Coframe energy-momentum current. Algebraic properties.

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Abstract. The coframe (teleparallel) description of gravity is known as a viable alternative to GR. One of advantages of this model is the existence of a conserved energy-momentum current which is covariant under all symmetries of the three-parameter Lagrangian. In this paper we study the relation between the covector valued current and the energy-momentum tensor. Algebraic properties of the conserved current for different values of parameters are derived. It is shown that the tensor corresponding to the coframe current is traceless and, in contrast to the electromagnetic field, has in general a non vanishing antisymmetric part. The symmetric part is also non zero for all values of the parameters. Consequently, the conserved current involves the energy-momentum as well as the rotational (spin) properties of the field.

Keywords: Teleparallel equivalent of GR, conserved current, energy-momentum tensor.

1. Introduction. Coframe gravity

The teleparallel description of gravity has been studied for a long time. It has recently evoked considerable interest for various reasons:

1. This theory is a natural application of gauge principles to spacetime symmetries [1]-[4].
2. The teleparallel Lagrangian constitutes a self consistent sector of metric-affine gravity [5], i.e., the gauge theory of the 4-dimensional affine group in the presence of a metric.
3. It represents a 1-parameter family of viable models of gravity, all with the Schwarzschild solution for the spherically symmetric 1-body problem [6], [7].
4. In comparison with the standard GR the teleparallel theory has an improved behavior of its Lagrangian insofar as it admits a covariantly defined energy-momentum current [8], [9].
5. The canonical analysis of the teleparallel construction has remarkable advantages in comparison with that of standard GR [10], [11], [12].

6. The teleparallel technique was successfully applied for a transparent treatment of Ashtekar’s complex variables [13] and for the tensorial proof of the positivity of the energy in GR [14].

The teleparallel theory is usually considered in a tensorial representation. A frame and a non-symmetric teleparallel connection (via its torsion and contortion tensors) play the role of the basic dynamical variables. The connection between these tensors is given by a constraint equation which represents the vanishing of the Riemannian curvature. The teleparallel theory allows also for an alternative coframe representation [6], [9], which follows the general construction of metric-affine gravity [5]. In this representation the field equation and the conserved energy-momentum current turn out to be completely similar to those of the Maxwell-Yang-Mills theory. Such an analogy may be useful for the transformation of the Yang-Mills technique to gravity. Furthermore, it lays a common basic framework for gravitational and electromagnetic fields.

Let us briefly recall the coframe approach to gravity:

Consider a smooth coframe field \( \{ \vartheta^a(x), x \in M, a = 0, 1, 2, 3 \} \) defined on a differential manifold \( M \). The coframe constitutes at every point \( x \in M \) a set of four linearly independent 1-forms, i.e., a basis of the cotangent vector space \( T^*M_x \). The 1-forms \( \vartheta^a \) are considered to be pseudo-orthonormal. This assumption fixes a metric on \( M \) which is represented by the coframe as

\[
g = \eta_{ab} \vartheta^a \otimes \vartheta^b, \tag{1.1}\]

where \( \eta_{ab} = \text{diag}(-1, 1, 1, 1) \) is the Lorentzian metric. Thus, the coframe field \( \vartheta^a \) plays the role of a dynamical variable. This is in contrast to metric gravity of GR based on the metric tensor \( g \). The dimension of the coframe 1-forms are of length in accordance with the length square dimension of the metric (1.1). We will use the geometrized units system where \( G = c = \hbar = 1 \).

The most general Lagrangian 4-form for the coframe field (minimally coupled to matter) that is quadratic in the first order derivatives is

\[
\mathcal{L} = \frac{1}{2} \mathcal{C}^a \wedge \ast \mathcal{F}^a + \text{ (mat) } \mathcal{L}, \tag{1.2}\]

where \( \mathcal{C}^a := d\vartheta^a \) is a 2-form of the coframe field strength. The 2-form \( \mathcal{F}^a \) is a linear combination of three 2-forms

\[
\mathcal{F}^a := \rho_1 \mathcal{C}^a + \rho_2 e^a_m (\vartheta^m \wedge \mathcal{C}_m) + \rho_3 e_m^a (\vartheta^a \wedge \mathcal{C}_m). \tag{1.3}\]

For the representation of \( \mathcal{F}_a \) via the irreducible pieces see [5]. The free parameters \( \rho_i \) are dimensionless. For instance, the set

\[
\rho_1 = 0, \quad \rho_2 + 2\rho_3 = 0
\]

represents the teleparallel equivalent of general relativity - \( GR_{\parallel} \). It is well known [3], however, that even the more general set of parameters:

\[
\rho_1 = 0, \quad \rho_2 - \text{ arbitrary}, \quad \rho_3 \neq 0
\]
produces just a unique static spherically symmetric coframe solution. It corresponds to the Schwarzschild metric [7].

The Lagrangian (1.2) is manifestly diffeomorphic invariant. It is also invariant under global transformations of the coframe \( \vartheta^a \rightarrow A^a_b \vartheta^b \), where \( A^a_b \in SO(1,3) \). The GR is a unique local Lorentz invariant teleparallel model, where the transformation matrix \( A^a_b \) is permitted to be a function of a point \( x \in M \).

The variation of the Lagrangian (1.2) yields the field equation of the Yang-Mills type

\[
d \ast F_a = \mathcal{T}_a + \text{(mat)} \mathcal{T}_a, \tag{1.4}
\]

where \( \mathcal{T}_a \) is a covector-valued 3-form constructed from the first order derivatives of the coframe:

\[
\mathcal{T}_a = (e_a \mathcal{L}^m) \wedge \ast F_m - e_a \mathcal{L}. \tag{1.5}
\]

\((\text{mat}) \mathcal{T}_a\) represents the energy-momentum current of the material field. Again, this expression (1.5) is similar to the energy-momentum current of the Yang-Mills field.

A straightforward consequence of the field equation (1.4) is the conservation law

\[
d(\mathcal{T}_a + \text{(mat)} \mathcal{T}_a) = 0. \tag{1.6}
\]

Observe that, in contrast to standard GR, the conserved value is the total current of the system, not the material current itself. Certainly, this situation is more physical.

The conserved current (1.5) is local, i.e., constructed from the fields and their derivatives at a point. It is invariant under the diffeomorphisms of the manifold and transforms as a covector under global Lorentz transformations of the coframe. Certainly, it is not a tensor if local Lorentz transformations of the coframe are applied. Note, however, that for general values of parameters the coframe Lagrangian also does not invariant under such transformations. Thus the current \( \mathcal{T}_a \) obeys all the symmetries of the Lagrangian. It is proved [3] to be related to the diffeomorphism invariance symmetry of the Lagrangian (1.2). Consequently, (1.5) represents the energy-momentum current of the coframe field.

In the present paper we study the algebraic properties of the covector-valued 3-form current \( \mathcal{T}_a \) and its relation to the energy-momentum tensor. The tensor corresponding to the current (1.5) is shown to be traceless for all values of the parameters \( \rho_i \). This is similar to the energy-momentum current of the electromagnetic field. Correspondingly, the gravitons in all coframe models are massless as in standard GR. For the trace of the gauge energy-momentum, see [5].

It is proved, in contrast to electromagnetic field, that the tensor corresponding to the coframe current has in general non vanishing symmetric and antisymmetric parts. By the Poincaré gauge theory of gravity the antisymmetric part of the current is connected to the rotational (spin) properties of the field [5]. It is equivalent to a scalar valued 2-form \( S = e_a \mathcal{T}^a \).

The coframe current preserves the symmetries of the Lagrangian for all values of the parameters except for the special case \( \rho_2 + 2\rho_3 = 0 \), which corresponds to standard GR.

We briefly discuss how the Einstein theory (in the coframe representation) is embedded in the family of viable coframe models.
2. Algebraic relations

In this section we describe some algebraic relations which will be useful for a treatment of the coframe current (1.5). There is no real advantage in restricting to dimension 4 and to the Lorentzian signature, so we work in this section (only) on a manifold of an arbitrary dimension and signature.

2.1. Basic facts

Let an $n$-dimensional manifold $M$ endowed with a coframe field $\{\vartheta^a(x), \ x \in M, \ a = 1, ..., n\}$ be given. The coframe is accepted to be ”pseudo-orthonormal”, i.e., the metric on $M$ is represented via the coframe as

$$g = \eta_{ab} \vartheta^a \otimes \vartheta^b,$$

(2.1)

where $\eta_{ab} = \text{diag}(-1, ..., -1, 1, ..., 1)$ is the pseudo-Euclidean metric. Let the number of negative entries in $\eta_{ab}$ be equal to $i$, i.e., the signature of the manifold $M$ is $(i, n - i)$ (in the Lorentzian case $i = 1$). We use in this paper the coframe index notation. All indexed objects are diffeomorphic invariants and global Lorentz co variants. The metric $\eta_{ab}$ and its inverse $\eta^{ab}$ will be used to lower and raise the indices.

Denote by $\mathcal{F}(M)$ the algebra (commutative and associative) of functions on $M$ and by $\mathcal{X}(M)$ the $\mathcal{F}(M)$-module of vector fields on $M$. Let $\Omega^1(M)$ designates module of 1-forms dual to $\mathcal{X}(M)$, i.e., the set of $\mathcal{F}(M)$-linear maps $\mathcal{X}(M) \to \mathcal{F}(M)$. Denote by $\Omega^p(M)$ the module of differential $p$-forms on $M$.

The inner product operation $\cdot : \mathcal{X}(M) \times \Omega^p(M) \to \Omega^{p-1}(M)$ is linear in both operands and acts on an exterior product of forms by the modified Leibniz rule. Namely, for $A \in \Omega^p(M), B \in \Omega^q(M), X \in \mathcal{X}(M)$

$$X \cdot (A \wedge B) = (X \cdot A) \wedge B + (-1)^p A \wedge (X \cdot B).$$

(2.2)

The Hodge dual map $\ast : \Omega^p(M) \to \Omega^{n-p}(M)$ is defined via the metric (2.1) in a standard manner [5]. Its square is an identity operator (up to a sign):

$$\ast^2 A = (-1)^{p(n-p)+i} A.$$

(2.3)

Two forms $A$ and $B$ of the same degree satisfy the commutative rule

$$A \wedge \ast B = B \wedge \ast A.$$

(2.4)

Denote by $e_a$ the basis vectors dual to $\vartheta^a$. The duality is expressed by the inner product operation as $e_a \cdot \vartheta^b = \delta_a^b$. Recall the useful formulas [5] which will be frequently used subsequently. The relations

$$e_a \cdot \ast A = \ast (A \wedge \vartheta_a)$$

(2.5)

and

$$\vartheta_a \wedge \ast A = (-1)^{p+1} \ast (e_a \cdot A)$$

(2.6)
hold for an arbitrary form $\mathcal{A}$. Trivial consequences of the antisymmetry are
\[ \vartheta^a \wedge (\vartheta_a \wedge \mathcal{A}) = 0, \] (2.7)
and
\[ e_a \mathcal{J}(e^a \mathcal{J} \mathcal{A}) = 0. \] (2.8)
The successive actions of $\mathcal{J}$ and $\wedge$ operations obey the properties
\[ \vartheta^a \wedge (e_a \mathcal{J} \mathcal{A}) = p \mathcal{A}, \] (2.9)
and
\[ e_a \mathcal{J}(\vartheta^a \wedge \mathcal{A}) = (n - p) \mathcal{A}. \] (2.10)

2.2. Form - tensor equivalence

Let a covector-valued $(n-1)$-form $\mathcal{T}_a$ on $M$ be given. This object has $n^2$ independent components, exactly as a generic second-rank tensor. Certainly, the coincidence is not by chance. Consider the Hodge dual form $\ast \mathcal{T}_a$. This covector-valued 1-form may be regarded as a map:
\[ \ast \mathcal{T}_a : \mathcal{X}(M) \to \Omega^1(M), \] (2.11)
i.e., for a given vector field $X \in \mathcal{X}(M)$, $\ast \mathcal{T}_a(X)$ are components of a 1-form. This 1-form may be successively regarded as a map to functions on $M$:
\[ \ast \mathcal{T}_a(X) : \mathcal{X}(M) \to \mathcal{F}(M). \] (2.12)
The composition of the maps (2.11) and (2.12) defines a map
\[ T : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{F}(M), \] (2.13)
which represents a $(2,0)$-rank tensor $T(X,Y)$. Relative to the basis $e_a$ this tensor obtains the component-wise form $T_{ab} := T(e_a, e_b)$, which is similar to the ordinary coordinate-wise notation of a second-rank tensor. It should be noted, however, that the components $T_{ab}$ are invariants under a coordinate transformation. They constitute a tensor under the global pseudo-orthonormal transformations of the coframe $\vartheta^a$ (and the corresponding transformations of the basis vectors $e_a$).

The equivalence between the current $\mathcal{T}_a$ and the tensor $T_{ab}$ is given by
\[ \mathcal{T}_a = T_{ab} \ast \vartheta^b, \] (2.14)
i.e., tensor is represented as a matrix of the coefficients of the current in the odd basis $\ast \vartheta^a$ of $\Omega^3$. Observe that two sides of (2.14) are odd 3-forms, thus, $T_{ab}$ is an even 0-form. Invert (2.14) by applying (2.3) and (2.5) to obtain the explicit expression
\[ T_{ab} = (-1)^{n-1+i} e_b \mathcal{J} \ast \mathcal{T}_a = (-1)^{n-1+i} \ast (\mathcal{T}_a \wedge \vartheta_b). \] (2.15)

Instead of the coframe representation (2.14) an alternative coordinate representation of the current $\mathcal{T}_a$ can be introduced by $\mathcal{T}_a = T_{a\mu} \ast dx^\mu$. The object $T_{a\mu}$ is a covector relative to coordinate transformations and a covector relative to coframe transformations. It is
not, however, a tensor. Only on a flat manifold, where a closed pseudo-orthonormal coframe can be defined by $\vartheta^a = dx^a$, the distinction between the indices is disappeared and $T_{ab}$ turns to be a tensor.

An alternative current-tensor relation (for electromagnetic field) was recently proposed by Hehl and Obukhov [15], [16]. In their approach the equivalence is defined by the relation

$$T_a^b = \diamond (\vartheta^b \wedge T_a),$$

(2.16)

where $\diamond$ denotes the dual with respect to the Levi-Civita density. The relation (2.16) is, certainly, in a more general setting than (2.14), (2.15), because it allows to manage a current defined on a manifold without metric, i.e., for coframes transformed under $GL(n)$. Such framework is basic for Hehl-Obukhov axiomatic construction of electromagnetic theory. In our pseudo-orthonormal coframe approach to gravity the metric and the Hodge dual are defined from the beginning.

On the other hand the relation (2.16) defines the $(1,1)$-type tensor and in order to consider its symmetric properties the contraction with some metric tensor have to be taken also here. It should noted, however, that the trace of the tensor can be extracted irreducibly under $GL(n)$ already in (2.16).

### 2.3. Irreducible decomposition of a current

A covector-valued $(n-1)$-form admits an irreducible decomposition under the action of the pseudo-orthonormal group $SO(i, n-i)$. For a given $(n-1)$-form $T_a$ define two pseudo-orthonormal (and diffeomorphic) invariants: an $n$-form

$$\mathcal{T} = \vartheta^a \wedge T_a,$$

(2.17)

and an $(n-2)$-form

$$\mathcal{S} = e_a \mathcal{J}^a.$$

(2.18)

The scalar-valued $n$-form $\mathcal{T}$ is an invariant linear combination of the components of the tensor $T_{ab}$. The only such invariant of a tensor is its trace.

**Proposition 2.1:** The $n$-form $\mathcal{T} = \vartheta^a \wedge T_a$ of an arbitrary vector-valued $(n-1)$-form $T_a$ satisfies the relation

$$\mathcal{T} = T_{ba}^a * 1,$$

(2.19)

where $T_{ba}^a = \eta^{ab}T_{ab}$ is the trace of the tensor.

**Proof:** Insert the definition (2.14) into (2.19) to obtain

$$\vartheta^a \wedge T_a = T_{ab} \vartheta^a \wedge * \vartheta^b$$

(2.20)

Use the relation (2.6) to get $\vartheta^a \wedge * \vartheta^b = \eta^{ab} * 1$. Thus (2.20) yields (2.19).

The scalar valued $(n-2)$-form $\mathcal{S}$ has $n(n-1)/2$ independent components exactly as a generic antisymmetric tensor.
Proposition 2.2: The relation
\[ S = -T_{[ab]} \ast (\vartheta^a \wedge \vartheta^b) \] holds for an arbitrary vector-valued \((n-1)\)-form \(T^a\).

Proof: Insert (2.14) into the LHS of the (2.21) and use (2.5) to obtain
\[ e_a J^a = T_{ab} e^a \ast \vartheta^b = T_{ab} \ast (\vartheta^b \wedge \vartheta^a). \] (2.22)

Proposition 2.3: The irreducible decomposition of a covector-valued \((n-1)\)-form \(T^a\) under the (pseudo) orthonormal group is
\[ T_a = (\text{sym}) T_a + (\text{ant}) T_a + (\text{tr}) T_a, \] (2.23)
where the trace part is
\[ (\text{tr}) T_a = \frac{1}{n} e_a J, \] (2.24)
the antisymmetric part is
\[ (\text{ant}) T_a = \frac{1}{2} \vartheta_a \wedge S, \] (2.25)
and the symmetric traceless part is
\[ (\text{sym}) T_a = T_a - \frac{1}{n} e_a J - \frac{1}{2} \vartheta_a \wedge S. \] (2.26)

Proof: Let us express the irreducible parts of the current via the tensor components \(T_{ab}\). Using (2.19) the trace part takes the form
\[ (\text{tr}) T_a = \frac{1}{n} T^m_{m} \ast \vartheta_a. \] (2.27)
As for the antisymmetric part by (2.21)
\[ (\text{ant}) T_a = T_{[ab]} \ast \vartheta^b. \] (2.28)
Finally, the traceless symmetric part is
\[ (\text{sym}) T_a = T_{ab} \ast \vartheta^b - T_{[ab]} \ast \vartheta^b - \frac{1}{n} T^m_{m} \ast \vartheta_a \]
\[ = (T_{(ab)} \ast \vartheta^b - \frac{1}{n} \eta_{ab} T^m_{m}) \ast \vartheta^b. \] (2.29)
The irreducible decomposition of the current (2.23) can now be regarded as
\[ T_{ab} = (T_{(ab)} - \frac{1}{n} \eta_{ab} T^m_{m}) + T_{[ab]} + \frac{1}{n} \eta_{ab} T^m_{m}, \] (2.30)
i.e., as an ordinary decomposition of a second rank tensor to a trace, antisymmetric, and symmetric traceless parts. \[\square\]
2.4. Quadratic relations

The structure of the coframe current (1.3) is similar to the Yang-Mills-Maxwell energy-momentum current. Both expressions are quadratic in first order derivatives of the corresponding fields. Consider a scalar-valued $p$-form $\mathcal{A}$ which represents a generalized strength of a model based on a $(p-1)$-form field. In an analogy to (1.3), the $(n-1)$-form of current for this field is of the form

$$\tilde{T}_a = (e_a \llcorner \mathcal{A}) \wedge * \mathcal{A} - \frac{1}{2} e_a \llcorner (\mathcal{A} \wedge * \mathcal{A}).$$

(2.31)

The symmetry of the tensor corresponding to this current is governed by a scalar valued $(n-2)$-form

$$\tilde{S} = (e_a \llcorner \mathcal{A}) \wedge (e_a \llcorner * \mathcal{A}).$$

(2.32)

As it will be shown in the consequence, the vanishing of this form guarantees the symmetry of the corresponding tensor. Let us consider the $(n-2)$-form $\tilde{S}$ in a general setting (without connection to some quadratic Lagrangian).

**Theorem 2.4:** The $(n-2)$-form $\tilde{S}$ is vanishing for an arbitrary $p$-form $\mathcal{A}$, i.e.,

$$(e_a \llcorner \mathcal{A}) \wedge (e_a \llcorner * \mathcal{A}) = 0.$$  

(2.33)

**Proof:** Suppose the LHS of (2.33) is nonzero. So, the $n$-form

$$B^{mn} := \partial^m \wedge (e_a \llcorner \mathcal{A}) \wedge \partial^n \wedge (e_a \llcorner * \mathcal{A})$$

(2.34)

is antisymmetric for the permutation of the indices $m$ and $n$. Use (2.2) to rewrite (2.34) as

$$B^{mn} = \left( \delta^m_a \mathcal{A} - e_a \llcorner (\partial^m \wedge \mathcal{A}) \right) \wedge \left( \eta^n_a * \mathcal{A} - e_a \llcorner (\partial^n \wedge * \mathcal{A}) \right)$$

(2.35)

Thus, $B^{mn}$ is expressed as a sum of four terms. The first one is:

$$\delta^m_a \mathcal{A} \wedge \eta^n_a * \mathcal{A} = \eta^{mn} \mathcal{A} \wedge * \mathcal{A} \quad \text{symmetric.}$$

The second term is

$$- e^n \llcorner (\partial^m \wedge \mathcal{A}) \wedge * \mathcal{A} = - \mathcal{A} \wedge * \left( e^n \llcorner (\partial^m \wedge \mathcal{A}) \right)$$

$$= (-1)^{k_1} (\partial^n \wedge \mathcal{A}) \wedge * (\partial^m \wedge \mathcal{A}) \quad \text{symmetric,}$$

where the value of the integer $k_1$ is defined by (2.3) and (2.3).

The third term is

$$- \mathcal{A} \wedge e^m \llcorner (\partial^n \wedge * \mathcal{A}) = - * \left( e^m \llcorner (\partial^n \wedge * \mathcal{A}) \right) \wedge * \mathcal{A}$$

$$= (-1)^{k_2} (\partial^m \wedge * \mathcal{A}) \wedge * (\partial^n \wedge * \mathcal{A}) \quad \text{symmetric.}$$

Finally, the fourth term is

$$\left( e_a \llcorner (\partial^m \wedge \mathcal{A}) \right) \wedge \left( e^a \llcorner (\partial^n \wedge * \mathcal{A}) \right)$$

$$= (-1)^{k_1} * \left( \partial_a \wedge * (\partial^m \wedge \mathcal{A}) \right) \wedge * \left( \partial^a \wedge * (\partial^n \wedge * \mathcal{A}) \right)$$

$$= (-1)^{k_2} \partial^a \wedge * (\partial^n \wedge * \mathcal{A}) \wedge \partial_a \wedge * (\partial^m \wedge \mathcal{A}) = 0$$

(2.35)
Therefore $B^{mn}$ is symmetric. The contradiction proves that the LHS of (2.33) is zero. Observe that (2.33) is trivial for $A$ be a wedge product of basis forms. The nonlinearity of the relation, however, seems to put an obstacle to restrict the proof.

In the case of a strength $A$ to be a vector-valued $p$-form the corresponding current involves the term $(e_\alpha \lhd A) \wedge *B$, where $B$ is a $p$-form constructed from $A$ by inner and wedge product with basis forms. Although in this case the term $(e_\alpha \lhd A) \wedge (e^a \lhd * B)$ is not zero, in general, some useful relations involving this expression may be established.

**Proposition 2.5:** Two forms $A$ and $B$ of the same degree satisfy

$$(e_\alpha \lhd A) \wedge (e^a \lhd * B) = -(e_\alpha \lhd B) \wedge (e^a \lhd * A).$$

*Proof:* It is enough to open the brackets in the relation

$$(e_\alpha \lhd (A + B)) \wedge (e^a \lhd * (A + B)) = 0.$$

**Proposition 2.6:** A form $A$ of the degree $n$ in $2n$-dimensional space satisfies

$$(e_\alpha \lhd * A) \wedge (e^a \lhd A) = (-1)^{n+i+1}(e_\alpha \lhd A) \wedge (e^a \lhd A).$$

*Proof:* It is enough to replace $B$ by $*A$ in (2.36).

Certainly the two sides of (2.37) are nonzero only for an odd $n$, i.e., in the dimensions: 2, 6, 10, etc.

For the vector field theory the strength is a form of the second degree. In this case the following useful rule to deal with the Hodge star is valid.

**Theorem 2.7:** Two forms $A$ and $B$ of the second degree satisfy

$$(e_\alpha \lhd A) \wedge (e^a \lhd * B) = * \left( (e_\alpha \lhd A) \wedge (e^a \lhd B) \right).$$

*Proof:* Write down the 2-form $A$ via the components relative to the pseudo-orthonormal basis $A = 1/2 A_{mn} \vartheta^m \wedge \vartheta^n$. Consequently $e_\alpha \lhd A = A_{am} \vartheta^m$. Compute the LHS of (2.38)

$$(e_\alpha \lhd A) \wedge (e^a \lhd * B) = A_{am} \vartheta^m \wedge (e^a \lhd * B)$$

$$= A_{am} \vartheta^m \wedge *(B \wedge \vartheta^a)$$

$$= (-1)^i A_{am} \wedge (\vartheta^m \wedge *(B \wedge \vartheta^a))$$

$$= A_{am} \wedge *(e^m \lhd (B \wedge \vartheta^a))$$

$$= A_{am} \wedge *(e^m \lhd B \wedge \vartheta^a + B \eta^a)$$

$$= A_{am} \wedge *(e^m \lhd B \wedge \vartheta^a) + * (e_\alpha \lhd A) \wedge (e^a \lhd B)$$
Observe that two sides of the equation (2.38) are \((n-2)\)-forms. In the case \(\mathcal{A} = \mathcal{B}\) the RHS is zero as a wedge square of a 1-form. Certainly, it is a special case of (2.33).

3. Coframe current

The form-tensor equivalence described above allows to study the algebraic properties of the current in parallel to the corresponding algebraic properties of the tensor. In this section we deal with the coframe current (1.5) defined on a 4\(D\)-manifold of Lorentzian signature.

3.1. Traceless property

The current (1.5) involves free parameters \(\rho_i\). It is natural to look for which values of these parameters the corresponding tensor is traceless. By (2.27) the traceless tensor corresponds to a current satisfied \(\mathcal{T} := \vartheta_a \wedge \mathcal{T}^a = 0\).

**Proposition 3.1:** The coframe current (1.5) is traceless for an arbitrary choice of the parameters \(\rho_1, \rho_2, \rho_3\).

**Proof:** Calculate

\[
\mathcal{T}^a \wedge \vartheta_a = -\vartheta^a \wedge (e_a \mathcal{J}^m) \wedge \ast \mathcal{F}_m + \vartheta^a \wedge e_a \mathcal{L} \tag{3.1}
\]

Use the relation (2.9) to obtain

\[
\mathcal{T}^a \wedge \vartheta_a = -2 \mathcal{C}_m \wedge \ast \mathcal{F}^m + 4 \mathcal{L} = 0. \tag{3.2}
\]

The coframe field equation (1.4) incorporates the current (1.5) as a source term. This current is traceless for an arbitrary coframe field \(\vartheta^a(x)\), even for this that does not satisfy the field equation. Let us look now how this traceless property influences upon the algebraic features of the pure coframe field equation. Take the material Lagrangian to be zero. Thus from (1.2) \(\mathcal{L} = \frac{1}{2} \mathcal{C}_a \wedge \ast \mathcal{F}^a\). Construct the exterior product in two sides of (1.4) to obtain

\[
\vartheta_a \wedge d \ast \mathcal{F}^a = 0, \tag{3.3}
\]

or, equivalently,

\[
d(\vartheta_a \wedge \ast \mathcal{F}^a) = 2 \mathcal{L}. \tag{3.4}
\]

Thus the on-shell value of the Lagrangian is an exact form. Insert in the LHS of (3.4) the definition (1.3) of the strength \(\mathcal{F}^a\) to obtain

\[
(\rho_1 - 2 \rho_3) d(\vartheta_a \wedge \ast \mathcal{C}^a) = 2 \mathcal{L}. \tag{3.5}
\]

Observe some conclusions from the equation (3.5) which valid for a pure coframe field in vacuum.
i) Consider the coframe model with parameters \( \rho_1 = 2 \rho_3, \rho_2 \) arbitrary. For all solutions of the corresponding field equation the on-shell value of the Lagrangian is zero.

ii) Conversely, consider the models with zero on-shell value of the Lagrangian. Let \( \rho_1 \) be different from \( 2 \rho_3 \). By (3.3) the 3-form \( \vartheta_a \wedge \ast \mathcal{C}^a \) is exact. Via the Poincaré lemma it means the (local) existence of a 2-form \( A \) satisfying \( dA = \vartheta_a \wedge \ast \mathcal{C}^a \). Consequently, for such models the 2-form \( A \) is an integral invariant.

iii) Let two different models (with different \( \rho \)'s) have a joint solution \( \vartheta^a(x) \). Thus the corresponding Lagrangians have the same on-shell value in both models, up to the coefficient \( (\rho_1 - 2 \rho_3) \).

iv) Consider the set of viable models: \( \rho_1 = 0, \rho_3 \neq 0, \rho_2 \) arbitrary. Let two models with different \( \rho_2 \) have a joint solution \( \vartheta^a(x) \). This solution have to satisfy

\[
(\vartheta_a \wedge \ast \mathcal{C}^a) \wedge \ast (\vartheta_b \wedge \ast \mathcal{C}^b) = 0,
\]

i.e., the pseudo-norm of the 3-form \( \vartheta_a \wedge \ast \mathcal{C}^a \) is zero. The well known solution of a such type is the Schwarzschild coframe which appears in all viable models (for all values of \( \rho_2 \)). This solution satisfies \( \vartheta_a \wedge \ast \mathcal{C}^a = 0 \), thus also (3.6).

3.2. Symmetric property of the current

The coframe current (1.5) is formally similar to the electromagnetic energy-momentum current expression. The important distinctions are:

i) The coframe Lagrangian and the corresponding current involve the covector-valued strengths, while the electromagnetic theory is based on the scalar-valued strength.

ii) Two coframe strengths \( \mathcal{C}^a \) and \( \mathcal{F}^a \) are different for a generic choice of parameters. Consequently, the tensor corresponding to the coframe current has not to be symmetric in general. By (2.23) the tensor is symmetric if and only if the 2-form \( \mathcal{S} \) vanishes.

Let us examine for which values of the parameters the coframe current produces a pure symmetric tensor.

**Proposition 3.2:** The 2-form \( \mathcal{S} \) vanishes identically if and only if \( \rho_2 = \rho_3 = 0 \). Consequently, for all viable models \( \mathcal{S} \) and, correspondingly, the antisymmetric part of the tensor are non-zero.

**Proof:** Calculate the 2-form \( \mathcal{S} \) for the coframe current (1.5) using the global \( SO(1,3) \) covariants [17]

\[
\mathcal{S} = -(e_a \mathcal{J}_m) \wedge (e^a \mathcal{J}^m) = * \left( (e_a \mathcal{J}^F_m) \wedge (e^a \mathcal{J}^C^m) \right).
\]

(3.7)

The formula (2.38) was applied in the last equation. Use the definition (1.3) of \( \mathcal{F}_a \) to obtain

\[
\mathcal{S} = \rho_1 * \left( (e_a \mathcal{J}_m) \wedge (e^a \mathcal{J}^C^m) \right) + \rho_2 * \left( (e_a \mathcal{J}_m) \wedge (e^a \mathcal{J}(\vartheta^k \wedge (e_m \mathcal{J}^C^k))) \right) +
\]
Coframe energy-momentum current. Algebraic properties.

\[ \rho_3 \ast \left( (e_a \mathcal{C}_m) \wedge (e^a \mathcal{J}(\mathcal{J}_m \wedge (e_k \mathcal{C}^k))) \right). \]  
(3.8)

The \( \rho_1 \)-term vanishes as a square of a 1-form (more generally, such type expressions are zero by (2.33)). Calculate the \( \rho_2 \)-term using the component-wise expression

\[ \mathcal{C}^a = \frac{1}{2} C_{abc} \psi^b \wedge \psi^c. \]

We obtain

\[ \rho_2 \ast (\cdots) = \rho_2 C_{ma} C_{ams} \left( \mathcal{J}^q \wedge (e_a \mathcal{J}(\mathcal{J}_m \wedge \mathcal{J}^s)) \right) \]

Use the 2-indexed \( SO(1,3) \) covariants (see the Appendix) to rewrite this term as

\[ \rho_2 \ast (\cdots) = \rho_2 \left( (4) A_{pq} + (5) A_{pq} \right) \ast (\mathcal{J}^p \wedge \mathcal{J}^q) \]  
(3.9)

As for the \( \rho_3 \)-term

\[ \rho_3 \ast (\cdots) = \rho_3 C_{ma} \right. \left. C_{ks} \left( \mathcal{J}^q \wedge (e_a \mathcal{J}(\mathcal{J}_m \wedge \mathcal{J}^s)) \right) \]

The first coefficient in this expression is symmetric thus

\[ \rho_3 \ast (\cdots) = -\rho_3 C_{pq} \ast (\mathcal{J}^p \wedge \mathcal{J}^q) = -\rho_3 (1) A_{pq} \ast (\mathcal{J}^p \wedge \mathcal{J}^q). \]  
(3.12)

The terms (3.10) and (3.12) are algebraic independent, thus the 2-form \( \mathcal{S} \) vanishes if and only if \( \rho_2 = \rho_3 = 0. \)

The 2-form \( \mathcal{S} \) is a diffeomorphic and a global \( SO(1,3) \) invariant. Certainly, it is not transforms invariantly under local \( SO(1,3) \) transformations of the coframe field. However, the Lagrangian itself does not have such invariance for a generic choice of the parameters. So, for all values of parameters (excepting the case of the teleparallel equivalent of GR) the 2-form \( \mathcal{S} \) is a well defined object. The antisymmetric part of the energy-momentum tensor is known by Poincaré gauge theory of gravity \[ \text{to represent the rotational (spin) properties of the field. The corresponding 2-form \( \mathcal{S} \) of the electromagnetic field vanishes identically. Also the 2-form \( \mathcal{S} \) being calculated for the Schwarzschild solution is zero \[. A rotational solution of the general free parametric coframe field equation may produce an example of a model with a non-zero 2-form \( \mathcal{S}. \]

3.3. Antisymmetric property of the current

To complete the consideration let us examine the possibility to have a model with a current corresponding to a pure antisymmetric tensor. Certainly, such model (if it exists) can not be related to a some viable physical situation because it have to describe a non-trivial dynamics with zero energy. Via (2.26) the current corresponding to a pure antisymmetric tensor have to satisfy the relation \( T_a = \frac{1}{2} \mathcal{J}_a \wedge \mathcal{S}. \)

Proposition 3.3: The symmetric part of a tensor corresponding to the coframe current \[ (1.3) \] in non-zero (in general) for all values of the parameters \( \rho_i, \) i.e., for all coframe models.
Proof: The coframe current \( (1.5) \) is traceless for arbitrary values of the parameters \( \rho_1, \rho_2, \rho_3 \). Thus, by \((2.29)\), the symmetric part of the tensor satisfy \((\text{sym}) T_a = T_{(ab)} \ast \vartheta^b\). Consequently, it is zero if and only if

\[
(\text{sym}) T_a = T_a - \frac{1}{2} \vartheta_a \wedge S = 0
\]
or using \((1.4)\)

\[
(e_a \mathcal{L} L - \frac{1}{2} \vartheta_a \wedge S) = 0.
\] (3.13)

Let us represent the LHS of this equation via the global \( SO(1,3) \) covariants \([17]\). All terms of \((3.13)\) are linear in the parameters \( \rho_i \). Thus, the contributions corresponding to different parameters may be computed separately.

Calculate the first term of \((3.13)\). The \( \rho_1 \)-contribution takes the form (the notation \( \vartheta_{ab} \cdots = \vartheta^a \wedge \vartheta^b \wedge \cdots \) is used)

\[
\rho_1 \left( e_a \mathcal{L} C_m \right) \wedge *C_m = \frac{1}{2} \rho_1 C_{m a q} C_{m r s} \vartheta_q \wedge * (\vartheta^{rs}) = -\rho_1 C_{m a} C_{m q s} * \vartheta^s
\]

\[
= \rho_1 (6) A_{ab} \ast \vartheta^b
\] (3.14)

The \( \rho_2 \)-contribution is

\[
\rho_2 \left( e_a \mathcal{L} C_m \right) \wedge \left( e_n \mathcal{L} (\vartheta^m \wedge C_n) \right) = \frac{1}{2} \rho_2 C_{m a q} C_{m r s} \vartheta^q \wedge * (\delta^r_m \vartheta^{rs} - 2 \delta^r_m \vartheta^{ns})
\]

\[
= -\rho_2 C_{m a} \left( C_{m qs} + C_{q sm} + C_{smq} \right) \ast \vartheta^s
\]

\[
= \rho_2 \left( (6) A_{ab} - (5) A_{ab} - (4) A_{ab} \right) \ast \vartheta^b
\] (3.15)

The \( \rho_3 \)-contribution is

\[
\rho_3 \left( e_a \mathcal{L} C_m \right) \wedge \left( e_n \mathcal{L} (\vartheta^m \wedge C_n) \right) = \frac{1}{2} \rho_3 C_{m a q} C_{m r s} \left( e_n \mathcal{L} \vartheta^{pq} \right) \wedge * (e_m \mathcal{L} \vartheta^{mr s})
\]

\[
= -\rho_3 \left( C_{m a q} C^{m q}_s - C^{m a q}_m C^{m r n}_s + C_{s q a} C^{m n}_q \right) \ast \vartheta^s
\]

\[
= \rho_3 \left( (6) A_{ab} - (7) A_{ab} - (1) A_{ab} \right) \ast \vartheta^b
\] (3.16)

Summing \((3.14), (3.15), (3.16)\) we obtain the first term of \((3.13)\). As for a contribution of the second term of \((3.13)\) it is of the form:

\[
e_a \mathcal{L} L = -(\eta_{ab} \ast L) \wedge * \vartheta^b.
\] (3.17)

This term is expressed by scalar valued invariants \((A.13)-(A.17)\).

The contribution of the third term of \((3.13)\) is represented by \((3.10)\) and \((3.12)\) as

\[
\vartheta_a \wedge S = \vartheta_a \wedge \left( \rho_2 (4) A_{pq} + \rho_2 (5) A_{pq} - \rho_3 (1) A_{pq} \right) \ast (\vartheta^p \wedge \vartheta^q)
\]

\[
= \left( \rho_3 (1) A_{[ab]} - \rho_2 (4) A_{[ab]} \right) \ast \vartheta^b
\] (3.18)

Recall, we are looking for which values of the parameters the LHS of \((3.13)\) is vanished identically. Using the algebraic independence of the covariants \((i) A_{ab}\) we obtain that \( \rho_2, \rho_3 \) should vanish in order to illuminate the contribution of \((5) A_{ab}\) and \((7) A_{ab}\).
correspondingly. Now, \( \rho_1 \) have also to be zero in order to illuminate the contribution of \( (1)A_{ab} \). Consequently, the coframe current can not be pure antisymmetric in any of coframe models.

4. Symmetric reduction of the field equation

The 2-form \( S \) and consequently the antisymmetric part of the current are not vanished in all viable models. Thus, the field equation (1.4) represents a system of 16 independent PDE. In the case \( 2\rho_2 + \rho_3 \neq 0 \) it is a well determined system for 16 independent components of the coframe.

As for the case \( 2\rho_2 + \rho_3 = 0 \) (the teleparallel equivalent of GR) the situation is rather different. The corresponding Lagrangian accepts an additional invariant transformation - local Lorentz transformation of the coframe field. Such invariance of the Lagrangian certainly preserves on the field equation level. The coframe variable, however, has only 10 independent components, related to the components of the metric tensor. Consider what is the situation with the field equation. Also here we are dealing with pure coframe field in vacuum, i.e., in (1.4) the energy-momentum current of the material field \( (\text{mat})T_a \) is taken to be zero. Rewrite the field equation (1.4) in the form

\[
\mathcal{E}_a := d \star F_a - T^a = 0.
\]  
(4.1)

The vector-valued 3-form \( \mathcal{E}_a \) accepts the decomposition to the symmetric and antisymmetric parts

\[
\mathcal{E}_a = (\text{sym})\mathcal{E}_a + (\text{ant})\mathcal{E}_a.
\]  
(4.2)

Similarly to the current \( T_a \) the antisymmetric part represents as

\[
(\text{ant})\mathcal{E}_a = \vartheta_a \wedge \mathcal{E},
\]  
(4.3)

where \( \mathcal{E} := e_a \mathcal{E}^a \) is a scalar-valued 2-form. Using the coframe invariant notations (see Appendix): we express the 2-form \( \mathcal{E} \) as

\[
\mathcal{E} = \left( -2(\rho_1 - 2\rho_2 - \rho_3)(^{(1)}B_{[ab]} + ^{(1)}A_{[ab]}) + (2\rho_2 + \rho_3) \left( ^{(3)}B_{[ab]} + ^{(2)}A_{[ab]} \right) \right) \star (\vartheta^a \wedge \vartheta^b).
\]  
(4.4)

The RHS of this equation is identically zero if and only if

\[
\rho_1 = 0, \quad 2\rho_2 + \rho_3 = 0.
\]  
(4.5)

This is the case of the teleparallel equivalent of the Einsteinian gravity. Consequently, the system of 16 field equations is restricted to the symmetric system of 10 independent equations. Thus, it represents a well defined system of PDE. This result shows also that the Einstein equation (in the coframe versus) is the unique symmetric field equation that can be derived from the quadratic coframe Lagrangian.
5. Concluding remarks

We consider a class of coframe models determined by the values of three dimensionless free parameters. In the case $\rho_1 \neq 0$ the field equation has no a spherical symmetric solution with Newtonian behavior at infinity. However, in the case $\rho_1 = 0$ all the models have the same Schwarzschild solution for arbitrary values of the remaining parameters. Consequently all these models can be considered as viable [9], [11]. The most of viable coframe Lagrangians (with $\rho_2 + 2\rho_3 \neq 0$) represent gravity models alternative to GR. A local conservative 3-form of energy-momentum current is well defined for such models. This object preserves all the invariance transformations of the corresponding Lagrangian. The tensor-form correspondence produces a diffeomorphic invariant and global Lorentz covariant energy-momentum tensor of coframe field. This tensor is traceless for all values of parameters. Consequently, the gravitons in quantum extensions of all such models have to be massless.

The antisymmetric part of the tensor corresponds to an invariant 2-form $S$, which is related to the rotation properties of the field. The tensor is proved to have in general non vanishing symmetric and antisymmetric parts in all viable models.

The exceptional case $\rho_2 + 2\rho_3 = 0$ describes a coframe model with additional local Lorentz invariance of the Lagrangian. It is an alternative coframe (teleparallel) description of GR, not an alternative gravity model. In this case the coframe is defined only up to local pseudo-rotations. The corresponding system of field equations is restricted to a system of 10 independent PDE for 10 independent components of the coframe. Thus, it is a well determined system.

The conserved current, however, does not preserves local Lorentz transformations, as in standard GR.

Two possibilities is open in this situation.

The first topic is to study the coframe models as an alternative to GR. The known obstacle to this is the particle analysis [18], [19], which shows the existence of non-physical modes (ghosts, tachyons). We plan to examine if these modes appear also in the coframe models in a way which is slightly different from the Einstein gravity.

The second topic is to consider standard GR as a limit of the free parametric teleparallel model with $\rho_2 + 2\rho_3 \to 0$. Such a limit produces one more symmetry of the Lagrangian. It is the local (pointwise) pseudo-rotations of the coframe. This symmetry is transported to the field equations. Unfortunately the metric construction of GR prevents $\mathcal{T}^a$ to share this property. Still the proper defined integrals of the current may be invariant. Also meaningful asymptotic invariants may be properly defined by the coframe current.

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Appendix A. Coframe invariants

We present a list of $SO(1,3)$ covariants \cite{17}. All of them are invariant under the diffeomorphisms of the manifold. The first order derivative covariant is defined as

$$C^a_{\ b\ c} = e_c \mathcal{J}(e_b \mathcal{J}d\vartheta^a). \quad (A.1)$$

The contraction of this 3-indexed object with Lorentz metric produces a 1-indexed object

$$C_a = C^m_{\ ma} = e_a \mathcal{J}(e_m \mathcal{J}d\vartheta^m). \quad (A.2)$$

This first order derivative $SO(1,3)$ covariant is proportional to the coderivative of the coframe $d^i \vartheta^a = \ast d \ast \vartheta^a$.

The second order derivatives of the coframe are expressed in $SO(1,3)$ covariant form via a 4-indexed object:

$$B^a_{\ bcd} = e_d \mathcal{J}C^a_{\ bc} = e_d \mathcal{J}\left(e_c \mathcal{J}(e_b \mathcal{J}d\vartheta^a)\right). \quad (A.3)$$

This 4-indexed covariant object is a coframe analog of the Riemannian curvature tensor. The field equation can involve only 2-indexed object. Three possible contractions of $B^a_{\ bcd}$:

1. $B_{ab} = B_{abm}^\ m$,
2. $B_{ab} = B^m_{\ mab}$,
3. $B_{ab} = B^m_{\ amb}$ \quad (A.4 - A.6)

are coframe analogs to the Ricci curvature tensor.

The unique scalar valued second order invariant is

$$B = B^a_{\ ab} = B^a_{\ abc} \eta^{bc} = (1)\ B^a_{\ a} = (2)\ B^a_{\ a}. \quad (A.7)$$

This is an analog to the Riemannian curvature scalar.

The field equation as well as the energy-momentum tensor of the coframe field also involve terms quadratic in the first order derivatives. The possible two-indexed $SO(1,3)$ covariants quadratic in first order derivatives are:

1. $A_{ab} := C_{abm}^m$,
2. $A_{ab} := C_{mab}^m$ \quad antisymmetric object, \quad (A.8 - A.9)
3. $A_{ab} := C_{amn}^mC^m_{\ b\ mn}$ \quad symmetric object, \quad (A.10)
4. $A_{ab} := C_{amn}^mC^m_{\ b\ n}$ \quad (A.11)
5. $A_{ab} := C_{man}^mC^m_{\ b\ m}$ \quad symmetric object, \quad (A.12)
6. $A_{ab} := C_{man}^mC^m_{\ b\ n}$ \quad symmetric object, \quad (A.13)
7. $A_{ab} := C_aC_b$ \quad symmetric object. \quad (A.14)

In addition to the 2-indexed $A$-objects the general field equation may also include their traces multiplied by $\eta_{ab}$. These traces of 2-indexed objects are scalar $SO(1,3)$ invariants:

1. $A := (1)\ A^a_{\ a} = -(7)\ A^a_{\ a}$, \quad (A.15)
2. $A := (3)\ A^a_{\ a} = (6)\ A^a_{\ a}$, \quad (A.16)
3. $A := (4)\ A^a_{\ a} = (5)\ A^a_{\ a}$. \quad (A.17)
Coframe energy-momentum current. Algebraic properties.

Three scalars \( (i) \) A constitute three independent parts of the coframe Lagrangian.

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