THE ENTIRE CYCLIC COHOMOLOGY OF NONCOMMUTATIVE 2-TORI

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ABSTRACT. Our aim in this paper is to compute the entire cyclic cohomology of noncommutative 2-tori. First of all, we clarify their algebraic structure of noncommutative 2-tori as a $F^*$-algebra, according to the idea of Elliott-Evans. Actually, they are the $F^*$-inductive limit of subhomogeneous $F^*$-algebras. Using such a result, we compute their entire cyclic cohomology, which is isomorphic to their periodic one as a complex vector space.

1. INTRODUCTION

Elliott and Evans [3] show that the irrational rotation $C^*$-algebras (or noncommutative 2-tori) $T^2_\theta$ are isomorphic to certain inductive limits, which are now called AT-algebras, 

$$\lim (C(T) \otimes (M_{\mathbb{Q}_{2n}}(\mathbb{C}) \oplus M_{\mathbb{Q}_{2n-1}}(\mathbb{C})), \pi_n).$$

To compute the entire cyclic cohomology of their smooth parts $(T^2_\theta)^\infty$, we need to know their algebraic structure. In this paper, we elaborate Elliott and Evans’ result cited above, and show that $(T^2_\theta)^\infty$ are isomorphic to inductive limits 

$$\lim (C^\infty(T) \otimes (M_{\mathbb{Q}_{2n}}(\mathbb{C}) \oplus M_{\mathbb{Q}_{2n-1}}(\mathbb{C})), \pi_n^\infty)$$

as Fréchet $^*$-algebras (or $F^*$-algebras). Using this fact, we can compute their entire cyclic cohomology quite easily.

In Sect.2, we prepare the notations needed for $(T^2_\theta)^\infty$ and review the definition of entire cyclic cohomology. In Sect.3, we determine the algebraic structure of $(T^2_\theta)^\infty$ by using appropriate smooth functions to construct projections based on Connes [1] instead of the original ones due to Rieffel [6]. In Sect.4, it is shown that the functor of entire cyclic cohomology $H^*$ is continuous in some sense. More precisely, 

$$H^*_e(\lim A_n) \simeq \lim H^*_e(A_n)$$

(cf. Meyer [5]), where the right hand side means the projective limit of $H^*_e(A_n)$ which will be defined in the same section.

Our main result is stated in Sect.5.
2. Preliminaries

First of all, we define some notations for our discussion in this section.

Given an irrational number \( \theta \), let us treat the noncommutative 2-tori \((T^2_\theta)^\infty\) generated by two unitaries \( u, v \) with relation
\[
uv = e^{2\pi i \theta} vu
\]
as a Fréchet \( * \)-algebra (or \( F^* \)-algebra). In some cases, we regard each element of \((T^2_\theta)^\infty\) as an operator on the Hilbert space \( L^2(T) \) of the square integrable complex valued functions on the 1-torus \( T \). For instance,
\[
(uf)(t) = tf(t), \quad (vf)(t) = f(e^{-2\pi i \theta} t)
\]
for \( f \in L^2(T), \ t \in T \).

There is a smooth action \( \alpha \) of \( T^2 \) on \((T^2_\theta)^\infty\) defined by
\[
\alpha_{t,s}(u) = tu, \ \alpha_{t,s}(v) = sv
\]
for \( t, s \in T \). Moreover, we have the two \( * \)-derivations \( \delta_1, \delta_2 \) on \((T^2_\theta)^\infty\) associated with \( \alpha \) satisfying
\[
\delta_1(u) = iu, \ \delta_2(u) = 0, \ \delta_1(v) = 0, \ \delta_2(v) = iv.
\]
Using these derivations, we define seminorms \( \| \cdot \|_{k,l} \) on \((T^2_\theta)^\infty\) by
\[
\| x \|_{k,l} = \| \delta_1^k \circ \delta_2^l(x) \|
\]
where \( \| \cdot \| \) is the usual \( C^* \)-norm on \( T^2_\theta \).

Here, we briefly review the definition of entire cyclic cohomology. For any unital \( F^* \)-algebra \( \mathfrak{A} \) and any integer \( n \geq 0 \), we put \( C^n \) be the set of all \((n+1)\)-linear functionals on \( \mathfrak{A} \). For \( n < 0 \), let \( C^n = \{0\} \). Moreover, we define
\[
C^{ev} = \{ (\varphi_{2n})_n \mid \varphi_{2n} \in C^{2n} (n \geq 0) \},
\]
\[
C^{od} = \{ (\varphi_{2n+1})_n \mid \varphi_{2n+1} \in C^{2n+1} (n \geq 0) \}.
\]

We call \( (\varphi_{2n}) \) an entire even cochain if for each bounded subset \( \Sigma \subset \mathfrak{A} \), we can find a constant \( C > 0 \) such that
\[
|\varphi_{2n}(a_0, \ldots, a_{2n})| \leq C \cdot n!
\]
for all \( n \geq 1 \) and \( a_j \in \Sigma \). In odd case, we define entire odd cochains by the same way as in even case. We denote by \( C^{ev}_\varepsilon \) (resp. \( C^{od}_\varepsilon \)) the set of all entire even (resp. odd) cochains. Then we define the entire cyclic cohomology of \( \mathfrak{A} \) by the cohomology of the short complex
\[
C^{ev}_\varepsilon \xrightarrow{\partial} C^{od}_\varepsilon,
\]
where \( \partial \) are certain derivativivions defined by Connes [2].
3. \((T^2_\theta)^\infty\) is a Fréchet Inductive Limit

In this section, we prove the key lemma which states that noncommutative 2-tori \((T^2_\theta)^\infty\) as \(F^*\)-algebras are isomorphic to inductive limits

\[
\varprojlim_n (C^\infty(T) \otimes (M_{q_n}(\mathbb{C}) \oplus M_{q_{2n-1}}(\mathbb{C})), \pi_n^\infty),
\]

where the sequence \(\{q_{2n-1}\}_n\) appears in the continued fraction expansion of \(\theta\).

Let \(\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in SL(2, \mathbb{Z})\) with \(p/q < \theta < p'/q'\), \(q > 0\) and \(q' > 0\) for each fixed \(\theta \in (0, 1)\). We write \(\beta = p' - q' \theta, \beta' = q \theta - p\). First of all, we construct two projections \(e_\beta\) and \(e_{\beta'}\) in \((T^2_\theta)^\infty\) with traces \(\beta\) and \(\beta'\) respectively using the functions \(f_\beta\) and \(g_\beta\) defined below. We regard the 1-torus \(T\) as the interval \([0, 1]\).

Since \(\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in SL(2, \mathbb{Z})\), we note that \(q \beta + q' \beta' = 1\). In particular, we have \(0 < \beta < 1/q, 0 < \beta' < 1/q'\). When \(\beta \geq 1/2q\), we put

\[
\begin{align*}
f_1(x) &= e^{-\alpha/x} & f_2(x) &= 1 - f_1(1/q - \beta - x) \\
f_3(x) &= f_2(1/q - x) & f_4(x) &= f_1(1/q - x),
\end{align*}
\]

where \(\alpha = (1/q - \beta) \log \sqrt{2}\). Using the functions described above, we define the functions \(f, g\) defined by

\[
f_\beta(x) = \begin{cases} 
    f_1(x) & (0 \leq x \leq 1/2q - \beta/2) \\
    f_2(x) & (1/2q - \beta/2 \leq x \leq 1/q - \beta) \\
    1 & (1/q - \beta \leq x \leq \beta) \\
    f_3(x) & (\beta \leq x \leq \beta/2 + 1/2q) \\
    f_4(x) & (\beta/2 + 1/2q \leq x \leq 1/q) \\
    0 & (1/q \leq x < 1),
\end{cases}
\]

\[
g_\beta(x) = \chi_{[\beta, 1/q]}(x) \sqrt{f(x)} - f(x)^2,
\]

where \(\chi\) stands for the characteristic function. In the case when \(\beta < 1/2q\), we put

\[
\begin{align*}
f_1(x) &= e^{-\alpha'/x} & f_2(x) &= 1 - f_1(1/q - \beta - x) \\
f_3(x) &= f_2(\beta - x) & f_4(x) &= f_1(\beta - x),
\end{align*}
\]
where \( \alpha' = \beta \log \sqrt{2} \), and define
\[
\begin{align*}
    f_1(x) &= (1/2q - \beta \leq x \leq 1/2q - \beta/2) \\
    f_2(x) &= (1/2q - \beta/2 \leq x \leq 1/2q) \\
    f_3(x) &= (1/2q \leq x \leq 1/2q + \beta/2) \\
    f_4(x) &= (1/2q + \beta/2 \leq x \leq 1/2q + \beta) \\
    0   &= (\text{otherwise}),
\end{align*}
\]
\[
g_\beta(x) = \chi_{[1/2q,1/2q+\beta]}(x) \sqrt{f(x) - f(x)^2}.
\]
We note that, in either case, \( f \) and \( g \) are infinitely differentiable functions. Putting \( e_\beta \) by
\[
e_\beta = v^{-q'} g(u) + f(u) + g(u)v',
\]
where \( f(u) \) and \( g(u) \) belong to the Fréchet *-algebra \( F^*(u) \) generated by \( u \), we have the following lemma:

**Lemma 3.1.** \( e_\beta \) cited above is a projection in \((T^2_\beta)^\infty\).

**Proof.** This follows from Connes [1]. \( \square \)

Another projection \( e_{\beta'} \) is constructed by the similar way as \( v \) and \( u^{-1} \) in place of \( u \) and \( v \), and as \( q' \) and \( \beta' \) in place of \( q \) and \( \beta \) respectively.

**Lemma 3.2.** The projections \( e_\beta, \alpha_{e^{-2\pi i q'/q}}(e_\beta), \ldots, \alpha_{e^{-2\pi i q'/q}}^{q-1}(e_\beta) \) are mutually orthogonal. So are the projections \( e_{\beta'}, \alpha_{e^{-2\pi i q'/q}}(e_{\beta'}), \ldots, \alpha_{e^{-2\pi i q'/q}}^{q-1}(e_{\beta'}) \).

**Proof.** We have that
\[
\alpha_{e^{-2\pi i q'/q},1}(e_\beta) = v^{-q'} g(e^{2\pi i q'/q}u) + f(e^{2\pi i q'/q}u) + g(e^{2\pi i q'/q}u)v'.
\]
Since the supports of \( g \) and \( g(e^{2\pi i q'/q}) \) are disjoint, we see for example that
\[
e_{\beta} \alpha_{e^{-2\pi i q'/q},1}(e_\beta) = v^{-q'} g(u)v^{-q'} g(e^{2\pi i q'/q}u) + f(u)v^{-q'} g(e^{2\pi i q'/q}u) \\
+ g(u)v' f(e^{2\pi i q'/q}u) + g(u)v' g(e^{2\pi i q'/q}u)v' \\
= v^{-2q'} g(e^{-2\pi i q'/q}u)g(e^{2\pi i q'/q}u) + v^{-q'} g(e^{2\pi i q'/q}u)f(e^{2\pi i q'/q}u) \\
+ v^{-q'} f(e^{-2\pi i q'/q}u) + v^{-q'} g(e^{2\pi i q'/q}u)g(e^{2\pi i q'/q}u)v' \\
= v^{-2q'} g(e^{2\pi i \beta}u)g(e^{2\pi i q'/q}u) + v^{-q'} f(e^{2\pi i \beta}u)g(e^{2\pi i q'/q}u) \\
+ v^{-q'} f(e^{-2\pi i \beta}u)f(e^{2\pi i q'/q}u) + v^{-q'} g(e^{-2\pi i \beta}u)g(e^{2\pi i q'/q}u)v'.
\]
When \( \beta \geq 1/2q \), since \( \operatorname{supp} f = [0,1/q] \) and \( \operatorname{supp} g = [\beta,1/q] \), we have
\[
\operatorname{supp} g(e^{2\pi i \beta}) = [2\beta,1/q + \beta], \quad \operatorname{supp} g(e^{-2\pi i \beta}) = [0,1/q - \beta] \\
\operatorname{supp} g(e^{-2\pi i q'/q}) = [\beta + p/q,(p+1)/q], \quad \operatorname{supp} f(e^{2\pi i \beta}) = [\beta,\beta + 1/q] \\
\operatorname{supp} f(e^{2\pi i q'/q}) = [p/q,(p+1)/q].
\]
Using the fact that $p$ and $q$ are mutually prime, we conclude that the supports of $g(e^{2\pi i\beta})$ and $g(e^{2\pi i\pi/q})$ are disjoint and so on, which implies that $\varepsilon\beta\alpha e^{2\pi i\pi/q}c(e\beta) = 0$. By the analogous argument, we also have that the above equation holds when $\beta < 1/2q$. By the same way, we see that

$$\alpha^k e^{2\pi i\pi/q}c(e\beta)\alpha^l e^{2\pi i\pi/q}c(e\beta) = 0$$

for $k, l \in \{0, 1, \cdots, q - 1\}$ with $k \neq l$, as desired. Similarly, we can prove that the projections $e\beta', c_{1,e-2\pi i\pi/q}(e\beta'), \cdots, c_{1,e-2\pi i\pi/q}(e\beta')$ are also mutually orthogonal.

Now we define the elements $e_1$ and $e_2$ by

$$e_1 = \sum_{k=0}^{q-1} (\alpha')^k(e\beta), \quad e_2 = 1 - \sum_{k=0}^{q-1} \alpha^k(e\beta),$$

where $\alpha = \alpha_{e^{2\pi i\pi/q}c, e^{2\pi i\pi/q}c}$, $\alpha' = \alpha_{e-2\pi i\pi/q}$. By the previous proposition, both $e_1$ and $e_2$ are projections in $(T^2_0)^\infty$. Furthermore, we have that $\tau(e\beta) = \beta, \tau(e\beta) = \beta'$, where $\tau(x)$ is the canonical trace of $x \in T^2_0$.

**Lemma 3.3.** The projections $e_1$ and $e_2$ are unitarily equivalent in $(T^2_0)^\infty$.

**Proof.** First of all, we show that $(T^2_0)^\infty$ is algebraically simple. Let $\mathcal{I}$ be a non-zero $*$-ideal of $(T^2_0)^\infty$. Since the closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ in $T^2_0$ is a closed $*$-ideal of $T^2_0$, it follows by the algebraic simplicity of $T^2_0$ that $\overline{\mathcal{I}}$ must be equal to $T^2_0$. Then, there is an element $x \in \mathcal{I}$ such that $\|1 - x\| < 1$, so that the spectrum of $x$ does not include the origin of $\mathbb{C}$. Since the function $h(t) = 1/t$ is holomorphic on the spectrum of $x$, it follows that $h(x) = x^{-1} \in (T^2_0)^\infty$. Hence, $1 = x^{-1}x \in \mathcal{I}$, which implies that $\mathcal{I} = T^2_0$, as claimed.

Next, we have to verify that stable rank of $(T^2_0)^\infty$ is equal to one, i.e., the set of all invertible elements of $(T^2_0)^\infty$ is dense in $(T^2_0)^\infty$. If we would have this fact, $(T^2_0)^\infty$ has cancellation property (cf. Rieffel [7, 8]). Take any element $a \in (T^2_0)^\infty$. We may assume that $a \geq 0$. Then, for $\forall \varepsilon > 0$, there exists an invertible element $b \geq 0$ in $T^2_0$ such that $\|a - b\| < \varepsilon/2$ (note that $T^2_0$ is of stable rank one). By the density of $(T^2_0)^\infty$, we can find an element $c \in (T^2_0)^\infty$ with $c \geq 0$ and $\|b - c\| < \varepsilon/2$. We act $(T^2_0)^\infty$ on $L^2(T)$ defined before. Let us show that $c$ is invertible as an operator on $L^2(T)$. If $\xi \in \ker c$ and $\|b - c\| < \varepsilon/2$, we have

$$\|(b - c)\xi\| = \|b\| < \frac{\varepsilon}{2}\|\xi\|.$$  

Since $\varepsilon$ is arbitrary, we see that $\xi = 0$, which means that $c$ is an injective operator. We note that we can find a positive number $\varepsilon/2 > \delta > 0$ such that $\|b\xi\| > \delta\|\xi\|$ for any $\xi \in L^2(T)$. We then have for any $\xi \in L^2(T)$,

$$\|c\xi\| > \|(b - c)\xi\| - \|b\xi\| \geq \delta - \frac{\varepsilon}{2}\|\xi\|,$$

which implies that $c^{-1}$ is bounded. By triangle inequality, $\|a - c\| \leq \|a - b\| + \|b - c\| < \varepsilon$. Consequently, the stable rank of $(T^2_0)^\infty$ is one.
Now recall that $\tau(e_1) = \tau(e_2)$, we thus have $[e_1] = [e_2] \in K_0((T^2_\theta)^\infty)$. Since $(T^2_\theta)^\infty$ has cancellation property, they are unitarily equivalent in $(T^2_\theta)^\infty$. □

Let $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ be the continued fraction expansion and define the matrices $P_1, P_2, \ldots$ by

$$P_n = \begin{pmatrix} a_{4n} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-3} & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geq 1$. Moreover, we put

$$\begin{pmatrix} q_{2n} \\ q_{2n-1} \end{pmatrix} = P_n P_{n-1} \cdots P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\mathfrak{A}_n = M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T)).$$

For each $n \geq 1$, we construct homomorphisms $\pi_n^\infty : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ as follows: we write $P_{n+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $z \in C^\infty(T)$ be the canonical unitary generator of $C^\infty(T)$. The element

$$\begin{pmatrix} z \\ \vdots \\ z \end{pmatrix} \oplus O_{q_{2n-1}} \in \mathfrak{A}_n = M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T))$$

should be mapped to the element

$$\begin{pmatrix} J_a \\ \vdots \\ J_a \end{pmatrix} \oplus \begin{pmatrix} J'_c \\ \vdots \\ J'_c \end{pmatrix} \oplus \begin{pmatrix} O_b \\ \vdots \\ O_b \end{pmatrix} \oplus \begin{pmatrix} J'_c \\ \vdots \\ J'_c \end{pmatrix} \oplus \begin{pmatrix} O_d \\ \vdots \\ O_d \end{pmatrix} \in \mathfrak{A}_{n+1}$$

$$\left( = \left( J_a \oplus \cdots \oplus J_a \oplus O_b \oplus \cdots \oplus O_b \right) \oplus \left( J'_c \oplus \cdots \oplus J'_c \oplus O_d \oplus \cdots \oplus O_d \right) \right),$$

where

$$J_k = \begin{pmatrix} 0 \\ \vdots \\ z \\ \vdots \\ 1 \end{pmatrix}, \quad J'_k = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in M_k(C^\infty(T))$$
and \( O_l \) means the \( l \times l \) zero matrix. Any element \((a_{ij}) \oplus O_{q_{n-1}} \in \mathcal{M}_{q_n}(\mathbb{C}) \oplus \mathcal{M}_{q_{n-1}}(\mathbb{C}) \subset \mathfrak{A}_n \) should be mapped to
\[
\begin{pmatrix}
a_{11}I_a & \cdots & a_{1q_{2n}}I_a \\
\vdots & & \vdots \\
a_{q_{2n},1}I_a & \cdots & a_{q_{2n},q_{2n}}I_a
\end{pmatrix}_{O_{q_{2n-1}}},
\]
where \( I_a, I_c \) are the \( a \times a, c \times c \) identity matrices respectively. The second direct summand of \( \mathfrak{A}_n \) should be mapped into \( \mathfrak{A}_{n+1} \) by the similar way as \( q_{2n} \) replaced by \( q_{2n-1} \), \( a \) and \( c \) by \( b \) and \( d \) respectively, and interchanging the places to whose elements are mapped from upper left-hand side to lower right-hand side. It is easily verified that these \( \pi_n^\infty \) are smooth inclusions.

Next, we need the following proposition. We define
\[
e_{kk} = \alpha^{k-1}(e_\beta) \quad (k = 1, 2, \ldots, q - 1)
\]
and
\[
e'_{kk} = (\alpha')^{k-1}(e_{\beta'}) \quad (k = 1, 2, \ldots, q' - 1).
\]

**Lemma 3.4.** Let \( e_{22}ve_{11} = e_{21}|e_{22}ve_{11}| \) be the polar decomposition of \( e_{22}ve_{11} \). Then, \( e_{21} = e_{22}ve_{11} \).

**Proof.** We write \( x = ve_{11} \). Since \( x^*x = e_{11}^*vve_{11} = e_{11} \), we have \( |x| = e_{11} \). Thus, \( x = ve_{11} \) is the polar decomposition of \( x \), which implies that it is a surjective operator since \( v \) is unitary. Hence, it follows that \( \operatorname{Ran}e_{22} = \operatorname{Ran}e_{22}ve_{11} \), where \( V \) is the closure of a linear subspace \( V \) of the Hilbert space \( L^2(T) \). Furthermore, it is also verified that \( \operatorname{Ran}e_{11} = \operatorname{Ran}|e_{22}ve_{11}| \). Note that \( e_{22}ve_{11} = (e_{22}ve_{11})e_{11} \). By uniqueness of polar decomposition, we deduce that \( e_{21} = e_{22}ve_{11} \), as desired. \( \square \)

By the similar way, we put \( e'_{21} = e'_{22}ve'_{11} \). Our goal in this section is to construct the \( F^* \)-subalgebras generated by some unitaries, which is isomorphic to \( M_{q_{2n}}(\mathcal{C}(\infty(T))) \oplus M_{q_{n-1}}(\mathcal{C}(\infty(T))) \). For this, since \( q_{2n-1} \) and \( q_{2n} \) are mutually prime, we can find an integer \( p_{2n-1}, p_{2n} \) with \( \left( \begin{array}{c} p_{2n-1} \\ p_{2n} \end{array} \right) \in SL(2, \mathbb{Z}) \) and \( p_n/q_n \to \theta \) as \( n \to \infty \). With the same notations as above, we set
\[
\left( \begin{array}{c} p' \\ p \\ q' \\ q \end{array} \right) = \left( \begin{array}{c} p_{2n} \\ p_{2n-1} \\ q_{2n} \\ q_{2n-1} \end{array} \right)
\]
and \( \beta = \beta_n = p_{2n-1} - q_{2n-1}\theta, \beta' = \beta'_n = q_{2n}\theta - p_{2n}, \) and so on. First of all, we check the following fact although it seems to be known:

**Lemma 3.5.** For arbitrary \( h \in C^\infty(T) \), \( \delta_j(h(u)) = h'(u)\delta_j(u) \) \((j = 1, 2)\), where \( h' \) is the first derivative of \( h \).
Proof. If \( h(x) = \sum_{\nu=-n}^{n} a_{\nu}x^{\nu} \) is a Laurent polynomial, we have
\[
\delta(h(u)) = \delta_1 \left( \sum_{\nu=-m}^{n} a_{\nu}u^{\nu} \right) = \sum_{\nu=-m}^{n} a_{\nu}uv^{\nu} = \left( \sum_{\nu=-m}^{n} a_{\nu}v^{\nu-1} \right) iu = h'(u)\delta_1(u).
\]

For any \( h \in C^\infty(T) \), we can find a family of Laurent polynomials \( \{p_n\}_{n \geq 1} \) such that \( p_n \to h \) with respect to the seminorms \( \{\| \cdot \|_{k,l}\} \). For \( m, n \geq 1 \), we have
\[
\delta_1(p_n(u) - p_m(u)) = (p'_n(u) - p'_m(u))\delta_1(u) = (p'_n(u) - p'_m(u))u.
\]
Since \( \{p_n(u)\}_n \) is Cauchy, \( \{\delta_1(p_n(u))\}_{n \geq 1} \) is also a Cauchy sequence. Using the fact that \( \delta_1 \) is a closed operator, we get
\[
\delta_1(h(u)) = \lim_{n \to \infty} \delta_1(p_n(u)) = \lim_{n \to \infty} p'_n(u)\delta_1(u) = h'(u)\delta_1(u).
\]
As \( \delta_2(u) = 0 \), it is clear that \( \delta_2(h(u)) = 0 = h'(u)\delta_2(u) \). This completes the proof. \( \square \)

In what follows, we use the notations \( e_{11}^{(n)} = e_{\beta_n}, (e'_{11})^{(n)} = e'_{\beta_n} \) and so on for \( n \geq 1 \). Denoting \( r_m = p_m/q_m \) for any integer \( m \geq 1 \), we define \( u_n = u_{n,1} + u_{n,2} \) and \( v_n = v_{n,1} + v_{n,2} \), where
\[
u_n,1 = \sum_{j=0}^{\nu_n-1} (-2\pi i r_{2n-1})^j e_{\nu_n,1}(e_{21}^{(n)}),
\]
\[
u_n,2 = \sum_{j=0}^{\nu_n-1} (-2\pi i r_{2n-1})^j e_{\nu_n,2}(e_{21}^{(n)}),
\]
\[
u_{n,1} = \sum_{j=0}^{\nu_n-1} (-2\pi i r_{2n-1})^j e_{\nu_n,1}(e_{21}^{(n)}),
\]
\[
u_{n,2} = \sum_{j=0}^{\nu_n-1} (-2\pi i r_{2n-1})^j e_{\nu_n,2}(e_{21}^{(n)}).
\]

We note that since
\[
\alpha^{\nu_n-1}(e_{21}^{(n)}) \in e_{11}^{(n)}(T^{2})^\infty e_{\nu_n,2n},
\]
\[
(e'_{21})^{\nu_n-1}(e_{21}^{(n)}) \in e_{11}^{(n)}(T^{2})^\infty (e'_{\nu_n-1,2n-1})^{(n)},
\]
where \( e_{\nu_n,2n} = \alpha^{\nu_n-1}_{2\pi i r_{2n-1}}(e_{11}) \) and \( e'_{\nu_n-1,2n-1}^{(n)} = e'_{\nu_n-1,2n-1}^{(n)} \), we can find a unitary \( v_{1,2n} \in e_{11}^{(n)}(T^{2})^\infty e_{11}^{(n)} \) (resp. \( u'_{1,2n} \in e_{11}^{(n)}(T^{2})^\infty e_{11}^{(n)} \)).
such that $\alpha_{q_{2n}}^{-1}(e_{21}^{(n)}) = v_{1q_{2n}} e_{1q_{2n}}^{(n)}$ (resp. $(\alpha')_{q_{2n}-1}^{-1}((e_{21}^{(n)})^{(n)} = u_{1q_{2n}-1}^{(n)}((e_{1q_{2n}}^{(n)})^{(n)}$).

By Lemma 3.2 we have

$$u_{n,1} u_{n,1}^* = \left( \sum_{j=0}^{q_{2n}-1} e^{2\pi i 2^{j+1}} \alpha_{2^{j+1}} e_{2^{j+1}} \varepsilon_{11}^{(n)} \right) \cdot \left( \sum_{j=0}^{q_{2n}-1} e^{-2\pi i 2^{j+1}} \alpha_{2^{j+1}} e_{2^{j+1}} \varepsilon_{11}^{(n)} \right) = \sum_{j,m} e^{2\pi i m(n-j+m)} \alpha_{2^{j+1}} e_{2^{j+1}} \varepsilon_{11}^{(n)} = \sum_{j=0}^{q_{2n}-1} \alpha_{2^{j+1}} e_{2^{j+1}} \varepsilon_{11}^{(n)} = 1 - e_{2}^{(n)}.
$$

Similarly, $u_{n,1}^* u_{n,1} = 1 - e_{2}^{(n)}$, $v_{n,2}^* v_{n,2} = v_{n,2} v_{n,2} = e_{1}^{(n)}$. Moreover, we have

$$u_{n,2} u_{n,2}^* = \left( \sum_{j=0}^{q_{2n}-1} (e_{2^{j+1}+1}^{(n)} + u_{1q_{2n}-1}^{(n)}(e_{1q_{2n}-1}^{(n)})) \right) \cdot \left( \sum_{j=0}^{q_{2n}-1} (e_{2^{j+1}+1}^{(n)} + (e_{2q_{2n}-1} - (e_{1q_{2n}-1}^{(n)}))^{(n)}) \right) = \left( (e_{21}^{(n)})^{(n)} + \cdots + (e_{q_{2n}-1,q_{2n}-1}^{(n)})(e_{q_{2n}-1,q_{2n}-1}^{(n)}) \right) \cdot \left( (e_{12}^{(n)})^{(n)} + \cdots + (e_{q_{2n}-1,q_{2n}-1}^{(n)})(e_{q_{2n}-1,q_{2n}-1}^{(n)}) \right) + \left( (e_{21}^{(n)})^{(n)} + \cdots + (e_{q_{2n}-1,q_{2n}-1}^{(n)})(e_{q_{2n}-1,q_{2n}-1}^{(n)}) \right) u_{1q_{2n}-1}^{(n)} \left( (e_{12}^{(n)})^{(n)} + \cdots + (e_{q_{2n}-1,q_{2n}-1}^{(n)})(e_{q_{2n}-1,q_{2n}-1}^{(n)}) \right) + \left( (e_{q_{2n}-1,q_{2n}-1}^{(n)})^{(n)}(u_{1q_{2n}-1})^{(n)} \right) u_{1q_{2n}-1}^{(n)} \left( (e_{1q_{2n}-1})^{(n)} \right),$$

where

$$(e_{k,k-1}^{(n)})^{(n)} = \alpha_{k+1}^{(n)} \cdot e^{-2\pi i^{k+1}} (e_{k}^{(n)}), \quad (e_{k,k-1}^{(n)})^{(n)} = ((e_{k,k-1}^{(n)})^{(n)})^{*}$$

for $k = 2, \ldots, q_{2n}-1$. Since $u_{1q_{2n}-1}^{(n)}$ is a unitary in $(e_{11}^{(n)})(T_{0}^{2})^{(n)}(e_{11}^{(n)})$, it follows that the second and the third terms above are 0 and

$$(e_{q_{2n}-1,q_{2n}-1}^{(n)})^{(n)}(u_{1q_{2n}-1})^{(n)} u_{1q_{2n}-1}^{(n)} (e_{1q_{2n}-1}^{(n)})^{(n)} = (e_{q_{2n}-1})^{(n)}(e_{11}^{(n)})^{(n)}(e_{1q_{2n}-1}^{(n)})^{(n)} = (e_{q_{2n}-1,q_{2n}-1}^{(n)})^{(n)}.$$
Moreover, we have

\[ v_{n,1}u_{n,1} = \left( e^{(n)}_{21} + \cdots + e^{(n)}_{q_{2n},q_{2n}}^{-1} + u_{1q_{2n}}e^{(n)}_{1q_{2n}} \right) \left( e^{(n)}_{11} + \cdots + \omega q_{2n}^{-1}e^{(n)}_{q_{2n},q_{2n}} \right) \]

\[ = e^{(n)}_{21} + \cdots + \omega q_{2n}^{-2}e^{(n)}_{q_{2n}q_{2n}-1} + \omega q_{2n}^{-1}u_{1q_{2n}}e^{(n)}_{1q_{2n}} \]

and

\[ u_{n,1}v_{n,1} = \left( e^{(n)}_{11} + \cdots + \omega q_{2n}^{-1}e^{(n)}_{q_{2n}q_{2n}} \right) \left( e^{(n)}_{21} + \cdots + e^{(n)}_{q_{2n}q_{2n}}^{-1} + u_{1q_{2n}}e^{(n)}_{1q_{2n}} \right) \]

\[ = e^{(n)}_{11}u_{1q_{2n}}e^{(n)}_{1q_{2n}} + \omega e^{(n)}_{21} + \cdots + \omega q_{2n}^{-1}e^{(n)}_{q_{2n}q_{2n}-1}, \]

where

\[ e^{(n)}_{k} = \alpha^{-k-1}\pi_{r_{2n},1}(e_{\beta_{n}}, \ (k = 2, \ldots, q_{2n} - 1), \]

\[ e^{(n)}_{k,k-1} = \alpha^{-k-1}\pi_{r_{2n},1}(e_{\beta_{n}}, \ (k = 2, \ldots, q_{2n}) \]

and \( \omega = e^{2\pi r_{2n}}. \) Using the fact that \( u_{1q_{2n}} \in e^{(n)}_{11}(T_{\theta}^2)_{\infty}e^{(n)}_{11} \) and \( \omega q_{2n} = 1, \) we have

\[ v_{n,1}u_{n,1} = e^{-2\pi r_{2n}}u_{n,1}v_{n,1}. \]

To sum up, we get the following:

**Lemma 3.6.** The following hold:

1. \( u_{n,1} \) and \( u_{n,2} \) are unitaries in \( (1 - e^{(n)}_{2})(T_{\theta}^2)_{\infty}(1 - e^{(n)}_{2}) \) and so are \( u_{n,2} \) and \( v_{n,2} \) in \( e^{(n)}_{1}T_{\theta}e^{(n)}_{1}. \)
2. \( u_{n,1}v_{n,1} = e^{2\pi r_{2n}}u_{n,1}v_{n,1}, \) \( u_{n,2}v_{n,2} = e^{2\pi r_{2n}}u_{n,2}v_{n,2}. \)

Now we construct subalgebras isomorphic to \( M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T)). \) Let \( \{e^{(n)}_{ij}\}_{1 \leq i, j \leq q_{2n}} \) be the matrix units constructed by

\[ \{e^{(n)}_{11}, e^{(n)}_{22}, \ldots, e^{(n)}_{q_{2n},q_{2n}}, e^{(n)}_{q_{2n},q_{2n}-1}\}. \]

We then see the following lemma:

**Lemma 3.7.** The \( F^*- \) algebras \( F^*\{\{e^{(n)}_{ij}\}_{1 \leq i, j \leq q_{2n}}, v_{1q_{2n}}\} \) generated by \( \{e^{(n)}_{ij}\}_{1 \leq i, j \leq q_{2n}} \) and \( v_{1q_{2n}} \) are isomorphic to \( M_{q_{2n}}(C^\infty(T)) \) for all integers \( n \geq 1. \)

**Proof.** Consider the continuous field \( S \ni t \mapsto e_{\beta_{n}} \) defined by Elliott and Evans [3], where \( S \) is a closed subinterval in \( (0, \infty). \) The functions \( f \) and \( g \) appeared in the construction of \( e_{\beta_{n}} \) are depend on \( t \in S, \) so that we write \( f = f_{t}, \) \( g = g_{t}. \) It is not difficult to verify that

\[ \|f_{t}^{(\nu)} - f_{t_{0}}^{(\nu)}\|_{\infty}, \|g_{t}^{(\nu)} - g_{t_{0}}^{(\nu)}\|_{\infty} \to 0 \]

as \( t \to t_{0} \) for any integer \( \nu \geq 0, \) where \( f^{(\nu)} \) stands for the \( \nu \)-th derivatives of \( f \in C^\infty(T) \) and \( \|\cdot\|_{\infty} \) is the supremum norm on \( C^\infty(T). \) Then our statement of this lemma follows immediately. \( \square \)
By the same way, it follows that the $F^*$-algebra $F^*\{\{e'_{ij}(n)\}_{1 \leq i,j \leq q_{2n-1}}\}$ generated by $\{e'_{ij}(n)\}_{1 \leq i,j \leq q_{2n-1}}$ and $u'_{1q_{2n-1}1}$ is isomorphic to $M_{q_{2n-1}}(C^\infty(T))$, where $\{e'_{ij}(n)\}_{1 \leq i,j \leq q_{2n-1}}$ are the matrix units generated by

$$\{(e'_{11}(n), \ldots, e'_{q_{2n-1},q_{2n-1}}(n)), (e'_{21}(n), \ldots, e'_{q_{2n-1},q_{2n-1}-1}(n))\}.$$

**Lemma 3.8.** For each $h \in C^\infty(T)$ and any integer $k \geq 1$, there exist $\{a_{\nu,k}\} \subset \mathbb{R}$ such that

$$\delta^k_1(h(u)) = \sum_{\nu=1}^{k} a_{\nu,k} h^{(\nu)}(u) u^\nu \quad (\nu = 1, \ldots, k).$$

**Proof.** For $k = 1$, by Proposition 3.5. If this statement holds for some $k \geq 1$, one has

$$\delta^{k+1}_1(h(u)) = \delta_1^{k+1} \left( \sum_{\nu=1}^{k} a_{\nu,k} h^{(\nu)}(u) u^\nu \right)$$

$$= \sum_{\nu=1}^{k} a_{\nu,k} \delta_1^{k+1} (h^{(\nu)}(u) u^\nu)$$

$$= \sum_{\nu=1}^{k} a_{\nu,k} \left( h^{(\nu+1)}(u) u^{\nu+1} + i\nu h^{(\nu)}(u) u^\nu \right)$$

$$= \sum_{\nu=1}^{k} a_{\nu,k} \left( h^{(\nu+1)}(u) u^{\nu+1} + i\nu h^{(\nu)}(u) u^\nu \right)$$

$$= \sum_{\nu=1}^{k} a_{\nu,k} \nu h^{(\nu)}(u) u^\nu + \sum_{\nu=1}^{k} i a_{\nu,k} \nu h^{(\nu)}(u) u^\nu.$$

Thus, we have

$$a_{\nu,k+1} = \sum_{\nu=2}^{k+1} a_{\nu-1,k} + \sum_{\nu=1}^{k} i a_{\nu,k},$$

this ends the proof.

We note that the coefficients $a_{\nu,k}$ do not depend on the choice $h$.

By Lemma 3.8 we have

$$\|\delta^k_1(f_n(u)) - \delta^k_1(f_m(u))\| = \left\| \sum_{\nu=1}^{k} a_{\nu,k} \left( f^{(\nu)}_n(u) - f^{(\nu)}_m(u) \right) u^\nu \right\|$$

$$\leq \sum_{\nu=1}^{k} |a_{\nu,k}| \| f^{(\nu)}_n(u) - f^{(\nu)}_m(u) \| \to 0 \quad (n, m \to \infty),$$

which means that $\{\delta^k_1(f_n(u))\}_n$ is a Cauchy sequence. Analogously, we see that $\{\delta^k_1(g_n(u))\}_n$ is also Cauchy.

By construction, the following fact follows:
Lemma 3.9. Let \( F^*(u_n, v_n) \) be the \( F^* \)-algebras generated by \( u_n \) and \( v_n \). Then, they are equal to \( F^*(\{e^n_{ij}\}, u_1 u_{2n-1}) \oplus F^*(\{(e^{(n)}_{ij})\}, u'_1 u_{2n-1}) \).

Proof. Since \( u_{n,j} \) and \( v_{n,j} \) \((j = 1, 2)\) are all periodic unitaries, their spectra are finite. Then the projections appeared in the spectral decompositions of \( u_{n,j}, v_{n,j} \) are unitarily equivalent to \( e^n_{ij} \)'s by the properties that \( F^*(u_{n,j}) \) and \( F^*(v_{n,j}) \) are closed under the holomorphic functional calculus.

Lemma 3.10. For any integers \( k, l \geq 0 \),
\[
\lim_{n \to \infty} \|u - u_n\|_{k,l} = \lim_{n \to \infty} \|v - v_n\|_{k,l} = 0.
\]

Proof. At first, we have to verify that the sequence \( \{\delta^k_1(e_{\beta_n})\} \) is Cauchy. By construction of \( e_{\beta_n} \), we have, for \( n, m \geq 1 \),
\[
\|\delta^k_1(e_{\beta_n}) - \delta^k_1(e_{\beta_m})\| \leq \|\delta^k_1(v^{q_2m-1}g_n(u) - v^{q_2m-1}g_m(u))\|
+ \|\delta^k_1(f_n(u) - f_m(u))\| + \|\delta^k_1(g_n(u)v^{q_2m-1} - g_m(u)v^{q_2m-1})\|
= \|v^{q_2m-1}\delta^k_1(g_n(u)) - v^{q_2m-1}\delta^k_1(g_m(u))\|
+ \|\delta^k_1(f_n(u)) - \delta^k_1(f_m(u))\|
+ \|\delta^k_1(g_n(u))v^{q_2m-1} - \delta^k_1(g_m(u))v^{q_2m-1}\|.
\]

Since \( p_{2n-1}/q_{2n-1} \to \theta \), the last term of the above calculation tends to 0 as \( n, m \to \infty \). Therefore, \( \{\delta^k_1 \circ \delta_2^l(u(1 - e_2^{(n)}) - u_{1,n})\} \) is Cauchy. Similarly, the sequence \( \{\delta^k_1 \circ \delta_2^l(ue_1^{(n)} - u_{n,2})\} \) is also a Cauchy sequence. Hence, by [6],
\[
u(1 - e_2^{(n)}) - u_{1,n} \to 0, \quad ue_1^{(n)} - u_{n,2} \to 0
\]
as \( n \to \infty \). Using the fact that \( \delta^k_1 \circ \delta_2^l \) are closed, the sequences above tend to 0 as \( n \to \infty \). Consequently,
\[
\|u - u_n\|_{k,l} \leq \|u(1 - e_2^{(n)}) - u_{1,n}\|_{k,l} + \|ue_1^{(n)} - u_{n,2}\|_{k,l}
\to 0 \quad (n \to \infty).
\]
By the similar argument, we have \( \|v - v_n\|_{k,l} \to 0 \) as \( n \to \infty \), this ends the proof. □

Combining all together in this section, we conclude that our key fact follows:

Proposition 3.1. Given an irrational number \( \theta \in (0, 1) \), \( (T^2_{\theta})^\infty \) is isomorphic to the Fréchet *-inductive limit
\[
\lim_{\varphi \to \infty} M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T), \pi_n^\infty).
\]

4. Entire Cyclic Cohomology of Fréchet Inductive Limits

Let \( \{A_n, i_n\}_{n \geq 1} \) be a family of Fréchet *-algebras and \( i_n : A_n \to A_{n+1} \) Fréchet *-embeddings. We can form the Fréchet *-inductive limit \( \varinjlim A_n \), which is denoted by \( A \). In this section, we prove that the projective limit \( \varinjlim H^*_e(A_n) \) of the entire cyclic cohomologies \( \varinjlim H^*_e(A_n) \) is isomorphic to \( H^*_e(A) \). Let [\cdot]_A be the entire
cyclic cohomology classes on $\mathfrak{A}_n$, and the maps $i_n^*: H^\text{cy}_e(\mathfrak{A}_{n+1}) \to H^\text{cy}_e(\mathfrak{A}_n)$ are defined by
\[
i_n^* ([(\varphi^{(n+1)}_{2k})_n])_{\mathfrak{A}_{n+1}} = [(i_n^{(2k+1)})^* \varphi^{(n+1)}_{2k})]_{\mathfrak{A}_n},
\]
where
\[
(i_n^{(2k+1)})^* \varphi^{(n+1)}_{2k})_n(a_0, \ldots, a_{2k}) = \varphi^{(n+1)}_{2k}(i_n(a_0), \ldots, i_n(a_{2k}))
\]
for $a_0, \ldots, a_{2k} \in \mathfrak{A}_n$. First of all, we define the notion of projective limit as follows:

**Definition 4.1.** The projective limit $\lim_{\leftarrow} H^\text{cy}_e(\mathfrak{A}_n)$ of $H^\text{cy}_e(\mathfrak{A}_n)$ is the space of sequences $\{[(\varphi^{(n)}_{2k})_n]_{\mathfrak{A}_n}\}_n \in \prod_{n \geq 1} H^\text{cy}_e(\mathfrak{A}_n)$ such that for any $n \geq 1$,
\[
i_n^* ([(\varphi^{(n+1)}_{2k})_n])_{\mathfrak{A}_{n+1}} = [(\varphi^{(n)}_{2k})_n]_{\mathfrak{A}_n}
\]
with the property that for any $k \geq 0, l \geq 1$,
\[
\sup_{n \geq 1} \|\varphi^{(n)}_{2k}\|_l < \infty,
\]
where
\[
\|\varphi^{(n)}_{2k}\|_l = \sup_{a_j \in \mathfrak{A}_n, \|a_j\|_l \leq 1} |\varphi^{(n)}_{2k}(a_0, \ldots, a_{2k})|.
\]

We define $\lim_{\leftarrow} H^\text{od}_e(\mathfrak{A}_n)$ in the similar way as in the even case. $\{[(\varphi^{(n)}_{2k})_n]_{\mathfrak{A}_n}\}_n = \{[(\psi^{(n)}_{2k})_n]_{\mathfrak{A}_n}\}_n$ if and only if there exists $\{[(\theta^{(n)}_{2k+1})_n]_{\mathfrak{A}_n}\}_n \in \lim_{\leftarrow} H^\text{od}_e(\mathfrak{A}_n)$ such that
\[
\varphi^{(n)}_{2k} - \psi^{(n)}_{2k} = b\theta^{(n)}_{2k-1} + B\theta^{(n)}_{2k+1}
\]
for any $n \geq 1, k \geq 0$.

Let us construct two maps between $\lim_{\leftarrow} H^\text{cy}_e(\mathfrak{A}_n)$ and $H^\text{cy}_e(\mathfrak{A})$. First of all, we define $\Phi: H^\text{cy}_e(\mathfrak{A}) \to \lim_{\leftarrow} H^\text{cy}_e(\mathfrak{A}_n)$ by
\[
\Phi(\varphi_{2k}) = \{(\varphi_{2k}|\mathfrak{A}_n)_n\}_{\mathfrak{A}_n},
\]
where $[\cdot]_n$ means the same symbol as $[\cdot]_{\mathfrak{A}_n}$. Actually it is well-defined. In fact, if $[(\varphi_{2k})]_n = [(\varphi'_{2k})]_n$ then there exists an odd entire cyclic cocycle $\theta = (\theta_{2k+1})_k$ such that $(\varphi_{2k} - \varphi'_{2k})_k = (b + B)(\theta_{2k+1})_k$, where $b + B$ is the derivation on entire cyclic cocycles. It is trivial that $(\varphi_{2k}|\mathfrak{A}_n - \varphi'_{2k}|\mathfrak{A}_n)_k = (b + B)(\theta_{2k+1}|\mathfrak{A}_n)_k$ for each integer $n \geq 1$. This means that $\{[(\varphi_{2k}|\mathfrak{A}_n)_n\}_{\mathfrak{A}_n} = \{(\varphi'_{2k}|\mathfrak{A}_n)_n\}_{\mathfrak{A}_n}$. Moreover,
\[
\sup_{n \geq 1} \|\varphi^{(n)}_{2k}|\mathfrak{A}_n\|_l = \|\varphi_{2k}\|_l < \infty,
\]
which implies $\{(\varphi^{(n)}_{2k}|\mathfrak{A}_n)_n\}_{\mathfrak{A}_n} \in H^\text{cy}_e(\mathfrak{A}_n)$.

Now we construct the inverse map $\Psi$ of $\Phi$. For any $\{[(\varphi^{(n)}_{2k})_n]_{\mathfrak{A}_n}\}_n \in \lim_{\leftarrow} H^\text{cy}_e(\mathfrak{A}_n)$ and $a_0, \ldots, a_{2k} \in \mathfrak{A}$, we can take sequences $\{b_j^{(m)}\}_m$ for $j = 0, \ldots, 2k$ which converge to $a_j$ as $m \to \infty$ with respect to the seminorms $\|\cdot\|_l$ on $\lim_{\leftarrow} \mathfrak{A}_n$. Choose integers $N(m) \geq 1$ such that $b_j^{(m)} \in \mathfrak{A}_{N(m)}$ for any $0 \leq j \leq 2k$. We may assume
Lemma 4.1. By Hahn-Banach theorem, we can extend \( \varphi \) to \( \tilde{\varphi} \) such that

\[
\varphi_{2k}^m(b_0^m, \ldots, b_{2k}^m) = \varphi_{2k}^{m'}(b_0^{m'}, \ldots, b_{2k}^{m'}) = (b_{2k-1}^{m'} + B\theta_{2k+1}^{m'})(b_0^{m'}, \ldots, b_{2k}^{m'}).
\]

By Hahn-Banach theorem, we can extend \( \varphi_{2k}^m \) and \( \varphi_{2k}^{m'} \) to \( \tilde{\varphi}_{2k}^m \) and \( \tilde{\varphi}_{2k}^{m'} \) on \( \mathfrak{A} \) such that

\[
\|\tilde{\varphi}_{2k}^m\|_l = \|\varphi_{2k}^m\|_l, \quad \|\tilde{\varphi}_{2k}^{m'}\|_l = \|\varphi_{2k}^{m'}\|_l
\]

for any \( l \geq 1 \).

**Lemma 4.1.** For any \( a_0, \ldots, a_{2k} \in \mathfrak{A} \), the sequence

\[
\{\tilde{\varphi}_{2k}^m(a_0, \ldots, a_{2k})\}_m
\]

is bounded.

**Proof.** We have

\[
\|\tilde{\varphi}_{2k}^m(a_0, \ldots, a_{2k})\| = \|\tilde{\varphi}_{2k}^m(a_0 - b_0^m, a_1, \ldots, a_{2k})\|
\]

By the above equation \( 1 \),

\[
\tilde{\varphi}_{2k}^m(b_0^m, \ldots, b_{2k}^m)
= \varphi_{2k}^m(b_0^m, \ldots, b_{2k}^m)
= \varphi_{2k}^{m'}(b_0^{m'}, \ldots, b_{2k}^{m'}) + (b_{2k-1}^{m'} + B\theta_{2k+1}^{m'})(b_0^{m'}, \ldots, b_{2k}^{m'})
\]

is a constant independent of \( m \). Using the hypothesis in Definition \( 4.1 \) and Hahn-Banach theorem, it follows that \( \lim_{m \to \infty} \|\tilde{\varphi}_{2k}^m(a_0, \ldots, a_{2k})\| \) is dominated by the constant \( \|\tilde{\varphi}_{2k}^{m'}(b_0^{m'}, \ldots, b_{2k}^{m'}) + (b_{2k-1}^{m'} + B\theta_{2k+1}^{m'})(b_0^{m'}, \ldots, b_{2k}^{m'})\| \). In particular, the sequence \( \{\|\tilde{\varphi}_{2k}^n(a_0, \ldots, a_{2k})\|\}_m \) is bounded.

Therefore, by taking the subsequence of \( \{\|\tilde{\varphi}_{2k}^n(a_0, \ldots, a_{2k})\|\}_N \), we may assume that

\[
\lim_{N \to \infty} \tilde{\varphi}_{2k}^N(a_0, \ldots, a_{2k}) = \lim_{N \to \infty} \tilde{\varphi}_{2k}^N(a_0, \ldots, a_{2k})
\]

exists, so that we define

\[
\tilde{\varphi}_{2k}(a_0, \ldots, a_{2k}) = \lim_{N \to \infty} \tilde{\varphi}_{2k}^N(a_0, \ldots, a_{2k}).
\]
Here we note that
\[ \tilde{\varphi}_{2k}(a_0, \cdots, a_{2k}) = \lim_{m \to \infty} \tilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \cdots, b_{2k}^{(m)}). \]

In fact, by the same reason as before, we have
\[
|\tilde{\varphi}_{2k}^{(m)}(a_0, \cdots, a_{2k}) - \tilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \cdots, b_{2k}^{(m)})| \\
\leq |\tilde{\varphi}_{2k}^{(m)}(a_0 - b_0^{(m)}, a_1, \cdots, a_{2k})| \\
+ \cdots \\
+ |\tilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \cdots, b_{2k-1}^{(m)}, a_{2k} - b_{2k}^{(m)})| \to 0
\]
as \( m \to \infty \). Using the above preparation, we shall show the following fact:

**Lemma 4.2.** \((\tilde{\varphi}_{2k})_k\) is an entire cyclic cocycle on \( \mathfrak{A} \).

**Proof.** Let \( \Sigma \) be a bounded subset of \( \mathfrak{A} \) and \( a_0, \cdots, a_{2k} \in \Sigma \). Then we can choose sequences \( \{b_j^{(m)}\}_m \subset \bigcup \mathfrak{A}_n \) for \( j = 0, \ldots, 2k \) such that \( b_j^{(m)} \to a_j \) as \( m \to \infty \) with respect to the topology induced by the seminorms \( \| \cdot \| \) on \( \mathfrak{A} \). In this case, the set
\[ \Sigma_0 = \{b_j^{(m)} \in \bigcup \mathfrak{A}_n \mid j = 0, \ldots, 2k, m \in \mathbb{N}\} \]
is bounded in \( \mathfrak{A} \). So, by the equation (11).
\[
|\tilde{\varphi}_{2k}(a_0, \cdots, a_{2k})| = \lim_{m \to \infty} |\tilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \cdots, b_{2k}^{(m)})| \\
\leq |\tilde{\varphi}_{2k}^{(1)}(b_0^{(1)}, \cdots, b_{2k}^{(1)})| \\
+ |(b_0^{(1)} - b_0^{(m)}) + B\theta_2^{(1)} + B\theta_1^{(1)})(b_0^{(1)}, \cdots, b_{2k}^{(1)})|.
\]
As \((\varphi_{2k}^{(1)})_k\) and \((b\theta_2^{(1)} + B\theta_1^{(1)})_k\) are entire on \( \mathfrak{A}_1 \),
\[ |\tilde{\varphi}_{2k}(a_0, \cdots, a_{2k})| \leq Ck! \]
for some constant \( C > 0 \) independent of \( m \), which implies that \((\tilde{\varphi}_{2k})_k\) is entire. \( \square \)

Now we are ready to define a map \( \Psi : \lim_{\mathfrak{A}_n} H_c^{ev}(\mathfrak{A}_n) \to H_c^{ev}(\mathfrak{A}) \) in the following fashion:
\[
\Psi\{[(\varphi_{2k}^{(n)})_k]|_{\mathfrak{A}_n}\} = [(\tilde{\varphi}_{2k})_k]|_{\mathfrak{A}}.
\]
We have to verify that the definition is well-defined. Let
\[ \{[(\varphi_{2k}^{(n)})_k]|_{\mathfrak{A}_n}\}_n = \{[(\psi_{2k}^{(n)})_k]|_{\mathfrak{A}_n}\}_n \in \lim_{\mathfrak{A}_n} H_c^{ev}(\mathfrak{A}_n). \]
Then for any \( n \geq 1 \), there exists an odd entire cyclic cocycles \( \theta^{(n)} = (\theta_{2k+1}^{(n)})_k \) on \( \mathfrak{A}_n \) such that
\[
\varphi_{2k}^{(n)}(b_0, \cdots, b_{2k}) - \psi_{2k}^{(n)}(b_0, \cdots, b_{2k}) = (b\theta_{2k-1}^{(n)} + B\theta_{2k+1}^{(n)})(b_0, \cdots, b_{2k})
\]
for \( b_0, \ldots, b_{2k} \in \mathfrak{A}_n \). By the above argument, there exists an odd entrie cyclic cocycle \( \bar{\theta} = (\bar{\theta}_{2k+1})_k \) on \( \mathfrak{A} \). Then by the definition of \( b + B \), we have that

\[
(b\theta^{(n)}_{2k-1} + B\theta^{(n)}_{2k+1})(a_0, \ldots, a_{2k}) \\
= \lim_{m \to \infty} (b\theta^{(m)}_{2k-1} + B\theta^{(m)}_{2k+1})(b_0^{(m)}, \ldots, b_{2k}^{(m)}) \\
= \lim_{m \to \infty} \left( \overline{\psi_2^{(m)}}(b_0^{(m)}, \ldots, b_{2k}^{(m)}) - \overline{\psi_2^{(m)}}(b_0^{(m)}, \ldots, b_{2k}^{(m)}) \right) \\
= \overline{\psi_2}(a_0, \ldots, a_{2k}) - \overline{\psi_2}(a_0, \ldots, a_{2k}),
\]

which implies that \( [(\overline{\psi_2})_k]_\mathfrak{A} = [(\overline{\psi_2})_k]_\mathfrak{A} \).

**Proposition 4.1.** The following isomorphism holds as a vector space over \( C \):

\[
\lim_{\epsilon \to 0} H^*_C(\mathfrak{A}_n) \cong H^*_C(\mathfrak{A}).
\]

**Proof.** We prove just in the even case. For any \( [(\varphi_{2k})_k]_\mathfrak{A} \in H^*_C(\mathfrak{A}) \), we have

\[
\Psi \circ \Phi([(\varphi_{2k})_k]_\mathfrak{A}) = \Psi([(\varphi_{2k} | \mathfrak{A}_n)_n]_\mathfrak{A} = [(\overline{\varphi_{2k}})_{k}]_\mathfrak{A}.
\]

For any \( a_0, \ldots, a_{2k} \in \mathfrak{A} \), we take sequences \( \{b_j^{(m)}\}_m \ (j = 0, \ldots, 2k) \) which converge to \( a_j \) as \( m \to \infty \) and \( b_j^{(m)} \in \mathfrak{A}_m \) for \( j = 0, \ldots, 2k \). Then,

\[
\overline{\varphi_{2k}}_{\mathfrak{A}_n}(a_0, \ldots, a_{2k}) = \lim_{m \to \infty} \varphi_{2k} | \mathfrak{A}_m(b_0^{(m)}, \ldots, b_{2k}^{(m)}) = \varphi_{2k}(a_0, \ldots, a_{2k}).
\]

This implies that \( \varphi_{2k} | \mathfrak{A}_n = \varphi_{2k} \), which means that \( \Psi \circ \Phi \) is the identity on \( H^*_C(\mathfrak{A}) \).

On the other hand, for any \( \{(\varphi_{2k}^{(m)}(n)_k | \mathfrak{A}_n)_n \in \lim_{\epsilon \to 0} H^*_C(\mathfrak{A}_n) \), we have

\[
\Phi \circ \Psi([(\varphi_{2k}^{(n)})_k]_\mathfrak{A}_n) = \Phi([(\overline{\varphi_{2k}})_{k}]_\mathfrak{A} = \{(\overline{\varphi_{2k}})_{k}]_\mathfrak{A}_n \}.
\]

Since for \( b_0, \ldots, b_{2k} \in \mathfrak{A}_n \), we have

\[
\overline{\varphi_{2k}}_{\mathfrak{A}_n}(b_0, \ldots, b_{2k}) = \lim_{m \to \infty} \overline{\varphi_{2k}}^{(m)}(b_0, \ldots, b_{2k}) = \lim_{m \to \infty} \varphi_{2k}^{(m)}(b_0, \ldots, b_{2k}) = \varphi_{2k}^{(n)}(b_0, \ldots, b_{2k}).
\]

Thus \( \Phi \circ \Psi([(\varphi_{2k}^{(n)})_k]_\mathfrak{A}_n) = \{(\varphi_{2k}^{(n)})_k]_\mathfrak{A}_n \}. \) Hence \( \Phi \circ \Psi \) is also the identity on \( \lim_{\epsilon \to 0} H^*_C(\mathfrak{A}_n) \). Therefore, the proof is completed.

**Remark.** We here prefer the original definition by Connes \[2\] to prove our main result although Meyer \[3\] obtained the above Proposition by means of analytic cyclic theory.
5. Entire Cyclic Cohomology of \((T^2_θ)^\infty\)

Summing up the argument discussed in the previous sections, we are ready to obtain the next main result.

**Theorem 5.1.** The entire cyclic cohomology \(H^*_ε((T^2_θ)^\infty)\) of the noncommutative 2-torus \((T^2_θ)^\infty\) is isomorphic to \(\mathbb{C}^4\) as linear spaces, especially

\[
\begin{align*}
H^*_ε((T^2_θ)^\infty) & = H^{ev}_ε((T^2_θ)^\infty) \cong \mathbb{C}^2 \\
H^*_ε((T^2_θ)^\infty) & = H^{od}_ε((T^2_θ)^\infty) \cong \mathbb{C}^2,
\end{align*}
\]

where \(HP^*((T^2_θ)^\infty)\) is the periodic cyclic cohomology of \((T^2_θ)^\infty\).

**Proof.** By Lemma 4.1, we have

\[
H^*_ε((T^2_θ)^\infty) \cong H^*_ε(\lim_{\to}(C^\infty(T) \otimes (M_{2^n}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C})), \pi_n^\infty))
\]

\[
\cong \lim_{\to} H^*_ε((C^\infty(T) \otimes (M_{2^n}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C})), (\pi_n^\infty)^*)
\]

We have the following decomposition by applying Khalkhali [4]'s Proposition 7 in the case of \(F^*\)-algebras:

\[
H^*_ε(C^\infty(T) \otimes (M_{2^n}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C}))) \cong H^*_ε(C^\infty(T) \otimes (M_{2^n}(\mathbb{C}) \oplus H^*_ε(C^\infty(T) \otimes M_{2^{n-1}}(\mathbb{C})))
\]

We also deduce applying Khalkhali [4]'s Theorem 6 in the case of \(F^*\)-algebras that

\[
H^*_ε(C^\infty(T) \otimes (M_q(\mathbb{C}))) \cong H^*_ε(C^\infty(T)) \quad (q \geq 1)
\]

Since the above two phenomena are shown for \(HP^*((T^2_θ)^\infty)\) as well and we can see that

\[
H^*_ε(C^\infty(T)) = HP^1(C^\infty(T)) \cong \mathbb{C} \quad (j = ev, od)
\]

(Connes [2], Thm 2(page 208) and Thm 25(page 382)), then we obtain that

\[
H^*_ε(C^\infty(T) \otimes (M_q(\mathbb{C}))) \cong HP^j(C^\infty(T) \otimes (M_q(\mathbb{C}))) \quad (j = ev, od)
\]

We then have the following commutative diagram:

\[
\begin{array}{ccc}
HP^ev(\mathfrak{A}_{n+1}) & \xrightarrow{\cong} & H^*_ε(\mathfrak{A}_{n+1}) \\
\downarrow (\pi_n^\infty)^* & & \downarrow (\pi_n^\infty)^*
\end{array}
\]

\[
\begin{array}{ccc}
HP^ev(\mathfrak{A}_n) & \xrightarrow{i^*} & H^*_ε(\mathfrak{A}_n) \\
\xrightarrow{\cong} & & \xrightarrow{\cong}
\end{array}
\]

where \(i^*\) is the canonical inclusion map. Then we work on the periodic cyclic cohomology in what follows: we consider homomorphisms

\[
(\pi_n^\infty)^* : HP^ev(C^\infty(T) \otimes (M_{2^{n+2}}(\mathbb{C}) \oplus M_{2^{n+1}}(\mathbb{C})))
\]

\[
\rightarrow HP^ev(C^\infty(T) \otimes (M_{2^n}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C}))).
\]
Now we note that
\[
\text{HP}^{\text{ev}}(C^\infty(T) \otimes (M_{q_{2n+2}}(\mathbb{C}) \oplus M_{q_{2n+1}}(\mathbb{C}))) \\
\simeq \text{HP}^{\text{ev}}(C^\infty(T) \otimes M_{q_{2n+2}}(\mathbb{C})) \oplus \text{HP}^{\text{ev}}(C^\infty(T) \otimes M_{q_{2n+1}}(\mathbb{C}))
\]
and moreover, we have seen that
\[
\text{HP}^{\text{ev}}(C^\infty(T) \otimes M_q(\mathbb{C})) \simeq \text{HP}^{\text{ev}}(C^\infty(T)) \otimes \text{HP}^{\text{ev}}(M_q(\mathbb{C})) \\
\simeq \mathbb{C} \left[ \int_T \otimes \text{Tr}_q \right]
\]
where \(f_T\) and \(\text{Tr}_q\) are the usual integral on \(C^\infty(T)\) and the trace on \(M_q(\mathbb{C})\) respectively. Here, we consider the following diagram:
\[
\begin{array}{ccc}
\text{HP}^{\text{ev}}(\mathfrak{A}_{n+1}) & \overset{\simeq}{\longrightarrow} & \mathbb{C} \left[ \int_T \otimes \text{Tr}_{q_{2n+2}} \right] \oplus \mathbb{C} \left[ \int_T \otimes \text{Tr}_{q_{2n+1}} \right] \\
(\pi_n^\infty)^* & \downarrow & \downarrow (\pi_n^\infty)^* \\
\text{HP}^{\text{ev}}(\mathfrak{A}_n) & \overset{\simeq}{\longrightarrow} & \mathbb{C} \left[ \int_T \otimes \text{Tr}_{q_{2n-1}} \right] \oplus \mathbb{C} \left[ \int_T \otimes \text{Tr}_{q_{2n-1}} \right],
\end{array}
\]
where the horizontal isomorphisms are defined by
\[
\varphi \mapsto \varphi |_{(C^\infty(T) \otimes M_{q_{2n}}(\mathbb{C})) \oplus 0} \oplus \varphi |_{0 \oplus (C^\infty(T) \otimes M_{q_{2n-1}}(\mathbb{C}))}.
\]
We check that the diagram above is also commutative.

So, we regard \((\pi_n^\infty)^*\) as the linear map from \(\mathbb{C}[\int_T \otimes \text{Tr}_{q_{2n+2}}] \oplus \mathbb{C}[\int_T \otimes \text{Tr}_{q_{2n+1}}]\) into \(\mathbb{C}[\int_T \otimes \text{Tr}_{q_{2n}}] \oplus \mathbb{C}[\int_T \otimes \text{Tr}_{q_{2n-1}}]\). Let us recall that we write the matrix \(P_{n+1}\) by \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) used in the definition of \(\pi_n^\infty\). Then we have
\[
(\ast) \quad \left( \left( \int_T \otimes \text{Tr}_{q_{2n+2}} \right) \oplus 0 \right) (\pi_n^\infty(\xi)) = a \left( \int_T \otimes \text{Tr}_{q_{2n}} \right) (1 \otimes (x_{ij})) \\\n+ b \left( \int_T \otimes \text{Tr}_{q_{2n-1}} \right) (1 \otimes (y_{ij}))
\]
for each \(\xi = (1 \otimes (x_{ij})) \oplus (1 \otimes (y_{ij})) \in (C^\infty(T) \otimes M_{q_{2n}}(\mathbb{C})) \oplus (C^\infty(T) \otimes M_{q_{2n-1}}(\mathbb{C}))\), where \(1\) is the function which evaluates 1 at each point of \(T\). In fact, by the
definition of $\pi_n$, we have

$$\pi_n^\infty((1 \otimes (x_{ij}) \oplus (1 \otimes (y_{ij}))) = \begin{pmatrix}
  x_{11}I_a & \ldots & x_{1q'}I_a \\
  \vdots & & \vdots \\
  x_{q'1}I_a & \ldots & x_{q'q'}I_a \\
  y_{11}I_b & \ldots & y_{1q}I_b \\
  \vdots & & \vdots \\
  y_{q}I_b & \ldots & y_{qq}I_b
\end{pmatrix} \oplus 
\begin{pmatrix}
  x_{11}I_c & \ldots & x_{1q'}I_c \\
  \vdots & & \vdots \\
  x_{q'1}I_c & \ldots & x_{q'q'}I_c \\
  y_{11}I_d & \ldots & y_{1q}I_d \\
  \vdots & & \vdots \\
  y_{q}I_d & \ldots & y_{qq}I_d
\end{pmatrix},$$

where $q = q_{2n-1}$, $q' = q_{2n}$ and so on. Then, it follows that

$$\left(\left(\int_T \otimes \text{Tr}_{q_{2n+2}} + 0\right) (\pi_n^\infty(\xi))\right) = a \sum_{i=1}^{q_{2n}} x_{ii} + b \sum_{t=1}^{q_{2n-1}} y_{tt} = a \left(\int_T \otimes \text{Tr}_{q_{2n}}(1 \otimes (x_{ij}))\right) + b \left(\int_T \otimes \text{Tr}_{q_{2n-1}}(1 \otimes (y_{ij}))\right).$$

Similarly, we have

$$\left(0 \oplus \left(\int_T \otimes \text{Tr}_{q_{2n+1}}\right)\right)(\pi_n^\infty(\xi)) = c \left(\int_T \otimes \text{Tr}_{q_{2n}}(1 \otimes (x_{ij}))\right) + d \left(\int_T \otimes \text{Tr}_{q_{2n-1}}(1 \otimes (y_{ij}))\right).$$

On the other hand, we check that

$$\left(\left(\int_T \otimes \text{Tr}_{q_{2n+2}} + 0\right) (\pi_n^\infty((z^k \otimes I_{q_{2n}}) \oplus 0))\right) = 0$$

$$\left(0 \oplus \left(\int_T \otimes \text{Tr}_{q_{2n+1}}\right)\right)(\pi_n^\infty((z^k \otimes I_{q_{2n}}) \oplus 0)) = 0$$

$$\left(\left(\int_T \otimes \text{Tr}_{q_{2n+2}} + 0\right) (\pi_n^\infty(0 \oplus (z^k \otimes I_{q_{2n-1}})))\right) = 0$$

and

$$\left(0 \oplus \left(\int_T \otimes \text{Tr}_{q_{2n+1}}\right)\right)(\pi_n^\infty(0 \oplus (z^k \otimes I_{q_{2n-1}}))) = 0$$
for each integer \( k \geq 1 \). Indeed, for example, it is easily verified that if
\[
\begin{pmatrix}
0 & z \\
1 & \\
\vdots & \\
& \\
1 & 0
\end{pmatrix}
\in M_q(C^\infty(T)),
\]
then
\[
\begin{pmatrix}
0 & z \\
1 & \\
\vdots & \\
& \\
1 & 0
\end{pmatrix}^k = \begin{cases} 
\begin{pmatrix}
z^\nu \otimes I_q \\
0 & * \\
\end{pmatrix} & (k \equiv 0 \mod q) \\
\begin{pmatrix}
* \\
0 \\
\end{pmatrix} & (k \not\equiv 0 \mod q)
\end{cases}
\]
for some integer \( \nu \geq 1 \). Thus, we have that
\[
(\int_T \otimes \text{Tr}_q)
\begin{pmatrix}
0 & z \\
1 & \\
\vdots & \\
& \\
1 & 0
\end{pmatrix}^k = \begin{cases} 
\begin{pmatrix}
z^\nu dz \\
0 \\
\end{pmatrix} & (k \equiv 0 \mod q) \\
0 & (k \not\equiv 0 \mod q)
\end{cases}
\]
\[= 0.
\]

Since the space of Laurent polynomials are dense in \( C^\infty(T) \) with respect to Fréchet topology, we then conclude that (31) and (33) hold for every \( \xi \in A_n \). Hence, it is verified that \( (\pi_\infty^\infty)^* \) is an isomorphism by the fact that
\[
\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det P_{n+1} = \det \begin{pmatrix} a_{4n+4} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+1} & 1 \\ 1 & 0 \end{pmatrix} = 1 \neq 0
\]
Finally, we conclude that
\[
H^\text{ev}_e((T^2_\theta)^\infty) \simeq \lim_{\xi} (\mathbb{C} \oplus \mathbb{C}, (\pi_\infty^\infty)^*) \simeq \mathbb{C}^2.
\]
Analogously, the same consequence is obtained in the odd case. We note that
\[
H^\text{odd}(C^\infty(T) \otimes M_q(\mathbb{C})) \simeq H^\text{odd}(C^\infty(T)) \otimes H^\text{even}(M_q(\mathbb{C})) \simeq \mathbb{C} [\psi \otimes \text{Tr}_q],
\]
where \( \psi(f, g) = \int_T f(t)g'(t)dt \) for \( f, g \in C^\infty(T) \). This ends the proof. \( \square \)
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REFERENCES

[1] A. Connes, C∗ algebrès et géométrie differentielle, C.R. Acad. Sci. Paris, Ser. A, 290, (1980), 599–604.
[2] A. Connes, Noncommutative Geometry, Academic Press (1994).
[3] G. A. Elliott, D. E. Evans, The structure of the irrational rotation C∗-algebra, Annals. of Math 138 (1993), 477–501.
[4] M. Khalkhali, On the entire cyclic cohomology of Banach algebras, Comm. in Alg. 22(14), 5861–5874.
[5] R. Meyer, Analytic cyclic cohomology, Ph.D. Thesis, Münster, 1999, arXiv. math. KT/9906205.
[6] M. A. Rieffel, C∗-algebras associated with irrational rotations, Pacific J. Math. 93 (1981), 415–429.
[7] M. A. Rieffel, The cancellation theorem for projective modules over irrational rotation C∗-algebras, Proc. London Math. Soc. (3) 47 (1983), no. 2, 285–302.
[8] M. A. Rieffel, Dimension and stable rank in the K-theory of C∗-algebras, Proc. London Math. Soc. (3) 46 (1983), no. 2, 301–333.