RC-positivity and scalar-flat metrics on ruled surfaces

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Abstract. Let $X$ be a ruled surface over a curve of genus $g$. We prove that $X$ has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $m(X) > 2 - 2g$ where $m(X)$ is an intrinsic number depends on the complex structure of $X$.

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1. Introduction

In his “Problem section”, S.-T. Yau proposed the following classical problem ([Yau82, Problem 41]), which is investigated intensively in the last forty years.

Problem 1.1. Classify all compact Kähler surfaces with zero scalar curvature.

By the celebrated Calabi-Yau Theorem ([Yau78]), all Kähler surfaces with vanishing first Chern class (e.g. $K3$ surfaces) admit Kähler metrics with zero scalar curvature. Such metrics are usually called scalar-flat Kähler metrics and it is a special class of constant scalar curvature Kähler (cscK) metrics or extremal metrics. Obstructions to the existence of such metrics have been known since the pioneering works of S.-T. Yau [Yau74] and E. Calabi [Cal85]. For comprehensive discussions on this rich topic, we refer to [Yau74, Yau78, Fut83, BD88, Tian90, Sim91, Fuj92, LS93, LS94, Tian97, Don01, RS05, AP06, AT06, RT06, Ross06, CT08, Sto08, AP09, ACGT11, Sze14, Sze17] and the references therein.

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In this paper, we study the geometry of compact complex manifolds with scalar-flat Hermitian metrics (with respect to the Chern connection), which is a generalization of Problem 1.1. We begin with a characterization of compact complex manifolds with scalar-flat Hermitian metrics, which can be regarded as a Hermitian analogue of Kazdan-Warner-Bourguignon’s classical work in Riemannian geometry, and we refer to [Bes86] and [Fut93] for more details.

**Theorem 1.2.** A compact complex manifold $X$ admits a scalar-flat Hermitian metric if and only if $X$ is Chern Ricci-flat, or both $K_X$ and $K_X^{-1}$ are RC-positive. Recall that, a line bundle $\mathcal{L}$ is called *RC-positive* if it has a smooth Hermitian metric $h$ such that its curvature $-\sqrt{-1}\partial \bar{\partial} \log h$ has at least one positive eigenvalue everywhere. By using a remarkable theorem in [TW10] established by Tosatti-Weinkove (which is a Hermitian analogue of Yau’s theorem [Yau78]), the anti-canonical bundle $K_X^{-1}$ is RC-positive if and only if $X$ has a smooth Hermitian metric $\omega$ such that its Ricci curvature $\text{Ric}(\omega)$ has at least one positive eigenvalue everywhere. A complex manifold $X$ is called *Chern Ricci-flat* if there exists a smooth Hermitian metric $\omega$ such that the Chern-Ricci curvature $\text{Ric}(\omega) = -\sqrt{-1}\partial \bar{\partial} \log \omega^n = 0$. On the other hand, we proved in [Yang17, Theorem 1.4] that a line bundle $\mathcal{L}$ is RC-positive if and only if its dual line bundle $\mathcal{L}^*$ is not pseudo-effective. By taking this advantage, we can verify the RC-positivity of $K_X$ or $K_X^{-1}$ by adapting methods in differential geometry as well as algebraic geometry.

As a straightforward application of Theorem 1.2, we obtain

**Corollary 1.3.** Let $X$ be a compact Kähler manifold. If $X$ has a scalar-flat Kähler metric $\omega$, then either $X$ is a Calabi-Yau manifold, or both $K_X$ and $K_X^{-1}$ are RC-positive.

For instance, if $X$ is the blowing-up of $\mathbb{P}^2$ along $m$-points ($m \leq 9$), it is well-known that the anti-canonical bundle $K_X^{-1}$ is effective (e.g. [Fri98, p. 125-p. 129]) and so it is pseudo-effective. In this case, $K_X$ can not be RC-positive and $X$ has no scalar-flat Hermitian (or Kähler) metrics.

**Corollary 1.4.** Let $\mathbb{P}^2 \# m \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ along $m$ points. If $X$ admits a scalar-flat Hermitian metric, then $m \geq 10$.

Indeed, it is proved by Rollin-Singer in [RS05, Theorem 1] (see also [Leb86, Leb91, LS93]) that: a complex surface $X$ obtained by blowing-up $\mathbb{P}^2$ at 10 suitably chosen points admits a scalar-flat Kähler metric and any further blowing-up of $X$ also admits a scalar-flat Kähler metric.
A compact complex surface $X$ is called a \textit{ruled surface} if it is a holomorphic $\mathbb{P}^1$-bundle over a compact Riemann surface $C$. It is well known that any ruled surface $X$ can be written as a projective bundle $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a rank two vector bundle over $C$. Moreover, two ruled surfaces $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ over $C$. The existence of cscK metrics on ruled surfaces are extensively studied, and we refer to [Yau74, BD88, Tian90, Sim91, Fuj92, LS93, LS94, RS05, AP06, AT06, RT06, Ross06, Sto08, ACGT11, Sze14] and the references therein. A remarkable result (e.g. [AT06, BD88, ACGT11]) asserts that: A ruled surface $\mathbb{P}(\mathcal{E})$ admits a cscK metric if and only if $\mathcal{E}$ is poly-stable.

In the following, we aim to classify ruled surfaces with scalar-flat Hermitian metrics. Let $\mathcal{E}$ be a rank two vector bundle over a smooth curve $C$. One can define a number $m(\mathcal{E})$ (e.g. [Fri98, p. 122]) which is equal to the minimal degree of $\mathcal{E} \otimes \mathcal{L}$ if there exists a sheaf extension of $\mathcal{E} \otimes \mathcal{L}$:

$$0 \to \mathcal{O}_C \to \mathcal{E} \otimes \mathcal{L} \to \mathcal{F} \to 0$$

for some line bundle $\mathcal{L}$. It is obvious that $m(\mathcal{E}) = m(\mathcal{E} \otimes \mathcal{L})$ for any line bundle $\mathcal{L}$. Hence, we can define an intrinsic number $m(X)$ for a ruled surface $X$: $m(X) = m(\mathcal{E})$ if $X$ can be written as $\mathbb{P}(\mathcal{E})$. It is obvious that $m(X)$ is independent of the choices of $\mathcal{E}$. Let’s explain the geometric meaning of $m(X)$ by the example $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C) \to C$ where $\mathcal{L}$ is a line bundle. In this case, $m(X) = -|\deg(\mathcal{L})| \leq 0$. As another application of Theorem 1.2, we obtain

**Theorem 1.5.** Let $X$ be a ruled surface over a smooth curve $C$ of genus $g$. Then $X$ has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $m(X) > 2 - 2g$.

In particular, we have

**Corollary 1.6.** Let $\mathcal{L} \to C$ be a line bundle over a smooth curve of genus $g$ and $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)$. Then $X$ has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $|\deg(\mathcal{L})| < 2g - 2$.

For instance, if $C$ is a smooth curve of degree $d > 4$ in $\mathbb{P}^2$, then the genus of $C$ is $g = \frac{1}{2}(d - 1)(d - 2)$ and the degree of $\mathcal{O}_C(1)$ is $d < 2g - 2$. Hence, $X = \mathbb{P}(\mathcal{O}_C(1) \oplus \mathcal{O}_C)$ has scalar-flat Hermitian metrics. Note also that, in Corollary 1.6, if $\deg(\mathcal{L}) = 0$, the vector bundle $\mathcal{L} \oplus \mathcal{O}_C$ is poly-stable and $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)$ admits scalar-flat Kähler metrics; however, when $0 < |\deg(\mathcal{L})| < 2g - 2$, it has no scalar-flat Kähler metrics. Moreover, we construct such examples in higher dimensional ruled manifolds.
Proposition 1.7. Let $C$ be a smooth curve with genus $g \geq 2$ and $\mathcal{L}$ be a line bundle over $C$. Suppose $\mathcal{E} = \mathcal{L} \oplus \mathcal{O}_C^{(n-1)}$ and $X = \mathbb{P}(\mathcal{E}^*) \to C$ is the projective bundle. If $0 < \deg(\mathcal{L}) < \frac{2g-2}{n-1}$, then $\mathbb{P}(\mathcal{E}^*)$ cannot support scalar-flat Kähler metrics, but it does admit scalar-flat Hermitian metrics.

As motivated by previous results, we propose the following question.

Question 1.8. Let $X$ be a compact Kähler manifold. Suppose $X$ has a scalar-flat Hermitian metric. Are there any geometric conditions on $X$ which can guarantee the existence of scalar-flat Kähler metrics?

Finally, we classify minimal compact complex surfaces with scalar-flat Hermitian metrics.

Theorem 1.9. Let $X$ be a minimal compact complex surface. If $X$ admits a scalar-flat Hermitian metric, then $X$ must be one of the following

1. an Enriques surface;
2. a bi-elliptic surface;
3. a $K3$ surface;
4. a 2-torus;
5. a Kodaira surface;
6. a ruled surface $X$ over a curve $C$ of genus $g \geq 2$ and $m(X) > 2 - 2g$;
7. a class $\text{VII}_0$ surface with $b_2 > 0$.

Remark 1.10. It is proved that surfaces in (1) to (6) all have scalar-flat Hermitian metrics. On the other hand, since class $\text{VII}_0$ surfaces with $b_2 > 0$ are not completely classified, we do not prove each class $\text{VII}_0$ surface with $b_2 > 0$ can support scalar-flat Hermitian metrics. Non-minimal surfaces with scalar-flat Hermitian metrics will also be studied in the sequel.

The rest of the paper is organized as follows. In Section 3, we give a characterization of compact complex manifolds with scalar-flat Hermitian metrics and prove Theorem 1.2. In Section 5, we classify ruled surfaces with scalar-flat Hermitian metrics and establish Theorem 1.5. In Section 6, we classify minimal complex surfaces with scalar-flat Hermitian metrics and obtain Theorem 1.9. In Section 7, we give some precise examples with scalar-flat Hermitian metrics (Proposition 1.7).

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2. Background materials

2.1. Scalar curvature and total scalar curvature on complex manifolds.
Let $(\mathcal{E}, h)$ be a Hermitian holomorphic vector bundle over a complex manifold $X$ with Chern connection $\nabla$. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on $X$ and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of $\mathcal{E}$. The curvature tensor $R^\mathcal{E} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \text{End}(\mathcal{E}))$ has components

$$R^\mathcal{E}_{\alpha\beta} = -\frac{\partial^2 h_{\alpha\beta}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\delta} \frac{\partial h_{\alpha\delta}}{\partial z^i} \frac{\partial h_{\beta\gamma}}{\partial \overline{z}^j}.$$ 

(Here and henceforth we sometimes adopt the Einstein convention for summation.) If $(X, \omega_g)$ is a Hermitian manifold, then $(T_X, g)$ has Chern curvature components

$$R_{ij\ell} = -\frac{\partial^2 g_{i\ell}}{\partial z^i \partial \overline{z}^j} + g^{pq} \frac{\partial g_{i\ell}}{\partial z^p} \frac{\partial g_{qj}}{\partial \overline{z}^j}.$$ 

The Chern-Ricci curvature $\text{Ric}(\omega_g)$ of $(X, \omega_g)$ is represented by

$$R_{ij} = g^{k\ell} R_{ijk\ell}.$$ 

The (Chern) scalar curvature $s$ of $(X, \omega_g)$ is given by

$$s = \text{tr}_{\omega_g} \text{Ric}(\omega_g) = g^{ij} R_{ij}.$$ 

The total (Chern) scalar curvature of $\omega_g$ is

$$\int_X s_{\omega_g^n} = n \int \text{Ric}(\omega_g) \wedge \omega_g^{n-1},$$

where $n$ is the complex dimension of $X$.

1. A Hermitian metric $\omega_g$ is called a Gauduchon metric if $\partial \overline{\partial} \omega_g^{n-1} = 0$. It is proved by Gauduchon ([Gau77]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to constant scaling).

2. A projective manifold $X$ is called uniruled if it is covered by rational curves.

2.2. Positivity of line bundles. Let $(X, \omega_g)$ be a compact Hermitian manifold, and $\mathcal{L} \to X$ be a holomorphic line bundle.

1. $\mathcal{L}$ is said to be positive (resp. semi-positive) if there exists a smooth Hermitian metric $h$ on $\mathcal{L}$ such that the curvature form $R^\mathcal{L} = -\sqrt{-1} \partial \overline{\partial} \log h$ is a positive (resp. semi-positive) $(1, 1)$-form.
(2) \( L \) is said to be \textit{nef}, if for any \( \varepsilon > 0 \), there exists a smooth Hermitian metric \( h_\varepsilon \) on \( L \) such that
\[-\sqrt{-1} \partial \bar{\partial} \log h_\varepsilon \geq -\varepsilon \omega_g.\]

(3) \( L \) is said to be \textit{pseudo-effective}, if there exists a (possibly) singular Hermitian metric \( h \) on \( L \) such that
\[-\sqrt{-1} \partial \bar{\partial} \log h \geq 0\]
in the sense of distributions. (See [Dem] for more details.)

(4) \( L \) is said to be \textit{Q-effective}, if there exists some positive integer \( m \) such that
\[H^0(X, L^\otimes m) \neq 0.\]

(5) \( L \) is called \textit{unitary flat} if there exists a smooth Hermitian metric \( h \) on \( L \) such that
\[-\sqrt{-1} \partial \bar{\partial} \log h = 0.\]

(6) The Kodaira dimension \( \kappa(L) \) of \( L \) is defined to be
\[\kappa(L) := \limsup_{m \to +\infty} \frac{\log \dim \mathcal{O}(X, L^\otimes m)}{\log m}\]
and the \textit{Kodaira dimension} \( \kappa(X) \) of \( X \) is defined as
\[\kappa(X) := \kappa(K_X)\]
where the logarithm of zero is defined to be \(-\infty\).

2.3. Positivity of vector bundles. The points of the projective bundle \( \mathbb{P}(\mathcal{E}^*) \)
of \( \mathcal{E} \to X \) can be identified with the hyperplanes of \( \mathcal{E} \). Note that every hyperplane \( \mathcal{V} \) in \( \mathcal{E}_z \) corresponds bijectively to the line of linear forms in \( \mathcal{E}_z \) which vanish on \( \mathcal{V} \). Let \( \pi : \mathbb{P}(\mathcal{E}^*) \to X \) be the natural projection. There is a tautological hyperplane subbundle \( \mathcal{I} \) of \( \pi^*\mathcal{E} \) such that \( \mathcal{I}_\xi = \xi^{-1}(0) \subset \mathcal{E}_z \) for all \( \xi \in \mathcal{E}_z \setminus \{0\} \). The quotient line bundle \( \pi^*\mathcal{E}/\mathcal{I} \) is denoted \( \mathcal{O}(1) \) and is called the \textit{tautological line bundle} associated to \( \mathcal{E} \to X \). Hence there is an exact sequence of vector bundles over \( \mathbb{P}(\mathcal{E}^*) \),

\[0 \to \mathcal{I} \to \pi^*\mathcal{E} \to \mathcal{O}(1) \to 0.\]

A holomorphic vector bundle \( \mathcal{E} \to X \) is called \textit{ample} (resp. \textit{nef}) if the line bundle \( \mathcal{O}(1) \) is ample (resp. nef) over \( \mathbb{P}(\mathcal{E}^*) \). (\textbf{Caution:} In general, \( \mathbb{P}(\mathcal{E}) \) and \( \mathbb{P}(\mathcal{E}^*) \) are not isomorphic! \( \mathcal{E}(1) \) is the tautological line bundle of \( \mathbb{P}(\mathcal{E}^*) \), and \( \mathcal{O}(1) \) is the tautological line bundle of \( \mathbb{P}(\mathcal{E}) \).) A Hermitian holomorphic vector bundle \( (\mathcal{E}, h) \) over a complex manifold \( X \) is called \textit{Griffiths positive} if at each point \( q \in X \) and for any nonzero vector \( v \in \mathcal{E}_q \), and any nonzero vector \( u \in T_qX, R^\mathcal{E}(u, \overline{u}, v, \overline{v}) > 0.\)

2.4. RC-positive line bundles. Let’s recall that

\textbf{Definition 2.1.} A line bundle \( \mathcal{L} \) is called \textit{RC-positive} if it has a smooth Hermitian metric \( h \) such that its curvature \( R^{(\mathcal{L}, h)} = -\sqrt{-1} \partial \bar{\partial} \log h \) has at least one positive eigenvalue everywhere.

In [Yang17, Theorem 1.4], we obtained an equivalent characterization for RC-positive line bundles.
Theorem 2.2. Let \( \mathcal{L} \) be a holomorphic line bundle over a compact complex manifold \( X \). The following statements are equivalent.

1. \( \mathcal{L} \) is RC-positive;
2. the dual line bundle \( \mathcal{L}^* \) is not pseudo-effective.

Hence, we obtain

Corollary 2.3. A line bundle \( \mathcal{L} \) is unitary flat if and only if neither \( \mathcal{L} \) nor \( \mathcal{L}^* \) is RC-positive.

Proof. It is easy to see that \( \mathcal{L} \) is unitary flat if and only if both \( \mathcal{L} \) and \( \mathcal{L}^* \) are pseudo-effective (e.g. [Yang17a, Theorem 3.4]). Hence, Corollary 2.3 follows from Theorem 2.2. \( \square \)

By using Theorem 2.2, the classical result of [BDPP13, Theorem] and Yau’s theorem [Yau78], we obtain in [Yang17, Corollary 1.9] that

Theorem 2.4. A projective manifold \( X \) is uniruled if and only if \( K_X^{-1} \) is RC-positive, i.e. \( X \) has a smooth Hermitian metric \( \omega \) such that the Ricci curvature \( \text{Ric}(\omega) \) has at least one positive eigenvalue everywhere.

3. Characterizations of complex manifolds with scalar-flat metrics

In this section, we shall prove Theorem 1.2. Let \( \omega \) be a smooth Hermitian metric on a compact complex manifold \( X \). For simplicity, we denote by \( \mathcal{F}(\omega) \) the total (Chern) scalar curvature of \( \omega \), i.e.

\[
\mathcal{F}(\omega) = \int_X s_\omega^n = n \int_X \text{Ric}(\omega) \wedge \omega^{n-1}.
\]

Note that, when \( X \) is not Kähler, the total scalar curvature differs from the total scalar curvature of the Levi-Civita connection of the underlying Riemannian metric (e.g. [LY17]). Let \( \mathcal{W} \) be the space of smooth Gauduchon metrics on \( X \). We obtained in [Yang17a, Theorem 1.1] a complete characterization on the image of the total scalar curvature function \( \mathcal{F} : \mathcal{W} \rightarrow \mathbb{R} \) following [Gau77, Mi82, La99] (see also some special cases in [Tel06, Gau77, HW12]). By Theorem 2.2, we obtain the following result.

Theorem 3.1. The image of the total scalar function \( \mathcal{F} : \mathcal{W} \rightarrow \mathbb{R} \) has exactly four different cases:

1. \( \mathcal{F}(\mathcal{W}) = \mathbb{R} \) if and only if both \( K_X \) and \( K_X^{-1} \) are RC-positive;
(2) \( \mathcal{F}(\mathcal{W}) = \mathbb{R}^{>0} \) if and only if \( K_X^{-1} \) is RC-positive but \( K_X \) is not RC-positive;
(3) \( \mathcal{F}(\mathcal{W}) = \mathbb{R}^{<0} \) if and only if \( K_X \) is RC-positive but \( K_X^{-1} \) is not RC-positive;
(4) \( \mathcal{F}(\mathcal{W}) = \{0\} \) if and only if \( X \) is Ricci-flat; or equivalently, neither \( K_X \) nor \( K_X^{-1} \) is RC-positive.

**Proof.** We obtained in [Yang17a, Theorem 1.1] that the image of the total scalar function \( \mathcal{F} : \mathcal{W} \to \mathbb{R} \) has exactly four different cases:

(1) \( \mathcal{F}(\mathcal{W}) = \mathbb{R} \), if and only if neither \( K_X \) nor \( K_X^{-1} \) is pseudo-effective;
(2) \( \mathcal{F}(\mathcal{W}) = \mathbb{R}^{>0} \), if and only if \( K_X^{-1} \) is pseudo-effective but not unitary flat;
(3) \( \mathcal{F}(\mathcal{W}) = \mathbb{R}^{<0} \), if and only if \( K_X \) is pseudo-effective but not unitary flat;
(4) \( \mathcal{F}(\mathcal{W}) = \{0\} \), if and only if \( K_X \) is unitary flat.

By [TW10, Corollary 2], \( K_X \) is unitary flat if and only if \( X \) is Ricci-flat, i.e. there exists a Hermitian metric \( \omega \) on \( X \) such that \( \text{Ric}(\omega) = 0 \). Hence Theorem 3.1 follows from Theorem 2.2 and Corollary 2.3. \( \square \)

**Remark 3.2.** It is easy to see that Theorem 3.1 also holds for Bott-Chern classes ([Yang17a, Theorem 3.4])

As an application of Theorem 3.1, we establish Theorem 1.2, that is,

**Theorem 3.3.** Let \( X \) be a compact complex manifold. Then \( X \) admits a scalar-flat Hermitian metric if and only if \( X \) is Ricci-flat, or both \( K_X \) and \( K_X^{-1} \) are RC-positive.

**Proof.** If \( X \) has a scalar-flat Hermitian metric \( \omega \), in the conformal class of \( \omega \), there exists a Gauduchon metric \( \omega_f = e^f \omega \). Then the total scalar curvature \( s_f \) of the Gauduchon metric \( \omega_f \) is

\[
(3.1) \quad s_f = n \int_X \text{Ric}(\omega_f) \wedge \omega_f^{n-1} = n \int_X \left( \text{Ric}(\omega) - n\sqrt{-1} \partial \bar{\partial} f \right) \wedge \omega_f^{n-1}.
\]

Since \( \omega_f \) is Gauduchon, i.e. \( \partial \bar{\partial} \omega_f^{n-1} = 0 \), an integration by part yields

\[
\begin{align*}
    s_f &= n \int_X \text{Ric}(\omega) \wedge \omega_f^{n-1} \\
    &= n \int_X \text{Ric}(\omega) \wedge e^{(n-1)f} \omega^{n-1} \\
    &= \int_X e^{(n-1)f} \cdot \text{tr}_\omega \text{Ric}(\omega) \cdot \omega^n.
\end{align*}
\]
Since $\omega$ has zero scalar curvature, i.e. $\text{tr}_\omega \text{Ric}(\omega) = 0$, we deduce that the total scalar curvature $s_f$ of the Gauduchon metric $\omega_f$ is zero. By Theorem 3.1, we conclude that either $X$ is Ricci-flat, or both $K_X$ and $K_X^{-1}$ are RC-positive.

On the other hand, suppose either $X$ is Ricci-flat, or both $K_X$ and $K_X^{-1}$ are RC-positive, by Theorem 3.1 again, we know $X$ has a Gauduchon metric $\omega_G$ with zero total scalar curvature. By a conformal perturbation method, it is easy to see that there exists a Hermitian metric $\omega$ with zero scalar curvature (e.g. [Yang17a, Lemma 3.2]). Indeed, let $s_G$ be the scalar curvature of $\omega_G$. It is well-known (e.g. [Gau77] or [CTW16, Theorem 2.2]) that the following equation

$$s_G - \text{tr}_{\omega_G} \sqrt{-1} \partial \bar{\partial} f = 0$$

has a solution $f \in C^\infty(X)$ since $\omega_G$ is Gauduchon and its total scalar curvature $\int_X s_G \omega^n_G$ is zero. Let $\omega = e^{f} \omega_G$. Then the scalar curvature $s$ of $\omega$ is,

$$s = \text{tr}_\omega \text{Ric}(\omega) = -\text{tr}_\omega \sqrt{-1} \partial \bar{\partial} \log(\omega^n)$$

$$= -e^{-f} \text{tr}_{\omega_G} \sqrt{-1} \partial \bar{\partial} \log(e^f \omega^n_G)$$

$$= -e^{-f} \left(s_G - \text{tr}_{\omega_G} \sqrt{-1} \partial \bar{\partial} f\right)$$

$$= 0.$$

The proof of Theorem 1.2 is completed.

The proof of Corollary 1.3. It is a special case of Theorem 1.2 since Kähler manifolds with unitary flat $K_X$ are Kähler Calabi-Yau.

Corollary 3.4. Let $X$ be a compact Kähler manifold. Suppose $X$ has a scalar-flat Hermitian metric, or a Gauduchon metric with zero total scalar curvature. If $K_X$ or $K_X^{-1}$ is pseudo-effective, then $X$ is a Kähler Calabi-Yau manifold.

4. Projective bundles with scalar-flat metrics

In this section, we prove the following result.

Theorem 4.1. Let $E$ be a nef vector bundle of rank $r \geq 2$ over a smooth curve $C$ with genus $g \geq 2$ and $X = \mathbb{P}(E)$. If $0 \leq \text{deg}(E) < 2g - 2$, then both $K_X$ and $K_X^{-1}$ are RC-positive. In particular, $X$ has scalar-flat Hermitian metrics.

Let’s recall some elementary settings. Suppose $\dim C = n$ and $r = \text{rank}(E)$. Let $\pi$ be the projection $\mathbb{P}(E^*) \to Y$ and $\mathcal{L} = \mathcal{O}_E(1)$. Let $(e_1, \cdots, e_r)$ be the local holomorphic frame on $E$ and the dual frame on $E^*$ is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on $E^*$ are denoted by
(W_1, \cdots, W_r). If \((h_{\alpha \beta})\) is the matrix representation of a smooth metric \(h^E\) on \(E\) with respect to the basis \(\{e_\alpha\}_{\alpha=1}^r\), then the induced Hermitian metric \(h^L\) on \(L\) can be written as \(h^L = \frac{i}{\sum h_{\alpha \beta} W_\alpha W_\beta}.\) The curvature of \((L, h^L)\) is

\[
4.1 \quad R^L = \sqrt{-1} \partial \bar{\partial} \log \left( \sum h_{\alpha \beta} W_\alpha W_\beta \right)
\]

where \(\partial\) and \(\bar{\partial}\) are operators on the total space \(\mathbb{P}(E^*)\). We fix a point \(p \in \mathbb{P}(E^*)\), then there exist local holomorphic coordinates \((z^1, \cdots, z^n)\) centered at point \(q = \pi(p) \in Y\) and local holomorphic basis \(\{e_1, \cdots, e_r\}\) of \(E\) around \(q\) such that

\[
4.2 \quad h_{\alpha \beta} = \delta_{\alpha \beta} - R^E_{ij\alpha \beta} z^i \bar{z}^j + O(|z|^3)
\]

Without loss of generality, we assume \(p\) is the point \((0, \cdots, 0, [a_1, \cdots, a_r])\) with \(a_r = 1\). On the chart \(U = \{W_r = 1\}\) of the fiber \(\mathbb{P}^{r-1}\), we set \(w^A = W_A\) for \(A = 1, \cdots, r - 1\). By formula (4.1) and (4.2)

\[
4.3 \quad R^E(p) = \sqrt{-1} \sum R^E_{ij\alpha \beta} \frac{a_\alpha a_\beta}{|a|^2} dz^i \wedge d\bar{z}^j + \omega_{FS}
\]

where \(|a|^2 = \sum_{\alpha=1}^r |a_\alpha|^2\) and \(\omega_{FS} = \sqrt{-1} \sum_{A,B=1}^{r-1} \left( \frac{\delta_{AB}}{|a|^2} - \frac{a_\beta a_\bar{\beta}}{|a|^2} \right) dw^A \wedge d\bar{w}^B\) is the Fubini-Study metric on the fiber \(\mathbb{P}^{r-1}\).

**Lemma 4.2.** If \(E\) is Griffiths-positive, then \(O_{E^*}(-1)\) is RC-positive.

**Proof.** It follows from formula (4.3). Indeed, by (4.3), the induced metric on \(O_{E^*}(-1)\) over \(\mathbb{P}(E^*)\) has curvature form

\[
R^{O_{E^*}(-1)} = - \left( \sqrt{-1} \sum R^{E^*}_{ij\alpha \beta} \frac{a_\alpha a_\bar{\beta}}{|a|^2} dz^i \wedge d\bar{z}^j + \omega_{FS} \right).
\]

On the other hand, \(R^{E^*} = (R^{E^*})^t\) and so

\[
R^{O_{E^*}(-1)} = \sqrt{-1} \sum R^{E^*}_{ij\alpha \beta} \frac{a_\alpha a_\bar{\beta}}{|a|^2} dz^i \wedge d\bar{z}^j - \omega_{FS}.
\]

Hence, \(O_{E^*}(-1)\) is RC-positive if \((E, h^E)\) is Griffiths-positive. \(\square\)

**Lemma 4.3.** If \(E\) is a nef vector bundle over a smooth curve \(C\). Then for any ample line bundle \(\mathcal{A}\) over \(C\) and any \(k \geq 0\), \(O_{E^*}(-k) \otimes \pi^*: \mathcal{A}\) is RC-positive.

**Proof.** It is easy to see that \(\text{Sym}^{\otimes k} E \otimes \mathcal{A}\) is an ample vector bundle over \(C\). By [CF90], \(\text{Sym}^{\otimes k} E \otimes \mathcal{A}\) has a smooth Griffiths-positive metric. In particular, by Lemma 4.2, the dual tautological line bundle

\[
4.4 \quad O_{\text{Sym}^{\otimes k} E^* \otimes \mathcal{A}^*}(-1)
\]
is RC-positive. More precisely, the base curve $C$ direction is a positive direction of the curvature tensor of $O_{\text{Sym}^k E^*}(-1)$. On the other hand, we have the following commutative diagram

\[
\begin{array}{c}
\mathbb{P}(\mathcal{E}) \\
\downarrow \pi_k \\
C \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\nu_k \\
f \\
\end{array} \quad \begin{array}{c}
\mathbb{P}(\text{Sym}^k \mathcal{E}) \\
\downarrow \pi_k \\
f \\
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\end{array} \quad \begin{array}{c}
P(\text{Sym}^k \mathcal{E} \otimes \mathcal{A}) \\
\downarrow \\
C \\
\end{array}
\]

where $\nu_k : \mathcal{E} \rightarrow \text{Sym}^k \mathcal{E}$ is the $k$-th Veronese map, $f$ = Identity and $i$ is an isomorphism. It is easy to see that $O_{\mathcal{E}^*}(-k) \otimes \pi^*(\mathcal{A})$ is RC-positive, i.e., the induced curvature has a positive direction along the base $C$ direction. □

The proof of Theorem 4.1. By using the projection formula on $X = \mathbb{P}(\mathcal{E})$,

\[K_X = O_{\mathcal{E}^*}(-n) \otimes \pi^*(K_C \otimes \mathcal{E}^*),\]

where $\pi : X \rightarrow C$ is the projection. If $\deg(\mathcal{E}) < 2g - 2 = \deg(K_C)$, then $\deg(K_C \otimes \mathcal{E}^*) > 0$ and so $K_C \otimes \mathcal{E}^*$ is ample. By Lemma 4.3, $K_X$ is RC-positive. On the other hand, by Theorem 2.4, it is easy to see that $K_X^{-1}$ is RC-positive. Hence, by Theorem 1.2, $X$ has scalar-flat Hermitian metrics. □

5. Classification of ruled surfaces with scalar-flat Hermitian metrics

In this section, we classify ruled surfaces with scalar-flat Hermitian metrics and prove Theorem 1.5. It is well-known that any ruled surface $X$ can be written as a projective bundle $\mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a rank two vector bundle over a smooth curve $C$ with genus $g$. Moreover, two ruled surfaces $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ over $C$. Since $\mathcal{E}$ has rank two and $X \cong \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}^*)$, we shall use projection formulas

\[K_X = O_{\mathcal{E}^*}(-2) \otimes \pi^*(K_C \otimes \mathcal{E}^*), \quad \pi : \mathbb{P}(\mathcal{E}^*) \rightarrow C\]

and

\[K_X = O_{\mathcal{E}^*}(-2) \otimes \pi^*(K_C \otimes \mathcal{E}^*), \quad \pi : \mathbb{P}(\mathcal{E}) \rightarrow C\]

alternatively.

When $g = 0$, $C \cong \mathbb{P}^1$ and each rank two vector bundle can be written as $\mathcal{E} = O_{\mathbb{P}^1}(a) \oplus O_{\mathbb{P}^1}(b)$. We can write a ruled surface over $\mathbb{P}^1$ as $X = \mathbb{P}(O_{\mathbb{P}^1}(-k) \oplus O_{\mathbb{P}^1})$. 11
Proposition 5.1. Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1})$ be a Hirzebruch surface. Then the anti-canonical line bundle $K_X^{-1}$ is effective and $X$ has no scalar-flat Hermitian metrics.

Proof. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$ and $X = \mathbb{P}(\mathcal{E}^*)$. We have $K_X^{-1} = \mathcal{O}_\mathcal{E}(2) \otimes \pi^* (\mathcal{O}_{\mathbb{P}^1}(2 - k))$. By the direct image formula (e.g. [Laz04, p.90]), we have

$$H^0(X, K_X^{-1}) = H^0(X, \mathcal{O}_\mathcal{E}(2) \otimes \pi^* (\mathcal{O}_{\mathbb{P}^1}(2 - k)))$$

$$= H^0(\mathbb{P}^1, \text{Sym}^2 \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2 - k))$$

$$= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k + 2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2 - k))$$

for any $k$. Therefore, $K_X^{-1}$ is effective and $K_X$ is not RC-positive. By Theorem 1.2, $X$ has no scalar-flat Hermitian metrics. □

Theorem 5.2. Let $X = \mathbb{P}(\mathcal{E}^*) \to C$ be a projective bundle over an elliptic curve $C$ where $\mathcal{E} \to C$ is a rank two vector bundle. Then the $K_X$ is not RC-positive and $X$ has no scalar-flat Hermitian metrics.

Proof. We divide the proof into three different cases.

Case 1. Suppose $\mathcal{E}$ is indecomposable and $\deg \mathcal{E} = 0$. A well-known result of Atiyah asserts that an indecomposable vector bundle over an elliptic curve is semi-stable and so $\mathcal{E}$ is semi-stable (e.g. [Tu93, Appendix A]). On the other hand, a semi-stable vector bundle over a curve is nef if $\deg(\mathcal{E}) \geq 0$ (e.g. [Laz04, Theorem 6.4.15]). Hence $\mathcal{E}$ is nef. By using the projection formula,

$$K_X^{-1} = \mathcal{O}_\mathcal{E}(2) \otimes \pi^* (K_C^{-1} \otimes \det \mathcal{E}^*) = \mathcal{O}_\mathcal{E}(2) \otimes \pi^* (\det \mathcal{E}^*)$$

we deduce $K_X^{-1}$ is nef.

Case 2. Suppose $\mathcal{E}$ is indecomposable and $\deg(\mathcal{E}) \neq 0$. There exists an étale base change $f : C' \to C$ of degree $k$ where $k$ is an integer such that $2|k$, and $C'$ is also an elliptic curve. Suppose $X' = \mathbb{P}(f^* \mathcal{E}^*)$, then we have the commutative diagram

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\pi' \downarrow & & \downarrow \pi \\
C' & \xrightarrow{f} & C.
\end{array}$$

Let $\ell$ be an integer defined as

$$\ell = \frac{\deg(f^* \mathcal{E})}{2} = \frac{k \deg(\mathcal{E})}{2},$$

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and \( \mathcal{F} \) be a line bundle over \( Y \) such that \( \deg(\mathcal{F}) = -\ell \). Now we set
\[
\mathcal{E} = f^* \mathcal{E} \otimes \mathcal{F},
\]
then \( \deg(\mathcal{E}) = 0 \). Since \( \mathcal{E} \) is indecomposable, it is semi-stable. Therefore \( f^* \mathcal{E} \) is semi-stable (e.g. [Laz04, Lemma 6.4.12]) and so \( \mathcal{E} \) is semi-stable. Therefore, \( \mathcal{E} \) is nef since \( \deg(\mathcal{E}) = 0 \). By projection formula again, we have
\[
K^{-1}_{X'} = \mathcal{O}_{\tilde{E}}(2) \otimes \pi^*(\det \mathcal{E}).
\]
We deduce \( K^{-1}_{X'} \) is nef. Hence \( K^{-1}_X \) is nef.

**Case 3.** If \( \mathcal{E} \) is decomposable, then there exits a line bundle \( \mathcal{L} \) such that
\[
\mathcal{E} = \mathcal{L} \oplus (\mathcal{L}^{-1} \otimes \det \mathcal{E}).
\]
By the projection formula (5.1) again, we have
\[
H^0(X, K_X^{-1}) = H^0(X, \mathcal{O}_\mathcal{E}(2) \otimes \pi^*(\det \mathcal{E}^*)) \cong H^0(C, \text{Sym}^2 \mathcal{E} \otimes \det \mathcal{E}^*))
\]
\[
= H^0(C, (\mathcal{L}^2 \otimes \det \mathcal{E}^*) \oplus \mathcal{O}_C \oplus (\mathcal{L}^{-2} \otimes \det \mathcal{E}))
\]
\[
\neq 0
\]
So \( K^{-1}_X \) is effective.

In summary, we conclude that the anti-canonical line bundle \( K^{-1}_X \) is pseudo-effective, i.e. \( K_X \) is not RC-positive. By Theorem 1.2, \( X \) has no scalar-flat Hermitian metrics.

Finally, we deal with ruled surfaces over curves of genus \( g \geq 2 \). For a rank two vector bundle \( \mathcal{E} \) over a curve \( C \), in general, it is not clear whether \( \mathcal{E} \) has an extension by \( \mathcal{O}_C \):
\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0
\]
where \( \mathcal{F} \) is a coherent sheaf over \( C \). However, one can obtain such an extension for \( \mathcal{E} \otimes \mathcal{L} \) where \( \mathcal{L} \) is some suitable line bundle. This enables us to make the following definition (see [Fri98, p.121-p.124] for more details).

**Definition 5.3.** Let \( \mathcal{E} \) be a rank two vector bundle over a smooth curve \( C \). The number \( m(\mathcal{E}) \) is defined to be the minimal degree of \( \mathcal{E} \otimes \mathcal{L} \) where there exists a sheaf extension of \( \mathcal{E} \otimes \mathcal{L} \):
\[
0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E} \otimes \mathcal{L} \rightarrow \mathcal{F} \rightarrow 0
\]
for some line bundle \( \mathcal{L} \) over \( C \).
It is easy to see that for a sufficiently ample line bundle $L$, $H^0(C, E \otimes L) \neq 0$ and a global section of $E \otimes L$ gives an extension (5.4). Hence, $m(E)$ is well-defined. It is obvious that $m(E) = m(E \otimes \widehat{L})$ for any line bundle $\widehat{L}$. Nagata proved in [Nag70, Theorem 1] (see also [Fri98, p. 123]) that

**Theorem 5.4.** $m(E) \leq g$.

(Note that, in [Fri98, p. 123], the notion $c(E)$ is exactly $-m(E)$.)

As we pointed out before, any ruled surface $X$ can be written as a projective bundle $\mathbb{P}(E)$ and two ruled surfaces $\mathbb{P}(E)$ and $\mathbb{P}(E')$ are isomorphic if and only if $E \cong E' \otimes L$ for some line bundle $L$, then we can define $m(X)$ by $m(E)$ for any ruled surface $X = \mathbb{P}(E)$.

One can see that the definition of $m(E)$ is related to stability of coherent sheaves. If $m(E) > 0$, then $E$ is stable. Indeed, for any rank one sub-sheaf $L$ of $E$, we have the short exact sequence:

$$0 \to L \to E \to F \to 0.$$ 

Since $E$ is torsion free, $L$ is torsion free and we know $L$ is a line bundle. Therefore,

$$0 \to \mathcal{O}_C \to E \otimes L^{-1} \to F \otimes L^{-1} \to 0.$$ 

By the definition of $m(E)$, we have $\deg(E \otimes L^{-1}) \geq m(E) > 0$ which is equivalent to $\deg L < \frac{\deg E}{2}$. This implies $E$ is stable. Conversely, if $E$ is stable, by a similar argument, we can conclude $m(E) > 0$. Hence, we obtain a fact pointed out in [Fri98, Proposition 12, p. 123].

**Proposition 5.5.** If $E$ is a rank two vector bundle over a Riemann surface $C$, then $E$ is stable if and only if $m(E) > 0$.

The proof of Theorem 1.5. Let $X$ be a ruled surface which can support scalar-flat Hermitian metrics. We can write $X = \mathbb{P}(E_0)$ for some rank 2 vector bundle $E_0$ over a smooth curve $C$. Note that, since $E_0$ has rank 2, $E_0 \cong E_0^* \otimes \det E_0$ and so $X \cong \mathbb{P}(E_0) \cong \mathbb{P}(E_0^*)$. By Proposition 5.1 and Theorem 5.2, we know the genus $g(C) \geq 2$. On the other hand, by the above discussion, we can write $X = \mathbb{P}(E)$ where $\deg(E) = m(X)$ and $E$ has an extension

(5.5)  

$$0 \to \mathcal{O}_C \to E \to F \to 0.$$ 

Hence, $\deg(E) = \deg(F) = m(X)$. 


(1). If \( m(X) = \deg(\mathcal{F}) \leq 2 - 2g \), \( X \cong \mathbb{P}(\mathcal{E}^*) \cong \mathbb{P}(\mathcal{E}) \) has no scalar-flat Hermitian metrics. Indeed, we consider \( X = \mathbb{P}(\mathcal{E}^*) \). By the exact sequence (5.5), we have

\[
0 \to H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{E}) \to \cdots
\]

Therefore, \( H^0(C, \mathcal{E}) \neq 0 \). By the Le Potier isomorphism ([LeP75]), we have

\[
H^0(\mathbb{P}(\mathcal{E}^*), \mathcal{O}_{\mathcal{E}}(1)) \cong H^0(C, \mathcal{E}) \neq 0.
\]

Hence, \( \mathcal{O}_{\mathcal{E}}(1) \) is effective and so it is pseudo-effective. On the other hand, since \( \deg(\mathcal{E}) \leq 2 - 2g = -\deg(K_C) \), we deduce \( K_C^{-1} \otimes \det \mathcal{E}^* \) is semi-positive. By the projection formula \( K^{-1}_X = \mathcal{O}_{\mathcal{E}}(2) \otimes \pi^*(K_C^{-1} \otimes \det \mathcal{E}^*) \), we know \( K^{-1}_X \) is pseudo-effective. By Theorem 2.2, \( K_X \) is not RC-positive. By Theorem 1.2, \( X \) has no scalar-flat Hermitian metrics.

(2). If \( 2 - 2g < m(X) = \deg(\mathcal{E}) = \deg(\mathcal{F}) \leq 0 \), we know \( 0 \leq \deg(\mathcal{E}^*) < 2g - 2 \). Since \( \mathcal{O}_C \) and \( \mathcal{F}^* \) are nef, by the dual exact sequence of (5.5),

\[
0 \to \mathcal{F}^* \to \mathcal{E}^* \to \mathcal{O}_C \to 0,
\]

we deduce \( \mathcal{E}^* \) is nef with \( 0 \leq \deg(\mathcal{E}^*) < 2g - 2 \). By Theorem 4.1, \( X \cong \mathbb{P}(\mathcal{E}^*) \) can support scalar-flat Hermitian metrics.

(3). If \( 0 < m(X) = \deg(\mathcal{E}) = \deg(\mathcal{F}) < 2g - 2 \), by the exact sequence (5.5), \( \mathcal{E} \) is nef with \( 0 < \deg(\mathcal{E}) < 2g - 2 \). By Theorem 4.1, \( X \cong \mathbb{P}(\mathcal{E}) \) admits scalar-flat Hermitian metrics. Note that \( \mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}^*) \).

(4). Suppose \( m(X) \geq 2g - 2 \). By Theorem 5.4, \( m(X) \leq g \). Hence, in this case, we have \( g = 2 \) and \( m(X) = \deg(\mathcal{E}) = 2 \). We work on \( X = \mathbb{P}(\mathcal{E}) \). By Proposition 5.5, \( \mathcal{E} \) is a stable vector bundle and \( \deg(\mathcal{E}) = 2 \). By ([Laz04, Theorem 6.4.15]), we know \( \mathcal{E} \) is an ample vector bundle over a smooth curve. According to [CF90], \( \mathcal{E} \) has a smooth Griffiths-positive metric. By using Lemma 4.2, \( \mathcal{O}_{\mathcal{E}^*}(-1) \) is RC-positive. By the projection formula again, we have

\[
K_X = \mathcal{O}_{\mathcal{E}^*}(-2) \otimes \pi^*(K_C \otimes \det \mathcal{E}^*).
\]

Since \( \deg(K_C) = \deg(\mathcal{E}) = 2 \), we know \( K_C \otimes \det \mathcal{E}^* \) and \( \pi^*(K_C \otimes \det \mathcal{E}^*) \) are unitary flat. Hence, we deduce \( K_X \) is RC-positive. Since \( X \) is uniruled, by Theorem 2.4, \( K_X^{-1} \) is RC-positive. Then we can apply Theorem 1.2 and assert that \( X \) has scalar-flat Hermitian metrics.
In summary, we prove that a ruled surface $X$ over a smooth curve $C$ admits scalar-flat Hermitian metrics if and only if $g(C) \geq 2$ and $m(X) > 2 - 2g$. The proof of Theorem 1.5 is completed. \hfill \square

6. Classification of minimal surfaces with scalar-flat Hermitian metrics

In this section, we classify minimal surfaces with scalar-flat Hermitian metrics and prove Theorem 1.9.

Proposition 6.1. Let $X$ be a compact complex manifold. If $X$ admits a scalar-flat Hermitian metric, then the Kodaira dimension $\kappa(X) = 0$ or $\kappa(X) = -\infty$.

Proof. According to the proof of Theorem 1.2, if $X$ admits a scalar-flat Hermitian metric, then $X$ has a Gauduchon metric with zero total scalar curvature. By Theorem [Yang17a, Theorem 1.4], $\kappa(X) = 0$ or $\kappa(X) = -\infty$. \hfill \square

If $X$ is a minimal surface with Kodaira dimension $\kappa(X) = 0$, $X$ is exactly one of the following (e.g. [BHPV04])

1. an Enriques surface;
2. a bi-elliptic surface;
3. a K3 surface;
4. a torus;
5. a Kodaira surface.

In this case, it is well-known that $X$ has torsion canonical line bundle, i.e. $K_X^{\otimes 6} = \mathcal{O}_X$ (e.g. [BHPV04, p. 244]). Hence, $X$ admits scalar-flat Hermitian metrics.

If $X$ is a minimal surface with Kodaira dimension $\kappa(X) = -\infty$, then $X$ lies in one of the following classes:

1. minimal rational surfaces;
2. ruled surfaces of genus $g \geq 1$;
3. minimal surfaces of class VII$_0$.

Minimal rational surfaces are either $\mathbb{P}^2$ or Hirzebruch surfaces. Hence, by Proposition 5.1, they can not support scalar-flat Hermitian metrics.

If $X$ is a minimal ruled surfaces of genus $g \geq 1$, by Theorem 1.9, $X$ has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $m(X) > 2 - 2g$.

If $X$ is a minimal surface of class VII$_0$, then $X$ is one of the following

- class VII$_0$ surfaces with $b_2 > 0$;
- Inoue surfaces: a class VII$_0$ surface has $b_2 = 0$ and contains no curves;
• Hopf surfaces: its universal covering is $\mathbb{C}^2 - \{0\}$, or equivalently a class VII$_0$ surface has $b_2 = 0$ and contains a curve.

According to the proof of [Tel06, Remark 4.2] (see also [TW13] or [HLY18, Theorem 5.1]), we know Inoue surfaces all have $K_X$ semi-positive but not unitary flat, and so it can not support scalar-flat Hermitian metrics. Similarly, it is proved in [Tel06, Remark 4.3], all Hopf surfaces have semi-positive anticanonical bundle, and so it has no scalar-flat Hermitian metrics. For class VII$_0$ surfaces with $b_2 > 0$, they are not completely classified, and it is possible that some of them can support scalar-flat Hermitian metrics (see the discussion in [Tel06, p. 977-p. 979]). The proof of Theorem 1.9 is completed.

7. Examples

In this section, we exhibit several examples on ruled manifolds with scalar-flat Hermitian metrics. As a straightforward application of Theorem 1.5, we get the following result.

**Corollary 7.1.** Let $\mathcal{L} \to C$ be a line bundle over a smooth curve of genus $g$ and $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}_C)$. Then $X$ has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $|\deg(\mathcal{L})| < 2g - 2$.

We can also construct higher dimensional ruled manifolds with scalar-flat metrics.

**Theorem 7.2.** Let $C$ be a smooth curve with genus $g \geq 2$ and $\mathcal{L}$ be a line bundle over $C$. Suppose $\mathcal{E} = \mathcal{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$ and $X = \mathbb{P}(\mathcal{E}^*) \to C$ is the projective bundle. If $0 \leq \deg(\mathcal{L}) < \frac{2g-2}{n-1}$, then both $K_X$ and $K_X^{-1}$ are RC-positive.

**Proof.** By using the projection formula, we know

\begin{equation}
K_X = \mathcal{O}_C(-n) \otimes \pi^*(K_C \otimes \det \mathcal{E}),
\end{equation}

where $\pi : X \to C$ is the projection. Fix an arbitrary smooth Hermitian metric $h^{\mathcal{L}}$ on $\mathcal{L}$ and the trivial metric on $\mathcal{O}_C$. Let $\{z\}$ be the local holomorphic coordinate on $C$. The curvature form of $(\mathcal{L}, h^{\mathcal{L}})$ is

\begin{equation}
R^{\mathcal{L}} = -\sqrt{-1} \partial \bar{\partial} \log h^{\mathcal{L}} = \sqrt{-1} \kappa dz \wedge d\bar{z}.
\end{equation}

Similarly, fix a smooth metric $h^{K_C}$ on $K_C$, and its curvature form is

\begin{equation}
R^{K_C} = -\sqrt{-1} \partial \bar{\partial} \log h^{K_C} = \sqrt{-1} \gamma dz \wedge d\bar{z}.
\end{equation}
Hence, $E$ has the curvature form

$$R^E = \sqrt{-1} \kappa dz \wedge d\bar{z} \otimes e^1 \otimes e^1 + \sum_{i=2}^{\infty} \sqrt{-1} \cdot 0 \cdot dz \wedge d\bar{z} \otimes e^i \otimes e^i,$$

where $e^1 = e_L$ is the local frame of $L$ and for $i \geq 2$, $e^i = e$ is the local holomorphic frame on $O_C$ with the order in the direct sum $E = L \oplus O_C^{\oplus(n-1)}$. Therefore, by (4.3), $O_E(1)$ has the curvature form at some point

$$R^{O_E(1)} = \sqrt{-1} \left( -\kappa |a_1|^2 dz \wedge d\bar{z} \right) - n\omega_{FS}.$$

Hence, by formula (7.1), the curvature of $K_X$ is given by

$$R^{K_X} = \sqrt{-1} \left( \left( \kappa + \gamma \right) - n\kappa |a_1|^2 \right) dz \wedge d\bar{z} - n\omega_{FS}.$$

Since $\deg(L) \geq 0$, we can choose the smooth metric $h^L$ such that its curvature is semi-positive, i.e. $\kappa \geq 0$. Therefore,

$$R^{K_X} \geq \sqrt{-1} \left( \left( \gamma - (n-1)\kappa \right) dz \wedge d\bar{z} \right) - n\omega_{FS}.$$

The condition $0 \leq \deg(L) < \frac{2g-2}{n-1}$ implies $\deg(K_C \otimes L^{1-n}) > 0$. Therefore, we can choose the Hermitian metric $h^{K_C}$ on $K_C$ such that $h^{K_C} \otimes (h^L)^{1-n}$ has positive curvature, i.e.

$$\gamma - (n-1)\kappa > 0.$$

By (7.5), we know the curvature of $K_X$ is positive along the base direction, i.e., $K_X$ is RC-positive. The RC-positivity of $K_X^{-1}$ follows from Theorem 2.4. □

**Example 7.3.** Let $n \geq 2$ be an integer. Let $C$ be a smooth curve of degree $d \geq n+3$ in $\mathbb{P}^2$. It is easy to see that $\deg(O_C(1)) = d$ and $C$ is a curve of genus

$$g = \frac{(d-1)(d-2)}{2}.$$

Let $L = O_C(1)$ and $E = L \oplus O_C^{\oplus(n-1)}$ and $X := \mathbb{P}(E^*) \rightarrow C$ be the projective bundle. Note that $\dim C X = n$. Then

$$\frac{2g-2}{n-1} = \frac{d(d-3)}{n-1} \geq \frac{d \cdot n}{n-1} > d = \deg(L) > 0.$$

Hence, the pair $(X, C, L, E)$ satisfies the conditions in Theorem 7.2. In particular, both $K_X$ and $K_X^{-1}$ are RC-positive.

The proof of Proposition 1.7. By Theorem 7.2 and Theorem 1.2, $X$ admits a scalar-flat Hermitian metric. On the other hand, by [ACGT11, Theorem 1], $X$ has no scalar-flat Kähler metrics since $E = L \oplus O_C^{\oplus(n-1)}$ is not polystable. □
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