ON SYMMETRIZABILITY AND PERFECTNESS OF
SECOND-COUNTABLE SPACES

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Abstract. A symmetrizability criterion of Arhangel’skii implies that a second-countable Hausdorff space is symmetrizable if and only if it is perfect. We present an example of a non-symmetrizable second-countable submetrizable space of cardinality \(\mathfrak{c}\) and study the smallest possible cardinality \(\mathfrak{q}_i\) of a non-symmetrizable second-countable \(T_i\)-space for \(i \in \{1, 2\}\).

Let us recall that a function \(d : X \times X \to [0, \infty)\) on a set \(X\) is a metric if for every points \(x, y, z \in X\) the following conditions are satisfied:

1. \(d(x, y) = 0\) if and only if \(x = y\);
2. \(d(x, y) = d(y, x)\);
3. \(d(x, z) \leq d(x, y) + d(y, z)\).

A function \(d : X \times X \to [0, \infty)\) is called a premetric (resp. a symmetric) on \(X\) if it satisfies the condition (1) (resp. the conditions (1) and (2)).

We say that the topology of a topological space \(X\) is generated by a premetric \(d\) if a subset \(U \subseteq X\) is open if and only if for every \(x \in U\) there exists \(\varepsilon > 0\) such that \(B_d(x; \varepsilon) \subseteq U\) where \(B_d(x; \varepsilon) \stackrel{\text{def}}{=} \{y \in X : d(x, y) < \varepsilon\}\) is the \(\varepsilon\)-ball centered at \(x\).

A topological space is symmetrizable (resp. metrizable) if its topology is generated by some symmetric (resp. metric). Symmetrizable spaces satisfy the separation axiom \(T_1\).

By the classical Urysohn Metrization Theorem [9, 4.2.9], each second-countable regular space is metrizable.

In this paper we address the following question (asked by the first author at Mathoverflow [2]).

Problem 1. Is every second-countable Hausdorff space symmetrizable?

To our surprise we have discovered that Problem [11] has negative answer, contrary to Theorem 2.9 [1] of Arhangel’skii claiming that a first-countable \(T_1\) space with a \(\sigma\)-discrete network is symmetrizable. In fact, the proof of this theorem works only under an additional restriction that the \(\sigma\)-discrete network consists of closed sets. Let us recall that a family \(\mathcal{F}\) of subsets of a topological space \(X\) is called a network for \(X\) if for every open set \(U \subseteq X\) and point \(x \in U\) there exists a set \(F \in \mathcal{F}\) such that \(x \in F \subseteq X\). A network \(\mathcal{F}\) is closed if every set \(F \in \mathcal{F}\) is closed in \(X\). So, the correct version of Arhangel’skii’s Theorem 2.9 in [1] reads as follows.

Theorem 2 (Arhangel’skii). Every first-countable \(T_1\) space with a \(\sigma\)-discrete closed network is symmetrizable.

This theorem implies

\[2020\] Mathematics Subject Classification. 54A35, 54E35, 54H05.

Key words and phrases. symmetrizable space, submetrizable space, \(Q\)-space, cardinal characteristic of the continuum.
Corollary 3. Every first-countable $T_1$-space with a countable closed network is symmetrizable.

We shall apply this corollary to prove that for second-countable Hausdorff spaces the symmetrizability is equivalent to the perfectness.

A topological space $X$ is called perfect if every closed subset $F$ of $X$ is of type $G_δ$, i.e., $F$ is the intersection of countably many open sets. The following proposition is known, see the discussion before Theorem 9.8 in [10].

Proposition 4. Every first-countable symmetrizable Hausdorff space $X$ is perfect.

Proof. We present a short proof for the convenience of the reader. Let $d$ be a symmetric generating the topology of $X$. First we show that for every $x \in X$ and $ε > 0$ the point $x$ is contained in the interior $B_d(x;ε)^o$ of the ball $B_d(x;ε)$. Indeed, in the opposite case we can use the first-countability of $X$ and find a sequence $S = \{x_n\}_{n \in ω} \subseteq X \setminus B_d(x;ε)$ that converges to $x$. The Hausdorff property of $X$ implies that the compact subset $K = \{x\} \cup S$ is closed in $X$ and hence for every $y \in X \setminus K$ there exists $ε_y$ such that $B_d(y;ε_y) \cap S \subseteq B_d(y;ε_y) \cap K = \emptyset$. Since also $B_d(x;ε) \cap S = \emptyset$, the set $S$ is closed in $X$, which is not possible as the sequence $(x_n)_{n \in ω}$ converges to $x \notin S$. This contradiction shows that $x$ is an interior point of the ball $B_d(x;ε)$.

Now we are ready to prove that $X$ is perfect. Given any closed set $F \subseteq X$, for every $n \in N$ consider the open neighborhood $U_n \overset{\text{def}}{=} \bigcup_{x \in F} B_d(x;\frac{1}{n})^o$ of $F$ in $X$ and observe that $F = \bigcap_{n \in ω} U_n$. □

Proposition 5. Each perfect second-countable $T_1$-space $X$ is symmetrizable.

Proof. Let $B$ be a countable base of the topology of $X$. By the perfectness of $X$, every open set $B \in B$ is equal to the union $\bigcup F_B$ of a countable family $F_B$ of closed sets in $X$. Then $F \overset{\text{def}}{=} \bigcup_{B \in B} F_B$ is a countable closed network for $X$. By Corollary 3 the space $X$ is symmetrizable. □

Propositions 4 and 5 imply the following criterion.

Theorem 6. A second-countable Hausdorff space is symmetrizable if and only if it is perfect.

Next, we prove that second-countable (submetrizable) $T_1$-spaces of sufficiently small cardinality are symmetrizable.

A topological space is submetrizable if it admits a continuous metric. Each submetrizable space is functionally Hausdorff in the sense that for any distinct elements $x, y \in X$ there exists a continuous function $f : X \to ω$ such that $f(x) \neq f(y)$. By [3], a second-countable space is submetrizable if and only if it is functionally Hausdorff.

A topological space $X$ is called a $Q$-space if every subset of $X$ is of type $G_δ$ in $X$. Every $Q$-space is perfect.

Let $q_0$ be the smallest cardinality of a second-countable metrizable space which is not a $Q$-space. For properties of the cardinal $q_0$, see [11 §4], [2], [4].

For $i \in \{1, 2, 3\}$, let $q_i$ be the smallest cardinality of a second-countable $T_i$-space which is not a $Q$-space. It is clear that $ω_1 \leq q_1 \leq q_2 \leq q_3 = q_0 \leq ε$, where $ε$ stands for the cardinality of continuum. By [3], $p \leq q_1$, where $p$ is the smallest cardinality of a subfamily $B \subseteq [ω]^ω$ such that for every finite subfamily $F \subseteq B$ the intersection $\bigcap F$ is infinite but for every infinite set $I \subseteq ω$ there exists a set $F \in F$ such that $I \cap F$ is finite. It is well-known (see [5] or [13]) that $p = ε$ under Martin’s Axiom. By [3], every submetrizable space of cardinality $< q_0$ is a $Q$-space. This fact combined with Proposition 5 implies
Proposition 7. Every second-countable submetrizable space of cardinality \(< q_0\) is symmetrizable.

Since functionally Hausdorff second-countable spaces are submetrizable, Proposition 7 implies another criterion of symmetrizability.

Proposition 8. Every second-countable functionally Hausdorff space of cardinality \(< q_0\) is symmetrizable.

Proposition 8 combined with the definition of the cardinals \(q_i\) implies the following semimetrizability criterion.

Proposition 9. Let \(i \in \{1, 2\}\). Every second-countable \(T_i\)-space \(X\) of cardinality \(|X| < q_i\) is symmetrizable.

Since \(p = q_1 = c\), we obtain the following corollary.

Corollary 10. Under Martin’s Axiom, every second-countable \(T_1\)-space of cardinality \(< c\) is symmetrizable.

Finally, we show that the cardinals \(q_0\) and \(q_2\) in Propositions 7, 8, 9 are the best possible.

Example 11. There exists a second-countable submetrizable space of cardinality \(q_0\) which is not symmetrizable.

**Proof.** By the definition of the cardinal \(q_0\), there exists a second-countable metrizable space \(X\), which is not a \(Q\)-space. Then \(X\) contains a subset \(A\) which is not of type \(G_\delta\) in \(X\). Let \(\tau’\) be the topology on \(X\) generated by the subbase \(\tau \cup \{A\}\) where \(\tau\) is the topology of the space \(X\). Since \(\tau \subseteq \tau’\), the space \(X’\) is submetrizable. Assuming that \(X’\) is perfect, we conclude that the closed set \(A\) is equal to the intersection \(\bigcap_{n \in \omega} W_n\) of some open sets \(W_n \in \tau’\). By the choice of the topology \(\tau’\), for every \(n \in \omega\) there exists open sets \(U_n, V_n \in \tau\) such that \(W_n = U_n \cup (V_n \setminus A)\). It follows from \(A \subseteq W_n = U_n \cup (V_n \setminus A)\) that \(A = A \cap W_n = A \cap U_n \subseteq U_n\).

\[
A = \bigcap_{n \in \omega} W_n = A \cap \bigcap_{n \in \omega} W_n = \bigcap_{n \in \omega} (A \cap W_n) = \bigcap_{n \in \omega} (A \cap U_n) \subseteq \bigcap_{n \in \omega} U_n \subseteq \bigcap_{n \in \omega} W_n = A
\]

and hence \(A = \bigcap_{n \in \omega} U_n\) is a \(G_\delta\)-set in \(X\), which contradicts the choice of \(A\). This contradiction shows that the submetrizable second-countable space \(X’\) is not perfect. By Proposition 4, \(X’\) is not symmetrizable.

By analogy, we can prove that the cardinal \(q_2\) in Proposition 9 is the best possible.

Example 12. There exists a second-countable Hausdorff space of cardinality \(q_2\) which is not symmetrizable.

However this argument does not work for the cardinal \(q_1\) (because Proposition 5 is applicable only for Hausdorff spaces).

Problem 13. Is \(q_1\) equal to the smallest cardinality of a second-countable \(T_1\)-space which is not symmetrizable?

Problem 14. Is \(q_1 = q_2 = q_0\)?

By Proposition 4, every symmetrizable first-countable Hausdorff space is perfect. On the other hand, by [6], [8], [12], there exists a non-perfect symmetrizable Hausdorff (even regular) spaces. However those examples are not first-countable.
Question 15. Is every second-countable symmetrizable space perfect?

Example 16. Consider the set \( X = \{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \) endowed with the symmetric

\[
d(x, y) = \begin{cases} 
\max\{x, y\} & \text{if } \max\{x, y\} > 0 \text{ and } \min\{x, y\} < 0; \\
|x - y| & \text{if } \max\{x, y\} \leq 0; \\
0 & \text{if } x = y; \\
1 & \text{otherwise.}
\end{cases}
\]

This symmetric generates a first-countable non-Hausdorff topology in which the unit ball \( B_d(0, 1) = \{-\frac{1}{n} : n \geq 2\} \cup \{0\} \) is nowhere dense. So, the proof of the perfectness of symmetrizable first-countable Hausdorff spaces from Proposition 4 does not work in this case. Nonetheless, the symmetrizable space \( X \) is countable and hence perfect, so this example does not answer Question 15.

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