SPECTRAL ANALYSIS OF NON-UNITARY TWO-PHASE QUANTUM WALKS IN ONE DIMENSION

CHUSEI KIUMI, KEI SAITO, AND YOHEI TANAKA

Abstract. It is recently shown by Asahara-Funakawa-Seki-Tanaka that existing index theory for chirally symmetric (discrete-time) quantum walks can be extended to the setting of non-unitary quantum walks. More precisely, they consider a certain non-unitary variant of the two-phase split-step quantum walk as a concrete one-dimensional example, and give a complete classification of the associated index in their study. Note, however, that it remains uncertain whether or not their index gives a lower bound for the number of so-called topologically protected bound states unlike the setting of unitary quantum walks. In fact, the spectrum of a non-unitary operator can be any subset of the complex plane, and so the definition of such bound states is ambiguous in the non-unitary case. The purpose of the present article is to show that the simple use of transfer matrices naturally allows us to obtain an explicit formula for a topologically bound state associated with the non-unitary split-step quantum walk model mentioned above.

1. Introduction

The major theme of the present article belongs to the broad class of spectral analysis for non-unitary quantum walks. Quantum walk theory is a quantum-mechanical counterpart of the classical random walk theory [Gud88, ADZ93, Mey96, ABN01], and many useful applications of this ubiquitous concept can be found in [Por13]. From a purely mathematical point of view, we may regard (discrete-time) quantum walks as periodically driven systems. Here, the one-step evolution after one driving period, known as a time-evolution operator, is given by a fixed unitary operator $U$ on a Hilbert space $\mathcal{H}$. We may consider various symmetry types for $U$. For example, the operator $U$ is said to have chiral symmetry, if the following equality holds true;

$$U^* = \Gamma U \Gamma,$$

where $\Gamma$ is a fixed unitary self-adjoint operator on $\mathcal{H}$. In the presence of symmetries of this kind, index theory for unitary quantum walks on the one-dimensional integer lattice $\mathbb{Z}$ has been a particularly active theme of recent mathematical studies of quantum walks [CGS16, CGG18, CGS18]. In particular, the chiral symmetry condition (1) alone has attracted tremendous attention [Suz19, ST19, Mat20, Tan21, CGWW21], and it is also the main subject of the present article.

Suzuki’s split-step quantum walk [FFS17, FFS18, FFS19, Tan21, NOW21, FNSS21] can be viewed as a standard example of a one-dimensional unitary quantum walk satisfying
The time-evolution operator of this model is given by the following $2 \times 2$ block-operator matrix on $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2) \simeq \ell^2(\mathbb{Z}; \mathbb{C}) \oplus \ell^2(\mathbb{Z}; \mathbb{C})$, where $\ell^2(\mathbb{Z}; \mathbb{C})$ is the Hilbert space of square-summable $\mathbb{C}$-valued sequences indexed by $\mathbb{Z}$:

$$U_{\text{suz}} := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}, \quad (2)$$

where $L$ is the bilateral left-shift operator on $\ell^2(\mathbb{Z}) := \ell^2(\mathbb{Z}; \mathbb{C})$ defined by $L \Psi = \Psi(\cdot + 1)$ for each $\Psi \in \ell^2(\mathbb{Z})$, and where $p = (p(x))_{x \in \mathbb{Z}}$ and $a = (a(x))_{x \in \mathbb{Z}}$ are real-valued sequences assuming values in the closed interval $[-1, 1]$. Note that any bounded sequence indexed by $\mathbb{Z}$ is identified with the corresponding multiplication operator on $\ell^2(\mathbb{Z})$ throughout the present article. The unitary operator $U_{\text{suz}}$ can then be decomposed into the product $U_{\text{suz}} = S_{\text{suz}} C_{\text{suz}}$, where the shift operator $S_{\text{suz}}$ and coin operator $C_{\text{suz}}$ are given respectively by

$$S_{\text{suz}} := \begin{pmatrix} 1 & 0 \\ 0 & L^* \end{pmatrix} \begin{pmatrix} p & \sqrt{1-p^2} \\ \sqrt{1-p^2} & -p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix}, \quad C_{\text{suz}} := \begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & -a \end{pmatrix}.$$

If we set $(\Gamma, U) := (S_{\text{suz}}, U_{\text{suz}})$ or $(\Gamma, U) := (C_{\text{suz}}, U_{\text{suz}})$, then it is easy to verify that the chiral symmetry condition (1) holds true, since $S_{\text{suz}}, C_{\text{suz}}$ are unitary self-adjoint. It is well-known that in either case we can assign to the given pair $(\Gamma, U)$ a certain well-defined Fredholm index, say ind $(\Gamma, U)$ (see §4.1 for precise definition). Moreover, the following estimate holds true:

$$|\text{ind} (\Gamma, U)| \leq \dim \ker (U - 1) + \dim \ker (U + 1), \quad (3)$$

provided that the essential spectrum of $U$, denoted by $\sigma_{\text{ess}}(U)$, contains neither $-1$ nor $+1$. As can be seen from (3), if $\text{ind} (\Gamma, U)$ is non-zero, then at least one of $\ker (U - 1), \ker (U + 1)$ contains a non-trivial eigenstate known as a topologically protected bound state. To put (3) into context, we may assume the existence of the following limit for each $\xi = p, a$ and each $\ast = -\infty, +\infty$:

$$\xi (\ast) := \lim_{x \to \ast} \xi (x). \quad (4)$$

It is shown in [Tan21, Theorem B(ii)] that under the assumption (4), we have $-1, 1 \notin \sigma_{\text{ess}}(U_{\text{suz}})$ if and only if $|p(\ast)| \neq |a(\ast)|$ for each $\ast = -\infty, +\infty$. Moreover, if we set $(\Gamma, U) := (S_{\text{suz}}, U_{\text{suz}})$, then

$$\text{ind} (\Gamma, U) = \begin{cases} 0, & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+ \text{sgn} p(+\infty), & |p(-\infty)| < |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \\
- \text{sgn} p(-\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| < |a(+\infty)|, \\
+ \text{sgn} p(+\infty) - \text{sgn} p(-\infty), & |p(-\infty)| > |a(-\infty)| \text{ and } |p(+\infty)| > |a(+\infty)|, \end{cases} \quad (5)$$

where $\text{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ denotes the sign function. Note that (5) is robust in the sense that it depends only on the asymptotic values (4). An analogous index formula for the case $(\Gamma, U) := (C_{\text{suz}}, U_{\text{suz}})$ can be found in [Tan21, Theorem B(i)].

The index formula (5) is revisited in [AFST21] with $U_{\text{suz}}$ replaced by a certain non-unitary version, say, $\tilde{U}_{\text{suz}}$ (see §2 for details). The shift operator of $\tilde{U}_{\text{suz}}$ remains unchanged, but the new coin operator is non-unitary as it contains an additional $\mathbb{R}$-valued sequence...
\(\gamma = (\gamma(x))_{x \in \mathbb{Z}}\). From a physical point of view, the new parameter \(\gamma\) represents the gain-loss effects of photons in an optical network experiment setup for the non-unitary model introduced by Mochizuki-Kim-Obuse [MKO16] (see [AFST21, Theorem B] for details). If \(\gamma\) is identically zero, then such effects are deemed to be negligible, and we recover the unitary case \(\tilde{U}_{suz} = U_{suz}\). Otherwise, \(\tilde{U}_{suz}\) is non-unitary in general. Under the assumption that limits of the form (4) exist for each \(\xi = p, a, \gamma\) and each \(* = -\infty, +\infty\), an analogous index formula for \(\tilde{U}_{suz}\) can be proved (see [AFST21, Theorem C(i)] for details).

Note, however, that it is not entirely obvious if this index formula plays any physically important role. In fact, it remains unclear whether or not (3) still holds true for non-unitary \(U\). Note that the spectrum of a non-unitary operator can be any subset of the complex plane, and so the definition of topologically protected bound states is ambiguous in general. The purpose of the present article is to show that the use of transfer matrices naturally allows us to obtain an explicit formula for such a bound state associated with \(\tilde{U}_{suz}\). The utility of transfer matrices is known to be particularly useful in characterising eigenvalues and eigenstates, but they can also be used to clarify various properties of quantum walks, such as stationary measures, dispersive estimate, or spectral types of time-evolution [CFO21, MSS+22, KK22, KKK18].

The rest of the present article is organised as follows. In §2 we give the precise definition of the non-unitary split-step quantum walk \(\tilde{U}_{suz}\), and transform the associated eigenvalue equation into a first-order difference equation by making use of transfer matrices. §3 is devoted to proving the main theorem of the present article (Theorem 3.3). The paper concludes with some discussions and remarks in §4.

On a final note, the scope of the present article is beyond that of [AFST21] which is in essence a study of topological invariants on the integer lattice \(\mathbb{Z}\), namely, mathematical quantities which depend only on asymptotic values of the form (4). On the other hand, eigenstates of \(\tilde{U}_{suz}\) depend also on local information of \(\xi = p, a, \gamma\), and this is precisely why we need to take a completely different approach in this paper.

2. Preliminaries

By an operator we shall always mean an everywhere-defined linear bounded operator on a Hilbert space. The underlying Hilbert space of the present article is \(H = \ell^2(\mathbb{Z}; \mathbb{C}^2) \cong \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\). The non-unitary version of Suzuki’ split-step quantum walk (2) we shall consider in this paper is an operator \(U\) on \(H\) of the form \(U = SC\), where the shift operator \(S\) and coin operator \(C\) are given respectively by the following formulas:

\[
S := \begin{pmatrix} p(\cdot) & q(\cdot) L \\ L^* q^*(\cdot) & -p(-1) \end{pmatrix}, \\
C := \begin{pmatrix} e^{-2\gamma x a(\cdot)} & e^{\gamma - \gamma x b^*(\cdot)} \\ e^{\gamma x - \gamma x b(\cdot)} & -e^{2\gamma x a(\cdot)} \end{pmatrix},
\]

where we assume that three real-valued sequences \(\gamma = (\gamma(x))_{x \in \mathbb{Z}}, p = (p(x))_{x \in \mathbb{Z}}, a = (a(x))_{x \in \mathbb{Z}}\), and two complex-valued sequences \(q = (q(x))_{x \in \mathbb{Z}}, b = (b(x))_{x \in \mathbb{Z}}\) satisfy the following two conditions for each \(x \in \mathbb{Z}\):

\[
\begin{align*}
p(x), a(x) & \in (-1, 1), \\
p^2(x) + |q(x)|^2 & = a^2(x) + |b(x)|^2 = 1.
\end{align*}
\]
We also assume that the coin operator $C$ has a \textit{two-phase} in the sense that for each $x \in \mathbb{Z}$ the two parameters $a(x), b(x)$ are of the following forms:

\[
    a(x) = \begin{cases} 
    a_p, & x \geq 0, \\
    a_m, & x < 0,
\end{cases} \\
    b(x) = \begin{cases} 
    b_p, & x \geq 0, \\
    b_m, & x < 0,
\end{cases}
\]

where $a_p, a_m \in (-1, 1)$. For simplicity, we assume that all parameters of the shift operator $S$ are uniform. That is, for each $\xi = \gamma, p, q$ and each $x \in \mathbb{Z}$, we set $\xi(x) = \xi$.

We consider an eigenvalue equation of the form $U \Psi = \lambda \Psi$ in what follows, where we may assume $\lambda \neq 0$ without loss of generality. Indeed, the set of eigenvalues of $U$, denoted by $\sigma_p(U)$, does not contain 0, since $U = SC$ is invertible.

Any solution $\Psi \in \ker(U - \lambda)$ admits the following representation as in [KS21]:

\[
\Psi(x) = \begin{cases} 
    (T_p)^x T_{-1} \varphi, & x \geq 0, \\
    \varphi, & x = -1, \\
    (T_m)^{x+1} \varphi, & x \leq -2,
\end{cases}
\]  \hspace{1cm} (6)

for some $\varphi \in \mathbb{C}^2$. Of course, $\varphi$ becomes the zero vector, if $\lambda \notin \sigma_p(U)$. Here, the $2 \times 2$ matrices $T_p, T_m, T_{-1}$ defined below are called \textit{transfer matrices}:

\[
T_p = \frac{1}{b_* q \lambda} \begin{pmatrix} 
    \lambda^2 - 2a_* p \cosh(2\gamma)\lambda + a_*^2 \\
    -b_* (p\lambda - a_* e^{-2\gamma}) / |b_*|^2
\end{pmatrix}, \quad \star \in \{p, m\}
\]

\[
T_{-1} = \frac{1}{b_p q \lambda} \begin{pmatrix} 
    \lambda^2 - (a_*^2 e^{2\gamma} + a_m e^{-2\gamma}) p\lambda + a_m a_p \\
    -b_p (p - a_m e^{-2\gamma}) / b_m b_p
\end{pmatrix}.
\]

Note that the existence of a non-trivial vector $\varphi \in \mathbb{C}^2$ in (6) is a necessary and sufficient condition for $\lambda \in \sigma_p(U)$. To ensure the existence of such a vector $\varphi$, we will impose some restrictions on the eigenvalue $\lambda$.

Firstly, we can easily check $\det |T_*| = 1$ for each $\star \in \{p, m\}$, and so the two eigenvalues of the transfer matrix $T_*$, denoted by $\zeta_\star^>$ and $\zeta_\star^<$, must satisfy $|\zeta_\star^>| \neq |\zeta_\star^<|$. If not, the sequence $\Psi$ in (6) fails to be square-summable unless $\varphi = 0$. By a direct calculation, the eigenvalues of each $T_*$ are given by

\[
\zeta_\star^> = \frac{(\lambda + \lambda^{-1} - 2a_* \cosh(2\gamma)) + s_\star \sqrt{(\lambda + \lambda^{-1} - 2a_* \cosh(2\gamma))^2 - 4|b_* q|^2}}{2b_* q},
\]

\[
\zeta_\star^< = \frac{(\lambda + \lambda^{-1} - 2a_* \cosh(2\gamma)) - s_\star \sqrt{(\lambda + \lambda^{-1} - 2a_* \cosh(2\gamma))^2 - 4|b_* q|^2}}{2b_* q},
\]

where $s_\star$ is a sign function such that $|\zeta_\star^>| \geq |\zeta_\star^<|$ holds. We note that $\lambda$ is always real according to the later discussion in this paper (see Lemma 3.2 for details), and so this sign function is given explicitly by $s_\star = \text{sgn} \left( \text{Re} \left( \lambda + \lambda^{-1} - 2a_* \cosh(2\gamma) \right) \right)$, where $\text{Re} \left( x \right)$ denotes the real part of a complex number $x$. It follows that the following set contains the point spectrum $\sigma_p(U)$:

\[
\Lambda := \{ \lambda \in \mathbb{C} \mid |\zeta_\star^>| \neq |\zeta_\star^<|, \ \star \in \{p, m\} \}.
\]
Lemma 2.1. For each \( \lambda \in \mathbb{C} \setminus \{0\} \) and each \(* \in \{p, m\} \), we have \( \lambda \not\in \Lambda \) if and only if 
\[ \lambda + \lambda^{-1} \in \mathbb{R} \quad \text{and} \quad (\lambda + \lambda^{-1} - 2pa_* \cosh(2\gamma))^2 - 4|b_*q|^2 \leq 0. \]

Proof. Let \( X = \lambda + \lambda^{-1} - 2pa_* \cosh(2\gamma) \). If \( |\zeta_+| = |\zeta_-| \), then these equal to 1 since ||det(T_*)|| = 1. Thus, holding this equation if and only if
\[ |X \pm \sqrt{X^2 - 4|b_*q|^2}|^2 = 4|b_*q|^2 \]
for both signs + and -. Moreover, these conditions are equivalent to holding following three conditions (i) to (iii) by taking square for both side after a deformation.

(i) \( |X^2 - 4|b_*q|^2|^2 = |X|^2 - 4|b_*q|^2|^2 \),
(ii) \( \text{Re} \left( \frac{X}{\sqrt{X^2 - 4|b_*q|^2}} \right) = 0 \),
(iii) \( |X|^2 - 4|b_*q|^2 \leq 0 \).

Here, the above condition (i) is equivalent to \( X \in \mathbb{R} \). If (i) and (iii) hold, then (ii) holds because \( X^2 = |X|^2 \). Thus, we get desired conclusion. \( \square \)

Noting that \( \varphi \in \ker(T_m - \zeta_+^m) \cap \ker((T_p - \zeta_-^p) T_{-1}) \) is a necessary condition for \( \Psi \) in (6) to be square-summable, we conclude the current section with the following lemma;

Lemma 2.2. We have \( \lambda \in \sigma_p(U) \) if and only if the following two conditions hold simultaneously:

(i) \( \lambda \in \Lambda \),
(ii) \( \dim(\ker(T_m - \zeta_+^m) \cap \ker((T_p - \zeta_-^p) T_{-1})) \neq 0 \).

3. Spectral analysis of the non-unitary split-step quantum walk

In this section, we focus on a non-trivial characterisation of the condition (ii) in Lemma 2.2 to find an explicit formula for \( \lambda \). To do so, we set the following notation for each \(* \in \{m, p\} : \)

\[ K_* = p\lambda - a_*e^{-2\gamma}, \quad K_*' = a_*e^{2\gamma} - p\lambda^{-1}, \quad J_* = \lambda - \lambda^{-1} + 2pa_* \sinh(2\gamma). \]

Here, we note that
\[ (\lambda + \lambda^{-1} - 2pa_* \cosh(2\gamma))^2 - 4|b_*q|^2 = J_*^2 - 4K_*K_*'. \]

For \( \lambda \in \mathbb{C} \setminus \{0\} \), let the two eigenvectors of \( T_* \) associated with \( \zeta_+, \zeta_- \) be \( v_+^*, v_-^* \) respectively. We may assume without loss of generality that \( J_* \neq 0 \). Otherwise, \( \zeta_+ = \zeta_- \) implies \( \lambda \not\in \Lambda \), and so \( \lambda \not\in \sigma_p(U) \) according to Lemma 2.2. In the case of \( K_* \neq 0 \), the two vectors \( v_+^* \) and \( v_-^* \) can be written as
\[ v_+^* = \left( \frac{-\lambda}{2b_*} \left( J_* + s_* \sqrt{J_*^2 - 4K_*K_*'} - 2a_*e^{2\gamma}\lambda^{-1}K_* \right) \right) \]
and
\[ v_-^* = \left( \frac{-\lambda}{2b_*} \left( J_* - s_* \sqrt{J_*^2 - 4K_*K_*'} - 2a_*e^{2\gamma}\lambda^{-1}K_* \right) \right). \]
In the case of \( K_s = 0 \), we have \( J_s \in \mathbb{R} \) and
\[
T_s = \frac{1}{b_s q \lambda} \left( \begin{array}{cc}
\lambda J_s + |b_s|^2 & 2a_s b_s \sinh(2\gamma) \\
0 & |b_s|^2 
\end{array} \right).
\]
Moreover, the eigenvalues of the transfer matrix \( T_s \) become
\[
\zeta_s^> = \frac{(1 + s_s \text{sgn}(J_s)) \lambda J_s + 2 |b_s|^2}{2b_s q \lambda}, \quad \zeta_s^< = \frac{(1 - s_s \text{sgn}(J_s)) \lambda J_s + 2 |b_s|^2}{2b_s q \lambda}.
\]
Note that the \((2, 2)\)-element of the matrix \( T_s - \zeta_s^> \) (resp. \( T_s - \zeta_s^< \)) equals to zero if and only if \( s_s \text{sgn}(J_s) < 0 \) (resp. \( s_s \text{sgn}(J_s) > 0 \)). Thus, \( v_s^> \) and \( v_s^< \) can be written as below:
\[
v_s^> = \begin{cases} 
\left( \begin{array}{c}
1 \\
0
\end{array} \right), & s_s \text{sgn}(J_s) > 0, \\
\left( 2a_s b_s \sinh(2\gamma), -\lambda J_s \right), & s_s \text{sgn}(J_s) < 0,
\end{cases}
\]
\[
v_s^< = \begin{cases} 
\left( \begin{array}{c}
2a_s b_s \sinh(2\gamma) \\
-\lambda J_s
\end{array} \right), & s_s \text{sgn}(J_s) > 0, \\
\left( 1, 0 \right), & s_s \text{sgn}(J_s) < 0.
\end{cases}
\]

\[ (10) \]

**Lemma 3.1.** For \( \lambda \in \mathbb{C} \setminus \{0]\), the condition (ii) in Lemma 2.2, i.e.,
\[
\dim \left( \ker \left( T_m - \zeta_s^> \right) \cap \ker \left( (T_p - \zeta_p^<) T_{-1} \right) \right) \neq 0,
\]
is satisfied if and only if the following three conditions hold simultaneously:

(i) \( (K_m J_p - K_p J_m) (K'_m J_p - K'_p J_m) + (K'_m K'_p - K'_p K'_m)^2 = 0 \),
(ii) \( \text{sgn} \left( s_p K_m \sqrt{J_p^2 - 4 K_p K'_p} + s_m K_p \sqrt{J_m^2 - 4 K_m K'_m} \right) = \text{sgn} (K_m J_p - K_p J_m) \),
(iii) \( \text{sgn} \left( 2 K_m K'_p + 2 K_p K'_m - J_p J_m \right) = \text{sgn} \left( s_p s_m \sqrt{J_p^2 - 4 K_p K'_p} \sqrt{J_m^2 - 4 K_m K'_m} \right) \).

**Proof.** The statement will be proved by checking the condition of linear dependency between \( T_{-1} v_m^> \) and \( v_p^< \) in (8), (9) and (10). In the case \( K_m \neq 0 \) and \( K_p \neq 0 \),
\[
T_{-1} v_m^> = \zeta_m^> \left( \begin{array}{c}
-\frac{\lambda}{2 b_p} \\
\frac{s_p K_m \sqrt{J_p^2 - 4 K_p K'_p} + s_m K_p \sqrt{J_m^2 - 4 K_m K'_m}}{K_m}
\end{array} \right)
\]
holds. Therefore, \( T_{-1} v_m^> \) and \( v_p^< \) are linear dependent if and only if
\[
s_p K_m \sqrt{J_p^2 - 4 K_p K'_p} + s_m K_p \sqrt{J_m^2 - 4 K_m K'_m} = K_m J_p - K_p J_m.
\]
By squaring both sides, we can check that this equation is equivalent to
\[
2 K_m K'_p + 2 K_p K'_m - J_p J_m = s_p s_m \sqrt{J_p^2 - 4 K_p K'_p} \sqrt{J_m^2 - 4 K_m K'_m}. \quad (11)
\]
and (ii). Furthermore, by squaring both sides and removing the square root again, (11) is converted to the equations (i) and (iii).
Next, we consider the case \( K_m = 0 \) and \( K_p \neq 0 \). According to (10), we can see that

\[
T_{-1}v_m^> = \begin{cases}
\begin{pmatrix}
\frac{q\lambda}{b_q} & 1 \\
\frac{b_m}{b_q} & 0
\end{pmatrix}, & s_m \text{sgn}(J_m) > 0,
\end{cases}
\]

\[
\begin{cases}
\begin{pmatrix}
\frac{b_m}{b_q} & \lambda K_m' - a_p e^{2\gamma J_m} \\
\frac{b_m}{b_q} & -b_p J_m
\end{pmatrix}, & s_m \text{sgn}(J_m) < 0.
\end{cases}
\]

If \( s_m \text{sgn}(J_m) > 0 \), it can be easily checked that \( T_{-1}v_m^> \) and \( v_p^<= \) are always not linear dependent since \( K_p \neq 0 \). Thus, in \( K_m = 0 \) and \( K_p \neq 0 \) case, \( T_{-1}v_m^> \) and \( v_p^<= \) are linear dependent if and only if

\[
\left( J_p - s_p \sqrt{J_p^2 - 4K_p K_p'} \right) J_m = 2K_p K_m'
\]

and

\[
s_m \text{sgn}(J_m) < 0,
\]

which are equivalent to (11) and (ii), respectively. As mentioned above, (11) is equivalent to (i) and (iii), so the proof of this case is completed. Considering similarly, we can also prove the statement with respect to the case \( K_m \neq 0, K_p = 0 \).

Finally, we consider the case \( K_m = K_p = 0 \). Since \( a_m = a_p \) holds, we can write \( s = s_m = s_p \) and \( J = J_m = J_p \). From (10), \( T_{-1}v_m^> \) and \( v_p^<= \) become

\[
(T_{-1}v_m^>, v_p^<=) = \begin{cases}
\begin{pmatrix}
\frac{q\lambda}{b_q} & 1 \\
\frac{b_m}{b_q} & 0
\end{pmatrix}, & (2ab_p \sinh(2\gamma)) (-\lambda J), & s \text{sgn}(J) > 0,
\end{cases}
\]

\[
\begin{cases}
\begin{pmatrix}
\frac{b_m}{b_q} & \lambda K_m' - a e^{2\gamma J} \\
\frac{b_m}{b_q} & -b_p J
\end{pmatrix}, & (1) 0), & s \text{sgn}(J) < 0.
\end{cases}
\]

Since \( \lambda, J \neq 0 \), these vectors are not linear dependent. Moreover, the equation (iii) does not hold in this case because this equation becomes \(-1 = s_p s_m\), but \( s_p = s_m \) gives \( s_p s_m = 1 \). From the above discussions, the proof is complete. \( \square \)

By analysing the equations (i), (ii) and (iii) in Lemma 3.1, we can explicitly calculate the eigenvalues of \( U \).

\textbf{Lemma 3.2.} Let \( p \cosh(2\gamma)/|q| \neq s_2a_*/|b_*| \) for each \( \ast \in \{m, p\} \), and let

\[
\lambda(s_1, s_2) := s_1 p \sinh(2\gamma) + s_2 \sqrt{1 + p^2 \sinh^2(2\gamma)}, \quad (s_1, s_2) \in \{-1, +1\}^2.
\]

Then we have the following inclusions:

\[
\sigma_p(U) \subset \{ \lambda(s_1, s_2) | (s_1, s_2) \in \{-1, +1\}^2 \} \subset \Lambda.
\]

Moreover, \( \lambda = \lambda(s_1, s_2) \) is equivalent to the condition (i) in Lemma 3.1.

\textbf{Proof.} Firstly, we will show \( \{ \lambda(s_1, s_2) | (s_1, s_2) \in \{-1, +1\}^2 \} \subset \Lambda \). By the Lemma 2.1, for \( \lambda = \lambda(s_1, s_2) \), it is sufficient to prove \( (\lambda + \lambda^{-1} - 2pa_\ast \cosh(2\gamma))^2 - 4|b_\ast q|^2 > 0 \). Here, (7)
and direct calculation give
\[
(\lambda + \lambda^{-1} - 2pa_\ast \cosh(2\gamma))^2 - 4|b_\ast q|^2 = J_\ast^2 - 4K_\ast K_\ast'
\]
\[
= 4 \left( p \cosh(2\gamma) - s_2a_\ast \sqrt{|q|^2 + p^2 \cosh^2(2\gamma)} \right)^2.
\]
Thus, the above inequality is always true except \( p \cosh(2\gamma)/|q| \neq s_2a_\ast/|b_\ast| \) case, so \{\lambda(s_1, s_2) | (s_1, s_2) \in \{-1, +1\}^2\} \subset \Lambda is proved.

Next, we will show \( \sigma_p(U) \subset \{\lambda(s_1, s_2) | (s_1, s_2) \in \{-1, +1\}^2\} \). Lemma 2.2 and Lemma 3.1 say the condition (i) in Lemma 3.1 is a necessary condition for \( \lambda \in \sigma_p(U) \). By direct calculation, we have
\[
K_mK'_p - K'_mK_p = p(a_m - a_p) \left( e^{-2\gamma} \lambda^{-1} - e^{2\gamma} \lambda \right),
\]
\[
K_mJ_p - K_pJ_m = (a_m - a_p) \left( e^{-2\gamma} \lambda^{-1} - (2p^2 \sinh(2\gamma) + e^{2\gamma}) \lambda \right),
\]
\[
K'_mJ_p - K'_pJ_m = (a_m - a_p) \left( e^{2\gamma} \lambda + (2p^2 \sinh(2\gamma) - e^{2\gamma}) \lambda^{-1} \right).
\]
If \( a_p = a_m \) holds, then \( K_p = K_m \) and \( J_p = J_m \) give (iii) in Lemma 3.1 becomes \( s_ms_p = -1 \), and it does not hold because of \( s_m = s_p \). Hence, \( \lambda \notin \sigma_p(U) \) for \( a_p = a_m \) case. Thus, we assume \( a_p \neq a_m \), then (i) is calculated as follows:
\[
(\lambda - \lambda^{-1})^2 - 4p^2 \sinh^2(2\gamma) = 0.
\]
We obtain \( \lambda(s_1, s_2) \) for any \((s_1, s_2) \in \{-1, +1\}^2\) is a solution to the equation. \( \square \)

We are now in a position to state the main theorem of the present article;

**Theorem 3.3.** Let \( \mathbb{R}_+, \mathbb{R}_- \) denote the set of positive and negative real numbers respectively, and let \( \lambda_\pm : \mathbb{R} \to \mathbb{R}_\pm \) be two increasing bijections defined by \( \lambda_\pm(x) = x \pm \sqrt{1 + x^2} \) for each \( x \in \mathbb{R} \). Let \( \sigma_\pm(U) \) be two subsets of \( \mathbb{R}_\pm \) defined by following formula;

\[
\sigma_\pm(U) = \begin{cases} 
\{\lambda_\pm(+)p \sinh(2\gamma))\}, & a_m < \pm p'_\gamma < a_p, \\
\{\lambda_\pm(-)p \sinh(2\gamma))\}, & a_p < \pm p'_\gamma < a_m, \\
\emptyset, & \text{otherwise},
\end{cases}
\]

where the constant \( p'_\gamma \in (-1, 1) \) is defined by
\[
p'_\gamma := \frac{p}{\sqrt{p^2 + |q|^2 \cosh^2(2\gamma)}}.
\]

Then \( \sigma_p(U) = \sigma_-(U) \cup \sigma_+(U) \).

**Proof.** From Lemma 2.2 and Lemma 3.2, it is sufficient to clarify the conditions to satisfy (ii) and (iii) in Lemma 3.1 with \( \lambda = \lambda(s_1, s_2) \). We set a new sign function as follows:
\[
s'_\ast = \text{sgn} \left( p \cosh(2\gamma) - s_2a_\ast \sqrt{1 + p^2 \sinh^2(2\gamma)} \right).
\]
Here, we do not have to consider \( s_m' s_p' \neq 0 \) case because \( s_*' = 0 \) gives \( \zeta_+^* = \zeta_-^* \) and \( \lambda(s_1, s_2) \not\in \Lambda \). Since \( \lambda(s_1, s_2) \in \mathbb{R} \), we can explicitly determine \( s_* \) as follows:

\[
 s_* = \text{sgn} \left( \lambda + \lambda^{-1} - 2p a_* \cosh(2\gamma) \right) \\
= \text{sgn} \left( s_2 \sqrt{p^2 \cosh^2(2\gamma) + |q|^2} - p a_* \cosh(2\gamma) \right) \\
= s_2.
\]

Therefore, we get

\[
2K_m K'_p + 2K_p K'_m - J_p J_m \\
= -4 \left( p \cosh(2\gamma) - s_2 a_p \sqrt{1 + p^2 \sinh^2(2\gamma)} \right) \left( p \cosh(2\gamma) - s_2 a_m \sqrt{1 + p^2 \sinh^2(2\gamma)} \right) \\
= -s_m' s_p' \sqrt{J_p^2 - 4K_p K'_p} \sqrt{J_m^2 - 4K_m K'_m}.
\]

It gives (iii) becomes \( s_m' s_p' = -1 \). Here, \( s_*' = +1 \) means

\[
p \cosh(2\gamma) > s_2 a_* \sqrt{p^2 \cosh^2(2\gamma) + |q|^2}.
\]

If \( s_2 a_* \geq 0 \), we see that \( s_*' = +1 \) holds if and only if \( p > 0 \) and the following holds:

\[
(|b_*| p \cosh(2\gamma) - a_* |q|)(|b_*| p \cosh(2\gamma) + a_* |q|) > 0.
\]

(13)

On the other hand, if \( s_2 a_* < 0 \), then we see that \( s_*' = +1 \) holds if and only \( p \geq 0 \) holds or the following holds with \( p < 0 \):

\[
(|b_*| p \cosh(2\gamma) - a_* |q|)(|b_*| p \cosh(2\gamma) + a_* |q|) < 0.
\]

(14)

Thus, (13) and (14) are equivalent to \( |b_*| p \cosh(2\gamma) > |a_* |q| \) and \( |b_*| p \cosh(2\gamma) > -|a_* |q| \), respectively. By the same way, we know similar condition for \( s_*' = -1 \) version, and these are summarized as follows.

\[
s_*' = +1 \iff \frac{p \cosh(2\gamma)}{|q|} > \frac{s_2 a_*}{|b_*|}, \quad s_*' = -1 \iff \frac{p \cosh(2\gamma)}{|q|} < \frac{s_2 a_*}{|b_*|}.
\]

where \( \iff \) denotes “if and only if”. Additionally, we define monotonically increasing and bijective function \( f \) as

\[
f(x) = \frac{x}{\sqrt{1 + x^2}}, \quad x \in \mathbb{R}.
\]

Since \( f \left( \frac{s_2 a_*}{|b_*|} \right) = s_2 a_* \) and \( f \left( \frac{a_*}{|q|} \cosh(2\gamma) \right) = p'_x \), applying this function to above inequalities gives the condition of (iii) in Lemma 3.1, i.e., \( s_m' s_p' = -1 \), is equivalent to the following:

- \( s_p' = +1 \) case: \( s_2 a_p < p'_x < s_2 a_m \),
- \( s_p' = -1 \) case: \( s_2 a_m < p'_x < s_2 a_p \).

Note that these inequalities guarantees \( \lambda(s_1, s_2) \in \Lambda \) is certain to hold by Lemma 3.2.
Next, we focus the condition of (ii). We have
\[ K_m J_p - K_p J_m = 2(a_p - a_m)p \sinh(2\gamma) \left( s_1 \left( e^{-2\gamma} + p^2 \sinh(2\gamma) \right) + s_2 p \sqrt{1 + p^2 \sinh^2(2\gamma)} \right) \]
and
\[ s_p K_m \sqrt{J_p^2 - 4K_p K'_p} + s_m K_p \sqrt{J_m^2 - 4K_m K'_m} \]
\[ = -2s_1 s_2 s'_p (a_p - a_m)p \sinh(2\gamma) \left( s_1 \left( e^{-2\gamma} + p^2 \sinh(2\gamma) \right) + s_2 p \sqrt{1 + p^2 \sinh^2(2\gamma)} \right). \]

Thus, (ii) holds only if \( p \sinh(2\gamma) = 0 \) or \( s'_p = -s_1 s_2 \). At first, we consider \( s'_p = -s_1 s_2 \) case. Then, above mentioned condition of (iii) derives the following list:
\[ \lambda(+1, +1) \in \sigma_p(U) \iff a_m < p'_\gamma < a_p, \]
\[ \lambda(+1, -1) \in \sigma_p(U) \iff -a_p < p'_\gamma < -a_m, \]
\[ \lambda(-1, +1) \in \sigma_p(U) \iff a_p < p'_\gamma < a_m, \]
\[ \lambda(-1, -1) \in \sigma_p(U) \iff -a_m < p'_\gamma < -a_p. \]

This list derives desired conclusion of the theorem. The rest of the proof is \( p \sinh(2\gamma) = 0 \) case. In this case, we see \( \lambda(s_1, s_2) = s_2 \), so the independence of \( s_1 \) gives the same conclusion of the other case. Thus, the proof is completed. \( \square \)

4. Discussion

4.1. Symmetry protection of eigenstates. What follows is a brief summary of the existing index theory for chirally symmetric unitary operators (see, for example, \[ MST21, \§2 \]). If \( U \) is an abstract unitary operator on a Hilbert space \( \mathcal{H} \) satisfying (1), then it admits the following block-operator matrix representation with respect to \( \mathcal{H} = \ker(\Gamma - 1) \oplus \ker(\Gamma + 1) \):
\[
U = \begin{pmatrix} R_1 & iQ_2 \\ iQ_1 & R_2 \end{pmatrix}_{\ker(\Gamma - 1) \oplus \ker(\Gamma + 1)},
\]
where \( R_j \) is self-adjoint for each \( j = 1, 2 \), and where \( Q_1^* = Q_2 \). With (15) in mind, we introduce the following three formal indices:
\[ \text{ind}_{\pm}(\Gamma, U) := \dim \ker(R_1 \mp 1) - \dim \ker(R_2 \mp 1), \]
\[ \text{ind}(\Gamma, U) := \dim \ker(Q_1) - \dim \ker(Q_2), \]
where the index defined by (17) is the one we have previously discussed in \( \§1 \). A key step in making these formal indices precise lies in the following equality:
\[ \ker(U \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1). \]
Indeed, it follows from (18) that if the eigenspace \( \ker(U \mp 1) \) is finite-dimensional, then \( \text{ind}_{\pm}(\Gamma, U) \) defined by (16) is a well-defined integer, and we obtain the following estimate:
\[ |\text{ind}_{\pm}(\Gamma, U)| \leq \dim \ker(U \mp 1). \]

\[ \text{ind}_{\pm}(\Gamma, U) := \dim \ker(R_1 \mp 1) - \dim \ker(R_2 \mp 1), \]
\[ \text{ind}(\Gamma, U) := \dim \ker(Q_1) - \dim \ker(Q_2), \]
where the index defined by (17) is the one we have previously discussed in \( \§1 \). A key step in making these formal indices precise lies in the following equality:
\[ \ker(U \mp 1) = \ker(R_1 \mp 1) \oplus \ker(R_2 \mp 1). \]
Indeed, it follows from (18) that if the eigenspace \( \ker(U \mp 1) \) is finite-dimensional, then \( \text{ind}_{\pm}(\Gamma, U) \) defined by (16) is a well-defined integer, and we obtain the following estimate:
\[ |\text{ind}_{\pm}(\Gamma, U)| \leq \dim \ker(U \mp 1). \]
In particular, the condition $\pm 1 \not\in \sigma_{\text{ess}}(U)$ ensures that $\text{ind}_\pm(\Gamma, U)$ is well-defined. Note also that (19) is a refinement of (3), since $\text{ind}(\Gamma, U)$ defined by (17) is given by the following equality (see [MST21, Lemma 2.1 (ii)] for details):

$$\text{ind}(\Gamma, U) = \text{ind}_-(\Gamma, U) + \text{ind}_+(\Gamma, U),$$

provided that $\ker(U - 1)$ and $\ker(U + 1)$ are both finite-dimensional.

The estimate (19) can be understood as an abstract form of the symmetry protection of eigenstates [CGS+16, CGG+18, CGS+18]. The present article is motivated by the desire to obtain a variant of (19) for non-unitary $U$. The hindrance of this non-unitary setting is two-fold. Firstly, the two indices $\text{ind}_\pm(\Gamma, U)$ on the left hand side of (19) become ill-defined, if $U$ is non-unitary. That is, we need to extend the formula (16) to the non-unitary setting in a mathematically rigorous fashion, but this is an open problem to the best of authors’ knowledge. Secondly, the eigenspace $\ker(U \mp 1)$ on the right hand side of (19) also requires a certain non-trivial modification. The main theorem of the present article, Theorem 3.3, gives us some insight as to how this generalisation can be done under the setting of the two-phase non-unitary split-step quantum walk on the one-dimensional integer lattice.

4.2. The spectral mapping theorem for chirally symmetric non-unitary operators. Let us briefly recall the spectral mapping theorem for chirally symmetric unitary operators [HKSS14, SS16, SS19]. Let $\mathcal{H}$ be an underlying separable Hilbert space, and let $U$ be an abstract unitary operator on $\mathcal{H}$ satisfying the chiral symmetry condition (1). We can then canonically decompose the unitary self-adjoint operator $\Gamma' := \Gamma U$ as the difference $\Gamma' = \partial^* \partial - (1 - \partial^* \partial)$ for some operator $\partial$ from $\mathcal{H} = \ker(\Gamma' - 1) \oplus \ker(\Gamma' + 1)$ into an auxiliary Hilbert space $\mathcal{K}$, satisfying $\partial \partial^* = 1$ (see, for example, [Suz19, Lemma 3.3]). Here, $\partial^* \partial$ (resp. $1 - \partial^* \partial$) is the orthogonal projection onto $\ker(\Gamma' - 1)$ (resp. $\ker(\Gamma' + 1)$). The spectral mapping theorem states that for each eigenvalue $z$ of $U$, we have

$$\dim \ker(U - z) = \begin{cases} \dim \ker(\partial \Gamma \partial^* \mp 1) + \dim (\ker(\Gamma \pm 1) \cap \ker \partial), & z = \pm 1, \\ \dim \ker(\partial \Gamma \partial^* - \frac{z + z^*}{2}), & \text{otherwise}, \end{cases} \quad (20)$$

where the spectrum of the self-adjoint operator $T := \partial \Gamma \partial^*$ is a subset of $[-1, 1]$, since $\|T\| \leq 1$ immediately follows from $\|\Gamma\| = 1$ and from $\|\partial\|^2 = \|\partial^*\|^2 = \|\partial^* \partial\| = 1$. If $z \neq \pm 1$, then the formula (20) has a visual interpretation (see Figure 1).

It is not known whether or not (20) can be naturally extended to the setting of non-unitary $U$, since the spectrum of $U$ can be any subset of the complex plane $\mathbb{C}$. This is a gap in the existing mathematics literature. With the notation introduced in Theorem 3.3, we can view $\gamma$ as a continuous variable. If $\gamma = 0$, then $U$ is unitary, and we can choose and fix $p, a_m, a_p \in (-1, 1)$ to ensure that either $-1$ or $+1$ becomes an eigenvalue of the unitary operator $U$. As we continuously alter the value of $\gamma$ from $\gamma = 0$, the operator $U$ becomes non-unitary, but we can still keep track of the continuous movement of this eigenvalue on the real axis according to the explicit formula (12).
Figure 1. Given the chirally symmetric unitary operator $U = \Gamma \Gamma'$, where $\Gamma' = \partial^* \partial - (1 - \partial^* \partial)$, we have that $z \in \mathbb{T}$ is an eigenvalue of $U$ if and only if so is $z^*$. In this case, their real part $(z + z^*)/2$ turns out to be an eigenvalue of the self-adjoint operator $T = \partial \Gamma \partial^*$, provided that $z \neq \pm 1$.

REFERENCES

[ABN+01] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous. One-dimensional quantum walks. In *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing*, pages 37–49, 2001.

[ADZ93] Y. Aharonov, L. Davidovich, and N. Zagury. Quantum random walks. *Phys. Rev. A*, 48:1687–1690, 1993.

[AFST21] K. Asahara, D. Funakawa, M. Seki, and Y. Tanaka. An index theorem for one-dimensional gapless non-unitary quantum walks. *Quantum Inf. Process.*, 20(9):287, 2021.

[CFO21] C. Cedzich, J. Fillman, and D. C. Ong. Almost everything about the unitary almost Mathieu operator. *arXiv:2112.03216*, 2021.

[CGG+18] C. Cedzich, T. Geib, F. A. Grünbaum, C. Stahl, L. Velázquez, A. H. Werner, and R. F. Werner. The Topological Classification of One-Dimensional Symmetric Quantum Walks. *Ann. Henri Poincaré*, 19(2):325–383, 2018.

[CGS+16] C. Cedzich, F. A. Grünbaum, C. Stahl, L. Velázquez, A. H. Werner, and R. F. Werner. Bulk-edge correspondence of one-dimensional quantum walks. *J. Phys. A*, 49(21):21LT01, 2016.

[CGS+18] C. Cedzich, T. Geib, C. Stahl, L. Velázquez, A. H. Werner, and R. F. Werner. Complete homotopy invariants for translation invariant symmetric quantum walks on a chain. *Quantum*, 2:95, 2018.

[CGWW21] C. Cedzich, T. Geib, A. H. Werner, and R. F. Werner. Chiral floquet systems and quantum walks at half-period. *Ann. Henri Poincaré*, 22(2):375–413, 2021.

[FFS17] T. Fuda, D. Funakawa, and A. Suzuki. Localization of a multi-dimensional quantum walk with one defect. *Quantum Inf. Process.*, 16(8):203, 2017.

[FFS18] T. Fuda, D. Funakawa, and A. Suzuki. Localization for a one-dimensional split-step quantum walk with bound states robust against perturbations. *J. Math. Phys.*, 59(8):082201, 2018.

[FFS19] T. Fuda, D. Funakawa, and A. Suzuki. Weak limit theorem for a one-dimensional split-step quantum walk. *Rev. Roum. Math. Pures Appl.*, 64(2-3):157–165, 2019.

[FNSS21] T. Fuda, A. Narimatsu, K. Saito, and A. Suzuki. Spectral analysis for a multi-dimensional split-step quantum walk with a defect. *Quantum Stud.: Math. Found.*, 9(1):93–112, 2021.

[Gud88] S. Gudder. *Quantum Probability*. Academic Press, New York, 1988.

[HKSS14] Y. Higuchi, N. Konno, I. Sato, and E. Segawa. Spectral and asymptotic properties of Grover walks on crystal lattices. *J. Funct. Anal.*, 267(11):4197–4235, 2014.

[KK22] T. Komatsu and N. Konno. Stationary measure induced by the eigenvalue problem of the one-dimensional Hadamard walk. *J. Stat. Phys.*, 187(1):24, 2022.
SPECTRAL ANALYSIS OF NON-UNITARY TWO-PHASE QUANTUM WALKS IN ONE DIMENSION

[HKK18] H. Kawai, T. Komatsu, and N. Konno. Stationary measure for two-state space-inhomogeneous quantum walk in one dimension. *Yokohama Math. J.*, 64:111–130, 2018.

[KS21] C. Kiumi and K. Saito. Eigenvalues of two-phase quantum walks with one defect in one dimension. *Quantum Inf. Process.*, 20(5):171, 2021.

[Mat20] Y. Matsuzawa. An index theorem for split-step quantum walks. *Quantum Inf. Process.*, 19(8):227, 2020.

[Mey96] D. A. Meyer. From quantum cellular automata to quantum lattice gases. *J. Statist. Phys.*, 85(5-6):551–574, 1996.

[MKO16] K. Mochizuki, D. Kim, and H. Obuse. Explicit definition of PT symmetry for nonunitary quantum walks with gain and loss. *Phys. Rev. A*, 93(6):062116, 2016.

[MSS+22] M. Maeda, H. Sasaki, E. Segawa, A. Suzuki, and K. Suzuki. Dispersive estimates for quantum walks on 1d lattice. *J. Math. Soc. Japan*, 74(1):217–246, 2022.

[MST21] Y. Matsuzawa, M. Seki, and Y. Tanaka. The bulk-edge correspondence for the split-step quantum walk on the one-dimensional integer lattice. *arXiv:1410.5093*, 2021.

[NOW21] A. Narimatsu, H. Ohno, and K. Wada. Unitary equivalence classes of split-step quantum walks. *Quantum Inf. Process.*, 20(11):368, 2021.

[Por13] R. Portugal. *Quantum Walks and Search Algorithms*. Springer, New York, 2013.

[SS16] E. Segawa and A. Suzuki. Generator of an abstract quantum walk. *Quantum Stud.: Math. Found.*, 3(1):11–30, 2016.

[SS19] E. Segawa and A. Suzuki. Spectral mapping theorem of an abstract quantum walk. *Quantum Inf. Process.*, 18(11):333, 2019.

[ST19] A. Suzuki and Y. Tanaka. The Witten index for 1d supersymmetric quantum walks with anisotropic coins. *Quantum Inf. Process.*, 18(12):377, 2019.

[Suz19] A. Suzuki. Supersymmetry for chiral symmetric quantum walks. *Quantum Inf. Process.*, 18(12):363, 2019.

[Tan21] Y. Tanaka. A constructive approach to topological invariants for one-dimensional strictly local operators. *J. Math. Anal. Appl.*, 500(1):125072, 2021.

Graduate School of Engineering Science, Yokohama National University, Hodogaya, Yokohama, 240-8501, Japan

Email address: kiumi-chusei-bf@ynu.jp

Department of Information Systems Creation, Faculty of Engineering, Kanagawa University, Kanagawa, Yokohama, 221-8686, Japan

Email address: ft102130ev@jindai.jp

Department of Mathematics, Shinshu University, Matsumoto 390-8621, Japan

Email address: tana35@shinshu-u.ac.jp