STOCHASTIC FLOWS WITH REFLECTION
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Summary: Some topological properties of stochastic flow $\varphi_t(x)$ generated by stochastic differential equation in a $\mathbb{R}^d_+$ with normal reflection at the boundary are investigated. Sobolev differentiability in initial condition is received. The absolute continuity of the measure-valued process $\mu \circ \varphi_t^{-1}$, where $\mu \ll \lambda^d$, is studied.

Flows generated by SDEs in Euclidean space is a well-studied topic nowadays. It is well known for example (cf. [1] and ref. therein) that if the coefficients of SDE are Lipschitzian then the SDE generates a flow of homeomorphisms, if coefficients are of the class $C^{n+\varepsilon}$ then SDE generates $C^n$-flow of diffeomorphisms, equations for derivatives are obtained by formal differentiation of the SDE etc.

Note that the similar questions for SDEs with reflection is much harder to answer. Even the problems about coalescence of two reflecting Brownian motions [2, 3, 4, 5] or differentiability of the Brownian reflecting flow ($\sigma(x) = const$) [6, 7] need accurate and non-trivial considerations.

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Assume that functions $a_k : \mathbb{R}^d_+ \to \mathbb{R}^d$ satisfy the Lipschitz condition. Here $\mathbb{R}^d_+ = \mathbb{R}^d - 1 \times [0, \infty)$. Consider an SDE in $\mathbb{R}^d_+$ with normal reflection from the boundary:

$$
\begin{cases}
    d\varphi_t(x) = a_0(\varphi_t(x))dt + \sum_{k=1}^{m} a_k(\varphi_t(x))dw_k(t) + \\
    \quad + n\mathbf{1}(dt, x), \\ 
    \varphi_0(x) = x, \xi(0, x) = 0, \quad x \in \mathbb{R}^d,
\end{cases}
$$

where $\{w_k(t), k = 1, \ldots, m\}$ are independent Wiener processes, $n = (0, \ldots, 0, 1)$ is a normal to hyperplane $\mathbb{R}^{d-1} \times \{0\}$, for each fixed $x \in \mathbb{R}^d_+$ a process $\xi(t, x)$ is non-decreasing in $t$, and

$$
\xi(t, x) = \int_0^t \mathbf{1}_{\{\varphi_s(x) \in \mathbb{R}^{d-1} \times \{0\}\}} \xi(ds, x),
$$

i.e. $\xi(t, x)$ is increasing only on those instants of time when $\varphi_t(x) \in \mathbb{R}^{d-1} \times$
Lipschitz property of the coefficients ensures the existence and the uniqueness to the solution of (1), cf. \cite{9}.

1. Existence of continuous modification.

Theorem 1 \cite{10}. There exists a modification of the processes $\varphi_t(x), \xi(t, x)$ (it will be denoted in the same way) such that

1) for any $x \in \mathbb{R}^d_+$, the pair $(\varphi_t(x), \xi(t, x)), t \geq 0$, is a solution of (1);

2) for any $\omega \in \Omega$ processes $\varphi_t(x), \xi(t, x)$ are continuous in a pair of arguments $(t, x), t \geq 0, x \in \mathbb{R}^d_+$.

The Theorem 1 is proved in a way similar to the corresponding proof used for the solution of SDE without reflection, cf. \cite{11}, with the use of Kolmogorov’s theorem on existence of continuous modification.

It will be assumed further that $\varphi_t(x), \xi(t, x)$ are already continuous.

2. The joint motion of solutions started from different initial points. It is well known \cite{11} that a solution of an SDE (without reflection) generates a flow of diffeomorphisms. However, the injectivity for reflecting flow can be failed as the following example shows.

Example 1. Let $d = 1, m = 1, a_0 = 0, a_1 = 1$, i.e. $\varphi_t(x)$ is the reflected Brownian motion in $\mathbb{R}^1_+$ started from $x \geq 0$:

$$\varphi_t(x) = x + w(t) + \xi(t, x), x \geq 0.$$ 

It is easy to see that $\varphi_t(x), \xi(t, x)$ is of the form

$$\varphi_t(x) = \begin{cases} w(t) - \min_{0 \leq s \leq t} w(s), & x = 0, \\ w(t) + x, & x > 0 \quad \tau(x) \geq t, \\ \varphi_t(0), & x > 0 \quad \tau(x) < t, \end{cases}$$

$$\xi(t, x) = \begin{cases} -\min_{\tau(x) \leq s \leq t} w(s), & \tau(x) < t, \\ 0, & \tau(x) \geq t, \end{cases}$$

where $\tau(x)$ is a moment, when the process $x + w(t)$ gets zero for the first time.

In other words, $\varphi_t(x)$ is moving as $x + w(t)$ before hitting 0, and then a motion of $\varphi_t(x)$ coincides with the reflected Brownian motion $\varphi_t(0)$ started from zero.

The similar situation takes place in multi-dimensional space.
Theorem 2. [10] Denote by \( \tau(x) = \inf \{t \geq 0 : \varphi_t(x) \in \mathbb{R}^{d-1} \times \{0\}\} \) the moment of the first hitting the hyperplane \( \mathbb{R}^{d-1} \times \{0\} \) by a solution started from \( x \in \mathbb{R}_+^d \).

Then there exists a set \( \Omega_0 \) of probability 1 such that for all \( \omega \in \Omega_0 \) the following statements hold true:

1) for all \( x, y \in \mathbb{R}_+^d, x \neq y \) and \( t < \max \{\tau(x), \tau(y)\} \) the inequality \( \varphi_t(x) \neq \varphi_t(y) \) is satisfied;

2) for any \( x \in \mathbb{R}_+^d \) there exists \( y = y(x, \omega) \in \mathbb{R}^{d-1} \times \{0\}, \) such that \( \varphi_{\tau(x)}(x) = \varphi_{\tau(x)}(y) \) if \( \tau(x) < \infty \). Moreover,

\[
\varphi_t(x) = \varphi_t(y) \text{ for } t \geq \tau(x).
\]

Remark. Informally this theorem can be formulated in the following way. A particle started from a point \( x \in \mathbb{R}^{d-1} \times (0, \infty) \) does not hit any other particle before getting the hyperplane \( \mathbb{R}^{d-1} \times \{0\} \). At the instant \( \tau(x) \) it coalesces with some other particle, which started from \( \mathbb{R}^{d-1} \times \{0\} \). After this both particles moves together.

3. Characterization of inner and boundary points of random set \( \varphi_t(\mathbb{R}_+^d) \).

Theorem 3. [11] For almost all \( \omega \) and all \( t \in [0, T] \) the following equality of random sets takes place

\[
\partial \varphi_t(\mathbb{R}_+^d) = \varphi_t(\partial \mathbb{R}^d_+) = \varphi_t\{x \in \mathbb{R}_+^d : \tau(x) \leq t\},
\]

where \( \tau(x) = \inf \{s \geq 0 : \varphi_s(x) \in \mathbb{R}^{d-1} \times \{0\}\} \) is the moment of the first hitting the hyperplane \( \mathbb{R}^{d-1} \times \{0\} \) by the solution started from \( x \).

Moreover, for all \( R > 0 \) Hausdorff measure \( H^{d-1} \) of the set \( \partial \varphi_t(\mathbb{R}_+^d) \cap \{x \in \mathbb{R}_+^d : \|x\| \leq R\} \) is finite.

4. Differentiability with respect to initial condition.

As in Example 1, there is no reasons to expect that a solution of (1) is continuously differentiable in \( x \) even when coefficients of the SDE are infinite differentiable. However, it can be proved that for any \( t \) the mapping \( x \to \varphi_t(x) \) belongs to a Sobolev space \( W^{1,p}_{p,\text{loc}}(\mathbb{R}_+^d, \mathbb{R}^d) \).

The equations for \( \nabla \varphi_t \) are not classical equations of stochastic analysis. We need the following definition.
Definition 1. Let \( w_1(t), \ldots, w_m(t) \) be independent Wiener processes, \( \mathcal{F}_t = \sigma(w_k(s), k = 1, m, s \leq t) \), \( a_k : \mathbb{R}^l \times \mathbb{R}^p \rightarrow \mathbb{R}^l \), \( b_k : \mathbb{R}^l \times \mathbb{R}^p \rightarrow \mathbb{R}^p \), \( k = 0, \ldots, m \), and \( x_t \) be a continuous \( \mathcal{F}_t \)-measurable stochastic process. Consider a random measure-valued process \( \nu(t) = \delta_0 I\{x(t) = 0\} \), where \( \delta_0 \) is a probability measure on \( \mathbb{R} \), assigned unit mass to a point zero.

A pair \((y_t, z_t)\) of \( \mathcal{F}_t \)-adapted processes satisfies the equation

\[
\begin{aligned}
    dy_t &= a_0(y_t, z_t)dt + \sum_{k=1}^{m} a_k(y_t, z_t)dw_k(t) - y_t - d\nu(t), \\
    dz_t &= b_0(y_t, z_t)dt + \sum_{k=1}^{m} b_k(y_t, z_t)dw_k(t), \ t \geq 0,
\end{aligned}
\]

if:

1) \( y_t, t \geq 0 \) has cadlag trajectories;
2) \( z_t, t \geq 0 \) has continuous trajectories;
3) \( z_t = z_0 + \int_0^t b_0(y_s, z_s)ds + \sum_{k=1}^{m} \int_0^t b_k(y_s, z_s)dw_k(s), t \geq 0 \) a.s.;
4) for almost all \( \omega \) the set \( \{t \geq 0 : x_t = 0\} \) is contained in \( \{t \geq 0 : y_t = 0\} \);
5) for any stopping time \( \tau \) such that \( x_\tau \neq 0 \) a.s., the following equality holds true

\[
y_t = y_\tau + \int_\tau^t a_0(y_s, z_s)ds + \sum_{k=1}^{m} \int_\tau^t a_k(y_s, z_s)dw_k(s)
\]

for all \( t \in [\tau, \tau^\circ) \), \( \tau^\circ = \inf\{t \geq \tau : x_t = 0\} \).

Theorem 4. Assume that functions \( a_k, b_k, k = \overline{0, m} \) satisfy Lipschitz condition. Then there exists a unique solution of (2) for any non-random initial condition \((y_0, z_0)\).

Theorem 5. I. If functions \( a_k : \mathbb{R}^d_+ \rightarrow \mathbb{R}^d, k = \overline{0, m} \) satisfy Lipschitz condition then for a.a. \( \omega \) a mapping \( \mathbb{R}^d_+ \ni u \rightarrow \varphi_t(u) \in \mathbb{R}^d \) belongs to the space \( \cap_{p>1} W^1_{p, \text{loc}}(\mathbb{R}^d_+, \mathbb{R}^d) \) for a.a. \( t \geq 0 \).

II. Assume that functions \( a_k : \mathbb{R}^d_+ \rightarrow \mathbb{R}^d, k = \overline{0, m} \), are continuously differentiable and their derivatives are bounded. Suppose also that \( \sum_{k=1}^{m} (a_{k,d}(x))^2 > 0 \) for all \( x \in \mathbb{R}^{d-1} \times \{0\} \), where \( a_{k,d} \) is the \( d \)-th coordinate of a function \( a_k = (a_{k,1}, \ldots, a_{k,d})^T \).
Then the Sobolev derivative $\nabla \varphi_t(x)$ satisfies the SDE

$$\begin{cases}
    d\nabla \varphi_t(x) = \nabla a_0(\varphi_t(x))\nabla \varphi_t(x)dt + \sum_{k=1}^{m} \nabla a_k(\varphi_t(x))\nabla \varphi_t(x)dw_k(t) - \\
    -P\nabla \varphi_t(x)n(dt, x), \\
    \nabla \varphi_0(x) = \mathbf{I},
\end{cases}$$

(3)

where $\mathbf{I}$ is an identity matrix, $P$ is a matrix corresponding to the orthoprojection on the $d$-th coordinate of the space $\mathbb{R}^d$, $n(dt, x)$ is a point random measure such that $n(\{t\}, x) = 1$ iff $\varphi_t(x)$ belongs to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$.

**Remark.** Equation (3) is understood in the sense of Definition 1. In this case we take the $d$-th coordinate of $\varphi_t(x)$ as $x_t$, the $d$-th row of $\nabla \varphi_t(x)$ as $y_t$, the first $(d-1)$ rows of $\nabla \varphi_t(x)$ and $\varphi_t(x)$ as the process $z_t$.

**Remark.** The process $\nabla \varphi_t(x)$ can be chosen measurable in $t, x, \omega$.

**Remark.** The similar result for constant diffusion coefficient was obtained in [7]. Moreover, it was proved that for all $x$ and a.a. $\omega$ the mapping $\varphi_t$ is continuously differentiable in some neighborhood of $x$.

Let us compare Sobolev differentiability and usual differentiability of the mapping $\varphi_t(\cdot, \omega)$. It is well known that if the diffusion matrix is a constant then for a.a. $\omega$ and all $t$ the mapping $x \to \varphi_t(x, \omega)$ satisfies Lipschitz condition. Therefore $\varphi_t(x)$ is differentiable for $\lambda^d$-a.a. $x \in G$ by Rademacher’s theorem [14]. Since the usual and Sobolev derivatives are equal (if they exist), so equation for usual derivative coincides with that for Sobolev. It is not difficult to prove that the usual derivatives exist not only for a.a. $x$, a.a. $\omega$, but for all $x$ and a.a. $\omega$. However almost everywhere local continuous differentiability is not evident.

It should be noted that [7] does not imply that for a.a. $\omega$ the mapping $x \to \varphi_t(x, \omega)$ is continuously differentiable. Really, for the process from Example 1:

$$\mathbb{P}(x \to \varphi_t(x) \text{ is continuously differentiable}) = 0$$

but for each $x_0 > 0$:

$$\mathbb{P}(x \to \varphi_t(x) \text{ is continuously differentiable in some neighborhood of } x_0) = 1!$$

The fact that $\mathbb{R}_+^d \ni x \to \varphi_t(x)$ is not continuously differentiable seems to be typical, because $\text{rank}\nabla \varphi_t(x) \leq d - 1$ if $\tau(x) \leq t$ and $\varphi_t(x)$, $\tau(x) > t$,.
coincides with the flow without reflection, so \( \det \nabla \varphi_t(x) \), \( \tau(x) > t, \|x\| \leq r \), is separated from zero for any \( r > 0 \).

5. Absolute continuity of image-measures driven by \( \varphi_t \). Let \( \mu \) be a finite measure in \( \mathbb{R}^d_+ \) which is absolute continuous w.r.t. Lebesgue measure. Consider a measure-valued process \( \mu_t = \mu \circ \varphi_t^{-1}, t \geq 0 \).

Let us introduce a random set \( O_t(\omega) = \{x \in \mathbb{R}^d_+ : t < \tau(x)\} \), where \( \tau(x) = \inf\{s \geq 0 : \varphi_s(x) \in \mathbb{R}^d-1 \times \{0\}\} \) is the moment of the first hitting the hyperplane \( \mathbb{R}^d-1 \times \{0\} \) by the process \( \varphi_s(x) \).

**Theorem 6.** [13] For a.a. \( \omega \) and every \( t \geq 0 \) a measure \( \mu_t \) is represented as a sum of orthogonal measures \( \mu_t = \mu|_{O_t} \circ \varphi_t^{-1} + \mu|_{\mathbb{R}^d_+ \setminus O_t} \circ \varphi_t^{-1} \), such that

a) the first measure is absolute continuous w.r.t. \( d \)-dimensional Lebesgue measure and the second one is singular;

b) the support of measure \( \mu|_{\mathbb{R}^d_+ \setminus O_t} \circ \varphi_t^{-1} \) is contained in the set \( \varphi_t(\mathbb{R}^d-1 \times \{0\}) \) of the \( \sigma \)-finite \( (d - 1) \)-dimensional Hausdorff measure \( H^{d-1} \).

The proof of the first part of the theorem follows from [16], and the second part follows from Theorem 3.

The next theorem gives a sufficient condition that ensures the absolute continuity of \( \mu|_{\mathbb{R}^d_+ \setminus O_t} \circ \varphi_t^{-1} \) with respect to \( H^{d-1}|_{\partial \varphi_t(\mathbb{R}^d_+)} \), which is the restriction of \( H^{d-1} \) to the set \( \partial \varphi_t(\mathbb{R}^d_+) \).

**Theorem 7.** [13] Assume that for \( \mu \)-a.a. \( x \in \mathbb{R}^d_+ \):

\[
P(\text{rank} \nabla \varphi_t(x) \geq d - 1, t \geq 0) = 1. \tag{4}
\]

Then with probability 1 the absolute continuity

\[
\mu|_{\mathbb{R}^d_+ \setminus O_t} \circ \varphi_t^{-1} \ll H^{d-1}|_{\partial \varphi_t(\mathbb{R}^d_+)} \tag{5}
\]

holds for all \( t \geq 0 \).

Here \( \mu|_{\mathbb{R}^d_+ \setminus O_t} \), \( H^{d-1}|_{\partial \varphi_t(\mathbb{R}^d_+)} \) are restrictions of measures \( \mu \), \( H^{d-1} \) to the sets \( \mathbb{R}^d_+ \setminus O_t \), \( \partial \varphi_t(\mathbb{R}^d_+) \), respectively.

**Remark.** In contrast to Hausdorff measure \( H^{d-1} \) in \( \mathbb{R}^d \), its restriction to the set \( \partial \varphi_t(\mathbb{R}^d_+) = \varphi_t(\mathbb{R}^d_+ \setminus O_t) \) is a \( \sigma \)-finite measure (Theorem 3). So the notion of absolute continuity in (5) does not require any specification.

The proof is provided by using the co-area formula [14] similarly to the case \( m = n \), cf. [16, 15].
The verification of Theorem 7 conditions is quite difficult. Let us give more simple sufficient conditions that ensure (4). Assume that functions $a_k, 0 \leq k \leq m,$ are continuously differentiable.

Denote by $U_{st}(x), s \leq t,$ a solution of the following linear SDE:

$$
\begin{align*}
 & \left\{ 
    \begin{array}{l}
    dU_{st}(x) = \nabla a_0(\varphi_t(x))U_{st}(x)dt + \sum_{k=1}^{m} \nabla a_k(\varphi_t(x))U_{st}(x)dw_k(t), t \geq s, \\
    U_{ss}(x) = I.
    \end{array}
    \right.
\end{align*}
$$

Let us represent a random set $A = A_t(x) = \{ s \in [0, t] : \varphi_s^d(x) > 0 \}$ as a disjoint union of random intervals $A = [\alpha_0(x), \beta_0(x)) \cup (\alpha_1(x), \beta_1(x)] \cup \bigcup_{k=2}^{\infty} (\alpha_k(x), \beta_k(x)),$ where $\alpha_0(x) = 0, \beta_1(x) = t.$ Let $P$ be the same as in Theorem 5.

**Theorem 8.** [13] Let the conditions of the second part of Theorem 5 be satisfied, $t \geq 0.$ Assume that for all $k \geq 0$ and a.a. $x \in U$

$$
P \left( \text{rank}((1 - P)U_{\alpha_k\beta_k}(x)(1 - P)) = d - 1 \right) = 1. \tag{6}
$$

Then relation (4) holds true.

**Corollary.** Assume that for all $x \in \mathbb{R}_+^d, s \leq t :$

$$
P \left( \varphi_s^d(x) = 0, \det \| \tilde{U}_{st}(x) \| = 0 \right) = 0,$$

where $\tilde{U}_{st}(x)$ is the matrix getting out from $U_{st}(x)$ by deleting last row and last column. Then (6) fulfills.

Observe that the assumption of the Corollary is the requirement of non-hitting zero by two-dimensional Ito process, and this is usually easier to check than (6).

**Example 2.** Let $d = 2,$ then assumptions of the Corollary to the Theorem 8 are satisfied if for all $x \in \mathbb{R}_+^2, y \in \mathbb{R}^2, y \neq 0$ vectors $(a_{k, 1}(x))_{1 \leq k \leq m}$ and $(\nabla_y a_{k, 2}(x))_{1 \leq k \leq m},$ are linear independent.

6. Flows generated by SDE with reflection in arbitrary set. Let $G \subset \mathbb{R}_+^d$ be a closed set with smooth boundary such that the following SDE with normal reflection on the boundary of $G$ has a unique strong solution defined for all $x \in G, t \geq 0 :$

$$
\begin{align*}
 & \left\{ 
    \begin{array}{l}
    d\varphi_t(x) = a_0(\varphi_t(x))dt + \sum_{k=1}^{m} a_k(\varphi_t(x))dw_k(t) + \\
    + \n\pi(\varphi_t(x))\xi(dt, x), t \geq 0, x \in G,
    \end{array}
    \right. \\
    \varphi_0(x) = x, \xi(0, x) = 0, \xi(t, x) = \int_0^t \mathbb{I}_{\varphi_s(x) \in \partial G} \xi(ds, x),
\end{align*}
$$
where \( \overline{n}(x) \) is the inward normal at a boundary point \( x \in \partial G \), \( \xi(t,x) \) is continuous and non-decreasing in \( t \) process for every fixed \( x \in G \).

Assume that for any \( x \in \partial G \) there exists a neighborhood \( O(x) \) and \( C^2 \)-diffeomorphism \( \alpha_x \) which transforms the set \( O(x) \cap G \) into \( \{ x \in \mathbb{R}^d : \|x\| \leq 1, x^d \geq 0 \} \) in such a manner that \( \nabla \alpha_x(y) = \overline{n}(y) = (0, \ldots, 0, 1), y \in \partial G \cap O(x) \).

Applying a localization of solutions, all statements on flows in half-space can be easily generalized (of course with natural changes) to the case of SDE in \( G \). For example, relation (6) will be of the form:

\[
P \left( \text{rank} \left( P(\varphi_{\beta_k})U_{\alpha_k\beta_k}(x)P(\varphi_{\alpha_k}) \right) = d - 1 \right) = 1,
\]

where \( P(x) \) is orthoprojection on the orthogonal complement to \( \overline{n}(x) \) where \( x \in \partial G \).

**Example 3.** Let \( G = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \} \) be a unit disk, \( \varphi_t(x), t \geq 0, x \in G \) be a Brownian motion in \( G \) with reflection on the boundary. I.e. \( \varphi_t(x) \) is a solution of the SDE

\[
\begin{cases}
d\varphi_t(x) = dw(t) + \overline{n}(\varphi_t(x))\xi(dt,x), t \geq 0, \\
\varphi_0(x) = x, \xi(0,x) = 0, x \in G,
\end{cases}
\]

where \( w(t) \) is a two-dimensional Wiener process.

Let us describe inner and boundary points of the set \( \varphi_t(G) \). Now, the stopping time \( \tau(x) \) from Theorem 2 is of the form \( \tau(x) = \inf \{ t \geq 0 : x + w(t) \in \partial G \} \). So, the set of inner points of \( \varphi_t(G) \) is equal to \( \{ x + w(t) : x \in G, t < \tau(x) \} \).

Introduce a stopping time \( \sigma = \inf \{ t \geq 0 : \|w(t)\| = 2 \} \). Observe that \( \sup_x \tau(x) \leq \sigma \). Therefore the analogues of Theorems 2,3 imply that for every \( t \geq \sigma \) the random set \( \varphi_t(G) \) coincides with the nowhere dense set \( \varphi_t(\partial G) \) of finite Hausdorff measure \( H^1 \). Moreover,

\[
P \left( \forall t \geq \sigma \ \forall x, \|x\| < 1 \ \exists y \in \partial G, x \neq y : \varphi_t(x) = \varphi_t(y) \right) = 1.
\]

It is interesting to compare this result with [2], where it is proved that any two solutions of (8) started from different initial points of \( G \) never meet each other with probability 1, that is

\[
\forall x, y \in G, x \neq y : \ P \left( \exists t \geq 0 : \varphi_t(x) = \varphi_t(y) \right) = 0.
\]
Note that in this example $U_{st}(x)$ is an identity matrix, so condition (7) is obviously satisfied. Thus for any absolute continuous measure $\mu$ on $G$ we have the absolute continuity
\[ \mu|_{G \setminus \partial^t} \circ \varphi_t^{-1} \ll H^1|_{\partial \varphi_t(G)} \] (9)
with probability 1. In particular, for a.a. $\omega$ and all $t \geq \sigma$:
\[ \mu \circ \varphi_t^{-1} \ll H^1|_{\partial \varphi_t(G)}. \]

Observe that if $G$ is not a unit disk but any domain with ”nice” boundary, and the boundary does not contain any two perpendicular segments, then (7) and so (4),(9) are also satisfied. Exactly the same condition on the boundary appears in [6]. Moreover, it can be easily shown that if the boundary contains two perpendicular segments then (4),(7), and (9) are false.

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