EREMENKO’S CONJECTURE,
WANDERING LAKES OF WADA,
AND MAVERICK POINTS

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Dedicated to Alex Eremenko and Misha Lyubich.

Abstract. We develop a general technique for realising full closed subsets of the complex plane as wandering sets of entire functions. Using this construction, we solve a number of open problems.

(1) We construct a counterexample to Eremenko’s conjecture, a central problem in transcendental dynamics that asks whether every connected component of the set of escaping points of a transcendental entire function is unbounded.

(2) We prove that there is a transcendental entire function for which infinitely many Fatou components share the same boundary. This resolves the long-standing problem whether Lakes of Wada continua can arise in complex dynamics, and answers the analogue of a question of Fatou from 1920 concerning Fatou components of rational functions.

(3) We answer a question of Rippon concerning the existence of non-escaping points on the boundary of a bounded escaping wandering domain, that is, a wandering Fatou component contained in the escaping set. In fact, we show that the set of such points can have positive Lebesgue measure.

(4) We give the first example of an entire function having a simply connected Fatou component whose closure has a disconnected complement, answering a question of Boe Thaler.

In view of (3), we introduce the concept of maverick points: points on the boundary of a wandering domain whose accumulation behaviour differs from that of internal points. We prove that the set of such points has harmonic measure zero, but that it can nonetheless be rather large. For example, it may have positive planar Lebesgue measure.

1. Introduction

The iteration of transcendental entire self-maps of the complex plane was initiated by Fatou in 1926 [Fat26] and has received much attention in recent years. A central object, introduced by Eremenko [Ere89] in 1989, is the escaping set of a transcendental entire function $f$,

$$I(f) := \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \}.$$ 

The escaping set is significant in the study of transcendental dynamics since it often contains structures that can be used to understand the global dynamics of $f$. Indeed,
already Fatou [Fat26, p. 369] noticed the existence of curves to infinity in the escaping sets of certain transcendental entire functions. Eremenko proved that, for every transcendental entire function, the connected components of $\mathcal{I}(f)$ are all unbounded [Ere89, Theorem 3]. He also states [Ere89, p. 343] that it is plausible that the set $\mathcal{I}(f)$ itself has no bounded connected components; this is known as Eremenko’s conjecture and has remained an open problem since then; see also [Ber93, Question 15].

**Conjecture 1.1** (Eremenko’s conjecture). Every connected component of the escaping set of a transcendental entire function is unbounded.

The escaping set and Eremenko’s conjecture have been at the centre of much of the progress that transcendental dynamics has seen during the past two decades. For example, Rippon and Stallard [RS05] showed that a certain subset of $\mathcal{I}(f)$ that had been introduced by Bergweiler and Hinkkanen [BH99], the fast escaping set $\mathcal{A}(f)$, see (3.1), has only unbounded components. (In particular, there is always at least one unbounded connected component of $\mathcal{I}(f)$.) The fast escaping set has since become an important object in the field. In particular, it plays an important role in Bishop’s construction of a transcendental entire function whose Julia set has Hausdorff dimension one [Bis18]; this solved a long-standing open question. Work on the fast escaping set has also had a substantial impact in quasiregular dynamics; see [BFLM09, BDF14].

Furthermore, investigation of the escaping set has led to major advances in the study of two important classes of transcendental entire functions: the Eremenko–Lyubich class $\mathcal{B}$ and the Speiser class $\mathcal{S}$. These consist of those transcendental entire functions $f$ for which the set $\mathcal{S}(f)$ of singular values is bounded resp. finite; compare [EL87]. For instance, in [RRRS11], it was shown that a stronger version of Eremenko’s conjecture, also stated in [Ere89] – that every escaping point can be connected to infinity by a curve consisting of escaping points – holds for a large subclass of $\mathcal{B}$, but fails for general $f \in \mathcal{B}$. According to Bishop (personal communication), the question whether such counterexamples can also be constructed in $\mathcal{S}$ was one of the original motivations for his technique of quasiconformal folding [Bis15], which has revolutionised the study of the classes $\mathcal{B}$ and $\mathcal{S}$.

Ideas introduced to study the escaping set have had significant impact also outside of transcendental dynamics; for example, techniques from [RRRS11] and [Rem09] were applied by Dudko and Lyubich [DL23] to achieve progress on the question of local connectivity of the Mandelbrot set, while quasiconformal folding has led to new results on the geometry of Riemann surfaces [BR21].

Eremenko’s conjecture has been confirmed in a number of cases, while stronger versions have been disproved. In addition to the results already referred to above, we mention [SZ03] [Rem07], [RS13] [OHS19], [NRS21] and [Rem16] Theorem 1.6. On the other hand, it is known that for an entire function $f$, $\mathcal{I}(f) \cup \{\infty\}$ is always connected [RS11], and that unbounded connected components are dense in $\mathcal{I}(f)$ [RS05]. Moreover, there is a quasiregular map $f : \mathbb{C} \to \mathbb{C}$ such that $\overline{\mathcal{I}(f)}$ has bounded components [BFLM09], in contrast to the entire case. So the problem is not of a purely topological nature.

The above results show that any counterexample to Eremenko’s conjecture has to be quite subtle, and explain why it has remained open until now despite considerable research efforts. In this article, we overcome these difficulties and resolve the conjecture.
in the negative, constructing a wide range of counterexamples. Recall that a compact set \( X \subseteq \mathbb{C} \) is full if \( \mathbb{C} \setminus X \) is connected.

**Theorem 1.2** (Counterexamples to Eremenko’s conjecture). Let \( X \subseteq \mathbb{C} \) be a non-empty full and connected compact set. Then there exists a transcendental entire function \( f \) such that \( X \) is a connected component of \( I(f) \).

We remark that a bounded connected component of \( I(f) \) need not be compact, but any compact connected component is necessarily full. See Proposition 7.6.

The examples in Theorem 1.2 are constructed using approximation theory; more precisely, we use a famous theorem of Arakelyan on approximation by entire functions (see Theorem 2.2). The use of approximation theory to construct examples in transcendental dynamics has a long history, going back to work of Eremenko and Lyubich from 1987 [EL87]. We shall prove Theorem 1.2 by developing a new general method for prescribing the dynamical behaviour of an entire function on certain closed subsets of the complex plane. We use this procedure to answer a number of further open problems in transcendental dynamics, which we describe in the following.

Let \( f: S \to S \) be a transcendental entire function or a rational map, where \( S = \mathbb{C} \) or \( S = \mathbb{C} := \mathbb{C} \cup \{\infty\} \), respectively. The Fatou set \( F(f) \subseteq S \) is the set of points whose orbits under \( f \) remain stable under small perturbations. More formally, \( F(f) \) is the largest open set on which the iterates of \( f \) are equicontinuous with respect to spherical distance; equivalently, it is the largest forward-invariant open set that omits at least three points of the Riemann sphere [Ber93, Section 2.1]. The complement \( J(f) := S \setminus F(f) \) is called the Julia set.

These two sets are named after Pierre Fatou and Gaston Julia, who independently laid the foundations of one-dimensional complex dynamics in the early 20th century. In his seminal memoir on the iteration of rational functions, Fatou posed the following question concerning the structure of the connected components of \( F(f) \), which are today called the Fatou components of \( f \).

**Question 1.3** (Fatou [Fat20, pp. 51–52]). If \( f \) has more than two Fatou components, can two of these components share the same boundary? \(^1\)

Fatou asked this question in the context of rational self-maps of the Riemann sphere, but it makes equal sense for transcendental entire functions, whose dynamical study Fatou initiated in 1926 [Fat26]. We give a positive answer to Question 1.3 in this setting.

**Theorem 1.4** (Fatou components with a common boundary). There exists a transcendental entire function \( f \) and an infinite collection of Fatou components of \( f \) that all share the same boundary. In particular, the common boundary of these Fatou components is a Lakes of Wada continuum.

Here a Lakes of Wada continuum is a compact and connected subset of the plane that is the common boundary of three or more disjoint domains (see Figure 1). That such continua exist was first shown by Brouwer [Bro10, p. 427]; the name “Lakes of Wada” arises from a well-known construction that Yoneyama [Yon17, p. 60] attributed to his

\(^1\)“S’il y a plus de deux régions contiguës distinctes (et par suite une infinité), deux régions contiguës peuvent-elles avoir la même frontière, sans être identiques?”
advisor, Takeo Wada; compare [HY61, p. 143], [HO95, Section 8] or [Ish17, Section 4].

In this construction, one begins with the closure of a bounded finitely connected domain, which we may think of as an island in the sea. We think of the complementary domains, which we assume to be Jordan domains, as bodies of water: the sea (the unbounded domain) and the lakes (the bounded domains). One then constructs successive canals, which ensure that the maximal distance from any point on the remaining piece of land to any body of water tends to zero, while keeping the island connected; see Figure 2. The land remaining at the end of this process is the boundary of each of its complementary domains; in particular, if there are at least three bodies of water, it is a Lakes of Wada continuum. One may even obtain infinitely many complementary domains, all sharing the same boundary, by introducing new lakes throughout the construction.

Lakes of Wada continua may appear pathological, but they occur naturally in the study of dynamical systems in two real dimensions; see [KY91]. For example, Figure 1 is (a projection of) the Plykin attractor [Ply74] (see also [Cou06]), which is the attractor of a perturbation of an Anosov diffeomorphism of the torus. Hubbard and Oberste-Vorth [HO95, Theorem 8.5] showed that, under certain circumstances, the basins of attraction of Hénon maps in $\mathbb{R}^2$ form Lakes of Wada.

Theorem 1.4 provides the first example where such continua arise in one-dimensional complex dynamics, answering a long-standing open question. The second author learned of this problem for the first time in an introductory complex dynamics course by Bergweiler in Kiel during the academic year 1998–99; see [Ber13, p. 27].

A Fatou component $U$ is called periodic if $f^p(U) \subseteq U$ for some $p \in \mathbb{N}$, and $U$ is preperiodic if there exists $q \in \mathbb{N}$ such that $f^q(U)$ is contained in a periodic Fatou component; otherwise $U$ is called a wandering domain of $f$. The work of Fatou and Julia left open the question whether wandering domains exist. In 1976, Baker [Bak76] showed that transcendental entire functions can have wandering domains, while Sullivan [Sul85] proved in 1985 that they do not occur for rational functions. If $f$ is a polynomial, a
bounded periodic Fatou component $U$ is either an immediate basin of a finite attracting or parabolic periodic point, or a Siegel disc, on which the dynamics is conjugate to rotation by an irrational angle. In the former case, it is known that $U$ is a Jordan domain $[RY08]$. Less is known about the possible topology of Siegel disc boundaries. However, suppose that $f$ is a quadratic polynomial with a Siegel disc $\Delta$ of rotation number $\alpha$. Petersen and Zakeri $[PZ04]$ proved that $\partial\Delta$ is a Jordan curve when $\alpha$ belongs to a positive measure set of rotation numbers. Shishikura and Yang $[SY24]$ and Cheraghi $[Che22]$ showed that the same holds for all high-type rotation numbers (for which all coefficients of the continued fraction expansion exceed a certain universal constant). Dudko and Lyubich $[DL22, Theorem 1.2]$ recently announced that, without restriction on the rotation number $\alpha$, the function $f$ is injective on the orbit of $\Delta$. This rules out Lakes of Wada boundaries, and provides a negative answer to Question 1.3 for quadratic polynomials. Both problems remain open for polynomials of degree at least three, but it seems reasonable to expect that, for polynomials and rational maps, the answer to Question 1.3 is negative. The problem whether the whole Julia set can be a Lakes of Wada continuum is related to Makienko’s conjecture concerning completely invariant domains and buried points of rational functions; compare $[SY03]$ and $[CMMR09]$.

In contrast, our motivation for considering Question 1.3 comes from the study of wandering domains, which have been the focus of much recent research; compare $[Bis15, BRS16, Bis18, MS20, BEF+21]$. They have also recently been studied in higher-dimensional complex dynamics; compare $[ABD+16, Boc21a, HP21, ABFP20]$. A wandering domain $U$ of an entire function $f$ is called escaping if it is contained in the escaping set $I(f)$ and it is called oscillating if it is contained in the bungee set

$$BU(f) := \{ z \in \mathbb{C} \setminus I(f) : \limsup_{n \to \infty} |f^n(z)| = +\infty \}.$$ 

We are interested in establishing whether the behaviour of points on $\partial U$ is determined by that of points in $U$. In all previously known examples of escaping wandering domains, the iterates $f^n$ in fact tend to infinity uniformly on $U$, and hence on $\partial U$. It is known $[RS12, Theorem 1.2]$ that this must be the case whenever $U$ is contained in the fast escaping set $A(f)$. This suggests the following question, which is Problem 2.94 in Hayman and Lingham’s Research Problems in Function Theory.
Question 1.5 (Rippon [HL19, p. 61]). If $U$ is a bounded escaping wandering domain of a transcendental entire function $f$, is $\partial U \subseteq I(f)$?

Here and subsequently, a bounded wandering domain is a wandering domain $U$ that is bounded as a subset of the complex plane. One should note the distinction with orbitally bounded wandering domains: wandering domains whose points have bounded orbits. The existence of orbitally bounded wandering domains is a famous open problem; see [HL19 Problem 2.67]. We answer Question 1.5 in the negative.

Theorem 1.6 (Non-escaping points in the boundary of escaping wandering domains). There exists a transcendental entire function $f$ with a bounded escaping wandering domain $U$ such that $\partial U \setminus I(f) \neq \emptyset$.

We prove Theorem 1.6 by showing that the function from Theorem 1.4 can be constructed such that at least one of these domains, $U_1$, is an escaping wandering domain while another, $U_2$, is an oscillating wandering domain. The set of non-escaping points in $\partial U_1 = \partial U_2$ has full harmonic measure seen from $U_2$ by a result of Rippon and Stalard [RS11, Theorem 1.2]; in particular, it is non-empty.

Both Theorem 1.4 and Theorem 1.6 are consequences of a much more general result, which concerns wandering compact sets of transcendental entire functions having arbitrary shapes.

Theorem 1.7 (Entire functions with wandering compacta). Let $K \subseteq \mathbb{C}$ be a full compact set. Let $Z_I, Z_{BU} \subseteq K$ be disjoint finite or countably infinite sets such that no connected component of $\mathrm{int}(K)$ intersects both $Z_I$ and $Z_{BU}$. Then there exists a transcendental entire function $f$ such that

(i) $\partial K \subseteq J(f)$;
(ii) $f^n(K) \cap f^m(K) = \emptyset$ for $n \neq m$;
(iii) every connected component of $\mathrm{int}(K)$ is a wandering domain of $f$;
(iv) $Z_I \subseteq I(f)$ and $Z_{BU} \subseteq BU(f)$.

Theorem 1.7 is inspired by a previous result of Boc Thaler [Boc21b]. Let $U \subseteq \mathbb{C}$ be a bounded domain such that $\mathbb{C} \setminus \overline{U}$ is connected, and such that furthermore $U$ is regular in the sense that $U = \mathrm{int}(\overline{U})$. Using a version of Runge’s approximation theorem, Boc Thaler proved that there is a transcendental entire function $f$ for which $U$ is a wandering domain on which the iterates of $f$ tend to $\infty$ [Boc21b, Theorem 1]. He then asked the following.

Question 1.8 (Boc Thaler [Boc21b, p. 3]). Is it true that the closure of any bounded simply connected Fatou component of an entire function has a connected complement?

If $U$ is a bounded domain such that $\partial U$ is a Lakes of Wada continuum, then the complement of $\overline{U}$ has at least two connected components by definition. Hence the closure of the Fatou component from Theorem 1.4 has a disconnected complement, answering Question 1.8 in the negative.

Remark 1.9. (i) We do not know whether in Theorem 1.7 one can additionally ensure that every connected component of $\partial K$ is a connected component of $J(f)$. This would only be possible if $f$ had multiply connected wandering domains limiting on $\partial K$. 
(ii) Certain unbounded domains can also be realised as wandering domains using similar techniques; see Remark 7.4 for the case of a half-strip.

In the case where \( Z_{BU} = \emptyset \) (which suffices for Theorem 1.4), we can prove Theorem 1.7 by a similar proof as Boc Thaler’s, but applying a subtle change of point of view: instead of beginning with a simply connected domain and approximating its closure, as in \( [\text{Boc21b}] \), we start the construction with the full compact set \( K \subseteq \mathbb{C} \) and consider its interior components. (See Theorem 3.1.) In this case, we can even ensure that \( K \) belongs to the fast escaping set \( A(f) \); see Proposition 3.3. The only previously known examples of fast escaping bounded simply connected wandering domains are due to Bergweiler [Ber11] and Sixsmith [Six12]. This construction can be used to provide new counterexamples to the strong Eremenko conjecture, mentioned above. It was resolved in the negative in [RRRS11, Theorem 1.1], but our construction is much simpler. (See Theorem 3.4.)

The iterates of the function \( f \) resulting from this argument converge to infinity uniformly on \( K \), so additional ideas are needed to prove Theorems 1.6 and 1.7. A key new ingredient in the proof is to use Arakelyan’s theorem (instead of Runge’s theorem) to ensure that there is a sequence of unbounded domains that are mapped conformally over one another by any function involved in the construction (see Section 4 and Figure 6). The presence of these domains allows us to let the image of the compact set \( K \) be stretched (horizontally) at certain steps during the construction, ensuring that the spherical diameter of \( f^n(K) \) does not tend to zero as \( n \to \infty \), and allowing \( K \) to contain points of both the escaping set and the bungee set.

In order to obtain a counterexample to Eremenko’s conjecture, we develop the method even further, now applying it to an unbounded ray \( K \) that connects a finite endpoint to \( \infty \). The unboundedness of \( K \) allows us to ensure that \( K \) is surrounded by connected components of an attracting basin, which means that any connected component of the escaping set intersecting \( K \) is also contained in \( K \). By additionally ensuring that the finite endpoint of \( K \) escapes, while some other point is in \( BU(f) \), we obtain a counterexample to Eremenko’s conjecture. The stronger statement in Theorem 1.2 is obtained by a more careful application of similar ideas.

Recall that the example from Theorem 1.6 has an escaping wandering domain and an oscillating wandering domain, and their (shared) boundary contains both escaping points and bungee points. It seems interesting to investigate, more generally, those points on the boundary of a wandering domain whose orbits have different accumulation behaviour than the interior points.

**Definition 1.10** (Maverick points). Let \( f \) be a transcendental entire function and suppose that \( U \) is a wandering domain of \( f \). We say that a point \( z \in \partial U \) is maverick if there is a sequence \( (n_k) \) such that \( f^{n_k}(z) \to w \in \hat{\mathbb{C}} \) as \( k \to \infty \), but \( w \) is not a limit function of \( f^{n_k}|_U \).

Equivalently, a point \( z \in \partial U \) is maverick if for any (and hence all) \( w \in U \), one has \( \limsup_{n \to \infty} \text{dist}^\#(f^n(z), f^n(w)) > 0 \), where \( \text{dist}^\# \) denotes spherical distance. (Compare Lemma 8.1.) If \( U \) is an escaping wandering domain, then \( z \in \partial U \) is maverick if and only if \( z \notin I(f) \). However, if \( U \) is oscillating, then \( \partial U \) may contain maverick points that are also in the bungee set \( BU(f) \), but with a different accumulation pattern. Indeed, the
function $f$ that is constructed in the proof of Theorem 1.7 satisfies
$$\text{dist}^#(f^n(z), f^n(w)) > 0$$
for any distinct $z, w \in Z_{BU}$ not belonging to the same interior component of $K$; see Remark 5.3. In particular, if $z$ was chosen in some component $U$ of $\text{int}(K)$ and $w \in \partial U$, then $w$ will be a maverick point of the wandering domain $U$.

As far as we are aware, Theorems 1.6 and 1.7 provide the first examples of wandering domains whose boundaries contain maverick points. The following result shows that, as suggested by their name, most points on $\partial U$ are not maverick points.

**Theorem 1.11** (Maverick points have harmonic measure zero). Let $f$ be a transcendental entire function and suppose that $U$ is a wandering domain of $f$. The set of maverick points in $\partial U$ has harmonic measure zero with respect to $U$.

If $U \subseteq I(f)$, Theorem 1.11 reduces to [RS11, Theorem 1.1]. For non-escaping wandering domains, the theorem strengthens [OS16, Theorem 1.3], which states that, for points $z \in \partial U$ from a set of full harmonic measure, the $\omega$-limit set $\omega(z, f) = \bigcap_{n=1}^{\infty} \{f^n(z) : k > n\}$ agrees with the $\omega$-limit set of points in $U$. Observe that maverick points may have the same $\omega$-limit set as points in $U$. Indeed, the function in Theorem 1.7 can be constructed so that all points in $Z_{BU}$ share the same $\omega$-limit set, but have different accumulation behaviour; see Remark 5.7. So the set of maverick points may indeed be larger than the set considered by Osborne and Sixsmith.

Benini et al [BEF+24, Theorem 9.3] independently prove a result that implies the following weaker version of Theorem 1.11: if $U$ is a wandering domain of a transcendental entire function, and $Z \subseteq \partial U$ is a set of maverick points such that additionally $\text{dist}^#(f^n(z), f^n(w)) \to 0$ as $n \to \infty$ for all $z, w \in Z$, then $Z$ has harmonic measure zero. However, their result applies in a setting (sequences of holomorphic maps $F_n : U \to U_n$ from a simply connected domain $U$ to simply connected domains $U_n$) in which the stronger conclusion of Theorem 1.11 becomes false in general; compare [BEF+24, Example 7.6].

Generalising a question of Bishop [Bis14, p. 133] for escaping wandering domains, we may ask whether Theorem 1.11 can be strengthened as follows. (See [Bis14, Section 5.2] for the definition and a discussion of logarithmic capacity.)

**Question 1.12.** Let $U$ be a simply connected wandering domain of a transcendental entire function. Does the set of maverick points in $\partial U$ have zero logarithmic capacity when seen from $U$? That is, let $\varphi : \mathbb{D} \to U$ be a conformal isomorphism between the unit disc $\mathbb{D}$ and $U$, and consider the set $\Xi \subseteq \partial \mathbb{D}$ of points at which the radial limit of $\varphi$ exists and is a maverick point. Does $\Xi$ have zero logarithmic capacity?

Our proof of Theorem 1.7 can be adapted to show that logarithmic capacity zero is the best one may hope for in the above question.

**Theorem 1.13** (Maverick points of logarithmic capacity zero). Let $\Xi \subseteq \partial \mathbb{D}$ be a compact set of zero logarithmic capacity. Then there exists a transcendental entire function $f$ such that $\mathbb{D}$ is a wandering domain of $f$, and every point in $\Xi$ is maverick. The wandering domain in this example can be chosen to be either escaping or oscillating.
Sets of zero harmonic measure (or zero logarithmic capacity) need not be small in an absolute geometric sense. Indeed, recall that the set of maverick points in our proof of Theorem 1.6 has full harmonic measure when seen from another complementary domain of the Lakes of Wada continuum, and hence has Hausdorff dimension at least 1 by a result of Makarov; see [GM05, Section VIII.2]. By a further application of our construction, we can strengthen the example as follows.

**Theorem 1.14 (Maverick points of positive Lebesgue measure).** There exists a transcendental entire function \( f \) with a wandering domain \( U \) such that the set of non-maverick points on \( \partial U \) has Hausdorff dimension 1, while the set of maverick points contains a continuum of positive Lebesgue measure. The wandering domain in this example can be chosen to be either escaping or oscillating.

**Further questions.** We conclude the introduction with several questions arising from our work. While Eremenko’s conjecture is false in general, it is shown in [RRRS11, Theorem 1.6] that it holds for all functions of finite order in the Eremenko–Lyubich class \( B \). (See [RRRS11] for definitions.) We may ask whether one of these two conditions can be omitted.

**Question 1.15.** Can \( I(f) \) have a bounded connected component if \( f \in B \) has infinite order, or if \( f \notin B \) has finite order?

The function \( f \) constructed in the proof of Theorem 1.2 is of infinite order and does not belong to the class \( B \) (see Remark 1.5).

The following question is a modification of Question 1.8 that takes into account the new types of wandering domains provided by Theorem 1.7. Recall that if \( A \subseteq \mathbb{C} \) is compact, then \( \text{fill}(A) \) denotes the fill of \( A \), that is, the complement of the unbounded connected component of \( \mathbb{C} \setminus A \).

**Question 1.16.** Suppose that \( U \) is a bounded simply connected Fatou component of a transcendental entire function, and let \( K = \text{fill}(U) \). Is it true that \( \partial U = \partial \text{fill}(U) \)?

In view of Theorem 1.7 a positive answer to Question 1.16 would imply that a bounded simply connected domain \( U \) can arise as a Fatou component of an entire function if and only if \( \partial U = \partial \text{fill}(U) \).

Our results imply the existence of both escaping and oscillating simply connected wandering domains with Lakes of Wada boundaries. In contrast, it appears much more difficult to construct invariant Fatou components with Lakes of Wada boundaries, if this is at all possible.

**Question 1.17.** Let \( f \) be a transcendental entire function and suppose that \( U \) is an invariant Fatou component of \( f \). Must every connected component of \( \mathbb{C} \setminus \overline{U} \) intersect \( J(f) \)?

If a Fatou component \( U \) is completely invariant (i.e. \( f^{-1}(U) = U \)), then \( \partial U = J(f) \). Hence a positive answer to Question 1.17 would imply that a transcendental entire function has at most one such Fatou component; compare [RS19]. The previously-mentioned partial results in the polynomial case motivate the following strengthening of Question 1.17 for bounded \( U \).

**Question 1.18.** Let \( f \) be an entire function and suppose that \( U \) is a bounded invariant Fatou component of \( f \). Must \( \partial U \) be a simple closed curve?
Meromorphic functions. For transcendental meromorphic functions \( f: \mathbb{C} \to \mathbb{C} \), the analogue of Eremenko’s conjecture is false for elementary reasons. Indeed, for the function \( f(z) = \tan(z)/2 \), we have \( I(f) \subseteq J(f) \) and \( J(f) \) is totally disconnected; see e.g. [KK97, Theorem 5.2]. However, the study of wandering domains of transcendental meromorphic functions is of significant interest. All entire functions constructed in this paper can easily be modified to be meromorphic with infinitely many poles. Even more is true: any (not necessarily full) compact set \( K \) can be realised as a wandering compactum of a transcendental meromorphic function \( f \), in such a way that every connected component of \( \partial K \) is a connected component of \( J(f) \); see [MRW24, Theorem 1.3] and recall Remark 1.9 (i). For a meromorphic function having a wandering domain whose orbit consists only of simply connected Fatou components, Theorem 1.11 goes through with the same proof. It is plausible that Theorem 1.11 is true also for general wandering domains of transcendental meromorphic functions, but this requires further investigation.

Notation. For a set \( X \subseteq \mathbb{C} \), let \( \partial X \), \( \text{int}(X) \), and \( \overline{X} \) denote, respectively, the boundary, the interior, and the closure of \( X \) in \( \mathbb{C} \). We write \( \text{dist} \), \( \text{dist}^\# \), and \( \text{dist}_H \) for the Euclidean, spherical and hyperbolic distance in a domain \( U \subseteq \mathbb{C} \), respectively. We use \( \ell_U(\gamma) \) to denote the hyperbolic length of a curve \( \gamma \) in a domain \( U \subseteq \mathbb{C} \). Moreover, \( \text{diam} \) and \( \text{diam}^\# \) denote the Euclidean and spherical diameter, respectively. The Euclidean disc of radius \( \delta > 0 \) around \( z \in \mathbb{C} \) is denoted by \( D(z, \delta) \), and the unit disc is denoted by \( \mathbb{D} = D(0, 1) \). Finally, we use \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \).

Structure of the paper. In Section 2 we recall the results from approximation theory needed for our constructions, and prove a number of useful lemmas about behaviour that is preserved under approximation. In Section 3, we give a simplified proof of Theorem 1.7 in the special case that the wandering compact set \( K \) escapes uniformly, that is, \( Z_{BU} = \emptyset \) (see Theorem 3.1). We use this to obtain wandering domains that form Lakes of Wada (Theorem 1.4) and new counterexamples to the strong Eremenko conjecture (Theorem 3.4). The results of Section 3 are not strictly required for the proofs of the main results stated in the introduction (Theorem 1.4 also follows from Theorem 1.7), but the proof of Theorem 3.1 already contains a number of ideas that are also present in the more involved constructions relating to Theorems 1.7 and 1.2.

In Section 4, we set up the basic structure of the examples constructed in both Theorem 1.7 and Theorem 1.2. The proof of Theorem 1.7 in full generality, and the deduction of Theorem 1.6 from it, is in Section 5. Section 6 briefly outlines how the proof of Theorem 1.7 can be modified to prove Theorems 1.13 and 1.14. The techniques and results of Section 6 are not used elsewhere in the paper.

Our counterexamples to Eremenko’s conjecture are constructed in Section 7. This construction is similar to that in Section 5 but can be read independently of it. Thus, a reader primarily interested in the proof of Theorem 1.2 should study Sections 2, 4 and 7, but could choose to omit Sections 5, 6 and 8.

Theorem 1.11 is proved in Section 8. We conclude the paper with Section 9, which discusses and proves an auxiliary result used in Sections 5 and 6.
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Note from the authors. This work follows in the footsteps of two world-leading Ukrainian mathematicians, Alex Eremenko and Misha Lyubich. Their collaboration, which pioneered the use of approximation theory in complex dynamics, took place in the fall of 1983 in Kharkiv. At that time, Alex Eremenko was based at the Institute of Low Temperature Physics and Engineering, and it was there that he formulated what is now known as Eremenko’s conjecture. The city of Kharkiv has been devastated during the ongoing invasion of Ukraine, and the Institute has been severely damaged [NDGP22]. We dedicate this paper to Profs. Eremenko and Lyubich, to the people of Kharkiv, and to all victims of the invasion of Ukraine.

2. Preliminary results on approximation

We recall two classical results of approximation theory. The classical theorem of Runge [Run85] (see also [Gai87, Theorem 2 in Chapter II §3]) concerns the approximation of functions on compact and full sets by polynomials\(^2\). This is sufficient to prove Theorem 1.7 when \(Z_{BU} = \emptyset\), and hence Theorem 1.4. In contrast, Arakelyan’s theorem [Ara64] (see also [Gai87, Theorem 3 in Chapter IV §2]) allows us to approximate functions defined on (potentially) unbounded sets by entire functions; this will be crucial in our remaining constructions and, in particular, in the proof of Theorem 1.2.

**Theorem 2.1** (Runge’s theorem on polynomial approximation). Let \(A \subseteq \mathbb{C}\) be a compact set such that \(\mathbb{C} \setminus A\) is connected. Suppose that \(g: A \to \mathbb{C}\) is a continuous function that extends holomorphically to an open neighbourhood of \(A\). Then for every \(\varepsilon > 0\), there exists a polynomial \(f\) such that

\[ |f(z) - g(z)| < \varepsilon \quad \text{for all } z \in A. \]

**Theorem 2.2** (Arakelyan’s theorem). Let \(A \subseteq \mathbb{C}\) be a closed set such that

(i) \(\mathbb{C} \setminus A\) is connected;
(ii) \(\mathbb{C} \setminus A\) is locally connected at \(\infty\).

Suppose that \(g: A \to \mathbb{C}\) is a continuous function that is holomorphic on \(\text{int}(A)\). Then for every \(\varepsilon > 0\), there exists an entire function \(f\) such that

\[ |f(z) - g(z)| < \varepsilon \quad \text{for all } z \in A. \]

Here \(\mathbb{C} \setminus A\) is locally connected at \(\infty\) if \(\infty\) has a neighbourhood base consisting of connected sets. Since \(A\) is closed, this is equivalent to the following: For every \(R > 0\) there is an \(R' \geq R\) such that any point \(z \in \mathbb{C} \setminus A\) with \(|z| \geq R'\) can be connected to

\(^2\)Runge’s theorem in fact applies more generally to approximation of holomorphic functions on arbitrary compact sets by rational maps [Gai87, Theorem 2 in Chapter III §1]. The key case for us is when the compact set is full, in which case the approximating function can be chosen to be a polynomial.
infinity by a curve in $\mathbb{C} \setminus A$ consisting of points of modulus greater than $R$ (see [Gal87, Lemma on p. 138]). In particular, the conditions of Arakelyan’s theorem are satisfied whenever $A = \bigcup_{k=0}^{\infty} A_k$, where the $A_k \subseteq \mathbb{C}$ are pairwise disjoint non-empty closed subsets such that

- if $A_k$ is unbounded, then $A_k \cup \{\infty\}$ is locally connected;
- all connected components of $\mathbb{C} \setminus A_k$ are unbounded;
- the $A_k$ tend to infinity in the Hausdorff metric as $k \to \infty$; that is, if $(z_k)_{k=0}^{\infty}$ is a sequence with $z_k \in A_k$ for all $k$, then $z_k \to \infty$ as $k \to \infty$.

In this article, we will only apply Arakelyan’s theorem to sets of this form; in fact each $A_k$ will be either a full compact set, a topological half-strip (bounded by a single arc tending to infinity in both directions) or a topological strip (bounded by two arcs tending to infinity in both directions).

In each of our constructions, we use two simple facts concerning approximation. The first is essentially a uniform version of Hurwitz’s theorem, while the second is an elementary exercise. For the reader’s convenience, we include the proofs.

**Lemma 2.3** (Approximation of univalent functions). Let $U, V \subseteq \mathbb{C}$ be open, and let $\varphi : U \rightarrow V$ be a conformal isomorphism.

Let $A \subseteq U$ be a closed set such that $\text{dist}(A, \partial U) > 0$ and $\eta := \inf_{z \in A} |\varphi'(z)| > 0$. Then there is $\varepsilon > 0$ with the following property: if $f : U \rightarrow \mathbb{C}$ is holomorphic with $|f(z) - \varphi(z)| \leq \varepsilon$ for all $z \in U$, then $f$ is injective on $A$, with $f(A) \subseteq V$.

Moreover, $|f'(z) - \varphi'(z)| < \eta/2$ for $z \in A$. In particular, $|f'(z)| > \eta/2$, and if $|\varphi'(z)|$ is bounded from above on $A$, then so is $|f'(z)|$.

**Remark 2.4.** When $A \subseteq U$ is compact, the hypotheses on $A$ are automatically satisfied.

**Proof.** Let $\delta < \text{dist}(A, \partial U)/2$. By Koebe’s 1/4-theorem,

$$\varphi(D(z_0, \delta)) \supseteq D(\varphi(z_0), \rho)$$

for all $z_0 \in A$, where $\rho := \delta \cdot \eta/4$. In particular,

$$\text{dist}(\varphi(A), \partial V) \geq \rho > 0 \quad (2.1)$$

and, since $\varphi$ is injective,

$$|\varphi(z) - \varphi(z_0)| \geq \rho \quad (2.2)$$

for all $z_0 \in A$ and $z \in U$ with $|z - z_0| \geq \delta$.

Now let $f$ be as in the statement of the lemma, where $\varepsilon < \rho/2$. Then

$$f(z) \neq f(z_0) \quad \text{when } |z - z_0| \geq \delta, \quad (2.3)$$

by [2.2]. Moreover, $f(A) \subseteq V$ by (2.1).

We must show that also $f(z) \neq f(z_0)$ for $z \neq z_0$ when $z \in D := D(z_0, \delta)$. In other words, we claim that $f(z) - f(z_0) = 0$ has a unique solution $z \in D$. According to the argument principle, the number of such solutions is given by the winding number of $f(\partial D)$ around $f(z_0)$. By choice of $\varepsilon$ and [2.2], the curves $\varphi(\partial D) - \varphi(z_0)$ and $f(\partial D) - f(z_0)$ are homotopic in $\mathbb{C} \setminus \{0\}$. Thus, they have the same winding number around 0. As $\varphi$ is injective, that winding number is 1, and the claim is proved. Together with [2.3], we see that $f$ is injective on $A$. 

Finally, by Cauchy’s theorem, we have
\[
|f'(z_0) - \varphi'(z_0)| = \frac{1}{2\pi} \left| \int_{\partial D(z_0, \delta)} \frac{f(\zeta) - \varphi(\zeta)}{(\zeta - z_0)^2} \, d\zeta \right| \leq \frac{2\pi \delta \varepsilon}{2\pi \delta^2} = \frac{\varepsilon}{\delta} < \frac{\eta}{2}
\]  
(2.4)
for \(z_0 \in A\). This proves the final statement of the lemma.

\[\text{Lemma 2.5 (Approximation of iterates). Let } U \subseteq \mathbb{C} \text{ be open, and let } g: U \to \mathbb{C} \text{ be continuous. Suppose that } K \subseteq U \text{ is closed and } n \in \mathbb{N} \text{ is such that } g^k(K) \text{ is defined and a subset of } U \text{ for all } k < n. \]

Suppose furthermore that \(\hat{K} := \bigcup_{k=0}^{n-1} g^k(K)\) satisfies \(\text{dist}(\hat{K}, \partial U) > 0\) and that \(g\) is uniformly continuous at every point of \(\hat{K}\), with respect to Euclidean distance.

Then for every \(\varepsilon > 0\), there is \(\delta > 0\) with the following property. If \(f: U \to \mathbb{C}\) is continuous with \(|f(z) - g(z)| < \delta\) for all \(z \in U\), then
\[
|f^k(z) - g^k(z)| < \varepsilon
\]
(2.5)
for all \(z \in K\) and all \(k \leq n\).

\[\text{Remark 2.6. By the uniform continuity assumption in the second paragraph, we mean that there exists a modulus of continuity that is valid at all points of } \hat{K}. \text{ That is, for every } \varepsilon > 0 \text{ there is } \delta'(\varepsilon) < \text{dist}(\hat{K}, \partial U) \text{ such that } |g(\zeta) - g(\omega)| < \varepsilon/2 \text{ whenever } \omega \in \hat{K} \text{ and } \zeta \in U \text{ with } |\zeta - \omega| \leq \delta'(\varepsilon). \text{ This assumption is satisfied, in particular, if } g \text{ is holomorphic and } |g'| \text{ is uniformly bounded above on such a neighbourhood } V. \]

In particular, the assumptions on \(\hat{K}\) are automatically satisfied when \(K\) is compact.

\[\text{Proof. Fix } U, g \text{ and } K \text{ as in the statement of the lemma. For every } n \in \mathbb{N} \text{ that satisfies the hypotheses, we prove the existence of a function } \delta_n \text{ such that } \delta = \delta_n(\varepsilon) \text{ has the desired property. The proof proceeds by induction on } n; \text{ for } n = 1 \text{ we may set } \delta_1(\varepsilon) = \varepsilon. \text{ Suppose that the induction hypothesis holds for } n, \text{ and let } \varepsilon > 0. \text{ Define}
\]
\[
\delta_{n+1}(\varepsilon) = \min \{\delta_1(\varepsilon), \delta_n(\delta'(\varepsilon)), \varepsilon/2\}
\]
where \(\delta'(\varepsilon)\) is as in Remark 2.6. Let \(z \in K\). Then [2.5] holds for \(k \leq n\) by the induction hypothesis. Setting \(\zeta := f^n(z)\) and \(\omega := g^n(z)\), we have \(|\zeta - \omega| \leq \delta'(\varepsilon)\) by the induction hypothesis, and, in particular, \(\zeta \in U\). Thus,
\[
|f^{n+1}(z) - g^{n+1}(z)| \leq |f(\zeta) - g(\zeta)| + |g(\zeta) - g(\omega)| < \varepsilon,
\]
as required. \]

\[\text{Corollary 2.7 (Approximating univalent iterates). Let } U \subseteq \mathbb{C} \text{ be open and } g: U \to \mathbb{C} \text{ be holomorphic. Suppose that } G \subseteq U \text{ is open and } K \subseteq G \text{ is closed with the following properties for some } n \geq 1:
\]
(a) \(g^n\) is defined and univalent on \(G\);
(b) \(|g'|\) is bounded from above and below by positive constants on \(U\);
(c) \(\text{dist}(K, \partial G) > 0\).

Then for every \(\varepsilon > 0\), there is \(\delta > 0\) with the following property. For any holomorphic \(f: U \to \mathbb{C}\) with \(|f(z) - g(z)| \leq \delta\) for all \(z \in U\), \(f^n\) is defined and injective on \(K\), \(|f^k(z) - g^k(z)| \leq \varepsilon\) on \(K\) for \(k \leq n\), \(|f'|\) is bounded from above and below by positive
constants on \( \hat{K} = \bigcup_{k=0}^{n-1} f^k(K) \), and \( f \) is uniformly continuous at every point of \( \hat{K} \) in the sense of Remark 2.6.

**Remark 2.8.** If \( K \) is compact, then hypotheses (b) and (c) can be omitted. Indeed, in this case (c) is trivial and (b) is automatically satisfied for the restriction of \( f \) to a neighbourhood of \( \hat{K} \).

**Proof.** Note that \( g^k \) is univalent on \( G \) for \( 1 \leq k \leq n \). Furthermore, by Koebe’s theorem and the assumption on \( g' \), \( g \) is uniformly continuous at every point of \( g^{k-1}(K) \) and \( \text{dist}(g^k(K), \partial U) > 0 \), for all \( k \geq 1 \). So \( \text{dist}(\hat{K}, \partial U) > 0 \) and the hypotheses of Lemma 2.5 are satisfied. Thus, there is \( \delta > 0 \) such that \( |f^k(z) - g^k(z)| \leq \varepsilon \) for all \( z \in K \) and all \( k \leq n \) if \( |f(z) - g(z)| \leq \delta \) on \( U \).

If \( \varepsilon \) is chosen small enough, then for \( k \leq n \) the hypotheses of Lemma 2.3 are satisfied for \( \varphi = g^k \) on \( G \). It follows that \( f^k \) is injective on \( K \) with \(|(f^k)'|\) bounded above and below by positive constants. Since this holds for all \( k \leq n \), we see that \( |f'| \) is bounded from above and below by positive constants on each \( f^k(K) \). This completes the proof of the lemma, apart from the statement about uniform continuity.

For that final statement, observe that for \( \eta > 0 \), the set \( K(\eta) := \{ z \in \mathbb{C} : \text{dist}(z, K) \leq \eta \} \) also satisfies the hypotheses of the corollary, if \( \eta < \text{dist}(K, \partial G) \). Now we apply the part of the corollary that we just proved to \( K(\eta) \), and conclude that the derivative of \( f \) is bounded from above on \( \hat{K}(\eta) = \bigcup_{k=0}^{n-1} f^k(K(\eta)) \). By (b), \( \text{dist}(\partial \hat{K}(\eta), \hat{K}) > 0 \). As noted in Remark 2.6 this implies that \( f \) is uniformly continuous at every point of \( \hat{K} \), as desired.

In Sections 3 and 5, we also require the following fact about approximating compact and full sets from above by finite collections of Jordan domains; see Figure 3. This is a classical fact of plane topology, used already by Runge to prove Theorem 2.1 [Run85, pp. 230–231] (see also [Wal69, Theorem 2 on pp. 7–8]). For the reader’s convenience, we provide a simple proof.

**Lemma 2.9** (Approximation by unions of Jordan domains). Let \( K \subseteq \mathbb{C} \) be compact and full. Then there exists a sequence \( (K_j)_{j=0}^{\infty} \) of compact and full sets such that \( K_j \subseteq \text{int}(K_{j-1}) \) for all \( j \in \mathbb{N} \) and

\[
\bigcap_{j=0}^{\infty} K_j = K.
\]

Each \( K_j \) may be chosen to be bounded by a finite disjoint union of closed Jordan curves.

**Proof.** Define

\[
\hat{K}_j := \text{fill}(\{ z \in \mathbb{C} : \text{dist}(z, K) \leq \frac{1}{j} \}).
\]

Each \( \hat{K}_j \) is compact and full by definition, and clearly \( K \subseteq \hat{K}_{j+1} \subseteq \text{int}(\hat{K}_j) \) for all \( j \). Any \( z \in \mathbb{C} \setminus K \) can be connected to \( \infty \) by a curve \( \gamma \) disjoint from \( K \), and hence we have \( z \notin \hat{K}_j \) for all sufficiently large \( j \in \mathbb{N} \).

Now fix \( j \) and let \( V_1, \ldots, V_m \) be the finitely many connected components of \( \text{int}(\hat{K}_j) \) that intersect \( K \). Then each \( V_\ell \) is a simply connected domain, and hence (say by the Riemann mapping theorem) there is a Jordan domain \( U_\ell \subseteq V_\ell \) with \( K \cap V_\ell \subseteq U_\ell \). The sets \( K_j := \bigcup_{\ell=1}^{m} U_\ell \) then have the required property. \( \blacksquare \)
Figure 3. Nested sequences of compacta \((K_j)\) shrinking down to the filled-in Julia set \(K = K(p)\) of \(p(z) = z^2 + c\) with \(c \approx -0.12 + 0.74i\).

3. Uniform escape

In this section, we prove Theorem 1.7 when \(Z_{BU} = \emptyset\), in which case we can choose \(f\) such that the iterates converge to infinity uniformly on all of \(K\).

**Theorem 3.1** (Wandering compacta with uniform escape). Let \(K \subseteq \mathbb{C}\) be a full compact set. Then there exists a transcendental entire function \(f\) such that

1. \(f^n(K) \cap f^m(K) = \emptyset\) when \(n \neq m\);
2. \(f^n|_K \to \infty\) uniformly as \(n \to \infty\);
3. \(\partial K \subseteq J(f)\);
4. every connected component of \(\text{int}(K)\) is a wandering domain of \(f\);

This result was proved by Boc Thaler [Boc21b] in the case where \(K\) is the closure of a simply connected domain \(U\). Apart from a change of perspective – starting with the compact set \(K\) instead of the domain \(U\) – our proof follows similar lines, but with some modifications that will become important later. For example, [Boc21b] uses a stronger version of Runge’s theorem, due to Eremenko and Lyubich, in which the approximating function is required to agree with the original one at finitely many given points. This allows Boc Thaler to prescribe exactly the orbits of a sequence of points accumulating on the boundary \(\partial U\), ensuring that this boundary is contained in the Julia set. We instead use the original, unmodified version of Runge’s theorem, which yields less control over the exact orbits, but now allows us to ensure that the union of a sequence of potentially uncountable sets (the sets \(P_j\) below), accumulating on \(\partial K\), belongs to a basin of attraction. While this does not play an essential role in the proof of Theorem 3.1 (other than to simplify it), it is of crucial importance in the proof of Theorem 1.2 in Section 7.
where the $P_j$ are replaced by a sequence of unbounded connected sets that will separate the desired point component from other points in the escaping set. This approach is also essential for Theorem 3.4 below, which provides simple new counterexamples to the strong Eremenko conjecture.

To set up the proof of Theorem 3.1, let $K \subseteq \mathbb{C}$ be a full compact set and choose a sequence $(K_j)$ of approximating sets according to Lemma 2.9. By applying an affine transformation, we may assume without loss of generality that $K_0 \subseteq \mathbb{D}$. For $j \geq 0$, also choose a non-separating compact and $2^{-j}$-dense subset

$$P_j \subseteq \partial K_j.$$

That is, $\text{dist}(z, P_j) \leq 2^{-j}$ for all $z \in \partial K_j$; in particular, $\partial K$ is the Hausdorff limit of the sets $P_j$. We shall construct a function $f$ such that all $P_j$ are contained in a basin of attraction, while $f^n$ tends to infinity on $K$ itself; this ensures that $\partial K \subseteq J(f)$. In most of our applications, we may choose the $P_j$ as finite sets, but for the proof of Theorem 3.4 below we shall use larger sets $P_j$.

For $j \geq -1$, consider the discs

$$D_j := D(3j, 1) = \{ z \in \mathbb{C} : |z - 3j| < 1 \}.$$  

Our main goal now is to prove the following proposition, which implies Theorem 3.1.

**Proposition 3.2.** There exists a transcendental entire function $f$ with the following properties:

(a) $f(\overline{D_{-1}}) \subseteq D_{-1}$;

(b) $f^{j+1}(P_j) \subseteq D_{-1}$ for all $j \geq 0$;

(c) $f^j$ is injective on $K_j$ for all $j \geq 0$, with $f^j(K_j) \subseteq D_j$.

**Proof of Theorem 3.1 using Proposition 3.2.** Let $j \geq 0$; then $K \subseteq K_j$ by choice of $K_j$ and hence $f^j(K) \subseteq D_j$ by (c). This establishes (i) and (ii).

On the other hand, by (a) and (b), we have $f^k(P_j) \subseteq D_{-1}$ for $k > j$. Since every point $z \in \partial K \subseteq K$ is the limit of a sequence of points $p_j \in P_j$, it follows that the family $(f^k)_{k=1}^{\infty}$ is not equicontinuous at $z$, and thus $z \in J(f)$. This proves (iii). If $U$ is a connected component of $\text{int}(K)$, then $U \subseteq F(f)$ by (ii) and the definition of the Fatou set, while $\partial U \subseteq J(f)$ by (iii). So $U$ is a Fatou component, and it is wandering by (i). \qed

**Proof of Proposition 3.2.** We construct $f$ as the limit of a sequence of polynomials $(f_j)_{j=0}^\infty$, which are defined inductively using Runge’s theorem. More precisely, for $j \geq 1$, the function $f_j$ approximates a function $g_j$, defined and holomorphic on a neighbourhood of a compact set $A_j \subseteq \mathbb{C}$, up to an error of at most $\varepsilon_j > 0$. The function $g_j$ in turn is defined in terms of the previous function $f_{j-1}$.

Define

$$\Delta_j := \overline{D(-3, 1 + 3j)} \supseteq D_{j-1}$$

for $j \geq 0$. The inductive construction ensures the following properties:

(i) For every $j \geq 0$, $f_j$ is injective on $K_j$ and $f_j(K_j) \subseteq D_j$.

(ii) For $j \geq 1$, $\Delta_{j-1} \subseteq A_j \subseteq \Delta_j$.

(iii) $\varepsilon_1 < 1/2$ and $\varepsilon_j \leq \varepsilon_{j-1}/2$ for $j \geq 2$. 

To anchor the induction, we set $f_0(z) = -3$ for $z \in \mathbb{C}$ and note that $(i)$ holds trivially for $j = 0$.

Let $j \geq 0$ and suppose that $f_j$ has been defined, and that $\varepsilon_j$ has been defined if $j \geq 1$. Applying Lemma 2.9 to $K_{j+1}$, we find a full compact set $L_j \subseteq K_j$ with $K_{j+1} \subseteq \text{int}(L_j)$. Set $Q_j := f^j_j(P_j)$. By $(i)$, $Q_j \subseteq D_j$ and $Q_j \cap (\partial D_j \cup f^j_j(L_j)) = \emptyset$. Let $R_j$ be a compact full neighbourhood of $Q_j$ disjoint from $\partial D_j$ and $f^j_j(L_j)$, and set

$$A_{j+1} := \Delta_j \cup R_j \cup f^j_j(L_j).$$

We have $R_j \subseteq D_j$ and $f^j_j(L_j) \subseteq f^j_j(K_j) \subseteq D_j$ by the inductive hypothesis $(i)$. In particular, neither set intersects $\Delta_j$. It follows that $A_{j+1}$ satisfies $(ii)$ and the hypotheses of Runge’s theorem. We define

$$g_{j+1} : A_{j+1} \rightarrow \mathbb{C}; \quad z \mapsto \begin{cases} 
    f_j(z), & \text{if } z \in \Delta_j, \\
    -3, & \text{if } z \in R_j, \\
    z + 3, & \text{if } z \in f^j_j(L_j).
\end{cases}$$

(See Figure 4.) By definition, the function $g_{j+1}$ extends analytically to a neighbourhood of $A_{j+1}$. Observe that $g_{j+1}^j(P_j) = g_{j+1}(Q_j) \subseteq g_{j+1}(R_j) = \{-3\} \subseteq D_{-1}$, and that $g_{j+1}^j$ is defined and univalent on $\text{int}(L_j)$ by $(i)$.

Now choose $\varepsilon_{j+1}$ according to $(iii)$ and sufficiently small that any entire function $f$ with $|f(z) - g_{j+1}(z)| \leq 2\varepsilon_{j+1}$ on $A_{j+1}$ satisfies:

1. $f^j_{j+1}(P_j) \subseteq D_{-1}$;
2. $f^j_{j+1}$ is injective on $K_{j+1} \subseteq \text{int}(L_j)$;
3. $f^j_{j+1}(K_{j+1}) \subseteq D_{j+1}$.

Here $(1)$ is possible by Lemma 2.5 and $(2)$ and $(3)$ are possible by Corollary 2.7. We now let $f^j_{j+1} : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial approximating $g_{j+1}$ up to an error of at most $\varepsilon_{j+1}$, according to Lemma 2.1. This completes the inductive construction.
Moreover, the disc replacing the discs $D_j$ slowly, but we can easily modify the construction to increase the rate of escape, by $X$

Hence the limit function $f$ satisfies $|f(z) - g_k(z)| \leq 2\varepsilon_k$ for all $z \in A_k$ and $k \geq 1$.

Since $g_k(D_{-1}) = f_0(D_{-1}) = \{-3\}$, and $2\varepsilon_1 < 1$, it follows that $f(D_{-1}) \subseteq D_{-1}$. Moreover, $f^{j+1}(P_j) \subseteq D_{-1}$ for $j \geq 0$ by (1). Finally, by (2) and (3), $f^j$ is injective on $K_j$ and $f^j(K_j) \subseteq D_j$. This completes the proof of Proposition 3.2.

We now prove Theorem 1.4 regarding the existence of Fatou components with a common boundary.

**Proof of Theorem 1.4.** Let $X \subseteq \mathbb{C}$ be a continuum such that $\mathbb{C} \setminus X$ has infinitely many connected components and the boundary of every such component coincides with $X$. As mentioned in the introduction, such a continuum was first constructed by Brouwer [Bro10, p. 427], and can be obtained using the construction described by Yoneyama [Yon17, p. 60]. Let $U$ be the unbounded connected component of $\mathbb{C} \setminus X$.

The set $K := \text{fill}(X) = \mathbb{C} \setminus U$ satisfies $\partial K = \partial U = X$. Apply Theorem 1.7 to the full continuum $K$ to obtain a transcendental entire function $f$ for which $X = \partial K \subseteq J(f)$, and every connected component of $\text{int}(K) \subseteq \text{int}(A)$ is a simply connected wandering domain. Since there are infinitely many such components, each of which is bounded by $X$, Theorem 1.4 is proved.

Note that the set $K$ constructed in the proof of Theorem 3.1 escapes to infinity rather slowly, but we can easily modify the construction to increase the rate of escape, by replacing the discs $D_j$ by any other disjoint sequence of discs tending to infinity (and the map $z + 3$ used in the definition of $g_{j+1}$ by an affine map taking $D_j$ to $D_{j+1}$). Moreover, the disc $D_{j+1}$ need not be chosen in advance; it may be chosen to depend on the map $f_j$. In this manner, we can ensure that $K$ belongs to the fast escaping set

$$A(f) := \{z \in J(f) : \text{there is } n_0 \geq 0 \text{ such that } |f^{n_0+j}(z)| \geq M^j(r,f) \text{ for all } j \geq 0\}. \tag{3.1}$$

Here $r$ is any number sufficiently large that $D(0,r) \cap J(f) \neq \emptyset$, and $M^j(r,f)$ denotes the $j$-th iterate of the maximum modulus function $r \mapsto M(r,f) := \max_{|z| \leq r}|f(z)|$.

**Proposition 3.3.** The function $f$ in Theorem 3.1 can be chosen such that $K \subseteq A(f)$.

**Proof.** We modify the inductive construction of $g_{j+1}$ by choosing both the discs $\Delta_j$ and $D_j$ inductively during the construction. Each $\Delta_j$ is a closed disc of radius $r_j$ centred at the origin, where $r_j$ is a rapidly increasing sequence, while all $D_j$ are open discs of radius 1. As before, they satisfy $D_j \subseteq \Delta_{j+1}$ and $D_j \cap \Delta_j = \emptyset$. The initial choices of $D_{-1}$, $D_0$ and $\Delta_0$ remain unchanged. The radius $r_1$ is chosen large enough to ensure that $\Delta_0 \cup D_0 \subseteq \Delta_1$.

Prior to defining $g_{j+1}$, we choose $r_{j+1}$ (if $j > 0$) greater than $M(r_j, f_j) + 2$, and then choose the disc $D_{j+1}$ to be a disc of radius 1 that is disjoint from $\Delta_{j+1}$. This construction ensures that $D_j \subseteq \Delta_{j+1}$: this holds by assumption for $j = 0$, and follows inductively since $D_{j+1}$ contains the image of a point in $D_j$, which has modulus less than $M(r_j, f_j) < r_{j+1} - 2$. 

Condition (iii) implies that $(f_j)_{j=k}^\infty$ forms a Cauchy sequence on every set $A_k$, and by (ii) $\bigcup_{k=1}^\infty A_k = \mathbb{C}$. So the functions $f_j$ converge locally uniformly to an entire function $f$. For $1 \leq k \leq j$,

$$|f_j(z) - g_k(z)| \leq \varepsilon_k + \cdots + \varepsilon_j \leq 2\varepsilon_k \text{ for all } z \in A_k.$$
The remainder of the construction remains unchanged, except that on $f_j^j(L_j)$, we define $g_{j+1}$ to be the restriction of an affine function mapping $D_j$ to $D_{j+1}$.

Let $f$ be the resulting function. Since $|f - f_j| < 2\varepsilon_j < 2$ on $\Delta_j$, we see that $r_{j+1} > M(r_j, f_j) + 2 > M(r_j, f)$ for all $j \geq 1$. In particular, $r_j > M_j^{-1}(r_1, f)$. If $z \in K$, then $f^{j+1}(z) \in D_{j+1}$, and hence $|f^{j+1}(z)| > r_{j+1} > M_j(r_1, f)$ for all $j \geq 0$. Since $D(0, r_1)$ contains $K$, and hence intersects the Julia set, it follows that $K \subseteq A(f)$, as claimed. ■

As mentioned in the introduction, we can use Proposition 3.2 to give new counterexamples to the strong version of Eremenko’s conjecture.

**Theorem 3.4** (Counterexamples to the strong Eremenko conjecture). Let $X \subseteq \mathbb{C}$ be a full continuum. Then there exists a transcendental entire function $f$ such that every path-connected component of $X$ is a path-connected component of the escaping set $I(f)$, and every path-connected component of $\partial X$ is a path-connected component of $J(f)$. In particular, no point of $X$ can be connected to $\infty$ by a curve in $I(f)$.

**Proof.** If $X$ is a singleton, then we may consider a continuum $X'$ of which $X$ is a path-connected component and continue the proof with $X'$ instead of $X$. (For example, we can take $X'$ to be as shown in Figure 5.) From now on we therefore suppose that $X$ is not a singleton.

Construct a function $f$ as in Proposition 3.2, where the set $P_j \subseteq \partial K_j$ is chosen such that no point of $K$ is accessible from $\mathbb{C} \setminus (K \cup \bigcup_{j=0}^{\infty} P_j)$. For example, let $z_0, z_1$ be distinct points of $\partial K$, and for each $j$ choose an open arc $\gamma_j \subseteq \partial K_j$ in such a way that $\operatorname{diam}(\gamma_j) \leq 2^{-j}$ for all $j$ and such that, furthermore, $\gamma_{2j} \to z_0$ and $\gamma_{2j+1} \to z_1$ as $j \to \infty$. Then the sets $P_j := \partial K_j \setminus \gamma_j$ have the desired property.

Now apply Proposition 3.2 to obtain a function $f$ satisfying the conclusions of Theorem 3.1. Each $P_j$ is contained in an iterated preimage of the forward-invariant disc $D_{-1}$, and hence is disjoint from $J(f) \cup I(f)$. By the choice of $P_j$, this implies that there is no curve in $J(f) \cup I(f)$ connecting a point of $\mathbb{C} \setminus K$ to a point of $K$. Since $K \subseteq I(f)$, every path-connected component of $K$ is a path-connected component of $I(f)$; similarly, since $\partial K = K \cap J(f)$, every path-connected component of $\partial K$ is a path-connected component of $J(f)$.

**Remark 3.5.** As in Proposition 3.3 we can again choose $f$ so that $X \subseteq A(f)$. 

**Figure 5.** A continuum $X'$ having a singleton path-connected component. $X'$ is obtained as a countable union of progressively smaller copies of the $\sin(1/x)$ continuum, accumulating on a single point (the left-most point in the figure).
4. Scaffolding

The functions whose existence is asserted in Theorems 1.2 and 1.7 will both be constructed using a sequence of unbounded horizontal strips \((S_j)_{j=0}^\infty\) and \((T_j)_{j=0}^\infty\), such that \(S_j\) is mapped univalently over \(S_{j+1}\) and \(T_{j+1}\). (See Figure 6) Our wandering set \(K\) starts out in the strip \(T_0\), maps into \(S_0\) and from there into \(T_1\), then back into \(S_0\) and on to \(T_2\) via \(S_1\), and so on. The crucial step happens at the time

\[
N_j := \sum_{\ell=1}^{j} (\ell + 1) = \frac{j \cdot (j + 3)}{2},
\]

when \(f^{N_j}(K)\) is in \(T_j\), and is mapped back inside \(S_0\) by \(f\). (By convention, \(N_0 := 0\).

Similarly as in the proof of Theorem 3.1, \(f\) is the limit of a sequence of functions \((f_j)_{j=0}^\infty\) constructed using approximation theory. We have to ensure that all these functions share the mapping behaviour on the strips \((S_j)\) described above. This can be achieved using Lemma 2.3 together with Arakelyan’s theorem.

To provide the details of this set-up, consider the affine map

\[
\Phi : \mathbb{C} \rightarrow \mathbb{C}; \quad z \mapsto 5z.
\]

For \(j \geq 0\), set

\[
S_j := \{ z \in \mathbb{C} : 5^{j+1} - 1 < 4 \text{Im} z < 5^{j+1} + 3 \}, \quad 
T_j := S_j + 2i = \{ z \in \mathbb{C} : 5^{j+1} + 7 < 4 \text{Im} z < 5^{j+1} + 11 \}.
\]

Observe that

\[
\Phi(S_j) = \{ z \in \mathbb{C} : 5^{j+2} - 5 < 4 \text{Im} z < 5^{j+2} + 15 \} \supseteq S_{j+1} \cup T_{j+1}
\]

for \(j \geq 0\). (Compare Figure 6) Define \(S := \bigcup_{j=0}^\infty S_j\).

**Lemma 4.1.** There is an \(\varepsilon > 0\) with the following property. Suppose that \(f : S \rightarrow \mathbb{C}\) is analytic and \(|f(z) - \Phi(z)| \leq \varepsilon\) for all \(z \in S\).

Then \(|\text{Re} f(z)| > 4|\text{Re} z|\) for all \(z \in S\) with \(|\text{Re} z| \geq 1\). Furthermore, for every \(j \geq 1\), there is a domain \(V_j = V_j(f) \subseteq S_0\) such that \(f^j : V_j \rightarrow T_j\) is a conformal isomorphism and \(f^k(V_j) \subseteq S_k\) for \(k = 0, \ldots, j-1\). Moreover, \(\text{Re} z\) is unbounded from above and below on \(V_j\), and \(2 \leq |f'| \leq 8\) on \(\bigcup_{k=0}^{j-1} f^k(V_j)\).

**Proof.** For \(k \geq 0\), define

\[
\tilde{S}_k := \{ z \in S_k : \text{dist}(z, \partial S_k) \geq 1/10 \}
\]

\[
= \{ z \in \mathbb{C} : 5^{k+1} - 3/5 \leq 4 \text{Im} z \leq 5^{k+1} + 3/5 \}
\]

\[
= \Phi^{-1} \left( \{ z \in \mathbb{C} : 5^{k+2} - 3 \leq 4 \text{Im} z \leq 5^{k+2} + 3 \} \right).
\]

Set \(U := S, V := \Phi(U)\) and \(A := \bigcup_{k=0}^\infty \tilde{S}_k\). Then the hypotheses of Lemma 2.3 are satisfied. Let \(\varepsilon\) be as given in that lemma, chosen additionally such that \(\varepsilon < 1/2\), and let \(f\) be \(\varepsilon\)-close to \(\Phi\) on \(U\). Then

\[
|\text{Re} f(z)| \geq |\text{Re} \Phi(z)| - \frac{1}{2} = 5|\text{Re} z| - \frac{1}{2} > 4|\text{Re} z|
\]

for all \(z \in S\) with \(\text{Re} z \geq 1\).
Moreover, $f$ is injective on each $\tilde{S}_k$ with $f(\tilde{S}_k) \supseteq S_{k+1} \cup T_{k+1}$. So we may define inductively $V_j^3 = T_j$ and, for $k = 0, \ldots, j - 1$,

$$V_j^k := (f|_{\tilde{S}_k})^{-1}(V_j^{k+1}).$$

Then $V_j := V_j^0$ has the desired properties. Finally, $\Phi'(z) = 5$ for all $z \in \mathbb{C}$, which implies the final claim by Lemma 2.3 (observe that $\eta = \inf_{z \in A} |\Phi'(z)| = 5$).

Define

$$D := D\left(0, \frac{1}{2}\right)$$
and observe that \( \overline{D} \cap \overline{S} = \emptyset \). The disc \( D \) will play the role of the disc \( D_{-1} \) from Section 3. We define \( A_0 := \overline{D} \cup \overline{S} \) and

\[
g_0: A_0 \to \mathbb{C}; \quad z \mapsto \begin{cases} \Phi(z), & \text{if } z \in \overline{S}, \\ 0, & \text{if } z \in \overline{D}. \end{cases}
\]

Set

\[
\varepsilon_0 := \min \left\{ \varepsilon, \frac{1}{2}, \frac{1}{5} \right\},
\]

where \( \varepsilon \) is the number from Lemma 4.1. By the discussion in Section 2, \( A_0 \) satisfies the hypotheses of Arakelyan’s theorem. Hence there exists an entire function \( f_0 \) such that

\[
|f_0(z) - g_0(z)| \leq \varepsilon_0 \quad \text{for } z \in A_0.
\] (4.3)

The set \( A_0 \) and the function \( f_0 \) will remain fixed with these properties throughout the following sections. Any entire function \( f \) that is \( \varepsilon_0 \)-close to \( f_0 \) (and hence \( \varepsilon \)-close to \( g_0 \)) on \( A_0 \) satisfies the hypothesis and hence the conclusion of Lemma 4.1 and also

\[
f(\overline{D}) \subseteq D.
\] (4.4)

5. Entire functions with wandering compacta

We now prove Theorem 1.7. Let \( K, Z_I \) and \( Z_{BU} \) be as in Theorem 1.7. Without loss of generality, suppose that \( K \subseteq T_0 \), where \( T_0 \) is the strip from the previous section. We may also assume that \( Z_{BU} \neq \emptyset \), since the case \( Z_{BU} = \emptyset \) is covered already by Theorem 3.1. (Alternatively, replace \( K \) by \( K \cup \{ \zeta \} \), where \( \zeta \not\in K \), and set \( Z_{BU} := \{ \zeta \} \).) We further assume that no component of \( \text{int}(K) \) contains more than one point of \( Z := Z_I \cup Z_{BU} \).

Indeed, any Fatou component that intersects \( I(f) \) is contained in \( I(f) \), and likewise for \( BU(f) \), so any additional points in the same component of \( \text{int}(K) \) can be omitted without affecting the conclusion of the theorem. Let \( (Z_j)_{j=0}^{\infty} \) be a sequence of finite subsets of \( Z \) with \( Z_j \subseteq Z_{j+1} \) and such that \( Z = \bigcup_{j=0}^{\infty} Z_j \).

Remark 5.1. We may always augment \( Z \) to be infinite. However, it turns out that the proof is slightly simpler in the case where \( \#Z = 2 \), which is the case required for the proof of Theorem 1.6. So we allow \( Z \) to be either finite or infinite, and discuss those instances where the assumption \( \#Z = 2 \) leads to simplifications.

Let \( (\zeta_j)_{j=0}^{\infty} \) be a sequence in \( Z_{BU} \) such that for every \( \zeta \in Z_{BU} \), there are infinitely many \( j \geq 0 \) such that \( \zeta_j = \zeta \).

Again choose a sequence \( (K_m)_{m=0}^{\infty} \) of compact and full subsets as in Lemma 2.9, as well as non-separating and \( 2^{-m} \)-dense subsets \( P_m \subseteq \partial K_m \). We may choose these sets such that \( K_0 \subseteq T_0 \). Our goal is to prove the following version of Proposition 3.2. Here and in the following, we retain all notation from Section 4, including the disc \( D \), the functions \( g_0 \) and \( f_0 \), the set \( A_0 \) and the number \( \varepsilon_0 \).

Proposition 5.2. Let \( (N_j)_{j=0}^{\infty} \) be defined as in (4.1): \( N_0 = 0 \) and \( N_j = N_{j-1} + (j + 1) \) for \( j \geq 1 \). There is a transcendental entire function \( f \) and an increasing sequence \( (m_j)_{j=0}^{\infty} \) with the following properties:

(a) \( |f(z) - g_0(z)| \leq 2\varepsilon_0 \) for \( z \in A_0 \); in particular, \( f(\overline{D}) \subseteq D \) and the domains \( V_j(f) \) from Lemma 4.1 are defined for all \( j \geq 1 \);
for all $j \leq h$ hence $1$. Let $\zeta$ be maximal with $|\Re f^n(\zeta)| = 1$ for all $j \geq 0$;
(e) if $N_j + 1 \leq n \leq N_{j+1}$ and $\zeta \in Z_j \setminus \{\zeta_j\}$, $|\Re f^n(\zeta)| \geq j$.

Most of the remainder of the section is dedicated to the proof of Proposition 5.2. However, we first show how this proposition implies Theorems 1.6 and 1.7.

**Proof of Theorem 1.7 using Proposition 5.2.** Similarly as in the proof of Theorem 3.1, all points in $P_{m_j}$ eventually map to $D$ and remain there by (a) and (b) while by (c),
\[ f^{N_j+1}(K) = f^{N_j+j+2}(K) \subseteq f^{j+1}(V_{j+1}(f)) = T_{j+1} \]
for all $j \geq 0$, and hence all points in $K$ have unbounded orbits. So $\partial K \subseteq J(f)$, proving (i).

Furthermore, if $N_j + 1 \leq n < N_{j+1}$, then
\[ f^n(K) \subseteq f^{n-N_j-1}(V_{j+1}(f)) \subseteq S_{n-N_j-1}, \]
and, in particular, $f^n(K) \cap T_{j+1} = \emptyset$ for $n < N_{j+1}$. Hence, $K$ is a wandering compactum, and (ii) holds. Together with (i), this also shows that every connected component of $\text{int}(K)$ is a wandering domain.

It remains to prove (iv). First, let $\zeta \in Z_{BU}$. As noted above, all points in $K$ have unbounded orbits. On the other hand, there are infinitely many $j \geq 0$ such that $\zeta_j = \zeta$, and hence $|\Re f^{N_j+1}(\zeta)| \leq 1$ by (d). Furthermore, by (e), $f^{N_j+1}(\zeta) \in V_{j+1} \subseteq S_0$ and hence $1 \leq |\Im f^{N_j+1}(\zeta)| \leq 2$. So $\liminf_{n \to \infty} |f^n(\zeta)| \leq 3$, and $\zeta \in BU(f)$ as required.

On the other hand, let $\zeta \in Z_I$. Let $j_0 \geq 0$ be large enough that $\zeta \in Z_j$ for $j \geq j_0$. If $n \geq N_{j_0} + 1$, let $j \geq j_0$ be maximal with $n \geq N_j + 1$. Then $|\Re f^n(\zeta)| \geq j$ by (e). Hence $|\Re f^n(\zeta)| \to +\infty$ as $n \to \infty$, and $\zeta \in I(f)$.

**Remark 5.3.** Let $\zeta \in Z_{BU}$, and $\zeta' \in Z \setminus \{\zeta\}$. Then (d) and (e) show that
\[ \limsup_{j \to \infty} \text{dist}#(f^{N_j+1}(\zeta), f^{N_j+1}(\zeta')) > 0. \tag{5.1} \]

Recall that, at the beginning of the section, we simplified the set $Z = Z_{BU} \cup Z_I$ by assuming that no interior component of $K$ contains more than one point of $Z$. If two points $z$ and $\tilde{z}$ belong to the same wandering Fatou component, then $\text{dist}#(f^n(z), f^n(\tilde{z})) \to 0$ as $n \to \infty$. Hence, as noted following Definition 1.10 (5.1) holds for the function in Theorem 1.7 as long as $\zeta$ and $\zeta'$ do not belong to the same connected component of $\text{int}(K)$.

**Proof of Theorem 1.6 using Theorem 1.7.** Let $X$ be a Lakes of Wada continuum with complementary components $U_0, U_1, U_2$ whose boundary agrees with $X$. We may assume that $U_0$ is the unbounded connected component of $C \setminus X$. Let $K = C \setminus U_0 = \text{fill}(X)$. Choose $\zeta_1 \in U_1$ and $\zeta_2 \in U_2$, and set $Z_I = \{\zeta_1\}$ and $Z_{BU} = \{\zeta_2\}$. Now apply Theorem 1.7.

Then $U_1$ is an escaping wandering domain, and $U_2$ is an oscillating wandering domain. By [RS11] Theorems 1.1 and 1.2(a) (or our stronger Theorem 1.11),
\[ I(f) \cap \partial U_1 = I(f) \cap X \subseteq X \]
has full harmonic measure when viewed from $U_1$, while
\[ \partial U_2 \setminus I(f) = X \setminus I(f) \subseteq X \]
has full harmonic measure when viewed from $U_2$. In particular, these sets are non-empty.

We will require the following preliminary observation concerning the conformal geometry of the sets $K_m$. For $\zeta \in Z$ and $m \geq 0$, let $U_m^\zeta$ denote the connected component of $\text{int}(K_m)$ containing $\zeta$.

**Lemma 5.4.** Suppose that $\omega \neq \zeta$ and $\zeta, \omega$ belong to the same connected component of $K$, but $\zeta, \omega$ do not belong to the same connected component of $\text{int}(K)$. Then the hyperbolic distance $\text{dist}_{U_m^\zeta}(\zeta, \omega)$ tends to $\infty$ as $m \to \infty$.

**Proof.** Let $m \geq 0$ and set $\delta_m := \max_{z \in \partial K} \text{dist}(z, \partial K_m)$. (Note that $\delta_m \to 0$ as $m \to \infty$.) Suppose that $\gamma$ is a curve connecting $\zeta$ and $\omega$ in $U_m^\zeta = U_m^\omega$. Such a curve must contain a point $\alpha \in \partial K$. (This is trivial if one of the points belongs to $\partial K$, and otherwise it follows from the fact that $\partial K$ separates $\zeta$ and $\omega$.) At least one of $\zeta$ or $\omega$ has distance at least $|\zeta - \omega|/2$ from $\alpha$. Without loss of generality, we can assume that this point is $\omega$. Parametrise the subcurve $\tilde{\gamma} \subseteq \gamma$ from $\alpha$ to $\omega$ by arc-length as $\tilde{\gamma} : [0, T] \to U_m^\zeta$. Then
\[
T \geq |\alpha - \omega| \geq \frac{|\zeta - \omega|}{2} \quad \text{and} \quad \text{dist}(\tilde{\gamma}(t), \partial U_m^\zeta) \leq \text{dist}(\alpha, \partial U_m^\zeta) + t \leq \delta_m + t
\]
for $t \in [0, T]$. By the standard estimate on the hyperbolic metric in a simply connected domain, the hyperbolic density of $U_m^\zeta$ at a point $z$ is bounded below by $1/(2 \text{dist}(z, \partial U_m^\zeta))$. It follows that
\[
\ell_{U_m^\zeta}(\gamma) \geq \ell_{U_m^\zeta}(\tilde{\gamma}) \geq \int_0^T \frac{dt}{2(\delta_m + t)} = \frac{1}{2} \ln \left(1 + \frac{T}{\delta_m}\right) \geq \frac{1}{2} \ln \left(1 + \frac{|\zeta - \omega|}{2\delta_m}\right),
\]
and hence
\[
\text{dist}_{U_m^\zeta}(\zeta, \omega) \geq \frac{1}{2} \ln \left(1 + \frac{|\zeta - \omega|}{2\delta_m}\right) \to +\infty \quad \text{as} \quad m \to \infty.
\]

Recall that $U_m^\zeta$ is a Jordan domain; let $\pi_m^\zeta : U_m^\zeta \to \mathbb{D}$ be a conformal map with $\pi_m^\zeta(\zeta) = 0$. Let $\omega \in Z \setminus \{\zeta\}$ belong to the same connected component of $K$ as $\zeta$. By the assumption on $Z$ made at the beginning of this section, the points $\zeta$ and $\omega$ do not both belong to the same interior component of $K$. Hence, by Lemma 5.4, $|\pi_m^\zeta(\omega)| \to 1$ as $m \to \infty$.

Passing to a subsequence of $(K_m)_{m=0}^\infty$, we may assume that for every pair of distinct points $\zeta, \omega \in Z$, there is a point $\xi^\omega(\omega) \in \partial \mathbb{D}$ such that $\pi_m^\zeta(\omega) \to \xi^\omega(\omega)$ as $m \to \infty$.

**Remark 5.5.** Suppose that $Z$ consists of exactly two points $\zeta$ and $\omega$ both belong to the same connected component of $K$. Then we may normalise $\pi_m^\zeta$ and $\pi_m^\omega$ so that they map the other point to the positive real axis, and the above property is automatic with $\xi^\zeta(\omega) = \xi^\omega(\zeta) = 1$.

Recall from Proposition 5.2 that, at time $N_j + 1$, the point $\zeta_j$ should map to a bounded part of the set $V_{j+1}(f)$, while other points of $Z_j$ should map sufficiently far to the right. In order to achieve this, we shall use the following fact about conformal mappings.
Figure 7. Illustration of Proposition 5.6. Here $V$ is a half-strip, $\nu \in V$ and $\Delta \subseteq V$ is a further half-strip, far to the right of $\nu$. The set $\Xi \subseteq \partial \mathbb{D}$ consists of five points, which are mapped into $\Delta$ by $\varphi: \mathbb{D} \to W \subseteq V$.

**Proposition 5.6** (Conformal maps with prescribed behaviour). Let $V \subseteq \mathbb{C}$ be a domain, let $\Delta \subseteq V$ be non-empty and open, and let $\nu \in V$. Let $\Xi \subseteq \partial D$ be a compact set of zero logarithmic capacity.

Then there exist a bounded Jordan domain $W \subseteq V$ and a conformal isomorphism

$$\varphi: \mathbb{D} \to W$$

with $\varphi(0) = \nu$ whose continuous extension to $\overline{\mathbb{D}}$ satisfies $\varphi(\xi) \in \Delta$ for all $\xi \in \Xi$.

Proposition 5.6 follows from a result of Bishop [Bis06] concerning interpolation using boundary values of conformal maps. It will be applied in settings where $\nu$ and $\Delta$ are very far apart in $V$ (see Figure 7). To avoid interrupting the flow of ideas, we postpone the proof of Proposition 5.6 until Section 9, but remark that most of the applications in this paper do not use the full strength of the result. For the proof of Theorem 1.7 we use only the case where $\# \Xi < \infty$. Moreover, the case when $\# \Xi = 1$ is trivial: choose $W \subseteq V$ to be a Jordan domain containing $\nu$ and whose boundary passes through $\Delta$, and let $\varphi$ be an appropriately normalised conformal isomorphism. This trivial case corresponds to the setting where $\# Z = 2$ which, as noted above, is sufficient to prove Theorem 1.6. We also remark that Proposition 5.6 is not used in the proof of Theorem 1.2.

**Proof of Proposition 5.2**. The construction follows a similar pattern as the proof of Proposition 3.2. We inductively construct a sequence $(f_j)_{j=0}^{\infty}$ of entire functions, where $f_j$ approximates a function $g_j$ – continuous on a closed (but unbounded) set $A_j$ and holomorphic on its interior – up to a uniform error of at most $\varepsilon_j$. (Recall that $f_0$, $g_0$, $A_0$ and $\varepsilon_0$ were already chosen in Section 4.) For $j \geq 0$, define

$$\Sigma_j := \overline{B} \cup \{ z \in \mathbb{C} : |4 \text{Im } z| \leq 5^{j+1} + 3 \}.$$ 

Then $A_0 \subseteq \Sigma_j$, $\Sigma_j \cap T_j = \emptyset$, $T_\ell \subseteq \Sigma_j$ for $\ell < j$ and $\bigcup_{j=0}^{\infty} \Sigma_j = \mathbb{C}$.

The sequence $(m_j)_{j=0}^{\infty}$ is also chosen as part of the inductive construction, which ensures the following inductive hypotheses.

(i) For every $j \geq 0$, $f_j^{N_j}$ is injective on $K_{m_j}$, and $f_j^{N_j}(K_{m_j}) \subseteq T_j$.

(ii) For every $j \geq 1$, $\Sigma_{j-1} \subseteq A_j \subseteq \Sigma_j$. 


(iii) $\varepsilon_j \leq \varepsilon_{j-1}/2$ for $j \geq 1$.

Set $m_0 := 0$; the inductive hypothesis holds trivially for $j = 0$.

Suppose that $f_j$ and $m_j$ have been constructed. Let $R_j \subseteq T_j$ be a compact full neighbourhood of $Q_j := f_j^{N_j}(P_{m_j})$ disjoint from $f_j^{N_j}(K_{m_{j+1}})$.

Let $Z_j'$ consist of those $\omega \in Z_j \setminus \{\zeta_j\}$ that belong to the same connected component of $K$ as $\zeta_j$. Define $\Xi_j := \{\xi_j^\omega : \omega \in Z_j'\} \subseteq \partial \mathbb{D}$.

The inductive hypothesis implies, in particular, that $f_j$ is $2\varepsilon_0$-close to $g_0$ on $A_0$. Therefore the domains $V_{j+1} = V_{j+1}(f_j)$ from Lemma 4.1 are defined. Define $\Delta_j := \{z \in V_{j+1} : \text{Re } z > j\}$ and $\tilde{\Delta}_j := \{z \in V_{j+1} : \text{Re } z < 1\}$. Choose any $\nu_j \in \tilde{\Delta}_j$ and apply Proposition 5.6 with $V = V_{j+1}$, $\Delta = \Delta_j$, $\nu = \nu_j$ and $\Xi = \Xi_j$. We obtain a Jordan domain $W_j \subseteq V_{j+1}$ and a conformal map $\varphi_j : \mathbb{D} \to W_j$, extending continuously to $\partial \mathbb{D}$, such that $\varphi_j(0) = \nu_j \in \tilde{\Delta}_j$ and $\varphi_j(\xi) \in \Delta_j$ for all $\xi \in \Xi_j$. 

Figure 8. The definition of $g_{j+1}$ in the proof of Proposition 5.2. The set $L_j \subseteq K_{m_j} \subseteq T_0$ (shown in light grey) has two connected components, one of which is $U_j$, containing $\zeta_j$. On $f_j^{N_j}(U_j)$, the map $g_{j+1}$ is the composition of three conformal isomorphisms: $(f_j^{N_j}|_{U_j})^{-1}$, the Riemann map $\pi_j : U_j \to \mathbb{D}$ and $\varphi_j : \mathbb{D} \to W_j \subseteq V_{j+1}$, as obtained from Proposition 5.6.
Now choose \( m_{j+1} \geq m_j + 2 \) sufficiently large that
\[
\varphi_j(\pi^{\xi_j}_{m_{j+1}-1}(\omega)) \in \Delta_j
\]
for all \( \omega \in Z'_j \), and such that \( \omega \notin U^{\xi_j}_{m_{j+1}-1} \) for \( \omega \in Z_j \setminus \{Z'_j \cup \{\zeta_j\}\} \). (Recall that \( \pi^\xi_m(\omega) \to \xi^\xi(\omega) \in \Xi_j \) as \( m \to \infty \) for \( \omega \in Z'_j \)). We set \( L_j := K_{m_{j+1}-1}, U_j := U^{\xi_j}_{m_{j+1}-1} \) and \( \pi_j := \pi^{\xi_j}_{m_{j+1}-1} \). Recall that \( L_j \) is a finite disjoint union of closed Jordan domains, one of which is \( \overline{U}_j \). In particular, the conformal map \( \pi_j : U_j \to \mathbb{D} \) extends continuously to \( \overline{U}_j \).

Let \( \alpha_j \) be an affine map such that
\[
\alpha_j(f^N_j(L_j)) \subseteq \{z \in V_{j+1} \setminus \overline{W}_j : \text{Re } z > j\} = \Delta_j \setminus \overline{W}_j.
\]
Set
\[
A_{j+1} := \Sigma_j \cup R_j \cup f^N_j(L_j) \subseteq \Sigma_j \cup T_j \subseteq \Sigma_{j+1}.
\]
The set \( \Sigma_j \) is the disjoint union of a large closed central horizontal strip and the strips \( \overline{S}_k \) for \( k > j \). It is disjoint from \( T_j \), and hence from \( R_j \) and \( f^N_j(L_j) \). The latter two sets are compact and full, and disjoint from each other by choice of \( R_j \). Hence \( A_{j+1} \) satisfies the hypotheses of Arakelyan’s theorem. We define
\[
g_{j+1} : A_{j+1} \to \mathbb{C}; \quad z \mapsto \begin{cases} f_j(z), & \text{if } z \in \Sigma_j; \\ 0, & \text{if } z \in R_j; \\ \varphi_j((f^N_j|_{L_j})^{-1}(z)), & \text{if } z \in f^N_j(U_j), \\ \alpha_j(z), & \text{if } z \in f^N_j(L_j \setminus U_j). \end{cases}
\]
(See Figure 8.) Then \( g_{j+1} \) is continuous on \( A_{j+1} \) and holomorphic on its interior.

Finally, choose \( \varepsilon_{j+1} \) according to \( \text{(iii)} \) and sufficiently small that any entire function \( f \) with \( |f(z) - g_{j+1}(z)| \leq 2\varepsilon_{j+1} \) on \( A_{j+1} \) satisfies:
\begin{enumerate}
  \item \( f^N_{j+1}(P_{m_j}) \subseteq D; \)
  \item \( f^N_{j+1} \) is injective on \( K_{m_{j+1}} \subseteq \text{int}(L_j); \)
  \item \( f^N_{j+1}(K_{m_{j+1}}) \subseteq S_{j+1} \) for \( \ell = 1, \ldots, j+1; \)
  \item \( f^N_{j+1}(K_{m_{j+1}}) \subseteq T_{j+1}; \)
  \item \( f^N_{j+1}(\zeta_j) \in \Delta_j; \)
  \item \( f^N_{j+1}(\omega) \in \Delta_j \) for \( \omega \in Z_j \setminus \{\zeta_j\}. \)
\end{enumerate}
Note that \( g_{j+1} \) itself satisfies these properties, and they are preserved under sufficiently close approximation by Lemma 2.5 and Corollary 2.7. We complete the inductive construction by applying Arakelyan’s theorem to find an entire function \( f_{j+1} \) that is \( \varepsilon_{j+1} \)-close to \( g_{j+1} \) on \( A_{j+1} \). The inductive hypothesis \( \text{(ii)} \) follows from \( \text{(2)} \) and \( \text{(4)} \) while \( \text{(ii)} \) and \( \text{(iii)} \) hold by our choice of \( A_{j+1} \) and \( \varepsilon_{j+1} \).

By \( \text{(ii)} \) and \( \text{(iii)} \),
\[
|f_k(z) - g_j(z)| \leq \sum_{\ell=j}^{k} \varepsilon_\ell < 2\varepsilon_j \tag{5.2}
\]
whenever \( k \geq j \geq 0 \) and \( z \in A_j \supseteq \Sigma_{j-1} \). It follows that \( f_k \) converges locally uniformly to a limit function \( f : \mathbb{C} \to \mathbb{C} \), which satisfies claim \( \text{(a)} \) of Proposition 5.2 by \( \text{(5.2)} \). Claim \( \text{(b)} \) holds by \( \text{(1)} \). Claim \( \text{(c)} \) holds by \( \text{(2)} \), \( \text{(3)} \) and \( \text{(4)} \). Claim \( \text{(d)} \) follows from \( \text{(5)} \). Finally, \( \text{(e)} \) follows from \( \text{(6)} \) together with Lemma 4.1.
Remark 5.7. With a slight modification of the proof, we may achieve that all points \( \zeta \in Z_{BU} \) have the same \( \omega \)-limit set. Indeed, let \( f \) be any map as in Lemma 4.1 and consider the curves

\[
\gamma_j^f: \mathbb{R} \to V_j; \quad t \mapsto (f^j|_{V_j})^{-1}\left(5^j \cdot t + \frac{5^{j+1} + 9}{4} i\right).
\]

Since \( f \) and \( \Phi \) are close to each other and both are uniformly expanding on \( S \), it is straightforward to see that \( \gamma_j^f \) converges uniformly to a curve \( \gamma_f: \mathbb{R} \to S_0 \), which consists precisely of the points \( z \) with \( f^j(z) \in S_j \) for all \( j \geq 0 \). Moreover, again because of the expansion of \( f \), the distance \( |\gamma_j^f(t) - \gamma_f(t)| \) is bounded by a constant that depends only on \( j \) (not on \( f \) or \( t \)), and tends to zero as \( j \to \infty \).

In the proof of Proposition 5.2, when defining \( g_{j+1} \), we chose \( \nu_j \) to have small real part, but could have chosen it to be any point of \( V_{j+1} \), as long as for every \( \zeta \in Z_{BU} \), there is an infinite set of \( j \) with \( |\gamma_j^f| \) such that \( \Re \nu_j \) is bounded independently of \( j \). Let \( (t_j)_{j=0}^{\infty} \) be a sequence of rational numbers so that, for every \( t \in \mathbb{Q} \) and every \( \zeta \in Z_{BU} \), there are infinitely many \( j \) such that \( t_j = t \) and \( \zeta_j = \zeta \). In the proof of Proposition 5.2, we then adjust the definition of \( \nu_j \) and \( \tilde{\Delta}_j \), taking \( \nu_j := \gamma_j^{f+1}(t_j) \) and \( \tilde{\Delta}_j := D(\nu_j, 1/j) \cap V_{j+1} \) for all \( j \). If \( \varepsilon_{j+1} \) is chosen sufficiently small, and \( f \) is any function with \( |f(z) - g_{j+1}(z)| \leq 2\epsilon_{j+1} \) on \( A_{j+1} \), then \( f^{N_j+1}(\zeta_j) \) and \( \gamma_j^{f+1}(t_j) \) both belong to \( \tilde{\Delta}_j \).

For the resulting limit function \( f \), the \( \omega \)-limit set of any \( z \in K \) is contained in the union \( \Omega \) of \( \gamma_f \), its forward iterates \( f^n(\gamma_f) \subseteq S_n \), and \( \infty \). On the other hand, let \( \zeta \in Z_{BU} \). By construction, the \( \omega \)-limit set of \( \zeta \) contains \( \gamma_f(t) \) for all \( t \in \mathbb{Q} \), and hence agrees with \( \Omega \).

Observe that, if \( Z_{BU} \) contains a point of \( \text{int}(K) \), then the resulting function \( f \) has a wandering domain on which the set of (constant) limit functions of the iterates contains unbounded closed connected sets. Lazebnik [Laz17] previously showed that the set of limit functions in a wandering domain can be uncountable, answering a question raised by Osborne and Sixsmith [OS16 Question 2].

6. LARGER SETS OF MAVERICK POINTS

In this section, we indicate how we may modify the proof of Proposition 5.2 to obtain examples with larger sets of maverick points. In particular, we prove Theorems 1.13 and 1.14. Recall that Theorem 1.13 says that in the case that \( U = \mathbb{D} \), we may obtain any compact subset of \( \partial \mathbb{D} \) of logarithmic capacity zero as the set of maverick points of \( U \). On the other hand, Theorem 1.14 shows that for some domains, the set of maverick points may have positive planar Lebesgue measure.

As already mentioned, in the proof of Proposition 5.2, we do not use the full strength of Proposition 5.6 which we apply only to a finite set \( \Xi_j \). Recall that, in Proposition 5.6, the compact set \( \Xi \) may be chosen to be infinite, as long as it has zero logarithmic capacity. Another way in which we may modify the proof of Proposition 5.2 is that we might choose to reverse the roles of \( \Delta \) and \( \tilde{\Delta} \), so that the points of the sequence \( \zeta_j \) become escaping points rather than elements of \( BU(f) \).

To investigate the additional flexibility this gives us in the proof, let us follow the set-up and notation as in Section 5 but modify the parts of the construction that were
related to the sets $Z_I$ and $Z_{BU}$. As before, we suppose that $K \subseteq T_0$ is a compact, full set. For simplicity, we restrict here to the case where $K$ is the closure of a simply connected domain $U = \text{int}(K)$, which is the case of interest for Theorems 1.13 and 1.14. We also fix a point $\zeta_0 \in U$ and $k \in \{0,1\}$. Our goal is to construct a function $f$ for which $U$ is an oscillating (if $k = 0$) or escaping (if $k = 1$) wandering domain, and for which $\partial U$ contains a set of maverick points on its boundary that is “large” in some sense. Theorem 1.7 shows that we may realise any countable set $Z_{\text{mav}} \subseteq \partial U$ as a set of maverick points.

Indeed, we may set $Z_{BU} = \{\zeta_0\}$ and $Z_I = Z_{\text{mav}}$ if $k = 0$, or $Z_{BU} = Z_{\text{mav}}$ and $Z_I = \{\zeta_0\}$ if $k = 1$.

Now, instead of $Z_I$ and $Z_{BU}$, suppose that we start with a given $\sigma$-compact set $Z_{\text{mav}} \subseteq \partial U$. For $j \geq 0$, also fix

(i) non-empty compact sets $Z_j \subseteq Z_{\text{mav}}$ with the property that, for every $z \in Z_{\text{mav}}$, there are infinitely many $j$ such that $z \in Z_j$;

(ii) compact sets $\Xi_j \subseteq \partial \mathbb{D}$ of zero logarithmic capacity.

Once again, we choose a nested sequence $K_m \subseteq T_0$ of closed Jordan domains shrinking to $K$, and a conformal map $\pi_m: U_m \to \mathbb{D}$ with $\pi_m(\zeta_0) = 0$, where $U_m = \text{int}(K_m)$. The key assumption is that this choice can be made in such a way that

(*) for every $j \geq 0$, we have $\pi_m(Z_j) \to \Xi_j$ in the Hausdorff metric as $m \to \infty$.

With this setup, we follow the proof of Proposition 5.2 almost word-for-word, except for the following modifications.

- We take $\zeta_j := \zeta_0$ for all $j$.
- The role of $Z'_j$ is taken by $Z_j$ (since $\zeta_j \notin Z_j$ and $K$ is connected).
- The set $\Xi_j$ is the set from (ii).
- If $k = 1$, then the roles of $\Delta_j$ and $\tilde{\Delta}_j$ are exchanged. That is, in this case $\tilde{\Delta}_j := \{z \in V_{j+1}: \text{Re } z > j\}$ and $\Delta_j := \{z \in V_{j+1}: |\text{Re } z| < 1\}$.
- The map $\alpha_j$ is not required, since $L_j = U_j$.

**Proposition 6.1** (General construction of maverick points). Using the above notation, and assuming (*) there exists a transcendental entire function $f$ with the following properties.

(a) $U$ is an oscillating wandering domain if $k = 0$, and an escaping wandering domain if $k = 1$.

(b) Every point of $Z_{\text{mav}} \subseteq \partial U$ is a maverick point.

(c) If $k = 0$, then every point $z \in Z_{\text{mav}}$ such that $z \in Z_j$ for all but finitely many $j$ belongs to the escaping set.

**Proof.** That $U$ is a wandering domain follows exactly as in the proof of Theorem 1.7. We have $f^{N_j+1}(\zeta_0) \in \tilde{\Delta}_j$ for all $j$. If $k = 0$, then $|\text{Re } f^{N_j+1}(\zeta_0)| < 1$ for all $j$; hence $\zeta_0 \in BU(f)$ and $U$ is an oscillating wandering domain. If $k = 1$, then $\text{Re } f^{N_j+1}(\zeta_0) > j$ for all $j$, and by Lemma 4.1 also $\text{Re } f^{n}(\zeta_0) > j$ for $N_j + 1 \leq n \leq N_{j+1}$. Hence $\zeta_0 \in I(f)$ and $U$ is an escaping wandering domain.

For $j \geq 0$, we have $f^{N_j+1}(\zeta_0) \in \Delta_j$, but $f^N(\zeta_0) \in \Delta_j$ for all $z \in Z_j$. In particular, $\limsup_{j \to \infty} \text{dist}^\#(f^{N_j+1}(z), f^{N_j+1}(\zeta_0)) > 0$ for all $z \in Z_{\text{mav}}$. Hence every such $z$ is a maverick point (see Lemma 8.1 below).
If \( k = 0 \) and \( z \in Z_j \) for all but finitely many \( j \), then \( \text{Re} f^{N_j+1}(z) > j \) for all such \( j \), and it follows as in the proof of part \( (e) \) of Proposition 5.2 that \( \text{Re} f^n(z) > j \) for \( N_j + 1 \leq n \leq N_{j+1} \). Hence \( z \in I(f) \). 

We now apply this observation to prove Theorems 1.13 and 1.14.

**Proof of Theorem 1.13.** Let \( K_m = \overline{D(\zeta_0, r_m)} \) be a nested sequence of concentric closed discs shrinking to a disc \( K = \overline{D(\zeta_0, r)} \), where \( \zeta_0 \in T_0 \) and \( r_0 \) is chosen sufficiently small such that \( K_0 \subseteq T_0 \). Let \( \pi_m(z) = (z-\zeta_0)/r_m \); then \( \pi_m \) maps \( U_m := \text{int}(K_m) \) to \( \mathbb{D} \). Define

\[
Z_{\text{mav}} := \{ \zeta_0 + r\xi : \xi \in \Xi \}.
\]

Then \( \pi_m(Z_{\text{mav}}) = \Xi \) for all \( m \), so property \( (*) \) is satisfied.

Let \( k \in \{0, 1\} \), and apply Proposition 6.1 with \( Z_j = Z_{\text{mav}} \) and \( \Xi_j = \Xi \) for all \( j \). We obtain an entire function for which \( U = D(\zeta_0, r) \) is a wandering domain (oscillating or escaping depending on \( k \)) and all points of \( Z_{\text{mav}} \) are maverick points. Conjugating with an affine map that takes \( K \) to \( \mathbb{D} \), the proof is complete.

**Proof of Theorem 1.14.** Let \( Z \subseteq \mathbb{C} \) be a Jordan arc of positive Lebesgue measure, let \( \gamma : (0, 1) \to \mathbb{C} \setminus Z \) be an injective analytic curve such that as \( t \to 0 \), \( \gamma(t) \) accumulates on all of \( Z \) from one side, while as \( t \to 1 \), \( \gamma(t) \) accumulates on all of \( Z \) from the other side. Then \( \mathbb{C} \setminus (Z \cup \gamma) \) has exactly two connected components by a generalised version of the Jordan curve theorem; see e.g. [Tim10, Theorem 5.7]. Let \( U \) be the bounded connected component of \( \mathbb{C} \setminus (Z \cup \gamma) \). Then \( U \) is a regular simply connected domain with \( \partial U = Z \cup \gamma \); we set \( K := \overline{U} \). Let \( \zeta_0 \in U \), and observe that \( Z \) has zero harmonic measure in \( U \). (Indeed, by construction all points of \( Z \), with the exception of one endpoint, are inaccessible from \( U \).)

Choose a nested sequence of closed Jordan domains \( (K_m) \) shrinking down to \( K \). Set \( U_m = \text{int}(K_m) \) and let \( \pi_m : U_m \to \mathbb{D} \) be a conformal isomorphism with \( \pi_m(\zeta_0) = 0 \). Then the Euclidean diameter of \( \pi_m(Z) \) tends to zero as \( m \to \infty \), so we may normalise \( \pi_m \) in such a way that \( \pi_m(Z) \to \{1\} \) as \( m \to \infty \).

We assume that the above sets were chosen such that \( K_0 \subseteq T_0 \). For \( j \geq 0 \), set \( Z_j := Z_{\text{mav}} := Z \), \( \Xi_j := \{1\} \). Let \( k \in \{0, 1\} \), and apply Proposition 6.1 with either \( k = 0 \) or \( k = 1 \). We obtain a function \( f \) for which \( U \) is an escaping (if \( k = 1 \)) or oscillating (if \( k = 0 \)) wandering domain, and for which the set of maverick points contains \( Z \) and hence has positive Lebesgue measure. The set of non-maverick points is contained in the analytic curve \( \gamma = \partial U \setminus Z \), and hence has Hausdorff dimension 1.

**Remark 6.2.** For the functions constructed in the proofs of Theorems 1.13 and 1.14, if the wandering domain is oscillating, then all points of \( Z_{\text{mav}} \) are in fact escaping by Proposition 6.1\((e)\).

7. **Counterexamples to Eremenko’s conjecture**

We prove Theorem 1.2 first in the case where \( X = \{0\} \), and then indicate how to modify the proof in order to obtain the more general case.

**Theorem 7.1.** There is a transcendental entire function \( f \) such that

(a) \([0, +\infty) \subseteq J(f)\);
(b) \(0 \in I(f)\);
Figure 9. The unbounded closed set $K = \bigcap_{j=0}^{\infty} K_j$ in the proof of Theorem 7.1.

(c) $(0, +\infty) \subseteq BU(f)$;
(d) $[0, +\infty)$ is a connected component of $J(f) \cup I(f) \cup BU(f)$.
In particular, $\{0\}$ is a connected component of $I(f)$.

In the proofs of Theorems 7.1 and 1.2, we shall use conformal maps between domains that agree with strips close to infinity. The following simple fact will allow us to control their derivatives.

Lemma 7.2 (Conformal maps between extensions of half-strips). Let $U_1, U_2 \subseteq \mathbb{C}$ be simply connected domains such that, for sufficiently large $R > 0$ and $j \in \{1, 2\}$,

$$\{\text{Im } z: z \in U_j \text{ and } \text{Re } z = R\} = (-\pi/2, \pi/2).$$

Let $\varphi: U_1 \to U_2$ be a conformal map such that $\text{Re } \varphi(z) \to +\infty$ as $\text{Re } z \to +\infty$. Then

$$\varphi'(z) = 1 + O(\exp(-\text{Re } z)) \quad \text{and} \quad \varphi(z) = z + \rho + O(\exp(-\text{Re } z))$$
as $\text{Re } z \to +\infty$, where $\rho \in \mathbb{R}$ is a constant.

Proof. Restricting $\varphi$, if necessary, we may assume that $U_1$ and $U_2$ are contained in the strip of height $\pi$ centred at the real axis. Write $w = \varphi(z)$, $\zeta = \exp(-z)$ and $\omega = \exp(-w)$. Since $\exp$ is injective on $U_1$ and $U_2$, setting $\psi(\zeta) := \omega$ yields a well-defined conformal map $\psi: \exp(-U_1) \to \exp(-U_2)$. By the Schwarz reflection principle, $\psi$ extends conformally to a neighbourhood of 0; set $\lambda := \psi'(0) > 0$ and $\rho := -\log \lambda$.

We have $\omega = \lambda \zeta + O(|\zeta|^2)$, $1/\omega = 1/(\lambda \zeta) + O(1)$ and $\psi'(\zeta) = \lambda + O(|\zeta|)$ as $\zeta \to 0$ (and hence as $\text{Re } z \to +\infty$). Thus

$$\varphi'(z) = -\zeta \cdot \psi'(\zeta) \cdot \frac{-1}{\omega} = \frac{\lambda \zeta}{\omega} \cdot (1 + O(|\zeta|)) = 1 + O(|\zeta|) = 1 + O(\exp(-\text{Re } z))$$
as claimed. Similarly,

$$\varphi(z) = \log \left( \frac{1}{\omega} \right) = \log \left( \frac{1}{\lambda \zeta} + O(1) \right) = z + \rho + O(|\zeta|).$$

$\blacksquare$
Proof of Theorem 7.1. We shall prove the theorem with \([0, \infty)\) replaced by \(K := [0, \infty) + 7i/2\); the result then follows by conjugating with a translation. Let the strips \((S_j), (T_j)\), the disc \(D\), the map \(f_0\) and the numbers \(N_j\) and \(\varepsilon_0\) be as in Section 4. Also recall the definition of the domains \(V_j(f)\) from Lemma 4.1. Observe that \(K \subseteq T_0\).

Similarly as in Proposition 5.2 we construct an entire function \(f\) and a sequence \((K_j)_{j=0}^{\infty}\) with the following properties for all \(j \geq 0\), where \(\zeta := 7i/2\) and \(t_j := j + 2\) (see Figure 9):

- (a) \(K_j\) is a closed horizontal half-strip with \(K \subseteq K_{j+1} \subseteq \text{int}(K_j) \subseteq T_0\) for all \(j \geq 0\), and \(\bigcap_{j=0}^{\infty} K_j = K\);
- (b) \(f(D) \subseteq D\);
- (c) \(f^{N_j+1}(\partial K_j) \subseteq D\);
- (d) \(f^{N_j+1}\) is injective on \(K_{j+1}\), with \(f^{N_j+1}(K_{j+1}) \subseteq V_{j+1}(f)\);
- (e) if \(z = t + 7i/2\) with \(1/t_j \leq t \leq t_j\), then \(|\text{Re} f^{N_j+1}(z)| \leq 1\);
- (f) if \(N_j + 1 \leq n \leq N_{j+1}\), then \(|\text{Re} f^n(\zeta)| \geq j\).

Observe that this proves the theorem. Indeed, let \(z \in K \setminus \{\zeta\}\). Then by (d) and (e) for \(j \geq 0\), \(|\text{Re} f^{N_j+1}(z)| \leq 1\) and \(1 \leq \text{Im} f^{N_j+1}(z) \leq 2\), while \(f^{N_j+1}(z) \in T_{j+1}\), and hence \(f^{N_j+1}(z) \to \infty\) as \(j \to \infty\). So \(z \in BU(f)\). On the other hand, \(\zeta \in I(f)\) by (f).

Furthermore, by (b) and (c) \(\partial K_j\) belongs to \(F(f) \setminus (I(f) \cup BU(f))\). So, by (a) \(\bigcup_{j=0}^{\infty} \partial K_j\) separates \(K\) from every other point of \(J(f) \cup I(f) \cup BU(f)\).

To construct the function \(f\), we proceed similarly as in the proof of Proposition 5.2 but will need to take some care because now the sets \(K\) and \(K_j\) are unbounded. Thus, we once more construct a sequence of entire functions \(f_j\), each of which approximates a function \(g_j\) (defined inductively in terms of \(f_{j-1}\)) on a set \(A_j\) up to an error \(\varepsilon_j\), where \(A_j\) satisfies the hypotheses of Arakelyan’s theorem. Again define

\[\Sigma_j := \mathfrak{S} \cup \{z \in \mathbb{C} : |4 \text{Im } z| \leq 5^{j+1} + 3\} .\]

Along with \(f_j\), we construct a half-strip \(K_j\) as in (a) in such a way that the following inductive hypotheses are satisfied:

- (i) \(f_j^{N_j}(K_j) \subseteq T_j\) and \(\text{dist}(f_j^{N_j}(K_j), \partial T_j) > 0\);
- (ii) \(\text{Re } f_j^{N_j}(z) \to +\infty\) as \(z \to \infty\) in \(K_j\);
- (iii) on the set \(\tilde{K}_j := \bigcup_{k=0}^{N_j-1} f_j^k(K_j)\), the derivative \(|f_j^k|\) is bounded from above and below by positive constants, and \(f_j\) is uniformly continuous at every point of \(\tilde{K}_j\) in the sense of Remark 2.6
- (iv) \(\Sigma_{j-1} \subseteq A_j \subseteq \Sigma_j\) if \(j \geq 1\);
- (v) \(\varepsilon_j \leq \varepsilon_{j-1}/2\) if \(j \geq 1\).

We choose \(K_0 \subseteq \text{int}(T_0)\) to be an arbitrary closed half-strip with \(K \subseteq \text{int}(K_0)\) (see Figure 9). The inductive hypotheses hold trivially for \(j = 0\) (recall that \(N_0 = 0\)).

Suppose that \(f_j\) and \(K_j\) have been constructed. Let \(L_j\) be a closed horizontal half-strip contained in \(\text{int}(K_j)\) such that \(\zeta \in \partial L_j\) and \(K \setminus \{\zeta\} \subseteq \text{int}(L_j)\). Let \(\delta_j\) denote the hyperbolic length, in the hyperbolic metric of \(\text{int}(L_j)\), of the segment \([\zeta + 1/t_j, \zeta + t_j]\).
Figure 10. The definition of $g_{j+1}$ in the proof of Theorem 7.1. The set $\tilde{L}_j \subseteq K_j \subseteq T_0$ is the light grey half-strip containing the ray $K$. On \text{int}(f_j^{N_j}(\tilde{L}_j))$, the map $g_{j+1}$ is the composition of three conformal maps: $(f_j^{N_j}|_{\text{int}(L_j)})^{-1}$, $\hat{\varphi}_j$, and $(f_{j+1}|_{V_j+1(f_j)})^{-1}$.

By (i) and (ii), $Q_j := f_j^{N_j}(\partial K_j)$ is a Jordan arc with both ends at infinity and dist($Q_j, \partial T_j$) > 0. By (iii) also dist($Q_j, f_j^{N_j}(L_j)$) > 0. Let $R_j \subseteq T_j$ be a closed neighbourhood of $Q_j$ homeomorphic to a bi-infinite closed strip (with real part tending to $+\infty$ in both directions), such that dist($Q_j, \partial R_j$) > 0 and dist($R_j, f_j^{N_j}(L_j)$) > 0.

Now let $W_j \subseteq T_{j+1}$ be a bi-infinite horizontal strip. If $W_j$ is chosen sufficiently thin, then any hyperbolic ball of radius $\delta_j$ (in the hyperbolic metric of $W_j$) has Euclidean diameter less than 1. Let $\varphi_j: \text{int}(L_j) \to W_j$ be a conformal isomorphism with $\text{Re}\varphi_j(\zeta + 1) = 0$ whose continuous extension to the boundary satisfies $\varphi_j(\zeta) = -\infty$ and $\varphi_j(+\infty) = +\infty$. Then $|\text{Re}\varphi_j(\zeta + t)| < 1$ for $1/t_j \leq t \leq t_j$. Observe that $\varphi_j(z)$ is finite for all $z \in \partial L_j \setminus \{\zeta\}$.
Let \( \eta > 0 \) be small (see below); define \( \tilde{L}_j := L_j - \eta/2 \) and \( \tilde{\varphi}_j(z) := \varphi_j(z + \eta) \). Then \( \tilde{\varphi}_j : \tilde{L}_j \to \mathbb{W}_j \) is continuous. Set
\[
A_{j+1} := \Sigma_j \cup R_j \cup f_{N_j}^j(\tilde{L}_j)
\]
and
\[
g_{j+1} : A_{j+1} \to \mathbb{C}; \quad z \mapsto \begin{cases} f_j(z), & \text{if } z \in \Sigma_j, \\ 0, & \text{if } z \in R_j, \\ (f_{j+1}^j)'(z)^{-1}(\tilde{\varphi}_j((f_{N_j}^j|_{\tilde{L}_j})^{-1}(z))), & \text{if } z \in f_{N_j}^j(\tilde{L}_j). \end{cases} \tag{7.1}
\]
(See Figure 10.) If \( \eta \) is chosen sufficiently small, then
- \( \tilde{L}_j \subseteq K_j \) and \( f_{N_j}^j(\tilde{L}_j) \) is disjoint from \( R_j \);
- \( |\text{Re } g_{N_j}^{j+1}(\zeta)| = |\text{Re }((f_{j+1}^j|_{V_{j+1}(f_j)})^{-1}(\tilde{\varphi}_j(\zeta)))| > j \);
- \( |\text{Re } g_{N_j}^{j+1}(\zeta + t)| \leq 1 \) for \( 1/|t_j| \leq t \leq t_j \).

(For the equality in the final bullet point, recall that \( N_{j+1} = N_j + j + 2 \).) Let \( K_j+1 \supseteq K \) be any closed half-strip contained in the interior of \( \tilde{L}_j \) and with \( K \subseteq \text{int}(K_{j+1}) \), chosen sufficiently small such that
\[
\max_{z \in \partial K_{j+1}} \text{dist}(z, K) \leq \frac{1}{j+1}. \tag{7.2}
\]

The set \( A_{j+1} \) is the union of three sequences of pairwise disjoint topological closed strips and half-strips, each tending uniformly to \( \infty \), and therefore satisfies the hypotheses of Arakelyan’s theorem.

Observe that \( g_{N_j}^{j+1} \) is defined and injective on \( \tilde{L}_j \). We claim that \( |g_{j+1}'| \) is bounded from above and below by positive constants on \( g_{j+1}^k(\tilde{L}_j) \) for \( 0 \leq k < N_{j+1} \).

This holds by the inductive hypothesis for \( 0 \leq k < N_j \). Since \( g_{N_j}^{j+1}(\tilde{L}_j) \subseteq V_{j+1}(f_j) \), the claim also holds for \( N_j + 1 \leq k < N_{j+1} \), by Lemma 4.1. So it remains to establish it for \( k = N_j \). The derivatives of the iterated inverse branches of \( f_j \) used in the definition of \( g_{j+1} \) on \( f_{N_j}^j(\tilde{L}_j) \) are bounded from above and below by positive constants, again by the inductive hypothesis and Lemma 4.1. For \( |\varphi_j'| \), this follows from Lemma 7.2 applied to a map obtained from \( \tilde{\varphi}_j \) by pre- and post-composition with affine maps.

In particular, \( \text{dist}(g_{j+1}^k(\partial K_{j+1}), g_{N_j}^{j+1}(\partial \tilde{L}_j)) > 0 \) for \( 0 \leq k \leq N_j \), and \( g_{j+1} \) is uniformly continuous at every point of \( g_{N_j}^{j+1}(K_{j+1}) \) for \( 0 \leq k < N_j \). Observe furthermore that \( g_{j+1} \) is uniformly continuous at every point of \( g_{j+1}^k(K_{j+1}) \) for \( 0 \leq k < N_j \) by \( \text{[iii]} \) and also, trivially, at every point of \( g_{N_j}^{j+1}(\partial K_{j+1}) = Q_j \subseteq R_j \).

We now claim that, if \( \varepsilon_{j+1} \leq \varepsilon_j/2 \) is sufficiently small, then any entire function \( f \) with
\[
|f(z) - g_{j+1}(z)| \leq 2\varepsilon_{j+1} \quad \text{on } A_{j+1} \text{ satisfies the following properties.}
\]
(1) \( f_{N_j}^{j+1} \) is injective on \( K_{j+1} \), with \( |f'| \) bounded above and below by positive constants on \( \bigcup_{k=0}^{N_j-1} f^k(K_{j+1}) \), and \( f \) is uniformly continuous at every point of this set;
(2) \( f_{N_j}^{j+1}(\partial K_j) \subseteq D \);
(3) \( f_{N_j}^{j+1}(K_{j+1}) \subseteq V_{j+1}(f) \) and \( \text{dist}(f_{N_j}^{j+1}(K_{j+1}), \partial T_{j+1}) > 0 \);
(4) \( |\text{Re } f_{N_j}^{j+1}(\zeta)| > j \);
Lemma 2.5, which we may apply because of the above facts concerning the uniform continuity of $g_{j+1}$. Property (1) follows from Corollary 2.7 applied with $G = \bigcup_{k=0}^{\infty}(\text{int}(\bar{L}_j))$.

Apply Arakelyan’s theorem to obtain a function $f_{j+1}$, with $|f_{j+1}(z) - g_{j+1}(z)| \leq \varepsilon_{j+1}$ on $A_{j+1}$. Then (i) holds by (1) and (3). Property (ii) holds by definition of $g_{j+1}$. Property (iii) also holds by (1) while (iv) and (v) are immediate from the definitions of $A_{j+1}$ and $\varepsilon_{j+1}$. This concludes the inductive construction.

By (iv) we have $A_{j+1} \supseteq A_j$ and $\bigcup_{j=1}^{\infty} A_j = \mathbb{C}$. By (v) the functions $(f_j)$ form a Cauchy sequence on every $A_j$, and thus converge locally uniformly to an entire function $f$ with $|f(z) - g_j(z)| \leq 2\varepsilon_j$ for all $j \geq 0$ and all $z \in A_j$.

Properties (a) to (d) follow directly from the construction. Indeed, Property (a) holds by definition of $K_n$ and by choice of $\varepsilon_0$ and Lemma 4.1. Claim (e) is a consequence of (2) while (f) follows from (1) and (3).

Finally, property (e) follows from (5) by the statement on real parts in Lemma 4.1. Likewise, (f) follows from (4).

**Remark 7.3.** Every point of $J(f)$ is an accumulation point of unbounded connected components of $I(f)$ [RS05, Theorem 1]. Using the notation of the proof of Theorem 7.1, it follows that $X = \{\zeta\}$ is accumulated on by unbounded components of $I(f)$ lying in strips of the form $\text{int}(K_n) \setminus K_{n+1}$.

**Remark 7.4.** We can modify the proof of Theorem 7.1 by letting the height of the half-strips $(K_n)$ tend to a positive constant rather than to 0, so that $K = \bigcap K_n$ is a closed half-strip. In this way, we obtain a transcendental entire function $f$ for which $U = \text{int}(K)$ is an (unbounded) oscillating wandering domain such that $\partial U \cap I(f) \neq \emptyset$.

The set $\partial U \cap I(f)$ has zero harmonic measure in $\partial U$ by [RS11, Theorem 1.2(a)], and is therefore totally disconnected. Hence $\partial U$ contains a singleton component of $I(f)$.

We now prove Theorem 1.2 which is a more general version of Theorem 7.1.

**Proof of Theorem 1.2.** Let $X \subseteq \mathbb{C}$ be a full plane continuum. As in the proof of Theorem 7.1, we may assume that $X \subseteq T_0$. Let $\zeta \in X$ be a point with maximal real part, and let $K$ be the union of $X$ with the horizontal ray $\zeta + [0, +\infty)$.

We proceed with the recursive construction as before, but must adjust the definition of $L_j$ and $K_{j+1}$. These sets can no longer be chosen to be straight horizontal strips. Instead, each such set will be what we will term a *decorated horizontal half-strip*, that is, a topological half-strip whose intersection with some right half-plane is a straight horizontal half-strip. Let $K_0$ be any decorated horizontal half-strip such that $K_0 \subseteq T_0$ and $K \subseteq \text{int}(K_0)$.

In the recursive step of the construction (when $K_j$ has been defined), we first let $L_j \subseteq \text{int}(K_j)$ be a closed horizontal half-strip with $\zeta + \varepsilon \in \partial L_j$ and $\zeta + (\varepsilon, +\infty) \subseteq \text{int}(L_j)$, where $\varepsilon < 1/t_j$. Exactly as in the proof of Theorem 7.1, we find a conformal map $\varphi_j$ such that $\text{int}(L_j) \to W_j$, where $W_j \subseteq T_{j+1}$ is a horizontal strip such that $\varphi_j(\zeta + \varepsilon) = +\infty$, $\varphi_j(\zeta + \varepsilon) = -\infty$, and $|\text{Re} \varphi_j(\zeta + t)| < 1$ for $1/t_j \leq t \leq t_j$.

Now let $\hat{L}_j \supseteq L_j$ be obtained from $L_j$ by adding a small neighbourhood of $X$ to $L_j$, chosen in such a way that $\hat{L}_j$ is a decorated horizontal half-strip. (That this is possible
follows from Lemma 2.9.) Let \( \hat{\varphi}_j : \text{int}(\hat{L}_j) \to W_j \) be a conformal isomorphism with \( \hat{\varphi}_j(\zeta + 1) = \varphi_j(\zeta + 1) \) and \( \hat{\varphi}_j(+\infty) = +\infty \). We suppose that \( \text{int}(\hat{L}_j) \) is sufficiently close to \( \text{int}(L_j) \) in the sense of Carathéodory kernel convergence (seen from \( \zeta + 1 \)); compare [Pom92] Section 1.4. Then \( \hat{\varphi}_j \) still maps \( \zeta + [1/t_j, t_j] \) to points at real parts less than 1, while

\[
|\text{Re}(f_j^{j+1}|_{V_{j+1}(f_j)})^{-1}(\hat{\varphi}_j(z))| > j \tag{7.3}
\]

for all \( z \in X \).

We now let \( K_{j+1} \subseteq \hat{L}_j \subseteq \hat{L}_j \) be slightly smaller topological half-strips and let \( \hat{\varphi}_j \) be the restriction of \( \hat{\varphi}_j \to \hat{L}_j \). We define \( g_{j+1} \) as in (7.1). Just as in the proof of Theorem 7.1, the derivative \( |\hat{\varphi}_j'| \) is bounded from above and below on \( \hat{L}_j \) by Lemma 7.2. The remainder of the proof proceeds as before, except that (4) is replaced by the condition that \( |\text{Re} f^{N_j+1}(z)| > j \) for all \( z \in X \), which is possible by (7.3).

For the resulting function \( f \), it follows that \( X \subseteq I(f) \), while \( \zeta + (0, \infty) \subseteq BU(f) \). Furthermore, every point of \( K \) is separated from any other point of \( \mathbb{C} \) by some element of \( \partial K_j \). As all points of \( \partial K \) are eventually mapped to \( D \), we see that \( K \) is a connected component of \( I(f) \cup BU(f) \), and \( X \) is a connected component of \( I(f) \), as claimed.

\[\square\]

Remark 7.5. For the function \( f \) constructed in the proof of Theorem 1.2, the set \( f^{-1}(\mathbb{C} \setminus D) \) has infinitely many components (at least one inside each strip \( T_n \)). Hence \( f \) is of infinite order by the Denjoy–Carleman–Ahlfors theorem. On the other hand, the fact that \( |f'| \) is bounded above on an unbounded set for which \( |f| \) is large contradicts the expansivity property of functions in the class \( B \) [EL92 Lemma 1] (see also [Rem22, Theorem 1.1]), so \( f \notin B \).

To conclude the section, we show that – as mentioned in the introduction – the condition that \( K \) is full in Theorem 1.2 is necessary, but the condition that \( K \) is compact is not.

Proposition 7.6. Let \( f \) be a transcendental entire function, and suppose that \( K \subseteq I(f) \) is a compact connected component of \( I(f) \). Then \( K \) is full.

On the other hand, there exists a transcendental entire function for which \( I(f) \) has a bounded connected component that is not compact.

Proof. Suppose that \( K \subseteq I(f) \) is compact and connected, and let \( U \) be a bounded connected component of \( \mathbb{C} \setminus K \). There are two possibilities: either \( U \) is a component of the Fatou set of \( f \), or \( U \) intersects the Julia set. In the first case, \( U \subseteq I(f) \) since all boundary points of \( U \) escape [RST11, Theorem 1.2 (a)]. In the second, \( U \) intersects the fast escaping set \( A(f) \) defined by [3.1] [BH99, Lemma 3]. Every connected component of \( A(f) \) is unbounded [RS05, Theorem 1], and hence intersects \( K \). We conclude that, in either case, \( K \) is not a connected component of \( I(f) \), as claimed.

To prove the second part of the proposition, we can apply the proof of Theorem 1.2 with \( K = \overline{D} \), but modified similarly as in Section 5 so that a countable subset \( Y \subseteq \partial D = \partial K \) belongs to \( BU(f) \). In other words, \( D \subseteq I(f) \), but \( Y \cap I(f) = \emptyset \). So the connected component of \( I(f) \) containing \( D \) is a proper subset of \( \overline{D} \). (We omit the details.)

\[\square\]
8. Maverick Points and Harmonic Measure

In this section, we prove Theorem [1,11] showing that the set of maverick points of a wandering domain has zero harmonic measure with respect to the wandering domain. We begin with the following observation.

Lemma 8.1 (Maverick points and spherical diameter). Let $f$ be a transcendental entire function and suppose that $U$ is a wandering domain of $f$. Let $\zeta \in U$ and $z \in \partial U$. Then $z$ is maverick if and only if $\text{dist}^\#(f^n(\zeta), f^n(z)) \not\to 0$ as $n \to \infty$. In particular, if $\text{diam}^\#(f^n(U)) \to 0$ as $n \to \infty$, then $U$ has no maverick points.

Proof. The first claim is a simple restatement of the definition of a maverick point. Indeed, if $f^n_k(z) \to w \in \hat{\mathbb{C}}$ and $f^{n_k}(\zeta) \not\to w$ as $k \to \infty$, then $\text{dist}^\#(f^{n_k}(z), f^{n_k}(\zeta)) \not\to 0$ as $k \to \infty$. Conversely, if $(n_k)_{k=0}^\infty$ is a subsequence such that $\text{dist}^\#(f^{n_k}(z), f^{n_k}(\zeta)) \geq \varepsilon > 0$ for all $k \in \mathbb{N}$, then we may pass to a further subsequence $(n_{k_j})_{j=0}^\infty$ such that $f^{n_{k_j}}(z)$ and $f^{n_{k_j}}(\zeta)$ diverge to distinct points on the sphere.

The second claim follows immediately from the first.

Our proof of Theorem [1,11] is similar to that of [OS16, Theorem 1.5], but somewhat simpler, despite giving a stronger result. As with [OS16, Theorem 1.5], we use the following lemma, which is [OS16, Lemma 4.1] and is related to [RS11, Theorem 1.1].

Lemma 8.2. Let $(G_k)_{k=0}^\infty$ be a sequence of pairwise disjoint simply connected domains in $\mathbb{C}$. Suppose that, for each $k \in \mathbb{N}$, $g_k : \overline{G_{k-1}} \to \overline{G_k}$ is analytic in $G_{k-1}$, continuous in $\partial G_{k-1}$, and satisfies $g_k(\partial G_{k-1}) \subseteq \partial G_k$. Let

$$h_k = g_k \circ \cdots \circ g_2 \circ g_1$$

for $k \in \mathbb{N}$.

Let $\xi \in \hat{\mathbb{C}}$, $\delta \in (0, 1)$, $c > 1$, and $z_0 \in G_0$. Then

$$H = \{ z \in \partial G_0 : \text{dist}^\#(h_k(z), \xi) \geq c\delta \text{ and } \text{dist}^\#(h_k(z_0), \xi) < \delta \text{ for infinitely many } k \}$$

has harmonic measure zero relative to $G_0$.

Proof of Theorem [7,11]. Let $f$ be a transcendental entire function with a wandering domain $U$. As mentioned in the introduction, if $U$ is multiply connected, then $U \subseteq I(f)$; see [RS05, Theorem 2]. Hence, we assume that $U$ is simply connected and fix some $z_0 \in U$.

For $0 < \delta < 1$ and $\xi \in \hat{\mathbb{C}}$ define $H(\xi, \delta)$ to consist of all points $z \in \partial U$ for which there are infinitely many $n \in \mathbb{N}$ such that $\text{dist}^\#(f^n(z_0), \xi) < \delta$ and $\text{dist}^\#(f^n(z), \xi) > 2\delta$. We claim that the harmonic measure of $H(\xi, \delta)$ relative to $U$ is zero.

Let $(n_k)$ be the sequence of all times for which $\text{dist}^\#(f^{n_k}(z_0), \xi) < \delta$. (If such a sequence does not exist, then $H(\xi, \delta) = \emptyset$ and there is nothing to prove.)

Set $G_0 := U$, $g_k := f^{n_k-n_k-1}$, and let $G_k$ be the Fatou component of $f$ that contains the image $f^{n_k}(U)$. Then, each $G_k$ is a simply connected wandering domain and satisfies $g_k(\partial G_{k-1}) \subseteq \partial G_k$. By Lemma 8.2, the harmonic measure of $H(\xi, \delta)$ relative to $U$ is zero as claimed.

We claim that the union of the sets $H(\xi, \delta)$, where $\delta \in \mathbb{Q}$ and $\xi \in \mathbb{Q}(i)$ are rational, contains all maverick points. By Lemma 8.1, if $z \in \partial U$ is a maverick point, then there exists $\xi' \in \omega(U, f)$ and an increasing sequence $(n_k)_{k=0}^\infty$ such that $f^{n_k}(z_0) \to \xi'$ but
Passing to a further subsequence if necessary, we may assume that there exists $\delta \in \mathbb{Q}$ with $0 < \delta < 1$ such that $\text{dist}^\#(f^k(z_0), \xi') < \delta/2$ and $\text{dist}^\#(f^k(z), \xi') > 3\delta$ for all $k$. Hence, if $\xi \in \mathbb{Q}(i)$ with $\text{dist}^\#(\xi, \xi') < \delta/2$, then $z \in H(\xi, \delta)$. Thus,

$$\{ z \in \partial U : z \text{ is a maverick point} \} \subseteq \bigcup_{\xi \in \mathbb{Q}(i)} \bigcup_{\delta \in \mathbb{Q} \cap (0,1)} H(\xi, \delta).$$

Since $H(\xi, \delta)$ has zero harmonic measure relative to $U$ and by countable additivity of the harmonic measure, the set of maverick points has zero harmonic measure relative to $U$.

**Remark 8.3.** Note that the proof does not require us to distinguish whether $U$ is escaping, oscillating or orbitally bounded. (Recall from the introduction that the latter means that every point in $U$ has bounded orbit, and that it is unknown whether orbitally bounded wandering domains exist.)

### 9. Proof of Proposition 5.6

Proposition 5.6 is a consequence of the following fact.

**Proposition 9.1** (Conformal maps taking boundary points to infinity). Let $\Xi \subseteq \partial \mathbb{D}$ be a compact set of zero logarithmic capacity. Then there exists a simply connected domain $\Omega \subseteq \Sigma := \{ x + iy : x > 0 \text{ and } |y| < \pi/2 \}$ and a conformal isomorphism $\varphi : \mathbb{D} \to \Omega$ such that $\varphi$ extends continuously to $\partial \mathbb{D}$ with $\varphi(\xi) = \infty$ for all $\xi \in \Xi$.

**Proof of Proposition 5.6 using Proposition 9.1.** First observe that we may assume that $V = \Sigma := \{ x + iy : |y| < \pi \}$, that $\Delta$ contains all points of $V$ of modulus greater than some $R > 0$, and that $\nu = 0$. Indeed, otherwise let $\tilde{V} \subseteq V$ be a Jordan domain that contains $\nu$ and whose boundary passes through $\Delta$, and map $\tilde{V}$ conformally onto $\tilde{\Sigma}$, taking $\nu$ to zero and some point of $\partial \tilde{V} \cap \Delta$ to $+\infty$.

Now let $\varphi$ and $\Omega$ be as in Proposition 9.1. For $0 < \lambda < 1$, define

$$\tilde{\varphi} : \mathbb{D} \to \tilde{\Sigma}; \quad z \mapsto \varphi(\lambda z) - \varphi(0).$$

Since $|\text{Im} \varphi(0)| < \pi/2$, this map does indeed take values in $\tilde{\Sigma}$, and clearly $\tilde{\varphi}(0) = 0$. Moreover, if $\lambda$ is chosen sufficiently close to 1, then $\text{Re} \tilde{\varphi}(\xi) > R$ for all $\xi \in \Xi$. Setting $W := \tilde{\varphi}(\mathbb{D})$ for such $\lambda$, the proof is complete.

Proposition 9.1 follows from a powerful result of Bishop about boundary interpolation sets for conformal maps [Bis06].

**Proof of Proposition 9.1 using [Bis06, Theorem 1].** Fix an interval $J \subseteq \partial \mathbb{D}$ that contains $\Xi$. We may choose a homeomorphism $g : \mathbb{D} \to \Sigma$ such that $g$ extends continuously to $\partial \mathbb{D}$ with $g(\xi) = \infty$ for all $\xi \in J$. Of course, $g$ cannot be conformal since it collapses a whole interval of the unit circle to a point. However, by [Bis06, Theorem 1], there is a domain $\Omega \subseteq \Sigma$ and a conformal isomorphism $f : \mathbb{D} \to \Omega$ that extends continuously to $\partial \mathbb{D}$ and $f|_{\Xi} = g|_{\Xi}$.

In fact, Proposition 9.1 is considerably weaker than [Bis06, Theorem 1]. Indeed, it is a preliminary step in the proof of [Bis06, Theorem 6], which in turn is used in the proof of [Bis06, Theorem 1]. For the reader’s convenience, and in order to be able to discuss
how the proof simplifies when $\Xi$ is finite, we sketch a direct proof of Proposition 9.1,
following \[Bis06\].

**Proof of Proposition 9.1.** Let us move from the unit disc $D$ to the upper half-plane $\mathbb{H}^+$. We are given a compact set $E \subseteq \mathbb{R}$ of zero logarithmic capacity, and wish to construct a conformal isomorphism between $\mathbb{H}^+$ and a subset of $\Sigma$ that maps all points of $E$ to $\infty$.

When $E$ is finite, such a conformal isomorphism can be explicitly written down. Let $p_1, \ldots, p_n$ be the elements of $E$, in increasing order. Set $\Sigma' = \{ x + iy : 0 < y < \pi \}$ and define

$$
\psi : \mathbb{H}^+ \to \Sigma' ; \quad z \mapsto \pi i - \frac{1}{n} \sum_{j=1}^{n} \log(z - p_j),
$$

(9.1)

where Log is the principal branch of the logarithm. Since $0 < \arg(z - p_j) < \pi$ for $z \in \mathbb{H}^+$ and all $j$, we see that $\psi$ does indeed take values in $\Sigma'$. Moreover,

$$
\psi'(z) = -\frac{1}{n} \sum_{j=1}^{n} \frac{1}{z - p_j}.
$$

Since $\Im(z - p_j) = \Im z > 0$ for all $z \in \mathbb{H}^+$ and all $j$, it follows that $\Im \psi'(z) > 0$ for all $z \in \mathbb{H}^+$. Hence $\psi$ is a conformal map defined on $\mathbb{H}^+$ (see \[EY12\, Lemma 1\]). Clearly $\Re \psi(z) \to +\infty$ as $z \to p_j$ for any $j$, and $\psi$ extends continuously to $\mathbb{R} \cup \{ \infty \}$ (with the convention that it takes the value $+\infty$ for all $z \in E$, and the value $-\infty$ at $\infty$). We obtain the desired map by postcomposing $\psi$ with a conformal isomorphism from $\Sigma'$ to $\Sigma$ that fixes $+\infty$, and the proof (in the case of finite $E$) is complete.

For the more general case, where $E$ has zero logarithmic capacity, we replace the sum (9.1) by an integral, using a theorem of Evans; see \[GM05\, Theorem E.2\]. This states that there exists a probability measure $\mu$ supported on $E$ such that the potential

$$
u(z) = \int_{\mathbb{R}} -\log|z - x| \, d\mu(x)
$$
tends to infinity at every point of $E$. Now define

$$
\psi : \mathbb{H}^+ \to \Sigma' ; \quad z \mapsto \pi i - \int_{\mathbb{R}} \log(z - x) \, d\mu(x).
$$

(9.2)

Then $u = \Re \psi$, and in particular $\psi(z) \to +\infty$ as $z \to E$. The proof now proceeds exactly as in the case of finite $E$.

**Remark 9.2.** Observe that the formula (9.1) for the finite case is a special case of (9.2). Conversely, the function $\psi$ in (9.2) is the limit of the corresponding functions (9.1) for a suitably chosen sequence of finite subsets of $E$. (See \[GM05\, Proof of Theorem E.2\].) That this can be done in such a way that $\psi(z) \to \infty$ at all points of $E$ follows from (and is indeed equivalent to) the fact that $E$ has logarithmic capacity zero; see \[GM05\, Theorem E.1\].

The formula (9.1) is a Schwarz-Christoffel mapping for a degenerate polygon with $\#E + 1$ zero angles (at the images of $-\infty$ and the points of $E$) and $\#E - 1$ angles of $2\pi$ (at the images of the critical points of $\psi$, of which there is one in every bounded complementary interval of $E$). For further discussion of these maps, and their relationship to real polynomials with only real critical points, see \[EY12\, Section 2\].
Figure 11. The conformal map $\psi : \mathbb{H}^+ \to \Sigma'$ in the proof of Proposition 9.1 with the points $p_j \in E$ such that $\psi(p_j) = \infty$.

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