A Variational Fock-Space Treatment of Quarkonium

L. Di Leo and J.W. Darewych
Department of Physics and Astronomy
York University
Toronto, ON
M3J 1P3 Canada

Abstract

The variational method and the Hamiltonian formalism of QCD are used to derive relativistic, momentum space integral equations for a quark-antiquark system with an arbitrary number of gluons present. As a first step, the resulting infinite chain of coupled equations is solved in the nonrelativistic limit by an approximate decoupling method. Comparison with experiment allows us to fix the quark mass and coupling constant, allowing for the calculation of the spectra of massive systems such as charmonium and bottomonium. Studying the results with and without the nonAbelian terms, we find that the presence of the nonAbelian factors yields better agreement with the experimental spectra.

PACS number(s): 11.10.Ef,12.38.-t,14.40.-n
I. INTRODUCTION

Although the hadron spectrum is understood quite well in the context of the quark model, completely ab initio treatments have proved to be difficult to implement, particularly outside the realm of lattice gauge theory. In this paper we shall consider the description of quark-antiquark states using a variational approach within the canonical Hamiltonian formalism of QCD. This approach has been used with good effect for describing relativistic two and three body bound and quasi-bound states in various QFTs, including QED [1-6], the Wick-Cutkosky model [7], model QCD [8], and other models. An overview of the variational approach in QFT up to 1988 is given in the conference proceedings mentioned in [1]. A brief review of the description of few-particle bound and quasi-bound states in QFT by means of the variational approach in the Hamiltonian formalism is given in reference [9].

The present work is similar, in some respects, to the approach used by Zhang and Koniuk [8], except that they used a model Hamiltonian in which a scalar confining potential is put in by hand. In the present work we use the full QCD Hamiltonian in the Lorentz gauge as explained in Section II. The variational trial state for the quarkonium system is a Fock-space expansion that includes terms of the type $|\phi_0 q q\rangle$, $|\phi_1 q q g\rangle$, $|\phi_2 q q g g\rangle$, where the $\phi_\beta$ are variational coefficient functions and $q$, $\bar{q}$ and $g$ represent quarks, antiquarks and gluons respectively. We use a Gupta-Bleuler type constraint to implement the Lorentz gauge condition.

The variational principle then leads to an infinite chain of coupled integral equations for the functions $\phi_\beta$, which are given in Section III. Such a system of equations is impossible to solve, so the rest of the paper deals with some approximate solutions. Thus, in Section IV we consider the limit of heavy quark masses, and approximate solutions of the nonrelativistic equations are discussed in Section V, as would be adequate for the heavy $c\bar{c}$ and $b\bar{b}$ systems. A comparison with some observed charmonium and bottomonium states is presented and discussed. Concluding remarks are given in Section VI.

II. LAGRANGIAN, HAMILTONIAN AND CANONICAL QUANTIZATION

Suppressing the flavor indices, the QCD Lagrangian density is [11]

$$L_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + i \bar{\psi}^i \gamma^\mu (D_\mu)_{ij} \psi^j - m \bar{\psi}^i \psi^i - (m_0 - m) \bar{\psi}^i \psi^i$$

(1)

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu,$$

(2)

and

$$(D_\mu)_{ij} = \delta_{ij} \partial_\mu - ig_s t^a_{ij} A^a_\mu.$$  

(3)

The QCD coupling constant is denoted as $g_s$, $f_{abc}$ are the structure constants, the fermion mass $m$ is an adjustable parameter not necessarily equivalent to the bare fermion mass $m_0$, and

$$t^a_{ij} = \frac{\lambda^a_{ij}}{2},$$

(4)

where the $\lambda$s are the Gell-Mann matrices. As usual, repeated indices are summed over, with the color indices $i, j = 1, 2, 3$ and $a, b, c = 1, ..., 8$ for the gluon fields.

Upon expansion of the terms above, the Lagrangian density may also be written as

$$L_{QCD} = L_\psi + L_g + L_{g\psi} + L_{3g} + L_{4g}$$

(5)
where
\[ L_\psi = \bar{\psi}^i \left( i \gamma^\mu \partial_\mu - m \right) \psi^i - (m_0 - m) \bar{\psi}^i \psi^i \] (6)
\[ L_g = -\frac{1}{2} \partial_\mu A_\nu^a (\partial^\mu A^{a \nu} - \partial^\nu A^{a \mu}) \] (7)
\[ L_{g \psi} = g_s t^a_{ij} \bar{\psi}^i A^a \psi^j \] (8)
\[ L_{3g} = -\partial^\mu A^{a \nu} g_s f_{abc} A^b_\mu A^c_\nu \] (9)
\[ L_{4g} = -\frac{1}{4} g_s^2 f_{abc} f_{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu \] (10)

Since the Lagrangian yields
\[ \Pi_0^a (x) = \frac{\partial L_{QCD}}{\partial \dot{A}_{0}^a} = 0 \] (11)
we cannot quantize unless we use the Lorentz condition
\[ \partial_\mu A^a \mu = 0 \] (12)
and add a gauge fixing term [12] to \( L_g \) such that
\[ L_g \rightarrow -\frac{1}{2} \partial_\mu A_\nu^a \partial^\mu A^{a \nu} \] (13)

We will use in the quantized theory, in place of (12), the weaker Gupta-Bleuler (GB) constraint [12]
\[ \partial_\mu A^a \mu (x) |\psi\rangle = 0. \]

The Hamiltonian density is canonically derived from
\[ H_{QCD} = \Pi_k \dot{\psi}^k + \Pi^a_\alpha \dot{A}_\alpha^a - L_{QCD} \] (14)
where
\[ \Pi^k = \frac{\partial L_{QCD}}{\partial \dot{\psi}^k} = \bar{\psi}^i \left( i \gamma^0 \delta^i_k \right), \quad \Pi^a_\alpha = \frac{\partial L_{QCD}}{\partial \dot{A}_{\alpha}^a} = -\dot{A}_{\alpha}^a - g_s f_{abc} A_0^b A^c_\alpha \] (15)
so that using (5) we can write
\[ H_{QCD} = H_\psi + H_g + H_{g \psi} + H_{3g} + H_{4g} \] (16)
where
\[ H_\psi = \bar{\psi}^i \left( -i \gamma^\mu \partial_\mu + m_0 - m \right) \psi^i, \] (17)
\[ H_g = -\frac{1}{2} \Pi_\nu^a \Pi^a_\nu - \frac{1}{2} \partial_\mu A^a_\nu \partial^\mu A^{a \nu}, \] (18)
\[ H_{g \psi} = g_s t^a_{ij} \bar{\psi}^i A^a \psi^j, \] (19)
\[ H_{3g} = -g_s f_{abc} \Pi^a_\nu A_0^b A^c_\nu + g_s f_{abc} \partial^j A^{a \nu} A^b_\mu A^c_\nu, \] (20)
and
\[ H_{4g} = -\frac{1}{4} g_s^2 f_{abc} f_{ade} A^c_\nu A^e_\nu \left[ A_0^b A^d_0 + A_l^b A^d_l \right]. \] (21)
We can expand the Dirac field \((t = 0, \text{color index } j)\) and gluon field \((t = 0, \text{gluon index } a)\) in the standard Fourier decomposition:

\[
\psi^j(x) = \sum_s \int d^3p \sqrt{\frac{m}{(2\pi)^3\omega_p}} \left[ c^j(p, s) u(p, s) e^{i p \cdot x} + d^j(p, s) v(p, s) e^{-i p \cdot x} \right], \tag{22}
\]

\[
\bar{\psi}^j(x) = \sum_s \int d^3p \sqrt{\frac{m}{(2\pi)^3\omega_p}} \left[ d^j(p, s) \bar{v}(p, s) e^{i p \cdot x} + c^j(p, s) \bar{u}(p, s) e^{-i p \cdot x} \right], \tag{23}
\]

\[
A^a \mu(x) = \sum_{\lambda=0}^3 \int \frac{d^3k}{\sqrt{(2\pi)^3 2|k|}} \bar{c}_\lambda^a(k) \left[ a_\lambda^a(k) e^{i k \cdot x} + a_\lambda^{a \dagger}(k) e^{-i k \cdot x} \right], \tag{24}
\]

and

\[
\Pi^a \nu(x) = i \sum_{\lambda=0}^3 \int \frac{d^3k}{\sqrt{(2\pi)^3 2|k|}} \bar{e}_\lambda^a(k) \left[ a_\lambda^a(k) e^{i k \cdot x} - a_\lambda^{a \dagger}(k) e^{-i k \cdot x} \right], \tag{25}
\]

where \(c^\dagger, d^\dagger\) and \(a_\lambda^a\) are the creation operators of a (free) fermion, antifermion and gluon respectively, and \(\omega_p = \sqrt{p^2 + m^2}\). Furthermore, the anticommutation and commutation relations are

\[
\{c^j(p, s), c^{j \dagger}(p', s')\} = \{d^j(p, s), d^{j \dagger}(p', s')\} = \delta_{ij} \delta_{ss'} \delta^3(p - p') \tag{26}
\]

\[
[a_\lambda^a(k), a_{\lambda'}^{a \dagger}(k')] = \zeta_\lambda \delta_{\alpha \beta} \delta_{\lambda \lambda'} \delta^3(k - k') \tag{27}
\]

where \(\zeta_1 = \zeta_2 = \zeta_3 = 1\), \(\zeta_0 = -1\), and all other commutation relations vanish.

Since we are not interested in the vacuum energy, we will normal-order the Hamiltonian

\[
: H_{QCD} := \int d^3x : \mathcal{H}_{QCD} :
\]

and expand its component parts using equations (17) to (25) in (28).

III. COUPLED EQUATIONS

For a quark-antiquark system and an arbitrary number of gluons, we will use a Fock-space variational ansatz (rest frame \(P_{\text{total}} = 0\)) that is a linear combination of a quark-antiquark state, a quark-antiquark-gluon state, a quark-antiquark-gluon-gluon state, \textit{ad infinitum}. Without the terms that contain Fock states with at least two gluons, we would not be able to sample the non-Abelian terms of the Hamiltonian. Explicitly,

\[
\left| \psi \right> = \sum_{i,s_1,s_2} \int d^3p d^3q \phi_0(p, n, s_1, s_2) c_i \dagger (p, s_1) d^i \dagger (n, s_2) \delta^3(p + n) |0\rangle
\]

\[
+ \sum_{i,j,s_1,s_2} \sum_{z_1,z_2} \int d^3p d^3q_1 \phi_1(p, n, s_1, s_2, q_1, z_1) c^i \dagger (p, s_1) t_{z_1}^j \dagger (n, s_2) a_{z_1}^{z_2} \dagger (q_1) \delta^3(p + n + q_1) |0\rangle
\]

\[
+ \sum_{i,j,l,s_1,s_2} \sum_{z_1,z_2,\lambda_1,\lambda_2} \int d^3p d^3q_1 d^3q_2 \phi_2(p, n, s_1, s_2, q_1, q_2, \lambda_1, \lambda_2) c^i \dagger (p, s_1) \delta^3(p + n + q_1 + q_2) |0\rangle + \ldots
\]
\[
\begin{align*}
&= \sum_{s_1, s_2} \sum_{i, j} \int d^3 p \, d^3 n \, c_i^\dagger(p, s_1) \, d_j^\dagger(n, s_2) \\
&\times \sum_{\beta=0}^\infty \sum_{\lambda_1, \ldots, \lambda_\beta} \sum_{l_1, \ldots, l_{\beta+1}} \sum_{z_1, \ldots, z_\beta} \int d^3 q_1 \cdots d^3 q_\beta \, \phi_\beta(p, n, s_1, s_2, q, \lambda) \, \delta_{l_1, i} \, \delta_{l_{\beta+1}, j} \\
&\quad \times \left\{ t_{l_1} t_{l_2} t_{l_3} \cdots t_{l_{\beta+1}} \right\} a_{\lambda_1}^{z_1} (q_1) \, a_{\lambda_2}^{z_2} (q_2) \cdots a_{\lambda_\beta}^{z_\beta} (q_\beta) \, \delta^3 (p + n + q_1 + \cdots + q_\beta) |0\rangle,
\end{align*}
\]

where the \(\lambda_k\)'s are summed from 0 to 3, \(i, j, k, l, m\) and the \(l_k\)'s from 1 to 3, and the \(z_k\)'s from 1 to 8, and the \(\{\cdots\}\) represent all \((\beta!)\) permutations of the Gell-Mann matrices. The total momentum is zero in the rest frame and the annihilation operators \(c_i^\dagger, d_j^\dagger\) and \(a_\lambda^a\) have the property

\[
c_i^\dagger |0\rangle = d_j^\dagger |0\rangle = a_\lambda^a |0\rangle = 0,
\]

where \(|0\rangle\) is our trial vacuum state. In our ansatz, we have used the simplified notation [13]

\[
q = q_1, \ldots, q_\beta, \quad \lambda = \lambda_1, \ldots, \lambda_\beta
\]

and define \(\tilde{q}_\beta\) as the set of all the \(q\)'s excluding \(q_\beta\). We will use this same definition for the \(z_k\)'s and \(t^z\)'s in what follows. The symmetry of \(\phi_\beta(p, n, s_1, s_2, q_1, \ldots, q_\beta, \lambda_1, \ldots, \lambda_\beta)\) is apparent under the exchange of any \((q_\kappa, \lambda_\kappa)\) pair.

The GB condition removes the redundant gluon degrees of freedom by requiring that our trial state \(|\psi\rangle\) obeys the condition

\[
\left( a_{\lambda=3}^{z_1}(q_1) - a_{\lambda=0}^{z_1}(q_1) \right) |\psi\rangle = 0
\]

for all \(q_1\), which forces the variational coefficients \(\phi_\beta\) to obey

\[
\begin{align*}
\phi_1(p, n, s_1, s_2, q_1, \lambda_1 = 3) &= -\phi_1(p, n, s_1, s_2, q_1, \lambda_1 = 0) \\
\phi_2(p, n, s_1, s_2, q_1, q_2, \lambda_1, \lambda_2 = 3) &= -\phi_2(p, n, s_1, s_2, q_1, q_2, \lambda_1, \lambda_2 = 0) \\
\phi_2(p, n, s_1, s_2, q_1, q_2, \lambda_1 = 3, \lambda_2) &= -\phi_2(p, n, s_1, s_2, q_1, q_2, \lambda_1 = 0, \lambda_2)
\end{align*}
\]

etc.

The variational coefficients \(\phi_0, \phi_1, \ldots, \phi_\beta\) in (29) are determined from the variational principle

\[
\langle \delta \psi | : H_{QCD} - E : |\psi \rangle = 0
\]

which, with our trial state and the GB condition, gives the infinite chain of coupled, multidimensional integral equations:
We have defined with the condition that we need to decouple them (which we need to do to make the problem tractable). The set of equations that is not as tractable as the present case; that is, it's not obvious how to simplify that will provide an approximate solution. One can start with the fixed or heavy mass limit, which will be adequate in dealing with very heavy quark systems such as \( \psi_3 \) and \( \eta_3 \), in Appendix B for completeness.

If, in our ansatz (29), we remove the \( \{ \cdots \} \) permutation condition on the Gell-Mann terms such that

\[
\{ t^{z_1}_{i_1 j_1} t^{z_2}_{i_2 j_2} \cdots t^{z_{\beta+1}}_{i_{\beta+1} j_{\beta+1}} \} \rightarrow t^{z_1}_{i_1 j_1} t^{z_2}_{i_2 j_2} \cdots t^{z_{\beta+1}}_{i_{\beta+1} j_{\beta+1}}
\]

then we still retain the symmetry of \( \phi_\beta \) under the exchange of any \((q_k, \lambda_k)\) pair, but now we obtain a set of equations that is not as tractable as the present case; that is, it's not obvious how to decouple them (which we need to do to make the problem tractable). The \( H_{3g} \) contribution is now present and these equations are written explicitly in Appendix B for completeness.

IV. HEAVY-MASS LIMIT

The coupled equations of (31) are obviously very difficult to solve so we will begin with a simplification that will provide an approximate solution. One can start with the fixed or heavy mass limit, which will be adequate in dealing with very heavy quark systems such as \( J/\psi \) (\( c \bar{c} \)) or \( \Upsilon \) (\( b \bar{b} \)) that have a largely nonrelativistic behavior.

In the lowest order of \( \frac{p}{m} \), that is, letting \( \omega_p \rightarrow m + \frac{p^2}{2m} \) in the kinetic energy terms and letting \( \omega_p \rightarrow m \) in the potential energy terms, (31) becomes

\[
+ g' m \sum_{s} \sum_{k=1}^{\beta} \frac{Q_4(\beta, k, n, s_2, s)}{\sqrt{|q_k|}} \phi_{\beta-1}(p, n + q_k, s_1, s, q_k, \tilde{\lambda}_k) \\
- g' m \sum_{s} \sum_{k=1}^{\beta} \frac{Q_4(\beta, k, p, s_1, s)}{\sqrt{|q_k|}} \phi_{\beta-1}(p + q_k, n, s_2, q_k, \tilde{\lambda}_k) \\
- \frac{g^2}{8} \sum_{\lambda', \lambda''} (\sum_{k \neq j} \int \frac{d^3k'}{\sqrt{|q_k| |q_j| |k'| |q_k + q_j - k'|}} \\
\times \phi_{\beta}(p, n, s_1, s_2, q_k + q_j - k', k', q_k, q_j) = 0
\]

with the condition

\[
p + n + \sum_{k=1}^{\beta} q_k = 0.
\]

We have defined

\[
g' = \frac{g_s}{\sqrt{2(2\pi)^3}}
\]

and the definitions for \( X, I^\beta \) and the \( Q' \)s are given in Appendix A. The factor \( X \) is composed of the gluon polarization vectors, and \( Q \) depends on the Dirac spinors as well as relativistic factors; \( I^\beta \) is a color sum dependent on the Gell-Mann matrices. We should point out that the contribution from the “cubic” \( H_{3g} \) term of equation (20) vanishes because of the properties of the Gell-Mann matrices and our choice of variational ansatz (29) which is symmetrized in the Gell-Mann matrices. Consequently, the only surviving potential terms are of a color “Coulombic” type and a “quartic” contribution from the \( H_{4g} \) term (equation (21)); the nonAbelian effects are present only for \( \beta \geq 2 \).
\[
\times \left[ J_1^{\beta+1} \phi_{\beta+1}(p, n - k, q', k, \lambda, \lambda'') - J_2^{\beta+1} \phi_{\beta+1}(p - k, n, q, k, \lambda, \lambda'') \right]
\]

\[
+ g' \sum_{k=1}^{\beta} \frac{\zeta_{\lambda'} \zeta_{\lambda''}}{\sqrt{|q_k|}} \left[ J_3^{\beta-1}(z_k) \phi_{\beta-1}(p + q_k, \tilde{q}_k, \tilde{\lambda}_k) - J_4^{\beta-1}(z_k) \phi_{\beta-1}(p + q_k, n, \tilde{q}_k, \tilde{\lambda}_k) \right]
\]

\[
- \frac{g'^2}{8} \sum_{\lambda', \lambda''=0}^{2} \zeta_{\lambda'} \zeta_{\lambda''} \sum_{k \neq j}^{\beta} \int \frac{d^3 k'}{|q_k| |q_j| |k' - q_j|} \times \phi_{\beta}(p, n, q_k + q_j - k', k', \tilde{q}_k, \tilde{q}_j, \lambda', \lambda', \tilde{\lambda}_k, \tilde{\lambda}_j) \tilde{X}(\beta, k, j, \lambda''', \lambda', k') = 0 \tag{33}
\]

where

\[
\epsilon_{\lambda''}(k) = \epsilon_0(k) + \epsilon_{0}(k) \tag{34}
\]

for \( \lambda'' = 0 \), and

\[
\tilde{\epsilon}_{\lambda''}(k) = \epsilon_{\lambda''}(k) \tag{35}
\]

for \( \lambda'' = 1, 2 \).

In the QED fixed-mass case, the kinetic energy terms are neglected and all the \( I' \)'s and \( J' \)'s in (33) collapse to 1, thus allowing the infinite sequence of coupled equations to have an exact solution \([13]\): an ansatz, \( \phi_{\beta} \), may be found (along with a specific mass renormalization condition) that decouples the equations. However, in the QCD case the Gell-Mann matrices hamper such an exact solution, but we will follow the spirit of that approach in at least getting an approximate solution. Thus we choose an ansatz

\[
I^{\beta} \phi_{\beta}(p, n, q, \lambda) = -g' \frac{\epsilon_{\lambda}(q_{\beta})}{\sqrt{|q_{\beta}|}}
\]

\[
\times \left[ J_3^{\beta-1}(z_{\beta}) \phi_{\beta-1}(p + q_{\beta}, \tilde{q}_{\beta}, \tilde{\lambda}_{\beta}) - J_4^{\beta-1}(z_{\beta}) \phi_{\beta-1}(p + q_{\beta}, n, \tilde{q}_{\beta}, \tilde{\lambda}_{\beta}) \right], \tag{36}
\]

substitute it in (33), and impose a mass-renormalization condition (\( \beta \)-dependent)

\[
2 (m_0 - m) = -\frac{\alpha}{4 \pi^2} \left\{ \frac{J_1^{\beta+1} J_3^{\beta}(z_{\beta+1})}{I^{\beta} I^{\beta+1}} + \frac{J_2^{\beta+1} J_4^{\beta}(z_{\beta+1})}{I^{\beta} I^{\beta+1}} \right\} \int \frac{d^3 k'}{|k'|^2} \phi_{\beta}(p + k, n - k, q, \lambda) \tag{37}
\]

This uncouples the chain of equations (33) and results in a sequence of bound-state equations

\[
\left[ 2 m + \frac{p^2}{2 m} + \frac{n^2}{2 m} - E \right] \phi_{\beta}(p, n, q, \lambda)
\]

\[
- \frac{\alpha}{4 \pi^2} \left\{ \frac{J_1^{\beta+1} J_3^{\beta}(z_{\beta+1})}{I^{\beta} I^{\beta+1}} + \frac{J_2^{\beta+1} J_4^{\beta}(z_{\beta+1})}{I^{\beta} I^{\beta+1}} \right\} \int \frac{d^3 k'}{|k'|^2} \phi_{\beta}(p + k, n - k, q, \lambda)
\]

\[
- \frac{\alpha}{32 \pi^2 I^{\beta}} \sum_{\lambda', \lambda''}^{2} \zeta_{\lambda'} \zeta_{\lambda''} \sum_{k \neq j}^{\beta} \int \frac{d^3 k'}{|q_k| |q_j| |k' - q_j|} \times \phi_{\beta}(p, n, q_k + q_j - k', k', \tilde{q}_k, \tilde{q}_j, \lambda', \lambda', \tilde{\lambda}_k, \tilde{\lambda}_j) \tilde{X}(\beta, k, j, \lambda'', \lambda', k') = 0 \tag{38}
\]

with the correct rest-plus-kinetic energy for a two-particle system. It is understood that the divergent integral in (37) is controlled by a suitable regulator (cut-off) which does not appear, subsequently, in (38).
Equation (38), for any given $\beta$, is a momentum space Schrödinger-like equation for the stationary states of the heavy quark-antiquark system; there is an obvious color “Coulomb” term present plus a “quartic” contribution from the $\mathcal{H}_{4g}$ term. On the surface, this “quartic” term also seems to share a Coulomb-type $(\frac{1}{|k|^2})$ behavior. An explicit $\frac{1}{|k|^3}$ or $\frac{1}{|k|^4}$ term in the “quartic” part does not appear, a term that would signal a confining-type potential.

Note that we have chosen a specific representation [12]:

\[
\begin{align*}
\epsilon_0^\mu(k) &= \eta^\mu = (1, 0, 0, 0) \\
\epsilon_r^\mu(k) &= (0, \epsilon_r(k)), \quad r = 1, 2, 3 \\
\epsilon_1 \cdot \epsilon_2 &= 0, \quad \epsilon_3(k) = \frac{k}{|k|}, \\
k \cdot \epsilon_r(k) &= 0, \quad r = 1, 2
\end{align*}
\]

with

\[
\alpha = \frac{g_s^2}{4 \pi}
\]

to simplify the color “Coulomb” term. We’ll work out the terms for $X$ in the “quartic” term shortly.

The sequence of equations (38) represents different approximations for the $q\bar{q}$ system with various numbers, $\beta$, of “spectator” gluons. Note that nonAbelian effects begin only with $\beta \geq 2$. Explicitly, for $\beta = 0, 1, 2$, our mass-renormalization conditions, bound-state equations and momenta constraints are

\[
\begin{align*}
2 (m_0 - m) &= -\frac{\alpha}{2\pi^2} \frac{4}{3} \int \frac{d^3k}{|k|^2} \\
\left[ 2m + \frac{p^2}{m} - E \right] \phi_0(p, n) - \alpha \frac{4}{3} \int \frac{d^3k}{|k|^2} \phi_0(p + k, n - k) &= 0
\end{align*}
\]

(40a)

\[
\begin{align*}
p + n &= 0
\end{align*}
\]

(40c)

for $\beta = 0$,

\[
\begin{align*}
2 (m_0 - m) &= -\frac{\alpha}{2\pi^2} \frac{7}{12} \int \frac{d^3k}{|k|^2} \\
\left[ 2m + \frac{p^2}{2m} + \frac{n^2}{2m} - E \right] \phi_1(p, n, q_1, \lambda_1) - \alpha \frac{7}{12} \int \frac{d^3k}{|k|^2} \phi_1(p + k, n - k, q_1, \lambda_1) &= 0
\end{align*}
\]

(41b)

\[
\begin{align*}
p + n + q_1 &= 0
\end{align*}
\]

(41c)

for $\beta = 1$, and

\[
2 (m_0 - m) = -\frac{\alpha}{2\pi^2} \frac{10}{21} \int \frac{d^3k}{|k|^2}
\]

(42a)
\[
\left[ 2m + \frac{p^2}{2m} + \frac{n^2}{2m} - E \right] \phi_2(p, n, q_1, q_2, \lambda_1, \lambda_2) \\
- \frac{\alpha}{2\pi^2} \frac{10}{21} \int \frac{d^3k}{|k|^2} \phi_2(p + k, n - k, q_1, q_2, \lambda_1, \lambda_2) \\
- \frac{3\alpha}{32\pi^2 56} \sum_{\lambda', \lambda''=0}^2 \zeta_{\lambda'} \zeta_{\lambda''} \int \frac{d^3k'}{\sqrt{|q_1|} |q_2| |k'| |q_1 + q_2 - k'|} \\
\times \phi_2(p, n, q_1 + q_2 - k', k', \lambda'', \lambda') \left[ \tilde{X}(2, 1, 2, \lambda'', \lambda', k') + \tilde{X}(2, 2, 1, \lambda'', \lambda', k') \right]
\]

\[
= \left[ 2m + \frac{p^2}{2m} + \frac{n^2}{2m} - E \right] \phi_2(p, n, q_1, q_2, \lambda_1, \lambda_2) \\
- \frac{\alpha}{2\pi^2} \frac{10}{21} \int \frac{d^3k}{|k|^2} \phi_2(p + k, n - k, q_1, q_2, \lambda_1, \lambda_2) \\
- \frac{6\alpha}{32\pi^2 56} \sum_{\lambda', \lambda''=0}^2 \zeta_{\lambda'} \zeta_{\lambda''} \int \frac{d^3k'}{\sqrt{|q_1|} |q_2| |k'| |q_1 + q_2 - k'|} \\
\times \phi_2(p, n, q_1 + q_2 - k', k', \lambda'', \lambda') \tilde{X}(2, 1, 2, \lambda'', \lambda', k') = 0, \quad (42b)
\]

\[
p + n + q_1 + q_2 = 0 \quad (42c)
\]

for \(\beta = 2\). There are no gluon kinetic energy terms in (41b) and (42b) because of the ansatz (36).

All three equations have a one-gluon exchange “Coulomb” term, modified by color factors stemming from the sums explicitly written in APPENDIX A; that is, we have the coefficients \(\frac{4}{3}, \frac{7}{12}\) and \(\frac{10}{21}\) for \(\beta = 0, 1, 2\) respectively. The \(\beta = 0\) equation has only the virtual one-gluon exchange and its familiar \(\frac{4}{3}\) factor, while the \(\beta = 1, 2\) bound state equations have spectator gluons present. However it is only for \(\beta = 2\) (more generally \(\beta \geq 2\)) that there appears a nonAbelian contribution to the interquark potential energy, and so we expect that the \(\beta = 2\) equation is the more realistic representation of quarkonium from among the three \((\beta = 0, 1, 2)\) approximations. It is evident, though, that the equations become increasingly more complicated, and so more difficult to solve, as \(\beta\) increases.

V. APPROXIMATE SOLUTIONS

A. Suppression of NonAbelian terms

If we ignore the nonAbelian terms in equation (38), we have

\[
\left[ 2m + \frac{p^2}{2m} + \frac{n^2}{2m} - E \right] \phi_\beta(p, n, q, \lambda) - \frac{\alpha}{4\pi^2} \gamma(\beta) \int \frac{d^3k}{|k|^2} \phi_\beta(p + k, n - k, q, \lambda) = 0 \quad (43)
\]

with

\[
p + n + \sum_{k=1}^{\beta} q_k = 0, \quad (44)
\]

and where (see APPENDIX A)

\[
\gamma(\beta) = \left\{ \frac{J_1^{\beta+1} J_3^{\beta} (z_{\beta+1})}{I^\beta I^{\beta+1}} + \frac{J_2^{\beta+1} J_3^{\beta} (z_{\beta+1})}{I^\beta I^{\beta+1}} \right\} = \left\{ \frac{J_1^{\beta+1} J_3^{\beta} (z_{\beta+1})}{I^\beta I^{\beta+1}} + \frac{J_2^{\beta+1} J_4^{\beta} (z_{\beta+1})}{I^\beta I^{\beta+1}} \right\}
\]

\[
= 2 \frac{I^{\beta+1}}{(\beta + 1) I\beta}, \quad (45)
\]
Furthermore, unlike similar equations in QED, we note that $\gamma(\beta)$ is not a constant but $\beta$-dependent; that is, it is determined by the number of “spectator” gluons. The first few terms are

$$
\begin{align*}
\gamma(0) &= 2 \cdot \frac{4}{3} = 2 \times 1.3333 \\
\gamma(1) &= 2 \cdot \frac{7}{12} = 2 \times 0.5833 \\
\gamma(2) &= 2 \cdot \frac{10}{21} = 2 \times 0.4762 \\
\gamma(3) &= 2 \cdot \frac{47}{96} = 2 \times 0.4896 \\
\gamma(4) &= 2 \cdot \frac{62}{141} = 2 \times 0.4397 \\
\gamma(5) &= 2 \cdot \frac{1621}{3720} = 2 \times 0.4357
\end{align*}
$$

which reveal a screening effect caused by the spectator gluons, with $\gamma(\beta)$ apparently tending to a limiting value. In the QED case, we would have $\gamma(\beta) = 2$ for all $\beta$.

Equation (43) has exact solutions for each $\beta$, as could be found easily in coordinate space. For example, the $\beta = 0$ case of equation (43) is presented in equation (40b) and may be rewritten as

$$
\left[ \frac{p^2}{m} - \varepsilon \right] \phi_0(p) - \frac{\alpha}{4\pi^2} \gamma(0) \int \frac{d^3k}{|k|^2} \phi_0(p + k) = 0
$$

Equation (47) has the standard (modified) solutions

$$
\phi_0(r) = \frac{\mu \alpha \gamma(0)}{2 n^2} 
$$

The $\phi_0(p)$ of (47) correspond to the momentum space wave functions

$$
\phi_0(p) \propto \frac{p^l}{(p^2 + a^2)^{2+l}} C^{l+1}_{n-l-1} \left( \frac{p^2 - a^2}{p^2 + a^2} \right)
$$

with

$$
a = \frac{\mu \alpha \gamma(0)}{2 n}
$$

and the $C^{l+1}_{n-l-1}$’s are the Gegenbauer functions.
Now, for $\beta = 1$ (which corresponds to a term with the $q \overline{q}$ pair plus an explicit gluon; i.e., an $H_2^+$-like configuration), equation (43) yields equation (41b), which we rewrite as

$$
\left[ \frac{p^2}{2m} + \frac{n^2}{2m} - \epsilon \right] \phi_1(p, n, \lambda_1) - \frac{\alpha}{4\pi^2} \gamma(1) \int \frac{d^3k}{|k|^2} \phi_1(p + k, n - k, \lambda_1) = 0
$$

(51)

with $\phi_1(p, n, \lambda_1) \equiv \phi_1(p, n, q_1, \lambda_1)$ (recall that $p + n + q_1 = 0$ for this case). If we assume that $\phi_1(p, n, \lambda_1) = \phi_1(p, n) Z(\lambda_1)$ then the $Z$ factors out and in coordinate space (51) is

$$
\left( \frac{1}{2m} \left( - \nabla_{r_1}^2 - \nabla_{r_2}^2 \right) - \frac{\alpha \gamma(1)}{2 |r_1 - r_2|} \right) \phi_1(r_1, r_2) = \epsilon \phi_1(r_1, r_2)
$$

(52)

If we transform to new coordinates

$$
r = r_1 - r_2,
$$

$$
R = \frac{r_1 + r_2}{2}
$$

(53)

and let

$$
\phi_1(r_1, r_2) = \phi_1(r) \phi_1(R)
$$

(55)

equation (52) reduces to the form (47), with a solution

$$
\epsilon = E - 2m = -\frac{\mu \alpha^2 \gamma^2(1)}{8 n^2}.
$$

(56)

Here, $\phi_1(r)$ is the typical hydrogenic wavefunction, and $\phi_1(R)$ is a plane wave solution. In momentum space,

$$
\phi_1(r) \phi_1(R) \rightarrow \phi_1(p_r) \delta(P_R)
$$

(57)

where $\phi_1(p_r)$ is the typical momentum space hydrogenic wavefunction (see (49)). Noting the transformations (53) and (54), the analogous procedure in momentum space is to let

$$
p_r = \frac{p - n}{2}
$$

(58)

and

$$
P_R = p + n.
$$

(59)

This will be of use in the next section.

Lastly, for $\beta = 2$ (a system that is composed of a $q \overline{q}$ pair plus two explicit gluons; i.e., an $H_2$-molecule-like configuration), equation (43) yields (equation (42b) without the nonAbelian terms)

$$
\left[ \frac{p^2}{2m} + \frac{n^2}{2m} - \epsilon \right] \phi_2(p, n, q_1) - \frac{\alpha}{4\pi^2} \gamma(2) \int \frac{d^3k}{|k|^2} \phi_2(p + k, n - k, q_1, q_2) = 0
$$

(60)

with $\phi_2(p, n, q_1) \equiv \phi_2(p, n, q_1, q_2)$ (recall $p + n + q_1 + q_2 = 0$) and we have factored out the $\lambda$ terms as in the last case. In coordinate space, equation (60) is

$$
\left( \frac{1}{2m} \left( - \nabla_{r_1}^2 - \nabla_{r_2}^2 \right) - \frac{\alpha \gamma(2)}{2 |r_1 - r_2|} \right) \phi_2(r_1, r_2, r_3) = \epsilon \phi_2(r_1, r_2, r_3).
$$

(61)
Note that $r_3$ appears as a parameter in equation (61); it has no effect on $\epsilon$. As for the $\beta = 1$ case, we can use the transformations (53) and (54) to get

$$\epsilon = E - 2m = -\frac{\mu \alpha^2 \gamma^2(2)}{8 n^2},$$  \hspace{1cm} (62)

where the wave functions have the same structure as $\beta = 1$. We expect the same result to hold for $\beta > 2$, with arbitrary functions present. That is, in general, we have

$$\epsilon = E - 2m = -\frac{\mu \alpha^2 \gamma^2(\beta)}{8 n^2}$$  \hspace{1cm} (63)

(with an ‘inverse Bohr radius’ $a = \frac{\mu \alpha \gamma(\beta)}{2 n}$ used in the respective momentum space wave functions).

Let us apply this modified Balmer formula to the heavy quark mesons such as charmonium and bottomonium to obtain a prediction for the low-lying bound states. Using the experimental values for the lowest-lying $1S$ and $2S$ states of $c\bar{c}$ (i.e., $J/\psi(1S)$, $J/\psi(2S)$; [11]), we can fix two of our parameters; that is, we can set our mass for the charm quark ($m_c$) as well as the coupling constant $\alpha$ particular to this system. Once this is done, we use our expressions for $E$ (equations (48), (56), and (62)) to get the predicted values for the $3S$, $4S$, $5S$ and $6S$ states. We repeat the process for the $b\bar{b}$ mesons, using $\Upsilon(1S)$, $\Upsilon(2S)$, and present the results in Tables I and II for charmonium and bottomonium respectively.

In this approximation suppressing the nonAbelian terms, we note that the predicted values for charmonium and bottomonium are independent of $\beta$, the number of “spectator” gluons present. The mass values are constant, as can be easily seen from equation (61) for $n = 1, 2$, as is the value $\alpha^2 \gamma^2(\beta)$; only $\alpha$ changes with $\beta$. As expected, the predicted masses are not particularly close to the observed ones, though the values for $b\bar{b}$ are better than for charmonium, since this is a heavier, more nonrelativistic system for which the mass spectrum is influenced more by the Coulomb potential. One would expect better results still for $t\bar{t}$, were the data available.

B. NonAbelian terms present: $\beta = 2$ case

We now turn to the more realistic approximation in which we have an explicit manifestation of the nonAbelian nature of the interaction. Recalling our transformations (58) and (59), our equation (42b), for $\beta = 2$, may be rewritten in the new coordinates as

$$\left[ \frac{p_r^2}{2 \mu} + \frac{P_R^2}{8 \mu} - \epsilon \right] \phi_2(p_r, P_R, q_1, q_2, \lambda_1, \lambda_2)$$

$$- \frac{\alpha}{4\pi^2} \gamma(2) \int \frac{d^3k}{|k|^2} \phi_2(p_r - k, P_R, q_1, q_2, \lambda_1, \lambda_2)$$

$$- \frac{6 \alpha}{32 \pi^2 56} \sum_{\lambda', \lambda''} \zeta_{\lambda'} \zeta_{\lambda''} \int \frac{d^3k'}{\sqrt{|q_1| |q_2| |k'| |q_1 + q_2 - k'|}}$$

$$\times \phi_2(p_r, P_R, q_1 + q_2 - k', k', \lambda'', \lambda') \tilde{X}(2, 1, 2, \lambda'', \lambda', k') = 0$$  \hspace{1cm} (64)

If we make the transformation

$$\sqrt{|q_1|} \sqrt{|q_2|} \phi_2(p_r, P_R, q_1, q_2, \lambda_1, \lambda_2) = \Phi(p_r, P_R, q_1, q_2) Z(\lambda_1, \lambda_2)$$  \hspace{1cm} (65)
then equation (64) becomes

\[
\left[ \frac{p_r^2}{2 \mu} + \frac{P_R^2}{8 \mu} - \epsilon \right] \Phi(p_r, P_R, q_1, q_2) Z(\lambda_1, \lambda_2)
- \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3k}{|k|^2} \Phi(p_r - k, P_R, q_1, q_2) Z(\lambda_1, \lambda_2)
- \frac{6 \alpha}{32 \pi^2 56} \sum_{\lambda', \lambda'' \lambda'''} \zeta_{\lambda'} \zeta_{\lambda'''} \int \frac{d^3k'}{|k'| |q_1 + q_2 - k'|} \times \Phi(p_r, P_R, q_1 + q_2 - k', k') Z(\lambda'', \lambda') \tilde{X}(2, 1, 2, \lambda'', \lambda', k') = 0
\] (66)

For an approximate solution, let’s multiply (66) by \( \Phi(p_r, P_R, q_1, q_2) \) and integrate over \( p_r, P_R, q_1 \) such that

\[
\int d^3p_r \frac{d^3P_R}{d^3q_1} \left[ \frac{p_r^2}{2 \mu} + \frac{P_R^2}{8 \mu} - \epsilon \right] \Phi^2(p_r, P_R, q_1, q_2) Z(\lambda_1, \lambda_2)
- \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3k}{|k|^2} d^3p_r \frac{d^3P_R}{d^3q_1} \Phi(p_r, P_R, q_1, q_2) \Phi(p_r - k, P_R, q_1, q_2) Z(\lambda_1, \lambda_2)
- \frac{6 \alpha}{32 \pi^2 56} \sum_{\lambda', \lambda'' \lambda'''} \zeta_{\lambda'} \zeta_{\lambda'''} \int \frac{d^3k'}{|k'| |q_1 + q_2 - k'|} \times \Phi(p_r, P_R, q_1 + q_2 - k', k') \Phi(p_r, P_R, q_1) Z(\lambda'', \lambda') \tilde{X}(2, 1, 2, \lambda'', \lambda', k') = 0
\] (67)

As a trial function, we will use the nonAbelian solutions of the last section

\[
\Phi(p_r, P_R, q_1, q_2) = \phi_c(p_r) \delta(P_R) \psi(q_1)
\] (68)

where \( \phi_c(p_r) \) is the classic hydrogenic wave function and \( \psi(q_1) \) is an arbitrary function which we choose to regulate the integral in momentum space. We can treat \( a \) in the wave function as a variational parameter to optimize our results (see (49),(50)). Integrating over \( P_R \) leads us to

\[
\int d^3p_r \left[ \frac{p_r^2}{2 \mu} - \epsilon \right] \phi_c^2(p_r) Z(\lambda_1, \lambda_2) \int d^3q_1 \psi^2(q_1)
- \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3k}{|k|^2} d^3p_r \phi_c(p_r) \phi_c(p_r - k) Z(\lambda_1, \lambda_2) \int d^3q_1 \psi^2(q_1)
- \frac{6 \alpha}{32 \pi^2 56} \sum_{\lambda', \lambda'' \lambda'''} \zeta_{\lambda'} \zeta_{\lambda'''} \int d^3p_r \phi_c^2(p_r) Z(\lambda'', \lambda') \int \frac{d^3k'}{|k'|^2} \psi(-k') \times \int d^3q_1 \psi(q_1) \tilde{X}(2, 1, 2, \lambda'', \lambda', k') = 0
\] (69)

with

\[ q_1 + q_2 + P_R = q_1 + q_2 = 0; \]

i.e.,

\[ q_2 = -q_1 \]
which simplifies the polarization vector terms in \(X\) considerably (see APPENDIX A, equation (A8)). Summing over \(\lambda_1, \lambda_2\) in (69), we obtain

\[
\int d^3 p_r \left[ \frac{p_r^2}{2 \mu} - \epsilon \right] \phi_c^2(p_r) \int d^3 q_1 \psi^2(q_1) \\
- \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3 k}{|k|^2} d^3 p_r \phi_c(p_r) \phi_c(p_r - k) \int d^3 q_1 \psi^2(q_1) \\
- \frac{6 \alpha}{32 \pi^2} \sum_{\lambda', \lambda''} \sum_{\lambda_1, \lambda_2 = 0} \zeta_{\lambda_1} \bar{\zeta}_{\lambda_2} \int d^3 p_r \phi_c^2(p_r) \int \frac{d^3 k'}{|k'|^2} \int d^3 q_1 \psi(q_1) \psi(-k') \times \\
\times \int d^3 q_1 \psi(q_1) \bar{X}(2, 1, 2, \lambda'', \lambda', \lambda'')_{\lambda'' \leftrightarrow \lambda_1, \lambda' \leftrightarrow \lambda_2} = 0
\]  

(70)

Summing again to remove the \(\lambda_1, \lambda_2\) terms still present in the “quartic” term of (70), that is, working out

\[
\sum_{\lambda', \lambda'' = 0} \sum_{\lambda_1, \lambda_2 = 0} \zeta_{\lambda_1} \bar{\zeta}_{\lambda_2} \bar{X}(2, 1, 2, \lambda'', \lambda', \lambda'')_{\lambda'' \leftrightarrow \lambda_1, \lambda' \leftrightarrow \lambda_2} = 0
\]  

(71)

using the convention of [14]

\[
\epsilon_1(-k) = -\epsilon_1(k), \quad \epsilon_2(-k) = \epsilon_2(k)
\]  

(72)

and recalling (39), we can simplify (70) to

\[
\int d^3 p_r \left[ \frac{p_r^2}{2 \mu} - \epsilon \right] \phi_c^2(p_r) \int d^3 q_1 \psi^2(q_1) \\
- \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3 k}{|k|^2} d^3 p_r \phi_c(p_r) \phi_c(p_r - k) \int d^3 q_1 \psi^2(q_1) \\
- \frac{12 \cdot 36 \alpha}{32 \pi^2} \sum_{\lambda'} \sum_{\lambda_2 = 0} \zeta_{\lambda_1} \bar{\zeta}_{\lambda_2} \int d^3 p_r \phi_c^2(p_r) \int \frac{d^3 k'}{|k'|^2} \int d^3 q_1 \psi(q_1) \psi(-k') \times \\
\times \left\{ \left( \epsilon_1(k') \cdot \epsilon_1(q_1) + \epsilon_2(k') \cdot \epsilon_2(q_1) - \epsilon_1(q_1) \cdot \frac{k'}{|k'|} \right) \right\} = 0
\]  

(73)

or,

\[
\epsilon = \left\{ \int d^3 p_r \frac{p_r^2}{2 \mu} \phi_c^2(p_r) - \frac{\alpha}{4 \pi^2} \gamma(2) \int \frac{d^3 k}{|k|^2} d^3 p_r \phi_c(p_r) \phi_c(p_r - k) \right\} \int d^3 p_r \phi_c^2(p_r) \\
- \frac{18 \alpha}{7} \int dk' \psi(-k') \int dq_1 q_1^2 \psi(q_1) \int d^3 q_1 \psi^2(q_1)
\]  

(74)

We will use trial functions of the scaled hydrogenic-type (equation (49)); for example, for the ground state we have

\[
\phi_c(p_r) \propto \frac{1}{(p_r^2 + a^2)^2},
\]
where \( a \) is an arbitrary scale parameter. For the trial function \( \psi(q_1) \), we choose a form
\[
\psi(a, q_1) = \frac{1}{(q_1^2 + a^2)^2} e^{-\frac{q_1}{a}},
\]
using the same scale \( a \) as for \( \phi_c(p_r) \). In the second part of (74), changing the variable by letting \( q'_1 = \frac{q_1}{a} \) allows us to pull out the \( a \). That is,
\[
\epsilon = a^2 \frac{a \alpha \gamma(2)}{m} - \frac{9 a \alpha}{14 \pi} Y,
\]
where we define
\[
Y = \int dq'_1 \psi(-q'_1) \int dq'_1 q'_1^2 \psi(q'_1).
\]
If we optimize (75) with respect to \( a \) we obtain the optimum value
\[
a_n = \frac{m \alpha}{2} \left( \frac{10}{21 n} + \frac{9 Y}{14 \pi} \right)
\]
and so the corresponding
\[
\epsilon_n = -\frac{m \alpha^2}{4} \left( \frac{10}{21 n} + \frac{9 Y}{14 \pi} \right)^2.
\]
Note that the binding energy is non-zero as \( n \to \infty \), quite unlike the Abelian (QED-like) case.

We try a function
\[
\psi(q'_1) = \frac{1}{(q'_1^2 + 1)^N}
\]
in (78) and use the 1\( S \) and 2\( S \) levels to fix \( \alpha \) and \( m \). Thus we get the predicted values for the 3\( S \), 4\( S \), 5\( S \) and 6\( S \) states for \( c\bar{c} \) which are listed in Table III and for \( b\bar{b} \) in Table IV. Alternatively, trying a function of type
\[
\psi(q'_1) = e^{-\gamma q'^N}
\]
produces the results in Table V and Table VI, for \( c\bar{c} \) and \( b\bar{b} \) respectively. These results are for the \( \beta = 2 \) case only and we find that the inclusion of the nonAbelian terms improves the agreement with experiment substantially over the results of Tables I and II which were derived using only the modified Balmer formula (nonAbelian terms suppressed). Once again, the bottomonium values are better than the charmonium ones. The trial function (80) seems somewhat better than (79), but the difference is not large, suggesting that the approximate variational solutions are reasonably accurate. Our results vary with our choice of parameter \( N \), with \( N = 1.75 \) giving the best results (note that for our trial function (79), we must have \( N > 1.5 \) to insure convergence of our integrals in (76); pushing down to this limit improves the results slightly. For (80), we need \( N > 0 \).

C. NonAbelian terms present: \( \beta = 3, 4, 5 \) cases

For the \( \beta = 3, 4, \) and 5 cases, we follow the same procedure as in the previous section. Our equations become more complicated and one finds that it is easier to deal with the multidimensional calculations by resorting to Monte Carlo integration techniques [15]. Using the trial function (79), convergence is achieved much more quickly for the \( N = 3, 4 \) cases. Comparing these results with the \( \beta = 2 \) calculation for \( N = 3, 4 \) is sufficient to give us an idea of the effect of increasing the number of gluons. From Table VII, for \( c\bar{c} \) and \( N = 3 \), we can see that the \( \beta = 5 \) results are better than the \( \beta = 4 \) results, which are in turn an improvement over \( \beta = 3 \). However, none improve on the answers for \( \beta = 2 \). The same pattern holds for \( N = 4 \) and for \( b\bar{b} \).
VI. CONCLUSIONS

Using the variational method and the Hamiltonian formalism of QCD, we have derived an infinite chain of equations for a quark-antiquark system interacting via an arbitrary number of gluons. These coupled equations are in principle exact, but not tractable. We attempt an approximation in which we decouple these equations using an ansatz borrowed from QED-type calculations. This leads to a sequence of (increasingly more complex) equations for the $q\bar{q}$ system with $\beta = 0, 1, 2, ...$ gluons present. NonAbelian effects appear only for $\beta \geq 2$. For $\beta = 0, 1$ we have only the Abelian Coulomb-type interaction present, and this leads to a Balmer-like mass-spectrum formula. We solve the $\beta = 2$ equation variationally, and work out predictions for the low-lying energy levels of charmonium and bottomonium (at least in the nonrelativistic limit). The results are encouraging since they show a substantial improvement over the Abelian approximation. We perform the same procedure for $\beta = 3, 4, 5$ but find that the results do not improve upon the ones for $\beta = 2$. It would be useful to see to what extent the results are improved if the relativistic versions of these calculations are performed.

ACKNOWLEDGMENTS

The support of the Natural Sciences and Engineering Research Council of Canada for this project is gratefully acknowledged.
APPENDIX A

In Equation (31), we use the following definitions:

\[
\begin{align*}
\tilde{Q}_1(\beta, n, s_2, s, \lambda'', k) &= Q_1(\beta, n, s_2, s, 0, k) + Q_1(\beta, n, s_2, s, 3, k) \\
\tilde{Q}_2(\beta, p, s_1, s, \lambda'', k) &= Q_2(\beta, p, s_1, s, 0, k) + Q_2(\beta, p, s_1, s, 3, k)
\end{align*}
\]  

for \( \lambda'' = 0 \), and

\[
\begin{align*}
\tilde{Q}_1(\beta, n, s_2, s, \lambda'', k) &= Q_1(\beta, n, s_2, s, \lambda'', k) \\
\tilde{Q}_2(\beta, p, s_1, s, \lambda'', k) &= Q_2(\beta, p, s_1, s, \lambda'', k)
\end{align*}
\]  

for \( \lambda'' = 1 \). And,

\[
\begin{align*}
\tilde{X}(\beta, k, j, \lambda''', \lambda', k') &= X(\beta, k, j, 0, 0, k') + X(\beta, k, j, 3, 3, k') \\
&+ X(\beta, k, j, 0, 3, k') + X(\beta, k, j, 3, 0, k') \quad \text{for} \quad \lambda', \lambda''' = 0 \\
\tilde{X}(\beta, k, j, \lambda''', \lambda', k') &= X(\beta, k, j, 0, \lambda', k') + X(\beta, k, j, 3, \lambda', k') \quad \text{for} \quad \lambda' = 1, 2; \lambda''' = 0 \\
\tilde{X}(\beta, k, j, \lambda''', \lambda', k') &= X(\beta, k, j, \lambda''', 0, k') + X(\beta, k, j, \lambda''', 3, k') \quad \text{for} \quad \lambda' = 0; \lambda''' = 1, 2 \\
\tilde{X}(\beta, k, j, \lambda''', \lambda', k') &= X(\beta, k, j, \lambda''', \lambda', k') \quad \text{for} \quad \lambda', \lambda''' = 1, 2
\end{align*}
\]  

where

\[
\begin{align*}
Q_1(\beta, n, s_2, s, \lambda'', k) &= \frac{v(n, s_2) \ell''(k)}{\omega(n) \omega(n-k)} \overline{\lambda''(n-k, s)} J_1^{\beta+1} \\
Q_2(\beta, p, s_1, s, \lambda'', k) &= \frac{\bar{u}(p, s_1) \ell''(k)}{\omega(p) \omega(p-k)} \overline{\lambda''(p-k, s)} J_2^{\beta+1} \\
Q_3(\beta, k, n, s_2, s) &= \frac{v(n, s_2) \ell''(q_k)}{\omega(n) \omega(n+q_k)} \overline{\lambda''(n+q_k, s)} J_3^{\beta-1}(z_k) \\
Q_4(\beta, k, p, s_1, s) &= \frac{\bar{u}(p, s_1) \ell''(q_k)}{\omega(p) \omega(p+q_k)} \overline{\lambda''(p+q_k, s)} J_4^{\beta-1}(z_k)
\end{align*}
\]  

and

\[
X(\beta, k, j, \lambda''', \lambda', k') = 2 K_6^{\beta'}(z_k, z_j) \left[ \epsilon_{\lambda'}(k') \cdot \epsilon_{\lambda_k}(q_k) \epsilon_{\lambda_j}(q_j) \cdot \epsilon_{\lambda'''}(q_k + q_j - k') \\
- \epsilon_{\lambda_k}(q_k) \cdot \epsilon_{\lambda_j}(q_j) \epsilon_{\lambda'}(k') \cdot \epsilon_{\lambda'''}(q_k + q_j - k') \right]
\]  

And, lastly, summing over repeated indices,

\[
I^\beta = \left\{ t_{1,2}^{z_1} \cdots t_{1,2}^{z_1} t_{1,2}^{z_1} \right\} \delta_{1,2}^{z_1} \delta_{1,2}^{z_1} \delta_{1,2}^{z_1} \\
\times (\beta)  \left\{ t_{1,2}^{z_1} \cdots t_{1,2}^{z_1} t_{1,2}^{z_1} \right\}
\]

\[
J_1^{\beta+1} = \left\{ t_{1,2}^{z_1} \cdots t_{1,2}^{z_1} t_{1,2}^{z_1} \right\} \delta_{1,2}^{z_1} \delta_{1,2}^{z_1} \delta_{1,2}^{z_1} \\
\times (\beta+1)  \left\{ t_{1,2}^{z_1} \cdots t_{1,2}^{z_1} t_{1,2}^{z_1} \right\}
\]
\[ J_2^\beta+1 = \left\{ t_{l_1'}^{z_{1'}} \ldots t_{l_{\beta+1}'}^{z_{\beta+1}} \right\} t_{l_i'}^{\beta_i} \delta l_{i_i} \beta \delta \beta_{\beta+2} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} t_{l_{\beta+2}} \times (\beta + 1)! \left\{ t_{l_1} t_2 t_{l_2} t_3 \ldots t_{l_{\beta+1}} t_{l_{\beta+2}} \right\} \]

\[ J_3^\beta-1(z_k) = \left\{ t_{l_1'}^{z_{1'}} \ldots t_{l_{\beta}'}^{z_{\beta}} t_{l_{\beta+1}'}^{z_{\beta+1}} \right\} t_{l_i'}^{\beta_i} \delta l_{i_i} \beta \delta \beta_{\beta} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} t_{l_{\beta+2}} \times (\beta - 1)! \left\{ t_{l_1} t_2 t_{l_2} t_3 \ldots t_{l_{\beta-1}} t_{l_{\beta}} \right\} \]

\[ J_4^\beta-1(z_k) = \left\{ t_{l_1'}^{z_{1'}} \ldots t_{l_{\beta}}^{z_{\beta}} t_{l_{\beta+1}'}^{z_{\beta+1}} \right\} t_{l_i'}^{\beta_i} \delta l_{i_i} \beta \delta \beta_{\beta} \delta \beta_{\beta} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} t_{l_{\beta+2}} \times (\beta - 1)! \left\{ t_{l_1} t_2 t_{l_2} t_3 \ldots t_{l_{\beta-1}} t_{l_{\beta}} \right\} \]

\[ K_0^\beta(z_k, z_j) = f_a z_j a f_a z_k e \left\{ t_{l_1'}^{z_{1'}} \ldots t_{l_{\beta}}^{z_{\beta}} t_{l_{\beta+1}'}^{z_{\beta+1}} \right\} \delta l_{i_i} \beta \delta \beta_{\beta} \delta \beta_{\beta} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} \delta \beta_{\beta+1} t_{l_{\beta+2}} \times \beta! \left\{ t_{l_1} t_2 t_{l_2} t_3 t_{l_4} \ldots t_{l_{\beta}} t_{l_{\beta+1}} \right\} \]

(A9)

A closer look at our sums (A9) reveals the following relationships:

\[ J_1^\beta+1 = J_2^\beta+1 \]
\[ J_3^\beta(z_{\beta+1}) = J_4^\beta(z_{\beta+1}) \]
\[ I^{\beta+1} = (\beta + 1) J_1^{\beta+1} = (\beta + 1)^2 J_3^\beta(z_{\beta+1}) \]

(A10)

so that the expressions in (37) and (38) (i.e., \( \gamma(\beta) \)) may be simplified to

\[ \gamma(\beta) = \left\{ \frac{J_1^{\beta+1} J_2^\beta(z_{\beta+1})}{I^\beta I^{\beta+1}} + \frac{J_2^\beta J_3^\beta(z_{\beta+1})}{I^\beta I^{\beta+1}} \right\} = \left\{ \frac{J_1^{\beta+1} J_3^\beta(z_{\beta+1})}{I^\beta I^{\beta+1}} + \frac{J_2^{\beta+1} J_4^\beta(z_{\beta+1})}{I^\beta I^{\beta+1}} \right\} \]

\[ = 2 \frac{J_1^{\beta+1} J_4^\beta(z_{\beta+1})}{I^\beta I^{\beta+1}} = 2 \frac{I^{\beta+1}}{(\beta + 1)^3 I^\beta} \]

(A11)
APPENDIX B

Removing the permutation terms in (29) leads us to the set of equations:

\[
\begin{align*}
&\left[\omega_p + \omega_n + m (m_0 - m) \left(\frac{1}{\omega_p} + \frac{1}{\omega_n}\right) + \sum_{k=1}^{\beta} |q_k| - E\right] I^\beta \phi^\beta(p, n, s_1, s_2, q, \lambda) \\
+ g' m \sum_{\lambda'' = 0}^{2} \sum_{s}^{} \zeta_{\lambda''} \int d^3 k \frac{\tilde{g}_1(\beta, n, s_2, \lambda'', k)}{|k|} \phi^{\beta+1}(p, n - k, s_1, s, q, \lambda, \lambda'') \\
- g' m \sum_{\lambda'' = 0}^{2} \sum_{s}^{} \zeta_{\lambda''} \int d^3 k \frac{\tilde{g}_2(\beta, p, s_1, s, \lambda'', k)}{|k|} \phi^{\beta+1}(p - k, n, s_2, q, \lambda, \lambda'') \\
+ g' m \sum_{k=1}^{\beta} \frac{Q_3(\beta, n, s_2, s)}{|q_k|} \phi^{\beta-1}(p, n + q_k, s_1, s, q_k, \lambda_k) \\
- g' m \sum_{s}^{} \sum_{k=1}^{\beta} \frac{Q_4(\beta, n, s_1, s)}{|q_k|} \phi^{\beta-1}(p + q_k, n, s_2, q_k, \lambda_k) \\
- \frac{g'}{2} \sum_{k \neq j \neq l}^{\beta} T_1(\beta, k, j, l) \phi^{\beta-3}(p, n, s_1, s_2, q_k, q_j, q_l, \lambda_k, \lambda_j, \lambda_l) \frac{\delta^3(q_k + q_j + q_l)}{|q_k| |q_j| |q_l|} \\
+ \frac{g'}{2} \sum_{k \neq j} \zeta_{\lambda''} \sum_{k \neq j}^{\beta} T_2(\beta, k, j, \lambda'') \phi^{\beta-4}(p, n, s_1, s_2, q_k, q_j, q_l, \lambda_k, \lambda_j, \lambda_l) \frac{\delta^3(q_k + q_j + q_l)}{|q_k| |q_j| |q_l|} \\
- \frac{g^2}{8} \sum_{k \neq j \neq l \neq m}^{\beta} F_1(\beta, k, j, l, m) \phi^{\beta-4}(p, n, s_1, s_2, q_k, q_j, q_l, q_m, \lambda_k, \lambda_j, \lambda_l) \frac{\delta^3(q_k + q_j + q_l + q_m)}{|q_k| |q_j| |q_l| |q_m|} \\
- \frac{g^2}{8} \sum_{k \neq j \neq l \neq m}^{\beta} F_2(\beta, k, j, l, \lambda') \phi^{\beta-3}(p, n, s_1, s_2, q_k + q_j + q_l, q_k, q_j, q_l, \lambda', \lambda_k, \lambda_j, \lambda_l) \frac{\delta^3(q_k + q_j + q_l)}{|q_k| |q_j| |q_l| |q_k + q_j + q_l|} \\
- \frac{g^2}{8} \sum_{\lambda'' = 0}^{2} \sum_{\lambda'' = 0}^{\beta} \zeta_{\lambda''} \zeta_{\lambda'''} \sum_{k \neq j} \frac{d^3 k'}{|q_k| |q_j| |k'| |q_k + q_j - k'|} \times \tilde{X}(\beta, k, j, \lambda'', \lambda', k') \phi^\beta(p, n, s_1, s_2, q_k + q_j - k', k', q_k, q_j, \lambda'', \lambda', \lambda_k, \lambda_j) = 0
\end{align*}
\]

where the \(Q\)'s and \(\tilde{X}\) are defined in APPENDIX A and

\[
\begin{align*}
\tilde{T}_2(\beta, k, j, \lambda'') &= T_2(\beta, k, j, 0) + T_2(\beta, k, j, 3) \\
\tilde{F}_2(\beta, k, j, l, \lambda') &= F_2(\beta, k, j, l, 0) + F_2(\beta, k, j, l, 3)
\end{align*}
\]

for \(\lambda'', \lambda' = 0\), and

\[
\begin{align*}
\tilde{T}_2(\beta, k, j, \lambda'') &= T_2(\beta, k, j, \lambda'') \\
\tilde{F}_2(\beta, k, j, l, \lambda') &= F_2(\beta, k, j, l, \lambda')
\end{align*}
\]

for \(\lambda'', \lambda' = 1, 2\).
Furthermore,

\[
T_1(\beta, k, j, l) = G_2^{\beta-3}(z_k, z_j, z_l) \epsilon^\nu_{\lambda}(q_k) \epsilon^\nu_{\lambda}(q_l) \left[ q_k \cdot \epsilon_{\lambda}(q_j) - |q_k| \epsilon^0_{\lambda}(q_j) \right] \tag{B4}
\]

\[
T_2(\beta, k, j, \lambda'') = G_3^{\beta-1}(z_k, z_j)
\times \left[ -\epsilon^\nu_{\lambda}(q_k + q_j) \epsilon^\nu_{\lambda}(q_k) \left[ (q_k + q_j) \cdot \epsilon_{\lambda}(q_j) - |q_k + q_j| \epsilon^0_{\lambda}(q_j) \right] \\
+ \epsilon^\nu_{\lambda}(q_k) \epsilon^\nu_{\lambda}(q_j) \left[ q_k \cdot \epsilon_{\lambda}(q_k) - |q_k| \epsilon^0_{\lambda}(q_k) \right] \\
- \epsilon^\nu_{\lambda}(q_k) \epsilon^\nu_{\lambda}(q_k + q_j) \left[ q_k \cdot \epsilon_{\lambda}(q_k) - |q_k| \epsilon^0_{\lambda}(q_k) \right] \right] \tag{B5}
\]

\[
F_1(\beta, k, j, l, m) = K_2^{\beta-4}(z_k, z_j, z_l, z_m) \epsilon_{\lambda k} \epsilon^\nu(q_k) \epsilon^\nu_{\lambda}(q_j) \left[ \epsilon^0_{\lambda}(q_l) \epsilon^0_{\lambda}(q_m) + \epsilon_{\lambda}(q_l) \cdot \epsilon_{\lambda}(q_m) \right] \tag{B6}
\]

\[
F_2(\beta, k, j, l, \lambda') = K_3^{\beta-2}(z_k, z_j, z_l)
\times \left[ 2 \epsilon_{\lambda k} \epsilon^\nu(q_k) \epsilon^\nu_{\lambda}(q_k + q_l) \left[ \epsilon^0_{\lambda}(q_j) \epsilon^0_{\lambda}(q_l) + \epsilon_{\lambda}(q_j) \cdot \epsilon_{\lambda}(q_l) \right] \\
+ 2 \epsilon_{\lambda j} \epsilon^\nu(q_j) \epsilon^\nu_{\lambda}(q_j) \left[ \epsilon^0_{\lambda}(q_k + q_l) + \epsilon_{\lambda}(q_k + q_l) \cdot \epsilon_{\lambda}(q_k) \right] \right]. \tag{B7}
\]

Finally, (summing over repeated indices),

\[
I^\beta = t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z \delta^\nu_{l_1} j^\nu \delta_{l_1} i^\nu \delta_{l_1+1} j^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z | \text{permutations}[z_1, \ldots, z_\beta] \right]
\]

\[
J_1^{\beta+1} = t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z \delta^\nu_{l_1} j^\nu \delta_{l_1} i^\nu \delta_{l_1+2} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+2}^z t_{l_\beta+1}^z | \text{permutations}[a, z_1, \ldots, z_\beta] \right]
\]

\[
J_2^{\beta+1} = t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z t_{l_\beta+1}^z \delta^\nu_{l_1} j^\nu \delta_{l_1} i^\nu \delta_{l_1+2} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+2}^z t_{l_\beta+1}^z | \text{permutations}[a, z_1, \ldots, z_\beta] \right]
\]

\[
J_3^{\beta-1}(z_k) = t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z t_{l_\beta}^z \delta_{l_1} j^\nu \delta_{l_1+1} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+1}^z | \text{permutations}[z_1, \ldots, z_k, \ldots, z_\beta] \right]
\]

\[
J_4^{\beta-1}(z_k) = t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z t_{l_\beta+1}^z \delta_{l_1} j^\nu \delta_{l_1+1} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+1}^z | \text{permutations}[z_1, \ldots, z_k, \ldots, z_\beta] \right]
\]

\[
G_2^{\beta-3}(z_k, z_j, z_l) = i f z_k z_j z_l t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z \delta_{l_1} j^\nu \delta_{l_1+1} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+1}^z | \text{permutations}[z_1, \ldots, z_k, \ldots, z_l, \ldots, z_\beta] \right]
\]

\[
G_3^{\beta-1}(z_k, z_j) = i f z_k z_j t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta}^z t_{l_\beta+1}^z \delta_{l_1} j^\nu \delta_{l_1+1} j^\nu \delta_{l_1+1} i^\nu \\
\times \left[ t_{l_1}^z t_{l_2}^z \cdots t_{l_\beta+1}^z | \text{permutations}[z_1, \ldots, z_k, \ldots, z_j, \ldots, z_\beta] \right]
\]
\[
K_2^{-4}(z_k, z_l, z_1, z_m) = f_{a z_k} f_{a z_l} f_{a z_1} f_{a z_m} \times [t_{l_1}^{\beta_1} t_{l_2}^{\beta_2} \ldots t_{l_1}^{\beta_{l_1-1}} t_{l_2}^{\beta_{l_2-1}} + \text{permutations}\{c, z_1, \ldots, z_k, z_j, \ldots, z_m\}]
\times [t_{l_1}^{\beta_1} t_{l_2}^{\beta_2} \ldots t_{l_1}^{\beta_{l_1-1}} t_{l_2}^{\beta_{l_2-1}} + \text{permutations}\{z_1, \ldots, z_k, z_j, z_l, z_m, \ldots, z_m\}]
\]

\[
K_3^{-2}(z_k, z_l, z_1) = f_{a z_k} f_{a z_l} f_{a z_1} \times \left[ \sum_{j' \neq j} t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} \ldots t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} t_{l_1}^{\beta_{l_1-1}} t_{l_2}^{\beta_{l_2-1}} + \text{permutations}\{e, z_1, \ldots, z_k, z_j, \ldots, z_l\} \right]
\times \left[ \sum_{j' \neq j} t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} \ldots t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} t_{l_1}^{\beta_{l_1-1}} + \text{permutations}\{z_1, \ldots, z_k, z_j, \ldots, z_l\} \right]
\]

\[
K_6^{-1}(z_k, z_l) = f_{a z_k} f_{a z_l} e \times \left[ \sum_{j' \neq j} t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} \ldots t_{l_1}^{1} t_{l_2}^{1} t_{l_3}^{1} t_{l_1}^{\beta_{l_1-1}} t_{l_2}^{\beta_{l_2-1}} + \text{permutations}\{d, e, z_1, \ldots, z_k, z_j, \ldots, z_l\} \right]
\]

\[(B8)\]
References

[1] W. Dykshoorn, R. Koniuk and J. Darewych, in *Variational Calculations in Quantum Field Theory*, edited by L. Polley and DEL Pottinger (Singapore, World Scientific, 1988) pp. 188-192.
[2] W. Dykshoorn and R. Koniuk, Phys. Rev. A **41**, 64 (1990).
[3] J.W. Darewych and M. Horbatsch, J. Phys. B **22**, 973 (1989) and **23**, 337 (1990).
[4] T. Zhiang, L. Xiao and R. Koniuk, Can. J. Phys. **70**, 670 (1992).
[5] J. W. Darewych, M. Horbatsch and R. Koniuk, Phys. Rev. D **45**, 675 (1992).
[6] W.C. Berseth and J.W. Darewych, Phys. Lett. A **178**, 347 (1993) and Erratum **185**, 503 (1994).
[7] L. Di Leo and J.W. Darewych, Can. J. Phys. **70**, 412 (1992) and **71**, 365 (1993).
[8] T. Zhang and R. Koniuk, Phys. Rev. D **43**, 1688 (1991) and **48**, 5382 (1993).
[9] J.W. Darewych, Ukr. J. Phys **41**, 41 (1996).
[10] J.R. Spence and J.P. Vary, Phys. Rev. C **52**, 1668 (1995).
[11] Review of Particle Properties, 1998, European Physical Journal.
[12] F. Mandl and G. Shaw, *Quantum Field Theory*, (John Wiley & Sons, Cichester, 1984).
[13] J.W. Darewych *et al*, Phys Rev C **47**, 1885 (1993).
[14] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, (McGraw-Hill, New York, 1965).
[15] W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery, *Numerical Recipes*. (Cambridge University Press, 1992) pp. 309-314.
Table I. Predicted energies for $c\bar{c}$ (nonAbelian terms suppressed)

| $n$ | $\beta = 0$ | $\beta = 1$ | $\beta = 2$ | $\text{Exp}'(\text{MeV})$ |
|-----|-------------|-------------|-------------|-------------------------|
| 1   | $\alpha = 0.9542$ | $\alpha = 2.181$ | $\alpha = 2.672$ | 3096.88 |
| 2   | $m_c = 1941$ | $m_c = 1941$ | $m_c = 1941$ | 3686 |
| 3   | 3795        | 3795        | 3795        | 4040 |
| 4   | 3833        | 3833        | 3833        | 4415 |
| 5   | 3851        | 3851        | 3851        | $-- --$ |
| 6   | 3860        | 3860        | 3860        | $-- --$ |

Table II. Predicted energies for $b\bar{b}$ (nonAbelian terms suppressed)

| $n$ | $\beta = 0$ | $\beta = 1$ | $\beta = 2$ | $\text{Exp}'(\text{GeV})$ |
|-----|-------------|-------------|-------------|-------------------------|
| 1   | $\alpha = 0.575135$ | $\alpha = 1.31459$ | $\alpha = 1.61038$ | 9.46037 |
| 2   | $m_b = 5.10547$ | $m_b = 5.10547$ | $m_b = 5.10547$ | 10.02330 |
| 3   | 10.1275      | 10.1275      | 10.1275      | 10.3553 |
| 4   | 10.1640      | 10.1640      | 10.1640      | 10.5800 |
| 5   | 10.1809      | 10.1809      | 10.1809      | 10.865 |
| 6   | 10.1901      | 10.1901      | 10.1901      | 11.019 |

Table III. Predicted energies for $c\bar{c}$ (nonAbelian terms present, $\beta = 2$) with trial function (79)

| $n$ | $N = 1.75$ | $N = 2$ | $N = 3$ | $\text{Exp}'(\text{MeV})$ |
|-----|------------|--------|--------|-------------------------|
| 1   | $\alpha = .7372$ | $\alpha = 1.053$ | $\alpha = 1.550$ | 3096.88 |
| 2   | $m_c = 3370$ | $m_c = 2717$ | $m_c = 2267$ | 3686 |
| 3   | 3871       | 3863    | 3848    | 4040 |
| 4   | 3961       | 3948    | 3923    | 4415 |
| 5   | 4014       | 3998    | 3965    | $-- --$ |
| 6   | 4050       | 4031    | 3993    | $-- --$ |
| $\infty$ | 4223      | 4190    | 4121    | $-- --$ |

Table IV. Predicted energies for $b\bar{b}$ (nonAbelian terms present, $\beta = 2$) with trial function (79)

| $n$ | $N = 1.75$ | $N = 2$ | $N = 3$ | $\text{Exp}'(\text{GeV})$ |
|-----|------------|--------|--------|-------------------------|
| 1   | $\alpha = 0.5200$ | $\alpha = .7016$ | $\alpha = .9804$ | 9.46037 |
| 2   | $m_b = 6.4710$ | $m_b = 5.847$ | $m_b = 5.4169$ | 10.02330 |
| 3   | 10.1999    | 10.1928 | 10.1781 | 10.3553 |
| 4   | 10.2862    | 10.2742 | 10.2494 | 10.5800 |
| 5   | 10.3372    | 10.3219 | 10.2902 | 10.865 |
| 6   | 10.3710    | 10.3533 | 10.3166 | 11.019 |
| $\infty$ | 10.5366 | 10.5046 | 10.4386 | $-- --$ |
Table V. Predicted energies for $\bar{c}c$ (nonAbelian terms present, $\beta = 2$) with trial function (80)

| $n$ | $N = .5$ | $N = 1$ | $N = 2$ | $Exp'(MeV)$ |
|-----|---------|---------|---------|-------------|
| 1   | $\alpha = .1014$ | $\alpha = .9197$ | $\alpha = 1.592$ | 3096.88     |
| 2   | $m_c = 18119$    | $m_c = 9234$    | $m_c = 2244$    | 3686        |
| 3   | 3881           | 3867           | 3846           | 4040        |
| 4   | 3979           | 3954           | 3920           | 4415        |
| 5   | 4037           | 4006           | 3962           | -- -- --     |
| 6   | 4076           | 4040           | 3989           | -- -- --     |
| $\infty$ | 4270  | 4205           | 4114           | -- -- --     |

Table VI. Predicted energies for $b\bar{b}$ (nonAbelian terms present, $\beta = 2$) with trial function (80)

| $n$ | $N = .5$ | $N = 1$ | $N = 2$ | $Exp'(GeV)$ |
|-----|---------|---------|---------|-------------|
| 1   | $\alpha = 0.0931$ | $\alpha = .6258$ | $\alpha = 1.0037$ | 9.46037     |
| 2   | $m_b = 20.56$    | $m_b = 6.0538$    | $m_b = 5.3948$    | 10.02330    |
| 3   | 10.2098        | 10.1960         | 10.1767         | 10.3553     |
| 4   | 10.3029        | 10.2796         | 10.2470         | 10.5800     |
| 5   | 10.3586        | 10.3288         | 10.2871         | 10.865      |
| 6   | 10.3958        | 10.3612         | 10.3130         | 11.019      |
| $\infty$ | 10.5812  | 10.5190       | 10.4322         | -- -- --     |

Table VII. Predicted energies for $e\bar{c}$ (nonAbelian terms present, $\beta = 2, 3, 4, 5$) with $N = 3$ and trial function (79)

| $n$ | $\beta = 5$ | $\beta = 4$ | $\beta = 3$ | $\beta = 2$ | $Ex(MeV)$ |
|-----|-------------|-------------|-------------|-------------|-----------|
| 1   | $\alpha = 1.886 - 1.883$ | $\alpha = 1.930 - 1.928$ | $\alpha = 1.799 - 1.798$ | $\alpha = 1.550$ | 3096.88   |
| 2   | $m_c = 2179 - 2180$    | $m_c = 2155 - 2156$    | $m_c = 2129$    | $m_c = 2267$    | 3686      |
| 3   | 3841 - 3842       | 3839           | 3836           | 3848         | 4040      |
| 4   | 3912           | 3908           | 3903           | 3923         | 4415      |
| 5   | 3951           | 3946           | 3940           | 3965         | -- -- --  |
| 6   | 3977           | 3971           | 3964           | 3993         | -- -- --  |
| $\infty$ | 4091 - 4092  | 4081           | 4069           | 4121         | -- -- --  |