A COMBINATORIAL CALCULATION OF THE LANDAU-GINZBURG MODEL $M = \mathbb{C}^3, W = z_1z_2z_3$

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Abstract. The aim of this paper is to apply ideas from the study of Legendrian singularities to a specific example of interest within mirror symmetry. We calculate the Landau-Ginzburg A-model with $M = \mathbb{C}^3, W = z_1z_2z_3$ in its guise as microlocal sheaves along the natural singular Lagrangian thimble $L = \text{Cone}(T^2) \subset M$. The description we obtain is immediately equivalent to the $B$-model of the pair-of-pants $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ as predicted by mirror symmetry.

Contents

1. Introduction 1
   1.1. Acknowledgements 3

2. Arborealization lemma 3
   2.1. Preliminaries 3
   2.2. Statement and proof 5
   2.3. Toy application 5

3. Landau-Ginzburg model 7
   3.1. Background 7
   3.2. Superpotential 8
   3.3. Mirror symmetry 8

References 10

1. Introduction

The aim of this paper is to apply ideas from study of Legendrian singularities to a specific example of interest within mirror symmetry. In the papers [12, 13, 14], we introduced a class of Legendrian singularities of a simple combinatorial nature, called arboreal singularities for their relations to trees, then presented an algorithm to deform any Legendrian singularity to a nearby Legendrian with arboreal singularities. Furthermore, we showed that the category of microlocal sheaves on the original Legendrian singularity is equivalent to that on the nearby Legendrian.

We will not directly use the above theory, but will rather extract and apply one of its key constructions, formalized in Lemma 2.3 below. It enables the calculation of microlocal sheaves on an $n$-dimensional Legendrian singularity in terms of constructible sheaves on an $n$-dimensional ball. For general Legendrian singularities, an inductive application of this construction leads to the algorithm presented in [14]. But for Legendrian singularities that are simply cones over smooth manifolds, a single application suffices to give appealing results.

In this paper, we will implement this on a single example: the Landau-Ginzburg A-model with background $M = \mathbb{C}^3$ and superpotential $W = z_1z_2z_3$. (Natural generalizations will appear in forthcoming work [15].) Beyond the specific calculation, our aim is to advocate for the
elementary tools used to describe the symplectic geometry. Due to the fact that the critical locus \( \{ dW = 0 \} \subset M \) is not smooth or proper, there has not yet appeared a definitive account of this Landau-Ginzburg A-model. Nevertheless, it should not be too complicated since it is expected to be mirror to the B-model of the pair-of-pants \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \). For further discussion, we recommend the beautiful paper [2] establishing mirror symmetry in the other direction, between the Landau-Ginzburg B-model with \( M = \mathbb{C}^3 \), \( W = z_1 z_2 z_3 \) and the A-model of \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \). (There is also work in progress [1] pursuing results parallel to those of this paper but from a more traditional perspective.)

Our starting point will be the following viewpoint on the A-model. The superpotential \( W = z_1 z_2 z_3 \) suggests a natural conic Lagrangian skeleton

\[
L = \{(z_1, z_2, z_3) \in M \mid W(z_1, z_2, z_3) \in \mathbb{R}_{\geq 0}, |z_1| = |z_2| = |z_3| \} \subset M
\]

which can be regarded as a singular thimble over a nearby vanishing two-torus \( T^2 \subset W^{-1}(1) \). We would like to study a model of \( A \)-branes running along \( L \) (as found in the infinitesimal Fukaya-Seidel category [10]) or transverse to \( L \) (as found in the partially wrapped Fukaya category [3, 4]). For the moment, let us suggest a model for the first version, but see Remark 1.4 for the modifications that lead to a model of the second.

**Ansatz 1.1.** The Landau-Ginzburg A-model of \( M = \mathbb{C}^3 \) with superpotential \( W = z_1 z_2 z_3 \) is given by the dg category of microlocal sheaves along the conic Lagrangian skeleton \( L \subset M \).

**Remark 1.2.** The ansatz is compatible with the broad expectation that given \( L \subset M \) a conic Lagrangian skeleton of an exact symplectic manifold, there should be many equivalent approaches to its “quantum category” of \( A \)-branes: the Floer-Fukaya-Seidel theory of Lagrangian intersections and pseudo-holomorphic disks [16] (analysis); the Kashiwara-Schapira [11] theory of microlocal sheaves [11] (topology); and the theory of holonomic modules over deformation quantizations, exemplified by \( D \)-modules [9]. The close relation between the last two stems from the Riemann-Hilbert correspondence [2, 8, 9, 10], and there are numerous confirmations of their close relation with the first, including proposals within the context of mirror symmetry [17] to regard A-branes as microlocal sheaves as in the ansatz.

Now we can state the main theorem of this paper. It is a combinatorial calculation of microlocal sheaves along the conic Lagrangian skeleton \( L \subset M \) leading to a verification of its expected mirror symmetry with the B-model of the pair-of-pants \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \).

Fix a field \( k \) of characteristic zero.

Introduce the base space \( X = \mathbb{R}^3 \times \mathbb{R} \), its cotangent bundle \( T^*X \), and its spherically projectivized cotangent bundle \( S^*X = (T^*X \setminus X)/\mathbb{R}_{>0} \).

We explain in Section 3 how standard constructions identify the conic Lagrangian skeleton \( L \subset M \) with a Legendrian subvariety \( \Lambda \subset S^*X \).

Let \( \mu \mathcal{Sh}_\Lambda(X) \) denote the dg category of microlocal sheaves of \( k \)-vector spaces supported along the Legendrian subvariety \( \Lambda \subset S^*X \). See [11] and the discussion of Section 2 for a precise definition.

Let \( \text{Coh}_{\text{tors}}(\mathbb{P}^1 \setminus \{ 0, 1, \infty \}) \) denote the bounded dg category of finitely-generated torsion complexes on \( \mathbb{P}^1 \setminus \{ 0, 1, \infty \} \). It is nothing more than the dg enhancement of the bounded derived category of finite-dimensional \( k \)-vector spaces \( V \) equipped with an automorphism \( m : V \to V \) such that 1 is not an eigenvalue of \( m \).

**Theorem 1.3** (see Theorem 3.2). There is an equivalence

\[
\mu \mathcal{Sh}_\Lambda(X) \simeq \text{Coh}_{\text{tors}}(\mathbb{P}^1 \setminus \{ 0, 1, \infty \})
\]
Remark 1.4. This comment falls within the “categorical functional analysis” underpinning observed dualities between infinitesimal Fukaya-Seidel categories and partially wrapped Fukaya categories, and similarly between properly-supported coherent complexes and perfect complexes (as established in [5]).

We have selected the formulation of the theorem from among a collection of closely related assertions. It is the version that matches with the definition of microlocal sheaves that includes the traditional local finiteness of constructibility. Alternatively, one can take the left hand side to consist of arbitrarily large complexes but nevertheless with the prescribed singular support. Our arguments then provide an equivalence of this with a right hand side comprising all quasi-coherent complexes QCoh($\mathbb{P}^1 \setminus \{0,1,\infty\}$). Going further, for example by passing to compact objects, one can arrive at a version involving specifically coherent complexes Coh($\mathbb{P}^1 \setminus \{0,1,\infty\}$).

Taking a step back, our constructions offer a simple combinatorial model of the geometry of the Lagrangian skeleton that could be substituted for any prior definition of the left hand side.

1.1. Acknowledgements. I thank D. Auroux and E. Zaslow for suggesting the application studied in this paper. I also thank them as well as D. Ben-Zvi, M. Kontsevich, J. Lurie, N. Rozenblyum, V. Shende, N. Sheridan, D. Treumann, and H. Williams for their interest, encouragement, and valuable comments. Finally, I am grateful to the NSF for the support of grant DMS-1502178.

2. Arborealization lemma

2.1. Preliminaries. Let $X$ be a real analytic manifold, with cotangent bundle $T^*X \to X$, and spherically projectivized cotangent bundle $\pi : S^*X = (T^* \setminus X)/\mathbb{R}_{>0} \to X$. For convenience, fix a Riemannian metric on $X$, so that in particular $S^*X$ is identified with the unit cosphere bundle $U^*X \subset T^*X$.

Let $\Lambda \subset S^*X$ be a closed Legendrian subvariety with image $H = \pi(\Lambda) \subset X$. We will always work in the generic situation where the projection $\pi|_\Lambda : \Lambda \to H$ is finite so that $H \subset X$ is a hypersurface. For convenience, fix $S = \{X_\alpha\}_{\alpha \in A}$ a Whitney stratification of $X$ such that $H \subset X$ is a union of strata. Thus in particular $\Lambda \subset S_\delta^*X = \coprod_{\alpha \in A} S^*_\alpha X$, where $S^*_\alpha X \subset S^*X$ denotes the spherically projectivized conormal bundle to the stratum $X_\alpha \subset X$.

Fix a field $k$ of characteristic zero. Let $Sh(X)$ denote the dg category of constructible complexes of sheaves of $k$-vector spaces on $X$. Let $Sh_S(X) \subset Sh(X)$ denote the full dg subcategory of complexes constructible with respect to $S$. Recall to any $F \in Sh(X)$, we can assign its singular support $ss(F) \subset S^*X$ which is a closed Legendrian subvariety. Let $Sh_\Lambda(X) \subset Sh(X)$ denote the full dg category of complexes with singular support in $\Lambda$. The inclusion $\Lambda \subset S_\delta^*X$ implies the full inclusion $Sh_\Lambda(X) \subset Sh_S(X)$.

Let $x \in X$ be a point. Let $B_x(\rho), C_x(\rho) \subset X$ be the open ball and sphere of radius $\rho > 0$ centered at $x \in X$. For small enough $\rho > 0$, the sphere $C_x(\rho) \subset X$ will be transverse to the stratification $S = \{X_\alpha\}_{\alpha \in A}$ in the sense that it is transverse to each stratum $X_\alpha \subset X$.

Fix small enough $\rho_2 > \rho_1 > 0$, and for simplicity set $B = B_x(\rho_2), C = C_x(\rho_1)$. Let $S_B = \{X_\alpha \cap B\}_\alpha, S_C = \{X_\alpha \cap C\}_\alpha$ denote the respective induced Whitney stratifications.

Let $\Lambda_B = \Lambda \times X B \subset S^*B$ denote the induced closed Legendrian subvariety. The inclusion $C \subset X$ induces a correspondence

$$S^*X \leftarrow\kern-0.8cm\leftarrow \quad S^*X \times_X C \quad \rightarrow\kern-0.8cm\rightarrow \quad S^*C$$

Let $\Lambda_C = q(p^{-1}(\Lambda)) \subset S^*C$ denote the induced closed Legendrian subvariety.

Restriction along the inclusion $i_C : C \to X$ provides functors

$$i_C^* : Sh_S(X) \xrightarrow{\sim} Sh_{S,C}(C) \quad \quad \quad i_C^* : Sh_\Lambda(X) \xrightarrow{\sim} Sh_{\Lambda,C}(C)$$
Suppose in addition the fiber $\Lambda_x = \Lambda \times_X \{x\}$ is a single codirection. Let $\mu Sh_{\Lambda_B}(B)$ denote the dg category of microlocal sheaves on $B$ supported along $\Lambda_B$. Within our working context, it admits two simple equivalent descriptions. (In general, the induced map $\pi_0(\Lambda_x) \to \pi_0(\Lambda_B)$ is a bijection, and one can treat each connected component of $\Lambda_B$ separately.)

On the one hand, the natural projection functor is an equivalence

$$Sh_{\Lambda_B}(B)/\mathcal{Loc}(B) \xrightarrow{\sim} \mu Sh_{\Lambda_B}(B)$$

where $\mathcal{Loc}(B) \subset Sh(B)$ denotes the full dg subcategory of local systems, or in other words complexes with empty singular support.

On the other hand, let $i_x : \{x\} \to X$ denote the inclusion. Let $Sh(B)^0 \subset Sh(X)$ denote the full dg subcategory of $F \in Sh(X)$ such that $i_x^* F \simeq 0$. Let $Sh_{\Lambda_B}(B)^0 \subset Sh_{\Lambda_B}(B)$ denote the full dg subcategory of $F \in Sh_{\Lambda_B}(B)$ such that $i_x^* F \simeq 0$. Alternatively, let $\Gamma_c : Sh(B) \to \text{Mod}_k$ denote the functor of global sections with compact support. Under the natural identification $Sh(\{x\}) \simeq \text{Mod}_k$, for any $F \in Sh_{\mathcal{S}h_B}(B)$, in particular for $F \in Sh_{\Lambda_B}(B)$, there is a natural equivalence $i_x^* F \simeq \Gamma_c(B, F)$, and hence $Sh_{\Lambda_B}(B)^0 \subset Sh_{\Lambda_B}(B)$ is also the full dg subcategory of $F \in Sh_{\Lambda_B}(B)$ such that $\Gamma_c(B, F) \simeq 0$. Finally, the natural projection functor is an equivalence

$$Sh_{\Lambda_B}(B)^0 \xrightarrow{\sim} \mu Sh_{\Lambda_B}(B)$$

**Example 2.1.** Let $H \subset X$ be a smooth hypersurface given as the zero-locus of a submersion $f : X \to \mathbb{R}$. Let $\Lambda \subset S^*X$ be the smooth Legendrian subvariety given by the codirection of the differential $df$ along $H$. Thus $\Lambda \subset S^*X$ is one of the two connected components of $S^*_H X \subset S^*X$, and the restriction $\pi|_{\Lambda} : \Lambda \to H$ is a diffeomorphism.

Let $x \in H \subset X$ be a point, and $B \subset X$ a small ball around $x$. Let $j : B_+ \to B$ be the inclusion of the open subset where $f > 0$. Then $Sh_{\Lambda_B}(B)$ is generated by the constant sheaf $k_B$ and the standard extension $j_* k_{B+}$. The only nontrivial morphism between them is the canonical map $k_B \to j_* j^* k_B \simeq j_* k_{B+}$. Thus $Sh_{\Lambda_B}(B)$ is equivalent to the dg category of finitely-generated complexes of $k$-modules over the directed $A_2$-quiver

$$\bullet \longrightarrow \bullet$$

Furthermore, $Sh_{\Lambda_B}(B)^0$ is generated by the standard extension $j_* k_{B+}$. Thus $Sh_{\Lambda_B}(B)^0$ is equivalent to the dg category of finitely-generated complexes of $k$-modules. Note that for $F \in Sh_{\Lambda_B}(B)^0$, the equivalence is realized by taking the stalk of $F$ at any point of $B_+$.

**Example 2.2.** For $a = 1, 2$, let $H_a \subset X$ be a smooth hypersurface given by the zero-locus of a submersion $f_a : X \to \mathbb{R}$. Let $\Lambda_a \subset S^*X$ be the smooth Legendrian subvariety given by the codirection of the differential $df_a$ along $H_a$. Thus $\Lambda_a \subset S^*X$ is one of the two connected components of $S^*_H X \subset S^*X$, and the restriction $\pi|_{\Lambda_a} : \Lambda_a \to H_a$ is a diffeomorphism.

Suppose in addition that $H_1$ and $H_2$ are transverse. Let $x \in H_1 \cap H_2 \subset X$ be a point, and $B \subset X$ a small ball around $x$. For $a = 1, 2$, let $j_i : B_{a+} \to B$ be the inclusion of the open subset where $f_a > 0$. Then $Sh_{\Lambda_B}(B)$ is generated by the constant sheaf $k_B$ and the standard extensions $j_* k_{B_{a+}}$, for $a = 1, 2$. The only nontrivial morphisms between them are the canonical maps $k_B \to j_a j_a^* k_B \simeq j_a k_{B_{a+}}$, for $a = 1, 2$. Thus $Sh_{\Lambda_B}(B)$ is equivalent to the dg category of finitely generated complexes of $k$-modules over the directed $A_3$-quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

Furthermore, $Sh_{\Lambda_B}(B)^0$ is generated by the standard extensions $j_* k_{B_{a+}}$, for $a = 1, 2$. Thus $Sh_{\Lambda_B}(B)^0$ is equivalent to the direct sum of two copies of the dg category of finitely generated complexes of $k$-modules. Note that for $F \in Sh_{\Lambda_B}(B)^0$, the equivalence is realized by taking the stalks $F_{b_1}, F_{b_2}$ at any points $b_1 \in B_{1+} \setminus (B_{1+} \cap \overline{B}_{2+})$, $b_2 \in B_{2+} \setminus (B_{2+} \cap \overline{B}_{1+})$ respectively.
Moreover, the stalk $F_b$ at any point $b \in B_{1+} \cap B_{2+}$ comes equipped with a canonical presentation as the direct sum $F_b \simeq F_{b_1} \oplus F_{b_2}$.

2.2. **Statement and proof.** Let us continue with the above setup.

Now we can state a simple-minded but effective construction (at the heart of the arborealization algorithm of [14] for calculating $\mu$-stabilization algorithm of [14$^+$]) for calculating $F$ as the direct sum

$$\oplus \Lambda_{b_1} \oplus \Lambda_{b_2}.$$

Moreover, the stalk $F_b$ at any point $b \in B_{1+} \cap B_{2+}$ comes equipped with a canonical presentation as the direct sum $F_b \simeq F_{b_1} \oplus F_{b_2}$.

Recall that we have assumed the fiber $\Lambda_{|x} = \Lambda \times X \{x\}$ is a single codirection. Fix a smooth function $f : X \to \mathbb{R}$ such that $df_x \neq 0$ and represents the codirection $\Lambda_{|x}$. Then for small enough $\rho_2 > \rho_1 > 0$ as above, within the ball $B = B_x(\rho_2)$, the hypersurface $H = \pi(\Lambda)$ will lie arbitrarily close to the zero-locus of $f$.

Choose a point $c \in C = C_x(\rho_1)$ such that $f(c) < e$ for any $e \in f(H \cap B)$. Let $A = C \setminus \{c\}$ be the open ball, and note that $\dim A = \dim B - 1$. Write $\Lambda_A \subset S^* A$ for the closed Legendrian subvariety $\Lambda_{C} \subset S^* C$ regarded within $S^* A = S^* C \times_{C} A$. Let $Sh\Lambda_{A}(A)_c \subset Sh\Lambda_{A}(A)$ denote the full dg subcategory of complexes with compact support.

Recall that for $F \in Sh\Lambda_{B}(B)^0$, the support of $F$ will be disjoint from $c$.

**Lemma 2.3 (Arborealization Lemma).** Restriction induces an equivalence

$$\mu Sh\Lambda_B(B) \simeq Sh\Lambda_B(B)^0 \sim Sh\Lambda_A(A)_c.$$

**Proof.** Let $j_U : U = B \setminus \{x\} \to B$ denote the open inclusion.

For $F \in Sh(B)$, there is a natural triangle

$$i_{\ast} ! F \xrightarrow{\sim} F \xrightarrow{j_U \ast j_U^\ast F}$$

For $F \in Sh(B)^0_0$, the left term vanishes so there is a natural equivalence

$$F \sim j_U \ast j_U^\ast F$$

or in other words, pushforward induces an equivalence

$$j_U : Sh\Lambda_B(U) \sim Sh\Lambda_B(B)^0.$$

Thus the assertion follows from the evident restriction equivalence

$$Sh\Lambda_U(U) \sim Sh\Lambda_A(C)$$

and the fact that the support of any $F \in Sh\Lambda_B(B)^0_0$ is disjoint from $c \in C$. 

2.3. **Toy application.** Here we will apply Lemma 2.3 in a simple toy situation, but one which will arise subsequently in the proof of Theorem 3.2.

Let $D \subset \mathbb{R}^3$ denote the two-dimensional open disk

$$D = \{(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \mid \theta_1^2 + \theta_2^2 + \theta_3^2 < 1, \theta_1 + \theta_2 + \theta_3 = 0\}$$

Let $W \subset D$ denote the subset where at least one of the coordinates $\theta_1$, $\theta_2$, or $\theta_3$ is zero. Note that $W$ is the union of three lines intersecting at the origin.

Introduce the maps given by

$$f : D \to \mathbb{R}^2 \quad f(\theta_1, \theta_2, \theta_3) = (\cos(\theta_1) - \cos(\theta_2), \cos(\theta_2) - \cos(\theta_3))$$

$$g : D \to \mathbb{R} \quad g(\theta_1, \theta_2, \theta_3) = \sum_{a=1}^{3} \cos(\theta_a) \sin(\theta_a)$$

$$F : D \to \mathbb{R}^3 \quad F = (f, g)$$

Here are some of their easily-verified properties:
(1) $f$ is the quotient map for the $\mathbb{Z}/2$-action: $(\theta_1, \theta_2, \theta_3) \mapsto (-\theta_1, -\theta_2, -\theta_3)$;

(2) $g$ is odd: $g(-\theta_1, -\theta_2, -\theta_3) = -g(\theta_1, \theta_2, \theta_3)$;

(3) $W$ is the zero-locus of $g$, and $g$ is a submersion along $W$.

Property (1) implies $f$, and hence $F$, is an immersion over $D \setminus \{(0, 0, 0)\}$. Properties (2) and (3) then imply $F$ is an embedding over $D \setminus W$ and self-transverse along $W$.

Set $H = F(D) \subset \mathbb{R}^3$, and $H^o = H \setminus \{(0, 0, 0)\} \subset \mathbb{R}^3$. Note that $H^o \subset \mathbb{R}^3$ is immersed with a consistent codirection that is positive on the last coordinate vector field $\partial/\partial x_3$. Let $\Lambda^o \subset S^*\mathbb{R}^3$ denote the corresponding smooth Legendrian submanifold.

It is straightforward to check that the closure $\Lambda = \overline{\Lambda^o} \subset S^*\mathbb{R}^3$ is a smooth Legendrian submanifold with fiber at the origin the single codirection $dx_3$ corresponding to the last coordinate.

Let $B \subset \mathbb{R}^3$ be a small ball around the origin, and let $\Lambda_B = \Lambda \times_{\mathbb{R}^3} B \subset S^*B$ denote the induced closed Legendrian subvariety.

Let us recall the further ingredients of Lemma 2.3 used to calculate the dg category of microlocal sheaves $\mu Sh\Lambda_B(B)$. (In the abstract, this is silly: general theory tells us that since $\Lambda_B$ is smooth, we will find $\mu Sh\Lambda_B(B)$ is equivalent to the dg category of finitely generated complexes of $k$-modules. But Lemma 2.3 will give an attractive presentation of it that will be compatible with its interaction with other constructions.)

Let $C \subset B$ be a smaller sphere around the origin, and let $c \in C$ be its intersection with the ray $\{(0, 0, -r)\} \subset \mathbb{R}^3$, for $r > 0$.

Let $A = C \setminus \{c\}$ be the open complementary disk, and let $\Lambda_A \subset S^*A$ denote the induced closed Legendrian subvariety.

It is straightforward to see that $\Lambda_A$ is diffeomorphic to a circle, and projects to the immersed “trefoil diagram” curve $K \subset A$ as pictured in Figure 1. Moreover, one recovers $\Lambda_A$ by taking the “inward pointing” codirection along $K \subset A$.

Applying Lemma 2.3 provides an equivalence

$$\mu Sh\Lambda_B(B) \simeq Sh\Lambda_B(B)^0 \xrightarrow{\sim} Sh\Lambda_A(A)_c$$

Figure 1. Disk $A$ with immersed “trefoil diagram” curve $K \subset A$. 
Now the calculation of $Sh_{\Lambda}(A)_c$ is appealingly simple. As pictured in Figure 1, the connected components of $A \setminus K$ consist of a compact contractible central component $U$, three compact contractible neighboring components $U_1, U_2, U_3$, and a non-compact complementary component. Any $F \in Sh_{\Lambda}(A)_c$ vanishes on the non-compact complementary component, and has a respective stalk $V$ and $V_1, V_2, V_3$ along each of the compact contractible components.

Similarly to Examples 2.1 and 2.2 we find that $Sh_{\Lambda}(A)_c$ is thus equivalent to the bounded derived category of the category of four finite-dimensional $k$-modules $V$ and $V_1, V_2, V_3$, with maps $V_1 \to V, V_2 \to V, V_3 \to V$ that induce a direct sum decomposition

$$V_a \oplus V_b \sim V,$$

for any distinct pair of indices $a \neq b$.

As expected, this is nothing more than the dg category of finitely-generated complexes of $k$-modules, since any such data is equivalent to the data of say $V_1$ alone (starting from the direct sum decomposition $V \simeq V_1 \oplus V_2$, we can view $V_3$ as the graph of an isomorphism $V_1 \simeq V_2$).

3. Landau-Ginzburg model

We apply here the constructions of Section 2 in particular adopting its notation, to calculate microlocal sheaves along the natural Lagrangian skeleton of the Landau-Ginzburg model $M = \mathbb{C}^3, \lambda = z_1 z_2 z_3$.

We begin by reviewing some basic constructions in the natural generality of the Landau-Ginzburg model $M = \mathbb{C}^n, \lambda = z_1 \cdots z_n$. For $n > 3$, its combinatorics are more involved and we postpone the analogous calculation of microlocal sheaves along its natural Lagrangian skeleton to [15].

3.1. Background. Let $M = \mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$, where we also write $z_a = x_a + iy_a = r_a e^{i \theta_a}$, for $a = 1, \ldots, n$. Equip $M$ with the exact symplectic form

$$\omega_M = \sum_{a=1}^n dx_a dy_a = \sum_{a=1}^n r_a dx_a d\theta_a$$

with primitive

$$\alpha_M = \sum_{a=1}^n (x_a dy_a - y_a dx_a) = \sum_{a=1}^n r_a^2 d\theta_a$$

The associated Liouville vector field $v_M = \sum_{a=1}^n r_a \partial_{x_a}$ generates the positive real dilations.

Let $Y = \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$, and introduce the exact Lagrangian fibration

$$p : M \longrightarrow Y \quad p(z_1, \ldots, z_n) = (x_1, \ldots, x_n)$$

There is the evident section $s : Y \to M, s(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$, and the function

$$f : M \longrightarrow \mathbb{R} \quad f(z_1, \ldots, z_n) = \sum_{a=1}^n x_a y_a$$

is a primitive for the restriction of $\alpha_M$ to each fiber of $p$.

Let $N = M \times \mathbb{R} = \mathbb{C}^n \times \mathbb{R}$ with coordinates $(z_1, \ldots, z_n, t)$. Equip $N$ with the contact form

$$\lambda_N = \alpha_M - dt = \sum_{a=1}^n (x_a dy_a - y_a dx_a) - dt = \sum_{a=1}^n r_a^2 d\theta_a - dt$$

and in particular the cooriented contact structure $\xi_N = \ker(\lambda_N) \subset TN$.

Let $X = \mathbb{R}^n \times \mathbb{R}$ with coordinates $(x_1, \ldots, x_n, t)$, and introduce the Legendrian fibration

$$q : N \longrightarrow X \quad q(z_1, \ldots, z_n, t) = (x_1, \ldots, x_n, t + \sum_{a=1}^n x_a y_a)$$

Let $T^* X \to X$ be the cotangent bundle of $X$, and $\pi : S^* X = (T^* \setminus X)/\mathbb{R}_{>0} \to X$ its spherically projectivized cotangent bundle. The Euclidean metric on $X$ provides an identification of $S^* X$ with the unit cosphere bundle $U^* X \subset T^* X$. 

Let us regard $\pi : S^*X \to X$ as a Legendrian fibration and compare it to the Legendrian fibration $q : N \to X$. By the Darboux Theorem for Legendrian fibrations, any points $n \in N$ and $\lambda \in S^*X$ admit open neighborhoods $U \subset N$ and $V \subset S^*X$ and a cooriented contactomorphism $\varphi : U \to V$ such that $\varphi(n) = \lambda$ and $\pi \circ \varphi = q$. Thus the contact geometry of a small ball $U \subset N$ centered at the origin $0 \in N$ may be recovered from the differential topology of a small ball $B \subset X$ centered at the origin $0 \in X$.

3.2. Superpotential. Introduce the superpotential

$$W : M \longrightarrow \mathbb{C} \quad W(z_1, \ldots, z_n) = z_1 \cdots z_n$$

Its critical locus $\{dW = 0\} \subset M$ is the union of the codimension two coordinate planes where two or more coordinates vanish. For $n \geq 3$, the critical locus is not proper. For $n = 1$, there are no critical values, and for $n \geq 2$, the critical values consist of the single point $0 \in \mathbb{C}$.

The fiber $T = W^{-1}(1)$ is a complex torus of rank $n - 1$ with respect to coordinate-wise multiplication, and the restriction $W|_{C \setminus \{0\}} : M|_{C \setminus \{0\}} \to \mathbb{C} \setminus \{0\}$ is likewise a $T$-bundle. Introduce the maximal compact torus

$$K = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in M \mid \theta_1 + \cdots + \theta_n = 0\} \subset T$$

and its positive real cone

$$L = \{(re^{i\theta_1}, \ldots, re^{i\theta_n}) \in M \mid \theta_1 + \cdots + \theta_n = 0, r \in \mathbb{R}_{\geq 0}\} \subset M$$

Lemma 3.1. $L \subset M$ is a conic Lagrangian subvariety.

Proof. Clearly $L$ is invariant under positive real dilations. Let $v = c_r \partial_r + c_1 \partial_{\theta_1} + \cdots + c_n \partial_{\theta_n}$, with $c_1 + \cdots + c_n = 0$, be a tangent vector to the smooth locus $L \setminus \{0\}$. Then $i_v \omega = \sum_{a=1}^n (c_r r_a d\theta_a - c_a r_a dr_a) = 0$, since along $L \setminus \{0\}$, we have $\sum_{a=1}^n c_r r_a d\theta_a = c_r \sum_{a=1}^n d\theta_a = 0$ and $n \sum_{a=1}^n c_a r_a dr_a = (\sum_{a=1}^n c_a) r dr = 0$, where $r = r_1 = \cdots = r_n$. \hfill $\square$

Since $L \subset M$ is a conic Lagrangian subvariety, $L \times \{t\} \subset M \times \{t\} \subset N$ is a Legendrian subvariety for any choice of $t \in \mathbb{R}$. Any lift of $L \subset M$ to a Legendrian subvariety $L \subset N$ is of this form, and we will write $L = L \times \{0\} \subset M \times \{0\} \subset N$ for the lift where $t = 0$.

3.3. Mirror symmetry. Now let us focus on the case $n = 3$ so that we have the Landau-Ginzburg model $M = \mathbb{C}^3$, $W = z_1 z_2 z_3$.

Recall the base space $Y = \mathbb{R}^3$, and the Lagrangian fibration

$$p : M \longrightarrow Y \quad q(z_1, z_2, z_3) = (x_1, x_2, x_3)$$

given by taking the real parts of vectors. Note that $p$ is equivariant for real dilations.

Let us focus on its restriction to the Lagrangian subvariety

$$L = \{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}) \in M \mid \theta_1 + \theta_2 + \theta_3 = 0, r \in \mathbb{R}_{\geq 0}\} \subset M$$

and in particular its link at the origin

$$K = \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \in M \mid \theta_1 + \theta_2 + \theta_3 = 0\} \subset M$$

Note that $K$ is a two-torus, and $p|_K$ is the quotient map for the $\mathbb{Z}/2$-action

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \longrightarrow (e^{-i\theta_1}, e^{-i\theta_2}, e^{-i\theta_3})$$

Thus $p|_K$ is a two-fold cover ramified at the four points

$$R = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\} \subset K$$
and its image \( p(K) \subset X \) is a two-sphere.

Next recall the contact manifold \( N = M \times \mathbb{R} = \mathbb{C}^3 \times \mathbb{R} \), the base space \( X = Y \times \mathbb{R} = \mathbb{R}^3 \times \mathbb{R} \), and the Legendrian fibration

\[
q : N \longrightarrow X \quad q(z_1, z_2, z_3, t) = (x_1, x_2, x_3, t + \sum_{a=1}^{3} x_ay_a)
\]

Note that \( q \) is equivariant under simultaneous real dilations of the first three coordinates with real squared-dilations of the last.

Let us focus on its restriction to the Legendrian subvariety

\[
\mathcal{L} = \{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}, 0) \in N \mid \theta_1 + \theta_2 + \theta_3 = 0, \, r \in \mathbb{R}_{\geq 0}\} \subset N
\]

and in particular its link at the origin

\[
\mathcal{K} = \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, 0) \in N \mid \theta_1 + \theta_2 + \theta_3 = 0\} \subset N
\]

Note that \( \mathcal{K} \) is a two-torus, and \( q|_{\mathcal{K}} \) is an immersion away from

\[
\mathcal{R} = \{(1,1,1,0), (1,-1,-1,0), (-1,1,-1,0), (-1,-1,1,0)\} \subset \mathcal{K}
\]

Moreover, note that \( q|_{\mathcal{K}} \) is an embedding away from

\[
\mathcal{W} = \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, 0) \in \mathcal{K} \mid \theta_a = 0, \text{for some } a = 1,2,3\} \subset \mathcal{K}
\]

and is self-transverse along \( \mathcal{W} \).

Altogether, we see that \( H = q(\mathcal{L}) \subset X \) is a cone (with respect to simultaneous real dilations of the first three coordinates and real squared-dilations of the last) over the surface \( S = q(\mathcal{K}) \subset X \), and so in particular is a hypersurface.

Now recall that we can identify as contact manifolds a small ball around the origin \( 0 \in N \) with a small ball around the codirection \([dt|_0] \in S_0^*X\) at the origin \( 0 \in X \) so that the Legendrian fibrations \( q : N \rightarrow X, \pi : S^*X \rightarrow X \) agree.

Let us write \( \Lambda \subset S^*X \) for the natural closed conic (with respect to simultaneous real dilations of the first three coordinates and real squared-dilations of the last) Legendrian subvariety that agrees with the transport of \( \mathcal{L} \subset N \) near the origin. To be precise, we recover \( \Lambda \subset S^*X \) from the hypersurface \( H \subset X \) as follows. Let \( H^{sm} \subset H \) be any dense open smooth locus, and \( S^*_H \backslash \Lambda \subset S^*X \) its spherically projectivized conormal bundle. There is a distinguished codirection \( \sigma : H^{sm} \rightarrow S^*X \) such that \( \Lambda|_{H^{sm}} = \sigma(H^{sm}) \), and the closure of \( \sigma(H^{sm}) \subset S^*X \) recovers \( \Lambda \subset S^*X \).

Now we are in the situation of Section 2 and can apply Lemma 3.2 to calculate the dg category of microlocal sheaves \( \mu Sh_{\Lambda}(X) \). We will henceforth adopt the constructions and notations introduced therein. Note that our current situation is slightly simpler: the closed Legendrian subvariety \( \Lambda \subset S^*X \) is conic (with respect to simultaneous real dilations of the first three coordinates and real squared-dilations of the last) so there is no difference between working over all of \( X \) and in a small open ball \( B \subset X \) around the origin.

**Theorem 3.2.** There is an equivalence

\[
\mu Sh_{\Lambda}(X) \simeq \text{Coh}_{\text{tors}}(\mathbb{P}^1 \setminus \{0,1,\infty\})
\]

**Proof.** Let \( C \subset B \) be a three-dimensional sphere around the origin, and let \( c \in C \) be its intersection with the ray \( \{(0,0,0,-r)\} \subset \mathbb{R}^4 \), for \( r > 0 \).

Let \( A = C \setminus \{c\} \) be the open complementary three-dimensional ball, and let \( \Lambda_A \subset S^*A \) denote the induced closed Legendrian subvariety.
Observe that \( \Lambda_A \) is diffeomorphic to a two-torus, and projects to a surface \( \Sigma = A \cap H \) along the natural map \( \pi : S^* A \to A \). We recover \( \Lambda_A \) by taking the closure of the “inward pointing” codirection along the smooth locus of \( \Sigma \).

Observe that the key properties of \( \pi|_{\Lambda_A} \), and hence of \( \Sigma \subset A \), coincide with those of \( q|_{K} \) described above. Namely, identifying \( \Lambda_A \) with the two-torus \( \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1 + \theta_2 + \theta_3 = 0 \} \), the projection \( \pi|_{\Lambda_A} \) is an immersion away from the four points

\[ \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1) \} \]

an embedding away from the locus

\[ \{(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mid \theta_1 + \theta_2 + \theta_3 = 0, \theta_a = 0, \text{for some } a = 1,2,3 \} \]

and self-transverse where immersed. Furthermore, the local geometry of \( \Sigma \subset A \) near the four ramified points has been described in the toy application of Section 2.3 above.

Now applying Lemma 2.3 provides an equivalence

\[ \mu Sh_A(X) \simeq Sh_A(X)^! \longrightarrow Sh_{\Lambda_A}(A)_c \]

Following Examples 2.1 and 2.2 and the calculation of Section 2.3, we find the following simple description of \( Sh_{\Lambda_A}(A)_c \). The connected components of \( A \setminus \Sigma \) consist of a compact contractible central component \( U \), four compact contractible neighboring components \( U_1, U_2, U_3, U_4 \), and a non-compact complementary component. Any \( \mathcal{F} \in Sh_{\Lambda_A}(A)_c \) vanishes on the non-compact complementary component, and has a respective stalk \( V \) and \( V_1, V_2, V_3, V_4 \) along each of the compact contractible components. Similarly to Examples 2.1 and 2.2 and the calculation of Section 2.3, we find that \( Sh_{\Lambda_A}(A)_c \) is equivalent to the bounded derived category of the category of five finite-dimensional \( k \)-modules \( V \) and \( V_1, V_2, V_3, V_4 \), with maps \( V_1 \to V, V_2 \to V, V_3 \to V, V_4 \to V \) that induce a direct sum decomposition

\[ V_a \oplus V_b \sim V, \quad \text{for any distinct pair of indices } a \neq b. \]

Starting from the direct sum decomposition \( V \simeq V_1 \oplus V_2 \), we can view \( V_3 \) and \( V_4 \) as the graphs of isomorphisms

\[ m_3 : V_1 \sim V_2 \quad \text{and} \quad m_4 : V_1 \sim V_2 \]

Up to equivalence, the data are determined by the automorphism

\[ m = m_4 \circ m_3^{-1} : V_1 \sim V_1 \]

and the direct sum decomposition \( V_3 \oplus V_4 \simeq V \), imposes the only constraint that 1 is not an eigenvalue of \( m \).

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