BIPARTITE COMPLEMENTS OF CIRCLE GRAPHS

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ABSTRACT. Using an algebraic characterization of circle graphs, Bouchet proved in 1999 that if a bipartite graph $G$ is the complement of a circle graph, then $G$ is a circle graph. We give an elementary proof of this result, based on the ham sandwich theorem.

A graph is a circle graph if it is the intersection graph of the chords of a circle. Using an algebraic characterization of circle graphs proved by Naji [5] (as the class of graphs satisfying a certain system of equalities over GF(2)), Bouchet proved the following result in [1].

Theorem 1 (Bouchet [1]). If a bipartite graph $G$ is the complement of a circle graph, then $G$ is a circle graph.

The known proofs of Naji’s theorem are fairly involved [5, 3, 6], and Bouchet [1] (see also [2]) asked whether, on the other hand, Theorem 1 has an elementary proof. The purpose of this short note is to present such a proof.

We will need two simple lemmas. Given a finite set of points $X \subset \mathbb{R}^2$ of even cardinality, a line $\ell$ bisects the set $X$ if each open half-plane defined by $\ell$ contains precisely $|X|/2$ points. The following lemma is the 2-dimensional discrete ham sandwich theorem (for more details, see e.g. [4]).

Lemma 2. Let $X, Y \subset \mathbb{R}^2$ be disjoint finite point sets of even cardinality, in general position. Then there exists a line $\ell$ simultaneously bisecting both $X$ and $Y$.

Lemma 3. Consider a set of pairwise intersecting chords $c_1, \ldots, c_n$ of a circle $C$, with pairwise distinct endpoints. Then any line $\ell$ that bisects the $2n$ endpoints of the chords intersects all the chords $c_1, \ldots, c_n$.

Proof. Assume for the sake of contradiction that some chord $c_i$ does not intersect $\ell$. Then $c_i$ lies in one of the two open half-planes defined by $\ell$, say to the left of $\ell$. Since $\ell$ bisects the $2n$ endpoints of the chords, it follows that there is another chord $c_j$ that does not intersect $\ell$ and which lies in the half-plane to the right of $\ell$. This implies that $c_i$ and $c_j$ do not intersect, which is a contradiction.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Consider a bipartite graph $G$ such that its complement $\overline{G}$ is a circle graph. In particular, for any vertex $v_i$ of $\overline{G}$ there is a chord

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$c_i$ of some circle $C$ such that any two vertices $v_i$ and $v_j$ are adjacent in $\overline{G}$ (equivalently, non-adjacent in $G$) if and only if the chords $c_i$ and $c_j$ intersect. Since $G$ is bipartite, the vertices $v_1, \ldots, v_n$ (and the corresponding chords $c_1, \ldots, c_n$) can be colored with colors red and blue such that any two chords of the same color intersect. We can assume without loss of generality that the endpoints of the $n$ chords are pairwise distinct, so the coloring of the chords also gives a coloring of the $2n$ endpoints with colors red or blue (with an even number of blue endpoints and an even number of red endpoints).

Since the $2n$ endpoints lie on the circle $C$, they are in general position and it follows from Lemma 2 that there exists a line $\ell$ simultaneously bisecting the set of blue endpoints and the set of red endpoints.

On one side of $\ell$, reverse the order of the endpoints of the chords $c_1, \ldots, c_n$ along the circle $C$. Observe that crossing chords intersecting $\ell$ become non-crossing, and vice versa. By Lemma 3, $\ell$ intersects all the chords $c_1, \ldots, c_n$, and thus the resulting circle graph is precisely $G$. It follows that $G$ is a circle graph, as desired.

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