CKM Matrix Elements

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The current analysis of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix uses the standard parametrisation by 3 mixing angles and the CP-violating KM phase. However it would be more convenient to express these mixing angle parameters in terms of the known CKM matrix elements like $V_{ud}$, $V_{us}$, $V_{ub}$, $V_{cb}$ and the CP-violating phase $\delta$. The other CKM matrix elements are then expressed in terms of these known matrix elements instead of the standard mixing angles. In this paper, using $V_{ud}$, $V_{us}$, $V_{ub}$ and $V_{cb}$ from the current global fit, we show that the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$ imply $\gamma = (68 \pm 3)^{\circ}$ at which $\sin \beta$ and $\sin \alpha$ are near its maximum, consistent with the global fit.

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The current global fit of the CKM matrix elements in the Standard Model with imposed unitarity constraints[1, 2] seems to allow a rather precise determination of some of the less known CKM matrix elements like $V_{cb}$ and $V_{ub}$. It would then be possible to directly express the mixing angle parameters in terms of the known CKM matrix elements like $V_{ud}$, $V_{us}$, $V_{ub}$, $V_{cb}$ and the phase $\delta$, rather than using a parametrization for the mixing angles as usually done in the current studies of the CKM matrix elements. The remaining CKM matrix elements $V_{cd}$, $V_{cs}$, $V_{td}$, $V_{ts}$ and $V_{tb}$ are thus completely determined directly in terms of these known quantities and the CP-violating phase $\delta$, the main feature of our approach. Furthermore, the CP-violating phase $\delta$ can also be expressed in terms of the known CKM matrix elements and the angle $\gamma$, $\gamma$ being one of the angle of the $(db)$ unitarity triangle[2, 3] as shown in Fig. (1). The other angles $\beta$ and $\alpha$ as well as the ratio $|V_{td}/V_{ts}|$ are then given as functions of $\gamma$. In this paper we shall present our calculation of ratio $|V_{td}/V_{ts}|$, $\sin \beta$ and $\sin \alpha$ in terms of $\gamma$ using $V_{ud}$, $V_{us}$, $V_{ub}$, $V_{cb}$ from the global fit. We find that the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$ imply that $\gamma = (68.6 \pm 3)^{\circ}$ and $\alpha \approx (89 \pm 3)^{\circ}$ consistent with the global fit values for these quantities.

We begin by writing down the Cabibbo-Kobayashi-Maskawa (CKM) [4, 5] quark mixing matrix as:

$$V_{CKM} = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}$$

(1)
To impose unitarity for the CKM matrix, we take the standard parametrization as used in [2] which is given as [6]:

\[
V_{\text{CKM}} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}\exp(-i\delta) \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}\exp(i\delta) & c_{12}c_{23} - s_{12}s_{23}s_{13}\exp(i\delta) & s_{23}c_{13} \\
s_{12}s_{23} - c_{12}c_{23}s_{13}\exp(i\delta) & -c_{12}s_{23} - s_{12}c_{23}s_{13}\exp(i\delta) & c_{23}c_{13}
\end{pmatrix}
\]

where \(s_{ij} = \sin(\theta_{ij})\), \(c_{ij} = \cos(\theta_{ij})\) and \(\delta\) is the CP-violating KM phase. The angles \(\theta_{ij}\) can be chosen to be in the first quadrant, so \(s_{ij}\) and \(c_{ij}\) can be taken as positive [2]. The current global fit gives for the magnitudes of all 9 CKM matrix elements [1, 2]:

\[
|V_{\text{CKM}}| = \begin{pmatrix}
  0.97428 \pm 0.00015 & 0.2253 \pm 0.0007 & 0.00347^{+0.00016}_{-0.00012} \\
  0.2252 \pm 0.0007 & 0.97345^{+0.00015}_{-0.00016} & 0.0410^{+0.0011}_{-0.00007} \\
  0.00862^{+0.00026}_{-0.00020} & 0.0403^{+0.0011}_{-0.00007} & 0.99915^{+0.000030}_{-0.000045}
\end{pmatrix}
\]

Instead of using the current parametrization [2, 7], which is a form of Wolfenstein parametrization [8] and is unitary to all order in \(s_{12} = \lambda\), we shall now express \(s_{ij}\) and \(c_{ij}\) in terms of \(V_{ud}, V_{us}, V_{ab}\) and \(V_{cb}\). As these quantities are assumed to be positive, they can be obtained directly from the measured absolute values. For simplicity, we shall put \(s_{13} = V_{ub}\) and the matrix element \(V_{ab}\) of the CKM matrix in Eq. (1) is now written with the phase \(\delta\) taken out \((V_{ab} \to V_{ab}\exp(-i\delta))\). We have:

\[
\begin{align*}
s_{12} &= \frac{V_{us}}{\sqrt{(V_{ud}^2 + V_{us}^2)}}, & c_{12} &= \frac{V_{ud}}{\sqrt{(V_{ud}^2 + V_{us}^2)}} \\
s_{13} &= V_{ub}, & c_{13} &= \sqrt{(1 - V_{ub}^2)} \\
s_{23} &= \frac{V_{cb}}{\sqrt{(1 - V_{ub}^2)}}, & c_{23} &= \sqrt{(1 - V_{ub}^2 - V_{cb}^2)} \sqrt{(1 - V_{ub}^2)}.
\end{align*}
\]

Since \(s_{12}^2 + c_{12}^2 = 1\), we have

\[
V_{ud}^2 + V_{us}^2 = c_{13}^2 = 1 - V_{ub}^2
\]

The matrix elements of the CKM matrix in Eq. (1) are then given by:

\[
\begin{align*}
V_{us} &= \frac{V_{us}\sqrt{(1 - V_{ub}^2)}}{\sqrt{(V_{ud}^2 + V_{us}^2)}}, & V_{ud} &= \frac{V_{ud}\sqrt{(1 - V_{ub}^2)}}{\sqrt{(V_{ud}^2 + V_{us}^2)}}, & V_{ab} &\to V_{ab}\exp(-i\delta) \\
V_{cd} &= -\frac{V_{us}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}} - \frac{V_{ud}V_{cb}V_{ub}\exp(i\delta)}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}}, \\
V_{cs} &= \frac{V_{ud}\sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}} - \frac{V_{us}V_{cb}V_{ub}\exp(i\delta)}{\sqrt{(V_{ud}^2 + V_{us}^2)}\sqrt{(1 - V_{ub}^2)}}, & V_{cb} &\to V_{cb}
\end{align*}
\]
\[ V_{td} = \frac{V_{us} V_{cb}}{\sqrt{(V_{ud}^2 + V_{us}^2)/(1 - V_{ub}^2)}} - \frac{V_{ud} \sqrt{(1 - V_{ub}^2 - V_{cb}^2)V_{ub} \exp(i\delta)}}{\sqrt{(V_{ud}^2 + V_{us}^2)/(1 - V_{ub}^2)}} \]
\[ V_{ts} = -\frac{V_{ud} V_{cb}}{\sqrt{(V_{ud}^2 + V_{us}^2)/(1 - V_{ub}^2)}} - \frac{V_{us} \sqrt{(1 - V_{ub}^2 - V_{cb}^2)V_{ub} \exp(i\delta)}}{\sqrt{(V_{ud}^2 + V_{us}^2)/(1 - V_{ub}^2)}} \]
\[ V_{tb} = \sqrt{(1 - V_{ub}^2 - V_{cb}^2)} \] (6)

Since \( s_{12}^2 + c_{12}^2 = 1 \), we have
\[ V_{ud}^2 + V_{us}^2 = c_{13}^2 = 1 - V_{ub}^2 \] (7)

We note that this expression could be used to obtain \( V_{ub} \) provided that \( V_{ud} \) and \( V_{us} \) could be measured with great precision which could be achieved in future experiments. With this relation, we recover the expressions for \( V_{ud} \) and \( V_{us} \) in Eq. (6), given here with the factor \( \sqrt{(1 - V_{ub}^2)/(V_{ud}^2 + V_{us}^2)} \) included so that unitarity for the CKM matrix is explicitly satisfied.

FIG. 1: The \((db)\) unitarity triangle with the sides represent \( R_u = |V_{ud} V_{ub}^*/V_{cd} V_{cb}^*| \), \( R_t = |V_{td} V_{tb}^*/V_{cd} V_{cb}^*| \) and \( R_c = 1 \)

The above expressions allow a direct determination of \( V_{cd}, V_{cs}, V_{td}, V_{ts} \) and \( V_{tb} \) in terms of \( V_{ud}, V_{us}, V_{ub}, V_{cb} \). In the following we shall present expressions and results of our analysis for \( |V_{td}/V_{ts}|, \) \( \sin\beta, \sin\alpha \) and \( \sin\gamma \) in terms of the known CKM matrix elements provided by the global fit and the CP-violating phase \( \delta \).

As seen from the above expressions, the CP-violating phase \( \delta \) originating from \( V_{ub} \) enters in the quantities \( V_{td}, V_{ts} \) and also in \( V_{cd}, V_{cs} \) and manifests itself in the CP-asymmetries in \( B^0 - \bar{B}^0 \) mixing and in charmless \( B \) decays, for example.
Consider now the quantity $|V_{td}/V_{ts}|$. We have from Eq. (6):

$$|V_{td}/V_{ts}| = \frac{V_{us}}{V_{ud}} \left( \sqrt{(1 - 2 K_d \cos \delta + K_d^2)} \right)$$

with

$$K_d = \frac{V_{ud} V_{ub}}{V_{us} V_{cb}} \sqrt{1 - V_{ub}^2 - V_{cb}^2}, \quad K_s = \frac{V_{us} V_{ub}}{V_{ud} V_{cb}} \sqrt{1 - V_{ub}^2 - V_{cb}^2},$$

For the $(db)$ unitarity triangle, the angles $\alpha, \beta, \gamma$ are given by

$$\alpha = \arg(-V_{td} V_{tb}^*/V_{ud} V_{ub}^*), \quad \beta = \arg(-V_{cd} V_{cb}^*/V_{td} V_{tb}^*), \quad \gamma = \arg(-V_{ud} V_{ub}^*/V_{cd} V_{cb}^*)$$

with the sides:

$$R_u = |V_{ud} V_{ub}^*/V_{cd} V_{cb}|, \quad R_t = |V_{td} V_{tb}^*/V_{cd} V_{cb}|, \quad R_c = 1$$

Using the expressions in Eq. (6), we find:

$$\sin \alpha = \sin \delta \frac{\sin \delta}{\sqrt{(1 - 2 K_t \cos \delta + K_t^2)}}, \quad \sin \gamma = \sin \delta \frac{\sin \delta}{\sqrt{(1 + 2 K_c \cos \delta + K_c^2)}}$$

$$\sin \beta = \frac{(K_t + K_c) \sin \delta}{\sqrt{(1 - 2 K_t \cos \delta + K_t^2)} \sqrt{(1 + 2 K_c \cos \delta + K_c^2)}}$$

with

$$K_t = \frac{V_{ud} V_{ub}}{V_{us} V_{cb}} \sqrt{1 - V_{ub}^2 - V_{cb}^2}, \quad K_c = \frac{V_{ud} V_{ub} V_{cb}}{V_{us} \sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}$$

$$(K_t + K_c = V_{ud} V_{ub} (1 - V_{ub}^2)/(V_{us} V_{cb} \sqrt{(1 - V_{ub}^2 - V_{cb}^2)}))$$

Similarly, we have:

$$R_u = \frac{(K_c + K_t)}{\sqrt{(1 + 2 K_c \cos \delta + K_c^2)}}, \quad R_t = \frac{\sqrt{(1 - 2 K_t \cos \delta + K_t^2)}}{\sqrt{(1 + 2 K_c \cos \delta + K_c^2)}}, \quad R_c = 1.$$  

From the expressions for $\sin \alpha$, $\sin \beta$ and $\sin \gamma$ in Eq. (12), we recover the relation

$$\sin \beta = R_u \sin \alpha, \quad \sin \gamma = R_t \sin \alpha$$

obtained previously in Eq. (7).

Eq. (12) for $\sin \gamma$ can be used to express $\delta$ in terms of $\gamma$. We have:

$$\cos \delta = \cos \gamma \left( \sqrt{(1 - K_c^2 \sin^2 \gamma)} \right) - K_c \sin^2 \gamma$$
As can be seen from Eq. (13), $K_c$ is very much suppressed compared with $K_t$ ($K_c = 0.006$, $K_t = 0.366$), $\sin \delta \approx \sin \gamma$ and $\cos \delta \approx \cos \gamma$ with $\sin^2 \gamma$ and $\cos \gamma$ correction terms of $O(10^{-4})$. Hence $\delta$ can be replaced by $\gamma$ in the computed quantities with no great loss of accuracy. Thus, with $\sin \delta = \sin \gamma$ and $\cos \delta = \cos \gamma$, we obtain, with the global fit central value $V_{ub} = 0.00347$:

\[
V_{cd} = -0.225112 - 0.000138 \exp(i\gamma), \quad V_{cs} = 0.973467 - 0.000032 \exp(i\gamma) \quad (17)
\]

\[
V_{td} = 0.009237 - 0.003378 \exp(i\gamma), \quad V_{ts} = -0.039946 - 0.000781 \exp(i\gamma) \quad (18)
\]

We see that the contribution from the suppressed CKM matrix elements $V_{cb}$ and $V_{ub}$ produces only a small contribution to $V_{cd}$ and $V_{cs}$. Thus unitarity of the CKM matrix in the standard model with 3 generations implies that $V_{cd}$ and $V_{cs}$ are rather well determined and therefore could be used in the semi-leptonic and non-leptonic decays of charmed mesons and charmed baryons. In particular the semi-leptonic $D_s$ and $D$ decays can be used to obtain the decays constants $f_D$ and $f_{D_s}$.

![Graph](image)

**FIG. 2:** $|V_{td}/V_{ts}|$, $\sin \beta$ and $\sin \alpha$ plotted against the angle $\gamma$. The middle curves represent the computed quantities for $V_{ub} = 0.00347$, the global fit central value; the upper and lower curves for for $V_{ub} = 0.00363$ and $V_{ub} = 0.00335$ respectively. The deviation from the central value for $|V_{td}/V_{ts}|$ and $\sin \alpha$ is barely visible. The solid straight lines are the measured values with experimental errors represented by the gray area.

The matrix element $V_{td}$, gets a large imaginary part from the phase $\delta$ in $V_{ub}$ as given in Eq. (18). This large imaginary part is responsible for the large time-dependent CP-asymmetry in $B$...
decays. For the $|V_{td}/V_{ts}|$, Eq. (8) gives, with $\delta$ replaced by $\gamma$:

$$
|V_{td}/V_{ts}| = \frac{V_{us}}{V_{ud}} \frac{\sqrt{(1 - 2 K_d \cos \gamma + K_d^2)}}{\sqrt{(1 + 2 K_s \cos \gamma + K_s^2)}}
$$

(19)

and by neglecting the suppressed $K_c$ terms, from Eq. (15):

$$
\sin \beta = \frac{B_u \sin \gamma}{\sqrt{(1 - 2 K_t \cos \gamma + K_t^2)}}, \quad B_u = \frac{V_{ud} V_{ub} (1 - V_{ub}^2)}{V_{us} V_{cb} \sqrt{(1 - V_{ub}^2 - V_{cb}^2)}}
$$

$$
\sin \alpha = \frac{\sin \gamma}{\sqrt{(1 - 2 K_t \cos \gamma + K_t^2)}}.
$$

(20)

It is clear from Eq. (20) that $\sin \beta$ is a measure of $\gamma$, $V_{ub}$ and $V_{ub}/V_{cb}$. Numerically, $B_u = 0.366$, $\sin \beta \approx 0.366 \sin \alpha$ as seen from the plot in Fig. 2. On the other hand, the curve for $|V_{td}/V_{ts}|$ in the plot implies that $\gamma = (68.6 \pm 3)^\circ$. The middle curve for $\sin \beta$ shows the measured $\sin \beta$ at almost the same value of $\gamma$. The lower curve for $\sin \beta$ is a bit below the measured value but consistent with experiment. The upper curve is slightly above the measured value. This indicates that a value for $V_{ub}$ not too far from the global fit is favored.

In conclusion, by expressing the CKM matrix elements in terms of $V_{ud}$, $V_{us}$, $V_{ub}$, $V_{cb}$ and the CP-violating phase $\delta$ instead of using the usual mixing angle parametrization, we have obtained analytical expressions for the CKM matrix elements and the angles of the ($db$) unitarity angle. Using the global fit for $V_{ud}$, $V_{us}$, $V_{ub}$, $V_{cb}$ and the measured values for $|V_{td}/V_{ts}|$ and $\sin \beta$, we find $\gamma = (68.6 \pm 3)^\circ$ at which $\sin \beta$ and $\sin \alpha$ are near its maximum. Our simple and direct approach could be used in any further analysis of the CKM matrix elements with more precise measured values for $V_{ub}$ and $V_{cb}$.

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