Primitive permutation groups of degree $3p$

by Peter M. Neumann

This paper presents an analysis of primitive permutation groups of degree $3p$, where $p$ is a prime number, analogous to H. Wielandt's treatment [19] of groups of degree $2p$. It is also intended as an example of the systematic use of combinatorial methods as surveyed in §6 for distilling information about a permutation group from knowledge of the decomposition of its character. The work is organised into three parts. Part I contains the lesser half of the calculation, the determination of the decomposition of the permutation character. Part II contains a survey of the combinatorial methods and, based on these methods, the major part of the calculation. Part III ties up loose ends left earlier in the paper and gives a tabulation of detailed numerical results.

Part I

1 Introduction

The results can be summarised as follows: if $p$ is prime and $p \geq 5$, then every primitive permutation group of degree $3p$ is 2-fold transitive unless, for some integer $a$,

\begin{align*}
(i) & \quad p = 3a^2 + 3a + 1 \quad (a \geq 1) \\
\text{or} & \quad (ii) \quad p = 48a^2 + 30a + 5 \quad (a \geq 0) \\
\text{or} & \quad (iii) \quad p = 48a^2 + 66a + 23 \quad (a \geq 0) \\
\text{or} & \quad (iv) \quad p = 7, 19 \text{ or } 31.
\end{align*}

It will be seen later why cases (i) and (iv) are allowed to overlap.

In cases (ii) and (iii) simply transitive primitive groups must have rank 3 (in the sense of D. G. Higman [9]), and in all cases the rank of our group is at most 4. The calculation gives the lengths of the suborbits, and other numerical information of a similar kind. Details are tabulated at the end of the paper.

Simply transitive, primitive groups of degree $3p$ do exist for $p = 5, 7$ and 19:

Example 1. $A_6$, $S_6$ operating on unordered pairs are primitive groups of degree 15 and rank 3 (case (ii) with $a = 0$).

Example 2. $A_7$, $S_7$ operating on unordered pairs are primitive groups of degree 21 and rank 3 (case (i) with $a = 1$).
Example 3. PGL(2, 7) operating by conjugation on its set of Sylow 2-subgroups is a primitive group of degree 21 and rank 4, exemplifying case (iv).

Example 4. PSL(2, 19) operating by right multiplication on the cosets of one of its subgroups $A_5$ is primitive of degree 57 and rank 4, again an instance of case (iv).

How these four examples fit into the more detailed scheme of Table 5.2 is shown in Table 14.1. I have no idea whether there are more. The possibility $p = 31$ in case (iv) is of particular interest, and I hope to show in a later paper how the combinatorial data collected in this work can be used to decide whether or not it does actually correspond to a group.

The context of the results is the problem of classifying as far as is possible the primitive permutation groups of degree $kp$ where $l$ is small (compared with $p$). I had hoped that the methods I use would extend to higher values of $k$, but even for $k = 4$ the complexity of the system of simultaneous diophantine equations which arises seems to become prohibitive. Nevertheless, if one is concerned to test only relatively weak hypotheses, then the methods may just be sufficient, for example,

(A) for each value of $k$ ($k \geq 2$) there is a finite set $p_{k,1}(a), \ldots, p_{k,f(k)}(a)$ of quadratic polynomials with integer coefficients such that, if $p \neq p_{k,i}(a)$ for some integers $i, a$, then all primitive groups of degree $kp$ are 2-fold transitive;

or the weaker

(B) for each value of $k$ ($j \geq 2$) the set $P_k$ defined by

$$P_k = \{ p \mid p \text{ prime and there exists a simply transitive, primitive permutation group of degree } kp \}$$

is no more “dense” than the set of all perfect squares.

That (A) is true for $k = 2$ is proved by Wielandt [19, 20]; and for $k = 3$ it is still quite easy to show: much of the detailed calculation of §§7–10 goes to prove the refined version stated in the first paragraph of this paper. And even before I had been able to complete the proof of Theorem 9.2, B. J. Birch had shown me how to prove (B) (see §13).

Of course, this is still a very far cry from the classification of the groups themselves. Partial results along these lines have been obtained by Ito [12, 13] and Rowlinson [15] only for $k = 2, 3, 4$ under very restrictive conditions on the normalisers of Sylow $p$-subgroups.

The main part of the calculation is based on the language and notations introduced in §6. Little of this is new, except perhaps the point of view, but it unifies and simplifies various pieces of information scattered throughout the literature. Apart from this the work contains no new ideas and the proof is similar to Wielandt’s proof for groups of degree $2p$ [19, 20]. The difference lies
in the fact that, for groups of degree 3\(p\), even once the decomposition of the permutation character is known, there is still a great deal of calculation to be done, far heavier than the corresponding calculation for groups of degree 2\(p\) (cf. [20], pp.102, 103). I hope that in spite of the weight of the calculations, the proof will support my thesis that the point of view proposed in §6 is a sensible one.

In the unpublished part of his thesis, which he kindly made available to me, Tamaschke has done a great deal of work on groups of degree 3\(p\), but apart from the initial reductions of the problem his calculations proceed along quite different lines from mine. L. L. Scott [16] has also found partial results – but I do not have details yet.

It is a pleasure to acknowledge with thanks the help of Mr J. E. Stoy, whose calculations on the Oxford University computer showed me how to complete the proof of Theorem 9.2; and of Mr P. Rowlinson who read the first draft of this paper and helped with discussion and informed comment.

2 Notation

Throughout the paper \(p\) is prime, \(G\) is a primitive but not 2-fold transitive permutation group of degree 3\(p\), \(H\) denotes a stabiliser in \(G\), and \(P\) a Sylow \(p\)-subgroup of \(G\). The suborbits of \(G\) are the orbits of \(H\), the subdegrees are the lengths of the suborbits, and the rank \(r\) of \(G\) is the number of suborbits.

We will identify \(G\) with the linear group given by the appropriate permutation matrices. The field over which this linear group is defined will depend on the context – usually it will be implicitly assumed to be a suitable cyclotomic extension of the rational field \(\mathbb{Q}\). The principal character of a group \(X\) will be denoted \(1_X\). We use \(F\) to denote the cyclotomic field \(\mathbb{Q}(\theta)\), where \(\theta\) is a primitive \(|G|\)-root of 1. It will be proved later, and assumed now, that \(|G| = p.m\) where \(p \nmid m\). Thus \(F\) is generated by a primitive \(p\)-th root \(\theta^m\) of 1 and a primitive \(m\)-th root \(\theta^p\). Let \(\Gamma\) be the automorphism group of \(F\) (over \(\mathbb{Q}\)) and \(\Gamma^* \leq \Gamma\) the subgroup fixing \(\theta^p\). Thus \(\Gamma^*\) is cyclic of order \(p - 1\), and permutes the primitive \(p\)-th roots of unity in \(F\) regularly. Every complex character of \(G\) has its values in \(F\), and so \(\Gamma\) operates on this set: if \(\chi\) is a character of \(G\) and \(\gamma \in \Gamma\) we define \(\chi^\gamma\) by

\[
\chi^\gamma(g) = \chi(g)\gamma
\]

for all \(g \in G\). It is well known that \(\chi^\gamma\) is again a character. The character \(\chi\) is said to be \(p\)-rational if \(\chi^\gamma = \chi\) for all \(\gamma \in \Gamma^*\).

The names used for certain special groups are more or less standard: \(S_n\), \(A_n\) for the symmetric and alternating groups on the set \(\{1, 2, \ldots, n\}\); \(\text{GL}(m, q)\), \(\text{SL}(m, q)\) for the general and special (unimodular) linear groups respectively over the field \(\text{GF}(q)\) with \(q\) elements; \(\text{PGL}(m, q)\), \(\text{PSL}(m, q)\) for the corresponding projective groups. Other notation and terminology is explained in §6.
3 Preliminary lemmas

Primitive permutation groups of degrees 6 and 9 are known. In fact those of degree 6 are all 2-fold transitive: $S_6$, $A_6$, $\text{PGL}(2,5)$, $\text{PSL}(2,5)$, the last two operating on the points of the projective line over $GF(5)$. If $G$ is primitive of degree 9, then by a theorem of Jordan [20, p.32], the only primes dividing $|G|$ are 2 and 3. Consequently $G$ is soluble, and contains a regular elementary abelian normal subgroup of order 9. The stabiliser $H$ is a complement for $T$ in $G$, and, as it is represented faithfully as a group of automorphisms of $G$, it can be identified as a subgroup of $\text{GL}(2,3)$. There are just two such primitive groups which are not 2-fold transitive: a Frobenius group in which $H$, the Frobenius complement, is cyclic of order 4; and a group in which $H$ is dihedral of order 8. Both these groups are of rank 3 and have subdegrees 1, 4, 4.

From now on we will suppose that $p \geq 5$.

Lemma 3.1 $p^2 \nmid |G|$.

Proof If $p^2 \mid |G|$ then certainly $p \mid |H|$ and $G$ would contain an element $g$ of order $p$ having at least one fixed point. Then $g$ has degree $p$ (and $2p$ fixed point) or degree $2p$ (and $p$ fixed points) and, by theorems of Jordan and Manning (see [20], p.39) $G$ would be alternating or symmetric of degree $3p$. This contradicts our assumption that $G$ is not 2-fold transitive, and proves the lemma: and the Sylow $p$-subgroup $P$ is cyclic of order $p$.

Next a triviality about the normal structure of $G$:

Lemma 3.2 $G$ contains no non-trivial abelian normal subgroup.

Proof If $K$ were a non-trivial abelian normal subgroup then since $G$ is primitive $K$ would be transitive, hence regular. But then the $p$-primary constituent of $K$ would be normal in $G$ but intransitive, and $G$ could not be primitive.

Remark 3.3 Tamaschke observes in [18] that in fact any non-trivial normal subgroup in $G$ is again primitive except in case $p = 7$, $G = \text{PGL}(2,7)$ as in Example 3 (on page 2). This is a genuine exception, for $\text{PSL}(2,7)$ operating as a transitive group of degree 21 is imprimitive (it is a group of rank 6 and (see below) Type VIII, with subdegrees 1, 2, 2, 4, 4, 8).

Thirdly, a well-known fact about the permutation character of $G$.

Lemma 3.4 Every non-principal constituent of $\pi$ is faithful.

Proof Let $\chi$ be a non-principal character of $G$ which appears as a constituent of $\pi$, and let $K$ be the kernel of $\chi$. Then certainly $K$ is a proper normal subgroup of $G$. The restriction $\pi|_K$, which is the appropriate permutation character of $K$, contains $(1 + \chi)|_K$ which is a proper multiple of the principal character of $K$. Consequently $K$ is intransitive on $\Omega$, and since $K$ is normal, the orbits of $K$ form a system of blocks for $G$. As $G$ is primitive each of these blocks contains just 1 element, and so $K = 1$. That is, $\chi$ is a faithful character.
4 A theorem of Feit and the groups $\text{PSL}(2, q)$

Our aim in this section is to prove the crucial

**Lemma 4.1** If $f$ is the degree of a non-principal irreducible constituent of $\pi$, then $f \geq p - 1$.

This depends on a deep theorem of Feit [3] who proved that a finite group whose centre is trivial and which has a faithful character of degree less than $p - 1$ either has a normal Sylow $p$-subgroup or is isomorphic to $\text{PSL}(2, p)$ or to $\text{SL}(2, q)$ where $q$ is a power of 2 and $p = q + 1$. In case $\pi$ did contain a non-principal constituent of degree less than $p - 1$, by Lemmas 3.2 and 3.4 Feit’s theorem would be applicable, and since by (3.2) our Sylow $p$-subgroup is not normal, we would have either $G = \text{PSL}(2, p)$ or $G = \text{SL}(2, q)$. The proof is therefore completed by the following two lemmas.

**Lemma 4.2** If $q$ is a power of 2, $p = q + 1$ and $p \geq 5$, then $\text{SL}(2, q)$ (i.e. $\text{PSL}(2, q)$) has just one representation as a transitive permutation group of degree $3p$, but this is imprimitive.

**Proof** Let $H$ be the stabilizer in a representation of $\text{SL}(2, q)$ as a transitive group of degree $3p$. Since the index of $H$ is odd, $H$ contains a Sylow 2-subgroup $U$ of $\text{SL}(2, q)$. If $U \not\triangleleft H$ then $H$ contains at least one further Sylow 2-subgroup, but since $U$ permutes the other Sylow 2-subgroups of $\text{SL}(2, q)$ transitively by conjugation, it would follow that $H$ contained all Sylow 2-subgroups and would be the whole group. Hence $U \triangleleft H$, and $H \leq N(U)$. Now $N(U)$ has index $p$ in $\text{SL}(2, q)$ and so $|N(U) : H| = 3$. In fact, $N(U)$ is the split extension of $U$ by a cyclic group of order $q - 1$. Since $q$ and $q + 1$ are not multiples of 3 it follows that 3 does divide $p - 1$, therefore $N(U)$ contains precisely one subgroup of index 3, and the lemma is proved.

**Lemma 4.3** If $\text{PSL}(2, p)$ has a representation as a transitive permutation group of degree $3p$ then $p$ is 5, 7 or 19. In case $p$ is 5 or 7 the groups are imprimitive. $\text{PSL}(2, 19)$ can be represented in two (similar) ways as a primitive group of degree 3.19; the stabilizer is isomorphic to $A_5$; the rank is 4; the subdegrees are 1, 6, 20, 30; the degree of the irreducible constituents of the permutation character are 1, 18, 18, 20.

**Proof** We take $G$ to be $\text{PSL}(2, p)$. and hunt for the subgroup $H$ of index $3p$ by comparing with the list of all subgroups in $\text{PSL}(2, p)$ given, for example, in Burnside [1]. We have the possibilities

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1. This group is a counterexample of rank 4 to a conjecture of Frame, conjecture (B) on p.89 of [20]; it gives a negative answer to the implicit question raised in [20], p.93, 11.3,4. Compare the footnote on p.25 and Ito [14].
(i) $H$ is contained in a dihedral group of order $p - 1$;

or (ii) $H$ is contained in a dihedral subgroup of order $p + 1$;

or (iii) $H$ is contained in a maximal subgroup isomorphic to $A_4$, $S_4$ or $A_5$.

Possibility (i) gives that $\frac{1}{2}p(p+1)$ divides $3p$, whence $p = 5$; case (ii) can only arise if $\frac{1}{2}p(p-1)$ divides $3p$, that is, if $p = 7$; and in case (iii), if $H$ is a proper subgroup of the relevant maximal subgroup, then either $H$ is isomorphic to $A_4$ or it is already covered by cases (i) and (ii). Thus we may assume that $H$ has order 12, 24 or 60. These give

\[ 3p.12 = \frac{1}{2}p(p^2 - 1) \]
\[ 3p.24 = \frac{1}{2}p(p^2 - 1) \]
\[ 3p.60 = \frac{1}{2}p(p^2 - 1) \]

respectively. Of these only the last is possible and it gives $p = 19$.

Since $19 \equiv -1 \pmod{5}$, $\text{PSL}(2, 19)$ contains two conjugate classes of maximal subgroups isomorphic to $A_5$ (cf. Burnside [1], §324), and therefore $\text{PSL}(2, 19)$ has two representations as a primitive group of degree 3.19. The two classes of $A_5$-subgroups are interchanged by the outer automorphism of $\text{PSL}(2, 19)$, and our two primitive groups of degree 3.19 are therefore similar.

A straightforward calculation from the character table of $\text{PSL}(2, 19)$ yields that the degrees of the irreducible constituents of the permutation character are 1, 18, 18, 20. Knowing that the rank therefore is 4 and knowing that the stabilizer is $A_5$, it is easy to compute that the subdegrees must be 1, 6, 20, 30 (compare §10). In any case, since there is no constituent of degree less than $p - 1$ in the permutation character of this group, it follows that Lemma 4.1 is correct.

\textbf{N.B.} Even as imprimitive groups of degrees 15, 21 respectively $\text{PSL}(2, 5)$ and $\text{PSL}(2, 7)$ are not counterexamples to Lemma 4.1. For in the case of $\text{PSL}(2, 5)$ one finds that $\pi = 1 + 2\chi_1 + \chi_2$ where $\chi_1$ is irreducible of degree 5, $\chi_2$ irreducible of degree 4; and in the case of $\text{PSL}(2, 7)$, $\pi = 1 + \chi_1 + 2\chi_2$ where $\chi_1$ is irreducible of degree 8, $\chi_2$ irreducible of degree 6.

\section{Reduction of the permutation character of $G$}

Suppose that $\pi = 1_G + \sum e_i \chi_i$, where the characters $\chi_i$ are distinct non-principal irreducible characters of $G$. Since $\pi|_P = 3\rho_P$, where $\rho_P$ is the character of the regular representation of $P$, we have

\[ \left( \sum e_i \chi_i \right)|_P = 2.1_P + 3 \sum_{\lambda \in \Lambda^*} \lambda \]
where $\Lambda^*$ is the set of non-principal linear characters of $P$. We may assume the numbering to be chosen so that, for $1 \leq i \leq t$, $\chi_i|P$ contains $1_P$, and for $i > t$, $\chi_i|P$ does not contain $1_P$. Clearly then $t \leq 2$. The operation of the Galois group $\Gamma$ leaves $\pi$ invariant and therefore permutes the irreducible constituents of $\pi$: the characters $\chi_i$, $i \leq t$, are permuted among themselves, and the characters $\chi_i$, $i > t$, are permuted among themselves. Thus if $\chi_a = \sum_{i \leq t} e_i \chi_i$ and $\chi_b = \sum_{i > t} e_i \chi_i$ then $\chi_a$ is a rational character and $\chi_b$ is either 0 or a rational character of G. Furthermore $\chi_a$ is faithful (Lemma 3.4) and so $\chi_a|P$ contains the non-principal linear characters of $P$ each at least once and all with the same multiplicity. Similarly, $\chi_b$ is either 0, or of degree $p - 1$, or of degree $2(p - 1)$. We take these possibilities in turn.

Suppose first that $\chi_b = 0$. Since $G$ is not 2-fold transitive $\chi_a$ cannot be irreducible, and therefore is the sum of two irreducible constituents, $\chi_1$ and $\chi_2$. (We know that $\chi_1 \neq \chi_2$ because $\chi_1|P$ is not the double of any character of $P$.) If $\chi_1$ is not $p$-rational then $\chi_1$ and $\chi_2$ must be interchanged by $\Gamma^*$. In this case therefore $\chi_1$ and $\chi_2$ have the same degree, $(3p - 1)/2$, and if $\Gamma^{**}$ denotes the (unique) subgroup of index 2 in $\Gamma^*$, then

$$\chi_i|P = \rho P + \sum \{ \lambda | \lambda \in \Lambda_i^* \}$$

where $\Lambda_1^*$, $\Lambda_2^*$ are the two orbits of $\Gamma^{**}$ operating on $\Lambda^*$. This is case I. If $\chi_1$ is $p$-rational then so is $\chi_2$, and by suitable choice of the labels we have case II: $\deg \chi_1 = p$, $\deg \chi_2 = 2p - 1$.

Next suppose that $\deg \chi_b = p - 1$. By Lemma 4.1 the constituents of $\chi_1$ have degree at least $p - 1$, and in fact it is clear that $\chi_1$ cannot have a constituent of degree $p - 1$. Thus there are just three possibilities: either $\chi_a$ is irreducible, or $\chi_1 = \chi_1 + \chi_2$ where $\chi_1 = \chi_2$ and $\chi_1$, $\chi_2$ both have degree $p$, or $\chi_1 = 2\chi_1$ where $\chi_1$ has degree $p$. These are cases III, IV, V respectively.

Finally, if $\deg \chi_b = 2(p - 1)$, then $\chi_a$ has degree $p + 1$ and, by Lemma 4.1 must be irreducible. Again by 4.1, $\chi_b$ has at most two irreducible constituents, and if it does have two, then they both have degree $p - 1$. The three possibilities, $\chi_b$ irreducible of degree $2(p - 1)$, $\chi_b$ the sum of two distinct irreducible characters each of degree $p - 1$, $\chi_b$ twice an irreducible character of degree $p - 1$, are cases VI, VII, VIII respectively in Table 5.2.

Further information about the irreducible characters which appear in the decomposition of $\pi$, for example that even in cases IV and VII the characters of degree $p$, $p - 1$ respectively are $p$-rational, can be obtained directly from Suzuki’s theory of exceptional characters or (equivalently in this case) from Brauer’s theory of modular characters. However, I have not been able to make use of such facts (though for groups of degree $4p$ the analogous information does seem to be useful – see [15]), and details are left to the reader. Only in case I do we need a small amount of additional information:

**Lemma 5.1** Suppose that $\pi = 1 + \chi_1 + \chi_2$ where $\chi_1$ and $\chi_2$ are irreducible characters of degree $(3p - 1)/2$. Then

(i) $\chi_1 = \chi_2$ if $p \equiv 3 \pmod{4}$;
(ii) \( \chi_1 = \chi_1 \) if \( p \equiv 1 \pmod{4} \).

Proof: Complex conjugation operates on \( \{\chi_1, \chi_2\} \) in the same way as the (unique) element of order 2 in \( \Gamma^* \). Since \( \Gamma^* \) has order \( p - 1 \), if \( p \equiv 3 \pmod{4} \) then \( \Gamma^{**} \) has odd order and complex conjugation interchanges \( \chi_1 \) and \( \chi_2 \); while if \( p \equiv 1 \pmod{4} \) then \( \Gamma^{**} \) contains the element of order 2 in \( \Gamma^* \) and conjugation leaves \( \chi_1 \) and \( \chi_2 \) unchanged.

The entries in the main part of the following table are the degrees \( f_i \) of the irreducible constituents \( \chi_i \) of \( \pi \). All multiplicities \( e_i \) are 1 except in the two cases shown. \( \chi_0 \) is the principal character.

| Type | \( r \) | \( \chi_0 \) | \( \chi_1 \) | \( \chi_2 \) | \( \chi_3 \) |
|------|------|------|------|------|------|
| I    | 3    | 1    | \((3p - 1)/2\) | \((3p - 1)/2\) | |
| II   | 3    | 1    | \( p \) | \( 2p - 1 \) | |
| III  | 3    | 1    | \( 2p \) | \( p - 1 \) | |
| IV   | 4    | 1    | \( p \) | \( p \) | \( p - 1 \) |
| V    | 6    | 1    | \( p \) | \( p - 1 \) | |
| (mult. 2) | |
| VI   | 3    | 1    | \( p + 1 \) | \( 2p - 2 \) | |
| VII  | 4    | 1    | \( p + 1 \) | \( p - 1 \) | \( p - 1 \) |
| VIII | 6    | 1    | \( p + 1 \) | \( p - 1 \) | (mult. 2) |

Table 5.2: the decomposition of \( \pi \)

The second part of this paper is devoted to showing that cases I, V, VI, VIII cannot arise – thus, in particular, the rank of \( G \) is at most 4; that case II corresponds to possibilities (ii), (iii) of page 1; that cases III, IV give possibility (i) of page 1 and that case VII cannot arise unless \( p \) is 7, 19 or 31.

Part II

6 Survey of combinatorial methods

Suppose now for convenience that \( \Omega \), the set on which \( G \) operates, is \( \{0, 1, \ldots, n - 1\} \) (in our case \( n = 3p \)), and that \( H = G_0 \). If \( \Delta_0, \Delta_1, \ldots, \Delta_{r - 1} \) are the orbits of \( G \) operating in the natural way on \( \Omega \times \Omega \), we define the corresponding basic adjacency matrices \( B_0, \ldots, B_{r - 1} \) by

\[
((B_i)_{\alpha\beta}) = \begin{cases} 
1 & \text{if } (\alpha, \beta) \in \Delta_i, \\
0 & \text{if } (\alpha, \beta) \notin \Delta_i.
\end{cases}
\]

Then (cf. Wielandt [20], p.80) the matrices \( B_0, \ldots, B_{r - 1} \) span the algebra \( V \) of all matrices which commute with the permutation matrices representing \( G \).
The orbits $\Delta_i$ correspond in a natural way to the orbits of $H$ in $\Omega$; the length of the corresponding $H$-orbit can be read off as the number of entries equal to 1 in each row (or column) of $B_i$, that is, the subdegree is the row (or column) sum of $B_i$; the matrix $B_i$ is the transpose of $B_j$ if and only if the corresponding orbits are paired \([20]\) p.83, and in particular, self-paired orbits correspond to symmetric basic adjacency matrices.

Since $G$ is transitive on $\Omega$ the diagonal is one $G$-orbit in $\Omega \times \Omega$, and if we choose the numbering correctly we can take it to be $\Delta_0$: the corresponding suborbit is \(\{0\}\), and $B_0$ is the identity matrix. We write $\Delta^*$ for the orbit of $G$ in $\Omega \times \Omega$ obtained by reversing all the pairs in $\Delta$:

$$\Delta^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in \Omega\}.$$ 

Then $\Delta^*$ and $\Delta$ correspond to paired suborbits of $G$ in $\Omega$. We will also use $^*$ to denote an involution of $\{1, \ldots, r-1\}$ in such a way that

$$\Delta^*_i = \Delta_{i^*}.$$ 

Thus $B^*_i$ is the transpose of $B_i$.

There is a convenient and suggestive geometrical interpretation of the orbits $\Delta_i$. As Sims has pointed out \([17]\), if $i \neq 0$, the orbit $\Delta_i$ is a directed graph with vertex set $\Omega$, and $G$ is a group of graph automorphisms. As a graph $\Delta_i$ has the special property that every point has edges emanating from it, and furthermore $G$ is transitive on the directed edges. We will say that $\Delta_i$ admits $G$ as a flag-transitive automorphism group. Since we are interested not only in each individual suborbit, that is, each individual graph, but also in the way that these graphs fit together to form the complete directed graph $(\Omega \times \Omega) - \Delta_0$, it is appropriate to use the notions of edge-coloured graphs. We choose colours $c_1, \ldots, c_{r-1}$ and colour the edge $(\alpha, \beta)$ with colour $c_i$ if and only if $(\alpha, \beta) \in \Delta_i$, to make a coloured directed graph $C(G, \Omega)$. This ensures of course that the monochrome subgraphs – i.e. the subgraphs defined by all the vertices and all the edges of one given colour – are just the original graphs $\Delta_1, \ldots, \Delta_{r-1}$. As an edge-coloured directed graph $C(G, \Omega)$ has several special properties. For example, its automorphism group contains $G$ and is therefore transitive on the flags of each colour; and if $\Delta$ is a monochrome subgraph, then so is $\Delta^*$, the graph obtained by reversing all edges of $\Delta$. If all the suborbits of $G$ are self-paired then each monochrome subgraph is completely determined by the underlying non-directed graph, and we have an edge-coloured complete graph in the usual sense (see, for example, \([8]\), ch. 6).

The basic adjacency matrices $B_1, \ldots, B_{r-1}$ are the adjacency matrices in the usual sense for the monochrome subgraphs of $C(G, \Omega)$. Products of these matrices enumerate paths in $C(G, \Omega)$ in the following way:

$$B_{i_1}B_{i_2} \cdots B_{i_k} = (m_{\alpha\beta})$$

where $m_{\alpha\beta}$ is the number of paths of length $k$ from $\alpha$ to $\beta$ whose first step is an edge of colour $c_{i_1}$, second step an edge of colour $c_{i_2}$, \ldots, last step an edge of colour $c_{i_k}$.
If we amalgamate two or more colours in \( C(G, \Omega) \) the resulting coloured graph will still admit \( G \) as a group of automorphisms, though now no longer flag-transitive, and the property that if \( \Delta \) is a monochrome subgraph then so is \( \Delta^* \) may not persist. This operation of combining colours corresponds to adding the appropriate basic adjacency matrices. Accordingly we define a general adjacency matrix for \( C(G, \Omega) \) to be any matrix of the form \( \sum_{i \in I} B_i \) where \( I \) is a non-empty subset of \( \{1, 2, \ldots, r-1\} \), and we define the (generalized) subdegree of this adjacency matrix to be its row – or column – sum. A special case will be used in the proof of Lemma 6.13 if \( \Gamma, \Delta \) are monochrome subgraphs (possibly after some amalgamation of colours) we define a subgraph \( \Gamma \circ \Delta \) by

\[
\Gamma \circ \Delta = \{ (\alpha, \beta) \mid \alpha \neq \beta \text{ and there exists } \gamma \in \Gamma \text{ such that } (\alpha, \gamma) \in \Gamma \text{ and } (\gamma, \beta) \in \Delta \}.
\]

That is, \( (\alpha, \beta) \in \Gamma \circ \Delta \) if and only if \( \alpha \neq \beta \) and there is a path of length 2 directed from \( \alpha \) to \( \beta \), whose first leg is coloured with the \( \Gamma \)-colour and whose second leg is the colour corresponding to \( \Delta \). It can happen that \( \Gamma \circ \Delta \) is empty, but only if \( \Gamma = \Delta^* \) and \( \Delta \) has subdegree 1. A primitive group which is not regular (i.e. not cyclic of prime order) never has non-trivial suborbits of subdegree 1, and so in such a case \( \Gamma \circ \Delta \) is never empty. It is of course a very special property of the coloured graph \( C(G, \Omega) \) that \( \Gamma \circ \Delta \) is a union of monochrome subgraphs.

It has been remarked above that the basic adjacency matrices \( B_0, B_1, \ldots, B_{r-1} \) span an algebra \( V \). Therefore there exist complex numbers \( a_{ijk} \) such that

\[
B_i B_j = \sum_{k=0}^{r-1} a_{ijk} B_k.
\]

(6.1)

In fact, these multiplication constants \( a_{ijk} \) are non-negative integers, for, if the \( (\alpha, \beta) \) entry of \( B_k \) is 1 then the \( (\alpha, \beta) \) entries of all the other basic adjacency matrices are 0, therefore \( a_{ijk} \) is the \( (\alpha, \beta) \) entry in \( B_i B_j \); and the coefficients of \( B_i B_j \) are obviously non-negative integers. The multiplication constants have an obvious geometric significance:

\[
\begin{align*}
a_{0jk} &= \delta_{jk} \\
a_{j0k} &= \delta_{ik} \\
a_{ijk} &= \delta_{ij} n_k
\end{align*}
\]

(where \( n_i \) is the appropriate subdegree), and, if \( i \geq 1, j \geq 1, k \geq 1 \), then \( a_{ijk} \) is the number of oriented triangles \( (\alpha, \beta, \gamma) \) on a fixed base \( (\alpha, \beta) \) of colour \( c_k \), whose other two sides, \( (\alpha, \gamma) \) and \( (\gamma, \beta) \) are coloured \( c_i \) and \( c_j \) respectively. Notice that \( n_k a_{ijk} \) is the number of oriented triangles at each vertex having edges of colours \( c_i, c_j, c_k \) in that order.

**N.B.** These structure constants, and numbers closely related to them, have appeared many times before in the literature: see, for example, Frame [4, 5, 6, 7] and D.G. Higman [9, 10]. In this last paper [10] Higman’s \( \mu^{(\alpha)}_{ij} \) is our \( a_{\alpha^*, j} \).
Consequently, if we define (as Higman does) $M_\alpha = (\mu_{ij}^{(\alpha)})_{ij}$, then (6.1) shows that $M_\alpha^*$ is the matrix representing $B_\alpha$ with respect to the basis $\{B_0, \ldots, B_{r-1}\}$ of $V$ in the regular representation of $V$ (compare [10], p.31). There are many relations between the constants $a_{ijk}$ (compare [5] Theorem 2.9(b), (c), or see [10], (4.1), (4.2)) which are geometrically almost obvious, or which can be read off from equation (6.4) below. The less obvious relations,

$$\sum_\nu a_{ij\nu}a_{\nu kl} = \sum_\nu a_{jk\nu}a_{i\nu l},$$

which express the associativity of $V$, also have geometric significance. The left side enumerates quadrilaterals $(\alpha, \gamma, \delta, \beta)$ on a fixed edge $(\alpha, \beta)$ of colour $c_i$, whose other sides $(\alpha, \gamma)$, $(\gamma, \delta)$, $(\delta, \beta)$ are coloured $c_i$, $c_j$, $c_k$ respectively, by counting those in which the diagonal $(\alpha, \delta)$ is coloured $c_\nu$, and summing. The right side of the equation enumerates the same quadrilaterals, but by counting those in which the diagonal $(\gamma, \beta)$ has colour $c_\nu$ and summing.

The constants $a_{ijk}$ can be computed from trace relations. From the definition of the matrices $B_i$ it is clear that

$$\text{tr}(B_i) = \begin{cases} n & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}. \quad (6.2)$$

From (6.2) and the defining relations (6.1) of $V$ it follows that

$$\text{tr}(B_iB_j) = na_{ij0} = \begin{cases} 0 & \text{if } j \neq i^* \\ nn_i & \text{if } j = i^* \end{cases}, \quad (6.3)$$

(see [20], Theorem 28.10). Furthermore,

$$B_iB_jB_{k^*} = \sum_l a_{ijl}B_lB_{k^*},$$

and so (compare [6], Equation (4.2)) from the above equation,

$$\text{tr}(B_iB_jB_{k^*}) = n n_k a_{ijk}. \quad (6.4)$$

This again may be seen geometrically, for $B_iB_jB_{k^*}$ has as its $(\alpha, \alpha)$ coordinate the number of paths of length 3 from $\alpha$ back to $\alpha$ coloured $c_i$, $c_j$, $c_{k^*}$ in that order. This number is $n_k a_{ijk}$ for all $\alpha \in \Omega$, and so the trace of $B_iB_jB_{k^*}$ is $n n_k a_{ijk}$.

The equation

$$\sum_{i=0}^{r-1} B_i = W, \quad (6.5)$$

where $W$ is the matrix all of whose entries are 1, expresses the fact that the monochrome subgraphs $\Delta_i$ cover the whole complete graph on $\Omega \times \Omega$ without overlapping.
We shall find a slight generalisation of equations (6.2), (6.3) and (6.5) useful. Namely, let \( C \) be a complete directed graph obtained from \( C(G, \Omega) \) by amalgamating colours according to the rule\(^2\) that, if \( \Delta \) is a monochrome subgraph in \( C \), then so also is \( \Delta^* \). The adjacency matrices \( A_1, A_2, \ldots, A_{r-1} \) (\( t \leq r \)) are some of the general adjacency matrices of \( C(G, \Omega) \) defined on page 10. If we put \( A_0 = I \), and extend the convention used before so that \( A_i^* = A_i^\top \) then it is easy to see that the linear and quadratic trace relations hold (compare [15]):

\[
\text{tr}(A_i) = \begin{cases} n & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}, \quad (6.2#)
\]

\[
\text{tr}(A_i A_j) = \begin{cases} 0 & \text{if } j \neq i^* \\ nm_i & \text{if } j = i^* \end{cases}, \quad (6.3#)
\]

where \( m_i \) now is the row sum (or column sum) of \( A_i \), that is, it is the number of edges of \( C \) of colour \( c_i \) emanating from each point. Of course we will also have

\[
\sum_{i=0}^{t-1} A_i = W. \quad (6.5#)
\]

Since \( V \) and the algebra spanned by \( G \) (as a set of permutation matrices) are centralizer algebras of one another we know that they are simultaneously reducible\(^3\) and there is an invertible matrix \( U \) (see [20], §29) such that

\[
U^{-1} gU = \begin{pmatrix}
1 & & & \\
\varepsilon_1 \{ D_1(g) & \cdots & D_1(g) \\
\varepsilon_2 \{ D_2(g) & \cdots & D_2(g) \\
& \cdots & \\
& \cdots & D_s(g)
\end{pmatrix}
\]

for all \( g \in G \); and, for any adjacency matrix \( A_i \),

\[
U^{-1} A_i U = \begin{pmatrix}
m_i & \Theta_{i1} \times I_{f_1} \\
& \Theta_{i2} \times I_{f_2} \\
& \cdots \\
& \Theta_{is} \times I_{f_s}
\end{pmatrix}.
\]

\(^2\)Such a collection of the resulting matrices will be called an admissible set of adjacency matrices.

\(^3\)We assume that the coefficient field is the splitting field for \( G \), or at least, that the matrix representations \( D_\lambda \) are absolutely irreducible.
Here $D_\lambda$ is a matrix representation of $G$ of degree $f_\lambda$ which affords the character $\chi_\lambda$ (where $\pi = 1 + \sum s_1 e_\lambda \chi_\lambda$); $m_i$ is the subdegree of $A_i$; $\Theta_\lambda$ is an $e_\lambda$ by $e_\lambda$ matrix, $I_\lambda$ is the $f_\lambda$ by $f_\lambda$ identity matrix, and $\times$ denotes Kronecker product. Since conjugation by $U$ gives an algebra isomorphism which leaves traces invariant, equations (6.2#), (6.3#), (6.4) become

\[
\sum_{\lambda=0}^s f_\lambda \text{tr}(\Theta_{i\lambda}) = 0 \quad (i \geq 1)
\]

(6.6)

\[
\sum_{\lambda=0}^s f_\lambda \text{tr}(\Theta_{i\lambda} \Theta_{j\lambda}) = \begin{cases} 
0 & \text{if } j \neq i^* \\
nm_i & \text{if } j = i^*
\end{cases}
\]

(6.6)

\[
\sum_{\lambda=0}^s f_\lambda \text{tr}(\Theta_{i\lambda} \Theta_{j\lambda} \Theta_{k\lambda}) = nn_k a_{ijk},
\]

(6.8)

where, in (6.6) and (6.7) the matrices $\Theta_{i\lambda}$ can be interpreted as coming from an admissible set of general adjacency matrices $A_1, \ldots, A_{t-1}$ as on page 12, but in the third equation the matrices $\Theta_{i\lambda}$ must come from the basic adjacency matrices $B_1, \ldots, B_{r-1}$. Since

\[
U^{-1}WU = \begin{pmatrix} n & 0 & \ldots & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \end{pmatrix},
\]

(6.5#) gives

\[
\sum_{i=1}^{t-1} \Theta_{i\lambda} = -I_\lambda \quad (\lambda = 1, \ldots, s)
\]

(6.9)

where the $\Theta_{i\lambda}$ can be interpreted as coming from an admissible set of general adjacency matrices $A_1, \ldots, A_{t-1}$.

If $e_\lambda = 1$ then $\Theta_\lambda$ is just an eigenvalue $\theta_\lambda$ of the adjacency matrix $A$. In any case, if $\theta$ is an eigenvalue of $\Theta_\lambda$ then $\theta$ is an eigenvalue of $A$: we shall call the eigenvalues of $\Theta_\lambda$ the eigenvalues of $A$ associated with $\chi_\lambda$, and of course it can happen that one eigenvalue of $A$ is associated with several different constituents of $\pi$. In case all the multiplicities $e_\lambda$ are 1, so that $V$ is commutative (and $s = r - 1$, the relations (6.6)(6.9), (6.7), (6.8) are just linear, quadratic and cubic relations on the eigenvalues of the adjacency matrices. This is the case in which they are the most useful. Two small facts about this case are useful:

\begin{itemize}
  \item[(6.10)] If $r = 3$ then $\theta_{i1} \neq \theta_{i2}$ \quad ($i = 1, 2$);
  \item[(6.11)] If $V$ is commutative (i.e. $e_\lambda = 1$ for all $\lambda$), if the degrees of $f_\lambda$ are all different, and if the eigenvalues of the adjacency matrix $A$ associated with distinct constituents $\chi_\lambda$ are different, then $A$ is symmetric (i.e. the corresponding colour is self-paired).
\end{itemize}

13
Proof If \( r = 3 \) then \( V \) is automatically commutative. If \( \theta_{11} = \theta_{12} \) then (6.9) gives
\[
\theta_{21} = -1 - \theta_{11} = -1 - \theta_{12} = \theta_{22}.
\]
But then the matrices \( O, U^{-1}B_1U, U^{-1}B_2U \) would be linearly dependent, which is not so. Thus \( \theta_{11} \neq \theta_{12} \), and similarly \( \theta_{21} \neq \theta_{22} \).

If \( V \) is commutative then \( U^{-1}AU \) and \( U^{-1}A^\top U \) are diagonal matrices. Since \( A \) and \( A^\top \) have the same eigenvalues with the same multiplicities, the hypotheses of (6.11) ensure that \( U^{-1}AU = U^{-1}A^\top U \). That is, \( A = A^\top \).

Lemma 6.12 Let \( \chi, \psi \) be irreducible constituents of \( \pi \) whose multiplicities are 1, and let \( \theta, \phi \) be associated eigenvalues of an adjacency matrix \( A \). If \( \gamma \in \Gamma \) (the Galois group of a suitable field, cf. §2) and \( \chi^\gamma = \psi \), then \( \theta^\gamma = \phi \).

Proof Use a superscript \( \gamma \) to indicate the natural operation of \( \Gamma \) on \( F^n \), the vector space of \( 1 \times n \) row vectors over \( F \), and on \( M_n(F) \), the set of all \( n \times n \) matrices over \( F \). Since \( G \subseteq M_n(F) \), \( G \) operates on \( F^n \) by right multiplication and \( \pi \) is the character of this representation of \( G \). Let \( E(\chi) \) be the \( G \)-invariant subspace of \( F^n \) which affords \( \phi \). If \( \{x_i \mid i = 1, \ldots, f\} \) is a basis for \( E(\chi) \), and if for \( g \in G \),
\[
x_i^\gamma g = \sum_j \lambda_{ij}(g)x_j^\gamma,
\]
then, since \( g \) is a matrix with rational coefficients,
\[
x_i^\gamma g = \sum_j \lambda_{ij}(g)\gamma x_j^\gamma.
\]
We read off from this that \( \{x_i^\gamma \mid i = 1, \ldots, f\} \) is a basis for the subspace of \( F^n \) affording the character \( \chi^\gamma \) of \( G \). That is, \( E(\chi)^\gamma = E(\psi) \). Now
\[
xAx = \theta x
\]
for all \( x \in E(\chi) \). Therefore
\[
x^\gamma A = \theta^\gamma x^\gamma
\]
for all \( x \in E(\chi) \), since \( A \) is a rational matrix. Thus \( \theta^\gamma \) is the eigenvalue of \( A \) on \( E(\chi)^\gamma \), that is, by definition of \( \phi \) we have \( \theta^\gamma = \phi \).

N.B. This lemma has an obvious generalisation in case the multiplicities of \( \chi \), \( \psi \) in \( \pi \) are not 1, namely, the set of eigenvalues of \( A \) associated with \( \chi \) is carried by \( \gamma \) to the set of eigenvalues associated with \( \psi \). And it has an easy converse: is \( \theta, \phi \) are eigenvalues of \( A \) such that \( \theta^\gamma = \phi \), then \( \chi^\gamma = \psi \) where
\[
\chi = \sum \theta \chi_i | \theta \text{ is an eigenvalue associated with } \chi_i, \\
\psi = \sum \phi \chi_i | \phi \text{ is an eigenvalue associated with } \chi_i.
\]
However, for both the generalisation and its converse we may need to work in a larger field than \( F \), namely a normal extension which contains all the eigenvalues of the appropriate matrices \( \Theta_\lambda \) as on page 13.

Our last lemma in this section is the filter which sorts primitive from imprimitive groups in the calculation of §§7–10. D. G. Higman [10] and C. C. Sims [17] have pointed out that \( G \) is primitive on \( \Omega \) if and only if all the monochrome subgraphs in \( C(G, \Omega) \) are connected. This can be translated into a criterion applying to the eigenvalues of the adjacency matrices, a slight modification of a special case of the Perron–Frobenius Theorem (see, for example, [24]). For completeness a proof is included.

**Lemma 6.13** Suppose that \( G \) is primitive on \( \Omega \). Let \( A \) be an adjacency matrix of subdegree \( m \) where \( m > 1 \). Then

(i) \( m \) is an eigenvalue of \( A \) of (algebraic) multiplicity 1;

(ii) if \( \theta \) is any other eigenvalue of \( A \) then \( |\theta| < m \).

**Proof** The matrix \( A \) operates by right multiplication as a linear transformation of \( \mathbb{C}^n \), the space of \( 1 \) by \( n \) row vectors \( x = (x_0, \ldots, x_{n-1}) \) with complex coordinates. If \( t = (1, 1, \ldots, 1) \) then, since every column of \( A \) contains \( m \) entries 1 and \( n - m \) entries 0, we have

\[ tA = mt. \]

Hence \( m \) is an eigenvalue of \( A \) with (algebraic) multiplicity at least 1. Let \( U = \{ x \mid \sum x_\alpha = 0 \} \). Then \( \mathbb{C}^n = (t) \oplus U \), and \( U \) is invariant under \( A \). Thus it will be sufficient to show that if \( \theta \) is an eigenvalue of \( A \) on \( U \) then \( |\theta| < m \).

Now let \( x \) be any vector in \( \mathbb{C}^n \), and let \( y = xA \). If the \((\alpha, \beta)\) coordinate of \( A \) is \( m_{\alpha\beta} \) then

\[ |y|^2 = \sum_\beta \left| \sum_\alpha x_\alpha m_{\alpha\beta} \right|^2. \]

Now \( m_{\alpha\beta} \) is 0 or 1, so \( m_{\alpha\beta}^2 = m_{\alpha\beta} \), and

\[ \left| \sum_\alpha x_\alpha m_{\alpha\beta} \right|^2 = \left| \sum_\alpha x_\alpha m_{\alpha\beta}^2 \right|^2 \]

\[ \leq \left( \sum_\alpha |x_\alpha m_{\alpha\beta}|^2 \right) \left( \sum_\alpha |m_{\alpha\beta}|^2 \right) \]

by Cauchy’s Inequality. Each column of \( A \) contains precisely \( m \) entries equal to 1, the rest 0, and so \( \sum_\alpha |m_{\alpha\beta}|^2 = m \).

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\(^4\)I am grateful to Dr J. A. D. Welsh for introducing me to relevant literature.
Thus
\[
\|y\|^2 \leq m \sum_{\beta} \sum_{\alpha} |x_\alpha m_{\alpha\beta}|^2 \\
= \sum_{\alpha} \sum_{\beta} m_{\alpha\beta}^2 |x_\alpha|^2 \\
= m \sum_{\alpha} m |x_\alpha|^2 \\
= m^2 \|x\|^2,
\]
and so \(\|y\| \leq m \|x\|\).

If equality is to hold then, returning to our application of Cauchy’s Inequality, we require that for every \(\beta\) the coordinates \(x_\alpha\) for which \(m_{\alpha\beta} = 1\) are all equal. That is, \(x_\alpha = x_\gamma\) if there exists \(\beta\) such that \((\alpha, \beta) \in \Delta\) and \((\gamma, \beta) \in \Delta\), where \(\Delta\) is the graph of which \(A\) is adjacency matrix. In other words, \(x_\alpha = x_\gamma\) if \((\alpha, \beta) \in \Delta \circ \Delta^*\). Now \(\Delta \circ \Delta^*\) is not empty since the subdegree of \(\Delta\) is not 1 (cf. p. 10), and therefore \(\Delta \circ \Delta^*\) is a union of monochrome subgraphs of \(C(G, \Delta)\). These are all connected since \(G\) is primitive, and consequently \(\Delta \circ \Delta^*\) is connected. It now follows that \(x_\alpha = x_\gamma\) for all \(\alpha, \gamma \in \Omega\), and \(x\) is a multiple of \(t\). Thus if \(x \notin \langle t \rangle\) then \(\|xA\| < m \|x\|\). In particular, if \(\theta\) is an eigenvalue of \(A\) on \(U\) then \(|\theta| < m\), and the lemma is proved.

**N.B.** The condition \(m > 1\) is necessary but not restrictive, for, if \(m = 1\) then \(G\) must be cyclic of prime order \(p\) and \(A\) must be one of the basic adjacency matrices. In this case \(A\) is a permutation of order \(p\) whose eigenvalues are \(p\)-th roots of 1. Primitivity of \(G\) is also necessary, for, if \(G\) is imprimitive, having \(k\) blocks of size \(l\), then by suitably re-numbering the elements of \(\Omega\) we can ensure that \(A\) is a matrix in block form \((B_{ij})\), where \(N_{ij}\) is an \(l \times l\) zero matrix if \(i \neq j\), and \(B_{ii}\) is an \(l \times l\) matrix of zeros and ones having \(m\) entries equal to 1 in each row and column. It is easy to see that in this case \(A\) has at least \(k\) linearly independent eigenvectors with eigenvalue \(m\).

The linear and quadratic equations in (6.6), (6.7), (6.9) were used by Frame and D. G. Higman \cite{4, 5, 6, 7, 9, 10} to obtain information about the degrees \(f_\lambda\) of the irreducible constituents of \(\pi\) from information about the subdegrees \(n_i\). They have been used by Wielandt and Ito \cite{19, 13} the other way round, to deduce the subdegrees \(n_i\) from knowledge of the numbers \(f_\lambda\). This is the program to be carried out now. In most cases \(V\) turns out to be commutative, and we take the linear and quadratic equations (6.6), (6.7), (6.9) as equations for the eingenvalues of the adjacency matrices. As such we have about \(r^2\) unknowns and only about \(\frac{1}{2}r(r+3)\) equations, where \(r\) is the rank – and these equations seem not to be independent in general. On the other hand, since the adjacency matrices have integer coordinates, their eigenvalues are algebraic integers, and (6.6)–(6.9) is a system of diophantine equations. The cubic equations (6.8) introduce new unknowns \(a_{ijk}\), one for each equation. Nevertheless they are useful as divisibility conditions. There are of course higher trace relations, but
these give no new information, because once equations (6.6)–(6.8) have been solved to give the eigenvalues, they give the structure constants $a_{ijk}$ and hence full information about the algebra $V$. The policy therefore will be to find all solutions of (6.6)–(6.9) for which the numbers $a_{ijk}$ are non-negative rational integers.

In §§7–10 there are never more than 4 eigenvalues of the matrices $B_i$ (or $A_i$). Of these one is $n_i$ (or $m_i$). The others will be re-named $\lambda_i$, $\mu_i$, $\nu_i$ for simplicity.

7 Case I does not arise

In this case $r$ is 3 and (6.6) gives

$$n_i + \left(\frac{3p-1}{2}\right) (\lambda_i + \mu_i) = 0.$$  

Now $\lambda_i$, $\mu_i$ are algebraic integers, and so $\lambda_i + \mu_i$, being rational, is a rational integer. Hence $n_1, n_2$ are both multiples of $(3p-1)/2$. But $n_1 + n_2 = 3p - 1$, and therefore $n_1 = n_2 = (3p-1)/2$ (compare [20], Theorem 30.2).

Suppose now that $\Delta_1$ is self-paired. Then, from (6.6), (6.7),

$$\lambda_1 + \mu_1 = -1,$$

$$\lambda_1^2 + \mu_1^2 = \frac{3p+1}{2}.$$  

This yields $\lambda_1\mu_1 = -(3p-1)/4$, and since $\lambda_1, \mu_1$ are algebraic integers we must have $3p \equiv 1 \pmod{4}$, that is,

$$p \equiv 3 \pmod{4}.$$  

It follows (Lemma 5.1) that $\chi_1 = \chi_2$, and so, from Lemma 6.12 $\lambda_1, \mu_1$ are complex conjugate. But this is certainly not the case – the equations above give $\lambda_1, \mu_1 = (-1 \pm \sqrt{3p})/2$.

Suppose therefore that $\Delta_1$ is paired with $\Delta_2$. Then

$$\lambda_1 + \mu_1 = -1,$$

$$\lambda_1^2 + \mu_1^2 = \frac{3p-1}{2}.$$  

In this case therefore $\lambda_1\mu_1 = (3p+1)/4$, so that $p \equiv 1 \pmod{4}$. Applying Lemmas 5.1 and 6.12 we see that $\lambda_1, \mu_1$ must both be real whereas the equations clearly give complex conjugate values. These two contradictions prove that Case I cannot occur.

8 Groups of type II

Theorem 8.1 If a group of type II exists, then either
\( (i) \quad p = 48a^2 + 30a + 5 \\
and the subdegrees are 1, p + 4a + 1, 2p - 4a - 2; \ or \\

(ii) \quad p = 48a^2 + 66a + 23 \\
and the subdegrees are 1, p - 4a - 3, 2p + 4a + 2. \)

Here \( a \geq 0 \) and \( a \) is an integer. The groups \( A_6, S_6 \) as in Example 1 (p. 1) are instances of case (i) above with \( a = 0 \). In §12 we will show that in case (i) \( a \) is even, and in case (ii) \( a \) must be odd.

**Proof** Groups of type II are of rank 3 and both their non-trivial suborbits are self-paired ((6.10) and (6.11)). We have (6.6)

\[
    n_i + p\lambda_i + (2p - 1)\mu_i = 0
\]

and \( \lambda_i, \mu_i \) are rational integers. Consequently

\[
    n_i \equiv \mu_i \pmod{p}
\]

and so \( n_i = \epsilon_i + \mu_i \). Since \( n_i > 0 \), from Lemma [6.13] we know that \( \epsilon_i > 0 \), and from (6.9), \( \epsilon_1 + \epsilon_2 = 3 \). Choosing the notation suitably we may therefore suppose that \( \epsilon_1 = 1, \epsilon_2 = 2 \). Thus \( n_1 = p + \mu_1 \) and \( \lambda_1 = -2\mu_1 - 1 \). Now (6.7) gives

\[
    (p + \mu_1)^2 + p(1 + 2\mu_1)^2 + (2p - 1)\mu_1^2 = 3p(p + \mu_1),
\]

which simplifies:

\[
    6\mu_1^2 + 3\mu_1 + 1 - 2p = 0.
\]

\[
    \mu_1 = \frac{1}{4} \left( -1 \pm \sqrt{16p - 5} \right).
\]

Since \( \mu_1 \) is a rational integer it follows that \( 16p - 5 = 3b^2 \) for some positive integer \( b \). We require \( p \equiv 3, 5, 11 \) or 13 (mod 16) if \( 3b^2 + 5 \equiv 0 \), and \( \mod 3\) if \( 3b^2 + 5 \) is not to be divisible by 32. Hence \( b = 16a + 5 \) or \( b = 16a + 11, \quad a \geq 0 \). In the former case \( \mu_1 = (-1 + b)/4 \) since \(-1 - b \) is not divisible by 4, and in the latter case \( \mu_1 = (-1 - b)/4 \). Now computing \( p \) and \( n_1 \) in terms of \( a \) gives the statement of the theorem.

**9 Cases III, IV, V**

**Theorem 9.1** If \( G \) is of Type III then \( p = 3a^2 + 3a + 1 \) for some integer \( a \), and the subdegrees are either

\( (i) \quad 1, p - 2a - 1, 2p + 2a; \ or \)

\( (ii) \quad 1, p + 2a + 1, 2p - 2a - 2. \)
The groups $A_7$, $S_7$ of degree 21 as in Example 2 (p. 1) are instances of Type III, case (ii) with $a = 1$.

**Theorem 9.2** If $G$ is of Type IV, then $p = 3a^2 + 3a + 1$ where $a$ is an integer. In case two suborbits are paired, say $\Delta_1^i = \Delta_3^i$, then

(i) $a$ is even and $n_1 = p - 2a - 1$, $n_2 = n_3 = p + a$; or

(ii) $a$ is odd and $n_1 = p + 2a + 1$, $n_2 = n_3 = p - a - 1$.

Otherwise all suborbits are self-paired, in which case

(iii) $a$ is even and the subdegrees are $1, p + 2a + 1, p - a - 1, p - a - 1$.

**Theorem 9.3** Case V does not arise.

**Proof of 9.1** Again our group has rank 3 and both its non-trivial suborbits are self-paired. Using (6.6),

$$n_i + 2p\lambda_i + (p - 1)\mu_i = 0$$

so that $n_i = \epsilon_i p + \mu_i$. As before, (6.13) gives $\epsilon_i \geq 1$, and from (6.9), $\epsilon_1 + \epsilon_2 = 3$. We choose the notation so that $\epsilon_1 = 1$, $n_1 = p + \mu_1$, $\lambda_1 = (-\mu_1 - 1)/2$. Now from (6.7),

$$(p + \mu_1)^2 + 2p\left(\frac{\mu_1 + 1}{2}\right)^2 + (p - 1)\mu_1^2 = 3p(p + \mu_1),$$

$$3\mu_1^2 = 4p - 1.$$ 

Since again $\mu_1$ is a rational integer, we may put $(4p - 1)/3 = (2a + 1)^2$ where $a$ is an integer, $a \geq 0$, and then $p = 3a^2 + 3a + 1$, $\mu_1 = \pm(2a + 1)$. Hence $n_1 = p \pm (2a + 1)$, and Theorem 9.1 follows.

**Proof of 9.3** A group of type V would have rank 6, and there are 5 basic adjacency matrices $B_1, \ldots, B_5$ to be considered. The matrices $\Theta_{i,1}$ are size $2 \times 2$, and their eigenvalues $\lambda_i, \mu_i$ are eigenvalues of $B_i$ of multiplicity $p$. The fourth eigenvalue $\nu_i$ of $B_i$ is necessarily a rational integer. The equation (6.6) gives

$$n_i + p(\lambda_i + \mu_i) + (p - 1)\nu_i = 0.$$ 

Now $\lambda_i + \mu_i$ is an algebraic integer which is rational, hence a rational integer, and we get $n_i = \epsilon_i p + \nu + i$. As before, by Lemma 6.13 we must have $\epsilon_i \geq 1$ for all $i$ and so $\sum \epsilon_i \geq 5$. On the other hand, by (6.9) $\sum \nu_i = -1$, and

$$3p - 1 = \sum_{i=1}^{5} \nu_i = (\sum \epsilon_i)p + \sum \nu_i = (\sum \epsilon_i)p - 1,$$ 

so that $\sum \epsilon_i = 3$. This contradiction proves that case V cannot arise.
Proof of 9.2  Groups of type IV have rank 4 and the three basic adjacency matrices $B_i$ have eigenvalues $n_i, \lambda_i, \mu_i, \nu_i$ where $\nu_i$ is a rational integer, and either $\lambda_i, \mu_i$ are rational integers or they are algebraically conjugate algebraic integers. Using the equation (6.6) we get $n_i = \epsilon_i p + \nu_i$ where, from Lemma 6.13 $\epsilon_i > 1$, and from (6.9), $\sum \epsilon_i = 3$. Thus $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and

$$n_i = p + \nu_i \quad i = 1, 2, 3. \quad (9.4)$$

Suppose now that $\Delta_3 = \Delta_2^*$. Then $B_3 = B_2^*$ and the eigenvalues of $B_2, B_3$ must be the same. Thus $\nu_3 = \nu_2$ and either $\lambda_3 = \lambda_2, \mu_3 = \mu_2$ or $\lambda_3 = \mu_2, \mu_3 = \lambda_2$. However, if $\lambda_3 = \lambda_2$ and $\mu_3 = \mu_2$ then $U^{-1} B_3 U = U^{-1} B_2 U$ (where $U$ is as on p.13) so that $B_3 = B_2$ which is not so. Therefore $\lambda_3 = \mu_2$ and $\mu_3 = \lambda_2$. Now amalgamating colours $c_3$ and $c_2$ gives admissible adjacency matrices $A_1 = B_1$ and $A_2 = B_2 + B_3$, and produces the situation which we have analysed in the proof of 9.1 above. The result is that $p = 3a^2 + 3a + 1$ and either

(i) $n_1 = p - 2a - 1, n_2 = n_3 = p + a$, or

(ii) $n_1 = p + 2a + 1, n_2 = n_3 = p - a - 1$.

We wish to show now that in case (i) $a$ must be even, and in case (ii) $a$ is odd.

In case (i) the eigenvalues of the matrices $B_i$ are as shown:

| $B_i$ | $n_i$ | $\lambda_i$ | $\mu_i$ | $\nu_i$ |
|-------|-------|-------------|--------|--------|
| $B_1$ | $p - 2a - 1$ | $a$ | $a$ | $-2a - 1$ |
| $B_2$ | $p + a$ | $\lambda$ | $\mu$ | $a$ |
| $B_3$ | $p + a$ | $\mu$ | $\lambda$ | $a$ |

where $\lambda + \mu = -a - 1$ and, from (6.7)

$$\lambda \mu = \frac{1}{2}(2p - a - a^2) = \frac{1}{2}(5a + 2)(a + 1).$$

One of the equations (6.8) is

$$3p(p + a) a_{223} = (p + a)^3 + p \lambda \mu (\lambda + \mu) + (p - 1)a^3.$$

A straightforward calculation gives

$$a_{223} = a^2 + (3a)/2,$$

from which we deduce, since $a_{223}$ is an integer, that $a$ is even.

Case (ii) is similar: in this case the eigenvalues of the matrices $B_i$ are

| $B_i$ | $n_i$ | $\lambda_i$ | $\mu_i$ | $\nu_i$ |
|-------|-------|-------------|--------|--------|
| $B_1$ | $p + 2a + 1$ | $-a - 1$ | $-a - 1$ | $2a + 1$ |
| $B_2$ | $p - a - 1$ | $\lambda$ | $\mu$ | $-a - 1$ |
| $B_3$ | $p - a - 1$ | $\mu$ | $\lambda$ | $-a - 1$ |

and (6.6),(6.7) give $\lambda + \mu = 1, \lambda \mu = \frac{1}{2}a(5a + 3)$. Then (6.8):

$$3p(p - a - 1) a_{223} = (p - a - 1)^3 + p \lambda \mu (\lambda + \mu) - (p - 1)(a + 1)^3.$$
This yields $a_{223} = a^2 + (a - 1)/2$ and therefore $a$ must be odd. This completes our treatment of Theorem 9.2 for the case when two suborbits are paired: in fact no further restrictions on $a$ can be deduced by the methods of §6.

For the remainder of this section we suppose that all three non-trivial suborbits are self-paired. Using equations (9.4) we eliminate $n_i$ from equations (6.6),(6.7):

\[
\lambda_i + \mu_i = -1 - \nu_i \\
\lambda_i^2 + \mu_i^2 = -\nu_i^2 + \nu_i + 2p.
\]

This gives $2\lambda_i \mu_i = 1 + 3\nu_i + 2\nu_i^2 - 2p$, from which we deduce that

\[
\nu_i \text{ is odd}
\]

and $\lambda_i, \nu_i$ are respectively $\frac{1}{2}(1 - \nu_i \pm \sqrt{4p - 1 - 3\nu_i^2})$. Adjusting the sign of the square root where necessary, we may take it that

\[
\lambda_i = \frac{1}{2}(-1 - \nu_i + \sqrt{4p - 1 - 3\nu_i^2}) \\
\mu_i = \frac{1}{2}(-1 - \nu_i - \sqrt{4p - 1 - 3\nu_i^2}).
\]

The adjacency matrices $A_i$ being symmetric, we know that their eigenvalues are real. Hence

\[
3\nu_i^2 \leq 4p - 1.
\]

And, from (6.9) we have

\[
\left\{ \begin{array}{c} 
\nu_1 + \nu_2 + \nu_3 = -1 \\
\sqrt{4p - 1 - 3\nu_1^2} + \sqrt{4p - 1 - 3\nu_2^2} + \sqrt{4p - 1 - 3\nu_3^2} = 0.
\end{array} \right.
\]

Eliminating $\nu_3$ and rationalising gives

\[
\nu_i^2(3\nu_2 + 2p + 1) + \nu_1(3\nu_2^2 + 2\nu_2 + 4\nu_2 + 2p + 1) \\
+ (2p + 1)(\nu_2^2 + \nu_2) - 2p(p - 1) = 0.
\]

Now

\[
3\nu_2^2 + 2p\nu_2 + 4\nu_2 + 2p + 1 = (3\nu_2 + 2p + 1)(\nu_2 + 1),
\]

and therefore $3\nu_2 + 2p + 1$ must divide $(2p + 1)(\nu_2^2 + \nu_2) - 2p(p - 1)$. Calculating modulo $\nu_2 + 2p + 1$ we eliminate $p$ from the equation

\[
2(2p + 1)(\nu_2^2 + \nu_2) - 4p(p - 1) = 0
\]

to get $3(\nu_2 + 1)^2(2\nu_2 + 1) \equiv 0 \pmod{3\nu_2 + 2p + 1}$. There is complete symmetry between $\nu_1, \nu_2, \nu_3$, and so

\[
(2p + 3\nu_i + 1) \text{ divides } 3(\nu_i + 1)^2(2\nu_i + 1).
\]

So far we have worked only with the equations (6.6),(6.7),(6.9), but at this point we need also the cubic equations (6.8). They give, among other relations,
that $3p\nu_i$ divides $n_i^3 + p(\lambda_i^3 + \mu_i^3) + (p - 1)\nu_i^3$. Substitute for $n_i, \lambda_i, \mu_i$ in terms of $\nu_i$, divide by $p$, multiply by $2$, and rearrange terms:

$$6(p + \nu_i) \text{ divides } 2p^2 - 6p - 6\nu_i^2 + 2\nu_i^3 + (1 + \nu_i)(4\nu_i^2 - \nu_i + 1).$$

Now calculation modulo $2(p + \nu_i)$ yields $2p^2 - 6p \equiv 2\nu_i(\nu_i + 3)$ and after a little simplification we find that

$$2(p + \nu_i) \text{ divides } (\nu_i + 1)(2\nu_i + 1)(3\nu_i + 1). \quad (9.9)$$

Since (9.7) $\nu_1 + \nu_2 + \nu_3 = -1$ not all of $\nu_1, \nu_2, \nu_3$ can be negative. Let us choose $b$ to be one of $\nu_1, \nu_2, \nu_3$ such that $b \geq 0$. Put

$$(b + 1)(2b + 1)(3b + 1) = u.2(p + b)$$
$$(b + 1)(2b + 1)(3b + 3) = v.(2p + 3b + 1)$$

so that, by (9.8) and (9.9), $u$ and $v$ are integers. Subtraction gives

$$2(b + 1)(2b + 1) = w.2(p + b) + v(b + 1)$$

where $w = v - u$. We wish to show that $w = 0$. To do this, observe first that

$$w = v - u = (b + 1)(2b + 1) \left( \frac{3b + 3}{2p + 3b + 1} - \frac{3b + 1}{2(p + b)} \right)$$
$$= \frac{(b + 1)(2b + 1)(4p - 1 - 3b^2)}{2(2p + 3b + 1)(p + b)}, \quad (9.10)$$

so that $w \geq 0$ by (9.6).

To obtain an upper bound for $w$ we first eliminate $v$,

$$3(b + 1)^3(2b + 1) = (2p + 3b + 1)(2(b + 1)(2b + 1) - 2w(p + b)$$

and then put $p + b = x$:

$$4wx^2 - 2(b + 1)(4b + 2 - w)x + (b + 1)^2(2b + 1)(3b + 1) = 0.$$  

Since $x$ is certainly real the discriminant of this quadratic function cannot be negative. Thus

$$4(b + 1)^2(4b + 2 - w)^2 - 16w(b + 1)^2(2b + 1)(3b + 1) \geq 0,$$

and so

$$(4b + 2 - w)^2 \geq 4w(2b + 1)(3b + 1) = w(4b + 2)(6b + 2). \quad (9.11)$$

However, from equation (9.10) we have that

$$w = \frac{(b + 1)(2b + 1)(4p - 1 - 3b^2)}{2(2p + 3b + 1)(p + b)} < 2b + 1.$$
And now, since \( w \geq 0 \) it follows that
\[
2b + 1 < 4b + 2 - w \leq 4b + 2.
\]
Therefore (9.11) clearly implies that \( w \leq 0 \).

We have proved now that \( w = 0 \). Consequently, by (9.10), \( 4p - 1 = 3b^2 \).

Since \( b \) must therefore be odd we put \( b = 2a + 1 \) where \( a \geq 0 \), and we have \( p = 3a^2 + 3a + 1 \). Furthermore, if we suppose it was \( \nu_1 \) that was \( b \) then from (9.7),
\[
\nu_2^2 = \nu_3^2 \\
\nu_2 = \pm \nu_3.
\]
But \( \nu_2 + \nu_3 = -1 - \nu_1 \neq 0 \) and it follows that
\[
\nu_2 = \nu_3 = (-1 - \nu_1)/2.
\]
Thus \( \nu_1 = 2a + 1, \nu_2 = \nu_3 = -a - 1 \). Finally since by (9.5) \( \nu_2 \) and \( \nu_3 \) must be odd, there is the restriction that \( a \) is even. And
\[
\begin{align*}
n_1 &= p + 2a + 1 \\
n_2 = n_3 &= p - a - 1
\end{align*}
\]
as was claimed.

## 10  Cases VI, VII and VIII

**Theorem 10.1** Case VI cannot arise.

**Theorem 10.2** If \( G \) is of type VII then \( p \) is 7, 19 or 31.

(i) If \( p = 7 \), the subdegrees are 1, 4, 8, 8;

(ii) if \( p = 19 \), the subdegrees are 1, 6, 20, 30;

(iii) if \( p = 31 \), the subdegrees are 1, 20, 32, 40.

Examples 3 and 4 (page 11) are instances with \( p = 7, 19 \) respectively. I do not know if a primitive group of Type VII and degree 93 exists.

**Theorem 10.3** Case VIII cannot arise.

We begin the proofs with a lemma:

**Lemma 10.4** No symmetric matrix for \( G \) (other than \( W - I \)) has just three eigenvalues with multiplicities 1, \( p + 1 \), \( 2(p - 1) \).
Proof Suppose on the contrary that $A_1$ is a symmetric adjacency matrix whose eigenvalues $m_1$ (the subdegree), $\lambda_1$, $\mu_1$ have multiplicities $1$, $p + 1$, $2(p - 1)$ respectively. Let $A_2 = W - I - A_1$. Then $A_2$ is also a symmetric adjacency matrix whose eigenvalues $m_2$, $\lambda_2$, $\mu_2$ have multiplicities $1$, $p + 1$, $2(p - 1)$, and $\{A_1, A_2\}$ is an admissible set.

Furthermore, $A_1$ and $A_2$ commute and therefore there is a matrix $U$ such that $U^{-1}A_1U$, $U^{-1}A_2U$ are both diagonal. Now if $\lambda_1 = \mu_1$ then $\lambda_2 = \mu_2$ and $I$, $A_1$, $A_2$ would not be linearly independent. But in fact $I$, $A_1$, $A_2$ are linearly independent (we are assuming that $A_1 \neq W - I$, $A_1 \neq I$) and it must follow that $\lambda_1 \neq \mu_1$, $\lambda_2 \neq \mu_2$. Notice that $\lambda_i$, $\mu_i$ are rational integers.

Equations (6.6), (6.7) give

$$
\begin{align*}
\begin{cases}
m_i + (p + 1)\lambda_i + 2(p - 1)\mu_i = 0, \\
m_i^2 + (p + 1)\lambda_i^2 + 2(p - 1)\mu_i^2 = 3pm_i.
\end{cases}
\end{align*}
$$

Calculating modulo $p$:

$$
\begin{align*}
\begin{cases}
m_i \equiv 2\mu_i - \lambda_i \\
m_i^2 \equiv 2\mu_i^2 - \lambda_i^2
\end{cases} \pmod{p}.
\end{align*}
$$

Thus

$$
2\mu_i - \lambda_i^2 \equiv 2\mu_i^2 - \lambda_i^2 \pmod{p}
$$

and this gives

$$
\begin{align*}
\text{and this gives } m\mu_i & \equiv \lambda_i, \\
\text{hence also } m_i & \equiv \lambda_i \pmod{p}.
\end{align*}
$$

Put $m_i = \epsilon_i p + \lambda_i$, $\mu_i = \eta_i p + \lambda_i$, substitute in (10.4.1) and compute modulo $p^2$:

$$
\begin{align*}
\begin{cases}
\epsilon_i p + \lambda_i + (p + 1)\lambda_i + 2(p - 1)\lambda_i - 2\eta_i p \equiv 0, \\
2\epsilon_i p\lambda_i^2 + \lambda_i^2 + (p + 1)\lambda_i^2 + 2(p - 1)\lambda_i^2 - 4\lambda_i \eta_i p \equiv 3p\lambda_i
\end{cases} \pmod{p^2}.
\end{align*}
$$

Collect terms and divide by $p$:

$$
\begin{align*}
\begin{cases}
\epsilon_i + 3\lambda_i - 2\eta_i \equiv 0, \\
3\lambda_i^2 + \lambda_i(2\epsilon_i - 4\eta_i - 3) \equiv 0
\end{cases} \pmod{p}.
\end{align*}
$$

Since $1 + \lambda_1 + \lambda_2 = 0$ not both $\lambda_1$ and $\lambda_2$ are divisible by $p$, and interchanging $A_1$ and $A_2$ if necessary, we may therefore assume that $\lambda_1 \equiv 0 \pmod{p}$. Then

$$
\begin{align*}
\begin{cases}
3\lambda_i \equiv 2\eta_i - \epsilon_i, \\
3\lambda_i \equiv 4\eta_i - 2\epsilon_i + 3
\end{cases} \pmod{p}.
\end{align*}
$$

Eliminating $2\eta_i - \epsilon_i$ gives $\lambda_1 \equiv -1 \pmod{p}$. It follows that $m_1 = p - 1$ or $m_1 = 2p - 1$. If $m_1 = p - 1$, then, since $\lambda_1 \equiv \mu_1 \equiv -1 \pmod{p}$ and by Lemma 6.13 $|\lambda_1| < m_1$, $|\mu_1| < m_1$, the only possibility is $\lambda_1 = \mu_1 = -1$, which contradicts our earlier observation that $\lambda_1 \neq \mu_1$. But, if $m_1 = 2p - 1$ then $m_2 = p$, and again there is only one possibility, namely $\lambda_2 = \mu_2 = 0$, consistent with Lemma 6.13. This also gives a contradiction, and the lemma is proved.
Proof of 10.1  In case VI the basic adjacency matrices would have three eigenvalues with multiplicities $1, p + 1, 2(p - 1)$, and they would be symmetric by (6.10) and (6.11). Thus the lemma shows that case VI cannot arise.

Lemma 10.5  If VII is the case then all three non-trivial suborbits are self-paired.

Proof  If not then the three basic adjacency matrices would be $B_1, B_2, B_3$ where $B_3 = B_2^T$ say. Then the characteristic equation of $B_3$ would be the same as that of $B_2$ and so the eigenvalues of $B_3$ would be the same as those of $B_2$ with the same multiplicities. Since $U^{-1}B_3U \neq U^{-1}B_2U$ (where $U$ is on p.13) it would follow that $\mu_3 = \nu_2$ and $\mu_2 = \nu_3$. But then, since
\[
\mu_1 + \mu_2 + \mu_3 = -1
\]
and
\[
\nu_1 + \nu_2 + \nu_3 = -1
\]
we would have $\mu_1 = \nu_1$. Thus $B_1$ would be a matrix of the kind which Lemma 10.4 forbids. Hence all three non-trivial suborbits are self-paired.

From here on we treat cases VII and VIII together. In both cases the linear trace relation for an adjacency matrix $A$ becomes
\[
m + (p + 1)\lambda + (p - 1)(\mu + \nu) = 0.
\]
Now $\lambda$ is a rational integer and $\mu, \nu$ are algebraic integers. We know that for a suitable invertible matrix $U$,
\[
U^{-1}B_iU = \begin{pmatrix}
n_i & \lambda_i & \cdots & \lambda_i \\
p + 1 & \ddots & \ddots & \ddots \\
 & & \lambda_i & \Theta_i
\end{pmatrix}
\]
and therefore if we form a new adjacency matrix by amalgamating colours then we must add the appropriate eigenvalues $\lambda_i$ to get the corresponding eigenvalue $\lambda$ of $A$. In particular, if $\Delta_i$ is not self-paired then we amalgamate the colours $c_i, c_i^*$ to obtain a symmetric matrix whose eigenvalues include $2\lambda_i$ with multiplicity $p + 1$, and $m_i = 2n_i = \epsilon_i(p - 1) - 2\lambda_i$ where $\epsilon_i = 2\eta_i$.

For the rest of this section we use the following notation: $A_1, \ldots, A_{t-1}$ are the symmetric adjacency matrices obtained by amalgamating paired colours; the eigenvalues of $A_i$ are $m_i, \lambda_i, \mu_i, \nu_i$ with multiplicities $1, p + 1, p - 1, p - 1$ respectively; and $\epsilon_i$ is defined by the equation
\[
m_i = \epsilon_i(p - 1) - 2\lambda_i.
\]
We recall that $\lambda_i$ is a rational integer, $\mu_i, \nu_i$ are algebraic integers and since $A_i$ is symmetric they are real. Moreover, if $A_i$ is not a basic adjacency matrix, that is, if $A_i$ has arisen from an amalgamation of paired colours (by 10.5) this
can happen only in case VIII), then \( \epsilon_i, m_i, \lambda_i \) are all even. In case VII there are 3 such matrices and they are all basic (by Lemma 10.5); in case VIII there may be 3, 4 or 5 matrices. In any case, \( \{A_1, \ldots, A_{t-1}\} \) is an admissible set of adjacency matrices.

**Lemma 10.6** \( \sum \epsilon_i = 3 \) and \( \epsilon_i \geq 0 \) for all \( i \).

**Proof**

\[
3p - 1 = \sum m_i = (p - 1) \sum \epsilon_i - 2 \sum \lambda_i
\]

and \( \sum \lambda_i = -1 \) by (6.9). Hence \( (p - 1)(\sum \epsilon_i) = 3p - 3 \) and so \( \sum \epsilon_i = 3 \).

To prove the second assertion suppose on the contrary that \( \epsilon_i < 0 \). Then, since \( m_i \geq 0 \) we must have \( \lambda_i < 0 \) and (Lemma 6.13) \( -\lambda_i < \epsilon_i(p - 1) - 2 \lambda_i \) so that \( \lambda_i < \epsilon_i(p - 1) \). In particular, in this case \( |\lambda_i| > p - 1 \) and so \( |\lambda_i| \geq p \).

Since \( \mu_i, \nu_i \) are real, \( \mu_i^2 \geq 0 \) and \( \mu_i^2 \geq 0 \). Therefore from the quadratic trace relation (6.7) we get the inequality

\[
m^2_i + (p + 1)\lambda^2_i \leq 3pm_i,
\]

that is,

\[
(p + 1)\lambda^2_i \leq m_i(3p - m_i) \leq (3p/2)^2.
\]

Hence \( \lambda_i^2 < 9p/4 \) and so \( |\lambda_i| < 3\sqrt{p}/2 < p \). This contradicts our earlier inequality and proves the lemma.

Substituting \(-2\lambda_i \) for \( m_i \) in the equation (6.7) and calculating modulo \( p - 1 \) easily yields that \( p - 1 \) divides \( 6\lambda_i(\lambda_i + 1) \). We will need a refinement:

**Lemma 10.7** \( p - 1 \) divides \( 3\lambda_i(\lambda_i + 1) \).

**Proof** In this case equations (6.6), (6.7) give

\[
\mu_i + \nu_i = -\epsilon_i - \lambda_i
\]

and \( (p - 1)(\mu_i^2 + \nu_i^2) = 3p(\epsilon_i(p - 1) - 2\lambda_i) - (\epsilon_i(p - 1) - 2\lambda_i)^2 - (p + 1)\lambda_i^2 \).

Now \( \mu_i\nu_i \) is a rational integer, and \( 2\mu_i\nu_i = (\mu_i + \nu_i)^2 - (\lambda_i^2 + \mu_i^2) \). Calculating modulo \( 2(p - 1) \) therefore gives

\[
\begin{align*}
0 & \equiv (p - 1)(\epsilon_i + \lambda_i)^2 - 3p(\epsilon_i(p - 1) - 2\lambda_i)^2 + (p + 1)\lambda_i^2 \\
& \equiv (p - 1)(\epsilon_i + \lambda_i)^2 - \epsilon_i(p - 1) + 6\lambda_i + 4\lambda_i^2 + (p - 1)\lambda_i^2 + 2\lambda_i^2 \\
& \equiv (p - 1)(\epsilon_i^2 - \epsilon_i) + 6\lambda_i + 6\lambda_i^2
\end{align*}
\]

(mod \( 2(p - 1) \)).

Since \( \epsilon_i^2 - \epsilon_i \) is even we see that \( 2(p - 1) \) divides \( 6\lambda_i(\lambda_i + 1) \) and therefore \( p - 1 \) divides \( 3\lambda_i(\lambda_i + 1) \) as claimed.
The last paragraph of the proof of 10.6 was to show that $|\lambda_i| < 3\sqrt{p}/2$. Again we need a finer estimate:

**Lemma 10.8**

$$\epsilon_i(p - 1)(6p - 2\epsilon_ip + \epsilon_i) - 6\lambda_i(2p - \epsilon_ip + \epsilon_i) - (3p + 9)\lambda_i^2 \geq 0.$$  

**Proof** We look more closely at the discriminant of the quadratic equation whose roots are $\mu_i, \nu_i$. Since these are real,

$$2(\mu_i^2 + \nu_i^2) - (\mu_i + \nu + i^2) = (\mu_i - \nu_i)^2 \geq 0,$$

and so using equations (6.6), (6.7),

$$6p(\epsilon_i)p - 2\lambda_i - 2(\epsilon_i(p - 1) - 2\lambda_i)^2 - 2(p_1)\lambda_i^2 - (p - 1)(\epsilon_i + \lambda_i)^2 \geq 0,$$

and this can be rearranged as given in the statement of the lemma.

From Lemma [10.6] we know that either one of the numbers $\epsilon_i$ is 0, or there are just three adjacency matrices and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. We deal now with the former possibility; notice that this must be the situation if we are dealing with case VIII.

**Lemma 10.9** If $\epsilon_1 = 0$ then $p$ is 7 or 19, and the rank of our group is 4. If $p = 7$ then $m_1 = 4$ and if $p = 19$ then $m_1 = 6$.

**Proof** If $\epsilon_1 = 0$ then $m_1 = -2\lambda_1$ and $\lambda_1$ must be negative. From Lemma [10.8] we get

$$-12p\lambda_i - (3p + 9)\lambda_i^2 \geq 0,$$

and so $4p + (p + 3)\lambda_i \geq 0$. Thus $\lambda_i \geq (-4p)/(p + 3) > -3$.

Consequently $\lambda_i$ is $-3, -2$ or $-1$, and $m_1$ is 6, 4 or 2. Since $G$ is primitive, $m_1$ cannot be 2 (cf. [20], Theorem 18.7). Hence $\lambda_i$ is $-3$ or $-2$. Now from Lemma [10.7] $p - 1$ divides 18, or $p - 1$ divides 6 respectively. In the former case $p$ is 19 or 7, but if $p - 7$ and $\lambda_1 = -3$, returning to Lemma [10.8] we see that the necessary inequality is not satisfied. Therefore

$$p = 19, \quad \lambda_1 = -3, \quad m_1 = 6$$

or

$$p = 7, \quad \lambda_1 = -2, \quad m_1 = 4$$

are the only possibilities, and we are left with proving that the rank of our group is 4 and not 6. To do this, put $A = \sum \{A_i \mid \epsilon_i = 0\}$. Then $A$ is a symmetric adjacency matrix of subdegree $m = \sum m_i$ and with eigenvalue $\lambda = \sum \lambda_i$ of multiplicity $p + 1$. The analysis which we have already given applies as well to $A$ as to any of the matrices $A_i$. Thus if $p = 19$ then $\lambda = -3, m = 6$; and if $p = 7$ then $\lambda = -2, m = 4$. Consequently there can be no more than
one summand, namely \( A_1 \), going in to the making of \( A \). So \( \epsilon_i > 0 \) for \( i > 1 \).

Since \( \sum \epsilon_i = 3 \) there can be at most 3 further matrices \( A_i \), and reordering if necessary, the possibilities are \( t = 5, \epsilon_2 = \epsilon_3 = \epsilon_4 = 1 \), or \( t = 3 \) and \( \epsilon_2 = 1, \epsilon_3 = 2 \).

Remembering that, if \( A_i \) arose from amalgamation of paired colours, then \( \epsilon_i \) is even, we see that in case VIII either \( A_1 \) or both \( A_1 \) and \( A_3 \) must have arisen from amalgating paired colours. But if \( A_1 \) comes from amalgamating paired colours then \( \lambda_1 \) is even. Only \( p = 7, \lambda_1 = 2 \) satisfies this condition, and so the subdegree corresponding to the original colour from which \( A_1 \) was obtained would be \( m_1/2 = 2 \). This is impossible in a primitive group \( (20, 18.7) \), and this completes the proof of Lemma 10.9.

It is now quite easy to show that if \( p = 7 \) then the only possibilities for \( \lambda_2, \lambda_3 \) consistent with (6.9) and with

\[ \text{Lemma 10.8 are } (1, 0), (0, 1) \text{ or } (-1, 2), \]

of these only the last gives values of \( \mu_1, \mu_2, \nu_1, \nu_2, \nu_3 \) which can satisfy (6.9). Thus for \( p = 7 \) we get \( n_1 = 4, n_2 = 8, n_3 = 8m \) and with suitable choice of notation,

\[
\begin{align*}
\mu_1 &= 1 + \sqrt{2} & \nu_1 &= 1 - \sqrt{2} \\
\mu_2 &= -2\sqrt{2} & \nu_2 &= 2\sqrt{2} \\
\mu_3 &= -2 + \sqrt{2} & \nu_3 &= -2 - \sqrt{2}.
\end{align*}
\]

Similarly, if \( p = 19 \) the possibilities for \( \lambda_2, \lambda_3 \) consistent with (6.9) and with

\[ \text{Lemmas 10.7, 10.8 are } (3, -1), (2, 0), (0, 2), (-1, 3). \]

Only the last gives values of \( \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 \) which can satisfy (6.9), and so in this case \( n_1 = 6, n_2 = 20, n_3 = 30 \), and with suitable choice of notation,

\[
\begin{align*}
\mu_1 &= \frac{3 + \sqrt{5}}{2} & \nu_1 &= \frac{3 - \sqrt{5}}{2} \\
\mu_2 &= -2\sqrt{5} & \nu_2 &= 2\sqrt{5} \\
\mu_3 &= \frac{-5 + 3\sqrt{5}}{2} & \nu_3 &= \frac{-5 - 3\sqrt{5}}{2}.
\end{align*}
\]

Thus we have proved Theorem 10.3 and to complete the proof of Theorem 10.2 we need to deal with the case where the rank is 4 and \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 1 \).

In this case the matrices \( A_i \) are in fact just the basic adjacency matrices, \( B_1, B_2, B_3 \).

**Lemma 10.10** If \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 1 \) then \( p = 31 \) and the subdegrees are

\[ 1, 20, 32, 40. \]

The proof will be given as a series of numbered steps.

**10.11** The numbers \( \lambda_1, \lambda_2, \lambda_3 \) are all different.

For suppose not, \( \lambda_1 = \lambda_2 \) say. Then, since \( \epsilon_1 = \epsilon_2 \) we have \( n_1 = n_2 \) and the quadratic equation derived from (6.6), (6.7) whose roots are \( \mu_1, \nu_1 \) is the same as that whose roots are \( \mu_2, \nu_2 \). Thus either \( \mu_1 = \mu_2, \nu_1 = \nu_2 \) or \( \mu_1 = \nu_2, \]

\[ \text{[Footnote lost from scan]} \]

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\(\mu_2 = \nu_1\). Now since it is case VII which we are considering we know that \(B_1\) and \(B_2\) are simultaneously diagonalisable. Therefore the argument used for proving Lemma 10.5 applies to complete the proof.

10.12 Put \(a_i = 3\lambda_i(\lambda_i + 1)/(p - 1)\) so that, by Lemma 10.7, \(a_i\) is an integer. Then if \(\lambda_i \geq 0\) we have \(a_i \leq 3\), and in any case \(a_i \leq 4\).

**Proof** Since \(\epsilon_i = 1\), Lemma 10.8 gives

\[
(p - 1)(4p + 1) - 6\lambda_i(p + 1) - (3p + 9)\lambda_i^2 \geq 0.
\]

Hence

\[
(3p + 9)(\lambda_i^2 + \lambda_i) \leq (p - 1)(4p + 1) - (3p - 3)\lambda_i,
\]

and

\[
a_i = \frac{3\lambda_i(\lambda_i + 1)}{p - 1} \leq \frac{4p + 1}{p + 3} \cdot \frac{3\lambda_i}{p + 3} = 4 - \frac{11}{p + 3} \cdot \frac{3\lambda_i}{p + 3}.
\]

If \(\lambda_i \geq 0\) then we have \(a_i \leq 4\) and so \(a_i \leq 3\). If \(\lambda_i < 0\) and \(p \geq 19\) then, as in the proof of Lemma 10.6, we have \(-3\lambda_i/(p + 3) \leq 1\), hence \(a_i < 5\) and so \(a_i \leq 4\). Finally, if \(p < 19\) and \(\lambda_i \leq 0\) it is a simple matter to check that \(a_i \leq 3\) must hold.

10.13 If none of \(a_1, a_2, a_3\) is zero, then \(a_1, a_2, a_3\) are all different.

Suppose, for example, that \(a_1 = a_2\). Then both \(\lambda_1, \lambda_2\) are roots of the equation

\[
3\lambda(\lambda + 1) - a_1(p - 1) = 0.
\]

Since (10.11) \(\lambda_1 \neq \lambda_2\) it follows that \(\lambda_1 + \lambda_2 = -1\). But from (6.9) we know that \(\lambda_1 + \lambda_2 + \lambda_3 = -1\), and so \(\lambda_3 = 0\). Then \(a_3 = 0\) and (10.13) is proved.

10.14 If \(a > 0\) and \(\lambda\) is a root of the equation

\[
x^2 + x - a = 0
\]

then \(\lambda = -\frac{1}{2} \pm \sqrt{a + \eta}\), where \(|\eta| < 1/(8\sqrt{a})\).

For \((\lambda + \frac{1}{2})^2 = \lambda^2 + \lambda + \frac{1}{4} = a + \frac{1}{4}\).

It is trivial to check that the positive square root of \(a + \frac{1}{4}\) lies strictly between \(\sqrt{a}\) and \(\sqrt{a} + 1/(8\sqrt{a})\), and that the negative square root lies strictly between \(-a - 1/(8\sqrt{a})\) and \(-\sqrt{a}\).

We shall now show that one of the numbers \(a_i\) is zero by using the fact that otherwise \(p\) has three different representations in the form \(1 + 3\lambda_i(\lambda_i + 1)/a_i\), with three different (10.13) values of \(a_i\), where \(a_i\) is 1, 2, 3 or 4 and \(\lambda_1 + \lambda_2 + \lambda_3 = -1\):

10.15 One of \(a_1, a_2, a_3\) must be zero.
Proof Suppose not: then (10.13) $a_1, a_2, a_3$ are all different. We know that $\lambda_i$ is a root of the equation

$$x^2 + x - a_i(p - 1)/3 = 0$$

and so, by 10.14, we have

$$\lambda_i = -\frac{1}{2} \pm \sqrt{\frac{a_i(p - 1)}{3}} + \eta_i$$

where

$$|\eta_i| < \frac{1}{8} \sqrt{\frac{3}{a_i(p - 1)}} < \frac{1}{8}.$$ 

Now by (6.9) we have

$$\lambda_1 + \lambda_2 + \lambda_3 = -1,$$

and so

$$-\frac{3}{2} + \sqrt{\frac{p - 1}{3}} (\pm \sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3}) + \eta_1 + \eta_2 + \eta_3 = -1.$$ 

This gives

$$\left| \sqrt{\frac{p - 1}{3}} (\pm \sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3}) \right| < \frac{7}{8}.$$ 

We know (by 10.13 and 10.12) that $a_1, a_2, a_3$ are three among the numbers 1, 2, 3, 4, and crude approximations to $\sqrt{2}$ and $\sqrt{3}$ give the estimate

$$|\pm \sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3}| > \frac{4}{10}.$$ 

So we have

$$\frac{4}{10} \sqrt{\frac{p - 1}{3}} < \frac{7}{8}.$$ 

This yields $p < 15$: but elementary arithmetic is enough to show that no prime less than 15 has three representations in the form $1 + 3\lambda_i(\lambda_i + 1)/a_i$ with $\lambda_i, a_i$ integral and $1 \leq a_i \leq 4$. Thus our supposition leads to a contradiction, and one of $a_1, a_2, a_3$ must be zero.

Choose the notation so that $a_1 = 0$. Then

10.16 $\lambda_1 = -1$.

Proof Certainly $\lambda_1$ is 0 or $-1$. Assume therefore that $\lambda_1 = 0$. Not both $a_2, a_3$ are zero, for otherwise $\lambda_1, \lambda_2, \lambda_3$ would all be roots of the equation $x^2 + x = 0$ and they would not all be different as we know (10.11) they must. Suppose therefore that $a_2 \neq 0$. Then

$$p = \frac{3\lambda_2(\lambda_2 + 1)}{a_2} + 1.$$ 

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If $3 \nmid a_2$ then clearly $p \equiv 1 \pmod{3}$. If $3 \mid a_2$ then by 10.12 $a_2 = 3$, $p = \lambda^2 + \lambda + 1$ and again we see that $p \equiv 1 \pmod{3}$. Thus $p \equiv 1 \pmod{3}$ in all cases.

From equations (6.6), (6.7) we have

$$\begin{align*}
\mu_1 + \nu_1 &= -1, \\
\mu_1^2 + \nu_1^2 &= 3p = (p-1) = 2p+1,
\end{align*}$$

and (6.8) gives

$$3p(p-1)a_{111} = (p-1)^3 + (p-1)(\mu_1^3 + \nu_1^3).$$

Therefore

$$\begin{align*}
3pa_{111} &= (p-1)^2 + \frac{3}{2}(\mu_1 + \nu_1)(\mu_1^2 + \nu_1^2) - \frac{1}{2}(\mu_1 + \nu_1)^3 \\
&= (p-1)^2 - \frac{3}{2}(2p+1) + \frac{1}{2} \\
&= p^2 - 5p \\
3a_{111} &= p - 5.
\end{align*}$$

Since $a_{111}$ is an integer we see that $p \equiv 5 \pmod{3}$, which contradicts the congruence we derived in the previous paragraph. Hence $\lambda_1 \neq 0$, so $\lambda_1 = -1$.

At last we are in a position to prove that $p = 31$. Since $\lambda_1 = -1$ and $\lambda_1 + \lambda_2 + \lambda_3 = -1$ we know that $\lambda_2 = -\lambda_3 = \lambda$, say. In this case

$$\begin{align*}
p - 1 \text{ divides } 3\lambda(\lambda+1) \\
p - 1 \text{ divides } 3(-\lambda)(-\lambda+1).
\end{align*}$$

Therefore $p - 1$ divides $6\lambda$.

By interchanging $\lambda_2$, $\lambda_3$ if necessary we may assume that $\lambda \geq 0$. Then $\lambda \neq 0$ since (10.11) $\lambda_2 \neq \lambda_3$, and $\lambda \neq 1$ since $\lambda_3 \neq \lambda_1$. Thus $\lambda \geq 2$. Moreover, we know from (10.12) that

$$\frac{6\lambda}{p-1} \cdot \frac{\lambda + 1}{2} \leq 3.$$ 

Thus $\lambda + 1 \leq 6$ and if $t\lambda \neq p - 1$ then $\lambda + 1 \leq 3$. It is now easy to see that the only possibilities are

$$\begin{align*}
6\lambda &= p - 1, & \lambda &= 5, & p &= 31; \\
or &\lambda &= 3, & p &= 19; \\
or &3\lambda &= p - 1, & \lambda &= 2, & p &= 7.
\end{align*}$$

In case $p = 7$ we have $\lambda_2 = 2$ and $n_2 = 2$, but a primitive group of degree 21 cannot have a suborbit of length 2 ([20, p51]). This case therefore does not arise.

In case $p = 19$ we use (6.6), (6.7) to calculate $\mu_1$, $\mu_2$, $\nu_2$, $\mu_3$, $\nu_3$. The result is

$$\begin{align*}
\mu_1, \nu_1 &\text{ are } \pm 2\sqrt{5} \\
\mu_2, \nu_2 &\text{ are } -2 \pm \sqrt{6} \\
\mu_3, \nu_3 &\text{ are } 5, -3.
\end{align*}$$
Equation (6.9), \( \mu_1 + \mu_2 + \mu_3 = -1 \) is now impossible to satisfy.

Finally, in case \( p = 31 \), no such contradiction arises. With suitable choices of roots one finds

\[
\begin{align*}
\mu_1 &= 4\sqrt{2} \\
\mu_2 &= -3 - \sqrt{2} \\
\mu_3 &= 2 - 3\sqrt{2}
\end{align*}
\]

\[
\begin{align*}
\nu_1 &= -4\sqrt{2} \\
\nu_2 &= -3 + \sqrt{2} \\
\nu_3 &= 2 + 3\sqrt{2}
\end{align*}
\]

And since \( \lambda_1 = -1, \lambda_2 = 5, \lambda_3 = -5 \), the subdegrees are \( n_1 = 32, n_2 = 20, n_3 = 40 \). This is the third possibility in Theorem 10.2 and the proof is now complete.

**Part III**

**Appendices**

11 A remark on imprimitive groups

The analysis presented in §§4.5 does not apply to imprimitive groups in general, but it is not hard to see that it is valid in case \( p \geq 5 \) and \( G \) is an insoluble group of degree \( 3p \) which has no blocks of size \( p \) in \( \Omega \). Such a group either is primitive, or, if imprimitive, then it has \( p \) blocks of size 3, and, by a well-known theorem of Burnside, the group \( G^* \) induced by \( G \) on the set of blocks in \( \Omega \) is 2-fold transitive. If the stabilizer of a block operates as \( S_3 \) on that block then \( G \) is non-canonically embeddable in the monomial group (non-standard wreath product) \( S_3 \wr G^* \); and if the stabilizer of a block operates as \( A_3 \) then \( G \) is canonically embeddable in \( A_3 \wr G^* \).

The irreducible character \( \chi \) such that \( 1 + \chi \) is the permutation character of \( G \) operating on the set of blocks in \( \Omega \) is necessarily a constituent of our permutation character \( \pi \) of degree \( 3p \). Since \( \chi \) has degree \( p - 1 \), \( G \) is of one of the Types III, IV, V, VII, VIII. It is not hard to prove the facts listed below by methods similar (in most cases) to those used in Part II of this paper. We shall use \( G \) to denote an insoluble group which is imprimitive of degree \( 3p \), and which has no blocks of size \( p \).

### 11.1 If \( G \) is of Type III then its subdegrees are \( 1, 2, 3(p - 1) \).

Examples exist without restriction on \( p \). For instance, if \( G^* \) is any insoluble group of degree \( p \) then \( S_3 \wr G^* \), as a group of degree \( 3p \), has Type III. \( SL(3, 3) \) operating as a group of degree 39 is also of this kind.

### 11.2 If \( G \) is of Type IV then either

(i) its subdegrees are \( 1, 2, p - 1, 2(p - 1) \); or

(ii) its subdegrees are \( 1, 1, 1, 3(p - 1) \).
In case (i) the stabilizer of a block operates on that block as the full symmetric group $S_3$, and in case (ii) the stabilizer of a block operates on it as $A_3$. Examples of case (i) occur whenever $p$ is a Fermat prime – the smallest case being $S_5$ acting by conjugation on its set of 15 Sylow 2-subgroups – and I do not know examples for other primes. Examples of case (ii) occur for all $p$. For example, if $G^*$ is any insoluble group of degree $p$ then $A_3 \wr G^*$ has type IV and subdegrees $1, 1, 1, 3(p - 1)$.

**11.3 If $G$ is of Type V then its subdegrees are $1, 1, 1, p - 1, p - 1$ and the stabilizer of a block operates as $A_3$ on that block.**

Many examples exist. The groups $SL(2, p - 1)$, where $p$ is a Fermat prime, operating as described in Lemma 4.2 are of this kind. Similarly, if $q$ is a prime, $q \equiv 1 \pmod{3}$, and $p = (q^n - 1)/(q - 1)$ (this requires $n \equiv \pm 1 \pmod{6}$), then $PGL(n, q)$, $PSL(n, q)$ can be faithfully represented as imprimitive groups of degree $3p$ and Type V. By a theorem of Ito [11] these, the group $SL(3, 3)$ mentioned above, and $PSL(2, 7)$ are essentially the only groups $PSL(n, q)$ of degree $3p$.

**11.4 There are no imprimitive groups of type VII.**

Thus Theorem 10.2 is exhaustive for groups of this kind.

**11.5 If $G$ is of type VIII then $p = 7$, the subdegrees are $1, 2, 2, 4, 4, 8$, and $G$ is $PSL(2, 7)$ operating by conjugation on its set of Sylow 2-subgroups.**

This is the group appearing in Lemma 4.3 for $p = 7$; it is the imprimitive normal subgroup of index 2 in the group of Example 3 (p. 2).

**12 The index of $P$ in $N(P)$**

N. Ito, in [13], has achieved a complete classification of transitive simple permutation groups of degree $3p$ under the assumption that $N(P)$, the normaliser of a Sylow $p$-subgroup $P$, has order $2p$. This raises the question whether one can find a good bound for $N(P)$ in general. Of course, there will be no significant such bound for 2-fold transitive groups, but it is easy to see that for primitive groups which are not 2-fold transitive, the results of Part II imply that $|N(P) : P|$ is always bounded by a function whose order of magnitude is $4\sqrt{3p}$. Indeed, a little more precisely:

**12.1 If $G$ is primitive of degree $3p$ and Type III or IV, then $|N(P) : P| = q$ where $q$ divides one of $12, 18, 8a, 12a, 8(a + 1)$ or $12(a + 1)$.**

---

6 This provides another, rather small, example to settle the point raised in [20], p.93, 11.3.4.

7 A similar deduction can be made from Wielandt’s results for simply transitive, primitive groups of degree $2p$. 
The proof is similar to the proof given below.
Curiously, for groups of Type II one gets far better information – and in view of Ito’s work there is good hope that this is significant:

**Theorem 12.2** If $G$ is of Type II and contains no odd permutations then $|N(P) : P|$ divides 8. If $G$ does contain odd permutations then $|N(P) : P|$ divides 16.

**Proof** The second statement follows from the first since if $G$ contains odd permutations then $G \cap A_3^p$ is a subgroup of index 2 in $G$ which must be primitive (cf. Tamashke’s remark, 3.3 above; or compare §11), and of Type II. So let us suppose that $G$ contains no odd permutations. Then the centraliser $C(P)$ is $P$, for, if $C(P)$ contained a cycle of length $3p$ then $G$ would be 2-fold transitive by a theorem of Schur (see [20], p.65), and if $C(P)$ had order $2p$ then it would contain an element of order 2 having $p$ fixed points and $p$ transpositions, an odd permutation. Since $C(P) = P$ we know that $N(P) = P.Q$ where $Q$ is a cyclic group of order $q$, say, $q$ divides $p - 1$ and $Q$ acts faithfully by conjugation as a group of automorphisms of $P$. A consequence of the theorem of §8 is that $p \equiv 2 \mod 3$, hence $3 \nmid |N(P)|$ and $N(P)$ cannot be transitive on $\Omega$. Therefore $N(P)$ either has one orbit of length $p$ and one of length $2p$, or three orbits of length $p$ in $\Omega$. In either case $q$ is a stabiliser for an $N(P)$ orbit of length $p$, and so we may suppose that $Q \leq H$. Either $Q$ has a further 2 fixed points, or it has an orbit of length 2; in both cases the remaining $3p - 3$ points fall into $Q$-orbits of length $q$. A non-trivial $H$-orbit is a union of $Q$-orbits, and it follows that

$$|\Gamma| \equiv 0, 1 \text{ or } 2 \mod q.$$  

We know also that $p - 1 \equiv 0 \mod q$. In case (i) (see §8) we may take $|\Gamma| = p + 4a + 1$, and we have:

- $q$ divides $(48a^2 + 34a + 6, 48a^2 + 30a + 4)$
- or $q$ divides $(48a^2 + 34a + 5, 48a^2 + 30a + 4)$
- or $q$ divides $(48a^2 + 32a + 4, 48a^2 + 30a + 4)$.

The Euclidean algorithm and a little computation yield that $q$ divides 2, 1 and $(2a + 4, 8)$ respectively. That is, $q \mid 8$, and if $a$ is odd then $q \mid 2$. Similarly, in case (ii) we may take $|\Gamma| = p - 4a - 3$, and we find that $q$ divides 2, 1 or $(8, 2a + 6)$. That is, $q \mid 8$, and if $a$ is even then $q \mid 2$. This proves the theorem.

If $|N(P) : P| = 2$ it is not hard to show that $p = 5$ and $G$ is $A_5$ as in Example 1 (see also Ito [13]). Thus we may in fact suppose that for groups of type II(i) $a$ is even, and for groups of type II(ii) $a$ is odd. However, I would guess that if $|N(P) : P| = 4$ then $G$ is $S_6$ as in Example 1, and that groups of Type II with $|N(P) : P| = 8$ or 16 do not exist.

*$(b, c)$ denotes the highest common factor of $b$ and $c.$
13 A proof by Dr B. J. Birch

The proof of Theorem 9.2 in the case where all orbits are self-paired eluded me for some time. Equations (9.7) and inequality (9.6) have two infinite families of solutions. One of these arises in case \( \nu_1 = -\nu_2 = \nu_3 = -1 \), in which case \( p = \nu^2 \), solutions of which are of no interest to us since \( p \) is a prime. The other is the family of solutions found in §9, where \( p = 3a^2 + 3a + 1 \), \( \nu_1 = 2a + 1 \), \( \nu_2 = \nu_3 = -a - 1 \). In fact, as Mr J. E. Stoy showed by computation, (9.6) and (9.7) do have other solutions, but these appear to be relatively few and far between. At this point Dr Birch proved for me that other solutions are indeed rare, and as his argument may well extend for groups of degree \( kp \) with \( k > 3 \), I reproduce the relevant part of his letter here.

“You consider the equations

\[
\begin{align*}
\gamma_1 + \gamma_2 + \gamma_3 + 1 &= 0, \\
\sqrt{4p - 1 - 3\gamma_1^2} + \sqrt{4p - 1 - 3\gamma_2^2} &= \sqrt{4p - 1 - 3\gamma_3^2}
\end{align*}
\] (1)

Removing square roots, one gets

\[
\begin{align*}
4p - 1 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 + 2T \\
\text{with } T &\geq 0
\end{align*}
\] (2)

and

\[
\begin{align*}
T^2 &= \gamma_1^4 + \gamma_2^4 + \gamma_3^4 - \gamma_1^2\gamma_2^2 - \gamma_2^2\gamma_3^2 - \gamma_3^2\gamma_1^2 \\
\gamma_1 + \gamma_2 + \gamma_3 &= 0
\end{align*}
\] (3)

Note that (3) is soluble with any two of \( \gamma_1^2, \gamma_2^2, \gamma_3^2 \) equal – leading to your two families of solutions.

Suppose from now on that \( |\gamma_1| > |\gamma_2| > |\gamma_3| > 0 \), so that \( \gamma_2, \gamma_3, \gamma_2 - \gamma_3, \gamma_2 + \gamma_3 \) all have the same sign (allowing \( |\gamma_1| = |\gamma_2| \) or \( |\gamma_2| = |\gamma_3| \) if you like]. Write \( \gamma_2 + \gamma_3 = S, \gamma_2 - \gamma_3 = D \). Then

\[
4T^2 = (2\gamma_1^2 - \gamma_2^2 - \gamma_3^2)^2 + 3(\gamma_2^3 - \gamma_3^3)^2
\]

\[
= (2(S + 1)^2 - \frac{1}{2}S^2 - \frac{1}{2}D^2)^2 + 3S^2D^2
\]

\[
[2T + (2(S + 1)^2 - \frac{1}{2}S^2 - \frac{1}{2}D^2)][2T - (2(S + 1)^2 - \frac{1}{2}S^2 - \frac{1}{2}D^2)] = 3S^2D^2
\]

\[
4(S + 1)^2 - S^2 - D^2 = P - Q \text{ with } PQ = 3S^2D^2.
\] (4)

Write \( P = XY \) with \( X \mid 3S^2, Y \mid D^2 \); then \( Q = ZT \) with \( ZX = 3S^2, YT = D^2 \), and we get

\[
-8S - 4 = XZ - YT - XY + ZT = (X + T)(Z - Y).
\] (5)

Take any large \( N_0 \) and small \( \epsilon > 0 \). If now \( p < N_0 \), then by (2) \( S^2 < \frac{4}{3}N_0 \) (approximately). By (5) and \( XZ = 3S^2 \), for fixed \( S \) there are only \( O(S^\epsilon) \) possibilities for \( X, Y, Z, T \); so there are only \( O(N_0^{4+\epsilon}) \) primes \( p < N_0 \) for which (1) is soluble.

In fact ‘sporadic’ solutions of (4) appear to be extremely rare – but I don’t know how to prove this.”
14 Tabulation of results

The first of the following tables summarises the results of Part II. The remaining ones give the eigenvalues of the basic adjacency matrices for every case in which equations (6.6)–(6.9) are soluble. In these tables the columns are indexed by the degrees of the irreducible constituents of \( \pi \); the rows are indexed by the basic adjacency matrices. The multiplication constants are not tabulated; they can be calculated directly from the tables of eigenvalues by means of (6.8).
| Type  | $p$       | rank | degrees of irreducible constituents of $\pi$ | subdegrees                      | Comments                                      |
|-------|-----------|------|---------------------------------------------|----------------------------------|-----------------------------------------------|
| VII (i) | 7         | 4    | $1, p + 1, p - 1, p - 1$                    | 1, 4, 8, 8                      | PSL(2,7) – Example 3                         |
|       | (ii)      |      |                                              | 1, 6, 20, 30                    | PSL(2,19) – Example 4                        |
|       | (iii)     |      |                                              | 1, 20, 32, 40                   |                                                |
| II (i) | $48a^2 + 30a + 5$ | $a \geq 0$ |                                              | $1, 2(8a + 3)(3a + 1), 8(4a + 1)(3a + 1)$ | $A_6, S_6$ – Example 1, are examples for $a = 0$ |
|        |           | $a$ even |                                              |                                  | $N(P)$ is small, see §12                     |
|        | $48a^2 + 66a + 23$ | $a \geq 0$ |                                              | $1, 2(8a + 5)(3a + 2), 8(4a + 3)(3a + 2)$ |                                                |
|        |           | $a$ odd |                                              |                                  |                                                |
| III (i) | $3a^2 + 3a + 1$ | $a \geq 2$ |                                              | $1, a(3a + 1), 2(a + 1)(3a + 1)$ | For $a = 1$, $a_{111} = -1$, which is not possible |
|        |           | $a \geq 1$ |                                              | $1, (a + 1)(3a + 2), 2a(3a + 2)$ | $A_7, S_7$ as in Example 2 arise for $a = 1$  |
|        |           | $a \geq 1$ |                                              |                                  |                                                |
| IV (i) | $3a^2 + 3a + 1$ | $a \geq 1$ |                                              | $1, a(3a + 1), (a + 1)(3a + 1), (a + 1)(3a + 1)$ | $\Delta_3 = \Delta_2$ I know no examples of these kinds of group |
|        |           | $a$ even |                                              |                                  |                                                |
|        |           | $a \geq 1$ |                                              | $1, (a + 1)(3a + 2), a(3a + 2), a(3a + 2)$ |                                                |
|        | (ii)      | $a \geq 1$ |                                              |                                  |                                                |
|        | (iii)     | $a \geq 1$ |                                              |                                  |                                                |

Table 14.1: Summary of primitive groups of degree $3p$
Table 14.2: Type VII(i), \( p = 7 \)

| \( B_i \setminus f_i \) | 1 | 8 | 6 | 6 |
|-------------------------|---|---|---|---|
| \( B_0 \)               | 1 | 1 | 1 | 1 |
| \( B_1 \)               | 4 | −2 | 1 + \( \sqrt{2} \) | 1 − \( \sqrt{2} \) |
| \( B_2 \)               | 8 | −1 | −2\( \sqrt{2} \) | 2\( \sqrt{2} \) |
| \( B_3 \)               | 8 | 2 | −2 + \( \sqrt{2} \) | −2 − \( \sqrt{2} \) |

Table 14.3: Type VII(ii), \( p = 19 \)

| \( B_i \setminus f_i \) | 1 | 20 | 18 | 18 |
|-------------------------|---|----|----|----|
| \( B_0 \)               | 1 | 1  | 1  | 1  |
| \( B_1 \)               | 6 | −3 | \( \frac{1}{2}(3 + \sqrt{5}) \) | \( \frac{1}{2}(3 - \sqrt{5}) \) |
| \( B_2 \)               | 20 | −1 | −2\( \sqrt{5} \) | 2\( \sqrt{5} \) |
| \( B_3 \)               | 30 | 3  | \( \frac{1}{2}(-5 + 3\sqrt{5}) \) | \( \frac{1}{2}(-5 - 3\sqrt{5}) \) |

Table 14.4: Type VII(iii), \( p = 31 \)

| \( B_i \setminus f_i \) | 1 | 32 | 30 | 30 |
|-------------------------|---|----|----|----|
| \( B_0 \)               | 1 | 1  | 1  | 1  |
| \( B_1 \)               | 32 | −1 | 4\( \sqrt{2} \) | −4\( \sqrt{2} \) |
| \( B_2 \)               | 20 | 5  | −3 − \( \sqrt{2} \) | −3 + \( \sqrt{2} \) |
| \( B_3 \)               | 40 | −5 | 2 − 3\( \sqrt{2} \) | 2 + 3\( \sqrt{2} \) |

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Table 14.5: Type II(i), $p = 48a^2 + 30a + 5$

$$
\begin{array}{|c|ccc|}
\hline
B_i \setminus j_i & 1 & 48a^2 + 30a + 5 & 96a^2 + 60a + 9 \\
B_0 & 1 & 1 & 1 \\
B_1 & 2(8a + 3)(3a + 1) & -8a - 3 & 4a + 1 \\
B_2 & 8(4a + 1)(3a + 1) & 8a + 2 & -4a - 2 \\
\hline
\end{array}
$$

Table 14.6: Type II(ii), $p = 48a^2 + 66a + 23$

$$
\begin{array}{|c|ccc|}
\hline
B_i \setminus j_i & 1 & 48a^2 + 66a + 23 & 96a^2 + 132a + 45 \\
B_0 & 1 & 1 & 1 \\
B_1 & 2(8a + 5)(3a + 2) & 8a + 5 & -4a - 3 \\
B_2 & 8(4a + 3)(3a + 2) & -8a - 6 & 4a + 2 \\
\hline
\end{array}
$$

Table 14.7: Type III(i), $p = 3a^2 + 3a + 1$

$$
\begin{array}{|c|ccc|}
\hline
B_i \setminus j_i & 1 & 6a^2 + 6a + 2 & 3a^2 + 3a \\
B_0 & 1 & 1 & 1 \\
B_1 & a(3a + 1) & a & -2a - 1 \\
B_2 & 2(a + 1)(3a + 1) & -a - 1 & 2a \\
\hline
\end{array}
$$

Table 14.7: Type III(i), $p = 3a^2 + 3a + 1$
Table 14.8: Type III(ii), $p = 3a^2 + 3a + 1$

| $B_i \setminus f_i$ | 1   | $6a^2 + 6a + 2$ | $3a^2 + 3a$ |
|---------------------|-----|-----------------|-------------|
| $B_0$               | 1   | 1               | 1           |
| $B_1$               | $(a + 1)(3a + 2)$ | $-a - 1$ | $2a + 1$ |
| $B_2$               | $2a(3a + 2)$ | $a$ | $-2a - 2$ |

Table 14.9: Type IV(i), $p = 3a^2 + 3a + 1$, $a$ even

| $B_i \setminus f_i$ | 1   | $3a^2 + 3a + 1$ | $3a^2 + 3a + 1$ | $3a^2 + 3a$ |
|---------------------|-----|-----------------|-----------------|-------------|
| $B_0$               | 1   | 1               | 1               | 1           |
| $B_1$               | $a(3a + 1)$ | $a$ | $a$ | $-2a - 1$ |
| $B_2$               | $(a + 1)(3a + 1)$ | $\alpha$ | $\beta$ | $a$ |
| $B_3$               | $(a + 1)(3a + 1)$ | $\beta$ | $\alpha$ | $a$ |

$\alpha + \beta = -a - 1$  
$\alpha \beta = \frac{1}{2}(5a + 2)(a + 1)$

Table 14.10: Type IV(ii),(iii), $p = 3a^2 + 3a + 1$

Type IV(ii): $a$ is odd and $\alpha + \beta = a$, $\alpha \beta = \frac{1}{2}a(5a + 3)$

Type IV(iii): $a$ is even and $\alpha + \beta = a$, $\alpha \beta = -\frac{1}{2}a(4a + 3)$

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Afterword

By Peter J. Cameron, University of St Andrews

As said above, this paper was written in the academic year 1968–69. For various reasons, including the fact that other mathematicians (Olaf Tamaschke and Leonard Scott) had proved overlapping results, it was never published. I would like to explain why I am making it available now. There are three main reasons.

Historical significance The paper was written at a time when mathematical objects from three different areas (association schemes in statistics, coherent configurations in permutation groups, and cellular algebras in computer science), were being recognised as essentially the same thing, and to provide a framework for using combinatorial methods in the study of permutation groups. It gives a very good introduction to this material and would have been very influential had it been published at the time. As it was, its effect was indirect: for example, I read it as a doctoral student, and the ideas were present in my thesis.

Mathematical content The arguments are much more difficult and subtle than those used by Wielandt, and the mathematics there which could be valuable in other contexts.

Mathematical significance The main theorem of the paper has been subsumed by results proved using the Classification of Finite Simple Groups (which was still in the future when the paper was written): for example, we now have a classification of primitive groups of odd degree\(^9\). But it has other implications which have not been drawn out. It is known that Wielandt’s arguments can be used to prove a purely combinatorial result, concerning strongly regular graphs with prescribed eigenvalue multiplicities\(^10\), and it is likely that those in this

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\(^9\)W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes, *J. Algebra* **106** (1987), 15–45; Martin W. Liebeck and Jan Saxl, The primitive permutation groups of odd degree, *J. London Math. Soc.* (2) **31** (1985), 250–264.

\(^10\)P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Math. Soc. Student Texts **22**, Cambridge University Press, Cambridge, 1991, Theorem 2.20.
paper will have similar implications for strongly regular graphs and, more generally, for association schemes. So new mathematics will result from the paper, and it will be important to have it available to researchers.

As a fourth, more personal reason, I want it to be a tribute to Peter Neumann, an important mathematician and historian of mathematics, an inspiring teacher, and a fine and generous person. I was his sixth doctoral student and regard myself as in his debt.

I have re-typed the paper in LaTeX from a scan of the original typescript. In a couple of places the text was illegible; specifically footnote 2 on page 12 of the present document and the surrounding text (where I have made a guess as to what was written) and footnote 5 on page 28 (which was cut off in the scan but seems to me to be unnecessary).

I am grateful to Leonard Scott for providing me with a scan of the paper and of related historical material on his proposed collaboration with Peter Neumann (which did not materialise), to Sylvia Neumann for giving permission to post this paper, and to Cheryl E. Praeger and others for encouraging me to do so.

Final note This second arXiv version corrects some typographic errors (both by Neumann in the original and by me while re-typing). I am very grateful to Marina Anagnostopoulou-Merkouri for spotting these.