Criterion and Regions of Stability for Quasi-Equidistant Soliton Trains.

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Using the complex Toda chain (CTC) as a model for the propagation of the $N$-soliton pulse trains of the nonlinear Schrödinger (NLS) equation, we can predict the stability and the asymptotic behavior of these trains. We show that the following asymptotic regimes are stable: (i) asymptotically free propagation of all $N$ solitons; (ii) bound state regime where the $N$ solitons move quasi-equidistantly; and (iii) various different combinations of (i) and (ii). We show with examples of $N = 2$ and $3$ how this analysis can be used to determine analytically the set of initial soliton parameters corresponding to each of these regimes. We also compare these analytical results against the corresponding numerical solutions of the NLS and find excellent agreement for each of the regimes described above. We pay special attention to the regime (ii) because the quasi-equidistant propagation of all $N$ solitons is of importance for optical fiber soliton communication. Our studies show that such propagation can be realized for $N = 2$ to $8$. For the case of 3-soliton trains we also show how one can determine the instability regions and the transition regions between the various regimes. Finally we propose realistic configurations for the sets of the amplitudes, for which the trains show quasi-equidistant behavior to very large run lengths.

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I. INTRODUCTION

One of the important problems in optical fiber soliton communication is to achieve as high of a bit rate as possible. In order to do this, one needs to be able to pack the solitons into as short a space as possible. However, if the solitons are too close together, then their mutual (linear and nonlinear) interactions can cause them to collide and/or separate, thereby corrupting the signal. The current solution of this problem is simply to require each soliton to be sufficiently far apart from all others (usually 6 or so soliton widths) so that such interactions can be totally neglected. However, at the same time, it was predicted and experimentally confirmed that for certain values of relative soliton parameters, this separation can be reduced, and at the same time, still maintain signal integrity. Our main purpose here is to analytically and numerically detail the soliton parameter regime, inside of which, signal integrity can be maintained. In particular, we are interested to determine how one may use this inter-soliton interaction for stabilizing a soliton train.

Any optical communication signal will be composed of "random" combinations of 0’s and 1’s. Such a signal can also be viewed as being composed of a random collection of N-soliton trains, with varying widths of 0’s between them. Thus it is then adequate for us to simply analyze the stability of individual N-soliton trains, for $N = 1, 2, 3, \ldots$. This we will do and will study a nonperiodic and finite train (chain) of soliton pulses by both analytical and numerical methods using and developing the ideas in [4,5].

The basic model and description of N-soliton trains in optical fibers is provided by the nonlinear Schrödinger (NLS) equation and its perturbed version:

$$i u_t + \frac{1}{2} u_{xx} + |u|^2 u(x, t) = i R[u].$$

This equation describes a variety of wave interactions, including solitons in nonlinear fiber optics [6,7] and spatial solitons in nonlinear refractive media [11].

The inverse scattering method [6,7] allows one to solve exactly Eq. (1) when $R[u] = 0$ and to calculate explicitly its N-soliton solutions. However for our purposes, this method is impractical for two reasons. First, there will be important physical problems in this system in need of addressing where $R[u]$ will be nonzero, for which no explicit solutions are known. Second, an approximate method can serve much better than an exact approach since the N-soliton trains that we need to study are rather special and can only be approximated by N-soliton solutions. Such trains are actually sums of 1-soliton pulses, which are spaced almost equally, have almost equal amplitudes, and move with essentially the same velocity. More specifically, they are the solutions to Eq. (1) with $R[u] = 0$ satisfying the following initial conditions:

$$u(x, 0) = \sum_{k=1}^{N} u_{1s,k}(x, 0),$$  

where $u_{1s,k}(x, t)$ is the 1-soliton solution of the NLS given by:

$$u_{1s,k}(x, t) = \frac{2v_k e^{i \phi_k}}{\cosh(2\nu_k(x - \xi_k(t)))},$$

$$\phi_k(x, t) = 2\mu_k(x - \xi_k(t)) + \delta_k(t),$$

$$\xi_k(t) = 2\mu_k t + \xi_{k0},$$

$$\delta_k(t) = 2(\mu_k^2 + \nu_k^2) t + \delta_{k0}$$

Here by $\delta_k(t), \xi_k(t)$ we have denoted the phase and position respectively of the $k$-th soliton, $2\nu_k$ is soliton’s amplitude while $2\mu_k$ is its velocity. The quantities $\delta_{k0}, \xi_{k0}, \nu_{k0},$ and $\mu_{k0}$ all denote the initial values at $t = 0$.

An effective method for studying the interaction of such trains of soliton pulses was first proposed by Karpman and Solov’ev (KS), for the simplest non-trivial case of a 2-soliton interaction [14]. Further developments and analysis for different physically important perturbations can be found in [1, 6–15] and the references therein. For alternative approaches see e.g. [10, 22]: for a review see Ref. [6].

The KS method is based on the adiabatic approximation. It is valid for any collection of well separated solitons, such that their mutual interactions will lead to a slow deformation in the soliton parameters. These conditions are met provided the soliton parameters satisfy:

$$|\mu_{k0} - \mu_{n0}| \ll \mu_0,$$

$$|\nu_{k0} - \nu_{n0}| \ll \nu_0,$$

$$\nu_0|\xi_{k0} - \xi_{n0}| \gg 1,$$

$$|\nu_{k0} - \nu_{n0}| |\xi_{k0} - \xi_{n0}| \ll 1,$$

where $2\nu_0$ and $2\mu_0$ are the average initial amplitude and velocity.

How fast adjacent solitons will interact can be measured by a control parameter, $\epsilon_0$, which is a measure of both the overlap between the neighboring solitons and also the strength of their interaction. This is given by $\epsilon_0 = \nu_0 e^{-2\nu_0 r_0}$, where $r_0$ is the average initial pulse separation.

Gorshkov [23] and Arnold [24], have conjectured that an infinite train of out-of-phase soliton pulses, with equal amplitudes and velocities could be described by the real Toda chain:

$$\frac{d^2 q_k}{dx^2} = e^{q_{k+1} - q_k} - e^{q_k - q_{k-1}}.$$
where $\tau = 4\nu_0 t$, $k = 0, \pm 1, \pm 2, \ldots$ and $q_k$ are real functions related to the soliton positions. This system will be referred to as the real Toda chain (RTC). The next step towards the physically more realistic case with finite number of solitons was proposed in $[23]$. In it the RTC was shown to describe the propagation of $N$ equal amplitudes out-of-phase solitons numerically.

Recently in Refs. $[4,5,25,26]$, the Karpman–Solov’ev method was extended to $N$-soliton pulses, and then with additional approximations, was reduced to the complex Toda chain equations (CTC) $[3]$ with $N$ sites. The corresponding system of equations is $[4]$, but with $k = 1, \ldots, N$ and $e^{-q_0} = e^{q_{N+1}} = 0$. Meanwhile, the complex valued functions $q_k(t)$ are related to the soliton parameters by:

$$q_{k+1} - q_k = -2\nu_0 (\xi_{k+1} - \xi_k) + \ln 4\nu_0^2 + i(\pi + 2\nu_0 (\xi_{k+1} - \xi_k) - (\delta_{k+1} - \delta_k)). \quad (6)$$

With this, the problem of determining the evolution of an NLS $N$-soliton train has been reduced to the problem of solving the CTC for $N$ sites. Since $[3]$ is also integrable, then we may use the special techniques valid for integrable lattices (and chains) to study this problem. It has already been shown that one may determine the asymptotic behavior of asymptotically separating $N$-soliton trains, simply by analyzing the eigenvalues of a certain matrix $[3]$.

Our main results, briefly reported in $[27]$ are the following:

(i) we show by analyzing the exact analytic solutions of CTC, that it has several qualitatively different classes of asymptotic regimes. Besides the asymptotically free motion (which is the only possibility for the RTC), CTC allows also for: (a) bound state regime when all the $N$ particles move quasi-equidistantly; (b) all possible intermediate regimes when one (or several) group(s) of particles form bound state(s) and the rest of them go into free motion asymptotics. In addition to these relatively stable regimes of motion there are also less stable regions in the space of soliton parameters, where one regime switches into another one. There one can find (c) singular solutions, which tend to infinity for finite values of $\tau$, and (d) various types of degenerate solutions when two or more of the eigenvalues become equal.

(ii) we show, by comparing the predictions from the CTC model with the numerical solutions of the NLS, that regimes of type (a) and (b) indeed take place in the soliton interactions and are very well described by the CTC model. Our analytic approach allows us to predict the set of initial parameters, for which each of the asymptotic regimes takes place. We put special stress on the bound state and quasi-equidistant regimes (a) since such solutions would be optimum in long distance fiber optics communications.

II. ASYMPTOTIC REGIMES OF THE CTC

As in $[3]$, one can generalize the RTC $[24,25]$ to the complex CTC case. We list the four most important points concerning this below:

S1) The CTC Lax representation is the same as for the RTC:

$$L = [B, L], \quad (7a)$$

$$L = \sum_{k=1}^{N} (b_k E_{kk} + a_k (E_{k,k+1} + E_{k+1,k})), \quad (7b)$$

$$B = \sum_{k=1}^{N} a_k (E_{k,k+1} - E_{k+1,k}). \quad (7c)$$

Here the matrices $(E_{kn})_{pq} = \delta_{kp}\delta_{nq}$, and $E_{kn} = 0$ whenever one of the indices becomes 0 or $N+1$; the other notations in $[3]$ are as follows:

$$a_k = \frac{1}{2} e^{(q_{k+1} - q_k)/2}, \quad (8a)$$

$$b_k = \frac{1}{2} (\mu_k + iv_k). \quad (8b)$$

S2) The integrals of motion in involution are provided by the eigenvalues, $\zeta_k$, of $L$.

S3) The solutions of both the CTC and the RTC are determined by the scattering data for $L_0 = L(\tau = 0)$. When the spectrum of $L_0$ is nondegenerate, i.e. $\zeta_k \neq \zeta_j$ for $k \neq j$, then this scattering data consists of $\{\zeta_k, r_k\}_{k=1}^{N}$. Here $r_k$ are the first components of the corresponding eigenvectors $\xi^{(k)}$ of $L_0$:

$$L_0 \xi^{(k)} = \zeta_k \xi^{(k)}, \quad (9)$$

which are uniquely determined (up to an overall sign) by the normalization condition

$$\sum_{s=1}^{N} (\xi^{(k)}_s)^2 = 1, \quad (10)$$

[see $[30,32]$. This scattering data uniquely determines both $L_0$ and the solution of the CTC.

S4) Lastly, the eigenvalues of $L_0$ uniquely determine the asymptotic behavior of the solutions of the CTC; these eigenvalues can be calculated directly from the initial conditions. We will extensively use this fact for the description of the different classes of asymptotic behavior.

In addition to the dynamical variables becoming complex valued, there are other, important differences between the RTC and CTC, and their spectra. For the RTC, one has that $[30,31]$ both the eigenvalues, $\zeta_k$, and
the coefficients, \( r_k \), are always real-valued. Moreover, one can prove that \( \zeta_k \neq \zeta_j \) for \( k \neq j \), i.e. no two eigenvalues can be exactly the same. As a direct consequence of this, it follows that the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the solitons.

This situation is different for the CTC. In addition to all dynamical variables being possibly complex, the eigenvalues can also become complex, \( \zeta_k = \kappa_k + i n_k \), as well as the coefficients, \( r_k \), as given above. Furthermore, one could now have multiple eigenvalues. However, the collection of eigenvalues, \( \zeta_k \), still determines the asymptotic behavior of the solitons. In particular, it is \( \kappa_k \) that determines the asymptotic velocity of any soliton. For simplicity, here we shall always take \( \zeta_k \neq \zeta_j \) for \( k \neq j \).

(However, this condition does not necessarily mean that \( \kappa_k \neq \kappa_j \).) We also may assume that the \( \kappa_k \)'s are ordered as: \( \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_N \). Once this is done, then in any train of solitons, there are three possible general configurations:

D1) \( \kappa_k \neq \kappa_j \) for \( k \neq j \). Since the asymptotic velocities are all different, one has the well known asymptotically separating, free solitons.

D2) \( \kappa_1 = \kappa_2 = \ldots = \kappa_N \). In this case, all \( N \) solitons will move with the same mean asymptotic velocity, and therefore will form a “bound state”. The key question now will be the nature of the internal motions in such a bound state. In particular, one would want any two adjacent solitons to move quasi-equidistantly.

D3) One may have also a variety of intermediate situations when only one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Obviously the cases D2) and D3) have no analogies in the RTC and are qualitatively different from D1). The same is also true for the special degenerate cases, where two or more of the \( \zeta_k \)'s may become equal. These cases will be considered elsewhere.

Another type of solutions of the CTC which should be dealt with separately are the singular solutions. They appear in a special non-euclidean formulation of the RTC \[33\]; they can be obtained from CTC by taking one or several of the \( a_k \)'s purely imaginary and the others – real-valued functions of \( \tau \).

In order to avoid complicated and long formulas, we will skip most of the technical details and will limit ourselves with the simplest nontrivial cases: \( N = 2 \) and \( N = 3 \). We also assume without loss of generality that \( \text{tr} \ L_0 = \sum_{k=1}^{N} \zeta_k = 0 \).

**A. The \( N = 2 \) case**

The general solution of \( N = 2 \) CTC can always be chosen so that \( q_1(\tau) = -q_2(\tau) \) (the center-of-mass is at rest and at the origin of the coordinate system). Then (see also [33]):

\[ q_1(\tau) = \ln \frac{\cosh(2\zeta_1 \tau + \gamma)}{2\zeta_1} \] \hspace{1cm} (11)

where \( \gamma = -\ln(r_1/r_2) \). From formula (11) we get:

\[ q_1(\tau) = i\nu_0 r(\tau) - \ln 2\nu + i \frac{\pi - \delta(\tau)}{2} , \] \hspace{1cm} (12)

where \( r(\tau) = \zeta_2(\tau) - \zeta_1(\tau) \) and \( \delta(\tau) = \delta_2(\tau) - \delta_1(\tau) \) are respectively the distance between the solitons and their phase difference. From (11), (12) and (13), we find the following expression for \( L_0 \), in terms of the initial soliton parameters:

\[ L_0 = \left( \frac{-\frac{1}{2}(\Delta \mu_0 + i\Delta \nu_0)}{i\nu_0 e^{-\nu_0 r_0 - i\delta_0/2}} \right) \] \hspace{1cm} (13)

where \( \Delta \mu_0 = \mu_2 - \mu_1 \), \( \Delta \nu_0 = \nu_2 - \nu_1 \) and \( \delta_0 = \delta(0) \).

For simplicity, from now on, we will consider only those trains where the solitons have vanishing initial velocities: \( \mu_0 = 0 \) in some moving coordinate system. Then solving the characteristic equation for \( L_0 \), we find:

\[ \zeta_{1,2} = \pm \frac{i}{4} \sqrt{D} = \pm \kappa_1 \pm i\eta_1 \], \hspace{1cm} (14a)

\[ D = (\Delta \nu_0)^2 + 16\nu_0^2 e^{-2\nu_0 r_0 - i\delta_0} \] \hspace{1cm} (14b)

When \( D \) is complex in general, with a nonzero imaginary part, then by the above, \( \kappa_1 \) will be nonzero and the two solitons will always asymptotically separate. However, when \( D \) is real, then it is possible for a bound state of two solitons to form. This requires the relative phase between the solitons, \( \delta_0 \), to be either 0 or \( \pi \). If it is zero, then \( \kappa_1 \) is always zero, regardless of the size of any amplitude variations. If it is \( \pi \), then \( \kappa_1 \) will again be zero, provided that \( |\Delta \nu_0| > \Delta \nu_{cr,2} \), where

\[ \Delta \nu_{cr,2} = 4\nu_0 e^{-\nu_0 r_0} \] \hspace{1cm} (15)

and \( 2\Delta \nu_{cr,2} \) is a critical value for amplitude variations in a 2-soliton train. This means that if two adjacent solitons have a sufficiently large initial difference in their amplitudes of \( 2\Delta \nu_0 > 2\Delta \nu_{cr,2} \), and if the phase difference is \( \pi \), then these two solitons will never asymptotically separate. In other words, a slight variation in the amplitudes can stabilize and prevent two solitons from asymptotically separating. Obviously, in this case the distance, \( r(t) \) (\( t = 4\nu_0 t \)), between the solitons will be periodic in \( t \) with period:

\[ T_2^\pm = \frac{\pi}{2\nu_0 \sqrt{(\Delta \nu_0)^2 \pm (\Delta \nu_{cr,2})^2}} \] \hspace{1cm} (16)

where the signs plus and minus correspond to \( \delta_0 = 0 \) and \( \delta_0 = \pi \) respectively.
Most importantly, this motion is then bounded, and optimally its range should be significantly smaller than the initial spacing between the solitons.

Define
\[ A_k = \frac{\tilde{r}_{k,\text{max}} - \tilde{r}_{k,\text{min}}}{r_0}, \]
where \( \tilde{r}_{k,\text{max}} = \max(\xi_{k+1} - \xi_k) \) and \( \tilde{r}_{k,\text{min}} = \min(\xi_{k+1} - \xi_k) \) are the maximal and minimal values of \( \xi_{k+1}(\tau) - \xi_k(\tau) \). When \( A_k \ll 1 \), we will call such motion “quasi-equidistant”. For such motion, the solitons will not asymptotically separate, but instead will slowly oscillate with some small amplitude.

For \( N = 2 \) from the explicit form of the solution, we can easily recover (see also [10]):

\[
\begin{align*}
\tilde{r}_{1,\text{min}} &= \frac{1}{2\nu_0} \ln \frac{\nu_0^2 (\cosh(2\gamma_0) - 1)}{2\eta_0^2}, \quad (18a) \\
\tilde{r}_{1,\text{max}} &= \frac{1}{2\nu_0} \ln \frac{\nu_0^2 (\cosh(2\gamma_0) + 1)}{2\eta_0^2}. \quad (18b)
\end{align*}
\]

Therefore \( A_1 \) equals to:
\[ A_1 = \frac{1}{2\nu_0 r_0} \ln \frac{\cosh(2\gamma_0) + 1}{\cosh(2\gamma_0) - 1}. \quad (18c) \]

where
\[ \gamma = \gamma_0 + i\gamma_1 = \frac{1}{2} \ln \frac{e^{-i\theta_0} + y_0^* - y_0}{\sqrt{e^{-i\theta_0} + y_0^* + y_0}}. \quad (19)\]

and \( y_0 = |\Delta
\nu_0|/\Delta\nu_{cr,2} \).

Let us look at our two cases. For \( \delta_0 = 0 \), one obtains \( \tilde{r}_{1,\text{max}} = r_0 \) and:
\[ \tilde{r}_{1,\text{min}} = r_0 + \frac{1}{2\nu_0} \ln \frac{y_0^2}{y_0^* - 1}. \quad (20) \]

Note that for even moderate values of \( r_0 \), almost any small, nonzero value of \( \Delta\nu_0 \) prevents the singularity in the CTC model; taking \( 2\Delta\nu_0 \) appropriately large (say 0.2 for \( r_0 = 8 \), see Table IV) leads to a quasi-equidistant motion.

For \( \delta_0 = \pi \) analogously we find \( \tilde{r}_{1,\text{min}} = r_0 \) and:
\[ \tilde{r}_{1,\text{max}} = r_0 + \frac{1}{2\nu_0} \ln \frac{y_0^2}{y_0^* + 1}. \quad (21) \]

Note that in this case, the quasi-equidistant regime holds only for \( y_0 > 1 \). The value \( y_0 = 1 \) is a critical point, at which the quasi-equidistant regime switches over into the asymptotically separating regime.

Thus the motion will be quasi-equidistant if \( e^{2|\gamma_0|} \gg 1 \). Both formulas (20), (21) show that an increase in \( |\Delta\nu_0| \) diminishes the oscillations of \( r(\tau) \). Another way to diminish \( A_1 \) for fixed \( \Delta\nu_0 \) and \( r_0 \) is to increase the average amplitude \( 2\nu_0 \); this would diminish \( \Delta\nu_{cr,2} \) and as a consequence, also \( A_1 \).

From here on, we will need to refer frequently to various sets of initial conditions for the \( N \)-soliton trains; for clarity and brevity we will denote them by quadruples \( N|\nu_0|\cdot 10^2|k_0| \), where \( N \) is the number of pulses, \( r_0 \) is the distance between neighboring pulses. In most of our runs, listed in the tables, the pulses are initially equidistant (i.e. \( \xi_{k+1,0} - \xi_k,0 = r_0 \)), the initial phases are chosen to be of the form \{0, \delta_{0,0}, \delta_{2,0}, 0, \ldots \} and the amplitudes are either equal \( (\Delta\nu_0 = 0) \) or \( \Delta\nu_0 = \nu_{k+1,0} - \nu_{k,0} \) and such that the average amplitude equals \( 2\nu_0 \). The initial velocities, \( 2\nu_{k,0} \), in all of our runs are zero.

**B. The \( N = 3 \) case**

We assume that \( q_1(\tau) + q_2(\tau) + q_3(\tau) = 0 \), i.e. the center of mass is fixed at the origin. This is compatible with \( tr\ L_0 = \zeta_1 + \zeta_2 + \zeta_3 = 0 \). Our analysis below is not exhaustive since we will consider only the particular physically important submanifold of soliton parameters \( R \), which is restricted by: a) zero initial velocities of the solitons; b) initially equidistant solitons \( \xi_{k+1,0} - \xi_k,0 = r_0 \). Condition a) means that the diagonal elements in \( L_0 \) are purely imaginary \( b_k = i\kappa_k/4 \); from b) there follows that \( |a_1| = |a_2| \) for \( \tau = 0 \).

It is clear that the initial soliton parameters determine \( L_0 \), but the regimes of propagation are determined by the eigenvalues of \( L_0 \). Thus we consider various possible combinations of the latter.

**(i) Asymptotically free propagation.** Choose \( \kappa_1 < \kappa_2 < \kappa_3, \eta_k - \text{generic} \). Then we have:

\[
\begin{align*}
\lim_{\tau \to \infty} (q_1(\tau) + 2\zeta_1\tau) &= \frac{2}{3} \ln \frac{\rho_1}{\Delta_{12}\Delta_{13}\Delta_{23}}, \quad (22a) \\
\lim_{\tau \to \infty} (q_2(\tau) + 2\zeta_2\tau) &= \frac{2}{3} \ln \frac{\rho_2}{\Delta_{12}\Delta_{13}\Delta_{23}}, \quad (22b) \\
\lim_{\tau \to \infty} (q_3(\tau) + 2\zeta_3\tau) &= \frac{2}{3} \ln \frac{\rho_3}{\Delta_{12}\Delta_{13}\Delta_{24}}, \quad (22c)
\end{align*}
\]

where
\[ \rho_k = \frac{\nu_k^3}{r_{1,0}r_{2,0}r_{3,0}}. \quad (22d) \]

\[ \Delta_{ik} = 2(\zeta_i - \zeta_k). \quad (22e) \]

Obviously case (i) corresponds to asymptotically free motion of the pulses; their velocities are determined by \( \kappa_k \). This case was investigated in [23] for \( N|\nu_0| \cdot 10^2|k_0| \) type trains. We note that for this type of initial conditions the CTC reduces to the RTC, for which the asymptotically free regime is the only possible one.
(ii) Three-soliton bound state. Choose \( \kappa_1 = \kappa_2 = \kappa_3 = 0 \) and \( \eta_1 = -\eta_2, \eta_2 = 0 \). Then we find the following periodic solutions of the CTC:

\[
q_1(\tau) = \ln \frac{2 \cosh(2\zeta_1 \tau + \gamma) + \rho_2}{8\zeta_1^2} - \frac{1}{3} \ln \left( \frac{\rho_2}{2} \right),
\]

\[
q_2(\tau) = \ln \frac{2 \cosh(2\zeta_1 \tau + \gamma) + \frac{4}{\rho_2}}{2 \cosh(2\zeta_1 \tau + \gamma) + \rho_2} + \frac{2}{3} \ln \left( \frac{\rho_2}{2} \right),
\]

\[
q_3(\tau) = \ln \frac{8\zeta_1^2}{2 \cosh(2\zeta_1 \tau + \gamma) + \frac{4}{\rho_2}} - \frac{1}{3} \ln \left( \frac{\rho_2}{2} \right),
\]

where \( \zeta_1 = i\eta_1^\pm \)

\[
\eta_1^\pm = \sqrt{\frac{(\Delta \nu_0)^2 \pm (\Delta \nu_{\text{cr},3})^2}{2}}.
\]

\[
\Delta \nu_{\text{cr},3} = 2 \sqrt{2} \nu_0 e^{-\nu_0 r_0}.
\]

In terms of the soliton parameters we have:

\[
\gamma_3,0 + i \gamma_{3,1} = \ln \frac{e^{-i \delta_0} + z_0^2}{e^{-i \delta_0} + z_0^2 - z_0},
\]

where \( z_0 = |\Delta \nu_0|/\Delta \nu_{\text{cr},3} \). In particular, for \( \delta_0 = 0 \) one gets \( \tilde{r}_{1,\text{max}} = r_0 \) and:

\[
\tilde{r}_{1,\text{min}} = r_0 + \frac{1}{2 \nu_0} \ln \frac{\gamma_0^2}{z_0^2 - 1}.
\]

For \( \delta_0 = \pi \) analogously we get \( \tilde{r}_{1,\text{min}} = r_0 \) and:

\[
\tilde{r}_{1,\text{max}} = r_0 + \frac{1}{2 \nu_0} \ln \frac{\gamma_0^2}{z_0^2 - 1}.
\]

These formulas are very similar to the ones for the \( N = 2 \) case; note however that the value of \( \Delta \nu_{\text{cr},3} \) is different from the one of \( \Delta \nu_{\text{cr},2} \). Note also that the symmetrical case is a special one and takes place only when \( \rho_2 = 2 \).

The same property is shared also by the more complicated solution for which \( \eta_1 \neq \eta_2 \neq 0 \) take generic values constrained only by \( \eta_1 + \eta_2 + \eta_3 = 0 \). As we mentioned this solution of CTC will be periodic only if the ratio \( \eta_1 - \eta_2)/(\eta_1 - \eta_3) \) is a rational number.

In the symmetrical case, i.e. for \( q_1(\tau) = -q_3(\tau) \), \( q_2(\tau) = 0 \), we can consider \( A_{1,2} \) like in (23) and find that \( A_1 = A_2 \). The quasi-equidistant regime requires again \( A_1 \ll r_0 \).

From the explicit form of the solution we find:

\[
\tilde{r}_{1,\text{min}} = \tilde{r}_{2,\text{min}} = \frac{1}{2 \nu_0} \ln \frac{\nu_0^2 (\cosh(\gamma_{3,0}) - 1)}{\eta_1^\pm},
\]

\[
\tilde{r}_{1,\text{max}} = \tilde{r}_{2,\text{max}} = \frac{1}{2 \nu_0} \ln \frac{\nu_0^2 (\cosh(\gamma_{3,0}) + 1)}{\eta_1^\pm},
\]

and consequently

\[
A_1 = A_2 = \frac{1}{2 \nu_0 r_0} \ln \frac{\cosh(\gamma_{3,0} + 1)}{\cosh(\gamma_{3,0} - 1)}.
\]

In the quasi-equidistant regime requires again \( A_1 \ll r_0 \).

Thus we conclude that this case corresponds to a bound state of all three particles. The same property is shared also by the more complicated solution for which \( \eta_1 \neq \eta_2 \neq 0 \) take generic values constrained only by \( \eta_1 + \eta_2 + \eta_3 = 0 \). As we mentioned this solution of CTC will be periodic only if the ratio \( \eta_1 - \eta_2)/(\eta_1 - \eta_3) \) is a rational number.

In the symmetrical case, i.e. for \( q_1(\tau) = -q_3(\tau) \), \( q_2(\tau) = 0 \), we can consider \( A_{1,2} \) like in (23) and find that \( A_1 = A_2 \). The quasi-equidistant regime requires again \( A_1 \ll r_0 \).

From the explicit form of the solution we find:

\[
\tilde{r}_{1,\text{min}} = \tilde{r}_{2,\text{min}} = \frac{1}{2 \nu_0} \ln \frac{\nu_0^2 (\cosh(\gamma_{3,0}) - 1)}{\eta_1^\pm},
\]

\[
\tilde{r}_{1,\text{max}} = \tilde{r}_{2,\text{max}} = \frac{1}{2 \nu_0} \ln \frac{\nu_0^2 (\cosh(\gamma_{3,0}) + 1)}{\eta_1^\pm},
\]

and consequently

\[
A_1 = A_2 = \frac{1}{2 \nu_0 r_0} \ln \frac{\cosh(\gamma_{3,0} + 1)}{\cosh(\gamma_{3,0} - 1)}.
\]
\[
\lim_{\tau \to \infty} (q_\ell(\tau) - \zeta_\ell \tau) = \frac{1}{3} \ln \left( \frac{\Delta_{12} \Delta_{13} \Delta_3^4}{\rho_1} \right) - \ln(2\cos(\eta \tau + \Gamma)),
\]
\[(29c)\]

where
\[
\eta = 2(q_2 - \eta_3),
\]
\[(29d)\]

\[
\Gamma = i \ln \left( \frac{r_2 \Delta_{12}}{r_3 \Delta_{13}} \right) = \Gamma_0 + i \Gamma_1.
\]
\[(29e)\]

As it is easy to see now, the second and the third particles asymptotically form a bound state. They both move with the same mean velocity and the distance between them is bounded (as long as \(\Gamma_1 \neq 0\)) and is asymptotically a periodic function of time.

In the particular case when \(\Gamma_1 = 0\), we get a singular solution. Indeed, if \(\tau_k\) is such that \(\eta \tau_k + \Gamma_0 = (k + \frac{1}{2})\pi\) for some integer \(k\), then the right hand sides of the equations (29b) and (29c) become infinite.

If the CTC model predicts collisions, this always indicates that the NLS solitons are becoming dangerously close together, see e.g. [5]. There is an excellent match between CTC and NLS except in the vicinities of the points where the CTC develops singularities since here, (4) is becoming violated.

Let us now describe the particular choices of the soliton parameters, which lead to the regimes described above. Such analysis must be based on the solution of the characteristic equation for \(L_0\), which for \(N = 3\) with \(\text{tr} \ L_0 = 0\) is:
\[
\zeta^3 + \zeta p + q = 0,
\]
\[(30a)\]

\[
p = \frac{1}{32} (d_1^2 + d_2^2 + d_3^2) - a_1^2 - a_2^2,
\]
\[(30b)\]

\[
q = \frac{i}{4} (a_1^2 d_3 + a_2^2 d_1) + \frac{i}{64} d_1 d_2 d_3,
\]
\[(30c)\]

where \(d_k = 2(\nu_{k,0} - \nu_0)\), \(\nu_0 = \sum_{k=0}^{N} \nu_{k,0}/N\) is the average amplitude. For the initial sets of soliton parameters specified above we find:
\[
a_k = i\xi_0 e^{i(\delta_{k,0} - \delta_{k+1,0})/2},
\]
\[(31a)\]

\[
\xi_0 = \nu_0 e^{-i\nu_0 \tau_0},
\]
\[(31b)\]

and
\[
p = \xi_0^2 \left( e^{-i\delta_{2,0}} + e^{i(\delta_{2,0} - \delta_{3,0})} \right) + \frac{1}{32} (d_1^2 + d_2^2 + d_3^2),
\]
\[(31c)\]

\[
q = \frac{i\xi_0^2}{4} \left( d_3 e^{-i\delta_{2,0}} + d_1 e^{i(\delta_{2,0} - \delta_{3,0})} \right) + \frac{i}{64} d_1 d_2 d_3.
\]
\[(31d)\]

Now we can use Cardano formulas to determine the roots \(\zeta_k\). For each concrete set of soliton parameters it is easy to evaluate \(\zeta_k\) and to determine the asymptotic regime predicted by CTC. The complete analytic analysis for generic complex valued \(p\) and \(q\) is rather lengthy. For brevity and simplicity we limit ourselves to two special choices of the initial amplitudes:

\[
A) \quad d_2 = 0, \quad d_1 = -d_3 = 2\Delta \nu_0,
\]
\[(32a)\]

\[
B) \quad d_1 = d_3 = \frac{2\Delta \nu_0}{3}, \quad d_2 = -d_1 - d_3.
\]
\[(32b)\]

In case A) it is possible to adjust the set of initial phases of the solitons so that \(p\) and \(q\) will take real values:

\[
\text{Ph}_1 : \quad \{\delta_{k,0}\}_{k=1}^3 \equiv \{0, \delta_{2,0}, 0\},
\]
\[(33a)\]

\[
\text{Ph}_2 : \quad \{\delta_{k,0}\}_{k=1}^3 \equiv \{0, \delta_{2,0}, 2\delta_{2,0}\},
\]
\[(33b)\]

\[
\text{Ph}_3 : \quad \{\delta_{k,0}\}_{k=1}^3 \equiv \{0, \delta_{2,0}, 2\delta_{2,0} - \pi\},
\]
\[(33c)\]

For real \(q\) and \(p\) the properties of the roots are determined by the sign of the discriminant \(Q\) of (30a) [24]:
\[
Q = \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2.
\]
\[(34)\]

Skipping the details, we find that each of the regimes occurs for the following choices of the soliton parameters. For regimes (i) and (iii) we will study only case A): regime (ii) will be studied exhaustively for both cases A) and B).

**Regime (i)** for real-valued \(p\) and \(q\) requires
\[
Q < 0.
\]
\[(35a)\]

Obviously this requires \(p < 0\). The condition (35a) is satisfied for each one of the following sets of parameters:

1. a) \(\Delta \nu_0 = 0\), \(\text{Ph}_1\) with \(\frac{\pi}{2} < \delta_2 < \frac{\pi}{4}\);

1. b) \(\text{Ph}_1\) with \(\delta_2 = \pi\) and
\[
|\Delta \nu_0| < \Delta \nu_{\text{cr},3},
\]
\[(35b)\]

where \(\Delta \nu_{\text{cr},3}\) is defined by (23).

1. c) \(|\Delta \nu_0| \geq 0\), \(\text{Ph}_2\) with \(\delta_{2,0} \neq 0\) and \(\pi\). Note that in this and in the next subcases \(q = 0\) but \(p\) is complex.

1. d) \(\Delta \nu_0 = 0\), \(\{\delta_{k,0}\}_{k=1}^3 \equiv \{0, \delta_{2,0}, \delta_{3,0}\}\) with \(\delta_{3,0} \neq 0\) and \(\delta_{2,0} - \frac{\delta_{3,0}}{2} \neq \pm \frac{\pi}{2}\) and \(\pm \frac{\pi}{4}\).

**Regime (ii)** - three solitons form a bound state if all the three roots of the characteristic polynomial (30a) are purely imaginary, i.e. \(\zeta_k = i\kappa_k\). Then by changing \(\zeta = i\zeta_1\) we get a polynomial
\[
\zeta_1^3 - p\zeta_1 + iq = 0
\]
\[(36)\]

with the purely real roots \(\kappa_k\); therefore its coefficients \(p\) and \(iq\) must also be real and the well known classification
of the solutions of cubic equations (see [33]) solves the problem exhaustively. In terms of the coefficients of (39) this situation takes place when

\[ Q_1 = -Q < 0, \quad p > 0. \]  

(37)

In terms of the soliton parameters (37) is satisfied in the following cases.

**Case A** (32a). For this choice of initial amplitudes we get soliton bound states in the following subcases:

- ii.a) \( \text{Ph} \equiv \{0,0,0\} \) and \( |\Delta \nu_0| \geq 0; \)
- ii.b) \( \text{Ph} \equiv \{0,\pm\pi,0\} \) and

\[ |\Delta \nu_0| > 2\sqrt{2}\epsilon_0 = \Delta \nu_{cr,3}; \]  

(38a)

- ii.c) \( \text{Ph} \equiv \{0,\pi_0,\pi_0\} \) and \( \text{Ph} \equiv \{0,0,\pi_0\} \) where \( \pi_0 = \pm\pi \) and

\[ |\Delta \nu_0| > 2\sqrt{3/4}\epsilon_0; \]  

(38b)

**Case B** (32b). For this choice of initial amplitudes we get soliton bound states in the following subcases:

- ii.d) \( \text{Ph} \equiv \{0,0,0\} \) and \( |\Delta \nu_0| \geq 0; \)
- ii.e) \( \text{Ph} \equiv \{0,\pm\pi,0\} \) and

\[ |\Delta \nu_0| > 4\sqrt{2}\epsilon_0; \]  

(38c)

- ii.f) \( \text{Ph} \equiv \{0,\delta_{2,0},0\} \) and \( |\Delta \nu_0| \geq 0 \) if \( \cos \delta_{2,0} > 0 \); if \( \cos \delta_{2,0} < 0 \) then the bound state regime holds for

\[ |\Delta \nu_0| > 4\sqrt{2}\sqrt{-\cos \delta_{2,0}\sigma_0}; \]  

(38d)

As one may expect, the asymptotic behavior depends very much on the initial choice of phases. Thus, in the cases ii.a and ii.d the bound state regime is obtained for any value of \( \Delta \nu_0 \), while in all other cases this regime is entered only if \( |\Delta \nu_0| \) is larger than some critical value, which of course, depends on the initial parameters, compare e.g. (38a), (38b), (38c) and (38d).

**Regime (iii)** requires one of the two conditions:

\[ Q > 0, \quad q \neq 0, \]  

(39a)

or

\[ Q = 0, \]  

(39b)

i.e.,

\[ p < 0, \quad q = \pm 2\sqrt{-\frac{p}{3}}. \]  

(39c)

For the soliton parameters (39a), (39b) respectively mean:

- iii.a) \( \text{Ph}_1 \) with \( \frac{\pi}{2} < \delta_{2,0} < \frac{3\pi}{2} \) and \( |\Delta \nu_0| = \Delta \nu_{cr,3}\sqrt{-\cos \delta_{2,0}}. \) Then \( p = 0 \) and \( q \) is real.
- iii.b) \( |\Delta \nu_0| > 0, \text{Ph}_1 \) with \( \delta_{2,0} = \frac{\pi}{2} \).
- iii.c) \( |\Delta \nu_0| \geq \Delta \nu_{cr,3}, \text{Ph}_1 \) with \( \delta_{2,0} \neq 0, \pi \).

We also note several critical sets of soliton parameters when one regime switches into another one. For example for \( \Delta \nu_0 = 0 \) and \( \text{Ph}_1 \) with \( \delta_{2,0} = \frac{\pi}{2} \) (or \( \delta_{2,0} = \frac{3\pi}{2} \approx -\frac{\pi}{2} \)) regime (i) switches over into regime (ii). The same happens also for \( |\Delta \nu_0| = \Delta \nu_{cr,3} \) and \( \text{Ph}_1 \) with \( \delta_{2,0} = \pi \).

The complete study of the characteristic equation (30a) in the general case, when both \( p \) and \( q \) are complex and nonvanishing will be given elsewhere.

These three main types of asymptotic behavior exist also for cases with \( N > 3 \) particles. Then it is possible to choose the parameters \( \zeta_k \) and \( r_k \) in such a way that the corresponding solution of the CTC will describe \( N \)-particle bound state; if, in addition, all the ratios \( (\eta_k - \eta_j)/(\eta_k - \eta_m) \) are rational, the corresponding solution will be periodic in time. One may also find the conditions under which this solution will develop singularities for finite \( \tau \) (like \( \Gamma_1 = 0 \) or \( \gamma_{3,0} = 0 \) above).

Obviously, there exist also all intermediary types of asymptotic. For example, if we have \( \kappa_1 < \ldots \kappa_k = \kappa_{k+1} = \ldots = \kappa_{k+m-1} < \ldots < \kappa_{N} \) then we will get in the asymptotic a \( m \)-particle bound state and all the other particles will have free-motion asymptotic.

### III. COMPARISON BETWEEN CTC AND NLS

In this section we compare the analytical results obtained in the Sections II with the numerical solutions of the unperturbed NLS \((R[u] = 0)\) (3) with initial conditions (3).

We have collected in tables, variety of runs. We will consider separately cases with different \( N \) and different type of asymptotic regimes. The quantity showing how well the solution of the CTC fits the NLS solution is given by the root mean square:

\[ \delta \chi_k = \sqrt{\frac{\sum_{\alpha=1}^{N_1} (\xi_k(t_\alpha)^{NLS} - \xi_k(t_\alpha)^{CTC})^2}{N_1}}. \]  

(40)

By \( N_1 \) above we have denoted the number of experimental points \( t_\alpha \), at which the numerical values are calculated.

In the case of regime (iii) we have also compared the mean values of the asymptotic velocities of the solitons calculated from NLS and CTC.

### N = 2.

Initial conditions assuring different type of regimes were described in Sec. (4)

Results comparing the NLS and the CTC are collected in table I and the first rows of tables III and IV. It is seen that the agreement is very good and improves with the increasing of \( \Delta \nu_0 \) and \( \nu_0 \). Transition from regime (i) to regime (ii) is also shown in table I. In addition to these results we show that the factor \( A_1 \) (17), see table IV, has very low values, which is a reason to call such regime quasi-equidistant.

**Degenerated** case. It is realized when the two eigenvalues of \( L_0 \) are equal. This happens when \( \Delta \nu_0 = \Delta \nu_{cr,2} \).
and \( \delta_{2,0} = \pi \). This case is very difficult for observing numerically, however our data show that the value of \( \Delta \kappa_{\text{cr,2}} \) is quite well predicted by CTC.

**Remark.** We mention again that under initial conditions \( \delta_0 = 0 \) and \( \Delta \nu_0 = 0 \) the CTC possesses singular solutions. It is known that the N-soliton solution of (1) is always analytic, i.e. does not possess any singularities. Our numerical checks show that such sets of initial conditions correspond to solitons colliding (for small \( r_0 \leq 8 \)) or coming very close to each other (for \( r_0 > 8 \)). In the regions, where this happens, the adiabatic approximation is no longer valid and the CTC does not match with the numeric solutions. For \( N = 2 \) our numerical calculations show that the first coalescence takes place at \( t = T_2^+|\Delta \nu_0=0/2 \) and the next ones tend to repeat periodically with period very close to \( T_2^+|\Delta \nu_0=0 \). Similar behavior is also found for \( N = 3 \) with \( T_3^+|\Delta \nu_0=0 \) replaced by \( T_3^+|\Delta \nu_0=0 \). For \( N > 3 \) the picture quickly becomes rather complicated.

Our conclusion is that for such initial conditions the quasi-equidistant propagation of the solitons is maintained for a restricted period of \( t \leq T_n^+|\Delta \nu_0=0/2 \).

\[ N = 3. \] Numerical data for regime (i) for \( \Delta \nu_0 = 0 \) and \( \delta_0 \) close to \( \pi \) was given in [4,5,25,26]. In Table II we have collected data from runs for a variety of values for \( \delta_{2,0} \). From it we see a very good agreement between CTC and NLS when \( \delta_{2,0} \) is far from \( \pi \); this value is critical for a transition from regime (i) to regime (ii). In the vicinity of \( \pi \) the agreement is not too good, see case 3. It is also seen that agreement is better for \( r_0 = 8 \) than for \( r_0 = 7 \).

**Regime (ii) - bound state solitons.** Data collected in table III shows very good agreement between NLS and CTC. The values of \( A_k \) collected in Table IV are rather low, especially for \( 2\Delta \nu_0 = 0.15 \) and 0.20 and again we may call such propagation quasi-equidistant.

As in the \( N = 2 \) case, the increasing of \( \Delta \nu_0 \) and \( r_0 \) leads to better agreement between NLS and CTC (see Table III), and also leads to lower value of \( A_k \) (see Table IV). Another fact illustrated in these tables is the better agreement for \( \delta_0 = 0 \) than for \( \delta_0 = \pi \).

**Regime (iii).** This type of regime is only available when \( N > 2 \). In this case two of the solitons form (nearly) bound state and the third one has free propagation. Examples of initial conditions leading to such type of regime are iii.a), iii.b) and iii.c). In Table V we show the quantity \( \delta_{\chi_k} \) as a criterion of agreement between NLS and CTC. The soliton parameters for these runs correspond to the case iii.c) and regime (iii) must take place for all values of \( \delta_{2,0} \neq 0 \) and \( \pi \).

Another possibility to compare CTC and NLS is given in table VI where we show the mean values of the asymptotic velocities of each soliton. We see that CTC always shows that two of the solitons form a bound state. This is not always what NLS shows, however the asymptotic values of the velocities of 2-nd and 3-rd solitons are quite close, so that we could consider it as a "quasi-bound" state.

If \( \Delta \nu_0 = \Delta \nu_{\text{cr,3}} \) and \( \delta_{2,0} = \pi \) then all eigenvalues of \( L_0 \) are equal and thus a degenerated case is realized. It is again very difficult to observe it numerically. The CTC gives logarithmically growing distance between solitons i.e. unbound state.

\[ N > 3. \] The cases listed above can also be extended to larger number of solitons showing the same type of regimes; of course now there are more mixed regimes possible. To analyze them one should solve the characteristic polynomial of \( L_0 \) and determine the soliton parameters, for which two or more of the \( \kappa_i \)’s are different (regime (i)) or equal (regimes (iii) and (ii)). More details about them will be published elsewhere. Cases when the spectrum of \( L \) becomes degenerate, i.e. when two (or more) of the eigenvalues \( \xi_k \) become equal, deserve special attention and further investigations.

Here we concentrate only on regime (ii) as the one which might be important in long range fiber optics communications. In tables III and IV we compare CTC and NLS for soliton trains \( N\pi(2|\Delta \nu_0|k \) with \( k = 0 \) and 1, and \( N = 2 \) to 5. The results listed in the table IV are from run lengths equal to \( t = 300 \) in dimensionless units; for regime (ii) we find periodic behavior with periods much less than 300 and small amplitudes of oscillations given by \( A_k \). On several occasions, we extended the runs to lengths of \( t = 600 \) and 996, the same type of behavior and the same results for \( A_k \). Thus we conclude that these sets of initial parameters lead to a very stable behavior. Indeed, taking the amplitudes of the solitons to be increasing by the same amount of \( \Delta \nu_0 \) we find for \( 2\Delta \nu_0 = 0.2 \) and \( r_0 = 8 \), an equidistant behavior with error \( A_k \lesssim 10\% \).

Regime (ii) is characteristic also for such trains with \( N > 5 \). However in such cases where one has to consider soliton trains with large number of solitons, a configuration with continually increasing amplitudes becomes impractical. Therefore we explore other possibilities for sets of initial soliton amplitudes.

In fact, the first step towards the solution of this problem was to use a soliton train of in-phase solitons with alternating amplitudes \( 2\nu_1 = 2\nu_3 = 1.0 \) and \( 2\nu_2 = 2\nu_4 = 1.25 \) in [3] and verified experimentally by [3]. In [3] the run length was equal to 141. This idea was experimentally verified in [3] where 20 Gbit/s single channel soliton transmission over 11 500 km using alternating amplitude solitons was reported.

To check this idea we made a series of runs of \( N \) in-phase soliton trains with alternating amplitudes and \( N = 3 \) to 8, see Table VII. What we find is that the quasi-equidistant propagation takes place for \( t \leq T_{\text{qed}} \) with \( T_{\text{NLS}} \sim 250 \). For larger values of \( t \), some of the solitons come rather close to each other. CTC also predicts the same type of behavior but with a larger value for \( T_{\text{CTC}} \sim 460 \). This is the reason why, in Table VII, the numerical data from NLS shows non equidistant propagation for a run length of 300. We extended some of these runs to lengths of 660 and 996. The results for \( N = 3 \) show a structure close to a periodic one with a
period of about $2T_{\text{qied}}^{\text{NLS}}$. So if the goal is to achieve a quasi-equidistant propagation to lengths $t \leq T_{\text{qied}}^{\text{NLS}}$ then such configurations can be used. As an illustration of this fact, we have provided Fig. 2 where a train of 8 solitons with alternating amplitudes is shown. We see, that in the regions where the solitons tend to coalesce, the match between CTC and NLS is on a qualitative level.

As one can see from Figs. 2-5, there are two distinct time scales in these figures. First, there are the oscillations with periods on the order of 10 or so. But at the same time, one notes that there is in general a much longer time scale, one that is on the order of 250 or larger. It is the motion on this longer time scale that violate the equidistant propagation, see Fig. 2. Note that in Fig. 4, when one can set the initial values so that the very long periodic motion is absent, then one has a very stable and equidistant motion. For larger run lengths, other configurations must be used, see also [1].

The next possibility which we explored was a "saw-like" configuration for the soliton amplitudes with three and four different amplitudes. The results are collected in Table VIII for $N = 4$ to 8. We see that configurations with only three different amplitudes show basically the same behavior as the alternating amplitudes case; the value of $T_{\text{qied}}^{\text{NLS}}$ here is slightly larger, and again in the regions where the solitons tend to coalesce the match between CTC and NLS is only qualitative, see Fig. 3.

Finally, for "saw-like" configurations of the amplitudes with four different amplitudes we get a substantially different picture. It seems that here the value of $T_{\text{qied}}^{\text{NLS}}$ is much larger than in the previous two cases. The CTC model shows a quasi-equidistant behavior with the same small values of $A_k$ like in the last column of Table VIII to run lengths of 12 000. The numeric solution of NLS shows an excellent match with CTC to lengths of 1 200, see Fig. 4.

IV. CONCLUSIONS

A method for the description of the asymptotic behavior of the $N$-soliton pulse trains of the NLS equation is proposed, based on the CTC model for the soliton interaction. It describes correctly several qualitatively different classes of asymptotic regimes. Several sets of soliton parameters have been described for which the propagation is quasi-equidistant. Such behavior with a conveniently low value for $A_k$ can be achieved by: a) taking soliton trains with $\Delta \nu_0$ large enough. Increasing $\Delta \nu_0$ too much may lead to violation of the condition (4c) of the adiabatic approximation. Somewhat surprisingly, this nevertheless leads to a better agreement between the CTC and the NLS; b) increasing the value $2\nu_0$ of the average amplitude; and c) increasing the distance $r_0$ between the neighboring solitons.

The critical values of the soliton parameters, for which one regime switches over to another one have been evaluated. We find that near these critical values, the match between CTC and NLS becomes worse, as one would expect.

These results have a natural generalization also for $N > 3$ soliton trains. The CTC-model provides one with a tool for constructing sets of initial data for the $N$-soliton trains which will possess a given asymptotic trait, which is also determined by the eigenvalues $\zeta_k$ of $L_0$.

The monotonically increasing amplitudes cannot be used in case one wants to send trains with larger number of solitons. That is why we investigated also other possibilities for amplitudes for which the propagation is quasi-equidistant: alternating and "saw-like" configurations with three and four different values for $2\nu_k$. We find that the alternating amplitudes configuration and the "saw-like" one with three different amplitudes may provide quasi-equidistant propagation up to run lengths on the order of about 230 or 300 correspondingly. The "saw-like" configurations with four different amplitudes show quasi-equidistant behavior to even much larger run lengths.

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[1] C. Desem and P. L. Chu, In Optical Solitons – Theory and Experiment, ed. by J. R. Taylor (University Press, Cambridge, 1992), p. 127.
TABLE I. Comparison between NLS and CTC for each soliton ($\delta x_k$): transition from regime (i) to regime (ii). Run length equals 300.

| $k$ | Regime (i) | Regime (ii) |
|-----|------------|-------------|
| 1   | 2/7|08|1 | 2/7|10|1 | 2/7|12|1 | 2/7|14|1 | 2/7|16|1 |
| 0.31 | 0.56 | 1.58 | 0.38 | 0.22 |
| 2   | 0.54 | 0.84 | 1.90 | 0.58 | 0.36 |
| 3   | 2/8|4|1 | 2/8|6|1 | 2/8|8|1 | 2/8|10|1 |
| 0.040 | 0.11 | 0.35 | 0.093 | 0.44 | 0.16 |

TABLE II. Comparison between NLS and CTC for each soliton ($\delta x_k$): regime (i). Run length equals 300.

| $k$ | 3/8|0|1/20 | 3/8|0|1/17 | 3/8|0|1/14 | 3/8|0|1/11 |
|-----|-----------|-----------|-----------|-----------|
| 1   | 0.89 | 0.23 | 0.16 | 0.10 |
| 2   | 0.00 | 0.00 | 0.00 | 0.00 |
| 3   | 0.89 | 0.23 | 0.16 | 0.10 |

| $k$ | 3/7|0|1/20 | 3/8|0|1/17 | 3/8|0|1/14 |
|-----|-----------|-----------|-----------|
| 1   | 2.29 | 0.68 | 0.42 |
| 2   | 0.00 | 0.00 | 0.00 |
| 3   | 2.29 | 0.68 | 0.42 |
### TABLE III. Comparison of NLS and CTC for each soliton ($\delta \chi_k$): regime (ii). Run length equals 300.

| $k$ | 3$|n_1|$ | 3$|n_1|$ | 3$|n_1|$ | 3$|n_1|$ | 3$|n_1|$ |
|-----|-----|-----|-----|-----|-----|
| 1   | 0.24 | 0.26 | 0.68 | 1.03 | 1.16 |
| 2   | 0.83 | 0.90 | 0.68 | 0.25 | 0.84 |
| 3   | 0.28 | 0.40 | 0.50 | 0.25 | 0.13 |

### TABLE V. Comparison between NLS and CTC for each soliton ($\delta \chi_k$): regime (iii). Here amplitudes have the values: $n_1 = (0.9146, 1.0, 1.0854)$ ($\Delta \nu_0 = \Delta \nu_{cr,x,|n_0=7}$), $n_2 = (0.9482, 1.0, 1.0518)$ ($\Delta \nu_0 = \Delta \nu_{cr,x,|n_0=8}$). Run length equals 300.

| $k$ | NLS | CTC | NLS | CTC | NLS | CTC |
|-----|-----|-----|-----|-----|-----|-----|
| 1   | 0.034 | 0.032 | 0.045 | 0.042 | 0.063 | 0.058 |
| 2   | 0.011 | 0.016 | 0.015 | 0.021 | 0.024 | 0.029 |
| 3   | 0.018 | 0.016 | 0.024 | 0.021 | 0.032 | 0.029 |

### TABLE VI. Comparison between NLS (mean asymptotic velocities) versus $-4\chi_k$ from CTC and for each soliton: regime (iii). Notations $n_1, n_2$ here mean the same as in the table above. Run length equals 300.

| $k$ | NLS | CTC | NLS | CTC | NLS | CTC |
|-----|-----|-----|-----|-----|-----|-----|
| 1   | 0.020 | 0.020 | 0.011 | 0.010 | 0.011 | 0.010 |
| 2   | 0.020 | 0.020 | 0.021 | 0.021 | 0.020 | 0.020 |
| 3   | 0.021 | 0.020 | 0.020 | 0.020 | 0.020 | 0.020 |
TABLE VII. Comparison between NLS and CTC for each pair of neighbor solitons: value of \( A_k \) for \( r_0 = 8 \). Amplitudes are ordered alternatingly: by "sl" we have denoted trains with \( 2\nu_1 = 1.0 \) and \( 2\nu_2 = 1.25 \) and "ls" means \( 2\nu_1 = 1.25 \) and \( 2\nu_2 = 1.00 \). Run length equals 300.

| \( k \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( \epsilon \) | \( \zeta \) | \( \eta \) |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.62  | 0.08  | 0.012 | 0.026 | 0.33  | 0.005 |
| 2     | 0.62  | 0.08  | 0.012 | 0.026 | 0.42  | 0.078 |
| 3     | 0.31  | 0.005 | 0.026 | 0.026 | 0.42  | 0.078 |
| 4     | 0.23  | 0.014 | 0.59  | 0.006 | 0.59  | 0.006 |
| 5     | 0.23  | 0.014 | 0.59  | 0.006 | 0.59  | 0.006 |
| 6     | 0.23  | 0.014 | 0.59  | 0.006 | 0.59  | 0.006 |

TABLE VIII. Comparison between NLS and CTC for each pair of neighbor solitons: value of \( A_k \). The train is a saw-like one with amplitudes \( 3\nu_1 \) → \( \{0.85, 1.00, 1.15, 0.85, \ldots\} \) and \( 4\nu_1 \) → \( \{0.85, 1.00, 1.15, 1.30, 0.85, \ldots\} \). Run length equals 1200.

| \( k \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( \epsilon \) | \( \zeta \) | \( \eta \) |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.19  | 0.07  | 0.47  | 0.07  | 0.05  | 0.04  |
| 2     | 0.09  | 0.05  | 0.25  | 0.06  | 0.06  | 0.04  |
| 3     | 1.13  | 0.04  | 0.39  | 0.05  | 0.02  | 0.04  |
| 4     | 0.41  | 0.06  | 0.01  | 0.00  | 0.01  | 0.00  |
| 5     | 0.40  | 0.22  | 0.05  | 0.02  | 0.66  | 0.35  |
| 6     | 0.15  | 0.20  | 0.08  | 0.02  | 0.54  | 0.45  |
| 7     | 0.52  | 0.15  | 0.02  | 0.00  | 0.02  | 0.00  |

| \( k \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( \epsilon \) | \( \zeta \) | \( \eta \) |
|-------|-------|-------|-------|-------|-------|-------|
| 1     | 0.09  | 0.03  | 0.84  | 0.19  | 0.09  | 0.03  |
| 2     | 0.06  | 0.04  | 0.38  | 0.30  | 0.06  | 0.03  |
| 3     | 0.02  | 0.03  | 0.26  | 0.13  | 0.02  | 0.03  |
| 4     | 0.05  | 0.02  | 0.36  | 0.28  | 0.05  | 0.02  |
| 5     | 0.08  | 0.03  | 0.22  | 0.33  | 0.08  | 0.03  |
| 6     | 0.05  | 0.03  | 0.49  | 0.06  | 0.05  | 0.03  |
| 7     | 0.52  | 0.15  | 0.02  | 0.00  | 0.02  | 0.00  |
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FIG. 1. Plot of the functions $A_{1,km}(\Delta \nu_0)$ in Eq. (28).
FIG. 2. Propagation of 8 solitons with alternating amplitudes $2\nu_{2k-1} = 1.0$, $2\nu_{2k} = 1.25$, $r_0 = 8$. 
FIG. 3. Propagation of 8 solitons with a "saw"-like configuration of the amplitudes: $2\nu_{3k-2} = 0.85$, $2\nu_{3k-1} = 1.0$, $2\nu_{3k} = 1.15$, $r_0 = 8$. 
FIG. 4. Propagation of 8 solitons with a "saw"-like configuration of the amplitudes: $2\nu_{4k-3} = 0.85$, $2\nu_{4k-2} = 1.0$, $2\nu_{4k-1} = 1.15$, $2\nu_{4k} = 1.30$, $r_0 = 8$. 
FIG. 5. Propagation of 3 solitons with alternating amplitudes: $2\nu_{1,0} = 2\nu_{3,0} = 1.00$, $2\nu_{2,0} = 1.25$, $r_0 = 8$. 
