Black hole evaporation in an expanding universe

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Abstract
We calculate the quantum radiation power of black holes which are asymptotic to the Einstein–de Sitter universe at spatial and null infinities. We consider two limiting mass accretion scenarios, no accretion and significant accretion. We find that the radiation power strongly depends on not only the asymptotic condition but also the mass accretion scenario. For the no accretion case, we consider the Einstein–Straus solution, where a black hole of constant mass resides in the dust Friedmann universe. We find negative cosmological correction besides the expected redshift factor. This is given in terms of the cubic root of ratio in size of the black hole to the cosmological horizon, so that it is currently of order 10^{-5} (M/10^6 M_\odot)^{1/3} (t/14 Gyr)^{-1/3} but could have been significant at the formation epoch of primordial black holes. Due to the cosmological effects, this black hole has not settled down to an equilibrium state. This cosmological correction may be interpreted in an analogy with the radiation from a moving mirror in a flat spacetime. For the significant accretion case, we consider the Sultana–Dyer solution, where a black hole tends to increase its mass in proportion to the cosmological scale factor. In this model, we find that the radiation power is apparently the same as the Hawking radiation from the Schwarzschild black hole of which mass is that of the growing mass at each moment. Hence, the energy loss rate decreases and tends to vanish as time proceeds. Consequently, the energy loss due to evaporation is insignificant compared to huge mass accretion onto the black hole. Based on this model, we propose a definition of quasi-equilibrium temperature for general conformal stationary black holes.

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1. Introduction

Precisely speaking, in our universe there is no black hole which is asymptotically flat. We call black holes asymptotic to an expanding universe cosmological black holes. A particularly important example of such black holes are primordial black holes [1] which may have formed in the early universe. They would play a unique role as probes into currently unknown physics in various aspects. The key observable phenomenon is the Hawking radiation. See [2] and references therein. The effect of cosmological expansion on the dynamics of black holes has been studied [3]. However, the cosmological effect on the Hawking radiation has not been seriously investigated yet. It is generally believed that the cosmological expansion will not affect the evaporation process if the cosmological horizon is much larger than the black hole horizon. Although this is very plausible from a physical point of view, this should be verified from a definite argument and it is also important to estimate the possible cosmological corrections. Moreover, this assumption might not be the case in some cosmological situations. For example, the black hole horizon may have been of the same order as the cosmological horizon immediately after primordial black holes formed and/or if matters around the black hole continued accreting so rapidly that self-similar growth of a black hole horizon might be possible [3].

The evaporation of cosmological black holes is also important from the point of view of thermodynamics. The black hole thermodynamics [4] offers a unique possibility of understanding the theory of gravity through the laws of thermodynamics. It is argued that a black hole is described as an object in thermal equilibrium (black body) with temperature $T_H = \frac{\kappa}{2\pi}$ [5, 6], where $\kappa$ is the surface gravity of the event horizon, and evaporates by radiating its mass energy according to the Stefan–Boltzmann law [7]. This argument has been so far established only for black holes which are asymptotic to flat, dS and AdS spacetimes at spatial and null infinities. See [4] for asymptotically flat case, and [8] for asymptotically dS and AdS cases. It might be reasonable that black hole thermodynamics requires the asymptotically static nature of spacetimes, because an equilibrium state corresponds to a static condition. However, if black holes can be regarded as thermodynamic systems even for nonstationary cases, it is natural to ask what kind of nonequilibrium states correspond to dynamical black holes. An interesting example is cosmological black holes. The spectrum of its Hawking radiation has already been studied for the case of no mass accretion [9]. However, the radiation power or luminosity has not been explicitly calculated yet.

In the present paper, we calculate the power of the Hawking radiation through the quantum expectation value of a stress–energy tensor in an expanding universe based on the two limiting scenarios about mass accretion. To do this, we use two interesting models for cosmological black holes. One is the Einstein–Straus black hole and the other is the Sultana–Dyer black hole. The former has no mass accretion, while the latter has significant mass accretion. Our first model was raised by Einstein and Straus [10]. This is the exact solution of the Einstein equation with timelike dust, which is obtained by pasting a Schwarzschild spacetime with a dust Friedmann universe on a timelike hypersurface. These kinds of models are often termed as of ‘Swiss-cheese’ type. The existence of a black hole event horizon in the Einstein–Straus solution is guaranteed by construction and the radius of the event horizon is constant. All energy conditions are of course satisfied in this spacetime. Our second model was given by Sultana and Dyer [12, 13]. This is obtained by a conformal transformation operated on the Schwarzschild spacetime, whose conformal factor is carefully chosen so that the metric is the exact solution with the combination of timelike and null dusts and the spacetime is asymptotic to the Einstein–de Sitter (or flat dust Friedmann) universe at spatial and null infinities. Since this transformation does not affect the causal structure, the existence of an event horizon is
guaranteed, although this spacetime has some trouble with energy conditions. This spacetime has a feature that the physical radius of the event horizon increases due to accretion, and it approaches infinity as time proceeds although its growth rate tends to be much slower than the growth rate of the cosmological horizon.

This paper is organized as follows. Sections 2 and 3 are devoted to the calculation of Hawking radiation from Einstein–Straus and Sultana–Dyer black holes, respectively. Summary and discussions are given in section 4. Throughout this paper, we use the Planck units, \( c = \hbar = G = k_B = 1 \).

2. Hawking radiation from the Einstein–Straus black hole

2.1. The Einstein–Straus black hole and its formation

The Einstein–Straus black hole is constructed by pasting the Schwarzschild and the Friedmann solutions at a spherically symmetric timelike hypersurface \( \Sigma_1 \) (see [10] or appendix A in [9]).

The Schwarzschild metric is given by

\[
d_{\text{BH}}^2 = -C(R) \, dt^2 + \frac{dR^2}{C(R)} + R^2 \, d\Omega^2,
\]

(2.1)

where

\[
C(R) := 1 - \frac{2M}{R},
\]

(2.2)

\( M \) is the mass of this black hole, \( d\Omega \) is the line element of a unit two-dimensional sphere, and \( T \) and \( R \) are the time coordinate and areal radius in the Schwarzschild coordinates, respectively. We can get the double null form of the metric as

\[
d_{\text{BH}}^2 = -C(R) \, dU \, dV + R^2 \, d\Omega^2,
\]

(2.3)

where

\[
dR^* := \frac{dR}{C(R)},
\]

(2.4a)

\[
U := T - R^*, \quad V := T + R^*.
\]

(2.4b)

The Friedmann metric is given by

\[
d_{F}^2 = -dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 \, d\Omega^2 \right),
\]

(2.5)

where \( a \) is the scale factor, \( k \) is the spatial curvature, \( r \) is the comoving radius and \( t \) is the proper time of the comoving observer or the cosmological time. Hereafter we assume the open or flat Friedmann metric (\( k = 0 \) or \( -1 \)) in order to guarantee the existence of future null infinity. The double null form of the metric is given by

\[
d_{F}^2 = -a^2 \, du \, dv + R^2 \, d\Omega^2,
\]

(2.6)

where

\[
\eta := \frac{dt}{a}, \quad \chi := \frac{dr}{\sqrt{1 - kr^2}},
\]

(2.7a)

\[
u := \eta - \chi, \quad \chi := \eta + \chi.
\]

(2.7b)

The junction surface \( \Sigma \) is given by a constant comoving radius \( r = r_\Sigma \) and is located at \( R = a(t)r_\Sigma \).

(2.8)
Figure 1. The Einstein–Straus black hole formed in an expanding universe. The zig-zag lines
denote spacetime singularities. H, I, B, S and Σ denote the black hole event horizon, the null
infinity, the starting spacelike surface of gravitational collapse, the surface of collapsing dust ball
and the junction surface between the Schwarzschild and Friedmann spacetimes, respectively. The
empty region surrounded by S, Σ and the black hole singularity is described by the Schwarzschild
solution. The collapsing region surrounded by B, S and the regular centre is described by some
regular dynamical metric. The expanding region surrounded by B, Σ, I, the regular centre and the
big bang singularity is described by the Friedmann solution with dust.

The Israel junction condition [11] with no singular hypersurface reduces the continuity of
the first and second fundamental forms on the junction surface Σ. Then, we obtain two
independent equations,

\[ \frac{dT(t)}{dt} = \frac{\sqrt{1 - kr^{2}}}{C_{|Σ}}, \]  
\[ \left( \frac{\dot{a}}{a} \right)^2 + k \frac{a}{a^2} = \frac{2M}{r_{Σ}^3} \frac{1}{a^3}, \]  
where \( \dot{a} := da(t)/dt \) and

\[ C_{|Σ} = 1 - \frac{2M}{ar_{Σ}}. \]  

Equation (2.9a) gives the relation between the time coordinates T and t. Equation (2.9b) determines the time evolution of the scale factor a(t) and is identical with the dust Friedmann equation. Thus, it is required that the Friedmann metric is that of the dust Friedmann universe. By comparing equation (2.9b) with the dust Friedmann equation,

\[ \left( \frac{\dot{a}}{a} \right)^2 + k \frac{a}{a^2} = \frac{8\pi \rho_{s}}{3 a^3}, \] 

where \( \rho_{s} \) is a constant, we find a relation between \( \rho_{s} \) and M,

\[ M = \frac{\frac{4}{3} \pi r_{Σ}^3 \rho_{s}}{3}. \]

The Einstein–Straus black hole describes a cosmological black hole spacetime with no mass
accretion.

To calculate the power of Hawking radiation from the Einstein–Straus black hole, we need
to specify how it has formed in the expanding universe. Here we assume that an overdense
region of which the comoving radius is \( r_{Σ} \) and mass M begins to contract at the moment \( t = t_{B} \) and collapse to form a black hole. The Penrose diagram of this gravitational collapse is shown in figure 1. In this figure, H, I, B, S denote the event horizon, the null infinity, the spacelike hypersurface \( t = t_{B} \) for \( r < r_{Σ} \) and the surface of the collapsing dust ball, respectively. We can assume that the collapsing region surrounded by B, S and the regular centre is described by some regular dynamical metric,

\[ ds_{\text{col}}^2 = A(τ, λ)(−dτ^2 + dλ^2) + R(τ, λ)^2 dΩ^2, \]
where \( \tau \) and \( \lambda \) are respectively appropriate temporal and radial coordinates. We can set that \( \lambda = 0 \) corresponds to the regular centre. The double null form of the metric is given by

\[
d s_{\text{col}}^2 = -A \, d\alpha \, d\beta + R^2 \, d\Omega^2.
\] (2.14)

in the collapsing region, where

\[
\alpha := \tau - \lambda, \quad \beta := \tau + \lambda.
\] (2.15)

As assumed above, the starting surface \( B \) of collapse is given by a spacelike hypersurface in the expanding region,

\[
t = t_0(r) \quad \text{and} \quad 0 \leq r \leq r_\Sigma.
\] (2.16)

This surface is also described in the collapsing region by

\[
\tau = \tau_0(r) \quad \text{and} \quad \lambda = \lambda_0(r),
\] (2.17)

using the coordinates in the collapsing region.

2.2. Redshift and Hawking radiation

We introduce a matter field \( \phi \) which describes quantum radiation from a black hole. For simplicity, let \( \phi \) be a massless scalar field with minimal coupling, which satisfies the Klein–Gordon equation, \( \Box \phi = 0 \). In manipulating quantum field theory in curved spacetimes, especially on black hole spacetimes, we need to estimate the redshift.

The wave mode of \( \phi \) propagates along a null geodesic \( \gamma \) passing near the event horizon. As shown in the upper panel in figure 2, this mode is ingoing at the initial surface and becomes outgoing after passing through the centre. The function \( \bar{v} = G(u) \) relates the ingoing null coordinate \( \bar{v} \) of \( \gamma \) at the initial surface and the outgoing null coordinate \( u \) of \( \gamma \) at late times. Here note that, although both the initial surface and the spacetime region where a distant observer is are given by the Friedmann solution, we need to distinguish the null coordinates by \( (\bar{v}, \bar{u}) \) at the initial surface and by \( (u, v) \) for the observer because their values are different from each other in this construction. The function \( G(u) \) is obtained by the junction of null coordinates at the intersections of \( \gamma \) with the surfaces \( B, S \) and \( \Sigma \).

The junction at \( B \) gives a relation between \( \bar{v} \) and \( \beta \) as \( \bar{v} = \bar{v}(\beta) \). Since we can assume that the gravitational collapse begins with sufficient smoothness, \( \bar{v} = \bar{v}(\beta) \) is \( C^1 \) for the relevant ingoing null rays, i.e.,

\[
\bar{v} = \bar{v}_1 \beta + \bar{v}_0, \quad \text{(2.18)}
\]

where \( \bar{v}_0 \) and \( \bar{v}_1 (>0) \) are constants.\(^5\) The reflection of \( \gamma \) at the regular centre is given by a simple replacement of \( \beta \) by \( \alpha \) as \( \bar{v} = \bar{v}(\alpha) \). This expresses the redshift along \( \gamma \) from the initial surface to the collapsing region.

The junction at \( S \) gives a relation between \( \alpha \) and \( U \) as \( \alpha = \alpha(U) \). This junction is completely the same as one gets for an asymptotically flat black hole (see [5] and section 8.1 in [6]). Hence, we do not show the details of the calculation but only quote the result:

\[
\alpha = \alpha_1 \exp(-\kappa U) + \alpha_0, \quad \text{(2.19)}
\]

where \( \alpha_0 \) and \( \alpha_1 (>0) \) are constants and

\[
\kappa := \frac{1}{2} \frac{dC}{dR} \Big|_{R=2M} = \frac{1}{4M}. \quad \text{(2.20)}
\]

This expresses the redshift from the collapsing region to the empty region.

\(^5\) A similar discussion is already in section 8.1 in [6].
The junction on $\Sigma$ gives a relation between $U$ and $u$ as $U = U(u)$. To obtain this, we need to consider the junction of metrics, $d_{\text{BH}}^2|_{\Sigma} = d_{\text{F}}^2|_{\Sigma}$. Since $\Sigma$ is the timelike hypersurface given by $r = r_{\Sigma}$ in the comoving coordinates, the null coordinates on $\Sigma$ are regarded as functions of $t$. Furthermore, from relation (2.7a), these null coordinates are instead regarded as functions of $\eta$. Then the junction of the null coordinates on $\Sigma$ can be discussed using the partial derivative with respect to $\eta$, i.e., the junction of metrics gives the following relation on $\Sigma$,

$$CV_{\eta}U_{\eta} = a^2. \quad (2.21)$$

On the other hand, from equations (2.4) and (2.8), we get on $\Sigma$

$$V_{\eta} - U_{\eta} = 2\frac{r_{\Sigma}a'}{C}. \quad (2.22)$$

where the prime $'$ denotes the argument differential, $a' := da(\eta)/d\eta$. Therefore, equation (2.21) becomes a quadratic equation for $U_{\eta}$ on $\Sigma$. We get the positive root and obtain $dU/du$ as

$$\frac{dU}{du} = \left. \frac{U_{\eta}}{u_{\eta}} \right|_{\Sigma} = \frac{1}{C}\left(-r_{\Sigma}a' + \sqrt{r_{\Sigma}^2a'^2 + a^2} \right)|_{\Sigma}. \quad (2.23)$$

This expresses the redshift along $\gamma$ from the empty region to a comoving observer in the expanding region.

Thus, we obtain the function $G(u)$ by combining three functions and the reflection at the centre.

$$\bar{v} = G(u) = \bar{v}'_1 \exp[-\kappa U(u)] + \bar{v}'_0, \quad (2.24)$$

where $\bar{v}'_0$ and $\bar{v}'_1 (>0)$ are constants. Furthermore, for later convenience, we will extend the background to include the negative radial coordinate region. The extended Penrose diagram is shown in the lower panel in figure 2. Then we do not need to consider the reflection of $\gamma$ at the centre. By this virtual extension, equation (2.18) becomes $\bar{u} = \bar{u}_0 + \bar{u}_0'$, where $\bar{u}_0$ and
\( \bar{u}_{1}(>0) \) are constants. Hence, the redshift in the extended background spacetime is given by the same function \( G(u) \) obtained above with replacing \( \bar{v} \) by \( \bar{u} \) in the left-hand side,

\[
\bar{u} = G(u) = \bar{u}_{1} \exp[-\kappa U(u)] + \bar{u}_{0},
\]

(2.25)

where \( \bar{u}_{0} \) is a constant. Note that this function (2.25) also applies to the dimensionally reduced spacetime introduced below.

### 2.3. Hawking radiation from a cosmological black hole with no accretion

The quantum expectation value must be renormalized. The regularization technique in two dimensions has been well established. The appendix summarizes the calculation of the vacuum expectation value of the stress–energy tensor \( \langle \text{vac} | T_{\mu\nu} | \text{vac} \rangle \) and its renormalized value \( \langle T_{\mu\nu} \rangle \), where \( | \text{vac} \rangle \) is an appropriate initial vacuum state. For simplicity, here we reduce the gravitational collapse spacetime described by figure 2 to a two-dimensional one by cutting out the two-dimensional angular part from metrics (2.1), (2.5) and (2.13). It has been known that the two-dimensional Schwarzschild black hole gives a qualitatively correct power of Hawking radiation in four dimensions, since the so-called grey body factor in the Hawking radiation disappears due to the absence of curvature scattering of matter fields in two dimensions. The curvature scattering of matter fields also does not occur in two-dimensional Einstein–Straus spacetime. We can expect that the two-dimensional Einstein–Straus black hole gives a qualitatively correct radiation power.

The thermal radiation in asymptotically flat black hole spacetimes has been obtained under the following three procedures [5, 6]; to neglect the curvature scattering, to define an initial vacuum state on a spacelike hypersurface before the black hole formation, and to observe particles at sufficiently late times. The first is automatically done if we work in two dimensions. The second implies that, in the Heisenberg picture, the quantum state which has been vacuum initially is no longer vacuum after the gravitational collapse. The third implies that a wave mode detected by a distant observer should pass the neighbourhood of the event horizon and hence it has been strongly redshifted before it is observed. This means that it has been of very high frequency near the event horizon, and the geometrical optics approximation is valid, being consistent with neglecting the curvature scattering even for the four-dimensional case.

To calculate the Hawking radiation from the Einstein–Straus black hole, we adopt the vacuum state \( | \text{vac} \rangle \) associated with a comoving observer at the initial surface as a physical initial vacuum state and calculate \( \langle T_{\mu\nu} \rangle \) for a distant comoving observer at sufficiently late times. To be precise, \( | \text{vac} \rangle \) is defined by the quantization of \( \phi \) using the normal modes obtained in the coordinate system of equation (2.7b) on the initial surface, and the components of \( \langle T_{\mu\nu} \rangle \) is calculated in the same coordinates. Moreover, in two dimensions there is no genuine cosmological particle creation for a massless scalar field (see section 3.4 in [6] for example), and hence \( \langle T_{\mu\nu} \rangle \) expresses purely the Hawking radiation from cosmological black holes.

First, we calculate \( \langle T_{\bar{u}\bar{v}} \rangle^{(\text{in})} \) at the initial surface using equations (A.4) and (A.5) given in the appendix. The metric suitable for this purpose is given by \( ds_{\text{E}}^{2} = -a^{2} \bar{u} \, d\bar{v} \), for which

\[
R = -\frac{2[(a')^{2} - aa'']}{a^{4}},
\]

(2.26)

in equation (A.4) and \( D = a^{2} \) in equation (A.5). Therefore we obtain

\[
\langle T_{\bar{u}\bar{v}} \rangle^{(\text{in})} = \langle T_{\bar{v}\bar{u}} \rangle^{(\text{in})} = -\frac{2(a')^{2} - aa''}{48\pi a^{2}},
\]

(2.27a)
The observed power for a comoving observer on the initial surface is proportional to the $\bar{\eta} - \bar{\chi}$ component of $\langle T_{\bar{\mu} \bar{\nu}} \rangle^{(in)}$. By coordinate transformation from $(\bar{u}, \bar{v})$ to $(\bar{\eta}, \bar{\chi})$, we find

$$\langle T_{\bar{\eta} \bar{\chi}} \rangle^{(in)} = 0.$$  \hspace{1cm} (2.28)

This indicates that no energy flux is observed at the initial surface.

Next we calculate $\langle T_{\bar{\mu} \bar{\nu}} \rangle^{(obs)}$ measured by a comoving observer at late times. The metric suitable for this purpose is $d\bar{s}^2_{\bar{\eta} \bar{\chi}} = -a^2 du dv$. This gives the same $\mathcal{R}$ as for $\langle T_{\bar{\mu} \bar{\nu}} \rangle^{(in)}$. In order to find the function $D$ in equation (A.5), null coordinates $(u, v)$ of a late-time observer should be expressed in terms of null coordinates $(\bar{u}, \bar{v})$ with which the initial vacuum state $\ket{\text{vac}}$ is defined. The relation between $(u, v)$ and $(\bar{u}, \bar{v})$ reflects the time evolution of the background spacetime in between and is given by the redshift along outgoing and ingoing null geodesics which connect the late-time observer and the initial surface. For the extended background spacetime (lower panel in figure 2), the relation between $u$ and $v$ is given by the redshift (2.25) of the outgoing null geodesic $\gamma$. The relation between $v$ and $\bar{v}$ is a simple one $v = \bar{v}$, because the relevant ingoing null geodesic lies in the expanding region without passing any surfaces $B$, $S$ and $\Sigma$ during propagating from the initial surface to the late-time observer. Consequently, $D$ which gives the metric at late times as $d\bar{s}^2_{\bar{\eta} \bar{\chi}} = -D du dv$ is obtained as

$$D = a^2 \frac{du}{d\bar{u}} = \frac{a^2}{G''},$$  \hspace{1cm} (2.29)

where the coordinate transformation (2.7b) is used and $G' := dG(u)/du$. This gives

$$D_{, \bar{u}} = \frac{aa'}{G''} = \frac{a^2 G''}{G''},$$  \hspace{1cm} (2.30a)

$$D_{, u} = \frac{(a a')'}{2G^3} - \frac{a a' G''}{G^4} - \frac{a^2 G'''}{G^4} + \frac{3}{2} \frac{a^3 G'''}{G^5},$$  \hspace{1cm} (2.30b)

$$D_{, \bar{v}} = \frac{aa'}{G''},$$  \hspace{1cm} (2.30c)

$$D_{, v} = \frac{(a a')'}{2G''},$$  \hspace{1cm} (2.30d)

where $a_{, u} = (du/d\bar{u})(\partial \eta / \partial u)a' = a'/2G'$ and $a_{, \bar{v}} = (dv/d\bar{v})(\partial \eta / \partial v)a' = a'/2$ are used. Hence equation (A.4) gives

$$\langle T_{\bar{u} \bar{v}} \rangle^{(obs)} = \frac{1}{24 \pi} \left[ \frac{3}{2} \left( \frac{G''}{G'} \right)^2 - \frac{G'''}{G'} \right] + \langle T_{\bar{v} \bar{v}} \rangle,$$  \hspace{1cm} (2.31a)

$$\langle T_{\bar{v} \bar{v}} \rangle^{(obs)} = -\frac{2(a')^2 - a a''}{48 \pi a^2},$$  \hspace{1cm} (2.31b)

$$\langle T_{\bar{u} \bar{u}} \rangle^{(obs)} = \frac{(a')^2 - aa''}{48 \pi a^2}.$$  \hspace{1cm} (2.31c)

The observed power $P_{\text{obs}}$ is given by the tetrad component $\langle T_{\eta \chi}^{(x)} \rangle^{(\text{obs})}$. We obtain $P_{\text{obs}}$ from the above calculations,

$$P_{\text{obs}} := \langle T_{\eta \chi}^{(x)} \rangle^{(\text{obs})} = -\frac{1}{a_0^2} \langle T_{\bar{\eta} \bar{\chi}} \rangle^{(\text{obs})} = \frac{1}{24 \pi a_0^2} \left[ \frac{3}{2} \left( \frac{G''}{G'} \right)^2 - \frac{G'''}{G'} \right],$$  \hspace{1cm} (2.32)
where \( a_0 := a(\eta_0) \) and \( \eta_0 \) is the conformal time at the moment of observation. By substituting expression (2.25) for \( G(u) \) into \( P_{\text{obs}} \),

\[
P_{\text{obs}} = \frac{\kappa^2}{48\pi} \left( \frac{U'}{a_0} \right)^2 + \frac{1}{24\pi a_0^2} \left[ \frac{3}{2} \left( \frac{U''}{U'} \right)^2 - \frac{U'''}{U'} \right],
\]

(2.33)

where \( U' := \frac{dU}{du} \) is given by equation (2.23).

By comparing equation (2.33) with equation (2.28), it is obvious that the quantum creation of energy flow occurs due to the forming black hole. Here, recall that the power of the Hawking radiation \( P_{\text{H}}(2D) \) in an asymptotically flat two-dimensional black hole is

\[
P_{\text{H}}(2D) = \frac{1}{2\pi} \int_0^\infty d\omega \frac{\omega \exp(\frac{2\pi \omega}{\kappa}) - 1}{\exp(2\pi \omega/\kappa) - 1} = \frac{\kappa^2}{48\pi}.
\]

(2.34)

Comparing \( P_{\text{obs}} \) with \( P_{\text{H}}(2D) \), we find that the factor \( (U'/a_0)^2 \) in the first term and the whole of the second term in equation (2.33) are the effects of cosmological expansion.

Here we should recall that our calculation is performed on a two-dimensional background spacetime. That is, in calculating \( P_{\text{H}}(2D) \) in equation (2.34), the state density \( N/2\pi \) at energy level \( \omega \) is appropriate to a two-dimensional case (one spatial dimension), where \( N \) is the effective degrees of freedom and \( N = 1 \) for a scalar field. Therefore the numerical factor in equation (2.33) will be valid only for the two-dimensional case. However, we expect that equation (2.33) is qualitatively correct even for a four-dimensional case if we neglect the curvature scattering and the cosmological particle creation.

2.4. Application to a two-dimensional Einstein–Straus black hole

Here we apply equation (2.33) to our collapse model shown in figure 2. In the following, we assume \( k = 0 \) for simplicity. Then, from equation (2.9b) and the relation \( a d\eta = dt \), the scale factor becomes

\[
a = \left( t/t_{\text{in}} \right)^{2/3} = \left( \frac{\eta}{\eta_{\text{in}}} \right)^2,
\]

(2.35)

where \( \eta_{\text{in}} = 3t_{\text{in}} \), and \( t_{\text{in}} \) and \( \eta_{\text{in}} \) are respectively the cosmological and conformal times at the initial surface. We normalize the scale factor at the initial surface. Furthermore, the Friedmann equation (2.11) relates \( t_{\text{in}} \) and \( \eta_{\text{in}} \) with \( \rho_* \),

\[
\rho_* = \frac{3}{2\pi \eta_{\text{in}}^2}.
\]

(2.36)

Then equation (2.12) gives

\[
M = \frac{2r^2_{\Sigma}}{\eta_{\text{in}}^2}.
\]

(2.37)

Substituting equation (2.35) into the right-hand side of equation (2.23), we obtain

\[
U' = \frac{a}{F} \bigg|_{\Sigma},
\]

(2.38)

where

\[
F := 1 + \frac{2r_{\Sigma}}{\eta},
\]

(2.39)
where equations (2.37) and \( \eta_{in} = 3t_{in} \) are used. Hence, substituting equation (2.38) into equation (2.35), we obtain

\[
P_{obs} = \left( \frac{a_{ret}}{a_0} \right)^2 \frac{\kappa^2}{48\pi} \frac{1}{F(\eta_{ret})^2} + \left( \frac{a_{ret}}{a_0} \right)^2 \frac{1}{24\pi a_{ret}^2} \times \left[ \frac{3}{2} \frac{a'}{a} - \frac{a''}{a} - \frac{a'}{a} \frac{F'}{F} + \frac{F''}{2F} - \frac{1}{2} \left( \frac{F'}{F} \right)^2 \right]_{\eta_{ret}},
\]

(2.40)

where \( Q_{ret} \) denotes the evaluation of \( Q \) at \( \eta = \eta_{ret} := \eta_0 - (r_{obs} - r_{\Sigma}) \) when the ray \( \gamma \) intersects \( \Sigma \). This \( P_{obs} \) is regarded as a function of the cosmological time \( t_0 \) of the observer by using equation (2.35), which gives

\[
t_{ret} = \frac{t_{in}}{\eta_{in}^3} \eta_{ret} = \frac{t_{in}}{\eta_{in}^3} \left[ \left( \frac{\eta_{in}}{\eta_{in}^3} \right)^{1/3} - (r_{obs} - r_{\Sigma}) \right] = \left[ t_{in}^{1/3} - \left( \frac{M}{6} \right)^{1/3} \left( \frac{r_{obs}}{r_{\Sigma}} - 1 \right) \right]^3,
\]

(2.41)

where equation (2.37) is used in the last equality.

Furthermore the observed power \( P_{obs} \) can be expressed in a more convenient form. Using equation (2.37) and the Hubble parameter

\[
H := \frac{a'}{a^2} = \frac{2}{\eta a},
\]

(2.42)

we can get

\[
2r_{\Sigma} \left( \frac{H_{ret}}{\eta_{ret}} \right)^{1/3} = \epsilon^{1/3},
\]

(2.43)

where \( \epsilon := 2M H_{ret} \) is the ratio of the black hole horizon radius to the Hubble horizon radius when \( \gamma \) intersects \( \Sigma \). Furthermore, we observe the cosmological redshift \( z \) of the photon emitted from \( \Sigma \),

\[
1 + z := \frac{a_0}{a_{ret}} = \left( \frac{\eta_0}{\eta_{ret}} \right)^2.
\]

(2.44)

This \( z \) will be regarded as the redshift of the host galaxy of the black hole, and the ratio \( \epsilon \) can be expressed as \( \epsilon = 2M H_0 (1 + z)^{3/2} \), where \( H_0 \) is the present Hubble parameter. It is very natural that the cosmological correction is given in terms of the ratio \( \epsilon \). Using this ratio, we can express \( P_{obs} \) in equation (2.40) simply as

\[
P_{obs} = \frac{\kappa^2}{48\pi (1 + z)^2} \left[ (1 + \epsilon^{1/3})^{-2} + 8\epsilon^2 \left\{ 1 + \frac{\epsilon^{1/3}}{1 + \epsilon^{1/3}} - \frac{1}{8} \left( \frac{\epsilon^{1/3}}{1 + \epsilon^{1/3}} \right)^2 \right\} \right],
\]

(2.45)

where \( \kappa = 1/(4M) \).

Note that the observed power \( P_{obs} \) is not intrinsic but cosmologically redshifted. The intrinsic power \( P_{ES(2D)} \) is then given by

\[
P_{ES(2D)} := (1 + z)^2 P_{obs} = \frac{\kappa^2}{48\pi} \left[ (1 + \epsilon^{1/3})^{-2} + 8\epsilon^2 \left\{ 1 + \frac{\epsilon^{1/3}}{1 + \epsilon^{1/3}} - \frac{1}{8} \left( \frac{\epsilon^{1/3}}{1 + \epsilon^{1/3}} \right)^2 \right\} \right].
\]

(2.46)

The evaporation should be described by this intrinsic power. Up to \( O(\epsilon^{1/3}) \), we get \( P_{ES(2D)} \approx P_{H(2D)} (1 - 2\epsilon^{1/3}) \). This implies that the intrinsic power is suppressed by the cosmological expansion. The physical interpretation of this effect is proposed in section 4.

We are interested in two distinct limits from a physical point of view. In the first, the event horizon is much smaller than the Hubble horizon at present. This corresponds to the limit
\( \epsilon \to 0 \) with keeping \( z \) constant, and we obtain \( P_{ES(2D)} \to P_{H(2D)} \) and \( P_{obs} \to P_{H(2D)}/(1+z)^2 \). In the second, we consider a very late phase of the cosmological evolution, i.e., \( \eta_0 \to \infty \). This corresponds to the limit \( \epsilon \to 0 \) and \( z \to 0 \) simultaneously as seen from equations (2.42) and (2.44). Then we obtain \( P_{ES(2D)} \to P_{H(2D)} \) and \( P_{obs} \to P_{H(2D)} \).

In black hole thermodynamics [4], the Schwarzschild black hole is regarded as in thermal equilibrium, and the temperature \( T_H \) is assigned to the black hole (zeroth law). This temperature is given by \( T_H = \kappa/2\pi = 1/(8\pi M) \) which satisfies the Stefan–Boltzmann law in two dimensions \( P_{H(2D)} = (\pi/12)T_H^4 \) as seen from equation (2.34). Then one might also want to assign the temperature \( T_H = 1/(8\pi M) \) to the Einstein–Straus black hole. However, the radiation power \( P_{ES(2D)} \) deviates from the Stefan–Boltzmann law due to the correction term of \( O(\epsilon^{1/3}) \). This suggests that the Einstein–Straus black hole deviates from thermal equilibrium in a finite cosmological time. Only in the limit \( \eta_0 \to \infty \), this black hole settles down to thermal equilibrium.

### 2.5. Evaporation of the Einstein–Straus black hole in four dimensions

The power \( P_{H(2D)} = \kappa^2/48\pi \) is obtained for asymptotically flat two-dimensional black holes. Therefore, we simply replace the factor \( \kappa^2/48\pi \) in equation (2.46) by the four-dimensional counterpart \( P_{H(4D)} \). This \( P_{H(4D)} \) is given by the Stefan–Boltzmann law in four dimensions,

\[
P_{H(4D)} = \sigma T_H^4 = \frac{N}{30720\pi M^2},
\]

where \( A_H = 4\pi(2M)^2 \) is the spatial area of the event horizon and \( \sigma = N\pi^2/120 \) is the Stefan–Boltzmann constant for the massless matter field with the effective degrees of freedom \( N \). Here \( N \) is given by

\[
N := n_b + \frac{7}{8}n_f,
\]

where \( n_b \) and \( n_f \) are the numbers of helicities of massless bosonic and fermionic fields, respectively, and the factor \( 7/8 \) comes from the difference of statistics of fermions from bosons (see for example [7] for derivation). Then it is appropriate to estimate the order of \( N \) by the standard particles (inner states of quarks, leptons and gauge particles of four fundamental interactions), \( N \simeq 100 \) if the black hole temperature is lower than \( \simeq 1 \text{ TeV} \). Next we consider the correction terms in square brackets in equation (2.46). These terms come from the factors \( U', U'' \) and \( U''' \) in equation (2.33). Here recall that the function \( G(u) \) in equation (2.25) is valid even for four dimensions. Hence we can expect that the same correction terms appear as well for the four-dimensional case. From the above consideration, we expect that the four-dimensional intrinsic power \( P_{ES(4D)} \) is given by

\[
P_{ES(4D)} = \frac{N}{30720\pi M^2} \left[ (1+\epsilon^{1/3})^{-2} + 8\epsilon^2 \left( 1 + \frac{\epsilon^{1/3}}{1+\epsilon^{1/3}} - \frac{1}{8} \left( \frac{\epsilon^{1/3}}{1+\epsilon^{1/3}} \right)^2 \right) \right].
\]  

Finally we estimate the evaporation time of the Einstein–Straus black hole \( t_{es} \). Equations (2.35) and (2.43) give \( \epsilon = (4M)/(3t_{es}) \). Hence, equating \( P_{ES(4D)} \) to \(-dM/dt_{es} \) in the left-hand side of equation (2.49), we can regard it as the evolution equation of mass \( M \) as a function of the cosmological time \( t \). Up to the first correction term of order \( O(\epsilon^{1/3}) \), equation (2.49) gives the semiclassical evolution equation of \( M(t_{es}) \) as

\[
\frac{dM}{dt} \simeq \frac{N}{30720\pi M^2} \left[ 1 - 2 \left( \frac{4M}{3t} \right)^{1/3} \right],
\]

where we denote \( t_{es} \) as \( t \), representing the cosmological time of the evaporating black hole. Since the correction is negative, the emission is suppressed and the life time is prolonged.
Assuming that the correction term is small, we get the order estimate for the deviation of $t_{ES}$ from the evaporation time of the Schwarzschild black hole $t_H$ as

$$\frac{t_{ES}}{t_H} - 1 = O(M^{-2/3})$$

(2.51)

or

$$t_{ES} - t_H = O(M^{7/3})$$

(2.52)

where $t_H$ is given by neglecting the correction term in equation (2.50),

$$t_H \simeq \frac{30720\pi}{N} M^3$$

(2.53)

We can see from equation (2.51) that as the initial mass is larger, the evaporation time is better estimated by $t_H$. This is reasonable since the Hubble parameter of the Einstein–de Sitter universe becomes small as time proceeds and the cosmological effect on the evaporation becomes negligible. It should be noted that the cosmological correction on the evaporation time is relatively small even for a primordial black hole unless its mass is of order the Planck mass. On the other hand, the deviation $(t_{ES} - t_H)$ itself can be very large if the black hole is very massive.

3. Hawking radiation from the Sultana–Dyer black hole

3.1. The Sultana–Dyer black hole

The Sultana–Dyer black hole is obtained by the conformal transformation of the Schwarzschild black hole [12]. Its metric is given by

$$ds^2_{SD} = a(\eta)^2 \left[ -d\eta^2 + dr^2 + r^2 d\Omega^2 + \frac{2M}{r}(d\eta + dr)^2 \right].$$

(3.1)

where $M$ is a positive constant, $a(\eta) = (\eta/\eta_*)^2$ and $\eta_*$ is a constant. This spacetime is asymptotic to the Einstein–de Sitter universe as $r \to \infty$. Here we consider the following coordinate transformation,

$$\eta := t + 2M \ln \left( \frac{r}{2M} - 1 \right).$$

(3.2)

This transforms the metric (3.1) to the conformal Schwarzschild one,

$$ds^2_{SD} = a(t, r)^2 \left[ -dt^2 + \left( 1 - \frac{2M}{r} \right) dr^2 + \left( 1 - \frac{2M}{r} \right)^{-1} d\Omega^2 \right].$$

(3.3)

$r = 2M$ remains an event horizon because the conformal transformation preserves the causal structure. The Penrose diagram of this spacetime is shown in figure 3. There are curvature singularities at $\eta = 0$ and $r = 0$. The singularity at $\eta = 0$ is spacelike for $r > 2M$, timelike for $r < 2M$ and null for $r = 2M$. The central singularity at $r = 0$ is spacelike and surrounded by the event horizon. Hereafter we consider the spacetime given by regions I and II shown separately in the right panel of figure 3.

This spacetime is conformally static, since there exists a conformal Killing vector $\xi = \partial_t$ which is the Killing vector on the Schwarzschild spacetime and satisfies the following relation

$$\mathcal{L}_\xi g_{\mu\nu} = (\mathcal{L}_\xi \ln a^2) g_{\mu\nu},$$

(3.4)

where $\partial_t = \xi_0$ due to the coordinate transformation (3.2) and $\mathcal{L}_\xi \ln a^2 = 4/\eta$. The hypersurface at $r = 2M$ is the conformal Killing horizon which is a null hypersurface where $\xi$ becomes
null. This coincides with the event horizon of the Sultana–Dyer black hole. The Misner–Sharp mass \( m \) at an arbitrary spacetime point is given by

\[
m(\eta, r) = Ma - 2Mr a' + \frac{r^3 (a')^2}{2a} \left( 1 + \frac{2M}{r} \right),
\]

(3.5)

where \( a' := \frac{da}{d\eta} = 2\eta/\eta^2 \). Then the Misner–Sharp mass at the event horizon is

\[
m_{EH} := Ma \left( 1 - \frac{8M}{\eta} + \frac{32M^2}{\eta^2} \right).
\]

(3.6)

This means that the mass of the event horizon tends to increase in proportion to the scale factor \( a \propto \eta^2 \) as \( \eta \to \infty \).

Substituting the metric (3.1) into the Einstein equation, we get

\[
T^{(SD)}_{\mu \nu} = \rho_m u_\mu u_\nu + \rho_r k_\mu k_\nu,
\]

(3.7)

where \( u_\mu \) is a unit timelike vector, \( k_\mu \) is a null vector normalized by \( k_\mu u_\mu = -1 \), \( \rho_m \) is the density of timelike dust and \( \rho_r \) is the density of null dust. The vectors \( u^\mu \) and \( k^\mu \) are given in \((\eta, r, \theta, \phi)\) coordinates as

\[
\begin{align*}
u^\mu &= \begin{pmatrix} \eta^2 r^2 + M(2r - \eta) - \frac{\eta^2}{r} M(2r - \eta) & 0 & 0 \end{pmatrix}, \\
k^\mu &= \begin{pmatrix} \eta^2 \sqrt{r^2 + 2M(r - \eta)} & -\eta^2 \frac{\sqrt{r^2 + 2M(r - \eta)}}{\eta^2} & 0 \end{pmatrix}.
\end{align*}
\]

(3.8a, 3.8b)

The densities of dusts are given by

\[
\begin{align*}
\rho_m &= \eta^4 \frac{12[r^2 + 2M(r - \eta)]}{8\pi r^2 \eta^6}, \\
\rho_r &= \eta^4 \frac{4M[4r^2 + 3M(2r - \eta)]}{8\pi r^2 \eta^3 [r^2 + 2M(r - \eta)]}.
\end{align*}
\]

(3.9a, 3.9b)

These densities imply that energy conditions are satisfied only when \( \eta < r(r + 2M)/2M \) where \( \rho_m > 0 \) and \( \rho_r > 0 \). Furthermore the velocities of dusts (3.8) denote that the null dusts
in the region $\eta < r(r + 2M)/2M$ fall towards the black hole, and also that the timelike dusts in the region $\eta < r(r + 2M)/2M$ and $\eta < 2r$ do so. We can describe a physical picture that the accretion of timelike and null dusts increases the mass of the black hole as shown in equation (3.6). For $\eta > r(r + 2M)/2M$, the source matter fields of the Einstein equation get unphysical. However, the Sultana–Dyer metric is featured with the global structure of a cosmological black hole as seen in figure 3 and also with the conformally static nature which makes the physical interpretation and calculation of the quantum stress–energy tensor most tractable. Hence, we adopt it as not only workable but also a physically interesting model for a cosmological black hole with significant mass accretion.

One may consider the future outer trapping horizon as a local definition of a black hole horizon [14]. The trapping horizon is given by $2m(\eta, r) = R(\eta)$, where $R = ra$ is an areal radius. Thus the trapping horizon in this spacetime is obtained by the following algebraic equation:

$$1 = \frac{2M}{r} - \frac{8M}{\eta} + \frac{4r^2}{\eta^2} \left( 1 + \frac{2M}{r} \right).$$

(3.10)

This has two roots, $r = r_1$ and $r = r_2$ ($r_1 < r_2$), where

$$r_1 := -M + \frac{-\eta + \sqrt{\eta^2 + 24M\eta + 16M^2}}{4},$$

(3.11a)

$$r_2 := \frac{\eta}{2}.$$  

(3.11b)

Regions $0 < r < r_1, r_1 < r < r_2$ and $r_2 < r$ are future trapped, untrapped and past trapped, respectively.

### 3.2. Power of Hawking radiation from the Sultana–Dyer black hole

We introduce a matter field which describes quantum radiation from the black hole. Let $\phi$ be a massless scalar field with conformal coupling, satisfying $(\Box - R/6)\phi = 0$, where $R$ is the Ricci scalar, for which we will see that the renormalized stress–energy tensor $\langle T_{\mu\nu} \rangle$ can be expressed in terms of $\langle T_{\mu\nu} \rangle_{\text{Sch}}$ for the Schwarzschild spacetime. The appendix summarizes how to derive $\langle T_{\mu\nu} \rangle$. We use equation (A.15) given in the appendix. For the Sultana–Dyer spacetime, the metric $\tilde{g}_{\mu\nu}$ and the tensors $\tilde{X}_{\mu\nu}$ and $\tilde{Y}_{\mu\nu}$ in equation (A.15) are all for the Schwarzschild spacetime, and $\langle T_{\mu\nu} \rangle$ is $\langle T_{\mu\nu} \rangle_{\text{Sch}}$. Obviously, $\tilde{X}_{\mu\nu} = \tilde{Y}_{\mu\nu} = 0$ since $\tilde{R}_{\mu\nu} = 0$.

Then equation (A.15) becomes

$$\langle T_{\mu\nu} \rangle = \frac{1}{4\pi} \langle T_{\mu\nu} \rangle_{\text{Sch}} - \frac{1}{2880\pi^2} \left( \frac{1}{6} X_{\mu\nu} - Y_{\mu\nu} \right),$$

(3.12)

where $X_{\mu\nu}$ and $Y_{\mu\nu}$ are obtained by substituting the metric (3.1) into equations (A.16). The first term in $\langle T_{\mu\nu} \rangle$ expresses purely the Hawking radiation from the black hole, and the second term includes cosmological particle creation. Hereafter we consider an observer at $r = \text{const}$, i.e., we calculate $\langle T_{\mu\nu} \rangle$ in $(\eta, r, \theta, \phi)$ coordinates of equation (3.1), which implies that the vacuum state $|\text{vac}\rangle$ in $\langle T_{\mu\nu} \rangle$ is defined with respect to the mode function of $\phi$ in the same coordinates.

For the observer distant from the black hole, the observed energy flux $F_{\text{obs}}$ is given by

$$F_{\text{obs}} := \langle T_{0r} \rangle = -\frac{1}{a^2} \langle T_{0r} \rangle_{\text{Sch}} + \frac{1}{a^2} \frac{1}{2880\pi^2} \left( \frac{1}{6} X_{0r} - Y_{0r} \right),$$

(3.13)

where $\langle T_{0r} \rangle$ is a tetrad component evaluated in the Sultana–Dyer spacetime, and equation (3.12) and $T_{0r} = -a^{-2} T_{0r}$ at the distant region are used in the second equality.
The distant observer is comoving with the timelike dust as seen in equation (3.8). From equations (A.16), the second term in the right-hand side of equation (3.13) becomes

\[
\frac{1}{2880\pi^7} \left( \frac{1}{6} Y_{y^r} - Y_{y^r} \right) = -\frac{M^2 \eta^4_s [-9M \eta^8 + (21M + \eta) \eta^4_s \eta^4 + 12M \eta^8_s]}{30\pi^2 \eta^8 r^6} + \frac{M^2 \eta^4_s [(-28M - \eta) \eta^4_s \eta^4 + 160M \eta^8_s]}{90\pi^2 \eta^4 r^5}
\]

\[
- \frac{M \eta^4_s [2M (65M - 37\eta) \eta^8 + (-240M^2 - 64M \eta + 11\eta^2) \eta^4_s \eta^4 - 120M (5M - \eta) \eta^8_s]}{120\pi^2 \eta^6 r^4}
\]

\[
+ \frac{M \eta^4_s [-4M (65M - 27\eta) \eta^8 + 5(96M + 11\eta) \eta^4_s \eta^4 + 1200M \eta^8_s]}{360\pi^2 \eta^6 r^2}
\]

\[
- \frac{M \eta^4_s [13 \eta^8 - 24 \eta^4_s \eta^4 - 60 \eta^8_s]}{72\pi^2 \eta^6 r^2}
\]

(3.14)

This falls off very rapidly for a distant observer, and hence we get \( F_{\text{obs}} = -a^4 \langle T_{y^r} \rangle_{\text{Sch}} \) for the distant observer. Here the coordinate transformation (3.2) gives \( \langle T_{y^r} \rangle_{\text{Sch}} = \langle T_{y^r} \rangle_{\text{Sch}} (1 - r/M)^{-1} \langle T_{y^r} \rangle_{\text{Sch}} \). Hence we obtain

\[
F_{\text{obs}} = -\frac{1}{a^4} \langle T_{y^r} \rangle_{\text{Sch}} = \frac{1}{a^4} \langle T_{y^r} \rangle_{\text{Sch}}.
\]

(3.15)

where \( \langle T_{y^r} \rangle_{\text{Sch}} \) is a tetrad component evaluated in the Schwarzschild spacetime. This \( F_{\text{obs}} \) is the flux (energy flow per unit time and unit area) detected by the distant comoving observer.

It should be pointed out that the flux \( F_{\text{obs}} \) contains only the Hawking radiation from the black hole but no cosmological particle creation, since \( F_{\text{obs}} \) is proportional only to \( \langle T_{y^r} \rangle_{\text{Sch}} \) as shown in equation (3.15). This does not imply the absence of cosmological particle creation at our distant observer. The energy density \( \sigma \) of the quantum field \( \phi \) in the distant region indicates the cosmological particle creation at the distant observer. We get from equations (3.12) and (A.16),

\[
\sigma := \langle T_{(y^r)} \rangle = -\frac{1}{a^4} \langle T_{y^r} \rangle
\]

\[
= -\frac{1}{a^4} \langle T_{y^r} \rangle_{\text{Sch}} + \frac{59 \eta^8_s \eta^8 + 70 \eta^12_s \eta^4 - 100 \eta^16_s}{240\pi^2 \eta^20 r^2}
\]

(3.16)

for the distant observer. The second term does not include \( M \) and expresses purely the cosmological particle creation in the distant region. However this raises no energy flow as shown in equation (3.15).

Hence \( F_{\text{obs}} \) in equation (3.15) is the flux of the Hawking radiation from the Sultana–Dyer black hole. Then the intrinsic power of the Hawking radiation \( P_{SD} \) should be given by

\[
P_{SD} := 4\pi (ra)^2 F_{\text{obs}} = \frac{1}{a^4} [4\pi r^2 \langle T_{y^r} \rangle_{\text{Sch}}].
\]

(3.17)

where the right-hand side should be evaluated for the distant observer. Here note that the factor \( 4\pi r^2 \langle T_{y^r} \rangle_{\text{Sch}} \) is the observed power of the Hawking radiation in the Schwarzschild spacetime, and it should equal \( P_{H(4D)} \) given in equation (2.47) if the geometrical optics approximation is valid. Therefore, under this approximation, we get

\[
P_{SD} = \frac{1}{a^4} \left[ \frac{N \kappa^2}{1920\pi} \right]
\]

(3.18)

where \( \kappa = 1/(4M) \) and \( N \) is given by equation (2.48).

The geometrical optics approximation gets very good for late times, and the mass \( m_{EH}(\eta) \) in equation (3.6) becomes \( m_{EH} \rightarrow Ma \) as \( \eta \rightarrow \infty \). Hence, by comparing our result \( P_{SD} \)
with the Schwarzschild one \( P_{\text{H}}(4D) \), it is suggested that the effective temperature \( T_{\text{eff}} \) of the Sultana–Dyer black hole at late times is given by

\[
T_{\text{eff}} = \frac{1}{8\pi M a} = \frac{\kappa}{2\pi a}. \tag{3.19}
\]

So, both the intrinsic power and temperature of the radiation from the Sultana–Dyer black hole are the same as those of the Hawking radiation from the Schwarzschild black hole of which mass is the momentary mass of the growing event horizon. This temperature \( T_{\text{eff}} \) and the intrinsic power \( P_{\text{SD}} \) decrease as time proceeds. This result is reasonable since the Sultana–Dyer black hole describes significant mass accretion. This black hole does not lose but gains mass due to the accretion of timelike and null dusts, and can be regarded as an object in quasi-equilibrium with temperature \( T_{\text{eff}} \).

### 3.3. Conformal dynamics at infinity and temperature of the Sultana–Dyer black hole

A stationary spacetime is defined by a timelike Killing vector \( \xi \) satisfying \( \mathcal{L}_\xi g_{\mu\nu} = 0 \). The Killing horizon is a null hypersurface where \( \xi \) becomes null, and the surface gravity \( \kappa \) is defined by \( \xi^\alpha \nabla_\alpha \xi^\mu = \kappa \xi^\mu \) at the Killing horizon. The value of \( \kappa \) changes according to the normalization of \( \xi \) by definition. For asymptotically flat stationary black hole spacetimes, the Killing horizon coincides with the event horizon. The surface gravity \( \kappa \) of the stationary black hole is constant everywhere on the event horizon. This is the zeroth law of black hole thermodynamics [4, 5] which states that a unique temperature can be assigned to the stationary black hole. Then the thermal spectrum of Hawking radiation from the stationary black hole, which is a quantum phenomenon, determines the value of the temperature to be \( \kappa / 2\pi \) under the normalization of the Killing vector as \( \xi^\mu \xi_\mu \rightarrow -1 \) at null and spatial infinities [5].

Several generalizations of the zeroth law have already been discussed for general conformal stationary black hole spacetimes whose metric \( g_{\mu\nu} \) is given by

\[
g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu},
\]

where \( \Omega^2 \) is the conformal factor and \( \tilde{g}_{\mu\nu} \) is the metric of an asymptotically flat stationary black hole [15–17]. A natural generalization of the surface gravity \( \kappa_{\text{DH}} \) can be introduced by the following relation at the conformal Killing horizon [15, 16],

\[
\xi^\mu \nabla_\alpha \xi^\alpha = \kappa_{\text{DH}} \xi^\mu, \tag{3.20}
\]

where \( \xi \) is a conformal Killing vector satisfying \( \mathcal{L}_\xi g_{\mu\nu} = (\mathcal{L}_\xi \ln \Omega^2) g_{\mu\nu} \), and the conformal Killing horizon is the hypersurface where \( \xi \) becomes null. Under the conditions \( \Omega \rightarrow 1 \) (or constant) and \( \xi^\mu \xi_\mu \rightarrow -1 \) at null infinity, Sultana and Dyer conjectured that the temperature of conformal stationary black holes \( T_{\text{SD}} \) is given by [16]

\[
T_{\text{SD}} := \frac{1}{2\pi} (\kappa_{\text{DH}} - \mathcal{L}_\xi \ln \Omega^2). \tag{3.21}
\]

\( T_{\text{SD}} \) is constant everywhere on the conformal Killing horizon, while \( \kappa_{\text{DH}} \) is not. On the other hand, Jacobson and Kang independently introduced a generalized surface gravity \( \kappa_{\text{JK}} \) as [17],

\[
\nabla_\mu (\xi^\mu \xi_\nu) = -2\kappa_{\text{JK}} \xi_\mu. \tag{3.22}
\]

\( \kappa_{\text{JK}} \) is invariant under the conformal transformation, while \( \kappa_{\text{DH}} \) is not. Then under the conditions \( \Omega \rightarrow 1 \) and \( \xi^\mu \xi_\mu \rightarrow -1 \) at null infinity, they conjectured that the temperature of conformal stationary black holes \( T_{\text{JK}} \) is given by

\[
T_{\text{JK}} := \frac{\kappa_{\text{JK}}}{2\pi}. \tag{3.23}
\]

In fact, it can be shown that the relation \( \kappa_{\text{JK}} = \kappa_{\text{DH}} - \mathcal{L}_\xi \ln \Omega^2 \) holds [17], and hence \( T_{\text{SD}} = T_{\text{JK}} = T_{\text{JKSD}} \). Thus, although Sultana–Dyer [16] and Jacobson–Kang [17]...
independently considered the surface gravity and temperature of conformal stationary black holes, they reached the same conjecture.

For the Sultana–Dyer black hole, the conjectured temperature $T_{JKSD}$ has already been calculated [12]. The conformal Killing vector is $\xi = \partial_\eta$ and the conformal Killing horizon coincides with the event horizon $r = 2M$. The norm of $\xi$ is $\xi^\mu \xi_\mu \to -a^2 \neq -1$ at null infinity, not satisfying the unit norm condition for $T_{JKSD}$. Sultana and Dyer still assumed in [12] the temperature $T_{JKSD}$ should be assigned to the Sultana–Dyer black hole. Then, substituting $\xi$ into equation (3.20) or (3.22), they obtained $\kappa_{JK} = \kappa_{DH} - 4/\eta = 1/(4M)$ and

$$T_{JKSD} = \frac{1}{8\pi M},$$

(3.24)

which is equal to the Hawking temperature of the Schwarzschild black hole of mass $M$, but not to our effective temperature $T_{\text{eff}}$ in equation (3.19). However, the temperature of the black hole should be given based on the spectrum and/or the power of the Hawking radiation. Here, we propose that the physically reasonable temperature of black holes which are conformal stationary and asymptotically dynamical is not $T_{JKSD}$ but $T_{\text{eff}} = T_{JKSD}/\Omega$.

(3.25)

So the effective temperature depends on space and time through the conformal factor. This might be understood in an analogy with Tolman’s law for thermal equilibrium in the presence of a gravitational field [18].

4. Summary and discussions

We have calculated the intrinsic power of the Hawking radiation from cosmological black holes for two cases, no mass accretion and significant mass accretion.

For the no mass accretion case, we have considered the Einstein–Straus black hole. Our result $P_{ES(4D)}$ in equation (2.49) indicates $P_{ES(4D)} < P_{H(4D)}$, i.e., the black hole evaporation is suppressed by the cosmological expansion. The ratio $P_{ES(4D)}/P_{H(4D)} < 1$ is given in terms of $\epsilon^{1/3}$ where $\epsilon$ is the ratio in size of the black hole to the cosmological horizon. The first correction term is $O(\epsilon^{1/3})$ and therefore currently as small as $10^{-5}(M/10^6 M_\odot)^{1/3}(t/14\text{Gyr})^{-1/3}$, but could be significant for the formation epoch of primordial black holes. The evaporation time is essentially the same as that of the Schwarzschild black hole as long as its mass is greater than the Planck mass. Furthermore, by comparing the functional form of $P_{ES(4D)}$ with that of $P_{H(4D)}$ in equation (2.47), we can see that the Einstein–Straus black hole has not settled down to thermal equilibrium in a finite cosmological time.

For the significant mass accretion case, we have considered the Sultana–Dyer black hole. This has very different properties. Our result $P_{SD}$ in equation (3.18) indicates that the Sultana–Dyer black hole does not evaporate away. Furthermore the Sultana–Dyer black hole can be regarded as an object in quasi-equilibrium, since the effective temperature $T_{\text{eff}}$ can be assigned as equation (3.19). The intrinsic power $P_{SD}$ of the Hawking radiation is consistent with the Stefan–Boltzmann law for a black body with temperature $T_{\text{eff}}$. We propose a new definition (3.25) for the temperature for general conformal stationary black holes.

Finally we try to interpret $P_{ES(4D)}$ for the Einstein–Straus black hole in an analogy with quantum radiation of a slowly moving mirror in a flat spacetime. For simplicity, we consider a moving mirror $x = x(t)$ in the two-dimensional Minkowski spacetime with the Cartesian coordinates $(t, x)$. Then it is well known that the moving mirror emits quantum radiation of
a massless scalar field $\phi$. When an observer at rest is in the region $x > x(t)$, the observed power $P_{\text{mir}}$ of quantum radiation from the mirror is given by [6, 19]

$$P_{\text{mir}} := \langle T^{(x)} \rangle_{\text{mir}} = -\frac{1}{12\pi} \sqrt{1 - v^2} \frac{d\alpha_{\text{mir}}}{dt_{\text{ret}}},$$

where $\langle T^{(x)} \rangle_{\text{mir}}$ is a tetrad component, $t_{\text{ret}}$ is the retarded time when the observed particle of $\phi$ was emitted from the mirror, $v = dx(t)/dt|_{t_{\text{ret}}} := \dot{x}(t_{\text{ret}})$ is positive when the mirror is approaching towards the rest observer, and $\alpha_{\text{mir}}$ is the proper acceleration of the mirror. In the Minkowski spacetime, the observed power $P_{\text{mir}}$ is equal to the intrinsic power of quantum radiation from the mirror. Here we consider the case that the mirror moves slowly, i.e., $|v| \ll 1$. Then the power $P_{\text{mir}}$ is given as

$$P_{\text{mir}} \approx -\frac{\dot{x}(t_{\text{ret}})}{12\pi} (1 + 2v) + O(v^2).$$

Note that the kinematic effect comes in the radiation power in the form of $(1 + 2v)$ in the lowest order. On the other hand for the Einstein–Straus black hole, the relative ‘velocity’ $v_{\text{ES}}$ of the junction surface/$\Sigma_1$ to the black hole at the retarded time may be written as

$$v_{\text{ES}} = -\frac{\dot{r}_{\Sigma}}{r_{\Sigma} \dot{a}_{\text{ret}}},$$

where $a_{\text{ret}} := a(t_{\text{ret}})$, and the minus sign means the increase of the relative distance. Equation (2.9b) with $k = 0$ gives $\dot{a}_{\text{ret}}^2 = (2MH_{\text{ret}})/r_{\Sigma}^3 = \epsilon/r_{\Sigma}^3$, i.e., $v_{\text{ES}} = -\epsilon^{1/3}$. Therefore, the kinematic correction factor in equation (4.2) coincides with the cosmological correction factor in equation (2.49) up to this order. Hence, we can interpret that the correction factor $(1 - 2\epsilon^{1/3})$ in $P_{\text{ES}(4D)}$ is some kinematic effect from the cosmological expansion in an analogy with radiation from a moving mirror.

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Appendix. Vacuum expectation value of a quantum stress–energy tensor

A.1. Two-dimensional case

It has already been recognized for a few decades that many different methods of renormalization give equivalent results (see for example, chapters 6 and 7 in [6]). We consider a minimally coupled massless scalar field $\phi$, whose stress–energy tensor is given by

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi\phi^{\alpha\beta}.$$  \hspace{1cm} (A.1)

The background spacetime is described in double null coordinates $(u, v)$ as

$$ds^2 = -D(u, v) du dv.$$  \hspace{1cm} (A.2)

The field $\phi$ satisfies the Klein–Gordon equation $\Box \phi = 0$. When a coordinate system (not necessarily null) is specified to describe the differential operator $\Box$, we can find a complete orthonormal set $\{f_\omega\}$ for arbitrary solutions of $\Box \phi = 0$, where $\omega$ denotes the frequency...
of the mode function. The positive frequency mode is the mode function \( f_\omega \) which is constructed to satisfy the conditions, \( \omega > 0 \), \( f_\omega, f_\omega' = \delta(\omega - \omega') \), \( f_\omega, f_\omega' = 0 \) and \( (f_\omega^*, f_\omega) = -\delta(\omega - \omega') \), where \( (f, g) \) is the inner-product defined from the Noether charge of time translation of \( \phi \) and \( f_\omega^* \) is a complex conjugate to \( f_\omega \), called the negative frequency mode. In two-dimensional spacetimes, the positive frequency modes can be decomposed with respect to the direction of propagation. In the double null coordinates, they are \( f_\omega(u) = \exp(-i\omega u)/\sqrt{4\pi \omega} \) and \( f_\omega(v) = \exp(-i\omega v)/\sqrt{4\pi \omega} \). Then, the quantum operator \( \phi \) is expanded by the complete orthonormal set of the positive and negative frequency modes as

\[
\phi(u, v) = \int_0^\infty d\omega [a_\omega f_\omega(u) + a_\omega^* f_\omega^*(u) + b_\omega f_\omega(v) + b_\omega^* f_\omega^*(v)].
\]

(A.3)

The canonical quantization presumes the simultaneous commutation relation between \( \phi \) and its conjugate momentum, so that \( [a_\omega] \) and \( [b_\omega] \) are harmonic operators satisfying the commutation relations; \( [a_\omega, a_\omega^*] = \delta(\omega - \omega') \) and \( [b_\omega, b_\omega^*] = \delta(\omega - \omega') \) and all others vanish. They define the Fock space of quantum states and give particle interpretation. The vacuum state \( |\text{vac}\rangle \) is defined as a quantum state satisfying \( a_\omega|\text{vac}\rangle = b_\omega|\text{vac}\rangle = 0 \) for all \( \omega \).

If we choose different coordinates \((\bar{u}, \bar{v})\), a natural orthonormal set of mode functions is \( \{f_{\omega}\} \), where \( f_{\omega}(\bar{u}) = \exp(-i\omega \bar{u})/\sqrt{4\pi \omega} \) and \( f_{\omega}(\bar{v}) = \exp(-i\omega \bar{v})/\sqrt{4\pi \omega} \). Then the expansion (A.3) gives different harmonic operators \( \{a_\omega\} \) and \( \{b_\omega\} \). These harmonic operators define another vacuum state \( |\text{vac}^{\prime}\rangle \) if there arises the mixing of positive and negative frequency modes \( f_{\omega}, f_{\omega}^* \neq 0 \) between the two coordinate systems. Thus, even if a quantum state is initially set to be a vacuum state, this does not remain vacuum but corresponds to an excited state associated with the coordinate system natural to an observer at the final time if the mixing of positive and negative modes arises. This will be interpreted as quantum particle creation in curved spacetimes.

The quantum expectation value of the stress–energy tensor \( \langle \text{vac}|T_{\mu\nu}|\text{vac}\rangle \) is calculated by substituting the quantum operator (A.3) (after replacing \( a_\omega \) and \( b_\omega \) with \( \bar{a}_\omega \) and \( \bar{b}_\omega \)) into the stress–energy tensor (A.1). However, \( \langle \text{vac}|T_{\mu\nu}|\text{vac}\rangle \) diverges even for flat background cases. Therefore, we need to renormalize the stress–energy tensor. We do not get into the details of the regularization method but only quote the result [20],

\[
\langle T_{\bar{\mu}\bar{\nu}} \rangle = \theta_{\bar{\mu}\bar{\nu}} + \frac{\mathcal{R}}{48\pi} g_{\bar{\mu}\bar{\nu}},
\]

(A.4)

where \( \langle T_{\bar{\mu}\bar{\nu}} \rangle \) is the renormalized expectation value of \( \langle \text{vac}|T_{\bar{\mu}\bar{\nu}}|\text{vac}\rangle \), \( \mathcal{R} \) is the Ricci scalar of the background spacetime, and \( \theta_{\bar{\mu}\bar{\nu}} \) is a symmetric tensor whose components in the coordinate system \((\bar{u}, \bar{v})\) on which the vacuum \( |\text{vac}\rangle \) is defined is given by

\[
\theta_{\bar{u}\bar{u}} := -\frac{1}{24\pi\overline{s}} \left[ \frac{1}{2} \frac{D_{\bar{u}}}{D} D_{\bar{u}} - \frac{D_{\bar{u}\bar{u}}}{D} \right],
\]

(A.5a)

\[
\theta_{\bar{v}\bar{v}} := -\frac{1}{24\pi\overline{s}} \left[ \frac{1}{2} \frac{D_{\bar{v}}}{D} D_{\bar{v}} - \frac{D_{\bar{v}\bar{v}}}{D} \right],
\]

(A.5b)

\[
\theta_{\bar{u}\bar{v}} = \theta_{\bar{v}\bar{u}} = 0,
\]

(A.5c)

where \( D(\bar{u}, \bar{v}) = -2\overline{s}_{\bar{u}\bar{v}} \). The renormalized expectation value \( \langle T_{\mu\nu} \rangle \) of \( \langle \text{vac}|T_{\mu\nu}|\text{vac}\rangle \) in the other coordinates \((u, v)\) is calculated from the above components through the usual coordinate transformation for tensor components,

\[
\langle T_{\mu\nu} \rangle = \frac{\partial x^\alpha}{\partial \bar{u}^a} \frac{\partial x^\beta}{\partial \bar{v}^b} \langle T_{ab} \rangle.
\]

(A.6)
A.2. Four-dimensional case

The renormalized vacuum expectation value of the stress–energy tensor $\langle T_{\mu\nu} \rangle$ of some matter field in four dimensions may also be calculated along with the canonical quantization formalism as shown for the two-dimensional case in the previous section. However the path integral quantization formalism is more convenient to summarize $\langle T_{\mu\nu} \rangle$ on a four-dimensional conformal spacetime. The effective action $W$ of a quantum matter field $\phi$ on a spacetime of metric $g_{\mu\nu}$ gives the vacuum expectation value of quantum stress–energy tensor. $W$ can be evaluated by the path integral method and the vacuum state $\langle \text{vac} \rangle$ is specified by the Green function of $\phi$ used in evaluating the path integral. However the precise path integral form of $W$ is not important here. $W$ is decomposed into two parts as $W = W_{\text{ren}} + W_{\text{div}}$, where $W_{\text{ren}}$ is the renormalized part and $W_{\text{div}}$ is the divergent part. The functional differentiation of $W_{\text{ren}}$ gives the renormalized vacuum expectation value $\langle T_{\mu\nu} \rangle$,

$$\langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{ren}}}{\delta g^{\mu\nu}}. \quad (A.7)$$

We consider the case that the metric $g_{\mu\nu}$ is conformal to the other one as

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \quad (A.8)$$

and the matter field $\phi$ is a conformally coupled massless scalar field satisfying $((\Box - \mathcal{R}/6)\phi = 0$. On the other hand, we get by definition of functional differentiation,

$$W_{\text{ren}} - \tilde{W}_{\text{ren}} = \int \frac{\delta W_{\text{ren}}}{\delta g^{\alpha\beta}} \frac{\delta g^{\alpha\beta}}{\delta \Omega} d^4x, \quad (A.9)$$

where $\tilde{W}_{\text{ren}}$ is the renormalized effective action obtained from $W_{\text{ren}}$ with replacing $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu}$. Then considering the functional differentiation only by the conformal transformation, $\delta g^{\mu\nu} = -2g^{\mu\nu}/\Omega^2 \delta \Omega$, the effective action is expressed as

$$W_{\text{ren}} = \tilde{W}_{\text{ren}} - \int g^{\alpha\beta} \langle T_{\alpha\beta} \rangle \frac{\delta \Omega}{\Omega} \sqrt{-g} d^4x. \quad (A.10)$$

Substituting this into equation (A.7), we get

$$\langle T_{\mu\nu} \rangle = \frac{1}{\Omega^2} \langle \tilde{T}_{\mu\nu} \rangle - \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int g^{\alpha\beta} \langle T_{\alpha\beta} \rangle \frac{\delta \Omega}{\Omega} \sqrt{-g} d^4x, \quad (A.11)$$

where $\tilde{T}_{\mu\nu} = \tilde{g}_{\mu\sigma} \tilde{g}^{\sigma\nu}$, $g_{\mu\sigma} \tilde{g}^{\sigma\nu} = \Omega^2 \tilde{g}_{\mu\nu}$ and the general relation,

$$g^{\mu\alpha} \frac{\delta}{\delta g^{\nu\sigma}} = \tilde{g}^{\mu\alpha} \frac{\delta}{\delta \tilde{g}^{\nu\sigma}}, \quad (A.12)$$

are used to get the first term of the right-hand side of equation (A.11). The trace $g^{\alpha\beta} \langle T_{\alpha\beta} \rangle$ is usually called the conformal anomaly or the trace anomaly, and it is well known that the divergent part $W_{\text{div}}$ gives the conformal anomaly as (see section 6.3 in [6] for example)

$$g^{\alpha\beta} \langle T_{\alpha\beta} \rangle = \frac{\Omega}{\sqrt{-g}} \frac{\delta W_{\text{div}}}{\delta \Omega}. \quad (A.13)$$

Hence substituting this expression of the conformal anomaly into equation (A.11) and using equations (A.12) and (A.9) with replacing $W_{\text{ren}}$ by $W_{\text{div}}$, we obtain

$$\langle T_{\mu\nu} \rangle = \frac{1}{\Omega^2} \langle \tilde{T}_{\mu\nu} \rangle - \frac{2}{\sqrt{-g}} \frac{\delta W_{\text{div}}}{\delta g^{\mu\nu}} + \frac{2\Omega^2}{\sqrt{-g}} \frac{\delta \tilde{W}_{\text{div}}}{\delta g^{\mu\nu}}. \quad (A.14)$$

The divergent part $W_{\text{div}}$ can be evaluated from the Green function of the matter field $\phi$. We do not follow the details of the calculation of $W_{\text{div}}$, but quote only the result for $\langle T_{\mu\nu} \rangle$ for
the conformally coupled massless scalar field $\phi$ on the spacetime of metric (A.8) (see [21] or sections 6.2 and 6.3 in [6] for detail),

$$\langle T_{\mu\nu} \rangle = \frac{1}{\Omega^2} \langle \tilde{T}_{\mu\nu} \rangle - \frac{1}{2880 \pi^2} \left( \frac{1}{6} X_{\mu\nu} - Y_{\mu\nu} \right) + \frac{1}{2880 \pi^2 \Omega^2} \left( \frac{1}{6} \tilde{X}_{\mu\nu} - \tilde{Y}_{\mu\nu} \right),$$  \hspace{1cm} (A.15)

where

$$X_{\mu\nu} := 2 \nabla_\mu \nabla_\nu \mathcal{R} - 2 g_{\mu\nu} \Box \mathcal{R} + \frac{1}{2} \mathcal{R}^2 g_{\mu\nu} - 2 \mathcal{R} R_{\mu\nu},$$  \hspace{1cm} (A.16a) 

$$Y_{\mu\nu} := - R^a_\mu R_{a\nu} + \frac{2}{3} \mathcal{R} R_{\mu\nu} + \frac{1}{2} R_{ab} R^{ab} g_{\mu\nu} - \frac{1}{4} \mathcal{R}^2 g_{\mu\nu},$$  \hspace{1cm} (A.16b)

where $R_{\mu\nu}$ and $\mathcal{R}$ are the Ricci tensor and scalar with respect to $g_{\mu\nu}$ respectively, and $\tilde{X}_{\mu\nu}$ and $\tilde{Y}_{\mu\nu}$ are defined similarly with respect to the metric $\tilde{g}_{\mu\nu}$. Equation (A.15) is the generalization of equation (6.141) in [6] to the general conformal spacetimes of metric (A.8).

References

[1] Zeldovich Y B and Novikov I D 1967 Sov. Astron. A. J. 10 602
  Hawking S W 1971 Mon. Not. R. Astron. Soc. 152 75
  Carr B J 1975 Astrophys. J. 201 1
[2] Carr B J 2004 Proc. 22nd Texas Symp. on Relativistic Astrophysics (Stanford University, Stanford, California, 13–17 December 2004) p 204 (Preprint astro-ph/0504034)
[3] Harada T and Carr B J 2005 Phys. Rev. D 71 104009
  Harada T and Carr B J 2005 Phys. Rev. D 71 104010
  Harada T and Carr B J 2005 Phys. Rev. D 72 044021
  Harada T, Maeda H and Carr B J 2006 Phys. Rev. D 74 024024
  Carr B J 2005 Preprint astro-ph/0511743
[4] Hawking S W 1971 Phys. Rev. Lett. 26 1344
  Bekenstein J D 1973 Phys. Rev. D 7 2333
  Bekenstein J D 1974 Phys. Rev. D 9 3292
  Bardeen J M, Carter B and Hawking S W 1973 Commun. Math. Phys. 31 161
  Hawking S W 1976 Phys. Rev. D 13 191
  Israel W 1986 Phys. Rev. Lett. 57 397
  Iyer V and Wald R M 1994 Phys. Rev. D 50 846
[5] Hawking S W 1975 Commun. Math. Phys. 43 199
[6] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press)
[7] Saada H 2005 Physica A 356 481
  Saada H 2006 Class. Quantum Grav. 23 6227
  Saada H 2007 Class. Quantum Grav. 24 691
[8] Gibbons G W and Hawking S W 1977 Phys. Rev. D 10 2738
  Denardo G and Spallucci E 1979 Nuovo Cimento B 53 334
  Denardo G and Spallucci E 1980 Nuovo Cimento B 55 97
  Hawking S W and Page D N 1983 Commun. Math. Phys. 87 577
[9] Saada H 2002 Class. Quantum Grav. 19 3179
[10] Einstein A and Straus E G 1945 Rev. Mod. Phys. 17 120
  Einstein A and Straus E G 1946 Rev. Mod. Phys. 18 148
  Dyer C and Oliwa C 2000 Preprint astro-ph/0004090
[11] Israel W 1966 Nuovo Cimento B 44 1
  Israel W 1967 Nuovo Cimento B 48 463
[12] Sultana J and Dyer C C 2005 Gen. Rel. Grav. 37 1349
[13] McClure M L and Dyer C C 2006 Gen. Rel. Grav. 38 1347
  McClure M L and Dyer C C 2006 Class. Quantum Grav. 23 1971
[14] Hayward S A 1994 Phys. Rev. D 49 6467
[15] Dyer C C and Honig E 1979 J. Math. Phys. 20 409
[16] Sultana J and Dyer C C 2004 J. Math. Phys. 45 4764
[17] Jacobson T and Kang G 1993 Class. Quantum Grav. 10 L201
[18] Tolman R C 1987 Relativity, Thermodynamics and Cosmology (New York: Dover)
  Tolman R C 1934 Original (Oxford: Oxford University Press)
[19] Fulling S A and Davies P C W 1976 Proc. R. Soc. A 348 393
[20] Davies P C W, Fulling S A and Unruh W G 1976 Phys. Rev. D 13 2720
  Davies P C W and Fulling S A 1977 Proc. R. Soc. A 354 59
  Davies P C W 1977 Proc. R. Soc. A 354 529
[21] Brown L S and Cassidy J P 1977 Phys. Rev. D 15 2810
  Bunch T S 1979 J. Phys. A. Math. Gen. 12 517