Algorithms for the Minimum Dominating Set Problem in Bounded Arboricity Graphs: Simpler, Faster, and Combinatorial

Adir Morgan* \hspace{1cm} Shay Solomon‡ \hspace{1cm} Nicole Wein‡
Tel Aviv University \hspace{1cm} Tel Aviv University \hspace{1cm} MIT

Abstract

We revisit the minimum dominating set problem on graphs with arboricity bounded by \( \alpha \). In the (standard) centralized setting, Bansal and Umboh [BU17] gave an \( O(\alpha) \)-approximation LP rounding algorithm, which also translates into a near-linear time algorithm using general-purpose approximation results for explicit mixed packing and covering or pure covering LPs [KY14, You14, AZO19, Qua20]. Moreover, [BU17] showed that it is NP-hard to achieve an asymptotic improvement for the approximation factor. On the other hand, the previous two non-LP-based algorithms, by Lenzen and Wattenhofer [LW10], and Jones et al. [JLR+13], achieve an approximation factor of \( O(\alpha^2) \) in linear time.

There is a similar situation in the distributed setting: While there is an \( O(\log^2 n) \)-round LP-based \( O(\alpha) \)-approximation algorithm implied in [KMW06], the best non-LP-based algorithm by Lenzen and Wattenhofer [LW10] is an implementation of their centralized algorithm, providing an \( O(\alpha^2) \)-approximation within \( O(\log n) \) rounds.

We address the questions of whether one can achieve an \( O(\alpha) \)-approximation algorithm that is elementary, i.e., not based on any LP-based methods, either in the centralized setting or in the distributed setting. We resolve both questions in the affirmative, and en route achieve algorithms that are faster than the state-of-the-art LP-based algorithms. More specifically, our contribution is two-fold:

1. In the centralized setting, we provide a surprisingly simple combinatorial algorithm that is asymptotically optimal in terms of both approximation factor and running time: an \( O(\alpha) \)-approximation in linear time. The previous state-of-the-art \( O(\alpha) \)-approximation algorithms are (1) LP-based, (2) more complicated, and (3) have super-linear running time.

2. Based on our centralized algorithm, we design a distributed combinatorial \( O(\alpha) \)-approximation algorithm in the CONGEST model that runs in \( O(\alpha \log n) \) rounds with high probability. Not only does this result provide the first nontrivial non-LP-based distributed \( o(\alpha^2) \)-approximation algorithm for this problem, it also outperforms the best LP-based distributed algorithm for a wide range of parameters.

*Email: adirmorgan@gmail.com
†Partially supported by the Israel Science Foundation grant No.1991/19. Email: solo.shay@gmail.com
‡Supported by NSF Grant CCF-1514339. Email: nwein@mit.edu
1 Introduction

1.1 Background

The minimum dominating set (MDS) problem is a classic combinatorial optimization problem. Given a graph \( G \) we want to find a minimum cardinality set \( D \) of vertices, such that every vertex of the graph is either in \( D \) or has a neighbor in \( D \). Besides its theoretical implications, solving this basic problem efficiently has many practical applications in domains ranging from wireless networks to text summarizing (see, e.g., [WAF02, NA16, SL10]). The MDS problem was one of the first problems recognized as NP-complete [Gar79]. It was also one of the first problems for which an approximation algorithm was analyzed: a simple greedy algorithm achieves a \( \ln n \)-approximation in general graphs [Joh74]. This approximation factor is optimal up to lower order terms unless \( P = NP \) [DS14].

Distributed MDS in general graphs The first efficient distributed approximation algorithm for MDS was given by Jia, Rajaraman, and Suel [JRS02], who gave a randomized \( O(\log \Delta) \)-approximation in \( O(\log^2 n) \) rounds in the CONGEST model. This was improved by Kuhn, Moscibroda and Wattenhofer [KMW06], who gave a randomized \( (1 + \varepsilon)(1 + \ln(\Delta + 1)) \)-approximation in \( O(\log^2 \Delta / \varepsilon^4) \) rounds in the CONGEST model and in \( O(\log n / \varepsilon^2) \) rounds in the LOCAL model. Ghaffari, Kuhn, and Maus [GKM17] showed that by allowing exponential-time local computation, one can get a randomized \( (1 + o(1)) \)-approximation in a polylogarithmic number of rounds in the LOCAL model. This result was derandomized by the network decomposition result of Rozhoň and Ghaffari [RG20]. From the lower bounds side, Kuhn, Moscibroda, and Wattenhofer [KMW16] showed that getting a polylogarithmic approximation ratio requires \( \Omega(\sqrt{\log n / \log \log n}) \) and \( \Omega(\log \Delta / \log \log \Delta) \) rounds in the LOCAL model.

For deterministic distributed algorithms, improving over previous work, Deurer, Kuhn, and Maus [DKM19] recently gave two algorithms in the CONGEST model with approximation factor \( (1 + \varepsilon) \ln(\Delta + 1) \) for \( \varepsilon > 1 / \text{polylog} \Delta \), running in \( 2^{O(\sqrt{\log n / \log \log n})} \) and \( O((\Delta + \log^* n) \log \log \Delta) \) rounds, respectively; the running time of the former CONGEST algorithm [GK18], achieving approximation factor \( O(\log^2 n) \), is dominated by the time needed for deterministically computing a network decomposition in the CONGEST model, which, due to [GGR21], is thus reduced to \( O(\text{polylog} n) \).

Graphs of bounded arboricity The MDS problem has been studied on a variety of restricted classes of graphs, such as graphs with bounded degree (e.g., [CC08]), planar and bounded genus graphs (e.g., [Bak94, CHW08, ASS19]), and graphs of bounded arboricity — which is the focus of this paper. The class of bounded arboricity graphs is a wide family of uniformly sparse graphs, defined as follows:

Definition 1.1. Graph \( G \) has arboricity bounded by \( \alpha \) if for every \( S \subseteq V \), it holds that \( m_s / n_s - 1 \leq \alpha \), where \( m_s \) and \( n_s \) are the number of edges and vertices in the subgraph induced by \( S \), respectively.

The class of bounded arboricity graphs contains the other graph classes mentioned above as well as bounded treewidth graphs, and in general all graphs excluding a fixed minor. Moreover, many natural and real world graphs, such as the world wide web graph, social networks and transaction networks, are believed to have bounded arboricity. Consequently, this class of graphs has been subject to extensive research, which led to many algorithms for bounded arboricity graphs in both the (classic) centralized setting (e.g. [Epp94, GG06, CN85]) and in the distributed setting (e.g. [CHS09, BE10, GS17, SV20]); there are also many algorithms in other settings, such as dynamic graph algorithms, sublinear algorithms and streaming
In distributed settings, one cannot always assume that all processors know the arboricity of the graph, so it is important to devise robust algorithms, which can perform correctly also when the arboricity is unknown to the processors (see e.g. [BE10, LW10]).

1.2 Approximating MDS on graphs of arboricity \(\alpha\)

**Centralized setting**

In the centralized setting, there are two non-LP-based algorithms for MDS for graphs of arboricity (at most) \(\alpha\) (for brevity, in what follows we may write graphs of “arboricity \(\alpha\)” instead of arboricity at most \(\alpha\)). One is by Lenzen and Wattenhofer [LW10], the other is by Jones, Lokshtanov, Ramanujan, Saurabh, and Suchý [JLR+13], and both achieve an \(O(\alpha^2)\)-approximation in deterministic linear time.\(^1\) There is also a very simple LP rounding algorithm by Bansal and Umboh that gives a \(3\alpha\)-approximation [BU17]. This algorithm is very simple, after the LP has been solved. To solve the LP, there are near-linear time general-purpose approximation algorithms for explicit mixed packing and covering or pure covering LPs [KY14, You14, AZO19, Qua20]. Combining such an algorithm with [BU17] yields an \(O(\alpha)\)-approximation for MDS, either deterministically within \(O(m \log n)\) time [You14] or randomly (with high probability) within \(O(n \log n + m)\) time [KY14]. The latter bound is super-linear in the entire (non-degenerate) regime of arboricity \(\alpha = o(\log n)\); the regime \(\alpha = \Omega(\log n)\) is considered degenerate, since in that case one can use the greedy linear-time \(\ln n\)-approximation algorithm. Bansal and Umboh [BU17] also proved that achieving asymptotically better approximation is NP-hard.\(^2\)

**Distributed setting**

In the distributed setting, there are two non-LP-based algorithms for MDS for graphs of arboricity \(\alpha\), both by Lenzen and Wattenhofer [LW10]. The first is a randomized \(O(\alpha^2)\)-approximation algorithm in the CONGEST model that runs in \(O(\log n)\) rounds with high probability. This algorithm was made deterministic by Amiri [Ami21], and uses an LP-based subroutine of Even, Ghaffari, and Medina [EGM18]. The second algorithm of Lenzen and Wattenhofer is a deterministic \(O(\alpha \log \Delta)\)-approximation algorithm in the CONGEST model that runs in \(O(\log \Delta)\) rounds, where \(\Delta\) is the maximum degree.

Regarding LP-based algorithms, Kuhn, Moscibroda, and Wattenhofer [KMW06] developed a general-purpose method for solving LPs of a particular structure in the distributed setting. It seems that by applying their method (specifically, Corollary 4.1 of [KMW06]) to the LP approximation result of Bansal and Umboh in bounded arboricity graphs [BU17], one can get a deterministic \(O(\alpha)\)-approximation algorithm for MDS in the CONGEST model that runs in \(O(\log^2 \Delta)\) rounds, but such a result has not been explicitly claimed in the literature.

**A natural question**

The aforementioned results demonstrate a significant gap for MDS algorithms in bounded arboricity graphs when comparing LP-based methods to elementary combinatorial approaches. It is natural to ask whether this gap can be bridged.

---

\(^1\)Note that the theorem statement of [LW10] has a typo suggesting that the approximation factor is \(O(\alpha)\).

\(^2\)More specifically, achieving an \((\alpha - 1 - \varepsilon)\)-approximation is NP-hard for any \(\varepsilon > 0\) and any fixed \(\alpha\); achieving an \(\lfloor \alpha/2 \rfloor - \varepsilon\)-approximation is NP-hard for any \(\varepsilon > 0\) and any \(\alpha = 1, \ldots, \log^\delta n\), for some constant \(\delta\) [BU17, DGKR05]. These hardness of approximation results are achieved by applying a reduction by [BU17] from the \(k\)-hypergraph vertex cover \((k\text{-HVC})\) problem (where we need to find a minimum vertex cover of a \(k\)-uniform hypergraph) to the MDS problem in arboricity-\(k\) graphs, in conjunction with NP-hardness results by [DGKR05] for the \(k\)-HVC problem.
• In the centralized setting, is there any efficient non-LP-based $O(\alpha)$-approximation algorithm for MDS (even one that is slower than the aforementioned $O(m \log n)$ time deterministic and $O(n \log n + m)$ time randomized LP-based algorithms)? Further, can one achieve an $O(\alpha)$-approximation in linear time using any (even LP-based) algorithm?

• In the distributed setting, is there any efficient non-LP-based distributed $O(\alpha)$-approximation algorithm for MDS? Further, can one achieve an $O(\alpha)$-approximation in the CONGEST model within $o(\log^2 \Delta)$ rounds using any (even LP-based) algorithm?

We note the caveat that there is no clear-cut distinction between combinatorial and non-combinatorial algorithms, but we operate under the premise that an algorithm is combinatorial if all its intermediate computations have a natural combinatorial interpretation in terms of the original problem. While all algorithms presented in this paper are certainly combinatorial under this premise, it is far less clear whether prior work is. In particular, the previous state-of-the-art LP-based approaches are based on general-purpose primal/dual methods; when restricted to the MDS problem, it is possible that these methods could reduce, after proper adaptations, into simpler combinatorial algorithms. Nonetheless, even if possible, it is unlikely that the resulting algorithm would be as simple and elementary as ours. In the distributed setting, Kuhn and Wattenhofer [KW05] give an LP-based algorithm specifically for MDS that is simpler than the subsequent general-purpose LP-based algorithm of Kuhn, Moscibroda, and Wattenhofer [KMW06]; however, [KW05] is inferior to [KMW06] in both approximation ratio and running time.

1.3 Our Contributions

We answer all parts of the above question in the affirmative. In particular, we give algorithms that achieve the asymptotically optimal approximation factor of $O(\alpha)$, and are not only simple and elementary, but also run faster than all known algorithms, including LP-based algorithms.\footnote{\textit{O}(\alpha) is the asymptotically optimal approximation factor for polynomial time algorithms in the centralized setting and also in distributed settings where processors are assumed to have polynomially-bounded processing power.}

Centralized Setting

Our core contribution is an asymptotically optimal algorithm in the centralized setting.

\textbf{Theorem 1.2.} For graphs of arboricity $\alpha$, there is an $O(m)$ time $O(\alpha)$-approximation algorithm for MDS.

We note that our algorithm works even when $\alpha$ is not known a priori, since there is a linear time 2-approximation algorithm for computing the arboricity of a graph [AMZ97].

Our algorithm is asymptotically optimal in both running time and approximation factor: it runs in linear time, and asymptotically improving the approximation factor it gets is proved to be NP-hard [BUT17]. (The constant in the approximation ratio is not tight; our algorithm gives an $8\alpha$-approximation.) While the quantitative improvement in running time over prior work is admittedly minor (a logarithmic factor over the deterministic algorithm, and $\log n/\alpha$ over the randomized algorithm), still getting a truly linear time algorithm is qualitatively very different than an almost-linear time. Indeed, the study of linear time algorithms has received much attention over the years, even when it comes to shaving factors that grow as slowly as inverse-Ackermann type functions. This line of work includes celebrated breakthroughs in computer science: For example, for the Union-Find data structure, efforts to achieve a linear time algorithm led to a lower bound showing that inverse-Ackermann function dependence is necessary [FS89], matching the...
upper bound [Tar75], which is a cornerstone result in the field. Another example is MST, where the inverse-Ackermann function was shaved from the upper bound of [Cha00] to achieve a linear time algorithm either using randomization [KKT95] or when the edge weights are integers represented in binary [FW90], but it remains a major open problem whether or not there exists a linear time deterministic comparison-based MST algorithm.

**Distributed Setting**

We demonstrate the applicability of our centralized algorithm, by using its core ideas to develop a distributed algorithm.

**Theorem 1.3.** For graphs of arboricity \( \alpha \), there is a randomized distributed algorithm in the CONGEST model that gives an \( O(\alpha) \)-approximation for MDS and runs in \( O(\alpha \log n) \) rounds. The bound on the number of rounds holds with high probability (and in expectation). The algorithm works even when either \( \alpha \) or \( n \) is unknown to each processor.

For the “interesting” parameter regime where \( \Delta \) is polynomial in \( n \), and \( \alpha = o(\log n) \), the number of rounds in our algorithm beats the prior work obtained by combining [KMW06] and [BU17] which appears to run in \( O(\log^2 \Delta) \) rounds; as noted already, such an algorithm has not been claimed explicitly before. We note the caveat that our algorithm is randomized while their algorithm appears to be deterministic.

In the process of obtaining our distributed algorithm, we also obtain a deterministic algorithm in the LOCAL model (with polynomial message sizes) in a polylogarithmic number of rounds, via reduction to the maximal independent set (MIS) problem:

**Theorem 1.4.** Suppose there is a deterministic (resp., randomized) distributed algorithm in the LOCAL model for computing an MIS on a general graph in \( R(n) \) rounds. Then, for graphs of arboricity \( \alpha \), there is a deterministic (resp., randomized) distributed algorithm in the LOCAL model that gives an \( O(\alpha) \)-approximation for MDS in \( O(R(n) \cdot \alpha^2 \log n) \) rounds. The algorithm works even when either \( \alpha \) or \( n \) is unknown to each processor.

While [Theorem 1.4] is the first deterministic non-LP-based algorithm to achieve an \( O(\alpha) \)-approximation, we note that the LP-based approach obtained by combining [KMW06] and [BU17] appears to achieve fewer rounds and work in the CONGEST model. [Theorem 1.4] is not our main result and is used as a stepping stone towards our \( O(\alpha \log n) \) round algorithm in the CONGEST model.

We finally note that unlike in the centralized setting, handling unknown \( \alpha \) in the distributed setting it is not trivial and requires special treatment; in [Section 5.3] we demonstrate that all of our distributed algorithms can cope with unknown \( \alpha \) without increasing the approximation factor and running time beyond constant factors.

**Wider applicability** We have demonstrated the applicability of our centralized algorithm to the distributed setting. We anticipate that the core idea behind our centralized algorithm could be applied more broadly, to other settings that involve locality. Perhaps the prime example in this context is the standard (centralized) setting of dynamic graph algorithms, where the graph undergoes a sequence of edge updates (a single edge update per step), and the algorithm should maintain the graph structure of interest (\( O(\alpha) \)-approximate MDS in our case) with a small update time — preferably poly \( \log(n) \) and ideally \( O(1) \).
1.4 Technical overview

Centralized algorithm As a starting point, we consider the algorithm of Jones, Lokshtanov, Ramanujan, Saurabh, and Suchy [JLR+13], which achieves an $O(\alpha^2)$-approximation in linear time. Their algorithm is as follows. They iteratively build a dominating set $D$ and maintain a partition of the remaining vertices into the dominated vertices $B$ (the vertices that have a neighbor in $D$), and the undominated vertices $W$. This partition of the vertices, as well as further partitioning described later, is shown in Fig. 1. The basic property of arboricity $\alpha$ graphs used by their algorithm is that every subgraph contains a vertex of degree $O(\alpha)$. They begin by choosing a vertex $v$ with degree $O(\alpha)$ and adding $v$ along with $v$’s entire neighborhood $N(v)$ to $D$. The intuition behind this is that at least one vertex in $\{v\} \cup N(v)$ must be in $OPT$ (an optimal dominating set), since $OPT$ must dominate $v$. Hence, they add at least one vertex in $OPT$ and use that to pay for adding $O(\alpha)$ vertices not in $OPT$. We say that a vertex $w$ witnesses $v$ and the vertices in $N(v)$ that are added to $D$, if $w \in OPT \cap (\{v\} \cup N(v))$. Now, the goal of the algorithm is to iteratively choose vertices $v$ to add to $D$ along with $O(\alpha)$ many of $v$’s neighbors so that each vertex in $OPT$ witnesses $O(\alpha)$ vertices $v$ along with $O(\alpha)$ neighbors for each such vertex $v$. That is, each vertex in $OPT$ witnesses $O(\alpha^2)$ vertices in $D$, which yields an $O(\alpha^2)$-approximation.

To choose which vertices $v$ and which $O(\alpha)$ of $v$’s neighbors to add to $D$, they partition the set $B$ into two subsets $B_{low}$ and $B_{high}$, which are the sets of vertices in $B$ with low and high degree to $W$, respectively, where the degree threshold is $\delta \alpha$ for some constant $\delta$. We also define $W_{low} \subseteq W$ (differently from the notation of [JLR+13]) as the subset of vertices with degree at most $\delta \alpha$ in the subgraph induced by $W \cup B_{high}$. They add a vertex $w \in W_{low}$ to $D$ along with $w$’s $O(\alpha)$ neighbors that are in $W \cup B_{high}$. In the interest of brevity, we will not motivate why this scheme achieves the desired outcome that each vertex in $OPT$ witnesses $O(\alpha^2)$ vertices in $D$.

The key innovation in our algorithm that allows us to reduce the approximation factor from $O(\alpha^2)$ to $O(\alpha)$ is a simple but powerful idea. After choosing a vertex $w$ to add to $D$, we do not immediately add $O(\alpha)$ of $w$’s neighbors to $D$. Instead $w$ casts a “vote” for these $O(\alpha)$ neighbors, and only once a vertex gets $\delta \alpha$ many votes is it added to $D$. With this modification, we can argue that each vertex in $OPT$ still witnesses $O(\alpha)$ such vertices $w$ as in the previous approach, but the catch here is that each such vertex $w$ contributes only $O(1)$ neighbors to $D$ on average, so each vertex in $OPT$ only witnesses a total of $O(\alpha)$ vertices in $D$, rather than $O(\alpha^2)$. Moreover, it is straightforward to implement this algorithm in linear time.

Figure 1: The partition of vertices
**Distributed algorithms using MIS** This section concerns the proof of Theorem 1.4, our reduction from MDS to MIS in the LOCAL model. This section also concerns a modification of this reduction that gives an $O(\alpha^2 \log^2 n)$ round algorithm in the CONGEST model. We use this algorithm as a stepping stone towards obtaining our main distributed algorithm (Theorem 1.3), which runs in $O(\alpha \log n)$ rounds in the CONGEST model.

We adapt our centralized algorithm to the distributed setting as follows. Recall that in our centralized algorithm, we repeatedly choose a vertex $w \in W_{low}$, add $w$ to $D$, and cast a vote for each vertex in $N(w) \cap (W \cup B_{high})$. For our distributed algorithms, we would like to choose many such vertices $w$ and process them in parallel. In fact, a constant fraction of the vertices in $W \cup B_{high}$ could be chosen as our vertex $w$ since a constant fraction of vertices in a graph of arboricity $\alpha$ have degree $O(\alpha)$. However, we cannot simply process all of these vertices in parallel. In particular, if a vertex $v$ has many neighbors being processed in parallel, $v$ might accumulate many votes during a single round. This would invalidate the analysis of the algorithm, which relies on the fact that once a vertex $v$ receives $\delta \alpha$ votes, $v$ enters $D$.

To overcome this issue, we compute an MIS with respect to a 2-hop graph built from a subgraph of “candidate” vertices, and only process the vertices in this MIS in parallel. This MIS has two useful properties: 1. Its maximality implies that in any 2-hop neighborhood of a candidate vertex there is a vertex in the MIS; this helps to bound the number of rounds, and 2. Its independence implies that every vertex has at most one neighbor in the MIS, which ensures that any vertex can only receive one vote per round. To conclude, this approach gives a reduction from distributed MDS to distributed MIS in the LOCAL model. This approach can be made to work in the CONGEST model by replacing the black-box MIS algorithm with a 2-hop version of Luby’s algorithm. This approach of running the 2-hop version of Luby’s algorithm was also used in [LW10] for their distributed $(\alpha^2)$-approximation for MDS.

**Faster randomized distributed algorithm** In the CONGEST model, our distributed algorithm using MIS runs in $O(\alpha^2 \log^2 n)$ rounds with high probability. We devise a new, more nuanced algorithm that decreases the number of rounds to $O(\alpha \log n)$ with high probability. Our new algorithm is based on our previous algorithm, but with two key modifications, which save factors of $\log n$ and $\alpha$, respectively.

Our first key modification, which shaves a $\log n$ factor from the number of rounds, is that we do not run an MIS algorithm as a black box. Instead, we run only a single phase of a Luby-like MIS algorithm before updating the data structures. Intuitively, this saves a $\log n$ factor because we are running just one phase of a $O(\log n)$-phase algorithm, but it is not clear a priori whether we achieve the same progress as Luby’s algorithm in a single phase. We demonstrate that this is indeed the case via more refined treatment of the behavior of each edge.

Our second key modification, which shaves an $\alpha$ factor from the number of rounds, concerns the Luby-like algorithm. Recall that in Luby’s algorithm, each vertex $v$ picks a random value $p(v)$ and then joins the MIS if $p(v)$ is the local minimum. In our algorithm, a vertex $v$ instead joins the dominating set if $p(v)$ is an $\alpha$-minimum, which roughly means that $p(v)$ is among the $\alpha$ smallest values that it is compared to. We show that with this relaxed definition, we still have the desired property that no vertex receives more than $\delta \alpha$ votes in a single round.

The main technical challenge is the analysis of the number of rounds. It is tempting to use an analysis similar to that of Luby’s algorithm, where we count the expected number of “removed edges” over time. However, our above modifications introduce several complications that preclude such an analysis. Instead, we use a carefully chosen function to measure our progress. Throughout the algorithm, we add “weight” to particular edges, and our function measures the “total available weight”. Specifically, whenever a vertex $v$ is added to the dominating set, $v$ adds weight to a particular set of edges in its 2-hop neighborhood. We
show that the total amount of weight added in a single iteration of the algorithm decreases the total available weight substantially, which allows us to bound the total number of iterations.

All of our distributed algorithms so far have assumed that $\alpha$ is known to each processor but that $n$ is unknown. We additionally show that all of them can be made to work in the setting where $\alpha$ is unknown but $n$ is known. The idea of this modification is to guess $\log n$ values of $\alpha$ and run a truncated version of the algorithm for each guess. However, it is impossible for an individual processor to know which guess of $\alpha$ is the most accurate without knowing the whole graph, so the processors cannot coordinate their guesses globally. We end up with different processors using different guesses of $\alpha$, but we show that we can nonetheless obtain an algorithm whose approximation factor and running time are in accordance with the correct $\alpha$.

1.5 Organization
Section 2 is for preliminaries. In Section 3, we present our centralized algorithm (Theorem 1.2). In Section 4, we present our distributed algorithms using MIS: in the LOCAL model we prove Theorem 1.4 and in the CONGEST model we give a randomized algorithm with $O(\alpha^2 \log^2 n)$ rounds that serves as a warm-up for the faster algorithm of Theorem 1.3. In Section 5, we prove Theorem 1.3 and show that all of our distributed algorithms can be made to handle unknown $\alpha$.

2 Preliminaries
Let $G = (V, E)$ be an unweighted undirected graph. For any $S \subseteq V$, let $G[S]$ be denote the subgraph induced by $S$. For any $v \in V$, $N_G(v)$ denotes the neighborhood of $v$, and $\deg_G(v) = |N_G(v)|$ denotes the degree of $v$. When the graph $G$ is clear from context, we omit the subscript.

We define the LOCAL and CONGEST models (cf. [Lin87, Lin92, Pel00]):

Definition 2.1. The LOCAL model: given a graph $G$ on $n$ vertices, every vertex is a separate processor running one process. Every vertex starts knowing only $n$ and it’s own unique identifier. The algorithm works in synchronous rounds, and in every round each vertex performs some computation based on its own current information, then it sends a message to its neighbors, and finally it receives the messages sent to it by its neighbors in that round.

Definition 2.2. The CONGEST model: given a graph $G$ on $n$ vertices, every vertex is a separate processor running one process. Every vertex starts knowing only $n$ and it’s own unique identifier. The algorithm works in synchronous rounds, and in every round each vertex performs some computation based on its own current information, then it sends a message to its neighbors of at most $B = O(\log n)$ bits on each of its edges (possibly a different message to each neighbor), and finally it receives the messages sent to it by its neighbors in that round.

For the problem of MDS in both models, the requirement is that at the end of the computation, every vertex knows whether or not it belongs to the dominating set.

The following two claims about graphs of bounded arboricity will be useful. Simple proofs of both can be found in [AMZ97].

Claim 2.3. In a graph of arboricity $\alpha$, every subgraph contains a vertex of degree at most $2\alpha$.

Claim 2.4. In a graph $G$ with arboricity $\alpha$, at least half of the vertices in any subgraph have degree at most $4\alpha$. 

7
3 Linear time $O(\alpha)$-approximation for MDS

In this section we will prove [Theorem 1.2] which we recall:

**Theorem 1.2.** For graphs of arboricity $\alpha$, there is an $O(m)$ time $O(\alpha)$-approximation algorithm for MDS.

3.1 Algorithm

A description of our algorithm is as follows. See [Algorithm 1] for the pseudocode.

We first introduce some notation. Since our algorithm builds off of [JLR+13], we stick to their notation for the most part. See [Fig. 1]. We define a constant $\delta$ and let $\delta\alpha$ be our degree threshold. We will set $\delta = 2$, but we use the variable $\delta$ so that our analysis also applies to our distributed algorithms, where $\delta$ is a different constant. We maintain a partition of the vertices into three sets: $D$, $B$, and $W$, where initially $D = \emptyset$, $B = \emptyset$, and $W = V$. The set $D$ is our current dominating set, the set $B$ is the vertices not in $D$ with at least one neighbor in $D$, and the set $W$ is the remaining vertices, i.e. the undominated vertices. The set $B$ is further partitioned into two sets based on the degree of each vertex to $W$. Let $B_{\text{low}} = \{v \in B : |N(v) \cap W| \leq \delta\alpha\}$ and let $B_{\text{high}} = B \setminus B_{\text{low}}$. Let $W_{\text{low}} = \{v \in W : |N(v) \cap (W \cup B_{\text{high}})| \leq \delta\alpha\}$ Also, each vertex $v$ has a counter $c_v$ initialized to 0. (The counter $c_v$ counts the number of “votes” that $v$ receives, for the notion of “votes” introduced in the technical overview.)

First we claim that while $W$ is nonempty, $W_{\text{low}}$ is also nonempty. By [Claim 2.3] $G[W \cup B_{\text{high}}]$ contains a vertex $v$ of degree at most $2\alpha$. Since $\delta = 2$, $v$ cannot be in $B_{\text{high}}$ by the definition of $B_{\text{high}}$, so $v$ must be in $W$, and hence in $W_{\text{low}}$.

The algorithm proceeds as follows. While there still exists an undominated vertex (i.e. while $W \neq \emptyset$), we do the following. First, we pick an arbitrary vertex $w \in W_{\text{low}}$ (we have shown that $W_{\text{low}}$ is nonempty). Then, for all $v \in N(w) \cap (W \cup B_{\text{high}})$, we increment $c_v$, and if $c_v = \delta\alpha$, then we add $v$ to $D$. Then, we add $w$ to $D$. Lastly, we update the sets $B_{\text{low}}, B_{\text{high}}, W$, and $W_{\text{low}}$ according to their definitions. This concludes the description of the algorithm.

**Algorithm 1** Linear time $O(\alpha)$-approximation for MDS

1: Initialize partition: $D \leftarrow \emptyset$, $B = \emptyset$, $B_{\text{high}} \leftarrow \emptyset$, $B_{\text{low}} \leftarrow \emptyset$, $W \leftarrow V$, $W_{\text{low}} = \{v \in V : \deg(v) \leq \delta\alpha\}$
2: Initialize counters: $\forall v \in V : c_v \leftarrow 0$
3: while $W \neq \emptyset$ do
4: \hspace{1em} $w \leftarrow$ a vertex in $W_{\text{low}}$
5: \hspace{1em} for each $v \in N(w) \cap (W \cup B_{\text{high}})$ do
6: \hspace{2em} $c_v \leftarrow c_v + 1$
7: \hspace{2em} if $c_v = \delta\alpha$ then
8: \hspace{3em} $D \leftarrow D \cup v$
9: \hspace{2em} $D \leftarrow D \cup w$
10: \hspace{2em} // Bookkeeping to update partition:
11: \hspace{3em} $B = \{v : N(v) \cap D \neq \emptyset\}$
12: \hspace{3em} $B_{\text{low}} = \{v \in B : |N(v) \cap W| \leq \delta\alpha\}$
13: \hspace{3em} $B_{\text{high}} = B \setminus B_{\text{low}}$
14: \hspace{3em} $W = V \setminus (D \cup B)$
15: \hspace{3em} $W_{\text{low}} = \{v \in W : |N(v) \cap (W \cup B_{\text{high}})| \leq \delta\alpha\}$
16: End while
17: Return $D$
3.2 Analysis

First, we note that $D$ is indeed a dominating set because the algorithm only terminates once the set $W$ of vertices that are not dominated, is empty.

3.2.1 Approximation ratio analysis

Let $OPT$ be an optimal MDS. We will prove that the set $D$ returned by [Algorithm 1] is of size at most $4\delta \alpha \cdot |OPT|$.

We first make the following simple claim about the behavior of the partition of vertices over time.

Claim 3.1.

1. No vertex can ever leave $D$.
2. No vertex can ever enter $W$ from another set.
3. No vertex can ever leave $B_{low}$.

Proof. Item 1 is by definition. Item 2 follows from item 1 combined with the fact that $W$ is defined as the set of vertices with no neighbors in $D$. Now we prove item 3. A vertex from $B_{low}$ cannot enter $W$ by item 2. A vertex from $B_{low}$ cannot enter $B_{high}$ since the degree partition of $B$ is based on degree to $W$, and by item 2 the degree of any vertex to $W$ can only decrease over time. A vertex from $B_{low}$ cannot enter $D$ because there are two ways a vertex can enter $D$: on Line 8 a vertex can only enter $D$ from $W \cup B_{high}$, and on Line 9 a vertex can only enter $D$ from $W$.

In order to show that $|D| \leq 4\delta \alpha \cdot |OPT|$, we partition $D$ into two sets, $D_{active}$ and $D_{passive}$, and bound each of these sets separately. The set $D_{active}$ consists of the vertices added to $D$ due to being chosen as the vertex $w$; that is, the vertices added to $D$ in Line 9 of [Algorithm 1]. The set $D_{passive}$ consists of the vertices added to $D$ as a result of their counters reaching $\delta \alpha$; that is, the vertices added to $D$ in Line 8 of [Algorithm 1]. We will first bound $|D_{active}|$.

Claim 3.2. $|D_{active}| \leq 2\delta \alpha \cdot |OPT|$.

Proof. For each vertex $v \in D_{active}$, we assign $v$ to an arbitrary vertex $u \in N(v) \cap OPT$, and we say that $u$ witnesses $v$. Such a vertex $u$ exists since $OPT$ is a dominating set. For each vertex $u \in OPT$, let $D_u \subseteq D_{active}$ be the set of vertices that $u$ witnesses. Our goal is to show that for each $u \in OPT$, $|D_u| \leq 2\delta \alpha$.

Fix a vertex $u \in OPT$. We partition the vertices $v \in D_u$ into two sets $D_u[B_{low}]$ and $D_u[B_{high} \cup W]$. Let $D_u[B_{low}] \subseteq D_u$ be the vertices that enter $D$ while $u$ is in $B_{low}$. Let $D_u[B_{high} \cup W] \subseteq D_u$ be vertices that enter $D$ while $u$ is in $B_{high} \cup W$. We note that no vertex in $D_u$ can enter $D$ while $u$ is in $D$, because by definition, every vertex in $D_{active} \supseteq D_u$ moves directly from $W$ to $D$. Therefore, $D_u = D_u[B_{low}] \cup D_u[B_{high} \cup W]$.

We first bound $|D_u[B_{low}]|$. By definition, while $u$ is in $B_{low}$, $u$ has at most $\delta \alpha$ neighbors in $W$. Since no vertex can ever enter $W$ by [Claim 3.1] no vertex can ever enter $N(u) \cap W$. Therefore, starting from the time that $u$ first enters $B_{low}$, the total number of vertices ever in $N(u) \cap W$ is at most $\delta \alpha$. Every vertex $v \in D_u[B_{low}]$ is in $N(u) \cap W$ right before moving to $D$, so $|D_u[B_{low}]| \leq \delta \alpha$.

Now, we bound $D_u[B_{high} \cup W]$. By the specification of the algorithm, whenever a vertex $v \in D_u[B_{high} \cup W]$ enters $D$, the counter $c_u$ is incremented. Once $c_u$ reaches $\delta \alpha$, $u$ is added to $D$. Therefore, $|D_u[B_{high} \cup W]| \leq \delta \alpha$.

Putting everything together, we have $|D_u| = |D_u[B_{low}]| + |D_u[B_{high} \cup W]| \leq 2\delta \alpha$. \qed
Now we bound $D_{\text{passive}}$.

**Claim 3.3.** $|D_{\text{passive}}| \leq |D_{\text{active}}|$.

*Proof.* We will show that every vertex in $D_{\text{passive}}$ has at least $\delta \alpha$ neighbors in $D_{\text{active}}$, while every vertex in $D_{\text{active}}$ has at most $\delta \alpha$ neighbors in $D_{\text{passive}}$. Then, by the pigeonhole principle, it follows that $|D_{\text{passive}}| \leq |D_{\text{active}}|$.

First, we will show that every vertex in $D_{\text{passive}}$ has at least $\delta \alpha$ neighbors in $D_{\text{active}}$. By definition, every vertex $v \in D_{\text{passive}}$ has had its counter $c_v$ incremented $\delta \alpha$ times. Every time $c_v$ is incremented, one of $v$’s neighbors (the vertex $w$ from Algorithm 1) is added to $D$, joining $D_{\text{active}}$. Each such neighbor of $v$ that joins $D_{\text{active}}$ is distinct since every vertex can be added to $D$ at most once by Claim 3.1. Therefore, every vertex in $D_{\text{passive}}$ has at least $\delta \alpha$ neighbors in $D_{\text{active}}$.

Now we will show that every vertex in $D_{\text{active}}$ has at most $\delta \alpha$ neighbors in $D_{\text{passive}}$. Fix a vertex $w \in D_{\text{active}}$. By definition, when $w$ enters $D$, $w$ is moved straight from $W$ to $D$. Thus, by Claim 3.1, $w$ is never in $B$. Therefore, $w$ is added to $D$ before any of its neighbors are added to $D$, as otherwise $w$ would enter $B$. Therefore, when $w$ enters $D$, all of $w$’s neighbors that will enter $D_{\text{passive}}$ are in $B \cup W$. By Claim 3.1 no vertex in $B_{\text{low}}$ can ever enter $D$, so actually, when $w$ enters $D$ all of $w$’s neighbors that will enter $D_{\text{passive}}$ are in $B_{\text{high}} \cup W$. By definition, when $w$ enters $D$, $w$ has at most $\delta \alpha$ neighbors in $B_{\text{high}} \cup W$. Therefore, $w$ has at most $\delta \alpha$ neighbors in $D_{\text{passive}}$.

Combining Claim 3.2 and Claim 3.3 we have that $|D| = |D_{\text{active}}| + |D_{\text{passive}}| \leq 4 \delta \alpha \cdot |OPT|$.

### 3.2.2 Running time analysis

Our goal is to prove that Algorithm 1 runs in $O(m)$ time.

Throughout the execution of the algorithm, we maintain a data structure that consists of the following:

- The partition of $V$ into $D$, $B$, $W$; with subsets $B_{\text{low}}$, $B_{\text{high}}$, $W_{\text{low}}$
- The induced graph $G[W \cup B_{\text{high}}]$ represented as an adjacency list
- For each vertex $v \in W \cup B_{\text{high}}$, the quantities $|N(v) \cap W|$ and $|N(v) \cap (W \cup B_{\text{high}})|$

First, we show that the data structure can be initialized in $O(m)$ time. Initially $D \cup B_{\text{high}} \cup B_{\text{low}} = \emptyset$, $W = V$, and the induced graph $G[W \cup B_{\text{high}}] = G$. For every vertex $v \in V$, initially $|N(v) \cap W| = |N(v) \cap (W \cup B_{\text{high}})| = \deg(v)$. Initially $W_{\text{low}} = \{v \in V : \deg(v) \leq \delta \alpha\}$.

Now, we show that the data structure can be maintained in $O(m)$ time. In particular, we will show that to maintain this data structure, it suffices to scan the neighborhood of a vertex every time it either leaves $W \cup B_{\text{high}}$ (and enters $B_{\text{low}} \cup D$), enters $D$, or leaves $W$. Note that by Claim 3.1, each of these events only happens once per vertex. As a consequence, the total amount of time spent scanning neighborhoods is $O(m)$.

We assume inductively that we have maintained the data structure so far, and we consider the next iteration of the for each loop. First, we consider maintenance of the partition of $V$ into $D$, $B_{\text{low}}$, $B_{\text{high}}$, and $W$. During an iteration, the only changes made to the partition are the addition of at least one vertex to $D$ (on Line 8) and/or (Line 9), and the resulting update of the rest of the partition. To maintain the partition we do the following. When we add a vertex $v$ to $D$, we remove $v$ from whichever set it was previously in. Then, we update $B$ by scanning $N(v)$ and adding every vertex $u \in N(v) \setminus D$ to $B$, removing $u$ from whichever set it was previously in. Updating $D$ and $B$ automatically updates $W$ since $W = V \setminus (D \cup B)$. Before updating $B_{\text{low}}$ and $B_{\text{high}}$, we first need to update $|N(v) \cap W|$. To do this, whenever a vertex $v$ leaves $W$,
we scan \( N(v) \) and for each \( u \in N(v) \), we decrement \( |N(u) \cap W| \). Whenever we decrement \( |N(u) \cap W| \) down to \( \delta \alpha \) for a vertex \( u \in B_{\text{high}} \), we move \( u \) to \( B_{\text{low}} \). This concludes the maintenance of the partition of \( V \) into \( D, B_{\text{low}}, B_{\text{high}}, \) and \( W \).

It remains to update \( G[W \cup B_{\text{high}}], |N(v) \cap (W \cup B_{\text{high}})| \), and \( W_{\text{low}} \). Whenever we remove a vertex \( v \) from \( W \cup B_{\text{high}} \), we scan \( N(v) \) and for each vertex \( u \in N(v) \), we remove the edge \((u, v)\) from \( G[W \cup B_{\text{high}}] \) and decrement \( |N(u) \cap (W \cup B_{\text{high}})| \). Whenever we decrement \( |N(u) \cap (W \cup B_{\text{high}})| \) down to \( \delta \alpha \) for \( u \in W \), we add \( u \) to \( W_{\text{low}} \). This concludes the running time analysis for maintaining the data structure.

Now we will show that maintaining the data structure allows the algorithm to run in time \( O(m) \). First, each iteration of the while loop adds at least one vertex to \( D \) (on Line 9), and by Claim 3.1 each vertex is added to \( D \) at most once, so the total number of iterations of the while loop is at most \( n \). Now we will go line by line through the body of the while loop. On Line 4 we let \( w \) be a vertex in \( W_{\text{low}} \). On Line 5 we loop through every vertex in \( |N(w) \cap (W \cup B_{\text{high}})| \). The number of iterations of this loop is at most \( \delta \alpha \) by choice of \( w \). Furthermore, identifying all of the vertices to loop through takes time \( O(\alpha) \) since our data structure explicitly maintains \( G[W \cup B_{\text{high}}] \). In Line 6 through Line 9 we update counters and then add vertices to \( D \), which takes constant time per iteration of the loop. In Line 10 through Line 13 we update \( B, B_{\text{low}}, B_{\text{high}}, \) and \( W \), which takes constant time since we store these sets in our data structure. Thus, given access to the data structure, the algorithm runs in time \( O(n\alpha) = O(m) \).

Previously we showed that maintaining the data structure takes time \( O(m) \), so we have that the entire algorithm takes time \( O(m) \).

4 Distributed \( O(\alpha) \)-approximation for MDS using MIS

In this section we will prove Theorem 1.4, which we recall:

**Theorem 1.4.** Suppose there is a deterministic (resp., randomized) distributed algorithm in the LOCAL model for computing an MIS on a general graph in \( R(n) \) rounds. Then, for graphs of arboricity \( \alpha \), there is a deterministic (resp., randomized) distributed algorithm in the LOCAL model that gives an \( O(\alpha) \)-approximation for MDS in \( O(R(n) \cdot \alpha^2 \log n) \) rounds. The algorithm works even when either \( \alpha \) or \( n \) is unknown to each processor.

In this section we also show how to modify the proof of Theorem 1.4 to get a bound in the CONGEST model:

**Theorem 4.1.** For graphs of arboricity \( \alpha \), there is a randomized distributed algorithm in the CONGEST model that gives an \( O(\alpha) \)-approximation for MDS that runs in \( O(\alpha^2 \log^2 n) \) rounds with high probability. The algorithm works even when either \( \alpha \) or \( n \) is unknown to each processor.

In Section 5 we will use the algorithm of Theorem 4.1 as a starting point to get an improved algorithm with \( O(\alpha \log n) \) rounds.

The algorithms presented in this section assume that \( \alpha \) is known to each processor but \( n \) is unknown. We defer discussion of handling unknown \( \alpha \) to the next section.

4.1 Algorithm

**Overview.** Our algorithm is an adaptation of our centralized algorithm from Theorem 1.2 to the distributed setting. Recall that in our centralized algorithm, we repeatedly choose a vertex \( w \in W_{\text{low}} \), add \( w \) to the dominating set, and increment the counter of \( w \)'s neighbors that are in \( W \cup B_{\text{high}} \). For our distributed
algorithms, we would like process many vertices in $W_{\text{low}}$ in parallel. There are in fact many vertices in $W_{\text{low}}$ (if $\delta \geq 4$) since Claim 2.4 implies that at least half of the vertices in any subgraph has degree at most $4\alpha$. However, we cannot simply process all of $W_{\text{low}}$ at once. In particular, if a vertex $v$ has many neighbors being processed in parallel, $v$ might have its counter incremented once for each of these neighbors. This is undesirable because the analysis of our centralized algorithm relies on the fact that once a vertex has its counter incremented to $\delta\alpha$, it is added to the dominating set. Therefore, we would like to guarantee that only a limited number of $v$’s neighbors are processed in parallel.

This is where the MIS problem becomes relevant: we ensure that no vertex has more than one neighbor being processed in parallel by taking an MIS $I$ with respect to the graph $G_{\text{low}}$ defined as follows: the vertex set of $G_{\text{low}}$ is $W_{\text{low}}$. There is an edge $(u,v)$ in $G_{\text{low}}$ if there is a path of length 2 between $u$ and $v$ in $G[W \cup B_{\text{high}}]$. Note that because no vertex has more than one neighbor in $I$, we can process all vertices in $I$ in parallel and only increase the counter of each vertex by at most one.

The algorithms for [Theorem 1.4] and [Theorem 4.1] are identical except for the MIS subroutine. [Theorem 1.4] is for the LOCAL model so we can simply run any distributed MIS algorithm that works in the LOCAL model on $G_{\text{low}}$ as a black box. On the other hand, [Theorem 4.1] is for the CONGEST model and because $G_{\text{low}}$ can have higher degree than $G$, running an MIS algorithm directly on $G_{\text{low}}$ could result in messages that become too large after translating the algorithm to run on $G$. To bypass this issue, we use a simple modification of Luby’s algorithm that computes $I$ using only small messages, without increasing the number of rounds.

### Algorithm description
We provide a description of the algorithms here, and include the pseudocode in Algorithm 2. The only difference between the algorithms for [Theorem 1.4] and [Theorem 4.1] is the MIS subroutine, which we will handle separately later.

The sets $D$, $B$, $W$, $B_{\text{high}}$, $B_{\text{low}}$, and $W_{\text{low}}$ are defined exactly the same as in our centralized algorithm, except we set $\delta = 4$ instead of $\delta = 2$ so that we can apply Claim 2.4 instead of Claim 2.3. We repeat the definitions here for completeness. The set $D$ is our current dominating set, the set $B$ is the vertices not in $D$ with at least one neighbor in $D$, and the set $W$ is the remaining vertices, i.e. the undominated vertices. The set $B$ is further partitioned into two sets based on the degree of each vertex to $W$. Let $B_{\text{low}} = \{v \in B : |N(v) \cap W| \leq \delta\alpha\}$ and let $B_{\text{high}} = B \setminus B_{\text{low}}$. Also, let $W_{\text{low}} = \{v \in W : |N(v) \cap (W \cup B_{\text{high}})| \leq \delta\alpha\}$.

Lastly, each vertex $v$ has a counter $c_v$.

Each vertex $v$ maintains the following information:

- The set(s) among $D$, $B$, $W$, $B_{\text{high}}$, $B_{\text{low}}$, and $W_{\text{low}}$ that $v$ is a member of.
- The quantity $|N(v) \cap W|$.
- The quantity $|N(v) \cap (W \cup B_{\text{high}})|$.
- The counter $c_v$.

At initialization, every vertex $v$ is in $W$ (so $D$ and $B$ are empty). Consequently, the quantities $|N(v) \cap W|$ and $|N(v) \cap (W \cup B_{\text{high}})|$ are both equal to $\deg(v)$. For each vertex $v$, if $\deg(v) \leq \delta\alpha$, then $v \in W_{\text{low}}$. Each counter $c_v$ is initialized to 0.

It will be useful to define the graph $G_{\text{low}}$, which changes over the execution of the algorithm:

**Definition 4.2.** Let $G_{\text{low}}$ be the graph with vertex set $W_{\text{low}}$ such that there is an edge $(u,v)$ in $G_{\text{low}}$ if there is a path of length 2 between $u$ and $v$ in $G[W \cup B_{\text{high}}]$. 

12
The algorithm proceeds as follows. Repeat the following until \(W\) is empty. Compute an MIS \(I\) with respect to \(G_{low}\). This step is implemented differently for Theorem 1.4 and Theorem 4.1 and we describe the details of this step later.

Then, each vertex in \(I\) adds itself to \(D\) and tells its neighbors to increment their counters. Whenever the counter of a vertex reaches \(\delta \alpha\), it enters \(D\) (and does not tell its neighbors to increment their counters).

Whenever a vertex moves from one set of the partition to another, it notifies each of its neighbors \(v\) so that \(v\) can update the quantities \(|N(v) \cap W|\) and \(|N(v) \cap (W \cup B_{high})|\), and move to the appropriate set. When no more vertices are left in \(W\), \(B_{high}\) is also empty, and all processors terminate. This concludes the description of the algorithm. See Algorithm 2 for the precise ways that vertices react to the messages that they receive.

**MIS subroutine** Theorem 1.4 is a reduction from MDS to MIS, while Theorem 4.1 is not, so we need to describe the MIS subroutine (in the CONGEST model) only for Theorem 4.1. Recall that we cannot use a reduction to MIS in the CONGEST model because running an MIS algorithm directly on \(G_{low}\) could result in messages that become too large after translating the algorithm to run on \(G\).

Our goal is to compute an MIS with respect to \(G_{low}\), using small messages sent over \(G\). We use a simple adaptation of Luby’s algorithm. Recall that Luby’s algorithm builds an MIS \(I\) as follows. While the graph is non-empty, do the following: Add all singletons to \(I\). Then, each vertex \(v\) picks a random value \(p(v) \in [0, 1]\). Then, all vertices whose value is less than that of all of their neighbors are added to \(I\). Then, all vertices that are in \(I\) or have a neighbor in \(I\) are removed from the graph for the next iteration of the loop.

We use the following adaptation of Luby’s algorithm. See Algorithm 3 for the pseudocode. Initially, the set \(L\) of live vertices is the set \(W_{low}\). While \(L \neq \emptyset\), do the following: Each vertex \(v \in L\) picks a random value \(p(v) \in [0, 1]\). In the first round each \(v \in L\) sends \(p(v)\) to its neighbors. In the second round, each vertex that receives one or more values \(p(v)\), forwards to its neighbors the minimum value that it received. Then, for each vertex \(v \in W_{low}\), if \(p(v)\) is equal to the minimum value that \(v\) receives in the second round, \(v\) is added to \(I\). When \(v\) is added to \(I\), \(v\) notifies its neighbors, and each neighbor of \(v\) that is in \(W \cup B_{high}\), forwards this notification to their neighbors. Note that each vertex has at most one neighbor in \(I\), so forwarding this notification only takes one round. Now, every vertex knows whether it has a neighbor with respect to \(G_{low}\) that is in \(I\), and every vertex that does is removed from \(L\) for the next iteration of the loop.

The proof that this algorithm runs in \(O(\log n)\) rounds with high probability and produces an MIS with respect to \(G_{low}\) is the same as the analysis of Luby’s algorithm and we will not include it here.

### 4.2 Analysis

The proof that Algorithm 2 achieves an \(O(\alpha)\)-approximation is precisely the same as that of the centralized algorithm (see Section 3.2.1) given that no counter \(c_v\) ever exceeds \(\delta \alpha\). This is true because in a single iteration of the while loop each vertex can only have its counter incremented once since only vertices in the MIS \(I\) send INCREMENT COUNTER messages, and each vertex in \(W \cup B_{high}\) only has at most one neighbor in \(I\). This bound on the number of neighbors in \(I\) holds, since otherwise there is a path of length 2 between two vertices in \(G[W \cup B_{high}]\), making \(I\) not an independent set in \(G_{low}\). Once \(c_v\) reaches \(\delta \alpha\), the vertex \(v\) enters \(D\), which prevents \(c_v\) from increasing in the future.

Our goal in this section is to prove that if the MIS subroutine takes \(R(n)\) rounds, then Algorithm 2 takes \(O(R(n) \cdot \alpha^2 \log n)\) rounds. First, we note that the body of the while loop besides the MIS subroutine takes a constant number of rounds. Thus, our goal is to show that the number of iterations of the while loop is \(O(\alpha^2 \log n)\).
**Algorithm 2** Distributed $O(\alpha)$-approximation for MDS using MIS

1: Initialize partition: $D \leftarrow \emptyset, B_{\text{high}} \leftarrow \emptyset, B_{\text{low}} \leftarrow \emptyset, W \leftarrow V, W_{\text{low}} \leftarrow \{v \in V : \deg(v) \leq \delta \alpha\}$
2: Initialize counters: $\forall v \in V : c_v \leftarrow 0$
3: Initialize degrees: $\forall v \in V : |N(v) \cap W| = \deg(v), |N(v) \cap (W \cup B_{\text{high}})| = \deg(v)$
4: while $W \neq \emptyset$ do
5: Find an MIS $I$ with respect to the graph $G_{\text{low}}$
6: Each vertex $v$ runs the following procedure:
7: if $v \in I$ then
8: Move $v$ to $D$
9: Send INCREMENT COUNTER message to neighbors
10: Send MOVED FROM $W$ TO $D$ message to neighbors
11: if $v \in W \cup B_{\text{high}}$ and $v$ receives INCREMENT COUNTER then
12: Increment $c_v$
13: if $c_v = \delta \alpha$ then
14: if $v \in W$ then
15: Send MOVED FROM $W$ TO $D$ message to neighbors
16: if $v \in B_{\text{high}}$ then
17: Send MOVED FROM $B_{\text{high}}$ TO $D$ message to neighbors
18: Move $v$ to $D$
// The rest of the algorithm is bookkeeping
19: if $v$ receives MOVED FROM $W$ TO $D$ then
20: Decrement $|N(v) \cap W|$
21: if $v \in B_{\text{high}}$ and $|N(v) \cap W| = \delta \alpha$ then
22: Move $v$ to $B_{\text{low}}$
23: if $v$ receives MOVED FROM $W$ TO $D$ or MOVED FROM $B_{\text{high}}$ TO $D$ then
24: Decrement $|N(v) \cap (W \cup B_{\text{high}})|$
25: if $v \in W$ and $|N(v) \cap W| \leq \delta \alpha$ then
26: Move $v$ to $B_{\text{low}}$
27: Send MOVED FROM $W$ TO $B_{\text{low}}$ message to neighbors
28: else if $v \in W$ and $|N(v) \cap W| > \delta \alpha$ then
29: Move $v$ to $B_{\text{high}}$
30: Send MOVED FROM $W$ TO $B_{\text{high}}$ message to neighbors
31: if $v$ receives MOVED FROM $W$ TO $B_{\text{low}}$ or MOVED FROM $W$ TO $B_{\text{high}}$ then
32: Decrement $|N(v) \cap W|$
33: if $v \in B_{\text{high}}$ and $|N(v) \cap W| = \delta \alpha$ then
34: Move $v$ to $B_{\text{low}}$
35: if $v$ receives MOVED FROM $W$ TO $B_{\text{low}}$ then
36: Decrement $|N(v) \cap (W \cup B_{\text{high}})|$
37: if $v \in W$ and $|N(v) \cap (W \cup B_{\text{high}})| = \delta \alpha$ then
38: Add $v$ to $W_{\text{low}}$
Algorithm 3 Distributed MIS with respect to $G_{\text{low}}$ in the CONGEST model

1: $L = W_{\text{low}}$
2: while $L \neq \emptyset$ do
3: Each vertex $v$ runs the following procedure:
4: if $v \in L$ then
5: $p(v) \leftarrow$ a value in $[0, 1]$ chosen uniformly at random
6: Send $p(v)$ message to neighbors
7: Send $m_v = \min_{y \in N(v) \cap L} p(y)$ message to neighbors
8: if $p(v) = \min_{y \in N(v)} m_y$ then
9: Add $v$ to $I$
10: Send ADDED message to neighbors
11: if $v \in W \cup B_{\text{high}}$ and $v$ receives ADDED then
12: Send NEIGHBOR ADDED message to neighbors
13: if $v$ receives NEIGHBOR ADDED and $v \in L$ then
14: Remove $v$ from $L$

We begin with a simple claim about the behavior of the partition of vertices over time:

Claim 4.3.

1. No vertex can ever enter $W$ from another set.
2. No vertex can ever move from $W_{\text{low}}$ to $W_{\text{high}}$.

Proof. The proof of item 1 is the same as in the proof of Claim 3.1. For item 2, it is impossible for a vertex to move from $W_{\text{low}}$ to $W_{\text{high}}$ since for all $v$ the quantity $N(v) \cap (W \cup B_{\text{high}})$ that determines membership in $W_{\text{low}}$ versus $W_{\text{high}}$, can only decrease over time (in Algorithm 2, this quantity is only decremented).

We begin with the following claim, which when combined with Claim 4.3 implies that each vertex only spends a limited number of rounds in $W_{\text{low}}$.

Claim 4.4. For every vertex $v$ that is ever in $W_{\text{low}}$, within $(\delta \alpha)^2$ iterations of the while loop after $v$ joins $W_{\text{low}}$, $v$ leaves $W$.

Proof. First we note that by Claim 4.3 no vertex can ever move from $W_{\text{low}}$ to $W_{\text{high}}$. Thus, if $v$ is in $W_{\text{low}}$, $v$ will remain in $W_{\text{low}}$ until $v$ leaves $W$. Suppose $v$ is in $W_{\text{low}}$ at the beginning of an iteration of the while loop. Because $I$ is an MIS with respect to $G_{\text{low}}$, if $v$ does not join $I$ during this iteration, then $v$ has a neighbor $y \in W \cup B_{\text{high}}$ such that a neighbor $z$ of $y$ joins $I$. As a result, $z$ immediately joins $D$ and $c_y$ is incremented. Thus, during every iteration that $v$ remains in $W_{\text{low}}$, a vertex in $N(v) \cap (W \cup B_{\text{high}})$ has its counter incremented. Recall that whenever a vertex has its counter incremented $\delta \alpha$ times, it joins $D$. Because $v \in W_{\text{low}}$, we have that $|N(v) \cap (W \cup B_{\text{high}})| \leq \delta \alpha$. Therefore, the event that a vertex in $N(v) \cap (W \cup B_{\text{high}})$ has its counter incremented can only happen at most $(\delta \alpha)^2$ times. Thus, $v$ can only remain in $W_{\text{low}}$ for $(\delta \alpha)^2$ iterations of the while loop.

We will complete the analysis using the fact that enough vertices are in $W_{\text{low}}$ at any given point in time. In particular, Claim 2.4 implies that at least half of the vertices in $W \cup B_{\text{high}}$ are in $W_{\text{low}}$. This implies that at least half of the vertices in $W$ are in $W_{\text{low}}$. Formally, we divide the execution of the algorithm into phases where each phase consists of $(\delta \alpha)^2$ iterations of the while loop. At the beginning of any phase, at least half
of the vertices in \( W \) are in \( W_{low} \). By the end of the phase, all of these vertices have left \( W \) by Claim 4.4. Therefore, each phase witnesses at least half of the vertices in \( W \) leaving \( W \). By Claim 4.3, no vertex can re-enter \( W \), so there can only be \( O(\log n) \) phases.

Putting everything together, there are \( O(\log n) \) phases, each consisting of \((\delta \alpha)^2\) iterations of the while loop, and one iteration of the while loop takes \( O(R(n)) \) rounds. Therefore, the total number of rounds is \( O(R(n) \cdot \alpha^2 \log n) \).

For Theorem 4.1, \( R(n) = O(\log n) \), so the number of rounds is \( O(\alpha^2 \log^2 n) \).

5 Faster Randomized Distributed \( O(\alpha) \)-approximation for MDS

In this section we will prove Theorem 1.3, which we recall:

**Theorem 1.3.** For graphs of arboricity \( \alpha \), there is a randomized distributed algorithm in the CONGEST model that gives an \( O(\alpha) \)-approximation for MDS and runs in \( O(\alpha \log n) \) rounds. The bound on the number of rounds holds with high probability (and in expectation). The algorithm works even when either \( \alpha \) or \( n \) is unknown to each processor.

We first present an algorithm that assumes that \( \alpha \) is known to each processor but \( n \) is unknown. In Section 5.3, we handle the case of unknown \( \alpha \).

5.1 Algorithm

**Overview** We use our \( O(\alpha^2 \log^2 n) \) round algorithm from Theorem 4.1 as a starting point (though the algorithm description and analysis are self-contained). Our goal is to shave both a \( \log n \) factor and an \( \alpha \) factor from the number of rounds. To do so, we use a combination of two key modifications, which respectively address the two factors that we wish to shave.

Our first key modification, which shaves a \( \log n \) factor from the number of rounds, is that instead of using a Luby-style algorithm as a black box, we open the box and run only one phase of a Luby-style algorithm at a time. Here, one phase means that each participating vertex \( v \) picks a single random value \( p(v) \) and enters \( D \) if \( p(v) \) is a local minimum. Between each such phase, we update the dominating set \( D \) as well as the information stored by each vertex. This way, we can embed the analysis of the Luby-style algorithm into our analysis instead of repeatedly paying for \( \log n \) phases of a black-box algorithm.

Our second key modification, which shaves an \( \alpha \) factor from the number of rounds, is that instead of adding \( v \) to \( D \) only when \( p(v) \) is the single local minimum, we allow \( v \) to be added to \( D \) when \( p(v) \) is an \( \alpha \)-minimum. The definition of an \( \alpha \)-minimum is slightly nuanced due to the fact that we need to be able to compute it using small messages, but it roughly means that \( p(v) \) is among the \( \alpha \) smallest values that it is compared to. Using this modification we can ensure that during each iteration of our algorithm each vertex only has its counter incremented by \( O(\alpha) \). Even though each vertex in our previous \( O(\alpha^2 \log^2 n) \)-round algorithm only had its counter incremented by at most 1 during each iteration, this change does not asymptotically increase the approximation factor.

The main technical part of the argument is the probabilistic analysis of the number of rounds. We would like to use an analysis similar to that of Luby’s algorithm, however there are a few obstacles. Recall that to analyze Luby’s algorithm, one can argue that after a single phase, a constant fraction of the edges in the graph are removed in expectation. Our first obstacle is that we are running a phase of a Luby-style algorithm on an auxiliary graph that is different from our original graph; in particular, an edge in the auxiliary graph can represent a 2-hop path in the original graph, and it is not clear how removing an edge from the auxiliary
graph enters counters (in the third round). Whenever the counter $c$ then that $\alpha$.

$\cup$ $\alpha$ $\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$

$\cup$ $\cup$ $\cup$$\cup$
Whenever a vertex \( v \) moves from one set of the partition to another, \( v \) notifies each vertex \( u \in N(v) \) so that \( u \) can update the quantities \( |N(u) \cap W| \) and \( |N(u) \cap (W \cup B_{high})| \), and move to the appropriate set. The bookkeeping for updating this information is identical to that of Algorithm 2. When no more vertices are left in \( W \), \( B_{high} \) is also empty, and all processors terminate. This concludes the description of the algorithm.

Algorithm 4 Faster Randomized Distributed \( O(\alpha) \)-approximation for MDS

1. Initialize partition: \( D \leftarrow \emptyset, B_{high} \leftarrow \emptyset, B_{low} \leftarrow \emptyset, W \leftarrow V, W_{low} \leftarrow \{ v \in W : |N(v) \cap (W \cup B_{high})| \leq \delta \alpha \} \), \( W_{high} \leftarrow W \setminus W_{low} \)
2. Initialize counters: \( \forall v \in V : c_v \leftarrow 0 \)
3. Initialize degrees: \( \forall v \in V : |N(v) \cap W| = \deg(v), |N(v) \cap (W \cup B_{high})| = \deg(v) \)
4. while \( W \neq \emptyset \) do
5. Each vertex \( v \) runs the following procedure:
6. if \( v \in W_{low} \) then
7. \( p(v) \leftarrow \) a value in \([0, 1]\) chosen uniformly at random
8. Send \( p(v) \) message to neighbors
9. if \( v \in B_{high} \cup W_{high} \) then
10. Send ACK to each \( u \in N_{G_{\delta \alpha}}(v) \) such that \( p(u) \) is among the \( \alpha \) smallest values \( v \) received
11. if \( v \in W_{low} \) and \( v \) receives ACK from all \( u \in N_{G_{\delta \alpha}}(u) \) then
12. Move \( v \) to \( D \)
13. Send INCREMENT COUNTER message to neighbors
14. Send MOVED FROM \( W \) TO \( D \) message to neighbors
15. Run Algorithm 2 starting from Line 11

5.2 Analysis

We begin with a simple claim about the behavior of the partition of vertices over time:

Claim 5.2.

1. No vertex can ever leave \( D \).
2. No vertex can ever enter \( W \cup B_{high} \) from another set.
3. No vertex in \( W_{low} \) can ever at a later point be in \( B_{high} \cup W_{high} \).

Proof. The proofs of items 1 and 2 follow from the proof of Claim 3.1. Item 3 holds because if a vertex \( v \) is in \( B_{high} \) then \( |N(v) \cap W| > \delta \alpha \) and if \( v \) is in \( W_{high} \) then \( |N(v) \cap (W \cup B_{high})| > \delta \alpha \). For any \( v \), the quantities \( |N(v) \cap W| \) and \( |N(v) \cap (W \cup B_{high})| \) can only decrease over time (they are only decremented in Algorithm 4).

Next, we prove a simple claim that upper bounds the counter of each vertex:

Claim 5.3. For all \( v \in V \), at all times \( c_v < 2 \delta \alpha \).

Proof. If for any \( v \in V \), it happens that \( c_v \geq \delta \alpha \), then during the same iteration of the while loop, \( v \) enters \( D \) (on Line 18), after which point \( v \) never leaves \( D \) (by Claim 5.2) so \( c_v \) cannot ever change again. Thus, it suffices to show that for all vertices \( v \in V \setminus D \), during a single iteration of the while loop, \( c_v \) can be incremented at most \( \delta \alpha \) times, leading to a maximum value of at most \( 2 \delta \alpha - 1 \). This is true for \( v \in W_{low} \).
because \(|N(v) \cap (W \cup B_{\text{high}})| \leq \delta \alpha\), and only vertices in \(|N(v) \cap (W \cup B_{\text{high}})|\) can tell \(v\) to increment \(c_v\). On the other hand, if \(v \in W \cup B_{\text{high}}\), then a neighbor \(u\) of \(v\) only sends \text{INCREMENT COUNTER} if \(p(u)\) is among the \(\alpha\) smallest values in \(v\)’s neighborhood, so \(c_v\) is only incremented \(\alpha\) times during a single iteration.

The analysis of correctness is the same as that of our centralized algorithm (see Section 3.2.1), with one technicality: By Claim 5.3, the counter of each vertex has maximum value \(2\delta \alpha\) instead of \(\delta \alpha\), causing an increase in the leading constant in the \(O(\alpha)\) approximation factor.

Our goal in the rest of this section is to prove that Algorithm 4 runs in \(O(\alpha \log n)\) rounds with high probability. Note that one iteration of the while loop takes a constant number of rounds. Thus, our goal is to show that there are \(O(\alpha \log n)\) total iterations of the loop.

For any vertex \(v \in V(G_{bi})\), let \(N^2(v)\) be the set of vertices in the 2-hop neighborhood of \(v\) with respect to \(G_{bi}\), and let \(E^2(v)\) be the set of edges within 2 hops of \(v\) with respect to \(G_{bi}\); that is, \(E^2(v)\) contains the edge \((v, u)\) for all \(u \in N_{G_{bi}}(v)\), and the edge \((u, y)\) for all \(y \in N_{G_{bi}}(u)\).

We divide the iterations of the while loop into two types. We say that an iteration is of type low degree if at least half of the vertices \(v \in W_{\text{low}}\) have \(|N^2(v)| \leq \alpha\). Otherwise, we say that an iteration is of type high degree.

It is simple to deterministically bound the number of iterations of type low degree:

**Claim 5.4.** The total number of iterations of type low degree is \(O(\log n)\).

**Proof.** By the specification of the algorithm, all vertices \(v\) with \(|N^2(v)| \leq \alpha\) are added to \(D\) (on Line 12). Thus, by definition, during an iteration of type low degree, at least half of the vertices in \(W_{\text{low}}\) enter \(D\). By Claim 2.4, at least half of the vertices in \(W \cup B_{\text{high}}\) are in \(W_{\text{low}}\). Thus, during an iteration of type low degree, at least 1/4 of the vertices in \(W \cup B_{\text{high}}\) enter \(D\). By Claim 5.2, no vertex can ever enter \(W \cup B_{\text{high}}\) from another set. Therefore, during every iteration of type low degree, \(W \cup B_{\text{high}}\) shrinks by a factor of at least 4. Thus, there are only \(O(\log n)\) iterations of type low degree.

It remains to bound the number of iterations of type high degree. Fix an iteration \(I\) of type high degree.

For the purpose of analysis, we assign each edge \(e \in E\) a weight \(w(e)\) that increases over the execution of the algorithm. The rule for updating the weight of edges is as follows. Whenever a vertex \(v \in W_{\text{low}}\) is moved to \(D\) on Line 12 (as a result of \(p(v)\) being an \(\alpha\)-minimum), \(v\) increments the weight of every edge in \(E^2(v)\).

We will define the available weight at iteration \(I\) as a function that will capture the total amount of weight that could ever be added over all iterations starting from iteration \(I\). Our goal is to provide:

1. an upper bound of \(2\delta \alpha^2 \cdot |V(G_{bi})|\) for the available weight at iteration \(I\), and
2. a lower bound of \(\alpha \cdot |V(G_{bi})|/4\) for the expected total weight added to edges during iteration \(I\).

Combining these upper and lower bounds yields the result that in each iteration the expected total amount of weight added is a \(1/O(\alpha)\) fraction of the total available weight. This allows us to bound the number of iterations by \(O(\alpha \log n)\) with high probability.

**Definition 5.5.** The available weight at iteration \(I\), denoted \(A(I)\) is given by

\[
A(I) = \sum_{e \in E(G[W \cup B_{\text{high}}])} 2\delta \alpha - w(e)
\]

where the parameters in the expression are taken to be their values at the beginning of iteration \(I\).
Claim 5.6. \( A(I) \leq 2\delta \alpha^2 \cdot |V(G_{bi})| \).

Proof. By definition all edge weights are non-negative, so \( A(I) \leq \sum_{v \in E(G[W \cup B_{high}])} 2\delta \alpha \). Since \( G[W \cup B_{high}] \) has arboricity at most \( \alpha \), \( |E(G[W \cup B_{high}])| \leq \alpha \cdot |W \cup B_{high}| = \alpha \cdot |V(G_{bi})| \). This completes the proof.

Now, we consider item 2 of our above goal. Let \( w(I) \) be a random variable denoting the aggregate total weight added to edges during iteration \( I \). The randomness is over the choice of \( p(v) \) for each vertex \( v \in W_{low} \). Towards lower bounding \( w(I) \), for every vertex \( v \in W_{low} \) we define the random variable \( R_v \) as the number of edges whose weight is incremented by \( v \) during iteration \( I \). That is, \( w(I) = \sum_{v \in W_{low}} R_v \).

We now calculate \( \mathbb{E}[R_v] \).

Claim 5.7. For all \( v \in W_{low} \) with \( |N^2(v)| \geq \alpha \), it holds that \( \mathbb{E}[R_v] \geq \alpha \).

Proof. Note that \( E^2(v) \) and \( N^2(v) \) are taken to mean these value at the beginning of iteration \( I \). A vertex \( v \in W_{low} \) increments the weight of each edge in \( E^2(v) \) if \( v \) is an \( \alpha \)-minimum, and otherwise \( v \) does not increment the weight of any edges. A sufficient condition for \( v \) to be an \( \alpha \)-minimum is that \( p(v) \) is among the \( \alpha \) smallest values in \( N^2(v) \). Since \( |N^2(v)| \geq \alpha \), the probability that \( p(v) \) is among the \( \alpha \) smallest values in \( N^2(v) \) is \( \alpha/|N^2(v)| \geq \alpha/|E^2(v)| \) since each vertex chooses its value uniformly at random. Therefore, \( \mathbb{P}[R_v = |E^2(v)|] \geq \alpha/|E^2(v)| \). Thus, \( \mathbb{E}[R_v] \geq \alpha \).

We now give a lower bound on the expected aggregate weight \( \mathbb{E}[w(I)] \):

Claim 5.8. \( \mathbb{E}[w(I)] \geq \alpha \cdot |V(G_{bi})|/4 \)

Proof. Recall that \( w(I) = \sum_{v \in W_{low}} R_v \). Thus,

\[
\mathbb{E}[w(I)] = \mathbb{E}\left[ \sum_{v \in W_{low}} R_v \right] \\
= \sum_{v \in W_{low}} \mathbb{E}[R_v] \\
\geq \sum_{v \in W_{low}, |N^2(v)| \geq \alpha} \mathbb{E}[R_v] \\
\geq \sum_{v \in W_{low}, |N^2(v)| \geq \alpha} \alpha \quad \text{(by Claim 5.7)} \\
\geq \alpha \cdot |W_{low}|/2 \quad \text{(since iteration \( I \) is of type high degree)} \\
\geq \alpha \cdot |B_{high} \cup W|/4 \quad \text{(by Claim 2.4)} \\
= \alpha \cdot |V(G_{bi})|/4.
\]

Before combining the above upper and lower bounds, we need to show that our function \( A(I) \) accurately measures the progress of our algorithm by proving the following properties:

Claim 5.9.
1. If $I'$ is the iteration right after $I$, then $A(I') \leq A(I) - w(I)$.

2. $A(I) > 0$.

Proof. For item 1, it suffices to observe that the weight of an edge can only increase and the set of edges we sum over in the definition of $A(I)$ can only decrease. This is true because by Claim 5.2 no vertex can ever enter $W \cup B_{high}$, from another set.

For item 2, it suffices to show that $2\delta\alpha - w(e)$ is always positive. Suppose for contradiction that there is an edge $(u, v)$ with $w(u, v) \geq 2\delta\alpha$. Consider the point at which $w(u, v)$ was incremented to $\delta\alpha$. Consider $G_{bi}$ at this point in time. Only the weight of edges in $G_{bi}$ can be incremented, so $(u, v) \in E(G_{bi})$. Without loss of generality, $u \in B_{high} \cup W_{high}$ and $v \in W_{low}$. Ever since the edge $(u, v)$ entered $G_{bi}$, $u$ has been in $B_{high} \cup W_{high}$ and $v$ has been in $W_{low}$, since no vertex in $W_{low}$ can later be in $B_{high} \cup W_{high}$ by Claim 5.2.

Each time $w(u, v)$ is incremented, it is caused by an INCREMENT COUNTER message sent by some vertex $y$ that moved from $W_{low}$ to $D$, for which $(u, v) \in E^2(y)$. Then since $y$ was in $W_{low}$, $u \in B_{high} \cup W_{high}$, and $(u, v) \in E^2(y)$, we have $y \in N(u)$. Thus $y$ sends INCREMENT COUNTER to $u$. Therefore, every time $w(u, v)$ is incremented, $c_u$ is also incremented. Since $w(u, v) = 2\delta\alpha$, we have $c_u = 2\delta\alpha$, which is a contradiction by Claim 5.3.

We are now ready to put everything together to complete the analysis. For all $j$, let $I_j$ denote the $j^{th}$ iteration of type high degree. Then, by Claim 5.6 we have that for all $j$, $A(I_j) \leq 2\delta\alpha^2 \cdot |V(G_{bi})|$, and by Claim 5.8 we have that for all $j$, $\mathbb{E}[w(I_j)] \geq \alpha \cdot |V(G_{bi})|/4$ (where $G_{bi}$ is taken to be its value at the beginning of iteration $I_j$). Thus, $\mathbb{E}[w(I_j)] \geq A(I_j)/(8\delta\alpha)$.

Thus, by item 1 of Claim 5.9, we have that if for all $j$, $A(I_{j+1}) \leq A(I_j) - w(I_j)$, so for all $j$, we have

$$\mathbb{E}[A(I_{j+1})] \leq \mathbb{E}[A(I_j) - w(I_j)]$$

$$\quad \leq (1 - \frac{1}{8\delta\alpha}) \mathbb{E}[A(I_j)]$$

$$\quad \leq (1 - \frac{1}{8\delta\alpha})^j \cdot A(I_1)$$

$$\quad = (1 - \frac{1}{8\delta\alpha})^j \cdot m(2\delta\alpha)$$

$$\quad = (1 - \frac{1}{32\alpha})^j \cdot 8\alpha^2 n.$$

We note that the above inequalities use the fact that the randomness in each iteration is independent.

For any constant positive integer $c$, by setting $j = 50c\alpha \log n$, we have $\mathbb{E}[A(I_{j+1})] < 1/n^c$. By Markov’s inequality and the fact that the available weight is always integral and non-negative, we have that $A(I_j) = 0$ with high probability. By item 2 of Claim 5.9 the algorithm terminates before it reaches an iteration $I^*$ with $A(I^*) = 0$. Thus, the number of iterations of type high degree is $O(\alpha \log n)$ with high probability.

### 5.3 Handling Unknown arboricity

All of our distributed algorithms so far have assumed that $\alpha$ is known to each processor but that $n$ is unknown. In this section we will show that all of our distributed algorithms (Theorem 1.3, Theorem 1.4 and Theorem 4.1) still work if the arboricity $\alpha$ is unknown to each processor, but $n$ is known.

Let $A$ be any one of our three distributed algorithms. Let $R^*_A(\alpha)$ be the upper bound on the number of rounds of algorithm $A$ for a graph on $n$ vertices of arboricity $\alpha$ when alpha is known, where this upper
bound is provided by Theorem 1.3, Theorem 1.4, or Theorem 4.1 depending on which algorithm \( \mathcal{A} \) refers to. We will run \( \mathcal{A} \) guessing at most \( \log n \) different values for \( \alpha \). For all \( i \) from 0 to \( \log n \), let \( \mathcal{A}_i \) be \( \mathcal{A} \) run with \( 2^i \) as the guessed value of \( \alpha \), where we stop after \( R^\alpha_\mathcal{A}(2^i) \) rounds. We note that each processor can compute the value of \( R^\alpha_\mathcal{A}(2^i) \) since \( n \) is known.

The algorithm is as follows. We do the following until every vertex is dominated (i.e., until \( W \) is empty). For each \( i \) from 0, 1, 2 \ldots in order, we run \( \mathcal{A}_i \). Importantly, we begin running \( \mathcal{A}_i \) with the conditions present at the end of \( \mathcal{A}_{i-1} \). That is, we begin \( \mathcal{A}_i \) with the partition into \( D \), \( B \), and \( W \) from the end of \( \mathcal{A}_{i-1} \), and the values of all counters \( c_v \) from the end of \( \mathcal{A}_{i-1} \).

We will now discuss the number of rounds. The goal is to show that the number of rounds is \( O(R^\alpha_\mathcal{A}(\alpha)) \). If the algorithm terminates right after \( \mathcal{A}_j \), then by definition the number of rounds is \( \sum_{i=0}^{j} R^\alpha_\mathcal{A}(2^i) \). Since \( R^\alpha_\mathcal{A} \) has at least linear dependence on its input for all of our algorithms \( \mathcal{A} \), this is a geometric series with sum \( O(R^\alpha_\mathcal{A}(2^j)) \). Thus, it suffices to show that the algorithm terminates (i.e., \( W \) becomes empty) by the time \( \mathcal{A}_{[\log \alpha]} \) terminates. This would be true by definition if \( \mathcal{A}_{[\log \alpha]} \) were run from scratch, without the initial conditions imposed by the previous \( \mathcal{A}_i \)s. When we do have these initial conditions however, we are instead running \( \mathcal{A}_{[\log \alpha]} \) on an instance of the problem where some vertices have already been added to \( D \). That is, \( \mathcal{A}_{[\log \alpha]} \) is given an initial partition into \( D \cup B \cup W \) and we only require \( \mathcal{A}_{[\log \alpha]} \) to add vertices in \( D \) to dominate the vertices in \( W \) (i.e., the vertices that are not yet dominated). In other words, we are running \( \mathcal{A}_{[\log \alpha]} \) on a graph where some progress is built into the initial conditions, so it is intuitive this algorithm satisfies the same running time guarantees as running \( \mathcal{A}_{[\log \alpha]} \) from scratch, but this needs to be formally verified. Indeed, one can verify that all of our claims in the running time analyses of our three algorithms still hold given this initial condition. That is, the entire argument for each of our algorithms can be applied verbatim, after replacing \( \alpha \) with \( 2^{[\log \alpha]} \). As a result, our algorithm terminates by the time \( \mathcal{A}_{[\log \alpha]} \) terminates.

We will now analyze the approximation factor. Let \( OPT \) be a minimum dominating set. For all \( i \), let \( W_i \) be the value of the set \( W \) right before executing \( \mathcal{A}_i \). We define \( OPT_i \) as the smallest set of vertices that dominates \( W_i \). By definition, for all \( i \) we have \( OPT_i \leq OPT \). Let \( D_i \) be the set of vertices added to \( D \) during the execution of \( \mathcal{A}_i \). Our goal is to show that for all \( i \), \( |D_i| = O(2^i \cdot |OPT_i|) \). The consequence of this is that since the algorithm terminates by the time \( \mathcal{A}_{[\log \alpha]} \) terminates, we have that \( D = O(\sum_{i=0}^{[\log \alpha]} 2^i \cdot |OPT_i|) = O\left( \frac{\alpha}{\log \alpha} \cdot |OPT| \right) \).

We will apply the analysis of correctness from our centralized algorithm (Section 3.2.1) separately for each \( \mathcal{A}_i \). Fix \( i \). The analysis of correctness of the centralized algorithm partitions \( D \) into two sets: \( D_{active} \) and \( D_{passive} \). Let \( D_{active} = D_i \cap D_{active} \) and let \( D_{passive} = D_i \cap D_{passive} \). The proof of correctness for our centralized algorithm has two claims which together imply that \( |D| = O(\alpha \cdot |OPT|) \); Claim 3.2 \([|D_{active}| \leq 2\delta \alpha \cdot |OPT|] \), and Claim 3.3 \([|D_{passive}| \leq |D_{active}|] \). Both of these claims remain true for the analysis of \( \mathcal{A}_i \) up to constant factors, after replacing \( D_{active} \) and \( D_{passive} \) with \( D_{active} \) and \( D_{passive} \) respectively, replacing \( |OPT| \) with \( |OPT_i| \), and replacing \( \alpha \) with \( 2^i \). After these replacements, the proof of Claim 3.2 is identical to its original proof (except if the algorithm \( \mathcal{A} \) is from Theorem 1.3 then each counter \( c_v \) is at most \( 2\delta \alpha \) instead of \( \delta \alpha \)). In particular, the original proof uses the fact that each vertex in \( D_{active} \) is adjacent to a vertex in \( OPT \), while the proof for \( \mathcal{A}_i \) uses the fact that each vertex in \( D_{active} \) is adjacent to a vertex in \( OPT_i \). The proof of Claim 3.3 for \( \mathcal{A}_i \) has a slightly different constant factor than the original. In particular, in the original proof of Claim 3.3 we show that every vertex in \( D_{passive} \) has at least \( \delta \alpha \) neighbors in \( D_{active} \), while every vertex in \( D_{active} \) has at most \( \delta \alpha \) neighbors in \( D_{passive} \). The latter is still true in \( \mathcal{A}_i \), but the bound for the former becomes \( \delta \alpha /2 \) since the counter of a vertex that reaches \( \delta \alpha \) could have reached at most \( \delta \alpha /2 \) during \( \mathcal{A}_1, \ldots, \mathcal{A}_{i-1} \), and thus, must have incremented at least \( \delta \alpha /2 \) times during \( \mathcal{A}_i \).
Acknowledgements  The authors would like to thank Yosi Hezi and Quanquan Liu for fruitful discussions. We would also like to thank Neal E. Young and Kent Quanrud for correspondence about their prior work.

References

[Ami21] Saeed Akhoondian Amiri. Deterministic congest algorithm for mds on bounded arboricity graphs. arXiv preprint arXiv:2102.08076, 2021.

[AMZ97] Srinivasa R Arikati, Anil Maheshwari, and Christos D Zaroliagis. Efficient computation of implicit representations of sparse graphs. Discrete Applied Mathematics, 78(1-3):1–16, 1997.

[ASS19] Saeed Akhoondian Amiri, Stefan Schmid, and Sebastian Siebertz. Distributed dominating set approximations beyond planar graphs. ACM Transactions on Algorithms (TALG), 15(3):1–18, 2019.

[AZO19] Zeyuan Allen-Zhu and Lorenzo Orecchia. Nearly linear-time packing and covering lp solvers. Mathematical Programming, 175(1):307–353, 2019.

[Bak94] Brenda S Baker. Approximation algorithms for np-complete problems on planar graphs. Journal of the ACM (JACM), 41(1):153–180, 1994.

[BCG20] Suman K Bera, Amit Chakrabarti, and Prantar Ghosh. Graph coloring via degeneracy in streaming and other space-conscious models. corr, abs/1905.00566. In ICALP, volume 2019, page 4, 2020.

[BE10] Leonid Barenboim and Michael Elkin. Sublogarithmic distributed mis algorithm for sparse graphs using nash-williams decomposition. Distributed Computing, 22(5-6):363–379, 2010.

[BF99] Gerth Stølting Brodal and Rolf Fagerberg. Dynamic representation of sparse graphs. In Frank K. H. A. Dehne, Arvind Gupta, Jörg-Rüdiger Sack, and Roberto Tamassia, editors, Algorithms and Data Structures, 6th International Workshop, WADS ’99, Vancouver, British Columbia, Canada, August 11-14, 1999, Proceedings, volume 1663 of Lecture Notes in Computer Science, pages 342–351. Springer, 1999.

[BPS20] Suman K. Bera, Noujan Pashanasangi, and C. Seshadhri. Linear time subgraph counting, graph degeneracy, and the chasm at size six. In Thomas Vidick, editor, 11th Innovations in Theoretical Computer Science Conference, ITCS 2020, January 12-14, 2020, Seattle, Washington, USA, volume 151 of LIPIcs, pages 38:1–38:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[BS20] Suman K. Bera and C. Seshadhri. How the degeneracy helps for triangle counting in graph streams. In Dan Suciu, Yufei Tao, and Zhewei Wei, editors, Proceedings of the 39th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2020, Portland, OR, USA, June 14-19, 2020, pages 457–467. ACM, 2020.

[BU17] Nikhil Bansal and Seeun William Umboh. Tight approximation bounds for dominating set on graphs of bounded arboricity. Information Processing Letters, 122:21–24, 2017.
[CC08] Miroslav Chlebík and Janka Chlebíková. Approximation hardness of dominating set problems in bounded degree graphs. Information and Computation, 206(11):1264–1275, 2008.

[Cha00] Bernard Chazelle. A minimum spanning tree algorithm with inverse-ackermann type complexity. Journal of the ACM (JACM), 47(6):1028–1047, 2000.

[CHS09] Andrzej Czygrinow, Michał Hańćkowiak, and Edyta Szymańska. Fast distributed approximation algorithm for the maximum matching problem in bounded arboricity graphs. In International Symposium on Algorithms and Computation, pages 668–678. Springer, 2009.

[CHW08] Andrzej Czygrinow, Michał Hańćkowiak, and Wojciech Wawrzyniak. Fast distributed approximations in planar graphs. In International Symposium on Distributed Computing, pages 78–92. Springer, 2008.

[CN85] Norishige Chiba and Takao Nishizeki. Arboricity and subgraph listing algorithms. SIAM Journal on computing, 14(1):210–223, 1985.

[DGKR05] Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. A new multilayered PCP and the hardness of hypergraph vertex cover. SIAM J. Comput., 34(5):1129–1146, 2005.

[DKM19] Janosch Deurer, Fabian Kuhn, and Yannic Maus. Deterministic distributed dominating set approximation in the congest model. In Proceedings of the 2019 ACM Symposium on Principles of Distributed Computing, pages 94–103, 2019.

[DS14] Irit Dinur and David Steurer. Analytical approach to parallel repetition. In Proceedings of the forty-sixth annual ACM symposium on Theory of computing, pages 624–633, 2014.

[EGM18] Guy Even, Mohsen Ghaffari, and Moti Medina. Distributed set cover approximation: Primal-dual with optimal locality. In 32nd International Symposium on Distributed Computing (DISC 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[ELR18] Talya Eden, Reut Levi, and Dana Ron. Testing bounded arboricity. In Artur Czumaj, editor, Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 2081–2092. SIAM, 2018.

[Epp94] David Eppstein. Arboricity and bipartite subgraph listing algorithms. Information processing letters, 51(4):207–211, 1994.

[ERR19] Talya Eden, Dana Ron, and Will Rosenbaum. The arboricity captures the complexity of sampling edges. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, 46th International Colloquium on Automata, Languages, and Programming, ICALP 2019, July 9-12, 2019, Patras, Greece, volume 132 of LIPIcs, pages 52:1–52:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.

[ERS20] Talya Eden, Dana Ron, and C. Seshadhri. Faster sublinear approximation of the number of $k$-cliques in low-arboricity graphs. In Shuchi Chawla, editor, Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms, SODA 2020, Salt Lake City, UT, USA, January 5-8, 2020, pages 1467–1478. SIAM, 2020.
Michael Fredman and Michael Saks. The cell probe complexity of dynamic data structures. In Proceedings of the twenty-first annual ACM symposium on Theory of computing, pages 345–354, 1989.

Michael L Fredman and Dan E Willard. Trans-dichotomous algorithms for minimum spanning trees and shortest paths. In Proceedings [1990] 31st Annual Symposium on Foundations of Computer Science, pages 719–725. IEEE, 1990.

Michael R Garey. A guide to the theory of np-completeness. Computers and intractability, 1979.

Gaurav Goel and Jens Gustedt. Bounded arboricity to determine the local structure of sparse graphs. In International Workshop on Graph-Theoretic Concepts in Computer Science, pages 159–167. Springer, 2006.

Mohsen Ghaffari, Christoph Grunau, and Václav Rozhoň. Improved deterministic network decomposition. In Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 2904–2923. SIAM, 2021.

Mohsen Ghaffari and Fabian Kuhn. Derandomizing distributed algorithms with small messages: Spanners and dominating set. In 32nd International Symposium on Distributed Computing (DISC 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

Mohsen Ghaffari, Fabian Kuhn, and Yannic Maus. On the complexity of local distributed graph problems. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 784–797, 2017.

Mohsen Ghaffari and Hsin-Hao Su. Distributed degree splitting, edge coloring, and orientations. In Philip N. Klein, editor, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 2505–2523. SIAM, 2017.

Meng He, Ganggui Tang, and Norbert Zeh. Orienting dynamic graphs, with applications to maximal matchings and adjacency queries. In Hee-Kap Ahn and Chan-Su Shin, editors, Algorithms and Computation - 25th International Symposium, ISAAC 2014, Jeonju, Korea, December 15-17, 2014, Proceedings, volume 8889 of Lecture Notes in Computer Science, pages 128–140. Springer, 2014.

Mark Jones, Daniel Lokshtanov, MS Ramanujan, Saket Saurabh, and Ondřej Suchý. Parameterized complexity of directed steiner tree on sparse graphs. In European Symposium on Algorithms, pages 671–682. Springer, 2013.

David S Johnson. Approximation algorithms for combinatorial problems. Journal of computer and system sciences, 9(3):256–278, 1974.

Lujun Jia, Rajmohan Rajaraman, and Torsten Suel. An efficient distributed algorithm for constructing small dominating sets. Distributed Computing, 15(4):193–205, 2002.

David R Karger, Philip N Klein, and Robert E Tarjan. A randomized linear-time algorithm to find minimum spanning trees. Journal of the ACM (JACM), 42(2):321–328, 1995.
[KMW06] Fabian Kuhn, Thomas Moscibroda, and Roger Wattenhofer. The price of being near-sighted. In SODA, volume 6, pages 1109557–1109666. Citeseer, 2006.

[KMW16] Fabian Kuhn, Thomas Moscibroda, and Roger Wattenhofer. Local computation: Lower and upper bounds. Journal of the ACM (JACM), 63(2):1–44, 2016.

[KS21] Haim Kaplan and Shay Solomon. Dynamic representations of sparse distributed networks: A locality-sensitive approach. ACM Transactions on Parallel Computing (TOPC), 8(1):1–26, 2021.

[KW05] Fabian Kuhn and Roger Wattenhofer. Constant-time distributed dominating set approximation. Distributed Computing, 17(4):303–310, 2005.

[KY14] Christos Koufogiannakis and Neal E. Young. A nearly linear-time ptas for explicit fractional packing and covering linear programs. Algorithmica, 70(4):648–674, 2014.

[Lin87] Nathan Linial. Distributive graph algorithms global solutions from local data. In 28th Annual Symposium on Foundations of Computer Science (sfcs 1987), pages 331–335. IEEE, 1987.

[Lin92] Nathan Linial. Locality in distributed graph algorithms. SIAM Journal on computing, 21(1):193–201, 1992.

[LW10] Christoph Lenzen and Roger Wattenhofer. Minimum dominating set approximation in graphs of bounded arboricity. In Nancy A. Lynch and Alexander A. Shvartsman, editors, Distributed Computing, 24th International Symposium, DISC 2010, Cambridge, MA, USA, September 13-15, 2010. Proceedings, volume 6343 of Lecture Notes in Computer Science, pages 510–524. Springer, 2010.

[MV18] Andrew McGregor and Sofya Vorotnikova. A simple, space-efficient, streaming algorithm for matchings in low arboricity graphs. In Raimund Seidel, editor, 1st Symposium on Simplicity in Algorithms, SOSA 2018, January 7-10, 2018, New Orleans, LA, USA, volume 61 of OASICS, pages 14:1–14:4. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[NA16] Jose C Nacher and Tatsuya Akutsu. Minimum dominating set-based methods for analyzing biological networks. Methods, 102:57–63, 2016.

[OSSW18] Krzysztof Onak, Baruch Schieber, Shay Solomon, and Nicole Wein. Fully dynamic MIS in uniformly sparse graphs. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic, volume 107 of LIPIcs, pages 92:1–92:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018.

[Pe100] David Peleg. Distributed computing: a locality-sensitive approach. SIAM, 2000.

[PPS16] Merav Parter, David Peleg, and Shay Solomon. Local-on-average distributed tasks. In Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, pages 220–239. SIAM, 2016.
[PS16] David Peleg and Shay Solomon. Dynamic (1+\(\epsilon\))-approximate matchings: A density-sensitive approach. In Robert Krauthgamer, editor, Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016, pages 712–729. SIAM, 2016.

[Qua20] Kent Quanrud. Nearly linear time approximations for mixed packing and covering problems without data structures or randomization. In Symposium on Simplicity in Algorithms, pages 69–80. SIAM, 2020.

[RG20] Václav Rozhoň and Mohsen Ghaffari. Polylogarithmic-time deterministic network decomposition and distributed derandomization. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, pages 350–363, 2020.

[SL10] Chao Shen and Tao Li. Multi-document summarization via the minimum dominating set. In Proceedings of the 23rd International Conference on Computational Linguistics (Coling 2010), pages 984–992, 2010.

[SV20] Hsin-Hao Su and Hoa T. Vu. Distributed dense subgraph detection and low outdegree orientation. In Hagit Attiya, editor, 34th International Symposium on Distributed Computing, DISC 2020, October 12-16, 2020, Virtual Conference, volume 179 of LIPIcs, pages 15:1–15:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.

[SW18] Shay Solomon and Nicole Wein. Improved dynamic graph coloring. In 26th Annual European Symposium on Algorithms (ESA 2018). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[Tar75] Robert Endre Tarjan. Efficiency of a good but not linear set union algorithm. Journal of the ACM (JACM), 22(2):215–225, 1975.

[WAF02] Peng-Jun Wan, Khaled M Alzoubi, and Ophir Frieder. Distributed construction of connected dominating set in wireless ad hoc networks. In Proceedings. Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies, volume 3, pages 1597–1604. IEEE, 2002.

[You14] Neal E Young. Nearly linear-work algorithms for mixed packing/covering and facility-location linear programs. arXiv preprint arXiv:1407.3015, 2014.