AN EFFICIENT NUMERICAL METHOD FOR FRACTIONAL MODEL OF ALLELOPATHIC STIMULATORY PHYTOPLANKTON SPECIES WITH MITTAG-LEFFLER LAW

BEHZAD GHANBARI
Department of Engineering Science
Kermanshah University of Technology, Kermanshah, Iran
Department of Mathematics, Faculty of Engineering and Natural Sciences
Bahçeşehir University, 34349 Istanbul, Turkey

DEVENDRA KUMAR
Department of Mathematics
University of Rajasthan, Jaipur-302004, Rajasthan, India

JAGDEV SINGH*
Department of Mathematics
JECRC University, Jaipur-303905, Rajasthan, India

Abstract. The principal aim of the present article is to study a mathematical pattern of interacting phytoplankton species. The considered model involves a fractional derivative which enjoys a nonlocal and nonsingular kernel. We first show that the problem has a solution, then the proof of the uniqueness is included by means of the fixed point theory. The numerical solution of the mathematical model is also obtained by employing an efficient numerical scheme. From numerical simulations, one can see that this is a very efficient method and provides precise and outstanding results.

1. Introduction, background, and preliminaries. The investigation of phytoplankton is a very crucial issue in ecology. These are freely hovering and fragile swimming creatures inside the aquatic environment (AE). These single-celled breeds play a crucial part in the foundation of the food cycle and also the universal carbon cycle. Many aspects manage the constant alteration and immediate variation of phytoplankton density inside the AE. For more information, one can refer to the work of Edvarsen and Paasche [16]. The ecologist has noticed that an extreme enhancement in phytoplankton population, up by many times, which is abruptly resulting into an immediate breakdown. In the past twenty years, an extensive empirical observation has been presented in the direction of damage algal blooms in the latest years [2, 26]. The terminology allelopathy was as stated in a book of Rice [26], is the consequence of a plant kind on the expansion of further plant species by supplying a chemical into the nearby surroundings. These kinds of chemical compounds are termed as allelochemicals [33]. Allelochemicals may have both favorable and adverse consequences on the expansion of other breeds. In particular,
Enteromorpha linza and the green alga yield allelochemicals, which is stimulating the growth of another phytoplankton species (PS) known as enteromorpha species (ES) [36, 11]. An excellent work on the impact of the size of the inoculum on the growth of Chlorella vulgaris was reported by Pratt [25]. One of the possible mathematical modeling for this problem is as follows.

\[ \frac{dx}{dt} = x(t) \left( k_1 - \alpha_1 x(t) - \beta_{12} y(t) \right), \]
\[ \frac{dy}{dt} = y(t) \left( k_2 - \alpha_2 x(t) - \beta_{21} y(t) \right). \]  

In Eq. (1) \( x \) and \( y \) indicates corresponding populations of two considered PS at a time \( t \), and \( k_1, k_2 \) stand for the cell proliferation rates, \( \alpha_1, \alpha_2 \) describe the intrapsychic competition rates, respectively, while \( \beta_{12}, \beta_{21} \) are the constants.

Maynard-Smith [34] modified the mathematical model (1) in the subsequent form

\[ \frac{dx}{dt} = x(t) \left( k_1 - \alpha_1 x(t) - \beta_{12} y(t) - \sigma_1 x(t) y(t) \right), \]
\[ \frac{dy}{dt} = y(t) \left( k_2 - \alpha_2 x(t) - \beta_{21} x(t) - \sigma_2 x(t) y(t) \right). \]  

In Eq. (2) \( \sigma_1, \sigma_2 \) represent the toxic related values for both types.

The fractional extension of the mathematical model of allelopathic stimulatory PS was studied by Abbas et al. [1] by using the Caputo approach. In recent years, the fractional approach has become the essential tools in mathematical modeling of phenomena. Many important applications of mathematical models of fractional order in diverse fields have been discussed [35, 23, 7, 18, 19, 8, 9, 29, 30]. There are many definitions used to introduce fractional-order derivatives [13, 28, 14, 15, 4, 3, 12]. Some of the very common and famous descriptions are given below.

First, let us define the Riemann–Liouville (RL) integral and Caputo derivative of non-integer order \( \rho \), respectively, as follows [28]

\[ C^\rho_0 I^\rho_t g(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t - \omega)^{\rho - 1} g(\omega)d\omega, \quad 0 < \rho \leq 1. \]
\[ C^\rho_0 D^\rho_t g(t) = \frac{1}{\Gamma(k - \rho)} \int_0^t (t - \omega)^{k - \rho - 1} g^{(k)}(\omega)d\omega, \quad k - 1 < \rho \leq k, \quad k \in \mathbb{Z}^+. \]  

Caputo and Fabrizio give another definition for a fractional operator using the concept of exponential kernel and is presented as [15]:

\[ C^{ABC}_0 D^\rho_t g(t) = \frac{M(\rho)}{1 - \rho} \int_0^t \exp\left[\frac{\rho}{1 - \rho}(t - \omega)\right]g'(\omega)d\omega, \quad 0 < \rho \leq 1. \]

Here \( M(\cdot) \) is a weight function with \( M(0) = M(1) = 1 \).

As another important definition, the Atangana-Baleanu integral and derivative in Caputo sense (ABC) of a function \( g(t) \) are as follows [4]

\[ AB^\rho_0 I^\rho_t g(t) = \frac{1 - \rho}{B(\rho)} g(t) + \frac{\rho}{\Gamma(\rho)B(\rho)} \int_0^t \frac{g(\omega)(t - \omega)^{\rho - 1}}{B(\rho)} d\omega, \quad 0 < \rho \leq 1, \]
\[ ABC^\rho_0 D^\rho_t g(t) = \frac{B(\rho)}{1 - \rho} \int_0^t E_\rho\left[\frac{-\rho}{1 - \rho}(t - \omega)\right]g'(\omega)d\omega, \quad 0 < \rho \leq 1. \]
where $E_\rho(t)$ is the well-known Mittag-Leffler (ML) function and is given as:

$$E_\rho(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + 1)}, \quad \rho > 0.$$ 

and $B(\rho)$ is a function which has a value of one at two points $\rho = 0$ and $\rho = 1$. We can see that the ABC derivative is defined based on the ML function. This is the main reason why this definition is capable of modeling many different models in real applications. The ABC operator has been used by many scientists and engineers to portray memory effects in the mathematical formulation of physical, chemical, and engineering processes [5, 22, 31, 20, 6, 21, 32, 17, 24, 27, 10]. Given great usability and efficiency of ABC fractional operator in scientific and technological fields, we are motivated to apply this new approach of fractional calculus to investigate the dynamic behavior of allelopathic stimulatory PS.

The main structure of this article is given as: In Section 2, fractional model of the problem is discussed. In Section 3, existence and uniqueness analysis of the solution is demonstrated. In Section 4, expression of numerical solution method is investigated. In section 5, numerical experiments are shown. Lastly in Section 6, the concluding remarks are presented.

2. Fractional model of the problem. Here, we study the dynamic behavior of allelopathic stimulatory PS by using the ABC fractional derivative as

$$\begin{align*}
\ABC D^\rho_0 x(t) &= x(t) (k_1 - \alpha_1 x(t) - \beta_{12} y(t)), \\
\ABC D^\rho_0 y(t) &= y(t) (k_2 - \alpha_2 y(t) - \beta_{21} x(t) + \gamma x(t) y(t)).
\end{align*}$$

(6)

In Eq. (6), $x$ and $y$ indicate the 1st and 2nd density of phytoplankton species population at a time $t$, respectively, $k_1$, $k_2$ stand for the cell proliferation rates per unit time of the 1st and 2nd PS, $\alpha_1$, $\alpha_2$ represent the rates of intra-specific competition of the 1st and 2nd PS. While $\beta_{12}$, $\beta_{21}$ are the constants and represent the rates of inter specific competition among the 1st and 2nd PS and vice-versa and $\gamma$ stands for the rate of allelopathic substance supplied by the 1st PS which invigorate the enhancement of 2nd PS.

3. Existence and uniqueness analysis. Here, we will study the existence and uniqueness of the solutions for the considered mathematical model of allelopathic stimulatory PS with the aid of fixed-point theory. Utilizing the AB operator presented in (5) on both sides of (6), one gets

$$\begin{align*}
x(t) &= x(0) + \frac{(1 - \rho)}{B(\rho)} \{ x(t) (k_1 - \alpha_1 x(t) - \beta_{12} y(t)) \}, \\
&\quad + \frac{\rho}{B(\rho) \Gamma(\rho)} \int_0^t (t - s)^{\rho - 1} \{ x(s) (k_1 - \alpha_1 x(s) - \beta_{12} y(s)) \} ds, \\
y(t) &= y(0) + \frac{(1 - \rho)}{B(\rho)} \{ y(t) (k_2 - \alpha_2 y(t) - \beta_{21} x(t) + \gamma x(t) y(t)) \} \\
&\quad + \frac{\rho}{B(\rho) \Gamma(\rho)} \int_0^t (t - s)^{\rho - 1} \{ y(s) (k_2 - \alpha_2 y(s) - \beta_{21} x(s) + \gamma x(s) y(s)) \} ds.
\end{align*}$$

(7)

To simplify the Eq. (7), we set

$$\begin{align*}
\Lambda_1(t, x) &= x(t) (k_1 - \alpha_1 x(t) - \beta_{12} y(t)), \\
\Lambda_2(t, y) &= y(t) (k_2 - \alpha_2 y(t) - \beta_{21} x(t) + \gamma x(t) y(t)).
\end{align*}$$

(8)
Firstly, we demonstrate that the Lipschitz condition holds for the kernels \( \Lambda_1(t,x) \) and \( \Lambda_2(t,y) \).

**Theorem 3.1.** The kernels \( \Lambda_1 \) and \( \Lambda_2 \) hold the Lipschitz condition and also contractions if \( 0 \leq \vartheta_1 < 1 \) and \( 0 \leq \vartheta_2 < 1 \), where \( \vartheta_1 = (k_1 + \alpha_1(\eta + \eta_1) + \beta_12\lambda) \) and \( \vartheta_2 = (k_2 + (\alpha_2 + \gamma\eta)(\lambda + \lambda_1) + \beta_21\eta) \).

**Proof.** Let us start to show that the kernel \( \Lambda_1 \) satisfy the Lipschitz condition. In order to demonstrate it we assume that \( x \) and \( x^* \) are two bounded functions i.e. \( \|x\| \leq \eta \) and \( \|x^*\| \leq \eta_1 \).

It can be easily achieved the below result

\[
\|\Lambda_1(t,x) - \Lambda_1(t,x^*)\| \\
= \| \{ x(t) (k_1 - \alpha_1 x(t) - \beta_12y(t)) \} - \{ x^*(t) (k_1 - \alpha_1 x^*(t) - \beta_12y(t)) \} \| \\
= \| k_1 (x(t) - x^*(t)) - \alpha_1 (x_2(t) - x_2^*(t)) - \beta_12y(t) (x(t) - x^*(t)) \| \\
= \| k_1 (x(t) - x^*(t)) - \alpha_1 (x(t) - x^*(t)) (x(t) + x^*(t)) - \beta_12y(t) (x(t) - x^*(t)) \|. 
\]

(9)

On employing the property of norm in Eq. (9), we achieve the subsequent result

\[
\|\Lambda_1(t,x) - \Lambda_1(t,x^*)\| \\
\leq k_1 \|x(t) - x^*(t)\| + \alpha_1 \|x(t) - x^*(t)\| (x(t) + x^*(t)) \| + \beta_12 \|y(t)(x(t) - x^*(t))\| \\
\leq k_1 \|x(t) - x^*(t)\| + \alpha_1(\eta + \eta_1) \|x(t) - x^*(t)\| + \beta_12 \lambda \|x(t) - x^*(t)\| \\
\leq (k_1 + \alpha_1(\eta + \eta_1) + \beta_12\lambda) \|x(t) - x^*(t)\|. 
\]

(10)

Taking \( (k_1 + \alpha_1(\eta + \eta_1) + \beta_12\lambda) = \vartheta_1 \), we have

\[
\|\Lambda_1(t,x) - \Lambda_1(t,x^*)\| \leq \vartheta_1 \|x(t) - x^*(t)\|. 
\]

(11)

Thus, the Leibniz condition for \( \Lambda_1 \) holds. Also if \( 0 \leq \vartheta_1 < 1 \), then it is contraction. Similarly the below result can be achieved

\[
\|\Lambda_2(t,y) - \Lambda_2(t,y^*)\| \leq \vartheta_2 \|y(t) - y^*(t)\|. 
\]

(12)

It shows that \( \Lambda_2 \) holds the Lipschitz condition, moreover if \( 0 \leq \vartheta_2 < 1 \), then it also contraction.

On using the notions presented in Eq. (8), the Eq. (7) is written as

\[
\begin{align*}
x(t) &= x(0) + \frac{(1 - \rho)}{B(\rho)} \Lambda_1(t,x) + \frac{\rho}{B(\rho) \Gamma(\rho)} \int_0^t (t - s)^{\rho - 1} \Lambda_1(s,x)ds, \\
y(t) &= y(0) + \frac{(1 - \rho)}{B(\rho)} \Lambda_2(t,y) + \frac{\rho}{B(\rho) \Gamma(\rho)} \int_0^t (t - s)^{\rho - 1} \Lambda_2(s,y)ds.
\end{align*}
\]

(13)

If we take \( \frac{(1 - \rho)}{B(\rho)} = \phi(\rho) \) and \( \frac{\rho}{B(\rho) \Gamma(\rho)} = \psi(\rho) \), then Eq. (13) takes the subsequent from

\[
\begin{align*}
x(t) &= x(0) + \phi(\rho) \Lambda_1(t,x) + \psi(\rho) \int_0^t (t - s)^{\rho - 1} \Lambda_1(s,x)ds, \\
y(t) &= y(0) + \phi(\rho) \Lambda_2(t,y) + \psi(\rho) \int_0^t (t - s)^{\rho - 1} \Lambda_2(s,y)ds.
\end{align*}
\]

(14)
The recursive formulae are presented as
\[
\begin{align*}
  x_n(t) &= \phi(\rho)\Lambda_1(t,x_{n-1}) + \psi(\rho) \int_0^t (t-s)^{\rho-1}\Lambda_1(s,x_{n-1})ds, \\
  y_n(t) &= \phi(\rho)\Lambda_2(t,y_{n-1}) + \psi(\rho) \int_0^t (t-s)^{\rho-1}\Lambda_2(s,y_{n-1})ds.
\end{align*}
\] (15)

The initial conditions are presented below
\[
\begin{align*}
  x_0(t) &= x(0), \\
  y_0(t) &= y(0).
\end{align*}
\] (16)

Next, we present the difference formula as
\[
\begin{align*}
  F_n(t) &= x_n(t) - x_{n-1}(t) = \phi(\rho)\left(\Lambda_1(t,x_{n-1}) - \Lambda_1(t,x_{n-2})\right) \\
  &\quad + \psi(\rho) \int_0^t (t-s)^{\rho-1}\left(\Lambda_1(s,x_{n-1}) - \Lambda_1(s,x_{n-2})\right)ds, \\
  G_n(t) &= y_n(t) - y_{n-1}(t) = \phi(\rho)\left(\Lambda_2(t,y_{n-1}) - \Lambda_2(t,y_{n-2})\right) \\
  &\quad + \psi(\rho) \int_0^t (t-s)^{\rho-1}\left(\Lambda_2(s,y_{n-1}) - \Lambda_2(s,y_{n-2})\right)ds.
\end{align*}
\] (17)

It can be observed that
\[
\begin{align*}
  x_n(t) &= \sum_{i=0}^{n} F_i(t), \\
  y_n(t) &= \sum_{i=0}^{n} G_i(t).
\end{align*}
\] (18)

The following assumptions are used
\[
\begin{align*}
  x_{-1}(0) &= 0, y_{-1}(0) = 0.
\end{align*}
\] (19)

The following results can be easily derived
\[
\begin{align*}
  \|F_n(t)\| &\leq \phi(\rho)\vartheta_1 \|F_{n-1}(t)\| + \psi(\rho)\vartheta_1 \int_0^t \|F_{n-1}(s)\| (t-s)^{\rho-1}ds, \\
  \|G_n(t)\| &\leq \phi(\rho)\vartheta_2 \|G_{n-1}(t)\| + \psi(\rho)\vartheta_2 \int_0^t \|G_{n-1}(s)\| (t-s)^{\rho-1}ds.
\end{align*}
\] (20)

Now, with the aid of Eq. (20), we demonstrate the existence of the solution of the fractional model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law.

**Theorem 3.2.** The solution of the mathematical model of allelopathic stimulatory phytoplankton species with fractional derivative given in (6) exists if for \(t_0\), we have
\[
\phi(\rho)\vartheta_1 + \frac{\psi(\rho)\vartheta_1 t_0^\rho}{\rho} < 1.
\] (21)

**Proof.** We assume that \(x(t)\) and \(y(t)\) are bounded functions. In view of Eq. (20) and using recursive method, we can obtain the following results
\[
\begin{align*}
  \|F_n(t)\| &\leq \|x(0)\| \left[\phi(\rho)\vartheta_1 + \frac{\psi(\rho)\vartheta_1 t_0^\rho}{\rho}\right]^n, \\
  \|G_n(t)\| &\leq \|y(0)\| \left[\phi(\rho)\vartheta_2 + \frac{\psi(\rho)\vartheta_2 t_0^\rho}{\rho}\right]^n.
\end{align*}
\] (22)
Thus, both of the given functions in Eq. (17) exist and smooth. Next to verify that the expression expressed in the form of Eq. (15) is the solution of the fractional model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law, we let

\[
\begin{align*}
x(t) - x(0) &= x_n(t) - \varepsilon_n(t), \\
y(t) - y(0) &= y_n(t) - \mu_n(t).
\end{align*}
\]

(23)

So, one gets

\[
\|\varepsilon_n(t)\| \leq \left( \phi(\rho) + \frac{\psi(\rho)t^\rho}{\rho} \right)^{n+1} \vartheta_1^{n+1} \eta. 
\]

(24)

Now \( t = t_0 \), we have

\[
\|\varepsilon_n(t)\| \leq \left( \phi(\rho) + \frac{\psi(\rho)t_0^\rho}{\rho} \right)^{n+1} \vartheta_1^{n+1} \eta. 
\]

(25)

On taking the limit \( n \to \infty \), Eq. (25) yields

\[
\|\varepsilon_n(t)\| \to 0. 
\]

(26)

On following the same kind of process, we see that

\[
\|\mu_n(t)\| \to 0. 
\]

(27)

It reveals that the studied fractional model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law given in (6) has a solution. Next, we demonstrate that the solution of the fractional model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law is surely unique.

**Theorem 3.3.** The mathematical model of allelopathic stimulatory PS of fractional order presented in Eq. (6) have a unique solution if

\[
\left( 1 - \phi(\rho)\partial_1 - \frac{\psi(\rho)t^\rho}{\rho} \right) > 0.
\]

(28)

**Proof.** To investigate the uniqueness of the solution of fractional model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law, we assume that there exists another solution \( x^*(t) \) and \( y^*(t) \) of Eq. (6).

We can easily see that

\[
x(t) - x^*(t) = \phi(\rho) (\Lambda_1(t, x) - \Lambda_1(t, x^*)) + \psi(\rho) \int_0^t (\Lambda_1(s, x) - \Lambda_1(s, x^*)) (t-s)^{\rho-1} ds.
\]

(29)

On utilizing of property of norm in Eq. (29), we have

\[
\|x(t) - x^*(t)\| \left( 1 - \phi(\rho)\partial_1 - \frac{\psi(\rho)t^\rho}{\rho} \right) \leq 0.
\]

(30)

If the result given in Eq. (28) holds, then from Eq. (30) we have

\[
\|x(t) - x^*(t)\| = 0.
\]

(31)

From the Eq. (31), we have

\[
x(t) = x^*(t).
\]

(32)

On utilizing the similar methodology, we have

\[
y(t) = y^*(t).
\]

(33)

Therefore, the mathematical model of allelopathic stimulatory PS of arbitrary order presented in Eq. (6) has a unique solution. \( \square \)
4. Expression of numerical solution method. In this part, we will discuss how to use the numerical procedure to obtain approximate solution of the problem. To this end, let us take into account the below equation

$$A_{ABC} \int_0^t \mathcal{Y}(t) = \mathcal{S}(t, \mathcal{Y}(t)).$$

(34)

By using the integral operator one can write

$$\mathcal{Y}(t) - \mathcal{Y}(0) = \frac{1 - \rho}{B(\rho)} \mathcal{S}(t, \mathcal{Y}(t)) + \frac{\rho}{\Gamma(\rho) B(\rho)} \int_0^t \mathcal{S}(\omega, \mathcal{Y}(\omega))(t - \omega)^{\rho - 1} d\omega.$$  

(35)

Taking $t = t_n = n \Delta t$ in (35), one achieves

$$\mathcal{Y}(t_n) = \mathcal{Y}(0) + \frac{1 - \rho}{B(\rho)} \mathcal{S}(t_n, \mathcal{Y}(t_n)) + \frac{\rho}{\Gamma(\rho) B(\rho)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathcal{S}(\omega, \mathcal{Y}(\omega))(t_n - \omega)^{\rho - 1} d\omega.$$  

(36)

Now, with the help of linear interpolation of $\mathcal{S}(t, \mathcal{Y}(t))$, one gets

$$\mathcal{S}(t, \mathcal{Y}(t)) \approx \mathcal{S}(t_{i+1}, \mathcal{Y}_{i+1}) + \frac{t - t_{i+1}}{\Delta t} (\mathcal{S}(t_{i+1}, \mathcal{Y}_{i+1}) - \mathcal{S}(t_{i}, \mathcal{Y}_{i})), \quad t \in [t_{i}, t_{i+1}],$$  

(37)

where the notation of $\mathcal{Y}_{i} = \mathcal{Y}(t_{i})$ is used.

Substituting (37) in (36), the approximate solution of the problem will obtain as [17]

$$\mathcal{Y}_n = \mathcal{Y}_0 + \frac{\rho \Delta t^\rho}{B(\rho)} \left( \eta_n \mathcal{S}(t_0, \mathcal{Y}_0) + \sum_{i=1}^{n} \theta_{n-i} \mathcal{S}(t_i, \mathcal{Y}_i) \right),$$  

(38)

where

$$\eta_n = \frac{(n - 1)^{\rho + 1} - n^\rho (n - \rho - 1)}{\Gamma(\rho + 2)},$$

$$\theta_j = \begin{cases} \frac{1}{\Gamma(\rho + 2)} + \frac{1 - \rho}{B(\rho)}, & j = 0 \\ \frac{(j-1)^{\rho + 1} - (j+1)^{\rho + 1}}{\Gamma(\rho + 2)}, & j = 1, 2, \ldots, n - 1 \end{cases}$$  

(39)

Using the numerical scheme (38) presented above, the approximate solution of the problem (6) will be achieved recursively as

$$x_n = x_0 + \frac{\rho \Delta t^\rho}{B(\rho)} \left( \eta_n x_0 (k_1 - \alpha_1 x_0 - \beta_2 y_0) + \sum_{i=1}^{n} \theta_{n-i} x_i (k_1 - \alpha_1 x_i - \beta_2 y_i) \right),$$

$$y_n = y_0 + \frac{\rho \Delta t^\rho}{B(\rho)} \left( \eta_n y_0 (k_2 - \alpha_2 y_0 - \beta_2 x_0 + \gamma x_0 y_0) + \sum_{i=1}^{n} \theta_{n-i} y_i (k_2 - \alpha_2 y_i - \beta_2 x_i + \gamma x_i y_i) \right).$$  

(40)

Approximate solutions to the problem can be determined using the numerical method described above.

5. Numerical experiments. In this portion, we will apply the outlined approximate method in (40) to obtain numerical simulations for solving the fractional-order model of allelopathic stimulatory PS (6).
In the numerical experiments we have used the values for the parameters as belloows

\[ k_1 = 2.00, \alpha_1 = 0.07, \alpha_2 = 0.08, \beta_{12} = 0.05, k_2 = 1.00, \beta_{21} = 0.015, \]

and using \((x_0, y_0) = (3, 30)\) the time step size of \(h = 1.0 \times 10^{-3}\).

In what follows we will present some graphs that describe the nature of the solutions for distinct values of some parameters. Numerical solutions to the problem are shown in the Figures 1-6 for different values of \(\gamma\) and \(\rho\). One can observer that the numerical outcomes are fully consistent with the theoretical results stated in the preceding sections.

**Figure 1:** Influence of \(\rho\) on response behavior of solutions when \(\gamma = 0.001\).

**Figure 2:** Influence of \(\rho\) on response behavior of solutions when \(\gamma = 0.002\).

**Figure 3:** Influence of \(\rho\) on response behavior of solutions when \(\gamma = 0.003\).
6. **Conclusions.** In this work, we have studied a novel fractional-order mathematical model of interacting PS. The model has been developed by involving a new nonsingular and nonlocal fractional derivative with the ML kernel. Mathematical proof of the existence and uniqueness of solution of fractional order model of allelopathic stimulatory phytoplankton species with Mittag-Leffler law is demonstrated by employing the fixed-point theory into account. A numerical approximation is also utilized to solve the model. We see that the numerical outcomes are consistent with the assumed theoretical results. The employed numerical approach in the present manuscript can be adopted to examine other problems in the field.

**REFERENCES**

[1] S. Abbas, L. Mahto, A. Favini and M. Hafayed, *Dynamical study of fractional model of allelopathic stimulatory phytoplankton species*, *Differ. Equ. Dyn. Syst.*, 24 (2016), 267–280.
[2] D. M. Anderson, Toxic algae blooms and red tides: A global perspective, In: Okaichi, T., Anderson, D.M., Nemoto, T. (eds.) Red Tides: Biology, Environmental Science and Toxicology, Elsevier, New York (1989) 11–21.

[3] C. Arora, V. Kumar and S. Kant, Dynamics of a high-dimensional stage-structured prey predator model, Int. J. Appl. Comput. Math., 3 (2017), 427–445.

[4] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model, Therm. Sci., 20 (2016), 763–769.

[5] A. Atangana and R. T. Alqahtani, New numerical method and application to Keller-Segel model with fractional order derivative, Chaos, Solitons & Fractals, 116 (2018), 14–21.

[6] A. Atangana and I. Koca, Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order, Chaos, Solitons & Fractals, 89 (2016), 447–454.

[7] D. Baleanu, A. Jajarmi, S. S. Sajjadi and D. Mozyrska, A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator, Chaos, 29 (2019), 083127.

[8] D. Baleanu, B. Shiri, H. M. Srivastava and M. Al Qurashi, A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel, Advances in Difference Equations, 2018 (2018), 353.

[9] D. Baleanu and B. Shiri, Collocation methods for fractional differential equations involving non-singular kernel, Chaos, Solitons & Fractals, 116 (2018), 136–145.

[10] R. G. Batogna and A. Atangana, Generalised class of time fractional Black Scholes equation and numerical analysis, Discrete & Continuous Dynamical Systems - Series S, 12 (2019), 435–445.

[11] H. Berglund, Stimulation of growth of two marine green algae by organic substances excreted by enteromorphilina in unialgal and axenic cultures, Physiol. Plant, 22 (2006), 1069–1073.

[12] R. Garrappa, Numerical solution of fractional differential equations: A survey and a software tutorial, Mathematics, 6 (2018), 16.

[13] R. Caponetto, G. Dongola, L. Fortuna and I. Petráš, Fractional Order Systems Modeling and Control Applications, World Scientific Series on Nonlinear Science Series A, (2010).

[14] M. Caputo, Elasticita e Dissipazione, Zani-Chelli, Bologna, 1969.

[15] M. Caputo and M. Fabrizio, A new Definition of Fractional Derivative without Singular Kernel, Progr. Frac. Differ. Appl., (2015) 73–85.

[16] B. Edvarsen and E. Pausche, Bloom dynamics and physiology of Primesium and Chrysochro- mulina, Physiological Ecology of Harmful Algal Bloom, Springer, Berlin (1998).

[17] B. Ghanbari and D. Kumar, Numerical solution of predator-prey model with Beddington-DeAngelis functional response and fractional derivatives with Mittag-Leffler kernel, Chaos, 29 (2019), 063103.

[18] A. Jajarmi, S. Arshad and D. Baleanu, A new fractional modelling and control strategy for the outbreak of dengue fever, Physica A, 535 (2019) 122524.

[19] A. Jajarmi, D. Baleanu, S. S. Sajjadi and J. H. Asad, A new feature of the fractional Euler-Lagrange equations for a coupled oscillator using a nonsingular operator approach, Frontiers in Physics, 7 (2019), 196.

[20] D. Kumar, J. Singh, D. Baleanu and Sushila, Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel, Physica A, 492 (2018) 155–167.

[21] D. Kumar, J. Singh and D. Baleanu, A new analysis of Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler type kernel, European Journal of Physical Plus, 133 (2018), 70.

[22] D. Kumar, J. Singh, K. Tanwar and D. Baleanu, A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler Laws, International Journal of Heat and Mass Transfer, 138 (2019), 1222–1227.

[23] K. M Owolabi and Z. Hammouch, Spatiotemporal patterns in the Belousov–Zhabotinskii reaction systems with Atangana–Baleanu fractional order derivative, Physica A, 523 (2019), 1072–1090.

[24] K. M. Owolabi, Numerical patterns in reaction–diffusion system with the Caputo and Atangana–Baleanu fractional derivatives, Chaos, Solitons & Fractals, 115 (2018), 160–169.

[25] R. Pratt, Influence of the size of the inoculum on the growth of Chlorella vulgaris in freshly prepared culture medium, Am. J. Bot., 27 (1940), 52–67.

[26] E. Rice, Allelopathy, Academic Press, New York, 1984.
[27] K. M Saad, M. M. Khader, J. F. Gómez-Aguilar and D. Baleanu, Numerical solutions of the fractional Fisher’s type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, Chaos, 29 (2019), 023116.

[28] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.

[29] B. Shiri and D. Baleanu, System of fractional differential algebraic equations with applications, Chaos, Solitons & Fractals, 120 (2019), 203–212.

[30] B. Shiri and D. Baleanu, Numerical solution of some fractional dynamical systems in medicine involving non-singular kernel with vector order, Results in Nonlinear Analysis, 2 (2019), 160–168.

[31] J. Singh, D. Kumar and D. Baleanu, New aspects of fractional Biswas-Milovic model with Mittag-Leffler law, Mathematical Modelling of Natural Phenomena, 14 (2019), 303.

[32] J. Singh, A new analysis for fractional rumor spreading dynamical model in a social network with Mittag-Leffler law, Chaos, 29 (2019), 013137.

[33] T. Smayda, Novel and nuisance phytoplankton blooms in the sea: evidence for a global epidemic. In: Granéli, E., Sundström, B., Edler, L., Anderson, D.M. (eds.) Toxic Marine Phytoplankton, Elsevier, New York (1990), 29–40.

[34] J. M. Smith, Mathematical Models in Biology, Cambridge University Press, Cambridge, 1968.

[35] S. Uçar, E. Uçar, N. Özdemir and Z. Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana–Baleanu derivative, Chaos, Solitons & Fractals, 118 (2019), 300–306.

[36] R. H. Whittaker and P. P. Feeny, Allelochemics: chemical interactions between species, Science, 171 (1971), 757–770.