Reduction of torsional vibrations of drivetrains of machines and plants using impulsive torques and characterization of related energy transfer effects

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Abstract
The reduction of torsional vibrations of drivetrains of machines and plants is of considerable interest in various fields of engineering, as they can have a negative influence not only on the drivetrain itself, but also on the driven machinery. Hence, a variety of countermeasures were developed to minimize such vibrations. In the present contribution, the possibilities of reducing torsional vibrations of drivetrains by introducing impulsive torques are investigated. The impulsive strength is chosen in a way that fully elastic impacts of a point mass with a rigid wall are mimicked, i.e., energy is neither fed to nor extracted from the mechanical system by the impulses. It is shown that therewith a transfer of vibration energy to higher modes is possible, where it is dissipated more effectively utilizing the enhanced damping capabilities of higher modes. A modal damping amplification factor is introduced allowing to characterize the energy transfer based on a mapping of the state-vector from one instant of time where an impulse is applied, to the next. It is demonstrated with a numerical example that the damping amplification factor allows to analyze modal energy transfer effects in mechanical systems exhibited to impulsive torques in an efficient manner.

Introduction
Mechanical systems with impacts have been investigated for a long time from the engineering community and the efforts to understand such systems and related effects are
still going on. This is due to the fact that they are of high practical relevance if we think about forging machines using impacts to plastically form hot or cold steel, or impact drilling machines where impacts are used to enhance the drilling process. In this class of machines, impacts are part of the machining process itself. By contrast, in a large number of cases, impacts are often unwanted and have to be avoided or at least their effects have to be minimized to ensure a proper functionality of machines. Examples are the phenomenon of gear rattling in drivetrains of cars, see [1] and [2], for example, or impacts of rotors with stationary parts in the field of rotordynamics, see [3].

Analytical as well as numerical treatment of systems with impacts can be very challenging due to the rich dynamical behavior they may exhibit. A comprehensive collection of corresponding literature can be found in [4] and references therein. Impacts in mechanical systems are frequently modeled by contracting the impact process to an instant of time resulting in an instantaneous change of the motion characteristics which is expressed by the well known coefficient of restitution. Such systems are classified as non-smooth mechanical systems, see [5] and the definition therein.

The reduction of torsional vibrations of drivetrains was investigated extensively in various fields of engineering and a variety of countermeasures were developed. This was also motivated by the fact that torsional vibrations of drivetrains can have a negative influence not only on the drivetrain itself, but also on the driven machinery. A classical countermeasure are torsional vibration dampers which are utilizing oil or elastomers to dissipate vibration energy. Moreover, they allow to detune the drivetrain and to avoid or reduce the excitation of resonance frequencies. In the automotive industry, typical devices are torsional clutch dampers or dual mass flywheels, see [1] and [2], for example.

A totally different concept is that utilizing the damping capabilities of higher modes of vibration of the mechanical structure. If vibration energy is transferred from lower to higher modes using any kind of active or passive device or mechanism, it can be dissipated in a shorter period of time. In [6], for example, continuous and periodic stiffness excitation at certain combination frequencies which were discovered by A. Tondl, see [7], was used to induce a periodic transfer of energy to higher modes of a classical torsional vibration system. The transfer of vibration energy to higher modes triggered by vibro-impacts was investigated in [8] by introducing an additional moveable mass to the mechanical structure, which was allowed to collide with the structure itself during the motion. It was shown that these collisions can lead to a scattering of vibration energy to higher modes. In [9], the effects of impacts to a multi-story structure using a rotating mass were addressed. In [10] and [11], the concept of nonlinear energy sink, see [12], was applied to an automotive drivetrain, where an additional inertia is coupled strongly nonlinear to the primary structure resulting in a broadband redistribution of vibration energy in the modal space of the structure, as well as a local dissipation of energy.

The suppression of fluid-flow induced self-excited vibrations of mechanical systems by introducing impulsive force excitation is investigated in [13]. Stability charts of a self-excited torsional system with two degrees of freedom which is subjected to impulsive torsional moments are presented in [14], showing the stability borders for variation of system parameters (moments of inertia, stiffness parameters). Moreover, a concept of a test-rig is presented in [14] suggesting two permanent excited synchronous electrical engines to realize the impulsive excitation. It is shown in [13] and [14], that unstable self-excited systems can be effectively stabilized by impulsive force respectively impulsive torque excitation. The capability of impulsive stiffness excitation to transfer vibration energy to higher modes was investigated in [15]. For such impulsive parametrically excited systems, a global modal damping coefficient was derived based on [8], representing a measure for the modal energy transfer induced by the parametric excitation, where a limitation to initial conditions representing shock loads was assumed. Possible modal energy redistributions induced by the impulsive parametric excitation were investigated numerically in [15]. A torsional system subjected to impulsive torques was investigated in [16], where a priori a limitation to impulses which mimic fully elastic impacts was made. The global modal damping measures were calculated for different values of the timespan between adjacent impulses, indicating that this measure is probably well-suited to identify modal energy flows. For one selected value of the timespan between impulses, the time series of the modal energy contents were provided.

In the present contribution, which is a further development of the work presented in [16], an analytical description of the effect of impulsive torques on the state of the mechanical system is presented first. A comparison with the classical theory of point masses impacting rigid walls is given, and it is shown that a corresponding coefficient of restitution for the impulsively excited system can be identified. Therefrom, a simple condition for the impulsive strength is presented which results in a conservation of the kinetic energy across an impulse, i.e., purely elastic impacts are mimicked where neither energy is fed to the mechanical system by the impulses, nor is extracted from it. Utilizing the classical impact theory provides detailed insights in the physical behaviour and the effect of the impulsive strength. To quantify the benefit of the impulsive excitation, a modal damping amplification factor is introduced based on the concept of equivalent damping measures, which was first presented in [8, 17] and [18], and a general derivation of
this measure by utilizing the variation of the total modal energy content due to the power of the modal dissipative forces is given. At the end of the analytical section, it is shown that the modal damping measures can be calculated using a discrete mapping of the state-vector from one impact to the next, which is computationally less expensive than integrating the equations of motion. A generic example of a drivetrain is introduced and the effective damping measures are numerically calculated for a wide range of the timespan between impulses. It is shown that a comparative assessment of the damping measures of all modes allows to draw conclusions about the modal energy flow induced by the impulsive excitation. This is in detail verified for several representative cases which underline that the timespan between impulses has to be selected carefully to achieve an efficient energy flow to higher modes and a faster decay of vibrations.

### 2 Analytical Investigations

In the following, a linear torsional mechanical system with \( n \) degrees of freedom, subjected to a series of impulsive torques of Dirac-delta form is investigated. The equations of motion of such a non-smooth system are of the form

\[
\ddot{q}(t) + C\dot{q}(t) + Kq(t) = T(t) = \sum_{k=1}^{K} \varepsilon_k \delta(t - t_k)f_i,
\]

where \( I, C \) and \( K \) represent the symmetric mass moments of inertia, damping, and stiffness matrices and \( q \) the vector of rotational degrees of freedom. Table 1 shows the corresponding nomenclature where all parameters and variables are listed including their physical units.

At instants of time \( t_k \), where \( 0 < t_1 < t_2 < \ldots \), and \( \Delta t = t_k - t_{k-1} = \) const. holds, Dirac-delta impulses \( \delta(t - t_k) \) with a strength of \( \varepsilon_k \) are applied to mass moments of inertia specified by the vector \( f = [f_1, f_2, \ldots f_n]^T \), \( f_i \in \{0,1\}, i = 1 \ldots n \).

First, the effect of a single impulse \( T(t) = \varepsilon_k \delta(t - t_k)f_i \) on the dynamics of the mechanical system is investigated. The equality of left and right side of Eq. (1) requires a Dirac-delta impulse behavior of \( \ddot{q}(t) \) at \( t = t_k \), resulting in a finite jump of \( \dot{q}(t_k) \), see also [19] and [20], for example. Hence, \( q(t) \) remains continuous at \( t_k \), i.e.

\[
q(t_{k+}) = q(t_{k-}),
\]

where the +/- signs indicate instants of time just after/before the impulse. Integrating Eq. (1) from \( t_{k-} \) to \( t_{k+} \) leads to the relation

\[
I(\dot{q}(t_{k+}) - \dot{q}(t_{k-})) = \varepsilon_k f_i.
\]

The impulsive strength \( \varepsilon_k \) is selected to be proportional to the velocity \( \dot{q}(t_{k-}) \) by introducing

\[
\varepsilon_k = -\mu_k \Phi \ddot{q}(t_{k-}).
\]
where $\mu_k$ represents the proportionality constant. With Eqs. (3) and (4), $\dot{q}(t_{k+})$ can be expressed as

$$\dot{q}(t_{k+}) = -(\mu_k I_j - 1) \dot{q}(t_{k-}).$$

(5)

where $E$ represents the identity matrix. For the sake of simplicity, it is assumed that in the following only one mass is subjected to a Dirac-delta impulse. Therewith, the velocity $\dot{q}_{k,j,+}$ of the mass moment of inertia $I_j$ just after the impulse at the instant of time $t_k$ is given by

$$\dot{q}_{k,j,+} = -\left(\frac{\mu_k}{I_j} - 1\right) \dot{q}_{k,j,-}.$$ 

(6)

Equation (6) is very similar to the equation describing the velocity of a mass impacting a rigid wall if we interpret $(\mu_k/I_j - 1)$ as coefficient of restitution. If

$$\frac{\mu_k}{I_j} - 1 = 0,$$

(7)

the velocity $\dot{q}_{k,j,+}$ just after the Dirac-delta impulse is equal to zero, which is also the case for purely plastic impacts, where all of the kinetic energy is converted into plastic deformation of the impacting mass.

In the case the condition

$$\frac{\mu_k}{I_j} - 1 = 1,$$

(8)

is fulfilled, the velocity just before and just after the impulse are equal in magnitude, but change their sign, i.e. if $\mu_k = 2I_j$ the Dirac-delta impulse mimics a purely elastic impact conserving the kinetic energy of the respective mass. In the following, it is assumed that for each impulse in Eq. (1), the condition Eq. (8) holds.

Although the total energy content is not changed by the impulses, transfers of kinetic energy in the modal space possibly occur, see also [14]. Therefore, the modal transformation $q = \Phi u$, where $\Phi = [\phi_1, \phi_2, \ldots \phi_n]$ comprises the mass-normalized eigenvectors $\phi_i$ of the undamped system and $u = [u_1, u_2, \ldots u_n]^T$ the modal coordinates $u_i$, is applied to Eq. (1) resulting in

$$\ddot{u}(t) + \alpha \Omega^2 \dot{u}(t) + \Omega^2 u(t) = T^+(t).$$

(9)

where a stiffness proportional damping $C = \alpha K$ was assumed. The diagonal matrix $\Omega$ consists of the natural frequencies $\Omega_i$ of the undamped mechanical system, i.e. $\Omega = \text{diag}(\Omega_1, \Omega_2, \ldots \Omega_n)$.

At this point the question about a measure which allows to quantify the effect of a sequence of impulses on the energy content and the vibration behavior of the mechanical system arises. In [17], effective stiffness and damping measures for mechanical systems consisting of a linear primary structure and an additional, strongly nonlinear coupled mass were presented. It is shown that these measures characterize very well the energy redistribution and the accompanied enhancement of the effective damping induced by the nonlinear attachment. This concept was adapted in [8] for investigating the redistribution and mitigation of shock-induced energy of a mechanical system consisting of a primary structure and an additional moveable mass which is allowed to collide with the primary structure. It was shown in [15] that these measures can be used similarly for systems with repeated impulsive stiffness excitation. In the following, this concept is described in a short manner, and thereafter, the focus is on the further development of this measure for non-smooth systems, especially taking into account that the state-vector of systems according to Eq. (1) can be simply mapped from one discrete instant of time to another.

Following [8] and [17], effective modal oscillators are introduced for each modal coordinate according to

$$\ddot{u}_i(t) + \gamma_{e,i} \dot{u}_i(t) + \sigma_{e,i} u_i(t) = 0, \quad i = 1, \ldots, n,$$

(10)

to approximate the behavior of the impulsively excited original system by a proper choice of the parameters $\gamma_{e,i}$ and $\sigma_{e,i}$, which are denoted as effective modal damping and effective modal stiffness. The energy content of each mode $i$ of the original system (9) changes due to dissipation and due to sudden energy transfers at the instants of time $t_k$ where impulses are applied. The basic idea is now to find a value for the effective damping measure $\gamma_{e,i}$ which allows the effective modal oscillators to exhibit (in an averaged manner) a similar characteristics of the total modal energy content compared to the original impulsively excited system. Moreover, to ensure that the effective oscillators initially possess the same total energy content than the impulsively excited system, the same initial conditions as well as an identical stiffness $\sigma_{e,i} = \Omega_i^2$ is assumed.

The corresponding total mechanical energy of the effective modal oscillators (10) is given by

$$E_{i,\text{tot}}(t) = T_i(t) + U_i(t),$$

(11)

with the kinetic energy $T_i(t)$ and the potential energy $U_i(t)$ according to

$$T_i(t) = \frac{1}{2} \dot{u}_i^2(t) \quad \text{and} \quad U_i(t) = \frac{1}{2} \sigma_{e,i} u_i^2(t).$$

(12)

The energy content of the effective oscillator Eq. (10) is related to the power of the dissipative forces by the equation

$$\dot{E}_{i,\text{tot}}(t) = -\gamma_{e,i} \dot{u}_i^2(t).$$

(13)
In the following, the objective is to calculate the effective damping coefficient $\gamma_{e,i}$ and thereby taking into account the abrupt change of the modal velocities at the instants of time where impulses are applied. Therefore, Eq. (13) is integrated over an initially arbitrary interval of time $[t_a, t_b]$ which gives

$$\int_{t_a}^{t_b} \dot{E}_{i,\text{tot}}(t) dt = -\gamma_{e,i} \int_{t_a}^{t_b} \dot{u}_i^2(t) dt$$

(14)

where $\gamma_{e,i} = \text{const}$, within the interval was assumed. Executing the integration on the left side of Eq. (14) and rearranging yields the effective damping coefficient in the form

$$\gamma_{e,i} = -\frac{E_{i,\text{tot}}(t_b) - E_{i,\text{tot}}(t_a)}{\int_{t_a}^{t_b} \dot{u}_i^2(t) dt}.$$  

(15)

Now, Eq. (15) is applied to a time interval starting just after an impulse and ending just after the next impulse, i.e. $t_a \rightarrow t_{k-1,+}$ and $t_b \rightarrow t_{k,+}$. Moreover, the structural modal damping $\alpha \Omega_i^2$ is introduced. Therewith, Eq. (15) can be written as

$$\gamma_{e,i} = -\frac{E_{i,\text{tot}}(t_{k,+}) - E_{i,\text{tot}}(t_{k-1,+})}{\int_{t_{k-1,+}}^{t_{k,+}} \alpha \Omega_i^2 \dot{u}_i^2(t) dt} \alpha \Omega_i^2.$$  

(16)

Thus, the effective damping coefficient results from an approximation which includes the autonomous phase from $t_{k-1,+}$ to $t_{k,+}$, where the equations of motion are decoupled and no energy transfers across the modes are possible, as well as the sudden change of the modal energy content at $t_k$ caused by the impulse. From the energy balance equation for the interval $[t_{k-1,+}, t_{k,+}]$, the difference of the total energy at the boundaries of the interval is given by

$$E_{i,\text{tot}}(t_{k,+}) - E_{i,\text{tot}}(t_{k-1,+}) = \Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) + \Delta E_i(t_k).$$  

(17)

where $\Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+})$ represents the energy dissipated by structural damping and $\Delta E_i(t_k)$ the change of the modal energy due to the impulse. Using Eq. (17) and taking into consideration that

$$\Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) = \int_{t_{k-1,+}}^{t_{k,+}} \alpha \Omega_i^2 \dot{u}_i^2(t) dt,$$

(18)

Eq. (16) can be written in the form

$$\gamma_{e,i} = \frac{\Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) + \Delta E_i(t_k)}{\Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) \alpha \Omega_i^2} \alpha \Omega_i^2.$$  

(19)

$$= h_i \alpha \Omega_i^2,$$

where the abbreviation $h_i$ was introduced, see also [16]. Depending on the sign of $\Delta E_i(t_k)$, $h_i$ becomes larger or smaller than one, see also Fig. 1. If, for example, energy is extracted from mode $i$ by an impulse, i.e. $\Delta E_i(t_k) < 0$, then $h_i > 1$. In this case, the energy content has decreased not only by structural damping ($\Delta E_{i,\text{diss}} < 0$), but also by the impulse, which requires a damping coefficient $\gamma_{e,i}$ of the effective oscillator that is larger than the structural damping $\alpha \Omega_i^2$ to approximate this change of the total mechanical energy.

As $\gamma_{e,i}$ is valid within the interval $[t_{k-1,+}, t_{k,+}]$, it has a kind of a local character. To characterize the global vibration behavior of the mechanical system, a single effective modal oscillator is introduced for each mode covering the overall timespan from $t = 0$ to an instant of time $t_N \gg 0$. Introducing Eq. (13) from $0$ to $t_N$ leads to the corresponding global effective damping coefficient

$$\tilde{\gamma}_{e,i} = -\frac{E_{i,\text{tot}}(t_N) - E_{i,\text{tot}}(0)}{\int_0^{t_N} \dot{u}_i^2(t) dt}.$$  

(20)

Provided that $E_{i,\text{tot}}(t_N) \approx 0$ at $t_N$, i.e. the vibrations have almost vanished, Eq. (20) simplifies to

$$\tilde{\gamma}_{e,i} \approx \frac{E_{i,\text{tot}}(0)}{\int_0^{t_N} \dot{u}_i^2(t) dt},$$  

(21)

see also [8]. Analog to Eqs. (16)–(19), the global effective damping coefficient $\tilde{\gamma}_{e,i}$ can be written in the form

$$\tilde{\gamma}_{e,i} = \frac{\sum_{k=1}^{N} \left[ \Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) + \Delta E_i(t_k) \right]}{\sum_{k=1}^{N} \Delta E_{i,\text{diss}}(t_{k-1,+}, t_{k,+}) \alpha \Omega_i^2} \alpha \Omega_i^2,$$

(22)

$$= \tilde{h}_i \alpha \Omega_i^2.$$
Therein, \( \tilde{h}_i \) is denoted as global damping amplification factor of mode \( i \). For the numerical evaluation of Eqs. (19) and (22), it is sufficient to calculate the state-vector of the mechanical system at the instants of time where impulses are applied, which can easily be done using an appropriate mapping. With Eqs. (2), (5) and (8), the state-vector \([q(t_k^\pm),\tilde{q}(t_k^\pm)]^T\) just after the impulse can be related to the state-vector just before by

\[
\begin{bmatrix}
q(t_k^-) \\
\dot{q}(t_k^-)
\end{bmatrix} = J \begin{bmatrix}
q(t_{k-1}^+) \\
\dot{q}(t_{k-1}^+)
\end{bmatrix}, \quad \text{with} \quad J = \begin{bmatrix} E & 0 \\ 0 & -2\pi^2 T \end{bmatrix},
\]

(23)

where the matrix \( J \) is denoted as jump transfer matrix, following [21] and [22]. The matrix \( E \) represents the identity matrix. Between adjacent impulses, the right side of the equations of motion (1) is equal to zero, i.e. the well-known relation

\[
\begin{bmatrix}
q(t_{k-1}^-) \\
\dot{q}(t_{k-1}^-)
\end{bmatrix} = D \begin{bmatrix}
q(t_{k-1}^+) \\
\dot{q}(t_{k-1}^+)
\end{bmatrix}, \quad \text{where} \quad D = e^{A\Delta t},
\]

(24)

with

\[
A = \begin{bmatrix} 0 & E \\ -I^1 K & -I^1 C \end{bmatrix},
\]

(25)

holds. Combining Eqs. (23) and (24) results in

\[
\begin{bmatrix}
q(t_k^-) \\
\dot{q}(t_k^-)
\end{bmatrix} = DJ \begin{bmatrix}
q(t_{k-1}^+) \\
\dot{q}(t_{k-1}^+)
\end{bmatrix}.
\]

(26)

With Eqs. (23) and (26), the state-vector \([q, \dot{q}]^T\) can be calculated at the instants of time \( t_k^\pm, k = 1, 2, ..., \) by simple mappings, which are computationally less expensive and allow a fast calculation of the modal damping measures \( y_{e,i} \) and \( \tilde{y}_{e,i} \). This is especially of importance if the effect of system parameters on modal energy transfers have to be investigated in order to optimize a mechanical system.

In the following, it is demonstrated with a numerical example that the modal damping amplification factors \( h_i \) and \( \tilde{h}_i \) are well suited to analyze non-smooth systems with periodic impulsive excitation.

### 3 Example

#### 3.1 Mechanical Model

Fig. 2 shows a schematic of the investigated mechanical system with five rotary degrees of freedom \( q_1, \ldots, q_5 \). Disk 1 is subjected to impulsive torsional moments \( T_1(t) \) which are equidistant in time, i.e. the equations of motion are of the form of Eq. (1), where \( f = [1, 0, ..., 0]^T \) holds.

The strength \( \varepsilon_k \) of the impulses is selected according to Eq. (4) with \( \mu_k = 2I_1 \) (see Eq. 8), i.e. each impulse just reverses the velocity of disk 1 mimicking an elastic impact and does neither extract energy from, nor feed energy to the disk.

To simplify the investigations, a sub-class of drivetrains characterized by identical mass-, damping- and stiffness-parameters, i.e., \( I_i = I, k_{i-1,i} = k \) and \( c_{i-1,i} = c, i = 1, \ldots, 5 \), is considered. Furthermore, for the sake of generality, the equations of motion are scaled by introducing a transformation to a dimensionless time using a reference mass moment of inertia \( I_{\text{ref}} \), and a reference torsional stiffness \( k_{\text{ref}} \). Relating the system parameters to this reference quantities leads to dimensionless equations of motion, i.e. all symbols in Eq. (1) now denote dimensionless quantities. The non-dimensional system parameters

\[
I = 1, k = 1, c = \alpha k, \quad \alpha = 0.01,
\]

(27)

are used for the following numerical simulations.

The initial conditions \( q(0) = 0 \) and \( \dot{q}(0) = [0, 0, 0, 0, 1]^T \) are used for all numerical investigations, representing some disturbance of the continuous operation of the drivetrain.

In the following it will be shown that the concept of effective modal oscillators as well as the introduced damping amplification factor which were proposed in section 2, are well suited to characterize the effect of a sequence of torsional impulses on the global dynamic behavior of mechanical systems.

#### 3.2 System Response for \( \Delta t = 2 \)

First, the drivetrain according to Fig. 2 was subjected to torsional moments mimicking fully elastic impacts, where the pulse-pause \( \Delta t \) was arbitrarily selected to be equal to 2. Fig. 3 depicts the corresponding results. Therein, the jagged lines represent the total modal energy content \( E_{\text{tot},i} \) of modes \( i = 1, \ldots, 5 \). Comparing \( E_{\text{tot},i} \) with the energy content \( E_{\text{tot},i}^{(\text{aut})} \) of the autonomous system, see grey-colored line, reveals the notable effect of the impulsive excitation on the modal energy content of the system. The autonomous sys-

![Fig. 2 Schematic of the investigated drivetrain with five degrees of freedom \( q_i, i = 1, \ldots, 5 \), and impulsive torsional excitation \( T_1(t) \) of disk 1](image-url)
and therewith, the energy contents $E_{\text{tot},i}$ recognize that the effective oscillators reflect the long-term oscillators (see dashed-dotted lines) were calculated. One by the impulsive excitation, which will be investigated in conclusions about the modal energy redistribution induced manner. A comparison of the impulsively excited system very well in an averaged N numerical evaluation of the impulsively excited system, and initial conditions, but without impulsive excitation. Af- tem is defined as system with identical system parameters and initial conditions, but without impulsive excitation. Af- ter numerical evaluation of the impulsively excited system, the global effective damping measures $\tilde{\gamma}_{c,i}$ (acc. to Eq. 22) and therewith, the energy contents $E_{\text{tot},i}^{(\text{eo})}$ of the effective oscillators (see dashed-dotted lines) were calculated. One recognizes that the effective oscillators reflect the long-term (or global) behavior of the modal energy content $E_{\text{tot},i}$ of the impulsively excited system very well in an averaged manner. A comparison of $E_{\text{tot},i}^{(\text{eo})}$ with $E_{\text{tot},i}^{(\text{aut})}$ allows to draw conclusions about the modal energy redistribution induced by the impulsive excitation, which will be investigated in the following. The global damping amplification factors $\tilde{h}_i$ according to Eq. (22) are

$$
\tilde{h}_1 = 3.07, \tilde{h}_2 = 1.50, \tilde{h}_3 = 0.99, \tilde{h}_4 = 0.38, \tilde{h}_5 = 0.26. \quad (28)
$$

The first mode is characterized by $\tilde{h}_1 = 3.07$, which means that the damping coefficient of the corresponding effective oscillator, $\tilde{\gamma}_{c,1}$, is about 3-times higher than the structural modal damping $\alpha \Omega_i^2$. Accordingly, the energy content of the effective oscillator of the first mode, $E_{\text{tot},1}^{(\text{eo})}$, decreases much faster compared to the energy content $E_{\text{tot},1}^{(\text{aut})}$ of the system without impulsive excitation (compare dashed-dotted and grey-colored lines in Fig. 3a). This means that the impulsive excitation efficiently extracts vibration energy from the first mode. A similar behavior, but to a smaller extent, is observed for the second mode ($\tilde{h}_2 = 1.50$), see Fig. 3b. The energy content of the effective oscillator of the third mode, $E_{\text{tot},3}^{(\text{eo})}$, almost coincides with that of the related autonomous system, $E_{\text{tot},3}^{(\text{aut})}$, see Fig. 3c. This is expressed by a global damping amplification factor $\tilde{h}_3$, which is close to 1 ($\tilde{h}_3 = 0.99$). Hence, on average, the third mode is unaffected by the impulsive excitation. The energy contents of the effective oscillators of the fourth and fifth mode are larger than that of the associated autonomous systems ($E_{\text{tot},4}^{(\text{eo})} > E_{\text{tot},4}^{(\text{aut})}$ and $E_{\text{tot},5}^{(\text{eo})} > E_{\text{tot},5}^{(\text{aut})}$, see Figs. 3d and e), i.e. energy extracted from the lower modes is transferred to fourth and fifth mode. Accordingly, the global damping amplification factors $\tilde{h}_4$ and $\tilde{h}_5$ are both significantly smaller than 1 ($\tilde{h}_4 = 0.38, \tilde{h}_5 = 0.26$).

In summary, the sequence of impulses induces a redistribution of vibration energy from lower to higher modes. Therewith, the damping capabilities of the higher modes of the mechanical structure can be exploited more effectively, resulting in a faster decay of vibrations. It has to be pointed out that, due to the fact that the impulses mimic fully elastic impacts, no energy crosses the system boundary during the impulses.

In the present case, the timespan $\Delta t$ between adjacent impulses was selected to be equal to 2, which was a more or less arbitrary choice just to demonstrate the characterization of modal energy transfers using the proposed modal damping amplification factor. Hence, it is obvious to investigate if there exist distinguished values of the pulse-pause $\Delta t$ for which the impulsive excitation is more efficient than for others, i.e. causes a faster decay of vibrations.

### 3.3 Effect of $\Delta t$ on the Modal Energy Transfer

In the following, the effect of the pulse-pause $\Delta t$ on the modal energy transfer is investigated in detail. To allow a direct comparison of the energy content of the impulsively excited system to the energy content of the mechani-
The total relative energy content of the mechanical system in the autonomous case, the total relative energy content according to

\[
E_{\text{tot,rel}} = \sum_{i=1}^{5} E_{\text{tot},i}/\sum_{i=1}^{5} E_{\text{tot},i}^{(\text{aut})},
\]

is introduced. Fig. 4 depicts \(E_{\text{tot,rel}}\) for \(\Delta t \in [0.1,12]\) (step-size of 0.1) and \(t \in [0,500]\). Isolines for \(E_{\text{tot,rel}} = 0.2, 0.4, 0.6\) and 0.8 are included to facilitate the interpretation. One observes that \(E_{\text{tot,rel}}\) is characterized by pronounced minima (dark-shaded areas) where \(E_{\text{tot,rel}} \ll 1\), i.e. the energy content of the impulsively excited system is much smaller than that of the autonomous system, which is indicating an efficient transfer of vibration energy to higher modes. The minima are emerging around certain values of the pulse-pause \(\Delta t\) (see dashed lines in Fig. 4). To localize the minima more exactly, \(E_{\text{tot,rel}}\) is calculated for constant instants of time \(t = 0, 125, 250, 375\) and 500 for \(\Delta t \in [0,12]\), see Fig. 5. Local minima of \(E_{\text{tot,rel}}\) are observed near \(\Delta t = 2.90, 3.72, 7.31\) and 8.50 for example. Selecting a pulse-pause equal to one of these values results in a decay of vibrations much faster than that of the autonomous system. An opposite behavior is observed for \(\Delta t = 5.6\) and \(\Delta t = 11.3\), where \(E_{\text{tot,rel}} > 1\) within the timespan under investigation. This means that the energy-content of the impulsively excited system is (for a limited period of time) higher than that of the autonomous system. This is due to the fact that the impulsive excitation initially transfers energy from higher to lower modes, where it takes longer to be dissipated.

In the following, the cases where \(\Delta t = 3.72, 5.60\) and 7.31, corresponding to pronounced minima and maxima of \(E_{\text{tot,rel}}\) are investigated in detail.

A pulse-pause of \(\Delta t = 3.72\) or \(\Delta t = 7.31\) results in a similar behavior of \(E_{\text{tot,rel}}\), see Fig. 6. The total energy content drops very fast in both cases, where the most efficient transfer of energy is observed for \(\Delta t = 7.31\). In this case, \(E_{\text{tot,rel}}\) has a negligible value of about 0.018 at \(t = 300\), i.e. only 1.8% of the energy content of the autonomous system. The damping capabilities of the mechanical structure, especially those of the higher modes, are utilized more effectively than without impulsive excitation. To underline this, Fig. 7 provides a comparison of the corresponding total power of the dissipative forces of the mechanical system without (a), and with impulsive excitation (b), and of the total energy content (c). It can be seen that the impulsively excited system possesses an exceptional higher dissipative power than the autonomous system up to \(t \approx 50\), compare Fig. 7a and b. Thereafter, up to \(t \approx 125\), \(P_{\text{diss,tot}}\) is still much higher than \(P_{\text{diss,aut}}\).

As mentioned before, impulsive excitation can also cause a temporal increase of the total vibration energy compared to the autonomous case, if the pulse-pause is chosen in an unfavorable way, see \(E_{\text{tot,rel}}\) for \(\Delta t = 5.6\) in Fig. 6, for example.

The present investigations demonstrated that the pulse-pause has to be selected very carefully to achieve the desired behavior of the mechanical system. Hence, it could be useful to have a measure at hand that provides some information about the global effect of an equidistant sequence of torsional impulses. In the following, the modal damping amplification factors \(\tilde{h}_i\) are discussed in detail. It will be...
shown that they provide a solid basis to assess impulsively excited mechanical systems.

Fig. 8 shows the global damping amplification factor $\tilde{h}_i$ for modes $i = 1, \ldots, 5$, for different values of the pulse-pause $\Delta t$. For each $\Delta t$, the mechanical system was numerically evaluated until the vibrations have decayed to a negligible value, and afterwards, the damping amplification factors $\tilde{h}_i$ were calculated according to Eq. (22). As described in section 2, it is therefore sufficient to calculate the state-vector of the system at the instants of time impulses are applied using the described mapping.

From a practical point of view, it is often desirable to reduce lower modes vibrations as fast as possible, as they can cause serious damages of mechanical structures. A fast decay of vibrations of a certain mode $i$ correlates with a value of the corresponding damping amplification factor $\tilde{h}_i$ which is much higher than 1. In Fig. 8, $\tilde{h}_1$ has a local maximum at $\Delta t = 3.72$, where $\tilde{h}_1 = 33.280 \gg 1$, see also Tab. 2.

The corresponding vibrations $u_1$ of the first mode, as well as the energy content $E_{\text{tot}1}$ are shown in Fig. 9a and b. Therein, the gray colored lines represent the behavior of the autonomous system. The impulsive excitation extracts very efficiently energy from the first mode, as $E_{\text{tot}1}$ decreases very fast compared to the autonomous system. The first mode vibrations $u_1$ have almost vanished at $t \approx 500$, whereas the amplitude of the autonomous system is still about 80% of the amplitude at the beginning. The energy content $E_{\text{tot}2}$ of the second mode decreases also significantly faster than the energy content $E_{\text{tot}2}^{(\text{aut})}$ of the corresponding autonomous system, see Fig. 9d, but not to that extent compared to the first mode. Accordingly, the damp-
**Fig. 9** Modal vibrations $u_i$ and total modal energy content $E_{\text{tot},i}$ of modes $i = 1, \ldots, 5$ of impulsively excited system where the pulse-pause $\Delta t = 3.72$ (black) and comparison with autonomous system (grey colored).

**Fig. 10** Modal vibrations $u_i$ and total modal energy content $E_{\text{tot},i}$ of modes $i = 1, \ldots, 5$ of impulsively excited system where the pulse-pause $\Delta t = 7.31$ (black) and comparison with autonomous system (grey colored).
ing amplification factor $1 < \bar{h}_2 \ll \bar{h}_1$, see Tab. 2. The third mode vibrations are characterized by a global damping amplification factor of $\bar{h}_3 = 0.794 < 1$, i.e. the third mode receives energy from the other modes. A quite similar behavior shows the fourth mode, characterized by $\bar{h}_4 = 0.498$.

Most of the energy extracted from the lower modes is transferred to the fifth mode. The energy content $E_{tot}$ is significantly larger than that of the autonomous system at almost any point in time, which is reflected by a low damping amplification factor $\bar{h}_5 = 0.071 \ll 1$. Accordingly, fifth mode vibrations $u_5$ with higher amplitudes than in the autonomous case are observed, see Fig. 9i and j.

The second case which will be investigated in detail is that with a pulse-pause of $\Delta t = 7.31$. Fig. 6 implies that in this case more energy is transferred to higher modes compared to the previous one, as the total energy content $E_{tot}$ drops faster. Comparing the global damping amplification factors $\bar{h}_i$ in Tab. 2 for both cases confirms this assumption. For the first mode, $\bar{h}_1(\Delta t = 7.31) \approx \bar{h}_1(\Delta t = 3.72)$ holds. The modal vibrations $u_1$ and the total energy content $E_{tot}$ show a very similar behavior, compare Fig. 9a and b with Fig. 10a and b. On the contrary, for the second mode $\bar{h}_2(\Delta t = 7.31) > \bar{h}_2(\Delta t = 3.72)$ holds, i.e. more energy is extracted from the second mode if $\Delta t = 7.31$. Accordingly, the total energy content of the second mode, $E_{tot}$, and therewith the modal vibrations $u_2$ show a faster decay, compare Fig. 9c and d and Fig. 10c and d. The third mode shows, from a global point of view, a qualitatively different behavior for both values of $\Delta t$. For $\Delta t = 3.72$, the third mode receives energy from other modes, as $\bar{h}_3 < 1$, whereas for $\Delta t = 7.31$, $\bar{h}_3 > 1$, i.e. energy is extracted from this mode, see Tab. 2. As in both cases, $\bar{h}_3$ is relatively close to 1, the contribution of the third mode on the global modal energy redistribution is of minor importance, see also Figs. 9 and 10.

The damping amplification factors of the fourth mode, $\bar{h}_4$, are nearly identical, see Tab. 2, whereas those of the fifth mode, $\bar{h}_5$, show a significant difference. As $\bar{h}_5(\Delta t = 7.31) \ll \bar{h}_5(\Delta t = 3.72)$, the fifth mode receives much more energy from the other modes in the case $\Delta t = 7.31$. This is underlined by the timeseries of $E_{tot}$ and $u_5$, see Figs. 9 and 10.

The last case which will be investigated in detail is that with a pulse-pause of $\Delta t = 5.60$. The modal displacements $u_i$ and energy contents $E_{tot}$ can be found in Fig. 11. The sequence of elastic impulses causes $E_{tot} > E_{tot}^{(aut)}$ up to about $t \approx 450$, i.e. vibration energy is transferred

**Table 2** Global damping amplification factor $\bar{h}_i$ for modes $i = 1 \ldots 5$ for $\Delta t = 3.72, 5.60$ and 7.31

| $\Delta t$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $h_5$ |
|-----------|-------|-------|-------|-------|-------|
| 3.72      | 33.280| 2.804 | 0.794 | 0.498 | 0.071 |
| 5.60      | 1.361 | 0.863 | 1.123 | 1.556 | 0.220 |
| 7.31      | 31.980| 7.621 | 1.353 | 0.543 | 0.048 |
Comparing the damping amplification factors $\hat{h}_i$ of all three investigated cases, a pulse-pause of $\Delta t = 7.31$ is possibly most interesting from a practical point of view. In this case, the first three modes are characterized by damping amplification factors which are larger than one, i.e. energy is extracted from these modes and transferred to modes 4 and 5. Moreover, the extraction of energy – especially from the first and second mode – is very efficient, as $\hat{h}_1 \gg 1$ and $\hat{h}_2 \gg 1$ holds. Fig. 12 depicts the corresponding physical coordinates $q_i$. One observes that all vibrations $q_1...q_5$ decrease much faster as without impulsive excitation.

4 Conclusion

It was shown in this contribution that impulsively applied torsional moments mimicking fully elastic impacts are capable of transferring vibration energy efficiently to higher modes of vibration, where the instant of time between adjacent impulses is of significant importance. This allows to utilize the damping capabilities of higher modes of the drivetrain more effectively, resulting in a much faster decay of vibrations compared to the impulse-free case. A modal damping amplification factor is introduced based on the modal energy change due to the impulsive excitation. It was shown that this damping amplification factor allows to characterize if energy was extracted from the respective mode, or if energy was fed to the mode, and that a comparative assessment of the damping amplification factors of all modes allows to draw conclusions about the modal energy flow induced by the impulsive excitation. It was also shown that the timespan between impulses has to be selected very carefully to induce the desired energy-flow to higher modes, and not to initiate the opposite effect – an energy transfer from higher to lower modes. Moreover, it was pointed out that for the calculation of the damping amplification factor it is sufficient to compute the state of the mechanical system at the instants of time where impulses are applied. This can be done in a numerically efficient way using the described mapping of the state-vector.

The practical realization of the proposed impulsive excitation of drivetrains is in any case challenging. Actuators with the ability to apply impulsive torques with a duration smaller than the time-period of the highest relevant torsional mode of vibration of the mechanical system are required, as well as appropriate controllers. A basic idea which is currently investigated is to use the magnetic field between rotor and stator of electric drive engines to generate the required impulsive excitation. This could probably open up the wide field of electrically driven machines for the proposed technique.
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Conflict of interest  I herewith state that there is no conflict of interest.

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