A note on the spectrum of the Neumann Laplacian in periodic waveguides

Carlos R. Mamani and Alessandra A. Verri
Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil
August 30, 2017

Abstract
We study the Neumann Laplacian \(-\Delta_N\) restricted to a periodic waveguide. In this situation its spectrum \(\sigma(-\Delta_N)\) presents a band structure. Our goal and strategy is to get spectral information from an analysis of the asymptotic behavior of these bands provided that the waveguide is sufficiently thin.

1 Introduction
Let \(\Lambda\) be a periodic strip (in \(\mathbb{R}^2\)) or a periodic tube (in \(\mathbb{R}^3\)). Denote by \(-\Delta\) the Laplacian operator restricted to \(\Lambda\). At the boundary \(\partial\Lambda\), consider the Dirichlet or Neumann conditions. An interesting point is to know something about the spectrum \(\sigma(-\Delta)\) which has a band structure.

In [17] the author studied the band gap of the spectrum of the Dirichlet Laplacian in a periodic strip in \(\mathbb{R}^2\). In a more particular situation, in [9] the authors studied the band lengths as the diameter of the strip tends to zero. In [15] the authors proved the absolute continuity for \(-\Delta\) in a periodic strip with either Dirichlet or Neumann conditions.

In the case of periodic tubes, the absolute continuity was proven in [3, 7, 16]. In [3, 16] only the Dirichlet boundary condition was considered. In [7] the boundary conditions are more general, but a symmetry condition is required. In [13], the author established the existence of gaps in the essential spectrum of the Neumann Laplacian in a periodic tube.

Consider the Neumann Laplacian \(-\Delta^N\) restricted to a periodic waveguide in \(\mathbb{R}^3\). This work has two goals. The first one, is to obtain information about the absolutely continuous spectrum of \(-\Delta^N\). The second, is to prove the existence of band gaps in \(\sigma(-\Delta^N)\); although this result is proven in [13], we give an alternative proof in this text. We highlight that our purpose is to prove the results above from an analysis of the asymptotic behavior of the bands of \(\sigma(-\Delta^N)\) provided that the waveguide is sufficiently thin. Ahead, we give more details.

Let \(r : \mathbb{R} \to \mathbb{R}^3\) be a simple \(C^3\) curve in \(\mathbb{R}^3\) parametrized by its arc-length parameter \(s\). Suppose that \(r\) is periodic, i.e., there exists \(L > 0\) and a nonzero vector \(\bar{u}\) so that \(r(s + L) = \bar{u} + r(s), \forall s \in \mathbb{R}\). Denote by \(k(s)\) and \(\tau(s)\) the curvature and torsion of \(r\) at the position \(s\), respectively. Pick \(S \neq \emptyset\); an open, bounded, smooth and connected subset of \(\mathbb{R}^2\). Build a waveguide \(\Lambda\) in \(\mathbb{R}^3\) by properly moving the region \(S\) along \(r(s)\); at each point \(r(s)\) the cross-section region \(S\) may present a (continuously differentiable) rotation angle \(\alpha(s)\). Suppose that \(\alpha(s)\) is \(L\)-periodic. For each \(\varepsilon > 0\) (small enough), one can perform this same construction with the region \(\varepsilon S\) and so obtaining a thin waveguide \(\Lambda_{\varepsilon}\).
Now, let $h : \mathbb{R} \to \mathbb{R}$ be a $L$-periodic and $C^2$ function satisfying
\[ 0 < c_1 \leq h(s) \leq c_2, \forall s \in \mathbb{R}. \] (1)

We consider the thin waveguide, as presented above, but we deform it by multiplying their cross sections by the function $h(s)$. Thus, we obtain a deformed thin tube $\Omega_\varepsilon$; see Section 2 for details of this construction.

Let $-\Delta^N_{\Omega_\varepsilon}$ be the Neumann Laplacian in $\Omega_\varepsilon$, i.e., the self-adjoint operator associated with the quadratic form
\[ \psi \mapsto \int_{\Omega_\varepsilon} |\nabla \psi|^2 d\vec{x}, \quad \psi \in H^1(\Omega_\varepsilon). \] (2)

The first result of this work states that

**Theorem 1.** For each $E > 0$, there exists $\varepsilon_E > 0$ so that the spectrum of $-\Delta^N_{\Omega_\varepsilon}$ is absolutely continuous in the interval $[0, E]$, for all $\varepsilon \in (0, \varepsilon_E)$.

In [7] the absolute continuity for $-\Delta^N_{\Omega_\varepsilon}$ was proven under the condition of invariance under the reflection $s \mapsto -s$.

At first, in this introduction, we present the main steps of the proof of Theorem 1; the details will be presented along the work. Then, we comment our strategy to guarantee the existence of gaps in the spectrum $\sigma(-\Delta^N_{\Omega_\varepsilon})$.

Fix a number $c > 0$. Denote by 1 the identity operator. For technical reasons, we are going to study the operator $-\Delta^N_{\Omega_\varepsilon} + c \cdot 1$; see Section 4.

A change of coordinates shows that $-\Delta^N_{\Omega_\varepsilon} + c \cdot 1$ is unitarily equivalent to the operator
\[ T_\varepsilon \psi := -\frac{1}{h^2 \beta_\varepsilon} \left[ \left( \partial_s + \text{div}_y R^h \right) \frac{h^2}{\beta_\varepsilon} \partial_{s,y}^R \psi + \frac{1}{\varepsilon^2} \text{div}_y (\beta_\varepsilon \nabla_y \psi) \right] + c \psi, \] (3)
\[ \text{dom } T_\varepsilon := \left\{ \psi \in H^2(\mathbb{R} \times S) : \frac{\partial \text{Rh}_\psi}{\partial N} = 0 \text{ on } \partial(\mathbb{R} \times S) \right\}, \] (4)
acting in the Hilbert space $L^2(\mathbb{R} \times S, h^2 \beta_\varepsilon \text{dsd}y)$. Here, $y := (y_1, y_2) \in S$, div$_y$ denotes the divergent of a vector field in $S$,
\[ \beta_\varepsilon(s,y) := 1 - \varepsilon k(s)(y_1 \cos \alpha(s) + y_2 \sin \alpha(s)), \] (5)
\[ \langle \partial_{s,y}^R \psi(s,y) \rangle := \partial_s \psi(s,y) + \langle \nabla_y \psi(s,y), R^h(s,y) \rangle, \] (6)
\[ R^h(s,y) := (R y)(\tau + \alpha')(s) - y \frac{h'(s)}{h(s)}, \] (7)
where $\partial_s \psi := \partial \psi/\partial s$, $\nabla_y \psi := (\partial \psi/\partial y_1, \partial \psi/\partial y_2)$, and $R$ is the rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Furthermore,
\[ \frac{\partial \text{Rh}_\psi}{\partial N}(s,y) := \frac{h^2(s)}{\beta_\varepsilon(s,y)} \langle R^h(s,y), N(y) \rangle \langle \partial_{s,y}^R \psi(s,y) \rangle + \frac{\beta_\varepsilon(s,y)}{\varepsilon^2} \langle \nabla_y \psi(s,y), N(y) \rangle; \] (8)

$N$ denotes the outward point unit normal vector field of $\partial S$.

Since the coefficients of $T_\varepsilon$ are periodic with respect to $s$, we utilize the Floquet-Bloch reduction under the Brillouin zone $\mathcal{C} := [-\pi/L, \pi/L]$. More precisely, we show that $T_\varepsilon$ is unitarily equivalent to the operator $\int_{\mathcal{C}}^{\oplus} T_\varepsilon^\theta d\theta$, where
\[ T_\varepsilon^\theta \psi := -\frac{1}{h^2 \beta_\varepsilon} \left[ \left( \partial_s + \text{div}_y R^h + i \theta \right) \frac{h^2}{\beta_\varepsilon} \left( \partial_{s,y}^R + i \theta \right) \psi + \frac{1}{\varepsilon^2} \text{div}_y (\beta_\varepsilon \nabla_y \psi) \right] + c \psi, \] (9)
with domain
\[ \text{dom } T^\varepsilon_\theta = \{ \psi \in H^2([0, L] \times S) : \psi(0, \cdot) = \psi(L, \cdot) \text{ and } \partial_{\varepsilon,y}^2 \psi(0, \cdot) = \partial_{\varepsilon,y}^2 \psi(L, \cdot) \text{ in } L^2(S) , \] \[ \frac{\partial R^\varepsilon \theta}{\partial N} = -i \theta \frac{h^2}{\beta^2} \partial (R^\varepsilon, N) \psi \text{ in } L^2([0, L] \times \partial S) \} . \]

Although acting in the Hilbert space \( L^2([0, L] \times S, h^2 \beta \text{d}s \text{d}y) \), \( \partial_{\varepsilon,y}^2 \psi \) and \( \partial R^\varepsilon \theta / \partial N \) have action given by (6), (7) and (8), respectively. Furthermore, for each \( \theta \in \mathcal{C} \), \( T^\varepsilon_\theta \) is self-adjoint; see Lemma 1 in Section 4 for this decomposition.

Each \( T^\varepsilon_\theta \) has compact resolvent and is bounded from below. Thus, \( \sigma(T^\varepsilon_\theta) \) is discrete. Denote by \( \{ E_n(\varepsilon, \theta) \}_{n \in \mathbb{N}} \) the family of all eigenvalues of \( T^\varepsilon_\theta \) and by \( \{ \psi_n(\varepsilon, \theta) \}_{n \in \mathbb{N}} \) family of the corresponding normalized eigenfunctions, i.e.,

\[ T^\varepsilon_\theta \psi_n(\varepsilon, \theta) = E_n(\varepsilon, \theta) \psi_n(\varepsilon, \theta), \quad n = 1, 2, 3, \cdots , \quad \theta \in \mathcal{C} . \] (10)

We have

\[ \sigma(-\Delta_N^{\Omega_\varepsilon}) = \bigcup_{n=1}^{\infty} \{ E_n(\varepsilon, \mathcal{C}) \} , \quad \text{where } E_n(\varepsilon, \mathcal{C}) := \bigcup_{\theta \in \mathcal{C}} \{ E_n(\varepsilon, \theta) \} . \] (11)

Thus, in order to study the spectrum \( \sigma(-\Delta_N^{\Omega_\varepsilon}) \), we need to analyze each \( E_n(\varepsilon, \mathcal{C}) \) which is called \( n \)th band of \( \sigma(-\Delta_N^{\Omega_\varepsilon}) \).

For each \( \theta \in \mathcal{C} \), consider the unitary operator \( \mathcal{W}_\theta \) given by (20) in Section 5. Define \( \tilde{T}^\varepsilon_\theta := \mathcal{W}_\theta T^\varepsilon_\theta \mathcal{W}_\theta^{-1}, \text{ dom } \tilde{T}^\varepsilon_\theta = \mathcal{W}_\theta (\text{dom } T^\varepsilon_\theta) \). Due to the definition of \( \mathcal{W}_\theta \), each domain \( \text{dom } \tilde{T}^\varepsilon_\theta \) is independent of \( \theta \). Thus, in that same section, we prove that \( \{ \tilde{T}^\varepsilon_\theta, \theta \in \mathcal{C} \} \) is a type A analytic family. This fact ensures that \( E_n(\varepsilon, \theta), \ n = 1, 2, 3, \cdots , \) are real analytic functions. In addition to this information, another important point to prove Theorem 1 is to know an asymptotic behavior of the eigenvalues \( E_n(\varepsilon, \theta) \) as \( \varepsilon \) tends to 0. For each \( \theta \in \mathcal{C} \), consider the one dimensional self-adjoint operator

\[ T^\theta_w := (-i \partial_s + \theta)^2 w + \frac{h''(s)}{h(s)} w + c w, \quad \text{in } L^2[0, L) , \] (12)

where the functions in \( \text{dom } T^\theta_w \) satisfy the conditions \( w(0) = w(L) \) and \( w'(0) = w'(L) \). For simplicity, write \( Q := [0, L] \times S \). Define the closed subspace \( \mathcal{L} := \{ w(s) : w \in L^2[0, L) \} \subset L^2(Q) \). Note that this subspace is directly related to the fact that the first eigenvalue of the Neumann Laplacian in a bounded region is zero (and the constant function is the corresponding eigenfunction). Consider the unitary operators \( \mathcal{X}_\varepsilon \) and \( \Pi_\varepsilon \) defined by (22) and (33), respectively, in Section 7. Our main tool to find an asymptotic behavior for \( E_n(\varepsilon, \theta) \) is given by

**Theorem 2.** There exists a number \( K > 0 \) so that, for all \( \varepsilon > 0 \) small enough,

\[ \sup_{\theta \in \mathcal{C}} \left\{ \left\| \mathcal{X}_\varepsilon^{-1} \left( T^\varepsilon_\theta \right)^{-1} \mathcal{X}_\varepsilon - \left( \mathcal{X}_\varepsilon^{-1} T^\theta \mathcal{X}_\varepsilon^{-1} \Pi_\varepsilon \oplus 0 \right) \right\| \right\} \leq K \varepsilon , \]

where 0 is the null operator on the subspace \( \mathcal{L}^\perp \).

Note that the effective operator \( T^\theta_w \) depends only on a potential induced by the deformation \( h(s) \). The bend and twist effects do not influence \( T^\varepsilon_\theta \). This situation change if the Dirichlet condition is considered at the boundary \( \partial \Omega_\varepsilon \); see [16] for a comparison of results.
The spectrum of $T^\theta$ is purely discrete; denote by $\nu_n(\theta)$ its $n$th eigenvalue counted with multiplicity. Let $\mathcal{K}$ be a compact subset of $\mathcal{C}$ which contains an open interval and does not contain the points $\pm\pi/L$ and $0$. Given $E > 0$, without lost of generality, we can suppose that, for all $\theta \in \mathcal{K}$, the spectrum of $T^\theta$ below $E$ consists of exactly $n_0$ eigenvalues $\{E_n(\varepsilon, \theta)\}_{n=1}^{n_0}$. As a consequence of Theorem 2

**Corollary 1.** For any $n_0 \in \mathbb{N}$, there exists $\varepsilon_{n_0} > 0$ so that, for all $\varepsilon \in (0, \varepsilon_{n_0})$,

$$E_n(\varepsilon, \theta) = \nu_n(\theta) + O(\varepsilon),$$

(13)

holds for each $n = 1, 2, \cdots, n_0$, uniformly in $\mathcal{K}$.

**Proof of Theorem 1:** Given $E > 0$ we can suppose that, for all $\theta \in \mathcal{K}$, the spectrum of $T^\theta$ below $E$ consists of exactly $n_0$ eigenvalues $\{E_n(\varepsilon, \theta)\}_{n=1}^{n_0}$. As already mentioned, the considerations of Section 5 ensure that $E_n(\varepsilon, \theta)$, $n = 1, 2, \cdots, n_0$, are real analytic functions. The next step is to show that each $E_n(\varepsilon, \theta)$ is nonconstant. Consider the functions $\nu_n(\theta)$, $\theta \in \mathcal{K}$. By Theorem XIII.89 in [14], they are nonconstant. By Corollary 2 there exists $\varepsilon_{E} > 0$ so that (14) holds true for $n = 1, 2, \cdots, n_0$, uniformly in $\theta \in \mathcal{K}$, for all $\varepsilon \in (0, \varepsilon_{E})$. Note that $\varepsilon_{E} > 0$ depends on $n_0$, i.e., the thickness of the tube depends on the length of the energies to be covered. By Section XIII.16 in [14], the conclusion follows.

As already mentioned, the spectrum of $-\Delta^N_{\partial \Omega}$ coincides with the union of bands; see [11]. It is natural to question the existence of gaps in its structure. This subject was studied in [13]. In that work, the author ensured the existence of gaps. However, we give an alternative proof for this result.

At first, it is possible to organize the eigenvalues $\{E_n(\varepsilon, \theta)\}_{n \in \mathbb{N}}$ of $T^\theta$ in order to obtain a non-decreasing sequence. We keep the same notation and write

$$E_1(\varepsilon, \theta) \leq E_2(\varepsilon, \theta) \leq \cdots \leq E_n(\varepsilon, \theta) \cdots, \quad \theta \in \mathcal{C}.$$  

In this step the functions $E_n(\varepsilon, \theta)$ are continuous and piece-wise analytic in $\mathcal{C}$ (see Chapter 7 in [11]); each $E_n(\varepsilon, \mathcal{C})$ is either a closed interval or a one point set. In this case, similar to Corollary 1 we have

**Corollary 2.** For any $n_0 \in \mathbb{N}$, there exists $\varepsilon_{n_0} > 0$ so that, for all $\varepsilon \in (0, \varepsilon_{n_0})$,

$$E_n(\varepsilon, \theta) = \nu_n(\theta) + O(\varepsilon),$$

(14)

holds for each $n = 1, 2, \cdots, n_0$, uniformly in $\mathcal{C}$.

As a consequence

**Theorem 3.** Suppose that $h''(s)/h(s)$ is not constant. Then, there exist $n_1 \in \mathbb{N}$, $\varepsilon_{n_1+1} > 0$ and $C_{n_1} > 0$ so that, for all $\varepsilon \in (0, \varepsilon_{n_1+1})$,

$$\min_{\theta \in \mathcal{C}} E_{n_1+1}(\varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n_1}(\varepsilon, \theta) = C_{n_1} + O(\varepsilon).$$

Theorem 2 ensures that at least one gap appears in the spectrum $\sigma(-\Delta^N_{\partial \Omega})$, for all $\varepsilon > 0$ small enough. We highlighted that the deformation at the boundary $\partial \Omega^\varepsilon$ caused by $h(s)$ generates this effect. The proof of Theorem 3 is based on arguments of [4] 17.

**Remark 1.** Due to the characteristics of $h$, if $h$ is not constant, we always have that $h''/h$ is not constant. In fact, suppose $h''/h = C$. Without loss of generality, assume $C > 0$. By condition (1), we must have $h'' > 0$, i.e., $h'$ is strictly increasing. But this does not occur because $h'$ is $L$-periodic.
Remark 2. Under conditions of Theorems 1 and 3 we have the existence at least one gap in the absolutely continuous spectrum of $-\Delta^N_{H_0}$. In fact, it is enough to choose $\varepsilon > 0$ small enough and an appropriate $E > 0$.

Although we have proved Theorem 1 in this Introduction, the proof of Theorem 3 will be presented in Section 8.

This work is written as follows. In Section 2 we construct with details the tube $\Omega_\varepsilon$. In Section 3 we perform a change of coordinates so that $\Omega_\varepsilon$ is homeomorphic to the straight tube $\mathbb{R} \times S$; as well as the expression for the quadratic form (2) in the new variables. In Section 4 we realize the Floquet-Bloch decomposition mentioned in (9). In Section 5 we discuss analyticity properties of the functions $E_n(\varepsilon, \theta)$ an $\psi_n(\varepsilon, \theta)$, $n = 1, 2, 3, \ldots$. Section 6 is dedicated to study the Neumann problem in the cross section $S$. Section 7 is intended at proofs of Theorem 2 and Corollary 2 (the proof of Corollary 1 is similar to the proof of Corollary 2 it will omitted in this text). In Section 8 we prove Theorem 3. Along the text, the symbol $K$ is used to denote different constants and it never depends on $\theta$.

2 Geometry of the domain

Let $r : \mathbb{R} \to \mathbb{R}^3$ be a simple $C^3$ curve in $\mathbb{R}^3$ parametrized by its arc-length parameter $s$. We suppose that $r$ is periodic, i.e., there exists $L > 0$ and a nonzero vector $\vec{u}$ so that

$$r(s + L) = \vec{u} + r(s), \quad \forall s \in \mathbb{R}.$$  

The curvature of $r$ at the position $s$ is $k(s) := \|r''(s)\|$. We choose the usual orthonormal triad of vector fields $\{T(s), N(s), B(s)\}$, the so-called Frenet frame, given the tangent, normal and binormal vectors, respectively, moving along the curve and defined by

$$T = r'; \quad N = k^{-1}T'; \quad B = T \times N.$$  \quad (15)

To justify the construction (15), it is assumed that $k > 0$, but if $r$ has a piece of a straight line (i.e., $k = 0$ identically in this piece), usually one can choose a constant Frenet frame instead. It is possible to combine constant Frenet frames with the Frenet frame (15) and so obtaining a global $C^2$ Frenet frame; see [12], Theorem 1.3.6. In each situation we assume that a global Frenet frame exists and that the Frenet equations are satisfied, that is,

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$  \quad (16)

where $\tau(s)$ is the torsion of $r(s)$, actually defined by (16). Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a $L$-periodic and $C^2$ function so that $\alpha(0) = 0$, and $S$ an open, bounded, connected and smooth (nonempty) subset of $\mathbb{R}^2$. Let $h : \mathbb{R} \to \mathbb{R}$ be a $L$-periodic and $C^2$ function satisfying (1); see Introduction. For $\varepsilon > 0$ small enough and $y = (y_1, y_2) \in S$, write

$$\tilde{x}(s, y) = r(s) + \varepsilon h(s)y_1 N_\alpha(s) + \varepsilon h(s)y_2 B_\alpha(s)$$

and consider the domain

$$\Omega_\varepsilon = \{\tilde{x}(s, y) \in \mathbb{R}^3 : s \in \mathbb{R}, y = (y_1, y_2) \in S\},$$

where

$$\begin{align*}
N_\alpha(s) &:= \cos \alpha(s)N(s) + \sin \alpha(s)B(s), \\
B_\alpha(s) &:= -\sin \alpha(s)N(s) + \cos \alpha(s)B(s).
\end{align*}$$
Roughly speaking, this tube \( \Omega_\varepsilon \) is obtained by putting the region \( \varepsilon h(s)S \) along the curve \( r(s) \), which is simultaneously rotated by an angle \( \alpha(s) \) with respect to the cross section at the position \( s = 0 \).

### 3 Change of coordinates

Consider the Neumann Laplacian \(-\Delta^N_\omega\), i.e., the self-adjoint operator associated with the quadratic form

\[
b_\varepsilon(\psi) := \int_{\Omega_\varepsilon} |\nabla \psi|^2 \, d\varepsilon, \quad \text{dom } b_\varepsilon = H^1(\Omega_\varepsilon).
\]

Fix a number \( c > 0 \). For technical reasons, we consider the quadratic form

\[
d_\varepsilon^c(\psi) := \int_{\Omega_\varepsilon} \left(|\nabla \psi|^2 + c|\psi|^2\right) \, ds dy, \quad \text{dom } d_\varepsilon^c = H^1(\Omega_\varepsilon).
\]

For simplicity of notation, the symbol \( c \) will be omitted; \( d_\varepsilon(\psi) := d_\varepsilon^c(\psi) \).

In this section we perform a change of the variables so that the integration region in (17), and consequently the domain of the quadratic form \( d_\varepsilon(\psi) \), does not depend on \( \varepsilon \). For this, consider the mapping

\[
F_\varepsilon : \mathbb{R} \times S \rightarrow \Omega_\varepsilon \quad (s, y_1, y_2) \mapsto r(s) + \varepsilon h(s)y_1 N_\alpha(s) + \varepsilon h(s)y_2 B_\alpha(s).
\]

Since \( h \in L^\infty(\mathbb{R}) \), \( F_\varepsilon \) will be a (global) diffeomorphism for \( \varepsilon > 0 \) small enough.

In the new variables the domain of \( d_\varepsilon(\psi) \) turns to be \( H^1(\mathbb{R} \times S) \). On the other hand, the price to be paid is a nontrivial Riemannian metric \( G = G^\alpha_\varepsilon \) which is induced by \( F_\varepsilon \) i.e.,

\[
G = (G_{ij}), \quad G_{ij} = (e_i, e_j), \quad 1 \leq i, j \leq 3,
\]

where

\[
e_1 = \frac{\partial F_\varepsilon}{\partial s}, \quad e_2 = \frac{\partial F_\varepsilon}{\partial y_1}, \quad e_3 = \frac{\partial F_\varepsilon}{\partial y_2}.
\]

Some calculations show that in the Frenet frame

\[
J := \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = \begin{pmatrix}
\beta_\varepsilon & \sigma_\varepsilon & \delta_\varepsilon \\
0 & \varepsilon h \cos \alpha & \varepsilon h \sin \alpha \\
0 & -\varepsilon h \sin \alpha & \varepsilon h \cos \alpha
\end{pmatrix},
\]

where \( \beta_\varepsilon(s, y) \) is given by (15) in the Introduction, and

\[
\sigma_\varepsilon(s, y) := -\varepsilon h(s)(\tau + \alpha')(s)\alpha_\varepsilon(s, y) + \varepsilon h'(s)\eta_\varepsilon(s, y),
\]

\[
\delta_\varepsilon(s, y) := \varepsilon h(s)(\tau + \alpha')(s)\alpha_\varepsilon(s, y) + \varepsilon h'(s)\eta_\varepsilon(s, y),
\]

\[
\alpha_\varepsilon(s) := (\cos \alpha(s), -\sin \alpha(s)),
\]

\[
\eta_\varepsilon(s) := (\sin \alpha(s), \cos \alpha(s)).
\]

The inverse matrix of \( J \) is given by

\[
J^{-1} = \begin{pmatrix}
\beta_\varepsilon^{-1} & \bar{\sigma}_\varepsilon & \bar{\delta}_\varepsilon \\
0 & (\varepsilon h)^{-1} \cos \alpha & -(\varepsilon h)^{-1} \sin \alpha \\
0 & (\varepsilon h)^{-1} \sin \alpha & (\varepsilon h)^{-1} \cos \alpha
\end{pmatrix},
\]
where
\[
\tilde{\sigma}(s, y) := \frac{1}{\beta \epsilon} \left[ (\tau + \alpha')(s) y_2 - \frac{h'(s)}{h(s)} y_1 \right], \quad \tilde{\delta}(s, y) := -\frac{1}{\beta \epsilon} \left[ (\tau + \alpha')(s) y_1 - \frac{h'(s)}{h(s)} y_2 \right].
\]

Note that $JJ' = G$ and $\det J = |\det G|^{1/2} = \epsilon^2 h^2(s)\beta(s, y) > 0$. Thus, $F_\epsilon$ is a local diffeomorphism. By requiring that $F_\epsilon$ is injective (i.e., the tube is not self-intersecting), a global diffeomorphism is obtained.

Introducing the notation
\[
\|\psi\|_G^2 := \int_{\mathbb{R} \times S} |\psi(s, y)|^2 h^2(s)\beta(s, y) ds dy,
\]
we obtain a sequence of quadratic forms
\[
t_\epsilon(\psi) = \|J^{-1}\nabla \psi\|_G^2 + c\|\psi\|_G, \quad \text{dom } t_\epsilon = H^1(\mathbb{R} \times S).
\]  

(18)

More precisely, the change of coordinates above is obtained by the unitary transformation
\[
\Psi_\epsilon : \quad L^2(\Omega^c) \rightarrow L^2(\mathbb{R} \times S, h^2\beta ds dy),
\]
\[
\Psi_\epsilon \rightarrow \epsilon \Psi \circ F_\epsilon.
\]

After the norms are written out, by (18) we obtain
\[
t_\epsilon(\psi) = \int_{\mathbb{R} \times S} \left( \frac{h^2}{\beta \epsilon} \left| \partial_{s,y} \psi \right|^2 + \frac{\beta \epsilon}{\epsilon^2} |\nabla_y \psi|^2 + c h^2 \beta_\epsilon |\psi|^2 \right) ds dy,
\]
\[
\text{dom } t_\epsilon = H^1(\mathbb{R} \times S); \text{ recall the definition of } \partial_{s,y} \psi \text{ in the Introduction. Note that dom } t_\epsilon
\]
\[
is a subspace of the Hilbert space $L^2(\mathbb{R} \times S, h^2\beta ds dy)$.
\]

Denote by $T_\epsilon$ the self-adjoint operator associated with the quadratic form $t_\epsilon(\psi)$. In fact, $\Psi_\epsilon(- \Delta \Omega^c + c \epsilon) \Psi_\epsilon^{-1} \psi = T_\epsilon \psi$, dom $T_\epsilon = \Psi_\epsilon(\text{dom } (- \Delta \Omega^c))$. Some calculations show that $T_\epsilon$ has action and domain given by (3) and (4), respectively. See Appendix A of this work for a discussion about quadratic forms and operators associated with them.

4 Floquet-Bloch decomposition

Since the coefficients of $T_\epsilon$ are periodic with respect to $s$, we perform the Floquet-Bloch reduction over the Brillouin zone $\mathcal{C} = [-\pi/L, \pi/L]$. For simplicity of notation, we write $\Omega := \mathbb{R} \times S$ and
\[
\mathcal{H}_\epsilon := L^2(\Omega, h^2\beta ds dy), \quad \mathcal{H}'_\epsilon := L^2(\mathbb{R} \times S, h^2\beta ds dy).
\]

Recall $Q = [0, L) \times S$ and, for each $\theta \in \mathcal{C}$, the operator $T_\epsilon^{\theta}$ given by (2) in the Introduction.

Lemma 1. There exists a unitary operator $U_\epsilon : \mathcal{H}_\epsilon \rightarrow \int_\mathcal{C} \mathcal{H}'_\epsilon d\theta$, so that
\[
U_\epsilon T_\epsilon U_\epsilon^{-1} = \int_\mathcal{C} T_\epsilon^{\theta} d\theta.
\]  

(19)

Furthermore, for each $\theta \in \mathcal{C}$, $T_\epsilon^{\theta}$ is self-adjoint.

Proof. For $(\theta, s, y) \in \mathcal{C} \times [0, L) \times S$ and $f \in \mathcal{H}_\epsilon$ consider the unitary operator
\[
U_\epsilon f(\theta, s, y) := \sum_{n \in \mathbb{Z}} \sqrt{\frac{L}{2\pi}} e^{-inL\theta - isy} f(s + Ln, y).
\]

Some calculations, which will be omitted here, lead to the formula (19). For the claim that each $T_\epsilon^{\theta}$ is self-adjoint, see Appendix A.
Remark 3. For each \( \theta \in \mathcal{C} \), the quadratic form \( t^\theta_\varepsilon(\psi) \) associated with the operator \( T^\theta_\varepsilon \) is given by
\[
t^\theta_\varepsilon(\psi) = \int_Q \left( \frac{h^2}{\beta_\varepsilon} \partial_y R^h \psi + i \theta |\psi|^2 + \frac{\beta_\varepsilon}{\varepsilon^2} \nabla_y |\psi|^2 + c h^2 |\psi|^2 \right) ds dy,
\]
\[\text{dom } t^\theta_\varepsilon = \{ \psi \in H^1(Q) : \psi(0, \cdot) = \psi(L, \cdot) \text{ in } L^2(S) \}.
\]
Again, see Appendix A of this work for a discussion about this subject.

5 Analyticity properties

The goal of this section is to ensure that, for each \( n = 1, 2, \cdots \), the functions \( E_n(\varepsilon, \theta) \) and \( \psi_n(\varepsilon, \theta) \), defined by (10) in the Introduction, are real analytic functions.

The first step is to perform a change of variables in order to turn the domain \( \text{dom } T^\theta_\varepsilon \) independent of the parameter \( \theta \).

Recall the definitions of \( \partial_x R^h \) and \( R^h \) given by (8) and (7), respectively; see Introduction. Based on [7], let \( \mu : Q \to \mathbb{R} \) be a real function, smooth in the closed set \( \overline{Q} \), satisfying
1. \( \mu \) is \( L \)-periodic with respect to \( s \), i.e., \( \mu(0, y) = \mu(L, y) \), for all \( y \in S \);
2. \( \frac{\partial_x R^h \mu}{\partial N} = \frac{h^2}{\beta_\varepsilon} \langle R^h, N \rangle \).

Now, define the unitary operator
\[
\mathcal{W}_\theta : \mathcal{H}^\varepsilon_\eta \to \mathcal{H}^\varepsilon_\eta, \quad \eta \mapsto e^{i\theta \mu} \eta, \tag{20}
\]
and the self-adjoint operator
\[
\tilde{T}^\theta_\varepsilon = \mathcal{W}_\theta T^\theta_\varepsilon \mathcal{W}^{-1}_\theta, \quad \text{dom } \tilde{T}^\theta_\varepsilon = \mathcal{W}_\theta(\text{dom } T^\theta_\varepsilon).
\]

Recall the action of \( \partial_{s,y} R^h \psi \) by (6) (again, see Introduction of this work). Some straightforward calculations show that
\[
\tilde{T}^\theta_\varepsilon \psi = -\frac{1}{h^2 \beta_\varepsilon} \left( \partial_s + \text{div}_y R^h + i \theta (1 - \partial_{s,y} \mu) \right) \frac{h^2}{\beta_\varepsilon} \left( \partial_{s,y} \psi + i \theta (1 - \partial_{s,y} \mu) \psi \right)
\]
\[\quad - \frac{1}{\varepsilon^2 h^2 \beta_\varepsilon} \sum_{j=1}^2 (\partial_{y_j} - i \theta \partial_{y_j} \mu) \beta_\varepsilon (\partial_{y_j} - i \theta \partial_{y_j} \mu) \psi + c \psi,
\]
and,
\[
\text{dom } \tilde{T}^\theta_\varepsilon = \left\{ \psi \in H^2(Q) : \psi(0, \cdot) = \psi(L, \cdot) \text{ and } \partial^\varepsilon R^h \psi(0, \cdot) = \partial^\varepsilon R^h \psi(L, \cdot) \text{ in } L^2(S), \right. \]
\[\left. \frac{\partial R^h \psi}{\partial N} = 0 \text{ in } L^2([0, L] \times \partial S) \right\}.
\]

Since the domains \( \text{dom } \tilde{T}^\theta_\varepsilon \) do not depend on \( \theta \), we have

Lemma 2. \( \{ \tilde{T}^\theta_\varepsilon, \theta \in \mathcal{C} \} \) is a type A analytic family.
Denote by $\lambda$ associated with the quadratic form $\langle u, v \rangle_S$ equipped with the inner product $\langle u, v \rangle_{L^2(S)} := \int_S u \overline{v} \beta \, dy$. Define $\psi$ an important step to prove Theorem 2. Consequently, the analyticity of $\psi$ ensures the analyticity of the functions $E_n(\varepsilon, \theta)$, $n = 1, 2, 3, \cdots$. The proof of Lemma 2 follows the same steps of the proof of Lemma 1 in [16].

Since the operators $T_\varepsilon^\theta$ and $\tilde{T}_\varepsilon^\theta$ are unitarily equivalent, they have the same spectrum. Thus, the eigenvalues of $\tilde{T}_\varepsilon^\theta$ are given by $E_n(\varepsilon, \theta)$, $n = 1, 2, 3, \cdots$. For each $n = 1, 2, 3, \cdots$, the corresponding eigenfunction is

$$\tilde{\psi}_n(\varepsilon, \theta) := e^{i\theta \mu} \psi_n(\varepsilon, \theta).$$

Lemma 2 ensures the analyticity of the functions $E_n(\varepsilon, \theta)$, $\tilde{\psi}(\varepsilon, \theta)$, $n = 1, 2, 3, \cdots$. Consequently, the analyticity of $\tilde{\psi}_n(\varepsilon, \theta)$, $n = 1, 2, 3, \cdots$.

### 6 Cross section problem

In this section we investigate the Neumann problem in the cross section $S$ which is an important step to prove Theorem 2.

For each $s \in [0, L)$ and $\varepsilon > 0$ consider the Hilbert space $H^s_\varepsilon := L^2(S, \beta \, dy)$ which is equipped with the inner product $\langle u, v \rangle_{H^s_\varepsilon} := \int_S u \overline{v} \beta \, dy$. Define the quadratic form

$$q^s_\varepsilon(u) := \int_S |\nabla_y u|^2 \beta \, dy,$$

and denote by $Q^s_\varepsilon$ the self-adjoint operator associated with it. The geometric features of $S$ ensure that $Q^s_\varepsilon$ has compact resolvent. Denote by $\lambda_1^s(n)$ the $n$th eigenvalue of $Q^s_\varepsilon$ counted with multiplicity and $u^s_\varepsilon(n)$ the corresponding normalized eigenfunction, i.e.,

$$0 = \lambda^s_1(1) \leq \lambda^s_2(2) \leq \lambda^s_3(3) \leq \cdots,$$

and

$$Q^s_\varepsilon u^s_\varepsilon(n) = \lambda^s_\varepsilon(n) u^s_\varepsilon(n), \quad n = 1, 2, 3, \cdots.$$

We pay attention that, for each $s \in [0, L)$ and $\varepsilon > 0$, $\lambda^s_1(1) = 0$ and its corresponding eigenfunction $u^s_\varepsilon(1)$ is constant.

Introduce the unitary operator

$$V^s_\varepsilon : L^2(S) \rightarrow H^s_\varepsilon, \quad u \mapsto \beta^{-1/2}_\varepsilon u,$$

and define

$$q^s_\varepsilon(V^s_\varepsilon u) := q^s_\varepsilon(V^s_\varepsilon u), \quad \text{dom } q^s_\varepsilon := H^1(S).$$

Some calculations show that

$$q^s_\varepsilon(u) := \int_S |\nabla_y u|^2 \beta \, dy,$$

and define

$$q^s_\varepsilon(u) := q^s_\varepsilon(V^s_\varepsilon u), \quad \text{dom } q^s_\varepsilon := H^1(S).$$

Let $-\Delta^N_S$ be the Neumann Laplacian operator in $S$, i.e., the self-adjoint operator associated with the quadratic form

$$q(u) := \int_S |\nabla_y u|^2 \, dy, \quad \text{dom } q = H^1(S).$$

Denote by $\lambda^n$ the $n$th eigenvalue of $-\Delta^N_S$ counted with multiplicity and by $u_n$ the corresponding normalized eigenfunction, i.e.,

$$0 = \lambda^1 < \lambda^2 \leq \lambda^3, \cdots,$$

and

$$-\Delta^N_S u^n = \lambda^n u^n, \quad n = 1, 2, 3, \cdots.$$
Theorem 4. Fix $c_3 > 0$. There exists $K > 0$ so that, for all $\varepsilon > 0$ small enough,
\[
\sup_{s \in [0, L]} \{ \| (V^\varepsilon)^{-1} (Q^\varepsilon + c_3 1)^{-1} V^\varepsilon - (-\Delta_S^\varepsilon + c_3 1)^{-1} \| \} \leq K \varepsilon.
\]

Proof. At first, we add the constant $c_3 > 0$ only due to a technical detail. Some calculations show that there exists a number $K > 0$ so that, for all $\varepsilon > 0$ small enough,
\[
\| (q^\varepsilon(u) + c_3 \| u \|_{L^2(S)}) - (q(u) + c_3 \| u \|_{L^2(S)}) \| \leq \varepsilon K (q(u) + c_3 \| u \|_{L^2(S)}),
\]
\[\forall u \in H^1(S), \forall s \in [0, L].\] Now, the result follows by Theorem 3 in \cite{2}. \hfill \Box

As a consequence of Theorem \[4\] for all $\varepsilon > 0$ small enough,
\[
\left| \frac{1}{\lambda_2^\varepsilon(s) + c_3} - \frac{1}{\lambda_2^\varepsilon(s) + c_3^2} \right| \leq \varepsilon K, \forall s \in [0, L).
\]
Then,
\[0 < \gamma(\varepsilon) \leq \lambda_2^\varepsilon(s), \forall s \in [0, L),\]
where $\gamma(\varepsilon) := (\lambda^2 - \varepsilon c_3 K (\lambda^2 + c_3^2)) / (1 + \varepsilon K (\lambda^2 + c_3^2)) \to \lambda^2 > 0$, as $\varepsilon \to 0$. Thus, there exists $\tilde{\gamma} > 0$ so that, for all $\varepsilon > 0$ small enough,
\[0 < \tilde{\gamma} \leq \gamma(\varepsilon) \leq \lambda_2^\varepsilon(s), \forall s \in [0, L). (21)\]

7 Proof of Theorem \[2\] and Corollary \[2\]

Recall $\mathcal{H}_x^\varepsilon = L^2(Q, h^2 \beta_x ds dy)$. Consider the Hilbert space $\tilde{H}_x := L^2(Q, \beta_x ds dy)$ equipped with the inner product $\langle \psi, \varphi \rangle_{\tilde{H}_x} = \int_Q \overline{\psi} \varphi \beta_x ds dy$. At first, we perform a change of variables in order to work in $\tilde{H}_x$. This change is given by the unitary operator
\[
\mathcal{X}^\varepsilon : \tilde{H}_x \rightarrow \mathcal{H}_x^\varepsilon, \quad \psi \mapsto h^{-1} \psi.
\]

We start to study the quadratic form
\[
s^\theta_x(\psi) := t^\theta_x(\mathcal{X}^\varepsilon(\psi)), \quad \text{dom } s^\theta_x := \mathcal{X}^{-1}_x(\text{dom } t^\theta_x).
\]

One can show
\[
s^\theta_x(\psi) = \int_Q h^2 \beta_x \left| \partial_{s,y} h^{-1} \psi \right|^2 ds dy + \int_Q \frac{\beta_x}{\varepsilon^2} |\nabla_y (h^{-1} \psi)|^2 ds dy + c \int_Q |h^{-1} \psi|^2 h^2 \beta_x ds dy
\]
\[= \int_Q \frac{1}{\beta_x} \left| \partial_{s,y} h^{-1} \psi + h_\theta(s) \psi \right|^2 ds dy + \int_Q \beta_x \left| \nabla_y \psi \right|^2 ds dy + c \int_Q |\psi|^2 \beta_x ds dy,
\]
where $h_\theta(s) := i \theta - (h'(s) / h(s))$.

Since $h$ is a bounded and $L$-periodic function,
\[\text{dom } s^\theta_x = \{ \psi \in H^1(Q) : \psi(0, \cdot) = \psi(L, \cdot) \text{ in } L^2(S) \}.\]
Here, $H^1(Q)$ is a subspace of the Hilbert space $\tilde{\mathcal{H}}_\varepsilon$.

Denote by $S^\theta_\varepsilon$ the self-adjoint operator associated with the quadratic form $s^\theta_\varepsilon(\psi)$. Actually, dom $S^\theta_\varepsilon \subset$ dom $s^\theta_\varepsilon$ and

$$X^{-1}_\varepsilon(T^\theta_\varepsilon)X_\varepsilon = S^\theta_\varepsilon.$$

On the other hand, we define

$$m^\theta_\varepsilon(\psi) := \int_Q \beta_\varepsilon \left| \partial_\nu \psi + h_\theta(s)\psi \right|^2 \text{d} s \text{d} y$$

$$+ \int_Q \frac{\beta_\varepsilon}{\varepsilon^2 h^2} |\nabla_y \psi|^2 \text{d} s \text{d} y + c \int_Q |\psi|^2 \beta_\varepsilon \text{d} s \text{d} y,$$

dom $m^\theta_\varepsilon :=$ dom $s^\theta_\varepsilon$. Denote by $M^\theta_\varepsilon$ the self-adjoint operator associated with $m^\theta_\varepsilon(\psi)$.

**Proposition 1.** There exists a number $K > 0$ so that, for all $\varepsilon > 0$ small enough,

$$\sup_{\theta \in \mathcal{C}} \left\{ \| (S^\theta_\varepsilon)^{-1} - (M^\theta_\varepsilon)^{-1} \| \right\} \leq K \varepsilon.$$

The main point in this proposition is that $\beta_\varepsilon \to 1$ uniformly as $\varepsilon \to 0$. Its proof is very similar to the proof of Theorem 3.1 in [6] and will be omitted here. For technical reasons, we start to study the sequence of operators $M^\theta_\varepsilon$.

Consider the closed subspace $L = \{ w(s) 1 : w \in L^2[0, L] \}$ of the Hilbert space $\tilde{\mathcal{H}}_\varepsilon$. Take the orthogonal decomposition $\tilde{\mathcal{H}}_\varepsilon = L \perp L^\perp$. Thus, for $\psi \in$ dom $m^\theta_\varepsilon$, one can write

$$\psi(s, y) = w(s) 1 + \eta(s, y), \quad w \in H^1[0, L], \eta \in$ dom $m^\theta_\varepsilon \cap L^\perp. \quad (23)$$

Furthermore, $w(0) = w(L)$.

Define $a_\varepsilon(s) := \int_s L \beta_\varepsilon(s, y) \text{d} y$ and introduce the Hilbert space $\mathcal{H}_{a_\varepsilon} := L^2([0, L], a_\varepsilon \text{d} s)$ equipped with the inner product $\langle w_1, w_2 \rangle_{\mathcal{H}_{a_\varepsilon}} = \int_0^L \overline{w_1(s)} w_2 a_\varepsilon \text{d} s$. Acting in $\mathcal{H}_{a_\varepsilon}$, consider the one dimensional quadratic form

$$n^\theta_\varepsilon(w) := m^\theta_\varepsilon(w 1) = \int_Q \beta_\varepsilon \left( \left| \left| \partial_s + h_\theta \right| w \right|^2 + c |w|^2 \right) \text{d} s \text{d} y$$

$$= \int_0^L \left( a_\varepsilon(s) \left| \left| \partial_s + h_\theta \right| w \right|^2 + c a_\varepsilon(s) |w|^2 \right) \text{d} s,$$

dom $n^\theta_\varepsilon := \{ w \in \mathcal{H}^1[0, L]; w(0) = w(L) \}$. Denote by $N^\theta_\varepsilon$ the self-adjoint operator associated with $n^\theta_\varepsilon(w)$.

**Proof of Theorem 2.** We begin with some observations. If $\eta \in$ dom $m^\theta_\varepsilon \cap L^\perp$,

$$\int_Q w(s) \eta(s, y) \beta_\varepsilon \text{d} s \text{d} y = 0, \quad \forall w \in L. \quad (24)$$

Consequently,

$$\int_S \eta(s, y) \beta_\varepsilon(s, y) \text{d} y = 0 \quad \text{a.e. } s, \quad (25)$$

and

$$\int_S \beta_\varepsilon(s, y) \partial_s \eta(s, y) \text{d} y = - \int_S \partial_s \beta_\varepsilon(s, y) \eta(s, y) \text{d} y \quad \text{a.e. } s. \quad (26)$$

Furthermore, for each $s \in [0, L)$, the Min Max Principle ensures that

$$\int_S |\nabla_y \eta(s, y)|^2 \beta_\varepsilon \text{d} y \geq \lambda^2_1(s) \int_S |\eta|^2 \beta_\varepsilon \text{d} y; \quad (27)$$
Thus, Proposition 3.1 in [8], ensures that, for all \( \varepsilon > 0 \) for some number \( K > c \). By defining \( m_\varepsilon^\theta(w) \) and with

\[
m_\varepsilon^\theta(\psi) = n_\varepsilon^\theta(w) + m_\varepsilon^\theta(w, \eta) + m_\varepsilon^\theta(\eta).
\]

We are going to check that there are functions \( c(\varepsilon) \), \( 0 \leq p(\varepsilon) \) and \( 0 \leq q(\varepsilon) \), which do not depend on \( \theta \in \mathcal{C} \), so that \( n_\varepsilon^\theta(w) \), \( m_\varepsilon^\theta(w, \eta) \) and \( m_\varepsilon^\theta(\eta) \) satisfy the following conditions:

\[
n_\varepsilon^\theta(w) \geq c(\varepsilon)\|w\|_{\mathcal{H}_{\varepsilon}}^2, \quad \forall w \in \text{dom } n_\varepsilon^\theta, \quad c(\varepsilon) \geq c_0;
\]

\[
m_\varepsilon^\theta(\eta) \geq p(\varepsilon)\|
\]

\[
|m_\varepsilon^\theta(w, \eta)|^2 \leq q(\varepsilon)^2 n_\varepsilon^\theta(w) m_\varepsilon^\theta(\eta), \quad \forall \psi \in \text{dom } m_\varepsilon^\theta;
\]

and with

\[
p(\varepsilon) \to \infty, \quad c(\varepsilon) = O(p(\varepsilon)), \quad q(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.
\]

Thus, Proposition 3.1 in [8], ensures that, for all \( \varepsilon > 0 \) small enough,

\[
\sup_{\theta \in \mathcal{C}} \left\{ \|(M_\varepsilon^\theta)^{-1} - ((N_\varepsilon^\theta)^{-1} \oplus 0)\| \right\} \leq p(\varepsilon)^{-1} + K q(\varepsilon)c(\varepsilon)^{-1},
\]

for some number \( K > 0 \). Recall \( \mathbf{0} \) is the null operator on the subspace \( \mathcal{L}^\perp \).

Clearly,

\[
n_\varepsilon^\theta(w) \geq c\|w\|_{\mathcal{H}_{\varepsilon}}^2, \quad \forall w \in \text{dom } n_\varepsilon^\theta.
\]

By defining \( c(\varepsilon) := c \), it follows the condition (28).

Recall the condition (11) in the Introduction. Note that

\[
m_\varepsilon^\theta(\eta) \geq \frac{1}{\varepsilon^2} \int_Q \frac{\beta^\varepsilon|\nabla_y \eta|^2}{(\text{div}_\varepsilon + h_\varepsilon)} \text{dsdy} \geq \frac{1}{\varepsilon^2 c_2^2} \int_Q \beta^\varepsilon|\nabla_y \eta|^2 \text{dsdy}, \quad \forall \eta \in \text{dom } m_\varepsilon^\theta \cap \mathcal{L}^\perp.
\]

By (21) and (27), for all \( \varepsilon > 0 \) small enough,

\[
m_\varepsilon^\theta(\eta) \geq \frac{\tilde{\gamma}}{\varepsilon c_2^2} \int_Q |\eta|^2 \beta^\varepsilon \text{dsdy}, \quad \forall \eta \in \text{dom } m_\varepsilon^\theta \cap \mathcal{L}^\perp.
\]

Just to take \( p(\varepsilon) := \tilde{\gamma}/\varepsilon^2 c_2^2 \) and then condition (29) is satisfied.

By polarization identity,

\[
m_\varepsilon^\theta(w, \eta) \geq \int_Q \beta^\varepsilon(\text{div}_\varepsilon + h_\varepsilon) w \text{dsdy} + \int_Q \frac{\beta^\varepsilon}{\varepsilon^2 c_2^2} (\nabla_y w, \nabla_y \eta) \text{dsdy},
\]

which, by (21) and (27), is simplified to

\[
m_\varepsilon^\theta(w, \eta) = \int_Q \beta^\varepsilon(\partial_\varepsilon w + h_\varepsilon w) \partial_\varepsilon \eta \text{dsdy} + \int_Q \beta^\varepsilon(\partial_\varepsilon w + h_\varepsilon w) (\nabla_y \eta, R^h) \text{dsdy}.
\]

By (28),

\[
m_\varepsilon^\theta(w, \eta) = -\int_Q \partial_\varepsilon(\beta^\varepsilon)(\partial_\varepsilon w + h_\varepsilon w) \eta \text{dsdy} + \int_Q \beta^\varepsilon(\partial_\varepsilon w + h_\varepsilon w) (\nabla_y \eta, R^h) \text{dsdy}.
\]
Thus, Theorem 3 in [2] ensures that, for all \( \varepsilon > \theta \),
the self-adjoint operator associated with \( t \) has bounded coordinates, by Hölder inequality,

\[
|m_{\varepsilon}^\theta(w, 1, \eta)| \leq K \left( \varepsilon \int_Q |\partial_s w + h_\theta w| \, |\eta| \, ds \, dy \right) + \left( \int_Q |\partial_s w + h_\theta w| \, |\nabla_y \eta| \, ds \, dy \right)
\]

Therefore, we finish the proof of (32) where the upper bound in that inequality is

\[
K \varepsilon (m_{\varepsilon}^\theta(w))^{1/2} (m_{\varepsilon}^\theta(\eta))^{1/2},
\]

for all \( w \in \text{dom} \, n_{\varepsilon}^\theta \), for all \( \eta \in \text{dom} \, m_{\varepsilon}^\theta \cap \mathcal{L}^\perp \), for some \( K > 0 \), for all \( \varepsilon > 0 \) small enough.

Now, we can see that

\[
|m_{\varepsilon}^\theta(w, 1, \eta)| \leq K \varepsilon (n_{\varepsilon}^\theta(w))^{1/2} (m_{\varepsilon}^\theta(\eta))^{1/2}, \quad \forall w \in \text{dom} \, n_{\varepsilon}^\theta, \forall \eta \in \text{dom} \, m_{\varepsilon}^\theta \cap \mathcal{L}^\perp,
\]

for some \( K > 0 \), for all \( \varepsilon > 0 \) small enough.

Then, by taking \( q(\varepsilon) := K \varepsilon \), it is found that conditions (30) and (31) are satisfied. Therefore, we finish the proof of (32) where the upper bound in that inequality is \( K \varepsilon \).

The next step is to study the sequence of one-dimensional operators \( N_{\varepsilon}^\theta \).

In order to work in \( L^2[0, L] \) with the usual measure, we define the unitary operator

\[
\Pi_{\varepsilon} : \quad L^2[0, L] \quad \rightarrow \quad \mathcal{H}_{a_{\varepsilon}} w \quad \mapsto \quad a_{\varepsilon}^{-1/2} w,
\]

and the quadratic form

\[
o_{\varepsilon}^\theta(w) := \langle n_{\varepsilon}^\theta(w) \Pi_{\varepsilon} w \rangle
\]

\[
= \int_0^L \left( (\partial_s w + h_\theta w - (2 a_{\varepsilon})^{-1} \partial_s(a_{\varepsilon})w)^2 + c|w|^2 \right) ds,
\]

\( \text{dom} \, o_{\varepsilon}^\theta = \{ w \in \mathcal{H}^1[0, L] ; w(0) = w(L) \} \). Denote by \( O_{\varepsilon}^\theta \) the self-adjoint operator associated with \( o_{\varepsilon}^\theta(w) \). Note that \( O_{\varepsilon}^\theta = \Pi_{\varepsilon}^{-1} N_{\varepsilon}^\theta \Pi_{\varepsilon} \).

Finally, we define

\[
t^\theta(w) := \int_0^L \left( (|\partial_s w + h_\theta w|^2 + c|w|^2) \right) ds, \quad \text{dom} \, t^\theta := \text{dom} \, o_{\varepsilon}^\theta.
\]

The self-adjoint operator associated with \( t^\theta(w) \) is given by \( T^\theta \); see [12] in the Introduction.

One can show that there exists \( K > 0 \) so that, for all \( \varepsilon > 0 \) small enough,

\[
|o_{\varepsilon}^\theta(w) - t^\theta(w)| \leq K \varepsilon t^\theta(w), \quad \forall w \in \text{dom} \, t^\theta, \forall \theta \in \mathcal{C}.
\]

Thus, Theorem 3 in [2] ensures that, for all \( \varepsilon > 0 \) small enough,

\[
\sup_{\theta \in \mathcal{C}} \left\{ \| (O_{\varepsilon}^\theta)^{-1} - (T^\theta)^{-1} \| \right\} \leq K \varepsilon.
\]

It is important to mention that the constants \( K \)'s, in all this proof, do not depend on \( \theta \in \mathcal{C} \).
By Proposition \(1\) estimates \((52)\) and \((54)\), Theorem \(2\) is proven.

**Proof of Corollary 2**: Theorem \(2\) in the Introduction and Corollary 2.3 of \([10]\) imply

\[
\left| \frac{1}{E_n(\epsilon, \theta)} - \frac{1}{\nu_n(\theta)} \right| \leq K \epsilon, \quad \forall n \in \mathbb{N}, \forall \theta \in \mathcal{C},
\]

(35)

for all \(\epsilon > 0\) small enough. Then,

\[
|E_n(\epsilon, \theta) - \nu_n(\theta)| \leq K \epsilon |E_n(\epsilon, \theta)| |\nu_n(\theta)|, \quad \forall n \in \mathbb{N}, \forall \theta \in \mathcal{C},
\]

for all \(\epsilon > 0\) small enough.

The functions \(\nu_n(\theta)\) are continuous in \(\mathcal{C}\) and consequently bounded (see Theorem XIII.89 in \([14]\)). This fact and the inequality \((35)\) ensure that, for each \(\tilde{n}_0 \in \mathbb{N}\), there exists \(K_{\tilde{n}_0} > 0\), so that

\[
|E_n(\epsilon, \theta)| \leq K_{\tilde{n}_0}, \quad \forall \theta \in \mathcal{C},
\]

for all \(\epsilon > 0\) small enough.

Finally, for each \(n_0 \in \mathbb{N}\), there exists \(K_{n_0} > 0\) so that

\[
|E_n(\epsilon, \theta) - \nu_n(\theta)| \leq K_{n_0} \epsilon, \quad n = 1, 2, \ldots, n_0, \forall \theta \in \mathcal{C},
\]

for all \(\epsilon > 0\) small enough.

**8 Existence of band gaps; proof of Theorem 3**

This section is dedicated to the proof of Theorem \(3\). The steps are similar to those in \([17]\). In that work, the author studied the band gap of the spectrum of the Dirichlet Laplacian in a planar periodically curved strip.

Consider the operator

\[
T w = -w'' + \frac{h''(s)}{h(s)} w + cw, \quad \text{dom} \ T = H^2(\mathbb{R}).
\]

Recall we have denoted by \(\nu_n(\theta)\) the \(n\)th eigenvalue of \(T^\theta\). By Theorem XIII.89 in \([14]\), each \(\nu_n(\theta)\) is a continuous function in \(\mathcal{C}\). Furthermore,

(a) \(\nu_n(\theta) = \nu_n(-\theta)\), for all \(\theta \in \mathcal{C}, n = 1, 2, 3, \ldots\).

(b) For \(n\) odd (resp. even), \(\nu_n(\theta)\) is strictly monotone increasing (resp. decreasing) as \(\theta\) increases from 0 to \(\pi/L\). In particular,

\[
\nu_1(0) < \nu_1(\pi/L) \leq \nu_2(\pi/L) < \nu_2(0) \leq \cdots \leq \nu_{2n-1}(0) < \nu_{2n-1}(\pi/L) \leq \nu_{2n}(\pi/L) < \nu_{2n}(0) \leq \cdots.
\]

Now, for each \(n = 1, 2, 3, \ldots\), define

\[
B_n := \begin{cases} 
(\nu_n(0), \nu_n(\pi/L)), & \text{for } n \text{ odd,} \\
(\nu_n(\pi/L), \nu_n(0)), & \text{for } n \text{ even,}
\end{cases}
\]

and

\[
G_n := \begin{cases} 
(\nu_n(\pi/L), \nu_{n+1}(\pi/L)), & \text{for } n \text{ odd so that } \nu_n(\pi/L) \neq \nu_{n+1}(\pi/L), \\
(\nu_n(0), \nu_{n+1}(0)), & \text{for } n \text{ even so that } \nu_n(0) \neq \nu_{n+1}(0), \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

14
By Theorem XIII.90 in [14], we have \( \sigma(T) = \bigcup_{n=1}^{\infty} B_n \); \( B_n \) is called the \( j \)th band of \( \sigma(T) \), and \( G_n \) the gap of \( \sigma(T) \) if \( B_n \neq \emptyset \).

Corollary 2 implies that for any \( n_0 \in \mathbb{N} \), there exists \( \varepsilon_{n_0} > 0 \) so that, for all \( \varepsilon \in (0, \varepsilon_{n_0}), \)

\[
\max_{\theta \in \mathcal{C}} E_n(\varepsilon, \theta) = \begin{cases} 
\nu_n(\pi/L) + O(\varepsilon), & \text{for } n \text{ odd,} \\
\nu_n(0) + O(\varepsilon), & \text{for } n \text{ even,}
\end{cases}
\]

and

\[
\min_{\theta \in \mathcal{C}} E_n(\varepsilon, \theta) = \begin{cases} 
\nu_n(0) + O(\varepsilon), & \text{for } n \text{ odd,} \\
\nu_n(\pi/L) + O(\varepsilon), & \text{for } n \text{ even,}
\end{cases}
\]

hold for each \( n = 1, 2, \cdots, n_0 \). Thus, we have

**Corollary 3.** For any \( n_2 \in \mathbb{N} \), there exists \( \varepsilon_{n_2+1} > 0 \) so that, for all \( \varepsilon \in (0, \varepsilon_{n_2+1}), \)

\[
\min_{\theta \in \mathcal{C}} E_{n+1}(\varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_n(\varepsilon, \theta) = |G_n| + O(\varepsilon),
\]

holds for \( n = 1, 2, \cdots, n_2 \), where \(| \cdot |\) is the Lebesgue measure.

Besides Corollary 3, another important point to prove Theorem 3 is the following result due to Borg [4].

**Theorem 5.** (Borg) Suppose that \( W \) is a real-valued, piecewise continuous function on \([0, L]\). Let \( \lambda_n^+ \) be the \( n \)th eigenvalue of the following operator counted with multiplicity respectively

\[
-\frac{d^2}{ds^2} + W(s), \quad \text{in } L^2(0, L),
\]

with domain

\[
\{ w \in H^2(0, L); w(0) = \pm w(L), w'(0) = \pm w'(L) \}. \tag{36}
\]

We suppose that

\[
\lambda_n^+ = \lambda_{n+1}^+, \quad \text{for all even } n,
\]

and

\[
\lambda_n^- = \lambda_{n+1}^-, \quad \text{for all odd } n.
\]

Then, \( W \) is constant on \([0, L]\).

**Proof of Theorem 3:** For each \( \theta \in \mathcal{C} \), we define the unitary transformation \((u_\theta w)(s) = e^{-i\theta s} w(s)\). In particular, consider the operators \( T^0 : = u_0 T^0 u_0^{-1} \) and \( T^\pi/L : = u_{\pi/L} T^\pi/L u_{\pi/L}^{-1} \) whose eigenvalues are given by \( \{ \nu_n(0) \}_{n \in \mathbb{N}} \) and \( \{ \nu_n(\pi/L) \}_{n \in \mathbb{N}} \), respectively. Furthermore, the domains of these operators are given by \( \tag{36} \) \( T^0 \) (resp. \( T^\pi/L \)) is called operator with periodic (resp. antiperiodic) boundary conditions.

Since \( h_n(s)/h(s) \) is not constant in \([0, L]\), by Borg’s Theorem, without loss of generality, we can say that there exists \( n_1 \in \mathbb{N} \) so that \( \nu_{n_1}(0) \neq \nu_{n_1+1}(0) \). Now, the result follows by Corollary 3.

### A Appendix

Let \( \mathcal{J} \) be a Hilbert space and \( b : \text{dom } b \times \text{dom } b \to \mathbb{C} \) a sesquilinear form in \( \mathcal{J} \). Denote by \( b(\psi) = b(\psi, \psi) \) the quadratic form associated with it. We say that \( b(\psi) \) is lower bounded.
if there is \( \beta \in \mathbb{R} \) with \( b(\psi) \geq \beta \|\psi\|^2 \), for all \( \psi \in \text{dom} \, b \). If \( \beta > 0 \), \( b \) is called positive. A sesquilinear form \( b \) is called hermitian if \( b(\psi, \eta) = b(\eta, \psi) \), for all \( \psi, \eta \in \text{dom} \, b \).

Let \( b \) be a hermitian form and \( (\psi_n) \subset \text{dom} \, b \). Even though \( b \) is not necessarily positive, this sequence is called a Cauchy sequence with respect to \( b \) (or in \( (\text{dom} \, b, b) \)) if \( b(\psi_n - \psi_m) \to 0 \) as \( n, m \to \infty \). It is said that \( (\psi_n) \) converges to \( \psi \) with respect to \( b \) (or in \( (\text{dom} \, b, b) \)) if \( \psi \in \text{dom} \, b \) and \( b(\psi_n - \psi) \to 0 \) as \( n \to \infty \).

A sesquilinear form \( b \) is closed if for each Cauchy sequence \( (\psi_n) \) in \( (\text{dom} \, b, b) \) with \( \psi_n \to \psi \) in \( \mathcal{F} \), one has \( \psi \in \text{dom} \, b \) and \( \psi_n \to \psi \) in \( (\text{dom} \, b, b) \).

Given a sesquilinear form \( b \), the operator \( T_b \) is associated with \( b \) is defined as

\[
\text{dom} \, T_b := \{ \psi \in \text{dom} \, b : \exists \zeta \in \mathcal{F} \text{ with } b(\eta, \psi) = \langle \eta, \zeta \rangle, \forall \eta \in \text{dom} \, b \},
\]

\[
T_b \psi := \zeta, \quad \psi \in \text{dom} \, T_b.
\]

Thus, \( b(\eta, \psi) = \langle \eta, T_b \psi \rangle \), for all \( \eta \in \text{dom} \, b \), for all \( \psi \in \text{dom} \, T_b \). Such operator is well defined when \( \text{dom} \, b \) is dense in \( \mathcal{F} \).

Recall the quadratic form \( t_\varepsilon^b(\psi) \) and the operator \( T_\varepsilon^b \) defined in Section 4.2.6 in [5], there exists a self-adjoint operator, denoted by \( T_\varepsilon^b \), so that,

\[
t_\varepsilon^b(\eta, \psi) = \langle \eta, T_\varepsilon^b \psi \rangle, \quad \forall \eta \in \text{dom} \, t_\varepsilon^b, \forall \psi \in \text{dom} \, T_\varepsilon^b.
\]

Second, we show that \( T_\varepsilon^b = T_\varepsilon^b \).

**Proposition 2.** For each \( \theta \in \mathcal{C} \), the quadratic form \( t_\varepsilon^b(\psi) \) is closed.

**Proof.** We are going to consider the particular case where \( \theta = 0 \) and \( k(s) = 0 \), i.e., \( \beta_e(s, y) = 1 \). The general case is similar.

Let \( (\psi_n) \) be a Cauchy sequence in \( (\text{dom} \, t_\varepsilon^b, t_\varepsilon^b) \) with \( \psi_n \to \psi \) in \( L^2(Q, h^2 dsdy) \). In particularly, since \( h \) is a bounded function, \( (\psi_n) \) is a Cauchy sequence in \( L^2(Q) \). We also note that

\[
\int_Q |\nabla_y(\psi_n - \psi_m)|^2 dsdy \leq \varepsilon^2 t_\varepsilon^b(\psi_n - \psi_m),
\]

and

\[
\int_Q |\partial_s(\psi_n - \psi_m)|^2 dsdy \leq \frac{1}{(\inf h(s))^2} \int_Q h^2 |\partial_s(\psi_n - \psi_m)|^2 dsdy
\]

\[
\leq \frac{2}{(\inf h(s))^2} \int_Q h^2 \left| \partial_s R_h(\psi_n - \psi_m) \right|^2 dsdy
\]

\[
+ 2 \int_Q \left| \nabla_y(\psi_n - \psi_m), R_h \right|^2 dsdy
\]

\[
\leq K \left( t_\varepsilon^b(\psi_n, \psi_m) + \int_Q \left( |\nabla_y(\psi_n - \psi_m)|^2 + |\psi_n - \psi_m|^2 \right) dsdy \right),
\]

for some \( K > 0 \).

With theses inequalities, we can see that \( (\psi_n) \) is a Cauchy sequence in the Hilbert space \( \mathcal{H}^1(Q) \). Thus, there exists \( \eta \in \mathcal{H}^1(Q) \), so that, \( \psi_n \to \eta \) in \( \mathcal{H}^1(Q) \). We conclude that \( \eta = \psi \) in \( L^2(Q) \). Furthermore, \( \partial_s \psi_n \to \partial_s \psi, \nabla_y \psi_n \to \nabla_y \psi \) in \( L^2(Q) \).

Now, we are going to show that \( \psi(0, y) = \psi(L, y) \) in \( L^2(S) \). Define

\[
V_n(y) := \int_0^L \partial_s \psi_n(s, y) ds, \quad V(y) := \int_0^L \partial_s \psi(s, y) ds,
\]

16
and note that
\[
\int_s |V_n(y) - V(y)|dy \leq \int_Q |\partial_s \psi_n - \partial_s \psi| dsdy \\
\leq |Q|^{1/2} \left( \int_Q |\partial_s \psi_n - \partial_s \psi|^2 dsdy \right)^{1/2} \to 0, \ n \to \infty.
\]

Thus, \( V_n \to V \) in \( L^1(S) \). Therefore, there exists a subsequence \( (V_{n_k}) \) of \( (V_n) \), so that, \( V_{n_k}(y) \to V(y) \), a.e. \( y \). More exactly,

\[
\lim_{k \to \infty} \int_0^L \partial_s \psi_{n_k}(s,y)ds = \int_0^L \partial_s \psi(s,y)ds, \ a.e. \ y.
\]

Recall \( \psi_{n_k}(L,y) = \psi_{n_k}(0,y) \). By Fundamental Theorem of Calculus

\[
0 = \lim_{k \to \infty} (\psi_{n_k}(L,y) - \psi_{n_k}(0,y)) = \psi(L,y) - \psi(0,y), \ a.e. \ y.
\]

Thus, \( \psi \in \text{dom} \ t_0^\varepsilon \).

Finally, we can see that there exists \( K > 0 \), so that,

\[
t_0^\varepsilon(\psi_n - \psi) \leq K \|\psi_n - \psi\|^2_{H^1(Q)} \to 0, \ n \to \infty,
\]
i.e., \( \psi_n \to \psi \) in \( \text{dom} \ t_0^\varepsilon \), \( t_0^\varepsilon \).

\[\square\]

**Proposition 3.** For each \( \theta \in C \), \( T_0^\varepsilon = T_\varepsilon^\theta \).

*Proof.* Again, consider the particular case \( \theta = 0 \) and \( k(s) = 0 \). Write \( R^h = (R_1^h, R_2^h) \), denote by \( N = (N_1, N_2) \) the outward pointing unit normal to \( S \) and \( dA \) the measure of area of the region \( \partial S \).

By identity polarization we obtain the sesquilinear form \( t_0^\varepsilon(\eta, \psi) \) associated with the quadratic form \( t_0^\varepsilon(\psi) \). Namely,

\[
t_0^\varepsilon(\eta, \psi) = \int_Q \left( h^2 \partial_{s,y} R^h \eta \partial_{s,y} R^h \psi + \frac{1}{\varepsilon^2} \langle \nabla_y \eta, \nabla_y \psi \rangle \right) dsdy \\
= \int_Q h^2 \partial_s \eta \partial_{s,y} R^h \psi dsdy + \int_Q h^2 \langle \nabla_y \eta, R^h \partial_{s,y} R^h \psi \rangle dsdy \\
+ \int_Q \frac{1}{\varepsilon^2} \langle \nabla_y \eta, \nabla_y \psi \rangle dsdy + c \int_Q h^2 \psi dsdy.
\]

For each \( \eta \in \text{dom} \ t_0^\varepsilon \) and \( \psi \in \text{dom} \ t_0^\varepsilon \cap H^2(Q) \), the Fubini Theorem and an integration by parts show that

\[
\int_Q h^2 \partial_s \eta \partial_{s,y} R^h \psi dsdy = -\int_Q \nabla \eta \partial_s \left( h^2 \partial_{s,y} R^h \psi \right) dsdy + \int_S \left( \nabla \eta h^2 \partial_{s,y} R^h \psi \right)|_S^{L} dy = \]

\[
-\int_Q \nabla \eta \partial_s \left( h^2 \partial_{s,y} R^h \psi \right) dsdy + \int_S \nabla \eta(0,y)h^2(0) \left( \partial_{s,y} R^h \psi(L,y) - \partial_{s,y} R^h \psi(0,y) \right) dy.
\]
Furthermore,
\[
\int_Q h^2 \langle \nabla_y \eta, R^h \rangle \partial_{s,y}^R \psi \, dy = \\
- \int_Q \frac{1}{h} \partial_y \left(R^h \partial_{s,y}^R \psi \right) \, dy + \int_Q \frac{1}{h} \partial_y R^h \partial_{s,y}^R \psi \, dy,
\]
and
\[
\int_Q \frac{1}{\varepsilon^2} \langle \nabla_y \eta, \nabla_y \psi \rangle \, dy = - \int_Q \frac{1}{\varepsilon^2} \nabla_y \psi \, dy + \int_Q \frac{1}{\varepsilon^2} \nabla_y \psi \, dy.
\]

Thus,
\[
\ell_\varepsilon^0(\eta, \psi) = - \int_Q \frac{1}{h^2} \left( \partial_s + \text{div}_y R^h \right) \partial_{s,y}^R \psi + \frac{1}{\varepsilon^2} \Delta_y \psi \right) \, dy \\
+ \int_S \frac{1}{h^2} \left( \partial_{s,y}^R \psi (L, y) - \partial_{s,y}^R \psi (0, y) \right) \, dy \\
+ \int_0^L \int_{\partial S} \frac{1}{h^2} \left( \partial_{s,y}^R \psi (L, y) - \partial_{s,y}^R \psi (0, y) \right) \, dA + \int_Q h^2 \partial \psi \, dy.
\]

For \( \psi \in \text{dom} \, \ell_\varepsilon^0 \cap H^2(Q) \), we define
\[
Z_\varepsilon^0 := - \frac{1}{h^2} \left( \partial_s + \text{div}_y R^h \right) \partial_{s,y}^R \psi + \frac{1}{\varepsilon^2} \Delta_y \psi + c \psi.
\]

Therefore,
\[
\ell_\varepsilon^0(\eta, \psi) = \langle \eta, Z_\varepsilon^0(\psi) \rangle_H + \int_S \frac{1}{h^2} \left( \partial_{s,y}^R \psi (L, y) - \partial_{s,y}^R \psi (0, y) \right) \, dy \\
+ \int_0^L \int_{\partial S} \frac{1}{h^2} \partial_{s,y}^R \psi \, dA,
\]
for all \( \eta \in \text{dom} \, \ell_\varepsilon^0 \), for all \( \psi \in \text{dom} \, \ell_\varepsilon^0 \cap H^2(Q) \).

**Step 1:** Given \( \psi \in \text{dom} \, T_\varepsilon^0 \), we have \( \partial^R \psi / \partial N = 0 \) on \([0, L] \times \partial S\) and,
\[
\ell_\varepsilon^0(\eta, \psi) = \langle \eta, T_\varepsilon^0(\psi) \rangle_{\mathcal{H}_\varepsilon}, \quad \forall \eta \in \text{dom} \, \ell_\varepsilon^0.
\]

Thus, \( \psi \in \text{dom} \, T_{\varepsilon^0} \) and \( T_{\varepsilon^0} \psi = T_\varepsilon^0 \psi \).

**Step 2:** Conversely, take \( \psi \in \text{dom} \, T_{\varepsilon^0} \subset \text{dom} \, \ell_\varepsilon^0 \). Then, there exists \( \zeta \in \mathcal{H} \), so that,
\[
\ell_\varepsilon^0(\eta, \psi) = \langle \eta, \zeta \rangle_{\mathcal{H}_\varepsilon}, \quad \forall \eta \in \text{dom} \, \ell_\varepsilon^0.
\]

This implies that \( \psi \in H^2(Q) \) (see Chapter 7 in [1]) and, by [2],
\[
\langle \eta, \zeta - Z_\varepsilon^0(\psi) \rangle_{\mathcal{H}_\varepsilon} = \int_S \frac{1}{h^2} \left( \partial_{s,y}^R \psi (L, y) - \partial_{s,y}^R \psi (0, y) \right) \, dy + \int_0^L \int_{\partial S} \partial_{s,y}^R \psi \, dA.
\]
In particular,
\[ \langle \eta, \zeta - Z_0^\epsilon \psi, \rangle_{\mathcal{H}_\epsilon} = 0, \quad \forall \eta \in C_0^\infty(Q) \subset \text{dom } t_\epsilon^0. \]

Therefore, \( \zeta = Z_0^\epsilon \psi \). It remains to show that \( \psi \in \text{dom } T_\epsilon^0 \).

We know that \( \psi(0, y) = \psi(L, y) \) in \( L^2(S) \). On the other hand, since \( \zeta = Z_0^\epsilon \psi \),
\[
\int_S \eta(0, y) h^2(0) \left( \partial_{s,y} \psi(L, y) - \partial_{s,y} \psi(0, y) \right) dy + \int_0^L \int_{\partial S} \eta \frac{\partial R^h \psi}{\partial N} dAdS = 0,
\]
for all \( \eta \in \text{dom } t_\epsilon^0 \). By taking \( \eta(s, y) = w(s) u(y) \), with \( w \in C_0^\infty(0, L) \) and \( u \in H^1(S) \),
\[
\int_0^L w(s) \int_{\partial S} u(y) \frac{\partial R^h \psi}{\partial N} dAdS = 0, \quad \forall w \in C_0^\infty(0, L), \forall u \in H^1(S).
\]

Thus,
\[
\frac{\partial R^h \psi}{\partial N} = 0, \quad \text{in } L^2(Q). \tag{38}
\]

Consequently,
\[
\int_S \eta(0, y) h^2(0) \left( \partial_{s,y} \psi(L, y) - \partial_{s,y} \psi(0, y) \right) dy = 0, \quad \forall \eta \in \text{dom } t_\epsilon^0.
\]

With suitable choices of \( \eta \), one can show
\[
\partial_{s,y} \psi(L, y) = \partial_{s,y} \psi(0, y), \quad \text{in } L^2([0, L] \times \partial S). \tag{39}
\]

The fact that \( \psi(0, y) = \psi(L, y) \) in \( L^2(S) \), together with the conditions \( \text{[38]} \) and \( \text{[39]} \), ensures that \( \psi \in \text{dom } T_\epsilon^0 \).

**Remark 4.** Recall the quadratic form \( t_\epsilon(\psi) \) and the operator \( T_\epsilon \) defined in Section 3. Similarly, one can show that \( t_\epsilon(\psi) \) is a closed quadratic form and \( T_\epsilon \) is the self-adjoint operator associated with it. The proof will be omitted in this text.

**References**

[1] Baiocchi, C. and Capelo, A.: Variational and quasivariational inequalities: applications to free boundary problems. Published by John Wiley & Sons Ltd (1984)

[2] Bedoya, R., de Oliveira, C. R. and Verri A. A.: Complex \( \Gamma \)-convergence and magnetic Dirichlet Laplacian in bounded thin tubes. J. Spectr. Theory 4 (2014), 621-642.

[3] Bentosela, F., Duclos, P. and Exner, P.: Absolute continuity in periodic thin tubes and strongly coupled leaky wires. Lett. in Math. Phys. 65 (2003), 75-82.

[4] Borg, G.: Eine Umkehrung der Sturm–Liouvillschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte, Acta Math. 78 (1946), 1–96.

[5] de Oliveira, C. R.: Intermediate Spectral Theory and Quantum Dynamics. Birkhäuser, Basel (2009)

[6] de Oliveira, C. R. and Verri, A. A.: Asymptotic spectrum for Dirichlet Laplacian in thin Deformed tubes with scaled geometry. J. Phys. A: Math. and Theor., 45 (2012) p.435201.
[7] Friedlander, L.: Absolute continuity of the spectra of periodic waveguides. Contemporary Mathematics, 339 (2003), 37-42.

[8] Friedlander, L. and Solomyak, M.: On the spectrum of the Dirichlet Laplacian in a narrow infinite strip. Amer. Math. Soc. Transl. 225 (2008), 103–116.

[9] Friedlander, L. and Solomyak, M.: On the spectrum of narrow periodic waveguide. Russ. J. Math. Phys. 15 (2008), 238-242.

[10] Gohberg, I. C. and Krein, M. G.: Introduction to the theory of linear nonselfadjoint operators. Translations of Mathematical Monographs 18, American Mathematical Society (1969)

[11] Kato, T.: Perturbation Theory for linear Operators. Springer-Verlag, Berlin (1995)

[12] Klingenberg, W.: A Course in Differential Geometry. Springer-Verlag, New York (1978)

[13] Nazarov, S. A.: A Gap in the Essential Spectrum of the Neumann Problem for an Elliptic System in a Periodic Domain. Functional Analysis and Its Applications, 43, No. 3, (2009), 239-241. Translated from Funktsional’nyi Analiz i Ego Prilozheniya, 43, No. 3, (2009), 92–95.

[14] Reed, M, and Simon, B.: Methods of Modern Mathematical Physics, IV. Analysis of Operator. Academic Press, New York (1978)

[15] Sobolev, A. V. and Walthoe, J.: Absolute continuity in periodic waveguides. Proc. London Math. Soc. 85 (2002), 717-741.

[16] Verri, A. A. and Mamani, C. R.: Absolute continuity in periodically bent and twisted tubes. http://arxiv.org/abs/1508.02574

[17] Yoshitomi, K.: Band gap of the spectrum in periodically curved quantum waveguides. J. Differ. Equations 142 (1998), 123-166.