Coherent States for Transparent Potentials

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Darboux transformation operators that produce multisoliton potentials are analyzed as operators acting in a Hilbert space. Isometric correspondence between Hilbert spaces of states of a free particle and a particle moving in a soliton potential is established. It is shown that the Darboux transformation operator is unbounded but closed and can not realize an isometric mapping between Hilbert spaces. A quasispectral representation of such an operator in terms of continuum bases is obtained. Different types of coherent states of a multisoliton potential are introduced. Measures that realize the resolution of the identity operator in terms of the projectors on the coherent states vectors are calculated. It is shown that when these states are related with free particle coherent states by a bounded symmetry operator the measure is defined by ordinary functions and in the case of a semibounded symmetry operator the measure is defined by a generalized function.

I. INTRODUCTION

The concept of coherent states (CS) is widely used in different fields of physics and mathematics (see for example Refs. [1] - [3]). In particularly, it plays an important role in the Berezin quantization scheme [4], in the analysis of growth of functions holomorphic in a complex domain [5], in a general theory of phase space quasiprobability distributions [6], and in a quantum state engineering [7]. It is necessary to note that in present no a unified definition of such states exists in the literature and different authors mean different things when speaking about them. Nevertheless, a careful analysis (see for example Ref. [8]) shows that almost all definitions have some common points that can be taken as a general definition. Following Klauder [8] I mean by coherent states such states that satisfy the following defining properties: (1) CS are defined by vectors $\psi_z(x,t)$ which belong to a Hilbert space $H$ of the states of a quantum system with scalar product $\langle \cdot | \cdot \rangle$; (2) The parameter $z$ takes continuous values from a domain $D$ of an $n$-dimensional complex space; (3) There exists a measure $\mu = \mu(z,\bar{z})$ (the bar over a symbol indicates complex conjugation) that realizes the resolution of the identity operator $\mathbb{I}$ acting in $H$ in terms of the projectors on the vectors $\psi_z$

$$\int_D d\mu(\psi_z)\langle \psi_z | = \mathbb{I}; \quad (1)$$

(4) CS have to prove the property of temporal stability. By temporal stability I mean that the vectors $\psi_z(x,t)$ remain coherent at all times i.e. satisfy the properties 1-3. at all times. To satisfy this condition I shall assume that the functions $\psi_z(x,t)$ are solutions of the Schrödinger equation

$$(i\partial_t - h_0)\psi_z(x,t) = 0$$

where $h_0$ is the Hamiltonian of a given quantum system which in general can depend on time. Operator $h_0$ is supposed to be Hermitian in $H$ and to have a unique self-adjoint extension. The Eq. (1) should be understood in a weak sense. This means that it is equivalent to the following relation

$$\int_D d\mu(\psi_a|\psi_z)\langle \psi_z |\psi_b = \langle \psi_a |\psi_b)$$

which should hold for all $\psi_{a,b}$ from a dense set in $H$.

Transparent potentials have many remarkable properties. For instance, a quantum particle prove no reflection in the scattering process on such a potential. Another remarkable property is that each level in the discrete spectrum of such a potential occupies a preassigned position, which is controlled by values of the parameters the potential depend on. The discrete spectrum levels may even be situated in the middle of the continuous spectrum. In the latter case we have completely transparent potentials [9]. Transparent potentials find a more significant application in soliton theory. There is a marvelous vast literature on this subject. I cite here only a monograph [10]. Because of their remarkable properties transparent potentials would find an application in pseudopotential theories. Recently they have been used to describe relaxation processes in Fermi liquid [11].

CS for transparent potentials are very far from being explored. It may be explained by the fact that up to now no systematic way is known for their investigation. No a clear algebraic structure related to these potentials is known and therefore well known algebraic methods prove to be a little suitable in this context. No simple ladder operators for the discrete spectrum eigenfunctions of transparent potentials are known as well and therefore we can not use the approach of Ref. [8] for this purpose. An approach based on the uncertainty relation [12] is not consistent with the property 3. mentioned above and therefore it should be rejected.

A conjecture has been advanced recently [13,14] to use Darboux transformation operator approach for investigating the CS of those system that is related by Darboux transformation with that for which the CS are known.
Let us have an exactly soluble Hamiltonian $h_0 = -\partial_x^2 + V_0(x,t)$ for which the CS $\psi_L(x,t)$ are known and we want to obtain the CS for another Hamiltonian $h_1 = -\partial_x^2 + V_1(x,t)$ related with $h_0$ by the Darboux transformation operator that I shall denote by $L$. In general it should be a nonstationary Darboux transformation operator defined by the following intertwining relation \[ L(i\partial_t - h_0) = (i\partial_t - h_1)L. \]

If such an operator $L$ is known then solutions of the transformed Schrödinger equation (i.e. the Schrödinger equation with the Hamiltonian $h_1$) can easily be obtained by the action of the operator $L$ on solutions of the initial Schrödinger equation (i.e. the Schrödinger equation with the Hamiltonian $h_0$). It is clear that the functions $\varphi_z(x,t) = L\psi_z(x,t)$ will satisfy all the properties of the CS enumerated above except may be for the property 3. One of the main goal of this paper is to prove that in the case of soliton potentials this property is fulfilled. I would like to mention that this approach has been successfully applied to study CS of anharmonic oscillator Hamiltonians with equidistant and quasi-equidistant spectra [3] and CS of nonstationary soliton potentials [6] that are related with soliton solutions of the Kadomtsev-Petviashvili equation. With the help of this approach a classical counterpart of the Darboux transformation has been formulated and shown that at classical level this transformation leads to a distortion of a phase space [7]. CS of a one-soliton potential have been investigated as well and their supercoherent structure has been revealed [14]. In this paper I generalize these results to a multisoliton case.

The paper is organized as follows. In the Section 2 I give well-known results for CS of the free particle in preparation for their application in the following sections. In the Section 3 the Darboux transformation operator from the solutions of the Schrödinger equation with zero potential to the solutions of the same equation with solitons potential is analyzed as an operator acting in the Hilbert space of the states of a free particle. It is shown that it can not realize a mapping of Hilbert spaces since it is not defined in the whole Hilbert space and can not be extended to the whole Hilbert space. Isomeric operators expressed in terms of continuous bases similar to these previously proposed by L.D. Faddeev [13] and analyzed by D.L. Pursey [12] for the case of purely discrete basis sets are introduced. These operators realize a polar decomposition of Darboux transformation operators. A quasizpectral representation of the Darboux transformation operator and its inverse in terms of continuous bases are obtained. In the Section 4. different systems of CS are introduced for soliton potentials. It is established that the resolution of the identity operator exists in every case. Explicit expressions for measures that realize this equality are found. A brief conclusion brings a paper to a close.

II. FREE PARTICLE COHERENT STATES

In this section I give a brief overview of well-known properties of the Hilbert space of states a free particle (see Ref. [21] and references therein) and corresponding CS [8] we need for subsequent analysis.

Annihilation $a$ and creation $a^+$ operators

$$a = (i-t)\partial_x + ix/2, \quad a^+ = (i+t)\partial_x - ix/2$$

form the Heisenberg-Weil subalgebra of the six-dimensional Schrödinger algebra which is a symmetry algebra of the equation with zero potential. Solutions of the free particle Schrödinger equation which are square integrable over full real axis $\mathbb{R} = (-\infty, +\infty)$ with respect to the Lebesgue measure are the eigenstates of the symmetry operator $K_0 = aa^+ + a^+a$, $K_0\psi_n(x,t) = (2n+1)\psi_n(x,t)$. Their coordinate representation is as follows

$$\psi_n(x,t) = (-i)^n(n!)^{1/2}\sqrt{2\pi}^{-1/2}(1+it)^{-1/2} \times \exp[-in \arctan t + \frac{\pi}{2}(it - 1)] \ H_n(y),$$

$$y = \frac{x}{\sqrt{2 + t^2}}.$$

Operators $a$ and $a^+$ are the ladder operators for the basis functions $\psi_n$: $a\psi_n = \sqrt{n}\psi_{n-1}$, $a^+\psi_n = \sqrt{n+1}\psi_{n+1}$, and $a\psi_0 = 0$.

By $L_0$ I denote the lineal of the functions $\psi_n$, $n = 0, 1, \ldots$ which is the space of all finite linear combinations of the functions $\psi_n$ with the coefficients from the field $\mathbb{C}$. The operators $a$ and $a^+$ being linear are defined for all elements from $L_0$ and $L_0$ is invariant with respect to the action of these operators. Since the momentum operator $p_x = -i\partial_x$ and the initial Hamiltonian $h_0$ are expressed in terms of $a$ and $a^+$: $p_x = -(a + a^+)/2$, $h_0 = p_x^2$, these operators are defined in $L_0$ and map this space into itself.

The Hilbert space of the states of the free particle, $H$, is defined as a closure of the lineal $L_0$ with respect to the measure generated by the scalar product $\langle \psi_a | \psi_b \rangle$, $\psi_{a,b} \in L_0$, which is defined by the Lebesgue integral. The functions $\psi_n$ form an orthonormal basis in $H$, $\langle \psi_n | \psi_k \rangle = \delta_{nk}$. It is well-known [21][22] that the operators $p_x$ and $h_0$ initially defined on $L_0$ have unique self-adjoint extensions and consequently they are essentially self-adjoint in $H$.

The spectrum of $h_0$ and $p_x$ is purely continuous. They have common eigenfunctions $\psi_p = \psi_p(x,t)$: $p_x\psi_p = p\psi_p$, $h_0\psi_p = p^2\psi_p$, $p \in \mathbb{R}$, which do not belong to $H$ but belong to a more wide space $H_\infty$ of the linear functionals over $H_\infty$, $H_\infty \subset H \subset H_\infty$ (so called Gelfand triplet). We can choose the Hilbert-Schmidt equipment of the space $H$ by letting $H_\infty = K_0^{-1}H$ since $K_0^{-1}$ is a Hilbert-Schmidt operator. We refer a reader to Refs. [21][23] for more details on the nested Hilbert space. The coordinate representation of the functions $\psi_p(x,t)$ is well-known and I omit it here.

The functions $\psi_p$ form an orthonormal and complete (in the sense of generalized functions) basis in $H$,
\[ \langle \psi_p | \psi_q \rangle = \delta(p - q). \] The completeness condition is expressed symbolically in terms of the projectors onto these functions as follows

\[ \int dp |\psi_p\rangle \langle \psi_p| = 1. \]  

(2)

I do not indicate the limits of integration in the integrals along the whole real axis. This equality should be understood in a weak sense. This means that it is equivalent to

\[ \int dp |\psi_j\rangle \langle \psi_p| \langle \psi_p| \psi_k \rangle = \delta_{jk}, \quad j, k = 0, 1, \ldots \]

where \( \psi_k, k = 0, 1, \ldots \) are orthonormal basis functions in the space \( H \).

The free particle CS may be obtained by applying a displacement operator in the Heisenberg-Weil group to the vacuum vector \( \psi_0 \):

\[ \psi_z(x, t) = \exp \left( z a^+ - \bar{z} a \right) \psi_0(x, t), \quad z \in \mathbb{C}. \]

These vectors are also the eigenvectors of the annihilation operator \( a \psi_z = \psi_z \). The vectors \( \psi_z \in H \) belong to a more wide set than \( \mathcal{L}_0 \). Their Fourier decomposition in terms of the basis \( \psi_n \) has the form

\[ \psi_z = \Phi \sum_n a_n z^n \psi_n, \]

\[ \Phi = \Phi(z, \bar{z}) = \exp(-z\bar{z}/2), \]

\[ a_n = (n!)^{-1/2}, \quad z \in \mathbb{C}. \]

The vectors \( \psi_z(x, t) \) satisfy all the properties enumerated in the Introduction. In particular, the measure \( d\mu = d\mu(z, \bar{z}) \) from the relation (1) is well-known: \( d\mu = dx dy/\pi \), \( z = x + iy \) and the domain of integration \( \mathcal{D} \) is the whole complex plane \( \mathbb{C} \). In what follows I will not indicate the domain of integration in the integrals over the measures. Integration will be always extended over the whole complex plane. Finally I give a coordinate representation of the free particle CS

\[ \psi_z(x, t) = (2\pi)^{-1/4}(1 + it)^{-1/2} \times \exp \left[ -\frac{1}{4}(z + \bar{z})^2 + \frac{(x + 2iz)^2 (it - 1)}{4(1 + t^2)} \right]. \]

I use the notation \( x \) as the spatial coordinate and as the real part of a complex number \( z \). I hope that it will not cause a confusion since these quantities will never appear in the same formula.

III. DARBOUX TRANSFORMATIONS AND ISOMETRIC OPERATORS

In this section I give an analysis of Darboux transformation operator \( L \) as an operator defined in the Hilbert space \( H \). I would like to stress that this operator is unbounded and cannot be defined over the whole space \( H \). It has a domain of definition which is a subset of \( H \) and will be specified. Moreover, it domain of values does not coincide with \( H \). Therefor this operator can not realize shifting between Hilbert spaces contrary to published assertion [20].

To obtain \( N \)-soliton potential I use the Darboux transformation operator approach elaborated in details in Ref. [14]. The action of this operator on a sufficiently smooth function is defined by the formula

\[ L\psi = W^{-1}(u_1, \ldots, u_N)W(u_1, \ldots, u_N, \psi) \]

where \( W \) stands for the usual symbol of a Wronskian.

In the case when the initial potential \( V_0 \) does not depend on time, the functions \( u_k = u_k(x, t) \) being solutions of the initial Schrödinger equation may be eigenfunctions of the initial Hamiltonian as well \( h_0u_k = a_k u_1 \) and in general are not supposed to satisfy any boundary conditions. In this case the transformation operator \( L \) does not depend on time and transforms solutions of the initial Schrödinger equation onto solutions of the Schrödinger equation with the potential

\[ V_1 = V_0 - 2\beta x \log W(u_1, \ldots, u_N) \]

which is independent on time. In this paper we need not to use time dependent Darboux transformation which was proposed by V. Matveev and M. Salle (see Ref. [10]) and advanced by V. Bagrov and B. Samsonov [22].

To obtain an \( N \)-soliton potential we should take \( V_0 = 0 \) and specify the transformation functions \( u_k \) as follows [10]:

\[ u_{2k-1} = \cosh(a_{2k-1} x + b_{2k-1}), \]

\[ u_{2k} = \sinh(a_{2k} x + b_{2k}), \]

\[ h_0u_k = -a_k^2 u_k, \quad k = 1, 2, \ldots, N; \]

\[ a_1 < a_2 < \ldots < a_N. \]

The time dependent phase factors are omitted from these functions since they do not affect all the results. In general the Wronsky determinant contains \( N! \) summands. I would like to stress that in a special case of soliton potentials this determinant may be substantially simplified and presented as a sum of \( 2^{N-1} \) hyperbolic cosines [28]

\[ W(u_1, \ldots, u_N) = 2^{1-N} \sum_{(\varepsilon_1, \ldots, \varepsilon_N)} \varepsilon_2 \varepsilon_4 \cdots \varepsilon_p \times \prod_{j > i} \left( x_j a_j - x_i a_i \right) \varepsilon_i \prod_{l=1}^N \varepsilon_l (a_l x + b_l), \]

where \( \varepsilon_i = \pm 1 \), the value of the subscript \( p \) at \( \varepsilon_p \) should be taken equal to \( N \) for even \( N \) values and to \( N - 1 \) for odd \( N \) values, the summation runs over all ordered and nonidentical sets \( (\varepsilon_1, \ldots, \varepsilon_N) \) (the sets \( (\varepsilon_1, \ldots, \varepsilon_N) \) and \( (-\varepsilon_1, \ldots, -\varepsilon_N) \) are declared to be identical).

It can be shown [10] that the potential so obtained is regular and bounded from below. This implies that the Hamiltonian \( h_1 = -\partial_x^2 + V_1 \) is essentially self-adjoint in \( H \). It has a mixed spectrum. The position of the discrete spectrum levels is defined by the values of the parameters
and the operator \( g_1 \) is strictly positive. I conserve the same notations for these operators as operators acting in \( H_1 \). Taking into account the spectral decomposition of these operators

\[
\begin{align*}
    h_1 &= \int dp |\varphi_p\rangle \langle \varphi_p|, \\
    g_1 &= \int dp N_p^2 |\varphi_p\rangle \langle \varphi_p|
\end{align*}
\]

we can specify their domain of definitions. For the operator \( h_1 \) it consists of all \( \varphi \in H_1 \) such that the integral

\[
\|h_1 \varphi\|^2 = \int dp p^2 |\langle \varphi | \varphi_p \rangle|^2
\]

converges and for the operator \( g_1 \) we should demand the convergence of the integral

\[
\|g_1 \varphi\|^2 = \int dp N_p^4 |\langle \varphi | \varphi_p \rangle|^2.
\]

It is clear that the operator \( g_1 \) is defined on \( \mathcal{L}_{1p} \) and maps this space into itself. Using this fact and the factorization property (3) we can define the action of the operator \( L^+ \) onto the functions \( \varphi_p \),

\[
L^+ \varphi_p = N_p^{-1} L \psi_p
\]

and extend this operator by linearity on the whole space \( \mathcal{L}_{1p} \).

It is not difficult to see that the following equality

\[
\langle L \psi_p | \varphi_q \rangle = \langle \psi_p | L^+ \varphi_q \rangle
\]

holds for all \( \psi_p \in \mathcal{L}_{0p} \) and \( \varphi_q \in \mathcal{L}_{1p} \). Nevertheless, this fact does not mean that \( L^+ \) is an operator conjugate with respect to the scalar product to \( L \) which domain of definition is \( \mathcal{L}_{0p} \). To find such an operator we have to specify correctly its domain of definition. I shall not look for this domain. Instead I shall give a closed extension \( \bar{L} \) of the operator \( L \) and then find its conjugate \( \bar{L}^+ \).

Once we know the bases \( \psi_p \) and \( \varphi_p \) in \( H \) and \( H_1 \) respectively we can consider isometric operators

\[
\begin{align*}
    U &= \int dp |\varphi_p\rangle \langle \psi_p|, \\
    U^{-1} &= U^+ = \int dp |\psi_p\rangle \langle \varphi_p|.
\end{align*}
\]

Similar operators have been introduced by L.D. Faddeev [5] and considered by L. Pursey [10] for purely discrete bases. These operators are bounded and defined for all elements from \( H \) and \( H_1 \) respectively.

Consider now the following operators

\[
\begin{align*}
    \bar{L} &= \int dp N_p |\varphi_p\rangle \langle \psi_p|, \\
    \bar{L}^+ &= \int dp N_p |\psi_p\rangle \langle \varphi_p|.
\end{align*}
\]

It is not difficult to specify their domains of definition. For this purpose I use the spectral decompositions of the operator \( g_0 \) and its square root

\[
\begin{align*}
    g_0 &= \int dp N_p^2 |\psi_p\rangle \langle \psi_p|,
\end{align*}
\]

where \( W^k(u_1, \ldots, u_N) \) is the Wronskian of the functions \( u_1, \ldots, u_N \) except for the function \( u_k \) and the factor \( N_k \). is introduced to ensure the normalization of the functions \( \varphi_k \). Therefor, consider the orthogonal decomposition of the space \( H : H = H_0 \oplus H_1 \) where \( H_0 \) is an \( N \)-dimensional space with the basis \( \varphi_k \), \( k = 1, \ldots, N \). The functions \( \varphi_p, p \in \mathbb{R} \) form a basis (in the sense of generalized functions) in \( H_1 \). In what follows I shall not consider the space \( H_0 \) and restrict the consideration only by the space \( H_1 \). The operators \( h_1 \) and \( g_1 \) being restricted to this space have only a continuous spectrum

\[
\begin{align*}
    &a_k : E_k = -a_k^2, \text{ Corresponding eigenfunctions have the form } \var\phi_k = N_k W^{k}(u_1, \ldots, u_N)/W(u_1, \ldots, u_N), \\
    &N_k = \left( \frac{1}{2} a_k \prod_{j=1(j\neq k)}^N (|a_k^2 - a_j^2|)^{1/2}, \\
    &h_1 \var\phi_k = -a_k^2 \var\phi_k, \quad k = 1, \ldots, N
\end{align*}
\]

and the space \( H \) restricted to this space have only a continuous spectrum numbers}
\[ g_0^{1/2} = \int dpN_p |\psi_p\rangle \langle \psi_p |. \] (9)

It follows that
\[ \|L\psi\|^2 = \|g_0^{1/2} \psi\|^2 = \int dpN_p^2 |\langle \psi | \psi_p \rangle|^2. \]

This means that the domain of definition of \( L \) coincides with that of \( g_0^{1/2} \) and consists of all \( \psi \in H \) such that the integral in the right hand side of this equation converges. The domain of definition of \( L^+ \) coincides with that of the operator \( g_1^{1/2} \). It is worthwhile to mention that these domains may be described in terms of conditions imposed on functions that are comprised in these domains (see for example [31]) since \( h_0 \) and \( h_1 \) are operators bounded from below and essentially self-adjoint.

It is clear from the formulæ (7) and (8) that the operator \( \bar{L} \) is conjugate to \( L \) with respect to the scalar product and their domains of definition are well specified. Moreover, \( \bar{L}^+ = L \). This implies that the operator \( \bar{L} \) is closed. The formulæ (6) and (8) give quasi-spectral representation of the closed operators \( \bar{L} \) and \( \bar{L}^+ \).

It follows from the formulæ (7) and (8) that \( L\psi = \psi = N_p\psi_p \), and \( L^+ \psi_p = N_p^0 \psi_p \). This means that \( L \) is a closed extension of the operator \( L \) and \( L^+ \) is a similar extension of the operator \( \bar{L} \) when the domains \( L_{0\psi} \) and \( L_{1\psi} \) are taken as their initial domains of definitions.

From the spectral decomposition of the operators \( g_0^{1/2 \psi} \) and \( g_1^{1/2 \psi} \),
\[ g_0^{1/2} = \int dpN_p |\varphi_p\rangle \langle \varphi_p |, \]
we obtain the following representations for \( \bar{L} \) and \( \bar{L}^+ \):
\[ \bar{L} = U g_0^{1/2} = g_1^{1/2} U, \quad \bar{L}^+ = g_0^{1/2} U^+ = U^+ g_1^{1/2}. \]

Such representations are known as polar decompositions or canonical representations of closed operators (see for example Refs. [22, 24]).

The action of the operator \( U \) on the basis \( \psi_n \) gives an orthonormal basis in \( H_1 \): \( \zeta_n = U \psi_n, \langle \zeta_n | \zeta_k \rangle = \delta_{nk} \). The functions \( \varphi_n = g_1^{1/2} \zeta_n = \bar{L} \psi_n = L \psi_n \), hence, form a basis in \( H_1 \) equivalent to an orthonormal (so called Riesz basis, see for example Ref. [33]). The operator \( U \) is non-local and rather complicated. Therefor there is no way in which simple explicit expressions can be derived for the functions \( \zeta_n \). The functions \( \varphi_n(x, t) = L \psi_n(x, t) \) are much simpler but they are not orthogonal to each other: \( \langle \varphi_n | \varphi_k \rangle = S_{nk} \). I shall denote by \( S \) the infinite matrix with the entries \( S_{nk} \). The elements of this matrix can easily be expressed in terms of the elements of another matrix \( S^0(a) \) with the entries \( S^0_{nk}(a) = \langle \psi_n | h_0 + a^2 | \psi_k \rangle \):
\[ S_{nk} = [S^0(a_1) S^0(a_2) \ldots S^0(a_N)]_{nk}, \]
where the use of the factorization property (3) has been made. Taking into account that \( h_0 \) is expressed in terms of the ladder operators \( a \) and \( a^\dagger \) for the basis functions \( \psi_n \), \( h_0 = \frac{1}{2} (a + a^\dagger) \), we derive the nonzero elements of the matrix \( S^0(a) \). \( S^0_{nn}(a) = n/2 + 1/4 + a^2 \), \( S^0_{nn+2}(a) = \frac{1}{4} \sqrt{(n+1)(n+2)} \). All the other matrix elements are zero. We see, hence, that the number of nonzero elements in each row and column of the matrix \( S \) is finite.

Consider now bounded operators
\[ M = \int dpN_p^{-1} |\varphi_p\rangle \langle \varphi_p |, \]
\[ M^+ = \int dpN_p^{-1} |\psi_p\rangle \langle \psi_p | \]
defined in \( H \) and \( H_1 \) respectively. It is not difficult to see that \( M \bar{L}^+ = 1 \) is the unit operator in \( H_1 \) and \( M^+ \bar{L} = 1 \) is the unit operator in \( H \). Using the spectral resolutions of the operators \( g_{0^{1/2}} \) and \( g_{1^{1/2}} \),
\[ g_{0^{1/2}} = \int dpN_p^{-1} |\psi_p\rangle \langle \psi_p |, \]
\[ g_{1^{1/2}} = \int dpN_p^{-1} |\varphi_p\rangle \langle \varphi_p | \]
we derive the polar decompositions of the operators \( M \) and \( M^+ \):
\[ M = U g_{0^{-1/2}} = g_{1^{-1/2}} U, \]
\[ M^+ = g_{0^{-1/2}} U^+ = U^+ g_{1^{-1/2}}. \]

It is easily seen that these operators factorise the operators inverse to \( g_0 \) and \( g_1 \): \( M^+ + M = g_1^{-1}, \quad MM^+ = g_{1^{-1}} \).

The functions \( \eta_n = g_{1^{-1/2}} \zeta_n = M \zeta_n \), form another basis in \( H_1 \) equivalent to an orthonormal. This basis is biorthogonal to \( \varphi_n, \langle \varphi_n | \eta_k \rangle = \delta_{nk} \). It follows the representation for the elements \( S_{nk}^{-1} \):
\[ S_{nk}^{-1} = \langle \eta_n | \eta_k \rangle = \langle \psi_n | g_{0^{-1}} | \psi_k \rangle = \int dpN_p^{-2} |\psi_n\rangle \langle \psi_p | \langle \psi_p | \psi_k \rangle \]

As a final remark of this section I would like to notice the following. The space \( H_1 \) can be obtained as a closure of the lineal \( L_1 \) of all finite linear combinations of the functions \( \varphi_n = L \psi_n \) with respect to the norm generated by the scalar product which is a restriction of the given scalar product in \( H \) to the lineal \( L_1 \). The set of functions of the form \( \varphi = L \psi \) when \( \psi \) run through the whole domain of definition of the operator \( L \) (i.e. the domain \( D_{\sqrt{\bar{M}}} \) of definition of the operator \( \sqrt{\bar{M}} \)) cannot give the whole space \( H_1 \). Nevertheless, if we define a new scalar product in \( L_1 \), \( \langle \varphi_a | \varphi_b \rangle \equiv \langle \varphi_a | \varphi_b \rangle = \langle \psi_a | g_0 | \psi_b \rangle \), \( \psi_{a,b} \in L_0, \varphi_{a,b} \in L_1 \) then the closure of \( L_1 \) with respect to the norm generated by this scalar product will coincide with the set \( \varphi = L \psi, \psi \in D_{\sqrt{\bar{M}}} \). This space is embedded in \( H_1 \).
IV. COHERENT STATES OF SOLITON POTENTIALS

The operator $g_0$ is a symmetry operator for the Schrödinger equation. Therefor it commutes with the Schrödinger operator $i\partial_t - \mathcal{H}_0$ when applied to the solutions of the Schrödinger equation. It follows that the operator $U = \mathcal{L}_{g_0}^{-1/2}$ is an intertwining operator for the Schrödinger operators $U(i\partial_t - \mathcal{H}_0) = (i\partial_t - \mathcal{H}_1)U$ and therefor it is a transformation operator. Hence, being applied to a solution of the initial Schrödinger equation (in our case the free particle Schrödinger equation) it gives a solution of the transformed equation (in our case the Schrödinger equation) of the transformed operator (in our case the $N$-soliton potential). The functions $\zeta_n = U\psi_n$ and $\zeta_z = U\psi_z$ are then solutions of the Schrödinger equation with soliton potential. The Fourier decomposition of the function $\zeta_z$ in terms of the basis $\{\zeta_n\}$ has the same form as that of the function $\psi_z$ in terms of $\{\psi_n\}$:

$$
\zeta_z = \Phi \sum_n a_n \zeta_n.
$$

The vectors $\zeta_z, z \in \mathbb{C}$ satisfy all the conditions formulated for CS in the Introduction because of the isometric nature of the operator $U$. The resolution of the identity operator (1) in the space $H_1$ in terms of the projectors on $\zeta_z$ takes place with the same measure $d\mu = dx dy/\pi$, $z = x + iy$. One of the deficiencies of these coherent states is that no a simple explicit expression for such vectors exists. This deficiency may be cured by acting to them by a symmetry operator for the Schrödinger equation with soliton potential.

Consider the vectors

$$
\varphi_z = g_1^{1/2} \zeta_z = \mathcal{L}\psi_z = \Phi \sum_n a_n \varphi_n.
$$

It is not difficult to see that the value $\langle \psi_z | g_0 | \psi_z \rangle$ is finite. This means that $\psi_z$ belong to the domain of definition of the operator $L$ and the above equality has a sense. Moreover, these functions are sufficiently smooth and we can apply to them directly the differential operator $L$. Thus we obtain a coordinate representation of $\varphi_z$. For instance, in the case of the one-soliton potential this representation reads

$$
\varphi_z(x, t) = \frac{1}{2} (2\pi)^{-1/4} (1 + it)^{-3/2} \times \left[ x + 2iz + 2a(1 + it) \tanh(ax) \right] \times \exp\left\{ -\frac{(x + 2iz)^2}{4 + 4it} - \frac{1}{2}(z + \bar{z})^2 \right\}.
$$

We see thus that these functions are much simpler then $\zeta_z$ and may be analyzed without difficulties. For example it is easily seen that $|\mathcal{L}\varphi_z(x, t)|^2 = |\varphi_z(-x, -t)|^2$. This property reflects a transparent nature of the one-soliton potential.

Another system of states may be obtained with the help of the transformation operator $M$. Consider the vectors

$$
\eta_z = g_1^{-1/2} \zeta_z = M \psi_z = \Phi \sum_n a_n \eta_n.
$$

The operator $M$ being inverse to $L$ has an integral nature. For the case of the one-soliton potential the integration may be carried out analytically [4]. This yields

$$
\langle \eta_z(x, t) | \varphi_z(x, t) \rangle = \frac{1}{2} \sqrt{\pi} (2\pi)^{-1/4} \text{sech}(ax) \times \exp\left\{ -\frac{1}{2}(z + \bar{z})^2 + a^2(1 + it) \right\} \times \left[ \text{erfc}\left( a\sqrt{1 + it} + x/2 + i\bar{z} \right) - \text{erfc}\left( a\sqrt{1 + it} - x/2 + i\bar{z} \right) \right].
$$

Where the parameter $b$ is taken to be zero.

It is worthwhile to mention that all the states $\psi_z(x, t)$, $\varphi_z(x, t)$, $\eta_z(x, t)$, and $\zeta_z(x, t)$ cannot represent non-spread wave packets. Nevertheless, we can interpret them as coherent states since they satisfy all the properties of such states enumerated in the Introduction. I shall show now that for the vectors $\varphi_z$ and $\eta_z$ there exist measures $\mu_\varphi = \mu_\varphi(z, \bar{z})$ and $\mu_\eta = \mu_\eta(z, \bar{z})$ that realize the resolution of the identity operator in $H_1$ in terms of the projectors on these vectors.

First consider another continuous basis in $H_1$: $\eta_p = N_p M \psi_p$, $\langle \eta_p | \eta_p \rangle = \delta(p - q)$, $p, q \in \mathbb{R}$. Since $\{\varphi_p\}$ and $\{\eta_p\}$ are bases in $H_1$, the resolutions of the identity operator of the type (1) in terms of the vectors $\eta_p$ and $\varphi_z$ are equivalent to the equations

$$
\int d\mu_p(z, \bar{z}) \langle \eta_p | \eta_p \rangle \langle \eta_p | \eta_q \rangle = \delta(p - q),
$$

$$
\int d\mu_\varphi(z, \bar{z}) \langle \varphi_\varphi | \varphi_\varphi \rangle \langle \varphi_\varphi | \varphi_q \rangle = \delta(p - q).
$$

Taking into account that the functions $\psi_p$ are the eigenfunctions of $g_0$ and $g_0^{-1}$, $g_0 \psi_p = N_p^2 \psi_p$, $g_0^{-1} \psi_p = N_p^{-2} \psi_p$ we arrive at equations for the measures $\mu_\eta$ and $\mu_\varphi$

$$
(N_p N_q)^{-1} \int d\mu_\eta \langle \eta_p | \eta_p \rangle \langle \eta_p | \eta_q \rangle = \delta(p - q),
$$

$$
N_p N_q \int d\mu_\varphi \langle \varphi_\varphi | \varphi_\varphi \rangle \langle \varphi_\varphi | \varphi_q \rangle = \delta(p - q).
$$

Note that the integrals involved in these equations are time-independent and hence can be calculated at $t = 0$. Therefore in what follows I let $t = 0$ and look for the measures independent on time.

The momentum representation of the CS $\psi_z$ is well-known

$$
\langle \psi_p | \psi_z \rangle = (2/\pi)^{1/4} \Phi \psi_p(z),
$$

$$
\psi_p(z) = \exp(-p^2 + 2zp - z^2/2), z = x + iy.
$$

Let us look for the measure $\mu_\eta$ in the form $d\mu_\eta = \omega_\eta(x) dx dy$, $z = x + iy$. After performing the integration with respect to $y$ in the Eq. (10) we arrive at an equation for $\omega_\eta(x)$

$$
(2\pi)^{1/2} \int d\omega_\eta(x) F_p(x) = N_p^2 \exp(2p^2),
$$

$$
F_p(x) = \text{erfc}\left( a\sqrt{1 + it} + x/2 + i\bar{z} \right) - \text{erfc}\left( a\sqrt{1 + it} - x/2 + i\bar{z} \right).
$$

Thus we obtain a coordinate representation of $\varphi_z$ in terms of $\zeta_z$.
$F_p(x) = \exp(4px - 2x^2).$

The function $N_p^2$ is a polynomial in $p$ which is known. We conclude then that $\omega_p(x)$ is a polynomial in $x$ whose coefficients are uniquely defined by the coefficients of the polynomial $N_p^2$. For instance, for the one-soliton potential we have

$$\omega_p(x) = (x^2 + a^2 - 1/4)/\pi.$$  

This proves that the states $\eta_z$ may be interpreted as CS.

We note that the states $\eta_z$ are defined with the help of the bounded operator $g_0^{1/2}$. This is the reason for which the measure $\mu_n$ is expressed in terms of ordinary (non generalized) functions. An other case takes place for the states $\varphi_z$ which are defined by the semibounded operator $g_1^{1/2}$. I shall show now that the measure $\mu_\varphi$ is expressed in terms of generalized functions.

Let us look for the measure $\mu_\varphi$ in the form $d\mu_\varphi = dyd\omega_\varphi(x)$. The integration in the equation (13) with respect to $y$ leads us to an equation for the measure $d\omega_\varphi(x)$

$$(2\pi)^{1/2} \int d\omega_\varphi(x) F_p(x) = N_p^{-2} \exp(2p^2).$$  

First we note that $|F_p(x + iy)| \leq \exp(-dx^2 + by^2)$ where $2 \leq d \leq b$. This means that $F_p(x)$ belongs to a subspace of the space $S^{1/2}_1$ of entire functions $F$ such that $|F(x + iy)| \leq \exp(-dx^2 + by^2)$, $0 \leq d \leq b$. We look for $\omega_\varphi$ as a functional (i.e. a generalized function) over $S^{1/2}_1$. (We will see that really this is a functional over a subspace $S^{1/2}_1 \subset S^{1/2}_1$.)

As it is known positive definite functionals (we look for such a functional) over $S^{1/2}_1$ are specified by their Fourier transforms. Let $\tilde{\omega}_\varphi$ be the Fourier transform of the measure $\omega_\varphi(x)$. This means that an integration of a function $F(x) \in S^{1/2}_1$ with respect to the measure $\omega_\varphi(x)$ should be replaced by the integration of the Fourier transform $\tilde{F}(t)$ of this function with respect to the measure $\tilde{\omega}_\varphi$. In particular

$$\int d\omega_\varphi(x) F_p(x) = \int d\tilde{\omega}_\varphi(t) \tilde{F}_p(t)$$  

where $\tilde{F}_p(t)$ is the Fourier image of the function $F_p(x)$ which in our case can easily be found

$$\tilde{F}_p(t) = \sqrt{\pi}/2 \exp(2p^2 + i pt - t^2/8).$$  

As a result the Eq. (13) yields the equation for $\tilde{\omega}_\varphi$

$$\pi \int d\tilde{\omega}_\varphi(t) \exp(-t^2/8 + i pt) = N_p^{-2}.$$  

It is an easy exercise to see that $\tilde{\omega}_\varphi(t)$ may be expressed in terms of elementary functions. For this purpose we look for $\tilde{\omega}_\varphi(t)$ in the form $d\omega_\varphi(t) = \rho_\varphi(t) dt$ and use the following representation for the function $N_p^{-2}$:

$$N_p^{-2} = \sum_{k=1}^{N} \frac{A_k}{\tau + \frac{a_k^2}{\tau^2}}, \quad \tau = p^2;$$  

$$A_k = \frac{\int dN_p^2/d\tau|_{\tau=-a_k^2}^{1}.}$$

After some algebra we obtain a formula for $\rho_\varphi(t)$

$$\rho_\varphi(t) = (2\pi)^{-1} \sum_{k=1}^{N} \frac{A_k}{a_k^2} \exp((t^2/8 - a_k^2).$$  

Note that for the function $\rho_\varphi(t)$ of the form (12) there exist in $S^{1/2}_1$ such functions $F(p)$ that the integral in the right hand side of the Eq. (13) diverges. The convergence condition for this integral imposes a restriction on the decrease of the integrand function $F(x)$ in the left hand side of the Eq. (12) as $|x| \to \infty$. This function should satisfy an inequality $|F(x)| \geq \exp(-2x^2 - Ax)$ where $A$ is a nonnegative constant own to every function $F(x) \in S^{1/2}_1$. I denote the set of functions satisfying this condition by $S^{1/2}_1(\subset S^{1/2}_1)$ which obviously is a linear space.

Thus, we have found the measure $\mu_\varphi$ in terms of the generalized function $\omega_\varphi(x)$ over the space $S^{1/2}_1$, $d\mu_\varphi = dyd\omega_\varphi(x)$, $z = x + iy$ which is defined by its Fourier transform $\tilde{\omega}_\varphi$. The integrals with respect to this measure should be calculated as follows

$$\int d\mu_\varphi(\varphi_a|\varphi_z)(\varphi_z|\varphi_b) = \int dt \tilde{\mu}_\varphi(t) \tilde{F}_{ab}(t)$$

where $\tilde{F}_{ab}(t)$ is the Fourier transform of the function

$$F_{ab}(x) = \int dy (\varphi_a|\varphi_z)(\varphi_z|\varphi_b), \quad z = x + iy.$$  

Finally I give comments on the calculation of the norms of the functions $\eta_z$ and $\varphi_z$. The square of the norm of $\eta_z$ may be calculated with the aid of the formula (13) for the function $N_p^{-2}$ and the factorization property of the operator $g_0^{-1}$ in terms of the operators $M$ and $M^+$

$$\langle \eta_z|\eta_z \rangle = \langle \psi_z|g_0^{-1}|\psi_z \rangle = \int dp N_p^{-2} |\langle \psi_z|\psi_p \rangle|^2.$$  

After some algebra we obtain

$$\langle \eta_z|\eta_z \rangle = \sum_{k=1}^{N} A_k F_k, \quad z = x + iy;$$

$$F_k = \frac{\sqrt{2\pi}}{a_k^2 \exp(2(a_k^2 - x^2)] \times Re [\exp(4ia_k x) \text{erfc}(a_k \sqrt{2} + i \sqrt{2}x)].}$$

Similarly, the square of the norm of the function $\varphi_z$ coincides with the expectation value of the operator $g_0$ in the state $\psi_z$. For instance, for the one-soliton potential we obtain

$$\langle \varphi_z|\varphi_z \rangle = \langle \psi_z|g_0|\psi_z \rangle = 1/4 + a^2 + x^2, \quad z = x + iy.$$
A classical particle proves no reflection in the scattering process on a potential well. For a quantum particle in general this is not the case. Nevertheless, there exists a wide class of potentials called transparent potentials for which the scattering process of the quantum particle comes in some sense about in a similar way that those of the classical particle i.e. without reflection. In my opinion this mysterious phenomena up to now has no any perspicuous explanation. From a practical point of view the answer to this question is rather important. If at quantum level we would be able to force a signal to propagate without reflection we could decrease the output of the emitted signal. All transparent potentials known at present have a remarkable property. They are related with zero potential (free particle) by Darboux transformations. Up to recent times it was believed that such potentials have a finite number of discrete spectrum levels. Nevertheless a method based on an infinite chain of Darboux transformations with the help of which one can create transparent potentials with infinite number of discrete spectrum levels has been proposed recently \[ \text{[34]} \]. To understand better the nature of transparent potentials we should investigate them in all details.

As it is well known the quantum theory gives a more detailed description of the nature than the classical one. Therefore different quantum systems may correspond to the same classical system. Furthermore, the quantization procedure is not unique (canonical quantization, Berezin quantization, geometric quantization, etc.). In this respect the following question is of interest. What are common points between two classical systems a quantization of which gives the quantum systems that are related to each other by a Darboux transformation operator? In particular, what are common points between the classical free particle and the particle that moves in a potential quantization of which gives a transparent potential? The CS approach make it possible to formulate clear steps in the direction of obtaining an answer to this question. It permits one to construct a classical mechanics counterpart of a given quantum system and analyze properties of such a system. This approach has been realized recently for the potential of the form \( x^2 + gx^{-2} \). It was established that at classical level the Darboux transformation consists in a distortion of a phase space of the classical system. Moreover, this distortion is consistent with the transformation of the Hamilton function in such a way that the equations of motion remain unchanged.

Up to now no any approach for analysis of CS of transparent potentials has been proposed. In this paper I show that the Darboux transformation operator approach is suitable for this purpose. A next step in this direction would be an analysis of the classical counterpart of the quantum system that moves in a transparent potential.

**V. CONCLUSION**

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