COHERENT STATES AND THE MEASUREMENT PROBLEM

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The convenience of coherent state representation is discussed from the viewpoint of what is in a broad sense called the measurement problem in quantum mechanics. Standard quantum theory in coherent state representation is intrinsically related to a number of earlier concepts reconciling quantum and classical processes. From a natural statistical interpretation, free of collapses or measurements, the usual von Neumann-Lüders collapse as well as its quantum state diffusion interpretation follow. In particular, a theory of coupled quantum and classical dynamics arises, containing the fluctuation corrections versus the phenomenological mean-field theories.

Key words: quantum measurement, wave function collapse, open quantum systems.

1. Introduction. The state of a canonical quantum system is characterized by its state vector \( |\psi\rangle \) satisfying the Schrödinger evolution equation. The Schrödinger-equation, with Hamiltonian \( \mathcal{H}(\hat{x}, \hat{p}) \), takes the equivalent form

\[
\frac{d}{dt} \hat{\rho} = -i [\mathcal{H}(\hat{x}, \hat{p}), \hat{\rho}]
\]

if we use the density operator \( \hat{\rho} = |\psi\rangle \langle \psi| \) to represent the system’s quantum state.

In a certain sense, clarified ultimately by von Neumann [1], one may speak about the probability distribution of the coordinate operator \( \hat{x} \), which is given by the diagonal elements \( \langle x|\hat{\rho}|x\rangle \) of the density operator in coordinate representation. This distribution contains no information about the quantum system’s momentum \( \hat{p} \).
Vice versa, in momentum representation the distribution \( \langle p|\hat{\rho}|p \rangle \) lacks information about the statistics of \( \hat{x} \). Apparently, the proper statistical interpretation of the quantum state needs the distinguished notion of quantum measurement and assumes abrupt changes, i.e., collapses, of the wave function \([1,2]\). Quantum theory becomes dichotomic: the evolution of the quantum state is governed by the Schrödinger-equation (1) while in quantum measurements the wave function performs random jumps.

Why couldn’t one incorporate the statistical interpretation of the wave function into a suitable quantum dynamics free of collapses? In this talk I suggest that this can perhaps be done within standard quantum mechanics merely by choosing a proper representation.

As the proper one, I suggest the well-known coherent state representation \(|x, p\rangle \) \([3]\). Then the diagonal elements \( \langle x, p|\hat{\rho}|x, p \rangle \) may, up to a well-defined precision, be interpreted as the joint probability distribution \( \rho(x, p) \) of the coordinate and momentum. Opposite to the coordinate or momentum representations, the distribution \( \rho(x, p) \) contains complete information about the system’s quantum state and thus satisfies a closed evolution equation \([5]\).

In Part 2., I recapitulate the coherent state representation in phase space coordinates. A most natural statistical interpretation, free of any reference to measurements or collapses (cf. Ref. \([6]\)), will be proposed. In Part 3., we extend our results to the case of composite systems. In the subsequent parts I stress that typical theories of the emergence of classicality in quantum theory, including the von Neumann-Lüders’ one, could be derived in coherent state representation.

2. QM in coherent state representation. Coherent states \(|x, p\rangle\), labelled by the canonical variables \(x\) and \(p\), form a non-orthogonal overcomplect basis in the system’s Hilbert-space. Coherent states are eigenstates of the particular non-Hermitian combination \(\hat{x} + i\hat{p}\) of canonical operators:

\[
(\hat{x} + i\hat{p})|x, p\rangle = (x + ip)|x, p\rangle.
\]

(2)

The influence of the adjoint operator \(\hat{x} - i\hat{p}\) brings derivative terms in:

\[
(\hat{x} - i\hat{p})|x, p\rangle = (\partial_x - i\partial_p)|x, p\rangle + \frac{1}{2}(x - ip)|x, p\rangle.
\]

(3)

To fix the normalization of the coherent states, we use the completeness relation in the following form:

\[
\int |x, p\rangle\langle x, p|dx dp = I.
\]

(4)

Then, we consider the diagonal elements of the density operator in coherent state representation:

\[
\rho(x, p) = \langle x, p|\hat{\rho}|x, p\rangle.
\]

(5)
Obviously, it is a mere technical task to rewrite the abstract Schrödinger-equation (1) into the above coherent state representation. It is, however, much less obvious that the diagonal part will satisfy a closed evolution equation [7]:

\[ \frac{d}{dt}\rho(x,p) = -i\mathcal{H} \left( x + \frac{\partial_x + i\partial_p}{2}, p + \frac{\partial_p - i\partial_x}{2} \right) \rho(x,p) + \text{c.c.} \]  

(6)

A few examples are given in the Appendix.

One might think of \( \rho(x,p) \) as the joint probability distribution of the phase space coordinates \( x \) and \( p \) associated with \( \hat{x} \) and \( \hat{p} \) or, saying briefly, of the operators \( \hat{x} \) and \( \hat{p} \) themselves. The case is, however, more complicated because, for all composite variables \( \mathcal{F}(x,p) \), one should specify with what Hermitian operators could they be associated. We suggest the following correspondence rule. Let any classical dynamic variable \( \mathcal{F}(x,p) \) be associated with the fully symmetrized [8] Hermitian operator \( \{\mathcal{F}(\hat{x},\hat{p})\}_{\text{sym}} \). Let us, furthermore, introduce a Gaussian coarse-graining of the dynamic variables over the Planckian phase space cell:

\[ \bar{\mathcal{F}}(x,p) \equiv \left\langle \mathcal{F}(x+\xi/\sqrt{2}, p+\eta/\sqrt{2}) \right\rangle_{\xi,\eta}, \]  

(7)

where \( \xi, \eta \) are standard Gaussian variables over which an average is understood on the r.h.s. of the above equation. After these preparations, one can state an equivalence theorem [9]. The quantum expectation value in state \( \hat{\rho} \) of a coarse-grained fully symmetrized Hermitian dynamic variable coincides exactly with the classical expectation value of the corresponding fine-grained classical dynamic variable, using \( \rho(x,p) \) for the phase space probability distribution, i.e.:

\[ \text{tr} \left( \hat{\rho} \{\mathcal{F}(\hat{x},\hat{p})\}_{\text{sym}} \right) = \int \mathcal{F}(x,p)\rho(x,p)dxdp \]  

(8)

satisfies for all \( \mathcal{F}(x,p) \). This is the statistical interpretation which emerges naturally in coherent state representation. To cut it short, we shall say that \( \rho(x,p) \) expresses the coarse-grained joint probability distribution of \( \hat{x} \) and \( \hat{p} \).

3. QM in hybrid representation. Let us consider a composite quantum system \( C \times Q \), with canonical variables \( \hat{x}, \hat{p} \) and \( \hat{X}, \hat{P} \), respectively. We introduce a coherent state basis \( |x,p\rangle \) for the subsystem \( C \) while we keep the abstract Hilbert-space notation for the subsystem \( Q \). This will be called hybrid representation [9]. Accordingly, by projecting the density operator \( \hat{\rho} \) of the composite system \( C \times Q \), we introduce the hybrid density [9] (cf. also [10,11])

\[ \hat{\rho}(x,p) \equiv \text{tr}\left(|x,p\rangle\langle x,p| \otimes I\right)\hat{\rho}. \]  

(9)
This mathematical object might seem strange for the first sight. It characterizes the quantum state of the original composite system while, formally, it is a phase space distribution for the subsystem $C$ and density operator for the subsystem $Q$.

Similarly to the case in the previous Part, a closed evolution equation exists for the hybrid density, too [12]:

$$
\frac{d}{dt} \hat{\rho}(x,p) = -i \hat{\mathcal{H}} \left( x + \frac{\partial_x + i \partial_p}{2}, p + \frac{\partial_p - i \partial_x}{2} \right) \hat{\rho}(x,p) + h.c. \quad (10)
$$

where the compact notation $\mathcal{H}(\hat{x}, \hat{X}, \hat{p}, \hat{P}) = \hat{\mathcal{H}}(\hat{x}, \hat{p})$ has been applied to the composite system’s Hamiltonian.

The hybrid density has a rich structure to interpret statistically. First of all,

$$\hat{\rho}_Q \equiv \int \hat{\rho}(x,p) dxdp \quad (11)$$

is the reduced density operator of subsystem $Q$. Alternatively,

$$\rho_C(x,p) \equiv tr \hat{\rho}(x,p) \quad (12)$$

is the coarse-grained phase-space distribution of $\hat{x}, \hat{p}$ of subsystem $C$, in the very sense of the statistical interpretation of the previous Part. And there is an additional possibility, common in statistics, that is to introduce

$$\hat{\rho}_{Q|x} \equiv \frac{\hat{\rho}(x,p)}{\rho_C(x,p)} \quad (13)$$

as the conditional quantum state [9] of the subsystem $Q$ when the subsystem $C$’s coordinates take given values $x, p$. This interpretation makes the so-called quantum measurement and collapse simple consequences of the standard quantum theory as we shall see in the forthcoming Part.

4. Application: Collapse. Consider von Neumann’s dynamic model [1] of quantum measurement. $Q$ will denote the measured system and $C$ will denote the measuring apparatus. Both of them are quantum systems hence we apply standard quantum mechanics to the composite system $Q \times C$. We do so in hybrid representation. For simplicity, the measured system $Q$ is a two-state system initially in the superposition: $|\psi_{Q_i}\rangle = \alpha |0\rangle + \beta |1\rangle$. The apparatus $C$ is a canonical quantum system; let the pointer’s position be $\hat{x}$. In coherent state representation, $C$’s initial state $\rho_{Ci}(x, p)$ is concentrated at pointer position $x \approx 0$ with precision $\Delta x \approx 1$. Then the initial state of the composite system $Q \times C$ has the following hybrid density (9):

$$\hat{\rho}_i(x, p) = |\psi_{Q_i}\rangle \langle \psi_{Q_i}| \rho_{Ci}(x, p). \quad (14)$$
According to von Neumann, the interaction during the act of measurement can be modelled by the interaction Hamiltonian

$$\hat{H}(x, p) = -g\delta(t)|1\rangle\langle 1|p,$$

(15)

where, again, we use hybrid representation. The coupling constant $g$ satisfies $g \gg 1$.

To calculate the final entangled state of the composite system, we apply the hybrid equation of motion (10) with the above interaction Hamiltonian:

$$\frac{d}{dt}\hat{\rho}(x, p) = -g\delta(t)|1\rangle\langle 1|\partial_x\hat{\rho}(x, p).$$

(16)

Then the initial state (14) evolves into a calculable final state which, after integrating out the irrelevant momentum variable $p$, takes the form:

$$\hat{\rho}_f(x) = |\alpha|^2|0\rangle\langle 0|\rho_{C_i}(x) + |\beta|^2|1\rangle\langle 1|\rho_{C_i}(x-g)$$

$$+ (\alpha^*\beta|1\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1|)\rho_{C_i}(x-g/2).$$

(17)

Let us invoke the statistical interpretation proposed in Part 3.: the coarse-grained probability distribution (12) of the pointer’s position $\hat{x}$ in the final state (17) is

$$\rho_{C_f}(x) \equiv \text{tr}\hat{\rho}_f(x) = |\alpha|^2\rho_{C_i}(x) + |\beta|^2\rho_{C_i}(x-g),$$

(18)

i.e. $x \approx 0$ with probability $|\alpha|^2$ and $x \approx g$ with probability $|\beta|^2$. The two outcomes separate well since we assumed $g \gg 1$. The corresponding conditional final quantum states (13) $\hat{\rho}_{xf} \equiv \hat{\rho}_f(x)/\rho_{C_f}(x)$ are $|0\rangle\langle 0|$ if $x \approx 0$ and $|1\rangle\langle 1|$ if $x \approx g$, respectively. This is the von Neumann-Lüders collapse [1,2].

5. Application: Markovian open systems. Our example will be the composite quantum system $Q \times C$, where $Q$ is a two-state atom interacting with the infinite number of electromagnetic field oscillators $C$ (see, e.g., in Ref.[3]). For each eigenfrequency $\omega$, the canonical oscillator variables $\hat{x}_\omega, \hat{p}_\omega$ are, for convenience, expressed by the usual non-Hermitian ones $\hat{a}_\omega, \hat{a}_\omega^\dagger$. If $|\psi_{Q_i}\rangle$ is the atomic initial state then, in hybrid representation, the composite system’s initial state (9) takes the form

$$\hat{\rho}_i(a, a^*) = |\psi_{Q_i}\rangle\langle \psi_{Q_i}|\exp(-|a|^2),$$

(19)

assuming vacuum state for the radiation field initially. The interaction Hamiltonian is, in interaction picture, modelled by

$$\hat{H}(a, a^*) = \sum_\omega g_\omega e^{-i(\omega-\omega_0)t}\hat{a}_\omega + h.c. \equiv \Gamma\hat{a}_+ + h.c.$$

(20)
where \( g_\omega \) are coupling constants, \( \hat{\sigma}_+ = |1\rangle\langle 0| \), and \( \omega_0 \) is the atomic energy split. (The time dependence of \( \hat{H}(a,a^*) \) is suppressed in notation.) For later convenience, we have introduced the (complex) effective field \( \Gamma \) to which the atom is really coupled. In interaction picture, the hybrid equation of motion (10) reads:

\[
\frac{d}{dt} \hat{\rho}(a,a^*) = -i[\hat{H}(a,a^*), \hat{\rho}(a,a^*)] - \frac{i}{\omega} \sum_\omega (g_\omega e^{-i(\omega-\omega_0)t} \hat{\sigma}_+ - \hat{\rho}(a,a^*) - \text{h.c.}).
\] (21)

It is well-known that in Markovian approximation the atomic reduced density operator satisfies the master equation (see, e.g., in Ref.[4])

\[
\frac{d}{dt} \hat{\rho}_Q = \gamma \left( \hat{\rho}_Q \hat{\sigma} - \frac{1}{2} \hat{\rho}_Q \hat{\sigma}_+ \hat{\sigma}_+ - \frac{1}{2} \hat{\sigma}_+ \hat{\rho}_Q \hat{\sigma} - \text{h.c.} \right) = L \hat{\rho}_Q ,
\] (22)

where \( \gamma = \rho_\omega^2 |\Gamma_{\omega}|^2 \); the spectral density \( \rho_\omega \) is defined by \( \sum_\omega \rightarrow \int \rho_\omega d\omega/2\pi \). Consistent with this reduced dynamics, the Markovian limit of the composite system dynamics exists as well. The hybrid equation of motion (21) leads to coupled Wiener (diffusion) processes for the conditional pure state \( \hat{\rho}_\Gamma = |\psi_\Gamma\rangle\langle \psi_\Gamma| \) of the atom and the effective field strength \( \Gamma \), described respectively by the following Ito-Langevin-equations:

\[
\frac{d}{dt} \hat{\rho}_\Gamma = \sqrt{\gamma} \left( \hat{\sigma}_- \hat{\rho}_\Gamma - \hat{\rho}_\Gamma \hat{\sigma}_+ - \frac{1}{2} \hat{\rho}_\Gamma \hat{\sigma}_+ \hat{\sigma}_+ - \text{h.c.} \right) = L \hat{\rho}_\Gamma ,
\] (23)

\[
i\Gamma = \gamma \langle \hat{\sigma}_- \hat{\rho}_\Gamma \rangle + \sqrt{\gamma} \xi_t
\] (24)

where \( \langle \hat{\sigma}_- \rangle = \text{tr}(\hat{\sigma}_- \hat{\rho}_\Gamma) \) and \( \xi_t \) is the standard complex white-noise [13].

Eq.(23) is the heuristic equation of the quantum state diffusion model [14], obtained originally to describe the wave function of individual quantum systems during their interactions with measuring apparatuses [15].

6. Application: Coupling quantum to classical. This time we make the crucial observation that the hybrid equation of motion (10) could formally be taken as the equation of motion for the composite system \( Q \times C \) where \( Q \) is a quantum system with canonical coordinates \( \hat{X}, \hat{P} \) but the other subsystem \( C \) is genuine classical with coordinates \( x \) and \( p \). So, the equation (10) could in principle describe the interaction of a true quantum and a true classical subsystem. The shorthand notation \( \mathcal{H}(x,p,\hat{X},\hat{P}) \equiv \mathcal{H}(x,p) \) will be understood again.

To illustrate things, let \( C \) be a classical harmonic oscillator with Hamilton-function \( (p^2 + x^2)/2 \), coupled linearly to a certain quantum system \( Q \) with Hamiltonian \( \hat{H} \). The total Hamiltonian will be the following:

\[
\hat{\mathcal{H}}(x,p) = \hat{H} + \frac{p^2 + x^2}{2} + \hat{J} x ,
\] (25)
where $\hat{J}$ is a certain Hermitian operator for $Q$. For our purpose it will be useful to represent the state of the composite system by the classical phase space density (12) of $C$ and by the conditional quantum state (13) of $Q$. Using the Hamiltonian (25), the hybrid equation of motion (10) leads to a couple of equations for $\rho_C(x,p)$ and $\hat{\rho}_{Qxp}$. The first equation reads:

$$\frac{d}{dt}\rho_C(x,p) = (x\frac{\partial}{\partial p} - p\frac{\partial}{\partial x})\rho_C(x,p) + \partial_p\langle\hat{J}\rangle_{xp}\rho_C(x,p)$$  \hspace{1cm} (26)$$

where $\langle\hat{J}\rangle_{xp} = tr(\hat{J}\hat{\rho}_{Qxp})$. This is a classical Liouville-equation. The phase space distribution changes as if the Hamilton function were the quantum expectation value of the hybrid Hamiltonian (25):

$$\dot{x} = \partial_p\langle\hat{H}(x,p)\rangle_{xp}, \hspace{1cm} \dot{p} = -\partial_x\langle\hat{H}(x,p)\rangle_{xp}.$$  \hspace{1cm} (27)$$

These equations are identical to the phenomenological mean-field equations widely used to calculate the evolution of a classical system under the influence of a quantized one (see, e.g., in Ref. [11]). The second equation will govern the quantized system’s state under the influence of the classical subsystem. Our hybrid equation of motion (10) yields, after some elaboration, the following form:

$$\frac{d}{dt}\hat{\rho}_{Qxp} + (\hat{x}\partial_x + \hat{p}\partial_p)\hat{\rho}_{Qxp} + i[\hat{H}(x,p),\hat{\rho}_{Qxp}] =$$

$$= -\frac{i}{2}(\partial_x s)[\hat{J},\hat{\rho}_{Qxp}] - \frac{i}{2}[\hat{J},\partial_x \hat{\rho}_{Qxp}] + \frac{1}{2}\partial_p\{\hat{J} - \langle\hat{J}\rangle_{xp},\hat{\rho}_{Qxp}\} + \frac{1}{2}\{\partial_p s\}{\hat{J} - \langle\hat{J}\rangle_{xp},\hat{\rho}_{Qxp}}$$  \hspace{1cm} (28)$$

where $s(x,p) \equiv \ln\rho_C(x,p)$. This equation differs from the corresponding equation of the standard mean-field theory by the terms on the r.h.s which are absent in the phenomenological theory. The standard theory cannot describe the stochastic fluctuations induced in the classical subsystem $C$ by the quantum subsystem $Q$. The terms on the r.h.s. take these fluctuations into the account [16].

**Moral.** The classical content, if there is any, of a given quantum state displays itself in coherent state representation. I tend to believe that this is the general case for canonical quantum systems. It follows and is important to realize that the classical content is a relative and apparent feature. We can read it out if we use the proper framework — the coherent state representation, presumably. On the other hand, such classicality might be a robust feature against changes of the framework, here I mean canonical transformations on the first place. Obviously, the coherent state representation is not invariant canonically. But, due to the well-defined amount of coarse-graining used in the statistical interpretation, one hopes a satisfactory robustness of classicality whenever the dynamic variables $\mathcal{F}(x,p)$ have no structure at short scales $\Delta x \approx \Delta p \approx 1$. 

7
It is rather promising that the concept, presented in this talk, seems to cover all alternative concepts, including the wave function collapse, the quantum state diffusion, and the interpretation based on true classical subsystems [17]. I think that a Bohm-type interpretation, too, follows naturally (cf., maybe, Ref. [18]). Though I did not mention the decoherent history interpretation I have no doubts that hybrid quantum dynamics turn out to build up decoherent histories (we do not need the Markovian limit [19] either).

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Harmonic oscillator, $\mathcal{H} = (p^2 + x^2)/2$:

$$\frac{d}{dt}\rho(x, p) = (x \partial_p - p \partial_x)\rho(x, p).$$

Free particle, $\mathcal{H} = p^2/2$:

$$\frac{d}{dt}\rho(x, p) = -p \partial_x \rho(x, p) + \frac{1}{2} \partial_x \partial_p \rho(x, p).$$

Quartic oscillator, $\mathcal{H} = (p^2 + x^2)/2 + \lambda x^4$:

$$\frac{d}{dt}\rho(x, p) = (x \partial_p - p \partial_x)\rho(x, p) - \lambda \left(x + \frac{\partial_x}{2}\right)^3 \partial_p \rho(x, p)$$

$$+ \frac{\lambda}{4} \left(x + \frac{\partial_x}{2}\right)^3 \partial_p^3 \rho(x, p).$$

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7. This equation preserves the positivity of $\rho(x, p)$ provided certain analyticity conditions are satisfied initially. It preserves the pure state property of $\hat{\rho}$ though all
forthcoming formulae will be valid for mixtures as well. The proofs will be given elsewhere. The eq. (10) has an equivalent compact form:

$$\frac{d}{dt} \rho = \mathcal{H} \frac{\partial_x \partial_p - \partial_p \partial_x}{2} + \frac{\partial_x \partial_x + \partial_p \partial_p}{2} \exp \frac{\partial_x \partial_x + \partial_p \partial_p}{2} \rho = \{\mathcal{H}, \rho\}_{\text{Poisson}} + \text{h.o.t.}$$

On the very right, one may observe the classical Liouville evolution equation to appear in the lowest order of the derivatives.

8. Full symmetrization (Weyl-ordering) is defined by the recursive rules

$$\{\hat{x} \mathcal{F}(\hat{x}, \hat{p})\}_{\text{sym}} = \frac{1}{2} \left\{ \hat{x}, \{ \mathcal{F}(\hat{x}, \hat{p}) \}_{\text{sym}} \right\}, \quad \{\hat{p} \mathcal{F}(\hat{x}, \hat{p})\}_{\text{sym}} = \frac{1}{2} \left\{ \hat{p}, \{ \mathcal{F}(\hat{x}, \hat{p}) \}_{\text{sym}} \right\},$$

while $\{\hat{x}\}_{\text{sym}} = \hat{x}$ and $\{\hat{p}\}_{\text{sym}} = \hat{p}$.

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12. Expansion in the derivatives yields the form

$$\frac{d}{dt} \hat{\rho} = -i[\hat{\mathcal{H}}, \hat{\rho}] + \frac{1}{2} \{\hat{\mathcal{H}}, \hat{\rho}\}_{\text{Poisson}} - \frac{1}{2} \{\hat{\rho}, \hat{\mathcal{H}}\}_{\text{Poisson}}$$

$$- \frac{i}{2} [\partial_x \hat{\mathcal{H}}, \partial_x \hat{\rho}] - \frac{i}{2} [\partial_p \hat{\mathcal{H}}, \partial_p \hat{\rho}] + \text{h.o.t.}$$

showing the tricky combination of Liouville's classical and Schrödinger's quantum evolutions. The first line of the r.h.s. is identical to the Aleksandrov-bracket [10] which, in itself, would violate the positivity of $\hat{\rho}$ [11].

13. Proofs will be given elsewhere.

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16. Obviously, if the quantum fluctuation of the "current" $\dot{J}$ is small enough then the last two r.h.s. terms go away. If, furthermore, the states $\rho_C(x,p)$ and $\hat{\rho}_{Qxp}$ are smooth enough functions of $x,p$ then all r.h.s. terms can be ignored and we are left with the standard mean-field equation

$$\frac{d}{dt} \hat{\rho}_{Qxp} + i[\hat{\mathcal{H}}(x,p), \hat{\rho}_{Qxp}] = 0.$$

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