On the difference between Poincaré and Lorentz gravity

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Abstract

The Poincaré invariance of GR is usually interpreted as Lorentz invariance plus diffeomorphism invariance. In this paper, by introducing the local inertial coordinates (LIC), it is shown that a theory with Lorentz and diffeomorphism invariance is not necessarily Poincaré invariant. Actually, the energy-momentum conservation is violated there. On the other hand, with the help of the LIC, the Poincaré invariance is reinterpreted as an internal symmetry. In this formalism, the conservation law is derived, which has not been sufficiently explored before.

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1 Introduction

It is known from Utiyama’s paper [1] that GR can be viewed as a gauge theory of the Lorentz group. Later the diffeomorphism invariance is identified to the translational invariance, then GR is found to be a gauge theory of the Poincaré group [2]. This interpretation of Poincaré invariance as a combination of Lorentz invariance and diffeomorphism invariance becomes the standard one in the Poincaré gauge theory of gravity [3–5]. However, the diffeomorphism invariance is not always equal to the translational invariance. For example, they are different in the de Sitter (dS) gauge theory of gravity. In fact, the diffeomorphism symmetry does not correspond to any conservation law in dS gravity [6]. Then one would wonder whether the identification of diffeomorphism with translation in Poincaré gravity is also problematic.

In this paper, we show that a theory with Lorentz invariance and diffeomorphism invariance is not necessarily Poincaré invariant. We call such a theory the Lorentz gauge theory of gravity. Just like the case of dS gravity, the diffeomorphism symmetry does not correspond to any conservation law in Lorentz gravity. There exists only the angular momentum (AM) conservation with respect to the Lorentz symmetry, in other words, the energy-momentum (EM) conservation is absent.

Then how to interpret the Poincaré invariance when it is present? The answer lies in the construction of the Lorentz gravity. That is to introduce a vector field whose components may be called the local inertial coordinates (LIC). The prototype of the LIC is Cartan’s radius-vector field [4, 7], which is $GL(n, \mathbb{R})$ covariant. The dS/AdS/Poincaré-covariant LIC are first given by Guo [8], West [9] and Pilch [10], respectively. See also

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Refs. [11, 12] for a proof of their existence on an arbitrary spacetime. With the help of the LIC, the Poincaré invariance can be interpreted as an internal symmetry, just like the Lorentz symmetry. In this formalism, the fundamental variables are the Poincaré connection and LIC. In the Lorentz gauge, the LIC are fixed, and the Poincaré connection turns out to be a combination of the Lorentz connection and tetrad field, which are the traditional variables for Poincaré gravity. For completeness, we also compute the conservation law with respect to the Poincaré symmetry in this formalism, which reduces to the ordinary one [2] in the Lorentz gauge. Although Kawai [13] has attempted to do this, his result does not completely respect the spirit of this formalism. The key point is that the Poincaré-covariant derivative has not entered Kawai’s conservation law, and thus the AM conservation has not been formulated into a neat form, which will be given here.

The paper is organized as follows. In section 2, we construct the Lorentz gravity and derive the conservation law. In section 3, the same thing is done except changing the Lorentz group to the Poincaré group. In section 4, some remarks on the different formalisms and gauge groups of gravity are presented.

2 Lorentz gravity

2.1 Lorentz gravity from the gauge principle

The Lorentz gravity is a gauge theory of the Lorentz group. Recall that in Weyl’s gauge theory, the gauge field is introduced to localize a global symmetry [14]. First consider a classical field theory with global Lorentz invariance and diffeomorphism invariance. The action integral and Lagrangian function of this matter field are as follows:

\[ S_M = \int_{\Omega} d^4y \sqrt{-g} L_M, \quad L_M = L_M(\psi, \hat{\nabla}_a \psi, c.c., x^\alpha, \hat{\nabla}_a x^\alpha), \]

where \( \Omega \) is an arbitrary domain of the flat spacetime \( \mathcal{M}_0 \), \( \{y^\mu\} \) is an arbitrary coordinate system on \( \Omega \), \( g \) is the determinant of the Minkowski metric \( g_{\mu\nu} \), \( \psi \) is the matter field, \( \hat{\nabla}_a \) is the metric-compatible and torsion-free derivative, \( a \) is an abstract index [15, 16], which shows that the quantity is independent of coordinate choice, and can be transformed into any tetrad or coordinate index by taking the corresponding component, \( c.c. \) denotes the complex conjugate, and \( x^\alpha \) are inertial coordinates. The theory is assumed to be Lorentz invariant, i.e., \( S_M \) is invariant under the transformation:

\[ \psi \rightarrow T(h)\psi, \quad x^\alpha \rightarrow h^\alpha_\beta x^\beta, \]

where \( h = h^\alpha_\beta \) is an element of the Lorentz group \( SO(1,3) \), and \( T \) is the representation of \( SO(1,3) \) associated with the matter field \( \psi \). Note that the Minkowski metric

\[ g_{ab} = \eta_{\alpha\beta}(\hat{\nabla}_a x^\alpha)(\hat{\nabla}_b x^\beta), \]

where \( \eta_{\alpha\beta} = \text{diag}(-1,1,1,1) \). It is Lorentz invariant, so as \( \sqrt{-g} \). Also, the theory is supposed to be diffeomorphism invariant in the sense that \( S_M \) is independent of the choice of \( \{y^\mu\} \) and invariant under the transformation:

\[ \Omega \rightarrow \phi[\Omega], \quad \psi \rightarrow \phi_\ast \psi, \quad x^\alpha \rightarrow \phi_\ast x^\beta, \]
where $\phi$ is a diffeomorphism, and $\phi_*$ denotes the pushforward by it: $(\phi_\ast \psi)(\phi x) = \psi(x)$, $\forall x \in M_0$. We give an example of such a theory: the Dirac Lagrangian

$$L_M = -\frac{1}{2}\hat{i}(\bar{\psi}\gamma^a\nabla_a\psi - \text{c.c.}) + i m \bar{\psi}\psi,$$  \hspace{1cm} (5)$$

where $\hat{i}$ is the imaginary unit, $\gamma^a = \gamma^\alpha(\partial/\partial x^\alpha)^a$, and $\gamma^\alpha$ are Dirac matrices.

The localization of the above theory is to replace $h$ in Eq. (2) by a function valued at $SO(1,3)$. To achieve this, introduce a connection 1-form $\Gamma^\alpha_{\beta a}$ valued at $so(1,3)$, i.e., subject to $\Gamma^\alpha_{\beta a} = -\Gamma^\alpha_{\beta a}$. Then modify $\nabla_a\psi$ and $\nabla_a x^\alpha$ to be

$$D_a\psi = \tilde{\nabla}_a\psi + T_{\beta}^\alpha\Gamma^\alpha_{\beta a}\psi,$$  \hspace{1cm} (6)$$

$$D_a x^\alpha = \tilde{\nabla}_a x^\alpha + \Gamma^\alpha_{\beta a}x^\beta,$$  \hspace{1cm} (7)$$

where $T_{\alpha}^\beta$ are representations of the Lorentz generators. It can be checked that $S_M$ is invariant under Eq. (2) together with the connection transformation:

$$\Gamma^\alpha_{\beta a} \rightarrow h^\alpha_{\gamma} \Gamma^\gamma_{\beta a}(h^{-1})^\delta_{\beta} + h^\alpha_{\gamma} \tilde{\nabla}_a(h^{-1})^\gamma_{\beta}.$$  \hspace{1cm} (8)$$

Then we say that the theory is locally Lorentz invariant. Moreover, $S_M$ is still independent of the choice of $\{y^\mu\}$, and invariant under Eq. (4) together with $\Gamma^\alpha_{\beta a} \rightarrow \phi_*\Gamma^\alpha_{\beta a}$. In this sense, we say that the theory is still diffeomorphism invariant. Also note that the metric (3) is modified to be

$$g_{ab} = \eta_{\alpha\beta}(D_a x^\alpha)(D_b x^\beta),$$  \hspace{1cm} (9)$$

which is not necessarily flat. Accordingly, the $x^\alpha$ are no longer inertial coordinates. They become the components of some vector field and may be called the local inertial coordinates (LIC). The geometrical meaning of $\Gamma^\alpha_{\beta a}$ and $x^\alpha$ can be read off from Eqs. (8)–(9): $e^\alpha_a \equiv D_a x^\alpha$ is an orthonormal co-tetrad field, and $\Gamma^\alpha_{\beta a}$ is just the spacetime connection which defines a metric-compatible derivative $\nabla_a$ by $e^a_b\nabla_a e^b_b = \Gamma^\alpha_{\beta a}$.

The last step of the construction of Lorentz gravity is to determine $\Gamma^\alpha_{\beta a}$ dynamically, i.e., to write down its action integral $S_G$, which may be defined as

$$S_G = \int_\Omega d^4y\sqrt{-g} L_G, \quad L_G = L_G(x^\alpha, D_a x^\alpha, R^\alpha_{\beta ab}),$$  \hspace{1cm} (10)$$

where

$$R^\alpha_{\beta ab} = d_a \Gamma^\alpha_{\beta b} + \Gamma^\alpha_{\gamma a} \wedge \Gamma^\gamma_{\beta b}$$  \hspace{1cm} (11)$$

is the curvature 2-form of $\Gamma^\alpha_{\beta a}$, and $d_a$ is the exterior derivative defined by, e.g., $d_a \Gamma^\alpha_{\beta b} = 2\tilde{\nabla}_a \Gamma^\alpha_{[\beta b]} - \tilde{\nabla}_b \Gamma^\alpha_{\beta a}$. The gravitational field equations are given by $V^\alpha_{\beta a} \equiv \delta S/\delta \Gamma^\alpha_{\beta a} = 0$ and $V \equiv \delta S/\delta x^\alpha = 0$, where $S = S_M + S_G$. It follows from a direct computation that

$$V^\alpha_{\beta a} = \frac{\partial L}{\partial D_a \psi} T_{\alpha}^\beta \psi + \text{c.c.} + 2D_b \frac{\partial L}{\partial R^\alpha_{\beta ab}} + \left( \frac{\partial L}{\partial D_a x^\alpha} + L D^a x^\alpha \right) \cdot x^\beta,$$  \hspace{1cm} (12)$$

$$V^\alpha_a = \frac{\partial L}{\partial x^\alpha} - D_a \left( \frac{\partial L}{\partial D_a x^\alpha} + L D^a x^\alpha \right),$$  \hspace{1cm} (13)$$

where $L = L_M + L_G$, and $(D^a x^\alpha) \cdot x^\beta = (D^a x^\alpha) \cdot x_\gamma \eta^{\gamma \beta}$. 


2.2 Noether’s theorem in Lorentz gravity

Now let us generalize Noether’s theorem [17] to Lorentz gravity, i.e., find out the conservation laws corresponding to the Lorentz and diffeomorphism symmetries. Summarizing the results in the last subsection, the action integral and Lagrangian function of the coupling system of a matter field and a Lorentz gravitational field are as follows:

\[
S = \int_{\Omega} d^4y \sqrt{-g} \cdot \mathcal{L} = \mathcal{L}(\psi, D_a \psi, c.c., x^\alpha, D_a x^\alpha, R^\alpha_{\beta ab}).
\]  

(14)

The action \(S\) is independent of \(\{y^\mu\}\), and invariant under the transformation

\[
\Omega \rightarrow \phi[\Omega], \quad \psi \rightarrow T(h)\phi_\ast \psi, \quad x^\alpha \rightarrow h^\alpha_\beta \phi_\ast x^\beta,
\]

\[
\Gamma^\alpha_{\beta a} \rightarrow h^\alpha_\gamma (\phi_\ast \Gamma_{\gamma \beta a})(h^{-1})^\delta_\beta + h^\alpha_\gamma \hat{\nabla}_a (h^{-1})^\gamma_\beta.
\]

(15)

To derive the conservation law, vary \(\phi\) and \(h\) to the one-parameter groups \(\{\phi_t\}\) and \(\{h_t\}\). Denote \((d/dt)|_{t=0}\) by \(\delta\), then it follows from the chain rule that

\[
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial D_a \psi} \delta D_a \psi + c.c. + \frac{\partial \mathcal{L}}{\partial x^\alpha} \delta x^\alpha
\]

\[+ \frac{\partial \mathcal{L}}{\partial D_a x^\alpha} \delta D_a x^\alpha + \frac{\partial \mathcal{L}}{\partial R^\alpha_{\beta ab}} \delta R^\alpha_{\beta ab}.
\]

(16)

The variations \(\delta \mathcal{L}, \delta \psi, etc.\) can be expressed in terms of the generators of \(\{\phi_t\}\) and \(\{h_t\}\), denoted by \(v^a\) and \(A^a_\beta\). The vector field \(v^a\) at any point \(p\) of the spacetime is equal to the tangent vector of the curve \(\phi_t p\), and the \(so(1,3)\)-valued function \(A^a_\beta = \delta h^a_\beta\). Note that the diffeomorphism in Eq. (15) is gauge dependent, i.e., \(\phi\) and \(h\) do not commute. It seems that a gauge-independent diffeomorphism is more natural to be a fundamental symmetry transformation, which can be defined by Eq. (15) with \(A^a_\beta = -\Gamma^a_{\beta a}v^a\). This transformation is interpreted as a translation in Poincaré gravity [3]. As will be shown later, this interpretation does not hold in the present framework. Generally, set \(A^a_\beta = B^a_\beta - \Gamma^a_{\beta a}v^a\), where \(B^a_\beta\) is an \(so(1,3)\)-valued function. Then \(B^a_\beta\) stands for a Lorentz rotation, and \(v^a\) a gauge-independent diffeomorphism. Now it is ready to write down the variations \(\delta \mathcal{L}, \delta \psi, etc.\) in Eq. (16). The result is:

\[
\delta \mathcal{L} = -v^a \hat{\nabla}_a \mathcal{L},
\]

\[
\delta \psi = B^a_\alpha T^\alpha_\beta \psi - v^a D_a \psi, \quad \delta x^a = B^a_\beta x^\beta - v^a D_a x^\alpha,
\]

\[
\delta D_a \psi = B^a_\alpha T^\alpha_\beta D_a \psi - v^b D_b D_a \psi - (D_b \psi) \hat{\nabla}_a v^b,
\]

\[
\delta D_a x^\alpha = B^a_\beta D_a x^\beta - v^b D_b D_a x^\alpha - (D_b x^\alpha) \hat{\nabla}_a v^b,
\]

\[
\delta R^\alpha_{\beta ab} = [B^c_\gamma, R^\gamma_{\beta ab}] - v^c D_c R^\alpha_{\beta ab} - R^\alpha_{\beta cb} \hat{\nabla}_a v^c - R^\alpha_{\beta ac} \hat{\nabla}_b v^c.
\]

(17)

Suppose that the matter field equation \(\delta S/\delta \psi = 0\) is satisfied, then substitution of Eq. (17) into Eq. (16) leads to

\[
\hat{\nabla}_b \mathcal{L} = \left( D_a \frac{\partial \mathcal{L}}{\partial D_a \psi} \right) D_b \psi + \frac{\partial \mathcal{L}}{\partial D_a \psi} D_b D_a \psi + c.c. + \frac{\partial \mathcal{L}}{\partial x^\alpha} D_b x^\alpha
\]

\[+ \frac{\partial \mathcal{L}}{\partial D_a x^\alpha} D_b D_a x^\alpha + \frac{\partial \mathcal{L}}{\partial R^\alpha_{\beta ac}} D_b R^\alpha_{\beta ac}.
\]

(18)
\[ 0 = \frac{\partial L}{\partial D_\alpha} D_\beta \psi + c.c. + \frac{\partial L}{\partial D_\alpha x^\alpha} D_\beta x^\alpha + 2 \frac{\partial L}{\partial R^\alpha_{\beta\gamma}} R^\alpha_{\beta\gamma c.c.}, \]  
(19)

\[ 0 = \left( D_\alpha \frac{\partial L}{\partial D_\beta \psi} \right) T_\alpha^\beta \psi + \frac{\partial L}{\partial D_\alpha \psi} T_\alpha^\beta D_\beta \psi + c.c. + \frac{\partial L}{\partial x^{[\alpha}} x^{\beta]} + \frac{\partial L}{\partial R^\alpha_{\gamma\beta a}} R^\beta_{\gamma a} - \frac{\partial L}{\partial R^\gamma_{\beta a} R^\alpha_{\beta a}}, \]  
(20)

where the arbitrariness of \( v^a \), \( \nabla_a v^b \) and \( B^\alpha_{\beta} \) at any given point is used.

The conservation law is just hidden in the identities (18)-(20). To see this, define the EM tensor \( \Sigma^a_b = (\partial L / \partial D_\alpha x^\alpha) D_\beta x^\alpha + \delta^a_b \), and the spin tensor \( \tau^\beta_a = (\partial L / \partial D_\alpha \psi) T_\alpha^\beta \psi + c.c. + 2D_\alpha (\partial L / \partial R^\alpha_{\beta\gamma}) \). Then Eq. (19) implies that \( \Sigma^a_b = -(\partial L / \partial D_\alpha \psi) D_\beta \psi + c.c. - 2(\partial L / \partial R^\alpha_{\beta\gamma}) R^\alpha_{\beta\gamma c.c.} + \delta^a_b \). Also note that \([D_\alpha, D_\beta] T = \sum_i R^c_{\beta\gamma} T^d - \sum_i T^c_i R^c_{\beta\gamma} + \sum U \mathcal{R}_{\gamma}\alpha \beta j T^j - \sum L \mathcal{U}_{\gamma}\alpha \beta j T^j \), where \( T \) is a tensor field valued at some tensor space of \( V_R \), \( V_R \) is a representation space of \( SO(1, 3) \), \( i, j \) are the indices of \( V_R \), \( \sum_U \) denotes a sum for the upper indices of \( T \), and \( \sum_L \) for the lower indices. \( R^c_{\beta\gamma} \) is the curvature tensor of \( \nabla_a \), \( \mathcal{R}_{\gamma}\alpha \beta j \) is the representation of \( R^\alpha_{\beta\gamma} \), and the indices of \( T \) are omitted except for those interacting with the curvature. It is also instructive to note that \( D_\alpha T_\beta^\gamma = \nabla_\alpha T_\beta^\gamma + [\omega_\alpha, T_\beta^\gamma] = -2T_\alpha^{[\gamma} T_\beta^{\gamma]} \), \( D_\alpha D_\beta x^\alpha = K^a_{\alpha b} \), and \( D_\alpha R^\gamma_{\beta\gamma a} = 0 \), where \( \omega_\alpha = \Gamma^\alpha_{\beta\gamma} T_\beta^\gamma, K^c_{\alpha b} = (S^c_{\alpha b} + S^c_{b\alpha} + S^c_{\alpha b})/2 \) is the contorsion tensor of \( \nabla_a \), and \( S^a_{\alpha b} = d_a e^a_{\beta} + \Gamma^\alpha_{\beta\gamma} \wedge e^\beta_{\gamma} \) is the torsion 2-form. With the help of the above definitions and formulas, Eqs. (18) and (20) can be reformulated into

\[ \nabla_a \Sigma^a_b = -\Sigma^a_c K^c_{\alpha b} - \tau^c_{\alpha d} R^c_{\beta d} + \frac{\partial L}{\partial x^{[\alpha}} x^{\beta]} e^a_{\beta}, \]  
(21)

\[ D_\alpha \tau^\beta_a = -\Sigma^\alpha_{[\beta} - \frac{\partial L}{\partial x^{[\alpha}} x^{\beta]}. \]  
(22)

In the SR limit with \( R^\alpha_{\beta\gamma a} = 0 \) and so \( S^a_{\alpha b} = 0 \), Eq. (21) reduces to \( \partial_a \Sigma^a_b = \partial L / \partial x^\alpha \), which shows that the EM current is not conserved unless \( \partial L / \partial x^\alpha = 0 \). Note that \( \partial L / \partial x^\alpha = 0 \) is not forced to hold by the diffeomorphism invariance defined here. On the other hand, the AM current is conserved. To prove this, first define the orbital AM current \( \Sigma^a_b = \Sigma^a_{[\alpha} x^{\beta]} \). Then from Eq. (12), \( V^a_{\alpha} = \tau^\beta_a + \Sigma^\alpha_{[\beta} \) is just the total AM current. Moreover, the combination of Eqs. (21) and (22) leads to \( D_\alpha V^a_{\alpha} = (\Sigma^a_{[\alpha} S^c_{\beta a} + \tau^c_{\alpha d} R^c_{\beta d}) e^b_{[\alpha} x^{\beta]} \). In the SR limit, this equation reduces to \( \partial_a V^a_{\alpha} = 0 \), which is just the AM conservation law. Finally, according to Eqs. (13) and (21), we have \( V_a = \partial / \partial x^\alpha - D_\alpha \Sigma^a_{\alpha} = (\Sigma^a_{[\alpha} S^c_{\beta a} + \tau^c_{\alpha d} R^c_{\beta d}) e^b_{[\alpha} x^{\beta]} \), and hence the AM conservation law has the following elegant form:

\[ D_\alpha V^a_{\alpha} = x_{[\alpha} V_{\beta]}. \]  
(23)

In the SR limit, \( V_a = 0 \) is automatically satisfied, and thus the \( x^\alpha \) can be viewed as an auxiliary field. In conclusion, the Lorentz and diffeomorphism symmetries in Lorentz gravity only result in the AM conservation (23). This implies that the diffeomorphism invariance defined here does not lead to the EM conservation, and therefore it should not stand for the translational invariance.
3 Poincaré gravity

3.1 Poincaré gravity from the gauge principle

The Poincaré gravity is a gauge theory of the Poincaré group. First consider a classical field theory with global Poincaré invariance and diffeomorphism invariance. The action integral and Lagrangian function of this matter field are as follows:

\[ S_M = \int \Omega d^4 y \mathcal{L}_M \sqrt{-g}, \quad \mathcal{L}_M = \mathcal{L}_M(\psi, \nabla_a \psi, c.c., \xi^A, \nabla_a \xi^A), \]  

(24)

where \( \xi^A = (\xi^\alpha, l) \), \( \xi^\alpha \) are inertial coordinates of the flat spacetime \( \mathcal{M}_0 \), and \( l \) is a constant with the dimension of length\(^1\), which may be seen as the 5th coordinate of \( \mathcal{M}_0 \) as a plane embedded in the 5d flat space. The theory is assumed to be Poincaré invariant, i.e., \( S_M \) is invariant under the transformation:

\[ \psi \to T(h)\psi, \quad \xi^A \to H_B^A \xi^B, \]  

(25)

where \( h = h^\alpha_\beta \) is an element of the Lorentz group \( SO(1, 3) \), \( T \) is the representation of \( SO(1, 3) \) associated with the matter field \( \psi \), and \( H^A_B \) is a 5d representation of the Poincaré group, which satisfies \( H^\alpha_\beta = h^\alpha_\beta, \ H^4_\alpha = 0, \) and \( H^4_4 = 1 \), such that \( H^A_B \xi^B = (\xi^\alpha, l) \) with \( \xi^\alpha = h^\alpha_\beta \xi^\beta + H^4_\alpha \cdot l \). Note that the matter field is valued at a representation space of \( SO(1, 3) \), rather than that of the full Poincaré group. We make this choice just for naturalness. The reader who is interested at the latter kind of matter fields may refer to Ref. [18]. Notice that the Minkowski metric

\[ g_{ab} = \eta_{AB}(\nabla_a \xi^A)(\nabla_b \xi^B) \]  

(26)

is Poincaré invariant, where \( \eta_{AB} = \text{diag}(-1, 1 \cdots 1) \). Also, the theory is supposed to be diffeomorphism invariant in the sense that \( S_M \) is independent of the choice of \( \{y^\mu\} \) and invariant under the transformation:

\[ \Omega \to \phi[\Omega], \quad \psi \to \phi_\ast \psi, \quad \xi^A \to \phi_\ast \xi^A, \]  

(27)

where \( \phi \) is a diffeomorphism. Recall that an example (5) is given for the special theory of Lorentz gravity. As a matter of fact, the Dirac theory (5) is also Poincaré invariant and diffeomorphism invariant in the above sense.

The localization of the above theory is to replace \( H^A_B \) in Eq. (25) by a function valued at the Poincaré group. To that end, introduce a connection 1-form \( \Omega^A_{Ba} \) valued at the Poincaré algebra, i.e., subject to the condition that \( \Omega^\alpha_{\beta a} = -\Omega^\alpha_{\beta a} \) and \( \Omega^4_{Ba} = 0 \). Then modify \( \nabla_a \psi \) and \( \nabla_a \xi^A \) to be

\[ D_a \psi = \nabla_a \psi + T^\beta_\alpha \Omega^\alpha_{\beta a} \psi, \]  

(28)

\[ D_a \xi^A = \nabla_a \xi^A + \Omega^A_{Ba} \xi^B, \]  

(29)

where \( T^\beta_\alpha \) are representations of the Lorentz generators. It can be checked that \( S_M \) is invariant under Eq. (25) together with the connection transformation:

\[ \Omega^A_{Ba} \to H^A_C \Omega^C_{Da}(H^{-1})^D_B + H^A_C \nabla_a (H^{-1})^C_B. \]  

(30)

\(^1\)In dS gravity, the analog of \( l \) is the radius of the internal dS space, which is linked to the cosmological constant by \( \Lambda = 3/l^2 \) [11]. But here \( l \) is an arbitrary constant without physical meaning.
Then we say that the theory is locally Poincaré invariant. Moreover, $S_M$ is still independent of the choice of $\{y^\mu\}$, and invariant under Eq. (27) together with $\Omega^A_{Ba} \rightarrow \phi_s\Omega^A_{Ba}$. In this sense we say that the theory is still diffeomorphism invariant. Also note that the metric (26) is modified to be [10, 11]

$$g_{ab} = \eta_{AB}(D_a\xi^A)(D_b\xi^B),$$

which is not necessarily flat. Accordingly, the $\xi^A$ are no longer inertial coordinates. They become the components of some 5-vector field and may be called the (5d) LIC. The geometrical meaning of $\Omega^A_{Ba}$ and $\xi^A$ can be read off from Eqs. (30)–(31): $e^a_c \equiv D_a\xi^a$ is an orthonormal co-tetrad field, and $\Omega^a_{\beta a}$ is just the spacetime connection which defines a metric-compatible derivative $\nabla_a$ by $e^a_b \nabla_a e^b = \Omega^a_{\beta a}$. Notice that in the Lorentz gauge with $\xi^A = (0 \cdots 0, l)$, $D_a\xi^a = \Omega^a_{4a} = l$ and so $\Omega^a_{4a} = e^a_a \cdot l^{-1}$.

The last step of the construction of Poincaré gravity is to determine $\Omega^A_{Ba}$ dynamically, i.e., to write down its action integral $S_G$, which may be defined as

$$S_G = \int_\Omega d^4y \mathcal{L}_G \sqrt{-g}, \quad \mathcal{L}_G = \mathcal{L}_G(\xi^A, D_a\xi^A, \mathcal{F}^A_{Bab}),$$

where

$$\mathcal{F}^A_{Bab} = D_a\Omega^A_{Bb} + \Omega^A_{Ca} \wedge \Omega^C_{Bb}$$

is the curvature 2-form of $\Omega^A_{Ba}$. It can be verified that $\mathcal{F}^a_{\beta ab} = R^a_{\beta ab}$ is the Lorentz curvature (11), and in the Lorentz gauge $\mathcal{F}^a_{4ab} = S^a_{ab} \cdot l^{-1}$ is the torsion 2-form. Finally, the gravitational field equations are given by $V^a_{A_{Ba}} \equiv \delta S/\delta \Omega^A_{Ba} = 0$ and $V^a_4 \equiv \delta S/\delta \xi^A = 0$, where $S = S_M + S_G$. On account of $\Omega^4_{Ba} = 0$ and $\xi^4 = l$, it follows that $V^a_4 = 0$ and $V^4 = 0$. Moreover, it can be shown that

$$V^a_{\beta a} = \frac{\partial \mathcal{L}}{\partial D_a\psi} T^\beta_{\psi} + c.c. + 2D_b \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\beta ab}} + \left( \frac{\partial \mathcal{L}}{\partial D_a\xi^a} + \mathcal{L}_e_{[\alpha}^a \cdot \xi_{\beta]} \right),$$

$$V^a_{4a} = 2D_b \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{4ab}} + \left( \frac{\partial \mathcal{L}}{\partial D_a\xi^a} + \mathcal{L}_e^a \right) \cdot l,$$

$$V^a_a = \frac{\partial \mathcal{L}}{\partial \xi^a} - D_a \left( \frac{\partial \mathcal{L}}{\partial D_a\xi^a} + \mathcal{L}_e^a \right),$$

where $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_G$, and $D_b(\partial \mathcal{L}/\partial \mathcal{F}_{[\alpha \beta ab]} = D_b(\partial \mathcal{L}/\partial \mathcal{F}_{\gamma ab})\eta_{\beta \gamma} \eta_{\delta \beta}$. Note that although $\partial \mathcal{L}/\partial \mathcal{F}_{\beta ab}$ is anti-symmetric about $\alpha$ and $\beta$, its Poincaré-covariant derivative is not necessarily anti-symmetric.

### 3.2 Noether’s theorem in Poincaré gravity

Now let us generalize Noether’s theorem to Poincaré gravity, i.e., find out the conservation laws corresponding to the Poincaré and diffeomorphism symmetries. Summarizing the results in the last subsection, the action integral and Lagrangian function of the coupling system of a matter field and a Poincaré gravitational field are as follows:

$$S = \int_\Omega d^4y \mathcal{L} \sqrt{-g}, \quad \mathcal{L} = \mathcal{L}(\psi, D_a\psi, c.c., \xi^A, D_a\xi^A, \mathcal{F}^A_{Bab}).$$

The action $S$ is independent of $\{y^\mu\}$, and invariant under the transformation

$$\Omega \rightarrow \phi[\Omega], \ \psi \rightarrow T(h)\phi, \ \xi^A \rightarrow H^A B \phi \xi^B,$$

$$\Omega^A_{Ba} \rightarrow H^A C (\phi \Omega^C_{Da})(H^{-1})^D_B + H^A C \nabla_a (H^{-1})^C_B.$$  \hspace{1cm} (38)

To derive the conservation law, vary $\phi$ and $H$ to the one-parameter groups $\{\phi_t\}$ and $\{H_t\}$. Then it follows from the chain rule that

$$\delta S = \frac{\partial S}{\partial \psi} \delta \psi + \frac{\partial S}{\partial D_a \psi} \delta D_a \psi + c.c. + \frac{\partial S}{\partial \xi^A} \delta \xi^A + \frac{\partial S}{\partial \mathcal{F}^A_{Bab}} \delta \mathcal{F}^A_{Bab}.$$  \hspace{1cm} (39)

The variations $\delta S$, $\delta \psi$, etc. can be expressed in terms of the generators of $\{\phi_t\}$ and $\{H_t\}$, denoted by $v^a$ and $A^4 B$. As before, the diffeomorphism in Eq. (38) is gauge dependent, i.e., $\phi$ and $H$ do not commute. The gauge-independent diffeomorphism can be defined by Eq. (38) with $A^4_B = -\Omega^A_{Ba} v^a$. Generally, set $A^4_B = B^4_B - \Omega^A_{Ba} v^a$, where $B^4_B$ is a function valued at the Poincaré algebra. Then $B^4_B$ stands for a Poincaré transformation, and $v^a$ a gauge-independent diffeomorphism. Now it is ready to write down the variations $\delta S$, $\delta \psi$, etc. in Eq. (39). The result is: $\delta S = -v^a \nabla_a S$,

$$\delta \psi = B^a_{\alpha} T^a_{\alpha} \psi - v^a D_a \psi, \ \delta \xi^A = B^A_{B} \xi^B - v^a D_a \xi^A,$$

$$\delta D_a \psi = B^a_{\alpha} T^a_{\alpha} D_a \psi - v^b D_b D_a \psi - (D_b \psi) \nabla_a v^b,$$

$$\delta D_a \xi^A = B^A_{B} D_a \xi^B - v^b D_b D_a \xi^A - (D_b \xi^A) \nabla_a v^b,$$

$$\delta \mathcal{F}^A_{Bab} = [B^A_{C} \mathcal{F}^C_{Bab}] - v^c D_c \mathcal{F}^A_{Bab} - \mathcal{F}^A_{Bcb} \nabla_a v^c - \mathcal{F}^A_{Bac} \nabla_b v^c.$$  \hspace{1cm} (40)

Suppose that the matter field equation $\delta S/\delta \psi = 0$ is satisfied, then substitution of Eq. (40) into Eq. (39) leads to

$$\nabla_b S = \left( D_a \frac{\partial S}{\partial D_a \psi} \right) D_b \psi + \frac{\partial S}{\partial D_a \psi} D_b D_a \psi + c.c. + \frac{\partial S}{\partial \xi^A} D_b \xi^A + \frac{\partial S}{\partial \mathcal{F}^A_{Bac}} D_b \mathcal{F}^A_{Bac},$$  \hspace{1cm} (41)

$$0 = \frac{\partial S}{\partial D_a \psi} D_b \psi + c.c. + \frac{\partial S}{\partial \xi^A} D_b \xi^A + 2 \frac{\partial S}{\partial \mathcal{F}^A_{Bac}} \mathcal{F}^A_{Bbc},$$  \hspace{1cm} (42)

$$0 = \left( D_a \frac{\partial S}{\partial \xi^A} \right) T^a_{\alpha} \psi + \frac{\partial S}{\partial \xi^A} T^a_{\alpha} D_a \psi + c.c. + \frac{\partial S}{\partial \mathcal{F}^{\alpha}_{[\xi^A]}},$$

$$+ \frac{\partial S}{\partial D_a \xi^A} D_a \xi^A + \frac{\partial S}{\partial \mathcal{F}^{\alpha}_{[\xi^A]}} \mathcal{F}^{\alpha}_{[\xi^A]},$$

$$0 = \frac{\partial S}{\partial \xi^A} \cdot l - \frac{\partial S}{\partial \mathcal{F}^{\alpha}_{[\xi^A]}} \mathcal{F}^{\alpha}_{[\xi^A]},$$  \hspace{1cm} (44)

where the arbitrariness of $v^a$, $\nabla_a v^b$, $B^a_{\alpha}$ and $B^a_{\alpha}$ at any given point is used.

The conservation law is hidden in the identities (41)-(44). To see this, define the orbital EM tensor $\Sigma^a_b = (\partial S/\partial D_a \xi^A) D_b \xi^A + \partial S/\partial \xi^A$, and the Poincaré spin current $\tau^a_{Ba}$.
\[ \tau_{a}^{\beta a} = (\partial L / \partial D_a \psi) T_{a}^{\beta} \psi + \text{c.c.} + 2 D_b (\partial L / \partial F^{[\alpha}_{\beta a} b), \ \tau_{a}^{4a} = 2 D_b (\partial L / \partial F^{a}_{\beta 4a}, \ \tau_{a}^{4a} = 0. \] 

Then Eq. (42) implies that \[ \Sigma_{a}^{\alpha} = - (\partial L / \partial D_a \psi) D_b \psi + \text{c.c.} - 2 (\partial L / \partial F^{A}_{B a}) F^{A}_{B c d} + \delta \alpha_{b}. \] 

Also note that \[ [D_a, D_b] T = \sum_{i} R_{i}^{c d a b} T^{i} - \sum_{d} T_{d} R_{i}^{c d a b} + \sum_{i} \mathcal{F}_{j a b} T^{j} - \sum_{d} T_{d} \mathcal{F}_{j a b}, \] 

where \( T \) is a tensor field valued at some tensor space of \( V_{R} \). \( V_{R} \) is a representation space of the Poincaré group, \( i, j \) are the indices of \( V_{R} \), and \( \mathcal{F}_{j a b} \) is the representation of \( \mathcal{F}^{A}_{B a b} \). It is also instructive to note that \[ D_a T_{a}^{\beta} = - 2 T_{[a} \gamma \Omega_{b]}^{\beta} \gamma a, \] 

\[ D_{a} D_{b} \xi^{A} = (K_{b}^{a} \alpha, 0), \] 

and \( D_{[c} \mathcal{F}^{A}_{B a b]} = 0. \) With the help of the above definitions and formulas, Eqs. (41), (43) and (44) can be reformulated into

\[ \nabla_{a} \Sigma_{b}^{\alpha} = - \Sigma_{c}^{a} K_{c a b} - \tau_{A}^{B a} F^{A}_{B a b} + \frac{\partial L}{\partial \xi^{A}} D_{b} \xi^{A}, \] 

(45)

\[ D_{a} \tau_{[\alpha}^{\beta a} = - \Sigma_{[\alpha}^{\beta a} - \frac{\partial L}{\partial \xi^{a}} \xi^{\beta}], \] 

(46)

\[ D_{a} \tau_{a}^{4a} = - \frac{\partial L}{\partial \xi^{a}} \cdot l. \] 

(47)

Furthermore, define the 5d orbital AM current \( \Sigma_{A}^{B a} \): \[ \Sigma_{a}^{\alpha} = \Sigma_{a}^{\alpha} \xi^{\beta}, \] 

\[ \Sigma_{a}^{4a} = \xi_{a}, \] 

and \( \Sigma_{4}^{a} = 0. \) Then from Eqs. (34)–(35), \( V_{a}^{A} = \tau_{A}^{B a} + \Sigma_{a}^{B a} \) is just the total AM current. The combination of Eqs. (45) and (46) leads to \[ D_{a} V_{a}^{[\beta a} = (\Sigma_{c}^{a} S_{c a a}^{a} + \tau_{A}^{B a} f_{A a}^{a} F^{A}_{B a a} \cdot l. \] 

In the SR limit with \( F^{A}_{B a b} = 0 \), this equation reduces to \[ \partial_{a} V_{\alpha}^{\beta a} = 0, \] 

which is just the AM conservation. Moreover, the combination of Eqs. (45) and (47) yields \[ D_{a} V_{a}^{4a} = (\Sigma_{c}^{a} S_{c a a}^{a} + \tau_{A}^{B a} f_{A a}^{a} F^{A}_{B a a} \cdot l. \] 

In the SR limit, this equation becomes \( \partial_{a} \Sigma_{a}^{a} = 0, \) which is just the EM conservation. Finally, according to Eqs. (36) and (45), we have \[ V_{a} = \frac{\partial L}{\partial \xi^{a} - D_{a} \Sigma_{a}^{a} = \Sigma_{c}^{a} S_{C a a}^{a} + \tau_{A}^{B a} F^{A}_{B a a} = V_{a}^{a} F^{A}_{B a a}, \] 

and so the AM and EM conservation have the following elegant form:

\[ D_{a} V_{a}^{[\beta a} = - V_{[\alpha} \cdot \xi^{\beta}], \] 

(48)

\[ D_{a} V_{a}^{4a} = - V_{a} \cdot l. \] 

(49)

In the SR limit, \( V_{a} = 0 \) is automatically satisfied. In the Poincaré gravity, \( V_{a} = 0 \) as long as \( V_{a}^{B a} = 0 \). As a result, the \( \xi^{A} \) can be viewed as an auxiliary field in both cases. Generally, two gauge invariants can be defined from \( V_{a}^{B a} \) and \( V_{a} \): the EM tensor \( V_{b}^{a} = V_{a}^{4a} D_{b} \xi^{a} \cdot l^{-1} \), and \( V_{a} = V_{a} D_{a} \xi^{a} \). Note that \( V_{a}^{B a} \) is a vector field valued at the dual space of the Poincaré algebra, and hence \( V_{a}^{B a} \) is not Poincaré covariant. But in the Lorentz gauge, \( V_{a}^{B a} = \tau_{a}^{B a} \) is Lorentz covariant, and so a Lorentz-invariant spin tensor can be defined: \( \tau_{c}^{d a} = \tau_{c}^{d a} e_{c a} \xi^{a}. \) Then the conservation laws (48)–(49) reduce to \[ \nabla_{a} \tau_{a}^{B a} = - V_{a}^{B a} \] 

and \[ \nabla_{a} V_{a}^{B a} = - V_{a}^{B a} K^{c a b} - \tau_{c}^{d a} R^{c a b}, \] 

where \( \nabla_{a} \) is the Lorentz-covariant derivative, and \( V_{a} = V_{a} \cdot l. \) These two equations are the AM and EM conservation in the ordinary form [2–4]. In conclusion, Eqs. (48)–(49) generalize the ordinary conservation laws from the Lorentz gauge to the general gauge.

4 Remarks

It can be concluded from our analysis that the diffeomorphism invariance defined in the LIC formalism is different from the translational invariance. In the LIC formalism, the translational invariance is treated as an internal symmetry. This makes the gravitational
theory look more like a matter gauge theory. Also, the LIC formalism presents a unified framework for the Poincaré and dS gravity, such that both the Poincaré and dS invariance are independent of the diffeomorphism invariance.

Besides from the standard tetrad formalism and the LIC formalism presented here, there also exist the nonlinear realization [19, 20]/Cartan geometry [21] formalisms for the Poincaré gravity. In these formalisms, the dynamical variables are the nonlinear connection/Cartan connection, which project the linear connection into the Lorentz gauges. These formalisms may be useful for the situation with symmetry breaking. But in this paper, the Poincaré gravity is discussed in the Lagrangian level, in which there exists no symmetry breaking. Then the LIC formalism, with the dynamical variable being the linear connection, becomes a better choice.

Finally, I would like to give a remark on the choice of the gravitational gauge group. If the AM and EM conservation are required, the gauge group should be the dS/AdS/Poincaré group. Comparing the dS/AdS gravity and Poincaré gravity, I find that the former is more elegant because the AM and EM currents can be unified there. Actually, the AM and EM conservation (48)–(49) can be combined into a neat form as $D_a V^{aB} = \xi^A V_B$ in the dS/AdS gravity [6].

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