Parameters for which the Lawrence-Krammer representation is reducible

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Abstract

We show that the representation, introduced by Lawrence and Krammer to show the linearity of Braid groups, is generically irreducible, but for that for some values of its two parameters when these are specialized to complex numbers, it becomes reducible. To do so, we construct a representation of degree $\frac{n(n-1)}{2}$ of the BMW algebra of type $A_{n-1}$ inside the Lawrence-Krammer space. As a representation of the Braid group on $n$ strands, it is equivalent to the Lawrence-Krammer representation where the two parameters of the BMW algebra are related to the two parameters of the Lawrence-Krammer representation. We give the values of the parameters for which the representation is reducible and give the proper invariant subspaces in some cases. We use this representation to show that for these special values of the parameters and other values, the BMW algebra of type $A_{n-1}$ is not semisimple.

1 Introduction

1.1 Introduction and main results

In [5], Daan Krammer constructed a representation of the Braid group in order to show that it is linear. Since this representation was earlier introduced by Ruth Lawrence in [6], it is called the Lawrence-Krammer representation. In this paper, we examine a representation of degree $\frac{n(n-1)}{2}$ of the BMW algebra of type $A_{n-1}$ in the Lawrence-Krammer space. As a representation of the Braid group on $n$ strands, it is equivalent to the Lawrence-Krammer representation (abbreviated L-K representation). By studying this representation we show that the L-K representation is generically irreducible but that for some values of its two parameters when these are specialized to complex numbers, it becomes reducible. Throughout the paper, we let $l$, $m$ and $r$ be three nonzero complex
parameters, where $m$ and $r$ are related by $m = \frac{1}{r} - r$. We define $H_{F, r^2}(n)$ as the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ over the field $F = \mathbb{Q}(l, r)$ with generators $g_1, \ldots, g_{n-1}$, that satisfy the Braid relations and the relation $g_i^2 + m g_i = 1$ for all $i$. Our definition is the same as the definition of $[8]$. After the generators have been rescaled by a factor $\frac{1}{r}$. Our main result is as follows.

**Theorem 1. (Main theorem)**

Let $n$ be an integer with $n \geq 3$ and let $m, l$ and $r$ be three nonzero complex parameters, where $m$ and $r$ are related by $m = \frac{1}{r} - r$. Assume that $H_{F, r^2}(n)$ is semisimple, and so assume that $r^{2k} \neq 1$ for every integer $k \in \{1, \ldots, n\}$.

When $n \geq 4$, the Lawrence-Krammer representation of the BMW algebra of type $A_{n-1}$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is irreducible, except when $l \in \{r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, when it is reducible.

When $n = 3$, the Lawrence-Krammer representation of the BMW algebra of type $A_2$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is irreducible, except when $l \in \{-r^3, \frac{1}{r}, 1, -1\}$, when it is reducible.

A consequence of this result and of the method that we use is the following.

**Theorem 2.**

Let $n$ be an integer with $n \geq 4$ and let $l$, $m$ and $r$ be three nonzero complex parameters, where $m$ and $r$ are related by $m = \frac{1}{r} - r$.

Suppose $n \geq 4$. If $r^{2k} = 1$ for some $k \in \{2, \ldots, n\}$ or if $l$ belongs to the set of values $\{r, -r^3, \frac{1}{r}, -\frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r}, -\frac{1}{r}\}$, the BMW algebra of type $A_{n-1}$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is not semisimple.

(Case $n = 3$). If $r^6 = 1$ or $r^9 = 1$ or if $l \in \{-r^3, \frac{1}{r}, 1, -1\}$, the BMW algebra of type $A_2$ with parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$ is not semisimple.

In [11], Wenzl states that the BMW algebra of type $A_{n-1}$ is semisimple except possibly if $r$ is a root of unity or $l = r^n$, for some $n \in \mathbb{Z}$. Here, Theorem 2 gives instances of when the algebra is not semisimple. The result of this theorem is also contained in the recent work of Hebing Rui and Mei Si (see [10]). They use the representation theory of cellular algebras.

### 1.2 The method

We show that the action on a proper invariant subspace of the Lawrence-Krammer space must be an Iwahori-Hecke algebra action.

First, we study the Iwahori-Hecke algebra representations of small degrees and investigate whether they may occur inside the L-K space and if so for which values of $l$ and $r$. We will denote by $V^{(n)}$ the L-K space. We show that if there exists a one-dimensional invariant subspace inside $V^{(n)}$, it forces the value $\frac{1}{r}$ for $l$, except when $n = 3$ when it forces $l \in \{-r^3, \frac{1}{r}\}$. Conversely, for these values of $l$ and $r$, there exists a one-dimensional invariant subspace of $V^{(n)}$ and the representation is thus reducible. Similarly, we show that if there exists an irreducible $(n-1)$-dimensional invariant subspace inside $V^{(n)}$, it forces $l = \frac{1}{r}$ or $l = -\frac{1}{r}$ in the case when $n \neq 4$ and $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}\}$ in the case when $n = 4$. Conversely, for each of these values of $l$ and $r$, there exists an irreducible $(n-1)$-dimensional subspace of $V^{(n)}$, which shows the reducibility of the representation in these cases as well.
Second, we identify a proper invariant subspace of $\mathcal{V}^{(n)}$ which is nontrivial when $l = r$ (case $n \geq 4$) or $l = -r^3$. This shows that the representation is also reducible in these cases.

Third, we study in detail the small cases $n \in \{3, 4, 5, 6\}$.

At last, when $n \geq 7$, we use a result from representation theory: the irreducible representations of $H_{F,r}(n)$ have degrees $1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}$, or degree greater than $\frac{(n-1)(n-2)}{2}$, except in the case $n = 8$, when they have degrees $1, 7, 14, 20, 21$ or degrees greater than $21$. We use this fact, and proceed by induction on $n \geq 5$ to show that if $\mathcal{V}^{(n)}$ is reducible, it forces $l \in \{r, -r^3, \frac{1}{r-2}, \frac{1}{r+1}, -\frac{1}{r-2}, -\frac{1}{r+1}\}$. To do so, we use the fact that if the dimension of a proper invariant subspace $W$ of $\mathcal{V}^{(n)}$ is large enough, then the intersections $W \cap \mathcal{V}^{(n-1)}$ and $W \cap \mathcal{V}^{(n-2)}$ are nontrivial.

### 1.3 Definitions

#### 1.3.1 The BMW algebra

We recall below the defining relations of the BMW algebra $B(A_{n-1})$ (or simply $B$) of type $A_{n-1}$ with nonzero complex parameters $l$ and $m$ over the field $\mathbb{Q}(l, r)$, where $r$ is a root of the quadratic $X^2 - mX + 1$. This algebra has two sets of $(n-1)$ elements, namely the invertible $g_i$’s that satisfy the Braid relations (1) and (2) and generate the algebra and the $e_i$’s that generate an ideal. For nodes $i$ and $j$ with $1 \leq i, j \leq n - 1$, we will write $i \sim j$ if $|i - j| = 1$ and $i \not\sim j$ if $|i - j| > 1$.

\[
\begin{align*}
g_i g_j &= g_j g_i & \text{if } i \not\sim j \\
g_i g_j g_i &= g_j g_i g_j & \text{if } i \sim j \\
e_i &= \frac{1}{m} (g_i^2 + mg_i - 1) & \text{for all } i \\
g_i e_i &= l^{-1} e_i & \text{for all } i \\
e_i g_j e_i &= l e_i & \text{if } i \sim j
\end{align*}
\]

We will also use some direct consequences of these defining relations (see [2], Proposition 2.1):

\[
\begin{align*}
e_i g_i &= l^{-1} e_i & \text{for all } i \\
g_i^2 &= 1 - mg_i + ml^{-1} e_i & \text{for all } i \\
g_i^{-1} &= g_i + m - m e_i & \text{for all } i
\end{align*}
\]

as well as the following ”mixed Braid relations” (see [2], Proposition 2.3):

\[
\begin{align*}
g_i g_j e_i &= e_j e_i & \text{if } i \sim j \\
g_i e_j e_i &= g_j e_i + m(e_i - e_j e_i) & \text{if } i \sim j
\end{align*}
\]

This algebra was shown by Morton and Wassermann to be isomorphic to the tangle algebra of Morton and Traczyk (see [9]). All the algebraic relations given in this paper have a geometric formulation in terms of tangles. In particular, we will use the tangles in §3.4.
1.3.2 The Lawrence-Krammer space

We now recall some terminology associated with root systems of type $A_{n-1}$. Let $M = (m_{ij})_{1 \leq i, j \leq n-1}$ be the Coxeter matrix of type $A_{n-1}$.

Let $(\alpha_1, \ldots, \alpha_{n-1})$ be the canonical basis of $\mathbb{R}^{n-1}$ and let’s define a bilinear form $B_M$ over $\mathbb{R}^{n-1}$ by:

$$B_M(\alpha_i, \alpha_j) = -\cos \left( \frac{\pi}{m_{ij}} \right)$$

By the theory in [1], $B_M$ is an inner product that we will simply denote by $(\cdot, \cdot)$.

Let $r_i$ denote the reflection with respect to the hyperplane $\text{Ker}(\alpha_i)$ of $\mathbb{R}^{n-1}$ and so:

$$\forall x \in \mathbb{R}^{n-1}, r_i(x) = x - 2(\alpha_i|x)\alpha_i$$

Finally, let $\phi^+$ denote the set of $\frac{n(n-1)}{2}$ positive roots:

$$\phi^+ = \{\alpha_1, \alpha_2 + \alpha_1, \alpha_3 + \alpha_2, \alpha_3 + \alpha_2 + \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1} + \alpha_{n-2}, \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1\}$$

We define $V^{(n)}$ as the vector space over $\mathbb{Q}(l, r)$ with basis the $x_\beta$’s, indexed by the positive roots $\beta \in \phi^+$. Thus, $\text{dim}_F V^{(n)} = |\phi^+| = \frac{n(n-1)}{2}$. The so-defined space $V^{(n)}$ is the Lawrence-Krammer space.

To each positive root, we associate an element of the BMW algebra in the following way:

- To $\alpha_1$ we associate $e_1$.
- To $\alpha_i = r_{i-1} \cdots r_1 r_i \cdots r_2(\alpha_1)$, we associate the algebra element $g_{i-1} \cdots g_1 g_i \cdots g_2 e_1$, which after using the defining rules (1) and (9) above simplifies to $e_i e_1$.
- To $\alpha_j + \cdots + \alpha_i = r_j \cdots r_{i+1}(\alpha_i)$ where $j \geq i + 1$, we associate the algebra element $g_j \cdots g_{i+1} e_i \cdots e_1$.

2 The representation

2.1 The BMW left module

In what follows, $F$ still denotes the field $\mathbb{Q}(l, r)$ and $H$ denotes the Hecke algebra of the symmetric group $\text{Sym}(n-2)$ over the field $F$ with generators $g_3, \ldots, g_{n-1}$ and relations the Braid relations and the relations $g_i^2 + mg_i = 1$ for each $i$. As $r^2 + mr - 1 = 0$, our base field $F$ is a one-dimensional $H$-module for the action given by $g_i.1 = r$ for every integer $i$ with $3 \leq i \leq n-1$. We define $B_1$ as the quotient of two left ideals of $B$:

$$B_1 = Be_1/ < Be_i e_1 >_{i=3 \ldots n-1}$$

Since $e_i$ commutes with $e_j$ for any $i \neq j$, we have for each node $i$ with $3 \leq i \leq n-1$:

$$e_1(g_i^2 + mg_i - 1) = \frac{1}{m}e_1 e_i = 0 \text{ in } B_1$$
Then, \( B_1 \) is a right \( H \)-module. Thus, \( B_1 \) is a left \( B \)-module and a right \( H \)-module. Since \( F \) is an \( H \)-module, we get a left \( B \)-module by considering the tensor product

\[ B_1 \otimes_H F \]

This \( B \)-module is precisely the left representation of \( B \) that we study in this paper. Its degree is \( \frac{n(n-1)}{2} \) since by the forthcoming computations, we have:

\[
B_1 = \text{Span}_P(e_1, e_2 e_1, g_2 e_1, e_3 e_2 e_1, g_3 e_2 e_1, \ldots, \\
e_{n-1} \ldots e_1, g_{n-1} e_{n-2} \ldots e_1, g_{n-1} \ldots g_2 e_1, \ldots, \\
e_{n-1} \ldots e_1 \otimes_H 1, g_{n-1} e_{n-2} \ldots e_1 \otimes_H 1, g_{n-1} \ldots g_2 e_1 \otimes_H 1, \ldots)
\]

We will denote by \( B \) the spanning set above. The action of the \( g_k \)'s on the elementary tensors \( b \otimes_H 1 \), where \( b \in B \), was computed in section 2.2 below. These computations show in particular that if \( G_1(n) \) denotes the matrix of the left action of \( g_1 \) on the vectors

\[
e_1 \otimes_H 1, e_2 e_1 \otimes_H 1, g_2 e_1 \otimes_H 1, e_3 e_2 e_1 \otimes_H 1, g_3 e_2 e_1 \otimes_H 1, g_3 g_2 e_1 \otimes_H 1, \ldots, \\
e_{n-1} \ldots e_1 \otimes_H 1, g_{n-1} e_{n-2} \ldots e_1 \otimes_H 1, g_{n-1} \ldots g_2 e_1 \otimes_H 1
\]

of \( B_1 \otimes_H F \), and if \( \det G_1(n) \) denotes its determinant, we have:

\[
\det G_1(3) = -1 \\
\det G_1(n) = -r^{n-3} \det G_1(n-1) \quad \forall n \geq 4
\]

Thus, the determinant of \( G_1(n) \) is nonzero, which shows that these vectors are linearly independent.

We notice that there is a bijection between \( B \) and the set of positive roots \( \phi^+ \), as described in the previous section. Let's name this bijection \( u \).

### 2.2 The action by the \( g_k \)'s

We describe further the representation by computing the action of the \( g_k \)'s on the elementary tensors \( b \otimes_H 1 \), where \( b \) is an algebra element in the spanning set \( B \). An element \( b \) of \( B \) is of the form:

\[
g_j \ldots g_{i+1} e_i \ldots e_1 \quad \text{with} \quad j > i \geq 1 \quad \text{(or simply} \quad g_{j,i+1} e_{i,1}) \quad (I)
\]

that we will refer to as of type \( I \), or of the form:

\[
e_i \ldots e_1 \quad \text{with} \quad i \geq 1 \quad \text{(or simply} \quad e_{i,1}) \quad (II)
\]

referred to as of type \( II \). For \( i \geq j \), we set \( g_{i,j} = g_i \ldots g_j \) and \( e_{i,j} = e_i \ldots e_j \), where \( g_{i,j} \) and \( e_{i,j} \) are simply \( g_i \) and \( e_i \) respectively. When \( i < j \), we define \( g_{i,j} \) to be the identity.

In what follows, we fix \( i \) and \( j \) as in \( (I) \) and \( (II) \). There are several cases.

#### 2.2.1 Action by \( g_{i-1} \) (Case \( A \))

Let's first compute the action of \( g_{i-1} \) for \( i \geq 2 \) on elements of both types. We have for the type \( I \):

\[
g_{i-1,b} = g_{j,i+1}g_{i-1,e_{i,1}} \quad \text{by} \ (1) \\
= g_{j,i}e_{i-1,1} + m e_{i-1,1}g_{j,i+1} - m g_{j,i+1}e_{i,1} \quad \text{by} \ (10) \quad \text{and} \ (1)
\]
And for the type (II):

\[ g_{i-1,b} = g_i e_{i-1,1} + m e_{i-1,1} - m e_{i,1} \quad \text{by (10)} \]

Thus, we get:

\[ g_{i-1}(b \otimes_H 1) = \begin{cases} 
  g_{j,i} e_{i-1,1} \otimes_H 1 + m r^{-1} e_{i-1,1} \otimes_H 1 - m g_{j,i+1} e_{i,1} \otimes_H 1 \\
  g_{i-1,1} \otimes_H 1 + m e_{i-1,1} \otimes_H 1 - m e_{i,1} \otimes_H 1
\end{cases} \]

where the first line refers to type (I) and the second line to type (II). For future references, we name these two equalities \( A(I) \) and \( A(II) \) respectively.

### 2.2.2 Action by \( g_i \) (Case B)

We have for types (I) and (II) respectively:

\[ g_i.b = g_{j,i} e_{i+1,1} \quad \text{by (9)} \]

\[ g_i.b = i^{-1} e_{i,1} \quad \text{by (4)} \]

Thus, we get:

\[ g_i.(b \otimes_H 1) = \begin{cases} 
  g_{j,i+2} e_{i+1,1} \otimes_H 1 & B(I) \\
  i^{-1} e_{i,1} \otimes_H 1 & B(II)
\end{cases} \]

Notice that if \( j = i + 1 \), expression \( B(I) \) reduces to \( e_{i+1,1} \otimes_H 1 \).

### 2.2.3 Action by \( g_j \) (Case C)

Let’s first deal with Type (I). We have by (7):

\[ g_j.b = g_{j-1,i+1} e_{i,1} - m g_{j,i+1} e_{i,1} + ml^{-1} e_{j,1} e_{j-1,i+1} e_{i,1} \]

We will rearrange the last term of the sum and to do so, we will need more mixed braid relations, as in the following lemma.

**Lemma 1.**

\[ e_j e_i = e_i e_j^{-1} \text{ when } i \sim j \quad (11) \]

\[ e_i e_j e_i = e_i \text{ when } i \sim j \quad (12) \]

**Proof.** These are equalities (8) and (10) of Proposition 2.3 in [2]. □

Using the relations of Lemma 1, we now give a new expression for \( e_j g_{j-1,i+1} e_{i,1} \).

**Lemma 2.**

\[ e_j g_{j-1,i+1} e_{i,1} = e_{j-1} g_{j-1} \cdots g_{i+2}^{-1} \quad (13) \]

**Proof.** Using Lemma 1, we replace \( e_j \) with \( e_j e_{j-1} e_j \), then replace \( e_{j-1} e_j g_{j-1} \) with \( e_{j-1} g_{j-1}^{-1} \) to get:

\[ e_j g_{j-1,i+1} e_{i,1} = e_j e_{j-1} g_{j-2,i+1} e_{i,1} g_{j}^{-1} \]

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Proceeding inductively, we obtain (13).

When $b$ is of the second type, we simply have:

$$g_{j,b} = \begin{cases} e_{i,1}g_j & \text{if } j > i + 1 \\ g_{i+1}e_{i,1} & \text{if } j = i + 1 \end{cases}$$

In the following expressions, the first line is for type (I) and the next two are for type (II).

$$g_j(b \otimes_H 1) = \begin{cases} g_{j-1,i+1}e_{i,1} \otimes_H 1 - m g_{j,i+1}e_{i,1} \otimes_H 1 + \frac{m}{r_{j-i-1}} e_{j,1} \otimes_H 1 & \text{when } j > i + 1 \\ r e_{i,1} \otimes_H 1 & \text{when } j = i + 1 \end{cases}$$

We will refer to these equations as $(CI), (CII)$ and $(CIII)$ respectively.

### 2.2.4 Action by $g_{j+1}$ (Case D)

Since

$$g_{j+1,b} = \begin{cases} g_{j+1,j+1}e_{i,1} & \text{when } b \text{ is of type (I)} \\ e_{i,1}g_{j+1} & \text{when } b \text{ is of type (II)} \end{cases}$$

we simply get:

$$g_{j+1,b} = \begin{cases} g_{j+1,i+1}e_{i,1} \otimes_H 1 & (DI) \\ r e_{i,1} \otimes_H 1 & (DII) \end{cases}$$

### 2.2.5 Action by $g_k$ where $k \not\in \{i - 1, i, j, j + 1\}$ (Case E)

- Suppose first $k < i - 1$ and $b$ of type (I). We compute:

$$g_k.b = g_{j,i+1}e_{i,k+2}g_{k+1}e_{k,1}$$

Expanding $g_k e_{k+1}e_k$ with (10) yields:

$$g_k.b = g_{j,i+1}e_{i,k+2}g_{k+1}e_{k,1} + m g_{j,i+1}e_{i,k+2}g_{k+1}e_{k,1} - m g_{j,i+1}e_{i,1}$$

Since $e_{k+2}e_k = 0$ in $B_1$, this expression simplifies as follows:

$$g_k.b = g_{j,i+1}e_{i,k+2}(g_{k+1} - m e_{k+1})e_{k,1}$$

Replacing $g_{k+1} - m e_{k+1} = g_{k+1} - m$ with (8) and simplifying with $e_{k+2}e_k = 0$, we then obtain:

$$g_k.b = g_{j,i+1}e_{i,k+2}g_{k+1}e_{k,1}$$

Applying equality (11) to $e_{k+2}g_{k+1}$ now yields the new expression for $g_k.b$:

$$g_k.b = g_{j,i+1}e_{i,k+1}g_{k+2}$$

which is also after commutation of $g_{k+2}$:

$$g_k.b = g_{j,i+1}e_{i,1}g_{k+2}$$
Thus, the action of $g_k$ on the tensor $g_{j,i+1}e_{i,1} \otimes_H 1$ is simply a multiplication by $r$. After inspection, the computation for type (II) is identical and the action of $g_k$ on the tensor $e_{i,1} \otimes_H 1$ is also a multiplication by $r$.

- Suppose now that $k > j + 1$. Visibly, $g_k$ commutes with $g_{j,i+1}e_{i,1}$ and with $e_{i,1}$, so that in both cases, the action by $g_k$ on the tensor $b \otimes_H 1$ is simply a multiplication by $r$.

- Finally, suppose $k$ belongs to $\{i+1, \ldots, j-1\}$ where $j \geq i+2$. We look at the action of $g_k$ on $g_{j,i+1}e_{i,1}$. We move $g_k$ to the right, then use the Braid relation $g_k g_{k+1} g_k = g_{k+1} g_k g_{k+1}$, then move $g_{k+1}$ this time to the end of the expression. After doing these moves, we get:

$$g_k g_{j,i+1} e_{i,1} = g_{j,i+1} e_{i,1} g_k$$

It follows in particular that

$$g_k b \otimes_H 1 = r b \otimes_H 1,$$

as in the previous cases. It remains to look at the action of $g_k$ on $e_{i,1}$. We have:

$$g_k b = \begin{cases} 
g_{i+1} e_{i,1} & \text{if } k = i+1 \\
e_{i,1} g_k & \text{otherwise}
\end{cases}$$

We summarize Case E in the following two equalities:

$$\forall k \not\in \{i-1, i, j, j+1\}, g_k b \otimes_H 1 = \begin{cases} g_{i+1} b \otimes_H 1 & \text{if } k = i+1 \text{ and } b \text{ of type (II)} \\
r b \otimes_H 1 & \text{in all the other cases}
\end{cases}$$

We note that the top equality is $(CII^\prime)$. Let’s name the bottom equality $(E)$.

With Cases A, B, C, D, E, the action of the $g_i$’s on the vector space $B_1 \otimes_H F$ is entirely described. The object of the next part is to give an expression of the representation in terms of roots.

### 2.3 Expression of the representation in the Lawrence-Krammer space

#### 2.3.1 The Lawrence-Krammer representation

Following our discussion at the end of §1.3.2 and in §2.1, there is a bijection:

$$u : \begin{array}{c} \phi^+ \\ \beta \end{array} \longrightarrow B \begin{array}{c} u(\beta) \\ \otimes_H 1 \end{array},$$

where $b$ is the algebra element associated with the positive root $\beta$, as in §1.3.2. It follows that there is a natural isomorphism $\varphi$ of vector spaces over $F$, defined on the basis vectors by:

$$\varphi : \begin{array}{c} \psi^{(n)} \\ x_{\beta} \end{array} \longrightarrow B_1 \otimes_H F \begin{array}{c} u(\beta) \otimes_H 1 \end{array}$$

We now get a representation of the BMW algebra inside the Lawrence-Krammer space as follows.
Theorem 3. The map on the generators

\[ \nu^{(n)}: B(A_{n-1}) \xrightarrow{g_i} \text{End}_F(V^{(n)}) \]

where each \( \nu_i \) is defined on the basis vectors of \( V^{(n)} \) by

\[ \nu_i(x_\beta) = \varphi^{-1}(g_i(u(\beta) \otimes_H 1)) \]
defines a representation of degree \( \frac{n(n-1)}{2} \) of the BMW algebra \( B(A_{n-1}) \) in the L-K space \( V^{(n)} \). Once irreducibility over \( \mathbb{Q}(l, r) \) has been established, as a representation of the Braid group on \( n \) strands, it is equivalent to the Lawrence-Krammer representation.

Proof. By definition, \( \nu^{(n)} \) is a representation of \( B \) in \( V^{(n)} \). We notice that \( \nu^{(n)} \) factors through the quotient \( B/I_2 \), where \( I_2 \) is the two sided ideal of \( B \) generated by all the products \( e_ie_j \) with \( |i-j| > 2 \). Indeed, in \( B_1 \), the algebra element \( e_ie_j \) is zero and so in \( B_1 \otimes_H F \), the vector \( e_ie_jb \otimes_H 1 \) is zero. Thus, we have:

\[ \nu^{(n)}(e_ie_j) = 0 \quad \text{when } |i-j| > 2 \]

Then by [2], as a representation of the Braid group on \( n \) strands, \( \nu^{(n)} \) must be equivalent to the Lawrence-Krammer representation of the Artin group of type \( A_{n-1} \) based on the two parameters \( t \) and \( r \), as defined in [3]. The \( r \) of this paper is the \( \ell \) of [3]; the parameter \( t \) of [3] is related to the parameters \( l \) and \( r \) of this paper by \( lt = r^3 \). \( \square \)

2.3.2 An explicit form of the representation in terms of roots

Given a positive root

\[ \beta = \alpha_i + \ldots + \alpha_j \text{ with } i < j, \]
we read on the expressions \((AI), (BI), (CI), (DI)\) and \((E)\) of §2.2 an expression of \( \nu_k(x_\beta) \) for \( k \in \{i-1, i, j, j+1\} \) and for \( k \notin \{i-1, i, j, j+1\} \). We define the height \( \text{ht}(\beta) \) of a positive root \( \beta \) as the sum of its coefficients with respect to the simple roots \( \alpha_1, \ldots, \alpha_{n-1} \). We have:

\[
\begin{align*}
\nu_{i-1}(x_\beta) &= x_{\beta+\alpha_{i-1}} + m \frac{\nu^{\text{ht}(\beta)-1}}{\nu^{\text{ht}(\beta)-2}} x_{\alpha_{i-1}} - m x_\beta & \text{(AI)} \\
\nu_i(x_\beta) &= x_{\beta-\alpha_i} & \text{(BI)} \\
\nu_{j}(x_\beta) &= x_{\beta-\alpha_i} + \frac{m}{\nu^{\text{ht}(\beta)-2}} x_{\alpha_j} - m x_\beta & \text{(CI)} \\
\nu_{j+1}(x_\beta) &= x_{\beta+\alpha_j+1} & \text{(DI)} \\
\nu_k(x_\beta) &= r x_\beta & \forall k \notin \{i-1, i, j, j+1\} & \text{(E)}
\end{align*}
\]

Similarly for type \((II)\), if \( \beta = \alpha_i \) is a simple root, we have:

\[
\begin{align*}
\nu_{i-1}(x_\beta) &= x_{\beta+\alpha_{i-1}} + m x_{\alpha_{i-1}} - m x_\alpha & \text{(AII)} \\
\nu_i(x_\beta) &= l^{-1} x_{\alpha_i} & \text{(BII)} \\
\nu_{i+1}(x_\beta) &= x_{\beta+\alpha_{i+1}} & \text{(CII)} \\
\nu_k(x_\beta) &= r x_\beta & \forall k \notin \{i-1, i, i+1\}
\end{align*}
\]
The last equation is obtained with (CII) when \( k > i + 1 \) and with (E) when \( k < i - 1 \).
For each node \( i \), we summarize the action of \( \nu_i \) on \( x_\beta \) as follows.

\[
\nu_i(x_\beta) = \begin{cases} 
  x_\beta & \text{if } (\beta|\alpha_i) = 0 \\
  1 x_\beta & \text{if } (\beta|\alpha_i) = 1 \\
  x_\beta + m_1 x_\alpha_i & \text{if } (\beta|\alpha_i) = 1 - \frac{1}{2} \text{ and (c)} \\
  x_\beta + m_1 x_\alpha_i - m x_\beta & \text{if } (\beta|\alpha_i) = -1 - \frac{1}{2} \text{ and (c')} \\
  x_\beta - m x_\beta & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \text{ and (d)} \\
  x_\beta - m x_\beta & \text{if } (\beta|\alpha_i) = 0 \text{ and (d')} \\
\end{cases}
\]

where (c), (c’), (d), (d’) are the following conditions:

\[
\begin{align*}
(c) \quad \beta &= \alpha_i + \cdots + \alpha_{i-1} & \text{with } t \leq i - 1 \\
(c') \quad \beta &= \alpha_{i+1} + \cdots + \alpha_s & \text{with } s \geq i + 1 \\
(d) \quad \beta &= \alpha_i + \cdots + \alpha_t & \text{with } t \leq i - 1 \\
(d') \quad \beta &= \alpha_i + \cdots + \alpha_s & \text{with } s \geq i + 1 \\
\end{align*}
\]

We note that:

(a) is equivalent to \( \text{Supp}(\beta) \cap \{i - 1, i, i + 1\} = \emptyset \) or \( \{i - 1, i, i + 1\} \subseteq \text{Supp}(\beta) \).

(b) is equivalent to \( \beta = \alpha_i \).

(c) and (c’) are the two ways the inner product \( (\beta|\alpha_i) \) can be \( \frac{1}{2} \).

(d) and (d’) are the two ways the inner product \( (\beta|\alpha_i) \) can be \( -\frac{1}{2} \).

We deduce from these equalities an expression for \( \nu^{(n)}(e_i) \):

\[
\nu^{(n)}(e_i)(x_\beta) = \begin{cases} 
  0 & \text{if } (\beta|\alpha_i) = 0 \\
  \left(1 - \frac{t}{r-1}\right) x_\alpha_i & \text{if } (\beta|\alpha_i) = 1 \\
  \frac{1}{1 - r^{h_1(\beta) - 1}} x_\alpha_i & \text{if } (\beta|\alpha_i) = -1 - \frac{1}{2} \text{ and (c)} \\
  \frac{1}{1 - r^{h_1(\beta) - 2}} x_\alpha_i & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \text{ and (c')} \\
  \frac{1}{1 - r^{h_1(\beta)}} x_\alpha_i & \text{if } (\beta|\alpha_i) = \frac{1}{2} \text{ and (d)} \\
  \frac{1}{1 - r^{h_1(\beta) - 1}} x_\alpha_i & \text{if } (\beta|\alpha_i) = \frac{1}{2} \text{ and (d')} \\
\end{cases}
\]

Notice \( \nu^{(n)}(e_i)(x_\beta) \) is always a multiple of \( x_\alpha_i \). This is easily pictured on the tangles.

The next part establishes Theorem 1, following the discussion of §1.2.

3 Reducibility of the representation

3.1 Action on a proper invariant subspace of the L-K space

We show the following result:

Proposition 1.

For any proper invariant subspace \( \mathcal{U} \) of \( \mathcal{V}^{(n)} \), we have \( \nu^{(n)}(e_i)(\mathcal{U}) = 0 \) for all \( i \).
Proof. If \( \mathcal{U} \) is trivial, there is nothing to prove. Otherwise, let \( u \) be a nonzero vector of \( \mathcal{U} \) such that \( \nu^{(n)}(e_i)(u) \neq 0 \). Since \( \nu^{(n)}(e_i)(u) \) is a multiple of \( x_{\alpha_i} \), we see that \( x_{\alpha_i} \) is in \( \mathcal{U} \). From there, we have:

\[
\nu_{i-1}(x_{\alpha_i}) = x_{\alpha_i+\alpha_{i-1}} + m x_{\alpha_{i-1}} \quad \text{modulo } F x_{\alpha_i}
\]

Hence \( x_{\alpha_i+\alpha_{i-1}} + m x_{\alpha_{i-1}} \) is in \( \mathcal{U} \). Another application of \( \nu_{i-1} \) now yields:

\[
\nu_{i-1}(x_{\alpha_i+\alpha_{i-1}}) + m x_{\alpha_{i-1}} = x_{\alpha_i} + \frac{m}{l} x_{\alpha_{i-1}},
\]

from which we derive that \( x_{\alpha_{i-1}} \) is in \( \mathcal{U} \). By induction, we see that all the \( x_{\alpha_i} \)'s for \( i \leq t \) are in \( \mathcal{U} \). In particular, \( x_{\alpha_i} \) is in \( \mathcal{U} \). But since \( e_1 \otimes_H \mathbf{1} \) spans \( B_1 \otimes_H F_{r_{2\times n}} x_{\alpha_1} \) spans \( V^{(n)} \). Then \( \mathcal{U} \) is the whole L-K space \( V^{(n)} \), in contradiction with our assumption that \( \mathcal{U} \) is proper. □

Corollary 1. Let \( \mathcal{W} \) be a proper irreducible invariant subspace of \( V^{(n)} \). Then, \( \mathcal{W} \) is an irreducible \( \mathcal{H}_{F,r_{2\times n}} \)-module.

Proof. By Proposition 1 and (3), we have

\[
\left[ g_i^2 + m g_i - 1 \right].\mathcal{W} = 0 \quad \text{for all } i.
\]

Hence \( \mathcal{W} \) is an \( \mathcal{H}_{F,r_{2\times n}} \)-module. Since the \( e_i \)'s are polynomials in the \( g_i \)'s, \( \mathcal{W} \) is an irreducible \( \mathcal{H}_{F,r_{2\times n}} \)-module. □

The next part investigates the existence of a one-dimensional invariant subspace of \( V^{(n)} \). We define for two nodes \( i \) and \( j \) with \( i < j \)

\[
w_{ij} = x_{\alpha_i + \cdots + \alpha_j}
\]

We will sometimes write \( w_{i,j} \) instead of \( w_{ij} \). Below is how \( w_{ij} \) is represented in the tangle algebra:

![Tangle Algebra Diagram]

It has two horizontal strands: one that joins nodes \( i \) and \( j \) at the top, and one that joins nodes 1 and 2 at the bottom and \( (n - 2) \) vertical strands that don’t cross within each other. The top horizontal strand over-crosses the vertical strands that it intersects.

### 3.2 The case \( l = \frac{1}{2^{n-2}} \)

We will prove the theorem:

**Theorem 4.** Let \( n \) be an integer with \( n \geq 3 \) and assume \( (r^2)^2 \neq 1 \). Suppose \( n \geq 4 \). There exists a one-dimensional invariant subspace of \( V^{(n)} \) if and only if \( l = \frac{1}{2^{n-2}} \). If so, it is spanned by \( \sum_{1 \leq s < t \leq n} r^{s+t} w_{st} \). (Case \( n = 3 \)) There exists a one-dimensional invariant subspace of \( V^{(3)} \) if and only
if $l = \frac{1}{r}$ or $l = -r^3$.

Moreover, if $r^6 \neq -1$, it is unique and

when $l = \frac{1}{r}$, it is spanned by $w_{12} + r w_{13} + r^2 w_{23}$

when $l = -r^3$, it is spanned by $w_{12} - \frac{1}{r} w_{13} + \frac{1}{r^3} w_{23}$

If $r^6 = -1$, there are exactly two one-dimensional invariant subspaces of $V(3)$ and they are respectively spanned by the vectors above.

**Proof.** Let $U$ be a one-dimensional invariant subspace of $V(n)$ and $u$ a spanning vector of $U$. For each $i$, let $\gamma_i$ be the scalar such that $\nu_i(u) = \gamma_i u$. Since $(v^2 + m \nu_i - id_{V(n)})(u) = 0$ by Proposition 1, it follows that $\gamma_i^2 + m \gamma_i - 1 = 0$. Hence $\gamma_i \in \{r, -\frac{1}{r}\}$. Further, since $(r^2)^2 \neq 1$, the Braid relation $\nu_i \nu_j \nu_i = \nu_j \nu_i \nu_j$ when $i \sim j$ forces that $\gamma_i$ takes the same value as $\gamma_j$. Let’s denote by $\gamma$ the common value of the $\gamma_i$’s. So, for each $i$, we have $\nu_i(u) = \gamma u$, where $\gamma \in \{r, -\frac{1}{r}\}$.

A general form for $u$ is:

$$u = \sum_{1 \leq i < j \leq n} \mu_{ij} w_{ij}, \quad \text{where} \quad \mu_{ij} \in F$$

We look for relations between these coefficients.

**Lemma 3.** Let $i$ be some node. Suppose $v = \sum_{1 \leq k < j \leq n} \mu_{kj} w_{kj}$ is a vector of $V(n)$ with $\nu_i(v) = \gamma v$ where $\gamma \in \{r, -\frac{1}{r}\}$. Then the following equalities hold for the coefficients of $v$:

$$\forall s \geq i + 2, \mu_{i+1,s} = \gamma \mu_{i,s} \quad (14)$$

$$\forall t \leq i - 1, \mu_{t,i+1} = \gamma \mu_{t,i} \quad (15)$$

When $i = 1$, only (14) holds and when $i = n - 1$, only (15) holds.

**Proof.** To show (14), we look at the coefficient of $w_{i+1,s}$ in $\nu_i(v) = \gamma v$, where $s \geq i + 2$. We get: $\mu_{i,s} - m \mu_{i+1,s} = \gamma \mu_{i+1,s}$. Since $\gamma + m = \frac{1}{\gamma}$, this equality is equivalent to $\mu_{i+1,s} = \gamma \mu_{i,s}$. Similarly, by equating the coefficients of $w_{t,i+1}$ in $\nu_i(v) = \gamma v$, we obtain (15). □

Applying these equalities to the coefficients of $u$, we see that all the coefficients of $u$ must be nonzero. In particular, when $n \geq 4$, the coefficient $\mu_{34}$ of $u$ is nonzero. Because an action of $g_1$ on $w_{34}$ is a multiplication by $r$ and an action on $g_1$ on the other terms $w_{ij}$ does not create any term in $w_{34}$, this forces $\gamma = r$. Thus, without loss of generality, we have:

$$u = \sum_{1 \leq i < j \leq n} r^{i+j} w_{ij}$$

From there, let’s look at the action of $g_1$ on $u$ and the resulting coefficient in $w_{12}$. The action of $g_1$ on $w_{12}$ is a multiplication by $l^{-1}$ and an action of $g_1$ on the $w_{2j'}$’s for $3 \leq j' \leq n$ creates new terms in $w_{12}$ with respective coefficients $m r^{j'-3}$. Thus, we get the equation:

$$\frac{r^3}{l} + \sum_{j=3}^{n} (r^2)^j = r^4,$$

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from which we derive that \( l = \frac{1}{r^{2n-3}} \).

Conversely, if \( l = \frac{1}{r^{2n-3}} \), we define \( u \) as \( \sum_{1 \leq i < j \leq n} r^{i+j} w_{ij} \) and check that \( \nu_i(u) = ru \) for each \( i \). First we show that the coefficient \( r^{2i+1} \) of \( w_{i,i+1} \) is multiplied by \( r \) when acting by \( g_i \). There are three contributions. One comes from the terms \( w_{i,i+1} \), with coefficient \( r^{2i+1} \frac{m r^{2n-3}}{r^{2n-2}} \). Another one from the terms \( w_{i,i+1} \), with coefficient \( r^{s+i+1} m r^{s-i-2} \), and a third one simply from the term \( w_{i,i+1} \), with coefficient \( r^{2i+1} r^{2n-3} \). Now, we have:

\[
m r^{2n-2} \sum_{i=1}^{i-1} r^{2i} + \frac{m}{r} \sum_{i=1}^{n} r^{2i} + r^{2n-2} = r r^{2i+1}
\]

Next, given a positive root \( \beta \), if none of the nodes \( i-1, i, i+1 \) is in the support of \( \beta \) or if all three nodes \( i-1, i, i+1 \) are in the support of \( \beta \), then it comes:

\[
\nu_i(x_\beta) = r x_\beta
\]

Thus, we only need to study the action of \( \nu_i \) on \( w_{k,i}, w_{k,i+1} \), with \( k \leq i-1 \) on one hand and \( w_{i,j}, w_{i+1,j} \), with \( l \geq i+2 \) on the other hand.

We have:

\[
\begin{align*}
\nu_i(w_{k,i}) & = r^{k+i} w_{k,i} \\
\nu_i(w_{k,i+1}) & = r^{k+i} w_{k,i+1} + m r^{k+i+1} w_{k,i+1} \mod F x_{\alpha_i}
\end{align*}
\]

So we get:

\[
\nu_i(r^{k+i} w_{k,i} + r^{k+i+1} w_{k,i+1}) = r^{k+i+1} w_{k,i} + r^{k+i+2} w_{k,i+1} \mod F x_{\alpha_i}
\]

Similarly, we have:

\[
\begin{align*}
\nu_i(w_{i,j}) & = r^{l+i} w_{i,j} \\
\nu_i(w_{i+1,j}) & = r^{l+i} w_{i+1,j} + m r^{l+i+1} w_{i+1,j} \mod F x_{\alpha_i}
\end{align*}
\]

so that:

\[
\nu_i(r^{l+i} w_{i,j} + r^{l+i+1} w_{i+1,j}) = r^{l+i+1} w_{i,j} + r^{l+i+2} w_{i+1,j} \mod F x_{\alpha_i}
\]

This ends the proof of the Theorem when \( n \geq 4 \).

Suppose now \( n = 3 \). So,

\[
u_1(u) = \gamma^3 w_{12} + \gamma^4 w_{13} + \gamma^5 w_{23}\]

Let’s compute \( \nu_1(u) \) and \( \nu_2(u) \):

\[
\begin{align*}
\nu_1(u) & = (\frac{\gamma^3}{3} + m \gamma^5) w_{12} + \gamma^5 w_{13} + \gamma^6 w_{23} \\
\nu_2(u) & = \gamma^4 w_{12} + \gamma^5 w_{13} + (\frac{\gamma^6}{3} + m \gamma^4) w_{23}
\end{align*}
\]

Since \( \nu_1(u) = \gamma u \), we must have:

\[
\frac{\gamma^3}{3} + m \gamma^5 = \gamma^4, \quad i.e. \quad \frac{1}{3} = \gamma(1 - m \gamma), \quad i.e. \quad l = \frac{1}{r^3}, \quad as \quad 1 - m \gamma = \gamma^2
\]

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Thus, if there exists a one dimensional invariant subspace of \( \mathcal{V}^{(3)} \) then \( l \) must take the values \( \frac{1}{r} \) or \( -r^3 \).

Conversely, let's consider the two vectors:

\[
\begin{align*}
    u_r &= w_{12} + r w_{13} + r^2 w_{23} \\
    u_{-\frac{1}{r}} &= w_{12} - \frac{1}{r} w_{13} + \frac{1}{r^2} w_{23}
\end{align*}
\]

We read on the equations giving the expressions for \( \nu_1(u) \) and \( \nu_2(u) \) that

- If \( l = \frac{1}{r} \), then \( \nu_1(u_r) = \nu_2(u_r) = r u_r \)
- If \( l = -r^3 \), then \( \nu_1(u_{-\frac{1}{r}}) = \nu_2(u_{-\frac{1}{r}}) = -\frac{1}{r} u_{-\frac{1}{r}} \)

The next section investigates the existence of an irreducible \((n-1)\)-dimensional invariant subspace of \( \mathcal{V}^{(n)} \).

### 3.3 The case \( l \in \{ \frac{1}{r^n}, -\frac{1}{r^n} \} \)

In Theorem 4, the case \( n = 3 \) was special. Likewise, in the following Theorem 5, the case \( n = 4 \) needs to be formulated separately.

**Theorem 5.**

Let \( n \) be a positive integer with \( n \geq 3 \) and \( n \neq 4 \). Assume \( \mathcal{H}_{F,r^2}(n) \) is semisimple. Then, there exists an irreducible \((n-1)\)-dimensional invariant subspace of \( \mathcal{V}^{(n)} \) if and only if \( l = \frac{1}{r^n} \) or \( l = -\frac{1}{r^n} \).

If so, it is spanned by the \( v_i^{(n)} \)'s, \( 1 \leq i \leq n-1 \), where \( v_i^{(n)} \) is defined by the formula:

\[
v_i^{(n)} = \left( \frac{1}{r} - \frac{1}{r^n} \right) w_{s,i+1} + \sum_{s=i+2}^{n} r^{s-i-2} (w_{s,i} - \frac{1}{r} w_{s+1,i}) + \epsilon_t \sum_{t=1}^{n-2-i} r^{n-i-2+t} (w_{t,i} - \frac{1}{r} w_{t+1,i})
\]

with \( \epsilon_{\frac{1}{r^n}} = 1 \) and \( \epsilon_{-\frac{1}{r^n}} = -1 \)

(Case \( n = 4 \)) Assume \( \mathcal{H}_{F,r^2}(4) \) is semisimple. Then, there exists an irreducible 3-dimensional invariant subspace of \( \mathcal{V}^{(4)} \) if and only if \( l \in \{ \frac{1}{r^2}, -\frac{1}{r^2}, -r^3 \} \).

If \( l \in \{ -\frac{1}{r}, \frac{1}{r^2} \} \), it is spanned by \( v_1^{(4)}, v_2^{(4)}, v_3^{(4)} \).

If \( l = -r^3 \), it is spanned by the vectors:

\[
\begin{align*}
    u_1 &= r w_{23} + w_{13} + \left( \frac{1}{r} + \frac{1}{r^2} \right) w_{34} - w_{24} - \frac{1}{r} w_{14} \\
    u_2 &= -r w_{12} - r^2 w_{13} - \frac{1}{r} w_{34} - \frac{1}{r^2} w_{24} + \left( r + \frac{1}{r^2} \right) w_{14} \\
    u_3 &= (r + \frac{1}{r}) w_{12} + \frac{1}{r} w_{23} - w_{13} + w_{24} - r w_{14}
\end{align*}
\]

**Proof.** Suppose that there exists an irreducible \((n-1)\)-dimensional invariant subspace \( \mathcal{U} \) of \( \mathcal{V}^{(n)} \).
Claim 1. Except in the case when \( n = 6 \), there exists a basis \( (v_1, \ldots, v_{n-1}) \) of \( \mathcal{U} \) such that one of the following two sets of relations holds:

\[
(\Delta) \quad \begin{align*}
\nu_t(v_1) &= r v_i & \forall t \notin \{i - 1, i, i + 1\} \\
\nu_t(v_i) &= -\frac{1}{r} v_i & \forall 1 \leq i \leq n - 1 \\
\nu_{i+1}(v_i) &= r(v_i + v_{i+1}) & \forall 1 \leq i \leq n - 2 \\
\nu_{i-1}(v_i) &= r v_i + \frac{1}{r} v_{i-1} & \forall 2 \leq i \leq n - 1
\end{align*}
\]

\[
(\Upsilon) \quad \begin{align*}

\nu_t(v_1) &= -1/r v_i & \forall t \notin \{i - 1, i, i + 1\} \\
\nu_t(v_i) &= r v_i & \forall 1 \leq i \leq n - 1 \\
\nu_{i+1}(v_i) &= -1/r(v_i + v_{i+1}) & \forall 1 \leq i \leq n - 2 \\
\nu_{i-1}(v_i) &= -1/r v_i - r v_{i-1} & \forall 2 \leq i \leq n - 1
\end{align*}
\]

Proof. We first recall some general fact about the irreducible representations of the Iwahori-Hecke algebra of the symmetric group. The following result was established by James for the irreducible representations of the symmetric group, but applies here to the Iwahori-Hecke algebra \( \mathcal{H}_{F,r,z}(n) \) since we work in characteristic zero and assumed \( \mathcal{H}_{F,r,z}(n) \) semisimple. By Theorem 6, point (i) in [4], when the characteristic of the field \( F \) is zero and for \( n \geq 7 \), an irreducible \( F \text{Sym}(n) \)-module is either one of the Specht modules \( S^{(n)} \), \( S^{(1^n)} \), \( S^{(n-1,1)} \), \( S^{(2,1^{n-2})} \) or has dimension greater than \( (n - 1) \). The statement is also true when \( n = 3 \) and \( n = 5 \). When \( n = 4 \), the statement does not hold as \( S^{(2,2)} \) has dimension 2 and when \( n = 6 \), the statement also fails since \( S^{(3,3)} \) and \( S^{(2,2,2)} \) both have dimension 5. In any case, there are exactly two inequivalent irreducible representations of \( F \text{Sym}(n) \) of degree \( (n - 1) \), except in the case \( n = 6 \), when there are exactly four inequivalent irreducible representations of \( F \text{Sym}(6) \) of degree 5. The same statement holds for \( \mathcal{H}_{F,r,z}(n) \) when the algebra is semisimple.

Consider now the set of relations \((\Delta)\) (resp \((\Upsilon)\)). For each \( i \), let \( M_i \) (resp \( N_i \)) be the matrix of the endomorphism \( \nu_i \) in the basis \( (v_1, \ldots, v_{n-1}) \). It is a direct verification that the \( M_i \)'s (resp \( N_i \)'s) satisfy the Braid relations and the relation \( M_i^2 + m M_i = I_{n-1} \) (resp \( N_i^2 + m N_i = I_{n-1} \)) for each \( i \), where \( I_{n-1} \) is the identity matrix of size \( (n - 1) \). Hence the \( M_i \)'s (resp the \( N_i \)'s) yield a matrix representation of \( \mathcal{H}_{F,r,z}(n) \) of degree \( (n - 1) \).

To show that these two matrix representations are irreducible, relying on James' statement above, it suffices to check that there is no one-dimensional invariant subspace of \( F^{n-1} \) when \( n \neq 4 \) and that there is no one-dimensional or irreducible two-dimensional invariant subspace of \( F^3 \) when \( n = 4 \). This is the case if \( r^{2n-1} \neq 1 \) when \( n \neq 4 \) and if \( (r^2)^2 \neq 1 \) and \( (r^2)^4 \neq 1 \) when \( n = 4 \).

When \( n = 3 \), the two matrix representations are equivalent. When \( n \geq 4 \), they are not: visibly, the matrices of one representation all have the same trace \( -(n-2)r^{-1} + r \) and the matrices of the other one all have the same trace \( (n-2)r - \frac{1}{r} \). These two values are distinct when \( (r^2)^2 \neq 1 \) and \( n \geq 4 \). We conclude that these are the two inequivalent irreducible representations of \( \mathcal{H}_{F,r,z}(n) \) when \( n \geq 4 \) and \( n \neq 6 \). \( \square \)

In what follows, we assume \( n \geq 4 \). We will show that it is impossible to have the second set of relations, except in the case \( n = 4 \) when it forces \( l = -r^3 \). Suppose the \( \nu_i \)'s satisfy \((\Upsilon)\). The relation \( \nu_{n-1}(v_1) = -\frac{1}{r} v_1 \) implies that in \( v_1 \) there are no terms in \( w_{s,t} \) for integers \( s, t \in \{1, \ldots, n - 2\} \) such that \( s < t \).
Hence we may write:

\[
v_1 = \sum_{j=1}^{n-2} \mu_{j,n-1} w_{j,n-1} + \sum_{j=1}^{n-1} \mu_{j,n} w_{j,n}
\]

Moreover, the relation \( \nu_3(v_1) = -\frac{1}{r} v_1 \) implies that there are no terms in \( w_{j,k} \) in \( v_1 \) for \( j \geq 5 \). Further, the relations

\[
\begin{align*}
\nu_1(w_{2,n-1}) &= w_{1,n-1} + mw^{n-4} w_{12} - m w_{2,n-1} \\
\nu_1(w_2) &= w_1 + mw^{n-3} w_{12} - m w_2
\end{align*}
\]

imply that: \( mw^{n-4} \mu_{2,n-1} + mw^{n-3} \mu_{2,n} = 0 \), i.e. \( \mu_{2,n} = -\frac{1}{r} \mu_{2,n-1} \), as there is no term in \( w_{12} \) in \( v_1 \). Furthermore, an application of (14) with \( \gamma = r \) and \( i = 1 \) yields for \( s = n-1 \) and \( s = n \) respectively: \( \mu_{2,n-1} = r \mu_{1,n-1} \) and \( \mu_{2,n} = r \mu_{1,n} \).

So, up to a multiplication by a scalar,

\[
v_1 = w_{1,n-1} + rw_{2,n-1} - \frac{1}{r} w_{1,n} - w_{2,n} \\
+ \mu_{3,n-1} w_{3,n-1} + \mu_{4,n-1} w_{4,n-1} + \mu_{3,n} w_{3,n} + \mu_{4,n} w_{4,n}
\]

(16)

or

\[
v_1 = \mu_{3,n-1} w_{3,n-1} + \mu_{4,n-1} w_{4,n-1} + \mu_{3,n} w_{3,n} + \mu_{4,n} w_{4,n}
\]

(17)

If \( n \geq 5 \), the relation \( \nu_3(v_1) = -\frac{1}{r} v_1 \) implies that \( \mu_{2,n} = 0 \). Then an expression for \( v_1 \) is given by (17) and not (16). Assume \( n > 5 \). Then there is no term in \( w_{34} \) in \( v_1 \). Then it comes \( \mu_{4,n} = -\frac{1}{r} \mu_{4,n-1} \). Moreover, by (14) applied with \( \gamma = -\frac{1}{r} \) and \( i = 3 \), we have \( \mu_{4,n} = -\frac{4}{3} \mu_{3,n} \) and \( \mu_{4,n-1} = -\frac{2}{3} \mu_{3,n-1} \). Further, when \( n > 5 \), \( \nu_4(w_{3,n}) = r w_{3,n} \) and an action of \( g_4 \) on the other terms of \( v_1 \) in (17) won’t create any term in \( w_{3,n} \). Thus, the relation \( \nu_4(v_1) = -\frac{1}{r} v_1 \) forces \( \mu_{3,n} = 0 \). Then, by the relations previously established, all the coefficients of \( v_1 \) are actually zero, which is impossible. The case \( n = 5 \) also leads to a contradiction and details appear in [7], §8.3.

Suppose now \( n = 4 \). By (16) and (17), \( v_1 = w_{13} + r w_{23} - \frac{1}{r} w_{14} - w_{24} + \mu_{34} w_{34} \) or \( v_1 = w_{34} \). Suppose \( v_1 \) is of the second type. Then, by \( \nu_3(v_1) = -\frac{1}{r} v_1 \), we must have \( i = r \). Since \( \nu_2(v_1) = -\frac{1}{r} (v_1 + v_2) \), we must have:

\[
v_2 = -r w_{24} + (r^2 - 1) w_{23} - r^2 w_{34}
\]

Since

\[
\begin{align*}
\nu_2(w_{34}) &= w_{24} + mw_{23} - mw_{34} \\
\nu_2(w_{23}) &= -\frac{1}{r} w_{23}
\end{align*}
\]

and since \( \nu_2(v_2) = r v_2 \), we get:

\[-m r^2 - \frac{1}{r} (r^2 - 1) = r (r^2 - 1), \] which reads \( m = 0 \) after simplification.

As \( m \) is nonzero, this is a contradiction. Thus, \( v_1 \) is of the first type. Then, denoting by \( \lambda_{ij} \) the coefficient of \( w_{ij} \) in \( v_2 \), we get by looking at the coefficient of \( w_{12} \) in \( \nu_2(v_1) = -\frac{1}{r} v_1 - \frac{1}{r} v_2 \) that \( \lambda_{12} = -r \). Since by (15) with \( \gamma = r \) and \( i = 2 \), we have \( \lambda_{13} = r w_{12} \), it follows that \( \lambda_{13} = -r^2 \). Next, by looking at the coefficient of \( w_{14} \) in the relation \( \nu_2(v_1) = -\frac{1}{r} v_1 - \frac{1}{r} v_2 \), we get:

\[-1 = -\frac{\lambda_{14}}{r} + \frac{1}{r^2} \] i.e. \( \lambda_{14} = r + \frac{1}{r} \).
Also, by looking at the coefficient of \( w_{24} \) in the same relation, we obtain:

\[
\mu_{24} = \frac{1}{r} - \frac{\lambda_{24}}{r} \tag{18}
\]

Next, we use the relation \( \nu_1(v_2) = -\frac{1}{r} v_2 - r v_3 \). First, we look at the coefficient of \( w_{13} \) to get \( \lambda_{23} = 0 \) and by looking at the coefficient of \( w_{14} \), we get \( \lambda_{24} = -\frac{r}{r^2} \).

By \( \nu_2(v_2) = r v_2 \) and (14), it comes \( \mu_{24} = r \lambda_{24} = -\frac{1}{r} \). Plugging the value of \( \lambda_{24} \) in (18) now yields \( \mu_{24} = \frac{1}{r} + \frac{1}{r^2} \).

Finally, by looking at the coefficient of \( w_{12} \) in \( \nu_1(v_2) = -\frac{1}{r} v_2 - r v_1 \), we get \( -\frac{1}{r} - \frac{w_1}{r} = 1 \), from which we derive \( l = -r^2 \).

Also, gathering all the results above, we see that the vectors \( v_1 \) and \( v_2 \) are exactly the vectors \( u_1 \) and \( u_2 \) of the Theorem. Similar computations would also lead to \( v_3 = u_3 \) (see [7]). Conversely, we can show that if \( l = -r^2 \), then the vectors \( u_1, u_2 \) and \( u_3 \) form a free family of vectors that satisfy the relations (\( \triangle \)).

This shows that their linear span over \( F \) is an irreducible 3-dimensional invariant subspace of \( \mathcal{V}^{(4)} \).

Suppose now that the \( v_i \)'s satisfy (\( \triangle \)). The relation \( \nu_i(v_i) = -\frac{1}{r} v_i \) implies that in \( v_i \) there are no terms in \( w_{is} \) for \( s \leq i - 1 \) or \( t \geq i + 2 \) or \( t \leq i - 1 \) and \( s \geq i + 2 \).

Thus, a general form for \( v_i \) must be:

\[
v_i = \mu_{i,i+1} w_{i,i+1} + \sum_{s=i+1}^n \mu_{i,s} w_{i,s} + \sum_{s=i+2}^n \mu_{i+1,s} w_{i+1,s} + \sum_{t=1}^{i-1} \mu_{t,i} w_{t,i} + \sum_{t=1}^{i-1} \mu_{t,i+1} w_{t,i+1} \tag{19}
\]

Since \( \nu_i(v_i) = -\frac{1}{r} v_i \), both equalities (14) and (15) hold with \( \gamma = -\frac{1}{r} \). Further, since \( \nu_q(v_i) = r v_i \) for \( q \not\in \{i - 1, i, i + 1\} \), applying (14) and (15) with \( i = q \) and \( \gamma = r \) yields:

\[
\forall j \geq q + 2, \mu_{q+1,j} = r \mu_{q,j} \tag{20}
\]

\[
\forall k \leq q - 1, \mu_{k,q+1} = r \mu_{k,q} \tag{21}
\]

Apply (20) with \( q \leq i - 2 \) and \( j \in \{i, i + 1\} \) to get:

\[
\forall q \leq i - 2, \mu_{q+1,i} = r \mu_{q,i} \quad \& \quad \mu_{q+1,i+1} = r \mu_{q,i+1} \tag{22}
\]

Apply (21) with \( q \geq i + 2 \) and \( k \in \{i, i + 1\} \) to get:

\[
\forall q \geq i + 2, \mu_{i,q+1} = r \mu_{i,q} \quad \& \quad \mu_{i+1,q+1} = r \mu_{i+1,q} \tag{23}
\]

Expression (19) may now be rewritten:

\[
v_i = \zeta^{(i)} w_{i,i+1} + \delta^{(i)} \sum_{s=i+1}^n r^{s-i-2} (w_{i,s} - \frac{1}{r} w_{i+1,s}) + \lambda^{(i)} \sum_{t=1}^{i-1} r^{t-1} (w_{t,i} - \frac{1}{r} w_{t,i+1}),
\]

where \( \zeta^{(i)}, \delta^{(i)} \) and \( \lambda^{(i)} \) are three coefficients to determine. First, we show that all the \( \delta^{(i)} \) with \( i \in \{1, \ldots, n - 2\} \) may be set to the value one. Notice that if \( v_1, v_n \) satisfy (\( \triangle \)), then \( \delta v_1, \ldots, \delta v_n \) also satisfy (\( \triangle \)), where \( \delta \) is any nonzero scalar. Then, without loss of generality, we set \( \delta^{(1)} = 1 \). Suppose \( \delta^{(i)} = 1 \) for some node \( i \) with \( 1 \leq i \leq n - 2 \). We will show that \( \delta^{(i+1)} = 1 \).
Notice that \( \delta^{(i+1)} \) is the coefficient of \( w_{i+1,i+3} \) in \( v_{i+1} \). Since an action of \( g_{i+1} \) on \( v_i \) never creates a term in \( w_{i+1,i+3} \), by looking at the coefficient of \( w_{i+1,i+3} \) in \( \nu_{i+1}(v_i) = r v_i + \frac{1}{r} v_i, \) we get \( 0 = -r \delta^{(i)} + r \delta^{(i+1)} \). After replacing \( \delta^{(i)} \) by 1, this yields \( \delta^{(i+1)} = 1 \). Thus, all the \( \delta^{(i)} \) may be set to the value 1. It remains to find the coefficients \( \zeta^{(i)} \) and \( \lambda^{(i)} \). By looking at the coefficient of \( w_{i,i+1} \) in \( \nu_{i+1}(v_i) = r (v_i + \frac{1}{r} v_{i-1}), \) we get:
\[
r \zeta^{(i)} + r^i \lambda^{(i+1)} = 1, \quad \text{for each } i \text{ with } 1 \leq i \leq n - 2
\] (22)

Also, by looking at the coefficient of the same term \( w_{i,i+1} \) in the relation \( \nu_{i-1}(v_i) = r v_i + \frac{1}{r} v_{i-1}, \) we get:
\[
-m \zeta^{(i)} - r^{i-3} \lambda^{(i)} = r \zeta^{(i)} - \frac{1}{r^2}, \quad \text{for each } i \text{ with } 2 \leq i \leq n - 1
\]

After multiplication by a factor \( r^2 \), we obtain:
\[
r \zeta^{(i)} + r^{i-1} \lambda^{(i)} = 1, \quad \text{for each } i \text{ with } 2 \leq i \leq n - 1
\] (23)

By (22) and (23), we get \( \lambda^{(i)} = \frac{1}{2r^2} \lambda^{(2)}, \) for all \( i \geq 2 \). Let’s do a change of indices in (22) to get:
\[
r \zeta^{(i-1)} + r^{i-1} \lambda^{(i)} = 1 \quad \text{for each } i \text{ with } 2 \leq i \leq n - 1
\] (24)

(23) and (24) show that \( \zeta^{(i)} = \zeta^{(i-1)} \) for each \( i \) with \( 2 \leq i \leq n - 1 \). In other words, all the \( \zeta^{(i)} \) are equal to a certain scalar \( \zeta \). The relation between \( \zeta \) and \( \lambda^{(2)} \) is given by equation (24) with \( i = 2 \):
\[
\lambda^{(2)} = \frac{1}{r} - \zeta
\] (25)

Thus, by determining \( \zeta \), we will get a complete expression for all the vectors \( v_i \)'s. Since we have
\[
v_1 = \zeta w_{12} + \sum_{s=3}^{n} r^{s-3} (w_{1,s} - \frac{1}{r} w_{2,s}),
\]
by looking at the coefficient of \( w_{12} \) in the relation \( \nu_1(v_1) = -\frac{1}{r} v_1, \) we get the equation:
\[
\zeta \left( \frac{1}{r} + \frac{1}{r^2} \right) = \frac{1}{r^2} - (r^2)^{n-3}
\] (26)

Further, by looking at the coefficient of \( w_{i,i+1} \) in \( \nu_i(v_i) = -\frac{1}{r} v_i, \) we have:
\[
\zeta \left( \frac{1}{r} + \frac{1}{r^2} \right) = \sum_{s=i+2}^{n} r^{s-i-3} m r^{s-i-2} + \lambda^{(i)} \sum_{t=1}^{i-1} r^{t-2} \frac{m}{t^{r^{i-t-1}}}
\]
\[i.e.
\zeta \left( \frac{1}{r} + \frac{1}{r^2} \right) = \frac{1}{r^2} - (r^2)^{n-i-2} + \lambda^{(i)} \left( \frac{1}{r^2} - r^{i-2} \right) \quad (\ast)_i
\]

Let’s write down \((\ast)_2\) and \((\ast)_3\):
\[
\zeta \left( \frac{1}{r} + \frac{1}{r^2} \right) = \frac{1}{r^2} - (r^2)^{n-4} + \lambda^{(2)} \left( \frac{1}{r^2} - 1 \right) \quad (\ast)_2
\]
\[
\zeta \left( \frac{1}{r} + \frac{1}{r^2} \right) = \frac{1}{r^2} - (r^2)^{n-5} + \lambda^{(2)} \left( \frac{1}{r^3} - r \right) \quad (\ast)_3
\]
where $\lambda^{(3)}$ has been replaced by $\frac{\lambda^{(2)}}{r}$. Let’s subtract these two equalities:

$$\frac{\lambda^{(2)}}{l} \left( \frac{1}{r^2} - \frac{1}{r^3} \right) = (r^2)^{n-4} \left( 1 - \frac{1}{r^2} \right) \quad (\ast)_2 - (\ast)_3$$

After multiplying this equality by $\frac{1}{r^2}$ and dividing it by $\frac{1}{r} - \frac{1}{r^2}$ (licit as $m \neq 0$), we obtain:

$$\lambda^{(2)} = l (r^2)^{n-3}$$

Hence, by (25), $\zeta = \frac{1}{r} - l (r^2)^{n-3}$. Plugging this value for $\zeta$ into (26) now yields:

$$l^2 = \frac{1}{(r^2)^{n-3}}, \quad \text{hence} \quad l \in \left\{ \frac{1}{(r^2)^{n-3}}, \frac{1}{r} \right\}$$

If $l = \frac{1}{(r^2)^{n-3}}$, we get successively $\lambda^2 = r^{n-3} = \frac{1}{r} \zeta = \frac{1}{r} - \frac{1}{l}$ and $\lambda^3 = r^{n-i-1}$. If $l = -\frac{1}{(r^2)^{n-3}}$, $\lambda^2$ and $\lambda^3$ are still respectively $\frac{1}{l}$ and $\frac{1}{l} - \frac{1}{r}$ and $\lambda^3 = -r^{n-i-1}$. We obtain the formula announced in Theorem 5.

Conversely, if $l \in \left\{ \frac{1}{(r^2)^{n-3}}, \frac{1}{r} \right\}$, we can show that the $v^{(n)}_i$'s defined in Theorem 5 satisfy the relations (Δ) (see [Z], §8.3). In particular, their linear span over $F$ is a proper invariant subspace of $V^{(n)}$, hence is an $\mathcal{H}_{F,r^{(n)}}$-module by Corollary 1. When $n \neq 4$, if the vectors $v^{(n)}_i$'s were linearly dependent, then their span would either be one-dimensional or would contain a one-dimensional $\mathcal{H}_{F,r^{(n)}}$-submodule, as there is no irreducible $\mathcal{H}_{F,r^{(n)}}$-module of dimension between 1 and $(n-1)$. In any case, by Theorem 4, that would force $l = \frac{1}{(r^2)^{n-3}}$ when $n \neq 3$ and $l \in \left\{ -r^3, \frac{1}{r} \right\}$ when $n = 3$. This is impossible with our assumption that $l \in \left\{ \frac{1}{(r^2)^{n-3}}, \frac{1}{r} \right\}$ and the fact that $r^{2n} \neq 1$. As for $n = 4$, the freedom over $F$ of the family of vectors $(v^{(4)}_1, v^{(4)}_2, v^{(4)}_3)$ is a direct verification or is a consequence of Theorem 4 and forthcoming Proposition 3 (See §3.4). We are now able to conclude: the vector space $\text{Span}_F(v^{(n)}_1, \ldots, v^{(n)}_{n-1})$ is $(n-1)$-dimensional, is invariant under the action of the $g_i$'s and is an $\mathcal{H}_{F,r^{(n)}}$-module since it is a proper invariant subspace of $V^{(n)}$. Then, by the relations satisfied by the $v^{(n)}_i$'s, it must be irreducible.

To complete the proof of Theorem 5, we show that there does not exist any irreducible 5-dimensional invariant subspace of $V^{(6)}$ that is isomorphic to one of the Specht modules $S^{(3,3)}$ or $S^{(2,2,2)}$. Indeed, suppose such a subspace exists and name it $\mathcal{W}$. Since we have assumed that $\mathcal{H}_{F,r^{(6)}}$ is semisimple, it is licit to use the branching rule as it is described in Corollary 6.2 of [F]. We have:

$$S^{(3,3)} \downarrow_{\mathcal{H}_{F,r^{(5)}}} \simeq S^{(3,2)} \downarrow_{\mathcal{H}_{F,r^{(5)}}} \simeq S^{(2,2,2)} \downarrow_{\mathcal{H}_{F,r^{(5)}}} \simeq S^{(2,2,1)}$$

We will show that the restriction of $\mathcal{W}$ to $\mathcal{H}_{F,r^{(5)}}$ cannot be isomorphic to $S^{(3,2)}$ or $S^{(2,2,1)}$, hence a contradiction. A proof of the following fact is in [Z], §8.3

**Fact 1.** Suppose $\mathcal{H}_{F,r^{(5)}}$ is semisimple. Then, up to equivalence, the two irreducible matrix representations of degree 5 of $\mathcal{H}_{F,r^{(5)}}$ are respectively defined by the matrices $P_1, P_2, P_3, P_4$ and $Q_1, Q_2, Q_3, Q_4$ given by:

$$P_1 := \begin{bmatrix} r & r \\ 1 & -r^2 - \frac{1}{r} \\ 1 & -\frac{1}{r} \end{bmatrix}, \quad P_2 := \begin{bmatrix} -\frac{1}{r} & 1 & 1 \\ -\frac{1}{r} & 1 & 1 \\ r & 1 & r \end{bmatrix}$$
Indeed, suppose that such a basis of vectors exists. Let’s denote by \( w_\text{coefficient of } g_\) in which the matrices of the left action by the \( w_\text{of } g_\) where the blanks must be filled with zeros.

and for the conjugate representation:

We look at the coefficient of \( \lambda \) in \( g_\) since \( \lambda \sim 0 \). Thus, \( \lambda \) is also zero by Lemma 3. Then, by looking at the coefficient of \( w_\text{in } w_\) we also have \( \lambda \) is also zero by Lemma 3. Then, by looking at the coefficient of \( w_\text{action on } w_\) and Lemma 3, we also get:

Finally, we show that it is impossible to have a basis \( \{w_1, w_2, w_3, w_4, w_5\} \) of \( W \) in which the matrices of the left action by the \( g_i \)’s, \( i = 1, \ldots, 4 \) are the \( Q_i \)’s. Indeed, suppose that such a basis of vectors exists. Let’s denote by \( \lambda_{ij}^{(k)} \) the coefficient of \( w_\text{in } w_\). Since \( g_4.w_4 = w_5 \) and \( g_3.w_5 = -\frac{1}{r^2} w_4 - \frac{1}{r} w_5 \), we get

We look at the coefficient of \( w_1 \) in this equation to get

Since \( (r^2)^3 \neq 0 \), we have \( r^2 + 1 + \frac{1}{r^2} \neq 0 \) and so \( \lambda^{(4)}_{12} = 0 \). Now since \( g_3.w_1 = -\frac{1}{r} w_1 + w_4 \), we get \( r \lambda^{(1)}_{12} = -\frac{1}{r^2} \lambda^{(4)}_{12} \) and so \( \lambda^{(1)}_{12} = 0 \). This implies that \( \lambda^{(1)}_{13} \) is also zero by \( g_2.w_1 = r w_1 \) and Lemma 3. Then, by looking at the coefficient of \( w_\text{in } g_2.w_\) we get \( \lambda^{(4)}_{13} = \frac{1}{r^2} \lambda^{(4)}_{13} \), where we used that \( \lambda^{(4)}_{12} = 0 \). Thus, \( \lambda^{(1)}_{13} = 0 \). Since \( g_4.w_4 = r w_4 \) and \( g_1.w_4 = r w_4 \), by Lemma 3, we also get:

Let’s now look at the term in \( w_2 \) in \( g_3.g_4 = -\frac{1}{r} g_4.w_4 - \frac{1}{r^2} w_4 \). We have:

where we used that \( \lambda^{(4)}_{25} = r \lambda^{(4)}_{15} \). Then, \( \lambda^{(1)}_{15} = 0 \) and also \( \lambda^{(4)}_{25} = 0 \). Further, since \( \lambda^{(4)}_{12} = 0 \), in \( g_1.w_4 \), a term in \( w_1 \) is created only when \( g_1 \) acts on \( w_6 \), with coefficient \( mr^3 \). Thus the relation \( g_1.w_4 = r w_4 \) yields \( \lambda^{(4)}_{26} = 0 \). Then by \( g_1.w_4 = r w_4 \), we also have \( \lambda^{(4)}_{16} = 0 \).
Furthermore, on one hand, by looking at the coefficient of $w_{34}$ in $g_1.w_1 = -\frac{1}{r}w_1 + w_4$, we get $r\lambda_{34}^{(4)} = -\frac{1}{r}\lambda_{34}^{(1)} + \lambda_{34}^{(4)}$, i.e.

$$\lambda_{34}^{(4)} = (r + \frac{1}{r})\lambda_{34}^{(1)}$$

On the other hand, by looking at the coefficient of $w_{34}$ in $g_2.w_4 = w_1 - \frac{1}{r}w_4$ and remembering that $\lambda_{24}^{(4)} = 0$, we have $-m\lambda_{34}^{(4)} = -\frac{1}{r}\lambda_{34}^{(4)} + \lambda_{34}^{(1)}$, i.e.

$$\lambda_{34}^{(4)} = \frac{1}{r}\lambda_{34}^{(1)}$$

The two relations binding $\lambda_{34}^{(4)}$ and $\lambda_{34}^{(1)}$ now yield $\lambda_{34}^{(4)} = \lambda_{34}^{(1)} = 0$. So $w_4$ reduces to

$$w_4 = r\lambda_{35}^{(4)}w_{45} + \lambda_{35}^{(4)}w_{35} + \lambda_{56}^{(4)}w_{56} + r\lambda_{36}^{(4)}w_{46} + \lambda_{36}^{(4)}w_{36}$$

At this point, it is of interest to derive the following result.

**Result 1.** The irreducible matrix representation of degree 5 of $\mathcal{H}_{r,\ast}(5)$ defined by the matrices $Q_i$’s is not a constituent of the Lawrence-Krammer representation of degree 10 of the BMW algebra of type $A_4$.

**Proof.** If $W$ is a subspace of $V^{(5)}$ instead, then we simply have

$$w_4 = r\lambda_{35}^{(4)}w_{45} + \lambda_{35}^{(4)}w_{35}$$

Then, by looking at the coefficient of $w_{34}$ in $g_3.w_4 = r w_4$, we get $m\lambda_{35}^{(4)} = 0$, so that $\lambda_{35}^{(4)} = \lambda_{35}^{(4)}$. Then $w_4 = 0$, which is impossible. □

Let’s go back to the main proof. By looking at the coefficient of $w_{34}$ in $g_3.w_4 = r w_4$ and using the complete expression for $w_4$ this time, we get: $m r\lambda_{35}^{(4)} + m r^2\lambda_{36}^{(4)}$, i.e $\lambda_{36}^{(4)} = -\frac{1}{r}\lambda_{35}^{(4)}$. We see that all the coefficients in $w_4$ except $\lambda_{36}^{(4)}$ are multiples of $\lambda_{35}^{(4)}$. Moreover, we claim that $w_4$ may not be a multiple of $w_{56}$. Indeed, recall the formula

$$g_3g_4.w_4 = -\frac{1}{r}g_1.w_4 = \frac{1}{r^2}w_4$$

and observe that in $-\frac{1}{r}g_1.w_4 = \frac{1}{r^2}w_4$, there is no term in $w_{36}$ while in $g_3g_4.w_4$, there is one. Thus, without loss of generality, we may set $\lambda_{35}^{(4)} = 1$. So,

$$w_4 = r w_{45} + w_{35} + \lambda_{56}^{(4)}w_{56} - w_{46} - \frac{1}{r}w_{36}$$

We then deduce a complete expression for $w_1$ by using the relation $w_1 = g_2.w_4 + \frac{1}{r}w_4$:

$$w_1 = (1 + r^2)w_{45} + r w_{35} + w_{25} + \left(r + \frac{1}{r}\right)\lambda_{56}^{(4)}w_{56} - \left(r + \frac{1}{r}\right)w_{46} - w_{36} - \frac{1}{r}w_{26}$$

A contradiction now arises when looking at the coefficient of $w_{26}$ in $g_3.w_1 = -\frac{1}{r}w_1 + w_4$. Indeed, this yields $-1 = \frac{1}{r}$ and contradicts $(r^2)^2 \neq 1$. 

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Suppose now that there exists a basis \((w_1, w_2, w_3, w_4, w_5)\) of \(W\) in which the matrices of the left action by the \(g_i\)'s, \(i = 1, \ldots, 4\) are the \(P_i\)'s. We read on the matrices \(P_1\) and \(P_3\) that \(g_1.w_4 = -\frac{1}{r} w_4\) and \(g_3.w_4 = -\frac{1}{r} w_4\). Thus, we have:

\[
w_4 = \mu_{13}^{(4)} w_{13} + \mu_{23}^{(4)} w_{23} + \mu_{24}^{(4)} w_{24} + \mu_{14}^{(4)} w_{14},
\]

where the coefficients are related by \(\mu_{14}^{(4)} = \mu_{23}^{(4)} = -\frac{1}{r} \mu_{13}^{(4)} = -r \mu_{24}^{(4)}\). In particular, all these coefficients are nonzero. As \(g_1.w_4 = g_3.w_4\), by looking at the coefficient in \(w_{23}\), we obtain

\[
\mu_{13}^{(1)} - m \mu_{23}^{(1)} = \mu_{24}^{(1)}
\]

Moreover, by looking at the same coefficient \(w_{23}\) in \(g_3.w_4 = r w_1 + w_4\), we get

\[
\mu_{24}^{(1)} = r \mu_{23}^{(1)} + \mu_{23}^{(4)}
\]

Combining both equations yields

\[
\mu_{13}^{(1)} - \frac{1}{r} \mu_{23}^{(1)} = \mu_{23}^{(4)}
\]

Further, by looking at the coefficient in \(w_{13}\) in the same equation, we get \(0 = r \mu_{13}^{(1)} + \mu_{13}^{(4)}\), as there is no term in \(w_{14}\) in \(w_1\) by \(g_2.w_4 = -\frac{1}{r} w_1\). Then we get \(\mu_{23}^{(1)} = 0\). Looking at the coefficient of \(w_{23}\) in \(g_2.w_4 = w_1 + r w_4\) now yields \(r \mu_{23}^{(4)} = \frac{1}{r} \mu_{23}^{(4)} + m \mu_{13}^{(4)}\), which by using the relations from the beginning reads

\[
\left( \frac{m}{l} + \frac{1}{r} \left( 1 - \frac{1}{l} \right) \right) \mu_{13}^{(4)} = 0
\]

As the coefficient \(\mu_{13}^{(4)}\) is nonzero, this forces \(l = r\). It will be useful to derive the following result along the way:

**Result 2.** Assume \(\mathcal{H}_{F,r^2}(5)\) is semisimple. If there exists an irreducible 5-dimensional invariant subspace of \(V(5)\) then \(l = r\).

**Proof.** Indeed, if we assume \(W \subset V(5)\) instead of \(W \subset V(6)\) in the computations above, they are unchanged and lead to the same conclusion. \(\square\)

As seen along the way, \(w_4\) is, up to a multiplication by a scalar that can be set to 1 without loss of generality,

\[w_4 = w_{23} + w_{14} - r w_{13} - \frac{1}{r} w_{24}\]

Then the other spanning vectors of \(W\) must be:

\[
\begin{align*}
w_5 &= g_4.w_4 = r w_{23} + w_{15} - r^2 w_{13} - \frac{1}{r} w_{25} \\
w_1 &= g_2.w_4 - r w_4 = w_{24} - r w_{12} + w_{13} - \frac{1}{r} w_{34} \\
w_2 &= g_4.w_1 = w_{25} - r^2 w_{12} + r w_{13} - \frac{1}{r} w_{35} \\
w_3 &= g_3.w_2 - r w_2 = r w_{14} - r^2 w_{13} + w_{35} - \frac{1}{r} w_{45}
\end{align*}
\]

where we replaced \(l\) by \(r\). But \(W\) is an invariant subspace of \(V(6)\). In particular, it must be invariant under the action by \(g_5\). This is not compatible with the spanning set above. We conclude that it is impossible to have

\[W \downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(3,2)} \quad \text{or} \quad W \downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(2,2,1)}\]
and so \( W \) cannot be isomorphic to \( S^{(3,3)} \) or \( S^{(2,2,2)} \). Thus, by previous work, the existence of an irreducible 5-dimensional invariant subspace of \( \mathcal{V}^{(6)} \) implies that \( l \in \{ \frac{1}{r}, -\frac{1}{r^3} \} \). This completes the proof of the Theorem. \( \square \)

3.4 The cases \( l = r \) and \( l = -r^3 \)

In this section, we show that when \( l = r \) the representation \( \nu^{(n)} \) is reducible for all \( n \geq 4 \) and when \( l = -r^3 \), the representation is reducible for all \( n \geq 3 \). The latter point is true by Theorem 4 when \( n = 3 \) and by Theorem 5 when \( n = 4 \). To do so, we show that some proper invariant subspace \( K(n) \) of \( \mathcal{V}^{(n)} \), defined in the Proposition below, is nontrivial.

**Proposition 2.** For any two nodes \( i \) and \( j \) with \( 1 \leq i < j \leq n \), define

\[
 c_{ij} = \begin{cases} 
 g_{j-1} \cdots g_{i+1} e_i g_{i+1}^{-1} \cdots g_j^{-1} & \text{if } j \geq i + 2 \\
 e_i & \text{if } j = i + 1
\end{cases}
\]

Then, \( K(n) = \bigcap_{1 \leq i < j \leq n} \ker \nu^{(n)}(c_{ij}) \) is a proper invariant subspace of \( \mathcal{V}^{(n)} \). Moreover, any proper invariant subspace of \( \mathcal{V}^{(n)} \) must be contained in \( K(n) \).

**Proof.** \( K(n) \) is proper, as is visible on the expressions for \( \nu^{(n)}(e_i) \). Further, if an \( x_\beta \) is annihilated by all the \( g_i \) conjugates of the \( e_i \)'s, then \( \nu_k(x_\beta) \) is also annihilated by these same elements. Verification of this fact is tedious and can be found in [7], §2. Hence \( K(n) \) is invariant. Let \( W \) be a proper invariant subspace of \( \mathcal{V}^{(n)} \). By Proposition 1, we have \( \nu^{(n)}(c_{i,i+1})(W) = 0 \) for all \( i \) with \( 1 \leq i \leq n - 1 \). This fact is also true for the other conjugates \( c_{ij} \)'s. Hence \( W \) must be contained in \( K(n) \) \( \square \)

This is how an element \( c_{ij} \) is represented in the tangle algebra:

![Tangle Diagram](image)

It has two horizontal strands: one at the top and one at the bottom, each joining nodes \( i \) and \( j \) and moreover, when \( j \geq i + 2 \), such horizontal strands over-cross all the vertical strands that they intersect.

To show that \( \nu^{(n)} \) is reducible, it will suffice to exhibit a nontrivial element in \( K(n) \) when \( l = r \) or \( l = -r^3 \). The following Proposition shows that \( K(4) \) is nontrivial and irreducible when \( l = r \) and \( \mathcal{H}_{F,r^2}(4) \) is semisimple.

**Proposition 3.** Assume \( \mathcal{H}_{F,r^2}(4) \) is semisimple. There exists an irreducible 2-dimensional invariant subspace of \( \mathcal{V}^{(4)} \) if and only if \( l = r \). If so it is unique and it is \( K(4) \). Moreover, it is spanned over \( F \) by the two linearly independent vectors:

\[
 v_1 = w_{13} - \frac{1}{r} w_{23} + \frac{1}{r^2} w_{24} - \frac{1}{r} w_{14} \tag{28}
\]

\[
 v_2 = w_{12} - \frac{1}{r} w_{13} - \frac{1}{r} w_{24} + \frac{1}{r^2} w_{34} \tag{29}
\]
Proof. When $\mathcal{H}_{F,\tau}(4)$ is semisimple, the following three matrices

$$H_1 = \begin{bmatrix} -\frac{1}{r} & 1 \\ 0 & r \end{bmatrix}, \quad H_2 = \begin{bmatrix} r & 0 \\ 1 & -\frac{1}{r} \end{bmatrix}, \quad H_3 = \begin{bmatrix} -\frac{1}{r} & 1 \\ 0 & r \end{bmatrix}$$

define an irreducible matrix representation of degree 2 of $\mathcal{H}_{F,\tau}(4)$. Suppose $\mathcal{W}$ is an irreducible 2-dimensional invariant subspace of $\mathcal{V}(4)$. Then $\mathcal{W}$ has a basis $(v_1, v_2)$ of vectors such that the matrix of $\nu_i$ in this basis is $H_i$. Since $\nu_1(v_1) = -\frac{1}{r} v_1$ (resp. $\nu_2(v_1) = -\frac{1}{r} v_1$), there is no term in $w_{34}$ (resp. $w_{12}$) in $v_1$ and since $\nu_2(v_2) = -\frac{1}{r} v_2$, there is no term in $w_{14}$ in $v_2$. Let the $\lambda_{ij}$'s (resp. $\mu_{ij}$'s) denote the coefficients of the $w_{ij}$'s in $v_1$ (resp. $v_2$). We have:

$$\begin{cases} 
\lambda_{23} = -\frac{1}{r} \lambda_{13} & \text{and} \quad \lambda_{24} = -\frac{1}{r} \lambda_{14} \quad \text{by (14) with} \ i = 1 \text{ and } \gamma = -\frac{1}{r} \\
\lambda_{24} = -\frac{1}{r} \lambda_{23} & \text{and} \quad \lambda_{14} = -\frac{1}{r} \lambda_{13} \quad \text{by (15) with} \ i = 3 \text{ and } \gamma = -\frac{1}{r} \\
\mu_{34} = -\frac{1}{r} \mu_{24} & \text{by (14) with} \ i = 2 \text{ and } \gamma = -\frac{1}{r} \\
\mu_{13} = -\frac{1}{r} \mu_{12} & \text{by (15) with} \ i = 2 \text{ and } \gamma = -\frac{1}{r} 
\end{cases}$$

Hence, without loss of generality, $v_1 = w_{13} - \frac{1}{r} w_{23} + \frac{1}{r} w_{24} - \frac{1}{r} w_{14}$ and $v_2$ is a multiple of $w_{12} - \frac{1}{r} w_{13} + \mu (w_{24} - \frac{1}{r} w_{34}) + \mu' w_{23}$, where $\mu$ and $\mu'$ are scalars to determine. The relation $\nu_2(v_1) = r v_1 + v_2$ sets $v_2 = w_{12} - \frac{1}{r} w_{13} + \mu (w_{24} - \frac{1}{r} w_{34}) + \mu' w_{23}$, by just looking at the coefficient in $w_{12}$. The same relation yields $\mu = -\frac{1}{r}$ by looking at the coefficient in $w_{24}$. Next, by looking at the coefficient of $w_{23}$ in $\nu_3(v_2) = v_1 + r v_2$, we get $r \mu' - \frac{1}{r} = \mu$. Replacing $\mu$ by $-\frac{1}{r}$ now yields $\mu = 0$. We thus get the expressions in (28) and (29) for $v_1$ and $v_2$ respectively. Also, by looking at the coefficient in $w_{12}$ in $\nu_1(v_2) = v_1 + r v_2$, we have $\frac{1}{r} - \gamma = r$, hence $l = r$. Conversely, if $l = r$, it is a direct verification that the vectors $v_1$ and $v_2$ given by the formulas (28) and (29) are linearly independent and that they verify the relations:

$$\begin{align*}
\nu_i(v_i) &= -\frac{1}{r} v_i \quad \text{when} \ i \in \{1, 2\} \quad \& \quad \nu_3(v_1) = -\frac{1}{r} v_1 \\
\nu_1(v_2) &= v_1 + r v_2 = \nu_3(v_2) \quad \& \quad \nu_2(v_1) = r v_1 + v_2
\end{align*}$$

Thus, their linear span over $F$ is an irreducible 2-dimensional invariant subspace of $\mathcal{V}(4)$. It remains to show that it is in fact $K(4)$. By Proposition 2, $Span_F(v_1, v_2)$ is contained in $K(4)$. If $K(4)$ is three-dimensional, either it is irreducible and so $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r} \}$ by Theorem 5. This is impossible as $l = r$. Or it is reducible and it must contain a one-dimensional invariant subspace. Then by Theorem 4, it forces $l = -\frac{r^3}{r}$, which is again impossible. If $K(4)$ is four-dimensional, then $K(4)$ is not irreducible as its dimension is not 1, 2 or 3. Since we just saw that there exists only one irreducible 2-dimensional invariant subspace of $\mathcal{V}(4)$ when $l = r$, $K(4)$ must then contain a one-dimensional invariant subspace, which is again impossible. For similar reasons, it is also impossible to have $k(4) = 5$, hence the only possibility that is left is to have $k(4) = 2$. We conclude that $K(4) = Span_F(v_1, v_2)$ and $K(4)$ is thus irreducible. \square

The next Proposition shows the reducibility of the representation when $l = r$ and $n \geq 4$, where we still assume that $\mathcal{H}_{F,\tau}(n)$ is semisimple.

**Proposition 4.** Assume $l = r$ and $\mathcal{H}_{F,\tau}(n)$ is semisimple.

Then the vector $v_1 = w_{13} - \frac{1}{r} w_{23} + \frac{1}{r} w_{24} - \frac{1}{r} w_{14}$ of Proposition 3 belongs to $K(n)$ for all $n \geq 4$. Thus, $\nu(n)$ is reducible when $l = r$ and $n \geq 4$. 

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Proof. For $n = 4$, the result is contained in Proposition 3. When $i \geq 5$, we simply have for any $j \geq i + 2$, $\nu_{i+1}^{-1} \cdots \nu_{j-1}^{-1}(v_1) = \frac{1}{r^j - 1} v_1$ and since $\nu^{(n)}(c_{ij})v_1 = 0$, we see that $v_1$ is thus annihilated by all the $\nu^{(n)}(c_{ij})$ with $i \geq 5$. Also, since we just saw in Proposition 3 that $v_1$ is in $K(4)$, $v_1$ is annihilated by all the $\nu^{(n)}(c_{ij})$'s with $j \leq 4$. Thus, it suffices to check that $v_1$ is annihilated by $\nu^{(n)}(c_{ij})$, $\nu^{(n)}(c_{2j})$, $\nu^{(n)}(c_{3j})$ and $\nu^{(n)}(c_{4j})$ for any $j \geq 5$. We will use the following formulas that give the action of the $c_{ij}$'s on the basis vectors of the L-K space in some relevant cases here:

$$\nu^{(n)}(c_{ij})(w_{i,j-k}) = \frac{1}{r^k - 1} w_{i,j}$$

$$\nu^{(n)}(c_{ij})(w_{i-k,i}) = \nu^{(k)}(c_{ij})(w_{i,j})$$

$$\nu^{(n)}(c_{ij})(w_{i-t,j-s}) = \nu^{(t,s)}(c_{ij})(w_{i,j})$$

These formulas can be shown and pictured easily by using the tangles. Let’s take an example in $\nu^{(12)}$. The product tangle $c_{4,9} w_{2,7}$ as represented in the figure below:

![Tangle Diagram](image)

expands as follows, where we use the Kauffman skein relation:

![Expanded Tangle Diagram](image)

After doing a Reidemeister’s move of type II (as it is described in §2.2 of [9]) on the first tangle of the sum above, we see that it is zero. After “delooping” the second tangle of the sum and using Reidemeister’s move II twice, we see that it is obtained from the basis vector $w_{4,9}$ with non-crossed vertical strands.
by multiplying it to the right by $g_i^{-1}g_s^{-1}$. The resulting coefficient is $\frac{mr}{r} = \frac{1}{l}\left(\frac{1}{r} - \frac{1}{r}\right)$. If we call over-crossing a multiplication at the bottom by a $g_i$ in $H$ and under-crossing a multiplication at the bottom by a $g_i^{-1}$ in $H$, we see that in the last tangle of the sum, there are five under-crossings and two over-crossings. Thus, the resulting coefficient is $-\frac{mr}{r} = \frac{1}{l} - \frac{1}{r}$. When adding these two coefficients, we get $(\frac{1}{r} - \frac{1}{r})(\frac{1}{r} - \frac{1}{r})$, which is the coefficient in $(C_{2,2})$. Using this example as a support, it is easy to see that, more generally, for any $i$, the coefficient of $\nu^{(n)}(c_{ij})(w_{i-t,j-s})$, where $1 \leq t < i-1$ and $1 \leq s < j - i - 1$, is:

$$m \left( \frac{1}{l} \frac{1}{r(t-1)+(s-1)} - \frac{r((j-s)-i-1)}{r(t-1)+(j-i-1)} \right)$$

which after simplification is the coefficient in $(C_{1,s})$. Thus, we obtain the family of equations $(C_{1,s})_{1 \leq t \leq i-1, 1 \leq s \leq j-1}$. Similarly for fixed nodes $i$ and $j$, the two families of equations $(R_k)_{1 \leq k \leq i-1}$ and $(L_{j-k})_{1 \leq k \leq j-1}$ can be pictured easily by using the tangles, or can be established by using the definition of $c_{ij}$ and the expression for the representation in §2.3.2.

When $l = r$, we note that the action of $c_{ij}$ on $w_{i-t,j-s}$ is zero. From there, we have for $j \geq 5$, where we replaced $l$ by $r$:

$$\begin{align*}
\nu^{(n)}(c_{1,j})(v_1) = \nu^{(n)}(c_{1,j})(w_{13} - \frac{1}{r}w_{14}) = 0 & \text{ by } (R_{j-3}) \text{ and } (R_{j-4}) \\
\nu^{(n)}(c_{2,j})(v_1) = \nu^{(n)}(c_{2,j})(-\frac{1}{r}w_{23} + \frac{1}{r}w_{24}) = 0 & \text{ by } (R_{j-3}) \text{ and } (R_{j-4}) \\
\nu^{(n)}(c_{3,j})(v_1) = \nu^{(n)}(c_{3,j})(w_{13} - \frac{1}{r}w_{23}) = 0 & \text{ by } (L_{j-3,2}) \text{ and } (L_{j-3,1}) \\
\nu^{(n)}(c_{4,j})(v_1) = \nu^{(n)}(c_{4,j})(\frac{1}{r}w_{24} - \frac{1}{r}w_{14}) = 0 & \text{ by } (L_{j-4,2}) \text{ and } (L_{j-4,3})
\end{align*}$$

So $v_1$ is in $K(n)$ for all $n \geq 4$, as announced. It will be useful to note on the way that by the game of the coefficients, the equalities to the right of the first two lines of equations still hold when $l = -r^3$. \(\square\)

When $l = -r^3$, we have a similar result. This is the object of the next proposition.

**Proposition 5.** Assume $H_{P_r}$ is semisimple. When $l = -r^3$, the vector $u_1$ defined as in Theorem 5 by $u_1 = r w_{23} + w_{13} + (\frac{1}{r} + \frac{1}{r})w_{34} - w_{24} - \frac{1}{r}w_{14}$ belongs to $K(n)$ for all $n \geq 4$. Thus, when $l = -r^3$, the representation $\nu^{(n)}$ is reducible for every $n \geq 3$.

**Proof.** When $l = -r^3$, $\nu^{(3)}$ is reducible by Theorem 4 and $\nu^{(4)}$ is also reducible by Theorem 5. Suppose now $n \geq 5$. To show that $u_1$ is in $K(n)$, like in the case $l = r$, it will suffice to check that $\nu^{(n)}(c_{ij})(u_1) = 0$ for all $i \leq 4$ and $j \geq 5$. With $l = -r^3$, the coefficients of type $(C_{1,s})$ are no longer zero. But we have: $\nu^{(n)}(c_{2,j})(w_{13} - \frac{1}{r}w_{14}) = 0$ by $(C_{1,j-3})$ and $(C_{1,j-4})$. For $\nu^{(n)}(c_{3,j})(u_1)$, there is no shortcut and a complete evaluation must be performed. We have, where we respected the same order of the terms in Proposition 5 for the coefficients:

$$\nu^{(n)}(c_{3,j})(u_1) = r \frac{1}{r^{j-2}} + \frac{1}{r^{j-3}} + \left(\frac{1}{r} + \frac{1}{r}\right) \left(-\frac{1}{r^{j-5}}\right) + \left(\frac{1}{r^{j-4}} - \frac{1}{r^{j-6}}\right) \left(\frac{1}{r^{j-1}} + \frac{1}{r}\right) + \left(\frac{1}{r^{j-3}} - \frac{1}{r^{j-5}}\right) \left(\frac{1}{r^{j-3}} + \frac{1}{r}\right) w_{3,j}$$

The rules used are, in the same order: $(L_{j-3,1})$, $(L_{j-3,2})$, $(R_{j-4})$, $(C_{1,j-4})$ and $(C_{2,j-4})$. All the coefficients cancel nicely to give $\nu^{(n)}(c_{3,j})(u_1) = 0$. 

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Finally, for \( \nu^{(n)}(c_{4,l})(u_1) \), only the terms in \( u_1 \) whose last node is node number 4 yield a nonzero contribution, the first one contributing with a coefficient \( \frac{1}{r_0} \), the second one with a coefficient \(-\frac{1}{r_0} \) and the third one with a coefficient \(-\frac{1}{r_0} \) by rules \((L_{j-4,1}), (L_{j-4,2})\) and \((L_{j-4,3})\) respectively. The sum of these three coefficients is zero.

Thus, we are done with all the cases and \( u_1 \) belongs to \( K(n) \) for all \( n \geq 4 \). □ At this stage, we have shown that when \( l \) and \( r \) take the values of Theorem 1, the representation \( \nu^{(n)} \) is reducible. In the next section, we show conversely that if \( \nu^{(n)} \) is reducible, then \( l \) and \( r \) must related in the way described in Theorem 1.

4 Proof of the main theorem

We recall from Proposition 2 that any proper irreducible invariant subspace of \( V^{(n)} \) is an irreducible \( H_{F,r}(n) \)-module. When \( n = 3 \), the irreducible \( H_{F,r}(3) \)-modules have dimension 1 or 2. We showed in Theorem 4 that there exists a one-dimensional invariant subspace of \( V^{(3)} \) if and only if \( l \in \{r^3, \frac{1}{r^3}\} \) and we saw in Theorem 5 that there exists an irreducible 2-dimensional invariant subspace of \( V^{(3)} \) if and only if \( l \in \{1, -1\} \). Hence the main theorem is proven for \( n = 3 \). When \( n = 4 \), the irreducible \( H_{F,r}(4) \)-modules have dimensions 1, 2 or 3. By Theorem 4 (resp Theorem 5, resp Proposition 3), there exists a one-dimensional (resp an irreducible 2-dimensional, resp an irreducible 3-dimensional) invariant subspace of \( V^{(4)} \) if and only if \( l = \frac{1}{r^3} \) (resp \( l = r \), resp \( l \in \{-r^3, \frac{1}{r^3}, -\frac{1}{r^3}\}\)). Thus, the main theorem also holds for \( n = 4 \). Suppose now \( n \geq 5 \). By §3, it suffices to prove that if \( \nu^{(n)} \) is reducible, then \( l \in \{r, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}\} \) and the proof of the main theorem will be complete. We show it for \( n = 5 \) and \( n = 6 \), then proceed by induction on \( n \).

4.1 The case \( n = 5 \)

If \( W \) is an irreducible proper invariant subspace of \( V^{(5)} \), then \( \dim(W) \in \{1, 4, 5, 6\} \), as \( W \) is an irreducible \( H_{F,r}(5) \)-module by Corollary 1 of §3.1. If \( \dim(W) = 5 \), it forces \( l = r \) by Result 2. If \( \dim(W) = 1 \), it forces \( l = \frac{1}{r^3} \) by Theorem 4 and if \( \dim(W) = 4 \), it forces \( l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\} \) by Theorem 5. From now on, assume that \( l \notin \{r, \frac{1}{r^3}, -\frac{1}{r^3}\} \). We will show that \( l = -r^3 \). By choice for \( l \) and \( r \), we have \( \dim(W) = 6 \). Then \( W \cap V^{(4)} \neq \{0\} \), as otherwise \( W \oplus V^{(4)} \subseteq V^{(5)} \), which implies on the dimensions \( \dim W \leq 10 - 6 = 4 \). Since \( W \cap V^{(4)} \) is a proper invariant subspace of \( V^{(4)} \), the representation \( \nu^{(4)} \) is then reducible, which yields \( l \in \{-r^3, \frac{1}{r^3}, -\frac{1}{r^3}\} \). We will show that it is impossible to have \( l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\} \), unless \( l = -r^3 \).

Let’s first assume that \( l = \frac{1}{r^3} \) and show that under our assumptions, it forces \( l = -r^3 \). When \( l = \frac{1}{r^3} \), there exists a one-dimensional invariant subspace, say \( U_1 \), of \( V^{(4)} \) by Theorem 4 and by Proposition 2 it is contained in \( K(4) \). In particular the dimension \( k(4) \) of \( K(4) \) is 1, 2, 3, 4 or 5. But since \( W \cap V^{(4)} \) is a proper invariant subspace of \( V^{(4)} \), it must be contained in \( K(4) \) by Proposition 2. Hence we have \( k(4) \geq \dim(W \cap V^{(4)}) \). Also, since \( \dim(W \cap V^{(4)}) = \dim(W) + \dim(V^{(4)}) - \dim(W + V^{(4)}) \geq 12 - \dim(V^{(5)}) = 2 \), we get \( k(4) \geq 2 \). By semisimplicity of \( H_{F,r}(4) \) and by uniqueness of a one-dimensional invari-
stant subspace of $\mathcal{V}^{(4)}$ when it exists, $\mathcal{U}_l$ then has a summand $S$ in $K(4)$ that is of dimension greater than or equal to 2. $S$ is an invariant subspace of $\mathcal{V}^{(4)}$ and it cannot contain a one-dimensional invariant subspace. Nor can it contain an irreducible 2-dimensional invariant subspace by Proposition 3 since we assumed $l \neq r$. Also, since $1 + \dim(S) = k(4) \leq 5$, we note that $\dim(S) \leq 4$. Then, by the same arguments already exposed, $\dim(S) = 3$ and $S$ is irreducible. By Theorem 5, we now get $l \in \{-\frac{1}{r}, \frac{1}{r}, -r^3\}$. But since $l = \frac{1}{r}$, it forces $l = -r^3$ as it is impossible to have $(r^2)^4 = 1$ when $\mathcal{H}_{r, r^2}(5)$ is semisimple.

Assume next that $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. We show that these values lead to a contradiction. First, by choice for $l$ and Theorem 5 (case $n = 4$), $\mathcal{V}^{(4)}$ contains an irreducible 3-dimensional invariant subspace and by Proposition 2, this proper invariant subspace must be contained in $K(4)$. Hence $k(4) \geq 3$. Since there cannot exist any one-dimensional invariant subspace of $\mathcal{V}^{(4)}$ (as $l \neq \frac{1}{r}$ when $(r^2)^4 \neq 1$) or any irreducible 2-dimensional invariant subspace of $\mathcal{V}^{(4)}$ (as $l \neq r$), we cannot have $k(4) \in \{4, 5\}$. Thus, we have $k(4) = 3$ and so $K(4)$ is irreducible. Now the irreducibility of $K(4)$ and the fact that $0 \subset \mathcal{W} \cap \mathcal{V}^{(4)} \subseteq K(4)$ implies that

$$\mathcal{W} \cap \mathcal{V}^{(4)} = K(4) \quad (30)$$

We show that this is impossible.

Consider first the case when $l = \frac{1}{r}$. When $l = \frac{1}{r}$, the vector $w_{14} - w_{23}$ belongs to $K(4)$. Indeed, this vector is

$$\frac{1}{r^2 + \frac{1}{r^4}} \left( r v_1^{(4)} + \left( r - \frac{1}{r} \right) v_2^{(4)} - \frac{1}{r} v_3^{(4)} \right)$$

Then it also belongs to $\mathcal{W}$. It follows that $\mathcal{V}^{(5)}(e_4)(w_{14} - w_{23}) = \frac{1}{r} x_{\alpha_4} \in \mathcal{W}$, as $\mathcal{W}$ is an invariant subspace of $\mathcal{V}^{(5)}$. But then, by the same argument as in the proof of Proposition 1, $\mathcal{W}$ is the whole space $\mathcal{V}^{(5)}$, in contradiction with $\mathcal{W}$ is proper. Thus, it is impossible to have (30) and so $l$ cannot take the value $\frac{1}{r}$.

Consider now the case when $l = -\frac{1}{r}$. The vector $w_{14} - w_{23} + w_{12} + w_{34}$ belongs to $K(4)$ since it is

$$\frac{1}{r + \frac{1}{r}} \left( v_1^{(4)} + v_3^{(4)} \right)$$

By (30), this vector also belongs to $\mathcal{W}$. But then

$$\mathcal{V}^{(5)}(e_4)(w_{14} - w_{23} + w_{12} + w_{34}) = \left( 1 + \frac{1}{r^2} \right) x_{\alpha_4} \in \mathcal{W}$$

As $(r^2)^2 \neq 1$, this implies in turn that $x_{\alpha_4}$ is in $\mathcal{W}$. Then $\mathcal{W}$ is the whole space $\mathcal{V}^{(5)}$, a contradiction. Thus, it is also impossible to have $l = -\frac{1}{r}$.

We have now shown that if $\mathcal{V}^{(5)}$ is reducible and $l \not\in \{r, \frac{1}{r}, \frac{1}{r^4}, -\frac{1}{r^4}\}$, then $l = -r^3$. Thus, if $\mathcal{V}^{(5)}$ is reducible, then $l \in \{r, -r^3, \frac{1}{r^4}, -\frac{1}{r^4}\}$.

### 4.2 The case $n = 6$

Let $\mathcal{W}$ be an irreducible proper invariant subspace of $\mathcal{V}^{(6)}$. So $\mathcal{W}$ is an irreducible $\mathcal{H}_{r, r^2}(6)$-module. The irreducible representations of $\mathcal{H}_{r, r^2}(6)$ have degrees 1, 5, 9, 10, 16. The vector space $\mathcal{V}^{(6)}$ is 15-dimensional. Hence $\dim(\mathcal{W}) \in$
If \( \dim(V) = 1 \), then \( l = \frac{1}{r^2} \) by Theorem 4 and if \( \dim(V) = 5 \), then \( l \in \left\{ \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \right\} \) by Theorem 5. Suppose now \( l \not\in \left\{ \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \right\} \). Then \( \dim(V) \geq 9 \), which implies in particular that \( \mathcal{W} \cap \mathcal{V}^{(5)} \neq \{0\} \). Moreover, \( \mathcal{W} \cap \mathcal{V}^{(5)} \) is a proper subspace of \( \mathcal{V}^{(5)} \), as otherwise \( \mathcal{W} \) would contain \( \mathcal{V}^{(5)} \) and would in fact be the whole space \( \mathcal{V}^{(6)} \). Hence we see that \( \nu^{(5)} \) is reducible, which implies that

\[
l \in \left\{ r, r^{-3}, \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \right\}
\]  

(31)

by the case \( n = 5 \). Also, if \( \mathcal{W} \cap \mathcal{V}^{(4)} = \{0\} \), then \( \mathcal{W} \oplus \mathcal{V}^{(4)} \subseteq \mathcal{V}^{(6)} \) and so \( \dim(\mathcal{V}) \leq 15 - 6 = 9 \). Then \( \dim(\mathcal{W}) = 9 \). We notice that \( \dim(\mathcal{V}) + \dim(\mathcal{V}^{(4)}) = \dim(\mathcal{V}^{(6)}) \). Thus, we get \( \mathcal{W} \oplus \mathcal{V}^{(4)} = \mathcal{V}^{(6)} \). But since \( \mathcal{V} \subseteq \mathcal{K}(6) \) by Proposition 2, we must have in particular \( \nu^{(6)}(e_5)(\mathcal{V}) = 0 \). But \( e_5 \) also acts trivially on \( \mathcal{V}^{(4)} \).

It follows that \( e_5 \) acts trivially on the direct sum \( \mathcal{W} \oplus \mathcal{V}^{(4)} \), hence acts trivially on \( \mathcal{V}^{(6)} \). This is a contradiction. Hence, we have \( \mathcal{W} \cap \mathcal{V}^{(4)} \neq \{0\} \). Also, \( \mathcal{W} \cap \mathcal{V}^{(4)} \) is a proper invariant subspace of \( \mathcal{V}^{(4)} \). Consequently, \( \nu^{(4)} \) is reducible and by the case \( n = 4 \), we have

\[
l \in \left\{ r, r^{-3}, \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \right\}
\]  

(32)

Since \( r^2 \neq 1 \), \( (r^2)^3 \neq 1 \) and \( (r^2)^6 \neq 1 \) when \( \mathcal{H}_{F,r^2}(6) \) is semisimple, (31) and (32) imply that \( l \in \{ r, r^{-3} \} \). Thus, if \( \nu^{(6)} \) is reducible and \( l \not\in \{ \frac{1}{r}, \frac{1}{r^3}, -\frac{1}{r} \} \), then \( l \in \{ r, r^{-3} \} \). So if \( \nu^{(6)} \) is reducible, then \( l \in \{ r, r^{-3}, \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \} \).

### 4.3 Proof of the main theorem when \( n \geq 7 \)

By the work from previous parts, the main theorem holds for \( n \in \{ 3, 4, 5, 6 \} \). When \( n \geq 7 \), we proceed by induction to prove the theorem. Given an integer \( n \) with \( n \geq 7 \), suppose the main theorem holds for \( \nu^{(n-1)} \) and \( \nu^{(n-2)} \). We already saw that when \( l \in \{ r, r^{-3}, \frac{1}{r^2}, \frac{1}{r}, -\frac{1}{r} \} \), the representation \( \nu^{(n)} \) is reducible. We will show conversely that if \( \nu^{(n)} \) is reducible, it forces these values for \( l \) and \( r \). The theorem will then be proven. Suppose \( \nu^{(n)} \) is reducible and let \( \mathcal{W} \) be an irreducible nontrivial proper invariant subspace of \( \mathcal{V}^{(n)} \). By Corollary 1, we know that \( \mathcal{W} \) is an irreducible \( \mathcal{H}_{F,r^2}(n) \)-module. The following proposition is part of the author’s work in [4].

**Proposition 6.** Let \( K \) be a field of characteristic zero. Let \( n \) be an integer with \( n \geq 9 \). Every irreducible \( K \text{Sym}(n) \)-module is either isomorphic to one of the Specht modules \( S^{(n)}, S^{(n-1,1)}, S^{(n-2,2)}, S^{(n-2,1,1)} \) or to one of their conjugates, or has dimension greater than \( \frac{(n-1)(n-2)}{2} \).

We have the Corollary on the dimensions:

**Corollary 2.** Assume \( \mathcal{H}_{F,r^2}(n) \) is semisimple.

1. Let \( \mathcal{D} \) be an irreducible \( F \text{Sym}(n) \)-module with \( n = 7 \) or \( n \geq 9 \), where \( F \) is a field of characteristic zero. Then, there are two possibilities:
   
   \[
either \quad \dim \mathcal{D} \in \{ 1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2} \} 
   \]

   or
   
   \[
   \dim \mathcal{D} > \frac{(n-1)(n-2)}{2} 
   \]
Let’s go back to the proof of the Main Theorem. Suppose first

\[ W \in \{ \text{dim} \; H \} \]

of the irreducible \( F \) Sym \( m \) dimensions of the irreducible \( H \) in characteristic zero, when the Hecke algebra statement also holds by direct investigation, using for instance the Hook formula. Proof. (ii) can be seen directly by using the Hook formula. Point (i) is for \( n \geq 9 \) a direct consequence of Proposition 6 after noticing that \( S(n-2, 2) \) has dimension \( \frac{n(n-3)}{2} \) and \( S(n-2, 1, 1) \) dimension \( \frac{(n-1)(n-2)}{2} \). When \( n = 7 \), the statement also holds by direct investigation, using for instance the Hook formula. In characteristic zero, when the Hecke algebra \( H_{F,r^2}(n) \) is semisimple, the dimensions of the irreducible \( F \) Sym \( n \)-modules are the same as the dimensions of the irreducible \( H_{F,r^2}(n) \)-modules, hence (iii). □

Claim 2. Let \( W \) be a subspace of \( V(n) \).
If \( \dim W > n - 1 \), then \( W \cap V^{(n-1)} \neq \{0\} \).
If \( \dim W > 2n - 3 \), then \( W \cap V^{(n-2)} \neq \{0\} \)

Proof. If \( W \cap V^{(n-1)} = \{0\} \), the L-K space \( V(n) \) contains the direct sum \( W \oplus V^{(n-1)} \), which yields on the dimensions: \( \dim W + \frac{(n-1)(n-2)}{2} \leq \frac{n(n-1)}{2} \). Then \( \dim W \leq n - 1 \). Similarly, if \( W \cap V^{(n-2)} = \{0\} \), we get
\[ \dim W \leq \frac{n(n-1)}{2} - \frac{(n-2)(n-3)}{2} = 2n - 3. \] □

Lemma 4. When \( n > 6 \), we have \( \frac{n(n-3)}{2} > 2n - 3 \) and \( \frac{n(n-3)}{2} > n - 1 \).

By the claim and the lemma, the intersections \( W \cap V^{(n-1)} \) and \( W \cap V^{(n-2)} \) are nontrivial. Since \( W \) is proper in \( V(n) \), \( W \) cannot contain \( V^{(n-1)} \). Nor can it contain \( V^{(n-2)} \). Hence \( W \cap V^{(n-1)} \) (resp \( W \cap V^{(n-2)} \)) is a proper nontrivial invariant subspace of \( V^{(n-1)} \) (resp \( V^{(n-2)} \)). Now \( V^{(n-1)} \) and \( V^{(n-2)} \) are both reducible. Since we assumed the main theorem to be true for \( V^{(n-1)} \) and \( V^{(n-2)} \), we get:
\[ l \in \left\{ r, -r^3, \frac{1}{r^{2n-3}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-3}} \right\} \cap \left\{ r, -r^3, \frac{1}{r^{2n-7}}, \frac{1}{r^{n-6}}, -\frac{1}{r^{n-5}} \right\} \]

Since \( r^2 \neq 1 \), \( r^{2(n-3)} \neq 1 \) and \( r^{2n} \neq 1 \) when \( H_{F,r^2}(n) \) is semisimple, it only leaves the possibility \( l \in \{ r, -r^3 \} \).

When \( n = 8 \), if \( l \notin \left\{ \frac{1}{r^{2n-3}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-3}} \right\} \), then we have \( \dim W \geq 14 > 13 = 2 \times 8 - 3 \).

Hence the same method applies and yields again \( l \in \{ r, -r^3 \} \).

Thus, we have shown that if the representation is reducible and \( l \notin \left\{ \frac{1}{r^{2n-3}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-3}} \right\} \), then \( l \in \{ r, -r^3 \} \). □

5 Non semisimplicity of the BMW algebra for some specializations of its parameters

In §2.1, we let the Hecke algebra \( H \) act on the base field \( F \) by \( g_i \cdot 1 = r \) for all \( i \in \{ 3, \ldots, n - 1 \} \). If we consider the action given by \( g_i \cdot 1 = -\frac{1}{r} \) instead, \( F \) is
again a left $H$-module for this action and we get another left $B$-module of dimension $\frac{n(n-1)}{2}$ by considering again the tensor product $B_1 \otimes_H F$. We call this representation the conjugate L-K representation. By the symmetry of the roles played by $r$ and $-\frac{1}{r}$, when $n \geq 4$ and $\mathcal{H}_{F,r}(n)$ is semisimple, the conjugate L-K representation is reducible exactly when $l \in \{-\frac{1}{r}, \frac{1}{r}, -r^{2n-3}, r^{n-3}, -r^{n-3}\}$. In particular, when $n \geq 6$, since $\frac{1}{r} \notin \{r, -r^3, \frac{1}{r}, \frac{1}{r^3}, -\frac{1}{r^3}\}$, the two representations are not equivalent. This is also true when $n \in \{4, 5\}$. For instance, for the L-K representation, the trace of the matrix of the left action by $g_{n-1}$ is $\frac{(n-2)(n-3)}{2} r + \frac{1}{r} - (n-2)m$. For the conjugate representation it is $\frac{(n-2)(n-3)}{2} (-\frac{1}{r}) + \frac{1}{r} - (n-2)m$.

We note that Proposition 1 remains valid for the conjugate L-K representation. A consequence of this Proposition is that when the representation is reducible, it is indecomposable. Then the BMW algebra is not semisimple for the values of $l$ and $r$ for which the L-K representation or its conjugate representation are reducible. This is the statement of Theorem 2.

6 Conclusion and future developments

In [2], it is established that $I_1/I_2$ is generically semisimple where $I_1$ is the two-sided ideal $B e_1 B$ and $I_2$ the two-sided ideal generated by all the products $e_i e_j$ with $|i-j| > 2$. For each irreducible representation of the Hecke algebra of type $A_{n-2}$ of degree $\theta$, the authors build a generically irreducible representation of $B/I_2$ of degree $\frac{n(n-1)}{2} \theta$ and show that these are all the inequivalent generically irreducible representations of $I_1/I_2$. One of the two so-built inequivalent representations of $B/I_2$ of degree $\frac{n(n-1)}{2}$ is the Lawrence-Krammer representation. The other one is obtained from the first one by replacing $r$ by $-\frac{1}{r}$. Since the representation $B_1 \otimes_H F$ built in this paper is a generically irreducible representation of $B/I_2$ and its kernel does not contain $I_1$, it must be equivalent to the Lawrence-Krammer representation. We think that the other generically irreducible $I_1/I_2$-modules are those with $F$ replaced by an irreducible $H$-module and that by studying these representations we could show that $I_1/I_2$ is semisimple if and only if $l$ and $r$ don’t take the specializations of the Main Theorem.

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