Including off-diagonal anisotropies in anisotropic hydrodynamics

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Abstract

In this paper we present a method for efficiently including the effects of off-diagonal local rest frame momentum anisotropies in leading-order anisotropic hydrodynamics. The method relies on diagonalization of the space-like block of the anisotropy tensor and allows one to reduce the necessary moments of the distribution function in the off-diagonal case to a linear combination of diagonal-anisotropy integrals. Once reduced to diagonal-anisotropy integrals, the results can be computed efficiently using techniques described previously in the literature. We present a general framework for how to accomplish this and provide examples for off-diagonal anisotropy moments entering into the energy-momentum tensor and viscous update equations which emerge when performing anisotropic pressure matching.

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I. INTRODUCTION

Ultra-relativistic heavy ion collision (URHIC) experiments, e.g. RHIC at BNL and LHC at CERN, aim to study the dynamics and properties of matter at extremely high-energy density. In these experiments, matter is heated up to temperatures above the QCD pseudo-critical temperature, $T_{pc} \simeq 155$ MeV, using ultra-relativistic collisions among heavy nuclei, protons, deuterons, etc. The strongly interacting droplet of matter produced during high-energy and high-multiplicity URHICs is called the quark-gluon plasma (QGP). In high-multiplicity events, the QGP demonstrates strong collective behavior during evolution from hydrodynamization ($\tau \sim 1$ fm/c) to hadronic freeze out ($\tau \sim 10$ fm/c). During this time period it has been found that relativistic fluid dynamics formalisms can effectively describe the evolution of the system and one finds that information about initial state geometry of the target (average eccentricity and fluctuations) is reflected in final state observables, e.g. the azimuthal dependence of hadron production. In other words, one can track the correlations between the eccentricity of the initial state’s geometry and the flow harmonics observed in the final state hadron spectra using dissipative hydrodynamics. The success of relativistic dissipative hydrodynamics [1–4] has inspired theoreticians to make the underlying formalisms more complete and robust with respect to large deviations from isotropic thermal equilibrium using standard fixed-order viscous hydrodynamics (vHydro) treatments [5–32] and resummed anisotropic hydrodynamics (aHydro) treatments [4, 33–54].

The introduction of the aHydro formalism was driven by the fact that, due to the strong early-stage longitudinal expansion of the QGP, one finds large momentum-space anisotropy in the local rest frame (LRF) of the QGP which persists for many fm/c. The magnitude of the momentum-space anisotropy has cast some doubt on the quantitative accuracy of standard vHydro which assumes that one can linearize around isotropic equilibrium. aHydro is a non-equilibrium hydrodynamics model which takes into account the strong momentum-space anisotropy of the QGP at leading order and in doing so resums an infinite number of terms in inverse Reynolds number [55]. In contrast to standard vHydro, aHydro is based on Taylor expansion about an anisotropic distribution function instead of an isotropic one. This allows one to capture the dominant anisotropic contributions to the distribution function in the leading order term, thereby guaranteeing positivity of the one-particle distribution at all space-time points at leading-order. aHydro and vHydro have been tested against exact
solutions of the Boltzmann equation for systems subject to Bjorken [55–60] and Gubser flows [61–64]. In all cases, it was found that aHydro provided the best approximation to the exact solutions for both hydrodynamic and non-hydrodynamic moments of the distribution function [60].

This provided motivation to compare the aHydro framework with experimental results. Despite the success of these early comparisons, in all phenomenological applications of aHydro to date, aHydro codes have been implemented using an anisotropy tensor which possesses only diagonal (elliptical) anisotropies (see Ref. [4] for a recent review). This was done mainly because of the difficulty of efficiently evaluating the necessary moment integrals in the presence of off-diagonal anisotropies $\xi_{ij}$ with $i \neq j$. However, to be complete, one must also include the possibility of off-diagonal leading-order anisotropies. Near equilibrium, this is equivalent to including off-diagonal components in the LRF shear viscous tensor $\pi^{ij}$.

In this paper, we present a technique that can be used to efficiently include non-vanishing $\xi_{ij}$. This is done by a change of variables in the generic moment integrals which diagonalizes the anisotropy tensor. Once cast into diagonal form, a previously developed technique for the efficient application of diagonal moment integrals can be used to compute the necessary off-diagonal moment integrals (see Appendix B of Ref. [53]). We present the general method of diagonalization and provide some concrete examples for the application to aHydro frameworks which use the so-called anisotropic-pressure- or Tinti-matching [43, 48].

**CONVENTIONS AND NOTATION**

The Minkowski metric tensor is taken to be “mostly minus”, i.e. $g^{\mu\nu} = \text{diag}(+,-,-,-)$. The transverse projection operator $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$ is used to project four-vectors and/or tensors into the space orthogonal to $u^\mu$. Parentheses and square brackets on indices denote symmetrization and anti-symmetrization, respectively, i.e. $A^{(\mu\nu)} \equiv \frac{1}{2} (A^{\mu\nu} + A^{\nu\mu})$ and $A^{[\mu\nu]} \equiv \frac{1}{2} (A^{\mu\nu} - A^{\nu\mu})$. Angle brackets on indices indicate projection with a four-index transverse projector, $A^{(\mu\nu)} \equiv \Delta^{\mu\nu}_{\alpha\beta} A^{\alpha\beta}$, where $\Delta^{\mu\nu}_{\alpha\beta} \equiv \Delta^{[\mu}_{\alpha} \Delta^{\nu]}_{\beta} - \Delta^{\mu\nu} \Delta_{\alpha\beta}/3$ projects out the traceless and $u^\mu$-transverse components of a rank-two tensor. The Lorentz-invariant momentum-space integration measure is indicated as $dP = \tilde{N} d^3 p / E$, with $\tilde{N} = N_{\text{dof}}/(2\pi)^3$ where $N_{\text{dof}}$ is the number of degrees of freedom.
II. LEADING-ORDER ANISOTROPIC HYDRODYNAMICS

In leading-order aHydro, the one-particle distribution function is parametrized by an anisotropy tensor which results in the deformation of the argument of an isotropic distribution function into an anisotropic one [37, 41]

\[ f(x,p) = f_{\text{iso}}\left(\frac{1}{\lambda} \sqrt{p_\mu \Xi^{\mu\nu} p_\nu}\right), \]

(1)

where \( \lambda \) has dimensions of energy and can be identified with temperature only in the isotropic equilibrium limit. In practice, \( f_{\text{iso}} \) can be a Bose-Einstein, Fermi-Dirac, or Maxwell-Boltzmann distribution depending on particle statistics and/or energy. In the non-conformal (massive) case, the rank-2 tensor \( \Xi^{\mu\nu} \) specifying the shape of the distribution in momentum space is defined as [37, 41]

\[ \Xi^{\mu\nu} = u^\mu u^\nu + \xi^{\mu\nu} - \Phi \Delta^{\mu\nu}, \]

(2)

where \( u^\mu \) is the flow velocity four-vector and \( \xi^{\mu\nu} \) denotes a symmetric traceless anisotropy tensor, i.e. \( \xi_x + \xi_y + \xi_z = 0 \).

The quantities \( \lambda, u^\mu, \) and \( \xi^{\mu\nu} \) are understood to be functions of spacetime and satisfy the following identities

\[ u^\mu u_\mu = 1, \]

(3)

\[ \xi^\mu_\mu = 0, \]

(4)

\[ u_\mu \xi^{\mu\nu} = 0. \]

(5)

The third condition, indicating orthogonality of \( \xi^{\mu\nu} \) to \( u^\mu \) implies that in the LRF \( \xi^{\mu\nu} \) obeys the following conditions

\[ \xi^{00} = \xi^{0i} = \xi^{i0} = 0. \]

(6)

Working in the LRF, this allows us to focus on the non-trivial space-like components of \( \xi^{\mu\nu} \) as \( \xi \), which is a \( 3 \times 3 \) matrix. The argument of distribution function subject to the mass-shell
condition can be simplified as

\[ p \cdot \Xi^{\mu\nu} \cdot p = p \cdot \kappa \cdot p + m^2, \tag{7} \]

where

\[ \kappa \equiv I(1 + \Phi) + \xi, \tag{8} \]

with \( I \) being a \( 3 \times 3 \) identity matrix.

If \( \xi \) is diagonal, i.e.

\[ \xi = \text{diag}(\xi_x, \xi_y, \xi_z), \tag{9} \]

one has \( \kappa = \text{diag}(1/\alpha_x^2, 1/\alpha_y^2, 1/\alpha_z^2) \), which gives

\[ f(x, p) = f_{iso}\left(\frac{1}{\lambda} \sqrt{\sum_i \frac{p_i^2}{\alpha_i^2} + m^2}\right), \tag{10} \]

with \( \alpha_i = (1 + \xi_i + \Phi)^{-1/2} \) [41].

In general, however, \( \xi \) can include off-diagonal components, as well. Using the definition above for \( \alpha_i \), one can parametrize \( \kappa \) as

\[ \kappa = \begin{pmatrix} 1/\alpha_x^2 & \xi_{xy} & \xi_{xz} \\ \xi_{xy} & 1/\alpha_y^2 & \xi_{yz} \\ \xi_{xz} & \xi_{yz} & 1/\alpha_z^2 \end{pmatrix}, \tag{11} \]

and the distribution function is

\[ f_\alpha(x, p) = f_{iso}\left(\frac{1}{\lambda} \sqrt{p \cdot \kappa \cdot p + m^2}\right). \tag{12} \]

III. DIAGONALIZATION

Calculating the bulk variables in aHydro requires computing momentum-space moments of the distribution function. However, the distribution function in Eq. (12) is a complicated function of momentum and there is no way to perform the integrals analytically except in some special cases. In this section, we introduce an algebraic method to diagonalize the \( \kappa \) matrix so that we can reduce the computation of moment-integrals including off-diagonal
anisotropies to a linear combination of diagonal momentum-space moment integrals.

For any $N \times N$ square matrix $\kappa$, which has $N$ linearly independent eigenvectors, there exists an orthogonal matrix $A$ ($A^T A = A A^T = 1$) such that

$$\kappa = A \kappa_D A^{-1}. \tag{13}$$

The columns of the $A$ are given by the eigenvectors of $\kappa$. The matrix $\kappa_D$ is the diagonal representation of $\kappa$ whose diagonal elements, in principle, are the eigenvalues of $\kappa$. One can rewrite the equation above as

$$\kappa_D = A^{-1} \kappa A. \tag{14}$$

We know from linear algebra that any real symmetric square matrix, i.e. $\kappa = \kappa^\dagger$, has real eigenvalues, i.e. $\kappa_D = \kappa_D^\dagger$. Using this, one can prove that $A$ is unitary: Taking the Hermitian conjugate of both sides of Eq. (13) one has

$$\kappa^\dagger = [A \kappa_D A^{-1}]^\dagger = (A^\dagger)^{-1} \kappa_D^\dagger A^\dagger = (A^\dagger)^{-1} \kappa_D A^\dagger = \kappa. \tag{15}$$

This relation is consistent with (13) only if $A^\dagger = A^{-1}$, indicating that $A$ is unitary. On the other hand, with real eigenvalues there are always a set of purely real eigenvectors of the matrix $\kappa$. This indicates that matrix $A$ is real as well (and indeed not symmetric generally). Taken together, this indicates that $A$ is orthogonal, i.e. $A^T = A^{-1}$. The combination $p \cdot \kappa \cdot p$ can be written as

$$p \cdot \kappa \cdot p = p^T \kappa p = \left[p^T A \right]\left[A^{-1} \kappa A \right] \left[A^{-1} p \right] = \tilde{p}^T \kappa_D \tilde{p} = \tilde{p} \cdot \kappa_D \cdot \tilde{p}, \tag{16}$$

with $\tilde{p} \equiv A^{-1} p$. By definition we have

$$p = A \tilde{p} \quad \Rightarrow \quad p_i = \sum_j A_{ij} \tilde{p}_j. \tag{17}$$

For example

$$p_i = \sum_j A_{ij} \tilde{p}_j = \sum_j v_i^{(j)} \tilde{p}_j, \tag{18}$$
where the vector \( v^{(i)} = (v_x^{(i)}, v_y^{(i)}, v_z^{(i)}) \) is the \( i \)th eigenvector of \( \kappa \). Therefore, we have two frames, i.e. initial frame and tilde frame, where the components of momentum vector are \( p_i \) and \( \tilde{p}_i \), respectively. These two frames are connected by rotations through a set of Euler angles. Note that the Jacobian, \(|\text{det} A|\), for transforming between the two frames is unity.

It is obvious that the length of \( p \) is invariant under the coordinate transformation. Accordingly, as expected, \( E \) is the same in both coordinates

\[
E = \sqrt{p^2 + m^2} = \sqrt{\tilde{p}^2 + m^2} = \tilde{E}.
\]

Using Eq. (18), one can define the diagonal anisotropic distribution function as

\[
f_a(x, p) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{p \cdot \kappa \cdot p + m^2} \right) = f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{\tilde{p} \cdot \kappa_D \cdot \tilde{p} + m^2} \right) \equiv f_a^D(x, \tilde{p}).
\]

IV. THE ENERGY-MOMENTUM TENSOR

We begin by demonstrating how this method can be used to efficiently evaluate the components of the energy-momentum tensor including off-diagonal anisotropies. In the general case, we have six anisotropy parameters \( (\alpha_x, \alpha_y, \alpha_z, \xi_{xy}, \xi_{xz}, \text{and} \xi_{yz}) \), one energy-scale parameter \( (\lambda) \), and the three independent components of the fluid four-velocity \( (u^i) \), resulting in ten space-time fields for which we must obtain equations of motion. In the LRF, the non-vanishing components of the energy-momentum tensor are

\[
\mathcal{E} = T^{00} = \int dP E^2 f_a(x, p),
\]

\[
T^{ij} = \int dP p_i p_j f_a(x, p).
\]

Using the techniques introduced in the previous section one finds

\[
\mathcal{E} = \int dP E^2 f_a(x, p)
\]

\[
= \tilde{N} \int d^3 \tilde{p} \sqrt{\tilde{p}^2 + m^2} f_{\text{iso}} \left( \frac{1}{\lambda} \sqrt{\tilde{p} \cdot \kappa_D \cdot \tilde{p} + m^2} \right)
\]

\[
= \tilde{\alpha} \lambda^4 Q_3(\tilde{\alpha}^2, \tilde{m}),
\]
and

\[ T_{ij} = \int dP p_i p_j f_\alpha(x, p) = \tilde{\alpha} \lambda^4 \sum_{k=1}^{3} v_i^{(k)} v_j^{(k)} \tilde{\alpha}_k Q_3^k(\tilde{\alpha}^2, \tilde{m}). \]  

(24)

The \( Q \)-functions appearing above only depend on the diagonal anisotropies \( \tilde{\alpha} \) and are defined in Appendix A. The second argument of the \( Q \)-functions is the scaled mass \( \tilde{m} \equiv m/\lambda \) and we have introduced the compact notation that \( \tilde{\alpha} \equiv \tilde{\alpha}_x \tilde{\alpha}_y \tilde{\alpha}_z \). Note that for the diagonal terms (pressures) one obtains

\[ \mathcal{P}_i = T_{ii} = \tilde{\alpha} \lambda^4 \sum_{k=1}^{3} \left[ v_i^{(k)} \right]^2 \tilde{\alpha}_k^2 Q_3^k(\tilde{\alpha}^2, \tilde{m}). \]  

(25)

In all cases above, we have reduced the problem to computing \( Q \)-functions with only diagonal anisotropies. The diagonal anisotropy tensor integrals can be well-approximated by Taylor expanding to high-order around an isotropic point, e.g. \( \tilde{\alpha}_{\text{iso}} = (\alpha_0, \alpha_0, \alpha_0) \). At each order in this expansion the required integrals can be performed analytically. In order to cover the space using truncated Taylor expansions, one can utilize multiple expansion points which are then pieced together to accurately span the range of diagonal anisotropies which are generated in typical simulations. Using modern computerized algebra systems one can extend the Taylor expansion expressions described above to high order. In practice, phenomenological codes have used 12\textsuperscript{th} order truncations in \( \tilde{\delta} = \tilde{\alpha} - \tilde{\alpha}_{\text{iso}} \) (see Appendix B of Ref. [53]).

V. DYNAMICAL EQUATIONS - ANISOTROPIC PRESSURE MATCHING

To further demonstrate the utility of this method, we now consider equations for the viscous tensor obtained by anisotropic pressure matching [43]. In relaxation-time approximation (RTA) the dynamical equations for the shear and bulk viscous corrections based on
anisotropic matching are

\[ \partial_\mu T^{\mu\nu} = 0, \quad (26) \]

\[ D\pi^{(\mu\nu)} + \frac{1}{\tau_{eq}} \pi^{\mu\nu} = -\left( \sigma_{\rho\sigma} + \frac{1}{3} \theta \Delta_{\rho\sigma} \right) \int dP \frac{p^{(\mu} p^{\nu)} p^\rho p^\sigma f_a}{(p \cdot U)^2} - 2\pi^{(\mu}_{\alpha} \sigma^{\nu\alpha} \]

\[ + 2 P \sigma^{\mu\nu} - \frac{5}{3} \theta \pi^{\mu\nu} + 2 \pi^{\mu}_{\alpha} \omega^{\nu\alpha}, \quad (27) \]

\[ DP + \frac{1}{\tau_{eq}} (P - P_{eq}) = \frac{1}{3} \left( \sigma_{\rho\sigma} + \frac{1}{3} \theta \Delta_{\rho\sigma} \right) \int dP \frac{(p \cdot \Delta \cdot p) p^\rho p^\sigma f_a}{(p \cdot U)^2} \]

\[ + \frac{2}{3} \pi_{\mu\nu} \sigma^{\mu\nu} - \frac{5}{3} P \theta. \quad (28) \]

In above relations, one has

\[ D = u^\mu \partial_\mu, \quad (29) \]

\[ \sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \partial_\alpha u^\beta, \quad (30) \]

\[ \theta = \nabla_\mu u^\mu, \quad (31) \]

\[ \nabla^\mu = \Delta^{\mu\nu} \partial_\nu, \quad (32) \]

\[ \omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2, \quad (33) \]

\[ P = P_{eq} + \Pi, \quad (34) \]

\[ \Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2} \left( \Delta^\mu_{\alpha} \Delta^\nu_{\beta} + \Delta^\nu_{\alpha} \Delta^\mu_{\beta} - \frac{2}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) . \quad (35) \]

The aim of this section is to expand and simplify the equations above for the case of a non-ellipsoidal anisotropic distribution function. To do so, let’s define two useful functions that repeatedly appear

\[ F_{ij} \equiv \int dP \ p_i p_j f_a(x,p), \quad (36) \]

\[ F_{ijkl} \equiv \int dP \ p_i p_j p_k p_l f_a(x,p). \quad (37) \]

There are two terms in Eqs. (27) and (28) needing detailed expansion. The first one is

\[ \left( \sigma_{\rho\sigma} + \frac{1}{3} \theta \Delta_{\rho\sigma} \right) p^\rho p^\sigma = \left[ p^\alpha - (p \cdot u) u^\alpha \right] \left[ p^\beta - (p \cdot u) u^\beta \right] \partial_\alpha u_\beta = p^\beta \left[ \partial_\alpha p^\beta - (p \cdot u) \Delta u_\beta \right] . \quad (38) \]

\[ ^1 \text{Note that } F_{ij} \text{ corresponds to the space-like components of the energy-momentum tensor discussed in the previous section.} \]
and also
\[ p^{(\mu p^\nu)} = \Delta_{\alpha \beta} p^\alpha p^\beta = p^\mu p^\nu + (p \cdot u)^2 u^\mu u^\nu - 2 u^{(\mu p^\nu)}(p \cdot u) + \frac{1}{3} \Delta_{\mu \nu} p^2. \]  

(39)

From \( p^{(\mu p^\nu)} \) we only need the spatial block which is the result of contraction with \( X^\mu_i X^\nu_j \) which yields
\[ p^{(i p^j)} = p_i p_j - \frac{1}{3} \delta_{ij} p^2, \]  

(40)

and
\[ p \cdot \Delta \cdot p = -p^2. \]  

(41)

Using the above relations, one can expand the following integrals
\[ -(\sigma_{\rho \sigma} + \frac{1}{3} \theta \Delta_{\rho \sigma}) \int \frac{dP}{E^2} p^\rho p^\sigma p^{(\mu p^\nu)} f_a = -\int \frac{dP}{E^2} f_a \left[ p_i p_j - \frac{1}{3} \delta_{ij} p^2 \right] p^\beta \left[ p^\alpha \partial_\alpha - (p \cdot u) D \right] u_\beta \]  

\[ = \int \frac{dP}{E^2} f_a \left[ -p_i p_j p^\beta p^\alpha \partial_\alpha + p_i p_j p^\beta E D + \frac{1}{3} \delta_{ij} p^2 p^\beta p^\alpha \partial_\alpha - \frac{1}{3} \delta_{ij} p^2 p^\beta E D \right] u_\beta \]  

= sum of four terms.

(42)

Using the functions defined in Eqs. (36)-(37)
\[ \text{term1} = -\int dP f_a p_i p_j \partial_0 u_0 + \int \frac{dP}{E^2} f_a p_i p_j p_k \partial_k u_k = -\mathcal{F}_{ij} \partial_0 u_0 + \mathcal{F}_{ijkl} \partial_k u_l, \]  

(43)

\[ \text{term2} = \int dP f_a p_i p_j D u_0 = \mathcal{F}_{ij} D u_0, \]  

(44)

\[ \text{term3} = \frac{1}{3} \delta_{ij} \int \frac{dP}{E^2} f_a p^2 \left[ E^2 \partial_0 u_0 - p_i p_k \partial_k u_k \right] = \frac{1}{3} \delta_{ij} \sum_{l=1}^{3} \mathcal{F}_{il} \partial_0 u_0 - \frac{1}{3} \delta_{ij} \sum_{n=1}^{3} \mathcal{F}_{mnkl} \partial_k u_l, \]  

(45)

\[ \text{term4} = -\frac{1}{3} \delta_{ij} \int dP f_a p^2 D u_0 = -\frac{1}{3} \delta_{ij} \sum_{l=1}^{3} \mathcal{F}_{il} D u_0. \]  

(46)

Putting everything together one obtains
\[ -(\sigma_{\rho \sigma} + \frac{1}{3} \theta \Delta_{\rho \sigma}) \int \frac{dP}{E^2} p^\rho p^\sigma p^{(\mu p^\nu)} f_a = \]  

\[ \mathcal{F}_{ij} (D u_0 - \partial_0 u_0) + \mathcal{F}_{ijkl} \partial_k u_l + \frac{1}{3} \delta_{ij} \sum_m \left[ -\mathcal{F}_{mnkl} \partial_k u_l + \mathcal{F}_{mm} (\partial_0 u_0 - D u_0) \right]. \]  

(47)
Similarly, the non-trivial term appearing in the bulk viscous equation of motion (28) is
\[
\frac{1}{3} \left( \sigma_{\rho\sigma} + \frac{1}{3} \theta \Delta_{\rho\sigma} \right) \int \frac{dP}{E^2} p^\rho p^\sigma (p \cdot \Delta \cdot p) f_a = -\frac{1}{3} \int \frac{dP}{E^2} \frac{f_a}{p^2} p^\beta \left[ p^\alpha \partial_\alpha - (p \cdot u) D \right] u_\beta
\]
\[
= -\frac{1}{3} \int \frac{dP}{E^2} \frac{f_a}{p^2} \left[ E^2 \partial_0 u_0 - p l p k \partial_k u_k - E^2 Du_0 \right]
\]
\[
= \frac{1}{3} \sum_i F_{ii} \left[ Du_0 - \partial_0 u_0 \right] + \frac{1}{3} \sum_m F_{mmkl} \partial_k u_l. \quad (48)
\]

To proceed, we can write the final expression for \( F_{ij} \) obtained previously (24)
\[
F_{ij} = \tilde{\alpha} \lambda^4 \sum_{k=1}^3 v_i^{(k)} v_j^{(k)} \tilde{\alpha}^2 Q^k_3 (\tilde{\alpha}^2, \hat{m}). \quad (49)
\]

Based on the symmetry of \( F_{ij} \) under exchanging the indices, out of 9 possible values there are only 6 unique terms that must be calculated. Next we consider the four-index \( F \) integral. Using similar techniques as used for the two-index version, one obtains
\[
F_{ijkl} = \tilde{N} \int \frac{d^3p}{E^3} p_i p_j p_k p_l f_a(x, p) = \tilde{N} \int \frac{d^3p}{E^3} f_a(x, \tilde{p}) \sum_{m,n=1}^3 \mathcal{P}_{mn} \left[ v_i^{(m)} v_j^{(m)} v_i^{(n)} v_j^{(n)} \right] \tilde{p}_m^2 \tilde{p}_n^2,
\]
\[
= \tilde{\alpha} \lambda^4 \sum_{m,n=1}^3 \mathcal{P}_{mn} \left[ v_i^{(m)} v_j^{(m)} v_i^{(n)} v_j^{(n)} \right] \tilde{\alpha}_m^2 \tilde{\alpha}_n^2 Q^m_3 (\tilde{\alpha}, \hat{m}). \quad (50)
\]
The operator \( \mathcal{P}_{mn} \) introduced above is the permutation operator which sums over all possible permutation of \( m \) and \( n \) in the operand. Based on the symmetry of \( F_{ijkl} \) under exchanging the indices, out of 81 possible values there are only 15 unique terms that must be calculated. The two-index \( Q \)-function, \( Q^{mn} \), introduced above is defined in Appendix A.

VI. DISCUSSION

As we demonstrated in the previous two sections, one can reduce the problem of evaluating complicated off-diagonal anisotropy moment integrals to a sum of diagonal anisotropy integrals. In practice, one can use Eqs. (26), (27), and (28) to evolve the energy-momentum tensor, shear viscous tensor, and bulk viscous correction, respectively. Given an initial condition for \( T^{\mu\nu} \) specified in terms of all anisotropies and the momentum scale, \( \lambda \), one can construct the full energy-momentum tensor at the initial time. One can then evolve the cou-
pled partial differential equations (26), (27), and (28) forward in time by one infinitesimal step making use of the methods explained in the previous section to evaluate the non-trivial integrals involving $f_a$ in Eqs. (27) and (28).

Once the update is complete, one can solve a set of seven non-linear equations to extract the updated LRF anisotropies and scale parameter. These can then we used to compute the non-trivial integrals involving $f_a$ in the next time step. Repeating this procedure, one can evolve all dynamical fields using Eqs. (26), (27), and (28). Critical to accomplishing this is the efficient evaluation of the integrals involving $f_a$ in Eqs. (27) and (28) and the subsequent extraction of the local anisotropy tensor from the full energy-momentum tensor. The diagonalization method described in the previous two sections solves this problem by removing the bottleneck of evaluating complicated three dimensional integrals on demand.

VII. CONCLUSIONS

In this paper we presented a method for efficiently including the effects of off-diagonal local rest frame momentum anisotropies in leading-order anisotropic hydrodynamics. The method relies on diagonalization of the space-like block of the anisotropy tensor and allows one to reduce the necessary moments of the distribution function in the off-diagonal case to a linear combination of diagonal-anisotropy integrals. Once reduced to diagonal-anisotropy integrals, the results can be computed efficiently using techniques described previously in the literature [53]. We presented a general framework for how to accomplish this and provided examples for off-diagonal anisotropy moments entering into the energy-momentum tensor and viscous update equations which emerge when performing anisotropic pressure matching [43]. With this method in hand one can implement a leading-order anisotropic hydrodynamics code that takes into account off-diagonal anisotropies non-perturbatively. Additionally, since the equations are formulated at the level of the energy-momentum tensor and shear viscous tensor, this more easily allows for the use of advanced numerical techniques for solving the necessary partial differential equations (see e.g. [65]).
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Appendix A: Q-functions

The Q-functions used in expanding the equations are defined as follows

\[
Q_3(\alpha, \hat{m}) = \tilde{N} \int d^3p \sqrt{\sum_k \alpha_k^2 p_k^2 + \hat{m}^2} f_{\text{iso}} \left( \sqrt{p^2 + \hat{m}^2} \right), \quad (A1)
\]

\[
Q_i^3(\alpha, \hat{m}) = \tilde{N} \int d^3p \frac{p_i^2}{\sqrt{\sum_k \alpha_k^2 p_k^2 + \hat{m}^2}} f_{\text{iso}} \left( \sqrt{p^2 + \hat{m}^2} \right), \quad (A2)
\]

\[
Q_{ij}^3(\alpha, \hat{m}) = \tilde{N} \int d^3p \frac{p_i^2 p_j^2}{\left( \sum_k \alpha_k^2 p_k^2 + \hat{m}^2 \right)^{3/2}} f_{\text{iso}} \left( \sqrt{p^2 + \hat{m}^2} \right), \quad (A3)
\]

where \( \alpha = \{\alpha_x, \alpha_y, \alpha_z\} \). We note that the functions above functions are related, e.g.

\[
Q_i^3(\alpha, \hat{m}) = 2 \frac{\partial Q_3}{\partial \alpha_i^2}, \quad (A4)
\]

\[
Q_{ij}^3(\alpha, \hat{m}) = -2 \frac{\partial Q_3}{\partial \alpha_j^2} = -4 \frac{\partial^2 Q_3}{\partial \alpha_i^2 \partial \alpha_j^2}. \quad (A5)
\]

This fact allows us to reduce the number of underlying Q-functions that have to be computed to the “master function” \( Q_3 \).

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