THE RATE OF GROWTH OF MOMENTS OF CERTAIN COTANGENT SUMS

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Abstract. We consider cotangent sums associated to the zeros of the Estermann zeta function considered by the authors in their previous paper [5]. We settle a question on the rate of growth of the moments of these cotangent sums left open in [5], and obtain a simpler proof of the equidistribution of these sums.

Key words: Cotangent sums; equidistribution; Estermann zeta function; moments; continued fractions; measure.

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1. Introduction

The authors in joint work and the second author in his thesis, investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi mr}{b}\right),$$

as $r$ ranges over the set

$$\{r : (r, b) = 1, A_0 b \leq r \leq A_1 b\},$$

where $A_0, A_1$ are fixed with $1/2 < A_0 < A_1 < 1$ and $b$ tends to infinity.

Especially, they considered the moments

$$H_k = \lim_{b \to +\infty} \phi(b)^{-1} b^{-2k} (A_1 - A_0)^{-1} \sum_{\substack{A_0 b \leq r \leq A_1 b \\ (r, b) = 1}} c_0\left(\frac{r}{b}\right)^{2k}, \ k \in \mathbb{N},$$

where $\phi(\cdot)$ denotes the Euler phi-function.

They could show that all the moments $H_k$ exist and that

$$\lim_{k \to +\infty} H_k^{1/k} = +\infty$$

Thus the series $\sum_{k \geq 0} H_k x^{2k}$ converges only for $x = 0$.

It was left open, whether the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

converges for values of $x$ different from 0. This fact would considerably simplify the proof for the distribution of the cotangent sums $c_0(r/b)$ (uniqueness of measures determined by their moments, see [1], Section 30, The Method of Moments,
Theorem 30.1).
Crucial for the investigation was the result:

\[ H_k = \int_0^1 \left( \frac{g(x)}{2\pi} \right)^{2k} dx, \]

where

\[ g(x) = \sum_{l \geq 1} \frac{1 - 2\{lx\}}{l}. \]

The function \( g \) has been also investigated in the paper [2] of R. de la Bretèche and G. Tenenbaum. Their ideas will be crucial in our paper. We shall show the following theorems.

**Theorem 1.1.** There exists a constant \( C_0 > 0 \), such that

\[ \int_0^1 |g(x)|^L dx \leq C_0^L L^L, \]

for all \( L \in \mathbb{N} \).

**Theorem 1.2.** The series

\[ \sum_{k \geq 0} H_k \frac{x^k}{(2k)!} \]

diverges for \( |x| > \pi^2 \), where \( x \in \mathbb{C} \).

From Theorem 1.1, an affirmative answer regarding the question of the positive radius of convergence of (*) follows. From Theorem 1.2 it follows that the radius of convergence of the series (*) is finite.

**Conjecture 1.3.** The radius of convergence of the series (*) is \( \pi^2 \).

### 2. Continued fractions

**Definition 2.1.** Let \( \alpha \in [0, 1] \setminus \mathbb{Q} \). Assume that

\[ \alpha = [0; a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} \]

is its continued fraction expansion with integers \( a_i \geq 1 \) for \( i = 1, 2, \ldots \)

We denote the partial quotients by \( p_r/q_r \), i.e.

\[ [0; a_1, a_2, \ldots, a_r] = \frac{p_r}{q_r}, \quad \text{with} \ (p_r, q_r) = 1. \]

We set \( p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1 \).

**Definition 2.2.** The map \( T : (0, 1) \rightarrow (0, 1), \alpha \mapsto \frac{1}{\alpha} - \left[ \frac{1}{\alpha} \right] \) is called the continued fraction map (or Gauss map).

**Lemma 2.3.** The partial quotients \( p_r, q_r \) satisfy the recursion:

(1) \[ p_{r+1} = a_{r+1} p_r + p_{r-1} \quad \text{and} \quad q_{r+1} = a_{r+1} q_r + q_{r-1}. \]

**Proof.** (cf. [3], p. 7).
Lemma 2.4. For $\alpha = [0; a_1, a_2, \ldots, a_r, a_{r+1}, \ldots]$, we have
\begin{equation}
(2) 
T^r \alpha = [0; a_{r+1}, a_{r+2}, \ldots]
\end{equation}
The map $T$ preserves the measure
\begin{equation}
(3) 
\omega(\mathcal{E}) = \frac{1}{\log 2} \int_{\mathcal{E}} \frac{dx}{1 + x},
\end{equation}
\text{i.e.} $\omega(T(\mathcal{E})) = \omega(\mathcal{E})$, for all measurable sets $\mathcal{E} \subset (0, 1)$.
\begin{proof}
The result (2) is well known and can be easily confirmed by direct computation. For (3) cf. [3], p. 119.
\end{proof}

Lemma 2.5. There is a constant $A_0 > 1$, such that
\begin{equation}
q^r \geq A_0^r,
\end{equation}
for all $r \in \mathbb{N}$.
\begin{proof}
This is well known and easily follows from (1) of Lemma 2.3.
\end{proof}

Definition 2.6. Let $\alpha \in (0, 1) \setminus \mathbb{Q}$, $r \in \mathbb{N}$. Then, we set
\begin{equation*}
c(\alpha, r) = \sum_{j=0}^{r} \frac{\log q_{j+1}}{q_j},
c(\alpha, +\infty) = \sum_{j=0}^{+\infty} \frac{\log q_{j+1}}{q_j} \in \mathbb{R} \cup \{+\infty\}
\end{equation*}
We define the constant $c_0 > 0$, by
\begin{equation*}
c_0 \sum_{r \geq 0} A_0^{-r/2} = \frac{1}{4}
\end{equation*}
and define the sequence $(w^{(r)})$ by
\begin{equation*}
w^{(r)} = \frac{1}{2} + c_0 \sum_{j=0}^{r} A_0^{-j/2}.
\end{equation*}
For $z \in (0, +\infty)$, we define
\begin{equation*}
\mathcal{E}(z, 0) := \{ \alpha \in (0, 1) \setminus \mathbb{Q} : c(\alpha, 1) \geq w^{(0)} z \}, \ (w^{(0)} = 1/2)
\end{equation*}
\begin{equation*}
\mathcal{E}(z, r) := \{ \alpha \in (0, 1) \setminus \mathbb{Q} : c(\alpha, r - 1) < w^{(r-1)} z, c(\alpha, r) \geq w^{(r)} z \}
\end{equation*}
\begin{equation*}
\mathcal{E}(z, +\infty) := \{ \alpha \in (0, +\infty) \setminus \mathbb{Q} : c(\alpha, +\infty) \geq z \}.
\end{equation*}

Lemma 2.7. For $z \in (0, +\infty)$, it holds
\begin{equation*}
\text{meas}(\mathcal{E}(z, +\infty)) \leq \sum_{r \geq 0} \text{meas}(\mathcal{E}(z, r)),
\end{equation*}
where \text{meas} stands for the Lebesgue measure.
\begin{proof}
Assume that $\alpha \notin \mathcal{E}(z, r)$, for every $r \in \mathbb{N} \cup \{0\}$. Then it follows by induction on $r$, that
\begin{equation*}
c(\alpha, r) \leq w^{(r)} z
\end{equation*}
and thus
\begin{equation*}
c(\alpha, +\infty) = \lim_{r \to +\infty} c(\alpha, r) \leq \frac{3}{4} z.
\end{equation*}
Therefore, if $\alpha \in \mathcal{E}(z, +\infty)$ we have $\alpha \in \mathcal{E}(z, r)$ for at least one value of $r \in \mathbb{N} \cup \{0\}$. Thus
\[
\mathcal{E}(z, +\infty) \subset \bigcup_{r=0}^{+\infty} \mathcal{E}(z, r),
\]
which proves Lemma 2.7.

**Lemma 2.8.** There are absolute constants $z_0 > 0$ and $c_0 > 0$, such that
\[
\text{meas}(\mathcal{E}(z, r)) \leq \exp\left(-\frac{1}{2}c_0 A_0^{r/2} z\right),
\]
for all $z \geq z_0$.

**Proof.** Assume that $\alpha \in \mathcal{E}(z, r)$. We have
\[
c(\alpha, r) = c(\alpha, r - 1) + \log \frac{q_{r+1}}{q_r}.
\]
The inequalities
\[
c(\alpha, r - 1) < w^{(r-1)} z \quad \text{and} \quad c(\alpha, r) \geq w^r z,
\]
imply that
\[
\log \frac{q_{r+1}}{q_r} \geq \left(w^r - w^{(r-1)}\right) z = c_0 A_0^{-r/2} z
\]
and
\[
q_{r+1} \geq \exp(c_0 A_0^{-r/2} q_r z) \geq \exp\left(c_0 q_r^{1/2} z\right).
\]
From
\[
q_{r+1} = a_{r+1} q_r + q_{r-1} \leq (a_{r+1} + 1) q_r
\]
we obtain
\[
a_{r+1} \geq q_{r+1} q_r^{-1} - 1 \geq \exp\left(c_0 q_r^{1/2} z\right) q_r^{-1} - 1
\]
\[
\geq \exp\left(\frac{3}{4} c_0 q_r^{1/2} z\right) \geq \exp\left(\frac{3}{4} c_0 A_0^{r/2} z\right),
\]
if $z_0$ is sufficiently large.

We have for all $w > 0$:
\[
T'\{\alpha = [0; a_1, \ldots, a_{r+1}, \ldots], a_{r+1} \geq w\} = \{\alpha = [0; a_{r+1}, \ldots], a_{r+1} \geq w\},
\]
by Lemma 2.4.

Since $T$ preserves the measure $\omega$, we have:
\[
\omega\{\alpha = [0; a_1, \ldots, a_{r+1}, \ldots], a_{r+1} \geq w\} = \omega\{\alpha = [0; a_{r+1}, \ldots], a_{r+1} \geq w\}.
\]
Therefore
\[
[0; a_{r+1}, \ldots] \leq w^{-1}
\]
and thus
\[
\omega\{\alpha = [0; a_1, \ldots, a_{r+1}, \ldots], a_{r+1} \geq w\} \leq \frac{1}{\log 2} \int_0^{w^{-1}} \frac{dx}{1 + x} \leq 2w^{-1}.
\]
Applying (6) and (7) we obtain
\[
\text{meas } (\mathcal{E}(z, r)) \leq 2w^{-1}.
\]
We set in (8): 

\[ w = \exp \left( \frac{3}{4} c_0 A_0^{r/2} z \right). \]

Then 

\[ \text{meas}(\mathcal{E}(z, r)) \leq \exp \left( -\frac{1}{2} c_0 A_0^{r/2} z \right). \]

\[ \square \]

**Lemma 2.9.** There is a constant \( c_1 > 0 \), such that 

\[ \text{meas}(\mathcal{E}(z, +\infty)) \leq \exp(-c_1 z), \] if \( z \geq z_0 \).

**Proof.** This follows from Lemmas 2.7 and 2.8. \( \square \)

3. Results of R. de la Bretèche and G. Tenenbaum

R. de la Bretèche and G. Tenenbaum [2] prove the following result (Théorème 4.4):

**Theorem 3.1.** The function 

\[ g(\alpha) = \sum_{l \geq 1} \frac{1 - 2\{l \alpha\}}{l} \]

converges for \( \alpha \in \mathbb{Q} \) if and only if 

\[ \sum_{r \geq 1} (-1)^r \frac{\log q_{r+1}}{q_r} \]

converges. In this case 

\[ (**\) \quad g(\alpha) = -\sum_{m \geq 1} \frac{\tau(m)}{\pi m} \sin(2\pi m \alpha), \]

where \( \tau \) stands for the divisor function.

The following definitions are adopted from [2], p. 8.

**Definition 3.2.** For a multiplicative function \( g \) and \( x, y \) with \( 1 \leq y \leq x \) and \( \theta \in \mathbb{R} \) we denote by 

\[ Z_g(x, y; \theta) :\! = \sum_{n \in S(x, y)} g(n) \sin(2\pi n \theta), \]

where 

\[ S(x, y) = \{ n \leq x : P(n) \leq y \}, \]

\( P(n) \) being the largest prime factor of \( n \).

We set 

\[ \mu(\theta; Q) :\! = \min_{1 \leq m \leq Q} \| m \theta \| \leq \frac{1}{Q} \]

and 

\[ q(\theta; Q) :\! = \min\{ q : 1 \leq q \leq Q, \text{ with } \|q \theta\| = \mu(\theta; Q)\}, \]

where \( \| \cdot \| \) denotes the distance to the nearest integer.

We have:
Lemma 3.3. Let $A > 0$. For $x \geq 2$,

$$Q_x := \frac{x}{(\log x)^{44+24}},$$

$$q := q(\theta; Q_x), \ a \in \mathbb{Z}, \ (a, q) = 1,$$

$$|q\theta - a| \leq \frac{1}{Q_x}, \ \theta_q := \theta - \frac{a}{q}, \ \theta \in \mathbb{R},$$

one has uniformly

$$Z_\tau(x, x; \theta) = x(\log x) \left\{ \frac{\sin^2(\pi \theta_q x)}{\pi \theta_q x} + O \left( \frac{(\log q) \log(1 + (\theta_q x)^2)}{q \theta_q |x \log x|} \right) + \frac{1}{(\log x)^A} \right\}$$

Proof. This is Lemma 11.2 of [2], pp. 64-65. □

Definition 3.4. For $\theta \in \mathbb{R} \setminus \mathbb{Q}$ let $(q_m)_{m \geq 1} = (q_m(\theta))_{m \geq 1}$ denote the sequence of the denominators of the partial fractions of $\theta$. Let $a_m/q_m$ denote the $m$-th partial fraction of $\theta$.

We set

$$\varepsilon_m := \theta - \frac{a_m}{q_m}.$$ 

The set of all real numbers for which $q(\theta; Q_x) = q_m$ is an interval defined by the conditions $q_m \leq Q_x < q_m+1$. We denote it by $[\xi_m, \xi_{m+1}]$.

Then, we have:

Lemma 3.5. For a positive real constant $B$, we have:

$$\xi_m \approx q_m(\log q_m)^B,$$

$$|\varepsilon_m| \xi_m \approx \frac{(\log q_m)^B}{q_m+1},$$

$$|\varepsilon_m| \xi_{m+1} \approx \frac{(\log q_{m+1})^B}{q_m},$$

where $K \asymp L$ denotes $K = O(L)$ and $L = O(K)$.

Proof. This is equation (6.3) of [2], p. 22. □

Lemma 3.6. Let $\alpha \in (0, 1) \setminus \mathbb{Q}$. There are constants $c_2, c_3 > 0$, such that

$$|g(\alpha)| \leq c_2 c(\alpha, +\infty) + c_3.$$

Proof. We closely follow [2], p. 65. By partial summation, we obtain:

$$g(\alpha) = \sum_{n \geq 1} \frac{\tau(n)}{n} \sin(2\pi n \alpha) = \int_1^{+\infty} Z_\tau(t, t; \alpha) \frac{dt}{t^2}$$

$$= \sum_{m \geq 1} \left( \int_{\xi_m}^{\xi_{m+1}} Z_\tau(t, t; \alpha) \frac{dt}{t^2} \right).$$

By equation 11.5 of [2], p. 65, we have

$$\int_{\xi_m}^{\xi_{m+1}} Z_\tau(t, t; \alpha) \frac{dt}{t^2} = \frac{1}{2} \pi \sgn(\varepsilon_m) \frac{\log q_{m+1}}{q_m} + O \left( \frac{1}{q_{m+1}} + \int_{\xi_m}^{\xi_{m+1}} \frac{dt}{t \log t} \right),$$

where $\sgn(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0 \end{cases}$. 

Therefore,

$$\int_{\xi_m}^{\xi_{m+1}} Z_\tau(t, t; \alpha) \frac{dt}{t^2} = \frac{1}{2} \pi \sgn(\varepsilon_m) \frac{\log q_{m+1}}{q_m} + O \left( \frac{1}{q_{m+1}} + \int_{\xi_m}^{\xi_{m+1}} \frac{dt}{t \log t} \right).$$

This proves the lemma.
where $A$ is fixed, but arbitrarily large.

Therefore

$$g(\alpha) = \int_1^{+\infty} Z_r(t, t; \alpha) \frac{dt}{t^2}$$

$$\leq c_2 \sum_{m \geq 1} \frac{\log q_{m+1}}{q_m} + \sum_{m \geq 1} q_m^{1/B} + \int_1^{+\infty} \frac{dt}{t(\log t)^A}$$

$$\leq c_2 c(\alpha, +\infty) + c_3,$$

since the sequence $(q_m)_{m \geq 1}$ is growing exponentially and the integral converges if $A > 1$. This completes the proof.

Proof of Theorem 1.1. Let $L \in \mathbb{N}$ and assume that $\alpha$ satisfies (***) (Théorème 4.4. of [2]) and $|g(\alpha)| \geq 4L$.

We apply Lemmas 2.9 and 3.6 and obtain

$$\text{meas}\{\alpha : |g(\alpha)| \geq yL\} \leq \exp(-c_1 yL).$$

Therefore

$$\int_0^1 |g(\alpha)|^k d\alpha \leq \sum_{j \geq 0} ((2^{j+1}L)^L \text{meas}\{\alpha : 2^j L \leq |g(\alpha)| \leq 2^{j+1}L\})$$

$$\leq \sum_{j \geq 0} (2^{j+1}L)^L \exp(-c_1 2^j L) \leq C_0^L L^L.$$

However,

$$H_k = \int_0^1 \left(\frac{g(x)}{2\pi}\right)^{2k} dx = (2\pi)^{-2k} \int_0^1 g(x)^{2k} dx$$

$$\leq (2\pi)^{-2k} C_0^{2k} (2k)^{2k}$$

$$= \left(\frac{C_0}{2\pi}\right)^{2k} (2k)^{2k},$$

because of Theorem 1.1 with $L = 2k$, $k \in \mathbb{N}$.

Also,

$$(2k)^{2k} \leq (2k)! 3^{2k},$$

for $k \geq k_0$, for some $k_0 \in \mathbb{N}$.

Hence

$$\frac{H_k}{(2k)!} \leq \left(\frac{C_0}{2\pi}\right)^{2k} 3^{2k}$$

$$= \left(\frac{3 C_0}{2\pi}\right)^{2k},$$

for $k \geq k_0$, for some $k_0 \in \mathbb{N}$.

Hence, the radius of convergence of the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$
is positive.

For the proof of Theorem 1.2 the following definitions and lemmas will be used.

**Definition 3.7.** For \( k \in \mathbb{N} \cup \{0\} \) we set

\[
I := I(k) = [0, e^{-2k}] \quad \text{and} \quad l_0 := l_0(k) = e^{2k}.
\]

We fix \( \delta > 0 \) arbitrarily small and set

\[
g_1(\alpha) := \sum_{l \leq l_0^{-2s}} \frac{B(l\alpha)}{l}, \quad g_2(\alpha) := \sum_{l_0^{-2s} < l \leq l_0^{1+s}} \frac{B(l\alpha)}{l}, \quad g_3(\alpha) := \sum_{l > l_0^{1+s}} B(l\alpha) l.
\]

where \( B(u) = 1 - 2\{u\}, \ u \in \mathbb{R} \).

In the sequel, we assume \( k \geq k_0 \), where \( k_0 \in \mathbb{N} \), sufficiently large.

**Lemma 3.8.** We have

\[
g(\alpha) = g_1(\alpha) + g_2(\alpha) + g_3(\alpha),
\]

for every \( \alpha \in \mathbb{R} \).

**Proof.** It is obvious by the definition of \( g(\alpha), g_1(\alpha), g_2(\alpha), g_3(\alpha) \). \( \square \)

**Lemma 3.9.** For \( \alpha \in I \), we have

\[
g_1(\alpha) \geq (1 - 8\delta)2k,
\]

for \( k \in \mathbb{N} \cup \{0\} \).

**Proof.** For \( \alpha \in I, l \leq l_0^{1-2s} \) we have \( l\alpha \leq \delta \) and therefore

\[
B(l\alpha) \geq 1 - 4\delta
\]

because of Definition 3.7. Thus

\[
g_1(\alpha) \geq (1 - 4\delta) \sum_{l \leq l_0^{-2s}} \frac{1}{l}.
\]

From the formula

\[
\sum_{m \leq u} \frac{1}{m} = \log u + O(1) \quad (u \to +\infty),
\]

we have

\[
g_1(\alpha) \geq (1 - 8\delta)2k.
\]

\( \square \)

**Lemma 3.10.** It holds

\[
|g_2(\alpha)| \leq 16\delta k,
\]

for \( k \in \mathbb{N} \cup \{0\} \) and sufficiently small \( \delta > 0 \).

**Proof.** We have

\[
|g_2(\alpha)| \leq \sum_{l_0^{-2s} < l \leq l_0^{1+s}} \frac{1}{l} \leq 2 \left( \log(l_0^{1+2\delta}) - \log(l_0^{1-2\delta}) \right) \leq 16\delta k.
\]

\( \square \)
Lemma 3.11. For all $\alpha \in I$ that do not belong to an exceptional set $E$ with measure
$$\text{meas}(E) \leq e^{-2k(1+\delta)},$$
we have
$$|g_3(\alpha)| \leq \delta k.$$

Proof. The function $g_3$ has the Fourier expansion:
$$g_3(\alpha) = \sum_{l > l_0^{1+2\delta}} c(l) e(l\alpha),$$
where $c(l) = O(l^{-1+\epsilon})$ for $\epsilon$ arbitrarily small, by Lemma 5.6 of [5].

By Parseval’s identity we have
$$\int_0^1 g_3(\alpha)^2 d\alpha = \sum_{l > l_0^{1+2\delta}} c(l)^2 = O \left( \sum_{l > l_0^{1+2\delta}} l^{-2+2\epsilon} \right) = O \left( l_0^{1-3\delta/2} \right).$$

Let
$$E = \{ \alpha : |g_3(\alpha)| > \delta k \}.$$

Then
$$(\text{meas}(E))(\delta k)^2 \leq \int_E g_3(\alpha)^2 d\alpha \leq \int_0^1 g_3(\alpha)^2 d\alpha = O \left( l_0^{1-3\delta/2} \right).$$

Therefore
$$\text{meas}(E) \leq O \left( (\delta k)^{-2} l_0^{1-3\delta/2} \right) = O \left( e^{-2k(1+\delta)} \right).$$

This completes the proof of the Lemma. \qed

Proof of Theorem 1.2.
By Lemmas 3.9, 3.10 and 3.11, we have
$$|g(\alpha)| \geq |g_1(\alpha)| - |g_2(\alpha)| - |g_3(\alpha)| \geq (1 - 20\delta)2k,$$
for all $\alpha \in I$ except for those values of $\alpha$ that belong to an exceptional set $E(I) := E \cap I \subset I$
with
$$\text{meas}(E(I)) \leq \frac{1}{2} |I|,$$
where $|I|$ stands for the length of $I$. Hence, we obtain
$$H_k = \int_0^1 \left( \frac{g(\alpha)}{\pi} \right)^{2k} d\alpha \geq \frac{1}{2} |I| \left( \frac{1 - 20\delta}{\pi} \right)^{2k} \left( \frac{e^{-2k \log 2k}}{\pi} \right)^{2k}.$$

By Stirling’s formula we have
$$(2k)! \geq \exp(2k \log 2k) \exp(-(1-\delta)2k)$$
and therefore
$$\frac{H_k}{(2k)!} \geq \frac{1}{2} e^{-\delta k} \left( \frac{1 - 20\delta}{\pi} \right)^{2k}.$$
Since $\delta > 0$ can be fixed arbitrarily small, we have

$$\limsup_{k \to +\infty} \left( \frac{H_k}{(2k)!} \right)^{1/k} \geq \frac{1}{\pi^2}.$$ 

Therefore, the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

diverges for $|x| > \pi^2$, where $x \in \mathbb{C}$. This completes the proof of Theorem 1.2.

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Therefore, the series

$$\sum_{k \geq 0} \frac{H_k}{(2k)!} x^k$$

diverges for $|x| > \pi^2$, where $x \in \mathbb{C}$. This completes the proof of Theorem 1.2.

4. The distribution of the cotangent sums $c_0 \left( \frac{r}{b} \right)$

We now give a simpler proof of Theorem 5.2 of [5] regarding the equidistribution of $c_0(r/b)$ for fixed large positive integer values of $b$ and $A_0b \leq r \leq A_1b$, where $1/2 < A_0 < A_1 < 1$. We need the following Lemmas and Definitions from [1].

**Lemma 4.1.** Let $\mu$ be a probability measure on the line having finite moments

$$\alpha_k = \int_{-\infty}^{+\infty} x^k \mu(dx)$$

of all orders. If the power series

$$\sum_{k \geq 1} \frac{\alpha_k}{k!} x^k$$

has a positive radius of convergence, then $\mu$ is the only probability measure with the moments $\alpha_1, \alpha_2, \ldots$

**Proof.** This is Theorem 30.1 of [1], pp. 388-389. □

**Definition 4.2.** A probability measure satisfying the conclusion of Lemma 4.1 is said to be determined by its moments.

**Definition 4.3.** A sequence $(F_n)_{n \geq 1}$ of distribution functions is said to converge weakly to the distribution function $F$ (denoted by $F_n \Rightarrow F$) if

$$\lim_{n \to +\infty} F_n(x) = F(x)$$

for every point $x$ of continuity of $F(x)$.

A sequence $(X_n)_{n \geq 1}$ of random variables is said to converge in distribution (or in law) towards a random variable $X$ (denoted by $X_n \Rightarrow X$) with distribution function $F$, if and only if $F_n \Rightarrow F$, that is $X_n \Rightarrow X$ if and only if $F_n \Rightarrow F$.

**Lemma 4.4.** For a sequence $(X_n)_{n \geq 1}$ of random variables and a random variable $X$, we have $X_n \Rightarrow X$ if and only if

$$\lim_{n \to +\infty} P[X_n \leq x] = P[X \leq x]$$

for every $x \in \mathbb{R}$, such that $P[X = x] = 0$.

**Proof.** This follows immediately from Definition 4.3. □
Lemma 4.5. Suppose that the distribution of $X$ is determined by its moments and that the $X_n$ have moments of all orders, as well as

$$
\lim_{n \to +\infty} E(X_n^r) = E(X^r)
$$

for $r = 1, 2, 3, \ldots$. Then $X_n \Rightarrow X$.

Proof. This is Theorem 30.2 of [1], p. 390. □

We now recall the Definition 5.1 and Theorem 5.2 from [5].

Definition 4.6. For $z \in \mathbb{R}$, let

$$
F(z) = \text{meas}\{\alpha \in [0, 1] : g(\alpha) \leq z\},
$$

where “meas” denotes the Lebesgue measure,

$$
g(\alpha) = \sum_{l=1}^{\infty} \frac{1 - 2\{\alpha\}}{l}
$$

and

$$
C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \forall \varepsilon > 0, \exists a \text{ compact set } K \subset \mathbb{R}, \text{such that } |f(x)| < \varepsilon, \forall x \notin K\}.
$$

Theorem 4.7. i) $F$ is a continuous function of $z$.

ii) Let $A_0, A_1$ be fixed constants, such that $1/2 < A_0 < A_1 < 1$. Let also

$$
H_k = \int_0^1 \left( \frac{g(x)}{\pi} \right)^{2k} dx,
$$

where $H_k$ is a positive constant depending only on $k$, $k \in \mathbb{N}$.

There is a unique positive measure $\mu$ on $\mathbb{R}$ with the following properties:

(a) For $\alpha < \beta \in \mathbb{R}$ we have

$$
\mu([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha)).
$$

(b)

$$
\int x^k d\mu = \begin{cases} (A_1 - A_0)H_{k/2}, & \text{for even } k \\ 0, & \text{otherwise} \end{cases}
$$

(c) For all $f \in C_0(\mathbb{R})$, we have

$$
\lim_{b \to +\infty} \frac{1}{\phi(b)} \sum_{r : (r, b) = 1, A_0b \leq r \leq A_1b} f \left( \frac{1}{b} c_0 \left( \frac{r}{b} \right) \right) = \int f d\mu,
$$

where $\phi(\cdot)$ denotes the Euler phi-function.

We now state and give a new proof of a special case of Theorem 4.7 (c) from which the complete Theorem 4.7 follows by the definition of the abstract Lebesgue integral.

Theorem 4.8. Let $A_0, A_1$ be fixed constants, such that $1/2 < A_0 < A_1 < 1$, then we have for $\alpha < \beta \in \mathbb{R}$:

$$
\lim_{b \to +\infty} \frac{1}{\phi(b)} \left| \left\{ r : (r, b) = 1, A_0b \leq r \leq A_1b, \frac{r}{b} \leq \beta b \right\} \right| = (A_1 - A_0)(F(\beta) - F(\alpha)).
$$
Proof. Let \((b_n)_{n \geq 1}\) be a sequence of positive integers with \(b_n \to +\infty \) as \(n \to +\infty\). We set
\[
X_n = \frac{1}{b_n} c_0 \left( \frac{r}{b_n} \right)
\]
and consider \(X_n\) as a random variable on the probability space
\[
\Omega_n = \{ r : (r, b_n) = 1, A_0 b_n \leq r \leq A_1 b_n \}
\]
with the counting measure
\[
\mu_n(\mathcal{E}) = \frac{|\mathcal{E}|}{|\Omega_n|}
\]
for all \(\mathcal{E} \subset \Omega_n\).

By Lemma 5.13 of \cite{MaierRassias}, we have
\[
\lim_{n \to +\infty} \mu_n([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha))
\]
for all \(\alpha < \beta \in \mathbb{R}\).

By Theorem 1.1, Lemma 4.1 and Definition 4.2 the measure \(\mu\) given by
\[
\mu([\alpha, \beta]) = (A_1 - A_0)(F(\beta) - F(\alpha)),
\]
is determined by its moments. By Theorem 4.7 we have
\[
\lim_{n \to +\infty} E(X_n^r) = E(X^r).
\]
Thus, Lemma 4.5 implies \(X_n \Rightarrow X\), where \(X = g(\alpha)\) is a random variable on the probability space \([0, 1]\). Since \(F\) is a continuous function by Theorem 5.2(i) of \cite{MaierRassias}, the claim of Theorem 4.8 follows. \(\square\)

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