2n + 2\} \), and hence \( \Phi^\alpha_{i, \sigma(i)} \not\equiv 0 \) for all \( \alpha \in \mathbb{Z}^n \) with \( |\alpha| \leq 1 \). By Lemma 4.5 we have

\[
T(r) \geq \sum_{u=1}^{3} \sum_{t=i, \sigma(i)} N_{(f^u, H_t), \leq k_t}(r) - (2n + 1)N_{(f, H_i), \leq k_i}(r) - (n + 1)N_{(f, H_{\sigma(i)}), \leq k_{\sigma(i)}}(r)
+ 2 \sum_{t=1}^{2n+2} N_{(f, H_t), \leq k_t}(r) - \sum_{u=1}^{3} \left( \frac{2n + 1}{3} N_{(f^u, H_i), > k_i}(r) + \frac{n + 2}{3} N_{(f^u, H_{\sigma(i)}), > k_{\sigma(i)}}(r) \right)
+ o(T(r)).
\]
Summing-up both sides of the above inequalities over all \( i \in \{2, \ldots, 2n + 2\} \), we get

\[
(2n + 1)T(r) \geq 2 \sum_{i=2}^{2n+2} \sum_{u=1}^{3} N_{(f^u, H_i), \leq k_i}(r) + (n - 4) \sum_{i=2}^{2n+2} N_{(f, H_i), \leq k_i}(r)

+ 2(2n + 1)N_{(f, H_1), \leq k_1}(r) - (n + 1) \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i), > k_i} + o(T(r))

\geq 2 \sum_{i=2}^{2n+2} \sum_{u=1}^{3} N_{(f^u, H_i), \leq k_i}(r) + \frac{5n - 6}{3} \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i), \leq k_i}(r)

- \frac{4n + 2}{3} \sum_{u=1}^{3} N_{(f^u, H_1), > k_1}(r) - (n + 1) \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i), > k_i} + o(T(r)) + o(T(r))

\geq \frac{11n - 6}{3n} \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i)}(r)

- \frac{4n + 2}{3} \sum_{u=1}^{3} N_{(f^u, H_1), > k_1}(r) - \frac{14n - 3}{3} \sum_{u=1}^{3} \sum_{i=2}^{2n+2} N_{(f^u, H_i), > k_i} + o(T(r)) + o(T(r))

\geq \frac{11n - 6}{3} T(r) - \frac{14n - 3}{3} \sum_{i=1}^{2n+2} \frac{1}{k_i + 1} T(r) + o(T(r)).

Letting \( r \to +\infty \), we get

\[
\frac{5n - 6}{14n - 3} \leq \sum_{i=1}^{2n+2} \frac{1}{k_i + 1}.
\]

This is a contradiction.

**Case 3:** \( P = \emptyset \). Then for all \( i \neq j \), by Lemma 4.3 we have

\[
T(r) \geq 3 \sum_{u=1}^{3} N_{(f^u, H_i), \leq k_i}(r) + \sum_{k=1}^{3} N_{(f^k, H_j), \leq k_j}(r) + 2 \sum_{i=1}^{2n+2} N_{(f, H_i), \leq k_i}(r)

- (2n + 1)N_{(f, H_1), \leq k_1}(r) - (n + 1)N_{(f, H_j), \leq k_j}(r) + N(r, \nu_j)

- \sum_{u=1}^{3} \left( (1 + \frac{n - 1}{3})N_{(f^u, H_j), > k_j}(r) + (1 + \frac{2n - 2}{3})N_{(f^u, H_i), > k_i}(r) \right) + o(T(r)).
\]
Summing-up both sides of the above inequalities over all pairs \((i, j)\) we get

\[
(2n + 2)T(r) \geq 2 \sum_{u=1}^{3} \sum_{t=1}^{2n+2} N^{(n)}_{(f^u, H_t), \leq k_t}(r) + (n - 2) \sum_{t=1}^{2n+2} N^{(1)}_{(f, H_t), \leq k_t}(r) + \sum_{t=1}^{2n+2} N(r, \nu_t)
\]

\[
- (n + 1) \sum_{u=1}^{3} \sum_{t=1}^{2n+2} N_{(f^u, H_t), \geq k_t} + o(T(r))
\]

\[
\geq (2 + \frac{n - 2}{3n}) \sum_{u=1}^{3} \sum_{t=1}^{2n+2} N^{(n)}_{(f^u, H_t), \leq k_t}(r) - \frac{10n + 1}{3} \sum_{u=1}^{3} \sum_{t=1}^{2n+2} N^{(1)}_{(f^u, H_t), > k_t}(r)
\]

\[
+ \sum_{t=1}^{2n+2} N(r, \nu_t) + o(T(r))
\]

\[
\geq \frac{(7n - 2)(n + 1)}{3n} T(r) - \frac{10n + 1}{3} \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} T(r) + \sum_{t=1}^{2n+2} N(r, \nu_t)
\]

(4.13) + o(T(r)).

We now consider two cases where \(n \geq 3\) and \(n = 2\).

(a) If \(n \geq 3\), from (4.13) we get

\[
(2n + 2)T(r) \geq \frac{(7n - 2)(n + 1)}{3n} T(r) - \frac{10n + 1}{3} \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} T(r) + o(T(r)).
\]

Letting \(r \to +\infty\), we get

(4.14) \[
\sum_{t=1}^{2n+2} \frac{1}{k_t + 1} \geq \frac{(n - 2)(n + 1)}{n(10n + 1)}.
\]

The inequality (4.14) contradicts to the assumption.

(b) If \(n = 2\), from (4.13) we get

(4.15) \[
\sum_{i=1}^{6} N(r, \nu_i) \leq 7 \sum_{t=1}^{2n+2} \frac{1}{k_t + 1} T(r) + o(T(r)).
\]

We see that \(V_i \neq V_{i+3}\) \((1 \leq i \leq 3)\). By Lemma 4.2, we see that \(V_i \not\sim V_j\) for all \(i \neq j\). Hence, we have

\[P_{st}^{ij} \buildrel \text{Def} \over = (f^s, H_t)(f^t, H_j) - (f^t, H_j)(f^s, H_t) \neq 0 \ (s \neq t, i \neq j).\]

Claim 4.16. With \(i \neq j \neq l \neq i\), for every \(z \in T_i\) we have

\[
\sum_{1 \leq s < t \leq 3} \nu_{P_{st}^{ij}}(z) \geq 4\chi_{T_i}(z) - \chi_{\nu_i}(z).
\]
Indeed, for \( z \in T_i \setminus \nu_i \), we may assume that \( \nu(f_1, H_i)(z) < \nu(f_2, H_i)(z) \leq \nu(f_3, H_i)(z) \). Since \( f^1 \wedge f^2 \wedge f^3 \equiv 0 \), we have \( \det(V_i, V_j, V_l) \equiv 0 \), and hence

\[
(f^1, H_i) P_{23}^{ij} = (f^2, H_i) P_{13}^{ij} - (f^3, H_i) P_{12}^{ij}.
\]

This yields that

\[
\nu_{P_{23}^{ij}}(z) \geq 2
\]

and hence \( \sum_{1 \leq s < t \leq 3} \nu_{P_{st}^{ij}}(z) \geq 4 = 4 \chi_{T_i}(z) - \chi_{\nu_i}(z) \).

Now, for \( z \in \nu_i \), we have \( \sum_{1 \leq s < t \leq 3} \nu_{P_{st}^{ij}}(z) \geq 3 = 4 \chi_{T_i}(z) - \chi_{\nu_i}(z) \). Hence, the claim is proved.

On the other hand, with \( i = j \) or \( i = l \), for every \( z \in \{ \nu(f, H_i) \leq k_i(z) > 0 \} \) we see that

\[
\nu_{P_{st}^{ij}}(z) \geq \min\{ \nu(f^{(1)}, H_i) \leq k_i(z), \nu(f^{(2)}, H_i) \leq k_i(z) \}
\]

\[
\geq \nu^{(2)}(f^{(1)}, H_i) \leq k_i(z) + \nu^{(2)}(f^{(1)}, H_i) \leq k_i(z) - 2 \nu^{(1)}(f^{(1)}, H_i) \leq k_i(z).
\]

and hence

\[
\sum_{1 \leq s < t \leq 3} \nu_{P_{st}^{ij}}(z) \geq \sum_{u=1}^{3} \nu^{(2)}(f^{(1)}, H_i) \leq k_i(z) - 6 \nu^{(1)}(f^{(1)}, H_i) \leq k_i(z).
\]

Combining this inequality and the above claim, we have

\[
\sum_{1 \leq s < t \leq 3} \nu_{P_{st}^{ij}}(z) \geq 2 \sum_{i=s,l}^{3} \left( \sum_{u=1}^{3} \nu^{(2)}(f^{(1)}, H_i) \leq k_i(z) - 6 \nu^{(1)}(f^{(1)}, H_i) \leq k_i(z) \right) + \sum_{i \neq s,l}^{3} \left( 4 \nu^{(1)}(f^{(1)}, H_i) \leq k_i(z) - \chi_{\nu_i}(z) \right).
\]

This yields that

\[
\sum_{1 \leq s < t \leq 3} N_{P_{st}^{ij}}(z) \geq 2 \sum_{i=s,l}^{3} \left( \sum_{u=1}^{3} N^{(2)}(f^{(1)}, H_i) \leq k_i(r) - 6 N^{(1)}(f^{(1)}, H_i) \leq k_i(r) \right) + \sum_{i \neq s,l}^{3} \left( 4 N^{(1)}(f^{(1)}, H_i) \leq k_i(r) - N(r, \nu_i) \right).
\]

(4.17)

On the other hand, be Jensen formula, we easily see that

\[
N_{P_{st}^{ij}}(z) \leq T^{(1)}(r) + T^{(2)}(r) + o(T(r)) \quad (1 \leq s < t \leq 3).
\]

Then the inequality (4.17) implies that

\[
2T(r) \geq 2 \sum_{i=s,l}^{3} \left( \sum_{u=1}^{3} N^{(2)}(f^{(1)}, H_i) \leq k_i(r) - 6 N^{(1)}(f^{(1)}, H_i) \leq k_i(r) \right) + \sum_{i \neq s,l}^{3} \left( 4 N^{(1)}(f^{(1)}, H_i) \leq k_i(r) - N(r, \nu_i) \right).
\]

Summing-up both sides of the above inequalities over all pair \((j, l)\), we obtain

\[
2T(r) \geq \frac{4}{6} \sum_{u=1}^{3} \sum_{i=1}^{6} N^{(2)}(f^{(1)}, H_i) \leq k_i(r) + \frac{10}{3} \times 6 \sum_{u=1}^{3} \sum_{i=1}^{6} N^{(1)}(f^{(1)}, H_i) \leq k_i(r) - \frac{2}{6} \sum_{i=1}^{6} N(r, \nu_i) + o(T(r)).
\]
Thus
\[
\|2T(r)\| \geq \frac{4}{6} \sum_{u=1}^{3} \sum_{i=1}^{6} N_{(f^{u}, H_{i})}^{(2)} < k_{i}(r) + \frac{5}{18} \sum_{u=1}^{3} \sum_{i=1}^{6} N_{(f^{u}, H_{i})}^{(2)} \leq k_{i}(r)
- \frac{14}{6} \sum_{i=1}^{6} \frac{1}{k_{i} + 1} T(r) + o(T(r))
\leq \frac{17}{18} \sum_{u=1}^{3} \sum_{i=1}^{6} N_{(f^{u}, H_{i})}^{(2)} (r) - \frac{14}{6} \sum_{i=1}^{6} \frac{1}{k_{i} + 1} T(r) + o(T(r))
\geq \frac{17}{18} \sum_{u=1}^{3} \sum_{i=1}^{6} N_{(f^{u}, H_{i})}^{(2)} (r) - \frac{17}{18} \sum_{u=1}^{3} \sum_{i=1}^{6} N_{(f^{u}, H_{i})}^{(2)} > k_{i}(r)
- \frac{14}{6} \sum_{i=1}^{6} \frac{1}{k_{i} + 1} T(r) + o(T(r))
\geq \frac{17}{18} (6 - 3) T(r) - \frac{34}{18} \sum_{i=1}^{6} \frac{1}{k_{i} + 1} T(r) - \frac{14}{6} \sum_{i=1}^{6} \frac{1}{k_{i} + 1} T(r) + o(T(r)).
\]

Letting \(r \to +\infty\), we get
\[
\frac{1}{k_{i} + 1} \leq \frac{15}{76}.
\]

This is a contradiction.

Hence the supposition is impossible. Therefore, \(\|\mathcal{F}(f, \{H_{i}, k_{i}\}_{i=1}^{2n+2}, 1)\| \leq 2\). We complete the proof of the theorem. \(\square\)

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