Chiral Tomonaga-Luttinger Liquids and
the Calogero-Sutherland Model with Boundaries †

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Abstract

Applying boundary conformal field theory we study the low-energy critical behavior of the Calogero-Sutherland model of $BC_N$-type. The universality class of the model is found to be a chiral Tomonaga-Luttinger liquid. Various correlation exponents depending on the interaction strength are obtained.

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I. INTRODUCTION

Recently the one-dimensional quantum integrable models with long-range interactions have received renewed interest. The original model of this type is the Calogero-Sutherland (CS) model which consists of \( N \) particles repelling each other via inverse-square interactions on a circle of circumference \( L \). The Hamiltonian is given by

\[
\mathcal{H}_A = -\sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda - 1) \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \frac{1}{L} \sin \left( \frac{\pi}{L} (q_j - q_k) \right),
\]

where \((q_1, q_2, \cdots, q_N)\) denote particle coordinates and \( \lambda \) is a coupling constant. The CS model has various significant features: (i) This model has relatively simple eigen functions which enable us to perform the exact computation of physical quantities such as dynamical correlation functions \[3\]; (ii) The energy spectrum is shown to be reproduced exactly with the use of the asymptotic Bethe-ansatz (ABA) method \[2\]. (iii) The excitations are described by quasiparticles that obey exclusion statistics \[4\]. (iv) The CS model is a typical example of the Tomonaga-Luttinger liquid \[5\]. These simple, but non-trivial natures of the CS model originate from the integrability of the system and affirm its paradigmatic role as the anyonic analog of the free boson or fermion gas. In addition the CS model is related to various branches of physics and contains many interesting aspects in mathematical physics (see for reviews \[6\] and references therein). Especially it is established that this model is the universal Hamiltonian for disordered systems (typifying quantum chaos) \[7\].

In the original formulation of the CS model, the interaction among the particles is simply given in a pairwise form depending on the difference of particle coordinates. This simple pairwise interaction is understood as reflecting the underlying structure of the root system of type \( A_{N-1} \). Thus the integrability of the model has its connection with the theory of Lie algebras. Similarly to the case of integrable soliton theories which are systematically constructed on the basis of the Lie algebra, the integrability still holds for many-body systems with inverse-square potentials which are obtained by means of root systems associated to the general Weyl groups \[8\]. Although these models have nice properties which reflect the
Weyl group symmetries, it seems that little attention from the physics point of view has been paid to the models except the \( A_{N-1} \)-CS model.

More recently, it has been revealed that the so-called CS model of \( BC_N \)-type (\( BC_N \)-CS model) which is the most general one among the extended CS models is relevant to one-dimensional physics with boundaries, for example, the boundary quantum sine-Gordon model [9] and the Dorokhov-Mell-Pereyra-Kumar equation [10] which describes evolution of an ensemble of quasi one-dimensional disordered wires. Therefore it is now important to clarify the critical properties of the \( BC_N \)-CS model toward our further understanding in one-dimensional physics including boundary effects.

Our purpose in this contribution is to analyze the long-distance critical properties of the \( BC_N \)-CS model [11], following the earlier consideration of the \( A_{N-1} \)-CS model [5]. In particular it will be shown that the critical behavior of the \( BC_N \)-CS model is described by \( c = 1 \) Gaussian conformal field theory (CFT) with boundaries [13] in which we have only left (or right) moving sector of CFT. Hence the universality class of the \( BC_N \)-CS model will be identified as a chiral Tomonaga-Luttinger liquid [14].

II. \( BC_N \)-CS MODEL AND ASYMPTOTIC BETHE-ANSATZ SPECTRUM

We recall the \( BC_N \)-CS model [8]. The \( BC_N \)-CS model is related to the root system of type \( BC_N \) and invariant under the action of the Weyl group of type \( B_N \). That is, the model is invariant under the exchange of particle coordinates and the sign change of coordinates. The latter is regarded as the exchange of the particle and its mirror image particle with respect to the origin. The model has the periodic Hamiltonian

\[
\mathcal{H}_{BC} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda - 1) \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \left\{ \frac{1}{\sin^2 \frac{\pi}{L}(q_j - q_k)} + \frac{1}{\sin^2 \frac{\pi}{L}(q_j + q_k)} \right\} + \mu(\mu - 1) \left( \frac{\pi}{L} \right)^2 \sum_{j=1}^{N} \frac{1}{\sin^2 \frac{\pi}{L}q_j} + \nu(\nu - 1) \left( \frac{\pi}{L} \right)^2 \sum_{j=1}^{N} \frac{1}{\cos^2 \frac{\pi}{L}q_j},
\]

(2)
where $\lambda, \mu, \nu$ ($>0$) are coupling constants. It is clearly seen that the Hamiltonian (2) is invariant under the action of the Weyl group of type $B_N$, however, is not translationally invariant. Therefore the total momentum is not a good quantum number for the $BC_N$-CS model.

The interaction terms which violate translationally invariance are required by invariance under the action of the Weyl group of type $B_N$. These terms have clear physical interpretation: the term $1/\sin^2(\pi/L)(q_j + q_k)$ represents the two-body interaction between the $j$-th particle and the mirror-image of the $k$-th particle ($j \neq k$). The terms $1/\sin^2(\pi/L)q_j$ and $1/\cos^2(\pi/L)q_j$ can be regarded as the potential due to impurities located at the origin.

The spectrum of the Hamiltonian (2) of the $BC_N$-CS model has already been obtained [12]. Let us present the result in the ABA form. The energy eigenvalue of the system then takes the non-interacting form

$$E_N = \sum_{j=1}^{N} k_j^2,$$

where pseudomomenta $k_j$’s satisfy $k_1 > k_2 > \cdots > k_N > 0$ and obey the ABA equations

$$k_j L = 2\pi I_j + \pi(\lambda - 1) \sum_{l=1, l \neq j}^{N} \{ \text{sgn}(k_j - k_l) + \text{sgn}(k_j + k_l) \}$$
$$+ \pi(\mu - 1)\text{sgn}(k_j) + \pi(\nu - 1)\text{sgn}(k_j), \quad j = 1, \cdots, N,$$

with $\text{sgn}(x) = 1$ for $x > 0$, $= 0$ for $x = 0$ and $= -1$ for $x < 0$. The quantum numbers $I_j \in \mathbb{Z}_{>0}$ ($j = 1, \cdots, N$) characterize the excited states. Notice that the form of Bethe-ansatz equations (4) is common to the Bethe-ansatz solvable models with boundaries, for example, the nonlinear Schrödinger equation on the half line [13] and the $XXZ$ model with open boundary conditions [17].

It is immediate to solve the ABA equations (4), obtaining

$$k_j = \frac{2\pi}{L} [I_j - (N - j + 1)] + k_j^{(0)}, \quad j = 1, \cdots, N,$$

where

$$k_j^{(0)} = \frac{2\pi}{L} \left[ \lambda(N - j) + \frac{\mu + \nu}{2} \right].$$
We see that the ground state is specified by the quantum numbers $I_j^{(0)} = N - j + 1, \ (j = 1, \cdots, N)$. Thus, we get the Fermi point $I_1^{(0)} = N$ and the Fermi momentum $k_F = \max \{ k_j^{(0)} \} = 2\pi \lambda N/L + \pi (\mu + \nu - 2\lambda)/L$. It is important to note that the Fermi surface of the $BC_N$-CS model consists of a single point.

For later use, we shall evaluate the Fermi velocity of the elementary excitations for the $BC_N$-CS model. The Fermi velocity of the $BC_N$-CS model cannot be determined from the dispersion relation, because the momentum is not a good quantum number. In order to circumvent this apparent difficulty we assume that CFT for the $BC_N$-CS model has the central charge $c = 1$. This assumption seems to be legitimate since the $A_{N-1}$-CS model which may be regarded as the bulk counterpart of the $BC_N$-CS model is described in terms of $c = 1$ CFT [5]. Having the value of the central charge we can determine the Fermi velocity $v_F$ from the low-temperature expansion formula of the free energy $F(T)$,

$$F(T) \simeq F(T = 0) - \frac{\pi T^2}{6v_F}c, \quad (7)$$

where $T$ is the temperature. Now, applying the method of Yang and Yang for the elementary excitation at finite temperatures [17] to the ABA formula (4), one can perform the low-temperature expansion of the free energy $F(T)$ of the $BC_N$-CS model. We get

$$F(T) \simeq F(T = 0) - \frac{\pi T^2}{6(4\pi \lambda d)}, \quad (8)$$

where $d = N/L$ is the particle density. Then comparing (8) and (7) with $c = 1$ we obtain $v_F = 4\pi \lambda d$. In what follows the validity of our assumption will be confirmed.

III. FINITE-SIZE SCALING ANALYSIS

When the principle of conformal invariance is applicable we have a powerful way of obtaining the exponents of correlation functions in the long-wavelength limit. As is mentioned in the previous section the Fermi surface of the $BC_N$-CS model consists of a single point. This implies that the low-energy critical behavior of the $BC_N$-CS model will be effectively
described by a left (or right)-moving sector of CFT. Hence we expect that boundary CFT will play a role in our study of the $BC_N$-CS model. To begin with, we summarize several fundamental formulas in boundary CFT [13] which will be needed to analyze the energy spectrum.

Using conformal invariance under free boundary conditions [18] we have the finite-size scaling form of the ground-state energy

$$E^{(0)} = L\epsilon^{(0)} + 2f - \frac{\pi v_p}{24L}c,$$

(9)

where $\epsilon^{(0)}$ and $f$ are, respectively, the bulk limits of the ground-state energy density and the boundary energy. The central charge $c$ which labels the universality class of the system appears as the universal amplitude of the $1/L$ term in (9).

Consider a critical system on the half-plane $\{(y, \tau) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\}$ with a surface at $y = 0$. ($\tau$ means the imaginary time.) Let $A(y, \tau)$ be a local operator. We consider its two-point correlation function $G(y_1, y_2, \tau) = \langle A(y_1, \tau_1)A(y_2, \tau_2)\rangle$, which is a function of $\tau = \tau_1 - \tau_2$ because of translational invariance along the surface. For $|\tau| \gg y_1, y_2$, the two-point function $G$ has the asymptotic form

$$G(y_1, y_2, \tau) \sim \frac{1}{\tau^{2x_b}},$$

(10)

where $x_b$ is called the boundary critical exponent.

The boundary critical exponents $x_b$ can be read off from the scaling behavior of the excitation energy [13]

$$E - E^{(0)} = \frac{\pi v_p}{L}x_b,$$

(11)

where $E$ is the excitation energy. The value of $x_b$ is generically distinct from that of the bulk exponent for a corresponding scaling operator. Note also that, in terms of CFT, the bulk exponent is expressed as the sum of left and right conformal weights, while the boundary exponent is equal to the left (or right) conformal weight.

Let us now calculate the finite-size corrections to the spectrum of the $BC_N$-CS model. From (3) and (6) the ground-state energy is obtained as

From (3) and (6) the ground-state energy is obtained as
\[ E_{N}^{(0)} = \sum_{j=1}^{N} \left( k_{j}^{(0)} \right)^{2} = \left( \frac{2\pi}{L} \right)^{2} \left[ \frac{1}{3} \lambda N^{3} + \frac{1}{2} \lambda (\mu + \nu - \lambda) N^{2} + \frac{1}{12} \left( 3(\mu + \nu - \lambda)^{2} - \lambda^{2} \right) N \right]. \quad (12) \]

The formula (12) leads to the finite-size corrections to the ground-state energy

\[ E_{N}^{(0)} = \epsilon^{(0)} L + 2f + \frac{\pi v_{F}}{L} \lambda (\Delta N_{b})^{2} - \frac{\pi v_{F}}{12L} \lambda, \quad (13) \]

where \( \epsilon^{(0)} = 4\pi^{2} \lambda^{2} d^{3}/3, \ f = \pi^{2} \lambda (\mu + \nu - \lambda) d^{2} \) and

\[ \Delta N_{b} = \frac{1}{2} \left( 1 - \frac{\mu + \nu}{\lambda} \right). \quad (14) \]

The quantum number \( \Delta N_{b} \) physically represents the phase shift due to the scattering by the impurity- and boundary-potentials [11]. Thus the ground state which has the energy \( E_{N}^{(0)} \) is regarded as the phase-shifted ground state [19]. If we define a hypothetical system which does not include these boundary contributions, the corresponding ground-state energy \( \tilde{E}_{N}^{(0)} \) is given by

\[ \tilde{E}_{N}^{(0)} = E_{N}^{(0)} - \frac{2\pi v_{F}}{L} \frac{\lambda}{2} (\Delta N_{b})^{2} = \epsilon^{(0)} L + 2f - \frac{\pi v_{F}}{12L} \lambda. \quad (15) \]

Comparing this with (9) one would find a curious value of the central charge which differs from \( c = 1 \). The similar phenomenon was observed for the \( A_{N-1-CS} \) model [1]. This discrepancy may be traced back to the finite-size approximation to the long-range potential. Then this will not affect our assumption that \( c = 1 \).

We next calculate the finite-size corrections to the excited states. Excited state are created as follows: (a) we add \( \Delta N \) extra particles to get the ground-state configuration for \( N + \Delta N \), and (b) we create particle-hole excitation near the Fermi point labeled by non-negative integers \( n \). Notice that any excitations which carry currents with large momentum transfer are prohibited due to the absence of translational invariance in the \( BC_{N-CS} \) model. Let us first create an excited state corresponding to (a). In this case, we can solve ABA equations [4] to obtain the pseudomomenta

\[ k_{j} = \frac{2\pi}{L} \left[ \lambda (N + \Delta N - j) + \frac{\mu + \nu}{2} \right]. \quad (16) \]
Then the finite-size corrections to leading order in $1/L$ read

$$E_{N+\Delta N}^{(0)} - E_N^{(0)} \simeq \mu_c^{(0)} \Delta N + \frac{\pi}{L} \left[ 4\pi \lambda (\mu + \nu - \lambda) d\Delta N + 4\pi \lambda^2 d(\Delta N)^2 \right],$$

where $\mu_c^{(0)} = \partial e^{(0)} / \partial d = (2\pi \lambda d)^2$ is the chemical potential. This is rewritten as

$$E_{N+\Delta N}^{(0)} - \tilde{E}_N^{(0)} = \frac{2\pi v_F}{L} \frac{\lambda}{2} (\Delta N - \Delta N_b)^2,$$

where we have redefined $E_N^{(0)}$ as $E_N^{(0)} - \mu_c^{(0)} N$. We notice that this expression for the finite-size spectrum is essentially the same as that for the charge sector in the Kondo problem (see formula (49) in [20]).

The other possible low-energy excitations are provided by particle-hole excitations (b). The corresponding energy is obtained by adding $2\pi v_F n/L$ to (18). Hence we finally have

$$E - \tilde{E}_N^{(0)} = \frac{2\pi v_F}{L} \left[ \frac{\lambda}{2} (\Delta N - \Delta N_b)^2 + n \right],$$

where $E$ stands for the energy of the excited state specified by the set of quantum numbers $(\Delta N, \Delta N_b, n)$.

### IV. BOUNDARY CRITICAL EXponents

We now wish to calculate various critical exponents using the scaling relation (11). Notice that when comparing our result (19) with (11) we have to replace $L$ with $2L$ since $L$ has been defined as the periodic length of the system.

Let us first introduce an operator $\psi_b$ which corresponds to the phase-shifted ground state. This operator can be assumed to be the boundary changing operator [19] which plays a fundamental role in boundary CFT. Then the phase-shifted ground state is an excited state relative to $\tilde{E}_N^{(0)}$ defined in (15). The scaling dimension of $\psi_b$ is thus obtained as

$$x_{\psi_b} = \frac{L}{\pi v_F} \left( E_N^{(0)} - \tilde{E}_N^{(0)} \right) = \frac{1}{2\xi^2} (\Delta N_b)^2,$$

where we have put $\xi = 1/\sqrt{\lambda}, \; \zeta = 1/\sqrt{\mu + \nu}$, and hence $\Delta N_b = (1 - \xi^2/\zeta^2)/2$. 

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We next consider an operator $\phi$ which induces the particle number change as well as the particle-hole excitation in the phase-shifted ground state. From (19) and (11) we have

$$x_\phi = \frac{L}{\pi v_F} \left( E - \tilde{E}_N^{(0)} \right) = \frac{1}{2\xi^2} \left( \Delta \hat{N} \right)^2 + n,$$

(21)

where

$$\Delta \hat{N} = \Delta N - \Delta N_b.$$  

(22)

Scaling dimensions (20) and (21) take the form of conformal weights characteristic of $c = 1$ CFT. This is consistent with our starting assumption that $c = 1$. Note that it is crucial to take the fictitious ground-state energy (15) in order to obtain the right scaling dimensions.

The radius $R$ of compactified $c = 1$ free boson is taken to be $R = \xi$. Let us concentrate on the self-dual point $R = 1/\sqrt{2}$ (i.e. $\lambda = 2$) where the symmetry is known to be enhanced to the level-1 $SU(2)$ Kac-Moody (KM) algebra. In the $BC_N$-CS model we have the other continuous parameters $\mu, \nu$ which should also be adjusted to achieve the $SU(2)$ point. It turns out that $\mu + \nu = 0, 1, 2, 3$ and $4$ with $\lambda = 2$ are the desired points. Results are summarized in table I. We note that several $SU(2)$ points appeared in [22] are in agreement with our result.

We have thus shown that the low-energy critical behavior of the $BC_N$-CS model is described by $c = 1$ boundary CFT, i.e. the universality class of a chiral Tomonaga-Luttinger liquid where the Tomonaga-Luttinger liquid parameter $K$ is given by $K = 1/(2\xi^2)$.

V. DISCUSSIONS

When thinking of the application of our present results we may take into account at least two possible physical situations depending on how the boundary effect is switched on in the system.

First, suppose that we suddenly turn on the boundary effects in the ground state just like the X-ray absorption in metallic systems. In this case, to describe the long-time asymptotic
behavior of correlation functions, we need a set of quantum numbers \((\Delta N, \Delta N_b, n)\) as is considered in the Kondo problem \([13, 20]\). The boundary changing operator \(\psi_b\), for instance, is represented by \((\Delta N, \Delta N_b, n) = (0, \Delta N_b, 0)\).

The other case is to compute the critical exponents of ordinary correlation functions with boundary effects. For this we need a set of quantum numbers \((\hat{\Delta} \bar{N}, n)\) where \(\hat{\Delta} \bar{N}\) is regarded as the ordinary particle number change in (21) (forgetting about \(\Delta N_b\) in (22)). In table II, we summarize boundary critical exponents for the one-particle Green function in the above two cases and the density-density correlation function. It will be interesting to apply our analysis to the Haldane-Shastry model of \(BC_N\)-type \([22]\) and the dynamical (or spin) \(BC_N\)-CS model \([23]\).

In the picture corresponding to the set \((\hat{\Delta} \bar{N}, n)\), the \(\xi\)-dependent exponents appear only in the correlation functions whose intermediate states are related to the change of the number of particles. Thus, the density-density correlation function which is controlled by the excitations without the particle number change should exhibit the asymptotic behavior given in table II, that is, intermediate states contributing to this correlation function are all particle-hole type. This fact was confirmed in the special choice of coupling constants \([24]\) (see also \([11]\)). This is one of the characteristic features of chiral Tomonaga-Luttinger liquids \([14]\).

We finally remark that, as the \(A_{N-1}\)-CS model is related to the Gaussian random matrix theory, the \(BC_N\)-CS model with appropriate coupling constants is intimately related to the Laguerre random matrix theory \([11]\). Our results on the asymptotic behavior of correlation functions will also be useful in the context of the Laguerre random matrix theory which finds interesting applications in physics \([25]\).
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TABLES

| $\mu + \nu$ | $x_\phi$ | Representation |
|-------------|----------|----------------|
| 2           | $\frac{1}{4}(2\Delta N)^2 + n$ | spin 0 rep. of the level-1 $SU(2)$ KM algebra |
| 0, 4        | $\frac{1}{4}(2\Delta N + 1)^2 + n$ | spin 1/2 rep. of the level-1 $SU(2)$ KM algebra |
| 1, 3        | $\frac{1}{16}(4\Delta N + 1)^2 + n$ | unique rep. of the level-1 twisted $SU(2)$ KM algebra \[21\] |

TABLE I. $SU(2)$ points ($\lambda = 2$).

| Quantum numbers | Two-point function | Exponent |
|----------------|-------------------|----------|
| $(\Delta N, \Delta N_b, n) = (0, \Delta N_b, 0)$ | $\langle \psi_b(\tau) \psi_b(0) \rangle \sim \frac{1}{\tau^{2x_{\psi_b}}}$ | $x_{\psi_b} = \frac{1}{8\xi^2} \left(1 - \frac{\xi^2}{\zeta^2}\right)^2$ |
| $(\Delta N, \Delta N_b, n) = (1, \Delta N_b, 0)$ | $\langle \Psi^\dagger(\tau) \Psi(0) \rangle_{\text{sudden}} \sim \frac{1}{\tau^{2x_G}}$ | $x_G = \frac{1}{8\xi^2} \left(1 + \frac{\xi^2}{\zeta^2}\right)^2$ |
| $(\tilde{\Delta N}, n) = (1, 0)$ | $\langle \Psi^\dagger(\tau) \Psi(0) \rangle \sim \frac{1}{\tau^{2x_g}}$ | $x_g = \frac{1}{2\xi^2}$ |
| $(\tilde{\Delta N}, n) = (0, k)$ | $\langle \rho(\tau) \rho(0) \rangle \sim \frac{1}{\tau^x}$ | $x = k$ |

TABLE II. Two-point correlation functions. $\Psi$ is the particle annihilation operator, $\rho$ is the density operator and $\langle \cdot \cdot \rangle_{\text{sudden}}$ stands for the expectation value when the boundary potential is suddenly switched on.