Elastic Plate Deformation with Transverse Variation of Microrotation

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Abstract

The purpose of this paper is to present a new mathematical model for the deformation of thin Cosserat elastic plates. Our approach, which is based on a generalization of the classical Reissner plate theory, takes into account the transverse variation of microrotation of the plates. The model assumes polynomial approximations over the plate thickness of asymmetric stress, couple stress, displacement, and microrotation, which are consistent with the elastic equilibrium, boundary conditions and the constitutive relationships. Based on the generalized Hellinger-Prange-Reissner variational principle and strain-displacement relation we obtain the complete theory of Cosserat plate. We also proved the solution uniqueness for the plate boundary value problem.

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1 Introduction

The well known classical bending theory of elastic plates [6], [7], [17], was first presented by Kirchhoff in his thesis (1850) and is described by a bi-harmonic differential equation [2],[17]. The usual assumption of this theory is that the normal to the middle plane remains normal during deformation. Thus the theory neglects transverse shear strain effects. A system of equations, which takes into account the transverse shear deformation, has been developed by E. Reissner (1945) [13], [14].

One of the advantages of Reissner’s model is that it is able to determine the reactions along the edges of a simply supported rectangular plate, where classical theory leads to a concentrated reaction at the corners of the plate. The Reissner theory has been applied to thin walled structures with moderate
thickness. The study of the relationships between these two models has proved that the solution of the clamped Reissner plate approaches the solution of the Kirchhoff plate as the thickness approaches zero \([1]\) and that the maximum bending can reach up to 20% for moderate plate thickness \([2]\). The numerical calculations of bending behavior of the plate of moderate thickness, \([16]\) show high level agreement between 3D and Reissner models. More remarks on the history of the modeling of classic linear elastic plates can be found in \([6]\), \([16]\), \([15]\).

In order to describe deformation of elastic plates with microstructure that possess grains, particles, fibers, and cellular structures \([10]\), \([11]\). A. C. Eringen (1967) was the first to propose a theory of plates in the framework of Cosserat (micropolar) Elasticity \([3]\). His theory is based on a direct technique of integration of the Cosserat Elasticity. The Eringen plate theory does not consider a transverse variation of the microrotation over the thickness, which might be necessary for rather thick plates under vertical load and pure twisting momentum. In order to develop a theory of plates, which can be used for thin wall structures with moderate thickness, we propose to use the classic Reissner plate theory as a foundation for the modeling of Cosserat elastic plates. Our approach, in addition to the traditional model, takes into account the second order approximation of couple stresses and the variation of three components of microrotation in the thickness direction.

2 Micropolar (Cosserat) Linear Elasticity

2.1 Fundamental Equations

Before proceeding some notation convention should be explained. We use the usual summation conventions and all expressions that contain Latin letters as subindices are understood to take values in the set \([1, 2, 3]\). When Greek letters appear as subindices then it will be assumed that they can take the values 1 or 2.

The Cosserat elasticity equilibrium equations without body forces represent the balance of linear and angular momentums of micropolar elasticity and have the following form \([3]\):

\[
\begin{align*}
\text{div}\sigma &= 0, \\
\varepsilon \cdot \sigma + \text{div}\mu &= 0,
\end{align*}
\]

where the quantity \(\sigma = \{\sigma_{ji}\}\) is the stress tensor, \(\mu = \{\mu_{ji}\}\) the couple stress tensor, \(\varepsilon = \{\varepsilon_{ijk}\}\) is the Levi–Civita tensor, where \(\varepsilon_{ijk}\) equals 1 or -1 according as \((i, j, k)\) is an even or odd permutation of 1,2,3 and zero otherwise, and \(\varepsilon \cdot \sigma = \{\varepsilon_{ijk}\sigma_{jk}\}\).

The constitutive equations can be written in the form \([12]\):

\[
\begin{align*}
\sigma &= (\mu + \mu_c)\gamma + (\mu - \mu_c)\gamma^T + \lambda(\text{tr}\gamma)1, \\
\mu &= (\gamma + \epsilon)\chi + (\gamma - \epsilon)\chi^T + \beta(\text{tr}\chi)1,
\end{align*}
\]
and the strain-displacement and torsion-rotation relations

$$\gamma = (\nabla \mathbf{u})^T + \mathbf{e} \cdot \phi$$

and

$$\chi = \nabla \phi,$$

where quantities $\gamma$ and $\chi$, are the micropolar strain and torsion tensors, $\mathbf{u}$ and $\phi$ the displacement and rotation vectors respectively, $\mathbf{1}$ the identity tensor, and $\mu, \lambda$ are the symmetric, $\mu_c, \beta, \gamma, \epsilon$ the asymmetric Cosserat elasticity constants.

In the reversible form:

$$\gamma = (\mu' + \mu_c')\sigma + (\mu' - \mu_c')\sigma^T + \lambda'(\text{tr} \sigma)\mathbf{1},$$

$$\chi = (\gamma' + \epsilon')\mu + (\gamma' - \epsilon')\mu^T + \beta(\text{tr} \mu)\mathbf{1},$$

where $\mu' = \frac{1}{4\mu}$, $\mu_c' = \frac{1}{4\mu_c}$, $\gamma' = \frac{1}{4\gamma}$, $\epsilon' = \frac{1}{4\epsilon}$, $\lambda' = \frac{-\lambda}{6\mu(\lambda + 2\mu)}$, and $\beta' = \frac{-\beta}{6\mu(\beta + 2\gamma)}$.

We consider a Cosserat elastic body $B_0$. In this case the equilibrium equations (1) - (2) with constitutive formulas (3) - (4) and kinematics formulas (5) should be accompanied by the following mixed boundary conditions

$$\mathbf{u} = \mathbf{u}_o, \phi = \phi_o \text{ on } G_1 = \partial B_0 \setminus \partial B_\sigma,$$

$$\sigma_n = \sigma \cdot n = \sigma_o, \mu_n = \mu \cdot n = \mu_o \text{ on } G_2 = \partial B_\sigma,$$

where $\mathbf{u}_o, \phi_o$ are prescribed on $G_1$, $\sigma_o$ and $\mu$ on $G_2$, and $n$ denotes the outward unit normal vector to $\partial B_0$.

### 2.2 Cosserat Elastic Energy

The strain stored energy $U_C$ of the body $B_0$ is defined by the integral [12]:

$$U_C = \int_{B_0} W \{\gamma, \chi\} \, dv,$$

where

$$W \{\gamma, \chi\} = \frac{\mu + \mu_c}{2} \gamma_{ij} \gamma_{ij} + \frac{\mu - \mu_c}{2} \gamma_{ij} \gamma_{ji} + \frac{\lambda}{2} \gamma_{kk} \gamma_{nn}$$

$$+ \frac{\gamma + \epsilon}{2} \chi_{ij} \chi_{ij} + \frac{\gamma - \epsilon}{2} \chi_{ij} \chi_{ji} + \frac{\beta}{2} \chi_{kk} \chi_{nn},$$

then the constitutive relations (3) - (4) can be written in the form:

$$\sigma = C_\sigma [W] = \nabla \gamma W \text{ and } \mu = C_\mu [W] = \nabla \chi W.$$

The function $W$ is positive if and only if [12]

$$\mu > 0, \quad 3\lambda + 2\mu > 0,$$

$$\gamma > 0, \quad 3\beta + 2\gamma > 0,$$

$$\mu_c > 0, \quad \mu + \mu_c > 0, \quad \epsilon > 0.$$
The following conditions \[9\] for the Cosserat elastic energy

\[
\begin{align*}
\mu & > 0, \quad 3\lambda + 2\mu > 0, \\
\gamma & > 0, \quad 3\beta + 2\gamma > 0, \\
\mu_c & \geq 0, \quad \epsilon \geq 0.
\end{align*}
\]

are enough to provide the uniqueness of static problems.

For future convenience, we present the stress energy

\[ U_K = \int_{B_0} \Phi \{ \sigma, \mu \} \, dv, \]

where

\[
\Phi \{ \sigma, \mu \} = \frac{\mu'}{2} \sigma_{ij} \sigma_{ij} + \frac{\mu'}{2} \sigma_{ij} \sigma_{ji} + \frac{\lambda'}{2} \sigma_{kk} \sigma_{nn} \\
+ \frac{\gamma'}{2} \mu_{ij} \mu_{ij} + \frac{\beta'}{2} \mu_{ij} \mu_{ji} + \frac{\epsilon'}{2} \mu_{kk} \mu_{nn}.
\]

The reversible constitutive relation \((6) - (7)\) can be also written in form:

\[ \gamma = K_\gamma [\sigma] = \frac{\partial \Phi}{\partial \sigma}, \quad \chi = K_\chi [\mu] = \frac{\partial \Phi}{\partial \mu}. \]

The total internal work done by the stresses \(\sigma\) and \(\mu\) over the strains \(\gamma\) and \(\chi\) for the body \(B_0\) \[12\] is

\[ U = \int_{B_0} [\sigma \cdot \gamma + \mu \cdot \chi] \, dv \]

and

\[ U = U_K = U_C \]

provided the constitutive relations \[3\] - \[4\] hold.

### 2.3 The Generalized Hellinger-Prange-Reissner (HPR) Principle

The HPR principle \[5\] in the case of Cosserat elasticity states, that for any set \(A\) of all admissible states \(s = [u, \varphi, \gamma, \chi, \sigma, \mu]\) that satisfy the strain-displacement and torsion-rotation relations \[5\], the zero variation

\[ \delta \Theta(s) = 0 \]

of the functional

\[ \Theta(s) = U_K - \int_{B_0} [\sigma \cdot \gamma + \mu \cdot \chi] \, dv \]

\[ + \int_{\partial_1} [\sigma_n \cdot (u - u_o) + \mu_n (\varphi - \varphi_o)] \, da + \int_{\partial_2} [\sigma_o \cdot u + m_o \cdot \varphi] \, da \]
at \( s \in A \) is equivalent of \( s \) to be a solution of the system of equilibrium equations (1) - (2), constitutive relations (6) - (7), which satisfies the mixed boundary conditions (8) - (9). The proof is similar to the proof for HPR principle for classic linear elasticity [5].

3 The Cosserat Plate Assumptions

In this section we formulate our stress, couple stress and kinematic assumptions of the Cosserat plate. The set of points \( P = \{ \Gamma \times [-h/2, h/2] \} \cup T \cup B \) forms the entire surface of the plate and \( \{ \Gamma_u \times [-h/2, h/2] \} \) is the lateral part of the boundary where displacements and microrotations are prescribed. The notation \( \Gamma_\sigma = \Gamma \setminus \Gamma_u \) of the remainder we use to describe the lateral part of the boundary edge \( \{ \Gamma_\sigma \times [-h/2, h/2] \} \) where stress and couple stress are prescribed. We also use notation \( P_0 \) for the middle plane internal domain of the plate.

In our case we consider the vertical load and pure twisting momentum boundary conditions at the top and bottom of the plate, which can be written in the form:

\[
\begin{align*}
\sigma_{33}(x_1, x_2, h/2) &= \sigma^t(x_1, x_2), \quad \sigma_{33}(x_1, x_2, -h/2) = \sigma^b(x_1, x_2), \\
\sigma_{3\beta}(x_1, x_2, \pm h/2) &= 0, \\
\mu_{33}(x_1, x_2, h/2) &= \mu^t(x_1, x_2), \quad \mu_{33}(x_1, x_2, -h/2) = \mu^b(x_1, x_2), \\
\mu_{3\beta}(x_1, x_2, \pm h/2) &= 0,
\end{align*}
\]

where \((x_1, x_2) \in P_0\).

3.1 Stress and Couple Stress Assumptions

Our approach, which is in the spirit of the Reissner’s theory of plates [13], assumes that the variation of stress \( \sigma_{kl} \) and couple stress \( \mu_{kl} \) components across the thickness can be represented by means of polynomials of \( x_3 \) in such a way that it will be consistent with the equilibrium equations (1) and (2). First, as it is assumed in the standard theory of plates, we use expressions for the stress components in the following form [18]:

\[
\sigma_{\alpha\beta} = n_{\alpha\beta}(x_1, x_2) + \frac{h}{2}\zeta_3 m_{\alpha\beta}(x_1, x_2),
\]

where \( \zeta_3 = \frac{2}{h}x_3 \), and \( \alpha, \beta \in \{1, 2\} \). The only difference between our assumptions and those of Reissner’ [13] is that the functions \( n_{\alpha\beta} \) and \( m_{\alpha\beta} \) are not symmetric. Based on (23) and by means of the first two equations of written in the component form stress equilibrium (11)

\[
\sigma_{j\beta,j} = 0
\]

we obtain for the shear stress components
\[ \sigma_{33} = q_3(x_1, x_2) \left( 1 - \zeta_3^2 \right), \]  
\hfill (24)

It is natural to assume that the expressions for the remaining shear stress component are in the form similar to (24), i.e.

\[ \sigma_{3\beta} = q^\beta_3(x_1, x_2) \left( 1 - \zeta_3^2 \right). \]  
\hfill (25)

Here, as is usual for the asymmetric elasticity, the functions \( q_3, q^\beta_3 \) can be different.

Substituting equations (25) in the remaining equilibrium differential equation for stress

\[ \sigma_{j3,j} = 0 \]

we obtain the expression for the transverse normal stress

\[ \sigma_{33} = \zeta_3 \left( \frac{1}{3} \zeta_3^2 - 1 \right) k^*(x_1, x_2) + m^*(x_1, x_2). \]  
\hfill (26)

The next step is to accommodate approximations (26) to the boundary conditions (19). By direct substitution to (19) it easy to obtain that

\[ \sigma_{33} = -\frac{3}{4} \left( \frac{1}{3} \zeta_3^2 - \zeta_3 \right) p + \sigma_0, \]  
\hfill (27)

where \( p = \sigma^t(x_1, x_2) - \sigma^b(x_1, x_2) \) and \( \sigma_0 = \frac{1}{2} \left( \sigma^t(x_1, x_2) + \sigma^b(x_1, x_2) \right) \) satisfy the boundary condition requirements. We note that expression (27) is identical to the expression of \( \sigma_{33} \) given in [13] in the case of \( \sigma^b = 0 \).

It is also assumed that the couple stress \( \mu_{\alpha\beta} \) should have expression similar to the shear stress \( \sigma_{3\beta} \) expressions (24) - (25):

\[ \mu_{\alpha\beta} = (1 - \zeta_3^2) r_{\alpha\beta}(x_1, x_2). \]  
\hfill (28)

Finally we assume that couple stress \( \mu_{3\beta} \) expression is similar to \( \sigma_{3\beta} \) [23]:

\[ \mu_{3\beta} = \zeta_3 s_\beta(x_1, x_2) + m_\beta(x_1, x_2). \]  
\hfill (29)

Note that the first two equations of (2) can be written in the form

\[ \epsilon_{\betajk}\sigma_{jk} + \mu_{j3\beta,j} = 0, \]  
\hfill (30)

and substituting the couple stress (28) in (30) and taking into account (24) and (25) we obtain the expression for the transverse shear couple stress:

\[ \mu_{3\beta} = \left( \frac{1}{3} \zeta_3^2 - \zeta_3 \right) s_\beta(x_1, x_2) + m_\beta(x_1, x_2). \]  
\hfill (31)

Substituting (31) to boundary conditions (21) we obtain that

\[ s_\beta(x_1, x_2) = 0 \quad \text{and} \quad m_\beta(x_1, x_2) = 0, \]
i.e. the transverse shear couple stress
\[ \mu_{33} = 0. \]  
(32)

Now, substituting the couple stress (29) and stress (23) in the remaining differential equation of the equilibrium of angular momentum (2)
\[ \epsilon_{3jk}\sigma_{jk} + \mu_{3j, j} = 0, \]
(33)
we obtain the transverse normal couple stress to be in the form:
\[ \mu_{33} = \frac{1}{2}\zeta_3^2 a^*(x_1, x_2) + \zeta_3 b^*(x_1, x_2) + c^*(x_1, x_2). \]
(34)

The next step is to accommodate boundary conditions (21) to (34). At this stage we restrict the form of (34), which could allow us to determine couple stress \( \mu_{33} \) directly from boundary conditions (21). To this end we make an additional assumption that \( \mu_{33} \) must be a first order polynomial
\[ \mu_{33} = \zeta_3 b^*(x_1, x_2) + c^*(x_1, x_2). \]
(35)

This assumption is also consistent with the equilibrium equation (33) and allows us to proceed as we did for the determination of transverse loading stress (27) from the stress boundary conditions. Now boundary conditions (21) are sufficient to determine \( \mu_{33} \), which must be of the form
\[ \mu_{33} = \zeta_3 v + t, \]
(36)
where \( v(x_1, x_2) = \frac{1}{2} (\mu^t(x_1, x_2) - \mu^b(x_1, x_2)) \) and \( t(x_1, x_2) = \frac{1}{2} (\mu^t(x_1, x_2) + \mu^b(x_1, x_2)) \).

### 3.2 Kinematic Assumptions

The choice of kinematic assumptions is based on simplicity and their compatibility with the constitutive relationships of stress and couple stress assumptions (3). As in the standard theory of thin plates, it is assumed that \( u_\alpha \) displacements are distributed linearly over the thickness of the plate (3) and that \( u_3 \) does not vary over the thickness of the plate, i.e.
\[ u_\alpha = U_\alpha(x_1, x_2) - \frac{h}{2}\zeta_3 V_\alpha(x_1, x_2), \]
(37)
\[ u_3 = w(x_1, x_2), \]

The terms \( V_\alpha(x_1, x_2) \) in (37) represent the rotations in middle plane.

In order to accommodate the transverse microrotations to the constitutive relations (3) we propose the variation of microrotation with respect to \( x_3 \) by means of the second and third order polynomials:
\[ \varphi_\alpha = \Theta^0_\alpha(x_1, x_2) (1 - \zeta_3^2), \]
(38)
\[ \varphi_3 = \Theta^0_3(x_1, x_2) + \zeta_3 \left( 1 - \frac{1}{3}\zeta_3^2 \right) \Theta_3(x_1, x_2). \]
(39)
The constitutive formulas (3) - (4) motivate us to chose the forms (38) and (39), which produce expressions for $\phi_{\alpha,\beta}$ and $\phi_{3,3}$ similar to what we have for couple stress approximations (28).

The functions $\Theta^0_i$ in (38) and (39) describe microrotation components in the middle plane of the plate and $\Theta_3(x_1,x_2)$ the slope at the middle plane. Thus, in the assumptions (38) and (39) the transverse variation effect of microrotations is not neglected.

4 Specification of HPR Variational Principle for the Cosserat Plate

The HPR variational principle for a Cosserat plate is most appropriately expressed in terms of corresponding integrands calculated across the whole thickness. We also introduce the weighted characteristics of displacements, microrotations, strains and stresses of the plate, which will be used to produce the explicit forms of these integrands.

4.1 The Cosserat plate stress energy density

We define the plate stress energy density by the formula;

$$\Phi(S) = \frac{h}{2} \int_{-1}^{1} \Phi \{ \sigma, \mu \} d\zeta_3.$$ (40)

Taking into account the stress and couple stress assumptions (23) - (36) and by the integrating $\Phi \{ \sigma, \mu \}$ with respect $\zeta_3$ in $[-1,1]$ we obtain the explicit plate stress energy density expression in the form:
\[ \Phi(S) = \frac{\lambda + \mu}{2h\mu(3\lambda + 2\mu)} \left[ N_{aa}^2 + \frac{12}{h^2} M_{aa}^2 \right] \\
- \frac{\lambda}{2h\mu(3\lambda + 2\mu)} \left[ N_{11} N_{22} + \frac{12}{h^2} M_{11} M_{22} \right] \\
+ \frac{\mu}{8h\mu,\mu} \left[ (1 - \delta_{\alpha\beta}) \left( N_{\alpha\beta}^2 + \frac{12}{h^2} M_{\alpha\beta}^2 \right) + \frac{6}{5} (Q_\alpha Q_\alpha + Q_\beta Q_\beta) \right] \\
+ \frac{3(\mu - \mu)}{10h,\mu,\mu} \left[ Q_\alpha Q_\alpha^* + \frac{5}{6} N_{12} N_{21} + \frac{10}{h^2} M_{12} M_{21} \right] - \frac{3\lambda}{5h\mu(3\lambda + 2\mu)} pM_{\beta\beta} \\
+ \frac{3}{5h(3\beta + 2\gamma)} [(\beta + \gamma) R_{\alpha\alpha} - \beta R_{11} R_{22}] + \frac{3}{10h} \left( \frac{1}{\gamma} - \frac{1}{\epsilon} \right) R_{12} R_{21} \\
+ \frac{17h(\lambda + \mu)}{280(3\lambda + 2\mu)^2} \frac{1}{N_{\alpha\alpha}} \sigma_0 \\
+ \frac{h(\lambda + \mu)}{2\mu(3\lambda + 2\mu)} \frac{\sigma_0}{\gamma + \epsilon} \left[ \frac{1}{8} M_{\alpha\alpha}^* M_{\alpha\alpha}^* + \frac{3}{2h^2} S_{\alpha\beta}^* S_{\alpha\beta}^* + \frac{3}{20} (1 - \delta_{\alpha\beta}) R_{\beta\gamma}^2 \right] \\
- \frac{\beta}{2\gamma(3\beta + 2\gamma)} R_{\alpha\alpha} t + \frac{h(\beta + \gamma)}{2\gamma(3\beta + 2\gamma) t^2} \frac{1}{6\gamma(3\beta + 2\gamma)} v^2, \quad (41) \]

where the Cosserat stress set

\[ S = [M_{\alpha\beta}, Q_\alpha, Q_\alpha^*, R_{\alpha\beta}, S_{\alpha\beta}^*, N_{\alpha\beta}, M_{\alpha}^*], \quad (42) \]

where

\[ M_{\alpha\beta} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 \sigma_{\alpha\beta} d\zeta_3 = \frac{h^3}{12} m_{\alpha\beta}, \quad (43) \]

\[ Q_\alpha = \frac{h}{2} \int_{-1}^{1} \sigma_{3\alpha} d\zeta_3 = \frac{2h}{3} q_\alpha, \quad Q_\alpha^* = \frac{h}{2} \int_{-1}^{1} \sigma_{3\alpha} d\zeta_3 = \frac{2h}{3} q_\alpha^* \]

\[ R_{\alpha\beta} = \frac{h}{2} \int_{-1}^{1} \mu_{\alpha\beta} d\zeta_3 = \frac{2h}{3} r_{\alpha\beta}, \]

\[ S_{\alpha}^* = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \zeta_3 \mu_{\alpha\beta} d\zeta_3 = \frac{h^2}{6} s_{\alpha}, \]

\[ N_{\alpha\beta} = \frac{h}{2} \int_{-1}^{1} \sigma_{\alpha\beta} d\zeta_3 = h n_{\alpha\beta}, \quad M_{\alpha}^* = \frac{h}{2} \int_{-1}^{1} \mu_{\alpha\beta} d\zeta_3 = h m_{\alpha}^*, \]

Here \( M_{11} \) and \( M_{22} \) are the bending moments, \( M_{12} \) and \( M_{21} \) the twisting moments, \( Q_\alpha \) the shear forces, \( Q_\alpha^* \) the transverse shear forces, \( R_{11} \) and \( R_{22} \) the micropolar bending moments, \( R_{12} \) and \( R_{21} \) the micropolar twisting moments, \( S_{\alpha}^* \) the micropolar couple moments, all defined per unit length, \( N_{11} \) and \( N_{22} \) are the bending forces, \( N_{12} \) and \( N_{21} \) the twisting forces, \( M_{\alpha}^* \) the micropolar shear couple-stress resultants.
Then the stress energy of the plate $P$

$$U^S_K = \int_{P_0} \Phi(S) da,$$  \hspace{1cm} (44)

where $P_0$ is the internal domain of the middle plane of the plate $P$.

4.2 The density of the work done over the Cosserat plate boundary

In the following consideration we also assume that the proposed stress, couple stress, and kinematic assumptions are valid for the lateral boundary of the plate $P$ as well.

We evaluate the density of the work over the boundary $\Gamma_u \times [-h/2, h/2]$

$$W_1 = \frac{h}{2} \int_{-1}^{1} \left[ \sigma_n \cdot u + \mu_n \nu \right] d\zeta_3.$$  \hspace{1cm} (45)

Taking into account the stress and couple stress assumptions (23) - (36) and kinematic assumptions (37) - (39) we are able to represent $W_1$ by the following expression:

$$W_1 = S_n U = \tilde{M}_a \Phi_\alpha + \tilde{Q}^* W + \tilde{R}_\alpha \Omega^0_\alpha + \tilde{S}^* \Omega^3 + \tilde{N}_\alpha U_\alpha + \tilde{M}^* \Omega^0_3,$$  \hspace{1cm} (46)

where the sets $S_n$ and $U$ are defined as

$$S_n = \left[ \tilde{M}_a, \tilde{Q}^*, \tilde{R}_\alpha, \tilde{S}^*, \tilde{N}_\alpha, \tilde{M}^* \right],$$

$$U = \left[ \Phi_\alpha, W, \Omega^0_\alpha, \Omega^3, U_\alpha, \Omega^0_3 \right]$$

and

$$\tilde{M}_a = M_{a\beta} n_\beta, \quad \tilde{Q}^* = Q^*_{\beta} n_\beta, \quad \tilde{R}_\alpha = R_{a\beta} n_\beta,$$

$$\tilde{S}^* = S^*_{\beta} n_\beta, \quad \tilde{N}_\alpha = N_{a\beta} n_\beta, \quad \tilde{M}^* = M^*_{\beta} n_\beta,$$

In the above $n_\beta$ is the outward unit normal vector to $\Gamma_u$, and
\[\Psi_{\alpha} = \frac{3}{h} \int_{-1}^{1} \zeta_3 u_\alpha d\zeta_3,\]
\[W = \frac{3}{4} \int_{-1}^{1} (1 - \zeta^2) u_3 d\zeta_3,\]
\[\Omega^0_{\alpha} = \frac{3}{4} \int_{-1}^{1} (1 - \zeta^2) \varphi_\alpha d\zeta_3,\]
\[\Omega_3 = \frac{3}{h} \int_{-1}^{1} \zeta_3 \varphi_3 d\zeta_3,\]
\[U_{\alpha} = \frac{1}{2} \int_{-1}^{1} u_\alpha d\zeta_3,\]
\[\Omega^0_3 = \frac{1}{2} \int_{-1}^{1} \varphi_3 d\zeta_3.\]

Here \(\Psi_{\alpha}\) are the rotations of the middle plane around \(x_\alpha\) axis, \(W\) the vertical deflection of the middle plate, \(\Omega^0_{\alpha}\) the microrotations in the middle plate around \(x_k\) axis, \(U_{\alpha}\) is the in-plane displacements of the middle plane along \(x_\alpha\) axis, \(\Omega_3\) the rate of change of the microrotation \(\varphi_3\) along \(x_3\).

We also obtain the correspondence between the weighted displacement and the microrotations (47) and the kinematic variables by applying (37) and (39) in integration of expressions (47):

\[\Psi_{\alpha} = V_{\alpha}(x_1, x_2), \quad W = w(x_1, x_2),\]
\[\Omega^0_{\alpha} = k_1 \Theta^0_{\alpha}(x_1, x_2), \quad \Omega_3 = \frac{k_2}{h} \Theta_3(x_1, x_2),\]
\[U_{\alpha} = U_{\alpha}(x_1, x_2), \quad \Omega^0_3 = \Theta^0_3(x_1, x_2),\]

where coefficients \(k_1\) and \(k_2\) depend on the variation of microrotations. Under the conditions (39) we have that \(k_1 = \frac{4}{5}\) and \(k_2 = \frac{8}{5}\).

The density of the work over the boundary \(\Gamma_\sigma \times [-h/2, h/2]\)

\[\mathcal{W}_2 = \frac{h}{2} \int_{-1}^{1} (\sigma_{\alpha\beta} u_\alpha + m_{\alpha\beta} \varphi_\alpha) n_\alpha d\zeta_3\]

can be presented in the form

\[\mathcal{W}_2 = S^{\star} U = \Pi_{\alpha\alpha} \Psi_{\alpha} + \Pi_{\alpha3} W + M_{\alpha\alpha} \Omega^0_{\alpha} + M^{\star}_{\alpha3} \Omega_3 + \Sigma_{\alpha,\alpha} U_{\alpha} + \Upsilon_{\alpha3} \Omega^0_3,\]

where

\[M_{\alpha\beta} n_\beta = \Pi_{\alpha\alpha}, \quad R_{\alpha\beta} n_\beta = M_{\alpha\alpha},\]
\[Q^{\star}_{\alpha} n_\alpha = \Pi_{\alpha3}, \quad S^{\star}_{\alpha} n_\alpha = M^{\star}_{\alpha3}.\]
Now \( n_3 \) is the outward unit normal vector to \( \Gamma_\sigma \), and

\[
\Pi_{o\alpha} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \xi_3 \sigma_{o\alpha} d\xi_3, \quad M_{o\alpha} = \frac{h}{2} \int_{-1}^{1} \mu_{o\alpha} d\xi_3,
\]

\[
\Pi_{o3} = \frac{h}{2} \int_{-1}^{1} (\sigma_{o3} - \sigma_0) d\zeta_3, \quad M_{*o3} = \frac{h}{2} \int_{-1}^{1} (\mu_{o3} - \mu_3) d\zeta_3,
\]

\[
\Sigma_{o,\alpha} = \frac{h}{2} \int_{-1}^{1} \sigma_{o\alpha} d\zeta_3, \quad \Upsilon_{o3} = \left( \frac{h}{2} \right)^2 \int_{-1}^{1} \xi_3 (\mu_{o3} - \mu_3) d\zeta_3.
\]

\[\tag{52}\]

We are able to evaluate the work done at the top and bottom of the Cosserat plate by using boundary conditions \((19)\) and \((21)\).

\[
\int_{T \cup B} (\sigma_{o3} u_3 + m_{o3} \varphi_{o3}) n_3 da = \int_{P_0} (pW + vQ_3^0) da.
\]

### 4.3 The Cosserat plate internal work density

Here we define the density of the work done by the stress and couple stress over the Cosserat strain field:

\[
W_3 = \frac{h}{2} \int_{-1}^{1} (\sigma \cdot \gamma + \mu \cdot \chi) d\zeta_3.
\]

Substituting stress and couple stress assumptions \((23) - (36)\) and integrating expression \((53)\) we obtain the following expression:

\[
W_3 = \mathcal{S} \cdot \mathcal{E} = M_{\alpha\beta} e_{\alpha\beta} + Q_{\alpha} \omega_{\alpha} + Q_{3\alpha}^{*} \omega_{\alpha}^{*} + R_{\alpha\beta} \tau_{\alpha\beta} + S_{\alpha}^{*} \tau_{3\alpha} + N_{\alpha\beta} v_{\alpha\beta} + M_{\alpha}^{*} \tau_{3,\alpha}^{0},
\]

\[\tag{54}\]

where \( \mathcal{E} \) is the Cosserat plate strain set of the the weighted averages of strain and torsion tensors

\[
\mathcal{E} = \left[ e_{\alpha\beta}, \omega_{\alpha}, \omega_{\alpha}^{*}, \tau_{3\alpha}, \tau_{\alpha\beta}, v_{\alpha\beta}, \tau_{3,\alpha}^{0} \right].
\]
Here the components of $\mathcal{E}$ are

$$e_{\alpha\beta} = \frac{3}{h} \int_{-1}^{1} \zeta_3 \gamma_{\alpha\beta} d\zeta_3,$$

(55)

$$\omega_\alpha = \frac{3}{4} \int_{-1}^{1} \gamma_{\alpha3} (1 - \zeta^2) d\zeta_3,$$

(56)

$$\omega^*_\alpha = \frac{3}{4} \int_{-1}^{1} \gamma_{3\alpha} (1 - \zeta^2) d\zeta_3,$$

(57)

$$\tau_{3\alpha} = \frac{3}{h} \int_{-1}^{1} \chi_{3\alpha} d\zeta_3,$$

(58)

$$\tau_{\alpha\beta} = \frac{3}{4} \int_{-1}^{1} \chi_{\alpha\beta} (1 - \zeta^2) d\zeta_3,$$

(59)

$$v_{\alpha\beta} = \frac{1}{2} \int_{-1}^{1} \gamma_{\alpha\beta} d\zeta_3,$$

(60)

$$\tau^0_{3\alpha} = \frac{1}{2} \int_{-1}^{1} \chi_{3\alpha} d\zeta_3.$$

(61)

The components of Cosserat plate strain (55)-(61) can also be represented in terms of the components of set $\mathcal{U}$ by the following formulas:

$$e_{\alpha\beta} = \Psi_{\beta,\alpha} + \varepsilon_{3\alpha\beta} \Omega_3,$$

$$\omega_\alpha = \Psi_\alpha + \varepsilon_{3\alpha\beta} \Omega^0_{\beta},$$

$$\omega^*_\alpha = W_\alpha + \varepsilon_{3\alpha\beta} \Omega^0_{\beta},$$

$$\tau_{3\alpha} = \Omega_{3,\alpha},$$

$$\tau_{\alpha\beta} = \Omega^0_{\beta,\alpha},$$

$$v_{\alpha\beta} = U_{\beta,\alpha} + \varepsilon_{3\alpha\beta} \Omega^0_3,$$

$$\tau^0_{3\alpha} = \Omega^0_{3,\alpha}.$$

(62)

We call the relation (62) the Cosserat plate strain-displacement relation.

## 5 Cosserat Plate HPR Principle

It is natural now to reformulate HPR variational principle for the Cosserat plate $P$. Let $\mathcal{A}$ denote the set of all admissible states that satisfy the Cosserat plate strain-displacement relation (62) and let $\Theta$ be a HPR functional on $\mathcal{A}$ defined by

$$\Theta(s) = U^S_K - \int_{\Gamma_0} (S \cdot \mathcal{E} - pW + v \Omega^0_3) da + \int_{\Gamma_s} S_n \cdot (\mathcal{U} - \mathcal{U}_0) ds + \int_{\Gamma_s} S_n \cdot d\mathcal{U} ds,$$

(63)

for every $s = [\mathcal{U}, \mathcal{E}, \mathcal{S}] \in \mathcal{A}$. Then

$$\delta \Theta(s) = 0$$

13
is equivalent to the following plate bending (A) and twisting (B) mixed problems.

A. The bending equilibrium system of equations:

\[ M_{\alpha\beta,\alpha} - Q_\beta = 0, \]  
\[ Q^*_\alpha + p = 0, \]  
\[ R_{\alpha\beta,\alpha} + \varepsilon_{3\beta\gamma} (Q^*_\gamma - Q_\gamma) = 0, \]  
\[ S^*_{\alpha,\alpha} + \varepsilon_{3\beta\gamma} M_{\beta\gamma} = 0, \]

with the resultant traction boundary conditions:

\[ M_{\alpha\beta} n_\beta = \Pi_{\alpha\alpha}, \quad R_{\alpha\beta} n_\beta = M_{\alpha\alpha}, \]  
\[ Q^*_\alpha n_\alpha = \Pi_{33}, \quad S^*_{\alpha,\alpha} = \Upsilon_{33}, \]

at the part \( \Gamma_\sigma \) and the resultant displacement boundary conditions

\[ \Psi_\alpha = \Psi_{\alpha\alpha}, \quad W = W_\alpha, \quad \Omega^0_{33} = \Omega^0_{\alpha\alpha}, \quad \Omega_3 = \Omega_{33}, \]

at the part \( \Gamma_u \).

The constitutive formulas:

\[ \varepsilon_{\alpha\alpha} = \frac{\partial \Phi}{\partial M_{\alpha\alpha}} = \frac{12(\lambda + \mu)}{h^3\mu(3\lambda + 2\mu)} M_{\alpha\alpha} - \frac{6\lambda}{h^3\mu(3\lambda + 2\mu)} M_{\beta\beta} - \frac{3\lambda}{5h\mu(3\lambda + 2\mu)} p, \]  
\[ \varepsilon_{\alpha\beta} = \frac{\partial \Phi}{\partial M_{\alpha\beta}} = \frac{3(\mu_c + \mu)}{h^3\mu_c\mu} M_{\alpha\beta} + \frac{3(\mu_c - \mu)}{h^3\mu_c\mu} M_{\beta\alpha}, \, \alpha \neq \beta \]

\[ \omega_\alpha = \frac{\partial \Phi}{\partial Q_\alpha} = \frac{3(\mu_c - \mu)}{10h\mu_c\mu} Q^*_\alpha + \frac{3(\mu_c + \mu)}{10h\mu_c\mu} Q_\alpha, \]  
\[ \omega^*_\alpha = \frac{\partial \Phi}{\partial Q^*_\alpha} = \frac{3(\mu_c - \mu)}{10h\mu_c\mu} Q_\alpha + \frac{3(\mu_c + \mu)}{10h\mu_c\mu} Q^*_\alpha, \]

\[ \tau^0_{\alpha\alpha} = \frac{\partial \Phi}{\partial R_{\alpha\alpha}} = \frac{6(\beta + \gamma)}{5h\gamma(3\beta + 2\gamma)} R_{\alpha\alpha} - \frac{3\beta}{5h\gamma(3\beta + 2\gamma)} R_{\beta\beta} - \frac{\beta}{2\gamma(3\beta + 2\gamma)} t, \]

\[ \tau^0_{\alpha\beta} = \frac{\partial \Phi}{\partial R_{\beta\alpha}} = \frac{3(\epsilon - \gamma)}{10h\gamma\epsilon} R_{\alpha\beta} + \frac{3(\gamma + \epsilon)}{10h\gamma\epsilon} R_{\beta\alpha}, \, \alpha \neq \beta \]

\[ \tau_{3\alpha} = \frac{\partial \Phi}{\partial S^*_\alpha} = \frac{3(\gamma + \epsilon)}{h^3\gamma\epsilon} S^*_\alpha. \]
B. The twisting equilibrium system of equations:

\begin{align}
N_{\alpha\beta,\alpha} &= 0, \quad (76) \\
M^*_{\alpha,\alpha} + \varepsilon_{3\beta\gamma} N_{\beta\gamma} + v &= 0, \quad (77)
\end{align}

with the resultant traction boundary conditions at \( \Gamma_\sigma \):

\begin{align}
N_{\alpha\beta} n_\beta &= \Sigma_\alpha, \quad (78) \\
M^*_{\alpha} n_\alpha &= M^*_{\alpha3}, \quad (79)
\end{align}

and the resultant displacement boundary conditions at \( \Gamma_u \):

\begin{align}
U_\alpha &= U_{\alpha o}, \quad \Omega^0_3 = \Omega^0_{\alpha3}. \quad (80)
\end{align}

The constitutive formulas:

\begin{align}
\omega_{\alpha\alpha} &= \frac{\partial \Phi}{\partial N_{\alpha\alpha}} = \frac{\lambda + \mu}{h\mu(3\lambda + 2\mu)} N_{\alpha\alpha} \\
&- \frac{\lambda}{2h\mu(3\lambda + 2\mu)} N_{(\alpha+1)(\alpha+1)} - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_0, \quad (81) \\
\omega_{\alpha\beta} &= \frac{\partial \Phi}{\partial N_{\alpha\beta}} = \frac{\alpha + \mu}{4h\alpha\mu} N_{\alpha\beta} + \frac{\alpha - \mu}{4h\alpha\mu} N_{3\alpha3}, \quad \alpha \neq \beta \quad (82) \\
\tau^0_{3\alpha} &= \frac{\partial \Phi}{\partial M^*_{\alpha}} = \frac{\gamma + \varepsilon}{4\gamma\varepsilon} M^*_{\alpha}. \quad (83)
\end{align}

We also represent the above constitutive relation in the compact form:

\[ \mathcal{E} = \mathcal{K} [\mathcal{S}] = \mathcal{K} \cdot \mathcal{S}, \]

where we call \( \mathcal{K} \) the compliance Cosserat plate tensor.

Proof of the principle. The variation of \( \Theta(s) \)

\[ \delta \Theta(s) = \int_{P_0} \left\{ (\mathcal{K} [\mathcal{S}] - \mathcal{E}) \cdot \delta \mathcal{S} - \mathcal{S} \delta \mathcal{E} + p \delta W + \nu \delta \Omega^0_3 \right\} da + \int_{\Gamma_s} \delta \mathcal{S}_n \cdot (U - U_o) + \mathcal{S}_n \delta \mathcal{U} \right\} ds + \int_{\Gamma_u} \mathcal{S}_o \delta \mathcal{U} ds. \]

We apply Green’s theorem and integration by parts for \( \delta \mathcal{U} \) to the expression:

\begin{align}
\int_{P_0} \mathcal{S} \cdot \delta \mathcal{E} da &= \int_{P_0} \mathcal{S}_o \delta \mathcal{U} ds - \int_{P_0} \left\{ (M_{\alpha\beta,\alpha} - Q_\beta) \delta \Psi_\beta + Q^*_{\alpha,\alpha} \delta W \right. \\
&+ (R_{\alpha\beta,\alpha} + \varepsilon_{33\gamma} (Q^*_\gamma - Q_\gamma) R_{\alpha\beta,\alpha}) \delta \Omega^0_\beta \\
&+ (S^*_{\alpha,\alpha} + \varepsilon_{33\gamma} M_{3\gamma}^* \delta \Omega_3 + N_{\alpha\beta,\alpha} \delta U_\beta \\
&+ (M^*_{\alpha,\alpha} + \varepsilon_{33\gamma} N_{\alpha\gamma}^* \delta \Omega^0_3) da. \]
\end{align}
Then based on the fact that $\delta U$ and $\delta E$ satisfy the Cosserat plate strain-displacement relation (62), we obtain

\[
\delta \Theta(s) = \int_{P_0} \left\{ (K[S] - E) \cdot \delta S - S \delta E \right\} da 
+ \int_{P_0} \left\{ (M_{\alpha\beta,\alpha} - Q_\beta) \delta \Psi_\beta + (Q^*_{\alpha\alpha} + p) \delta W 
+ (R_{\alpha\beta,\alpha} + \varepsilon_{3337} (Q^*_\gamma - Q_\gamma) R_{\alpha\beta,\alpha}) \delta \Omega^0_{\beta} 
+ (S^*_{\alpha\alpha} + \varepsilon_{3337} M_{3\beta}) \delta \Omega_3 + N_{\alpha\beta,\alpha} \delta U_\beta + (M^*_{\alpha\alpha} + \varepsilon_{3337} N_{3\beta} + v) \delta \Omega^0_3 \right\} da 
+ \int_{\Gamma_s} \delta S_n \cdot (U - U_o) ds + \int_{\Gamma_s} (S_o - S_n) \cdot \delta U ds.
\]

If $s$ is a solution of the mixed problem, then

\[
\delta \Theta(s) = 0.
\]

On the other hand, some extensions of the fundamental lemma of calculus of variations [5] together with the fact that $U$ and $E$ satisfy the Cosserat plate strain-displacement relation (62) imply that $S$ is a solution of the A and B mixed problems.

**Remark.** In the case of $T_{33} = \Omega_{3,\alpha} = 0$ we obtain that $S^*_{\alpha} = 0$. and $M$ is symmetric, i.e. $M_{\alpha\beta} = M_{\beta\alpha}$, and the corresponding constitutive relation is

\[
\epsilon_{\alpha\beta} = \frac{\partial \Phi}{\partial M_{\alpha\beta}} = \frac{6}{h^3 \mu} M_{\alpha\beta}
\]

The bending system for this case (64)-(67) is reduced to the following:

\[
M_{\alpha\beta,\alpha} - Q_\beta = 0, \quad \text{ (84)}
\]
\[
Q^*_{\alpha\alpha} + p = 0, \quad \text{ (85)}
\]
\[
R_{\alpha\beta,\alpha} + \varepsilon_{3337} (Q^*_\gamma - Q_\gamma) = 0. \quad \text{ (86)}
\]

We also notice

\[
\omega - \omega^* = \frac{3}{5h \mu} (Q_\alpha - Q^*_\alpha)
\]
\[
\omega + \omega^* = \frac{3}{5h \mu} (Q_\alpha + Q^*_\alpha)
\]

and the case $\mu_c = 0$ is consistent with the requirements

$Q^*_\alpha = Q_\alpha$ and $\Psi_\alpha = W_{\alpha}$

Thus we obtain the equilibrium system in the decoupling form:

\[
M_{\alpha\beta,\alpha} - Q_\beta = 0, \quad \text{ (87)}
\]
\[
Q^*_{\alpha\alpha} + p = 0, \quad \text{ (88)}
\]
\[
R_{\alpha\beta,\alpha} = 0. \quad \text{ (89)}
\]
6 Solution Uniqueness

Here we prove that if there is a solution for the deformation of a Cosserat elastic plate, which satisfies the equilibrium equations (64) - (67), (76) - (77), constitutive (71) - (75), (81) - (83) and kinematics formulas (62) with boundary conditions (68), (69), (79), (78) at $\Gamma_\sigma$ and (70) and (80) at $\Gamma_u$ then this elastic solution must be unique. We also assume that all functions and the plate middle plane region $P_0$ satisfy Green - Gauss theorem requirements.

The proof will be based on contradiction. Let us assume that the solution of the Cosserat plate is not unique in terms of the stresses and strains, i.e. there would be two different solutions of (64) - (67) and (76) - (77), both of which satisfy the same boundary conditions (20) and (22) at $\Gamma_\sigma$ and (70) and (80) at $\Gamma_u$. Due to linearity of the proposed model, the difference between these two different solutions is also a solution of the same system of equations with the following zero boundary conditions:

$$M_{\alpha\beta} n_{\beta} = 0, \quad R_{\alpha\beta} n_{\beta} = 0, \quad (90)$$

$$Q^*_{\alpha} n_{\alpha} = 0, \quad S^*_{\alpha} n_{\alpha} = 0, \quad (91)$$

or

$$\Psi_{\alpha} = 0, \quad W = 0, \quad V_{\beta} = 0, \quad \Omega_0^0 = 0, \quad \Omega_3 = 0, \quad U_{\alpha} = 0. \quad (92)$$

It can be shown that for zero loads, the internal work $U$ can be expressed by applying integration by parts as follows:

$$U = \int_{P_0} S \cdot \varepsilon da = \int_{P_0} \left\{ (M_{\alpha\beta} \Psi_{\alpha} + Q^*_{\beta} W + R_{\alpha\beta} \Omega_0^0 + S^*_{\beta} \Omega_3 + N_{\alpha\beta} U_{\alpha} + M^*_{\beta} \Omega_3)_{\beta} - (M_{\alpha\beta,\alpha} - Q_{\beta}) \Psi_{\beta} - Q^*_{\alpha,\alpha} W - (R_{\alpha\beta,\gamma} + \varepsilon_{3\beta\gamma} (Q^*_{\gamma} - Q_{\gamma}) R_{\alpha\beta,\alpha}) \Omega_0^0 - (S^*_{\alpha,\alpha} + \varepsilon_{3\alpha\gamma} M_{\beta\gamma}) \Omega_3 - N_{\alpha\beta,\alpha} U_{\beta} - (M^*_{\alpha,\alpha} + \varepsilon_{3\beta\gamma} N_{\beta\gamma}) \Omega_3^0 \right\} da. \quad (93)$$

Taking into account Green’s theorem, the equilibrium equations (64) - (67) and (76) - (77), expression (93) is reduced to the following line integral:

$$U = \oint_{\Gamma} S_n ds = \oint_{\Gamma} (\tilde{M}_{\alpha} \Psi_{\alpha} + \tilde{Q}^* W + \tilde{R}_{\alpha} \Omega_0^0 + \tilde{S}^* \Omega_3 + \tilde{N}_{\alpha} U_{\alpha} + \tilde{M}^* \Omega_3^0) ds = 0, \quad (94)$$

which vanishes because of the zero boundary conditions (90) - (92).

Using the constitutive equation (71) - (75), (81) - (83) in a reversible form, the positive definite quadratic form strain energy density (41) can be represented in terms of the Cosserat plate strain set $\varepsilon$, which components in this case should be zeros. Then from the Cosserat plate strain-displacement relation (62) we obtain the system:
\[ U_{1,1} = 0, \quad U_{2,2} = 0, \quad U_{2,1} - \Omega_3^0 = 0, \quad U_{1,2} + \Omega_3^0 = 0, \]
\[ W_{1} + \Omega_2^0 = 0, \quad W_{2} - \Omega_1^0 = 0, \quad \Psi_{1,1} = 0, \quad \Psi_{2,2} = 0, \quad \Omega_{3,1} = 0, \quad \Omega_{3,2} = 0, \]
\[ \Psi_{2,1} - \Omega_3^0 = 0, \quad \Psi_{1,2} + \Omega_3^0 = 0, \quad \Psi_1 - \Omega_2^0 = 0, \quad \Psi_2 + \Omega_1^0 = 0, \]
\[ \Omega_{1,1}^0 = 0, \quad \Omega_{2,2}^0 = 0, \quad \Omega_{1,2}^0 = 0, \quad \Omega_{2,1}^0 = 0, \quad \Omega_{3,1}^0 = 0, \quad \Omega_{3,2}^0 = 0, \]

which has the following solution:
\[ U_\alpha = \varepsilon_3 \beta \alpha \Omega_3^0 x_\beta + U_\alpha^0, \quad W = \varepsilon_3 \beta \alpha \Omega_3^0 x_\beta + W^0, \quad \Psi_\alpha = \varepsilon_3 \beta \alpha \Omega_3^0, \quad \Omega_3 = 0, \]

where constant parameters \( U_\alpha^0, W^0 \) are the rigid translations of the middle plane, \( \Omega_3^0 \) the component of the rigid rotation of the plane around \( x_3 \) axis, and \( \Omega_3 \) the slope of the rigid rotation around \( x_3 \). Thus the difference between any two deformations and microrotations of the plate, having the same boundary conditions, represents changes of the plate as a rigid body.

### 7 Conclusion

We generalized Hellinger-Prange-Reissner (HPR) principle in order to derive the new equilibrium equations in the middle plane and constitutive relationships for the plate. The polynomial approximations of the variation of couple stress and micropolar rotations in the thickness direction in order higher than one allowed us, based on the generalized HPR principle, to project Cosserat 3D Elasticity equilibrium equations into the new form of equilibrium equations and the constitutive relations in the middle plane of the plate.

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