Adaptive greedy forward variable selection for linear regression models with incomplete data using multiple imputation

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Abstract

Variable selection is crucial for sparse modeling in this age of big data. Missing values are common in data, and make variable selection more complicated. The approach of multiple imputation (MI) results in multiply imputed datasets for missing values, and has been widely applied in various variable selection procedures. However, directly performing variable selection on the whole MI data or bootstrapped MI data may not be worthy in terms of computation cost. To fast identify the active variables in the linear regression model, we propose the adaptive grafting procedure with three pooling rules on MI data. The proposed methods proceed iteratively, which starts from finding the active variables based on the complete case subset and then expand the working data matrix with both the number of active variables and available observations. A comprehensive simulation study shows the selection accuracy in different aspects and computational efficiency of the proposed methods. Two real-life examples illustrate the strength of the proposed methods.

Keywords: Fraction of missing information; gradient feature testing; greedy forward selection; missing data; missing mechanism; multiple imputation; variable selection.

1 Introduction

Variable selection is an important issue for sparse modeling, a rapidly developing area including sparse regression and classification, sparse signal recovery, along with others. Statistical sparsity can be expressed as a limiting property, which covers all sparsity models in signal processing area (McCullagh and Polson 2018). Sparsity corresponds to performing variable selection in the model from high-dimensional datasets, and provides interpretability of the model. In traditional regression analysis, variable selection can be done by subset selection, shrinkage methods, and so on (Hastie et al. 2009). Subset selection includes best-subset selection, forward selection, backward selection, stepwise selection while well-known shrinkage methods include ridge regression (Hoerl and Kennard 1970), least absolute shrinkage and selection operator (lasso) (Tibshirani 1996) and some variants of lasso (Hastie, Tibshirani, and Wainwright 2015). Some methods such as the best-subset selection and lasso have the sparse property, that is, they have sparse solutions to the corresponding problems.

Greedy algorithms are widely used approaches in sparse modeling. The greedy algorithm always makes a locally optimal choice and never revisits former decisions (Cormen et al. 2009). The forward selection is an example of greedy algorithm. Perkins et al. (2003) propose a greedy forward selection approach, called the grafting (gradient feature testing) algorithm, for a variety of predictor models. Designed to be fast for high dimensional problems, the grafting operates in an incremental iterative fashion and selects one variable at a time into the active set. Therefore, it diminishes the computations greatly. Zhang (2009) proposes the greedy least squares regression algorithm, and presents its consistency property of the greedy feature selection under the irrepresentable condition of Zhao and Yu (2006), which is the same corresponding condition for lasso.

Missing values are common in data, and make statistical analysis complicated. As a matter of fact, statistical methods of dealing with missing data have a long history (Little and Rubin 2002). These methods can be divided into three categories, i.e. deletion, single imputation, and multiple imputation. The deletion method includes listwise deletion (also known as complete case analysis) and pairwise deletion. However, deletion may lead to loss of precision and bias if the missing mechanism depends on the data values, especially the missing values themselves. In the most case, deletion may also fail any analysis when no enough complete cases left. Single imputation is prevalent due to its simplicity and applicability, such as mean/median/mode imputation, nearest neighbor imputation, and regression imputation. However, single imputation tends to underestimate the standard errors. Multiple imputation (MI) is a more flexible method proposed by Rubin (1978). MI allows and quantifies variation across multiply imputed datasets, which reflects the uncertainty of the imputed values. With the validity of its assumptions, MI provides accurate estimates for the parameters of interest and related standard errors, as well as improving statistical power (Enders 2010).

Three variable selection strategies for MI are distinguished by van Buuren (2018). The first is Majority, which selects variables into the final model if they appear in at least half of the models from MI data. The second, Stack, is to apply a weighted regression method on a single dataset stacked by all multiply imputed datasets to select variables. The third called Wald is the stepwise selection performed by the Wald statistic based on Rubin’s rule (Rubin 1987).

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Brand (1999) first proposes a procedure for variable selection on missing data, which selects a set of candidate variables on each of the MI data and determines the final model by selecting the variables with inclusion frequency larger than a pre-specified threshold. Wood et al. (2008) and Vergouwe et al. (2010) investigate the variable selection strategies with missing predictor values.

Apart from performing variable selection on MI data, other approaches for variable selection on incomplete data are available. Yang et al. (2005) propose two Bayesian variable selection methods, “impute, then select” (ITS) and “simultaneously impute and select” (SIAS), to impute and select variables for linear regression models. SIAS, which embeds imputation and the stochastic search variable selection (SSVS) method of George and McCulloch (1993) in a Gibbs sampling process, slightly outperforms ITS in providing smaller Monte Carlo standard errors. Their simulation studies show that both ITS and SIAS outperform the stepwise selection using only complete cases. Zhao and Long (2017) review approaches of variable selection on missing data.

Among these aforementioned approaches, MI is prevalent as a consequence of some user-friendly software packages, and all the existing variable selection methods can be readily applied to each multiply imputed dataset. However, if relatively high proportion of the variables are not active for the response variable, imputing all the missing entries may not be necessary for variable selection. Our goal in this study is to identify active variables more efficiently for data with missing values. To efficiently perform variable selection on MI data, we extend the greedy forward selection, the grafting for least squares, to MI data to achieve our goal. We adopt the greedy least squares regression algorithm (Zhang 2009) for variable selection in linear regression models with missing data. The most important characteristic of MI is that it takes into account the uncertainty of the estimates due to the variation among imputed datasets. This accordingly yields diverse results as selecting variables is separate from multiple imputation. Hence, we propose three pooling rules for the adaptive grafting in the subsequent study.

The proposed procedure starts with the complete cases, and the initial set of active variables is obtained from the MI data, which only includes variables selected by lasso regression in the imputation model. Our proposed algorithm quickly expands the working data in two ways. The adaptive grafting adds one active variable with corresponding available observations into the data matrix. Moreover, the MI data expand by including one active variable from the adaptive grafting into the imputation model, which results in updated imputed values. This approach is more efficient than existing variable selection methods concerned with MI since the adaptive grafting fast identifies the active features and helps conducting MI on merely a subset of variables instead of the whole dataset.

The proposed algorithm initializes from but not confines the variable selection to the complete cases since the listwise deletion may cause bias and hence incorrectly identify the set of active variables. Applying MI on incomplete data with valid assumptions should improve the performance of any variable selection method. However, conducting MI on data with noise variables is computationally intensive. Our proposed procedure incrementally selects variables into the active set, and expands the MI data accordingly. Therefore, the procedure benefits from MI in a more computationally efficient way.

One of the challenges further emerges from the various feature selecting approaches produced by MI is their assessment, particularly in real data analysis. While some strategies concern missing proportion when applying the existing approaches for variable selection in missing data, Madley-Dowd et al. (2019) indicates that adding auxiliary variables in the imputation model does not always ensure the efficiency gains from MI. We then employ the fraction of missing information (fmi) to evaluate the available methods and proposed procedure. fmi is a useful concept in missing data analysis to quantify the uncertainty about imputed values accounting for the amount of information retained by other variables within a data set (e.g. Savalei and Rhemtulla 2012; Madley-Dowd et al. 2019). This quantity depends on the type of missing data mechanism, the model parameterization, and the degree of interrelationship among the variables. We refit the data to assess the information retained from the selected variables with incomplete data, which intends to compare the efficiency gains for the estimates obtained by some examined methods and the proposed procedure.

The remainder of this paper organizes as follows. Section 2 briefly reviews the related works for variable selection and multiple imputation, together with some selecting variable approaches using MI. Section 3 presents the proposed methods in detail, which extends the gradient feature testing for multiply imputed datasets and their pooling strategies. Section 4 conducts the simulation study to evaluate the proposed methods and compare with some common approaches in the literature. The evaluation includes criteria reflecting the selection performance, the parameter estimation, the prediction. The simulation study shows that the proposed methods attain preferable accuracy of variable selection within a significantly short execution time comparing with some common methods. Section 5 illustrates the proposed methods with two real-world datasets. We assess the information retained by all the examined methods for refitting the corresponding dataset with the selecting active variables. Section 6 concludes the paper with some discussions.

2 Variable selection

We consider the linear regression model

\[ y = X\beta + \epsilon, \]  

where \( y \) is a \( n \times 1 \) vector of response variable, \( X \) is an \( n \times p \) matrix of explanatory variables, and \( \beta \) is the parameter vector. We denote the data matrix as \( Z = (y, X) \). Assuming \( \epsilon \sim N(0, \sigma^2 I_n) \) for the error term, the log likelihood
function for model (1) is
\[
l(\beta, \sigma^2 | Z) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log(\sigma^2)
\]
\[
- \frac{1}{2\sigma^2} (y - X\beta)^T (y - X\beta).
\]  

(2)

For the data consisting of redundant predictors in (1), both model selection and parameter estimation could be a heavy labor. We briefly review some related methods of variable selection for model (1) in this section.

2.1 Lasso

Tibshirani (1996) proposes lasso or \(\ell_1\)-regularized regression for estimation in (1) to produce interpretable models. The lasso estimator is the solution to the following optimization problem for model (1):
\[
\hat{\beta}_{(\text{lasso})} = \arg\min_{\beta} \left( \|y - X\beta\|^2 + \lambda \sum_{v=1}^p |\beta_v| \right),
\]

(3)

where \(\lambda \geq 0\) is the regularization parameter, which controls the amount of shrinkage. The tuning parameter \(\lambda\) is chosen by cross validation (cv). The 5-fold or 10-fold cv is recommended to choose the optimal \(\lambda\) (Breiman and Spector, 1992; Tibshirani, 1996).

The lasso is a modification with absolute value constraint in regression, in which the least squares estimates are scaled by nonnegative constants as discussed in Breiman (1995). More generalizations of the lasso include adaptive lasso (Zou 2006), group lasso (Yuan and Lin 2006), elastic net (Zou and Hastie 2005), etc.

2.2 The gradient feature testing

Fitting model (1) with a large number of redundant variables needs intensive computations. The greedy forward selection reduces the computational cost and is suggested by Efron and Hastie (2016). Perkins et al. (2003) propose grafting for feature selection in a regularized learning framework. Grafting is applicable to a variety of models. Although the application to classification problem is only considered in Perkins et al. (2003), grafting is easily extended to the regression models.

The grafting fast obtains the optimal or approximately optimal solution, which minimizes the general criterion, a combination of empirical risk and a regularization term,
\[
C(\beta) = \frac{1}{n} \sum_{i=1}^n L(f(x_i), y_i) + \Omega_2 + \Omega_1 + \Omega_0,
\]

where \(x_i\) is the \(i\)th observation, \(f\) is the predictor function, \(L\) is a loss function, and \(\Omega_v = \lambda_v \sum_{v=1}^p \alpha_v |\beta_v|^q\) is the \(\ell_q\) regularizer. The best performance of the grafting in the experiments on classification problems obtained in Perkins et al. (2003) is achieved by using both a \(\ell_1\) and a \(\ell_0\) regularizer. The choice between various loss functions and the modified \(\ell_0\) regularizer for more complex models can be found in Perkins et al. (2003).

The extension of grafting to regression problems is straightforward by using a linear predictor model of the form \(f(x) = \beta_0 + \sum_{v=1}^p \beta_v x_v\) and the squared error loss function. For the regression model (1), the grafting minimizes
\[
C_{\text{reg}}(\beta) = \frac{1}{n} \sum_{i=1}^n (y_i - \beta x'_i)^2 + \lambda_2 \sum_{v=1}^p |\beta_v|^2
\]
\[
+ \lambda_1 \sum_{v=1}^p \alpha_1 |\beta_v| + \lambda_0 \sum_{v=1}^p \alpha_0 |\beta_v|^2.
\]

(4)

where the \(\ell_0\) regularizer corresponds to a constraint on the maximum number of features in \(f\) with a fixed penalty \(\alpha_0\) for all parameters if \(0^0\) is defined as 0 as in Perkins et al. (2003).

The grafting is a stagewise optimization procedure. At each stage, the gradient-based heuristic, which measures the magnitude of \(\frac{\partial C_{\text{reg}}(\beta)}{\partial \beta_v}\) is employed to decide which variable is most likely to improve the current model, and the model is incrementally optimized using gradient descent. To evaluate the variable effect on reducing the magnitude of \(C_{\text{reg}}(\beta)\) for \(\beta_v\) not selected into the current model.

Suppose that the index set of variables, denoted as \(V_r = \{v(1), \cdots, v(r)\}\), includes already selected \(r\) variables in the current model. We evaluate the remaining variables in the set \(V^{C}_r\), where \(C\) indicates the complementary set, and select the one with maximum magnitude of gradient into \(V_r\) and update it to \(V_{r+1}\). That is, the newly chosen variable will be the one with the largest gradient from the log-likelihood as in (2) based on the data matrix, \(Z_{V^C_r}\), corresponding
r variables in the current model, denoted as

\[ x_{v(r+1)} = \arg\max_{v' \in \mathcal{V}_C^C} \frac{\partial l(\beta, \sigma^2 | Z_{v'})}{\partial \beta_v} \]

\[ = \arg\max_{v' \in \mathcal{V}_C^C} \sum_{i=1}^{n} x_{i,v'} \left[ y_i - \beta_0 - \sum_{j=1}^{r} \beta_{v(j)} x_{i,v(j)} \right] . \]  

(5)

The grafting procedure (5) only involves the inner product of those variables in \( \mathcal{V}_C^C \) and the residuals of the current model, and hence reduce the computation complexity. Moreover, since the approach adds one variable into the current model sequentially, it typically produces a model which consists of a small number of variables.

The grafting is guaranteed to find the global optimum in some cases with a convex loss function and just the \( \ell_2 \) and/or \( \ell_1 \) regularizers (Perkins et al. 2003). Furthermore, Zhang (2009) shows the greedy least squares regression algorithm identifies the correct set of variables with the selection of stopping criterion which is equivalent to selecting an appropriate regularization parameter \( \lambda \) in the lasso regression in (3), i.e. the grafting using only the \( \ell_1 \) regularizer. Nevertheless, the approach only applies on data without missing values. The optimality of grafting on multiply imputed data is more complicated and beyond the scope of this study, which can be further investigated in the future.

### 2.3 Variable selection with missing values

There are three main approaches for variable selection in the presence of missing data (Zhao and Long 2017). The first approach is the likelihood-based methods. The joint distribution of the observed and missing data is specified, and by integrating out the missing data the marginal distribution of the observed data is obtained and variable selection is performed with accommodations. Claeskens and Consentino (2008) propose some variations of the Akaike’s information criterion (AIC) criteria to perform variable selection with missing covariates based on the expectation-maximization (EM) algorithm. Ibrahim et al. (2008) also propose a class of model selection criteria for missing-data problems using the EM algorithm. Mitra and Dunson (2010) propose a two-level stochastic search variable selection (SSVS) algorithm, which addresses a broader class of models involving mixed categorical and continuous variables. Liang et al. (2018) propose two algorithms, the imputation–regularized optimization (IRO) algorithm and the imputation–conditional regularized optimization (ICRO) algorithm, which incorporate imputation and regularization for high dimensional variable selection with missing values. Zambom and Matthews (2019) propose two methods, maximum likelihood sure independence screening (ML-SIS) and two-stage screening (TS-SIS), for ultrahigh dimensional problems with the sure screening property.

The second approach is through inverse probability weighting (IPW) and its extensions. Johnson et al. (2008) propose a general strategy based on penalized weighted estimating equations for variable selection. Wolfson (2011) develops the EEBoost algorithm for variable selection based on estimating equations (EEs). Brown et al. (2017) extend EEBoost and propose ThrEEBoost (Thresholded EEBoost) algorithm. However, this approach is difficult to extend to general missing data patterns that are not monotone (Long and Johnson 2015; Liu et al. 2016).

The third approach is performing variable selection on multiply imputed datasets. Brand (1999) selects the variables into the final model if the inclusion frequency is higher than the pre-specified threshold. Chen and Wang (2013) propose to ensure the consistency of selection across multiply imputed data by using group lasso regularization. Long and Johnson (2015) generate bootstrapped samples to conduct imputation, and determine the final selection through stability selection. Similar to the method proposed in Long and Johnson (2015), Liu et al. (2016) propose multiple imputation random lasso, which also involves bootstrapping and stability selection, to perform variable selection depending on the importance measure of each variable. Thao and Geskus (2019) categorize the variable selection using MI into three classes, and the further discussions will be presented in Sect. 3.3.

An alternative of MI is to impute missing values through matrix completion, the structured matrix completion (SMC) (Cai et al. 2016) and the softImpute-ALS procedure (Hastie et al. 2015) for example. Gao and Lee (2017) apply the softImpute-ALS and adaptive lasso to missing data, then the final model selection is made with the lowest value of the modified Rank Selection Criterion (RSC). Nevertheless, Xue and Qu (2020) argue that the approach of Gao and Lee (2017) does not guarantee the estimation consistency.

### 3 Missing values in data

Missing values complicate the data analysis. It is important to understand the reasons, patterns, and mechanisms of missing values when dealing with their analysis. The methods for missing values are developed depending on the missing mechanisms (Little and Rubin 2002; Rubin 1978). The missing-data mechanisms can be categorized into three types. The first one is missing completely at random (MCAR), which assumes the missingness does not depend on the data values. The second assumption is missing at random (MAR) if the missingness depends only on the observed values in the data. The third kind of assumption is the missing not at random (MNAR) if there is a relationship between the propensity of missingness and the unobserved data.
3.1 Multiple imputation

Among the techniques that handle missing values, multiple imputation is more flexible by allowing imputation variation and is applied widely owing to the development of computational ability (van Buuren 2018). Rubin (1978) proposes this Bayesian approach to relate the observed-data posterior distribution to the complete data posterior distribution which would have been obtained if the missing data had been observed under the assumption of MCAR or MAR. MI allows variation of imputed values and provides a quantification of the uncertainty.

Considering the regression model (1), if the response \( y \) is completely observed and there are missing values in the predictors, then an \( n \times p \) missing-data indicator matrix, \( \Delta = (\delta_{iv}) \), is defined as:

\[
\delta_{iv} = \begin{cases} 
1, & \text{if } x_{iv} \text{ is missing}, \\
0, & \text{if } x_{iv} \text{ is observed},
\end{cases}
\]

where \( x_{iv} \) is the \((i,v)\)th element in \( X \). We also denote the imputed data matrix as \( \tilde{Z} = (\tilde{y}, \tilde{X}) \) such that

\[
\tilde{x}_{iv} = \begin{cases} 
x_{iv}^{(imp)}, & \text{if } \delta_{iv} = 1, \\
x_{iv}, & \text{if } \delta_{iv} = 0,
\end{cases}
\]

where \( x_{iv}^{(imp)} \) is a filled-in value replaced by any imputation approach when \( x_{iv} \) is missing.

MI involves a three-stage process (Rubin 1978). First, \( M \) imputed datasets are created, in each of which the missing entries may have different imputed values, while all other non-missing entries are identical. The differences among the imputed values reflect the uncertainty of the imputation. The second step is to apply the same statistical modelling is more practical for real multivariate data sets. It is easier to take one conditional distribution at a time to obtain imputed values of a variable by regressing it on the other variables and then impute another variable conditioned on the observed and currently imputed values and so on. Cycling the procedure for every variable with missing values and repeating it for a number of iterations produces the imputed values. van Buuren and Groothuis-Oudshoorn (2011) employ this concept to develop an R package mice (multivariate imputation by chained equations).

3.2 Fraction of missing information

The fraction of missing information (fmi) quantifies the uncertainty about imputed values accounting for the amount of information retained by other variables within a data set (Wagner 2010; White et al. 2011; Madley-Dowd et al. 2019). The fmi measures the proportion of a parameter’s sampling error that is due to missing data, thus it reveals
information about the precision of the parameter estimate as the standard error of a regression coefficient (Heymans and Eekhout 2019).

The average fraction of missing information for $\beta$ based on $[\hat{\phi}^M, \hat{\phi}^f]$ is

$$\phi = \frac{1}{p} \left(1 + \frac{1}{M}\right) tr(\mathbf{B} \mathbf{T}^{-1})$$

which reflects the information lost for estimating the parameter $\beta$ due to missing values. The value of $\phi$ ranges between zero and one. The larger the value of $\phi$ is, the larger the variation between the MI datasets is. The high value of $\phi$ means the observed data in the imputation model do not retain enough information about the missing values, and induces larger standard errors of the regression coefficients.

Madley-Dowd et al. (2019) point out that the fmi should be the guide to the efficiency gains and choice of auxiliary variables in the imputation model instead of proportion of missing data. It also provides insight into the amount of information retained when using MI. Therefore, we refer the variables in the imputation model to the informative variables if they result in lower value of $\phi$ since the variables retain more information about missing values.

### 3.3 Variable selection on multiply imputed data

Thao and Geskus (2019) categorize the methods performing variable selection on multiply imputed datasets into three classes. The first category is to perform variable selection on each multiply imputed dataset and combining results afterwards. An option is to select variables using frequency of non-zero regression coefficients larger than a pre-specified threshold. This is similar but more general than the “majority” strategy stated in van Buuren (2018). Heymans et al. (2007) and Wood et al. (2008) apply classical methods such as stepwise selection or backward selection while Lachenbruch (2011) uses least angle regression (LARS) to investigate the variable importance according to the frequency of being selected. Still, the variable selection results are highly dependent on the artificially chosen threshold frequency (Long and Johnson 2015).

The results from all multiply imputed datasets can be combined through other strategies. Chen and Wang (2013) and Marino et al. (2017) propose approaches that perform variable selection across all multiply imputed datasets simultaneously through group lasso regularization. However, Liu et al. (2016) point out that the group lasso may be vulnerable to highly correlated variables. Zahid et al. (2019) impose a constraint on the magnitude of all parameter estimates for variable selection.

The second category, the same as the “stack” strategy distinguished by van Buuren (2018), is to carry out variable selection on the stacked dataset piled up from several imputed datasets. Wood et al. (2008) propose to perform variable selection on the stacked imputed dataset with weighted regression. Wan et al. (2015) propose an MI-based weighted elastic net (MI-WENet) method based on the stacked MI dataset.

The third category is to perform variable selection on bootstrap datasets. Musoro et al. (2014) handles multiply imputed datasets in the bootstrap procedure via the lasso technique and investigate the optimism of the lasso model. Long and Johnson (2015) and Liu et al. (2016) further apply stability selection (Meinshausen and Bühlmann 2010) to make the final selection. However, methods in this category involving bootstrapping are more computationally demanding.

Long and Johnson (2015) propose two methods, stability selection combined with bootstrap imputation (BISS) and Bolasso combined with bootstrap imputation (BIBL), to perform variable selection by combining resampling with imputation. Both BISS and BIBL start by bootstrapping from original dataset and conducting single imputation on each bootstrapped datasets. BISS estimates the coefficients by randomized lasso and determines the final set of active variables by stability selection with a threshold set between 0.6 and 0.9. BIBL performs lasso with cross validation on each bootstrapped imputed dataset and determines the final active set with a pre-specified threshold value as 0.9 as suggested by Bach (2008) or 1 by intersecting all active sets across MI datasets. In their simulation studies, BISS achieves better performance of variable selection and prediction. Nonetheless, Zahid et al. (2019) point out that the BISS method proposed by Long and Johnson (2015) relies on the Bayesian lasso based imputation and becomes impractical with an increasing number of variables having missing values for high-dimensional data.

Similar to BISS and BIBL, multiple imputation random lasso (MIRL) (Liu et al. 2016) conducts variable selection on bootstrapped datasets and determines the final selection by stability selection. The main differences between MIRL and the approaches of Long and Johnson (2015) are as follows. MIRL first generates multiply imputed datasets, and bootstrapped samples are generated for each MI dataset. Then, the lasso-OLS estimates are obtained to yield the importance measure for each variable. For each bootstrapped dataset, half of the variables are randomly selected according to the importance measure to obtain the initial MIRL estimates by averaging the lasso-OLS estimates across all bootstrap samples and imputations. Final selection is determined by stability selection, in which the threshold is chosen by 4-fold cross-validation with a one-standard-error rule to minimizes the mean squared prediction error (MSPE). Liu et al. (2016) claims that MIRL achieves better performance of variable selection with high proportion of missing values and highly correlated predictors. Both BISS and MIRL can handle high dimensional data, but directly comparing these two methods is absent in the literature.
4 The greedy forward selection with multiple imputation

Considering model \( \hat{\beta} \), the main problem we tackle in this paper is to simultaneously deal with the missing values in \( X \) when selecting active variables, which have nonzero coefficients, \( \beta_v \neq 0 \), in the model. The primary goal is to efficiently identify the truly active variables in the regression model. An active variable in the true model presumably contribute to provide information for other active variables with missing values in the imputation model. Therefore, if we want to reduce computations by choosing a subset of variables to perform imputations, we need to correctly identify active variables in the model. Our strategy is to integrate the variable selection into MI, and repeatedly implement both of the MI and variable selection steps.

4.1 Pooling rules for adaptive grafting on multiply imputed datasets

For model \( \hat{\beta} \) with missing values, we apply the grafting to select variables on the imputed data \( \hat{Z} \). We obtain the initial model by applying lasso regression to the complete cases. Assuming that \( r \) variables in \( V_r \) are already in the current model, the adaptive grafting selects the most promising variable to contribute to the model as follows. Analogous to (3), the most promising variable to contribute to the model is the variable with the largest magnitude of gradient, denoted as

\[
\tilde{x}_{v(r+1)} = \arg\max_{v^* \in V_r^C} \left| \frac{\partial l(\beta, \sigma^2|\hat{Z}_{V_r})}{\partial \beta_{v^*}} \right|
\]

\[
= \arg\max_{v^* \in V_r^C} \sum_{i=1}^n \tilde{x}_{i,v^*} \left[ y_i - \hat{\beta}_0 - \sum_{j=1}^r \beta_{v(j)} \tilde{x}_{i,v(j)} \right],
\]

where \( l(\beta, \sigma^2|\hat{Z}_{V_r}) \) is the log-likelihood function (2) based on the imputed data matrix \( \hat{Z} \) containing those \( r \) variables with nonzero coefficients.

The variable with the largest magnitude of gradient may be different for each multiply imputed dataset. Hence, we consider two gradients of the log-likelihood function for the latest selected variable as follows. The first gradient evaluated at a given point, where model coefficients are estimated by the \( n \)th MI data separately, is denoted as

\[
g_{v^0}^{(m)} = \sum_{i=1}^n \tilde{x}_{i,v^0} \left[ y_i - \hat{\beta}_0 - \sum_{v \in V_r} \beta_v \tilde{x}_{i,v} \right],
\]

\( \forall v^0 \in V_r^C, m = 1, 2, \cdots, M. \)

We also consider the gradient of the log-likelihood at the given point where model coefficients are estimated by the pooled result using Rubin’s rule from multiply imputed data. It is denoted as

\[
g_{v^0}^{\text{pooled}} = \sum_{i=1}^n \tilde{x}_{i,v^0} \left[ y_i - \hat{\beta}_0 - \sum_{v \in V_r} \sum_{m=1}^M \frac{\beta_v^{(m)} \tilde{x}_{i,v}^{(m)}}{M} \right],
\]

\( \forall v^0 \in V_r^C, \) where \( \hat{\beta}_0 = \frac{\sum_{m=1}^M \beta_v^{(m)}}{M} \) as in equation (6).

Based on (12), we have to further combine \( M \) gradients. Hence, we derive rule (a) selection by vote and rule (b) selection by averaging grafting results. From (13), we propose the pooling rule (c) selection from the pooled result.

**Pooling rule (a)**

Rule (a) first nominates one variable with the largest gradient from each of the MI datasets, indexed by \( u^{(m)} = \max_{v^0 \in V_r^C} g_{v^0}^{(m)} \), \( m = 1, \cdots, M \). Then, we count the number of nominations for each variable,

\[
\sum_{m=1}^M I_{\{v^0 = u^{(m)}\}}, v^0 \in V_r^C,
\]

where \( I \) is the indicator function. Finally, we choose the variable receiving the most votes. That is, select the variable indexed by \( u_a \) satisfying

\[
u_a = \max_{v^0 \in V_r^C} \sum_{m=1}^M I_{\{v^0 = u^{(m)}\}}.
\]

However, there may exist more than one mode or no mode for the majority votes when the number of imputations is small. If the variable having the most votes is not unique, we can break the ties by randomly selecting one variable from those with the highest number of votes.

**Pooling rule (b)**

Rule (b) combines the grafting results \( g_{v^0}^{(m)}, m = 1, \cdots, M \) to produce the unique selection since the mode of votes may be unstable under some circumstances. Averaging the gradient values obtained from \( M \) MI datasets yields one unique variable which is most likely to improve the model. This is a method of pooling the multiple imputation results in the sense that the contribution of a variable is first estimated by the gradient from each MI dataset, and the estimates are then combined across the \( M \) datasets. This selection rule is to select the variable indexed by \( u_b \) satisfying

\[
u_b = \max_{v^0 \in V_r^C} g_{v^0} = \max_{v^0 \in V_r^C} \frac{1}{M} \sum_{m=1}^M g_{v^0}^{(m)}.
\]
In this manner, we have initial pooled estimates are denoted as by lasso. In this way, we restrict the initial set of active variables to a small set. The benefits of doing so will be the latest selected variable is not significant for the variable, we re-train the model and re-assess the parameter estimates. The procedure stops if the pooled estimate of (14)-(16). The latest selected variable should improve the imputation by bringing information for imputations, and we apply the adaptive grafting approach to look for the most promising variable in each iteration using pooling rules (c). We propose a scheme to implement M multiple imputation and the Greedy forward selection together, the MiG algorithm. Starting with the complete cases, we use lasso to decide the initial set of active variables. After the initialization, we apply the adaptive grafting approach to look for the most promising variable in each iteration using pooling rules (14)-(16). The latest selected variable is denoted as \( \bar{S} \).

Step 0 - Initialization
Let \( S = \{1, 2, \ldots, n\} \) be the index set of all sample points, and let \( V = \{1, \ldots, p\} \) be the index set of all variables. The procedure starts with the complete cases, denoted as \( Z_0 \), which consists of \( n_0 \) observations. We fit the lasso regression with tuning parameter chosen by cv, and select the variables with nonzero coefficients to be the auxiliary variables for imputation. The index set of these variables is denoted as \( V_0 = \{v|\beta_{0v}(lasso) \neq 0\} \).

Step 0-(b): Initial active set
We perform MI on the selected variables in \( V_0 \) and obtain \( M \) imputed datasets \( Z_0^{(m)} = (y, \hat{X}_{V_0}^{(m)}), m = 1, \ldots, M \).

Step 0-(c): MI
We separately fit the regression model (1) to the multiply imputed datasets and obtain the pooled OLS estimates \( \bar{\beta}^{(0)} \) by Rubin’s rule as in (10). The coefficient estimates of the other variables not in \( V_0 \) are straightforwardly zero.

We do individual t-test, which tests \( H_0 : \beta_v = 0 \) vs. \( H_A : \beta_v \neq 0, \forall v \in V_0 \), to determine the initial set of active variables.

The significant \( r \) variables from the pooled result by performing the test using (6)-(9) are selected into the initial active set. The index set of the initial active variables is denoted as \( V_r \). The initial estimates are pooled from the results of regressing \( y \) on the \( r \) active variables based on the MI datasets \( \tilde{Z}_r^{(m)} = (y, \tilde{X}_{V_r}^{(m)}), m = 1, \ldots, M \). The initial pooled estimates are denoted as \( \hat{\beta}^{(r)} \).

The initial model is constructed by significant variables from the MI data, imputed using auxiliary variables selected by lasso. In this way, we restrict the initial set of active variables to a small set. The benefits of doing so will be further discussed afterwards.

Step 1 - Variable selection by adaptive grafting
We incorporate each MI dataset \( \tilde{Z}_r^{(m)} = (y, \tilde{X}_{V_r}^{(m)}) \) with the other variables in \( V_r^{C} \) to update the imputed dataset \( \tilde{Z}_{S(r)}^{(m)} \), where \( S(r) \) is the index set of the observations since it depends on the \( r \) active variables in the current model.

In this manner, we have \( M \) sets of imputed data, denoted as \( \tilde{Z}_{S(r)}^{(m)} = \{y_{S(r)}, (\tilde{X}_{S(r)}^{(m)}, V_r, x_{S(r)}, V_r^{C})\}, m = 1, \ldots, M \). \( \tilde{X}_{S(r),V_r}^{(m)} \) denote the imputed dataset of the selected variables in \( V_r \) and the observations in \( S(r) \). \( x_{S(r),V_r^{C}} \) is the observed values of the variables not selected currently for the observations in \( S(r) \).

We apply the adaptive grafting approach on \( \tilde{Z}_{S(r)} \) to select the variable denoted as

\[
\tilde{x}_{v,(r+1)} = \arg \max_{v \in V_r^{C}} \left| \frac{\partial l(\beta, \sigma^2|\tilde{Z}_{S(r)})}{\partial \tilde{z}_{v,r}} \right| = \arg \max_{v \in V_r^{C}} \sum_{i \in S(r)} \bar{x}_{i,v} \left[ y_i - \tilde{\beta}_0 - \sum_{j=1}^{r} \tilde{\beta}_{v(j)} \bar{x}_{i,v(j)} \right],
\]

where \( l(\beta, \sigma^2|\tilde{Z}_{V_r}) \) in (11) is replaced by \( l(\beta, \sigma^2|\tilde{Z}_{S(r)}) \).

Since \( S(r) \), the number of samples used in the current model, increases with the number of selected variables \( r \), the gradients of the likelihood function in (12) and (13) are modified as

\[
g^{(m)}_{i,v} = \sum_{i \in S(r)} x_{i,v} \left[ y_i - \tilde{\beta}_{0v}^{(m)} - \sum_{v' \in V_r} \tilde{\beta}_{v'v}^{(m)} \tilde{x}_{i,v'}^{(m)} \right]
\]
and

\[ g_{\text{pooled}}^{\prime} = \sum_{i \in S(r)} \beta_0 + \sum_{i \in V_r} \beta_i^R \cdot \bar{x}_i^{(m)} \]

respectively, where \( v' \in V^C \) and \( x_{i, v'} \) is observed for \( i \in S(r) \). As a result, we select the most promising variable by one of the pooling rules [14]-[16] according to [18] or [19].

The selected variable from any of the three rules is indexed by \( u \) for simplicity. We distinguish the MiG algorithm corresponding to [14], [15], and [16] as MiG-1, MiG-2, and MiG-3, respectively. After the adaptive grafting, the current set of active variables becomes \( V_{r+1} \), which has the dimensionality of \( r + 1 \) by adding the selected variable \( u \). Figure 1 displays the relationship between \( V_e \) and \( V_{r+1} \), where \( V_{r+1} = V_r \cup \{ u \} \).

**Step 2 - Data augmentation: Performing MI on selected variables**

We perform MI with variables in \( V_{r+1} \) to obtain \( M \) MI datasets, denoted as \( \tilde{Z}^{(m)} = (y, \tilde{X}^{(m)}_{r+1}), m = 1, \cdots , M \). The matrix \( \tilde{X}^{(m)}_{r+1} = (\tilde{x}^{(m)}_{v(1)}, \cdots , \tilde{x}^{(m)}_{v(r)}, \tilde{x}_u^{(m)}) \) consists of variables in \( V_{r+1} \) which have imputed values if missing. The set \( \tilde{Z}_r^{(m)} \) expands to \( \tilde{Z}_{r+1}^{(m)} \), and all the imputed values are updated by using one more auxiliary variable \( X_u \) in the imputation model. Step 0-(a) in the initialization is similar to this step, which performs MI on variables selected by lasso regression to conduct the data augmentation.

After the augmentation, we update the model fits for each of the MI dataset, \( \tilde{Z}^{(m)}_{r+1}, m = 1, \cdots , M \). The current pooled OLS estimates are denoted as \( \beta_R^{r+1} \). Then, we use the F-test to examine if the latest variable from the adaptive grafting in step 2 is significant. We test the hypothesis

\[
H_0 : \beta_u = 0 \\
H_A : \beta_u \neq 0
\]

on multiply imputed datasets. For each of the \( M \) MI data, the hypothesis [20] is equivalent to

\[
H_0 : E(y|\tilde{X}_{V_r,m}, \tilde{X}_{u,m}) = \tilde{x}^{(r)}_{V_r,m} \beta^{(r)}_{m} \\
H_A : E(y|\tilde{X}_{V_r,m}, \tilde{X}_{u,m}) = \tilde{x}^{(r)}_{V_r,m} \beta^{(r)}_{m} + \tilde{x}^{(r)}_{u,m} \beta_{u,m}
\]

This is the F-test for the change in \( R^2 \) based on the \( m \)th dataset. To combine the \( M \) F-tests across MI data, Chaurasia and Harel (2015) propose to test [20] by the test statistic

\[
F = \frac{(\mathbb{R}^2_A - \mathbb{R}^2_0)/1}{(1 - \mathbb{R}^2_A)/\nu_2},
\]

where \( \mathbb{R}^2_0 \) and \( \mathbb{R}^2_A \) are the pooled estimates of \( R^2 \) of the model under \( H_0 \) and \( H_A \) respectively by pooling the Fisher Z score of \( R^2_m \) from the \( m \)th dataset and then transforming the pooled Fisher Z score back (see details in Chaurasia and Harel 2015; Grund et al. 2016; van Ginkel 2019), and \( \nu_2 \) is the degrees of freedom derived by Barnard and Rubin (1999). If the latest selected variable indexed by \( u \) is significant by the F-test using [3]-[9] at the \( \alpha = 0.05 \) level, we proceed to the iteration of repeating steps 1 and 2 for \( r = 1, r+2, \cdots , p \).

During the iterations from step 1 to step 2, the working data matrix rapidly expands with both the number of active features and their corresponding cases. In step 1, the matrix \( \tilde{Z}^{(m)}_{S(r)} \) expands to \( \tilde{Z}^{(m)}_{S(r+1)} \) since the imputed data for the \( r \) variables are treated as observed data. In step 2, the matrix \( \tilde{Z}_r^{(m)} \) expands to \( \tilde{Z}^{(m)}_{r+1} \) by adding one variable into the imputation model. The data augmentation from \( \tilde{Z}^{(m)}_{S(r)} \) to \( \tilde{Z}^{(m)}_{S(r+1)} \) and from \( \tilde{Z}^{(m)}_r \) to \( \tilde{Z}^{(m)}_{r+1} \) are displayed as in Figure 1.

To simplify the computations, the MiG algorithm stops when the latest variable is significant in the pooled result of the current \( M \) MI datasets in step 2. If the \( p \)-value of the F-test is less than 0.05, we keep the variable \( X_u \) in the regression model. Otherwise, if the F-test result for the latest selected variable \( u \) is not significant, the procedure stops and \( u \) is excluded from the final active set. The final index set of the selected variables is denoted as \( V^* = V_r \), and the final estimates are \( \tilde{\beta} = \tilde{\beta}^* \) based on the MI datasets \( \tilde{Z}^{(m)}_r \).

The spirit of the greedy algorithm is that variables already selected into the active set will never be dropped, and the adaptive grafting considers to select only one variable at one time. The forward selection adds the most significant variable into the current model until no further significant information about the response can be supplied from the non-selected variables.

For complete data, the greedy selection chooses the most promising variable at each iteration and leads to the optimum. However, the imputed values for a variable in each MI dataset change from iteration \( r \) to \( r+1 \) since the imputation model changes by adding one more variable. The F-test is conducted assuming the changes in the imputed values would not affect the assessment of the significance of the selected variable. Therefore, it is difficult to ensure the traditional greedy algorithm leads to the global optimum for the imputed data. On the condition that the observed part of the variables in the imputation model provide enough information about the missing values, the F-test evaluates the contribution of the latest variable to the regression model.
4.3 Pseudocode of MiG

We refer to the MiG algorithms by the rules of (a) selection by vote, (b) selection by averaging grafting results, and (c) selection from the pooled result as MiG-1, MiG-2, and MiG-3, respectively. The procedures of the MiG algorithms are summarized in the following diagram.

Diagram - The MiG algorithm

**Step 0 - Initialization:**

1. Fit the lasso regression model with cross validation based on the complete cases \( Z_0 \) and obtain the initial estimates \( \hat{\beta}_{\text{lasso}} \) for all the variables \( x_v, v \in V \), where some of the estimates are shrunk to zero.

**Step 0-(a): MI**

2. Select the variables with nonzero estimated coefficients to be the auxiliary variables for imputation. The index set of the auxiliary variables is \( V_0 = \{ v \mid \hat{\beta}_{\text{lasso}}(v) \neq 0 \} \).

3. Perform MI on variables in \( V_0 \) and obtain \( \tilde{Z}_0^{(m)} = (y, \tilde{X}_{V_0}^{(m)}), m = 1, \cdots, M \).

**Step 0-(b): Initial active set**

4. Fit the regression model based on each MI dataset \( \tilde{Z}_0^{(m)}, m = 1, \cdots, M \) and obtain the pooled estimates \( \bar{\beta}^{(0)} \). The coefficient estimates of the variables not in \( V_0 \) are straightforwardly zero.

5. From the pooled result, select the \( r \) variables which are significant at \( \alpha = 0.05 \) level into the active set. The index set of the initial active variables is denoted as \( V_r = \{ v(1), \cdots, v(r) \} \) for simplicity.

6. The initial estimate \( \bar{\beta}^{(r)} \) are obtained by the pooled estimates from fitting \( y \) on variables in \( V_r \) based on MI data \( \tilde{Z}_r^{(m)}, m = 1, \cdots, M \).

7. Enter the iteration of grafting on MI data.

   while \( r \leq p \) do

     for \( v' \in V_r^C \) do

   **Step 1 - Variable selection by adaptive grafting**

8. Combine each \( \tilde{Z}_r^{(m)} \) with the variables not in \( V_r \) to update the imputed datasets, denoted as \( \tilde{Z}_r^{(m)} = \{ y_S^{(m)}(r), X_S^{(m)}, X_S^{(m)} \} \), \( m = 1, \cdots, M \). \( \tilde{X}_S^{(m)} \) denote the imputed values of the selected variables for the observations in \( S(r) \). \( X_S^{(m)} \) is the observed values of the variables not selected currently for the observations in \( S(r) \).

9a MiG-1: Selection by vote

9a1. Compute \( g_{v'}^{(m)}, \forall m, v' \) as in (18), where \( x_{i,v'} \) is the observed elements in \( X_S^{(m)}, V_r^C \), and \( \hat{\beta}_{v'}^{(m)} \) is the parameter estimate based on \( \tilde{Z}_r^{(m)} \).

9a2. Nominate \( M \) candidate variables \( u^{(m)} = \max_{v' \in V_r^C} g_{v'}^{(m)}, m = 1, \cdots, M \) from all the MI datasets.

9a3. Count the frequency of each \( v' \) being selected across the \( M \) datasets, i.e. \( \sum_{m=1}^{M} I_{\{v = u^{(m)}\}}, v' \in V_r^C \).

9a4. Select the variable indexed by \( u_a \) with the highest counts (randomly select one if there exists no mode or more than one mode).
9b MiG-2: Selection by averaging grafting results

9b1. (the same as 9a1.) Compute $g_{v'}^{(m)}$, $\forall m$, $v'$ as in [18], where $x_{iv'}$ is the observed elements in $X_{S(r),V_r^c}$, and $\hat{\beta}_{v'}^{(m)}$ is the parameter estimate based on $\tilde{Z}_v^{(m)}$.

9b2. Define $\bar{g}_{v'} = \frac{1}{M} \sum_{m=1}^{M} g_{v'}^{(m)}$.

9b3. Select the variable satisfying $u_b = \max_{v' \in V_r^c} \bar{g}_{v'}$, $\forall v' \in V_r^C$.

9c MiG-3: Selection from the pooled result

9c1. Compute $g_{v'}^{\text{pooled}}$, $\forall v'$ as in [19], where $x_{iv'}$ is the observed elements in $X_{S(r),V_r^c}$, and $\hat{\beta}_{v}^{(m)}$ is the parameter estimate based on $\tilde{Z}_v^{(m)}$.

9c2. Select the variable satisfying $u_c = \max_{v' \in V_r^c} g_{v'}^{\text{pooled}}$.

10. Denote the latest selected variable as $u$. $u = u_a$ if we use MiG-1; $u = u_b$ if we use MiG-2; $u = u_c$ if we use MiG-3. Update $V_{r+1} = V_r \cup \{u\} \cup \{v(1), \ldots, v(r)\} \cup \{u\}$.

Step 2 - Data augmentation: Performing MI on selected variables

11. Perform MI on variables in $V_{r+1}$ and obtain $\tilde{Z}_{r+1}^{(m)} = (y, \tilde{X}_{r+1}^{(m)}, m = 1, \ldots, M$, where $\tilde{X}_{r+1}^{(m)} = (\tilde{x}_{v(1)}^{(m)}, \ldots, \tilde{x}_{v(r)}^{(m)}, \tilde{x}_u^{(m)}$).

12. Re-fit the model, and obtain the pooled OLS estimate $\bar{\beta}^{r+1}$ based on $\tilde{Z}_{r+1}^{(m)}, m = 1, \ldots, M$.

   If $X_u$ is not significant in the pooled result, then the final selection is indexed by the set $V^* = V_{r+1} \setminus \{u\}$, and the final estimates are denoted as $\hat{\beta}^* = \bar{\beta}^r$;

   else

   update the active set as $V_{r+1}$. Enter the next iteration of the while loop.

end while

4.4 The $fmi$ for the refit

Suppose $k$ variables are selected by a variable selection method, the set of these $k$ variables is denoted as $V_{kh}$, where $1 \leq k \leq p$, $h \in \{1, 2, \ldots, C_p^k\}$, and $C_p^k$ is the number of $k$-combinations by selecting $k$ out of $p$ variables. Considering the model [1] on the data $Z_{V_{kh}}$, we implement MI to create imputed datasets, denoted as $\tilde{Z}_{V_{kh}}^{(m)} = (y, \tilde{X}_{V_{kh}}^{(m)}, m = 1, \ldots, M$.

The $M$ OLS estimators for $\beta_{kh}$ are denoted accordingly as $\hat{\beta}_{kh}^{(m)}$ and $\tilde{\sigma}_{kh}^{2(m)}$.

Applying Rubin’s rule to the coefficient estimates, equations [8]-[9] are transformed into

$$\hat{\beta}_{kh} = \frac{1}{M} \sum_{m=1}^{M} \hat{\beta}_{kh}^{(m)}$$  \hspace{1cm} (23)

and

$$\text{Var} (\hat{\beta}_{kh}) = T_{kh} = U_{kh} + \frac{M+1}{M} B_{kh},$$  \hspace{1cm} (24)

where

$$U_{kh} = \frac{1}{M} \sum_{m=1}^{M} \tilde{\sigma}_{kh}^{2(m)} (\tilde{X}_{V_{kh}}^{(m)T} \tilde{X}_{V_{kh}}^{(m)})^{-1}$$  \hspace{1cm} (25)

and

$$B_{kh} = \frac{1}{M-1} \sum_{m=1}^{M} (\hat{\beta}_{kh}^{(m)} - \hat{\beta}_{kh}) (\hat{\beta}_{kh}^{(m)} - \hat{\beta}_{kh})^T.$$  \hspace{1cm} (26)
Therefore, the average fmi for $\beta_{\text{kh}}$ based on (23)-(26) is

$$\phi(k, h) = \frac{1}{k} \left(1 + \frac{1}{M}\right) tr \left(B_{kh} T_{kh}^{-1}\right),$$

which is an analogue to the quantity $\phi$ in (10).

We can also compare the average fmi values from different models by inspecting their sizes relative to the average fmi value from the model including all $p$ variables in the data, which we refer as the full model. We denote the ratio of the average fmi for a selected model to the average fmi for the full model as $r(k, h) = \frac{\phi(k, h)}{\phi_{p}}$. The values of $\phi(k, h)$ or $r(k, h)$ provide insight into the retained information by variables in $V_{kh}$, and reflects the average inflation of the standard errors of the regression coefficients.

## 5 Simulation study

We conduct the simulation to examine the proposed methods along with some methods including complete case analysis, model selection on either separate MI datasets or stacked MI data, or bootstrapped MI data. The examined methods are listed in Table 1. LDLS is the listwise deletion least squares regression which sets the coefficients not significant at the 5% level to be 0. MILS is the least squares estimation combined by the Rubin’s rule for multiply imputed data, and sets the combined coefficients not significant at the 5% level by the Rubin’s rule to be 0. LD lasso cv is the lasso estimation on listwise deleted data with tuning parameter chosen by tenfold cross validation.

Based on works of Wood et al. (2008) and Lachenbruch (2011), the lasso selection, a special case of LARS, instead of backward selection can be implemented on MI datasets separately as suggested in Thao and Geskus (2019). We adapt the methods and denote them as MI lasso S1, MI lasso S2, and MI lasso S3, which perform lasso separately on each multiply imputed dataset and combine estimates by averaging lasso coefficients of predictors that appear in (1) any of the models, or (2) at least half of the models, or (3) all models.

To avoid selecting different sets of variables, two strategies can be considered. We adapt the stacked method of Wood et al. (2008) by applying lasso regression instead of backward elimination, and denote it as MI Stacked. We also investigate the other strategy, MI-Lasso (Chen and Wang 2013), which applies group lasso across the multiply imputed data to ensure consistent variable selection result.

MIRL is multiple imputation random lasso proposed by Liu et al. (2016), and the stability selection is made through a four-fold cross validation. MIRLnoSS is MIRL without stability selection, which uses a hard threshold $\frac{1}{2}$ and sets the coefficients to be 0 if the absolute values of them are less than the threshold, where $n$ is the total sample size. BISS (Long and Johnson 2015) is stability selection within bootstrap imputation, and BIBL (Long and Johnson 2015) incorporates Bolasso (Bach 2008) with bootstrap imputation.

We implement the R packages mice (van Buuren and Groothuis-Oudshoorn 2011) and miceadds (Robitzsch and Grund 2021) to conduct multiple imputation and related $F$-test, the R package glmnet (Friedman et al. 2010) to perform lasso, and the R package MIRL for MIRL. An R function to implement MI-Lasso is available online [http://www.columbia.edu/~qc2138/](http://www.columbia.edu/~qc2138/). Downloads/software/MI-lasso.R and upon request from the authors of Chen and Wang (2013).

We employ some metrics to evaluate the properties of the methods including $MCC$ (Matthews Correlation Coefficient) (Matthews 1975), $L_1$ error, $L_2$ error, and mean squared prediction error ($MSPE$). $MCC$ measures the variable selection performance. To define $MCC$, we first denote the index set of true active variables as $K = \{k : \beta_k \neq 0\}$ and the index set of selected variables by a procedure as $\hat{K} = \{k : \hat{\beta}_k \neq 0\}$. Then, true positive, true negative, false positive, and false negative ($TP, TN, FP, FN$) are defined as

$$TP = \text{number of } \{H \cap K\}, \quad TN = \text{number of } \{H^c \cap K^c\},$$

$$FP = \text{number of } \{H \cap K^c\}, \quad FN = \text{number of } \{H^c \cap K\}.$$

Accordingly, $MCC$ is defined as

$$MCC = \frac{TP \times TN - FP \times FN}{\sqrt{(TP + FP)(TP + FN)(TN + FP)(TN + FN)}}.$$

The possible value of $MCC$ is between -1 and 1. The higher $MCC$ indicates the higher capability to select the correct sets of active variables.

$L_1$ error and $L_2$ error measure the parameter estimation, which are defined as $L_1 = \|\hat{\beta} - \beta\|_1$ and $L_2 = \|\hat{\beta} - \beta\|_2$, respectively. The larger they are, the less the estimation accuracy is. The $MSPE$ is used as a measure of prediction ability, which is defined as

$$MSPE = \frac{1}{n} \sum_{i=1}^{n} (\hat{y}_i - \tilde{y}_i)^2,$$

where $\tilde{y}_i$ is the predicted value for the observation $y_i$. A higher $MSPE$ represents a lower prediction accuracy.
5.1 Data generating process

The data generating mechanisms adapt that of Liu et al. (2016) for the simulation studies. Each simulated data set is of size 400 with p variables, and is separated into a training set and a testing set with 200 observations each. 10 out of the p variables are active variables, and the others are noise variables of size 25, 50, or 100. Therefore, the total number of variables p is 35, 60, or 110. X is generated from the multivariate normal distribution with a pairwise correlation of ρ and ε ∼ N(0, Ip). The pairwise correlation is ρ = 0.2 or ρ = 0.6. The 10 nonzero coefficients of the active variables are 1, 2, 3, 4, 5, -1, -2, -3, -4, and -5. We generate 100 replicates of data sets from the model.

The missing elements are 1%, 3%, or 5% of all entries in the design matrix, X. The missing data generation process is as follows. X1 and X6 are complete while X2, X5, and X10 are missing with probabilities \(1 + \exp(-X_6 + 2.5)^{-1}\), \(1 + \exp(-X_1 - X_6 + 2)^{-1}\), and \(1 + \exp(X_1 + 0.5X_6 + 2)^{-1}\), respectively. The other variables are MCAR such that the overall missing proportion is as the specified percentage. To avoid some computational difficulties for some methods, we randomly retain 25% observations as complete. Table 2 provides the mean number of complete cases available in the training sets in the simulation for all scenarios.

5.2 Simulation result

Table 3 presents means and standard errors (in parentheses) of the metrics evaluating the selection result, parameter estimation error, prediction error, and execution time for simulated data consisting of 35 explanatory variables with pairwise correlation of 0.2. The L1 and L2 columns show the \(L_1\) error and \(L_2\) error, which measure the errors of parameter estimation. The MSPE column presents the prediction error from each method. The TP, TN, FP, FN, and MCC columns are details and a summary of selection results. We focus on discussing the MCC value to evaluate the selection result as follows since it measures the overall performance by considering all of the metrics.

The last column is the average CPU time in seconds and the standard error. The bold number in each column indicates the best performance.

When missing proportion is only 1%, BIBL with threshold \(\pi = 1\) has the highest mean MCC. MILS produces the most accurate estimation and prediction with the lowest mean \(L_1\) error, mean \(L_2\) error, and mean MSPE. With the increase of missing proportion, MiG-2 and MiG-1 obtain the highest mean MCC for 3% and 5% missing values, respectively. For estimations, MiG-2 has the lowest mean \(L_1\) and \(L_2\) errors for 3% missing values, and MiG-3 has the lowest mean \(L_1\) and \(L_2\) errors for 5% missing values. For predictions, MILS has the lowest mean MSPEs for all of the three scenarios with 1%, 3%, and 5% missing elements.

Table 4 shows results for simulated data of 35 variables with pairwise correlation of 0.6. With 1% missing values, MiG-1 obtains the highest mean MCC while BIBL is more vulnerable to highly correlated data. MILS still has the best performances of estimation and prediction for data with 1% missing values. For 3% missing elements, MiG-3 has the best selection performances, MiG-2 has the most accurate estimations, and MILS has the most accurate predictions. With 5% missing values, MiG-3 obtains the best overall performances in selection, estimation, and prediction. LDLS is the fastest under all circumstances in Table 3 and 4. The standard errors of MCC for the methods MI lasso S1-S3 and MI-Lasso tend to be larger. MIIRL, BISe, and BIBL tend to have bigger standard errors of the \(L_1\) error, the \(L_2\) error, and the MSPE.

Similar to Tables 3 and 4, Tables 5 and 6 present the results for \(p = 60\) variables with pairwise correlation among variables of 0.2 and 0.6, respectively. For selection, BIBL with threshold of 0.9 shows the best performance under most scenarios, but MiG-1 and MiG-2 produce the highest mean MCC for data with 3% and 5% missing values respectively when the pairwise correlation is 0.6. For estimation and prediction, MiG-1 or MiG-2 has the lowest mean estimation errors and mean MSPEs in most situations. However, MiG-2 tends to be better when the proportion of missing values is higher. LDLS is the fast but produces the worst overall performances and the largest standard errors under all scenarios. LDLS, MI lasso S1-S3, MIIRL, and MI-Lasso seem to have higher standard errors of MCC. MIIRL, BISe, and BIBL tend to have bigger standard errors for estimation and prediction. The standard errors of MCC, estimation, and prediction from the MiGs are all small.

Tables 7 and 8 display the simulation results for data with 110 variables, which have pairwise correlations of 0.2 and 0.6, respectively. NA indicates the missing result due to the failure when applying LDLS to the high dimensional data. LD lasso cv takes the shortest mean execution time in Tables 7 and 8. For variable selection, MiG-2 obtains the highest mean MCCs in most situations while MiG-1 or MiG-3 performs best in the other situations. The differences between the MiGs are not significant. MIIRL has the highest standard errors of MCC for data with 110 variables. For estimation and prediction, the MiGs (MiG-2 mostly) generally obtain the lowest mean \(L_1\) errors, mean \(L_2\) errors, and mean MSPEs with smallest standard errors.

From Table 3 to Table 8, the selection results of the examined methods are generally worse when the pairwise correlation is 0.6, especially for MIIRL and BIBL while the selections of the MiGs are more stable. When the number of variables is small and the missing proportion is low, MILS shows good overall performances with little computation time. With large number of variables and high missing proportion, MILS becomes more time consuming and obtains poorer results while MiG-2 mostly presents the best overall performances within fairly short computation time.

Figure 2 displays the mean \(L_1\) error in Tables 3-8 according to the missing proportion and the number of variables. Figure 2(a) and (b) show the results for data with pairwise correlation of 0.2 and 0.6, respectively. The pattern of the mean \(L_2\) errors is similar to that of the mean \(L_1\) errors, and thus we only examine the estimation error using the \(L_1\) error. When the number of variables are small and the missing proportion is low, LDLS has accurate estimations. However, with the increase of variables and the missing values, LDLS clearly produces higher estimation error than...
other methods. When the number of variables is 110, LDLS fails since the number of complete cases is too small as in Table 2. LDLS only works for data with few variables and small proportion of missing values. MILS produces accurate estimations under all scenarios. However, MILS has slightly biased estimation if the pairwise correlation between variables are higher when we compare results from data with \( p = 110 \) and 5\% missing values. That is, if the dimension and missing proportion are both high, the correlation between variables have more influence on the estimation of MILS. LD lasso cv, which is a listwise-deletion method as well as LDLS, still works for the data with large \( p \). The accuracy of the estimation from LD lasso cv is at the intermediate level among all methods. MI lasso S1, S2, and S3 select variables with the threshold of low, median, and high level, respectively. As a result, the estimation accuracy of MI lasso S1-S3 are at low, median, and high level in order. Unlike most methods, the performances of MI lasso S1-S3 are better for higher correlated variables with pairwise correlation of 0.6 than variables with pairwise correlation of 0.2.

MI Stacked, MIRLnoSS, and BISS with threshold \( \pi = 0.9 \) produce larger mean \( L_1 \) errors among all methods under all circumstances. However, a different performance of BISS shows for data with \( p = 110 \), whose mean \( L_1 \) error declines with the increase of missing proportion, while the others tend to increase. The standard errors of the \( L_1 \) errors for BISS are large, thus the decrease in mean \( L_1 \) error is not significant though. MIRL obtains higher mean \( L_1 \) errors compared to other methods. For data with higher pairwise correlation, mean \( L_1 \) errors of MIRL are even larger. BIBL with threshold \( \pi = 0.9 \) obtains lower mean \( L_1 \) errors than BIBL with \( \pi = 1 \). The levels of threshold have different impacts on the estimation accuracy for MI lasso S1-S3 and BIBL. MI-Lasso produces smaller mean \( L_1 \) errors among all methods, but not as small as those of MILS and the MiGs. The estimation errors of the MiGs are the smallest under all scenarios.

Figure 3(a) and (b) show the mean \( \text{MCC} \) with respect to the mean execution time in Tables 3-8 on condition of dimension and missing proportion for data with pairwise correlation of 0.2 and 0.6, respectively. The upper left area in each diagram show the methods with best overall performances, i.e. higher selection accuracy with lower computation time. LDLS obtains high mean \( \text{MCC} \) values for small data with 35 variables within little time no matter what missing proportion is. When number of variables and missing proportion increase, the performance of LDLS declines sharply although it still takes little time. LDLS is not applicable for data with \( p = 110 \). MILS has relatively high mean \( \text{MCC} \) values in all situations compared to the other methods. However, with the increase of \( p \) or missing values, the mean \( \text{MCC} \) values drops and the mean CPU time noticeably rises. LD lasso cv obtains the intermediate level of mean \( \text{MCC} \) values ranging between 0.6 to 0.8, but it takes the least time except for LDLS. Applying MI lasso S1-S3 takes the same time. The mean \( \text{MCC} \) values of MI lasso S1-S3 clearly elevate by using the threshold of higher value. With similar execution time, MI lasso S1-S3 have distinctly lower selection accuracy than MILS.

MI Stacked and MIRLnoSS both produce very low mean \( \text{MCC} \) values under all scenarios. MIRL and BIBL (with both \( \pi = 0.9 \) and \( \pi = 1 \)) have relatively high mean \( \text{MCC} \) values, while BISS has low mean \( \text{MCC} \) values. More specifically, BIBL has better selection performances than MIRL since mean \( \text{MCC} \) of MIRL decreases when number of variables is large (\( p = 110 \)) and the pairwise correlation is higher. Nevertheless, BIBL needs much more computation time than MIRL or most of other methods. MI-Lasso obtains the mean \( \text{MCC} \) values at the intermediate level, but slightly lower than those values of LD lasso cv. Considering \( \text{MCC} \) along with the execution time, LD lasso cv performs better than MI-Lasso. The MiGs are stable to select the correct set of variables under different scenarios, and the MiGs only take a little more mean execution time than LDLS and LD lasso cv. When the number of variables and the missing proportion are both high, most methods need more time to some degree, while the mean CPU time of the MiGs remains little.

Although the MiGs rely on the lasso regression to select the initial imputation model, we restrict the initial model in the proposed algorithm to a smaller one by the pooled result from the initial MI data. Therefore, we do not always keep all the variables selected by lasso in the initial active set. The proposed algorithm depends more on the adaptive grafting for variable selection, and hence obtains the set of variables with higher \( \text{MCC} \) values than LD lasso cv.

Figure 4 demonstrates mean \( \text{MSPE} \) from 100 replicates with respect to mean CPU time for results in Tables 3-8. LDLS is excluded since it has poor overall performance. The lower left areas of Figure 4(a) and (b) represent a combination of the lowest prediction error and the lowest computation time. MILS has good performance when the data is small with 35 variables. As the dimension and missing proportion elevate, MILS produces larger and larger prediction errors with more and more time. LD lasso cv always takes the least time, but the mean \( \text{MSPE} \) values grow much higher than MILS if the data are larger with more missing values.

For most methods, the prediction ability decreases when the number of variables or the percentage of missing values increases. MI lasso S1-S3, MI Stacked, and MI-Lasso produce relatively better predictions while MIRLnoSS, MIRL, BISS, and BIBL produce comparatively worse predictions. The MiGs have the lowest mean \( \text{MSPE} \) values within relatively little time under all circumstances.

From Tables 3-8 and Figures 2-4, the differences in \( \text{MCC} \), \( L_1 \) error, \( L_2 \) error, and \( \text{MSPE} \) values between the three rules of MiG are not significant. The MiGs (mainly MiG-2) have the most stable and the best performances of selection, parameter estimates, and predictions. Since the pooling rule of MiG-2 is the selection of variable with the largest mean gradient across all MI data as in [15], the variable selected by MiG-2 at each iteration has the most significant effect in the regression model from any one of the MI datasets. Hence, MiG-2 obtains more precise estimations and predictions. In the simulated data, MiG-2 has the best overall performances within quite little computation time.
5.3 Refit

To better compare the methods, we refit the data by the approach suggested in Chen and Wang (2013) and Liu et al. (2016) by including the variables selected into the imputation model. The coefficients estimated by some methods, such as MIRL, BISS, and BIBL, are averaged across bootstrapped datasets and affected by the shrinkage effect of lasso. Furthermore, MI is not involved in listwise deletion methods, and hence we can not measure how much information is retained by the selected variables and compare it with other methods. We perform MI for $M = 5$ times on the subset of data using the selected variables from each method for the simulated missing data in the previous section. Afterwards, we refit the selected model on all the MI datasets and obtain $M$ OLS estimates of the parameters. The regression coefficients are obtained using Rubin’s rule.

Figure 5 shows the mean value of $\phi$ based on the 100 replicates of simulated data in the previous section. Figure 5(a) and (b) display results of data with pairwise correlation 0.2 and 0.6, respectively. The LDLS method is excluded from the figures since it fails for high dimensions. The methods such as MI lasso S2, MIRLnoSS, MIRL, and BIBL with threshold $\pi = 0.9$ have lower mean $\phi$ values under scenarios of high dimensionality and high missing proportion. This indicates that the imputation models consisting variables selected by these methods retain more information for MI. MiG-1 has slightly higher mean $\phi$ values than these methods, but the values are lower than MiG-2, MiG-3, and other methods. MiG-1 tends to select variables which retain more information into the imputation model for missing data compared to MiG-2 and MiG-3.

To summarize the simulation results, although the differences between MiG-1, 2, and 3 are not significant, MiG-2 performs the best and is also stable and computationally cost effective. By examining the mean $\phi$ values from refitting the simulated data, the variables selected by MiG-1 are more informative for imputation than MiG-2 and MiG-3. This probably results from the differences between the three rules in MiG algorithm. The pooling rules of MiG-2 and MiG-3 use the average gradient values and the gradient of log-likelihood with averaged parameter estimates. Taking average eliminates some information provided by MI data concerning the variation. However, averaging out the estimated potential improvement of each variable for the regression model by MiG-2 delivers the stable and accurate result.

6 Real data examples

This section illustrates the proposed algorithms to two real data sets, and compares the selection results with some feature selection methods. The wages and hours data is a small one. Difficulties arise from the facts that some of the variables have high correlations and the number of complete cases is small. The pulp lignin content data have higher percentage of missing values.

6.1 The wages and hours data

The wages and hours data consist of 39 demographic groups and 10 variables. Variable descriptions are listed in Table 9. The goal of the study is to estimate the response of average hours of labor supply during the year to increasing hourly wages. The dataset is available in the R package GSE. Figure 6(a) demonstrates the missing proportion of each variable on the left panel, and displays the missing pattern of the data on the right panel with missing frequencies by the proportion of each pattern in the data. Figure 6(b) is the pairwise correlation of the variables based on the complete pairs of observations. The sample size for complete case analysis is 28.

We apply three methods of complete-case analysis including LDLS, LD Step (listwise deletion stepwise regression), and LD lasso cv. However, different number of folds used for cross validation (cv) leads to unstable selection results. The small sample size (28 complete cases) results in this inconsistency. Hastie et al. (2009, Chapter 7) discuss the behavior of cross validation and indicate the importance of the estimated standard error of the cv estimate. In this study, we use 4-fold and 5-fold cv and name the methods as LD lasso 4cv and LD lasso 5cv, respectively.

The initialization of MiG is to fit lasso to the complete cases. Since we use the lasso regression with cross validation in the initialization of our algorithm, different choices of cv folds may produce different selections. To investigate the influence of different numbers of cv, we also perform 4-fold and 5-fold cv for lasso to obtain the initial imputation model in our algorithm and distinguish them as MiG 4-fold and MiG 5-fold. Normalizing all variables first is important to compare the gradient magnitude heuristic with other selection methods, and is suggested by Perkins et al. (2003). Therefore, we also consider to apply MiG on the scaled data, of which all variables are normalized to have a mean of 0 and a standard deviation of 1. We name this method as MiGnorm, and distinguish the methods using different initial imputation model as MiGnorm 4-fold and MiGnorm 5-fold. Since the data in simulation study are generated from multivariate normal distribution, the results of applying MiG and MiGnorm do not differ much, and thus we omit the investigations of MiGnorm for simulated data.

6.1.1 Selection results

We present the variable selection results from the above mentioned methods in Table 10, where the selected variables are shown by the checkmarks. We use the selected variables to refit the data by the approach suggested in Chen and Wang (2013) and Liu et al. (2016) in order to compare all the examined methods. The heatmap of Figure 7 shows the significance level of each variable in the refit results based on the selections from all methods. Colors in the heatmap demonstrate the $p$-values of the parameter estimates in the refit models. Lighter blue stands for coefficients that are...
more significant, and deeper blue represents a larger \( p \)-value. The plus sign and the negative sign in each cell displays the sign of the parameter estimate while zero represents the status of the variable not being selected.

LDLS selects five variables, and four of them are significant in the refit result. The LD Step selects one extra variable, \( \text{nein} \), into the model. The LD lasso 5cv selects nearly all variables in the data except for \( \text{asset} \), but only \( \text{erno} \) is significant in the refit result. The LD lasso 4cv selects only one variable \( \text{race} \). Refit result shows that \( \text{race} \) is significant.

Comparing the refit results by using those variables selected from the three listwise deletion methods, most of the variables are not significant. The only significant variable for both LDLS and LD lasso 4cv is \( \text{race} \); the significant variables for LD Step are \( \text{erno} \) and \( \text{nein} \); \( \text{erno} \) is the only significant variable for LD lasso 5cv. This suggests that the variable selection using complete-case analysis is inconsistent between different methods. Also, different numbers of cv result in unstable selections, and even a selection of redundant variables.

Surprisingly, none variable is selected by MILS and MIRL. The best probability threshold for MIRL is 0.85. From MILR, two variables, \( \text{nein} \) and \( \text{race} \), have the selection probability of 0.768 and 0.784, respectively, which are close to this threshold. If these two variables are selected, it is the same with the outcome of the selection from BIBL with a selection threshold of 0.9.

With a high threshold value of 1, BIBL only selects \( \text{race} \), which is the same as the result of LD lasso 4cv and MI lasso S3. The other extreme result comes from BISS with the threshold of 0.9, it selects all the variables into the model. From the simulation results in the previous section, BISS tends to select more variables than other methods. The same phenomenon shows for this dataset. Therefore, if considering to implement variable selection through bootstrapped MI data, we suggest to apply BIBL with proper determined threshold over BISS.

MI lasso S1, MI lasso S2, and MI Stacked produce number of selected variables between the two extremes of selecting none or all variables. Moreover, the three methods select \( \text{asset} \) while most of the examined methods no not. The variable \( \text{asset} \) is also selected by MILS, BISS, and MI-Lasso, but they select more and even all variables.

MI lasso S1 selects \( \text{erno} \), \( \text{asset} \), and \( \text{race} \) while MI lasso S2 selects \( \text{asset} \) and \( \text{race} \). Nevertheless, the refit results contradict each other. With the selection of only one variable \( \text{race} \), it is significant in the refit result. With \( \text{asset} \) and \( \text{race} \) in the model, \( \text{asset} \) is not significant. Refitting with \( \text{erno} \), \( \text{asset} \) and \( \text{race} \), is not significant anymore, but the other two are significant. This may be due to a negative correlation of -0.739 between \( \text{asset} \) and \( \text{race} \) calculated by using all complete pairs of observations in the original data.

MI Stacked and MI-Lasso selects the same 7 out of 9 variables except for \( \text{rate} \) and \( \text{age} \). However, only \( \text{erno} \) is significant in the refit result. Compared with the refit results of MI lasso S1-S3, the variables \( \text{asset} \) and \( \text{race} \) are not significant in the refit model, which seems to be contradictory.

Our proposed methods MiG-1, 2, and 3 select the same subset of variables if using cross validation with the same number of folding at the initialization stage. Hence, we simply refer them as MiG 4-fold and MiG 5-fold regardless of the choice between pooling rules for the MiGs, and refer to MiGnorm 4-fold and MiGnorm 5-fold for applying the MiGs on the scaled data in the following discussion. MiG 4-fold selects \( \text{erno} \) and \( \text{nein} \). Moreover, they are significant by refitting the model. MiG 5-fold selects \( \text{race} \), which is significant in the refit model. MiGnorm 4-fold selects \( \text{erno} \) and \( \text{race} \). The variable \( \text{race} \) is significant by refitting the model while \( \text{erno} \) is not. The correlation coefficient between complete pairs of \( \text{nein} \) and \( \text{race} \) is -0.687, representing a moderate negative correlation. This explains the selection result of MiGnorm 4-fold is a replacement for the selection result of MiG 4-fold by selecting \( \text{race} \) instead of \( \text{nein} \), only \( \text{erno} \) is no longer significant if refitting along with \( \text{race} \). Moreover, the signs of the refit estimates are different, negative for \( \text{race} \) and positive for \( \text{nein} \). MiGnorm 5-fold only selects \( \text{race} \) and it is still significant in the refit result.

Comparing the results of MiG with different folds of cv to obtain initial imputation models with and without normalization, our proposed method is more stable than only applying lasso to the complete cases if the number of complete cases is small. Furthermore, all variables selected from MiG are significant while LD lasso cv select redundant variables, which complicates the imputation and refitting. Hence, it turns out that most variables selected by LD lasso cv are not significant.

Although the survey is conducted to estimate the response of labor supply to increasing hourly wages, \( \text{rate} \) is not selected by most of the methods. At the first glance of Figure 6(b), \( \text{rate} \) seems to have strong linear relationship with the response variable \( \text{hrs} \). Nevertheless, \( \text{rate} \) also has high correlation with \( \text{nein} \), \( \text{asset} \), and \( \text{school} \) (0.702, 0.778, and 0.884 respectively). It may cause collinearity and result in the situation where a selection between these variables must be made.

There are no missing values for variables \( \text{nein} \) and \( \text{asset} \), which have a strong positive correlation, 0.988, between each other. Therefore, intuitively to avoid multicollinearity they should not be selected into the model at the same time. MI Stacked, BISS, and MI-Lasso are inappropriate in this sense. Relationship exists between \( \text{rate} \) and \( \text{school} \) for having a correlation of 0.884. BISS is the only method that selects both of them since all variables are included.

### 6.1.2 The fraction of missing information in refit model

Table 10 displays the average \( \phi \), \( \phi \), from all the refit models. Table 10 shows the total number of selected variables, \( k \), from each method. The bold numbers indicates the two smallest \( \phi \), obtained from BISS with threshold \( \pi = 0.9 \) and MiG 4-fold. However, MiG 4-fold only selects two variables while BISS \( \pi = 0.9 \) selects 9 variables, which is the full model. Also, we notice that including more variables in the imputation model does not necessarily retain more information for the missing values. For example, LD Step selects 6 variables while LD lasso 5cv selects two more variables \( \text{age} \) and \( \text{school} \). However, the value of \( \phi(LDlasso5cv) \) is 0.0410, which is larger than the value of \( \phi(LDStep) \).
as 0.0201. Similar effect shows between results of MI lasso S1 and MiGnorm 4-fold. Including asset in the imputation model when ern0 and race are already in the model does not elevate the average \( fmi \).

If all variables are included in the imputation model as the result of BISS \( \pi = 0.9 \), the refit standard errors of ern0, nein, and race are 0.1144, 0.5164, and 1.1501, respectively. For MiG 4-fold, the refit standard errors of ern0 and nein are 0.0701 and 0.0452, which result in the ratios of 0.61 and 0.09 if they are compared to those in the full model. The refit standard errors of ern0 and race for MiGnorm 4-fold are 0.1004 and 0.0922, and the ratios of them to those in the full model are 0.88 and 0.08. When race is the only selected variable, the refit standard error is 0.3300, which is less than 30% of the standard error in the full model. This coincides with one of the conclusions of Madley-Dowd et al. (2019). That is, adding auxiliary variables into the imputation model does not always elevate the efficiency.

Nevertheless, if we only consider to impute the variable race, which has the highest number of missing values among all variables and has a strong correlation of -0.832 with the dependent variable hrs, the imputation is made based on the information of dependent variable, not from the other predictors. This is a potential weakness for prediction if new cases in the testing dataset have missing values in the variable race. That is, if we only select race, the model is a simple regression model. When the new case has missing value in race, we do not observe race nor hrs, and hence cannot conduct imputation and make prediction for hrs.

To sum up, MiG 4-fold selects a model consisting of only two variables, ern0 and nein. The variable nein has strong correlation with asset and rate, 0.988 and 0.702 respectively, and has moderate correlation with dep, race, and school, -0.518, -0.687, and 0.539 respectively. Hence, nein contributes not only to the regression model as an explanatory variable and also to the imputation model as an informative variable.

### 6.2 The pulp lignin content data

The pulp lignin content data concern with pulp quality by lignin content remaining, and consist of 301 observations and 22 variables. The dataset is available in the R package VIM. The response variable is the Kappa number, denoted as \( Y.Kappa \), which measures the pulp quality. The other 21 variables are explanatory variables. The left panel of Figure 8(a) shows the missing proportion of each variable, and the right panel of Figure 8(a) demonstrates the missing pattern of this dataset with proportion of each pattern. We demonstrate the pairwise correlation of the variables based on the complete pairs of observations of the data in Figure 8(b). The number of complete cases is 131, which is 44% of the whole data.

Compared with the wages and hours data, the number of complete cases in the data is larger. There are few pairs of variables having high correlations. Therefore, the variable selection results are more consistent across all examined methods although the listwise deletion methods are deficient in identifying the correct set of variables. Figure 9 displays the heatmap of the variable selection results from all examined methods. The zero in a cell represents that the variable is not selected, and the plus sign or minus sign shows the sign of the parameter estimate for a selected variable. Most selected variables are identified consistently by all methods. In this example, MiG-1, 2, and 3 yield the same selection results so that we simply refer them as MiG for briefing as follows.

The three listwise deletion methods, LDLS, LD Step, and LD lasso cv, obtain different results. LDLS does not select ChipRate and ChipLevel4. These two variables along with ChipMoisture4 and T.Top.Chips4 are selected but not significant in the refit result by LD Step. Variables BF.CMratio, UCZAA, ChipMass4, and SteamHeatF3 are not so significant as they are in the models selected by other methods related to MI. LD lasso cv selects Upper.HeatT3 instead of Lower.HeatT3 due to their high correlation of 0.93. Five variables, BlowFlow, ChipLevel4, ChipMoisture4, SteamFlow4, and WeakWashF, are selected by LD lasso cv, but are not significant or only significant with high \( p \)-values in the refit. This indicates that the listwise deletion methods produce confusing selections.

Besides the listwise deletion methods, other methods incorporating variable selection with MI produce more consistent results. However, most of them tend to select more variables, among which several are not significant in the refit results. Four methods, MILS, MIRL, BIBL with the threshold of 1, and MiG, obtain smaller models, in which all variables are significant. MiGnorm, which conducts MiG algorithm on the scales data, selects one more variable ChipMoisture4 than MiG. The variable ChipMoisture4 is not significant in the refit with the \( p \)-value of 0.1. MILS and MIRL select the same set of variables, but these two methods do not select ChipLevel4, a significant variable in most models. In addition, BIBL with the threshold of 1 excludes two variables, ChipLevel4 and ChipMass4, which are significant in most models.

The only difference between the selection by MiG and MiGnorm is the variable ChipMoisture4, which is not significant in the refit. All other variables selected by MiG or MiGnorm are significant in their refit results. Table 11 presents the ratio of the average fraction of missing information to the full model, \( r \), along with the number of selected variables from every examined method. The bold numbers are the three smallest values, which are obtained by MI lasso S1, MiGnorm, and MI lasso S3 in order. However, the number of variables selected by MiGnorm is the smallest among the three. This suggests that selecting the variable ChipMoisture4 helps retain information for the missing values although it is near-marginal significance (\( p \)-value of 0.1). The results of MI lasso S1-S3 also coincide with the conclusion of Madley-Dowd et al. (2019) that adding auxiliary variables does not always result in efficiency gains. Since variables are selected by MI lasso S1-S3 using different levels of threshold, the set of variables selected by MI lasso S3 is a subset of the variable set selected by MI lasso S2. Nevertheless, \( r(MIlassoS2) \) is 1.0609, which is higher than 0.2830, the value of \( r(MIlassoS3) \). Table 11 suggests that MiG out only selects the correct set of active variables, but also selects the informative variables for imputation.
7 Conclusion and discussion

There are some challenges to variable selection for missing data, especially when there are redundant variables in the dataset. Furthermore, the noninformative variables with missing values slow down the execution of multiple imputation. In this paper, we propose to apply the greedy forward search on multiply imputed data in a computationally effective manner. We implement the adaptive grafting approach to identify the active variables incrementally, and conduct multiple imputation on the subset of the missing data, which expands iteratively by including both variables and observations. To tackle the selections across multiply imputed datasets, we propose three pooling rules, namely selection by vote, selection by averaging grafting results, and selection from the pooled result.

The simulation study shows that our proposed methods achieve the overall performances within little computation time when comparing with some other common methods. Among the three proposed rules, MiG-2 mostly performs the best at obtaining correct set of active variables, good estimations and good predictions. MiG-2 selects the variables which are the most promising based on the mean gradient of the likelihood from all of the MI datasets, and hence obtains more stable and accurate estimations and predictions. The differences of variable selection between the three rules are not remarkable in both of the simulation and the real data analysis. According to the average fmi in the imputation model, MiG-1 selects the set of variables which are more informative among the three MiG procedures.

The applications to two real-life data examples confirm the strengths of MiGs on different conditions. If the total number of complete cases is small, the listwise deletion methods are not reliable, but the impact on MiGs is smaller. When the explanatory variables are highly correlated between each other, the multicollinearity causes difficulties in variable selection for some methods as in the wages and hours data example. In the situation, MiGs select few active variables which also provide information for imputation. However, the MiGs produces different selection results depending on if the data are scaled or the choice of the number of folds for cross validation used in initialization. For data in which fewer variables are highly correlated and there are more complete cases as in the pulp lignin content example, the results of the methods performing variable selection on MI data are more consistent. The selections of the three rules of the MiGs are identical. The MiG and the MiGnorm select the same set of variables except that the MiGnorm selects one more variable, which is only slightly non-significant when refitting the selected model while the other variables are all significant. However, with this variable in the imputation model, the average fmi diminishes.

Figure 1: Updates on observations, variables, and imputed datasets in the iteration of MiG algorithm
Figure 2: Mean $L_1$ error for simulated data: (a) pairwise correlation = 0.2; (b) pairwise correlation = 0.6
Figure 3: Selection performance vs computation time for simulated data: (a) pairwise correlation = 0.2; (b) pairwise correlation = 0.6
Figure 4: Prediction performance vs computation time for simulated data: (a) pairwise correlation = 0.2; (b) pairwise correlation = 0.6
Figure 5: Mean $\phi$ for simulated data: (a) pairwise correlation = 0.2; (b) pairwise correlation = 0.6
Figure 6: Wages and hours data: (a) Missing pattern; (b) Pairwise correlation
Figure 7: Heatmap of variable selection results for the wages and hours data
Figure 8: Pulp lignin content data: (a) Missing pattern; (b) Pairwise correlation
Figure 9: Heatmap of variable selection results for the pulp lignin content data

Table 1: Model selection methods used in the simulation.

| Method         | Description                                                                 |
|----------------|-----------------------------------------------------------------------------|
| LDLS           | Listwise deletion least squares regression combined by Rubin’s Rule         |
| MILS           | Multiple imputation with least squares regression                          |
| LD lasso cv    | Listwise deletion lasso with cross validation                              |
| MI lasso S1    | Separate imputations and select predictors that appear in any model replacing backward elimination in Wood et al. (2008) with lasso |
| MI lasso S2    | Separate imputations and select predictors that appear in at least half of the models replacing backward elimination in Wood et al. (2008) with lasso |
| MI lasso S3    | Separate imputations and select predictors that appear in all models replacing backward elimination in Wood et al. (2008) with lasso |
| MI Stacked     | The method of stacked imputed data sets in Wood et al. (2008) replacing the stepwise procedure by lasso |
| MIRLnoSS      | Multiple imputed random lasso without stability selection                  |
| MIRL          | Multiple imputed random lasso with stability selection                     |
| BISS, $\pi = 0.9$ | Bootstrap imputation and stability selection, threshold = 0.9             |
| BIBL, $\pi = 0.9$ | Bootstrap imputation and BoLasso, threshold = 0.9                      |
| BIBL, $\pi = 1$ | Bootstrap imputation and BoLasso, threshold = 1                          |
| MI-Lasso       | Multiple imputation with lasso, treating the coefficients of the same variable across all MI sets as a group by the group lasso penalty |
| MiG-1          | Multiple imputation with Greedy forward selection, selection by vote     |
| MiG-2          | Multiple imputation with Greedy forward selection, selection by averaging grafting results |
| MiG-3          | Multiple imputation with Greedy forward selection, selection from the pooled result |
Table 2: Average number of complete cases in the training set used in simulation

| Missing proportion | 1% | 3% | 5% |
|--------------------|----|----|----|
| Pairwise correlation | 0.2 | 0.6 | 0.2 | 0.6 | 0.2 | 0.6 |
| p = 35 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
| p = 60 | 113.20 | 111.90 | 62.30 | 61.92 | 52.28 | 52.22 |
| p = 110 | 81.69 | 81.26 | 50.72 | 50.77 | 49.40 | 49.42 |

Table 3: Simulation result for the case with p = 35 variables and pairwise correlation p = 0.2

| Variables and pairwise correlation | 1% | 3% | 5% |
|-----------------------------------|----|----|----|
| p = 0.2 | 27 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
| p = 0.3 | 27 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
| p = 0.5 | 27 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
| p = 0.7 | 27 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
| p = 0.9 | 27 | 137.78 | 136.14 | 83.48 | 83.02 | 62.25 | 62.02 |
### Table 4: Simulation result for the case with p = 35 variables and pairwise correlation ρ = 0.6

| L1   | L2   | L3   | MSPE | TP   | TN   | FP   | FN   | MCC  | Time |
|------|------|------|------|------|------|------|------|------|------|
| LDLS | 1.44 (0.58) | 0.26 (0.14) | 1.22 (0.18) | 10.00 (0.00) | 47.32 (1.99) | 2.48 (2.18) | 0.00 (0.00) | 0.88 (0.09) | 2.18 (0.78) |
| MILS | 1.33 (0.74) | 0.18 (0.11) | 2.07 (0.36) | 10.00 (0.00) | 46.73 (2.48) | 3.26 (2.48) | 0.00 (0.00) | 0.91 (0.06) | 5.67 (1.57) |
| MiL3 | 3.06 (0.93) | 0.05 (0.16) | 1.56 (0.17) | 10.00 (0.00) | 48.64 (1.40) | 1.36 (1.29) | 0.00 (0.00) | 0.91 (0.06) | 3.91 (1.29) |
| MI stacked | 4.56 (1.09) | 0.73 (0.31) | 1.62 (0.28) | 10.00 (0.00) | 13.14 (4.91) | 36.86 (4.91) | 0.00 (0.00) | 0.24 (0.06) | 10.72 (1.16) |
| MI lasso S2 | 3.26 (1.05) | 0.53 (0.23) | 1.45 (0.22) | 10.00 (0.00) | 27.37 (9.22) | 22.63 (9.22) | 0.00 (0.00) | 0.43 (0.14) | 10.91 (1.19) |
| MI lasso S1 | 3.57 (1.02) | 0.54 (0.23) | 1.46 (0.22) | 10.00 (0.00) | 7.62 (6.53) | 42.38 (6.53) | 0.00 (0.00) | 0.17 (0.08) | 10.91 (1.19) |

### Table 5: Simulation result for the case with p = 60 variables and pairwise correlation ρ = 0.2

| L1   | L2   | L3   | MSPE | TP   | TN   | FP   | FN   | MCC  | Time |
|------|------|------|------|------|------|------|------|------|------|
| LDLS | 2.79 (1.32) | 1.27 (0.21) | 1.58 (0.97) | 10.00 (0.00) | 46.69 (1.99) | 1.31 (0.91) | 0.32 (0.57) | 0.94 (0.00) | 2.18 (0.78) |
| MILS | 2.90 (1.34) | 0.80 (0.14) | 2.04 (0.26) | 10.00 (0.00) | 45.98 (2.48) | 3.26 (2.48) | 0.00 (0.00) | 0.91 (0.06) | 5.67 (1.57) |
| MiL3 | 3.36 (0.97) | 0.05 (0.16) | 1.56 (0.17) | 10.00 (0.00) | 48.64 (1.40) | 1.36 (1.29) | 0.00 (0.00) | 0.91 (0.06) | 3.91 (1.29) |
| MI stacked | 4.56 (1.09) | 0.73 (0.31) | 1.62 (0.28) | 10.00 (0.00) | 13.14 (4.91) | 36.86 (4.91) | 0.00 (0.00) | 0.24 (0.06) | 10.72 (1.16) |
| MI lasso S2 | 3.26 (1.05) | 0.53 (0.23) | 1.45 (0.22) | 10.00 (0.00) | 27.37 (9.22) | 22.63 (9.22) | 0.00 (0.00) | 0.43 (0.14) | 10.91 (1.19) |
| MI lasso S1 | 3.57 (1.02) | 0.54 (0.23) | 1.46 (0.22) | 10.00 (0.00) | 7.62 (6.53) | 42.38 (6.53) | 0.00 (0.00) | 0.17 (0.08) | 10.91 (1.19) |

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Note: L1 L2 MSPE TP TN FP FN MCC Time
### Table 6: Simulation result for the case with 60 variables and pairwise correlation ρ = 0.6

| Method      | L1 | L2 | MSPE | TP  | TN  | FP  | FN  | MCC  | Time  |
|-------------|----|----|------|-----|-----|-----|-----|------|-------|
| MI stacked  | 1.19 | 1.05 | 0.95 | 0.95 | 1.09 | 0.15 | 0.15 | 0.95 | 9.95 |
| MI lasso S2 | 2.02 | 2.02 | 0.01 | 0.00 | 2.02 | 0.02 | 0.02 | 0.01 | 9.99 |
| MI lasso S3 | 3.89 | 3.89 | 0.11 | 0.11 | 3.89 | 0.12 | 0.12 | 0.11 | 9.99 |
| MI stacked  | 5.66 | 5.66 | 0.17 | 0.17 | 5.66 | 0.18 | 0.18 | 0.17 | 9.99 |
| MI lasso S2 | 3.63 | 3.63 | 0.78 | 0.78 | 3.63 | 0.79 | 0.79 | 0.78 | 9.99 |
| MI lasso S3 | 5.52 | 5.52 | 1.47 | 1.47 | 5.52 | 1.48 | 1.48 | 1.47 | 9.99 |
| MI stacked  | 10.39 | 10.39 | 4.21 | 4.21 | 10.39 | 4.22 | 4.22 | 4.21 | 9.99 |
| MI LassoSS | 6.81 | 6.81 | 3.56 | 3.56 | 6.81 | 3.57 | 3.57 | 3.56 | 9.99 |
| MI stacked  | 14.94 | 14.94 | 7.09 | 7.09 | 14.94 | 7.10 | 7.10 | 7.09 | 9.99 |
| MI lasso S2 | 5.10 | 5.10 | 0.39 | 0.39 | 5.10 | 0.40 | 0.40 | 0.39 | 9.99 |
| MI lasso S3 | 7.19 | 7.19 | 0.64 | 0.64 | 7.19 | 0.65 | 0.65 | 0.64 | 9.99 |
| MI stacked  | 11.87 | 11.87 | 1.58 | 1.58 | 11.87 | 1.59 | 1.59 | 1.58 | 9.99 |
| MI LassoSS | 7.64 | 7.64 | 1.87 | 1.87 | 7.64 | 1.89 | 1.89 | 1.88 | 9.99 |
| MI stacked  | 13.74 | 13.74 | 3.86 | 3.86 | 13.74 | 3.87 | 3.87 | 3.86 | 9.99 |

### Table 7: Simulation result for the case with 60 variables and pairwise correlation ρ = 0.2

| Method      | L1 | L2 | MSPE | TP  | TN  | FP  | FN  | MCC  | Time  |
|-------------|----|----|------|-----|-----|-----|-----|------|-------|
| MI stacked  | 2.10 | 2.10 | 0.21 | 0.21 | 2.10 | 0.21 | 0.21 | 0.21 | 9.99 |
| MI lasso S2 | 3.67 | 3.67 | 0.88 | 0.88 | 3.67 | 0.89 | 0.89 | 0.88 | 9.99 |
| MI lasso S3 | 5.54 | 5.54 | 1.64 | 1.64 | 5.54 | 1.65 | 1.65 | 1.64 | 9.99 |
| MI stacked  | 11.58 | 11.58 | 4.17 | 4.17 | 11.58 | 4.18 | 4.18 | 4.17 | 9.99 |
| MI LassoSS | 7.48 | 7.48 | 2.93 | 2.93 | 7.48 | 2.94 | 2.94 | 2.93 | 9.99 |
| MI stacked  | 14.94 | 14.94 | 8.89 | 8.89 | 14.94 | 8.90 | 8.90 | 8.89 | 9.99 |
| MI lasso S2 | 5.10 | 5.10 | 0.39 | 0.39 | 5.10 | 0.40 | 0.40 | 0.39 | 9.99 |
| MI lasso S3 | 7.19 | 7.19 | 0.64 | 0.64 | 7.19 | 0.65 | 0.65 | 0.64 | 9.99 |
| MI stacked  | 11.87 | 11.87 | 1.58 | 1.58 | 11.87 | 1.59 | 1.59 | 1.58 | 9.99 |
| MI LassoSS | 7.64 | 7.64 | 1.87 | 1.87 | 7.64 | 1.89 | 1.89 | 1.88 | 9.99 |
| MI stacked  | 13.74 | 13.74 | 3.86 | 3.86 | 13.74 | 3.87 | 3.87 | 3.86 | 9.99 |

**Notes:**
- L1, L2: L1 and L2 regularization coefficients
- MSPE: Mean Squared Prediction Error
- TP, TN, FP, FN: True Positive, True Negative, False Positive, False Negative counts
- MCC: Matthews Correlation Coefficient
- Time: Computation time in seconds
### Table 8: Simulation result for the case with $p = 110$ variables and pairwise correlation $\rho = 0.6$

| L1 | L2 | MDD | TP | TN | FP | FN | MCC | Time |
|----|----|-----|----|----|----|----|-----|------|
| MILS | NA (NA) | NA (NA) | NA (NA) | NA (NA) | NA (NA) | NA (NA) | NA (NA) | NA (NA) |
| MI | 3.40 (1.68) | 1.67 (0.94) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |
| MI Lasso cv | 3.73 (2.10) | 1.67 (0.94) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |
| MI Lasso S1 | 2.98 (1.56) | 2.01 (1.33) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |
| MI Lasso S2 | 2.95 (1.53) | 2.01 (1.33) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |
| MI Lasso S3 | 2.95 (1.53) | 2.01 (1.33) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |
| MI Lasso | 3.46 (1.78) | 1.67 (0.94) | 9.44 (0.92) | 99.35 (1.69) | 0.65 (0.07) | 0.97 (0.07) | 4.55 (1.76) |

### Table 9: Description of the wages and hours data

| Variable | Description |
|----------|-------------|
| hrs | Average hours worked during the year |
| rate | Average hourly wage (USD) |
| ersp | Average yearly earnings of spouse (USD) |
|erno | Average yearly earnings of other family members (USD) |
|nein | Average yearly non-earned income |
|asset | Average family asset holdings (Bank account, etc.) (USD) |
|age | Average age of respondent |
|dep | Average number of dependents |
|race | Percent of white respondents |
|school | Average highest grade of school completed |
Table 10: Variable selection results for the wages and hours data

| Method          | Variable | k | φ  | r  |
|-----------------|----------|---|----|----|
| LDLS            | rate, ersp, erno, nein, asset, age, dep, race, school | 5 | 0.0429 | 3.4061 |
| LD Step         | rate, ersp, erno, nein, asset, age, dep, race, school | 6 | 0.0201 | 1.5986 |
| LD lasso 4cv    | rate, ersp, erno, nein, asset, age, dep, race, school | 1 | 0.0129 | 1.0198 |
| LD lasso 5cv    | rate, ersp, erno, nein, asset, age, dep, race, school | 8 | 0.0410 | 3.2522 |
| MILS            |          | 0 | -  | -  |
| MI lasso S1     |          | 3 | 0.0633 | 5.0238 |
| MI lasso S2     |          | 2 | 0.0743 | 5.8995 |
| MI lasso S3     |          | 1 | 0.0129 | 1.0198 |
| MI Stacked      |          | 7 | 0.0313 | 2.4861 |
| MI-Lasso        |          | 0 | -  | -  |
| BISS π = 0.9    |          | 9 | 0.0126 | 1    |
| BIBL π = 0.9    |          | 2 | 0.0442 | 3.5079 |
| BIBL π = 1      |          | 1 | 0.0129 | 1.0198 |
| MI-Lasso        |          | 7 | 0.0313 | 2.4861 |
| MiG 4-fold      |          | 2 | 0.0114 | 0.9048 |
| MiGnorm 4-fold  |          | 2 | 0.0469 | 3.7196 |
| MiG 5-fold      |          | 1 | 0.0129 | 1.0198 |
| MiGnorm 5-fold  |          | 1 | 0.0129 | 1.0198 |

* The checkmark symbolizes the variable being selected.

Table 11: Fraction of missing information from imputation models selected by examined methods for the pulp lignin content data

| Method          | k | r  |
|-----------------|---|----|
| LDLS            | 8 | 0.3795 |
| LD Step         | 12| 0.4027|
| LD lasso cv     | 13| 0.4298|
| MILS            | 9 | 1.2783|
| MI lasso S1     | 19| 0.2451|
| MI lasso S2     | 18| 1.0609|
| MI lasso S3     | 15| 0.2830|
| MI Stacked      | 18| 1.0609|
| MIRL            | 9 | 1.2783|
| BISS π = 0.9    | 18| 2.3572|
| BIBL π = 0.9    | 16| 0.9793|
| BIBL π = 1      | 8 | 0.6018|
| MI-Lasso        | 18| 1.0609|
| MiG             | 10| 0.3106|
| MiGnorm         | 11| 0.2817|

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