Dimension-Free Matrix Spaces

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Abstract—Based on various types of semi-tensor products of matrices, the corresponding equivalences of matrices are proposed. Then the corresponding vector space structures are obtained as the quotient spaces under equivalences, which are called the dimension-free Matrix spaces (DFESs). Certain structures and properties are investigated. Finally, the Lie bracket structure of general linear algebra is extended to DFMSs to make them Lie algebras, called dimension-free general linear algebra (DFGLA). Inspire of the fact that the DFGLAs are of infinite dimension, they have most properties of finite dimensional Lie algebras, which are studied in the paper.

Index Terms—Quotient space, dimension-free matrix space (DFES), dimension-free general linear algebra (DFGLA), semi-tensor product (STP) of matrices.

I. INTRODUCTION

Lie group was firstly proposed by Sophus Lie (1840-1899) to describe the continuous symmetries of algebraic or geometric objects, and the corresponding Lie algebra is considered as the infinitesimal generator of the Lie group. Since the objects might be of finite or infinite dimensions, the related Lie groups and Lie algebras can also be of either finite or infinite dimensions. As pointed in [13]: “Starting with the works of Lie, Killing, and Cartan, the theory of finite-dimensional Lie groups and Lie algebras has developed systematically in depth and scope. On the other hand, works on simple infinite dimensional Lie algebras have been virtually forgotten until the mod-sixties” (of last century). By the author’s observation, the reason for this may lies on the big differences between Lie groups and Lie algebras of finite vs infinite dimensions.

Finite dimensional Lie algebra has many nice properties, such as Killing form, complete reducibility of representations, root systems, etc. [13], [19], which can not be easily extended to infinite dimensional case.

The STP of matrices was proposed firstly by the first author as follows: Let \( A \in \mathcal{M}_{m \times n} \) and \( B \in \mathcal{M}_{p \times q} \), \( t = n \lor p \) be the least common multiple of \( n \) and \( p \). The STP of \( A \) and \( B \) is defined as follows:

\[
A \times B := (A \otimes I_{t/n}) (B \otimes I_{t/p}),
\]

where \( \otimes \) is the Kronecker product.

STP generalizes the conventional matrix product to general case where two factor matrices are of arbitrary dimensions [4], [5]. Fortunately, STP keeps major properties of the conventional matrix product available. The last two decades have witnessed a fast progress in the theory and applications of STP of matrices. We refer to some survey papers for its some major applications: (i) application to logical dynamic (control) networks [11], [17], [18]; (ii) application to game theory [9]; (iii) application to finite automata [20]; (iv) application to engineering problems [16].

Based on STP, certain equivalences of matrices are obtained. Then the quotient spaces are produced as infinite dimensional vector space of matrices, called DFMSs. The addition and scalar product are posed on quotient space to make it a vector space. The lattice and the topology are also obtained. The structure of general linear algebras (GLAs) (\( gl(n, \mathbb{R}) \) or \( gl(n, \mathbb{C}) \)) can be extended to DFMSs, which turns the DFMS into an infinite dimensional Lie algebra, called the DFGLA. Then we prove that many properties of GLA can be transferred to DFGLA. Finally, the Killing structure is also built for DFMSs.

The outline of this paper is as follows: In section 2 several general STPs of matrices are presented by using matrix multiplier. Section 3 poses addition and scalar product of matrices to make certain sets of matrices vector spaces. In Section 4 a kind of equivalence relation is proposed for matrices, which lead to the lattice structure on the set of matrices. The quotient spaces, called the DFMSs, are obtained by the equivalences, which are discussed in Section 5. Section 6 poses the Lie algebraic structure onto the DFMSs to build an infinite dimensional Lie algebra on DFMS. The Killing form is introduced for the DFMSs in Section 7. Section 8 is a brief conclusion.

II. GENERAL STP OF MATRICES

The STP of matrices defined by [11] is the classical one. There are some other useful STPs [6]. A general way to
construct STPs is via matrix multiplier, which is defined as follow:

**Definition 2.1:** A set of nonzero matrices

\[ \Gamma := \{ \Gamma_n \in M_{n \times n} \mid n \geq 1 \} \]

is called a matrix multiplier, if it satisfies:

(i) \[ \Gamma_n \Gamma_n = \Gamma_n; \] (2)

(ii) \[ \Gamma_p \otimes \Gamma_q = \Gamma_{pq}. \] (3)

It follows from the definition immediately that the matrix multiplier has some elementary properties as follows.

**Proposition 2.2:** Consider a matrix multiplier \( \Gamma \).

(i) The spectrum (set of eigenvalues) of \( \Gamma_n \) is either \( \sigma(\Gamma_n) = \{1\} \) or \( \sigma(\Gamma_n) = \{0,1\} \). Hence

\[ \sigma_{\max}(\Gamma_n) = 1, \quad n = 1, 2, \ldots \] (4)

(ii) \[ \Gamma_1 = 1. \] (5)

Using a matrix multiplier, the corresponding STP of matrices can be defined as following.

**Definition 2.3:** Given a matrix multiplier \( \Gamma \), and let \( A \in M_{m \times n} \) and \( B \in M_{p \times q} \), \( t = n \lor p \) be the least common multiple of \( n \) and \( p \).

(i) The left \( \Gamma \)-STP of \( A \) and \( B \) is defined by

\[ A \prec \Gamma B := (A \otimes \Gamma_{t/n}) (B \otimes \Gamma_{t/p}). \] (6)

(ii) The right \( \Gamma \)-STP of \( A \) and \( B \) is defined by

\[ A \succ \Gamma B := (\Gamma_{t/n} \otimes A) (\Gamma_{t/p} \otimes B). \] (7)

Using the properties of \( \Gamma \), it is easy to verify the following proposition.

**Proposition 2.4:** Both left and right \( \Gamma \)-STP, defined by (6) and (7) respectively, satisfy the following condition.

(i) If \( n = p \), then

\[ A \prec \Gamma B = A \succ \Gamma B = AB. \]

That is, they are the generalizations of the convenient matrix product.

(ii) (Associativity)

\[ (A \preceq \Gamma B) \preceq \Gamma C = A \preceq \Gamma (B \preceq \Gamma C), \] (8)

where \( \preceq \Gamma \) can be either \( \prec \Gamma \) or \( \succ \Gamma \).

(iii) (Distributivity)

\[ (A + B) \preceq \Gamma C := A \preceq \Gamma C + B \preceq \Gamma C \]

\[ A \preceq \Gamma (B + C) = A \preceq \Gamma B + A \preceq \Gamma C. \] (9)

(iv) (Transpose)

\[ (A \preceq \Gamma B)^T = B^T \preceq \Gamma A^T. \] (10)

(v) (Inverse)

Assume \( \Gamma_n, \quad n \geq 1 \) are invertible. If \( A \) and \( B \) are invertible, then \( A \preceq \Gamma B \) is invertible. Moreover,

\[ (A \preceq \Gamma B)^{-1} = B^{-1} \preceq \Gamma A^{-1}. \] (11)

The following example shows some matrix multipliers.

**Example 2.5:**

(i) (MM-1 STP): Set

\[ \Gamma^1 := \{ I_n \mid n = 1, 2, \ldots \}. \]

Then we denote

\[ \times_{\Gamma} = \times; \quad \times_{\Gamma} = \times. \]

This set is called the type-1 STP. Eq. (1) is of this type.

(ii) (MM-2 STP): Set

\[ \Gamma^2 := \{ J_n \mid n = 1, 2, \ldots \}, \]

where

\[ J_n := \frac{1}{n} I_{n \times n}, \quad n = 1, 2, \ldots. \] (12)

Then we denote

\[ \times_{\Gamma} = \circ_n; \quad \times_{\Gamma} = \circ_n. \]

This set is called the type-2 STP.

(iii) Set

\[ \Gamma^3 := \{ U_n \mid n = 1, 2, \ldots \}, \]

where,

\[ U_n := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_{n \times n}. \] (13)

(iv) Set

\[ \Gamma^4 := \{ L_n \mid n = 1, 2, \ldots \}, \]

where

\[ L_n := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in M_{n \times n}. \] (14)
III. VECTOR SPACE STRUCTURE ON MATRICES

Denote by

\[ \mathcal{M} := \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{M}_{m \times n}. \]

Then the following proposition is obvious.

**Proposition 3.1:** Consider the product \( \otimes_{r} \) on \( \mathcal{M} \). Then, \( (\mathcal{M}, \otimes_{r}) \) is a semi-group. Particularly, set \( \Gamma = \Gamma^1 = \{ I_n \mid n = 1, 2, \ldots \} \), then \( (\mathcal{M}, \otimes_{r}) \) is a monoid, i.e., a semi-group with identity.

Define

\[ \mathcal{M}_{\mu} := \{ A \in \mathcal{M}_{m \times n} \mid m/n = \mu \}. \]

Then we have the following partition:

\[ \mathcal{M} = \bigcup_{\mu \in \mathbb{Q}^+} \mathcal{M}_{\mu}, \quad (15) \]

where \( \mathbb{Q}^+ \) is the set of positive natural numbers.

Assume

\[ \mu = \mu_y/\mu_x, \]

where \( \mu_y \wedge \mu_x = 1 \), i.e., \( \mu_y \) and \( \mu_x \) are co-prime. Then \( \mu_y \) and \( \mu_x \) are called the \( y \) and \( x \) component of \( \mu \) respectively. 

Henceforth, we always assume \( \mu_y \) and \( \mu_x \) are co-prime. Then they are uniquely determined by \( \mu \).

Next, we pose a vector space structure on \( \mathcal{M}_{\mu} \). To this end, we define the addition on it.

**Definition 3.2:** Consider \( \mathcal{M}_{\mu} \), where \( \mu \in \mathbb{Q}^+ \). Let \( A, B \in \mathcal{M}_{\mu} \) and \( A \in \mathcal{M}_{m \times n}, B \in \mathcal{M}_{p \times q} \) respectively, and \( t = m \vee p \). Then

(i) the left \( \Gamma \) addition is defined by

\[ A \oplus \Gamma B := (A \otimes \Gamma_{t/m}) + (B \otimes \Gamma_{t/p}); \quad (16) \]

(ii) the right \( \Gamma \) addition is defined by

\[ A \oplus \Gamma B := (\Gamma_{t/m} \otimes A) + (\Gamma_{t/p} \otimes B). \quad (17) \]

Corresponding to (10) and (17), the subtraction can be defined respectively as

\[ A - \Gamma B := A \oplus \Gamma(-B); \quad (18) \]

and

\[ A - \Gamma B := A \oplus \Gamma(-B). \quad (19) \]

By definition, it is easy to verify the following.

**Proposition 3.3:** With the addition defined by (16) (or (17)) and a conventional scalar product, \( \mathcal{M}_{\mu} \) is a pseudo vector space.

Henceforth, we always assume \( \mu = \mu_y/\mu_x \) where \( \mu_y \wedge \mu_x = 1 \), i.e., \( \mu_y \) and \( \mu_x \) are co-prime. Particularly, set \( \Gamma = \Gamma^1 = \{ I_n \mid n = 1, 2, \ldots \} \), then \( \mathcal{M} \) is a semi-group such that \( \mu_y/\mu_x \) is the set of positive natural numbers.

**Remark 3.4:** A pseudo vector space satisfies all the requirements for a vector space except one: the “zero” is not unique [11]. Hence the inverse \(-x\) for each \( x \) is also not unique. In our case,

\[ \bar{0} := \{ 0_{k \mu_y \times k \mu_x} \mid k = 1, 2, \ldots \}. \]

IV. EQUIVALENCE AND LATTICE STRUCTURE ON \( \mathcal{M}_{\mu} \)

**Definition 4.1:** Let \( A, B \in \mathcal{M}_{\mu} \).

(i) \( A \) is said to be left \( \Gamma \) equivalent to \( B \), denoted by \( A \sim_{\ell} B \), if there exist \( \Gamma_{\alpha} \) and \( \Gamma_{\beta} \) such that

\[ A \otimes \Gamma_{\alpha} = B \otimes \Gamma_{\beta}. \quad (20) \]

(ii) \( A \) is said to be right \( \Gamma \) equivalent to \( B \), denoted by \( A \sim_{r} B \), if there exist \( \Gamma_{\alpha} \) and \( \Gamma_{\beta} \) such that

\[ \Gamma_{\alpha} \otimes A = \Gamma_{\beta} \otimes B. \quad (21) \]

**Remark 4.2:** Hereafter, we consider only

- **Case 1:**

\[ \Gamma = \Gamma^1 = \{ I_n \mid n = 1, 2, \ldots \}. \]

Then \( A \sim_{\ell} B \) is simply denoted by \( A \sim_{\ell} B \).

The equivalence class is denoted by

\[ (A)_{\ell} := \{ B \mid B \sim_{\ell} A \}. \]

\( A \sim_{r} B \) is simply denoted by

\[ A \sim_{r} B. \]

The equivalence class is denoted by

\[ (A)_{r} := \{ B \mid B \sim_{r} A \}. \]

- **Case 2:**

\[ \Gamma = \Gamma^2 = \{ I_n \mid n = 1, 2, \ldots \}. \]

Then \( A \sim_{\ell} B \) is simply denoted by \( A \sim_{\ell} B \).

The equivalence class is denoted by

\[ \vartriangleleft A \triangleright_{\ell} := \{ B \mid B \sim_{\ell} A \}. \]

\( A \sim_{r} B \) is simply denoted by \( A \sim_{r} B \).

The equivalence class is denoted by

\[ \langle A \rangle_{r} := \{ B \mid B \sim_{r} A \}. \]

For statement ease, in the following we consider only \( \Gamma = \Gamma^1 \) and the equivalence

\[ \sim := \sim^{\Gamma^1}. \]

Hence

\[ (A) := (A)_{\ell}. \]

All the arguments are applicable to the other three cases.
Then it is reasonable to pose an order on \( M \). Now if \( A, B \in \langle A \rangle \), if there exists an \( I_\alpha \) such that \( A \otimes I_\alpha = B \), then \( A \) is said to be preceding to \( B \), denoted by \( A \prec B \). Then \( \langle A \rangle \) becomes a partially ordered set.

Proposition 4.4: [8] Let \( A, B \in M_\mu \). Assume \( A \sim B \), that is, (20) holds. Then there exist a \( \Lambda \in M_\mu \) and two identity matrices \( I_\alpha \) and \( I_\beta \), such that
\[
A = \Lambda \otimes I_\beta, \quad B = \Lambda \otimes I_\alpha.
\]
(22)

Consider (20). Without loss of generality, we can assume \( \alpha \land \beta = 1 \). In this case we define
\[
\Theta := A \otimes I_\alpha = B \otimes I_\beta.
\]
(23)

Proposition 4.5: [8] Assume \( \alpha \land \beta = 1 \). Then the least common upper boundary of \( A \) and \( B \) is \( \Theta \), and the greatest common lower boundary of \( A \) and \( B \) is \( \Lambda \).

Definition 4.6: [2] Consider a non-empty set \( L \neq \emptyset \), if there is a partial order \( \prec \) on it such that for any two elements \( a, b \) in \( L \) there exist a least common upper boundary \( \inf(a, b) \in L \) and a greatest common lower boundary \( \sup(a, b) \in L \), then \( (L, \prec) \) is called a lattice.

Recall Proposition 4.5 one sees easily that \( \langle \langle A \rangle, \prec \rangle \) is a lattice.

Fig. 1, which is called the Hasse diagram, shows the lattice structure of \( \langle A \rangle \).

Next, we consider \( M_\mu \). Define
\[
M_k := M_{k_\mu, \cdot \cdot \cdot, k_\mu}.
\]

Then we have a partition as
\[
M_\mu = \bigcup_{k=1}^{\infty} M_k.
\]
(24)

Now if \( s = rt \), then
\[
M_r \otimes I_t \subset M_s.
\]
(25)

Then it is reasonable to pose an order on \( M_\mu \) as
\[
M_r \prec M_s, \quad \text{if } r \mid s.
\]
(26)

Then we have the lattice structure on \( M_\mu \) as follows.

Proposition 4.7: \( M_\mu \) with the partial order determined by (26) is a lattice, where
\[
\inf (M^p, M^q) = M^{p \cap q},
\]
\[
\sup (M^p, M^q) = M^{p \cup q}.
\]

Fig. 2 shows the lattice structure of \( M_\mu \).

V. QUOTIENT SPACES OF \( M_\mu \)

For statement ease, this section focuses on \( \Gamma = \Gamma^1, \sim = \sim^{1} \), and \( \Sigma_\mu = \Sigma^\mu_\mu \). We leave the parallel discussion for other quotient spaces to reader.

A. Topology on Quotient Space

We consider the topology of the quotient space \( \Sigma_\mu = M_\mu / \sim \). First, the topology of \( M_\mu \) should be clarified.

Definition 5.1: A natural topology on \( M_\mu \), denoted by \( N \), is constructed as follows.

(i) For each \( M^k \), the natural topology of Euclidean space \( R^{k_\mu, \cdot \cdot \cdot, k_\mu} \) is posed on \( M^k, k = 1, 2, \ldots \).

(ii) Each \( M_k, k = 1, 2, \ldots \), is considered as a clopen set of \( M_\mu \).

Then it is clear that under this natural topology the \( M_\mu \) has following property. We refer to any classical text book for the related concepts, e.g., [10].

Proposition 5.2: The topological space \( (M_\mu, N) \) is a Hausdorff space. It is also second countable.

Next, we consider the quotient spaces of \( M_\mu \) with respect to equivalences.
**Definition 5.3:** Some quotient spaces of \( M_\mu \) under equivalences are defined as follows:
\[
\begin{align*}
\Sigma_\mu^e & := M_\mu / \sim^e; \\
\Sigma_\mu^r & := M_\mu / \sim^r; \\
\Theta_\mu & := M_\mu / \sim^e; \\
\Theta_\mu^r & := M_\mu / \sim^r.
\end{align*}
\]  

We pose the quotient topology on quotient spaces. To be specific, we focus on \( \Sigma_\mu := \Sigma_\mu^e \) and denote by \( \langle A \rangle := \langle A \rangle^e \).

Consider \( \Sigma_\mu \). Assume \( \pi : M_\mu \to \Sigma_\mu \) is the natural projection, i.e., \( \pi : A \to \langle A \rangle \). Using the natural topology \( N \) on \( M_\mu \), then the quotient topology on \( \Sigma_\mu \) deduced by the mapping \( \pi \) is denoted by \( Q \). Precisely speaking, the open set on \( \Sigma_\mu \) is
\[
Q := \{ \pi^{-1}(O) \mid O \in N \}.
\]

Next, we consider the fiber bundle structure with \( M_\mu \) and \( \Sigma_\mu \).

**Definition 5.4:** Let \( T \) and \( B \) be two topological spaces. \( \pi : T \to B \) is a continuous surjective mapping. Then
\[
\begin{array}{ccc}
T & \longrightarrow & B \\
\pi \downarrow & & \downarrow \pi \\
(\Sigma_\mu, N) & \longrightarrow & (\Sigma_\mu, Q)
\end{array}
\]  
is called a fiber bundle, with \( T \) as the total space, with \( B \) as the base space. For each \( b \in B \), \( \pi^{-1}(b) \) is called the fiber over \( b \).

Then we have the following fiber bundle structure over \( M_\mu \) and \( \Sigma_\mu \).

**Proposition 5.5:** Consider \( (M_\mu, N) \) as total space and \( (\Sigma_\mu, Q) \) as base space, and
\[
\pi : A \to \langle A \rangle, \quad A \in M_\mu
\]
is the natural projection. Then
\[
(\Sigma_\mu, N) \longrightarrow (\Sigma_\mu, Q)
\]
is a fiber bundle. The fiber of \( \langle A \rangle \) is \( \{ B \mid B \in \langle A \rangle \} \).

**B. Vector Space Structure on Quotient Space**

**Definition 5.6:** Consider \( \Sigma_\mu \).

(i) Let \( \langle A \rangle, \langle B \rangle \in \Sigma_\mu \).
\[
\langle A \rangle + \langle B \rangle := \langle A + B \rangle,
\]
\[
\langle A \rangle \cdot \langle B \rangle := \langle A \cdot B \rangle.
\]  

(ii) Let \( \langle A \rangle \in \Sigma_\mu, r \in \mathbb{R} \).
\[
r \cdot \langle A \rangle := \langle rA \rangle.
\]

The following proposition can easily be verified by a straightforward computation.

**Proposition 5.7:** The addition \( + \) and scalar product \( \cdot \) defined by \( (30) \) and \( (31) \) for \( \Sigma_\mu \) are properly defined. Moreover, \( (\Sigma_\mu, +, \cdot) \) is a vector space.

**Definition 5.8:** Consider \( \Sigma_1 \). Assume \( \langle A \rangle, \langle B \rangle \in \Sigma_1 \), then
\[
\langle A \rangle \times \langle B \rangle := \langle A \times B \rangle.
\]  

It is easy to verify that \( (33) \) is properly defined. That is, the right hand side of \( (33) \) is independent of the choice of representatives \( A \in \langle A \rangle \) and \( B \in \langle B \rangle \).

**Definition 5.9:** A vector space \( V \) (over \( \mathbb{R} \)) with a binary operator (called a product) \(* : V \times V \to V\) is called an algebra, if the following distributive law is satisfied.
\[
x \ast (ay_1 + by_2) = ax \ast y_1 + bx \ast y_2,
\]
\[
(ay_1 + by_2) \ast x = ay_1 \ast x + by_2 \ast x.
\]

Then the following proposition is obvious.

**Proposition 5.10:** \( \Sigma_1 \) with the product define by \( (33) \) is an algebra.

The followings are some fundamental properties of the algebra \( (\Sigma_1, \times) \).

**Definition 5.11:** Consider \( \Sigma = M / \sim_\ell \).

(i) \( \langle A \rangle^T := \langle A^T \rangle \), \( \langle A \rangle \in \Sigma \). \( (33) \)

(ii) Let \( A \in M_1 \). If \( A \) is invertible (where \( \Gamma = \Gamma^1 \)), then
\[
\langle A \rangle^{-1} := \langle A^{-1} \rangle \), \( \langle A \rangle \in \Sigma \). \( (34) \)

(iii) Let \( \langle A \rangle, \langle B \rangle \in \Sigma_1 \). \( \langle A \rangle \) and \( \langle B \rangle \) are said to be equivalent, if (where \( \Gamma = \Gamma^1 \)) there exist nonsingular \( P \) and \( Q \), such that
\[
\langle P \rangle \times \langle A \rangle \langle Q \rangle = \langle B \rangle . \]  

(iv) Let \( \langle A \rangle, \langle B \rangle \in \Sigma_1 \). \( \langle A \rangle \) and \( \langle B \rangle \) are said to be congruent, if (where \( \Gamma = \Gamma^1 \)) there exists nonsingular \( P \), such that
\[
\langle P \rangle \times \langle A \rangle \times \langle P \rangle^{-1} = \langle B \rangle . \]  

(v) Let \( \langle A \rangle, \langle B \rangle \in \Sigma_1 \). \( \langle A \rangle \) and \( \langle B \rangle \) are said to be similar, if (where \( \Gamma = \Gamma^1 \)) there exists a nonsingular \( P \), such that
\[
\langle P \rangle^{-1} \times \langle A \rangle \times \langle P \rangle = \langle B \rangle . \]  

**Remark 5.12:** Using Definition \( 5.11 \) many related properties of matrices can be extended naturally to \( \Sigma_\mu \). For example:

(i) \( \langle A \rangle \times \langle B \rangle \times \langle A \rangle^T = \langle B \rangle^T \times \langle A \rangle \times \langle A \rangle^T \). \( (38) \)

(ii) If \( A, B \in M_1 \) are invertible, then
\[
\langle A \rangle \times \langle B \rangle \times \langle A \rangle^{-1} = \langle B \rangle^{-1} \times \langle A \rangle^{-1} . \]  

(iii) If \( \langle A \rangle \) is equivalent (congruent, similar) to \( \langle B \rangle \), and \( \langle B \rangle \) is equivalent (correspondingly, congruent, similar) to \( \langle C \rangle \), then \( \langle A \rangle \) is equivalent (correspondingly, congruent, similar) to \( \langle C \rangle \).
C. Analytic Functions on $\Sigma_1$

**Proposition 5.13:** Let $\langle A \rangle \in \Sigma_1$. Then

$$e^{\langle A \rangle} = \langle e^A \rangle.$$  (40)

**Proof:** We have only to prove that if $A \sim B$, then $e^A \sim e^B$. Let $\inf(A, B) = \Theta$. It is enough to prove that $e^A \sim e^\Theta$. Note that there exists an identity matrix, say $I_k$, such that $A = \Theta \otimes I_k$. Then it is enough to show that

$$e^{\Theta \otimes I_k} = e^\Theta \otimes I_k.$$  (41)

Assume $\Theta \in M_{n \times n}$, then we have

$$W(\Theta \otimes I_k)W^{-1} = I_k \otimes \Theta = \text{diag}(A, A, \ldots, A),$$

where $W = W_{[n,k]}$. Then we have

$$e^A = e^{W^{-1}(I_k \otimes \Theta)W} = W^{-1}e^{\text{diag}(\Theta, \Theta, \ldots, \Theta)}W = W^{-1}\text{diag}(e^\Theta, e^\Theta, \ldots, e^\Theta)W = W^{-1}I_k \otimes e^\Theta W = e^\Theta \otimes I_k.$$

**Remark 5.14:** Eq. (40) is of particular importance. In the sequel, it will be used to construct dimension-free general linear group (DFGLG) $GL(\mathbb{R})$ from dimension-free general linear algebra (DFGLA) $gl(\mathbb{R})$.

In general, an analytic function can be defined over $\Sigma_1$:

**Proposition 5.15:** Let $f$ be an analytic function. Then $f(\langle A \rangle) = \langle f(A) \rangle$.

**Proof:** Note that

$$\langle A \rangle \times \langle B \rangle = \langle A \times B \rangle,$$

and

$$\langle A \rangle \oplus \langle B \rangle = \langle A \oplus B \rangle.$$  (42)

Using Taylor expansion, we have

$$f(\langle A \rangle) = \langle \frac{d}{dt} \bigg|_{t=0} t^n A^n \rangle = \langle \frac{d}{dt} \bigg|_{t=0} \frac{t^n}{n!} A^n \rangle = \langle \frac{d}{dt} \bigg|_{t=0} \frac{t^n}{n!} A^n \rangle = \langle f(A) \rangle.$$  (43)

* $W_{[m,n]} = (I_n \otimes \delta_{m1}, I_n \otimes \delta_{m2}, \ldots, I_n \otimes \delta_{mn})$ is called the swap matrix.*

As aforementioned, $(\Sigma_1, \mid \cdot \mid)$ is a vector space, the Lie bracket on $\Sigma_1$ can be defined as follows:

$$[\langle A \rangle, \langle B \rangle] := \langle A \times \langle B \rangle \rangle, \quad <\langle A \rangle, \langle B \rangle > \in \Sigma_1.$$  (44)

It is easy to verify the following lemma.

**Lemma 6.1:** Let $A \in M^1_n, B \in M^1_n, C \in M^1_n$, and $n = r \vee s \vee t$. Then

$$(A \times B) \times C = A \times (B \times C) = (A \otimes I_{n/r}) (B \otimes I_{n/s}) (C \otimes I_{n/t}).$$  (45)

Using Lemma 6.1, one can prove the following result by a straightforward computation.

**Proposition 6.2:** $\Sigma_1$ with Lie bracket defined by 42 is a Lie algebra, called the DFGLA, denoted by $gl(\mathbb{R})$ (or $gl(\mathbb{C})$ as the vector space is considered over $\mathbb{C}$).

In the following, we consider only $gl(\mathbb{R})$, the results are all available for $gl(\mathbb{C})$, unless elsewhere is stated.

Define a function on $M_1$ as follows.

**Definition 6.3:** Let $A \in M^1_n$. Then

$$Tr(A) := \frac{1}{k} \text{trace}(A),$$

which is called the dimension-free trace.

**Remark 6.4:**

(i) It is easy to verify that if $A \sim^{\Gamma_1} B \sim^{\Gamma_1}$ stands for either $\sim^{g1}$ or $\sim^{r1}$, then

$$Tr(A) = Tr(B).$$  (46)

Hence we can define

$$Tr(\langle A \rangle) := Tr(A), \quad \langle A \rangle \in \Sigma_1.$$  (47)

(ii) Eq. (44) can be replaced by

$$Tr(A) := \frac{1}{\text{trace}(I_k)} \text{trace}(A).$$  (48)

Then we have that if $A \sim^{\Gamma} B$, then (45) holds. Hence we can also define

$$Tr(\langle A \rangle) := Tr(A), \quad \langle A \rangle \in M_{\mu/} \sim^{\Gamma}.$$  (49)

**Definition 6.5:**

(i) Let $(L, \langle \cdot, \cdot \rangle)$ be a Lie algebra, $V \subset L$ is a subspace of $L$. If $(V, \langle \cdot, \cdot \rangle)$ is also a Lie algebra, then it is called a Lie sub-algebra of $L$.

(ii) Assume $V \subset L$ is a Lie sub-algebra. $V$ is called an ideal of $L$, if

$$[X, A] \in V, \quad X \in L, \quad A \in V.$$  (50)

**Definition 6.6:**
(i) Let \((L, [\cdot, \cdot]_L)\) and \((H, [\cdot, \cdot]_H)\) be two Lie algebras. \(\pi : L \to H\) is called a homomorphism, if
\[
\pi([X,Y]_L) = [\pi(X), \pi(Y)]_H, \quad X, Y \in L.
\] (49)

(ii) Assume \(\pi : L \to H\) is a Lie algebra homomorphism, if \(\pi\) is a bijective mapping, \(\pi\) is called an isomorphism.

In the following example some Lie sub-algebras of \(gl(\mathbb{R})\) are constructed.

**Example 6.7:**

1) Let
\[
A := \{\langle A \rangle \in gl(\mathbb{R}) \mid Tr(\langle A \rangle) = 0\}.
\] (50)
Then \(A\) is an ideal of \(gl(\mathbb{R})\).

2) Given an \(\langle M \rangle \in gl(\mathbb{R})\), define
\[
gl(\langle M \rangle, \mathbb{R}) := \{\langle A \rangle \in gl(\mathbb{R}) \mid \langle A \rangle \times \langle M \rangle + \langle M \rangle \times \langle A \rangle^T = 0\}.
\] (51)
Then \(gl(\langle M \rangle, \mathbb{R})\) is a Lie sub-algebra of \(gl(\mathbb{R})\).

(i) Assume
\[
M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]
then the \(gl(\langle M \rangle, \mathbb{R})\) is called the orthogonal algebra.

(ii) Assume
\[
M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},
\]
then the \(gl(\langle M \rangle, \mathbb{R})\) is called the symplectic algebra.

(iii) If \(\langle M \rangle\) and \(\langle N \rangle\) are congruence, then \(gl(\langle M \rangle, \mathbb{R})\) and \(gl(\langle N \rangle, \mathbb{R})\) are isomorphic.

VII. KILLING FORM ON \(gl(\mathbb{R})\)

Next, we define the Killing form on \(\Sigma_1 = gl(\mathbb{R})\).

**Definition 7.1:** Let \(G\) be a lie algebra and \(A \in G\). Define
\[
ad_A X := [A, X], \quad X \in G.
\]
Then \(ad_A \in \text{End}(G)\) is an endomorphism. (Equivalently, if \(\dim(G) = n, ad_A \in gl(n, \mathbb{R})\).)

**Proposition 7.2:** Let \(A \in gl(n, \mathbb{R})\). Then \(ad_A \in gl(n^2, \mathbb{R})\) is expressed as
\[
ad_A = I_n \otimes A - A^T \otimes I_n.
\] (52)

**Proof:** Since
\[
ad_A X = [A, X] = AX -XA.
\]
Express it into standard linear mapping form, we have \[V_c(ad_A X) = (I_n \otimes A - A^T \otimes I_n)V_c(X),\] where \(V_c(A)\) is the column stacking form of \(A\).

Now we can define
\[
ad : \Sigma_1 \to \text{End}(\Sigma_1)
\]
as follows:
\[
ad(\langle A \rangle \langle X \rangle) := \langle [A], \langle X \rangle \rangle = \langle [A], \langle X \rangle \rangle.
\] (53)

Using Proposition 7.2 we have
\[
ad(\langle A \rangle \langle X \rangle) = \langle [A], \langle X \rangle \rangle = \langle \langle A \rangle \times \langle X \rangle \rangle = \langle [ad_A], \langle X \rangle \rangle.
\]

Hence
\[
ad(\langle A \rangle) = \langle ad_A \rangle.
\] (54)

Hence \(ad_A\) itself can be considered as an element in \(gl(\mathbb{R})\).

From this perspective, we have the following result.

**Proposition 7.3:** \(ad : gl(\mathbb{R}) \to gl(\mathbb{R})\) is a homomorphism.

**Proof:**
\[
ad(\langle [A, B] \rangle \langle X \rangle) = \langle [\langle A \rangle \times \langle B \rangle + \langle B \rangle \times \langle A \rangle], \langle X \rangle \rangle = \langle \langle A, B \rangle X \rangle
\]
\[
= \langle ad_A(ad_B)X - ad_B(ad_A)X \rangle
\]
\[
= \langle [ad_A, ad_B] \rangle \langle X \rangle.
\]

Hence, we have
\[
ad(\langle [A, B] \rangle) = [ad(\langle A \rangle), ad(\langle B \rangle)].
\]

**Definition 7.4:**

(i) Let \(A, B \in M_1\). Then the Killing form of \(A\) and \(B\) is define by
\[
(A, B) := Tr(ad_A \times ad_B).
\] (55)

(ii) Let \(\langle A \rangle, \langle B \rangle \in \Sigma_1\). Then the Killing form of \(\langle A \rangle\) and \(\langle B \rangle\) is define by
\[
\langle (A), (B) \rangle := (A, B).
\] (56)

We have to prove the Killing form \(\langle (A), (B) \rangle\) is properly defined.

**Proposition 7.5:** The Killing form \(\langle (A), (B) \rangle\) of \(\Sigma_1\) is properly defined.

**Proof:** Denote by \(A_1 \in \langle A \rangle\) and \(B_1 \in \langle B \rangle\) the minimum elements in \(\langle A \rangle\) and \(\langle B \rangle\) respectively. It is enough to prove that for any \(A \in \langle A \rangle\) and \(B \in \langle B \rangle\)
\[
(A, B) = (A_1, B_1).
\] (57)

Assume \(A_1 \in M_1^*, B_1 \in M_1^*,\) and \(r \vee s = t.\)
Using formula \( \text{tr}(AB) = \text{tr}(BA) \); (b) \( \text{tr}(A^T) = \text{tr}(A) \); (c) \( \text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B) \), we calculate that

\[
\text{Tr}(ad_{A_1} \times ad_{B_1}) = \text{Tr}\left\{ \left( \left( ad_{A_1} \otimes I_{\ell_2/\ell_1^2} \right) \left( ad_{B_1} \otimes I_{\ell_2/\ell_1^2} \right) \right) \right\}
= \text{Tr}\left\{ \left( \left( I_\ell \otimes A_1 - A_1^T \otimes I_\ell \right) \otimes I_{\ell_2/\ell_1^2} \right) \right\}
\]

Finally, assume \( A = A_1 \otimes I_\ell \) and \( B = B_1 \otimes I_q \), then we have

\[
\text{Tr}(ad_A \times ad_B) = \text{Tr}\left\{ \left( \left( ad_{A_1} \otimes I_{\ell_2/\ell_1^2} \right) \otimes I_p \right) \right\}
= \frac{1}{\ell} \text{tr}(A_1 \times B_1) - \frac{1}{\ell q} \text{tr}(A_1)\text{tr}(B_1).
\]

The following properties of Killing form on \( gl(\mathbb{R}) \) follow the definition immediately.

**Proposition 7.6:** The Killing form on \( gl(\mathbb{R}) \) has the following propositions:

(i)

\[
\langle \langle A \rangle, \langle B \rangle \rangle = \langle \langle B \rangle, \langle A \rangle \rangle, \quad \langle A \rangle, \langle B \rangle \in gl(\mathbb{R}). \tag{58}
\]

(ii)

\[
(a \langle A \rangle_1 \cup b \langle A \rangle_2, \langle B \rangle) = a \langle \langle A \rangle_1, \langle B \rangle \rangle \cup b \langle \langle A \rangle_2, \langle B \rangle \rangle, \quad a, b \in \mathbb{R}.
\]

(iii)

\[
\langle ad_{\langle A \rangle} \langle X \rangle, \langle Y \rangle \rangle + \langle \langle X \rangle, ad_{\langle A \rangle} \langle Y \rangle \rangle = 0. \tag{60}
\]

**VIII. Conclusions**

In this paper the DFES is proposed. The vector space structure is posed. Based on the equivalence, the quotient topology is constructed. Then the Lie bracket is defined, which turns the DFES a Lie algebra, which is of infinite dimension. It is then shown that this algebra has certain properties of finite dimensional Lie algebra. Particularly, the Killing form is constructed for it. Its properties are also investigated.