Some Growth Analysis of Entire Functions in the Form of Vector Valued Dirichlet Series on the Basis of Their $(p, q)$-th Relative Ritt Order and $(p, q)$-th Relative Ritt Type

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Abstract. In this paper we wish to study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of $(p, q)$-th relative Ritt order, $(p, q)$-th relative Ritt type and $(p, q)$-th relative Ritt weak type where $p \geq 0$ and $q \geq 0$.

1. Introduction and Definitions

Suppose $f(s)$ be an entire function of the complex variable $s = \sigma + it$ ($\sigma$ and $t$ are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series briefly known as VVDS

\[ f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \]  

(1.1)

where $a_n$'s belong to a Banach space $(E, \| \cdot \|)$ and $\lambda_n$'s are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}$ ($n \geq 1$), $\lambda_n \to +\infty$ as $n \to +\infty$ and satisfy the conditions $\lim_{n \to +\infty} \frac{\log n}{\lambda_n} = D < +\infty$ and $\lim_{n \to +\infty} \frac{\log \| a_n \|}{\lambda_n} = -\infty$. If $\sigma_c$ and $\sigma_a$ denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a = \sigma_c = +\infty$. The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1.1), is written as follows

\[ M_f(\sigma) = \sup_{-\infty < t < +\infty} \| f(\sigma + it) \| . \]

Now we state the following two notations which are frequently use in our subsequent study:

\[ \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \cdots ; \]

\[ \log^{[0]} x = x, \log^{-1} x = \exp x. \]

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and
\[
\begin{align*}
\exp^{[k]} x &= \exp \left( \exp^{[k-1]} x \right) \text{ for } k = 1, 2, 3, \ldots; \\
\exp^{[0]} x &= x, \exp^{[-1]} x = \log x.
\end{align*}
\]

Juneja, Nandan and Kapoor [4] first introduced the concept of \((p, q)\)-th order and \((p, q)\)-th lower order of an entire Dirichlet series where \(p \geq q + 1 \geq 1\). In the line of Juneja et al. [4], one can define the \((p, q)\)-th Ritt order and \((p, q)\)-th Ritt lower order of an entire function \(f\) represented by VVDS in the following way:

\[
\rho_f (p, q) = \lim_{\sigma \to +\infty} \sup_{\sigma \to +\infty} \frac{\log^{[p]} M_f (\sigma)}{\log^{[q]} \sigma},
\]

\[
\lambda_f (p, q) = \lim_{r \to +\infty} \sup_{r \to +\infty} \frac{\log^{[p]} \sigma}{\log^{[q]} M_f^{-1} (\sigma)}.
\]

where \(p \geq q + 1 \geq 1\).

In this connection let us recall that if \(0 < \rho_f (p, q) < \infty\), then the following properties hold

\[
\rho_f (p - n, q) = \infty \text{ for } n < p, \quad \rho_f (p, q - n) = 0 \text{ for } n < q, \quad \text{and}
\]

\[
\rho_f (p + n, q + n) = 1 \text{ for } n = 1, 2, \ldots
\]

Similarly for \(0 < \lambda_f (p, q) < \infty\), one can easily verify that

\[
\lambda_f (p - n, q) = \infty \text{ for } n < p, \quad \lambda_f (p, q - n) = 0 \text{ for } n < q, \quad \text{and}
\]

\[
\lambda_f (p + n, q + n) = 1 \text{ for } n = 1, 2, \ldots
\]

Recalling that for any pair of integer numbers \(m, n\) the Kroenecker function is defined by \(\delta_{m,n} = 1\) for \(m = n\) and \(\delta_{m,n} = 0\) for \(m \neq n\), the aforementioned properties provide the following definition.

**Definition 1.** An entire function \(f\) represented by VVDS is said to have index-pair \((1, 1)\) if \(0 < \rho_f (1, 1) < \infty\). Otherwise, \(f\) is said to have index-pair \((p, q) \neq (1, 1)\), \(p \geq q + 1 \geq 1\), if \(\delta_{p-q,0} < \rho_f (p, q) < \infty\) and \(\rho_f (p - 1, q - 1) \notin \mathbb{R}^+\).

**Definition 2.** An entire function \(f\) represented by VVDS is said to have lower index-pair \((1, 1)\) if \(0 < \lambda_f (1, 1) < \infty\). Otherwise, \(f\) is said to have lower index-pair \((p, q) \neq (1, 1)\), \(p \geq q + 1 \geq 1\), if \(\delta_{p-q,0} < \lambda_f (p, q) < \infty\) and \(\lambda_f (p - 1, q - 1) \notin \mathbb{R}^+\).

An entire function \(f\) (represented by VVDS) of index-pair \((p, q)\) is said to be of regular \((p, q)\)-Ritt growth if its \((p, q)\)-th Ritt order coincides with its \((p, q)\)-th Ritt lower order, otherwise \(f\) is said to be of irregular \((p, q)\)-Ritt growth.

Now to compare the relative growth of two entire functions represented by VVDS having same non zero finite \((p, q)\)-th Ritt order, one may introduce the definition of \((p, q)\)-th Ritt type (resp. \((p, q)\)-th Ritt lower type) in the following manner:

**Definition 3.** The \((p, q)\)-th Ritt type and \((p, q)\)-th Ritt lower type respectively denoted by \(\Delta_f (p, q)\) and \(\Delta_f (p, q)\) of an entire function \(f\) represented by VVDS when \(0 < \rho_f (p, q) < +\infty\) are defined as follows:

\[
\Delta_f (p, q) = \lim_{\sigma \to +\infty} \sup_{\sigma \to +\infty} \frac{\log^{[p-1]} M_f (\sigma)}{\log^{[q-1]} \sigma^{\rho_f (p, q)}}.
\]
where $p \geq q + 1 \geq 1$.

Analogously to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same non zero finite $(p, q)$-th Ritt lower order, one may introduce the definition of $(p, q)$-th Ritt weak type in the following way:

**Definition 4.** The $(p, q)$-th Ritt weak type denoted by $\tau_f (p, q)$ of an entire function $f$ represented by VVDS is defined as follows:

$$
\tau_f (p, q) = \lim_{\sigma \to +\infty} \frac{\log^{[p-1]} M_f (\sigma)}{\log^{[q-1]} \sigma}^{\lambda_f (p, q)}, \quad 0 < \lambda_f (p, q) < +\infty .
$$

Also one may define the growth indicator $\overline{\tau}_f (p, q)$ of an entire function $f$ represented by VVDS in the following manner:

$$
\overline{\tau}_f (p, q) = \lim_{\sigma \to +\infty} \frac{\log^{[p-1]} M_f (\sigma)}{\log^{[q-1]} \sigma}^{\lambda_f (p, q)}, \quad 0 < \lambda_f (p, q) < +\infty,
$$

where $p \geq q + 1 \geq 1$.

The above definitions are extended the generalized Ritt growth indicators of an entire function $f$ represented by VVDS for each integer $p \geq 2$ and $q = 0$. Also for $p = 2$ and $q = 0$, the above definitions reduces to the classical definitions of an entire function $f$ represented by VVDS.

G. S. Srivastava [8] introduced the relative Ritt order between two entire functions represented by VVDS to avoid comparing growth just with $\exp \exp z$. In the case of relative Ritt order, it therefore seems reasonable to define suitably the $(p, q)$-th relative Ritt order of entire function represented by VVDS. Recently, Datta and Biswas [3] introduce the concept of $(p, q)$-th relative Ritt order $\rho^{(p,q)}_g (f)$ of an entire function $f$ represented by VVDS with respect to another entire function $g$ which is also represented by VVDS, in the light of index-pair which is as follows:

**Definition 5.** [3] Let $f$ and $g$ be any two entire functions represented by VVDS with index-pair $(m, q)$ and $(m, p)$, respectively, where $p, q, m$ are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then the $(p, q)$-th relative Ritt order and $(p, q)$-th relative Ritt lower order of $f$ with respect to $g$ are defined as

$$
\rho^{(p,q)}_g (f) = \lim_{\sigma \to +\infty} \sup \frac{\log^{[p]} M_g^{-1} M_f (\sigma)}{\log^{[q]} \sigma} = \lim_{r \to +\infty} \inf \frac{\log^{[p]} M_g^{-1} (\sigma)}{\log^{[q]} M_f^{-1}(\sigma)} .
$$

In this connection, we intend to give a definition of relative index-pair of an entire function with respect to another entire function (both of which represented by VVDS) which is relevant in the sequel:

**Definition 6.** Let $f$ and $g$ be any two entire functions both represented by VVDS with index-pairs $(m, q)$ and $(m, p)$ respectively where $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then the entire function $f$ is said to have relative index-pair $(p, q)$ with respect to another
Let $f$ and $g$ be any two entire functions represented by VVDS with index-pair $(m, q)$ and $(m, p)$, respectively, where $p, q, m$ are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$ and $0 < \rho_g^{(p,q)}(f) < \infty$. Then the $(p,q)$-th relative Ritt type and $(p,q)$-th relative Ritt lower order of $f$ with respect to $g$ are defined as

$$
\Delta_g^{(p,q)}(f) = \lim_{\sigma \to +\infty} \sup \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\inf \left[ \log^{[q-1]} \sigma \right]^{\rho_g^{(p,q)}(f)}}.
$$

Analogously to determine the relative growth of two entire functions represented by VVDS having same non zero finite $p, q$-th relative Ritt lower order with respect to another entire function represented by VVDS, one may introduce the definition of $(p,q)$-th relative Ritt weak type in the following way:

**Definition 8.** Let $f$ and $g$ be any two entire functions represented by VVDS with index-pair $(m, q)$ and $(m, p)$, respectively, where $p, q, m$ are positive integers such that $m \geq q + 1 \geq 1$ and $m \geq p + 1 \geq 1$. Then $(p,q)$-th relative Ritt weak type denoted by $\tau_g^{(p,q)}(f)$ of an entire function $f$ with respect to another entire function $g$ is defined as follows:

$$
\tau_g^{(p,q)}(f) = \lim_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(\sigma)}{\left[ \log^{[q-1]} \sigma \right]^{\lambda_g^{(p,q)}(f)}}, \quad 0 < \lambda_g^{(p,q)}(f) < +\infty.
$$
Similarly the growth indicator \( \tau_{g}^{(p,q)} (f) \) of an entire function \( f \) with respect to another entire function \( g \) both represented by \textit{VVDS} in the following manner:

\[
\tau_{g}^{(p,q)} (f) = \lim_{\sigma \to +\infty} \frac{\log^{[p-1]} M_{g}^{-1} M_{f} (\sigma)}{\log^{[q-1]} \lambda_{g}^{(p,q)} (f)} = 0 < \lambda_{g}^{(p,q)} (f) < +\infty.
\]

If \( f \) and \( g \) (both \( f \) and \( g \) are represented by \textit{VVDS}) have got index-pair \((m,0)\) and \((m,l)\), respectively, then Definition 5 and Definition 8 reduce to the definition of \textit{generalized relative Ritt growth indicators}, such as \( \rho_{g}^{[m]} (f) \), \textit{generalized relative Ritt type} \( \Delta_{g}^{[m]} (f) \) etc. If the entire functions \( f \) and \( g \) (both \( f \) and \( g \) are represented by \textit{VVDS}) have the same index-pair \((p,0)\) where \( p \) is any positive integer, we get the definitions of \textit{relative Ritt growth indicators} such as \( \rho_{g} (f) \), \textit{relative Ritt type} \( \Delta_{g} (f) \) etc. introduced by Srivastava [8] and Datta et al. [1]. Further if \( g = \exp^{[m]} z \), then Definition 5, Definition 7 and Definition 8 reduce to the \((m,q)\)th \textit{relative Ritt growth indicators} of an entire function \( f \) represented by \textit{VVDS}. Also for \( g = \exp^{[m]} z \), \textit{relative Ritt growth indicators} reduces to the definition of \textit{generalized Ritt growth indicators}, such as \( \rho_{g}^{[m]} (f) \), \textit{generalized Ritt type} \( \Delta_{g}^{[m]} (f) \) etc. Moreover, if \( f \) is an entire function with index-pair \((2,0)\) and \( g = \exp^{[2]} z \), then Definition 5, Definition 7 and Definition 8 becomes the classical definitions of \( f \) represented by \textit{VVDS}.

During the past decades, several authors \{cf. [1], [2], [5], [6], [7], [9], [10], [11]\} made closed investigations on the properties of entire \textit{Dirichlet series} in different directions using the \textit{growth indicator} as \textit{relative Ritt order}. In the present paper we wish to establish some basic properties of entire functions represented by a \textit{VVDS} on the basis of \((p,q)\)-th \textit{relative Ritt order}, \((p,q)\)-th \textit{relative Ritt type} and \((p,q)\)-th \textit{relative Ritt weak type} where \( p \geq 0 \) and \( q \geq 0 \).

2. Main Results

In this section we state the main results of the paper.

**Theorem 1.** Let \( f \), \( g \) and \( h \) be any three entire functions represented by \textit{vector valued Dirichlet series} and \( p \geq 0 \), \( q \geq 0 \) and \( m \geq 0 \). If \((m,q)\)-th \textit{relative Ritt order} \((m,q)\)-th \textit{relative Ritt lower order} of \( f \) with respect to \( h \) and \((m,p)\)-th \textit{relative Ritt order} \((m,p)\)-th \textit{relative Ritt lower order} of \( g \) with respect to \( h \) are respectively denoted by \( \rho_{h}^{(m,q)} (f) \) \((\text{resp.} \lambda_{h}^{(m,q)} (f)) \) and \( \rho_{h}^{(m,p)} (g) \) \((\text{resp.} \lambda_{h}^{(m,p)} (g)) \), then

\[
\frac{\lambda_{h}^{(m,q)} (f)}{\rho_{h}^{(m,p)} (g)} \leq \frac{\lambda_{g}^{(p,q)} (f)}{\rho_{h}^{(m,p)} (g)} \leq \min \left\{ \frac{\lambda_{h}^{(m,q)} (f)}{\rho_{h}^{(m,p)} (g)} : \frac{\rho_{h}^{(m,q)} (f)}{\rho_{h}^{(m,p)} (g)} \right\} \leq \max \left\{ \frac{\lambda_{h}^{(m,q)} (f)}{\rho_{h}^{(m,p)} (g)} : \frac{\rho_{h}^{(m,q)} (f)}{\rho_{h}^{(m,p)} (g)} \right\} \leq \frac{\rho_{g}^{(p,q)} (f)}{\lambda_{h}^{(m,p)} (g)} \leq \frac{\rho_{g}^{(m,q)} (f)}{\lambda_{h}^{(m,p)} (g)}.
\]

**Proof.** From the definitions of \((p,q)\)-th \textit{growth indicators} \( \rho_{g}^{(p,q)} (f) \) and \((p,q)\)-th \textit{growth indicators} \( \lambda_{g}^{(p,q)} (f) \) we get that
\[
\log \rho^{(p,q)}_g (f) = \lim_{\sigma \to +\infty} \left[ \log^{[p+1]} M^{[-1]}_q (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right], \quad (2.1)
\]
\[
\log \lambda^{(p,q)}_g (f) = \lim_{\sigma \to +\infty} \left[ \log^{[p+1]} M^{[-1]}_q (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right]. \quad (2.2)
\]
Now from the definitions of \( \rho^{(m,q)}_h (f) \) and \( \lambda^{(m,q)}_h (f) \), it follows that
\[
\log \rho^{(m,q)}_h (f) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right], \quad (2.3)
\]
\[
\log \lambda^{(m,q)}_h (f) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right]. \quad (2.4)
\]
Similarly, from the definitions of \( \rho^{(m,p)}_h (g) \) and \( \lambda^{(m,p)}_h (g) \), we obtain that
\[
\log \rho^{(m,p)}_h (g) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right], \quad (2.5)
\]
\[
\log \lambda^{(m,p)}_h (g) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right]. \quad (2.6)
\]
Therefore from (2.2), (2.4) and (2.5), we get that
\[
\log \lambda^{(p,q)}_g (f) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma)
\]
\[
- \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right) \right]
\]
i.e., \( \log \lambda^{(p,q)}_g (f) \geq \left[ \lim_{\sigma \to +\infty} \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right)
\]
\[
- \lim_{\sigma \to +\infty} \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right) \right]
\]
i.e., \( \log \lambda^{(p,q)}_g (f) \geq \left( \log \lambda^{(m,q)}_h (f) - \log \rho^{(m,p)}_h (g) \right) \). \quad (2.7)

Similarly, from (2.1), (2.3) and (2.4), it follows that
\[
\log \rho^{(p,q)}_g (f) = \lim_{\sigma \to +\infty} \left[ \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma)
\]
\[
- \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right) \right]
\]
i.e., \( \log \rho^{(p,q)}_g (f) \leq \left[ \lim_{\sigma \to +\infty} \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[q+1]} M^{[-1]}_f (\sigma) \right)
\]
\[
- \lim_{\sigma \to +\infty} \left( \log^{[m+1]} M^{[-1]}_h (\sigma) - \log^{[p+1]} M^{[-1]}_g (\sigma) \right) \right]
\]
i.e., \( \log \rho^{(p,q)}_g (f) \leq \left( \log \rho^{(m,q)}_h (f) - \log \lambda^{(m,p)}_h (g) \right) \). \quad (2.8)
Again, in view of (2.2) we obtain that
\[
\log \lambda_{g}^{(p,q)}(f) = \lim_{\sigma \to \infty} \left[ \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[q+1]} M_{f}^{-1}(\sigma) \right.
\]
\[
\left. - \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[p+1]} M_{g}^{-1}(\sigma) \right) \right]
\]
By taking \( A = \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[q+1]} M_{f}^{-1}(\sigma) \right) \) and \( B = \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[p+1]} M_{g}^{-1}(\sigma) \right) \), we get from above that
\[
\log \lambda_{g}^{(p,q)}(f) \leq \min \left[ \lim_{\sigma \to \infty} A + \lim_{\sigma \to \infty} B, \lim_{\sigma \to \infty} A + \lim_{\sigma \to \infty} B \right]
\]
i.e., \log \lambda_{g}^{(p,q)}(f) \leq \min \left[ \lim_{\sigma \to \infty} A - \lim_{\sigma \to \infty} B, \lim_{\sigma \to \infty} A - \lim_{\sigma \to \infty} B \right].
\]
Therefore in view of (2.3), (2.4), (2.5) and (2.6) we get from above that
\[
\log \lambda_{g}^{(p,q)}(f) \leq \min \left\{ \log \lambda_{h}^{(m,q)}(f) - \log \lambda_{h}^{(m,p)}(g), \log \rho_{h}^{(m,q)}(f) - \log \rho_{h}^{(m,p)}(g) \right\}.
\]
Further from (2.1) it follows that
\[
\log \rho_{g}^{(p,q)}(f) = \lim_{\sigma \to \infty} \left[ \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[q+1]} M_{f}^{-1}(\sigma) \right.
\]
\[
\left. - \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[p+1]} M_{g}^{-1}(\sigma) \right) \right]
\]
By taking \( A = \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[q+1]} M_{f}^{-1}(\sigma) \right) \) and \( B = \left( \log^{[m+1]} M_{h}^{-1}(\sigma) - \log^{[p+1]} M_{g}^{-1}(\sigma) \right) \), we obtain from above that
\[
\log \rho_{g}^{(p,q)}(f) \geq \max \left[ \lim_{\sigma \to \infty} A + \lim_{\sigma \to \infty} B, \lim_{\sigma \to \infty} A + \lim_{\sigma \to \infty} B \right]
\]
i.e., \log \rho_{g}^{(p,q)}(f) \geq \max \left[ \lim_{\sigma \to \infty} A - \lim_{\sigma \to \infty} B, \lim_{\sigma \to \infty} A - \lim_{\sigma \to \infty} B \right].
\]
Therefore in view of (2.3), (2.4), (2.5) and (2.6), it follows from above that
\[
\log \rho_{g}^{(p,q)}(f) \geq \max \left\{ \log \lambda_{h}^{(m,q)}(f) - \log \lambda_{h}^{(m,p)}(g), \log \rho_{h}^{(m,q)}(f) - \log \rho_{h}^{(m,p)}(g) \right\}.
\]
Thus the theorem follows from (2.7), (2.8), (2.9) and (2.10).
\]

In view of Theorem 1, one can easily verify the following corollaries:

**Corollary 1.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) be an entire function with regular relative \((m,q)\)-Ritt growth with respect to entire function \( h \) and \( g \) be entire having relative index-pair \((m,p)\) with respect to another entire function \( h \) where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\lambda_{g}^{(p,q)}(f) = \frac{\rho_{h}^{(m,q)}(f)}{\rho_{h}^{(m,p)}(g)} \quad \text{and} \quad \rho_{g}^{(p,q)}(f) = \frac{\rho_{h}^{(m,q)}(f)}{\lambda_{h}^{(m,p)}(g)}.
\]
In addition, if \( \rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g) \), then
\[
\lambda_y^{(p,q)}(f) = \rho_f^{(q,p)}(g) = 1.
\]

**Corollary 2.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) be an entire function with relative index-pair \((m,q)\) with respect to entire function \( h \) and \( g \) be entire of regular relative \((m,p)\)-Ritt growth with respect to another entire function \( h \) where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\lambda_y^{(p,q)}(f) = \frac{\lambda^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_y^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.
\]

In addition, if \( \rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g) \), then
\[
\rho_y^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1.
\]

**Corollary 3.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) and \( g \) be any two entire functions with regular relative \((m,q)\)-Ritt growth and regular relative \((m,p)\)-th Ritt growth with respect to entire function \( h \) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\lambda_y^{(p,q)}(f) = \frac{\lambda^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.
\]

**Corollary 4.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) and \( g \) be any two entire functions with regular relative \((m,q)\)-Ritt growth and regular relative \((m,p)\)-th Ritt growth with respect to entire function \( h \) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Also suppose that \( \rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g) \). Then
\[
\lambda_y^{(p,q)}(f) = \rho_y^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1.
\]

**Corollary 5.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) and \( g \) be any two entire functions with relative index-pairs \((m,q)\) and \((m,p)\) with respect to entire function \( h \) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \) and either \( f \) is not of regular relative \((m,q)\) - Ritt growth or \( g \) is not of regular relative \((m,p)\) - Ritt growth, then
\[
\rho_y^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g) \geq 1.
\]

If \( f \) and \( g \) are both of regular relative \((m,q)\)- Ritt growth and regular relative \((m,p)\) - Ritt growth with respect to entire function \( h \) respectively, then
\[
\rho_y^{(p,q)}(f) \cdot \rho_f^{(q,p)}(g) = 1.
\]

**Corollary 6.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series. Also let \( f \) and \( g \) be any two entire functions with relative index-pairs \((m,q)\) and \((m,p)\) with respect to entire function \( h \) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \) and either \( f \) is not of regular relative \((m,q)\) - Ritt growth or \( g \) is not of regular relative \((m,p)\) - Ritt growth, then
\[
\lambda_y^{(p,q)}(f) \cdot \lambda_f^{(q,p)}(g) \leq 1.
\]
If \( f \) and \( g \) are both of regular relative \((m, q)\) - Ritt growth and regular relative \((m, p)\) -Ritt growth with respect to entire function \( h \) respectively, then

\[
\lambda_g^{(p, q)}(f) \cdot \lambda_f^{(q, p)}(g) = 1.
\]

**Corollary 7.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series and \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Also let \( f \) be an entire function with relative index-pair \((m, q)\), Then

(i) \( \lambda_g^{(p, q)}(f) = \infty \) when \( \rho_h^{(m, p)}(g) = 0 \),

(ii) \( \rho_g^{(p, q)}(f) = \infty \) when \( \lambda_h^{(m, p)}(g) = 0 \),

(iii) \( \lambda_g^{(p, q)}(f) = 0 \) when \( \rho_h^{(m, p)}(g) = \infty \)

and

(iv) \( \rho_g^{(p, q)}(f) = 0 \) when \( \lambda_h^{(m, p)}(g) = \infty \).

**Corollary 8.** Let \( f, g \) and \( h \) be any three entire functions represented by vector valued Dirichlet series and \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Also let \( g \) be an entire function with relative index-pair \((m, q)\), Then

(i) \( \rho_g^{(p, q)}(f) = 0 \) when \( \rho_h^{(m, q)}(f) = 0 \),

(ii) \( \lambda_g^{(p, q)}(f) = 0 \) when \( \lambda_h^{(m, q)}(f) = 0 \),

(iii) \( \rho_g^{(p, q)}(f) = \infty \) when \( \rho_h^{(m, q)}(f) = \infty \)

and

(iv) \( \lambda_g^{(p, q)}(f) = \infty \) when \( \lambda_h^{(m, q)}(f) = \infty \).

**Remark 1.** Under the same conditions of Theorem 1, one may write \( \rho_g^{(p, q)}(f) = \rho_h^{(m, q)}(f) \left( \frac{\lambda_h^{(m, q)}(f)}{\lambda_h^{(m, p)}(g)} \right) \) and \( \lambda_g^{(p, q)}(f) = \frac{\lambda_h^{(m, q)}(f)}{\rho_h^{(m, p)}(g)} \) when \( \lambda_h^{(m, p)}(g) = \rho_h^{(m, p)}(g) \). Similarly \( \rho_g^{(p, q)}(f) = \frac{\lambda_h^{(m, q)}(f)}{\rho_h^{(m, p)}(g)} \) and \( \lambda_g^{(p, q)}(f) = \frac{\rho_h^{(m, q)}(f)}{\rho_h^{(m, p)}(g)} \) when \( \lambda_h^{(m, q)}(f) = \rho_h^{(m, q)}(f) \).

Next we prove our theorem based on \((p, q)\)-th relative Ritt type and \((p, q)\)-th relative Ritt weak type of entire functions represented by VVDS.

**Theorem 2.** Let \( f \) and \( g \) be any two entire functions VVDS defined by \([1, 1]\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function \( h \) VVDS defined by \([1, 1]\) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then

\[
\max \left\{ \frac{\Delta_h^{(m, q)}(f)}{\rho_h^{(m, p)}(g)}, \frac{\Delta_h^{(m, q)}(f)}{\lambda_h^{(m, p)}(g)} \right\} \leq \Delta_g^{(p, q)}(f) \leq \frac{\Delta_h^{(m, q)}(f)}{\Delta_h^{(m, p)}(g)} \frac{1}{\rho_h^{(m, p)}(g)}.
\]
Proof. From the definitions of $\Delta_h^{(m,q)}(f)$ and $\overline{\Delta}_h^{(m,q)}(f)$, we have for all sufficiently large values of $\sigma$ that

$$M_f(\sigma) \leq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_h^{(m,q)}(f) + \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \right], \tag{2.11}$$

$$M_f(\sigma) \geq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_h^{(m,q)}(f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \right], \tag{2.12}$$

and also for a sequence of values of $\sigma$ tending to infinity, we get that

$$M_f(\sigma) \geq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_h^{(m,q)}(f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \right], \tag{2.13}$$

$$M_f(\sigma) \leq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_h^{(m,q)}(f) + \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \right]. \tag{2.14}$$

Similarly from the definitions of $\Delta_h^{(m,p)}(g)$ and $\overline{\Delta}_h^{(m,p)}(g)$, it follows for all sufficiently large values of $\sigma$ that

$$M_g(\sigma) \leq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_h^{(m,p)}(g) + \varepsilon \right) \left[ \log^{[p-1]} \sigma \right] \rho_h^{(m,p)}(g) \right] \right], \tag{2.15}$$

$$i.e., M_h(\sigma) \geq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\Delta_h^{(m,p)}(g) + \varepsilon} \right) \frac{1}{\rho_h^{(m,p)}(g)} \right] \quad \text{and} \quad \tag{2.15}$$

$$M_h(\sigma) \leq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\overline{\Delta}_h^{(m,p)}(g) - \varepsilon} \right) \frac{1}{\rho_h^{(m,p)}(g)} \right]. \tag{2.16}$$

Also for a sequence of values of $\sigma$ tending to infinity, we obtain that

$$M_h(\sigma) \leq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\Delta_h^{(m,p)}(g) - \varepsilon} \right) \frac{1}{\rho_h^{(m,p)}(g)} \right] \quad \text{and} \quad \tag{2.17}$$

$$M_h(\sigma) \geq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\overline{\Delta}_h^{(m,p)}(g) + \varepsilon} \right) \frac{1}{\rho_h^{(m,p)}(g)} \right]. \tag{2.18}$$
From the definitions of $\tau_h^{(m,q)}(f)$ and $\tau_h^{(m,q)}(f)$, we have for all sufficiently large values of $\sigma$ that

$$M_f(\sigma) \leq M_h \left[ \exp^{m-1} \left( \tau_h^{(m,q)}(f) + \varepsilon \right) \left[ \log^{[p-1]} \sigma \right] \lambda_h^{(m,q)}(f) \right], \quad (2.19)$$

and also for a sequence of values of $\sigma$ tending to infinity, we get that

$$M_f(\sigma) \geq M_h \left[ \exp^{m-1} \left( \tau_h^{(m,q)}(f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \lambda_h^{(m,q)}(f) \right], \quad (2.20)$$

Similarly from the definitions of $\tau_h^{(m,p)}(g)$ and $\tau_h^{(m,p)}(g)$, it follows for all sufficiently large values of $\sigma$ that

$$M_g(\sigma) \leq M_h \left[ \exp^{[m-1]} \left( \tau_h^{(m,p)}(g) + \varepsilon \right) \left[ \log^{[p-1]} \sigma \right] \lambda_h^{(m,p)}(g) \right] \quad \text{and} \quad (2.21)$$

$$M_h(\sigma) \geq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\tau_h^{(m,p)}(g) + \varepsilon} \right) \lambda_h^{(m,p)}(g) \right] \quad \text{and} \quad (2.22)$$

Also for a sequence of values of $\sigma$ tending to infinity, we obtain that

$$M_h(\sigma) \leq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\tau_h^{(m,p)}(g) - \varepsilon} \right) \lambda_h^{(m,p)}(g) \right] \quad \text{and} \quad (2.23)$$

$$M_h(\sigma) \geq M_g \left[ \exp^{[p-1]} \left( \frac{\log^{[m-1]} \sigma}{\tau_h^{(m,p)}(g) + \varepsilon} \right) \lambda_h^{(m,p)}(g) \right]. \quad (2.24)$$

Now from (2.13) and in view of (2.23), we get for a sequence of values of $\sigma$ tending to infinity that

$$M_g^{-1} M_f(\sigma) \geq M_g^{-1} M_h \left[ \exp^{[m-1]} \left( \Delta_h^{(m,q)}(f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \quad \text{i.e.,} \quad M_g^{-1} M_f(\sigma) \geq$$
\[
\exp^{[p-1]} \left( \log^{[m-1]} \exp^{[m-1]} \left( \frac{\left( \Delta_h^{(m,q)} (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f)}{(\tau_h^{(m,p)} (g) + \varepsilon)} \right) \right) \right) \]

i.e., \( \log^{[p-1]} M_g^{-1} M_f (\sigma) \geq \left[ \left( \Delta_h^{(m,q)} (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \frac{1}{\lambda_h^{(m,p)}(g)} . \)

Since in view of Theorem 1, \( \rho_h^{(m,q)}(f) \geq \rho_g^{(p,q)}(f) \) and as \( \varepsilon (> 0) \) is arbitrary, therefore it follows from above that

\[
\lim_{\sigma \to \infty} \log^{[p-1]} M_g^{-1} M_f (\sigma) \geq \left[ \frac{\Delta_h^{(m,q)} (f)}{\tau_h^{(m,p)} (g)} \right] \frac{1}{\lambda_h^{(m,p)}(g)} \cdot \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) . \]

i.e., \( \Delta_g^{(p,q)} (f) \geq \left[ \frac{\Delta_h^{(m,q)} (f)}{\tau_h^{(m,p)} (g)} \right] \frac{1}{\lambda_h^{(m,p)}(g)} . \) \quad (2.27)

Similarly from (2.12) and in view of (2.26), it follows for a sequence of values of \( \sigma \) tending to infinity that

\[
M_g^{-1} M_f (\sigma) \geq M_g^{-1} M_h \left[ \exp^{[m-1]} \left( \sum_h^{(m,q)} (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \]

i.e., \( M_g^{-1} M_f (\sigma) \geq \)

\[
\exp^{[p-1]} \left( \log^{[m-1]} \exp^{[m-1]} \left( \frac{\sum_h^{(m,q)} (f) - \varepsilon \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f)}{(\tau_h^{(m,p)} (g) + \varepsilon)} \right) \right) \right) \]

i.e., \( \log^{[p-1]} M_g^{-1} M_f (\sigma) \geq \left[ \left( \sum_h^{(m,q)} (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho_h^{(m,q)}(f) \right] \frac{1}{\lambda_h^{(m,p)}(g)} . \)
Since in view of Theorem 11, it follows that \( \frac{\rho^{(m,q)}(f)}{\lambda^{(m,p)}(g)} \geq \rho^{(p,q)}(f) \). Also \( \varepsilon (> 0) \) is arbitrary, so we get from above that

\[
\lim_{\sigma \to \infty} \frac{\log^{[p-1]} M_g^{-1} M_f (\sigma)}{[\log^{[q-1]} \sigma]^{\rho^{(p,q)}(f)}} \geq \frac{\Delta^{(m,q)}(f)}{\Delta^{(m,p)}(g)} \cdot \frac{1}{\rho^{(m,p)}(g)}.
\]

i.e., \( \Delta^{(p,q)}(f) \geq \frac{\Delta^{(m,q)}(f)}{\Delta^{(m,p)}(g)} \cdot \frac{1}{\rho^{(m,p)}(g)} \). \hspace{1cm} (2.28)

Again in view of (2.16), we have from (2.11) for all sufficiently large values of \( \sigma \) that

\[
M_g^{-1} M_f (\sigma) \leq M_g^{-1} M_h \left[ \exp^{[m-1]} \left( \Delta^{(m,q)}(f) + \varepsilon \right) \cdot [\log^{[q-1]} \sigma]^{\rho^{(m,q)}(f)} \right]
\]

i.e., \( M_g^{-1} M_f (\sigma) \leq \left[ \frac{\exp^{[m-1]} \left( \Delta^{(m,q)}(f) + \varepsilon \right) \cdot [\log^{[q-1]} \sigma]^{\rho^{(m,q)}(f)}}{\Delta^{(m,p)}(g) - \varepsilon} \right] \]

\[
\Delta^{(p,q)}(f) \leq \frac{\Delta^{(m,q)}(f)}{\Delta^{(m,p)}(g)} \cdot \frac{1}{\rho^{(m,p)}(g)} \cdot [\log^{[q-1]} \sigma]^{\rho^{(m,q)}(f)}.
\]

As in view of Theorem 1 there follows that \( \frac{\rho^{(m,q)}(f)}{\rho^{(m,p)}(g)} \leq \rho^{(p,q)}(f) \). Since \( \varepsilon (> 0) \) is arbitrary, we get from (2.29) that

\[
\lim_{\sigma \to \infty} \frac{\log^{[p-1]} M_g^{-1} M_f (\sigma)}{[\log^{[q-1]} \sigma]^{\rho^{(p,q)}(f)}} \leq \frac{\Delta^{(m,q)}(f)}{\Delta^{(m,p)}(g)} \cdot \frac{1}{\rho^{(m,p)}(g)}.
\]

i.e., \( \Delta^{(p,q)}(f) \leq \frac{\Delta^{(m,q)}(f)}{\Delta^{(m,p)}(g)} \cdot \frac{1}{\rho^{(m,p)}(g)} \). \hspace{1cm} (2.30)

Thus the theorem follows from (2.27), (2.28) and (2.30). \( \square \)

The conclusion of the following corollary can be carried out from (2.16) and (2.19); (2.19) and (2.24) respectively after applying the same technique of Theorem 2 and with the help of Theorem 11 Therefore its proof is omitted.
Corollary 9. Let $f$ and $g$ be any two entire functions VVDS defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function $h$ VVDS defined by \((1.1)\) respectively where $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then

$$
\Delta^{(p,q)}_y (f) \leq \min \left\{ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)}, \frac{\rho^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right\}.
$$

Similarly in the line of Theorem \(2\) and with the help of Theorem \(1\) one may easily carried out the following theorem from pairwise inequalities numbers \((2.20)\) and \((2.23)\); \((2.17)\) and \((2.19)\), \((2.16)\) and \((2.22)\) respectively and therefore its proofs is omitted:

Theorem 3. Let $f$ and $g$ be any two entire functions VVDS defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function $h$ VVDS defined by \((1.1)\) respectively where $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then

$$
\tau^{(p,q)}_y (f) \geq \max \left\{ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)}, \frac{\rho^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right\}.
$$

Corollary 10. Let $f$ and $g$ be any two entire functions VVDS defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function $h$ VVDS defined by \((1.1)\) respectively where $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then

$$
\tau^{(p,q)}_y (f) \geq \max \left\{ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)}, \frac{\rho^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right\}.
$$

With the help of Theorem \(1\) the conclusion of the above corollary can be carry out from \((2.12)\), \((2.15)\) and \((2.12)\), \((2.23)\) respectively after applying the same technique of Theorem \(2\) and therefore its proof is omitted.

Theorem 4. Let $f$ and $g$ be any two entire functions VVDS defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function $h$ VVDS defined by \((1.1)\) respectively where $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then

$$
\tau^{(p,q)}_y (f) \geq \max \left\{ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)}, \frac{\rho^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right\}.
$$

Proof. From \((2.12)\) and in view of \((2.23)\), we get for all sufficiently large values of $\sigma$ that

$$
M^{-1}_g M_f (\sigma) \geq M^{-1}_g M_h \left[ \exp^{[m-1]} \left( \Delta^{(m,q)}_h (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho^{(m,q)}_h (f) \right]
$$

i.e., $M^{-1}_g M_f (\sigma) \geq

$$
\exp^{[p-1]} \left( \log^{[m-1]} \exp^{[m-1]} \left( \Delta^{(m,q)}_h (f) - \varepsilon \right) \left[ \log^{[q-1]} \sigma \right] \rho^{(m,q)}_h (f) \right) \tau^{(m,p)}_h (g) + \varepsilon.
$$
i.e., \( \log^{[p-1]} M_g^{-1} M_f (\sigma) \geq \left[ \frac{\Delta_{h \cdot g} \left( f^m, q^m, p, q \right)}{\Delta_{h \cdot g} \left( g \right)} + \varepsilon \right] \cdot \log^{[q-1]} \sigma \). \]

Now in view of Theorem 1, it follows that \( \rho \geq (p, q) \). Since \( \varepsilon > 0 \) is arbitrary, we get from above that

\[
\lim_{r \to \infty} \frac{\log^{[p-1]} M_g^{-1} M_f (\sigma)}{\log^{[q-1]} \sigma} \geq \left[ \frac{\Delta_{h \cdot g} \left( f^m, q^m, p, q \right)}{\Delta_{h \cdot g} \left( g \right)} \right] \cdot \log^{[q-1]} \sigma \] \]

i.e., \( \Delta_{h \cdot g} \left( f^m, q^m, p, q \right) \geq \left[ \frac{\Delta_{h \cdot g} \left( g \right)}{\rho^m \left( g \right)} \right] \). \( (2.31) \)

Further in view of (2.17), we get from (2.11) for a sequence of values of \( \sigma \) tending to infinity that

\[
M_g^{-1} M_f (\sigma) \leq M_h \left[ \exp^{[m-1]} \left[ \left( \Delta_{h \cdot g} \left( f^m, q^m, p, q \right) + \varepsilon \right) \cdot \log^{[q-1]} \sigma \right] \right]^{(m, q) \left( f \right)} \] \]

i.e., \( M_g^{-1} M_f (\sigma) \leq \left[ \exp^{[p-1]} \left[ \frac{\log^{[m-1]} \exp^{[m-1]} \left[ \Delta_{h \cdot g} \left( f^m, q^m, p, q \right) + \varepsilon \right] \cdot \log^{[q-1]} \sigma \right]^{(m, q) \left( f \right)}}{\Delta_{h \cdot g} \left( g \right)} \right] \] \]

i.e., \( \log^{[p-1]} M_g^{-1} M_f (\sigma) \leq \left[ \frac{\Delta_{h \cdot g} \left( f^m, q^m, p, q \right) + \varepsilon}{\Delta_{h \cdot g} \left( g \right)} \right] \cdot \log^{[q-1]} \sigma \). \( (2.32) \)

Again as in view of Theorem 1 \( \rho \geq (p, q) \), and \( \varepsilon > 0 \) is arbitrary, therefore we get from (2.32) that

\[
\lim_{r \to \infty} \frac{\log^{[p-1]} M_g^{-1} M_f (\sigma)}{\log^{[q-1]} \sigma} \leq \left[ \frac{\Delta_{h \cdot g} \left( f^m, q^m, p, q \right)}{\Delta_{h \cdot g} \left( g \right)} \right] \cdot \log^{[q-1]} \sigma \] \]

i.e., \( \Delta_{h \cdot g} \left( f^m, q^m, p, q \right) \leq \left[ \frac{\Delta_{h \cdot g} \left( g \right)}{\rho^m \left( g \right)} \right] \). \( (2.33) \)
Likewise from (2.14) and in view of (2.16), it follows for a sequence of values of \( \sigma \) tending to infinity that
\[
M_g^{-1}M_f(\sigma) \leq M_g^{-1}M_h \left[ \exp^{[m-1]} \left( \frac{\Delta_h^{(m,q)}(f) + \epsilon}{\Delta_h^{(m,p)}(g) - \epsilon} \right) \left[ \log[q-1] \sigma \right] \rho_h^{(m,q)}(f) \right]
\]
i.e., \( M_g^{-1}M_f(\sigma) \leq \left[ \frac{\Delta_h^{(m,q)}(f) + \epsilon}{\Delta_h^{(m,p)}(g) - \epsilon} \right] \frac{1}{\rho_h^{(m,p)}(g)} \left[ \log[q-1] \sigma \right] \rho_h^{(m,q)}(f) \left[ \log[q-1] \sigma \right] \rho_h^{(m,p)}(g) \). (2.34)

Analogously, we get from (2.34) that
\[
\lim_{r \to \infty} \frac{\log^{[p-1]} M_g^{-1}M_f(\sigma)}{\log[q-1] \sigma} \leq \frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)} \frac{1}{\rho_h^{(m,p)}(g)} \left[ \log[q-1] \sigma \right] \rho_h^{(m,q)}(f) \left[ \log[q-1] \sigma \right] \rho_h^{(m,p)}(g) \).
\]
i.e., \( \Delta_g^{(p,q)}(f) \leq \frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)} \frac{1}{\rho_h^{(m,p)}(g)} \), (2.35)

since in view of Theorem 11 \( \Delta_h^{(m,q)}(f) \leq \rho_h^{(p,q)}(f) \) and \( \epsilon > 0 \) is arbitrary.

Thus the theorem follows from (2.31), (2.33) and (2.35).

**Corollary 11.** Let \( f \) and \( g \) be any two entire functions VVDS defined by (1.1) with relative index-pairs \( (m, q) \) and \( (m, p) \) with respect to another entire function \( h \) VVDS defined by (1.1) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\Delta_g^{(p,q)}(f) \leq \min \left\{ \frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}, \frac{\delta_h^{(m,q)}(f)}{\delta_h^{(m,p)}(g)}, \frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}, \frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}, \frac{\tau_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)} \right\}.
\]

The conclusion of the above corollary can be carried out from pairwise inequalities no (2.16) and (2.22); (2.17) and (2.19); (2.22) and (2.24); (2.19) and (2.25) respectively after applying the same technique of Theorem 4 and with the help of Theorem 11. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1 one may easily carried out the following theorem from pairwise inequalities no (2.21) and (2.23); (2.20) and (2.26); (2.16) and (2.19) respectively and therefore its proofs is omitted:
Theorem 5. Let \( f \) and \( g \) be any two entire functions \( \text{VVDS} \) defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function \( h \) \( \text{VVDS} \) defined by \((1.1)\) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\max \left\{ \frac{\tau_h(m,q)}{\tau_h(m,p)}(g), \frac{\lambda_h(m,q)}{\lambda_h(m,p)}(g) \right\} \leq \tau_g^{(p,q)}(f) \leq \frac{\tau_h(m,q)}{\tau_h(m,p)}(g). 
\]

Corollary 12. Let \( f \) and \( g \) be any two entire functions \( \text{VVDS} \) defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function \( h \) \( \text{VVDS} \) defined by \((1.1)\) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Then
\[
\tau_g^{(p,q)}(f) \geq \max \left\{ \frac{\Delta_h^{(m,q)}}{\Delta_h^{(m,p)}}(g), \frac{\lambda_h^{(m,q)}}{\lambda_h^{(m,p)}}(g) \right\}.
\]

The conclusion of the above corollary can be carried out from pairwise inequalities
no (2.13) and (2.15); (2.12) and (2.11); (2.14) and (2.20); (2.12) and (2.20) respectively
after applying the same technique of Theorem 3 and with the help of Theorem 4. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because they can be derived easily using the same technique or with some easy reasoning by the help of with the help of Remark II and therefore left to the readers.

Theorem 6. Let \( f \) and \( g \) be any two entire functions \( \text{VVDS} \) defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function \( h \) \( \text{VVDS} \) defined by \((1.1)\) respectively where \( p \geq 0, q \geq 0 \) and \( m \geq 0 \). Also let \( \lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g) \). Then
\[
\frac{\Delta_h^{(m,q)}}{\Delta_h^{(m,p)}}(g) \leq \tau_g^{(p,q)}(f) \leq \min \left\{ \frac{\Delta_h^{(m,q)}}{\Delta_h^{(m,p)}}(g), \frac{\lambda_h^{(m,q)}}{\lambda_h^{(m,p)}}(g) \right\} 
\]

and
\[
\tau_h^{(m,q)}(f) \leq \tau_g^{(p,q)}(f) \leq \min \left\{ \tau_h^{(m,q)}(f), \frac{\lambda_h^{(m,q)(m,p)}}{\lambda_h^{(m,p)}}(g) \right\} 
\]
\[
\leq \max \left\{ \tau_h^{(m,q)}(f), \frac{\lambda_h^{(m,q)(m,p)}}{\lambda_h^{(m,p)}}(g) \right\} \leq \tau_g^{(p,q)}(f) \leq \frac{\tau_h^{(m,q)}}{\tau_h^{(m,p)}}(g).
\]

Theorem 7. Let \( f \) and \( g \) be any two entire functions \( \text{VVDS} \) defined by \((1.1)\) with relative index-pairs \((m, q)\) and \((m, p)\) with respect to another entire function \( h \) \( \text{VVDS} \) defined by
(1.1) respectively where $p \geq 0$, $q \geq 0$ and $m \geq 0$. Also let $\lambda^{(m,q)}_h (f) = \rho^{(m,q)}_h (f)$. Then
\[
\left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}} \leq \tau^{(p,q)}_g (f) \leq \min \left\{ \left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}}, \left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}} \right\}
\leq \max \left\{ \left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}}, \left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}} \right\} \leq \tau^{(p,q)}_g (f) \leq \left[ \frac{\Delta^{(m,q)}_h (f)}{\Delta^{(m,p)}_h (g)} \right]^{\frac{1}{\rho^{(m,p)}_h (g)}}
\]

and
\[
\left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}} \leq \Delta^{(p,q)}_g (f) \leq \min \left\{ \left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}}, \left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}} \right\}
\leq \max \left\{ \left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}}, \left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}} \right\} \leq \Delta^{(p,q)}_g (f) \leq \left[ \frac{\tau^{(m,q)}_h (f)}{\tau^{(m,p)}_h (g)} \right]^{\frac{1}{\lambda^{(m,p)}_h (g)}}
\]

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