Invariant measures for bipermutative cellular automata

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Abstract

A right-sided, nearest neighbour cellular automaton (RNNCA) is a continuous transformation $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ determined by a local rule $\phi : \mathcal{A}^{\{0,1\}} \to \mathcal{A}$ so that, for any $a \in \mathcal{A}^\mathbb{Z}$ and any $z \in \mathbb{Z}$, $\Phi(a)_z = \phi(a_z, a_{z+1})$. We say that $\Phi$ is bipermutative if, for any choice of $a \in \mathcal{A}$, the map $\mathcal{A} \ni b \mapsto \phi(a, b) \in \mathcal{A}$ is bijective, and also, for any choice of $b \in \mathcal{A}$, the map $\mathcal{A} \ni a \mapsto \phi(a, b) \in \mathcal{A}$ is bijective.

We characterize the invariant measures of bipermutative RNNCA. First we introduce the equivalent notion of a quasigroup CA, to expedite the construction of examples. Then we characterize $\Phi$-invariant measures when $\mathcal{A}$ is a (nonabelian) group, and $\phi(a, b) = a \cdot b$. Then we show that, if $\Phi$ is any bipermutative RNNCA, and $\mu$ is $\Phi$-invariant, then $\Phi$ must be $\mu$-almost everywhere $K$-to-1, for some constant $K$.

We use this to characterize invariant measures when $\mathcal{A}^\mathbb{Z}$ is a group shift and $\Phi$ is an endomorphic CA.

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1 Introduction

If $\mathcal{A}$ is a (discretely topologized) finite set, then $\mathcal{A}^\mathbb{Z}$ is compact in the Tychonoff topology. Let $\sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ be the shift map: $\sigma(a) = [b_z]_{z \in \mathbb{Z}}$, where $b_z = a_{z-1}$, $\forall z \in \mathbb{Z}$. A cellular automaton (CA) is a continuous map $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ which commutes with $\sigma$. Equivalently, $\Phi$ is defined by a local rule $\phi : \mathcal{A}^{[-\ell,\ldots,\ell]} \to \mathcal{A}$ (for some $\ell, r \geq 0$) so that, for any $a \in \mathcal{A}^\mathbb{Z}$ and any $z \in \mathbb{Z}$, $\Phi(a)_z = \phi(a_{z-\ell}, \ldots, a_{z+r})$. We say $\Phi$ is right-permutative if, for any fixed $a \in \mathcal{A}^{[-\ell,\ldots,\ell]}$, the map $\mathcal{A} \ni b \mapsto \phi(a,b) \in \mathcal{A}$ is bijective. Likewise, $\Phi$ is

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left-permutative if, for any fixed \( b \in \mathcal{A}^{(-\ell...r]} \), the map \( \mathcal{A} \ni a \mapsto \phi(a, b) \in \mathcal{A} \) is bijective, and \( \Phi \) is bipermutative if it is both left- and right-permutative.

**Example 1:**

(a) If \((\mathcal{A}, +)\) is an abelian group, \( \ell = 0 \) and \( r = 1 \), and \( \phi(a_0, a_1) = a_0 + a_1 \), then \( \Phi \) is called a nearest neighbour addition CA, and is bipermutative.

(b) If \( \mathcal{A} = \mathbb{Z}/p \), and let \( c_0, c_1 \in [1..p) \) be constants. If \( \phi(a_0, a_1) = c_0a_0 + c_1a_1 \), then \( \Phi \) is called a Ledrappier CA, and is bipermutative.

We say that \( \Phi \) is a right-sided, nearest neighbour cellular automaton (RNNCA) if \( \ell = 0 \) and \( r = 1 \) (as in Examples 1a and 1b). It is easy to show:

**Lemma 2** Let \( \Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \) be a CA and let \( \mathcal{B} = \mathcal{A}^{\ell+r} \). There is an RNNCA \( \Gamma : \mathcal{B}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z} \) so that the topological dynamical system \((\mathcal{A}^\mathbb{Z}, \Phi)\) is isomorphic to the system \((\mathcal{B}^\mathbb{Z}, \Gamma)\).

Furthermore \( \Phi \) is bipermutative if and only if \( \Gamma \) is bipermutative. \( \square \)

Let \( \lambda \) be the uniform Bernoulli measure on \( \mathcal{A}^\mathbb{Z} \). Thus, for any \( c_1, \ldots, c_M \in \mathcal{A} \), and any \( z_1, \ldots, z_M \in \mathbb{Z} \),

\[ \lambda\{a \in \mathcal{A}^\mathbb{Z} ; a_{z_1} = c_1, \ldots, a_{z_M} = c_M\} = \frac{1}{|\mathcal{A}|^M}. \]

Any permutative CA is surjective, and any surjective CA preserves \( \lambda \). What other \( \sigma \)-invariant measures on \( \mathcal{A}^\mathbb{Z} \) are also invariant under permutative CA? Let \( h_\mu(\Phi) \) denote the entropy \([3] \S 5.2\) of the measure-preserving dynamical system \((\mathcal{A}^\mathbb{Z}, \Phi, \mu)\). Host, Maass, and Martinez \([1]\) have shown:

**Proposition 3** Let \( \mathcal{A} = \mathbb{Z}/p \), where \( p \) is prime, and let \( \Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z} \) be a Ledrappier CA. Suppose \( \mu \) is a measure which is \( \Phi \)-invariant and \( \sigma \)-ergodic. If \( h_\mu(\Phi) > 0 \), then \( \mu = \lambda \).

This paper provides generalizations of Proposition 3 to a variety of contexts. In \([2]\) we introduce quasigroups, which provide a convenient formulation of bipermutative RNNCA as quasigroup cellular automata, and suggest a natural generalization of Proposition 3 (Conjecture 7). In \([3]\) we characterize invariant measures for nearest-neighbour multiplication CA (when \( \mathcal{A} \) is a nonabelian group), and construct an explicit counterexample to Conjecture 7. In \([4]\) we will extend the method of \([1]\) to prove that, if \( \mu \) is \( \Phi \)-invariant, then there is some \( K \leq |\mathcal{A}| \) so that \( \Phi \) is \( K \)-to-1 \((\mu\text{-ae})\) (Theorem 22). In \([5]\) we will provide a generalization of Proposition 3 to endomorphic CA on group shifts (Theorem 34).

**Notation:** If \( \mu \) is a measure, then ‘\( \forall \mu \text{-ae} x \)’ means ‘for \( \mu \)-almost all \( x \),’ and ‘\( \mu \text{-ae} \)’ means ‘\( \mu \)-almost everywhere’. If \( U \) and \( V \) are measurable sets, then ‘\( U \subseteq \mu V \)’ means \( \mu[U \setminus V] = 0 \), and ‘\( U \overset{\mu}{=} V \)’ means \( U \subseteq \mu V \) and \( V \subseteq \mu U \). If \( \mathcal{G} \) is a sigma algebra and \( U \) is a measurable set, then \( \mathbb{E}_\mu[U|\mathcal{G}] \) is the conditional expectation of \( U \) given \( \mathcal{G} \).
2 Quasigroup Cellular Automata

A quasigroup $\mathcal{A}$ is a finite set $\mathcal{A}$ equipped with a binary operation ‘$\ast$’ which has the left- and right-cancellation properties. In other words, for any $a, b, c \in \mathcal{A}$,

$$(a \ast b = a \ast c) \implies (b = c), \quad \text{and} \quad (b \ast a = c \ast a) \implies (b = c).$$

If we identify $\mathcal{A}$ with $[1..N]$ in some arbitrary way, then the ‘multiplication table’ for $\ast$ is the $N \times N$ matrix $M^\ast = [m_{i,j}]_{i,j=1}^N$ where $m_{i,j} = i \ast j$. We say $M^\ast$ is a Latin square if every column and every row of $M^\ast$ contains each element of $[1..N]$ exactly once. It follows:

$$\left( (\mathcal{A}, \ast) \text{ is a quasigroup } \right) \iff \left( M^\ast \text{ is a Latin square } \right).$$

Note that the operator ‘$\ast$’ is not necessarily associative. Indeed, it is easy to show:

$$\left( \text{‘$\ast$’ is associative} \right) \iff \left( (\mathcal{A}, \ast) \text{ is a group } \right).$$

A quasigroup cellular automaton (QGCA) is a right-sided, nearest neighbour cellular automaton $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ with local rule $\phi : \mathcal{A}^{\{0,1\}} \to \mathcal{A}$ given: $\phi(a_0, a_1) = a_0 \ast a_1$, where ‘$\ast$’ is a quasigroup operation. For example, any Ledrappier automaton is a QGCA. It follows:

Proposition 4 $\left( \Phi \text{ is a bipermutative RNNCA} \right) \iff \left( \Phi \text{ is a quasigroup CA} \right)$. $\square$

The obvious generalization of Proposition 3 fails for arbitrary quasigroup CA. If $\mathcal{B} \subset \mathcal{A}$, then we call $\mathcal{B}$ a subquasigroup (and write ‘$\mathcal{B} \prec \mathcal{A}$’) if $\mathcal{B}$ is closed under the ‘$\ast$’ operation.

Lemma 5 If $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is a QGCA, and $\mathcal{B} \prec \mathcal{A}$, then $\mathcal{B}^\mathbb{Z}$ is a $\Phi$-invariant subshift. If $\mu$ is the uniform Bernoulli measure on $\mathcal{B}^\mathbb{Z}$, then $\mu$ is $\Phi$-invariant and $\sigma$-ergodic. If $|\mathcal{B}| = K$, then $h_\mu(\Phi) = \log(K)$ and $\Phi$ is $K$-to-1 ($\mu$-æ).

If $\mathcal{B} \prec \mathcal{A}$ and $(\mathcal{A}, \ast)$ is a finite group, then $\mathcal{B}$ is a subgroup. Thus, if $|\mathcal{A}|$ is prime, then $\mathcal{A}$ can’t have nontrivial subquasigroups. However, other prime cardinality quasigroups can:

Example 6: Let $\mathcal{D} = \{a_1, a_2; b_1, b_2; c_1, c_2, c_3\}$; thus, $|\mathcal{D}| = 7$ is prime. Let $\ast$ have the following multiplication table:

$$
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\ast & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & c_3 \\
\hline
a_1 & a_1 & a_2 & c_1 & c_2 & b_1 & b_1 & c_3 \\
\hline
a_2 & a_2 & a_1 & c_2 & c_1 & b_1 & b_1 & c_3 \\
\hline
b_1 & c_1 & c_3 & b_1 & b_2 & c_2 & a_1 & a_2 \\
\hline
b_2 & c_3 & c_1 & b_2 & b_1 & a_1 & a_2 & c_2 \\
\hline
c_1 & b_1 & b_2 & c_3 & a_1 & a_2 & c_2 & c_1 \\
\hline
c_2 & b_2 & c_2 & a_1 & a_2 & c_3 & c_1 & b_1 \\
\hline
c_3 & c_2 & b_1 & a_2 & c_3 & c_1 & b_2 & a_1 \\
\hline
\end{array}
$$

Clearly, the quasigroup $(\mathcal{D}, \ast)$ has two subquasigroups: $\mathcal{A} = \{a_1, a_2\}$ and $\mathcal{B} = \{b_1, b_2\}$. $\square$
This suggests that the correct generalization of Proposition 3 is:

**Conjecture 7:** Let $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ be a QGCA. If $\mu$ is a $\Phi$-invariant and $\sigma$-ergodic measure, and $h_\mu(\Phi) > 0$, then $\mu$ is the uniform measure on $\mathcal{B}^\mathbb{Z}$, for some $\mathcal{B} \prec \mathcal{A}$.

Conjecture 7 is false, as we will show with Example (12b) of §3.

**Unilateral vs. Bilateral Cellular Automata:** Any right-sided CA $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ induces a unilateral CA $\tilde{\Phi} : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ with the same local rule. Any $\Phi$-invariant measure on $\mathcal{A}^\mathbb{Z}$ projects to a $\tilde{\Phi}$-invariant measure on $\mathcal{A}^\mathbb{N}$; conversely, any $\tilde{\Phi}$- and $\sigma$-invariant measure on $\mathcal{A}^\mathbb{N}$ extends to a unique $(\Phi, \sigma)$-invariant measure on $\mathcal{A}^\mathbb{Z}$. In what follows, we will abuse notation and write $\tilde{\Phi}$ as $\Phi$. Thus, Conjecture 7 is equivalent to:

**Conjecture $\tilde{7}$:** Let $\Phi : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ be a (unilateral) QGCA. If $\mu$ is $\Phi$-invariant and $\sigma$-ergodic, and $h_\mu(\Phi) > 0$, then $\mu$ is the uniform measure on $\mathcal{B}^\mathbb{N}$, for some $\mathcal{B} \prec \mathcal{A}$.

**Dual Cellular Automata:** There is a well-known conjugacy between any right-permutative unilateral CA and a full shift. Define $\Xi : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ by $\Xi(a) = [a_0, \Phi(a)_0, \Phi^2(a)_0, \Phi^3(a)_0, \ldots]$.

**Lemma 8** If $\Phi$ is right-permutative, then $\Xi$ is a topological conjugacy from the dynamical system $(\mathcal{A}^\mathbb{N}, \Phi)$ to the system $(\mathcal{A}^\mathbb{N}, \sigma)$ (ie. $\Xi$ is a homeomorphism and $\Xi \circ \Phi = \sigma \circ \Xi$). □

Let $(\mathcal{A}, *)$ be a quasigroup. The dual quasigroup is the set $\mathcal{A}$ equipped with binary operator $\hat{*}$ defined: $a \hat{*} b = c$, where $c$ is the unique element in $\mathcal{A}$ such that $a * c = b$. If $(\mathcal{A}, *)$ is a group, then $a \hat{*} b = a^{-1} * b$. If $\Phi : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ is a QGCA (with local map $\phi(a, b) = a * b$), then the dual of $\Phi$ is the CA $\hat{\Phi} : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ having local map $\hat{\phi}(a, b) = a \hat{*} b$.

**Lemma 9** Let $(\mathcal{A}, *)$ be a quasigroup and let $\Phi$ be the corresponding QGCA. Then:

(a) $(\mathcal{A}, \hat{*})$ is a quasigroup, and $\hat{\Phi}$ is a QGCA. The dual of $\hat{*}$ is $*$; the dual of $\hat{\Phi}$ is $\Phi$.

(b) $\Xi$ is a topological conjugacy from the dynamical system $(\mathcal{A}^\mathbb{N}, \sigma)$ to the system $(\mathcal{A}^\mathbb{N}, \hat{\Phi})$, so that we have the following commuting cube:
(c) If $B \subset A$, then $\left( (B, *) \preceq (A, *) \right) \iff \left( (\hat{B}, \hat{*}) \preceq (\hat{A}, \hat{*}) \right)$.

Let $\mu$ be a measure on $A^\mathbb{Z}$, and let $\hat{\mu} = \Xi(\mu)$. Then:

(a) \( \left( \mu \text{ is } \Phi\text{-invariant} \right) \iff \left( \hat{\mu} \text{ is } \sigma\text{-invariant} \right) \).

(b) \( \left( \mu \text{ is } \sigma\text{-ergodic} \right) \iff \left( \hat{\mu} \text{ is } \hat{\Phi}\text{-ergodic} \right) \).

(c) If $\mu$ is $\Phi$- and $\sigma$-invariant, then $h(\Phi, \mu) = h(\hat{\Phi}, \hat{\mu}) = h(\sigma, \mu) = h(\sigma, \hat{\mu})$.

Thus, Conjecture 7 is equivalent to:

Conjecture 7: Let $\Phi : A^\mathbb{N} \to A^\mathbb{N}$ be a QGCA. If $\mu$ is a $\Phi$-ergodic and $\sigma$-invariant measure, and $h(\mu, \sigma) > 0$, then $\mu$ is the uniform measure on $B^\mathbb{N}$, for some $B \subset A$.

It is Conjecture 7 which we’ll refute in §3.

3 Multiplication CA on Nonabelian Groups

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, and let $\mu$ be a $\sigma$-invariant measure on $A^\mathbb{N}$. Let $\hat{\mathbb{N}} = \{1, 2, 3, \ldots\}$ For any $a \in A^{\hat{\mathbb{N}}}$, let $\mu_a$ be the conditional measure induced by $a$ on the zeroth coordinate. That is, for any $b \in A$,

$$
\mu_a(b) = \mu \left[ x_0 = b \mid x_{\hat{\mathbb{N}}} = a \right]. \quad (\text{where } x \in A^\mathbb{N} \text{ is a } \mu\text{-random sequence})
$$

Let $\hat{\mu}$ be the projection of $\mu$ onto $A^{\hat{\mathbb{N}}}$. Then we have the following disintegration:

$$
\mu = \int_{A^{\hat{\mathbb{N}}}} (\mu_a \otimes \delta_a) \, d\hat{\mu}[a]. \quad (1)
$$

Suppose $A$ is a finite (possibly nonabelian) group, and let $C \subset A$ be a subgroup. We call $\mu$ a $C$-measure if, for $\forall a \in A^{\hat{\mathbb{N}}}$, $\text{supp}(\mu_a)$ is a right coset of $C$, and $\mu_a$ is uniformly distributed on this coset. It follows:

Lemma 10

(a) If $\mu$ is a $C$-measure, then $h(\mu, \sigma) = \log_2 |C|$.

(b) Let $\{e\}$ be the identity subgroup. Then $\left( h(\mu, \sigma) = 0 \right) \iff \left( \mu \text{ is an } \{e\}\text{-measure} \right)$.

(c) \( \left( \mu \text{ is an } A\text{-measure} \right) \iff \left( \mu \text{ is the uniform measure on } A^\mathbb{N} \right) \).
Let $\Phi : \mathcal{A}^N \to \mathcal{A}^N$ be the nearest neighbour multiplication $\text{CA}$, having local map $\phi(a_0, a_1) = a_0 \cdot a_1$. This type of CA was previously studied in [7, 10]. Our goal is to prove:

**Theorem 11** If $\mu$ is $\sigma$-invariant and $\Phi$-ergodic, then $\mu$ is a $\mathcal{C}$-measure for some $\mathcal{C} \prec \mathcal{A}$. □

**Example 12:**

(a) Let $\mathcal{C} \prec \mathcal{A}$ be any subgroup, and let $\mu$ be the uniform measure on $\mathcal{C}^N$. Then $\mu$ is a $\mathcal{C}$-measure (for any $a \in \mathcal{A}^N$, $\mu_a$ is uniform on $\mathcal{C}$), and $\mu$ is $\sigma$-invariant and $\Phi$-ergodic.

(b) Let $\mathcal{Q} = \{\pm1, \pm i, \pm j, \pm k\}$ be the Quaternion group $[2, §1.5]$, and let $\Phi_{\mathcal{Q}} : \mathcal{Q}^N \to \mathcal{Q}^N$ be the nearest neighbour multiplication $\text{CA}$. It follows:

\[
\begin{align*}
\text{If } \ p & = [i, j, k, i, j, k, i, j, k, \ldots] \\
\text{then } \Phi_{\mathcal{Q}}(p) & = [k, i, j, k, i, j, k, i, j, \ldots], \\
\text{and } \Phi_1^2(p) & = [j, k, i, j, k, i, j, k, i, \ldots], \\
\text{and } \Phi_2^3(p) & = [i, j, k, i, j, k, i, j, k, \ldots] = p.
\end{align*}
\]

Let $\mu_{\mathcal{Q}}$ be the probability measure on $\mathcal{Q}^N$ assigning probability $1/3$ to each of $p$, $\Phi_{\mathcal{Q}}(p)$ and $\Phi_1^2(p)$. Then $\mu_{\mathcal{Q}}$ is $\sigma$-invariant and $\Phi_{\mathcal{Q}}$-ergodic.

Now, let $\mathcal{C}$ be any other group, and let $\mathcal{A} = \mathcal{C} \times \mathcal{Q}$. Identify $\mathcal{C}$ with $\mathcal{C} \times \{1\} \prec \mathcal{A}$; then $\mathcal{C}$ is a normal subgroup of $\mathcal{A}$, and $\mathcal{Q} = \mathcal{A}/\mathcal{C}$. The cosets of $\mathcal{C}$ all have the form $\mathcal{C} \times \{q\}$ for some $q \in \mathcal{Q}$. There is a natural identification $\mathcal{A}^N \cong \mathcal{C}^N \times \mathcal{Q}^N$, given:

\[
\left( [c_0, q_0), (c_1, q_1), (c_2, q_2), \ldots ] \right) \leftrightarrow \left( [c_0, c_1, c_2, \ldots ]; [q_0, q_1q_2, \ldots ] \right)
\]

Let $\mu_{\mathcal{C}}$ be the uniform Bernoulli measure on $\mathcal{C}^N$, and let $\mu = \mu_{\mathcal{C}} \otimes \mu_{\mathcal{Q}}$.

**Claim 1:** $\mu$ is a $\mathcal{C}$-measure.

**Proof:** Suppose $a \in \mathcal{A}^N$ is a $\mu$-random sequence. Then $a = (c, q)$, where $q \in \{p, \Phi_{\mathcal{Q}}(p), \Phi_1^2(p)\}$, (with probability $1/3$ each), and $c = (c_0, c_1, c_2, \ldots )$ is a sequence of independent, uniformly distributed random elements of $\mathcal{C}$. The coordinates

\[
[a_1, a_2, a_3, \ldots ] = (c_1, q_1), (c_2, q_2), (c_2, q_2), \ldots
\]

determine $q$, and thus, determine $q_0$. Thus, $\mu_{\{a_1, a_2, a_3, \ldots \}}$ is uniformly distributed on the coset $\mathcal{C} \times \{q_0\}$. .......................................................... ◊ [Claim 1]

Let $\Phi : \mathcal{A}^N \to \mathcal{A}^N$ be the nearest neighbour multiplication map on $\mathcal{A}^N$.

**Claim 2:** $\mu$ is $\Phi$-ergodic and $\sigma$-invariant.
Corollary 13 Let $h_{\text{max}} = \max \{ \log_2 |C| ; C \text{ a proper subgroup of } A \}$ (In particular, if $A$ has no nontrivial proper subgroups, then $h_{\text{max}} = 0$.) If $\mu$ is $\sigma$-invariant and $\Phi$-ergodic, and $h(\mu, \sigma) > h_{\text{max}}$, then $\mu$ is the uniform measure.

Proof: Theorem 11 says $\mu$ must be a $C$-measure for some subgroup $C \triangleleft A$. But if $B$ is any proper subgroup, then $h(\mu, \sigma) > h_{\text{max}} \geq \log_2 |B|$, so Lemma 11(a) says $C$ can’t be $B$. Thus, $C = A$. Then Lemma 11(c) says that $\mu$ is the uniform measure. □

Example 14: If $p$ and $q$ are prime and $p$ divides $q - 1$, then there is a unique nonabelian group of order $pq$ \cite{2} §5.5. For example, let $p = 3$ and $q = 7$ and let $A$ be the unique nonabelian group of order 21. Then $h_{\text{max}} = \log_2(7) \approx 2.807 < 4.392 \approx \log_2(21)$. Hence, if $\mu$ is $\sigma$-invariant and $\Phi$-ergodic and $h(\mu, \sigma) \geq 2.81$, then $\mu$ is the uniform measure.

If $b \in A$, then we define (left) scalar multiplication by $b$ upon $A^\mathbb{N}$ in the obvious way: if $c = [c_0, c_1, c_2, \ldots] \in A^\mathbb{N}$, then $b \cdot c = [bc_0, bc_1, bc_2, \ldots]$. For any sequence $a = [a_1, a_2, a_3, \ldots]$ in $A^\mathbb{N}$ and any $b \in A$, let $[b, a]$ denote the sequence $[b, a_1, a_2, a_3, \ldots]$ in $A^\mathbb{N}$. Recall the conjugacy $\Xi : A^\mathbb{N} \rightarrow A^\mathbb{N}$ and the dual cellular automaton $\Phi : A^\mathbb{N} \rightarrow A^\mathbb{N}$ introduced in \cite{2}

Lemma 15 Let $a \in A^\mathbb{N}$, and suppose $\Xi(a) = [b_0, b_1, b_2, \ldots]$. Then:

(a) $\Xi[e, a] = [e, b_0, b_0b_1, b_0b_1b_2, b_0b_1b_2b_3, \ldots]$.

(b) For any $b \in A$, $\Xi[b, a] = b \cdot \Xi[e, a]$. □

Say that an element $g \in A^\mathbb{N}$ is $(\Phi, \mu)$-generic if, for any cylinder set $U \subset A^\mathbb{N}$,

$$\mu[U] = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} 1_U(\Phi^n(g)).$$

The Birkhoff Ergodic Theorem says that $\mu$-almost all points in $A^\mathbb{N}$ are $(\Phi, \mu)$-generic.

Let $\nu = \Xi(\mu)$. It follows that $\nu$ is $\Phi$-invariant and $\sigma$-ergodic. Lemma 8 implies:
Lemma 16 Let $g \in A^N$. Then \(g\) is \((\Phi, \mu)\)-generic $\iff$ \((\Xi(g)\) is \((\sigma, \nu)\)-generic. $\square$

Lemma 17 Let $a \in A^{\tilde{N}}$, and let $[b, a] \in A$. Suppose both $[b, a]$ and $[b', a]$ are \((\Phi, \mu)\)-generic. If $c = b' \cdot b^{-1}$, then $\nu$ is invariant under (left) scalar multiplication by $c$. In other words, for any measurable subset $U \subseteq A^N$, $\mu[c \cdot U] = \mu[U]$.

Proof: Let $g = \Xi[b, a] \text{ and } g' = \Xi[b', a]$. Then $b' = c \cdot b$, so
\[
g' = \Xi[b', a] \cdot \Xi[e, a] = cb \cdot \Xi[e, a] \cdot \Xi[b, a] = c \cdot g, \quad (2)
\]
where (L15) is by Lemma 15(b). Next, Lemma 16 says that $g$ and $g'$ are both \((\sigma, \nu)\)-generic. Thus, for any cylinder set $U \subseteq A^N$,
\[
\nu[U] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_U(\sigma^n(g)) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_U(\sigma^n(c^{-1} \cdot g')) = \nu[c \cdot U].
\]

(g1) is because $g$ is generic, (eq2) is by eqn. (2), and (g2) is because $g'$ is generic. $\square$

We next show that the hypothesis of Lemma 17 is not vacuous. Let
\[
\tilde{F}_2 = \left\{a \in A^\tilde{N} \mid \text{card} \supp(\mu_a) \geq 2\right\}.
\]

Lemma 18 If $h(\mu, \sigma) > 0$, then $\tilde{\mu}[\tilde{F}_2] > 0$.

Proof: Let $\tilde{F}_1 = \left\{a \in A^\tilde{N} \mid \text{card} \supp(\mu_a) \geq 1\right\}$.

Claim 1: $\tilde{\mu}[\tilde{F}_1] = 1$.

Proof: $1 = \mu[A^{\tilde{N}}] \cdot \int_{A^{\tilde{N}}} \mu_a[A] \, d\tilde{\mu}[a] = \int_{\tilde{F}_1} 1 \, d\tilde{\mu}[a] = \tilde{\mu}[\tilde{F}_1]. \quad \square$ [Claim 1]

If $\rho$ is a measure on $A$, define $H(\rho) = -\sum_{b \in A} \rho(b) \log_2 (\rho(b))$. Recall [8, Prop. 5.2.12] that
\[
h(\mu, \sigma) = \int_{A^{\tilde{N}}} H(\mu_a) \, d\tilde{\mu}[a]. \quad (3)
\]

Claim 2: If $\tilde{\mu}[\tilde{F}_2] = 0$, then $H(\mu_a) = 0$ for $\forall \tilde{\mu} \in A^{\tilde{N}}$. 8
Proof: Let $\tilde{F}_* = \tilde{F}_1 \setminus \tilde{F}_2 = \left\{ a \in \mathcal{A}^\mathbb{N} ; \text{card} [\text{supp} (\mu_a)] = 1 \right\}$. If $\tilde{\mu}(\tilde{F}_2) = 0$, then $\mu[\tilde{F}_*] = \mu[\tilde{F}_1] - \mu[\tilde{F}_2] = 1$. Thus, there is measurable function $\gamma : \mathcal{A}^\mathbb{N} \to \mathcal{A}$ so that $\mu_a(\gamma(a)) = 1$ for $\forall \mu_a$ $a \in \mathcal{A}^\mathbb{N}$. Hence, $H(\mu_a) = 0$, for $\forall \mu_a$ $a \in \mathcal{A}^\mathbb{N}$. ...... $\diamond$ [Claim 2]

Claim 2 and equation (3) imply that $h(\mu, \sigma) = 0$, contradicting our hypothesis. 

Let $\tilde{G} = \left\{ a \in \mathcal{A}^\mathbb{N} ; [b, a] \text{ is } (\Phi, \mu)\text{-generic for every } b \in \text{supp} (\mu_a) \right\}.$

Lemma 19 $\tilde{\mu}(\tilde{G}) = 1.$

Proof: Suppose not. Let $H = \mathcal{A}^\mathbb{N} \setminus \tilde{G} = \left\{ a \in \mathcal{A}^\mathbb{N} ; [b, a] \text{ is not } (\Phi, \mu)\text{-generic for some } b \in \text{supp} (\mu_a) \right\}.$

For every $h \in H$, let $B_h = \left\{ b \in \text{supp} (\mu_h) ; [b, a] \text{ is not } (\Phi, \mu)\text{-generic.} \right\}$. Thus,

$$\forall h \in H, \quad \mu_h[B_h] > 0. \quad (4)$$

Let $H = \left\{ [b, h] ; h \in \tilde{H}, b \in B_h \right\}$. If $\tilde{\mu}(\tilde{G}) < 1$, then $\tilde{\mu}(H) > 0$. Thus,

$$\mu[H] \geq \int_H \mu_h[B_h] \, d\tilde{\mu}[h] > 0,$$

Now let $G = \left\{ g \in \mathcal{A}^\mathbb{N} ; g \text{ is } (\Phi, \mu)\text{-generic} \right\}$. Then the Birkhoff Ergodic Theorem says $\mu[G] = 1$. But clearly $G \subset \mathcal{A}^\mathbb{N} \setminus H$, so if $\mu[H] > 0$, then $\mu[G] < 1$. Contradiction. $\square$

Let $\tilde{I} = \left\{ a \in \mathcal{A}^\mathbb{N} ; \text{there are distinct } b, b' \in \mathcal{A} \text{ so that } [b, a] \text{ and } [b', a] \text{ are } (\Phi, \mu)\text{-generic} \right\}.

Lemma 20 If $h(\mu, \sigma) > 0$, then $\tilde{\mu}(\tilde{I}) > 0$ (so the hypothesis of Lemma 17 is nonvacuous).

Proof: Observe that $\tilde{I} \supset \tilde{F}_2 \cap \tilde{G}$. Now combine Lemmas 18 and 19 $\square$

Lemma 21 If $\nu$ is invariant under scalar multiplication by $c$, then, for $\forall \mu_a \in \mathcal{A}^\mathbb{N}$, $\mu_a$ is invariant under left multiplication by $c$.

Proof: Let $b \in \mathcal{A}$ and let $b' = c \cdot b$. Define $\beta : \mathcal{A}^\mathbb{N} \to \mathbb{R}$ by $\beta(a) = \mu_a(b)$. Likewise, let $\beta'(a) = \mu_a(b')$. Then $\beta$ and $\beta'$ are measurable, and we want to show that $\beta = \beta'$, $\tilde{\mu}$-a.e.

Define $\gamma : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ by $\gamma[a_0, a_1, a_2, ...] = [c \cdot a_0, a_1, a_2, ...]$. Define $\Gamma : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ by $\Gamma[a] = c \cdot a$.

Claim 1: $\Xi \circ \gamma = \Gamma \circ \Xi$. 

9
Proof: Generalize the reasoning behind equation [2]. ................. ◇ [Claim 1]

Claim 2: \( \mu \) is \( \gamma \)-invariant.

Proof: For any measurable subset \( U \subset \mathcal{A} \),
\[
\mu \left[ \gamma(U) \right] = \nu \left[ \Xi \circ \gamma(U) \right] = \nu \left[ \Gamma \circ \Xi(U) \right] = \mu [U].
\]
(D) is by definition of \( \nu \). (C1) is Claim \( \square \) (I) is because \( \nu \) is \( \Gamma \)-invariant. ◇ [Claim 2]

Claim 3: For any measurable subset \( \tilde{W} \subset \mathcal{A} \), \[ \int_{\tilde{w}} \beta(w) \, d\tilde{\mu}[w] = \int_{\tilde{W}} \beta'(w) \, d\tilde{\mu}[w]. \]

Proof: Let \( U = \{b \times \tilde{w} \} \), and let \( U' = \gamma(U) = \{cb \times \tilde{w} \} = \{b' \times \tilde{W} \} \). Then:
\[
\int_{\tilde{w}} \beta(w) \, d\tilde{\mu}[w] = \mu[U] \quad \text{(c1)} \quad \mu[U'] \quad \text{(c2)} \quad \int_{\tilde{W}} \beta'(w) \, d\tilde{\mu}[w],
\]
where (c1) is by equation (11), and (c2) is by Claim 2. ................. ◇ [Claim 3]

It follows from Claim 3 that \( \beta = \beta', \tilde{\mu}\text{-ae.} \)

Proof of Theorem 11: Let \( C \) be the set of all \( c \in \mathcal{A} \) so that there is some \( a \in \mathcal{A} \) and \( b \in \mathcal{A} \) with both \( [b,a] \) and \( [(cb),a] \) being \( (\Phi,\mu) \)-generic.

If \( h(\mu,\sigma) = 0 \), then \( \mu \) is an \( \{e\} \)-measure by Lemma 10(b). So, assume \( h(\mu,\sigma) \neq 0 \); then Lemma 20 implies that \( C \) is nontrivial.

Claim 1: \( C \) is a group, and \( \mu_a \) is invariant under (left) \( C \)-multiplication for \( \forall \tilde{a} \in \tilde{A} \).

Proof: Lemma 17 says that \( \nu \) is invariant under \( C \)-scalar multiplication. Let \( D \) be the group generated by \( C \). Then \( C \subseteq D \), and \( \nu \) is also invariant under \( D \)-scalar multiplication. Lemma 21 implies that \( \mu_a \) is invariant under (left) \( D \)-multiplication for \( \forall \tilde{a} \in \tilde{A} \). It follows from Lemma 19 that \( D \subseteq C \), and hence, \( C = D \). ................. ◇ [Claim 1]

Claim 2: \( \forall \tilde{a} \in \tilde{A} \), \( \text{supp} \ (\mu_a) \) is a (right) coset of \( C \).

Proof: For \( \forall \tilde{a} \in \tilde{A} \), Claim 1 implies that \( \text{supp} \ (\mu_a) \) is a disjoint union of cosets of \( C \), and that \( \mu_a \) is uniformly distributed on each of these cosets. Let
\[
\tilde{M} = \left\{ a \in \tilde{A} \colon \text{supp} \ (\mu_a) \text{ contains more than one coset of } C \right\}.
\]
We claim that \( \tilde{\mu}[\tilde{M}] = 0 \). Suppose not. Then Lemma 19 implies that \( \mu[\tilde{M} \cap \tilde{G}] > 0 \). So let \( \tilde{m} \in \tilde{M} \cap \tilde{G} \), and find elements \( b, b' \in \text{supp} \ (\mu_m) \) living in different cosets, such that \( [b,\tilde{m}] \) and \( [b',\tilde{m}] \) are both \( (\Phi,\mu) \)-generic. If \( c = b^{-1}b' \), then \( b' = cb \), so \( c \in C \). But \( b \) and \( b' \) are in different cosets of \( C \); hence, \( c \notin C \). Contradiction. ____ ◇ [Claim 2] □
4 Degree of QGCA relative to invariant measures

If \( \mu \) is a \( \Phi \)-invariant measure, then \( \Phi \) is \( K \)-to-1 (\( \mu \)-\( \approx \)) if there is a subset \( \mathcal{U} \subset A^\mathbb{Z} \) such that:

1. \( \mu[\mathcal{U}] = 1 \).
2. \( \Phi^{-1}(\mathcal{U}) \equiv \mathcal{U} \).
3. \( \mu \)-almost every element \( u \in \mathcal{U} \) has exactly \( K \) preimages in \( \mathcal{U} \) —ie. \( |\mathcal{U} \cap \Phi^{-1}\{u\}| = K \).

We will generalize the methods of [1] to prove:

**Theorem 22** Let \( \Phi : A^\mathbb{Z} \to A^\mathbb{Z} \) be a quasigroup CA, and let \( \mu \) be a measure which is \( \Phi \)-invariant and \( \sigma \)-ergodic. Let \( |A| = N \). Then there is some \( K \in [1..N] \) so that

(a) \( h_\mu(\Phi) = \log_2(K) \).

(b) \( \Phi \) is \( K \)-to-1 (\( \mu \)-\( \approx \)).

\( \blacksquare \)

**Example 23:** Let \( \lambda \) be the uniform Bernoulli measure on \( A^\mathbb{Z} \). Then \( \lambda \) is invariant for any QGCA, \( h_\lambda(\Phi) = \log_2(N) \), and \( \Phi \) is \( N \)-to-1 (\( \lambda \)-\( \approx \)). Indeed, \( \lambda \) is the only \( (\Phi, \sigma) \)-invariant measure with entropy \( \log_2(N) \). Thus, Proposition 3 is proved in [1] by first proving a special case of Theorem 22 (when \( \Phi \) is a Ledrappier CA) and then showing that \( K = N \).

Let \( \mu \) be a measure on \( A^\mathbb{Z} \). If \( q \) is any partition of \( A^\mathbb{Z} \), and \( \mathcal{S} \) is any sigma-algebra, define

\[
H_\mu(q | \mathcal{S}) = \sum_{Q \in q} \int_Q \log_2 \left( E_\mu[Q | \mathcal{S}] \right)(x) \, d\mu(x).
\]

(5)

Let \( p_0 \) be the partition of \( A^\mathbb{Z} \) generated by zero-coordinate cylinder sets, and let \( p_{[\ell,n]} = \bigvee_{m=\ell}^{n} \sigma^{-m}(p_0) \). Thus, \( \mathcal{B} = p_{[-\infty,\infty]} \) is the Borel sigma-algebra of \( A^\mathbb{Z} \). Let \( \mathcal{B}^1 = \Phi^{-1}(\mathcal{B}) \).

If \( \mu \) is a \( \Phi \)-invariant measure, then

\[
h_\mu(\Phi) = \lim_{r \to \infty} h_\mu(\Phi, p_{[-r,r]}),
\]

where

\[
h_\mu(\Phi, p_{[-r,r]}) = H_\mu \left( p_{[-r,r]} \mid \bigvee_{t=1}^{\infty} \Phi^{-t}(p_{[-r,r]}) \right).
\]

**Lemma 24** If \( \Phi : A^\mathbb{Z} \to A^\mathbb{Z} \) is a QGCA, and \( \mu \) is \( (\Phi, \sigma) \)-invariant, then

\[
h_\mu(\Phi) = H_\mu (p_0 | \mathcal{B}^1).
\]

**Proof:** Let \( x \in A^\mathbb{Z} \) be an unknown sequence. Because \( \Phi \) is bipermutative, complete information about \( (\Phi^t(x))_{[-r,r]} \) (for \( t \in [0..T] \)) is sufficient to reconstruct \( x_{[-T-r,T+r]} \), and vice versa. In other words, we have an equality of partitions:

\[
\bigvee_{t=0}^{T-1} \Phi^{-t}(p_{[-r,r]}) = p_{[-T-r,T+r]}.
\]
Letting \( T \to \infty \), we get an equality of sigma-algebras: \( \bigvee_{t=0}^{\infty} \Phi^{-t}(p_{[-r,r]}) = p_{[-\infty,\infty]} = \mathcal{B} \).

Applying \( \Phi^{-1} \) to everything yields: \( \bigvee_{t=1}^{\infty} \Phi^{-t}(p_{[-r,r]}) = \Phi^{-1}(\mathcal{B}) = \mathcal{B}^{1} \). Hence,

\[
h_\mu(\Phi, p_{[-r,r]}) = H_\mu \left( p_{[-r,r]} \bigg| \bigvee_{t=1}^{\infty} \Phi^{-t}(p_{[-r,r]}) \right) = H_\mu(\Phi, p_{[-r,r]} | \mathcal{B}^{1})
\]

Now, \( \Phi \) is bipermutative, so if we have complete knowledge of \( \Phi(x) \), then we can reconstruct \( x \) from knowledge only of \( x_0 \). Thus, \( H_\mu(p_{[-r,r]} | \mathcal{B}^{1}) = H_\mu(p_0 | \mathcal{B}^{1}) \).

Thus, \( h_\mu(\Phi) = \lim_{r \to \infty} h_\mu(\Phi, p_{[-r,r]}) = \lim_{r \to \infty} H_\mu(p_0 | \mathcal{B}^{1}) = H_\mu(p_0 | \mathcal{B}^{1}). \) \( \square \)

For any \( x \in A^Z \), let \( F(x) = \Phi^{-1}\{\Phi(x)\} = \{y \in A^Z ; \Phi(y) = \Phi(x)\} \). Hence, the sets \( F(x) \) (for \( x \in A^Z \)) are the ‘minimal elements’ of the sigma algebra \( \mathcal{B}^{1} \).

The conditional expectation operator \( \mathbb{E}_\mu[\cdot | \mathcal{B}^{1}] \) defines ‘fibre’ measures \( \mu_x \) (for \( \forall \mu, x \in A^Z \)) having three properties:

(F1) For any measurable \( U \subset A^Z \) and for \( \forall \mu, x \in A^Z \), \( \mu_x(U) = \mathbb{E}_\mu[U | \mathcal{B}^{1}](x) \).

(F2) For any fixed \( x \in A^Z \), \( \mu_x \) is a probability measure on \( A^Z \), and \( \text{supp}(\mu) = F(x) \).

(F3) For any fixed measurable \( U \subset A^Z \), the function \( A^Z \ni x \mapsto \mu_x(U) \in \mathbb{R} \) is \( \mathcal{B}^{1} \)-measurable. Hence, \( \mu_x = \mu_y \) for any \( y \in F(x) \).

Our goal is to show that there is some constant \( K \) and, for \( \forall \mu, x \in A^Z \), there is a subset \( E \subset F(x) \) of cardinality \( K \) so that \( \mu_x \) is uniformly distributed on \( E \).

Lemma 25 For any measurable \( U \subset A^Z \) and for \( \forall \mu, x \in A^Z \), \( \mu_x(\sigma^{-1}(U)) = \mu_{\sigma(x)}(U) \).

Proof: For \( \forall \mu, x \in A^Z \), property (F1) says

\[
\mu_x(\sigma^{-1}(U)) = \mathbb{E}_\mu[\sigma^{-1}(U) | \mathcal{B}^{1}](x),
\]

and \( \mu_{\sigma(x)}(U) = \mathbb{E}_\mu[U | \mathcal{B}^{1}](\sigma(x)) \).

Thus, we must show that \( \mathbb{E}_\mu[\sigma^{-1}(U) | \mathcal{B}^{1}](x) = \mathbb{E}_\mu[U | \mathcal{B}^{1}](\sigma(x)) \) for \( \forall \mu, x \in A^Z \). But \( \mathbb{E}_\mu[\sigma^{-1}(U) | \mathcal{B}^{1}] \) and \( \mathbb{E}_\mu[U | \mathcal{B}^{1}] \) are \( \mathcal{B}^{1} \)-measurable functions, so it suffices to show that

\[
\int_{B} \mathbb{E}_\mu[\sigma^{-1}(U) | \mathcal{B}^{1}](x) \, d\mu[x] = \int_{B} \mathbb{E}_\mu[U | \mathcal{B}^{1}](\sigma(x)) \, d\mu[x], \text{ for any } B \subset \mathcal{B}^{1}.
\]

But

\[
\int_{B} \mathbb{E}_\mu[\sigma^{-1}(U) | \mathcal{B}^{1}](x) \, d\mu[x] \overset{(\text{meas})}{=} \int_{B} \mathbb{1}_{\sigma^{-1}(U)}(x) \, d\mu[x] = \mu[B \cap \sigma^{-1}(U)]
\]
\[
\begin{align*}
\mu \left[ \sigma(B) \cap U \right] &= \int_{\sigma(B)} \mathbb{1}_U(x') \, d\mu[x'] \\
\overset{(E)}{=} \int_{\sigma(B)} \mathbb{E}_\mu \left[ U | \mathfrak{B}^1 \right] (x') \, d\mu[x'] \\
\overset{(S)}{=} \int_B \mathbb{E}_\mu \left[ U | \mathfrak{B}^1 \right] (\sigma(x)) \, d\mu[x],
\end{align*}
\]

as desired. Here \((E)\) is the defining property of conditional expectation, \((I)\) is because \(\mu\) is \(\sigma\)-invariant, and \((S)\) is the substitution \(x' = \sigma(x)\) (again because \(\mu\) is \(\sigma\)-invariant). \(\square\)

For any \(x \in A^Z\), let \(\eta(x) = \mu_x \{ x \}\). Thus, if \(y\) is an unknown, \(\mu\)-random sequence, then \(\eta(x)\) represents the conditional probability that \(y = x\), given that \(\Phi(y) = \Phi(x)\).

**Lemma 26**

(a) \(\eta\) is \(\sigma\)-invariant (\(\mu\)-ae).

(b) If \(\mu\) is \(\sigma\)-ergodic, then there is some \(H \in \mathbb{R}\) such that \(\eta(x) = H\) for \(\forall \mu\) \(x \in A^Z\).

(c) If \(\mu\) is also \(\Phi\)-invariant, then \(\eta\) is \(\Phi\)-invariant (\(\mu\)-ae).

**Proof:**

(a) \(\eta(\sigma(x)) = \mu_{\sigma(x)} \{ \sigma(x) \} \overset{(25)}{=} \mu_x (\sigma^{-1}(\sigma(x))) = \mu_x \{ x \} = \eta(x)\).

(b) \(\mathbb{E}_\mu \left[ P | \mathfrak{B}^1 \right] (x) = \mu_x \{ x \}\) \(\overset{(26a)}{=} \mu_x \left( P \cap \mathcal{F}(x) \right) \overset{(26b)}{=} \eta(x) \overset{(26)}{=} H\).

Here, \(\text{(c2)}\) follows from Claim 2 below, and \(\text{(26b)}\) is by Corollary \(26\text{(b)}\). \(\diamond \) [Claim 1]

**Claim 2:** If \(x \in P \in p_0\), then \(P \cap \mathcal{F}(x) = \{ x \}\).

**Proof:** \(\Phi\) is bipermutative, so if \(y \in \mathcal{F}(x)\), then \(y\) is entirely determined by \(y_0\). Thus, \(\left( y \in P \right) \Rightarrow \left( y_0 = x_0 \right) \Rightarrow \left( y = x \right)\). \(\diamond \) [Claim 2] \(\square\)
Our goal is to show that $H = \frac{1}{K}$ for some $K$.

Suppose $|A| = N$, and identify $A$ with the group $\mathbb{Z}/N$ in an arbitrary way. Define $\tau: A^Z \to A^Z$ as follows. For any $x \in A^Z$, $\tau(x) = y$, where $y$ is the unique element in $F(x)$ such that $y_0 = x_0 + 1 \pmod{N}$. Existence/uniqueness of $y$ follows from bipermutativity.

Note that $\tau(\mu) \neq \mu$, so a statement which is true $\mu$-a.e. may not be true $\tau(\mu)$-a.e. For example, Lemma 26(c) does not imply that $\eta \left( \Phi \left( \tau(x) \right) \right) = \eta \left( \tau(x) \right)$ for $\forall \mu, x$.

Let $E_n = \left\{ x \in A^Z : \eta \left( \tau^n(x) \right) > 0 \right\}$. Let $\mu_n = \tau^n(1_{E_n} \cdot \mu)$.

**Lemma 28** $\mu_n$ is absolutely continuous relative to $\mu$.

**Proof:** Suppose $Z \subset A^Z$ is Borel-measurable, and $\mu[Z] = 0$. We want to show $\mu_n[Z] = 0$ also. But $\mu_n[Z] = (1_{E_n} \cdot \mu) [\tau^{-n}(Z)] = \mu [\tau^{-n}(Z) \cap E_n]$, so it suffices to show: \textbf{Claim 1}: For $\forall \mu, z \in \tau^{-n}(Z), \quad \eta \left( \tau^n(z) \right) = 0$; hence $z \notin E_n$.

**Proof:** \[ \int_{A^Z} \mu_n[Z] \, d\mu[x] = \mu[Z] = 0. \] Hence, for $\forall \mu, x \in A^Z$, $\mu_n[Z] = 0$.

But if $z \in \tau^{-n}(Z)$, then $\tau^n(z) \in F(z) \cap Z$, so we get $\eta \left( \tau^n(z) \right) = \mu_{\tau^n(z)} \left( \tau^n(z) \right) = \mu_{\tau^n(z)} \left( \tau^n(z) \right) \leq \mu_{z} \left( \tau^n(z) \right) \leq \mu_{z} \left[ Z \cap F(x) \right] \leq \mu_{z} \left[ Z \right] = 0. \quad \Box$ [Claim 1]

**Corollary 29** If $\mu$ is $\sigma$-ergodic and $\Phi$-invariant, then $\eta$ is $\Phi$-invariant ($\mu_n$-a.e).

**Proof:** Lemma 28 means that a statement which is true for $\forall \mu, x$ is also true for $\forall \mu_n, x$. Now apply Lemma 26(c). \[ \Box \]

**Corollary 30** For $\forall \mu, x \in E_n$, $\eta(x) = \eta \left( \tau^n(x) \right)$.

**Proof:** $\eta \left( x \right) \overset{\text{(26)}}{=} \eta \left( \Phi[x] \right) \overset{(*)}{=} \eta \left( \Phi \left[ \tau^n(x) \right] \right) \overset{\text{(29)}}{=} \eta \left( \tau^n(x) \right).$ Here, (26) is by Corollary 26(c), $(*)$ is because $\tau^n(x) \in F(x)$, and (29) is by Corollary 29. \[ \Box \]

Now, let $\mathcal{E}(x) = \{ y \in F(x) : \eta(y) > 0 \}$.

**Corollary 31** For $\forall \mu, x \in A^Z$, $\mu_x$ is equidistributed on $\mathcal{E}(x)$. If $\text{card} \left[ \mathcal{E}(x) \right] = K$, then $\mu_x$ assigns mass $\frac{1}{K}$ to each element in $\mathcal{E}(x)$. In particular, $\eta(x) = \frac{1}{K}$.
Proof: By definition, \( 1 = \mu_x \left( F(x) \right) = \sum_{y \in F(x)} \mu_x \{ y \} = \sum_{y \in F(x)} \mu_y \{ y \} = \sum_{y \in E(x)} \eta(y) \).

However, if \( y = \tau^n(x) \), then \( \left( y \in E(x) \right) \Longleftrightarrow \left( x \in E_n \right) \), in which case Corollary 30 implies that \( \eta(y) = \eta(x) \). Hence, \( 1 = \sum_{y \in E(x)} \eta(y) = \sum_{y \in E(x)} \eta(x) = K \cdot \eta(x) \), where \( K = \text{card} \left[ E(x) \right] \). We conclude that \( \eta(x) = \frac{1}{K} \). \( \square \)

**Corollary 32** There is some \( K \) so that, for \( \forall \mu \in \mathbb{A}^Z \), \( \text{card} \left[ E(x) \right] = K \), and \( \mu_x \) assigns mass \( \frac{1}{K} \) to each element of \( E(x) \). Thus, \( H = \frac{1}{K} \). Thus, \( h_\mu (\Phi) = \log_2(K) \).

Proof: Combine Corollaries 26(b) and 31. Then apply Lemma 27. \( \square \)

**Proof of Theorem 22** Let \( U = \{ x \in \mathbb{A}^Z : \text{card} \left[ E(x) \right] = K \} \). Then Corollary 32 says \( \mu(U) = 1 \). Since \( \mu \) is \( \Phi \)-invariant, it follows that \( \mu(\Phi^{-1}(U)) = 1 \) also; hence \( \Phi^{-1}(U) = \mu U \).

Thus, for \( \forall \mu \in U \), there is some \( x \in U \) so that \( \Phi(x) = u \). But then \( \Phi^{-1}(u) = F(x) \), and \( \Phi^{-1}(u) \cap U = F(x) \cap U = E(x) \) is a set of cardinality \( K \), by definition of \( U \). \( \square \)

## 5 Endomorphic Cellular Automata

A **group shift** is a sequence space \( \mathbb{A}^Z \) equipped with a topological group structure such that \( \sigma \) is a group automorphism. Equivalently, the multiplication operation \( \bullet \) on \( \mathbb{A}^Z \) is defined by some **local multiplication map** \( \psi : \mathbb{A}^{[-\ell..r]} \times \mathbb{A}^{[-\ell..r]} \rightarrow \mathbb{A} \) so that, if \( a, b \in \mathbb{A}^Z \) and \( c = a \bullet b \), then \( c_0 = \psi(a_{-\ell}, \ldots, a_r; b_{-\ell}, \ldots, b_r) \).

The most obvious group shift is a **product group**, where \( \mathbb{A} \) is a finite group and multiplication on \( \mathbb{A}^Z \) is defined componentwise. However, this is not the only group shift \( 5 \).

An **endomorphic cellular automaton** (ECA) is a cellular automaton \( \Phi : \mathbb{A}^Z \rightarrow \mathbb{A}^Z \) which is also a group endomorphism of \( \mathbb{A}^Z \). For example, it is easy to verify:

**Proposition 33** Let \( \mathbb{A} \) be an additive abelian group. Let \( \mathbb{A}^Z \) be the product group. Let \( \Phi : \mathbb{A}^Z \rightarrow \mathbb{A}^Z \) be a right-sided, nearest neighbour CA, with local map \( \phi : \mathbb{A}^{[0,1]} \rightarrow \mathbb{A} \). Then:

(a) \( \Phi \) is an ECA iff \( \phi(a_0, a_1) = \phi_0(a_0) + \phi_1(a_1) \), where \( \phi_0, \phi_1 \) are endomorphisms of \( \mathbb{A} \).

(b) \( \Phi \) is bipermutative iff \( \phi_0 \) and \( \phi_1 \) are automorphisms of \( \mathbb{A} \). \( \square \)

We will now apply the results of \( 4 \) to bipermutative ECA, to prove:
**Theorem 34** Let $\mathcal{A}^\mathbb{Z}$ be a group shift and let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a bipermutative ECA. Suppose $\ker(\Phi)$ contains no nontrivial $\sigma$-invariant subgroups.

If $\mu$ is $\Phi$-invariant and totally $\sigma$-ergodic, and $h_\mu(\Phi) > 0$, then $\mu = \lambda$. 

Recall from §4 that if $x \in \mathcal{A}^\mathbb{Z}$, then $F(x) = \Phi^{-1}\{\Phi(x)\}$.

**Lemma 35** Let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a bipermutative ECA on a group shift. Let $K = \ker(\Phi)$.

(a) For any $x \in \mathcal{A}^\mathbb{Z}$, $F(x) = x \cdot K$.

(b) Let $e \in \mathcal{A}^\mathbb{Z}$ be the identity element. Then $e$ is a constant sequence —ie. there is some $e \in \mathcal{A}$ so that $e = (...., e, e, e, ....)$.

(c) $K$ is $\sigma$-invariant. Also, if $k \in K$, then $k$ is entirely determined by $k_0$.

(d) There is a natural bijection $\zeta : \mathcal{A} \rightarrow K$, where $\zeta[a]$ is the unique element $k \in K$ with $k_0 = a$. In particular, $\zeta[e] = e$.

(e) There is a permutation $\rho : \mathcal{A} \rightarrow \mathcal{A}$ so that $\sigma(\zeta[a]) = \zeta[\rho(a)]$. In particular, $\rho(e) = e$.

It follows that every element of $K$ is $P$-periodic, for some $P < |\mathcal{A}|$.

(f) Any $\sigma$-invariant subgroup $J \triangleleft K$ is thus a disjoint union of periodic $\sigma$-orbits, which corresponds to a disjoint union of $\rho$-orbits in $\mathcal{A}$.

(g) In particular: $\left(\mathcal{A} \setminus \{e\} \text{ consists of a single } \rho\text{-orbit}\right)$

\[\iff \left( K \text{ has no nontrivial } \sigma\text{-invariant subgroups}\right)\]

**Proof:** (a) is a basic property of group homomorphisms. To see (b), recall that $\sigma$ is a group automorphism of $\mathcal{A}^\mathbb{Z}$. Thus, $\sigma(e) = e$, so $e$ must be constant. (c) follows from (b) and the fact that $\Phi$ is bipermutative. Then (c) implies (d) implies (e) implies (f).

If $(\mathcal{A}, +)$ is abelian and $\mathcal{A}^\mathbb{Z}$ is the product group, then Lemma 35 takes the form:

**Lemma 36** Let $(\mathcal{A}, +)$ be an abelian group and let $\mathcal{A}^\mathbb{Z}$ be the product group. Let $\Phi : \mathcal{A}^\mathbb{Z} \rightarrow \mathcal{A}^\mathbb{Z}$ be a bipermutative ECA and let $K = \ker(\Phi)$.

(a) For any $x \in \mathcal{A}^\mathbb{Z}$, $F(x) = x + K$.

(b) The map $\zeta : \mathcal{A} \rightarrow K$ from Lemma 35(d) is a group isomorphism.
Lemma 37  
(a) The map $\rho : \mathcal{A} \to \mathcal{A}$ from Lemma 35(e) is a group automorphism. To be precise, suppose $\Phi$ has local map $\phi(a_0, a_1) = \phi_0(a_0) + \phi_1(a_1)$, where $\phi_0$ and $\phi_1$ are automorphisms of $\mathcal{A}$, as in Proposition 35(b). Then $\rho = -\phi_1^{-1} \circ \phi_0$.

(b) If $\mathcal{J} \triangleleft \mathcal{K}$ is a $\sigma$-invariant subgroup, then $\mathcal{J} = \zeta(\mathcal{B})$, where $\mathcal{B} \triangleleft \mathcal{A}$ is a $\rho$-invariant subgroup of $\mathcal{A}$.

(c) In particular, $\left( \mathcal{A} \text{ has no nontrivial } \rho \text{-invariant subgroups} \right) \iff \left( \mathcal{K} \text{ has no nontrivial } \sigma \text{-invariant subgroups} \right)$.

Proof: We need only verify the claim in (b) that $\zeta$ is a group homomorphism. To see this, suppose $k = \zeta(a)$ and $k' = \zeta(a')$. Let $j = k + k'$ and let $i = \zeta(a + a')$; we want to show $j = i$. From Lemma 35(c), it suffices to show that $i_0 = j_0$. But the operation on $\mathcal{K}$ is componentwise addition. Thus, $j_0 = k_0 + k'_0 = a + a' = i_0$. Hence, $\zeta$ is a homomorphism; being bijective, $\zeta$ is thus an isomorphism. All other claims follow. \[ \square \]

Let $\eta$ be as in [41] and for any $k \in \mathcal{K}$, let $E_k = \{ x \in \mathcal{A}^\mathbb{Z} : \eta(x \cdot k) > 0 \}$.

Lemma 37  
(a) $\sigma(E_k) \equiv E_{\sigma(k)}$.

(b) Thus, if $\sigma^P(k) = k$, then $\sigma^P(E_k) \equiv E_k$.

Proof: To prove (a) it suffices to show that $\sigma(E_k) \subseteq E_{\sigma(k)}$ (and then, by symmetric reasoning, that $E_{\sigma(k)} \subseteq \sigma(E_k)$.) To show this, we define the measure $\mu(k)$ by $\mu(k)(U) = \mu(E_k \cap (U \cdot k^{-1}))$. Then $\mu(k)$ is absolutely continuous with respect to $\mu$ (by reasoning similar to Lemma 28). Lemma 26(a) says $\eta$ is $\sigma$-invariant ($\mu$-æ); hence $\eta$ is $\sigma$-invariant ($\mu(k)$-æ), by reasoning similar to Corollary 29. Thus, for $\forall \mu x \in E_k$, $0 < \eta(x \cdot k) = \eta\left( \sigma(x \cdot k) \right) = \eta\left( \sigma(x) \cdot \sigma(k) \right)$, and thus, $\sigma(x) \in E_{\sigma(k)}$. Hence $\sigma(E_k) \subseteq E_{\sigma(k)}$. \[ \square \]

Recall from [41] that $E(x) = \{ y \in F(x) : \eta(y) > 0 \}$.

Corollary 38 If $\mu$ is $\Phi$-invariant and totally $\sigma$-ergodic, then there is a $\sigma$-invariant subgroup $\mathcal{J} \subset \mathcal{K}$ so that, for $\forall \mu x \in \mathcal{A}^\mathbb{Z}$, $E(x) = x \cdot \mathcal{J}$.

Proof: Define $\mathcal{J} = \{ k \in \mathcal{K} : \mu(E_k) > 0 \}$.

Claim 1: For any $j \in \mathcal{J}$, $\mu(E_j) = 1$.

Proof: By Lemma 35(e), find $P \in \mathbb{N}$ so that $\sigma^P(j) = j$. But then Lemma 37(b) says that $\sigma^P(E_j) = E_j$. But $\mu$ is $\sigma^P$-ergodic, so this means that $\mu(E_j) = 1$. \[ \square \] [Claim 1]
Claim 2: For $\forall \mu \ x \in \mathbb{A}^Z$, $\mathcal{E}(x) = x \cdot J$.

Proof: First note that $\mathcal{E}(x) = \{x \cdot k : k \in \mathcal{K}, x \in E_k\}$. Thus, we want to show that, for $\forall \mu \ x \in \mathbb{A}^Z$, and all $k \in \mathcal{K}$, $(x \in E_k) \iff (k \in J)$. Observe that

$$\mu \left( \bigcup_{k \in \mathcal{K}\setminus J} E_k \right) = 0 \quad \text{(by definition of } J) \quad \text{and} \quad \mu \left( \bigcap_{j \in J} E_j \right) = 1 \quad \text{(by Claim 1)}.$$ 

Thus, $x \in \bigcap_{j \in J} E_j \setminus \bigcup_{k \in \mathcal{K}\setminus J} E_k$ for $\forall \mu \ x \in \mathbb{A}^Z$. The claim follows. ....... \[\square\] [Claim 2]

Let $\mathcal{U} = E_e = \{x \in \mathbb{A}^Z : \eta(x) > 0\}$.

Claim 3: If $k \in \mathcal{K}$, then $(k \in J) \iff (\mathcal{U} \cdot k \subset \mathcal{U}, \text{modulo a set of measure zero})$.

Proof: Claim 1 implies that $\mu(\mathcal{U}) = 1$. Thus,

$$(k \in J) \iff _{C1} (\mu(\mathcal{U} \cap E_k) = 1) \iff _{DE} (\text{For } \forall \mu \ u \in \mathcal{U}, \eta(u \cdot k) > 0)$$

$$\iff _{DU} (\text{For } \forall \mu \ u \in \mathcal{U}, u \cdot k \in \mathcal{U}) \iff (\mathcal{U} \cdot k \subset \mathcal{U}).$$

(C1) is Claim 1. (DE) is by definition of $E_k$. (DU) is by definition of $\mathcal{U}$. \[\square\] [Claim 3]

Claim 4: $J$ is a subgroup of $\mathcal{K}$.

Proof: Let $j_1, j_2 \in \mathcal{J}$, and let $j = j_1 \cdot j_2$. Then Claim 3 says $\mathcal{U} \cdot j = (\mathcal{U} \cdot j_1) \cdot j_2 \subset \mathcal{U} \cdot j_1 \subset \mathcal{U}$, (modulo sets of measure zero). Thus, Claim 3 implies $j \in \mathcal{J}$ also. Hence, $\mathcal{J}$ is closed under ‘$\cdot$’. Since $\mathcal{J}$ is finite, it is a subgroup. ......... \[\square\] [Claim 4]

It remains to show that $\sigma^{-1}(\mathcal{J}) = \mathcal{J}$. To see this, let $k \in \mathcal{K}$. Then

$$(k \in \mathcal{J}) \iff _{C1} (\mu(E_k) = 1) \iff _{(\sigma)} (\mu(\sigma^{-1}(E_k)) = 1)$$

$$\iff (\mu(E_{\sigma(k)}) = 1) \iff _{C1} (\sigma(k) \in \mathcal{J}) \iff (k \in \sigma^{-1}(\mathcal{J})).$$

Here, (C1) is by Claim 1, (\sigma) is because $\mu$ is $\sigma$-invariant, and (\dagger) is by Lemma 37(a). \[\square\]

Corollary 39 Let $J = |\mathcal{J}|$. Then $h_\mu(\Phi) = \log(J)$, and $\Phi$ is $J$-to-1 ($\mu$-æ).

Proof: Combine Corollary 38 with Corollary 32 \[\square\]
Proof of Theorem 34 If \( h_\mu(\Phi) > 0 \), then Corollary 39 says \( |J| > 1 \), so \( J \) is a nontrivial \( \sigma \)-invariant subgroup of \( K \). Thus, \( J = K \), which means \( |J| = |K| = |A| \), where the second equality is by Lemma 35(d). Thus, \( h_\mu(\Phi) = \log |A| \). Thus, \( h_\mu(\sigma) = \log |A| \), which means \( \mu \) must be the uniform measure. \( \square \)

Lemmas 35(g) and 36(e) provide conditions under which \( K \) has no \( \sigma \)-invariant subgroups. For example, suppose \( p \in \mathbb{N} \) is prime, and let \( A = (\mathbb{Z}/p\mathbb{Z})^N \) for some \( N > 0 \). Then \( A \) is a vector space over the field \( \mathbb{Z}/p\mathbb{Z} \), and \( \rho : A \rightarrow A \) is a group automorphism iff \( \rho \) is a \( \mathbb{Z}/p\mathbb{Z} \)-linear automorphism. Thus, \( \rho \) can be described by an \( N \times N \) matrix \( M \) of coefficients in \( \mathbb{Z}/p\mathbb{Z} \). Furthermore, \( B \subset A \) is a \( (\rho \)-invariant) subgroup iff \( B \) is a \( (\rho \)-invariant) subspace. The structure of \( \rho \)-invariant subspaces in \( A \) is described by the rational canonical form of \( \rho \); this is an \( N \times N \) matrix \( \tilde{M} \), similar to \( M \), having the block-diagonal form

\[
\tilde{M} = \begin{bmatrix}
M_1 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & M_L
\end{bmatrix}
\]

where each component matrix \( M_\ell \) has the form:

\[
M_\ell = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & m_1 \\
1 & 0 & 0 & \ldots & 0 & m_2 \\
0 & 1 & 0 & \ldots & 0 & m_3 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & m_{r-1} \\
0 & 0 & 0 & \ldots & 1 & m_r
\end{bmatrix}
\]

some \( r > 0 \) and \( m_1, \ldots, m_r \in \mathbb{Z}/p\mathbb{Z} \). Each component matrix corresponds to a \( \rho \)-invariant subspace of \( A \). If \( \tilde{M} \) has only one component, then we say \( \tilde{M} \) is simple. We call the automorphism \( \rho \) simple if its rational canonical form is simple. It follows:

**Lemma 40** \( \rho \) is simple \( \iff \) \( A \) has no nontrivial \( \rho \)-invariant subspaces. \( \square \)

**Corollary 41** Let \( A = (\mathbb{Z}/p\mathbb{Z})^N \) and let \( A^\mathbb{Z} \) be the product group. Let \( \Phi : A^\mathbb{Z} \rightarrow A^\mathbb{Z} \) be a bipermutative ECA with local map \( \phi(a_0, a_1) = \phi_0(a_0) + \phi_1(a_1) \). Suppose \( \rho = -\phi_1^{-1} \circ \phi_0 \) is simple. Then the conclusion of Theorem 34 holds.

**Proof:** Combine Lemma 40 with parts (c) and (e) of Lemma 36 \( \square \)

**Example 42:** Let \( A = (\mathbb{Z}/7)^4 \), and suppose \( \phi(a_0, a_1) = \phi_0(a_0) + a_1 \), where \( \phi_0 \) has matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Thus, \( \rho = -\phi_0 \) is simple. Hence, if \( \mu \) is \( \Phi \)-invariant and totally \( \sigma \)-ergodic, and \( h_\mu(\Phi) > 0 \), then \( \mu \) is the uniform measure.
Conclusion

We have characterized the invariant measures for several natural families of bipermutative cellular automata. Many questions remain unanswered. For example, in §3 and §5 we exploited an algebraic structure on $A^Z$ to study the $\Phi$-invariant measures. What other algebraic properties of the quasigroup structure of $A$ can be exploited in this way?

Also, if $\Phi$ is a QGCA on $A^Z$, then the system $(A^Z, \Phi, \lambda)$ is measurably isomorphic to the uniform Bernoulli shift $(A^N, \sigma, \lambda)^{[12]}$. Theorem 22 suggests that, if $\mu$ is any positive-entropy, $\Phi$-invariant measure, then the system $(A^Z, \Phi, \mu)$ is isomorphic to $(K^Z, \sigma, \kappa)$, where $K$ is an alphabet of $K$ letters and $\kappa$ is the uniform Bernoulli measure on $K^N$. Is this true?

Finally, Example 12b refuted Conjecture 7, but did so by using a structural decomposition $A = C \times Q$ to get an invariant measure without full support. This leaves us with the following:

Conjecture: Let $(A, \ast)$ be a quasigroup and let $\Phi : A^Z \rightarrow A^Z$ be the corresponding QGCA. Let $\mu$ be a $\Phi$-invariant and $\sigma$-ergodic measure.

1. If $\mu$ has full support, then $\mu = \lambda$.
2. If $(A, \ast)$ is simple (ie. has no nontrivial quotients), and $h_\mu(\Phi) > 0$, then $\mu = \lambda$.

References

[1] Bernard Host Alejandro Maass and Servet Martínez. Uniform Bernoulli measure in dynamics of permutative cellular automata with algebraic local rules. (in press), 2003.
[2] David S. Dummit and Richard M. Foote. Abstract Algebra. Prentice-Hall, Englewood Cliffs, NJ, 1991.
[3] G. Hedlund. Endomorphisms and automorphisms of the shift dynamical systems. Mathematical System Theory, 3:320–375, 1969.
[4] J. Dénes and A.D. Keedwell. Latin squares and their applications. Academic Press, New York, 1974.
[5] Bruce Kitchens. Expansive dynamics in zero-dimensional groups. Ergodic Theory & Dynamical Systems, 7:249–261, 1987.
[6] Rune Kleveland. Mixing properties of one-dimensional cellular automata. Proceedings of the AMS, 125(6):1755–1766, June 1997.
[7] Cris Moore. Quasi-linear cellular automata. Physica D, 103:100–132, 1997.
[8] Karl Petersen. Ergodic Theory. Cambridge University Press, New York, 1989.
[9] Hala O. Pflugfelder. Quasigroups and Loops: Introduction, volume 7 of Sigma Series in Pure Math B. Heldermann Verlag, Berlin, 1990.
[10] M. Pivato. Multiplicative cellular automata on nilpotent groups: Structure, entropy, and asymptotics. Journal of Statistical Physics, 110(1/2):247–267, January 2003.
[11] Laurent Schwartz. Lectures on disintegration of measures. Tata Institute of Fundamental Research, Bombay, 1975.
[12] Mark A. Shereshevsky. Ergodic properties of certain surjective cellular automata. Monatshefte für Mathematik, 114:305–316, 1992.