Counting Quiddities of Polygon Dissections

By Charles H. Conley and Valentin Ovsienko

A dissection is a partition of a convex polygon into subpolygons, or cells, by noncrossing diagonals. The simplest and most popular class of dissections is the triangulations, where the cells are all triangles. Polygon dissections have appealed to mathematicians since the time of ancient Greece, and they play roles in the study of such modern objects as associahedra and cluster algebras. The entry A033282 in the On-Line Encyclopedia of Integer Sequences (OEIS) gives an extensive list of references.

The enumeration of dissections was initiated by Euler [6], who already knew of the Catalan numbers and their main properties. The topic arises in several areas of mathematics, connecting Young tableaux, knot invariants, continued fractions, and many other notions (some discussed at the end of this article). It is still an active area; in particular, the enumeration of classes of dissections under various equivalence relations has been considered by a number of modern authors. In this brief note we advertise a new open problem of this nature.

Classical Formulas

Here are the most classical sequences arising in our context:

- The Catalan sequence [OEIS A000108] is
  \[
  C_n := \frac{1}{n+1} \binom{2n}{n},
  \]
  the number of triangulations of the \((n+2)\)-gon. For hundreds of interpretations and references, see Richard Stanley’s addendum [18] and Igor Pak’s webpage [12].

- The Kirkman–Cayley sequence [OEIS A033282] is
  \[
  D_{n,m} := \frac{1}{n+1} \binom{n-1}{m-1} \binom{n+m}{m},
  \]
  the number of dissections of the \((n+2)\)-gon into \(m\) cells. It contains the Catalan sequence as \(D_{n,n}\).

The Kirkman–Cayley formula (2) was conjectured by Thomas Kirkman in 1857 [9] and proved by Arthur Cayley in 1891 [3]. It was also stated as a question by Eugène Prouhet in 1866 [13]. Many modern proofs are available; to give only a few examples, short proofs using generating functions may be found in [13] and [7], and proofs by bijection are given in [17] (using Young tableaux), [14] (in relation to knot theory), and [8]. Later in this article we will explain the generating function approach.

Let us mention a generalization of the Catalan sequence: the Fuss sequence
The number of dissections of the \((n+2)\)-gon into \(m\) cells of equal size. Here \(n\) must be a multiple of \(m\), and the cells are all \(\left(\frac{n}{m} + 2\right)\)-gons. In light of Exercise 2, the Fuss sequence is not directly related to our subject.

**Conway–Coxeter Quiddities**

The most obvious equivalence classes of dissections arise from the action of the dihedral group by rotations and reflections. Their enumeration has received considerable attention; see, for example, [2, 15].

Here we propose the enumeration of the classes of a different type of equivalence, given by the notion of quiddity. The idea goes back to John Conway and Harold S. M. Coxeter, who used quiddities of triangulations to classify frieze patterns [5] (see also [1] and the video [19]). Their definition applies equally to arbitrary dissections:

**Definition.** The **quiddity** of a dissection of the \(N\)-gon is the \(N\)-tuple \((c_1, \ldots, c_N)\), where \(c_i\) is the number of cells contacting the \(i\)th vertex.

Here are some simple examples of dissections, depicted with their quiddities:

![Figure 1](image)

The quiddity gives a good deal of information about the dissection. For example, consider the following exercises:

**Exercise 1.** The quiddity sum \(c_1 + \cdots + c_N\) determines the total number of cells in the dissection.

**Exercise 2.** Dissections into cells of equal sizes are determined by their quiddities.

**Hint.** In any dissection, at least one cell is “exterior”: only one of its sides is a diagonal of the polygon. Check that the quiddity determines the locations of the exterior cells, and then base an inductive argument on the operation of removing an exterior cell.

Exercise 2 shows in particular that triangulations are determined by their quiddities. This is not true for arbitrary dissections. As can be seen in Figure 1, it begins to fail in the octagonal case. The two dissections in the figure are congruent by rotation, but this is not always the case.

**Exercise 3.** Construct distinct dissections with the same quiddity that are not congruent by any dihedral symmetry.

We formulate the following general problem. To the best of our knowledge, it is open and has not previously been considered. It seems (at least to us) to be difficult; at any rate, more difficult than enumerating the dissections themselves.

**Problem 1.** Enumerate the distinct quiddities of dissections of the \(N\)-gon into \(m\) cells.

**3-Periodic Quiddities**

Here we give the enumeration of the quiddities of a particular class of dissections.

**Definition.** An \(\ell\)-**periodic dissection** of a polygon is a dissection such that the number of vertices of every cell is congruent to 3 modulo \(\ell\).

For example, 1-periodic dissections are simply arbitrary dissections, 2-periodic dissections have only cells with odd numbers of sides, and in 3-periodic dissections, the number of vertices of every cell is a multiple of 3. The dissections depicted in the preceding section are all 3-periodic. Our result is as follows.

**Theorem 1.** ([4]) Let \(Q_{n,m}\) be the number of distinct quiddities of 3-periodic dissections of the \((n+2)\)-gon into \(m\) cells. Then \(Q_{n,m} = 0\) unless \(n \equiv m \mod 3\), in which case it is

\[
Q_{n,m} = \sum_{0 \leq s \leq m+1} \binom{n-s}{s} \binom{m-s-2}{m-s-1}.
\]

The proof is somewhat involved and relies heavily on 3-periodicity; we give a little of its flavor in the next two sections. As far as we can see, it does not adapt to the case of arbitrary dissections. The initial values of \(Q_{n,m}\) are shown in Table 1.

Note that the first row, \(Q_{n,n}\), is the Catalan sequence, and the second row, \(Q_{n,n-3}\), counts quiddities of dissections of the \((n+2)\)-gon into \(n - 4\) triangles and one hexagon. The column sums are the total number of quiddities of 3-periodic dissections of the \((n+2)\)-gon [OEIS A348666].

In the last section we will see that 3-periodic dissections have applications to other areas of mathematics.
The Mathematical Intelligencer

Table 1. The coefficients $Q_{n,m}$

| $m \setminus n$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----------------|----|----|----|----|----|----|----|----|----|
| $n$             | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $n-3$           | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $n-6$           | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $n-9$           | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| $n-12$          | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |

| $n$             | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|-----------------|----|----|----|----|----|----|----|----|----|
| $n-3$           | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| $n-6$           | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| $n-9$           | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| $n-12$          | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |

Surprisingly, they are also the only class of dissections whose quiddities we have been able to count.

### Generating Functions

It is always useful to organize sequences of integers as power series. Such series are called generating functions, and their analytic properties are powerful tools. For example, many classes of dissections may be enumerated as follows: find a functional equation for the generating function, and then apply Lagrange inversion. In this section we outline the derivation of (1), (2), and (3) via this strategy.

#### The Case of Catalan

For the Catalan sequence, the generating function is

$$C(z) := \sum_{n=0}^{\infty} C_n z^n,$$

the coefficient $C_n$ being the number of triangulations of the $(n+2)$-gon. Wikipedia gives six proofs of the formula (1) for $C_n$, including some using paths on grids, some using Dyck words, and one using nothing but ingenious markings of triangulations.

The most basic approach rests on the recurrence relation

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0,$$

where $C_0$ is defined to be 1. To prove this relation, designate arbitrarily one edge of the $(n+2)$-gon to be its base edge, a standard trick in this field. There are $n$ triangular cells containing this base edge. Choose one of them, as in Figure 2.

It splits the polygon into two pieces, an $(n+2)$-gon and an $(n+1)$-gon, where $n_0 + n_1 = n - 1$. Therefore, there are $C_{n_0} C_{n_1}$ triangulations containing it. Summing over the $n$ choices gives the relation.

The recurrence relation is equivalent to the functional equation

$$C(z) = 1 + zC(z)^2$$

for the generating function. To understand this, it suffices to observe that for $n > 0$, the right-hand side of the recurrence relation is the coefficient of $z^n$ in $C(z)^2$.

The functional equation is a quadratic in $C(z)$, whose solution is

$$C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}.$$

Computing the Taylor expansion of this formula gives (1).

#### Kirkman–Cayley

For the Kirkman–Cayley sequence, the generating function is bivariate: it is

$$D(z, w) := \sum_{n, m=0}^{\infty} D_{n,m} z^n w^m,$$

where $D_{n,m}$ is the number of dissections of the $(n+2)$-gon into $m$ cells. It satisfies the functional equation

$$D = 1 + \frac{wzD^2}{1-zD} = 1 + wzD^2 + wz^2D^3 + wz^3D^4 + \cdots.$$ (4)

To understand why, designate again a base edge and choose an $(r+2)$-cell containing it, as in Figure 3.

First question: how many dissections into $m$ cells contain the chosen $(r+2)$-cell? The cell splits the polygon into $(r+1)$ pieces: an $(n_0+2)$-gon, an $(n_1+2)$-gon, and so on, up to an $(n_r+2)$-gon, where $\sum_{r=0}^{m} n_r = n - r$ (the case $r = 2$ is depicted). Each of these pieces may itself be dissected, and if the $(n_r+2)$-gon is dissected into $m_r$ cells, then we must have $\sum_{r=0}^{m} m_r = m$. Therefore, the answer is

$$\sum_{m_0 + \cdots + m_r = m-1} D_{n_0,m_0} \cdots D_{n_r,m_r}.$$ (5)
(Care must be taken with the degenerate case in which the chosen cell shares some edges with the polygon. Here \( n_z = 0 \) for some values of \( s \), and it is necessary to use the convention that \( D_{0,0} = 1 \) and \( D_{0,m} = 0 \) for \( m > 0 \). To put it in words, the 2-gon has a unique dissection, which has 0 cells.)

Second question: in how many dissections into \( m \) cells is the base edge in some \( (r+2) \)-cell? The set of all \( (r+2) \)-cells containing the base is indexed by the \((r+1)\)-tuples \((n_0, \ldots, n_r)\) such that \( \sum_{i=0}^{r} n_i = n - r \), so the answer is

\[
\sum_{n_0 + \cdots + n_r = n - r} D_{n_0, n_1} \cdots D_{n_r, m_r}.
\]

(5)

Third question: what is the total number \( D_{n,m} \) of dissections into \( m \) cells? The base edge may be in a cell of any size, so the answer is the sum of (5) over \( r \):

\[
D_{n,m} = \sum_{r=1}^{\infty} \sum_{n_0 + \cdots + n_r = n - r} D_{n_0, n_1} \cdots D_{n_r, m_r}.
\]

Last question: how does this yield the recurrence relation (4)? It suffices to verify that (5) is the coefficient of \( z^n w^m \) in \( wz^{-1} D^{-1} \), or in other words, the coefficient of \( z^{n-1} w^{m-1} \) in \( D^{-1} \). We leave this as an exercise for the reader.

Lagrange Inversion

The strategy used in the Catalan case to obtain the explicit formula for \( \mathbb{C}_r \) from the functional equation for \( \mathbb{C}(z) \) does not work well for dissections: one can easily solve for \( D \) in (4), but it is difficult to use Taylor expansion to obtain (2). However, Lagrange’s inversion theorem makes things easy. We briefly recall it: suppose \( \phi(y) \) is a series in \( y \) with nonzero constant term. The theorem gives the terms of the series \( y(z) \) in \( z \) that inverts the function \( y \mapsto y / \phi(y) \), i.e., that satisfies \( z = y(z) / \phi(y(z)) \). Using the standard notation \( [z^n] y \) for the coefficient of \( z^n \) in \( y(z) \), the result is

\[
n[z^n] y = [y^{-1}] \phi^z.
\]

Exercise 4. Use Lagrange inversion to prove the Kirkman–Cayley formula (2).

Hint. Set \( y := zD(z, w) \) and rearrange (4) to see that one may take

\[
\phi(y) = 1 + w y / (1 - y - w y).
\]

Then use \((1-x)^{-n+1} = \sum_{n=0}^{\infty} \binom{n+m}{m} x^n\).

Let us remark that the proof of the inversion theorem rests on nothing but clever use of the residue theorem and integration by substitution. Writing \( f_y \), for integration around zero and substituting \( y / \phi(y) \) for \( z \) and \( dy \) for \( dy / dz \) at the appropriate moment, the core of the argument is

\[
n[z^n] y = [z^{n-1}] y' = \int_0^y \frac{y' \, dz}{2 \pi i z} = \int_0^y \frac{\phi'(y) \, dy}{2 \pi i y} = [y^{n-1}] \phi^z.
\]

Despite its short proof, Lagrange inversion is a powerful tool. As we have mentioned, it permits the enumeration of a wide variety of classes of dissections, such as, for example, the \( \ell \)-periodic dissections:

Exercise 5. The number of \( \ell \)-periodic dissections of the \((n+2)\)-gon into \( m \) cells is zero unless \( n \equiv m \mod \ell \), in which case it is

\[
L_{n,m} = \frac{1}{n+1} \left( m - 1 + \frac{(n-m)/\ell}{m-1} \right) \left( \frac{n+m}{m} \right).
\]

Hint. Adapt the proof of (4) to deduce that the corresponding bivariate generating function satisfies

\[
L(z, w) = 1 + \frac{w z L^2}{1 - z^\ell L^\ell},
\]

and then apply Lagrange inversion.

Quiddities

Enumerating quiddities appears to be significantly more difficult than enumerating dissections, and as noted earlier, we have been able to accomplish it only in the 3-periodic case. The difficulty rests in finding functional equations. Indeed, the 3-periodic generating function

\[
Q(z, w) := \sum_{n,m \geq 0} Q_{n,m} z^n w^m
\]

does not seem to satisfy a functional equation itself, but it can be given in terms of an auxiliary function \( P(z, w) \) that is defined by such an equation:

\[
P(z, w) := 1 + \frac{w z P^2}{1 - z P^3} = 1 + w z P^2 + w z^4 P^4 + w z^7 P^6 + \cdots.
\]

The main ingredient in the proof of Theorem 1 is the formula for \( Q \) in terms of \( P \):

\[
Q(z, w) = 1 + \frac{w z P^2}{1 - z^3 P^3} = 1 + w z P^2 + w z^4 P^5 + w z^7 P^8 + \cdots.
\]

From (6), a few preparatory tricks and a generalization of Lagrange inversion known as the Lagrange–Bürmann formula give (3).

Why Is 3-Periodicity Special?

Let us now describe some of the ideas behind the proof of (6), as well as some of the obstacles to counting the quiddities of other classes of dissections.

Surgery and Maximally Open Dissections

Surgery is an operation acting on a single cell of a dissection to produce a new dissection, as follows: choose two edges of the cell that are both diagonals of the dissection,
and moreover are separated from each other on both sides by at least two additional edges of the cell. Remove the two chosen edges from the dissection and replace them with the two other line segments having the same endpoints. For example, surgery transforms each octagonal dissection in Figure 1 into the other.

The crucial point is that surgery does not alter the quiddity: “surgery equivalence classes refine quiddity equivalence classes.” Our approach to (6) is to prove that the two equivalence classes are the same: any two 3-periodic dissections with the same quiddity can be transformed into one another by a sequence of surgeries.

We admit only surgeries preserving 3-periodicity: 3-periodic surgeries. We define a canonical dissection within each 3-periodic surgery equivalence class as follows. As usual, designate one edge of the polygon to be its base. This endows each cell with its own naturally defined base edge: the one closest to the base edge of the polygon. A surgery on a cell is said to be opening if it removes the cell’s base edge. A 3-periodic dissection is said to be maximally open if it admits no 3-periodic opening surgeries.

In order to transform a 3-periodic dissection into a maximally open one, apply repeated 3-periodic opening surgeries to all of its cells, beginning with the cells farthest from the base edge of the polygon. A simple example is shown in Figure 4.

This process leads to the conclusion that there exists a maximally open dissection in each 3-periodic surgery equivalence class. In order to prove that it is unique, and simultaneously that 3-periodic surgery equivalence classes are the same as quiddity equivalence classes, we prove that any two maximally open 3-periodic dissections with the same quiddity are identical. This is accomplished by induction on the size of the polygon, using an argument refining the initial idea of Conway and Coxeter [5]. The details are given in [4].

Thus in the 3-periodic case, the enumeration of quiddities is equivalent to the enumeration of maximally open dissections. From here, we obtain (6) by methods similar to those yielding the Kirkman–Cayley functional equation (4).

**In Search of More General Surgery**

Outside of the 3-periodic case, surgery equivalence classes and quiddity equivalence classes are not the same. For example, consider as a “toy model” the dissections in which all cells are either triangles or quadrilaterals. Such dissections admit no surgeries, since surgery is never possible on a cell smaller than a hexagon. However, we have the following exercise:

**Exercise 6.** Dissections into triangles and quadrilaterals are not determined by their quiddities.

**Hint.** The following dissection has less symmetry than its quiddity:

![Dissection](image)

Note the similarity to Figure 1, which admits a 3-periodic surgery. The dissection here can be transformed into its reflection on the vertical by a more general type of quiddity-preserving surgery that operates simultaneously on two adjacent cells. In the next figure one must operate simultaneously on three nonadjacent cells in order to transform the dissection into its reflection:

![Dissection](image)

Can one define a quiddity-preserving surgery on this type of dissection whose equivalence classes are equal to the quiddity equivalence classes? Do such techniques lead to a functional equation? We do not know. Let us formulate the relevant simplification of Problem 1:

![Dissection](image)

**Figure 4.** Consecutive 3-periodic opening surgeries producing a maximally open dissection.
Problem 2. Enumerate the quiddities of dissections into triangles and quadrilaterals.

Once again, the problem of enumerating the dissections themselves may be solved using standard techniques:

Exercise 7. The number of dissections of the \((n + 2)\)-gon into \(m\) cells all of which are either triangles or quadrilaterals is

\[
p_{n,m}^{3,4} = \frac{1}{n + 1} \begin{pmatrix} m \\ n - m \end{pmatrix} \begin{pmatrix} n + m \\ m \end{pmatrix}.
\]

Hint. As in Exercise 5, adapt the proof of (4) to show that the bivariate generating function satisfies

\[
D^{3,4} = 1 + wz(D^{3,4})^2 + wz^2(D^{3,4})^3
\]

and then apply Lagrange inversion.

Exercise 6 shows that for arbitrary dissections, quiddity equivalence classes are larger than “classical” surgery equivalence classes. This is true also in the 2-periodic case: here is a dissection into triangles and pentagons with the same quiddity as its reflection across the vertical:

Is there a type of quiddity-preserving surgery transforming the one into the other?

Unexpected Connections
We conclude with some connections between quiddities of 3-periodic dissections and topics in continued fractions, discrete analysis, and number theory.

Continued Fractions
Every rational number \(\frac{r}{s} > 1\) has two types of continued fraction expansions:

\[
\frac{r}{s} = a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{c_k}}}
\]

and

\[
\frac{r}{s} = c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_k}}}
\]

where \(a_i \geq 1\) and \(c_i \geq 2\). The second expansion, often called the Hirzebruch–Jung continued fraction, is useful in hyperbolic geometry and toric varieties.

Conway–Coxeter quiddities appear in a beautiful manner in this context. Construct the triangulation in which the integers \(a_i\) count the number of consecutive triangles in the same orientation, alternating base down and base up, as depicted in Figure 5.

Remarkably, the integers \(c_i\) then turn out to be the quiddity coefficients on the top of the triangulation! This should be attributed to [5]; for an explanation, see [10].

Example. The rational number \(\frac{7}{5}\) has the expansions

\[
\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}} = 2 - \frac{1}{2 - \frac{1}{3}},
\]

corresponding to the triangulation

The Discrete Schrödinger Equation
Let \((c_i)_{i \in \mathbb{Z}}\) be an \(N\)-periodic sequence of positive integers:

\(c_{i+N} = c_i\) for all \(i \in \mathbb{Z}\). Consider the linear recurrence

\[
5
\]

Figure 5. The triangulation in which the integers \(a_i\) count the number of consecutive triangles in the same orientation, alternating base down and base up.
\begin{equation}
    v_{i+1} = c_i v_i - v_{i-1},
\end{equation}

where the \(v_i\) are unknowns. This is the simplest second-order recurrence, often called the 1-dimensional discrete Schrödinger (or Sturm–Liouville) equation. It appears in many areas of mathematics, and of course its continuous analogue is the subject of a vast literature (see, for example, [16]).

**Problem 3.** Characterize and enumerate the sequences \((c_i)\) such that all solutions \((v_i)\) are either \(N\)-antiperiodic \((v_{i+N} = -v_i\) for all \(i)\) or \(N\)-periodic.

**The Modular Group**

Recall that \(\text{SL}(2, \mathbb{Z})\) is the group of \(2 \times 2\) integer matrices of determinant 1, and \(\text{PSL}(2, \mathbb{Z})\) is its central quotient \(\text{SL}(2, \mathbb{Z})/\{\pm \text{Id}\}\).

**Exercise 8.** Every element \(A\) of \(\text{SL}(2, \mathbb{Z})\) can be written (not uniquely!) as a product of “elementary” matrices: for some positive integers \(c_1, \ldots, c_N\),

\[
    A = \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_N & -1 \\ 1 & 0 \end{pmatrix}.
\]

**Hint.** Begin with the well-known generators \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\) of \(\text{SL}(2, \mathbb{Z})\).

The next problem describes the relations between elementary matrices in \(\text{PSL}(2, \mathbb{Z})\):

**Problem 4.** Characterize and enumerate the positive integers \(c_1, \ldots, c_N\) such that \((8)\) is \(\pm \text{Id}\).

In contrast with Problems 1 and 2, Problems 3 and 4 have in fact already been solved:

**Theorem 2.** ([11], Theorem 1.1(i)) Problems 3 and 4 have identical solution sets: the quiddities of the 3-periodic dissections of the \(N\)-gon.

One wonders whether the quiddities of other classes of polygon dissections also have nice interpretations!

**Acknowledgments**

C. H. C. was partially supported by Simons Foundation Collaboration Grant 519533. V. O. was partially supported by the ANR project PhyMath, ANR-19-CE40-0021.

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