Orthogonal-state-based and semi-quantum protocols for quantum private comparison in noisy environment

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Abstract

Private comparison is a primitive for many cryptographic tasks, and recently several schemes for the quantum private comparison (QPC) have been proposed, where two users can compare the equality of their secrets with the help of a semi-honest third party (TP) without knowing each other’s secret and without disclosing the same to the TP. In the existing schemes, secrecy is obtained by using conjugate coding, and considering all participants as quantum users who can perform measurement(s) and/or create states in basis other than computational basis. In contrast, here we propose two new protocols for QPC, first of which does not use conjugate coding (uses orthogonal states only) and the second one allows the users other than TP to be classical whose activities are restricted to either reflecting a quantum state or measuring it in computational basis. Further, the performance of the protocols is evaluated under various noise models.

Keywords: quantum private comparison, secure multiparty computation, socialist millionaire problem, quantum cryptography, noise models, quantum communication in noisy environment

1 Introduction

One of the most important branches of classical and quantum cryptography is secure multi-party computation (SMC) \cite{1,2} and references therein). SMC is a primitive for distributed computation. It enables the distributed computing of correct output of a function in a situation, where the inputs are given by a group of mutually distrustful users. A SMC is required to be fair, and secure. Specifically, it should not leak the secret inputs of the individual players. Efforts have been made to achieve this in various ways, both classically \cite{1,3,4} and quantum mechanically \cite{5,6,7,8,9,10}. However, in those efforts \cite{5,6,10}, it has been assumed that some of the users follow the protocol honestly (which implies that some of the users are semi-honest). Among the variants of SMC schemes, a specific variant having particular importance is “socialist millionaire problem” which was first introduced by Yao \cite{1} in 1982 as a computing task where two millionaires (Alice and Bob) wish to know who is richer without disclosing the amount of their wealth to each other. Subsequently, Boudot et al. \cite{3}, modified it to a task where the millionaires are only interested to know whether they are equally rich or not. Thus, they wish to compute a function \( f(i_A, j_B) : f(i_A, j_B) = a \neq b = f(i_A, i_B) \), where the subscripts \( A \) and \( B \) represent inputs from Alice and Bob, respectively. Thus, \( i_A \) and \( i_B \) are essentially classical information corresponding to the assets of Alice and Bob, and a scheme for Boudot’s version of the socialist millionaire problem is actually a scheme for private comparison of equality of information. A lot of work has been done on classical private comparison, but as the security of all classical cryptographic schemes are conditional, it can never lead to an unconditionally secure scheme of private comparison. In contrast, we can achieve unconditional security in the quantum world. This fact led to many proposals for private comparison of equality of information using quantum resources \cite{2,5,6,7,8,9,10,20}, we may refer to all such protocols as protocols for “quantum private comparison (QPC)”. Before we proceed further, it would be apt to note that QPC problem, the socialist millionaire problem and Tierce problem \cite{3} are equivalent, and in what follows we will mostly refer to all such problems as QPC problem.

In the original definition of socialist millionaire problem, it was a two-party computation task, but a pioneering work of Lo \cite{21} established that two-party secure QPC is not possible. This implies that to implement a secure QPC, we must have a third party (TP), who would assist the users to compare the equality of their secrets. Interestingly, the TP may be semi-honest \cite{5,6}, semi-honest having an intelligent robot \cite{10}, dishonest \cite{13}, or almost-dishonest \cite{11}. Despite, the strong proof of Lo, some efforts have been

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made to achieve QPC without TP \cite{22,23}, but they have been found to be insecure and/or unfair \cite{24}. Thus, in what follows, we will concentrate on three party protocols of QPC, where a TP helps Alice and Bob to compare the equality of their information. Such protocols for QPC have been proposed using different types of entangled states. For example, in Ref. \cite{5}, a scheme for QPC has been proposed using g-type state, W state was used in Ref. \cite{12}. Bell state was used in \cite{6,7,13}, GHZ state was used in \cite{11}. Further, in Ref. \cite{25}, a group theoretic structure of the protocols of quantum dialogue was proposed using a large number of different types of entangled states (e.g., W, GHZ, cluster, Q4, Q5, and Brown states), and it was shown that the quantum dialogue scheme proposed there can be converted to a protocol of the socialist millionaire problem, which is equivalent to QPC. Thus, in Ref. \cite{25}, several options for realization of protocols for QPC were provided. Further, in the similar line, in \cite{26}, a set of new options (e.g., 4-qubit $$\Omega$$ state, 4-qubit cat state, etc.) for realization of QD, and thus, QPC have been proposed.

It’s already established that the schemes for QPC have useful applications in private bidding and auctions, secret ballot elections, e-commerce, etc. (\cite{8} and references therein). Due to the fact that a scheme for QPC has applications in many fields, many variants of QPC have been studied in the recent past. For example, Huang et al., have recently proposed a GHZ-state-based QPC scheme for $$n$$ users \cite{11}. Huang et al.’s scheme considers an almost-dishonest TP and allows him to compare the equality of the secrets of a subset of users, too.

In what follows, we plan to propose two protocols for QPC in the line of modified Tseng-Lin-Hwang (TLH) protocol, which was proposed in its original form in 2012 \cite{6}. In the original TLH protocol a scheme for quantum private comparison was proposed using Bell states, but almost immediately after the publication of TLH scheme, Yang et al., \cite{13} had shown that there exist a security flaw in the original TLH scheme and other similar schemes, which assume TP to be semi-honest. Yang et al., \cite{13} also proposed a modified scheme which they claimed to be free from the limitations of the previous protocols as their scheme considered a dishonest TP. Subsequently, in an interesting work on cryptanalysis of Yang et al.’s scheme, it was observed by Zhang et al., \cite{14} that Yang et al.’s scheme was also not completely secure. Keeping these facts in mind, here we aim to propose two schemes of QPC that are free from the attacks described earlier and are fundamentally different from all the existing schemes of QPC as one of the proposed schemes is semi-quantum in nature and the other one is a completely orthogonal-state-based scheme. Here it would be apt to note that protocols based on orthogonal states alone and semi-quantum protocols are foundationally different from usual conjugate-coding based (BB84-type) protocols. This is so because, in contrast to common perception developed by BB84-type schemes, orthogonal-state-based schemes establish the fact that unconditional security does not originate from the conjugate coding (our inability to simultaneously measure a quantum state using two or more MUBs). Further, semi-quantum schemes establish that all parties involved in performing the cryptographic tasks do not require to be quantum. A brief review of the existing orthogonal-state-based and semi-quantum schemes for various quantum cryptographic task is provided in the next section. The review clearly shows that to the best of our knowledge, there does not exist any orthogonal-state-based or semi-quantum scheme for QPC. This fact further sets the motivation for designing such schemes for the first time.

In a practical situation, all communication schemes are affected by the noise present in the communication channel. So it’s extremely important to know how QPC schemes behave in the presence of noise. Recently, some efforts have been made to investigate the effect of noise on QPC schemes. For example, the effects of Pauli noise \cite{13}, and collective amplitude damping \cite{14} have been studied very recently. However, to the best of our knowledge, no serious effort has yet been made to rigorously investigate the effect of a complete set of possible noise models on the schemes of QPC. This letter aims to do that for the protocols proposed in this letter.

Remaining part of the letter is organized as follows. In Section 2 we have proposed two schemes of QPC that are fundamentally different from the existing schemes of QPC. In Section 3 the security and efficiency of the proposed schemes are analyzed. The effect of various types of noise models on the proposed schemes is investigated in Section 4. Finally, the letter is concluded in Section 5.

## 2 Protocols for quantum private comparison

In this section, we introduce two protocols for QPC. The first one is an orthogonal-state-based protocol, and the second one is a semi-quantum protocol. Here, we assume that Alice and Bob wish to compare their assets with the help of a TP.

### 2.1 Orthogonal-state-based protocol for quantum private comparison

In quantum cryptography, orthogonal-state-based protocols are of fundamental interest as in contrast to BB84 type conjugate coding based protocol, where the security arises from the conjugate coding or noncommutivity, here (i.e., in case of orthogonal-state-based schemes), the security arises from the duality (monogamy of entanglement) for single particle (multipartite entangled) states. The first ever orthogonal-state-based scheme of QKD was introduced in 1995 \cite{27}, and is now known as GV protocol. Since then several quantum cryptographic schemes based on orthogonal states have been proposed for various cryptographic tasks. For example, orthogonal-state-based schemes have been proposed for QKA \cite{28}, QSDC \cite{29,31}, DSQC \cite{29,31,32}, CDSQC \cite{34}, etc. A few of these schemes have also been experimentally realized \cite{34}. However, to the best of our knowledge, no effort has yet been made to design an orthogonal-state-based protocol for quantum private comparison. In what follows, the same is proposed in 8 steps, which are denoted by (OSB1, OSB2, … OSB8), where OSB stands for orthogonal-state-based. This notation is adopted to distinguish
steps of this protocol from the steps of our second protocol, which is a semi-quantum scheme, and whose steps are denoted by (SQ1, · · · ; SQ9), where SQ stands for semi-quantum. The proposed orthogonal-state-based scheme works as follows:

**OSB1:** TP prepares $2N$-EPR pairs randomly chosen from the set of four Bell states \( \left\{ |\psi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, |\phi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \right\} \). TP prepares two quantum sequences, namely \( S_A \) and \( S_B \), which contain all the first and second particles of the EPR pairs, respectively.

Thus, both \( S_A \) and \( S_B \), contain \( 2N \) qubits, and if a sequence is transmitted via quantum channel, we would require to check half of the travel qubits for eavesdropping. The same can be done by adding \( 2N \) decoy qubits in each sequence.

**OSB2:** TP prepares two copies of \( |\psi^{+}\rangle^N \). He uses the first (second) copy as a set of decoy qubits \( D_A \) \( (D_B) \) to be randomly inserted in the sequence \( S_A \) \( (S_B) \) to perform eavesdropping check using GV-subroutine (detail of GV-subroutine can be found in [30][31][35]). Specifically, TP randomly inserts \( D_A \) in \( S_A \) \( (D_B \) in \( S_B) \) to obtain a new enlarged sequence \( S'_A \) \( (S'_B) \). Later, he sends the quantum sequence \( S'_A \) to Alice and \( S'_B \) to Bob.

**OSB3:** After receiving quantum sequences \( S'_{A/B} \) Alice/Bob perform GV subroutine for eavesdropping check with the help of TP. They compute the error rate, if this error rate is more than a predetermined allowed threshold value, then they abort the protocol and restarts from OSB1 considering that an eavesdropper is present. Otherwise, they proceed to the next step.

As the eavesdropping is checked by measuring the decoy qubits, after this step, Alice and Bob are left with \( S_A \) and \( S_B \), respectively, because they remove the qubits measured for eavesdropping check.

**OSB4:** Qubits of \( S_A \) and \( S_B \) are now measured in the computational basis by Alice and Bob, respectively. Both would individually obtain \( 2N \)-bits key strings of bit values \( 0 \) and \( 1 \) corresponding to the measurement outcomes \( |0\rangle \) and \( |1\rangle \), respectively.

In half of the shared secret bits, they can check correlations to enhance security against disturbance attacks by Eve. The reduced bit string of \( N \)-bits with Alice (Bob) is denoted as \( K_A \) \( (K_B) \).

**OSB5:** Alice and Bob prepare a shared key \( K_{AB} \) of \( N \)-bits using an orthogonal-state-based scheme of QKD [27] or quantum key agreement (QKA) [28].

**OSB6:** Alice and Bob have the private information regarding their assets which they wish to compare. Consider that the private information of Alice and Bob is represented by the bit strings \( M_A \) and \( M_B \), respectively. Alice encrypts her secret \( M_A \) with her key string \( K_A \) and shared key \( K_{AB} \) by using an exclusive-OR operation to obtain \( C_A^i = (M_A \oplus K_A^i \oplus K_{AB}^i) \). Meanwhile, Bob also encrypts his message \( M_B \) with keys \( K_B \) and \( K_{AB} \) by using the same operation to obtain \( C_B^i = (M_B \oplus K_B^i \oplus K_{AB}^i) \).

Here, the superscript \( i \) denotes the \( i \)th bit of the \( N \)-bits string. Alice and Bob send the calculated strings \( C_A \) and \( C_B \) separately to TP via a public channel.

**OSB7:** TP generates a classical bit string \( C_{TP} \) of \( N \)-bits corresponding to the choice of the initial Bell states in OSB1, such that for the \( i \)th Bell state being \( |\psi^{\pm}\rangle \) \( (|\phi^{\pm}\rangle) \) he generates \( i \)th bit value in \( C_{TP} \) as \( 0 \) \( (1) \).

**OSB8:** TP computes \( R \), which is now exclusive-OR result of \( C_A, C_B, \) and \( C_{TP} \) as \( R = (C_A \oplus C_B \oplus C_{TP}) \). The bit value \( 0 \) \( (1) \) of \( R^i \) corresponds to the same (different) values of \( M_A^i \) and \( M_B^i \). Thus, if \( R^i = 0 \) : \( \forall i \in \{1, \ldots, N\} \) both Alice and Bob have the same amount of assets. Checking values of \( R^i \), TP announces whether Alice and Bob have equal amount of assets.

Interestingly, following similar argument, the existing controlled QD scheme proposed by some of the present authors in Ref. [36] can also be modified to design a completely orthogonal-state-based QPC protocol. However, we are not interested to elaborately discuss another orthogonal-state-based scheme. Instead, here, we prefer to propose another foundationally relevant scheme which can work with limited quantum resources.

### 2.2 Semi-quantum protocol for quantum private comparison

Now, we would try to design a semi-quantum protocol for QPC. In which, TP would be considered as the only quantum party (i.e., possess quantum resources) while both Alice and Bob are classical parties. In convention, a classical party can only access and process classical piece of information, i.e., can only work in the computational basis [37][40]. Thus, being classical user, Alice and Bob can only measure and prepare qubits in computational basis. Additionally, they can reflect the qubits without disturbing them. The first semi-quantum protocol of key distribution (SQKD) was introduced in 2007 [38][39]. In which an unconditionally secure quantum key was shared between a classical and quantum party by using measure and resend [38][40] and permutation of particles [40] techniques to avoid eavesdropping. In what follows, we refer to these two equivalent techniques (i.e., the schemes discussed in Refs. [38][40]) for eavesdropping checking as semi-quantum subroutine. The protocol proposed in Ref. [39] was widely discussed and was followed by a set of SQKD protocols ( [37] and references therein), semi-quantum DSQC [41], etc. A protocol for semi-quantum private comparison (SQPC) is described below. In the protocol proposed below, we will assume that Alice and
Bob's choice | Alice's choice | TP's measurement | outcome
--- | --- | --- | ---
R | R | $|\psi^+\rangle$ | $|\phi^+\rangle$
M | R | $|\psi^\pm\rangle$ | $|\phi^\pm\rangle$
M | R | $|\psi^\pm\rangle$ | $|\phi^\pm\rangle$
M | M | $|\psi^\pm\rangle$ | $|\phi^\pm\rangle$

Table 1: All possible cases that may appear in a semi-quantum private comparison scheme are summarized in this table. Here, both the classical users can either choose to measure or reflect half of the qubits, randomly and independently. We have explicitly shown all the cases when the initial state was $|\psi^+\rangle$ or $|\phi^+\rangle$. It is only Case 4, which is of interest, because only this case leads to successful key sharing between Alice and Bob. Case 1 is used for eavesdropping checking, i.e., TP’s announcement “1” for the qubits corresponding to Case 1 may be considered as the signature of disturbance caused by Eve.

Bob share a secure key $K_{AB}$, which has been distributed/produced using SQKD protocol proposed in Ref. [37], in which Alice and Bob restricted to classical resources can share a quantum key [37]. We further assume that Alice (Bob) prepares an unconditionally secure key $K_{AT}$ ($K_{BT}$) in collaboration with quantum enabled TP using a semi-quantum key agreement (SQKA) protocol [52]. In other words, we assume that Alice (Bob) knows $K_{AB}$ and $K_{AT}$ ($K_{AB}$ and $K_{BT}$) before the start of the following protocol for semi-quantum private comparison.

**SQ1:** Same as OSB1, with the only difference that the number of EPR states prepared is $\approx 8N$.

**SQ2:** TP sends the two sequences $S_A$ and $S_B$ to the classical users Alice and Bob, respectively. As mentioned beforehand, the classical users can only perform two operations, i.e., either “measure and resend” in the computational basis or “reflect” the qubit without any change.

**SQ3:** Both the users measure half of the received qubits (i.e., $\approx 4N$ qubits) randomly in computational basis and replace them with the freshly prepared qubits in the same basis in accordance with the measurement outcome. They also store the bit values obtained in their measurement outcome corresponding to each measured qubit.

**SQ4:** After receiving all the qubits from Alice and Bob, TP performs Bell measurements on the received pairs of qubits (i.e., $S_A^i$ and $S_B^i$). A particular measurement would yield one of the Bell states. If TP’s measurement yields the same Bell state as was initially prepared by him, he announces 0, otherwise he announces 1.

Out of $\approx 8N$ initial Bell states, in SQ3, both Alice and Bob randomly measure one half of the received qubits. Therefore, for a particular pair of qubits (ith Bell state, which was prepared initially), there are four possible cases as listed below and summarized in Table 1.

- **Case 1:** Both Alice and Bob reflect the incoming qubits undisturbed. This case may not be useful in sharing a key between Alice and Bob, but it can be used for eavesdropping checking. As in the absence (presence) of an eavesdropper and/or noise the initial state prepared by TP and the final state obtained by him as the outcome of his Bell measurement should be the same (different in 75% cases).
- **Case 2 and Case 3:** Either Alice or Bob measures the qubit while the other party reflects it. All these cases will be discarded as these cases are neither useful in sharing key nor for eavesdropping checking.
- **Case 4:** Both Alice and Bob perform measurements in computational basis on the qubits received by them, and store the measurement outcome (0 or 1) to be used in the future. This case would lead to a key.

**SQ5:** After the announcement of TP, both Alice and Bob disclose the coordinates of the qubits they have measured. Subsequently, both of them make two separate lists corresponding to qubits falling in Cases 1 and 4. Then, Alice and Bob check eavesdropping using semi-quantum subroutine using qubits of Case 1. In which, both the classical parties ensure that TP’s measurement outcome is 0 when both have reflected the qubits (Case 1). In this case, about one-fourth of the initial states (i.e., $\approx 2N$ Bell states) will be used for eavesdropping checking. If the errors are found to be lower than a tolerable limit, they proceed to the next step, otherwise they discard the scheme, and restart from SQ1.

On the successful completion of this step (i.e., if the error rate is found to be less than the tolerable limit), Alice and Bob obtain $K_{A_2N}$ and $K_{B_2N}$, respectively, where $K_{A_2N}$ and $K_{B_2N}$ are correlated strings of length $\approx 2N$-bits.

**SQ6:** On request of Alice and Bob, TP discloses his initial choices of Bell states for $N$ cases, that are selected randomly. Subsequently, using the values of $K_{A_2N}$ and $K_{B_2N}$, Alice and Bob compare the parity values for the Bell states prepared by TP and the bit values obtained by them for all $N$ cases. If it matches, they proceed, otherwise they abort.

This would check whether TP had genuinely prepared a Bell state or not, which in turn would ensure the correlation in the reduced $N$-bit strings $K_A$ and $K_B$. 

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4
3 Analysis of security and efficiency of the proposed protocols

Before we analyze the security of the proposed protocols, it would be beneficial to summarize the information known (or unknown) to each party. To be specific, in both the protocols proposed in the last section, TP has the information of the initial Bell states, which is unknown to both Alice and Bob. This ensures 1 bit of ignorance for both Alice and Bob. This is so because the initial Bell states to each party. To be specific, in both the protocols proposed in the last section, TP has the information of the initial Bell states, which may also choose to disturb the protocol. For example, in the orthogonal-state-based QPC protocol, she may apply a Pauli operation on all the qubits traveling from TP to Alice without disturbing Bob’s qubits. Alice would not be able to detect this attack using the relevant in some of the intercept and resend attacks of dishonest Alice or Bob. Instead of trying to extract the useful information, Eve may also choose to disturb the protocol. For example, in the orthogonal-state-based QPC protocol, she may apply a Pauli operation on all the qubits traveling from TP to Alice without disturbing Bob’s qubits. Alice would not be able to detect this attack using the relevant in some of the intercept and resend attacks of dishonest Alice or Bob. Instead of trying to extract the useful information, Eve cannot extract any information encoded by Alice or Bob through this attack, but she may force Alice and Bob to reach at a wrong conclusion, which is not desirable. This undesirable situation can be circumvented by introducing an additional security checking (as mentioned in OSB4), where Alice and Bob check correlations in a part of the secure strings shared by them with the help of TP. This correlation check would reveal whether Eve has adopted the above mentioned attack strategy.

Now, we can consider various possible attacks by each party to gain the information not accessible to them. For example, TP can try to take benefit of state preparation by preparing a state other than Bell state to gain advantage by entanglement swapping attack. In fact, he can prepare an arbitrary quantum state to exploit the state preparation, and try to extract correlated keys $K_{AB}$ and $K_B$. However, the unconditional security of the quantum key $K_{AB}$ would maintain security against any such attempt as both Alice and Bob encrypt their messages regarding asset information $M_A$ and $M_B$, respectively using the quantum key $K_{AB}$. In other words, it can be viewed as Alice and Bob performing communication in a secure manner by using a quantum key and the security of message can be attributed completely to the security of the quantum key. Additionally, Alice (Bob) may wish to extract Bob’s (Alice’s) key $K_B$ ($K_A$) by designing an intercept and resend attack. The eavesdropping checking mechanism adopted in both the protocols (by GV and semi-quantum subroutines, respectively) ensures security against this kind of an attack. It may be noted that the eavesdropping checking procedure adopted here would provide security against both insider and outsider attackers.

SQ7: Same as OSB6 with the only difference that Alice (Bob) encrypts her (his) secret $M_A$ ($M_B$) with her (his) key string $K_A$ ($K_B$), key shared with Bob (Alice) $K_{AB}$, and TP $K_{AT}$ ($K_{BT}$) by using an exclusive-OR operation to obtain $C_A = (M_A' + K_A' + K_{AB}' + K_{AT}')$, before announcing the strings $C_A$ and $C_B$ separately to TP via a public channel.

SQ8: Same as OSB7.

SQ9: Same as OSB8 with a minor change in TP’s calculation of $R^i$, which is exclusive-OR result of $C_A$, $C_B$, $C_{TP}$, $K_{AT}$, and $K_{BT}$ as $R = (C_A + C_B + C_{TP} + K_{AT} + K_{BT})$. 
A modified intercept and resend attack strategy by a classical Alice (Bob) in SQPC scheme could be to intercept all the qubits sent to Bob (Alice) and keep them in a memory. Though, possession of a quantum memory by a classical party is beyond his/her limits, but it is an interesting scenario to be investigated. Suppose Alice measures half of the qubits which TP has sent to her, and follows the same for the corresponding Bob’s qubits in her possession. Depending upon the outcomes of her measurement, Alice prepares new qubits and sends (the same number of qubits TP would have sent) them to Bob after adding the remaining auxiliary qubits. Bob proceeds with the protocol as expected, then Alice intercepts all the qubits Bob returns to TP and replaces them with the qubits (initially sent by TP to Bob) in her possession. Using this approach, if $K_{BT}$ is not used) she will get $K_B$ completely as it will be prepared using a subset of the string of qubits, she had measured and sent to Bob. Using a modification of this attack, she may decide to not to intercept qubits from TP to Bob and still perform a similar attack. Specifically, Alice (or equivalently Bob) can also extract all the information of Bob by intercepting all the qubits that Bob has sent to TP and measuring the pair qubits of each qubit that Alice had measured. None of these two attacks can be detected using the semi-quantum subroutine alone. However, these attacks can be circumvented by allowing Alice and Bob to prepare individual quantum keys with quantum enabled TP using SQKA protocol (as proposed in Ref. [52]). The use of SQKA scheme instead of a SQKD scheme would allow the protocol to succeed even when the TP is not honest. Now, using SQKA scheme, if Alice and TP (Bob and TP) prepare a key $K_{AT}$ ($K_{BT}$), and Alice and Bob use these additional keys $K_{AT}$ and $K_{BT}$ to encode their messages $M_A$ and $M_B$, respectively (as described in SQ7), then the above mentioned attacks can be circumvented. This is easy to observe that in the modified scheme, by performing one of the above mentioned attacks though Alice would obtain $K_B$, which would not help her to deduce $M_B$ as she is completely ignorant about $K_{BT}$.

Additionally, it is worth noting that the SQPC protocol intrinsically uses a scheme for QKD to obtain its security against the untrusted TP and external attackers. Thus, a QKD scheme, which is unconditionally secure [27,43] and composable [44], is used here to provide security from attackers other than Alice and Bob (primarily from the malicious TP). To ensure this security, Alice and Bob are required to be honest as they do not have a better strategy (to protect their individual secret from the malicious TP) than to honestly share a key. Here, we may note that compossibility [45] of QKD plays a crucial role here as it specifies additional security criteria that must be fulfilled in order for QKD to be composed with other tasks to form a larger application like QPC. Moreover, we may note that the criteria for compossibility would be more stringent in a situation (like QPC) that involves mutually mistrustful parties (see [46] for details). Further, unlike the orthogonal-state-based protocol for QPC which uses QKA involving both dishonest parties, in SQPC, Alice and Bob share a quantum key using a semi-quantum KD protocol [37], which does not allow any one of them to solely decide the entire key (thus, the QKD scheme used here can be viewed as a weaker version of QKA) as the QKD scheme considered here puts both Alice and Bob on the same footing. This is so because the untrusted TP prepares Bell states and shares among Alice and Bob to form a symmetric key. Consequently, neither Alice nor Bob can control the whole key. Finally, if both of them encrypt their information with this symmetric key $K_{AB}$, the result at the TP’s end will have a $R$ which would be free from the contribution of key. This fact has been exploited in some of the recent proposals where the competing parties share a quantum key [5,14,18] or a random number [13] honestly for security against untrusted TP.

The qubit efficiency of the proposed QPC protocols can be calculated using the quantitative measure proposed in Refs. [47]. The efficiency $\eta = \frac{c}{q+b}$, where $c$ number of classical bits are transmitted using the total number of $q$ qubits, and $b$-bits classical communication is involved. It should be noted that only the classical communication used for decoding the message is considered, not the classical communication required for eavesdropping checking. In both the orthogonal-state-based QPC and SQPC protocols, Alice and Bob share their $N$-bit secrets, contributing equally in $c = 2N$. The orthogonal-state-based QPC involves $4N$ qubits, which are shared in a secure manner using $4N$ number of additional decoy qubits. Finally, Alice and Bob announce $C_{AB}$ of $N$-bits each which is followed by 1 bit of classical communication by TP (i.e., the announcement that discloses whether the assets are equal (say, 0) or not (say, 1)). Additionally, Alice and Bob also share a quantum key and the qubits and bits used to obtain this shared key should also be counted in the computation of efficiency. Here, we have chosen orthogonal-state-based QKA scheme [28] for considering the resources involved in sharing the quantum key. Using Shukla et al.’s scheme [28], and exploiting the dense coding capacity of the quantum channel used, an $N$-bit quantum key can be shared by two parties using $4N$ qubits (2$N$ as quantum channel and 2$N$ decoy qubits), which involves $3N$-bit classical communication. Therefore, total number of qubits used are $q = 12N$ with additional classical communication $b = 5N + 1$. Thus, the efficiency of the orthogonal-state-based QPC protocol would be $\eta = \frac{2N}{17N+1}$, which becomes 11.76% for large $N$.

Similarly, in case of SQPC protocol, 16$N$ qubits are initially prepared by TP, subsequently 8$N$ qubits are prepared by Alice and Bob (4$N$ each) as a replacement of the qubits measured by them. TP announces 8$N$ bit measurement outcomes which Alice and Bob use to generate their keys with the help of 4$N$ bit of classical messages available with each of them. Finally, Alice and Bob both share an $N$-bit encrypted messages which helps TP to obtain and announce 1 bit final result. Additionally, both classical parties also share a secure quantum key [37] for which they use 24$N$ bits (i.e., TP initially prepares 16$N$ qubits, and Alice (Bob) prepares additional 4$N$ (4$N$) qubits). Similar to the SQPC protocol, during key distribution TP also announces 8$N$-bit information which is followed by 4$N$-bit announcement by both Alice and Bob. It is important to note that Alice and Bob keep the bit values only for specific announcements by TP. Here, we have considered the asymptotic case where equal probabilities for favorable and unfavorable measurement outcomes of TP has been considered. In the present SQPC protocol, TP had also shared one unconditionally secure quantum key with each classical user using a SQKA scheme [52], which involves 5$N$ qubits and 5$N$ bits for each $N$-bit shared key.
Using all these values, the efficiency of the SQPC protocol is \( \eta = \frac{2N}{N^2 + 1} \). For large values of \( N \), \( \eta \approx 1.96\% \). Thus, the protocol 1 proposed here is more efficient than the protocol 2, which is not surprising as we have considered two classical users in the second protocol.

Here, it would be worth mentioning that in analogy with the classical communication involved in eavesdropping checking process (which is not counted in computation of \( \eta \)), we have not counted the classical bits that are used to check correlations between \( K_A \) and \( K_B \).

4 Effect of different noise models on the proposed protocols

In this section, we aim to study the effect of a set of noise models on the feasibility of both the QPC schemes proposed here. For this we will use the operator sum representation of a transformed quantum state \((\rho')\), initially prepared in \( \rho = |\psi\rangle \langle \psi| \), which is

\[
\rho' = \sum_i K_i \rho K_i^\dagger.
\]

Here, the operators satisfying \( \sum_i K_i^\dagger K_i = I \) ensure the trace-preserving nature of the quantum channel. In what follows, first we will describe the Kraus operators of the noise channels and mathematical formulation used in the present letter. Subsequently, we will analyze both of the proposed schemes under the effect of various noise models. Specifically, we will consider the amplitude damping (AD) channel, bit flip channels, phase flip channels and depolarizing channels. Let us first describe an AD channel, which causes the loss of energy from the system to its surrounding considered as a reservoir at vacuum.

1. Amplitude damping: The Kraus operators for AD channels \([42]\) are \( E_0^{AD} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \) and \( E_1^{AD} = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \). Here, and in what follows, \( p \) is the probability of error due to the specific noise discussed.

We will also study the effect of Pauli noise, which affects the qubit independent of its initial state with a certain probability and leaves it unaffected with the remaining probability. The error caused due to this kind of noise can be studied as bit flip (BF), phase flip (PF), and depolarizing channel (DC) noise.

2. Bit flip channels: The BF channel, which flips the qubit with probability \( p \) and leaves it unchanged with remaining probability, can be given by the Kraus operators \([42]\) \( E_0^{BF} = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( E_1^{BF} = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

3. Phase Flip channels: Similar to the BF channels, the Kraus operators for a phase flip channel are given by \([42]\) \( E_0^{PF} = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( E_1^{PF} = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \), which flips the phase of the qubits with \( p \) probability while identity acts with the remaining probability. It is interesting that the effect of phase damping noise on certain scheme can be reduced from the obtained results for phase flip channels (\([43]\) and references therein), so we have not explicitly studied the phase damping kind of noise.

4. Depolarizing channels: If the noisy channels leaves the state unchanged with a certain probability and completely mixed with the remaining probability, the Kraus operators for such a channel would be \([42]\) \( E_0^{DC} = \sqrt{1-p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( E_1^{DC} = \sqrt{p} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( E_2^{DC} = \sqrt{p} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \), and \( E_3^{DC} = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

In both the QPC schemes proposed here, TP initially prepares one of the Bell states and shares that with two distant parties. Therefore, \( \rho \) will be a two qubit density matrix evolving under two independent quantum channels. Mathematically, the transformed density matrix can be written as

\[
\rho' = \sum_{i,j} E_i^k (p_1) \otimes E_j^l (p_2) \rho (E_i^k (p_1) \otimes E_j^l (p_2))^\dagger,
\]

where \( p_i \) corresponds to the probability of decoherence in the \( i \)th qubit and \( E_i^k \) are the Kraus operators of the noisy channel acting on the particular qubit, with \( k, l \in \{ AD, BF, PF, DC \} \). To quantify the effect of noise we will use a distance based measure, fidelity, between the quantum state received by the receivers (Alice and Bob here) after the effect of a specific noise channel and the quantum state expected in the ideal condition, when noisy channel do not alter the quantum state. In fact, in the ideal condition, the state at the receivers end should be the same prepared by TP. Therefore, the fidelity between the initial \( \rho \) and final \( \rho' \) state is

\[
F = \langle \psi | \rho^\dagger | \psi \rangle.
\]
Here, it is also worth mentioning that the fidelity expression used here is square of the conventional fidelity expression but is also used widely as a measure (36,49,50 and references therein). Now, we will discuss the effect of all these noise models on the proposed schemes.

4.1 Effect of noise on the protocols

It is noteworthy that the qubits used by Alice and Bob to prepare their respective keys undergo the same fate in both the schemes. Therefore, the discussion what follows is applicable to both orthogonal-state-based QPC and SQPC protocols. Specifically, when both the qubits of the initial state $|\psi^{\pm}\rangle$ are subjected to AD noise the compact analytic expression of the obtained fidelity is

$$F = \frac{1}{4} \left( 2 + 2 \sqrt{(1 - p_1)(1 - p_2)} - (p_2 + p_1) + 2p_1p_2 \right), \quad (4)$$

where $p_1$ and $p_2$ are the probabilities of errors in the first and second qubits, respectively. The same calculated for the different choice of initial state, i.e., $|\phi^{\pm}\rangle$ is

$$F = \frac{1}{4} \left( \sqrt{1 - p_1} + \sqrt{1 - p_2} \right)^2. \quad (5)$$

In what follows, the same fidelity expressions are obtained irrespective of the initial choice of Bell state under different noisy environment.

Similar to the earlier case, when the first qubit evolves under the effect of AD channel, whereas the second qubit is affected by BF channel the calculated fidelity turns out to be

$$F = \frac{1}{4} \left( -2 \left( 1 + \sqrt{1 - p_1} \right) (-1 + p_2) + p_1 (-1 + 2p_2) \right). \quad (6)$$

Similarly, when the first qubit is subjected to AD noise and the second qubit evolves under the effect of PF and DC noise models, we obtain the fidelity expressions as

$$F = \frac{1}{4} \left( 2 + 2 \sqrt{1 - p_1} - p_1 - 4 \sqrt{1 - p_1p_2} \right) \quad (7)$$

and

$$F = \frac{-2p_1 (-1 + p_2) + (1 + \sqrt{1 - p_1}) (-4 + 3p_2)}{4(-2 + p_2)}, \quad (8)$$

respectively.

Before we consider the effect of other kind of noise models (other than AD) on the first qubit, it is worth noting that the scheme is symmetric with respect to Alice and Bob (due to the symmetry of the Bell states used), and the similar fidelity expressions are expected whether the first (second) qubit is affected by AD (BF) or the first (second) qubit is affected by BF (AD) noise. In fact, if we interchange the values of $p_1$ and $p_2$ we can obtain the fidelity expression of one from the other. Due to this fact, we are enlisting here only the expressions for fidelity, for those cases which lack this kind of symmetry (i.e., which cannot be obtained from another expression by simply using a symmetry argument).

If the first qubit is subjected to BF noise, while the second qubit evolves under BF, PF and DC noises, then a systematic computation would yield the fidelity expressions as

$$F = 1 - p_2 + p_1 (-1 + 2p_2), \quad (9)$$

$$F = (-1 + p_1)(-1 + p_2), \quad (10)$$

and

$$F = \frac{3}{2} - 2p_1 + \frac{1 - 2p_1}{1 - 2 + p_2}, \quad (11)$$

respectively. It is interesting to report that the fidelity obtained while both the qubits evolve under PF environment is exactly the same as that under BF noise (9).

If we consider PF noise on the Alice’s qubit and DC noise on the Bob’s qubit we obtain a closed form analytic expression of fidelity as

$$F = \frac{(-1 + p_1)(-4 + 3p_2)}{2(2 - p_2)}. \quad (12)$$
Figure 1: The fidelity variation of the quantum states for the QPC protocols in specific cases of noisy environment is shown here. The smooth (blue), small-dashed (red) and dot dashed (cyan) curves correspond to the fidelity of the obtained states when only Alice’s (or Bob’s) qubit is subjected to AD, BF and DC noises, respectively. Similarly, the dotted (black), large-dashed (orange) and large-dot dashed (magenta) lines show variation in the fidelity of the obtained states with probability of errors, when both Alice’s and Bob’s qubits are subjected to the same kind of noise with equal probabilities of errors, i.e., AD, BF and DC noises, respectively. In both the cases of PF channels, the same fidelity as that under BF noise is obtained. As discussed in the text the choice of initial Bell states only matter when both the qubits are subjected to AD noise. Here, the dotted (black) line shows the fidelity variation for the initial state \( |\psi^\pm\rangle \) while the small-dashed (red) line corresponds to the initial state \( |\phi^\pm\rangle \).

Finally, when both the qubits are subjected to DC noise, the obtained fidelity between the quantum state affected by noise and the initial state is

\[
F = \frac{8 - 6p_2 + p_1(-6 + 5p_2)}{2(-2 + p_1)(-2 + p_2)}. \tag{13}
\]

Before we proceed with the analysis of the various fidelity expressions obtained under different combinations of noisy channels, it is worth establishing the motivation of this study. Specifically, we have already mentioned the fidelity between the initial and final state in the ideal conditions is expected to be unity. For the sake of argument, consider that one of the channel (either TP to Alice or TP to Bob) is noiseless, then this unit fidelity falls considerably (cf. Fig. 1). Specially, for the higher values of decoherence rate, it even becomes null for BF or PF channels. More realistic scenario would be where both the qubits undergo decoherence. If we consider that the rate of decoherence in both the channels is the same, then the fidelity of the obtained state evolving under BF or PF noise turned upside down to become unity for higher probability of errors. Though, a similar benefit appears when the qubits evolve under AD noise. However, no such advantage is seen in presence of the DC noise.

We have already reported ten fidelity expressions (Eqs. (4)-(13)) calculated for the quantum states shared between all the parties and evolving under different noise channels. All the analytic expressions depend on two independent variables (probability of error). Therefore, variation of fidelity with these variables can be explicitly illustrated using contour plots. Specifically, in Figs. 2-5 we have shown the contour variation of Eqs. (4)-(13). In Fig. 2 the fidelity as a function of both the probabilities \( p_i \)'s is shown when the initial choice of Bell state by TP was \( |\psi^\pm\rangle \) and \( |\phi^\pm\rangle \) in (a) and (b), respectively. A gradual decay in fidelity with each independent probability appears in (a), and when both the decoherence rates are higher, relatively better results in terms of fidelity are visible. Whereas (b) gives a contrasting picture, and fidelity is found to fall continuously with both the probabilities. The comparative analysis reveals that \( |\psi^\pm\rangle \) gets less affected than \( |\phi^\pm\rangle \). It may be easily observed in both of these plots (i.e., Figs. 2(a) and 2(b)) that the contour variation is symmetric along a diagonal from lower left to top right corner due to the fact that the same noisy channel is affecting both the qubits. It will remain valid in all such plots to follow (cf. Figs. 4(a) and 5(b)) where the same noise acts on both qubits and symmetric nature is visible in the contour plots.

If we consider that quantum channel for TP to Alice is characterized as AD while Bob’s qubit is affected by one of the BF, PF or DC noise, then the symmetry observed in Fig. 2 is lost (cf. Fig. 3(a), (b), and (c), respectively). The variations of fidelity in all these cases show that it is not sufficient to characterize one quantum channel, as the type of noise applied to the second qubit is also relevant. If we look closely at/in these three contour plots (i.e., Fig. 3(a), (b), and (c)), we can see the same seven contour lines at the starting point at/on the \( X \)-axis undergoing different variation due to the effect of the noisy channel of the second qubit. Among all these noise models, DC noise is found to minimally affect the fidelity, while PF noise is found to have most devastating effects.

Another possibility is that the first qubit is subjected to BF noise. In Fig. 4(a), another qubit is also traveling through the BF channel. Interestingly, the degrading of fidelity with an increase in probability of error in one of the channels can be controlled by...
Figure 2: QPC protocols subjected to AD channels, i.e., both the qubits evolve under AD noise. In (a) and (b), the choice of the initial Bell state by TP is $|\psi^\pm\rangle$ and $|\phi^\pm\rangle$, respectively.
Figure 3: QPC protocols subjected to a noisy environment, when the first qubit of the Bell state (Alice’s qubit) is subjected to AD, while the second (Bob’s) qubit evolves under different noisy channels. In (a), (b), and (c), it evolves under BF, PF, and DC, respectively.
increasing the decoherence rate for the other channel. The same intercepts on the X-axis for all three plots correspond to the same type of noise acting on the first qubit. Unlike the second qubit evolving under BF channel, PF noise does not check the decay in fidelity (cf. Fig. 4(b)). In fact, least fidelity is obtained in Fig. 4(b) when compared among the second qubit evolving under BF, PF, and DC noise. While the second qubit is subjected to DC noise (Fig. 4(c)), the fidelity can be further maintained, but the variation is quite different from Fig. 4(a).

It should be noted that when the first (second) qubit is subjected to BF (AD) noise contour variation of the fidelity can be obtained by interchanging the X and Y axes in Fig. 3(a). Due to this reason, these cases will not be further discussed. Similarly, Fig. 4(a) also shows the variation of fidelity when both qubits travel through the PF channels.

Another interesting possibility would be to consider the first and second qubits evolving under PF and DC noise, respectively. It can be observed from Fig. 5(a) that DC noise affects the fidelity comparatively less than other noises. Finally, we consider both the qubits subjected to DC noise in Fig. 5(b). The symmetric decay in fidelity shown in this case has a specific characteristic that even the lowest value of the obtained fidelity is an appreciable amount when compared with the remaining cases.

It is also important to observe a comparative analysis of the effect of different kinds of noisy channels. Specifically, if we consider the first qubit travels through a specific channel with the known rate of error than the effect of noisy environment due to the second qubit is studied here in two dimensional variation. Specifically, in Fig. 6 we consider two cases (namely, decoherence rate $p_1 = 0.2$ and 0.8) of variation of fidelity with/in error rate for the second channel. The effects of different values of $p_1$ are shown in the figure, as for its higher value, the fidelity remains the same for initial $|\psi^{\pm}\rangle$ state, while falls sharply for initial $|\phi^{\pm}\rangle$. It can be seen that for a higher rate of damping the effect of noise on the other qubit is hardly relevant (cf. Fig. 6(b)).

Similarly, when the first qubit evolves under the effect of BF channel with low error rate, the second qubit subjected to DC noise performed best as shown in Fig. 7(a). However, in Fig. 7(b) the second qubit under BF channel outperformed it for higher probability of error. Interestingly, the two dimensional cut shown in Fig. 7(b) the fidelity is seen to increase with increasing probabilities of errors in the second channel.

When the first qubit is fixed to be traveling through a PF channel with intermediate error rate, the second qubit evolving under DC noise suffers most (cf. Fig. 8). As the fidelity in this case is initially highest for lower values of error in the second channel, becoming second to the PF channel for higher errors. The obtained fidelity in the same case, i.e., second qubit under DC noise, falls comprehensively even below AD noise for sufficiently large values of error rates for the second channel.

Finally, we will consider when the first qubit subjected to DC noise in Fig. 9. Here, we can observe that for higher values of error in the first channel though fidelity in two of the cases appears increasing, but it is consistent with corresponding contour plots (cf. Figs. 2-5). It is worth commenting here that fidelity obtained when the second qubit is evolving under the BF channel improves considerably to be better than in damping and DC noise.

So far, we have been discussing the effect of noise on the qubits which are used by Alice and Bob to obtain random key strings. However, it is also desirable that during eavesdropping checking stage the presence of Eve and noise can be discriminated. This was the motivation of our last work [35], where we have reported that different suitable choices of decoy qubits and eavesdropping (cf. Figs. 2-5). It is worth commenting here that fidelity obtained when the second qubit is evolving under the BF channel improves considerably to be better than in damping and DC noise.

4.2 Effect of noise on semi-quantum subroutine

In this section, we discuss the effect of noise on the semi-quantum subroutine. The security can be inherently achieved by using one of them. Specifically, in a semi-quantum subroutine, a quantum party prepares the decoy qubits and classical party reflects them to quantum party, who later calculates the error rate for that. Depending upon the choice of decoy qubits BB84 or GV subroutine can be used.

However, from the point of view of the effect of noise on this subroutine, to and fro communication of decoy qubits should be considered while studying the effect of noisy environments. Here, for the simplicity of discussion, we will consider the same kind of noise with the same error rate acting on certain qubit while its to and fro travel.

Specifically, when both the qubits undergo damping during both rounds of travels, the obtained fidelity expressions are

$$F = \frac{1}{4} \left( (2 + p_2)^2 - 2p_1 (2 + p_2) (-1 + 2p_2) + p_1^2 (1 + 2 (2 + p_2) p_2) \right) \tag{14}$$

and

$$F = \frac{1}{4} (-2 + p_1 + p_2)^2 \tag{15}$$

for the initial choice of Bell state as $|\psi^{\pm}\rangle$ and $|\phi^{\pm}\rangle$, respectively. In the remaining cases, when the first qubit is subjected to damping and the second qubit is considered under BF, PF, and DC noises, the obtained fidelity is calculated as

$$F = \frac{1}{4} (-2 + p_1) \left( -2 + p_1 (1 - 2p_2)^2 - 4 (-1 + p_2) p_2 \right) \tag{16}.$$
Figure 4: QPC protocols subjected to noisy environment, when the first qubit of the Bell state (Alice’s qubit) is subjected to BF, while the second (Bob’s) qubit evolves under different noisy channels. In (a), (b) and (c) Bob’s qubits is affected by BF, PF, and DC, respectively. As mentioned in the text (a) also corresponds to the PF noise on both the qubits.
Figure 5: (a) QPC protocol subjected to a noisy environment, when the first qubit of the Bell state (Alice’s qubit) is subjected to PF while the second (Bob’s) qubit evolves under DC. (b) Similarly, when both the qubits of the Bell state (i.e., Alice’s and Bob’s qubits) evolve under the effect of DC noisy environment.
Figure 6: The effect of noise on the fidelity obtained in the QPC protocol when the first qubit is subjected to AD noise, while the second qubit is subjected to different noisy channels. The smooth (blue), large-dashed (orange) lines correspond to the effect of AD noise on the second qubit for the initial Bell state $|\psi^\pm\rangle$ and $|\phi^\pm\rangle$, respectively. Similarly, the dotted (red), small-dashed (magenta), large-dot dashed (cyan) curves correspond to the fidelity of the obtained states when Bob’s qubit is subjected to BF, PF and DC noises, respectively. In (a) and (b), the decoherence rate for the Alice’s channel is 0.2 and 0.8, respectively. In the plots, (also in the following figures) the probability of error $p_2$ is written as $p$. 
Figure 7: The effect of noise on the fidelity obtained in the QPC protocol when the first qubit is subjected to BF noise, while the second qubit is subjected to different noisy channels. The smooth (blue), dotted (red), dashed (magenta), dot dashed (cyan) curves correspond to the fidelity of the obtained states when Bob’s qubit is subjected to AD, BF, PF and DC noises, respectively. In (a) and (b), the probability of error in Alice’s channel is 0.2 and 0.8, respectively.
Figure 8: The effect of noise on the fidelity obtained in the QPC protocol when the first qubit is subjected to PF noise, while the second qubit is subjected to different noisy channels. The smooth (blue), dotted (red), dashed (magenta), dot dashed (cyan) curves correspond to the fidelity of the obtained states when Bob’s qubit is subjected to AD, BF, PF and DC noises, respectively. In (a) and (b), the probability of error in Alice’s channel is 0.4 and 0.6, respectively.
Figure 9: The effect of noise on the fidelity obtained in the QPC protocol when the first qubit is subjected to DC noise, while the second qubit is subjected to different noisy channels. The smooth (blue), dotted (red), dashed (magenta), dot dashed (cyan) curves correspond to the fidelity of the obtained states when Bob’s qubit is subjected to AD, BF, PF and DC noises, respectively. In (a) and (b), the probability of error in Alice’s channel is 0.2 and 0.6, respectively.
Figure 10: The effect of noise on the fidelity of the quantum states used by TP for eavesdropping checking in the SQPC protocol when the first both the qubits are subjected to AD noise for initial choice of Bell states as $|\psi^\pm\rangle$ and $|\phi^\pm\rangle$ in the smooth (blue) and dashed (magenta) lines, respectively. The dotted (red) and dot dashed (cyan) curves correspond to the fidelity of the obtained states when both qubits are affected by BF (equivalently PF) and DC noises, respectively.

\[
F = \frac{1}{4} \left( (-2 + p_1)^2 + 8 (-1 + p_1) p_2 - 8 (-1 + p_1) p_2^2 \right),
\]  
and

\[
F = \frac{(-2 + p_1) (-8 + 4 p_1 (-1 + p_2)^2 + (12 - 5 p_2) p_2)}{4 (-2 + p_2)^2},
\]

respectively. In another case, when the first qubit is subjected to BF channel and second qubit evolves under the effect of BF, PF, and DC noise the analytic expressions of fidelity are

\[
F = 1 - 2 p_1 (1 - 2 p_2)^2 + 2 p_1^2 (1 - 2 p_2)^2 + 2 (-1 + p_2) p_2,
\]

\[
F = -\frac{(1 + 2 (-1 + p_1) p_1) (1 + 2 (-1 + p_2) p_2)}{-1 + (-1 + p_1)^2 (-1 + p_2) p_2},
\]

and

\[
F = \frac{1}{2 (-2 + p_2)^2} \left[ 8 - 16 p_1 (-1 + p_2)^2 + 16 p_1^2 (-1 + p_2)^2 + p_2 (-12 + 5 p_2) \right],
\]

respectively. Similarly, when the first (second) qubit is affected by PF (DC) noise, then the fidelity can be calculated as

\[
F = \frac{(1 + 2 (-1 + p_1) p_1) (8 + p_2 (-12 + 5 p_2))}{2 (-2 + p_2)^2},
\]

Finally, when both the qubits are subjected to DC noise, we obtain the expression for fidelity as

\[
F = \frac{1}{2 (-2 + p_1)^2 (-2 + p_2)^2} \left[ 4 (8 + p_2 (-12 + 5 p_2)) - 4 p_1 (12 + p_2 (-20 + 9 p_2)) + p_1^2 (20 + p_2 (-36 + 17 p_2)) \right].
\]

The fidelity expressions for the remaining cases are not explicitly written here, as those expressions can be derived from the above expressions by symmetry argument. The analysis of the obtained fidelity expression is simplified here by considering the case when both the qubits are affected by the same kind of noise with the same error rate. In this particular case, as shown in Fig. 10 it is observed that the fidelity for the damping effect, for $|\phi^\pm\rangle$ as the initial choice of TP, gradually decays to zero. Further, for the state evolving over the BF or PF channel, a revival of with an increasing error rate is observed. For the remaining cases (i.e., for $|\psi^\pm\rangle$ state under AD channels and an arbitrary Bell state subjected to DC noise), fidelity falls gradually and becomes constant around $p = 0.5$. 

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5 Conclusion

Two protocols for QPC have been proposed. The essential beauty of the present work underlies in the fact that both of the proposed protocols for QPC are fundamentally different from all the existing protocols of QPC. Specifically, the first protocol is designed solely using orthogonal states and the security of the protocol does not rely on conjugate coding. This is interesting because of the fact that BB84 protocol [51] for QKD and early protocols of secure quantum communication (see Chapter 8 of Ref. [29] for a review), whose security arise from the fact that one cannot perform simultaneous measurements in mutually unbiased bases (conjugate coding), led to a perception that unconditional security of QKD and other quantum tasks arises from the conjugate coding (i.e., through the use of non-orthogonal states as the states prepared in mutually unbiased bases are not orthogonal to each other). Interestingly, in Refs. [27,43] it has been firmly established that QKD is possible without using non-orthogonal states, i.e., by using orthogonal states only. Subsequently, a set of orthogonal-state-based protocols for various secure quantum communication tasks has been proposed by some of the present authors [28–30,32], and those protocols have established that most of the quantum cryptographic tasks that can be performed using non-orthogonal states can also be performed using orthogonal states. Specifically, it has been established that one can perform quantum secure direct communication [29,30,32], QKA [28], etc., solely using orthogonal states. These orthogonal-state-based protocols are fundamentally different from the conjugate coding based schemes as their security does not arise from our inability to perform simultaneous measurement using two mutually unbiased bases. However, until now, no effort has been made to design an orthogonal-state-based protocol for QPC. Thus, the first protocol of the present letter is fundamentally different from all the existing protocols for QPC, and it is unique in that sense. Interestingly, the second protocol proposed here is also an orthogonal-state-based, but in the case of the second protocol we did not stress much on this characteristic, because it has another unique characteristic, which has not been explored in any of the existing schemes for QPC: the protocol is semi-quantum in nature, i.e., it allows some of the users to be classical. These two protocols answer couple of questions (in context of QPC) that are founditionally important. For example, both the protocol answer: Which quantum properties are essential for the implementation of schemes of QPC? It establishes that conjugate coding is not required and ensuring simultaneous non-availability of all the pieces of information to Eve after splitting it into several pieces is sufficient to ensure security of schemes for QPC. Specifically, conjugate-coding-based schemes achieve security by hiding the basis information of two mutually unbiased bases, whereas an orthogonal-state-based employs geographical information splitting using PoP technique. The second protocol answers another question: How much quantumness is needed (or in other words, how many quantum users are needed) to implement a scheme for QPC? It is established that only TP needs to be quantum and all other users can be classical. This is in sharp contrast to the existing protocols where all the users are required to be quantum. Thus, the second protocol of ours clearly uses reduced quantum resources, as two parties comparing their assets are completely classical in nature. However, there exists a trade-off between the amount of quantum resources used and the qubit efficiency achieved. Specifically, the qubit efficiency of the SQPC protocol (i.e., our second protocol) is found to be much lower compared to the first protocol where all the users are quantum in nature. Clearly, this happens because in SQPC, a lesser number of parties possess quantum resources and that increases the requirement of $q$ and $b$ for accomplishing the same task (i.e., to communicate the same amount of classical information $c$ by following the restrictions of the same cryptographic task).

The feasibility of both of the proposed schemes is analyzed under well-known noise models, such as AD, BF, PF, and DC. The study has led to many interesting conclusions. In general, the effect of any specific noise model is independent of the choice of the initial Bell state. However, it is observed that when both the qubits are subjected to AD noise, the fidelity expressions depend on the parity of the Bell state. Further, the study has established that the quantum state is least affected due to DC illustrated through a higher value of fidelity. Interestingly, it has also been observed that the higher error rates in one of the quantum channel can also lead to positive effects. Specifically, a few cases have been observed where the higher decoherence rate in one of the quantum channel had resulted in higher fidelity compared to the situation having lower error rates in the same channel. Another interesting result has been obtained when both the qubits were traveling over a BF channel, as in this case a revival in fidelity with increasing error rate in a quantum channel has been observed. The effect of noise in the eavesdropping checking for SQPC protocol is also considered, as it is necessary to differentiate between noise and Eve. It has been observed that the parity 1 Bell states have least fidelity when subjected to AD noise.

Keeping the above in mind, we conclude the letter by noting that a QPC scheme neither requires user other than TP to be quantum, nor it requires to use the quantum states prepared in mutually unbiased bases. However, to implement a scheme for QPC in a realistic situation, characterization of the channel (knowledge of noise(s) present in the quantum channel) would play a very important role, as in the absence of this knowledge, TP may reach to an incorrect conclusion.

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