Patterns of CP violation from mirror symmetry breaking in the $\eta \to \pi^+\pi^−\pi^0$ Dalitz plot

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A violation of mirror symmetry in the $\eta \to \pi^+\pi^−\pi^0$ Dalitz plot has long been recognized as a signal of C and CP violation. Here we show how the isospin of the underlying C- and CP-violating structures can be reconstructed from their kinematic representation in the Dalitz plot. Our analysis of the most recent experimental data reveals, for the first time, that the C- and CP-violating amplitude with total isospin $I = 2$ is much more severely suppressed than that with total isospin $I = 0$.

Introduction. The decay $\eta \to 3\pi$ first came to prominence after the observation of $K_L \to \pi^+\pi^-\pi^0$ decay and the discovery of CP violation in 1964 [1], because it could be used to test whether $K_L \to \pi^+\pi^-\pi^0$ was generated by CP violation in the weak interactions [2, 3]. Rather, CP violation could arise from the interference of the CP-conserving weak interaction with a new, “strong” interaction that breaks C and CP; this new interaction could be identified through the appearance of a charge asymmetry in the momentum distribution of $\pi^+$ and $\pi^-$ in $\eta \to \pi^+\pi^-\pi^0$ decay [4, 4, 5]. Since $\eta \to \pi^+\pi^-\pi^0$ breaks G parity, isospin $I$ and/or charge-conjugation C must be broken in order for the process to occur. Thus a charge asymmetry could arise from the interference of a C-conserving, but isospin-breaking amplitude with a isospin-conserving, but C-violating one [4]. Numerical estimates were made by assuming that the isospin-violating contributions were driven by electromagnetism [4–6]. Since that early work, our understanding of these decays within the Standard Model (SM) has changed completely: the weak interaction does indeed break CP symmetry, through flavor-changing transitions characterized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Moreover, isospin breaking in the strong interaction, mediated by the up-down quark mass difference [7–9], is now known to provide the driving effect in mediating $\eta \to 3\pi$ decay [10–13], with isospin-breaking, electromagnetic effects playing a much smaller role [14–17].

Modern theoretical studies of $\eta \to 3\pi$ decay focus on a complete description of the final-state interactions within the SM, in order to extract the isospin-breaking, light-quark mass ratio $Q^2 \equiv \sqrt{(m_u^2 - m_d^2)/(m_d^2 - m_u^2)}$, with $\bar{m} = (m_d + m_u)/2$, precisely [11–13, 18–23]. There has been no further theoretical study of CP violation in $\eta \to \pi^+\pi^-\pi^0$ decay since 1966. Since the $\eta$ meson carries neither spin nor flavor, searches for new physics in this system possess special features. For example, $\eta \to \pi^+\pi^-\pi^0$ decay must be parity P conserving if the $\pi$ and $\eta$ mesons have the same intrinsic parity, so that C violation in this process implies that CP is violated as well. There has been, moreover, much effort invested in the possibility of flavor-diagonal CP violation via a nonzero permanent electric dipole moment (EDM), which is P and time-reversal T violating, or P and CP violating if CPT symmetry is assumed. Studies of flavor-diagonal, C and CP violating processes are largely lacking. We believe that the study of the Dalitz plot distribution in $\eta \to \pi^+ (p_{\pi^+} + p_{\pi^0}) \pi^- (p_{\pi^-} - p_{\pi^0}) \pi^0 (p_{\pi^0})$ decay is an ideal arena in which to search for C and CP violation beyond the SM. Were we to plot the Dalitz distribution in terms of the Mandelstam variables $t \equiv (p_{\pi^+} + p_{\pi^0})^2$ and $u \equiv (p_{\pi^+} + p_{\pi^0})^2$, the charge asymmetry we have noted corresponds to a failure of mirror symmetry, i.e., of $t \leftrightarrow u$ exchange, in the Dalitz plot. In contrast to that C and CP violating observable, a nucleon EDM could be mediated by a minimal P- and T-violating interaction, the mass-dimension-four $\tilde{\theta}$ term of the SM, and not new weak-scale physics. Since the $\tilde{\theta}$ term is C even, it cannot contribute to the charge asymmetry, at least at tree level. Moreover, SM weak interactions do not support flavor-diagonal C and CP violation.

The appearance of a charge asymmetry and thus of C (and CP) violation in $\eta \to \pi^+ (p_{\pi^+} + p_{\pi^0}) \pi^- (p_{\pi^-} - p_{\pi^0}) \pi^0 (p_{\pi^0})$ decay can be probed experimentally through the measurement of a left-right asymmetry, $A_{LR}$ [24]:

$$A_{LR} \equiv \frac{N_+ - N_-}{N_+ + N_-} = \frac{1}{N_{tot}}(N_+ - N_-),$$  

where $N_\pm$ is the number of events with $u > t$, so that the $\pi^+$ has more (less) energy than the $\pi^-$ if $u > (<) t$ in the $\eta$ rest system. A number of experiments have been conducted over the years to test for a charge asymmetry in $\eta \to \pi^+\pi^-\pi^0$ decay, with early experiments finding evidence for a nonzero asymmetry [25–27], but with possible systematic problems becoming apparent only later, as, e.g., in Ref. [28]. Other experiments find no evidence for a charge asymmetry and C violation [24, 28–32], and we note that new, high-statistics experiments are planned [33–35]. It is also possible to form asymmetries that probe the isospin of the C-violating final state: a sextant asymmetry $A_S$, sensitive to the $I = 0$ state [4, 5], and a quadrant asymmetry $A_Q$, sensitive to the $I = 2$ final state [4, 24]. These asymmetries are more challenging to measure and are only poorly known [24]. In this paper we develop a method to discriminate between the possible $I = 0$ and $I = 2$ final states by considering the pattern of mirror-symmetry-breaking events they engender in the Dalitz plot. Mirror-symmetry breaking as a
probe of CP violation has also been studied in untagged, heavy-flavor decays [36–39], with Ref. [38] analyzing how different CP-violating mechanisms populate the Dalitz plot. We also note Refs. [40, 41] for Dalitz studies of CP violation in heavy-flavor decays.

**Theoretical Framework.** The $\eta \to 3\pi$ decay amplitude in the SM can be expressed as [10, 11]

$$A(s, t, u) = -\frac{1}{Q^2} \frac{M^2_R M^2_R - M^2_{\pi} - M^2_{\pi}}{3\sqrt{3} F^2_\pi} M(s, t, u), \quad (2)$$

where we employ the Mandelstam variables $u, t$, and $s = (p_{\pi^+} + p_{\pi^-})^2$ and work to leading order in strong-interaction isospin breaking, so that

$$Q^2 = \frac{m^2_{\pi} - m^2_u}{m^2_{\pi} - m^2_u}, \quad \tilde{m} = \frac{1}{2}(m_u + m_d). \quad (3)$$

Since $C = (-1)^I$ in $\eta \to 3\pi$ decay [4], the C- and CP-even transition amplitude with a $\Delta I = 1$ isospin-breaking prefactor must have $I = 1$. The amplitude $M(s, t, u)$ thus corresponds to the total isospin $I = 1$ component of the $\pi^+\pi^-\pi^0$ state and can be expressed as [11]

$$M^I_I(s, t, u) = M_0(s) + (s - u) M_1(t) + (s - t) M_1(u) + M_2(t) + M_2(u) - \frac{2}{3} M_2(s), \quad (4)$$

where $M_I(z)$ is the decay amplitude with $\pi - \pi$ rescattering in the $z$-channel with isospin $I$. This decomposition can be recovered under isospin symmetry in chiral perturbation theory (ChPT) up to next-to-next-leading order (NNLO), $O(p^6)$, because the only absorptive parts that can appear are in the $\pi - \pi S$- and $P$-wave amplitudes [13]. An analogous relationship exists in $\eta \to 3\pi^0$ decay [11].

Since we are considering C and CP violation, additional amplitudes can appear — here, namely, $I = 0$ and $I = 2$ amplitudes. These can be built from the $M_I(z)$ in Eq. (4), though they are accompanied by unknown C- and CP-violating low-energy constants. After using angular-momentum conservation and the Clebsch-Gordon coefficients for the addition of the possible isospin states we have

$$M^0_0(s, t, u) = (s-t)M_1(u) + (u-s)M_1(t) - (u-t)M_1(s) \quad (5)$$

and

$$M^2_2(s, t, u) = (s-t)M_1(u) + (u-s)M_1(t) + 2(u-t)M_1(s) + \sqrt{5}[M_2(u) - M_2(t)]. \quad (6)$$

The total amplitude is thus

$$A(s, t, u) = -\frac{1}{Q^2} \frac{M^2_R M^2_R - M^2_{\pi} - M^2_{\pi}}{3\sqrt{3} F^2_\pi} M^I_I(s, t, u) + \alpha M^0_0(s, t, u) + \beta M^2_2(s, t, u), \quad (7)$$

where $\alpha$ and $\beta$ are complex numbers to be determined by fits to the experimental event populations in the Dalitz plot. The amplitude now contains CP-violating terms that leave the total decay rate unchanged in $O(\alpha), O(\beta)$, because the interference terms vanish exactly after integration over the entire phase space since it is symmetric under $u \leftrightarrow t$ exchange.

We wish to study the possible patterns of C- and CP-violation across the Dalitz plot, so that we now turn to the explicit evaluation of Eq. (7) and its associated Dalitz distribution. Much effort has been devoted to the evaluation of the SM contribution, with work in ChPT [10, 13, 42], as well as in frameworks tailored to address various final-state-interaction effects [11, 18–23, 43–47]. In what follows we employ a next-to-leading-order (NLO) ChPT analysis [10, 13] because it is the simplest choice in which the C- and CP-violating coefficients $\alpha$ and $\beta$ can have both real and imaginary parts. A comparison of the NLO and NNLO analyses of Bijnen et al. [13], noting Table I of Ref. [28], shows that this is an acceptable choice. We thus think it is rich enough to give a basic view as to how our idea works.

To compute the C-violating amplitudes, we decompose the $I = 1$ amplitude into the isospin basis $M_I(z)$. As well known [13, 21, 22, 48], the isospin decompositions involving the $\pi - \pi$ rescattering functions $J_{\pi Q}(s)$ are unique, whereas the polynomial parts of the amplitude are not, due to the relation $s + t + u = 3s_0$, where $s_0 = (M^2_R + 2M^2_{\pi^+} + M^2_{\pi^0})/3$. Thus there are $M_I(z)$ redefinitions that leave the $I = 1$ amplitude invariant, as discussed in Ref. [48]. However, once we demand that the C-violating $I = 0, 2$ amplitudes remain invariant also, only the redefinition $M_0(s) - \frac{3}{5} \delta_1$ and $M_2(z) + \delta_1$, with $\delta_1$ an arbitrary constant, survives. In what follows we adopt the NLO analyses of Refs. [10, 13], and our isospin decomposition of Ref. [10] is consistent with that in Bijnen and Ghorbani [13] — its detailed form and further details can be found in the supplemental information. Small differences in the numerical predictions exist, however, due to small differences in the inputs used [10, 13], and we study their impact explicitly.

**Results.** The Dalitz distribution in $\eta \to \pi^+\pi^-\pi^0$ is usually described in terms of variables $X$ and $Y$ [49]:

$$X \equiv \sqrt{3} \frac{T_{\pi^+} - T_{\pi^-}}{Q_{\eta}} = \sqrt{\frac{3}{2M_{\eta}Q_{\eta}}} (u - t), \quad (8)$$

$$Y \equiv 3 \frac{T_{\pi^0}}{Q_{\eta}} - 1 = 3 \frac{3}{2M_{\eta}Q_{\eta}} [(M_{\eta} - M_{\pi^0})^2 - s] - 1, \quad (9)$$

where $Q_{\eta} = T_{\pi^+} + T_{\pi^-} + T_{\pi^0} = M_{\eta} - 2M_{\pi^+} - M_{\pi^0}$, and $T_{\pi^+}$, is the $\pi^+$ kinetic energy in the $\eta$ rest frame. The decay probability can be parametrized in a polynomial expansion around the center point $(X, Y) = (0, 0)$ [28]:

$$|A(s, t, u)|^2 = N (1 + aY + bY^2 + cX + dX^2 + eXY + fY^3 + gX^2Y + hXY^2 + iX^3 + \ldots). \quad (10)$$

Since the C transformation on the decay amplitude is equivalent to $t \leftrightarrow u$ exchange [38], we see that the appearance of terms that are odd in $X$ would indicate both
C and CP violation. The KLOE-2 collaboration [28] has provided a more precise estimate of the C-even parameters in Eq. (10) and bounded the C-odd ones. Returning to Eq. (7), we see that the C- and CP-violating contributions to the decay probability are

\[
\frac{1}{\xi}|A(s,t,u)|^2_C = M_0^C [\alpha M_0^G + \beta M_0^F]^* + H.c.
\]

\[
= 2\text{Re}(\alpha)[\text{Re}(M_0^C)\text{Re}(M_0^G) + \text{Im}(M_0^C)\text{Im}(M_0^G)]
\]

\[
- 2\text{Im}(\alpha)[\text{Re}(M_0^C)\text{Im}(M_0^G) - \text{Im}(M_0^C)\text{Re}(M_0^G)]
\]

\[
+ 2\text{Re}(\beta)[\text{Re}(M_0^C)\text{Re}(M_0^F) + \text{Im}(M_0^C)\text{Im}(M_0^F)]
\]

\[
- 2\text{Im}(\beta)[\text{Re}(M_0^C)\text{Im}(M_0^F) - \text{Im}(M_0^C)\text{Re}(M_0^F)],
\]

where \(\xi = -(M_0^2/M_0^G)(M_0^2 - M_0^2)/(3\sqrt{3} F_0^2 Q^2)\), and the existing experimental assessments of \(|A(s,t,u)|^2_C\) correspond to the set of odd \(X\) polynomials in \(|A(s,t,u)|^2\). The parameter \(N\) drops out in the evaluation of the asymmetries, and the parameters \(c, e, h,\) and \(l\) are taken from the first line of Table 4.6 in the Ph.D. thesis of Caldeira Balkestahl [50],

\[
c = (-4.34 \pm 3.39) \times 10^{-3} , \quad e = (2.52 \pm 3.20) \times 10^{-3} ,
\]

\[
h = (1.07 \pm 0.90) \times 10^{-2} , \quad l = (1.08 \pm 6.54) \times 10^{-3} ,
\]

which fleshes out Ref. [28]. There is a typographical error in the sign of \(c\) in Ref. [28]. We now turn to the extraction of \(\text{Re}(\alpha), \text{Im}(\alpha), \text{Re}(\beta),\) and \(\text{Im}(\beta)\) using the experimental data and Eqs. (4,5,6) using the \(M_1(z)\) amplitudes of \(O(p^4)\) ChPT [10, 13]. We evaluate the denominators of the possible charge asymmetries by computing \(\xi^2[M_1^C(z, t, u)]^2\) only.

Hereewith we collect the parameters needed for our analysis. We compute the phase space with physical masses, so that \(s + t + u = 3m_0\), but the decay amplitudes [10, 13] on which we rely, namely, \(M(s,t,u)\) in Eq. (2), should be in the isospin limit, implying some adjustment of the input parameters may be needed. We adopt the hadron masses and \(\sqrt{2} F_\pi = (130.2 \pm 1.7) \times 10^{-3}\) GeV from Ref. [51] for both amplitudes. For the Gasser and Leutwyler (GL) amplitude [10] we use \(M_\pi = \sqrt{(2 M_{\pi^+}^2 + M_{\pi^0}^2)/3}, M_K = \sqrt{(M_{K^+}^2 + M_{K^0}^2)/2}\), where we discuss our treatment of the two-particle thresholds in the supplement, \(F_0 = F_\pi, F_K/F_\pi = 1.1928 \pm 0.0026 [51],\) and \(L_\mu = (-3.82 \pm 0.30) \times 10^{-3}\) from the NLO fit with the scale \(\mu = 0.77\) GeV [52]. We use these parameters in the prefactor in Eq. (2) also, as well as \(Q = 22.0\) [22], to find \(\xi = -0.137\). For the Bijnen and Ghorbani (BG) amplitude through \(O(p^4)\) [13], we use \(M(s,t,u) = M^{(2)}(s) + M^{(4)}(s,t,u)\) and multiply the prefactor in Eq. (2) by \(-3 F_\pi^2/(M_\pi^2 - M_\pi^2)\) to yield that in Ref. [13]. In the \(O(p^2)\) term, which contributes to \(M_0(s),\)

\[
M^{(2)}(s) = (4 M_2^2 - s)/F_\pi^2 ,
\]

and we use \(M_\pi\) and \(F_\pi\) as defined for the GL amplitude [10]. In the \(O(p^4)\) term, we use \(M_{\pi^0}\) and \(M_{K^0}\) as indicated, as well as \(\Delta = M_{\pi^0} - M_{\pi^0}\) and \(L_\beta, L_5, L_7, L_8\) from fit 10 of Ref. [53].

\[
\text{Re}(\alpha) = 16 \pm 24 ,
\]

\[
\text{Re}(\beta) = (-1.5 \pm 2.7) \times 10^{-3} ,
\]

\[
\text{Im}(\alpha) = -20 \pm 29 ,
\]

\[
\text{Im}(\beta) = (-1.3 \pm 4.7) \times 10^{-3} .
\]

In the first row of Fig. 1 we compare the resulting assessment of Eq. (11) with the KLOE-2 results. Large discrepancies exist, particularly at large values of \(X\) and/or \(Y\), where the empirical Dalitz plot [28] shows considerable strength. Thus we turn to a second procedure, in which we make a global fit of \(\alpha\) and \(\beta\) in Eq. (11) to the KLOE-2 results. That is, we assess the Dalitz distribution \(N(X,Y)\) and its error by using the Dalitz plot parameters in Eq. (12), discretized onto a \((X,Y)\) mesh with 682 points. To determine \(N(X,Y)\) and its error we use the odd-\(X\) terms in Eq. (10) with the normalization factor \(N = 0.0474\) as per the GL amplitude [10] and compute the covariance matrix using Eq. (12) and the correlation matrix given in Table 4.3 of Ref. [50]. We then fit \(|A(s,t,u)|^2_C\) using the GL amplitude [10] to \(N(X,Y)\) using a \(\chi^2\) optimization to find

\[
\text{Re}(\alpha) = -0.65 \pm 0.80 ,
\]

\[
\text{Im}(\alpha) = 0.44 \pm 0.74 ,
\]

\[
\text{Re}(\beta) = (-6.3 \pm 14.7) \times 10^{-4} ,
\]

\[
\text{Im}(\beta) = (2.2 \pm 2.0) \times 10^{-3} .
\]
and we show the results of this method in the second row of Fig. 1. Enlarging the \((X, Y)\) mesh to 1218 points incurs changes within ±1 of the last significant figure. The comparison with experiment shows that the fitting procedure is the right choice. We draw the same conclusion from the use of the BG amplitude \([13]\), noting that the global fit in that case (with \(N = 0.0508\)) gives

\[
\begin{align*}
\text{Re}(\alpha) &= -0.79 \pm 0.91, \\
\text{Im}(\alpha) &= 0.61 \pm 0.93, \\
\text{Re}(\beta) &= (-1.4 \pm 2.3) \times 10^{-3}, \\
\text{Im}(\beta) &= (2.3 \pm 1.4) \times 10^{-3},
\end{align*}
\]

so that the results are compatible within errors. Using these solutions, we obtain \(A_{LR} = (-7.18 \pm 4.51) \times 10^{-4}\) using Ref. \([10]\) and \(A_{LR} = (-7.20 \pm 4.52) \times 10^{-4}\) using Ref. \([13]\). These compare favorably with \(A_{LR} = (-7.29 \pm 4.81) \times 10^{-4}\) that we determine using the complete set of Dalitz plot parameters and the covariance matrix we construct given the information in Ref. \([50]\).

We note that our \(A_{LR}\) as evaluated from the Dalitz plot parameters, which are fitted from the binned data, is a little different from the reported value using the unbinned data, i.e., \((-5.0 \pm 4.5 \, ^{+5.0}_{-1.0}) \times 10^{-4}\), reported by KLOE-2 \([28]\). The discrepancy is not significant, and we suppose its origin could arise from the slight mismatch between the theoretically accessible phase space and the experimentally probed one, or other experimental issues.

We have shown that the empirical Dalitz plot distribution can be used to determine \(\alpha\) and \(\beta\). These, in turn, limit the strength of C-odd and CP-odd operators that can arise from physics beyond the SM \([54–59]\). That \(\beta\) is so much smaller than \(\alpha\) can be understood from the isospin structure of the associated strong amplitudes in the SM, because the total isospin \(I = 2\) amplitude breaks isospin, whereas the \(I = 0\) amplitude does not. Crudely, the ratio of scales is that of the SM electromagnetic interactions that would permit the \(I = 2\) amplitude to contribute to \(\eta \to \pi^+\pi^-\pi^0\) decay. The utility of our Dalitz analysis is underscored by our results for the quadrant asymmetry \(A_Q\) and sextant asymmetry \(A_S\) defined in Fig. 2. Using Ref. \([10]\) and Eq. \((14)\), e.g., we find \(A_Q = (2.85 \pm 3.72) \times 10^{-4}\), and \(A_S = (3.87 \pm 4.04) \times 10^{-4}\); the asymmetries by themselves hide the nature of the underlying strong amplitudes. For reference we note the KLOE-2 results using unbinned data \([28]\): \(A_Q = (1.8 \pm 4.5 \, ^{+4.8}_{-2.3}) \times 10^{-4}\) and \(A_S = (-0.4 \pm 4.5 \, ^{+3.1}_{-3.5}) \times 10^{-4}\), with which our results are compatible within errors.

**Summary.** We propose an innovative way of probing C- and CP-violation in the \(\eta \to \pi^+\pi^-\pi^0\) Dalitz plot. Working to leading order in charge conjugation \(C\) and isospin \(I\) breaking, we have shown that the strong amplitudes associated with the appearance of C- and CP-violation can be constructed from the SM amplitude for \(\eta \to \pi^+\pi^-\pi^0\) if the decomposition of Eq. \((4)\) holds \([11]\).

**FIG. 2.** The Dalitz plot geometry in \(\eta \to \pi^+\pi^-\pi^0\) decay, broken into regions for probes of its symmetries. The solid line is the boundary of the physically accessible region. The asymmetry \(A_{LR}\), Eq. \((1)\), compares the population \(N_+ (X > 0)\) with \(N_- (X < 0)\). The quadrant asymmetry \(A_Q\) probes \(I = 2\) contributions, \(N_{\text{tot}}A_Q \equiv N(A) + N(C) - N(B) - N(D)\) \([4]\), and the sextant asymmetry \(A_Q\) probes \(I = 0\) contributions, \(N_{\text{tot}}A_S \equiv N(1) + N(III) + N(VI) - N(II) - N(IV) - N(VI)\) \([4, 5]\).

All asymmetries probe C and CP violation.

We have illustrated this in NLO ChPT, though the use of more sophisticated theoretical analyses would also be possible. New-physics contributions that differ in their isospin can thus be probed through the kinematic pattern they imprint in the Dalitz plot. Our method opens a new window on the study of C- and CP-violation in \(\eta \to \pi^+\pi^-\pi^0\) decay, and holds promise for the high-statistics experiments of the future \([33–35]\).

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Supplemental Information. Here is our isospin decomposition of the $\eta \to \pi^+\pi^-\pi^0$ amplitude of Gasser and Leutwyler through $O(p^4)$ [10]:

\[
M_0(s) = \left[ \frac{2(s - s_0)}{\Delta} + \frac{5}{3} \right] \frac{1}{2F^2_\pi} (2s - M^2_\pi) J^r_{\pi\pi}(s) + \frac{1}{6F^2_\pi} \left( M^2_K - 3M^2_\pi - M^2_\eta (s - 2M^2_\pi) J^r_{\pi\pi}(s) \right. \\
+ \frac{1}{4F^2_\pi} \left[ -6s^2 + 3(5M^2_\pi + 4M^2_\eta + 3M^2_\eta) - 2M^2_\eta M^2_K + \frac{1}{3} M^2_\pi \right] J^r_{\pi\pi}(s) \\
+ \frac{M^2_\pi}{3F^2_\pi} \left( 2s - \frac{11}{3} M^2_\pi + M^2_\eta \right) J^r_{\pi\pi}(s) - \frac{M^2_\pi}{2F^2_\pi} J^r_{\eta\eta}(s) \\
+ \frac{3s}{8F^2_\pi} \left( 3s - 4M^2_\eta \right) \left( J^r_{\eta\eta}(s) - J^r_{\eta\eta}(0) \right) - \frac{1}{8s^2} \\
\left. + \left[ 1 + a_1 + 3a_2\Delta + a_3(9M^2_\eta - M^2_\eta) + \frac{2}{3} d_1 \\
+ \frac{8M^2_\pi}{3s} d_2 \right] \left( 1 + \frac{3 - s - s_0}{\Delta} \right) + a_4+ \frac{8M^2_\pi}{3s} d_1 \right] \\
- \frac{3}{2} \Delta (2\mu_\pi + \mu_K) (s - s_0) \\
\left. + \frac{4L_\Delta}{F^2_\pi} - \frac{1}{64\pi^2 F^2_\eta} \left( \frac{4}{3}s^2 - 9s_0 s + 9s_0^2 \right) \right] \\
- \frac{1}{64\pi^2 F^2_\pi} 3(s - s_0)(4M^2_\pi + 2M^2_\eta), \tag{16}
\]

\[
M_1(z) = \frac{1}{4\Delta F^2_\pi} \left( (z - 4M^2_\eta) J^r_{\eta\eta}(z) \right. \\
\left. + (1 - z/2 - 2M^2_\eta) J^r_{\eta\eta}(K)(z) \right), \tag{17}
\]

and

\[
M_2(z) = \left( 1 - \frac{3 - s - s_0}{\Delta} \right) \left[ -\frac{1}{2F^2_\pi} (z - 2M^2_\eta) J^r_{\eta\eta}(z) \right. \\
+ \frac{1}{4F^2_\pi} (3z - 4M^2_\pi) J^r_{\eta\eta}(K) + \frac{M^2_\pi}{3F^2_\pi} J^r_{\eta\eta}(s) \right]\left. + \frac{4L_\Delta}{F^2_\pi} - \frac{1}{64\pi^2 F^2_\pi} \right\} \frac{1}{\Delta} z^2, \tag{18}
\]

where \( \Delta = M^2_\eta - M^2_\eta, M^2_\eta = (2M^2_\eta + M^2_\eta)/3, \) and \( M^2_\eta = (M^2_K + M^2_K)/2. \) We refer to Ref. [60] for \( J^r_{\pi\pi}(z) \) and \( k_{PQ}(z), \) where \( F \) and \( Q \) denote the mesons \( \pi, K, \text{ or } \eta, \) and to Ref. [10] for \( a_1 \) and \( d_1. \) For this choice of \( M^2_\eta \) and the use of physical phase space we need to evaluate the possible two-particle thresholds with care. The rescattering function \( J^r_{\eta\eta}(z) \) contains \( \sigma(s) = \sqrt{1 - 4M^2_\eta/z}. \) If we use \( m^2_\eta = M^2_\eta \), then for \( M_1(z) \) with \( z = t \) evaluated at its minimum value the argument of the square root is less than zero. To avoid this problem, we use \( \sigma(s) = \sqrt{1 - 4M^2_\eta/z} \) for \( M_2(s) \). Moreover, we note \( J^r_{\eta\eta}(s) \) contains \( \nu(s) = \sqrt{(s - (M^2 - M^2_\eta))(s - (M^2 + M^2_\eta))}. \) If we use \( m^2_\eta = M^2_\eta \), then for \( M_2(s) \) at the maximum of \( s \), we once again find the argument of the square root to be less than zero. To avoid this, we use \( \nu(s) = \sqrt{(s - (M^2 - M^2_\eta))(s - (M^2 + M^2_\eta))} \) for \( M_1(s) \). To be consistent, we use \( \nu(z) = \sqrt{(z - (M^2 - M^2_\eta))(z - (M^2 + M^2_\eta))} \) for \( M_1(z) \) with \( z = t \) or \( u. \) As a check of our assessments we have extracted the \( C \)- and CP-conserving Dalitz plot parameters from this amplitude. Describing the CP-conserving piece of \( |A(s, t, u)|^2 \) by \( N(1 + aY + bY^2 + dX^2 + fY^3 + gX^2Y) \), recalling Eq. (10), we find using a global fit that \( a = -1.326, b = 0.426, d = 0.086, f = 0.017, \) and \( g = -0.072. \) These results compare favorably to the global fit results of Ref. [21]; namely, \( a = -1.328, b = 0.429, d = 0.090, f = 0.017, \) and \( g = -0.081. \) That work also uses the decay amplitude of Ref. [10] through \( O(p^3) \) and the same value of \( L_3 \) but includes electromagnetic corrections through \( O(e^2p^3) \) as well.

In evaluating the BG amplitude [13] we note that an overall 2 should not appear on the second right-hand side of Eq. (3.23); this is needed for the result to agree with that of Ref. [10].

Values of the strong functions associated with the CP-violating parameters \( \text{Re}(\alpha), \text{Im}(\alpha), \text{Re}(\beta), \text{Im}(\beta) \) in Eq. (11) on our analysis grids in \((X, Y)\) are available upon request.

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[1] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Lett. 13, 138 (1964).
[2] T. D. Lee and L. Wolfenstein, Phys. Rev. 138, B1490 (1965). [3] J. Prentki and M. J. G. Veltman, Phys. Lett. 15, 88 (1965).
[4] T. D. Lee, Phys. Rev. 139, B1415 (1965).
[5] M. Nauenberg, Phys. Lett. 17, 329 (1965).
[6] B. Barrett, M. Jacob, M. Nauenberg, and T. N. Truong, Phys. Rev. 141, 1342 (1966).
[7] D. J. Gross, S. B. Treiman, and F. Wilczek, Phys. Rev. D19, 2188 (1979).
[8] P. Langacker and H. Pagels, Phys. Rev. D19, 2070 (1979).
[9] H. Leutwyler, Phys. Lett. B378, 313 (1996), arXiv:hep-ph/9602366 [hep-ph].
[10] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 539 (1985).
[11] A. V. Anisovich and H. Leutwyler, Phys. Lett. B375, 335 (1996), arXiv:hep-ph/9601237 [hep-ph].
[12] J. Bijnens and J. Gasser, Production, interaction and decay of the eta meson. Proceedings, Workshop, Uppsala, Sweden, October 25-27, 2001, Phys. Scripta T99, 34 (2002), arXiv:hep-ph/0202242 [hep-ph].
[13] J. Bijnens and K. Ghorbani, JHEP 11, 030 (2007), arXiv:0709.0230 [hep-ph].
[14] D. G. Sutherland, Phys. Lett. 23, 384 (1966).
[15] J. S. Bell and D. G. Sutherland, Nucl. Phys. B4, 315 (1968).
[16] R. Baur, J. Kambor, and D. Wyler, Nucl. Phys. B460, 127 (1996), arXiv:hep-ph/9510396 [hep-ph].
[17] C. Ditsche, B. Kubis, and U.-G. Meissner, Eur. Phys. J. C60, 83 (2009), arXiv:0812.0344 [hep-ph].
[18] G. Colangelo, S. Lanz, and E. Passemar, Proceedings, 6th International Workshop on Chiral dynamics: Bern, Switzerland, July 6-10, 2009, PoS CD09, 047 (2009), arXiv:0910.0765 [hep-ph].
[19] K. Kampf, M. Knecht, J. Novotny, and M. Zdrahal, Phys. Rev. D84, 114015 (2011), arXiv:1103.0982 [hep-ph].
[20] P. Guo, I. V. Danilkin, D. Schott, C. Fernández-Ramírez, V. Mathieu, and A. P. Szczepaniak, Phys. Rev. D92, 054016 (2015), arXiv:1505.01715 [hep-ph].
[21] M. Albaldadejo and B. Mousallam, Eur. Phys. J. C77, 508 (2017), arXiv:1702.04931 [hep-ph].
[22] G. Colangelo, S. Lanz, H. Leutwyler, and E. Passemar, Phys. Rev. Lett. 118, 022001 (2017), arXiv:1610.03494 [hep-ph].
[23] G. Colangelo, S. Lanz, H. Leutwyler, and E. Passemar, Eur. Phys. J. C78, 947 (2018), arXiv:1807.11937 [hep-ph].
[24] J. G. Layter, J. A. Appel, A. Kotlevski, W.-Y. Lee, S. Stein, and J. J. Thaler, Phys. Rev. Lett. 29, 316 (1972).
[25] C. Baltay, Phys. Rev. 149, 1044 (1966).
[26] M. Gornley, E. Hyman, W.-Y. Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein, Phys. Rev. D2, 501 (1970).
[27] M. Gornley, E. Hyman, W. Lee, T. Nash, J. Peoples, C. Schultz, and S. Stein, Phys. Rev. Lett. 21, 402 (1968).
[28] A. Anastasi et al. (KLOE-2), JHEP 05, 019 (2016), arXiv:1601.06985 [hep-ex].
[29] A. Larriebe et al., Phys. Lett. 23, 600 (1966).
[30] J. G. Layter, J. A. Appel, A. Kotlevski, W.-Y. Lee, S. Stein, and J. J. Thaler, Phys. Rev. D7, 2565 (1973).
[31] M. R. Jane et al., Phys. Lett. 48B, 260 (1974).
[32] F. Ambrosino et al. (KLOE), JHEP 05, 006 (2008), arXiv:0801.2642 [hep-ex].
[33] L. Gan, Proceedings, 14th International Conference on Meson-Nucleon Physics and the Structure of the Nucleon (MENU 2016): Kyoto, Japan, July 25-30, 2016, JPS Conf. Proc. 13, 020063 (2017).
[34] C. Gatto, B. Fabela Enriquez, and M. I. Pedraza Morales (REDTOP), Proceedings, 38th International Conference on High Energy Physics (ICHEP 2016): Chicago, IL, USA, August 3-10, 2016, PoS ICHEP2016, 812 (2016).
[35] J. Beacham et al., (2019), arXiv:1901.09966 [hep-ex].
[36] G. Burkard and J. F. Donoghue, Phys. Rev. D45, 187 (1992).
[37] S. Gardner, Phys. Lett. B553, 261 (2003), arXiv:hep-ph/0203152 [hep-ph].
[38] S. Gardner and J. Tandean, Phys. Rev. D69, 034011 (2004), arXiv:hep-ph/0308228 [hep-ph].
[39] A. A. Petrov, Phys. Rev. D69, 111901 (2004), arXiv:hep-ph/0403030 [hep-ph].
[40] I. Bediaga, I. I. Bigi, A. Gomes, G. Guerrer, J. Miranda, and A. C. d. Reis, Phys. Rev. D80, 096006 (2009), arXiv:0905.4233 [hep-ph].
[41] I. Bediaga, J. Miranda, A. C. dos Reis, I. I. Bigi, A. Gomes, J. M. Otalora Goicochea, and A. Veiga, Phys. Rev. D86, 036005 (2012), arXiv:1205.3036 [hep-ph].
[42] H. Osborn and D. J. Wallace, Nucl. Phys. B20, 23 (1970).
[43] A. Neveu and J. Scherk, Annals Phys. 57, 39 (1970).
[44] C. Roiesnel and T. N. Truong, Nucl. Phys. B187, 293 (1981).
[45] J. Kambor, C. Wiesendanger, and D. Wyler, Nucl. Phys. B465, 215 (1996), arXiv:hep-ph/9509374 [hep-ph].
[46] B. Borasoy and R. Nissler, Eur. Phys. J. A26, 383 (2005), arXiv:hep-ph/0510384 [hep-ph].
[47] S. P. Schneider, B. Kubis, and C. Ditsche, JHEP 02, 028 (2011), arXiv:1010.3946 [hep-ph].
[48] J. Bijnens, P. Dhonte, and F. Persson, Nucl. Phys. B648, 317 (2003), arXiv:hep-ph/0205341 [hep-ph].
[49] S. Weinberg, Phys. Rev. Lett. 4, 87 (1960), [Erratum: Phys. Rev. Lett.4,585(1960)].
[50] L. Caldeira Balkeståhl, Measurement of the Dalitz Plot Distribution for $\eta \to \pi^+ \pi^- \pi^0$ with KLOE, Ph.D. thesis, Uppsala U. (2016).
[51] M. Tanabashi and others. (Particle Data Group), Phys. Rev. D98, 076005 (2018), arXiv:1803.09697 [hep-ph].
[52] A. Anastasi et al. (KLOE-2), JHEP 05, 019 (2016), arXiv:1601.06985 [hep-ex].
[53] A. Larriebe et al., Phys. Lett. 23, 600 (1966).
[54] J. G. Layter, J. A. Appel, A. Kotlevski, W.-Y. Lee, S. Stein, and J. J. Thaler, Phys. Rev. D7, 2565 (1973).
[55] M. R. Jane et al., Phys. Lett. 48B, 260 (1974).
[56] F. Ambrosino et al. (KLOE), JHEP 05, 006 (2008), arXiv:0801.2642 [hep-ex].
[57] L. Gan, Proceedings, 14th International Conference on Meson-Nucleon Physics and the Structure of the Nucleon (MENU 2016): Kyoto, Japan, July 25-30, 2016, JPS Conf. Proc. 13, 020063 (2017).
[58] C. Gatto, B. Fabela Enriquez, and M. I. Pedraza Morales (REDTOP), Proceedings, 38th International Conference on High Energy Physics (ICHEP 2016): Chicago, IL, USA, August 3-10, 2016, PoS ICHEP2016, 812 (2016).
[59] J. Beacham et al., (2019), arXiv:1901.09966 [hep-ex].
[60] G. Burkard and J. F. Donoghue, Phys. Rev. D45, 187 (1992).