EQUIVARIANT SYMPLECTIC HOMOLOGY AND MULTIPLE CLOSED REEB ORBITS

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Abstract. We study the existence of multiple closed Reeb orbits on some contact manifolds by means of $S^1$-equivariant symplectic homology and the index iteration formula. It is proved that a certain class of contact manifolds which admit displaceable exact contact embeddings, a certain class of prequantization bundles, and Brieskorn spheres have multiple closed Reeb orbits.

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1. Introduction

After Weinstein’s famous conjecture [Wei79], the existence problem of a closed Reeb orbit has been extensively studied. It is a natural question to ask the number of (simple) closed Reeb orbits for contact manifolds which are known to have one closed Reeb orbit. This multiplicity problem has been studied and answered for some classes of contact manifolds as well. We refer to [HWZ98, HWZ03, GHHM12] for tight 3-spheres and to [HT09, CGH12] for general contact 3-manifolds. To the authors knowledge, there are few multiplicity result for general higher dimensional contact manifolds but there are a number of theorems [EL80, BLM85, EH87, LZ02, WHL07, Wan11] for pinched or convex hypersurfaces in $\mathbb{R}^{2n}$.

In the present paper we study the multiplicity problem of closed Reeb orbits for nondegenerate contact manifolds which admit displaceable exact contact embeddings, prequantization bundles, and Brieskorn spheres. Our approach is based on $S^1$-equivariant symplectic homology and the index iteration formula. Although we only treat those three cases, we expect that our method can apply for other contact manifolds for which the formulas of $S^1$-equivariant symplectic homology (or contact homology) are nice in a sense that will be explained below.

An embedding $i : \Sigma \hookrightarrow W$ of a contact manifold $(\Sigma, \xi)$ into a symplectic manifold $(W, \omega)$ is called an exact contact embedding if $i(\Sigma)$ is bounding, $\omega = d\lambda$ for some 1-form $\lambda$, and there exists a contact form $\alpha$ on $(\Sigma, \xi)$ such that $\ker \alpha = \xi$ and $\alpha - \lambda|_{\Sigma}$ is exact. Throughout this paper we identify $i(\Sigma)$ with $\Sigma$. Here by bounding we mean that $\Sigma$ separates $W$ into two connected components of which one is relatively compact. We denote by $W_0$ the relatively compact region. This embedding is said to be displaceable if there exists a perturbation
$F \in C_c^\infty(S^1 \times W)$ such that the associated Hamiltonian diffeomorphism $\phi_F$ defined below displaces $\Sigma$ from itself, i.e. $\phi_F(\Sigma) \cap \Sigma = \emptyset$. A symplectic manifold $(W, \omega)$ is called convex at infinity if there exists an exhaustion $W = \bigcup_k W_k$ of $W$ by compact sets $W_k \subset W_{k+1}$ with smooth boundaries such that $\lambda|_{\partial W_k}$, $k \in \mathbb{N}$ are contact forms. Simply speaking, we require by convex at infinity that $(W, \omega)$ is symplectomorphic to the positive part of the symplectization of a contact manifold at infinity.

The Reeb vector field $R$ on $(\Sigma, \alpha)$ is characterized by $\alpha(R) = 1$ and $i_R d\alpha = 0$. We recall that a closed Reeb orbit is nondegenerate if the linearized Poincaré return map associated to the orbit has no eigenvalue equal to 1. A contact form $\alpha$ on $(\Sigma, \xi)$ is called nondegenerate if every closed Reeb orbit is nondegenerate.

**Theorem A.** Suppose that a contact manifold $(\Sigma, \xi)$ of dimension $2n-1$ admits a displaceable exact contact embedding into $(W, \omega)$ which is convex at infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. If either

(i) $H_*(W_0, \Sigma; \mathbb{Q}) \neq 0$ for some $* \in 2\mathbb{N} - 1$ or,

(ii) $H_*(W_0, \Sigma; \mathbb{Q}) = 0$ for all even degree $* \leq 2n - 4$,

then there are at least two closed Reeb orbits contractible in $W$ for any nondegenerate contact form $\alpha$ on $(\Sigma, \xi)$.

One may ask if there are more than two closed Reeb orbits when both conditions (i) and (ii) are fullfilled. This question does not seem to be easily answered in general. However the Conley-Zehnder index of closed Reeb orbits on 3-dimensional contact manifolds is special enough to answer this question and the precise statement is given below.

The following list of examples meet the condition (ii) in the theorem.

1. $(\Sigma, \xi)$ is a rational homology sphere;
2. $(\Sigma, \xi)$ is a $\pi_1$-injective fillable 5-manifold;
3. $(\Sigma, \xi)$ is a Weinstein fillable 5-manifold;
4. $(\Sigma, \xi)$ is a subcritical Weinstein fillable 7-manifold.

It is worth pointing out that due to [FSvK12, Lemma 3.4]

$$H_*(\Sigma; \mathbb{Q}) \cong H_{*+1}(W_0, \Sigma; \mathbb{Q}) \oplus H_*(W_0; \mathbb{Q})$$

if $\Sigma$ is displaceable in $W$ and in particular $H_{*+1}(\Sigma; \mathbb{Q}) = 0$ implies $H_*(W_0, \Sigma; \mathbb{Q}) = 0$.

**Question.** Does every (nondegenerate) subcritical Weinstein fillable contact manifold has two closed Reeb orbits? More generally, every contact manifold admitting a displaceable exact contact embeddings possesses two closed Reeb orbits?

We expect that the above question will be answered positively. There is no particular reason that the conditions (i) and (ii) in Theorem A are essential. We include some examples in the appendix which do not meet such conditions but have two closed Reeb orbits.

Aforementioned, it turned out that every 3-dimensional contact manifold has two closed Reeb orbits [CGH12]. Moreover if a nondegenerate contact 3-manifold is not a lens space there are at least three closed Reeb orbits [HT09]. In the following we show that if a contact manifold $(\Sigma, \xi)$ in Theorem A is of dimension 3, we have not only two closed Reeb orbits but also $b_3(W_0, \Sigma; \mathbb{Q})$-many closed Reeb orbits.

**Corollary A.** Suppose that a 3-dimensional contact manifold $(\Sigma, \xi)$ admits an exact contact embedding into $(W, \omega)$ which is convex at infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. If $\Sigma$
displaceable in \((W, \omega)\), then for any nondegenerate contact form \(\alpha\),
\[
\#\{\text{closed Reeb orbits contractible in } W\} \geq b_3(W_0, \Sigma; \mathbb{Q}) + 2.
\]
Moreover, \(b_3(W, \Sigma; \mathbb{Q})\)-many simple closed Reeb orbits are of Conley-Zehnder index 2. In particular, if \((W, \omega)\) is subcritical Weinstein,
\[
\#\{\text{closed Reeb orbits contractible in } W\} \geq b_2(\Sigma; \mathbb{Q}) + 2.
\]

**Remark 1.1.** A closed Reeb orbit \(\gamma_0\) of Conley-Zehnder index 3 in Corollary 4.4 and \(b_3(W, \Sigma; \mathbb{Q})\)-many simple closed Reeb orbits \(\{\gamma_1, \ldots, \gamma_{b_3(W_0, \Sigma; \mathbb{Q})}\}\) of Conley-Zehnder index 2 in the above corollary have the following nice property. There exist gradient flow lines of the symplectic action functional which connect such closed Reeb orbits with Morse critical points in \((W_0, \Sigma)\). These gradient flow lines can be used to obtain finite energy planes. This will be discussed in the forthcoming paper [FK14]. For instance for subcritical Weinstein fillable contact 3-manifolds, using such finite energy planes, we are able to prove that if any closed Reeb orbit is linked with such \(\gamma_i\), then the linking number is always positive.

As a matter of fact, the proof of Theorem A heavily relies on the facts that the positive part of \(S^1\)-equivariant symplectic homology is periodic, i.e. \(\dim SH_1^{\alpha_1 + \epsilon}(W) = \dim SH_1^{\alpha_1 + \epsilon}(W)\) for not small \(\epsilon \in \mathbb{N}\) and that the positive part of \(S^1\)-equivariant symplectic homology vanishes for low degrees (condition (ii) in Theorem A guarantees this). In other words, we can find more than one closed Reeb orbits if (the positive part of) the \(S^1\)-equivariant symplectic homology of a fillable contact manifold is *nice* in such a sense. A certain class of prequantization bundles and Brieskorn spheres which are treated below have nice \(S^1\)-equivariant symplectic homologies and thus it is able to find more than one closed Reeb orbit.

Let \((Q, \Omega)\) be a symplectic manifold with an integral symplectic form \(\Omega\), i.e. \([\Omega] \in H^2(Q; \mathbb{Z})\). For each \(k \in \mathbb{N}\), there exists a corresponding *prequantization bundle* \(P\) over \(Q\) with \(c_1(P) = k[\Omega]\). Due to [BW58], such a prequantization bundle \((P, \xi := \ker a_{BW})\) is a contact manifold with a connection 1-form \(\alpha_{BW}\). The following theorem proves the existence of two closed Reeb orbits for a certain class of prequantization bundles which naturally arise from the Donaldson’s construction, see Remark 4.6.

**Theorem B.** Let \((P, \xi)\) be a prequantization bundle over a simply connected integral symplectic manifold \((Q, \Omega)\) with \(c_1(P) = k[\Omega]\) for some \(k \in \mathbb{N}\). Suppose that \([\Omega]\) is primitive in \(H^2(Q; \mathbb{Z})\) and \(c_1(Q) = c[\Omega]\) for some \(|c| > n - 1\) and that \((P, \xi)\) admits an exact contact embedding into \((W, \omega)\) with \(c_1(W)\big|_{\pi_1(W)} = 0\) which is \(\pi_1\)-injective. Then for a nondegenerate contact form \(\alpha\), there are two closed Reeb orbits contractible in \(W\) and hence in \(P\).

More generally, \(S^1\)-orbibundles over symplectic orbifolds provide more examples of contact manifolds. In particular Brieskorn spheres, one of the simplest examples, are our interest for which the positive part of equivariant symplectic homology (contact homology) is already computed in [Ust99]. For \(a = (a_0, \ldots, a_n) \in \mathbb{N}^{n+1}\), we define
\[
V_{\epsilon}(a) = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} \cdots z_n^{a_n} = \epsilon\}
\]
which is singular when \(\epsilon = 0\). Then a 1-form \(\alpha_a = \frac{i}{8} \sum_{j=0}^n a_j (\bar{z}_j dz_j - z_j d\bar{z}_j)\) on \(\Sigma_a = V_0(a) \cap S^{2n+1}\) is a contact form. We call \((\Sigma_a, \xi_a := \ker \alpha_a)\) a *Brieskorn manifold*. When \(n\) is odd and \(a_0 \equiv \pm 1 \mod 8\) and \(a_1 = \cdots = a_n = 2\), \(\Sigma_a\) is diffeomorphic to \(S^{2n-1}\) and called a
Brieskorn sphere. As mentioned, Brieskorn manifolds are generalized example of prequantization bundles. Indeed, all Reeb flows of \((\Sigma_a, \alpha_a)\) are periodic and thus Brieskorn manifolds can be interpreted as principal circle bundles over symplectic orbifolds. Furthermore a Brieskorn manifold is Weinstein fillable and in fact a filling symplectic manifold is \(V_\epsilon\) with \(\epsilon \neq 0\). We refer to [Gei08, Section 7.1] for detailed explanation about Brieskorn manifolds.

**Theorem C.** Brieskorn spheres with nondegenerate contact forms have two closed Reeb orbits.

Except 3-dimensional displaceable case, we only can find two closed Reeb orbits but we do not think that this lower bound is optimal. For instance, it is interesting to ask:

**Question.** Can one find more than two closed Reeb orbits on Brieskorn spheres with nonperiodic contact forms?

1.1. **Idea of the proof.**

We briefly explain the idea of the proof of Theorem A in the easiest case that a contact manifold is a rational homology sphere. Suppose that a \((2n-1)\)-dimensional contact manifold \((\Sigma, \xi)\) admits an exact displaceable contact embedding into \((W, \omega)\) which is convex at infinity and satisfies \(c_1(W)|_{\pi_2(W)} = 0\). Let \(\alpha\) be a corresponding nondegenerate contact form on \((\Sigma, \xi)\). If in addition \(H^*_s(\Sigma; \mathbb{Q}) = H^*_s(S^{2n-1}; \mathbb{Q})\) for all \(s \in \mathbb{Z}\), using the vanishing property of \(SH_{S^1}(W)\) and the Viterbo long exact sequence, we obtain

\[
SH^*_{S^1}(W) = \begin{cases} 
\mathbb{Q} & \ast = n - 1 + 2k, \quad k \in \mathbb{N}, \\
0 & \text{otherwise.}
\end{cases} \tag{1.1}
\]

This computation yields there exists at least one simple closed Reeb orbit \(\gamma\). Assume by contradiction that \((\Sigma, \alpha)\) has precisely one closed Reeb orbit.

**Case 1.** \(\mu_{CZ}(\gamma) \geq n + 1\).

In this case, \(\mu_{CZ}(\gamma^k) + 2 \leq \mu_{CZ}(\gamma^{k+1})\) for all \(k \in \mathbb{N}\). Thus \(\mu_{CZ}(\gamma) = n + 1\). But there exists \(k_0 \in \mathbb{N}\) such that \(\mu_{CZ}(\gamma^{k_0}) \neq n - 1 + 2k_0\). This contradicts to the computation (1.1).

**Case 2.** \(\mu_{CZ}(\gamma) < n + 1\).

Since \(SH^*_{\mu_{CZ}(\gamma)}(W) = 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\mu_{CZ}(\gamma^{k_0}) = \mu_{CZ}(\gamma) + 1\) or \(\mu_{CZ}(\gamma^{k_0}) = \mu_{CZ}(\gamma) - 1\). But this implies that \(\gamma^{k_0}\) is a bad orbit and does not contribute to the \(S^1\)-equivariant symplectic homology of \((W, \omega)\). Thus we have a contradiction \(SH^*_{\mu_{CZ}(\gamma)}(W) \supset \mathbb{Q} \langle \gamma \rangle\).

**Remark 1.2.** In the first case we can easily find another closed Reeb orbit in the same way using nonequivariant symplectic homology. However in the second case, i.e. \(\mu_{CZ}(\gamma) < n + 1\), the Conley-Zehnder index behaves badly under iteration, for instance \(\mu_{CZ}(\gamma^{k+1}) < \mu_{CZ}(\gamma^k)\) can occur for some \(k \in \mathbb{N}\). Thus it seems difficult to the author to find another closed Reeb orbit using nonequivariant symplectic homology.

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2. $S^1$-equivariant symplectic homology

2.1. Borel type construction.

$S^1$-equivariant symplectic homology theory was first introduced in [Vit99]. Recently $S^1$-equivariant symplectic homology theory was rigorously studied and written up in [BO09b, BO12b]. In the present paper following [Vit99, BO09a, BO09b] we use the Borel type construction of $S^1$-equivariant (Morse-Bott) symplectic homology and refer to [BO12b] for other constructions, their equivalences, and applications.

Let $(\Sigma, \xi)$ be a contact manifold which admits an exact contact embedding into a symplectic manifold $(W, d\lambda)$ being convex at infinity. Suppose that a corresponding contact form $\alpha$ is nondegenerate. We denote by $W$ of $(\Sigma, \xi)$ forms a discrete subset $\text{Spec}(\Sigma, \alpha)$ in $\mathbb{R}_+ := (0, \infty)$. We define a family of $S^1$-invariant admissible Hamiltonians $K_\tau \in C^\infty(\hat{W} \times S^{2N+1})$, $\tau \in \mathbb{R}_+ \setminus \text{Spec}(\Sigma, \alpha)$ to have the following properties:

(i) $K_\tau(x, z) = H_\tau(x) + h(z)$ for $(x, z) \in \hat{W} \times S^{2N+1}$;
(ii) $h \in C^\infty(S^{2N+1})$ is invariant under the $S^1$-action on $S^{2N+1}$;
(iii) On $W$, $H_\tau < 0$ and is a $C^2$-small Morse function;
(iv) On $\Sigma \times [1, \infty)$, $H_\tau(x) = h_\tau(r)$ for some strictly increasing function $h_\tau : [1, \infty) \to \mathbb{R}_+$ satisfying $h_\tau''(r) > 0$ on $(1, r_0)$ for some $r_0 > 0$;
(v) $h_\tau(r) = \tau r - \tau$ on $\Sigma \times (r_0, \infty)$.

With a family of admissible Hamiltonians $K_\tau \in C^\infty(\hat{W} \times S^{2N+1})$, we define a family of action functionals $A^N_{K_\tau} : \mathcal{L}^N_\hat{W} \times S^{2N+1} \to \mathbb{R}$, where $\mathcal{L}^N_\hat{W}$ denotes the free loop space of $\hat{W}$, by

$$A^N_{K_\tau}(v, z) := -\int_{S^1} v^*\hat{\lambda} - \int_{S^1} K_\tau(v, z)dt.$$  

We note that this action functional is $S^1$-invariant with respect to the diagonal action of $S^1$ on $\mathcal{L}^N_\hat{W} \times S^{2N+1}$, 

$$\theta \cdot (v(t), z) := (v(t - \theta), e^{2\pi i \theta}z), \quad t \in S^1, \quad z \in S^{2N+1} \subset \mathbb{C}^{N+1}.$$
That is \( \mathcal{A}^N_{K_r}(\theta v, \theta \lambda) = \mathcal{A}^N_{K_r}(v, \lambda) \), and thus the critical points set \( \text{Crit} \mathcal{A}^N_{K_r} \) is \( S^1 \)-invariant as well. Here \((v, z) \in \text{Crit} \mathcal{A}^N_{K_r} \) if and only if
\[
\begin{align*}
\partial_t v - X_{H_r}(v) &= 0, \\
d_z f(z) &= 0.
\end{align*}
\]

We denote an \( S^1 \)-family of critical points containing \((v, z)\) by
\[
S_{(v, z)} := \{ \theta \cdot (v, z) \mid (v, z) \in \text{Crit} \mathcal{A}^N_{K_r}, \theta \in S^1 \}.
\]

There are two types of critical points of \( \mathcal{A}^N_{K_r} \):

1) \((v, z)\) where \( z \in \text{Crit} f \) and where \( v \equiv x \in W_0 \) is a critical point of the Morse function \( H_r|_W \);

2) \((v, z)\) where \( z \in \text{Crit} f \) and where \( v \in \mathcal{L}_{\hat{W}} \), lying on levels \( \Sigma \times \{r\} \), \( r \in (1, r_0) \) is a solution of
\[
\partial_t v = -h'_r(\pi \circ v) R(v).
\]

Here \( \pi : \Sigma \times [1, \infty) \to [1, \infty) \) is the projection to the second factor.

The second type solutions correspond to closed Reeb orbits with period \( h'_r(\pi \circ v) \in (0, \tau) \). They are transversally nondegenerate (see [BO09a, Lemma 3.3]), i.e.
\[
\ker [d\varphi_{-h'_r(\pi \circ v)}(v(0)) - \mathbbm{1}_{T\pi(0)\hat{W}}] = \langle \partial_t v(0) \rangle.
\]

Suppose that \( c_1(W)|_{\pi_2(W)} = 0 \). Then we are able to associate the parametrized index function \( \mu \) to critical points of \( \mathcal{A}^N_{K_r} \)
\[
\mu : \text{Crit} \mathcal{A}^N_{K_r} \to \mathbb{Z},
\]
and due to the splitting property
\[
\mu(v, z) = \mu_{CZ}(v) + \text{ind}_f(z)
\]
where \( \mu_{CZ} \) and \( \text{ind}_f \) stand for the Conley-Zehnder index and the Morse index respectively, see [BO12a]. In particular if \((v, z) \in \text{Crit} \mathcal{A}^N_{K_r} \) is of the first type, i.e. \( v = x \in W_0 \),
\[
\mu(v, z) = \mu_{CZ}(v) + \text{ind}_f(z) = \text{ind}_{-H_r|_W}(x) - \frac{\dim W}{2} + \text{ind}_f(z).
\]

We also define a family of \( S^1 \)-invariant admissible almost complex structures \( J = (J^t_\lambda) \), \( \lambda \in S^{2N+1} \), \( t \in S^1 \) such that \( J^t_\lambda \) is an admissible almost complex structure on \( \hat{W} \) and is \( S^1 \)-invariant, i.e. \( J^t_\theta \lambda = J^{t-\theta}_\theta \) for \( \theta \in S^1 \). Together with a Riemannian metric \( g \) on \( S^{2N+1} \) invariant under the \( S^1 \)-action, a metric on \( \mathcal{L}_{\hat{W}} \times S^{2N+1} \) is defined by
\[
m_{(v, z)}((\xi_1, \xi_2), (\xi_3, \xi_4)) := \int_{S^1} \omega(\xi_1, J^t_\lambda \xi_2) dt + g(\xi_3, \xi_4), \quad (\xi_i, \xi_i) \in T_v \mathcal{L}_{\hat{W}} \times T_v S^{2N+1}.
\]

A gradient flow line \((u, y) : C^\infty(\mathbb{R} \times S^1, \hat{W}) \times C^\infty(\mathbb{R}, S^{2N+1}) \) of \( \mathcal{A}^N_{K_r} \) with respect to the metric \( m \) is a solution of
\[
\begin{align*}
\partial_s u + J^t_{y(s)}(\partial_t u - X_{H_r}(u)) &= 0, \\
\partial_s y - \nabla_g f(y) &= 0.
\end{align*}
\]
We denote by $\widehat{\mathcal{M}}(S_{(v_-,z_-)}, S_{(v_+,z_+)})$ the moduli space of gradient flow lines from $(v_-, z_-)$ to $(v_+, z_+)$, i.e.

$$\widehat{\mathcal{M}}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) = \widehat{\mathcal{M}}(S_{(v_-,z_-)}, S_{(v_+,z_+)}; K_\tau, J, g)$$

$$:= \left\{ (u,y) \mid (u,y) \text{ solves } (2.2) \text{ and } \lim_{s \to \pm \infty} (u,y)(s) \in S_{(v_\pm,z_\pm)} \right\}.$$ 

We divide out the $\mathbb{R}$-action on $\widehat{\mathcal{M}}(S_{(v_-,z_-)}, S_{(v_+,z_+)})$ defined by shifting gradient flow lines in the $s$-variable. Then we have the moduli space of unparametrized gradient flow lines denoted by

$$\mathcal{M}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) := \widehat{\mathcal{M}}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) / \mathbb{R}$$

We note that solutions of (2.2) is $S^1$-equivariant, that is if $(u, y)$ solves (2.2), then so does $\theta \cdot (u, y)$, and this $S^1$-action freely acts on the moduli space $\mathcal{M}(S_{(v_-,z_-)}, S_{(v_+,z_+)})$. Denote the quotient by

$$\mathcal{M}_{S^1}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) := \mathcal{M}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) / S^1.$$ 

It turned out that this moduli space is a smooth manifold of dimension

$$\dim \mathcal{M}_{S^1}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) = \mu(v_-, z_-) - \mu(v_+, z_+) - 1$$

for a generic $J$. For the detailed transversality analysis we refer to [BO10]. We define the $S^1$-equivariant chain group $SC_{*}^{S^1,N}(K_\tau)$ by the $\mathbb{Q}$-vector space generated by $S^1$-families of critical points of $A^N_{K_\tau}$ of $\mu$-index $* \in \mathbb{Z}$.

$$SC_{*}^{S^1,N}(K_\tau) = \bigoplus_{S_{(v,z)} \subset \text{Crit} A^N_{K_\tau}} \mathbb{Q}(S_{(v,z)}).$$

The boundary operator $\partial^{S^1} : SC_{*}^{S^1,N}(K_\tau) \to SC_{*+1}^{S^1,N}(K_\tau)$ is defined by

$$\partial^{S^1}(S_{(v_-,z_-)}) = \sum_{S_{(v_+,z_+)} \subset \text{Crit} A^N_{K_\tau}, \mu(v_-,-) - \mu(v_+,z_+) = 1} \# \mathcal{M}_{S^1}(S_{(v_-,z_-)}, S_{(v_+,z_+)}) S_{(v_+,z_+)}$$

where by $\#$ we mean a signed (via the coherent orientations) count of the number of the finite set $\mathcal{M}_{S^1}(S_{(v_-,z_-)}, S_{(v_+,z_+)})$. Then $\partial^{S^1} \circ \partial^{S^1} = 0$ and thus we are able to define

$$HF_{*}^{S^1,N}(K_\tau) = H_{*}(SC_{*}^{S^1,N}(K_\tau), \partial^{S^1}).$$

Taking direct limits, the $S^1$-equivariant symplectic homology of $(W, \omega)$ is defined by

$$SH_{*}^{S^1}(W) := \lim_{N \to \infty} \lim_{\tau \to \infty} HF_{*}^{S^1,N}(K_\tau).$$

Here the direct limit of $N$ is taken with respect to the embedding $S^{2N-1} \hookrightarrow S^{2N+1}$. The resulting homology depends only on $(W_0, \omega)$. In order to define the negative/positive part of $S^1$-equivariant symplectic homology, we consider

$$SC_{*}^{S^1,-N}(K_\tau) = \bigoplus_{S_{(v,z)} \subset \text{Crit} A^N_{K_\tau}} \mathbb{Q}(S_{(v,z)}) \quad SC_{*}^{S^1,+N}(K_\tau) = SC_{*}^{S^1,N}(K_\tau) / SC_{*}^{S^1,-N}(K_\tau)$$

where $\epsilon < \min \text{Spec}(\Sigma, \alpha)$. That is, $SC_{*}^{S^1,-N}$ resp. $SC_{*}^{S^1,+N}$ is generated by type 1 resp. type 2) critical points of $A^N_{K_\tau}$. Since the action values decrease along gradient flow lines, there
exist associated boundary operators $\partial_{\pm}^{S^1}$ induced by $\partial^{S^1}$, and hence we are able to define $SH_*^{S^1,\pm}(W)$ the negative/positive part of $S^1$-equivariant symplectic homology of $(W, \omega)$.

2.2. Morse-Bott spectral sequence.

This subsection is devoted to observe that bad orbits do not contribute to $S^1$-equivariant symplectic homology which is certainly expected to be true in $S^1$-equivariant theory. To see this feature we use a Morse-Bott spectral sequence. We refer to [Fuk96] for detailed explanation about the Morse-Bott spectral sequence. We should mention that this approach was used by [FSvK12] to study the non-existence of a displaceable exact contact embedding of Brieskorn manifolds.

There is a Morse-Bott spectral sequence which converges to $SH_*^{S^1,\pm}(W)$ whose first page is given by

$$E^1_{j,i} = \bigoplus_{\gamma \in P} H_j(\gamma \times_{S^1} ES^1; O_\gamma)$$

where $O_\gamma$ is a orientation rational bundle of $\gamma$. We note that if $\gamma$ is a $k$-fold cover of a simple Reeb orbit, $\gamma \times_{S^1} ES^1$ is the infinite dimensional lens space $B\mathbb{Z}_k$. We recall that parities of Conley-Zehnder indices of all even/odd multiple covers of a simple closed Reeb orbits are the same, i.e.

$$\mu_{CZ}(\gamma^{2k}) \equiv \mu_{CZ}(\gamma^{2\ell}), \quad \mu_{CZ}(\gamma^{2k+1}) \equiv \mu_{CZ}(\gamma^{2\ell+1}) \mod 2, \ k, \ell \in \mathbb{N}.$$ 

See [Vit89, Ust99] for instance. A closed Reeb orbit $\gamma$ is called bad if $\gamma = \gamma_0^k$ for a simple Reeb orbit $\gamma_0$ and some $k \in \mathbb{N}$ (if fact, $k \in 2\mathbb{N}$) and the parity of $\mu_{CZ}(\gamma)$ disagrees with the parity of $\mu_{CZ}(\gamma_0)$. A closed Reeb orbit which is not bad is called good. If $\gamma$ is a good orbit, the twist bundle $O_\gamma$ is trivial and $H_j(B\mathbb{Z}_k; \mathbb{Q})$ vanishes except degree zero. If $\gamma$ is a bad orbit, $O_\gamma$ is the orientation bundle of $B\mathbb{Z}_k$ and $H_j(B\mathbb{Z}_k; O_\gamma)$ vanishes for every degree, see [Vit89]. Therefore only good closed Reeb orbits contribute to the first page of the Morse-Bott spectral sequence and thus to the positive part of $S^1$-equivariant symplectic homology as well.

2.3. Resonance identity.

Following [vK05] we define the mean Euler characteristic by

$$\chi_m(W) := \lim_{n \to \infty} \frac{1}{N} \sum_{\ell=-N}^{N} (-1)^\ell \dim SH_\ell^{S^1,\pm}(W)$$

From the observation of the previous subsection we know the first page of the Morse-Bott spectral sequence converging to $SH^{S^1,\pm}(W)$ is given by

$$E^1_{j,i} = \bigoplus_{\gamma \in \mathcal{G}} \mathbb{Q}$$

where $\mathcal{G}$ is the set of good closed Reeb orbits contractible in $W$. Since the mean Euler characteristic of $E^1$ is the same as that of $SH^{S^1,\pm}(W)$, we have

$$\chi_m(W) = \lim_{n \to \infty} \frac{1}{N} \sum_{\gamma \in \mathcal{G}_N} (-1)^{\mu_{CZ}(\gamma)}$$
where $\mathcal{G}_N$ is the set of good closed Reeb orbits of Conley-Zehnder indices in $[-N, N]$. Let $\Delta(\gamma)$ be the mean Conley-Zehnder index of $\gamma$ which will be explained in the next section. From $|\mu_{\text{CZ}}(\gamma^k) - k\Delta(\gamma)| < n - 1$, see [SZ92], we have

$$k\Delta(\gamma) - (n - 1) < \mu_{\text{CZ}}(\gamma^k) < k\Delta(\gamma) + (n - 1),$$

$$\Delta(\gamma) = \lim_{k \to \infty} \frac{\mu_{\text{CZ}}(\gamma^k)}{k}.$$ 

Thus there exist constants $C_1, C_2 \in [-n - 1, n - 1]$ such that $\mu_{\text{CZ}}(\gamma^k) \in [-N, N]$ if and only if

$$\max \left\{ 0, \frac{-N + C_1}{\Delta(\gamma)} \right\} < k < \frac{N + C_2}{\Delta(\gamma)}.$$  

(2.3)

We abbreviate by $\mathfrak{S}_s$ the set of simple closed Reeb orbits contractible in $W$ whose multiple covers are all good and by $\mathfrak{B}_s$ the set of simple closed Reeb orbits contractible in $W$ whose even multiple covers are bad. Then (2.3) implies the following proposition. This idea is basically identical to [GK10].

**Proposition 2.1.** Let $(W, \omega)$ be as above. Then we have

$$\chi_m(W) = \sum_{\gamma \in \mathfrak{G}_s} \frac{(-1)^{\mu_{\text{CZ}}(\gamma)}}{\Delta(\gamma)} + \sum_{\gamma \in \mathfrak{B}_s} \frac{(-1)^{\mu_{\text{CZ}}(\gamma)}}{2\Delta(\gamma)}.$$  

(2.4)

3. **Index Iteration Formula**

In the present section, we first recall the Conley-Zehnder index of a closed Reeb orbit and then briefly explain how the Conley-Zehnder index varies under iteration. For detailed explanation we refer to Long’s book [Lon02], see also [CZ84, SZ92, Sal99, Gut12]. For the sake of compatibility, we will adopt the notation and terminology of [Lon02].

Let $\text{Sp}(2n)$ be the space of $2n \times 2n$ symplectic matrices and $\text{Sp}(2n)^*$ be a subset which consists of nondegenerate elements, i.e.

$$\text{Sp}(2n)^* := \{ M \in \text{Sp}(2n) \mid \det(M - I_{2n}) \neq 0 \}.$$ 

We observe that $\text{Sp}(2n)^* = \text{Sp}(2n)^* \cup \text{Sp}(2n)^-$ where

$$\text{Sp}(2n)^\pm := \{ M \in \text{Sp}(2n) \mid \pm \det(M - I_{2n}) > 0 \}.$$ 

An element $M \in \text{Sp}(2n)$ is called elliptic if the spectrum $\sigma(M)$ is contained in the unit circle $U := \{ z \in \mathbb{C} \mid |z| = 1 \}$. Since we are interested in the nondegenerate case, i.e. $M \in \text{Sp}(2n)^*$, $\sigma(M) \subset U \setminus \{1\}$. The elliptic height of $M$ is defined by the total algebraic multiplicity of all eigenvalues of $M$ in $U$ and denoted by $e(M)$. On the other hand if $\sigma(M) \cap U = \emptyset$, i.e. $e(M) = 0$, $M$ is called hyperbolic.

We abbreviate

$$\mathcal{P}(2n, \tau)^* := \{ \Psi : [0, \tau] \to \text{Sp}(2n)^* \mid \Psi(0) = I_{2n} \}.$$ 

For $\Psi \in \mathcal{P}(2n, \tau)^*$, we join $\Psi(\tau) \in \text{Sp}(2n)^\pm$ to

$$-I_{2n}, \text{ or } \text{diag}(2, 1/2, -1, \ldots, -1)$$

by a path $\psi : [0, 1] \to \text{Sp}(2n)^*$. We recall that there exists a continuous map

$$\rho : \text{Sp}(2n) \to S^1$$
which is uniquely characterized by the naturality, the determinant, and the normalization properties. For any path \( r : [0, c] \rightarrow \text{Sp}(2n) \), we choose a function \( \alpha_r : [0, c] \rightarrow \mathbb{R} \) such that \( \rho(r(t)) = e^{i\alpha_r(t)} \). Then the Maslov-type index for a path \( \Psi \in \mathcal{P}(2n, \tau)^* \) is defined by

\[
\mu(\Psi) := \frac{\alpha_\Psi(\tau) - \alpha_\Psi(0)}{\pi} + \frac{\alpha_\psi(1) - \alpha_\psi(0)}{\pi} \in \mathbb{Z}.
\]

In particular, we denote

\[
\Delta(\Psi) := \frac{\alpha_\Psi(\tau) - \alpha_\Psi(0)}{\pi} \in \mathbb{R}.
\]

and call the mean index of \( \gamma \). We remark that since \( \text{Sp}(2n)^* \) is simply connected, both \( \mu \) and \( \Delta \) are independent of the choice of a path \( \psi \).

Now we associate this Maslov-type index to each closed Reeb orbit contractible in a symplectic filling. Let \( \gamma \) be a \( \tau \)-period closed Reeb orbit on \((\Sigma, \alpha, \xi)\) contractible in \((W, \omega)\). We take a filling disk \( \bar{\gamma} : D^2 \rightarrow W \) such that \( \bar{\gamma}|_{\partial D^2} = \gamma \). Then a symplectic trivialization \( \Phi : \bar{\gamma}^*\xi \rightarrow D^2 \times \mathbb{R}^{2n-2} \) and the linearized flow \( T\phi_R(\gamma(0))|_{\xi} \) along \( \gamma \) induces a path of symplectic matrices

\[
\Psi_\gamma(t) := \Phi(\gamma(t)) \circ T\phi_R(\gamma(0))|_{\xi} \circ \Phi^{-1}(\gamma(0)) : [0, \tau] \rightarrow \text{Sp}(2n - 2).
\]

If \( \gamma \) is nondegenerate, \( \Psi_\gamma \in \mathcal{P}(2n - 2, \tau)^* \) and we are able to define the Conley-Zehnder index of \( \gamma \) by

\[
\mu_{\text{CZ}}(\gamma) := \mu(\Psi_\gamma).
\]

The mean Conley-Zehnder index \( \Delta(\gamma) \) is also defined in a trivial way. In order to prove our main results we need to study the Conley-Zehnder indices of \( \gamma^k, k \in \mathbb{N} \) where

\[
\gamma^k : [0, k\tau] \rightarrow \Sigma, \quad \gamma^k(t) := \gamma(t - j\tau) \quad \text{for} \quad t \in [j\tau, (j+1)\tau], \quad 1 \leq j \leq k - 1.
\]

We define the \( k \)-th iteration \( \Psi^k \in \mathcal{P}(2n - 2, k\tau)^* \) of \( \Psi \in \mathcal{P}(2n - 2, \tau)^* \) by

\[
\Psi^k(t) := \Psi(t - j\tau)\Psi(\tau)^{j}, \quad t \in [j\tau, (j+1)\tau], \quad 1 \leq j \leq k - 1
\]

so that \( \Psi_{\gamma^k} = \Psi^k \) and \( \mu_{\text{CZ}}(\gamma^k) = \mu(\Psi^k) \).

Let \( M_1 \) resp. \( M_2 \) be \( 2i \times 2i \) resp. \( 2j \times 2j \) matrix of the square block form as below.

\[
M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.
\]

The \( \circ \)-product of \( M_1 \) and \( M_2 \) is a \( 2(i+j) \times 2(i+j) \) matrix defined by

\[
M_1 \circ M_2 := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
\]
The following symplectic matrices are called basic normal forms.

- \( D(\pm 2) = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 1/2 \end{pmatrix} \),
- \( N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \), \( \lambda = \pm 1, b = \pm 1, 0 \),
- \( R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \), \( \theta \in (0, \pi) \cup (\pi, 2\pi) \),
- \( N_2(\theta, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix} \) for \( B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \), \( \theta \in (0, \pi) \cup (\pi, 2\pi), b_2 \neq b_3 \).

We note that \( D(\pm 2) \) are basic normal forms for eigenvalues outside \( U \) and \( N_1, R, \) and \( N_2 \) are basic normal forms for eigenvalues in \( U \). Therefore \( e(D) = 0 \), \( e(N_1) = e(R) = 2 \), and \( e(N_2) = 4 \).

The homotopy set \( \Omega(M) \) of \( M \in \text{Sp}(2n) \) is defined by
\[
\Omega(M) = \{ M' \in \text{Sp}(2n) \mid \sigma(M') \cap U = \sigma(M) \cap U, \nu_\lambda(M') = \nu_\lambda(M) \text{ for all } \lambda \in \sigma(M) \cap U \}
\]
where
\[
\nu_\lambda(M) := \dim C_\lambda M - \lambda I_{2n}.
\]
We denote by \( \Omega^0(M) \) the path connected component of \( \Omega(M) \) containing \( M \).

**Theorem 3.1.** For \( M \in \text{Sp}(2n) \), there exists a path \( h : [0, 1] \to \Omega^0(M) \) such that
\[
h(0) = M \quad \text{and} \quad h(1) = M_1 \circ \cdots \circ M_k \circ M_0
\]
where \( M_i \)'s, \( i \in \{1, \ldots, k\} \) are basic normal forms for eigenvalues in \( U \) and \( M_0 \) is either \( D(2)^{\ell} \) or \( D(-2) \circ D(2)^{\ell-1} \) for some \( \ell \in \mathbb{N} \).

**Proof.** The proof can be found in [Lon02, Theorem 1.8.10 & Corollary 2.3.8] \( \square \)

Since we are interested in the nondegenerate case, i.e. \( M \in \text{Sp}(2n) \) with \( \nu_1(M^k) = 0 \) for all \( k \in \mathbb{N} \), we can exclude the basic normal form \( N_1(\lambda, b) \) since
\[
\nu_1(N_1(1, b)^k) \geq 1, \quad \text{for some } k \in \mathbb{N}.
\]
Moreover,
\[
\nu_1(R(\theta)^k) = 2 - 2\varphi\left(\frac{k\theta}{2\pi}\right), \quad \nu_1(N_2(\theta, B)^k) = 2 - 2\varphi\left(\frac{k\theta}{2\pi}\right), \quad k \in \mathbb{N},
\]
where \( \varphi(a) = 0 \) if \( a \in \mathbb{Z} \) and \( \varphi(a) = 1 \) if \( a \notin \mathbb{Z} \). Thus \( \theta/2\pi \) should be irrational due to the nondegeneracy condition. Therefore in the case at hand, the endpoint of the path \( h : [0, 1] \to \Omega^0(M) \) in Theorem 3.1 is simply
\[
h(1) = R(\theta_1) \circ \cdots \circ R(\theta_p) \circ N_2(\theta_{p+1}, B_1) \circ \cdots \circ N_2(\theta_{p+q}, B_q) \circ M_0 \quad (3.1)
\]
for \( \theta_i/2\pi \in (0, 1) \setminus \mathbb{Q}, i \in \{1, \ldots, p + q\} \). Now we are ready to state the following theorem due to [Lon00] which will plays a crucial role.
\textbf{Theorem 3.2.} Let $\Psi \in \mathcal{P}(2n, \tau)$ with $\Psi(\tau)^k \in \text{Sp}(2n)^*$ for all $k \in \mathbb{N}$, i.e. $\Psi^k \in \mathcal{P}(2n, k\tau)^*$, and $h : [0, 1] \to \Omega^0(\Psi(\tau))$ such that $h(0) = \Psi(\tau)$ and $h(1)$ is as (3.1). Then the Maslov index of $\Psi^k$ is
\[
\mu(\Psi^k) = \sum_{1 \leq i \leq p} \left( k(P_i - 1) \right) + 2\left[ \frac{k\theta}{2\pi} \right] + 1 + \sum_{1 \leq j \leq q} kW_j + \sum_{1 \leq o \leq \ell} kQ_o
\]
where $P_i$'s are odd integers and $W_j$'s, $Q_o$'s are integers and they satisfy
\[
\sum_{i,j,o} P_i + W_j + Q_o = \mu(\Psi).
\]
Here, $[a] \in \mathbb{Z}$ is the biggest integer number smaller than or equal to $a \in \mathbb{R}$.

\textbf{Proof.} The proof can be found in [Lon00] or [Lon02, Chapter 8]. \hfill \square

\textbf{Example 3.3.} Let $\gamma$ be a simple closed Reeb orbit on a contact manifold of dimension 3 and suppose that all $\gamma^k$'s are nondegenerate. If $\gamma$ is elliptic, $\mu_{CZ}(\gamma) \in 2\mathbb{Z} + 1$ and
\[
\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 1) + 2[k\theta] + 1, \quad \theta \in (0, 1) \setminus \mathbb{Q}.
\]
If $\gamma$ is hyperbolic,
\[
\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).
\]
If $\gamma$ has a negative real Floquet multiplier, $\mu_{CZ}(\gamma)$ is odd. Otherwise, $\gamma$ has a positive real Floquet multiplier and $\mu_{CZ}(\gamma)$ is even.

\textbf{Example 3.4.} Let $\gamma$ be a simple closed Reeb orbit on a contact manifold of dimension 5 and suppose that all $\gamma^k$'s are nondegenerate. If $\gamma$ is elliptic, either $\mu_{CZ}(\gamma) \in 2\mathbb{Z}$ and
\[
\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 2) + 2[k\theta_1] + 2[k\theta_2] + 2, \quad \theta_1, \theta_2 \in (0, 1) \setminus \mathbb{Q}
\]
or
\[
\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).
\]
If $\gamma$ is hyperbolic,
\[
\mu_{CZ}(\gamma^k) = k\mu_{CZ}(\gamma).
\]
If $\gamma$ is neither elliptic nor hyperbolic, i.e. $e(\gamma) = 2$, then
\[
\mu_{CZ}(\gamma^k) = k(\mu_{CZ}(\gamma) - 1) + 2[k\theta] + 1, \quad \theta \in (0, 1) \setminus \mathbb{Q}.
\]

One can see that the Conley Zehnder index cannot decrease (resp. increase) under iteration if $\Sigma$ is 3-dimensional and $\mu_{CZ}(\gamma)$ is positive (resp. negative). Unfortunately this does not remain true for higher dimensions. However if the Conley-Zehnder index of a simple closed Reeb orbit is big or small enough, we still have that property. We recall that our contact manifold $(\Sigma, \alpha)$ is of dimension $2n - 1$.

\textbf{Proposition 3.5.} Let $\gamma$ be a simple closed Reeb orbit with $\mu_{CZ}(\gamma) \geq n - 1$. Then we have
\[
\mu_{CZ}(\gamma^k) \leq \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.
\]

\textbf{Proof.} According to Theorem 3.2, the Conley-Zehnder index of the $k$-fold cover of $\gamma$ is of the following form.
\[
\mu_{CZ}(\gamma^k) = kr + \sum_{i=1}^{j} 2[k\theta_i] + j, \quad r + j = \mu_{CZ}(\gamma) \geq n - 1.
\]
Since $j \in \{0, \ldots, n - 1\}$ and $\theta_i \in (0, 1) \setminus \mathbb{Q}$, $r \geq 0$ and thus the claim follows directly. \hfill \square
**Proposition 3.6.** If a simple closed Reeb orbit $\gamma$ has $\mu_{CZ}(\gamma) = n + 1$, we have

$$\mu_{CZ}(\gamma^k) + 2 \leq \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.$$ 

Moreover there exists $k_0 \in \mathbb{N}$ such that

$$\mu_{CZ}(\gamma^{k_0}) + 2 < \mu_{CZ}(\gamma^{k_0+1}).$$

**Proof.** The first inequality follows from that $r \geq 2$ in the following form again.

$$\mu_{CZ}(\gamma^k) = rk + \sum_{i=1}^j 2[k\theta_i] + j, \quad r + j = n + 1$$

where $1 \leq j \leq n - 1$. If $r \geq 3$, $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 3$ for all $k \in \mathbb{N}$. If $r = 2$, we pick $k_0 \in \mathbb{N}$ satisfying $[k_0\theta] = 1$ so that $\mu_{CZ}(\gamma^{k_0+1}) \geq \mu_{CZ}(\gamma^{k_0}) + 4$. \qed

**Proposition 3.7.** If a simple closed Reeb orbit $\gamma$ has $\mu_{CZ}(\gamma) \leq -n$,

$$\mu_{CZ}(\gamma^k) > \mu_{CZ}(\gamma^{k+1}), \quad k \in \mathbb{N}.$$ 

**Proof.** We note that the Conley-Zehnder index of $\gamma^k$ is

$$\mu_{CZ}(\gamma^k) = rk + \sum_{i=1}^j 2[k\theta_i] + j, \quad j \in \{0, 1, \ldots, n - 1\}$$

and $r + j = \mu_{CZ}(\gamma) \leq -n$. Thus $r \leq -n - j < -2j$ and the claim is proved. \qed

The following lemmas and corollary will be used in proving Theorem C.

**Lemma 3.8.** If for $(\theta_1, \ldots, \theta_j) \in (0, 1)^j$, $1, \theta_1, \ldots, \theta_j$ are rationally independent, then the sequence $(k\theta_1 - [k\theta_1], \ldots, k\theta_j - [k\theta_j])$ is dense in $(0, 1)^j$.

**Proof.** See [BCE07, Lemma 9]. \qed

**Lemma 3.9.** Suppose that $\theta_1, \ldots, \theta_j \in (0, 1)$, $j \geq 2$ are irrational and $\sum_{i=1}^j \theta_i$ is rational. Then there exists at least two $i_1, i_2 \in \{1, \ldots, j\}$ such that $1, \theta_{i_1}, \theta_{i_2}$ are rationally independent.

**Proof.** Assume by contradiction that there exists $p_i, q_i \in \mathbb{Q}$ such that $\theta_i = p_i\theta_1 + q_i$ for all $i \in \{1, \ldots, j\}$. Then we have

$$\sum_{i=1}^j \theta_i = \sum_{i=1}^j p_i\theta_1 + \sum_{i=1}^j q_i.$$ 

This contradict to that $\theta_1$ is irrational and proves the corollary. \qed

**Corollary 3.10.** Under the assumption of Lemma 3.9, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{i=1}^j [k_0\theta_i] + 2 \leq \sum_{i=1}^j [(k_0 + 1)\theta_i].$$

**Proof.** According to Lemma 3.8 and Lemma 3.9, there exists $k_0 \in \mathbb{N}$ and $i_1, i_2 \in \{1, \ldots, j\}$ such that

$$k_0\theta_{i_1} - [k_0\theta_{i_1}] < \theta_{i_1} \quad \text{and} \quad k_0\theta_{i_2} - [k_0\theta_{i_2}] < \theta_{i_2}.$$  

Thus $[k_0\theta_{i_1}] + 1 = [(k_0 + 1)\theta_{i_1}]$, $[k_0\theta_{i_2}] + 1 = [(k_0 + 1)\theta_{i_2}]$ and the claim is proved. \qed
4. Proofs of the main results

4.1. Displaceable case.

This subsection is concerned with a contact manifold \((\Sigma, \xi)\) which admits an exact contact embedding into a symplectic manifold \((W, \omega)\) which is convex at infinity and \(c_1(W)|_{\pi_2(W)} = 0\). We continue to assume that the corresponding contact form \(\alpha\) is nondegenerate.

**Theorem 4.1.** Suppose that \(\Sigma\) is displaceable in \((W, \omega)\). Then the \(S^1\)-equivariant symplectic homology of \((W, \omega)\) vanishes.

The above vanishing theorem can be proved by applying big theorems. Due to [CF09, AF10], displaceability of \(\Sigma\) in \(W\) implies vanishing of the Rabinowitz Floer homology of \((W, \Sigma)\). Then using a long exact sequence involving Rabinowitz Floer homology and Symplectic (co)homology in [CFO10] and a unit in symplectic cohomology, [Rit10] proved that vanishing of Rabinowitz Floer homology implies vanishing of symplectic homology. Since there exists a spectral sequence converging to \(\text{SH}^{S^1}(W)\) with second page given by

\[
E^2_{i,j} \cong \text{SH}_i(W) \otimes \text{H}_j(\mathbb{C}P^\infty; \mathbb{Q}),
\]

see [Vit99, BO12b], \(\text{SH}^{S^1}(W)\) vanishes as well provided that \(\Sigma\) is displaceable in \(W\). We remark that the last argument can be replaced by a different spectral sequence [Sei08, Section 8b]. However recently a direct relation between leafwise intersections and vanishing of \(\text{SH}(W)\) and \(\text{SH}^{S^1}(W)\) was studied in [Kan13] (also in [CO08]) in the case that \((W, \omega)\) is the completion of Liouville domain, i.e. \((W, \omega = \hat{W}, \hat{\omega})\). Therefore we have a direct proof of the theorem in that case and leave the following question.

**Question 4.2.** If \(\Sigma\) is displaceable in \((W, \omega)\), so is in \((\hat{W}, \hat{\omega})\)? or a counter example?

Combining the above theorem with the Viterbo long exact sequence we obtain the following computation which agrees with the contact homology computation [Yau04] in the subcritical Weinstein case.

**Proposition 4.3.** If \(\Sigma\) is displaceable in \((W, \omega)\), we have

\[
\text{SH}^{S^1}(W) \cong \bigoplus_{i+j=*=n-1} H_i(W_0, \Sigma; \mathbb{Q}) \otimes H_j(\mathbb{C}P^\infty; \mathbb{Q}).
\]

**Proof.** The \(S^1\)-equivariant version of the Viterbo long exact sequence is

\[
\cdots \to H_{*+n}^S(W_0, \Sigma; \mathbb{Q}) \to \text{SH}^S_*(W) \to \text{SH}^{S_1}_*(W) \to H_{*+n-1}^S(W_0, \Sigma; \mathbb{Q}) \to \cdots.
\]

According to the above theorem, \(\text{SH}^{S_1}_*(W) \cong H_{*+n-1}^S(W_0, \Sigma; \mathbb{Q})\). Since the \(S^1\)-action on \((W_0, \Sigma)\) is trivial and

\[
H_{*+n-1}^S(W_0, \Sigma; \mathbb{Q}) \cong \bigoplus_{i+j=*=n-1} H_i(W_0, \Sigma; \mathbb{Q}) \otimes H_j(\mathbb{C}P^\infty; \mathbb{Q}).
\]

\(\square\)

**Corollary 4.4.** If \((\Sigma, \alpha)\) is displaceable in \((W, \omega)\), there exists a contractible closed Reeb orbit \(\gamma\) such that \(\mu_{CZ}(\gamma) = n + 1\).

**Proof.** This directly follows from Proposition 4.3. We would like to mention that this result is not new and has been proved in various ways. \(\square\)
A direct consequence of Proposition 2.1 and Proposition 4.3 is:

**Corollary 4.5.** Suppose that \( \Sigma \) is displaceable in \((W, \omega)\). Then,

\[
\sum_{i=1}^{n} b_i(W_0, \Sigma; \mathbb{Q}) \frac{\Delta(\gamma)}{2} = \chi_m(W) = \sum_{\gamma \in \mathcal{G}_s} (-1)^{\mu_{\text{CZ}}(\gamma)} \frac{\Delta(\gamma)}{\Delta(\gamma)} \sum_{\gamma \in \mathcal{G}_s} (-1)^{\mu_{\text{CZ}}(\gamma)} \frac{\Delta(\gamma)}{2\Delta(\gamma)}.
\]

**Proof of Theorem A.**

**Case (i).** Suppose that \( H_{2\ell-1}(W_0, \Sigma; \mathbb{Q}) \neq 0 \) for some \( \ell \in \mathbb{N} \) and that \( \gamma \) in Corollary 4.4 is the only closed Reeb orbit on \((\Sigma, \alpha)\). Let \( \gamma_0 \) be a primitive of \( \gamma \). Applying Proposition 4.3, we have

\[
\begin{align*}
SH_{n+1}^{S^1,+}(W) &\cong \bigoplus_{i=1}^{n} H_{2i}(W_0, \Sigma; \mathbb{Q}), \\
SH_{2\ell-n}^{S^1,+}(W) &\cong \bigoplus_{i=1}^{\ell} H_{2i-1}(W_0, \Sigma; \mathbb{Q}),
\end{align*}
\]

and thus multiples of \( \gamma_0 \) represent nonzero homology classes in \( SH_{n+1}^{S^1,+}(W) \) and \( SH_{2\ell-n}^{S^1,+}(W) \). However the parity of \( n+1 \) and the parity of \( 2\ell-n \) are different. Assume that the parity of \( \mu_{\text{CZ}}(\gamma_0) \) is different from the parity of \( n+1 \). The other case follows in the same manner. Then all multiple covers of \( \gamma_0 \) with Conley-Zehnder index \( n+1 \) are bad orbits and thus do not contribute to \( SH_{n+1}^{S^1,+}(W) \). This contradiction implies the existence of a second orbit geometrically different from \( \gamma_0 \).

**Case (ii).** Suppose that \( H_{2\ell}(W_0, \Sigma; \mathbb{Q}) = 0 \) for all \( 0 \leq \ell \leq n-2 \) and that \( H_{2m-1}(W_0, \Sigma; \mathbb{Q}) = 0 \) for all \( m \in \mathbb{N} \). Assume on the contrary that there exists precisely one simple closed Reeb orbit \( \gamma_0 \) as above. According to Proposition 4.3, we have

\[
\begin{align*}
SH_{n-1}^{S^1,+}(W) &\cong H_{2n-2}(W_0, \Sigma; \mathbb{Q}), \\
SH_{n-1+2j}^{S^1,+}(W) &\cong H_{2n}(W_0, \Sigma; \mathbb{Q}) \oplus H_{2n-2}(W_0, \Sigma; \mathbb{Q}), \quad j \in \mathbb{N}, \\
SH_n^{S^1,+}(W) &\cong \{0\}, \quad \ast \in \mathbb{Z}\setminus\{n-3+2j \mid j \in \mathbb{N}\}.
\end{align*}
\]

**Subcase 1.** If \( \mu_{\text{CZ}}(\gamma_0) \geq n+1 \), \( \mu_{\text{CZ}} \) nondecreases under iteration, see Proposition 3.5, and thus \( \mu_{\text{CZ}}(\gamma_0) = n+1 \). But even in this case, \( \mu_{\text{CZ}}(\gamma_0)^{k} + 2 \leq \mu_{\text{CZ}}(\gamma_0^{k+1}) \) for all \( k \in \mathbb{N} \) and there exists \( k_0 \in \mathbb{N} \), \( \mu_{\text{CZ}}(\gamma_0^{k_0}) + 2 < \mu_{\text{CZ}}(\gamma_0^{k_0+1}) \) due to Proposition 3.6. This implies that there exist \( j \in \mathbb{N} \) such that multiple covers of \( \gamma_0 \) cannot generate \( SH_{n-1+2j}^{S^1,+}(W) \). Thus \( \gamma_0 \) cannot be the only one closed Reeb orbit.

**Subcase 2.** If \( \mu_{\text{CZ}}(\gamma_0) < n-1 \) or \( \mu_{\text{CZ}}(\gamma_0) = n \), there exists \( k \in \mathbb{N} \) such that \( \mu_{\text{CZ}}(\gamma_0^{k}) = \mu_{\text{CZ}}(\gamma_0) + 1 \) or \( \mu_{\text{CZ}}(\gamma_0^{k}) = \mu_{\text{CZ}}(\gamma_0) - 1 \) because of (4.1). But then \( \gamma_0^{k} \) is a bad orbit which does not contribute to \( S^1 \)-equivariant symplectic homology, we need another geometrically distinct closed Reeb orbit in this case as well.

**Subcase 3.** We assume that \( \mu_{\text{CZ}}(\gamma_0) = n-1 \). Due to the index iteration formula,

\[
\mu_{\text{CZ}}(\gamma_0^{r}) = kr + \sum_{i=1}^{j} 2[k\theta_i] + j, \quad r + j = n - 1
\]

for some \( \theta_i \in (0, 1) \setminus \mathbb{Q} \). Since \( 0 \leq j \leq n-1, r \geq 0 \). If \( r = 0 \),

\[
\mu_{\text{CZ}}(\gamma_0^{r}) = \sum_{i=1}^{n-1} 2[k\theta_i] + n - 1, \quad \gamma_0 \in \mathcal{G}_s
\]
and we have the following identity due to Corollary 4.5.
\[
\sum_{i=1}^{n-1} 2\theta_i = \Delta(\gamma_0) = \frac{2}{1 + b_{2n-2}(W_0, \Sigma; \mathbb{Q})}.
\]
(4.2)

However according to the computation (4.1),
\[
\mu_{CZ}(\gamma_0) = b_{2n-2}(W_0, \Sigma; \mathbb{Q}) + 1
\]

since \(\mu_{CZ}(\gamma_0) \leq \mu_{CZ}(\gamma_0^k)\) for all \(k \in \mathbb{N}\) as observed in Proposition 3.5. Therefore there exists \(i \in \{1, \ldots, n-1\}\) such that \([(1 + b_{2n-2}(W_0, \Sigma; \mathbb{Q}))) \theta_i = 1\). But this contradicts to (4.2) and the fact that \(\theta_i \in (0, 1) \setminus \mathbb{Q}\) for all \(1 \leq i \leq n - 1\).

If \(r \geq 1\), then there exists \(k_0 \in \mathbb{N}\) such that
\[
\mu_{CZ}(\gamma_0^{k_0} + 1) \geq \mu_{CZ}(\gamma_0^{k_0}) + 3
\]

which contradicts to (4.1). Hence there exists a closed Reeb orbit geometrically distinct from \(\gamma_0\) and this completes the proof. \(\square\)

Before proving Corollary A, we refer to Example 3.3 for the index iteration formula in the 3-dimensional case.

**Proof of Corollary A.**

According to Proposition 4.3,
\[
\begin{align*}
\text{SH}^{S^{1,+}}_{1}(W) & \cong H_2(W_0, \Sigma; \mathbb{Q}), \\
\text{SH}^{S^{1,+}}_{2k}(W) & \cong H_3(W_0, \Sigma; \mathbb{Q}), \\
\text{SH}^{S^{1,+}}_{2k+1}(W) & \cong H_2(W_0, \Sigma; \mathbb{Q}) \oplus H_4(W_0, \Sigma; \mathbb{Q}),
\end{align*}
\]

(4.3)

for all \(k \in \mathbb{N}\). If we write \(\text{SH}^{S^{1,+}}_{2k}(W) = \mathbb{Q} \langle \gamma_1, \ldots, \gamma_{b_2(W_0, \Sigma; \mathbb{Q})} \rangle\), all \(\gamma_i\)s are simple. Indeed if \(\gamma_i\) is not simple, it has to be a double cover of a simple one of Conley-Zehnder index 1 and thus bad. Therfore,
\[
\mu_{CZ}(\gamma_i^k) = 2k, \quad i \in \{1, \ldots, b_2(W_0, \Sigma; \mathbb{Q})\}, \quad k \in \mathbb{N}.
\]

Since \(\dim \text{SH}^{S^{1,+}}_{3}(W) \geq 1\), there exists another closed Reeb orbit \(v\) with \(\mu_{CZ}(v) = 3\). If \(v\) is simple, there exists another closed Reeb orbit due to Proposition 3.6 to satisfy (4.3). Suppose that \(v\) is a multiple cover of a simple one, say \(v_0\), and that there is no simple closed Reeb orbit except \(v_0\) and \(\gamma_1\)'s. Then \(\mu_{CZ}(v_0) = 1\). If \(v_0\) is hyperbolic, i.e. \(\mu_{CZ}(v_0^k) = k\), only odd multiple covers take into account. Then there exists another simple closed Reeb orbit since \(\dim \text{SH}^{S^{1,+}}_{3}(W) = \dim \text{SH}^{S^{1,+}}_{1}(W) + 1 \geq 2\). Suppose that \(v_0\) is elliptic, i.e. \(\mu_{CZ}(v_0^k) = 2|k\theta| + 1\) for some \(\theta \in (0, 1) \setminus \mathbb{Q}\). Since \(v_0\) generates all odd degrees of \(\text{SH}^{S^{1,+}}_{1}(W)\),
\[
2k + 1 = \mu_{CZ}(v_0^{kb_2(W_0, \Sigma; \mathbb{Q})}) = 2[(kb_2(W_0, \Sigma; \mathbb{Q}) + 1)\theta] + 1, \quad k \in \mathbb{N}.
\]

By dividing both sides by \(k\) and taking a limit \(k \to \infty\), we obtain a contradiction
\[
\theta = \frac{1}{b_2(W_0, \Sigma; \mathbb{Q})} \in \mathbb{Q}.
\]

This completes the proof. \(\square\)
4.2. Prequantization bundles.

Let $(Q, \Omega)$ be a symplectic manifold with an integral symplectic form $\Omega$, i.e. $[\Omega] \in H^2(Q; \mathbb{Z})$. Since the first Chern class classifies isomorphism classes of complex line bundles, we can find a principal $S^1$-bundle $p : P \to Q$ with $c_1(P) = k[\Omega]$ for $k \in \mathbb{N}$. Such a prequantization bundle $P$ carries a connection $1$-form $\alpha_{BW}$ such that the curvature form of $\alpha_{BW}$ is $-2k\pi \Omega$, i.e. $-2k\pi p^*\Omega = d\alpha_{BW}$, see [BW58] or [Gei08, Chapter 7.2]. Therefore a prequantization bundle $(P, \xi := \ker \alpha_{BW})$ is a contact manifold and especially $(P, \alpha_{BW})$ is periodic, i.e. all Reeb flows are periodic. Suppose that $c_1(Q) = c[\Omega]$ for some $c \in \mathbb{Z}$. Due to the Gysin sequence for $S^1 \to P \to Q$, $0 = p^*c_1(P) = kp^*[\Omega]$ and thus $c_1(\xi) = \pi^*c_1(Q) = cp^*[\Omega]$ is a torsion class.

Hence the Maslov indices for homologically trivial Reeb orbits are well defined. We remark that the generalized Maslov index due to [RS93] is well defined although the Conley-Zehnder index is not since $(P, \alpha_{BW})$ is Morse-Bott. These two indices agree in the nondegenerate case.

Suppose furthermore that $(Q, \omega)$ is simply connected and that $[\omega]$ is a primitive element in $H^2(Q)$. We denote by $\gamma$ a principal orbit in $P$. Then since $\pi_1(P) = \mathbb{Z}_k$, the $k$-fold cover of $\gamma$ is contractible and its Maslov index equals to $2c$, i.e. $\mu(\gamma^k) = 2c$, see [EGH00, Bou02].

We learned the following remark and proposition from Otto van Koert.

Remark 4.6. ([vK12]) In this remark we construct some examples which meet requirements in Theorem B. Let $(B, \omega)$ be a simply connected integral symplectic manifold such that $[\omega] \in H^2(B; \mathbb{Z})$ is primitive and $c_1(B) = a[\omega]$ for some $a \in \mathbb{Z}$. Let $Q_k$ be a symplectic Donaldson hypersurface in $(B, \omega)$ Poincaré dual to $k[\omega]$ for sufficiently large $k \in \mathbb{N}$, see [Don96] and [CDvK12, Section 6]. Then according to [Gir02, Proposition 11], $W := B - \nu_B(Q_k)$ is a compact Weinstein. Here $\nu_B(Q_k)$ is the the normal disk bundle over $Q_k$ in $B$ with $c_1(\nu_B(Q_k)) = k[\omega|_{Q_k}]$. Therefore the prequantization bundle $(P, \alpha_{BW})$ over $Q_k$ with $c_1(P) = k[\omega|_{Q_k}]$ has a Weinstein filling $(W, \omega|_W)$. Now we show that this example meets assumptions in Theorem B.

(i) $c_1(W)|_{\pi_2(W)} = 0$

since $c_1(W) = c_1(B)|_W = a[\omega]|_W = a[d\lambda]$ for some $1$-form $\lambda$ on a Weinstein manifold $W$.

(ii) $Q_k$ is simply connected

by (the analogue of) the Lefschetz hyperplane theorem, see [Don96, Proposition 39].

(iii) $c_1(Q_k) = (a - k)[\omega|_{Q_k}]$

due to $c_1(Q_k) = c_1(B) - c_1(\nu_B(Q))$. Moreover if $\dim W \geq 8$, we have

(iv) $\pi_1(W) \cong \pi_1(\partial W)$

since $\pi_2(W, \partial W)$ and $\pi_1(W, \partial W)$ are trivial. Indeed, since $W$ is Weinstein, the Liouville flow can be used to make null-homotopy of elements in $\pi_1(W, \partial W)$ and $\pi_2(W, \partial W)$.

Proposition 4.7. ([vK12]) Let $(P, \xi := \ker \alpha_{BW})$ be a prequantization bundle over a simply connected integral symplectic manifold $(Q, \omega)$ of dimension $(2n - 2)$ such that $[\omega]$ is primitive and $c_1(Q) = c[\omega]$ for $k \in \mathbb{N}$. Suppose that $c_1(Q) = c[\omega]$ for some $c > n - 1$ and that $(P, \xi)$ admits an exact contact embedding $i : (P, \xi) \hookrightarrow (W, d\lambda)$ with $c_1(W)|_{\pi_2(W) = 0}$ which is $\pi_1$-injective. Then

$$\text{SH}_{s}^{S^1, +}(W) \cong \bigoplus_{n=1}^{\infty} H_{s-(2Nc-n+1)}(Q; \mathbb{Q}).$$

Proof. We compute the symplectic homology for $(W, \omega, P, \alpha_{BW})$, i.e. $\lambda|_{i(P)} = \alpha_{BW}$, and the resulting homology is an invariant for $(W, \omega, P, \xi)$. We note that since $i$ is injective on
\( \pi_1 \)-level, Morse-Bott components are exactly \( kN \)-fold covered fibers which we denote by \( P_{kN} \). As in the subsection 2.4, there exists a Morse-Bott spectral sequence with \( E^1 \)-page

\[
E^1_{pq} = \bigoplus_{\mu(\gamma^{kN})=p} H^q_\mu(P_{kN}; \mathbb{Q})
\]

converging to \( SH_*^{S^1,+}(W) \). We observe that

\[
\mu(\gamma^{kN}) = \mu_{M\text{aslow}}(\gamma^{kN}) - \frac{1}{2} \dim Q = 2cN - (n - 1).
\]

Since \( P \) is a principal circle bundle and every contractible closed Reeb orbits is good, we have

\[
H^S_*(P_{kN}; \mathbb{Q}) \cong H_*(Q \times B\mathbb{Z}_{kN}; \mathbb{Q}) \cong H_*(Q; \mathbb{Q}).
\]

Since \( 2c \geq 2n \) and the height of the spectral sequence is \( \dim Q = 2n - 2 \), the spectral sequence stabilizes at the \( E^1 \)-page, see Figure 4.1, and thus we conclude

\[
SH_*^{S^1,+}(W, d\lambda) \cong \bigoplus_{N \in \mathbb{N}} H_*(-2Nc-(n-1))(Q; \mathbb{Q}).
\]

\[
\cdots
\]

| 2n - 2 | \* | \* |
|---|---|---|
| \* | \* | \* |
| \* | \* | \* |

\[
0 \quad -2c \quad \cdots \quad -2c \quad \cdots \quad -2c \quad \cdots \quad -2c
\]

\[
\cdots
\]

\[
\text{Figure 4.1. } E^1 \text{-page of Morse-Bott spectral sequence}
\]

\[
\Box
\]

**Proof of Theorem B.**

We observe that for all \( N \in \mathbb{N} \),

\[
SH_*^{S^1,+}(W) = H_0(Q; \mathbb{Q}) = \mathbb{Q}, \quad SH_*^{S^1,+}(2Nc+(n-1))(W) = H_{2n-2}(Q; \mathbb{Q}) = \mathbb{Q}.
\]

We first treat the case \( c \geq n \). Note that \( SH_*^{S^1,+}(W) = 0 \) for all \( * < 2c - (n - 1) \). Assume by contradiction that there is a precisely one simple closed Reeb orbit \( \gamma \). If \( \mu_{CZ}(\gamma) \) is smaller than \( 2c - (n - 1) \), there has to be another closed Reeb orbit \( v \) of Conley-Zehnder index \( \mu_{CZ}(\gamma) + 1 \) or \( \mu_{CZ}(\gamma) - 1 \) since \( SH_*^{S^1,+}(\mu_{CZ}(\gamma))(W) = 0 \). But \( v \) cannot be a multiple cover of \( \gamma \) since otherwise \( v \) is a bad orbit. We may assume that \( \mu_{CZ}(\gamma) \geq 2c - (n - 1) \geq n + 1 \). Then since \( \mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 2 \) for all \( k \), see Proposition 3.6, \( \mu_{CZ}(\gamma) \) has to be \( 2c - (n - 1) \) the first degree when \( SH_*^{S^1,+} \) does not vanish.

We first exclude the case \( \mu_{CZ}(\gamma^k) = k \mu_{CZ}(\gamma) \), \( k \in \mathbb{N} \). Indeed, if it does, for some \( k \in \mathbb{N} \),

\[
k(2c - (n - 1)) = \mu_{CZ}(\gamma^k) = 2c + (n - 1).
\]
If $k \geq 3$, $c \leq n - 1$. Let $k = 2$ and $c = (3n - 3)/2$. Moreover we know that for some $\ell \in \mathbb{N}$,
\[
(2n - 2) = \mu_{CZ}(\gamma^\ell) = 4c - (n - 1) = 5n - 5.
\]
This contradiction proves the claim. Therefore the index iteration formula for $\gamma$ has to be of the following form. Suppose that $\gamma$ is good and the bad case is proved in the same way.
\[
\mu_{CZ}(\gamma^k) = rk + \sum_{i=1}^{j} 2[k\theta_i] + j, \quad j \in \{1, \ldots, n - 1\}
\]
where $\theta_i \in (0, 1) \setminus \mathbb{Q}$ for all $i$. In particular we have
\[
\mu_{CZ}(\gamma) = r + j = 2c - (n - 1)
\]
Due to (4.4), there exist $k \in \mathbb{N}$ satisfying
\[
\mu_{CZ}(\gamma^k) = 2c + n - 1.
\]
Then since $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + 2$ and $SH_{S^1,+}^* (W) = 0$ for all $2c+n-1 < * < 4c-(n-1)$ according to Proposition 4.7,
\[
\mu_{CZ}(\gamma^{k+1}) = 4c - (n - 1).
\]
Since $SH_{S^1,+}^*$ is periodic according to Proposition 4.7 again, we have for $N \in \mathbb{N},$
\[
\mu_{CZ}(\gamma^{(N-1)k+1}) = 2Nc - (n - 1), \quad \mu_{CZ}(\gamma^{Nk}) = 2Nc + (n - 1).
\]
Since
\[
2Nc = \mu_{CZ}(\gamma^{Nk+1}) - \mu_{CZ}(\gamma) = Nkr + \sum_{i=1}^{j} 2[(Nk + 1)\theta_i],
\]
when $N = 1$, we obtain
\[
r = \frac{2c - \sum_{i=1}^{j} 2[(k + 1)\theta_i]}{k}.
\]
Again by (4.5), we have
\[
N \sum_{i=1}^{j} [(k + 1)\theta_i] = \sum_{i=1}^{j} [(Nk + 1)\theta_i], \quad \text{for all } N \in \mathbb{N}.
\]
But dividing out both sides by $N$ and taking a limit $N \to \infty$, we deduce
\[
\sum_{i=1}^{j} (k + 1)\theta_i = \sum_{i=1}^{j} k\theta_i
\]
and this contradiction proves the theorem in the case $c \geq n$.

Now we consider the case $c \leq -n$. Due to Proposition 3.7, $\mu_{CZ}(\gamma^k) > \mu_{CZ}(\gamma^{k+1})$ and thus $\mu_{CZ}(\gamma) = 2c + n - 1$, see (4.4). As above we have for $N \in \mathbb{N},$
\[
\mu_{CZ}(\gamma^{(N-1)k+1}) = 2Nc + (n - 1), \quad \mu_{CZ}(\gamma^{Nk}) = 2Nc - (n - 1)
\]
and this case is proved in a similar fashion. □
4.3. Brieskorn spheres.

Let \((\Sigma_a, \xi_a)\) be a Brieskorn sphere explained in the introductory section. The contact homologies of Brieskorn spheres were computed originally by [Ust99] and reproved using the Morse-Bott approach by [Bou02]. It is possible to compute the positive part of \(S^1\)-equivariant symplectic homology of \(V_\varepsilon(a)\) a natural Weinstein filling of \((\Sigma_a, \xi_a)\) in a similar way or using an isomorphism between those two homologies [BO12b]. Therefore we have

\[
\text{SH}^{S^1, +}_*(V_\varepsilon(a)) = \begin{cases} 
0 & * \in 2\mathbb{Z} + 1 \text{ or } * < n - 1, \\
\mathbb{Q} \oplus \mathbb{Q} & * \in 2\left[\frac{2N}{a_0}\right] + 2N(n - 2) + n + 1, \; N \in \mathbb{N}, \; 2N + 1 \notin a_0\mathbb{Z}, \\
\mathbb{Q} & \text{otherwise.}
\end{cases}
\]

Proof of Theorem C.

Since \(\text{SH}^{S^1, +}(V_\varepsilon(a))\) does not vanish, there exists a simple closed Reeb orbit \(\gamma\). Suppose that there is no another simple closed Reeb orbit except \(\gamma\).

Case 1. If \(\mu_{CZ}(\gamma) < n - 1\), there exists a closed Reeb orbit \(v\) with \(\mu_{CZ}(v) = \mu_{CZ}(\gamma) + 1\) or \(\mu_{CZ}(v) = \mu_{CZ}(\gamma) - 1\) since \(\text{SH}^{S^1, +}_{\mu_{CZ}(\gamma)}(V_\varepsilon(a)) = 0\). Since \(v\) has to be a good orbit, it cannot be a multiple cover of \(\gamma\).

Case 2. If \(\mu_{CZ}(\gamma) \geq n - 1\), \(\mu_{CZ}(\gamma) = n - 1\) since \(\text{SH}^{S^1, +}_{n-1}(V_\varepsilon(a)) = \mathbb{Q}\) and \(\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k)\) due to Proposition 3.5. Thus the iteration formula for the Conley-Zehnder index of \(\gamma\) is

\[
\mu_{CZ}(\gamma^k) = kr + \sum_{i=1}^{\lfloor j/2 \rfloor} 2[k\theta_i] + j, \quad r + j = n - 1
\]

where \(j \in \{0, \ldots, n - 1\}\). We claim that \(r = 0\). If \(r \in 2\mathbb{N}\), every multiple cover of \(\gamma\) is good and there exists \(k_0 \in \mathbb{N}\) such that \(\mu_{CZ}(\gamma^{k_0+1}) \geq \mu_{CZ}(\gamma^{k_0}) + 4\). This contradicts to that \(\dim \text{SH}^{S^1, +}_*(V_\varepsilon(a)) \geq 1\) for every even degree bigger than \(n - 2\). If \(r \in 2\mathbb{N} + 1\), only odd multiple covers are good and there exists \(k_0 \in 2\mathbb{N} + 1\) such that \(\mu_{CZ}(\gamma^{k_0+2}) \geq \mu_{CZ}(\gamma^{k_0}) + 4\). This is again a contradiction and proves the claim. Therefore a possible scenario is

\[
\mu_{CZ}(\gamma^k) = \sum_{i=1}^{n-1} 2[k\theta_i] + n - 1.
\]

We can easily compute the mean Euler characteristic

\[
\chi_m(V_\varepsilon(a)) = \lim_{N \to \infty} \frac{1}{N} \left( \frac{N}{2} + \frac{N}{4/p + 2(n - 2)} - \frac{N}{4 + 2p(n - 2)} \right) = \frac{1}{2} \left( \frac{1 + p(n - 1)}{2 + p(n - 2)} \right)
\]

and due to the resonance identity, Proposition 2.1, we have

\[
\sum_{i=1}^{\lfloor n/2 \rfloor} \theta_i = \frac{2 + p(n - 2)}{1 + p(n - 1)}.
\]

Since \(n - 1 \geq 2\) from the construction of the Brieskorn spheres, we can apply Corollary 3.10. That is, there exists \(k_0 \in \mathbb{N}\) such that

\[
\mu_{CZ}(\gamma^{k_0+1}) - \mu_{CZ}(\gamma^{k_0}) = \sum_{i=1}^{n-1} 2[(k_0 + 1)\theta_i] - \sum_{i=1}^{n-1} 2[k_0\theta_i] \geq 4
\]
This contradicts to the computation that $SH_*^{S^1,+}(V_\epsilon(a))$ is nonzero for every even degree bigger than $n - 2$. \hfill $\Box$

5. Appendix: More examples

In this appendix we give examples which have two closed Reeb orbits even though they do not meet the requirements of Theorem A. For simplicity we treat 5-dimensional case, see Example 3.4.

Example 5.1. Let $(\Sigma, \xi)$ be a contact 5-manifolds which has displaceable exact contact embedding into $(W, \omega)$ which is convex infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. Suppose that a corresponding contact form is nondegenerate. If $b_2(W_0, \Sigma; \mathbb{Q}) = 1$ and $b_4(W_0, \Sigma; \mathbb{Q}) = 0$, there are two closed Reeb orbits contractible in $W$.

**Proof.** We may assume that $b_3(W_0, \Sigma; \mathbb{Q}) = b_5(W_0, \Sigma; \mathbb{Q}) = 0$ since otherwise the assertion is covered by Theorem A. According to Proposition 4.3, we have

$$SH_0^{S^1,+}(W) = SH_2^{S^1,+}(W) = \mathbb{Q}, \quad SH_{2k+2}^{S^1,+}(W) = \mathbb{Q} \oplus \mathbb{Q}, \quad k \in \mathbb{N},$$

and

$$SH_{3^1,+}^{S^1}(W) = 0, \quad \ast \in \mathbb{Z} \setminus (2\mathbb{N} \cup \{0\}).$$

Suppose that there exists precisely one simple closed Reeb orbit $\gamma$. One can immediately see that $\gamma$ cannot be hyperbolic, see Example 3.4. If $e(\gamma) = 2$, it has to be

$$\mu_{CZ}(\gamma^k) = -k + 2[k\theta] + 1, \quad k \in \mathbb{N}, \quad \theta \in (0, 1) \setminus \mathbb{Q}.$$ 

But Corollary 4.5 implies

$$1 = \frac{1}{2\Delta(\gamma)} = \frac{1}{2(2\theta - 1)}$$

which implies a contradiction $\theta = 3/4$. The remaining case is that $\gamma$ is elliptic and

$$\mu_{CZ}(\gamma^k) = -2k + 2[k\theta_1] + 2[k\theta_2] + 2, \quad k \in \mathbb{N}, \quad \theta_i \in (0, 1) \setminus \mathbb{Q}.$$ 

Again by Corollary 4.5, we obtain

$$\theta_1 + \theta_2 = \frac{3}{2}.$$ 

Since both $\theta_1$ and $\theta_2$ are irrational, we have

$$[2k\theta_1] + [2k\theta_2] = [2k\theta_1] + [3k - 2k\theta_1] = 3k - 1, \quad k \in \mathbb{N}$$

and thus $\mu_{CZ}(\gamma^{2k}) = 2k$. From this we can derive $\mu_{CZ}(\gamma^{k+1}) = 2k + 2$ for all $k \in \mathbb{N}$ since $|\mu_{CZ}(\gamma^{k+1}) - \mu_{CZ}(\gamma^k)| \leq 2$. This yields that

$$[(2k + 1)\theta_1] + [(2k + 1)\theta_2] = [(2k + 1)\theta_1] + [3k + 1 + 1/2 - (2k + 1)\theta_1] = 3k + 1,$$

and thus we have the following contradictory inequality

$$(2k + 1)\theta_1 - [(2k + 1)\theta_1] < \frac{1}{2}, \quad k \in \mathbb{N}.$$ 

Indeed since $2\theta_1$ is irrational, there exists $k_0 \in \mathbb{N}$ such that $2k_0\theta_1 - [2k_0\theta_1] \approx 1 - \theta_1$. \hfill $\Box$

Example 5.2. Let $(\Sigma, \xi)$ be a contact 5-manifolds which has displaceable exact contact embedding into $(W, \omega)$ which is convex infinity and satisfies $c_1(W)|_{\pi_2(W)} = 0$. Suppose that a corresponding contact form is nondegenerate. If $b_2(W_0, \Sigma; \mathbb{Q}) = 1$ and $b_4(W_0, \Sigma; \mathbb{Q}) \geq 5$, there are two closed Reeb orbits contractible in $W$. 

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Proof. By the same reason as above we may assume that $b_3(W_0, \Sigma; \mathbb{Q}) = b_5(W_0, \Sigma; \mathbb{Q}) = 0$. According to Proposition 4.3, we have $SH_0^{S^1,+}(W) = \mathbb{Q}$ and $SH_2^{S^1,+}(W) = \mathbb{Q}\oplus(b_4(W_0, \Sigma; \mathbb{Q})+1)$. Assume by contradiction that there exists precisely one simple closed Reeb orbit $\gamma$. The case that $\gamma$ is not elliptic can be easily excluded. The only possible nontrivial case is that $\mu_{CZ}(\gamma) = 0$ and

$$\mu_{CZ}(\gamma^k) = -2k + 2[k\theta_1] + 2[k\theta_2] + 2, \quad \theta_i \in (0, 1) \setminus \mathbb{Q}, \ k \in \mathbb{N}.$$  

Corollary 4.5 yields that

$$\theta_1 + \theta_2 = \frac{b_4(W_0, \Sigma; \mathbb{Q}) + 2}{b_4(W_0, \Sigma; \mathbb{Q}) + 1} \leq \frac{7}{6}.$$  

We note that $\mu_{CZ}(\gamma^k) \geq 2$ for $k \geq 2$ because of $SH_0^{S^1,+}(W) = \mathbb{Q}$ and $SH_2^{S^1,+}(W) = 0$ for all $* < 0$. Since $\mu_{CZ}(\gamma^2) \geq 2$, both $\theta_1$ and $\theta_2$ are bigger than $1/2$. In addition $\mu_{CZ}(\gamma^3) \geq 2$ implies that one of $\theta_1$ and $\theta_2$ is bigger than $2/3$. Thus we deduce

$$\theta_1 + \theta_2 > \frac{2}{3} + \frac{1}{2} = \frac{7}{6}.$$  

This contradiction completes the proof. \(\square\)

References

[AF10] P. Albers, U. Frauenfelder, Leaf-wise intersections and Rabinowitz Floer homology, Journal of Topology and Analysis, 2 no. 1, (2010) 77–98.

[BC02] P. Biran, K. Cieliebak, Lagrangian embeddings into subcritical Stein manifolds, Israel J. Math, 127 (2002) 221–244.

[BCE07] F. Bourgeois, K. Cieliebak, T. Ekholm, A note on Reeb dynamics on the tight 3-sphere, J. Modern dynamics 1, No. 4 (2007) 597–613.

[BW58] W.M. Boothby, H.C. Wang, On contact manifolds, Ann. Math. 68, No. 3, (1958) 721–734.

[Bou02] F. Bourgeois, A Morse-Bott approach to contact homology, Ph.D. thesis, Stanford University, (2002).

[BLM85] H. Berestycki, J.M. Lasry, G. Mancini, Existence of Multiple periodic orbits on star-shaped Hamiltonian surfaces, Commun. Pure Appl. Math. Vol 38, Issue 3, (1985) 253–289.

[BO09a] F. Bourgeois, A. Oancea, Symplectic Homology, autonomous Hamiltonians, and Morse-Bott moduli spaces, Duke Math. J. 146 No.1 (2009), 71–174.

[BO09b] F. Bourgeois, A. Oancea, The Gysin exact sequence for $S^1$-equivariant symplectic homology, arXiv:0909.4526

[BO10] F. Bourgeois, A. Oancea, Fredholm theory and transversality for the parametrized and for the $S^1$-invariant symplectic action, 12, Issue 5, (2010) 1181-1229.

[BO12a] F. Bourgeois, A. Oancea, The index of Floer moduli problems for parametrized action functionals, arXiv:1207.5360.

[BO12b] F. Bourgeois, A. Oancea, $S^1$-equivariant symplectic homology and Linearized contact homology, arXiv:1212.3731.

[CDvK12] R. Chiang, F. Ding, O. van Koert, Open books for Boothby-Wang bundles, fibered Dehn twists and the mean Euler characteristics, (2012) arXiv:1211.0201.

[CF09] K. Cieliebak, U. Frauenfelder, A Floer homology for exact contact embeddings, Pacific J. Math. 239 (2009), 251-316.2

[CFO10] K. Cieliebak, U. Frauenfelder, A. Oancea, Rabinowitz Floer homology and symplectic homology, Annales scientifiques de l'ENS 43, fasc. 6, 957–1015 (2010).

[CGH12] D. Cristofaro-Gardiner, M. Hutchings, From one Reeb orbit to two, (2012) arXiv:1202.4839.

[CO08] K. Cieliebak, A. Oancea, Symplectic and contact homology revisited, preprint, version 2008.

[CZ84] C.C. Conley, E. Zehnder, Morse-type index theory for flows and periodic solutions of Hamiltonian equations, Commun. Pure Appl. Math 37 (1984) 207–253.

[Don96] S.K. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Diff. Geom. 44 (1996) 666–705.
[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, Introduction to Symplectic Field Theory, Geom. Funct. Anal., Special Volume, Part II:560-673, 2000.

[EH87] I. Ekeland, H. Hofer, Convex Hamiltonian Energy surfaces and their periodic trajectories, Commun. Math. Phys. 113, (1987) 419–469.

[EL80] I. Ekeland, J.M Lasry, On the Number of Periodic Trajectories for a Hamiltonian Flow on a Convex Energy Surface, Ann. Math. Vol. 112, No. 2 (1980) 283–319.

[Fuk96] K. Fukaya, Floer homology of connected sum of homology 3-spheres, Topology 35 No.1 (1996) 89–136.

[FK14] U. Frauenfelder, J. Kang, From gradient flow lines to finite energy planes, in preparation.

[FS12] U. Frauenfelder, F. Schlenk, A vanishing result for equivariant Rabinowitz Floer homology, in preparation.

[FSvK12] U. Frauenfelder, F. Schlenk, O. van Koert, Displaceability and the mean Euler characteristic, Kyoto J. Math. 52(4) (2012), 797–915.

[Gei08] H. Geiges, “An introduction to contact topology”, Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.

[Gir02] E. Giroux, Géométrie de Contact: de la Dimension Trois vers les Dimensions Supérieures, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 405–414, Higher Ed. Press, Beijing, 2002.

[Gut12] J. Gutt, The Conley-Zehnder index for a path of symplectic matrices, arXiv:1201.3728.

[GK10] V. Ginzburg, E. Kerman, Homological resonances for Hamiltonian diffeomorphisms and Reeb flows, Int. Math. Res. Not. (2010), no. 1, 53–68.

[GHMM12] V. Ginzburg, D. Hein, U. Hryniewicz, L. Macarini, Closed Reeb orbits on the sphere and symplectically degenerate maxima, Acta Mathematica Vietnamica 2013, 38(1), 55–78

[HT09] M. Hutching, C.H. Taubes, The Weinstein conjecture for stable Hamiltonian structures, Geom. Topol., 13 (2009), 901–941.

[HWZ03] H. Hofer, K. Wysocki, E. Zehnder, Finite energy foliations of tight three-spheres and Hamiltonian dynamics, Ann. Math., 157 (2003) 125–257.

[HWZ98] H. Hofer, K. Wysocki, E. Zehnder, The dynamics on a strictly convex energy surface in \( \mathbb{R}^4 \), Ann. Math., 148 (1998) 197–289.

[Kan13] J. Kang, Symplectic homology of displaceable Liouville domains and leafwise intersections,

[Lon00] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics, Adv. Math. 157 (2000), 76–131.

[Lon02] Y. Long, “Index Theory for Symplectic Paths with Applications”, Birkhäuser, 2002.

[LZ02] Y. Long, C. Zhu, Closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \), Ann. Math. 155 (2002) no. 2 317–368.

[Rit10] A. Ritter, Topological quantum field theory structure on symplectic cohomology, to appear in J. Topology doi: 10.1112/jtopol/jts038.

[RS93] J. Robbin, D. Salamon, The Maslov index for paths, Topology 32 827–844 1993.

[Sa99] D. Salamon, Lectures on Floer homology, in “Symplectic Geometry and Topology”, Eds: Y. Eliashberg and L. Traynor, IAS/Park City Mathematics series, 7, (1999), 143–230.

[Sei08] P. Seidel, A biased view of symplectic cohomology, Current Developments in Mathematics Volume 2006 (2008), 211–253.

[STZ92] D. Salamon, E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, Comm. Pure Appl. Math. 45 (1992) no.10, 1303–1360.

[Ust99] I. Ustilovsky, Infinitely many contact structures on \( S^{4m+1} \), Int. Math. Res. Notices, No. 14 (1999) 781–791.

[Vit89] C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems, Trans. Amer. Math. Soc., 311 (1989), 621-655.

[Vit99] C. Viterbo, Functors and computations in Floer homology with applications, II, Geom. Funct. Anal. 9 (1999), 985–1033.

[yK05] O. van Koert, Open books for contact five-manifolds and applications of contact homology, Ph.D. thesis, Universität zu Köln (2005).

[yK12] O. van Koert, Private communications, (2012).

[Wan11] W. Wang, Existence of closed characteristics on compact convex hypersurfaces in \( \mathbb{R}^{2n} \), (2011) arXiv:1112.5501.
[Wei79] A. Weinstein, *On the hypotheses of Rabinowitz periodic orbit theorems*, J. Differential Equations, 33 (1979), 353-358.

[WHL07] W. Wang, Y. Long, X. Hu, *Resonance identity, stability, and multiplicity of closed characteristics on compact convex hypersurfaces*, Duke Math. J. 139 (2007), no. 3, 411-462.

[Yau04] M.-L. Yau, *Cylindrical contact homology of subcritical Stein-fillable contact manifolds*, Geom.-Topo., 8 (2004) 1243–1280.