On Single-Input Controllable Linear Systems Under Periodic DoS Jamming Attacks

Hamed Shisheh Foroush and Sonia Martínez

Abstract—In this paper, we study remotely-controlled single-input controllable linear systems, subject to periodic Denial-of-Service (DoS) attacks. We propose a control strategy which can beat any partially identified jammer by properly placing the closed-loop poles. This is proven theoretically for systems of dimension 4 or less. Nevertheless, simulations show the practicality of this strategy for systems up to order 5.

I. INTRODUCTION

Cyber-physical systems consist of physical networked systems which are controlled and monitored remotely [8]. Novel advances in communications and sensing technologies have promoted the emergence of these systems, which bear numerous advantages ranging from ease of implementation to increased utility in infrastructure facilities [17]. However, the potential benefits of such systems may be overturned by several challenges that include a higher exposure to external attacks. This has motivated a renewed research effort in the area of system security [9], [1], which attempts to address system preservation issues.

In particular, the security of cyber-physical systems can especially be threatened by communication-signal jammers that are exogenous to the system. Common attacks include Denial-of-Service (DoS) and Deceptive attacks. In brief, a Deceptive attacker aims to change parceled data, whereas a DoS one tries to corrupt the transmitted data [31], [24]. According to [7], the most likely type of attack is a DoS attack. These attacks can be further categorized into periodic or Pulse-Width Modulated (PWM) jammers motivated by ease of implementation, detection avoidance, and energy constraints while; see the papers [19], [20], [10], [15]. Inspired by this facts, this work focuses on the compensation of PWM DoS jamming attacks whose periodicity has already been detected.

The subject of security of cyber-physical systems is receiving wide attention in the controls community. In the context of multiagent systems, the works [27], [22], [23], aim to identify malicious agents who are part of the network in order to cancel their contribution. The main goal of [6] and [5] is to maintain group connectivity despite the presence of a malicious agent, thus identification is off the ground. The paper [32] proposes a Receding Horizon Control methodology to deal with a class of deceptive Replay attackers inducing system delays in formation control missions. Our problem setup is related to these studies in the sense that the jammer is assumed to be detected and we aim to develop a method to overcome its effect.

In the framework of secure discrete LTI systems, [12] considers deceptive attacks where the observation channel is jammed. In [2], however, a DoS attack where the attacker corrupts the channel while obeying an Identically Independent Distributed (IID) assumption is considered. Similar research is conducted in [26].

Game Theory is a natural framework to study system security; some representative references include [16], [29], and [25]. These papers model the security problem as a (dynamic) zero-sum, non-cooperative game in order to predict the behavior of the attacker. Inspired by a leader-follower game-theoretic formulation, the paper [33] employs reinforcement learning to beat a deceptive attacker that can be modeled as a linear map. In this framework, the closest reference to our research is [16], which studies how the optimal control of linear system remotely under a strategic, energy-constrained DoS jammer. A main restriction of [16] is the consideration of scalar dynamics, which makes the presented analysis more tractable.

Motivated by the goal of maintaining “intelligent” and economic communications, here we address the problem of system resilience in the context of triggering control, i.e., control actions triggered only when it is necessary. This is inspired by recent work [28], [21], [30]. The distinctive feature in our study is the fact that communication is not always feasible.

We consider partially identified DoS attacks imposed by PWM jammers, along with single-input LTI systems. The current work follows upon [13], where we provide sufficient conditions on the jammer’s parameters which, in conjunction with a given triggering law, can ensure system stability. Despite the fact that it covers a broad class of continuous LTI systems, the previous strategy cannot cope with every given DoS periodic jammer. The current manuscript solves this issue by introducing a parameter-dependent control strategy which can tackle any DoS periodic jammer. Thus it ensures asymptotic stability under this class of

The authors are with Department of Mechanical and Aerospace Engineering, University of California, San Diego, 9500 Gilman Dr, La Jolla CA, 92093, hshisheh, soniamd@ucsd.edu
attacks, i.e., it renders the system safe and secure. The strategy is proven to work for a narrower class of continuous LTI systems of order 4, or less, and, based on simulations, we conjecture its validity for a large class of higher-order systems as well.

The rest of the paper is organized as follows. Section II includes the problem formulation and notations. In Section III some preliminaries are provided, where we propose a control design law to be employed later. Then we go on to Section IV where we discuss a novel attack-resilient triggering law consistent with the jammer signal. In Section V we analyze and prove the security of the system equipped with this control design and triggering law. In Section VI we demonstrate the functionality of our theoretical results on two academic examples. At last, in Section VII we summarize the results and state the future work.

II. PROBLEM FORMULATION

In this section, we state, both formally and informally, the main problem analyzed in the paper.

We consider a remote operator-plant setup, where the operator uses a control channel to send wirelessly a control command to an unstable plant, see Figure 1. We assume that the plant has no specific intelligence and is only capable of updating the control based on the data it receives. We also assume that the operator knows the plant dynamics and is able to measure its states at particular time-instants.

More precisely, consider the following closed-loop dynamics:

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad u(t) = Kx(t_k), \quad \forall t \in [t_k, t_{k+1}], \]

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R} \) is the input, \( A, B \) and \( K \) are matrices of proper dimensions, and \( \{t_k\}_{k \geq 1} \) is a triggering time sequence. Here, we also assume that: (i) the system (1a) is open-loop unstable, and (ii) the pair \((A, B)\) is controllable.

We consider an energy-constrained, periodic jammer whose signal can be represented as follows:

\[
u_{jmd}(t) = \begin{cases} 
1, & (n-1)T \leq t \leq (n-1)T + T_{\text{off}}, \\
0, & (n-1)T + T_{\text{off}} \leq t \leq nT,
\end{cases}
\]

where \( n \in \mathbb{N} \) is the period number, \( T \in \mathbb{R}_{>0} \), and \( T = [0, T] \) is the action-period of the jammer. Also, \( T_{\text{off}} \in \mathbb{R}_{>0}, T_{\text{off}} < T, \) and \( T_{\text{off}} = [0, T_{\text{off}}] \) is the time-period where it is sleeping, so communication is possible. We further denote \( T_{\text{on}} \in \mathbb{R}_{>0} \) and \( T_{\text{on}} = [T_{\text{on}}, T] \) to be the time-period where the jammer is active, thus no data can be sent. Accordingly, it holds that \( T_{\text{off}} + T_{\text{on}} = T \). We also note that the parameter \( T_{\text{off}} \) need not be time-invariant which recalls Pulse-Width Modulated (PWM) jamming. Finally, we denote by \( T_{\text{off}}^\gamma \) a uniform lower-bound for \( T_{\text{off}} \), i.e., \( T_{\text{off}}^\gamma \leq T_{\text{off}} \) which we assume holds for all the periods and we have identified as well.

In this paper, we assume that the type of jammer and the period of the jamming signal has been identified. Future work will be devoted to enlarge the triggering time sequence for identification purposes.

Putting these pieces together, we study the following problem:

[Problem formulation]: Consider any energy-constrained, periodic jammer described by (2) with parameters \( T \) and \( T_{\text{off}} \). Knowing \( T \) and \( T_{\text{off}}^\gamma \) a uniform lower bound on the jammer’s sleeping periods, find a control strategy of the form (1b) that is resilient to the action of this jammer.

III. PRELIMINARIES

In this section, we recall some useful properties of the systems that we study. These will be employed in the subsequent analysis.

Since \((A, B)\) is a controllable pair, the system (1a) can be put into a controllable canonical form by a proper similarity transformation [4]. Based on this fact, we narrow our study down to the systems of this form:

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- a_n & - a_{n-1} & - a_{n-2} & \cdots & - a_1 \\
0 & 0 & 0 & \cdots & 1 \\
-a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 
\end{bmatrix} x + \begin{bmatrix} 0 \\
0 \\
\vdots \\
1 \\
0 
\end{bmatrix} u, \\
u &= \left[ -k_n + a_n, -k_{n-1} + a_{n-1}, \ldots, -k_1 + a_1 \right] x.
\end{align*}
\]

Recalling the pole-placement assignment techniques, we obtain the following result:
Proposition 3.1: Consider \( \lambda \in \mathbb{R}_{>0} \) and system (3). By choosing:

\[
k_i = \left( \begin{array}{c} n \\ i \end{array} \right) \lambda^i, \quad i \in \{1, \ldots, n\},
\]

all the closed-loop system poles are placed at \(-\lambda\).

Proof: By plugging the proposed gains, \( K_\lambda = [k_1, \ldots, k_n] \), in the dynamics (3), the state matrix of the closed-loop system becomes:

\[
A + BK_\lambda = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-\left( \begin{array}{c} n \\ n-1 \end{array} \right) \lambda^n & -\left( \begin{array}{c} n \\ n-1 \end{array} \right) \lambda^{n-1} & \cdots & -n\lambda
\end{bmatrix}.
\]

Note that the characteristic polynomial of this matrix, thanks to its specific structure, is given by [4]:

\[p(s) = s^n + n\lambda s^{n-1} + \cdots + \left( \begin{array}{c} n \\ n-1 \end{array} \right) \lambda^{n-1} s + \left( \begin{array}{c} n \\ n \end{array} \right) \lambda^n.
\]

Observe that the latter polynomial is indeed \( p(s) = (s + \lambda)^n \). Recall also that the eigenvalues of the matrix \( A + BK_\lambda \), i.e., the poles of the closed-loop system, are the roots of \( p(s) \). Thus, we conclude that with this choice of gains, all the closed-loop poles are placed at \(-\lambda\).

The multiplicities of the eigenvalue \(-\lambda\), described in the previous result, are further characterized next:

Proposition 3.2: Consider the system (3) along with the gains given in Proposition 3.1. The eigenvalue \(-\lambda\) of the matrix \( A + BK_\lambda \) has algebraic multiplicity \( n \) and geometric multiplicity 1.

Proof: Since the characteristic polynomial of \( A + BK_\lambda \) is \( p(s) = (s + \lambda)^n \), the algebraic multiplicity of \(-\lambda\) is equal to \( n \). The geometric multiplicity of \(-\lambda\) is equal to the number of linearly independent eigenvectors of \( A + BK_\lambda \) associated to \(-\lambda\); in other words, the nullity of the matrix \( A + BK_\lambda + \lambda I \). Further, it holds that [4]:

\[
\text{nullity}(A + BK_\lambda + \lambda I) = n - \text{rank}(A + BK_\lambda + \lambda I).
\]

By construction, \( \text{rank}(A + BK_\lambda + \lambda I) \geq n - 1 \), as it has \( n - 1 \) linearly independent columns. Also, since \(-\lambda\) is an eigenvalue of \( A + BK_\lambda \), then it holds that \( \det(A + BK_\lambda + \lambda I) = 0 \), thus \( \text{rank}(A + BK_\lambda + \lambda I) < n \). From here, \( \text{rank}(A + BK_\lambda + \lambda I) = n - 1 \). According to [4], we have that \( \text{nullity}(A + BK_\lambda + \lambda I) = 1 \), hence the geometric multiplicity of \(-\lambda\) is 1.

Remark 3.3: Note that the matrix \( A + BK_\lambda \) has only one linearly independent eigenvector, therefore, it is not diagonalizable. This property holds for all values of \( \lambda \in \mathbb{R}_{>0} \).

Furthermore, since the matrix \( A + BK_\lambda + \lambda I \) depends on \( \lambda \) in a polynomial way, the components of this eigenvector are rational functions of \( \lambda \). In fact, \( \lambda \) can be found as the solution to the following equation:

\[(A + BK_\lambda + \lambda I)v = 0.
\]

IV. JORDAN DECOMPOSITION AND TRIGGERING STRATEGY

Our control strategy will consist of choosing an appropriate \( K_\lambda \) as in Proposition 3.1 and an associated triggering strategy \( \{t_k\}_{k \geq 1} \). In this section, we study the Jordan decomposition of the closed-loop system under \( K_\lambda \), and how to choose the corresponding \( \{t_k\}_{k \geq 1} \).

From Proposition 3.1 the eigenvalues of the matrix \( A + BK_\lambda \) are at \(-\lambda\). Thus, the Jordan decomposition of this matrix can be expressed as:

\[A + BK_\lambda = T_\lambda J_\lambda T_\lambda^{-1},
\]

where \( J_\lambda = -\lambda I + N \) and \( T_\lambda \) is a matrix built upon the linearly independent and generalized eigenvectors. We would like to remark the following:

- The matrix \( N \) has a unique structure for all values of \( \lambda \); this is because the geometric multiplicity of this eigenvalue remains unchanged. Moreover, by construction of Jordan decomposition, it does not depend on this parameter,

- As discussed in the Remark 3.3 the only linearly independent eigenvector of \( A + BK_\lambda \) depends in a rational way on \( \lambda \). Then, by construction of the generalized eigenvectors [11], they also rationally depend on \( \lambda \). Hence, the matrices \( T_\lambda \) and \( T_\lambda^{-1} \), they also depend on \( \lambda \) in a rational way.

Before presenting our triggering strategy, we introduce a family of coordinate transformations used in this paper. They are based on the Jordan decomposition of previous paragraphs. Let us consider the system (3), with the control \( u(t) = K_\lambda x(t_k) \). Then, the closed-loop dynamics is:

\[
\dot{x} = (A + BK_\lambda)x + BK_\lambda e,
\]

where \( e(t) = x(t_k) - x(t) \). Recalling (5), the latter dynamics under the static transformations \( e(t) = T_\lambda e_\lambda(t) \), and \( x(t) = T_\lambda x_\lambda(t) \), yields:

\[
\dot{x}_\lambda = J_\lambda x_\lambda + T_\lambda^{-1} BK_\lambda T_\lambda e_\lambda.
\]

The following result states our first attempt in developing the triggering strategy.

Proposition 4.1: Take \( \lambda > \|N\|^{1/2} + 1/2 \) and \( K_\lambda \) as in Proposition 3.1. Then \( V(x_\lambda) = x_\lambda^2 \) is an ISS-Lyapunov function for the system (9) and the event-triggering condition:

\[
|e_\lambda(t)|^2 \leq \frac{\sigma(\lambda - 1 - 2\|N\|)}{\|T_\lambda^{-1} BK_\lambda T_\lambda\|^2} |x_\lambda(t)|^2,
\]

guarantees the asymptotic stability of the system, for \( \sigma \in (0, 1) \).
Proof: Let \( \bar{B}_\lambda \triangleq T_\lambda^{-1}BK_\lambda T_\lambda \), computing the time-derivative of \( V(x_\lambda) \), and plugging from dynamics (6), we obtain:
\[
\dot{V} = \dot{x}_\lambda^T x_\lambda + e_\lambda^T B^T \bar{B} \lambda e_\lambda = (J_\lambda x_\lambda + B_\lambda e_\lambda)^T x_\lambda + x_\lambda^T (J_\lambda x_\lambda + B_\lambda e_\lambda).
\]
After some simplification, we obtain:
\[
\dot{V} = x_\lambda^T (J_\lambda^T + J_\lambda) x_\lambda + e_\lambda^T B^2 \lambda x_\lambda + x_\lambda^T B \lambda e_\lambda.
\]
The latter can be further bounded recalling the following inequality:
\[
e_\lambda^T B^2 \lambda x_\lambda + x_\lambda^T B \lambda e_\lambda \leq x_\lambda^T x_\lambda + e_\lambda^T B^2 \lambda B \lambda e_\lambda,
\]
which then yields:
\[
\dot{V} \leq x_\lambda^T (J_\lambda^T + J_\lambda + I) x_\lambda + e_\lambda^T B^2 \lambda B \lambda e_\lambda. \tag{8}
\]
Now, from our discussion on Jordan decomposition, it holds that \( J_\lambda^T + J_\lambda = -2\lambda I + N + N \). By plugging this back into (8), we obtain:
\[
\dot{V} \leq x_\lambda^T (N^T + N - (2\lambda - 1)I) x_\lambda + e_\lambda^T B^2 \lambda B \lambda e_\lambda.
\]
We further upper-bound the latter equation, noting that \( \|N^T\| = \|N\| \):
\[
\dot{V} \leq -(2\lambda - 1 - 2\|N\|) \|x_\lambda\|^2 + \|\bar{B}_\lambda\|^2 \|e_\lambda\|^2. \tag{9}
\]
Hence, for \( \lambda > \|N\| + 1/2 \), according to Equation (9), and that \( V(x_\lambda) = \|x_\lambda\|^2 \), \( \forall x_\lambda \), we conclude that \( V(x_\lambda) = x_\lambda^T x_\lambda \) is an ISS-Lyapunov function for the system (6). Moreover, let \( \sigma \in (0, 1) \) and that the triggering time-sequence be violated when the following condition is violated:
\[
|e_\lambda|^2 \leq \frac{\sigma(2\lambda - 1 - 2\|N\|)}{\|\bar{B}_\lambda\|^2} \|x_\lambda\|^2,
\]
it then holds that:
\[
\dot{V} \leq -(1 - \sigma)(2\lambda - 1 - 2\|N\|) \|x_\lambda\|^2.
\]
Hence, the event-triggering condition, described by (9), guarantees the asymptotic stability of the system. \( \square \)

Remark 4.2: Let \( t_k \) and \( t_{k+1} \) be two consecutive time-instants given by the triggering law (9). Then, for each \( \lambda \), the following holds:
\[
\exists \tau_\lambda > 0, \text{ such that } t_{k+1} - t_k \geq \tau_\lambda, \forall k \in \mathbb{N}.
\]
This is based on Theorem III.1, presented in [28]. In other words, the time-sequence generated by the triggering law (9) does not accumulate. This is an important observation used in our analysis.

For the parameter \( \tau_\lambda \), we show the following property:

Theorem 4.3: Consider the parameter \( \tau_\lambda \) introduced in Remark 4.2. Then, the following holds:
\[
\lim_{\lambda \to \infty} \tau_\lambda = 0. \tag{10}
\]

Proof: Recalling Remark 4.2 we first note that for the parameter \( \tau_\lambda \) it holds that \( \tau_\lambda \leq t_{k+1} - t_k, \forall k \in \mathbb{N} \). In particular, it holds that \( \tau_\lambda \leq t_2 - t_1 \). Let us, without loss of generality, set \( t_1 = 0 \), and denote \( t_k \triangleq t_k \). Then, it holds that \( 0 \leq \tau_\lambda \leq t_1 \). In this proof, we shall show \( \lim_{\lambda \to \infty} t_\lambda = 0 \), which implies that \( \lim_{\lambda \to \infty} \tau_\lambda = 0 \). By construction of the triggering law (9), the time-instant \( t_\lambda \) is when the following holds:
\[
|e_\lambda(t_\lambda)| = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|T_\lambda^{-1}BK_\lambda T_\lambda\|} \|x_\lambda(t_\lambda)\|. \tag{11}
\]
In the latter inequality and according to (9), it holds that \( BK_\lambda = T_\lambda J_\lambda T_\lambda^{-1} - A \), that is:
\[
T_\lambda^{-1}BK_\lambda T_\lambda = J_\lambda - T_\lambda^{-1}AT_\lambda.
\]
Hence, according to this equation, (11) can be written as follows:
\[
|e_\lambda(t_\lambda)| = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|J_\lambda - T_\lambda^{-1}AT_\lambda\|} \|x_\lambda(t_\lambda)\|.
\]
Let us denote:
\[
F(\lambda) \triangleq \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|J_\lambda - T_\lambda^{-1}AT_\lambda\|}. \tag{12}
\]
We continue by presenting the following result:

Claim 4.4: The following holds:
\[
\lim_{\lambda \to \infty} F(\lambda) = 0. \tag{13}
\]

Proof of Claim 4.4: Recalling that \( J_\lambda = -\lambda I + N \), Equation (12) can be written as:
\[
F(\lambda) = \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|(-\lambda I - T_\lambda^{-1}AT_\lambda) - (-N)\|},
\]
where by applying the triangular-inequality of the form \( \|(-\lambda I - T_\lambda^{-1}AT_\lambda) - (-N)\| \leq \|(-\lambda I - T_\lambda^{-1}AT_\lambda)\| + \|(-N)\| \), we obtain:
\[
0 \leq F(\lambda) \leq \frac{\sqrt{\sigma(2\lambda - 1 - 2\|N\|)}}{\|(-\lambda I - T_\lambda^{-1}AT_\lambda)\| - \|(-N)\|}.
\]
Computing the asymptotic limit of the latter inequality, noting that \( N \) is a constant matrix, and \( T_\lambda^{-1}T_\lambda = I \), yields:
\[
0 \leq \lim_{\lambda \to \infty} F(\lambda) \leq \lim_{\lambda \to \infty} \frac{\sqrt{2\|N\|}}{\|T_\lambda^{-1}(-\lambda I - A)T_\lambda\|}.
\]
On the other hand, for a matrix \( A \in \mathbb{R}^{n \times n} \) it holds that \( \rho(A) \leq \|A\| \), where \( \rho(A) \) is the spectral radius of \( A \); for further insight, refer to [4]. Hence, the latter equation can be further bounded as in the following:
\[
0 \leq \lim_{\lambda \to \infty} F(\lambda) \leq \lim_{\lambda \to \infty} \frac{\sqrt{2\|\rho(A)\|}}{\|T_\lambda^{-1}(-\lambda I - A)T_\lambda\|}.
\]
Case (ii): $t$ evolve. In other words there is no need for triggering, Claim 4.5: Case (i):

Recalling the triggering law and by

Proof of Claim 4.5:

Applying the latter equation on (17) gives:

Having proved $\lim_{\lambda \to \infty} F(\lambda) = 0$, we would like to show $\lim_{\lambda \to \infty} t_\lambda = 0$, which then, by the squeeze theorem, yields $\lim_{\lambda \to \infty} \tau_\lambda = 0$. The next claim goes along this direction.

Claim 4.5: The following holds:

Proof of Claim 4.5: Recalling the triggering law and by construction of $t_\lambda$, the following holds:

On the other hand, recall that for $t \in [0, T_\lambda]$

that yields:

Applying the latter equation on (17) gives:

In order to prove the result, we now consider two cases:

Case (i): $|x_\lambda(t_\lambda)| = 0$. In this case, by (18), it holds that $x_0 = 0$. Also since the system is linear and the chosen control law of the form $u = Kx_0$, the system does not evolve. In other words there is no need for triggering, $t_\lambda = 0$ which renders $\tau_\lambda = 0$.

Case (ii): $|x_\lambda(t_\lambda)| \neq 0$. In this case, dividing (18) by $|x_\lambda(t_\lambda)|$, bestows:

Computing the absolute-value of the latter equation, and applying the triangular inequality, yields the following result:

Now, according to the Claim 4.5 and by the squeeze theorem, we obtain:

Now we will show that $\lim_{\lambda \to \infty} t_\lambda = 0$ exploiting a contradiction argument. Hence, let us assume that $\lim_{\lambda \to \infty} t_\lambda \neq 0$. Then, the negation of the mathematical definition of $\lim_{\lambda \to \infty} t_\lambda = 0$ implies:

In (20), let us take the sequence $\{\lambda_k\}_{k \in \mathbb{N}}$, where $\lambda_k \to \infty$ as $k \to \infty$, and set $\tilde{\lambda} = \lambda_k$. Then we have the counterpart sequence $\{t_{\lambda_k}\}_{k \in \mathbb{N}}$, for which we have that $t_{\lambda_k} > t^*$. Because of this, we have that:

which follows from $V(t) < 0$ for $t < t_{\lambda_k}$—by the choice of our triggering law—and that $V(t) = |x_{\lambda_k}(t)|^2$. To obtain a deeper insight, we refer the reader to the proof of Theorem 5.1 Based on this observation, we derive the following inequality:

Since $t^* < t_{\lambda_k}$, and by Equation (20), we obtain:

which yields:

As $\sigma \in (0, 1)$, by properly letting $\lambda_k \geq 1/2 + \|N\|$ go to infinity, we obtain:

which induces:

from (21). From here we conclude that (22) is in contradiction with (19). Therefore, it must be that $\lim_{\lambda \to \infty} t_\lambda = 0$.

Henceforth, we observe that for both Case (i) and Case (ii), the Equation (16) holds, thus it completes the proof of this claim.
Finally, from the set of inequalities $0 \leq \tau_\lambda \leq t_\lambda$, which holds for all $\lambda \in \mathbb{R}_{>0}$, and by Claim 4.5 that shows that $\lim_{\lambda \to \infty} t_\lambda = 0$, we can conclude that $\lim_{\lambda \to \infty} \tau_\lambda = 0$.

In this paper, we assume that the jammer is imposing a "worst-case jamming scenario", i.e., $T_{\text{off}} = T_{\text{off}}^{\text{cr}}$. Now, having established the Proposition 4.1, and introduced the parameter $\tau_\lambda$ in Remark 4.2, we define the triggering strategy as follows.

**Definition 4.6:** The triggering strategy used in this paper, despite presence of the jammer, is as follows:

$$t_{k,n}^* \in \{ t_{\lambda} \mid t_{\lambda} \in [(n-1)T,(n-1)T + T_{\text{off}}^{\text{cr}}] \} \cup \{ nT \} .$$

(23)

We note that based on Theorem 4.3 and for a given $T$, we can find a $\lambda$, so that the multiples of $\tau_\lambda$ lie in the desired interval, i.e., the set introduced in (23) is never empty. The triggering law introduced in our recent paper [13] has inspired this strategy. The main difference between both laws is the choice of triggering times. While here we adapt the triggering sequence via an appropriate choice of $\tau_\lambda$ that depends on the jammer, in [13] we study when the time-sequence generated by (7) is sufficient to beat a given jammer.

In (23), $k \in \mathbb{N}$ denotes the number of triggering times occurring in the $n^{\text{th}}$ jammer action-period, and $l \in \mathbb{N}$ stands for the multiples of $\tau_\lambda$ starting from $l = 1$ in the first period and adding up afterwards.

**V. STABILITY ANALYSIS OF THE TRIGGERING STRATEGY**

In this section, we shall present the main result of this paper to guarantee the stability of the class of systems considered under the given type of jamming attacks.

The following bound is found in [18]:

$$\| \exp(M) \| \leq \exp(\mu(M)) , \quad M \in \mathbb{R}^{n \times n} ,$$

(24)

where the $\mu$ operator is defined as follows:

$$\mu(M) = \max \left\{ \mu \mid \mu \in \text{spec} \left( \frac{M + MT}{2} \right) \right\} .$$

(25)

In the proof of next result and in order to avoid the sign confusion, we shall use a variation of (25). Denote:

$$\mu_M \triangleq |\mu(M)| + 1 .$$

(26)

Then, the following holds:

$$\| \exp(M) \| \leq \exp(\mu_M) .$$

(27)

**Theorem 5.1:** Consider the system (3) of order lesser than 5, where $(A,B)$ is a controllable pair. Given a jammer signal $(T_{\text{off}},T)$, then $\exists \lambda^+ > \|N\| + 1/2$, such that $\forall \lambda \geq \lambda^+$, the system with control gain $K_\lambda$ as chosen in Proposition 4.1, and with triggering strategy (23), is asymptotically stable.

**Proof:** We shall focus on the first jammer action-period, i.e., $0 \leq t \leq T$. For the sake of brevity, we drop $n = 1$ in the $t_{k,n}$ annotation. Without loss of generality, let $t^*_k = kT_\lambda$, for $k \in \{1, \ldots, m\}$, be the time-sequence generated by (23), where $m$ is such that:

$$t^*_m = mT_\lambda \leq T_{\text{off}}^{\text{cr}} < t^*_{m+1} = (m + 1)T_\lambda .$$

We note that we can always assume this, since according to the Theorem 4.3, we can make $T_\lambda$ arbitrarily small by choosing $\lambda$ large enough. As $T_\lambda > 0$, the latter equation yields:

$$m \leq \frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} < m + 1 .$$

Thus, $\lfloor \frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} \rfloor = m$, where $\lfloor \cdot \rfloor$ is the floor operator, and

$$t^*_m = \left\lfloor \frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} \right\rfloor .$$

(28)

It is easy to see that for all $a > 0$, if $|a| \geq 1$, then $|a| \geq a/2$. Based on this observation, and as $\frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} \geq 1$, then it holds that $\frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} \geq \frac{T_{\text{off}}^{\text{cr}}}{T_\lambda}$, which by (28), gives:

$$t^*_m = \left\lfloor \frac{T_{\text{off}}^{\text{cr}}}{T_\lambda} \right\rfloor T_\lambda \geq \frac{T_{\text{off}}^{\text{cr}}}{2} .$$

(29)

The rest of the proof goes over the following steps:

1) We break the time-interval $[0,T]$ into two subintervals $[0,t^*_{m+1}]$, and $[t^*_{m+1},T]$; namely, when the jammer is sleeping and active, respectively.

2) Then, in order to find an estimate for $|x(t^*_{m+1})|$, and $|x(T)|$, we first transform the original system into new coordinates by the matrix $T_\lambda$; we perform some computations, and transform it back into its original coordinates, by $T_\lambda^{-1}$. This is done for each subinterval $[0,t^*_{m+1}]$, and $[t^*_{m+1},T]$. This way, the analysis becomes more tractable.

3) Finally, the theorem conclusion will follow by studying a coefficient $C(\lambda)$, appearing in $|x(T)| < C(\lambda)|x(0)|$. Due to $\lim_{\lambda \to \infty} C(\lambda) = 0$, we will be able to guarantee $\{ |x(nT)| \}$ is decreasing and use a Lyapunov argument to prove stability.

Let us consider the transformed system (6). We observe that for $t \in [0,t^*_{m+1}]$, and according to Remark 4.2, the event (7) introduced in Proposition 4.1 holds, and that also that $V = x_\lambda^T x_\lambda = |x_\lambda|^2$ is an ISS-Lyapunov function. Hence, resorting to the proof of this proposition, the following inequality holds:

$$V(x_\lambda(t)) \leq -(1 - \sigma)(2\lambda - 1 - 2\|N\|)|x_\lambda|^2 = - (1 - \sigma)(2\lambda - 1 - 2\|N\|)V(x_\lambda), \quad \forall t \in [0,t^*_{m+1}] .$$

The latter equation, by the comparison principle, yields:

$$V(x_\lambda(t)) \leq V(x_\lambda(0)) \exp \left( - (1 - \sigma)(2\lambda - 1 - 2\|N\|)t \right) .$$
which then, by recalling $V = x^T \lambda x = |x_\lambda|^2$, yields:

$$|x_\lambda(t)| \leq |x_\lambda(0)| \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| t)/2).$$  \hfill (30)

Now, we have to transform the latter equation into original coordinates. First, by using $x(t) = T_\lambda x_\lambda(t)$:

$$\lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1})) |x|^2 \leq |x_\lambda|^2 \leq \|T_\lambda^{-1}x\|^2 |x|^2. \hfill (31)
$$

The latter equation is obtained noting that (i) $|x_\lambda|^2 = x^T (T_\lambda^{-1})^T (T_\lambda^{-1}) x$, and (ii) the matrix $(T_\lambda^{-1})^T (T_\lambda^{-1})$ is a positive-definite symmetric matrix.

According to (31), Equation (30) implies:

$$|x(t^*_m)| \leq \|T_\lambda^{-1}\| \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| t^*_m/2)) |x_0| \sqrt{\lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))},$$

which is computed for $t = t^*_m$.

In addition, in an analogous way, this time considering $t \in [t^*_m, t^*_{m+1}]$, we can obtain the following result:

$$|x(t^*_{m+1})| \leq \|T_\lambda^{-1}\| \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| t^*_m/2)) \times \sqrt{\lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} |x(t^*_m)|,$$

where we note that $\tau_\lambda$ appears, as by our triggering law, $t^*_{m+1} - t^*_m = \tau_\lambda$.

Let us consider the transformed system (6), once more. We consider the time-interval $[t^*_m, T]$, then $x_\lambda(t) = x_\lambda(t^*_m) - x_\lambda(t)$ and so an equivalent form of (6) can be written as:

$$\dot{x}_\lambda = T_\lambda^{-1} A T_\lambda x_\lambda + T_\lambda^{-1} B K \lambda T_\lambda x_\lambda(t^*_m), \ \forall t \in [t^*_m, T].$$

Solving this dynamics for the initial condition $x_\lambda(t^*_m)$, we obtain the following:

$$x_\lambda(t) = \exp((t - t^*_m)T_\lambda^{-1} A T_\lambda)x_\lambda(t^*_m) + \int_{t^*_m}^t \exp((s - t)T_\lambda^{-1} A T_\lambda^{-1} B K \lambda T_\lambda x_\lambda(t^*_m)) \, ds,$$

which holds for $t \in [t^*_m, T]$. In order to further simplify the latter equation, we use the fact that for a given matrix $A \in \mathbb{R}^{n \times n}$, and invertible matrix $T \in \mathbb{R}^{n \times n}$, it holds: $exp(T^{-1}AT) = T^{-1}exp(A)T$.

Hence, Equation (34) is simplified as follows:

$$x_\lambda(t) = T_\lambda^{-1} \exp((t - t^*_m)A) T_\lambda x_\lambda(t^*_m) + \int_{t^*_m}^t T_\lambda^{-1} \exp((s - t)A) B K \lambda T_\lambda x_\lambda(t^*_m) \, ds,$$

which then results in the following equation:

$$T_\lambda x_\lambda(t) = \exp((t - t^*_m)A) T_\lambda x_\lambda(t^*_m) + \int_{t^*_m}^t \exp((s - t)A) B K \lambda T_\lambda x_\lambda(t^*_m) \, ds,$$

and using $x = T_\lambda x_\lambda$ to transform it back into the original dynamics, yields:

$$x(t) = \exp((t - t^*_m+1)A) x(t^*_m+1)+ \int_{t^*_m+1}^t \exp((s - t)A) B K \lambda x(t^*_m) \, ds. \hfill (35)$$

We upper-bound (35), using (27), which results in the following:

$$|x(t)| \leq |x(t^*_m)| \exp((t - t^*_m+1)\mu_A) + \|x(t^*_m)||B K \lambda\| \int_{t^*_m+1}^t \exp((s - t)\mu_A) \, ds.$$  \hfill (36)

Since $T - T^*_\text{off} = T^*_\text{on}$ and $T^*_\text{off} < t^*_{m+1}$, we have that:

$$T - t^*_{m+1} < T^*_\text{on}.$$  \hfill (37)

Thus, we can rewrite (36) as:

$$|x(T)| \leq |x(t^*_m)| \exp(T^*_\text{on}\mu_A) + \|x(t^*_m)||B K \lambda\| \exp(T^*_\text{on}\mu_A) - 1. \hfill (38)$$

Applying now Equation (33) on (37), we get:

$$\frac{|x(T)|}{|x(t^*_m)|} \leq \left( \frac{\|B K \lambda\|\exp(T^*_\text{on}\mu_A) - 1 + \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| \tau_\lambda/2) \exp(T^*_\text{on}\mu_A))}{\|T_\lambda^{-1}\|^{-1} \lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} \right). \hfill (39)$$

Now, combining (32) and (38), we obtain:

$$\frac{|x(T)|}{|x_0|} \leq \left( \frac{\exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| t^*_m/2))}{\|T_\lambda^{-1}\|^{-1} \lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} \right) \times \left( \frac{\|B K \lambda\|\exp(T^*_\text{on}\mu_A) - 1 + \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| \tau_\lambda) \exp(T^*_\text{on}\mu_A))}{\|T_\lambda^{-1}\|^{-1} \lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} \right).$$

To obtain the main equation, we shall use (29) to further bound (39), which then results in:

$$\frac{|x(T)|}{|x_0|} \leq \left( \frac{\exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| t^*_\text{off}/4))}{\|T_\lambda^{-1}\|^{-1} \lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} \right) \times \left( \frac{\|B K \lambda\|\exp(T^*_\text{on}\mu_A) - 1 + \exp(- (1 - \sigma) (2 \lambda - 1 - 2 \|N\| \tau_\lambda) \exp(T^*_\text{on}\mu_A))}{\|T_\lambda^{-1}\|^{-1} \lambda_{\min}((T_\lambda^{-1})^T (T_\lambda^{-1}))} \right) \leq C(\lambda). \hfill (40)$$
We present now the following result on the coefficient \( C(\lambda) \), introduced in the latter inequality.

Claim 5.2: In (40) following holds:
\[
\lim_{\lambda \to \infty} C(\lambda) = 0.
\] (41)

Proof of Claim 5.2 In order to complete the proof, we shall break \( C(\lambda) \)-expression as follows:
\[
C(\lambda) = C_1(\lambda)(C_2(\lambda) + C_3(\lambda)),
\]
where:
\[
C_1(\lambda) = \left( \frac{\exp\left(\left(-1 - \frac{1}{2}\right)(2\lambda - 1 - 2\|N\|)T_{\text{off}}^\tau / 4\right)}{\|T_\lambda^{-1}\|^{-1} \sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})} \right),
\]
\[
C_2(\lambda) = \left( \frac{|BK\lambda|}{\mu A} \right) (\exp (T_{\text{off}}^\tau \mu) - 1),
\]
and
\[
C_3(\lambda) = \left( \frac{\exp\left(\left(-1 - \frac{1}{2}\right)(2\lambda - 1 - 2\|N\|)T_{\text{off}}^\tau / 4\right)}{\|T_\lambda^{-1}\|^{-1} \sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})} \right) \exp (T_{\text{off}}^\tau \mu).
\]
Then, we shall show that \( \lim_{\lambda \to \infty} C_1(\lambda)C_2(\lambda) = 0 \), and \( \lim_{\lambda \to \infty} C_1(\lambda)C_3(\lambda) = 0 \).

According to (5), and recalling \( J_\lambda = -\lambda I + N \), we get \( BK_\lambda = -A + T_\lambda^{-1}(-\lambda I + N)T_\lambda \), which then results in \( BK_\lambda = -A - \lambda I + T_\lambda^{-1}NT_\lambda \), applying the 2-norm operator on both sides, we get \( \|BK_\lambda\| = \|(A + \lambda I) + T_\lambda^{-1}NT_\lambda\| \), which can be further upper-bounded as follows:
\[
\|BK_\lambda\| \leq \|A\| + \|\lambda I\| + \|T_\lambda^{-1}NT_\lambda\|.
\]

We shall employ this latter inequality, in order to obtain a new upper-bound for \( C_1(\lambda)C_2(\lambda) \):
\[
0 \leq C_1(\lambda)C_2(\lambda) \leq C_1(\lambda) \times \left( \frac{\|A\| + \|\lambda I\| + \|T_\lambda^{-1}NT_\lambda\|}{\mu A} \right) (\exp (T_{\text{off}}^\tau \mu) - 1). \] (42)

Now, in order to show that \( \lim_{\lambda \to \infty} C_1(\lambda)C_2(\lambda) = 0 \), we put together these two facts: (i) \( C_1(\lambda) \) decays exponentially, as \( \lambda \to \infty \), since \( \lambda > \|N\| + 1/2 \), and \( \sigma \in (0, 1) \), (ii) based on what we explained in Section [IV] the matrices \( T_\lambda \) and \( T_\lambda^{-1} \) depend on \( \lambda \) in a rational way, so the values \( \|T_\lambda\| \) and \( \|T_\lambda^{-1}\| \) depend on \( \lambda \) in a semi-algebraic form [3], hence the dependency of the coefficient of \( C_1(\lambda) \) appearing in upper-bound of \( C_1(\lambda)C_2(\lambda) \) in (42) on \( \lambda \) is of a semi-algebraic form, which is dominated by an exponential dependency. Therefore, we can conclude that the upper-bound of \( C_1(\lambda)C_2(\lambda) \) in (42) tends to zero as \( \lambda \to \infty \). Henceforth, as the lower-bound of \( C_1(\lambda)C_2(\lambda) \) is zero, then we conclude that:
\[
\lim_{\lambda \to \infty} C_1(\lambda)C_2(\lambda) = 0. \] (43)

In the following lines, we show that \( \lim_{\lambda \to \infty} C_1(\lambda)C_3(\lambda) = 0 \). First, we note that \( 0 \leq \tau_\lambda \leq T \), and \( -(1 - \sigma)(2\lambda - 1 - 2\|N\|) \leq 0 \), therefore, we get the following bounds for \( C_2(\lambda) \):
\[
\|T_\lambda^{-1}\| \exp\left(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T/2\right) \exp (T_{\text{off}}^\tau \mu) \leq \sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})\]
\[
C_3(\lambda) \leq \frac{\|T_\lambda^{-1}\| \exp\left(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T/2\right)}{\sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})} \exp (T_{\text{off}}^\tau \mu).
\]
Also, \( C_1(\lambda) > 0 \), \( \forall \lambda \), hence we can multiply the latter inequality by \( C_1(\lambda) \):
\[
\|T_\lambda^{-1}\| \exp\left(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T/2\right) \exp (T_{\text{off}}^\tau \mu) \leq \sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1}) \exp (T_{\text{off}}^\tau \mu) \frac{C_1(\lambda)C_3(\lambda)}{\sqrt{\lambda_{\min}}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})} C_1(\lambda). \] (44)

Then, we study the limit of upper- and lower-bounds of \( 44 \). Let us plug \( C_1(\lambda) \)-expression in the lower-bound of \( 44 \), we obtain:
\[
\text{LB}_{C_1C_3} \triangleq \frac{\|T_\lambda^{-1}\|^2 \exp\left(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T/2 + T_{\text{off}}^\tau / 4\right)}{\lambda_{\min}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})}.
\]
In order to show that \( \lim_{\lambda \to \infty} \text{LB}_{C_1C_3} = 0 \), we recall three facts: (i) Since \( \sigma \in (0, 1) \), \( \lambda \geq \|N\| + 1/2 \), then there is an exponentially decaying term in \( \text{LB}_{C_1C_3} \). (ii) Recalling our discussion in Section [IV] the matrix \( T_{\lambda}^{-1} \) depends on \( \lambda \) in a semi-algebraic way, which is dominated by exponential decay. (iii) Once again, referring to Section [IV] the matrix \( T_{\lambda}^{-1} \) depends on \( \lambda \) in a rational way. Hence, its characteristic polynomial depends on this parameter in a rational way, moreover, we note that this polynomial is of degree 4 or less, by assumption, and due to Galois theory, the dependency of the roots of this polynomial—including \( \lambda_{\min}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1}) \)—on \( \lambda \) is of semi-algebraic form, which is dominated by exponential decay.

Having discussed the behavior at infinity of the lower-bound of \( 44 \), we study the behavior of its upper-bound at infinity. Let us plug the \( C_1(\lambda) \) expression in the upper-bound of \( 44 \). We then obtain:
\[
\text{UB}_{C_1C_3} \triangleq \frac{\|T_\lambda^{-1}\|^2 \exp\left(-(1 - \sigma)(2\lambda - 1 - 2\|N\|)T_{\text{off}}^\tau / 4\right)}{\lambda_{\min}(T_{\lambda}^{-1})^T(T_{\lambda}^{-1})} \exp (T_{\text{off}}^\tau \mu).
\]
Similar to \( \text{LB}_{C_1C_3} \), it is easy to conclude that \( \lim_{\lambda \to \infty} \text{UB}_{C_1C_3} = 0 \). In the previous paragraphs, we have shown that the limit behavior as \( \lambda \) grows of the lower- and upper-bound of \( 44 \) is 0. Hence, we infer that:
\[
\lim_{\lambda \to \infty} C_1(\lambda)C_3(\lambda) = 0. \] (45)
Finally, having shown that the Equations (43), and (45), hold, we have proven (41). This completes the proof of this claim.

At this stage, we have proven that in $|x(T)| \leq |x_0|C(\lambda)$, it holds that $\lim_{\lambda \to \infty} C(\lambda) = 0$. The main consequence of this conclusion is that, based on the definition of limit, the following holds:

$\text{given } \epsilon > 0, \exists \lambda^* \text{ such that } \forall \lambda \geq \lambda^* \Rightarrow |C(\lambda)| < \epsilon, \quad (46)$

so, in other words, we can arbitrarily tune the decaying-rate of the states via $\lambda$ (and its effect on $C(\lambda)$). It is, nonetheless, worth mentioning that in order to paraphrase the asymptotic stability, the parameter $\epsilon$ has to be chosen such that $\epsilon < 1$, so that according to (46), $C(\lambda) < 1$, and so $V(T) < V(0)$, which ensures the asymptotic stability as demonstrated in Claim 4.3

Remark 5.3: As stated in the statement of this theorem, this is valid for systems of order 4, or less. An open question is the investigation the classes of systems of higher order for which the result still holds. In particular, we have observed the validity of the result for a system of order 5, and have presented the results in the Section VI.

In the following remark, we shall discuss that the results obtained so far are valid for an alternative problem formulation:

Remark 5.4: We recall from the Section III that in the current problem formulation, the jammer is corrupting the control channel signal, whereas the observation channel is safe. Indeed, we would like to point out that the results presented in this paper—including Theorem 5.1—are still valid under the other problem formulation, i.e., the observation channel is also corrupted by the same jammer. This is the case, because the measurement data is required at the same time-instant as the control is transmitted, which is successfully available, since the observation channel is jammed by the same jammer.

Remark 5.5: It is worth noting that our discussion in the Remark 5.4 is no longer valid for our previous results presented in [13], since therein the continuous measurement of the states was necessary.

Remark 5.6: At last, let us call (i) “frequency of communication” characterized by $T_{\lambda}$, and (ii) “actuation effort” characterized by $\lambda$ and $K_{\lambda}$, the two resources that the operator possesses in order to counteract the jammer. We would like to emphasize that in the method we are proposing, it is not feasible to decouple the utility of these two resources. In better words, the coupling between the utility of these resources yields the main results presented thus far. We, nonetheless, do not deny that an alternative approach may exist which encompasses this decoupling idea and yields the same assertion as in Theorem 5.1.

VI. Simulations

Having established the theoretical results of previous sections, here we demonstrate the functionality of these results on some representative academic examples.

A. Example 1: $3 \times 3$ system

We consider the following system:

$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$

$u = \left( \begin{bmatrix} 3 \\ 3 \end{bmatrix} \lambda^3 + 3, - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \lambda^2 + 2, -3\lambda - 3 \right) x.$

The state matrix of the closed-loop system is of the following form:

$A + BK_{\lambda} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda^3 & -3\lambda^2 & -3\lambda \end{bmatrix},$

where its only eigenvalue is $-\lambda$, which has algebraic and geometric multiplicity of 3, and 1, respectively, referring to Proposition 3.2. The only linearly independent eigenvector is given by solving the equation $(A + BK_{\lambda} + \lambda I)v_1 = 0$ for $v_1$, where we obtain:

$v_1 = \begin{bmatrix} 1 \\ -\lambda \\ \lambda^2 \end{bmatrix}.$

In order to build the matrix $T_{\lambda}$ (and $T_{\lambda}^{-1}$), we need to generate two other generalized eigenvectors, namely $v_2$, and $v_3$. They are, respectively, the solutions to $(A + BK_{\lambda} + \lambda I)v_2 = v_1$, and $(A + BK_{\lambda} + \lambda I)v_3 = v_2$ equations. After some algebraic manipulations, we get the following result:

$v_2 = \begin{bmatrix} \frac{\lambda}{\lambda^2} \\ -1 \\ 0 \end{bmatrix} , v_3 = \begin{bmatrix} \frac{1}{\lambda^2} \\ 1 \\ 0 \end{bmatrix}.$

Hence, matrix $T_{\lambda}$ is obtained as: $T_{\lambda} = [v_1, v_2, v_3]$. Moreover, given the multiplicities of $-\lambda$, the matrix $N$ is as follows:

$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

In order to perform the simulation, we have to “tune” some parameters related to the jammer and the triggering policy. We have chosen $\sigma = 0.1$, jammer action-period $T = 1$ sec, $T_{\sigma_{\text{on}, 1}} = 0.9T$, $T_{\sigma_{\text{off}, 1}} = 0.1T$, and $T_{\sigma_{\text{on}, 2}} = 0.5T$, $T_{\sigma_{\text{off}, 2}} = 0.5T$. We note that the first jammer is more malicious than the second one.

We use the procedure explained in Algorithm 1 to run the simulation. The result is presented in Figure 3.
Remark 6.1: We acknowledge these facts: (i) the 90% active jammer is more malicious than the 50% active jammer, and (ii) to maintain the asymptotic stability, we should at least guarantee $C(\lambda) < 1$. Hence, let us define:

$$\bar{\lambda} = \min_{1 \leq k \leq N^\prime} \{ \lambda_k | \forall \lambda \geq \lambda_k, C(\lambda) < 1 \}.$$ 

Then, we obtain $\bar{\lambda}_{90\%} = 1360$, and $\bar{\lambda}_{50\%} = 210$. Accordingly, we can induce that in order to guarantee the asymptotic stability, larger poles (in the absolute sense) are required in the case of 90% active jammer; this can be interpreted as larger control effort.

Remark 6.2: According to the intricacy of the $C(\lambda)$ equation; finding $\lambda$ analytically, for a given value of $C(\lambda)$ is not feasible. As an alternative way, one can use our proposed procedure, in order to numerically achieve this goal. So, e.g., having obtained the sequence $\{C(\lambda_k)\}_{k=1}^{N^\prime}$ for a given sequence $\{\lambda_k\}_{k=1}^{N^\prime}$, one can then obtain a polynomial or spline approximation for $C(\lambda)$.

For this system, we have also conducted a study on the evolution of the parameter $\tau_\lambda$. This time, we picked the sequence $\{\lambda_k = 0.01k\}_{k=1}^{1000}$ and for each $\lambda_k$, we run the procedure explained in Algorithm 1. The result is presented in Figure 3. The Figure 3 confirms our result in Theorem 4.3 on the evolution of $\tau_\lambda$.

B. Example 2: 5×5 system

Our main result in this paper, Theorem 5.1, is stated for the systems of order 4, or less. Nevertheless, this is based on the general condition provided by Galois Theory, leaving open the question of whether it holds for subclasses of systems of higher order. We have conducted a simulation study on a 5×5 system in canonical form, and as it comes later, our result is yet valid.

We consider the following system:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -7 & 10 & -3 & 4 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u,$$

$$u = [-\lambda^5 + 7, -5\lambda^4 - 10, -10\lambda^3 + 3, -10\lambda^2 - 4, -5\lambda + 6]x.$$
The state-matrix of the closed-loop system is of the following form:

\[
A + BK_\lambda = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
-\lambda^3 & -5\lambda^4 & -10\lambda^3 & -10\lambda^2 & -5\lambda \\
\end{bmatrix},
\]

where its only eigenvalue is $-\lambda$, which has algebraic and geometric multiplicity of 5, and 1, respectively - referring to Proposition 3.2. The only linearly independent eigenvector is given by solving the equation $(A + BK_\lambda + \lambda I)v_1 = 0$ for $v_1$, where we obtain:

\[
v_1 = \begin{pmatrix}
1 \\
-\lambda \\
\lambda^2 \\
-\lambda^3 \\
\lambda^4
\end{pmatrix}.
\]

In an analogous way as in Subsection VI-A we compute the generalized eigenvectors:

\[
v_2 = \begin{pmatrix}
\frac{3}{2} \\
-3 \\
2\lambda \\
-\lambda^2 \\
0
\end{pmatrix},
\quad
v_3 = \begin{pmatrix}
\frac{10}{2} \\
\frac{-3}{2} \\
3 \\
-\lambda \\
0
\end{pmatrix},
\quad
v_4 = \begin{pmatrix}
\frac{20}{3} \\
\frac{-1}{2} \\
\frac{-3}{2} \\
\frac{3}{2} \\
0
\end{pmatrix},
\quad
v_5 = \begin{pmatrix}
\frac{25}{4} \\
\frac{-15}{4} \\
-\lambda \\
0
\end{pmatrix}.
\]

Hence, matrix $T_\lambda$ is obtained as: $T_\lambda = [v_1, v_2, v_3, v_4, v_5]$. Moreover, given the multiplicities of $-\lambda$, the matrix $N$ is as follows:

\[
N = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

We “tune” the parameters related to the jammer and the triggering policy: $\sigma = 0.1$, jammer action-period $T = 1$ sec, $T_{\text{on,1}} = 0.9T$, $T_{\text{off,1}} = 0.1T$, and $T_{\text{on,2}} = 0.5T$, $T_{\text{off,2}} = 0.5T$. We note that the first jammer is more malicious than the second one. Then, we perform the simulation running the procedure explained in Algorithm [1] the result is shown in Figure 4.

Referring to Figure 4, we can list similar remarks, as in Subsection VI-A Furthermore, as promised before, in both cases, it holds that $\lim_{\lambda \to \infty} C(\lambda) = 0$, which could not be theoretically backed up.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we have considered single-input, of order 4 or less, continuous LTI systems, under periodic PWM DoS jamming attacks. We have proposed a resilient control design law, along with a triggering time-sequence to update the controller. In the main result, we demonstrated that this control design and triggering law is capable of countering the effect of any jammer. In other words, we show that the system is rendered asymptotically stable under our contributions. The functionality of the theoretical studies has been demonstrated in the simulation environment; where we have also shown that the result holds for a system of order 5, for which our theoretical result cannot be stretched.

In this work, we have assumed that the jammer signal has been previously detected and identified. We are currently studying how to exploit signal processing techniques to partly identify the jammer, that is to identify the parameter $T$: the jammer’s period. Moreover, as the title stands for, in this paper we have focused on single-input linear systems. In future work, we will plan to study nonlinear and multi-input classes of systems.

REFERENCES

[1] N. Adams. Workshop on future directions in cyber-physical systems security. Technical report, workshop organized by Department of Homeland Security (DHS), 2010.
[2] S. Amin, A. Cardenas, and S. S. Sastry. Safe and secure networked control systems under denial-of-service attacks. In Hybrid Systems: Computation and Control, pages 31–45, 2009.
[3] S. Basu, R. Pollack, and M.-F. Roy. Algorithms in Real Algebraic Geometry (Algorithms and Computation in Mathematics). Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.
[4] D. S. Bernstein. Matrix Mathematics: theory, facts, and formulas with application to linear system theory. Princeton University Press, 2005.
[5] S. Bhattacharya and T. Basar. Differential game-theoretic approach to a spatial jamming problem. In Proceedings of 14th International Symposium on Dynamic Games and Applications, Banff, Canada, June 2010.
[6] S. Bhattacharya and T. Basar. Graph-theoretic approach for connectivity maintenance in mobile networks in the presence of a jammer. In IEEE Conf. on Decision and Control, Atlanta, USA, December 2010.

[7] E. Byres and J. Lowe. The myths and facts behind cyber security risks for industrial control systems. In Proceedings of the VDE Congress, VDE Association for Electrical Electronics and Information Technologies, 2004.

[8] A. Cardenas, S. Amin, and S. Sastry. Secure control: Towards survivable cyber-physical systems. In First International Workshop on Cyber-Physical-Systems (WCPSP2008). IEEE, June 2008.

[9] A. Cardenas, S. Amin, B. Sinopoli, A. Giani, A. Perrig, and S. Sastry. Challenges for securing cyber physical systems. In Workshop on Future Directions in Cyber-physical Systems Security. DHS, July 2009.

[10] B. DeFrulh and P. Tague. Digital filter design for jamming mitigation in 802.15.4 communication. In Proceedings of 20th International Conference on Computer Communications and Networks (ICCCN), 2011, pages 1–6, 2011.

[11] V. N. Faddeeva. Computational Methods of Linear Algebra. Dover Publications, 1958.

[12] H. Fawzi, P. Tabuada, and S. Diggavi. Secure state-estimation for dynamical systems under active adversaries. In Proceedings of 49th Annual Allerton Conference on Communication, Control, and Computing, 2011.

[13] H. Shisheh Foroursh and S. Martinez. On event-triggered control of linear systems under periodic dos attacks. In Proc. of the 51st IEEE International Conference on Decision and Control, Maui, HI, December 2012. To appear.

[14] H. Shisheh Foroursh and S. Martinez. On single-input controllable linear systems under periodic dos jamming attacks. http://arxiv.org/abs/1209.4101.

[15] A. G. Fragkiadakis, V. A. Siris, and N. Petroulakis. Anomaly-based intrusion detection algorithms for wireless networks. In WWIC, pages 192–203, 2010.

[16] A. Gupta, C. Langbort, and T. Basar. Optimal control in the presence of an intelligent jammer with limited actions. In IEEE Conf. on Decision and Control, pages 1096–1101, Atlanta, USA, December 2010.

[17] J. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. Proceedings of IEEE Special Issue on Technology of Networked Control Systems, 95(1):138–162, 2007.

[18] C. V. Loan. The sensitivity of the matrix exponential. SIAM Journal of Numerical Analysis, 14(6):971–981, 1977.

[19] X. Luo, E. W. Chan, and R. K. C. Chang. Detecting pulsing denial-of-service attacks with nondeterministic attack intervals. EURASIP J. Adv. Signal Process, 2009:8:1–8:13, January 2009.

[20] X. Luo and R. K. C. Chang. On a new class of pulsing denial-of-service attacks and the defense. In In Network and Distributed System Security Symposium (NDSS), 2007.

[21] M. Mazo, A. Anta, and P. Tabuada. An ISS self-triggered implementation of linear controllers. Automatica, 46(8):1310 – 1314, 2010.

[22] F. Pasqualetti, A. Bicchi, and F. Bullo. Consensus computation in unreliable networks: A system theoretic approach. IEEE Transactions on Automatic Control, to appear, 2011.

[23] F. Pasqualetti, R. Carli, and F. Bullo. Distributed estimation and false data detection with application to power networks. Automatica. submitted.

[24] R. A. Poisel. Modern Communication Jamming Principles and Techniques. Artech, 2004.

[25] S. Roy, C. Ellis, S. Shiva, D. Dasgupta, V. Shandilya, and Q. Wu. A survey of game theory as applied to network security. In Proceedings of the 43rd Hawaii International Conference on System Sciences, pages 1–10, Hawaii, USA, 2010.

[26] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry. Foundations of control and estimation over lossy networks. Proceedings of IEEE Special Issue on Technology of Networked Control Systems, 95(1):163–187, 2007.

[27] S. Sundaram and C. N. Hadjicostis. Distributed function calculation via linear iterations in the presence of malicious agents - parts I, II. In American Control Conference, pages 1350–1362, June 2008.

[28] P. Tabuada. Event-triggered real-time scheduling of stabilizing control tasks. IEEE Transactions on Automatic Control, 52(9):1680–1685, 2007.

[29] G. Theodorakopoulos and J. S. Baras. Game theoretic modeling of malicious users in collaborative networks. IEEE Journal on Selected Areas in Communications, 7:1317–1327, 2008.

[30] X. Wang and M. D. Lemmon. Self-triggered feedback control systems with finite-gain L2 stability. Automatic Control, IEEE Transactions on, 54(3):452–467, 2009.

[31] W. Xu, W. Trapppe, Y. Zhang, and T. Wood. The feasibility of launching and detecting jamming attacks in wireless networks. In Proceedings of the 6th ACM international symposium on Mobile ad-hoc networking and computing, MobiHoc ’05, pages 46–57, 2005.

[32] M. Zhu and S. Martinez. On distributed constrained formation control in operator-vehicle adversarial networks. Automatica. Submitted, 2012.

[33] M. Zhu and S. Martinez. Attack-resilient distributed formation control via online adaptation. In IEEE International Conference on Decision and Control, pages 6624–6629, Orlando, FL, USA, December 2011.