MASS AND Riemannian Polyhedra

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Abstract. We show that the concept of the ADM mass in general relativity can be understood as the limit of the total mean curvature plus the total defect of dihedral angle of the boundary of large Riemannian polyhedra. We also express the $n$-dimensional mass as a suitable integral of geometric quantities that determine the $(n-1)$-dimensional mass.

1. Introduction

On a complete asymptotically flat manifold with nonnegative scalar curvature, the Riemannian positive mass theorem asserts the mass of the manifold is nonnegative, and is zero if the manifold is isometric to the Euclidean space. In dimension three, the theorem was first proved by Schoen and Yau [15], and by Witten [18]. Recently, a new proof was given by Bray, Kazaras, Khuri and Stern [3].

In [7], Gromov suggested a geometric comparison theory for scalar curvature via the use of Riemannian polyhedra. More precisely, given a convex polyhedron $P^n$ in the Euclidean space $(\mathbb{R}^n, \bar{g})$, Gromov [7, 8] conjectured that, if $g$ is a Riemannian metric on $P$ so that

- $g$ has nonnegative scalar curvature,
- each face of $\partial P$ is weakly mean convex in $(P, g)$, and,
- along any edge of $\partial P$, the dihedral angle of $(P, g)$ is no larger than the constant dihedral angle of $P$ in $(\mathbb{R}^n, \bar{g})$,

then $(P, g)$ is isometric to a Euclidean polyhedron.

Significant progress toward Gromov’s conjecture has been obtained by Li [9, 10]. In dimension three, Li [9] proved the conjecture for a large class of polyhedra that includes cubes and 3-simplices. In higher dimensions, Li [10] proved the conjecture for a class of prisms of dimensions up to 7.

In this paper, we present a result that ties the concept of mass of asymptotically manifolds to the polyhedra comparison theory of Gromov.

Theorem 1.1. Let $(M^n, g)$ be an asymptotically flat manifold with dimension $n \geq 3$. Let \( \{P_k\} \) denote a sequence of Euclidean polyhedra in a coordinate chart \( \{x_i\} \) that defines the end of \( (M, g) \). Suppose \( \{P_k\} \) satisfy the following conditions

a) $P_k$ encloses the coordinate origin and \( \lim_{k \to \infty} r_{P_k} = \infty \), where

$$r_{P_k} = \min_{x \in \partial P_k} |x|,$$

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b) $|F(\partial P_k)| = O(r_k^{n-1})$, where $F(\partial P_k)$ denotes the union of all the faces in $\partial P_k$, and $|F(\partial P_k)|$ is the Euclidean $(n-1)$-dimensional volume of $F(\partial P_k)$;

c) $|E(\partial P_k)| = O(r_k^{n-2})$, where $E(\partial P_k)$ denotes the union of all the edges in $\partial P_k$, and $|E(\partial P_k)|$ is the Euclidean $(n-2)$-dimensional volume of $E(\partial P_k)$; and

d) along each edge, the Euclidean dihedral angles $\bar{\alpha}$ of $P_k$ satisfies

$$|\sin \bar{\alpha}| \geq c$$

for some constant $c > 0$ that is independent on $k$.

Then, as $k \to \infty$, the mass of $(M, g)$, which we denote by $m(g)$, satisfies

$$(1.1) \quad m(g) = \frac{1}{(n-1)\omega_{n-1}} \left( -\int_{F(\partial P_k)} H\, d\sigma + \int_{E(\partial P_k)} (\alpha - \bar{\alpha})\, d\mu \right) + o(1).$$

Here $\omega_{n-1}$ is the volume of the standard $(n-1)$-dimensional sphere, $H$ denotes the mean curvature of the faces $F(\partial P_k)$ in $(P_k, g)$, $\alpha$ denotes the dihedral angle along the edges $E(\partial P_k)$ of $(P_k, g)$, and $d\sigma, d\mu$ denote the volume element on $F(\partial P_k), E(\partial P_k)$, respectively, with respect to the induced metric from $g$.

![Figure 1](image)

**Figure 1.** A sequence of polyhedra $\{P_k\}$ approaching $\infty$.

We give some remarks about Theorem [1.1]

**Remark 1.1.** $\{P_k\}$ is allowed to consist of different type of polyhedra. Moreover, elements in $\{P_k\}$ can be non-convex. (See Figure 1)

**Remark 1.2.** In 3-dimension, if $\{P_k\}$ is a family of coordinate cubes, [1.1] was one of the formula derived in [13].

**Remark 1.3.** Given any polyhedron $P$ enclosing the coordinate origin, let $P(r)$ denote the scaling of $P$ by a large constant $r$ in the coordinate space. The family $\{P(r)\}$ satisfies all conditions a) – d) above. As a result, formula [1.1] holds for any family of coordinate polyhedron $\{P(r)\}$.

**Remark 1.4.** The angle condition $|\sin \bar{\alpha}| \geq c$ is imposed because of the formula $y'(x) = -\frac{1}{\sin y}$, if $y = \arccos x$. In the proof of Theorem [1.1] we read dihedral angles from metric coefficients. A lower bound on $|\sin \bar{\alpha}|$ serves as a sufficient condition to convey estimates on the metric to estimates on the defect of dihedral angles.
We recall the definition of an asymptotically flat manifold and its mass.

**Definition 1.1.** A Riemannian manifold \((M^n, g)\) is asymptotically flat (with one end) if there exists a compact set \(K\) such that \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^n \setminus B_r(0)\), where \(B_r(0) = \{ |x| < r \}\) for some \(r > 0\), and with respect to the standard coordinates \(\{x_i\}\) on \(\mathbb{R}^n\), the metric \(g\) satisfies \(g_{ij} = \delta_{ij} + h_{ij}\), where

\[
(1.2) \quad h_{ij} = O(|x|^{-p}), \quad \partial h_{ij} = O(|x|^{-p-1}), \quad \partial^2 h_{ij} = O(|x|^{-p-2})
\]

for some \(p > \frac{n-2}{2}\). Moreover, the scalar curvature of \(g\) is integrable on \((M, g)\).

On an asymptotically flat manifold \((M^n, g)\), the ADM mass \([1]\) is given by

\[
(1.3) \quad m(g) = \lim_{r \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_r} (g_{ij,i} - g_{ii,j}) \tilde{\nu}^j d\tilde{\sigma}.
\]

Here \(S_r = \{x \mid |x| = r\}\) denotes a coordinate sphere, \(\tilde{\nu}\) is the Euclidean outward unit normal to \(S_r\), \(d\tilde{\sigma}\) is the Euclidean volume element on \(S_r\), and summation is applied over repeated indices. It was shown by Bartnik [2], and by Chruściel [6] independently, that \(m(g)\) is a geometric invariant of \((M, g)\), independent on coordinates satisfying \(1.2\).

Often it is convenient to compute the mass in \(1.3\) with \(\{S_r\}\) replaced by \(\{\partial D_k\}\), where \(\{D_k\}\) is another suitable exhaustion sequence of \((M, g)\). By Proposition 4.1 in [2] (more specifically, by its proof), it is known

\[
(1.4) \quad m(g) = \lim_{k \to \infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{\partial P_k} (g_{ij,i} - g_{ii,j}) \tilde{\nu}^j d\tilde{\sigma}.
\]

Here \(\{P_k\}\) is a sequence of polyhedra satisfying conditions a) and b) in Theorem 1.1.

The positive mass theorem in general dimensions was shown by Schoen-Yau [16]. (Also see the work of Lohkamp [11]). The following is a corollary of the positive mass theorem and Formula \(1.1\).

**Theorem 1.2.** Let \((M^n, g)\) be a complete, asymptotically flat manifold with nonnegative scalar curvature. Let \(\{P_k\}\) denote a sequence of Euclidean polyhedra satisfying conditions in Theorem \(1.1\). Then, for large \(k\),

\[
(1.5) \quad -\int_{\mathcal{E}(\partial P_k)} H d\sigma + \int_{\mathcal{F}(\partial P_k)} (\alpha - \bar{\alpha}) d\mu \geq 0.
\]

In particular, for any fixed Euclidean polyhedron \(P\),

\[
(1.6) \quad -\int_{\mathcal{E}(r)} H d\sigma + \int_{\mathcal{F}(r)} (\alpha - \bar{\alpha}) d\mu \geq 0,
\]

where \(\mathcal{F}(r)\) and \(\mathcal{E}(r)\) are the faces and edges of the polyhedron \(P(r)\) obtained by scaling \(P\) by a large constant factor \(r\).

The polyhedra \(P_k, P\) in Theorem \(1.2\) do not need to be convex (see Remark \(1.1\)). In this sense, Theorem \(1.2\) indicates that Gromov’s comparison theory of scalar curvature might extend to non-convex Euclidean polyhedra.
In [13], \( \{P_k\} \) was chosen as a family of coordinate cubes in 3-dimension, and a formula that represents the 3-dimensional mass via suitable integration of the angle defect, measured by the boundary term in the Gauss-Bonnet theorem, was obtained. In higher dimensions, by taking \( \{P_k\} \) as coordinate cubes, we give an induction-type formula that evaluates the \( n \)-dimensional mass via quantities that determine the \((n-1)\)-dimensional mass.

Theorem 1.3. Let \((M^n, g)\) be an asymptotically flat manifold with dimension \( n \geq 4 \). Given any \( k \in \{1, \cdots, n\} \), any large constant \( L \), and any \( t \in [-L, L] \), there is a quantity \( m_k^{(n-1)}(t, L) \), associated to the coordinate hyperplane \( \{x_k = t\} \), such that

\[
(1.7) \quad m(g|_{\{x_k = t\}}) = \lim_{L \to \infty} m_k^{(n-1)}(t, L),
\]

and

\[
(1.8) \quad m(g) = \frac{\omega_{n-2}}{(n-1) \omega_{n-1}} \lim_{L \to \infty} \sum_{k=1}^{n} \int_{-L}^{L} m_k^{(n-1)}(t, L) \, dt.
\]

We provide the precise definition of \( m_k^{(n-1)}(t, L) \) and the proof of Theorem 1.3 in Section 4.

2. Difference of the mean curvature

In this section, we derive a formula on the difference of the mean curvatures of a hypersurface with respect to two metrics that are close. The formula will be used in the next section to prove Theorem 1.1.

Setting: Let \( \Sigma^{n-1} \) be a hypersurface in a manifold \( U^n \). Let \( \bar{g}, g \) be two Riemannian metrics on \( U \). We view \( \bar{g} \) as a background metric. Let \( \bar{\gamma}, \gamma \) be the metrics on \( \Sigma \) induced from \( \bar{g}, g \), respectively, so that they point to the same side of \( \Sigma \). Let \( \bar{A}, \bar{H} \) and \( A, H \) be the second fundamental form, the mean curvature of \( \Sigma \) with respect to \( \bar{\nu} \) and \( \nu \), in \((M, \bar{g})\) and \((M, g)\), respectively.

Assumption: We write \( g = \bar{g} + h \), and assume

\[
(2.1) \quad |h|_{\bar{g}} < \epsilon(n) \text{ in } U.
\]

Here \( \epsilon(n) < \frac{1}{n-1} \) is a positive constant that depends only on \( n \), and \( |\cdot|_{\bar{g}} \) denotes the norm of a tensor with respect to \( \bar{g} \).

Proposition 2.1. The difference of the two mean curvatures satisfies

\[
(2.2) \quad 2(H - \bar{H}) = (d \text{tr}_g h - \text{div}_g(h)(\bar{\nu}) - \text{div}_\gamma X - \langle h, \bar{A} \rangle_\gamma) + |\bar{A}|_{\bar{g}} O(|h|^2_{\bar{g}}) + O(||Dh||_{\bar{g}} |h|_{\bar{g}}).
\]

Here \( \text{div}_g(\cdot) \) and \( \text{tr}_g(\cdot) \) denote the divergence and trace on \((U, \bar{g})\); \( \text{div}_\gamma(\cdot) \) denotes the divergence on \((\Sigma, \bar{\gamma})\); \( X \) is the vector field on \( \Sigma \) that is dual to the 1-form \( h(\bar{\nu}, \cdot) \) with respect to \( \bar{\gamma} \). Given two functions \( f \) and \( \phi \), we write \( f = O(\phi) \) to denote \( |f| \leq C|\phi| \) with a constant \( C \) that depends only on \( n \).
We remark that a linearized version of (2.2) can be found in [14, Equation (34)]. Below we verify (2.2). Given any \( p \in \Sigma \), let \( \{ x_i \}_{1 \leq i \leq n} \) be local coordinates around \( p \) in \( M \) such that \( \{ x_\alpha \}_{1 \leq \alpha \leq n-1} \) are coordinates near \( p \) on \( \Sigma \). By definition,

\[
H - \bar{H} = \gamma^{\alpha \beta} A_{\alpha \beta} - \bar{\gamma}^{\alpha \beta} \bar{A}_{\alpha \beta}
\]

\[
= \bar{\gamma}^{\alpha \beta} (A_{\alpha \beta} - \bar{A}_{\alpha \beta}) + (\gamma^{\alpha \beta} - \bar{\gamma}^{\alpha \beta}) \bar{A}_{\alpha \beta} + (\gamma^{\alpha \beta} - \bar{\gamma}^{\alpha \beta})(A_{\alpha \beta} - \bar{A}_{\alpha \beta}),
\]

where \( A_{\alpha \beta} = -\langle D_\partial, \partial_{\beta}, \nu \rangle_g \) and \( \bar{A}_{\alpha \beta} = -\langle \bar{D}_\partial, \partial_{\beta}, \bar{\nu} \rangle_g \).

We estimate \( \gamma^{\alpha \beta} - \bar{\gamma}^{\alpha \beta} \), \( g^{ij} - \bar{g}^{ij} \), and \( \nu - \bar{\nu} \) first. For convenience, suppose \( \{ \partial_i \} \) are orthonormal with respect to \( \bar{g} \) at \( p \). In particular, \( \partial_n = \bar{\nu} \) at \( p \).

**Lemma 2.1.** In the above coordinates,

\[
\gamma^{\alpha \delta} = \bar{\gamma}^{\alpha \delta} - h_{\alpha \delta} + O(|h|^2), \quad g^{ij} = \bar{g}^{ij} - h_{ij} + O(|h|^2),
\]

\[
\nu - \bar{\nu} = (-h_{n\beta} + O(|h|^2)) \partial_\beta + (-1/2 h_{nn} + O(|h|^2)) \partial_n.
\]

Here repeated indices denote summation over those indices. In terms of \( X \), this gives

\[
|\nu - \bar{\nu} + X + \frac{1}{2} h(\nu, \nu)\bar{\nu}|_g = O(|h|^2).
\]

**Proof.** Let \( \tau \) denote the tangential restriction of \( h \) to \( \Sigma \). At \( p \), \( \gamma^{\alpha \beta} = \delta^{\alpha \beta} + \tau_{\alpha \beta} \). Write \( \gamma^{\alpha \beta} = \delta^{\alpha \beta} - \eta^{\alpha \beta} \), then

\[
\tau_{\alpha \delta} - \eta^{\alpha \beta} \eta_{\beta \delta} = \eta^{\alpha \delta}.
\]

This and (2.1) imply \( \epsilon + (n - 1) M \epsilon \geq |\eta^{\alpha \delta}| \), where \( M = \max_{\alpha, \beta} |\eta^{\alpha \beta}| \). As \( \alpha, \delta \) are arbitrary, \( \epsilon + (n - 1) M \epsilon \geq M \). As a result, \( |\eta^{\alpha \delta}| \leq C_1 \). Here and below, \( \{ C_i \} \) denote positive constants that only depend on \( n \). By (2.7),

\[
|\eta^{\alpha \delta}| \leq C_2 \max_{\alpha, \delta} |\tau_{\alpha \delta}|.
\]

This in turn implies

\[
\gamma^{\alpha \delta} = \bar{\gamma}^{\alpha \delta} - \tau_{\alpha \delta} + O(|\tau|^2).
\]

For the same reason,

\[
g^{ij} = \bar{g}^{ij} - h_{ij} + O(|h|^2).
\]

Next, we consider \( \nu - \bar{\nu} \), where \( \bar{\nu} = \partial_n \). The condition \( \langle \nu, \nu \rangle_g = 1 \) gives

\[
\nu^i \nu^j g_{ij} = \nu^i (\delta_{ij} + h_{ij}) = 1,
\]

which implies \( \nu^i = O(1) \). The condition \( \langle \nu, \partial_{\beta} \rangle_g = 0 \) gives

\[
0 = \nu^\alpha g_{\alpha \beta} + \nu^n g_{n \beta} = \nu^\beta + \nu^\alpha h_{\alpha \beta} + \nu^n h_{n \beta}.
\]

Combined with \( \nu^i = O(1) \), this gives \( \nu^\beta = O(|h|^2) \). Coming back to (2.11), we have

\[
1 = \nu^\alpha \nu^\beta g_{\alpha \beta} + 2 \nu^\alpha \nu^n g_{\alpha n} + (\nu^n)^2 g_{nn}.
\]
Thus, \( 1 = O(|h|^2_g) + (\nu^n)^2 (1 + h_{nn}) \). This shows

\[
(2.14) \quad \nu^n = 1 - \frac{1}{2} h_{nn} + O(|h|^2_g).
\]

Plugging (2.14) into (2.12), we have

\[
(2.15) \quad \nu^\beta + \nu^n h_{\alpha \beta} + (1 + O(|h|^2_g) h_{\nu \beta} = 0,
\]

which gives \( \nu^\beta = -h_{\nu \beta} + O(|h|^2_g) \). This proves the lemma.

To estimate the difference of \( A \) and \( \bar{A} \), let \( D \) and \( \bar{D} \) denote the connection of \( g \) and \( \bar{g} \), respectively. Let \( T = D - \bar{D} \), then \( T \) is a \((1,2)\) tensor. With respect to \( \{x_i\} \), if we write \( D_{\delta_i} \partial_j - \bar{D}_{\delta_i} \partial_j = T_{ij}^k \partial_k \), then

\[
(2.16) \quad T_{ij}^k = \frac{1}{2} g^{kl} (h_{lj;i} + g_{li;j} - g_{ij;l}).
\]

Here \( ";" \) denotes covariant differentiation with respect to \( \bar{g} \). By Lemma 2.1

\[
T_{ij}^k = \frac{1}{2} g^{kl} (h_{lj;i} + h_{li;j} - h_{ij;l})
\]

\[
= \frac{1}{2} (\bar{g}^{kl} - h_{kl} + O(|h|^2_g) (h_{lj;i} + h_{li;j} - h_{ij;l})
\]

\[
= \frac{1}{2} (h_{kj;i} + h_{kj;j} - h_{ij;k}) - \frac{1}{2} h_{kl}(h_{lj;i} + h_{li;j} - h_{ij;l}) + O(|h|^2_g |Dh|_g).
\]

Lemma 2.2. The second fundamental forms \( A \) and \( \bar{A} \) satisfy

\[
(2.18) \quad A_{\alpha \beta} - \bar{A}_{\alpha \beta} = \frac{1}{2} \bar{A}_{\alpha \beta} h_{nn} - T_{\alpha \beta}^n + \bar{A}_{\alpha \beta} O(|h|^2_g) + O(|Dh|_g |h|_\bar{g}).
\]

Proof. By definition,

\[
(2.19) \quad A_{\alpha \beta} - \bar{A}_{\alpha \beta} = \langle D_{\delta_i} \partial_{\beta}, \nu \rangle_\bar{g} - \langle D_{\delta_i} \partial_{\beta}, \nu \rangle_g
\]

\[
= \langle D_{\delta_i} \partial_{\beta} \rangle^i (\nu^i - \bar{\nu}^i) \bar{g}_{ij} - \langle [D_{\delta_i} \partial_{\beta}]^i + T_{\alpha \beta}^i \rangle (\nu^i + \bar{\nu}^i - \bar{\nu}^i) (\bar{g}_{ij} + h_{ij}).
\]

For convenience, we can make the following simplification. In addition to assuming \( \{\partial_i\} \) are orthonormal at \( p \) with respect to \( \bar{g} \), we also assume that \( \{x_i\} \) are chosen so that their restrictions to \( \Sigma \), \( \{x_\alpha\} \), are normal at \( p \) on \( (\Sigma, \bar{g}) \). As a result, at \( p \),

\[
(2.20) \quad (D_{\delta_i} \partial_{\beta})^\delta = 0, \quad \text{and} \quad (D_{\delta_i} \partial_{\beta})^n = -\bar{A}_{\alpha \beta}.
\]

By Lemma 2.1 and (2.20), we have

\[
-(D_{\delta_i} \partial_{\beta})^i (\nu^i - \bar{\nu}^i) \bar{g}_{ij} = -(D_{\delta_i} \partial_{\beta})^n (\nu^n - \bar{\nu}^n)
\]

\[
= \bar{A}_{\alpha \beta} (-\frac{1}{2} h_{nn}) + \bar{A}_{\alpha \beta} O(|h|^2_g),
\]

\[
-T_{\alpha \beta}^i (\nu^i + \bar{\nu}^i - \bar{\nu}^i) \bar{g}_{ij} = -T_{\alpha \beta}^n - T_{\alpha \beta}^i (\nu^i - \bar{\nu}^i) \bar{g}_{ij},
\]

\[
-(D_{\delta_i} \partial_{\beta})^i (\nu^i - \bar{\nu}^i) h_{ij} = \bar{A}_{\alpha \beta} h_{nn},
\]

\[
-(D_{\delta_i} \partial_{\beta})^i (\nu^i - \bar{\nu}^i) h_{ij} = \bar{A}_{\alpha \beta} (-\bar{\nu}^i - \bar{\nu}^i) h_{nn} + \bar{A}_{\alpha \beta} (\nu^n - \bar{\nu}^n) h_{nn}
\]

\[
= \bar{A}_{\alpha \beta} (-h_{nn} h_{nn} + \bar{A}_{\alpha \beta} (-\frac{1}{2} h_{nn}) h_{nn} + \bar{A}_{\alpha \beta} O(|h|^2_g),
\]
(2.25) \[ T^i_{\alpha\beta}(\nu^i + \nu^j - \bar{\nu}^j)h_{ij} = T^i_{\alpha\beta}\bar{\nu}^j h_{ij} + O(|\bar{D}h|_\bar{g} |h|_\bar{g}^2). \]

Equation (2.18) follows from (2.21) - (2.25). \hfill \Box

**Proof of Proposition 2.1.** We recall from (2.3) that

(2.26) \[ H - \bar{H} = \bar{\gamma}^{\alpha\beta}(A_{\alpha\beta} - \bar{A}_{\alpha\beta}) + (\bar{\gamma}^{\alpha\beta} - \bar{\gamma}^{\alpha\beta})A_{\alpha\beta} + (\gamma^{\alpha\beta} - \bar{\gamma}^{\alpha\beta})(A_{\alpha\beta} - \bar{A}_{\alpha\beta}). \]

By Lemma 2.2,

(2.27) \[ \bar{\gamma}^{\alpha\beta}(A_{\alpha\beta} - \bar{A}_{\alpha\beta}) = \frac{1}{2} \bar{H}h_{nn} - \bar{\gamma}^{\alpha\beta}T^n_{\alpha\beta} + \bar{H}O(|h|_\bar{g}^2) + O(|\bar{D}h|_\bar{g} |h|_\bar{g}), \]

where, by (2.17),

(2.28) \[ -\bar{\gamma}^{\alpha\beta}T^n_{\alpha\beta} = - \bar{\gamma}^{\alpha\beta} \left( h_{n\alpha,\alpha} + h_{\alpha n;\beta} - h_{\alpha\beta;n} \right) + O(|h|_\bar{g} |\bar{D}h|_\bar{g}) \]

\[ = - (\text{div}_g h)_n + \frac{1}{2} (d \text{tr}_g h)_n + \frac{1}{2} h_{nn;n} + O(|h|_\bar{g} |\bar{D}h|_\bar{g}). \]

By Lemma 2.1

(2.29) \[ (\gamma^{\alpha\beta} - \bar{\gamma}^{\alpha\beta})A_{\alpha\beta} = - \langle h, \bar{A} \rangle_\bar{g} + O(|h|_\bar{g}^2 |\bar{A}|_\bar{g}). \]

Moreover, by Lemma 2.1 and Lemma 2.2

(2.30) \[ (\gamma^{\alpha\beta} - \bar{\gamma}^{\alpha\beta})(A_{\alpha\beta} - \bar{A}_{\alpha\beta}) = O(|h|_\bar{g}^2 |\bar{A}|_\bar{g}) + O(|\bar{D}h|_\bar{g} |h|_\bar{g}). \]

Therefore, it follows from (2.26) - (2.30) that

(2.31) \[ H - \bar{H} = \frac{1}{2} \bar{H}h_{nn} + \frac{1}{2} h_{nn;n} - \left( \text{div}_g h - \frac{1}{2} d \text{tr}_g h \right)_n - \langle h, \bar{A} \rangle_\bar{g} \]

\[ + |\bar{A}|_\bar{g} O(|h|_\bar{g}^2) + O(|\bar{D}h|_\bar{g} |h|_\bar{g}). \]

(A linearized version of (2.31) can be found in [12] Equation 42.)

On the other hand, we have an identity

(2.32) \[ (\text{div}_g h)_n = h_{nn;n} + \text{div}_g X + \bar{H}h_{nn} - \langle h, \bar{A} \rangle_\bar{g}. \]

(See [14, Equation (32)] for instance.) (2.2) now follows from (2.31) and (2.32). \hfill \Box

3. **Mass flux across the boundary of a polyhedron**

Let \((M^n, g)\) be an asymptotically flat manifold. Let \(K \subset M\) and \(B_r(0) \subset \mathbb{R}^n\) be given in Definition [1.1]. In what follows, we identify \(M \setminus K\) with \(\mathbb{R}^n \setminus B_r(0)\), and let \(\bar{g} = \delta_{ij}dx_i dx_j\) denote the background Euclidean metric.

We consider a polyhedron \(P \subset (\mathbb{R}^n, \bar{g})\) such that the interior of \(P\) contains \(B_r(0)\).

Let \(\partial P\) be the boundary of \(P\), which is a union of finitely many faces

\[ F_1, \ldots, F_{f(P)}, \]

where \(f(P)\) denotes the number of faces in \(\partial P\). Each face \(F_A, 1 \leq A \leq f(P)\), is an \((n-1)\)-dimensional polyhedron lying in a hyperplane in \((\mathbb{R}^n, \bar{g})\). We let

\[ r_P = \min_{x \in \partial P} |x|. \]
As $g$ is asymptotically flat, we assume $r_p$ is sufficiently large so that

\begin{equation}
|h|_g(x) < \epsilon(n), \quad \text{if } |x| > \frac{1}{2} r_p.
\end{equation}

Here $h = g - \bar{g}$ and $\epsilon(n)$ is the constant in (2.1).

Let $F$ be a face of $\partial P$. Let $d\sigma$, $d\bar{\sigma}$ denote the volume measure on $F$ with respect to the induced metrics $\gamma$, $\bar{\gamma}$, respectively. Let $\nu$, $\bar{\nu}$ be the outward unit normal to $F$ in $(M, g)$, $(\mathbb{R}^n, \bar{g})$, respectively. Let $H$ be the mean curvature of $F$ in $(M, g)$ with respect to $\nu$. By Proposition 2.1

\begin{equation}
2 \int_F H \, d\bar{\sigma} = \int_F (h_{ij,j} - h_{ij,i})\bar{\nu}^j \, d\bar{\sigma} - \int_F \text{div}_\gamma X \, d\bar{\sigma} + \int_F O(|Dh|_g |h|_g) \, d\bar{\sigma}.
\end{equation}

Here we made use of the fact that $F$ is totally geodesic in $(\mathbb{R}^n, \bar{g})$.

By (2.31), $H = O(r_p^{-p-1})$. By (1.2), $d\sigma = (1 + O(r_p^{-p})) \, d\bar{\sigma}$. Thus,

\begin{equation}
\int_F H \, d\bar{\sigma} = \int_F H \, d\sigma + |F|_\gamma O(r_p^{-2p-1}).
\end{equation}

Here $|F|_\gamma$ denotes the $(n-1)$-dimensional volume of $F$ in $(\mathbb{R}^n, \bar{g})$.

Similarly, by (1.2),

\begin{equation}
\int_F O(|Dh|_g |h|_g) \, d\bar{\sigma} = |F|_\gamma O(r_p^{-2p-1}).
\end{equation}

Integrating by parts, we have

\begin{equation}
\int_F \text{div}_\gamma X \, d\bar{\sigma} = \int_{\partial F} (X, \bar{n})_\gamma \, d\bar{\mu}.
\end{equation}

Here $\bar{n}$ denotes the outward unit normal to $\partial F$ in $(F, \bar{\gamma})$, and $d\bar{\mu}$ is the volume element on $\partial F$ induced from $\bar{\gamma}$. By the definition of $X$ and $h$,

\begin{equation}
(X, \bar{n})_\gamma = h(\bar{\nu}, \bar{n}) = g(\bar{\nu}, \bar{n}).
\end{equation}

Next we consider two adjacent faces $F_A$ and $F_B$. Let $\theta$ be the angle between $\nu_A$ and $\nu_B$ in $(M, g)$, and let $\bar{\theta}$ be the angle between $\bar{\nu}_A$ and $\bar{\nu}_B$ in $(\mathbb{R}^n, \bar{g})$.

The contribution from

\begin{equation}
\int_{F_A} \text{div}_\gamma X \, d\bar{\sigma} \quad \text{and} \quad \int_{F_B} \text{div}_\gamma X \, d\bar{\sigma}
\end{equation}

on the edge $F_A \cap F_B$ is

\begin{equation}
\int_{F_A \cap F_B} g(\bar{\nu}_A, \bar{n}_A) + g(\bar{\nu}_B, \bar{n}_B) \, d\bar{\mu}.
\end{equation}

In what follows, we write

\[ \bar{\nu}_A = (a_1, \ldots, a_n) \quad \text{and} \quad \bar{\nu}_B = (b_1, \ldots, b_n), \]

where $\sum_{i=1}^n a_i^2 = 1$ and $\sum_{i=1}^n b_i^2 = 1$. Using the fact $F_A$ and $F_B$ are, respectively, part of a level set of the functions

\[ f_A(x) = a_i x_i \quad \text{and} \quad f_B(x) = b_i x_i, \]
we know that the outward unit normal $\nu_A, \nu_B$ to $F_A, F_B$, respectively with respect to $g$, is given by

$$(3.8) \quad \nu_A = \frac{a^i e_i}{(a^i a^j g^{ij})^{\frac{1}{2}}} \quad \text{and} \quad \nu_B = \frac{b^i e_i}{(b^i b^j g^{ij})^{\frac{1}{2}}}.$$ 

Here $a^i = g^{ij} a_j$, $b^i = g^{ik} b_k$, and $e_i = \partial_x_i$.

By definition,

$$(3.9) \quad \cos \theta = g(\nu_A, \nu_B) = \frac{a_i b_j g^{ij}}{(a^i a^j g^{ij})^{\frac{1}{2}}(b^i b^j g^{ij})^{\frac{1}{2}}}.$$ 

By Lemma 2.1,

$$(3.10) \quad a_i b_j g^{ij} = \bar{g}(\bar{\nu}_A, \bar{\nu}_B) - a_i b_j h_{ij} + O(r_p^{-2p}),$$ 

$$(3.11) \quad (a_i a_j g^{ij})^{\frac{1}{2}} = 1 - \frac{1}{2} a_i a_j h_{ij} + O(r_p^{-2p}),$$ 

$$(3.12) \quad (b_i b_j g^{ij})^{\frac{1}{2}} = 1 - \frac{1}{2} b_i b_j h_{ij} + O(r_p^{-2p}).$$ 

Therefore,

$$\cos \theta = \left[ \bar{g}(\bar{\nu}_A, \bar{\nu}_B) - a_i b_j h_{ij} + O(r_p^{-2p}) \right] \left[ 1 + \frac{1}{2} a_i a_j h_{ij} + \frac{1}{2} b_i b_j h_{ij} + O(r_p^{-2p}) \right]$$

$$= \bar{g}(\bar{\nu}_A, \bar{\nu}_B) \left[ 1 + \frac{1}{2} a_i a_j h_{ij} + \frac{1}{2} b_i b_j h_{ij} \right] - a_i b_j h_{ij} + O(r_p^{-2p}).$$

Since $\cos \bar{\theta} = \bar{g}(\bar{\nu}_A, \bar{\nu}_B)$, this gives

$$(3.14) \quad \cos \theta - \cos \bar{\theta} = \frac{1}{2} \cos \bar{\theta} \left( a_i a_j h_{ij} + b_i b_j h_{ij} \right) - a_i b_j h_{ij} + O(r_p^{-2p}).$$

Next we consider two cases depending on whether $P$ is convex at the edge $F_A \cap F_B$.

**Case 1.** $P$ is convex at $F_A \cap F_B$. This means, if $\bar{\alpha}$ is the Euclidean dihedral angle of $P$ at $F_A \cap F_B$, then $0 < \bar{\alpha} < \pi$.

![Figure 2. $P$ is convex at $F_A \cap F_B$.](image)

In this case, we have

$$(3.15) \quad \bar{\alpha} + \bar{\theta} = \pi \quad \text{and} \quad (\sin \bar{\theta}) \bar{n}_A = -(\cos \bar{\theta}) \bar{\nu}_A + \bar{\nu}_B.$$
Hence,
\[
\sin \bar{\theta} g(\bar{\nu}_A, \bar{n}_A) = g(\bar{\nu}_A, -(\cos \bar{\theta}) \bar{\nu}_A + \bar{\nu}_B)
\]
\[
= - \cos \bar{\theta} (1 + a_i a_j h_{ij}) + \cos \bar{\theta} + a_i b_j h_{ij}
\]
\[
= \frac{1}{2} \cos \bar{\theta} (-a_i a_j h_{ij} + b_i b_j h_{ij}) + \cos \bar{\theta} - \cos \theta + O(r_p^{-2p}).
\]
(3.16)

Here we used (3.14) in the last step.

Similarly, we have
\[
\sin \bar{\theta} g(\bar{\nu}_B, \bar{n}_B) = \frac{1}{2} \cos \bar{\theta} (-b_i b_j h_{ij} + a_i a_j h_{ij}) + \cos \bar{\theta} - \cos \theta + O(r_p^{-2p}).
\]
(3.17)

Therefore, by (3.16) and (3.17), we conclude
\[
g(\bar{\nu}_A, \bar{n}_A) + g(\bar{\nu}_B, \bar{n}_B) = 2 \left[ \frac{1}{\sin \bar{\theta}} (\cos \bar{\theta} - \cos \theta) + \frac{1}{\sin \bar{\theta}} O(r_p^{-2p}) \right].
\]
(3.18)

For later use, we note the relation between \(\theta - \bar{\theta}\) and \(\alpha - \bar{\alpha}\) in this case. Here \(\alpha\) is the dihedral angle of \(P\) at \(F_A \cap F_B\) with respect to \(g\). Since \(g\) is a metric continuously defined in a neighborhood of \(\partial P\), \(\alpha\) also satisfies
\[
0 < \alpha < \pi \quad \text{and} \quad \alpha + \theta = \pi.
\]
(3.19)

As a result,
\[
\theta - \bar{\theta} = \bar{\alpha} - \alpha.
\]
(3.20)

**Case 2.** \(P\) is non-convex at \(F_A \cap F_B\). This means \(\pi < \bar{\alpha} < 2\pi\).

\[
\bar{\alpha} = \pi + \bar{\theta} \quad \text{and} \quad -(\sin \bar{\theta}) \bar{n}_A = -(\cos \bar{\theta}) \bar{\nu}_A + \bar{\nu}_B.
\]
(3.21)

By similar calculations, we have
\[
-g(\bar{\nu}_A, \bar{n}_A) - g(\bar{\nu}_B, \bar{n}_B) = 2 \left[ \frac{1}{\sin \bar{\theta}} (\cos \bar{\theta} - \cos \theta) + \frac{1}{\sin \bar{\theta}} O(r_p^{-2p}) \right].
\]
(3.22)

Also, \(\alpha = \pi + \theta\) in this case. Consequently,
\[
\theta - \bar{\theta} = \alpha - \bar{\alpha}.
\]
(3.23)
In either case, we can replace the term
\[ \frac{1}{\sin \theta} (\cos \theta - \cos \tilde{\theta}) \]
via \((\theta - \tilde{\theta})\). By definition, \(\theta\) and \(\tilde{\theta}\) satisfy \(0 < \theta, \tilde{\theta} < \pi\). Thus, by Taylor’s theorem,
\[ \theta - \tilde{\theta} = -\frac{1}{\sin \theta} (\cos \theta - \cos \tilde{\theta}) - \frac{1}{2} \frac{\cos \xi}{\sin^3 \theta} (\cos \theta - \cos \tilde{\theta})^2 \]
for some \(\xi\) between \(\theta\) and \(\tilde{\theta}\). Combined with (3.14), this shows
\[ \frac{1}{\sin \theta} (\cos \theta - \cos \tilde{\theta}) = \tilde{\theta} - \theta + \frac{1}{2} \frac{\sin \xi}{\sin^3 \theta} O\left(\frac{1}{r^{2p}}\right). \] (3.25)
Therefore,

i) if \(P\) is convex at \(F_A \cap F_B\), by (3.18), (3.25) and (3.20),
\[
g(\vec{v}_A, \vec{n}_A) + g(\vec{v}_B, \vec{n}_B) \\
= 2(\tilde{\theta} - \theta) + \frac{1}{(\sin \xi)^3} O\left(\frac{1}{r^{2p}}\right) + \frac{1}{\sin \theta} O\left(\frac{1}{r^{2p}}\right) \\
= 2(\tilde{\alpha} - \alpha) + \frac{1}{(\sin \xi)^3} O\left(\frac{1}{r^{2p}}\right) + \frac{1}{\sin \theta} O\left(\frac{1}{r^{2p}}\right); \\
\]

ii) if \(P\) is non-convex at \(F_A \cap F_B\), by (3.22), (3.25) and (3.23),
\[
g(\vec{v}_A, \vec{n}_A) + g(\vec{v}_B, \vec{n}_B) \\
= 2(\tilde{\theta} - \theta) + \frac{1}{(\sin \xi)^3} O\left(\frac{1}{r^{2p}}\right) + \frac{1}{\sin \theta} O\left(\frac{1}{r^{2p}}\right) \\
= 2(\tilde{\alpha} - \alpha) + \frac{1}{(\sin \xi)^3} O\left(\frac{1}{r^{2p}}\right) + \frac{1}{\sin \theta} O\left(\frac{1}{r^{2p}}\right).
\]
Thus, regardless of the convexity of \(P\) at \(F_A \cap F_B\), we always have
\[ g(\vec{v}_A, \vec{n}_A) + g(\vec{v}_B, \vec{n}_B) = 2(\tilde{\alpha} - \alpha) + \frac{1}{(\sin \xi)^3} O\left(\frac{1}{r^{2p}}\right) + \frac{1}{\sin \theta} O\left(\frac{1}{r^{2p}}\right). \] (3.26)
To proceed, we impose an angle assumption
\[ \sin \tilde{\theta} \geq c, \] (3.27)
where \(c \in (0, 1)\) is a constant independent on \(P\). This together with (3.14) implies, for sufficiently large \(r_P\),
\[ \sin \theta \geq \frac{1}{2} c. \]
As a result, \(\sin \xi \geq \frac{1}{2} c\), and
\[ g(\vec{v}_A, \vec{n}_A) + g(\vec{v}_B, \vec{n}_B) = 2(\tilde{\alpha} - \alpha) + c^{-3} O\left(\frac{1}{r^{2p}}\right). \] (3.28)
Moreover, by (3.20), (3.23) and (3.24),
\[ |\tilde{\alpha} - \alpha| = |\theta - \tilde{\theta}| = c^{-3} O\left(\frac{1}{r^{2p}}\right). \] (3.29)
Returning to (3.7), we have

\[
\int_{F_A \cap F_B} g(\bar{\nu}_A, \bar{n}_A) + g(\bar{\nu}_B, \bar{n}_B) \, d\bar{\mu}
\]

(3.30)

\[
= 2 \int_{F_A \cap F_B} (\bar{\alpha} - \alpha) \, d\bar{\mu} + |F_A \cap F_B|_\gamma c^{-3} O(r_p^{-2p})
\]

\[
= 2 \int_{F_A \cap F_B} (\bar{\alpha} - \alpha) \, d\mu + |F_A \cap F_B|_\gamma c^{-3} O(r_p^{-2p}).
\]

Here \(|F_A \cap F_B|_\gamma\) is the \((n-2)\)-dimensional volume of the edge \(F_A \cap F_B\) in \((\mathbb{R}^n, \tilde{g})\), and \(d\mu\) is the volume element with respect to the metric induced from \(g\).

Combining (3.2) – (3.7) and (3.30), we obtain the following proposition.

**Proposition 3.1.** Let \(c \in (0, 1)\) be a fixed constant. Suppose the polyhedron \(P\) satisfies

\[
\sin \bar{\theta} \geq c
\]

(3.31)

along each edge of \(P\). If \(r_p\) is sufficiently large, then

\[
\int_{\partial P} (g_{i,j,i} - g_{ii,j})d\bar{\sigma} = -2 \int_H d\sigma + 2 \int_E (\alpha - \bar{\alpha}) \, d\mu
\]

(3.32)

\[
+ c^{-3} |E|_\gamma O(r_p^{-2p}) + |F|_\gamma O(r_p^{-2p-1}) + |F'|_\gamma O(r_p^{-2p-1}).
\]

Here \(F\) and \(E\) are the union of all the faces and edges of \(P\), respectively.

Theorem 1.1 now follows from Proposition 3.1. Take \(P = P_k\), an element in \(\{P_k\}\).

Since

\[
|\sin \bar{\alpha}| = \sin \bar{\theta},
\]

condition d) is equivalent to \(\sin \bar{\theta} \geq c\). Therefore, by conditions a), b), c) and Proposition 3.1

\[
\int_{\partial P_k} (g_{i,j,i} - g_{ii,j})d\bar{\sigma} = -2 \int_\mathcal{F}(\partial P_k) H d\sigma + 2 \int_\mathcal{E}(\partial P_k) (\alpha - \bar{\alpha}) \, d\mu + o(1), \quad \text{as } k \to \infty.
\]

(3.33)

Here we also used the decay condition \(p > \frac{n-2}{2}\). Equation (1.1) follows from (1.4) and (3.33).

### 4. Integration of a Lower Dimensional Mass-Related Quantity

We next consider the case in which \(\{P_k\}\) is a sequence of large coordinate cubes. Cubes have a feature that, when sliced by hyperplanes parallel to a face, the resulting sections are \((n-1)\)-dimensional large cubes as well. The following formula was derived in [13, Equation (6)].
Theorem 4.1 ([13]). Let \((M^3, g)\) be an asymptotically flat 3-manifold. Let \(C_L^3\) denote a large coordinate cube centered at the coordinate origin, with coordinate side length \(2L\). For each \(k = 1, 2, 3\) and each \(t \in [−L, L]\), let \(S_t^{(k)}\) be the curve given by the intersection between \(\partial C_L^3\) and the coordinate plane \(\{x_k = t\}\). Then the mass of \((M^3, g)\) satisfies

\[
(4.1) \quad m(g) = \frac{1}{8\pi} \sum_{k=1}^{3} \int_{-L}^{L} m_k^{(2)}(t, L) \, dt + o(1), \text{ as } L \to \infty,
\]

where

\[
(4.2) \quad m_k^{(2)}(t, L) = 2\pi - \int_{S_t^{(k)}} \kappa^{(k)} ds - \beta_t^{(k)},
\]

\(\kappa^{(k)}\) is the geodesic curvature of \(S_t^{(k)}\) in \(\{x_k = t\}\), and \(\beta_t^{(k)}\) is the sum of the turning angle of \(S_t^{(k)}\) at its four vertices.

As noted in [13], the quantity \(m_k^{(2)}(t, L)\) can be interpreted as an angle defect of the surface delimited by \(S_t^{(k)}\) in \(\{x_k = t\}\). In the setting of asymptotically conical surfaces, it is known that this angle defect defines the 2-d “mass” of such surfaces (see [4] and [5] for instance).

Combined with the Gauss-Bonnet formula and the work of Stern [17], (4.1) can be used to explain the recent proof of the 3-dimensional positive mass theorem in [8]. Motivated by this, below we establish a higher dimensional analog of (4.1).

Theorem 4.2. Let \((M^n, g)\) be an asymptotically flat manifold with dimension \(n \geq 4\). Given any index \(k \in \{1, \cdots, n\}\), any large constant \(L\), and any \(t \in [−L, L]\), there is a quantity \(m_k^{(n−1)}(t, L)\), associated to the coordinate hyperplane \(\{x_k = t\}\), defined in (4.4) below, such that the mass of \((M^n, g)\) satisfies

\[
(4.3) \quad (n - 1)\omega_{n-1} m(g) = \omega_{n-2} \sum_{k=1}^{n} \int_{-L}^{L} m_k^{(n−1)}(t, L) \, dt + o(1), \text{ as } L \to \infty.
\]

To explain the quantity \(m_k^{(n−1)}(t, L)\), we first introduce some notations. Given a large constant \(L\), let \(C_L^n\) denote the coordinate cube in \((M, g)\), centered at the coordinate origin, with side length \(2L\). For each \(i \in \{1, \cdots, n\}\), let

\[
F^{(i)}_+ = \{x \in \partial C_L^n \mid x^i = L\} \quad \text{and} \quad F^{(i)}_- = \{x \in \partial C_L^n \mid x^i = -L\},
\]

which represent the front and back \(i\)-th face of \(\partial C_L^n\), respectively. Let \(H_i\) denote the mean curvature of \(F^{(i)}_{±}\) with respect to the outward normal \(\nu_i\) in \((M, g)\). As before, we use \(\alpha\) to denote the dihedral angle along every edge of \(\partial C_L^n\) in \((M, g)\).

For each \(t \in [−L, L]\), let \(S_t^{(k)}\) be the intersection of \(\partial C_L^n\) with the coordinate hyperplane \(\{x_k = t\}\), i.e.

\[
S_t^{(k)} = \partial C_L^n \cap \{x_k = t\}.
\]

\(S_t^{(k)}\) is the boundary of an \((n - 1)\)-dimensional cube in \(\{x_k = t\}\). (See Figure 4)
Figure 4. $S^{(k)}_i$ is the boundary of an $(n - 1)$-dimensional cube.

Within the hypersurface $\{x_k = t\}$, let $\tilde{H}^{(k)}$ denote the mean curvature of each face of $S^{(k)}_i$ with respect to the outward normal $\nu^{(k)}_i$. Let $\tilde{\alpha}^{(k)}$ denote the dihedral angle along every edge of $S^{(k)}_i$ in $\{x_k = t\}$, with respect to $g$.

For $m \in \{n - 1, n - 2\}$, let $d\sigma^m$, $d\mu^{m-1}$, $d\sigma^m_0$ and $d\mu^{m-1}_0$ denote the relevant volume forms, induced from the metrics $g$ and $\bar{g}$, on an $m$-dimensional face and an $(m - 1)$-dimensional edge of the corresponding cubes, respectively.

Associated to each $S^{(k)}_i$ in $\{x_k = t\}$, define

$$m^{(n-1)}_k(t, L) = \frac{1}{(n-2)\omega_{n-2}} \left( -\int_{S^{(k)}_i(t)} \tilde{H}^{(k)} d\sigma^{n-2} + \int_{\partial C^{(k)}_i(t)} (\tilde{\alpha}^{(k)} - \frac{\pi}{2}) d\mu^{n-3} \right).$$

**Remark 4.1.** For each fixed $k$ and $t$, as a special case of Theorem 1.1, we have

$$m(g|_{\{x_k = t\}}) = \lim_{L \to \infty} m^{(n-1)}_k(t, L),$$

where $g|_{\{x_k = t\}}$ denotes the induced metric on $\{x_k = t\}$ from $g$. In many cases, for instance if $g$ has a decay rate of $p = n - 2$, this limit will be zero as $m(g|_{\{x_k = t\}}) = 0$.

To prove Theorem 4.2, we first relate the mean curvatures and the dihedral angles of $S^{(k)}_i$ in $\{x_k = t\}$, $k = 1, \cdots, n$, to those of $\partial C^{(n)}_L$ in $(M, g)$.

**Lemma 4.1.** For any $i, k \in \{1, \cdots, n\}$ with $i \neq k$, let $\tilde{H}^{(k)}_i$ be the mean curvature of $S^{(k)}_i \cap F^{(i)}_{\pm}$ in $\{x_k = t\}$ with respect to $\nu^{(k)}_i$. Then

$$\sum_{k \in \{1, \cdots, n\}\setminus\{i\}} \tilde{H}^{(k)}_i = (n-2)H_i + O(L^{-2p-1}), \text{ as } L \to \infty.$$

Similarly, along each edge of $S^{(k)}_i$ in $\{x_k = t\}$,

$$\tilde{\alpha}^{(k)} = \alpha + O(L^{-2p}), \text{ as } L \to \infty.$$

**Proof.** It suffices to check (4.6) and (4.7) on $F^{(i)}_{\pm}$ and along $F^{(i)}_+ \cap F^{(j)}_+$, where $j \neq i$. For any $k \neq i$, at $S^{(k)}_i \cap F^{(j)}_+$, we have $\nu_i = \partial_t + O(L^{-p})$ and $\nu^{(k)}_i = \partial_t + O(L^{-p})$. Thus,

$$\nu_i = \nu^{(k)}_i + O(L^p).$$
By definition, we have

\[
(4.9) \quad \sum_{k \in \{1, \ldots, n\} \setminus \{i\}} \tilde{H}_i^{(k)} = \sum_{k \in \{1, \ldots, n\} \setminus \{i\}} \sum_{\alpha, \beta \in \{1, \ldots, n\} \setminus \{i, k\}} (-1)^{g^{\alpha\beta}} \langle \nabla_{\partial_\alpha} \partial_\beta, \nu_i^{(k)} \rangle = (n-2)g^{\alpha\alpha} \langle \nabla_{\partial_\alpha} \partial_\alpha, \nu_i \rangle + O(L^{-2p-1})
\]

Applying (4.8), we have

\[
(4.10) \quad \sum_{k \in \{1, \ldots, n\} \setminus \{i\}} \tilde{H}_i^{(k)} = \sum_{\alpha \in \{1, \ldots, n\} \setminus \{i\}} (n-2) (-1)^{g^{\alpha\alpha}} \langle \nabla_{\partial_\alpha} \partial_\alpha, \nu_i \rangle + O(L^{-2p-1}) = (n-2)H_i + O(L^{-2p-1}),
\]

which proves (4.6).

To prove (4.7), by (3.19), it suffices to check the corresponding relation for \(\theta^{(ij)}\) and \(\tilde{\theta}^{(ij)}\). Here \(\theta^{(ij)}\) is the angle between \(\nu_i\) and \(\nu_j\) along the edge \(F^{(i)}_+ \cap F^{(j)}_+\), and \(\tilde{\theta}^{(ij)}\) is the angle between \(\nu_i^{(k)}\) and \(\nu_j^{(k)}\) along \(F^{(i)}_+ \cap F^{(j)}_+ \cap S^{(k)}_{t}\) (see Figure 4).

Applying (3.14) to \(C^k_{L}\) along \(F^{(i)}_+ \cap F^{(j)}_+\) and noticing \(\tilde{\theta} = \pi/2\) in this case, we have

\[
(4.11) \quad \cos(\theta^{(ij)}) = -h_{ij} + O(L^{-2p}).
\]

The same reason applied to \(\{x_k = t\}\) also gives

\[
(4.12) \quad \cos(\tilde{\theta}^{(ij)}) = -h_{ij} + O(L^{-2p}).
\]

As a result, we have \(\theta^{(ij)} = \pi/2 + O(L^{-p})\) and

\[
(4.13) \quad \cos(\theta^{(ij)}) = \cos(\tilde{\theta}^{(ij)}) + O(L^{-2p}).
\]

These readily imply

\[
(4.14) \quad \theta^{(ij)} = \tilde{\theta}^{(ij)} + O(L^{-2p}).
\]

Equation (4.7) follows from (4.14) and (3.19). \(\square\)

We now prove Theorem 4.2. In what follows, we let \(F^{(i)} = F^{(i)}_+ \cup F^{(i)}_-\), and let \(E^{(ij)} = F^{(i)} \cap F^{(j)}\) for \(i \neq j\).
Proof of Theorem 4.2. By (4.6), we have
\[
\int_{F(\partial C_{L}^{n})} H d\sigma^{n-1} = \sum_{i} \int_{F^{(i)}} H_{i} d\sigma^{n-1}
\]
\[
= \frac{1}{n-2} \sum_{i} \sum_{k \in \{1, \ldots, n\} \backslash \{i\}} \int_{F^{(i)}} \tilde{H}_{i}^{(k)} d\sigma_{0}^{n-1} + O(L^{n-2-2p})
\]
(4.15)
\[
= \frac{1}{n-2} \sum_{k} \sum_{i \in \{1, \ldots, n\} \backslash \{k\}} \int_{-L}^{L} \left\{ \int_{F^{(i)} \cap S_{t}^{(k)}} \tilde{H}_{i}^{(k)} d\sigma_{0}^{n-2} \right\} dt + O(L^{n-2-2p})
\]
\[
= \frac{1}{n-2} \sum_{k} \int_{-L}^{L} \left\{ \int_{S_{t}^{(k)}} \tilde{H}^{(k)} d\sigma^{n-2} \right\} dt + O(L^{n-2-2p}).
\]

By (4.7), we have
\[
\int_{E(\partial C_{L}^{n})} (\alpha - \frac{\pi}{2}) d\mu^{n-2} = \frac{1}{2(n-2)} \sum_{i \neq j} \int_{E^{(i)}} \left( \tilde{\alpha}^{(k)} - \frac{\pi}{2} \right) d\mu^{n-2} + O(L^{n-2-2p})
\]
(4.16)
\[
= \frac{1}{(n-2)} \sum_{i \neq j} \sum_{k \in \{1, \ldots, n\} \backslash \{i, j\}} \int_{-L}^{L} \left\{ \int_{E^{(j)} \cap S_{t}^{(k)}} \left( \tilde{\alpha}^{(k)} - \frac{\pi}{2} \right) d\mu_{0}^{n-3} \right\} dt + O(L^{n-2-2p})
\]
\[
= \frac{1}{n-2} \sum_{k} \int_{-L}^{L} \left\{ \int_{S_{t}^{(k)}} \left( \tilde{\alpha}^{(k)} - \frac{\pi}{2} \right) d\mu^{n-3} \right\} dt + O(L^{n-2-2p}).
\]

Taking $L \to \infty$, we conclude from Theorem 1.1, (4.15) and (4.16) that
(4.17)
\[
(n-1)\omega_{n-1} m(g)
\]
\[
= - \int_{F(\partial C_{L}^{n})} H d\sigma^{n-1} + \int_{E(\partial C_{L}^{n})} (\alpha - \frac{\pi}{2}) d\mu^{n-2} + o(1)
\]
\[
= \frac{1}{n-2} \sum_{k} \int_{-L}^{L} \left\{ - \int_{S_{t}^{(k)}} \tilde{H}^{(k)} d\sigma^{n-2} + \int_{E(S_{t}^{(k)})} \left( \tilde{\alpha}^{(k)} - \frac{\pi}{2} \right) d\mu^{n-3} \right\} dt + o(1)
\]
\[
= \omega_{n-2} \sum_{k} \int_{-L}^{L} m_{k}^{(n-1)}(t, L) dt + o(1).
\]

This completes the proof. \qed

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