Asymptotic Estimation of Shift Parameter of a Quantum State

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1 Introduction

The rise of quantum estimation theory dates back to the late 1960s – 1970s (see [8], [9]). It was initially developed as an adequate mathematical framework for design of optimal receiver in quantum communication channels, and later turned out to be relevant also for clarification of some foundational issues of quantum measurement. New interest to quantum estimation theory was brought by the development of high precision experiments, in which researchers operate with elementary quantum systems. In such experiments quite important is the issue of extracting the maximum possible information from the state of a given quantum system. For example, in currently discussed proposals for quantum computations, the information is written into states of elementary quantum cells – qubits, and is read off via quantum measurements. From a statistical viewpoint, measurement gives an estimate for the quantum state, either as a whole, or for some of its components (parameters). The main concrete models of interest in quantum estimation theory considered up to now fall within one of the following classes:

1) Parametric models with a group of symmetries [9], [12]. In particular, the models with the shift or rotation parameter are strictly relevant to the issue of canonical conjugacy and nonstandard uncertainty relations, such as time-energy, phase-number of quanta, etc. It is these models which will be our main concern in this paper.

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2) The full model, in which the multidimensional parameter is the quantum state itself, i.e. we are interested in estimation of completely unknown quantum state. Although in finite dimensions it is a parametric model with a specific group of symmetries, it deserves to be singled out both because of its importance for physics and of its mathematical features. Especially interesting and mostly studied is the case of the qubit state, with the 3-dimensional parameter varying inside the Bloch sphere. Asymptotic estimation theory for the full model in the pure state case was developed in [5], [7], and for mixed states in [14].

On the other hand, the full model, especially in infinite dimensions, belongs rather to nonparametric quantum mathematical statistics, which is at present also in a stage of development. In this connection we would like to mention the method of homodyne tomography of a density operator in quantum optics [3], [6].

3) Estimation of the mean value of quantum Gaussian states. This is a quantum analog of the classical linear “signal+noise” problems, however with the noise having quantum-mechanical origin. This model was treated in detail e.g. in [9].

An important distinctive feature of quantum estimation appears in consideration of series of independent identical trials of a quantum system. In the paper [10], devoted to asymptotics of estimation of a shift parameter in a quantum state, it was observed that entangled covariant estimates in models with independent multiple observations can be more efficient than unentangled ones. For arbitrary locally unbiased observables this was demonstrated on the full model in [15], [5], [7], [4]. This property is a statistical counterpart of the strict superadditivity of Shannon information for quantum memoryless channels (see §5.1 in [12]).

In the present paper we develop further the asymptotic theory of estimation of a shift parameter in a quantum state to demonstrate the relation between entangled and unentangled covariant estimates in the analytically most transparent way. After recollecting basics of estimation of shift parameter in Sec. 2, we study the structure of the optimal covariant estimate in Sec. 3, showing how entanglement comes into play for several independent trials. In Secs. 4,5 we give the asymptotics of the performance of the optimal covariant estimate comparing it with the “semiclassical” unentangled covariant estimation in the regular case of finite variance of the generator of the shift group. Sec. 6 is devoted to estimation in the case where the regularity
assumption is violated.

In this paper we deal with the case of pure states. It is in fact yet another distinctive feature of quantum estimation that the complexity of the problem increases sharply with transition from pure to mixed states estimation. In fact, estimation theory for mixed states is an important field to a great extent still open for investigation. Another simplifying factor in our model is that the parameter is one-dimensional. It is well known that in the quantum case estimation of multidimensional parameter involves additional problems due to the non-commutativity of the algebra of quantum observables, see e. g. [9].

2 Covariant Estimates

Let \( \mathcal{H} \) be a Hilbert space of observed quantum system. The states of the system are described by density operators \( S \) in \( \mathcal{H} \), and observables with values in a measurable space \( \mathcal{X} \) by resolutions of the identity or probability operator valued measures \( M \) on \( \mathcal{X} \) (see, e. g. [9], [12]). Let \( G \) be either the real line \( \mathbb{R} \) (the case of a displacement parameter) or the unit circle \( \mathbb{T} \) (the case of a rotational parameter), and let \( x \rightarrow V_x = e^{-ixA}; \ x \in G, \) be a unitary representation of the one dimensional Abelian group \( G \) in \( \mathcal{H} \). The spectrum \( \Lambda \) of the operator \( A \) is contained in the dual group \( \hat{G} \), which is identified with \( \mathbb{R} \) in the case \( G = \mathbb{R} \) and with the group of integers \( \mathbb{Z} \) when \( G = \mathbb{T} \). We consider the problem of estimation of the shift parameter \( \theta \in G \) in the family of states

\[
S_{\theta} = e^{-i\theta A} S e^{i\theta A},
\]

where \( S \) is the initial state, assumed to be known. By estimate we call arbitrary observable with values in \( G \).

The estimate \( M \) is covariant if

\[
V_x^* M(B) V_x = M(B - x) \quad \text{for} \quad B \subset G, \ x \in G, \quad (1)
\]

where \( B - x := \{ y; y + x \in B \} \), and in the case \( G = \mathbb{T} \) addition is modulo \( 2\pi \).

A necessary and sufficient condition for existence of a covariant observable is that the spectrum of \( A \) is absolutely continuous with respect to the Haar measure in the dual group \( \hat{G} \) [11], which we assume from now on. In the case
$G = \mathbb{R}$ this means that $A$ has Lebesgue spectrum, while in the case $G = \mathbb{T}$ (with $\hat{G} = \mathbb{Z}$) this poses no restrictions. We introduce the operators

$$U_y := \int_G e^{iyx} M(dx); \quad y \in \hat{G}. \quad (2)$$

Then (1) reduces to the Weyl relation

$$U_y V_x = e^{ixy} V_x U_y; \quad x \in G, y \in \hat{G}, \quad (3)$$
in which, however, the operator $U_y$, is in general non-unitary. In this sense the observable $M$ is canonically conjugate to the observable $A$.

Introducing the characteristic function

$$\varphi^M_S(\lambda) = ESU_\lambda = \int_G e^{i\lambda x} \mu^M_S(dx), \quad \lambda \in \hat{G},$$

one has the following uncertainty relation for the generalized canonical pair $(A, M)$ [10]

$$\Delta^M_S(\lambda) \cdot D_S(A) \geq \frac{1}{4} \quad (4)$$

where $\Delta^M_S(\lambda) := \lambda^{-2}(|\varphi^M_S(\lambda)|^{-2} - 1), \lambda \neq 0$, is a functional measure of uncertainty of the covariant observable $M$ in the state $S$. If $G = \mathbb{R}$ and $M$ has finite variance $D_S(M)$, then $\lim_{\lambda \to 0} \Delta^M_S(\lambda) = D_S(M)$, so that from (4) follows the generalization of the Heisenberg uncertainty relation

$$D_S(M) \cdot D_S(A) \geq \frac{1}{4}. \quad (5)$$

Of the main interest are covariant estimates having the minimal uncertainty. The following result describes them in the case of a pure initial state $S = |\psi\rangle\langle\psi|$, where $\psi$ is a unit state vector in $\mathcal{H}$. Then

$$S_\theta = e^{-i\theta A}|\psi\rangle\langle\psi|e^{i\theta A}.$$

Since the spectrum of $A$ is absolutely continuous, we have the direct integral spectral decomposition

$$\mathcal{H} = \int_A \oplus \mathcal{H}(\lambda)d\lambda, \quad (6)$$
diagonalizing the unitary group \( \{ e^{i \theta A} \} \). This means that for any vector
\[
| \varphi \rangle = \int_\Lambda \oplus | \varphi(\lambda) \rangle d\lambda, \quad | \varphi(\lambda) \rangle \in \mathcal{H}(\lambda),
\]
one has
\[
e^{i \theta A} | \varphi \rangle = \int_\Lambda \oplus e^{i \theta \lambda} | \varphi(\lambda) \rangle d\lambda, \quad \theta \in G.
\]
We agree to extend \( \varphi(\lambda) \) to the whole of \( \hat{G} \) by letting it zero outside of \( \Lambda \).

**Theorem 1** [10]. For arbitrary covariant estimate \( M \)
\[
| \varphi^M_S(\lambda) \rangle \leq \varphi^*_S(\lambda) := \int_\hat{G} \| \psi(\lambda') \| \| \psi(\lambda + \lambda') \| \ d\lambda',
\]
\[
\Delta^M_S(\lambda) \geq \Delta^*_S(\lambda) := \lambda^{-2} (\varphi^*_S(\lambda) - 2 - 1).
\]
In the case \( G = \mathbb{R} \)
\[
D_S(M) \geq D^*_S := \int_\mathbb{R} \left( \frac{d}{d\lambda} \| \psi(\lambda) \| \right)^2 d\lambda,
\]
provided the right hand side is defined and finite.

The equalities are attained on the optimal covariant observable \( M^* \) which is given by the following kernel in the direct integral decomposition (6)
\[
M^*(dx) = \left[ e^{i x (\lambda' - \lambda)} \frac{\langle \psi(\lambda') | \psi(\lambda') \rangle}{\| \psi(\lambda) \| \| \psi(\lambda') \|} \right] \frac{dx}{2\pi}. \quad \Box 
\]

Notice that the operator
\[
P^*_\lambda := \int_G M^*(dx) = \left[ \delta(\lambda' - \lambda) \frac{| \psi(\lambda) \rangle \langle \psi(\lambda') |}{\| \psi(\lambda) \| \| \psi(\lambda') \|} \right]
\]
is a projection onto the invariant subspace of the group \( \{ e^{i \theta A} \} \) generated by the vectors \( e^{i \theta A} | \psi \rangle, \theta \in G \). Indeed, \( P^*_\lambda e^{i \theta A} | \psi \rangle = e^{i \theta A} | \psi \rangle \) and if the vector \( | \varphi \rangle \) is orthogonal to all of these vectors, then \( \int_\Lambda e^{i \lambda \theta} \langle \varphi(\lambda) | \psi(\lambda) \rangle d\lambda = 0, \theta \in G \), and \( \langle \varphi(\lambda) | \psi(\lambda) \rangle = 0 \) for \( \lambda \in \Lambda \), hence \( P^*_\lambda | \varphi \rangle = 0 \). Thus \( M^* \) is in general subnormalized, and one has to extend it to the orthogonal complement of \( \mathcal{H}_* = P^*_\lambda \mathcal{H} \) to obtain an observable. Independently of the extension, it has the
following probability density in the state $S_\theta$, given by the Fourier transform of the characteristic function (7),

$$p_\theta^*(x) = \frac{1}{2\pi} \left| \int e^{i\lambda(x-\theta)} \| \psi(\lambda) \| \, d\lambda \right|^2.$$

In [11] it is shown also that $M_\lambda$ minimizes the average deviation $R(M) = \int W(x-\theta)p_\theta^M(x)\,dx$ where $W$ is an arbitrary continuous conditionally negative definite function.

Similarly to (11), we obtain for future use

$$U_{sy} := \int_G e^{iyx} M_\lambda(dx) = \left[ \delta(y + \lambda - \lambda) \frac{\psi(\lambda)}{\| \psi(\lambda) \|} \| \psi(\lambda') \| \right].$$  \hspace{1cm} (12)

**Example** [8], [9]. Consider the case $\mathcal{H} = L^2(\mathbb{R})$, where $A$ acts as multiplication by the independent variable $\lambda$ (momentum representation). Then $\theta$ is the position displacement parameter. In this case $\mathcal{H}(\lambda) \simeq \mathbb{R}, \, \psi(\lambda)/\| \psi(\lambda) \| = e^{i\alpha(\lambda)}$ and (10) is an orthogonal resolution of the identity in $\mathcal{H}$. This is most easily seen by verifying that the operators (12) form a unitary group. This resolution of the identity is the spectral measure of the selfadjoint operator

$$Q_\lambda = \int x M_\lambda(dx) = \frac{1}{i} \frac{d}{dy} \bigg|_{y=0} \frac{d}{dy} \bigg|_{y=0} U_{sy}.$$  \hspace{1cm} (13)

According to (12) the action of this operator on the vector $\varphi$ is given by

$$Q_\lambda \varphi(\lambda) = i e^{i\alpha(\lambda)} \frac{d}{d\lambda} e^{-i\alpha(\lambda)} \varphi(\lambda),$$

where $Q = i \frac{d}{d\lambda}$ is the position operator in the momentum representation. Thus the optimal covariant observable is position observable up to the gauge transformation compensating the phase of the state vector $|\psi\rangle$ in the momentum representation. In case the argument $\alpha(\lambda)$ is absolutely continuous one has further

$$Q_\lambda \varphi(\lambda) = [Q + \alpha'(\lambda)] \varphi(\lambda)$$
on an appropriate domain of functions $\varphi(\lambda)$. 

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3 Multiple Observations

Now we consider the problem of estimation of the shift parameter $\theta \in \mathbb{R}$ in the family of states

$$ S_\theta \otimes_n = S_\theta \otimes \ldots \otimes S_\theta $$  \hspace{1cm} (14)

in the Hilbert space $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ ($n$-fold tensor product which corresponds to $n$ independent observations). Here $S_\theta = e^{-i\theta A}S e^{i\theta A}$, where $S = |\psi\rangle\langle\psi|$ is a pure state.

The family (14) is covariant with respect to the unitary representation of the group of shifts of $\mathbb{R}$

$$ \theta \rightarrow \exp(-i\theta A^{(n)}), \quad A^{(n)} = A \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes A $$

in $\mathcal{H}^{\otimes n}$. The corresponding direct integral decomposition reads

$$ \mathcal{H}^{\otimes n} = \int \oplus \mathcal{H}^{(n)}(\lambda) d\lambda, $$

where

$$ \mathcal{H}^{(n)}(\lambda) = \int \oplus [\mathcal{H}(\lambda_1) \otimes \ldots \otimes \mathcal{H}(\lambda_{n-1}) \otimes \mathcal{H}(\lambda - \lambda_1 - \ldots - \lambda_{n-1})] d\lambda_1 \ldots d\lambda_{n-1}. $$

For symmetry of notations we shall also denote this as

$$ \int_{\Gamma(\lambda)} \oplus [\mathcal{H}(\lambda_1) \otimes \ldots \otimes \mathcal{H}(\lambda_n)] d^{n-1} \sigma, $$

where

$$ \Gamma(\lambda) = \{ (\lambda_1, \ldots, \lambda_n) : \lambda_1 + \ldots + \lambda_n = \lambda \}. $$

Then $|\psi^{\otimes n}\rangle = \int \oplus |\psi^{(n)}(\lambda)\rangle d\lambda$, where

$$ |\psi^{(n)}(\lambda)\rangle = \int_{\Gamma(\lambda)} \oplus [ |\psi(\lambda_1)\rangle \otimes \ldots \otimes |\psi(\lambda_n)\rangle ] d^{n-1} \sigma, $$

so that

$$ \| \psi^{(n)}(\lambda) \|^2 = \int_{\Gamma(\lambda)} \oplus [ \| \psi(\lambda_1) \|^2 \otimes \ldots \otimes \| \psi(\lambda_n) \|^2 ] d^{n-1} \sigma. $$  \hspace{1cm} (15)
With these relations in mind, the optimal covariant observable is given by the formula (10)

\[ M^{(n)}_*(dx) = \left[ \frac{e^{ix(x-\lambda)} |\psi^{(n)}(\lambda)| \langle \psi^{(n)}(\lambda') | \rangle}{\|\psi^{(n)}(\lambda)\| \|\psi^{(n)}(\lambda')\|} \right] \frac{dx}{2\pi} \tag{16} \]

However in this case the projection

\[ P^{(n)}_* := \int M^{(n)}_*(dx) = \left[ \delta(x - \lambda) \frac{|\psi^{(n)}(\lambda)| \langle \psi^{(n)}(\lambda') | \rangle}{\|\psi^{(n)}(\lambda)\| \|\psi^{(n)}(\lambda')\|} \right], \quad n > 1, \tag{17} \]

cannot be equal to the identity operator: in any case it projects onto a subspace \( \mathcal{H}^{(n)}_* \) lying in the subspace of vector functions symmetrically depending on \( \lambda_1, \ldots, \lambda_n \).

Let us denote \( p(\lambda) = \|\psi(\lambda)\|^2 \) and \( p^{(n)}(\lambda) = \|\psi^{(n)}(\lambda)\|^2 \) the probability densities of the observables \( A \) and \( A^{(n)} \) in the states \( S \) and \( S^{\otimes n} \) respectively. Then (15) implies that \( p^{(n)}(\lambda) \) is the \( n \)-th convolution of \( p(\lambda) \), which we denote as \( p^{(n)}(\lambda) = p(\lambda)^n \). We assume that \( p(\lambda) \) and hence \( p^{(n)}(\lambda) \) are differentiable functions. Especially useful in this context is the concept of weak differentiability [13]: the probability density \( p(\lambda) \) is \textit{weakly differentiable} if there exists a function \( s(\cdot) \in L^2(p) \), such that for all \( f \) with \( \int |f(\lambda)|^2 p(\lambda + \theta)d\lambda < \infty \) the function \( g(\theta) = \int f(\lambda)p(\lambda + \theta)d\lambda \) has a derivative \( g'(\theta) = \int f(\lambda)s(\lambda + \theta)p(\lambda + \theta)d\lambda \).

To get more insight into the structure of the optimal covariant observable for \( n > 1 \), let us consider it in the situation of the Example. Then \( \mathcal{H}^{\otimes n} \) is isomorphic to \( L^2(\mathbb{R}^n) \), and \( A^{(n)} \) is just the operator of multiplication by \( \lambda = \lambda_1 + \ldots + \lambda_n \).

**Theorem 2.** The optimal covariant observable (16) is the spectral measure of the selfadjoint operator

\[ Q^{(n)}_* = P^{(n)}_* \frac{1}{n} (Q_* \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes Q_*) P^{(n)}_* \tag{18} \]

Observable \( \frac{1}{n} (Q_* \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes Q_*) \) corresponds to a “semi-classical” method of estimation, when the optimal quantum estimates for each of \( n \) components in \( \mathcal{H}^{\otimes n} \) are found and then used in a classical way to obtain the average over \( n \) observations. The theorem shows that projecting this average onto \( \mathcal{H}^{(n)}_* \) (and thus introducing entanglement) gives the optimal quantum estimate for \( n \) observations.
Proof. As in the Example, one shows that (16) is orthogonal resolution of the identity. As follows from (17) the subspace $\mathcal{H}_n = P_n^* \mathcal{H}$ consists of the functions of the form

$$\varphi(\lambda_1, \ldots, \lambda_n) = \frac{c(\lambda)}{\sqrt{p_n(\lambda)}} \psi(\lambda_1) \ldots \psi(\lambda_n),$$  \hspace{1cm} (19)

where $\int |c(\lambda)|^2 d\lambda = \|\varphi\|^2$. Consider the action on these functions of the selfadjoint operator

$$Q_n^* = \int x M_n^*(dx) = \left[ \int x \frac{e^{-ix\lambda} |\psi_n(\lambda)\rangle \langle \psi_n(\lambda')| e^{ix\lambda'} dx}{\|\psi_n(\lambda)\| \|\psi_n(\lambda')\|} \right] 2\pi.$$

(Notice a parallel between this expression and the Pitman formula

$$\theta^* = \frac{\int \theta \tilde{p}(x_1 - \theta) \ldots \tilde{p}(x_n - \theta) d\theta}{\int \tilde{p}(x_1 - \theta) \ldots \tilde{p}(x_n - \theta) d\theta}$$  \hspace{1cm} (20)

for the classical optimal covariant estimate of the shift parameter $\theta$ in the family $\{\tilde{p}(x_1 - \theta) \ldots \tilde{p}(x_n - \theta)\}$, see e.g. [17]). By using the analog of (13), one shows that for $\varphi \in \mathcal{H}_n$ \n
$$Q_n^* \varphi(\lambda_1, \ldots, \lambda_n) = i \frac{c'(\lambda)}{\sqrt{p_n(\lambda)}} \psi(\lambda_1) \ldots \psi(\lambda_n),$$

provided $c(\lambda)$ is absolutely continuous and $c'(\lambda)$ is square integrable. We have

$$c'(\lambda) = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j} c(\lambda_1 + \ldots + \lambda_n),$$

hence

$$Q_n^* \varphi(\lambda_1, \ldots, \lambda_n) = i \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j} \left[ \frac{c(\lambda_1 + \ldots + \lambda_n)}{\sqrt{p_n(\lambda_1 + \ldots + \lambda_n)}} \psi(\lambda_1) \ldots \psi(\lambda_n) \right]$$

$$- c(\lambda_1 + \ldots + \lambda_n) \frac{i}{n} \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j} \left[ \frac{\psi(\lambda_1) \ldots \psi(\lambda_n)}{\sqrt{p_n(\lambda_1 + \ldots + \lambda_n)}} \right].$$
Taking into account that \( \psi(\lambda_j) = \sqrt{p(\lambda_j)e^{i\alpha(\lambda_j)}} \) and performing differentiation, we obtain after some transformations

\[
Q^{(n)} \varphi(\lambda_1, \ldots, \lambda_n) = \sum_{j=1}^{n} \left( \frac{i}{n} \frac{\partial}{\partial \lambda_j} + \alpha'(\lambda_j) \right) \varphi(\lambda_1, \ldots, \lambda_n) + \frac{i}{2} F(\lambda_1, \ldots, \lambda_n) \varphi(\lambda_1, \ldots, \lambda_n),
\]

where

\[
F(\lambda_1, \ldots, \lambda_n) = \frac{[p^{(n)}(\lambda)]'}{p^{(n)}(\lambda)} - \frac{1}{n} \sum_{j=1}^{n} \frac{p'(\lambda_j)}{p(\lambda_j)}
\]

is a real function, or, briefly,

\[
Q^{(n)} \varphi = \frac{1}{n} \left( Q_\ast \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes Q_\ast \right) \varphi + \frac{i}{2} F \varphi, \quad \varphi \in \mathcal{H}^{(n)}_\ast.
\]

Taking inner product with \( \varphi \), and noticing that both \( Q^{(n)}_\ast \) and \( \frac{1}{n} \left( Q_\ast \otimes \ldots \otimes I + \ldots + I \otimes \ldots \otimes Q_\ast \right) \) are selfadjoint, we have \( \langle \varphi | F \varphi \rangle = 0 \), \( \varphi \in \mathcal{H}^{(n)}_\ast \), whence (18) follows.

Let us also show directly that \( \langle \varphi | F \varphi \rangle = 0 \), if \( \varphi \) is given by (19). We have

\[
\langle \varphi | F \varphi \rangle = \int \left\{ \frac{[p^{(n)}(\lambda)]'}{p^{(n)}(\lambda)} - \frac{1}{n} \sum_{j=1}^{n} \frac{p'(...
\]

But

\[
p^{(n)}(\lambda) = \int_{\Gamma(\lambda)} p(\lambda_1) \ldots p(\lambda_n) d^{n-1}\sigma \quad (22)
\]

\[
= p(\lambda)^n = \int p(\lambda_1) \ldots p(\lambda_{n-1}) p(\lambda - \lambda_1 - \ldots - \lambda_{n-1}) d\lambda_1 \ldots d\lambda_{n-1}.
\]
Differentiating the last equality with respect to \( \lambda \), we obtain

\[
[p^{(n)}(\lambda)]' = \int p(\lambda_1) \cdots p(\lambda_{n-1}) p'(\lambda - \lambda_1 - \cdots - \lambda_{n-1}) d\lambda_1 \cdots d\lambda_{n-1}
\]

\[
= \int_{\Gamma(\lambda)} \frac{\partial}{\partial \lambda_n} p(\lambda_1) \cdots p(\lambda_n) d^{n-1} \sigma,
\]

and similarly for all \( \lambda_j \). Hence

\[
[p^{(n)}(\lambda)]' = \frac{1}{n} \int_{\Gamma(\lambda)} \sum_{j=1}^{n} \frac{\partial}{\partial \lambda_j} p(\lambda_1) \cdots p(\lambda_n) d^{n-1} \sigma.
\]  

(23)

Taking into account (22), (23) shows that the inner integral in (21) is equal to zero. \( \square \)

4 The Limit Theorem

In this section we impose the regularity assumption:

(A) \( \psi \in D(A) \), or, equivalently, \( ||A\psi||^2 = \int \lambda^2 p(\lambda) d\lambda < \infty \), where \( p(\lambda) = ||\psi(\lambda)||^2 \) is the probability density of observable \( A \) in the state \( S \). Without loss of generality we assume \( E_\psi(A) = \langle \psi | A \psi \rangle = \int \lambda p(\lambda) d\lambda = 0 \), then \( D_S(A) = \int \lambda^2 p(\lambda) d\lambda \) is the variance of \( A \).

The uncertainty relation (5) together with (9) imply the lower bound for the variance of the optimal covariant estimate

\[
D^*_S = \int_R \left( \frac{d}{d\lambda} \sqrt{p(\lambda)^*} \right)^2 d\lambda \geq \frac{1}{4D^*_S(A^{(n)})} = \frac{1}{4nD_S(A)},
\]

with equality attained if and only if \( p(\lambda) \) is the Gaussian density. The quantity \( D^*_S \) represents the accessible minimum of variances of arbitrary (in general, entangled) estimates of the shift parameter, based on \( n \) independent observations.

By a variant of local central limit theorem [16],

\[
p_n(\lambda) := \sqrt{n} \sigma p(\sqrt{n} \sigma \lambda)^* \to p_0(\lambda) := \frac{1}{\sqrt{2\pi}} e^{-\lambda^2/2}
\]

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in the sense of $L^1$. Hence
\[
\int \left| \sqrt{p_0(\lambda)} - \sqrt{p_n(\lambda)} \right|^2 d\lambda \leq \int |p_0(\lambda) - p_n(\lambda)| d\lambda \to 0.
\]
Therefore the Fourier transform of $\sqrt{p_n(\lambda)}$ converges to that of $\sqrt{p_0(\lambda)}$ in $L^2$, and the probability density of the optimal covariant observable
\[
p^*_{S(n)}(x) = \frac{1}{2\pi} \left| \int e^{-ix\lambda} \sqrt{p(\lambda)n} d\lambda \right|^2
\]
satisfies the local limit theorem
\[
\int \left| \sqrt{\frac{2}{\pi} e^{-2x^2} - \frac{1}{\sqrt{n}\sigma} p^*_{S(n)} \left( \frac{x}{\sqrt{n}\sigma} \right)} \right| dx \to 0.
\]
Thus, under the assumption (A) the distribution of the optimal covariant observable $M_{s(n)}^*$ in the state $S_\theta$ is asymptotically normal with parameters $(\theta, \frac{1}{4n}\sigma^2)$. In this sense the bound of the uncertainty relation is asymptotically attainable. Moreover, by using the main result of [13] one has the asymptotic efficiency:

**Theorem 3.** If $p(\lambda)$ is weakly differentiable and $D_{S(n)}^*$ is finite for some $n$, then
\[
\lim_{n \to \infty} nD_{S(n)}^* = \frac{1}{4D_S(A)}.
\]

5 Semiclassical Estimation

Consider now the estimation strategy when the optimal covariant quantum estimate $M_{s(1)}(dx)$ is found for every of the $n$ components in the tensor product $H^{\otimes n}$, and then the classical estimation based on the obtained $n$ outcomes is made. The probability density of observable $M_{s(1)}(dx)$ in the state $S_\theta$ is
\[
\tilde{p}_\theta(x) = \frac{1}{2\pi} |\psi(x-\theta)|^2,
\]
where $\psi(x) = \int e^{ix\lambda} \|\psi(\lambda)\| d\lambda$. Under the assumption (A), $\psi(x)$ and hence $\tilde{p}_\theta(x)$ is differentiable, and for every unbiased estimate over $n$ observations the classical Cramér-Rao inequality holds:
\[
D_n \geq \left[ n \int \frac{|\tilde{p}_\theta(x)|^2}{\tilde{p}_\theta(x)} dx \right]^{-1}.
\]
Notice that
\[
\int |\tilde{p}_\theta(x)|^2 dx = 4 \int \frac{|\text{Re } \tilde{\psi}'(x) \tilde{\psi}(x)|^2}{|\tilde{\psi}(x)|^2} dx = 4 \int \left| \tilde{\psi}(x) \right|^2 dx
\]
One the other hand,
\[
\int \lambda^2 \|\psi(\lambda)\|^2 d\lambda = \int \left| \psi'(x) \right|^2 \frac{dx}{2\pi}.
\]
Comparing this with the asymptotically attainable quantum bound of the uncertainty relation, we have
\[
\int \left[ |\tilde{\psi}(x)|^2 \right] \frac{dx}{2\pi} = \int \left| \psi'(x) \right|^2 \frac{dx}{2\pi} = \int \left[ |\psi(x)|^2 \right] \frac{dx}{2\pi} + \int \left| \beta'(x) \psi(x) \right|^2 \frac{dx}{2\pi},
\]
if only \( \beta(x) := \text{arg } \psi(x) \neq \text{const.} \) Thus, under this condition, for \( n \) large enough
\[
n \min D_n > nD^*_S(n),
\]
where the minimum is over all unbiased classical estimates using unentangled quantum observables, which demonstrates superiority of the entangled quantum estimation.

### 6 Irregular Case

We now consider an instance of the Example where the regularity assumption (A) does not hold. Let \( S = |\psi\rangle \langle \psi| \) with
\[
\psi(\lambda) = \frac{1}{\sqrt{\pi a}} \frac{\sin a\lambda}{\lambda},
\]
so that \( D_S(A) = \infty \). In the coordinate representation this corresponds to the rectangular function
\[
\tilde{\psi}(x) = \int e^{-i\lambda x} \psi(\lambda) d\lambda = \begin{cases} \sqrt{\frac{2}{a}}, & \text{if } x \in [-a, a]; \\ 0, & \text{if } x \notin [-a, a]. \end{cases}
\]
that is, to the particle position uniformly distributed in \([-a,a]\), with the probability density \(\tilde{p}(x) = \frac{1}{2\pi} |\psi(x)|^2\) in \(\mathbb{R}\). Unfortunately the probability density of the optimal covariant observable \(Q^*\) which is

\[
p^*(x) = \frac{1}{2\pi} \left| \int e^{-i\lambda x} |\psi(\lambda)| d\lambda \right|^2,
\]

cannot be found in explicit form, therefore we shall consider semiclassical estimation based on unmodified position observable \(Q\) having the probability density \(\tilde{p}(x-\theta)\).

Turning to the case of \(n\) observations, we denote

\[
Q_1 = Q \otimes \ldots \otimes I, \ldots, Q_n = I \otimes \ldots \otimes Q.
\]

As is well known (see e. g. [2], n. 28.6), the Pitman estimate (20) in the case of the rectangular probability density \(\tilde{p}(x)\) has the form

\[
\theta_* = \frac{1}{2} \left[ \min (Q_1, \ldots, Q_n) + \max (Q_1, \ldots, Q_n) \right].
\]

Its variance is equal to

\[
D_S(\theta_*) = \frac{2a^2}{(n+1)(n+2)} \sim \frac{2a^2}{n^2},
\]

which shows faster decay than \(1/n\) characteristic to the regular case. The rectangular density is nondifferentiable violating the regularity assumption and making possible more efficient estimation than one which would follow from the Cramér-Rao bound (inapplicable in this case). Moreover, denoting by \(p_n(x)\) the probability density of \(\theta_*\), one has the limit law [2]

\[
\lim_{n \to \infty} n^{-1} p_n(x/n) = \frac{1}{2a} e^{-|x|/a}.
\] (24)

Now we shall find the asymptotics of the optimal quantum covariant estimate (18). Denoting by

\[
p(\lambda) = |\psi(\lambda)|^2 = \frac{1}{\pi a} \left( \frac{\sin a\lambda}{\lambda} \right)^2
\]
the probability density of $A$, and by $f(x) = \int e^{-i\lambda x} p(\lambda) d\lambda$, we have

$$f(x) = \begin{cases} 1 - \frac{|x|}{2a}, & \text{if } x \in [-2a, 2a]; \\ 0, & \text{if } x \notin [-a, a]. \end{cases}$$

Thus

$$p(\lambda)^n = \frac{1}{2\pi} \int e^{i\lambda x} f(x)^n dx,$$

whence

$$\lim_{n \to \infty} np_n(n\lambda)^n = \frac{1}{2\pi} \lim_{n \to \infty} \int e^{i\lambda x} f(x/n)^n dx$$

$$= \frac{1}{2\pi} \int \exp \left( i\lambda x - \frac{|x|}{2a} \right) dx = \frac{1}{2\pi a} \frac{1}{\lambda^2 + (2a)^{-2}}.$$

Taking into account that

$$\int e^{-i\lambda x} \frac{1}{\sqrt{\lambda^2 + (2a)^{-2}}} d\lambda = 2K_0 \left( \frac{|x|}{2a} \right),$$

where $K_0$ is the Macdonald (modified Bessel) function, and by using the limit law, we can show (see Appendix) that the renormalized probability density of the optimal observable

$$p_{S(n)}^* (x) = \frac{1}{2\pi} \left| \int e^{-i\lambda x} \sqrt{p(\lambda)^n} d\lambda \right|^2$$

obeys the limit law

$$\lim_{n \to \infty} \int \left| \frac{1}{n} p_{S(n)}^* \left( \frac{x}{n} \right) - \frac{2}{\pi a} K_0 \left( \frac{|x|}{2a} \right) \right|^2 dx = 0.$$ 

(25)

With some more effort we can show (see Appendix) that its variance satisfies

$$\lim_{n \to \infty} n^2 D_{S(n)}^* = \lim_{n \to \infty} \int_R \left( \frac{d}{d\lambda} \sqrt{np(n\lambda)^n} \right)^2 d\lambda$$

$$= \frac{1}{2\pi a} \int_R \left( \frac{d}{d\lambda} \frac{1}{\sqrt{\lambda^2 + (2a)^{-2}}} \right)^2 d\lambda = \frac{a^2}{2}, \quad (26)$$

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which is four times less than for the semiclassical estimate $\theta^*$. Moreover, $K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ for large positive $x$, whence

$$\frac{2}{\pi a} \left| K_0 \left( \frac{|x|}{2a} \right) \right|^2 \sim \frac{2}{|x|} e^{-\frac{|x|}{a}},$$

which shows that the tails of the asymptotic distribution of the optimal estimate has somewhat faster decay than (24).

7 Appendix

Proof of (26). Denoting $f_0(x) = e^{-|x|/2a}$, $p_0(\lambda) = \frac{1}{2\pi a} \frac{1}{\lambda^2 + (2a)^2}$,

$$\Delta f_n(x) = f_0(x) - f(x/n)^n; \quad \Delta p_n(\lambda) = p_0(\lambda) - np(n\lambda)^n,$$

we have

$$\Delta p_n(\lambda) = \frac{1}{2\pi} \int e^{i\lambda x} \Delta f_n(x) dx.$$  \hfill (27)

Now we observe that

$$\lim_{n \to \infty} \int_{x \neq 0} \left| \Delta f_n(x)^{(k)} \right| |x|^l \, dx = 0; \quad k, l = 0, 1, \ldots.$$  \hfill (27)

From (27) it follows in particular that

$$\max_{\lambda} |\Delta p_n(\lambda)| \leq \frac{1}{2\pi} \int |\Delta f_n(x)| \, dx \to 0, \quad \text{as } n \to \infty. \quad \hfill (28)$$

Let us also estimate the tails of $\Delta p_n(\lambda)$. Taking into account that $\Delta f_n(0) = (\Delta f_n(0))' = 0$, and making twice integration by parts in (27), we obtain

$$|\Delta p_n(\lambda)| \leq \frac{1}{\lambda^2} \int_{x \neq 0} \left| \Delta f_n(x)^{''} \right| \, dx = \frac{\varepsilon_n}{\lambda^2}, \quad \hfill (29)$$

where $\lim_{n \to \infty} \varepsilon_n = 0$. In the same way we obtain

$$\max_{\lambda} \left| \frac{d}{d\lambda} \Delta p_n(\lambda) \right| \leq \frac{1}{2\pi} \int |x \Delta f_n(x)| \, dx \to 0, \quad \text{as } n \to \infty. \quad \hfill (30)$$
and
\[ \left| \frac{d}{d\lambda} \Delta p_n(\lambda) \right| \leq \frac{1}{|\lambda|^3} \int_{x \neq 0} \left| (x \Delta f_n(x))''' \right| \, dx = \frac{\varepsilon_n'}{|\lambda|^3}. \quad (31) \]

From (28), (29) it follows that \( \int |\Delta p_n(\lambda)| \, d\lambda \to 0 \) as \( n \to \infty \), hence, arguing as before theorem 3, we have \( \sqrt{np(n\lambda)^m} \to \sqrt{p_0(\lambda)} \) in \( L^2 \), implying (25).

The estimates (28), (29), (30), (31) together with
\[ p_0(\lambda) \geq \frac{c}{1 + |\lambda|^2}; \quad \left| \frac{d}{d\lambda} p_0(\lambda) \right| \geq \frac{c}{1 + |\lambda|^3} \]

imply
\[ \left| \left( \frac{d}{d\lambda} \sqrt{p_0(\lambda)} \right)^2 - \left( \frac{d}{d\lambda} \sqrt{p_n(\lambda)} \right)^2 \right| = \frac{1}{4} \left| \frac{p_0'(\lambda)}{p_0(\lambda)} - \frac{p_n'(\lambda)}{p_n(\lambda)} \right|^2 \leq \frac{\varepsilon_n''}{1 + |\lambda|^2}, \]

whence (26) follows.

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References

[1] Barndorff-Nielsen O. E., Gill R. D., Jupp P. E. On quantum statistical inference. J. Royal Statist. Soc. B, 65, 1-31, 2003.
[2] Cramér H. Mathematical Methods of Statistics. Stockholm, 1946.

[3] D’Ariano G. M. Homodyning as universal detection. In: Quantum Communication, Computing and Measurement. Eds. Hirota O., Holevo A. S., Caves C. M. New York: Plenum Press, pp. 253-264, 1997; LANL e-print quant-ph/9701011.

[4] Derka A., Buzek V., Ekert A. Universal algorithm for optimal estimation of quantum states from finite ensembles; LANL e-print quant-ph/9707028.

[5] Gill R. D., Massar S. State estimation for large ensembles. Phys. Rev. A 61, 042312/1-16, 2000; LANL e-print quant-ph/9902063.

[6] Gill R., Guta M. I. An invitation to quantum tomography. LANL e-print quant-ph/0303020.

[7] Hayashi M. Asymptotic estimation theory for a finite dimensional pure state model. J. Phys. A 31, 4633-4655, 1998.

[8] Helstrom C.W. Quantum Detection and Estimation Theory, Acad. Press, New York, 1976. Russian translation: Moscow Mir, 344 pp (1979)

[9] Holevo A.S. Probabilistic and Statistical Aspects of Quantum Theory, Moscow, Nauka, 1980. English translation: North Holland, Amsterdam, (1982)

[10] Holevo A. S. Bounds for generalized uncertainty of shift parameter. Lect. Notes Math., 1021, 243-251, 1983.

[11] Holevo A. S. Generalized imprimitivity systems for Abelian groups, Izv VUZ. Matematika, N2, 49-71, 1983. English translation: J. Soviet Math., 53-80.

[12] Holevo A. S. Statistical Structure of Quantum Theory. Lect. Notes Phys. m67, Springer-Verlag: New York-Heidelberg-Berlin, 2001.

[13] Johnson O., Barron A. Fisher information inequalities and the central limit theorem. Preprint, 2003.
[14] Keyl M., Werner R. F. Estimating the spectrum of a density operator. Phys. Rev. A, 64, no.5, 052311, 2001.

[15] Massar S., Popescu S. Optimal extraction of information from finite quantum ensembles. Phys. Rev. Lett. 74, 1259-1263, 1995.

[16] Prokhorov Yu. V. Local theorem for densities. Doklady AN SSSR. 83, 797-800, 1952. (rus)

[17] Rukhin A. L. Some statistical and probabilistic problems on groups. Proc. Steklov Mathematical Institute. 111, 52-109, 1970. (rus)