Testing New Physics Effects in $B \rightarrow K^* \ell^+ \ell^-$

Rusa Mandal$^*$ and Rahul Sinha$^†$

The Institute of Mathematical Sciences, Taramani, Chennai 600113, India

Diganta Das$^‡$

Institut für Physik, Technische Universität Dortmund, D-44221 Dortmund, Germany

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It is generally believed that the decay mode $B \rightarrow K^* \ell^+ \ell^-$ is one of the best modes to search for physics beyond the standard model. The angular distribution enables the independent measurement of several observables as a function of the dilepton invariant mass. The plethora of observables so obtained enable unique tests of the standard model contributions. We start by writing the most general parametric form of the standard model amplitude for $B \rightarrow K^* \ell^+ \ell^-$, taking into account comprehensively all contributions within SM. These include all short-distance and long-distance effects, factorizable and non-factorizable contributions, complete electromagnetic corrections to hadronic operators up to all orders, resonance contributions and the finite lepton and quark masses. The parametric form of the amplitude in the standard model results a new relation involving all the $CP$ conserving observables. The derivation of this relation only needs the parametric form of the amplitude and not a detailed calculation of it. Hence, we make no approximations, however, innocuous. The violation of this relation will provide a smoking gun signal of new physics. We use the 1 fb$^{-1}$ LHCb data to explicitly show how our relation can be used to test standard model and search for new physics that might contribute to this decay.

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I. INTRODUCTION

It is a historical fact that several discoveries in particle physics were preceded by indirect evidence through quantum loop contributions. It is for this reason that significant attention is devoted to studying loop processes. The muon magnetic moment is one of the best examples of such a process where precision calculations have been done in order to search for new physics by comparing the theoretical expectation with experimental observation. It is a testimony to such searches for New Physics (NP) beyond the Standard Model (SM) that both theoretical estimates and experimental observation have reached a precision where the hadronic effects even for the lepton magnetic moment dominate the discrepancy between theory and observation. Indirect searches for new physics often involve precision measurement of a single quantity that is compared to a theoretical estimate that also needs to be very accurately calculated. Unfortunately, hadronic estimates involve calculation of long distance QCD effects which cannot easily be done accurately, limiting the scope of such searches. There exist, however, certain decay modes which involve the measurement of several observables that can be related to each other with minimal assumptions and completely calculable QCD contributions within the SM. The break down of such relation(s) between observables would unambiguously signal the presence of NP. Such tests are by nature not limited by incalculable hadronic effects and hence provide an unambiguous signal of NP. A well known example [1, 2] of such a process is the semileptonic penguin decay $B \rightarrow K^* \ell^+ \ell^-$, where $\ell$ is either the electron or the muon. In this paper we will show how this decay, which occurs in multiple partial waves, can be used to obtain reliable tests of NP.

Flavor changing neutral current transitions are well known to be sensitive to NP contributions. However, hadronic flavor changing neutral current receive short and long distance QCD contributions that are not easy to estimate reliably. It is evident from the data collected by the Belle, Babar and CMS collaborations at the B-factories, CLEO, CDF, Tevatron and LHCb that NP does not show up as a large and unambiguous effect. This has bought into focus the need for approaches that are theoretically cleaner i.e., where the hadronic uncertainties are much smaller than the effects of NP that are being probed. Hence, to effectively search for NP it is crucial to separate the effect of new physics from hadronic uncertainties that can contribute to the decay. The decay mode $B \rightarrow K^* \ell^+ \ell^-$ is regarded [2] as significant in this attempt. The full angular analysis of the final state gives rise to a multitude of observables [1, 3] that are related as they arise from the same decay mode. In addition, each of these observables can be measured as function of the dilepton invariant mass. In Ref. [2] an interesting relation between the various observables that can be measured in this mode was derived. The derivation was based on a few assumptions that are reasonable. These

$^*$ rusam@imsc.res.in
$^†$ sinha@imsc.res.in
$^‡$ diganta.das@tu-dortmund.de
included ignoring the mass of the lepton $\ell$ and the $s$-quark that appears in the short distance Hamiltonian describing the decay. The decay amplitude was assumed to be real, thereby ignoring the extremely tiny $CP$ violation, the small imaginary contribution to the amplitude that arises from the Wilson coefficient $C_9$ which is complex in general and the dilepton resonances which were presumed to be removed from the experimental analysis. These assumptions reduced the number of non-zero observables to only six. In this paper, we carefully redo the analysis without making any kind of approximation, however, innocuous. Our approach once again is to derive the most general parametric form of the decay amplitude, which results in a relation between the several related observables.

In this paper we generalize the derivation in Refs. [2] to incorporate a complex decay amplitude, eliminating the need to ignore imaginary contributions arising from $C_9$ and ensuring that the new relation is valid even when resonance contributions are not excluded from the (experimental) analysis. This implies that the new relation derived in this paper involves all the nine $CP$ conserving observables that can be measured using this mode. The derivation of the new relation does not depend on theoretical values of the Wilson coefficients and does not require making any assumptions on the form-factors; in particular we do not limit the form-factor to any power of $L_{\text{QCD}}/m_b$ expansion in heavy quark effect theory (HQET) [4]. In fact, the derivation of the new relation itself does not require HQET. The new derivation parametrically incorporates all short-distance and long-distance effects including resonance contributions, as well as, factorizable and non-factorizable contributions. We also include complete electromagnetic corrections to hadronic operators up to all orders. Finally, we retain the lepton mass and the $s$-quark mass. We envisage that the derivation to be exact in all respects and the new relation obtained here to be one of the cleanest tests of the SM in $B$ decays.

The LHCb collaboration has measured [5] all the possible $CP$ conserving observables through an angular analysis. These independent measurements consist of the differential decay rate with respect to the dilepton invariant mass, two independent helicity fractions and six angular asymmetries. Three of the asymmetries are zero unless there exist imaginary contributions to the decay amplitudes. If these asymmetries are measured to be zero in the future, the relation between the observables would be free from any hadronic parameter as derived in Ref. [2]. While these asymmetries are currently measured to be small and consistent with zero, there could, however, exist contributions from wide resonances which might still be permitted within statistical errors. Including these asymmetries in the analysis to account for complex amplitudes results in a modification of the relation purely between observables. The modifying terms now involve a single hadronic parameter in addition to being proportional to the three asymmetries. Hence, SM can be tested or equivalently NP contributions can be probed reliably with the knowledge of just one hadronic parameter. It is interesting that all effort to estimate long and short distance QCD contributions now need to be focused only on accurately estimating this single parameter. Since the asymmetries involved in modifying terms (which arise from complex amplitudes) are already constrained to be small, the results are not very sensitive to the single hadronic parameter. We find that the inclusion of imaginary contributions to the amplitude must always reduce the parameter space. This would enhance any discrepancy that may be observed even when the imaginary part of the amplitudes are ignored. We use the 1 fb$^{-1}$ LHCb data to show how our relation can be used to test Standard model and find new physics that might contribute to this decay.

In this paper we review the theoretical framework required to describe $B \to K^{\ast} \ell^+\ell^-$ and derive the most general parametric form of the amplitude describing the decay in Sec. II. The amplitude written is notionally exact in all respects. In Sec. III we construct all the observables in terms of the amplitude derived in Sec. II. Here we retain the lepton mass as well as the strange quark mass that appears in the short-distance Hamiltonian describing this decay. A new relation between observables is derived in Sec. IV under the assumption of massless lepton, but retaining all other effects and contribution. In Sec. V we generalize the new relation derived in Sec. IV to include the mass of the lepton that had been ignored earlier. We re-derive two simple limits of the relation between observables that hold at zero crossings of other asymmetries such as the forward-backward asymmetry. The values of all the observables at kinematic endpoints of the dilepton invariant mass are easily understood in Sec. VI. A numerical analysis is presented in Sec. VII that tests the validity of the relation derived assuming SM. We discuss the constraints already imposed by the 1 fb$^{-1}$ LHCb data [5], but refrain from drawing even the obvious conclusions given that results for 3 fb$^{-1}$ data will soon be presented. In Sec. VIII we summarize the significant results obtained in our paper.

II. THEORETICAL FRAMEWORK

In this section we will discuss the most general from of the amplitude that can describe the exclusive decay mode $B \to K^{\ast} \ell^+\ell^-$ in the SM. The description of the decay $B \to K^{\ast} \ell^+\ell^-$ requires as the first step the separation of short-distance effects which involve perturbative QCD and weak interaction from the long distance QCD contributions in an effective Hamiltonian. As is well explained in literature, exclusive decay modes are a challenge to describe theoretically. This difficulty arises not only in the need to know hadronic form-factors accurately but also from the existence of “non-factorizable” contributions that do not correspond to form-factors. These contributions originate from electromagnetic corrections to the
matrix element of purely hadronic operators in the effective Hamiltonian. It has been demonstrated [6] that these non-factorizable corrections can be computed allowing exclusive decay such as \( B \to K^\gamma \) and \( B \to K^{\ell^+ \ell^-} \) to be treated in a systematically much as their inclusive decay counterparts. It is based on this theoretical understanding that we will write the most general form of the amplitude for \( B \to K^{\ell^+ \ell^-} \) in the SM. Our approach will be to examine the various factorizable and non-factorizable contributions to the process and write the most general parametric form of the amplitude without making any attempt to evaluate it.

The decays \( B \to K^{\ell^+ \ell^-} \) occurs at the quark level via a \( b \to s\ell^+ \ell^- \) flavor changing neutral current transition. The short distance effective Hamiltonian for the inclusive process \( b \to s\ell^+ \ell^- \) is given in the SM by [7–9],

\[
\mathcal{H}_{\text{eff}} = - \frac{4 G_F}{\sqrt{2}} \left[ V_{ub} V_{cb}^* \left( C_1 O_1^* + C_2 O_2^* + \sum_{i=3}^{10} C_i O_i^* \right) + V_{ub} V_{ub}^* \left( C_1 (O_1^* - O_1) + C_2 (O_2^* - O_2) \right) \right].
\] (1)

The local operators \( O_i \) are as given in Ref. [8], however, for completeness we present the relevant operators that are dominant:

\[
\begin{align*}
O_7 &= \frac{e}{g^2} \left( \bar{s} \gamma_{\mu} (m_s P_R + m_b P_L) b \right) F^{\mu\nu}, \\
O_9 &= \frac{e^2}{g^2} \left( \bar{s} \gamma_{\mu} P_L b \right) \bar{\ell} \gamma^\mu \ell, \\
O_{10} &= \frac{e^2}{g^2} \left( \bar{s} \gamma_{\mu} P_L b \right) \bar{\ell} \gamma^\mu \gamma_5 \ell,
\end{align*}
\]

where \( g(\epsilon) \) is the strong (electromagnetic) coupling constant, \( P_{L,R} = (1 \mp \gamma_5)/2 \) are the left and right chiral projection operators and \( m_b (m_s) \) are the running \( b (s) \) quark mass in the \( \overline{\text{MS}} \) scheme. The Wilson coefficients \( C_i \) encode all the short distance effects and are calculated in perturbation theory at a matching scale \( \mu = M_W \) up to desired order in the strong coupling constant \( \alpha_s \) before being evolved down to the scale \( \mu = m_b \approx 4.8 \text{GeV} \). All NP contributions to \( B \to K^{\ell^+ \ell^-} \) contribute exclusively to \( C_7 \); this includes new Wilson coefficients corresponding to new operators that arise from NP.

Significant effort (see Ref. [10, 11] for reviews) has gone into evaluating the Wilson coefficients up to NNLO order. As has been stressed earlier [11] it is important to remember that “the construction of the effective Hamiltonian by means of operator product expansion and renormalization group methods can be done fully in the perturbative framework. The fact that the decaying hadron are bound states of quarks is irrelevant for this construction.” This implies that the \( C_i \) are decay mode independent. The dependence on the mode enters only through the matrix element of local bilinear quark operators \( O_i \), i.e. \( \langle f | O_i | B \rangle \), which encodes the long distance contributions. Since the decay amplitude cannot depend on the scale \( \mu \), \( \langle f | O_i | B \rangle \) must depend on the scale \( \mu \) as well. The cancellation of \( \mu \) dependence generally involves several terms in the operator product expansion. Since the calculation of the hadronic matrix element involves long distance contributions, non-perturbative methods are required. Much progress has been made in these calculations using HQET as a tool. However, the dominant theoretical error in the

\[ LHCb \] has observed a broad peaking structure [14, 15] in the dimuon spectrum of \( B \to K^{\ell^+ \ell^-} \). It would be of interest to see if this observation of broad resonances has implication on \( B \to K^{\ell^+ \ell^-} \) mode, since long distance effects would have to be included systematically.
The decay mode $B \rightarrow K^*\ell^+\ell^-$ carries more information [1, 3] on the dynamics as compared to the counterpart pseudoscalar mode $B \rightarrow K\ell^+\ell^-$, since the $K^*$ polarization can also be measured. In order to study the dependence of the amplitude on the helicity of the $K^*$ we further consider the decay $K^* \rightarrow K\pi$ or the decay process $B \rightarrow K^*\ell^+\ell^-$ to $(K\pi)_{\ell^+\ell^-}$. This further step itself does not complicate matters. The decay amplitude in terms of hadronic matrix elements must therefore include direct contributions proportional to $C_7$, $C_0$ and $C_{10}$ multiplied by $B \rightarrow K^*$ form-factors and contributions from non-local hadronic matrix elements $H_i$ such that [16, 17],

$$A(B(p) \rightarrow K^*(k)\ell^+\ell^-) = \frac{G_F\alpha}{\sqrt{2}\pi}V_{tb}V_{ts}^*\left\{ \tilde{C}_9(K^*)|\bar{s}\gamma^\mu P_Lb|\bar{B}\right\} - \frac{16\pi^2}{q^2} \sum_{i=\{-1,0,8\}} \tilde{C}_i\mathcal{H}_i^{\mu},$$  \hspace{1cm} (2)

where, $p = q + k$ with $q$ being the dilepton invariant momentum and the non-local hadron matrix element $\mathcal{H}_i^{\mu}$ is given by

$$\mathcal{H}_i^{\mu} = \langle K^*(k)|i\int d^4x \gamma^\nu T\{j_{em}(x),O_i(0)|\bar{B}(p)\rangle. $$

In Eq. (2), we have introduced new notional theoretical parameters $\tilde{C}_7$, $\tilde{C}_9$ and $\tilde{C}_{10}$ to indicate the true values of Wilson coefficients, which are by definition not dependent on the order of the perturbative calculation to which they are evaluated. Our definition is explicit and should not be confused with those defined earlier in literature. The amplitude expressed in Eq. (2) is notionally complete and free from any approximations. In this paper we do not attempt to estimate the hardronic matrix element involved in Eq. (2), instead we use Lorentz invariance to write out the most general form of the hadron matrix elements $\langle K^*|\bar{s}\gamma^\mu P_L\bar{b}|\bar{B}(p)\rangle$ and $\langle K^*|\bar{s}\sigma^{\mu\nu}q_\nu P_{L,R}\bar{b}|\bar{B}(p)\rangle$ which may be defined as

$$\langle K^*(\epsilon^*,k)|\bar{s}\gamma^\mu P_L\bar{b}|B(p)\rangle = \epsilon_\nu^*\left( X_0 g^{\mu\nu} + X_1 \left( g^{\mu\nu} - \frac{k\cdot q}{q^2} q^{\mu} q^{\nu}\right) + X_2 \left( k^{\mu} - \frac{k\cdot q}{q^2} q^{\mu}\right) q^{\nu} + iX_3 \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \right),$$  \hspace{1cm} (3)

$$\langle K^*(\epsilon^*,k)|i\bar{s}\sigma^{\mu\nu}q_\nu P_{L,R}\bar{b}|B(p)\rangle = \epsilon_\nu^*\left( \pm Y_1 \left( g^{\mu\nu} - \frac{k\cdot q}{q^2} q^{\mu} q^{\nu}\right) \pm Y_2 \left( k^{\mu} - \frac{k\cdot q}{q^2} q^{\mu}\right) q^{\nu} + iY_3 \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \right).$$  \hspace{1cm} (4)

We have written Eq. (3) such that the vector part of the current in $\langle K^*(\epsilon^*,k)|\bar{s}\gamma^\mu P_L\bar{b}|B(p)\rangle$ is conserved and only the $X_0$ term in the divergence of the axial part survives. Eq. (4) is also written so as to ensure that $\langle K^*(\epsilon^*,k)|i\bar{s}\sigma^{\mu\nu}q_\nu P_{L,R}\bar{b}|B(p)\rangle q_\mu = 0$. The relations between $X_{0,1,2,3}$ and $Y_{1,2,3}$ and the form-factors conventionally defined for on-shell $K^*$ are discussed in Appendix B. It should be noted that form-factors $X_{0,1,2,3}$ and $Y_{1,2,3}$ are functions of $q^2$ and $k^2$, but we suppress the explicit dependence for simplicity of notation. The subsequent decay of the $K^*$, i.e., $K^*(k) \rightarrow K(k_1)\pi(k_2)$ can be easily taken into account [1, 8] resulting in the hadronic matrix element $\langle [K(k_1)\pi(k_2)]_{\ell^+\ell^-}|\bar{s}\gamma^\mu P_L\bar{b}|B(p)\rangle$ being written as

$$\langle [K(k_1)\pi(k_2)]_{\ell^+\ell^-}|\bar{s}\gamma^\mu P_L\bar{b}|B(p)\rangle = D_{\nu\mu}(k^2)W_\nu\left( X_0 g^{\mu\nu} + X_1 \left( g^{\mu\nu} - \frac{k\cdot q}{q^2} q^{\mu} q^{\nu}\right) + X_2 \left( k^{\mu} - \frac{k\cdot q}{q^2} q^{\mu}\right) q^{\nu} + iX_3 \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \right),$$  \hspace{1cm} (5)

$$\langle [K(k_1)\pi(k_2)]_{\ell^+\ell^-}|i\bar{s}\sigma^{\mu\nu}q_\nu P_{L,R}\bar{b}|B(p)\rangle = D_{\nu\mu}(k^2)W_\nu\left( \pm Y_1 \left( g^{\mu\nu} - \frac{k\cdot q}{q^2} q^{\mu} q^{\nu}\right) \pm Y_2 \left( k^{\mu} - \frac{k\cdot q}{q^2} q^{\mu}\right) q^{\nu} + iY_3 \epsilon^{\mu\nu\rho\sigma} k_\rho q_\sigma \right),$$  \hspace{1cm} (6)

where, the subscript $K^*$ in $[K(k_1)\pi(k_2)]_{\ell^+\ell^-}$ indicates that the final state is produced by the decay of a $K^*$, $D_{\nu\mu}(k^2)$ is the $K^*$ propagator, so that

$$|D_{\nu\mu}(k^2)|^2 = \frac{g_{K^*K\pi}^2}{(k^2 - m^2_{K^*})^2 + (m_{K^*}\Gamma_{K^*})^2},$$  \hspace{1cm} (7)

with $g_{K^*K\pi}$ being the $K^*K\pi$ coupling and the other parameters introduced are

$$W_\nu = K_\nu - \xi k_\nu, \hspace{0.5cm} K = k_1 - k_2, \hspace{0.5cm} k = k_1 + k_2, \hspace{0.5cm} \xi = \frac{k_1^2 - k_2^2}{k^2}. $$

The most general expression for the hadronic matrix el-
emend $H_i^\mu$ can also be written using Lorentz invariance. Since this hadronic matrix element arises from non-local contributions at the quark level, it involves introducing

\[ H_i^\mu = (K^*(\epsilon^*,k)|i \int d^4x \epsilon^{iq} x T[j^\mu_{em}(x), O_i(0)]|B(p)) \]

\[ = \epsilon^*_\nu (Z_1^i (g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}) + Z_3^i (k^\mu - \frac{k \cdot q}{q^2} q^\mu) q^\nu + i Z_4^i \epsilon^{\mu\nu\rho\sigma} k^\rho q^\sigma). \]  

(8)

Our definition follows Ref. [6] of “non-factorizable” and includes those corrections that are not contained in the definition of form-factors introduced in Eqs. (3) and (4). Here the most general form of $H_i^\mu$ is written to ensure the conservation of EM current i.e., $q^\mu H_i^\mu = 0$.

The non-local effects represented by $H_i^\mu$ can be taken into account by absorbing the contributions into redefined $C_9$ and modifying the contribution from the electromagnetic dipole operator $O_i$. The electromagnetic corrections to operators $O_{1-6,8}$ can also contribute to $B \to K^*\gamma$ at $q^2 = 0$. Since, only the Wilson coefficient $\tilde{C}_7$ contributes to $B \to K^*\gamma$, the charm-loops at $q^2 = 0$ must contribute to $\tilde{C}_7$ in order for the Wilson coefficient to be process independent. It is easily seen that the effect of this is to modify the $\tilde{C}_7(K^\pi|s\sigma q_0(q_0 P_T + m_s P_L) b|B)$ terms such that the form-factors and Wilson coefficients mix in an essentially inseparable fashion. This holds true even for the leading logarithmic contributions [6, 18]. Both factorizable and non-factorizable contributions arising from electromagnetic corrections to hadronic operators up to all orders can in principle be included in this approach. The remaining contributions can easily be absorbed into a redefined “effective” Wilson coefficient $\tilde{C}_9$ defined such that

\[ \tilde{C}_9 \to \tilde{C}_9^{(j)} = C_9 + \Delta C_9^{(fac)}(q^2) + \Delta C_9^{(1), (non-fac)}(q^2) \]  

(9)

where, $j = 1, 2, 3$ and $\Delta C_9^{(fac)}(q^2), \Delta C_9^{(non-fac)}(q^2)$ correspond to factorizable and soft gluon non-factorizable contributions. Note that the non-factorizable contributions necessitates the introduction of new form-factors “new” form factors $Z_1^i, Z_2^i$ and $Z_3^i$ corresponding to non-factorizable contribution from each $H_i^\mu$ in analogy with those introduced in Eq. (3) as follows:

\[ Z_j \]  

and the explicit dependence on $Z_j/\lambda_j$ is absorbed in defining

\[ \Delta C_9^{(fac)} + \Delta C_9^{(1), (non-fac)} = -\frac{16\pi^2}{q^2} \sum_{i=1-6,8} \tilde{C}_i Z_i^{(1)} \lambda_j. \]  

(10)

resulting in the $j$ dependence of the term as indicated. We also mention that there is no non-factorizable correction term in Eq. (8) analogous to $\lambda_0$ (in Eq. (3)) due EM current conservation as discussed above.

The corresponding corrections to $\tilde{C}_7$ are taken into by the replacement,

\[ \frac{2(m_b + m_s)}{q^2} \tilde{C}_7 \gamma_j \to \tilde{\gamma}_j = \frac{2(m_b + m_s)}{q^2} \tilde{C}_7 \gamma_j + \cdots, \]  

(11)

where the dots indicate other factorizable and non-factorizable contributions and the factor $2(m_b + m_s)/q^2$ has been absorbed in the form-factors $\tilde{\gamma}_j$. Note that the $\tilde{\gamma}_j$’s are in general complex because of the non-factorizable contributions to the Wilson coefficient $\tilde{C}_7$, but on-shell quarks and resonances do not contribute to them. It should be noted that $\tilde{C}_9^{(1)}$ includes contributions from both factorizable and non-factorizable effects, whereas $\tilde{C}_{10}$ is unaffected by strong interaction effects coming from electromagnetic corrections to hadronic operators. The use of “widetilde” versus “widehat” throughout the paper is also meant as a notation to indicate this fact. It should be noted that $\tilde{C}_9$ is real in the SM, whereas, $\tilde{C}_9^{(1)}$ and $\tilde{\gamma}_j$ are in general complex within the SM. The amplitude in Eq. (2) can therefore be written as

\[ A \left( B(p) \to [K(k_1)\pi(k_2)]_{\mu\nu}^{\ell^+\ell^-} \right) = \frac{G_F \alpha}{\sqrt{2\pi}} V_{tb} V_{ts}^* D_{\ell\nu}(k^2) \]

\[ \left[ \left\{ C_L W_{q\lambda_0} q^\mu + C_L \chi_1 (k^\mu - \frac{W q}{q^2} q^\mu - \xi k^\mu) + C_L^2 W_{q\lambda_2} (k^\mu - \frac{k \cdot q}{q^2} q^\mu) + i C_L^3 \chi_3 \epsilon^{\mu\nu\rho\sigma} K_{\nu} k_\rho q_\sigma \right\} \tilde{\gamma}_\mu P_L \ell + L \to R \right], \]  

(12)

where, $C_{L,R} = \tilde{C}_9 \mp \tilde{C}_{10}, C_{L,R}^{(j)} = \tilde{C}_9^{(1)} \mp \tilde{C}_{10}$ and $\xi = (m_b - m_s)/(m_b + m_s)$. It may be noted that no
The angular coefficients \( A_I \) of the lepton is finite. The six transversity amplitudes excluding non-factorization contribution used widely in the literature are presented. These approximations are unnecessary for the discussions in this paper and are presented only as clarification of our notation and as ready reference for readers wanting to examine Eq. (12) in limiting conditions.

### III. ANGULAR DISTRIBUTION AND OBSERVABLES.

The decay \( B(p) \to K^*(k)\ell^+(q_1)\ell^-(q_2) \), with \( K^*(k) \to K(k_1)\pi(k_2) \) on the mass shell, is completely described by four independent kinematic variables. These kinematic variables are the lepton-pair invariant mass squared \( q^2 = (q_1 + q_2)^2 \), and the three angles \( \phi, \theta \) and \( \theta_K \). The angle \( \phi \) is the angle between the decay planes formed by \( \ell^+\ell^- \) and \( K^+ \). The angles \( \theta \) and \( \theta_K \) are defined as follows: assuming that the \( K^+ \) has a momentum along the positive \( z \) direction in \( B \) rest frame, \( \theta_K \) is the angle between the \( K \) and the \( +z \) axis and \( \theta \) is the angle of the \( \ell^- \) with the \( +z \) axis. The differential decay distribution of \( B \to K^+\ell^+\ell^- \) is written as

\[
d^3\Gamma(B \to K^+\ell^+\ell^-) = I(q^2, \theta, \theta_K, \phi) = \frac{9}{32\pi} [I_1^2 \sin^2 \theta_K + I_1^2 \cos^2 \theta_K + (I_2^2 \sin^2 \theta_K + I_2^2 \cos^2 \theta_K) \cos 2\theta \ell + I_3 \sin^2 \theta_K \sin^2 \theta \ell \cos 2\phi + I_3 \sin \theta_K \sin 2\theta \ell \sin 2\theta \ell \cos \phi + I_5 \sin \theta_K \sin \theta \ell \cos \phi + I_6 \sin^2 \theta_K \cos \theta \ell \sin 2\theta \ell + I_7 \sin 2\theta_K \sin \theta \ell \sin \phi + I_8 \sin \theta_K \sin 2\theta \ell \sin \phi + I_9 \sin^2 \theta_K \sin^2 \theta \ell \sin 2\phi].
\]

The angular coefficients \( I \)'s, which can be measured from the study of the angular distribution, are \( q^2 \) dependent. But for convenience we will suppress the explicit \( q^2 \) dependence.

The \( I \)'s are conveniently expressed in terms of “seven” amplitudes. These compromise of the six transversity amplitudes that survive in the massless lepton limit and an amplitude \( A_t \) that contributes only if the mass \( m \) of the lepton is finite. The six transversity amplitudes \( A_{L,R,\perp,\parallel} \), where \( \perp, \parallel \) and \( 0 \) represent the polarizations of the on shell \( K^* \) and \( L, R \) denote the chirality of the lepton current. The explicit expression for \( I \)'s in the transversity amplitudes \( A_{L,R,\perp,\parallel} \) and \( A_t \) are

\[
I_1 = \frac{(2 + \beta^2)}{4} [I_L^2|A_L^1|^2 + |A_L^2|^2 + (L \to R)],
\]

\[
I_2 = \frac{4m^2}{q^2} \text{Re}(A_L^L A_R^R + A_L^R A_R^L),
\]

\[
I_3 = \frac{\beta^2}{2} [I_L^L - I_L^R]^2 + (L \to R),
\]

\[
I_4 = \frac{\beta^2}{2} \text{Re}(A_L^L A_R^R) + (L \to R),
\]

\[
I_5 = \sqrt{2}\beta \text{Re}(A_L^L A_R^R) - (L \to R),
\]

\[
I_6 = 2\beta \text{Re}(A_L^L A_R^R) - (L \to R),
\]

\[
I_7 = \sqrt{2}\beta \text{Im}(A_L^L A_R^R) - (L \to R),
\]

\[
I_8 = \frac{1}{\sqrt{2}} \beta^2 \text{Im}(A_L^L A_R^R) + (L \to R),
\]

\[
I_9 = \beta^2 \text{Im}(A_L^L A_R^R) + (L \to R),
\]

where

\[
\beta = \sqrt{1 - \frac{4m^2}{q^2}}.
\]

We have dropped the explicit \( q^2 \) dependence of the transversity amplitudes \( A_{L,R,\perp,\parallel} \) and \( A_t \) for notational simplicity.

The seven amplitudes can be written in terms of the form-factors \( \chi_{0,1,2,3} \) and \( \chi_{1,2,3} \) as follows:

\[
A_{L,R}^{L,R} = N \sqrt{2} \lambda^{1/2}(m_B^2, m^2, q^2) \left( \tilde{C}_0^{(3)} + \tilde{C}_{10} \right) \chi_3 - \tilde{\eta}_3,
\]

\[
A_{\perp,\parallel}^{L,R} = 2 \sqrt{2} \lambda \left( \tilde{C}_0^{(3)} + \tilde{C}_{10} \right) \chi_1 - \zeta \tilde{\eta}_1,
\]
\[ A_{0}^{L,R} = \frac{N}{2m_{K}^{n}q^{2}} \left[ (\tilde{C}_{g}^{(2)} \chi + \tilde{C}_{10}) \{ 4k.q \chi_{1} + \lambda(m_{B}^{2}, m_{K}^{2}, q^{2}) \chi_{2} \} - \zeta \{ 4k.q \tilde{\chi}_{1} + \lambda(m_{B}^{2}, m_{K}^{2}, q^{2}) \tilde{\chi}_{2} \} \right] \].

(15c)

\[ A_{t} = -\frac{N}{m_{K}^{n}} q^{2} \lambda^{1/2}(m_{B}^{2}, m_{K}^{2}, q^{2}) \tilde{C}_{10} \chi_{0} \].

(15d)

where,

\[ \kappa = 1 + \frac{\tilde{C}_{g}^{(1)} - \tilde{C}_{g}^{(2)}}{\tilde{C}_{g}^{(2)}} \frac{4k.q \chi_{1}}{4k.q \chi_{1} + \lambda(m_{B}^{2}, m_{K}^{2}, q^{2}) \chi_{2}} \].

\[ \lambda(a, b, c) \equiv a^{2} + b^{2} + c^{2} - 2(ab + bc + ac) \] and \( N \) is the normalization constant. In the narrow width approximation for the \( K^{*} \), \( |D_{\nu}(k^{2})|^{2} \) simplifies to

\[ |D_{\nu}(k^{2})|^{2} = \frac{48\pi^{2}m_{K}^{2}}{\lambda^{3/2}(m_{K}^{2}, m_{K}^{2}, m_{Z}^{2})} \delta(k^{2} - m_{K}^{2}) \].

(16)

This result is obtained by simplifying \( N \) to

\[ N = V_{tb}V_{ts}^{*} \left[ \frac{G_{F}^{2}\alpha^{2}}{3 \cdot 210 \pi m_{B}^{2}} q^{2} \sqrt{\lambda(m_{B}^{2}, m_{K}^{2}, q^{2})} \right]^{1/2} \].

We note that in principle the effect of finite \( K^{*} \) resonance width can easily be taken into account, however, we make no attempt to do so as the value of the normalization constant is not going to be used anywhere in our calculation.

The six transversity amplitudes described by Eqs. (15a) – (15c) which survive in the massless lepton case, can be rewritten in a short-form notation by introducing new form-factors \( F_{\lambda} \) and \( \tilde{G}_{\lambda} \) as follows,

\[ A_{L,R}^{L,R} = C_{L,R}^{0} F_{\lambda} - \tilde{G}_{\lambda} = (\tilde{C}_{g}^{0} + \tilde{C}_{10}) F_{\lambda} - \tilde{G}_{\lambda} \].

(17)

The expressions of \( F_{\lambda} \) and \( \tilde{G}_{\lambda} \) can be obtained by comparing Eq. (17) with Eqs. (15a) – (15c) and are given in Appendix-B. \( F_{\lambda} \) and \( \tilde{G}_{\lambda} \) are \( g^{2} \) dependent form-factors, suitably defined to include both factorizable and non-factorizable corrections to all orders [2]. The form-factor dependence of \( \tilde{C}_{g}^{(j)} \) indicated by ‘\( j \)’ in Eqs. (15a) – (15c) is now translated to an effective helicity ‘\( \lambda \)’ dependence of Wilson coefficient \( C_{g}^{0} \) as

\[ \tilde{C}_{g}^{0} \equiv \tilde{C}_{g}^{(3)}, \tilde{C}_{g}^{0} \equiv \tilde{C}_{g}^{(1)}, \tilde{C}_{g}^{0} \equiv \tilde{C}_{g}^{(2)}. \]

(18)

It is easily seen that \( F_{\lambda} \) and \( \tilde{G}_{\lambda} \) are proportional to \( \chi_{j} \) and \( \tilde{\chi}_{j} \) respectively. Thus \( F_{\lambda} \)'s are completely real and \( \tilde{G}_{\lambda} \)'s are complex SM. All imaginary contributions to the amplitude arise from the complex \( C_{g}^{0} \) and \( \tilde{G}_{\lambda} \). An interesting observation is that \( A_{L,R}^{L,R} \) remains unchanged if the non-factorizable contributions between \( \tilde{G}_{\lambda} \) and \( \tilde{C}_{g}^{0} \) are rearranged. This observation differs from the conclusion obtained in Ref. [2] because \( C_{g}^{0} \) are now helicity dependent and implies that \( \tilde{G}_{\lambda} \) and \( \tilde{C}_{g}^{0} \) cannot be individually extracted.

Using very general arguments it is easy to see that the form of the amplitude described in Eq. (17) is the most general possible and the full decay amplitude can be completely described by them for the massless case. The amplitude must be described by the helicity of the \( K^{*} \) and can be divided into two parts one that depends on the chirality of the lepton and another that does not. It is easily noted that the term described by \( F_{\lambda} \) is chirality dependent whereas the contribution corresponding to the effective photon vertex \( \tilde{G}_{\lambda} \) is not. The form factors \( F_{\lambda} \) and \( \tilde{G}_{\lambda} \) depend only on the helicity and the chirality dependence is absorbed completely into the Wilson coefficients. The coefficient of chirality dependent terms proportional to \( F_{\lambda} \) can themselves either depend on helicity or be independent of it. Hence, the amplitudes in Eq. (17) are parameterized in terms of three terms. Throughout the rest of the paper we will use only the form of the amplitudes in Eq. (17), which is the most general possible in the SM.

It is obvious from Eq. (13) that a complete study of the angular distribution involves eleven orthogonal terms allowing us to measure ‘eleven’ observables. In the limit of massless lepton there exist two relations between the coefficient \( I_{s} \)’s, i.e. \( I_{7}^{0} = -I_{7}^{\pm} \) and \( I_{7}^{s} = 3I_{7}^{s} \). This reduces the number of independent observables to ‘nine’. We will divide our discussion into two parts. In Sec. IV we will restrict our discussion by assuming that the lepton is massless and in Sec. V we will generalize the discussion to the massive lepton case. In a previous paper [2] the mode \( B \to K^{*} \ell^{+} \ell^{-} \) was studied in the limit of massless lepton and under the assumption of vanishing \( CP \) violation and absence of resonance contributions in the \( g_{9}^{2} \) domains considered. Under these approximations \( I_{s,5,9} \) = 0 and the number of useful observables reduce to only ‘six’. In this paper we carefully examine each of these assumptions and in particular take into account resonance contributions and the effect of massless lepton. As emphasized in Sec. II we have taken into account charm loop effects. The charm loop effect and other resonance contributions can make the amplitude complex. In the discussions that pursue we will assume that the amplitude is complex and ensure that all SM contributions, both factorizable and non-factorizable, are taken into account completely when writing the most general parameterized amplitude.

Within SM, \( CP \)-violation is expected to be extremely tiny and essentially unobservable [1, 3] at the current level of experimental accuracy. In Ref. [1] the \( CP \) violating asymmetry was evaluated to be \( \sim 3 \times 10^{-4} \). This would imply that one need at the very least \( 10^{7} \) reconstructed events in this decay channel to observe the asymmetry at 1\( \sigma \). Given this we have justifiably ignored \( CP \)
violation in this channel and any observation of $CP$ violation at the current level of experimental sensitivity would constitute an unambiguous signal of NP. In view of this, we ignore $CP$ violation hence forth. It may be noted that $CP$ violation can be easily included in our approach. However, we ignore it because it is not central to our discussion and we do not wish to complicate our notation accounting for unobservable effects within the SM. Under the assumption of vanishing $CP$ violation the conjugate mode $B \rightarrow K^*\ell^+\ell^-$ has an identical decay distribution except that $I_{5,6,8,9}$ switch signs to become $-I_{5,6,8,9}$ in the differential decay distribution [1, 3].

Integration over $\cos \theta_K$, $\cos \theta_\ell$ and $\phi$ results in the differential decay rate with respect to the invariant lepton mass:

$$
\frac{d\Gamma}{dq^2} = \sum_{\lambda=0,\pm 1} \left( |A_{L\lambda}|^2 + |A_{R\lambda}|^2 \right). 
$$

(19)

We define the relevant observables to be the three helicity fractions defined as

$$
F_L = \frac{|A_{L0}|^2 + |A_{L\pm 1}|^2}{\Gamma_f},
$$

(20a)

and isolates the contribution from the $I_6$ term in Eq. (13).

Contributions from $I_4$ and $I_5$ in Eq. (13) are extracted by the two angular asymmetries,

$$
A_4 = \left[ \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{3\pi/2} d\phi \int_{-1}^{1} d\cos \theta_K \int_{-1}^{1} d\cos \theta_\ell \int_{-1}^{1} d\cos \theta_\ell \frac{d^4(\Gamma + \bar{\Gamma})}{dq^2 d\cos \theta_\ell d\cos \theta_K d\phi} \right],
$$

(22)

and

$$
A_5 = \left[ \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{3\pi/2} d\phi \int_{-1}^{1} d\cos \theta_K \int_{-1}^{1} d\cos \theta_\ell \int_{-1}^{1} d\cos \theta_\ell \frac{d^4(\Gamma - \bar{\Gamma})}{dq^2 d\cos \theta_\ell d\cos \theta_K d\phi} \right].
$$

(23)

The three new observables not considered in Ref. [2] are $A_7$, $A_8$ and $A_9$. These are non-zero if the amplitude is complex. They may be described in analogy as,

$$
A_7 = \left[ \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} d\cos \theta_K \int_{-1}^{1} d\cos \theta_\ell \int_{-1}^{1} d\cos \theta_\ell \frac{d^4(\Gamma + \bar{\Gamma})}{dq^2 d\cos \theta_\ell d\cos \theta_K d\phi} \right],
$$

(24)

$$
A_8 = \left[ \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} d\cos \theta_K \int_{-1}^{1} d\cos \theta_\ell \int_{-1}^{1} d\cos \theta_\ell \frac{d^4(\Gamma - \bar{\Gamma})}{dq^2 d\cos \theta_\ell d\cos \theta_K d\phi} \right],
$$

(25)

$$
A_9 = \left[ \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{1} d\cos \theta_K \int_{-1}^{1} d\cos \theta_\ell \int_{-1}^{1} d\cos \theta_\ell \frac{d^4(\Gamma + \bar{\Gamma})}{dq^2 d\cos \theta_\ell d\cos \theta_K d\phi} \right].
$$

(26)

The well known forward–backward asymmetry $A_{FB}$ and the five other angular asymmetries, $A_4$, $A_5$, $A_7$, $A_8$
and $A_9$ can be written directly in terms of the transversity amplitudes as follows:

$$A_{FB} = \frac{3}{2} \text{Re}(A_L^+ A_L^+ - A_L^R A_L^R) \frac{\Gamma_f}{\Gamma_f},$$  

$$A_4 = \frac{\sqrt{3}}{\pi} \text{Re}(A_L^0 A_L^+ - A_R^0 A_R^+ \frac{\Gamma_f}{\Gamma_f}),$$  

$$A_5 = \frac{3}{2\sqrt{2}} \text{Re}(A_L^+ A_L^+ - A_L^R A_L^R),$$  

$$A_7 = \frac{3}{2\sqrt{2}} \text{Im}(A_L^0 A_L^+ - A_R^0 A_R^+ \frac{\Gamma_f}{\Gamma_f}),$$  

$$A_8 = \frac{\sqrt{3}}{\pi} \text{Im}(A_L^0 A_L^+ + A_R^0 A_R^+) \frac{\Gamma_f}{\Gamma_f},$$  

$$A_9 = \frac{3}{2\pi} \text{Im}(A_L^0 A_L^+ + A_R^0 A_R^+) \frac{\Gamma_f}{\Gamma_f}.$$

The observables $A_4$, $A_5$, $A_{FB}$, $A_7$, $A_8$ and $A_9$ are related to the $CP$ averaged observables $S_4$, $S_5$, $A_{FB}^{\text{HCS}}$, $S_7$, $S_8$ and $S_9$ measured by LHCb [5] as follows respectively,

$$A_4 = -\frac{2}{\pi} S_4, \quad A_5 = \frac{3}{\pi} S_5, \quad A_{FB} = -A_{FB}^{\text{HCS}},$$

$$A_7 = \frac{3}{\sqrt{2}} S_7, \quad A_8 = -\frac{2}{\sqrt{2}} S_8, \quad A_9 = \frac{3}{2\pi} S_9.$$

We emphasize that our observables $A_4, A_5, A_{FB}, A_7, A_8$ and $A_9$ are related to the $CP$ violating asymmetries as $A_4, A_5, A_{FB}$ and $A_7, A_8$ as well as $A_9$ in our notation. We would refer to the $CP$ violating asymmetry measurements as $A_{4,5,7,8,9}^{L,R}$. The observables $F_L$ and $A_{FB}$ have been measured by different experiments Babar, Belle, CDF and LHCb [5, 20–26]. By doing an angular analysis in the angle $\phi$, LHCb has measured the observable $S_9$ [5]. $S_9$ is related to the transversity helicity fraction $F_\perp$ through the relation

$$S_9 = -\frac{1}{\pi} F_L - 2 F_\perp.$$

The observables $F_L$, $F_\perp$, $A_4$, $A_5$, $A_{FB}$, $A_7$, $A_8$ and $A_9$ defined in this section are not independent. In the subsequent sections we explore the relation between them.

### IV. THE MASSLESS LEPTON LIMIT.

In this section we generalize the approach developed in Refs. [2] to include all contribution from the SM that were ignored as their effects are sub-dominant, except that we still restrict our discussion to the limit where the lepton is massless. The corrections arising from massive leptons will be taken into account later in Sec V. In particular we will consider the possibility that the amplitudes $A_{L,R}^{\lambda}$ are in general complex. As already mentioned the imaginary contribution can be totally attributed to the complex $\tilde{C}_0^\lambda$ and $\tilde{G}_\lambda$. This would include loop contributions that are both factorizable and non-factorizable and all resonance contributions. We also take into account that the non-factorizable contributions can introduce an ‘effective helicity (A) dependence’ in the Wilson coefficient $\tilde{C}_0^\lambda$.

In Ref. [2] a new variable $r_\lambda$ was introduced that led to significant simplification. We once again introduce the same ‘real variable’ $r_\lambda$ defined as,

$$r_\lambda = \frac{\text{Re}(\tilde{G}_\lambda)}{\tilde{F}_\lambda} - \frac{\text{Re}(\tilde{C}_0^\lambda)}{\tilde{F}_\lambda}.$$

Since, we now consider $\tilde{C}_0^\lambda$ and $\tilde{G}_\lambda$ to be complex in general, we have modified $r_\lambda$ to include only the real contributions i.e, $\text{Re}(\tilde{C}_0^\lambda)$ and $\text{Re}(\tilde{G}_\lambda)$. The amplitude $A_{L,R}^\lambda$ in Eq. (17) can thus be written as,

$$A_{L,R}^\lambda = (\tilde{C}_0^\lambda + \tilde{C}_{10}) \tilde{F}_\lambda - \tilde{G}_\lambda$$

$$= (\mp \tilde{C}_{10} - r_\lambda) \tilde{F}_\lambda + i \epsilon_\lambda,$$

where $\epsilon_\lambda \equiv \text{Im}(\tilde{C}_0^\lambda) \tilde{F}_\lambda - \text{Im}(\tilde{G}_\lambda)$. The use of $\epsilon_\lambda$ is not necessarily meant to imply that the imaginary parts are negligibly small. We make no such assumption. It is, however, to be expected that the imaginary contributions are sub-dominant. The presence of the $\epsilon_\lambda$ term introduces three extra variables in comparison to the discussion in Ref. [2]. However, we now have three extra observables $A_7$, $A_8$ and $A_9$. Hence, dealing with complex amplitude introducing only a technical difficulty of solving for additional variables. We begin by expressing the observables $F_L$, $F_\parallel$, $F_\perp$, $A_4$, $A_5$, $A_{FB}$, $A_7$, $A_8$ and $A_9$ in terms of $\tilde{C}_{10}$, $r_\lambda$, $F_\perp$, $F_\parallel$ and $\epsilon_\lambda$ as follows:

$$F_L \parallel = 2 F_\parallel^2 (r_\parallel^2 + \tilde{C}_{10}^2) + 2 \epsilon_0,$$  

$$F_\parallel \parallel = 2 F_\perp^2 (r_\perp^2 + \tilde{C}_{10}^2) + 2 \epsilon_\parallel,$$  

$$F_\perp \parallel = 2 F_\perp^2 (r_\parallel^2 + \tilde{C}_{10}^2) + 2 \epsilon_\perp,$$  

$$\sqrt{2} \pi A_4 \parallel = 4 \pi F_0 F_\parallel (r_\parallel^2 + \tilde{C}_{10}^2) + 4 \epsilon_0 \epsilon_\parallel,$$  

$$\sqrt{2} A_5 \parallel = 3 \pi F_0 F_\parallel \tilde{C}_{10} (r_\parallel + r_\perp),$$  

$$\sqrt{2} A_8 \parallel = 3 \pi F_0 F_\parallel \tilde{C}_{10} (r_\parallel + r_\perp),$$  

$$\pi A_8 \parallel = 2 \sqrt{2} (F_0 r_\parallel \epsilon_0 - F_\parallel r_\parallel \epsilon_\parallel),$$  

$$\pi A_9 \parallel = 3 (F_\parallel r_\perp \epsilon_\parallel - F_\parallel r_\parallel \epsilon_\parallel).$$

One immediately concludes that

$$2\epsilon_0 \frac{i}{\Gamma_f} \leq F_L,$$  

$$2\epsilon_\parallel \frac{i}{\Gamma_f} \leq F_\parallel,$$  

$$2\epsilon_\perp \frac{i}{\Gamma_f} \leq F_\perp.$$

Eqs. (37)–(42) can be easily transformed to the form in Ref. [2] by the redefinition of the observables $F_L$, $F_\parallel$, $F_\perp$.
and $A_4$ as

$$F'_\lambda = F_\lambda - \frac{2\varepsilon^2}{\lambda^2},$$

$$A'_4 = A_4 - \frac{\sqrt{2}\varepsilon_0\varepsilon_\parallel}{\pi \lambda^2}.\quad (50)$$

It should be noted that $F'_L + F'_R + F'_\perp \leq 1$. Since only the ratios of the form-factors $F_\lambda$ play a role in the relations we wish to derive we define ratios of form-factors $P_1$, $P_2$ and $P_3$:

$$P_1 = \frac{F_\perp}{F_\parallel},$$

$$P_2 = \frac{F_\perp}{F_0},$$

$$P_3 = \frac{F_\perp}{F_0 + F_\perp} \equiv \frac{P_1 P_2}{P_1 + P_2}.\quad (53)$$

Following these redefinitions Eqs. (37)–(42) can be recast into three sets of equations just as done in Ref. [2]. The three sets of equation are:

○ **Set-I**

$$F'_\parallel \Gamma_f = 2 \frac{F^2_\perp}{P_1} \left( r_\parallel^2 + \tilde{C}_{10}^2 \right)\quad (54)$$

$$F'_\perp \Gamma_f = 2 \frac{F^2_\parallel}{P_2} \left( r_\perp^2 + \tilde{C}_{10}^2 \right)\quad (55)$$

$$A_{FB} \Gamma_f = 3 \frac{F^2_\perp}{P_1} \tilde{C}_{10} \left( r_\parallel + r_\perp \right)\quad (56)$$

○ **Set-II**

$$F'_\parallel \Gamma_f = 2 \frac{F^2_\perp}{P_1} \left( r_\parallel^2 + \tilde{C}_{10}^2 \right)\quad (57)$$

$$F'_\perp \Gamma_f = 2 \frac{F^2_\parallel}{P_2} \left( r_\perp^2 + \tilde{C}_{10}^2 \right)\quad (58)$$

$$\sqrt{2} A_5 \Gamma_f = 3 \frac{F^2_\perp}{P_2} \tilde{C}_{10} \left( r_\parallel + r_\perp \right)\quad (59)$$

○ **Set-III**

$$(F'_\parallel + 2\sqrt{2}\pi A'_1) \Gamma_f = 2 \frac{F^2_\perp}{P_3} \left( r_\lambda^2 + \tilde{C}_{10}^2 \right)\quad (60)$$

$$F'_\perp \Gamma_f = 2 \frac{F^2_\parallel}{P_3} \left( r_\perp^2 + \tilde{C}_{10}^2 \right)\quad (61)$$

$$(A_{FB} + \sqrt{2} A_5) \Gamma_f = 3 \frac{F^2_\perp}{P_3} \tilde{C}_{10} \left( r_\lambda + r_\perp \right)\quad (62)$$

In the above we have defined $r_\lambda$ as

$$r_\lambda = r_\parallel P_2 + r_\perp P_1\quad (63)$$

Of the nine equations defined in the three Sets only six of them are independent. These are the three equations Eqs. (54)–(56) in Set-I, two Eqs. (57) and (59) from Set-II and Eq. (60) of Set-III. It is easy to see that Set-II and Set-III can be obtained from Set-I by the following replacements:

- **Set-II from Set-I**

$$(F'_\parallel + F'_\perp + \sqrt{2}\pi A'_1) \Gamma_f = 2 \frac{F^2_\perp}{P_3} \left( r_\lambda^2 + \tilde{C}_{10}^2 \right)$$

- **Set-III from Set-I**

$$(F'_\parallel + F'_\perp + \sqrt{2}\pi A'_1) \Gamma_f = 2 \frac{F^2_\perp}{P_3} \left( r_\lambda^2 + \tilde{C}_{10}^2 \right)$$

It is obvious that we only need to solve Set-I to obtain $r_\parallel$ and $r_\perp$ in terms of $P_1$, $F'_\parallel$, $F'_\perp$ and $A_{FB}$. The solutions to Set-II and Set-III can be obtained by simple replacements.

The solution of Set-I gives (from Appendix A)

$$r_\parallel = \pm \frac{\sqrt{I_f}}{\sqrt{2} F_\perp} \frac{(P^2_1 F'_\perp + P_1 Z'_1)}{\sqrt{P^2_1 F'_\perp + F'_\perp + P_1 Z'_1}},\quad (64)$$

$$r_\perp = \pm \frac{\sqrt{I_f}}{\sqrt{2} F_\perp} \frac{(F'_\perp + P_1 Z'_1)}{\sqrt{P^2_1 F'_\perp + F'_\perp + P_1 Z'_1}},\quad (65)$$

where $Z'_1$ is defined as,

$$Z'_1 = \sqrt{4F'_\perp F'_\parallel - \frac{32}{9} A^2_{FB}}.$$  

The solution to Set-II is now easily seen to be

$$r_0 = \pm \frac{\sqrt{I_f}}{\sqrt{2} F_\perp} \frac{(P^2_2 F'_\perp + P_2 Z'_2)}{\sqrt{P^2_1 F'_\perp + F'_\perp + P_2 Z'_2}},\quad (67)$$

$$r_\perp = \pm \frac{\sqrt{I_f}}{\sqrt{2} F_\perp} \frac{(F'_\perp + P_2 Z'_2)}{\sqrt{P^2_2 F'_\perp + F'_\perp + P_2 Z'_2}},\quad (68)$$

with $Z'_2$ defined as,

$$Z'_2 = \sqrt{4F'_\perp F'_\parallel - \frac{32}{9} A^2_{FB}}.$$  

On comparing the solutions for $r_\perp$ in Eqs. (65) and (68) obtained from Set-I and Set-II respectively, we obtain a relation for $P_2$ in terms of $P_1$ and observables to be

$$P_2 = \frac{2P_1 A_{FB} F'_\perp}{s\sqrt{2} A_5 (2F'_\perp + Z'_1 P_1) - Z'_2 P_1 A_{FB}},\quad (70)$$

with $s \in \{-1, +1\}$. To remove the ambiguities in the $P_2$ solution let us divide Eq. (59) by Eq. (56) and using Eqs. (64) – (68) we get

$$A_{FB} = \frac{\sqrt{2} A_5}{P_2} \frac{P_1}{P_3} \frac{F'_\parallel F'_\perp + P_1 Z'_1}{F'_\perp + P_1 Z'_1} = \frac{P_1}{P_2} \frac{(2F'_\parallel + P_2 Z'_2)}{(2F'_\perp + P_1 Z'_1)}.$$  

Substituting it in Eq. (70) we have

$$s(2F'_\perp + P_2 Z'_2) - Z'_2 P_2 = 2F'_\perp,$$
which is valid for the whole $q^2$ region only for $s = 1$.

Finally we write the $r_\perp$ solution obtained from Set-III:

$$r_\perp = \pm \frac{\sqrt{F_\perp}}{\sqrt{2F_\perp}} \frac{(F'_\perp + \frac{1}{9} P_3 Z'_3)}{\sqrt{P_3^3 (F'_\perp + F'_\perp + \sqrt{2} A'_0) + F'_\perp + P_3 Z'_3}}, \quad (72)$$

where $Z'_3$ is defined as,

$$Z'_3 = \sqrt{4(F'_\perp + F'_\perp + \sqrt{2} A'_0) F'_\perp - \frac{16}{9}(A_{FB} + \sqrt{2} A_0)^2}. \quad (73)$$

Analogous comparison of solutions for $r_\perp$ in Eqs. (65) and (72) obtained from Set-I and Set-III respectively, results in a relation for $P_3$ in terms of $P_1$:

$$P_3 = \frac{2P_1 A_{FB} F'_\perp}{(A_{FB} + \sqrt{2} A_0)(2F'_\perp + Z'_1 P_1) - Z'_3 P_1 A_{FB}}. \quad (74)$$

The ambiguity in the $P_3$ solution is also taken to be positive for the same reason as the $P_2$ solution. The form factor ratio $P_3$ is not however independent of $P_1$ and $P_2$ and is related by Eq. (53). Substituting Eqs. (70) and (74) in Eq. (53) we obtain the relation between the observables as:

$$Z'_3 = Z'_1 + Z'_2. \quad (75)$$

The relations derived so far involve the primed observables that depend on $\varepsilon_\perp$, $\varepsilon_\parallel$ and $\varepsilon_0$. However, the $\varepsilon_\perp$'s can be solved using $A_7$, $A_8$ and $A_9$ from Eqs. (43)–(45) to give

$$\varepsilon_\perp = \frac{\sqrt{2} \pi F_\perp}{(r_0 - r_\perp)} \left[ A_0 P_1 \frac{3}{\sqrt{2}} + \frac{A_8 P_2}{4} - \frac{A_7 P_1 P_2 r_\perp}{3 \pi C_{10}} \right]. \quad (76)$$

$$\varepsilon_\parallel = \frac{\sqrt{2} \pi F_\parallel}{(r_0 - r_\parallel)} \left[ A_0 P_1 r_\perp + \frac{A_8 P_2 r_\parallel}{4 r_\parallel} - \frac{A_7 P_1 P_2 r_\parallel}{3 \pi C_{10}} \right], \quad (77)$$

$$\varepsilon_0 = \frac{\sqrt{2} \pi F_0}{(r_0 - r_\parallel)} \left[ A_0 P_1 r_\perp + \frac{A_8 P_2 r_\parallel}{4 r_\parallel} - \frac{A_7 P_1 P_2 r_\parallel}{3 \pi C_{10}} \right]. \quad (78)$$

A point to be noted that the $(\varepsilon_\lambda / \Gamma_f^{1/2})$'s are free from the form factor $F_\perp$ and $F_\parallel$ as can easily be seen from the expressions for $r_\parallel$, $r_\perp$ and $r_0$ (Eqs. (64), (65) and (67)), as well as $\hat{C}_{10}$ derived in Eq. (A12). Indeed, since $P_2$ can be expressed in terms of $P_1$ and observables using Eq. (70), it is easy to see that each of the $\varepsilon_\lambda$'s are completely expressed in terms of observables and the form factor ratio $P_1$. However, these solutions are essentially iterative, since the $r_\lambda$'s and $\hat{C}_{10}$ are derived in terms of the primed observables that depend on $\varepsilon_\perp$. If the $(\varepsilon_\lambda / \Gamma_f^{1/2})$ are small as should be expected, accurate solutions for them can be found with a few iterations.

Solving for $A_4$ from Eq. (75) the relation among the observables is,

$$A_4 = \frac{2 \sqrt{2} \varepsilon_\parallel \varepsilon_0}{\pi F_\parallel} - \frac{8 A_8 A_{FB}}{9 \pi \left( F_\perp - \frac{2 \varepsilon_0}{\pi F_\perp} \right)} + \sqrt{\frac{\left( F_\perp - \frac{2 \varepsilon_0}{\pi F_\perp} \right) \left( F_\parallel - \frac{2 \varepsilon_\parallel}{\pi F_\parallel} \right) - \frac{8}{9} A_8^2}{\pi \left( F_\perp - \frac{2 \varepsilon_0}{\pi F_\perp} \right)}}. \quad (79)$$

This relation for $A_4$ in terms of other observables $F_\perp$, $F_\parallel$, $A_5$, $A_{FB}$, $A_7$, $A_8$ and $A_9$ is a generalization of the relation derived in Ref. [2]. A point to be noted is that while we have solved for the observable $A_4$, we could have used Eq. (75) to derive an expression for any of the other observable. However, only the solution for $A_4$ is unique and hence the one we consider. The validity of this relation is a test of the consistency of the values of all measured observables. Unlike the expression obtained in Ref. [2], we now have a relation between observables that depends on only one hadronic parameter, the ratio of form-factors $P_1$. It is interesting to note that $P_1$ does not receive non-factorizable contributions and is corrected by charm loops effects. Since, $P_1$ is independent of the universal wave functions $[6, 27]$ in HQET, it can be reliably calculated as an expansion in both the strong coupling constant $\alpha_s$ and $\Lambda_{QCD}/m_b$. The dependence of $A_4$ on $P_1$ is rather weak, since the observables $A_7$, $A_8$ and $A_9$ are observed to be small and are currently consistent with zero as expected [5]. If $A_7$, $A_8$ and $A_9$ are all observed to be zero, it is easy to see from Eqs. (76)–(78) that $\varepsilon_\perp = \varepsilon_\parallel = \varepsilon_0 = 0$ reducing the relation in Eq. (79) to

$$A_4 = \frac{8 A_8 A_{FB}}{9 \pi F_\perp} + \sqrt{\frac{\left( F_\perp - \frac{2 \varepsilon_0}{\pi F_\perp} \right) \left( F_\parallel - \frac{2 \varepsilon_\parallel}{\pi F_\parallel} \right) - \frac{8}{9} A_8^2}{\pi F_\perp}}. \quad (80)$$

which was derived in Ref. [2]. Interestingly, in the limit of vanishing imaginary contributions, $A_4$ can be expressed purely in terms of observables and is free from any form factor or their ratio. In Appendix. B it is shown that both $P_1$ and $P_2$ are always negative. An interesting observation that $A_{FB}$ and $A_5$ always have same signs can
be then made from the relation in Eq. (71). Hence, we can arrive to a conclusion that, from Eq. (79) the observable \( A_4 \) is always positive unless the term proportional to \( \varepsilon_0 \) is negative and it dominates over the rest of the terms in the expression.

\( A_4 \) is an observable and hence must always be real. This places constraints on the arguments of the radicals, which are directly related to the fact that \( Z'_1 \), \( Z'_2 \) and \( Z'_3 \) are all real. The constraint that \( Z'_1 \) is real in turn implies that

\[
F_F F_L - \frac{4}{9} A_{\text{FB}}^2 \geq \frac{2}{9} A_{\text{FB}}^2 (2\varepsilon_\parallel^2 + 2\varepsilon_\perp^2 - 4\varepsilon_\parallel\varepsilon_\perp). \tag{81}
\]

In Eqs. (46)–(48), we showed that \( 0 \leq \frac{2\varepsilon_\parallel}{\Gamma F_\parallel} \leq 1 \), implying that the R.H.S of Eq. (81) must itself be greater than zero. This imposes the following constraint:

\[
F_F F_L - \frac{4}{9} A_{\text{FB}}^2 \geq 0. \tag{82}
\]

A similar constraint arising from \( Z'_2 \) and \( Z'_3 \) also being real implies that

\[
(F_F + F_L + \sqrt{2\pi A_4}) F_L - \frac{4}{9} (A_{\text{FB}} + \sqrt{2A_5})^2 \geq 0. \tag{83}
\]

The equality in the above three relations holds only when a minimum of two of the \( \varepsilon \)'s are zero. For example, \( \varepsilon_\parallel \) and \( \varepsilon_\perp \) are zero for the equality to hold in Eq. (82), whereas \( \varepsilon_0 \) and \( \varepsilon_\perp \) are zero for Eq. (83). The three inequalities in Eqs. (82)–(84) impose constraints on the parameter space of observables. It is obvious that non-zero \( \varepsilon \)'s will in general restrict the parameter space of observables even further. We emphasize that this conclusion is valid without any exception. We will come back to this point in Sec. VII when we discuss the tests of the relation for \( A_4 \) in Eq. (79).

V. GENERALIZATION TO INCLUDE LEPTON MASSES.

In this section we extend the model independent approach developed in the previous section (Sec. IV) to include the lepton mass \( m \). One of the consequences of retaining the lepton mass is the need to include an additional amplitude in order to describe the full decay rate, since the term proportional to \( q_0 \) in the amplitude cannot be dropped for the massive lepton case (for a review [5]). In addition to the six amplitude \( A_{\lambda}^{L,R} \) where \( \lambda \in \{0,\parallel,\perp\} \) the decay amplitude also depends on \( A_t \), resulting in a total of seven amplitudes. These amplitudes are given in Eqs. (15a) – (15d). In addition, since the massive leptons are no longer chirality eigenstates, terms involving admixtures of heicities that are proportional to \( m^2/q^2 \) (see Eqs. (14a) and (14b)) contribute to the differential decay rate.

These additional contributions complicate the extraction of the helicity amplitudes. The observables \( F_L, F_\parallel, A_4, A_5 \) and \( A_{\text{FB}} \) given in Sec. IV are modified because of the presence of the new transversity amplitude \( A_t \) and helicity admixture terms in the decay distribution. This in turn results in modifying the relations in Eqs. (79) and (80). The effect of the mass of the lepton is always included in the measured observables and it is not possible to measure any observable without the mass effects. In order to distinguish the “hypothetical observables without the mass effects” considered in Sec. IV from these true observables, we define them with a superscript “o” and relate to the massless limit observables as:

\[
\Gamma^oyo = \beta^oyo \Gamma + 3\Gamma_1, \tag{85a}
\]

\[
F^oyo_L = \frac{1}{\Gamma^oyo} (\beta^oyo F_L + \Gamma_1), \tag{85b}
\]

\[
F^oyo_\parallel = \frac{1}{\Gamma^oyo} (\beta^oyo F_\parallel + \Gamma_1), \tag{85c}
\]

\[
F^oyo_\perp = \frac{1}{\Gamma^oyo} (\beta^oyo F_\perp + \Gamma_1), \tag{85d}
\]

\[
A^oyo_4 = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_4, \tag{85e}
\]

\[
A^oyo_5 = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_5, \tag{85f}
\]

\[
A^oyo_{\text{FB}} = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_{\text{FB}}, \tag{85g}
\]

\[
A^oyo_8 = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_7, \tag{85h}
\]

\[
A^oyo_9 = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_8, \tag{85i}
\]

\[
A^oyo_9 = \frac{\Gamma^oyo}{\Gamma^oyo} \beta^oyo A_9. \tag{85j}
\]

In the above we have defined

\[
T_1 = (1 + E_1) \frac{m^2}{q^2} \Gamma^oyo \quad \text{where} \quad E_1 = \frac{|A_t|^2}{\Gamma^oyo} + 2 \frac{2}{\Gamma^oyo} \text{Re}[A^L_L A^R_R + A^L_\perp A^R_\perp + A^L_0 A^R_0].
\]

Using

\[
2 \text{Re}[A^_L L A^R_0] = |A^L_L + A^R_R|^2 - \Gamma^oyo F_\lambda
\]

and the Cauchy-Schwarz inequality, we find

\[
T_1 = (|A_t|^2 + \sum_{\lambda\in\{\parallel,\perp\}} |A^L_\lambda + A^R_\lambda|^2) \frac{m^2}{q^2} \leq (|A_t|^2 + 2\Gamma^oyo) \frac{m^2}{q^2}. \tag{86}
\]
which is always positive and bounded. This bound is important since $T_1$ has not been measured so far. $T_1$ can also be expressed in terms of angular coefficients as,

$$\frac{T_1}{\Gamma_f} = \frac{1}{3} - \frac{4I_2^L - I_2^L}{3\Gamma_f^L}$$

and measured in terms of two new observables $A_{10}$ and $A_{11}$, defined in terms of angular asymmetries as follows:

$$A_{10} = \int_0^{2\pi} d\phi \int_{-1}^1 d\cos \theta_K \left[ \int_{-1/2}^{-1/2} + \int_{1/2}^{1/2} \right] d\cos \theta_t \frac{d^4(\Gamma + \bar{\Gamma})}{d\cos \theta_t d\cos \theta_K d\phi}$$

$$A_{11} = \int_0^{2\pi} d\phi \int_{-1}^1 d\cos \theta_K \int_{-1}^1 d\cos \theta_t \frac{d^4(\Gamma + \bar{\Gamma})}{d\cos \theta_t d\cos \theta_K d\phi}$$

If the two asymmetries $A_{10}$ and $A_{11}$ are measured experimentally then we can get the estimate of the correction term arising due to lepton masses. However, from Eq. (86) it can be seen that $T_1$ is proportional to lepton mass (square) $m^2/q^2$ which is very small and difficult to measure except at small $q^2$. In the limit of zero lepton mass $T_1$ vanishes which gives a constraint on these two observables by,

$$A_{10} - \frac{4}{3}A_{11} = \frac{3}{16}$$

A deviation from this relation would indicate the effect of the non-zero lepton mass and provide an estimate of the size the mass corrections. The observables are re-expressed in terms of the variables $r_\lambda$ (defined in Eq. (35)) as follows:

$$F_{1\parallel}^\lambda = 2\beta^2 \frac{F_1^\parallel}{P_1^\parallel} (r_0^\parallel + \tilde{C}_1^2 + 2\beta \varepsilon_0^\parallel + T_1)$$

$$F_{1\perp}^\lambda = 2\beta^2 \frac{F_1^\perp}{P_1^\perp} (r_0^\perp + \tilde{C}_1^2 + 2\beta \varepsilon_0^\perp + T_1)$$

$$\sqrt{2}\pi A_{1\parallel}^\lambda = 4\beta^2 \frac{F_1^\parallel}{P_1^\parallel} (r_{0\parallel} + \tilde{C}_1^2) + 4\beta^2 \varepsilon_0^\parallel$$

$$\sqrt{2}\pi A_{1\perp}^\lambda = 3\beta^2 \frac{F_1^\perp}{P_1^\perp} (r_{0\perp} + \tilde{C}_1^2) + 4\beta^2 \varepsilon_0^\perp$$

$$A_{\parallel}^\lambda = 3\beta^2 \frac{F_1^\parallel}{P_1^\parallel} \tilde{C}_1^2 (r_0 + r_{\perp})$$

$$A_{\perp}^\lambda = 3\beta^2 \frac{F_1^\perp}{P_1^\perp} \tilde{C}_1^2 (r_0 + r_{\perp})$$

$$Z_3 = \sqrt{4 \left( (F_0^\parallel - \frac{T_0^\parallel}{\Gamma_f^\parallel}) (F_0^\perp - \frac{T_0^\perp}{\Gamma_f^\perp}) - \frac{16}{9} \beta^2 A_{\parallel}^2 \right)}$$

To simplify notation we have defined

$$T_\lambda = T_1 + 2\beta^2 \varepsilon_\lambda^2 ; \quad \lambda \in \{0, \perp, \parallel\}$$

Substituting Eqs. (101) and (102) in Eq.(53) we can get
the condition valid over whole $q^2$ range as:

$$Z'_0 = Z'_1 + Z'_2.$$  \hspace{1cm} (107)

The $\varepsilon_a$'s can be solved as was done in the previous section using Eqs. (98)–(100) to give

$$\varepsilon_\perp = \sqrt{2\pi\Gamma_f^o} \frac{A_0 P_1 + A_2^o P_2}{\beta^2 (r_0 - r_f) F_1} - \frac{A_0^2 P_1 P_2}{3\pi C_10} + \frac{A_2^o P_1 P_2 r_\perp}{3\pi C_10},$$  \hspace{1cm} (108)

$$\varepsilon_\parallel = \sqrt{2\pi\Gamma_f^o} \frac{A_0^o r_0 + A_2^o P_2 r_\parallel}{\beta^2 (r_0 - r_f) F_1} - \frac{A_0^o r_0}{3\pi C_10} - \frac{A_2^o P_2 r_\parallel}{3\pi C_10},$$  \hspace{1cm} (109)

Solving for $A_1^o$ from Eq. (107) the relation among the observables including lepton masses turns out

$$A_1^o = \frac{2\sqrt{2} \beta^2 \varepsilon_\parallel \varepsilon_0}{\pi \Gamma_f^o} + \frac{8 \beta^2 A_0^o A_1^o}{9\pi (F_\perp - \frac{7}{16} F_\parallel)} + \sqrt{2} \frac{(F_\perp - \frac{7}{16} F_\parallel)(F_\perp - \frac{7}{16} F_\parallel) - \frac{8}{9} \beta^2 A_0^2}{\pi (F_\perp - \frac{7}{16} F_\parallel)}.$$  \hspace{1cm} (111)

In analogy to the massless case, each of $Z'_1$, $Z'_2$ and $Z'_3$ are also real. A real $Z'_0$ implies that

$$F_\parallel F_\perp - \frac{4}{9} A_1^2 \geq F_\parallel F_\perp \left( \frac{\Gamma_f}{\Gamma_f F_\parallel} + \frac{\Gamma_f}{\Gamma_f F_\perp} - \frac{\Gamma_f}{\Gamma_f F_\parallel F_\perp} \right) = \frac{16 m^2 A_1^2}{9 q^2}.$$  \hspace{1cm} (112)

Since, $0 \leq \frac{\Gamma_f}{\Gamma_f F_\parallel} \leq 1$ as can be seen from Eqs. (92)–(94), we can obtain a bound on the L.H.S. of Eq. (112). The bounds arising from real $Z'_1$, $Z'_2$ and $Z'_3$ are

$$F_\parallel F_\perp - \frac{4}{9} A_1^2 \geq - \frac{16 m^2 A_1^2}{9 q^2},$$  \hspace{1cm} (113a)

$$F_\parallel F_\perp - \frac{8}{9} A_2^2 \geq - \frac{32 m^2 A_2^2}{9 q^2},$$  \hspace{1cm} (113b)

$$(F_\parallel F_\perp + \sqrt{2}\pi A_2^o) F_\perp - \frac{4}{9} (A_1^0 + \sqrt{2} A_2^o)^2 \geq - \frac{16 m^2 (A_1^0 + \sqrt{2} A_2^o)^2}{9 q^2}.$$  \hspace{1cm} (113c)

respectively. Clearly the L.H.S. of the above inequalities can, in the worst case, be a small negative number. Comparing this with the massless case we note that while the effect of the imaginary contributions is to restrict the parameter space further the effect of mass dependent terms is to oppose this restriction. The mass term should have the maximum effect at $q^2$ close to $4m^2$, but as we will see in the next section (Sec. VI) in the limit $q^2 \to 4m^2$ all the asymmetries approach zero. The contribution from the mass term should hence be insignificant, indicating that in practice the allowed parameter space of observables is not noticeably altered. This conclusion is borne out to be true in numerical estimates as we will see in Sec. VII. We conclude, therefore, that the most conservative allowed parameter space remains unaltered even if the small lepton mass term is dropped compared to $q^2$ and the imaginary contributions to the amplitudes are completely ignored.

The zero crossings of angular asymmetries $A_2^o$, $A_2^o$ and $A_0^o + \sqrt{2} A_2^o$ provide interesting limits where the relation in Eq. (111) simplifies to three independent relations with each of them providing an interesting test for NP. At the zero crossing of $A_2^o$, $A_0^o$, and $A_0^o + \sqrt{2} A_2^o$, Eq. (111) reduces to

$$\frac{8 A_3^2}{9 (F_\perp - \frac{7}{16} F_\parallel)} + \frac{\pi^2 \left( A_3 - \frac{2\sqrt{2} \beta^2 \varepsilon_\parallel \varepsilon_0}{\pi \Gamma_f^o} \right)^2}{2 (F_\perp - \frac{7}{16} F_\parallel)} = 1.$$  \hspace{1cm} (114a)
The zero-crossings of these observables are also interesting as the form factor ratios $P_1$, $P_2$ and $P_3$ can be related to the helicity fractions at those $q^2$ points. Eq. (97) implies that when $A^0_{FB} = 0$, $r_{\parallel} + r_{\perp}$ must be zero. Then, the expression for $r_{\parallel} + r_{\perp}$ (see Eq. (A10) for the massive case in Appendix A) gives,

$$r_{\parallel} + r_{\perp} |_{A^0_{FB}=0} = \pm \frac{\sqrt{F^0_{\perp}}}{\sqrt{2F_{\perp}}} \left( F^0_{\parallel} - \frac{T_1}{T_f} + P_1 \sqrt{F^0_{\parallel} - \frac{T_1}{T_f}} \right) = 0$$

$$\Rightarrow P_1 |_{A^0_{FB}=0} = - \frac{\sqrt{F^0_{\parallel} - \frac{T_1}{T_f}}}{\sqrt{F^0_{\perp} - \frac{T_1}{T_f}}}$$

(116)

$P_1$ can be iteratively solved from the above equation. We note that in order one has real positive form-factors by definition (Eq. (51)) $P_1$ is always negative. The zero crossing of $A^0_{FB}$ is observed at $q^2 = 4.9^{+1.1}_{-1.3}$ GeV$^2$ [5] which is in the large recoil region where it is believed that reliable calculations can be done in HQET. Hence, we can check the predictability of HQET in large recoil region, when enough data for all observables are available at this $q^2$ point.

Eqs. (101) and (102) can now be used to obtain $P_2$ and $P_3$ at the zero crossings $A^0_e = 0$ and $A^0_{FB} + \sqrt{2}A^0_\xi = 0$ respectively,

$$P_2 |_{A^0_e=0} = - \frac{\sqrt{F^0_{\parallel} - \frac{T_1}{T_f}}}{\sqrt{F^0_{\perp} - \frac{T_0}{T_f}}}$$

(117)

$$P_3 |_{A^0_{FB}+\sqrt{2}A^0_\xi=0} = - \frac{\sqrt{F^0_{\parallel} - \frac{T_1}{T_f}}}{\sqrt{(F^0_{\perp} - \frac{T_0}{T_f}) + (F^0_{\parallel} - \frac{T_1}{T_f}) + \sqrt{2}πA^0_9 - \frac{4β^2ε_0ε_0}{T_f}}}$$

(118)

The relation derived in Eq. (111) incorporates all the possible effects within SM. It includes a finite lepton
mass, electromagnetic correction to hadronic operators at all orders and all factorizable and non-factorizable contributions including resonances to the decay. It can be seen from the Eq. (106) the term $T_i/T_f^2$ contains $T_i/T_f$, which is expressed in Eq. (88) in terms of the asymmetries $A_{10}$ and $A_{11}$ which can be measured experimentally and the other term $(\epsilon_\lambda/T_f^{2/3})$ depends only on the observables and one form-factor ratio $P_1$. Thus, the relation in Eq. (111) is complete and exact in the sense that it involves all the eleven observables and only one hadronic input which can be reliably estimated using HQET.

VI. OBSERVABLES AT KINEMATIC EXTREME POINTS

In this section we will briefly discuss the limiting value of the observables at the two kinematic extremites of $q^2$, the dilepton invariant mass squared. The minimum $q^2$ value, $q^2 = q^2_{\text{min}} = 4m^2$ and the endpoint $q^2 = q^2_{\text{max}} = (m_B - m_K)^2$. The values of the observables we obtain below can be experimentally verified and any exception must imply NP.

- **Case-I:** $q^2 = 4m^2$

It is easy to see that at $q^2_{\text{min}}$ the two lepton carry equal momentum and recoil against the $K^*$. In the dilepton rest frame the two leptons carry zero momentum. Hence, angles $\theta$ and $\phi$ cannot be defined. The angular distribution in Eq. (13) thus implies that all asymmetries i.e $A_1, A_5, A_{FB}, A_7, A_8$ and $A_9$ must vanish in this limit. This implies that there is no preferred direction, leading to the conclusion that all helicities are equally probable.

Using the expressions of the observables derived in the previous section (Eqs. (85a) and (85b)) we can write

$$F_L = \frac{1}{\Gamma_f^\lambda} (\gamma^2 T_f + 1/3 (\Gamma_f^\beta - \beta^2 T_f))$$

$$\lambda = 0 \quad \frac{1}{3}$$

This limiting value holds for the other two helicity fractions as well. Hence, at the kinematic starting point we can write

$$F_L^\lambda = \frac{1}{3}, \quad \lambda \in \{L, \perp, \parallel\}.$$  \hspace{2cm} (120)

We conclude that each helicity fraction should be $1/3$ at $q^2_{\text{min}}$, which can be easily verified experimentally. The asymmetries defined in Eq. (89) and (90) also vanish at $q^2 = q^2_{\text{min}}$ implying $(T_i/T_f^2) \rightarrow \frac{1}{3} \text{ (from Eq. (88))}$. Thus the observable $A_4^\lambda$ from Eq.(111) at $q^2 = q^2_{\text{min}}$ is given by

$$A_4^\lambda = \frac{\sqrt{2}}{\pi} \sqrt{F_L - \frac{T_i}{T_f^2}} \sqrt{F_\parallel - \frac{T_i}{T_f^2}} - \frac{T_i}{T_f^2} = 0$$ \hspace{2cm} (121)

as it was expected above.

- **Case-II:** $q^2 = (m_B - m_K)^2$

In this kinematic limit the $K^*$ is at rest and the two leptons go back to back in the $B$ meson rest frame. Therefore, we can always choose the angle $\phi$ to be zero. The entire decay takes place in one plane, resulting in vanishing $F_\perp$. Also, the left and right chirality of the leptons contribute equally. These together result in only the angular asymmetry $A_4$ being finite with all other asymmetries vanishing. The relations among the various angular coefficients at this kinematical endpoint are derived in Ref. [28] where it is explicitly shown that

$$F_L(q^2_{\text{max}}) = \frac{1}{3}, \quad A_{FB}(q^2_{\text{max}}) = 0.$$

Solving for the other observables from Eq. (3.2) of Ref. [28] we can write

$$F_\perp(q^2_{\text{max}}) = 0,$$

$$A_4(q^2_{\text{max}}) = \frac{2}{3}, \quad A_{5,7,8,9}(q^2_{\text{max}}) = 0.$$ \hspace{2cm} (124)

These limiting values of the observables imply that $\epsilon_\lambda \rightarrow 0$ at the extremum $q^2 = q^2_{\text{max}}$ as can be seen from Eqs. (108) – (110). The lepton mass can be safely ignored at $q^2_{\text{max}}$ as it would have almost no effect at this endpoint hence, we have dropped the ‘o’ index from all the observables for this discussion only. Thus, in the limit $\epsilon_\lambda \rightarrow 0$, we find that Eq. (111) reduces to Eq. (80). Hence, the observable $A_4$ at $q^2 = q^2_{\text{max}}$ turns out to be

$$A_4 = \frac{8A_5 A_{FB}}{9\pi F_\perp} + \sqrt{2} \sqrt{F_L F_\perp - \frac{8}{7} A_4 F_\parallel - \frac{4}{7} A_{FB}}$$

$$= \frac{2}{\sqrt{2} \sqrt{F_L F_\parallel}} = \frac{2}{\pi F_L} = \frac{3}{3\pi F_L} \frac{1}{3\pi F_\perp} \frac{1}{2}$$

which exactly matches with the limit predicted in Eq. (124).

VII. NEW PHYSICS ANALYSIS

In this section, we demonstrate the possibility of how new physics could be tested using the relations derived in this paper. The basis of our analysis is the relation, which involves all the nine observables $F_L, F_\parallel, F_\perp, A_{FB}, A_4, A_5, A_7, A_8, A_9$ and a single form factor ratio $P_1$ derived in Eq. (79). Since the helicity fractions are related by $F_L + F_\parallel + F_\perp = 1$, we eliminate $F_\parallel$. All the observables have been measured by LHCb collaboration using $1 \text{fb}^{-1}$ data. However, currently the observables $A_7, A_8$ and $A_9$ are measured to be consistent with zero. Eqs. (108) – (110) therefore implies that $\epsilon_\lambda$ are all consistent with zero. In Sec. V we have shown that the most conservative allowed parameter space remains unaltered
FIG. 1. The $\chi^2$ projection onto the plane of observables $F_L$ and $F_{\perp}$. The experimental values of all the observables are taken from 1 fb$^{-1}$ LHCb measurements Ref.[5]. The green dots correspond to best fit value from $\chi^2$ minimization and the black squares corresponds to the measured central value. The pink (dark), yellow (light) and blue (darkest) correspond to the 1σ, 2σ and 3σ confidence level regions respectively. If the amplitudes are real, non-factorizable contributions vanish and the form-factors were reliably evaluated at leading order in HQET then using SM estimated values of Wilson coefficients we find $F_L - F_{\perp}$ are constrained to lie in the narrow region between the two solid black lines. See text for details.

FIG. 2. The $\chi^2$ projection onto the plane of observables $F_L$ and $A_{FB}$. The experimental values of all the observables are taken from Refs. [5]. The color codes are the same as in Fig. 1. If the amplitudes are real, non-factorizable contributions vanish and the form-factors were reliably evaluated at leading order in HQET then using SM estimated values of Wilson coefficients we find $A_{FB} - F_L$ are constrained to lie in the two solid black triangular region. See text for details.
FIG. 3. The $\chi^2$ projection onto the sets observables $A_S - A_{FB}$, $A_F - F_L$, $A_S - F_L$, and $A_{FB} - F_L$ for various $q^2$ bins going vertically from first to the sixth bin. The experimental values of all the observables are taken from Refs. [5]. The color codes are same as in Fig. 1.
even if the small lepton mass term is dropped compared to $q^2$ and the imaginary contributions to the amplitudes are completely ignored. Since the inclusion of $\varepsilon$ reduces the parameter space of observables, in order to check the consistency of measured observables we take a conservative approach and set all the $\varepsilon$'s to be equal to zero for the numerical analysis. Thus, the relation among the observables reduces to Eq. (80) which is in terms of six observables $F_L$, $\bar{F}_L$, $F_\perp$, $\bar{F}_\perp$, $A_{FB}$, $A_5$ and is completely free from any form factor dependence. If $A_7$, $A_8$ and $A_9$ are measured to be non zero in future experiments with reduced uncertainties, $\varepsilon$ can be solved iteratively using Eqs. (108)–(110) and an exact numerical analysis can always be done. We emphasize that a non-zero $\varepsilon$ would only restrict the allowed parameter space depicted in Figs. 1, 2 and 3 further as was already pointed out in Sec. V. Later in this section we will, nevertheless, solve for $\varepsilon$ in terms of $A_7$, $A_8$ and $A_9$ since the predicted value $A_4^{\text{pred}}$ depends on the values of $\varepsilon$.

We use the SM relation derived in Eq. (80), for $\varepsilon = 0$ and $4m^2/q^2 \to 0$ instead of Eq. (111), to check for consistency between measurements of all the observables. As noted above a finite value for $\varepsilon$ would provide a stronger constraint and since $\varepsilon$'s are consistent with zero, Eq. (80) provides a more conservative test. In order to preform the test we define a $\chi^2$ function

$$\chi^2 = \frac{1}{4} \left[ \left( \frac{A_4^{\text{exp}} - A_4^{\text{pred}}}{\Delta A_4^{\text{exp}}} \right)^2 + \left( \frac{F_L^{\text{exp}} - F_L}{\Delta F_L^{\text{exp}}} \right)^2 \right. \\
+ \left. \left( \frac{F_\perp^{\text{exp}} - F_\perp}{\Delta F_\perp^{\text{exp}}} \right)^2 + \left( \frac{A_{FB}^{\text{exp}} - A_{FB}}{\Delta A_{FB}^{\text{exp}}} \right)^2 \right] \\
+ \left( \frac{A_5^{\text{exp}} - A_5}{\Delta A_5^{\text{exp}}} \right)^2,$$

(125)

where $A_4^{\text{exp}}$, $F_L^{\text{exp}}$, $A_{FB}^{\text{exp}}$, $A_5^{\text{exp}}$, $F_\perp^{\text{exp}}$, $A_4^{\text{pred}}$, $F_L$, $A_{FB}$, $A_5$ indicate experimental central values of the observables and $\Delta A_4^{\text{exp}}$, $\Delta F_L^{\text{exp}}$, $\Delta F_\perp^{\text{exp}}$, $\Delta A_{FB}^{\text{exp}}$, $\Delta A_5^{\text{exp}}$ are the experimental uncertainties. The statistical and systematic uncertainties are added in quadrature for all the numerical analysis presented. We used Mathematica [29] to do all the numerical calculations presented in this paper. The $\chi^2$ function in Eq. (125) is minimized in the 4-dimensional parameter space of the observables by varying each of them simultaneously within the allowed region i.e $0 \leq F_L \leq 1$, $0 \leq F_\perp \leq 1$, $-1 \leq A_{FB} \leq 1$, $-1 \leq A_5 \leq 1$, while $A_4^{\text{pred}}$ is taken to be the theoretically calculated value for $A_4$ using Eq. (80). The minimized $\chi^2$ function is projected in different sets of planes of the observables, $(F_L, F_\perp)$, $(A_{FB}, A_5)$, $(A_{FB}, F_\perp)$, $(A_5, F_L)$, $(A_5, F_\perp)$ and $(A_{FB}, F_L)$ for the contour plots. In Fig. 1 we show the allowed domain of $F_L - F_\perp$ values for all the six $q^2$ bins corresponding to the $q^2$ values in the range $(0.1 - 2)$ GeV$^2$, $(2 - 4.34)$ GeV$^2$, $(4.34 - 8.68)$ GeV$^2$, $(10.09 - 12.86)$ GeV$^2$, $(14.0 - 16.0)$ GeV$^2$ and $(16.0 - 19.0)$ GeV$^2$. The pink, yellow and blue correspond to $1\sigma$, $2\sigma$ and $3\sigma$ confidence level regions. The black squares correspond to the experimentally measured central value and the green points correspond to best fit values obtained by minimizing $\chi^2$ using Eq. (125). As can be seen form Fig. 1 the bounds derived in this paper, involving only observables, have resulted in very significantly constraining the allowed parameters range of observables.

If it were true that there are no significant non-factorizable contributions to the decay mode, rendering $\tilde{C}_9$ independent of the helicity index $`a`, we can solve for $\tilde{C}_9$ as was shown in Ref.[2]. The ratio of $\tilde{C}_9/\tilde{C}_{10}$ so obtained could be inverted to solve for $A_{FB}$ resulting in the constraint between $F_L$ and $F_\perp$ given in Eq.(55) of Ref.[2]. The narrow constraint region between the two solid black lines depicted in $F_L - F_\perp$ plane in Fig. 1 is derived assuming real transversity amplitudes, form-factors are calculated at leading order in $\Lambda_{QCD}/m_b$ using HQET and the estimate that $\tilde{C}_9/\tilde{C}_{10} = -1$ is used. We emphasize that except for the two solid black lines for each of the $q^2$ bins all other information in Fig. 1 is completely free from any theoretical assumption. As can be seen from Fig. 1 the best fit values as well as the experimentally measured central values are largely not inside the narrow constraint region within two solid black lines. This indicates that there could exist any or all of the possibilities: imaginary contributions to the transversity amplitudes or sizable non-factorizable contributions or higher order corrections in HQET could also be relevant.

The allowed range for observables $A_{FB} - F_L$ is depicted in Fig. 2 for all the six bins. The color code and markers follow the same convention used in Fig. 1. The constraint of the allowed triangular region between two solid black line comes from Eq.(53) of Ref. [2]. Once again the constraint region within the solid black triangular depicted in $A_{FB} - F_L$ plane is derived assuming real transversity amplitudes, form-factors calculated at leading order in $\Lambda_{QCD}/m_b$ using HQET and the estimate that $\tilde{C}_9/\tilde{C}_{10} = -1$. However, note that the constraints depicted by the contour plots are completely free from any theoretical assumptions. The allowed region in the other four planes of observables i.e $A_{FB} - A_5$, $A_5 - F_L$, $A_5 - F_\perp$ and $A_{FB} - F_L$ are shown in Fig. 3. We emphasize once again that the plots are free from any theoretical uncertainty. In most of the contour plots depicted in Figs. 1, 2 and 3 the best fit points (green point) lie at the edge of the boundaries except for the third bin. The experimental measured central values (black squares) are mostly inside the allowed region perhaps validating the LHCb data set.
In Fig. 4 the measured Gaussian $A_4$ distribution is compared with the distribution of $A_4^{\text{pred}}$ computed using Eq. (80). In evaluating the right hand side of Eq. (80) we have used a Gaussian distribution of the observables $F_L$, $F_L$, $A_5$ and $A_{FB}$ with experimental central value as the mean and errors as the standard deviation from Ref. [5]. The plots correspond to a simulated theory sample of 144, 76, 281, 169, 114 and 124 events corresponding to first through sixth $q^2$ bins. These may be compared with 140, 73, 271, 168, 115 and 116 events obtained for the respective bins by LHCb using 1 fb$^{-1}$ data [5]. We have randomly chosen the number of events to be statistically consistent with the LHCb observation in each bin for this decay mode. As should be expected fewer events survive the constraint of Eq. (80) when the best fit points are at the edge of the permissible contour regions in Figs. 1, 2 and 3. The simulated $A_4$ values corresponding to the LHCb measurement for all six bins are shown in red (dark) histogram and the yellow (light) histogram corresponds to the values of $A_4^{\text{pred}}$ computed using Eq. (80). For a comparison, the probability distribution function (PDF) curves corresponding to 1000 times more events are also shown for theory using brown (light) curve and data using red (dark) curve.

The mean and 1σ regions for the theoretically calculated $A_4^{\text{pred}}$ distributions are shown in Fig. 5. We compare the two cases where lepton mass is ignored (Eq. (80)) with the case where lepton mass is finite (Eq. (111)). The purple (light) bands correspond to the massless case and the gray (dark) band correspond to the massive case. The error bars in red correspond to the experimentally measured [5] central values and errors in $A_4$ for the respective $q^2$ bins. The values of $A_4^{\text{pred}}$ obtained from the Eq. (80) seem to visually agree reasonably with the experimental measurements within the error bands in all the bins except for the first and the fifth bin. A large discrepancy in fifth bin can also be seen here. There is also a slight tension in first bin, which could be partly
FIG. 5. The mean values and 1σ regions for theoretically calculated \( A_4 \) distributions excluding lepton masses (Eq. (80)) and with massive leptons (Eq. (111)) are shown in purple (light) and gray (dark) bands respectively. The simulated samples consist of 50,000 events to start with, for each bin. The observables \( A_7, A_8 \) and \( A_9 \) are assumed to be zero. The error bars in red correspond to the experimentally measured [5] central values and errors in \( A_4 \) for the respective \( q^2 \) bins.

due to the lepton mass may affect the first bin. The corrections due to mass terms can be incorporated if the asymmetries \( A_{10} \) and \( A_{11} \) are measured in the future. In the absence of such measurements we have used the theoretical estimate of form-factors [13] to evaluate the effect of the finite mass contribution. Details are depicted in Fig. 5. The mass contributions only effects the first bin, the other bins are unaffected. As expected the agreement improves for the first bin if the mass contributions are added. While Fig. 5 indicates only a mild disagreement between the measured and predicted values of \( A_4 \), the distributions in Fig. 4 carry much more information than the mean and averages. We have compared the two simulated distributions shown in the Histograms using the Mathematica routine “DistributionFitTest” [30]. The \( P \)-values obtained by comparing the two are found to be less than \( 10^{-9} \) for each of the bins, except the second and fourth bins, where the \( P \)-values obtained are \( 2.54 \times 10^{-5} \) and \( 6.47 \times 10^{-6} \) respectively. A small \( P \)-value indicates that one should reject the hypothesis that all observables are consistent with the SM relation of Eq. (80).

In order to ascertain that the discrepancy in the \( A_4 \) enunciated using the \( P \)-values is not due to the imaginary contributions being ignored we have also preformed a simulation of all observables, including \( A_7, A_8 \) and \( A_9 \). We solved for \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_0 \) using Eqs. (108) – (110). These values of \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_0 \) depend only on observables and \( P_1 \). We assume \( P_1 \) values (see Ref. [2]) to be \( P_1 = -0.9395, -0.9286, -0.9034, -0.8337, -0.7156 \) and \(-0.4719 \) for the first through the sixth bin respectively. We only remark that if \( A_7, A_8 \) and \( A_9 \) are measured to be small the results are even more insensitive to the choice of the \( P_1 \) value. Nevertheless, we also studied the effect of varying \( P_1 \) within the range \( P_1 \pm 0.5 \), to ascertain our claim. Details will be presented elsewhere. The \( \varepsilon_\lambda \) were solved iteratively and it was found that they always converged in just a few iterations. If iteration led to a value of \( \varepsilon_\lambda \) larger than the derived constraints permitted, a smaller allowed value was assigned and the iteration continued. In some cases an oscillatory or randomly varying pattern was observed but in these cases the starting values of the observables could not be reproduced, indicating that further constraints imposed by the chosen values of \( A_7, A_8 \) and \( A_9 \) could not be satisfied. The solutions obtained for \( \varepsilon_\lambda / \sqrt{\Gamma_f} \) are shown for each of the six bins in Fig. 6. It can be seen that all the \( \varepsilon_\lambda \)'s are consistent with zero and even at extreme cases \( \varepsilon_\lambda^2 / \Gamma_f \) values are less than 0.2.

FIG. 6. The solutions for \( \varepsilon_\lambda / \sqrt{\Gamma_f}, \varepsilon_1 / \sqrt{\Gamma_f} \) and \( \varepsilon_0 / \sqrt{\Gamma_f} \) using distributions with 140, 78, 275, 175, 113 and 113 events for first through sixth \( q^2 \) bins. The number of events are chosen to be statistically consistent with the number of events observed by LHCb [5] in each bin for this decay mode. All the \( \varepsilon_\lambda \)'s are consistent with zero and even at extreme cases \( \varepsilon_\lambda^2 / \Gamma_f \) values are less than 0.2.
FIG. 7. A comparison of the measured and predicted $A_4$ values for the six $q^2$ bins considering all the measured observables. The simulated values of $A_4$ assuming Gaussian error in the LHCb data are shown in red (dark), whereas the Blue (light) distributions referred to as “Theory” correspond to the values of $A_4^{\text{pred}}$ computed using Eq. (111). The plots correspond to a simulated theory (LHCb 1 fb$^{-1}$ data [5]) sample of 140 (140), 78 (73), 275 (271), 175 (168), 113 (115) and 113 (116) events corresponding to first through sixth $q^2$ bins as depicted in the figure. The number of events are chosen to be consistent statistically with the number of events observed by LHCb in each bin for this decay mode. The values of all other observables used in the two equations are randomly generated using LHCb data assuming Gaussian measurements. We find that the $P$-values obtained using the Mathematica routine “DistributionFitTest” [30] comparing the two distributions are always less than $10^{-9}$ for all bins except the second bin where the $P$-value is $6.78 \times 10^{-3}$.

account all the contributions in SM. The asymmetries $A_{10}$ and $A_{11}$ (see Eqs. (89) and (90)) have not yet been measured and Fig. 5 indicates that the lepton mass effects are negligible for all but the first bin. We hence set $T_1 = 0$ in evaluating $A_4^{\text{pred}}$. This ensures that our results depend only on one theoretical parameter, the ratio of form-factors $P_4$ and that parameter resulting in unmeasurable tiny effects do not complicate the calculations. As predicted above, an even smaller number of events are now consistent with the constraints derived in the paper. Interestingly, $A_4^{\text{pred}}$ now fits better to a Gaussian distribution as indicated by a Kolmogorov-Smirnov test, compared to the previous case where transversity amplitudes were assumed to be real. This is indicative of the fact that the transversity amplitudes are complex. However, the values of $\varepsilon_\lambda / \sqrt{T_\lambda}$ are not large as indicted in Fig. 6. We have simulated numbers of events consistent statistically with the number of events observed by LHCb in each bin. The plots as depicted in Fig. 7 correspond to a simulated theory (LHCb 1 fb$^{-1}$ data [5]) sample of 140 (140), 78 (73), 275 (271), 175 (168), 113 (115) and 113 (116) events for the first through sixth $q^2$ bins. The PDF curves comparing the measured value of $A_4$ for each curve. If $\varepsilon_\lambda \neq 0$ only...
In this paper we have derived a new relation involving all the CP conserving observables that can be measured in the decay $B \to K^* \ell^+\ell^-$ using an angular study of the final state for the decay. The relation provides a very clean and sensitive way to test SM and search for NP by probing consistency between the measured observables. The relation reduces to the one derived in Ref. [2], when certain reasonable assumptions were made. Since, the relation is intended to be used as probe in search for NP, it is imperative that no avoidable assumptions be made. We have generalized previous results with this objective in mind. The new derivation is parametrically exact in the SM limit and incorporates finite lepton and quark masses, complex amplitudes enabling resonance contributions to be included, electromagnetic correction to hadronic operators at all orders and all factorizable and non-factorizable contributions to the decay.

We write the most general form factors and amplitudes in Sec. II based only on Lorentz invariance and gauge invariance. Our approach differs from what usually done in literature as we make no attempt to evaluate hadronic parameters but eliminate them in favour of measured observables to the extent possible. Hence, our conclusions are not limited in general by the order of accuracy up to which the calculations are done.

The decay is described by six transversity amplitudes which survive in the massless lepton case. If the mass of the lepton is finite yet another amplitude contributes to the decay. We have shown in Sec. V that the corrections to the amplitude arising from finite lepton mass can be determined completely from observables measured using angular analysis. These contributions are suppressed by $m^4/q^2$ and may be difficult to measure. A theoretical estimate also shows that they are insignificant in all but the first bin. We therefore began by focusing attention on the massless case which is described by the six transversity amplitudes alone. The massive lepton case was considered later to derive an exact relation valid in the SM limit. Even if the mass effects are too tiny to distinguish an attempt to measure them would ensure that the predictions are reliable and free from theoretical parameters.

We started by writing the most general parametric form of the transversity amplitude in the SM given in Eq. (17) that takes into account comprehensively all the contributions within SM. Unlike the derivations in Ref. [2] the general transversity amplitude is now allowed to be complex, by introducing three additional parameters $\varepsilon_\lambda$. This, however, poses no problem since there are three extra observables $A_\gamma$, $A_0$ and $A_4$ given in Sec. III, which are non-vanishing in the complex transversity amplitudes limit. Hence, dealing with complex amplitude introduces only a technical difficulty of solving for addi-

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**FIG. 8.** A PDF plot comparing the measured fifth bin ($14.0 \leq q^2 \leq 16.0$ GeV$^2$) value of $A_4$ with the two theoretically predicted values. One assuming $\varepsilon_\lambda = 0$ or completely real transversity amplitudes and the other with $\varepsilon_\lambda \neq 0$ or complex transversity amplitudes. The mean and errors for all the observables are assumed to be those measured by LHCb using 1 fb$^{-1}$ data set. All errors are assumed to be Gaussian. The PDF’s depicted in the figure are generated using $4 \times 10^5$ random events resulting in the simulated values of $A_4$ for each curve. If $\varepsilon_\lambda \neq 0$ only 6708 of the points survived the constraints. The plot corresponding to LHCb $A_4$ measurement is shown in left most red (dark) plot, whereas the central brown (lighter) distribution corresponds to the theoretically calculated $A_4$ using Eq. (79) and the right most blue (light) distribution is for $A_4$ predicted using Eq. (111).

6708 of the points survived the constraints of Eq. (111). LHCb data assuming Gaussian error is shown in left most red (dark) plot, whereas the central brown (lighter) distribution corresponds to the theoretically calculated $A_4$ using Eq. (79) and the right most blue (light) distribution is for $A_4$ predicted using Eq. (111). The values of all other observables used in the two equations are randomly generated assuming Gaussian measurements of the LHCb 1 fb$^{-1}$ data.

In this section we have discussed the constraints already imposed by the 1 fb$^{-1}$ LHCb data [5] on the parameter space of observables. We also compare the measured values of $A_4$ with those predicted using the new relations derived in this paper. We made several observations that indicate possibly sizable non-factorizable contributions and imaginary contribution and also possible higher order corrections in HQET to the transversity amplitudes. In addition, the $P$-values comparing the measured $A_4$ with the predicted value indicates new physics. However, we refrain from drawing even the obvious conclusions given that, results for $3 \text{ fb}^{-1}$ data will soon be presented by the LHCb collaboration. However, we emphasize that the approach developed in this paper could not only conclusively indicate presence of significant non-factorizable contributions and need for higher order power corrections to form-factors but also the presence of NP with larger statistics.

**VIII. CONCLUSIONS**
tional variables iteratively.

Using this general amplitude a new relation (see Eq. (80)) involving all the nine CP conserving observables is derived in Sec. IV, that is exact in the SM limit assuming massless leptons. The new derivation incorporates the effect of electromagnetic correction of hadronic operators to all orders and all factorizable and non-factorizable contributions including resonance effects to the decay. In addition to the nine observables, this new relation depends only on one form-factors ratio: \( P_1 \). The new relation becomes independent of \( P_1 \) and reduces to the one derived in Ref. [2] in the limit that the asymmetries \( A_7, A_8 \) and \( A_9 \) are all zero.

As mentioned repeatedly the inclusion of lepton mass contribution is trivial in our approach; the effect on all the observables is directly obtained in terms of asymmetries given in Eqs. (89) and (90) that can be measured as shown in Sec. V. The new relation obtained is generalized to include the lepton mass effects in Eq. (111). It is important to note that it involves only observables and the form-factor ratio \( P_1 \), and is free from any assumption within the SM framework. This relation also implies three inequalities given in Eqs. (113a)–(113c) which impose constraints on the parameter space of observables. We also presented three new relations between the observables that are exact at the zero crossings of angular asymmetries \( A_{2B}^\parallel, A_6^\parallel \) and \( A_{2B}^\perp + \sqrt{2} A_6^\perp \). These are particularly interesting if the mass effect and the imaginary contributions to the Wilson coefficients \( \tilde{C}_7 \) and \( \tilde{C}_9 \) are ignored, as they reduce to simple form presented in Eq. (115). Another interesting aspect is that the form-factor ratios \( P_1, P_2 \) and \( P_3 \) each can be written in terms of observables and \( P_1 \). In the limit of vanishing \( A_7, A_8 \) and \( A_9 \) (i.e. negligible imaginary contributions), the form-factor ratios can be measured purely in terms of helicity fractions.

The limiting values of the observables at the minimum and maximum values of \( q^2 \) are discussed in Sec. VI based on very general arguments. It is interesting to note that at \( q^2 = 4m^2 \) all angular asymmetries vanish and each of the helicity fraction approaches \( 1/3 \). At the maximum value of \( q^2_{\text{max}} \) similar results can be obtained.

In Sec. VII, we have highlighted the possible ways to check the consistency of the measured observables using the SM relation derived. It was noted that the inclusion of non-zero \( \varepsilon_\lambda \) indicating complex contributions to the amplitudes invarially reduces the allowed parameter space of the observables. Hence, in order to check the consistency of measured observables we take a conservative approach and set all the \( \varepsilon_\lambda \)'s equal to zero for the analysis. This was necessary since \( A_7, A_8 \) and \( A_9 \) are all consistent with zero. The relation among the observables, hence, reduces to Eq. (80) which is in terms of six observables \( F_L, F_L^\perp, A_{FB}, A_4, A_5 \) and is completely free from any form factor dependence. The \( \chi^2 \) function in Eq. (125) was minimized in the 4-dimensional parameter space of the observables \( F_L, F_L^\perp, A_{FB} \) and \( A_5 \) to check the consistency between the experimentally measured values by varying each of them simultaneously within the permissible domain and \( A_{FB}^{\text{pred}} \) was evaluated using the relation in Eq. (80). The projections of the minimized \( \chi^2 \) function are studied for the various pairs of observables as shown in the contour plots of Figs. 1–3. In most of the contour plots the best fit (green) points lie at the edge of the boundaries except for the third bin. The experimental measured central values (black squares) generally lie within the contours except for the fourth and sixth bin. It is interesting to note that the best fits are always in the 1σ region perhaps validating the LHCb data set.

We compared the two distributions generated by experimental measurement and theoretical prediction of the observable \( A_4 \), assuming that \( A_7, A_8 \) and \( A_9 \) are all zero in Fig. 4. The number of events for the “Theory” histogram are chosen to be consistent statistically with the number of events observed by LHCb in the 1 fb\(^{-1} \) [5] data set for each of the bins. The mean values together with 1σ error bands are shown in Fig. 5 with a comparison between the massless and massive lepton case. It is found that lepton mass can be ignored except for the first \( q^2 \) bin. The fifth bin shows a large discrepancy whereas the other bins are in reasonable agreement. Since the \( A_4 \) distributions in Fig. 4 carry much more information than the mean and averages, we compare the two simulated distributions shown in the Histograms using the Mathematica routine “DistributionFitTest” [30]. The \( P \)-values obtained by comparing the two are found to be less than \( 10^{-9} \) for each of the bins, except the second and fourth bins, where the \( P \)-values obtained are \( 2.54 \times 10^{-5} \) and \( 6.47 \times 10^{-6} \) respectively.

In order to understand better the role of the imaginary contributions that were earlier ignored, we have also performed a simulation of all observables including \( A_7, A_8 \) and \( A_9 \). The solutions for \( \varepsilon_\perp, \varepsilon_0 \) and \( \varepsilon_9 \) shown in Fig. 6 indicate that all the \( \varepsilon_\lambda \)'s are consistent with zero and even the tails of \( \varepsilon_\lambda^2 / \Gamma_\lambda \) do not cross 0.2. A comparison of the measured and predicted \( A_4 \) values for the six \( q^2 \) bins considering all the measured observables (including \( A_7, A_8 \) and \( A_9 \)) are shown in Fig. 7. Interestingly, \( A_{FB}^{\text{pred}} \) now fits better to a Gaussian distribution than the \( \varepsilon_\lambda = 0 \) case as indicated by a Kolmogorov-Smirnov test, implying possible imaginary contributions to the transversity amplitudes. The \( P \)-values still continue to be smaller than \( 10^{-9} \) for all the bins, except the second bin where the \( P \)-value is 6.78 \( \times 10^{-5} \), indicating that we reject the hypothesis that all observables are consistent with the exact SM relation of Eq. (111). Since the discrepancy seems to be the largest for the fifth bin (14.0 \( \leq q^2 \leq 16.0 \) GeV\(^2 \)), we have performed a detailed comparison of the PDF curves for both the theoretically predicted values using \( \varepsilon_\lambda = 0 \) and \( \varepsilon_\lambda \neq 0 \) with the measured value of \( A_4 \) as shown in Fig. 8.

In this paper we have derived a relation among the observables by taking into account all possible effects within Standard Model by restricting ourselves to rely only on one hadronic input. The violation of this relation will provide a smoking gun signal of New Physics. We have
explicitly shown how the relation can be used to test SM, and confirm our understanding of the hadronic effects. We used the 1 fb$^{-1}$ LHCb measured values of the observables to highlight the possible ways for the search of new physics that might contribute to this decay with the derived relations.

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Appendix A: Derivation of $r_\parallel$ Solutions

Here we present the derivation of $r_\parallel$, $r_\perp$ and $r_0$ solutions defined in Eq. (35). Starting with the first set of equations (Set-I) involving $r_\parallel$ and $r_\perp$ in terms of the observables given in Eqs. (54), (55) and (56) we have

\begin{align*}
    r_\parallel^2 + \bar{C}_10^2 & = \frac{F_1^\prime \Gamma_1 P_1}{2 F_\perp^2}, \quad (A1) \\
    r_\perp^2 + \bar{C}_10^2 & = \frac{F_2^\prime \Gamma_2}{2 F_\perp}, \quad (A2) \\
    \bar{C}_10(r_\parallel + r_\perp) & = \frac{A_{FB} \Gamma_1 P_1}{3 F_\perp^2}. \quad (A3)
\end{align*}

Multiplying Eq. (A1) and (A2) we can write

\begin{align*}
    \frac{F_\parallel^\prime F_\parallel^\prime \Gamma_1^2 P_1^2}{4 F_\perp^4} & = (r_\parallel r_\perp - \bar{C}_10^2)^2 + \bar{C}_10^2 (r_\parallel + r_\perp)^2 \\
    & = (r_\parallel r_\perp - \bar{C}_10^2)^2 + \frac{A_{FB}^2 \Gamma_1^2 P_1^2}{9 F_\perp^4}
\end{align*}

hence,

\begin{equation}
    r_\parallel r_\perp - \bar{C}_10^2 = \pm \frac{\Gamma_1 P_1}{2 F_\perp} \sqrt{F_\parallel^\prime F_\parallel^\prime} - \frac{4 A_{FB}^2}{9}. \quad (A4)
\end{equation}

Now expressing $\bar{C}_10^2$ in terms of $r_\parallel^2$ using Eq. (A1) and in terms of $r_\perp^2$ using Eq. (A2) we can write

\begin{align*}
    2r_\parallel r_\perp - 2\bar{C}_10^2 & = 2(r_\parallel - \frac{F_\parallel^\prime \Gamma_1 P_1}{2 F_\perp^2} - r_\parallel^2) - \frac{F_1^\prime \Gamma_1}{2 F_\perp^2} \\
    & = (r_\parallel + r_\perp)^2 - \frac{F_1^\prime \Gamma_1 P_1}{2 F_\perp^2} \quad (A5)
\end{align*}

Equating Eqs. (A4) and (A5) we get

\begin{equation}
    r_\parallel + r_\perp = \pm \left[ \frac{F_1^\prime \Gamma_1 P_1}{2 F_\perp^2} + \frac{F_1^\prime \Gamma_1}{2 F_\perp} \pm \frac{\Gamma_1 P_1}{2 F_\perp} Z_1' \right]^{1/2} \quad (A6)
\end{equation}

where $Z_1' = \sqrt{4 F_\parallel^\prime F_\parallel^\prime - \frac{16}{9} A_{FB}^2}$. Now, Eqs. (A1) and (A2) imply:

\begin{equation}
    r_\parallel^2 - r_\perp^2 = \frac{F_1^\prime \Gamma_1 P_1}{2 F_\perp^2} - \frac{F_1^\prime \Gamma_1}{2 F_\perp}, \quad (A7)
\end{equation}

which gives $r_\parallel - r_\perp$ to be,

\begin{equation}
    r_\parallel - r_\perp = \pm \sqrt{F_\parallel^\prime} \left[ \frac{P_1^2 F_\parallel^\prime - F_\parallel^\prime}{\sqrt{2 F_\perp^2}} \left( P_1 F_\parallel^\prime + F_\parallel^\prime \pm P_1 Z_1' \right) \right]^{1/2}. \quad (A8)
\end{equation}

To fix the sign ambiguity of the radical let us consider the zero crossing point of the observable $A_{FB}$ where,

\begin{equation}
    r_\parallel + r_\perp |_{A_{FB}=0} = \pm \frac{\sqrt{F_\parallel^\prime} (\sqrt{F_\parallel^\prime} \pm P_1 \sqrt{F_\parallel^\prime})}{\sqrt{2 F_\perp}} = 0 \quad (A9)
\end{equation}

It can be easily seen from Appendix. B that $P_1$ is always negative and thus the positive sign ambiguity has to be chosen within the radical. Solving Eqs. (A6) and (A8) we get the expressions for $r_\parallel$ and $r_\perp$ given in Eqs. (64) and (65). Similarly, following all the steps stated above for the other two sets of equations (Set-II and Set-III) we get the solutions for $r_0$ (in Eq. (67)) and two more expressions for the variable $r_\perp$ (Eqs. (68) and (72)).

Generalization of Eqs. (A6) and (A8) for the massive case in Sec. V is trivial from here. Below we present the explicit expressions for both massless and massive cases.

\begin{equation}
    r_\parallel + r_\perp = \begin{cases} 
    \frac{\pm \sqrt{F_\parallel^\prime}}{\sqrt{2 F_\perp}} \left[ P_1^2 \left( F_\parallel - \frac{2 \epsilon_\parallel}{\Gamma_1} \right) + \left( F_\perp - \frac{2 \epsilon_\perp}{\Gamma_1} \right) \pm P_1 Z_1' \right]^{1/2} & \text{massless case} \\
    \frac{\pm \sqrt{F_\parallel^\prime}}{\sqrt{2 F_\perp}} \left[ P_1^2 \left( F_\parallel + \frac{2 \epsilon_\parallel}{\Gamma_1} \right) + \left( F_\perp + \frac{2 \epsilon_\perp}{\Gamma_1} \right) \pm P_1 Z_1' \right]^{1/2} & \text{massive case}
\end{cases} \quad (A10)
\end{equation}
where these \(X_i\)'s and \(Y_i\)'s can be related to the well known form-factors \(V\), \(A_{0,1,2}\) and \(T_{1,2,3}\) by comparing with ref. [6] which are known up to order NNLO in HQET. However, it should be noted that the \(F_\lambda\) and \(\hat{G}_\lambda\) values are not directly used anywhere throughout our paper. Only the value of \(P_1\) is used to solve for \(\varepsilon_\lambda\) using Eqs. (108)–(110).

\[
\begin{align*}
F_\perp &= N\sqrt{2} \sqrt{\lambda(m_B^2, m_K^2, q^2)} X_3, \\
\tilde{G}_\perp &= N\sqrt{2} \sqrt{\lambda(m_B^2, m_K^2, q^2)} 2(m_B + m_s) \tilde{C}_7 Y_3 \\
&\quad + \cdots, \\
F_\parallel &= 2\sqrt{2} N X_1, \\
\tilde{G}_\parallel &= 2\sqrt{2} N \frac{2(m_B - m_s)}{q^2} \tilde{C}_7 Y_1 + \cdots, \\
F_0 &= \frac{N}{2m_K\sqrt{q^2}} [4k.q X_3 + \lambda(m_B^2, m_K^2, q^2) X_2], \\
\tilde{G}_0 &= \frac{N}{2m_K\sqrt{q^2}} \frac{2(m_B - m_s)}{q^2} \tilde{C}_7 [4k.q Y_1 \\
&\quad + \lambda(m_B^2, m_K^2, q^2) Y_2] + \cdots,
\end{align*}
\]

where these \(X_i\)'s and \(Y_i\)'s can be related to the well known form-factors \(V\), \(A_{0,1,2}\) and \(T_{1,2,3}\) by comparing with ref. [6] which are known up to order NNLO in HQET. However, it should be noted that the \(F_\lambda\) and \(\hat{G}_\lambda\) values are not directly used anywhere throughout our paper. Only the value of \(P_1\) is used to solve for \(\varepsilon_\lambda\) using Eqs. (108)–(110).

\[
\begin{align*}
X_0 &= -\frac{2m_K}{q^2} A_0(q^2), \\
X_1 &= -\frac{1}{2}(m_B + m_K) A_1(q^2), \\
X_2 &= \frac{A_2(q^2)}{m_B + m_K}, \\
X_3 &= \frac{V(q^2)}{m_B + m_K}, \\
Y_1 &= \frac{1}{2}(m_B^2 - m_K^2) T_2(q^2), \\
Y_2 &= -T_2(q^2) - \frac{q^2}{m_B^2 - m_K^2} T_3(q^2), \\
Y_3 &= -T_1(q^2).
\end{align*}
\]

Here a point to be noted that as the form-factors \(A_1\) and \(A_2\) are always positive the ratio

\[
\frac{2k.q(m_B + m_K)^2}{\lambda(m_B^2, m_K^2, q^2)} A_1 A_2 \geq 0
\]

giving rise to the fact that \(F_\parallel\) and \(F_0\) always have the same sign which is negative.

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