On the manifold of the Laughlin problem unique solutions

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Abstract
Solutions, exactly expressed in terms of elementary functions (unique Laughlin states), of the correlated motion problem for a pair of 2D-electrons in a constant and uniform magnetic field have been shown to exist for a certain relation between the magnetic field induction and the electron charge. Arguments that can help to understand the physical meaning of these remarkable magnetic field values have been provided. The special interest to this problem is justified by the importance of the new state of matter recently observed.

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Let us consider the motion of a pair of 2D-electrons in a constant and uniform magnetic field. In this problem, the motion of the electrons center of mass can be considered separately from their relative motion.

The stationary Schrödinger equation for the relative motion (the Laughlin problem) has the form

\[
\left\{ \frac{1}{m^*} \left[ \left( \hat{\mathbf{p}}_x - \frac{eB}{4c} \hat{y} \right)^2 - \left( \hat{\mathbf{p}}_y + \frac{eB}{4c} \hat{x} \right)^2 \right] + \frac{e^2}{\sqrt{x^2 + y^2}} \right\} \Psi = E \Psi.
\] (1)

Here \( m^* \) is the effective electron mass in the given heterostructure, \( e \) is the absolute value of the electron charge, \( B \) is the magnetic field induction. The symmetric gauge for the vector potential in the above equation has been used

\[
\mathbf{A} = -\frac{B}{2} y \hat{\mathbf{e}}_x + \frac{B}{2} x \hat{\mathbf{e}}_y = \frac{B}{2} r \hat{\mathbf{e}}_\phi.
\] (2)

Equation (1) formally corresponds to the Schrödinger equation for a mass \( m^*/2 \) and charge \(-e/2\) particle interacting with the uniform magnetic field (2) and with a fixed charge \(-2e\) placed at the origin.

Equation (1) is conveniently transformed to dimensionless variables. To this end, we introduce the magnetic length \( l_B = \sqrt{2\hbar c/(eB)} \) and the dimensionless energy eigenvalue \( \lambda = 2E/ (\hbar \omega_c) \), where \( \omega_c = eB/(m^*c) \) is the cyclotron frequency. Upon introduction of the new dimensional variables

\[
x = l_B \xi, \quad y = l_B \eta, \quad \rho = \sqrt{\xi^2 + \eta^2},
\] (3)

equation (1) takes the form

\[
\left[ \left( \hat{\mathbf{p}}_\xi - \frac{1}{2} \hat{\mathbf{v}} \right)^2 + \left( \hat{\mathbf{p}}_\eta + \frac{1}{2} \hat{\mathbf{v}} \right)^2 + \frac{a}{\rho} \right] \Psi = \lambda \Psi.
\] (4)

Here \( a = \sqrt{B_0/B} \) and \( B_0 = 2cm^*c^3/\hbar^3 \simeq 4.7 \times 10^9 (m^*/m)^2 \) is the critical magnetic field. In the polar coordinates, equation (4) reads

\[
\left[ -\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} - i \frac{\partial}{\partial \varphi} + \frac{1}{4} \rho^2 + \frac{a}{\rho} \right] \Psi = \lambda \Psi.
\] (5)

The wave function is looked for in the form

\[
\Psi = \exp(i l \varphi) R(\rho).
\] (6)

The radial part of the wave function is described by the equation

\[
\left[ -\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{l^2}{\rho^2} + l + \frac{1}{4} \rho^2 + \frac{a}{\rho} \right] R(\rho) = \lambda R(\rho).
\] (7)
Here \( l = 0 \pm 1, \pm 2, \ldots \) are eigenvalues of the operator \( L_z = -i \frac{\partial}{\partial \varphi} \).

In the following, only the case \( l = 0 \) is considered, and hence the equation takes the form

\[
\left[ -\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) + \frac{1}{4} \rho^2 + \frac{a}{\rho} \right] R(\rho) = \lambda R(\rho) .
\]  

(8)

The Hamiltonian operator in the left-hand side of (8) can be written in the form

\[
\hat{H} = \hat{p}_\rho^2 + \frac{1}{4} \rho^2 + \frac{1}{4} \rho^2 + \frac{a}{\rho} .
\]  

(9)

Here

\[
\hat{p}_\rho = \frac{1}{i} \left( \frac{\partial}{\partial \rho} + \frac{1}{2\rho} \right) .
\]  

(10)

is the radial momentum operator, selfadjoint in the Hilbert space \( L^2(0, \infty, \rho d\rho) \). It should be mentioned that the term \( 1/(4\rho^2) \) has appeared in (9).

Radial eigenfunctions are looked for in the form

\[
R(\rho) = \exp \left( -\rho^2/4 \right) f(\rho) .
\]  

(11)

The equation for the function \( f(\rho) \) reads

\[
\frac{d^2 f}{d\rho^2} + \left( \frac{1}{\rho} - \rho \right) \frac{df}{d\rho} + \left( \lambda - 1 - \frac{a}{\rho} \right) f(\rho) = 0 .
\]  

(12)

For magnetic fields of arbitrary strength (the parameter \( a \) takes arbitrary values), the regular at the origin solution of equation (12), ensuring that \( R(\rho) \) belongs to the Hilbert space, is given by a series with a complicated and almost unknown structure.

It is only for some unique values of the magnetic field that the functions \( f(\rho) \) are reduced to polynomials \([4] - [6]\), so that the radial functions \( R(\rho) \) take the form

\[
R_{nk}(\rho) = C_{nk} \exp \left( -\rho^2/4 \right) Q_{nk}(\rho) ,
\]  

(13)

where

\[
Q_{nk}(\rho) = \sum_{j=0}^{n} b_j \rho^j
\]  

(14)

is the order \( n \) polynomial with exactly \( k \) zeros (\( n \) and \( k \) are the principle and the radial quantum numbers respectively) in the physical region (\( \rho \geq 0 \)). The eigenvalues \( \lambda \) for all the states of this unique kind are given by the simple unified formula

\[
\lambda = n + 1, \quad n + 1, 2, \ldots
\]  

(15)
Coefficients of the polynomial \( Q_{nk} (\rho) \) are determined by the recurrence relations
\[
\begin{align*}
    b_0 &= 1, \\
    b_1 &= 2, \\
    b_j &= [a \ b_{j-1} + (j - n - 2) \ b_{j-2}] \ j^{-2}.
\end{align*}
\] (16)

The unique values of the magnetic fields are determined by the relations
\[
(n + 1)^2 b_{n+1} = ab_n - b_{n-1} = 0.
\] (17)

Several leading values of the parameter follow
\[
\begin{align*}
    a_{10} &= 1, \\
    a_{20} &= \sqrt{6}, \\
    a_{30} &= \sqrt{10 + \sqrt{73}}, \quad a_{31} = \sqrt{10 - \sqrt{73}}, \\
    a_{40} &= \sqrt{25 + 3\sqrt{33}}, \quad a_{41} = \sqrt{25 - 3\sqrt{33}}.
\end{align*}
\] (18)

It is of interest to remark that the energy levels for the unique states (13) can be obtained from the Bohr - Sommerfeld quantization rule, taking into consideration both the physical \((\rho \geq 0)\) and nonphysical \((\rho < 0)\) ranges of the variable \(\rho\). In this case, the effective potential energy should be put equal to (see (9))
\[
U_{\text{eff}} (\rho) = \frac{1}{4} \rho^2 + \frac{a}{\rho} + \frac{1}{4\rho^2}.
\] (19)

The Bohr - Sommerfeld quantization rule with allowance for the above remarks can be put into the form
\[
2 \left( \int_{\rho_4}^{\rho_2} d\rho \sqrt{\lambda - U_{\text{eff}} (\rho)} + \int_{\rho_3}^{\rho_4} d\rho \sqrt{\lambda - U_{\text{eff}} (\rho)} \right) = 2\pi (n + 1).
\] (20)

The appearance of unity in the right-hand side of (20) is explained by the fact that there exist four regular lower turning points, each of them contributing 1/4. It should be mentioned that, when the term \(1/(4\rho^2)\) is not included in expression (19), we have to assume that there is an impenetrable potential wall at the point \(\rho = 0\) with the contribution 1/2, instead of 1/4. Integration in the left-hand side of equation (20) is easily performed with the help of the residue technique (the integration contour is depicted in Fig. 3b). As a result, we obtain exactly formula (13) for \(\lambda\), irrespective of \(a\) values.

The physical meaning of the particular magnetic field values determined by relations (15) was unclear up to now. In the following, we present considerations that may elucidate the physical origin of this phenomenon.

Introduce the center of orbit operators
\[
\hat{X}_c = -\hat{p}_\eta + \frac{1}{2} \hat{\xi}, \quad \hat{Y}_c = \hat{p}_\xi + \frac{1}{2} \hat{\eta}.
\] (21)
Unlike [8], the symmetric gauge has been used here, and transformation to the dimensionless form has been performed by means of the magnetic length introduced. Then the operator of the square of the distance between the center of the orbit and the origin can be written in the form

$$\hat{R}_c^2 = \hat{p}_\xi^2 + \hat{p}_\eta^2 + \frac{1}{4}\hat{\rho}^2 + \hat{L}_z.$$  (22)

In the case of a pure magnetic field, these operator commute with the Hamiltonian, so that, in a stationary state, the value of $R_c^2$ is quite well defined. In our case $R_c^2$ can be written in the form

$$\hat{R}_c^2 = \hat{H} - 2\hat{L}_z - \frac{a}{\rho}.$$  (23)

Due to the presence of the Coulomb force, in our case, $R_c^2$ is not conserved even in a stationary state, and hence, only the quantum average of this physical quantity makes sense.

Introduce now the average value of the radius of orbit squared as it has been done in [9] for the pure magnetic field case

$$\langle \hat{R}^2 \rangle = \langle \rho^2 \rangle - \langle \hat{R}_c^2 \rangle.$$  (24)

With allowance for (4), (15) and for the fact that $l = 0$, this equation can be written in the form

$$\langle \hat{R}^2 \rangle = \langle \rho^2 \rangle + a \left\langle \frac{1}{\rho} \right\rangle - \lambda.$$  (25)

Thus, the average magnetic flux in a stationary state (ordinary units are again used) takes the form

$$\Phi = \pi \langle \hat{R}^2 \rangle \ B \ l_B^2 = \langle \hat{R}^2 \rangle \ \Phi_0,$$  (26)

where $\Phi_0 = 2\pi c h / e$ is the magnetic flux quantum.

Calculations demonstrated that, for stationary states (13), the quantity $\Phi$ is the integral multiple of $\Phi_0$

$$\Phi = (n + 1) \ \Phi_0, \quad n = 2, 3, ....$$  (27)

In the following, the results of calculations are illustrated by the examples for the cases of states $f_{10}$ and $f_{20}$. For the state $f_{10}$ we have $a = 1, \quad \lambda = 2$

$$f_{10}(\rho) = \exp \left( -\rho^2 / 4 \right) \ (1 + \rho),$$

$$\langle \rho^2 \rangle = \frac{10 + 3\sqrt{2\pi}}{(3 + \sqrt{2\pi})},$$

$$\langle \frac{1}{\rho} \rangle = \frac{2 + \sqrt{2\pi}}{(3 + \sqrt{2\pi})},$$

$$\langle \hat{R}^2 \rangle = \frac{(10 + 3\sqrt{2\pi})}{(3 + \sqrt{2\pi})}. $$
\[ \langle \hat{R}^2 \rangle = 2. \]

For the state \( f_{20} \) we have \( a = \sqrt{6}, \quad \lambda = 3 \)

\[ f_{20}(\rho) = \exp \left( -\rho^2 / 4 \right) \left( 1 + \sqrt{6} \rho + \rho^2 \right), \]

\[ \langle \rho^2 \rangle = \left( 114 + 36\sqrt{3}\pi \right) / 25 + 8\sqrt{3}\pi, \]

\[ \langle \frac{1}{\rho} \rangle = 6 \left( \sqrt{6} + \sqrt{2}\pi \right) / \left( 25 + 8\sqrt{2}\pi \right), \]

\[ \langle \hat{R}^2 \rangle = 3. \]

We believe that formula (26) is valid for all the unique states, though we are unable to present a general proof of this statement now.

We are grateful to V.Ch. Zhukovskii and A.V. Borisov for helpful discussions.

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Figure captions

Figure 1. The curves demonstrate qualitative behaviour of the relation $\lambda = 2E/\hbar\omega$ as a function of the parameter $a = \sqrt{B_0/B}$. The dots mark the value of parameters $\lambda$ and $a$ of unique states. The lower curve corresponds to the ground state, the next one is for the first exited state and so on.

Fig. 2. a) The curves of normalized radial functions of unique states $f_{nk}(r)$ in physical ($\rho \leq 0$) and nonphysical ($\rho < 0$) ranges at several values of principal ($n$) and radial ($k$) quantum numbers.

b) The curves of the normalized radial density of unique states $D_{nk}(r) = r f_{nk}^2(r)$ at several values of principal ($n$) and radial ($k$) quantum numbers.

The values of a radial variable are measured in the units of the effective Bohr radius $a_B^* = \hbar^2/(m^* e^2)$.

Fig. 3. a) On the curve of the effective potential $U_{eff}(\rho)$ in physical ($\rho \leq 0$) and nonphysical ($\rho < 0$) ranges at $\lambda = 2$ the turning points are shown.

b) The contour of circumvention of four branching points in the complex plane.
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