THE TOPOLOGY OF THE MOMENT-ANGLE MANIFOLDS
—–ON A CONJECTURE OF S.GITLER AND S.LÓPEZ

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Abstract. Let \( P \) be a simple polytope of dimension \( n \) with \( m \) facets and \( P_v \) be a polytope obtained from \( P \) by cutting off one vertex \( v \). Let \( Z = Z(P) \) and \( Z_v = Z(P_v) \) be the corresponding moment-angle manifolds. In [GL] S.Gitler and S.López conjectured that: \( Z_v \) is diffeomorphic to \( \partial((Z - \text{int}(D^{m-n})) \times D^2)^{m-n} \times (S^{m-n} \times S^{(m-n)-1}) \), and they have proved the conjecture in the case \( m < 3n \). In this paper we prove the conjecture in general case.

1. Introduction

The moment-angle manifold \( Z \) comes from two different ways:

1. The transverse intersections in \( \mathbb{C}^n \) of real quadrics of the form \( \sum_{i=1}^n a_i|z_i|^2 = 0 \) with the unit euclidean sphere of \( \mathbb{C}^n \).

2. An abstract construction from a simple polytope \( P^m \) with \( m \) facets (or a complex \( K \)).

The study of the first one led to the discovery of a new special class of compact non-kähler complex manifolds in the work of Lopez, Verjovsky and Meersseman ([LV],[Me],[MV]), now known as the LV-M manifolds, which helps us understand the topology of non-kähler complex manifolds.

The study of the second one is related to the quasitoric manifolds in the following way: for every quasitoric manifold \( \pi: M^{2n} \to P^n \), there is a principal \( T^{m-n} \)-bundle \( Z \to M^{2n} \) whose composite map with \( r \) makes \( Z \) a \( T^m \)-manifold with orbit space \( P^n \). The topology of the manifolds \( Z \) provides an effective tool for understanding inter-relations between algebraic and combinatorial aspects such as the Stanley-Reisner rings, the subspace arrangements and the cubical complexes etc.(see [BP]).

Following [BP], let \( P \) be an \( n \)-dimensional simple polytope with \( m \) facets, \( K_P \) be the dual of the boundary of \( P \). Obviously, \( K_P \) is a simplicial complex. Let \([m] = \{0, 1, \ldots, m-1\}\) represent the \( m \) vertices of the simplicial complex, \( \sigma \) be a simplex in the complex \( K_P \). Define

\[
(D^2)_\sigma \times T_{\sigma'} = \{(z_1, z_2, \ldots, z_m) \in (D^2)^m : |z_j| = 1 \text{ for } j \notin \sigma\}.
\]

and define the moment-angle complex \( Z(P) \) corresponding to \( P \) as

\[
Z(P) = \bigcup_{\sigma \in K_P} (D^2)_\sigma \times T_{\sigma'} \subset (D^2)^m.
\]

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A fundamental problem is to study the topology of \( Z(P) \) corresponding to some \( P \). One way to do this is to consider the change of the moment-angle complex \( Z \) after we do some ‘surgery’ on the polytope \( P \). These ‘surgeries’ include bistellar moves, cutting off faces, etc. (see [BP]). Obviously if we can obtain some information about the change of the topology of moment-angle complex after the ‘surgery’, we will get a better understanding of the moment-angle manifolds. However, cutting off vertices is the simplest ‘surgery’ and seems to be the only ‘surgery’ can be totally understood according to the conjecture of Gitler and López.

Let \( P \) be a simple polytope of dimension \( n \) with \( m \) facets, which is the convex hull of finitely many vertices in \( \mathbb{R}^n \). For any vertex \( v \), we can find a hyperplane \( H(x) = \sum a_i x_i = b \) satisfying that \( H(v) > b \) and \( H(\overline{v}) < b \) for any other vertex \( \overline{v} \). The set \( P \cap \{ x | H(x) \leq b \} \) is a new simple polytope \( P_v \), which is called to be obtained from \( P \) by cutting off the vertex \( v \). Let \( Z = Z(P) \) and \( Z_v = Z(P_v) \) be the corresponding moment-angle manifolds. From [BP] (6.9) we know that

\[
Z_v = \partial \left( \left( Z - T_{m-n} \times \text{int}(D^{2n}) \right) \times D^2 \right),
\]

where \( \sigma \) is the simplex dual to the vertex \( v \). S. Gitler and S. López conjectured that \( Z_v \) is diffeomorphic to

\[
\partial \left( \left( Z - \text{int}(D^{n+m}) \right) \times D^2 \right) \bigcup_{j=1}^{m-n} \left( m-n \atop j \right) (S^{j+2} \times S^{m+n-j-1})
\]

and they proved the conjecture in the case of \( m < 3n \) (see [GL]).

In the case \( m < 3n \) ([GL]), S. Gitler and S. López firstly proved that \( T_{m-n} \) is isotopic to a torus \( T_{m-n} \) inside an open disk in \( Z \). Then from the isotopy extension theorem (see [Ko]), one can prove that

\[
Z - T_{m-n} \times \text{int}(D^{2n}) \cong Z - T_{m-n} \times \text{int}(D^{2n}) \cong (Z - \text{int}(D^{n+m})) \cup (D^{n+m} - T_{m-n} \times \text{int}(D^{2n}))
\]

and

\[
Z_v = \partial \left( \left( Z - \text{int}(D^{n+m}) \right) \times D^2 \right) \bigcup_{j=1}^{m-n} \left( m-n \atop j \right) \partial \left( (S^{m+n} - T_{m-n} \times \text{int}(D^{2n})) \times D^2 \right).
\]

Secondly they considered the manifold \( \partial \left( (S^{m+n} - T_{m-n} \times \text{int}(D^{2n})) \times D^2 \right) \). They constructed spheres represent the Alexander dual homology of \( T_{m-n} \) in \( S^{m+n} \). According to a corollary of \( h \)-cobordism theorem, they proved that

\[
\partial \left( (S^{m+n} - T_{m-n} \times \text{int}(D^{2n})) \times D^2 \right) \cong \left( m-n \atop j \right) \bigcup_{j=1}^{m-n} \left( m-n \atop j \right) (S^{j+2} \times S^{m+n-j-1}).
\]

So the conjecture is true in this case.

In this paper, we firstly construct an isotopy of \( T_{m-n} \) in \( Z \) to move it to the regular embedding \( T_{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z \), thus \( Z_v \) is diffeomorphic to

\[
\partial \left( \left( Z - \text{int}(D^{n+m}) \right) \times D^2 \right) \bigcup_{j=1}^{m-n} \left( m-n \atop j \right) \partial \left( (S^{m+n} - T_{m-n} \times \text{int}(D^{2n})) \times D^2 \right)
\]

in general. Secondly by Lemma 2.13 in [Mc], we can prove that \( \partial \left( (S^{m+n} - T_{m-n} \times \text{int}(D^{2n})) \times D^2 \right) \) is diffeomorphic to \( \bigcup_{j=1}^{m-n} \left( m-n \atop j \right) (S^{j+2} \times S^{m+n-j-1}) \).

The main result of the paper is:
**Theorem 1.1.** Let $P$ be a simple polytope of dimension $n$ with $m$ facets and $P_v$ be a polytope obtained from $P$ by cutting off one vertex $v$. Let $Z = Z(P)$ and $Z_v = Z(P_v)$ be the corresponding moment-angle manifolds, then $Z_v$ is diffeomorphic to

$$\partial [(Z - \text{int}(D^{m+n})) \times D^2] \# \bigoplus_{j=1}^{m-n} (S^{j+2} \times S^{m+n-j-1}).$$

The manifold $\partial [(Z - \text{int}(D^{m+n})) \times D^2]$ is diffeomorphic to $(Z \times S^1 - \text{int}(D^{m+n}) \times S^1) \cup S^{m+n-1} \times D^2$, which can be obtained by doing a $(m + n, 1)$-type surgery on the manifold $Z \times S^1$ (see [M]).

**Proposition 1.1.** Let $[Z]$ and $[S^1]$ be the fundamental classes of $Z$ and $S^1$ respectively. Then the cohomology of $\partial [(Z - \text{int}(D^{m+n})) \times D^2]$ is isomorphic to

$$H^*(\partial [(Z - \text{int}(D^{m+n})) \times D^2]) \cong H^*(Z) \otimes H^*(S^1) / \{1 \otimes [S^1], [Z] \otimes 1\}$$

as a ring.

2. Construct the Isotopy of $T^m \times 0$ in $Z$

After cutting off a vertex $v$ on the simple polytope $P$, we obtain a new simple polytope $P_v$. Let $K_P$ and $K_{P_v}$ be the duals of the boundary of $P$ and $P_v$, $\sigma$ be the maximal simplex in $K_P$ dual to the vertex $v$ of the simple polytope $P$. Then we have $K_{P_v} = K_P /_{\sigma} \partial \Delta^n$ ($\Delta^n$ is the standard $n$-dimensional simplex, the choice of a maximal simplex in $\partial \Delta^n$ is irrelevant). By the definition, the moment-angle complex corresponding to $P$ (or $K_P$) is:

$$Z = \bigcup_{\sigma \in K_P} (D^2)_{\sigma} \times T_{\sigma} \subset (D^2)^m.$$  

Then we can express the moment-angle complex corresponding to $P_v$ (or $K_{P_v}$) as follows (see 6.4 in [MG]):

$$Z_v = (Z \times S^1 - T^m \times \text{int}(D_{\sigma}^2) \times S^1) \cup \bigcup_{\sigma \in K_P, \sigma \neq \sigma} T^m \times S^{m+n-1} \times D^2$$

$$\approx \partial [(Z - T^m \times \text{int}(D_{\sigma}^2)) \times D^2]$$

(1)

Without loss of generality, assume $\sigma$ correspond to the vertices $\{1, \ldots, m - n\}$, $\star$ be a point of $S^1_{m-n+1} \times \cdots \times S^1_{m-1}$, $y$ be a point of $S^1_0$. In this section, we construct an isotopy inductively to move the torus $\{y\} \times S^1_{m-n+1} \times \cdots \times S^1_{m-1} \times \{\star\}$ in $Z$ to the regular embedding $T^m \subseteq D^{m+n-1} \subseteq D^{m+n} \subseteq Z$.

**Remark 2.1.** We construct the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$ as follows: $S^1 \subseteq D^2 \subseteq \mathbb{R}^2$, assume that we have constructed the embedding of $T^{i-1}$ into $D^i \subseteq \mathbb{R}^i$. Represent $(i+1)$-sphere as $S^{i+1} = D^i \times S^1 \cup S^1_i \times D^1$. By the assumption, the torus $T^i = T^{i-1} \times S^1$ can be embedded into $D^i \times S^1$ and therefore into $S^{i+1}$. Since $T^i$ is compact and $S^{i+1}$ is the one-point compactification of $R^{i+1}$, we have $T^i \subseteq R^{i+1}$. Inductively, we can construct the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$ (or $D^{k+1}$). The regular embedding of $T^k$ into $\mathbb{R}^n$ is $T^k \subseteq \mathbb{R}^{k+1} \times \{0\} \subseteq \mathbb{R}^{k+1} \times \mathbb{R}^{n-k-1}$, where $T^k \subseteq \mathbb{R}^{k+1} \times \{0\}$ is the regular embedding of $T^k$ into $\mathbb{R}^{k+1}$. 
In terms of coordinates, we can express the regular embedding torus $T^k$ in $\mathbb{R}^{k+1}$ inductively as:

$$
T^k = \left\{ \begin{array}{c}
\sin \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\
\cos \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\
\frac{1}{2} \cos \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots)) \\
\vdots \\
\frac{1}{2^{k-1}} \cos \alpha_{k-1} \cdot (1 + \frac{1}{2} \sin \alpha_k) \\
\frac{1}{2^k} \cos \alpha_k
\end{array} \right\} \quad |0 \leq \alpha_i < 2\pi \quad (2)
$$

We shall call this the standard torus $T^k \subseteq R^{k+1}$. Consider the isotopy $F : T^k \times I \rightarrow D^{k+2}$ defined by:

$$
F(\alpha, t) = \left\{ \begin{array}{c}
\sin \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\
\cos \alpha_1 \cdot (1 + \frac{1}{2} \sin \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots))) \\
\frac{1}{2} \cos \alpha_2 \cdot (1 + \frac{1}{2} \sin \alpha_3 \cdot (\cdots (1 + \frac{1}{2} \sin \alpha_k) \cdots)) \\
\vdots \\
\frac{1}{2^{k-1}} \cos \alpha_{k-1} \cdot (1 + \frac{1}{2} \sin \alpha_k) \\
\frac{1}{2^k} \cos \alpha_k
\end{array} \right\} (1 - t) \sin \alpha_k + t|\sin \alpha_k| \quad (3)
$$

An examination of this isotopy proves the following:

**Lemma 2.1.** Let $T^k$ be the standard torus in $D^{k+1} = D^k \times D^1 \subseteq D^k \times S^1 \subseteq D^k \times D^2$. Then we may write $D^{k+2}$ as $D^k \times D^2$ so that $T^k$ is isotopic to $T^{k-1} \times (\partial D^2)$, where $T^{k-1}$ is a standard torus in $D^k$.

**Proof.** While $t = 0$, the last coordinate $|\sin \alpha_k|$ in (3) is omitted. \hfill $\Box$

Now we can construct an isotopy of torus $T_{\sigma}^{m-n} \times \{0\}$ in $Z$:

Obviously, the torus $\{0\} \times T_{\sigma}^{m-n} \times \{0\}$ in $Z$ is isotopic to $\{y\} \times T_{\sigma}^{m-n} \times \{\ast\}$.

As $n \geq 1$, the torus

$$
S_0^1 \times T_{\sigma}^{m-n} \times \{\ast\} \subseteq Z = \bigcup_{\sigma \in K} (D^2)_{\tau} \times T_{\sigma}^{m-n} \subseteq (D^2)^m.
$$

Assume the facet $\sigma_1$ contain vertex 1, so

$$
S_0^1 \times T_{\sigma}^{m-n} \times \{\ast\} \subseteq S_0^1 \times D^2_1 \times S_1^1 \times \ldots S_{m-n-1}^1 \times \{\ast\} \subseteq (D^2)_{\tau_1} \times T_{\sigma_1} \subseteq Z.
$$

For $\forall \ x \in S_2^1 \times \ldots \times S_{m-n}^1$, we construct an isotopy of $\{y\} \times S_1^1 \times \{x\} \times \{\ast\}$ in $S_0^1 \times D^2_1 \times \{x\} \times \{\ast\}$ as follows: Let $D^1_0$ be a closed interval of $S^1_0$ with original point $y = 0$. The coordinate of a point of $\{y\} \times S_1^1$ in $D^1_0 \times D^2_1 \subseteq S_0^1 \times D^1_1$ is $(0, \cos \alpha, \sin \alpha)$. Define

$$
F_1 : S_1^1 \times I \rightarrow D^1_0 \times D^2_1
$$

$$
F_1(\cos \alpha, \sin \alpha, t) = (t \sin \alpha, \cos \alpha, (1 - t) \sin \alpha + t|\sin \alpha|)
$$

when $t = 1$. $F_1(\cos \alpha, \sin \alpha, 1) = (\sin \alpha, \cos \alpha, |\sin \alpha|)$. Obviously, $(\sin \alpha, \cos \alpha, |\sin \alpha|)$ is the regular embedding $S_1^1 \times \partial D^2 \subseteq D^2 \subseteq S_0^1 \times S_1^1$, where $D^2$ is a smooth embedded disk in $S_0^1 \times S_1^1$. In this way, for $\forall \ x \in S_2^1 \times \cdots \times S_{m-n}^1$, we move $\{y\} \times S_1^1 \times \{x\} \times \{\ast\}$ to the regular embedding

$$
S_1^1 \subseteq D^2 \subseteq S_0^1 \times S_1^1 \times \{x\} \times \{\ast\}.
$$
This gives an isotopy to move \( \{y\} \times S_1 \times S_2 \times \cdots \times S_{m-n} \times \{\ast\} \) into

\[
D^2 \times (S_2^1 \times \cdots \times S_{m-n}^1) \times \{\ast\} \subseteq S_0^1 \times S_1^1 \times (S_2^1 \times \cdots \times S_{m-n}^1) \times \{\ast\},
\]
where \( D^2 \subseteq S_0^1 \times S_1^1 \).

Inductively suppose we have constructed an isotopy of \( \{y\} \times S_1 \times \cdots \times S_p^1 \times \{x\} \times \{\ast\} \) to move it to the regular embedding

\[
T^p \subseteq D^{p+1} \subseteq S_0^1 \times S_1^1 \times \cdots \times S_p^1 \times \{x\} \times \{\ast\}
\]
where \( x \) is a point of \( S_{p+1}^1 \times \cdots \times S_{m-n}^1 \) and the coordinate of the points of \( T^p \subseteq D^{p+1} \) is expressed as (2). Assume the facet \( \pi \) of \( S^1 \times \cdots \times S^1 \) chosen to be \( (\pi(x)) \times \{\ast\} \subseteq (D^2)_{\pi} \times T_{\pi} \subseteq Z \),

where \( \pi \) is the projection

\[
\pi : S_{p+1}^1 \times S_{p+2}^1 \cdots \times S_{m-n}^1 \to S_{p+2}^1 \times \cdots \times S_{m-n}^1.
\]

Using Lemma 2.1 above, we can construct an isotopy to move the torus

\[
T^p(\subseteq D^{p+1}) \times S_{p+1}^1(\subseteq D_{p+1}^2) \times \{\pi(x)\} \times \{\ast\}
\]
to the regular embedding

\[
T^{p+1} \subseteq D^{p+2} \subseteq S_0^1 \times S_1^1 \times \cdots \times S_{p+1}^1 \times \{\pi(x)\} \times \{\ast\}.
\]

In this way, we can construct an isotopy \( F_t \) of \( T_{\pi} \times \{0\} \subseteq Z \) to move it to the regular embedding \( T_{m-n} \subseteq D_{m-n+1} \subseteq D^{m+n} \subseteq Z \). According to the isotopy extension theorem, there exists an isotopy \( G_t \) of \( Z \) satisfying \( G_t \mid T_{m-n} \times \{0\} = F_t \). So the proper tubular neighborhood \( T_{\partial} \times \text{int}(D^{2n}) \) of \( T_{m-n} \times \{0\} \) in \( Z \) is isotopic to a proper tubular neighborhood \( N(T_{m-n}) \) of \( T_{m-n} \) \((T_{m-n} \text{ is a regular embedding } T_{m-n} \subseteq D^{m+n} \subseteq Z)\). By Theorem 3.5 in [Ko], \( N(T_{m-n}) \) can be chosen to be \( T_{m-n} \times \text{int}(D^{m+n}) \subseteq D^{m+n} \subseteq Z \). So

\[
Z - T_{\partial} \times \text{int}(D^{2n}) \cong Z\#(S^{m+n} - T_{m-n} \times \text{int}(D^{2n})),
\]

\[
Z_{0} \cong \partial((Z\#(S^{m+n} - T_{m-n} \times \text{int}(D^{2n}))) \times D^2).
\]

Recall Lemma 2 in [GL]:

**Lemma 2.2.** (Lemma 2 [GL]) Let \( M, N \) be connected \( n \)-manifolds, if \( M \) is closed but \( N \) has non-empty boundary, then \( \partial((M \# N) \times D^2) \) is diffeomorphic to \( \partial((M - \text{int}(D^n)) \times D^2)\#(N \times D^2) \).

According to the lemma, \( \partial((Z\#(S^{m+n} - T_{m-n} \times \text{int}(D^{2n}))) \times D^2) \) is diffeomorphic to

\[
\partial((Z - \text{int}(D^{m+n}) \times D^2)\#(S^{m+n} - T_{m-n} \times \text{int}(D^{2n}) \times D^2),
\]

where torus \( T_{m-n} \subseteq D_{m-n+1} \subseteq D^{m+n} \subseteq S^{m+n} \) is the regular embedding.
3. The manifold $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$

In order to prove the conjecture, it is sufficient to show that the manifold

$$\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2]$$

is diffeomorphic to $\bigcup_{j=1}^{m-n} (S^{j+2} \times S^{m+n-j-1})$.

Let $M$ and $N$ be $m$–manifolds with boundary, and $X$ be a closed $n$–manifold. Let $X \times D^{m-n-1}$ be embedded in $\partial M$ and $\partial N$. Then we can form $(M, F) \#_X (N, G) = M \cup_f N$ (denoted as $M \#_X N$), where $F, G$ are framings $(X, x_0) \to (S O(m - n - 1), \text{id})$,

$$f : X \times D^{m-n-1} (\subseteq \partial M) \to X \times D^{m-n-1} (\subseteq \partial N)$$

$$f(x, y) = (x, G(x) F^{-1}(x)(y))$$

It is not difficult to see that

$$\partial M \#_X \partial N = (\partial M - X \times \text{int}(D^{m-n-1})) \cup_{\partial f} (\partial N - X \times \text{int}(D^{m-n-1})) = \partial(M \#_X N)$$

So $\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2] (T^{m-n} \subseteq S^{m+n} \text{ is the regular embedding})$ is diffeomorphic to

$$\partial[(S^{m+n} \times S^1 - T^{m-n} \times \text{int}(D^{2n}) \times S^1) \cup T^{m-n} \times S^{2n-1} \times D^2]$$

$$\approx (S^{m+n} \times S^1 - T^{m-n} \times \text{int}(D^{2n}) \times S^1) \cup (T^{m-n} \times S^{2n+1} - T^{m-n} \times \text{int}(D^{2n}) \times S^1)$$

$$\approx S^{2n+1} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times S^{m+n}$$

$$\approx \partial(D^{2n+2} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m+n+1}),$$

where $T^{m-n} \times D^{2n}$ is embedded in $\partial(S^1 \times D^{m+n+1})$ through

$$S^1 \times (T^{m-n} \times D^4) \times D^{2n-4} \subseteq S^1 \times S^{m-n+4} \times D^{2n-4} \subseteq S^1 \times S^{m+n} = \partial(S^1 \times D^{m+n+1}),$$

and $T^{m-n} \times D^{2n}$ is embedded in $\partial(D^{2n+2} \times T^{m-n})$ through

$$T^{m-n} \times (S^1 \times D^4) \times D^{2n-4} \subseteq T^{m-n} \times S^5 \times D^{2n-4} \subseteq T^{m-n} \times S^{2n+1} = \partial(D^{2n+2} \times T^{m-n}).$$

Thus,

$$\partial(D^{2n+2} \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m+n+1})$$

$$\approx \partial(D^6 \times T^{m-n} \times D^{2n-4} \cup_{T^{m-n+1} \times D^4 \times D^{2n-4}} S^1 \times D^{m-n+5} \times D^{2n-4})$$

$$\approx \partial(D^6 \times T^{m-n} \#_{T^{m-n+1}} S^1 \times D^{m+n+5} \times D^{2n-4})$$

(5)

Recall Lemma 2.13 in [Me]:

Lemma 3.1. (Lemma 2.13 [Me]) Let $T^{n-4} = S^1 \times T^{n-5}$. Suppose we have product embeddings $S^1 \times (T^{n-5} \times D^4) \to S^1 \times D^n$ and $(S^1 \times D^4) \times T^{n-5} \to D^6 \times T^{n-5}$ (where $T^{n-5} \to D^n$ is the regular embedding). With these embeddings

$$D^6 \times T^{n-5} \#_{T^{n-4}} S^1 \times D^n \cong \bigcup_{j=1}^{n-5} \binom{n-5}{j} S^{j+2} \times D^{n-j-1}$$
By Lemma 3.1,
\[\partial[(D^6 \times T^{m-n} \# T^{m-n+1} S^1 \times D^{m-n+5}) \times D^{2n-4}] \]
\[\simeq \partial\left[ \bigoplus_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times D^{m-n-j} \right] \]
\[\simeq \bigoplus_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times S^{m+n-j-1}.\] (6)

Thus, \[\partial[(S^{m+n} - T^{m-n} \times \text{int}(D^{2n})) \times D^2] \simeq \bigoplus_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times S^{m+n-j-1}.\]

\[Z_v \simeq \partial((Z - \text{int}(D^{m+n})) \times D^2) \bigoplus_{j=1}^{m-n} \binom{m-n}{j} S^{j+2} \times S^{m+n-j-1}.\] (7)

Until now, we have proved the conjecture. In a subsequent paper we will discuss the more general problems: the topology of the moment-angle manifold corresponding to the connected sums, bistellar moves and cutting off high dimensional faces.

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