Symplectic Three-Algebra and $\mathcal{N} = 6$, $Sp(2N) \times U(1)$ Superconformal Chern-Simons-Matter Theory

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ABSTRACT: We introduce an anti-symmetric metric into a 3-algebra and call it a symplectic 3-algebra. The $\mathcal{N} = 6$, $Sp(2N) \times U(1)$ superconformal Chern-Simons-matter theory with $SU(4)$ R-symmetry in three dimensions is constructed by specifying the 3-brackets in a symplectic 3-algebra. We also demonstrate that the $\mathcal{N} = 6$, $U(M) \times U(N)$ theory can be recast into this symplectic 3-algebraic framework.

KEYWORDS: Symplectic Three-Algebra, Chern-Simons Theory, M2 branes.
1. Introduction

Recently the construction of superconformal Chern-Simons-matter (CSM) theories in three dimensions \cite{1,2} has attracted a lot of attention in string/M theory community, because they are natural candidates for the dual gauge description of M2 branes in M theory. It was already realized about twenty years ago \cite{3}, that generically Chern-Simons gauge theories in three dimensions are conformally invariant, both for pure gauge theories and for theories coupled to (massless) matter fields, even at the quantum level: in spite of a quantum shift at one loop order \cite{4}, the Chern-Simons gauge coupling does not run at all, because its $\beta$ function vanishes, as shown both by an explicit two-loop calculations for theories with matter \cite{5} and by a formal proof up to all orders in perturbation theory for pure gauge theories \cite{6}. In order to construct the dual gauge description of M2-branes, the relevant issue \cite{1} is then how to incorporate extended supersymmetries into CSM theories, since extended supersymmetry plays a crucial role in M-theory as it does in superstring theory.

Based on the totally anti-symmetric Nambu 3-brackets \cite{7,8}, the maximally (i.e. $\mathcal{N} = 8$) supersymmetric Chern-Simons-matter theory in $D = 3$ with $SO(4)$ gauge group and $SO(8)$ R-symmetry, was constructed independently by Bagger, Lambert \cite{9}, and Gustavsson \cite{10}. This theory, known as the BLG theory, is the dual gauge description of two M2 branes \cite{11,12}. It was also shown that the Nambu 3-algebra with a symmetric and positive define metric is unique \cite{13,14}: it generates an $SO(4)$ gauge symmetry. To generate arbitrary gauge group, the Nambu 3-algebras with a Lorentzian metric (Lorentzian
3-algebra) are introduced [13]. However, the BLG theory constructed from the Lorentzian 3-algebras turns out to be an $\mathcal{N} = 8$ super Yang-Mills theory [16, 17], which is not a supersymmetric Chern-Simons-matter theory.

Soon the BLG theory was generalized to the cases of reduced supersymmetries by Aharony, Bergman, Jafferis and Maldacena (ABJM) [18]. They have been able to construct, without consulting to the 3-algebra approach, an $\mathcal{N} = 6$ superconformal CSM theory with gauge group $U(N) \times U(N)$, $SU(4)$ R-symmetry and $U(1)$ global symmetry. They also argued that at level $k$, the theory describes the low energy limit of $N$ M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. At large-$N$ limit, it becomes the dual theory of M theory on $AdS_4 \times S^7/\mathbb{Z}_k$ [18]. Some further analysis of the ABJM theory can be found in Ref. [19, 20]. The superconformal gauge theories in $D = 3$ with more or less supersymmetries can be obtained by taking a conformal limit of $D = 3$ gauged supergravity theories [21, 22]. By generalizing Gaiotto and Witten’s construction [23] of $\mathcal{N} = 4$ CSM theories, not only was the ABJM theory re-derived (as a special case of $U(M) \times U(N)$ CSM theories), but also two new theories, $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $Sp(2M) \times O(2)$ CSM theories, were constructed in Ref. [24, 25]. Their M theory and string theory dualities were studied in Ref. [26].

Bagger and Lambert [27] have constructed the $\mathcal{N} = 6$, $U(M) \times U(N)$ theory in a modified 3-algebra approach, in which the structure constants are no longer required to be totally antisymmetric. By specifying the 3-brackets and taking a hermitian, gauge invariant metric on the 3-algebra, they are able to reproduce the $U(M) \times U(N)$ theory hence the ABJM theory. However, it remains unclear whether another important class of $\mathcal{N} = 6$ CSM theories, namely the ones with gauge group $Sp(2M) \times O(2)$, can be constructed in the 3-algebra approach or not. In this paper we will propose to solve this problem by introducing an anti-symmetric metric into a 3-algebra, which we call a symplectic 3-algebra. Then we will first present a construction of $\mathcal{N} = 6$ superconformal CMS theories with $SU(4)$ R-symmetry by utilizing the symplectic 3-algebras, and then to re-derive the $\mathcal{N} = 6$, $Sp(2N) \times U(1)$ superconformal CSM theories by specifying the 3-bracket. We will also demonstrate that the $\mathcal{N} = 6$, $U(M) \times U(N)$ theory can be recast into this symplectic 3-algebraic framework.

The paper is organized as follows. In section 2, we introduce the notion of symplectic three-algebra. In section 3.1, we study the supersymmetry transformations of physical fields valued in a symplectic 3-algebra, and construct the Lagrangian of the superconformal Chern-Simons-matter theories. In section 3.2, we derive the $\mathcal{N} = 6$, $Sp(2N) \times U(1)$ superconformal CMS theory from our Lagrangian by specifying the 3-brackets of the symplectic 3-algebra. In section 4, we recast the $\mathcal{N} = 6$, $U(M) \times U(N)$ theory into our framework. In section 5, we present conclusions and discussions. Our convention and useful identities are given in appendix A.

2. The Symplectic Three-Algebra

In this section, we will introduce the notion of the symplectic 3-algebra. Following [27], we
assume that the 3-algebra is a complex vector space, equipped with the 3-brackets

\[ [T^I, T^J; T^K] = f^{IJK} L T^L, \]  

(2.1)

where \( T^I \) are a set of generators. \((I = 1, 2, \cdots, M)\). Notice that our 3-brackets are not exactly the same as that of Ref. [27]. First, the third generator in the 3-brackets is not a complex conjugate one; secondly, we do not assume that the first two indices of the structure constants are necessarily anti-symmetric.

Further we assume the structure constants satisfy the fundamental identity (FI)

\[ f^{IJK} O f^{OLM} N - f^{ILM} O f^{OJK} N + f^{MJK} O f^{ILO} N = 0. \]  

(2.2)

In this way, the 3-algebra can be viewed as a generalization of the ordinary (Lie) 2-algebra, with the FI playing the role of the Jacobi identities. Note that our FI is also not the same as that of BL [27]. However, later we will see, one can ‘map’ the structure constants and the FI satisfied by the structure constants in Ref. [27] into ours (see section 4).

Since our goal is to construct a theory with gauge group \( Sp(2N) \times U(1) \), it is natural to introduce an antisymmetric metric \( \omega^{IJ} \) and its inverse \( \omega_{IJ} \) into the 3-algebra to raise or lower the indices:

\[ f^{IJKL} \equiv \omega^{LM} f^{IJK} M, \quad f^{IJ KL} \equiv \omega_{KM} f^{IJML}, \]  

(2.3)

where \( \omega^{IJ} = -\omega^{JI} \) and \( \omega_{IJ}\omega^{JK} = \delta^I_K \). The antisymmetric metric \( \omega^{IJ} \) will enter the theories via (2.3). We note that the existence of the inverse of \( \omega_{IJ} \) implies that \( \det(\omega_{IJ}) \neq 0 \), which in turn requires the complex dimension \( M \) of the 3-algebra be even \( M = 2L \).

Following BL [27], we define the global transformation of a 3-algebra valued field \( X_I \) as

\[ \delta_{\tilde{\Lambda}} X_I = -\Lambda^K_L f^{JL} K I X_J \equiv -\tilde{\Lambda}^{J} I X_J, \]  

(2.4)

where the parameter \( \Lambda^K_L \) is a 3-algebra tensor, independent of spacetime coordinate. The anti-symmetric metric must be invariant under such a global transformation:

\[ \delta_{\tilde{\Lambda}} \omega_{IJ} = -\Lambda^K L \omega_{KI} - \Lambda^K J \omega_{IK} \]  

\[ = -\Lambda^K M (f^{KM} LI \omega_{KI} + f^{KM} LJ \omega_{IK}) \]  

\[ = 0. \]  

(2.5)

From point of view of ordinary Lie group, the infinitesimal matrices \(-\tilde{\Lambda}^{K} I \) must form the Lie algebra \( Sp(2L, \mathbb{C}) \). Parts of the global symmetry (2.4) will be gauged in section 3.1 and 4. Eq. (2.3) imposes a strong constraint on \( f^{IJ KL} \). The FI (2.2) implies that the structure constants are also invariant under the global transformation [27]:

\[ \delta_{\tilde{\Lambda}} f^{IJ KL} = 0 \]  

(2.6)

We call the 3-algebra defined by the antisymmetric metric and the above Eqs. (2.1)-(2.3) a symplectic 3-algebra.
Moreover, since the 3-algebra is a complex vector space, the reality conditions and positivity are important in introducing gauge fields, matter fields and constructing invariant Lagrangians. Following BL [27], we define the gauge fields as

$$\tilde{A}_\mu{}^I_L = A_\mu{}^K_J f^{IJ}{}_{KL},$$

(2.7)

where $A_\mu{}^K_J$ is a 3-algebra tensor ($\mu = 0, 1, 2$), satisfying the anti-hermitian condition

$$A_\mu{}^{*K}{}_J = -A_\mu{}^J{}_K.$$  

(2.8)

To ensure the anti-hermiticity of the gauge field, i.e., $\tilde{A}_\mu^{*I}{}_L = -\tilde{A}_\mu{}^L{}_I$, the structure constants $f^{IJ}{}_{KL}$ are required to satisfy the reality condition

$$f^{*IJ}{}_{KL} = f^{LK}{}_{JI}.$$  

(2.9)

In accordance with (2.8), we also require that the parameter in (2.4) satisfies the anti-hermitian condition $\Lambda^{*K}{}_J = -\Lambda^J{}_K$. On the other hand, it obeys the natural reality condition $\Lambda^{*K}{}_J = -\omega_{KJI}\Lambda^{JL}{}_{LI}$, since it carries two symplectic 3-algebra indices. These two equations imply that $\Lambda_{KJ} = \Lambda_{JK}$. Re-examining the global transformation (2.4), we are led to require that the structure constants are symmetric in the middle two indices

$$f^{IJKL} = f^{IKLJ}.$$  

(2.10)

The reality conditions for the matter fields are a bit more complicated, since they involve additional indices associated with the $R$-symmetry, which will be discussed in next section. As for the positivity of invariant Lagrangians, the following hermitian bilinear form in the 3-algebra (as a complex vector space) is naturally positive-definite:

$$h(X, Y) = X^*_I Y_I,$$

(2.11)

where $*$ is the complex conjugation. Using it to construct the Lagrangians in our model will guarantee their positivity. But are they invariant under the transformations that preserve the anti-symmetric metric? Generally this is not true. But fortunately it is known that $Sp(2L, \mathbb{C})$ and $U(2L)$ have a non-empty intersection $Sp(2L)$, which can be selected by imposing certain reality conditions on the fields. In fact, eqn. (2.4) and reality condition (2.9) dictate the transformation property of $X_I^*$ to be $\delta_{\tilde{\Lambda}} X^*_I = \tilde{\Lambda}^I{}_J X^*_J$:

$$\delta_{\tilde{\Lambda}} X^*_I = -\Lambda^{*K}{}_L f^{*JL}{}_{KI} X^*_J$$

$$= -(-\Lambda^J{}_K) f^{JK}{}_{LI} X^*_J$$

$$= \tilde{\Lambda}^I{}_J X^*_J.$$  

(2.12)

The bilinear form (2.11) is therefore compatible with the antisymmetric metric $\omega^{IJ}$, in the sense that with the reality conditions respected, the complex conjugate $X^*_I$ transforms in the same way as a vector $\omega^{IJ} X_J$. Essentially this means that while the reality conditions are respected, it makes sense to denote $X^*_I$ as $\tilde{X}^I$, and rewrite eq. (2.11) in a manifestly invariant form: $h(X, Y) = \tilde{X}^I Y_I$. Therefore the terms in the action constructed in terms of the hermitian bilinear form will be $Sp(2L)$ invariant.
3. \( \mathcal{N} = 6 \), \( Sp(2N) \times U(1) \) CSM Theory from Symplectic 3-Algebra

3.1 Closure of the Super-algebra

In this subsection, we generalize the method of BL’s 3-algebra construction of \( \mathcal{N} = 6 \) CSM theories \([27]\). The formalism and computations are similar to theirs except for necessary changes arising from the fact that we introduce an antisymmetric metric \( \omega_{IJ} \) into the 3-algebra and hence the theories.

We first postulate that scalar and fermion fields are symplectic 3-algebra valued, carrying a vector index, while the gauge fields are defined by (2.7).

To generate a direct product of gauge group, such as \( Sp(2N) \times U(1) \), we split up an 3-algebra index into a pair of indices \( I \rightarrow a \pm \), where the index \( I \) runs from 1 to 4N, while the index \( a \) from 1 to 2N. The index \( a \) is an \( Sp(2N) \) index. And + or − is a \( U(1) = SO(2) \) index.

We then assume the theory has an \( SU(4) \) R-symmetry and a \( U(1) \) global symmetry. Combining the \( SU(4) \) R-symmetry and \( U(1) \) global symmetry, the complex scalar fields can be written as \( Z_{A}^{c_{+}} \). And \( \psi_{A_{+}} \) and \( \psi_{A_{-}} \). We also assign a unit global \( U(1) \) charge to \( Z_{A}^{c_{+}} \) and \( \psi_{A_{-}} \). The definition of the matter fields suggests that the parameter \( \Lambda_{KJ} \) (see (2.4)) takes the form

\[
\Lambda_{KJ}^{+} = \Lambda_{c_{+}b_{+}}^{+}. \tag{3.1}
\]

We will see that (3.1) is reasonable when we examine the closure of the super-algebra. (Actually, if we restore the \( U(1) \) index + in Eq. (3.1), i.e. replace \( b \) and \( c \) by \( b_{+} \) and \( c_{+} \), respectively, the parameter \( \Lambda \) will exactly have the above form.) Similarly, the 3-algebra tensor \( A_{\mu KJ}^{+} \) takes the form \( A_{\mu c_{+}b_{+}}^{+} \). So the gauge fields are given by \( \tilde{A}_{\mu c_{+}b_{+}}^{+} = A_{\mu c_{+}b_{+}}^{+} f_{a_{+}b_{+}c_{+}d_{+}}^{+} \).

We denote the antisymmetric metric \( \omega^{IJ} \) as

\[
\omega^{a_{+}b_{-}} \equiv \omega^{ab} h^{+-}, \tag{3.2}
\]

where \( \omega^{ab} \) is an antisymmetric bilinear form, and \( h^{+-} = h^{-+} = 1 \). The other components of the antisymmetric metric \( \omega^{IJ} \) vanish due to the fact that \( h^{++} = h^{--} = 0 \).

The hermitian bilinear form (2.11) becomes

\[
h(X, Y) = \tilde{X}^{a_{+}b_{+}c_{+}d_{+}} = X^{a_{+}b_{+}c_{+}d_{+}}. \tag{3.3}
\]

Because of the index structure of \( \Lambda_{c_{+}b_{+}}^{+} \), the two types of gauge parameters can be written as \( \tilde{\Lambda}_{a_{+}b_{+}} = \Lambda_{a_{+}b_{+}}^{+} f_{a_{+}b_{+}c_{+}d_{+}}^{+} \) and \( \tilde{\Lambda}_{a_{-}b_{-}} = \Lambda_{a_{-}b_{-}}^{+} f_{a_{-}b_{-}c_{+}d_{-}}^{+} \). At first it seems that we need both \( f_{a_{+}b_{+}c_{+}d_{+}}^{+} \) and \( f_{a_{-}b_{-}c_{+}d_{-}}^{+} \) to construct a gauge theory. But the gauge invariance condition of \( \omega \) (2.3) implies that

\[
f_{a_{+}b_{+}c_{+}d_{+}}^{+} = f_{d_{+}b_{+}c_{+}d_{+}}^{+} = 2 \Lambda_{a_{+}b_{+}}^{+} f_{a_{+}b_{+}c_{+}d_{+}}^{+} = \Lambda_{a_{+}b_{+}}^{+} \left( f_{a_{+}b_{+}c_{+}d_{+}}^{+} + f_{a_{+}b_{+}c_{+}d_{+}}^{+} \right). \tag{3.4}
\]
so $f^{a+b+}_{c+d+}$ alone will be sufficient to construct a gauge theory. As a result, we only need to consider the FI satisfied by $f^{a+b+}_{c+d+}$, or the gauge invariance condition

$$\delta_{\tilde{\Lambda}}f^{a+b+}_{c+d+} = 0$$

(3.5)

in (2.6).

Eq. (2.10) can be written as

$$f_{a-b-c+d+} = f_{a-c+b-d+}. \tag{3.6}$$

The reality condition (2.9) becomes

$$f^{*a+b+}_{c+d+} = f^{d+c+}_{b+c+}. \tag{3.7}$$

To close the $\mathcal{N} = 6$ super-algebra, we need to impose an additional constraint condition on the structure constants. Specifically, we will require the first two indices of the structure constants are anti-symmetric if they have the same gauge transformation property, or if they are on equal footing, i.e.

$$f_{a-b-c+d+} = -f_{b-a-c+d+}. \tag{3.8}$$

However, the first two pairs of $f_{d+b-c+a-}$ (see (3.4)) are not necessarily anti-symmetric, since $d+$ and $b-$ do not have the same gauge transformation property.

We now suppress the $SO(2)$ indices for the sake of brevity. We write the structure constants $f^{a+b+}_{c+d+}$ as $f^{ab}_{cd}$. Similarly, we write the scalar, fermion, gauge fields and the gauge parameter as $Z^A_c, \psi^A_c, \tilde{A}_\mu^A, a^f_{ab}, Z^A_c$ and $\Lambda^c_b$, respectively. The hermitian bilinear form (3.3) can be written as $\bar{X}^a Y_a$. Using the anti-symmetric metric (3.2) to raise $a-$ and $b-$ in Eq. (3.8), then Eq. (3.8) becomes

$$f^{ab}_{cd} = -f^{ba}_{cd}. \tag{3.9}$$

Now the reality condition (3.7) takes the following form:

$$f^{*ab}_{cd} = f^{dc}_{ba} = f^{cd}_{ab}. \tag{3.10}$$

And we write (3.3) as $\delta_{\tilde{\Lambda}}f^{ab}_{cd} = 0$, which is equivalent to

$$f^{ab}_{cd}f^{de}_{gf} + f^{ba}_{fd}f^{de}_{gc} - f^{ae}_{gd}f^{db}_{cf} - f^{be}_{gd}f^{da}_{fc} = 0. \tag{3.11}$$

Later we will use this form of the FI’s in proving the closure of the supersymmetry algebras. The FI (3.11) may also be written in some other forms; for example,

$$f^{ab}_{cd}f^{de}_{fg} - f^{ae}_{df}f^{db}_{cg} + f^{eb}_{cd}f^{da}_{gf} - f^{ea}_{gd}f^{db}_{cf} = 0. \tag{3.12}$$

For $\mathcal{N} = 6$ SUSY, the supersymmetry parameters are in the fundamental representation of $SO(6)$: $\epsilon^I$, $I = 1, ..., 6$. Since $SO(6) \cong SU(4)$, we can relabel these generators by two $SU(4)$ indices: $\epsilon^{AB} = -\epsilon^{BA}$. Namely, they transform as the 6 of $SU(4)$. The reality condition $\epsilon^{* AB} = \epsilon^{AB} = \frac{1}{2}\epsilon^{ABC} \epsilon_{CD}$ implies that they do not carry a global $U(1)$ charge.
To achieve conformal invariance, we assume that the local field theory is scale invariant. Under this assumption, we then propose the following manifest $SU(4)$ R-symmetry, $\mathcal{N} = 6$ SUSY transformations:\footnote{Our $f^{ab}_{cd}$ are actually $f^{a+b+c}_{d+} = \omega_{c+} f^{a+b+c}_{d+}$. If one replaces our $f^{ab}_{cd}$ with BL’s $f^{abc}_{d}$, many equations in this section take the same forms as those of BL’s $[27]$.}

\begin{align*}
\delta Z^A_d &= i \epsilon^{AB} \psi_{Bd} \\
\delta \bar{Z}^d_A &= i \epsilon^{AB} \bar{\psi}^B_A
\end{align*}

\begin{align*}
\delta \psi_{Bd} &= \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f^{ab}_{cd} Z^C_d \bar{Z}^C_{db} \epsilon_{AB} + f^{ab}_{cd} Z^A_d Z^B_d \epsilon_{CD} \\
\delta \bar{\psi}^B_d &= \gamma^\mu D_\mu \bar{Z}^d_A \epsilon^{AB} + f^{cd}_{AB} Z^B_d \bar{Z}^C_{db} \epsilon^{AC} + f^{cd}_{AB} Z^A_d Z^B_d \epsilon_{CD} \\
\delta \tilde{A}_\mu^c &= -i \epsilon^{AB} \gamma^\mu \bar{Z}^d_A \psi^B_{cd} f^{ca}_{bd} + i \epsilon^{AB} \gamma^\mu \tilde{Z}^a_A \bar{\psi}^B_{bd} f^{cb}_{ad}.
\end{align*}

The covariant derivatives are defined as

\begin{align*}
D_\mu Z^A_d &= \partial_\mu Z^A_d - \tilde{A}_\mu^c d Z^A_c \\
D_\mu \bar{Z}^d_A &= \partial_\mu \bar{Z}^d_A + \tilde{A}_\mu^c d \bar{Z}^d_A.
\end{align*}

and similar expressions for the fermionic fields.

We require the on-shell closure of the supersymmetry algebra. Namely, after imposing equations of motion, the commutator of two supersymmetry transformations must be equal to a translation plus a gauge term.

The commutator of two supersymmetry transformations acting on the scalar fields reads

\begin{equation}
[\delta_1, \delta_2] Z^A_d = v^\mu \partial_\mu Z^A_d + (\tilde{A}_\mu^c d - v^\mu \tilde{A}_\mu^c d) Z^A_c,
\end{equation}

where

\begin{align*}
v^\mu &= \frac{i}{2} \epsilon^2 C D_\alpha \rho_{1CD}, \\
\tilde{A}_\mu^c &= \Lambda^c_{bd} f^{ab}_{cd} \\
\Lambda^c_{bd} &= i(\epsilon^2 D E_{1CE} - \epsilon_1 D E_{2CE}) \bar{Z}^c D_\mu Z^D_c.
\end{align*}

The first term of eq. (3.14) is a translation, and the second represents a gauge transformation, as expected. In deriving (3.14), we have used Eq. (3.3): $f^{ba}_{cd} = -f^{ba}_{cd}$.

For the gauge field, using the FI (3.11) and some identities in the Appendix, we obtain

\begin{equation}
[\delta_1, \delta_2] \tilde{A}_\tilde{c} \tilde{d} = v^\nu \partial_\nu \tilde{A}_\tilde{c} \tilde{d} + D_\mu (\tilde{A}_\tilde{c} \tilde{d} - v^\nu \tilde{A}_\tilde{c} \tilde{d})
\end{equation}

\begin{equation}
+ v^\nu \left[ \tilde{F}_{\mu\nu} \tilde{c} \tilde{d} + \varepsilon_{\mu\nu\lambda} \left( D^\lambda Z^A \bar{Z}^b - Z^A D^\lambda \bar{Z}^b - i \bar{\psi}^A \gamma^\lambda \psi_{Aa} \right) f^{ac}_{bd} \right].
\end{equation}

where $\tilde{F}_{\mu\nu} \tilde{c} \tilde{d} = \partial_\mu \tilde{A}_\nu \tilde{c} \tilde{d} - \partial_\nu \tilde{A}_\mu \tilde{c} \tilde{d} + [\tilde{A}_\mu, \tilde{A}_\nu] \tilde{c} \tilde{d}$ is the field strength. We recognize the first term as a translation, and the second a gauge transformation. To achieve the closure, we need to impose the following equation of motion for the gauge field:

\begin{equation}
\tilde{F}_{\mu\nu} \tilde{c} \tilde{d} = -\varepsilon_{\mu\nu\lambda} \left( D^\lambda Z^A \bar{Z}^b - Z^A D^\lambda \bar{Z}^b - i \bar{\psi}^A \gamma^\lambda \psi_{Aa} \right) f^{ac}_{bd}.
\end{equation}
As BL discovered \cite{27}, the FI implies $D_\mu \Gamma^{ca}_{bd} = 0$, if one writes $\tilde{A}_\mu^c = A_\mu^b f^{ca}_{bd}$ in the expression of the covariant derivative. We have used this important equation to derive the second term in eq. (3.20): $f^{ca}_{bd} D_\mu \Lambda^b_a = D_\mu \tilde{A}_d^c$.

The commutator of two supersymmetry transformations acting on the fermionic fields reads

$$ \left[ \delta_1, \delta_2 \right] \psi_{Dd} = v^\mu \partial_\mu \psi_{Dd} + (\tilde{A}_d^c - v^\mu \tilde{A}_\mu^d) \psi_{Da}$$

$$- \frac{i}{2} (\epsilon_1^{AC} \epsilon_2^{2AD} - \epsilon_2^{AC} \epsilon_1^{AD}) E_{Cd}$$

$$+ \frac{i}{4} (\epsilon_1^{AB} \gamma_\nu \epsilon_2^{2AB}) \gamma^\nu E_{Dd},$$

(3.22)

where

$$E_{Cd} = \gamma^\mu D_\mu \psi_{Cd} + f^{ab}_{cd} \left( \psi_{Ca} Z_b^d Z_c^e - 2 \psi_{Da} Z_b^d Z_c^e - \epsilon_{CDEFG} \psi_{Dc} Z_a^e Z_b^F \right).$$

(3.23)

Again, the first two terms are a translation and a gauge transformation, respectively. To achieve the closure of the supersymmetry algebra, we have to impose the following equations of motion for the fermionic fields:

$$0 = E_{Cd} = \gamma^\mu D_\mu \psi_{Cd} + f^{ab}_{cd} \left( \psi_{Ca} Z_b^d Z_c^e - 2 \psi_{Da} Z_b^d Z_c^e - \epsilon_{CDEFG} \psi_{Dc} Z_a^e Z_b^F \right).$$

(3.24)

To derive the equations of motion of the scalar fields, we take the super-variation of the equations of motion of the fermionic fields: $\delta E_{Cd} = 0$. Two equations are obtained: One is

$$0 = D_\mu D^\mu Z_c^B - i f^{ab}_{cd} \left( \tilde{\psi}^A_d \tilde{\psi}^B_a Z_b^c Z_D^e - 2 \tilde{\psi}^B_d \tilde{\psi}^A_a Z_b^c Z_D^e - \epsilon^{ABCD} \tilde{\psi}^A_a \tilde{\psi}^B_b Z_D^c \right)$$

$$+ \frac{1}{3} \left( f^{ae}_{fd} f^{bd}_{cg} - 2 f^{ab}_{cd} f^{ed}_{fg} - 2 f^{ab}_{cd} f^{ed}_{fg} + 2 f^{ab}_{cd} f^{ed}_{cg} - 4 f^{ab}_{cd} f^{ed}_{cg} \right) Z_e^B Z_f^C Z_g^D Z_h^F. \tag{3.25}$$

The other equation is equivalent to the equation of motion of the gauge field (3.21).

The equations of motions of the gauge, fermion and scalar fields, Eqs. (3.21), (3.24) and (3.25) respectively, can be derived from the following Lagrangian:

$$\mathcal{L} = -D_\mu \tilde{Z}_A^a D^\mu Z_A^a - i \tilde{\psi}_A^a \gamma^\mu D_\mu \psi_A^a - V - \mathcal{L}_{CS}$$

$$- i f^{ab}_{cd} \tilde{\psi}_A^a \tilde{\psi}_B^b Z_B^c Z_C^d + 2 i f^{ab}_{cd} \tilde{\psi}_A^a \tilde{\psi}_B^b Z_B^c Z_C^d$$

$$- \frac{i}{2} \epsilon^{ABCD} f^{ab}_{cd} \tilde{\psi}_A^a \tilde{\psi}_B^b Z_C^c Z_D^d - \frac{i}{2} \epsilon^{ABCD} f^{ab}_{cd} \tilde{\psi}_A^a \tilde{\psi}_B^b Z_C^c Z_D^d. \tag{3.26}$$

Here the scalar potential $V$ is

$$V = 2 \left( f^{ab}_{cd} f^{ed}_{fg} - \frac{1}{2} f^{ab}_{cd} f^{ed}_{fg} \right) \tilde{Z}_A^a Z_B^c Z_C^d Z_D^e Z_f^g Z_h^l. \tag{3.27}$$

It can be recast into the following form \cite{27}:

$$V = \frac{2}{3} \gamma_B^a \gamma_C^b.$$. 

(3.28)
where

\[ \Upsilon_{Bd}^{CD} = f_{ab}^{\cd} \left( Z_a^C Z_b^D \bar{Z}_c^E - \frac{1}{2} \delta_a^{C} Z_b^D \bar{Z}_E^c + \frac{1}{2} \delta_b^{D} Z_a^E \bar{Z}_E^c \right), \]  

(3.29)

and the quantity \( \bar{\Upsilon}_{Bd}^{CD} \) is the complex conjugate of \( \Upsilon_{Bd}^{CD} \):

\[ \bar{\Upsilon}_{Bd}^{CD} = \bar{f}_{cd}^{ab} \left( \bar{Z}_a^C \bar{Z}_b^D Z_c^E - \frac{1}{2} \delta_c^{E} \bar{Z}_b^D \bar{Z}_E^a + \frac{1}{2} \delta_b^{D} \bar{Z}_a^E \bar{Z}_E^a \right), \]  

(3.30)

where the reality condition of the structure constants \( f^{*ab} = f^{cd} \) has been used.

The Chern-Simons term in the Lagrangian is

\[ L_{CS} = \frac{1}{2} \epsilon^{\mu \nu \lambda} \left( f^{ab}_{\mu c} A_{\mu}^c b \partial_\nu A_{\lambda}^a + \frac{2}{3} f^{ac}_{f d} f^{ge}_{f b} A_{\mu}^b A_{\nu}^c A_{\lambda}^d A_\epsilon^f \right). \]  

(3.31)

Notice that we have used the (positive definite) hermitian bilinear form (2.11) to construct the Lagrangian of the matter fields. This bilinear form is invariant under our gauge transformations which preserve the symplectic metric. Therefore our Lagrangian is gauge invariant. To derive the equations of motion of the scalar from the Lagrangian, one needs to use the FI (3.11). The equations of motion derived from the Lagrangian (3.26) are invariant under the 12 super-symmetries, and reproduce those we have imposed before for on-shell closure of the super-symmetries.

### 3.2 \( \mathcal{N} = 6, Sp(2N) \times U(1) \) CSM Theory

We first specify the structure constants as

\[ f_{a-,b-,c+,d+} = -k[(\omega_{ab}\omega_{cd} + \omega_{ad}\omega_{bc})h_{++}h_{++} + (\omega_{ad}\epsilon_{++})(\omega_{bc}\epsilon_{++})]. \]  

(3.32)

where \( h_{++} = h_{++} = 1 \) and \( \epsilon_{++} = -\epsilon_{++} = ih_{+-} \). The structure constants have the symmetry property (3.4), (3.6) and (3.8), obey the reality condition (3.7), and satisfy the FI (3.5). We observe that \( f_{d+, b-, c+, a-} \neq f_{b-, d+, c+, a-} = 0 \). However, this is not inconsistent with eq. (3.8), since \( d+ \) and \( b- \) do not have the same gauge transformation property. Using \( \omega^{a+,b-} \) to raise the first two pairs of indices of the structure constants, we get

\[ f^{a+b+c+d+} = k[(\omega_{ab}\omega_{cd} - \delta_{c}^{a} \delta_{d}^{b})\delta^{+-} + \delta^{+-} - (\delta_{d}^{a})(-i\delta^{+-})(\delta_{c}^{b})(-i\delta^{+-})], \]  

(3.33)

where we have used \( \epsilon_{++} = -ih_{+-} \). Suppressing the \( SO(2) \) indices gives

\[ f^{ab}_{cd} = k(\omega_{ab}\omega_{cd} - \delta_{c}^{a} \delta_{d}^{b} + \delta_{d}^{a} \delta_{c}^{b}). \]  

(3.34)

We notice that (3.34) takes the same form as the components of an embedding tensor in Ref. [22]. This is not just an accident, and we will investigate their connection further in a coming paper.
Substituting eq. (3.34) into eq. (3.26), and replacing \( A^b_a \) in the Lagrangian by \( \frac{1}{k} A^b_a \), we obtain the following Lagrangian:

\[
\mathcal{L} = -D_\mu \tilde{Z}^a_A D^\mu Z^a_A - \frac{i}{2} \tilde{\psi} A \gamma^\mu D_\mu \psi_{\bar{A}} - V - \mathcal{L}_{CS}
\]

\[
+ i k \left( \tilde{Z}^b_B \tilde{\psi}_{A b \bar{A}} \gamma^r Z^r_A - \tilde{Z}^b_B \tilde{\psi} A a \psi_{\bar{A} a} - \tilde{Z}^c_B \omega_{c d} \tilde{\psi} A d \psi_{\bar{A} c} Z^d_B \right)
\]

\[\text{Eqn. (3.35)}\]

\[
- 2 i k \left( \tilde{Z}^b_B \tilde{\psi}_{A b \bar{A}} Z^r_A - \tilde{Z}^b_B Z^c_B \tilde{\psi}_{A c a} \psi_{\bar{A} a} - \tilde{Z}^c_B \omega_{c d} \tilde{\psi} A d \psi_{\bar{A} c} Z^d_B \right)
\]

\[
-i k \varepsilon_{A B C D} \left( \tilde{Z}^a_A \tilde{\psi}_{B a} \tilde{Z}^b_C \tilde{\psi}_{D b} - \frac{1}{2} \tilde{Z}^a_A \omega_{c d} \tilde{Z}^c_B \tilde{\psi}_{D b} \psi_{\bar{A} c} \right)
\]

\[
-i k \varepsilon_{A B C D} \left( \tilde{\psi}_{B a} \tilde{Z}^a_A \tilde{\psi}_{D c} Z^c_B - \frac{1}{2} \tilde{Z}^a_A \omega_{c d} \tilde{Z}^c_B \tilde{\psi}_{D b} \psi_{\bar{A} c} \right)
\]

Here the potential is

\[
-V = -3 k^2 \tilde{Z}^a_A \omega_{c d} \tilde{Z}^b_B \tilde{Z}^c_{D a} \tilde{Z}^d_{B b} + \frac{5 k^2}{3} \tilde{Z}^a_A \omega_{c d} \tilde{Z}^c_{B b} \tilde{Z}^d_{B b} \tilde{Z}^a_A ,
\]

\[\text{Eqn. (3.36)}\]

and the Chern-Simons term is

\[
\mathcal{L}_{CS} = \frac{1}{2 k} \varepsilon^{\mu \nu \lambda} A_{\mu} \partial_\nu A_{\lambda} - \frac{1}{4 k} \varepsilon^{\mu \nu \lambda} tr \left( B_\mu \partial_\nu B_{\lambda} + \frac{2}{3} B_\mu B_\nu B_{\lambda} \right),
\]

\[\text{Eqn. (3.37)}\]

where the gauge fields \( B_{\mu}^c d \equiv - (A_{\mu} c d + A_{\mu}^c d) \) and \( A_{\mu} \equiv A_{\mu}^a a \) are defined by the following equation:

\[
\tilde{A}_{\mu}^c d = A_{\mu}^b a f_{c d}^{a b} = - (A_{\mu}^c d + A_{\mu}^a a d) + (A_{\mu}^a a) \delta^c_d.
\]

\[\text{Eqn. (3.38)}\]

The expression of the structure constants (3.34) has been used. We recognize that \( A_{\mu} \) is the \( U(1) \) part of the gauge potential, and \( B_{\mu}^c d \) the \( Sp(2N) \) part, since it can be written as \( B_{\mu}^c d = A_{\mu}^{a b} (t_{a b})^c_d \), where \( (t_{a b})^c_d \) is in the defining representation of the Lie 2-algebra of \( Sp(2N) \).

The Lie 2-algebra of the gauge group is generated by the FI (3.12). Formally, we can think of the structure constants \( f_{c d}^{a b} \) as ordinary matrix elements by defining \( (f_{c d}^{a b})^a_d \equiv f_{c d}^{b a} \). In other words, \( f_{c d}^{a b} \) is the matrix, while \( (f_{c d}^{a b})^a_d \) are its matrix elements. Then the Fundamental Identity,

\[
f_{c d}^{a b} f_{e f}^{d e} f_{g f} - f_{e f}^{a e} f_{d b}^{b g} + f_{e f}^{b d} f_{d g}^{a f} - f_{e d}^{a g} f_{c d}^{f e} = 0,
\]

\[\text{Eqn. (3.39)}\]

can be written as a commutator of two matrices:

\[
[f_{c d}^{a b} f_{e f}^{d e}]^a_g = f_{c d}^{b f} (f_{d f}^{a e})^a_g - f_{c d}^{e b} (f_{d f}^{a e})^a_g.
\]

\[\text{Eqn. (3.40)}\]

It is in this sense the FI generates an ordinary Lie 2-algebra. Specifying \( f_{c d}^{a b} \) amounts to choosing a particular set of structure constants of the ordinary Lie 2-algebra. In this way,
the Lie 2-algebra of the gauge group is completely determined through the FI. It turns out that the 3-algebra structure constants $f_{cd}^{ab}$ given by Eq. (3.34) precisely generate the Lie 2-algebra of the $Sp(2N) \times U(1)$ gauge group through the FI. 

Substituting the expression of the structure constants (3.34) into (3.13), the supersymmetry transformation law now reads

$$
\delta Z_d^A = i\epsilon^{AB} \psi_B^d, \\
\delta Z_A^d = i\epsilon_{AB} \psi^B_d.
$$

$$
\delta \psi_{Bd} = \gamma^\mu D_\mu Z_d^A \epsilon_{AB} - kZ_a^C \omega_{ab} Z_b^A \omega_{dc} \bar{Z}_C^e \epsilon_{AB} - kZ_a^C \omega_{ab} Z_b^A \omega_{dc} \bar{Z}_C^e \epsilon_{CD}
$$

$$
- kZ_a^C Z_d^A \epsilon_{AB} + k\bar{Z}_C^e Z_d^A \epsilon_{AB} - 2k\bar{Z}_B^e Z_a^C Z_d^A \epsilon_{CD}
$$

$$
\delta \psi_{Bd} = \gamma^\mu D_\mu Z_d^A \epsilon_{AB} - kZ_a^C \omega_{ab} \bar{Z}_d^\omega \omega_{dc} \epsilon^{AB} - kZ_a^C \omega_{ab} \bar{Z}_d^\omega \omega_{dc} \epsilon^{CD}
$$

$$
- kZ_a^C Z_d^A \epsilon_{AB} + k\bar{Z}_C^e Z_d^A \epsilon^{AB} - 2k\bar{Z}_B^e Z_a^C Z_d^A \epsilon^{CD}
$$

$$
\delta A_\mu = -i\epsilon_{AB} \gamma_\mu \psi_{Ba} Z_a^A + i\epsilon^{AB} \gamma_\mu \bar{Z}_a^A \psi_{Ba}
$$

$$
\delta B_\mu^c = i\epsilon_{AB} \gamma_\mu a \omega^{ca} Z_d^A \omega_{db} \psi^B_d - i\epsilon^{AB} \gamma_\mu \omega_{db} \bar{Z}_d^a \omega^{ca} \psi_{Ba}
$$

$$
+ i\epsilon_{AB} \gamma_\mu \bar{Z}_d^A \psi_{Bd} - i\epsilon^{AB} \gamma_\mu \bar{Z}_a^A \psi_{Bd}.
$$

(3.41)

The Lagrangian (3.33) and the corresponding supersymmetry transformation law (3.41) are in agreement with the $\mathcal{N} = 6, Sp(2M) \times O(2)$ superconformal CSM theory derived from ordinary Lie 2-algebra in Ref. [25].

4. Recasting $\mathcal{N} = 6, U(M) \times U(N)$ Theory into Symplectic 3-Algebra Framework

The $\mathcal{N} = 6, U(M) \times U(N)$ theory was derived by BL in a 3-algebra framework [27] with a hermitian metric. In this section, we will demonstrate that the $\mathcal{N} = 6, U(M) \times U(N)$ theory can be actually recast into our symplectic 3-algebraic framework.  

To generate the direct group $U(M) \times U(N)$, we decompose the 3-algebra index $I$ into $\alpha$, where $\alpha = 1, 2$, and $a = 1, \cdots, MN$ stands for the bi-fundamental index of $U(M) \times U(N)$. We then decompose the anti-symmetric metric $\omega^{IJ}$ as

$$
\omega^{IJ} = \begin{pmatrix}
0 & -\delta^a_b \\
\delta_a^b & 0
\end{pmatrix}.
$$

(4.1)

The component formalism of the above equation reads

$$
\omega^{IJ} = \omega^{\alpha, \beta} = -\delta^a_b \delta_{\alpha \beta} + \delta_a^b \delta_{\alpha \beta}.
$$

(4.2)

We then decompose the structure constants $f_{IKLJ}$ as

$$
f_{IKLJ} = f_{\alpha, \gamma, \delta, \beta} = f^{ac}_{\delta, \alpha, \gamma, \delta} \delta_{\beta} + f^{ac}_{\alpha, \gamma, \delta, \beta} \delta_{\gamma} + f^{bc}_{\alpha, \gamma, \delta, \beta} \delta_{\delta} + f^{bc}_{\delta, \alpha, \gamma, \delta} \delta_{\alpha} + f^{ac}_{\alpha, \gamma, \delta, \beta} \delta_{\beta} + f^{bc}_{\delta, \alpha, \gamma, \delta} \delta_{\gamma} + f^{bc}_{\alpha, \gamma, \delta, \beta} \delta_{\delta},
$$

(4.3)

<sup>1</sup>This section is inspired by the referee’s comment.

<sup>2</sup>In section 3 the index $a = 1 \cdots 2N$ is an $Sp(2N)$ index. We hope this does not cause any confusion.
One still need to impose an additional constraint condition for closing the $\mathcal{N} = 6$ superalgebra:

$$f^{ab}_{\ \ cd} = -f^{ba}_{\ \ cd}. \quad (4.4)$$

With these decompositions, the reality condition (2.9) becomes

$$f^{*ac}_{\ \ db} = f^{db}_{\ \ ac}, \quad (4.5)$$

and the FI (2.2) reads

$$f^{ab}_{\ \ cd}f^{de}_{\ \ gf} + f^{ba}_{\ \ fd}f^{de}_{\ \ gc} - f^{ae}_{\ \ gd}f^{db}_{\ \ cf} - f^{be}_{\ \ gd}f^{da}_{\ \ fc} = 0. \quad (4.6)$$

Using (4.5), it is not difficult to verify that the RHS of (4.3) satisfies the desired symmetry properties. The above three equations (4.4), (4.5) and (4.6) take exactly the same forms as Eqs (3.9), (3.10) and (3.11), respectively. They also take exactly the same forms as that of BL [27]. In other words, they must belong to the hermitian 3-algebra.

With the decomposition (4.1), it is natural to decompose a 3-algebra valued field $X_I$ as

$$X_I = \begin{pmatrix} \bar{X}^a \\ X_a \end{pmatrix}, \quad (4.7)$$

where $\bar{X}^a = X_a^*$ is the complex conjugate of $X_a$. Later we will see, with the decomposition (4.7), the complex conjugate $X_I^*$ transforms in the same way as $\omega^{IJ}X_J$ (see Eq. (4.8)). We then decompose the parameter $\Lambda^K_{\ L}$ in the global transformation (2.4) as

$$\Lambda^K_{\ L} = \frac{1}{2} \begin{pmatrix} -\Lambda^c_{\ d} & 0 \\ 0 & \Lambda^d_{\ c} \end{pmatrix}. \quad (4.8)$$

Here we require that $\Lambda^c_{\ d}$ is anti-hermitian, i.e., $\Lambda^{*c}_{\ d} = -\Lambda^d_{\ c}$, guaranteeing the anti-hermitian condition $\Lambda^{*K}_{\ L} = -\Lambda^{L}_{\ K}$. With the decompositions (4.1), (4.3) and (4.8), the global transformation (2.4) becomes

$$\begin{pmatrix} \delta_{\bar{\Lambda}}\bar{X}^a \\ \delta_{\bar{\Lambda}}X_a \end{pmatrix} = \begin{pmatrix} \Lambda^c_{\ d}f^{ad}_{\ \ cb} & 0 \\ 0 & -\Lambda^d_{\ c}f^{bd}_{\ \ ca} \end{pmatrix} \begin{pmatrix} \bar{X}^b \\ X_b \end{pmatrix} = \begin{pmatrix} \bar{\Lambda}^a_{\ b} & 0 \\ 0 & -\bar{\Lambda}^b_{\ a} \end{pmatrix} \begin{pmatrix} \bar{X}^b \\ X_b \end{pmatrix} = -\bar{\Lambda}^{I\ J}X_I, \quad (4.9)$$

and Eq. (2.5) is satisfied. Note that the reality conditions $\Lambda^{*c}_{\ d} = -\Lambda^d_{\ c}$ and (4.3) imply that $\bar{\Lambda}^{*a}_{\ b} = -\bar{\Lambda}^b_{\ a}$. So from ordinary Lie group point of view, the matrices $-\bar{\Lambda}^{I\ J}$ are in the $R \oplus R^*$ representation. To construct an $\mathcal{N} = 6$ theory, we need only to focus on the global transformation

$$\delta_{\bar{\Lambda}}X_a = -\bar{\Lambda}^a_{\ b}X_b. \quad (4.10)$$

One can easily obtain its complex conjugate. To gauge the above symmetry, we introduce the following gauge field

$$\tilde{A}_{\mu}^{\ a\ d} = A_{\mu}^{\ \ c\ b}f^{ab}_{\ \ cd}. \quad (4.11)$$
Here we require that the 3-algebra tensor $A^c_{\mu} \epsilon^a_b$ satisfies the reality condition $A^{a c}_{\mu} \epsilon^b_c = -A^{b c}_{\mu} \epsilon^a_c$. Eqs. (4.10) and (4.11) suggest us to define the scalar and fermion fields as follows

$$Z^A_c \quad \text{and} \quad \psi^{Ac}_a,$$  \hspace{1cm} (4.12)

where $A$ is an $SU(4)$ R-symmetry index. Their complex conjugates are denoted as $Z^*_c = \overline{Z^c}_A$ and $\psi^{*a}_c = \psi^{Ac}$. Note that all the fields also take exactly the same forms as those in section 3.1, though here the index $a$ runs from 1 to $MN$, and the fields do not carry the $U(1)$ gauge group index $+$. However, since $f$ becomes equals (3.8) or (3.9), and (4.4)). This condition can be understood as the $3$-algebra framework, by introducing the antisymmetric metric (4.2) and the ‘map’ (4.3).

So, if we also follow BL’s strategy to construct an $\mathcal{N} = 6$ theory, the supersymmetry transformations must take exactly the same forms as (3.13), i.e.

$$\delta Z^A_d = i\epsilon^{AB} \psi_{Bd},$$

$$\delta \psi_{Bd} = \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f_{\cd}^{ab} Z^C_b Z^C_d \epsilon_{AB} + f_{\cd}^{ab} Z^C_d \overline{Z^C}_B \epsilon_{CD},$$

$$\delta A^{c}_{\mu} \epsilon^a_b = -i\epsilon^{AB} \gamma_\mu Z^A_d \psi^{Bb}_a f^{ca}_{\cd} + i\epsilon^{AB} \gamma_\mu \overline{Z^c}_A \psi^{Bb}_a f^{cb}_{\cd},$$  \hspace{1cm} (4.13)

and their complex conjugates. And the equations of motion, required by the on-shell closure of the superalgebra, must also take exactly the same forms as those in section 3.1. Hence the Lagrangian of this section must also take the exact form as (3.26). Eqs (3.13) and (3.26) take exactly the same forms as BL’s general $\mathcal{N} = 6$ supersymmetry transformations and the Lagrangian [27], respectively. To derive the $U(M) \times U(N)$ theory, one just needs to adopt the specified 3-bracket

$$[X, Y; \bar{Z}] = k(XZ^\dagger Y - YZ^\dagger X)$$  \hspace{1cm} (4.14)

from Ref. [27], and write the hermitian bilinear form as

$$\bar{X}^a \epsilon_{\hat{a}} \epsilon_{\hat{a}} = \bar{X}^a \epsilon_{\hat{a}} \epsilon_{\hat{a}} = (X^a \epsilon_{\hat{a}} \epsilon_{\hat{a}})_{\hat{a}} \epsilon_{\hat{a}} - \epsilon_{\hat{a}} = \operatorname{tr}(X^a \epsilon_{\hat{a}} \epsilon_{\hat{a}}),$$  \hspace{1cm} (4.15)

Here $X^a \epsilon_{\hat{a}}$ is an $a \times n$ matrix, where $m = 1, \cdots, M$ is a fundamental index of $U(M)$ and $\hat{a} = 1, \cdots, N$ is an anti-fundamental index of $U(N)$, and $\overline{X}^\dagger$ is the hermitian conjugate of the $m \times \hat{n}$ matrix $X$. Substituting (4.14) and (4.13) into (3.26) reproduces the $U(M) \times U(N)$ theory [27].

In this way, one can ‘convert’ Bagger and Lambert’s framework into our symplectic $3$-algebraic framework, by introducing the antisymmetric metric (4.2) and the ‘map’ (4.3).

Finally, we would like to add one comment on the constraint condition $f^{\cd}_{\ab} = -f^{\ab}_{\cd}$ (see (3.8) or (3.9), and (4.4)). This condition can be understood as

$$f_{(IJK)L} = 0.$$

Since $f_{IJKL} = f_{IKJL}$ (see Eq. (2.10)) and Eq. (2.3) implies that $f_{IJKL} = f_{LJKI}$, the above equation is equivalent to $f_{(IJKL)} = 0$. Specifically, in the $Sp(2N) \times U(1)$ case, it becomes

$$f_{a-b-c+d} + f_{b-a-c+d} + f_{c+a-b-d} = 0. \quad (4.17)$$

However, since $f_{c+a-b-d} = 0$ (see Eq. (3.32) and the explanation below (3.32)), we obtain

$$f_{a-b-c+d} + f_{b-a-c+d} = 0,$$  \hspace{1cm} (4.18)
which is nothing but (3.8). In the $U(M) \times U(N)$ case, with the ‘map’ (4.3) and the reality condition (4.7), Eq. (4.16) becomes (4.4). The ordinary Lie algebra counterpart of (4.16), first discovered in Ref. [23], is the key requirement for enhancing the $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 4$.

5. Conclusions and Discussions

In this paper, we first introduce the notion of symplectic 3-algebras. We then give a formulation of $\mathcal{N} = 6$ superconformal Chern-Simons-matter (CSM) theory with $SU(4)$ R-symmetry based on the symplectic 3-algebras. By specifying the 3-brackets, we derive the $\mathcal{N} = 6, Sp(2N) \times U(1)$ superconformal CSM theory in our framework. We also recast the $\mathcal{N} = 6, U(M) \times U(N)$ into the symplectic 3-algebraic framework.

The $\mathcal{N} = 6$ superconformal CSM theories in three dimensions have been completely classified in Ref. [28] by using group theory. The $\mathcal{N} = 6$ CSM theories can also be classified by super Lie algebras [23, 25, 29]. Essentially, only two types are allowed: with gauge group $Sp(2N) \times U(1)$ and $U(M) \times U(N)$, respectively. Therefore our approach provides a unified 3-algebra framework to describe all known $\mathcal{N} = 6$ superconformal theories. Though our approach to the $\mathcal{N} = 6$ theories is essentially equivalent to that of BL [27], our formulation is slightly different from the latter, in that ours is more suited to the case with gauge group $Sp(2N) \times U(1)$.

The question of reformulating the known CMS models in a 3-algebra approach is not merely of mathematical interests. More important is whether or not the M2-branes physics would become more transparent if looked through a new mathematical framework such as 3-algebras.

It would be also nice to find the gravity dual of the $\mathcal{N} = 6, Sp(2N) \times U(1)$ CSM theory, and to investigate the integrability from both the gauge theory side and string/M theory side [23]. It would be interesting to generalize the symplectic 3-algebra model so that its gauge group has more general product structure, like those in quiver gauge theories.

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A. Conventions and Useful Identities

In $1+2$ dimensions, the gamma matrices are defined as $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. For the metric we use the $(-,+,+)$ convention. The gamma matrices can be defined as the Pauli matrices: $\gamma_\mu = (i\sigma_2, \sigma_1, \sigma_3)$, satisfying the important identity

$$\gamma_\mu \gamma_\nu = \eta_{\mu\nu} + \varepsilon_{\mu\nu\lambda} \gamma^\lambda. \quad (A.1)$$

We also define $\varepsilon^{\mu\nu\lambda} = -\varepsilon_{\mu\nu\lambda}$. So $\varepsilon_{\mu\nu\lambda} \varepsilon^{\rho\sigma\lambda} = -2\delta_{\mu}^{\rho}$. 
The following identities are adopted from Ref. [27]. In 1 + 2 dimensions the Fierz transformation is

\[ (\bar{\lambda} \chi) \psi = -\frac{1}{2} (\bar{\lambda} \psi) \chi - \frac{1}{2} (\bar{\lambda} \gamma_\nu \psi) \gamma^\nu \chi. \]  
(A.2)

Some useful $SU(4)$ identities are

\[
\begin{align*}
\frac{1}{2} \epsilon_1^{CD} \gamma_\nu \epsilon_{2CD} \delta_B^A &= \epsilon_1^{AC} \gamma_\nu \epsilon_{2BC} - \epsilon_2^{AC} \gamma_\nu \epsilon_{1BC} \\
2 \epsilon_1^{AC} \epsilon_{2BD} - 2 \epsilon_2^{AC} \epsilon_{1BD} &= \epsilon_1^{CE} \epsilon_{2DE} \delta_B^A - \epsilon_2^{CE} \epsilon_{1DE} \delta_B^A \\
&\quad - \epsilon_1^{AE} \epsilon_{2DE} \delta_B^C + \epsilon_2^{AE} \epsilon_{1DE} \delta_B^C \\
&\quad + \epsilon_1^{AE} \epsilon_{2BD} \delta_D^C - \epsilon_2^{AE} \epsilon_{1BD} \delta_D^C \\
&\quad - \epsilon_1^{CE} \epsilon_{2BE} \delta_D^A + \epsilon_2^{CE} \epsilon_{1BE} \delta_D^A \\
\frac{1}{2} \epsilon_{ABCD} \epsilon_1^{EF} \gamma_\mu \epsilon_{2EF} &= \epsilon_{1AB} \gamma_\mu \epsilon_{2CD} - \epsilon_{2AB} \gamma_\mu \epsilon_{1CD} \\
&\quad + \epsilon_{1AD} \gamma_\mu \epsilon_{2BC} - \epsilon_{2AD} \gamma_\mu \epsilon_{1BC} \\
&\quad - \epsilon_{1BD} \gamma_\mu \epsilon_{2AC} + \epsilon_{2BD} \gamma_\mu \epsilon_{1AC}. \nonumber \end{align*} \]  
(A.3)

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