Revisiting $k$-tuple Dominating Sets with Emphasis on Small Values of $k$

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Abstract
For any graph $G$ of order $n$ with degree sequence $d_1 \geq \cdots \geq d_n$, we define the double Slater number $s_{t \times 2}(G)$ as the smallest integer $t$ such that $t + d_1 + \cdots + d_{t-e} \geq 2n - p$ in which $e$ and $p$ are the number of end-vertices and penultimate vertices of $G$, respectively. We show that $\gamma_{t \times 2}(G) \geq s_{t \times 2}(G)$, where $\gamma_{t \times 2}(G)$ is the well-known double domination number of a graph $G$ with no isolated vertices. We prove that the problem of deciding whether the equality holds for a given graph is NP-complete even when restricted to 4-partite graphs. We also prove that the problem of computing $\gamma_{t \times 2}(G)$ is NP-hard even for comparability graphs of diameter two. Some results concerning these two parameters are given in this paper improving and generalizing some earlier results on double domination in graphs. We give an upper bound on the $k$-tuple domatic number of graphs with characterization of all graphs attaining the bound. Finally, we characterize the family of all full graphs, leading to a solution to an open problem given in a paper by Cockayne and Hedetniemi (Networks 7: 247–261, 1977).

Keywords Double domination number · Double Slater number · NP-complete · Tree · $k$-tuple domatic partition · Full graphs
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1 Introduction and Preliminaries

Throughout this paper, we consider $G$ as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [19] as a reference for terminology and notation which are not explicitly defined here. The open neighborhood of a vertex $v$ is denoted by $N(v)$, and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. We denote the degree of vertex $v$ by $\deg(v)$, and let $\deg_S(v) = |N(v) \cap S|$ in which $S \subseteq V(G)$. The minimum and maximum degrees of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. An end-vertex is a vertex of degree one and a penultimate vertex is a vertex adjacent to an end-vertex (they are called leaf and support vertex in the case of trees). Given the subsets $A$, $B \subseteq V(G)$, by $[A, B]$ we mean the set of all edges with one end point in $A$ and the other in $B$. Finally, for a given set $S \subseteq V(G)$, by $G[S]$ we represent the subgraph induced by $S$ in $G$.

A transitive orientation of a graph $G$ is an orientation $D$ such that whenever $(x, y)$ and $(y, z)$ are arcs in $D$, also there exists an edge $xz$ in $G$ that is oriented from $x$ to $z$ in $D$. A graph $G$ is a comparability graph if it has a transitive orientation. It is well known that the comparability graphs are perfect (see [19] for example).

A set $S \subseteq V(G)$ is a dominating set if each vertex in $V(G) \setminus S$ has at least one neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. For more information about domination and its related parameters, the reader can consult [15] and [16].

Slater [18] showed that the domination number of a graph of order $n$ with non-increasing degree sequence $d_1 \geq \cdots \geq d_n$ can be bound from below by the smallest integer $t$ such that $t$ added to the sum of the first $t$ terms of the above-mentioned sequence is at least $n$. This parameter was first called the Slater number and denoted by $s\ell(G)$ in [6]. This parameter and its properties have been investigated in [10] and [11].

A vertex subset $S$ of a graph $G$ with $\delta(G) \geq k - 1$ is said a $k$-tuple dominating set if $|N[v] \cap S| \geq k$ for each vertex $v$ of $G$. The $k$-tuple domination number $\gamma_{x_k}(G)$ of the graph $G$ is the minimum cardinality of a $k$-tuple dominating set in $G$. A $k$-tuple dominating set in $G$ of the minimum cardinality is called a $\gamma_{x_k}(G)$-set. This parameter was introduced by Harary and Haynes in [13]. For the especial case $k = 2$, it is common to write double dominating set and double domination number for the resulting set and graph parameter. Note that we emphasis on the small values of $k$ for various reasons given in this paper.

For a given graph $G$ of order $n$ with $e$ end-vertices, $p$ penultimate vertices and non-increasing degree sequence $d_1 \geq \cdots \geq d_n$, we define the double Slater number $s\ell_{x_2}(G)$ as follows:

$$s\ell_{x_2}(G) = \min\{t \mid t + d_1 + \cdots + d_{n-e} \geq 2n - p\}.$$ 

A vertex partition of a graph $G$ with $\delta(G) \geq k - 1$ is said to be a $k$-tuple domatic partition of $G$ if each partite set is a $k$-tuple dominating set in $G$. The $k$-tuple domatic
number \(d_{\times k}(G)\) is the maximum cardinality taken over all \(k\)-tuple domatic partitions of \(G\). A \(k\)-tuple domatic partition of \(G\) of maximum cardinality is called a \(d_{\times k}(G)\)-partition. The study of this kind of partitions was first begun by Harary and Haynes in [14] as a generalization of the well-known domatic partition in graphs ([4]). When \(k = 1\), \(d_{\times k}(G)\) and “\(k\)-tuple domatic partition” are simply written as \(d(G)\) and “domatic partition,” respectively.

This paper is organized as follows. We first present some properties of the double Slater number \(s_{\ell \times 2}(G)\) of graphs \(G\). In particular, we observe that \(s_{\ell \times 2}(G)\) is a lower bound on \(\gamma \times 2(G)\) for all graphs \(G\) with no isolated vertices. We then prove that the problem deciding whether the equality holds for a given graph \(G\) is NP-complete (even for 4-partite graphs) despite that fact that \(s_{\ell \times 2}(G)\) can be computed in linear-time. We also prove that the problem of computing \(\gamma \times k\) for \(k \geq 2\) is NP-hard for comparability graphs of diameter two. Some bounds on the double domination number in this paper improve the main results in [3] and [12]. An upper bound on the \(k\)-tuple domatic number of graphs with characterization of the extremal graphs for the bound is given in this paper. Finally, we give a complete characterization of full graphs (those graphs \(G\) for which \(d(G) = d_{\times 1}(G) = \delta(G) + 1\)), solving an open problem given in [4].

Note that for some other domination parameters, lower bounds similar to the double Slater number can be obtained. The reader can consult [1], [7] and [11] for more pieces of information about them.

2 Preliminary Properties of Double Slater Number

In this section, we prove some results on \(\gamma \times 2(G)\) and \(s_{\ell \times 2}(G)\) and discuss their relationship for general graphs \(G\). Let \(G\) be a graph of order \(n\) with \(\delta(G) \geq 2\) and degree sequence \(d_1 \geq \cdots \geq d_n\). Let \(A\) be a \(\gamma \times 2(G)\)-set. We then have

\[
|A| + \sum_{i=1}^{\frac{|A|}{2}} d_i \geq |A| + \sum_{v \in A} \deg(v) = |A| + \sum_{v \in A} |N(v) \cap A| \\
+ \sum_{v \in A} |N(v) \cap (V(G) \setminus A)| \\
\geq |A| + |A| + 2(n - |A|) = 2n.
\]

Therefore,

\[
\gamma \times 2(G) \geq s_{\ell \times 2}(G) = \min\{t \mid t + d_1 + \cdots + d_t \geq 2n\}. \tag{1}
\]

In spite of the fact that the lower bound given in (1) and its proof are simple, the family of graphs for which the equality holds cannot be characterized in polynomial time (even if we restrict the problem to some special families of graphs) unless \(P=NP\).

We will prove this later in Theorem 3.1. In the next two propositions, we discuss some properties of the double Slater number.
Proposition 2.1 Let $G$ be a graph of order $n$ with minimum degree $\delta \geq 2$ and maximum degree $\Delta$. Then,

$$\left\lceil \frac{2n}{1 + \Delta} \right\rceil \leq s\ell_{x2}(G) \leq \left\lceil \frac{2n}{1 + \delta} \right\rceil.$$

**Proof** Let $t = s\ell_{x2}(G)$. Then, $t + \Delta t \geq t + d_1 + \cdots + d_t \geq 2n$. So, $s\ell_{x2}(G) = t \geq \left\lceil 2n/(1 + \Delta) \right\rceil$. On the other hand,

$$\left\lceil \frac{2n}{1 + \delta} \right\rceil + d_1 + \cdots + d_t \geq \left\lceil \frac{2n}{1 + \delta} \right\rceil + \delta \left\lceil \frac{2n}{1 + \delta} \right\rceil = (1 + \delta)\left\lceil \frac{2n}{1 + \delta} \right\rceil \geq 2n.$$

Therefore, $s\ell_{x2}(G) \leq \left\lceil 2n/(1 + \delta) \right\rceil.$ \hfill \Box

As an immediate consequence of Proposition 2.1, we have $s\ell_{x2}(G) = \left\lceil 2n/(1 + r) \right\rceil$ for any $r$-regular graph $G$.

**Proposition 2.2** Let $G$ be a graph of order $n$ and size $m$ with minimum degree $\delta \geq 2$. Then, the following statements hold.

(i) $1 \leq s\ell_{x2}(G) - s\ell(G) \leq \lceil n/(\delta + 1) \rceil$. These bounds are sharp.

(ii) $2 \leq s\ell_{x2}(G) \leq n$. Moreover, $s\ell_{x2}(G) = 2$ if and only if $G$ has at least two vertices of degree $n - 1$, and $s\ell_{x2}(G) = n$ if and only if $m \leq \lfloor (n + \delta)/2 \rfloor$.

(iii) $s\ell_{x2}(G) = 2n/(1 + \Delta)$ if and only if $2n \equiv 0 \pmod{1 + \Delta}$ and $d_{2n/(1+\Delta)} = \Delta$.

**Proof** (i) Let $t = s\ell_{x2}(G)$. Then, $t + d_1 + \cdots + d_t \geq 2n$. So, $t - 1 + d_1 + \cdots + d_{t-1} \geq 2n - 1 - d_t \geq n$. Therefore, $s\ell(G) \leq s\ell_{x2}(G) - 1$ by the definition of $s\ell(G)$. We now let $t' = s\ell(G)$. We have

$$t' + \lceil n/(\delta + 1) \rceil + d_1 + \cdots + d_{t'} + d_{t'+1} + \cdots + d_{t'+\lfloor n/(\delta+1) \rfloor} \geq n + \lceil n/(\delta + 1) \rceil + \delta \lceil n/(\delta + 1) \rceil \geq 2n.$$

Thus, $s\ell_{x2}(G) \leq s\ell(G) + \lceil n/(\delta + 1) \rceil$.

The lower bound is sharp for the complete graph $K_n$ when $n \geq 3$. Moreover, we get from the definitions that $s\ell_{x2}(C_n) = \lceil 2n/3 \rceil$ and $s\ell(C_n) = \lceil n/3 \rceil$ for $n \geq 3$. Hence, the upper bound is sharp for the cycle $C_n$ in which $n \equiv 0 \pmod{2}$ (mod 3).

(ii) It is clear from the definition that $2 \leq s\ell_{x2}(G) \leq n$. If $s\ell_{x2}(G) = 2$, it happens that $2 + d_1 + d_2 \geq 2n$. This necessarily implies that $d_1 = d_2 = n - 1$. So, there exist at least two vertices $u$ and $v$ with $\deg(u) = \deg(v) = n - 1$. Conversely, if two vertices $x$ and $y$ are of degree $n - 1$, then $2 + d_1 + d_2 \geq 2 + \deg(x) + \deg(y) = 2n$. Hence, $s\ell_{x2}(G) = 2$.

We now prove the second “if and only if” part. Let $s\ell_{x2}(G) = n$. This implies that $d_1 + \cdots + d_{n-1} \leq n$. So, $2m \leq n + d_n = n + \delta$ which results in $m \leq \lfloor (n + \delta)/2 \rfloor$. Conversely, let $m \leq \lfloor (n + \delta)/2 \rfloor$. Then,

$$n - 1 + d_1 + \cdots + d_{n-1} = n - 1 + 2m - d_n = n - 1 + 2m - \delta \leq n - 1 + 2\lfloor (n + \delta)/2 \rfloor - \delta < 2n.$$
Therefore, \( s \ell \times 2(G) = n \).

(iii) We have \( s \ell \times 2(G) \geq 2n/(1 + \Delta) \) by Proposition 2.1. Suppose that the equality holds. Obviously, \( 2n \equiv 0 \pmod{1 + \Delta} \). On the other hand,

\[
2n \leq 2n/(1 + \Delta) + d_1 + \cdots + d_{2n/(1 + \Delta)} \leq 2n/(1 + \Delta) + 2n\Delta/(1 + \Delta) = 2n
\]

implies that \( d_{2n/(1 + \Delta)} = \Delta \). Conversely, we deduce that \( 2n/(1 + \Delta) + d_1 + \cdots + d_{2n/(1 + \Delta)} = 2n \) when \( 2n \equiv 0 \pmod{1 + \Delta} \) and \( d_{2n/(1 + \Delta)} = \Delta \).

\( \square \)

**Remark 2.3** Harary and Haynes [13] proved that \( \gamma \times 2(G) \geq 2n/(1 + \Delta) \) for all graphs \( G \) with no isolated vertices. So, the lower bound in Proposition 2.1 gives us an improvement of the result in [13]. In what follows, we show that the difference between \( s \ell \times 2(G) \) and \( 2n/(1 + \Delta) \) can be arbitrarily large. In fact, we claim that for any integer \( b \geq 1 \), the difference between these two graph parameters can be as large as \( b \). To see this, we begin with the star \( K_{1,n−1} \) with the central vertex \( u \) in which \( n = 4b + 4 \). Let \( G \) be obtained from \( K_{1,n−1} \) by constructing a path on the vertices in \( N_{K_{1,n−1}}(u) \). Note that all vertices in \( N_{K_{1,n−1}}(u) \) are of degree three except for two vertices of degree two. Clearly, \( 2n/(1 + \Delta) = 2 \). We have \( d_1 = n − 1 \) and \( d_i = 3 \) for all \( 2 \leq i \leq 4b + 2 \). We therefore have,

\[
b + 2 + d_1 + \cdots + d_{b+2} = 2n.
\]

Thus, \( s \ell \times 2(G) = b + 2 \). Therefore \( s \ell \times 2(G) − 2n/(1 + \Delta) = b \), as desired.

### 3 Complexity Results

Note that the non-increasing degree sequence of a graph \( G \) of order \( n \) can be made in linear-time by the counting sort algorithm. Therefore, sum of the first \( t \) degrees, and hence \( s \ell \times 2(G) \), can be computed in linear-time. In spite of this fact, the following theorem shows that the problem of determining whether the lower bound (1) holds with equality for a given graph \( G \) is NP-complete even if we restrict our attention to some special families of graphs.

**Theorem 3.1** The problem of deciding whether \( \gamma \times 2(G) = s \ell \times 2(G) \) for a given graph \( G \) is NP-complete even when restricted to 4-partite graphs of diameter two.

**Proof** We describe a polynomial transformation of the well-known 3-SAT problem to our problem. Consider an arbitrary instance of the 3-SAT problem, given by \( U = \{u_1, \ldots, u_a\} \) of variables (with the set of complements \( U' = \{u'_1, \ldots, u'_a\} \)) and a collection \( C = \{C_1, \ldots, C_b\} \) of three-variable clauses over \( U \cup U' \). Note that the 3-SAT problem remains NP-complete even if for each \( u_i \in U \), there are at most five clauses in \( C \) that contain either \( u_i \) or \( u'_i \) (see [9]).

For every variable \( u_i \), we associate a graph \( H_i \) of order \( 6a \) constructed from a copy \( P_{2}^{i} : q_iq'_i \) of the path \( P_2 \) by adding \( 6a - 2 \) independent vertices so that each of them is adjacent to both \( q_i \) and \( q'_i \). Corresponding to each three-variable clause \( C_j \), we create a vertex \( c_j \). The construction of graph \( H \) is completed by adding the edges \( c_jy \), where
Note that any vertex \( c \) corresponds to the vertices in Fig. 1. The graph \( G \) joining it to all vertices of \( H \) is a 4-partite graph of diameter two and order \( 2n + 2 \). Therefore, the equality (2) can be written as

\[
3b = \left| \{ (c_1, \cdots, c_b) \} \cap \{ V(H) \} \right| = \sum_{v \in V} \deg(c_1, \cdots, c_b)(v).
\]

We let \( \{ \deg(c_1, \cdots, c_b)(v) \} \cap V \) in which \( d_2' \geq \cdots \geq d_{2a+1}' \). Therefore, the equality (2) can be written as

\[
3b = \sum_{i=2}^{2a+1} d_i'.
\]

On the other hand, it is clear from the construction that the terms in \( \sum_{i=2}^{2a+1} d_i \) correspond to the vertices in \( W \). By renaming the vertices, we can write \( W = \{ w_2, \cdots, w_{2a+1} \} \) in such a way that \( d_i = \deg(w_i) \) for each \( 2 \leq i \leq 2a+1 \). Therefore,

\[
a + 1 + \sum_{i=1}^{a+1} d_i = a + 1 + d_1 + \sum_{i=2}^{a+1} \deg(w_i)
\]

\[
= a + 1 + d_1 + a + \sum_{i=2}^{a+1} \deg(H)(w_i) + \sum_{i=2}^{a+1} \deg(c_1, \cdots, c_b)(w_i)
\]

\[
= 2a + 1 + 6a^2 + b + a(6a - 1) + \sum_{i=2}^{a+1} d_i'
\]

\[
\geq 2a + 1 + 6a^2 + b + a(6a - 1) + 3b/2
\]
\[ s \ell \times_2 (G) = 12a^2 + a + 1 + 5b/2 \geq 2n. \]

This implies that \( s \ell \times_2 (G) \leq a + 1 \).

On the other hand, since every variable in \( W \) belongs to at most five clauses in \( C \), it follows that \( \deg(w_i) \leq 6a + 5 \) for all \( 2 \leq i \leq 2a + 1 \). So,

\[
a + \sum_{i=1}^{a} d_i = a + d_1 + \sum_{i=2}^{a} d_i = a + 6a^2 + b + (a - 1)(6a + 5) = 12a^2 + b - 5 < 2n.
\]

Therefore, \( s \ell \times_2 (G) = a + 1 \).

Now let the above-mentioned instance of 3-SAT be satisfiable. It follows that the subset of those vertices from \( W \) corresponding to a variables assigned TRUE along with the vertex \( z \) form a double dominating set in \( G \) of cardinality \( a + 1 \). Therefore, \( \gamma \times_2 (G) \leq a + 1 \). This shows that \( \gamma \times_2 (G) = s \ell \times_2 (G) = a + 1 \).

Conversely, if \( \gamma \times_2 (G) = s \ell \times_2 (G) \), then it must happen that \( \gamma \times_2 (G) = a + 1 \). Let \( S \) be a \( \gamma \times_2 (G) \)-set. Note that each graph \( H_i \) must have at least one vertex in \( S \) so that all vertices in \( V (H_i) \) can be double dominated by \( S \). Suppose to the contrary that \( |S \cap V (H_j)| \geq 2 \) for some \( 1 \leq j \leq a \). This implies that \( z \notin S \) and that \( |V (H_i) \cap S| = 1 \) for each \( 1 \leq i \neq j \leq a \). So, every vertex of \( H_i (i \neq j) \) different from \( q_i \) and \( q'_i \) is dominated by only one vertex in \( S \), a contradiction. The above argument guarantees that \( z \in S \) and that \( |S \cap V (H_i)| = 1 \) for each \( 1 \leq i \leq a \). We now assign the value TRUE to those variables \( u_i, u'_j \in U \cup U' \) for which \( q_i, q'_j \in S \). Since every vertex \( c_r \) has a neighbor \( q_i \) or \( q'_j \) in \( S \setminus \{z\} \), it follows that \( C_r \) contains at least one variable \( u_i \) or \( u'_j \) with the value TRUE. Thus, we have created a satisfying truth assignment for \( C \). This completes the proof. \( \square \)

For various reasons, the small values of \( k \) (especially \( k \in \{1, 2\} \)) regarding \( \gamma \times k \) attracts more attention from the experts in domination theory rather than the large ones. In particular, this parameter cannot be defined for some important families of graphs like trees when \( k > 2 \); many results for the case \( k \in \{1, 2\} \) can be generalized to the general case \( k \); one may obtain stronger results for the small values of \( k \) rather than the large ones. That is why we would rather emphasis on the case \( k = 2 \) in this paper (note that the case \( k = 1 \) leads to the usual domination in graphs). Notwithstanding this, some interesting papers treated this topic from the general point of view. For instance, Liao and Chang \[17\] proved that the decision problem \( k \)-TUPLE DOMINATING SET (associated with \( \gamma \times k \)) is NP-complete even for split graphs and for bipartite graphs. They also posed the question “what are the complexities of the \( k \)-tuple domination for other subclasses of perfect graphs?”

Here, we prove that this decision problem remains NP-complete even for a very restricted subfamily of comparability graphs, that is, comparability graphs of diameter two. To this aim, we first need to recall that the corona product \( G \odot H \) of graphs \( G \) (with \( V (G) = \{v_1, \ldots, v_n\} \)) and \( H \) is obtained from the disjoint union of \( G \) and \( n \) disjoint copies of \( H \), say \( H_1, \ldots, H_n \), such that for all \( i \in \{1, \ldots, n\} \), the vertex \( v_i \in V (G) \) is adjacent to every vertex of \( H_i \).
Theorem 3.2 For any integer \( k \geq 2 \), the \( k \)-TUPLE DOMINATING SET problem is NP-complete even when restricted to comparability graphs of diameter two.

Proof The problem clearly belongs to NP since checking that a given set is indeed a \( k \)-tuple dominating set of cardinality at most \( j \) can be done in polynomial time.

We set \( j = r + k - 1 \). Let \( V(H) = \{h_1, \ldots, h_{|V(H)|}\} \) and \( V(H_i) = \{h_i', \ldots, h_{|V(H)|}'\} \) for each \( 1 \leq i \leq |V(G)| \). Let \( S \) be a \( \gamma_{x,k}(G \circ H) \)-set. We set \( S_i = S \cap (V(H_i) \cup \{v_i\}) \) for each \( 1 \leq i \leq |V(G)| \). Suppose first that \( v_i \notin S_i \).

It follows that \( S_i \) is a \( k \)-tuple dominating set of \( H_i \). On the other hand, it turns out that \( S_i \setminus \{h_i'\} \) is a \((k-1)\)-tuple dominating set of \( H_i \), where \( h_i' \) is any vertex of \( S_i \). Therefore, \( |S_i| \geq \gamma_{x,(k-1)}(H_i) + 1 \). Suppose now that \( v_i \in S_i \). In such a situation, it is readily observed that \( S_i \setminus \{v_i\} \) is a \((k-1)\)-tuple dominating set of \( H_i \). Therefore, we have again \( |S_i| \geq \gamma_{x,(k-1)}(H_i) + 1 \). Consequently, \( \gamma_{x,k}(G \circ H) = |S| = \sum_{i=1}^{|V(G)|} |S_i| \geq |V(G)| (\gamma_{x,(k-1)}(H) + 1) \).

Conversely, let \( A \) be a \( \gamma_{x,(k-1)}(H) \)-set. Clearly, \( A_i = \{h_i \in V(H_i) \mid h_i \in A\} \) is a \( \gamma_{x,(k-1)}(H_i) \)-set for every \( 1 \leq i \leq |V(G)| \). So, every vertex of \( G \circ H \) is dominated by at least \( k-1 \) vertices of \( \bigcup_{i=1}^{V(G)} A_i \) (note that every vertex dominates itself). By the structure, every vertex of \( G \circ H \) is dominated by at least \( k \) vertices of \( A' = \bigcup_{i=1}^{V(G)} A_i \cup V(G) \), which means \( A' \) is a \( k \)-tuple dominating set in \( G \circ H \) of cardinality \( |V(G)| (\gamma_{x,(k-1)}(H) + 1) \). This results in the exact formula

\[
\gamma_{x,k}(G \circ H) = |V(G)| (\gamma_{x,(k-1)}(H) + 1)
\]

for any graphs \( G \) and \( H \). In particular, we have \( \gamma_{x,k}(K_1 \circ H) = \gamma_{x,(k-1)}(H) + 1 \) for any graph \( H \). We now define the sequence \( \{H_1, H_2, \ldots\} \) by \( H_1 = H \), \( H_t = K_1 \circ H_{t-1} \) for each \( t \geq 2 \). With this in mind and by using the equality (3) for \( G = K_1 \), we get

\[
\gamma_{x,k}(H_k) = \gamma_{x,(k-1)}(H_{k-1}) + 1 = \cdots = \gamma_{x,1}(H_1) + k - 1.
\]

Thus, \( \gamma_{x,k}(H_k) = \gamma(H) + k - 1 \) for every graph \( H \). Our reduction is now completed by taking into account that \( \gamma_{x,k}(H_k) \leq j \) if and only if \( \gamma(H) \leq r \).

On the other hand, let \( H = H_1 \) be a comparability graph. Hence, it has a transitive orientation \( D = D_1 \). Let \( V(H_2) = V(H) \cup \{x\} = V(H_1) \cup \{x\} \). We consider the orientation \( D_1 \) of \( H_1 \) along with the set of new arcs \( O = \{(x, v) \mid v \in V(H) = V(H_1)\} \). Since \( D_1 \) is transitive and because all arcs in \( O \) go from \( x \) to every vertex of \( H = H_1 \), we deduce that the resulting orientation of \( H_2 \) is also transitive. Therefore, \( H_2 \) is a comparability graph. Iterating this process, we have that all \( H_1, \ldots, H_k \) are comparability graphs.

Because the DOMINATING SET problem (associated with the usual domination number) is NP-complete for comparability graphs (see [5] or [8]) and since \( \text{diam}(H_k) \leq 2 \), we deduce that the \( k \)-TUPLE DOMINATING SET is NP-complete for comparability graphs of diameter two. This completes the proof. \( \square \)
4 Bounding the Double Domination Number

Let $H$ be a bipartite graph with partite sets $X$ and $Y$ such that every vertex in $X$ has degree two. We make a matching $M$ whose edges join some vertices in $Y$ which have neighbors in $X$. We also join at least one end-vertex to the vertices in $Y$ which are not saturated by $M$. Let $G$ be the constructed graph and let $\Omega$ be the family of such graphs $G$ (see Fig. 2 in the case of a tree $T$).

**Theorem 4.1** Let $G$ be a graph of order $n$ and size $m$ with no isolated vertices. Let $e$ and $p$ be the number of end-vertices and penultimate vertices in $G$, respectively. Then,

$$\gamma_{x2}(G) \geq \frac{4n - 2m + e - p}{3}.$$  

Furthermore, the equality holds if and only if $G \in \Omega$.

**Proof** Let $A$ be a $\gamma_{x2}(G)$-set. The definition of a double dominating set implies that every vertex not in $A$ is adjacent to at least two vertices in $A$, and every vertex in $A$ is adjacent to at least one vertex in $A$, necessarily. Therefore,

$$m = |E(G[A])| + |[A, V(G) \setminus A]| + |E(G[V(G) \setminus A])|$$

$$\geq e + (|A| - e - p)/2 + 2(n - |A|).$$  \hspace{1cm} (4)

Thus, $\gamma_{x2}(G) = |A| \geq (4n - 2m + e - p)/3$.

Suppose first that $G \in \Omega$. Then, $V(G)$ is the union of vertices of $|M|$ copies of $P_2$, $|Y| - 2|M|$ stars and $|X|$ independent vertices. Hence, $n = 2|M| + |X| + e + p$ and $m = |M| + 2|X| + e$. It is then easily checked that the set $S$ containing the saturated vertices by $M$ along with the vertices of the $|Y| - 2|M|$ stars is a double dominating set in $G$. Therefore, $\gamma_{x2}(G) \leq |S| = 2|M| + e + p = (4n - 2m + e - p)/3$. This ends up with equality in the lower bound.

Conversely, let the equality hold. So the inequality in (4) holds with equality, necessarily. In particular, $|E(G[A])| = e + (|A| - e - p)/2$ implies that the subgraph induced by $A$ is a disjoint union of $P_2$-copies and stars $T_i$ with partite sets $X_i$ and $Y_i$ with $|X_i| = 1$, for which $\cup_i Y_i$ and $\cup_i X_i$ are the sets of end-vertices and penultimate
vertices of \(G\), respectively. Moreover, \(|[A, V(G) \setminus A]| = 2(n - |A|)| \) implies that every vertex in \(V(G) \setminus A\) has precisely two neighbors in \(A\). Finally, the vertices not in \(A\) are independent as \(E(G[V(G) \setminus A])\) is empty. It is now easily observed that \(V(G) \setminus A, A\), the set of edges of \(P_2\)-copies, \(\cup_i X_i\) and \(\cup_i Y_i\) correspond to \(X, \cup_i Y_i \cup Y, M, M\) unsaturated vertices of \(Y\) and the end-vertices described in the process of defining \(\Omega\), respectively. Thus, \(G \in \Omega\).

Note that Theorem 4.4 implies the lower bound given in Theorem 4.1. In spite of this, we proved it by a different method so as to give the characterization of graphs attaining the lower bound.

As an immediate consequence of Theorem 4.1 for trees, we have the following result of Chellali in 2006.

**Theorem 4.2 ([3])** If \(T\) is a nontrivial tree of order \(n\) with \(\ell\) leaves and \(s\) support vertices, then \(\gamma_{x,2}(T) \geq (2n + \ell - s + 2)/3\).

Note that the whole of paper [3] is devoted to the bound given in Theorem 4.2 and a constructive characterization of all trees attaining it.

In order to characterize all trees for which the equality holds in the bound given in Theorem 4.2, it suffices to restrict the family \(\Omega\) to trees. Let us denote the resulting family by \(\Omega'\). In what follows, we describe a typical member of \(\Omega'\). We begin with \(a\) copies of \(P_2\) and \(s\) copies of stars \(T_i\) with partite sets \(X_i\) and \(Y_i\) such that \(|X_i| = 1\). We then add \(r = a + s - 1\) new vertices and join each of them to precisely two vertices in the \(P_2\)-copies and \(X_i\)'s such that all vertices of \(P_2\)-copies are incident with at least one of them and the resulting graph is connected. Note that \(m = a + \sum_i |Y_i| + 2r = n - 1\) guarantees that the resulting graph is a tree (see Fig. 2 for \((a, s, |Y_1|, |Y_2|) = (3, 2, 3, 2)\)).

Hajian and Jafari Rad in 2019 generalized Theorem 4.2 to connected graphs as follows. Note that they used the words “leaf” and “support vertex” instead of “end-vertex” and “penultimate vertex”, respectively.

**Theorem 4.3 ([12])** If \(G\) is a connected graph of order \(n \geq 2\) with \(k \geq 0\) cycles, \(e\) end-vertices and \(p\) penultimate vertices, then \(\gamma_{x,2}(G) \geq (2n + e - p + 2)/3 - 2k/3\).

They also characterized the family of all graphs achieving the equality in the above bound by extending the characterization given in [3].

Let \(G\) be a graph of size \(m\) given in Theorem 4.3 and let \(T\) be a spanning tree of it. Clearly, all end-vertices and penultimate vertices of \(G\) belong to \(V(T)\). Let \(k'\) be the number of edges of \(G\) which are not in \(T\). So, \(m = n - 1 + k'\). On the other hand, it is well known that for such an edge \(x'y\), \(T + xy\) contains a unique cycle \(C_{xy}\) containing \(xy\). Moreover, for any two such edges \(xy\) and \(x'y'\), \(C_{xy} = C_{x'y'}\) implies that \(xy = x'y'\). This shows that \(xy \rightarrow C_{xy}\) is a one-to-one function. Therefore, \(k' \leq k\). We then have

\[
\frac{4n - 2m + e - p}{3} = \frac{2n + e - p + 2}{3} - \frac{2k'}{3} \geq \frac{2n + e - p + 2}{3} - \frac{2k}{3}.
\]

In fact, Theorem 4.1 is an improvement of Theorem 4.3.
Theorem 4.4 For any graph $G$ of order $n$ and size $m$ with $e$ end-vertices, $p$ penultimate vertices and $\delta(G) \geq 1$,

$$\gamma_{x2}(G) \geq s\ell_{x2}(G) \geq (4n - 2m + e - p)/3.$$ \hspace{1cm}

Furthermore, $s\ell_{x2}(G) = (4n - 2m + e - p)/3$ if and only if $n + m + e - p \equiv 0 \pmod{3}$ and one of the following conditions holds:

(i) If $\delta(G) \geq 2$, then $d_q = 2$, where $q = (4n - 2m + 3)/3$.

(ii) If $\delta(G) = 1$, then $(n = 2m - e + p)$ or $(n < 2m - e + p$ and $d_q = 2$, where $q = (4n - 2m - 2e - p + 3)/3$.

Proof Let $A$ be a $\gamma_{x2}(G)$-set. Let $L$ and $P = \{u_1, \ldots, u_p\}$ be the sets of end-vertices and penultimate vertices of $G$, respectively. It is immediate from the definition that $P \cup L \subseteq A$. Moreover, each vertex in $A$ has at least one neighbor in $A$, and every vertex in $G \setminus A$ has at least two neighbors in $A$. Therefore,

\begin{align*}
|A| + \sum_{i=1}^{n} d_i & \geq |A| + \sum_{v \in A \setminus L} \deg(v) \\
& = |A| + \sum_{v \in A \setminus L} |N(v) \cap A| + \sum_{v \in A \setminus L} |N(v) \cap (V(G) \setminus A)| \\
& = |A| + \sum_{i=1}^{p} |N(u_i) \cap A| + \sum_{v \in A \setminus (P \cup L)} |N(v) \cap A| \\
& \quad + \sum_{v \in A \setminus L} |N(v) \cap (V(G) \setminus A)| \\
& \geq |A| + e + (|A| - e - p) + 2(n - |A|) \\
& = 2n - p.
\end{align*} \hspace{1cm} (5)

Thus, $\gamma_{x2}(G) = |A| \geq s\ell_{x2}(G).

Now let $t = s\ell_{x2}(G)$. If $t = n$, then $\gamma_{x2}(G) = s\ell_{x2}(G)$ and so the lower bound follows from Theorem 4.1. Therefore, we may assume that $t < n$. By the definition, we have

$$2n - p \leq t + d_1 + \cdots + d_{n-e} = t + 2m - \left( \sum_{i=t-e+1}^{n} d_i + \sum_{i=n-e+1}^{n} d_i \right) \leq t + 2m - (e + 2(n - t)).$$

Therefore, $s\ell_{x2}(G) = t \geq (4n - 2m + e - p)/3$.

Suppose now that $s\ell_{x2}(G) = (4n - 2m + e - p)/3$. Clearly, $n + m + e - p \equiv 0 \pmod{3}$. We distinguish two cases depending on the minimum degree of $G$.

Case 1. $\delta(G) \geq 2$. Then, $s\ell_{x2}(G) = (4n - 2m)/3$. If $s\ell_{x2}(G) = n$, then $n = 2m$ that contradicts the fact that $\delta(G) \geq 2$. Therefore, $s\ell_{x2}(G) < n$. Suppose to the contrary that $d_q > 2$, where $q = (4n - 2m + 3)/3$. Let $t = s\ell_{x2}(G)$. We then have

$$2n \leq t + d_1 + \cdots + d_t = t + 2m - \sum_{i=t+1}^{n} d_i < t + 2m - 2(n - t).$$

So, $t = s\ell_{x2}(G) > (4n - 2m)/3$. This is a contradiction. Therefore, $d_q = 2$. \hspace{1cm} \square
In this section, we give an upper bound on the $k$-tuple domatic number of graphs in terms of the order, size and $k$-tuple domination number. In order to characterize...
the extremal graphs which attain the bound, we introduce the family \( \Psi \) of graphs as follows. Let \( H_1, \ldots, H_r \) be \((k - 1)\)-regular graphs of the same order. Let \( G \) be obtained from \( H_1, \ldots, H_r \) by joining each vertex of \( H_i \) to precisely \( k \) vertices of \( H_j \) for all \( 1 \leq i \neq j \leq r \). Now let \( \Psi \) be the family of all such graphs \( G \).

**Theorem 5.1** For any graph \( G \) of order \( n \) and size \( m \) with \( \delta(G) \geq k - 1 \),

\[
d_{\times k}(G) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2m - (k - 1)n}{k\gamma_{\times k}(G)}}.
\]

Moreover, the equality holds if and only if \( G \in \Psi \).

**Proof** For the sake of convenience, we write \( d_{\times k} = d_{\times k}(G) \) and \( \gamma_{\times k} = \gamma_{\times k}(G) \). Let \( S = \{S_1, \ldots, S_{d_{\times k}}\} \) be a \( d_{\times k} \)-partition (that is, a \( k \)-tuple domatic partition of \( G \) of cardinality \( d_{\times k} \)). Without loss of generality, we may assume that \( |S_1| \leq \cdots \leq |S_{d_{\times k}}| \). By the definition, every vertex in \( S_i \) has at least \( k - 1 \) neighbors in \( S_i \) as well as \( k \) neighbors in \( S_j \) for each \( 1 \leq i \neq j \leq d_{\times k} \). Thus,

\[
m = \sum_{1 \leq i \leq d_{\times k}} |S_i| + \sum_{1 \leq i < j \leq d_{\times k}} |S_i| |S_j| \geq \sum_{1 \leq i \leq d_{\times k}} (k - 1) |S_i|/2 + \sum_{1 \leq i < j \leq d_{\times k}} |S_i| |S_j| \geq (k - 1)n + \sum_{i=1}^{d_{\times k}} k|S_i|(d_{\times k} - i) \geq (k - 1)n + k|S_1| \sum_{i=1}^{d_{\times k}} (d_{\times k} - i) \geq (k - 1)n + \left( \frac{d_{\times k}(d_{\times k} - 1)}{2} \right) k\gamma_{\times k}.
\]

This leads to \(-k\gamma_{\times k}d_{\times k}^2 + k\gamma_{\times k}d_{\times k} + 2m - (k - 1)n \geq 0 \). Solving this inequality for \( d_{\times k} \), we get \( d_{\times k} \leq \left( 1 + \sqrt{1 + 4(2m - (k - 1)n)/k\gamma_{\times k}} \right)/2 \).

Suppose now that \( G \in \Psi \). It is clear that \( \{H_1, \ldots, H_r\} \) is a \( k \)-tuple domatic partition of \( G \). In particular, this shows that \( \gamma_{\times k} \leq |H_1| \). Let \( D \) be a \( \gamma_{\times k} \)-set. We set

\[A = \{(d, h) \mid d \in D, h \in H_1 \text{ and } h \in N[d]\}.
\]

Notice that every vertex \( h \in H_1 \) is \( k \)-tuple dominated by \( D \). This shows that \(|A| \geq k|H_1|\). On the other hand, \(|N[d] \cap H_1| \leq k \) for every \( d \in D \). Therefore, \(|A| \leq k|D|\).

Together these two inequalities imply that \(|H_1| \leq |D| = \gamma_{\times k} \), and hence \( \gamma_{\times k} = |H_1| \). Therefore, \( \{H_1, \ldots, H_r\} \) is \( k \)-tuple domatic partition of \( G \) into \( \gamma_{\times k} \)-sets. In particular, \( n = r\gamma_{\times k} \) and \( r = d_{\times k} \). Furthermore, \( m = (kr - 1)n/2 \) because \( G \) is a \((kr - 1)\)-regular graph. It is now easy to compute that \( (1 + \sqrt{1 + 4(2m - (k - 1)n)/k\gamma_{\times k}})/2 = r = d_{\times k} \). So, we have the equality in the upper bound.

Conversely, let the upper bound hold with equality. Therefore, all inequalities in (6) hold with equality, necessarily. In particular, the first two resulting equalities imply that

(i) each vertex in \( S_i \) has precisely \((k - 1)\) neighbors in \( S_i \), and

(ii) each vertex in \( S_i \) \((1 \leq i \leq d_{\times k} - 1)\) has precisely \( k \) neighbors in \( S_j \) with \( j > i \).
Therefore, $G[S_i]$ is a $(k - 1)$-regular graph for each $1 \leq i \leq d_{\times k}$, and $|[S_i, S_j]| = k|S_i|$ for all $1 \leq i < j \leq d_{\times k}$. While the last two resulting equalities show that $|S_1| = \cdots = |S_{d_{\times k}}| = \gamma_{\times k}$. With this in mind, by the equality $|[S_i, S_j]| = k|S_i|$ for all $1 \leq i < j \leq d_{\times k}$ along with the fact that each $S_i$ is a $k$-tuple dominating set in $G$, we deduce that each vertex in $S_i$ has precisely $k$ neighbors in each $S_j$ for all $1 \leq i \neq j \leq d_{\times k}$. It is now easy to see that $d_{\times k}, G[S_1], \ldots, G[S_{d_{\times k}}]$ and $k$ have the same role as $r, H_1, \ldots, H_r$ and $k$ have in the process of introducing of $\Psi$, respectively. Therefore, $G \in \Psi$. This completes the proof.

Notice that for the special case when $k = 1$, we have the usual domination number $\gamma(G)$ and domatic number $d(G)$. In such a situation, the upper bound in Theorem 5.1 and its associated characterization become much simpler.

**Corollary 5.2** Let $G$ be a graph of size $m$. Then

$$d(G) \leq \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2m}{\gamma(G)}},$$

with equality if and only if $G$ is an $r$-partite graph with partite sets $H_1, \ldots, H_r$ in which the edge set of $G[H_i \cup H_j]$ is a perfect matching for each $1 \leq i \neq j \leq r$.

It is well known that $\delta(G) + 1$ is an upper bound on the domatic number of any graph $G$, and that a graph $G$ is called (domatically) full if $d(G) = \delta(G) + 1$ (see [4]). Cockayne and Hedetniemi in [4] posed the open problem of characterizing all full graphs. In what follows, we give a constructive characterization of all such graphs. To do so, we introduced the family $\Theta$ as follows. Let $H_1, \ldots, H_r$ be any graphs such that at least one of them, say $H_t$, has an isolated vertex $v$. Let $G$ be obtained from the disjoint union $H_1 + \cdots + H_r$ by

(i) joining $v$ to precisely one vertex in each $H_j$ with $j \neq i$, and

(ii) adding some edges among the vertices in $\bigcup_{t=1}^r V(H_t) \setminus \{v\}$ such that every vertex in $H_t$ has at least one neighbor in $H_{t'}$ for each $1 \leq t \neq t' \leq r$.

Let $\Theta$ be the family of all such graphs $G$.

**Theorem 5.3** A graph $G$ is full if and only if $G \in \Theta$.

**Proof** Suppose first that $G \in \Theta$. Clearly, $\deg(v) = \delta(G) = r - 1$. By the construction, we observe that $V(H_i)$ is a dominating set in $G$ for each $1 \leq t \leq r$. So, $\mathbb{H} = \{V(H_1), \ldots, V(H_r)\}$ is a domatic partition of $G$. On the other hand, $|\mathbb{H}| = r = \delta(G) + 1$ implies that $\mathbb{H}$ is a $d(G)$-set, and so $\delta(G) + 1 = r = d(G)$.

Suppose now that $d(G) = \delta(G) + 1$. Let $\mathcal{Q} = \{Q_1, \ldots, Q_{d(G)}\}$ be a $d(G)$-partition. Let $v \in Q_i$ be a vertex of minimum degree in $G$. Since every set in $\mathcal{Q}$ is a dominating set, $v$ is adjacent to at least one vertex in $Q_j$ for each $1 \leq i \neq j \leq d(G)$. Therefore, $\deg(v) \geq d(G) - 1$. With this in mind, the equality $d(G) = \delta(G) + 1$ implies that $v$ is an isolated vertex of $G[Q_i]$ having precisely one neighbor in $Q_j$ for each $1 \leq i \neq j \leq d(G)$. Furthermore, since $\mathcal{Q}$ is a domatic partition of $G$, every vertex different from $v$ in $Q_t$ has at least one neighbor in $Q_{t'}$ for each $1 \leq t \neq t' \leq d(G)$. It is now easily seen that $d(G)$ and $G[Q_1], \ldots, G[Q_{d(G)}]$ correspond to $r$ and $H_1, \ldots, H_r$ in the description of the family $\Theta$. Thus, $G \in \Theta$. $\square$
The characterization given in Theorem 5.3 would be much simpler in the case of regular graphs. In fact, if we restrict the family $\Theta$ to regular graphs, the following result will be immediate from Theorem 5.3. This result, by a different expression, was proved by Zelinka in [20].

**Corollary 5.4** Let $G$ be an $r$-regular graph. Then, $G$ is full if and only if it is an $(r + 1)$-partite graph with partite sets $H_1, \cdots, H_{r+1}$ in which the edges of $G[H_i \cup H_j]$ form a perfect matching for each $1 \leq i \neq j \leq r + 1$.

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**References**

1. Amos, D., Asplund, J., Brimkov, B., Davila, R.: The Slater and sub-$k$-domination number of a graph with applications to domination and $k$-domination. Discuss. Math. Graph Theory 40, 209–225 (2020)
2. Blidia, M., Chellali, M., Haynes, T.W., Henning, M.A.: Independent and double domination in trees. Utilitas Math. 70, 159–173 (2006)
3. Chellali, M.: A note on the double domination number in trees. AKCE J. Graphs. Combin. 3, 147–150 (2006)
4. Cockayne, E.J., Hedetniemi, S.T.: Towards a theory of domination in graphs. Networks 7, 247–261 (1977)
5. Corneil, D.G., Stewart, L.K.: Dominating sets in perfect graphs. Discrete Math. 86, 145–164 (1990)
6. Desormeaux, W.J., Haynes, T.W., Henning, M.A.: Improved bounds on the domination number of a tree. Discrete Appl. Math. 177, 88–94 (2014)
7. Desormeaux, W.J., Henning, M.A.: A new lower bound on the total domination number of a tree. Ars Combin. 138, 305–322 (2018)
8. Dewdney, A.K.: Fast Turing reductions between problems in NP 4, Report 71. University of Western Ontario, UK (1981)
9. Garey, M.R., Johnson, D.S.: Computers and intractability: a guide to the theory of NP-completeness. W.H. Freeman & Co., New York, USA (1979)
10. Gentner, M., Henning, M.A., Rautenbach, D.: Smallest domination number and largest independence number of graphs and forests with given degree sequence. J Graph Theory 88, 131–145 (2018)
11. Gentner, M., Rautenbach, D.: Some comments on the Slater number. Discrete Math. 340, 1497–1502 (2017)
12. Hajian, M., Jafari Rad, N.: A new lower bound on the double domination number of a graph. Discrete Appl. Math. 254, 280–282 (2019)
13. Harary, F., Haynes, T.W.: Double Domination in Graphs. Ars Combin. 55, 201–213 (2000)
14. Harary, F., Haynes, T.W.: The $k$-tuple domatic number of a graph. Math. Slovaca 48, 161–166 (1998)
15. Haynes, T.W., Hedetniemi, S.T., Henning, M.A. (eds.): Topics in Domination in Graphs. Springer International Publishing, Switzerland (2020)
16. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998)
17. Liao, C.S., Chang, G.J.: $k$-tuple domination in graphs. Inform. Process. Lett. 87, 45–50 (2003)
18. Slater, P.J.: Locating dominating sets and locating-dominating sets, In: Graph Theory, Combinatorics, and Applications, In: Proc. 7th Quadrennial Int. Conf. Theory Appl. Graphs, 2, 1073–1079 (1995)
19. West, D.B.: Introduction to Graph Theory, 2nd edn. Prentice Hall, USA (2001)
20. Zelinka, B.: Domatically critical graphs. Czechoslovak Math. J. 30, 486–489 (1980)

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