SURFACES IN 4-MANIFOLDS: SMOOTH ISOTOPY

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Abstract. In this paper we exhibit infinite families of embedded tori in 4-manifolds that are topologically isotopic but smoothly distinct. The interesting thing about these tori is that they are null-homologous, and in fact they each bound a (topologically embedded) solid handlebody in the 4-manifold.

1. Introduction

Just as a 4-manifold can have many inequivalent smooth structures, there can be many different smooth embeddings of surfaces into a 4-manifold which are topologically isotopic, but smoothly distinct. Two embeddings of the same surface that have this property are called exotic embeddings.

In this paper we will show that null-homologous tori first discovered by Fintushel and Stern in their knot surgery construction, in fact provide examples of exotic tori. Specifically,

Theorem 1.1. Let $X$ be a smooth 4-manifold with $b_2 \geq |\sigma| + 6$, non-trivial Seiberg-Witten invariant, and and embedded torus $T$ of self intersection $0$ such that $\pi_1(X \setminus T) = 1$. Then $X$ contains an infinite family of distinct tori that are topologically isotopic to the unknotted torus (a torus that bounds a solid handlebody in $X$), but not smoothly isotopic to it.

The first examples of exotic embeddings come from Fintushel and Stern’s “rim surgery” technique [3]. Their surfaces all have simply connected complement. A variation on rim surgery was given by Kim, and Kim-Ruberman which works in the case that the complement has non-trivial fundamental group ([7,8]). Tom Mark has used Heegaard-Floer homology to show that these constructions are also effective for constructing exotic embeddings of surfaces with negative self intersection ([10]). On the other hand, all of these constructions involve surfaces whose complement has finite first homology, and moreover all of these constructions essentially begin with symplectically embedded surfaces in a symplectic 4-manifold. Such surfaces can never be null-homologous. The significance of our examples is that they are null-homologous.
It is not difficult to satisfy the hypotheses of the theorem. For example, any elliptic surface contains such a torus and has non-trivial Seiberg-Witten invariant by virtue of being a symplectic manifold.

The strategy of proof is as follows: The knot surgery construction of Fintushel and Stern produces an infinite family of exotic smooth structures on a 4-manifold through a series of log-transforms on null-homologous tori. These are the tori we will focus on. We’ll define a gauge theoretic invariant to distinguish the tori smoothly. Finally, we’ll see that all such tori are topologically isotopic by a theorem of the second author:

**Theorem 1.2** ([12, Theorem 7.2]). Let $\Sigma_0$ and $\Sigma_1$ be locally flat embedded surfaces of the same genus in a simply connected 4-manifold $X$. The surfaces are topologically isotopic when $\pi_1(X \setminus \Sigma_i) = \mathbb{Z}$ and $b_2 \geq |\sigma| + 6$.

We conclude this introduction with an open question.

**Question.** Do there exist exotic embedded surfaces in $S^4$? In particular, is there an embedded $S^2$ that is topologically isotopic to the unknot but not smoothly isotopic to the unknot?

If one could produce such an exotic unknot, its complement would be an exotic $S^1 \times D^3$, and surgery along it would be an exotic $S^1 \times S^3$, two of the most elusive exotic creatures. The examples in this paper can be seen as prototype for answering this sort of question: Since the tori we construct bound solid handlebodies in $X$, they are close to being exotic surfaces in $S^4$ in the sense that they are topologically isotopic to a surface lying in a ball in $X$.

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2. Constructing the tori

Let $T$ be an embedded torus with self intersection zero in a 4-manifold $X$ such that $\pi_1(X \setminus T) = 1$. Such a torus is necessarily homologically essential. We will not construct exotic embeddings of $T$, but rather we will find exotic embeddings of nearby null-homologous tori which arise in the “knot surgery” construction of Fintushel and Stern ([2] and [4]). Knot surgery along torus $T$ using a knot $K \subset S^3$ is most straightforwardly defined as $X_K = (X \setminus \nu(T)) \cup (S^1 \times S^3 \setminus \nu(K))$ where the union is formed by taking the longitude of $K$ to the meridian of $T$ (apart from this requirement, the gluing is not, strictly speaking, well defined, but this is in general irrelevant). Fintushel and Stern proved that $X$ is homeomorphic to $X_K$ under the assumption that the complement of $T$ is simply connected, and they further proved
that their Seiberg-Witten invariants are related by \( SW_{X_K} = SW_X \cdot \Delta_K(2[T]) \).
Therefore, by varying \( K \), one can construct infinitely many smooth structures
on \( X \). The Seiberg-Witten formula is proved by viewing knot surgery as a
series of log-transforms on null-homologous tori. That is, rather than cutting out
\( \nu(T) = S^1 \times (S^1 \times D^2) \) and replacing it with \( S^1 \times S^3 \setminus \nu(K) \), we can view knot
surgery as a series of log-transforms on \( S^1 \times (S^1 \times D^2) \) which eventually lead to
\( S^1 \times S^3 \setminus \nu(K) \). Forgetting the extra \( S^1 \) direction for the moment, one can go from
\( S^3 \setminus \nu(K) \) to \( (S^1 \times D^2) \), the complement of the unknot, by doing \( \pm 1 \) surgery along
crossings of \( K \) to unknot it. See Figure 2. Crossing this whole picture with \( S^1 \)
gives the log-transforms needed for knot surgery.

Suppose for the moment, that \( K \) is a knot of unknotting number 1. Then knot
surgery is equivalent to a single log-transform on a null homologous torus \( T_K \). As
long as the complement of \( T \) is simply connected, then \( T_K \) will have \( \pi_1(X \setminus T_K) = \mathbb{Z} \).
This is because
\[
\pi_1(X_K \setminus T_K) = \pi_1(S^1 \times S^3 \setminus (\nu K \cup T_K)) \langle S^1 \times pt, \mu_K, \lambda_K \rangle
\]
where \( \mu_K \) and \( \lambda_K \) are respectively the meridian and longitude of \( K \). This implies
that all loops are homotopic to a multiple of the meridian to \( T_K \).

Already we see that this gives at least one exotically embedded torus. Specifically, \( T_K \) is topologically standard by Theorem 1.2 and moreover, performing a
log-transform on \( T_K \) will give an exotic smooth structure on \( X \), whereas performing
a log-transform on the standardly embedded torus, (i.e. the one that bounds a solid
handlebody), will not. Therefore these tori are smoothly distinct, but by Theorem
1.2 they must be topologically isotopic.

To construct infinite families of exotic surfaces, we need to be more careful. For
instance, supposing that \( K_i \) is the \( i \)-th twist knot, it might be possible to construct
\( X_{K_i} \) for any \( i \) via some log-transform on \( T_K \). (The effect of \( \frac{1}{n} \)-log transforms on
The Seiberg-Witten invariant of a 4-manifold $X$ is a map $SW_X : \mathcal{S} \rightarrow \mathbb{Z}$, where $\mathcal{S}$ is the set of isomorphism classes of spin$^c$ structures on $X$. The basic classes of $X$ are defined to be the spin$^c$ structures that map to non-zero integers. It is a well-known property of the Seiberg-Witten invariant that a 4-manifold has only a finite number of basic classes.

We will distinguish our null-homologous tori by computing an invariant that is, in a technical sense clarified below, related to the Seiberg-Witten basic classes of the complement of the tori. To do this we will need to understand how the Seiberg-Witten invariant of a 4-manifold is affected by log-transforms. Suppose we are given a 4-manifold with $T^3$ boundary, e.g. $X \setminus \nu T$, and suppose $H_1(T^3) = \mathbb{Z}[a,b,c]$. Denote the log-transformed 4-manifold constructed by gluing on a $D^2 \times T^2$, where $[D^2]$ is glued to $[pa + qb + rc]$ as $X_T(p,q,r)$, and denote the core torus in the $D^2 \times T^2$ part of this manifold as $T(p,q,r)$.

A formula of Morgan-Mrowka-Szabo from [11] give a formula relating the Seiberg-Witten invariants of various log-transforms:

$$\sum_i SW_{X_T(p,q,r)}(k(p,q,r) + i[T(p,q,r)]) = p \sum_i SW_{X_T(1,0,0)}(k(1,0,0) + i[T(1,0,0)])$$

$$+ q \sum_i SW_{X_T(0,1,0)}(k(0,1,0) + i[T(0,1,0)]) + r \sum_i SW_{X_T(0,0,1)}(k(0,0,1) + i[T(0,0,1)])$$

where the $k(a,b,c)$ are spin$^c$ structures that are equivalent on $X \setminus T$ and are trivial on the log-transformed torus $T(a,b,c)$. The sums here are intended to indicate summing over all spin$^c$ structures on $X_{(a,b,c)}$ which restrict to $k_{(a,b,c)}$ on $X \setminus T$. In particular, if $T(p,q,r)$ is null-homologous, then the left hand side of the equation has only one term.

Moreover, since there are a finite number (not depending on $(p,q,r)$) of $k_{(1,0,0)}$ such that the sum $\sum_i SW_{X_T(1,0,0)}(k_{(1,0,0)} + i[T_{(1,0,0)})]$ is not zero (respectively for $k_{(0,1,0)}$ and $k_{(0,0,1)}$), then according to the MMS formula there is a fixed, finite number of possible basic classes for $X_T(p,q,r)$ in the case that $T_{(p,q,r)}$ is null-homologous. Therefore, the following invariant is well defined:

**Definition.** Let $T$ be a null-homologous torus in $X$. Define $B(X,T)$ as the maximum divisibility of the first Chern class of a basic class of $X_{(p,q,r)}$ taken over all $(p,q,r)$ such that $[T_{(p,q,r)}] = 0$. 

$T_K$ is explored in [5]. To resolve this issue, we’ll have to look more deeply at how the Seiberg-Witten invariant changes under log-transforms on $T_K$, and restrict ourselves to certain classes of knots.
4. FAMILIES OF UNKNOTTING NUMBER ONE KNOTS, AND THE PROOF OF
THEOREM 1

Now that we have a better understanding of the smooth invariants needed to
distinguish potential infinite families of smooth tori, we can describe an explicit
family of knots that will give rise to smoothly distinct $\mathcal{T}_K$. For the purposes
of this paper, it will be sufficient to focus on a nice family of two-bridge knots. All
two bridge knots can be given in the form of Figure 1 where $a_i$ is the number of
right half-twists when $i$ is odd, and left half-twists when $i$ is even. We refer to two-
bridge knots using Conway’s notation, $C(a_1, \ldots, a_n)$, and we note that it is well
known (see [1] for instance), that two 2-bridge knots are equivalent if and only if
$[a_1, \ldots, a_n]$ and $[a'_1, \ldots, a'_m]$ are continued fraction expansions of the same rational
number.

Proposition 4.1 (Kanenobu-Murakami [6]). A two-bridge knot has unknotting
number one if and only if it can be expressed as

$$C(b, b_1, b_2, \ldots, b_k, \pm 2, -b_k, \ldots, -b_2, -b_1).$$

The following proposition of Burde-Zieschang is stated in terms of our convention
for presenting two bridge knots as in the Figure 1.

Proposition 4.2 (Burde-Zieschang [1, Theorem 12.22]). The Conway polynomial
of a two bridge knot expressed as $C(a_0, \ldots, a_{2n})$ has degree

$$\frac{1}{2} \sum_{j=0}^{n} |a_{2j}|.$$
Since there is a log-transform on $T_{K_i}$ that gives $X_{K_i}$, we have that

$$\lim B(X, T_{K_i}) \geq \lim(\text{max divisibility of basic classes of } X_{K_i}) = \infty.$$ 

Therefore, there are an infinite number of the $T_{K_i}$ that are smoothly distinguished by their $B$ invariant. 

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