Modified Fuzzy Data Envelopment Analysis Models

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Abstract
This paper examines the use of data envelopment analysis (DEA) in the conduct of efficiency measurement involving fuzzy (interval) input-output values. Data envelopment analysis is a linear programming method for comparing the relative productivity (or efficiency) of multiple service units. Standard DEA models assume crisp data for both the input and output values. In practice however, input and output values may be uncertain, vague, imprecise or incomplete. New pairs of fuzzy DEA (FDEA) models are presented which differ from existing fuzzy DEA models handling uncertain data. In this approach, upper bound interval data are used exclusively to obtain the upper frontier values while lower bound interval data are used exclusively to obtain the lower frontier values. The outcome, when compared with the outcome of existing approach, based on the same set of data, shows a swap in the upper and lower frontier values with exactly the same number of efficient decision making units (DMUs). This new approach therefore clears the ambiguity occasioned by the mixture of upper and lower bound values in the determination of the upper and lower frontier efficiency scores respectively. The modified FDEA models make application and interpretation of results easy. The most efficient units, for each of the models, have efficiency score of 1 with equivalent ranking score of 1. These efficient units also serve as reference sets to the inefficient units. The inefficient units have efficiency scores less than 1 for all the models. The most inefficient unit is S13 for all the models and it has the least efficiency score in each case and a ranking score of 25.

Keywords: Fuzzy, Data envelopment analysis, Modified, Models.
Introduction

This paper extends the technique of fuzzy (interval) data envelopment analysis (FDEA). In particular, the paper seeks to present modified fuzzy Charnes, Cooper and Rhodes (FCCR) and fuzzy Banker, Charnes and Cooper (FBCC) models for use with interval fuzzy numbers. This paper also compares the conventional DEA, the FDEA presented by Zeidan et al. (2016) and Demir (2014). For ease of comparison, the data set for 25 high schools in the 2012-2013 education year in Demir (2014) is used.

Data envelopment analysis (DEA) was first presented by Charnes, Cooper and Rhodes (1978) leveraging on the 1957 seminal paper of Farrell whose main purpose was the estimation of technical efficiency and efficiency frontiers. DEA has become one of the most widely used techniques for measuring the efficiency of decision making units (DMU). A basic assumption of DEA for the measurement of the total technical efficiency of a DMU is that of constant returns to scale (CRS). This was later modified by Banker, Charnes and Cooper (1984) to become variable returns to scale (VRS) (Demir, 2014). According to Zeidan et al., (2016), data envelopment analysis is a non-parametric technique for evaluating and measuring the relative efficiency of decision making units characterized by multiple inputs and multiple outputs.

The basic DEA works with crisp values for both the input and output values. Being a very responsive method, its efficiency is easily affected by errors bothering on imprecise data, incomplete data, judgment data, forecasting data or ambiguous data. In general, imprecise data can be presented in form of fuzzy numbers. It is therefore worthwhile to study how to evaluate the efficiency of a set of data in fuzzy form. In such a situation, FDEA becomes a useful method to overcome the shortcomings of basic DEA.

Wang et al. (2005) studied how to conduct efficiency assessment in interval and/or fuzzy input-output environments in a simple, rational and effective way using data envelopment analysis. They constructed a new pair of interval DEA models on the basis of interval arithmetic, which differs from the existing DEA models handling interval data.

Demir (2014) compared classical DEA and FDEA based on α-intercept method by means of an application for educational researches. He compared the relative activities of 25 high schools in the 2012-2013 education year by means of DEA and FDEA and strongly recommends that fuzzy theory be practiced for DEA problems with uncertain data in order to get more secure results in activity measurements.

Zeidan, et al. (2016) presented a technique to improve a statistical method based on arithmetic operations to solve fuzzy data envelopment analysis models. They transformed the original data into interval data in the form of lower and upper frontier data and used them to obtain the interval DEA efficiency scores. Their method requires that data should be distributed as a normal distribution. Thus, the technique assumes that the variables are normally distributed. This position is however at variance with the fact that DEA, being a non-parametric technique, does not assume any specific functional form relating inputs to outputs (Zhu, 2002).

Mahmudah and Lola (2016) applied the fuzzy DEA approach to measure the Indonesian universities performances under imprecise inputs and outputs. Their empirical results show that 36% of universities perform efficiently under the constant returns to scale model. For the variable returns to scale model, 52% of the universities were efficient. They discovered that the well-known universities obtained relatively low scores indicating the need for them to improve their performances in publishing scientific work in addition to providing useful information to the public through the official websites. They concluded that the results of the VRS model are better than the CRS model for both the DEA and FDEA methods.

Tlig and Hamed (2017) accessed the efficiency of commercial Tunisian Banks using two approaches of fuzzy data envelopment analysis, namely, the possibility approach and the approach based on relations between fuzzy numbers (BRONF). They evaluated the
efficiency of the banks in terms of several crisp and imprecise data. Their results indicate that in a competitive environment, no-financial inputs and outputs should be considered in order to have credible and realistic efficiency scores.

Gökşen et al. (2015) used Data Envelopment Analysis to determine the performance levels of departments in Dokuz Eylul University (Turkey). Their study discussed the technical scores and scale scores of departments and revealed the main cause of inefficiency. The input and output goals of departments were fixed for a better efficiency.

Fatimah and Mahmudah (2017) performed a two-stage DEA for the purpose of measuring the efficiency of elementary schools in Indonesia in the period 2014/2015 using 34 DMUs. Their results show that VRS model gives better results than CRS model in the first stage. They further showed that 12 provinces in Indonesia have efficient elementary schools under the CRS model, while 17 provinces have efficient elementary schools under the VRS model. Their study discussed the technical scores and scale scores of departments and revealed the main cause of inefficiency. The input and output goals of departments were fixed for a better efficiency.

Karimi (2019) analysed the technical efficiency of elementary schools in all 33 districts of Rajasthan, India from 2014 to 2016 using Data envelopment analysis VRS model. The result showed high average technical efficiency in 2016 as against 2014 and 2015. The paper further provided evidence that some high literacy rate but low technical efficiency scores were found after comparing literacy rates and technical efficiency scores of the districts, indicating that high literacy rate does not necessarily mean that districts are technically efficient.

The rest of the paper is organized as follows: Basic models of DEA and (FDEA) fuzzy Data Envelopment Analysis models, and the suggested modification to the fuzzy (interval) DEA, are discussed in the second, third and fourth sections covering the theoretical aspect of the study. Section five deals with the application of the modified fuzzy DEA model and its comparison with the model by Wang et al., (2005). Section six presents the summary of results and concludes the work.

Basic Models of Data Envelopment Analysis

Many authors have studied the technique of data envelopment analysis. Originally, DEA was designed to measure the relative efficiency of non-profit organizations. Due to its ability to model multiple input and multiple output relationships without a priori underlying functional form assumption, data envelopment analysis has also been applied to other areas which are profit oriented (Zhu, 2003). Development of new methods and models have evolved due to wide application. This paper will however, present only Charnes, Cooper and Rhodes (CCR) and Banker, Charnes and Cooper (BCC) DEA models for the purpose of understanding the fundamentals of DEA.

Charnes, Cooper, and Rhodes DEA model

The CCR DEA model by Charnes et al. (1978) is given below in fractional form.

Objective function:

\[
\max h_0 = \frac{\sum_{r=1}^{m} u_r y_{ro}}{\sum_{i=1}^{n} v_i x_{io}}
\]

Subject to:

\[
\sum_{r=1}^{m} u_r y_{rf} \leq 1; \quad j = 1, 2, \ldots, n
\]

\[
u_r, v_i \geq 0; \quad r = 1, 2, \ldots, s; \quad i 1, 2, \ldots, m
\]  

Transformation of fractional CCR DEA model (1) into linear form:

Objective function:

\[
\max h_0 = \sum_{r=1}^{m} u_r y_{ro}
\]

Subject to:

\[
\sum_{i=1}^{n} v_i x_{io} = 1
\]

\[
\sum_{r=1}^{m} u_r y_{rf} - \sum_{i=1}^{m} v_i x_{ij} \leq 0; \quad j = 1, 2, \ldots, n
\]
\( u_r, v_i \geq 0; \ r = 1, 2, \ldots, s; \ i = 1, 2, \ldots, m \)  

Efficiency Frontier of the CCR DEA Model

The Banker, Charnes and Cooper DEA Model

The BCC model was introduced by Banker, Charnes and Cooper in 1984. It is an extension of the CCR model. The major difference between the two models lies in the establishment of returns to scale. While constant returns to scale is assumed in CCR which means that increase in inputs results to commensurate increase in outputs, variable returns to scale is assumed in BCC implying that increase in inputs does not result to commensurate increase in outputs. Accordingly, the BCC model is more robust than the CCR model (Zeidan et al., 2016). The CCR and BCC radial models are depicted pictorially in Fig. 1 and Fig. 2, respectively.

The BCC model in fractional form differs from the CCR model (1) by an additional variable as presented below:

Objective function:

\[
\max h_o = \frac{\sum_{r=1}^{s} u_r y_{ro} - c_o}{\sum_{i=1}^{m} v_i x_{io}} \\
\]

Subject to:

\[
\frac{\sum_{r=1}^{s} u_r y_{rj} - c_o}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1; \ j = 1, 2, \ldots, n \\
\]

\( u_r, v_i \geq 0; \ r = 1, 2, \ldots, s; \ i = 1, 2, \ldots, m \ c_o \text{ unrestricted in sign} \)  

Where the new variable \( c_o \) separates scale efficiency from technical efficiency in CCR model. Model (3) can be transformed into linear form as follows:

\[
\max h_o = \sum_{r=1}^{s} u_r y_{ro} - c_o \\
\sum_{i=1}^{m} v_i x_{io} = 1 \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - c_o \leq 0; \ j = 1, 2, \ldots, n \\
u_r, v_i \geq 0; \ r = 1, 2, \ldots, s; \ i = 1, 2, \ldots, m \ c_o \text{ unrestricted in sign} 
\]

If \( c_o = 0 \), then constant returns to scale (CRS) is implied.
If \( c_o < 0 \), then increasing returns to scale (IRS) is implied.
If \( c_o > 0 \), then decreasing returns to scale (DRS) is implied, (Zeidan et al., 2016).
Efficiency Frontier of the BCC DEA Model

Fuzzy Data Envelopment Analysis (FDEA)

The work of Sengupta (1992, 1993) incorporated fuzziness into DEA. He suggested two membership functions, “Linear Membership Function” and “Non-linear Membership Function”, for fuzzy mathematical programming model (Demir, 2014).

The approach to change fuzzy data into offset data using $\alpha$-level mass to create a solution that could take advantage of a family of classical DEA models was made by Kao and Liu (2000, 2003). Leveraging on the approach, Saati et al., (2002) made fuzzy CCR model as an offset programming model through defining it as a programming problem using $\alpha$-level. An improvement on interval data DEA was made by Wang et al., (2005) by employing DEA technique in the offset data and established a fuzzy efficiency measurement. Cooper et al., (1999) created interval data envelopment analysis model (IDEA). The IDEA model can change the non-linear programming problems into linear programming problem through scale conversions and variable changes (Demir, 2014). Using Wang et al’s technique, interval data programming model can be solved like a definitive linear programming model for each DMU and an efficiency score can be made by means of each $\alpha$-level (Deniz, 2009).

The technique of DMU with fuzzy data which could convert FDEA model into certain DEA model series was improved by Kao and Liu (2000).

Existing FDEA Linear Programming Formulation

Given that all inputs and outputs are incomplete as a result of uncertainties, let these values be known as $x_{ij}^L > 0$ and $y_{rj}^L > 0$ and $[x_{ij}^L, x_{ij}^U]$ and $[y_{rj}^L, y_{rj}^U]$ and they are between these top-down limits. To deal with such uncertain situation, Kao and Liu, 2000, Wang et al., 2005, Güneş, 2006, and Demir, 2014 defined FDEA model with fuzzy interval data in which limited data is used for efficiency measurement to generate upper and lower bounds for each DMU, as follows:

Upper Bound

$$\text{Max } h^U_0 = \frac{\sum_{r=1}^{s} u_r y_r^U}{\sum_{i=1}^{m} v_i x_i^L}$$

subject to:

$$\frac{\sum_{r=1}^{s} u_r y_r^L}{\sum_{i=1}^{m} v_i x_i^L} \leq 1; \quad j = 1, 2, \cdots, n$$

$$u_r, v_i \geq 0; \quad r = 1, 2, \cdots, s; \quad i = 1, 2, \cdots, m$$  (5)

Lower Bound

$$\text{Max } h^L_0 = \frac{\sum_{r=1}^{s} u_r y_r^L}{\sum_{i=1}^{m} v_i x_i^U}$$

subject to:

$$\frac{\sum_{r=1}^{s} u_r y_r^U}{\sum_{i=1}^{m} v_i x_i^U} \leq 1; \quad j = 1, 2, \cdots, n$$

Fig. 2: Efficiency Frontier of the BCC Model
\[ u_r, v_i \geq 0; \quad r = 1, 2, \ldots, s; \quad i = 1, 2, \ldots, m \]  

(6)

Observe that, in the fractional programming model (5), a mixture of the upper output values and lower input values were used to obtain the upper bound of the best possible relative efficiency of DMU0, \( h_0^U \). Similarly, for model (6), a mixture of the lower output values and upper input values were used to obtain the lower bound of the best possible relative efficiency of DMU0, \( h_0^L \). However, the ratio of upper and lower bound values cannot logically give rise to \( h_0^U \), neither can the ratio of lower and upper bound values logically give rise to \( h_0^L \). Hence the need for a modification.

THE SUGGESTED MODIFICATION TO FUZZY (INTERVAL) DEA MODEL

In this study, \( h_0^U \), the upper bound of the best possible relative efficiency of DMU0 is obtained by using the ratio of upper bound values for both the output and input interval data. For \( h_0^L \), the lower bound of the best possible relative efficiency of DMU0 is obtained by using the ratio of lower bound values for both the output and input interval data. Models (5) and (6) are therefore modified as follows:

Max \( h_0^U = \sum_{i=1}^{s} u_r y_{r0} \)  

subject to: \[ \frac{\sum_{r=1}^{m} u_r y_{rj}^{U}}{\sum_{i=1}^{s} v_i x_{ij}} \leq 1; \quad j = 1, 2, \ldots, n \]  

(7)

Max \( h_0^L = \sum_{i=1}^{m} v_i x_{0i} \)  

subject to: \[ \frac{\sum_{r=1}^{s} u_r y_{rj}^{L}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1; \quad j = 1, 2, \ldots, n \]  

(8)

In linear programming form, models (7) and (8) become:

Max \( h_0^U = \sum_{i=1}^{s} u_r y_{r0} \)  

Subject to: \[ \sum_{r=1}^{s} u_r y_{rj}^{U} \leq \sum_{i=1}^{m} v_i x_{ij}; \quad j = 1, 2, \ldots, n \]  

(9)

Max \( h_0^L = \sum_{i=1}^{m} v_i x_{0i} \)  

Subject to: \[ \sum_{r=1}^{s} u_r y_{rj}^{L} \leq \sum_{i=1}^{m} v_i x_{ij}; \quad j = 1, 2, \ldots, n \]  

(10)

Application and Comparison of Classical DEA Models, existing Fuzzy Models and the Modified Models

Models (9) and (10) will be solved by first transforming the crisp data into interval data using the approach of Demir (2014). Standard errors for each variable will be added to obtain the upper frontier data, while standard errors for each variable will be subtracted to obtain the lower frontier data. For the upper frontier efficiency scores, the upper frontier values of both the output and input data will be used. To obtain the lower frontier efficiency scores, the lower frontier values of both the output and input data will be used.

To evaluate and compare results from classical DEA models, existing interval DEA models and the modified interval DEA models; real data set of 25 high schools in the 2012 – 2013 education year is taken from Demir (2014). The data description is as follows: inputs (numbers of students, teachers and classes), outputs (Transition to Higher Education Examination (YGS), Undergraduate Placement Exam (LYS) success (placement) rates, YGS point averages, all points of the LYS Maths-Science (MS), Turkish-Maths (TM), and Turkish-Social (TS) Sciences (Zeiden et al., 2016). See Appendix 1. The DEA models are solved using DEA-SOLVER-LV8.
The efficiency values for the classical DEA input oriented CCR and BCC models are presented in Tables 1 and 2.

Table 1: Efficiency scores for classical CCR DEA model

| No. | DMU | Score  | Rank |
|-----|-----|--------|------|
| 1   | S1  | 0.9701 | 5    |
| 2   | S2  | 0.4515 | 13   |
| 3   | S3  | 0.4737 | 12   |
| 5   | S5  | 0.3693 | 17   |
| 6   | S6  | 0.5488 | 10   |
| 7   | S7  | 0.5404 | 11   |
| 8   | S8  | 0.9643 | 7    |
| 9   | S9  | 0.6646 | 8    |
| 10  | S10 | 0.6864 | 6    |
| 11  | S11 | 1      | 1    |
| 12  | S12 | 0.2179 | 23   |
| 13  | S13 | 0.1259 | 25   |
| 14  | S14 | 0.2196 | 22   |
| 15  | S15 | 0.2727 | 20   |
| 16  | S16 | 0.2097 | 24   |
| 17  | S17 | 1      | 1    |
| 18  | S18 | 0.9365 | 9    |
| 19  | S19 | 0.392  | 15   |
| 20  | S20 | 0.4203 | 14   |
| 21  | S21 | 0.2397 | 21   |
| 22  | S22 | 0.276  | 19   |
| 23  | S23 | 0.6759 | 7    |
| 24  | S24 | 0.362  | 18   |
| 25  | S25 | 0.3863 | 16   |

Table 2: Efficiency scores for classical BCC DEA model

| No. | DMU | Score  | Rank |
|-----|-----|--------|------|
| 1   | S1  | 1      | 1    |
| 2   | S2  | 1      | 8    |
| 3   | S3  | 1      | 1    |
| 4   | S4  | 0.5106 | 13   |
| 5   | S5  | 0.4087 | 15   |
| 6   | S6  | 0.7185 | 11   |
| 7   | S7  | 0.5732 | 12   |
| 8   | S8  | 1      | 1    |
| 9   | S9  | 1      | 1    |
| 10  | S10 | 1      | 8    |
| 11  | S11 | 1      | 1    |
| 12  | S12 | 0.2221 | 22   |
| 13  | S13 | 0.1272 | 25   |
| 14  | S14 | 0.2303 | 23   |
| 15  | S15 | 0.2742 | 20   |
| 16  | S16 | 0.2143 | 24   |
| 17  | S17 | 1      | 1    |
| 18  | S18 | 1      | 1    |
| 19  | S19 | 0.3925 | 17   |
| 20  | S20 | 0.4483 | 14   |
| 21  | S21 | 0.2992 | 21   |
| 22  | S22 | 0.2851 | 19   |
| 23  | S23 | 0.8265 | 10   |
| 24  | S24 | 0.3815 | 16   |
| 25  | S25 | 0.3934 | 16   |
Thorough examination of Tables 1 and 2 indicate that twenty-one units are inefficient with only four units efficient for the classical CCR model. For the classical BCC model however, sixteen units are inefficient while nine units are efficient. The most efficient units have efficiency score of 1 with equivalent ranking score of 1. These efficient units also serve as reference sets to the inefficient units.

The most inefficient unit, S13, has least efficiency score of 0.1272 and a ranking score of 25. It has efficient units S3, S11 and S17 as reference set (Lambda). In order words, it should emulate what these efficient units are doing in order to become efficient. Notice that each efficient unit serves as its own reference (Lambda).

| No. | DMU | Score  | Rank | Reference (Lambda) |
|-----|-----|--------|------|--------------------|
| 1   | S1  | 0.7789 | 6    | S11 1.053 S17 1.35 |
| 2   | S2  | 0.338  | 13   | S11 0.807 S17 0.988 |
| 3   | S3  | 1      | 1    | S3 1 |
| 4   | S4  | 0.3607 | 12   | S11 1.064 S17 0.528 |
| 5   | S5  | 0.2816 | 17   | S3 0.312 S11 0.409 S17 0.658 |
| 6   | S6  | 0.4388 | 11   | S3 0.161 S11 0.671 S17 0.5 |
| 7   | S7  | 0.4384 | 10   | S3 0.399 S11 0.014 S17 0.654 |
| 8   | S8  | 0.8131 | 5    | S3 0.074 S17 1.38 |
| 9   | S9  | 0.601  | 7    | S17 1.367 |
| 10  | S10 | 0.5172 | 9    | S3 0.331 S17 1.058 |
| 11  | S11 | 1      | 1    | S11 1 |
| 12  | S12 | 0.1634 | 22   | S3 0.246 S11 0.192 S17 0.537 |
| 13  | S13 | 0.0898 | 25   | S3 0.107 S11 0.616 S17 0.266 |
| 14  | S14 | 0.1569 | 23   | S11 0.365 S17 0.688 |
| 15  | S15 | 0.1979 | 20   | S17 0.279 S17 0.812 |
| 16  | S16 | 0.1556 | 24   | S3 0.044 S11 0.365 S17 0.595 |
| 17  | S17 | 1      | 1    | S17 1 |
| 18  | S18 | 0.8325 | 4    | S17 1.045 |
| 19  | S19 | 0.2997 | 15   | S3 0.141 S11 0.068 S17 0.741 |
| 20  | S20 | 0.2925 | 18   | S3 0.037 S17 1.022 |
| 21  | S21 | 0.1723 | 21   | S11 0.635 S17 0.394 |
| 22  | S22 | 0.2059 | 19   | S3 0.076 S11 0.238 S17 0.63 |
| 23  | S23 | 0.5264 | 8    | S11 0.362 S17 0.768 |
| 24  | S24 | 0.2848 | 16   | S3 0.675 S11 0.017 S17 0.378 |
| 25  | S25 | 0.3021 | 14   | S11 1.141 |
Table 4: Upper bound CCR efficiency scores

| No. | DMU | Score | Rank | Reference (Lambda) |
|-----|-----|-------|------|--------------------|
| 1   | S1  | 1     | 1    | S1 1               |
| 2   | S2  | 1     | 1    | S2 1               |
| 3   | S3  | 1     | 1    | S3 1               |
| 4   | S4  | 0.458 | 13   | S1 0.399 S3 0.366 S11 0.235 |
| 5   | S5  | 0.3369 | 14  | S1 0.248 S3 0.512 S8 0.091 S17 0.149 |
| 6   | S6  | 0.6481 | 11  | S1 0.199 S3 0.575 S11 0.226 |
| 7   | S7  | 0.4726 | 12  | S1 0.043 S3 0.381 S8 0.099 S17 0.477 |
| 8   | S8  | 1     | 1    | S8 1               |
| 9   | S9  | 1     | 1    | S9 1               |
| 10  | S10 | 1     | 8    | S10 1              |
| 11  | S11 | 1     | 1    | S11 1              |
| 12  | S12 | 0.1641 | 22  | S3 0.198 S11 0.245 S17 0.557 |
| 13  | S13 | 0.09 | 25   | S3 0.085 S11 0.641 S17 0.275 |
| 14  | S14 | 0.1608 | 23  | S1 0.038 S3 0.323 S17 0.639 |
| 15  | S15 | 0.2073 | 20  | S1 0.064 S3 0.026 S11 0.18 S17 0.731 |
| 16  | S16 | 0.1558 | 24  | S3 0.053 S11 0.348 S17 0.599 |
| 17  | S17 | 1     | 1    | S17 1              |
| 18  | S18 | 1     | 1    | S18 1              |
| 19  | S19 | 0.3025 | 17  | S3 0.041 S11 0.174 S17 0.785 |
| 20  | S20 | 0.2935 | 18  | S3 0.021 S8 0.282 S17 0.697 |
| 21  | S21 | 0.178 | 21   | S1 0.015 S3 0.078 S11 0.554 S17 0.353 |
| 22  | S22 | 0.0111 | 19  | S1 0.036 S3 0.026 S11 0.326 S17 0.674 |
| 23  | S23 | 0.7498 | 10  | S3 0.513 S17 0.487 |
| 24  | S24 | 0.3033 | 16  | S1 0.044 S3 0.672 S8 0.081 S17 0.203 |
| 25  | S25 | 0.3108 | 15  | S3 0.239 S11 0.551 S17 0.21 |

Table 5: Lower bound efficiency scores for the modified BCC model

Source: Demir 2014

In Table 3, the efficient DMUs due to the modified FDEA lower bound CCR model are the same as the efficient DMUs due to the FDEA upper bound CCR model in Table 4 from Demir (2014).
Table 6: Upper bound BCC efficiency scores

Source: Demir 2014

In Table 5, the efficient DMUs due to the modified FDEA lower bound BCC model are the same as the efficient DMUs due to the FDEA upper bound BCC model in Table 6 from Demir (2014).

Table 7: Upper bound efficiency scores for the modified CCR model

Reference(Lambda)
Table 8: Lower bound CCR efficiency scores

| No. | DMU | Score | Reference (Lambda) |
|-----|-----|-------|--------------------|
| 1   | S1  | 1     | 1                  |
| 2   | S2  | 1     | 1                  |
| 3   | S3  | 1     | 1                  |
| 4   | S4  | 0.5538| 13                 |
| 5   | S5  | 0.4635| 16                 |
| 6   | S6  | 0.7655| 11                 |
| 7   | S7  | 0.6588| 12                 |
| 8   | S8  | 1     | 1                  |
| 9   | S9  | 1     | 1                  |
| 10  | S10 | 1     | 1                  |
| 11  | S11 | 1     | 1                  |
| 12  | S12 | 0.2754| 22                 |
| 13  | S13 | 0.1615| 25                 |
| 14  | S14 | 0.2737| 23                 |
| 15  | S15 | 0.3598| 19                 |
| 16  | S16 | 0.2654| 24                 |
| 17  | S17 | 1     | 1                  |
| 18  | S18 | 1     | 1                  |
| 19  | S19 | 0.4931| 15                 |
| 20  | S20 | 0.5481| 14                 |
| 21  | S21 | 0.9292| 21                 |
| 22  | S22 | 0.3464| 20                 |
| 23  | S23 | 0.8672| 10                 |
| 24  | S24 | 0.4451| 18                 |
| 25  | S25 | 0.4583| 17                 |

Source: Demir 2014

In Table 7, the efficient DMUs due to the modified FDEA upper bound CCR model are the same as the efficient DMUs due to the FDEA lower bound CCR model in Table 8 from Demir (2014).

Table 9: Upper bound efficiency scores for the modified BCC model

| No. | DMU | Score |
|-----|-----|-------|
| 1   | S1  | 1     |
| 2   | S2  | 1     |
| 3   | S3  | 1     |
| 4   | S4  | 0.5538|
| 5   | S5  | 0.4635|
| 6   | S6  | 0.7655|
| 7   | S7  | 0.6588|
| 8   | S8  | 1     |
| 9   | S9  | 1     |
| 10  | S10 | 1     |
| 11  | S11 | 1     |
| 12  | S12 | 0.2754|
| 13  | S13 | 0.1615|
| 14  | S14 | 0.2737|
| 15  | S15 | 0.3598|
| 16  | S16 | 0.2654|
| 17  | S17 | 1     |
| 18  | S18 | 1     |
| 19  | S19 | 0.4931|
| 20  | S20 | 0.5481|
| 21  | S21 | 0.9292|
| 22  | S22 | 0.3464|
| 23  | S23 | 0.8672|
| 24  | S24 | 0.4451|
| 25  | S25 | 0.4583|

Reference (Lambda):
- S1: 0.399, S3: 0.366, S11: 0.235
- S2: 0.25, S8: 0.143, S17: 0.133
- S3: 0.285, S8: 0.04, S17: 0.44
- S4: 0.399, S3: 0.366, S11: 0.235
- S5: 0.25, S8: 0.143, S17: 0.133
- S6: 0.285, S8: 0.04, S17: 0.44
- S7: 0.285, S8: 0.04, S17: 0.44
- S8: 0.143, S17: 0.133
- S9: 0.285, S8: 0.04, S17: 0.44
- S10: 0.285, S8: 0.04, S17: 0.44
- S11: 0.285, S8: 0.04, S17: 0.44
- S12: 0.285, S8: 0.04, S17: 0.44
- S13: 0.285, S8: 0.04, S17: 0.44
- S14: 0.285, S8: 0.04, S17: 0.44
- S15: 0.285, S8: 0.04, S17: 0.44
- S16: 0.285, S8: 0.04, S17: 0.44
- S17: 0.285, S8: 0.04, S17: 0.44
- S18: 0.285, S8: 0.04, S17: 0.44
- S19: 0.285, S8: 0.04, S17: 0.44
- S20: 0.285, S8: 0.04, S17: 0.44
- S21: 0.285, S8: 0.04, S17: 0.44
- S22: 0.285, S8: 0.04, S17: 0.44
- S23: 0.285, S8: 0.04, S17: 0.44
- S24: 0.285, S8: 0.04, S17: 0.44
- S25: 0.285, S8: 0.04, S17: 0.44

Modified Fuzzy Data Envelopment Analysis Models.
Table 10: Lower bound BCC efficiency scores

| DMU | Score | α | Efficient | α | Efficient | α | Efficient | α | Efficient | α | Efficient |
|-----|-------|---|-----------|---|-----------|---|-----------|---|-----------|---|-----------|
| F1  | 100.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F2  | 100.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F3  | 100.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F4  | 55.56 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F5  | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F6  | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F7  | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F8  | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F9  | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F10 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F11 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F12 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F13 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F14 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F15 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F16 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F17 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F18 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F19 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F20 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F21 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F22 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F23 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F24 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| F25 | 62.50 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Source: Demir 2014

In Table 9, the efficient DMUs due to the modified FDEA upper bound BCC model are the same as the efficient DMUs due to the FDEA lower bound BCC model in Table 10, from Demir (2014).

Table 11: Comparison of DEA, FDEA and modified FDEA Efficiency Results via CCR

| Classical DEA | Lower and upper efficient DEA (Demir) | Lower and upper efficient DEA (Modified) |
|---------------|----------------------------------------|----------------------------------------|
| S3            | S1                                     | S3                                     |
| S8            | S3                                     | S11                                    |
| S17           | S11                                    | S17                                    |

Results and Conclusion

Results of the efficient decision making units (DMUs) due to classical DEA models, fuzzy DEA models proposed by Wang et al., (2005) and adopted by Demir (2014) and the modified fuzzy DEA models are presented in summary form in tables 11 and 1.

Table 12: Comparison of DEA, FDEA and modified FDEA Efficiency Results via BCC

| Classical DEA | Lower and upper efficient DEA (Demir) | Lower and upper efficient DEA (Modified) |
|---------------|----------------------------------------|----------------------------------------|
| S1            | S1                                     | S1                                     |
| S2            | S2                                     | S2                                     |
| S3            | S3                                     | S3                                     |
| S8            | S8                                     | S8                                     |
| S9            | S9                                     | S9                                     |
| S10           | S10                                    | S10                                    |
| S11           | S11                                    | S11                                    |
| S17           | S17                                    | S17                                    |
| S18           | S18                                    | S18                                    |
Table 11 presents the efficient DMUs from the three DEA models. A major finding in the case of CCR, when the results of Demir and that of the modified model are compared is that, the efficient DMUs when the upper bound model (Model 5) is applied, corresponds to the efficient DMUs when the lower bound modified model (Model 10) is applied. Similarly, when the lower bound model (Model 6) is applied, the result corresponds to that of the upper bound modified model (Model 9).

The implication of this finding is that, the ambiguity created by the mixture of upper bound and lower bound values to generate efficiency scores in each of Models 5 and 6 can be avoided. Instead, the modified Models 9 and 10, where upper bound values are used exclusively to generate upper efficiency scores and lower bound values are used exclusively to generate lower efficiency scores can be adopted to avoid the ambiguity.

In the case of BCC, Table 12, the efficient DMUs are the same for all the models compared. This is not unexpected since BCC is more robust and adopts variable returns to scale (VRS) frontier as against the more restrictive CCR which adopts the constant returns to scale (CRS) frontier.

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