CATEGORICAL PROPERTIES OF SMOOTH UNFOLDINGS ON STRATIFIED SPACES

T. GUARDIA AND G. PADILLA

To Carely, Tomasito, Jolymar & Santi.

ABSTRACT. In a previous work we proved the uniqueness and functoriality of primary unfoldings on simple Thom-Mather spaces, which is a functor to the category of smooth manifolds. In this article we extend these results for any stratified Thom-Mather pseudomanifold with arbitrary finite length, through a new kind of intermediate desingularizations, the unbendings, which coincide with primary unfoldings in the simple case.

INTRODUCTION

The intersection homology was defined by Goresky and MacPherson in order to extend the Poincaré duality to the family of spaces with singularities [8]. Among the earliest works concerning smooth desingularizations and their relation with intersection cohomology, we find [6, 7, 22, 23]. In [2] Brasselet, Hector and Saralegi defined the intersection cohomology with differential forms on suitable smooth unfoldings and proved a stratified version of the De Rham theorem; the last author has continued a fruitful research in this direction [18]. Although the unfoldings are not uniquely determined, the intersection cohomology does not depend on their choice.

In [4] Dalmagro came back to the geometrical point of view; he worked with primary unfoldings, a simpler and slightly more restricted smooth desingularizations. In a previous article we proved the functorial behavior of primary unfoldings [9], so there is a canonical way to unfold simple Thom-Mather spaces. In this article we extend these results for Thom-Mather stratified spaces with arbitrary finite length, which is our first main result. This is accomplished as follows: We construct a new kind of intermediate desingularizations, the unbendings. They are recursive steps which can be used in order to obtain smooth primary unfoldings, and present nice functorial properties. The mutual incidence of tubular neighborhoods is avoided since, in a Thom-Mather stratified space, any family of non-comparable strata can be separated with a disjoint family of tubular neighborhoods. The unbending of a simple Thom-Mather space coincides with its primary unfolding so, in a certain sense, unbendings are more general. Our second main result is that our unfolding is a functorial construction.

This article has been organized as follows: Preliminary ideas are contained in §1. In §2 introduce the definition of unfoldings and unbendings. We devote §3 to

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show the functorial properties of the unbending process. In §4 we conclude with a proof of the existence and functoriality of smooth unfoldings. Each time we use the word manifold we mean a smooth differentiable manifold of class $C^\infty$ without boundary.

1. Stratified Pseudomanifolds

1.1. Stratified Spaces. In 1969 Thom \cite{21} introduced the notion of stratified spaces; they are metric spaces that can be decomposed in a locally finite disjoint union of smooth manifolds satisfying a certain incidence condition. Let $X$ be a 2nd countable metric space. A family of subsets $\mathcal{S}$ of $X$ is a stratification if and only if $\mathcal{S}$ is a locally finite partition of $X$ whose elements, with the induced topology, are disjoint nonempty locally closed smooth manifolds. A stratum of $X$ is an element $S \in \mathcal{S}$. Given any other stratum $S' \in \mathcal{S}$ we will say that $S'$ is incident over $S$ if $S \cap S' \neq \emptyset$. The strata of $X$ are required to satisfy the following incidence condition

\begin{equation}
\text{If } S' \text{ is incident over } S \text{ then } S \subset S'.
\end{equation}

If $\mathcal{S}$ is a stratification of $X$ we say that $(X, \mathcal{S})$ is a stratified space, though we will and talk about "a stratified space $X" whenever the choice of $\mathcal{S}$ is clear in the context.

For each stratified space $(X, \mathcal{S})$ the following properties are straightforward \cite{15},

1. The incidence condition is partial order relation.
2. There is at most a countable number of strata (i.e. $\mathcal{S}$ is countable).
3. For each stratum $S \in \mathcal{S}$;
   a. $S$ is maximal (resp. minimal) if and only if it is open (resp. closed).
   b. The closure of $S$ is the union of the strata over which it is incident, 
      \[ \overline{S} = \bigcup_{S' \leq S} S' \].
   c. The set $U_S = \bigcup_{S \leq S'} S'$ is open, we call it the incidence neighborhood
      of $S$.

A stratum $S \in \mathcal{S}$ is regular if it is open in $X$, otherwise we say that $S$ is singular. The singular part (resp. regular part) of $X$ is the union of the singular (resp. regular) strata, which we note $\Sigma$ (resp. $X - \Sigma$). The minimal part is the union of closed (and therefore minimal) strata, denoted $\Sigma_{\text{min}}$.

A stratified subspace of $(X, \mathcal{S})$ is a subset $Y \subset X$ such that 
\[ \mathcal{S}_Y = \{ S \cap Y : S \in \mathcal{S} \} \]
is a stratification of $Y$ with the induced topology.

Examples 1.1.1.

1. Each manifold $M$ is a stratified space with empty singular part $\Sigma = \emptyset$.
2. If $M$ is a manifold and $(X, \mathcal{S})$ is a stratified space then 
   \[ \mathcal{S}_{M \times X} = \{ M \times S : S \in \mathcal{S} \} \]
is a stratification of $M \times X$.\n
(3) Let \((L, S_L)\) be a compact stratified space. The **open cone** of \(L\) is the quotient space

\[ c(L) = \frac{L \times [0,1]}{\sim} \]

where \((l, 0) \sim (l', 0)\) for every \(l, l' \in L\). If \(r = 0\), we write \([l, r]\) for the equivalence class of a point \((l, r)\). The class of all points \((l, 0)\) is the **vertex** of the cone and will be denoted as \(v\). The stratification of \(c(L)\) is

\[ S_{c(L)} = \{v\} \sqcup \{S \times (0,1) : S \in S_L\} \]

(4) A **basic model** is a product of the form \(M \times c(L)\) where \(M\) is a manifold and \((L, S_L)\) is a compact stratified space.

Let us fix a stratified space \((X, S)\). Since \(S\) is locally finite, each point \(x \in S\) has a neighborhood which intersects a finite number of strata. We conclude that every strict incidence chain in \(S\) is finite. This motivates the next

**Definition 1.1.2.** The length of a stratum \(S \in S\) is the largest integer \(p \geq 0\) such that there is a strict incidence chain

\[ S = S_0 < S_1 < \cdots < S_p \]

in \(S\). The length of \(X\) is the supremum (possibly infinite) of the lengths of the strata. We will denote it by \(l(X)\). The **dimension** of \(X\), denoted as \(\text{dim}(X)\), is defined in a similar way.

**Definition 1.1.3.** A stratified morphism (resp. isomorphism) between two stratified spaces \((X, S_X)\) and \((Y, S_Y)\) is a continuous function (resp. homeomorphism) \(f : X \to Y\) that sends smoothly (resp. diffeomorphically) each stratum of \(X\) into a stratum of \(Y\). A stratified morphism \(f\) is an embedding if \(f(X) \subset Y\) is a stratified subspace and \(X \xrightarrow{f} f(X)\) is an isomorphism.

**Remark 1.1.4.** Each morphism \(M \times c(L) \xrightarrow{f} M' \times c(L')\) can be written as:

\[ f(u, [l, r]) = (a_1(u, l, r), [a_2(u, l, r), a_3(u, l, r)]) \]

Where \(a_1, a_2, a_3\) are maps defined on \(M \times L \times [0,1]\) and are piecewise smooth, i.e., smooth on \(M \times \{v\}\) and \(M \times S \times (0,1)\) for each \(S \in S_L\).

1.2. **Stratified Pseudomanifolds.** A stratified pseudomanifold is a stratified space together a family of conic charts which reflect the way in which we approach the singular part. The definition is given by induction on the length.

**Definition 1.2.1.** A 0-length **stratified pseudomanifold** is a smooth manifold with the trivial stratification. A stratified space \((X, S)\) with \(l(X) > 0\) is a stratified pseudomanifold if, for each singular stratum \(S\),

1. There is a compact stratified pseudomanifold \((L, S_L)\) with \(l(L) < l(X)\). We call \(L\) the **link** of \(S\) because
2. Each point \(x \in S\) has an open neighborhood \(x \in U \subset S\) and a stratified embedding \(U \times c(L) \xrightarrow{\alpha} X\) on an open neighborhood of \(x \in X\).
The image of $\alpha$ is called a **basic neighborhood** of $x$. Notice that $\exists (\alpha) \cap S = U$. Without loss of generality, we assume that $\alpha(u, v) = u$ for each $u \in U$ (where $v$ is the vertex of $c(L)$, see §1.1.1(4)). We summarize the above situation by saying that the pair $(U, \alpha)$ is a **chart** of $x$. The family of charts is an **atlas** of $(X, S)$. 

**Examples 1.2.2.**

1. A basic model $U \times c(L)$ is a stratified pseudomanifold if $(L, S_L)$ is a compact stratified pseudomanifold.
2. If $M$ is a manifold and $X$ is a stratified pseudomanifold then $M \times X$ is a stratified pseudomanifold.
3. Every open subset of a stratified pseudomanifold is again a stratified pseudomanifold.
4. Since algebraic manifolds satisfy the Whitney's conditions, every algebraic manifold is a stratified pseudomanifold [15].
5. The orbit space of a stratified pseudomanifold endowed with a suitable stratified action of a compact Lie group is again a stratified pseudomanifold [14, 16].
6. The foliation space of a suitably controlled locally conic foliated manifold is a stratified pseudomanifold [17, 19].
7. New examples of stratified pseudomanifolds are arising from the field of theoretical physics. See for instance [11].

2. **The Process of Removing Singularities**

The main feature of any suitable desingularization is its capability of preserving (co)homological or geometrical information near the singular strata. In this article we mention two kinds of desingularizations: smooth unfoldings [2, 4, 5, 10, 18] and the **unbendings** we are to introduce. The main difference between the last two objects is that the unbending removes only the minimal part of the stratified pseudomanifold, while the unfolding removes completely the singular part.

In [9] we proved the equivalence between the Thom-Mather conditions and the existence of unfoldings for simple spaces. In the following sections we will extend this result for any Thom-Mather stratified pseudomanifold with arbitrary finite length. In [4] Dalmagro works with transverse morphisms as an additional requirement which we do not ask here. We will show that after a finite number of iterated compositions of unbendings we get Dalmagro's "primary" unfoldings.

We fix a stratified pseudomanifold $(X, S)$ with finite length $l(X) = p < \infty$.

2.1. **Unfoldings.** It is in terms of this geometric tool that the intersection cohomology with smooth differential forms is defined. An **unfolding** of $X$ consists of a manifold $\tilde{X}$ and a continuous surjective proper map $\tilde{X} \xrightarrow{L} X$ satisfying the following conditions:

1. **Inductive condition:** There is a family of unfoldings of the links of the singular strata \( \{ L_s : \tilde{L}_s \xrightarrow{L_s} L_s : S \text{ is singular} \} \).
2. **Regularity:** The restriction $L^{-1}(X - \Sigma) \xrightarrow{L} X - \Sigma$ is a smooth finite trivial covering (hence, a diffeomorphism on each copy of the regular part).
(3) **Existence of unfoldable charts:** For each point \( z \in L^{-1}(\Sigma) \), there is a commutative diagram:

\[
\begin{array}{ccc}
U \times \tilde{L} \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \bar{X} \\
\downarrow c & & \downarrow L \\
U \times c(L) & \xrightarrow{\alpha} & X
\end{array}
\]

Such that:
(a) \((U, \alpha)\) is a chart at \( x = L(z) \).
(b) \(c(u, \tilde{l}, t) = (u, [\tilde{L}(\tilde{l}), |t|])\).
(c) \(\tilde{\alpha}\) is a diffeomorphism onto \( L^{-1}(\text{Im}(\alpha)) \).

The above diagram is an **unfoldable chart** on \( x \).

A stratified pseudomanifold \( X \) is **unfoldable** if it has an unfolding.

(4) **An unfoldable morphism** is a commutative diagram

\[
\begin{array}{ccc}
\bar{X} & \xrightarrow{\tilde{f}} & \bar{X}' \\
\downarrow L & & \downarrow L' \\
X & \xrightarrow{f} & X'
\end{array}
\]

such that the vertical arrows are unfoldings, \( f \) is a stratified morphism and \( \tilde{f} \) is a smooth map.

**Examples 2.1.1.**

(1) Any diffeomorphism \( f : M \to M \) is an unfolding, for any smooth manifold \( M \) considered as a 0-length stratified pseudomanifold.

(2) The map \( c \) given in (2) is an unfolding of the basic model \( U \times c(L) \).

(3) If \( \bar{X} \xrightarrow{L} X \) is an unfolding then

(a) For any open subset \( A \subset X \) the restriction \( L^{-1}(A) \xrightarrow{L} A \) is an unfolding.

(b) For any singular stratum \( S \) the restriction \( L^{-1}(S) \xrightarrow{L} S \) is a smooth bundle with typical fiber \( F = \tilde{L} \) the unfolding of the respective link.

(4) If \( M \supset \Sigma \) is a singular manifold then \( \Sigma \) has a smooth tubular neighborhood \( T \). The family \( S_M = \{ \Sigma, M - \Sigma \} \) is a stratification and \((M, S_M)\) is a stratified pseudomanifold. If we take off \( T \) and substitute it with the smooth fiber bundle \( \tilde{T} \) induced by the radial homotetia, we find an unfolding \( \tilde{M} \xrightarrow{L} M \). For more details see [3, 5].

(5) An iteration of the above example shows that any smooth manifold \( M \) endowed with a smooth Thom-Mather stratification has an unfolding [2].
Given a smooth effective action \( G \times M \rightarrow M \) of a compact Lie group \( G \) on a smooth manifold \( M \), the partition \( S_M \) of \( M \) in orbit types endows \((M, S_M)\) with a smooth Thom-Mather stratification. There is an unfolding \( \tilde{M} \rightarrow M \), such that \( \tilde{M} \) is endowed with the unique smooth free action of \( G \) such that \( L \) is \( G \)-equivariant. This structure passes in a canonical way to the respective orbit spaces \( B = M/G \) and \( \tilde{B} = \tilde{M}/G \) so that the orbit map \( M \rightarrow B \) is unfoldable morphism \([10]\). A completely analogous situation can be given for any stratified pseudomanifold that supports a suitable stratified action \([14]\).

### 2.2. Unbendings

The unbending of a stratified pseudomanifold is again a stratified pseudomanifold. In this section we study how the unfolding relates with a finite composition of unbendings (as many as the depth of the total space).

An unbending of \( X \) consists of a stratified pseudomanifold \( \hat{X} \) satisfying \( l(\hat{X}) = l(X) - 1 \); and continuous surjective proper map \( \hat{X} \rightarrow X \) such that:

1. The restriction \( \tilde{L}^{-1}(X - \Sigma_{\min}) \rightarrow \tilde{X} \rightarrow (X - \Sigma_{\min}) \) is a stratified double covering. The subset \( \tilde{L}^{-1}(X - \Sigma_{\min}) \) is an open dense in \( \hat{X} \), and it is the union of two disjoint isomorphic copies of \( (X - \Sigma_{\min}) \) which we denote \( (X - \Sigma_{\min})^{\pm} \) and the restriction of \( \tilde{L} \) to each of these copies is a stratified isomorphism.

2. For each \( z \in \tilde{L}^{-1}(\Sigma_{\min}) \), there is a commutative diagram:

   \[
   \begin{array}{ccc}
   U \times L \times \mathbb{R} & \xrightarrow{\hat{\alpha}} & \hat{X} \\
   \varepsilon \downarrow & & \tilde{L} \downarrow \\
   U \times c(L) & \xrightarrow{\alpha} & X
   \end{array}
   \]

   Such that:
   - (a) \((U, \alpha)\) is a chart of \( x = \tilde{L}(z) \).
   - (b) \( \hat{\alpha}(u, l, t) = (u, [l, |t|]) \).
   - (c) \( \hat{\alpha} \) is a stratified isomorphism on \( \tilde{L}^{-1}(\text{Im}(\alpha)) \).

   The above diagram is an unbendable chart on \( x \).

   A stratified pseudomanifold \( X \) is unbendable if it has an unbending.

3. An unbendable morphism is a commutative square diagram

   \[
   \begin{array}{ccc}
   \hat{X} & \xrightarrow{\hat{f}} & \hat{X}' \\
   \tilde{L} \downarrow & & \tilde{L}' \downarrow \\
   X & \xrightarrow{f} & X'
   \end{array}
   \]
such that the vertical arrows are unbendings and the horizontal arrows are stratified morphisms.

**Remark 2.2.1.** Every unfoldable stratified pseudomanifold is unbendable. It is enough to take $\hat{X}$ as the closure of the quotient space of $\hat{X} - L^{-1}(\Sigma - \Sigma_{\text{min}})$ with the following equivalence relation $z \sim z'$ if $L(z) = L(z')$. On the other hand, if $l(X) = 1$ then the links are compact smooth manifold, so the unbending is an unfolding. We leave the details to the reader.

**Examples 2.2.2.**

1. Any diffeomorphism $M \xrightarrow{f} M$ is an unbending, for any smooth manifold $M$ considered as a 0-length stratified pseudomanifold.
2. The map $\hat{c}$ given in 2.2-(2).(b) is an unbending of the basic model $U \times c(L)$.
3. If $\hat{X} \xrightarrow{\hat{L}} X$ is an unbending then
   
   (a) For any open subset $A \subset X$ the restriction $\hat{L}^{-1}(A) \xrightarrow{\hat{L}} A$ is an unbending.

   (b) For any minimal stratum $S$ the restriction $\hat{L}^{-1}(S) \xrightarrow{\hat{L}} S$ is a stratified bundle with typical fiber $F = L$.

3. The unbending of a Thom-Mather pseudomanifold

Tubular neighborhoods arose in riemannian geometry a useful tool for approaching to closed submanifold in a controlled way; they were given by means of a riemannian metric and a smooth transverse section of the closed submanifold called a "slice"; see for instance [3]. A generalization for stratified pseudomanifolds was first given by Thom in his historical article, while there are geometrical versions dealing with stratified slices [16, 21].

3.1. **Thom-Mather pseudomanifolds.** A tubular neighborhood $T$ around a singular stratum $S$ is a stratified fiber bundle with two main features, a conic fiber and a global tubular radium. A Thom-Mather pseudomanifold is a stratified pseudomanifold such that each singular stratum is contained in such a neighborhood.

We fix in the sequel a stratified pseudomanifold $(X, S)$.

**Definition 3.1.1.** Given a singular stratum $S$ in $X$, a **tubular neighborhood** on $S$ is a fiber bundle $\xi = (T, \tau, S, c(L))$ satisfying

1. $T$ is an open neighborhood of $S$ in $X$.
2. The fiber is $c(L)$, the open cone of the link of $S$.
3. The inclusion $S \subset T$ is a section: $\tau(x) = x$ for any $x \in S$.
4. The structure group is contained in $\text{Iso}(L, S_L)$. If $(U, \alpha), (V, \beta)$ are two bundle charts and $U \cap V \neq \emptyset$ then the cocycle is

   $$(U \cap V) \times c(L) \xrightarrow{\beta^{-1} \alpha} (U \cap V) \times c(L) \xrightarrow{\beta^{-1} (\alpha(u, [l, r]))} (u, [g_{\alpha \beta}(u)(l), r])$$

   where $g_{\alpha \beta}(u)$ is a stratified isomorphism of $(L, S_L)$ for all $u \in U \cap V$. 

Given a tubular neighborhood $T \xrightarrow{\tau} S$ by (3.1.1) (4) the cocycles of the fiber bundle preserve the conical radium, so it can be extended to a well defined tubular radium $T \xrightarrow{\rho} [0, \infty)$ which, locally, is given by $\rho(\alpha(u, [l, r])) = r$ in the image of a bundle chart $(U, \alpha)$. Notice that $\rho^{-1}(\{0\}) = S$ and $\rho^{-1}(\mathbb{R}^+) = (T - S)$. There is also an action $\mathbb{R}^+ \times T \xrightarrow{\lambda} T$, the radial stretching, given by $\lambda \cdot \alpha(u, [l, r]) = \alpha(u, [l, \lambda r])$.

We will say that $X$ is Thom-Mather provided that each singular stratum $S$ has a tubular neighborhood $T_S$. ■

We state now an easy and useful result we will use hereafter.

**Lemma 3.1.2.** In a Thom-Mather pseudomanifold, any family of non-comparable strata can be separated with a family of disjoint tubular neighborhoods.

**[Proof]** Because...

- **Tubular neighborhoods can be stretched:** Take a tubular neighborhood $T$ on a singular stratum $S$ and any other open neighborhood $O \supset S$. It is possible to find a smooth non-negative function so that the tubular radium $\rho$ of $T$ is suitably stretched in order to obtain a smaller tubular neighborhood $T'$ satisfying $S \subset T' \subset O$, with the same procedure employed by [3] for smooth tubes.

- **Non-comparable strata can be separated in any stratified space:** Notice that

  (a) It is enough to show it for minimal strata: If $F \subset S$ is a family of non-comparable strata, take the union of the incidence neighborhoods $Z = \bigcup_{s \in F} U_s$. Then $Z$ is open in $X$, therefore $S \in F$ iff $S$ is a minimal stratum in $Z$. Since $Z$ is open, it is enough to give a family of disjoint neighborhoods in $Z$ separating the strata in $F$.

  (b) Minimal strata can be separated by disjoint open subsets: Any two different minimal strata in $X$ are disjoint closed subsets, that can be separated with two disjoint open subsets because $X$ is $T^4$. The whole family of minimal strata can be separated because of §1.1(1), (3)-(b), and the facts that $X$ is $T^4$, and $S$ is locally finite [13].

□

**Remark 3.1.3.** Lemma 3.1.2 implies that we can now give a family of tubular neighborhoods for the singular strata, $\{\tau_S : T_S \xrightarrow{\tau} S : S$ is singular $\}$ such that non-comparable strata have disjoint tubes. If two tubular neighborhoods $T_S, T_R$ have non-empty intersection; then the corresponding strata $S, R$ are comparable:

$T_S \cap T_R \neq \emptyset \Rightarrow S \leq R$ or $R \leq S$

This incidence condition was quoted by Mather [12, pp.43-44].

**3.2. Local simplifications.** We briefly describe the unbending process that we will make explicit in section 3.3.
Take a family of disjoint tubes \( \{ T_s : S \text{ is minimal} \} \) for the minimal strata.

Unbend separately each tube obtaining a canonical map \( \hat{T}_s \xrightarrow{L_s} T_s \) for each minimal stratum \( S \). Each \( \hat{T}_s \) is a unique stratified fiber bundle \( \hat{T}_s \xrightarrow{\tau_s} S \) with fiber \( L_s \times \mathbb{R} \), where \( L_s \) is the link of \( S \). There is an unbending of the tubular radium \( \hat{T}_s \xrightarrow{\rho_s} \mathbb{R} \) satisfying
\[
|\rho_s(x)| = \rho_s \left( L_s(x) \right) \quad x \in \hat{T}_s
\]
so \( S \subset \hat{T}_s \) and the inclusion is the 0-section. They also satisfy \( l(T_s) = p - 1 \).

The difference \( \hat{T}_s = \left( \hat{T}_s - S \right) \) has two connected components, say \( \hat{T}_s^\pm \), which are again stratified fiber bundles over \( S \) with respective fibers \( L_s \times \mathbb{R}^\pm \). Define the global unbending of \( X \) as the stratified amalgamation of two copies of \( (X - \Sigma_{\text{min}}) \), say \( (X - \Sigma_{\text{min}})^\pm \), and the disjoint union of the unbended tubes. Again \( l(X) = p - 1 \) so the unbending process decreases the length of the total space.

Notice that in the process of unbending we do not touch the intermediate strata, but only disjoint tubes over non-comparable (minimal) strata. Hence, although the statements will remain as general as possible; in the context of the proofs and without loss of generality, by §3.1.2 we will make some or even all of the following assumptions:

1. \( X \) is a connected stratified pseudomanifold.
2. \( X \) has finite length \( l(X) = p < \infty \).
3. All strata in \( X \) are comparable, i.e. \( S \) is a well ordered set and there is a unique strict incidence chain \( S_0 < \cdots < S_p \). In particular, \( S_0 = \Sigma_{\text{min}} \).
4. \( X \) is Thom-Mather, and there is a family of tubular neighborhoods \( \{ T_k \xrightarrow{\tau_k} S_k : 0 \leq k \leq p - 1 \} \) such that \( T_{k+1} \subset (T_k - S_k) \) for all \( k \).

3.3. The unbending of a Thom-Mather pseudomanifold. Now, we study the connection between the tubular neighborhoods and the unbending. We follow now the main ideas of [4, 5].

**Lemma 3.3.1.** Each tubular neighborhood \( T \xrightarrow{\tau} S \) has an unbending \( \hat{T} \xrightarrow{L} T \) such that:

1. The composition \( \hat{T} \xrightarrow{\hat{\tau}} S \) given by \( \hat{\tau} = \tau L \) is a stratified fiber bundle.
2. The fiber of \( \hat{T} \) is \( L \times \mathbb{R} \) where \( L \) is the link of \( S \).
3. The cocycles of \( \hat{T} \) are the same of \( T \).
4. There is an unbending of tubular radium \( \rho \), i.e. a continuous function \( \hat{T} \xrightarrow{\hat{\rho}} \mathbb{R} \) such that \( |\hat{\rho}(x)| = \rho \left( L(x) \right) \) for all \( x \in \hat{T} \).

**[Proof]** Let us fix a bundle atlas \( \mathcal{U} = \{(U_\alpha, \alpha)\}_{\alpha \in \mathcal{A}} \) for \( T \).
(a) Unbending of a chart: For any chart \((U, \alpha) \in \mathcal{U}\), the unbending of \(\tau^{-1}(U)\) is just the composition

\[
U \times L \times \mathbb{R} \xrightarrow{\hat{c}} U \times c(L) \xrightarrow{\alpha} \tau^{-1}(U)
\]

where \(\hat{c}\) is the map defined in §2.2-(b).

(b) Definition of the bundle \(\hat{T}\): Take the quotient space

\[
\hat{T} = \bigsqcup_{\alpha} U_\alpha \times L \times \mathbb{R}
\]

\(\sim\)

\((u, l, t) \sim (u, g_{\alpha\beta}(u)(l), t) \forall \alpha, \beta \forall u \in U_\alpha \cap U_\beta\)

Denote by \([u, l, t]\) the equivalence class of \((u, l, t)\). Following [20, p.14] we get a unique fiber bundle

\[
\hat{T} \xrightarrow{\hat{\tau}} S \quad \hat{\tau}([u, l, t]) = u.
\]

with fiber \(F = L \times \mathbb{R}\) and the same structure group of \(T\).

(c) Unbending of \(\hat{T}\): Define

\[
\hat{T} \xrightarrow{\hat{l}} L \xrightarrow{\hat{l}} T \quad \hat{l}([u, l, t]) = \alpha(u, [l, t]) \quad \forall l \in L, \forall t \in L, \forall u \in U_\alpha, \forall \alpha
\]

Since the cocycles \(g_{\alpha\beta}\) are stratified isomorphisms of \(L\), we conclude that \(\hat{T}\) is a stratified pseudomanifold and \(l(\hat{T}) = l(T) - 1 = l(X) - 1\). In order to show that the above arrow is an unbending, let \((U, \alpha) \in \mathcal{U}\). Define

\[
U_\alpha \times L \times \mathbb{R} \xrightarrow{\hat{\alpha}} \hat{T} \quad \hat{\alpha}(u, l, r) = [u, l, r].
\]

Then \(\hat{\alpha}\) is stratified, because it is the restriction of the quotient map. The induced diagram §2.2 (2) commutes \(\forall \alpha\).

(d) Unbending of the tubular radium: The function

\[
\hat{T} \xrightarrow{\hat{\rho}} \mathbb{R} \quad \hat{\rho}(\hat{\alpha}(u, l, t)) = t \quad \forall l \in L, \forall t \in L, \forall u \in U_\alpha, \forall \alpha
\]

trivially satisfies the required property.

\[\square\]

**Proposition 3.3.2.** Let \(X\) be a stratified pseudomanifold with finite length. If every minimal stratum has a tubular neighborhood, then \(X\) is unbendable.

[Proof] By §3.2 assume that \(X\) has a unique minimal stratum \(S_0 = \Sigma_{\text{min}}\), with a tubular neighborhood \(T_0\). By §3.3.1 and since the cocycles are radium-preserving,

\[
\hat{L}^{-1}(T - S_0) = T_0^+ \cup T_0^-
\]

has two connected components. They are stratified bundles over \(S_0\) with respective fibers \(F^\pm = L_0 \times \mathbb{R}^\pm\). We obtain the unbending of the whole stratified pseudomanifold \(X\) by taking two copies \((X - S_0)^\pm\) of \(X - S_0\); and suitably gluing them together along \(\hat{T}\). In other words, we take

\[
\hat{X} = \frac{(X - S_0)^+ \cup \hat{T}_0 \cup (X - S_0)^-}{\sim}
\]

as the amalgamated sum by the inclusions \(T_0^\pm \subset (X - S_0)^\pm\).

\[\square\]
Corollary 3.3.3. Every Thom-Mather pseudomanifold is unbendable. The unbending of a connected Thom-Mather pseudomanifold of length $p$ is a connected Thom-Mather pseudomanifold of length $(p - 1)$.

[Proof] This is a consequence of §3.2-(4), §3.3.1-(4) and §3.3.2; the unbending process does not affect the tubular neighborhoods of non-minimal strata. □

3.4. Thom-Mather morphisms. A tube-morphism $T_S \xrightarrow{f} T_R$ between stratified tubular neighborhoods is a stratified morphism $f$ such that

1. It commutes with the tubular radius, $\rho_R f = \rho_S$.
2. It is a bundle-morphism, $\tau_R f = f \tau_S$.

Condition (1) implies that $\varphi(S) \subset R$ so (2) makes sense. Notice that a tube-morphism $f$ commutes with the respective bundle cocycles of $T_S, T_R$, see [20].

Example 3.4.1. If $X$ is a Thom-Mather stratified pseudomanifold and $X \xrightarrow{f} X$ is a stratified isomorphism, then $f$ is a Thom-Mather isomorphism. For each tubular neighborhood $T \xrightarrow{\tau} S$ it is enough to define $T'_S = f(T_S)$, $\tau' = f\tau$, and $\rho' = f\rho$. Then $T' \xrightarrow{\tau'} S$ is a tubular neighborhood and $T \xrightarrow{f} T'$ is a tube-isomorphism.

3.5. Functoriality of unbendings. In [9] we proved that the primary unfoldings are representative in the category of the unfoldable pseudomanifolds. In the more general context of this work unbendings have the same representation property and, for simple pseudomanifolds they coincide with the primary unfoldings.

Lemma 3.5.1. Let $X, X'$ be unbendable stratified pseudomanifolds with finite length, $\hat{X} \xrightarrow{\hat{L}} X$ and $\hat{X}' \xrightarrow{\hat{L}'} X'$ two unbendings. Then, for each stratified morphism $X \xrightarrow{f} X'$ there is a unique continuous function $\hat{X} \xrightarrow{\hat{f}} \hat{X}'$ such that

1. $f\hat{L} = \hat{L}' \hat{f}$ i.e. the diagram 2.2-(3) commutes;
2. The restriction of $\hat{f}$ to $\hat{L}^{-1}(X - \Sigma_{\min})$ is a stratified morphism.

[Proof] According to 2.2-(1) the open dense

$$(X - \Sigma_{\min})^\pm = \hat{L}^{-1}(X - \Sigma_{\min}) \subset \hat{X}$$

is the union of two isomorphic copies of $(X - \Sigma_{\min})$ and the restriction of $\hat{L}$ to each of these copies is a stratified isomorphism. A similar situation happens for $\hat{X}'$ and we write $(X' - \Sigma_{\min}')^\pm$ for the respective copies of $(X' - \Sigma_{\min}')$.

(a) Definition of $\hat{f}$ on $(X - \Sigma_{\min})^\pm$: Then the inverse maps

$$(X' - \Sigma_{\min}') \xrightarrow{\hat{L}'} (X' - \Sigma_{\min}')^+ \quad (X' - \Sigma_{\min}') \xrightarrow{\hat{L}^{-1}} (X' - \Sigma_{\min}')^-$$
are stratified isomorphisms. There are two ways in order to define the 
composition \( \hat{f} = L^{-1} \hat{f} \hat{L} \), depending on the copies of \((X' - \Sigma'_{\text{min}})\) where 
we take the inverse. Namely we can take \( \hat{f} = \hat{f}^+ \) as the "sign-preserving" 
lifting, i.e.
\[
\hat{f}^+ (X - \Sigma_{\text{min}})^+ \subset (X' - \Sigma'_{\text{min}})^+ \quad \hat{f}^+ (X - \Sigma_{\text{min}})^- \subset (X' - \Sigma'_{\text{min}})^-
\]
and \( \hat{f} = \hat{f}^- \) as the other possible definition. In any case of these cases, 
\( \hat{f} = \hat{f}^\pm \) is a well-defined stratified morphism and satisfies \(|3.5.1|\)

(b) **Continuous extension of \( \hat{f} \):** By the previous step, we have already defined a stratified morphism \( f \) on \((X - \Sigma_{\text{min}})\) satisfying \(|3.5.1|\). We define a global continuous extension of \( \hat{f} \) as follows.

Assume by \(|3.2|\) that \( \Sigma_{\text{min}} = S_0 \) is a single minimal stratum. Let 
\( \xi \in \hat{L}^{-1}(S_0) \). We must define \( \hat{f}(\xi) \). For this sake let 
\[
\{\xi_n\} \subset \hat{L}^{-1}(X - S_0)
\]
be any sequence converging to \( \xi \). Since \( L, f \) are continuous and \( \hat{L}, \hat{L}' \) is a 
continuous proper maps; by an argument of compactness and up to minor 
adjustments, we may suppose that the sequence \( \{\hat{f}(\xi_n)\} \) converges in \( \hat{X}' \).
We define
\[
\hat{f}(\xi) = \lim_{n \to \infty} \hat{f}(\xi_n)
\]
If our limit-definition makes sense then it is also continuous; so next we 
will show that \( \hat{f} \) is well defined. Since the above definition is local, we first study the

(c) **Lifting in terms of conics:** Assume that \( X = M \times c(L) \) and \( X' = M' \times c(L') \) are trivial basic models and their respective unbendings are the 
canonical ones - see \(|2.2.2|\). Then \( f \) can be written as in \(|1.1.4|\). The point 
\( \xi = (u, l, 0) \in M \times L \times \{0\} \subset \hat{X} = M \times L \times \mathbb{R} 
\]
is the limit of a sequence 
\[
\{\xi_n = (u_n, l_n, t_n)\} \subset M \times L \times (\mathbb{R} - \{0\})
\]
So the sequences \( \{u_n\}, \{l_n\} \) and \( \{t_n\} \) converge to \( u, l \) and \( 0 \) respectively.
Since \( \hat{L} = \hat{c}, L' = \hat{c}' \) and \( f \) are continuous maps, the sequence 
\[
w_n = f(\hat{c}(\xi_n)) = (a_1(u_n, l_n, |t_n|), a_2(u_n, l_n, |t_n|), a_3(u_n, l_n, |t_n|))
\]
converges to \( w = f(\hat{c}(\xi)) = (a_1(u, l, 0), v) \). By the continuity of the functions \( a_j \) for \( j = 1, 2, 3 \) and up to minor adjustments on \( a_i \) concerning the 
compactness arguments; we get that 
\[
\hat{w}_n = (a_1(u_n, l_n, |t_n|), a_2(u_n, l_n, |t_n|), a_3(u_n, l_n, |t_n|)) \]
converges to 
\[
\hat{w} = (a_1(u, l, 0), a_2(u, l, 0), 0) \in M' \times L' \times \{0\}
\]
(d) **The lifting is well defined:** From the continuity of the functions \( a_i, \ i = 1, 2, 3 \); it follows that the element \( \hat{w} \) does not depend on the choice of a 
particular sequence \( \{\xi_n\} \).
Definition 3.5.2. The $\hat{f}$ obtained at Proposition 3.5.1 is an almost-unbending of $f$.

Proposition 3.5.3. The almost-unbending

$$U \times L \times \mathbb{R} \xrightarrow{\hat{f}} U' \times L' \times \mathbb{R} \quad \hat{f}(u, l, t) = (\hat{a}_1(u, l, t), \hat{a}_2(u, l, t), \hat{a}_3(u, l, t))$$

of a stratified morphism between basic models

$$U \times c(L) \xrightarrow{f} U' \times c(L') \quad f(u, [l, r]) = (a_1(u, l, r), [a_2(u, l, r), a_3(u, l, r)])$$
is an unbending as in §2.2-(3), if and only if, for each stratum $S \in S_L$,

(a) $\hat{a}_j(u, l, 0) = a_j(u, l, 0) = 0$ for all $u \in U$, $l \in S$.

(b) With respect to the coordinate $t \in \mathbb{R}$, the functions $\hat{a}_1, \hat{a}_2$ are even and $\hat{a}_3$ is either odd or even.

(a) $\hat{a}_j$ is a smooth extension of $a_j$ for all $j$.

Proof If $\hat{f}$ is a unbending of $f$, then $fc = c'\hat{f}$ where $c$ and $c'$ are canonical unbendings as in §2.2-(2). Checking both sides of this equality we get

$$f(c(u, l, t)) = f(u, [l, |t|]) = (a_1(u, l, |t|), [a_2(u, l, |t|), a_3(u, l, |t|))]$$

and

$$c'(\hat{f}(u, l, t)) = c'(\hat{a}_1(u, l, t), \hat{a}_2(u, l, t), \hat{a}_3(u, l, t)) = (\hat{a}_1(u, l, t), [\hat{a}_2(u, l, t), \hat{a}_3(u, l, t)])$$

we conclude that

$$(a_1(u, l, |t|), [a_2(u, l, |t|), a_3(u, l, |t|)]) = (\hat{a}_1(u, l, t), [\hat{a}_2(u, l, t), \hat{a}_3(u, l, t)])$$

There are two cases; $t = 0$ and $t \neq 0$, from which we get 3.5.3.

Lemma 3.5.4. The cocycles of any tubular neighborhood are unbendable.

Proof For $f = \varphi = \beta^{-1}\alpha$ as in §3.1-(4); the functions $\hat{a}_3(u, l, t) = u$, $\hat{a}_2(u, l, t) = g_{\alpha, \beta}(u)|l\rangle$ and $\hat{a}_3(u, l, t) = t$ satisfy the hypothesis of 3.5.3.

Lemma 3.5.5. Consider a diagram of stratified morphisms

$$U \times c(L) \xrightarrow{f} U' \times c(L') \quad U \times c(L) \xrightarrow{f'} U' \times c(L')$$
such that $\varphi, \varphi'$ are as in §3.1-(4). Then $f'\varphi = \varphi'f$ if and only if:

$$a_1(u, l, r) = a'_1(u, g(u)|l\rangle, r)$$

$$g'(a_1(u, l, r))a_2(u, l, r) = a'_2(u, g(u)|l\rangle, r)$$

$$a_3(u, l, r) = a'_3(u, g(u)|l\rangle, r)$$

(4)

Proof Write $\varphi(u, [l, r]) = (u, g(u)|l\rangle, r)$ and $\varphi'(u', [l', r]) = (u', g'(u')|l\rangle, r)$ where $g(u), g'(u')$ are, respectively, isomorphisms on $L, L'$. This is a straightforward calculation.

Lemma 3.5.6. Each tube-morphism is unbendable.
Let $T_S \xrightarrow{f} T_R$ be a tube-morphism. By §3.5.1 we must show that the almost-unbending $\hat{f}$ is stratified, i.e. that

- $\hat{f}(L^{-1}(S)) \subset \hat{L}^{-1}(R)$: Since $\rho_R f = \rho_S$, we deduce that the unbended radii satisfy a similar property, $\hat{\rho}_R \hat{f} = \hat{\rho}_S$.

- $\hat{f}$ is stratified on $\hat{L}^{-1}(S)$: Since $f$ commutes with the respective cocycles of $T_S, T_R$, this is in fact a local matter. For any pair of bundle charts $(U, \alpha)$ of $T_S$, and $(U', \alpha')$ of $T_R$; the composition $h = \beta^{-1} f \alpha$ satisfies the requirements of §3.5.6.

Theorem 3.5.7.

1. **Each Thom-Mather morphism is unbendable.**
2. **The unbending process is a functor in the category of Thom-Mather stratified pseudomanifolds.**
3. **The unbending of a Thom-Mather stratified pseudomanifold is unique up to Thom-Mather isomorphisms.**

[Proof] This is a consequence of §3.4 and §3.5.6.

4. CATEGORICAL PROPERTIES OF SMOOTH DESINGULARIZATIONS

The aim of this section is to establish the sufficient and necessary conditions for the existence of a smooth unfolding. We will also prove that the unfoldings have similar functorial properties which they inherit from unbendings.

4.1. **The primary unfolding of a stratified Thom-Mather pseudomanifold.**

We start with an useful and easy result,

**Lemma 4.1.1.** **The unbending map of a Thom-Mather stratified pseudomanifold is a Thom-Mather morphism.**

[Proof] In order to prove this we can assume the simpler geometrical conditions of §3.2. By §3.3.3 (3) and (4), the tubular radium $\rho_o$ of the tubular neighborhood $T_o$ on the minimal stratum $S_o$ is unbendable. Since the other tubular neighborhoods are open subsets of $(T_o - S_o)$ and the preimage $\hat{L}^{-1}(X - S_o) = (X - S_o)\hat{L}$ is the union of two isomorphic copies of $(X - S_o)$; according to §3.1 and §3.4.1 $\hat{L}$ is tube-preserving on $\hat{L}^{-1}(X - S_o)$. We conclude that $\hat{L}$ is a Thom-Mather morphism.

[Proof] Let $p = l(X)$. We proceed in two steps.

- Definition of the unfolding: Corollary §3.3.3 implies that the unbending process does not change the assumptions §3.2 (1) and (2). Assumption §3.2 (3) also holds because any intermediate (non-minimal) stratum in $X$ is locally detected near $S_o$ at the link $L_o$. Since the unbending process does not touch the links we deduce that $\hat{X}$ still has only one incidence chain. This allows us to continue an iterative unbending operation.
Denote the first unbending of \( X \) by \( \hat{X} = X^1 \) and the corresponding map by \( \hat{L} = L^1 \). We obtain a finite sequence of unbendings

\[
X \xrightarrow{\hat{L}} X^1 \xrightarrow{L^2} \cdots \xrightarrow{L^{p-1}} X^p
\]

Notice that \( l(X^p) = 0 \), so \( X^p \) is a manifold. Take \( \tilde{X} = X^p \) and \( \tilde{L} = \hat{L} \cdots \hat{L}^1 \).

- **Existence of unfoldable charts:** In order to show that \( \tilde{X} \xrightarrow{L} X \) is an unfolding we check condition \([2.1](2)\). This can be done by induction on \( p \). For \( p = 0 \) is trivial and for \( p = 1 \) the minimal stratum \( S_0 = \Sigma \) coincides with the singular part. The link \( L_0 \) is a compact smooth manifold and the unbending is the unfolding; this case has been treated in \([9]\). We assume the inductive hypothesis so for any \( k \leq p \) the statement holds. This implies that

  1. For any singular stratum \( S \) the respective link \( L \) is unfoldable in the described way.
  2. Since \( l(X^1) = p - 1 \), the statement holds for \( X^1 \).

Take \( L' = \hat{L} \cdots \hat{L}^2 \). By inductive hypothesis

\[
X^1 \xrightarrow{L'} \tilde{X}
\]

is an unfolding of \( X^1 \). Consider the composition

\[
X \xrightarrow{\hat{L}} X^1 \xrightarrow{L'} \tilde{X} \xrightarrow{\tilde{L} \cdots \tilde{L}^1} \hat{X} = X^1
\]

Take a point \( z \in L^{-1}(\tilde{S}_0) \). Let us verify condition \([2.1](2)\), i.e. the existence of an unfoldable chart at \( z \).

Take an unbendable chart as in \([2.2](2)\) at \( z' = L'(z) \in X^1 \)

\[
U \times L_0 \times \mathbb{R} \xrightarrow{\hat{\alpha}} \tilde{X} = X^1 \quad \tilde{L} \xrightarrow{\tilde{\alpha}} X
\]

Since \( l(L_0) = p - 1 \), as we already remarked, by inductive argument the link \( L_0 \) can be unfolded with a finite sequence of \( p - 1 \) unbendings,

\[
L_0 \xrightarrow{\eta^1} L_0^1 \xrightarrow{\eta^2} \cdots \xrightarrow{\eta^{p-2}} L_0^{p-2} \xrightarrow{\eta^{p-1}} L_0^{p-1}
\]

By \([3.5.7]\) and \([4.1.3]\) these unbendings behave in a functorial way. Since \( U \times L_0 \times \mathbb{R} \) is a stratified Thom-Mather pseudomanifold and \( l(U \times L_0 \times \mathbb{R}) = l(L_0) = p - 1 \); we deduce that the composition of the maps \( \nu_j = id_U \times \eta^j \times id_{\mathbb{R}} \) provides a finite sequence of \( p - 1 \) unbendings

\[
U \times L_0 \times \mathbb{R} \xrightarrow{\nu_1} U \times L_0^1 \times \mathbb{R} \xrightarrow{\nu_2} \cdots U \times L_0^{p-2} \times \mathbb{R} \xrightarrow{\nu_{p-1}} U \times L_0^{p-1} \times \mathbb{R}
\]
Again by inductive hypothesis, we deduce that the composition 
\[ \nu = \nu_{p-1} \cdots \nu_1 = id_U \times L_{z_0} \times id_R \]
induces an unfolding 
\[ U \times L_0 \times \mathbb{R} \xrightarrow{\nu} U \times L_0 \times \mathbb{R} \]
Now, by definition, \( c = \tilde{c}\nu \). Applying §3.5.7(b) to the stratified morphism \( \hat{\alpha} \) we get a stratified morphism \( \tilde{\alpha} \) between 0-length stratified pseudomanifolds, from \( U \times L_0 \times \mathbb{R} \) to \( \tilde{X} \), so \( \tilde{\alpha} \) is smooth. We obtain a commutative diagram
\[
\begin{array}{ccc}
U \times L_0 \times \mathbb{R} & \xrightarrow{\tilde{\alpha}} & \tilde{X} \\
\downarrow c & & \downarrow L \\
U \times c(L_0) & \xrightarrow{\alpha} & X
\end{array}
\]
as desired. \( \square \)

4.2. Smooth liftings. We now study the smooth lifting of a Thom-Mather morphisms.

**Definition 4.2.1.** The primary unfolding of a given a finite length Thom-Mather stratified pseudomanifold is the one we obtain with the iterated unbending process described on §4.1.2.

**Proposition 4.2.2.** Every Thom-Mather morphism between finite-length stratified Thom-Mather pseudomanifolds is unfoldable, in the sense of §2.1(4).

**Proof** Let \( X \xrightarrow{f} X' \) be a Thom-Mather morphism. According to §4.1.2 \( X \) and \( X' \) are unbendable and unfoldable. By §3.5.7 \( f \) is unbendable, \( X_1 = \tilde{X} \) and \( X'_1 = \tilde{X}' \) are Thom-Mather stratified pseudomanifolds and the induced map \( X_1 \xrightarrow{f_1 = \tilde{f}} X'_1 \)
is a Thom-Mather morphism and satisfies \( f\tilde{L} = \tilde{L}'f_1 \).

After a finite number of iterated composition of unbendings, namely \( n = \max\{l(X), l(X')\} \), we get two smooth primary unfoldings \( \tilde{X} = X_{n} \xrightarrow{L} X \) and \( \tilde{X}' = X'_{n} \xrightarrow{L'} X' \); we deduce that the respective iterated \( n \)-th unbending \( \tilde{f} = f_n \) of \( f \) satisfies \( f\tilde{L} = \tilde{L}'\tilde{f} \) and is smooth. \( \square \)

**Theorem 4.2.3.**

1. The primary unfolding process is a functor in from the category of finite length Thom-Mather stratified pseudomanifolds to the category of smooth manifolds.
(2) The primary unfolding of a Thom-Mather stratified pseudomanifold is unique up to Thom-Mather isomorphisms.

[Proof] This is a consequence of §4.2.2. □

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E-mail address: tomas.guardia@ciens.ucv.ve

Departamento de Matemáticas, Universidad Nacional de Colombia, K30 con calle 45. Edificio 404, ofic. 316. Bogotá. (+571)3165000 ext 13166
E-mail address: gabrielpadillaleon@gmail.com