To See a World in a Grain of Sand: From WET-CMT to ETT-DMT to MTT-DMT to AMTT-DMT

Ren-Zun Lian, Long Li, and Kunyi Zhang
School of Electronic Engineering
Xidian University
Xi’an, China
Correspondence to: rzlian@vip.163.com (R.-Z. Lian); lilong@mail.xidian.edu.cn (L. Li)

Abstract—Some commonly used electromagnetic (EM) modal analysis theories are simply reviewed. The theories include the classical Sturm–Liouville eigen-mode theory (SL-EMT), the scattering matrix (SM)-based and integral equation (IE)-based characteristic mode theories (CMTs) for scatterers, the work-energy theorem (WET)-based CMTs for scatterers and lumped-port-driven structures, and the energy transport theorem (ETT)-based decoupling modal theory (DMT) for wave-port-fed structures. The applicable scopes and evolution progress of the theories are emphasized, and the core physical principles and mathematical formulas related to the ETT-based DMT (ETT-DMT) for wave-port-fed metallic transmitting antennas are given. Inspired by the evolution progress from the WET-based CMT (WET-CMT) to ETT-DMT and the principles and formulas of ETT-DMT, this paper focuses on proposing two novel modal analysis theories: the momentum transport theorem (MTT)-based and angular-momentum transport theorem (AMTT)-based DMTs for wave-port-fed metallic transmitting antennas. The engineering applications for the MTT-based DMT (MTT-DMT) and the AMTT-based DMT (AMTT-DMT) will be provided in our future papers. By outlining the evolution progress from WET-CMT to ETT-DMT to MTT-DMT to AMTT-DMT, this paper displays how to see a world in a grain of sand.

Keywords—energy, momentum, angular-momentum, conservation law, transport theorem, work-energy theorem, modal analysis, decoupled mode (DM), characteristic mode (CM)

I. INTRODUCTION

The main researching topics of information science involve the aspects of the transmitting, transferring, and receiving for information. The above aspects cannot be separated from the information carriers. In human society, electromagnetic (EM) energy had long been one of the most important information carriers. Recently, it had been found out that EM momentum [1]–[3] and EM angular-momentum [4]–[6] are also effective for carrying information.

The various EM devices are just the media used to control EM energy, momentum, and angular-momentum (and then to modulate information) by the human beings. The deep studies for the working modes of a device is indispensable for understanding how the device uses energy, momentum, and angular-momentum. For a linear device, it has many physically realizable working modes, and all the modes constitute a linear space—modal space [7], [8]. Evidently, to study all the modes in the modal space one by one is impracticable, because the modal space contains an infinite number of modes. A practicable way is to study the fundamental modes of the modal space, where the fundamental modes are decoupled and complete [7], [8]. Now, the theories for studying the fundamental modes are collectively referred to as modal analysis theory, or simply called modal theory/analysis.

A. Eigen-Mode Theories (EMTs)

Among the various modal theories, the earliest and the most classical one is the famous eigen-mode theory (EMT), which focuses on studying the energy-decoupled fundamental modes—eigen-modes. EMT has had a very long history, and dates back to the pioneering acoustic experiments done by Sauveur in the late 1600s. Through the unremitting efforts of generations of mathematicians (including D. Bernoulli, d’Alembert, Euler, Lagrange, and Fourier etc.), EMT was finally developed into its modern form—Sturm–Liouville EMT (SL-EMT)—by Sturm [9], [10] and Liouville [9] around 1836. The comprehensive summaries for the scalar-field-oriented SL-EMT (or called scalar SL-EMT) can be found in [11] and [12].

During the World War II, the various closed EM devices (such as waveguides and cavities etc.) were deeply studied and widely applied. To meet the requirements of scientific researches and engineering applications, the scientists (including Lord Rayleigh and Debye etc.) further generalized the scalar SL-EMT for acoustic problems to a more advanced version for the vectorial EM problems of closed EM devices. The comprehensive summaries and reviews for the vectorial SL-EMT of closed EM devices can be found in [13] and [14]. In [13] and [14], the variable separation method used to calculate the energy-decoupled eigen-modes of closed EM devices were discussed systematically.
In 1934, Eisenhart [15] proved that: in the 3-D Euclidean space, there are only 11 kinds of variable-separable coordinate systems, such as rectangular, cylindrical, and spherical systems. In the variable-separable coordinate systems, the closed-EM-device-oriented vectorial SL-EMT can be generalized to an alternative version for open EM devices, by using the variable separation method. The comprehensive summaries and reviews for the open-EM-device-oriented vectorial SL-EMT can be found in [16] and [17]. Unfortunately, when the boundary surface of the objective EM device does not coincide with the coordinate surfaces of the variable-separable coordinate systems, the variable separation method cannot be used.

B. Characteristic Mode Theories (CMTs)

To resolve the above problem on variable separation method, Garbacz [18], [19] and Garbacz and Turpin [20] generalized the classical concept of eigen-mode to a novel concept of characteristic mode (CM) in the late 1960s, and established a scattering matrix (SM)-based theory—CM theory (CMT)—to calculate the CMs. The SM-based CMT (SM-CMT) is a pioneering work in the realm of modern EM modal analysis. However, SM-CMT has some drawbacks, for example: SM-CMT is only applicable to the lossless scatterers, but not applicable to the lossy scatterers; the modal calculation method used in SM-CMT is very complicated.

To overcome the above-mentioned drawbacks on SM-CMT, Harrington and Mautz [21]–[23], Harrington et al. [24], and Chang and Harrington [25] developed another integral equation (IE)-based CMT. The IE-based CMT (IE-CMT) is suitable for both lossless and lossy scatterers, and also provided a more easily realized CM calculation method. Thus, IE-CMT got a rapid development and a wide application. The comprehensive summaries and reviews for IE-CMT can be found in [26]–[28]. However, some problems on the theoretical foundation of IE-CMT were also exposed recently [29], for example: it was not clarified whether the IE-based CMs are energy-decoupled.

Focusing on the problems on IE-CMT, some researchers proposed an alternative entire-structure-oriented work-energy theorem (ES-WET)-based CMT for metallic [30]–[32], [33, Chap. 3], material [33, Chap. 4], [34]–[36], and metal-material composite [33, Chap. 5], [37] scatterers. The ES-WET-based CMT (ES-WET-CMT) promotes the solving of the problems, and proves that the ES-WET-based CMs are energy-decoupled. Moreover, ES-WET-CMT also reveals that: SM-CMT, IE-CMT, and ES-WET-CMT are the modal analysis theories for scatterers, but not for transmitting antennas [36].

To realize an effective modal analysis for the lumped-port-driven transmitting antennas, [38] and [39, Secs. 4.2–4.5] proposed a novel CMT in the partial-structure-oriented work-energy theorem (PS-WET) framework. Later, the PS-WET-based CMT (PS-WET-CMT) was further applied to multi-coil wireless power transfer (WPT) systems in [39, Sec. 4.6] and [40]. (In fact, [30] tried to use ES-WET-CMT to physically explain the multi-coil WPT phenomenon, but did not succeed.) In addition, [38] proved that the PS-WET-based CMs are energy-decoupled, and emphasized that: the lumped-port-driven and wave-port-fed antennas have different working mechanisms, and PS-WET-CMT is not applicable to the latter.

In this paper, the ES-WET-CMT and PS-WET-CMT are collectively referred to as work-energy theorem (WET)-based CMT, and collectively denoted as WET-CMT.

![Fig. 1. A simple illustration for the evolution progress from ES-WET-CMT to PS-WET-CMT to ETT-DMT to MTT-DMT to AMTT-DMT. Some more detailed discussions and comparisons can be found in [36, Sec. I], [39, Sec. 1.5], and [42, Sec. 1.4.2].](image-url)
C. Decoupling Mode Theories (DMTs)

As a supplement of the PS-WET-CMT for lumped-port-driven transmitting antennas, [39, Sec. 3.2], [41], and [42, Chap. 6] proposed an alternative energy/power transport theorem (ETT/PTT)-based modal analysis theory—decoupling mode theory (DMT)—for wave-port-fed transmitting antennas. Afterwards, the ETT-based DMT (ETT-DMT) was further generalized to wave-guiding structures [39, Sec. 3.4], [42, Secs. 3.2–3.5], [43], free space [39, Sec. 3.5], [42, Sec. 3.6], and receiving antennas [39, Sec. 3.3], [42, Chap. 7]. In addition, the ETT-DMTs for waveguide-antenna transmitting systems [39, Sec. 3.6.2], [42, Sec. 8.2] and transmitting-receiving systems [39, Sec. 3.6.3], [42, Sec. 8.3] were also built. In fact, the modes derived from ETT-DMT are energy-decoupled.

Inspired by the above-mentioned ETT-DMT, this paper is dedicated to doing the following novel works:

1) The core physical principles and mathematical formulations related to the ETT-DMT for wave-port-fed metallic transmitting antennas are provided.

2) The momentum transport theorem (MTT) governing the momentum transport process related to wave-port-fed metallic transmitting antennas is derived, and the MTT-based DMT (MTT-DMT) for wave-port-fed metallic transmitting antennas is established, and then the momentum-decoupled fundamental modes are obtained.

3) The angular-momentum transport theorem (AMTT) governing the angular-momentum transport process related to wave-port-fed metallic transmitting antennas is derived, and a method to build the AMTT-based DMT (AMTT-DMT) for wave-port-fed metallic transmitting antennas is proposed, and then the angular-momentum-decoupled fundamental modes are obtained.

In our future papers, the MTT-DMT and AMTT-DMT for wave-port-fed metallic transmitting antennas will be further generalized to the other kinds of wave-port-fed structures, and the related engineering applications will also be provided.

The evolution progress from WET-CMT to ETT-DMT to MTT-DMT to AMTT-DMT is shown in Fig. 1. With the aid of the evolution progress, this paper exhibits how “to see a world in a grain of sand [44]” in scientific research.

D. Organization and Symbolic System of This Paper

This paper is organized as follows. Section II describes the geometry and topology of the EM transmitting problem discussed in this paper. Sections III, IV, and V discuss the ETT-DMT, MTT-DMT, and AMTT-DMT for wave-port-fed metallic transmitting antennas, respectively. Section VI provides some typical numerical examples. Section VII summarizes and concludes this paper. Some trivial formulations related to this paper are provided in the Appendix.

In this paper, the $e^{i\omega t}$ convention and the inner product form $\langle f, g \rangle = \int f^* \cdot g \, d\Omega$ are used throughout, where the superscript “ $^*$ ” is the conjugate transpose operation for a scalar/vector/matrix. The frequency-domain and time-domain EM fields are represented by $(\mathbf{E}, \mathbf{H})$ and $(\mathbf{E}, \mathbf{A})$, respectively. The frequency-domain power quadratic matrix and current expansion coefficient vector are denoted as $\mathbb{P}$ and $\mathbb{C}$, respectively.

II. PRELIMINARY

To avoid the risk that the core physical principle is covered by the complicated mathematical formulations, this paper focuses on one of the simplest (and also the most classical) wave-port-fed transmitting antennas—metallic horn. After finishing the establishments for ETT-DMT, MTT-DMT, and AMTT-DMT, the DMTs will be further generalized to the more complicated antennas in our future papers.

The horn antenna is shown in Fig. 2, and it has a thick metallic electric wall. The horn is fed by a feeding system constituted by a generator and a waveguide, and the horn is surrounded by the free-space environment with the constitutive parameters $(\mu_0, \varepsilon_0)$. The topological structure of the horn is shown in Fig. 3.

As shown in Fig. 3, the interactive port between the horn and the waveguide is denoted as $S_e$, and the normal direction of $S_i$ is denoted as $\hat{n}_i$, and $\hat{n}_i$ points to the surrounding environment. The boundary surface of the thick horn wall is divided into two parts—$S_h^i$ and $S_h^o$, and the interface between the feeding system and the surrounding environment is denoted as $S_f$. The union of the electric walls $S_e^i$, $S_h^i$, and $S_f$ is denoted as $S_e$, i.e., $S_e = S_h^i \cup S_h^o \cup S_f$.

The region occupied by the surrounding environment is denoted as $V_{en}$. Obviously, $S_e \cup S_i$ is a closed surface, and it is just the inner boundary of $V_{en}$, while the outer boundary of $V_{en}$ is denoted as $S_e$, which is usually a spherical surface with infinite radius. In addition, the surface $S_{en}$ shown in Fig. 3 is an arbitrary closed surface enclosing the whole generator-waveguide-horn system, and the region sandwiched between $S_e \cup S_i$ and $S_{en}$ is denoted as $V_{en}$. It is not difficult to find out that $(V_{en}, S_e)$ is only a special case of $(V_{en}, S_{en})$. 
III. ETT-BASED DMT (ETT-DMT)

In this section, the energy transport process of the horn-antenna problem shown in Fig. 3 and the ETT quantitatively governing the energy transport process are discussed. Under the ETT framework, an effective method to calculate the energy-decoupled modes (E-DMs) of the horn antenna is proposed.

A. Energy Transport Theorem (ETT)

In the generator, an energy is generated. Along the waveguide, the energy flows from the generator to the horn. With the modulation of the horn, the energy is released into the environment. By flowing through the environment, a part of the energy—the radiative energy—will be transported to $S_r$ finally, and the other part of the energy—the non-radiative/non-propagating energy—will be reactively stored in $V_{\text{env}}$.

From the above qualitative analysis for the energy transport process, it is not difficult to find out that: by modulating the energy inputted into it, the horn realizes to control the space distribution of the energy released into the environment. Below, a quantitative expression for the energy transport process will be derived from Maxwell’s equations.

The fields $(\mathbf{E}, \mathbf{H})$ on $V_{\text{env}}$ satisfy the homogeneous Maxwell’s equations $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, where $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$. From the Maxwell’s equations, a special form of energy conservation law (ECL)—ETT—can be obtained, and its mathematical expression is as follows:

$$
\begin{align*}
\frac{\partial}{\partial t} \int_{S_{\text{inh}}} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS + \int_{S_{\text{inh}}} (\mathbf{E} \times \mathbf{H}) \cdot \frac{dS}{dt} + \int_{S_{\text{inh}}} \left[ W_{\text{mag}}^{\text{inh}} + W_{\text{ele}}^{\text{inh}} \right] dS_{\text{inh}} \\
= \int_{S_{\text{inh}}} \left[ \frac{\partial}{\partial t} \left( \mathbf{E} \times \mathbf{H} \right) \right] \cdot \mathbf{n} dS + \int_{S_{\text{inh}}} \left[ \left( \mathbf{E} \times \mathbf{H} \right) \cdot \frac{dS}{dt} \right] + \int_{S_{\text{inh}}} \left[ W_{\text{mag}}^{\text{inh}} + W_{\text{ele}}^{\text{inh}} \right] dS_{\text{inh}}
\end{align*}
$$

(1)

The integral of (1) is equal to find out that: the energy $\mathcal{E}^{\text{inh}}$ passing through $S_{\text{inh}}$ is transformed by the horn into two parts—the energy $\int_{S_{\text{inh}}} (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} dS$ transported from the horn input port $S_i$ to the surface $S_{\text{inh}}$, and the energy $W_{\text{mag}}^{\text{inh}} + W_{\text{ele}}^{\text{inh}}$ stored in $V_{\text{inh}}$. Similarly, the second equality of (1) has the following physical meaning: the energy $\mathcal{E}^{\text{inh}}$ passing through $S_{\text{inh}}$ is transformed by the horn into two parts—the radiated energy $\mathcal{E}^{\text{rad}}$ transported from the horn to the infinity by $(\mathbf{E}, \mathbf{H})$ and the energy $\left[ W_{\text{mag}}^{\text{rad}} + W_{\text{ele}}^{\text{rad}} \right]$ stored in $V_{\text{inh}}$.

Evidently, (1) is a quantitative description for the energy transport process of the horn-transmitting problem, and this is just the reason to call it ETT.

For the convenience of the following discussions, we also provide the frequency-domain power version of the above time-domain ETT as follows:

$$
\begin{align*}
\frac{1}{2} \int_{S_{\text{inh}}} \left( \mathbf{E} \times \mathbf{H} \right)^* \cdot \mathbf{n} dS + \int_{S_{\text{inh}}} \left( \mathbf{E} \times \mathbf{H} \right) \cdot \mathbf{n} dS + j2\omega \left[ W_{\text{mag}}^{\text{inh}} - W_{\text{ele}}^{\text{inh}} \right] \\
= \int_{S_{\text{inh}}} \left( \mathbf{E} \times \mathbf{H} \right)^* \cdot \mathbf{n} dS + j2\omega \left[ W_{\text{mag}}^{\text{inh}} - W_{\text{ele}}^{\text{inh}} \right]
\end{align*}
$$

(2)

called PTT (because of its power dimension), where $W_{\text{mag}}^{\text{inh}} = (1/4) < \mathbf{H} >_{V_{\text{inh}}}^{2}$ and $W_{\text{ele}}^{\text{inh}} = (1/4) < \mathbf{D} >_{V_{\text{inh}}}^{2}$.

Obviously, the power source in (2) is just the power $\mathcal{P}^{\text{inh}}$ inputted into the horn, so it is correspondingly called input power in this paper.

B. Input Power Operator (IPO)

If the equivalent surface electric and magnetic currents on $S_i$ are defined as $J_i = \mathbf{n} \times \mathbf{H}$ and $M_i = \mathbf{E} \times \mathbf{n}$, the field-interaction formed input power $\mathcal{P}^{\text{inh}} = (1/2) \int_{S_i} \left( \mathbf{E} \times \mathbf{H} \right) \cdot \mathbf{n} dS$ can be equivalently rewritten as the field-current interaction forms $\mathcal{P}^{\text{inh}} = -(1/2) < J_i \times \mathbf{E} >_{V_{\text{inh}}} = -(1/2) < M_i, \mathbf{H} >_{V_{\text{inh}}}$. In addition, Huygens-Fresnel principle implies that the fields $\mathbf{E}$ and $\mathbf{H}$ on $V_{\text{inh}}$ can be expressed in terms of the functions of $J_i$, $J_s$, and $M_i$ as $\mathbf{E} = -j \mu \mathbf{\mathcal{L}}_0 (J_i + J_s) - \mathbf{\mathcal{K}}_0 (M_i)$ and $\mathbf{H} = \mathbf{\mathcal{K}}_0 (J_i + J_s) - j \mu \mathbf{\mathcal{L}}_0 (M_i)$, in which $\mathbf{\mathcal{L}}_0$ is the induced electric current on $S_i$ and the operators $\mathbf{\mathcal{L}}_0$ and $\mathbf{\mathcal{K}}_0$ are defined as $\mathbf{\mathcal{L}}_0 (X) = \left[ 1 - (1/k_c^2) \mathbf{V}_V \right] G_0 (r, r') X(r') d\Omega'$ and $\mathbf{\mathcal{K}}_0 (X) = \mathbf{V} \cdot G_0 (r, r') X(r') d\Omega'$, where the free-space scalar Green’s function is $G_0 (r, r') = e^{-j k_c |r - r'|} / (4\pi |r - r'|)$.

Thus, the input power $\mathcal{P}^{\text{inh}}$ can be formulated in terms of the functions of $J_i$, $J_s$, and $M_i$ as the following input power operators (IPOs):

$$
\begin{align*}
\mathcal{P}^{\text{inh}} &= -(1/2) \left< J_i, -j \mu \mathbf{\mathcal{L}}_0 (J_i + J_s) - \mathbf{\mathcal{K}}_0 (M_i) \right>_{S_i} \\
&= -(1/2) \left< M_i, \mathbf{\mathcal{K}}_0 (J_i + J_s) - j \mu \mathbf{\mathcal{L}}_0 (M_i) \right>_{S_i}
\end{align*}
$$

(3)

In the above (3), the right-hand side of the first line is called the J-E interaction form (or simply called JE form) of IPO, and the right-hand side of the second line is called the H-M interaction form (or simply called HM form) of IPO, and the integral surface $S_i^+$ is the right-side surface of the input port $S_i$ shown in Fig. 3.

In fact, the currents involved in (3) are not independent of each other, because they satisfy the following IEs:

$$
\begin{align*}
\left[ \mathbf{\mathcal{K}}_0 (J_i + J_s) - j \mu \mathbf{\mathcal{L}}_0 (M_i) \right]_{S_i^+} &= \mathbf{J} \times \mathbf{n}_i \\
\left[ -j \mu \mathbf{\mathcal{L}}_0 (J_i + J_s) - \mathbf{\mathcal{K}}_0 (M_i) \right]_{S_i^+} &= \mathbf{n} \times \mathbf{M}_i \\
\left[ -j \mu \mathbf{\mathcal{L}}_0 (J_i + J_s) - \mathbf{\mathcal{K}}_0 (M_i) \right]_{S_i^-} &= 0
\end{align*}
$$

(4) (5) (6)

where the superscripts “tan” represent that the IEs are satisfied by the tangential components, and the subscript “$S_i^+$ / $S_i^-$” means that the corresponding field is calculated on the surface $S_i^+ / S_i^-$. The IEs (4) and (5) are based on the definition of $\mathbf{J}$ (DoJ) and the definition of $\mathbf{M}_i$ (DoM), respectively, and they
are theoretically equivalent to each other. The IE (6) originates from the homogeneous tangential electric field boundary condition on the electric wall \( S_e \).

Expanding the currents involved in (3) in terms of some proper basis functions, the integral operators in (3) are immediately discretized into the following matrix forms:

\[
P^{\text{in}} = \oint_{R} \sum_{i=1}^{n} \left[ \begin{array}{c} J_i \cr \bar{J}_i \cr M_i \end{array} \right] \cdot \left[ \begin{array}{c} \tilde{J}_i \cr \tilde{\bar{J}}_i \cr \tilde{M}_i \end{array} \right] \cdot \left[ \begin{array}{c} J_{\text{HM}} \cr J_{\text{HM}} \cr M_{\text{HM}} \end{array} \right],
\]

where \( J_i \), \( \bar{J}_i \), and \( M_i \) are the basis function expansion coefficient vectors of \( J_i \), \( \bar{J}_i \), and \( M_i \), respectively.

Applying the method of moments (MoM) to (4)–(6), the IEs can be discretized into some matrix equations. By solving the matrix equations originated from “(6) and DoJ (4)” and the matrix equations originated from “(6) and DoM (5)”, the following transmutations

\[
\left[ \begin{array}{c} J_i \cr \bar{J}_i \cr M_i \end{array} \right] = \left[ \begin{array}{c} \tilde{J}_i \cr \tilde{\bar{J}}_i \cr \tilde{M}_i \end{array} \right] \cdot \left[ \begin{array}{c} J_{\text{HM}} \cr J_{\text{HM}} \cr M_{\text{HM}} \end{array} \right]
\]

(8)
can be derived, where the subscripts “DoJ” and “DoM” are to emphasize that \( \mathcal{T}_{\text{DoJ}} \) and \( \mathcal{T}_{\text{DoM}} \) are the DoJ and DoM originated from “(6) and DoJ (4)” and “(6) and DoM (5)”, respectively.

Substituting the above (8) into the previous (7), the following matrix-formed IPO

\[
P^{\text{in}} = \left[ \begin{array}{c} J_i \cr \bar{J}_i \cr M_i \end{array} \right] \cdot \left[ \begin{array}{c} \mathcal{T}_{\text{DoJ}} \cdot \tilde{J}_i \cr \mathcal{T}_{\text{DoM}} \cdot \tilde{M}_i \end{array} \right] \overset{\text{uniformly denoted}}{=} \mathcal{C}^{\text{in}} \cdot \mathcal{P}^{\text{in}} \cdot \mathcal{C}^{*}
\]

(9)
with only the independent current \( \mathcal{C}^{\text{in}} \) (either \( J_i \) or \( M_i \)) can be obtained.

### C. Diagonalizing IPO Method for Calculating E-DMs

The IPO \( \mathcal{P}^{\text{in}} \) is a square matrix, so there must be Toeplitz’s decomposition \( \mathcal{P}^{\text{in}} = [\mathcal{P}^{\text{in}} + (\mathcal{P}^{\text{in}})^{\dagger}] / 2 \) and \( \mathcal{P}^{\text{in}} = [\mathcal{P}^{\text{in}} - (\mathcal{P}^{\text{in}})^{\dagger}] / 2 j \), and it is obvious that both \( \mathcal{P}^{\text{in}} \) and \( \mathcal{P}^{\text{in}} \) are Hermitian [46, Sec. 0.2.5]. In addition, it is easy to prove that \( \mathcal{C}^{\text{in}} \cdot \mathcal{P}^{\text{in}} \cdot \mathcal{C}^{\dagger} = \mathcal{P}^{\text{in}} \), so the positive Hermitian part \( \mathcal{P}^{\text{in}} \) must be positive definite.

Because \( \mathcal{P}^{\text{in}} \) and \( \mathcal{P}^{\text{in}} \) are Hermitian, and \( \mathcal{P}^{\text{in}} \) is positive definite, there must be a set of weighted decoupled modal vectors \( \{ \mathcal{C}_{c_{\text{ij}}} \} \) which can diagonalize the weighted matrix \( \mathcal{P}^{\text{in}} \) [46, Theorem 7.6.4], and the modal vectors can be derived from solving the following modal power decoupling equation:

\[
\mathcal{P}^{\text{in}} \cdot \mathcal{C}_{c_{\text{ij}}} = \mathbf{\theta}_{c_{\text{ij}}} \cdot \mathcal{P}^{\text{in}} \cdot \mathcal{C}_{c_{\text{ij}}}
\]

(10)
which can be equivalently written as the following more beautiful form (which has a similar form to the energy eigen equation in quantum mechanics):

\[
\left[ \mathcal{P}^{\text{in}} \right]^{-1} \cdot \mathcal{P}^{\text{in}} \cdot \mathcal{C}_{c_{\text{ij}}} = \left( 1 + j \mathbf{\theta}_{c_{\text{ij}}} \right) \mathcal{C}_{c_{\text{ij}}}
\]

(11)
because the positive definite matrix \( \mathcal{P}^{\text{in}} \) must be invertible. Using (8), the modal vectors \( \{ J_{\text{c}_{\text{ij}}}, \bar{J}_{\text{c}_{\text{ij}}}, M_{\text{c}_{\text{ij}}} \} \) can be obtained from \( \{ \mathcal{C}_{c_{\text{ij}}} \} \); using the basis function expansions of \( J_{\text{c}_{\text{ij}}}, \bar{J}_{\text{c}_{\text{ij}}}, \) and \( M_{\text{c}_{\text{ij}}} \), the modal currents \( \{ J_{\text{c}_{\text{ij}}}, \bar{J}_{\text{c}_{\text{ij}}}, M_{\text{c}_{\text{ij}}} \} \) can be obtained immediately; using Huygens-Fresnel principle, the modal fields \( \{ E_{\text{c}_{\text{ij}}} \times H_{\text{c}_{\text{ij}}} \} \) on \( V_{\text{c}_{\text{ij}}} \) can be obtained easily.

By some simple derivations, it is not difficult to prove the following frequency-domain power decoupling relationship:

\[
\left( 1/2 \right) \mathcal{S}_{\text{in}} \left( E_{\text{c}_{\text{ij}}} \times H_{\text{c}_{\text{ij}}} \right) \cdot \mathbf{n} dS = \left( 1 + j \mathbf{\theta}_{c_{\text{ij}}} \right) \delta_{c_{\text{ij}}}
\]

(12)
and the following time-domain energy decoupling relationship:

\[
\left( 1/T \right) \mathcal{S}_{\text{in}} \left( E_{\text{c}_{\text{ij}}} \times H_{\text{c}_{\text{ij}}} \right) \cdot \mathbf{n} dS dt = \delta_{c_{\text{ij}}}
\]

(13)
In the above decoupling relationships, \( \delta_{c_{\text{ij}}} \) is the Kronecker delta symbol, and \( T \) is the time period of the time-harmonic EM field, and the fields have been normalized by using the method selected in [36, Sec. II-E]. Obviously, (13) has a very clear physical meaning: in an integral period, any two different modes do not have net energy exchange. Thus, the above-obtained modes are called E-DMs in this paper.

As proved in the Appendix, the energy-decoupling relationship (13), which is expressed as a field-field interaction on the input port \( S_e \), is equivalent to the following another energy-decoupling relationship:

\[
\left( 1/T \right) \mathcal{S}_{\text{in}} \left[ \mathcal{I}_{\text{in}} \left( E_{\text{c}_{\text{ij}}} \times H_{\text{c}_{\text{ij}}} \right) \right] dS dt = \delta_{c_{\text{ij}}}
\]

(14)
extended as a field-field interaction on an arbitrary closed surface \( \mathcal{S}_{\text{in}} \) enclosing the whole generator-waveguide-horn system. Similar to (13), (14) also has a very clear physical meaning: during the process of propagating outwards, any two different modes do not have net energy exchange in any integral period. Clearly, the limitation “\( \mathcal{S}_{\text{in}} \rightarrow \mathcal{S}_{\text{in}} \)” will lead to a special case of (14), and the special case is the following field-field orthogonality relationship at infinity, or simply called far-field orthogonality:

\[
\delta_{c_{\text{ij}}} = \left( 1/T \right) \mathcal{S}_{\text{in}} \left[ \mathcal{I}_{\text{in}} \left( E_{\text{c}_{\text{ij}}} \times H_{\text{c}_{\text{ij}}} \right) \right] dS dt = \frac{1}{T} \left[ \mathcal{S}_{\text{in}} \left( \frac{1}{\mathcal{\eta}_0} E_{\text{c}_{\text{ij}}} \times E_{\text{c}_{\text{ij}}} \right) \right] dS dt
\]

(15)
where the second and third equalities are based on the time-domain versions of the Sommerfeld’s radiation conditions \( \lim_{r \rightarrow \infty} \left[ \nabla \times E + jk_0 \mathcal{S} \times E \right] = 0 \) and \( \lim_{r \rightarrow \infty} \left[ \nabla \times H + jk_0 \mathcal{S} \times H \right] = 0 \) [47, Sec. 3-5].

### IV. MTT-BASED DMT (MTT-DMT)

In this section, the momentum transport process of the horn-transmitting problem and the MTT quantitatively governing the momentum transport process are discussed. In
the MTT framework, a method to calculate the momentum-decoupled modes (M-DMs) of the horn antenna is proposed.

A. Momentum Transport Theorem (MTT)

The fields \((\mathbf{E}, \mathbf{H})\) on \(V_{arb}\) satisfy the homogeneous Maxwell’s equations \(\nabla \times \mathbf{H} = \partial \mathbf{D} / \partial t\) and \(\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t\), so there exists the following MTT:

\[
\int_{\mathcal{S}_i} \left[ \int \mathbf{T}_i \cdot d\mathbf{S} \right] dt = \int_{\mathcal{S}_i} \left[ \int \mathbf{T} \cdot d\mathbf{S} \right] dt - \int_{\mathcal{V}_{arb}} \left[ \int \mathbf{D} \cdot d\mathbf{V} \right] dt \quad (16)
\]

In (16), \(\mathcal{S}_i^+\) is the outer boundary of \(\mathcal{S}_i\), and its normal direction points to the thick electric wall; \(\mathbf{T}\) is the famous Maxwell’s stress tensor, and it is as follows:

\[
\mathbf{T} = \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B}
\]

in which the operation \(\circ\) is defined as that

\[
a \circ b = \hat{x} \hat{x} \frac{1}{2} (a b_1 - a b_y - a b_z) + \hat{y} \hat{y} \frac{1}{2} (a b_y + a b_1 - a b_z) + \hat{z} \hat{z} \frac{1}{2} (a b_z - a b_y + a b_1)
\]

\[
+ \hat{x} \hat{x} a b_z + \hat{y} \hat{y} a b_y + \hat{z} \hat{z} a b_1 \quad (18)
\]

In (18), \(a = \hat{x} a, \hat{y} a, \hat{z} a\), and \(b = \hat{x} b, \hat{y} b, \hat{z} b\); \(\hat{x} \hat{x}, \hat{y} \hat{y}, \hat{z} \hat{z}\) are the dyads defined in [48].

The above MTT has the physical meaning: the momentum passing through \(\mathcal{S}_i\) is transformed by the horn into two parts—a part is transported from \(\mathcal{S}_i\) to \(\mathcal{S}_{arb} \cup \mathcal{S}_i^+\) and the other part is used to contribute the momentum in \(V_{arb}\).

The frequency-domain version of the time-domain Maxwell’s stress tensor (17) can be written as follows:

\[
\tilde{T} = \frac{(1/2)}{(1/2)} \left( \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B} \right)
\]

called frequency-domain Maxwell’s stress tensor, and

\[
\frac{(1/2)}{(1/2)} \left( \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B} \right) \cdot n_i dS = \left( \hat{e}_c + j \hat{\theta}_c \right) \delta_{zz} \quad (22)
\]

where the M-DMs have been normalized as follows:

\[
\text{Re} \left( \frac{(1/2)}{(1/2)} \left( \mathbf{D} \cdot \mathbf{E} + \mathbf{H} \cdot \mathbf{B} \right) \cdot n_i dS \right) = \hat{e}_c \quad (23)
\]

The above relationship (24) has a clear physical interpretation: in an integral period, any two different modes do not have net momentum exchange. This is just the reason to call the modes M-DMs in this paper.

V. AMTT-BASED DMT (AMTT-DMT)

In this section, the angular-momentum transport process of the horn-transmitting problem and the AMTT quantitatively governing the angular-momentum transport process are discussed. Under the AMTT framework, a method to calculate the angular-momentum-decoupled modes (AM-DMs) of the horn antenna is proposed.

A. Angular-Momentum Transport Theorem (AMTT)

Because of that \(\int_{\mathcal{S}_i} \nabla \times \mathbf{a} dV = \int_{\mathcal{V}_{arb}} \mathbf{n} \times d\mathbf{S}\) for any vector \(\mathbf{a}\) [48], there exists the following AMTT:

\[
\int_{\mathcal{S}_i} \left[ \int \mathbf{D} \times d\mathbf{V} \right] dV + \int_{\mathcal{V}_{arb}} \left[ \int \mathbf{D} \times d\mathbf{V} \right] dV = \int_{\mathcal{S}_i} \left[ \int \mathbf{n} \times d\mathbf{S} \right] dt + \int_{\mathcal{V}_{arb}} \left[ \int \mathbf{n} \times d\mathbf{S} \right] dt \quad (25)
\]

in time domain.

In fact, the time-domain AMTT (25) can also be rewritten as the following frequency-domain version:

\[
\frac{(1/2)}{(1/2)} \left( \left( \mathbf{D} \cdot \mathbf{B} \right) \times \mathbf{n}_i dS \right) + \frac{1}{2} \left( \left( \mathbf{D} \cdot \mathbf{B} \right) \times d\mathbf{S} \right) \quad (26)
\]

The above AMTT has the following physical meaning: the angular-momentum passing through \(\mathcal{S}_i\) is transformed by the horn into two parts—a part is transported from \(\mathcal{S}_i\) to \(\mathcal{S}_{arb} \cup \mathcal{S}_i^+\) and the other part is used to contribute the angular-momentum in \(V_{arb}\).
B. Angular-Momentum-Decoupled Modes (AM-DMs)

From the frequency-domain AMTT (26), it is easy to find out that the source used to sustain the steady angular-momentum transport is just the following area-averaged input angular-momentum:

\[ I^a = (1/2) \iint_{S_y} (\hat{D} \times \hat{B}) \times \hat{n} \, dS \]  

with the dimension [angular momentum] \( \cdot m^{-2} \).

Similar to calculating E-DMs and M-DMs, the AM-DMs can be calculated from diagonalizing the above \( I^a \), and the obtained AM-DMs satisfy the following frequency-domain decoupling relationship:

\[ (1/2) \iint_{S_y} (D_\theta \times B_\theta) \times \hat{n} \, dS = \hat{e}_\theta + j \hat{\theta}_\theta \delta_{\theta\theta} \]  

where the modes have been normalized as follows:

\[ \text{Re} \left( (1/2) \iint_{S_y} (D_\theta \times B_\theta) \times \hat{n} \, dS \right) = \hat{e}_\theta \]  

similar to (23). In fact, the frequency-domain decoupling relationship (29) implies the following time-domain decoupling relationship:

\[ (1/T) \int_0^T \iint_{S_y} (D_\theta \times B_\theta) \times \hat{n} \, dS \, dt = \hat{e}_\theta \delta_{\theta\theta} \]  

The above relationship (30) has a clear physical interpretation: in an integral period, any two different modes do not have net angular-momentum exchange. Thus, the above-obtained modes are called AM-DMs in this paper.

VI. NUMERICAL EXAMPLES

Here, we consider a special metallic horn antenna designed in [49, pp. 112]. The photograph and size of the horn are shown in Fig. 4. To reduce the computational burden, the metallic wall \( S_{\text{met}} \cup S_0 \) and the corresponding current are ignored in this section, i.e., this section uses the approximate condition \( S_0 \approx S_{\text{met}} \) as shown in Figs. 4(b) and 4(c).

The modal resistances of the E-DMs calculated from the formulations given in Section III are shown in Fig. 5. The JE-DoJ-based result is consistent with the HM-DoM-based result. In addition, the obtained effectively working frequency band is basically consistent with the one given in [49, pp. 112]. In our opinions, the main reason leading to the slight difference is to ignore the feeding system, and the method for incorporating the feeding system will be provided in our future papers.

The JE-DoJ-based E-DM 1 is “resonant” at 6.91 GHz, where the “resonance” is defined as that the resistance is maximum [50, pp. 440]. The far-field radiation pattern of the “resonant” E-DM 1 is shown in Fig. 6, and it is also consistent with [49]. The electric and magnetic field distributions of the “resonant” E-DM 1 are shown in Fig. 7.
In addition, as a reconfirmation for the conclusion given in [36, II-D] that the conventional IE-CMT [21–25] and ES-WET-CMT [30–37] are the modal analysis theories for scatterers but not for transmitting antennas, we use the IE-CMT to calculate the CMs of the horn, and show the associated modal significances (MSs) in Fig. 8. For the first 36 lower order IE-CMT-based CMs working at 6.91 GHz, their far-field radiation patterns are shown in Fig. 9. From the Figs. 8 and 9, it is not difficult to observe that: neither resonance frequencies nor radiation patterns of the IE-CMT-based CMs are consistent with the results given in [49, pp. 112]. A similar phenomenon for the ES-WET-CMT-based CMs can also be easily observed. Thus, it is reconfirmed that: the conventional IE-CMT and ES-WET-CMT indeed are not applicable to doing the modal analysis for the transmitting problem of antennas.

VII. SUMMARY AND CONCLUSION

Electromagnetic energy, momentum, and angular-momentum are the effective carries for EM information. By controlling the EM energy, momentum, and angular-momentum, the various EM devices achieve the modulation for the information. For a device, it has many different modes, and every mode corresponds to a way to control the EM energy, momentum, and angular-momentum. By constructing a set of complete and decoupled fundamental modes of the device, the above modal analysis theories can clearly reveal the energy/momentum/angular-momentum utilization mechanism and effectively guide the engineering design of the device.
In the realm of classical electrodynamics, there have existed some different modal analysis theories, such as SL-EMT, SM-CMT, IE-CMT, ES-WET-CMT, and PS-WET-CMT etc. The various modal analysis theories have their own applicable ranges. Beyond its applicable range, the theory will not be able to work. Specifically speaking,

1) The SL-EMT is a differential equation (DE)-based theory to do the modal analysis for the closed EM structures (such as waveguides and cavities). The SL-EMT-based analysis for the open free-space waveguide is only a direct generalization.

2) In essence, the SM-CMT is also a DE-based theory, and it is for the open scattering objects.

3) The IE-CMT is a IE-based theory to do the modal analysis for the open scattering objects.

4) The ES-WET-CMT is an energy-viewpoint-based theory to do the modal analysis for the open scattering objects.

5) The PS-WET-CMT is an energy-viewpoint-based theory to do the modal analysis for the lumped-port-driven EM structures (such as the antennas and WPT systems).

The wave-port-fed open EM structures (such as the horn transmitting antennas discussed in this paper) are the devices whose working mechanism is essentially different from the wave-guiding structures, scattering objects, and lumped-port-driven EM structures. Thus, the above-mentioned SL-EMT, SM-CMT, IE-CMT, ES-WET-CMT, and PS-WET-CMT fail to do the modal analysis for the wave-port-fed EM structures.

From three different viewpoints (energy viewpoint, momentum viewpoint, and angular-momentum viewpoint), this paper proposes some more advanced modal analysis theories for the wave-port-fed metallic horn transmitting antennas.

1) From the viewpoint of energy, the energy utilization mechanism of horns is discussed; the energy utilization mechanism manifests itself as an energy transport process; the energy transport process is quantitatively governed by ETT, and the ETT is a special form of ECL; the input power contained in ETT is just the source to sustain a steady energy transport process; the E-DMs of horns can be calculated from diagonalizing the input power; the obtained E-DMs do not have net energy exchange in any integral period.

2) From the viewpoint of momentum, the momentum utilization mechanism of horns is discussed; the momentum utilization mechanism manifests itself as a momentum transport process; the momentum transport process is quantitatively governed by MTT, and the MTT is a special form of momentum conservation law (MCL); the input force contained in MTT is just the source to sustain a steady momentum transport process; the M-DMs of horns can be calculated from diagonalizing the input force operator; the obtained M-DMs do not have net momentum exchange in any integral period.

3) From the viewpoint of angular-momentum, the angular-momentum utilization mechanism of horns is discussed; the angular-momentum utilization mechanism manifests itself as an angular-momentum transport process; the angular-momentum transport process is quantitatively governed by AMTT, and the AMTT is a special form of angular-momentum conservation law (AMCL); the input angular-momentum contained in AMTT is just the source to sustain a steady angular-momentum transport process; the AM-DMs of horns can be calculated from diagonalizing the input angular-momentum operator; the obtained AM-DMs do not have net angular-momentum exchange in any integral period.

The above ETT-DMT, MTT-DMT, and AMTT-DMT for metallic horns can be directly applied to the other wave-port-fed metallic transmitting antennas, such as metallic parabolic reflector antennas and metallic patch antennas etc. In our future papers, we will further generalize the ETT-DMT, MTT-DMT, and AMTT-DMT to the other more complicated wave-port-fed structures, such as the metasurface-inspired antennas.

Using the above evolution progress from WET-CMT to ETT-DMT to MTT-DMT to AMTT-DMT, this paper is dedicated to exhibiting how “to see a world in a grain of sand” in the process of scientific research.

APPENDIX

From (13), it is not difficult to derive the following relationships:

\[
\delta \varphi = \frac{1}{i} \int_0^{t+T} \delta \left[ \iiint_{S_{\text{eff}}} (\mathbf{E}_\tau \times \mathbf{H}_\tau) \cdot d\mathbf{S} \right] dt
\]

\[
= \frac{1}{i} \int_0^{t+T} \delta \left[ \iiint_{S_{\text{eff}}} (\mathbf{E}_\tau \times \mathbf{H}_\tau) \cdot d\mathbf{S} \right] dt
\]

\[
+ \frac{1}{T} \int_0^{t+T} \frac{d}{dt} \left( \frac{1}{2} \mathbf{H}_\tau \cdot \mu_0 \mathbf{H}_\tau \right) |_{S_{\text{eff}}} + \frac{1}{2} \left( \mathbf{E}_0 \mathbf{E}_\tau \cdot \mathbf{H}_\tau \right) |_{S_{\text{eff}}} \right) dt
\]

\[
= \frac{1}{i} \int_0^{t+T} \delta \left[ \iiint_{S_{\text{eff}}} (\mathbf{E}_\tau \times \mathbf{H}_\tau) \cdot d\mathbf{S} \right] dt
\]

(31)

where \( V_{\text{eff}} \) is the region sandwiched between \( S_0 \cup S_e \) and \( S_{\text{eff}} \), and \( S_{\text{eff}} \) is an arbitrary surface enclosing the whole generator-waveguide-horn system. In the above (31), the first equality is based on (13) and the homogeneous tangential electric field boundary condition on \( S_e \); the derivation for the second equality is similar to deriving the famous Poynting’s theorem; the third equality is because of the periodicity of the time-harmonic EM field.

REFERENCES

[1] R. Loudon, L. Allen, and D. F. Nelson, “Propagation of electromagnetic energy and momentum through an absorbing dielectric,” Phys. Rev. E, vol. 55, no. 1, pp. 1071–1085, Jan. 1997.

[2] S. Stallinga, “Energy and momentum of light in dielectric media,” Phys. Rev. E, vol. 73, pp. 026606-1–026606-12, Feb. 2006.

[3] T. G. Philbin, “Electromagnetic energy momentum in dispersive media,” Phys. Rev. A, vol. 83, pp. 013823-1–013823-6, Jan. 2011.

[4] B. Thidé, H. Then, J. Sjöholm, K. Palner, J. Bergman, T. D. Carozzi, Y. N. Isomoin, N. H. Braginov, and R. Khamitova, “Utilization of photon orbital angular momentum in the low-frequency radio domain,” Phys. Rev. Lett., vol. 99, no. 8, pp. 087701-1–087701-4, Aug. 2007.

[5] A. E. Willner, J. Wang, and H. Huang, “A different angle on light communications,” Science, vol. 337, no. 6095, pp. 655–656, Aug. 2012.

[6] O. Edfors and A. J. Johansson, “Is orbital angular momentum (OAM)
